Problem 5

Prove that for any integer n, at least one of the integers n, $n+2$, $n+4$ is divisible by 3.

PROOF In order to prove the statement, let's look at the division theorem: if $a, b \in \mathbb{Z}$, $b > 0$, then there exist a unique $q, r \in \mathbb{Z}$ such that $a = q \cdot b + r$, $0 \leq r < b$.

So, according to the theorem, it is possible to write a integer a in the form of $q \cdot b + r$.

To start the proof, it is necessary to know that 0, 1 and 2 are the only possible remainders for a division by 3.

$\exists\ k \in \mathbb{Z} \mid n = 3k + r$, where $r \in \mathbb{Z}$ and $0 \leq r < 3$.

Three cases.

$r = 0 \implies 3$ divides n.

$r = 1 \implies n + 2 = 3k + 1 + 2 = 3k + 3 \implies 3$ divides $n + 2$.

$r = 2 \implies n + 4 = 3k + 2 + 4 = 3k + 6 \implies 3$ divides $n + 4$.

So, by replacing n with $3k$, $3k + 1$ and $3k + 2$, I proved that for any integer n, at least one of the integers n, $n + 2$, $n + 4$ is divisible by 3.