Assignment

Homework HW1

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Musical Acoustics

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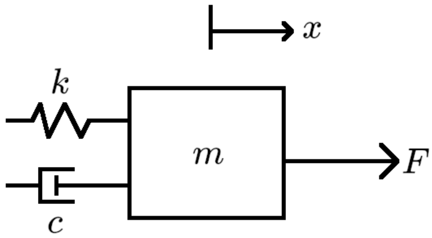
Characterization of the Resonator

The resonator consists of a mass $m = 0.1$ kg and a stiffness $K = 2.53 \cdot 10^4$ N/m.
Through experiments, it is observed that the motion of this system decays by 5 dB in a time $t_{-5} = 0.576$ s.

Compute the resonance frequency of such a resonator



The system in exam can be interpreted as a simple mass-spring-damper system. To find the natural frequency of the system, we use the formula:

$$\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{2.53 \cdot 10^4}{0.1}} = 502.99 \text{ rad/s}$$

Compute the decay time τ of the system

We know that the system's motion decays by 5 dB in a given time $t_{-5} = 0.576$ s.
We can find the decay time by means of the following procedure. By defining $A(t)$ as the amplitude of the displacement of the system at time t and using the definition of decibel, we can state that:

$$10 \log_{10} \left(\frac{A\left(t_1 + 0.576\right)}{A(t_1)} \right) = -5 \text{ dB}$$

By solving the equation, we obtain that:

$$\frac{A\left(t_1 + 0.576\right)}{A(t_1)} = 10^{-5/10} \implies A\left(t_1 + 0.576\right) = A(t_1) \cdot 10^{-1/2}$$

We know that the amplitude of the motion of a damped system can be described as follows:

$$A\left(t_1 + \bar{t}\right) = A(t_1) \cdot e^{-\bar{t}/\tau}$$

Therefore:

$$A\left(t_1 + 0.576\right) = A(t_1) \cdot e^{-0.576/\tau} = A(t_1) \cdot 10^{-1/2}$$

By solving the equation for tau, we obtain the damping time of the system:

$$\tau = \frac{-0.576}{-0.5 \log(10)} \approx 0.5 \text{ s}$$

Compute the quality factor Q

We can compute Q via the following expression:

$$Q = \frac{\omega_0}{2 \cdot \alpha} = \frac{502.99}{2 \cdot 1.99} \approx 125.82$$

where $\alpha = 1/\tau = 1.99$ 1/s.

Compute the resistance R associated to the system

The resistance associated with the system can be computed by starting from the definition of the parameter α:

$$\alpha = \frac{R}{2 \cdot m} \implies R = 2\alpha \cdot m = 2 \cdot 1.99 \cdot 0.1 \approx 0.39 \text{ kg/s}$$

Compute the -3 dB bandwidth of the resonance

The Quality Factor is defined as the ratio between the central frequency and the bandwidth, thus we can easily obtain the bandwidth:

$$BWH = 2\alpha = 2 \cdot 1.99 \approx 3.98 \text{ Hz}$$

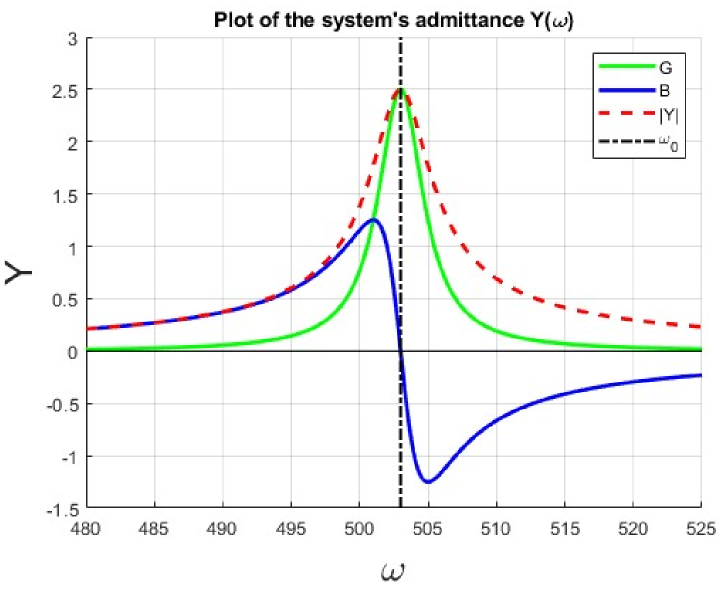
Plot the admittance of the system

The admittance is defined as the inverse of the impedance. We first compute the impedance as follows:

$$Z = \frac{\tilde{F}}{\tilde{V}} = R + j\left(\omega m - \frac{K}{\omega}\right) = 0.39 + j\left(0.1\omega - \frac{2.53 \cdot 10^4}{\omega}\right)$$

Thus:

$$Y = \frac{1}{Z} = G + jB$$



Time Response

The system is now subject to the external force $F(t) = 0.1 \sin(2\pi f_1 t)$ N, where f_1 = [60, 80, 100, 120, 140, 160] Hz. The external force starts at time $t_0 = 0$, while before $F = 0$ and the resonator is at rest.

Compute the time responses of the system for the different values of the input frequency and plot them

We're now going to compute the complete response of the system, given by the sum of the zero input response and the steady state response. To determine the steady state response, we first need the receptance of the system. We know that:

$$\Theta(\omega) = \tilde{X}(\omega)/\tilde{F}(\omega)$$

For a mass-spring-damper system, the force-displacement ratio in the frequency domain can be expressed as:

$$\tilde{X} = \frac{Fe^{j\omega t}}{K - \omega^2 m + j\omega R} = \tilde{F}(\omega) \cdot \Theta(\omega)$$

Therefore, the receptance can be defined as follows:

$$\Theta(\omega) = \frac{1}{K - \omega^2 m + j\omega R}$$

Finally, we can obtain the steady-state-response of the system, excited with a force at frequency f_1, as follows:

$$x_{SSR}(t) = 0.1 \cdot |\Theta(2\pi f_1)| \ \sin(2\pi f_1 t + \angle\Theta(2\pi f_1 t))$$

The zero input response for a slightly damped system can be written in the following form:

$$x_{ZIR}(t) = e^{-\alpha t}\left[A \cdot \cos(\omega_d t) + B \cdot \sin(\omega_d t)\right]$$

where A and B are two constant that will be determined by imposing the initial conditions of the system, while ω_d is the natural frequency of the damped system, defined as $\omega_d = \sqrt{\omega_0^2 - \alpha^2} \approx 502.98$. Since in our case α is really small, $\omega_d \approx \omega$. Thus, the complete response of the system is:

$$\begin{aligned} x(t) &= x_{ZIR}(t) \ + \ x_{SSR}(t) = \\ &= e^{-\alpha t}\left[A \cdot \cos(\omega_d t) + B \cdot \sin(\omega_d t)\right] \ + \ 0.1 \cdot |\Theta(2\pi f_1)| \ \sin(2\pi f_1 t + \angle\Theta(2\pi f_1 t)) \end{aligned}$$

We now proceed to find the values for the two constants A and B by imposing the given initial conditions. Since the system is at rest for $t < 0$, we can write:

$$\begin{cases} x(t=0) = x_0 = 0 \\ \dot{x}(t=0) = v_0 = 0 \end{cases}$$

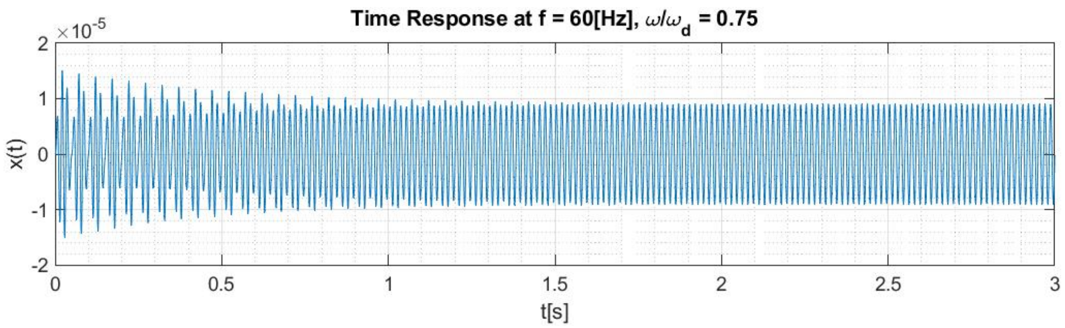
Imposing these conditions we get:

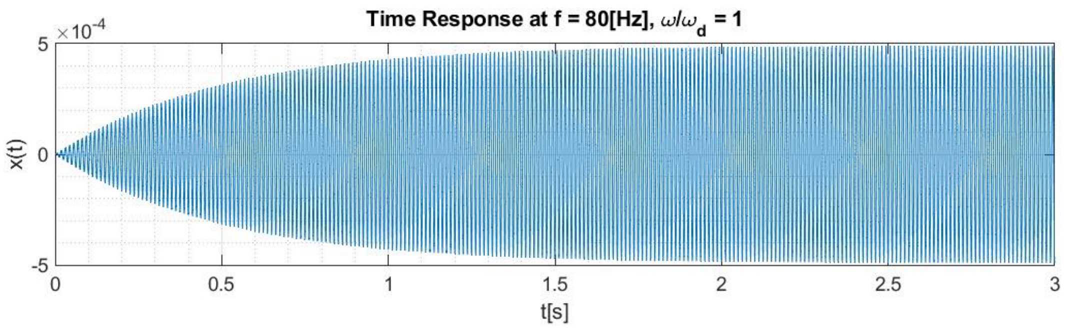
$$\begin{cases} x(t=0) = 0 = A + 0.1 \cdot |\Theta(2\pi f_1)| \sin(\angle\Theta(2\pi f_1)) \\ \dot{x}(t=0) = 0 = -\alpha A + B\omega_d + 2\pi f_1 \cdot |\Theta(2\pi f_1)| \cos(\angle\Theta(2\pi f_1)) \end{cases}$$

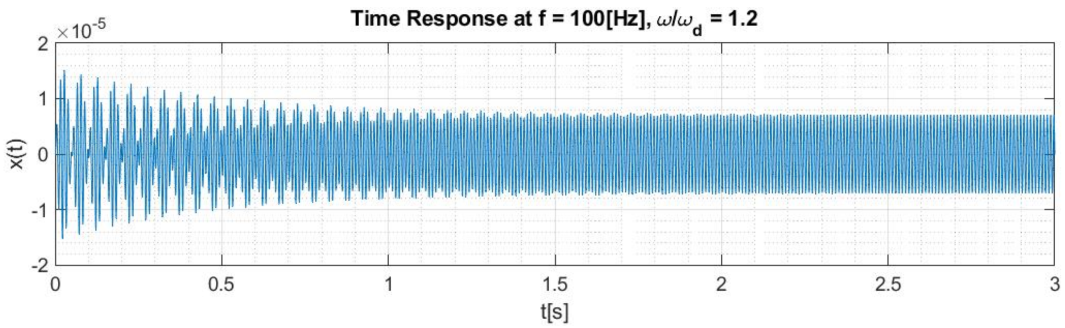
By solving for A and B, we obtain the two constant of the equation (which will obviously depend on the frequency of the exciting force).

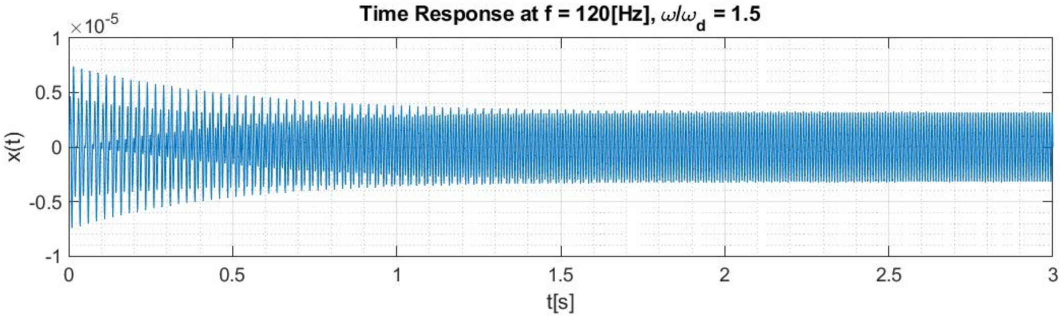
$$\begin{cases} A = -0.1 \cdot |\Theta(2\pi f_1)| \sin(\angle\Theta(2\pi f_1)) \\ B = [\alpha A - 2\pi f_1 \cdot |\Theta(2\pi f_1)| \cos(\angle\Theta(2\pi f_1))]/\omega_d \end{cases}$$

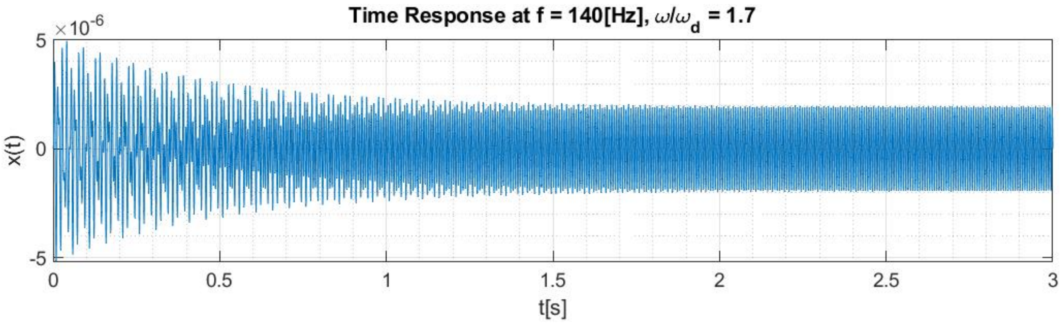
We will now plot the different time responses, according to the various frequency of the external force:

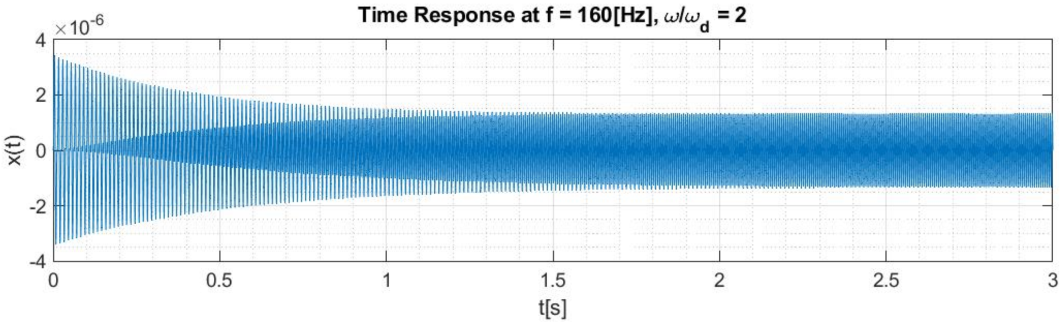












We can highlight that the system responds in different ways according to the input force frequency: this is due to the fact that the amplitude and phase of the receptance depend on such frequency.

It is important to notice that in the case of $f = 80\text{Hz}$, where the frequency of the input force is almost equal to f_0, the amplitude of oscillation grows much larger than in the other cases.

Conclusions

In conclusion, this study comprehensively characterized the resonator system, providing valuable insights into its dynamic behavior and response to external forces. The obtained results, including resonance frequency, decay time, quality factor, resistance, bandwidth, and time responses, contribute to a deeper understanding of the system's performance.