Mahindra École Centrale, Hyderabad

ES 211 (Numerical Methods), Problem Sheet−V

Tutorial problems

1. Prove the uniqueness of Cholesky decomposition.

2. Find a sufficient condition on the convergence of Thomas algorithm to solve a tri-diagonal matrix.

3. Using the definition of induced matrix norm, prove that

   (a) $\|A\|_1 = \sup_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$ (Maximum of column sum).

   (b) $\|A\|_\infty = \sup_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$ (Maximum of row sum).

4. (a) Find the relation between $\|A\|_2$ and $\|A\|_F$. (b) Prove that $K(Q) = 1$ with respect to l_2 norm, if Q is an orthogonal matrix.

5. Determine the condition number of (a) The Hilbert matrix, $H_3 = \dfrac{1}{i+j-1}, i,j = 1,2,3.$ (b) The Vandermonde matrix $V_3 = \begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix}$.

6. Let $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$ and A is non-singular. Consider the system of equations $Ax = b$. If there are perturbations in A and b. Then prove that

$$\frac{\|x-\hat{x}\|}{\|x\|} \le \left[\frac{K(A)}{1-\frac{K(A)\|\delta A\|}{\|A\|}}\right]\left(\frac{\|\delta A\|}{\|A\|}+\frac{\|\delta b\|}{\|b\|}\right), \quad \text{if} \quad \|\delta A\|\left\|A^{-1}\right\| < 1.$$

7. Find the effect of a disturbance $[\epsilon_1, \epsilon_2]^T$ on right hand side of the system of equations $Ax = b$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, $b = [5,\ 0]^T$, if $|\epsilon_1|, |\epsilon_2| \le 10^{-4}$.

8. Let $A(\alpha) = \begin{bmatrix} 0.1\alpha & 0.1\alpha \\ 1.0 & 1.5 \end{bmatrix}$. Using maximum norm determine $\alpha \in \mathbb{R}$ such that $K(A(\alpha))$ is minimized.

9. Develop a method to find an estimate of $K(A)$, without evaluating A^{-1}. Then find an approximate value of $K(A)$ when $A = \begin{bmatrix} 100 & -200 \\ -200 & 401 \end{bmatrix}$.

10. Prove that no eigenvalue of a matrix A exceeds the norm of a matrix, i.e., $||A|| \geq \rho(A)$. Then show that $K(A) \geq \dfrac{\lambda_{max.}}{\lambda_{min.}}$. Also find an lower bound of $K(A)$ when $A = \begin{bmatrix} 100 & -200 \\ -200 & 401 \end{bmatrix}$.

11. Let $A \in \mathbb{R}^{n\times n}$, then prove that $\lim\limits_{m\to\infty} A^m = \mathbf{0}$ (zero matrix), if $||A|| < 1$, or iff $\rho(A) < 1$.

12. If $A \in \mathbb{R}^{n\times n}$ is invertible such that $\|A\| < 1$, then $I - A$ is invertible, and the series $I + A + A^2 + \ldots$ converges to $(I - A)^{-1}$, if $\lim\limits_{m\to\infty} A^m = \mathbf{0}$.

13. If $A, B \in \mathbb{R}^{n\times n}$ are invertible matrices such that $\|I - AB\| < 1$, then prove that A and B are invertible. Further, $A^{-1} = B\sum\limits_{k=0}^{\infty}(I - AB)^k$ and $B^{-1} = \sum\limits_{k=0}^{\infty}(I - AB)^k A$.

14. Prove that the necessary and sufficient condition for the convergence of an iterative method of the form $\mathbf{x}^{(k+1)} = \mathbf{H}\mathbf{x}^{(k)} + \mathbf{c}, k = 0, 1, 2, 3, \ldots$ is that the eigenvalues of the iteration matrix satisfy $|\lambda_i(\mathbf{H})| < 1, i = 1, 2, 3, \ldots, n$.

15. Let $a \in \mathbb{R}$, consider $Ax = b$, where $A = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$, $b = [b_1, b_2]^T$. For which values of a, the Jacobi and Gauss-Seidel methods converge.

Problems for MATLAB

1. Solve the system of equations $3x_1 + 20x_2 - x_3 = -18, 2x_1 - 3x_2 + 20x_3 = 25, 20x_1 + x_2 - 2x_3 = 17$ using Gauss-Jacobi and Gauss-seidel methods.

2. Write a MATLAB code to find the inverse of A (Look at tutorial problem 13).