Paper notes

Target: BIGSSS Research Day

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In the agent-based model, an agent chooses between current location and one alternative location according to the ethnic utility and value utility attributed to neighborhoods. The aim is to fit the random utility framework for a discrete choice model and advance on the ACS paper (Paolillo and Lorenz 2018). I liked Bruch and Mare (2009)'s description disentangling between utility, probability and how they relate in the empirical data / agent-based models relation.

Probability that an agent will choose the alternative relies on the difference between the utility of the two neighborhood options:

$$P_{alterntive} = U_{alternative} - U_{current}$$

Where:

- $P_{alternative} = 1$ means moving to alternative, implying its utility is higher than the current one
- $P_{alternative} = 0$ means staying on the current location, implying its utility is higher than alternative

Definition of utility for a neighborhood, and its implication for the probability to relocate, fits the random utility framework:

$$U^i_{(ej,yj)} = \beta_e e_j + \beta_v v_j + \epsilon_j$$

With β_e being the importance of ethnic composition e_j of neighborhood j, β_v the importance of its value composition v_j, and ϵ_j the random term. The random term refers to other characteristics of neighborhoods or of decision makers that can influence the utility and probability of choice, but that are totally unknown. β regulates the systematic component of utility, and ϵ the random component of utility (Bruch and Mare 2009). Random component is assumed to be independent and identically distributed (i.i.d.) among options, while the systematic component of utility depends on the formalized functional form and how it interacts with β values. This condition allows that the higher is the parameter $\beta \in [0, \infty)$, the lower is the effect of random term and the more the utility derived from one option depends on their characteristics (here ethnic and value composition) and how they interact with the functional forms of utility implemented. The probability to choose one option over the others relies on the difference of utility so computed, with the option with the highest utility being selected. As a consequence, for high $\beta \in [0, \infty)$, the probability will depend on the systematic component of utility, selecting the best option. For low levels of β, the probability will be random due to the higher component of the random term.

In this implementation of the model, $beta_x$ follows a beta distribution. To avoid confusion between terms of parameter beta and beta distribution used in the simulation, β_x is called ethnic weight ew (e_w in R code), β_y is called value weight vw (v_w in R code), with $v_w = 1 - e_w$.

Following Zhang (2011), and also Bruch and Mare (2006), we want to implement and compare the consequences of adopting different functional forms in the definition of utility as individual behavior of agents. In particular, we want to compare a range of functional forms of individual behavior spanning from the single-peaked function as in Zhang (2011 graphic p. 174) to the threshold function as in the original Schelling's model and that we used in the ACS paper (Paolillo and Lorenz 2018). The idea was a third function as a combination between linear growing as in Zhang (2011) until the ideal point (proportion of similars) is reached, and then constant for higher proportions that include the ideal point as it would be in Schelling.

Since the task was to reduce number of parameters as much and reproduce the functions we need, I modified Zhang (2011)'s formula, assuming max Utility = 1 and introducing S parameter that combined with M generates the functions we need:

- $S[0,1]$, regulates the increasing left slope
- $M[0,1]$, regulates the decreasing right slope (same in Zhang)

Ethnic Utility

$$U_e^i = \begin{cases} 1, & \text{if } x = n \times i_e, \\ \\ (\frac{x}{n \times i_e}) \times S, & \text{if } x < n \times i_e, \\ \\ M + \frac{(1-\frac{x}{n}) \times (1-M)}{1-i_e}, & \text{if } x > n \times i_e \end{cases}$$

Where:

- U_e^i: Ethnic utility of agent i
- x: number of similar agents in the neighborhood
- n: total number of agents in the neighborhood
- i_e: desired ethnic composition

Fig.1 reports the example of the three functions, and how S and M are set to calculate them. The example refers to desired compostion 50% ($i_e = 0.5$). It was calculated out of a 8 agents neighborhood (with 4 agents of same ethnicity representing the 50%), I changed the units on x-axis to percentage.

IDEA/DOUBT: A fourth function we could include is for S=0 and M=0. In a computational framework,the aim of the paper is to explore and formalize the consequences of different functional forms, which represent different behaviors of how people attribute utility to a neighborhood, whatever might be the reason for this. Following this I think there would be no need to exclude one scenario out of the set and would be more sound to compare the four exteme cases. As in fig. 1:

- S=1,M=0, the symmetric single-peaked function with ideal point: people have a preference they value maximally, and devalue equally options that are below or above such preference
- S=1,M=1, people increasingly value options below their ideal point as long as they get closer to it, and are equally happy for those options that include and guarantee the ideal point
- S=0,M=1, the original threshold Schelling: people have a threshold preference, devalue and refuse (Utility = 0) options below it and are equally happy for those above it
- S=0,M=0 (to add): people devalue and refuse any option below their ideal point (Utility = 0), and decreasingly devalue those options that exceed their ideal point

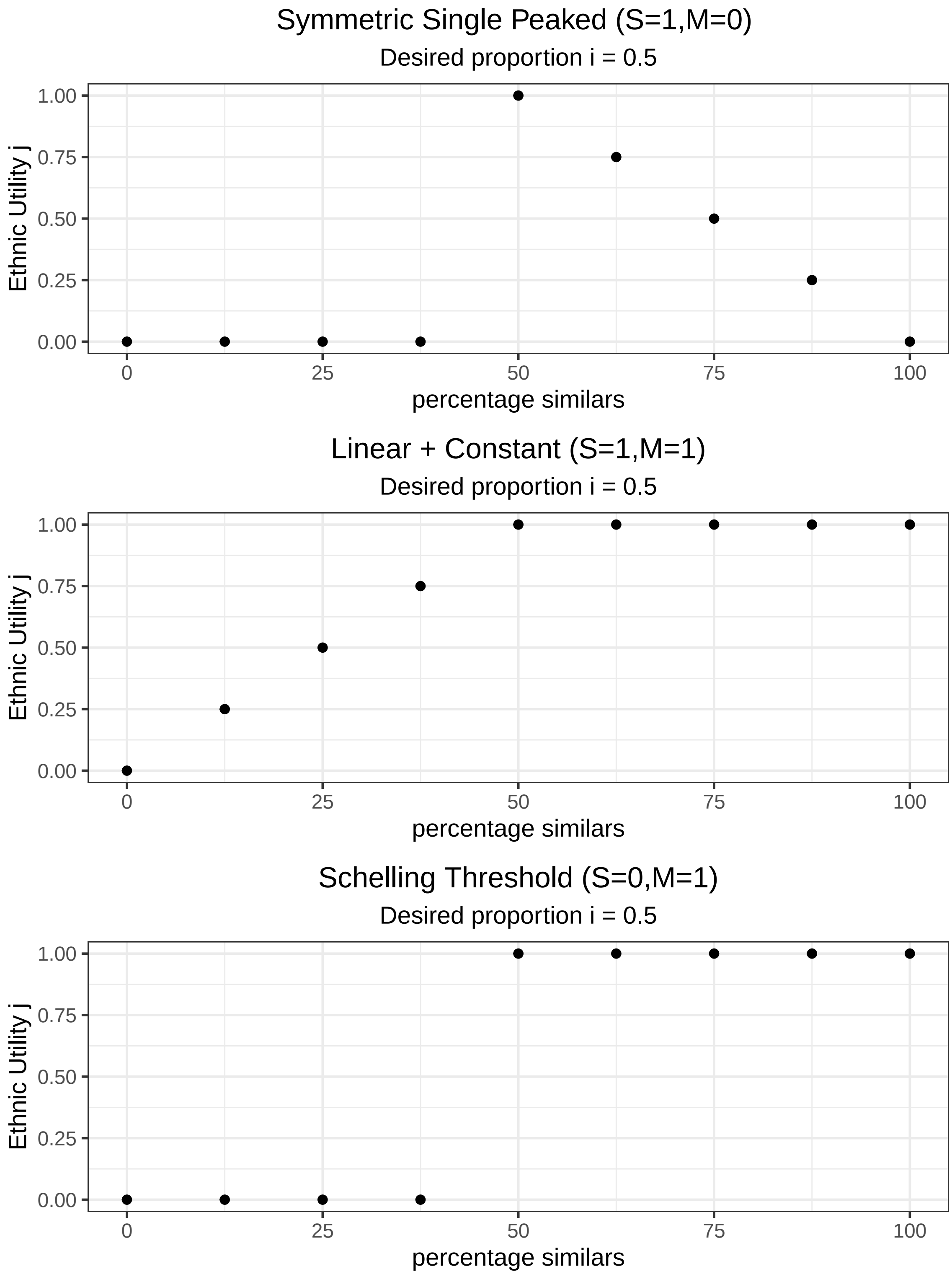

Figure 1: Ethnic Utility Function

In the ANNEX here you can see the function slope and values of ethnic utility given some parameteres. In the R-code you can set the parameters and test, e.g. comparing with NetLogo

Value Utility

The idea we discussed with Jan about value utility was to use the ethnic weight ew of each agent and how distant it is from the average of ethnic weights of other agents in the potential neighborhood. To do so, I include the parameter $i_v \in [0, 1]$, meaning the absolute desired distance between the agent's ethnic weight and the average of ethnic weights in its neighborhood. Values range from 0: only similars in the neighborhood are desired, to 1: any possible extrem distance is accepted. Here the ideal point becomes an interval, with average points outside of the ideal interval being equally distant. The function for ethnic utility as above with left increasing slope and right decreasing slope would not work. I elaborated another function using the same parameters S and M:

- $S[0, 1]$, regulates the extreme of the slope closer to the ideal point
- $M[0, 1]$, regulates the extreme of the slope further the ideal point

Fig.2 reports the 2 possible functions:

- S=1,M=0. Symmetric single peaked. Has a similar shape as ethnic utility, but due to different computation
- S=0, independent from the value of M: Schelling's Threshold function

$$U_v^i = \begin{cases} 1, & \text{if } \mid \overline{ew}_j - ew_i \mid \leq i_v, \\ \left(\frac{(1-|\overline{ew}_j - ew_i|)}{1-i_v}\right) \times S_v, & otherwise \end{cases}$$

Where:

- U_v^i: Value utility of agent i
- $\overline{ew}_j$: average ethnic weight in neighborhood j
- ew_i: ethnic weight of agent i
- i_v: desired value distance of agent i

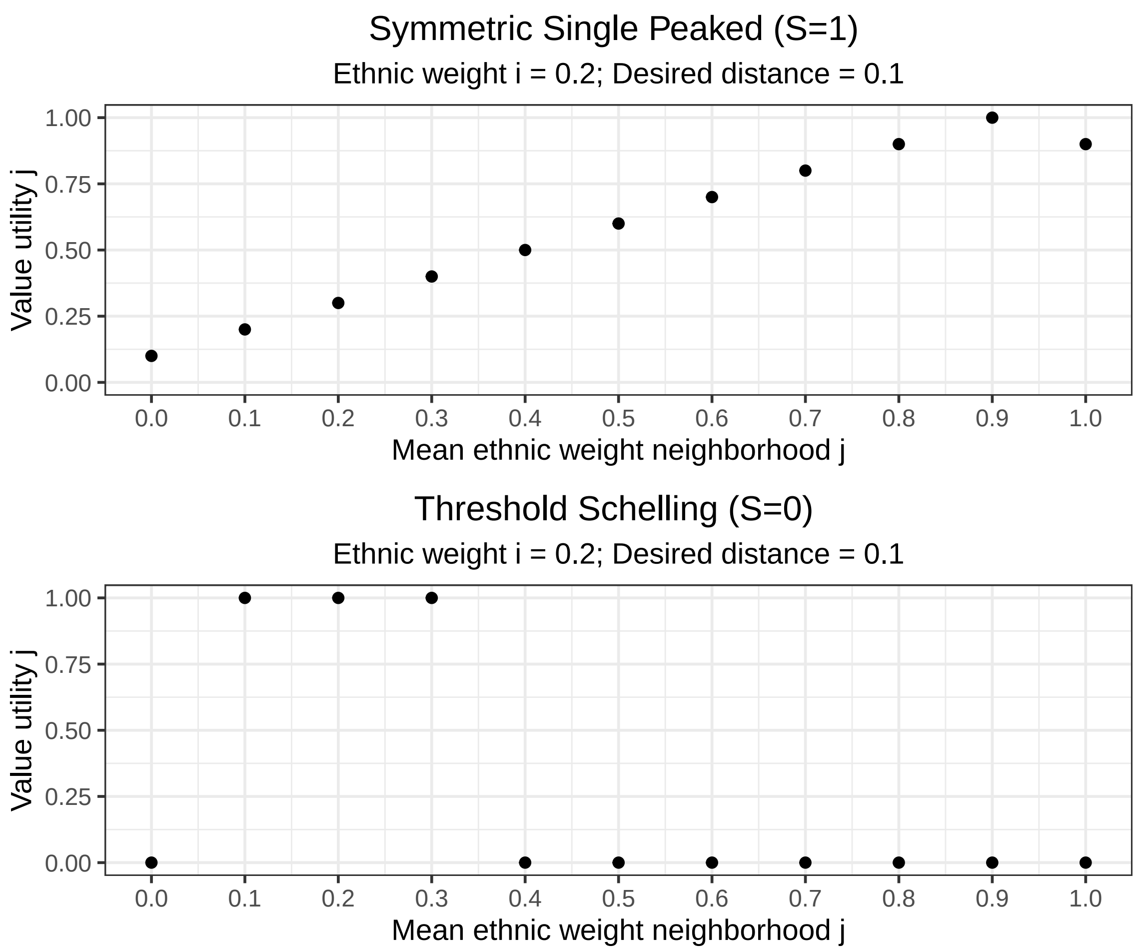

Figure 2: Comparison of the two function for the value utility of a neighborhood j

In the ANNEX here you can see the function slope and computation of value utility given some parameteres. In the R-code you can set the parameters and test.

DOUBT/IDEA: I have doubts with this definition of value utility (distance between agent and average ethnic weights of neighborhood), conceptually and for the computation. I will come back after implementing as above. As an extension of Schelling's model, I would focus on how similarity can be defined at the individual level between people as the distance between values/opinions, opposite to Schelling's condition of a "twofold, exhaustive and recognizable distinction" (Schelling 1969, 488). Defined the different types of similarity, the mechanism of threshold preferences and cascades would be the same, exploring how the value segregation and ethnic segregation interplay. This was the point of the ACS paper, I think it would better fit the ethnic boundary making's framework (Wimmer 2013) and Esser (2010)'s point along with the value homophily we mentioned in the ACS (McPherson, Smith-Lovin, and Cook 2001), and my dissertation project. In this terms, in the ACS there were 2 levels of values (tolerants vs intolerants) and related preferences (value-oriented vs ethnicity-oriented), with the ideal distance of value-oriented tolerants equal to 0 (similars are only other tolerants). Here we expand this and extend the residential preferences. For the number of parameters and simplicity, the desired distance i_v already must be included (it would only shift to distance between individual agents); to add there would be one continuous distribution of the values across the population (stylized parameter for norms, political orientations etc. in a population across the ethnic groups). The same functional form as for ethnic utility would then be used, making it more simple and any effect related to average computation/functional form avoided, focusing on the proportion as in Schelling. A simplest way is to use the distance between ethnic weights of agents as "values". I also have some doubts here both because they should be parameters of the choice and not definition of a person (Manski 1977), plus literature and choices of the model. To not overwhelm and create confusion now I will come to it later when more clear.

Probability to relocate to alternative

Calculated the value utility and ethnic utility, the agent chooses between current location and the alternative location. The probability is calcualted using the logistic function. We decided to limit the number of options to 2, both for computational reasons and because there is not need to do otherwise. This is the condition for the logit model for binary discrete choice model with two options (Liao 1994). The dependent variable in such a model is the probability that an event will occur ($P = 1$), or not ($P = 0$). In a residential choice model and for two options as the agent-based model here, the agent must make a choice between moving to alternative location ($P = 1$) or not ($P = 0$), where the latter outcome implies to stay on the current location. Fitting the paradigm of maximization of utility, the probability to relocate to alternative increases if the utility derived from the alternative location is higher than the current one, as described above from Bruch and Mare (2009). Translated in the logistic function:

$P_j = \frac{1}{1+\exp(-\lambda\times\Delta_{(U_j,U_i)})}$

An alternative to the logit model is the multinomial choice model, which is a generalization of the binary model where the choice is between more than two options (Train 2009). The probability to choose one option over the others is calculated according to the conditional logit model (McFadden 1994), depending on the characteristics of the options. Applied to utility maximization and random utility models (McFadden 1994; Van de Rijt, Siegel, and Macy 2009):

$P_j = \frac{\exp(U_j)}{\sum exp((U_k))}$

$P_j = \frac{1}{1+\exp((-\beta_e(e_j-e_i))+(-\beta_v(v_j-v_i))}$

$P_j = \frac{\exp(U_{(xj,yj)})}{\sum_l exp((U_{xl,yl}))}$

$P_j = \frac{1}{1+\exp((-\beta_x e_j-\beta_x e_i)}$

In Manski for probability to stay on current cell i$P_i = \frac{\exp(\beta_e e_j+\beta_v v_j)}{\exp(\beta_e e_j+\beta_v v_j)+\exp(\beta_e e_i+\beta_v v_i)}$

$P_i = \frac{1}{1+\exp((\beta_e e_j+\beta_v v_j)-(\beta_e e_i+\beta_v v_i))}$

$P_j = \frac{1}{1+\exp((-\beta_e(ej-ei))+(-\beta_v(vj-vi)))}$

To my understanding, the two forms to calculate probability with logit model (logistic function) or multinomial choice (conditional logit), would be equivalent in representing a random utility model, as long as the number of options is limited to 2. This is the case of our model where the choice of the agent is between the current location and an alternative location. Both guarantee that the probability computed is between 0 and 1, that the maximization of utility drives the agent to choose one option over the other. Both model a systematic component of utility and a random component. Both assume that the random term ϵ is independent and equally distributed among options (i.i.d.). This link synthetizes: logit_binary_choice, point C

The condition of random terms as independent and identically distributed

The random utility frame assumes an extreme-value distribution (Gumbel distribution) of the unknown random terms to fit this assumption. The difference of two Gumbel-distributed random variables has a logistic distribution, whose probability density function is the logistic function. The logistic function in the logit model implemented in this agent-based model calculates probability based on the difference of the two random utility

The implementation of the logit model here in the agent-based model fits this condition, since the probability to

Since the logit model as formulated in the agent-based model uses the difference in the utility between the two options,

(need of the exponential function), in the conditional logit through the Gumbel distribution, in the

have in our model where th

should be equivalent in representing a random utility model and what we need. They both are probabilities ranging from 0 to 1, the characteristic of the option regulates the probability to be selected, i.e. the utility derived from the alternative neighborhood.

This link synthetizes: logit_binary_choice, point C

Here is the logistic function implemented in the model for the choice between current location and alternative location:

$$P_j = \frac{1}{1 + \exp((-ew \cdot k \times \Delta_{U_e}) + (-vw \cdot k \times \Delta_{U_v}))}$$

Where:

- P_j: Probability to relocate to alternative neighborhood j leaving the current neighborhood
- ew: Ethnic weight of agent i
- Δ_{U_e}: Ethnic Utility Difference (alternative - current)
- vw: Value weight i
- Δ_{U_e}: Value Utility Difference (alternative - current)
- k: constant

P_j thus ranges from 0 = the agent remains in its current location, to 1 = the agent moves to the alternative location.

which uses the conditioanl logit to calculate the probability to choose j ($P_j = \frac{\exp(U_{(xjyj)})}{\sum_l exp((U_{xl,yl}))}$). As I wrote and as I understand so far, for two options the calculation of probability according to the multinomial choice formulation and the logistic function for binary choice can be considered equivalent as they both will assure the $i.i.d$ distribution of the random term, which is the priority of random utility models.\

k times ethnic weight ew and k times vw substitute the parameters in random utility for how important the ethnic composition and value composition are in the logistic function (λ in $P_j = \frac{1}{1+\exp(-\lambda \times \Delta_{U_e})+(-\lambda \times \Delta_{U_v}))}$ and $\beta_x e_j$ $\beta_y v_j$ in j ($P_j = \frac{\exp(U_{(xjyj)})}{\sum_l exp((U_{xl,yl}))}$. My idea is that ethnic weight and value weight range from 0 to 1, whose value is too low causing random choices, by multiplying by a constant the ratio in the distribution point stay the same.

Below I report some graphs where I implement the logistic function as above tuning $k \cdot ew$ and $k \cdot vw$, checking if it would reproduce what expected from random utility models, I think it does. For parameters equal to 0, the probability to choose the alternative is 50 in all conditions. The best option is taken when parameters are increased, always 50 when the actual difference is 0. I tested some conditions considering the 2 types of utility (some would be not possible in the simulation if we keep $vw = 1 - ew$, but here was to see how it worked).

On the x-axis is ethnic utility difference, on the y-axis the value utility difference. Each ranges from -1 (alternative = 0, current = 1), to 1 (alternative = 1, current = 0), with 0 = no difference between the options. I made up them out of possible conditions. For each condition (21x21= 441), the probability function is calculated using the logisticdunction above. Heatmaps in fig. 3

In the ANNEX of pdf is the sigmoid function for the logistic function, too long. Each vignette is one level of ethnic utility difference, the x-axis is the whole value utility difference range. You can manipulate ew, vw and k in the R code.

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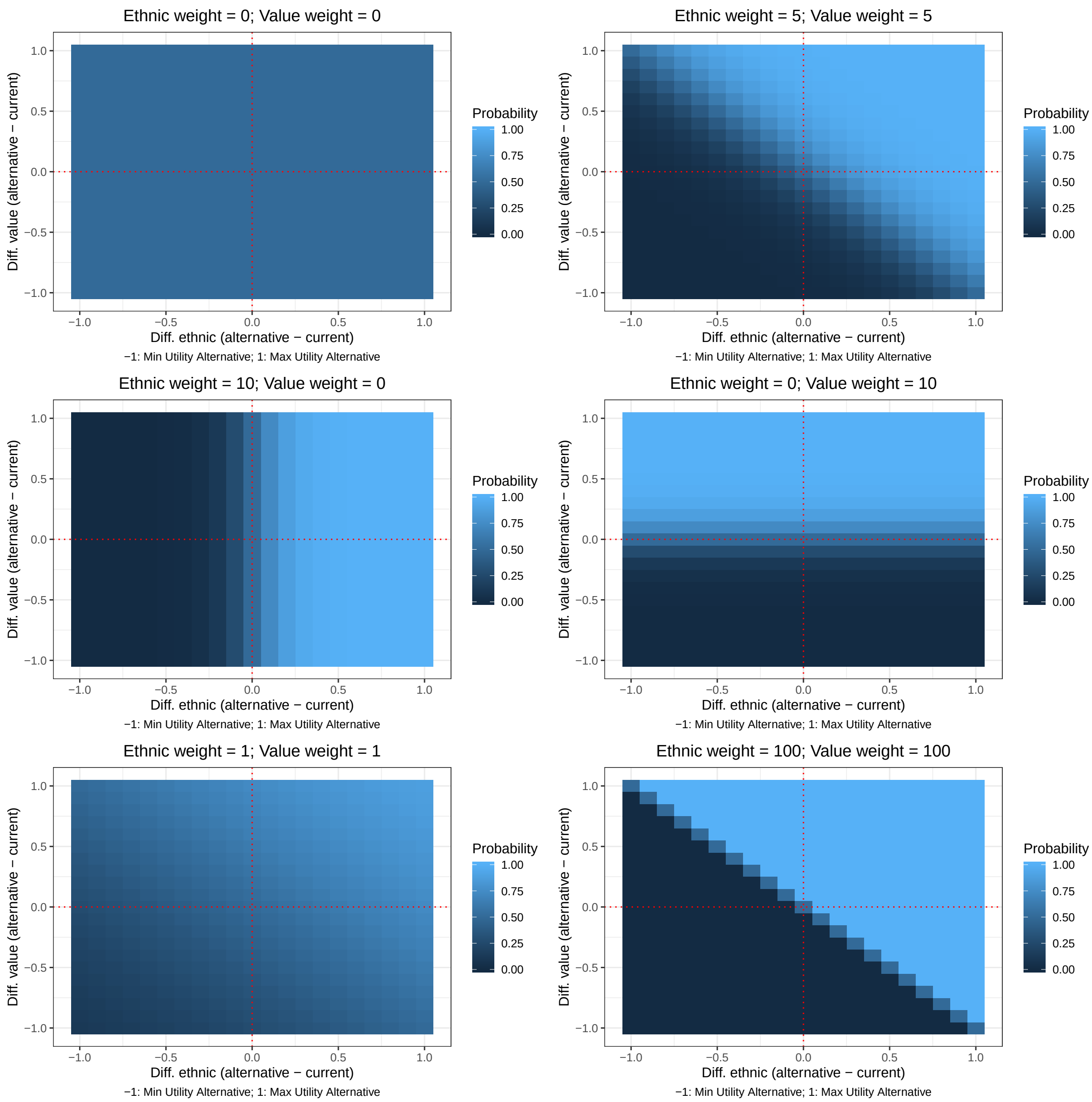


Figure 3: Probability to choose alternative location

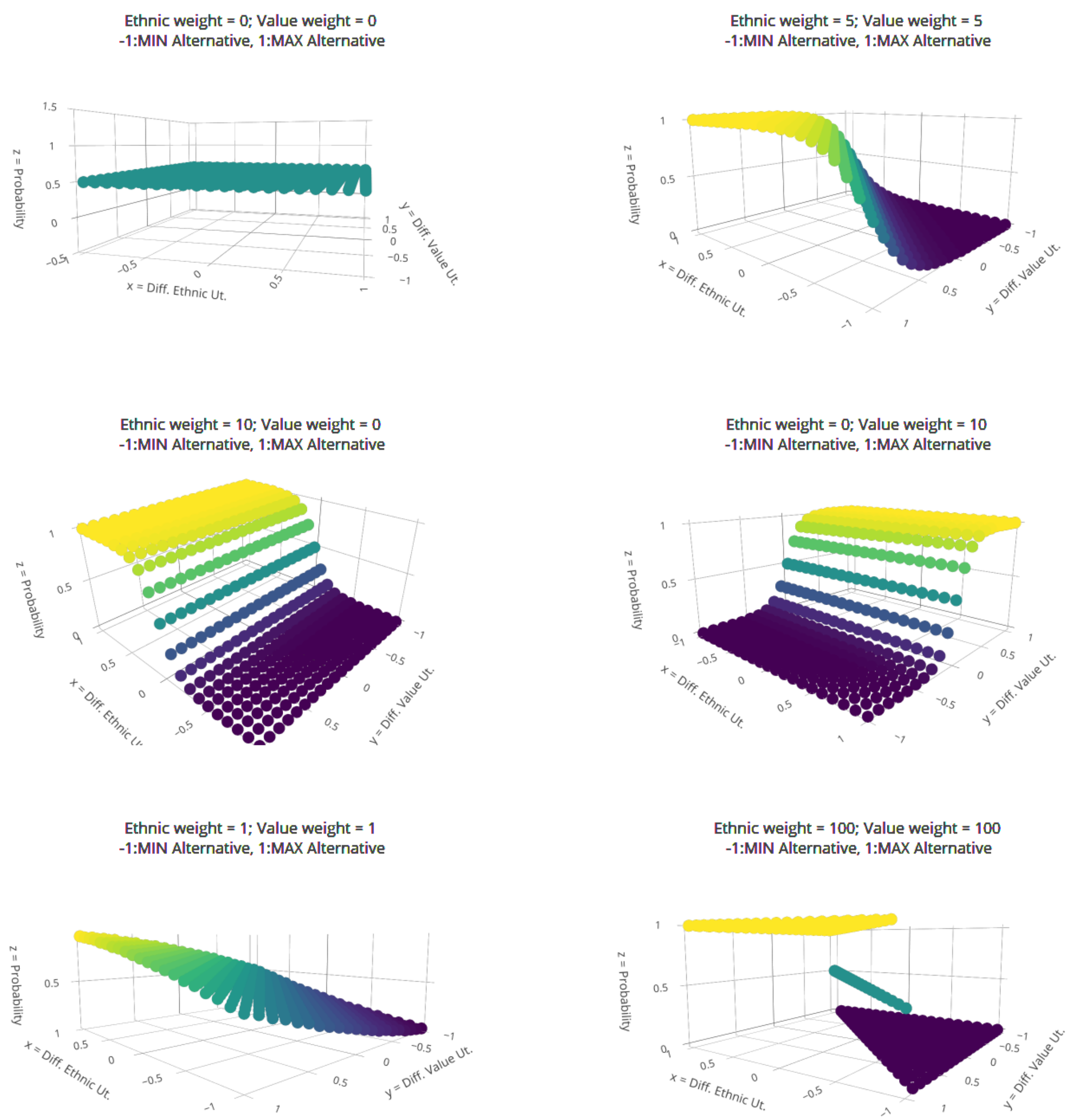


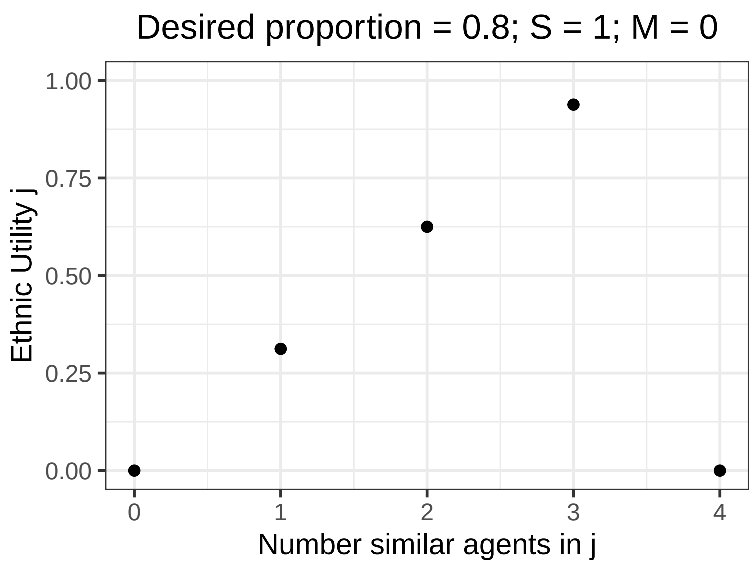
Figure 4: Probability to choose alternative location 3D

ANNEX

0.1 Single Run

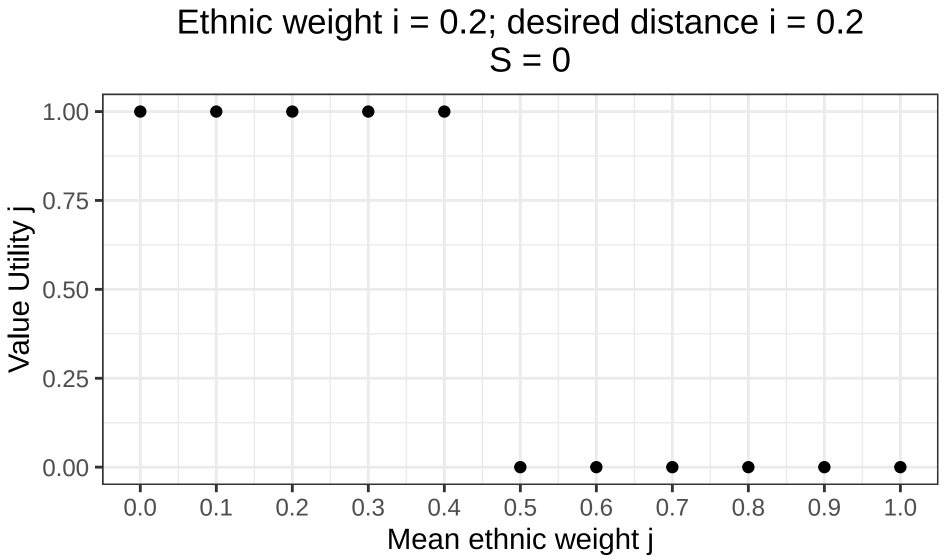
param_eth	param_val	i_e	i_v	S	M	dist_curr	dist_alt	conc_curr	conc_alt	uti_eth_curr	uti_eth_
1.75	0.25	0.5	0.1	1	0	0.058	0.031	0.625	0.5714286	0.75	0.857

0.2 Ethnic Utility Calculation



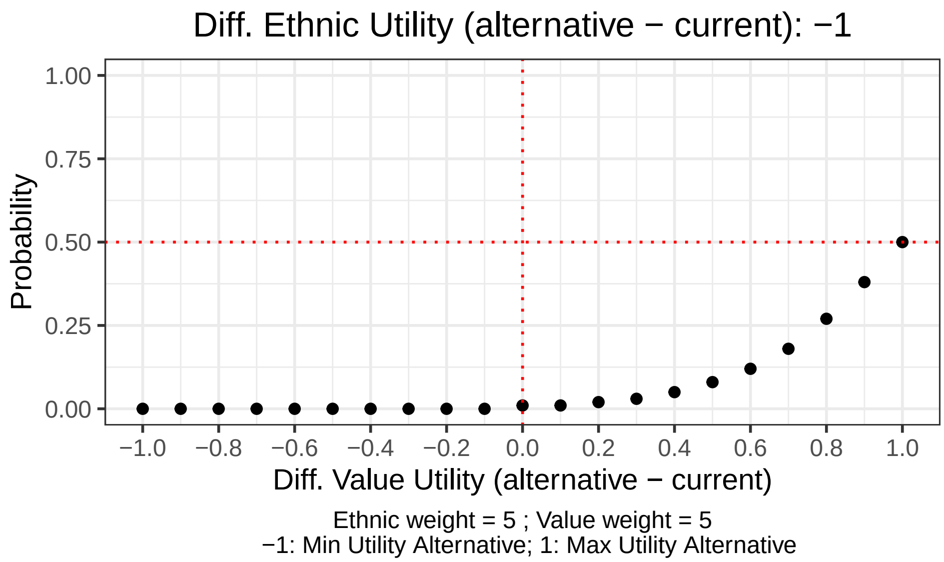
M.	S.	prop_des	num_tot	num_sim	Utility_E
0	1	0.8	4	0	0.000
0	1	0.8	4	1	0.312
0	1	0.8	4	2	0.625
0	1	0.8	4	3	0.938
0	1	0.8	4	4	0.000

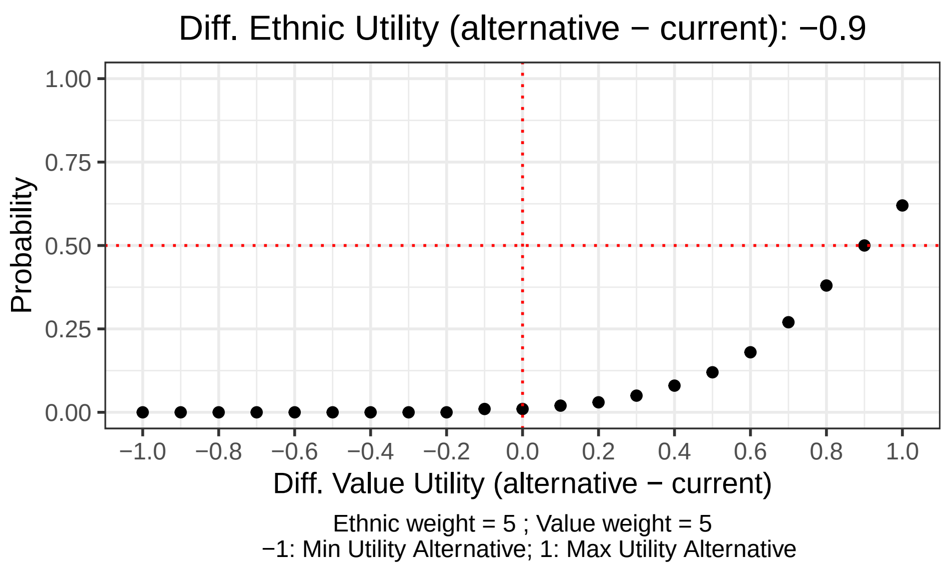
Value Utility Calculation

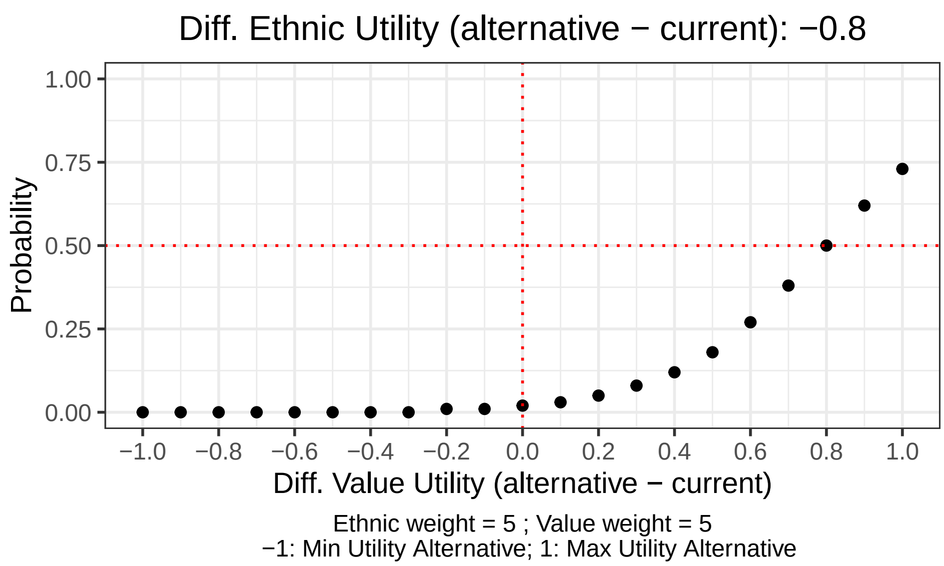


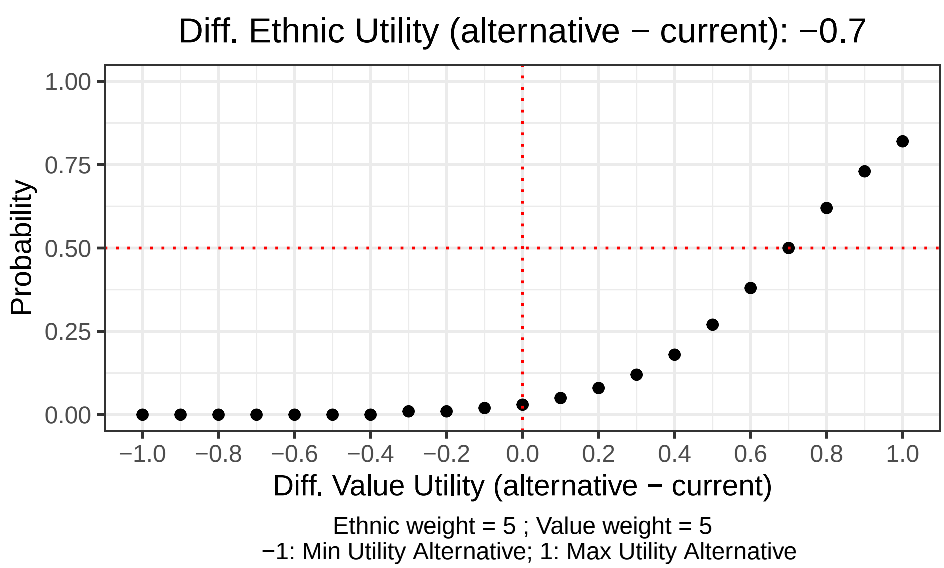
S_v	des_dist	ew_agent	mean_ew_ng	Utility_V
0	0.2	0.2	0.0	1
0	0.2	0.2	0.1	1
0	0.2	0.2	0.2	1
0	0.2	0.2	0.3	1
0	0.2	0.2	0.4	1
0	0.2	0.2	0.5	0
0	0.2	0.2	0.6	0
0	0.2	0.2	0.7	0
0	0.2	0.2	0.8	0
0	0.2	0.2	0.9	0
0	0.2	0.2	1.0	0

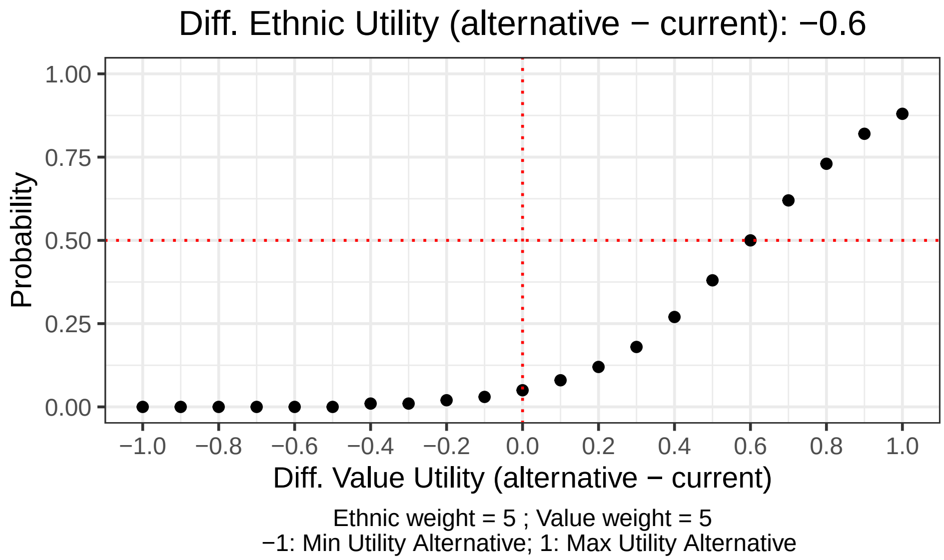
Logistic Function calculation

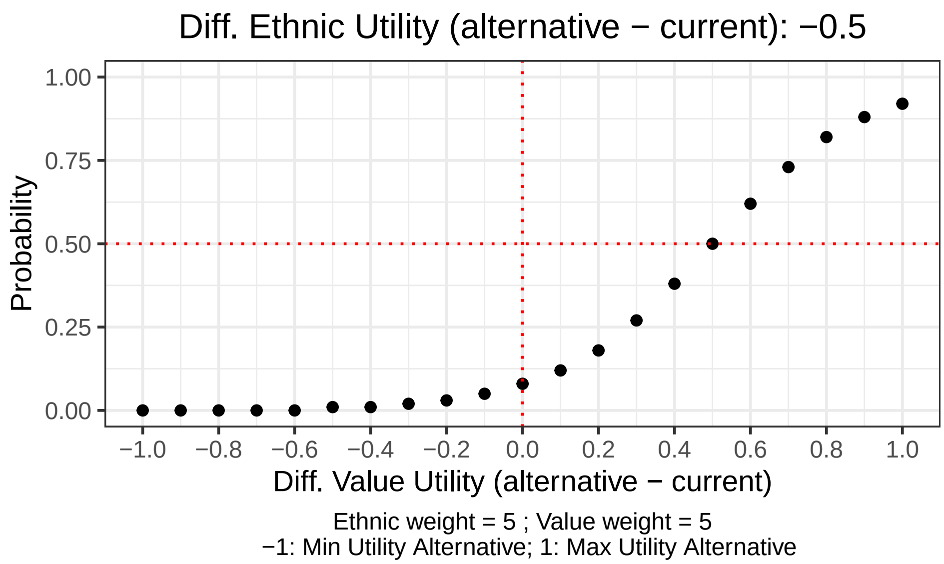


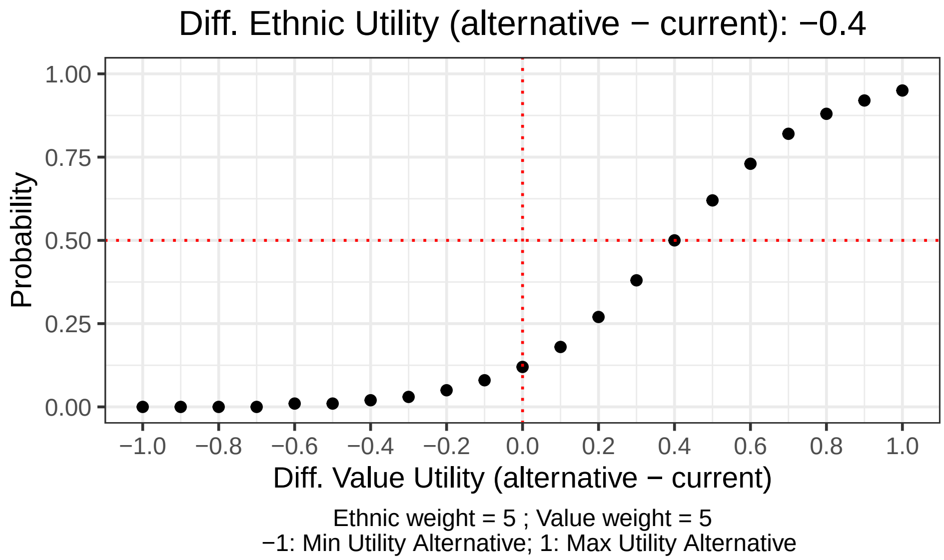


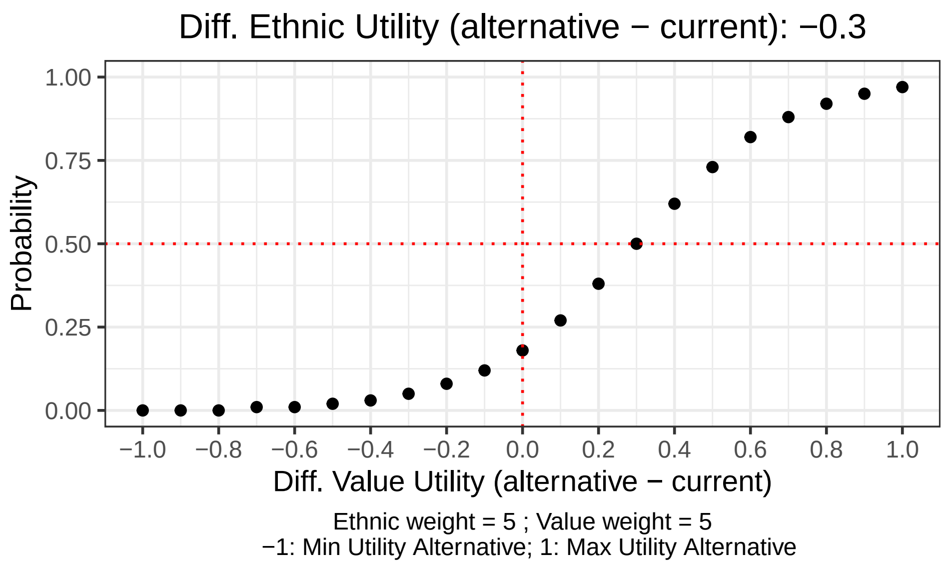

Diff. Ethnic Utility (alternative − current): −0.8
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative

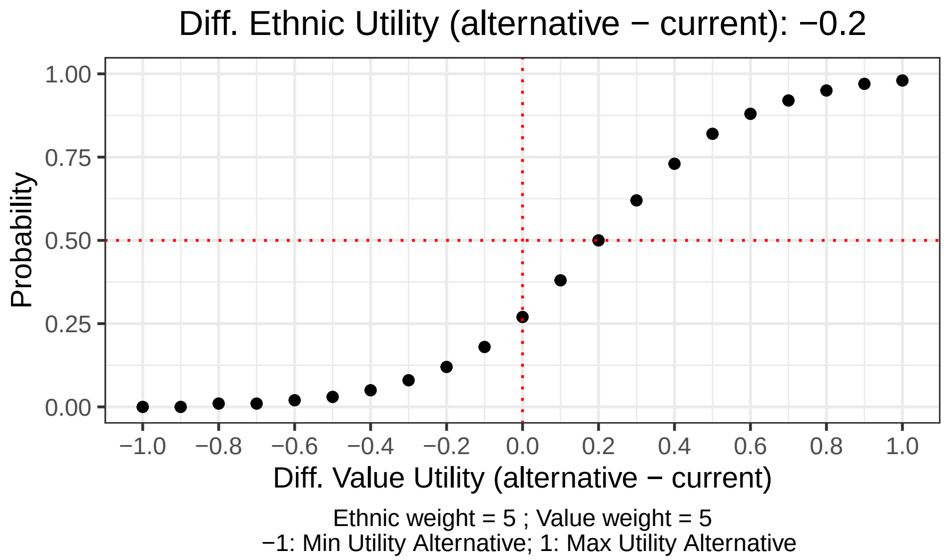

Diff. Ethnic Utility (alternative − current): −0.7
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): −0.6
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): −0.5
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): −0.4
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): −0.3
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): −0.2
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
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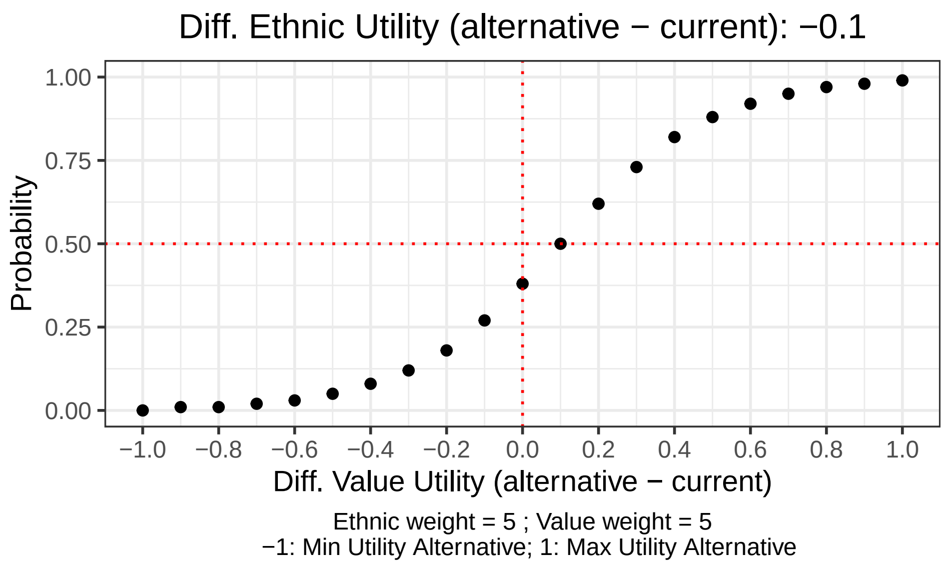
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): −0.1
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
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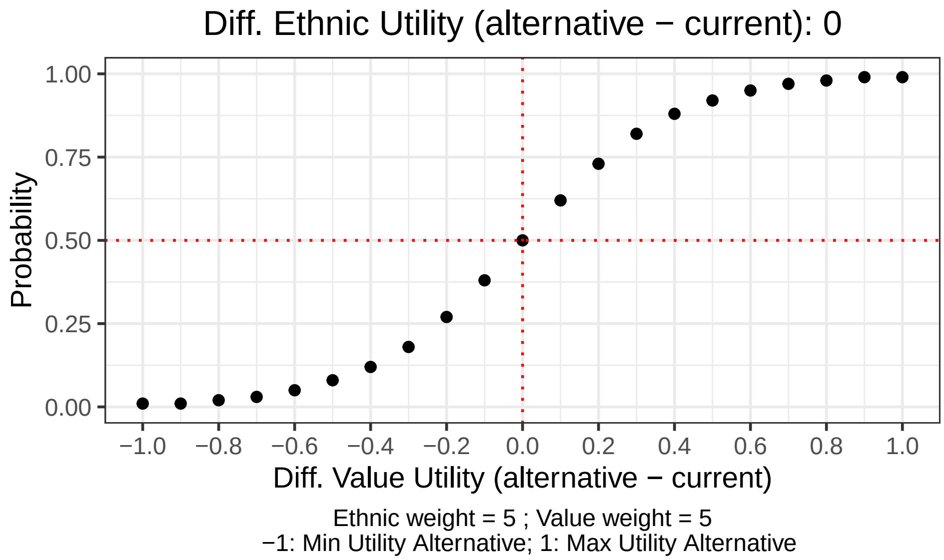
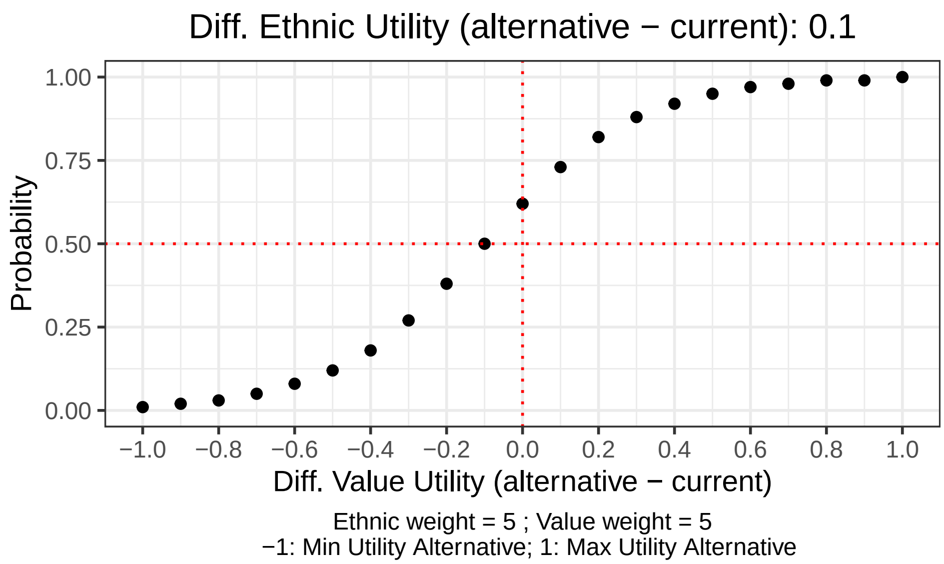
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): 0
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): 0.1
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
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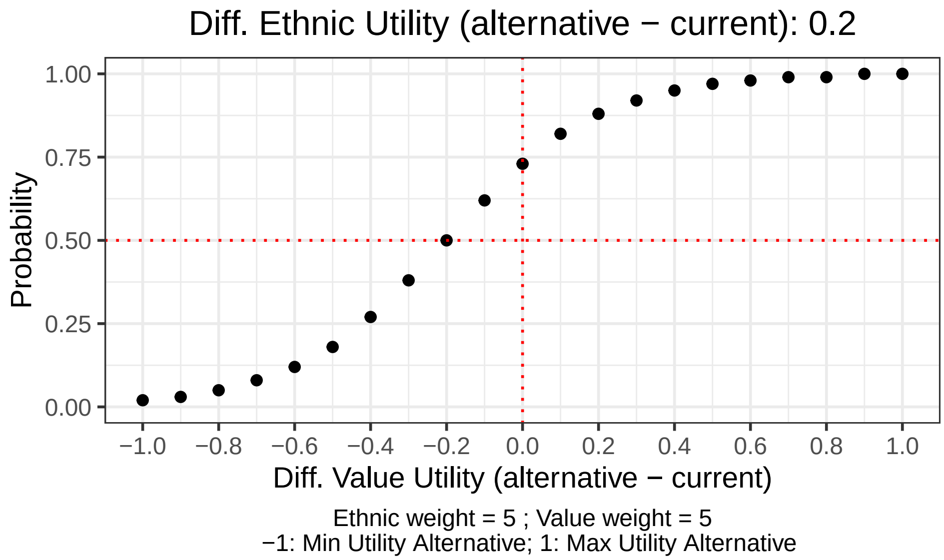
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): 0.2
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
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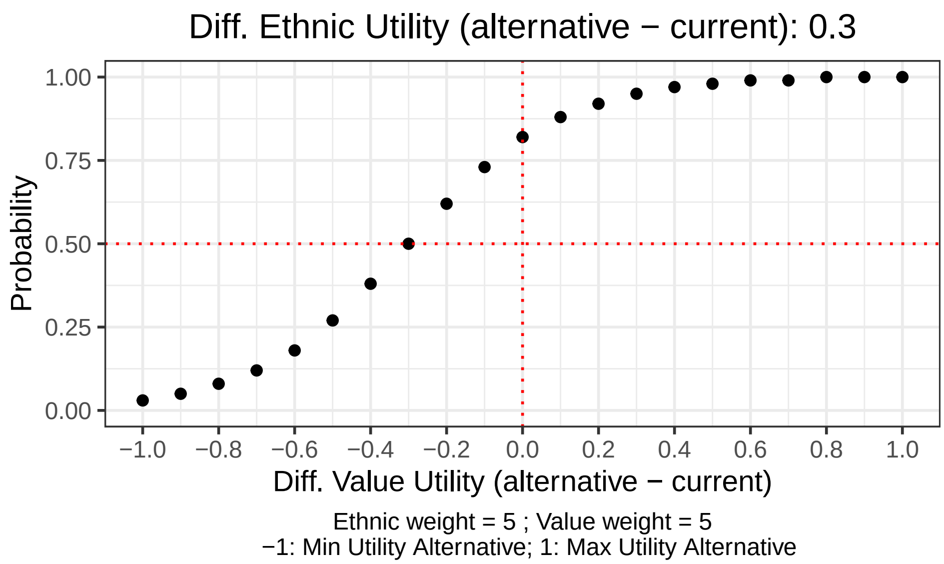
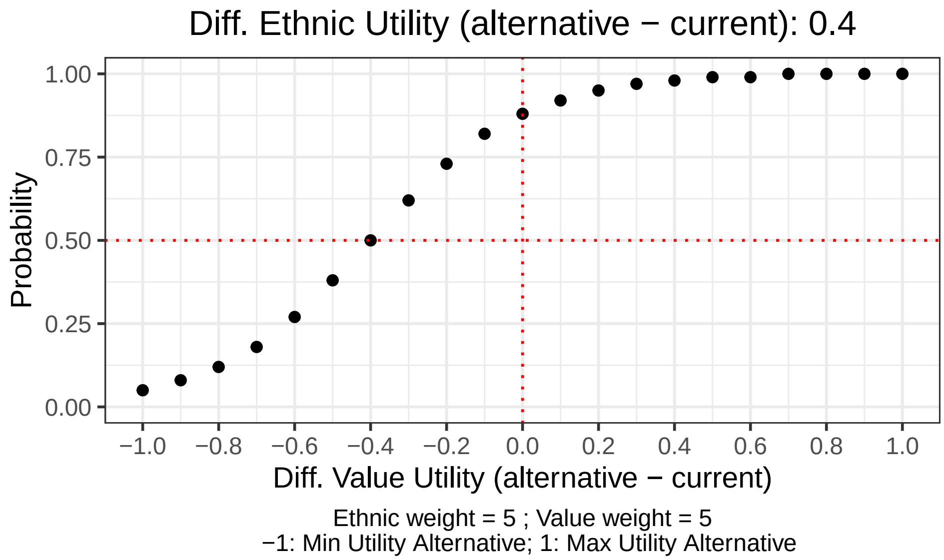
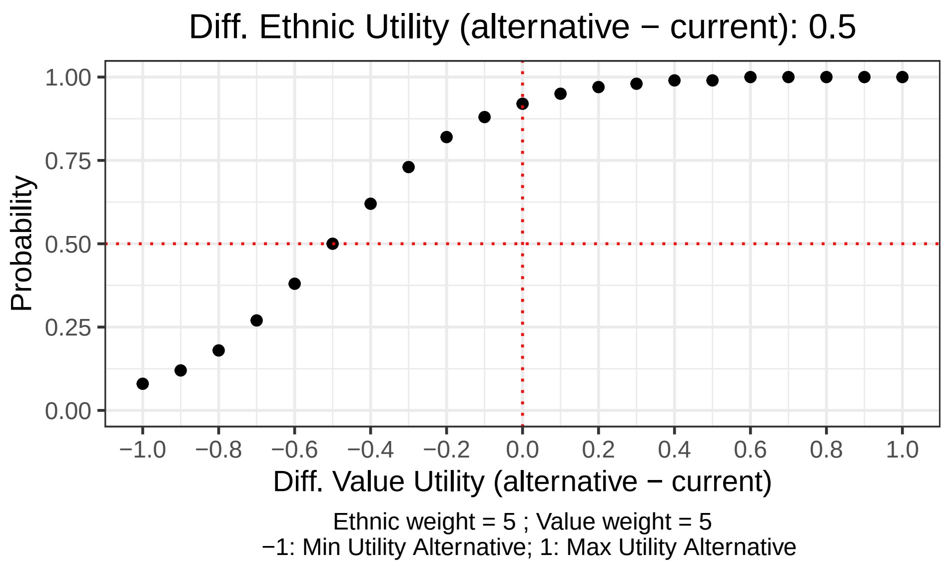
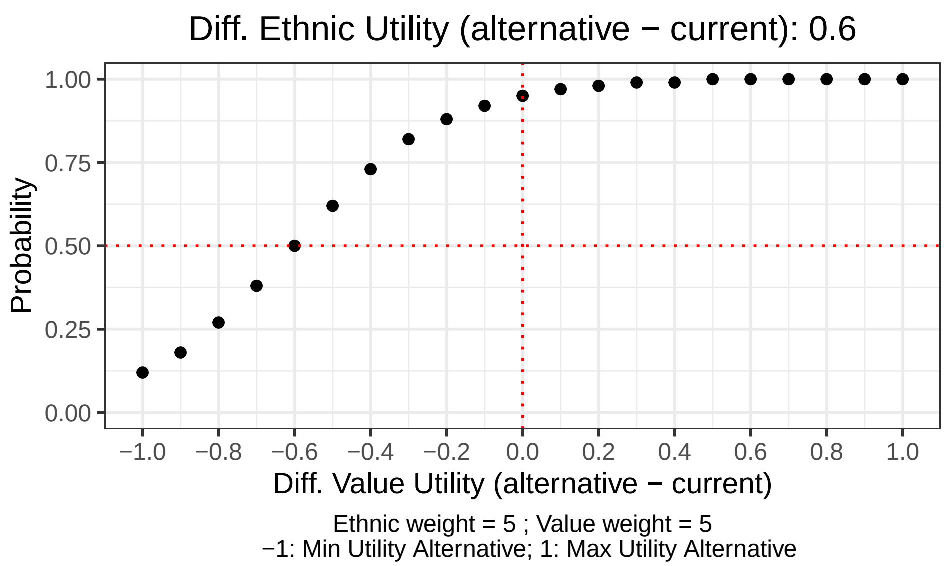
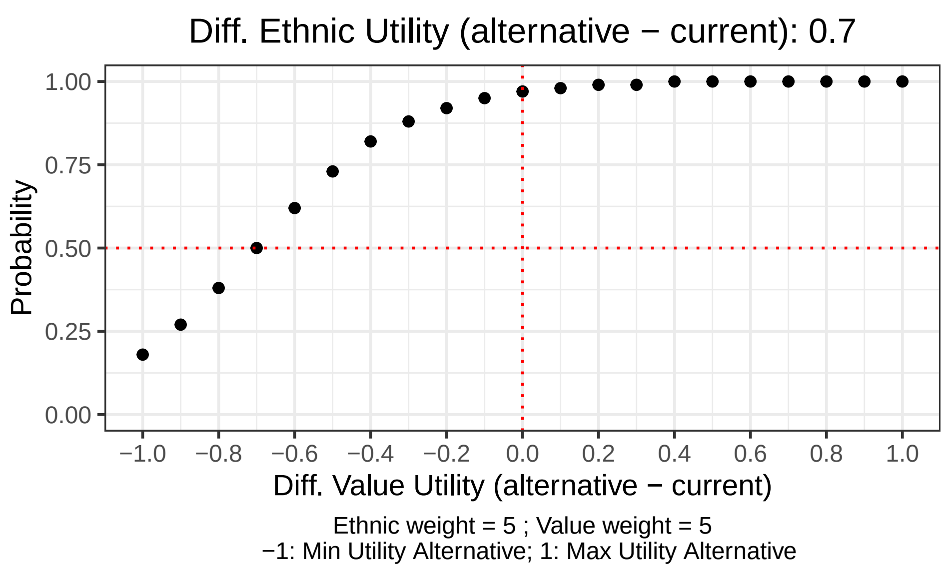
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative

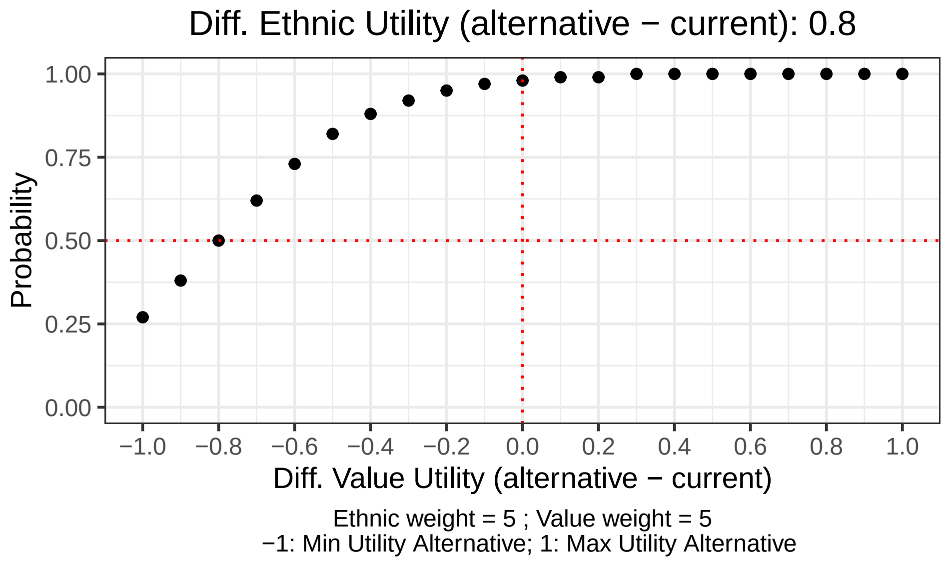

Diff. Ethnic Utility (alternative − current): 0.3
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative

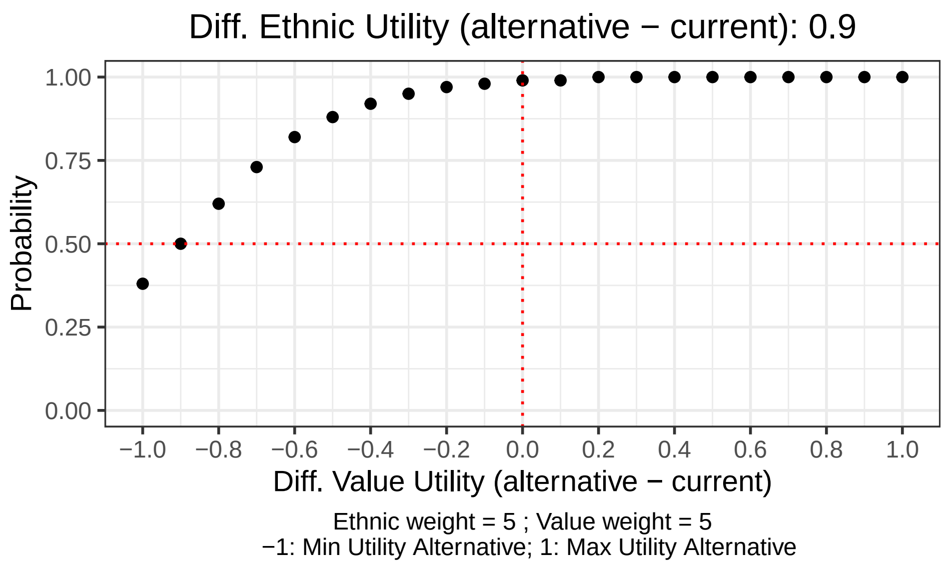

Diff. Ethnic Utility (alternative − current): 0.4
Probability
1.00
0.75
0.50
0.25
0.00
−1.0 −0.8 −0.6 −0.4 −0.2 0.0 0.2 0.4 0.6 0.8 1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative

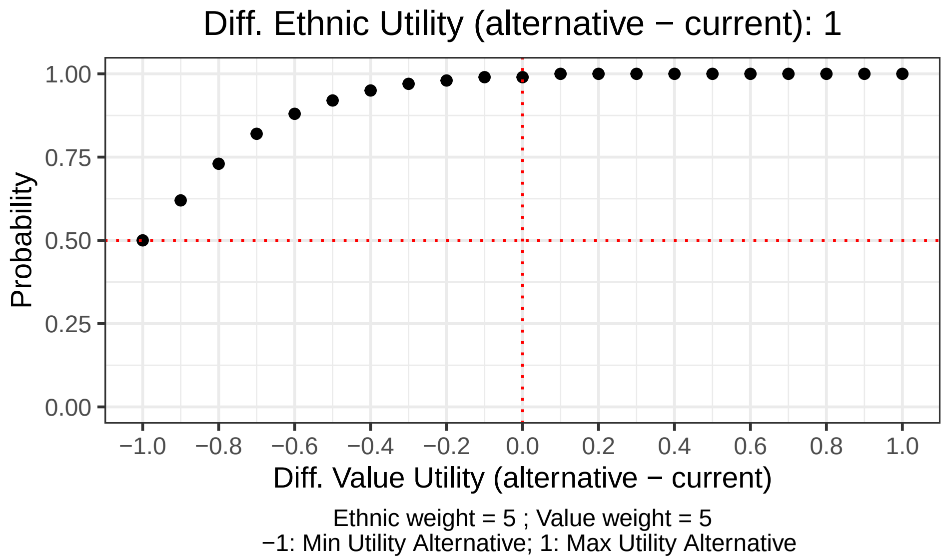

Diff. Ethnic Utility (alternative − current): 0.5
Probability
1.00
0.75
0.50
0.25
0.00
−1.0 −0.8 −0.6 −0.4 −0.2 0.0 0.2 0.4 0.6 0.8 1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): 0.6
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): 0.7
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): 0.8
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative


Diff. Ethnic Utility (alternative − current): 0.9
Probability
1.00
0.75
0.50
0.25
0.00
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Diff. Value Utility (alternative − current)
Ethnic weight = 5 ; Value weight = 5
−1: Min Utility Alternative; 1: Max Utility Alternative



References

Bruch, Elizabeth E, and Robert D Mare. 2006. "Neighborhood Choice and Neighborhood Change." *American Journal of Sociology* 112 (3): 667–709.

———. 2009. "Preferences and Pathways to Segregation: Reply to van de Rijt, Siegel, and Macy." *American Journal of Sociology* 114 (4): 1181–98.

Esser, Hartmut. 2010. "Assimilation, Ethnic Stratification, or Selective Acculturation?. Recent Theories of the Integration of Immigrants and the Model of Intergenerational Integration." *Sociologica* 4 (1): 0–0.

Liao, Tim Futing. 1994. *Interpreting Probability Models: Logit, Probit, and Other Generalized Linear Models*. 101. Sage.

Manski, Charles F. 1977. "The Structure of Random Utility Models." *Theory and Decision* 8 (3): 229–54.

McFadden, Daniel. 1994. *Conditional Logit Analysis of Qualitative Choice Behavior*. Edited by P. Zarembka. New York, NY:Academic Press.

McPherson, Miller, Lynn Smith-Lovin, and James M Cook. 2001. "Birds of a Feather: Homophily in Social Networks." *Annual Review of Sociology* 27 (1): 415–44.

Paolillo, Rocco, and Jan Lorenz. 2018. "How Different Homophily Preferences Mitigate and Spur Ethnic and Value Segregation: Schelling's Model Extended." *Advances in Complex Systems* 21 (06n07): 1850026.

Schelling, Thomas C. 1969. "Models of Segregation." *The American Economic Review* 59 (2): 488–93.

Train, Kenneth E. 2009. *Discrete Choice Methods with Simulation*. Cambridge university press.

Van de Rijt, Arnout, David Siegel, and Michael Macy. 2009. "Neighborhood Chance and Neighborhood Change: A Comment on Bruch and Mare." *American Journal of Sociology* 114 (4): 1166–80.

Wimmer, Andreas. 2013. *Ethnic Boundary Making: Institutions, Power, Networks*. Oxford University Press.

Zhang, Junfu. 2011. "Tipping and Residential Segregation: A Unified Schelling Model." *Journal of Regional Science* 51 (1): 167–93.