linear Stability analysis.

$$\frac{dy}{dx} = f(x,y) \quad y(x_0) = y_0$$

Consider

$$f(x,y) = f(x_0, y_0) + (x-x_0)\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + (y-y_0)\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} + \cdots$$

only these terms because linear

$$\frac{dy}{dx} = f(x_0, y_0) + (x-x_0)\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + (y-y_0)\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$$

coefficient of y?

consider test equation,

$$\frac{dy}{dx} = \lambda y \quad y(x_0) = x_0$$

$$\frac{dy}{dx} = \lambda y + g(x), \quad y(x_0) = y_0$$

$$\lambda = \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$$

<u>Note</u>: perturbed equation,

$$\frac{d\tilde{y}}{dx} = \lambda\tilde{y} + g(x)$$

Eg: $\frac{dy}{dx} = 1 + \frac{y}{x}$ $y(1) = 1$.

$f(x,y) = 1 + y/x$.

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \left.\frac{1}{x}\right|_{(1,1)} = 1.$$

for stability

Not stable & $Re(\lambda) > 0$

Test: $\frac{dy}{dx} = \lambda y$ $y(x_0) = y_0$.

$\frac{dy}{dx} = y$ $y(1) = 1$.

Stability of Euler's method.

Consider test eq. as $\frac{dy}{dx} = \lambda y$ $y(x_0) = y_0$ — ①

$$u_{i+1} = u_i + hf(x_i, u_i)$$

$$u_0 = A$$

$$u_i + h\lambda u_i$$

$(\because f(x_i, u_i) = \lambda u_i)$.

$$\boxed{u_{i+1} = (1 + h\lambda) u_i} \quad \text{— ②}$$

difference eq.

Eigen values?

let

sol of ②: $u_i = \mu^i$

$$\mu^{i+1} = (1 + h\lambda)\mu^i$$

$\Rightarrow \quad \mu = (1 + \lambda h)$

Sol: $\quad u_i = a(1+\lambda h)^i$

stability condition

$$\boxed{|(1+\lambda h)| < 1.}$$

Ex: $\frac{dy}{dx} = 1 + y/x \quad y(1) = 1.$

stability condition for this using Euler's method

$\Rightarrow |(1+h)| < 1$

$\because \lambda = 1$ from above

choose h according to difference eq.

RK-2 :

$u_{i+1} = u_i + \frac{1}{2}(k_1 + k_2).$

$k_1 = hf(x_i, u_i) \quad = h\lambda u_i$

$k_2 = hf(x_i + h, u_i + k_1)$

$\quad \hookrightarrow = h\lambda(u_i + k_1)$

$\quad = h\lambda(u_i + h\lambda u_i)$

$\quad = h\lambda u_i + (h\lambda)^2 u_i$

$\Rightarrow u_{i+1} = u_i + \frac{1}{2}(h\lambda u_i + h\lambda u_i + (h\lambda)^2 u_i)$

$$u_{i+1} = \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right) u_i$$

stability condition.

$$\left| 1 + h\lambda + \frac{(h\lambda)^2}{2} \right| < 1.$$

RK - 4 :

$$u_{i+1} = u_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).$$

$$k_1 = hf(x_i, u_i). \quad = h\lambda u_i$$

$$k_2 = hf\left(x_i + \frac{h}{2}, u_i + \frac{k_1}{2}\right)$$

$$= h\lambda\left(u_i + \frac{k_1}{2}\right).$$

$$= h\lambda\left(u_i + \frac{h\lambda u_i}{2}\right)$$

$$k_3 = hf\left[x_i + \frac{h}{2}, u_i + \frac{k_2}{2}\right)$$

$$= h\lambda\left[u_i + \frac{k_2}{2}\right).$$

$$= h\lambda\left(u_i + \frac{h\lambda\left(u_i + \frac{\lambda h u_i}{2}\right)}{2}\right)$$

$$k_4 = hf\left(x_i + h, u_i + k_3\right)$$

$$= h\lambda\left(u_i + k_3\right)$$

$$= h$$

stability condition :

$$\left| 1 + h\lambda + \frac{(h\lambda)^2}{2!} + \frac{(h\lambda)^3}{3!} + \frac{(h\lambda)^4}{4!} \right| < 1$$

check.

Recap.
24/4/19.

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0 \text{ --- (1)}$$

$$\frac{dy}{dx} \approx f(x_0, y_0) + (x - x_0) \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} + (y - y_0) \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$$

$$\frac{dy}{dx} = \lambda y + g(x) \quad y(x_0) = y_0 \text{ --- (2)}$$

where $g(x) = f(x_0, y_0) + (x - x_0) \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} - y_0 \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$

$$\frac{d\tilde{y}}{dx} = \lambda \tilde{y} + g(x), \quad \tilde{y}(x_0) = y_0 + \epsilon \text{ --- (3)}$$

(2) - (3)

$$\frac{dy}{dx} - \frac{d\tilde{y}}{dx} = \lambda (y - \tilde{y})$$

Scanned with CamScanner

$$\Rightarrow \frac{de}{dx} = \lambda e$$

$$e(x) = y(x) - \tilde{y}(x)$$

$$e(x_0) = y(x_0) - \tilde{y}(x_0)$$

$$= y_0 - (y_0 + \epsilon) = -\epsilon.$$

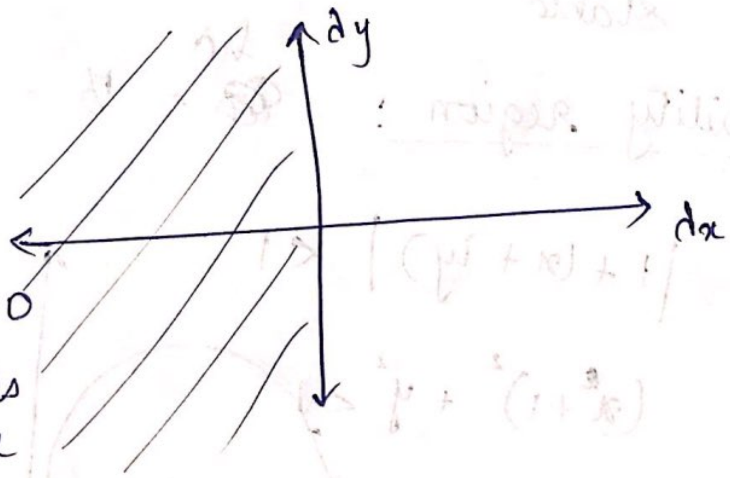
$$\frac{de}{dx} = \lambda e, \quad e(x_0) = -\epsilon.$$

$$e(x) = c_1 e^{\lambda x}$$

$$e(x_0) = c_1 e^{\lambda x_0} = -\epsilon. \Rightarrow c_1 = -\epsilon e^{-\lambda x_0}$$

Solution $\boxed{e(x) = -\epsilon e^{\lambda(x - x_0)}}$ $\quad \lambda \in \mathbb{C}$

we need to control the error.



If $Re(\lambda) < 0$ the IVP is stable

Euler's Method

$$\frac{dy}{dx} = \lambda y \qquad y(x_0) = y_0$$

$$u_{i+1} = u_i + h f(x_i, u_i) = u_i + h\lambda u_i$$

$$u_0 = A.$$

$$= (1 + h\lambda) u_i$$

$u_1 = (1 + h\lambda)u_0$

$u_2 = (1+h\lambda)u_1 = (1+h\lambda)^2 u_0$

$\vdots$

$u_{n+1} = (1+h\lambda)^{n+1} u_0.$

For stability

Eigen value $= \dfrac{u_{n+1}}{u_n}$

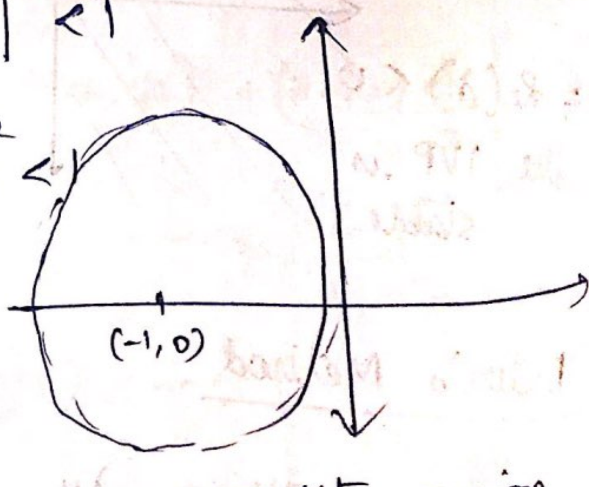
$= \dfrac{(1+h\lambda)^{n+1} u_0}{(1+h\lambda)^n u_0} = 1+h\lambda$

If $|1+h\lambda| < 1$, the Euler's method is stable

stability region : $h\lambda = z = (x+iy)$

$|1+(x+iy)| < 1$

$(x+1)^2 + y^2 < 1$



for differential eq, the stability region is the entire left half plane

for Euler's it is a unit circle

(this is because we truncate the terms and because of $h\lambda$)

RK - 2 Method.

$$u_{i+1} = u_i + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_i, u_i) = h\lambda u_i$$

$$k_2 = hf(x_i + h, u_i + k_1)$$

$$= h\lambda(u_i + k_1)$$

$$= h\lambda(u_i + h\lambda u_i)$$

$$u_{i+1} = \left[1 + h\lambda + \frac{(h\lambda)^2}{2}\right] u_i$$

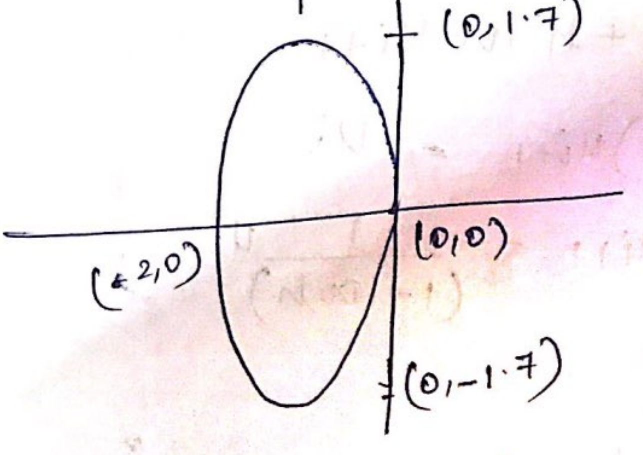
stability condition

$$\left|1 + h\lambda + \frac{(h\lambda)^2}{2}\right| < 1.$$

put $h\lambda = x + iy$.

$$\left|1 + x + iy + \frac{(x+iy)^2}{2}\right| < 1.$$

$$\left|1 + x + iy + \frac{x^2}{2} - \frac{y^2}{2} + \frac{2ixy}{2}\right| < 1.$$



RK - 4 Method

$h\lambda = x + iy$

$$\left|1 + h\lambda + \frac{(h\lambda)^2}{2!} + \frac{(h\lambda)^3}{3!} + \frac{(h\lambda)^4}{4!}\right| < 1$$

If λ is very high, then we have to tune h so that stability is maintained

Ex: $\frac{dy}{dx} = 100y \quad y(0) = 1.$

Eulers method,

$$|1 + h\lambda| < 1$$

$$|1 + 100h| < 1 \quad \Big| \; h \geq 0.$$

$$-1 < 1 + 100h < 1.$$

$$-2 < 100h < 0.$$

$$-\frac{1}{50} < h < 0.$$

But h cannot be < 0.

$$\Rightarrow \quad 0 < h < 1/50$$

Implicit Euler's Method

$$u_{i+1} = u_i + hf(x_{i+1}, u_{i+1}) \quad u_0 = A.$$

$$= u_i + h\,100\,u_{i+1}$$

$$(1 - 100h)u_{i+1} = u_i$$

$$u_{i+1} = \frac{1}{(1 - 100h)} u_i.$$

for stability

$$\left|\frac{1}{1 - 100h}\right| < 1.$$

$$|1 - 100h| > 1$$

25/4/19

Backward Euler's Method

$$u_{i+1} = u_i + hf(x_{i+1}, u_{i+1}) \quad \text{——} \ ①$$

$u_0 = A$.

$i = 0, 1, 2, \ldots$

let $\frac{dy}{dx} = \lambda y \quad y(x_0) = y_0$ —— ②

① $\Rightarrow \quad u_{i+1} = u_i + h\lambda u_{i+1}$

$$(1 - h\lambda)u_{i+1} = u_i$$

$$\Rightarrow \quad u_{i+1} = \frac{1}{1-h\lambda} u_i$$

Eigen value: $\quad \frac{u_{i+1}}{u_i} = \frac{1}{1-h\lambda}$

Stability region: let $z = h\lambda \quad z \in \mathbb{C}$

$$\left| \frac{1}{1-z} \right| < 1$$

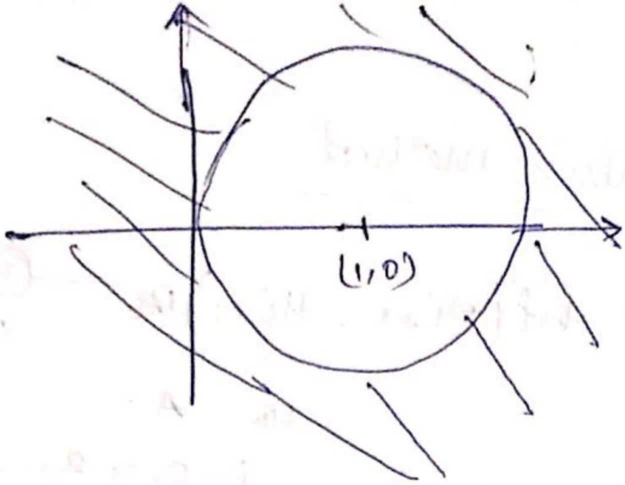
$$|1 - z| > 1.$$

let $z = x + iy$.

$$|1 - (x + iy)| > 1.$$

$$|1 - x - iy| > 1.$$

$$|(1-x)^2 + y^2|^{\frac{1}{2}} > 1.$$



Ex : $\frac{dy}{dx} = 100y \quad y(0) = 1.$

stability region

$$\left|\frac{1}{1-100h}\right| < 1.$$

$$|1-100h| > 1.$$

$$1 < 1-100h < \qquad h \in \left(0, \frac{1}{50}\right)$$

If we have non linear ODE, how do we solve using these methods

- Jacobian
- linearisation.

Ex : $\frac{dx}{dt} = -20x - 19y \quad x(0) = 2$ ①

$\frac{dy}{dt} = -19x - 20y \quad y(0) = 0$ ②

Solve using Euler's and Backward Euler's

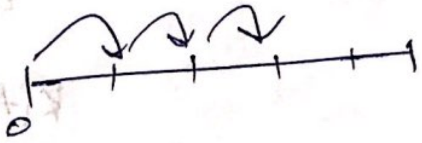
Euler's :

$$u_{i+1} = u_i + h f(x_i, u_i)$$

we apply this for both eq.

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$



$$\left.\begin{aligned} x_{i+1} &= x_i + h f(t_i, x_i, y_i) \\ y_{i+1} &= y_i + h g(t_i, x_i, y_i) \end{aligned}\right\} \text{General term}$$

eliminate x_i, y_i to get solution

$$\left\{\begin{aligned} x_{i+1} &= x_i + h(-20x_i - 19y_i) \\ y_{i+1} &= y_i + h(-19x_i - 20y_i) \end{aligned}\right.$$

Eq ① and ② in matrix form.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -20 & -19 \\ -19 & -20 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

eigen values of A.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -20-\lambda & -90 \\ -19 & -20-\lambda \end{vmatrix} = 0$$

$$(20+\lambda)^2 - 19^2 = 0.$$

$$(20+\lambda)^2 = 19^2$$

$$20+\lambda = \pm 19.$$

$$20+\lambda = 19$$

$$\lambda = -1$$

$$\lambda = -39.$$

$$K(A) = \frac{\lambda_{max}}{\lambda_{min}} = \frac{39}{1} = 39.$$

will effect round off errors.

<u>Solution</u> :

$$x_i = (1-39h)^i + (1-h)^i$$

$$y_i = (1-39h)^i - (1-h)^i$$

$$\begin{cases} x_{i+1} = x_i + h(-20x_i - 19y_i) \\ y_{i+1} = y_i + h(-19x_i - 20y_i). \end{cases}$$

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} 1-20h & -19h \\ -19h & 1-20h \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

→ to get solution,
find solution of
coupled difference equations.

stability condition

$$|1-39h|<1 \quad \text{and} \quad |1-h|<1.$$

$$h \in (0,2).$$

$$-1 < 1-39h < 1.$$

$$-2 < -39h < 0.$$

$$0 < 39h < 2$$

$$h \in \left(0, \frac{2}{39}\right).$$

$$\therefore \; h \in \left(0, \frac{2}{39}\right).$$ → we can control round off error now

When we have coupled equations,

1. find matrix
2. find eigen values
3. the step length is found out.

In general $h \in \left(0, \frac{2}{|\lambda|}\right)$ when λ is the eigen value of ODE.

Note: General sol of ODE $\{e^{-t}, e^{-39t}\}$

Basis because of eigen values

e^{-39t} will decrease decay will be fast

we choose h in such a way that it satisfies stability conditions.

Using Backward Euler's Method

$$u_{i+1} = u_i + hf(x_{i+1}, u_{i+1})$$

$$x_{i+1} = x_i + h(-20x_{i+1} - 19y_{i+1})$$

$$y_{i+1} = y_i + h(-19x_{i+1} - 20y_{i+1})$$

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1+20h & 19h \\ 19h & 1+20h \end{bmatrix}}_{A} \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = A^{-1} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1+20h & -19h \\ -19h & 1+20h \end{bmatrix} \frac{1}{(1+20h)^2 - (19h)^2}$$

$$(1+20h)^2 - (19h)^2 = 1 + 39h^2 + 40h$$

We can use Gauss Seidal or Gauss Jacobian Method

- condition - $\|A^{-1}\| < 1$.

$\therefore$ For stability : $\|A^{-1}\|_\infty < 1$

h has to be

$$\left|\frac{1+39h}{1+39h^2+40h}\right| < 1.$$

$$\left|\frac{1}{1+h}\right| < 1.$$

$$|1+h| > 1.$$

$\Rightarrow$ h can be any +ve no.

$\Rightarrow$ Unconditionally stable.

$\therefore$ The Backward Euler's method to solve the above IVP is unconditionally stable.

Lax Equivalence Theorem :

Consistency + stability $\Longleftrightarrow$ convergence

stability related to Round of error.

consistency " " Truncation error

TE + RE $\longleftrightarrow$

- Effect of error $\rightarrow 0$

exact sol & numerical sol will

converge properly.

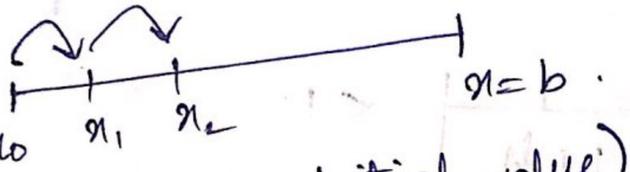
30/4/19

Lax Equivalence Theorem.

Consistency + Stability $\longleftrightarrow$ Convergence

T.E + R.E $\longleftrightarrow$ Cumulative error.

R.E $\downarrow$ Round off error.



(error due to initial value)

If TE & RE $\longrightarrow 0$ for large values of n then we can come to the exact solutions

Ex: Composite Trapezoidal rule.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_n)+2\left(f(x_1)+\ldots\right.\right.$$
$$\left.\left.\ldots+f(x_{n-1})\right)\right]$$

$$-\frac{h^2}{12}(b-a)f^{(2)}(c)$$

$\downarrow$ T.E. $\quad c \in [a,b)$

Let RE at $x_0, x_1, \ldots \ldots x_n$ be $\{\epsilon_0, \epsilon_1, \ldots \ldots \epsilon_n\}$

$$\Rightarrow \boxed{f(x_i) = f_i + \epsilon_i}$$

$i = 0, 1, \ldots, n$.

$$\int_a^b f(x)\,dx = \frac{h}{2}\Big[(f_0+\epsilon_0)+(f_n+\epsilon_n) + 2\left[f_1+\epsilon_1+\dots+f_{n-1}+\epsilon_{n-1}\right]\Big] - \frac{h^2}{12}(b-a)f^{(2)}(c) \quad c\in(a,b)$$

$$= \frac{h}{2}\Big[(f_0+f_n)+2(f_1+f_2+\dots+f_{n-1})\Big] + \frac{h}{2}\Big[(\epsilon_0+\epsilon_n)+2(\epsilon_1+\dots+\epsilon_{n-1})\Big] - \frac{h^2}{12}(b-a)f^{(2)}(c) \quad c\in(a,b)$$

let $\epsilon = \max_{0\le i\le n} |\epsilon_i| \quad i=0,1,\dots,n.$

$$\therefore |R.E| \le \frac{h}{\cancel{2}}\left[\cancel{2}n\epsilon\right] = hn\epsilon = \left[\frac{b-a}{\cancel{n}}\right]\cancel{n}\epsilon = (b-a)\epsilon.$$

$$\boxed{|R.E| \le (b-a)\epsilon} \longrightarrow \text{very small}$$

$$\lim_{h\to 0} |R.E| = 0.$$

Round off errors are under control because ϵ is very small.

T.E $\longrightarrow$ we get it from Basic trapezoidal rule

$$|T.E| \le \frac{h^2}{12}(b-a)M.$$

$$M = \max_{x \in (a,b)} |f''(x)|$$

$$\lim_{h \to 0} |T.E| = 0$$

Ex ②:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} + T.E.$$

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x)$$

$$\frac{y(x+h) - y(x)}{h} = \underbrace{y'(x)}_{\frac{dy}{dx}} + \frac{h}{2}y''(x).$$

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} - \frac{h}{2}y''(x)$$

$$\frac{dy}{dx} = \frac{(y_1 + G_1) - (y_0 + G_0)}{h} - \frac{h}{2}y''(x)$$

$$\frac{y_1 - y_0}{h} + \frac{G_1 - G_0}{h} - \frac{h}{2}y''(x)$$

x $\quad$ $x+h$.

Apply triangular inequality

$$|a-b| \le |a| + |b|$$

$$|R \cdot E| \le 2\epsilon / h.$$

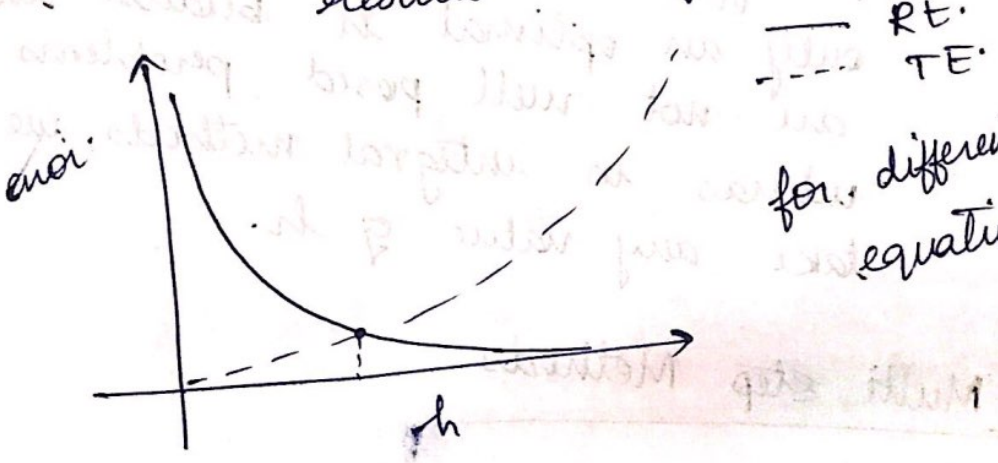
$$\epsilon = \max\{\epsilon_1, \epsilon_0\}$$

$$\lim_{h \to 0} |R \cdot E| = \infty$$

$$|T \cdot E| \le \frac{h}{2} M \qquad M = \max_{x \in (a,b)} |f''(x)|$$

$$\lim_{h \to 0} |T \cdot E| = 0.$$

$\therefore$ we will not convergent results always.



for differential equations

Case ①: $|R \cdot E| = |T \cdot E|$

$$\frac{2\epsilon}{h} = \frac{h}{2} M$$

$$\boxed{h = \sqrt{\frac{4\epsilon}{M}} = 2\sqrt{\frac{\epsilon}{M}}}$$

Case ②:

$E(h) = |R \cdot E| + |T \cdot E|$ is min

$$E(h) = \frac{2\epsilon}{h} + \frac{h}{2} M.$$

for it to be minimum,

$$\frac{dE}{dh} = 0 \Rightarrow \frac{-2\epsilon}{h^2} + \frac{M}{2} = 0.$$

$$\frac{2\epsilon}{h^2} = \frac{M}{2}.$$

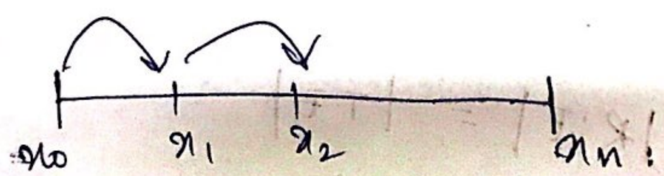
$$\boxed{h_{opt} = 2\sqrt{\frac{\epsilon}{M}}}$$

$\frac{d^2E}{dh}$ should be >0 for $\frac{dE}{dh}$ to [be] mi[nimum]

$$\frac{d^2E}{dh^2} = \frac{2\epsilon}{h^3} > 0.$$

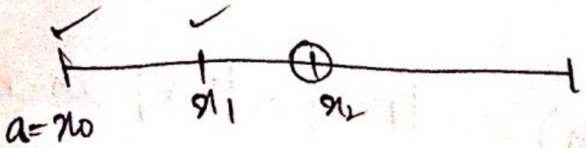
For differential equations we take only an optimal h because these are not well posed problems whereas in integral methods we can take any value of h.

Multi step Methods



in single step - we only need one previous value.

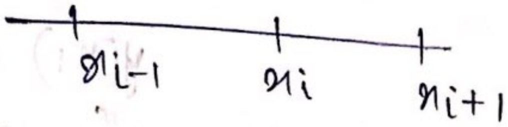
in Multistep you will need multiple previous step



IVP $\frac{dy}{dx} = f(x, y(x))$, $y(x_0) = y_0$

$x \in [a,b]$

let $g(x) = f(x, y(x))$



$$\int_{x_{i-1}}^{x_{i+1}} \frac{dy}{dx}\, dx = \int_{x_{i-1}}^{x_{i+1}} g(x)\, dx .$$

using mid point rule

$$\int_a^b f(x)\,dx = b-a\, f\left(\frac{a+b}{2}\right),$$

$$y(x_{i+1}) - y(x_{i-1}) = (x_{i+1} - x_{i-1})\, g(x_i)$$

$$y(x_{i+1}) = y(x_{i-1}) + 2h f(x_i, y(x_i))$$

$i = 1, 2, \ldots, n-1$

we need $y(x_0)$, $y(x_1)$.

1/5/19.

Multistep Methods.

IVP $\cancel{\frac{dy}{dx} = f(x, y(x))}$

Note : $$\int_{x_{i-1}}^{x_{i+1}} g(x)\,dx = 2h\, g(x_i) - \frac{(2h)^3}{24} f^{(2)}(c)$$

$c \in (x_{i-1}, x_{i+1})$

$$L.T.E = \frac{-h^3}{3} f^{(2)}(c) \qquad O(h^2)$$

$-\frac{h^3}{3} M \times \left(\frac{n}{2}\right)$.

$\Rightarrow \frac{-h^{\cancel{3}\,2}}{3} \times M \times \frac{b-a}{\cancel{h} \times 2}$

$y(x_1)$ is unknown

$y(x_0) = y_0 \longrightarrow$ given.

use corresponding single step method to find $y(x_1)$

NOTE.

Use RK-2 to generate y @ x_1 then use multistep method to generate others

Adams Bashforth Method (ABM)

$$u_{i+1} = u_i + \frac{h}{24}\Big[55 f(x_i, u_i) - 59 f(x_{i-1}, u_{i-1}) + 37 f(x_{i-2}, u_{i-2}) - 9 f(x_{i-3}, u_{i-3})$$

$i = 3, 4, 5, \ldots \ldots, n-1$

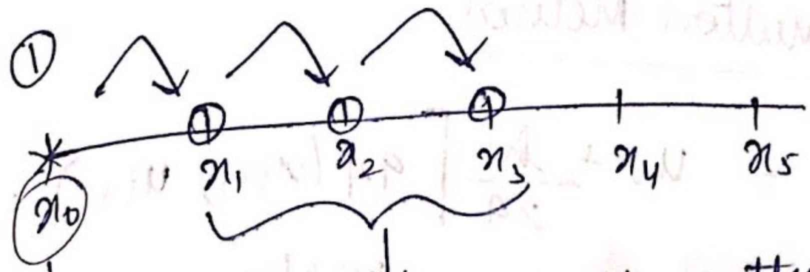
local Truncation Error.

$$\frac{251}{720} h^5 y^{(5)}(c) \quad c \in (x_{i-3}, x_i)$$

fourth order method

$O(h^4)$

stage ①



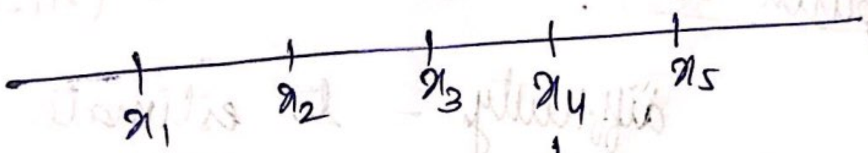
already known

generate these three values.

single step method.

we use RK-4.

stage ②



will be known from previous.

RK-4 ⟶ we evaluate function at 4 points

But this method we only need one functional evaluation.

complexity reduces

One functional evaluation is enough.

Order is same.

Implicit method - Backward Euler

drawback - at each level you should find inverse

Adams- Multon Method

$$u_{i+1} = u_i + \frac{h}{24}\Big[9f(x_{i+1}, u_{i+1}) + 19f(x_i, u_i) - 15f(x_{i-1}, u_{i-1}) + f(x_{i-2}, u_{i-2})\Big]$$

L.T.E $\quad \frac{-19}{720} h^5 y^{(5)}(c)$

$c \in (x_{i-2}, x_{i+1})$

fourth order.

difficulty - to estimate u_{i+1}

It is necessary to match the ($\geq$) order when we use two or more methods.

Adams- Multon predictor - corrector Method

$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0.$

$g_i = g(x_i) = f(x_i, y(x_i))$

and y_0, y_1, y_2 and y_3 are known

(should generate these values using fourth order methods)

while error $> 10^{-6}$

For $i = 3 : n-1$ $\quad g_i = f(x_i, u_i)$

$$u_{i+1}^{(p)} = u_i + \frac{h}{24}\Big[55g_i - 59g_{i-1} + 37g_{i-2} - 9g_{i-3}\Big]$$

$$u_{i+1}^{(c)} = u_i + \frac{h}{24}\left[9g_{i+1} + 19g_i - 15g_{i-1} + 9g_{i-2}\right]$$ ②

how to check accuracy at each ~~iteration~~

Do local iterations

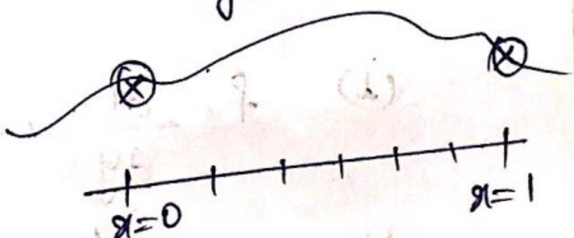
① — ② — ① — ②.

error = norm($u^P - u^c$);

end

end

Ex: $\frac{d^2y}{dx^2} + y = \sin(x)$ $y(0) = 0$, $y(1) = 1$



heights at the given points

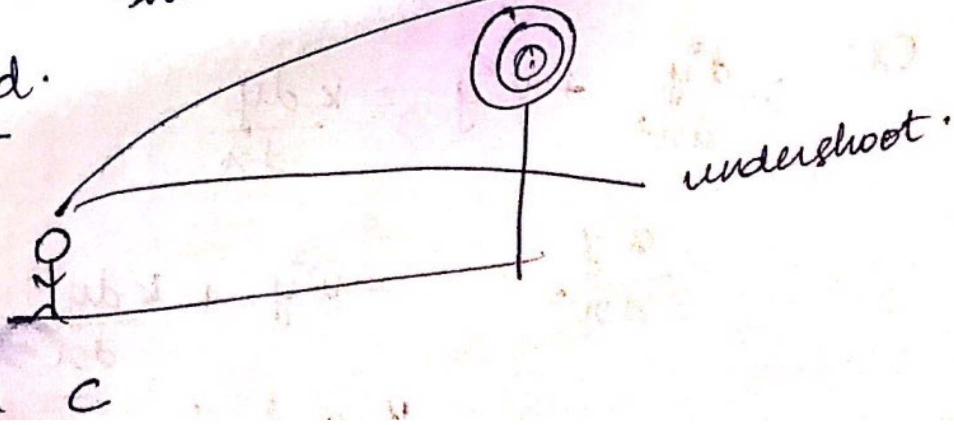
intial and boundary points given.

$y'(0) = c \longrightarrow$ IVP.

slope at initial point

how do you estimate c?

Shooting Method.



Bound C

2/5/19.

Boundary Value Problem (BVP)

Existence and Uniqueness (sufficient condition)

BVP $\frac{d^2y}{dx^2} = f(x, y(x), y'(x))$

$y(a) = \alpha$

$y(b) = \beta$.

and $x \in [a, b]$

and let

$D = \{(x, y, y') / x \in [a, b],$
$y \in \mathbb{R}, y' \in \mathbb{R}\}$

(i) $f, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'}$ are continuous on D

$\partial f/\partial y > 0$ OR

(ii) $\frac{\partial f}{\partial y} \cancel{\neq} 0, \forall (x, y, y') \in D$. $\frac{\partial f}{\partial y} < 0$

(iii) $\left|\frac{\partial f}{\partial y'}\right| \leq M \;\forall (x, y, y') \in D$

$M \in \mathbb{R}$

then there exists a unique solution

Ex: $\frac{d^2y}{dx^2} + w^2 y = k\frac{dy}{dx}$.

$\frac{d^2y}{dx^2} = -w^2 y + k\frac{dy}{dx} = f(x, y, y')$

y and y' are cont.

$\Rightarrow f$ is continuous.

$$f = -\omega^2 y + \frac{k\,dy}{dx}.$$

$$\frac{\partial f}{\partial y} = -\omega^2 \qquad \frac{\partial f}{\partial y'} = k.$$

$$\frac{\partial f}{\partial y} < 0 \quad \forall\ (x, y, y') \in D$$

<u>Note</u>: In particular

$$\frac{d^2y}{dx^2} = p(x)y' + q(x)y + r(x)$$

$$x \in [a, b]$$

Boundary conditions:

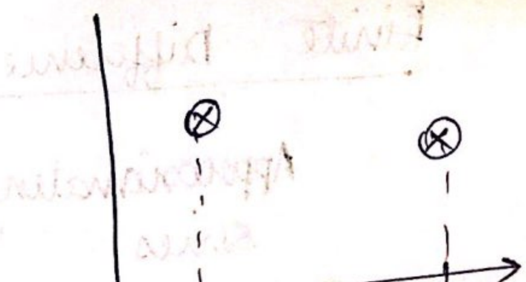
$y(a) = \alpha \qquad y(b) = \beta.$

second order linear equation.

$(y, y'$ and second order)

heights are specified at Boundaries ~~and~~

we need numerical sol to find data in between.



Existence and Uniqueness.

(i) $f, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial y'}$ are continuous

for f to be continuous, y & y'

$q(x)$ & $r(x)$ must be cont. on D

$$f = p(x)y' + q(x)y + r(x)$$

$$\frac{\partial f}{\partial y} = q(x)$$

$$\frac{\partial f}{\partial y'} = p(x)$$

$\therefore\ f,\ \partial f/\partial y$ and $\partial f/\partial y'$ are cont.

(ii) $\dfrac{\partial f}{\partial y} \neq 0 \qquad \dfrac{\partial f}{\partial y} > 0$ OR $\dfrac{\partial f}{\partial y} < 0$

$$\frac{\partial f}{\partial y} = q(x) > 0 \text{ OR } < 0$$

$$\forall (x, y, y') \in D$$

(iii) $\dfrac{\partial f}{\partial y'} = p(x)$

all continuous functions are bounded.

Finite Difference (FD) Method

Approximating derivatives using taylor series

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \ldots$$

$$y'(x) = \frac{y(x+h) - y(x)}{h} - \frac{h}{2}y''(x) \quad \text{—— (1)}$$

Forward difference approximation

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x)$$

$$y'(x) = \frac{y(x) - y(x-h)}{h} + \frac{h}{2!}y''(x) - \frac{h^2}{3!}y'''(x)$$

(2)

Central difference approximation

$$y(x+h) - y(x-h)$$

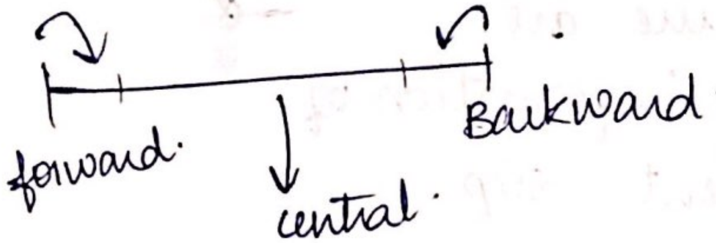
$$= 2hy'(x) + \frac{2h^3}{3!}y'''(x) + \ldots$$

$$\frac{y(x+h) - y(x-h)}{2h} = y'(x) + \frac{h^2}{3!}y'''(x) + \ldots$$

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} - \frac{h^2}{3!}y'''(x) - \ldots$$

difference b/w x+h & x-h is 2h.

Central difference gives us a better approximation because truncation error is the third derivative



Approximation for second order derivative

Consider

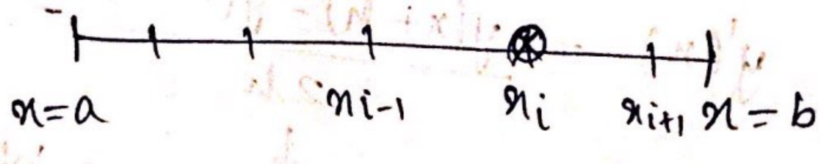
$$y(x+h) + y(x-h) = 2y(x) + \frac{2h^2}{2!}y''(x) + \frac{2h^4}{4!}y^{(4)}(x) + \dots$$

$$y''(x) = \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} - \frac{2h^2}{4!}y^{(4)}(x)$$

central difference approximation for second order derivative.

How to use these formulae to solve BVP.

- Discretize the domain



$$\left.\frac{dy}{dx}\right|_{x=x_i} = \frac{y(x_{i+1}) - y(x_{i-1})}{2h}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_i} = \frac{y(x_{i+1}) + y(x_{i-1}) - 2y(x_i)}{h^2}$$

Approximations of derivatives at x_i

No. of equations depend on no. of sub intervals.

Ex: $y(x) = \sin(\pi x)$

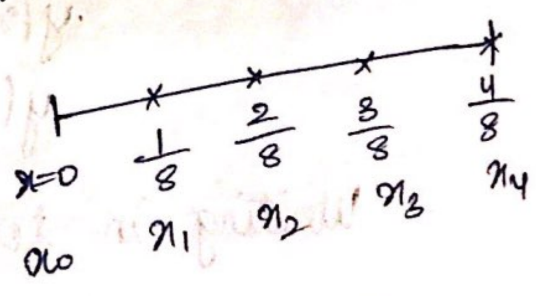
$$\frac{dy}{dx} = \pi\cos(\pi x).$$

$$\frac{d^2y}{dx^2} = -\pi^2\sin(\pi x).$$

forming differential eq.

BVP. $\frac{d^2y}{dx^2} + \pi^2 y = 0$ $\quad y(0) = 0.$ $\quad y(1/2) = 1$

interval — $[0, 1]$.



$$\left.\frac{d^2y}{dx^2}\right|_{x=x_i} + \left.\pi^2 y\right|_{x=x_i} = 0$$

should in general be

In general eq. satisfies intermediate points only

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + \pi^2 y(x_i) = 0$$

$i = 1, 2, 3.$

(internal nodes)

for $i = 1$:

$$\frac{y(x_2) - 2y(x_1) + y(x_0)}{h^2} + \pi^2 y(x_1) = 0$$

for $i = 2$

$$\frac{y(x_3) - 2y(x_2) + y(x_1)}{h^2} + \pi^2 y(x_2) = 0.$$

for $i = 3$

$$\frac{y(x_4) - 2y(x_3) + y(x_2)}{h^2} + \pi^2 y(x_3) = 0$$

$y(x_0) \longrightarrow$ known

$y(x_4) \longrightarrow$ known

Writing in terms of system of equations

$$\begin{bmatrix} -2/h^2 + \pi^2 & +1/h^2 & 0 \\ \frac{1}{h^2} & \left(-\frac{2}{h^2} + \pi^2\right) & 1/h^2 \\ 0 & 1/h^2 & -2/h^2 + \pi^2 \end{bmatrix} \begin{bmatrix} y(x_1) \\ y(x_2) \\ y(x_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0. \\ \frac{-1}{h^2} \end{bmatrix}$$

$h \rightarrow$ given $= 1/8$.

After solving the system of eq.

x	Exact	numerical	error
$1/8$	0.38	0.385	3×10^{-3}
$2/8$	0.707	0.710	3×10^{-3}
$3/8$	0.923	0.926	3×10^{-3}

Exact sol: $y(x) = \sin(\pi x)$.

why are the errors same?

Ex: $\dfrac{d^2y}{dx^2} + \pi^2 y = 0$, $y(0) = 0$

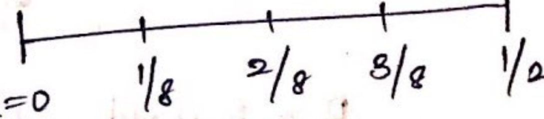
$y'(1/2) = 0$

do not want this

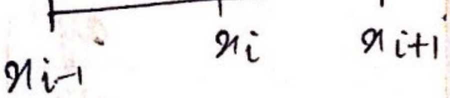
because RHS vector becomes 0 vector.

we need non-zero vector.

If we have differential operator, we can discritize the data at that points.



$$\frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{h^2} + \pi^2 y(x_i) = 0.$$



$\forall\ i = 1, 2, 3.$

when $i=1$.

$$\frac{y(x_0)-2y(x_1)+y(x_2)}{h^2}+\pi^2 y(x_1)=0 \quad \text{——} (2)$$

$i=2$.

$$\frac{y(x_1)-2y(x_2)+y(x_3)}{h^2}+\pi^2 y(x_2)=0 \quad \text{——} (3)$$

$i=3$.

$$\frac{y(x_2)-2y(x_3)+y(x_4)}{h^2}+\pi^2 y(x_3)=0. \quad \text{——} (4)$$

$y(x_0)=0 \quad \because x_0=0$.

$y(x_4)$ is not known

↓ to know this, we discritize the data at the boundary.

we use Backward approximation.

$x_3 = 3/8 \qquad x_4 = 1/2$

given: $\left.\frac{dy}{dx}\right|_{x=1/2} = 0$.

Backward difference approximation:

$$\left.\frac{dy}{dx}\right|_{x=x_i} = \frac{y(x_i)-y(x_{i-1})}{h}$$

$$\Rightarrow \frac{y(x_i)-y(x_{i-1})}{h}=0 \qquad i=4.$$

$$\Rightarrow \frac{y(x_4) - y(x_3)}{h} = 0.$$

$$y(x_4) = y(x_3) \quad \text{——— } (5).$$

Replace in eq (4)

But the equations are homogeneous.

To overcome that,

$$\frac{d^2y}{dx^2} + 2\pi^2 y = \pi^2 \sin(\pi x)$$

we get this by.

$$\frac{d^2y}{dx^2} + \pi^2 y = 0.$$

$$\frac{d^2y}{dx^2} + 2\pi^2 y = \pi^2 y.$$

$$\frac{d^2y}{dx^2} + 2\pi^2 y = \pi^2 \sin(\pi x).$$

$$y(0) = 0 \quad y'\left(\tfrac{1}{2}\right) = 0$$

∴ the solution remains the same.

we do this method for eq. for which we know the ~~answers~~ solution.

$$\begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \begin{bmatrix} y(x_1) \\ y(x_2) \\ y(x_3) \end{bmatrix} =$$

eq. become,

$i = 1.$

$$\frac{y(x_0) - 2y(x_1) + y(x_2)}{h^2} + 2\pi^2 y(x_1) = \pi^2 \sin(\pi x_1)$$

$i = 2.$

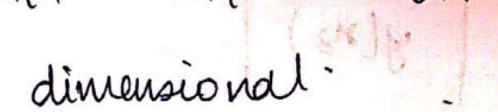
$$\frac{y(x_1) - 2y(x_2) + y(x_3)}{h^2} + 2\pi^2 y(x_2) = \pi^2 \sin(\pi x_2)$$

$i = 3$

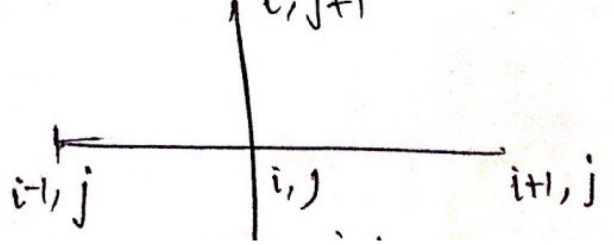
$$\frac{y(x_2) - 2y(x_3) + y(x_4)}{h^2} + 2\pi^2 y(x_3) = \pi^2 \sin(\pi x_3)$$

$$\begin{bmatrix} \frac{-2}{h^2} + 2\pi^2 & \frac{1}{h^2} & 0 \\ \frac{1}{h^2} & \frac{-2}{h^2} + 2\pi^2 & \frac{1}{h^2} \\ 0 & 1/h^2 & \frac{-1}{h^2} + 2\pi^2 \end{bmatrix} \begin{bmatrix} y(x_1) \\ y(x_2) \\ y(x_3) \end{bmatrix} = \begin{bmatrix} \pi^2 \sin(\pi x_1) \\ \pi^2 \sin(\pi x_2) \\ \pi^2 \sin(\pi x_3) \end{bmatrix}$$

This is one-dimensional.



two dimensional.

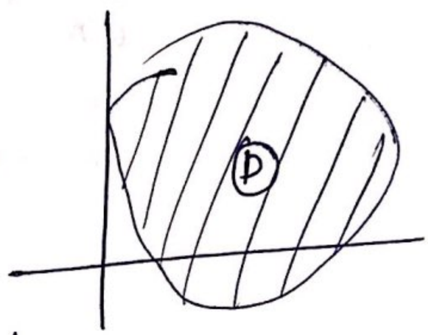


NDD — different centres
if the centres are the same we get
taylor series.

Revision

8/5/2019. IVP – existence and uniqueness

$\frac{dy}{dx} = f(x, y(x)) \quad y(x_0) = y_0 \quad x \in [a, b].$



$\max_{D} |f(x, y)| \leq M.$

$|x - x_0| \leq \min\left(a, \frac{b}{M}\right)$

$f(x, y)$, $\frac{\partial f}{\partial y}$ should be continuous.

↳ strong condition.

$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$

↳ sufficient condition

for existence you only need continuity of RHS func

↳ $f(x, y)$ should be continuous.

Backward Euler's

$u_{i+1} = u_i + h f(x_{i+1}, u_{i+1})$

$\frac{dy}{dx} = \lambda y \qquad \lambda \rightarrow$ eigen value

$\lambda = \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)}$

linearization of the function at that initial point

$$\frac{dy}{dx} = f(x,y) = f(x_0, y_0) + (x-x_0)\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + (y-y_0)\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$$

$$\frac{de}{dx} = \lambda e$$

$$e = y - \tilde{y}$$

$$u_{i+1} = u_i + h\lambda u_{i+1}$$

$$f(x,y) = \lambda y.$$

$$\Rightarrow (1-h\lambda)u_{i+1} = u_i$$

$$\frac{u_{i+1}}{u_i} = \frac{1}{1-h\lambda}$$

$\hookrightarrow$ eigen value.

for stability,

$$\left|\frac{1}{1-h\lambda}\right| < 1.$$

$$z = h\lambda$$

$$\left|\frac{1}{1-z}\right| < 1.$$

Tutorial 6.

⑧ (a)

$$|T.E| \le \frac{h^3}{9\sqrt{3}} f'''(c) \qquad c \in (a,b)$$

for quadratic interpolation

x : x_0 x_1 x_2
y : y_0 y_1 y_2

$$p(x) = f(x_0) + (x - x_0) f[x - x_0] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3]$$

Quadratic interpolation - first two terms.

$$\text{Error} = (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x]$$
$x \in (a, b)$

Equispaced data

x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$

let $t = x - x_0$.
$x - x_1 = t - h$.
$x - x_2 = t - 2h$.

$$\text{Error} = t(t-h)(t-2h).$$

if we have fourth order NDD,

$$g(t) = t(t-h)(t-2h)$$
$$= (t^2 - th)(t - 2h).$$
$$t^3 - t^2h - 2t^2h + 2th^2$$
$$t^3 - 3t^2h + 2th^2$$

$$g'(t) = 0 \quad 3t^2 - 6th + 2h^2 = 0.$$

$$= \frac{6h \pm \sqrt{36h^2 - 24h^2}}{6}.$$

x : x_0 x_1 x_2

y : y_0 y_1 y_2

$$p(x) = f(x_0) + (x-x_0) f[x - x_0] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3]$$

Quadratic interpolation - first two terms.

$$\text{Error} = (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x]$$

$x \in (a, b)$

Equispaced data

x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$

let $t = x - x_0$.

$x - x_1 = t - h$.

$x - x_2 = t - 2h$.

$$\text{Error} = t(t-h)(t-2h).$$

if we have fourth order NDD,

$$g(t) = t(t-h)(t-2h)$$

$$= (t^2 - th)(t-2h).$$

$$t^3 - t^2h - 2t^2h + 2th^2$$

$$t^3 - 3t^2h + 2th^2$$

$g'(t) = 0$ $3t^2 - 6th + 2h^2 = 0$.

$$= \frac{6h \pm \sqrt{36h^2 - 24h^2}}{6}.$$

$$\frac{6h \pm h2\sqrt{3}}{6} = \frac{3h \pm \sqrt{3}}{3}.$$

Scanned with CamScanner

$$t = h \pm \frac{1}{\sqrt{3}}.$$

$$g''(t) = 6t - 6h.$$

$h + \frac{1}{\sqrt{3}}$. $\qquad$ $h - \frac{1}{\sqrt{3}}$.

$6h + \frac{6}{\sqrt{3}}$ $\qquad$ $6h - \frac{1}{\sqrt{3}}$,

$$g\left(h - \frac{h}{\sqrt{3}}\right) = \left(h - \frac{h}{\sqrt{3}}\right)^3 - 3\left(h - \frac{h}{\sqrt{3}}\right)^2 h + 2\left(h - \frac{h}{\sqrt{3}}\right)h^2$$

$$= h^3\left(1 - \frac{1}{\sqrt{3}}\right)^3 - 3h^3\left(1 - \frac{1}{\sqrt{3}}\right)^2 + 2h^3\left(1 - \frac{1}{\sqrt{3}}\right).$$

$$h^3\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)^3 - 3h^3\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)^2 + 2h^3\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right).$$

$$\frac{h^3}{3\sqrt{3}}(\sqrt{3}-1)(3+1-2\sqrt{3}) - h^3(4-2\sqrt{3}) + \frac{2h^3}{\sqrt{3}}(\sqrt{3}-1)$$

$(4-2\sqrt{3})(\sqrt{3}-1)$

$4\sqrt{3} - 6 - 4 + 2\sqrt{3}$

$$\frac{h^3}{3\sqrt{3}}(6\sqrt{3}-10) - h^3(4-2\sqrt{3}) + 2h^3 - \frac{2h^3}{\sqrt{3}}.$$

$$\cancel{2h^3} - \frac{10h^3}{3\sqrt{3}} - \cancel{4h^3} + 2h^3\sqrt{3} + \cancel{2h^3} - \frac{2h^3}{\sqrt{3}}$$

$$-2\ h^3\left(-\frac{10}{3\sqrt{3}} + 2\sqrt{3} - \frac{2}{\sqrt{3}}\right)$$

$$h^3\left(\frac{-10}{3\sqrt{3}} + \frac{18}{3\sqrt{3}} - \frac{6}{3\sqrt{3}}\right) \qquad 18-6 = 12$$

$2\times 3\sqrt{3}$

$$= \frac{2h^3}{3\sqrt{3}}.$$

$$|T.E| \leq \frac{2h^3}{3\sqrt{3}} \times \frac{f'''(c)}{6}.$$

$$\leq \frac{2h^3}{18\sqrt{3}} f'''(c).$$

coming from

$$f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1 - x_0} = f'(c)$$

Generalized Mean value theorem

$$f[x_0, x_1, x_2] = \frac{f''(c)}{2}$$

If it is not equispaced data, then we have to use $(x-x_0)(x-x_1)(x-x_2)$ as it is

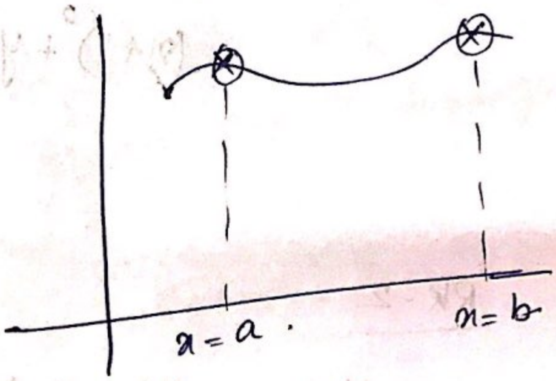
BVP

$$\frac{d^2y}{dx^2} = f(x, y(x), y'(x)) \qquad y(a) = A, \quad y(b) = B.$$

we have to convert into IVP

↓

condition only at starting point



define $y'(a) = c$ → slope at a.

↓

Root finding problem.

Eg : secant method

1) $f(x, y, y')$ is continuous

$\ast$ 2) $\left|\frac{\partial f}{\partial y}\right| > 0$ (or) $\left|\frac{\partial f}{\partial y}\right| < 0$

IVP. $\frac{dy}{dt} = \lambda y \qquad y(x_0) = y_0$

$f(x, y) = \lambda y.$

Euler's method

$$u_{i+1} = u_i + h f(x_i, u_i)$$
$$= u_i + h\lambda u_i$$
$$= (1 + h\lambda) u_i$$

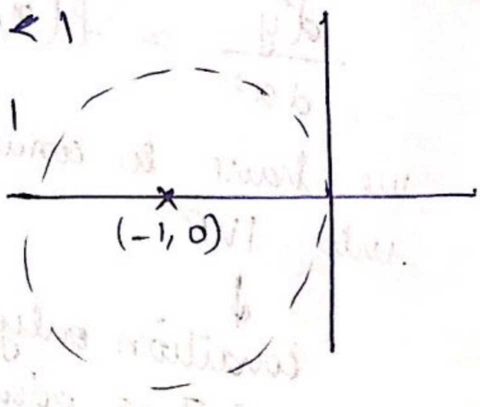
$$\left|\frac{u_{i+1}}{u_i}\right| = |1 + h\lambda| < 1$$

$\hookrightarrow$ eigen value.

let $h\lambda = z = x + iy$

$|1 + (x + iy)| < 1$

$(x+1)^2 + y^2 < 1$



RK-2

$$u_{i+1} = u_i + \frac{1}{2}(k_1 + k_2)$$
$$k_1 = hf(x_i, u_i) = \lambda u_i$$
$$k_2 = hf(x_i + h, u_i + k_1) = h\lambda (u_i + k_1)$$
$$= h\lambda [u_i + h\lambda u_i]$$

$= h\lambda(1+h\lambda)u_i$

$$u_{i+1} = u_i + \frac{1}{2}(k_1+k_2)$$

$$= u_i + \frac{1}{2}\left(h\lambda u_i + h\lambda(1+h\lambda)u_i\right)$$

$$= u_i\left[1 + \frac{h\lambda}{2} + \frac{h\lambda}{2} + \frac{(h\lambda)^2}{2}\right].$$

$$= u_i\left[1 + \frac{h\lambda}{2} + \frac{(h\lambda)^2}{2}\right].$$

$$\frac{|u_{i+1}|}{|u_i|} = \left|1 + h\lambda + \frac{(h\lambda)^2}{2}\right| < 1.$$

let $h\lambda = z = x+iy.$

$$\left|1 + (x+iy) + \frac{(x+iy)^2}{2}\right| < 1.$$

Tutorial 6.

84.

x:	x_0	x_1
y:	$f(x_0)$	$f(x_1)$.

NDD

$P_1(x)$ $f(x)$ $= f(x_0) + (x-x_0)f[x_0, x_1]$

if we equate this to lagrange's we are done.

$$= f(x_0) + x - x_0\left[\frac{f(x_1) - f(x_0)}{(x_1 - x_0)}\right].$$

$$= x_1 f(x_0) - \cancel{x_0 f(x_0)} + x f(x_1) - x f(x_0)$$

$$\frac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0} - x_0 f(x_1) + \cancel{x_0 f(x_0)} + x\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)$$

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$$\frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1} + \frac{x f(x_1)}{x_1 - x_0} - \frac{x f(x_0)}{x_1 - x_0}$$

$$f(x_0)\left[\frac{x_1}{x_1 - x_0} - \frac{x}{x_1 - x_0}\right] + f(x_1)\left[\frac{x - x_0}{x_1 - x_0}\right]$$

$$f(x_0)\left[\frac{x_1 - x}{x_1 - x_0}\right] + f(x_1)\left[\frac{x - x_0}{x_1 - x_0}\right]$$

$$= \left[\frac{x - x_0}{x_0 - x_1}\right] f(x_0) + \left[\frac{x - x_0}{x_1 - x_0}\right] f(x_1)$$

Hence proved.

(12)

$\frac{3}{8}$th rule.

$$\int_a^b f(x) = \frac{3}{8}(b-a)\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

* Basic rules

Least square method - sum of the squares of x from the polynomial is least.

$P(x) = a + bx$

Data $\{x_i, f(x_i)\}$

$$\begin{bmatrix} N & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma f(x_i) \\ \Sigma x f(x_i) \end{bmatrix}.$$

symmetric & +ve definite

$x^T A x > 0 \quad x \neq 0.$

$Ax = \lambda x$

$x^T A x = \lambda x^T x.$

→ Non singular

If eigen values are +ve, x vector is +ve.

(3) $f(x_0) = f'(x_0) = 0$

$f(x_1) = f'(x_1) = f''(x_1) = 0.$

multiplicity is > 1.

$f(x_0) = f'(x_0) = 0 \longrightarrow (x - x_0)^2$

$(x - x_1)^3.$

$P_5(x) \longrightarrow$ degree ≤ 5 and function of those two.

$\therefore \quad P_5(x) = A(x - x_0)^2 (x - x_1)^3$

$x_0 \quad x_1 \quad x_2$

$P(x_2)$

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impose $f(x_2)$ to determine the value of A.

$\Rightarrow \quad p_5(x_2) = f(x_2)$

$$f(x_2) = A(x_2 - x_0)^2 (x_2 - x_1)^3$$

$$A = \frac{f(x_2)}{(x_2 - x_0)^2 (x_2 - x_1)^3}$$