Problem 6

A classic unsolved problem in number theory asks if there are infinitely many pairs of "twin primes", pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

PROOF In order to prove the statement, it is important to pay attention in the properties of the number 3. The only possible remainders of this number are 0, 1 and 2.

It is possible to write any numbers with the difference of 2 in this way:

$$n,\ n+2,\ n+4$$

According to the Division Theorem, if $a, b \in \mathbb{Z}$, $b > 0$, then there exist a unique $q, r \in \mathbb{Z}$ such that $a = q \cdot b + r$, $0 \leq r < b$.

$\exists\ k \in \mathbb{Z} \mid n = 3k + r$, where $r \in \mathbb{Z}$ and $0 \leq r < 3$.

Three cases.

$r = 0 \implies 3$ divides n.

$r = 1 \implies n + 2 = 3k + 1 + 2 = 3k + 3 \implies 3$ divides $n + 2$.

$r = 2 \implies n + 4 = 3k + 2 + 4 = 3k + 6 \implies 3$ divides $n + 4$.

So, it is possible to know that at least of the numbers n, $n + 2$ and $n + 4$ will be a multiple of 3, and since one of them will be a multiple of 3, at least of them will not be a prime number, because:

A prime number is a positive integer that is not divisible without remainder by any integer except itself and 1.

With that in mind, the only possible prime triples would be 1,3, 5 and 3, 5, 7. However, 1 is not a prime number.

This proves the statement, the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.