a) Model the capabilities of these employees using a bipartite graph.

b) Find an assignment of responsibilites such that each employee is assigned a responsibility.

28. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

a) Model the possible marriages on the island using a bipartite graph.

b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.

29. How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 in this section is 4, 4, 4, 3, 2, 1, 0.

30. Find the degree sequences for each of the graphs in Exercises 21–25.

31. Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4 d) $K_{2,3}$ e) Q_3

32. What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

33. What is the degree sequence of K_n, where n is a positive integer? Explain your answer.

34. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

35. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

A sequence $d_1, d_2, \ldots, d_n$ is called *graphic* if it is the degree sequence of a simple graph.

36. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 b) 6, 5, 4, 3, 2, 1 c) 2, 2, 2, 2, 2, 2
d) 3, 3, 3, 2, 2, 2 e) 3, 3, 2, 2, 2, 2 f) 1, 1, 1, 1, 1, 1
g) 5, 3, 3, 3, 3, 3 h) 5, 5, 4, 3, 2, 1

37. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 3, 3, 3, 3, 2 b) 5, 4, 3, 2, 1 c) 4, 4, 3, 2, 1
d) 4, 4, 3, 3, 3 e) 3, 2, 2, 1, 0 f) 1, 1, 1, 1, 1

*38. Suppose that $d_1, d_2, \ldots, d_n$ is a graphic sequence. Show that there is a simple graph with vertices 1, 2, ..., n such that $\deg(i) = d_i$ for $i = 1, 2, \ldots, n$ and 1 is adjacent to 2, ..., $d_1 + 1$.

*39. Show that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.

*40. Use Exercise 39 to construct a recursive algorithm for determining whether a nonincreasing sequence of positive integers is graphic.

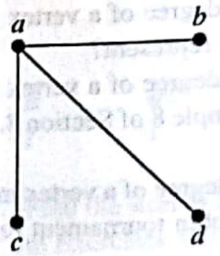
41. Show that every nonincreasing sequence of nonnegative integers with an even sum of its terms is the degree sequence of a pseudograph, that is, an undirected graph where loops are allowed. [*Hint:* Construct such a graph by first adding as many loops as possible at each vertex. Then add additional edges connecting vertices of odd degree. Explain why this construction works.]

42. How many subgraphs with at least one vertex does K_2 have?

43. How many subgraphs with at least one vertex does K_3 have?

44. How many subgraphs with at least one vertex does W_3 have?

45. Draw all subgraphs of this graph.



46. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$. b) $2e/v \leq M$.

A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called **n-regular** if every vertex in this graph has degree n.

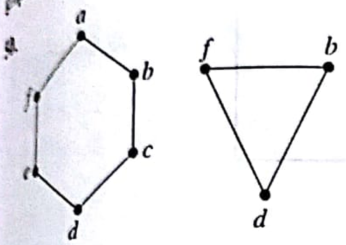
47. For which values of n are these graphs regular?
a) K_n b) C_n c) W_n d) Q_n

48. For which values of m and n is $K_{m,n}$ regular?

49. How many vertices does a regular graph of degree four with 10 edges have?

In Exercises 50–52 find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

50.



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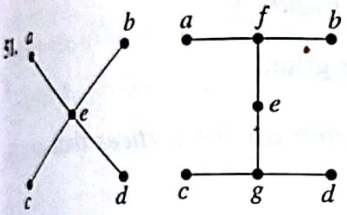

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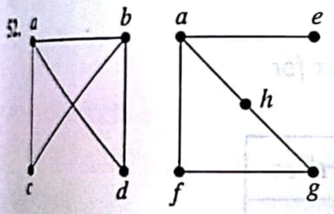


53. The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.
a) $\overline{K_n}$ b) $\overline{K_{m,n}}$ c) $\overline{C_n}$ d) $\overline{Q_n}$

54. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?

55. If the simple graph G has v vertices and e edges, how many edges does $\overline{G}$ have?

56. If the degree sequence of the simple graph G is 4, 3, 3, 2, 2, what is the degree sequence of $\overline{G}$?

57. If the degree sequence of the simple graph G is d_1, $d_2, \ldots, d_n$, what is the degree sequence of $\overline{G}$?

*58. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

59. Show that if G is a simple graph with n vertices, then the union of G and $\overline{G}$ is K_n.

*60. Describe an algorithm to decide whether a graph is bipartite based on the fact that a graph is bipartite if and only if it is possible to color its vertices two different colors so that no two vertices of the same color are adjacent.

The *converse* of a directed graph $G = (V, E)$, denoted by G^{conv}, is the directed graph (V, F), where the set F of edges of G^{conv} is obtained by reversing the direction of each edge in E.

61. Draw the converse of each of the graphs in Exercises 7–9 in Section 8.1.

62. Show that $(G^{conv})^{conv} = G$ whenever G is a directed graph.

63. Show that the graph G is its own converse if and only if the relation associated with G (see Section 7.3) is symmetric.

64. Show that if a bipartite graph $G = (V, E)$ is n-regular for some positive integer n (see the preamble to Exercise 47) and (V_1, V_2) is a bipartition of V, then $|V_1| = |V_2|$. That is, show that the two sets in a bipartition of the vertex set of an n-regular graph must contain the same number of vertices.

65. Draw the mesh network for interconnecting nine parallel processors.

66. In a variant of a mesh network for interconnecting $n = m^2$ processors, processor $P(i, j)$ is connected to the four processors $P((i \pm 1) \textbf{ mod } m, j)$ and $P(i, (j \pm 1) \textbf{ mod } m)$, so that connections wrap around the edges of the mesh. Draw this variant of the mesh network for 16 processors.

67. Show that every pair of processors in a mesh network of $n = m^2$ processors can communicate using $O(\sqrt{n}) = O(m)$ hops between directly connected processors.

8.3 REPRESENTING GRAPHS AND GRAPH ISOMORPHISM

Introduction There are many useful ways to represent graphs. As we will see throughout this chapter, in working with a graph it is helpful to be able to choose its most convenient representation. In this section we will show how to represent graphs in several different ways.

Sometimes, two graphs have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs are **isomorphic**. Determining whether two graphs are isomorphic is an important problem of graph theory that we will study in this section.

Links The best algori[...]morphic have exponential worst-case time complexity (in the number of vertices of the graphs). However, linear average-case time complexity algorithms are known that solve this problem, and there is some hope that an algorithm with polynomial worst-case time complexity for determining whether two graphs are isomorphic can be found. The best practical algorithm, called NAUTY, can be used to determine whether two graphs with as many as 100 vertices are isomorphic in less than 1 second on a modern PC. The software for NAUTY can be downloaded over the Internet and experimented with.

In graph theory, a circle graph is a graph whose vertices can be associated with chords of a circle such that two vertices are adjacent only if the corresponding chords in the circle intersect, for example, see Figure 13. As a result, we obtain Figure 14.

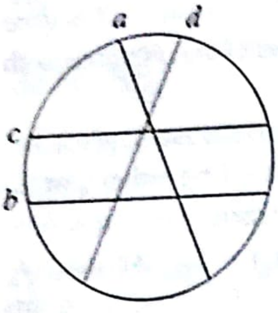


Figure 13

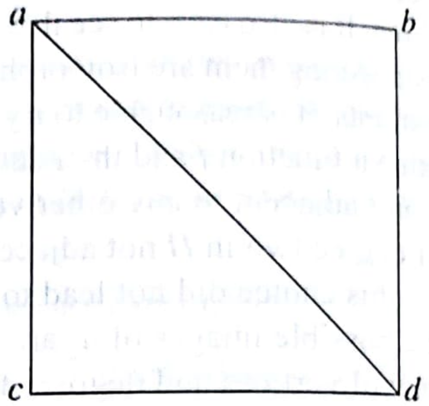


Figure 14

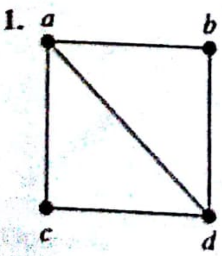
The adjacency matrix will be

$$
\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{cccc} a & b & c & d \end{array} \\
\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

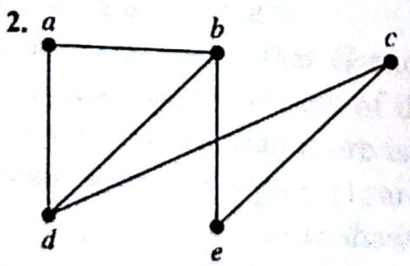
Exercises

In Exercises 1–4 use an adjacency list to represent the given graph.

1.



2.



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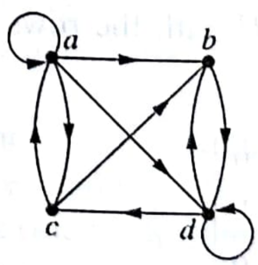
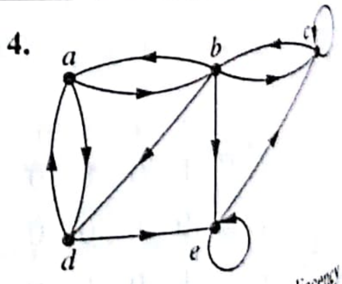


4.



5. Represent the graph in Exercise 1 with an adjacency matrix.
6. Represent the graph in Exercise 2 with an adjacency matrix.

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7. Represent the graph in Exercise 3 with an adjacency matrix.

8. Represent the graph in Exercise 4 with an adjacency matrix.

9. Represent each of these graphs with an adjacency matrix.
   a) K_4 b) $K_{1,4}$ c) $K_{2,3}$
   d) C_4 e) W_4 f) Q_3

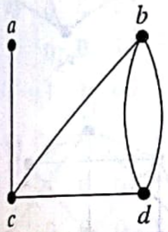
In Exercises 10–12 draw a graph with the given adjacency matrix.

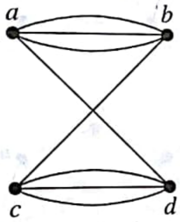
10. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

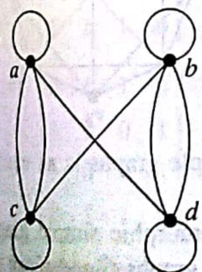
11. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

In Exercises 13–15 represent the given graph using an adjacency matrix.

13. 

14. 

15. 

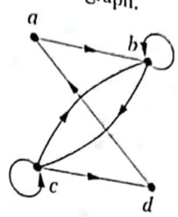
In Exercises 16–18 draw an undirected graph represented by the given adjacency matrix.

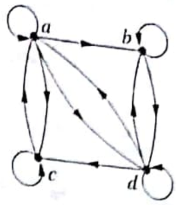
16. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

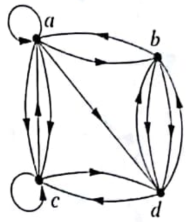
17. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

In Exercises 19–21 find the adjacency matrix of the given directed multigraph.

19. 

20. 

21. 

In Exercises 22–24 draw the graph represented by the given adjacency matrix.

22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

25. Is every zero–one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

26. Use an incidence matrix to represent the graphs in Exercises 1 and 2.

27. Use an incidence matrix to represent the graphs in Exercises 13–15.

*28. What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

*29. What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

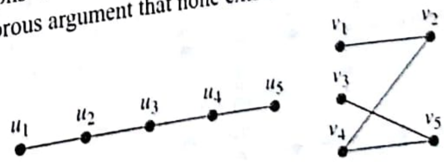
30. What is the sum of the entries in a row of the incidence matrix for an undirected graph?

31. What is the sum of the entries in a column of the incidence matrix for an undirected graph?

*32. Find an adjacency matrix for each of these graphs.
   a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

*33. Find incidence matrices for the graphs in parts (a)–(d) of Exercise 32.

In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

34. 

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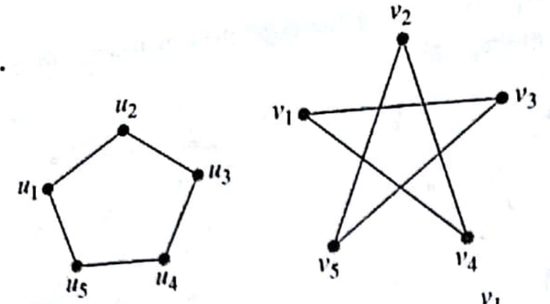

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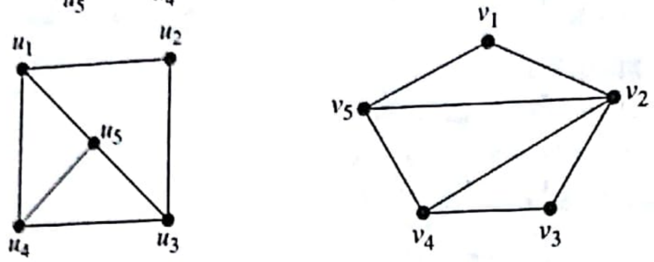
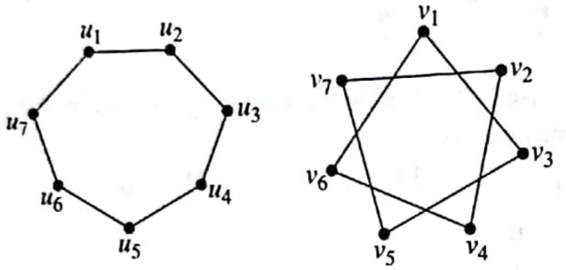


37.



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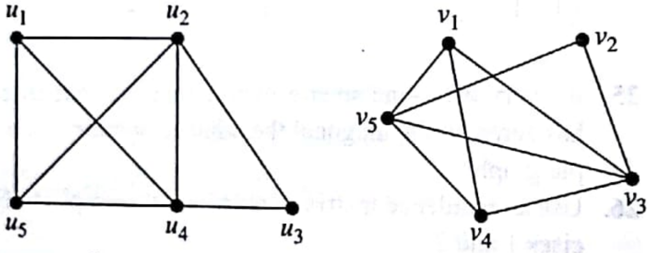

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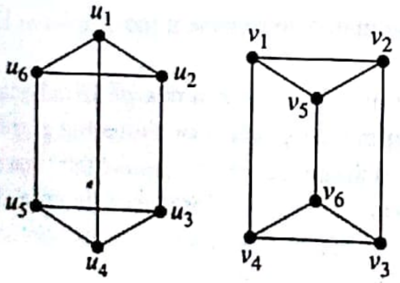

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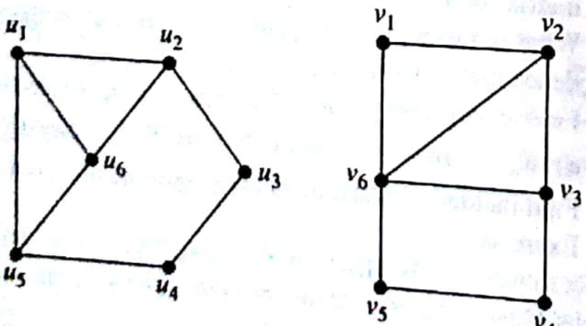

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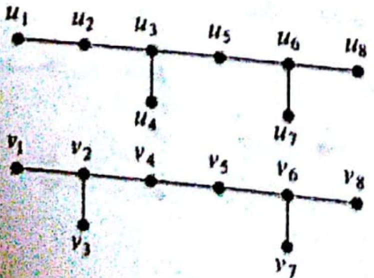

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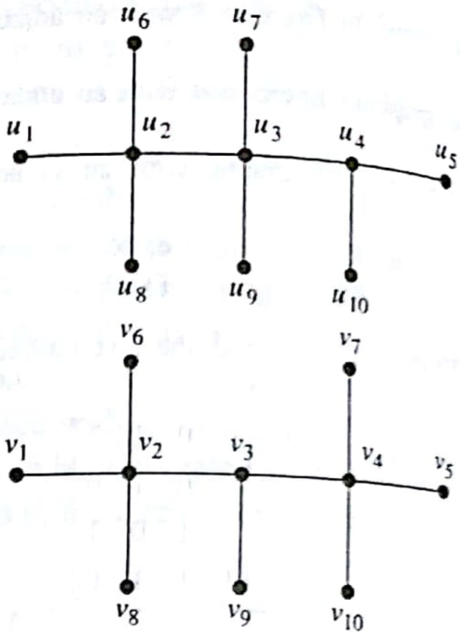

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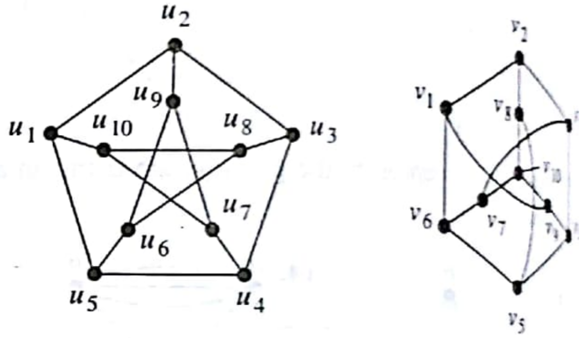

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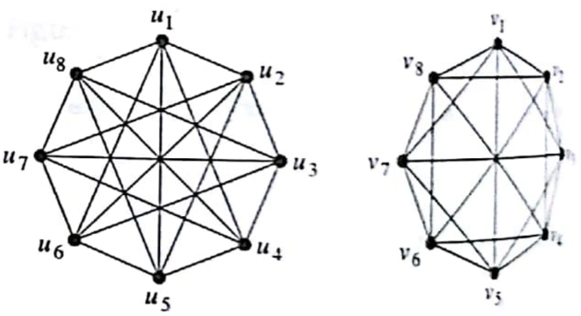


45. Show that isomorphism of simple graphs is an equivalence relation.

46. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

47. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex.

48. Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.

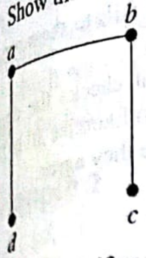
49. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix has the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix},$$

where the four entries shown are rectangular blocks.

A simple graph G is called **self-complementary** if G and $\overline{G}$ are isomorphic.

50. Show that this graph is self-complementary.



51. Find a self-complementary simple graph with five vertices.

52. Show that if G is a self-complementary simple graph with vertices, then $\equiv 0$ or $1 \pmod 4$.

53. For which integers n is C_n self-complementary?

54. How many nonisomorphic simple graphs are there with n vertices, when n is

a) 2? b) 3? c) 4?

55. How many nonisomorphic simple graphs are there with five vertices and three edges?

56. How many nonisomorphic simple graphs are there with six vertices and four edges?

57. Are the simple graphs with the following adjacency matrices isomorphic?

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

58. Determine whether the graphs without loops with these incidence matrices are isomorphic.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

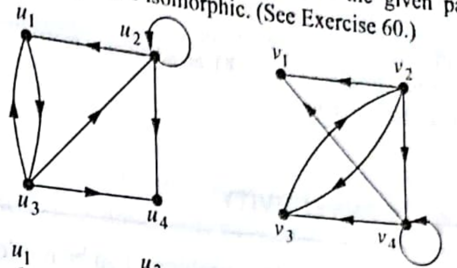
b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

59. Extend the definition of isomorphism of simple graphs to undirected graphs containing loops and multiple edges.

60. Define isomorphism of directed graphs.

In Exercises 61–64 determine whether the given pair of directed graphs are isomorphic. (See Exercise 60.)

61.



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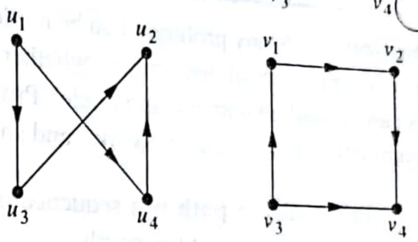

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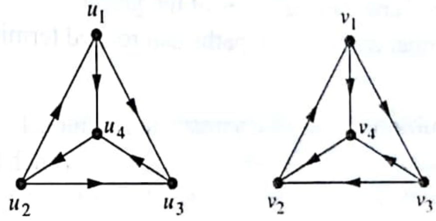

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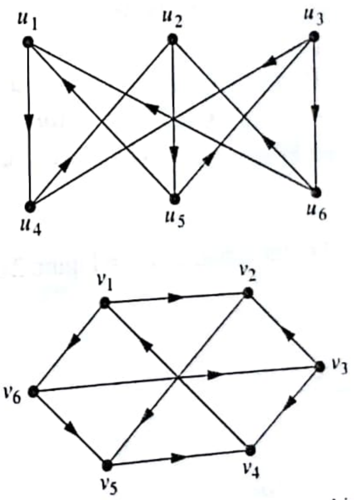


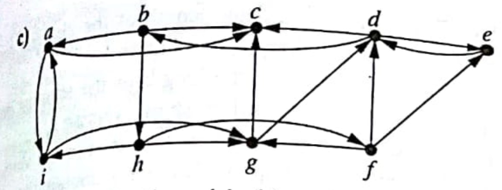
65. Show that if G and H are isomorphic directed graphs, then the converses of G and H (defined in the preamble of Exercise 61 of Section 8.2) are also isomorphic.

66. Show that the property that a graph is bipartite is an isomorphic invariant.

67. Find a pair of nonisomorphic graphs with the same degree sequence such that one graph is bipartite, but the other graph is not bipartite.

*68. How many nonisomorphic directed simple graphs are there with n vertices, when n is

a) 2? b) 3? c) 4?

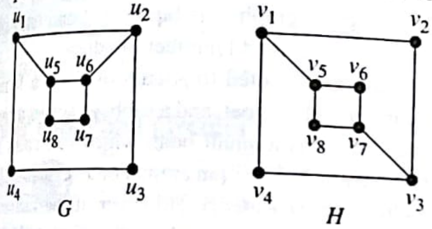


16. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in this strongly connected component.

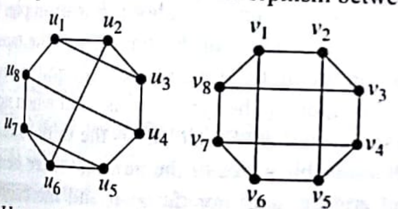
17. Find the number of paths of length n between two different vertices in K_4 if n is

a) 2. b) 3. c) 4. d) 5.

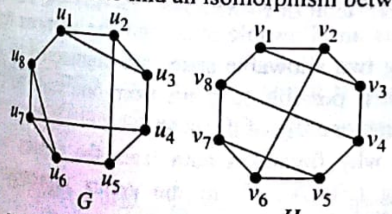
18. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.



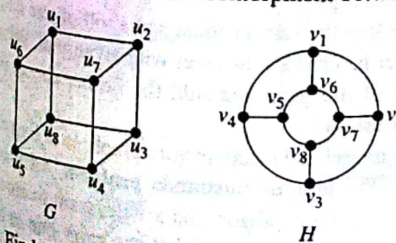
19. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



20. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



21. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



22. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in Exercise 17.

23. Find the number of paths of length n between any two nonadjacent vertices in $K_{3,3}$ for the values of n in Exercise 17.

24. Find the number of paths between c and d in the graph in Figure 1 of length

a) 2. b) 3. c) 4. d) 5. e) 6. f) 7.

25. Find the number of paths from a to e in the directed graph in Exercise 2 of length

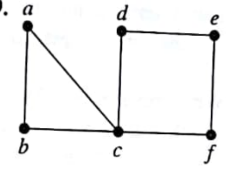
a) 2. b) 3. c) 4. d) 5. e) 6. f) 7.

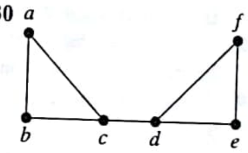
*26. Show that every connected graph with n vertices has at least $n - 1$ edges.

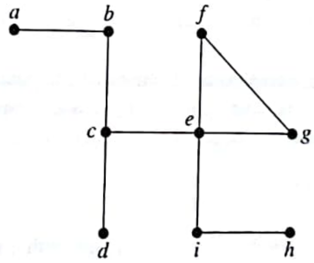
27. Let $G = (V, E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that $u = v$. Show that R is an equivalence relation.

*28. Show that in every simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

In Exercises 29–31 find all the cut vertices of the given graph.

29. 

30. 

31. 

32. Find all the cut edges in the graphs in Exercises 29–31.

*33. Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

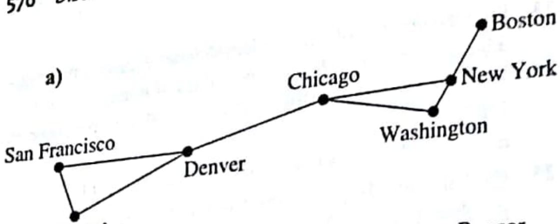
*34. Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

*35. Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.

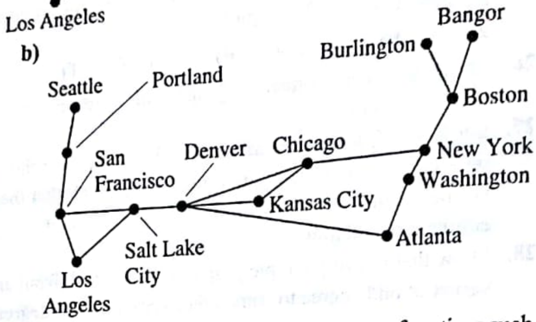
*36. Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

37. A communications link in a network should be provided with a backup link if its failure makes it impossible for some message to be sent. For each of the communications networks shown here in (a) and (b), determine those links that should be backed up.

a)



b)



A **vertex basis** in a directed graph is a set of vertices such that there is a path to every vertex in the directed graph not in the set from some vertex in this set and there is no path from any vertex in the set to another vertex in the set.

38. Find a vertex basis for each of the directed graphs in Exercises 7–9 of Section 8.2.

39. What is the significance of a vertex basis in an influence graph (described in Example 3 of Section 8.1)? Find a vertex basis in the influence graph in this example.

40. Show that if a connected simple graph G is the union of the graphs G_1 and G_2, then G_1 and G_2 have at least one common vertex.

***41.** Show that if a simple graph G has k connected components and these components have $n_1, n_2, \ldots, n_k$ vertices, respectively, then the number of edges of G does not exceed

$$\sum_{i=1}^{k} C(n_i, 2).$$

***42.** Use Exercise 41 to show that a simple graph with n vertices and k connected components has at most $(n-k)(n-k+1)/2$ edges. [*Hint:* First show that

$$\sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k),$$

where n_i is the number of vertices in the ith connected component.]

***43.** Show that a simple graph G with n vertices is connected if it has more than $(n-1)(n-2)/2$ edges.

44. Describe the adjacency matrix of a graph with n connected components when the vertices of the graph are listed so that vertices in each connected component are listed successively.

45. How many non isomorphic connected simple graphs are there with n vertices when n is

a) 2? **b)** 3? **c)** 4? **d)** 5?

46. Explain how Theorem 2 can be used to find the length of the shortest path from a vertex to a vertex w in a graph.

47. Use Theorem 2 to find the length of the shortest path between a and f in the graph in Figure 1.

48. Use Theorem 2 to find the length of the shortest path from a to c in the directed graph in Exercise 2.

49. Let P_1 and P_2 be two simple paths between the vertices u and in the simple graph G that do not contain the same set of edges. Show that there is a simple circuit in G.

50. Show that the existence of a simple circuit of length k, where k is a positive integer greater than 2, is an isomorphic invariant.

51. Explain how Theorem 2 can be used to determine whether a graph is connected.

52. Use Exercise 51 to show that the graph G_1 in Figure 2 is connected whereas the graph G_2 in that figure is not connected.

53. Show that a simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

54. In an old puzzle attributed to Alcuin of York, a farmer needs to carry a wolf, a goat, and a cabbage across a river. The farmer only has a small boat, which can carry the farmer and only one object (an animal or a vegetable). He can cross the river repeatedly. However, if the farmer is on the other shore, the wolf will eat the goat, and, similarly, the goat will eat the cabbage. We can describe each state by listing what is on each shore. For example, we can use the pair (FG, WC) for the state where the farmer and goat are on the first shore and the wolf and cabbage are on the other shore. [The symbol $\emptyset$ is used when nothing is on a shore, so that ($FWGC$, $\emptyset$) is the initial state.]

a) Find all allowable states of the puzzle, where neither the wolf and the goat nor the goat and the cabbage are left on the same shore without the farmer.

b) Construct a graph such that each vertex of this graph represents an allowable state and the vertices representing two allowable states are connected by an edge if it is possible to move from one state to the other using one trip of the boat.

c) Explain why finding a path from the vertex representing ($FWGC$, $\emptyset$) to the vertex representing ($\emptyset$, $FWGC$) solves the puzzle.

d) Find two different solutions of the puzzle, each using seven crossings.

e) Suppose that the farmer must pay a toll of one dollar whenever he crosses the river with an animal. Which solution of the puzzle should the farmer use to pay the least total toll?

***55.** Use a graph model and a path in your graph, as in Exercise 54, to solve the **jealous husbands problem.** Two married couples, each a husband and a wife, want to cross a river. They can only use a boat that can carry one or two people from one shore to the other shore. Each husband is extremely jealous and is not willing to leave his wife with the other husband, either in the boat or on shore. How can these four people reach the opposite shore?

Theorem 4 ORE'S THEOREM
If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

The proof of Ore's Theorem is outlined in Exercise 65 at the end of this section. Dirac's Theorem can be proved as a corollary to Ore's Theorem because the conditions of Dirac's Theorem imply those of Ore's Theorem.

Both Ore's Theorem and Dirac's Theorem provide sufficient conditions for a connected simple graph to have a Hamilton circuit. However, these theorems do not provide necessary conditions for the existence of a Hamilton circuit. For example, the graph C5 has a Hamilton circuit but does not satisfy the hypotheses of either Ore's Theorem or Dirac's Theorem, as the reader can verify.

Links The best algorithms known for finding a Hamilton circuit in a graph or determining that no such circuit exists have exponential worst-case time complexity (in the number of vertices of the graph). Finding an algorithm that solves this problem with polynomial worst-case time complexity would be a major accomplishment because it has been shown that this problem is NP-complete (see Section 3.3). Consequently, the existence of such an algorithm would imply that many other seemingly intractable problems could be solved using algorithms with polynomial worst-case time complexity.

Hamilton paths and circuits can be used to solve practical problems. For example, many applications ask for a path or circuit that visits each road intersection in a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once. Finding a Hamilton path or circuit in the appropriate graph model can solve such problems. The famous **traveling salesman problem** asks for the shortest route a traveling salesman should take to visit a set of cities. This problem reduces to finding a Hamilton circuit in a complete graph such that the total weight of its edges is as small as possible. We will return to this question in Section 8.6.

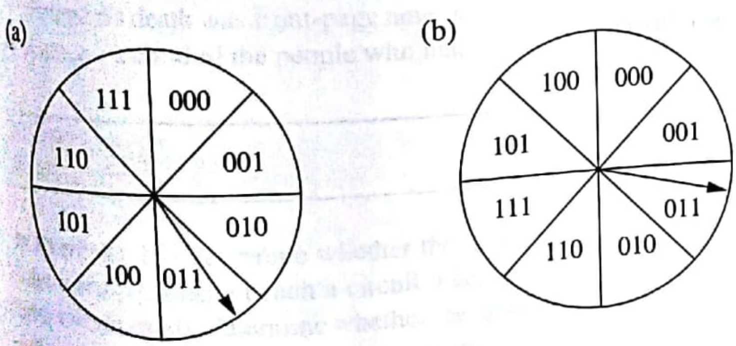


Figure 12 *Converting the Position of a Pointer into Digital Form.*

We now describe a less obvious application of Hamilton circuits to coding.

Example 8 **Gray Codes** The position of a rotating pointer can be represented in digital form. One way to do this is to split the circle into 2^n arcs of equal length and to assign a bit string of length n to each arc. Two ways to do this using bit strings of length three are shown in Figure 12.

The digital representation of the position of the pointer can be determined using a set of n contacts. Each contact is used to read one bit in the digital representation of the position. This is illustrated in Figure 13 for the two assignments from Figure 12.

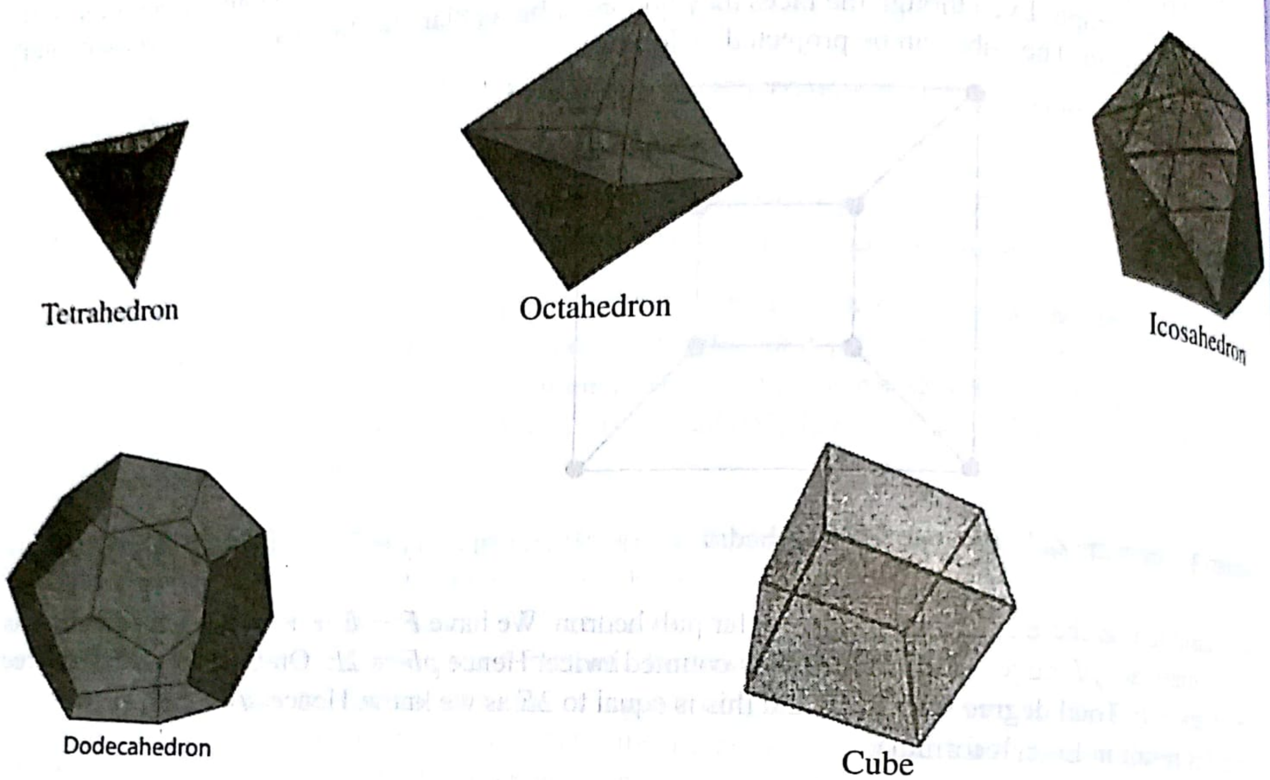
When the pointer is near the boundary of two arcs, a mistake may be made in reading its position. This [illegible] major error in the bit string read. For instance, in the coding scheme in Figure 12(a), if a

2. $p = 3, q = 4$. In this ...

3. $p = 3, q = 5$. In this case $E = 30, F = 20, V = 12$. We get an icosohedron.

4. $p = 4, q = 3$. In this case $E = 12, F = 6, V = 8$. We get a dodecahedron.

5. $p = 5, q = 3$. In this case $E = 30, F = 12, V = 20$. We get a cube.



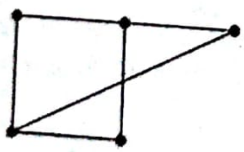
Thus it can be seen that there are only five regular polyhedra.

Exercises

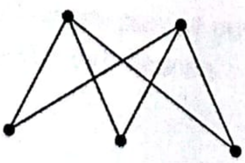
1. Can five houses be connected to two utilities without connections crossing?

In Exercises 2–4 draw the given planar graph without any crossings.

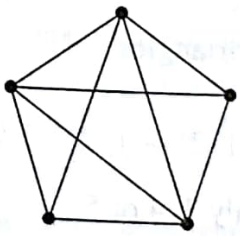
2.



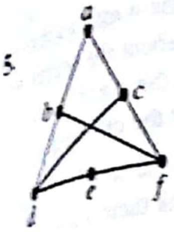
3.

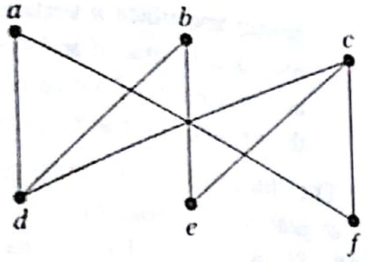


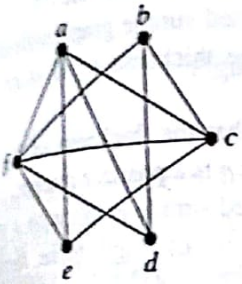
4.

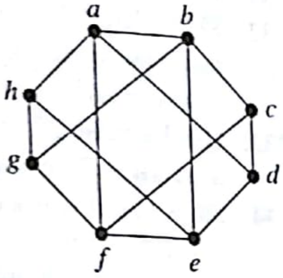


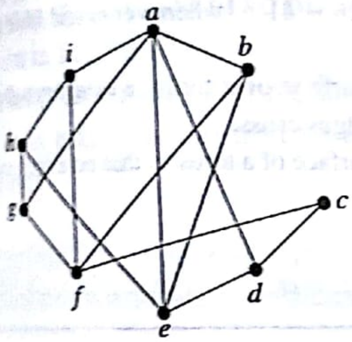
In Exercises 5–9 determine whether the given graph is planar. If so, draw it so that no edges cross.

5. 

6. 

7. 

8. 

9. 

10. Complete the argument in Example 3.

11. Show that K5 is nonplanar using an argument similar to that given in Example 3.

12. Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?

13. Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

14. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

15. Prove Corollary 3.

16. Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v - 4$ if $v \geq 3$.

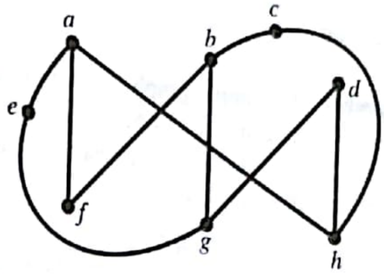
*17. Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)\,v - (10/3)$ if $v \geq 4$.

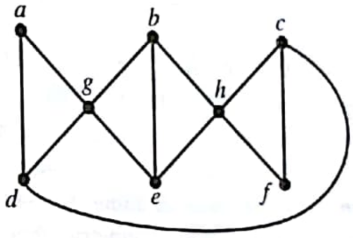
18. Suppose that a planar graph has k connected components, e edges, and v vertices. Also suppose that the plane is divided into r regions by a planar representation of the graph. Find a formula for r in terms of e, v, and k.

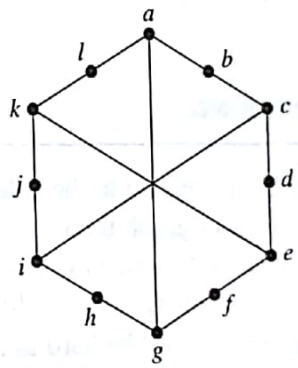
19. Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?

a) K_5 b) K_6 c) $K_{3,3}$ d) $K_{3,4}$

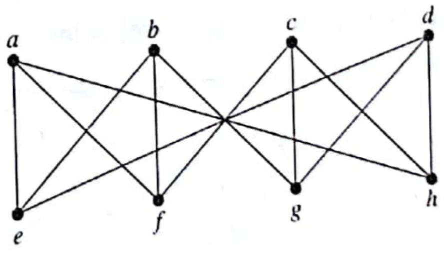
In Exercises 20–22 determine whether the given graph is homeomorphic to $K_{3,3}$.

20. 

21. 

22. 

In Exercises 23–25 use Kuratowski's Theorem to determine whether the given graph is planar.

23. 

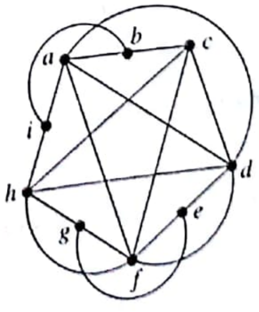
24. 

Figure 6 The Graph $K_{3,3}$.

(b)

Figure 7 Showing that $K_{3,3}$ Is Nonplanar.

Euler's Formula A planar representation of a graph splits the plane into **regions**, including an unbounded region. For instance, the planar representation of the graph shown in Figure 8 splits the plane into six regions. These are labeled in the figure. Euler showed that all planar representations of a graph split the plane into the same number of regions. He accomplished this by finding a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.

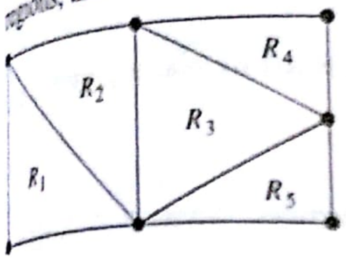


Figure 8 The Regions of the Planar Representation of a Graph.

Theorem 1 EULER'S FORMULA Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Proof: First, we specify a planar representation of G. We will prove the theorem by constructing a sequence of subgraphs $G_1, G_2, \ldots, G_e = G$, successively adding an edge at each stage. This is done using the following inductive definition. Arbitrarily pick one edge of G to obtain G_1. Obtain G_n from G_{n-1} by arbitrarily adding an edge that is incident with a vertex already in G_{n-1}, adding the other vertex incident with this edge if it is not already in G_{n-1}. This construction is possible because G is connected. G is obtained after e edges are added. Let r_n, e_n, and v_n represent the number of regions, edges, and vertices of the planar representation of G_n induced by the planar representation of G, respectively. The proof will now proceed by induction. The relationship $r_1 = e_1 - v_1 + 2$ is true for G_1, because $e_1 = 1$, $v_1 = 2$, and $r_1 = 1$. This is shown in Figure 9.

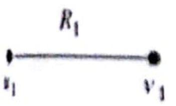


Figure 9 The Basis Case of the Proof of Euler's Formula.

Now assume that $r_n = e_n - v_n + 2$. Let $\{a_{n+1}, b_{n+1}\}$ be the edge that is added to G_n to obtain G_{n+1}. There are two possibilities to consider. In the first case, both a_{n+1} and b_{n+1} are already in G_n. These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a_{n+1}, b_{n+1}\}$ to G_n without two edges crossing (and G_{n+1} is planar). The addition of this new edge splits R into two regions. Consequently, in this case, $r_{n+1} = r_n + 1$, $e_{n+1} = e_n + 1$, and $v_{n+1} = v_n$. Thus, each side of the formula relating the number of regions, edges, and vertices increases by exactly one, so this formula is still true. In other words, $r_{n+1} = e_{n+1} - v_{n+1} + 2$. This case is illustrated in Figure 10(a).

In the second case, one of the two vertices of the new edge is not already in G_n. Suppose that a_{n+1} is in G_n but that b_{n+1} is not. Adding this new edge does not produce any new regions, because b_{n+1} must be in a region that has a_{n+1} on its boundary. Consequently, $r_{n+1} = r_n$. Moreover, $e_{n+1} = e_n + 1$ and $v_{n+1} = v_n + 1$. Each side of the formula relating the number of regions, edges, and vertices remains the same, so the formula is still true. In other words, $r_{n+1} = e_{n+1} - v_{n+1} + 2$. This case is illustrated in Figure 10(b).

We have completed the induction argument. Hence, $r_n = e_n - v_n + 2$ for all n. Because the original graph is the graph G_e, obtained after e edges have been added, the theorem is true. ◀

Euler's formula is illustrated in Example 4.

Example 4 *Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?*

Solution This graph has 20 vertices, each of degree 3, so $v = 20$. Because the sum of the degrees of the vertices, $3v = 3 \cdot 20 = 60$, is equal to twice the number of edges, $2e$, we have $2e = 60$, or $e = 30$. Consequently, from Euler's formula, the number of regions is

$$r = e - v + 2 = 30 - 20 + 2 = 12.$$

◀

Euler's formula can be used to establish some inequalities that must be satisfied by planar graphs. One such inequality is given in Corollary 1.

Corollary 1 If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Before we prove Corollary 1 we will use it to prove the following useful result.

Corollary 2 If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.