the final operator appears ... $x_0 \cdot x_1 \cdot \cdots \cdot x_k$ to determine the order in which these ... insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$ to determine the order in which these $n-k$ numbers are to be multiplied. Because this final operator can appear between any two of the $n+1$ numbers, it follows that

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0$$

$$= \sum_{k=0}^{n-1} C_kC_{n-k-1}.$$

Note that the initial conditions are $C_0 = 1$ and $C_1 = 1$. This recurrence relation can be solved using the method of generating functions, which will be discussed in Section 6.4. It can be shown that $C_n = C(2n, n)/(n+1)$. (See Exercise 41 at the end of that section.)

Links The sequence $\{C_n\}$ is the sequence of **Catalan numbers.** This sequence appears as the solution of many different counting problems besides the one considered here (see the chapter on Catalan numbers in [MiRo91] or [Ro84a] for details).

Exercises

1. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.
   a) $a_n = 6a_{n-1}, a_0 = 2$
   b) $a_n = a_{n-1}^2, a_1 = 2$
   c) $a_n = a_{n-1} + 3a_{n-2}, a_0 = 1, a_1 = 2$
   d) $a_n = na_{n-1} + n^2a_{n-2}, a_0 = 1, a_1 = 1$
   e) $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$
2. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.
   a) $a_n = -2a_{n-1}, a_0 = -1$
   b) $a_n = a_{n-1} - a_{n-2}, a_0 = 2, a_1 = -1$
   c) $a_n = 3a_{n-1}^2, a_0 = 1$
   d) $a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0$
   e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}, a_0 = 1, a_1 = 1, a_2 = 2$
3. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$.
   a) Find a_0, a_1, a_2, a_3, and a_4.
   b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.
   c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.
4. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if
   a) $a_n = 0$.
   b) $a_n = 1$.
   c) $a_n = (-4)^n$.
   d) $a_n = 2(-4)^n + 3$.
5. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if
   a) $a_n = 0$?
   b) $a_n = 1$?
   c) $a_n = 2^n$?
   d) $a_n = 4^n$?
   e) $a_n = n4^n$?
   f) $a_n = 2 \cdot 4^n + 3n4^n$?
   g) $a_n = (-4)^n$?
   h) $a_n = n^24^n$?
6. For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)
   a) $a_n = 3$
   b) $a_n = 2n$
   c) $a_n = 2n + 3$
   d) $a_n = 5^n$
   e) $a_n = n^2$
   f) $a_n = n^2 + n$
   g) $a_n = n + (-1)^n$
   h) $a_n = n!$
7. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if
   a) $a_n = -n + 2$.
   b) $a_n = 5(-1)^n - n + 2$.
   c) $a_n = 3(-1)^n + 2^n - n + 2$.
   d) $a_n = 7 \cdot 2^n - n + 2$.
8. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 5.
   a) $a_n = -a_{n-1}, a_0 = 5$
   b) $a_n = a_{n-1} + 3, a_0 = 1$
   c) $a_n = a_{n-1} - n, a_0 = 4$
   d) $a_n = 2a_{n-1} - 3, a_0 = -1$
   e) $a_n = (n+1)a_{n-1}, a_0 = 2$
   f) $a_n = 2na_{n-1}, a_0 = 3$
   g) $a_n = -a_{n-1} + n - 1, a_0 = 7$
9. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach such as that used in Example 5.
   a) $a_n = 3a_{n-1}, a_0 = 2$
   b) $a_n = a_{n-1} + 2, a_0 = 3$

c) $a_n = a_{n-1} + n, a_0 = 1$ d) $a_n = a_{n-1} + 2n + 3$, [illegible]
e) $a_n = 2a_{n-1} - 1, a_0 = 1$ f) $a_n = 3a_{n-1} + 1, a_0 =$ [illegible]
g) $a_n = na_{n-1}, a_0 = 5$ h) $a_n = 2na_{n-1}, a_0 = 1$

10. A person deposits $1000 in an account that yields 9% interest compounded anually.
   a) Set up a recurrence relation for the amount in the account at the end of n years.
   b) Find an explicit formula for the amount in the account at the end of n years.
   c) How much money will the account contain after 100 years?

11. Suppose that the number of bacteria in a colony triples every hour.
   a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.
   b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

12. Assume that the population of the world in 2002 was 6.2 billion and is growing at the rate of 1.3% a year.
   a) Set up a recurrence relation for the population of the world n years after 2002.
   b) Find an explicit formula for the population of the world n years after 2002.
   c) What will the population of the world be in 2022?

13. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month.
   a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.
   b) How many cars are produced in the first year?
   c) Find an explicit formula for the number of cars produced in the first n months by this factory.

14. An employee joined a company in 1999 with a starting salary of $50,000. Every year this employee receives a raise of $1000 plus 5% of the salary of the previous year.
   a) Set up a recurrence relation for the salary of this employee n years after 1999.
   b) What will the salary of this employee be in 2007?
   c) Find an explicit formula for the salary of this employee n years after 1999.

15. Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan of $5000 at a rate of 7% if a payment of $100 is made each month. [*Hint:* Express $B(k)$ in terms of $B(k-1)$; the monthly interest is $(0.07/12)B(k-1)$.]

16. a) Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate of r if a payment P is made on the loan each month. [*Hint:* Express $B(k)$ in terms of $B(k-1)$ and note that the monthly interest rate is $r/12$.]
   b) Determine what the monthly payment P should be so that the loan is paid off after T months.

17. Use mathematical induction to verify the formula derived in Example 5 for the number of moves required to complete the Tower of Hanoi puzzle.

18. a) Find a recurrence relation for the number of permutations of a set with n elements.
   b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

19. A vending machine dispensing books of stamps accepts only dollar coins, $1 bills, and $5 bills.
   a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.
   b) What are the initial conditions?
   c) How many ways are there to deposit $10 for a book of stamps?

20. A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of n pesos if the order in which the coins and bills are paid matters.

21. How many ways are there to pay a bill of 17 pesos using the currency described in Exercise 20, where the order in which coins and bills are paid matters?

*22. a) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \ldots, a_k$, where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \ldots, k-1$.
   b) What are the initial conditions?
   c) How many sequences of the type described in (a) are there when n is a positive integer with $n \geq 2$?

23. a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.
   b) What are the initial conditions?
   c) How many bit strings of length seven contain two consecutive 0s?

24. a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.
   b) What are the initial conditions?
   c) How many bit strings of length seven contain three consecutive 0s?

25. a) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.
   b) What are the initial conditions?
   c) How many bit strings of length seven do not contain three consecutive 0s?

*26. a) Find a recurrence relation for the number of bit strings that contain the string 01.

b) What are the initial conditions?

c) How many bit strings of length seven contain the string 01?

27. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions?

c) How many ways can this person climb a flight of eight stairs?

28. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

b) What are the initial conditions?

c) How many ways can this person climb a flight of eight stairs?

A string that contains only 0s, 1s, and 2s is called a **ternary string.**

29. a) Find a recurrence relation for the number of ternary strings that do not contain two consecutive 0s.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain two consecutive 0s?

30. a) Find a recurrence relation for the number of ternary strings that contain two consecutive 0s.

b) What are the initial conditions?

c) How many ternary strings of length six contain two consecutive 0s?

*31. a) Find a recurrence relation for the number of ternary strings that do not contain two consecutive 0s or two consecutive 1s.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain two consecutive 0s or two consecutive 1s?

*32. a) Find a recurrence relation for the number of ternary strings that contain either two consecutive 0s or two consecutive 1s.

b) What are the initial conditions?

c) How many ternary strings of length six contain two consecutive 0s or two consecutive 1s?

*33. a) Find a recurrence relation for the number of ternary strings that do not contain consecutive symbols that are the same.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain consecutive symbols that are the same?

**34. a) Find a recurrence relation for the number of ternary strings that contain two consecutive symbols that are the same.

b) What are the initial conditions?

c) How many ternary strings of length six contain consecutive symbols that are the same?

35. Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.

a) Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in n microseconds.

b) What are the initial conditions?

c) How many different messages can be sent in 10 microseconds using these two signals?

36. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).

b) In how many different ways can the driver pay a toll of 45 cents?

37. a) Find the recurrence relation satisfied by R_n, where R_n is the number of regions that a plane is divided into by n lines, if no two of the lines are parallel and no three of the lines go through the same point.

b) Find R_n using iteration.

*38. a) Find the recurrence relation satisfied by R_n, where R_n is the number of regions into which the surface of a sphere is divided by n great circles (which are the intersections of the sphere and planes passing through the center of the sphere), if no three of the great circles go through the same point.

b) Find R_n using iteration.

*39. a) Find the recurrence relation satisfied by S_n, where S_n is the number of regions into which three-dimensional space is divided by n planes if every three of the planes meet in one point, but no four of the planes go through the same point.

b) Find S_n using iteration.

40. Find a recurrence relation for the number of bit sequences of length n with an even number of 0s.

41. How many bit sequences of length seven contain an even number of 0s?

42. a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. [*Hint:* Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

b) What are the initial conditions for the recurrence relation in part (a)?

c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

43. a) Find a recurrence relation for the number of ways to lay out a walkway with slate tiles if the tiles are red, green,

$a_n = 3.3^{\log_2} - 2.2$

$= 3.3^{\log_2^n} - 2n$

But $3^{\log_2^n} = n^{\log_2^3}$

So $a_n = 3n^{\log_2^3} - 2n.$

Exercises

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.
   a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
   b) $a_n = 2na_{n-1} + a_{n-2}$
   c) $a_n = a_{n-1} + a_{n-4}$
   d) $a_n = a_{n-1} + 2$
   e) $a_n = a^2_{n-1} + a_{n-2}$
   f) $a_n = a_{n-2}$
   g) $a_n = a_{n-1} + n$
2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.
   a) $a_n = 3a_{n-2}$
   b) $a_n = 3$
   c) $a_n = a^2_{n-1}$
   d) $a_n = a_{n-1} + 2a_{n-3}$
   e) $a_n = a_{n-1}/n$
   f) $a_n = a_{n-1} + a_{n-2} + n + 3$
   g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$
3. Solve these recurrence relations together with the initial conditions given.
   a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
   b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
   c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
   d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
   e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$
   f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
   g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
4. Solve these recurrence relations together with the initial conditions given.
   a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
   b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
   c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
   d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
   e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
   f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
   g) $a_n + 2 = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$
5. How many different messages can be transmitted in n microseconds using the two signals described in Exercise 35 in Section 6.1?
6. How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?
7. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?
8. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.
   a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.
   b) Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.
9. A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.
   a) Find a recurrence relation for $\{P_n\}$, where P_n is the amount in the account at the end of n years if no money is ever withdrawn.
   b) How much is in the account after n years if no money has been withdrawn?

*10. Prove Theorem 2.

11. The Lucas numbers satisfy the recurrence relation
$$L_n = L_{n-1} + L_{n-2},$$
and the initial conditions $L_0 = 2$ and $L_1 = 1$.
   a) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, \ldots$, where f_n is the nth Fibonacci number.
   b) Find an explicit formula for the Lucas numbers.
12. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.
13. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.
14. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.
15. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

*16. Prove Theorem 3.

17. Prove this identity relating the Fibonacci numbers and the binomial coefficients:
$$f_{n+1} = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$$

where n is a positive integer and $k = \lfloor n/2 \rfloor$. [*Hint:* Let $a_n = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$. Show that the sequence $\{a_n\}$ satisfies the same recurrence relation and initial conditions satisfied by the sequence of Fibonacci numbers.]

18. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5$, $a_1 = 4$, and $a_2 = 88$.
19. Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.
20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.
21. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, −2, −2, −2, 3, 3, −4?
22. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots −1, −1, −1, 2, 2, 5, 5, 7?
23. Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.
    a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.
    b) Use Theorem 5 to find all solutions of this recurrence relation.
    c) Find the solution with $a_0 = 1$.
24. Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.
    a) Show that $a_n = n2^n$ is a solution of this recurrence relation.
    b) Use Theorem 5 to find all solutions of this recurrence relation.
    c) Find the solution with $a_0 = 2$.
25. a) Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$.
    b) Use Theorem 5 to find all solutions of this recurrence relation.
    c) Find the solution of this recurrence relation with $a_0 = 4$.
26. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if
    a) $F(n) = n^2$?
    b) $F(n) = 2^n$?
    c) $F(n) = n2^n$?
    d) $F(n) = (-2)^n$?
    e) $F(n) = n^2 2^n$?
    f) $F(n) = n^3(-2)^n$?
    g) $F(n) = 3$?
27. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if
    a) $F(n) = n^3$?
    b) $F(n) = (-2)^n$?
    c) $F(n) = n2^n$?
    d) $F(n) = n^2 4^n$?
    e) $F(n) = (n^2 - 2)(-2)^n$?
    f) $F(n) = n^4 2^n$?
    g) $F(n) = 2$?
28. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.
    b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.
29. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.
    b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.
30. a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.
    b) Find the solution of this recurrence relation with $a_1 = 56$ and $a_2 = 278$.
31. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$. [*Hint:* Look for a particular solution of the form $qn2^n + p_1 n + p_2$, where q, p_1, and p_2 are constants.]
32. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.
33. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.
34. Find all solutions of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ with $a_0 = -2$, $a_1 = 0$, and $a_2 = 5$.
35. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$.
36. Let a_n be the sum of the first n perfect squares, that is, $a_n = \sum_{k=1}^{n} k^2$. Show that the sequence $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n^2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for an by solving this recurrence relation.
37. Let a_n be the sum of the first n triangular numbers, that is, $a_n = \sum_{k=1}^{n} t_k$, where $t_k = k(k+1)/2$. Show that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n(n+1)/2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for a_n by solving this recurrence relation.
38. a) Find the characteristic roots of the linear homogeneous recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$. (*Note:* These are complex numbers.)
    b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$ and $a_1 = 2$.
*39. a) Find the characteristic roots of the linear homogeneous recurrence relation $a_n = a_{n-4}$. (Note: These include complex numbers.)
    b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 1$.
*40. Solve the simultaneous recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + 2b_{n-1}$$