Assignment

Homework HL3

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Musical Acoustics

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Introduction

In this study, we're going to study the behaviour of the string-hammer interaction in a piano (in the first part) and a guitar's model (in the second part). In order to do this study, we will use the Finite Difference Method in Matlab and also the electric analogy thanks to simulink.

Part 1: Piano string FD Model

In this section, we are going to build a model for the interaction between a C2 piano string and a hammer, as shown in the following section.

Model and parameters of the system

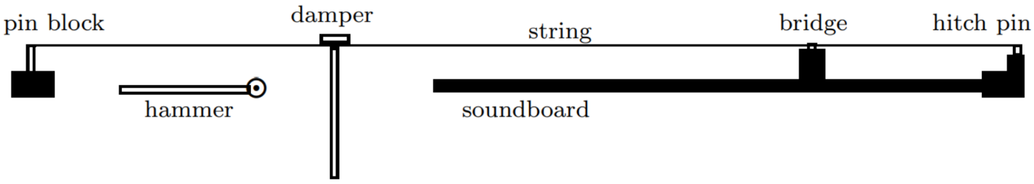


Figure 1: Model of the interaction between the piano string and the hammer.

In our study, we're going to neglect both the damping imposed by the damper and the coupling of the string with the sound board.
To characterize our system, we're going to use the following data:

- String Parameters:
  - fundamental frequency: $f_1 = 65.4$ Hz (C2).
  - string length: $L = 1.92$ m.
  - mass of the string : $M_s = 33.5 \times 10^{-2}$ Kg.
  - linear mass density : $\mu_{str} = M_s/L = 1.82 \times 10^{-2}$ Kg/m.
  - tension: $T_e = 4 \cdot \mu_{str} \cdot (L \cdot f_1)^2 = 1.15 \cdot 10^3$ N.
  - propagation speed : $c = \sqrt{T_e/\mu_{str}} = 251.13$ m/s.
  - string stiffness coefficient: $k = \epsilon = 7.5 \times 10^{-6}$
  - air damping coefficient: $b_1 = 0.5$ s^{-1}
  - internal friction coefficient : $b_2 = 6.2 \times 10^{-9}$ s.
- Hammer parameters:
  - mass: $M_H = 4.9 \cdot 10^{-3}$ Kg.
  - fluid damping coefficient: $b_H = 1 \times 10^{-4}$ s^{-1}.
  - relative striking position $a = 0.12$
  - initial hammer velocity: $V_{H0} = 2.5$ m/s.

- felt stiffness: $K = 4 \times 10^8$.
- felt stiffness exponent: $p = 2.3$
- width of hammer's spatial window g: $w = 0.2$.

Before going on, we shall now describe the interaction between the hammer and the string. The hammer consist of a wooden molding covered with several layers of compressed wool felt, whose hardness is carefully controlled.
The felt of the hammer is approximated by a nonlinear spring, whose stiffness increases with compression. We can now describe the force of the hammer F_H due to the felt as so:

$$F_H = K\xi^p$$

where $\xi(t)$ describes the felt compression in function of time. By defining $\eta(t)$ as the hammer's displacement with respect to the string's equilibrium position, the interaction between hammer and string can be modeled as:

$$M_H \frac{d^2\eta}{dt^2} = -F_H(t) - b_H \frac{d_\eta}{dt}$$

- Boundary Conditions:
  - left hinged end, normalized impedance : $\zeta_l = \frac{R_l}{\mu_{str}c} = 1 \times 10^{20}$ Ω m ^{2} s / kg.
  - bridge attached end, normalized impedance : $\zeta_b = \frac{R_b}{\mu_{str}c} = 1000$ Ω m ^{2} s / kg.

Sampling parameters

With the Finite Difference method, the goal is to compute the displacement of the whole string in time by subdividing it into several spatial samples. With this approach, we can in fact compute the shape of the string at the next time interval by considering the shape at current time and at the previous time intervals.
To apply this method, we first need to define the sampling frequency. For better computation, we've decided to use:

$$F_s = 44100 \text{ Hz} \rightarrow T_s = 1/F_s = 2.27 \times 10^{-5} \text{ s}$$

which satisfies the Nyquist theorem for our case of study ($F_{nyq} = F_s/2 > f_1$). In fact, considering $\gamma = F_{nyq}/f_1$, we can see that it has a value greater than one, therefore the anti-aliasing condition is satisfied.
We now proceed to compute the number of segments M of length X_s to subdivide the string into. Defining the courant number $\lambda = cT_s/X_s$, we can use the Courant-Friedrichs-Lewy condition (CFL) to obtain the maximum number of spatial steps M_max, as follows:

$$X_s \leq cT_s \rightarrow M_max = \lfloor \frac{L}{cT_s} \rfloor = 337$$

Although, according to the *Chaigne-Askenfelt* paper provided, in order to have stability and reduce numerical dispersion in the computation, the maximum number of segments must be defined with the following expression:

$$M_{max} = \lfloor\{[-1 + (1 + 16\epsilon\gamma^2]^{1/2}]/8\epsilon\}^{1/2}\rfloor = 217$$

Using this value, we're sure to avoid dispersion and instability of the solutions, and still satisfy the CFL condition. So our spatial samples become: $X_s = L/M_{max} = 0.0088$ m.

FD coefficients

We can now define all the parameters that will be used in the main loop.

Coefficients table		
Main loop	Left end	Right end
$a_1 = \frac{-\lambda^2\mu}{1+b_1T_s}$	$b_{L1} = \frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T_s+\zeta l\lambda}$	$b_{R1} = \frac{2-2\lambda^2\mu-2\lambda^2}{1+b_1T_s+\zeta b\lambda}$
$a_2 = \frac{\lambda^2+4\lambda^2\mu+\nu}{1+b_1T_s}$	$b_{L2} = \frac{4\lambda^2\mu+2\lambda^2}{1+b_1T_s+\zeta l\lambda}$	$b_{R2} = \frac{4\lambda^2\mu+2\lambda^2}{1+b_1T_s+\zeta b\lambda}$
$a_3 = \frac{2-2\lambda^2-6\lambda^2\mu-2\nu}{1+b_1T_s}$	$b_{L3} = \frac{-2\lambda^2\mu}{1+b_1T_s+\zeta l\lambda}$	$b_{R3} = \frac{-2\lambda^2\mu}{1+b_1T_s+\zeta b\lambda}$
$a_4 = \frac{-1+b_1T_s+2\nu}{1+b_1T_s}$	$b_{L4} = \frac{-1+b_1T_s+\zeta_l\lambda}{1+b_1T_s+\zeta l\lambda}$	$b_{R4} = \frac{-1+b_1T_s+\zeta_b\lambda}{1+b_1T_s+\zeta b\lambda}$
$a_5 = \frac{-\nu}{1+b_1T_s}$		
$a_F = \frac{T_s^2/\mu_{str}}{1+b_1T_s}$	$b_{LF} = \frac{T_s^2/\mu_{str}}{1+b_1T_s+\zeta l\lambda}$	$b_{RF} = \frac{T_s^2/\mu_{str}}{1+b_1T_s+\zeta b\lambda}$

Where

$$\mu = \frac{k^2}{c^2X_s} = 1.14 \times 10^{-11} \qquad \nu = \frac{2b_2T_s}{X^2} = 3.62 \times 10^{-9}$$

are some auxiliary parameters, useful for the computation.

Hammer Contact Window

We now proceed to model the interaction between the hammer and the string, so that we can take this phenomenon in consideration within the FD scheme. In fact, the equation governing the transfer motion of the piano string, in a plane perpendicular to the soundboard, is the following:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - kc^2L^2\frac{\partial^4 y}{\partial x^4} - 2b_1\frac{\partial y}{\partial t} + 2b_3\frac{\partial^3 y}{\partial t^3} + f(x, x_0, t)$$

where $f(x, x_0, t)$ represents the excitation produced by the hammer. We can separate the time and space dependance by using the following conversion:

$$f(x, x_0, t) = f_H(t)g(x, x_0)$$

where $g(x, x_0)$ is the spatial window taking in consideration the width of the hammer. For simpler computation, we're going to use an Hanning window of discretized length. The sample where the hammer hits the string is: $m_0 = \lfloor aL/X_s\rfloor = 26$. The number of spatial samples covered by the window are $w_s = \lfloor w/X_s \rfloor = 22$, and now $g_w(m_0)$ will finally represent the Hanning window centered around m_0 and of length w_s. We can now calculate the other coefficients to help us defining the hammer displacement in the FD scheme. In fact, we have that:

$$\eta(n+1) = d_1\eta(n) + d_2\eta(n-1) + d_F F_H(n)$$

The coefficients are calculated as so:

Hammer displacement parameters		
d_1	d_2	d_F
$\frac{2}{1+b_H T_S/2M_H}$	$\frac{-1+b_H T_S/2M_H}{1+b_H T_S/2M_H}$	$\frac{-T^2/M_H}{1+b_H T_S/2M_H}$

Computation loop

We can now describe the computation loop to define the displacement of the string at each time sample.

$$y_m{}^{n+1} = a_1(y^n_{m+2} + y^n_{m-2}) + a_2(y^n_{m+1} + y^n_{m-1}) + a_3 y^n_m + a_4 y^{n-1}_m + a_5(y^{n-1}_{m+1} + y^{n-1}_{m-1}) + a_F F^n_m$$

In this formula m is the spatial index and ranges from 0 to M, n is the time index. The problem is that this equation is valid only in the interior of the string, and it is not valid for $m = 0, m = 1, m = M-1, m = M$.
For these particular values of the index m we will set proper boundary conditions.

- $m = 1$:

  $$y_m{}^{n+1} = a_1(y^n_{m+2} - y^n_m + 2y^n_{m-1}) + a_2(y^n_{m+1} + y^n_{m-1}) + a_3 y^n_m + a_4 y^{n-1}_m + a_5(y^{n-1}_{m+1} + y^{n-1}_{m-1}) + a_F F^n_m$$

  with

  $$F^n_m = F_H(n)g(m, m_0)$$

- $m = M-1$:

  $$y_m{}^{n+1} = a_1(2y^n_{m+1} - y^n_m + y^n_{m-2}) + a_2(y^n_{m+1} + y^n_{m-1}) + a_3 y^n_m + a_4 y^{n-1}_m + a_5(y^{n-1}_{m+1} + y^{n-1}_{m-1}) + a_F F^n_m$$

- $m = 0$:

  $$y_m{}^{n+1} = b_{L_1} y^n_m + b_{L_2} y^n_{m+1} + b_{L_3} y^n_{m+2} + b_{L_4} y^{n-1}_m + b_{L_F} F^n_m$$

- $m = M$:

  $$y_m{}^{n+1} = b_{R_1} y^n_m + b_{R_2} y^n_{m+1} + b_{R_3} y^n_{m+2} + b_{R_4} y^{n-1}_m + b_{R_F} F^n_m$$

We can also set particular conditions for $n = 0, 1, 2$

- $n = 0$: both the hammer and the string are at the initial position
- $n = 1$: the hammer has travelled a distance $\eta(n) = V_{H0}T_s$ and hits the string at the sample m_0 with force $F_H = K|(\eta(n) - y(m_0, n))|^p$
- $n = 2$: we can write the displacement as:

$$y_m^n = y_{m-1}^n + y_{m+1}^n - y_{m-1}^n + \frac{T_s^2 M F_H g(m)}{M_s}$$

  And the hammer's travelled distance is:

$$\eta^n = 2\eta^{n-1} - \eta^{n-2} - \frac{T^2 F_H}{M_H}$$

From the third time sample to the last one, we have to check whether or not the hammer is still imposing force on the string. This can be done by comparing the hammer's displacement with the string's one at the spatial contact sample: if it's greater, then the hammer is still touching the string, so we have to update the current value of the hammer's force, otherwise the hammer's force can be set to 0. At every iteration, we update the hammer's displacement using the following formula:

$$\eta^{n+1} = d_1\eta^n + d_2\eta^{n-1} + d_F F_H$$

In the end, we average the string's displacement over 12 spatial samples. Such portion will be centered in a position specular with respect to the hammer's striking position.

Results

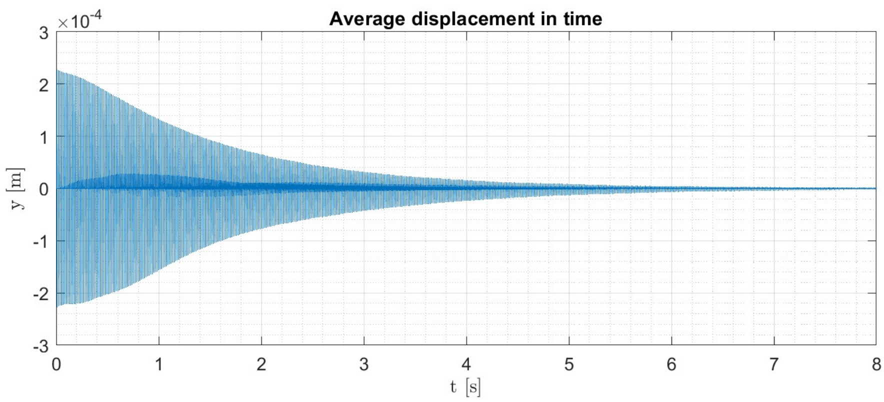


Figure 2: String's average displacement

Part 2: Guitar model

In this section, we're going to build a guitar model by using its electric analog to study the behaviour of the guitar soundboard when excited by a plucked string.
The guitar's body can be emulated via an electric analog, based on the mechanical *"Two-mass"* model, described in the following picture:

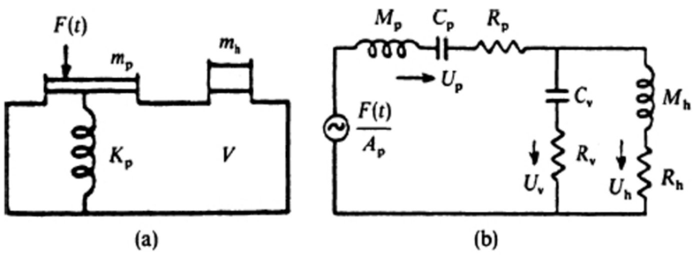


Figure 3: Guitar's mechanical and electric analog

where the components "M_p, C_p, R_p" describe the resonance of the top plate, while "C_v, R_v, M_h, R_h" entail the sound box resonance. We can notice how the resonances are modeled by inserting a filter bank in the circuit: in fact, each RLC sub-circuit models a particular filter with one associated resonance.
For our study, we were asked to model 20 resonances following this given template:

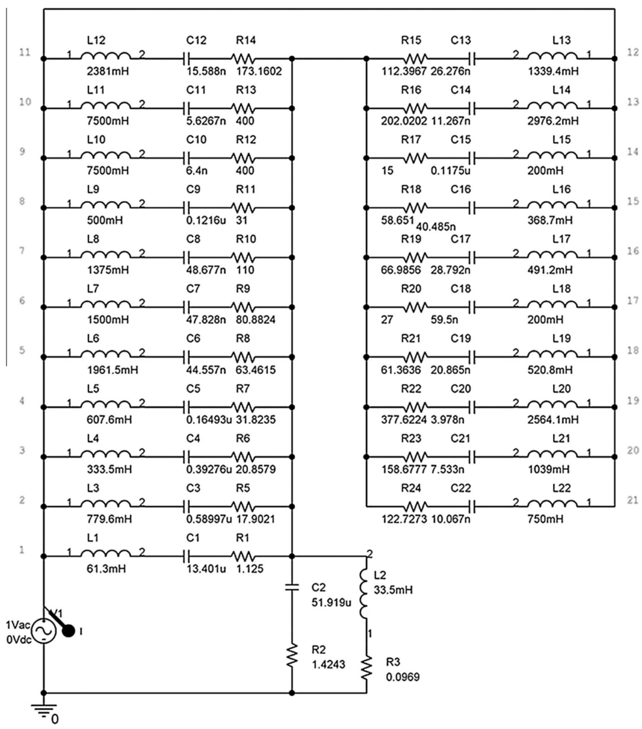


Figure 4: Guitar body, 20 resonances model

Part 1: Model without string's model

We're initially going to stimulate the circuit with a damped square wave with the following expression:

$$V_{in} = \text{sgn}(\sin(2\pi f_0 t))e^{-\beta t}, \text{with } f_0 = 300\text{Hz and } \beta = 3.$$

The circuit and the results are shown in the next figures.

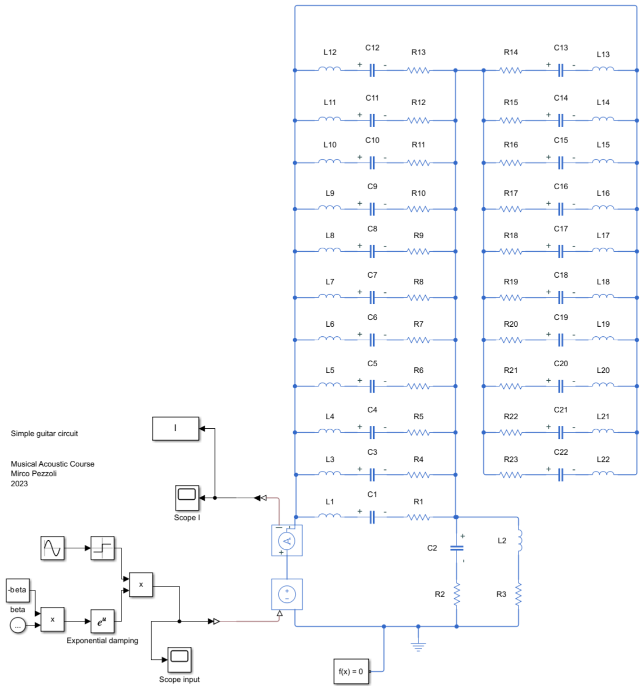


Figure 5: Guitar 20 resonances model with damped input

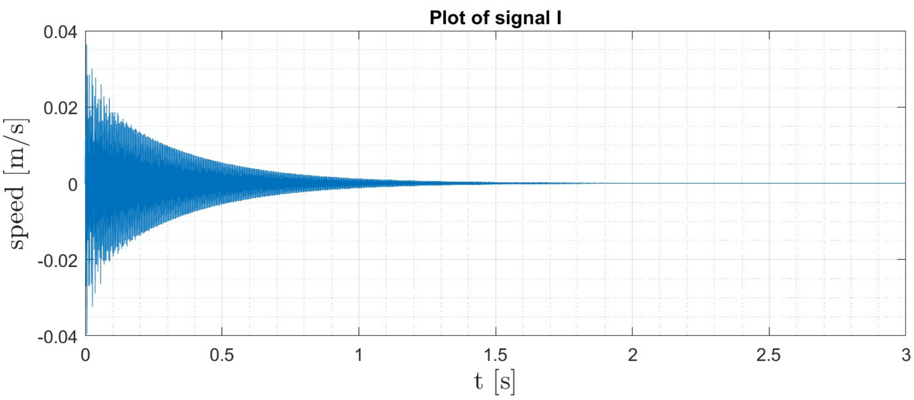


Figure 6: Signal in time

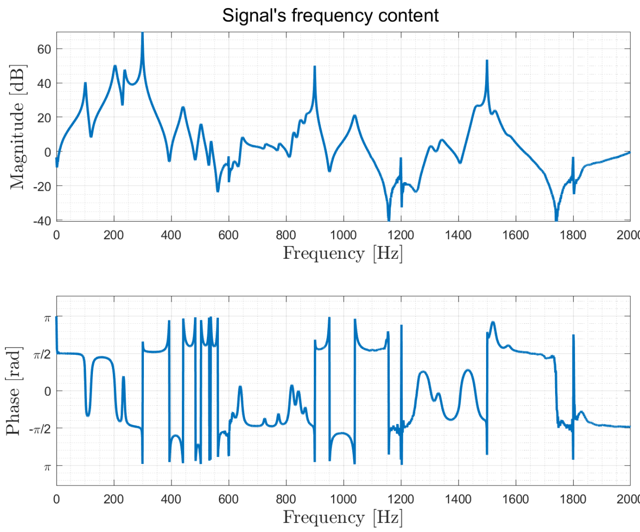


Figure 7: Signal's frequency spectra

Part 2: Model with string's model

We're now going to add a plucked string model to stimulate our circuit. In fact, a plucked string can be modelled with the following circuit:

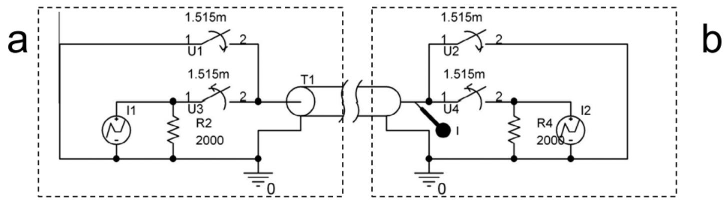


Figure 8: Plucked guitar model

We will use this signal as input signal for our model of the guitar body. Standing waves in the string are modeled through a transmission line, which allows to take into account wave reflections at the string boundaries. The transmission line is fed on both sides with two symmetrical triangular current pulses which duration is linked to half of the string fundamental frequency of vibration. These pulses correspond to two feasible propagating D'Alembert solutions for the transverse wave propagation on the string. Having a current signal as input means that we are directly analysing the velocity of the bridge when excited with the motion of the string.
The new circuit and its results are shown in the next figures:

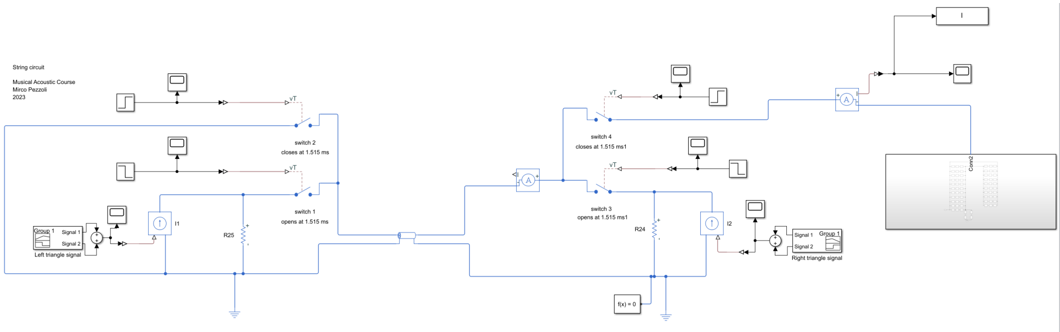


Figure 9: Guitar 20 resonances model with damped input

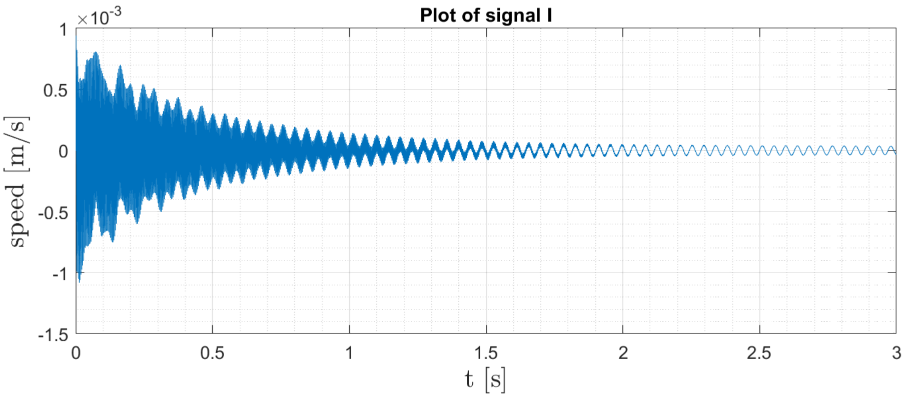


Figure 10: Signal in time

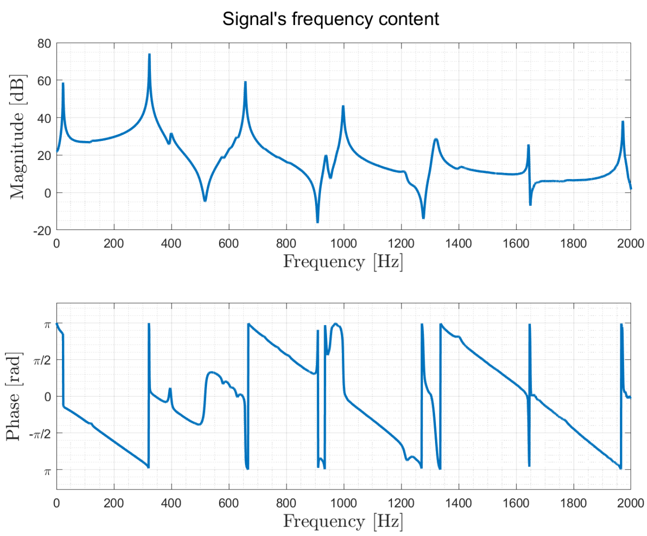
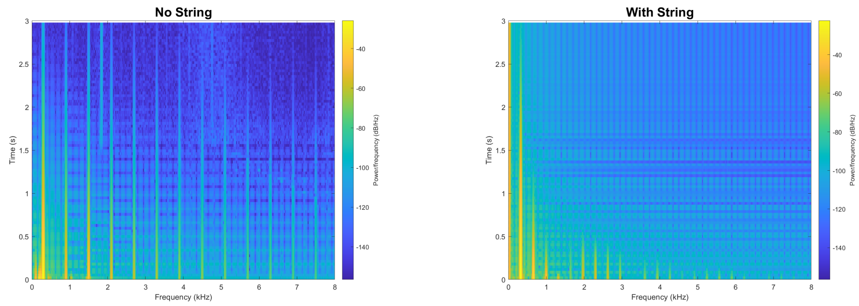


Figure 11: Signal's frequency spectra

Results

We can compare here the results from the previous simulations. The usage of the string model clearly results in a much more accurate approximation of a guitar sound: it looks much more rich from an harmonic point of view and its harmonics are better spread across the frequency.



(a) Signal's spectrogram, no string model

(b) Signal's spectrogram, string model

Figure 12: Spectrogram's comparison

From the spectrogram we can also notice how the harmonics are more distinct and defined with the usage of the string model, compared with the damped squared input in the left figure.

Conclusions

In conclusion, this study has provided a comprehensive examination of the behavior and vibration characteristics of piano and guitar strings. Employing the Finite Difference Method in Matlab, we analyzed the dynamic responses, shedding light on the intricate dynamics of these strings. Additionally, the application of the electric analogy model using Simulink further enriched our understanding of the acoustic properties under investigation. This work not only deepens our comprehension of string dynamics but also underscores the versatility of computational methods in advancing the study of acoustics and musical instrument mechanics. The synergy between computational methods and mechanical models also allowed us to get a first approach into synthesis of sound principles.