Assignment

Homework HW4

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Introduction

The objective of the present paper is to design a recorder flute and to predict its sound. We'll first design the bore dimension of the cone, then we will find the best position for each hole and, in the end, we'll compute a prediction of the input impedance for different configurations. We will use a simplification regarding the number of flute holes, which will be only 2.

Bore dimensioning

The resonator is shaped as a cone whose conical semi-angle is $\alpha = 0.75°$. The instrument is aimed at being a treble recorder, with a length of $L = 0.45m$. In this section we'll find the diameter of the cone at the resonator head and foot so that the note produced when all the finger holes are closed is E4, that has a frequency of 329.63 Hz. To ensure tone reproduction, the sum (in series) of the cone and mouth impedance must be minimized.

$$Z_{in} = Z_{cone} + Z_{mouth}$$

In order to compute the impedance of the cone Z_{cone} we can use the following formula

$$Z_{cone} = \frac{j\rho c}{S_1} \cdot \frac{\sin(kL)\sin(k\theta_2)}{\sin(k(L+\theta_2))}$$

the components of which are:

- $\rho = 1,255$ $[kg/m^3]$: density of the air;
- $c = 343$ $[m/s]$: speed of sound in air;
- $S_1 = r_1^2\pi$ $[m^2]$: cross section in the input (where $r_1 = r_2 + L \cdot tg(\alpha)$ $[m]$ where r_1 is the radius at the input and r_2 is the radius at the foot;
- $k = \frac{\omega}{c}$ $[1/m]$: wave number;
- $\omega = 2\pi f$ $[rad/s]$: radial frequency;
- $L = L_{cone} + 0,85 \cdot r_2$ $[m]$: bore length, including the end correction at the open end;
- $\theta_2 = tg^{-1}(kx_2)/k$ with $x_2 = r_2/tg(\alpha)$: distance between the resonator foot and the apex of the cone;
- M: air mass at the mouth window.

We can now proceed to compute M using the approximation valid for $k\Delta L << \pi/2$ and considering that for alto recorder, the typical value for ΔL is about $40mm$:

$$M = \frac{\Delta L \cdot \rho}{S_1}$$

In order to compute the total impedance Z_{tot} we need to compute also the impedance of the mouth .This mouth impedance is essentially an inertance, which can be evaluated from the mouth dimensions:

$$Z_{mouth} = j\omega M$$

In the context of wind instruments resonance and sound production are associated with a proper match between the impedance of the air within the instrument's cavity and the impedance of the surrounding air. When the impedance is minimized, it indicates a good match between the sound waves propagating through the instrument and the surrounding air. This facilitates efficient sound transmission and enhances the production of distinct tones. This is why we can now employ the formula of Z_{in} as a function of r_2 (found within the expression of S_1) to identify, through the impedance graph plot, the values of the radius r_2 corresponding to the impedance minima:

$$Z_{in}(r_2) = \frac{j\rho c}{S_1} \cdot \frac{\sin(kL)\sin(k\theta_2)}{\sin(k(L+\theta_2))} + j\omega M$$

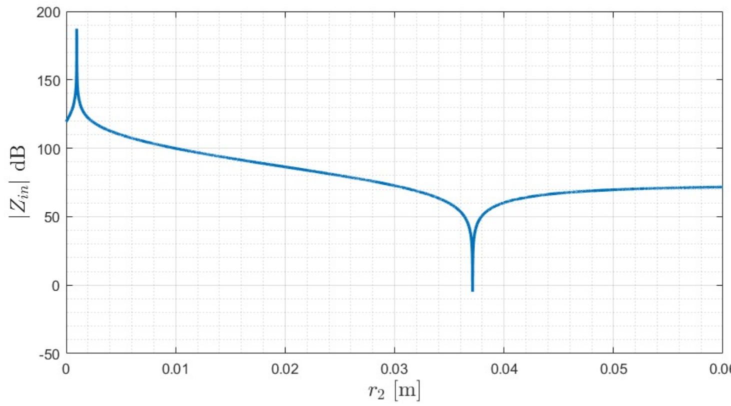


Figure 1: Impedance $Z_{in}(r_2)$

We can observe a significant minimum in the impedance graph, which precisely indicates the chosen value for r_2, also utilized in the calculation of r_1:

$$r_2 = 0.0371\ m$$

$$r_1 = r_2 + L \cdot tg(\alpha) = 0.04302\ m$$

We can compute the diameters $d_2 = 0.07425\ m$ and $d_1 = 0.08603\ m$.
A flute typically has a diameter of $\sim 19 - 20\ mm$. Based on the obtained results, we can infer that we are not referring to a real recorder.

First hole positioning

In this paragraph, we will proceed to find the position of the last finger hole, the one closest to the resonator foot, in order to produce the note F4 (349.23 Hz) when it is open. We will use a simplification, considering the finger hole diameter to be equal to the bore diameter at the resonator foot d_2. The insertion of a hole alters the acoustic behavior of the recorder. We have outlined the problem as follows, emphasizing the inclination of the cone for a clearer representation.

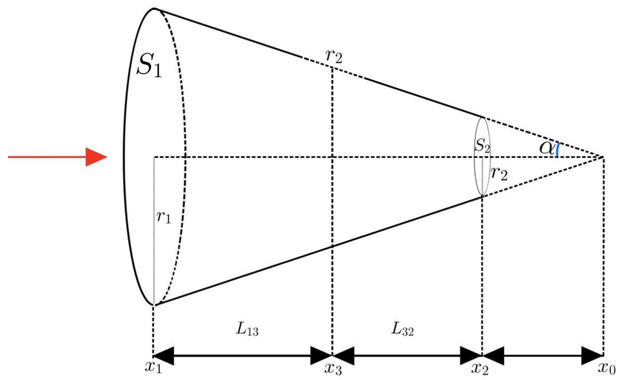


Figure 2: Recorder with one open finger hole

As we did in the previous paragraph, we can calculate the impedance of the cylinder as a function of the dimensioning distance, and then choose the length L_{32} corresponding to a minimum of the impedance. The input impedance of the recorder can be computed as the input of a cone, using the following formula:

$$Z_{in} = \frac{\rho c}{S_1} \cdot \left\{ \frac{jZ_L \frac{\sin(k'L_{13}-\theta_3)}{\sin(\theta_3)} + \frac{\rho c}{S_3}\sin(k'L_{13})}{Z_L \frac{\sin(k'L_{13}+\theta_3-\theta_1)}{\sin(\theta_1)\sin(\theta_3)} - j\frac{\rho c}{S_3}\frac{\sin(k'L_{13}+\theta_3)}{\sin(\theta_3)}} \right\} + j\omega' M$$

The components of which are:

- $S_3 = {r_3'}^2 \pi$ $[m^2]$;
- $f' = 349.23$ $[Hz]$;
- $\omega' = 2\pi f'$ $[rad/s]$: radial frequency;
- $k' = \frac{\omega'}{c}$ $[1/m]$: wave number;
- $l_3 = r_3' + 0,85 \cdot r_2$ $[m]$;
- $\theta_3 = tg^{-1}(k'x_3)/k$ with $x_3 = r_3/tg(\alpha)$;
- $\theta_1 = tg^{-1}(k'x_1)/k$ with $x_1 = r_1/tg(\alpha)$

We can compute the impedance Z_L by considering the two impedance in parallel, Z_{hole_3} and Z_{32}. The impedance related to the hole is computed using the formula for the impedance of a cylinder

$$Z_{hole_3} = \left(-j\frac{S_3}{\rho c}cot(k'l_3)\right)^{-1}$$

while for the cone section from the hole to the foot, we will use the formula for the impedance of a cone, indicated in the previous paragraph.

$$Z_{32} = \frac{j\rho c}{S_3} \cdot \frac{\sin(k'L_{32})\sin(k\theta_2)}{\sin(k'(L_{32}+\theta_2))}$$

By paralleling these two impedance, we will obtain $Z_L = \frac{Z_{hole_3} \cdot Z_{32}}{Z_{hole_3}+Z_{32}}$ which, when substituted into the formula for Z_{in}, will allow us to plot it as a function of L_{32} and thus dimension the position of the first hole at the minimum impedance.

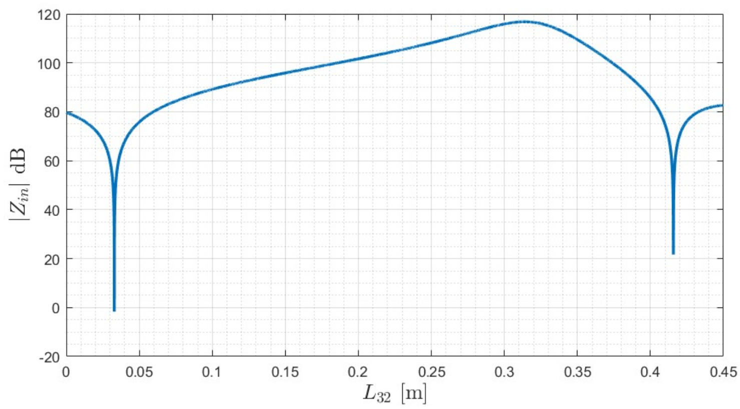


Figure 3: Impedance $Z_{in}(L_{32})$with one open finger hole

We can express the position of the first hole in two different ways:

$$L_{32} = 0.0329\ m$$

$$L_{13} = L - L_{32} = 0.4171\ m$$

To make a flute of this type feasible, one should either lengthen the flute or reduce the dimensions of the holes, thus negating the simplifying assumption expressed earlier

Second hole positioning

Following the same procedure outlined in the previous step, we proceed to calculate the position of the second last finger hole in order to produce the note G4 with a frequency of 392 Hz when the two finger holes are open. As before, we will consider the finger hole diameter to be equal to the bore diameter at the resonator foot d_2 as a simplification.

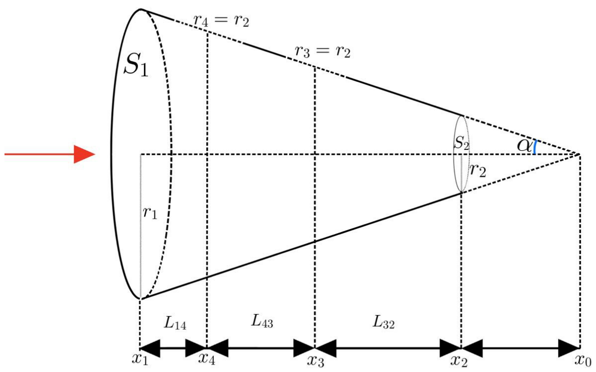


Figure 4: Recorder with one open finger hole

We can once again use the formula for the impedance of the cone:

$$Z_{in} = \frac{\rho c}{S_1} \cdot \left\{ \frac{jZ_4 \frac{\sin(k_2 L_{14} - \theta_1)}{\sin(\theta_1)} + \frac{\rho c}{S_4} \sin(k_2 L_{14})}{Z_4 \frac{\sin(k_2 L_{14} + \theta_4 - \theta_1)}{\sin(\theta_4)\sin(\theta_1)} - j\frac{\rho c}{S_4} \frac{sin(k_2 L_{14} + \theta_4)}{\sin(\theta_4)}} \right\} + j\omega_2 M$$

In this equation $Z_4 = \frac{Z_{hole_4} \cdot Z_{43}}{Z_{hole_4} + Z_{43}}$, where Z_{43} is computed following the same steps outlined in the previous paragraph used to calculate Z_{in} without considering the Z_{mouth} and Z_{hole_4} is computed in the same way of Z_{hole_3}.

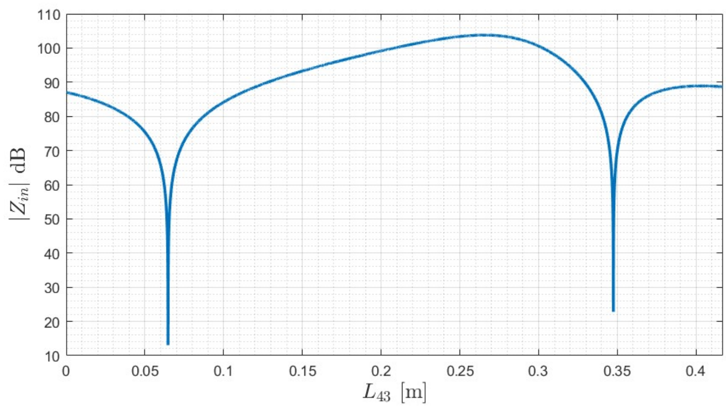


Figure 5: Impedance $Z_{in}(L_{43})$with two open finger holes

In the end, we can express the position of the second hole in two different ways:

$$L_{43} = 0.0647\ m$$

$$L_{14} = L - L_{43} - L_{32} = 0.3542\ m$$

Prediction of the input impedance

Now, using the transmission line model to represent the instrument, we'll try to predict the input impedance of the instrument in three different configurations:

- All holes closed
- First hole open
- Both holes open

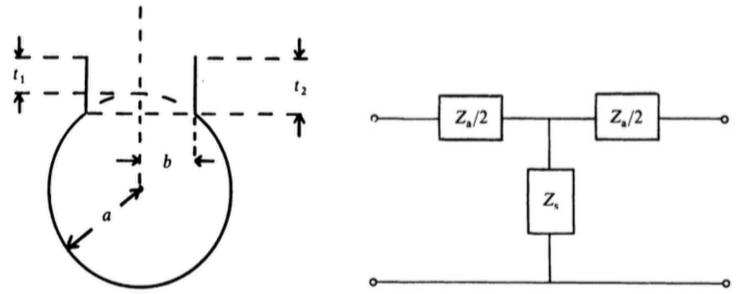
The idea behind this approach is to subdivide the bore of the instrument into a sequence of small parts, whose shape is approximately conical. Each section is represented as a four terminal component, with input $Z_1 = p_1/U_1$ and output $Z_2 = p_2/U_2$, being p_i the pressure and U_i the acoustic volume flow of such component. In particular:

$$\begin{cases} p_1 = Z_{11}U_1 + Z_{12}U_2 \\ \\ p_2 = Z_{21}U_1 + Z_{22}U_2 \end{cases}$$

Therefore, since $Z_{12} = Z_{21}$, the input impedance seen from Z_1 can be rewritten as:

$$Z_1 = Z_{11} + \frac{Z_{12}^2}{Z_2 - Z_{22}}$$

For a circular finger hole of radius $b = r_2$ and chimney height t in a tube of radius $a = r_h$, the equivalent component can be seen as:



(a) Finger hole representation (b) Network equivalent of the finger hole

Figure 6

and the imaginary part of the series and shunt elements have forms:

$$Z_s = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \times \begin{cases} (-j\cot kt_s) & \text{(closed)} \\ jkt_e & \text{(open)} \end{cases}$$

$$Z_a = \left(\frac{\rho c}{\pi a^2}\right)\left(\frac{a}{b}\right)^2 \times (-jkt_a) \text{ (closed or open)}$$

The form of these equations reflect the fact that an open hole behaves like a shunt inertance, while a closed hole behaves like a shunt compliance because of its enclosed volume. The series element Z_a, on the other hand, reflects the effective increase in tube cross section in the vicinity of the hole, whether it is open or closed.
The values of t, t_e and t_a are rather complicated on whether the hole is open or closed and, if it is open, whether or not there is a key pad or finger poised over it. To calculate such

values, we'll refer to **Keefe (1928b)**.

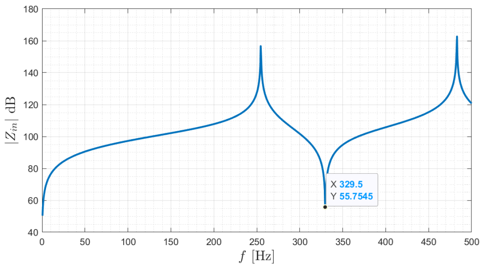
$$t_a^{(o)} = \frac{0.47b\delta^4}{\tanh(1.84t/b) + 0.62\delta^2 + 0.64\delta}$$

$$t_a^{(c)} = \frac{0.47b\delta^4}{\coth(1.84t/b) + 0.62\delta^2 + 0.64\delta}$$

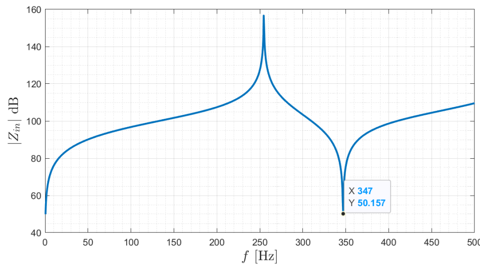
$$t_s = t + t_m = t + \frac{b\delta}{8}(1 + 0.207\delta^3)$$

$$t_e = t + (\pi/2)b$$

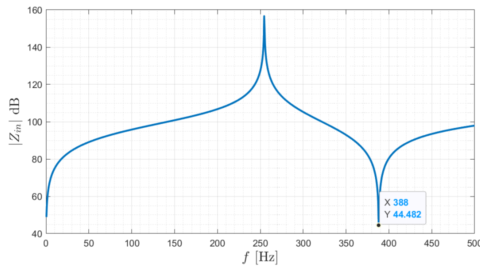
Following, we report the results obtained:



(a) Input impedance, all holes closed



(b) Input impedance, all holes open



(c) Input impedance, all holes open

Figure 7

We can see how the input impedance of the model presents local minimas at the required frequencies. The resonance frequencies aren't exactly the one requested, due to possible approximations and the equations used to calculate the parameters t, t_e, t_s and t_m.

Conclusions

In conclusion, this study focused on dimensioning an hypothetical recorder flute with two holes, aiming to optimize the sound of the notes based on various configurations of the flute. Through the application of impedance formulas for a cone, we determined the optimal length of the recorder as well as the positions of the first and second holes. The findings highlight the importance of considering impedance characteristics in the design process to achieve resonance frequencies that enhance the overall musical performance of the instrument.