Assignment

Homework HW3

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Musical Acoustics

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Introduction

The goal of the following study is to combine the frequency response function of the guitar body with the transfer function from the excitation point to the bridge, to synthesise the vibrational field measured on the body of the guitar. We'll first design the electric analog of the guitar body in *"Simulink"* environment to derive the impedance of the guitar's bridge. Next, we'll design a filter to represent the behaviour of a guitar string plucked at 1/5th of its length and, in the end, we'll compute the time domain response of the system.

Derivation of the Bridge Impedance

In order to obtain the bridge impedance, we're going to use electric analog of a guitar, based on the mechanical *"Two-mass"* model, described in the following picture:

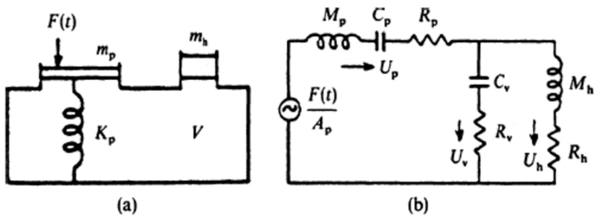

Figure 1: Mechanical and electric analog of a guitar

The value of such components is obtained with the following formulas, putting in relation the guitar attributes with the electric analog values:

- $M_p = \frac{m_p}{A_p^2}$: inheritance of the top plate $[kg/m^4]$
- $M_p = \frac{m_h}{A_h^2}$: inheritance of the air in the sound hole $[kg/m^4]$
- $C_p = \frac{A_p^2}{k_p}$: compliance of the top plate $[m^5/N]$
- $C_v = \frac{V}{\rho c^2}$: compliance of the enclosed air $[m^5/N]$
- U_p: volume velocity of the top plate $[m^3/s]$
- U_h: volume velocity of the air in the sound hole $[m^3/s]$
- U_v: volume velocity of the air into the cavity $[m^3/s]$
- R_p: loss in the top plate
- R_h: loss due to radiation from the soundhole
- R_v: loss due to the enclosure

We can now proceed to design the circuit in *Simulink* environment. In order to calculate the impedance of the circuit, which formula's is:

$$Z(\omega) = \frac{P(\omega)}{U(\omega)}$$

we're going to stimulate it with an impulse voltage signal, representing the input pressure, and measure the current U_p, which will represent the velocity of the top plate. To sample the response of the system, we're going to use a time step of $F_s = 44100$ Hz which allows us to correctly plot the impedance in the range between $0 - 500$ Hz, since for the Nyquist theorem $F_s \gg 2 \cdot f_{max} = 1000$ Hz. Having obtained the data, we send it to our workspace to perform the calculations, including a *"fast Fourier transform"* to obtain the Impedance's spectrum. In the end, we obtain the following graph:

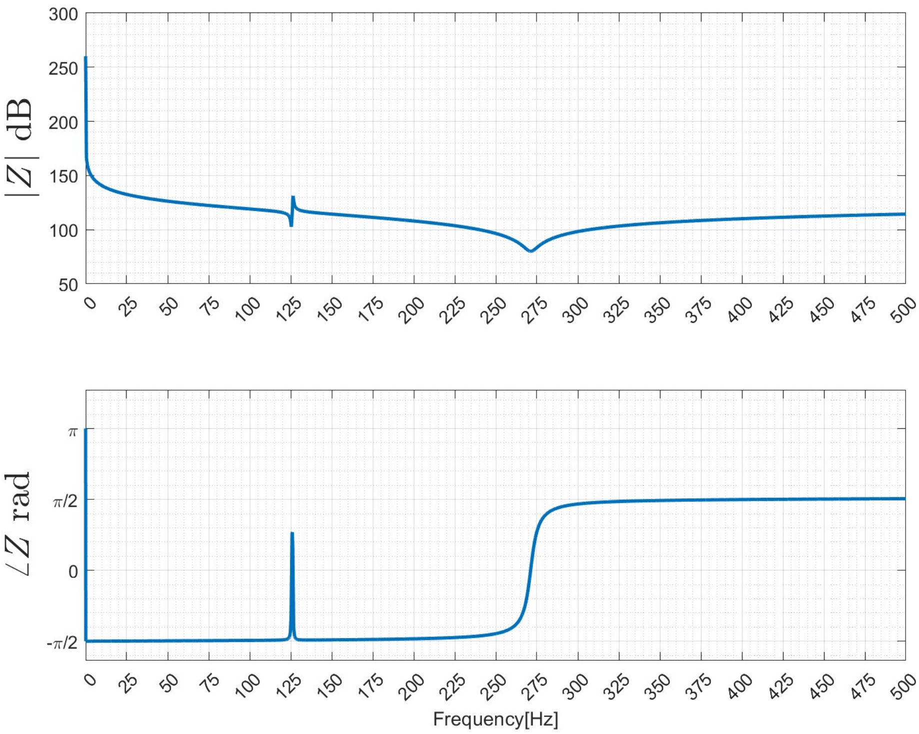

Figure 2: Impedance Z

We can immediately notice two *"trough"* in the impedance, which represent the resonance frequencies of our system, and a *peak* between the two representing an anti-resonance. Of course, this behaviour is due to the usage of the *"Two-mass model"* which has two resonance frequencies, one for each mass, and an anti resonance.

Derivation of the $H_{E,B}$ filter

In the following paragraph, we will proceed to calculate $H_{E,B}(\omega) = \frac{F(\omega)}{X(\omega)}$, the transfer function between the acceleration $X(j)$ at the excitation point and the force $F(j)$.
The behavior of a vibrating string with a plucked excitation can be described in terms of two traveling waves traversing the string in opposite directions and reflecting back at the string terminations. Assuming that during vibration the string is a LTI (Linear Time Invariant) system, we can model the plucked string with output at the bridge as follows:

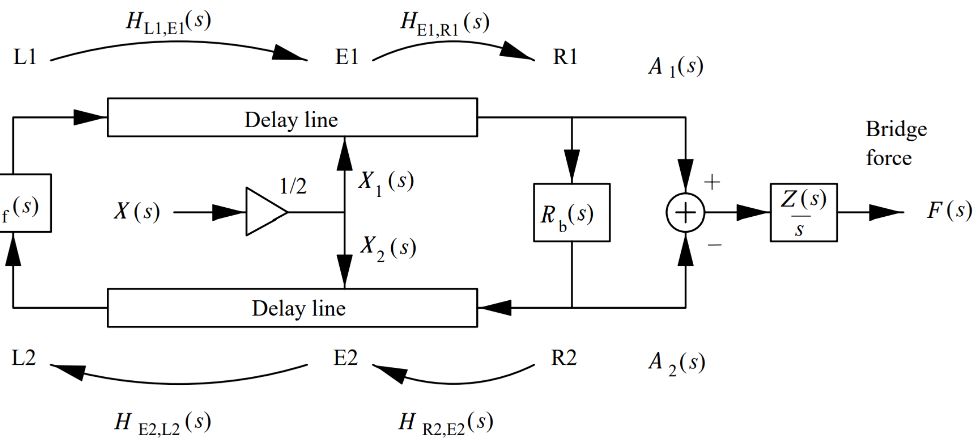

Figure 3: Dual delay-line waveguide model for a plucked string with output at the bridge.

where the two delay lines represent the travelling waves. Moreover, the other components of the system are:

- $H_{loop}(\omega)$ is the transfer function of the string
- $H_{E_1R_1}(\omega)$ is the transfer function from the excitation point to the bridge
- $H_{E_2R_1}(\omega)$ is the transfer function from the excitation point to the bridge passing through the nut
- $Z(\omega)$ is the bridge impedance
- $R_b(\omega)$ is a filter that relates the ingoing and outgoing waves along the string, and therefore can be modeled in a first approximation as a phase inversion filter.

and the filter $H_{E,B}$ can be formulated as follows:

$$H_{E,B}(\omega) = \frac{1}{2}[1 + H_{E_2R_1}(\omega)]\frac{H_{E_1R_1}(\omega)}{1 - H_{loop}(\omega)}\frac{Z(\omega)}{\omega}[1 - R_b(\omega)]$$

using $Z(\omega) = Z(\omega)/\max|Z(\omega)|$.
For our study, we're going to assume a plucking happening at one fifth of the string's length ($\beta = 1/5$). We're going to analyze the behavior of the six strings of the guitar, considering a length of $l = 0.65m$ and the following array of resonance frequencies (one for each string,

according to the guitar's standard tuning notes: "E2 - A2 - D3 - G3 - B3 - E4"):

$$f_0 = [82.41, 110, 146.83, 196, 246.94, 329.63] \text{ Hz}$$

Using the Z-transform (defining $z = e^{j\omega/F_s}$), we can now set the number of steps to sample the two sections of the string, as:

- $N_{left} = \lfloor \beta \cdot \lfloor F_s/(2 \cdot f_0) \rfloor \rfloor$
- $N_{right} = \lfloor F_s/(2 \cdot f_0) \rfloor \rfloor - N_{left}$

and so, we can build our transfer functions:

- $H_{E_2R_1} = H_{E_2L_2} \cdot (-R_f) \cdot H_{E_2E_1} \cdot H_{E_1R_1} = z^{-N_{left}} \cdot (-0.99) * z^{-N_{left}} \cdot z^{-N_{right}}$
- $H_{E_1R_1} = z^{-N_{right}}$
- $H_{loop} = (-R_b) \cdot H_{R_2E_2} \cdot H_{E_2E_1} \cdot H_{E_1R_1} = (-R_b) \cdot z^{-N_{right}} \cdot (-R_f) \cdot z^{-2 \cdot N_{left}} \cdot z^{-N_{right}}$

We're now able to compute our filters, of which we plot the amplitude graphs:

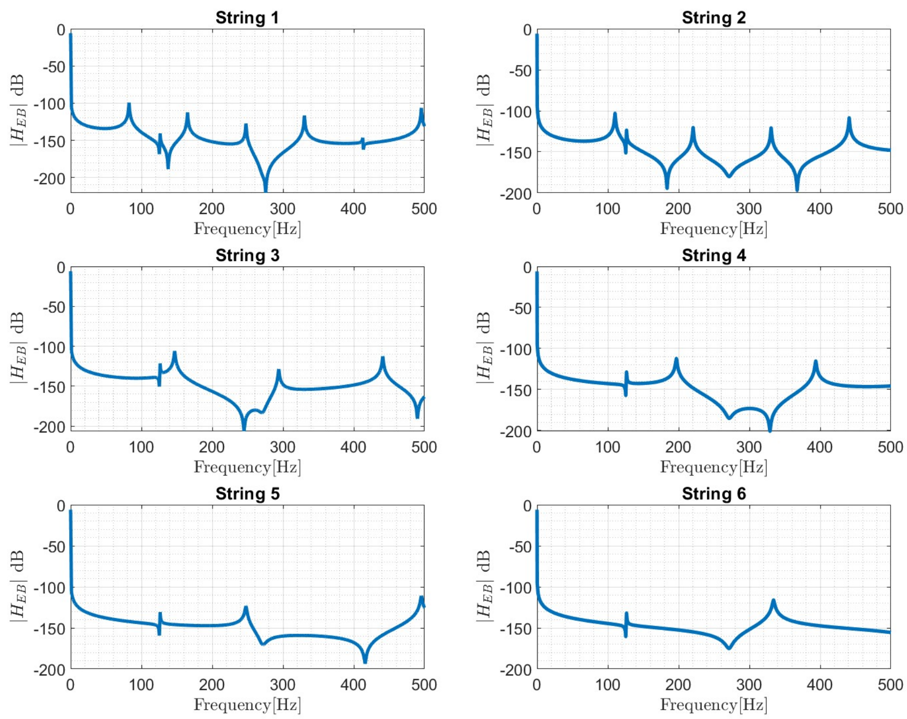

Figure 4: $|H_{EB}|$, for the 6 strings

Time domain computation

In this paragraph, we will proceed to conduct the final study of our interest: the time domain computation. We will compute the time domain response of the system to a plucking at time $t_0 = 0$ s happening as before at one fifth of the length of the string (which is $L = 0.65$ m long) and with a maximum displacement of $d_0 = 3$ mm. A representation of the string's configuration can be seen in the following graph:

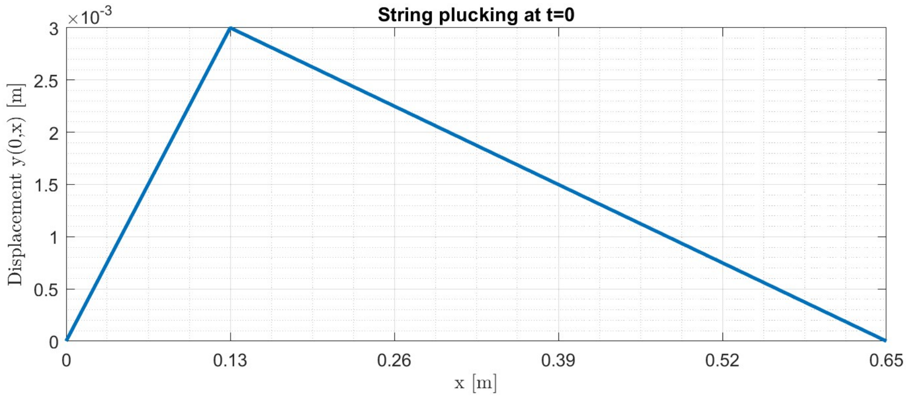

Figure 5: Caption

Since the transfer function H_{EB} is obtained as the ratio of between the force and the acceleration, we need to calculate the acceleration of the string at the plucking point in order to obtain the force applied on the bridge.

To do so, we compute the acceleration as the second derivative of displacement at the excitation point. This is given by the difference between the slopes of the two segments of the triangular waveform, resulting in an impulse of amplitude $a = -0.0288[m/s^2]$.

So in our case, the plucking can be modeled as an impulse of amplitude a. To compute the time response of the bridge, we need to consider the spectrum of such impulse: $X(\omega)$, which multiplied with the transfer function gives us the spectrum of the force $F(\omega)$. Finally, we go back to the time domain by performing an inverse fast Fourier transform, giving us the time domain response of the force applied on the bridge. The result is show in the following graphs.

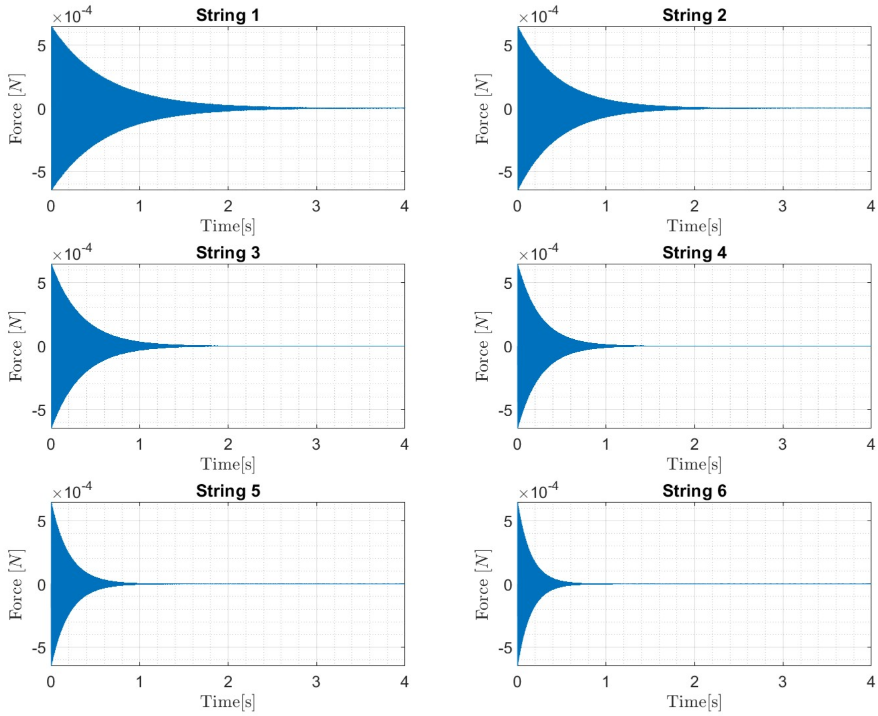

Figure 6: $F(t)$ for the 6 strings

We can see how the signals behave as expected, decaying exponentially over time. The plots have been limited up to 4 seconds for better visualization.

Conclusions

In conclusion, our investigation enabled us to accurately synthesize the vibrational field of the guitar body through the application of the delay line model, studying the interaction between the guitar strings and the guitar's bridge. The systematic progression of steps has not only enabled a nuanced comprehension of the system's behaviors but has also provided valuable insights for audio synthesis.