POLITECNICO
MILANO 1863

Homework Vibration Analysis and Vibroacoustics 22/23

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Contents

Introduction

The objective of this assignment is to analyze the functioning of a grand piano. To achieve this, we have divided the problem into three different, interconnected, simpler problems. By putting all the analysis together we will arrive at the global resolution of the problem.

1 Vibration of a String

First we study the transverse vibration of the string separated from the plate. In order to do so we adopt the model represented in figure (1), in which the spring on the left side simulates the bridge between the string and the harmonic plate. The first thing that we have to do in order to describe the string's vibration is to calculate the natural frequencies and the mode shapes in the frequency range of our interest [0-2000Hz].

Figure 1: Simplified scheme for a vibrating string.

1.1 Natural frequencies and mode shapes

To calculate natural frequencies and mode shapes we can use the "standing wave solution".

$$w(x,t) = \Phi(x) \cdot G(t) \tag{1}$$

To find the natural frequencies and the mode shapes we have to combine this equation with the boundary conditions.

$$\begin{cases} w(x,t) = \Phi(x) \cdot G(t) \\ \\ T sen(\alpha) = k_1 w(x,t)|_{x=0} \\ \\ T sen(\alpha) = 0 \end{cases} \tag{2}$$

We are in the hypothesis of small displacements, so $sen(\alpha) \approx \alpha$, and $\alpha = \frac{\partial \Phi(x)}{\partial x}$. We can also simplify the dependency w.r.t. time because the boundary conditions must be valid for every time instant.

$$\begin{cases} \Phi(x) = A sin(kx) + B cos(kx) \\ \\ T\dfrac{\partial \Phi(x)}{\partial x}|_{x=0} = k_1 \Phi(x)|_{x=0} \\ \\ T\dfrac{\partial \Phi(x)}{\partial x}|_{x=L} = 0 \end{cases} \tag{3}$$

$$\begin{bmatrix} Tk & -k_1 \\ Tkcos(kL) & -ksin(kL) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = 0 \Rightarrow [H] \begin{Bmatrix} A \\ B \end{Bmatrix} = 0 \tag{4}$$

The matrix [H] contains the values relating the displacement of the string to the tension, varying the frequency. The solution of this problem is $det([H]) = 0$ Remembering that $k = \frac{\omega}{c}$, $k = k(\omega)$ and so we have an [H] matrix for every ω Using Matlab we can plot the determinant of [H] matrix for different k_i values and we can find the minimums that correspond to the natural frequencies of the system.

$$H(\omega) = \begin{bmatrix} \dfrac{T\omega}{c} & -k_1 \\ \dfrac{T\omega}{c}cos(\dfrac{\omega}{c}L) & -\dfrac{\omega}{c}sin(\dfrac{\omega}{c}L) \end{bmatrix} \tag{5}$$

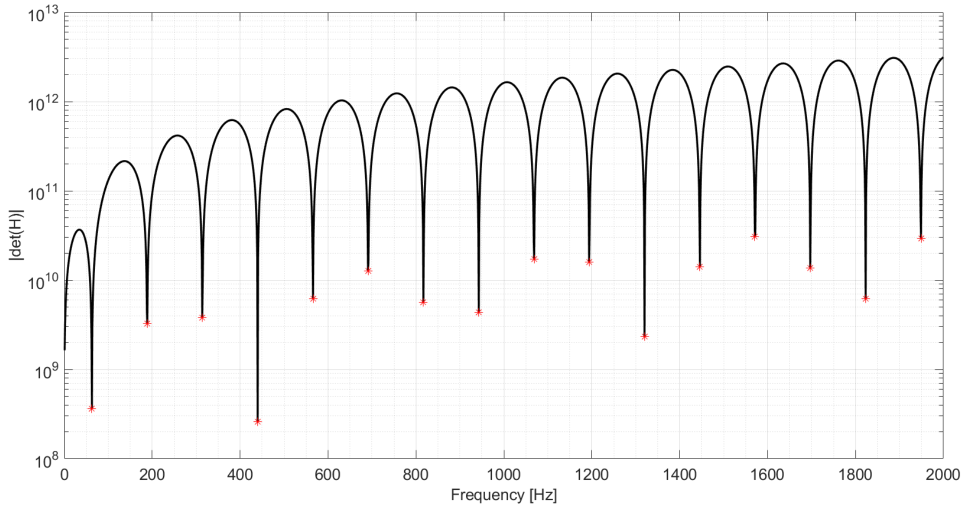

Figure 2: Magnitude of the determinant of the H matrix and identified natural frequencies f_0

We can now compute and draw the mode shapes of the string imposing A=1 and calculating B through the previous system:

$$B = \frac{Tk_i}{k_1} = \frac{T\omega_{0i}}{ck_1} \tag{6}$$

And so the mode shapes of the system in function of the i-esimal natural frequency ω_{0i} can be described by:

$$\Phi(x) = sin(\frac{\omega_{0i}}{c}x) + \frac{T\omega_{0i}}{ck_1}cos(\frac{\omega_{0i}}{c}x) \tag{7}$$

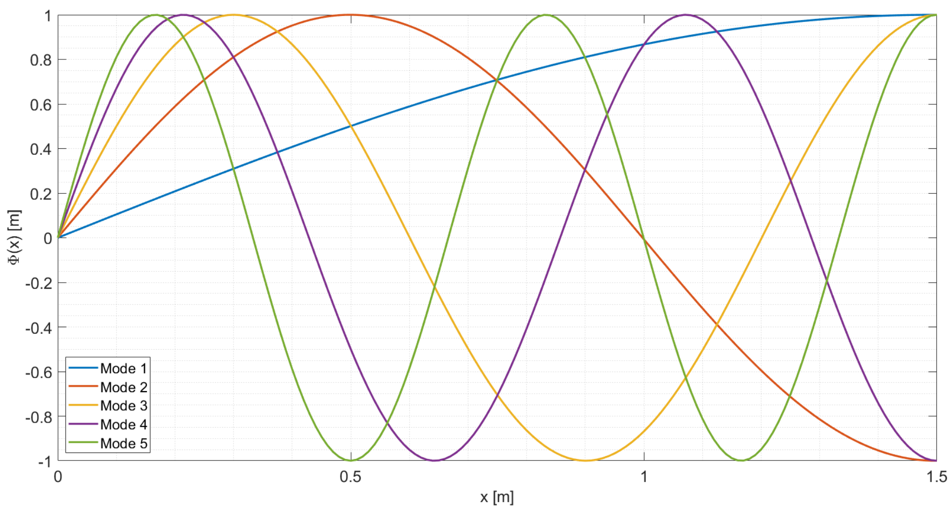

Figure 3: Mode shapes of string vibration for the first five vibrating modes.

1.2 Drive Point Receptance and Transfer Receptance

Now we have to calculate the receptance, a typology of the frequency response function of the system. Depending on where we calculate it, the receptance is called "driving point receptance" (x=L) when the measurement point and the force application point coincide, or "transfer receptance" (x=O) when it is calculated in a different point from the point of application of the force. The receptance can be defined as:

$$G(j\omega) = \frac{w(x,t)}{F(t)} \tag{8}$$

In this case we have to consider also the force $F = F_0 e^{(j\omega t)}$ and the damping force in the second boundary condition at x=L, neglected before because we have studied the natural vibration modes of the system:

$$\begin{cases} w(x,t) = \Phi(x) \cdot G(t) = [Asin(kx) + Bcos(kx)]e^{j\omega t} \\ \\ T\dfrac{\partial w(x,t)}{\partial x}|_{x=0} - k_1 w(x,t)|_{x=0} = 0 \\ \\ T\dfrac{\partial w(x,t)}{\partial x}|_{x=L} - c_1 \dfrac{\partial w(x,t)}{\partial t}|_{x=L} + F = 0 \end{cases} \tag{9}$$

$$\begin{cases} w(x)|_{x=L} = Asin(kL) + Bcos(kL) \\ \\ TAk - Bk_1 = 0 \\ \\ TAkcos(kL) - TBksin(kL) - c_1 j\omega Asin(kL) + c_1 j\omega Bcos(kL) + F = 0 \end{cases} \tag{10}$$

$$\begin{cases} w(x)|_{x=L} = Asin(kL) + Bcos(kL) \\ \\ A = \left[Tkcos(kL) - \dfrac{k}{k_1}sin(kL) - c_1 j\omega\left(sin(kL) + T\dfrac{k}{k_1}cos(kL)\right)\right]^{-1} \\ \\ B = \dfrac{ATk}{k_1} \end{cases} \tag{11}$$

Now we can compute the drive point receptance:

$$G(j\omega) = \frac{w(L,t)}{F(t)} \tag{12}$$

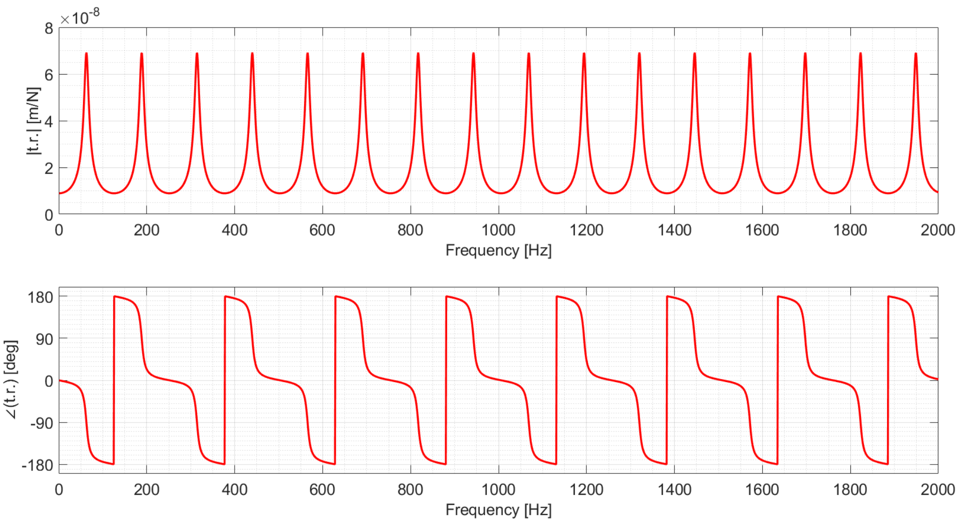

Figure 4: Drive point recepance represented in module and phase.

In the same way we can compute the transfer receptance:

$$G(j\omega) = \frac{w(0,t)}{F(t)} \tag{13}$$

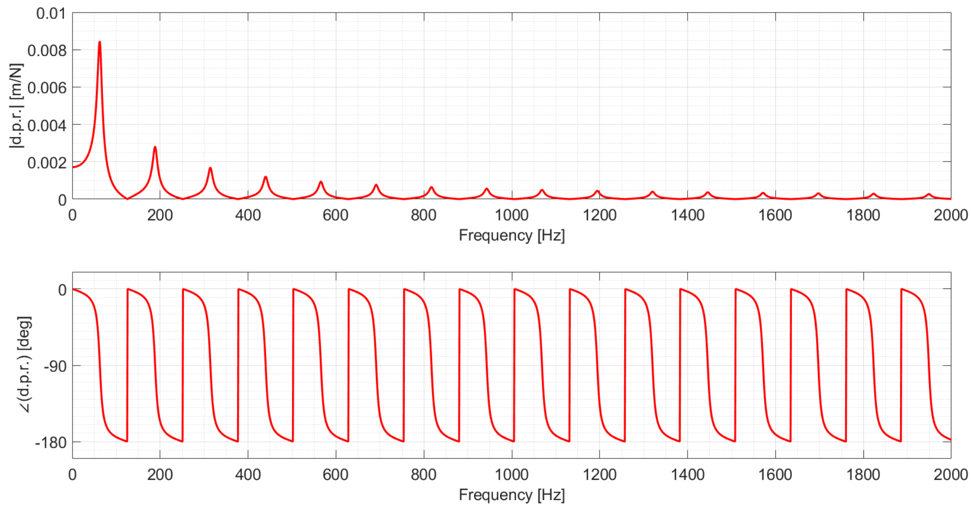

Figure 5: Transfer recepance represented in module and phase.

1.3 Reaction Force transmitted to the plane

In order to find the reaction force transmitted to the plate (the starting point of the second part of the analysis) we have to calculate the "Amplification factor", the ratio between the reaction and the force:

$$A = \frac{R(t)}{F(t)} \tag{14}$$

We can impose $A = R_0$ if we consider a unitary force module:

$$A = \frac{R(t)}{F(t)} = \frac{R_0 e^{(j\omega t)}}{1 \cdot e^{(j\omega t)}} = R_0 = k_1 \cdot G(j\omega) \tag{15}$$

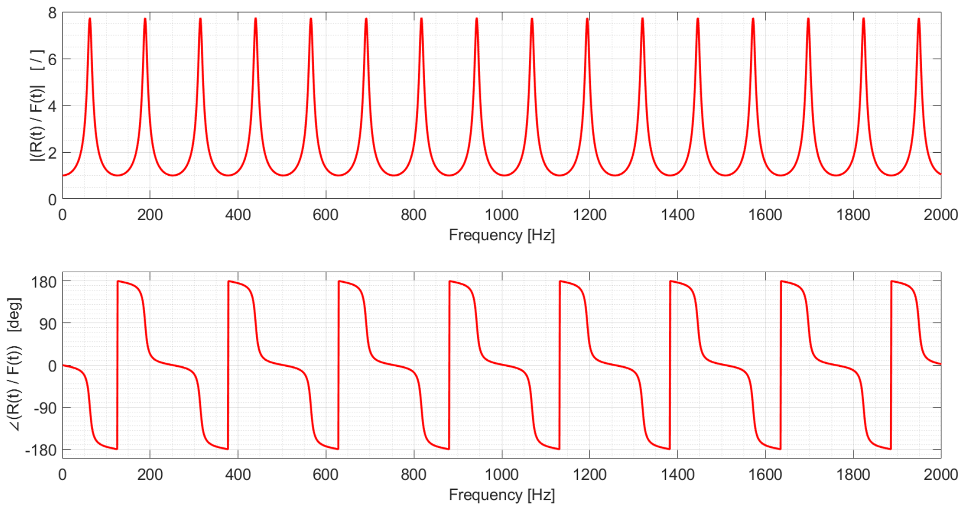

Figure 6: Module and phase of the amplification factor A

2 Vibrations of the harmonic plate

Having found the reaction force transmitted to the plate, the attention of our analysis moves to the study of the vibration of a thin plate:

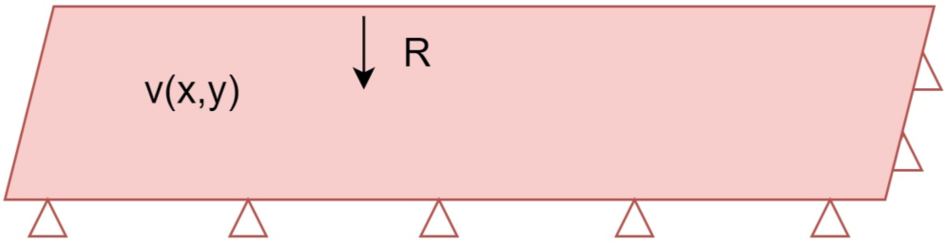

Figure 7: The model for the vibrating plate (the soundboard of the piano).

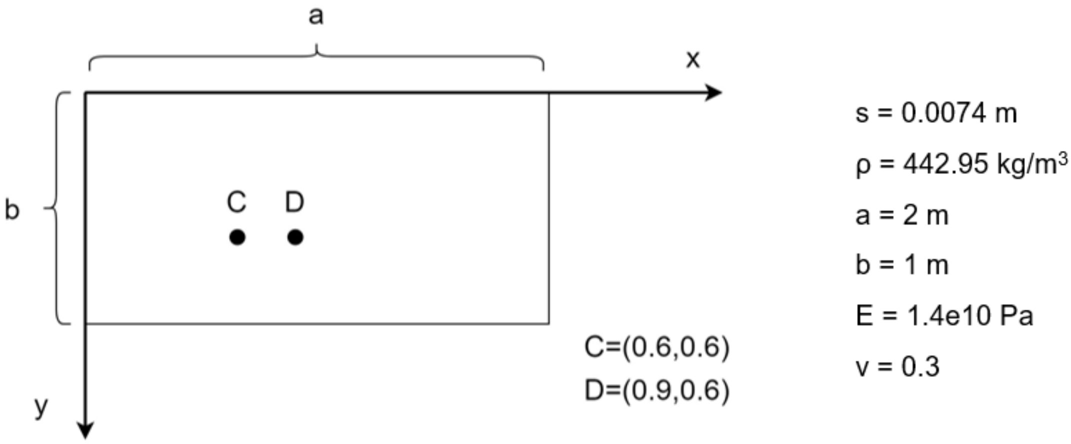

Figure 8: Structural data of the plate.

2.1 Natural frequencies

The first thing to do to study the plate problem is, as for the string, finding the natural frequencies of the system. We can use this formula:

$$\omega_{n,m} = \left((\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \right) \sqrt{\frac{B}{\rho s}} \tag{16}$$

$$\text{with } B = \frac{E}{1-\nu^2} \frac{s^3}{12} \tag{17}$$

Where ρ is the density of the plate and B is the plate's bending stiffness, which depends on the Poisson's ratio ν, the Young's modulus of the material E and the thickness of the plate s.

We can therefore change only the values of n and m from 1 to 10 to compute the table of frequencies of our interest:

[Hz]	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7	m = 8	m = 9	m = 10
n = 1	24,72	84,05	182,92	321,35	499,33	716,86	973,94	1270,57	1606,76	1982,49
n = 2	39,55	98,88	197,75	336,18	514,16	731,69	988,77	1285,41	1621,59	1997,32
n = 3	64,27	123,60	222,47	360,90	538,88	756,41	1013,49	1310,13	1646,31	2022,04
n = 4	98,88	158,20	257,08	395,51	573,49	791,02	1048,10	1344,73	1680,92	2056,65
n = 5	143,37	202,70	301,58	440,00	617,98	835,51	1092,60	1389,23	1725,41	2101,14
n = 6	197,75	257,08	355,96	494,39	672,37	889,90	1146,98	1443,61	1779,79	2155,53
n = 7	262,03	321,35	420,23	558,66	736,64	954,17	1211,25	1507,88	1844,06	2219,80
n = 8	336,18	395,51	494,39	632,82	810,79	1028,32	1285,41	1582,04	1918,22	2293,96
n = 9	420,23	479,56	578,43	716,86	894,84	1112,37	1369,45	1666,08	2002,27	2378,00
n = 10	514,16	573,49	672,37	810,79	988,77	1206,30	1463,39	1760,02	2096,20	2471,93

Figure 9: Natural frequencies related to the modes 10x-10y

Now we can also find the modes for a certain frequency using the following formula:

$$\phi_{n,m}(x,y) = sin(\frac{n\pi}{a}x)sin(\frac{m\pi}{b}y) \tag{18}$$

We choose $n = 5$ and $m = 4$ in order to evaluate the mode shape corresponding to the frequency of 440 Hz (as we can see from the previous table). At this point we can use the `meshgrid` and `surf` commands of Matlab to plot this mode shape:

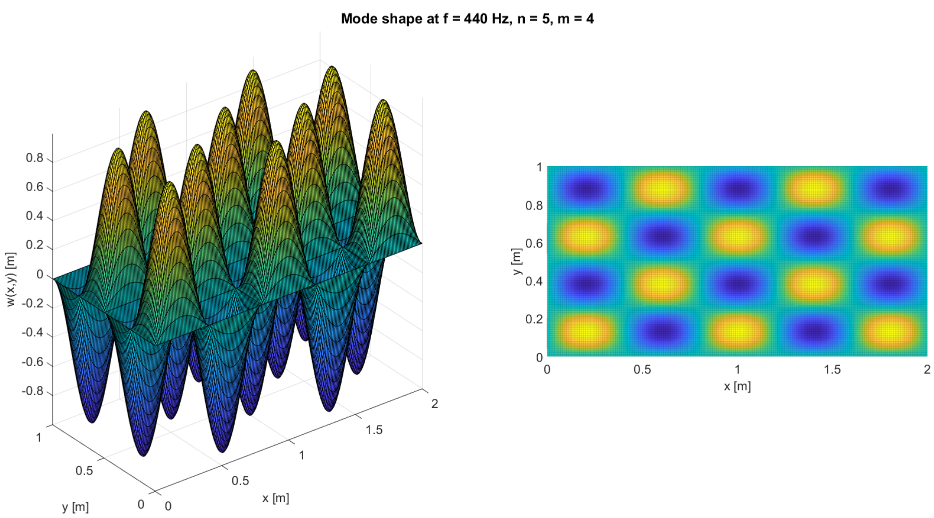

Figure 10: Mode shape for 5x - 4y.

2.2 Modal Approach

In order to study the system by means of the modal approach we first need to compute the modal mass matrix:

$$[M] = \begin{bmatrix} m_{1,1} & \dots & m_{1,100} \\ \dots & \dots & \dots \\ m_{100,1} & \dots & m_{100,100} \end{bmatrix} \tag{19}$$

with:

$$m_{i,j} = \iint_A \mu \phi_i \phi_j \, dA \tag{20}$$

where μ is the mass per unit area and ϕ_i and ϕ_j are the mode shapes calculated before. The modal stiffness matrix can now be computed starting from the modal mass matrix $[M]$ by calculating the $k_{i,j}$ elements as follows:

$$k_{i,j} = m_{i,j}\omega_{i,j}^2 \tag{21}$$

Now that we have $[M]$ and $[K]$, which are diagonal matrices, we can compute the modal damping matrix $[C]$ as a linear combination of them following the Rayleigh damping formula:

$$[C] = \alpha[M] + \beta[K] \tag{22}$$

In our case we'll assume $\alpha = 0.01$ and $\beta = 10^{-5}$. This way we successfully diagonalize $[C]$.

2.3 Point Mobility and Transfer Mobility

Now, having the modal matrices, we can compute the vibration velocity of a generic point belonging to the plate as a superposition of all the 100 modes of our interest:

$$v(\tilde{x}, \tilde{y}) = j\Omega R_0 e^{j\Omega t} \sum_{i=1}^{100} \frac{\phi_i(\tilde{x}, \tilde{y})\phi_i(x_D, y_D)}{-\Omega^2 m_i + j\Omega c_i + k_i} \tag{23}$$

To obtain the mobility w.r.t. the input force applied to the string, we need to use the force amplification ratio $A = \frac{R(t)}{F(t)}$ obtained in 1.3:

$$\frac{v(\tilde{x}, \tilde{y})}{F_0 e^{(j\Omega t)}} = \frac{v(\tilde{x}, \tilde{y})}{R_0 e^{(j\Omega t)}} \frac{R}{F} \tag{24}$$

We can particularize the previous formula (23) to calculate the **Driving point mobility** and the **Transfer mobility** by replacing $\tilde{x}$ and $\tilde{y}$ with the coordinates of D (for driving point) and C (for transfer):

$$v(x_D, y_D) = j\Omega R_0 e^{j\Omega t} \sum_{i=1}^{100} \frac{\phi_i^2(x_D, y_D)}{-\Omega^2 m_i + j\Omega c_i + k_i} \tag{25}$$

$$v(x_C, y_C) = j\Omega R_0 e^{j\Omega t} \sum_{i=1}^{100} \frac{\phi_i(x_C, y_C)\phi_i(x_D, y_D)}{-\Omega^2 m_i + j\Omega c_i + k_i} \tag{26}$$

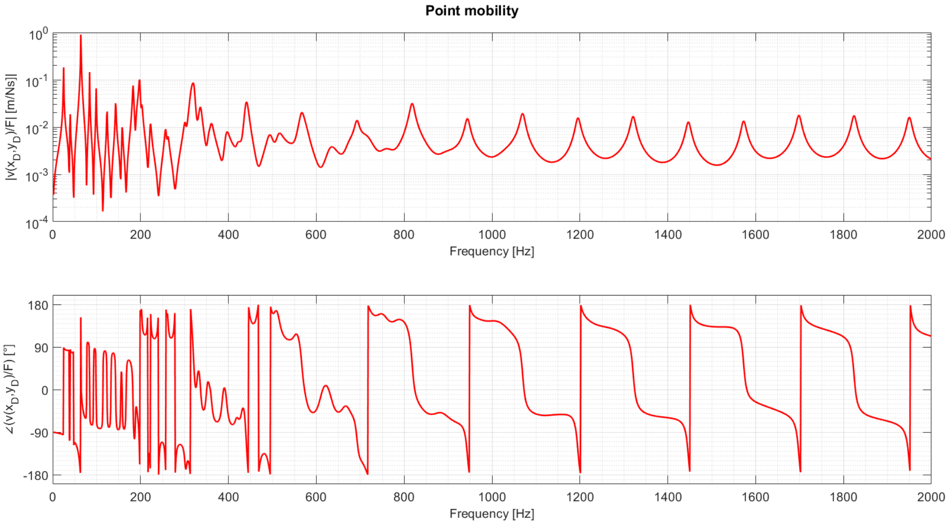

Figure 11: Point mobility.

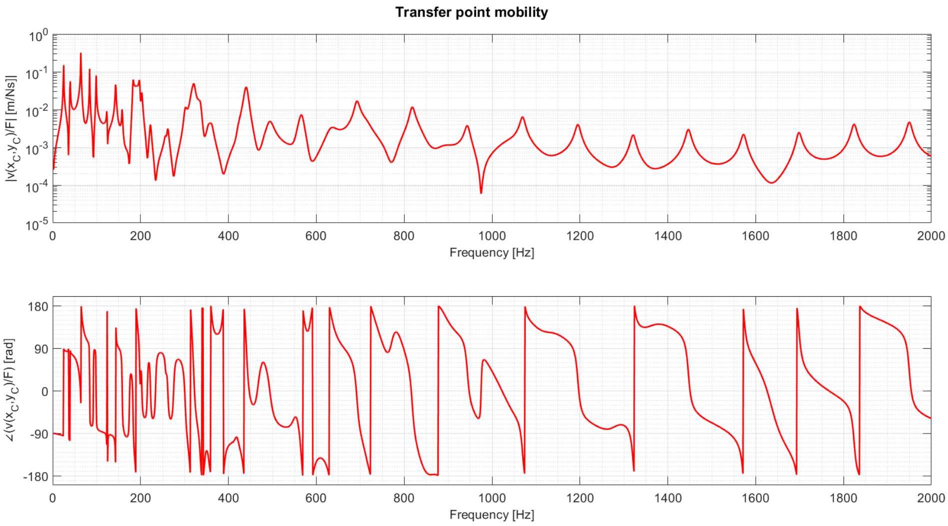

Figure 12: Transfer mobility.

2.4 Vibration velocity of the plate at 440 Hz

Let's now describe the vibration velocity of every point of the plate when excited by a unitary force with frequency of 440 Hz. In order to so we just need to evaluate the relation (23) for every point of the plate. The results are shown in the figures (13) and (14) below.

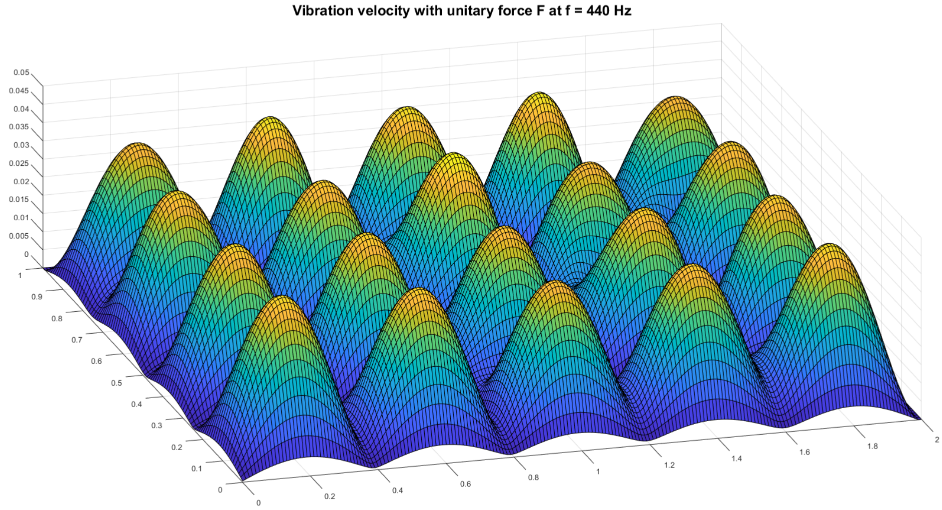

Figure 13: Vibration velocity for a unitary force at 440Hz.

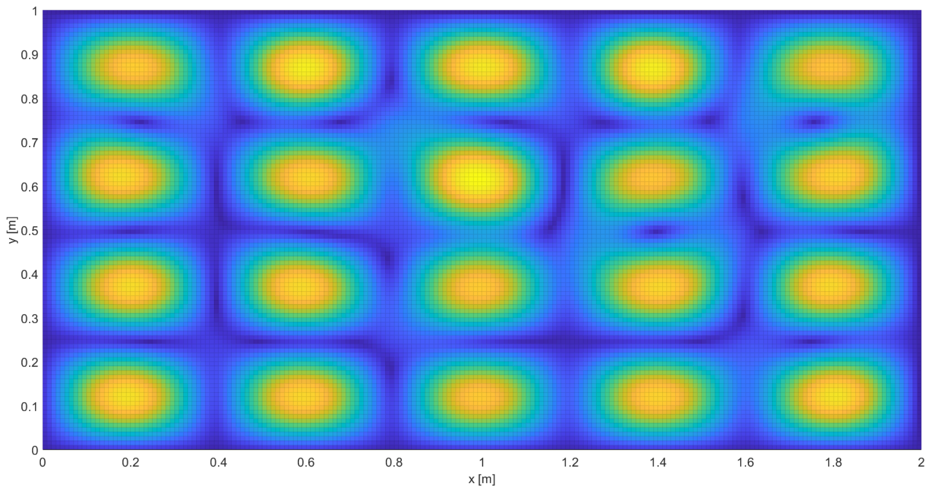

Figure 14: Vibration velocity for a unitary force at 440Hz.

3 Sound Radiation from Plate-like Structures

Now, having previously computed all the values of the vibration velocity for all the points of the plate and for all the excitation frequencies, let's move on to the third and last point of our analysis: the study of sound radiation. We can synthesize our system in the simplified model of Fig. (15), which includes the elements considered in all of the three points of this study.

3.1 Sound Radiation - Single Measurement Position

We will consider a sphere-shaped space with a radius of 4 meters around the plate. To carry out this analysis we have to calculate the SPL (sound pressure level) at each point of the hemisphere, to obtain a precise graph of the pressure trend in space.

In order to derive the value of the SPL in a certain point at a certain frequency we need to evaluate this double integral derived from the Kirchhoff-Helmholts integral equation:

$$p(\vec{r}) = j\omega\rho_0 \iint_S v(x_0, y_0) G(x, y, z|x_0, y_0, 0) dx_0 dy_0 = j\omega\rho_0 \iint_S v(x_0, y_0) \frac{e^{-jkr}}{2\pi r} dx_0 dy_0 \tag{27}$$

In this formula, r is the distance of the point with coordinates x, y, z from an infinitesimal

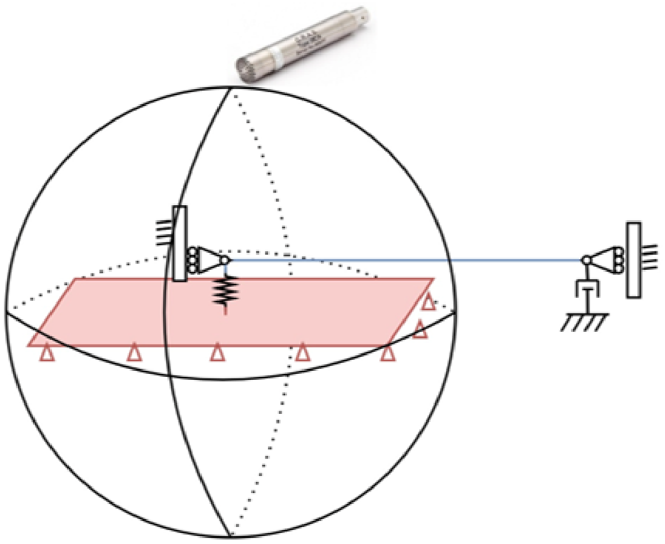

Figure 15: Sound radiation simplified model

surface area element of the plate (x_0 y_0), ω is the considered frequency and ρ_0 is the density of air.

3.2 Multiple microphones measurement

Now we will evaluate the SPL percieved by ten microphones positioned on the semisphere accordingly to the EN 60704-1:2021 standard, as shown in figure (16).

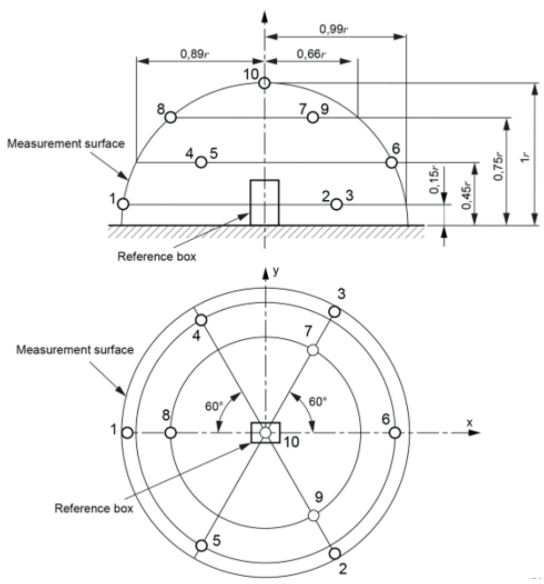

Figure 16: Measurement positions.

In order to do so we can use equation (27) which will allow us to evaluate the SPL in the microphones positions. Starting from the SPL defined for each frequency, we want to compute for each microphone the SPL of a one-third octave frequency band. These bands are reported in figure (17).

Octave band centre frequency (Hz)	One-third-octave band centre frequency (Hz)	Band frequency limits (Hz) Lower	Upper
	25	22	28
31.5	31.5	28	35
	40	35	44
	50	44	57
63	63	57	71
	80	71	88
	100	88	113
125	125	113	141
	160	141	176
	200	176	225
250	250	225	283
	315	283	353
	400	353	440
500	500	440	565
	630	565	707
	800	707	880
1000	1000	880	1130
	1250	1130	1414
	1600	1414	1760
2000	2000	1760	2250
	2500	2250	2825
	3150	2825	3530
4000	4000	3530	4400
	5000	4400	5650
	6300	5650	7070
8000	8000	7070	8800
	10 000	8800	11 300
	12 500	11 300	14 140
16 000	16 000	14 140	17 600
	20 000	17 600	22 500

Figure 17: Frequencies reference for computing SPL.

The SPL for each microphone position relative to the i-esimal band can be computed with the following relation:

$$L_{p,i} = 10log_{10}\left(\frac{\sum_{f_{i,min}}^{f_{i,max}} \frac{|p_j|^2}{2}}{p_{ref}^2}\right) \tag{28}$$

where

$$p_{ref} = 2 \cdot 10^{-5} Pa \tag{29}$$

The results for frequency bands centered at 250 Hz and 500 HZ are reported in the following table. These results will later be used to compute the whole hemisphere radiation pattern.

SPL [dB]	**Mic 1**	**Mic 2**	**Mic 3**	**Mic 4**	**Mic 5**	**Mic 6**	**Mic 7**	**Mic 8**	**Mic 9**	**Mic 10**
250 Hz	82.83	79.22	79.50	76.63	79.37	83.66	76.79	81.40	79.72	79.19
250 Hz	85.98	82.06	88.48	86.61	86.15	87.86	82.40	82.74	84.08	71.10

Table 1: Measured SPL at the ten microphones.

3.3 SPL 3D representation of one-third octave bands

Finally, we plot the SPL of the one-third octave bands for each point of the semisphere. In order to do so, the 10 measurements obtained in the previous point and listed in Table 1 are interpolated through a weighted average of the inversed distance.

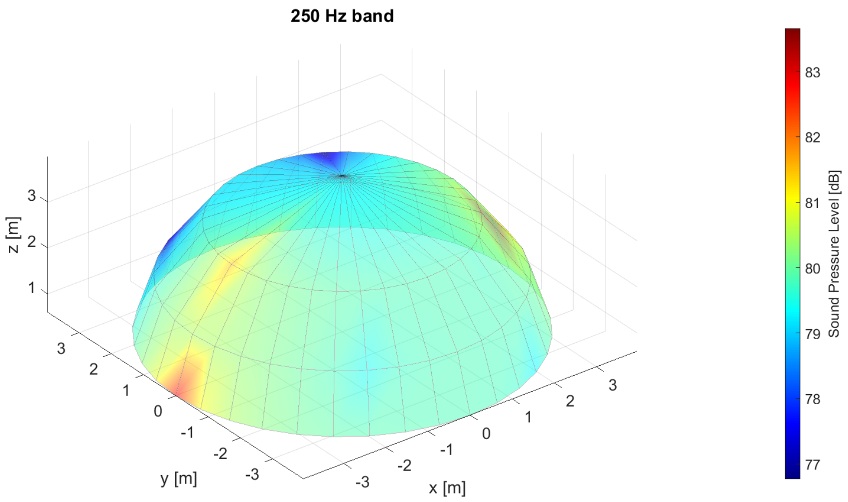

Figure 18: SPL on the semisphere at 250 Hz

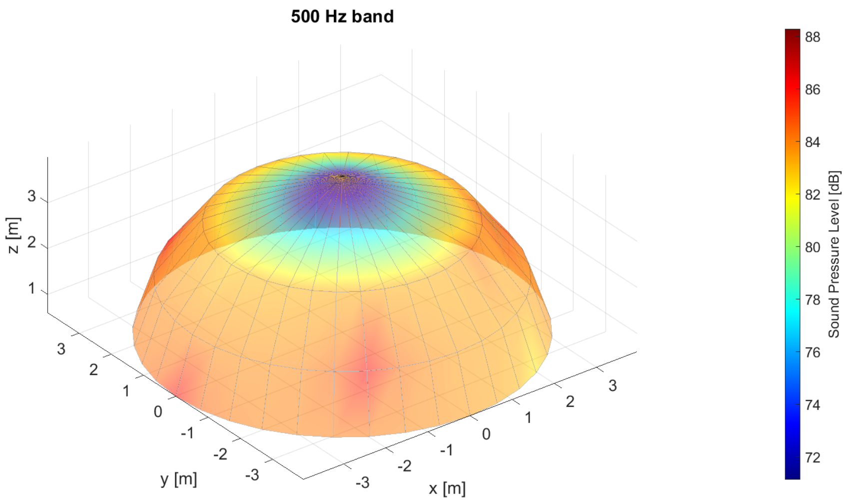

Figure 19: SPL on the semisphere at 500 Hz