Tut sheet-8

1.

$X = v + N$; v is constant voltage ; $N \sim N(0, 10)$

Sample mean : $\bar{X}_{30} = 100\ \mu V$

$E(X) = v$, $Var(X) = 10\ \mu V^2$.

95% confidence interval ; $1-\alpha = 0.95$

$$P\left(-z \leq \frac{\bar{X}_{30} - E(\bar{X}_{30})}{\sqrt{Var(\bar{X}_{30})}} \leq z\right) = 1-\alpha.$$

$$\Rightarrow P\left(-z \leq \frac{100 - v}{\sqrt{10/30}} \leq z\right) = 0.95$$

$$\Rightarrow 1 - 2Q(z) = 0.95$$

$$\Rightarrow 2Q(z) = 0.05$$

$$\Rightarrow Q(z) = 0.025$$

From Q-function table,

$$Q(z_{0.025}) = 0.025 \quad \Rightarrow z_{0.025} = 1.96$$

95% Confidence interval of v is

$$\left[\bar{X}_{100} - z_{0.025}\sqrt{\tfrac{10}{30}},\ \bar{X}_{100} + z_{0.025}\sqrt{\tfrac{10}{30}}\right]$$

$$= \left[100 - \frac{1.96}{\sqrt{3}},\ 100 + \frac{1.96}{\sqrt{3}}\right]$$

$$= [98.868,\ 101.132]$$

2. 225 light bulbs.

$\bar{X}_{225} = 223$ hr , $\hat{\sigma}^2_{225} = 100$ hr^2

(a) 95% confidence interval for $E(X) = \mu$.

$$P\left(-t \leq \frac{\bar{X}_{225} - E(\bar{X}_{225})}{\hat{\sigma}_{225}/\sqrt{225}} \leq t\right) = 1-\alpha$$

$$T_{225} = \frac{\bar{X}_{225} - E(\bar{X}_{225})}{\hat{\sigma}_{225}/\sqrt{225}}$$ is student's t-distribution

~~of~~ with 224 degrees of freedom.

$$E(\bar{X}_{225}) = \mu.$$

$$\therefore P\left(-t \le \frac{\bar{X}_{225} - \mu}{\hat{\sigma}_{225}/\sqrt{225}} \le t\right) = 2F_{224}(t) - 1$$

$$= P\left(\bar{X}_{225} - \frac{t\hat{\sigma}_{225}}{\sqrt{225}} \le \mu \le \bar{X}_{225} + \frac{t\hat{\sigma}_{225}}{\sqrt{225}}\right)$$

$$= 1 - \alpha$$

$$= 0.95$$

$$\therefore 2F_{224}(t) - 1 = 0.95$$

$$\Rightarrow F_{224}(t) = \frac{1 + 0.95}{2} = 0.975 = 1 - 0.025$$

From student's t-distribution table,

$$F_{224}(t_{0.025,224}) = 1 - 0.025$$

$$\therefore \cancel{t_{0.025,2}}\ t_{0.025,224} = 1.971$$

<u>Confidence interval of μ is</u> :

$$\left[\bar{X}_{225} - \frac{t_{0.025,224}\hat{\sigma}_{225}}{\sqrt{225}}, \bar{X}_{225} + \frac{t_{0.025,224}\hat{\sigma}_{225}}{\sqrt{225}}\right]$$

$$= \left[223 - \frac{1.971 \times \sqrt{100}}{\sqrt{225}}, 223 + \frac{1.971 \times \sqrt{100}}{\sqrt{225}}\right]$$

$$= \left[223 - \frac{1.971 \times 10}{15}, 223 + \frac{1.971 \times 10}{15}\right]$$

$$= [221.686, 224.314]$$

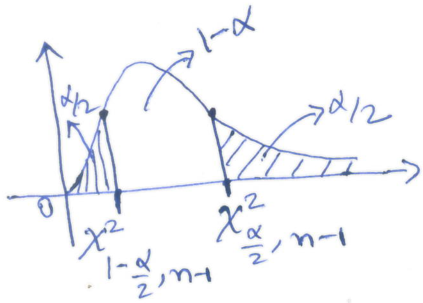
(b) 95% confidence interval for $Var(X) = \sigma_x^2$.

Unbiased ~~sample~~ variance estimator

$$\hat{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 , \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

let $$\chi^2_{n-1} = \frac{(n-1)\hat{\sigma}_n^2}{\sigma_x^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}_n}{\sigma_x} \right)^2$$

χ^2_{n-1} is chi-square distribution of $n-1$ degrees of freedom.



$$1-\alpha = P\left(\chi^2_{1-\frac{\alpha}{2},n-1} \le \frac{(n-1)\hat{\sigma}_n^2}{\sigma_x^2} \le \chi^2_{\frac{\alpha}{2},n-1} \right)$$

$$= P\left(\frac{(n-1)\hat{\sigma}_n^2}{\chi^2_{\frac{\alpha}{2},n-1}} \le \sigma_x^2 \le \frac{(n-1)\hat{\sigma}_n^2}{\chi^2_{1-\alpha/2,n-1}} \right)$$

$1-\alpha = 0.95$; $\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$; $1-\frac{\alpha}{2} = 0.975$

$n = 225$, $\hat{\sigma}_{225}^2 = 100 \text{ hr}^2$

$\chi^2_{0.025,224} =$

$\chi^2_{0.975,224} =$

Confidence interval for σ_x^2:

$$\left[\frac{224 \times 100}{\chi^2_{0.025,224}} , \frac{224 \times 100}{\chi^2_{0.975,224}} \right].$$

4. $H_0: E(X) = \mu = 30$; X is Poisson r.v.

$\bar{X}_8 = 32$; significance level: $\alpha = 0.05$.

$\alpha = P(\vec{X}_8 \in \tilde{R} \mid H_0)$; $\tilde{R}$ is rejection region for H_0.

$= P\left(\frac{\bar{X}_8 - 30}{\sqrt{Var(\bar{X}_8)}} > \frac{c}{\sqrt{Var(\bar{X}_8)}} \middle| H_0\right)$; $Var(\bar{X}_8) = \frac{Var(X)}{8} = \frac{30}{8}$

$= P\left(\frac{\bar{X}_8 - 30}{\sqrt{30/8}} > \frac{c}{\sqrt{30/8}}\right)$

$\simeq Q\left(\frac{c}{\sqrt{30/8}}\right)$, approx. by Central Limit Theorem.

$\therefore \alpha = Q\left(\frac{2c}{\sqrt{15}}\right)$

(a) $\Rightarrow Q\left(\frac{2c}{\sqrt{15}}\right) = 0.05$, $Z_{0.05} = 1.6449$.

$\therefore Z_{0.05} = \frac{2c}{\sqrt{15}} \Rightarrow c = \frac{\sqrt{15}\, Z_{0.05}}{2} = \frac{\sqrt{15} \times 1.6449}{2}$

$\therefore c = 3.185$

$\tilde{R} = \{\bar{x}_n : \bar{x}_n > 30 + c\}$; $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\tilde{R} = \{\bar{x}_n : \bar{x}_n > 33.185\}$.

Given that $\bar{X}_8 = 32$, so $\bar{X}_8 = 32 \in \tilde{R}^c$; ~~acc~~ acceptance region

So, the claim that μ has increased is not true.

(b) $Q\left(\frac{2c}{\sqrt{15}}\right) = 0.01$, $Z_{0.01} = 2.3263$; $\alpha = 1\%$

$c = \frac{\sqrt{15}\, Z_{0.01}}{2} = \frac{\sqrt{15} \times 2.3263}{2} = 4.5048$

$\tilde{R} = \{\bar{x}_n : \bar{x}_n > 30 + 4.505\} = \{\bar{x}_n : \bar{x}_n > 34.505\}$; $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\therefore \bar{X}_8 = 32 \in \tilde{R}^c$; acceptance region.

So, the claim that μ has increased is not true.

5 $H_0 : v_{in} = 0 ; \; X = N \sim N(0,4)$.

(a) $\alpha = P\left(\left| \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \right| > \frac{c - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \,\Big|\, H_0 \right)$

$E(\bar{X}_n | H_0) = E(X | H_0) = 0$.

$Var(\bar{X}_n | H_0) = \frac{Var(X | H_0)}{n} = \frac{4}{n}$.

$\alpha = P\left(\left| \frac{\bar{X}_n}{\sqrt{4/n}} \right| > \frac{c}{\sqrt{4/n}} \right) = 2Q\left(\frac{c}{\sqrt{4/n}} \right)$

$\Rightarrow Q\left(\frac{\sqrt{n}\,c}{2} \right) = \frac{\alpha}{2} = \frac{0.01}{2} = 0.005 ; \quad \alpha = 1\%$

$Z_{0.005} = \frac{\sqrt{n}\,c}{2} = 2.5758$

$\Rightarrow c = \frac{2 \times Z_{0.005}}{\sqrt{n}} = \frac{2 \times 2.5758}{\sqrt{n}}$

(b) $n = 10, \; \bar{X}_{10} = -0.75$

$c = \frac{2 \times 2.5758}{\sqrt{10}} = 1.629$

$\tilde{R} = \{ \bar{x}_n : |\bar{x}_n| > 1.629 \} = [-1.629, 1.629]^c$

$\therefore |\bar{x}_{10}| = 0.75 \notin \tilde{R}$

$\therefore |\bar{x}_{10}| = 0.75 \in \tilde{R}^c$; acceptance region

$\therefore$ the scientist's hunch that $v_{in} \neq 0$ is not true.

(c) $H_1 : v_{in} = 1\,V ; \; X = 1 + N \sim N(1,4)$

Alternative Hypothesis (simple).

$\beta = P(\bar{X}_n \in \tilde{R}^c | H_1) = P\left(\left| \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \right| \leq \frac{c - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \,\Big|\, H_1 \right)$

β is prob. of Type II error.

$\cancel{\beta =}$ $E(\bar{X}_n | H_1) = 1$, $Var(\bar{X}_n | H_1) = \frac{Var(X|H_1)}{n} = \frac{4}{n}$.

$\therefore \beta = P\left(\left| \frac{\bar{X}_n - 1}{\sqrt{4/n}} \right| \leq \frac{c-1}{\sqrt{4/n}} \right)$

$= 1 - 2\, Q\left(\frac{c-1}{\sqrt{4/n}} \right)$

$\beta = 1 - 2\, Q\left(\frac{\sqrt{n}(c-1)}{2} \right)$

For $n = 10$; $c = 1.629$ (see part (b))

$\therefore \beta = 1 - 2\, Q\left(\frac{\sqrt{10}(1.629 - 1)}{2} \right)$

$= 1 - 2\, Q\left(\frac{\sqrt{10} \times 0.629}{2} \right)$

$= 1 - 2\, Q(0.9945)$

$= 1 - 2(0.15999)$

$= 0.68$

6 H_0 : Company 1 ; $\mu = 8$, $\sigma^2 = \cancel{8}1$; $X \sim N(8,1)$

H_1 : Company 2 ; $\mu = 9$, $\sigma^2 = 1$; $X \sim N(9,1)$

Given :

$\alpha = 1\%$, $\cancel{\beta =}$ $1 - \beta = P_D = 99\%$

$\alpha = P\left(\left| \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \right| > \frac{c - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \middle| H_0 \right)$

$E(\bar{X}_n | H_0) = 8$, $Var(\bar{X}_n | H_0) = \frac{Var(X)}{n} = \frac{1}{n}$

$\therefore \alpha = P\left(\left| \frac{\bar{X}_n - 8}{\sqrt{1/n}} \right| > \frac{c-8}{\sqrt{1/n}} \right)$

$\Rightarrow 0.01 = 2\, Q(\sqrt{n}(c-8))$

$\Rightarrow Q(\sqrt{n}(c-8)) = 0.005$

$\therefore \sqrt{n}(c-8) = Z_{0.005} = 2.5758$ —— (1)

$$\beta = P\left(\left|\frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}}\right| \leq \frac{c - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} \,\middle|\, H_1\right)$$

$$E(\bar{X}_n | H_1) = 9, \quad Var(\bar{X}_n | H_1) = \frac{Var(X|H_1)}{n} = \frac{1}{n}$$

$$\therefore \beta = P\left(\left|\frac{\bar{X}_n - 9}{\sqrt{1/n}}\right| \leq \frac{c-9}{\sqrt{1/n}}\right)$$

$$\Rightarrow 0.01 = 1 - 2Q(\sqrt{n}(c-9))$$

$$\Rightarrow 0.99 = 2Q(\sqrt{n}(c-9))$$

$$\Rightarrow Q(\sqrt{n}(c-9)) = \frac{0.99}{2} = 0.495$$

$$\therefore \sqrt{n}(c-9) = Z_{0.495} = \qquad \text{—— (2)}$$

$$\sqrt{n}(c-8-1) = Z_{0.495}$$

$$\Rightarrow \sqrt{n}(c-8) - \sqrt{n} = Z_{0.495}$$

$$\Rightarrow \sqrt{n} = Z_{0.005} - Z_{0.495}, \quad \text{From (1)}$$

$$\therefore n = (Z_{0.005} - Z_{0.495})^2$$

$$c = 8 + \frac{Z_{0.005}}{\sqrt{n}} = 8 + \frac{Z_{0.005}}{Z_{0.005} - Z_{0.495}}$$

Here n and c are computed using two-sided test.

One-sided test :–

$$0.01 = \alpha = P\left(\frac{\bar{X}_n - 8}{\sqrt{1/n}} > \frac{c-8}{\sqrt{1/n}}\right) = Q(\sqrt{n}(c-8))$$

$$\therefore \sqrt{n}(c-8) = Z_{0.01} = 2.3263 \qquad \text{—— (1)}$$

$$0.99 = P_D = P\left(\frac{\bar{X}_n - 9}{\sqrt{1/n}} > \frac{c-9}{\sqrt{1/n}}\right) = Q(\sqrt{n}(c-9))$$

$$\therefore \sqrt{n}(c-9) = Z_{0.99} = -2.33 \qquad \text{—— (2)}$$

Solving (1) and (2), $n = (Z_{0.01} - Z_{0.99})^2 \approx 22$

$$\& \; c = 8 + \frac{Z_{0.01}}{\sqrt{n}} = 8.4996$$

One-sided test Decision Rule:

Accept H_0 if $\bar{x}_n \in \{x : x \leq 8.4996\}$

Accept H_1 if $\bar{x}_n \in \{x : x > 8.4996\}$

Here, $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ is any one realization of sample mean.

(7) $H_0 : m_0 = 2$; $X \sim \text{Exponential}(1/2)$; light internet user

$H_1 : m_1 = 4$; $X \sim \text{Exponential}(1/4)$; heavy internet user.

~~E(X₀)~~ $E(X|H_0) = m_0 = 2$, $\text{Var}(X|H_0) = m_0^2 = 2^2$.

$E(X|H_1) = m_1 = 4$, $\text{Var}(X|H_1) = m_1^2 = 4^2$

(a) Neyman-Pearson Test Decison Rule: Here $m_1 > m_0$

Accept H_0 if $\bar{x}_n \in \{x : x < \gamma\}$

Accept H_1 if $\bar{x}_n \in \{x : x \geq \gamma\}$

where $\gamma = m_0 + \frac{Z_\alpha \sigma_0}{\sqrt{n}} = 2 + \frac{2Z_{0.05}}{\sqrt{n}}$; $\alpha = 5\%$

(b) Prob. of detecting heavy user; $P_D = 1 - \beta$

$$P_D = P\left(\frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{\text{Var}(\bar{X}_n)}} > \frac{\gamma - E(\bar{X}_n)}{\sqrt{\text{Var}(\bar{X}_n)}} \,\Big|\, H_1\right)$$

$$= P\left(\frac{\bar{X}_n - 4}{\sqrt{4^2/n}} > \frac{\gamma - 4}{\sqrt{4^2/n}}\right)$$

$$\simeq Q\left(\frac{(\gamma - 4)\sqrt{n}}{4}\right)\text{, by CLT for } n \text{ large.}$$

$$\therefore P_D = Q\left(\frac{\sqrt{n}(\gamma - 4)}{4}\right);\ \gamma = 2 + \frac{2Z_{0.05}}{\sqrt{n}}.$$