Assignment

Homework HW2

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Musical Acoustics

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Introduction

In this study, we delve into the dynamic behavior of a vibrating plate, investigating various aspects of its mechanical response.
We'll first calculate the propagation speed of quasi-longitudinal and longitudinal waves within the plate, applying additionally a comprehensive analysis of the plate's bending waves and the resonance frequencies of the plate itself.
We'll then establish a novel coupling with a cord, aiming to understand the dynamics of this interaction. The coupling between the plate and the cord promises to unravel valuable information regarding the mutual influence between the two elements, allowing us to get a better understanding of the phenomena for many applications in various engineering and scientific domains.

Characterization of the Plate

In this project we will consider an aluminium square thin plate with clamped edges, which has dimensions $a = 0.15$ m and thickness of $s = 1$ mm.
Given the material composition, we can infer certain data, including $E = 69$ GPa,
$\rho = 2700$ kg/m^2, $\nu = 0.334$.

Propagation speed of quasi-longitudinal and longitudinal waves

Now we will consider longitudinal and quasi-longitudinal waves

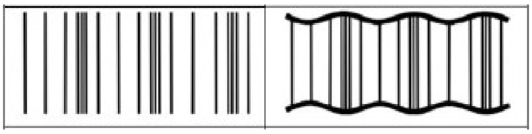

Figure 1: Longitudinal and quasi-longitudinal waves.

Longitudinal waves in plates do not propagate at the same speed as longitudinal waves in plates of the same material. This is due to the fact that when the wave propagates, it causes a slight expansion in the dimension orthogonal to the wavefront (due to poisson's ratio). These type of waves are called **"quasi-longitudinal waves"**. In order to calculate the speed of this kind of waves, we can use the following formula:

$$c_L = \sqrt{\frac{E}{\rho(1-\nu^2)}} \approx 5.36 \cdot 10^3 \text{ m/s}$$

While for longitudinal waves, which occur only in solids where all the dimensions are much greater than the wavelength, we can use the following expression of the velocity:

$$c_L' = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \approx 6.20 \cdot 10^3 \text{ m/s}$$

Propagation speed of the bending waves

The speed of bending waves for the considered plate is not constant for assigned material and plate thickness, but it depends on frequency. It obeys the following formula:

$$v(f) = \frac{\omega}{k} = \sqrt{\frac{\omega h c_L}{\sqrt{12}}} \text{ m/s}$$

As for bending waves in beams, bending waves travelling in thin plates are dispersive. We now proceed to plot it, highliting the direct proportionality between the bending wave velocity and the square root of the frequency:

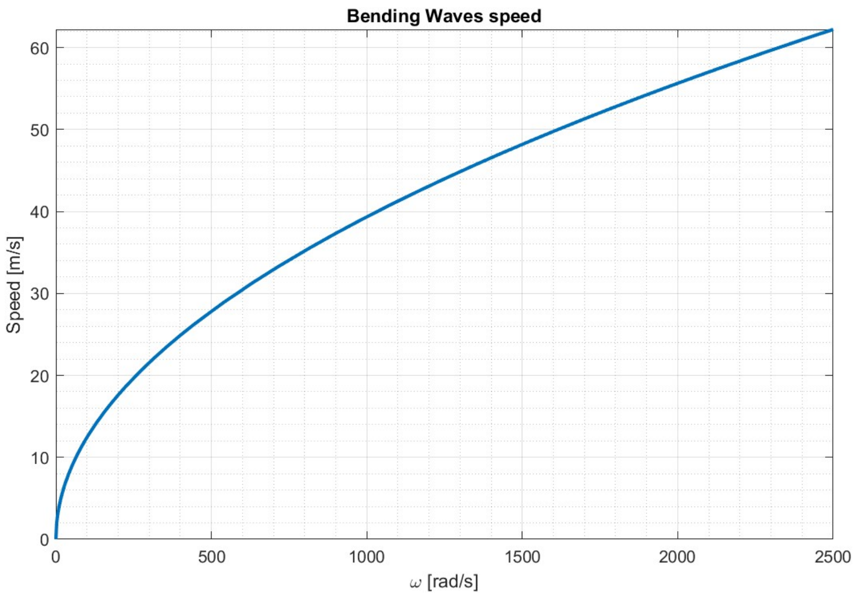

Figure 2: Bending Wave Speed.

Modal frequencies of the first six bending modes of the plate

In order to compute the modal frequencies of the first six bending modes of the plate, we first have to compute the natural frequency of the first vibration bending mode:

$$f_{00} = \frac{1.654 c_L h}{L^2} = 394.25 \text{ Hz}$$

We can now compute the first six bending modes frequencies of the plate by multiplying the natural frequency of the first mode by the following tabulated coefficients (3), one for every mode of the plate:

$$f_{Bi} = f_{00} \cdot C_i$$

Therefore:

$$f_{Bi} = [394.25, 804.28, 804.28, 1186.70, 1443, 1446.90] \text{ Hz for } i = 1, 2, ..., 6$$

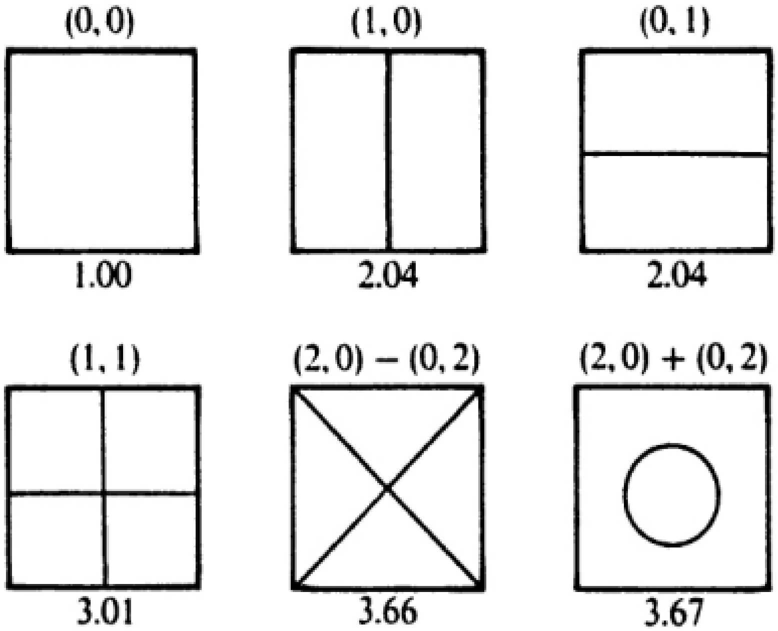

Figure 3: First 6 modes of a square plate.

Predict the length b of the side parallel to the fibers so that the ring and x modes can be observed, assuming the plate to be made by Sitka spruce and it is realized using the quarter-cut scheme.

The "quarter-cut scheme" is a wood cutting method, in this case used on Sitka spruce, a type of wood known for its lightweight and rigidity. In this method, the tree trunk is cut into quarters; this means that the wood is cut so that the annual growth rings are perpendicular to the surface of the plate. This type of cut is often used to obtain wood boards that are particularly stable and resistant to moisture, making it a common choice in the construction of musical instruments.
We know from theory that, in order to obtain the ring and x modes on a Sitka spruce plate realized using quarter-cut scheme, we need to have this ratio in dimensions:

$$\frac{L_x}{L_y} = 1.994$$

Since the main dimensions of the plate are aligned with the longitudinal and radial directions of the wood, we know that $L_y = a = 0.15$ m. Therefore we can find b as:

$$b = L_x = 1.994 \cdot L_y = 1.994 \cdot a = 0.285 \text{ m}$$

String-Plate Coupling

Consider now that a string is attached to the considered aluminum square plate, and its fundamental mode is tuned to the frequency of the first mode of the plate. The string is made with iron (ρ = 5000 kg/m^3), its cross section is circular with a radius of $r = 0.0011$ m, and its length is $L = 0.45$m. Due to internal losses and sound radiation, the plate at the frequency of the considered mode dissipates energy, and the merit factor is $Q = 25$.

Compute the tension of the string so that its fundamental mode is tuned with the first mode of the soundboard

To compute the tension of the string, we simply equal the formula for the resonance frequency of the first mode for a fixed edges string with the frequency of the first mode of the plate. From there, we can derive T as:

$$f_1 = \frac{1}{2L} \cdot \sqrt{\frac{T}{\mu}} = \frac{1.654 c_L h}{a^2} = f_{00} \implies T = (f_{00} \cdot 2L)^2 \cdot \mu = 2.39 \cdot 10^3 \text{ N}$$

Compute the frequencies of the modes of the string-soundboard system considering the coupling between the plate and the string

In order to define the frequencies of the system, we first need to calculate the resonance frequencies of the string neglecting its stiffness. For this study, we'll use the first 4 eigenfrequencies of the string to compare them with the ones of the plate, since the second and third mode and the fifth and sixth mode of the palte resonate at about the same frequency. These will be obtained with the following equation:

$$f_n = n \cdot f_1 = [394.25, 788.52, 1182.77, 1577.03] \text{ Hz, for } n = 1, 2, 3, 4$$

We now proceed to define the coupling between the string and the membrane. This is given by the following condition:

$$\begin{cases} \text{Weak Coupling} & \iff \frac{m}{M \cdot n^2} < \frac{\pi^2}{4 \cdot Q^2} \\ \text{Strong Coupling} & \iff \frac{m}{M \cdot n^2} > \frac{\pi^2}{4 \cdot Q^2} \end{cases}$$

where n is the is the mode number of the string and m and M are the mass of the string and the soundboard, respectively. Let's calculate them:

$$m = L \cdot \mu = L \cdot \pi r^2 \cdot \rho = 0.45 \text{ m} \cdot \pi \cdot (0.0011 \text{ m})^2 \cdot 5000 \text{ kg/m}^3 = 8.6 \cdot 10^{-3} \text{ kg}$$

$$M = a^2 \cdot s \cdot \rho = (0.15 \text{ m})^2 \cdot 10^{-3} \text{ m} \cdot 2700 \text{ kg/m}^3 = 6.0 \cdot 10^{-2} \text{ kg}$$

And now we can calculate the coupling for the various values of n. Since the coupling condition is inversely related to n, starting with $n = 4$, we obtain that:

$$\frac{m}{M \cdot n^2} = 8.8 \cdot 10^{-3} > 3.9 \cdot 10^{-3} = \frac{\pi^2}{4 \cdot Q^2}$$

Therefore, we have strong coupling for all the modes considered in this study, since for smaller values of n the strong coupling condition will still be satisfied.
To calculate the frequencies of the system, we can rely on the following graph, which shows the normal mode frequencies of a string coupled to a soundboard as a function of their uncoupled frequencies ω_S and ω_B for strong coupling.

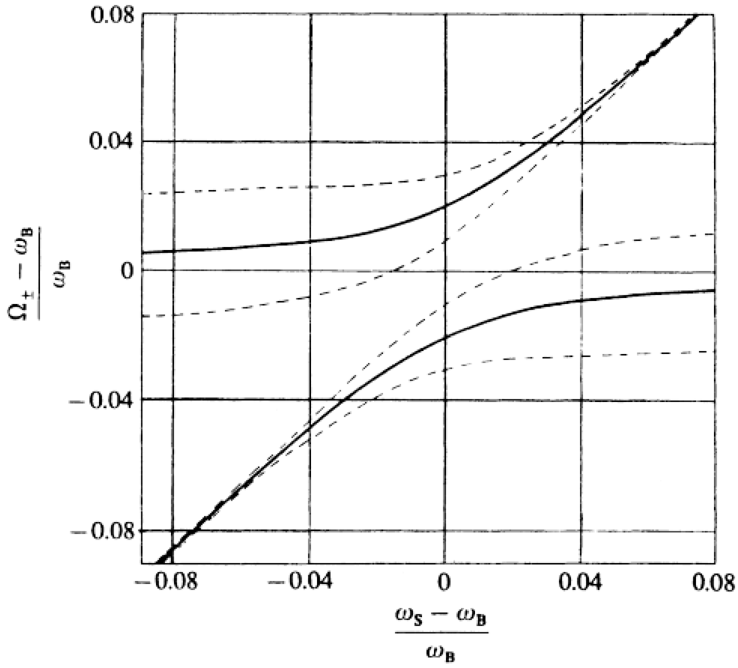

Figure 4: Normal mode frequencies of a string coupled to a soundboard for strong coupling.

We now calculate the relation between the string's and board's frequencies, approximating the results to the second significant figure, in accordance with the data on the graph (4).

$$\frac{f_{Si} - f_{Bi}}{f_{Bi}} \approx [0, -0.02, -0.02, 0, 0.09, 0.09] \text{ for } i = 1, 2, ..., 6$$

where we recall that (doubling the frequencies for the second and fourth mode of the string):

$$f_{Si} = [394.25, 788.52, 788.52, 1182.77, 1577.03, 1577.03] \text{ Hz}$$

and for the plate:

$$f_{Bi} = [394.25, 804.28, 804.28, 1186.70, 1443, 1446.90] \text{ Hz}$$

We can now obtain the frequencies $\Omega_\pm$ accordingly with the graph.

n=1

We consider the first mode of the string and the first mode of the board. Since the two modes have the same fundamental frequency, we obtain that the normal frequencies of the coupled system will be:

$$\begin{cases} \Omega_+ = f_{B1} + 0.02 \cdot f_{B1} \approx 402.14 \text{ Hz} \\ \Omega_- = f_{B1} - 0.02 \cdot f_{B1} \approx 386.37 \text{ Hz} \end{cases}$$

n=2

We consider the second mode of the string and the second or third mode of the board. The relation between the two frequencies gives a result of -0.02, so we can obtain from the graph:

$$\begin{cases} \Omega_+ = f_{B2} + 0.01 \cdot f_{B2} \approx 812,32 \text{ Hz} \\ \Omega_- = f_{B2} - 0.04 \cdot f_{B2} \approx 772.11 \text{ Hz} \end{cases}$$

n=3

We consider the third mode of the string and the fourth mode of the board. Since the two modes have about the same fundamental frequency, we obtain that the normal frequencies of the coupled system will be:

$$\begin{cases} \Omega_+ = f_{B3} + 0.02 \cdot f_{B3} \approx 1210.43 \text{ Hz} \\ \Omega_- = f_{B3} - 0.02 \cdot f_{B3} \approx 1162.96 \text{ Hz} \end{cases}$$

n=4

We consider the fourth mode of the string and the fifth or sixth mode of the board. The relation between the two frequencies gives a result of 0.09, that exceeds the range considered by the graph. We can notice how, for values higher than 0.08, the lower curve of the graph goes asymptotically to zero, while the upper curve acts as a straight line along the diagonal: this means that one of the frequencies of the system will be equal to the resonance frequency of the plate, while the other will be equal to the resonance frequency of the string itself.

Conclusions

Concluding, this study provides a comprehensive understanding of the dynamic behavior and interactions of the plate-string system, addressing the acoustic properties, resonance modes, and tuning requirements. The obtained results offer valuable insights into the design and performance of musical instruments and soundboards, as well as the physics of material interactions in such systems.