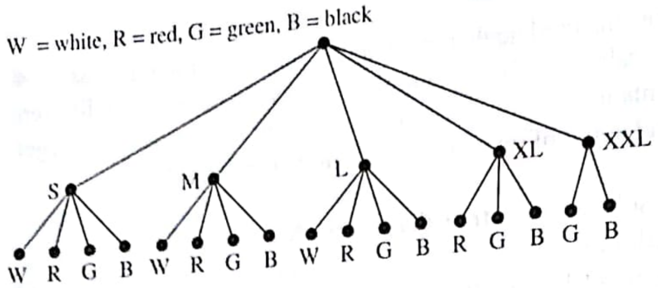

Figure 4 Counting Varieties of T-Shirts.

comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

Solution The tree diagram in Figure 4 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts. ◂

Exercises

1. There are 18 mathematics majors and 325 computer science majors at a college.
   a) How many ways are there to pick two representatives so that one is a mathematics major and the other is a computer science major?
   b) How many ways are there to pick one representative who is either a mathematics major or a computer science major?
2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?
3. A multiple-choice test contains 10 questions. There are four possible answers for each question.
   a) How many ways can a student answer the questions on the test if the student answers every question?
   b) How many ways can a student answer the questions on the test if the student can leave answers blank?
4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?
5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco? How many of these pairs involve more than one airline?
6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?
7. How many different three-letter initials can people have?
8. How many different three-letter initials with none of the letters repeated can people have?
9. How many different three-letter initials are there that begin with an *A*?
10. How many bit strings are there of length eight?
11. How many bit strings of length ten both begin and end with a 1?
12. How many bit strings are there of length six or less?
13. How many bit strings with length not exceeding n, where n is a positive integer, consist entirely of 1s?
14. How many bit strings of length n, where n is a positive integer, start and end with 1s?
15. How many strings are there of lowercase letters of length four or less?
16. How many strings are there of four lowercase letters that have the letter x in them?
17. How many strings of five ASCII characters contain the character @ ("at" sign) at least once? (*Note:* There are 128 different ASCII characters.)
18. How many positive integers between 5 and 31
    a) are divisible by 3? Which integers are these?
    b) are divisible by 4? Which integers are these?
    c) are divisible by 3 and by 4? Which integers are these?
19. How many positive integers between 50 and 100
    a) are divisible by 7? Which integers are these?
    b) are divisible by 11? Which integers are these?
    c) are divisible by both 7 and 11? Which integers are these?
20. How many positive integers less than 1000
    a) are divisible by 7?
    b) are divisible by 7 but not by 11?
    c) are divisible by both 7 and 11?
    d) are divisible by either 7 or 11?
    e) are divisible by exactly one of 7 and 11?
    f) are divisible by neither 7 nor 11?

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g) have distinct digits?
h) have distinct digits and are even?

21. How many positive integers between 100 and 999 inclusive
a) are divisible by 7?
b) are odd?
c) have the same three decimal digits?
d) are not divisible by 4?
e) are divisible by 3 or 4?
f) are not divisible by either 3 or 4?
g) are divisible by 3 but not by 4?
h) are divisible by 3 and 4?

22. How many positive integers between 1000 and 9999 inclusive
a) are divisible by 9?
b) are even?
c) have distinct digits?
d) are not divisible by 3?
e) are divisible by 5 or 7?
f) are not divisible by either 5 or 7?
g) are divisible by 5 but not by 7?
h) are divisible by 5 and 7?

23. How many strings of three decimal digits
a) do not contain the same digit three times?
b) begin with an odd digit?
c) have exactly two digits that are 4s?

24. How many strings of four decimal digits
a) do not contain the same digit twice?
b) end with an even digit?
c) have exactly three digits that are 9s?

25. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

26. How many license plates can be made using either three digits followed by three letters or three letters followed by three digits?

27. How many license plates can be made using either two letters followed by four digits or two digits followed by four letters?

28. How many license plates can be made using either three letters followed by three digits or four letters followed by two digits?

29. How many license plates can be made using either two or three letters followed by either two or three digits?

30. How many strings of eight English letters are there
a) if letters can be repeated?
b) if no letter can be repeated?
c) that start with X, if letters can be repeated?
d) that start with X, if no letter can be repeated?
e) that start and end with X, if letters can be repeated?
f) that start with the letters BO (in that order), if letters can be repeated?
g) that start and end with the letters BO (in that order), if letters can be repeated?
h) that start or end with the letters BO (in that order), if letters can be repeated?

31. How many strings of eight English letters are there
a) that contain no vowels, if letters can be repeated?
b) that contain no vowels, if letters cannot be repeated?
c) that start with a vowel, if letters can be repeated?
d) that start with a vowel, if letters cannot be repeated?
e) that contain at least one vowel, if letters can be repeated?
f) that contain exactly one vowel, if letters can be repeated?
g) that start with X and contain at least one vowel, if letters can be repeated?
h) that start and end with X and contain at least one vowel, if letters can be repeated?

32. How many different functions are there from a set with 10 elements to sets with the following numbers of elements?
a) 2 b) 3 c) 4 d) 5

33. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?
a) 4 b) 5 c) 6 d) 7

34. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$?

35. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$
a) that are one-to-one?
b) that assign 0 to both 1 and n?
c) that assign 1 to exactly one of the positive integers less than n?

36. How many partial functions (see the preamble to Exercise 73 in Section 2.3) are there from a set with five elements to sets with each of these number of elements?
a) 1 b) 2 c) 5 d) 9

37. How many partial functions (see the preamble to Exercise 73 in Section 2.3) are there from a set with m elements to a set with n elements, where m and n are positive integers?

38. How many subsets of a set with 100 elements have more than one element?

39. A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

40. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if
a) the bride must be in the picture?
b) both the bride and groom must be in the picture?
c) exactly one of the bride and the groom is in the picture?

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41. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if
    a) the bride must be next to the groom?
    b) the bride is not next to the groom?
    c) the bride is positioned somewhere to the left of the groom?
42. How many bit strings of length seven either begin with two 0s or end with three 1s?
43. How many bit strings of length 10 either begin with three 0s or end with two 0s?
*44. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?
**45. How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?
46. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?
47. How many positive integers not exceeding 100 are divisible either by 4 or by 6?
48. How many different initials can someone have if a person has at least two, but no more than five, different initials? Assume that each initial is one of the 26 letters of the English language.
49. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters $*$, $>$, $<$, !, $+$, and $=$.
    a) How many different passwords are available for this computer system?
    b) How many of these passwords contain at least one occurrence of at least one of the six special characters?
    c) If it takes one nanosecond for a hacker to check whether each possible password is your password, how long would it take this hacker to try every possible password?
50. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)
51. Suppose that at some future time every telephone in the world is assigned a number that contains a country code 1 to 3 digits long, that is, of the form X, XX, or XXX, followed by a 10-digit telephone number of the form NXX-NXX-$XXXX$ (as described in Example 8). How many different telephone numbers would be available worldwide under this numbering plan?
52. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.
53. How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?
54. Use a tree diagram to find the number of ways that the World Series can occur, where the first team that wins four games out of seven wins the series.
55. Use a tree diagram to determine the number of subsets of $\{3, 7, 9, 11, 24\}$ with the property that the sum of the elements in the subset is less than 28.
56. a) Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64-ounce bottles?
    b) Answer the question in part (a) using counting rules.
57. a) Suppose that a popular style of running shoe is available for both men and women. The woman's shoe comes in sizes 6, 7, 8, and 9, and the man's shoe comes in sizes 8, 9, 10, 11, and 12. The man's shoe comes in white and black, while the woman's shoe comes in white, red, and black. Use a tree diagram to determine the number of different shoes that a store has to stock to have at least one pair of this type of running shoe for all available sizes and colors for both men and women.
    b) Answer the question in part (a) using counting rules.
*58. Use the product rule to show that there are 2^{2^n} different truth tables for propositions in n variables.
59. Use mathematical induction to prove the sum rule for m tasks from the sum rule for two tasks.
60. Use mathematical induction to prove the product rule for m tasks from the product rule for two tasks.
61. How many diagonals does a convex polygon with n sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)
62. Data are transmitted over the Internet in **datagrams**, which are structured blocks of bits. Each datagram contains header information organized into a maximum of 14 different fields (specifying many things, including the source and destination addresses) and a data area that contains the actual data that are transmitted. One of the 14 header fields is the **header length field** (denoted by

Example 13 ... *two enemies. Show that there are either three mutual* ... *group.*

Solution Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from the generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements. In the former case, suppose that B, C, and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise, B, C, and D form a set of three mutual enemies. The proof in the latter case, when there are three or more enemies of A, proceeds in a similar manner. ◂

The **Ramsey number** $R(m, n)$, where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies. Example 13 shows that $R(3, 3) \le 6$. We conclude that $R(3, 3) = 6$ because in a group of five people where every two people are friends or enemies, there may not be three mutual friends or three mutual enemies (see Exercise 24).

It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Note that by symmetry it can be shown that $R(m, n) = R(n, m)$ (see Exercise 28). We also have $R(2, n) = n$ for every positive integer $n \ge 2$ (see Exercise 27). The exact values of only nine Ramsey numbers $R(m, n)$ with $3 \le m \le n$ are known, including $R(4, 4) = 18$. Only bounds are known for many other Ramsey numbers, including $R(5, 5)$, which is known to satisfy $43 \le R(5, 5) \le 49$. The reader interested in learning more about Ramsey numbers should consult [MiRo91] or [GrRoSp90].

Links

FRANK PLUMPTON RAMSEY (1903–1930) Frank Plumpton Ramsey, son of the president of Magdalene College, Cambridge, was educated at Winchester and Trinity Colleges. After graduating in 1923, he was elected a fellow of King's College, Cambridge, where he spent the remainder of his life. Ramsey made important contributions to mathematical logic. What we now call Ramsey theory began with his clever combinatorial arguments, published in the paper "On a Problem of Formal Logic." Ramsey also made contributions to the mathematical theory of economics. He was noted as an excellent lecturer on the foundations of mathematics. His death at the age of 26 deprived the mathematical community and Cambridge University of a brilliant young scholar.

Exercises

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.
2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.
3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
   a) How many socks must he take out to be sure that he has at least two socks of the same color?
   b) How many socks must he take out to be sure that he has at least two black socks?
4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
   a) How many balls must she select to be sure of having at least three balls of the same color?
   b) How many balls must she select to be sure of having at least three blue balls?
5. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

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6. Let d be a positive integer. Show that among any group of $d + 1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.
7. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.
8. Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.
9. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?
*10. Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.
*11. Let (x_i, y_i, z_i), $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, be a set of nine distinct points with integer coordinates in xyz space. Show that the midpoint of at least one pair of these points has integer coordinates.
12. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that a_1 **mod** $5 = a_2$ **mod** 5 and b_1 **mod** $5 = b_2$ **mod** 5?
13. a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.
    b) Is the conclusion in part (a) true if four integers are selected rather than five?
14. a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.
    b) Is the conclusion in part (a) true if six integers are selected rather than seven?
15. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?
16. How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?
17. A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle, location in the aisle, and shelf. There are 50 aisles, 85 horizontal locations in each aisle, and 5 shelves throughout the warehouse. What is the least number of products the company can have so that at least two products must be stored in the same bin?
18. Suppose that there are nine students in a discrete mathematics class at a small college.
    a) Show that the class must have at least five male students or at least five female students.
    b) Show that the class must have at least three male students or at least seven female students.
19. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.
    a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.
    b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.
20. Find an increasing subsequence of maximal length and a decreasing subsequence of maximal length in the sequence 22, 5, 7, 2, 23, 10, 15, 21, 3, 17.
21. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.
22. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.
*23. Describe an algorithm in pseudocode for producing the largest increasing or decreasing subsequence of a sequence of distinct integers.
24. Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.
25. Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.
26. Use Exercise 25 to show that among any group of 20 people (where any two people are either friends or enemies), there are either four mutual friends or four mutual enemies.
27. Show that if n is a positive integer with $n \geq 2$, then the Ramsey number $R(2, n)$ equals n.
28. Show that if m and n are positive integers with $m \geq 2$ and $n \geq 2$, then the Ramsey numbers $R(m, n)$ and $R(n, m)$ are equal.
29. Show that there are at least six people in California (population: 36 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials.
30. Show that if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars, then there are two who earned exactly the same amount of money, to the penny, last year.
31. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?
32. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

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33. A computer network consists of six computers. Each computer is directly connected to zero or more of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers. [*Hint*: It is impossible to have a computer linked to none of the others and a computer linked to all the others.]

34. Find the least number of cables required to connect eight computers to four printers to guarantee that four computers can directly access four different printers. Justify your answer.

35. Find the least number of cables required to connect 100 computers to 20 printers to guarantee that 20 computers can directly access 20 different printers. (Here, the assumptions about cables and computers are the same as in Example 9.) Justify your answer.

*36. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

37. An arm wrestler is the champion for a period of 75 hours.(Here, by an hour, we mean a period starting from an exact hour, such as 1, until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

*38. Is the statement in Exercise 37 true if 24 is replaced by
a) 2? b) 23? c) 25? d) 30?

39. Show that if f is a function from S to T, where S and T are finite sets and $m = \lceil |S| / |T| \rceil$, then there are at least m elements of S mapped to the same value of T. That is, show that there are distinct elements $s_1, s_2, \ldots, s_m$ of S such that $f(s_1) = f(s_2) = \cdots = f(s_m)$.

40. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

*41. Let x be an irrational number. Show that for some positive integer j not exceeding n, the absolute value of the difference between jx and the nearest integer to jx is less than $1/n$.

42. Let $n_1, n_2, \ldots, n_t$ be positive integers. Show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some i, $i = 1, 2, \ldots, t$, the ith box contains at least n_i objects.

*43. A proof of Theorem 3 based on the generalized pigeonhole principle is outlined in this exercise. The notation used is the same as that used in the proof in the text.

a) Assume that $i_k \leq n$ for $k = 1, 2, \ldots, n^2 + 1$. Use the generalized pigeonhole principle to show that there are $n + 1$ terms $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ with $i_{k_1} = i_{k_2} = \cdots = i_{k_{n+1}}$, where $1 \leq k_1 < k_2 < \cdots < k_{n+1}$.

b) Show that $a_{k_j} > a_{k_{j+1}}$ for $j = 1, 2, \ldots, n$. [*Hint*: Assume that $a_{k_j} < a_{k_{j+1}}$, and show that this implies that $i_{k_j} > i_{k_{j+1}}$, which is a contradiction.]

c) Use parts (a) and (b) to show that if there is no increasing subsequence of length $n + 1$, then there must be a decreasing subsequence of this length.

5.3 PERMUTATIONS AND COMBINATIONS

Introduction Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter. For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students? In this section we will develop methods to answers questions such as these.

Permutations We begin by solving the first question posed in the introduction to this section, as well as related questions.

Example 1 *In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?*

Extra Examples

Solution First, note that the order in ... There are five ways to sel...

1. List all the permutations of $\{a, b, c\}$.
2. How many different permutations are there of the set $\{a, b, c, d, e, f, g\}$?
3. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?
4. Let $S = \{1, 2, 3, 4, 5\}$.
   a) List all the 3-permutations of S.
   b) List all the 3-combinations of S.
5. Find the value of each of these quantities.
   a) $P(6, 3)$ b) $P(6, 5)$ c) $P(8, 1)$
   d) $P(8, 5)$ e) $P(8, 8)$ f) $P(10, 9)$
6. Find the value of each of these quantities.
   a) $C(5, 1)$ b) $C(5, 3)$ c) $C(8, 4)$
   d) $C(8, 8)$ e) $C(8, 0)$ f) $C(12, 6)$
7. Find the number of 5-permutations of a set with nine elements.
8. In how many different orders can five runners finish a race if no ties are allowed?
9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?
10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?
11. How many bit strings of length 10 contain
    a) exactly four 1s? b) at most four 1s?
    c) at least four 1s?
    d) an equal number of 0s and 1s?
12. How many bit strings of length 12 contain
    a) exactly three 1s? b) at most three 1s?
    c) at least three 1s?
    d) an equal number of 0s and 1s?
13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?
14. In how many ways can a set of two positive integers less than 100 be chosen?
15. In how many ways can a set of five letters be selected from the English alphabet?
16. How many subsets with an odd number of elements does a set with 10 elements have?
17. How many subsets with more than two elements does a set with 100 elements have?
18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
    a) are there in total?
    b) contain exactly three heads?
    c) contain at least three heads?
    d) contain the same number of heads and tails?
19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes
    a) are there in total?
    b) contain exactly two heads?
    c) contain at most three tails?
    d) contain the same number of heads and tails?
20. How many bit strings of length 10 have
    a) exactly three 0s? b) more 0s than 1s?
    c) at least seven 1s? d) at least three 1s?
21. How many permutations of the letters *ABCDEFG* contain
    a) the string *BCD*? b) the string *CFGA*?
    c) the strings *BA* and *GF*?
    d) the strings *ABC* and *DE*?
    e) the strings *ABC* and *CDE*?
    f) the strings *CBA* and *BED*?
22. How many permutations of the letters *ABCDEFGH* contain
    a) the string *ED*? b) the string *CDE*?
    c) the strings *BA* and *FGH*?
    d) the strings *AB*, *DE*, and *GH*?
    e) the strings *CAB* and *BED*?
    f) the strings *BCA* and *ABF*?
23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]
24. How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [*Hint:* First position the women and then consider possible positions for the men.]
25. One hundred tickets, numbered 1, 2, 3, ..., 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if
    a) there are no restrictions?
    b) the person holding ticket 47 wins the grand prize?

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c) the person holding ticket 47 wins one of the prizes?
d) the person holding ticket 47 does not win a prize?
e) the people holding tickets 19 and 47 both win prizes?
f) the people holding tickets 19, 47, and 73 all win prizes?
g) the people holding tickets 19, 47, 73, and 97 all win prizes?
h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?
i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?
j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

26. Thirteen people on a softball team show up for a game.
a) How many ways are there to choose 10 players to take the field?
b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

27. A club has 25 members.
a) How many ways are there to choose four members of the club to serve on an executive committee?
b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

28. A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

*29. How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers k, $k+1$, $k+2$, in the correct order
a) where these consecutive integers can perhaps be separated by other integers in the permutation?
b) where they are in consecutive positions in the permutation?

30. Seven women and nine men are on the faculty in the mathematics department at a school.
a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?
b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

31. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain
a) exactly one vowel?
a) exactly two vowels?
b) at least one vowel?
b) at least two vowels?

32. How many strings of six lowercase letters from the English alphabet contain
a) the letter a?
c) the letters a and b?
b) the letters a and b in consecutive positions with a preceding b, with all the letters distinct?
c) the letters a and b, where a is somewhere to the left of b in the string, with all the letters distinct?

33. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

34. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

35. How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

36. How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

37. How many bit strings of length 10 contain at least three 1s and at least three 0s?

38. How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

39. How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

40. How many ways are there to seat six people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

41. How many ways are there for a horse race with three horses to finish if ties are possible? (*Note:* Two or three horses may tie.)

*42. How many ways are there for a horse race with four horses to finish if ties are possible? (*Note:* Any number of the four horses may tie.)

*43. There are six runners in the 100-yard dash. How many ways are there for three medals to be awarded if ties are possible? (The runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals.)

*44. This procedure is used to break ties in games in the championship round of the World Cup soccer tournament. Each team selects five players in a prescribed order. Each of these players takes a penalty kick, with a player from the first team followed by a player from the second team and so on, following the order of players specified. If the score is still tied at the end of the 10 penalty kicks, this procedure is repeated. If the score is still tied after 20

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Exercises

1. Find the expansion of $(x+y)^4$
   a) using combinatorial reasoning, as in Example 1.
   b) using the Binomial Theorem.
2. Find the expansion of $(x+y)^5$
   a) using combinatorial reasoning, as in Example 1.
   b) using the Binomial Theorem.
3. Find the expansion of $(x+y)^6$.
4. Find the coefficient of x^5y^8 in $(x+y)^{13}$.
5. How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?
6. What is the coefficient of x^7 in $(1+x)^{11}$?
7. What is the coefficient of x^9 in $(2-x)^{19}$?
8. What is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$?
9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

*10. Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k is an integer.

*11. Give a formula for the coefficient of x^k in the expansion of $(x^2-1/x)^{100}$, where k is an integer.

12. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0\le k\le 10$, is:

    1 10 45 120 210 252 210 120 45 10 1

    Use Pascal's Identity to produce the row immediately following this row in Pascal's triangle.
13. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0\le k\le 9$?
14. Show that if n is a positive integer, then
    $1=\binom{n}{0}<\binom{n}{1}<\cdots<\binom{n}{\lfloor n/2\rfloor}=\binom{n}{\lceil n/2\rceil}>\cdots>\binom{n}{n-1}>\binom{n}{n}=1$
15. Show that $\binom{n}{k}\le 2^n$ for all positive n and all integers k with $0\le k\le n$.
16. a) Use Exercise 14 and Corollary 1 to show that if n is an integer greater than 1, then $\binom{n}{\lfloor n/2\rfloor}\ge 2^n/n$.
    b) Conclude from part (a) that if n is a positive integer, then $\binom{2n}{n}\ge 4^n/2n$.

☞17. Show that if n and k are integers with $1\le k\le n$, then $\binom{n}{k}\le n^k/2^{k-1}$.

18. Suppose that b is an integer with $b\ge 7$. Use the Binomial Theorem and the appropriate row of Pascal's triangle to find the base-b expansion of $(11)_b^4$ [that is, the fourth power of the number $(11)_b$ in base-b notation].
19. Prove Pascal's Identity, using the formula for $\binom{n}{r}$.
20. Suppose that k and n are integers with $1\le k\le n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k}=\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1},$$

which relates terms in Pascal's triangle that form a hexagon.

☞21. Prove that if n and k are integers with $1\le k\le n$, then

$$k\binom{n}{k}=n\binom{n-1}{k-1}$$

   a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]
   b) using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 5.3.

22. Prove the identity $\binom{n}{r}\binom{r}{k}=\binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r\le n$ and $k\le r$,
    a) using a combinatorial argument.
    b) using an argument based on the formula for the number of r-combinations of a set with n elements.
23. Show that if n and k are positive integers, then

$$\binom{n+1}{k}=(n+1)\binom{n}{k-1}\Big/k.$$

    Use this identity to construct an inductive definition of the binomial coefficients.
24. Show that if p is a prime and k is an integer such that $1\le k\le p-1$, then p divides $\binom{p}{k}$.
25. Let n be a positive integer. Show that

$$\binom{2n}{n+1}+\binom{2n}{n}=\binom{2n+2}{n+2}\Big/2.$$

*26. Let n and k be integers with $1\le k\le n$. Show that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}=\binom{2n+2}{n+1}\Big/2-\binom{2n}{n}.$$

*27. Prove that

$$\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+r+1}{r}$$

whenever n and r are positive integers,
   a) using a combinatorial argument.
   b) using Pascal's identity.

28. Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$

a) using a combinatorial argument.

b) by algebraic manipulation.

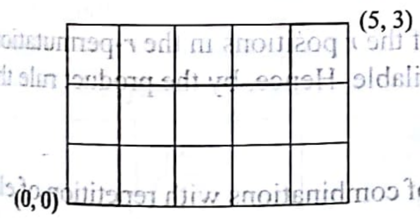
*29. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee and to then select a leader of the committee.]

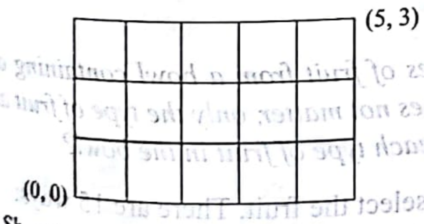
*30. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

31. Show that a nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

*32. Prove the Binomial Theorem using mathematical induction.

33. In this exercise we will count the number of paths in the xy plane between the origin $(0, 0)$ and point (m, n) such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.) Two such paths from $(0, 0)$ to $(5, 3)$ are illustrated here.





a) Show that each path of the type described can be represented by a bit string consisting of m 0s and n 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.

b) Conclude from part (a) that there are $\binom{m+n}{n}$ paths of the desired type.

34. Use Exercise 33 to prove that $\binom{n}{k} = \binom{n}{n-k}$ whenever k is an integer with $0 \le k \le n$. [*Hint:* Consider the number of paths of the type described in Exercise 33 from $(0, 0)$ to $(n - k, k)$ and from $(0, 0)$ to $(k, n - k)$.]

35. Use Exercise 33 to prove Theorem 4. [*Hint:* Count the number of paths with n steps of the type described in Exercise 33. Every such path must end at one of the points $(n - k, k)$ for $k = 0, 1, 2, \ldots, n$.]

36. Use Exercise 33 to prove Pascal's Identity. [*Hint:* Show that a path of the type described in Exercise 33 from $(0, 0)$ to $(n + 1 - k, k)$ passes through either $(n + 1 - k, k - 1)$ or $(n - k, k)$, but not through both.]

37. Prove the identity in Exercise 27 using Exercise 33. [*Hint:* First, note that the number of paths from $(0, 0)$ to $(n + 1, r)$ equals $\binom{n+1+r}{r}$. Second, count the number of paths by summing the number of these paths that start by going k units upward for $k = 0, 1, 2, \ldots, r$.]

38. Give a combinatorial proof that if n is a positive integer then $\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$. [*Hint:* Show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from this subset. Furthermore, express the right-hand side as $n(n-1)2^{n-2} + n2^{n-1}$.]

*39. Determine a formula involving binomial coefficients for the nth term of a sequence if its initial terms are those listed. [*Hint:* Looking at Pascal's triangle will be helpful. Although infinitely many sequences start with a specified set of terms, each of the following lists is the start of a sequence of the type desired.]

a) 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, ...

b) 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, ...

c) 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, ...

d) 1, 1, 2, 3, 6, 10, 20, 35, 70, 126, ...

e) 1, 1, 1, 3, 1, 5, 15, 35, 1, 9, ...

f) 1, 3, 15, 84, 495, 3003, 18564, 116280, 735471, 4686825, ...

5.5 GENERALIZED PERMUTATIONS AND COMBINATIONS

Introduction In many counting problems, elements may be used repeatedly. For instance, a letter or digit may be used more than once on a license plate. When a dozen donuts are selected, each variety can be chosen repeatedly. This contrasts with the counting problems discussed earlier in the

Links

Observe that distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in non increasing order. If $a_1 + a_2 + \cdots + a_j = n$, where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \geq \cdots \geq a_j$, we say that $a_1, a_2, \ldots, a_j$ is a **partition** of the positive integer n into j positive integers. We see that if $p_k(n)$ is the number of partitions of n into at most k positive integers, then there are $p_k(n)$ ways to distribute n indistinguishable objects into k indistinguishable boxes. No simple closed formula exists for this number.

Exercises

1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?
2. In how many different ways can five elements be selected in order from a set with five elements when repetition is allowed?
3. How many strings of six letters are there?
4. Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are six kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the seven days of a week if the order in which the sandwiches are chosen matters?
5. How many ways are there to assign three jobs to five employees if each employee can be given more than one job?
6. How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?
7. How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?
8. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?
9. A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose
   a) six bagels?
   b) a dozen bagels?
   c) two dozen bagels?
   d) a dozen bagels with at least one of each kind?
   e) a dozen bagels with at least three egg bagels and no more than two salty bagels?
10. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose
    a) a dozen croissants?
    b) three dozen croissants?
    c) two dozen croissants with at least two of each kind?
    d) two dozen croissants with no more than two broccoli croissants?
    e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?
    f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli croissants?
11. How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?
12. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?
13. A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?
14. How many solutions are there to the equation
    $$x_1 + x_2 + x_3 + x_4 = 17,$$
    where x_1, x_2, x_3, and x_4 are nonnegative integers?
15. How many solutions are there to the equation
    $$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$
    where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that
    a) $x_1 \geq 1$?
    b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
    c) $0 \leq x_1 \leq 10$?
    d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?
16. How many solutions are there to the equation
    $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$
    where x_i, $i = 1, 2, 3, 4, 5, 6$, is a nonnegative integer such that
    a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?
    b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$, and $x_6 \geq 6$?
    c) $x_1 \leq 5$?
    d) $x_1 < 8$ and $x_2 > 8$?
17. How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0s, three 1s, and five 2s?
18. How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?

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19. Suppose that a large family has 14 children, including two sets of identical triplets, three sets of identical twins, and two individual children. How many ways are there to seat these children in a row of chairs if the identical triplets or twins cannot be distinguished from one another?

20. How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

21. How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

22. How many ways are there to distribute 12 indistinguishable balls into six distinguishable bins?

23. How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

24. How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

25. How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

26. How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

27. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

28. Show that there are $C(n + r - q_1 - q_2 - \cdots - q_r - 1, n - q_1 - q_2 - \cdots - q_r)$ different unordered selections of n objects of r different types that include at least q_1 objects of type one, q_2 objects of type two, . . ., and q_r objects of type r.

29. How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of 12 0 bits, and must have at least two 0 bits following each 1 bit?

30. How many different strings can be made from the letters in *MISSISSIPPI*, using all the letters?

31. How many different strings can be made from the letters in *ABRACADABRA*, using all the letters?

32. How many different strings can be made from the letters in *AARDVARK*, using all the letters, if all three As must be consecutive?

33. How many different strings can be made from the letters in *ORONO*, using some or all of the letters?

34. How many strings with five or more characters can be formed from the letters in *SEERESS*?

35. How many strings with seven or more characters can be formed from the letters in *EVERGREEN*?

36. How many different bit strings can be formed using six 1s and eight 0s?

37. A student has three mangos, two papayas, and two kiwi fruits. If the student eats one piece of fruit each day, and only the type of fruit matters, in how many different ways can these fruits be consumed?

38. A professor packs her collection of 40 issues of a mathematics journal in four boxes with 10 issues per box. How many ways can she distribute the journals if
   a) each box is numbered, so that they are distinguishable?
   b) the boxes are identical, so that they cannot be distinguished?

39. How many ways are there to travel in xyz space from the origin (0, 0, 0) to the point (4, 3, 5) by taking steps one unit in the positive x direction, one unit in the positive y direction, or one unit in the positive z direction? (Moving in the negative x, y, or z direction is prohibited, so that no backtracking is allowed.)

40. How many ways are there to travel in $xyzw$ space from the origin (0, 0, 0, 0) to the point (4, 3, 5, 4) by taking steps one unit in the positive x, positive y, positive z, or positive w direction?

41. How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards?

42. In bridge, the 52 cards of a standard deck are dealt to four players. How many different ways are there to deal bridge hands to four players?

43. How many ways are there to deal hands of five cards to each of six players from a deck containing 48 different cards?

44. In how many ways can a dozen books be placed on four distinguishable shelves
   a) if the books are indistinguishable copies of the same title?
   b) if no two books are the same, and the positions of the books on the shelves matter? [*Hint:* Break this into 12 tasks, placing each book separately. Start with the sequence 1, 2, 3, 4 to represent the shelves. Represent the books by b_i, $i = 1, 2, \ldots, 12$. Place b_1 to the right of one of the terms in 1, 2, 3, 4. Then successively place $b_2, b_3, \ldots,$ and b_{12}.]

45. How many ways can n books be placed on k distinguishable shelves
   a) if the books are indistinguishable copies of the same title?
   b) if no two books are the same, and the positions of the books on the shelves matter?

46. A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen? [*Hint:* Represent the books that are chosen by bars and the books not chosen by stars. Count the

number of sequences of five bars and seven stars so that no two bars are adjacent.]

*47. Use the product rule to prove Theorem 4, by first placing objects in the first box, then placing objects in the second box, and so on.

*48. Prove Theorem 4 by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type i, $i = 1, 2, 3, \ldots, k$, and the distributions of n objects in k boxes such that n_i objects are placed in box i, $i = 1, 2, 3, \ldots, k$ and then applying Theorem 3.

*49. In this exercise we will prove Theorem 2 by setting up a one-to-one correspondence between the set of r-combinations with repetition allowed of $S = \{1, 2, 3, \ldots, n\}$ and the set of r-combinations of the set $T = \{1, 2, 3, \ldots, n + r - 1\}$.

a) Arrange the elements in an r-combination, with repetition allowed, of S into an increasing sequence $x_1 \le x_2 \le \ldots \le x_r$. Show that the sequence formed by adding $k - 1$ to the kth term is strictly increasing. Conclude that this sequence is made up of r distinct elements from T.

b) Show that the procedure described in (a) defines a one-to-one correspondence between the set of r-combinations, with repetition allowed, of S and the r-combinations of T. [*Hint:* Show the correspondence can be reversed by associating to the r-combination $\{x_1, x_2, \ldots, x_r\}$ of T, with $1 \le x_1 < x_2 < \cdots < x_r \le n + r - 1$, the r-combination with repetition allowed from S, formed by subtracting $k - 1$ from the kth element.]

c) Conclude that there are $C(n + r - 1, r)$ r-combinations with repetition allowed from a set with n elements.

50. How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

51. How many ways are there to distribute six distinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

52. How many ways are there to put five temporary employees into four identical offices?

53. How many ways are there to put six temporary employees into four identical offices so that there is at least one temporary employee in each of these four offices?

54. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

55. How many ways are there to distribute six indistinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

56. How many ways are there to pack eight identical DVDs into five indistinguishable boxes so that each box contains at least one DVD?

57. How many ways are there to pack nine identical DVDs into three indistinguishable boxes so that each box contains at least two DVDs?

58. How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

a) both the balls and boxes are labeled?

b) the balls are labeled, but the boxes are unlabeled?

c) the balls are unlabeled, but the boxes are labeled?

d) both the balls and boxes are unlabeled?

59. How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if

a) both the balls and boxes are labeled?

b) the balls are labeled, but the boxes are unlabeled?

c) the balls are unlabeled, but the boxes are labeled?

d) both the balls and boxes are unlabeled?

60. Suppose that a basketball league has 32 teams, split into two conferences of 16 teams each. Each conference is split into three divisions. Suppose that the North Central Division has five teams. Each of the teams in the North Central Divsion plays four games against each of the other teams in this division, three games against each of the 11 remaining teams in the conference, and two games against each of the 16 teams in the other conference. In how many different orders can the games of one of the teams in the North Central Division be scheduled?

*61. Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

62. How many different terms are there in the expansion of $(x_1 + x_2 + \ldots + x_m)^n$ after all terms with identical sets of exponents are added?

*63. Prove the **Multinomial Theorem:** If n is a positive integer, then

$$(x_1 + x_2 + \ldots + x_m)^n$$

$$= \sum_{n_1+n_2+\cdots+n_m=n} C(n; n_1, \cdots, n_m) x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m},$$

where

$$C(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1! n_2! \cdots n_m!}$$

is a **multinomial coefficient.**

64. Find the expansion of $(x + y + z)^4$.

65. Find the coefficient of $x^3 y^2 z^5$ in $(x + y + z)^{10}$.

66. How many terms are there in the expansion of $(x + y + z)^{100}$?

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