SIGNALS—CONTINUOUS-TIME AND DISCRETE-TIME

1

Learning Objectives

After going through this chapter, students will be able to

- define the terms: signal, continuous-time signal and discrete-time signal,
- determine whether a given signal–continuous-time or discrete time, is periodic and in case it is, be able to determine its fundamental period,
- determine the even and odd parts of any real signal and the conjugate symmetric and the conjugate anti-symmetric parts of any given complex signal,
- determine whether a given signal is a power signal or an energy signal or neither and also determine its average power or energy in case it is a power signal or an energy signal respectively,
- give the analytical and diagrammatic representation of commonly used continuous-time and discrete-time signals, and
- perform operations such as time-shifting and time-scaling on any given signal, and evaluate the convolution of any given pair of signals.

1.1 WHAT IS A SIGNAL?

All of us have an intuitive idea of what a signal is. In fact, signals constitute an important part of our daily lives. Human speech is a familiar example of a signal. When we speak, we create variations of acoustic pressure. These variations of acoustic pressure with time constitute the speech signal. A microphone converts these acoustic pressure variations into corresponding variations of electrical voltage. Here, the signal is a function of time.

If we take a record of the variations of the level of a road as we move along the center of the road, we get a signal representing the variations in the level with respect to distance. This signal is a function of distance.

Signals such as the above ones are called one-dimensional signals since they are functions of a single variable, time or distance, etc.

Now, consider a monochrome still picture or image. Any such picture is characterized by the variations in brightness or reflected light, from point to point. So it can be considered as a two-dimensional signal since I, the intensity of light is a function of x and y, the coordinates of any point in the image.

Suppose someone sends a string of numbers 9.8, 7.4, 10.1, 9.9, 10.3,...,where, the numbers represent the share value of a particular stock on consecutive days. This string of numbers also is a signal. Here, the share value is a function of n, the number of the day (first day, second day, etc.).

Thus, we may consider a signal to be a single-valued function of one or more variables that conveys some information.

1.2 TYPES OF SIGNALS

Signals may be classified in several ways as indicated below.

1. Continuous-time signals and discrete-time signals ***A continuous-time signal is one which is defined for all values of time*** At this point, it should be noted that a continuous-time signal need not be continuous (in the mathematical sense) at all points in time. Thus, even a rectangular wave such as the one shown in Fig. 1.1 which has discontinuities at several points, is a continuous-time signal.

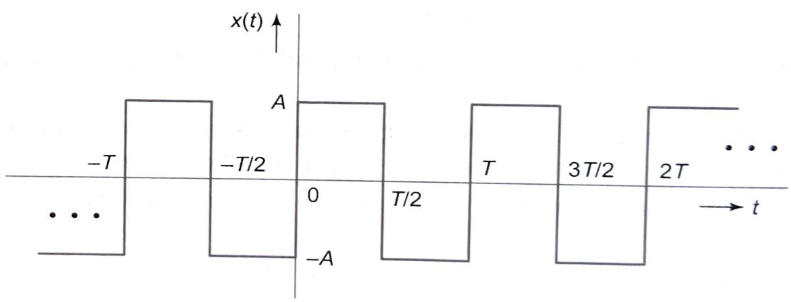

Fig. 1.1 *A rectangular wave*

Discrete-time signals are those which are defined only at discrete set of points in time. For example, see Fig. 1.2.

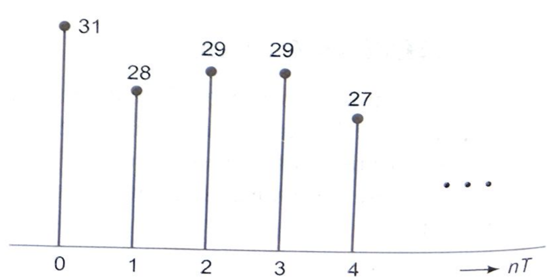

Fig. 1.2 *A discrete-time signal*

Suppose we note down the temperature at a particular place at 5 a.m. every day. We get a discrete-time signal as shown in Fig. 1.2, where n denotes the number of the day and the ordinate represents the temperature. Figure 1.2 is a typical diagrammatic representation of a discrete-time signal; the sample values being shown by pin-heads. One should not, however, conclude from this figure that the temperature, during the time between two samples, is zero. In fact, what all we can say is that we do not know, as the temperatures are recorded only at a discrete set of points in time.

Here, T represents the period between two successive samples. It should be noted that while time takes only discrete values, the amplitude, in this case, the temperature, takes a continuum of values.

Thus, the discrete-time signal is a sequence of numbers, real or complex. The time variable nT is generally represented only by n by assuming that T, the interval between two successive samples, is normalized to a value of 1; so that instead of denoting the DT signal as $x(nT)$, we denote it simply by $x(n)$.

2. Periodic signals and non-periodic signals A continuous-time signal $x(t)$ is said to be periodic in time if

$$x(t) = x(t + mT) \qquad \ldots (1.1)$$

for any t, and any integer m. The *smallest* value of the positive constant T, satisfying the above relation, is said to be the *fundamental* period of the periodic function $x(t)$.

For example, with k taking integer values, $x_k(t) = e^{j2\pi kt/T}$ has a fundamental period of T/k, while the set of signals $x_k(t)$ share a common period of T.

Therefore, T represents the duration of one complete cycle of $x(t)$. If $f \triangleq \frac{1}{T}$ then 'f' is in cycles/second/or Hertz (denoted by Hz).

The reader is urged to note the difference between the terms 'fundamental period' and 'period' of a periodic signal. If T_o is the *smallest* value of T for which $x(t) = x(t + mT)$ is satisfied for any integer value of m, then $x(t)$ repeats itself at regular intervals of not only the fundamental period T_o, but also at *periods* corresponding to integer multiples of T_o.

Any signal which does not satisfy Eq. (1.1), is said to be a non-periodic, or an aperiodic, signal. We may apply this classification to discrete-time signals also. A discrete-time signal, $x(n)$ is said to be periodic if

$$x(n) = x(n \pm m) \qquad \ldots (1.2)$$

for any integer n; where m is a positive integer. The smallest value of mM for which the above relation holds good, is called the fundamental period of the discrete-time signal $x(n)$. $1/M$ is then the fundamental frequency and $2\pi(1/M)$, the fundamental angular frequency of $x(n)$.

The reader can readily verify that for integer values of k and N, the signals $x_k(n) = e^{j2\pi nk/N}$ for various k, share a common period of N.

The reader should note the difference between Eq. (1.1) and Eq. (1.2). T in Eq. (1.1) can take any real positive value; while M in Eq. (1.2) can take only integer values.

Example 1.1 *Is the signal $x(t) = 10\cos^2(10\pi t)$ a periodic signal? If yes, what is its fundamental period?*

Solution

$$x(t) = 10\cos^2(10\pi t) = 5[1 + \cos 20\pi t]$$

Now, $\cos 20\pi t = \cos 2\pi(10t) = y(t)$

$\therefore$ $y(t + T) = \cos 2\pi(10t + 10T) = \cos 2\pi\, 10t . \cos 2\pi\, 10T - \sin 2\pi\, 10t \sin 2\pi\, 10T$

But $\cos 2\pi 10T = 1$ if $10T$ is an integer. Then $\sin 2\pi 10T = 0$.

$\therefore$ $y(t + T) = \cos 2\pi(10t + 10T) = \cos 2\pi 10t . \cos 2\pi 10T = \cos 2\pi 10t = y(t)$

$\therefore$ $y(t)$ is periodic with a fundamental period corresponding to the smallest value of T for which $10T$ is an integer.

$\therefore$ the fundamental period = 1/10 second, i.e., fundamental frequency = 10 c/s

Now, $x(t) = 5[1 + y(t)]$ $\therefore$ $x(t)$ also is periodic with a period of 1/10 second.

Example 1.2 *$x(t)$ and $y(t)$ are two periodic signals with $x(t)$ having a fundamental period T_x and $y(t)$ having a fundamental period T_y. If $z(t) = x(t) + y(t)$, what is the condition required to be satisfied for $z(t)$ to be periodic? If $z(t)$ is periodic, what is its fundamental period?*

Solution In case $z(t)$ is periodic, let us say, with a period T, and $z(t) = x(t) + y(t)$, both $x(t)$ as well as $y(t)$ must complete an integer number of their periods within the period T. Suppose $x(t)$ completes m periods and $y(t)$ completes n periods in the time T, where m and n are integers.

Then, $$T = mT_x = nT_y.$$

Hence, $$\frac{T_x}{T_y} = \frac{n}{m} = \text{a rational number}$$

Thus, for $z(t)$ to be periodic, the condition to be satisfied is that (T_x/T_y) must be a rational number.

Further, since $T = mT_x = nT_y$ where m and n are integers when $z(t)$ is periodic, if T is the fundamental period of $z(t)$, m and n must be the smallest integers satisfying the above relation. This means that

$$T = LCM\ (T_x, T_y)$$

Example 1.3 *Determine whether the following signals are periodic. For those which are periodic, determine the fundamental period.*

(a) $x(t) = 2\sin\left(\frac{2}{3}\right)t + 3\cos\left(\frac{2\pi}{5}\right)t$

(b) $y(t) = 3\sin t + 5\cos\left(\frac{4}{3}\right)t.$

Solution

(a) For $\sin\left(\frac{2}{3}\right)t$, $\omega_1 = \frac{2}{3}$ and its period $T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{2/3} = 3\pi$

For $\cos\left(\frac{2\pi}{5}\right)t$, $\omega_2 = \frac{2\pi}{5}$ $\therefore$ its period $T_2 = \frac{2\pi}{(2\pi/5)} = 5.$

$\therefore$ $\frac{T_1}{T_2} = \frac{3\pi}{5}$, which is not a rational number.

Hence, $x(t)$ is not periodic.

(b) T_1, the period of $\sin t$ is given by

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{1} = 2\pi$$

Similarly T_2, the period of $\cos\left(\frac{4}{3}\right)t$ is given by

$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{(4/3)} = \frac{3\pi}{2}$$

$$\therefore \quad \frac{T_1}{T_2} = 2\pi \times \frac{2}{3\pi} = \frac{4}{3}, \quad \text{a rational number.}$$

Therefore, $y(t)$ is periodic.

Its fundamental period = LCM (T_1, T_2)

= LCM$(2\pi, 1.5\pi) = 6\pi$.

Example 1.4 *Check whether the signal $x(t) = a \sin 4t + b \cos 7t$ is periodic. If it is periodic, determine its fundamental period.*

Solution The period T_1 of $a \sin 4t$ is given by

$$T_1 = \frac{2\pi}{4} = \frac{\pi}{2}$$

The period T_2 of $b \cos 7t$ is given by

$$T_2 = \frac{2\pi}{7}$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{\pi/2}{(2\pi/7)} = \frac{7}{4}, \quad \text{which is a rational number.}$$

Hence, the given $x(t)$ is periodic. Its fundamental period is given by

$$T = (\text{LM}\ (T_1, T_2) = 2\pi$$

Example 1.5 *Determine whether the following signal is periodic. If you find that it is periodic, determine its fundamental period.*

$$x(t) = \cos t + \sin \sqrt{2}\, t$$

Solution

$$\text{Period of } \cos t = T_1 = \frac{2\pi}{1} = 2\pi$$

$$\text{Period of } \sin \sqrt{2}\, t = T_2 = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\,\pi$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{2\pi}{\sqrt{2}\,\pi} = \sqrt{2}, \quad \text{which is an irrational number.}$$

Hence, $x(t)$ is not a periodic signal.

Example 1.6 *Check whether the following signals are periodic. If they are, find their fundamental periods.*

(a) $x(n) = 2 \sin 0.8\, \pi n$ (b) $x(n) = 3 \cos 4n$

Solution

(a)
$$x(n) = 2 \sin 0.8\, \pi n$$
$$\therefore \quad x(n+m) = 2\{\sin 0.8\, \pi n . \cos 0.8\, \pi m + \cos 0.8\, \pi n .\ \sin 0.8\, \pi m\}$$

This can be equal to 2 sin 0.8 πn only if cos 0.8 $\pi m = 1$

i.e., if 0.8 $\pi m = k\pi$ where $k = 0, 2, 4, \ldots.$

i.e., if $m = k/0.8$. The minimum value of k which makes m an integer is $k = 4$; and for this value of k, m takes the value 5.

Hence, $x(n)$ is periodic with a fundamental period of 5.

(b)
$$x(n) = 3 \cos 4n$$
$$\therefore \quad x(n+m) = 3[\cos 4n \cos 4m - \sin 4n . \sin 4m]$$

This can be equal to 3 cos $4n$ only if cos $4m = 1$, i.e., if $4m = k\pi$ where k is $0, 2, 4, \ldots$

This can be satisfied with integer values for m and k only if $k = 0$, i.e., $m = 0$. This implies that $x(n) =$ 3 cos $4n$ is not periodic.

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3. Even and odd signals

(a) Continuous time signals A real-valued continuous-time signal $x(t)$ is said to be ***even*** if

$$x(-t) = x(t) \text{ for all } t \qquad \ldots (1.3)$$

$$x(t) \text{ is said to be } \textbf{\textit{odd}} \text{ if } x(-t) = -x(t) \text{ for all } t \qquad \ldots (1.4)$$

For example, $x(t) = \cos t$ is an even signal since $\cos(-t) = \cos(t)$ for all t; and $x(t) = \sin t$ is an odd signal since $\sin(-t) =$ $\sin t$ for all t.

As can be seen from Figs. 1.3 and 1.4, the waveform of an even signal is symmetrical about the time origin while that of an odd signal is anti-symmetric with respet to the time origin.

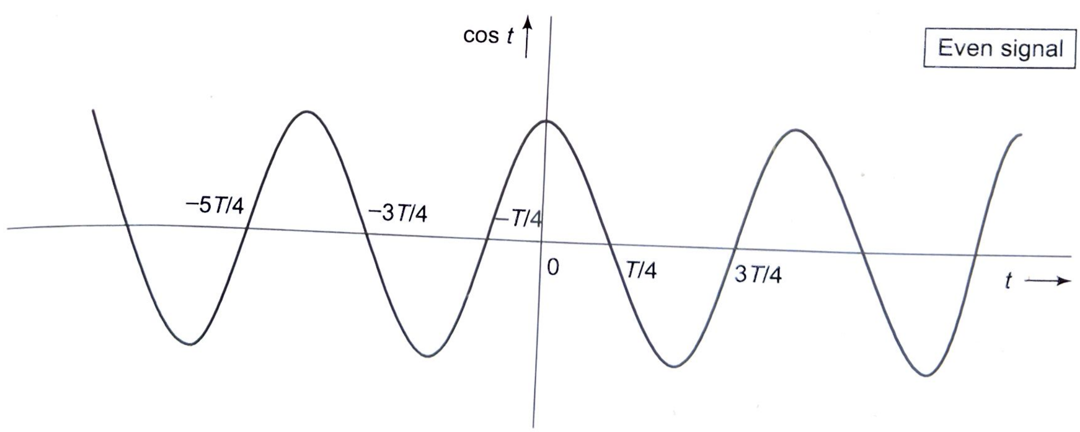


Fig. 1.3 *Cosinusoidal signal*

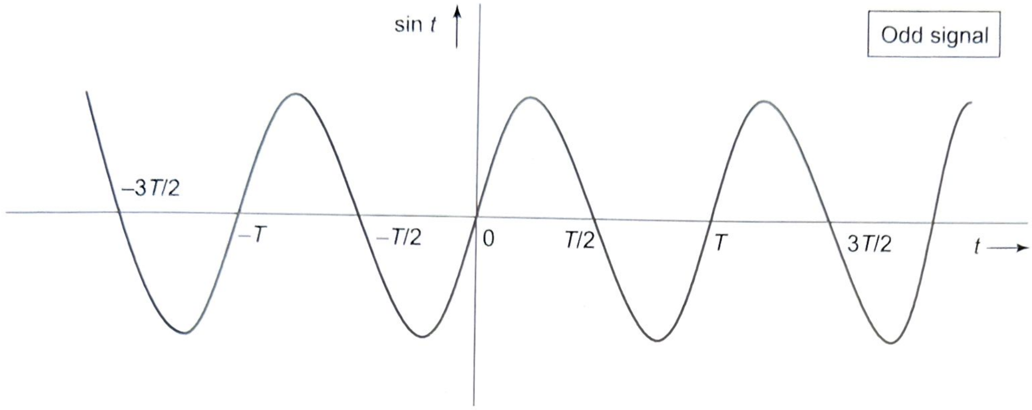


Fig. 1.4 *A sinusoidal signal*

> *It should be noted that every signal need not be purely even or purely odd.*

For example, if $x(t) = \cos t + \sin t$, it is neither purely even nor purely odd; it has an even component and an odd component.

Let $x(t) = x_e(t) + x_0(t)$ where $x_e(t)$ is the even component and $x_0(t)$ is the odd component, then $x(-t) = x_e(-t) + x_0(-t) = x_e(t) - x_0(t)$

$$\therefore \qquad x(t) + x(-t) = 2x_e(t)$$

or
$$x_e(t) = \tfrac{1}{2}[\,x(t) + x(-t)\,] \qquad \ldots (1.5)$$

and
$$x_0(t) = \tfrac{1}{2}[\,x(t) - x(-t)\,] \qquad \ldots (1.6)$$

In the case of complex-valued signals, we can also talk about conjugate symmetry

Let $x(t) = a(t) + jb(t)$ ...(1.7)

If both $a(t)$ and $b(t)$ are even, then $x(t)$ is even. If both $a(t)$ and $b(t)$ are odd, then $x(t)$ is odd.

But suppose $a(t)$ is even, while $b(t)$ is odd. Then $x(-t) = a(-t) + jb(-t) = a(t) - jb(t) = x^*(t)$

i.e., $x(t)$ is said to be possessing conjugate symmetry.

On the other hand, if $a(t)$ is odd while $b(t)$ is even, then $x(-t) = a(-t) + jb(-t) = -x^*(t)$. Then $x(t)$ is said to be conjugate anti-symmetric. For example, $x(t) = \sin t + j\cos t$ is a conjugate anti-symmetric signal.

Every complex-valued signal need not necessarily be either conjugate symmetric (CS) or conjugate anti symmetric (CAS). In general it can have a CS component and a CAS component.

Let a complex-valued signal $x(t)$ be given by

$$x(t) = x_{CS}(t) + x_{CAS}(t) \qquad \ldots (1.8)$$

Then,

$$x(-t) = x_{CS}(-t) + x_{CAS}(-t) = x^*_{CS}(t) - x^*_{CAS}(t)$$

$$\therefore \qquad x^*(-t) = x_{CS}(t) - x_{CAS}(t) \qquad \ldots (1.9)$$

From Eqs. (1.8) and (1.9), we have

$$\boxed{x_{CS}(t) = \frac{1}{2}[x(t) + x^*(-t)]} \qquad \ldots (1.10)$$

and $$\boxed{x_{CAS}(t) = \frac{1}{2}[x(t) + x^*(-t)]} \qquad ...(1.11)$$

Hence, a complex-valued signal is said to have conjugate symmetry if its real part is even and the imaginary part is odd. It is said to be conjugate anti-symmetric if its real part is odd and its imaginary part is even.

(*b*) *Discrete-time signals* A real-valued discrete-time signal $x(n)$ is said to be '*even*' if

$$x(-n) = x(n) \quad \text{for all } n \qquad ...(1.12)$$

and it is said to be having '*odd*' symmetry if

$$x(-n) = -x(n) \quad \text{for all } n \qquad ...(1.13)$$

For example, the sequence

$$x(n) = a^{-|n|},\ 0 < a < 1$$

has even symmetry, as can be seen from Fig. 1.5.

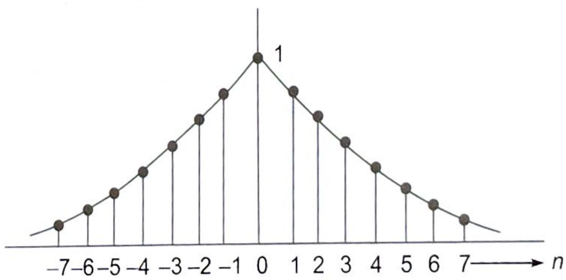


Fig. 1.5 *Sequence* $x(n) = a^{|n|};\ 0 < a < 1$

On the other hand, the sequence $x(n) = \sin 0.2\pi n$ sketched in Fig. 1.6, has odd symmetry.

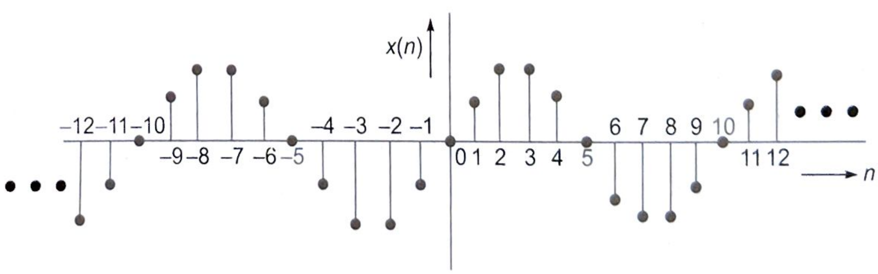


Fig. 1.6 *Sequence* $x(n) = \sin 0.2\pi n$

Every signal need not be either even or odd. It may not be either. In general, every sequence can be expressed as the sum of an even component and an odd component.

$\therefore$ Let $$x(n) = x_e(n) + x_o(n) \qquad ...(1.14)$$

where $x_e(n)$ is the even part, and $x_o(n)$ is the odd part of the sequence $x(n)$.

Then

$$x(-n) = x_e(-n) + x_o(-n) = x_e(n) - x_o(n)$$

$$\therefore \quad x(n) + x(-n) = 2x_e(n)$$

i.e.,
$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \quad ...(1.15)$$

Also,
$$x(n) - x(-n) = 2x_o(n)$$

i.e.,
$$x_o(n) = \frac{1}{2}[x(n) - x(-n)] \quad ...(1.16)$$

If $x(n)$ is a complex-valued sequence, let
$$x(n) = x_R(n) + jx_I(n) \quad ...(1.17)$$
where $x_R(n)$ and $x_I(n)$ are respectively the real and imaginary parts of the complex sequence $x(n)$. If $x_R(n)$ and $x_I(n)$ are both even sequences, the sequence $x(n)$ is even; when $x_R(n)$ and $x_I(n)$ are both odd, the sequence $x(n)$ is odd; when $x_R(n)$ is even and $x_I(n)$ is odd, $x(-n) = x^*(n)$, and the sequence $x(n)$ is said to be having conjugate symmetry (CS). On the other hand, if $x_R(n)$ is odd $x_I(n)$ is even, $x(-n) = -x^*(n)$, and the sequence $x(n)$ is said to be possessing conjugate anti-symmetry (CAS).

As in the case of continuous-time signals, here too any complex sequence in general will have a CS component and a CAS component:
$$x_{CS}(n) = \frac{1}{2}[x(n) + x^*(-n)] \quad ...(1.18)$$
$$x_{CAS}(n) = \frac{1}{2}[x(n) - x^*(-n)] \quad ...(1.19)$$

We may define **even and odd discrete-time signals** exactly in a similar way, and derive equations similar to Eqs. (1.5) and (1.6).

Example 1.7 *Determine and sketch the even and odd components of the continuous-time signal $x(t) = e^{-t}u(t)$.*

Solution The signal $x(t) = e^{-t}u(t)$ is sketched in Fig.1.7.

Now,
$$x_e(t) = \text{Even part of } x(t) = \frac{1}{2}[x(t) + x(-t)]$$
$$= \frac{1}{2}[e^{-t}u(t) + e^{t}u(-t)]$$
$$x_0(t) = \text{Odd part of } \frac{1}{2}[x(t) - x(-t)]$$
$$= \frac{1}{2}[e^{-t}u(t) - e^{t}u(-t)]$$

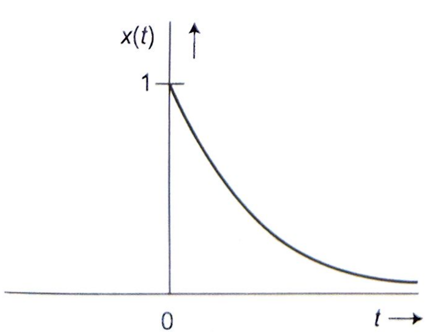


Fig. 1.7 *Signal $x(t) = e^{-t}u(t)$*

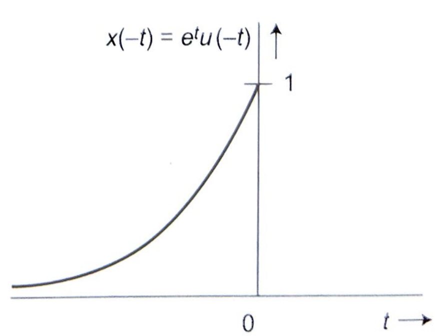


Fig. 1.8 *Signal $x(-t) = e^{t}u(-t)$*

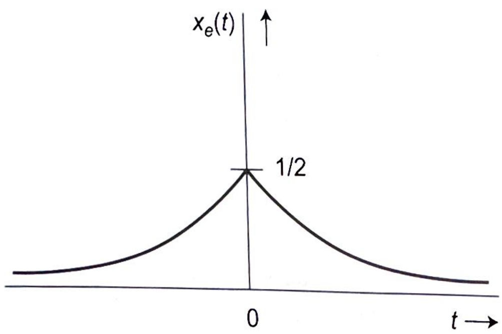


Fig. 1.9 *Signal $x_e(t)$, the even part of the signal $x(t)$*

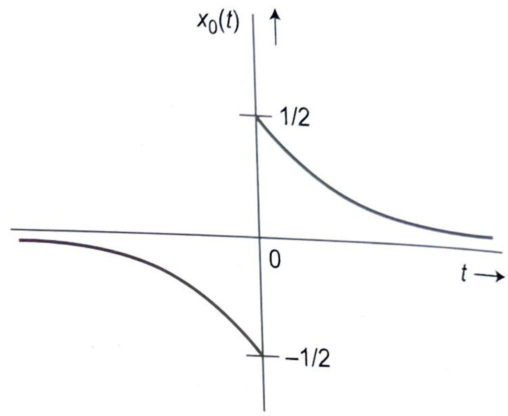


Fig. 1.10 *Signal $x_0(t)$, the odd part of signal $x(t)$*

Example 1.8 *Given the discrete-time signal shown in Fig.1.11, determine and sketch its even and odd components.*

Solution Figures 1.11 shows a sketch of the signal $x(n)$.

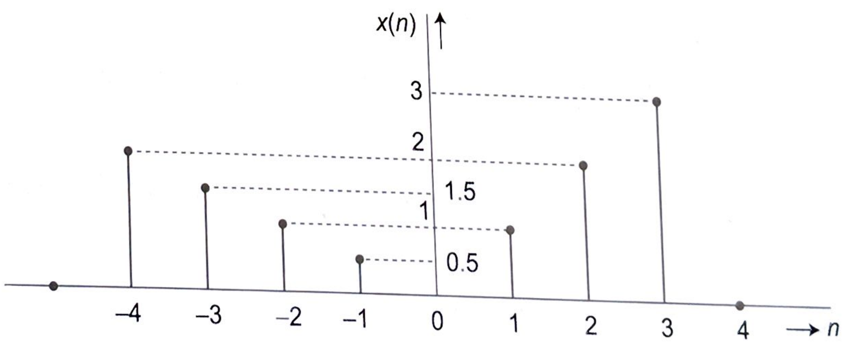


Fig. 1.11 *Sketch of the signal $x(n)$*

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \text{ and } x_0(n) = \frac{1}{2}[x(n) - x(-n)]$$

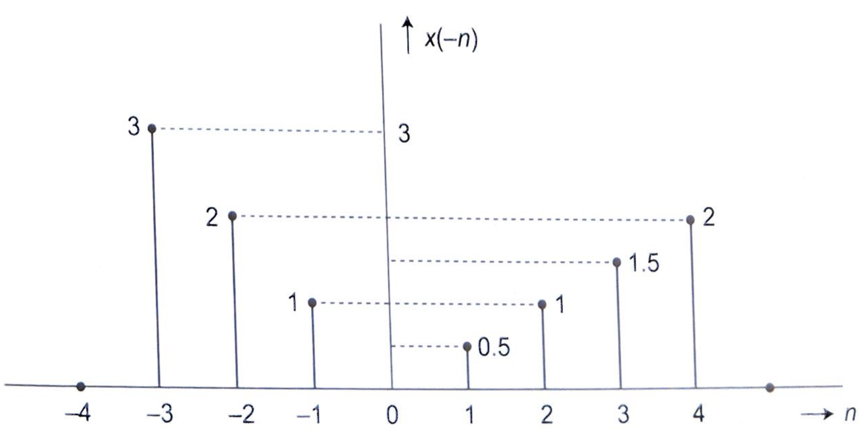


Fig. 1.12 *Sketch of the signal $x(-n)$*

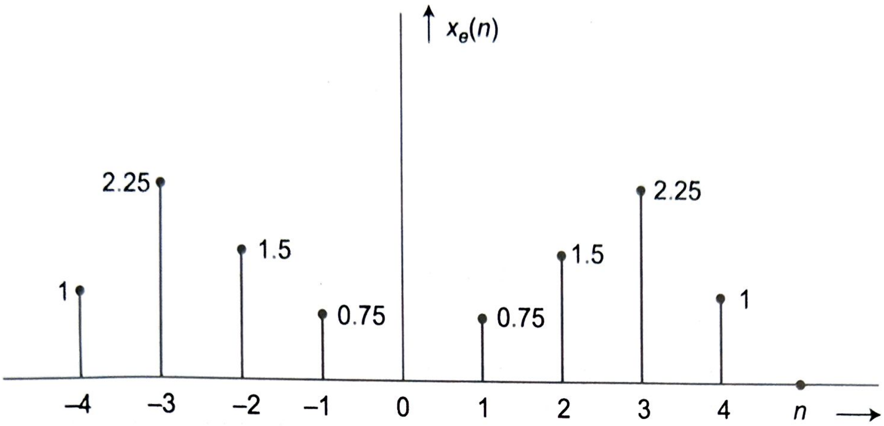


Fig. 1.13 *Sketch of the signal $x_e(n)$*

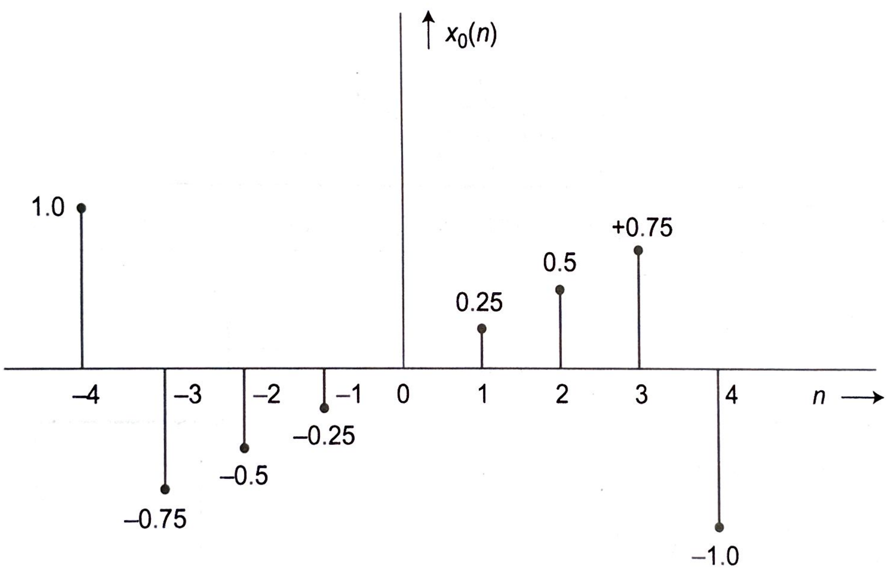


Fig. 1.14 *Sketch of the signal $x_0(n)$*

4. **Energy signals and power signals** Suppose a signal $x(t)$ represents some voltage as a function of time. If this signal is applied across the terminals of a resistor of R ohms, the instantaneous power delivered to the resistor is $x^2(t)/R$ watts. If $x(t)$ represents a current signal and if it is flowing through a resistance of R ohms, the instantaneous power is $x^2(t) \cdot R$ watts. In each case, this power is dependent upon the value of $x(t)$ as well as the value of R. If we make $R = 1$ ohm, then $x^2(t)$ represents the instantaneous power irrespective of whether $x(t)$ is a voltage or current; also, it depends only on the signal.

In the light of the above, it is customary to talk about the power or the energy of a signal, the underlying assumption being that the resistance across which the voltage signal $x(t)$ applied, or through which the current signal $x(t)$ is passed, is equal to 1 ohm.

Thus, the total energy of a continuous-time real-valued signal $x(t)$ may be written as

$$E = \underset{T\to\infty}{Lt} \int_{-T}^{T} x^2(t)dt \qquad \ldots (1.20)$$

The average power may be written as

$$P_{av} = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt \qquad \ldots (1.21)$$

In the case of a periodic signal $x(t)$ with a period T, it is readily seen that

$$P_{av} = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt \qquad \ldots (1.22)$$

Note *In case x(t) is a complex-valued signal.*

$$E = \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 dt \qquad \ldots (1.23)$$

and

$$P_{av} = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \qquad \ldots (1.24)$$

Definition *An energy signal is one whose total energy E is finite and non-zero; specifically $0<E<\infty$.*

Example 1.9

Consider $x(t) = \begin{cases} A\ ; & 0 < t < T_0 \\ 0\ ; & \text{otherwise} \end{cases}$

A plot of this signal is shown in Fig. 1.15.

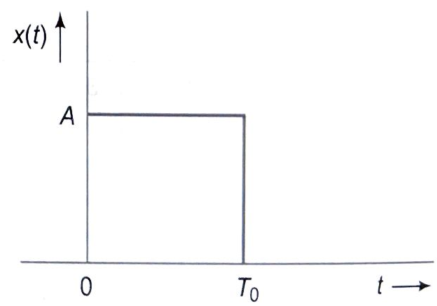


Fig. 1.15 *Plot of the signal given in Example 1.9*

Solution

The energy of this $x(t)$ is $E = \underset{T \to \infty}{Lt} \int_{-T}^{T} x^2(t)dt$

$$= \int_{0}^{T_0} x^2(t)dt = \int_{0}^{T_0} A^2 dt = A^2\, T_0.$$

The average power $= P_{av} = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt = \underset{T \to \infty}{Lt} \frac{1}{2T}\left[A^2 T_0\right] = 0$

$\therefore$ energy is finite and it is an energy signal.

Definition *A power signal is one whose average power P_{av} is finite and non-zero; specifically $0 < P_{av} < \infty$.*

Example 1.10 *Consider $x(t) = \cos 2\pi f_0 t$. Is it a power signal?*

Solution

$$E = \text{Total energy} = \underset{T\to\infty}{Lt} \int_{-T}^{T} x^2(t)dt = \underset{T\to\infty}{Lt} \int_{-T}^{T} \cos^2(2\pi f_0 t)dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2} \int_{-T}^{T} (1+\cos 4\pi f_0 t)dt = \underset{T\to\infty}{Lt} \frac{1}{2} \int_{-T}^{T} dt + \underset{T\to\infty}{Lt} \frac{1}{2} \int_{-T}^{T} \cos 4\pi f_0 t dt$$

The second integral is zero and the first one is ∞.

$$\therefore \qquad E = \infty.$$

The average power $= P_{av} = \underset{T\to\infty}{Lt} \dfrac{1}{2T} \displaystyle\int_{-T}^{T} x^2(t)dt = \underset{T\to\infty}{Lt} \dfrac{1}{2T}\left(\dfrac{1}{2}\right) \displaystyle\int_{-T}^{T} dt = \dfrac{1}{4T} \times 2T = \dfrac{1}{2}$

Thus, this signal has a non-zero and finite average power, but infinite total energy. Hence, it is a power signal.

Note: *Every signal need not be either an energy signal or a power signal. It can be neither, as in the case of signals such as $x(t) = e^{-t}$; $-\infty < t < \infty$. (The reader may check this.)*

In the case of a discrete-time signal $x(n)$, we replace the integrals in Eqs. (1.23) and (1.24) by summations and write

$$E = \underset{N\to\infty}{Lt} \sum_{n=-N}^{N} |x(n)|^2 \qquad \text{. . . (1.25)}$$

and

$$P_{av} = \underset{N\to\infty}{Lt} \frac{1}{2N} \sum_{n=-N}^{N} |x(n)|^2 \qquad \text{. . . (1.26)}$$

5. Deterministic signals and random signals Deterministic signals are those signals whose value at any instant of time is known for $-\infty < t < \infty$. For example, consider $x(t) = 10 \cos 100\pi t$. At any instant of time t, one can calculate the value of $x(t)$ from the given expression for $x(t)$. Thus, if $x(t)$ is specified by a function of t or through a look-up table, which will enable us to determine the value of $x(t)$ at any specified t, $-\infty < t < \infty$, then we call such a signal a deterministic signal.

As against this, there are some signals which are random in nature, i.e., their values cannot be determined or predicted. Noise signals and EEG signals are examples of such random signals. Random signals, if stationary, may be described in terms of certain average values only. Any observation of a random signal will be only one realization of a number of possible realizations, each having a certain probability of occurrence. Which one of the possible realizations actually occurs may be viewed as the outcome of a probabilistic experiment which may be represented by a variable ξ. Thus a random process is a function of t as well as ξ and is generally denoted by $X(t, \xi)$.

We shall assume throughout this text that the signals that we deal with are all only deterministic signals.

6. One-dimensional and multi-dimensional signals A signal which is a function of only one variable is referred to as a one-dimensional signal. For example, the signal, $x(t) = a \cos \omega_0 t$.

But signals may be multi-dimensional also. For example, if we consider a monochrome still image, the intensity of light reflected from any point on it is a function of x and y, the spatial coordinates, since I, the intensity, varies from point to point.

In this book, we shall restrict our attention only to one-dimensional signals. However, the extension of the concepts to 2-D signals is quite straight forward.

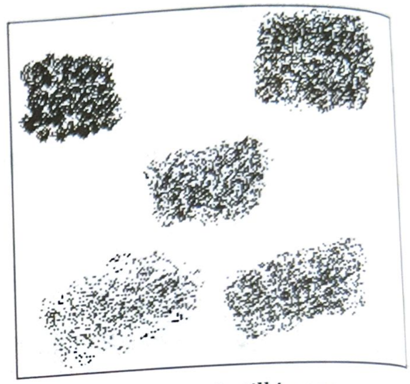


Fig. 1.16 *A still image*

1.3 REPRESENTATION OF SIGNALS

1. Continuous-time signals

Continuous-time signals are specified for all values of time by a function of time like $x(t) = e^{-|t|}; -\infty \le t \le \infty$, or by a look-up table which specifies its value for all time, as for example:

$$x(t) = \begin{cases} 2\ ; & -\infty < t \le 0 \\ 1\ ; & 0 < t < \infty \end{cases}$$

They are represented diagrammatically by their waveforms, i.e., graphs depicting their variation with time. In Chapters 2, 3 and 4, we will discuss some transform-domain representations of continuous-time signals.

2. Discrete-time signals A discrete-time signal, as mentioned earlier, is a sequence of numbers. These numbers are assumed to be occurring at regular intervals of T seconds. The nth sample of a discrete-time signal is generally denoted by $x(nT)$ or, simply $x(n)$; the sequence or the signal itself being denoted by $\{x(nT)\}$ or $\{x(n)\}$. In fact, since it is a cumbersome notation, we generally represent the sequence also by $x(n)$. Whether it is used to represent the sequence itself, or its nth sample value, will be clear from the context. This is similar to the use of $x(t)$ for referring to the function $x(.)$ as well as its value at the instant 't'.

A discrete-time signal $x(n)$ may be specified or described in one of the following ways:

(*a*) By means of a look-up table

(*b*) By means of an equation which specifies the nth sample value $x(n)$ as a function of n. For example, $x(n) = n^2; -\infty < n < \infty$

(*c*) By means of a recursive formula such as

$x(n) = 2x(n-1) + \delta(n)$ *with* $x(-1) = 0$, where $\delta(n)$ *is a special symbol used to represent what is called a unit sample sequence for which*

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

In addition, in Chapters 2 and 5, we will discuss some transform-domain representations of sequences.

1.4 SOME COMMONLY USED SIGNALS

1. Continuous-time signals

(a) Unit-impulse function $\delta(t)$ This is not a function in the usual sense. In fact, it comes under the category of generalized functions, or distributions, and is defined by the following:

$$\int_{t_1}^{t_2} x(t)\delta(t)dt = x(t)\,|_{t=0} = x(0) \qquad \text{. . . (1.27)}$$

where $x(t)$ is any function which is continuous at least at $t = 0$ and t_1 and t_2 are such that the interval t_1 to t_2 includes $t = 0$.

Property 1 *The area under a unit impulse function is equal to 1.*

Proof Let $x(t) = 1$, which is continuous at all points. Let $t_1 = -\infty$ and $t_2 = +\infty$. Then

$$\int_{-\infty}^{+\infty} 1.\delta(t)dt = 1 \Rightarrow \text{the area under } \delta(t) = 1.$$

Property 2 *The width of $\delta(t)$ along the time axis is zero.*

Proof $\int_{0-\epsilon}^{0+\epsilon} 1.\delta(t)dt = 1$. Let $\epsilon \to 0$. Still the total area under it does not change. Hence, it has zero width along the time axis around the time origin.

Property 3 (Sampling property) *From Eq.(1.27), if $x(t)$ is continuous at $t = \tau$*

$$\int_{t_1}^{t_2} x(t)\delta(t-\tau)dt = x(t)\Big|_{t=\tau} = x(\tau)$$

for any t_1 and t_2 such that the interval t_1 to t_2 includes $t = \tau$. But

$$\int_{t_1}^{t_2} x(\tau)\delta(t-\tau)dt = x(\tau)\int_{t_1}^{t_2} \delta(t-\tau)dt = x(\tau)$$

Therefore, $$\int_{t_1}^{t_2} x(t)\delta(t-\tau)dt = \int_{t_1}^{t_2} x(\tau)\delta(t-\tau)dt$$

for any $x(t)$ which is continuous at $t = \tau$ and for any t_1 and t_2, if their interval includes $t = \tau$. Thus, we conclude

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$$

This is called the *sampling property* of the impulse function.

From the above properties, it is clear that the unit impulse function $\delta(t)$, can be visualized as $\underset{\Delta \to 0}{Lt}\, x(t)$ where $x(t)$ is as shown in Fig. 1.17.

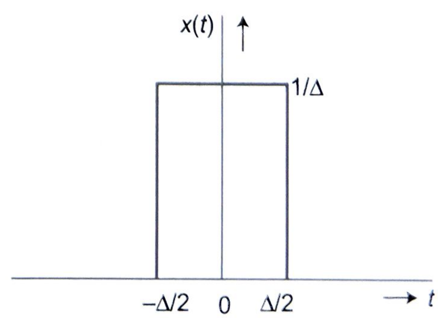


Fig. 1.17 *Rectangular pulse as an approximation to $\delta(t)$*

> **Note** *There is no particular significance to the rectangular shape of the pulse used to represent $x(t)$. We may even start with a triangular pulse of height $2/\Delta$ and base from $-\Delta/2$ to $+\Delta/2$ and allow Δ to tend to zero.*

We may thus visualize $\delta(t)$ as being located at $t = 0$, having an area of 1 and occupying zero width along the time axis. Because of this, it is generally represented in diagrams as shown in Fig. 1.18. The 1 marked on the diagram indicates that it is a unit impulse, i.e., it has an area of one.

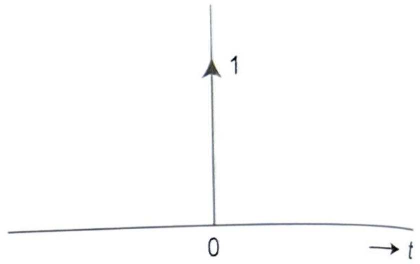


Fig. 1.18 *A unit impulse*

Following our usual notation, $\delta(t - t_0)$ denotes a delayed unit impulse, delayed by t_0 seconds. A $\delta(t)$ denotes an impulse function of strength A, i.e., the area under it is A.

Although $\delta(t)$ cannot be generated, it is useful as a mathematical tool in the analysis of systems. When this signal is applied to a circuit, its response is characteristic only of the circuit and not the signal applied (referred to as the natural response of the circuit). Further, the sampling property of $\delta(t)$ allows us to mathematically characterize sampling of signals in Chapter 6.

(b) Unit step function u(t) The unit step function, denoted by $u(t)$, is defined by the following:

$$u(t) \triangleq \begin{cases} 1 \ for \ t \geq 0 \\ 0 \ for \ t < 0 \end{cases} \quad \ldots (1.28)$$

It is diagrammatically represented as shown in Fig. 1.19.

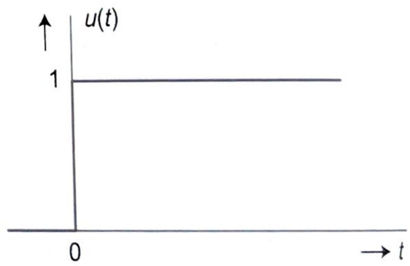


Fig. 1.19 *A unit step function*

There is an interesting and useful relationship between $\delta(t)$ and $u(t)$. Consider

$$x(t) = \int_{-\infty}^{t} \delta(\lambda) d\lambda \quad \ldots (1.29)$$

Obviously, since the right-hand side represents the area under the unit impulse function from $-\infty$ up to t, if $t < 0$, the area is zero. If $t \geq 0$, the area is 1. Hence,

$$x(t) = \begin{cases} 1 \ for \ t \geq 0 \\ 0 \ for \ t < 0 \end{cases}$$

But this is precisely how we have defined $u(t)$. Hence,

$$u(t) = \int_{-\infty}^{t} \delta(\lambda) d\lambda \quad \ldots (1.30)$$

and

$$\frac{d}{dt} u(t) = \delta(t) \quad \ldots (1.31)$$

A unit step function is an extremely useful signal in practice. Quite often, we suddenly apply a voltage of say V volts to a circuit at some instant (which is taken as the time origin, i.e., $t = 0$) and we will be interested in the response of the circuit for $t \geqslant 0$. In such cases, we may conveniently represent the applied voltage as $Vu(t)$. In general, if $x(t)$ is some signal and if we want to have a signal $x_1(t)$ such that

$$x_1(t) = \begin{cases} x(t) \text{ for } t \ge T \\ 0 \text{ otherwise} \end{cases}$$

then, all that we need to do is to multiply $x(t)$ by $u(t - T)$, i.e.,

$$x_1(t) = x(t)\, u(t - T).$$

Example 1.11 *If signal $x(t)$ is as shown is Fig. 1.20, sketch the waveforms of the signals $\dot{x}(t)$ and $\ddot{x}(t)$* (the derivative and double derivative respectively of $x(t)$).

Solution From $t = -3$ to $t = -1$, $x(t)$ is increasing linearly from 0 to 2, i.e., with a gradient of 1. Hence, $\dot{x}(t)$ remains constant at the value of 1 from $t = -3$ to $t = -1$.

From $t = -1$ to $t = +1$, $x(t)$ is constant and so $\dot{x}(t)$ is zero in this interval.

From $t = 1$ to $t = 3$, $x(t)$ is decreasing linearly with a gradient of -1. Hence a sketch of $\dot{x}(t)$ is as shown in Fig. 1.20(b).

Now, $\dot{x}(t)$ may be expressed in terms of $u(t)$ as

$$\dot{x}(t) = u(t + 3) - u(t + 1) - u(t - 1) + u(t - 3)$$

Differentiating the above and using Eq. (1.31), we have

$$\ddot{x}(t) = \delta(t + 3) - \delta(t + 1) - \delta(t - 1) + \delta(t - 3)$$

i.e., it consists of impulses of unit strength at $t = -3$ and $t = 3$ and of strength -1 at $t = -1$ and $t = 1$, as shown in Fig. 1.20(c).

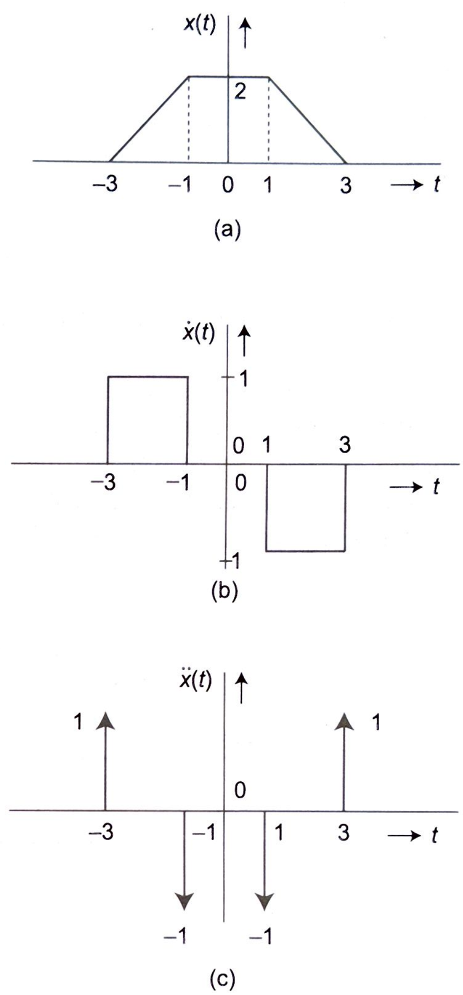


Fig. 1.20 *Waveforms of a signal x(t) and its first and second derivatives*

(c) Exponential signal As will be seen later, exponential signals are frequently encountered in the analysis of systems. The exponential signals may be represented as

$$x(t) = Ae^{bt} \qquad \ldots (1.32)$$

where A is the value of the signal at $t = 0$, and b is a number, real or complex. If b is real, depending on whether it is positive or negative, we get either an exponentially growing signal, or an exponentially decaying signal respectively. These are as shown in Figs. 1.21 (a) and (b) for different values of b.

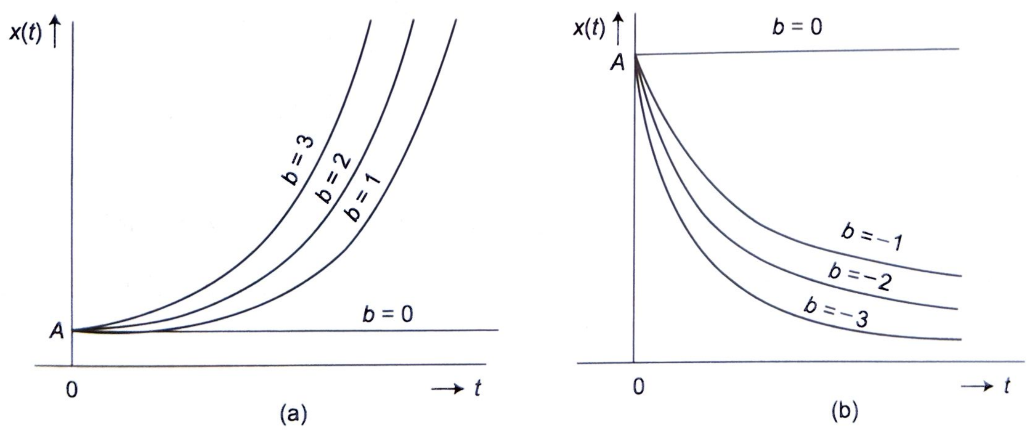


Fig. 1.21 *(a) A growing exponential (b) A decaying exponential*

Note ***In Figs. 1.21(a) and (b), the exponential signals have been shown only in the first quadrant.***

When $b = j\omega_o$, $x(t)$ becomes a complex exponential, which as we will show later, is very useful for system analysis and signal representation.

Let $$x(t) = e^{j\omega_o t} = A(\cos \omega_o t + j \sin \omega_o t)$$

$\therefore$ $$|x(t)| = A \quad \text{and} \quad \angle x(t) = \omega_o t = \text{say } \theta(t)$$

Since $\theta(t) = \omega_o t$, $\theta = 0$ at $t = 0$ and it is equal to 2π radians when $t = \dfrac{1}{f_o} = T_o$. This implies that the phasor of length A (equal to the magnitude of $A\,e^{j\omega_o t}$) will initially be along the real axis (since $\theta = 0$ at $t = 0$) and will be making one revolution every T_o seconds, in the counter-clock wise direction. Since $x(t) = A \cos \omega_o t + j \sin \omega_o t$, the real part of $x(t)$, i.e., the projection of the rotating length-A phasor on to the real axis, varies as $A \cos \omega_o t$ with respect to time as shown in Fig. 1.22. The projection of $x(t)$ on to the imaginary axis, which gives the imaginary part of $x(t)$, varies as $A \sin \omega_o t$, as shown in the same figure.

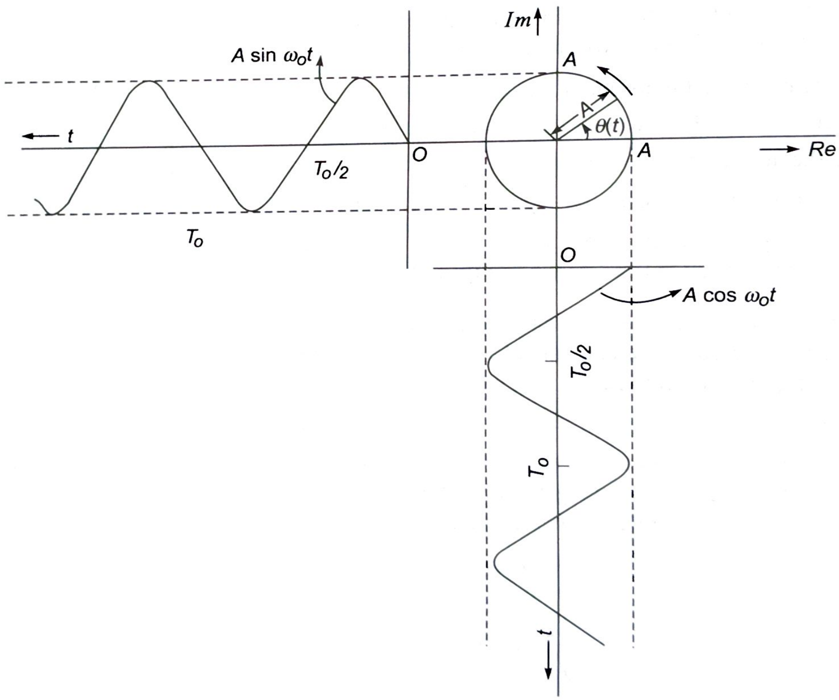


Fig. 1.22 *Rotating phasor representation of a complex exponential*

Let b be a complex number, say $b = \sigma_o + j\omega_o$. This complex b is generally called the *complex frequency*. Then $x(t) = Ae^{bt} = e^{(\sigma_o + j\omega_o)t} = e^{\sigma_o t} . e^{j\omega_o t}$

Two cases arise now depending on whether σ_o, the real part of b is positive or negative.

Case 1: $\sigma_o > 0$. In this case, $Ae^{\sigma_o t}$ goes on increasing with time exponentially. Hence, $x(t)$ represents a phasor whose magnitude goes on increasing exponentially with time at a rate that is determined by the magnitude of σ_o while the phasor itself rotates in the anti-clockwise direction at a uniform angular velocity of ω_o radians per second. Thus, its projection on to the real axis, which is the real part of $x(t)$, is a cosinusoidal signal with a constant frequency of $(\omega_o/2\pi)$ Hz and an exponentially growing amplitude. Similarly, the imaginary part of $x(t)$, which is the projection of the rotating phasor on the imaginary axis, will be an *exponentially growing sinusoid* with a constant frequency of $(\omega_o/2\pi)$ Hz.

Case 2: $\sigma_o < 0$. In this case, $Ae^{\sigma_o t}$ goes on decreasing with time, exponentially. So, the real part of $x(t)$, equal to $Ae^{\sigma_o t}\cos w_o t$ and given by the projection of the rotating phasor on to the real axis, is an *exponentially decaying cosinusoid*. Similarly, the imaginary part will be an *exponentially decaying sinusoid*

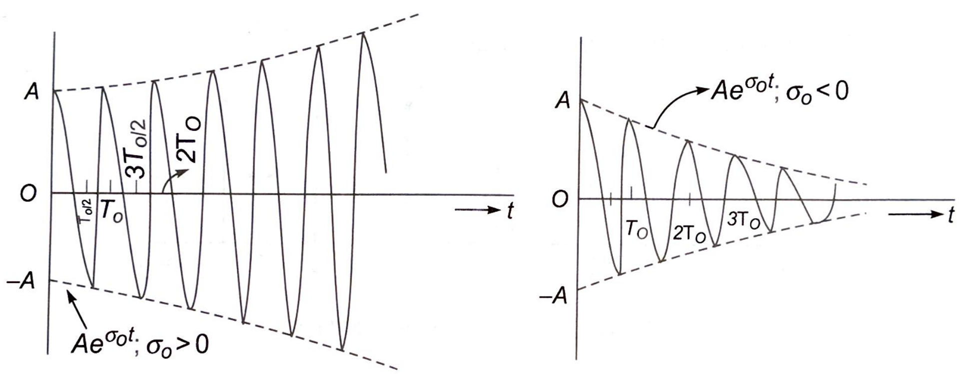


Fig. 1.23 *(a) An exponentially growing cosinusoid (b) An exponentially decaying cosinusoid*

(d) Rectangular pulse A rectangular pulse, symmetrically located with respect to the time origin, having an amplitude A and duration T is shown in Fig. 1.24 and is described by

$$x(t) = \begin{cases} A\,; -\dfrac{T}{2} \le t \le \dfrac{T}{2} \\ 0\,; \text{otherwise} \end{cases} \qquad \ldots (1.33)$$

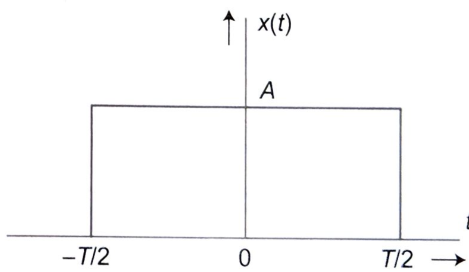


Fig. 1.24 *A rectangular pulse*

Since it is encountered quite frequently, it is given a special symbol: A rect (t/T) or $A \prod (t/T)$.

The lower case letter t in the symbol is used to indicate that it is a rectangular pulse **in time domain**. A represents its amplitude and T its duration.

(e) Sinusoidal signals These are the most extensively used signals, mainly because they are, as we will be showing later, eigensignals for an important class of systems (LTI systems). We may represent a sinusoidal signal $x(t)$ as

$$x(t) = A \sin(\omega t + \phi) \qquad \ldots (1.34)$$

where, A is its peak amplitude, $\omega = 2\pi f$, where f is the frequency of the sinusoid, and ϕ is its phase.

2. Discrete-time signals

(a) Unit sample sequence It is denoted by $\{\delta(n)\}$ and is defined by

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (1.35)$$

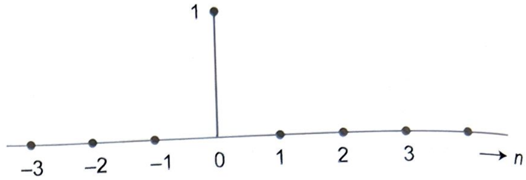


Fig. 1.25 *The unit sample sequence*

Diagrammatically, it is represented as shown in Fig. 1.25.

(b) Unit step sequence The unit step sequence, which is the discrete-time counterpart of the continuous-time signal, unit step function, is denoted by $\{u(n)\}$ and is defined by

$$u(n) = \begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ for } n < 0 \end{cases} \qquad \ldots (1.36)$$

It is diagrammatically represented as shown in Fig. 1.26.

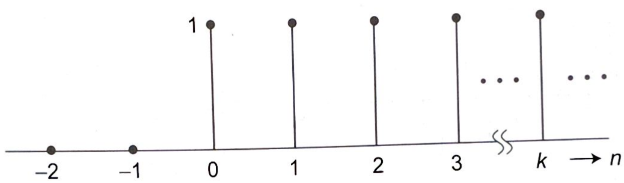


Fig. 1.26 *The unit step sequence*

(c) Unit ramp sequence A one-sided unit ramp sequence is denoted by $\{r(n)\}$ and is defined by

$$r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad \ldots (1.37)$$

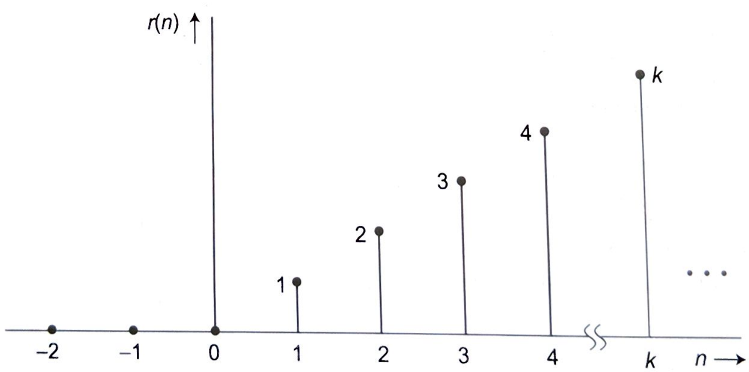


Fig. 1.27 *The one-sided unit ramp sequence*

(d) Exponential sequence A one-sided exponential is defined by

$$x(n) = a^n u(n) \qquad \ldots (1.38)$$

where a is a constant.

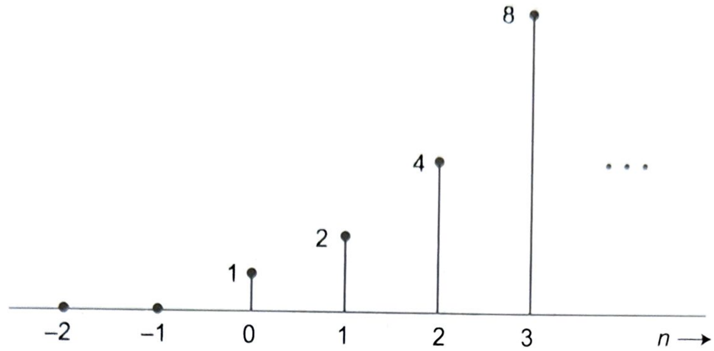


Fig. 1.28 *The growing exponential sequence with a = 2*

If a is real and greater than 1, we get a growing exponential. On the other hand, if a is real and positive but less than 1, we get a decaying exponential sequence.

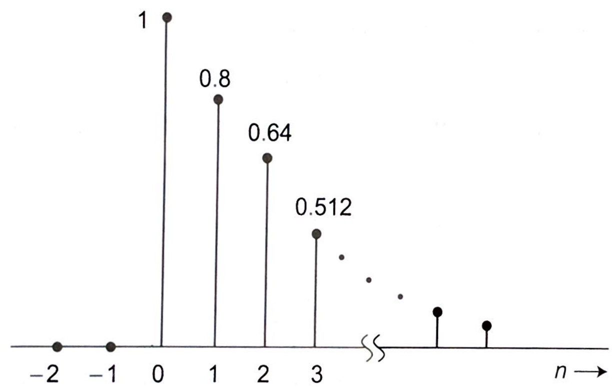


Fig. 1.29 *Decaying exponential sequence with a = 0.8*

(e) Cosinusoidal sequence If for a continuous-time cosinusoid, we write $x(t) = 2\cos \omega t$, we may write its discrete-time counterpart as $x(nT) = 2\cos \omega nT$ by replacing t in $x(t)$ by nT, where $T = 1/f_s$, f_s being the sampling frequency. If $f_s = 8f$, where f is the frequency of the continuous-time cosinusoid, then:

$$x(n) = 2\cos 2\pi\left(\frac{f}{f_s}\right)n = 2\cos\left(\frac{2\pi}{8}\right)n \qquad \ldots (1.39)$$

This is plotted in Fig. 1.30.

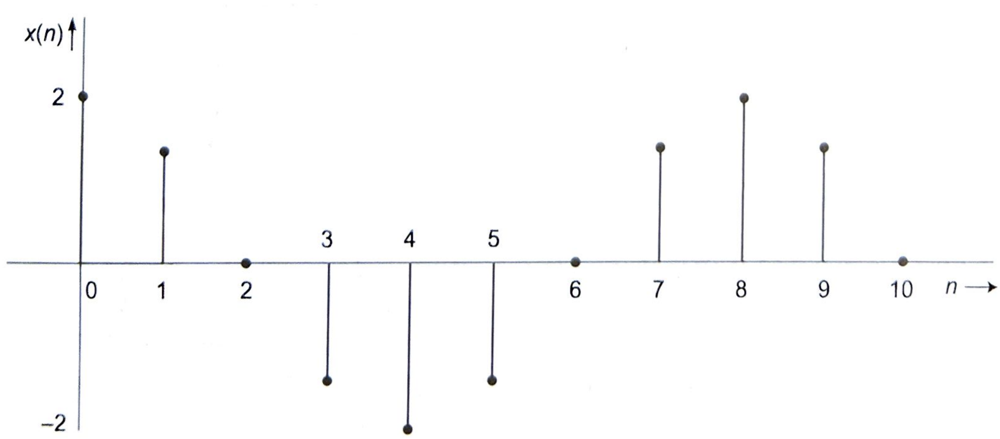


Fig. 1.30 *The discrete-time cosinusoidal signal*

1.5 OPERATIONS ON SIGNALS (INCLUDING TRANSFORMATION OF INDEPENDENT VARIABLES)

In our study of systems, we will find that systems operate upon, or manipulate, the signal(s) given as input(s) to produce one or more outputs. In this context, a study of the basic operations on signals will be very useful.

1. Continuous-time signals

(a) Addition and subtraction Let $x(t)$ and $y(t)$ be two signals. Then their sum $z(t)$ is defined by

$$z(t) = x(t) + y(t) \qquad \ldots (1.40)$$

(b) Multiplication of a signal by a constant

$$z(t) = a\,x(t) \qquad \ldots (1.41)$$

This is also known as *amplitude scaling*. If $|a| > 1$, the signal $a\,x(t)$ is an amplified version of $x(t)$ and if $|a| < 1$ it is an attenuated version.

(c) Multiplication of two signals This operation is defined by

$$z(t) = x(t)\,.\,y(t) \qquad \ldots (1.42)$$

Multiplication of two signals is an operation that is frequently resorted to in the field of communication engineering, as in the case of modulation.

(d) Differentiation and integration For example, the voltage $v(t)$ across an inductance L when a current $i(t)$ flows through it, is given by

$$v(t) = L\frac{di(t)}{dt} \qquad \ldots (1.43)$$

Similarly, if a current $i(t)$ is flowing through a capacitor C, then the voltage across it is

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \qquad \ldots (1.44)$$

(e) Shifting in time Consider a signal $x(t)$. Then, the signal $x(t - t_0)$ represents a delayed version of $x(t)$, delayed by t_0 seconds, as shown in Fig. 1.31.

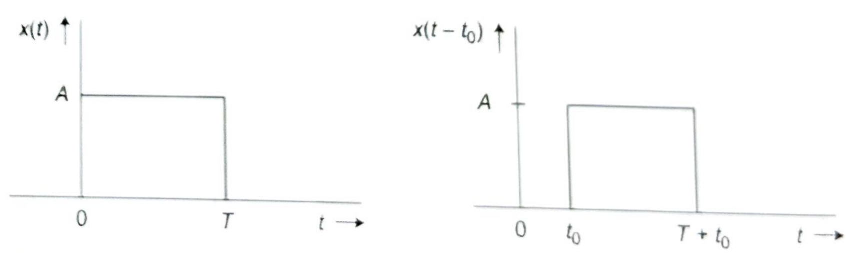


Fig. 1.31 *(a) A signal x(t)* *(b) Delayed version of x(t)*

The nature of the function $x(.)$ is such that when the argument $(.)$ is negative, the function is zero. When the argument is between 0 and T, $x(.)$ takes a constant value A. When the argument takes a value T, it drops down to zero and when$(.)$ is greater than T, it remains at zero. When we consider $x(t - t_0)$, its argument $(t - t_0) < 0$ as long as $t < t_0$ and at $t = t_0$, when the argument takes the value zero, $x(t - t_0)$ suddenly rises to a value A and remains there at that value as long as $(t - t_0)$ is between 0 and T i.e. as long as t lies between t_0 and $T + t_0$. Again when $t - t_0 > T$, i.e., when $t > T + t_0$, $x(t - t_0)$ takes zero value.

Following the same argument, $x(t + t_0)$ will be a time shifted version of $x(t)$, shifted to the *left* by t_0 seconds.

(f) Compressing/Expanding a signal in time Given a signal $x(t)$, let us examine what $x(at)$ represents, where a is a real number, and how it is relate to $x(t)$. Let $x(t)$ be a simple rectangular pulse, as shown in Fig. 1.32(a).

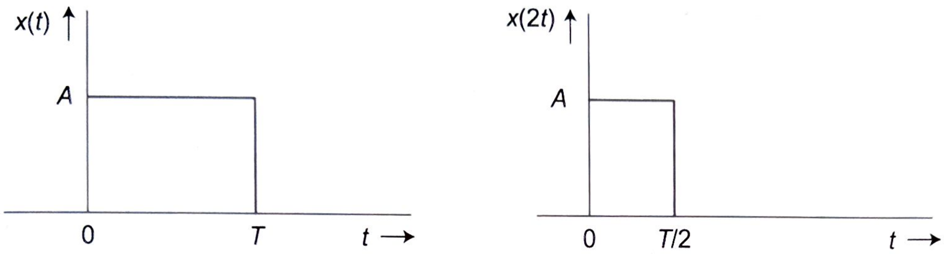
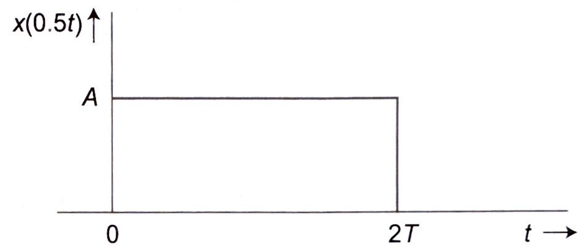


Fig. 1.32 *(a) Rectangular pulse* *(b) Compressed version*

With the same arguments as advanced with regard to time-shifted version of $x(t)$, we find that $x(2t)$ will be a rectangular pulse like $x(t)$, but compressed in time by a factor of 2, as shown in Fig. 1.32(b). Similarly, $x(0.5t)$ will be a time-expanded version of $x(t)$, expanded in time by a factor $1/0.5 = 2$.



(c) Expanded version of x(t).

Remark 1 *Quite often, we will be performing both time-shifting and time-scaling. For example, consider x(2t – 3) where x(t) is the same rectangular pulse shown in Fig. 1.32(a), with T = 2.*

Advancing the same arguments as earlier, $x(2t - 3) = y(t)$, will be zero when $(2t - 3) < 0$, i.e., when $t < 3/2$. It will have a constant value of 1 for $0 \le (2t - 3) \le 2$, i.e., $3/2 \le t \le 5/2$. Beyond $(2t - 3) > 2$, i.e., $t > 5/2$, it will be zero. A plot of $y(t) = x(2t - 3)$ vs t appears as shown in Fig. 1.33.

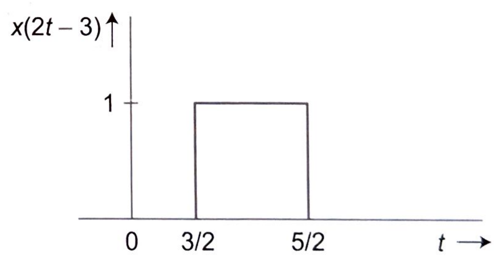
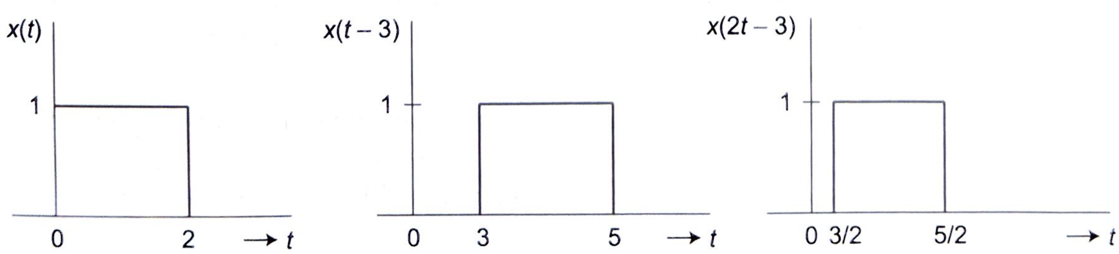


Fig. 1.33 *A plot of x(2t – 3)*

Now, recalling that in time-shifting by t_0 seconds, we replace t by $(t - t_0)$ while in time-scaling, we replace t by at, we can easily see that the steps involved in obtaining $x(2t - 3)$ are:

$$x(t)\big|_{t \to (t-3)} = x(t-3) \text{ and } x(t-3)\big|_{t \to 2t} = x(2t-3)$$

Thus, we should do the time-shifting first and then the time-scaling, as shown in Figs. 1.34 (a), (b) and (c).



(a) (b) (c)

Fig. 1.34 *(a),(b),(c) showing correct sequence of operations*

Doing the time-scaling first and then the time-shifting will lead to a *wrong* result:

$$x(t)\big|_{t\to 2t} = x(2t) \text{ and } x(2t)\big|_{t\to(t-3)} = x(2t-6) \neq x(2t-3)$$

> **Remark 2** *In $x(at)$, if a is negative, $x(at)$ will be a mirror-image of $x(t)$. Thus, if $x(t)$ is as shown in Fig. 1.35(a), then $x(-2t+1)$ would appear as shown in Fig. 1.35(b).*

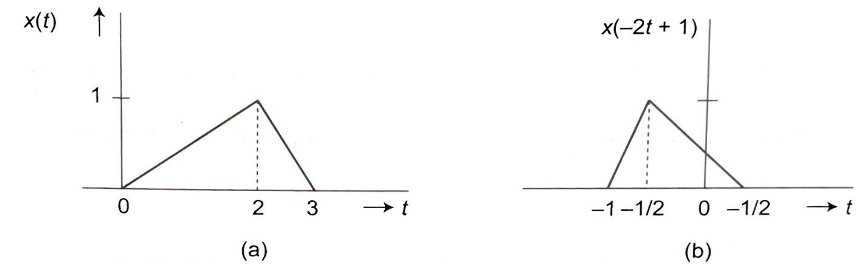


Fig. 1.35

(g) Time-scaling of step and delta functions Figure 1.34(a) shows a shifted step function.

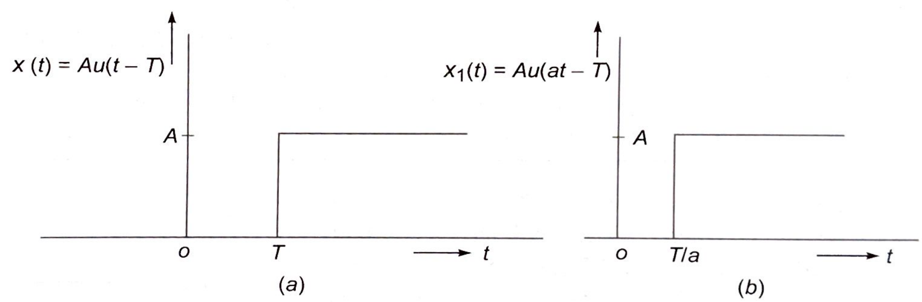


Fig. 1.34 *(a) A shifted step function*

A time-scaled version of $Au\,(t-T)$ is given by $x_1(t) = Au\,(at - T)$. Since $x(t) = A$ when the argument of $Au(t-T)$ is zero $x_1(t)$ also will be zero when the argument of $Au(at-T)$ equals zero, i.e., when $t = T/a$. Thus, the time-scaled version appears as shown in Fig. 1.34(b).

From the above figures, it is clear that in the special case when $T = 0$, $x(t) = u(t)$ and $x(at) = u(at)$ will look similar when plotted. However, they are not exactly equivalent (you can check that the derivative of the signals are not the same). Similarly, delta functions when scaled also change despite their infinitely small width. You can check by starting with a rectangular pulse of unit area (Fig. 1.17) that the area underneath the delta function changes when it is time-scaled.

(h) Time-domain aliasing operation It is an operation on signals that comes in handy for generation of a periodic signal from an arbitrary signal $x(t)$ by summing its shifted versions. Note that $x(t)$ itself is not periodic. This operation will be referred to as 'aliasing' in time domain. Specifically, an aliased signal is generated as follows:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(t - nT_o) \qquad \ldots(1.45)$$

When $x_a(t)$ is finite, it can readily be shown that it is periodic in time with period T_o.

Example 1.12 *A signal x(t) is given by* $x(t) = \begin{cases} t; & 0 \le t \le 1 \\ 1-t; & 1 \le t \le 2 \\ 0; & \text{otherwise} \end{cases}$ Sketch $x(t)$ and $x_a(t)$ for $T_o = 2$ and $T_o = 1$.

Solution

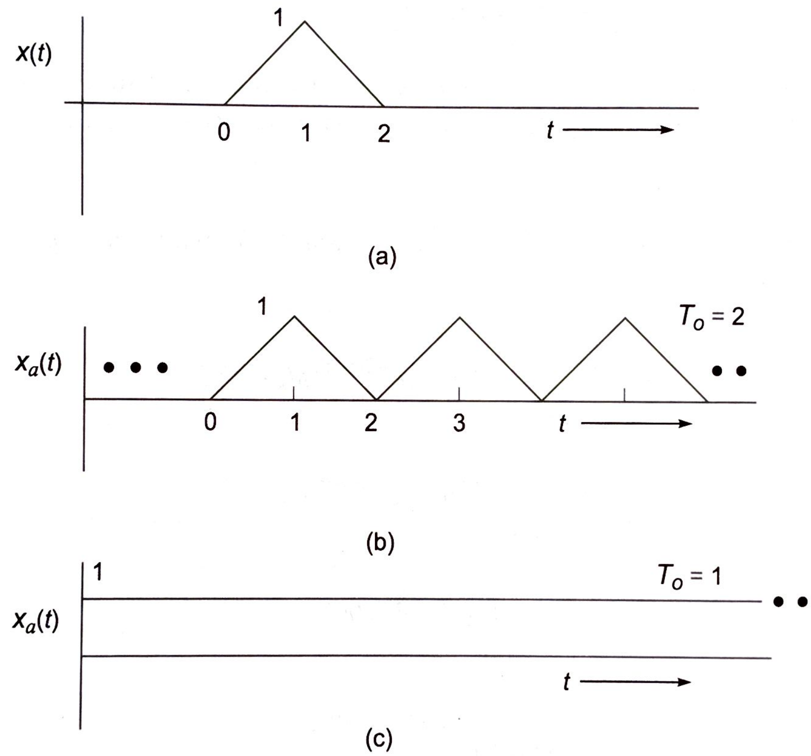


Fig. 1.36 *(a) Signal x(t), (b) $x_a(t)$ for $T_o = 2$ and (c) $x_a(t)$ for $T_o = 1$*

(i) Convolution Another operation defined on a pair of signals and encountered very often, is the convolution operation. The full physical significance of this operation will be discussed in Chapter 7. Given two signals, $x(t)$ and $y(t)$, the convolution $z(t)$ of these two signals is defined in the following two equivalent ways:

$$z(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau = \int_{\tau=-\infty}^{\infty} y(\tau)\, x(t-\tau)\, d\tau \qquad \ldots(1.46)$$

Such convolutions can be performed by integration when possible. This is demonstrated in the following examples.

Example 1.13 *Convolve y(t) with* $x(t) = \delta(t - T_o) + \delta(t + T_o)$.

Solution

$$z(t) = \int_{\tau=-\infty}^{\infty} y(\tau)x(t-\tau)d\tau$$

$$= \int_{\tau=-\infty}^{\infty} y(\tau)[\delta(t-\tau-T_o) + \delta(t-\tau+T_o)]d\tau$$

$$= \int_{-\infty}^{\infty} y(t-T_o)\delta(t-\tau-T_o)d\tau + \int_{-\infty}^{\infty} y(t+T_o)\delta(t-\tau+T_o)d\tau$$

$$= y(t-T_o) + y(t+T_o)$$

In the above, the third line follows from the sampling property of the delta function because of which $y(\tau)\,\delta(t-\tau-T_o) = y(t-T_o)\,\delta(t-\tau-T_o)$, and $y(\tau)\,\delta(t-\tau+T_o) = y(t+T_o)\,\delta(t-\tau+T_o)$. The last line follows from the fact that the area under the delta function is unity.

Example 1.14 *If $x(t) = e^{-\alpha t}u(t)$ and $y(t) = e^{-\beta t}u(t)$, evaluate the convolution of $x(t)$ and $y(t)$.*

Solution

$$z(t) = \int_{\tau=-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau$$

$$= \int_{\tau=-\infty}^{t} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}d\tau$$

$$= e^{-\beta t}\int_{\tau=0}^{t} e^{-(\alpha-\beta)\tau}d\tau$$

$$= \frac{1}{(\alpha-\beta)}\left[e^{-\beta t} - e^{-\alpha t}\right]u(t)$$

In the above, the second line follows from the fact that the product $u(\tau)\,u(t-\tau)$ is unity only when τ is between 0 to t and is zero otherwise.

Quite often, direct evaluation of the convolution integral is quite tedious and time consuming; and it may be more convenient to perform the convolutions graphically, as explained below in detail.

Graphical evaluation of convolution integral

Let $z(t) = x(t) * y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{\tau=-\infty}^{\infty} x(t-\tau)y(\tau)d\tau$ where * denotes the 'convolution operator'.

The graphical evaluation of convolution integral gives a better insight into the convolution operation. For this, let us consider the graphical evaluation of the first integral. Let $x(t) = y(t)$ be identical rectangular pulses as shown in Figs. 1.37(a).

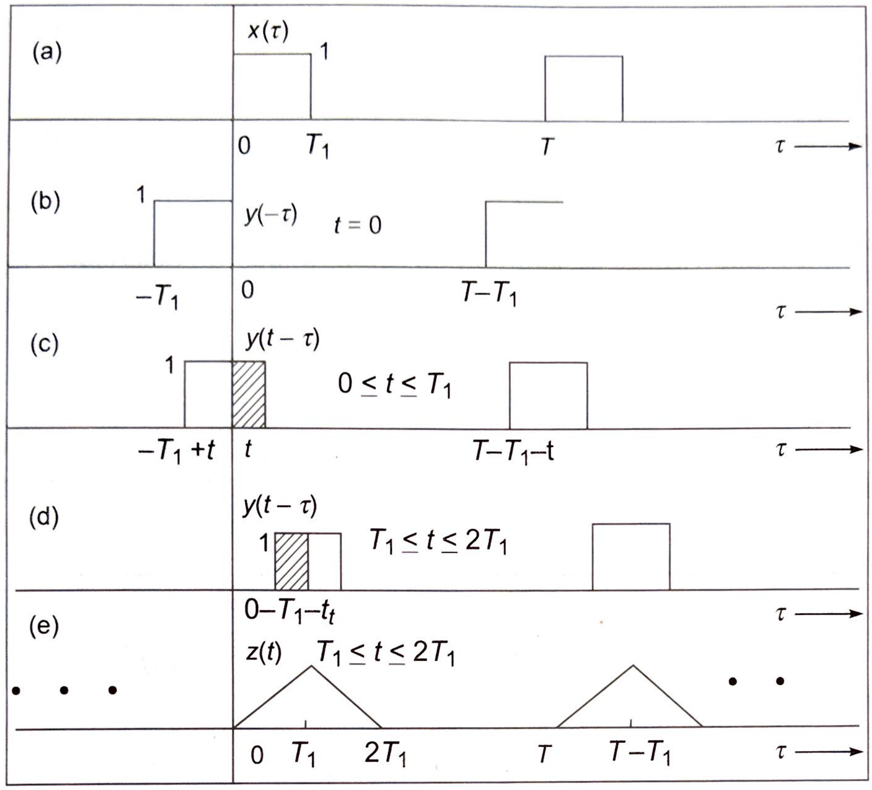


Fig. 1.37 *Graphical interpretation of convolution integral*

A plot of $x(\tau)$ vs τ will be the same as the plot of $x(t)$ vs t. Similarly, a plot of $y(\tau)$ vs τ will be same as the plot of $y(t)$ vs t. However, the plot of $y(-\tau)$ vs τ will be the mirror image of $y(t)$ vs. τ about the y-axis, as shown in Fig. 1.37(d). Note that in the convolution integral, the integration is with respect to τ and that t has to be regarded as a constant while carrying out the integration. Fig. 1.37(e) depicts $y(t - \tau)$ for a particular value of t; and as can be seen, it is obtained by sliding $y(-\tau)$ to the right by an amount of time equal to t. Since

$$z(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau\,,$$

the product of $x(t)$ and $y(t - \tau)$ is to be integrated with respect to τ.

That is, $z(t)$ for a particular value of t is obtained by finding out the area under the function $x(\tau)\,y(t - \tau)$ for that value of t. So, if $z(t)$ is to be evaluated for all values of t, imagine that the plot of $y(t - \tau)$ vs τ of Fig. (d) is superimposed on the plot of $x(\tau)$ vs τ of Fig. 1.37(c) and that the plot of $y(-\tau)$ vs τ is slid over the latter to the right by t seconds where, t starts from zero value and goes on increasing. Now, suppose $t = 0$. We find that the product of $x(\tau)$ and $y(-\tau)$ is zero as they do not overlap. Hence, for $t = 0$, the area under the product is zero. Now consider $y(t - \tau)$ for $t = T/2$. Then $y(t - \tau)$ and $x(\tau)$ will have an overlap from $\tau = 0$ to $\tau = T/2$. Hence, the area under the function $x(\tau)\,y(t - \tau)$ is $A^2T/2$. Thus, the value of $z(t)$ at $t = T/2$ is $A^2T/2$ as shown in Fig. 1.37(f). As t is increased further, the overlap and therefore the area under the function $x(\tau)\,y(t - \tau)$ increases linearly with t, taking a maximum value at $t = T$. For this value of t, the value of $\int x(\tau)h(t-\tau)\,d\tau$ takes the value A^2T as shown in Fig. 1.37(f). Any further increase in t will make the overlap, and therefore the area under the function $x(\tau)\,h(t - \tau)$ to decrease linearly with t. Hence, $z(t)$ starts decreasing from its maximum value of A^2T in a linear fashion, reaching the value zero at $t = 2T$ since the overlap between $x(\tau)$ and $y(t - \tau)$

becomes zero at that value of t. Now, it remains at zero for any further increase in t. Hence, the plot of $z(t)$ vs t will be as shown in Fig. 1.37(f).

Incidentally, the above illustration shows that the convolution of two rectangular signals, each of amplitude A and duration T seconds, results in a triangular signal of base width $2T$, and this triangular signal takes a maximum value of A^2T at $t = T$.

Example 1.15 *If $x(t)$ and $y(t)$ are as shown in Fig. 1.38 (a) and (b) determine graphically, the signal $z(t) \triangleq x(t) * y(t)$.*

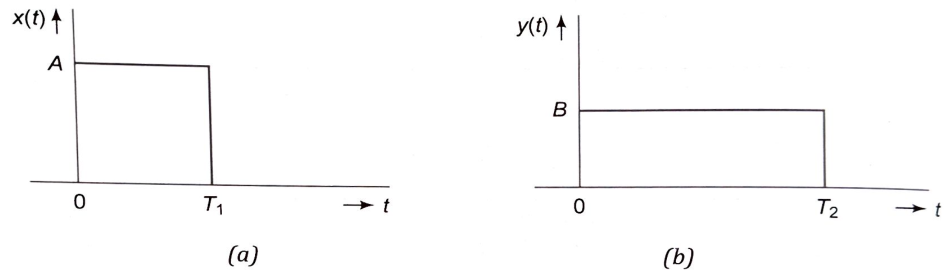
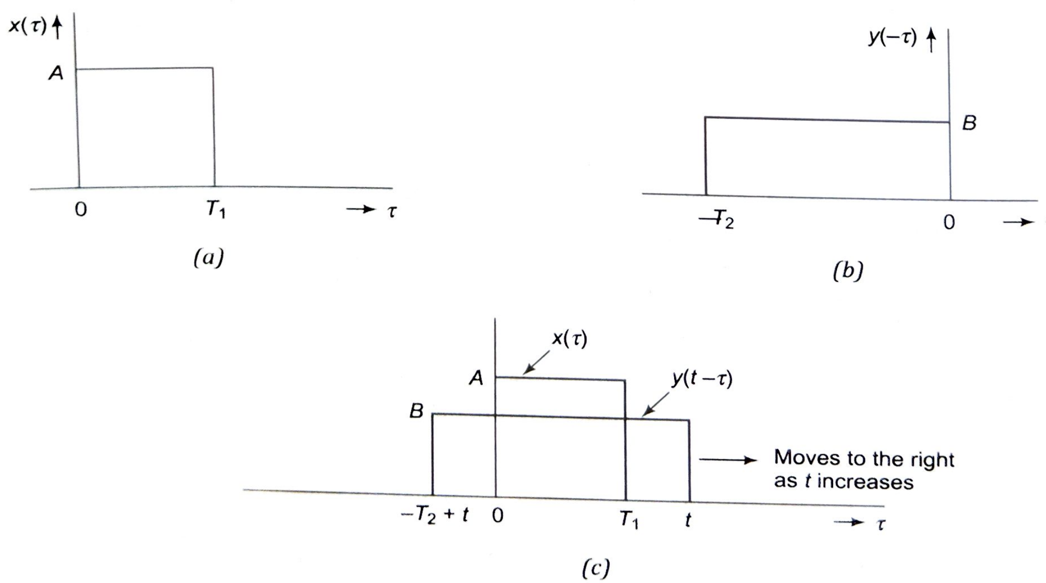


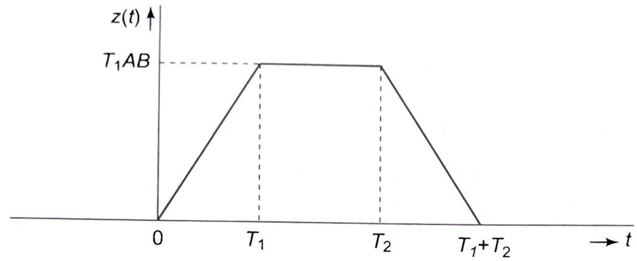
Fig. 1.38

Solution

$$z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \; = \text{Area under the product of } x(\tau) \text{ and } y(t-\tau) \text{ for any } t.$$

From Fig. 1.39, the following points are evident:





(d)

Fig. 1.39

1. When $t \le 0$, the product of $x(\tau)$ and $y(t-\tau)$ is zero, as there is no overlap of the two.
2. As t increases beyond zero, the overlap and hence the area under the product increases linearly with t. This continues till $t = T_1$; and at this value of t, the area under the product, i.e., $z(t)$, takes the maximum value equal to ABT_1.
3. As t increases beyond T_1 the overlap area and hence $z(t)$ will remain constant till $t = T_2$. When this value is reached, the left-side edge of $y(t-\tau)$ coincides with the y-axis, and any further increase in t beyond $t = T_2$ will make the overlap area to linearly decrease with time.
4. When t reaches the value $T_1 + T_2$, the left-side edge of $y(t-\tau)$ coincides with the right- side edge of $x(\tau)$. Hence the overlap area and hence $z(t)$ become zero and remain at that value for all $t > (T_1 + T_2)$.
5. Signal $z(t)$ will have a trapezoidal shape in this case, the height of the trapezium being ABT_1 (since $T_1 < T_2$). The total base width of the trapezium = $T_1 + T_2$.
6. In case $T_1 = T_2 = T$, $z(t)$ will have a triangular waveform with height equal to ABT and base width equal to $2T$.

Properties of convolution

1. **Distributive property** $x(t) * [y_1(t) + y_2(t)] = x(t) * y_1(t) + x(t) * y_2(t)$
2. **Associative property** $[x_1(t) * x_2(t)] * x_3(t) = x_1(t) * [x_2(t) * x_3(t)]$
3. **Commutative property** $x_1(t) * x_2(t) = x_2(t) * x_1(t)$.

Proofs for the above properties are very simple and are left to the readers as an exercise.

Periodic convolution

Given two *periodic* signals $x(t)$ and $y(t)$ of time-period T, their periodic convolution is defined in the following two equivalent ways:

$$z(t) \triangleq \int_{\tau=0}^{T} x(\tau)\, y(t-\tau)\, d\tau$$

$$\triangleq \int_{\tau=0}^{T} y(\tau)\, x(t-\tau)\, d\tau \qquad \ldots(1.47)$$

Consider $x(t) = e^{j2\pi kt/T}$ and $y(t) = e^{j2\pi kt/T}$ where k and m are integers. Their periodic convolution is given by

$$z(t) = \int_{\tau=0}^{T} e^{2\pi k\tau/T} e^{j2\pi m(t-\tau)/T} d\tau$$

$$= e^{j2\pi ktT} \int_{\tau=0}^{T} e^{j2\pi(k-m)\tau/T} d\tau \qquad \ldots(1.48)$$

$$= \begin{cases} 0 & k \neq m \\ Te^{j2\pi k t T} & k = m \end{cases}$$

In the above, the last line follows from the fact that the $\int_{\tau=0}^{T} e^{j2\pi l / T} d\tau = 0$ unless $l = 0$.

Such periodic convolutions can be performed by integrations when possible. However, it is often convenient to perform this graphically. This is demonstrated in the following example.

Example 1.16 *Let $x(t)$ be a periodic signal of period T with one time-period specified by $x(t) = u(t) - u(t - T_1)$ over $0 \le t \le T$ as shown in Fig. 1.40. We let $y(t) = x(t)$, and find $z(t)$, their periodic convolution (assume $T > 2T_1$).*

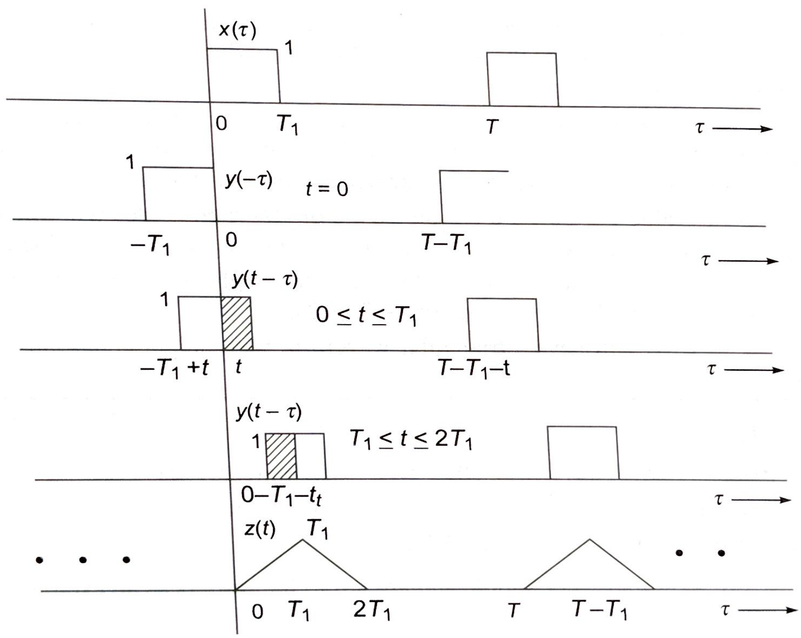


Fig. 1.40 *Graphical interpretation of periodic convolution*

We need to plot $z(t) = \int_{\tau=0}^{T} x(\tau)\, y(t - \tau) dt$. As with linear convolution, we start with sketching of $x(t)$ and $y(t - \tau)$ for various values of τ from 0 to T.

As with linear convolution, we first plot $x(\tau)$ and $y(t - \tau)$ for various values of t. The second plot is sketched for computing $z(t)$ at $t = 0$. Note that the area underneath the product of $x(\tau)$ and $y(-\tau)$ is zero over the interval 0 to T. Now consider the interval of time $0 \le t \le T_1$. As depicted in the third figure, the area underneath is t, which reaches a peak value of T_1 when $t = T_1$. For $T_1 \le 2T_2$, the area decreases linearly from T_1 to zero as depicted in the fourth figure. $z(t)$ is as depicted in the fifth figure.

2. Discrete-time signals

(a) Time-shifting This is done exactly in the same way as the time-shifting of CT signals.

(b) Time-scaling Since the time variable n in the case of discrete-time signals can take only integer values, time-scaling of these signals leads to peculiar results, as can be seen from the following example.

Example 1.17 *Consider $x(n)$ as shown in Fig. 1.41(a). Then $x(2n)$ will be as shown in Fig. 1.41(b).*

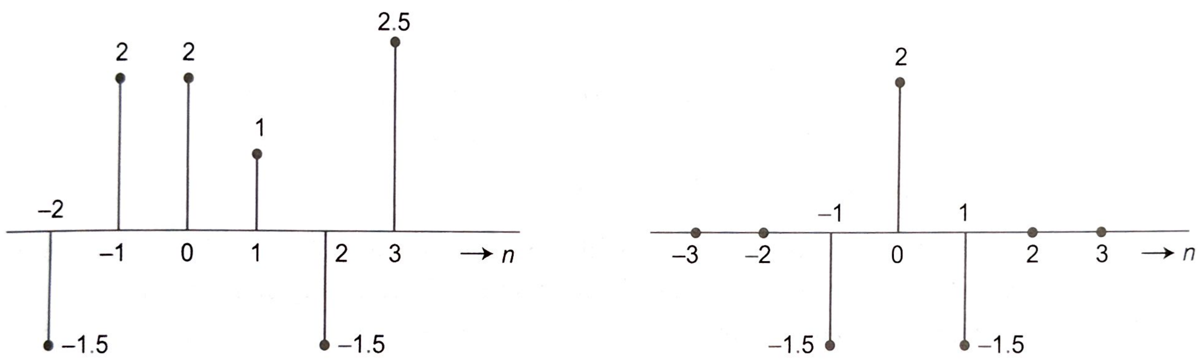


Fig. 1.41 *(a) Showing x(n) (b) Showing x(2n)*

As can be seen, the samples of $x(n)$ at $n = 3$, $n = 1$ and $n = -1$ are lost.

(c) Addition Addition of two D.T signals is performed by the addition of corresponding samples of the two signals. Hence $z(n) = x(n) + y(n)$.

(d) Multiplying the signal by a constant When a discrete-time signal $x(n)$ is to be multiplied by a constant, say 'a', we multiply every sample of $x(n)$ by that constant, i.e., $a\{x(n)\} = \{ax(n)\}$

(e) Accumulation The accumulated version $y(n)$ of $x(n)$ is defined as

$$y(n) = \sum_{m=-\infty}^{n} x(m) \qquad \ldots(1.49)$$

It is analogous to the integrated version of continuous-time signals. Clearly, $y(n) - y(n-1) = x(n)$.

(f) Convolution The convolution $z(n)$ between two sequences $x(n)$ and $y(n)$ can be defined in the following two equivalent ways:

$$z(n) = \sum_{m=-\infty}^{m=\infty} x(m)\,y(n-m)$$

$$= \sum_{m=-\infty}^{m=\infty} y(m)\,x(n-m) \qquad \ldots(1.50)$$

The precise physical significance of convolution will be clear in Chapter 7. Convolution sums can sometimes be evaluated directly. However, graphical evaluation is often convenient. In the following example, we evaluate the convolution directly.

Example 1.18 *Evaluate the convolution of a sequence $x(n)$ and another sequence $y(n) = \delta(n - N_o) + \delta(n + N_o)$*

Solution

$$z(n) = \sum_{m=-\infty}^{\infty} x(m)\, y(n-m)$$

$$= \sum_{m=-\infty}^{\infty} x(m)\,\delta(n - m - N_o) + \sum_{m=-\infty}^{\infty} x(m)\,\delta(n - m + N_o)$$

$$= x(n - N_o) \sum_{m=-\infty}^{\infty} \delta(n - N_o) + x(n + N_o) \sum_{m=-\infty}^{\infty} \delta(n - m + N_o) \qquad \ldots(1.51)$$

$$= x(n - N_o) + x(n + N_o)$$

In the above, the third line follows from the sampling property of the delta function, and the last line follows from the fact that the summation of a delta function over all its sampling points is unity.

As stated earlier, graphical evaluation of the convolution is often very convenient, as is evident from the following example.

Example 1.19 *Evaluate the convolution sum of $\{x(n)\}$ and $\{y(n)\}$ if $\{x(n)\} = \{2,-1,\underset{\uparrow}{1},0,2\}$ and $\{y(n)\} = \{\underset{\uparrow}{1},0,-1,2\}$. (Note: The arrow indicates the zeroth sample.)*

Solution Let $w(n) = x(n * y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$

Figures 1.42 (a) and (b) show $x(k)$ vs k and $y(k)$ vs k respectively. Figure 1.42 (c) is a plot of $y(-1-k)$ vs k. Figure 1.42 (d) is a plot of $y(-k)$ vs k. Figures (e), (f), (g), (h), (i) and (j) show $y(1-k)$ vs k, $y(2-k)$ vs k, $y(3-k)$ vs k, $y(4-k)$ vs k, $y(5-k)$ vs k, and $y(6-k)$ vs k respectively. From the diagrams, it is obvious that $w(n)$, the convolution sum of $x(n)$ and $y(n)$ has the following values for various values of n. $n = -2$, $w(-2) = 0$; as there is not going to be any overlap between $x(k)$ and $y(-2-k)$.

$$w(-1) = 2 \times 1 = 2$$

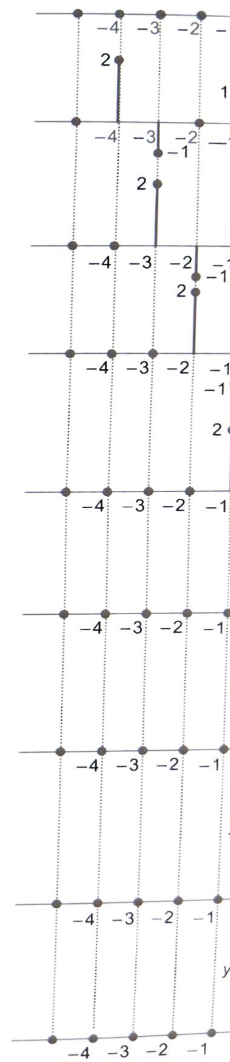
$$w(0) = (-1) \times 1 + 2 \times 0 + 0 \times (-1) + 0 \times 2$$
$$= -1$$

$$w(1) = 2 \times (-1) + (-1) \times 0 + 1 \times 1 = -1$$

$$w(2) = 5,\ w(3) = -1,\ w(4) = 2$$

$$w(5) = -2 \text{ and } w(6) = 4$$

$w(7)$, $w(8)$, etc., will all be zero.



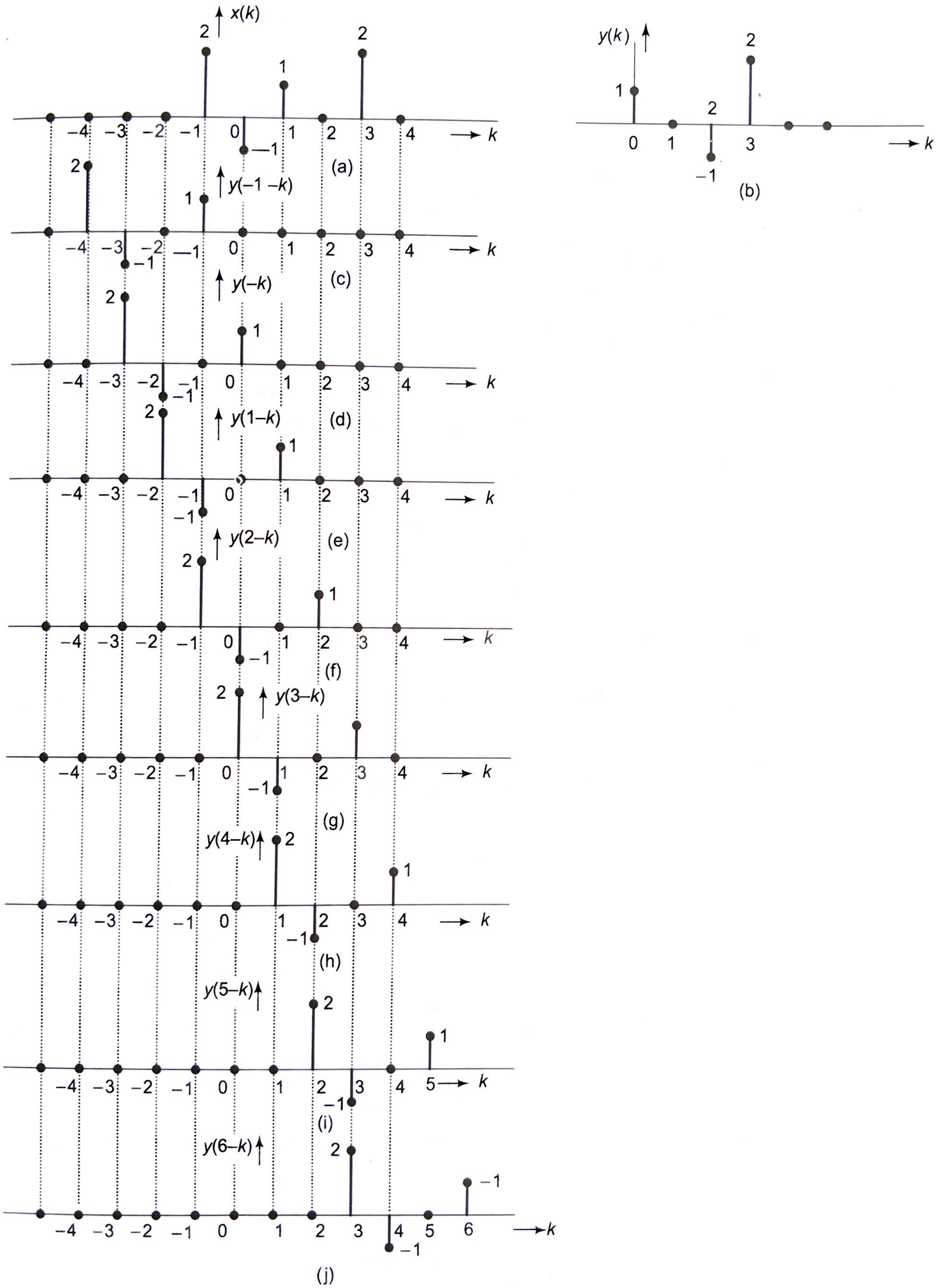


Fig. 1.42 *Graphical evaluation of discrete convolution*

$\therefore$

$$\{w(n)\} = \{2, -1, -1, 5, -1, 2\ -2, 4\}$$
$$\uparrow$$

Notes

1. *As can be seen from the above diagrams, if $x(n)$ is zero prior to $n = n_1$, and $y(n)$ is zero prior to $n = n_2$ $w(n)$ will be zero prior to $n = (n_1 + n_2)$. In this particular problem, n_1 is -1 and n_2 is zero, and so $w(n)$ is non-zero from $w(-1)$.*
2. *If $x(n)$ is a finite-length sequence of length N_1 and $y(n)$ is a finite-length sequence of length N_2, $w(n)$, the sequence obtained by convolving $x(n)$ and $y(n)$, will have a length of $(N_1 + N_2 - 1)$. In the example under consideration, N_1 is 5 and N_2 is 4, and so N the length of $w(n) = 5 + 4 - 1 = 8$.*

Periodic convolution Consider two discrete-time periodic sequences $x(n)$ and $y(n)$ of period N. Their periodic convolution $z(n)$ can be defined in the following two equivalent ways:

$$z(n) = \sum_{m=0}^{N-1} x(m)\, y(n-m)$$

$$= \sum_{m=0}^{N-1} y(m)\, x(n-m) \qquad \ldots(1.52)$$

Note that the summation is only applied over 0 to $N - 1$, and this ensures that $z(n)$ is finite.

Example 1.20 *Evaluate the periodic convolution $z(n)$ of $x(n) = e^{j2\pi nk/N}$ and $y(n) = e^{j2\pi nl/N}$ for integer k and l that take values between 0 and $N - 1$.*

Solution Note that $x(n)$ and $y(n)$ are periodic with periodic with N.

$$z(n) = \sum_{m=0}^{N-1} x(m)\, y(n-m)$$

$$= \sum_{m=0}^{N-1} e^{j2\pi km/N} e^{j2\pi l(n-m)/N}$$

$$= e^{j2\pi nl/N} \sum_{m=0}^{N-1} e^{j2\pi m(k-l)/N} \qquad \ldots(1.53)$$

$$= Ne^{j2\pi nl/N} \quad \text{if } \boldsymbol{k} = 1 \text{ and zero otherwise}$$

where the last line follows from the fact that a complex sinusoid summed over its time period is zero except when it has zero frequency:

$$\sum_{m=0}^{N-1} e^{j2\pi mk/N} = \begin{cases} N; & k = 0 \\ 0; & \text{Otherwise} \end{cases} \qquad \ldots(1.54)$$

Example 1.21 *Graphically evaluate the periodic convolution of a sequence x(n) depicted in the Fig. 1.42 with itself (so that y(n) = x(n)).*

Solution To compute $z(n) = \sum_{m=0}^{N-1} x(m)\, y(n-m)$ (as with convolution), we first sketch the sequence $x(m)$ with respect to the index m (*see Fig. 1.43(b)*). In Figs. (c), (d), (e), (f), and (g), the sequence $y(n-m)$ is sketched for $n = 0, 1, 2, 3$, and 4. We restrict our attention to the indices of m from 0 to 7, multiple $x(m)$ and $y(n-m)$ point by point and add to obtain $z(n)$ drawn in Fig. 1.43(h).

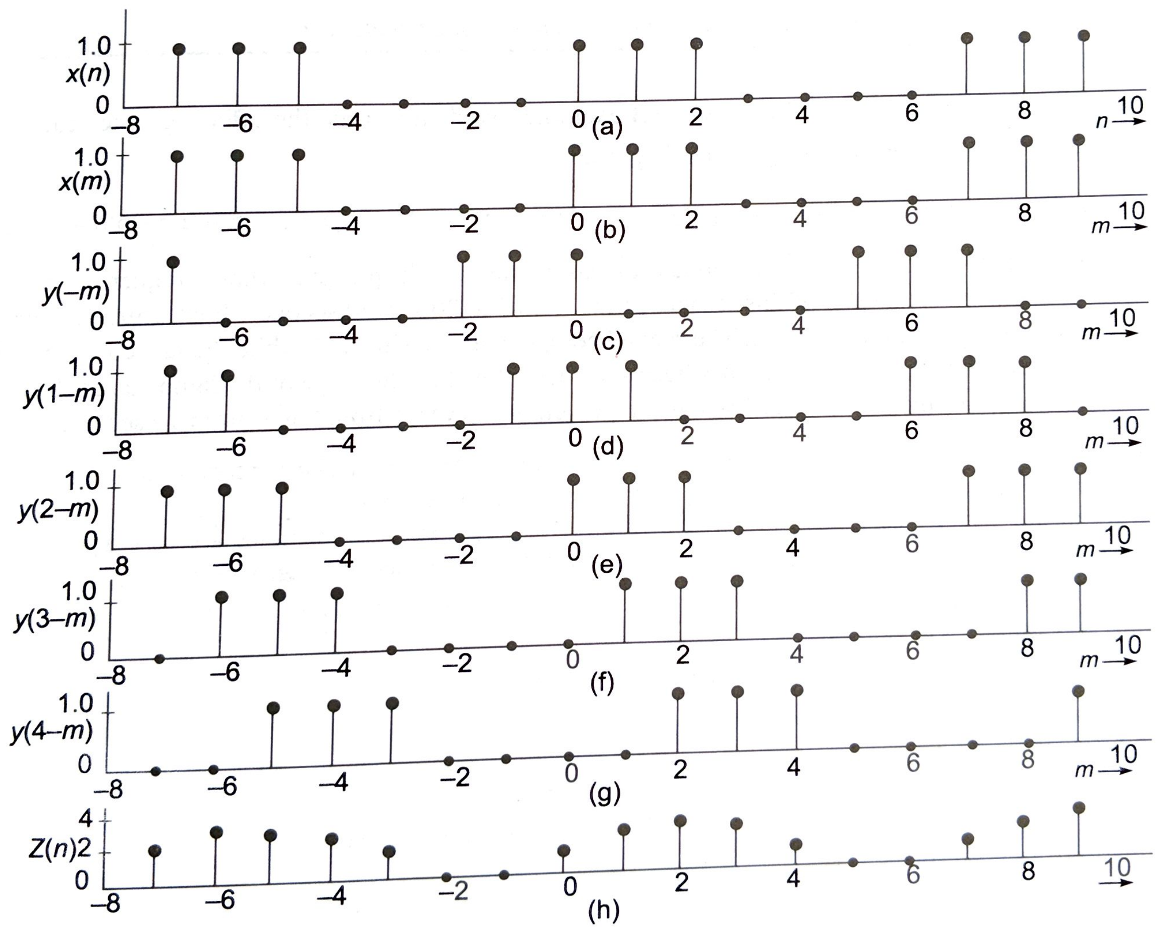


Fig. 1.43 *Graphical evaluation of the periodic convolution of x(n) with y(n) = x(n)*

Example 1.22 *Write down the expression for, and plot the sinusoidal discrete-time sequence whose peak amplitude is 10 and frequency is 100 Hz. The sampling frequency is 1000 samples per second.*

Solution $x(n) = 10 \sin 2\pi fnT = 10 \sin 2\pi.100.n(1/1000) = 10 \sin (2\pi/10)n$

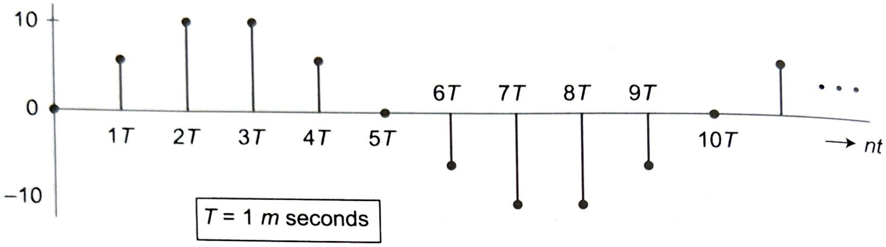


Fig. 1.44 *The discrete-time sinusoid of Example 1.22*

MATLAB Example 1.1 In this MATLAB example, we demonstrate the effect of time-scaling on signals. In particular, we consider a signal that is given by

$$x(t) = \cos(2\pi\,\alpha t^2)[u(t) - u(t-20)] + \cos[2\pi\,\beta(t-30)]\,(u(t-30) - u(t-50)) + (t-60)\,[u(t-60) - u(t-80)]$$

with $\alpha = 0.1$ and $\beta = 0.25$. The first component of this signal is a chirp signal whose frequency increases linearly with time. The frequency of this signal sweeps from 0 to 20α in 20 seconds. Such chirp signals find application in instrumentation and radar. The second component is cosinusoid, while the last component is a ramp. In the figure, the signal $y(t) = x(2t)$ has been sketched. Note that the slope of the ramp has doubled, and so has the frequency of the cosinusoid. The chirp however, now sweeps from 0 to 40α in 10 seconds.

```
t = 0:0.001:10;                  %the time duration of the signal and the sampling interval for plot are fixed
alpha=0.1;beta=0.25;             %frequencies of the two sinusoids are fixed
x = cos(2*pi*alpha*(t(1:10001)).^2); y = cos(2*pi*beta*t(1:10001)); %the two sinusoids are generated
sig = [x,zeros(1,5000),y,zeros(1,5000),t/10,zeros(1,5000-3)]; %the signal is generated
tt = [0:45000-1]/500;
subplot(211),plot(tt,sig,'k')
xlabel('Time in seconds'); ylabel('Magnitude'); title('Signal');
%Now, we generate and plot the time-scaled signal
t = 0:0.002:10;
x = cos(2*pi*alpha*(t(1:5001)).^2);y = cos(2*pi*beta*t(1:5001));
sig = [x,zeros(1,2500),y,zeros(1,2500),t/10,zeros(1,2500-3),zeros(1,22500)];
subplot(212),plot(tt,sig,'k');
xlabel'(Time in seconds'); ylabel('Magnitude');title('Time Scaled Signal')
axis([0,90,-1,1]);
```

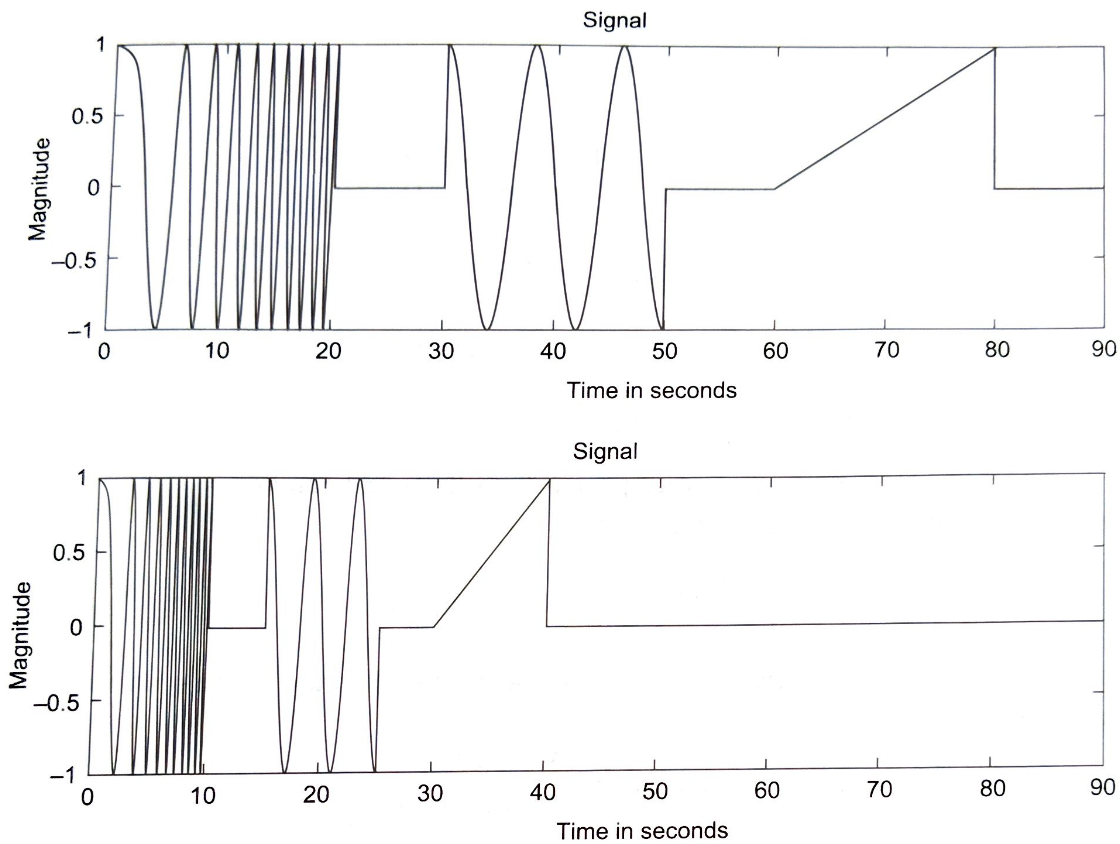


Fig. 1.45 *Plot of $x(t)$ (top), and its scaled version $y(t) = x(2t)$ (bottom)*

The frequency of the sinusoid increases, while the slope of the ramp dubles. For the chirp however, the maximum frequency to which it sweeps also increases.

Summary

1. We may consider a signal to be a single-valued function of one or more variables.
2. A continuous-time signal is one which is defined for all values of time.
3. A discrete-time signal is one which is defined only at a certain discrete set of points in time.
4. A continuous-time signal $x(t)$ is said to be periodic in time with a period T is $x(t) = x(t + mT)$ for any t and integer m. The smallest value of T satisfying this condition is called the fundamental period of a periodic signal $x(t)$.
5. Every real signal will have an even component and an odd component.
6. (a) The total energy of a continuous-time signal, $x(t)$, is given by

$$E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt$$

(b) The total energy of a discrete-time signal, $x\,(n)$, is given by

$$E = \underset{N\to\infty}{Lt} \sum_{n=-N}^{N} |x(n)|^2$$

(c) An energy signal is one whose total energy E is finite and non-zero.

7. (a) The average power, P_{av} of a continuous-time signal, $x\,(t)$ is

$$P_{av} = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

(b) The average power, P_{av} of a discrete-time signal, $x(n)$ is

$$P_{av} = \underset{N\to\infty}{Lt} \frac{1}{2N} \sum_{n=-N}^{N} |x(n)|^2$$

(c) A power signal is one whose average power, P_{av} is finite and non-zero.

8. A unit impulse function, $\delta(t)$ is defined by the following:

$$\int_{t_1}^{t_2} x(t)\, \delta(t)\, dt = x(t)\big|_{t=0} = x(0)$$

where $x(t)$ is any function which is continuous at least at $t = 0$, t_1 and t_2 are such that interval t_1 to t_2 includes $t = 0$.

9. A unit step function, $u(t)$, is defined by $u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

10. $\dfrac{d}{dt}u(t) = \delta(t)$ and $u(t) = \displaystyle\int_{-\infty}^{t} \delta(\alpha)\,d\alpha$

11. The unit sample sequence, $\delta(n)$ is defined by $\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$

12. A unit step sequence, $u(n)$ is defined by

$$u(n) = \begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ for } n < 0 \end{cases}$$

13. If a signal $x(t)$ is to be shifted in time and also time scaled, the time-shifting operation should be done first.

References and Suggested Reading

1. Haykin, Simon, Barry Van Veen, *Signals and Systems*, 2nd edition, John Wiley & Sons (Asia) Pte Ltd. 2004.
2. Roberts, M.J., *Signals and Systems*, Tata McGraw-Hill Edition, New Delhi, 2003.
3. Proakis, John G. and Dimitris G. Manolakis, *Digital Signal Processing – Principles, Algorithms and Applications*, 3rd edition; Prentice-Hall of India, New Delhi, 1997.

Review Questions

1. Define a continuous-time signal.
2. When do you say a signal $x(t)$ is periodic?
3. Show that the product of two odd signals is an even signal.
4. Show that the product of an even signal and an odd signal is an odd signal.
5. Define an energy signal.
6. Define a power signal.
7. Sketch the signal $x(t)=e^{-|t|}$; $-4 \le t \le 4$.
8. Determine the values of the following integrals:

   (i) $\displaystyle\int_{t=-2}^{t=2} \delta(t-8)dt$ (ii) $\displaystyle\int_{t=-3}^{t=3} e^{-2t}\delta(t-2)dt$.
9. Sketch the signal $x(n) = -2n$; $-4 \le n \le 4$.

Problems

1. Determine whether the following continuous-time signals are periodic or aperiodic. If they are periodic, determine their fundamental period.

   (a) $x(t) = \cos 3t$ (b) $x(t) = e^{j\omega_0 t}$ (c) $x(t) = \cos^2 10\pi t$

   (d) $x(t) = \sin 100\pi t + \sin 200\pi t$ (e) $x(t) = \cos t\, u(t)$ (f) $x(t) = \sin 3t + \cos \pi t$
2. If $x(t)$ and $y(t)$ are periodic signals with fundamental periods T_x and T_y respectively, find the condition required to be satisfied for $z(t) = x(t) + y(t)$ to be periodic. If $z(t)$ is periodic, what is its fundamental period?
3. For same integer k, is $x(t) = \cos\left(\dfrac{2\pi k t^2}{T}\right)$ periodic for any selected values of T? If so, specify those values of T.
4. Is $x(t) = \dfrac{\sin(N\omega_o t/2)}{\sin(\omega_o t/2)}$ periodic for some ω_o and integer N? If so, what is its, fundamental period?
5. Is $x(n) = \sin\left(\dfrac{2\pi k n^2}{N}\right)$ periodic for any value of N, given that k is an integer?
6. For any arbitrary signal $x(t)$, is $\displaystyle\sum_{n=-\infty}^{\infty} x(t+nT)$ periodic? How about $\displaystyle\sum_{n=0}^{\infty} x(t+nT)$?
7. Are the following signals periodic? If they are, find the fundamental periods.

   (a) $\displaystyle\sum_{n=-\infty}^{\infty} e^{-|4t-n|}$ (b) $\displaystyle\sum_{n=1}^{10} \cos 2\pi n t$
8. Sketch the following signals:

   (a) $[e^{-t}u(t)] \displaystyle\sum_{n=-\infty}^{\infty} \delta(t-nT)$ (b) $[a^n u(n)] \displaystyle\sum_{m=-\infty}^{\infty} \delta(n-mN)$.

9. Sketch the following signals:

(a) $u(t)u(t-2)$ (b) $u(t)u(1-t)$ (c) $r(3-t)u(t)$

(d) $e^{-2t}u(2-t)$ (e) $u(0.5t+2)$ (f) $u(t)-u(t-3)$

10. If $x(t)$ is as shown in the Fig. 1.46, sketch and label each of the following signals.

(a) $x(t\text{-}3)$ (b) $x(2t)$

(c) $x(t/2)$ (d) $x(\text{-}2t)$

(e) $x(3t\text{-}2)$

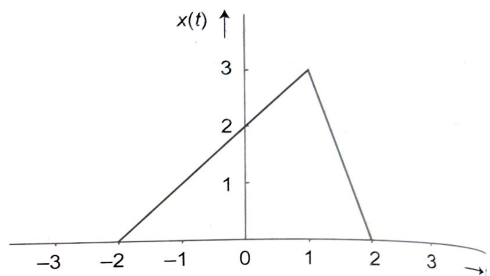


Fig. 1.46

11. Prove the following:

(a) The product of two even signals is even.

(b) The product of an even signal and an odd signal is odd.

(c) The product of two odd signals is even.

12. Sketch the following signals and determine and sketch their even and odd components:

(a) $x(t)=\begin{cases}0.8t\,;\,0\le t\le 4\\ 0\,;\,\text{otherwise}\end{cases}$ (b) $x(t)=10e^{-2t}u(t)$

(c) $x(n)=\begin{cases}1;\,0\le n\le 4\\ 0;\,\text{otherwise}\end{cases}$ (d) $x(n)=\begin{cases}n\text{ for }1\le n\le 3\\ 0\,;\,\text{otherwise}\end{cases}$

13. Are the following discrete-time signals periodic? If they are, what are their fundamental periods?

(a) $x(n)=2\cos\dfrac{\pi}{3}n+3\sin\dfrac{\pi}{4}n$. (b) $x(n)=\dfrac{1}{3}n$ (c) $x(n)=2\cos^2\dfrac{\pi}{6}n$

14. $x(n)$ and $y(n)$ are two periodic discrete-time signals with fundamental periods N_x and N_y respectively. What is the condition required to be satisfied, if $x(n)+y(n)=z(n)$ is to be periodic? When $z(n)$ is periodic, what is its fundamental period?

15. Find the value of T, the interval between two successive samples, if the following discrete-time signals are to be periodic:

(a) $x(n)=e^{j\omega_0 nT}$ (b) $x(n)=\cos 12nT$

16. Which of the following signals are power signals, and which of them are energy signals? Are there some signals which are neither power signals nor energy signals? In each case, justify your answer. For power/energy signals, find the average power or the total energy, which ever is appropriate.

(a) $(2-e^{-3t})u(t)$ (b) $e^{j\omega_0 t}$ (c) $u(t-2)-u(t-4)$ (d) $e^{-2t}\,u(t)$

(e) e^{-5t} (f) $e^{-|t|}\,\Pi[(t+1)/6]$ (g) $te^{-2t}u(t-1)$

17. Determine whether the following discrete-time signals are energy signals or power signals. In each case, justify your answer and determine the average power or the energy as the case may be.

(a) $x(n)=u(n)$ (b) $x(n)=u(n)-u(n-4)$

(c) $x(n)=(-0.2)^n u(n)$ (d) $x(n)=5e^{j2n}$

18. The signal $x(t)$ given by

$$x(t) = \begin{cases} \frac{1}{2}[\cos \omega t + 1]\,; & -\pi \le \omega t \le \pi \\ 0 & ;\text{otherwise} \end{cases}$$

is sketched in Fig. 1.47. Determine the total energy of this signal.

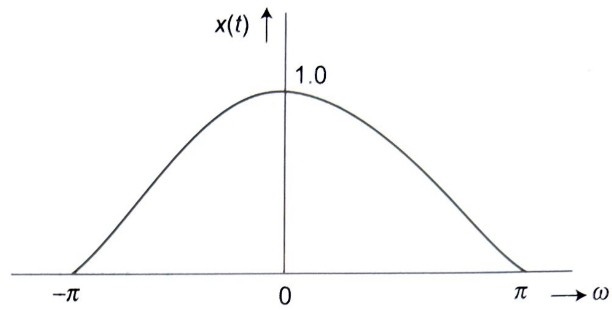


Fig. 1.47

19. Two signals $x(t)$ and $y(t)$ are as shown in Fig. 1.48 (a) and (b). Determine and sketch the following:

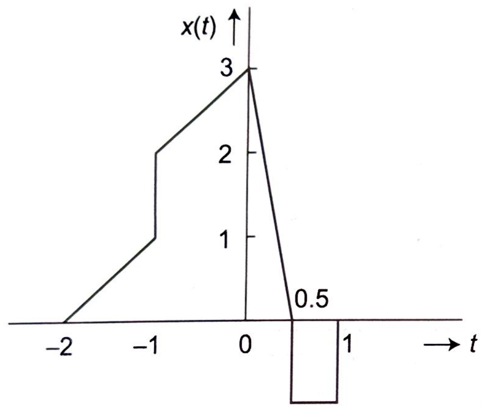


Fig. 1.48(a)

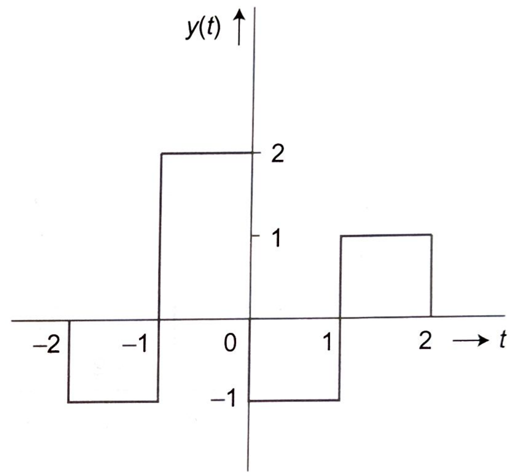


Fig. 1.48(b)

(a) $\frac{d}{dt}x(t)$ (b) $\frac{d}{dt}y(t)$ (c) $x(-t)$

(d) $y(-2t)$ (e) $x(-t+2)$ (f) $y(2t-3)$

(g) $x(t)\,y(t)$

20. The signal $x(t) = 5e^{-2t} \sin 10\pi t\; u(t)$ is called a damped sinusoid. Sketch $x(t)$ over the period $t = 0$ to $t = 2$ and determine its energy over that interval.

21. For the signal $x(t)$ shown in Fig. 1.49(a), determine the following.

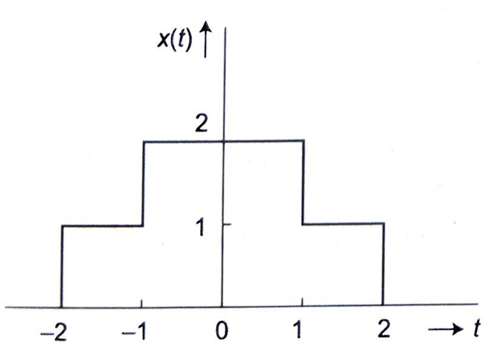


Fig. 1.49(a)

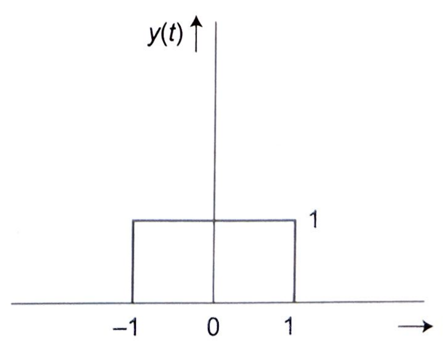


Fig. 1.49(b)

(a) A representation in terms of shifted versions of $u(t)$.

(b) A representation in terms of the rectangular pulse $y(t)$ and its scaled and shifted versions.

22. Two discrete-time signals $x(n)$ and $y(n)$ are shown in Fig. 1.50(a) and (b) respectively

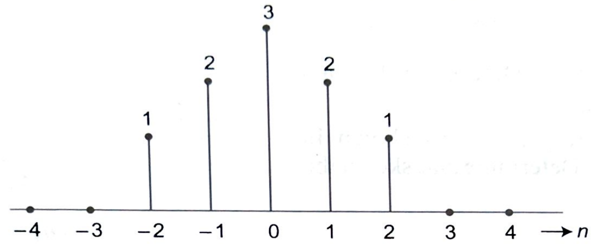


Fig. 1.50(a)

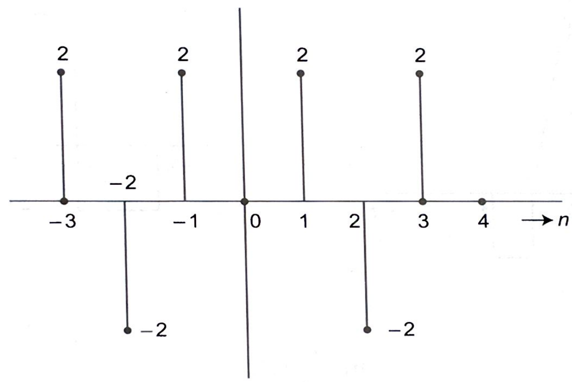


Fig. 1.50(b)

Determine:

(a) $x(2n-1)$

(b) $x(2n)$

(c) $y(2n)$

(d) $y(2n-2)$

(e) $x(-n+1)$

(f) $x(-n+1)\,y(n-1)$

(g) $x(n+1)$

(h) $x(2n+1)$

(i) $y(n+1)$

(j) $y(2n+1)$

(k) $x(n+1)\,y(-n+2)$

23. Determine the conjugate symmetric and the conjugate anti-symmetric parts of the sequence $\{x(n)\}$ if

$$\{x(n)\} = \{-4 - j5,\ 1 + j2,\ 4\}$$

24. Determine the linear convolution of the sequences $\{x(n)\}$ and $\{y(n)\}$ if $\{x(n)\} = \{1, \underset{\uparrow}{-1}, 2\}$ and $\{y(n)\} = \{\underset{\uparrow}{1}, 0, 1\}$.

25. Determine the circular convolution and also the linear convolution of the sequences $\{x(n)\} = \{\underset{\uparrow}{1}, 1, 1, 1, -1, -1\}$ and $\{y(n)\} = \{\underset{\uparrow}{0}, 0, 1, 0, -1, 0\}$.

26. If $z(t) = x(t) * y(t)$, show that $x(t - t_1) * y(t - t_2) = z(t - t_1 - t_2)$.
27. If $x(t) = u(t)$ and $y(t) = e^{-2t}\, u(t)$, determine $z(t) \triangleq x(t) * y(t)$.
28. Determine and sketch the periodic convolution of the periodic signal $x(t)$ shown in Fig. 1.51 with it self.

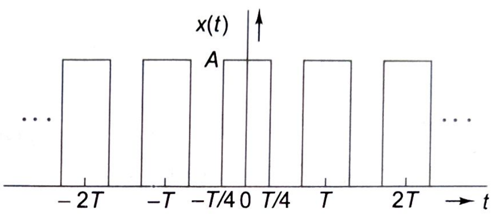


Fig. 1.51 *Signal of problem 28*

Multiple-Choice Questions

1. Discrete-time signals are those signals
   (a) which have non-zero values only at a discrete set of points in time
   (b) whose value is known only at a discrete set of points in time
   (c) which can take only a discrete set of values at any time
   (d) none of the above.

2. The fundamental period T, of a periodic continuous-time signal $x(t)$, is
   (a) the smallest positive constant satisfying the relation $x(t) = x(t + mT)$ for every t and any integer m
   (b) the positive constant satisfying the relation $x(t) = x(t + mT)$ for every t and any integer m
   (c) the largest positive constant satisfying the relation $x(t) = x(t + mT)$ for ant t and any integer m
   (d) the smallest positive integer satisfying the relation $x(t) = x(t + mT)$ for any t and any m.

3. For the signal $x(t) = 2 \sin^2 t + 1 \cos 2t$, which of the following is true?
   (a) It has frequency components with frequencies 0 Hz and $(1/2\pi)$ Hz.
   (b) It has frequency components with frequencies 0 Hz and $(1/\pi)$ Hz.
   (c) It has frequency components with frequencies $(1/2\pi)$ Hz and $(1/\pi)$ Hz.
   (d) It has frequency components with frequencies 0, $(1/2\pi)$ and $(1/\pi)$ Hz.

4. $x(t)$ is even while $y(t)$ is odd. If $z(t) = x(t) + y(t)$ and $w(t) = x(t).\, y(t)$ then $z(t)$ and $w(t)$ are respectively
   (a) even; odd (b) odd; even
   (c) neither; odd (d) neither; even

5. A complex-valued signal $x(t) = a(t) + jb(t)$ will be having conjugate symmetry if
   (a) $a(t)$ and $b(t)$ are both even (b) $a(t)$ and $b(t)$ are both odd.
   (c) $a(t)$ is even while $b(t)$ is odd (d) $a(t)$ is odd while $b(t)$ is even.

6. The value of $\int_{-\pi/4}^{\pi/4} \cos \omega t\, \delta(\omega)\, d\omega$ is
   (a) 0 (b) $\pi/2$ (c) $\sqrt{2}$ (d) 1

7. The value of the integral $\int_{-\infty}^{\infty} \delta(t) \cos^2\left(\frac{3t}{2}\right) dt$ is

(a) 1 (b) -1 (c) 0 (d) $\pi/2$

8. A signal $x(t)$ has energy E. The energy of the signal $x(2t)$ is

(a) E (b) $E/2$ (c) $2E$ (d) $4E$

9. $u(t)$ is

(a) a power signal (b) an energy signal
(c) neither power signal nor energy signal (d) both

10. $x(t) = 5e^{j20\pi t}$ is

(a) a power signal (b) an energy signal
(c) neither (d) both

12. The average power of the signal $x(t) = 12 \cos\left(40\pi t - \frac{\pi}{2}\right) + 8 \sin 15\pi t$ is

(a) 104 (b) 20 (c) 208 (d) 400

13. $x(t) = 10 \sin 10\pi t$

(a) is an even signal (b) is an odd signal
(c) has even and odd parts (d) none of the above

14. $x(t) = 10 \sin (10\pi t + 30°)$

(a) is an even signal (b) is an odd signal
(c) has an even part as well as an odd part (d) none of the above

15. The even and odd parts of a unit step function are respectively

(a) $-\frac{1}{2}, \frac{1}{2}$ sgn (t) (b) $\frac{1}{2}, \frac{1}{2}$ sgn (t)

(c) $\frac{1}{2}, -\frac{1}{2}$ sgn(t) (d) $-\frac{1}{2}, -\frac{1}{2}$ sgn (t)

16. $\{x(n)\} = \{4 - j5, 1 + j2, 4\}$ in the above, sgn(t) denoted the sign of t, and equals 1 if t is positive. t is negative. The conjugate anti-symmetric part of this sequence is

(a) $\{-4 - j2.5, j2, 4 - j2.5\}$ (b) $\{-j2.5, 1, j2.5\}$
(c) $\{-j5, j2, 0\}$ (d) $\{-4, 1, 4\}$

11. $e^{-t}u(t)$ is

(a) an energy signal (b) a power signal
(c) neither an energy signal nor a power signal (d) none of the above

17. $x(t) = \begin{cases} 1 \text{ for } 5 \le t \le 10 \\ -1 \text{ for } 10 \le t \le 15 \\ 0; \text{otherwise} \end{cases}$. Then $x(t)$ can be expressed as:

(a) $u(t + 5) - 2u(t + 10) + u(t + 15)$
(b) $u(t - 5) - u(t - 10) + u(t - 15)$
(c) $u(t - 5) - 2u(t - 10) + 2u(t - 15)$

(d) $u(t-5) - 2u(t-10) + u(t-15)$

18. A signal $x(t)$ is shown in Fig. 1.51 $x(t)$ can be expressed in terms of the unit-ramp function $r(t)$ and unit step function $u(t)$ as

(a) $r(t-2) - r(t-3) + u(t-3)$

(b) $r(t-2)u(t-2) - r(t-3)u(t-3) + u(t-3)$

(c) $r(t-2)u(t-2) - r(t-3)u(t-3)$

(d) $r(t-2) - r(t-3)$

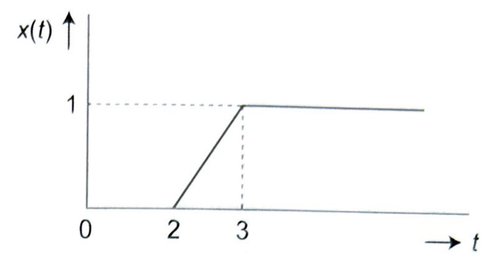


Fig. 1.51

19. Convolution of $x(t+6)$ with $\delta(t-4)$ gives

(a) $x(t-10)$ (b) $x(t+10)$ (c) $x(t+2)$ (d) $x(t-2)$

20. Convolution of $\delta(n-2)$ and $\delta(n-1)$ is

(a) $\delta(n-2) + \delta(n-1)$ (b) $\delta(n-3)$

(c) $\delta(n-4)$ (d) $\delta(n-1)\,\delta(n-2)$

MATLAB Exercises

1. Using appropriate basic MATLAB commands, generate the following continuous-time signals as per the specifications given.

(a) A sinusoidal wave: peak amplitude $A = 10$; $\omega_0 = 20\pi$ rad/sec phase $\phi = \pi/4$ radians; range of time = 0 to 1 second with 1000 samples.

(b) A square wave: amplitude = ±2 volts, fundamental frequency = 10 cycles per second duty cycle 0.5; range of time: 0 to 1 second. At $t = 0$, it should jump from –2V to +2V.

2. Using appropriate MATLAB commands, generate the following continuous-time signals as per the specifications given.

(a) Decaying exponential: Ae^{bt}, $A = 6$; $b = -5$; range of time: 0 to 1 second; number of samples: 1000, i.e, sampling period = 10^{-3} second.

(b) Growing exponential: $x(t) = Ae^{bt}$, $A = 1$; $b = 5$; range of time: 0 to 1 second; sampling frequency = 1000 samples per second.

3. Damped sinusoid: use MATLAB to generate the continuous-time signal, $x(t) = Ae^{-bt} \sin \omega_0 t$, where, $A = 100$; $b = 8$; $f_0 = 10$; range of time: 0 to 1 second; sampling frequency = 1000 samples per second.

4. Use MATLAB to generate the following sequences:

(a) $\delta(n)$, the unit sample sequence: 10 zeros to precede the unit sample which occurs at $n = 0$ and the unit sample should be followed by 10 more zero samples.

(b) $u(n)$, the unit-step sequence: Twenty zeros to be followed by 20 unit amplitude samples.

5. The sequence $x(n) = 10\cos\left(\frac{2\pi}{8}n + \frac{\pi}{3}\right)$ is to be generated by writing down a MATLAB program. Use the stem function of MATLAB to plot 64 samples.

6. Write MATLAB program to generate a continuous-time cosinusoid signal of some convenient frequency and amplitude and plot it. Plot its sampled version using a sampling frequency that is 8 times the frequency of the cosinusoid.

FOURIER SERIES OF CONTINUOUS-TIME SIGNALS

3

Jean Baptiste Joseph Fourier (1768–1830)

Fourier was a French mathematician who showed that a signal may be represented as a weighted sum of complex exponentials. He had developed what is now known after him as Fourier series while investigating the propagation of heat in solids. He lived during a very turbulent period in French history—the French revolution and the reign of Napolean Bonaparte.

Learning Objectives

After going through this chapter, students will be able to

- understand the basic vector space concepts and operations, apply these concepts to signals and thereby obtain a deeper understanding of the geometrical structure of signal spaces,
- understand that the Fourier series expansion of a periodic signal $x(t)$ with a finite energy over one period, is nothing but an orthogonal expansion of it making use of a complete set of orthonormal basis signals,
- understand that while complex exponential Fourier series and trigonometric Fourier series are both orthogonal expansions, the complete orthonormal sequences used as the basis vectors are different in the two cases,
- determine the Fourier series expansion of a given periodic signal and plot its magnitude and phase spectra, and
- understand and state the conditions for existence as well as convergence of the Fourier series of a periodic signal.

3.1 INTRODUCTION

Joseph Fourier (1768–1830) developed the theory for the study of signals in terms of their sinusoidal representation. Named after him as Fourier analysis, it is extensively useful in various branches of science and engineering, as it offers an insight into the frequency content of a signal.

This chapter deals with the Fourier series expansions of a continuous-time signal. The complex exponential and trigonometric Fourier series expansions of a continuous-time signal over a certain interval are derived as orthogonal expansions using pertinent Complete Orthonormal Sequences (CONs).

We start the chapter with a brief presentation of the basics of linear algebra required for this method of introducing the Fourier series. Incidentally, this brief introduction to linear algebra exposes the reader to topics such as Gram–Schmidt Orthogonalization, Schwarz's inequality and the best approximation problem, all of which will be extremely useful at a later stage in the study of communication engineering and control systems.

3.2 BASICS OF VECTOR SPACES

From elementary physics, we know that quantities like force, velocity, acceleration, etc., which possess both magnitude and direction are called vectors. A vector is represented by an oriented line segment, the length of the line representing the magnitude of the vector and the direction of the line, the direction of the vector.

To add two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we construct a parallelogram as shown in Fig. 3.1 in which AB and AD represent $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.

Then we know that the diagonal AC represents the vector $\boldsymbol{x}+\boldsymbol{y}$. We note here that

$$\begin{aligned} \boldsymbol{AB}+\boldsymbol{BC} &= \boldsymbol{x}+\boldsymbol{y}=\boldsymbol{AC} \\ =\boldsymbol{AD}+\boldsymbol{DC} &= \boldsymbol{y}+\boldsymbol{x} \\ \therefore \quad \boldsymbol{x}+\boldsymbol{y} &= \boldsymbol{y}+\boldsymbol{x}. \end{aligned}$$

Further, if CE represents another vector $\boldsymbol{z}$, we know that

$$\begin{aligned} \boldsymbol{AC}+\boldsymbol{CE} &= (\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} \\ &= \boldsymbol{AE}=\boldsymbol{AD}+\boldsymbol{DE}=\boldsymbol{AD}+(\boldsymbol{DC}+\boldsymbol{CE}) \\ &= \boldsymbol{y}+(\boldsymbol{x}+\boldsymbol{z}) \\ \therefore \quad (\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} &= \boldsymbol{y}+(\boldsymbol{x}+\boldsymbol{z}) \\ \text{But} \quad \boldsymbol{AE} &= \boldsymbol{AB}+(\boldsymbol{BC}+\boldsymbol{CE})=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z}) \\ \therefore \quad \boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z}) &= \boldsymbol{y}+(\boldsymbol{x}+\boldsymbol{z})=\boldsymbol{z}+(\boldsymbol{x}+\boldsymbol{y}) \end{aligned}$$

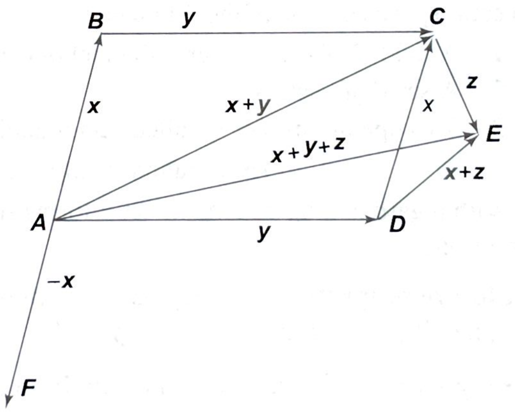


Fig. 3.1 *Addition of vectors*

Also, for any vector like $\boldsymbol{x}$ represented by $\boldsymbol{AB}$, there will always be another vector $\boldsymbol{AF}=-\boldsymbol{x}$ which is such that $\boldsymbol{AB}+\boldsymbol{AF}=\boldsymbol{0}$, the zero vector.

We also know that when a vector is multiplied by a scalar, the magnitude of the vector gets multiplied by the scalar but the direction of the vector does not change. Further, if a and b are two scalars and $\boldsymbol{x}$ and $\boldsymbol{y}$ are two vectors, we know that

1. $ab(\boldsymbol{x})=a(b\boldsymbol{x})$
2. $a(\boldsymbol{x}+\boldsymbol{y})=a\boldsymbol{x}+a\boldsymbol{y}$
3. $(a+b)\boldsymbol{x}=a\boldsymbol{x}+b\boldsymbol{x}$

The above properties related to vector addition and multiplication of a vector by a scalar, may be summarized as follows:

If a and b are any two scalars and $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ any arbitary vectors, then

1. $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$
2. $\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})=\boldsymbol{y}+(\boldsymbol{x}+\boldsymbol{z})=\boldsymbol{z}+(\boldsymbol{x}+\boldsymbol{y})$

3. There exists a vector $\mathbf{0}$ such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ for every vector $\boldsymbol{x}$.
4. For every vector $\boldsymbol{x}$, there exists a vector $\boldsymbol{w} = -\boldsymbol{x}$ such that $\boldsymbol{x} + \boldsymbol{w} = \mathbf{0}$, the zero vector.
5. $1 . \boldsymbol{x} = \boldsymbol{x}$
6. $(ab) \cdot \boldsymbol{x} = a \cdot (b\boldsymbol{x})$
7. $(a + b) \cdot \boldsymbol{x} = a \cdot \boldsymbol{x} + b \cdot \boldsymbol{x}$.
8. $a \cdot (\boldsymbol{x} + \boldsymbol{y}) = a \cdot \boldsymbol{x} + a \cdot \boldsymbol{y}$
9. When we talk of addition of vectors, it is of course assumed that the vectors concerned are all of the same kind, i.e., they are all velocity vectors, or all displacement vectors, etc., and that addition of two or more vectors results in a vector of the same kind. This we express by saying that all the vectors in question belong to the same set.

A generalization of the above ideas, together with some mathematical abstraction, leads to the very useful notion of a vector space, or a linear space, which we define below.

3.3 VECTOR SPACES

A vector space consists of the following:

1. A field F of scalars (generally real or complex numbers)
2. A set V of vectors
3. Two operations: (a) +, called vector addition
   (b) . , called multiplication of a vector by a scalar

With regard to + operation, the set V forms an abelian or, commutative group, i.e., it satisfies the following conditions:

1. Closure property For every pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ belonging to V, their sum $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ also belongs to V, i.e., if $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$, $\boldsymbol{z} \in V$ for $\forall$ $\boldsymbol{x}, \boldsymbol{y} \in V$.

2. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ for $\forall \boldsymbol{x}, \boldsymbol{y} \in V$, i.e., **addition of vectors is commutative.**

3. Addition is associative i.e., $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ for $\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in V$.

4. Zero vector There exists a unique vector, $\mathbf{0}$, in V, called the zero vector, such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ for every $\boldsymbol{x} \in V$.

5. Additive inverse For every vector $\boldsymbol{x} \in V$, there exists a vector $\mathbf{y} \in V$ such that $\boldsymbol{x} + \boldsymbol{y} = \mathbf{0}$, the zero vector. $\boldsymbol{y}$ is called the additive inverse of $\boldsymbol{x}$, and $\boldsymbol{x}$ is called the additive inverse of $\boldsymbol{y}$.

With regard to the multiplication of a vector by a scalar, using the operator '.', the following conditions hold. For every c, c_1 and c_2 belonging to F and every $\boldsymbol{x}$ and $\boldsymbol{y}$ belonging to V:

6. Closure property If $c . \boldsymbol{x} = z$ then z belongs to V for $\forall$ $\boldsymbol{x} \in V$ and $\forall$ $c \in F$.

7. Associativity $(c_1 c_2) . \boldsymbol{x} = c_1 . (c_2 . \boldsymbol{x})$ for $\forall$ $\boldsymbol{x} \in V$ and $\forall$ c_1 and $c_2 \in F$.

8. $\mathbf{1} . \boldsymbol{x} = \boldsymbol{x}$ for $\forall$ $\boldsymbol{x} \in V$ and where 1 is the multiplicative identity of F.

9. Distributive property (i) $c . (\boldsymbol{x} + \boldsymbol{y}) = c \cdot \boldsymbol{x} + c \cdot \boldsymbol{y}$ for $\forall$ $\boldsymbol{x}, \boldsymbol{y} \in V$ and $c \in F$.

10. Distributive property (ii) $(c_1 + c_2) . \boldsymbol{x} = c_1 \cdot \boldsymbol{x} + c_2 \cdot \boldsymbol{x}$ for $\forall$ $\boldsymbol{x} \in V$ and $c_1, c_2 \in F$.

From the above, it is clear that a vector space is a composite entity consisting of a field of scalars, a set of vectors and two operations—addition of two vectors and multiplication of a vector by a scalar, satisfying certain conditions. A vector space defined as above, is generally referred as a 'vector space V over a field F'. The elements of the set V, although called vectors, may bear little or no resemblance to our usual notion of a vector as an oriented line segment. As may be seen from the following examples, these vectors may be n-length sequences of scalars, $m \times n$ matrices with real or complex number entries, or polynomials, etc.

Examples

1. C^n It is the space of all n-tuples (i.e., sequences of length n) with entries drawn from the complex number field.

Let V be the set of all complex n-tuples, and

Let $$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$$

and $$\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$$

where x_i and y_i, $1 \le i \le n$, are complex numbers. Then $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to $V = C^n$ with the two operations '+' and '.' as defined below.

Let $$\boldsymbol{x} + \boldsymbol{y} \triangleq (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n)$$

and $$c.\boldsymbol{x} \triangleq (cx_1, cx_2, \ldots, cx_n)$$

It can easily be verified that the elements of the set V together with the '+' and '.' operations as defined above, will satisfy all the requirements of a vector space. In the above example, if the entries of the n-tuples are drawn from the field of real numbers, the vector space formed by them (with the operations of + and . defined exactly in the same way) would be called R^n.

2. The vector space $F^{m\times n}$ Let m and n be positive integers and let F be any field. Let $F^{m\times n}$ be the set of all $m \times n$ matrices with entries drawn from the field F. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any two vectors belonging to $F^{m\times n}$. Let us define addition of vectors by

$$(\boldsymbol{A} + \boldsymbol{B})_{i,j} = A_{ij} + B_{ij} \quad ; \quad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

And the multiplication of a vector by a scalar as

$$(c.\boldsymbol{A})_{i,j} = cA_{i,j} \quad ; \quad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

It can easily be verified that the set of vectors $F^{m\times n}$ together with the addition and multiplication operations as defined above, satisfies all the ten requirements of a vector space.

3. P(F) The Space of all polynomial functions over a field F A polynomial with coefficients drawn from a field F is an expression of the form

$$f(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n.$$

where, n is a non-negative integer and $c_0, c_1, c_2, \ldots, c_n$ are all elements of F.

Let $P(F)$ be the set of all polynomials with coefficients drawn from a field F. Let $\boldsymbol{f}(x)$ and $\boldsymbol{g}(x)$ be elements of $P(F)$ where,

$$\boldsymbol{f}(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

$$\boldsymbol{g}(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$$

Then we define vector addition by

$$(\boldsymbol{f} + \boldsymbol{g})(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_{n-1} + b_{n-1})x^{n-1} + (a_n + b_n)x^n.$$

and multiplication of a vector $\boldsymbol{f}(x)$ by c as

$$(c.\boldsymbol{f})(x) = ca_0 + ca_1x + ca_2x^2 + \ldots + ca_nx^n.$$

It can easily be verified that $P(F)$ forms a vector space.

3.4 SUBSPACES

Definition Let V be a vector space over a field F. A subset W of V is called a subspace of V provided it forms a vector space over F with the operations of addition of vectors and vector multiplication by a scalar as defined for V.

Thus, to determine whether a subset W of V forms a subspace or not, one has to check whether it is a vector space in its own right, with + and '.' operations, the same as on V. For this, one need not check whether all the ten properties of a vector space, as listed in Section 3.3, are satisfied or not. As W is a subset of V, vectors in W will automatically satisfy all the conditions like associativity and distributive properties except

1. closure property with respect to vector addition within W
2. existence of the zero vector in W
3. closure property with respect to multiplication of a vector by a scalar within W

Hence, it is enough if the above three properties are checked. In fact, the subspace definition and the above three requirements are expressed in a compact way through the following theorem.

Theorem *Let W be a non-empty subset of V. Then W is a subspace of V if for every pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in W and every scalar c in F, the vector $c.\mathbf{x}+\mathbf{y}$ again belongs to W.*

Examples

1. Consider the set of all $n \times n$ matrices with their entries drawn from the field of complex numbers. We know they form a vector space $C^{n\times n} = V$. The subset of V consisting of the set of all symmetrical $n \times n$ matrices is a subspace of V. (Supply proof.)

2. Let V be any vector space over a field F. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \ldots, \boldsymbol{x}_n$ be elements of V, and $c_1, c_2, c_3, c_4, \ldots, c_n$ be scalars belonging to F, Now consider

$$c_1 . \boldsymbol{x}_1 + c_2 . \boldsymbol{x}_2 + \ldots + c_n . \boldsymbol{x}_n = \sum_{i=1}^{n} c_i . \boldsymbol{x}_i$$

This is called a linear combination of the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \ldots, \boldsymbol{x}_n$.
Then, the set W of all such linear combinations of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \ldots, \boldsymbol{x}_n$ is a subspace of V. (Prove this.)

Remark *In the above example, the subspace W, each vector of which can be expressed as a linear combination of the elements $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n$ is said to be 'spanned by' or 'generated by' the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n$.*

3.5 LINEAR INDEPENDENCE, BASES AND DIMENSIONS

Definition *A set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n$ of distinct vectors in a vector space V over a field F is said to be linearly dependent if there exist scalars $c_1, c_2, \ldots, c_n$ belonging to F, not all of which are zero, such that*

$$c_1 . \boldsymbol{x}_1 + c_2 . \boldsymbol{x}_2 + c_3 . \boldsymbol{x}_3 + \ldots + c_n . \boldsymbol{x}_n = \sum_{i=1}^{n} c_i . \boldsymbol{x}_i = \mathbf{0} \text{ (the zero vector)}$$

A set of vectors which is not linearly dependent, is said to be linearly independent.

If $x_1, x_2, x_3, x_4, \ldots, x_n$ are linearly dependent,

$$x_1 = \frac{-1}{c_1}\left[c_2 . x_2 + c_3 . x_3 + \ldots + c_n . x_n\right]$$

i.e., x_1 is a linear combination of the rest of the vectors. Thus, to see if a given set S of vectors is linearly dependent, we check whether or not there is a vector in S which is a linear combination of the remaining vectors.

Example 3.1 *$\mathbf{x} = (-1,9,2)$, $\mathbf{y} = (2,1,-1)$, $\mathbf{z} = (0,2,1)$ and $\mathbf{w} = (1,-2,1)$ belong to R^3. Are these vectors linearly independent?*

Solution Suppose $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{w}$, where a, b and c are real numbers. Substituting for $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{w}$ we get the following equations:

$$a(-1,0,2) + b(2,1,-1) + c(0,2,1) = (1,-2,1)$$

i.e.,

$$-a + 2b + 0c = 1$$

$$0a + b + 2c = -2$$

$$2a - b + c = 1$$

Solving these, we get $a = -\dfrac{11}{5}$, $b = \dfrac{8}{5}$, $c = \dfrac{9}{5}$

$$\therefore \quad -\frac{11}{5}\mathbf{x} + \frac{8}{5}\mathbf{y} - \frac{9}{5}\mathbf{z} = \mathbf{w}$$

Hence the given vectors are linearly dependent.

Example 3.2 *Check whether the following three vectors belonging to R^3 are linearly independent.*

$\mathbf{x} = (1, 6, 5)$; $\mathbf{y} = (1, 1, 0)$; $\mathbf{z} = (7, 5, 2)$

Solution If the vectors are not linearly independent, any one of them can be expressed as a linear combination of some or all of the remaining vectors. Hence, in such a case, their determinant will be zero. Thus, the problem of checking whether the given vectors are linearly independent, reduces to checking whether their determinant is not, or is, equal to zero. Let us form a matrix A using these vectors.

$$[A] = \begin{bmatrix} 1 & 6 & 5 \\ 1 & 1 & 0 \\ 7 & 5 & 2 \end{bmatrix} \quad \therefore \quad |A| = \begin{vmatrix} 1 & 6 & 5 \\ 1 & 1 & 0 \\ 7 & 5 & 2 \end{vmatrix}$$

$$\therefore \quad |A| = 1(2-0) - 1(12-25) + 7(-5) = 2 + 13 - 35 = -20$$

Hence $|A| \neq 0$ and so the given vectors are linearly independent.

Basis

Definition *A basis for a vector space V is a set of linearly independent vectors in V which spans V.*

Note that to be a basis set for V, a set of vectors will have to be not only linearly independent, but also will have to span V, i.e., we should be able to express any arbitrary vector in V as a linear combination of the vectors in the basis set.

Note ***A basis set for any vector space is not unique.***

Examples

1. The set of vectors $x = (1,0,0)$, $y = (0,1,0)$ and $z = (0,0,1)$ is a basis set for R^3, the three-dimensional space with which we are all familiar.

(a) To show that they are linearly independent, we will show that a linear combination of x, y and z can be equal to the zero vector only if it is a trivial linear combination (i.e., all the coefficients of the linear combination are zero)

Let $$a.x + b.y + c.z = \mathbf{0}$$

i.e., $$a(1,0,0) + b(0,1,0) + c(0,0,1) = (a, b, c) = (0,0,0)$$

This implies that $$a = 0, b = 0, \text{ and } c = 0.$$

Hence, x, y and z are linearly independent.

(b) To show that x, y and z span R^3:

Consider any arbitrary vector $$w = (c_1, c_2, c_3) \in R^3$$

Then, $$w = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

i.e., $w = c_1.x + c_2.y + c_3.z$, a linear combination of the vectors x, y and z.

Hence $x = (1,0,0)$, $y = (0,1,0)$ and $z = (0,0,1)$ are linearly independent and they span R^3. $\therefore$ they form a basis for R^3.

In a similar way, we may show that the following n vectors $\in_1, \in_2, \ldots, \in_n$ form a basis for F^n, where

$$\in_1 = (1,0,0,\ldots,0,0)$$
$$\in_2 = (0,1,0,\ldots,0,0)$$
$$\in_3 = (0,0,1,\ldots,0,0)$$
$$\vdots \quad \vdots$$
$$\in_n = (0,0,0,\ldots,0,1)$$

The above basis set for F^n is called the 'standard basis set' for F^n.

There is an important theorem in linear algebra which we state here without proof.

Theorem *Let V be a vector space spanned by a finite set of vectors $x_1, x_2, \ldots, x_m$. Then there cannot be more than m linearly independent vectors in any subset of vectors from V.*

So this theorem states that if a vector space V is generated by m vectors, then any subset S of V can have at the most m linearly independent vectors in it, i.e., any set of $N(> m)$ vector belonging to S must be a linearly dependent set.

This important theorem leads to the concept of dimension of a vector space.

Consider a finite dimensional vector space V. Let $B_1 = (x_1, x_2, \ldots, x_m)$ be a basis set for V. Since B_1 spans V, there cannot be more than m linearly independent vectors in any subset of vectors from V. Let $B_2 = (y_1, y_2, \ldots y_n)$ also be a basis set for V. Being a basis set, B_2 must be a linearly independent set.

Since the 'm' vectors $(x_1, x_2, \ldots x_m)$ are spanning the space V, it follows from the above theorem that $n \le m$. Now let us reverse the roles of B_1 and B_2. Since B_2 is a basis set for V, its 'n' vectors $y_1, y_2, \ldots y_n$ must be spanning V. Hence no subset in V can have more than 'n' linearly independent vectors. However, B_1 being a basis set, has 'm' linearly independent vectors in it. Hence, from our theorem, $n \ge m$. Thus, $n \le m$ and also $n \ge m$ which implies $n = m$.

Thus, any two basis sets of a finite dimensional vector space must have the same number of vectors in them. Thus, we may define the dimension of a vector space as follows.

Definition *The dimension of a finite dimensional vector space V is the number of elements in a basis for V.*

3.6 INNER PRODUCTS AND INNER PRODUCT SPACES

In the Euclidean space R^3 with which we are quite familiar, we have been associating with our vectors two geometrical concepts—the length of a vector and the angle between two vectors. These quantities, the length and angle, we could define by making use of the algebraically defined 'scalar product' or 'dot product' of vectors in R^3. If A and B are two vectors in R^3 then we know that

$$A \cdot B = |A|\,|B| \cos\phi \quad \ldots (3.1)$$

where $|A|$ and $|B|$ are the magnitudes of A and B respectively and ϕ is the angle between them. These are given in terms of he scalar product by

$$|A| = (A \cdot A)^{1/2}, \quad |B| = (B \cdot B)^{1/2} \quad \ldots (3.2)$$

and,

$$\cos\phi = \frac{A \cdot B}{|A||B|} \quad \ldots (3.3)$$

We shall now extend these concepts to our generalized vector spaces so as to give them a geometrical structure. This we do by defining an 'inner product' of vectors which is a scalar product of two vectors, with properties similar to those of the familiar dot product of vectors in R^3.

It should be noted that in all our discussions of inner products and inner product spaces, we will be restricting ourselves to either real or complex vector spaces only, that is, vector spaces defined over the field of real or complex numbers.

Definition *Let V be a vector space over a field F of real or complex numbers. An inner product on V is a function that assigns a scalar in F, denoted by* $(\mathbf{x}, \mathbf{y})$, *for every ordered pair of vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *belonging to V. For every* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ *and every* $c \in F$ *the inner product should satisfy the following properties:*

(a) $((\mathbf{x} + \mathbf{y}), \mathbf{z}) = (\mathbf{x}, \mathbf{z}) + (\mathbf{y}, \mathbf{z})$

(b) $(c.\mathbf{x}, \mathbf{y}) = c(\mathbf{x}, \mathbf{y})$

(c) $(\mathbf{y}, \mathbf{x}) = \overline{(\mathbf{x}, \mathbf{y})}$ = *complex conjugate of* $(\mathbf{x}, \mathbf{y})$.

(d) $(\mathbf{x}, \mathbf{y}) \ge 0$, *the equality sign holding good if and only if* $\mathbf{x} = \mathbf{0}$, *the zero vector.*

Examples of Inner Product

1. *The Standard Inner Product on* F^n Let $V = F^n$, the space of all n-tuples with entries drawn from the field of real numbers, R or, from the field of complex numbers, C.

Let $\quad x = (x_1, x_2, x_3, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, y_3, \ldots, y_n)$

be any two vectors in V. Then, we define an innerproduct on V as follows.

$$(x, y) \triangleq \sum_{i=1}^{n} x_i \overline{y_i} \qquad \ldots (3.4)$$

Note

(a) *In the above, the complex conjugate of y_i is taken to ensure that $||\mathbf{x}||^2 \triangleq (\mathbf{x}, \mathbf{x})$ is purely real. Just as $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$, where $|\mathbf{A}|$ represents the magnitude of $\mathbf{A}$ (and therefore has to be purely real), $(\mathbf{x}, \mathbf{x}) = ||\mathbf{x}||^2$, called the norm-square of $\mathbf{x}$, should also be real.*

(b) *For $F = R$ and $n = 3$, we get the Euclidean space R^3 and for this, Eq. (3.4) reduces to*

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{3} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3,$$ *which is the familiar dot product in R^3.*

2. On V, the vector space of all continuous complex-valued functions defined over the interval $t_1 \le t \le t_2$, we define the inner product of $x(t)$ and $y(t)$ in V as follows:

$$(x(t), y(t)) = \int_{t_1}^{t_2} x(t)\overline{y(t)}dt \qquad \ldots (3.5)$$

Note *The inner products of the two examples given above satisfy all the properties of an inner product. (It is left to the reader to prove this.)*

Norm of a Vector As stated earlier, the inner product of a vector with itself is called the norm-square of the vector. Hence the positive square-root of (x, x) is called the norm of x.

$$\sqrt{(x,x)} = ||x|| \qquad (3.6)$$

Now, if $x \in R^3$, form Eq (3.4) we may write

$$||x|| = \sqrt{(x,x)} = \sqrt{\sum_{i=1}^{3} x_i \overline{x_i}} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

But $\quad \sqrt{x_1^2 + x_2^2 + x_3^2} = \text{length of the vector } x$

Because of this, we generally tend to associate $||x||$ of a vector x in any vector space with the length or magnitude of the vector x. However, what it actually means in relation to x will, of course, depend upon what physical quantity the vector x itself represents.

Inner Product Spaces When an inner product is defined over a real or complex vector space, the vector space, together with the inner product, forms what is called an 'Inner product Space'.

Note ***If two different inner products are defined over the same real or complex vector space, each of the inner products will, along with the vector space, from a separate inner product space.***

There are a few important relationships pertaining to inner product spaces, which we will discuss through the following examples.

Examples 3.3 *If V is an inner product space and* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *belong to V and if c belongs to the complex number field C on which V is defined, show that*

(a) $\|c.\boldsymbol{x}\| = |c|\ \|\boldsymbol{x}\|$ *(b)* $\|\boldsymbol{x}\| > 0$ *if* $\boldsymbol{x} \neq \boldsymbol{0}$ *(zero vector).*

Solution

(a) $\|c\cdot\boldsymbol{x}\|^2 = (c\cdot\boldsymbol{x}, c.\boldsymbol{x}) = c(\boldsymbol{x}, c\cdot\boldsymbol{x}) = c\cdot\,\overline{c}(\boldsymbol{x}, \boldsymbol{x}) = |c|^2\,\|\boldsymbol{x}\|^2$

$\therefore$ taking the positive square root on both sides, $\|c\cdot\boldsymbol{x}\| = |c|\,\|\boldsymbol{x}\|$

(b) $\|\boldsymbol{x}\|^2 = (\boldsymbol{x}, \boldsymbol{x})$ and the positive square root of$(\boldsymbol{x}, \boldsymbol{x})$ is called $\|\boldsymbol{x}\|$. Hence, $\|\boldsymbol{x}\| \geq 0$, the equality sign holding good if and only if $\boldsymbol{x} = \boldsymbol{0}$, the zeo vector. $\therefore$ $\|\boldsymbol{x}\| > 0$ if $\boldsymbol{x} \neq \boldsymbol{0}$.

Example 3.4 *If* $\boldsymbol{x}, \boldsymbol{y} \in V$*, an inner product space on complex number field, show that* $(\boldsymbol{x}, \boldsymbol{y}) \leq \|\boldsymbol{x}\|\ \|\boldsymbol{y}\|$ ***(Schwarz's inequality)***

Solution

(a) If either $\boldsymbol{x}$ or $\boldsymbol{y}$ is equal to the zero vector, the result follows, with the equality sign.

(b) If neither $\boldsymbol{x}$ nor $\boldsymbol{y}$ is zero, consider the vector

$$z = \boldsymbol{y} - \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x}$$

$$\|z\|^2 = (z, z) = \left(\boldsymbol{y} - \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x}, \boldsymbol{y} - \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x}\right).$$

$$= (\boldsymbol{y}, \boldsymbol{y}) - \left(\boldsymbol{y}, \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x}\right) - \left(\frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x}, \boldsymbol{y}\right) + \left(\frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x}, \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x}\right)$$

$$= \|\boldsymbol{y}\|^2 - \frac{\overline{(\boldsymbol{y}, \boldsymbol{x})}}{\|\boldsymbol{x}\|^2}(\boldsymbol{y}, \boldsymbol{x}) - \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}(\boldsymbol{x}, \boldsymbol{y}) + \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\frac{\overline{(\boldsymbol{y}, \boldsymbol{x})}}{\|\boldsymbol{x}\|^2}(\boldsymbol{x}, \boldsymbol{x})$$

$$= \|\boldsymbol{y}\|^2 - \frac{|(\boldsymbol{y}, \boldsymbol{x})|^2}{\|\boldsymbol{x}\|^2} - \frac{|(\boldsymbol{y}, \boldsymbol{x})|^2}{\|\boldsymbol{x}\|^2} + \frac{|(\boldsymbol{y}, \boldsymbol{x})|^2}{\|\boldsymbol{x}\|^2}$$

$$\therefore \quad \|z\|^2 = \|\boldsymbol{y}\|^2 - \frac{|(\boldsymbol{y}, \boldsymbol{x})|^2}{\|\boldsymbol{x}\|^2} \quad \text{but } \|z\|^2 \geq 0$$

$\therefore$ $\|\boldsymbol{y}\|^2\|\boldsymbol{x}\|^2 \geq |(\boldsymbol{y}, \boldsymbol{x})|^2$. Taking the positive square root on both sides, we have

$$|(\boldsymbol{y}, \boldsymbol{x})| \leq \|\boldsymbol{x}\|\,\|\boldsymbol{y}\|$$

In the above, the equality sign holds good only if $z = \boldsymbol{0}$ (zero vector). But, if $z = \boldsymbol{0}$, then

$$\boldsymbol{y} = \frac{(\boldsymbol{y}, \boldsymbol{x})}{\|\boldsymbol{x}\|^2}\cdot\boldsymbol{x} = c.\boldsymbol{x}$$

where c is a scalar

$\therefore \|(x, y)\| = \|\|x\|\|\ \|\|y\|\|$ if $y = c.x$	$\therefore \|(x, y)\| < \|\|x\|\|\ \|\|y\|\|$ if $y \neq c.x$

... (3

In R^3, we know that $A \cdot B = |A|\ |B| \cos \phi$, where A and B are any two vectors, $|A|$ is the magnitude of A a ϕ is the angle between the two vectors A and B. Since $|\cos \phi| \leq 1$, taking the magnitude on both sides, $|A \cdot B| \leq |A|\ |B|$, the equality sign holds only if $\phi = 0$, i.e., if A and B are in the same direction, i.e., if $A = c \cdot B$ where c is a constant. Schwarz's inequality is analogous to this.

Example 3.5 *(Triangle Inequality): If x and y belong to an inner product space, show that* $||x + y|| \leq ||x|| + ||y||$

Solution

$$\begin{aligned}
||x + y||^2 &= (x + y, x + y) \\
&= (x, x) + (x, y) + (y, x) + (y, y) \\
&= ||x||^2 + (x, y) + \overline{(x, y)} + ||y||^2 \\
&= ||x||^2 + 2\text{Re}\,(x, y) + ||y||^2 \\
&\leq ||x||^2 + 2\,||x||\ ||y|| + ||y||^2
\end{aligned}$$

This is because $\text{Re}(x, y) \leq |(x, y)|$ and from Schwarz's inequality, $|(x, y)| \leq ||x||\ ||y||$

$$||x + y||^2 \leq (||x|| + ||y||)^2. \qquad \ldots (3.8)$$

The triangle inequality follows if we take the positive square root on both sides.

3.7 ORTHOGONAL AND ORTHONORMAL SETS

In R^3 we know that two vectors A and B are orthogonal if their dot product $A \cdot B$ is equal to zero, i.e., $\underset{(\text{in } R^3)}{A \perp B}$ if $A \cdot B = 0$.

In an analogous manner, we say that two vectors x and y belonging to an inner product space are orthogonal if their inner product is zero,

i.e., $x \perp y$ in an inner product space if $(x, y) = 0$.

3.7.1 Orthogonal and Orthonormal Set of Vectors

A set S of vectors in V, an inner product space, is called an orthogonal set, provided all pairs of distinct vectors are orthogonal. An orthonormal set S of vectors is an orthogonal set in which every vector has a norm equal to 1.

We may visualize an orthonormal set of vectors as a set of mutually perpendicular vectors each of which has a unit length.

Example 3.6 *Show that the standard basis set of R^n (or C^n) is an orthonormal set with respect to the standard inner product on R^n.*

Solution The standard basis set of R^n has n vectors, x_k, the k^{th} of which $(1 \leq k \leq n)$ has a one only at the k^{th} position and zeros everywhere else. The standard inner product on R^n is

$$(x_k, x_l) = \sum_{i=1}^{n} x_{ik} x_{il};$$

x_{ik}, the i^{th} element of the k^{th} basis vector is equal to one only when $i = k$ or else it is equal to zero. Similarly, the i^{th} element of the l^{th} basis vector is equal to 1 only when $i = l$ otherwise it is equal to zero.

Since $k \neq l$, i cannot simultaneously be equal to k as well as l. Hence,

$$(x_k, x_l) = \sum_{i=1}^{n} x_{ik} x_{il} = \delta_{k,l} = \begin{cases} 1 \text{ for } k = l \\ 0 \text{ otherwise} \end{cases}$$

Hence the standard basis set on R^n is an orthonormal set.

Example 3.7 *Show that f(t) is orthonormal to signals cos t, cos 2t,..., cos nt,... for all non-zero integer values of n over the interval 0 to 2π if*

$$f(t) = \begin{cases} 1; 0 \leq t \leq \pi \\ -1; \pi \leq t \leq 2\pi \end{cases}$$

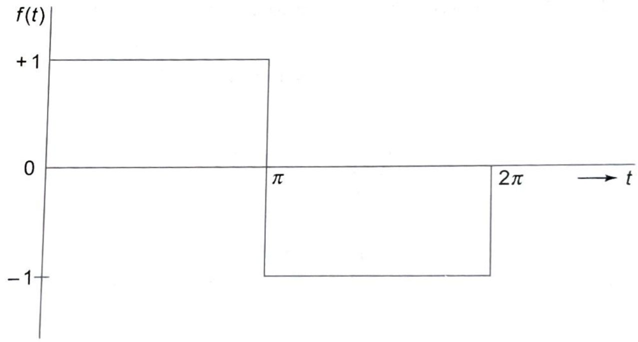


Fig. 3.2 *Waveform of f(t) of Example 3.7*

Solution

$f(t)$ is plotted in Fig. 3.2.

To show that $f(t)$ and the set of functions $\cos nt$, n an integer and $n \neq 0$, are orthogonal over the interval 0 to 2π, we show that their inner product, given by

$$f(t), \cos nt) \triangleq \int_0^{2\pi} f(t)\cos nt\, dt \text{ equals zero.}$$

$$\int_0^{2\pi} f(t)\cos nt\, dt = \int_0^{\pi} 1 \cdot \cos nt\, dt + \int_{\pi}^{2\pi} -1 \cdot \cos nt\, dt$$

$$= \frac{1}{n}\sin nt \Big|_{t=0}^{\pi} - \frac{1}{n}\sin nt \Big|_{\pi}^{2\pi} = 0 \text{ for any integer} \neq 0.$$

Hence, the set of functions, $\cos nt$ for all non-zero integer values of n, will be orthogonal to the given $f(t)$.

Example 3.8 *If $x_e(t)$ and $x_0(t)$ are respectively the even and odd parts of a signal $x(t)$, show that they are orthogonal over the interval $-T$ to T for any T.*

Solution To show that $x_e(t)$ and $x_0(t)$ are orthogonal to each other over $-T$ to T for any T, we have to show that their inner product, given by

$$(x_e(t), x_0(t)) = \int_{-T}^{T} x_e(t)x_0(t)dt \text{ equals zero.}$$

Now, $x_e(t)x_0(t)$ is the product of an even signal and an odd signal, and hence it is odd.

$$\therefore \quad \int_{-T}^{T} x_e(t)x_0(t)dt = 0 \text{ for any } T$$

Hence, $x_e(t)$ and $x_0(t)$ are orthogonal over $(-T, T)$ for any T.

Example 3.9 *V is the space of all continuous complex-valued functions defined over the interval $-T/2 \le t \le T/2$ with the inner product on it defined as $(x(t),y(t)) \triangleq \frac{1}{T}\int_{-T/2}^{T/2} x(t)\overline{y(t)}dt$. Show that the set of functions $x_n(t) = e^{j2\pi nt/T}$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ is an infinite set of orthonormal functions.*

Solution Let $x_k(t) = e^{j2\pi kt/T}$ and $x_l(t) = e^{j2\pi lt/T}$. These belong to V.

$$(x_k(t),x_l(t)) = \frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi kt/T} \cdot e^{-j2\pi lt/T} dt.$$

But,
$$\frac{1}{T}\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}(k-l)t} dt = 1 \text{ if } k = l$$

and,
$$\int_{-T/2}^{T/2} e^{j\frac{2\pi}{T}(k-l)t} dt = \int_{-T/2}^{T/2} e^{\frac{j2\pi mt}{T}} dt \qquad \text{if we put } k - l = m \ne 0$$

$$= \frac{T}{j2\pi m} e^{j\frac{2\pi}{T}mt}\Bigg|_{t=-T/2}^{t=T/2} = \left(\frac{T}{j2\pi m}\right)\left[e^{j\frac{2\pi}{T}m\cdot\frac{T}{2}} - e^{-j\frac{2\pi}{T}m\cdot\frac{T}{2}}\right]$$

$$= \frac{T}{j2\pi m}\cdot\left[e^{jm\pi} - e^{-jm\pi}\right] = \frac{T}{\pi m}\sin(m\pi) = 0,$$

since $\sin m\pi = 0$ for integer values of m (m is an integer, being equal to the difference between two distinct integers k and l.)

Hence, $$(x_k(t), x_l(t)) = \delta_{k,l}$$

$\therefore$ any pair of distinct vectors in V are orthogonal and every vector has a magnitude of 1.

$\therefore e^{j\frac{2\pi}{T}nt}, n = 0, \pm 1, \pm 2, \ldots$ from an infinite set of orthonormal vectors.

Example 3.10 *Show that the set of functions*

$$\frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}}\cos\omega_0 t, \sqrt{\frac{2}{T}}\cos 2\omega_0 t, \sqrt{\frac{2}{T}}\cos 3\omega_0 t, \ldots,$$

$$\sqrt{\frac{2}{T}}\sin\omega_0 t, \sqrt{\frac{2}{T}}\sin 2\omega_0 t, \ldots, \text{ where } \omega_0 = \frac{2\pi}{T}$$

forms an orthonormal set of functions over the interval −T/2 to +T/2 with respect to the inner product

$$\left(x(t), y(t)\right) \triangleq \int_{-T/2}^{T/2} x(t)\, y(t)dt$$

Solution

(a) ***To show that all these functions have unit norm***

(i) Consider $\frac{1}{\sqrt{T}}$: $\left\|\frac{1}{\sqrt{T}}\right\|^2 = \left(\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}\right) = \int_{-T/2}^{T/2} \frac{1}{\sqrt{T}} \cdot \frac{1}{\sqrt{T}} dt = 1$

(ii) Consider $\sqrt{\frac{2}{T}}\cos n\omega_0 t$ where $n = 1, 2, \ldots$

$$\left\|\sqrt{\frac{2}{T}}\cos n\omega_0 t\right\|^2 = \left(\sqrt{\frac{2}{T}}\cos n\omega_0 t, \sqrt{\frac{2}{T}}\cos n\omega_0 t\right) = \int_{-T/2}^{T/2} \frac{2}{T}\cos^2 n\omega_0 t dt$$

$$\frac{2}{2T}\int_{-T/2}^{T/2} (1 + \cos 2n\omega_0 t)dt = \frac{1}{T}\int_{-T/2}^{T/2} dt + \frac{1}{T}\int_{-T/2}^{T/2} \cos 2n\omega_0 t dt$$

In the above, the second integral is zero. Hence,

$$\left\|\sqrt{\frac{2}{T}}\cos n\omega_0 t\right\|^2 = \frac{1}{T}\int_{-T/2}^{T/2} dt = 1 \quad \text{for any integer } n \neq 0.$$

(iii) In a similar way, we can show that

$$\left\|\sqrt{\frac{2}{T}}\sin n\omega_0 t\right\|^2 = 1 \quad \text{for any integer } n \neq 0.$$

Hence, every function in the given set has a unit norm.

(b) ***To show that it forms an orthogonal set***

To prove this, we have to show that the inner product of any two *distinct* functions belonging to the set, is zero.

(i) Consider $\frac{1}{\sqrt{T}}$ and $\sqrt{\frac{2}{T}}\cos n\omega_0 t$

Their inner product is equal to $\int_{-T/2}^{T/2} \frac{\sqrt{2}}{T}\cos n\omega_0 t dt = 0$ for any integer $n \neq 0$.

(ii) Consider $\frac{1}{\sqrt{T}}$ and $\sqrt{\frac{2}{T}}\sin n\omega_0 t$ for any integer $n \neq 0$

$$\left(\frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}} \sin n\omega_0 t\right) = \int_{-T/2}^{T/2} \frac{\sqrt{2}}{T} \sin n\omega_0 t dt = 0$$

(iii) Consider $\sqrt{\frac{2}{T}}$ cos $n\omega_0 t$ and $\sqrt{\frac{2}{T}}$ cos $m\omega_0 t$ where m nd n are non-zero integers and $m \neq n$

$$\left(\sqrt{\frac{2}{T}} \cos n\omega_0 t, \sqrt{\frac{2}{T}} \cos m\omega_0 t\right) = \frac{2}{T} \int_{-T/2}^{T/2} \cos n\omega_0 t \cdot \cos m\omega_0 t dt$$

$$= \frac{2}{T} \int_{-T/2}^{T/2} \frac{1}{2}\left[\cos (n+m)\omega_0 t + \cos(n-m)\omega_0 t\right] dt$$

Since n and m are both integers and are not equal,
let $n + m = p$ and $(n - m) = q$. Obviously, $p \neq 0$ and $q \neq 0$ and both p and q are integers.

$$\left(\sqrt{\frac{2}{T}} \cos n\omega_0 t, \sqrt{\frac{2}{T}} \cos m\omega_0 t\right) = \frac{1}{T} \int_{-T/2}^{T/2} (\cos p\omega_0 t + \cos q\omega_0 t) dt = 0$$

(iv) Consider $\sqrt{\frac{2}{T}}$ cos $n\omega_0 t$ and $\sqrt{\frac{2}{T}}$ sin $m\omega_0 t$ where m and n are any two integers and not equal to zero.

$$\left(\sqrt{\frac{2}{T}} \cos n\omega_0 t, \sqrt{\frac{2}{T}} \sin m\omega_0 t\right) = \frac{2}{T} \int_{-T/2}^{T/2} \cos n\omega_0 t \cdot \sin m\omega_0 t dt$$

$$= \frac{2}{T} \int_{-T/2}^{T/2} \frac{1}{2}\left[\sin (m+n)\omega_0 t + \sin(m-n)\omega_0 t\right] dt = 0$$

Hence any two distinct functions of the set are orthogonal and all the functions have unit norm. Hence the given set is an orthonormal set.

We shall now discuss an important theorem which states that any set of orthogonal vectors that does not include the zero vector, will be a linearly independent set.

Theorem *Any set of orthogonal vectors, finite or infinite, which does not include the zero vector, is a linearly independent set.*

Proof Let V be an inner product space and let S be a finite or infinite set of non-zero orthogonal vectors in V. Suppose $x_1, x_2, x_3, \ldots, x_n$ are distinct vectors in S. Also let

$$y = c_1 \cdot x_1 + c_2 \cdot x_2 + c_3 \cdot x_3 + \ldots + c_k \cdot x_k + \ldots + c_n \cdot x_n \qquad \ldots (3.9)$$

Then taking the inner product of y with x_k, we get

$$(y, x_k) = (c_1 \cdot (x_1, x_k)) + (c_2 \cdot (x_2, x_k)) + \ldots + (c_k \cdot (x_k, x_k)) + \ldots + (c_n \cdot (x_n, x_k))$$

But $x_1, x_2, \ldots, x_k, \ldots x_n$ are mutually orthogonal.

$$\therefore \quad (x_i, x_j) = \begin{cases} 0 \text{ if } i \neq j \\ \|x_i\|^2 \text{ if } j = i \end{cases}$$

$$\therefore \quad (y, x_k) = c_k \|x_k\|^2$$

Since x_k is not the zero vector, we may write

$$\boxed{c_k = \frac{(y, x_k)}{\|x_k\|^2}; 1 \leq k \leq n} \qquad \ldots (3.10)$$

From the above equation, it is clear that if $y = \mathbf{0}$, i.e., if the linear combination of vectors $x_1, x_2, \ldots, x_n$ is equal to the zero vector then $c_k = 0$, for all k. Substituting the c_k in Eq. (3.9) using (3.10), we get

$$y = \frac{(y, x_1)}{\|x_1\|^2} \cdot x_1 + \frac{(y, x_2)}{\|x_2\|^2} \cdot x_2 + \ldots + \frac{(y, x_k)}{\|x_k\|^2} \cdot x_k + \ldots + \frac{(y, x_n)}{\|x_n\|} \cdot x_n \qquad \ldots (3.11)$$

Remark

1. *From Eq. (3.11) it follows that in case a vector* $\mathbf{y}$ *is expanded as an orthogonal expansion using orthogonal vectors* $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ *then*

   Component of $\mathbf{y}$ *along* $\mathbf{x_k}$ *is* $\dfrac{(y, x_k)}{\|x_k\|^2} \cdot x_k$ $\qquad \ldots (3.12)$

   And the coordinate of $\mathbf{y}$ *along* $\mathbf{x_k}$ *is* $\dfrac{(y, x_k)}{\|x_k\|^2}$ $\qquad \ldots (3.13)$

2. *In case* $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ *are not just orthogonal but orthonormal vectors, then Eq. (3.11) gets modified as*

$$y = (y, x_1).x_1 + (y, x_2).x_2 + \ldots (y, x_n).x_n \qquad \ldots (3.14)$$

   so that the

   component of $\mathbf{y}$ ***along*** $\mathbf{x_k}$ ***is*** $(y, x_k) \cdot x_k$ $\qquad \ldots (3.15)$

   and coordinate of $\mathbf{y}$ ***along*** $\mathbf{x_k}$ ***is*** (y, x_k) $\qquad \ldots (3.16)$

Example 3.11 *(University Examination Question)*

(a) *Show that the three functions given below are pairwise orthogonal over the interval [–2, 2].*

(b) *Determine the value of the constant A that makes the set of functions an orthonormal set.*

(c) *Express the waveform* $x(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq 2 \\ 0 \text{ otherwise} \end{cases}$ *in terms of the orthonormal set obtained in Part (b).*

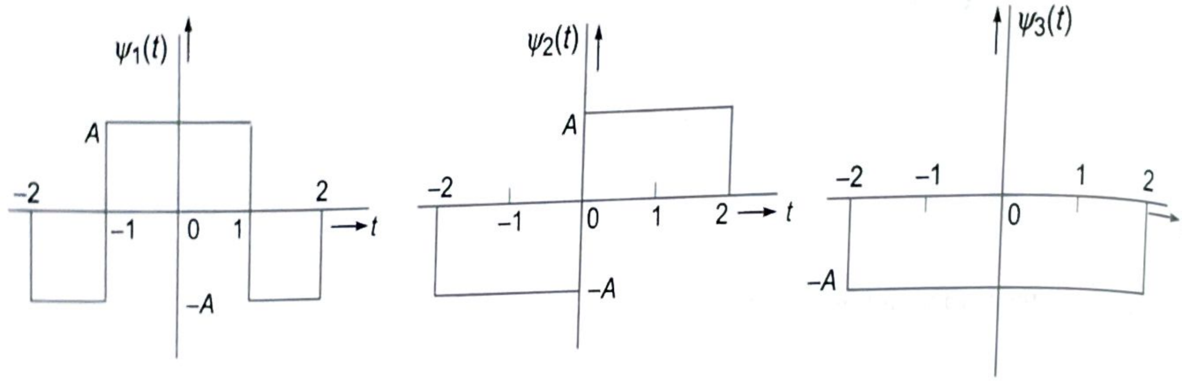


Fig. 3.3 (a) *Signals of Example 3.11*

Solution

(a) To show that the three functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are pairwise orthogonal, we have to show that pairwise, their inner products are zero, i.e.,

$$(\psi_1(t), \psi_2(t)) = (\psi_2(t), \psi_3(t)) = (\psi_3(t), \psi_1(t)) = 0$$

Now,

$$(\psi_1(t), \psi_2(t)) = \int_{-2}^{2} \psi_1(t), \psi_2(t)\,dt = \int_{-2}^{-1} \psi_1(t), \psi_2(t)\,dt + \int_{-1}^{0} \psi_1(t), \psi_2(t)\,dt + \int_{0}^{1} \psi_1(t), \psi_2(t)\,dt$$

$$+ \int_{1}^{2} \psi_1(t), \psi_2(t)\,dt$$

$$= \int_{-2}^{-1} (-A)(-A)\,dt + \int_{-1}^{0} (A)(-A)\,dt + \int_{0}^{1} (A)(A)\,dt + \int_{1}^{2} (-A)(A)\,dt = 0$$

In a similar way, it can be shown that $(\psi_2(t), \psi_3(t))$ and $(\psi_3(t), \psi_1(t))$ are each equal to zero. Hence. $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are pairwise orthogonal.

(b) If $\psi_i(t)$ is normalized then $\|\psi_i(t)\|^2 = 1 = (\psi_i(t)\ \psi_i(t))$, for $i = 1, 2$ and 3.

Taking, $(\psi_1(t), \psi_1(t)) = \|\psi_i(t)\|^2$, it is equal to

$$\int_{-2}^{2} \psi_1^2(t)\,dt = \int_{-2}^{-1} \psi_1^2(t)\,dt + \int_{-1}^{0} \psi_1^2(t)\,dt + \int_{0}^{1} \psi_1^2(t)\,dt + \int_{1}^{2} \psi_1^2(t)\,dt = 4\mathrm{A}^2$$

If $\psi_1(t)$ is normalized, this should be unity.

$\therefore$ $4A^2 = 1$ or $A = \pm 1/2$. Let us take it as 1/2.

$\therefore$ a value of $A = 1/2$ will normalize $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$, and make them orthonormal functions.

(c) $x(t) = \begin{cases} 1 \text{ for } 0 \le t \le 2 \\ 0 \text{ otherwise} \end{cases}$

Therefore $x(t)$ is as shown in Fig. 3.3.

Since $x(t)$ is to be expressed in terms of $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$, let

$$x(t) = K_1\,\tilde{\psi}_1(t) + K_2\,\tilde{\psi}_1(t) + K_3\,\tilde{\psi}_3(t)$$

Where $\tilde{\psi}_1(t)$, $\tilde{\psi}_2(t)$ and $\tilde{\psi}_3(t)$ are normalized functions, i.e., $A = 1/2$.

Then from Eq. (3.12),

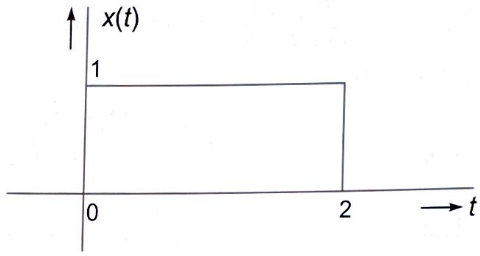


Fig. 3.3(b) *x(t) for Example 3.11*

$$K_1 = (x(t),\ \tilde{\psi}_1(t)) = \int_{-2}^{2} x(t),\ \tilde{\psi}_1(t)\,dt = \int_0^1 \frac{1}{2}\,dt + \int_1^2 \left(-\frac{1}{2}\right)dt = 0$$

$$K_2 = (x(t),\ \tilde{\psi}_2(t)) = \int_0^2 \left(\frac{1}{2}\right).1\,dt = 1$$

and

$$K_3 = (x(t),\ \tilde{\psi}_3(t)) = \int_0^2 \left(-\frac{1}{2}\right).1\,dt = -1$$

$\therefore$

$$x(t) = 0\,.\ \tilde{\psi}_1(t) + 1\,.\ \tilde{\psi}_2(t) + (-1)\ \tilde{\psi}_3(t)$$

where, $\tilde{\psi}_1(t)$, $\tilde{\psi}_2(t)$ and $\tilde{\psi}_3(t)$ are normalized functions (i.e., $A = 1/2$)

3.8 GRAM–SCHMIDT ORTHOGONALIZATION

Given n linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ in an inner product space V, this procedure enables us to derive from them a set $y_1, y_2, \ldots, y_n$ of orthogonal or orthonormal vectors in V.

Procedure Let $y_1 = \boldsymbol{x}_1$. Then y_2 is obtained from $\boldsymbol{x}_2$ by subtracting out from it the component of y_2 along y_1.

$\therefore$

$$y_2 = x_2 - \frac{(y_2, y_1)}{\|y_1\|^2}\cdot y_1 \qquad \text{(Refer to Eq. (3.12))}$$

$\therefore$

$$y_3 = x_3 - \sum_{k=1}^{2} \frac{(x_3, y_k)}{\|y_k\|^2}\cdot y_k\,;$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

In general,

$$y_{m+1} = x_{m+1} - \sum_{k=1}^{m} \frac{(x_{m+1}, y_k)}{\|y_k\|^2}\cdot y_k$$

Example 3.12 *From the vectors (3, 0, 4),(–1, 0, 7) and (2, 9, 11) in R^3, obtain an orthonormal set.*

Solution

$$x_1 = (3, 0, 4);\ x_2 = (-1, 0, 7) \text{ and } x_3 = (2, 9, 11)$$

$\therefore$ let
$$y_1 = x_1 = (3, 0, 4)$$

Then,
$$y_2 = x_2 - \frac{(x_2, y_1)}{\|y_1\|^2} \cdot y_1$$

$$= (-1,0,7) - \frac{(-3+0+28)}{(3^2+0^2+4^2)} .(3,0,4) = (-4,0,3)$$

$$y_3 = x_3 - \frac{(x_3, y_1)}{\|y_1\|^2} \cdot x_1 - \frac{(x_3, y_2)}{\|y_2\|^2} \cdot y_2$$

$$= (2,9,11) - \frac{(6+0+44)}{(3^2+0^2+4^2)} \cdot (3,0,4) - \frac{(-8+0+33)}{(4^2+0^2+3^2)} \cdot (-4,0,3)$$

$$= (2,9,11) - (6,0,8) - (-4, 0, 3)$$

$\therefore$
$$y_3 = (0,9,0)$$

To normalize y_1, y_2 and y_3 we divide each of these with its norm.

$\therefore$
$$\hat{y}_1 = \frac{(3,0,4)}{\sqrt{25}} = \left(\frac{3}{5}, 0, \frac{4}{5}\right);\ \hat{y}_2 = \frac{(-4,0,3)}{\sqrt{25}} = \left(-\frac{4}{5}, 0, \frac{3}{5}\right) \text{ and } \hat{y}_3 = \frac{(0,9,0)}{9} = (0,1,0)$$

Example 3.13 *Given the three signals $x_1(t)$, $x_2(t)$ and $x_3(t)$ as shown in Fig. 3.4 derive an orthonormal basis signal set for the signal space generated by them.*

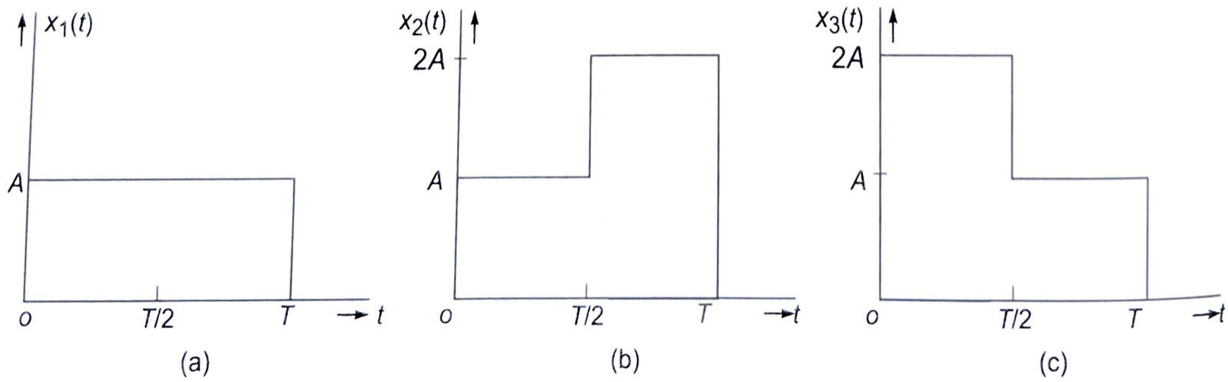


Fig. 3.4 *The given signals $x_1(t)$, $x_2(t)$ and $x_3(t)$*

Solution

We shall make use of Gram–Schmidt procedure to obtain an orthonormal set of signals from the given three signals.

First, let us define $s_1(t) = x_1(t)$.

Then,

$$s_2(t) = x_2(t) - \frac{(x_2(t), s_1(t))}{\|s_1(t)\|^2} . s_1(t)$$

$$(x_2(t), s_1(t)) = (x_2(t), x_1(t)) = \int_0^T x_2(t)\, x_1(t)\, dt = \int_0^{T/2} A^2\, dt + \int_{T/2}^T 2A^2\, dt = \frac{3}{2} A^2 T$$

$$\|s_1(t)\|^2 = \|x_1(t)\|^2 = (x_1(t), x_1(t)) = \int_0^T A^2\, dt = A^2 T$$

$$\therefore \quad s_2(t) = x_2(t) - \left(\frac{3}{2} A^2 T . \frac{1}{A^2 T}\right) x_1(t) = x_2(t) - \frac{3}{2}\, x_1(t)$$

A plot of $s_2(t)$ is shown in Fig. 3.5.

$$s_3(t) = x_3(t) - \frac{(x_3(t), x_1(t))}{\|x_1(t)\|^2}\, x_1(t) - \frac{(x_3(t), s_2(t))}{\|s_2(t)\|^2}\, s_2(t)$$

$$(x_3(t), x_1(t)) = \int_0^T x_3(t)\, x_1(t)\, dt = \int_0^{T/2} 2A^2\, dt + \int_{T/2}^T A^2\, dt = \frac{3}{2} A^2 T$$

$$(x_3(t), s_2(t)) = \int_0^T x_3(t)\, s_2(t)\, dt = \int_0^{T/2} 2A \times (-0.5A)\, dt$$

$$+ \int_{T/2}^T A \times (0.5A)\, dt = -\frac{0.5}{2} A^2 T$$

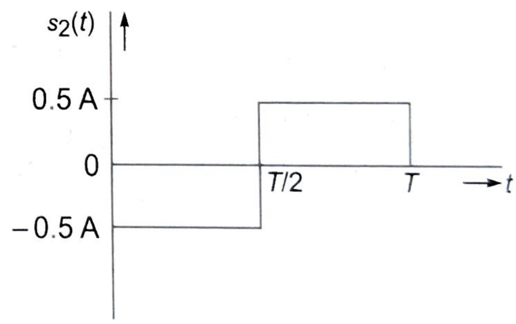


Fig. 3.5 *Signal $s_2(t)$*

$$\|s_2(t)\|^2 = (s_2(t), s_2(t)) = \int_0^{T/2} (-0.5A)(-0.5A)\, dt + \int_{T/2}^T (0.5A)(0.5A)\, dt$$

$$= \frac{0.5A^2T}{2} = 0.25A^2T$$

$$s_3(t) = x_3(t) - \left(\frac{3}{2} A^2 T\right)\left(\frac{1}{A^2 T}\right) x_1(t) - \frac{0.25A^2T}{0.25A^2T} . s_2(t)$$

$$= x_3(t) - \frac{3}{2} x_1(t) - s_2(t)$$

A plot of the signal $\left[\frac{3}{2} x_1(t) + s_2(t)\right]$ is shown in Fig. 3.6(a). We find that it is such that $x_3(t)$ minus this signal gives us what is shown in Fig. 3.6(b). We find that it is equal to $-2\, s_2(t)$, i.e., $s_3(t) = -2\, s_2(t)$. Hence, they are not linearly independent. This is due to the fact that the three given signals, $x_1(t)$, $x_2(t)$ and $x_3(t)$ a themselves not linearly independent. This is obvious from the fact that $x_1(t)$ can be obtained in terms of x_2 and $x_3(t)$.

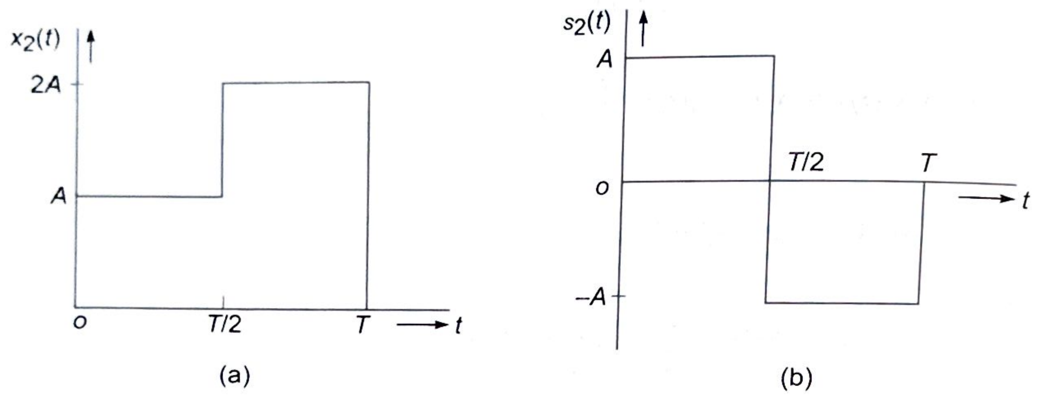


Fig. 3.6 *(a) Signal $x_2(t)$; (b) Signal $s_3(t)$*

Actually,
$$x_1(t) = \frac{1}{3}[x_2(t) + x_3(t)]$$

Hence the signal space generated by $x_1(t)$, $x_2(t)$ and $x_3(t)$ is not three dimensional—it is only two-dimensional and it has the orthogonal signals $s_1(t)$ and $s_2(t)$ as the basis signals. That they are orthogonal is clear from the fact their inner product, viz.,

$$(s_1(t), s_2(t)) = \int_0^{T/2} (-0.5A)A\,dt + \int_{T/2}^{T} (0.5A)A\,dt = 0$$

Now, to normalize them, we have to find their norms

$$||s_1(t)||^2 = (s_1(t), s_1(t)) = \int_0^T s_1^2(t)\,dt = A^2T = \text{energy } E$$

$$||s_2(t)||^2 = (s_2(t), s_2(t)) = \int_0^{T/2} (-0.5A)(-0.5A)dt + \int_{T/2}^{T} (0.5A)(0.5A)dt$$

$$= \frac{0.5A^2T}{2} = 0.25A^2\,T$$

$\therefore$ if $u_1(t)$ and $u_2(t)$ are the normalized versions of $s_1(t)$ and $s_2(t)$,

$$u_1(t) = \frac{s_1(t)}{\sqrt{A^2T}} = \frac{1}{\sqrt{E}}s_1(t) \text{ and } u_2(t) = \frac{s_2(t)}{||s_2(t)||} = \frac{s_2(t)}{\sqrt{0.25A^2T}} = \frac{2s_2(t)}{\sqrt{E}}$$

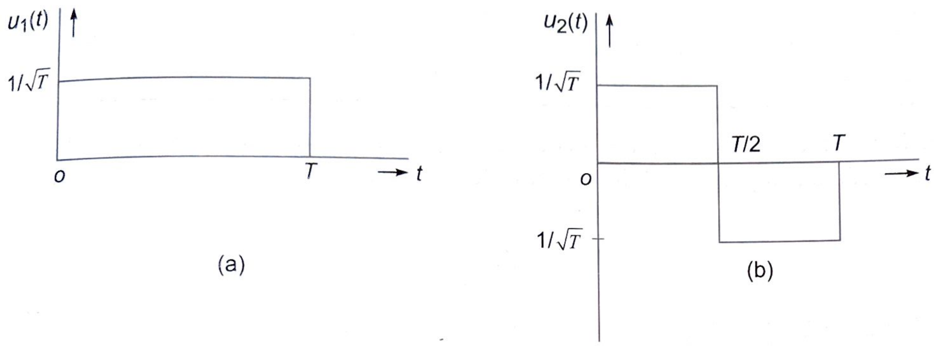


Fig. 3.7 *The orthonormal basis set*

Note that the energy of each of these signals is 1, as it should be.

3.9 THE BEST APPROXIMATION PROBLEM

Consider a vector $\boldsymbol{OA}$ in R^3, with components $\boldsymbol{OC}$, $\boldsymbol{OD}$ and **OE** along the X, Y and Z axes. Suppose we want to have the best approximation to this vector in the XY plane. We know that the best approximation to $\boldsymbol{OA}$ in the XY plane is simply $\boldsymbol{OB}$ which is the projection of $\boldsymbol{OA}$ onto the XY plane, as shown in Fig. 3.8. Since $\boldsymbol{OB} + \boldsymbol{BA} = \boldsymbol{OA}, \boldsymbol{OA} - \boldsymbol{OB} = \boldsymbol{BA}$. We say that $\boldsymbol{OB}$ is the best approximation to $\boldsymbol{OA}$ in the XY plane, because, if we take any other vector $\boldsymbol{OF}$ in the XY plane, $\boldsymbol{OA} - \boldsymbol{OF} = \boldsymbol{FA}$ will have a magnitude greater than that of $\boldsymbol{BA}$. This is because of the fact that $\boldsymbol{AB}$ is orthogonal to the XY plane. Finally, the best approximation $\boldsymbol{OB} = \boldsymbol{OC} + \boldsymbol{OD}$, where $\boldsymbol{OC}$ and $\boldsymbol{OD}$ are the components of $\boldsymbol{OA}$ along the co-ordinate axes for the two-dimensional plane in which the approximation to $\boldsymbol{OA}$ is sought. This problem of obtaining the best approximation to a three-dimensional vector in a two-dimensional plane which is a subspace of R^3, can be generalized and stated as follows:

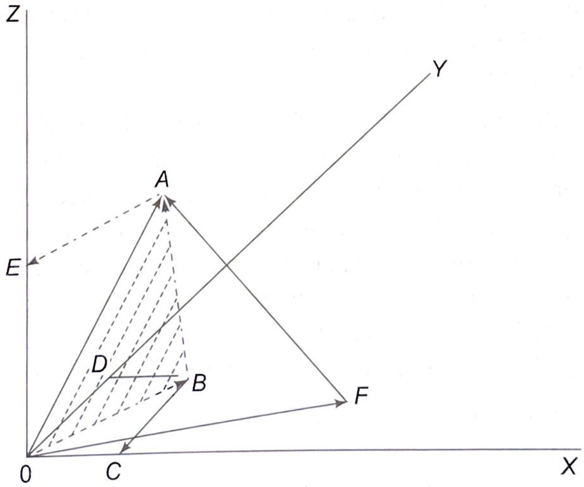


Fig. 3.8 *Best approximation in the xy plane*

Let V be a vector space and let **y** *be a vector in it. The problem is to find the best approximation to* **y** *in W, a subspace of V.*

The solution is also a generalization of the solution to the simpler problem discussed earlier and may be stated as follows.

*The vector **x** in W is a best approximation to the vector **y** in V if and only if (**y** – **x**) is orthogonal to every vector in W and the best approximation, when it exists, is unique.*

Further, if W is finite dimensional, and $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ is an orthogonal basis for W then the vector $\boldsymbol{x}$ is given by the sum of the components of the vector $\boldsymbol{y}$ along each of the basis vectors of the subspace W. Hence,

$$\boldsymbol{x} = \sum_{k=1}^{n} \frac{(\boldsymbol{y}, \boldsymbol{x}_k)}{\|\boldsymbol{x}_k\|^2} \cdot \boldsymbol{x}_k$$

3.9.1 Bessel's Inequality

Let $\{\boldsymbol{x}_k\}$ be a sequence of non-zero orthonormal vectors in an inner product space V. Let $\boldsymbol{y}$ be some vector in V. We then have

$$\sum_{k=1}^{\infty} |(\boldsymbol{y}, \boldsymbol{x}_k)|^2 \le \|\boldsymbol{y}\|^2 \qquad \ldots (3.17)$$

This is called Bessel's inequality. To understand what this inequality means, note that $(\boldsymbol{y}, \boldsymbol{x}_k)\cdot \boldsymbol{x}_k$ represents the component of $\boldsymbol{y}$ along $\boldsymbol{x}_k$. Hence, the inner product of this with itself, viz., $((\boldsymbol{y}, \boldsymbol{x}_k)\cdot \boldsymbol{x}_k, (\boldsymbol{y}, \boldsymbol{x}_k)\cdot \boldsymbol{x}_k)$ represents the norm-square of this component; and is equal to $|(\boldsymbol{y}, \boldsymbol{x}_k)|^2$ since $\boldsymbol{x}_k$ has unit norm. Then what Eq. (3.17) says is that the sum of the squares of the norms of the components of $\boldsymbol{y}$ along the various orthonormal vectors is either equal to, or, less than, the norm-square of the vector $\boldsymbol{y}$. Obviously, the equality sign holds good if vectors $\{\boldsymbol{x}_k\}$, $k = 1,2,\ldots$ span the space or subspace in which $\boldsymbol{y}$ is located; otherwise, the 'less than' sign holds.

Since the upper limit for the summation in Eq. (3.17) is infinity, x_k is an infinite sequence of non-zero orthonormal vectors, and the space spanned by these x_k's is therefore an infinite dimensional one. One may then wonder how they can fail to span the space in which the vector y is located. The fact is that even though $\{x_k\}$ sequence spans an infinite dimensional space, the vector y may be either completely outside this infinite dimensional space, or may have a component outside it. Example 3.7 illustrates a case where the given $f(t)$ is totally outside the infinite dimensional space spanned by the infinite set of orthogonal signals $\cos nt$; $n = 1, 2, 3, \ldots$. Infact, the given $f(t)$ is orthogonal to the infinite dimensional space spanned by $\cos nt$; $n = 1, 2, 3, \ldots$. In this case, the left hand side of Bessel's inequality given in Eq. (3.17) is just zero since $(y, x_k) = 0$ for every $x_k = 1, 2, 3, \ldots$.

3.10 SEQUENCES, CONVERGENCE AND LIMITS

1. Metric—Definition and examples A metric on a set is a measure of the distance or, the difference between two elements of the set. If x and y are two elements of a set X, the distance between the elements is represented by $d(x,y)$

Examples

1. In R, the set of real numbers, if k and l are any two elements of R,

$$d(k,l) \triangleq |k - l| \text{ for } \forall\, k,l \in R \qquad \ldots (3.18)$$

2. In an inner product space, the norm defined by the inner product may be used as the metric.

$$d(k,l) \triangleq \|x - y\| = (x - y, x - y)^{1/2} \qquad \ldots (3.19)$$

2. Metric spaces A set, together with a metric defined on it, is called a metric space.

3. Convergent sequence A sequence x_n in a metric space X is said to be convergent if there exists an $x \in X$ such that $\lim\limits_{n \to \infty} d(x_n, x) = 0$.

The sequence x_n is said to converge to the limit x, or $\lim\limits_{n \to \infty} x_n = x$

Example Consider the sequence $x_n = 1/n$; $n = 1, 2, \ldots$ This sequence is a convergent sequence in R (as the limit to which the sequence is tending to converge, viz, '0', is in R); but not in the metric space of all positive real numbers.

4. Cauchy sequences and complete space

(a) A sequence x_n in a metric space X is said to be a Cauchy sequence, if for every $\in > 0$, there is an $N = N(\in)$ such that $d(x_m, x_n) < \in$ for every $m, n > N$.

It may be noted here that while every convergent sequence must be a Cauchy sequence, every Cauchy sequence may not converge. This is because, whether a Cauchy sequence converges or not will depend on whether or not, the limit to which it is trying to converge is an element in that metric space on which the sequence is defined.

(b) A space X is said to be a complete space if every Cauchy sequence in it coverges.

Example The real line R is a complete metric space.

5. Hilbert space An inner product space which is complete is called a Hilbert Space.

Example The inner product space of all continuous, complex-valued functions with a finite energy in the interval $[-T/2, T/2]$ and with the iner product of any $x(t), y(t) \in V$ defined by

$$(x(t), y(t)) \triangleq \int_{-T/2}^{T/2} x(t) \cdot \overline{y(t)} dt \qquad \ldots (3.20)$$

3.11 COMPLETE ORTHONORMAL BASIS SETS

Let V be an inner product space and let $\{x_k\}$ be an orthonormal sequence of vectors in V. Then the sequence $\{x_k\}$ is said to be a complete orthonormal sequence if in Eq. (3.17) the equality sign holds good for every vector $y \in V$. In other words, $\{x_k\}$ is a complete orthonormal (CON) sequence provided *every* vector in V can be expressed as a linear combination of x_k's; i.e., if $\{x_k\}$ sequence of vectors spans V.

3.11.1 Examples of CON basis sets

1. Consider the inner product space V of all real-valued signals $x(t)$ which are continuous and are having a finite energy in the interval $[-T/2, T/2]$. If $x(t)$ and $y(t)$ belong to this space, let their inner product be defined as

$$(x(t), y(t)) \triangleq \int_{-T/2}^{T/2} x(t) y(t) dt$$

Now, consider the following two sets of infinite sequence of vectors belonging to V.

$$\frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}}\cos\omega_0 t, \sqrt{\frac{2}{T}}\cos 2\omega_0 t, \sqrt{\frac{2}{T}}\cos 3\omega_0 t, \ldots$$

$$\text{and } \sqrt{\frac{2}{T}}\sin\omega_0 t, \sqrt{\frac{2}{T}}\sin 2\omega_0 t, \sqrt{\frac{2}{T}}\sin 3\omega_0 t, \ldots$$

where $$\omega_0 \triangleq \frac{2\pi}{T}$$

Each one of the above two sequences has been shown, in Example 3.10, to form an infinite set of ortho normal functions, since in each set every function has a unit norm (positive square root of the inner product of the function with itself) and the inner product of any two distinct functions is zero. However, each set, taken separately, does not constitute a CON. This can easily be seen from the fact that a function like $x(t) = \sin\omega_0 t \in V$ cannot be expressed as a linear combination of the functions in the first set. Similarly, a function like $x(t) = \cos\omega_0 t$ which belongs to V cannot be expressed as a linear combination of the vectors in the second set. In fact, the two infinite dimensional spaces spanned by the two sets of infinite sequence of vectors are orthogonal to each other in the sense that every vector in one space is orthogonal to all the vectors in the other space. However, the two together constitute a CON and span V.

2. In $L_2(T)$, the space of all continuous, complex-valued functions defined over the interval $[-T/2, T/2]$, and having a finite energy over that interval, the sequence

$$x_n(t) = \frac{1}{\sqrt{T}} e^{jn\omega_0 t};\ n = 0, \pm 1, \pm 2, \ldots \text{ with } \omega_0 \triangleq \frac{2\pi}{T}$$

forms a CON with respect to the inner product

$$(x(t), y(t)) \triangleq \int_{-T/2}^{T/2} x(t)\,\overline{y(t)}\,dt$$

where $\overline{y(t)}$ denotes the complex conjugate of $y(t)$.

3.12 COMPLEX-EXPONENTIAL FOURIER SERIES

As we had already discussed, the sequence $\left\{\frac{1}{\sqrt{T}} e^{jn\omega_0 t}\right\}$, $n = 0, \pm 1, \pm 2, \ldots$ is a complete orthonormal sequence (CON) and forms a basis for the space of all continuous, complex-valued functions defined over the interval $[-T/2, T/2]$ and having a finite energy over that interval. Hence, if $x(t)$ belongs to this space, we can write it as a linear combination of this basis set.

$$\therefore \quad x(t) = \frac{1}{\sqrt{T}} \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_0 t};\ f_0 \triangleq \frac{1}{T};\ -\frac{T}{2} \le t \le \frac{T}{2} \qquad \ldots (3.21)$$

where, c_n's are the coefficients of the orthonormal expansion and are given by [refer to Eq. (3.16)]

$$c_n = \left(x(t), \frac{1}{\sqrt{T}} e^{j2\pi n f_0 t}\right) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} x(t) e^{-j2\pi n f_0 t}\,dt \qquad \ldots (3.22)$$

c_n's are generally known as *Fourier series coefficients*. The normalization factors $\frac{1}{\sqrt{T}}$ occurring on the right-hand side of Eqs. (3.21) and (3.22) can as well be combined to write these equations in a more convenient form as

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_0 t};\ f_0 \triangleq \frac{1}{T};\quad \frac{T}{2} \le t \le \frac{T}{2}} \qquad \ldots (3.23)$$

where the c_n's are given by

$$\boxed{c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi n f_0 t}\,dt} \qquad \ldots (3.24)$$

The reader might have observed that Eq. (3.23) implies that we have expanded $x(t)$ over the interval $[-T/2, T/2]$ using not the orthonormal set but only the orthogonal set $\{e^{j2\pi n f_0 t}\}$, $n = 0, \pm 1, \pm 2\ldots$ Hence, the coefficients of the linear combination, in this case, are given by (see Eq. (3.13))

$$c_n = \frac{(x(t), e^{j2\pi n f_0 t})}{\left\| e^{j2\pi n f_0 t}\right\|^2}. \qquad \ldots (3.25)$$

But

$$\left\| e^{j2\pi n f_0 t}\right\|^2 = \left(e^{j2\pi n f_0 t}, e^{j2\pi n f_0 t}\right)$$

$$= \int_{-T/2}^{T/2} e^{j2\pi n f_0 t}\, e^{-j2\pi n f_0 t}\,dt = T$$

$$\therefore \quad c_n = \frac{1}{T}(x(t), e^{+j2\pi n f_0 t}) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi n f_0 t}\,dt$$

This is precisely what we had obtained in Eq. (3.24). Thus, the complex exponential Fourier Series expansion of $x(t)$ over the interval $[-T/2, T/2]$ is generally given by Eq. (3.23) and the coefficients are given by Eq. (3.24). ***Note that the expansion of x(t) given by Eq. (3.23) is valid only over the interval [–T/2, T/2]. However, if x(t) is a periodic signal with a period T and is having a finite energy over a period, the expansion given in Eq. (3.23) is valid for all time.***

Hence, we generally express the complex-exponential Fourier series expansion as follows:

Let $x(t)$ be a periodic signal with a period T. Then we may write

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_0 t};\ f_0 \triangleq \frac{1}{T};\ -\infty \le t \le \infty} \qquad \ldots (3.26)$$

where c_n is given by

$$\boxed{c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi n f_0 t}\,dt} \qquad \ldots (3.27)$$

The above equation for c_n is generally referred to as Euler's formula.

Example 3.14 *If x(t) is a purely real-valued function, show that its complex-exponential Fourier series coefficients will have Hermitian symmetry.*

Solution

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi nf_0 t}\,dt$$

$$\therefore\ c_n^* = \text{complex conjugate of } c_n = \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)e^{j2\pi nf_0 t}\,dt$$

But $x(t)$, being a real-valued function, $x^*(t) = x(t)$

$$c_n^* = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi(-n)f_0 t}\,dt = c_{-n}$$

$$\therefore \qquad |c_{-n}| = |c_n^*| = |c_n| \quad \therefore \quad |c_{-n}| = |c_n|$$

and

$$\angle c_{-n} = \angle c_n^* = -\angle c_n \quad \therefore \quad \angle c_{-n} = -\angle c_n$$

$\therefore$ magnitude has even symmetry while the phase has odd symmetry.

3.12.1 Magnitude and Phase Spectra of Periodic Signals

A plot of $|c_n|$, the magnitude of c_n vs nf_0 or f is called the magnitude spectrum of $x(t)$ while a plot of $\angle c_n$ vs nf_0 or f is called the phase spectrum of the periodic signal $x(t)$. From Example 3.14, it is clear that the magnitude spectrum of a real-valued signal will have even symmetry, while its phase spectrum will have odd symmetry.

The term 'spectrum' is used in the above because of the fact that the complex-exponential Fourier series expansion of a periodic signal $x(t)$ is, indeed, a frequency domain representation of $x(t)$.

From Eq. (3.26), we find that $x(t)$ is expressed as a linear combination of the complex exponentials $e^{j2\pi nf_0 t}$, with frequencies nf_0, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ where f_0 is the fundamental frequency $= 1/T$. The coefficients of the linear combination are c_ns. As c_ns are in general, complex numbers, let us say $c_n = |c_n|\, e^{j\theta(n)}$ so that component of $x(t)$ along the complex exponential $e^{j2\pi nf_0 t}$ is given by $x_n(t) = |c_n|e^{j(2\pi nf_0 t + \theta(n))}$, i.e., this component has a magnitude of $|c_n|$, frequency of nf_0 and a phase angle of $\theta(n)$.

Thus, $x(t)$ is represented by the Fourier series as a linear combination of an infinite number of complex exponentials of positive and negative frequencies, the frequencies being harmonically related. Thus, the spectrum of a periodic function $x(t)$ with a period T consists of the zero frequency or DC component and components at $+f_0(=1/T)$ and $-f_0$, $+2f_0$ and $-2f_0$, $+3f_0$ and $-3f_0$, . . .

Thus, the spectrum of a continuous time periodic signal is a discrete one.

Example 3.15 *x(t) is a periodic signal and is shown in Fig. 3.9. Find its complex-exponential Fourier series expansion and plot its magnitude and phase spectra.*

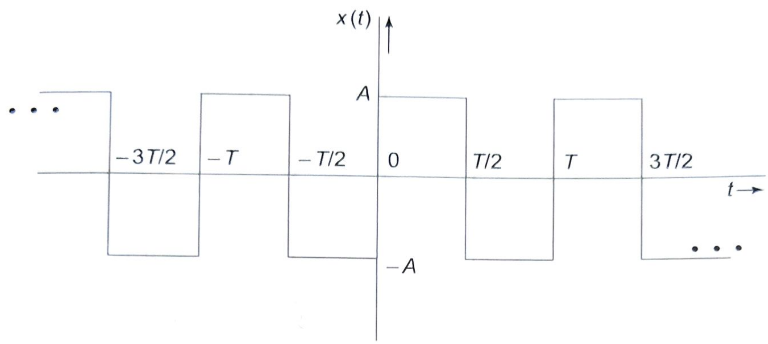


Fig. 3.9 *Signal of Example 3.15*

Solution

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = 0$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi nf_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{0}(-A)e^{-j2\pi nf_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2}(A)e^{-j2\pi nf_0 t}\,dt$$

$$= \frac{-A}{T}\int_{-T/2}^{0} e^{-j2\pi nf_0 t}\,dt + \frac{A}{T}\int_{0}^{T/2} e^{-j2\pi nf_0 t}\,dt$$

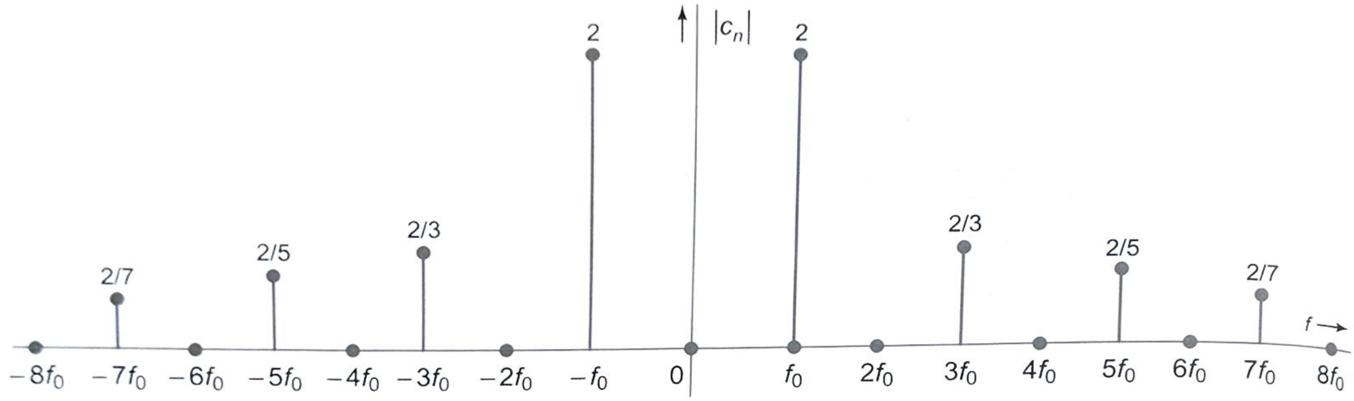
$$= \frac{-A}{T}\left.\frac{e^{-j2\pi nf_0 t}}{-j2\pi nf_0}\right|_{-T/2}^{0} + \frac{A}{T}\left.\frac{e^{-j2\pi nf_0 t}}{-j2\pi nf_0}\right|_{0}^{T/2} = \frac{A}{j2\pi n}\left[1-e^{j\pi n}\right] + \frac{A}{j2\pi n}\left[1-e^{-j\pi n}\right]$$

$$= \frac{A}{j2\pi n}\left[2-\left(e^{j\pi n}+e^{-j\pi n}\right)\right] = \frac{A}{j\pi n}\left[1-\cos\pi n\right]$$

But $\cos n\pi = \begin{cases} 1 \text{ for } n \text{ even} \\ -1 \text{ for } n \text{ odd} \end{cases}$

$$\therefore \quad c_n = \begin{cases} \dfrac{2A}{j\pi n} \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even} \end{cases}$$

For the purpose of plotting the magnitude and phase spectra of $x(t)$, we shall assume $A = \pi$. The magnitude and phase spectra are plotted in Fig 3.10.



(a) *Magnitude spectrum*

(b) *Phase spectrum*

Fig. 3.10

Example 3.16 *Determine the complex-exponential Fourier series expansion of the periodic signal shown in Fig. 3.11.*

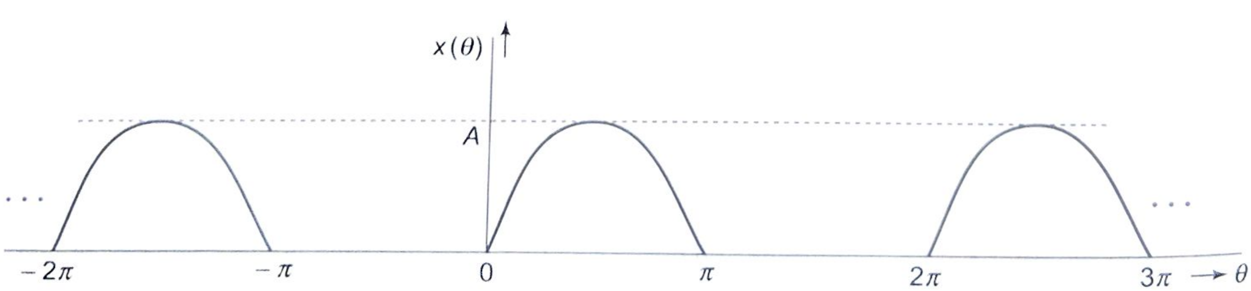


Fig. 3.11

Solution

$$x(\theta) = \begin{cases} A\sin\theta;\ 0 \le \theta \le \pi \\ 0;\ \pi \le \theta \le 2\pi \end{cases}$$

The complex-exponential Fourier series expansion for the given signal may be written as

$$x(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta};\ -\infty < \theta < \infty$$

where
$$c_n = \frac{A}{2\pi}\int_0^{\pi} \sin\theta e^{-jn\theta}\, d\theta$$

Since
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j},\text{ we have,}$$

$$c_n = \frac{A}{2\pi}\int_0^{\pi}\left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)e^{-jn\theta}\,d\theta = \frac{A}{4\pi j}\int_0^{\pi}\left[e^{j(\theta-n\theta)} - e^{-j(\theta+n\theta)}\right]d\theta$$

$$= \frac{A}{4\pi j}\left[\frac{e^{j(\theta-n\theta)}}{j(1-n)} + \frac{e^{-j(\theta+n\theta)}}{j(1+n)}\right]_0^{\pi}$$

$$= \left.\frac{-Ae^{j(1-n)\theta}}{4\pi(1-n)}\right|_0^{\pi} + \left.\frac{-Ae^{-j(1+n)\theta}}{4\pi(1+n)}\right|_0^{\pi}$$

$$= \frac{-A}{4\pi(1-n)}e^{j\pi(1-n)} + \frac{A}{4\pi(1-n)} - \frac{A}{4\pi(1+n)}e^{-j\pi(1+n)} + \frac{A}{4\pi(1+n)}$$

$$= \frac{A}{4\pi(n-1)}\left[e^{j\pi(1-n)} - 1\right] - \frac{A}{4\pi(1+n)}\left[e^{-j\pi(1+n)} - 1\right]$$

For n odd $n \ne 1$ or -1: $(1-n)$ and $(1+n)$ will both be even and hence $e^{j\pi(1-n)}$ and $e^{-j\pi(1+n)}$ will both be equal to 1.

$\therefore$
$$\underset{n\ odd}{c_n} = 0$$

For n even: $(1-n)$ and $(1+n)$ will both be odd and hence $e^{j\pi(1-n)}$ and $e^{-j\pi(1+n)}$ will both be equal to -1.

$\therefore$
$$\underset{n\ even}{c_n} = \frac{-2A}{4\pi(n-1)} + \frac{2A}{4\pi(1+n)} = \frac{A}{\pi(1-n^2)}$$

$\therefore$ for
$$n = 0: \quad c_0 = \frac{A}{\pi}$$

For $n = 1$: The second term reduces to zero but the first term takes the form of zero divided by zero. Hence, applying L'Hospital's rule to the first term, we get

$$c_1 = \frac{A}{4j}$$

For $n = -1$: The first term takes the value zero but the second term takes the form of zero by zero. Applying L'Hospital's rule to the second term, we get

$$c_{-1} = \frac{A}{4j}$$

Hence, the complex-exponential Fourier series expansion of the given waveform is

$$x(\theta) = \frac{A}{2}\sin\theta - \sum_{n=-\infty}^{\infty} \frac{A}{\pi(1-n^2)} e^{jn\theta}$$

Certain Useful Features of Fourier Series As an orthogonal expansion, the Fourier series expansion of a periodic signal has certain very useful features.

1. In general, the Fourier series expansion of a periodic signal $x(t)$ involves an infinite number of Fourier series coefficients, i.e., c_n's. However, if $x(t)$ is a bandlimited periodic signal, the number of coefficients will be finite.
2. When the Fourier series expansion of a periodic signal $x(t)$ involves the summation of an infinite number of terms, as happens in the case of a non-bandlimited periodic signal, we may try to approximate $x(t)$ by $x_N(t)$ where

   $$x_N(t) \triangleq \sum_{n=-N}^{N} c_n e^{j2\pi n f_0 t} \text{ ; where } N \text{ is an integer}$$

   In that case, the approximation becomes better and better as N tends to infinity. It may be noted that $x_N(t)$ as defined above, is in fact, the projection of $x(t)$, a vector in an infinite dimensional space, onto an N-dimensional subspace of that infinite dimensional space.
3. In case we wish to get a better approximation than the one given by $x_N(t)$, we may consider $x_M(t)$ where, M is an integer larger than N. In such a case, we need not again evaluate the first $(2N+1)$ Fourier series coefficients. They remain unaltered and we need to evaluate only $2(M-N)$ coefficients.
4. For any specified positive integer N, the signal

   $$x_N(t) \triangleq \sum_{n=-N}^{N} c_n e^{j2\pi n f_0 t}$$

   represents the best possible approximation of $x(t)$ in the sense that the error signal

   $$e_N(t) \triangleq [x(t) - x_N(t)]$$

   has the minimum energy over the time interval T, which is the period of $x(t)$. This is because, as discussed in Section 3.9 (the best approximation problem), $e_N(t)$, the error signal is orthogonal to the N-dimensional space in which $x_N(t)$ is located and so is orthogonal to $x_N(t)$ itself. Hence, its norm (i.e., positive square root of its energy) is minimum.

3.13 PROPERTIES OF COMPLEX-EXPONENTIAL FOURIER SERIES

Having derived the complex-exponential Fourier series, we shall now examine some of the important properties of these Fourier series coefficients. In what follows, we will be using the notation $x(t) \xleftrightarrow{FS} c_n^x$ frequently. This should be read as $x(t)$ has complex-exponential Fourier series, the nth coefficient being c_n^x.

1. **Linearity theorem** Let $x(t)\xleftrightarrow{FS} c_n^x$. Let $y(t)$ have the same fundamental period as $x(t)$, viz., T, and let $y(t)\xleftrightarrow{FS} c_n^y$. Further, if a and b are arbitrary constants, then this theorem states that

$$ax(t)+by(t)\xleftrightarrow{FS} ac_n^x+bc_n^y$$

Proof

$$c_n^x=\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi n f_0 t}\,dt$$

and

$$c_n^y=\frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-j2\pi n f_0 t}\,dt$$

$$\therefore\quad ac_n^x+bc_n^y=\frac{1}{T}\int_{-T/2}^{T/2} ax(t)e^{-j2\pi n f_0 t}\,dt+\frac{1}{T}\int_{-T/2}^{T/2} by(t)e^{-j2\pi n f_0 t}\,dt$$

$$=\frac{1}{T}\int_{-T/2}^{T/2}\left[ax(t)+by(t)\right]e^{-j2\pi n f_0 t}\,dt$$

$$\therefore\quad \boxed{\left[ax(t)+by(t)\right]\xleftrightarrow{FS} ac_n^x+bc_n^y} \qquad \ldots(3.28)$$

2. **Time-shift theorem** If $x(t)\xleftrightarrow{FS} c_n^x$ then for any real constant t_0, $x(t-t_0)\xleftrightarrow{FS} e^{-j2\pi n f_0 t_0}\cdot c_n^x$

Proof Let $y(t)\,\Delta\,x(t-t_0)$. Then,

$$c_n^y=\frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-j2\pi n f_0 t}\,dt=\frac{1}{T}\int_{-T/2}^{T/2} x(t-t_0)e^{-j2\pi n f_0 t}\,dt$$

Put $\quad t-t_0=\tau \quad \therefore\ dt=d\tau$ and $t=(\tau+t_0)$

$$\therefore\quad c_n^y=\frac{1}{T}\int_{-T/2-t_0}^{T/2-t_0} x(\tau)e^{-j2\pi n f_0(\tau+t_0)}\,d\tau=\left[\frac{1}{T}\int_T x(\tau)e^{-j2\pi n f_0\tau}\,d\tau\right]e^{-j2\pi n f_0 t_0}$$

$$\therefore\quad \boxed{c_n^y=e^{-j2\pi n f_0 t_0}c_n^x} \qquad \ldots(3.29)$$

3. **Frequency-shift theorem** If $x(t)\xleftrightarrow{FS} c_n^x$, then $e^{+j2\pi k f_0 t}x(t)\xleftrightarrow{FS} c_{n-k}^x$

Proof If $y(t)\,\Delta\,e^{j2\pi k f_0 t}x(t)$, then,

$$c_n^y=\frac{1}{T}\int_T\left\{e^{j2\pi k f_0 t}x(t)\right\}e^{-j2\pi n f_0 t}\,dt$$

$$=\frac{1}{T}\int_T x(t)e^{-j2\pi(n-k)f_0 t}\,dt=c_{n-k}^x$$

$$\therefore\quad \boxed{e^{j2\pi k f_0 t}x(t)\xleftrightarrow{FS} c_{n-k}^x} \qquad \ldots(3.30)$$

4. Periodic convolution theorem Let $z(t) = x(t) * y(t)$, where $x(t)$ and $y(t)$ are periodic signals with the same period T and $z(t) = \int_{t_o}^{t_o+T} x(t-\tau)\, y(\tau)\, d\tau$

If $x(t) \xleftrightarrow{FS} c_n^x$ and $y(t) \xleftrightarrow{FS} c_n^y$, then this theorem states that $c_n^z = T \cdot c_n^x \cdot c_n^y$

Proof $z(t) = \int_{t_o}^{t_o+T} x(t-\tau) y(\tau) d\tau$ = Periodic convolution of $x(t)$ and $y(t)$

Since $x(t+T) = x(t)$ for all t, from the above, it is evident that $z(t+T) = z(t)$ for all t, i.e., $z(t)$ is periodic with the same period T. Hence, we may write

$$c_n^z = \frac{1}{T}\int_T z(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T}\int_T \left[\int_T x(t-\tau) y(\tau) d\tau\right] e^{-j2\pi n f_0 t} dt$$

$$= \int_T y(\tau)\left[\frac{1}{T}\int_T x(t-\tau) e^{-j2\pi n f_0 t} dt\right] d\tau$$

$$= \int_T y(\tau) c_n^x e^{-j2\pi n f_0 \tau} d\tau = c_n^x \int_T y(\tau) e^{-j2\pi n f_0 \tau} d\tau$$

$$\therefore \quad \boxed{c_n^z = T \cdot c_n^x \cdot c_n^y} \qquad \ldots (3.31)$$

5. Multiplication theorem (*also known as Modulation theorem*) Let $x(t)$ and $y(t)$ be two periodic signals with the same period T and let $z(t) \triangleq x(t) \cdot y(t)$. Then

$$c_n^z = c_n^x * c_n^y = \sum_{k=-\infty}^{\infty} c_k^x c_{n-k}^y$$

Proof

$$c_n^z = \frac{1}{T}\int_T x(t) \,.\, y(t) e^{-j2\pi n f_0 t} dt$$

But

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{j2\pi k f_0 t}$$

$$c_n^z = \frac{1}{T}\int_T \left\{\sum_{k=-\infty}^{\infty} c_k^x e^{j2\pi k f_0 t}\right\} y(t) e^{-j2\pi n f_0 t} dt$$

Interchanging the order of integration and summation

$$c_n^z = \sum_{k=-\infty}^{\infty} c_k^x \left\{\frac{1}{T}\int_T y(t) e^{-j2\pi(n-k) f_0 t} dt\right\}$$

$$= \sum_{k=-\infty}^{\infty} c_k^x c_{n-k}^y$$

$$c_n^z = \sum_{k=-\infty}^{\infty} c_k^x c_{n-k}^y = c_n^x * c_n^y \qquad \ldots (3.32)$$

6. Differentiation theorem $x(t)$ is a periodic signal with period $T = 1/f_0$ and $x(t) \xleftrightarrow{FS} c_n^x$. If $y(t) \triangleq \frac{d}{dt}x(t)$, then this theorem states that

$$c_n^y = j2\pi n f_0 c_n^x$$

Proof

$$y(t) = \frac{d}{dt}x(t) = \frac{d}{dt}\left[\sum_{n=-\infty}^{\infty} c_n^x e^{j2\pi n f_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n^x \frac{d}{dt}\{e^{j2\pi n f_0 t}\} = \sum_{n=-\infty}^{\infty} (c_n^x j2\pi n f_0) e^{j2\pi n f_0 t}$$

Since $y(t)$ is also periodic with the same period T, we may write

$$y(t) = \sum_{n=-\infty}^{\infty} c_n^y e^{j2\pi n f_0 t}$$

But, we have earlier shown that

$$y(t) = \sum_{n=-\infty}^{\infty} (c_n^x j2\pi n f_0) e^{j2\pi n f_0 t}$$

Comparing the RHS of these two equations, we get

$$\boxed{c_n^y = j2\pi n f_0 c_n^x} \qquad \ldots (3.33)$$

7. Scaling theorem Let $x(t)$ be a periodic signal with a period of T seconds; and let $x(t) \xleftrightarrow{FS} c_n^x$. If we now define $y(t) = x(at)$ where a is a real number, this theorem gives us the relation between c_n^x and c_n^y where c_n^y's are the Fourier series coefficients of $y(t)$. The theorem says that $c_n^y = c_n^x$.

Proof We know that $x(at) = y(t)$ is a time-scaled version of $x(t)$ and therefore if $x(t)$ is periodic with a period T, $x(at)$ is also periodic and that its period is (T/a) and fundamental frequency is (af_0) where $f_0 \triangleq \frac{1}{T}$.

$$c_n^y = \frac{1}{(T/a)} \int_{-T/2a}^{T/2a} x(at) e^{-j2\pi n(af_0)t} dt$$

Put $at = \tau$ $\quad\therefore\quad$ $dt = (1/a)d\tau$

Also when $\quad t = -\frac{T}{2a},\ at = \tau = \frac{-T}{2}$ and when $t = \frac{T}{2a}, \tau = \frac{T}{2}$

$$\therefore \quad \boxed{c_n^y = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau) e^{-j2\pi n f_0 \tau} d\tau = c_n^x} \qquad \ldots (3.34)$$

Thus, we find that while the spacing between the spectral components is changed because the fu... mental frequency is now (af_0), the amplitudes of these spectral components remain unchanged.

8. Parseval's theorem This theorem states that the average power of a periodic signal $x(t)$ with a pe... is equal to the sum of the average powers of its components along the various basis functions, vz., e... $n = 0, \pm 1, \pm 2, \ldots$

ie.,
$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \text{average power of } x(t) = \sum_{n=-\infty}^{+\infty} |c_n|^2$$

Proof Average power of $x(t) = P_{av} = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \frac{1}{T}\|x(t)\|^2$

But
$$|x(t)|^2 = x(t)x^*(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \sum_{m=-\infty}^{+\infty} c_m^* e^{-jm\omega_0 t}$$

$$\|x(t)\|^2 = \int_{-T/2}^{T/2}\left[\left\{\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}\right\}\left\{\sum_{m=-\infty}^{+\infty} c_m^* e^{-jm\omega_0 t}\right\}\right] dt = \left[\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\left\{c_n c_m^* \int_{-T/2}^{T/2} e^{j(n-m)\omega_0 t}\right\}\right]$$

But
$$\int_{-T/2}^{T/2} e^{j(n-m)\omega_0 t}\, dt = \begin{cases} 0 \text{ for } m \neq n \\ T \text{ for } m = n \end{cases} = T\delta_{m,n}$$

where $\delta_{m,n} \triangleq$ Kronecker delta $= \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$

$$\therefore \quad P_{av} = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{\|x(t)\|^2}{T} = \left[\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} c_n c_m^* \delta_{n,m}\right]$$

$$\therefore \quad \boxed{\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |c_n|^2 = \text{Average power of } x(t)} \quad \ldots(\text{}$$

Now $|c_n|^2$ is the average power of $c_n e^{j2\pi n f_0 t}$ which is the component of $x(t)$ along the basis function $e^{j2\pi\ldots}$

This is because, the average power of this component $= \frac{1}{T}\int_{-T/2}^{T/2} |c_n e^{j2\pi n f_0 t}|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} |c_n|^2 dt = |c_n|^2$.

$\therefore$ *average power of x(t) = sum of the average powers of its components along the orthogon... functions.*

Remark *As the reader might have observed, this theorem directly follows from Bessel's inequ... for the case where the equality sign holds, since the basis functions, being a complete orthonorm... span the space in which x(t) lies.*

Symmetry Properties

Theorem 1 If $x(t)$ is a real-valued signal and $x(t) \overset{FS}{\longleftrightarrow} c_n$, then $c_{-n} = \overline{c_n}$, where the overbar indicates complex conjugation.

Proof

$$c_{+n} = \frac{1}{T}\int_{(T)} x(t)e^{-j2\pi n f_0 t}dt;\ f_0 \triangleq \frac{1}{T}$$

$$\therefore \quad \overline{c_{-n}} = \overline{\left[\frac{1}{T}\int_{(T)} x(t)e^{j2\pi n f_0 t}dt\right]} = \frac{1}{T}\int_{(T)} x(t)e^{-j2\pi n f_0 t}dt,$$

(since $x(t)$ is real valued, $\overline{x}(t) = x(t)$)

$$\therefore \quad \boxed{\overline{c_{-n}} = c_n, \textit{if } x(t) \text{ is } \textit{real-valued}} \qquad \ldots (3.36)$$

This implies that $|c_{-n}| = |c_n|$ and $\angle c_{-n} = -\angle c_n$

That is, c_n's will have Hermitian symmetry with respect to n.

Theorem 2 If a periodic signal $x(t)$ is purely imaginary and $x(t) \overset{FS}{\longleftrightarrow} c_n$, then $\overline{c_n} = -c_{-n}$.

Proof

$$c_{+n} = \frac{1}{T}\int_{(T)} x(t)e^{-j2\pi n f_0 t}dt$$

$$\therefore \quad \overline{c}_{+n} = \frac{1}{T}\int_{(T)} -x(t)e^{+j2\pi n f_0 t}dt, \text{ since } x(t) \text{ is purely imaginary.}$$

$$= -\left[\frac{1}{T}\int_{(T)} x(t)e^{-j2\pi(-n)f_0 t}dt\right] = -c_{-n}$$

$$\therefore \quad \overline{c_n} = -c_{-n} \textit{ if } x(t) \textit{ is purely imaginary.} \qquad \ldots (3.37)$$

This implies that $|c_n| = |c_{-n}|$ while $\angle c_{-n} = +\angle c_n$

Theorem 3 If a periodic signal $x(t)$ is real and even, c_n's are purely real and even with respect to n.

Proof

$$c_n = \frac{1}{T}\int_{(-T/2)}^{T/2} x(t)e^{-j2\pi n f_0 t}dt$$

$$\therefore \quad \overline{c}_n = \frac{1}{T}\int_{(-T/2)}^{T/2} \overline{x}(t)e^{+j2\pi n f_0 t}dt = \frac{1}{T}\int_{(-T/2)}^{T/2} x(t)e^{+j2\pi n f_0 t}dt = c_{-n}$$

If we now put $t = -\tau$,

$$\overline{c}_n = \frac{-1}{T}\int_{(T/2)}^{-T/2} x(-\tau)e^{-j2\pi n f_0 \tau}d\tau = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)e^{-j2\pi n f_0 \tau}d\tau = c_n$$

$\therefore\ \overline{c}_n = +c_n \quad \therefore$ c_n's are purely real. Further, we know that $c_{-n} = \overline{c}_n$ when $x(t)$ is purely real. $\therefore\ c_{-n} = c_n$.
Hence c_n's are real and have even symmetry.

∴ *c_n's are real and even, if $x(t)$ is real and even.*

Theorem 4 If a periodic signal $x(t)$ is real and odd, and if $x(t) \xleftrightarrow{FS} c_n$, then c_n's are purely imaginary and have odd symmetry.

Proof

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi nf_0t}\,dt$$

∴

$$\bar{c}_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{j2\pi nf_0t}\,dt = c_{-n} \text{ since } x(t) \text{ is real.}$$

However, since $x(t)$ is odd, we may write

$$c_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{j2\pi nf_0t}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(-t)e^{+j2\pi nf_0(-t)}\,dt$$

$$= \frac{-1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi nf_0t}\,dt = -c_n$$

∴

$$\boxed{c_{-n} = \bar{c}_n = -c_n}$$

Hence,

c_n's are purely imaginary and possess odd symmetry with respect to n, if $x(t)$ is real and odd.

Example 3.17 *A periodic signal $x(t)$ with fundamental period T_0 has complex-exponential Fourier series coefficients c_n^x. Express the following signals in terms of c_n^x.* (a) $x(t-t_0)$ (b) $\overline{x(t)}$ (c) $\frac{dx(t)}{dt}$

Solution

(a) Let $y(t) \triangleq x(t-t_0)$ and let c_n^y be the n^{th} complex-exponential Fourier series coefficient of $y(t)$. we know that

$$c_n^y = e^{-j2\pi nf_0t_0}c_n^x$$

∴

$$y(t) = \sum_{n=-\infty}^{\infty} c_n^y e^{j2\pi nf_0t} = \sum_{n=-\infty}^{\infty} c_n^x e^{-j2\pi nf_0t_0}\cdot e^{j2\pi nf_0t}$$

∴

$$y(t) = \sum_{n=-\infty}^{\infty} c_n^x e^{j2\pi nf_0(t-t_0)}$$

(b) Let $y(t) = \overline{x(t)}$

Then,

$$\overline{x(t)} = y(t) = \sum_{n=-\infty}^{\infty} \overline{c_n^x e^{j2\pi nf_0t}} = \sum_{n=-\infty}^{\infty} \overline{c_n^x}e^{-j2\pi nf_0t}$$

(c) Let $y(t) = \dfrac{dx(t)}{dt}$. If c_n^y is the n^{th} complex exponential Fourier series coefficient of $y(t)$, we know that

$$c_n^y = j2\pi n f_0 c_n^x$$

$$\therefore \quad y(t) = \sum_{n=-\infty}^{\infty} c_n^y e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{\infty} j2\pi n f_0 c_n^x e^{j2\pi n f_0 t}$$

3.14 TRIGONOMETRIC FOURIER SERIES

Let $x(t) \in L_2(T)$, i.e., the space of all complex-valued continuous-time signals defined over the interval $-T/2$ to $T/2$ and having a finite energy over that interval.

Then, we had seen in Section 3.11 that

$$\frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}}\cos\omega_0 t, \sqrt{\frac{2}{T}}\cos 2\omega_0 t, \ldots$$

$$\sqrt{\frac{2}{T}}\sin\omega_0 t, \sqrt{\frac{2}{T}}\sin 2\omega_0 t, \ldots, \qquad \text{with} \qquad \omega_0 \underline{\Delta} \frac{2\pi}{T}$$

form a CON sequence with respect to the inner product $\left(x(t), y(t)\right) \underline{\Delta} \int_{-T/2}^{T/2} x(t)\overline{y(t)}\,dt$ and can together be used as a basis set for $L_2(T)$. Hence $x(t) \in L_2(T)$ may be expressed as linear combination of the elements of this basis set, as follows:

$$x(t) = \alpha_0\left(\frac{1}{\sqrt{T}}\right) + \sum_{n=1}^{\infty}\alpha_n\left(\sqrt{\frac{2}{T}}\cos n\omega_0 t\right) + \sum_{n=1}^{\infty}\beta_n\left(\sqrt{\frac{2}{T}}\sin n\omega_0 t\right); -\frac{T}{2} \le t \le \frac{T}{2} \qquad \ldots (3.40)$$

Then, from Eq. (3.16) for the coefficients of an orthonormal expansion, we have

$$\alpha_0 = \left(x(t), \frac{1}{\sqrt{T}}\right) = \int_{-T/2}^{T/2} x(t)\frac{1}{\sqrt{T}}\,dt = \frac{1}{\sqrt{T}}\int_{-T/2}^{T/2} x(t)dt \qquad \ldots (3.41)$$

$$\alpha_n = \left(x(t), \sqrt{\frac{2}{T}}\cos n\omega_0 t\right) = \sqrt{\frac{2}{T}}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt \qquad \ldots (3.42)$$

and

$$\beta_n = \left(x(t), \sqrt{\frac{2}{T}}\sin n\omega_0 t\right) = \sqrt{\frac{2}{T}}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt \qquad \ldots (3.43)$$

If we define

$$\alpha_0 \frac{1}{\sqrt{T}} = a_0, \text{ then } a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt \qquad \ldots (344)$$

$$\alpha_n \sqrt{\frac{2}{T}} = a_n, \text{ then } a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt \qquad \ldots (3.45$$

and $$\beta_n\sqrt{\frac{2}{T}} = b_n \text{ then } b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt$$

Hence Eq. (3.40) may be rewritten as

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t;\ \omega_0 \triangleq \frac{2\pi}{T}$$

The above expansion is valid over the interval $-T/2 \le t \le T/2$. However, if $x(t)$ is periodic with a p… the expansion is valid for all t. Hence, we state

If $x(t)$ is a periodic function with period T and having a finite energy over a period, it may be expres… what is generally referred to as a trigonometric Fourier series, as follows:

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t;\ \omega_0 \triangleq \frac{2\pi}{T}; -\infty < t < \infty}$$

where
$$\boxed{\begin{aligned} a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \\ a_n &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt \end{aligned}}$$

3.14.1 Relation between Trigonometric and Complex-Exponential Fourier Series

If $x(t)$ is a periodic signal with finite energy in any one period,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} c_n e^{jn\omega_0 t}$$

In the above, put

$$c_0 = a_0;\ c_n = \frac{1}{2}(a_n - jb_n) \text{ and } c_{-n} = \frac{1}{2}(a_n + jb_n)$$

$$x(t) = c_0 + \sum_{n=1}^{+\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{+\infty} c_{-n} e^{-jn\omega_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right] + j\sum_{n=1}^{\infty} b_n \left[\frac{e^{-jn\omega_0 t} - e^{jn\omega_0 t}}{2}\right]$$

$$\therefore \quad \boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t} \quad \ldots (3.52)$$

Further, the coefficients a_0, a_n and b_n may be obtained from c_0 and c_n of the complex-exponential Fourier series.

We know

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt$$

$$c_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{+jn\omega_0 t}\,dt$$

But

$$a_n = c_n + c_{-n} \quad \therefore \quad a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt$$

and

$$b_n = \frac{1}{j}\left[c_{-n} - c_n\right] = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt$$

Also,

$$c_0 = a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

3.14.2 Symmetries and Trigonometric Fourier Series Coefficients

1. Even and odd symmetries If $x(t)$ is periodic with a period T, we know that $x(t)$ can be expressed as

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t; \qquad \omega_0 \triangleq \frac{2\pi}{T}$$

where,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt$$

and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt$$

Now, suppose $x(t)$ is a purely even function. Then $x(t)\sin n\omega_0 t$ is odd for all n.

$$\therefore \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt = 0$$

and

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt = \frac{4}{T}\int_{0}^{T/2} x(t)\cos n\omega_0 t\,dt$$

since $x(t)\cos n\omega_0 t$ is even for all n.

On the other hand, if $x(t)$ is odd, $x(t)\cos n\omega_0 t$ is odd,

and so
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t dt = 0 \text{ for all } n$$

and
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t dt = \frac{4}{T}\int_{0}^{T/2} x(t)\sin n\omega_0 t dt$$

since $x(t)\sin n\omega_0 t$ is even for all n.

Hence, the trigonometric Fourier Series of an even function will consist of only cosinusoids, i.e., $b_n = 0$ for all n. Further, the trigonometric Fourier series of an odd function will consist of only sinusoids, i.e., $a_n = 0$ for all n.

$$\boxed{\begin{aligned} &x(t) \text{ Even}: b_n = 0;\ a_n = \frac{4}{T}\int_{0}^{T/2} x(t)\cos n\omega_0 t dt \\ &x(t) \text{ Odd}: a_n = 0;\ b_n = \frac{4}{T}\int_{0}^{T/2} x(t)\sin n\omega_0 t dt \end{aligned}} \quad \ldots(3.5$$

2. Functions with rotational symmetry A periodic function $x(t)$ with period T is said to be having 'rotational symmetry' or 'half-wave symmetry' if $x(t \pm T/2) = -x(t)$ for all t.

We know hat

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t};\ \omega_0 \triangleq \frac{2\pi}{T}$$

where,
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t} dt$$

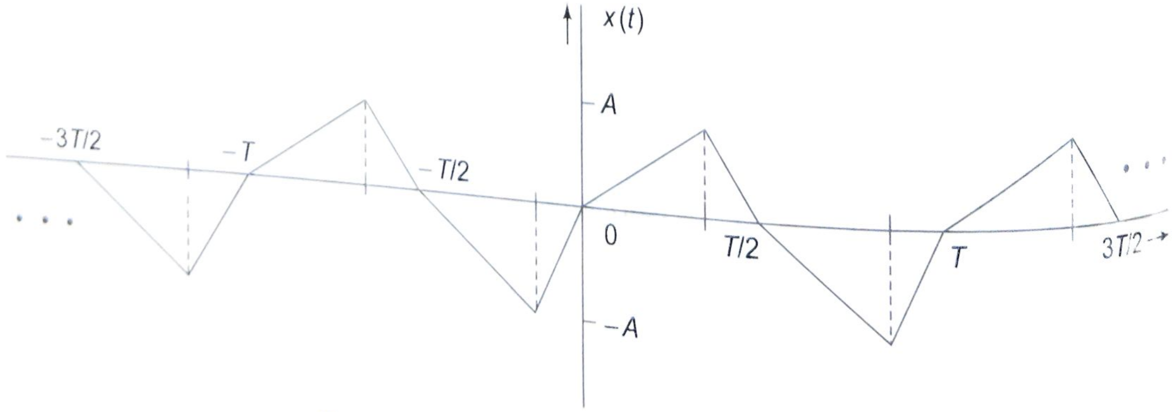


Fig. 3.12 *Waveform with rotational symmetry*

Let $x(t)$ have rotational symmetry as shown in Fig. 3.12 then

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{0} x(t)e^{-jn\omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2} x(t)e^{-jn\omega_0 t}\,dt$$

But $$x(t) = -x(t \pm T/2)$$

$\therefore$ in the second integral, let us replace t by $(t - T/2)$ then

$$c_n = \frac{1}{T}\int_{-T/2}^{0} x(t)e^{-jn\omega_0 t}\,dt + \frac{1}{T}\int_{-T/2}^{0} x(t - T/2)e^{-jn\omega_0 (t-T/2)}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{0} x(t)e^{-jn\omega_0 t}\,dt + \frac{1}{T}\int_{-T/2}^{0} -x(t)e^{-jn\omega_0 t}e^{jn\pi}\,dt$$

But $$e^{jn\pi} = \begin{cases} 1 \text{ for } n \text{ even} \\ -1 \text{ for } n \text{ odd} \end{cases}$$

$\therefore$ $$\underset{(n\,\text{even})}{c_n} = \frac{1}{T}\int_{-T/2}^{0} x(t)e^{-jn\omega_0 t}\,dt - \frac{1}{T}\int_{-T/2}^{0} x(t)e^{-jn\omega_0 t}\,dt = 0 \qquad \ldots (3.54)$$

and $$\underset{(n\,\text{odd})}{c_n} = \frac{2}{T}\int_{-T/2}^{0} x(t)e^{-jn\omega_0 t}\,dt \qquad \ldots (3.55)$$

Hence,

Periodic signals with rotational symmetry will have only odd harmonic components in their Fourier series expansion.

3.15 CONVERGENCE OF FOURIER SERIES

The foregoing may make it appear that every periodic function can be expanded in the form of a Fourier series. However, this is not true.

For Fourier series to exist, c_n must be finite for all n, i.e., $|c_n| < \infty$.

But $$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

Hence $$|c_n| = \frac{1}{T}\left|\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt\right| \le \frac{1}{T}\int_{-T/2}^{T/2} \left|f(t)e^{-jn\omega_0 t}\right|dt$$

i.e., $$|c_n| \le \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|\,dt \quad \text{since } \left|e^{-jn\omega_0 t}\right| = 1$$

Hence, it follows that

$$\text{if} \qquad \int_{-T/2}^{T/2} |f(t)|\,dt < \infty$$

then $|c_n|$ will be finite, i.e., Fourier series exists. The condition, stated in Eq. (3.56) is known as **weak** DIRICHLET'S CONDITION.

It must be noted here that satisfying weak Dirichlet's condition guarantees only the existence of the Fourier series; but not their convergence at every point. For guaranteeing the latter, the following conditions, known as **strong** Dirichlet's conditions must be satisfied:

1. $f(t)$ must remain finite at all points.
2. $f(t)$ must have only a finite number of maxima and minima in the period T.
3. $f(t)$ can have only a finite number of discontinuities and these discontinuities must be finite.

Example 3.18 *Find the trigonometric and complex-exponential Fourier series of the periodic waveform shown in Fig. 3.13.*

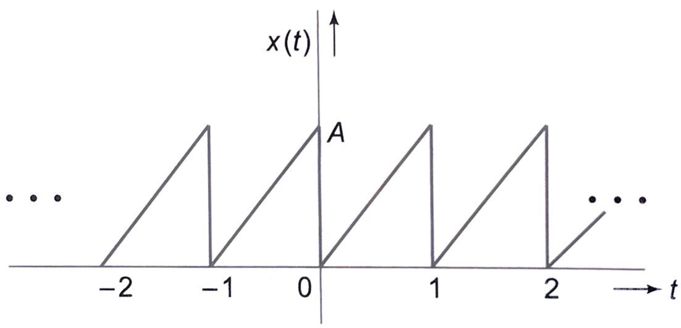


Fig. 3.13

Solution Here, $x(t) = At;\ 0 \le t \le 1$

Trigonometric Fourier series

$$\therefore \qquad a_0 = \frac{A}{1}\int_0^1 t\,dt = \frac{A}{2}$$

$$\therefore \qquad a_n = 2A\int_0^1 t\cos 2\pi nt\,dt = 2A\left[t\frac{1}{2\pi n}\sin 2\pi nt\Big|_0^1 - \frac{1}{2\pi n}\int_0^1 \sin 2\pi nt\,dt \right]$$

$$= 0$$

$$\therefore \qquad b_n = 2A\int_0^1 t\sin 2\pi nt\,dt = 2A\left[\frac{-t}{2\pi n}\cos 2\pi nt\Big|_0^1 + \frac{1}{2\pi n}\int_0^1 \cos 2\pi nt\,dt \right]$$

$$= \frac{-A}{\pi n}\cos 2\pi n + \frac{A}{\pi n}\cdot 0 = \frac{-A}{\pi n}$$

Complex-exponential Fourier series

$$\underset{(n\neq 0)}{c_n} = \frac{A}{1}\int_0^{t=1} t e^{-j2\pi nt}\,dt = A\left[\frac{t}{-j2\pi n}e^{-j2\pi nt}\Big|_0^{t=1} + \int_0^1 \frac{1}{+j2\pi n}e^{-j2\pi nt}\,dt\right]$$

$$= \frac{-A}{j2\pi n}e^{-j2\pi n} + 0 + \frac{A}{(2\pi n)^2}e^{-j2\pi nt}\Big|_0^1$$

$$= \frac{-Ae^{-j2\pi n}}{j2\pi n} + \frac{Ae^{-j2\pi n}}{(2\pi n)^2} - \frac{A}{(2\pi n)^2} = \frac{+jA}{2\pi n} \qquad (\because e^{-j2\pi n} = 1)$$

and

$$c_0 = \frac{A}{1}\int_0^1 t\,dt = \frac{A}{2}$$

Example 3.19 *The waveform shown in Fig. 3.14 is that of the signal at the output of a full-wave rectifier for a sinusoidal input voltage. Find the complex-exponential Fourier series expansion of this signal.*

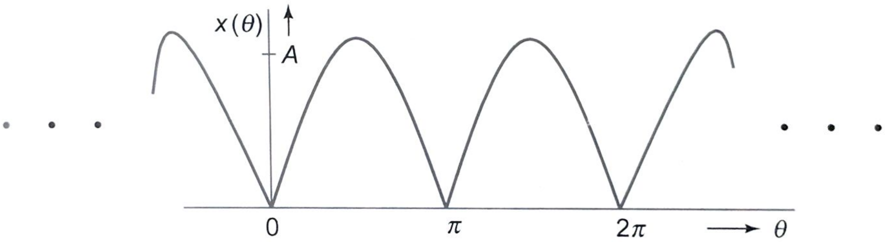


Fig. 3.14

Solution

$$x(\theta) = A\sin\theta\ ; \quad 0 \le \theta \le \pi$$

$$\therefore \quad x(\theta) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\theta}\ ; \quad -\infty < \theta < \infty$$

$$c_n = \frac{A}{\pi}\int_0^{\pi} \sin\theta e^{-jn\theta}\,d\theta = \frac{A}{\pi}\int_0^{\pi}\frac{1}{j2}\left(e^{j\theta} - e^{-j\theta}\right)e^{-jn\theta}\,d\theta$$

$$= \frac{A}{j2\pi}\int_0^{\pi} e^{j\theta(1-n)}\,d\theta - \frac{A}{j2\pi}\int_0^{\pi} e^{-j\theta(1+n)}\,d\theta$$

$$= \frac{A}{j2\pi}\left[\frac{1}{j(1-n)}e^{j\theta(1-n)}\Big|_0^{\pi} - \frac{1}{-j(1+n)}e^{-j\theta(1+n)}\Big|_0^{\pi}\right]$$

$$= \frac{A}{j2\pi}\left[\left\{\frac{1}{j(1-n)}e^{j\pi(1-n)} - \frac{1}{j(1-n)}\right\} + \left\{\frac{1}{j(1+n)}e^{-j\pi(1+n)} - \frac{1}{j(1+n)}\right\}\right]$$

$$= \begin{cases} \dfrac{A}{j2\pi}\left[\left\{\dfrac{-1}{j(1-n)} - \dfrac{1}{j(1-n)}\right\} + \left\{\dfrac{-1}{j(1+n)} - \dfrac{1}{j(1+n)}\right\}\right] & \text{if } n \text{ is even} \\ 0 \quad \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad c_n = \begin{cases} \dfrac{A}{j2\pi}\left[\dfrac{-2}{j(1-n)} - \dfrac{2}{j(1+n)}\right] & ; n \text{ even} \\ 0 & ; n \text{ odd} \end{cases}$$

or
$$c_n = \frac{A}{\pi}\frac{(1+n+1-n)}{(1-n^2)} = \begin{cases} \dfrac{2A}{\pi(1-n^2)} & \text{for } n \text{ even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

$$c_0 = \frac{A}{\pi}\int_0^{\pi} \sin\theta d\theta = \frac{A}{\pi}\left[-\cos\theta\Big|_0^{\pi}\right] = \frac{2A}{\pi}$$

$$\therefore \quad x(\theta) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} c_n e^{jn\theta} = \frac{2A}{\pi} - \sum_{\substack{n=2 \\ (n \text{ even})}}^{\infty} \frac{4A}{\pi(n^2-1)}\cos n\theta$$

$$\therefore \quad x(\theta) = \frac{2A}{\pi} - \sum_{n=1}^{\infty} \frac{4A}{\pi(4n^2-1)}\cos 2n\theta$$

Example 3.20 *For the waveform shown in Fig. 3.15, find the trigonometric Fourier series.*

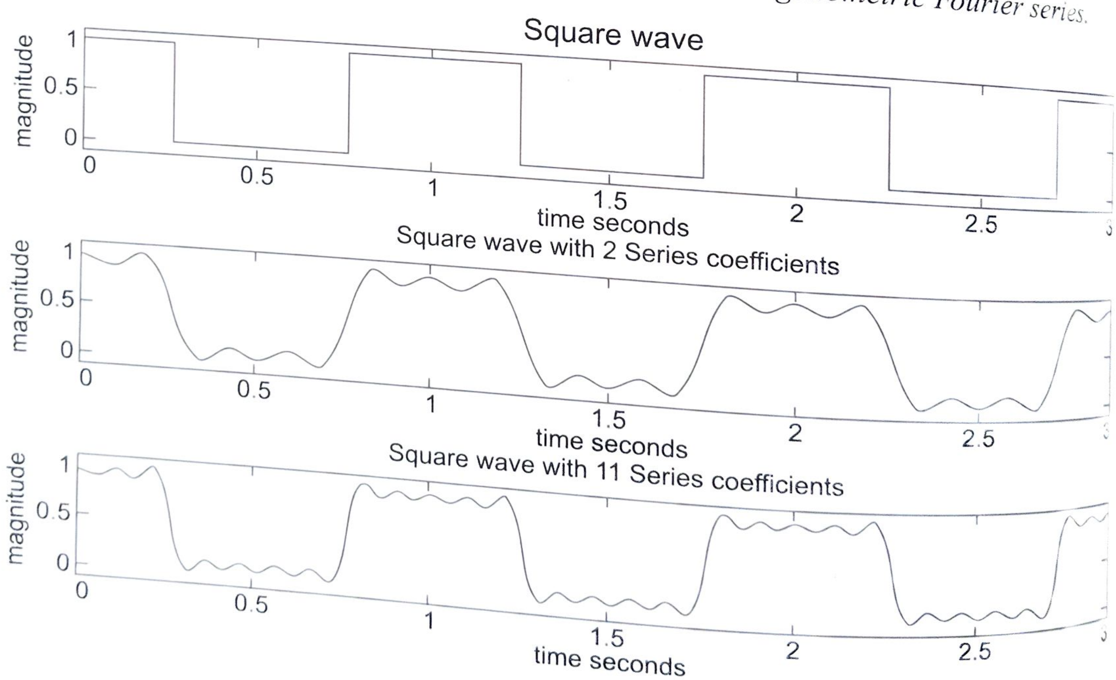


Fig. 3.15

Solution $$x(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

But because of the even symmetry of $x(\theta)$, we know that $b_n = 0$ for all n.

$$a_0 = \frac{A}{\pi}\int_0^{\pi} \sin\theta d\theta$$

$$\therefore \quad a_0 = \frac{A}{\pi}\left[-\cos\theta\Big|_0^{\pi}\right] = \frac{A}{\pi}[1+1] = \frac{2A}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\theta)\cos n\theta d\theta = \frac{2A}{\pi}\int_0^{\pi}\sin\theta\cos n\theta d\theta$$

But $$\sin\theta\cdot\cos n\theta = \frac{1}{2}\left[\sin(\theta+n\theta)+\sin(\theta-n\theta)\right]$$

$$\therefore \quad a_n = \frac{2A}{2\pi}\int_0^{\pi}\sin(1+n)\theta d\theta + \frac{A}{2\pi}\int_0^{\pi}\sin(1-n)\theta d\theta$$

$$= \frac{2A}{2\pi}\left[\frac{-\cos(1+n)\theta}{(1+n)}\Bigg|_0^{\pi} - \frac{\cos(1-n)\theta}{(1-n)}\Bigg|_0^{\pi}\right]$$

$$= \frac{2A}{2\pi}\left[\frac{1}{(1+n)}\{1-\cos(1+n)\pi\} - \frac{1}{(1-n)}\{\cos(1-n)\pi - \cos 0\}\right]$$

$$= \frac{2A}{2\pi}\left[\frac{2}{(1+n)} + \frac{2}{(1+n)}\right]_{n \text{ even}} = \frac{2A}{2\pi}\left[\frac{4}{(1-n^2)}\right]_{n \text{ even}}$$

$$\therefore \quad a_n = \begin{cases} \dfrac{4A}{\pi(1-n^2)} & ; n \text{ even} \\ 0 & ; n \text{ odd} \end{cases}$$

$$\therefore \quad x(\theta) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} a_{2n}\cos 2n\theta = \frac{2A}{\pi} + \sum_{n=1}^{\infty}\frac{4A}{\pi(1-4n^2)}\cos 2n\theta$$

Example 3.21 *Obtain the complex-exponential Fourier series expansion of the signal $x(t)$ = $2 + 3\cos 2\pi t + 4\sin 3\pi t$. Using these Fourier series coefficients, verify Parseval's theorem.*

Solution In $x(t)$, the component $3\cos 2\pi t$ has a frequency of 1 Hz, and the component $4\sin 3\pi t$ has a frequency of 3/2 Hz. Thus, these two components have periods of 1 second and 2/3 second respectively.

Hence, the fundamental period of $x(t) = \text{LCM}\left(1, \frac{2}{3}\right) = 2$

$\therefore$ the fundamental frequency of $x(t) = \frac{1}{2} \triangleq f_0$

Therefore, we my write

$$x(t) = \sum_{n=-3}^{3} c_n e^{j2\pi n f_0 t}$$

$$c_0 + c_1 e^{j2\pi f_0 t} + c_{-1} e^{-j2\pi f_0 t} + c_2 e^{j2\pi 2 f_0 t} + c_{-2} e^{-j2\pi 2 f_0 t} + c_3 e^{j2\pi 3 f_0 t} + c_{-3} e^{-j2\pi 3 f_0 t}$$

Since $$\cos 2\pi t = \cos 4\pi f_0 t = \frac{1}{2}\left[e^{j2\pi 2 f_0 t} + e^{-j2\pi 2 f_0 t}\right]$$

and $$\sin 3\pi t = \sin 2\pi 3 f_0 t = \frac{1}{2j}\left[e^{j2\pi 3 f_0 t} - e^{-j2\pi 3 f_0 t}\right]$$

Hence, $$x(t) = 2 + \frac{3}{2}\left[e^{j2\pi 2 f_0 t} + e^{-j2\pi 2 f_0 t}\right] + \frac{4}{2j}\left[e^{j2\pi 3 f_0 t} - e^{-j2\pi 3 f_0 t}\right]$$

$\therefore$ $$c_0 = 2;\ c_1 = c_{-1} = 0;\ c_2 = c_{-2} = \frac{3}{2};\ c_3 = 4/2j = -2j;\ c_{-3} = -4/2j = 2j$$

To verify Parseval's Theorem: Since the fundamental period of x(t) is 2, the average power of x(t) is given

$$P_{av} = \frac{1}{2}\int_0^2 (2 + 3\cos 2\pi t + 4\sin 3\pi t)^2\, dt = \frac{1}{2}\int_0^2 \frac{33}{2} dt = \frac{33}{2} = 16.5$$

$$\sum_{n=-3}^{3} |C_n|^2 = 2^2 + \frac{9}{4} + 0 + 2^2 + 0 + \frac{9}{4} + 2^2 = 12 + \frac{9}{2} = \frac{33}{2} = 16.5$$

$\therefore$ $$P_{av} = \sum_{n=-3}^{3} |C_n|^2$$

MATLAB Example 3.1 *In this example, we will study GIBB's phenomenon. For this, we will plot continuous-time Fourier series of a square wave and study the effect of truncation of Fourier series coefficients on the signal that is generated.*

```
T=1; %We start with a square wave of time period 1 second
T=linespace(0, 3, 3000);  % We oversample by 1000 with 3 time periods
x = [ones (1, 250), zeros (1, 500) ones (1, 250)]; %The square wave
xp = kron (ones(1, 3), x); %The signal is repeated
subplot (311), plot(t, xp, 'k');
axis ([0, 3, -0.1, 1.1]);
title ('Square wave'); xlabel ('time, seconds');
ylabel ('magnitude');
%First plot using only 11 Fourier Series Coefficients
xs1=zeros (1, 1000);
for k = -5:5
```

```
ek = cos (2*pi*k*t(1:1000)) + sqrt (-1) *sin (2*pi*k*t(1:1000));
xs1 = xs1+0.5*sinc (k/2)*ek;
end
subplot (312), plot (t, kron (ones (1, 3), real (xs1)), 'k');
title ('Square wave with 11 Series
Coefficients');xlabel ('time, seconds');
ylabel ('magnitude');
axis ([0, 3, -0.2, 1.2]);
%Repeat using 21 Fourier Series coefficients
xs1=zeros (1, 1000);
for k=-10 : 10
ek = cos (2*pi*k*t (1:1000)) + sqrt (-1)*sin (2*pi*k*t(1:1000));
xs1 = xs1+0.5*sinc (k/2) *ek;
end
subplot (313), plot (t, kron (ones (1, 3), real (xs1)), 'k');
title ('Square wave with 21 Series
coefficients');xlabel ('time, seconds');
ylabel ('magnitude');
axis ([0, 3, - 0.2, 1.2]);
```

Figure 3.15 shows the effect of truncation of Fourier Series on a square wave. The square wave (top) is plotted using only the coefficients for $|k| < 6$ (middle) and $|k| < 11$ (bottom). Note that the ripples do not decrease in magnitude, but only increase in frequency.

Summary

1. A vector space consists of the following:
   (a) A field F of scalars (generally, real or complex numbers)
   (b) A set V of vectors
   (c) Two operations: +, called vector addition •, called multiplication of a vector by a scalar. With regard to the '+' operation (i.e., vector addition), the set V forms an abelian, or commutative group and with regard to the '•' operation, it obeys closure property, associatively and distributive property.
2. *Subspace:* Let V a vector space over a field F. A subset W of V is called a subspace of V provided it forms a vector space over F with the operations of vector addition and multiplication of a vector by a scalar as defined on V.
3. *Linearly independent vectors:* A set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \mathbf{x}_n$ of distinct vectors in a vector space V defined over a field F, is said to be linearly dependent if there exist scalars $c_1, c_2, \ldots c_n$ belonging to F, not all of which are zero, such that

$$c_1 \cdot \mathbf{x}_1 + c_2 \cdot \mathbf{x}_2 + \ldots \quad + c_n \cdot \mathbf{x}_n = \sum_{i=1}^{n} c_i \cdot \mathbf{x}_i = 0 \text{ (zero vector)}.$$

   A set of vectors which is not linearly dependent, is said to be linearly independent

4. A basis for a vector space V is a set of linearly independent vectors in V which span V. A b
for a vector space is not unique.
5. Any two basis sets for a finite dimensional vector space must have the same number of
in them.
6. *Inner product:* Let V be a vector space over a field F of real or complex numbers. An inner
uct on V is a function that assigns a scalar in F denoted by $(\mathbf{x}, \mathbf{y})$, for every ordered pair
tors $\mathbf{x}$ and $\mathbf{y}$ belonging to V. For every $\mathbf{x}, \mathbf{y}, \mathbf{z}$ belonging to V and every c belonging to F, th
product should satisfy the following properties.
   (a) $((\mathbf{x} + \mathbf{y}), \mathbf{z}) = (\mathbf{x}, \mathbf{z}) + (\mathbf{y}, \mathbf{z}) + (c \cdot \mathbf{y}, \mathbf{z})$
   (b) $(c \cdot \mathbf{x}, \mathbf{y}) = c(\mathbf{x}, \mathbf{y})$
   (c) $(\mathbf{y}, \mathbf{x}) = \overline{(\mathbf{x}, \mathbf{y})}$ = complex conjugate of $(\mathbf{x}, \mathbf{y})$.
   (d) $(\mathbf{x}, \mathbf{x}) \geqslant 0$, the equality sign holding good if and only if $\mathbf{x} = 0$
7. *Norm:* The positive square-root of $(\mathbf{x}, \mathbf{x})$ is called the norm of $\mathbf{x} \cdot |\mathbf{x}| \triangleq \sqrt{(\mathbf{x}, \mathbf{x})}$
8. *Inner product space:* A real or complex vector space over which an inner product has be
fined, is called an inner product space.
9. *Schwarz's inequality:* $|(\mathbf{x}, \mathbf{y})| \leqslant ||\mathbf{x}||\ ||\mathbf{y}||$, the equality sign holding good if and only if $y = c$
10. *Triangle inequality:* $||\mathbf{x} + \mathbf{y}||^2 \leqslant ||\mathbf{x}||^2 + 2||\mathbf{x}||\ ||\mathbf{y}|| + ||\mathbf{y}||^2$
11. *Orthogonal and ortho-normal sets:* A set S of vectors in an inner-product space V, is cal
orthogonal set provided all pairs of distinct vectors in the set are orthogonal. An orthog
in which every vector has unit norm, is called an orthonormal set.
12. *Gram-Schmidt orthogonalization procedure:* Given n linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots$
in an inner product space V, this procedure enables us to derive n orthogonal (or orthon
vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ from them.
Let $\mathbf{y}_1 = \mathbf{x}_1$
then: $$\mathbf{y}_2 = \mathbf{x}_2 - \frac{(\mathbf{y}_2, \mathbf{x}_1)}{||\mathbf{y}_1||^2} \cdot \mathbf{y}_1$$
$$\vdots \quad \vdots \quad \vdots$$
And in general: $$\mathbf{y}_{m+1} = \mathbf{x}_{m+1} - \sum_{k=1}^{m} \frac{(\mathbf{x}_{m+1}, \mathbf{y}_k)}{||\mathbf{y}_k||^2} \cdot \mathbf{y}k$$
13. *Best approximation problem:* Let W be a subspace of an inner product space V. Then t
$\mathbf{x}$ in W is a best approximation to the vector $\mathbf{y}$ in V if and only if $(\mathbf{y} - \mathbf{x})$ is orthogonal
vector in W. The best approximation, when it exists, is unique.
14. *Bessel's inequality:* Let $\{\mathbf{x}_k\}$ be a sequence of non-zero orthonormal vectors in an inner
space V. Let $\mathbf{y}$ be some vector in V. Then
$$\sum_{k=1}^{\infty} |(\mathbf{y}, \mathbf{x}_k)|^2 \leqslant ||\mathbf{y}||^2$$
15. *A complete space:* A space X is said to be a complete space, if every Cauchy sequence in it co
16. *Hilbert space:* An inner product space which is complete, is called a Hilbert space.

17. *CON sequence:* Let $\{x_k\}$ be an orthonormal sequence in an inner product space V. Then $\{x_k\}$ is said to be a complete orthonormal sequence (CON sequence) provided the equality sign in Bessel's inequality holds good for every vector y in V.

18. *Complex-exponential Fourier series:* If $x(t)$ is a periodic function with period T, then it can be expanded as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp(j2pnf_o t)\,;\, f_o \triangleq \frac{1}{T};\ -\infty < t < \infty.$$

In which, the Fourier series coefficients, c_n s are given by

$$c_n = \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi nf_o t}\, dt$$

19. If $x(t)$ is a purely real-valued periodic function, its complex exponential Fourier series coefficients, c_n's will have Hermitian symmetry. That is

$$|c_{-n}| = |c_n| \text{ and } \angle c_{-n} = -\angle c_n \text{ for } n.$$

20. *Magnitude and phase spectra:* Plot of $|c_n|$ vs nf_o is magnitude spectrum. Plot of $\angle c_n$ vs nf_o is phase spectrum.

21. The spectrum of a continuous-time periodic signal is a discrete one.

22. *Parseval's theorem:* If $x(t)$ is a periodic signal with period T, then:

$$\frac{1}{T}\int_{-T/2}^{T/2} \left|x(t)^2\right| dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \text{Average power of } x(t).$$

23. *Same properties of c_n s:*
   (i) $c_{-n}^* = c_n$ if $x(t)$ is purely real-valued.
   (ii) $c_n^* = -c_{-n}$ if $x(t)$ is purely imaginary.
   (iii) c_n's are real and even if $x(t)$ is real and even.
   (iv) c_n's are purely imaginary and possess odd symmetry with respect to n, if $x(t)$ is real and odd.

24. *Trigonometric Fourier series:* $x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t\,;\ -\infty < t < \infty$
where,

$$a_o = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt\,;\ a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \cos n\omega_o t\, dt\,;\ b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \sin n\omega_o t\, dt\,.$$

25. *Symmetry properties of trigonometric Fourier series:*
   (i) if $x(t)$ is having even symmetry: $b_n = 0$ for all n; $a_n = \frac{4}{T}\int_0^{T/2} x(t) \cos n\omega_o t\, dt$.
   (ii) If $x(t)$ has odd symmetry: $a_n = 0$ for all n; $b_n = \frac{4}{T}\int_0^{T/2} x(t) \sin n\omega_o t\, dt$.
   (iii) If $x(t)$ has rotational symmetry, i.e., if $x(t \pm T/2) = -x(t)$ for all t, then it will have only odd harmonic components in its Fourier series.

26. (a) *Existence of Fourier series:* For a periodic signal $x(t)$ with period T, the magnitu... Fourier series coefficients will be finite and the Fourier series exists provided $x(t)$ ... lutely integrable over a period T.

(b) *Convergence of Fourier series (The strong Dirichlet's conditions):* For the Fourier s... $x(t)$ to converge at every point, the following conditions must be satisfied. (i) $x(t)$ ... remain finite at all points, (ii) $x(t)$ must have only a finite number of maxima and ... in the period T, (iii) $x(t)$ should have only a finite number of discontinuities and t... continuities should be finite discontinuities.

References and Suggested Reading

1. Hoffman, and Kunz, *Linear Algebra*, 2nd Edition; Prentice Hall of India, 2002.
2. Ziemer, R. Tranter, W., and Fannen, D., *Signals and Systems : Continuous and Discrete*; Prentic... 1998.
3. Van Valkenburg, M.E., *Network Analysis*, Asia Publishing House, 1960.
4. Papoulis, A., *Signal Analysis*, McGraw-Hill (New York), 1977.
5. Bruce, Carlson et.al., *Communication Systems—An Introduction to Signals and Noise in E... Communication*, 4th Edition, McGraw-Hill International Edition, 2002.

Review Questions

1. Define a 'vector space' and give an example of a vector space.
2. How do you check whether a given subset W forms a subspace of a vector space? Give an exa... a subspace.
3. State whether each of the following statements is TRUE or FALSE.
   (a) Any subspace must contain the zero vector.
   (b) Any set of linearly independent vectors must have the zero vector.
   (c) In a subspace spanned by n vectors, you can find n linearly independent vectors.
4. Define the dimension of a vector space.
5. What is an inner product space? Give an example.
6. Define norm of a vector in an inner product space.
7. When do you say the vectors $x_1, x_2, \ldots, x_n$ are linearly independent?
8. What is a complete metric space? Give an example.
9. Define 'orthogonal set' of vectors.
10. What is a complete orthonormal set? Give an example.
11. If two vectors are linearly dependent, show that one of them must be a scaled version of the o...
12. V is the vector space of all 2×2 matrices over some field F. Show that W, the set of all matrices ... type

$$\begin{bmatrix} a & -a \\ b & c \end{bmatrix};\ a, b, c \in F$$

forms a subspace of V.

13. Every convergent sequence must be a Cauchy sequence, but every Cauchy sequence may not be convergent. Explain why.
14. State and explain Dirichlet's conditions for convergence of Fourier series.
15. Write down the complex-exponential Fourier series expansion of the periodic signal $x(t) = 2\cos 2\pi t$.
16. By means of suitable examples, explain the need for usage of complete orthonormal sequences as the basis sets.
17. What is the physical meaning of Parseval's theorem pertaining to Fourier series?

Problems

1. V is the set of all complex-valued functions of time t, $-\infty < t < \infty$, satisfying the condition $f(-t) = \overline{f(t)}$ where the bar on $f(t)$ denotes complex conjugation. If the following operations are defined on the elements of V, show that V is a vector space over the field of real numbers.

   $(f+g)t = f(t) + g(t)$; for $\forall f, g \in V$

   and $(cf)(t) = cf(t)$; for $\forall\ c \in C$, the field of complex numbers.
2. Let F^n be the vector space formed by the set of all n-tuples of scalars belonging to a field F of scalars. Show that the set of all n-tuples of the form $(0, x_2, x_3, \ldots, x_n)$ forms a subspace of F^n. How about the set of all n-tuples of the form $(2, x_2, x_3, \ldots, x_n)$?
3. $C^{n\times n}$ is the vector space of all $n \times n$ matrices with entries drawn from C, the field of complex numbers. Show the set of all $n \times n$ symmetric matrices with entries from C forms a subspace of $C^{n\times n}$. How about the set of all $n \times n$ Hermitian matrices?
4. Let V be the vector space of all functions f from R into R. check whether the following sets of functions are subspaces of V:

   (a) All functions f such that $f(0) = f(2)$

   (b) All functions f which are continuous

   (c) All functions f such that $f(-3) = 0$
5. Show that the vectors $\boldsymbol{X} = (1,1,0,0)$, $\boldsymbol{Y} = (0,0,1,1)$, $\boldsymbol{Z} = (1,0,0,4)$ and $\boldsymbol{W} = (0,0,0,2)$ form a basis for R^4. Find the coordinates of each of the standard basis vectors of R^4 in the ordered basis set $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W})$
6. Do the following vectors form a basis for R^3?

$$\boldsymbol{X} = (1, 1, 0);\ \boldsymbol{Y} = (3, 0, 1);\ \boldsymbol{Z} = (5, 2, 1)$$

7. If (X, Y) is any inner product and if $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are vectors such that $(\boldsymbol{W}, \boldsymbol{Z}_1) = (\boldsymbol{W}, \boldsymbol{Z}_2)$ for every vector $\boldsymbol{W}$, show that $\boldsymbol{Z}_1 = \boldsymbol{Z}_2$.
8. Let V be an inner product space with $\{x_1, x_2, \ldots x_n\}$ as an orthonormal basis. Prove that

$$(Y, Z) = \sum_{i=1}^{n} (Y, x_i)(x_i, Z) \quad \text{for } \forall\ Y, Z \in V$$

9. Using Gram–Schmidt procedure, orthogonalize the following basis set for R^3.

$$\boldsymbol{X} = (2, -1, 2);\ \boldsymbol{Y} = (1, 1, 4);\ \boldsymbol{Z} = (6, 3, 9)$$

10. Functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are as shown in Figs. 3.16(a), (b) and (c)

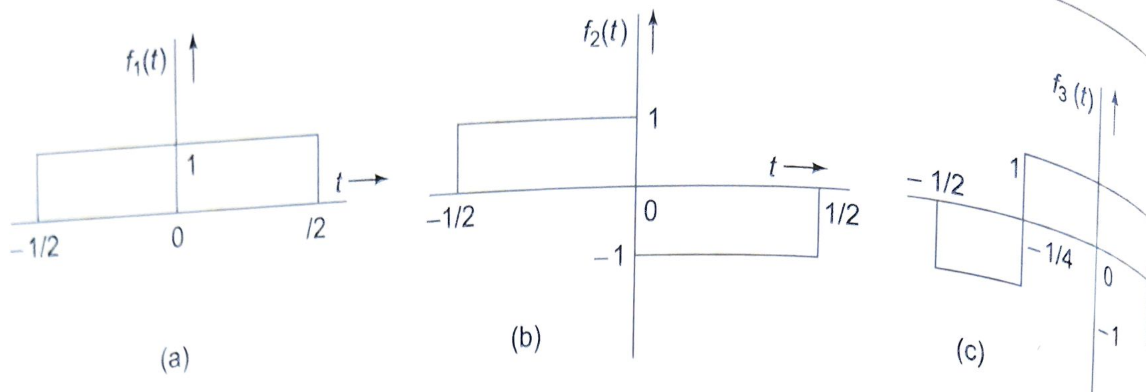


Fig. 3.16

(a) Show that the functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are orthogonal over the interval $(-1/2, 1/2)$.
(b) If the signal $x(t) = 2 \sin 2\pi t$ is expanded in terms of these functions, find the integral s
error of such a representation of $x(t)$.

11. A signal $x(t)$ is as show in Fig. 3.17.

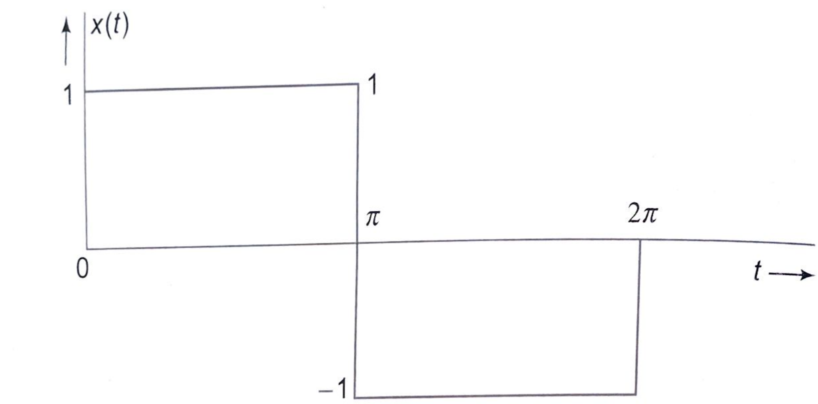


Fig. 3.17

Show that $x(t)$ is orthogonal to the signals $\cos t$, $\cos 2t$, $\cos 3t$, . . . , $\cos nt$ for all integer valu
$n \neq 0$, over the interval $(0, 2\pi)$.

12. Show that the standard inner product on F^n satisfies all the requirements of an inner product.
13. Let V be a vector space over the field R of real numbers. Let I_1 and I_2 be two inner products defined
check whether the following are also inner products on V.

(a) $I_1 + I_2$ (b) $I_1 - I_2$ (c) cI_1 where $c \in R$ and $c > 0$

14. Show that the standard basis set for R^n is an orthonormal set with respect to the standard inner pro
on R^n.
15. If $x_e(t)$ and $x_0(t)$ are respectively the even and odd parts of a function $x(t)$, show that they are
nal over an interval $(-T, T)$ for any value of T.
16. Let V be any vector space. Which is that unique vector in V which is such that it is orthogonal to
other vector in V?
17. $x_1(t), x_2(t), \ldots, x_n(t)$ are n mutually orthogonal signals defined over the interval $(-T, T)$. If a signal
is defined as

$y(t) \triangleq \sum_{i=1}^{n} x_i(t)$ show that the energy of the signal $y(t)$ over $(-T, T)$ is equal to the sum of the en
$x_i(t)$s, $i = 1$ to n.

18. The polynomials $P_n(t)$ defined by the formula

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2 - 1\right)^n, n = 0, 1, 2, \ldots$$

are called Legendre polynomials.

(a) Show that the Legendre polynomials form a set of orthogonal functions over the interval (–1, 1). Are they normalized?

(b) Express the signal $x(t) = \sin \pi t$ over the interval (–1, 1) in terms of $P_n(t)$s and obtain the squared-integral value of the error in this representation of $x(t)$.

19. Signal $x(t) = \begin{cases} t; 0 \le t \le 1 \\ 0; \text{elsewhere} \end{cases}$

Expand $x(t)$ over the interval (0,1) by

(a) trigonometric Fourier series

(b) complex exponential Fourier series

20. Expand the periodic function $x(\theta)$ shown in Fig. 3.18 using trigonometric Fourier series.

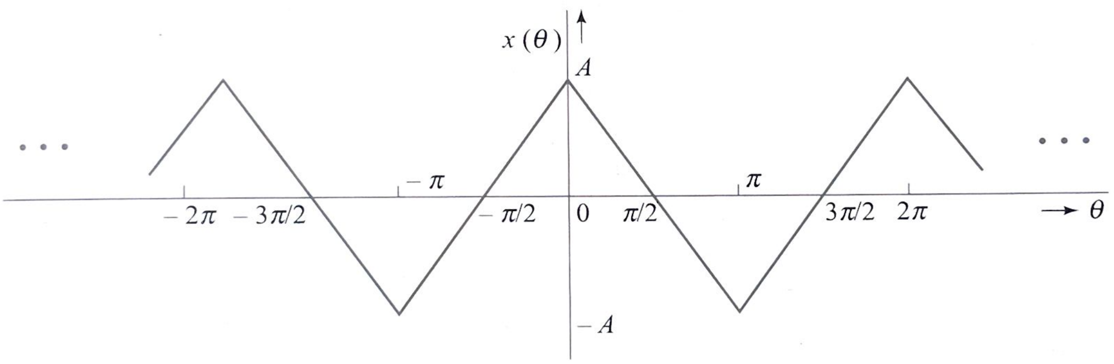


Fig. 3.18

21. Expand the periodic function $x(t)$ shown in Fig. 3.19 by

(a) trigonometric Fourier series

(b) complex exponential Fourier series

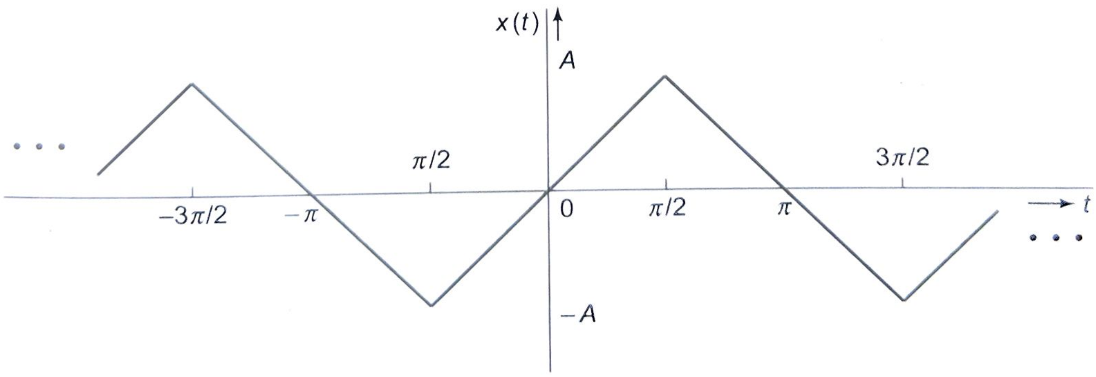


Fig. 3.19

22. A periodic function $x(t)$ has c_ns as its complex exponential Fourier series coefficients. If c_n's … complex exponential Fourier series coefficients of $x(t - t_0)$ where t_0 is some fixed time interval, … the relation between c_ns and c_n's.

23. Calculate and sketch the magnitude and phase spectra of the signal $x(t)$ given in Problem 20 up … fifth harmonic frequency component.

24. Expand the periodic waveform $x(t)$ shown in Fig. 3.20 by complex exponential as well as trigono… ric Fourier series.

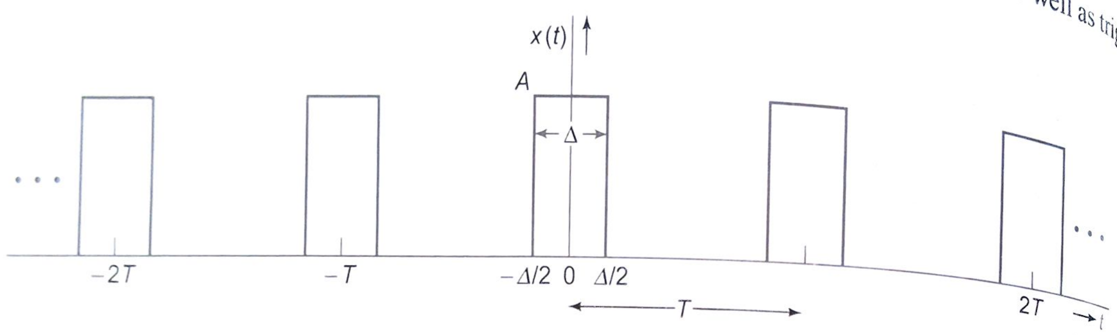


Fig. 3.20

25. (a) Find the complex exponential Fourier series expansion of the periodic waveform show… Fig. 3.21.

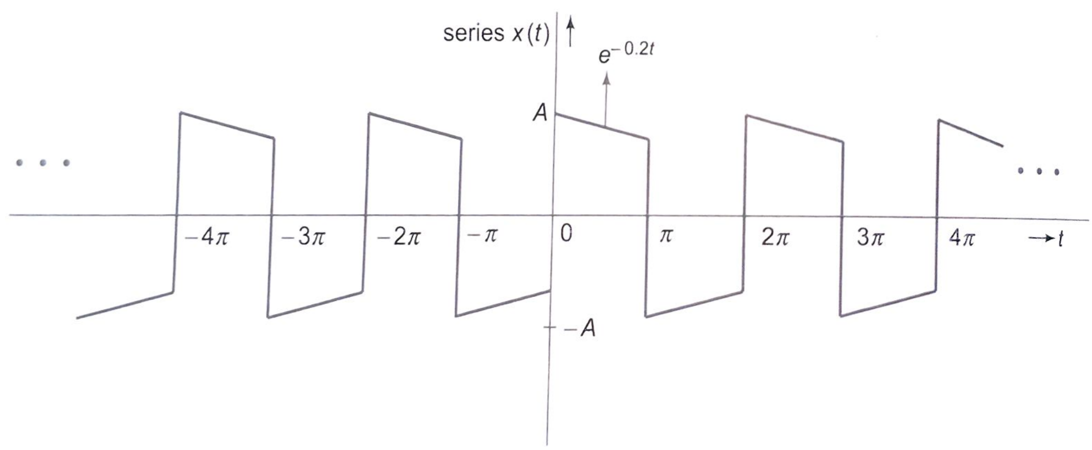


Fig. 3.21

(b) Calculate and plot the amplitude and phase spectra of $x(t)$ up to the 5th harmonic freq… component.

26. For the periodic waveform shown in Fig. 3.22, determine the complex exponential and trigonom… Fourier series expansions.

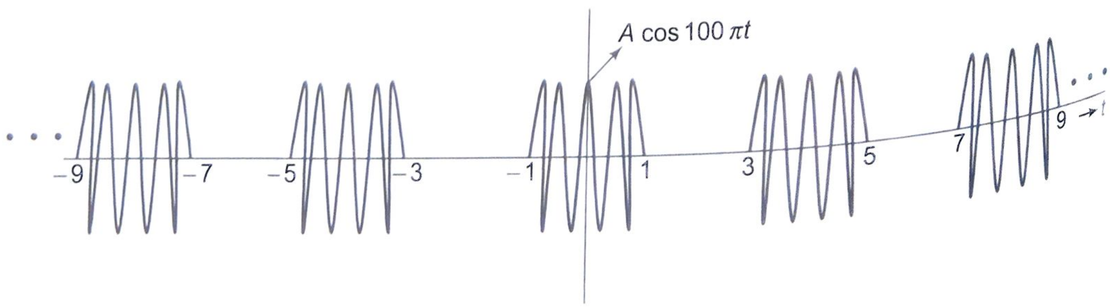


Fig. 3.22

27. $x(t)$ is a periodic function with a period $T_0 = 1/f_0$ and is continuous. Show that $\dot{x}(t)$ is also a periodic function with the same period. How are the complex exponential Fourier series coefficients of $\dot{x}(t)$ related to those of $x(t)$?

28. Express the signal $x(t) = 2 + \sin \omega_0 t + 3\cos(\omega_0 t + \pi/4) + 2\cos 2\omega_0 t$ as the sum of complex exponentials and plot its magnitude and phase spectra.

29. (a) $x_n(t) = e^{j2\pi nt/T}$, where n takes all integer values from $-\infty$ to $+\infty$. Show that the functions $x_n(t)$s are orthogonal over any interval of T seconds. Are they also orthonormal?

    (b) Are the functions $\sin n\omega_0 t$ and $\cos m\omega_0 t$ orthogonal over the interval $(0, T)$, where $\omega_0 = 2\pi/T$? Are they orthonormal? If they are not, normalize them.

30. In Section 3.14, we had shown that a periodic signal $x(t)$ having rotational, or, half-wave symmetry will have only odd harmonics. Prove the converse of it, i.e., if a periodic signal $x(t)$ with period T has only odd harmonic components, then it has half-wave symmetry, so that $x(t - T/2) = -x(t)$ for any t.

31. Let $x(t)$ be a rectangular pulse as shown in Fig. 3.23. Let its complex exponential Fourier series representation over the interval $|t| < 10$ be

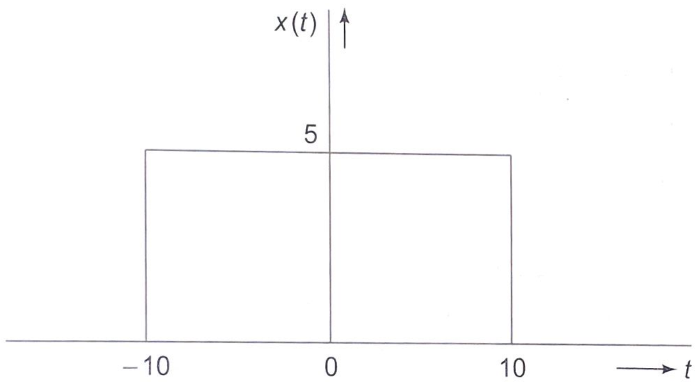


Fig. 3.23

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_0 t}; \; f_0 \triangleq \frac{1}{20}$$

if we limit this Fourier series expansion to say the k^{th} harmonic, we get a distorted version of $x(t)$, designated as $x_k(t)$ and given by

$$x_k(t) = \sum_{n=-k}^{k} c_n e^{j2\pi n f_0 t}$$

The distortion manifests itself as ripples at the points of discontinuity, in this case, near the leading and trailing edges of the pulse. This distortion arising out of taking partial sums of the Fourier series is called Gibbs's phenomenon. Plot $x(t)$ for $k = 3$ and $k = 10$. (See the MATLAB example)

Multiple-Choice Questions

1. A set of vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \ldots, \boldsymbol{X}_N$ is said to be linearly independent only if

   (a) the zero vector is one of the elements of the set

(b) the zero vector is not one of the elements of the set

(c) no linear combination of the vectors in the set is equal to the zero vector unless all the coefficients of linear combination are equal to zero.

(d) their linear combination is zero without all the coefficients being equal to zero.

2. Vectors $\boldsymbol{X} = (x_1, x_2, x_3)$, $\boldsymbol{Y} = (y_1, y_2, y_3)$ and $\boldsymbol{Z} = (z_1, z_2, z_3)$ are linearly independent only if

(a) they are orthogonal to each other

(b) their sum is not equal to zero

(c) their determinant is not equal to zero, i.e., $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$

(d) none of them is the zero vector

3. Let V be a vector space spanned by the vectors $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, and $\boldsymbol{X}_3$. Let S be any subset of the vectors in V. Then the number of linearly independent vectors in S is

(a) at the most 3 (b) at least 3 (c) equal to 3 (d) none of these

4. A basis for a vector space V may be defined as a set of vectors in V which

(a) are mutually orthogonal

(b) are linearly independent

(c) span the space V

(d) are linearly independent and span the space V

5. A basis set for an N-dimensional vector space will contain

(a) at least N vectors (b) at the most N vectors (c) N vectors (d) N^2 vectors

6. The angle between the vectors $A = (1, 0, 0)$ and $B = (1, 1, 0)$ is

(a) 0° (b) 45° (c) 30° (d) 60°

7. The dimension of a finite dimensional vector space V is equal to the number of

(a) vectors in V

(b) sets of linearly independent vectors in V

(c) elements in a basis set of V

(d) linearly independent vectors in any subset of V

8. An inner product space is

(a) obtained by taking the product of two vector spaces

(b) a vector space on which an inner product has been defined

(c) a vector space on which it is possible to define an inner product

(d) a vector space formed by taking the inner product

9. The norm of a finite energy signal, $x(t)$, represents the

(a) square-root of the energy of the signal $x(t)$

(b) energy of the signal $x(t)$

(c) square of the energy of the signal $x(t)$

(d) average power of the signal $x(t)$.

10. If x and y are two vectors belonging to an inner product space V, Schwarz's inequality states that
(a) $|(x, y)| \geq \|x\| \cdot \|y\|$ (b) $\|(x, y)\| \leq \|x\| \cdot \|y\|$ (c) $|(x, y)| \leq \|x\| \cdot \|y\|$ (d) $|(x, y)| \leq |x|\ |y|$
11. Any set S of orthogonal vectors is a linearly independent set.
(a) True
(b) False
(c) True only if S includes the zero vector also.
(d) True only if the zero vector is not included in S.
12. Let $y \in V$, an inner product space. Let x be any vector in V. Then the component of y along x is given by
(a) $\dfrac{(y, x)}{\|x\|} \cdot x$ (b) $\dfrac{|(y, x)|}{\|x\|} \cdot x$ (c) $\dfrac{(y, x)}{\|x\|^2} \cdot x$ (d) $\dfrac{|(y, x)|}{\|x\|^2} \cdot x$
13. Let x and $y \in v$, an inner product space and let (x, y) denote the inner product of x and y. If $a \in C$, the complex number field on which V is defined, (x, ay) equals
(a) $a(x, y)$ (b) $-a(x,y)$ (c) $|a|(x, y)$ (d) $\bar{a}(x, y)$
14. Let V be an inner product space and let $\{x_k\}$ be an orthonormal set in V. Then, if y be some vector in V, Bessel's inequality says

$$\sum_{k=1}^{\infty} |(y, x_k)|^2 \leq \|y\|^2$$

In this, for the equality sign to hold good,
(a) $\{x_k\}$must necessarily span V
(b) y should be in the subspace spanned by $\{x_k\}$
(c) y should not be in the subspace spanned by $\{x_k\}$
(d) V should be a finite-dimensional space.
15. A metric space is
(a) a vector space together with a metric defined on it
(b) a set on which a metric has been defined
(c) an inner product space on which a metric has been defined
(d) none of the above
16. A space is said to be a complete space if
(a) every Cauchy sequence in it converges
(b) no sequence in it converges
(c) it is an inner product space on which a metric has been defined
(d) none of the above
17. An orthonormal sequence of vectors, $\{x_k\}$ in an inner product space, V, is said to be a complete orthonormal sequence if
(a) $\left[\underset{k \to \infty}{Lt}\ x_k\right] < \infty$

(b) $\sum_{k=1}^{\infty} |(y, x_k)|^2 = \|y\|^2$ for every vector $y \in V$

(c) $\sum_{k=1}^{\infty} |(y, x_k)|^2 < \|y\|^2$ for every vector $y \in V$

(d) it has an infinite number of elements (vectors) in it

18. Let x and y belong to an inner product space V. Then the triangle inequality says $\|x\| + \|y\| \geq \|x + y\|$, the equality sign holds good only if.
(a) x and y are orthogonal vectors
(b) $y = c.x$, where $c \in C$
(c) $y = r.x$, where $r \in R$
(d) $y = x$

19. A Hilbert space is
(a) a metric space which is complete
(b) an inner product space in which no Cauchy sequence converges
(c) a metric space in which no Cauchy sequence converges
(d) an inner product space which is complete

20. Consider the inner product space V of all real-valued continuous-time functions defined over the interval $[-T/2, T/2]$ and having a finite energy over that interval, with the inner product

$$(x(t), y(t)) \triangleq \int_{-T/2}^{T/2} x(t)y(t)dt$$

In this space, the sequence of signals

$$1, \ \cos\omega_0 t, \ \cos 2\omega_0 t, \ \ldots \text{ where } \omega_0 = \frac{2\pi}{T}, \text{ form}$$

(a) a complete orthonormal set
(b) an orthonormal set but not complete
(c) only an orthogonal set which is complete
(d) an orthogonal set but not complete

21. Let V be the space of all continuous, complex-valued functions defined over the interval $[-T/2, T/2]$ and having a finite energy over that interval. Define an inner product as follows, on V.

$$(x(t), y(t)) \triangleq \frac{1}{T} \int_{-T/2}^{T/2} x(t)\overline{y(t)}dt$$

With respect to this inner product, the sequence $x_n = e^{j2\pi n f_0 t}$,

$$n = 0, \pm 1, \pm 2, \ldots \text{ where } f_0 = \frac{1}{T}, \text{ will form}$$

(a) a complete orthonormal set
(b) an orthonormal set but not complete
(c) only a complete orthogonal set
(d) an orthogonal set but not complete

22. The average power of the periodic signal $c_n e^{j2\pi n f_0 t}$ is
(a) c_n^2
(b) $|c_n|$
(c) $|c_n|^2$
(d) $c_n^2 e^{j4\pi n f_0 t}$

23. Parseval's theorem pertaining to Fourier series states that
(a) the signal $x(t)$ is equal to the sum of its components along each of the basis functions, $e^{j2\pi nf_0t}$, $n = 0, \pm 1, \pm 2, \ldots$
(b) the average power of $x(t)$ is equal to the sum of the average powers of its components along each of the basis functions, $e^{j2\pi nf_0t}$, $n = 0, \pm 1, \pm 2, \ldots$
(c) the energy of the signal $x(t)$ is equal to the sum of the energies of its components along each of the basis functions, $e^{j2\pi nf_0t}$, $n = 0, \pm 1, \pm 2, \ldots$
(d) energy of the signal may be obtained in the time domain or from the frequency domain

24. If $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nf_0t}$; $-\infty < t < \infty$ and $f_0 \triangleq \frac{1}{T}$ where T is the fundamental period of the periodic signal, $x(t)$, which is purely real-valued then
(a) $c_n = -c_{-n}$ (b) $c_n = c_{-n}$ (c) $c_{-n} = c_n^*$ (d) $c_n = -c_n^*$

25. $x(t)$ is a periodic signal with even symmetry. Then, its trigonometric Fourier series expansion
(a) will not have any sinusoidal components
(b) will not have any cosinusoidal components
(c) will not have any dc components
(d) will have only even harmonic components

26. If $x(t)$ is a periodic signal with odd symmetry, its trigonometric Fourier series expansion will have
(a) only cosinusoidal components
(b) only sinusoidal components
(c) a dc component
(d) only sinusoidal components having even harmonic frequencies

27. A periodic signal with fundamental period T, is said to possess 'rotational symmetry', or 'half-wave symmetry', if
(a) $x(t + T/2) = x(t)$ for any t (b) $x(t \pm T/2) = -x(t)$ for any t
(c) $x(t - T/2) = x(t)$ for any t (d) $x(t + T/2) = x(t - T/2)$ for any t

28. A periodic signal, $x(t)$, with 'rotational symmetry', can have, in its Fourier series expansion, only
(a) odd harmonic components of the cosinusoidal type
(b) even harmonic components
(c) even harmonic components of the sinusoidal type
(d) odd harmonic components

29. A periodic signal $x(t)$ with a fundamental period T can be expanded as Fourier series only if
(a) $|x(t)| < \infty$ only at a finite number of values of t within a period T
b) $\int_{-T/2}^{T/2} |x(t)|\, dt < \infty$
(c) $x(t)$ has rotational symmetry
(d) $x(t)$ has either even or odd symmetry

30. The Fourier series of a periodic signal $x(t)$ with period T will not converge if
   (a) $|x(t)|$ is not finite at all values of t
   (b) $x(t)$ has more than one maxima in one period T
   (c) $x(t)$ is not continuous at all points
   (d) $x(t)$ is not a bandlimited signal

31. The Fourier series expansion of the periodic signal $x(t) = |\sin 2\pi f_0 t|$ can have
   (a) only odd harmonics, i.e., components with frequency nf_0 where n is odd
   (b) no dc component
   (c) only even harmonics, i.e., components with frequency nf_0 where n is even.
   (d) both even and odd harmonics of the frequency f_0

32. The dc component of the periodic waveform shown is

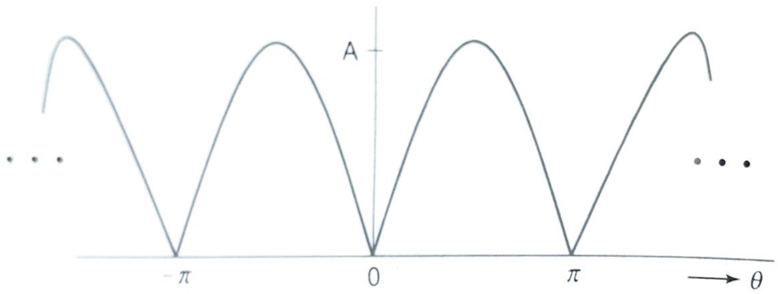


Fig. 3.24

(a) $\dfrac{A}{\pi}$ (b) $\dfrac{A}{2\pi}$ (c) $\dfrac{4A}{\pi}$ (d) $\dfrac{2A}{\pi}$

33. In the discrete spectrum of the periodic signal $x(t)$ shown in Fig. 3.25, the harmonic component having zero amplitude is

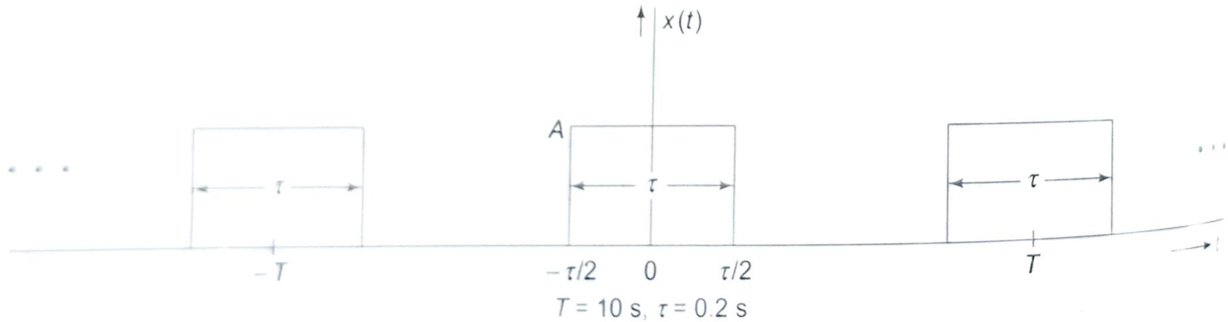


Fig. 3.25

(a) fifth (b) tenth (c) fiftieth (d) twentieth

34. $x(t)$ and $y(t)$ are both periodic signals with the same period T. It is found that $c_n^y = c_{(n+2)}^x$ where c_n^y and c_n^x are the n^{th} complex exponential Fourier series coefficients of $y(t)$ and $x(t)$ respectively. Then
   (a) $y(t) = x(t+2)$ (b) $y(t) = x(t-2)$
   (c) $y(t) = x(t)e^{-j4\pi t/T}$ (d) $y(t) = x(t)e^{j4\pi t/T}$

35. $x(t)$ and $y(t)$ are two periodic signals and $z(t) = x(t) * y(t)$. If c_n^x, c_n^y and c_n^z are the n^{th} complex exponential Fourier series coefficients of $x(t)$, $y(t)$ and $z(t)$ respectively. Then

(a) $c_n^z = Tc_n^x c_n^y$ (b) $c_n^z = c_n^x \cdot c_n^y$ (c) $c_n^z = \frac{1}{T} c_n^x \cdot c_n^y$ (d) none of these

36. $x(t)$ is a periodic signal and $x(t) \xleftrightarrow{FS} c_n^x$. $y(t) \triangleq x(at)$. The average powers in $x(t)$ and $y(t)$ are P_x and P_y respectively.

(a) $P_y = aP_x$ (b) $P_y = a^2 P_x$ (c) $P_y = \frac{1}{a^2} P_x$ (d) $P_y = P_x$

37. $x(t)$ is a periodic signal and $x(t) \xleftrightarrow{FS} c_n^x$. If $y(t) \triangleq x(at)$ and $y(t) \xleftrightarrow{FS} c_n^y$ then

(a) $c_n^y = c_n^x$ (b) $c_n^y = ac_n^x$

(c) $c_n^y = \frac{1}{a} c_n^x$ (d) none of the above

38. $x(t)$ is a periodic signal and $x(t) \xleftrightarrow{FS} c_n$. If $|c_n|$ has even symmetry and $\angle c_n$ has odd symmetry with respect to n,

(a) $x(t)$ has even symmetry
(b) $x(t)$ is complex-valued and $|x(t)|$ has even symmetry, while $\angle x(t)$ has odd symmetry
(c) $x(t)$ is purely real, or, purely imaginary
(d) $x(t)$ is purely imaginary

39. Which one of the following signals cannot be the Fourier series expansion of a periodic signal?

(a) $x(t) = 2 \cos t + B \cos 3t$ (b) $x(t) = 2 \cos \pi t + 3 \cos t$
(c) $x(t) = 3 \cos t + 0.5$ (d) $x(t) = 3 \cos 1.5 \pi t + \sin 3.5 \pi t$

40. An impulse train is represented by

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \text{ where } T_s = \frac{1}{f_s}$$

It Fourier series representation is given by

(a) $f_s \sum_{n=-\infty}^{\infty} \exp(-^{j2\pi nf_s t})$ (b) $f_s \sum_{n=-\infty}^{\infty} \exp(-j\pi nf_s t)$

(c) $f_s \sum_{n=-\infty}^{\infty} \exp(j\pi nf_s t)$ (d) $f_s \sum_{n=-\infty}^{\infty} \exp(j2\pi nf_s t)$

41. One period (O, T) each of two periodic waveforms W_1 and W_2 are shown in Fig. 3.26. The magnitudes of the n^{th} Fourier series coefficients of W_1 and W_2 for $n \geq 1$, n odd are respectively proportional to

Fig. 3.26

CONTINUOUS-TIME FOURIER TRANSFORM

4

Learning Objectives

After going through this chapter, students will be able to

- derive the Fourier transform from the Fourier series, as a limiting case,
- state the condition for the existence of the Fourier transform for a given signal $x(t)$ and also Dirichlet's conditions for the convergence of the inverse Fourier transform of $x(t)$,
- state and prove the various properties and theorems pertaining to the Fourier transform,
- determine the Fourier transform of a given signal by applying the Fourier transform theore... and sketch the magnitude and phase spectra of the signal,
- explain the physical meaning of the Fourier transform of a signal and state what it represen... relation to the signal,
- state the condition to be satisfied for obtaining the Fourier transform of a signal from its Lapl... transform $x(s)$ by simply replacing the s in $X(s)$ by $j\omega$, and
- determine the signal $x(t)$ given its Fourier transform $X(f)$.

4.1 INTRODUCTION

In the previous chapter, we had seen that any periodic function satisfying Dirichlet's conditions co... expressed as a Fourier series and that such an expression is valid over all time. We had also observed th... Fourier series coefficients provide information regarding the frequency content of the periodic signal.

The Fourier series representation does not provide an appropriate tool for the representation of aper... that is, non-periodic signals. This is because, it gives a true representation of the aperiodic signal only f... interval over which the Fourier series expansion of the signal is made; outside this interval, it repeats

In this chapter, we are interested in developing an appropriate mathematical tool for the analysis of a non-periodic signal for its spectral content, study its properties and some of its applications. We overcome the aforementioned difficulty in using the Fourier series by starting with the complex-exponential Fourier series expansion of a periodic signal with period T and then allowing T to tend to infinity. When we consider the effect of allowing T to tend to infinity on the Fourier series expansion, we arrive at what is called the Fourier transform.

The Fourier transform is a linear operator that maps a signal $f(t)$ satisfying certain conditions, into another function with 'ω', or 'f' as the independent variable. The physical meaning of the new function in relation to the original time function $f(t)$ is discussed in this chapter. In fact, it gives an indication of the spectral content of the aperiodic signal $f(t)$. This transform is invertible and the inverse Fourier transform provides a representation of $f(t)$ as a combination (integral) of weighted complex exponentials of all frequencies.

The Fourier transform is an extremely useful mathematical tool and is extensively used in the analysis of LTI systems, cryptography, signal processing, astronomy, etc. We will discuss some of these applications in the later chapters.

The Fourier transform derived in this chapter, is called the Continuous-time Fourier Transform (CTFT) to distinguish it from its several variants. However, for the sake of convenience, we shall call it here as only the 'Fourier transform'. Some of its variants are the Discrete-time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT). In the DTFT, the time parameter is discretised, making it useful for spectral analysis of discrete-time signals, or sequences. However, the frequency parameter is a continuous variable, thus making it unsuitable for the computation of the spectrum of a discrete-time signal using a digital computer. The DFT is a version of the Fourier transform in which both the time parameter, 't', and the frequency parameter 'f' are discretized, making it suitable for machine computation of the spectrum of a signal. Various algorithms for the fast computation of the DFT, called the Fast Fourier transforms (FFTs), exist. Some of these are discussed in detail in Chapter 5.

To find the spectral content of images, which are essentially two-dimensional signals, two-dimensional DFT (or 2-D DFT) and 2-D FFT are also available.

In the next section, we shall first derive the Fourier transform from the Fourier series, using the approach indicated earlier.

4.2 FOURIER TRANSFORM

In Chapter 3, we had derived the Fourier series of a periodic function. Now we shall try to arrive at the Fourier transform through the Fourier series.

For this purpose, let us start with a periodic function $f(t)$ with a fundamental period T. For the purpose of illustration, let $f(t)$ be some arbitrary periodic signal as shown in Fig. 4.1(a). We know that the spectrum of such a periodic signal will be a discrete one with spectral lines at frequencies of $0, f_0, 2f_0, 3f_0, \ldots$, with the adjacent spectral lines separated by a frequency interval of $f_0 = 1/T$. A plot of the amplitude spectrum of signal $f(t)$ is given in Fig. 4.1(b).

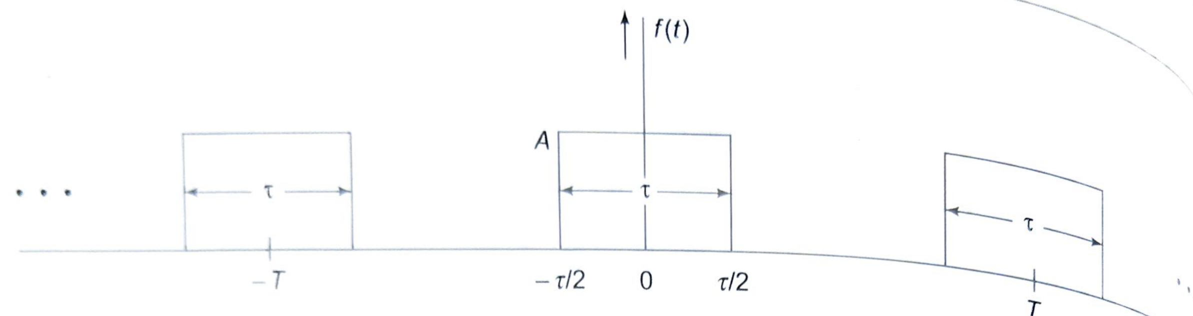


Fig. 4.1 (a) *A signal f(t)*

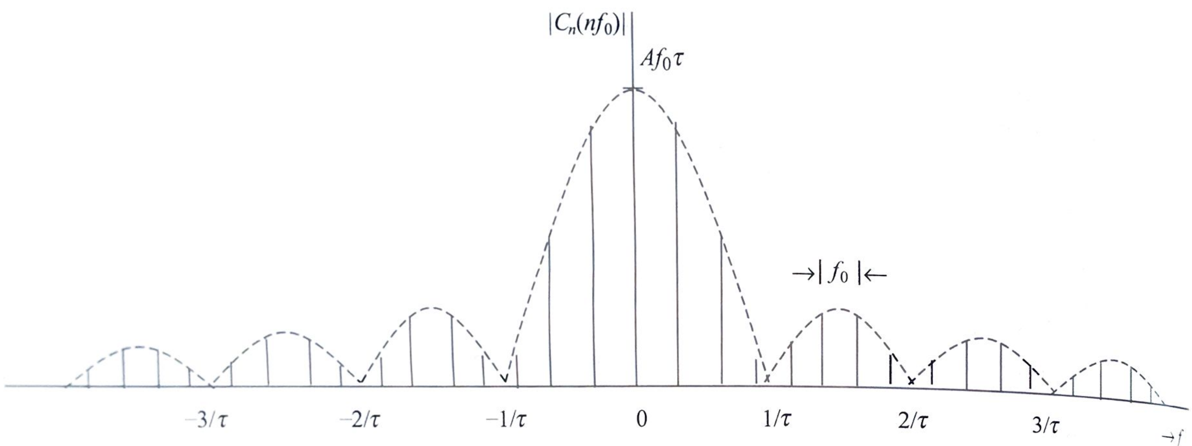


Fig. 4.1(b) *Amplitude spectrum of signal f(t) of Fig. 4.1(a)*

Suppose we now increase the fundamental period T of the periodic signal $f(t)$. Let us say we double the period T. Then the signal $f(t)$ and its amplitude spectrum will appear as shown in Figs. 4.2(a) and (b) respectively.

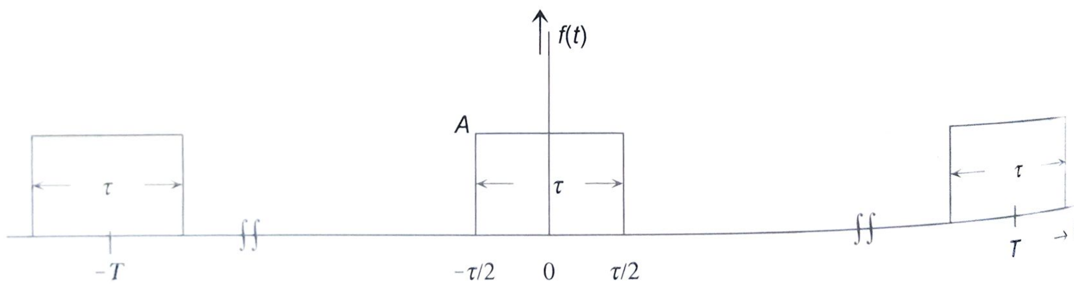


Fig. 4.2 (a) *Signal f(t) with fundamental period doubled*

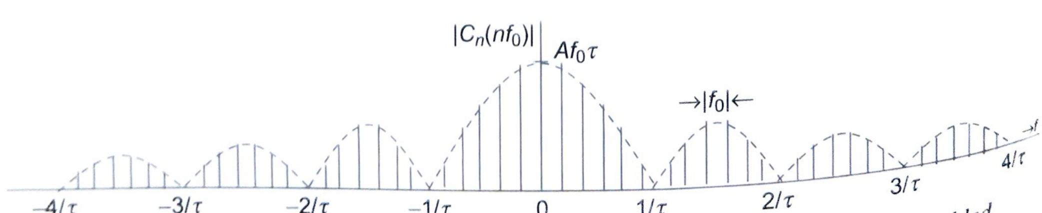


Fig. 4.2(b) *Amplitude spectrum of f(t) with the fundamental period doubled*

As can be seen from Fig. 4.2(b), an increase in the fundamental period T results in a spectrum in which the spectral lines become closer and closer. This is to be expected, since $f_0 = 1/T$ and so, as T goes on increasing, f_0 goes on decreasing and the spacing between adjacent spectral lines becomes smaller and smaller.

On the other hand, as the pulses in $f(t)$ are separated by T seconds, as T increases, the interval between adjacent pulses becomes larger and larger. Finally, as T tends to infinity, adjacent pulses become separated by an infinite amount of time, i.e., $f(t)$ becomes a non-periodic function. At the same time, separation between the spectral lines becomes infinitesimally small, or in other words, a non-periodic signal will be having a continuous spectrum.

With the above background, let us now derive the Fourier transform of a non-periodic signal making Fourier series of a periodic signal as our starting point.

Let $f(t)$ be a periodic signal with fundamental period T and let f_0 be defined as $1/T$. Then, we know that we can write

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_o t};\ \text{for} -\infty < t < \infty \qquad \ldots(4.1)$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{j2\pi n f_o t}\, dt \qquad \ldots(4.2)$$

Now, as $T \to \infty$, $f_o \to df$, an infinitesimally small quantity and nf_o becomes a continuous variable, which we shall represent by 'f'.

Then, from Eq. (4.2), it is clear that c_n becomes a function of f. Therefore, let us represent it by $c_n(f)$.

Then, Eq. (4.2) may be written as

$$TC_n(f) = \int_{-\infty}^{-\infty} f(t)\, e^{-j2\pi ft}\, dt \qquad \ldots(4.3)$$

The right hand side (of course, the left hand side too) of the above equation is a function only of f. Let us therefore represent it by $F(f)$

$$\therefore \quad F(f) \triangleq \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ft}\, dt \qquad \ldots(4.4)$$

Also, from Eq. (4.1) it follows that

$$f(t) = \frac{1}{T}\sum_{t=-\infty}^{+\infty} Tc_n(f) e^{j2\pi ft} \qquad \ldots(4.5)$$

But, $\frac{1}{T} = f_o$ and $f_o \to df$ as $T \to \infty$

$$\therefore \quad f(t) = \int_{-\infty}^{\infty} F(f)\, e^{j2\pi ft}\, dt \ \text{ for } -\infty < t < \infty \qquad \ldots(4.6)$$

Equation (4.4) transforms a function of time, $f(t)$, into a function of a new variable, f and is called the Fourier transform equation. $F(f)$ is called the Fourier transform of the non-periodic time function $f(t)$.

Equation (4.6) permits us to get back $f(t)$ from $F(f)$ and is therefore called the inverse Fourier transform. In fact, $f(t)$ and $F(f)$ are said to be forming a Fourier transform pair and this unique relationship between the two is symbolically represented by

$$f(t) \xleftrightarrow{F.T} F(f)$$

or as

$$F(f) = \mathcal{F}[f(t)]$$

or as

$$f(t) = \mathcal{F}^{-1}[F(f)]$$

Using the frequency variable 'f', we may write the Fourier transform Eq. (4.4) as

$$\boxed{F(f) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi ft}\,dt}$$

and the inverse Fourier transform Eq. (4.6) as

$$\boxed{F(t) = \int_{-\infty}^{+\infty} F(f)e^{j2\pi ft}\,df}$$

Sometimes it is more convenient to use 'ω' as the frequency variable instead of 'f'. In that case, Eqs. ... and (4.8) are written as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$

4.3 EXISTENCE OF THE FOURIER TRANSFORM

For any function $f(t)$, the Fourier transform is said to exist if it is finite. From Eq. (4.7), it follows that

$$|F(f)| = \left|\int_{-\infty}^{+\infty} f(t)e^{-j2\pi ft}\,dt\right| \le \int_{-\infty}^{+\infty}\left|f(t)e^{-j2\pi ft}\right|dt$$

i.e.,

$$|F(f)| \le \int_{-\infty}^{+\infty}|f(t)|\left|e^{-j2\pi ft}\right|dt$$

or

$$|F(f)| \le \int_{-\infty}^{+\infty}|f(t)|\,dt$$

since $\left|e^{-j2\pi ft}\right| = 1$. From Eq. (4.9), it therefore follows that $|F(f)| < \infty$ if

$$\boxed{\int_{-\infty}^{+\infty}|f(t)|\,dt < \infty}$$

Hence, the Fourier transform of a signal $f(t)$ exists if that signal is absolutely integrable. However, it should be noted that absolute integrability represents a sufficient condition rather than a necessary condition for the existence of the Fourier transform of a function $f(t)$. This is because, if we allow singularity functions or distributions, as we will be showing later on, we can derive the Fourier transforms of even such time functions as a unit-step function, a sinusoidal function, etc., which are definitely not absolutely integrable.

4.3.1 Convergence Concepts

The Fourier transform integral given by Eq. (4.7) and the inverse Fourier transform integral given by Eq. (4.8) may not converge for all functions $f(t)$ and $F(f)$ respectively.

A detailed analysis of the convergence of these integrals being beyond the scope of this book, we simply state that if a non-periodic function $f(t)$ satisfies the Dirichlet's conditions then pointwise convergence of the integral:

$$\int_{-\infty}^{+\infty} F(f)e^{j2\pi ft}\,df$$

is guaranteed for all values of t except those corresponding to discontinuities.

The Dirichlet's conditions are the following:

1. $f(t)$ should be absolutely integrable.
2. $f(t)$ should have only a finite number of maxima and minima in any finite interval of time. Further, the number of discontinuities in any finite interval of time should be finite.
3. The discontinuities, if any, must be finite discontinuities.

Fortunately, almost all the signals that we come across in physical problems satisfy all the above conditions except possibly the first one. Even if a signal is not absolutely integrable, as in the case of say, a unit-step function or a sinusoidal function, with the use of impulses, we can still obtain its Fourier transform; but these Fourier transforms do not converge.

Property 1 From Eqs. (4.7) and (4.8), it follows that

1. $f(t)\big|_{t=0} = f(0) = \int_{-\infty}^{\infty} F(f)\,dt$ = total area under the Fourier transform of $f(t)$.

2. $F(f)\big|_{t=0} = F(0) = \int_{-\infty}^{\infty} f(t)\,dt$ = Total area under the signal $f(t)$.

4.4 SOME SIMPLE PROPERTIES OF FOURIER TRANSFORM

Property 2 *Even if $x(t)$ is real valued, its Fourier transform, $X(f)$ is, in general, complex valued.* *This is obvious from the Fourier transform Eq. (4.7).*

Property 3 *If $x(t)$ is real valued, then $X(-f) = \overline{X(f)}$, the complex conjugate of $X(f)$.* *This property says that if $x(t)$ is real valued, then its spectrum $X(f)$ has Hermitian symmetry, i.e.,*

$$|X(f)| = |X(-f)| \quad \text{and} \quad \angle X(-f) = -\angle X(f)$$

Proof

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

$$\overline{X}(f) = \int_{-\infty}^{\infty} \overline{x(t)}e^{+j2\pi ft}dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi(-f)t}dt = X(-f)$$

$\therefore$
$$\overline{X}(f) = X(-f)$$

Since the magnitude remains unaltered due to complex conjugation we know that

$$\left|\overline{X}(f)\right| = |X(f)|$$

hence, from Eq. (4.11), we have

$$|X(-f)| = |X(f)|$$

Further, as the angle reverses due to conjugation,

$$\angle\overline{X(f)} = -\angle X(f)$$

$\therefore$ Combining this with Eq. (4.11), we have,

$$\angle X(-f) = -\angle X(f)$$

Property 4 *If x(t) is real and has even symmetry, then its Fourier transform X(f) is given by*

$$X(f) = 2\int_{0}^{\infty} x(t)\cos 2\pi ft\, dt$$

Also, if x(t) is real and has odd symmetry, then its Fourier transform X(f) is iven by

$$X(f) = -2j\int_{0}^{\infty} x(t)\sin 2\pi ft\, dt$$

Proof

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(t)\cos 2\pi ft\, dt - j\int_{-\infty}^{\infty} x(t)\sin 2\pi ft\, dt$$

Case 1 Let $x(t)$ be even. Then, since sin ωt is an odd function of time, the product of $x(t)$ and sin ωt is an odd function of time. Hence,

$$\int_{-\infty}^{\infty} x(t)\sin \omega t dt = 0$$

Also, $x(t)$ cos ωt will be an even function so that

$$\int_{-\infty}^{\infty} x(t)\cos \omega t dt = 2\int_{0}^{\infty} x(t)\cos \omega t dt$$

$\therefore$ if $x(t)$ is even,
$$X(f) = 2\int_{0}^{\infty} x(t)\cos 2\pi ft\, dt$$

Case 2 $x(t)$ is odd: The proof for this case is left as an exercise to the reader.

In general, if $x(t)$ is complex valued and is conjugate symmetric, i.e., $x(-t) = x^*(t)$, then

$$X(f) = \mathscr{F}[x(t)] = 2\int_{0}^{\infty}[x_r(t)\cos 2\pi ft + x_i(t)\sin 2\pi ft]\, dt$$

i.e., $X(f)$ will be purely real. In Eq 4.14(a), where $x_r(t)$ and $x_i(t)$ are respectively the real and imaginary parts of $x(t)$.
Also, if $x(t)$ is complex valued and is conjugate anti-symmetric, i.e., $x(-t) = -x^*(t)$, then we have

$$X(f) = \mathscr{F}[x(t)] = -2j \int_0^\infty [x_r(t)\sin 2\pi ft - x_i(t)\cos 2\pi ft]\,dt \qquad \ldots(4.15a)$$

i.e., $X(f)$ is purely imaginary.
Note that when $x(t)$ is real valued, Eq. (4.14a) reduces to Eq. (4.14) and Eq. (4.15a) reduces to Eq. (4.15).

4.5 MAGNITUDE AND PHASE SPECTRA OF SIGNALS

In property (1) above, we have noted that even if $x(t)$ is a real-valued function, $X(f)$ is, in general, a complex-valued function of frequency. Since $X(f)$ is called the spectrum of the signal $x(t)$, a plot of $|X(f)|$ vs f is called the magnitude spectrum of $x(t)$ and a plot of $\angle X(f)$ vs f is called the phase spectrum of $x(t)$.

Now, let us determine the magnitude and phase spectra of some simple signals

Example 4.1 $x(t) = \begin{cases} A & ; \quad t \le |\tau/2| \\ 0 & ; \quad \textit{otherwise} \end{cases}$

Solution

The above signal is a rectangular pulse and its plot is as shown in Fig. 4.3.

As we will be coming across this type of signal quite frequently, a special symbol is given to it.

$$x(t) = A\Pi(t/\tau) \quad \text{or} \quad A\ \text{rect}(t/\tau)$$

A indicates that it has an amplitude A; t indicates that it is in the time domain and τ indicates that the rectangular pulse has a total width of τ along the time axis.

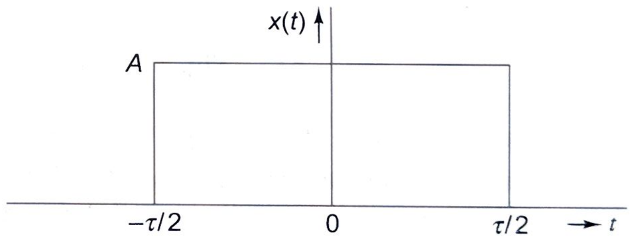


Fig. 4.3 *A rectangular pulse*

Note *It is always understood that the rectangular pulse represented by the above symbols, is symmetrically situated with reference to the time origin.*

As per the Fourier transform equation, we have

$$X(f) = \mathscr{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int_{-\tau/2}^{\tau/2} x(t)e^{-j2\pi ft}\,dt$$

$$= 2\int_0^{\tau/2} A\cos\omega t dt = \frac{2A}{\omega}\sin\omega t\Big|_0^{\tau/2} = A\tau\left(\frac{\sin\pi f\tau}{\pi f\tau}\right)$$

Defining

$$\frac{\sin\pi\lambda}{\pi\lambda} \ \underline{\Delta}\ \text{sinc}\,\lambda, \qquad \ldots(4.16)$$

we have

$$\boxed{X(f) = \mathscr{F}[A\Pi(t/\tau)] = A\tau\,\text{sinc}\,f\tau} \qquad \ldots(4.17)$$

Magnitude and phase spectra of the signal $x(t)$ are shown in the Figs. 4.4 and 4.5.

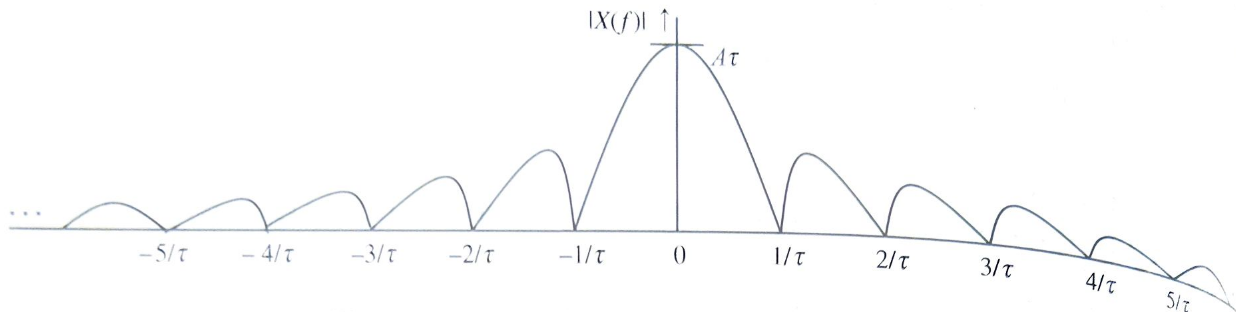


Fig. 4.4 *Magnitude spectrum of a rectangular pulse x(t)*

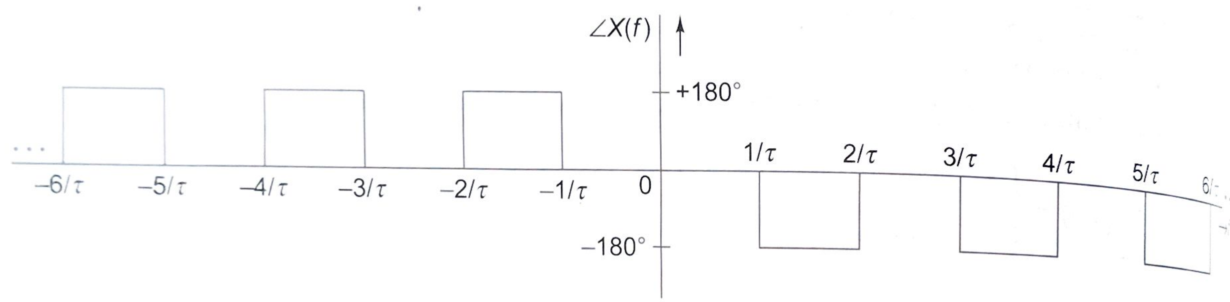


Fig. 4.5 *Phase spectrum of rectangular pulse x(t)*

In this case, $X(f) = A\tau \text{ sinc } f\tau$ is a purely real-valued function. However, this function changes its s whenever f takes the values $\pm 1/\tau, \pm 2/\tau, \pm 3/\tau \ldots$. This change of sign is interpreted here as a phase shi 180°. Actually, one need not distinguish between +180° phase shift and −180° phase shift. But we are de erately showing the +180° and −180° separately because we want to emphasize the fact that $x(t)$ being a re valued function, its $X(f)$ must have Hermitian symmetry, i.e., while its magnitude spectrum will have e symmetry, its phase spectrum will have odd symmetry.

Example 4.2 *Find and plot the magnitude and phase spectra of the signal* $x(t) = Ae^{-t/T}u(t)$ where A T are real-valued constants.

Solution

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{+\infty} Ae^{-t/T}u(t)e^{-j2\pi ft}\,dt$$

$$= A\int_{0}^{\infty} e^{\frac{-(1+j2\pi fT)t}{T}}\,dt = \frac{AT}{1+j2\pi fT}$$

Therefore,
$$|X(f)| = \frac{AT}{\sqrt{1+4\pi^2 f^2 T^2}} \text{ and } \angle X(f) = -\tan^{-1}(2\pi f T)$$

A plot of the magnitude spectrum is shown in Fig. 4.6 and that of the phase spectrum in Fig. 4.7.

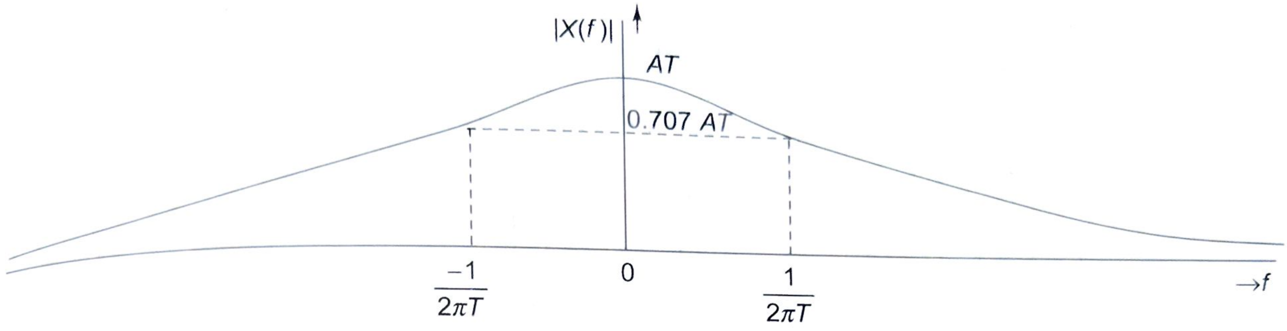


Fig. 4.6 *Magnitude spectrum of x(t)*

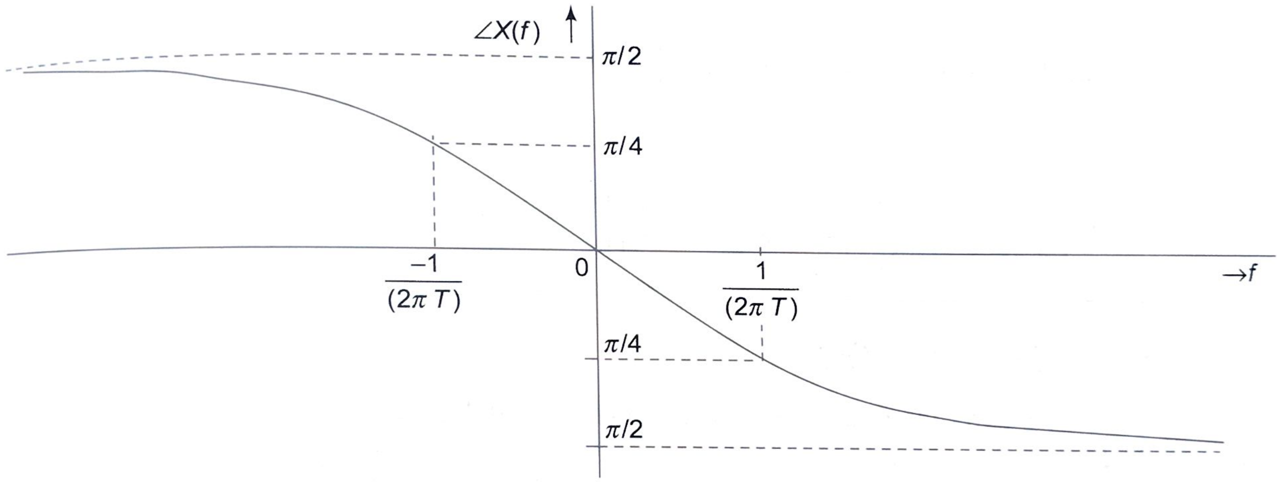


Fig. 4.7 *Phase spectrum of x(t)*

Example 4.3 *Given* $x(t) = (A \cos \omega_0 t)\Pi(t/\tau)$ *where* $\tau = \dfrac{1}{2f_0}$, *find* $X(f)$.

Solution

$$X(f) = A\int_{-\infty}^{\infty}\left[\cos(\omega_0 t)\Pi(t/\tau)\right]e^{-j2\pi ft}\,dt$$

$$= A\int_{-\tau/2}^{\tau/2}\cos(\omega_0 t)\,e^{-j2\pi ft}\,dt$$

$$= 2A\int_{0}^{\tau/2}\cos\omega_0 t\cdot\cos\omega\, t\;dt$$

$$= A\int_{0}^{\tau/2}[\cos(\omega-\omega_0)t+\cos(\omega-\omega_0)t]\,dt$$

$$= A\left[\frac{\sin(\omega-\omega_0)t}{(\omega-\omega_0)}\bigg|_0^{\tau/2}+\frac{\sin(\omega+\omega_0)t}{(\omega+\omega_0)}\bigg|_0^{\tau/2}\right]$$

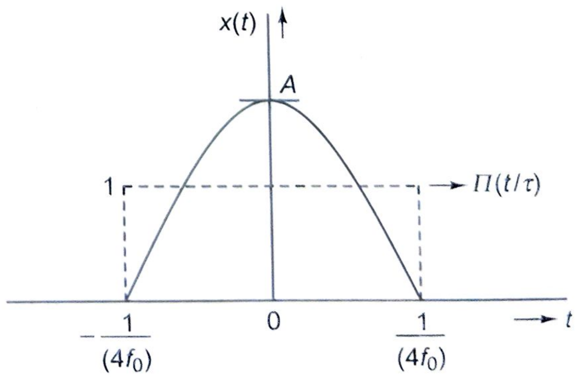


Fig. 4.8 *x(t) of Example 4.3*

$$= A\left[\frac{\sin 2\pi(f-f_0)\dfrac{1}{4f_0}}{2\pi(f-f_0)} + \frac{\sin 2\pi(f+f_0)\dfrac{1}{4f_0}}{2\pi(f+f_0)}\right]$$

$$= A\left[\frac{\sin \pi\left(\dfrac{f}{2f_0}-\dfrac{1}{2}\right)}{(4f_0)\pi\left(\dfrac{f}{2f_0}-\dfrac{1}{2}\right)} + \frac{\sin \pi\left(\dfrac{f}{2f_0}+\dfrac{1}{2}\right)}{(4f_0)\pi\left(\dfrac{f}{2f_0}+\dfrac{1}{2}\right)}\right]$$

$$= \left[\operatorname{sinc}\left(f\tau-\frac{1}{2}\right) + \operatorname{sinc}\left(f\tau+\frac{1}{2}\right)\right]\frac{2A}{\tau}$$

Hence,

$$X(f) = \frac{2A}{\tau}\left[\operatorname{sinc}\left(f\tau-\frac{1}{2}\right) + \operatorname{sinc}\left(f\tau+\frac{1}{2}\right)\right]$$

4.6 PARSEVAL'S THEOREM

All along, we have been using the Fourier transform equation and determining the Fourier transform of a given signal $x(t)$. At this stage, it would perhaps be appropriate to explore the physical meaning of this function $X(f)$ which we have been calling as the Fourier transform of the signal $x(t)$ and see what exactly it represents insofar as the signal $x(t)$ is concerned.

For this purpose, we shall now discuss 'Parseval's Theorem' or 'Rayleigh's Theorem' pertaining to the Fourier transform.

Parseval's Theorem If signals $x(t)$ and $y(t)$ have Fourier transforms $X(f)$ and $Y(f)$ respectively, then

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}\,dt = \int_{-\infty}^{\infty} X(f)\overline{Y(f)}\,df$$

Proof Since $Y(f)$ is the Fourier transform of $y(t)$, we may write

$$y(t) = \mathscr{F}^{-1}[Y(f)] = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}\,df$$

Therefore,

$$\overline{y(t)} = \int_{-\infty}^{\infty} \overline{Y(f)}e^{-j2\pi ft}\,df$$

Hence,

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}\,dt = \int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} \overline{Y(f)}\,e^{-j2\pi ft}\,df\right]dt$$

In the above, the inner integration is with respect to 'f' and the outer integration is with respect to the variable 't'. Now, interchanging the order of these integrations, we get

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}\,dt = \int_{-\infty}^{\infty} \overline{Y(f)}\left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt\right]df$$

$$= \int_{-\infty}^{\infty} \overline{Y(f)}.X(f)\,df$$

Thus,

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}\,dt = \int_{-\infty}^{\infty} X(f)\overline{Y(f)}\,df \qquad \ldots (4.18)$$

If now, $y(t)$ is the same as $x(t)$, i.e., $y(t) = x(t)$, then $Y(f) = X(f)$ and Eq. (4.18) may be written as

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2\,df \qquad \ldots (4.19)$$

But we know that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = E_x = \text{energy of the signal } x(t)$$

Therefore,

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2\,df} \qquad \ldots (4.20)$$

Equation (4.20) may be used to arrive at the physical meaning of $X(f)$. We find that integration of $|X(f)|^2$ with respect to frequency for all frequencies, gives the total energy in the signal. From this, we can deduce that $|X(f)|^2$ represents the energy density with respect to frequency, i.e., it gives the energy of the signal $x(t)$ in a unit interval of frequency, taken at the frequency 'f'. Further, Eq. (4.20) makes it clear that the energy of a signal $x(t)$ may be calculated in the time domain or in the frequency domain.

Example 4.4 *If the signal $x(t) = Ae^{-t/T}u(t)$ considered in Example 4.2 is given as input to an ideal low-pass filter whose cut-off frequency is $f_c = \dfrac{1}{(2\pi T)}$, what percentage of the energy of $x(t)$ will be available at the output of the filter?*

Solution As derived in Example 4.2, the spectrum of the signal $x(t)$ is given by its Fourier transform $X(f)$ and $|X(f)|^2$ is given by

$$|X(f)|^2 = \frac{A^2T^2}{1+4\pi^2 f^2T^2}$$

Hence, the total energy E_x of $x(t)$ is given by

$$E_x = \int_{-\infty}^{\infty} \frac{A^2T^2}{1+4\pi^2 f^2T^2}\,df$$

Putting $2\pi fT = \tan\theta$ in the above and noting that $df = (1/2\pi T)\sec^2\theta\, d\theta$, we have

$$E_x = \int_{-\infty}^{\infty} \frac{A^2T^2}{1+4\pi^2 f^2 T^2}\, df = \int_{-\pi/2}^{\pi/2} \frac{A^2T^2}{(1+\tan^2\theta)}\left(\frac{1}{2\pi T}\right)\sec^2\theta\, d\theta$$

$$\therefore \quad E_x = \int_{-\pi/2}^{\pi/2}\left(\frac{A^2T}{2\pi}\right)d\theta = \frac{A^2T}{2}$$

The filter passes on to the output only those frequency components of the input which are having frequencies up to $\frac{1}{2\pi T}$. Since $2\pi fT = \tan\theta$, when $f = \frac{1}{2\pi T}$, $\tan\theta = 1 \quad \therefore \theta = +\pi/4$.

When $f = -\frac{1}{2\pi T}$, $\tan\theta = -1 \quad \therefore \quad \theta = -\frac{\pi}{4}$

Hence, the energy contained in the output signal is given by

$$E_1 = \int_{-\pi/4}^{\pi/4}\left(\frac{A^2T}{2\pi}\right)d\theta = \frac{A^2T}{2\pi}\cdot\frac{\pi}{2} = \frac{A^2T}{4}$$

Hence, percentage of energy at the output $= \frac{E_1}{E_x} \times 100\%$

$$= \frac{A^2T}{4} \times \frac{2}{A^2T} \times 100 = 50\%$$

4.7 SOME FOURIER TRANSFORM THEOREMS

We will now discuss a few Fourier transform theorems. These theorems are very useful in finding the Fourier transforms of more complicated signals in terms of those of simpler signals.

1. Linearity theorem Fourier transform is a linear transform, in the sense that it obeys the superposition and homogeneity principles. That is, the Fourier transform of the sum of two signals is equal to the sum of their individual Fourier transforms and the Fourier transform of a constant times a signal is equal to the Fourier transform of the signal multiplied by that constant. Both superposition and homogeneity are implied in the following statement of the Linearity theorem.

If $x(t)$ and $y(t)$ have Fourier transforms given by $X(f)$ and $Y(f)$ respectively, and if α and β are any two arbitrary constants, then

$$\boxed{\mathcal{F}\left[\alpha x(t) + \beta y(t)\right] = \alpha X(f) + \beta Y(f)} \quad (4.2\ldots)$$

Proof of this theorem is trivial and left as an exercise to the reader.

2. Time-delay theorem This theorem gives us the Fourier transform of the time-shifted version of a signal $x(t)$ in terms of $X(f)$, the Fourier transform of $x(t)$. It states that

If $X(f)$ is the Fourier transform of $x(t)$, then the Fourier transform of $x(t-\tau)$, where τ is a real-valued constant (positive, or negative), is given by $X(f)e^{-j2\pi f\tau}$; that is, $\mathcal{F}[x(t-\tau)] = X(f)e^{-j2\pi f\tau}$

$$\mathscr{F}[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

Proof

$$\mathscr{F}[x(t-\tau)] = \int_{-\infty}^{\infty} x(t-\tau)\,e^{-j2\pi ft}\,dt$$

Hence,

Putting $t-\tau = t', t = t' + \tau$ and $dt = dt'$

$$\mathscr{F}[x(t-\tau)] = \int_{-\infty}^{\infty} x(t')\,e^{-j2\pi f(t'+\tau)}dt'$$

$$= \left[\int_{-\infty}^{\infty} x(t')e^{-j2\pi ft'}\,dt'\right]e^{-j2\pi f\tau} = X(f)e^{-j2\pi f\tau}$$

$$\boxed{x(t-\tau) \xleftrightarrow{F.T} X(f)e^{-j2\pi f\tau}} \quad \ldots (4.22)$$

3. **Modulation theorem** In communication engineering, it is quite common to translate or shift along the frequency scale, a low-frequency message signal, to facilitate its transmission over long distances. This process is called modulation and one easy way of accomplishing this, is by multiplying the low-frequency message signal $x(t)$ with a high-frequency carrier signal, say $e^{j2\pi f_c t}$, which has frequency of f_c. Of course, since $e^{j2\pi f_c t}$, which is a complex-valued signal, cannot be generated in practice, we use a cosine signal, $\cos \omega_c t$.

This theorem tells us that if $x(t)$ has a spectrum $X(f)$, then $x(t)e^{j\omega_c t}$ will have a spectrum given by $X(f-f_c)$. We know that $X(f-f_c)$ has exactly the same shape as $X(f)$, the only difference being that it is a shifted version of $X(f)$, shifted along the frequency axis to the right by an amount of frequency equal to the carrier frequency f_c.

Statement If $X(f) = \mathscr{F}[x(t)]$ then $\mathscr{F}[x(t)e^{j\omega_c t}] = X(f-f_c)$.

Proof

$$\mathscr{F}[x(t)e^{j\omega_c t}] = \int_{-\infty}^{\infty} [x(t)e^{j\omega_c t}]\,e^{-j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_c)t}\,dt = X(f-f_c)$$

$$\boxed{\mathscr{F}\left[x(t)e^{j2\pi f_c t}\right] = X(f-f_c)} \quad \ldots (4.23)$$

Suppose $x(t)$ is a low-frequency signal having frequency components from 0 Hz to W Hz. Let its spectrum, i.e., $X(f)$ be as shown in Fig. 4.9. The actual shape of $X(f)$ assumed here has no particular significance; however, since $x(t)$ is a real-valued signal, its Fourier transform should have a magnitude that has even symmetry.

As already mentioned earlier, in practice we have to use a real signal, say, $\cos \omega_c t$ as the carrier signal. So let us examine how the spectrum of $x(t) \cos \omega_c t$ would look like

$$x(t)e^{j\omega_c t} \xleftrightarrow{FT} X(f-f_c)$$

$$x(t)e^{-j\omega_c t} \xleftrightarrow{FT} X(f+f_c)$$

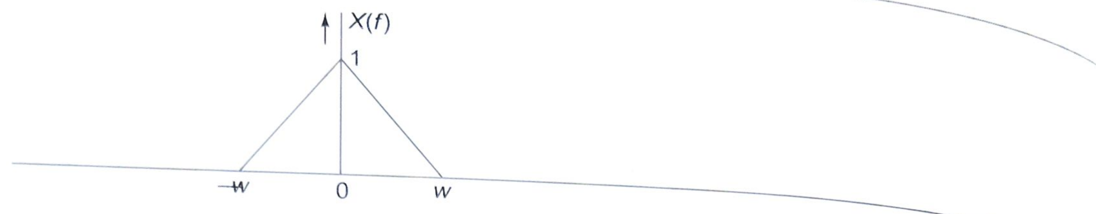


Fig. 4.9 *Spectrum of x(t)*

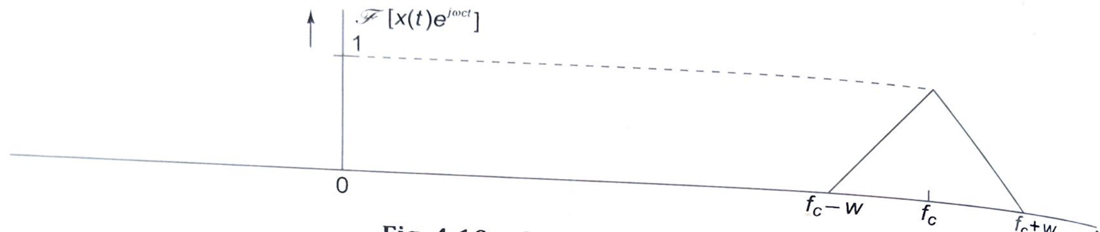


Fig. 4.10 *Spectrum of* $x(t)e^{j\omega_c t}$

Adding these two and invoking the linearity property of the Fourier transform, we get

$$x(t)\left[e^{j\omega_c t}+e^{-j\omega_c t}\right] \xleftrightarrow{FT} X(f-f_c)+X(f+f_c)$$

$\therefore$

$$x(t)\cos\omega_c t \xleftrightarrow{FT} \frac{1}{2}\left[X(f-f_c)+X(f+f_c)\right] \qquad \ldots(4.$$

Hence, the spectrum of $x(t)$ cos $\omega_c t$ would appear as shown in Fig. 4.11

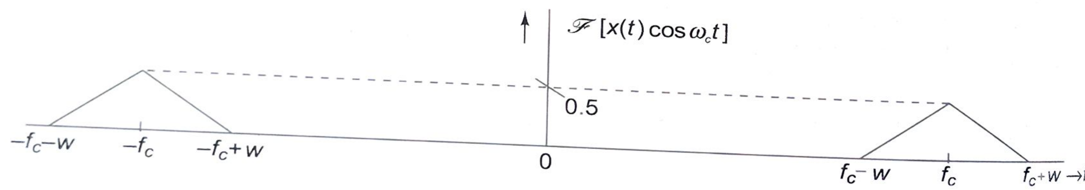


Fig. 4.11 *Spectrum of* $x(t)$ *cos* $\omega_c t$

It may be noted that whereas the spectrum of $x(t)$ cos $\omega_c t$ is having even symmetry, that of $x(t)\,e^{j\omega_c t}$ does not have even symmetry. This is because, while $x(t)$ cos $\omega_c t$ is a real-valued function, $x(t)\,e^{j\omega_c t}$ is not.

4. Scaling theorem In Section 1.5, wherein we had discussed various operations like time-shifting and time-scaling of signals, it was shown that $x(at)$ represents a time compressed version of $x(t)$ if the constant $a > 1$ and that it represents a time-expanded version of $x(t)$ if $0 < a < 1$.

The scaling theorem discusses the effect on the spectrum, of compressing or expanding a signal in time.

Statement If $X(f)$ is the spectrum of a signal $x(t)$, the spectrum of $x(at)$ is given by $\frac{1}{|a|}X\left(\frac{f}{a}\right)$

Proof We know that

$$\mathscr{F}[x(at)] = \int_{-\infty}^{\infty} x(at)\,e^{-j2\pi ft}\,dt$$

Case 1 Let $a > 0$

Putting $t' = at$, we have , $dt' = adt$

and

$$\mathscr{F}[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(t')\, e^{-j2\pi(f/a)t'}\, dt'$$

$$= \frac{1}{a} X(f/a) \qquad \ldots (4.25)$$

Case 2 Let $a < 0$.

Putting $t' = at$, we have, $dt' = a\, dt$

and

$$\mathscr{F}[x(at)] = \frac{1}{a}\int_{+\infty}^{-\infty} x(t')e^{-j2\pi(f/a)t'}\, dt'$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(t')e^{-j2\pi(f/a)t'}\, dt' = \frac{1}{|a|} X(f/a) \qquad \ldots (4.26)$$

From Eqs. (4.26) and (4.25), we may write in general, that

$$\boxed{\mathscr{F}[x(at)] = \frac{1}{|a|} X(f/a)} \qquad \ldots (4.27)$$

Remarks

1. *For $a > 1$, $x(at)$ represents a time-compressed version of the signal $x(t)$; but $X(f/a)$ represents a spectrum that has expanded in frequency. Hence, compressing a signal in time results in an expansion of its spectrum.*
2. *For $0 < a < 1$, $x(at)$ represents a signal that is expanded in time. But $X(f/a)$ represents a compression of the spectrum. Hence expansion of a signal in time results in a compression of its spectrum.*
3. *Compression of a signal in time leads to its expansion in frequency and vice versa. Hence, a signal cannot be compressed/expanded simultaneously in time as well as in frequency.*
4. *If a is negative, there will be lateral inversion of the spectrum accompanied by compression or expansion, depending upon whether $|a|$ is greater than or less than unity.*

Suppose, a gramophone disk which is to be played at 45 rpm is played at 72 rpm by mistake. A male voice that would have been recorded on it would sound like a female voice! The reason is that playing at 72 rpm a song that was recorded at 45 rpm amounts to compressing the song in time. Hence, its spectrum expands, producing a large number of high frequency components, making the voice sound like that of a female. Similarly, if a song rendered by a female and recorded at 72 rpm, is played at 45 rpm, the signal is expanded in time. Hence its spectrum is compressed, making the original high frequency components of the female voice to appear as low-frequency components. Thus, it would sound like a male voice.

5. Duality theorem As may be seen from Example 4.7, the duality theorem makes it possible for us to write down the Fourier transforms of certain signals just by inspection.

Statement If $X(f)$ is the Fourier transform of $x(t)$, the Fourier transform of $X(t)$ is given by $x(-f)$.

Proof

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

Interchanging t and f,

$$X(t) = \int_{-\infty}^{\infty} x(f)e^{-j2\pi ft}\,df$$

Now, putting $f' = -f$

$$X(t) = -\int_{+\infty}^{-\infty} x(-f')e^{j2\pi f't}\,df' = \int_{-\infty}^{\infty} x(-f')e^{j2\pi f't}\,df' = \mathcal{F}^{-1}[x(-f)]$$

$$\boxed{\mathcal{F}[X(t)] = x(-f)}$$

6. Convolution theorem As we are going to see later, the output signal of a linear time-invarian s can be obtained by the convolution of the input signal with the impulse response function $h(t)$ of the s However, evaluating the convolution integral is time-consuming and laborious.

(a) The convolution theorem that we are going to discuss presently converts a time-domain conv operation into a frequency domain multiplication operation. As multiplication is a much easier opera compute as compared to the convolution, the convolution theorem enables us to use the Fourier transf advantage in the analysis of linear time-invariant systems.

Statement Let $X(f)$ and $Y(f)$ be the Fourier transforms of $x(t)$ and $y(t)$ respectively. If $z(t) = x(t)*$ where * denotes convolution operation, and if $Z(f)$ is the Fourier transform of $z(t)$ then

$$Z(f) = X(f)\,.\,Y(f)$$

Proof

$$Z(f) = \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(\lambda)y(t-\lambda)\,d\lambda\right\}e^{-j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty} x(\lambda)\left\{\int_{-\infty}^{\infty} y(t-\lambda)e^{-j2\pi ft}\,dt\right\}d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)Y(f)e^{-j2\pi\lambda f}\,d\lambda = Y(f)\int_{-\infty}^{\infty} x(\lambda)e^{-j2\pi\lambda f}\,d\lambda$$

$$= Y(f)\cdot X(f)$$

$$\therefore \quad \boxed{Z(f) = Y(f)\,.\,X(f) \text{ if } z(t) = x(t) * y(t)}$$

(b) The convolution theorem also tells us that a time-domain product will be converted by the Fourier tr form into a convolution in the frequency domain.

Statement Let $X(f)$ and $Y(f)$ be the Fourier transforms of $x(t)$ and $y(t)$ respectively. If $z(t) = x(t).y(t)$, $Z(f)$ is the Fourier transform of $z(t)$, then $Z(f)$ is given by

$$Z(f) = X(f) * Y(f) = \int_{-\infty}^{\infty} X(\lambda) Y(f-\lambda) d\lambda$$

Proof

$$Z(f) = \mathscr{F}[z(t)] = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$

But

$$z(t) = x(t) \cdot y(t)$$

$\therefore$

$$Z(f) = \int_{-\infty}^{\infty} \{x(t).y(t)\} e^{-j2\pi ft} dt$$

However,

$$y(t) = \int_{-\infty}^{\infty} Y(\lambda) e^{j2\pi\lambda t} d\lambda$$

$\therefore$

$$Z(f) = \int_{-\infty}^{\infty} x(t) \cdot \left\{ \int_{-\infty}^{\infty} Y(\lambda) e^{j2\pi\lambda t} d\lambda \right\} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} Y(\lambda) \left\{ \int_{-\infty}^{\infty} [x(t) e^{j2\pi\lambda t}] e^{-j2\pi ft} dt \right\} d\lambda$$

But we know that $\mathscr{F}[x(t) e^{j2\pi f_c t}] = X(f - f_c)$

$$= \int_{-\infty}^{\infty} \left\{ x(t) e^{j2\pi\lambda t} \right\} e^{-j2\pi ft} dt = X(f-\lambda)$$

$$Z(f) = \int_{-\infty}^{\infty} Y(\lambda) X(f-\lambda) d\lambda = X(f) * Y(f)$$

$\therefore$

$$\boxed{Z(f) = X(f) * Y(f) \text{ if } z(t) = x(t) \cdot y(t)} \quad \ldots\ldots (4.30)$$

Example 4.5 *Find the Fourier transform of* $x(t) = \begin{cases} \cos \pi t; & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0; & \text{otherwise} \end{cases}$.

Solution We shall solve it by two methods.

(a) Using the defining equation of the Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{-1/2}^{1/2} \cos \pi t \left[\cos 2\pi ft - j \sin 2\pi ft\right] dt$$

$$= \int_{-1/2}^{1/2} \cos \pi t \cdot \cos 2\pi ft dt - j \int_{-1/2}^{1/2} \cos \pi t \cdot \sin 2\pi ft dt$$

In the above, the second integral is zero since $\cos \pi t$ is even while $\sin 2\pi ft$ is odd.

$$\therefore \quad X(f) = \int_{-1/2}^{1/2} \cos \pi t \cos 2\pi ft dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} \{\cos \pi(2f+1)t + \cos \pi(2f-1)t\}\, dt$$

$$= \frac{1}{2}\left[\frac{\sin \pi(2f+1)t}{\pi(2f+1)}\Bigg|_{t=-1/2}^{t=1/2} + \frac{\sin \pi(2f-1)t}{\pi(2f-1)}\Bigg|_{t=-1/2}^{1/2}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin \pi(f+1/2)}{2\pi(f+1/2)} + \frac{2\sin \pi(f-1/2)}{2\pi(f-1/2)}\right]$$

$$= \frac{1}{2}\left[\text{sinc}\left(f+\frac{1}{2}\right) + \text{sinc}\left(f-\frac{1}{2}\right)\right]$$

(b) Using the convolution theorem of Fourier transform:

The given $x(t)$ is shown in Fig. 4.12(a).

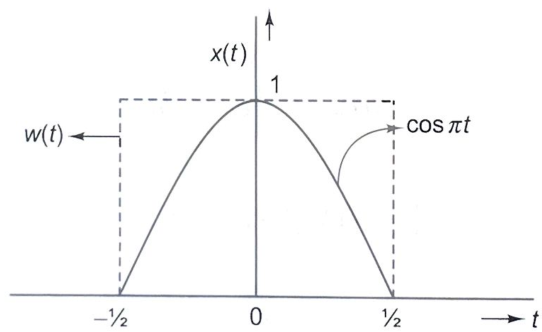


Fig. 4.12(a) *The given x(t)*

This may be viewed as the product of a signal $x_1(t) = \cos \pi t$ extending from $-\infty$ to $+\infty$ multiplied by a window function $w(t) = \Pi\,(t/1)$, which has a value 1 for $|t| \le 1/2$ and zero outside.

$$\therefore \quad x(t) = \cos \pi t . w(t)$$

$$X(f) = \mathcal{F}[\cos \pi t] * W(f) \quad \text{(by convolution theorem)}$$

But

$$\mathcal{F}[\cos \pi t] = \frac{1}{2}\left[\delta\left(f-\frac{1}{2}\right) + \delta\left(f+\frac{1}{2}\right)\right]$$

$$\therefore \quad X(f) = \frac{1}{2}\left[\delta\left(f-\frac{1}{2}\right) + \delta\left(f+\frac{1}{2}\right)\right] * W(f)$$

$$= \frac{1}{2}\left[W\left(f-\frac{1}{2}\right)+W\left(f+\frac{1}{2}\right)\right]$$

But $w(t) = \Pi(t/1) \quad \therefore \; W(f) = \text{sinc}\, f$

$$\therefore \quad X(f) = \frac{1}{2}\left[\text{sinc}\left(f-\frac{1}{2}\right)+\text{sinc}\left(f+\frac{1}{2}\right)\right]$$

Example 4.6 *Find the convolution of* $x(t) = \dfrac{\sin^2 \pi t}{\pi^2 t^2}$ *and* $y(t) = \dfrac{\sin 2\pi t}{\pi t}$.

Solution

$$x(t) = \frac{\sin^2 \pi t}{\pi^2 t^2} = \left(\frac{\sin \pi t}{\pi t}\right)^2 = \text{sinc}^2 t,$$

since $\text{sinc}\; t \triangleq \left(\dfrac{\sin \pi t}{\pi t}\right)$.

$$y(t) = \frac{\sin 2\pi t}{\pi t} = 2\left(\frac{\sin 2\pi t}{2\pi t}\right) = 2\;\text{sinc}\; 2t$$

Now, $\text{sinc}^2 t = \text{sinc}\; t \cdot \text{sinc}\; t$ and $\text{sinc}\; t \xleftrightarrow{FT} \Pi(f/1)$

$\text{sinc}^2 t \xleftrightarrow{FT} \Pi(f/1) * \Pi(f/1) = \Lambda(f/2)$ (Refer to Section 7.8)

Where, $\Lambda(f/2)$ denotes a symmetrical triangular pulse of unit height and base width of two units, as shown in Fig. 4.12(b).

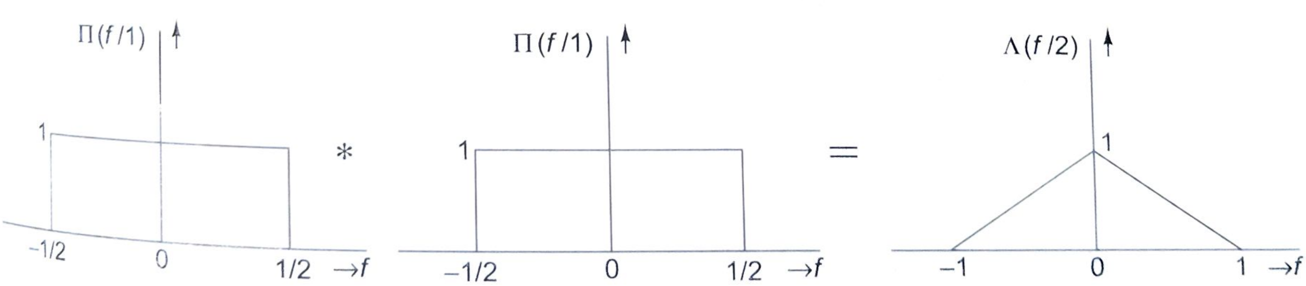


Fig. 4.12(b) *Convolution of two identical rectangular pulses results in a triangular pulse*

Next, $2\text{sinc}\; 2t \xleftrightarrow{FT} \Pi(f/2)$

$\therefore \; \mathcal{F}\left[\text{sinc}^2\, t * 2\text{sinc}2t\right] = \Lambda(f/2)\,.\,\Pi(f/2)$ (Convolution theorem)

This product of $\Lambda(f/2)$ and $\Pi(f/2)$ is shown pictorially in Fig. 4.12(c).

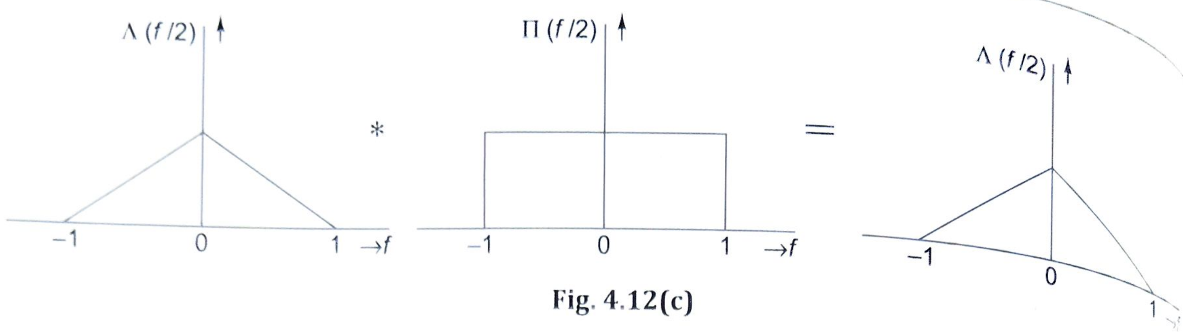


Fig. 4.12(c)

Taking the inverse Fourier transform on both sides, we get

$$\text{sinc}^2 t * 2\ \text{sinc}2t = \text{sinc}^2 t$$

i.e.,

$$x(t)*y(t) = x(t) \text{ for the given } x(t) \text{ and } y(t)$$

7. Differentiation-in-time theorem This theorem enables us to straightaway write down the Fourier tr... form of the derivative of a signal in terms of the Fourier Transform of the signal itself.

Statement If $X(f)$ is the Fourier transform of $x(t)$, then the Fourier transform of the time derivative of x... that is $\dot{x}(t)$, is given by $(j2\pi f)X(f)$.

Proof We know that

$$x(t) = \mathscr{F}^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df \quad \text{....(4.31)}$$

Taking the time derivative on both sides,

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df\right]$$

$$= \int_{-\infty}^{\infty} \{j2\pi f X(f)\}\, e^{j2\pi f t} df \quad \text{....(4.32)}$$

Comparing Eq.(4.32) with Eq.(4.31), we get

$$\boxed{\mathscr{F}\left[\frac{d}{dt}x(t)\right] = j2\pi f\ X(f)} \quad \text{....(4.33)}$$

n iterations of the above process yields

$$\boxed{\mathscr{F}\left[\frac{d^n}{dt^n}x(t)\right] = (j2\pi f)^n\ X(f)} \quad \text{....(4.34)}$$

Note *The fact that x(t) has finite energy and is Fourier transformable, does not in any way guarant... that its derivative or integral also is Fourier transformable.*

There is also an integration theorem pertaining to Fourier transform, but we shall take it up at a later st... in this chapter.

Remarks *Equation (4.33) clearly brings out the following two points:*

1. *The phase spectrum of $\dot{x}(t)$ is obtained by adding 90° at all frequencies to the phase spectrum of $x(t)$.*
2. *Multiplication of $X(f)$ by $2\pi f$ clearly shows that differentiation accentuates high frequencies.*

8. Differentiation in frequency theorem If $X(f)$ is the Fourier transform of $x(t)$, the inverse Fourier transform of $\frac{d}{df}X(f)$ is given by $-j2\pi t\,x(t)$

Proof

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

$$\therefore \quad \frac{d}{df}X(f) = \int_{-\infty}^{\infty} \{x(t)(-j2\pi t\}e^{-j2\pi ft}\,dt$$

$\therefore$ we may, by comparing the above equations, state that

$$\boxed{(-j2\pi t)x(t) \xleftrightarrow{FT} \frac{d}{df}X(f)} \qquad \ldots. (4.33(a))$$

Table 4.1 *Useful Fourier transform theorems*

Theorem	*Function*	*Transform*
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
Time-delay	$x(t-\tau)$	$X(f)\,e^{-j\omega\tau}$
Scale change	$x(at)$	$\frac{1}{\|a\|}X(f/a)$
Conjugation	$\bar{x}(t)$	$\bar{X}(-f)$
Duality	$X(t)$	$x(-f)$
Modulation	$x(t)e^{j2\pi f_c t}$	$X(f-f_c)$
Differentiation	$\frac{d}{dt}x(t)$	$j2\pi fX(f)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$
Convolution	$x(t) * y(t)$	$X(f).Y(f)$
Multiplication	$x(t)\,.\,y(t)$	$X(f) * Y(f)$
Parseval's or Rayleigh's theorem	$\int_{-\infty}^{\infty} \|x(t)\|^2\,dt = E_x = \int_{-\infty}^{\infty} \|X(f)\|^2\,df$	
Generalized Parseval's theorem	$\int_{-\infty}^{\infty} x(t)\overline{y(t)}dt = \int_{-\infty}^{\infty} X(f)\overline{Y(f)}df\,d$	

Example 4.7 *Determine the energy contained in the signal $x(t) = 20$ sinc $10t$*

Solution We shall solve this by applying Parseval's theorem.

We had seen that $A\Pi(t/\tau) \leftrightarrow A\tau$ sinc $f\tau$. Now, $A\tau$ sinc $f\tau$ is a **frequency function** with τ as a fixed time period. We may write a corresponding **time signal** as AWsinc Wt by replacing the fixed time interval τ by a fixed frequency interval W and by replacing f, the frequency variable, by the time variable t.

$\therefore$ let $\quad 20 \text{ sinc } 10t = AW \text{ sinc } Wt$

Hence $\quad AW = 20$ and $W = 10 \quad \therefore \quad A = 2$

We know from the duality theorem that

$$AW\text{sinc } Wt \leftrightarrow A\Pi(-f/w) = A\Pi(f/w)$$

$\therefore$ the FT of 20sinc$10t$ is $2\Pi(f/10) = X(f)$

This $X(f)$ is a rectangular pulse in frequency domain with an amplitude of 2 and base width of 10. Applying Parseval's theorem,

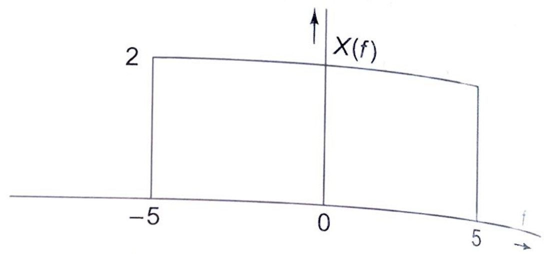


Fig. 4.13 *Spectrum of $x(t) = 20$ sinc $10t$*

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

$$= \int_{-5}^{5} 2^2\, df = 4 \times 10 = 40 \text{ units}$$

4.8 IMPULSE FUNCTION—DEFINITION AND PROPERTIES

We had, in Section 4.3 seen that Fourier transform of a function $x(t)$ exists if $x(t)$ is absolutely integrable. However, if we allow impulses, it is possible to get the Fourier transforms of even periodic signals, etc., which are not, in the strict sense, Fourier transformable.

Impulses are not functions in the normal sense. They come under the category of generalized functions or, distributions. Ordinary functions are mappings from a 'domain' to a 'range'. But generalized functions are not defined like that. They are defined by an assignment rule. An impulse function represented by the symbol $\delta(t)$, is defined by the following rule:

$$\boxed{\int_{t_1}^{t_2} x(t)\delta(t)dt = \begin{cases} x(0) & \text{if } t_1 < 0 < t_2 \\ o & \text{otherwise} \end{cases}} \qquad \ldots(4.35)$$

where $x(t)$ is any ordinary function which is continuous at $t = 0$; and $x(0)$ is a number representing the value of $x(t)$ at $t = 0$.

We can derive some very useful and interesting properties of an impulse function, by making use of Eq. (4.35). Since the only condition stipulated on $x(t)$ is that it should be continuous at $t = 0$, suppose we choose $x(t) = 1$, which, is a function of time, is continuous at all points, including at $t = 0$. Also, since the only condition imposed on the limits of integration, viz., t_1 and t_2, is that $t = 0$ must be included in the interval t_1 to t_2, suppose we make $t_1 = -\infty$ and $t_2 = +\infty$. Then, we have

$$\boxed{\int_{-\infty}^{+\infty} \delta(t)dt = 1} \qquad \ldots(4.36)$$

Equation (4.36) implies that the total area under the impulse function equals unity. This is one of the very portant properties of the impulse function.

For this reason, $\delta(t)$ is called a 'unit impulse function', implying that its area, or what is generally called e 'strength of the impulse', is equal to 1. Accordingly, an impulse of strength A is represented or denoted $A\delta(t)$.

Another important property of an impulse function, viz., that its width along the time axis is zero, can also derived from Eq. (4.35). For this, consider

$$\int_{-\in}^{+\in} \delta(t)dt$$

From Eq.(4.36), we know that the above quantity must be equal to 1. Now, if we allow $\in \to 0$, we find that ll the area under the function $\delta(t)$ between $t = -\in$ to $t = +\in$ continues to be equal to 1, however small $\in$ y be. This leads us to conclude that the width of $\delta(t)$ along the time axis is zero. Also, because of this, we that

$$\delta(t) = 0 \text{ if } t \neq 0 \qquad \ldots (4.37)$$

ecause of the above properties, the unit impulse function, $\delta(t)$, is usually visualized as the limiting case of rtain types of functions, when some parameter of those functions is allowed to tend to zero. For example, nsider a rectangular function of time, $x_\Delta(t)$, with a base width of Δ along the time axis and having an am- itude of $1/\Delta$ as shown in Fig. 4.14.

$$x_\Delta(t) = \frac{1}{\Delta}\Pi(t/\Delta) \qquad \ldots (4.38)$$

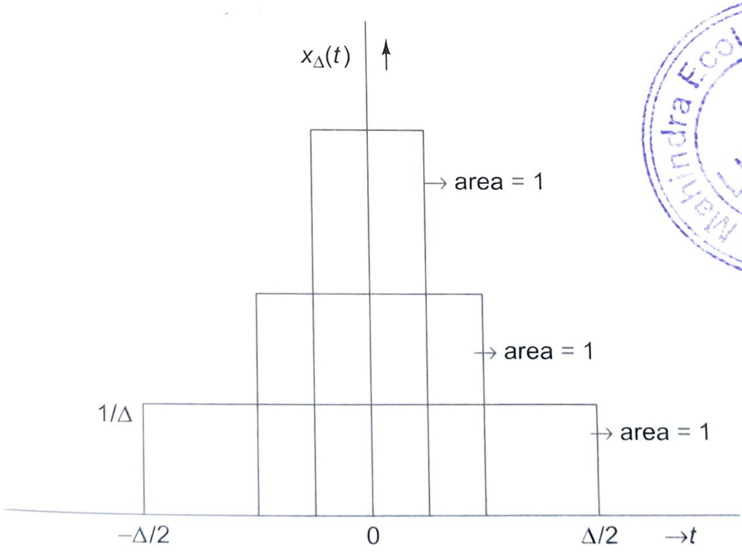


Fig. 4.14 *Impulse as limiting case of a rectangular pulse*

$$\lim_{\Delta \to 0} x_\Delta(t) = \delta(t) \qquad \ldots (4.39)$$

is for this reason that the unit impulse function is diagrammatically represented as shown in Fig. 4.15. nother example of such a function is $x_\in(t)$ given by

$$x_\in(t) = \frac{1}{\in}\text{sinc}\left(\frac{t}{\in}\right) \qquad \ldots (4.40)$$

A sketch of this function is given in Fig. 4.16.

$$\lim_{\in \to 0} x_{\in}(t) = \delta(t)$$

These two functions, $x_\Delta(t)$ and $x_\in(t)$ are such that ...(4.41)

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} f(t)\, x_\Delta(t)dt = f(0)$$

and

$$\lim_{\in \to 0} \int_{-\infty}^{\infty} f(t) x_\in(t)dt = f(0) \quad \text{...(4.4}$$

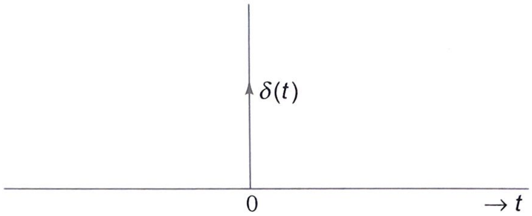


Fig. 4.15 *A unit impulse*

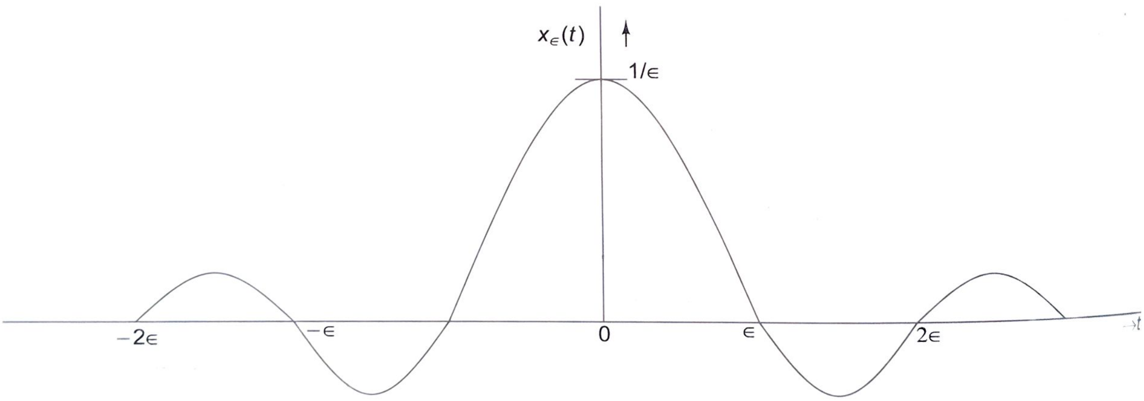


Fig. 4.16 *A sinc pulse that becomes an impulse in the limit*

As in the case of ordinary functions, for the impulses also, let us use $\delta(t - \tau)$ to represent an impulse lo-cated at τ, i.e., a time-delayed version of $\delta(t)$, delayed by τ seconds.

The two important properties of the impulse function are

1. Replication property If $x(t)$ is a continuous function, ...(4.43)

$$\boxed{x(t) * \delta(t - \tau) = x(t - \tau)}$$

2. Sampling property ...(4.44)

$$\boxed{x(t)\delta(t - \tau) = x(\tau)\delta(t - \tau) \text{ if } x(t) \text{ is continuous at } t = \tau}$$

The replication property can easily be proved by writing down the convolution integral and using the de-fining equation of an impulse function, viz., Eq. (4.35). It says that if we convolve a continuous function of time $x(t)$ with a unit impulse function which is delayed by τ seconds, $x(t-\tau)$, viz., a delayed version of $x(t)$ delayed by τ secs will result.

The sampling property given by Eq. (4.44) has been proved in Section 1.4.

There exists a relationship between the unit impulse function, $\delta(t)$, and the unit step function, $u(t)$, which is very useful in calculating the Fourier transforms of certain types of functions with discontinuities.

Consider $\int_{-\infty}^{t} \delta(\lambda)d\lambda$

This represents the area under the unit impulse function $\delta(\lambda)$ from $\lambda = -\infty$ up to $\lambda = t$. From the foregoing, we know that this area will be zero for $t < 0$ and will be equal to 1 for $t > 0$.

$$\int_{-\infty}^{t} \delta(\lambda)d\lambda = \begin{cases} 1 & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \qquad \ldots (4.45)$$

But we know that

$$u(t) \triangleq \begin{cases} 1 & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

Hence,

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda \qquad \ldots (4.46)$$

or differentiating both sides with respect to time

$$\boxed{\delta(t) = \frac{d}{dt} u(t)} \qquad \ldots (4.47)$$

4.9 FOURIER TRANSFORMS USING IMPULSES

In this section, we shall derive the Fourier transforms of periodic functions and certain other functions also that are not absolutely integrable, by making use of impulse functions and by considering transform in the limit.

4.9.1 Spectrum of an Impulse Function in Time

To find the spectrum of a unit impulse function in time, let us use the Fourier transform equation.

$$\mathscr{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft} dt \qquad \ldots (4.48)$$

In the right-hand side of Eq. (4.48), we have, inside the integral sign, the function $\exp(-j2\pi ft)$ which is a complex-valued continuous function of time.

Hence, using Eq. (4.35), the defining equation of an impulse function, we may write

$$\mathscr{F}[\delta(t)] = \int_{-\infty}^{\infty} e^{-j2\pi ft}\delta(t)\, dt = e^{-j2\pi ft}\Big|_{t=0} = 1$$

Hence,

$$\boxed{\delta(t) \xleftrightarrow{F.T} 1} \qquad \ldots (4.49)$$

What Eq. (4.49) says is that the spectrum $\Delta(f)$ of a unit impulse function of time consists of all frequency components $-\infty < f < \infty$ and that it has a value of unity at all frequencies, i.e., an impulse has all frequency components in equal proportion.

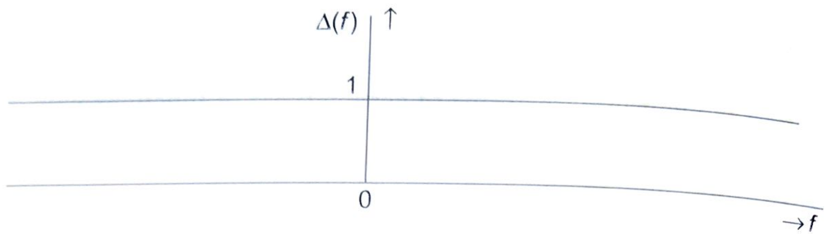


Fig. 4.17 *Spectrum of a unit impulse function in time, δ(t)*

In Eq. (4.49), the 1 on the right-hand side is the spectrum of $\delta(t)$ and hence is actually a function of frequency. If we now apply duality theorem and obtain the transform of 1 (considered as a function of time), we have

$$\mathscr{F}[1] = \delta(-f) = \delta(f)$$

Therefore,

$$\boxed{1 \xleftrightarrow{FT} \delta(f)} \quad \ldots(4.50)$$

Also form the time-delay theorem, we know that if $\mathscr{F}[x(t)] = X(f)$, then

$$\mathscr{F}[x(t-\tau)] = X(f)e^{-j2\pi f\tau}$$

Putting $\delta(t)$ in the place of $x(t)$, we get

$$\boxed{\delta(t-\tau) \xleftrightarrow{FT} e^{-j2\pi f\tau}} \quad \ldots(4.5$$

Further, from the modulation theorem, we know that

$$\mathscr{F}[x(t)e^{j2\pi f_0 t}] = X(f - f_0)$$

Taking $x(t)$ to be equal to 1 we get

$$\boxed{\mathscr{F}[1 \cdot e^{j2\pi f_0 t}] = \delta(f - f_0)} \quad \ldots(4.5$$

Since $\qquad e^{j2\pi f_0 t} \xleftrightarrow{FT} \delta(f - f_0),$

We have, $\qquad e^{-j2\pi f_0 t} \xleftrightarrow{FT} \delta(f + f_0)$

Then from the linearity theorem, we may write

$$\boxed{\cos 2\pi f_0 t \xleftrightarrow{FT} \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]} \quad \ldots(4.5$$

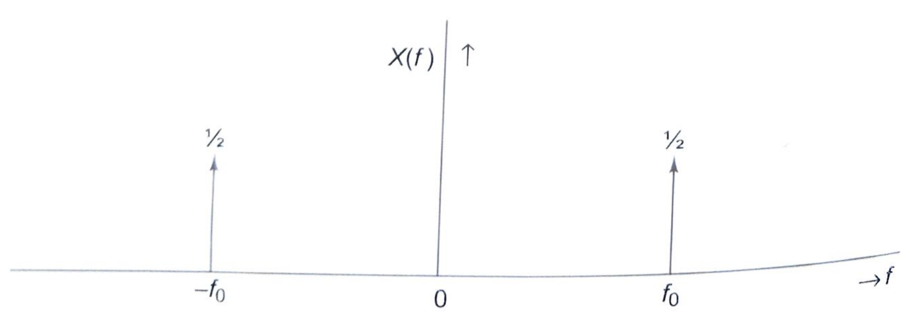


Fig. 4.18 *Spectrum of cos $2\pi f_0 t$*

Example 4.8 *Determine and sketch the spectrum of $x(t) = 10\sin 2\pi f_0 t$.*

Solution

$$10\sin 2\pi f_0 t = \frac{10}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

$$\mathscr{F}\left[10\sin 2\pi f_0 t\right] = \mathscr{F}\left[\frac{5}{j}\left\{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right\}\right]$$

Applying the linearity theorem, we get

$$X(f) = -5j\left[\delta(f - f_0) - \delta(f + f_0)\right]$$
$$= 5j\delta(f + f_0) - 5j\delta(f - f_0)$$

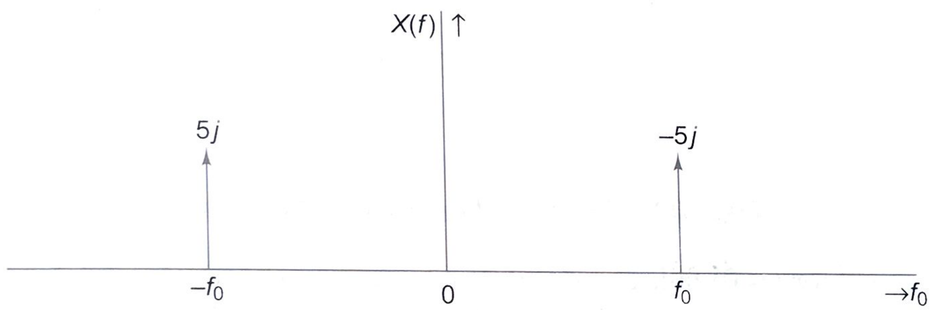


Fig. 4.19 *Spectrum of 10sin $2\pi f_0 t$*

4.9.2 Spectrum of Signum Function

The signum function in time, denoted by $sgn(t)$, is defined as follows:

$$sgn(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases} \qquad \ldots (4.54)$$

We shall derive the spectrum of the signum function as a transform in the limit. For this purpose, consider auxiliary function $x(t)$ given by

$$x(t) = \begin{cases} e^{-\alpha t} & \text{for } t > 0 \\ -e^{\alpha t} & \text{for } t < 0 \end{cases} ; \alpha > 0 \qquad \ldots (4.55)$$

A sketch of the signum function is shown in Fig. 4.20.

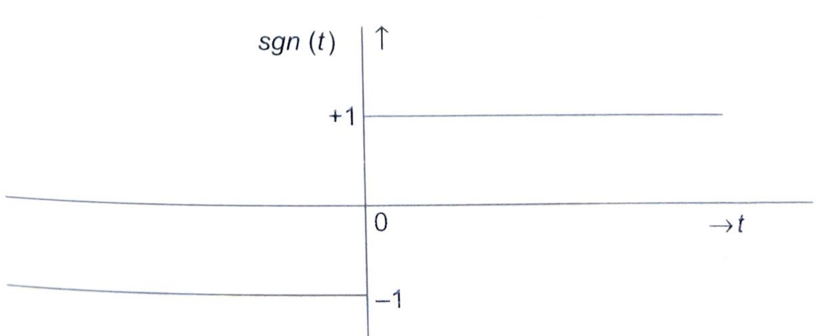


Fig. 4.20 *A signum function*

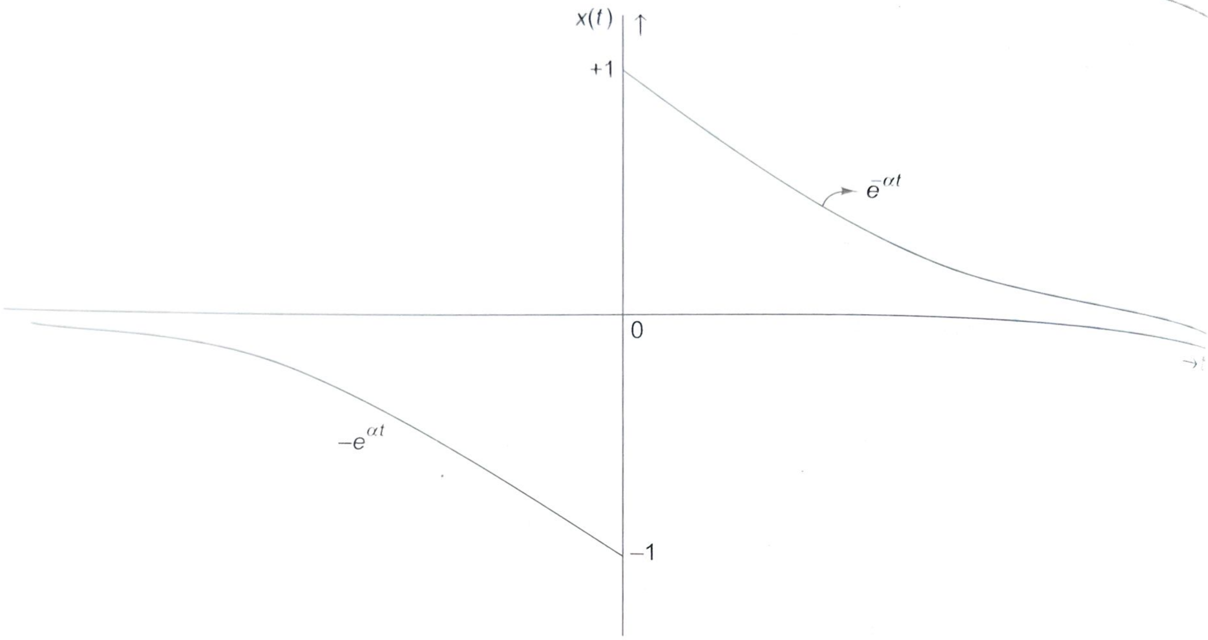


Fig. 4.21 *A sketch of the auxiliary function x(t)*

Obviously, $e^{-\alpha t}$ (or $e^{\alpha t}$ for $t < 0$) falls at a rate (at any given point of time) that depends on the numeri[...] value of α and as $\alpha \to 0$, the exponentially decaying curve approaches $x(t) = 1$ for $t > 0$ (and $-e^{\alpha t}$ approa[...] $x(t) = -1$ for $t < 0$)

Now,
$$x(t) = e^{-\alpha t}u(t) - e^{\alpha t}u(-t) \quad \ldots(4.5$$

$\therefore$
$$\mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi ft}\, dt - \int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi ft}\, dt$$

$$= \left.\frac{e^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)}\right|_{0}^{\infty} - \left.\frac{e^{(\alpha - j2\pi f)t}}{(\alpha - j2\pi f)}\right|_{-\infty}^{0}$$

$$= \frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f} = \frac{-j4\pi f}{\alpha^2 + 4\pi^2 f^2}$$

Hence,
$$\underset{\alpha \to 0}{\mathrm{Lt}}\ \mathcal{F}[x(t)] = \frac{1}{j\pi f}$$

Notes

1. *From the above equation, it should not be construed that $\mathcal{F}[sgn(t)] = S(f)$ is indeterminate a[t] $f = 0$. In fact, as sgn(t) is an odd function, its average value is zero and hence S(0), which must b[e] equal to the area under the sgn(t) function, must also be zero.*

Hence it is better to explicitly indicate this, by writing

$$\mathscr{F}[sgn(t)] = S(f) = \begin{cases} \dfrac{1}{j\pi f}; & f \neq 0 \\ 0; & f = 0 \end{cases} \qquad \ldots (4.57)$$

2. $\mathscr{F}[Sgn(t)]$ may be more easily derived by applying the differentiation theorem.
From Fig. 4.20, it is clear that

$$\frac{d}{dt} sgn(t) = 2\delta(t)$$

Therefore, from the differentiation theorem

$$\mathscr{F}\left[\frac{d}{dt} sgn(t)\right] = 2 = j2\pi f S(f)$$

where

$$S(f) = \mathscr{F}[sgn(t)]$$

$$\therefore \quad S(f) = \frac{1}{j\pi f}$$

The indeterminacy at $f = 0$ is removed by the observation made in NOTE(1) above.

Having found the Fourier transform of the signum function, it is an easy matter to determine the Fourier transform of a unit step function $u(t)$. From Fig. 4.20, we first note that

$$1 + sgn(t) = 2u(t)$$

$$u(t) = \frac{1}{2}[1 + sgn(t)] \qquad \ldots (4.58)$$

Applying the linearity property of Fourier transform and noting that

$\mathscr{F}[1] = \delta(f)$ and $\mathscr{F}[sgn(t)] = \dfrac{1}{j\pi f}$, we may write

$$\mathscr{F}[u(t)] = U(f) = \frac{1}{2}\left[\delta(f) + \frac{1}{j\pi f}\right]$$

$$\boxed{U(f) = \frac{1}{2}\left[\delta(f) + \frac{1}{j\pi f}\right]} \qquad \ldots (4.59)$$

4.9.3 Integration Theorem of Fourier Transform

In Section 4.7, wherein we had discussed the various theorems pertaining to Fourier transforms, we had not discussed the integration theorem. Now that the Fourier transform of a unit step function has been obtained using impulses, we are in a position to take up this theorem.

Statement Let
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Then,
$$Y(f) = \frac{1}{2}\left[X(f)\delta(f) + \frac{X(f)}{j\pi f}\right]$$

Proof Consider $x(t) * u(t)$. Writing down the convolution integral, we have
$$x(t)*u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$$

But
$$u(t-\tau) = \begin{cases} 1 \text{ for } \tau \le t \\ 0 \text{ for } \tau > t \end{cases}$$

Hence,
$$x(t)*u(t) = \int_{-\infty}^{t} x(\tau)d\tau = y(t)$$

$\therefore$
$$Y(f) = \mathcal{F}[x(t)*u(t)] = X(f).U(f)$$
$$= \frac{1}{2}\left[X(f)\delta(f) + \frac{X(f)}{j\pi f}\right]$$

$\therefore$
$$\boxed{\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{1}{2}\left[X(f)\delta(f) + \frac{X(f)}{j\pi f}\right]} \quad \ldots(4.60)$$

Making use of the sampling property of the impulse function, the above result may also be written as

$\therefore$
$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{1}{2}\left[X(0)\delta(f) + \frac{X(f)}{j\pi f}\right] \quad \ldots(4.61)$$

Example 4.9 *Find the Fourier transform of the RF pulse shown in Fig. 4.22.*

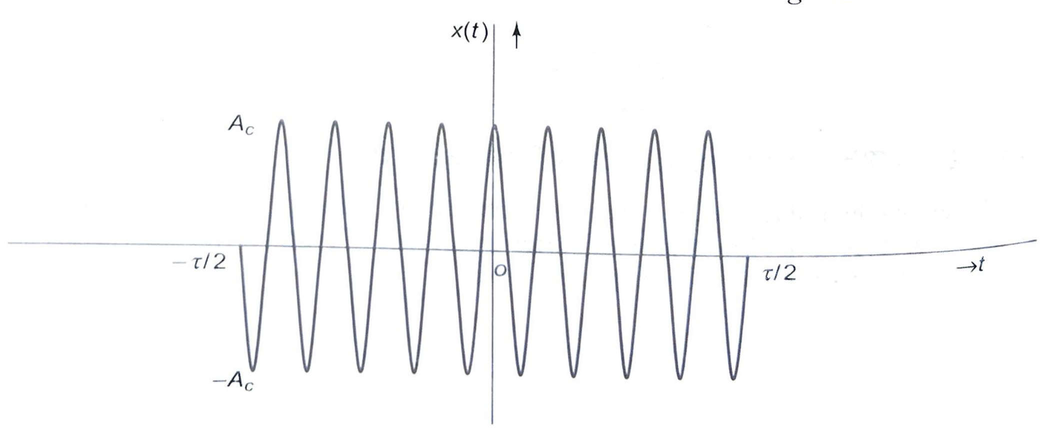


Fig. 4.22 *An RF pulse*

$$x(t) = \begin{cases} A_c \cos \omega_c t; & |t| \le \tau/2 \\ 0; & otherwise \end{cases} ; \frac{1}{f_c} << \tau \qquad \ldots (4.62)$$

...is rf pulse may also be represented as

$$x(t) = \Pi(t/\tau) A_c \cos \omega_c t \qquad \ldots (4.63)$$

As $x(t)$ is the product of two time functions $\Pi(t/\tau)$ and $A_c \cos \omega_c t$, its Fourier transform is the convolution ...the Fourier transforms of these two functions. Thus, from Eqs. (4.17) and (4.53), we have

$$X(f) = \frac{A_c \tau}{2} \text{sinc } (f - f_c)\tau + \frac{A_c \tau}{2} \text{sinc } (f + f_c)\tau \qquad \ldots (4.64)$$

...ketch of the magnitude of $X(f)$ is shown in Fig. 4.23.

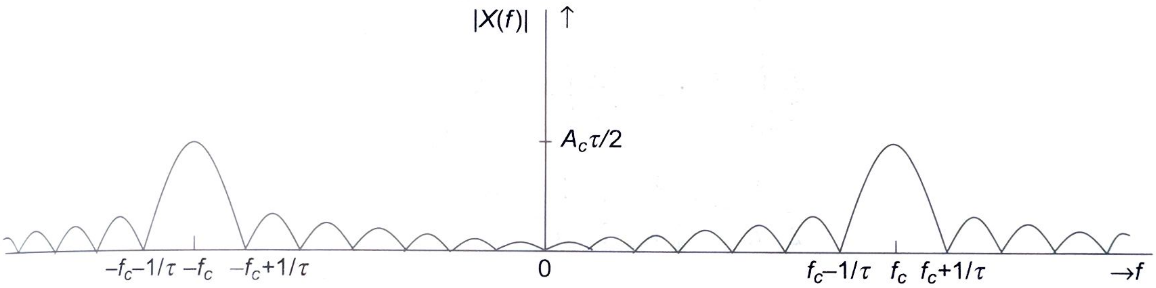


Fig. 4.23 *Magnitude spectrum of x(t), an rf pulse*

...mple 4.10 *Find the signal f(t) if its Fourier transform F(ω) is as shown in Figs. 4.24(a) and (b).*

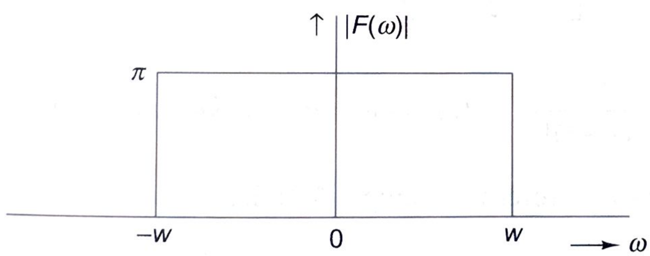


Fig. 4.24(a)

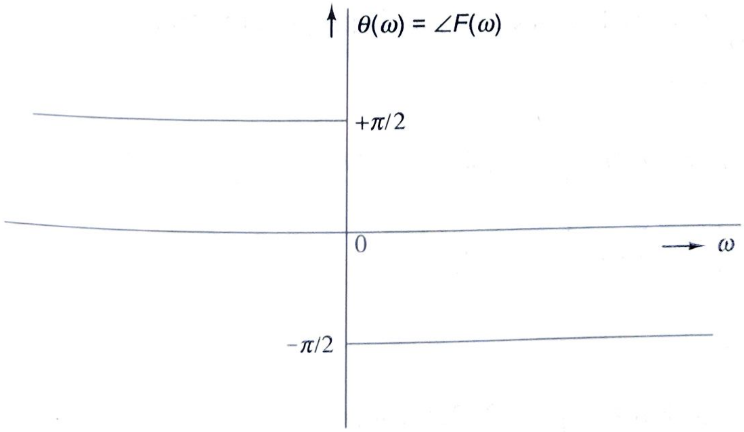


Fig. 4.24(b)

Solution We know that

$$F(\omega) = |F(\omega)|e^{j\theta(\omega)}$$

Here,

$$|F(\omega)| = \pi \text{ for } |\omega| \le W \text{ and } \theta(\omega) = \begin{cases} \pi/2 & \text{for } \omega < 0 \\ -\pi/2 & \text{for } \omega > 0 \end{cases}$$

The inverse Fourier transform of $F(\omega)$, say $f(t)$, is given by

$$f(t) = \mathscr{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j2\pi ft}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-W}^{0} \pi e^{j\pi/2}\cdot e^{j\omega t}\,d\omega + \int_{0}^{W} \pi e^{-j\pi/2}\cdot e^{j\omega t}\,d\omega\right]$$

$$= \frac{j}{2}\int_{-W}^{0} e^{j\omega t}\,d\omega + \frac{-j}{2}\int_{0}^{W} e^{j\omega t}\,d\omega$$

$$= \frac{j}{2}\cdot\frac{1}{jt}e^{j\omega t}\Big|_{-W}^{0} - \frac{j}{2}\cdot\frac{1}{jt}e^{j\omega t}\Big|_{0}^{W}$$

$$= \frac{1}{2t}(1-e^{-jWt}) - \frac{1}{2t}(e^{jWt}-1) = \frac{1}{t} - \frac{1}{t}\cos Wt$$

$$\therefore \quad f(t) = \left[\frac{(1-\cos Wt)}{t}\right]$$

Example 4.11 *If* $x(t) = \dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$. *where,* $a > 0$, *show that* $X(f) = \dfrac{1}{(a+j\omega)^n}$.

Solution We have seen earlier (refer to Example 4.2) that

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)}$$

$\therefore$ differentiating $\dfrac{1}{(a+j\omega)}$ once with respect to f and applying the frequency domain differentiation theorem given by Eq. (4.33(a)), we have

$$\frac{d}{df}\left[(a+j\omega)^{-1}\right] = -j2\pi(a+j\omega)^{-2} \xleftrightarrow{IFT} -j2\pi te^{-at}u(t)$$

or

$$\mathscr{F}^{-1}\left[\frac{1}{(a+j\omega)^{+2}}\right] = te^{-at}u(t).$$

Differentiating once again,

$$\frac{d}{df}\left[(a+j\omega)^{-2}\right] = -2j2\pi(a+j\omega)^{-3} \xleftrightarrow{IFT} (-j2\pi t)te^{-at}u(t)$$

$$\frac{1}{2}t^2 e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^3}$$

...ntinuing this process n times, we get

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{(a+j\omega)^n}$$

...ample 4.12 *A raised cosine pulse, described by* $x(t) = \left\{\frac{A}{2}\left(1+\cos\frac{\pi t}{\tau}\right)\right.$; $-\tau < t < \tau$ *is shown in ... 4.25. Determine its Fourier transform and sketch its magnitude spectrum. Compare its spectrum with ...t of a simple rectangular pulse.*

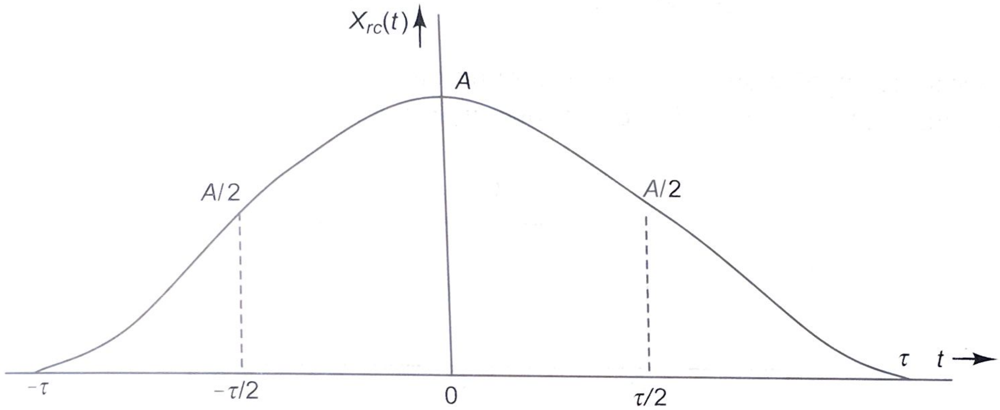


Fig. 4.25 *A raised cosine pulse*

...lution

$$x_{rc}(t) = \mathscr{F}\left[\frac{A}{2}\Pi(t/2\tau)\right] * \mathscr{F}\left[1+\cos\frac{\pi t}{\tau}\right]$$

$$= A\tau \text{ sinc } 2f\tau * \left[\delta(f) + \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right]$$

...ere $f_0 \triangleq \frac{1}{2\tau}$

$$X_{rc}(f) = A\tau \text{ sinc } 2f\tau + \frac{A\tau}{2}\text{ sinc } 2\tau(f-f_0) + \frac{A\tau}{2}\text{ sinc } 2\tau(f+f_0)$$

$$= A\tau\left(\frac{\sin 2\pi f\tau}{2\pi f\tau}\right) + \frac{A\tau}{2}\frac{\sin(2\pi f\tau - \pi)}{(2\pi f\tau - \pi)} + \frac{A\tau}{2}\frac{\sin(2\pi f\tau + \pi)}{(2\pi f\tau + \pi)}$$

$$= A\tau \sin 2\pi f\tau\left[\frac{1}{2\pi f\tau} - \frac{1/2}{2\pi f\tau - \pi} - \frac{1/2}{2\pi f\tau + \pi}\right]$$

...hich on simplification gives

$$X_{rc}(f) = \frac{A\tau \text{ sinc } 2f\tau}{1-(2f\tau)^2} \quad \ldots (4.65)$$

A plot of $|X_{rc}(f)|$ vs f is shown in Fig. 4.26.

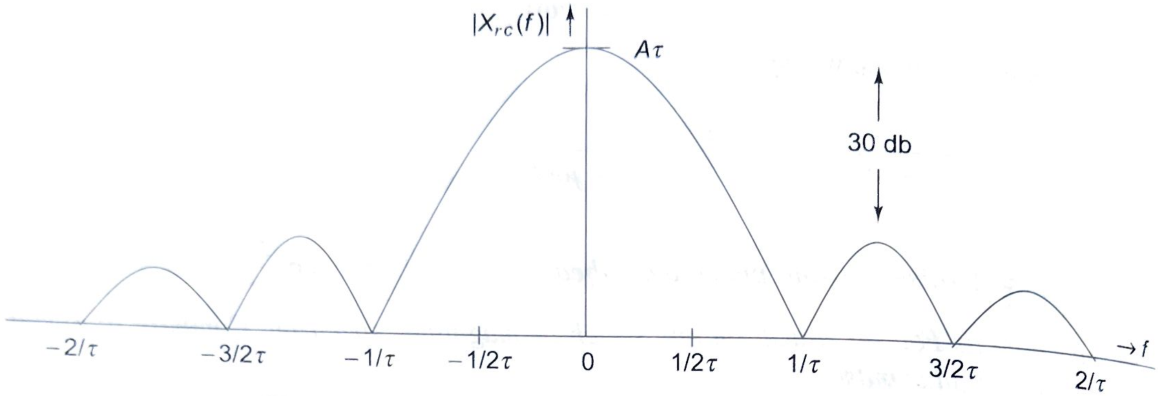


Fig. 4.26 *Magnitude spectrum of a raised cosine pulse*

If we consider a rectangular pulse $x_r(t)$ of the same duration and amplitude, as shown in Fig. 4.27, its magnitude spectrum $|2A\tau \text{ sinc } 2f\tau|$ will be as shown in Fig. 4.28.

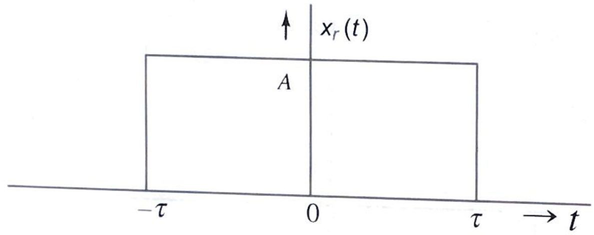


Fig. 4.27 *A rectangular pulse*

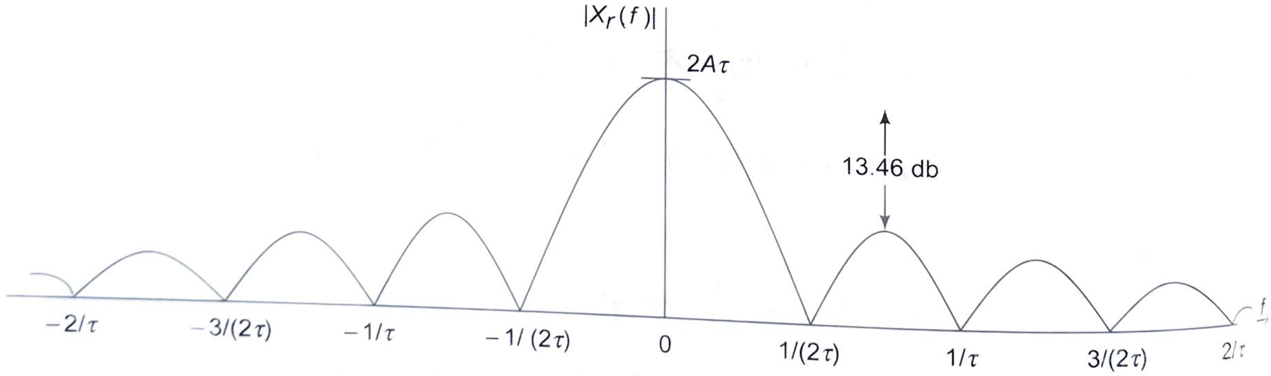


Fig. 4.28 *Magnitude spectrum of a rectangular pulse*

Note that while $|2A\tau \text{ sinc } 2f\tau|$ has only a first-order roll-off, the magnitude spectrum of the raised cosine pulse $\left|\dfrac{2A\tau \text{ sinc } 2f\tau}{1-4f^2\tau^2}\right|$ has a third order roll-off. That is why the raised cosine spectrum has a much better side-lobe suppression (–30db) than the magnitude spectrum of the rectangular pulse (–13.46 dB).

This particular feature of the raised cosine pulse makes it quite useful in practical applications. If raised-cosine pulses are used instead of rectangular pulses in data communications, we get a much lower adjacent channel interference. The raised cosine pulse shaped windows, are widely used in signal processing (Hanning and Hamming windows).

Example 4.13 *A particular signal, x(t), has a Fourier transform given by* $X(\omega) = \frac{1}{(1+\omega^2)} e^{-\frac{2\omega^2}{(1+\omega^2)}}$

Using appropriate Fourier transform theorems, write down the Fourier transforms of the following signals.

(a) $x(t-2)e^{jt}$ (b) $x(1-t)$ (c) $x\left(\frac{t}{2}-2\right)$

Solution

(a) $x(t-2)e^{jt} = x_1(t)$

$$x(t) \xleftrightarrow{FT} X(f)$$

$$x(t-2) \xleftrightarrow{FT} X(f)e^{-j4\pi f} \qquad \text{(Time-delay theorem)}$$

$$x_1(t) = x(t-2)e^{j2\pi(1/2\pi)t} \xleftrightarrow{FT} X\left(f - \frac{1}{2\pi}\right)e^{-j4\pi(f-1/2\pi)} \qquad \text{(Modulation theorem)}$$

$$X_1(\omega) = X(\omega-1)e^{j2(\omega-1)}$$

(b) $x_2(t) = x(1-t)$

$$x(t) \xleftrightarrow{FT} X(f)$$

$$x(-t) \xleftrightarrow{FT} X(-f) \qquad \text{(Scaling theorem)}$$

$$x(-t+1) \xleftrightarrow{FT} X(-f)e^{-j2\pi f} \qquad \text{(Time-delay)}$$

(c) $x_3(t) = x\left(\frac{t}{2}-2\right)$

$$x(t) \xleftrightarrow{FT} X(f)$$

$$x(t-2) \xleftrightarrow{FT} X(f)e^{-j4\pi f} \qquad \text{(Time-delay theorem)}$$

$$x\left(\frac{t}{2}-2\right) \xleftrightarrow{F.T} 2X(2f)e^{-j8\pi f} \qquad \text{(Scaling theorem)}$$

For all the three problems above, if we take

$$X(f) = \frac{1}{(1+4\pi^2 f^2)} e^{\left(\frac{-8\pi^2 f^2}{1+4\pi^2 f^2}\right)} \quad \text{as given,}$$

$$X_1(\omega) = \left[\frac{1}{1+(\omega-1)^2} e^{\frac{-2\omega^2}{1+\omega^2}}\right] e^{j2(\omega-1)}$$

$$X_2(\omega) = +\left[\frac{1}{1+\omega^2} e^{\frac{-2\omega^2}{1+\omega^2}}\right] e^{-j\omega}$$

$$X_3(\omega) = 2\left[\frac{1}{1+4\omega^2} e^{\frac{-8\omega^2}{1+4\omega^2}}\right] e^{-j4\omega}$$

Example 4.14 *Find the Fourier transform of the Gaussian pulse given by* $x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ ... (4.66)

Solution The Gaussian time-pulse given by Eq. (4.66) is sketched in Fig. 4.29.

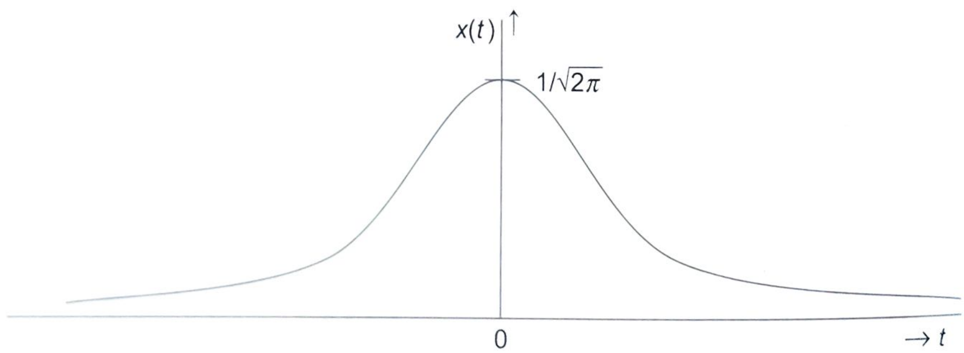


Fig. 4.29 *A Gaussian time pulse*

We find that $$\frac{d}{dt} x(t) = \frac{1}{\sqrt{2\pi}} (-t) . e^{\frac{-t^2}{2}} = -t x(t)$$ ... (4.67)

But $$\frac{d}{dt} x(t) \xleftrightarrow{FT} j2\pi f \, X(f)$$ ... (4.68)

$$-j2\pi \, t \, x(t) \xleftrightarrow{FT} \frac{d}{df} X(f)$$ ... (4.69)

...equency domain differentiation theorem)

...m Eqs. (4.67), (4.68) and (4.69), we have

$$-tx(t) \longleftrightarrow \frac{1}{j2\pi}\frac{d}{df}X(f) \qquad \ldots (4.70)$$

$$\frac{d}{dt}x(t) \longleftrightarrow \frac{1}{j2\pi}\frac{d}{df}X(f) \qquad \ldots (4.71)$$

...om Eqs. (4.68) and (4.71), we have

$$j2\pi f X(f) = \frac{1}{j2\pi}\frac{d}{df}X(f)$$

$$\frac{d}{df}X(f) = -4\pi^2 f\, X(f) \qquad \ldots (4.72)$$

...fferential equations (4.67) and (4.72) have the same mathematical form. Hence, $X(f)$ is also a Gaussian ...nction. Hence, let

$$X(f) = K_1 e^{-K_2 f^2} \text{ where } k_1 \text{ and } k_2 \text{ are constants to be determined}$$

$$\frac{d}{df}X(f) = -2k_2\, f k_1 e^{-k_2 f^2} \qquad \ldots (4.73)$$

...omparing right-hand sides of Eqs. (4.73) and (4.72), we have

$$K_2 = 2\pi^2 \qquad \ldots (4.74)$$

$$X(f) = k_1 e^{-2\pi^2 f^2}$$

...ow, to determine the constant k_1, we use the basic property of the Fourier transform that if

$$y(t) \xleftrightarrow{FT} Y(f), \text{ then}$$

$$Y(0) = \text{area under } y(t) = \int_{-\infty}^{\infty} y(t)dt$$

$$X(0) = \text{Area under } x(t) = \int_{-\infty}^{\infty} x(t)dt$$

But

$$x(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

which is a Gaussian density function. Hence, the total area under it must be equal to 1.

Hence,

$$X(0) = k_1 = 1$$

$$X(f) = e^{-2\pi^2 f^2} = e^{-\omega^2/2} \qquad \ldots (4.75)$$

Example 4.15 *Find the Fourier transform of the $x(t)$ shown in Fig. 4.30(a).*

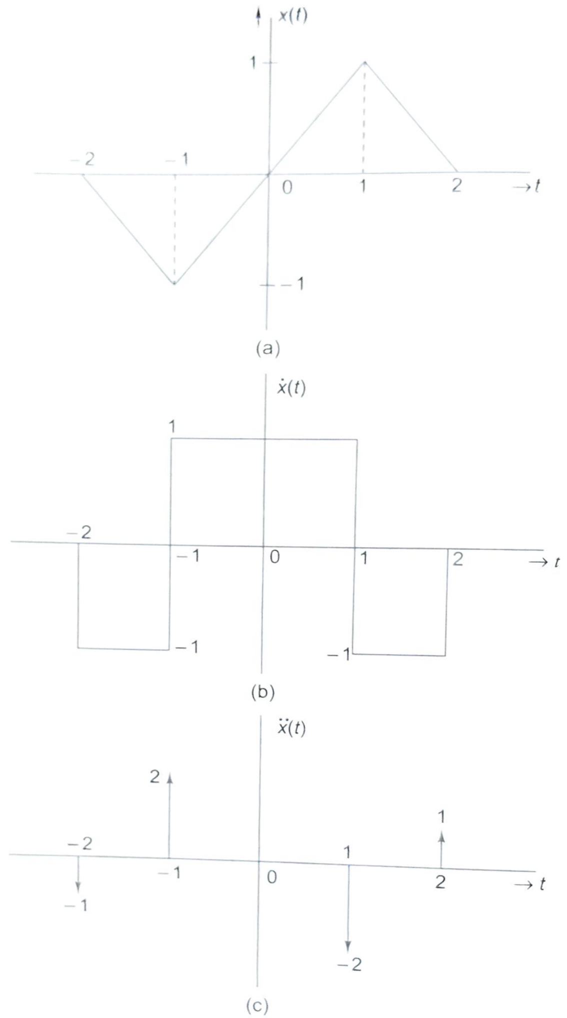


Fig. 4.30 *(a) $x(t)$, (b) $\dot{x}(t)$ and (c) $\ddot{x}(t)$*

Solution We shall use the differentiation theorem to find the FT of $x(t)$.

From Fig. 4.30 (c), we find that

$$\frac{d^2x(t)}{dt^2}=\ddot{x}(t)=-\delta(t+2)+2\delta(t+1)-2\delta(t-1)+\delta(t-2)$$

...from the differentiation theorem of FT, we know that if $x(t) \xleftrightarrow{FT} X(f)$. Then

$$\dot{x}(t) \xleftrightarrow{FT} j2\pi f X(f) \quad \text{and} \quad \ddot{x}(t) \xleftrightarrow{FT} -4\pi^2 f^2 X(f)$$

$$-4\pi^2 f^2 X(f) = \mathscr{F}\left[-\delta(t+2) + 2\delta(t+1) - 2\delta(t-1) + \delta(t-2)\right]$$

$$= -e^{+j4\pi f} + 2e^{+j2\pi f} - 2e^{-j2\pi f} + e^{-j4\pi f}$$

$$4\pi^2 f^2 X(f) = (e^{j4\pi f} - e^{-j4\pi f}) - 2\,(e^{j2\pi f} - e^{-j2\pi f})$$

$$4\pi^2 f^2 X(f) = 2j\sin 4\pi f - 4j\sin 2\pi f$$

$$X(f) = \frac{-1}{j2\pi^2 f^2}[\sin 4\pi f - 2\sin 2\pi f]$$

Example 4.16 *Find the Fourier transform of $x(t) = 1/t$.*

Solution We shall use the duality theorem for finding the Fourier transform of $1/t$.

We know that $sgn(t) \xleftrightarrow{FT} \dfrac{1}{j\pi f}$

Hence, from the linearity theorem of Fourier transform, we get

$$j\pi sgn(t) \xleftrightarrow{FT} \frac{1}{f},$$

Now, from the duality theorem, it follows that

$$\frac{1}{t} \xleftrightarrow{FT} j\pi sgn(-f) = -j\pi sgn(f)$$

Example 4.17 *Find the Fourier transform of $x(t) = \dfrac{1}{1+t^2}$.*

Solution To find this Fourier transform, we shall use duality theorem and the time-scaling theorem. We know that

$$e^{-|t|} \xleftrightarrow{FT} \frac{2}{1+4\pi^2 f^2}$$

Applying duality theorem,

$$\frac{2}{1+4\pi^2 t^2} \xleftrightarrow{FT} e^{-|f|}$$

Now, let

$$y(t) = \frac{2}{1+4\pi^2 t^2} \text{ and } y(at) = \frac{2}{1+t^2} = 2x(t)$$

Hence,

$$a = \frac{1}{2\pi}$$

Therefore, if

$$Y(f) = e^{-|f|}$$

$$2X(f) = \mathscr{F}[y(at)] = 2\pi e^{-|2\pi f|}$$

$$\therefore \quad \frac{1}{1+t^2} \xleftrightarrow{FT} \pi e^{-|\omega|}$$

Table 4.2 *Basic Fourier Transform Pairs*

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \quad \text{and} \quad x(t) = \int_{-\infty}^{\infty} X(f)e^{j\omega t}\,df$$

S.No.	Signal in time domain	Signal in Frequency domain
1.	$x(t) = \delta(t)$	$X(f) = 1$
2.	$x(t) = 1$	$X(f) = \delta(f)$
3.	$x(t) = u(t)$	$X(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
4.	$x(t) = e^{j(\omega_0 t + \phi)}$	$X(f) = e^{j\phi}\,\delta(f - f_0)$
5.	$x(t) = \text{sgn}\,(t)$	$X(f) = \frac{1}{j\pi f}$
6.	$x(t) = \cos(\omega_0 t + \phi)$	$X(f) = \frac{1}{2}[e^{j\phi}\delta(f - f_0) + e^{-j\phi}\delta(f - f_0)]$
7.	$x(t) = e^{-at}u(t)$	$X(f) = \frac{1}{a + j2\pi f}$
8.	$x(t) = e^{-a\|t\|}$	$X(f) = \frac{2a}{a^2 + (2\pi f)^2}$
9.	$x(t) = A\Pi(t/\tau)$	$X(f) = A\tau\,\text{sinc}\,f\tau$
10.	$x(t) = \text{sinc}2Wt$	$X(f) = \frac{1}{2W}\Pi(f/2W)$
11.	$x(t) = A\Lambda(t/\tau)$	$X(f) = A\tau\,\text{sinc}^2 f\tau$
12.	$x(t) = \text{sinc}^2\,2Wt$	$X(f) = \frac{1}{2W}\Lambda(f/2W)$

4.10 INVERSE FOURIER TRANSFORM

Just as in the case of Laplace transform, the inverse Fourier transform is generally obtained by making use of known transform pairs. For this purpose, we may use partial fraction expansion, or one or more of the Fourier transform theorems like the duality theorem or the convolution theorem. These methods are best illustrated through the following examples.

Example 4.18 *Determine the time-domain signal x(t) corresponding to the Fourier transform*
$X(j\omega) = \dfrac{1}{(j\omega)^2 + 7(j\omega) + 12}$.

Solution Expanding the given $X(j\omega)$ by partial fractions, we get

$$X(j\omega) = \frac{k_1}{j\omega + 3} + \frac{k_2}{j\omega + 4} = \frac{1}{(j\omega)^2 + 7(j\omega) + 12}$$

$\therefore$
$$k_1 = 1 \text{ and } k_2 = -1$$

and
$$X(j\omega) = \frac{1}{j\omega + 3} - \frac{1}{j\omega + 4}$$

We know that

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a + j\omega}$$

Hence,
$$x(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

Example 4.19 *Determine the time-domain signal corresponding to the Fourier transform.*
$X(f) = \dfrac{2 \text{ sinc } 2f}{(2 + j2\pi f)}$.

Solution We shall apply the convolution theorem of Fourier transforms to invert the above $X(f)$. For this purpose, let us define

$$X_1(f) = 2 \text{ sinc } 2f \quad \text{and} \quad X_2(f) = \frac{1}{2 + j2\pi f}$$

Then $x_1(t) = \mathcal{F}^{-1}[X_1(f)] = \Pi(t/2)$. This is because,

$$A\Pi(t/2) \xleftrightarrow{FT} \tau \text{ sinc } f\tau \quad \therefore \tau = 2 \text{ and } A = 1.$$

We know that
$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a + j2\pi f}$$

Comparing the right-hand side of the above with $X_2(f)$, we get $a = 2$.

$\therefore$
$$\mathcal{F}^{-1}\left[\frac{1}{2 + j2\pi f}\right] = e^{-2t}u(t)$$

Hence,
$$\mathcal{F}^{-1}[X_1(f) \cdot X_2(f)] = x_1(t) * x_2(t) = \Pi(t/2) * e^{-2t}u(t) = x(t)$$

$$= \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau = \int_{-1+t}^{1+t} e^{-2\tau}u(\tau)d\tau$$

As $u(\tau) = 0$ for $\tau < 0$, the lower limit will be taken as zero for $t < 1$ and as $(t - 1)$ for $t > 1$. Further, for the same reason, the upper limit will be taken as 0 for $t < -1$. Hence, three clear-cut cases arise.

$t < -1$ The lower and upper limits will both be zero and hence $x(t) = 0$.

$-1 \le t < 1$ The upper limit is not taken as zero; it is $(t + 1)$ and the lower limit will however be zero.

$$\therefore \qquad x(t) = \int_0^{t+1} e^{-2\tau} u(\tau)d\tau = -\frac{1}{2}e^{-2\tau}\Big|_0^{t+1} = \frac{1}{2} - \frac{1}{2}e^{-2(t+1)}$$

$t \ge 1$ In this case, $u(t)$ will not vanish at the lower as well as the upper limit. Hence, the lower limit is taken as $(-1 + t)$ and the upper limit is taken as $(t + 1)$

$$\therefore \qquad x(t) = \int_{t-1}^{t+1} e^{-2\tau} u(\tau)d\tau = -\frac{1}{2}e^{-2\tau}\Big|_{t-1}^{t+1} = \frac{1}{2}e^{-2(t-1)} - \frac{1}{2}e^{-2(t+1)}$$

Thus,

$$x(t) = \begin{cases} 0 \text{ for } t < -1 \\ \frac{1}{2}(1 - e^{-2(t+1)} \text{ for } -1 \le t < 1 \\ (e^{-2(t-1)} - e^{-2(t+1)})/2 \text{ for } t \ge 1 \end{cases}$$

4.11 POISSON'S SUM FORMULA

We will now derive 'Poisson's sum formula', which relates the sum of the time-domain samples of a time function with the sum of the frequency-domain samples of its Fourier transform. Specifically, it states that

$$\sum_{n=-\infty}^{\infty} x(nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(nf_s)$$

where $x(t)$ is any Fourier transformable time function; T_s is any period at which it is regularly sampled; f_s is the inverse of T_s and $X(f)$ is the Fourier transform of $x(t)$.

Proof Consider the impulse train given by

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nTs)$$

Being a periodic function with a period T_s, it can be expanded using complex exponential Fourier series

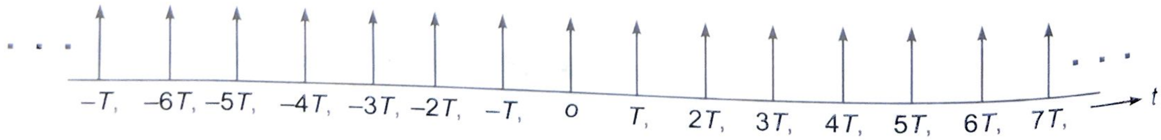


Fig. 4.31 *An impulse train*

$\therefore$ let

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nTs) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_s t}; -\infty < t < \infty$$

Then

$$c_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} s(t)\, e^{-j2\pi n f_s t} dt = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-j2\pi n f_s t} dt \text{ (since } s(t)$$

$$= \delta(t) \text{ for } -T_s/2 < t < T_s/2).$$

$$= f_s \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi n f_s t}\, dt = f_s \cdot \left[e^{-j2\pi n f_s t}\Big|_{t=0} \right] = f_s$$

$\therefore$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nTs) = f_s \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t}$$

Now, convolving the RHS and LHS of the above with any $x(t)$ that is Fourier transformable, we have

$$x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = f_s \cdot x(t) * \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t}$$

i.e.,

$$\sum_{n=-\infty}^{\infty} x(t - nT_s) = f^{-1}\left[f_s \sum_{n=-\infty}^{\infty} X(f) \cdot \delta(f - nf_s) \right]$$

$\therefore$

$$\sum_{n=-\infty}^{\infty} x(t - nT_s) = f_s \sum_{n=-\infty}^{\infty} X(nf_s) e^{j2\pi n f_s t}$$

For $t = 0$, the above equation reduces to

$$\boxed{\sum_{n=-\infty}^{\infty} x(nT_s) = f_s \sum_{n=-\infty}^{\infty} X(nf_s)} \quad (4.76)$$

This is referred to as poison's sum formula and it comes in handy while solving certain problems.

Example 4.20 *Show that for any* $\sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2 n^2} = \sum_{n=-\infty}^{\infty} e^{-\alpha|n|}$ $\alpha > 0$

Solution Dividing the numerator and denominator by α^2, and denoting $\left(\frac{1}{\alpha}\right)$ by T_s, the LHS may be written as

$$\sum_{n=-\infty}^{\infty} \frac{2T_s}{1 + 4\pi^2 n^2 T_s^2} = \sum_{n=-\infty}^{\infty} \frac{1}{1 + 4\pi^2 (nT_s)^2} \cdot 2T_s$$

which may immediately be recognized as $2T_s$ times the summation of the samples of $x(t)$ taken at regular intervals of T_s where

$$x(t) \triangleq \frac{2}{1 + 4\pi^2 t^2}.$$

Now, we know that (refer to Example 4.17)

$$\frac{2}{1+4\pi^2 t^2} \xleftrightarrow{FT} e^{-|f|}$$

Applying Poison's formula, we therefore get

$$\sum_{n=-\infty}^{\infty} \frac{2.T_s}{1+4\pi^2\left(n^2T_s^2\right)} = T_s \cdot \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{-|nf_s|} = \sum_{n=-\infty}^{\infty} e^{-f_s|n|}$$

But the above LHS is equal to $\sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2+4\pi^2 n^2}$ and $f_s = \alpha$

$\therefore$ we get the desired result as

$$\sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2+4\pi^2 n^2} = \sum_{n=-\infty}^{\infty} e^{-\alpha|n|}$$

4.12 LAPLACE AND FOURIER TRANSFORMS

Laplace transform $$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \;;\quad s=\sigma+j\omega \qquad \ldots(4.77)$$

Fourier transform $$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \ldots(4.78)$$

From the above equations for the Laplace and Fourier transforms, it is clear that the latter is a particular case of the former. While $s = \sigma + j\omega$, is what one may call as the complex frequency variable, ω is a real frequency variable. The Laplace transform maps a given function $x(t)$ into a function $X(s)$, called the Laplace transform. $X(s)$ is finite for values of the independent variable s lying in a region of the s-plane, called the Region of Convergence (ROC).

The Fourier transform, on the other hand, maps a given function $x(t)$ into a function $X(j\omega)$, whose independent variable ω has, as its domain, the set of points lying only on the $j\omega$ axis of the s-plane.

From Eqs (4.77) and (4.78), one is likely to be tempted to write down the Fourier transform $X(j\omega)$ of a given $x(t)$ by replacing the variable s in its Laplace transform $X(s)$, by $j\omega$. But, it should be realized that it may not always be correct to do so. It will be correct only if the $j\omega$-axis is within the region of convergence of the Lapalce transform $X(s)$ of the particular time function $x(t)$ under consideration. Hence, care should be exercised while trying to obtain the Fourier transform of any given $x(t)$ from its Laplace transform, by replacing s in $X(s)$ by $j\omega$. One should first make sure that the $j\omega$-axis is in the ROC of $X(s)$ before doing so.

For example, consider $x(t) = e^{at}\,u(t)$, where 'a' is a real constant. Its Laplace transform is

$$X(s) = \frac{1}{s-a};\ \mathrm{Re}(s) > a$$

s can be seen from Figs. 4.32 and 4.33, it turns out that whether the ROC includes $j\omega$ axis or not, will end on whether the constant 'a' is less than zero or greater than zero.

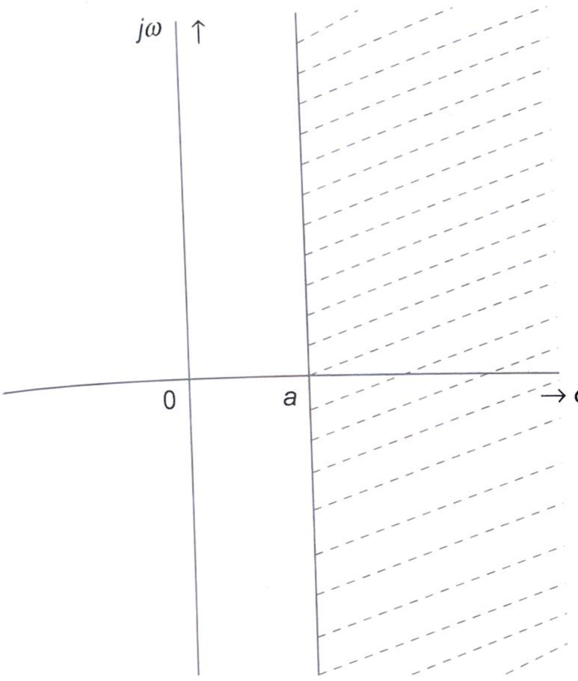


Fig. 4.32 *ROC of X(s) when a > 0*

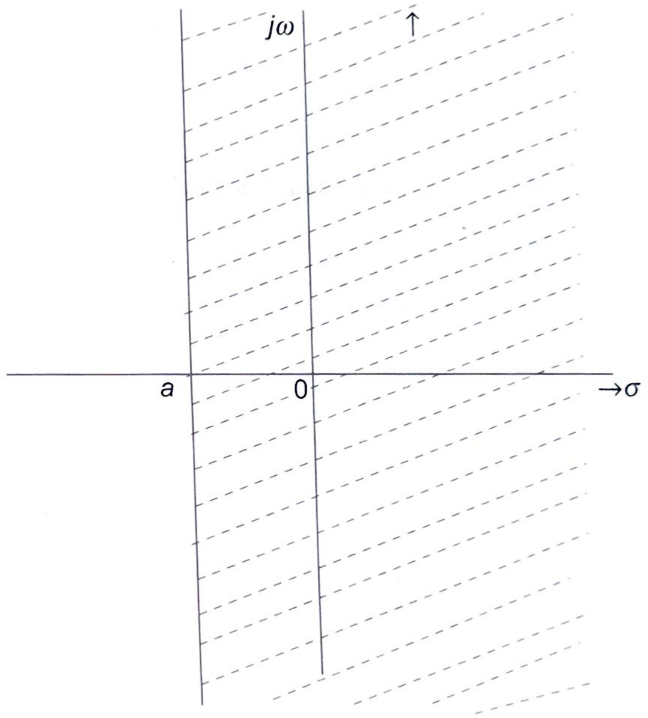


Fig. 4.33 *ROC of X(s) when a < 0*

hus, we may write

$$X(j\omega) = \frac{1}{j\omega - a}$$

s the Fourier transform of $x(t)$ only when a is negative but not when a is positive.

MATLAB Example 4.1 *Plot the magnitude spectrum and phase spectrum of the non-periodic signal shown in Fig. 4.34.*

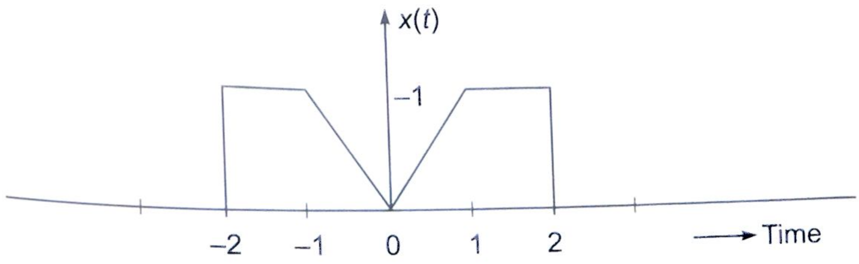


Fig. 4.34

MATLAB Program

```
clc
df = 0.01;
fs = 10;          % sampling frequency
fs = 1/fs         % sampling time
t = [-5:ts5]      % time scale
%
% Generation of nonperiodic signal
%
x = zeros(size(t));
x(32:41) = ones (size(x(32.41)));
for i = 1:1:10
x(41+i)= 1-0.1*1;
end
for i = 1:1:10
x(51+i)= 0.1*1;
end
x(61:70) = ones(size(x(61:70)));
subplot (3,1,1)
plot(t,x)
ylim([0 1.5]);
grid on
xlabel ('Time');
ylabel ('Amplitude');
title ('Given Signal');
%
% Finding magnitude spectrum and phase spectrum of the nonperiodic signal
%
[X,x1,df1] = fftseq(x,ts,df);
X1 = X/fs;
f = [0:df1:df1*(length(x1)-1)]-fs/2;
subplot(3,1,2)
plot(f,fftshift(abs(X1)));
grid on
xlabel ('frequency');
ylabel ('amplitude');
title ('Magnitude Spectrum');
subplot (3,1,3)
plot(f(412:612),ffshift(angle(X1(412:612))))
grid on
xlabel ('frequency');
ylabel ('radian');
title ('Phase Spectrum');
```

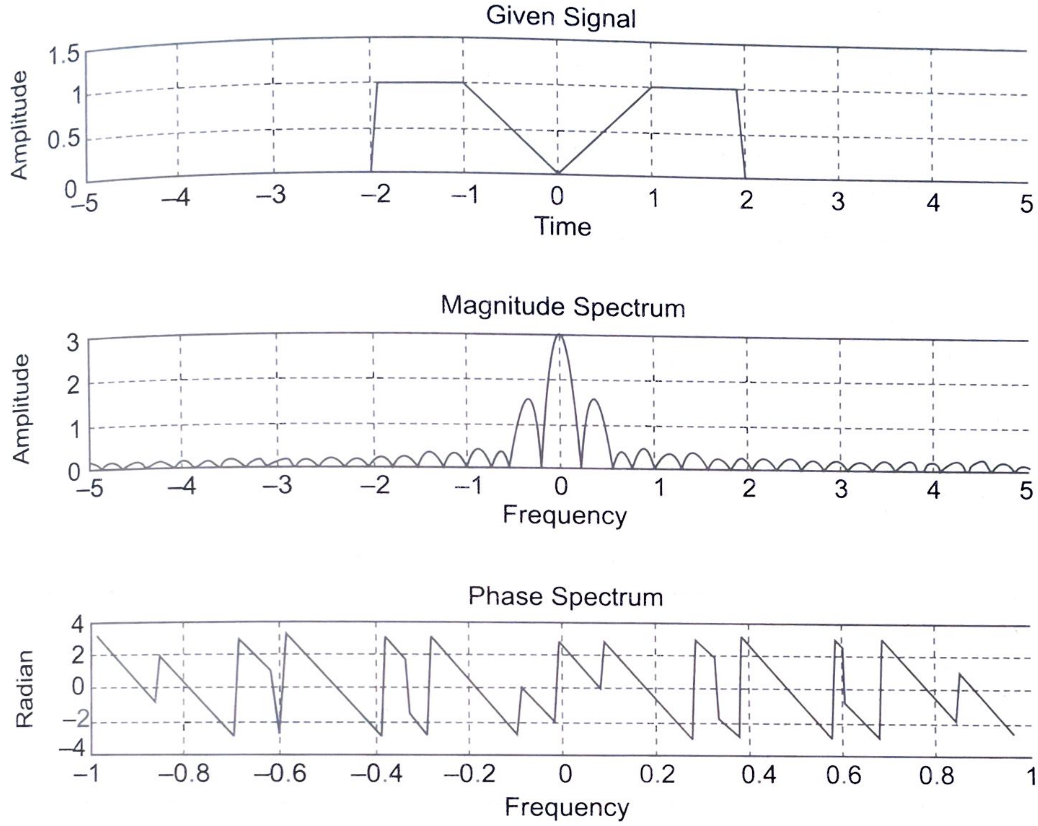


Fig. 4.35

MATLAB Example 4.2 *In this example, we demonstrate how the Fourier transform of a signal can be calculated using MATLAB. We use as an example the sum of two sinusoids that are closely spaced in frequency—one at 1.2566 rad/s and the other at 2.5133 rad/s.*

```
Ts = 0.02;     %A suitable time period that satisfies the aliasing condition is used.
t = 0 : Ts : 100 ;       %A suitable time interval is defined
f1 = 0.2 ; f2 = 0.4 ;    %the two frequencies are defined
x = cos(2*pi*f1*t) + cos(2*pi*0.4*t) ; % The signal is
generated
% We now proceed to estimate the Fourier Transform using the FFT, first
% using small number of points and then using larger number of points
N = 256 ; % Number of FFT points
ws = 2*pi/(N*Ts) ;        %Frequency separation obtained
fp = 0 : N/2 ;
y = Ts*fft(x(1 : N)) ;   %The fft is the algorithm for fast
computation of the DFT,
% which is used to find the transform of at a finite number
of points.
```

```
subplot (211), plot (fp*ws, abs(Y(fp + 1)), 'K') ; %The transform
is plotted for N points
axis ([0, 15, 0, 3]) ;
xlabel ('Frequency, rad. per sec.') ; ylabel ('Magnitude')
title ('Magnitude of Transform') ;
% Now estimate the transform using larger number of points
N = 1024 ; % Number FFT points
ws = 2*pi/(N*Ts) ; % Frequency separation obtained
fp = 0 : N/2 ;
y = Ts*fft (x(1 : N)) ;
subplot (212), plot (fp*ws, abs(y(fp + 1)), 'k') ; % The transform
is plotted for N = 1024 points
axis ([0, 15, 0, 12]) ;
xlabel ('Frequency, rad. per sec.') ; ylabel ('Magnitude')
title ('Magnitude of Transform') ;
```

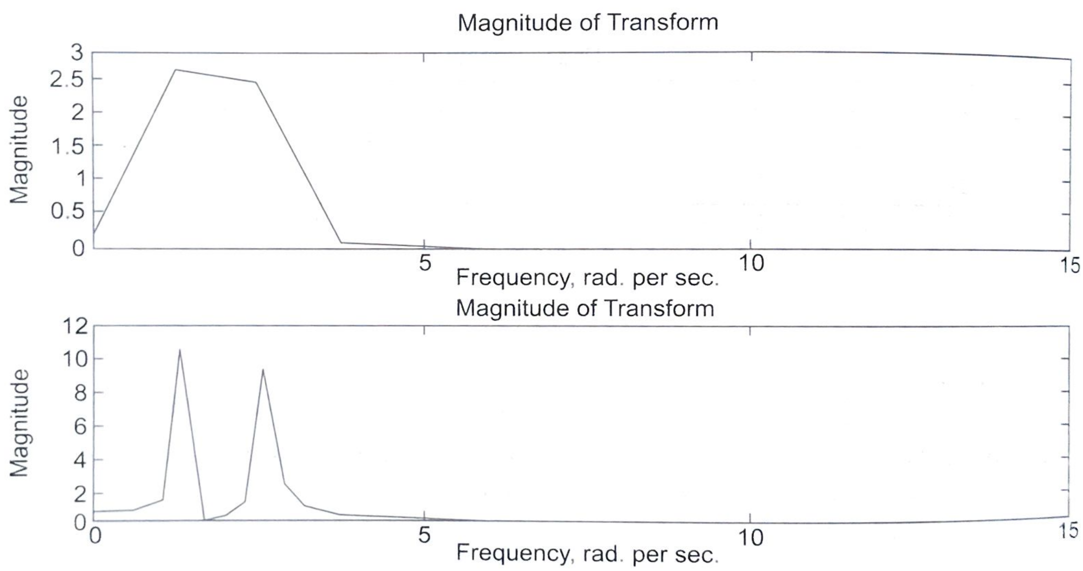


Fig. 4.36

The plot depicts magnitude of the transform obtained using $N = 256$ (top) and $N = 1024$ (bottom) points in the evaluation. As the number of points becomes infinite, impulses will be observed at the correct frequencies.

Summary

1. The Fourier transform of an a periodic signal $x(t)$ may be derived by starting with the Fourier series of a periodic repetition of $x(t)$ and then allowing the period of repetition to tend to infinity.

2. The Fourier transform of a signal $x(t)$ exists if $x(t)$ is absolutely integrable, i.e., if $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$

However, if the use of singularity functions like the delta function is permitted, we can write down the Fourier transforms of even signals which are not absolutely integrable.

3. If a non-periodic signal $x(t)$ satisfies the Dirichlet's conditions, then point-wise convergence of the integral

$$\int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

is guaranteed for all values of t except those corresponding to discontinuities.

4. The magnitude and phase spectra of an a periodic signal are continuous.
5. If $x(t)$ is real valued, its Fourier transform $X(f)$ has Hermitian symmetry.
6. $|X(f)|^2$ at any frequency 'f' represents the energy density of the signal $x(t)$ with respect to frequency at that frequency f.
7. While the Fourier transforms of simple functions can be obtained by directly using the Fourier transform equation, those of more complicated functions can be obtained by the application of appropriate Fourier transform theorems.
8. Given the Fourier transform $X(f)$, the signal $x(t)$ may be determined by using known transform pairs, by using the Fourier transform theorems, by partial fraction expansion, or more generally, a combination of some of these methods.

References and Suggested Reading

1. Papoulis, A., *The Fourier Integral and Its Applications*, McGraw-Hill, New York, 1962.
2. Guillemin, E.A., *The Mathematics of Circuit Analysis*, Wiley, New York, 1949 (Chapter 7).
3. Papoulis, A., *Signal Analysis*, McGraw-Hill (New York), 1977 (Chapter 3).
4. Lighthill, J., *Introduction to Fourier Analysis and Generalized Functions*, Cambridge University Press (New York).
5. Bracewell, R.N., *The Fourier Transform and Its Applications*, 2nd edition, McGraw- Hill (New York), 1978.
6. Ziemer, R.E., W.H.Tranter, and D.R. Fanin, *Signals and Systems*, 4th edition, 1998, Prentice Hall, NJ.

Review Questions

1. Write down the Fourier transform of $x(t) = \cos(\omega_c t + \theta)$.
2. If $x(t) \xleftrightarrow{FT} X(f)$, what is the Fourier transform of $x(at - t_0)$ where a and t_0 are real constants?
3. If $X(f)$, the Fourier transform of $x(t)$, has a Hermitian symmetry, what can you say about $x(t)$?
4. If $X(f)$ is the Fourier transform of $x(t)$, what does $|X(f)|^2$ represent in relation to the signal $x(t)$?
5. What is the energy contained in the signal $x(t) = 10 \operatorname{sinc} 5t$?
6. If $x(t)$ is real and even, show that $X(f)$ is purely real.
7. Sketch the magnitude and phase spectra of the signal $x(t) = 10 \sin(100\pi t + \pi/8)$.
8. If $X(f) = \cos(\pi f)\, \Pi(f/1)$, what is $x(t)$?
9. When Praveen was asked to find the Fourier transform of $u(t)$, he took its Laplace transform $X(s)$ and replaced 's' in it by $j\omega$. Is he justified in doing so? If not, why?

Problems

4.1 Find the Fourier transform of $x(t) = 5e^{-2|t|}$. Plot its magnitude and phase spectra.

4.2 Find the Fourier transforms of the following signals:

(a) $x(t) = e^{-3t}u(t-2)$

(b) $x(t) = e^{-2|t|}$

(c) $x(t) = 2te^{-2t}u(t)$

(d) $x(t)$ shown in Fig. 4.37

(e) $x(t) = \left[\exp\{j2\pi(t-1) - (t-1)\}\right]u(t-1)$

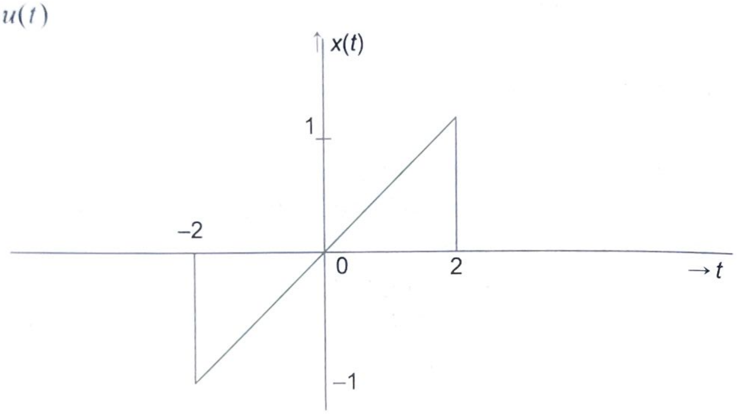


Fig. 4.37

4.3 Find the signal $x(t)$ whose Fourier transform $X(f)$ is given in (a) Fig. 4.38(a) and (b) Fig. 4.38(b).

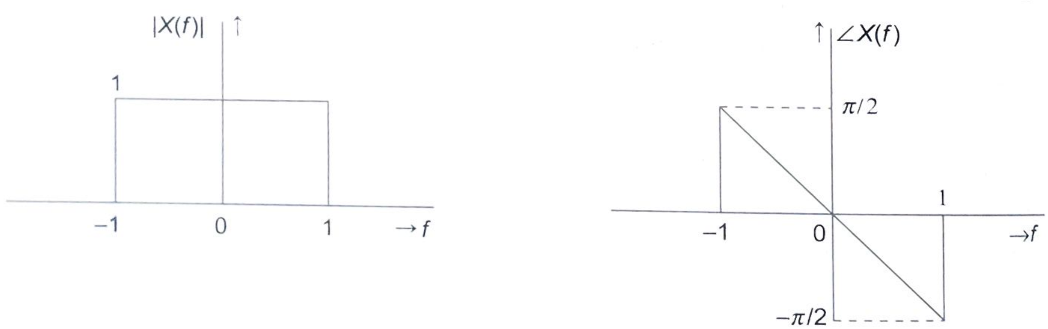


Fig. 4.38(a)

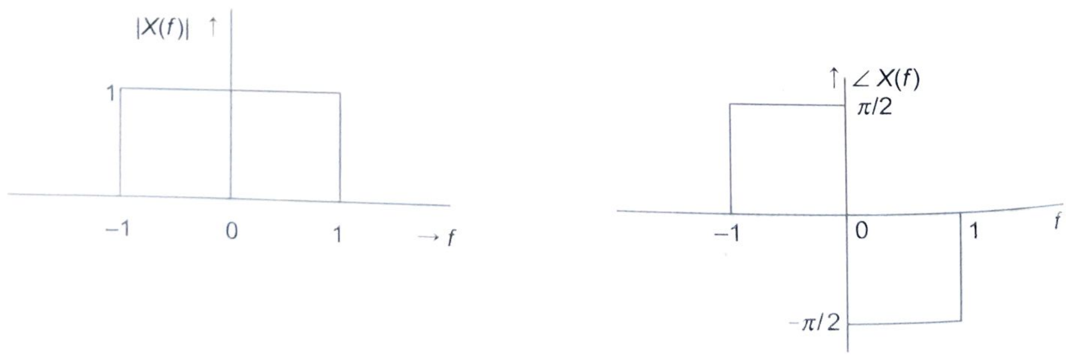


Fig. 4.38(b)

4.4 Use Parseval's theorem to calculate the energy in the signal $x(t) = 4 \text{ sinc } 40\,t$.

4.5 Calculate the energy contained in the signal $x(t) = 5e^{-3t}u(t)$ using time-domain equation and then using Parseval's theorem.

4.6 Calculate the energy contained in the signal in Problem 4.4 for $|f| \le \dfrac{3}{2\pi}$. Express it as a percentage of the total energy of the signal.

4.7 What is the energy contained in the signal $x(t) = 10 \exp(-2t)\, u(t)$ for frequencies $|f| < 1/\pi$? What percentage of the total energy of $x(t)$ is this?

4.8 If $y(t) = x(2t - 3)$, find $Y(f)$ in terms of $X(f)$.

4.9 Find the convolution of $x(t) = 5\Pi\,(t/4)$ with $y(t) = 5\Pi\,(t/4)$.

4.10 Find the Fourier transform of $z(t) = 100\,\Lambda(t/8)$ where $\Lambda(t/8)$ is a triangular pulse symmetrical about the $t = 0$ axis and having a peak amplitude of 100 and a total base width of 8 seconds. (*Hint: Use the result of Problem 4.9 and the convolution theorem of Fourier transform*).

4.11 Find and sketch $z(t) = x(t)*y(t)$ when $x(t) = 2 \text{ sinc } 4t$ and $y(t) = 3 \text{ sinc } 2t$

4.12 If $x(t) = \Pi(t/2\tau)$, find the Fourier transform of $y(t) = x(t - T) + x(t + T)$ where $T >> \tau$.

4.13 $z(t) = \delta(t - T) + \delta(t + T)$. If $w(t) = z(t)*x(t)$ where $x(t) = \Pi(t/2\tau)$. Find $W(f)$. Check with your result of Problem 4.12.

4.14 Use duality theorem to find the Fourier transform of $x(t) = 10 \text{ sinc } 20\,t$.

4.15 Given that $X(f)$ is the Fourier transform of $x(t)$, find the Fourier transform of the following:

(a) $y(t) = 2x(3t - 2)$ (b) $y(t) = x\left(\dfrac{t}{2} - 1\right)e^{j200\pi t}$ (c) $y(t) = x(1 - 2t)$

4.16 Find the Fourier transforms of the following signals, using transform theorems and properties.

(a) $x(t) = te^{-2|t|}$ (b) $x(t) = e^{-(2t+3)}u(t - 1)$ (c) $x(t) = \dfrac{d}{dt}\left[2te^{-2t}\sin(2t)\,u(t)\right]$

4.17 Find the Fourier transforms of the signals (a) $\exp[j2\pi(t-2) - (t-2)]\,u(t-2)$: (b) $\text{rect}\,[(t-1)/4]e^{j2\pi(t-1)}$

(a) By applying time-delay theorem first and then applying the modulation theorem.

(b) By applying the modulation theorem first and then applying the time-delay theorem.

4.18 Find the Fourier transforms of the signals shown in Figs. 4.39(a) to (e).

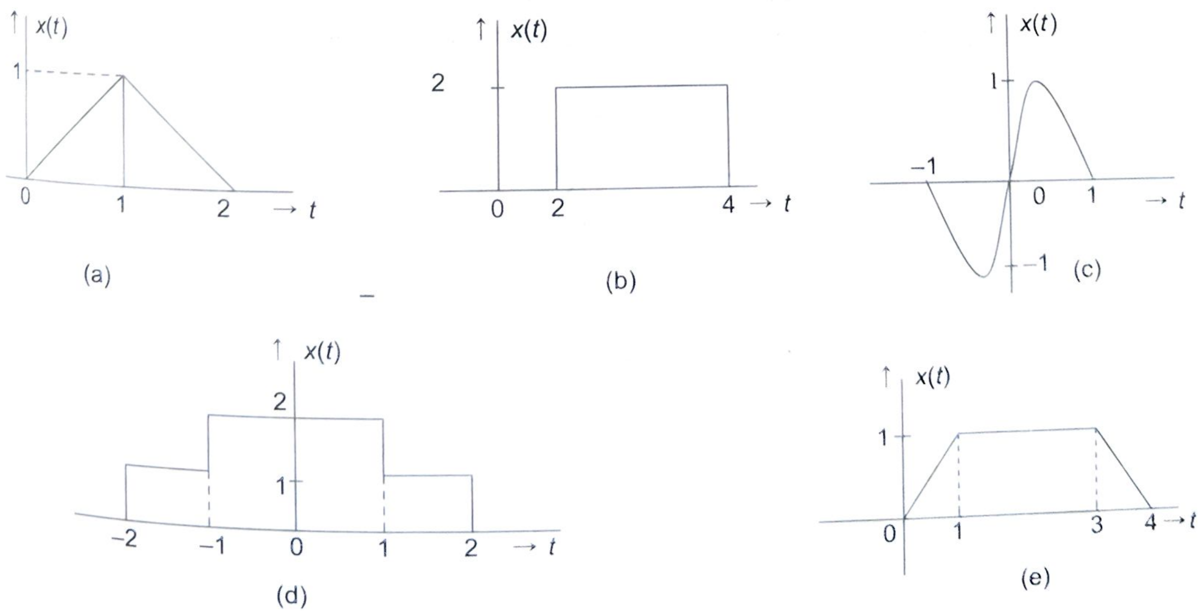


Fig. 4.39

4.19 Find $x(t)$ if its Fourier transform $X(f)$ is given by

(a) $X(f) = \dfrac{j2\pi f}{(1+j2\pi f)^2}$ (b) $X(f) = \dfrac{2\sin^2\omega}{\omega^2}$

(c) $X(f) = 5\left[\dfrac{\text{sinc } 4f}{(1+j2\pi f)}\right]$ (d)

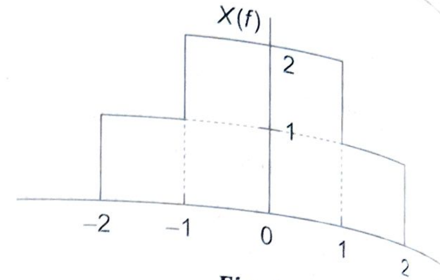


Fig. 4.40

4.20 A signal $x(t)$ whose spectrum is flat and extends from 1000 Hz to 5000 Hz, is recorded on a tape at a speed of 7 ½ inches per second. It is played back at a speed of 15 inches per second. Sketch the spect... of the recorded and played-back signals.

4.21 If the signal shown in Figs. 4.39 (a), (b) and (d) of Problem 4.18 are multiplied by $\cos 50\pi t$, determ... and sketch the magnitudes of the Fourier transforms of the resulting signals.

4.22 Given the signal $x(t) = 5$ for $|t| > 2$ and zero otherwise, find $X(f)$. Verify your result by writing an ex... pression for $x(t)$ in terms of a rectangular function and taking the Fourier transform of that expressio...

4.23 Obtain the Fourier transforms of the following signals:

(a) $x(t) = e^{-2|t|}\, sgn\ t$ (c) $x(t) = \text{sinc } t.u(t)$

(b) $x(t) = e^{-|t|}u(-t)$ (d) $x(t) = \text{sinc } t.\text{sgn}(-t)$

4.24 Taking $x(t) = e^{-at}u(t)$, verify the time-domain differentiation theorem by finding the Fourier transform of the derivative of $x(t)$ and comparing the result with that obtained by applying the time-domain differentiation theorem.

4.25 Obtain the Fourier transform of $1/t$.

(*Hint*: Apply duality theorem to the Fourier transform pair pertaining to a signum function)

4.26 Determine the Fourier transform of the $x(t)$ shown in Fig. 4.41.

(a) By applying time-domain differentiation theorem.

(b) By identifying $x(t)$ as having been obtained by the convolution of $\Pi(t/T)$ with itself and scaling down the magnitude by T and then applying convolution theorem.

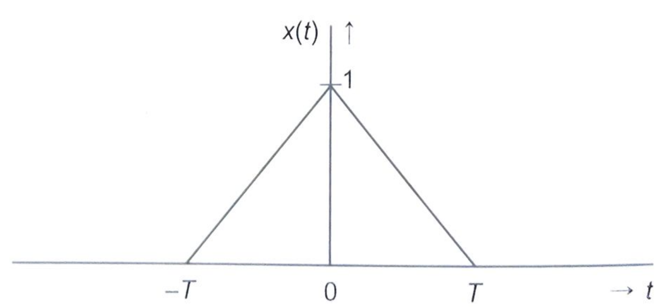


Fig. 4.41

Multiple-Choice Questions

1. Strictly speaking, which of the following signals is not Fourier transformable?
   (a) $e^{-|t|}$ (b) rect (τ/t) (c) tr (τ/t) (d) $\sin \omega_0 t$
2. If the signal $x(t)$ is real valued and its Fourier transform is $X(f)$ then
   (a) $X(f)$ is real valued (b) $|X(f)| = |X(-f)|$
   (c) $X(f)$ has even symmetry (d) $X(f)$ has odd symmetry
3. White noise is lowpass filtered by an ideal lowpass filter with a cutoff frequency of 1 KHz. The output of the filter is sampled at regular intervals of time. If uncorrelated samples are to be obtained, the sampling period is to be
   (a) 0.5 milli second (b) 1 milli second
   (c) 2 milli second (d) any value
4. The area under the signal 10 sinc f is
   (a) 5 (b) 20 (c) 10 (d) 10π
5. If $x(t) = 10$ rect $(t/2)$, the zero-frequency value of its spectrum is given by
   (a) 10 (b) 5 (c) 2 (d) 20
6. If $x(t) \xleftrightarrow{FT} X(f), |X(f)|$ represents
   (a) the energy density of $x(t)$ with respect to frequency, at the frequency f
   (b) the power density of $x(t)$ at f, with respect to frequency
   (c) positive square root of the energy density of $x(t)$ with respect to frequency, at the frequency f
   (d) positive square root of the power density of $x(t)$ with respect to frequency, at the frequency f.
7. Shifting a time signal along the time axis causes
   (a) a change in the amplitude spectrum
   (b) a change in both amplitude and phase spectrum
   (c) a change only in the phase spectrum
   (d) no change in amplitude as well as phase spectrum
8. If $x(t) = 10$ sinc $5t$, the energy contained in the signal is
   (a) 100 (b) 50 (c) 10 (d) 20
9. What is the time interval by which the signal $\cos 2000\,\pi t$ should be delayed in order to bring about a -90° phase shift in it?
   (a) 0.25×10^{-3}s (b) 0.5×10^{-3}s (c) $(4.5/\pi) \times 10^{-2}$s (d) 0.125×10^{-3}s
10. $x(t)$ is a low frequency modulating signal and $\cos \omega_c t$ is a carrier signal with frequency f_c. Then $x(t) \cos \omega_c t$ has
    (a) carrier and both the sidebands
    (b) both the sidebands but no carrier component
    (c) carrier and only the upper sideband
    (d) carrier and only the lower sideband

11. If $x(t) \xleftrightarrow{FT} X(f)$, the spectrum of $x(-3t)$ is a
 (a) frequency compressed version of $X(f)$ with a negative sign
 (b) frequency expanded version of $X(f)$ with a negative sign
 (c) frequency compressed version of $X(f)$ with lateral inversion
 (d) frequency expanded version of $X(f)$ with lateral inversion
12. The Fourier transform of $x(t) = [u(t) - u(t-10)]$ is
 (a) $10 \text{ sinc } 10f$
 (b) $\text{sinc } 10f$
 (c) $10(\text{sinc } 10f)e^{-j10\pi f}$
 (d) not Fourier transformable
13. If $y(t) \triangleq x(t) * \delta(t-\tau)$, $Y(f)$ is given by
 (a) $X(f)e^{+j2\pi f\tau}$
 (b) $X(f)e^{-j2\pi f\tau}$
 (c) $X(f-f_c)$ where, $f_c \triangleq \frac{1}{\tau}$
 (d) it is not Fourier transformable
14. If $y(t) \triangleq x(t)e^{-j2\pi f_0 t}$, $Y(f)$ is given by
 (a) $X(f+f_0)$
 (b) $X(f-f_0)$
 (c) $X(f)\, e^{j2\pi f_0}$
 (d) $X(f)\, \delta(f-f_0)$
15. If $x(t) = 5 \text{ sinc } 10t$, $X(f)$ is given by
 (a) $2\Pi(f/10)$
 (b) $0.5\Pi(f/5)$
 (c) $2\Pi(-f/5)$
 (d) $0.5\Pi(-f/10)$
16. If $y(t) \triangleq 10\Pi(t/5)e^{+j2\pi t}$, $Y(f)$ is given by
 (a) $2 \text{ sinc } [5(f-1/5)]$
 (b) $10 \text{ sinc } [5(f-1)]$
 (c) $2 \text{ sinc } [5(f+1/5)]$
 (d) $10 \text{ sinc } [5(f+1)]$
17. If $y(t) = x(0.1\,t)$, with reference to the spectrum of $y(t)$, which of the following is true?
 (a) Magnitude of the spectrum of $y(t)$ is 0.1 times the magnitude of the spectrum of $x(t)$.
 (b) Magnitude of the spectrum of $y(t)$ is 10 times the magnitude of the spectrum of $x(t)$ but the spectrum is compressed in frequency by a factor of 10.
 (c) Magnitude of the spectrum of $y(t)$ is 10 times the magnitude of the spectrum of $x(t)$ and the spectrum is expanded in frequency by a factor of 10.
 (d) Magnitude of the spectrum of $y(t)$ is 0.1 times the magnitude of the spectrum of $x(t)$ but the spectrum is expanded in frequency by a factor of 10.
18. If $x(t) = A\Pi\left(\frac{t-T/2}{T}\right)$, $\mathscr{F}[\dot{x}(t)]$ is equal to
 (a) $A[1+e^{-j2\pi fT}]$
 (b) $A[1-e^{j2\pi fT}]$
 (c) $A[1-e^{-j\pi fT}]$
 (d) $A[1-e^{-j2\pi fT}]$
19. A signal $x(t)$ is as shown in Fig. 4.42. The Fourier transform of $\ddot{x}(t)$ is given by
 (a) $(\cos 4\pi f + \cos 2\pi f)$
 (c) $(\cos 2\pi f - \cos \pi f)$
 (b) $(\cos 2\pi f - \cos 4\pi f)$
 (d) $(\cos 4\pi f - \cos 2\pi f)$
20. The Fourier transform of t sinc $10t$ is equal to
 (a) $\frac{1}{j20\pi}[\Pi(f/10)]$

(b) $\frac{1}{20j\pi}[\delta(f+5)-\delta(f-5)]$

(c) $\frac{f}{20\pi}[\Pi(f/10)]$

(d) $\frac{j}{20\pi}[\delta(f+5-\delta(f-5)]$

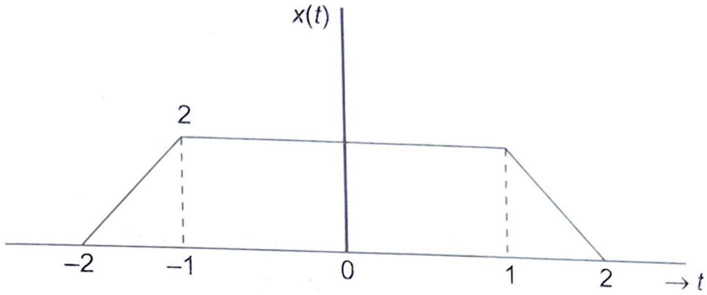


Fig. 4.42

1. The spectrum of a signal is $X(f) = 5j\delta(f+f_0) - 5j\delta(f-f_0)$. The corresponding signal $x(t)$ is given by

(a) $5 \cos 2\pi f_0 t$ (b) $10 \cos 2\pi f_0 t$ (c) $10 \sin 2\pi f_0 t$ (d) $5 \sin 2\pi f_0 t$

2. If $x(t) \xleftrightarrow{FT} X(f)$ and $y(t) \underline{\Delta} x(t-1)e^{jt}$, $Y(f)$ equals

(a) $X(\omega-1)\, e^{-j(\omega-1)}$ (b) $X(f-1)\, e^{j(f-1)2\pi}$

(c) $X(f-1)\, e^{-j2\pi(f-1)}$ (d) $X(\omega-1)\, e^{j(\omega-1)}$

3. The Fourier transform of $1/t$ is

(a) $j\pi f$ (b) $\frac{1}{j\pi f}$ (c) $-j\pi \operatorname{sgn}(f)$ (d) $j\pi \operatorname{sgn}(f)$

[*Hint: Use duality theorem.*]

4. If $y(t) = x(2-t)$, $Y(f)$ is given by

(a) $X(-f)\, e^{-j4\pi f}$ (b) $X(f)\, e^{-j4\pi f}$ (c) $X(-f)\, e^{j4\pi f}$ (d) $X(f)\, e^{j4\pi f}$

5. If $y(t) = x(t/3-2)$, $Y(f)$ is equal to

(a) $3X(f)\, e^{-j6\pi f}$ (b) $3X(3f)\, e^{-j12\pi f}$ (c) $3X(f)\, e^{-j12\pi f}$ (d) $3X(3f)\, e^{j12\pi f}$

6. The Fourier transform of $e^{-at}u(t)$ is

(a) $\frac{1}{a-j\omega}$ (b) $\frac{1}{-a+j\omega}$ (c) $\frac{1}{a+j\omega}$ (d) $\frac{-1}{a+j\omega}$

7. The Fourier transform of $e^{at}u(-t)$ is

(a) $\frac{1}{a-j\omega}$ (b) $\frac{1}{-a+j\omega}$ (c) $\frac{1}{a+j\omega}$ (d) $\frac{-1}{a-j\omega}$

8. A signal $x(t) = u(t-2) + u(t-1) - u(t+1) - u(t+2)$. Then $\int_{-\infty}^{\infty} X(f)\, df$ is equal to

(a) 1 (b) 2 (c) 0 (d) None of these

29. A signal $x(t) = a_1u(t-2) + a_2u(t-1) - a_3u(t+1) - a_4u(t+2)$ where a_1, a_2, a_3 and a_4 are real constants. Then $X(f)$ is

(a) necessarily real (c) may be real or complex

(b) necessarily complex (d) None of these

30. A signal $x(t) = a_1u(t-2) + a_2u(t-1) - a_2u(t+1) - a_1u(t+2)$, where, a_1 and a_2 are real constants. Then $X(f)$ is

(a) $\frac{1}{2}(a_1 + a_2)$ (b) $(a_1 + a_2)$ (c) 0 (d) None of these

31. A signal $x(t)$ has a Fourier transform $X(f)$. If $x(t)$ is real and even function of time t, then $X(f)$ is
   (a) a real and even function of 'f'
   (b) an imaginary and even function of 'f'
   (c) an imaginary and odd function of 'f'
   (d) a real and odd function of 'f'

32. The Fourier transform of the signal $x(t) = a\,e^{-bt^2}$ is of the form
   (a) $c\,e^{-\omega d}$ (b) $c\,|\omega|^d$ (c) $c\,e^{-\omega^2 d}$ (d) $c + d\,|\omega|^2$

33. The Fourier transform of $e^{-t/2}\,u(t)$ is $\frac{2}{1+j4\pi f}$. The Fourier transform of $\frac{1}{1+j4\pi t}$ is
   (a) $e^{-f/2}\,u(f)$ (b) $e^{+f/2}\,u(-f)$ (c) $e^{-2f}\,u(-f)$ (d) $e^{2f}\,u(-f)$

34. The Fourier transform of a conjugate anti-symmetric function is always
   (a) imaginary
   (b) conjugate symmetric
   (c) conjugate anti-symmetric
   (d) real

35. A signal $x(t)$ has a Fourier transform of $X(f)$. Then the inverse Fourier transform of $X(5f+3)$ is given by
   (a) $\frac{1}{3}x\left(\frac{t}{3}\right)e^{j5\pi t}$ (b) $\frac{1}{5}x\left(\frac{t}{5}\right)e^{-j6\pi t/5}$ (c) $3x(3t)\,e^{-j5\pi t}$ (d) $x(5t+3)$

36. Signal $x(t) = 3e^{-2t}\,u(t)$. Its 3-dB bandwidth is
   (a) $1/\pi$ Hertz (b) 2π Hertz (c) $\frac{2\pi}{3}$ Hertz (d) $\frac{2}{\pi}$ Hertz

37. $x(t) = \begin{cases} 2 \text{ for } -0.5 \le t \le 0.5 \\ 0 \text{ otherwise} \end{cases}$

   Two frequencies at which $X(f)$ takes zero value, are
   (a) 1, 2HZ (b) 0.5, 1 HZ (c) 0, 1 HZ
   (d) it does not become zero at any frequency.

MATLAB Exercises

1. Using MATLAB, study the changes in the spectrum of an $x(t)$ due to (a) time-shift (b) scaling and (c) modulation.
2. Given the signal $x(t) = e^{-t^2/2}\cos\pi t$, using MATLAB, evaluate its Fourier transform and determine its approximate band-limiting frequency.
3. Evaluate and plot the phase of the Fourier transform of the rectangular pulse of unit amplitude, and width T that is centered at $T/2$. Is the phase of the Fourier transform a linear function of frequency?