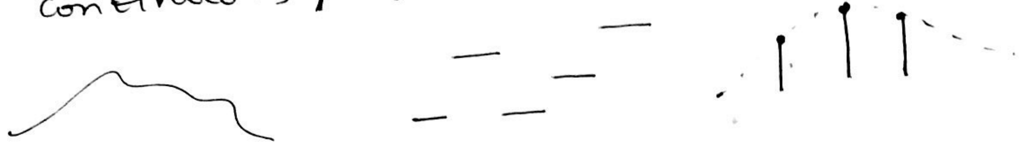
3 Aug.

Signals & Systems

Signal : mathematical function $f(\cdot)$ may be vector/scalar, multidimensional

independent variable(s): time, space, ... variable(s)

continuous / discontinuous / discrete / digital



even/odd : $f(x) = f(-x)$ / $f(x) = -f(-x)$

complex : $f = f_{real} + j f_{imag}$

for analytic signals : f_{real} & f_{imag} are related

periodic/aperiodic

causal / anticausal : $f(t) = 0 \quad (t<0)$

Gross characterizations : peak value / $\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, dt$ (mean)

$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\, dt$ (power) / $\sqrt{\text{power}}$: rms

$\frac{df}{dt} \uparrow\uparrow$ fast varying signal : large B.W.

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- single valued functions / signals

- $f(t) = f_{even}(t) + f_{odd}(t)$

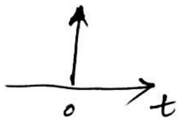
$f(-t) = f_{even}(t) - f_{odd}(t)$

$$\therefore \; f_{even}(t) = \frac{f(t) + f(-t)}{2}$$

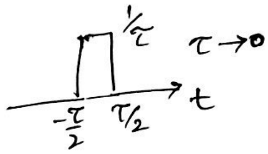
$$f_{odd}(t) = \frac{f(t) - f(-t)}{2}$$

Singularity functions (O. Heaviside ; L. Swartz)

Impulse function



$u_0(t)$

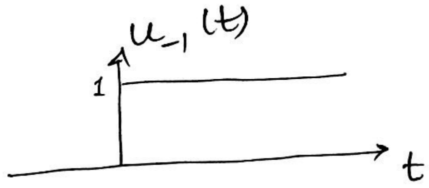


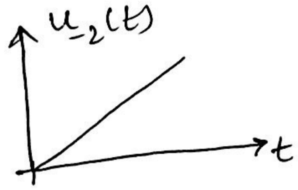
$$\int_{-\infty}^{\infty} u_0(t)\,dt = 1 \quad ; \quad \int_{-\infty}^{\infty} f(t)\,u_0(t)\,dt = f(0)$$

Sifting property: $\int_{-\infty}^{\infty} f(t)\,u_0(t-\tau)\,dt = f(\tau)$

unit step function: $\int_{-\infty}^{t} u_0(t)\,dt$

$u_{-1}(t)$





Ramp $u_{-2}(t)$

parabola $u_{-3}(t)$

⋮

* Signal representation using singularity functions
→ ∵ for linear systems superposition applies
→ Delayed signals' responses to systems are not a problem for Time Invariant systems.

e^{st} class of signals

stability : for $t \to \infty$ signal $\to 0$

s : real $\Rightarrow$ exponential

s : complex $\Rightarrow$ $e^{\sigma t} \; e^{j\omega t}$

[e^{st} is an eigen function of LTI systems]

Orthogonal Representation

$$f(t) = \sum_{n=1}^{\infty} f_n \phi_n(t)$$

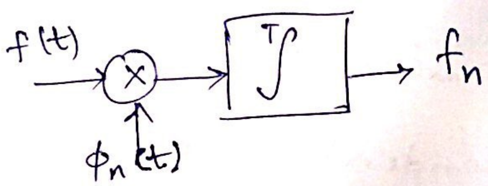
$0 < t < T$

$$\int_0^T \phi_j(t)\, \phi_k(t)\, dt = 0 \quad (\text{for } j \neq k)$$
$$= p \quad (\text{for } j = k)$$

(for $p = 1$: orthonormal)

$$f_n = \int_0^T f(t)\, \phi_n(t)\, dt$$

$$\because \int_0^T \sum_{k=1}^{\infty} f_k \phi_k(t)\, \phi_n(t)\, dt = f_n$$



f(t) periodic waveform $\longrightarrow$ set of numbers (f_n)

Time varying (complicated) signal $\to$ w/ some error : finite # of f_n s error (mean squared) : $\displaystyle\int_0^T \left(f(t) - \sum_{n=1}^{N} f_n \phi_n(t)\right)^2 dt$

uniformly, monotonically converging to zero

f_n is more accurate for m.s. error ↓↓

Gram-Schmitt procedure for obtaining orthonormal functions

$$f(t) \longrightarrow \boxed{} \longrightarrow g(t)$$

$$\phi_n(t) \longrightarrow \boxed{} \longrightarrow g_n(t)$$

when $\phi_n(t)$ is an eigen function of system : $g_n(t) = a_n \phi_n(t)$

for many (LTI) systems : sinusoids are eigen functions.

∴ choice of $\phi_n(t)$ depends on system

Digital signal : $\phi_n(t)$s ?

Random signal : $\phi_n(t)$ deterministic

f_ns : uncorrelated, independent $\phi_n(t)$s ?

K-L expansion

t

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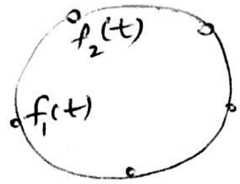
$f(t) \qquad 0 < t < T$ (function)

$\{f_n\} \qquad n : -\infty, \infty$ (vector)

$$E = \int_0^T f^2(t)\,dt = \sum_{n=-\infty}^{\infty} f_n^2$$



Waveforms of same energy

points on an n-dimensional sphere

distance between waveforms > rms value/energy of disturbance/noise

(shannon)

Channel capacity : sphere packing problem

$f_1(t)$ of noise energy around is a sphere

same energy fs : f_1, f_2 ... lie on surface of a sphere

Periodic signals: $f(t) = f(t + nT)$

orthonormal expansion of $f(t)$ using periodic functions $\phi_n(t)$

$-T/2 < t < T/2$; $\omega_0 = \frac{2\pi}{T}$

$$\phi_n(t) = e^{j\frac{2\pi}{T}nt} = e^{j\omega_0 nt}$$

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \phi_n(t)$$

for
Orthonormal functions:

$$\frac{1}{T}\int_{-T/2}^{T/2} \phi_n(t)\, \phi_m^*(t) = \frac{\sin (n-m)\,\omega_0 T/2}{(n-m)\,\omega_0 T/2}$$

$$= \frac{\sin (n-m)\pi}{(n-m)\pi} = \begin{cases} 1 & n=m \\ 0 & n\neq m \end{cases}$$

$\therefore\ \phi_n(t) = e^{j\frac{2\pi}{T}nt}$ are orthonormal functions

$\therefore\ f(t) = \sum f_n\, e^{jn\omega_0 t}$; $f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt$

(Fourier)

$$f_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{jn\omega_0 t}\, dt$$

$$f_n^* = \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\, e^{jn\omega_0 t}\, dt$$

$\therefore$ $\boxed{\text{when } f(t) \text{ is real : } f_{-n} = f_n^*}$

$$f(t) = f_0 + \sum_{n=1}^{\infty} f_{-n}\, e^{-jn\omega_0 t} + f_n\, e^{+jn\omega_0 t}$$

for $f_{-n} = f_n^*$:

$$f(t) = f_0 + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\{f_n\, e^{+jn\omega_0 t}\}$$

$$= f_0 + 2\sum_{n=1}^{\infty} \mathrm{Re}\{f_n\}\cos(n\omega_0 t) - \mathrm{Im}\{f_n\}\sin(n\omega_0 t)$$

t

for orthonormal expansion

rectangular
Cartesian symmetry : sines / cosines

circular " : Bessel functions

spherical " : Henkel "

(sinc diffraction fn)

— Linearity property

(1) $f_1(t)$, $f_2(t)$ are both periodic w/ period 'T'

$\hat{f}_{1n}$, $\hat{f}_{2n}$: coefficients

$a f_1(t) + b f_2(t) \longrightarrow a \hat{f}_{1n} + b \hat{f}_{2n}$ $\quad a, b \in R$

(2) $f(t)$ is real : $\hat{f}_{-n} = \hat{f}_n^*$

(3) $\boxed{f(t) \text{ is real \& even}}$: $\hat{f}_{-n} = \hat{f}_n^* = \hat{f}_n$ $\quad (\because \hat{f}_n \in R)$

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt = \sum_{n=-\infty}^{\infty} |f_n|^2 \quad (\text{Parseval})$$

($P_{avg.}$)

$$f(t) \approx \sum_{n=-N}^{N} f_n e^{jn\omega_0 t}$$

$$MSE = \sum_{n>N} |f_n|^2$$

Infinite series truncated : Gibb's phenomenon

finite # of discontinuities } expandable as Fourier series
" " " max/min. }

To reduce wiggles :

$$f(t) \approx \sum_{n=-N}^{N} f_n w_n e^{jn\omega_0 t}$$

w_n : windowing numbers

However, MSE may not be minimum/smallest

H.W. $f_1(t)$, $f_2(t)$: period = T

f_{1n} f_{2n} : Fourier coefficients

Fourier coeff.s of : $\begin{cases} f_1(t)\, f_2(t) \\ f_1(t)\, f_2^*(t) \end{cases}$

t

$$f_1(t) = \sum_m f_{1m}\, e^{jm\omega_0 t}$$

$$f_2(t) = \sum_p f_{2p}\, e^{jp\omega_0 t}$$

$$f_1 f_2 = \sum_n g_n\, e^{jn\omega_0 t}$$

$$g_n = \frac{1}{T}\int_{-T/2}^{T/2} f_1(t)\, f_2(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_m f_{1m}\, e^{jm\omega_0 t} \sum_p f_{2p}\, e^{jp\omega_0 t}\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_m \sum_p f_{1m}\, f_{2p}\, e^{j(m+p)\omega_0 t}\, e^{-jn\omega_0 t}\, dt$$

$m+p = n \Rightarrow p = n-m$

$$= \sum_m f_{1m} \sum_p f_{2p} \Big|_{m+p=n}$$

$$\therefore g_n = \sum_m f_{1m}\, f_{2(n-m)}$$

$m-p = n \Rightarrow p = m-n$

$$\therefore g_n = \sum_m f_{1m}\, f_{2(m-n)}$$

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Fourier Series

$f(t) \quad -T/2 < t < T/2$

Orthogonal CON set

$$f(t) = \sum f_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$f(t) = \sum_{n=-N}^{N} f_n e^{jn\omega_0 t}$$

$$f(t) = \sum_{-N}^{N}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\, e^{-jn\omega_0 \tau}\, d\tau\right] e^{jn\omega_0 t}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(\tau) \sum_{-N}^{N} e^{jn\omega_0(t-\tau)}\, d\tau$$

$$\Sigma = e^{-jN\omega_0(t-\tau)} + \cdots + 1 + \cdots + e^{jN\omega_0(t-\tau)}$$

$$= e^{-jN\omega_0(t-\tau)}\left[1 + \cdots + e^{jn\omega_0(t-\tau)} + \cdots e^{2jn\omega_0(t-\tau)}\right]$$

$$= e^{-jN\omega_0(t-\tau)} \frac{1 - e^{j(2N+1)\omega_0(t-\tau)}}{1 - e^{j\omega_0(t-\tau)}}$$

$$= \frac{\sin (2N+1)\frac{\omega_0(t-\tau)}{2}}{\sin \frac{\omega_0(t-\tau)}{2}}$$

$$\therefore f(t) = \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\ \underbrace{\frac{\sin \frac{(2N+1)\,\omega_0(t-\tau)}{2}}{\sin \frac{\omega_0 (t-\tau)}{2}}}_{G_T(t)\quad \text{Gibb's fn.}}\ d\tau$$

$$\lim_{N\to\infty} \int G(t)\,dt = 1$$

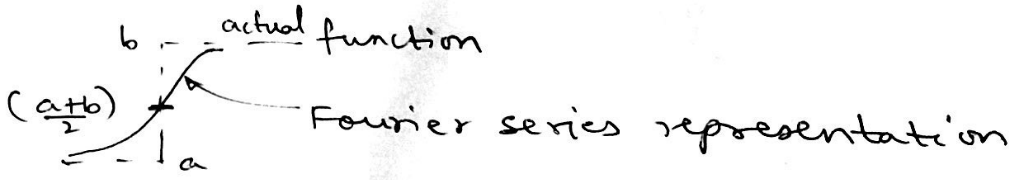
for $N\to\infty \qquad G(t-\tau) \to u_0(t-\tau)$

$$\text{So} \int_{-T/2}^{T/2} f(\tau)\,u_0(t-\tau)\,d\tau = f(t)$$

$$\underbrace{\int_{-T/2}^{T/2} f(\tau)\,G(t-\tau)\,d\tau}_{\text{Convolution integral}} \overset{?}{\approx} f(t)$$

For a discontinuous function :

@ the discontinuity Fourier series averages



$$= \frac{1}{T} \int_{-T/2} \overbrace{\sum g \quad e}^{g(t)}$$

5

Fourier series minimizes (mean square) error at every point not only overall avg. error

For finite 'N' for discontinuous function

Gibbs phenomenon -

(window functions can help w/ removing wiggles.)

Convolution :

$$\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

for periodic functions : $-\infty$ to ∞ convolution integral doesn't converge

$$\therefore z(t) \triangleq \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\, g(t-\tau)\, d\tau$$

(circular convolution)

$$z_n = g_n f_n$$

(Fourier coeffs $\because$ $z(t)$ is periodic)

$$z_n = \frac{1}{T}\int_{-T/2}^{T/2} z(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\, g(t-\tau)\, d\tau\; e^{-jn\omega_0 t}\, dt$$

$$f(t) = \sum_n f_n \, e^{jn\omega_0 t}$$

$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$g(t) = \sum_n g_n \, e^{jn\omega_0 t}$$

$$g_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, e^{-jn\omega_0 t}\, dt$$

$$\frac{1}{T}\frac{1}{T}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \sum_m f_m e^{jm\omega_0\tau} \sum_p g_p e^{jp\omega_0(t-\tau)} \, d\tau \, e^{-jn\omega_0 t}\, dt$$

$$\int_{-T/2}^{T/2}\left(\cdots + f_0 + f_1 e^{j\omega_0\tau} + f_2 e^{2j\omega_0\tau} + \cdots\right)\left(\cdots + g_0 + g_1 e^{j\omega_0(t-\tau)} + \cdots\right) d\tau$$

$$\int_{-T/2}^{T/2}\left(\cdots + f_0 g_0 + f_1 g_1 e^{j\omega_0 t} + \cdots\right) d\tau \quad = T\sum_m f_m g_m e^{mj\omega_0 t}$$

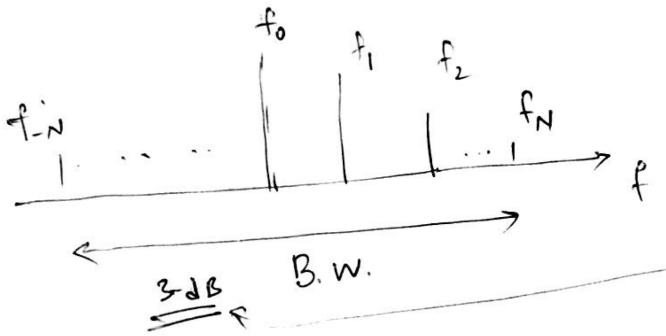
$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_m f_m g_m e^{mj\omega_0 t} \, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f_n g_n \, dt$$

$$= f_n g_n$$

Frequency spectrum :

- Representation of f_n s in f
- discrete for periodic signals
- for most signals as $n \to \infty$ f_n converges
- $\therefore$ beyond a certain N f_n s ≈ 0 to $\approx$ represent $f(t)$

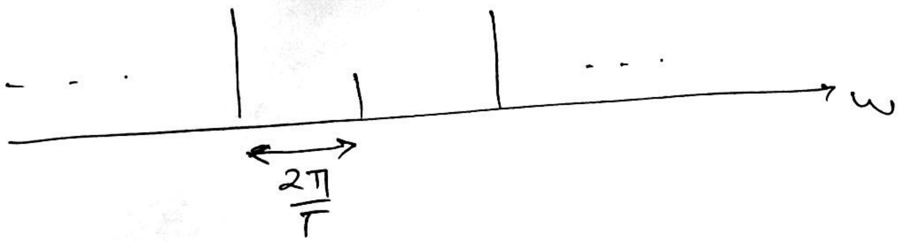


power of $f(t)$

$$= 2 \sum_{-N}^{N} f_n^2$$

Fourier series representation is useful to obtain responses to LTI systems

(sines/cosines are eigen fn.s)



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Aperiodic function

Def. of Fourier transform:

$$F(\omega) \triangleq \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

(limiting case of Fourier series for $T \to \infty$)

Def of Inverse Fourier transform

$$f(t) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Say: $F(\omega) = R(\omega) + j\, X(\omega)$

Properties of $F(\omega)$:

→ linear: $f_1(t) \longleftrightarrow F_1(\omega)$; $f_2(t) \longleftrightarrow F_2(\omega)$

$$\alpha f_1(t) + \beta f_2(t) \longleftrightarrow \alpha F_1(\omega) + \beta F_2(\omega)$$

→ $f(t)$ is real : $f(t) = f^*(t)$

$$F^*(\omega) = \int_{-\infty}^{\infty} f^*(t)\, e^{j\omega t}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt$$

$$= F(-\omega) \qquad (\cancel{F(\omega)\ \text{is}} \qquad)$$

→ $f(t)$ is real and even:

$$f(t) = f^*(t) = f(-t)$$

$$F(-\omega) = \int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt = \int_{-\infty}^{\infty} f(-t)\, e^{-j\omega t}\, dt \qquad t$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = F(\omega)$$

$$\therefore F^*(\omega) = F(-\omega) = F(\omega) \quad \left(F(\omega) \text{ is real \& even}\right)$$

$\rightarrow \quad g(t) = f_1(t)\, f_2(t)$

$$G(\omega) = \int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(x)\, e^{jxt}\, dx \; f_2(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(x)\, \underbrace{f_2(t)\, e^{-j(\omega - x)t}}\, dt\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(x)\, F_2(\omega - x)\, dx$$

$$= (F_1 * F_2)(\omega) \qquad f_1(t)\, f_2(t) \longleftrightarrow (F_1 * F_2)(\omega)$$

$\rightarrow \quad g(t) = \int_{-\infty}^{\infty} f_1(x)\, f_2(t-x)\, dx$

$$G(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x)\, f_2(t-x)\, dx\; e^{-j\omega t}\, dt$$

say, $t - x = z$

$$= \int_{-\infty}^{\infty} f_1(x) \int_{-\infty}^{\infty} f_2(z)\, e^{-j\omega(z+x)}\, dz\, dx = F_1(\omega)\, F_2(\omega)$$

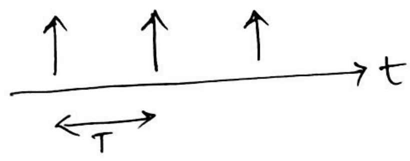
$$\therefore \left(f_1 * f_2\right)(t) \longleftrightarrow F_1(\omega)\, F_2(\omega)$$

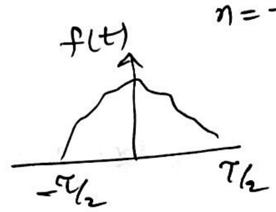
→ Impulse : $u_0(t)$

$$\mathcal{F}(u_0(t)) = \int_{-\infty}^{\infty} u_0(t)\, e^{-j\omega t}\, dt = 1$$

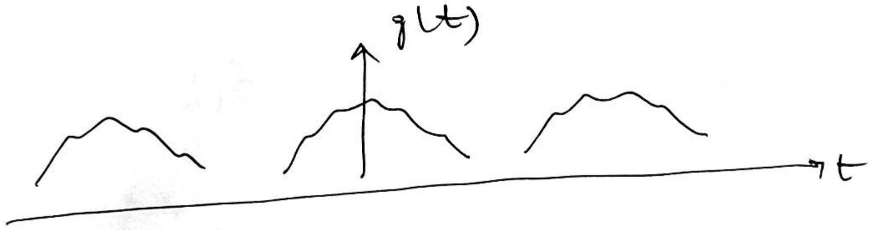
$$u_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{-jzt}\, dz$$

→ Periodic function : $g(t) = f(t) * \sum_{n=-\infty}^{\infty} u_0(t+nT)$





$\tau < T$



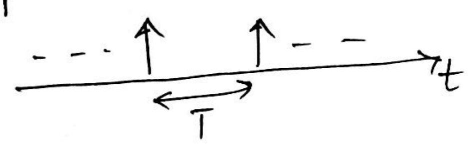
$$G(\omega) = F(\omega) \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_0(t+nT)\, e^{-j\omega t}\, dt \qquad \cancel{F(\omega) \sum e^{jn\omega T}}$$

$t + nT = z$

$$= F(\omega) \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_0(z)\, e^{-j\omega(z-nT)}\, dz = F(\omega) \sum_{n=-\infty}^{\infty} e^{j\omega nT}$$

$$= \sum_{n=-\infty}^{\infty} F(\omega)\, e^{jn\omega T}$$

$p(t)$: train of impulses $\longleftrightarrow \sum_{-\infty}^{\infty} e^{jn\omega T}$



Fourier series of $p(t)$:

$$p(t) = \sum_{-\infty}^{\infty} f_n e^{jn\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} u_0(t)\, e^{-jn\omega_0 t}\, dt = 1$$

$$\therefore\ p(t) = \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$P(\omega) = \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{jn\omega_0 t}\, e^{-j\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(\omega - n\omega_0)t}\, dt = \sum_{n=-\infty}^{\infty} 2\pi\, u_0(\omega - n\omega_0)$$

$$\left(\because\ u_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}\, d\omega\right)$$

$$f(t) * \sum_{n=-\infty}^{\infty} u_0(t + nT) \longleftrightarrow \underbrace{2\pi F(\omega) \sum_{n=-\infty}^{\infty} u_0(\omega - n\omega_0)}_{\text{discrete Fourier series}}$$

"<u>Effective</u>" width in time

" " " frequency
(Say 3 dB B.W)

Time Bandwidth product (Small vs large)

which waveforms produce small or large T. BW product.

$\boxed{\text{Time . BW} \geq \frac{1}{2}}$

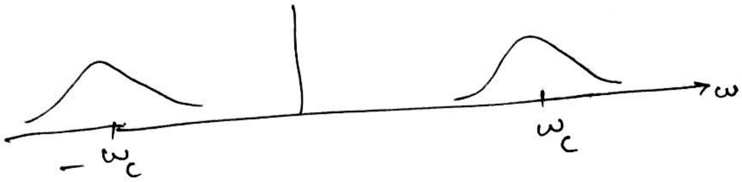
Time . BW $= \frac{1}{2}$ for Gaussian function

23 Aug

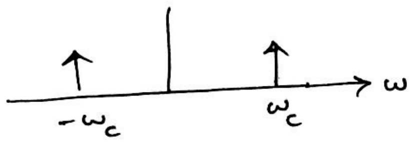
$$\cos \omega_0 t \longleftrightarrow \frac{1}{2}\, 2\pi\, u_0(\omega-\omega_0) + \frac{2\pi}{2}\, u_0(\omega+\omega_0)$$

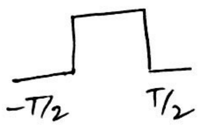
Modulation :

$f(t)\cos\omega_c t$



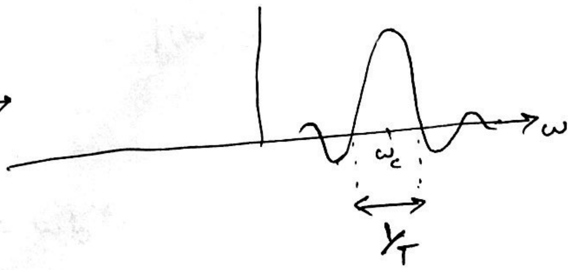
$\cos\omega_c t$: $-\infty < t < \infty$ $\longleftrightarrow$

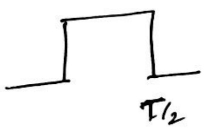




$\cos\omega_c t$: $-T/2 < t < T/2$

$\longleftrightarrow$





$$\longleftrightarrow \int_{-T/2}^{T/2} e^{-j\omega t}\, dt = \frac{\sin \omega T/2}{\omega T/2}$$

To determine freq. of waveform accurately : observation time must be as large as possible

To measure time instant(s) accurately :

observation waveform must be as narrow as possible

$\Rightarrow$ large B.W.

Radar requires large T. BW product waveforms

to get both range and doppler resolution

$\left(\frac{v}{c}\omega_0\right)$

Consider a waveform $f(t)$ with:

$$\int_{-\infty}^{\infty} f^2(t)\, dt = 1$$

1st moment (center of gravity)

$$\int_{-\infty}^{\infty} t\, f^2(t)\, dt = 0 \quad \leftarrow \text{shifting } f(t) \text{ to obtain it}$$

2nd moment

$$(\Delta t)^2 = \int_{-\infty}^{\infty} t^2 f^2(t)\, dt$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega = 1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega\, |F(\omega)|^2\, d\omega = 0$$

$$(\Delta\omega)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2\, d\omega$$

$$(T.B)^2 = (\Delta t)^2 (\Delta\omega)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^2 f^2(t)\, dt \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2\, d\omega$$

(Schwartz Inequality: $\left|\int_{-\infty}^{\infty} f\, g\, dt\right|^2 \leq \int_{-\infty}^{\infty} f^2\, dt \int_{-\infty}^{\infty} g^2\, dt$)

$$\therefore (T.B)^2 \geq \left| \int_{-\infty}^{\infty} t\, f(t)\, \frac{df}{dt}\, dt \right|^2$$

$$= \left| \int_{-\infty}^{\infty} t\, f\, df \right|^2$$

t

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{df}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, j\omega e^{j\omega t}\, d\omega$$

$$\therefore \frac{d^n f}{dt^n} \leftrightarrow (j\omega)^n F(\omega)$$

From Parseval's theorem: $\displaystyle\int_{-\infty}^{\infty} |j\omega F(\omega)|^2 d\omega = \int_{-\infty}^{\infty}\left(\frac{df}{dt}\right)^2 dt$

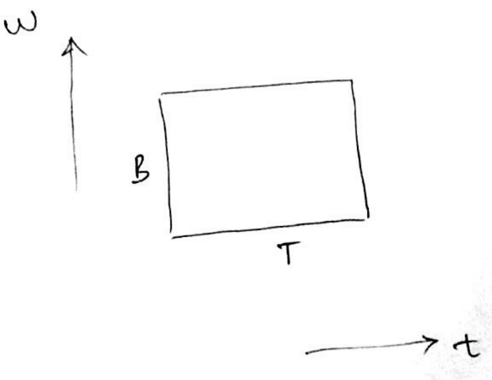
$$\therefore (T.B.)^2 \geq \left| \cancel{\frac{tf^2}{2}\Big|_{-\infty}^{\infty}}^{\,0} - \int \frac{f^2}{2}\, dt \right|^2 = \text{Constant}$$

(Uncertainty principle)

(Dennis Gaber Imperial College)

Schwarz inequality becomes equality for:

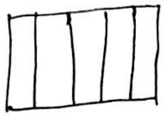
$$\pm t f = \frac{df}{dt} \Rightarrow f = e^{-t^2/2} \quad \text{(Gaussian function)}$$



(diff. freq.s are orthogonal $e^{j\omega_1 t}, e^{j\omega_2 t}$ $\omega_1 \neq \omega_2$)

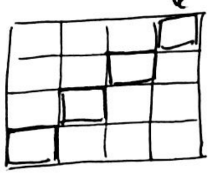
Freq. Div. Multiplex (FDM)

Time Div. Multiplex (TDM) (diff. time slots are orthogonal)



CDMA - Spread Spectrum Tech. (Heidi Lamar)

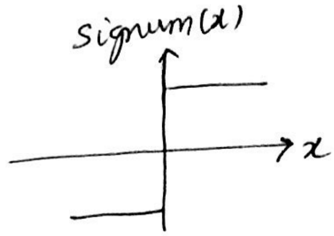
user 1



PN 

n = 10

28/08/'18

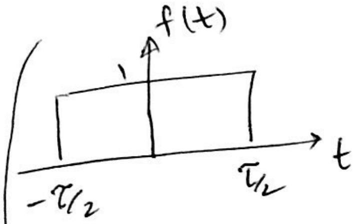


$$f(t)\quad \frac{1}{\pi t} \longleftrightarrow -j\ \text{Signum}(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} \frac{1}{\pi t} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \frac{1}{\pi t}\left(\cos\omega t - j\sin\omega t\right) dt$$

$$= -j\,\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega t}{t}\, dt$$

$$\because \int_{-\infty}^{\infty} \underbrace{\frac{\cos\omega t}{t}}_{\text{odd function}} dt = 0$$



$$\longleftrightarrow \int_{-\tau/2}^{\tau/2} 1\, e^{-j\omega t}\, dt = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-\tau/2}^{\tau/2} = \tau\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} \tau\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{(\omega\tau/2)}\, e^{j\omega t}\, d\omega = f(t)$$

Consider : for $t = 0$: $\quad f(0) = 1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\frac{\omega\tau}{2}}{\omega}\, d\omega$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin \omega(\tau/2)}{\omega}\, d\omega = \begin{cases} \pi & ; \tau > 0 \\ -\pi & ; \tau < 0 \end{cases}$$

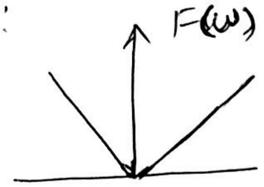
$\therefore F(\omega) =$

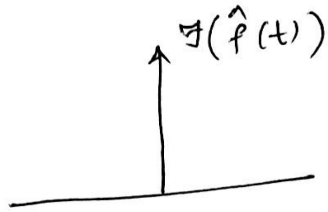
t

Hilbert transform: $\hat{f}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{t-x}\,dx$

$$\mathcal{F}\left(\hat{f}(t)\right) = F(\omega)\left(-j\,\text{signum}(\omega)\right)$$

say:





$$f_{analytic} = f(t) + j\hat{f}(t)$$

$$\mathcal{F}\left(f_{analytic}\right) = F(\omega) + \text{signum}(\omega)\,F(\omega)$$

$$= \begin{cases} 2F(\omega) & ; \ \omega > 0 \\ 0 & ; \ \omega < 0 \end{cases}$$

(Single side Band Modulation:

say: $f(t) = \cos(\omega_c t) \Rightarrow \hat{f}(t) = \sin(\omega_c t)$

now say: $f(t) = \sin\omega_c t \Rightarrow \hat{f}(t) = -\cos(\omega_c t)$

for $f(t) = 0 \quad t<0$ $f(t)$ is Causal

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[R(\omega) + jX(\omega)\right] (\cos\omega t + j\sin\omega t)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(R(\omega)\cos\omega t - X(\omega)\sin\omega t\right) d\omega$$

$$+ \frac{1}{2\pi} j \int_{-\infty}^{\infty} \left(R(\omega)\sin\omega t + X(\omega)\cos\omega t\right) d\omega$$

$\therefore$ when $f(t)$ is real :

$$\boxed{\int_{-\infty}^{\infty} R(\omega)\sin\omega t\, d\omega = -\int_{-\infty}^{\infty} X(\omega)\cos\omega t\, d\omega}$$

Real part sufficiency

further : $\boxed{X(\omega) = \hat{R}(\omega)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(\xi)\, d\xi}{\omega - \xi}$

and $R(\omega) = -\hat{X}(\omega)$

Say $f(t)$ is also causal : $f(t) = 0$, $t<0$

$\therefore f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} R(\omega)\cos\omega t\, d\omega = \frac{2}{\pi}\int_{0}^{\infty} R(\omega)\cos\omega t\, d\omega$
(for $t>0$)

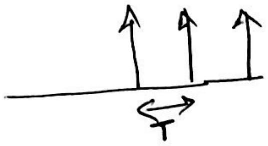
$R(\omega)$: even

$X(\omega)$: odd

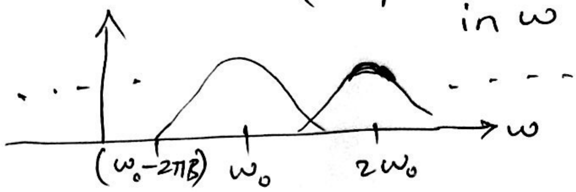
Sampling Principle (Shannon - Whittaker - Nyquist - Kotelinikov)

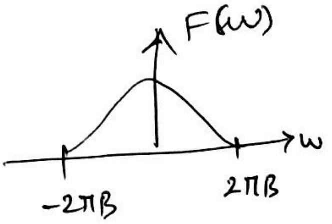
$f(t) \cdot p(t) \longleftrightarrow F(\omega) * P(\omega)$

(impulse train) (impulse train in ω)



$\omega_0 = \frac{2\pi}{T}$





$\therefore$ To recover $F(\omega)$ from $F(\omega) * P(\omega)$

$\omega_0 - 2\pi B > 2\pi B$

$\omega_0 > 4\pi B \Rightarrow \boxed{T < \frac{1}{2B}}$

$f(t) \cdot p(t)$ is like an orthogonal expansion of $f(t)$ using orthogonal functions as impulses

Signal duration : 'T' (essential)

" B.W : B (essential)

$\therefore$ # of samples : $(2BT + 1)$

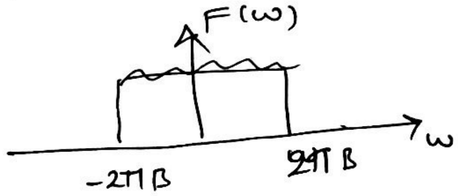
30/08/18

F(ω) of a 'real' signal satisfies:

$$\int_{-\infty}^{\infty} \frac{\ln|F(\omega)|}{1+\omega^2}\, d\omega < \infty \qquad \left(\text{Paley-Wiener Criterion}\right)$$

---

consider f(t) : strictly bandlimited



$$f(t) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} F(\omega)\, e^{j\omega t}\, d\omega$$

$e^{j\omega t}$ as a function of 'ω' $\underbrace{-2\pi B < \omega < 2\pi B}$

Fourier series expansion for $e^{j\omega t}$ in period

$$e^{j\omega t} = \sum_n c_n\, e^{jn\frac{2\pi}{4\pi B}\omega} = \sum_n c_n\, e^{jn\frac{\omega}{2B}}$$

& $$c_n = \frac{1}{4\pi B}\int_{-2\pi B}^{2\pi B} e^{j\omega t}\, e^{-jn\frac{\omega}{2B}}\, d\omega$$

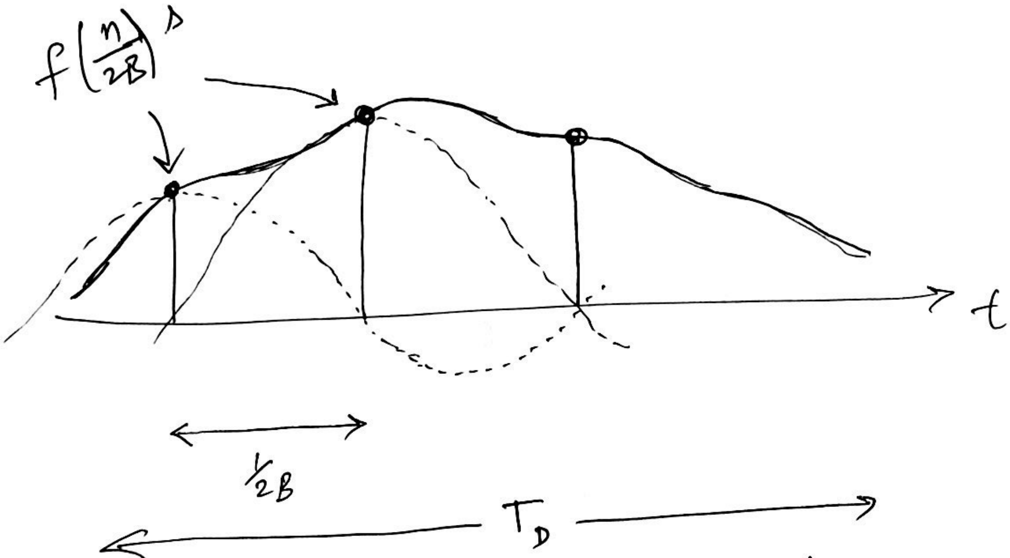
$$= \frac{1}{4\pi B}\, \frac{e^{j\omega(t-\frac{n}{2B})}}{j(t-n/2B)}\Bigg|_{-2\pi B}^{2\pi B} = \frac{\sin\left(2\pi B\left(t-\frac{n}{2B}\right)\right)}{2\pi B\left(t-\frac{n}{2B}\right)}$$

$$\therefore f(t) = \frac{1}{4\pi B}\int_{-2\pi B}^{2\pi B} F(\omega) \sum_n \frac{\sin 2\pi B\left(t-\frac{n}{2B}\right)}{2\pi B\left(t-\frac{n}{2B}\right)}\, e^{jn\frac{\omega}{2B}}$$

$$\left(\frac{1}{4\pi B} \int_{-2\pi B}^{2\pi B} F(\omega)\, e^{j\omega \frac{n}{2B}}\, d\omega = f\left(\frac{n}{2B}\right) \right.$$

$$\left. \because\ f(t) = \int_{-2\pi B}^{2\pi B} F(\omega)\, e^{j\omega t}\, d\omega \right)$$

$$\therefore\ f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2B}\right) \frac{\sin 2\pi B\left(t - \frac{n}{2B}\right)}{2\pi B\left(t - \frac{n}{2B}\right)}$$

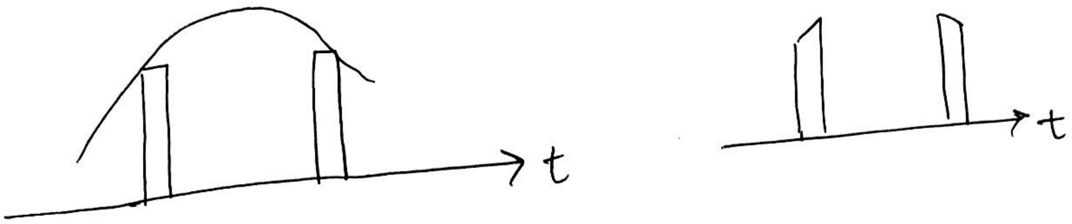


Time function is 'effectively' time-limited

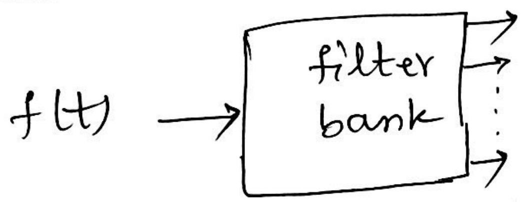
the # of samples, $N = 2BT + 1$

$$\left\{ f\left(\frac{n}{2B}\right) \right\}$$

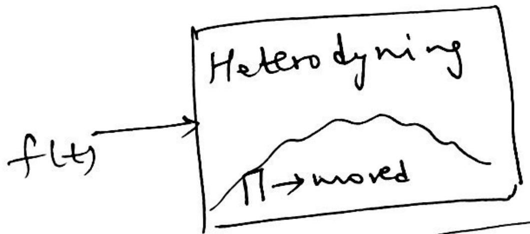
In practice:

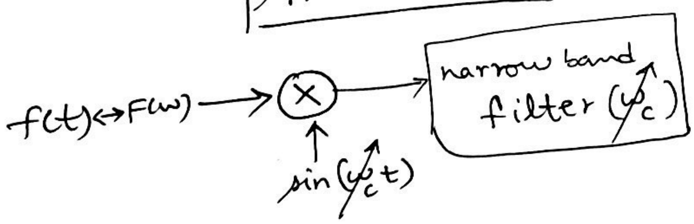


$\frac{\sin(t)}{\sinh(t)}$ has smaller side lobes than $\frac{\sin(t)}{t}$



F(ω)s centered around narrow band filters' center freq.

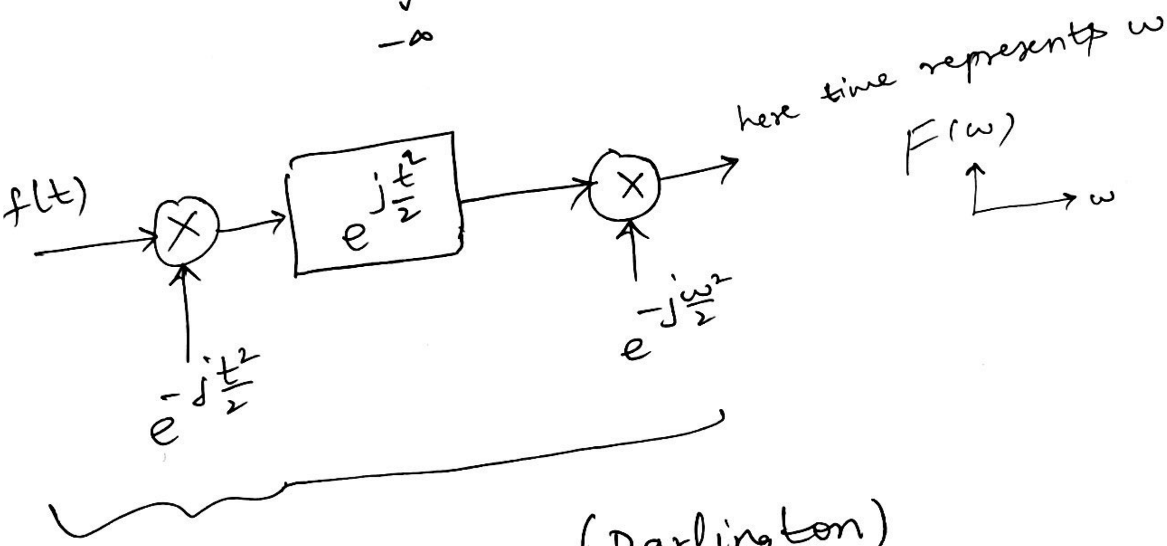




$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{j\frac{1}{2}(\omega - t)^2}\, e^{-j\frac{1}{2}\omega^2}\, e^{-j\frac{1}{2}t^2}\, dt$$

$$= e^{-j\frac{\omega^2}{2}} \int_{-\infty}^{\infty} f(t)\, e^{-j\frac{t^2}{2}}\, e^{j\frac{1}{2}(\omega - t)^2}\, dt$$

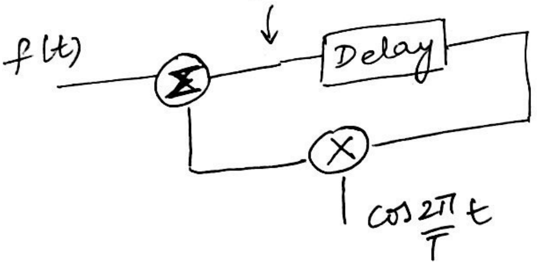


Chirp transform (Darlington)

31/08/18

Capon

$g(t)$ is $F(\omega)$ with t representing as ω



Coherent memory filter

FFT (software)

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$F\left(m \frac{2\pi}{T_0}\right) = \sum_{n=0}^{N-1} f\left(n \frac{2\pi}{4\pi B}\right) e^{-j m \frac{2\pi}{T_0} n T}$$

$$\sum_{n=0}^{N-1} f(n)\, e^{-j\omega n T}$$

$$F(m) = \sum_{n=0}^{N-1} f(n)\, e^{-j mn \frac{2\pi}{N}}$$

$$\left(N = \frac{T_0}{T}\right)$$

$m = 0, \ldots N-1$

$$F(m) = \sum_{n=0}^{N-1} f(n)\, e^{-j mn \frac{2\pi}{N}}$$

$$\left(e^{-j\frac{2\pi}{N}} = W_N\right)$$

$$\therefore F(m) = \sum_{n=0}^{N-1} f(n)\, W^{mn}$$

Sampling Theorem
- in time domain
  Band limited
- in frequency domain
  Time limited signal
  $0 \longrightarrow T_0$
  samples in frequency
  $\omega_0 = \frac{2\pi}{T_0}$; $m\omega_0$

repetitive nature of $e^{j\theta}$ is used in FFT

<u>FFT</u> :

<u>say $N=4$</u> :

$$\underbrace{\begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}}_{F(m)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{array}{c} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$W_4^6 = W_4^2$, $W_4^4 = 1$, $W_4^9 = W_4$ (cyclic)

$$\begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{P \text{ : permutation matrix}} \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix}$$

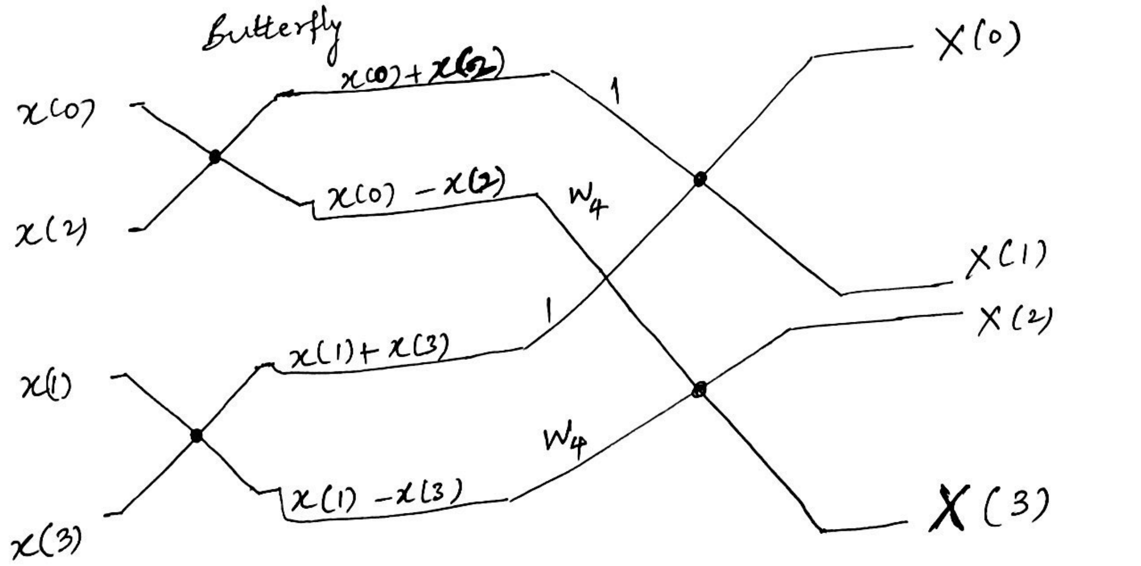
$$\begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} P \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & W_4^2 & W_4^1 & W_4^3 \\ \hline 1 & \cancel{W_4}\,1 & W_4^2 & W_4^2 \\ 1 & W_4^2 & W_4^3 & W_4 \end{array}\right] \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix}$$

$$= \left[\begin{array}{c|c} F_2 & D.F_2 \\ \hline F_2 & -D F_2 \end{array}\right] \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix}$$

FFT [Cooley / Tukey

$$D = \begin{bmatrix} 1 & 0 \\ 0 & W_4^1 \end{bmatrix}$$



At most we have $N \log_2 N$ complex computations without FFT algorithm : # of computations is N^2

$N \log_2 N < N^2$

12 Sept.

$f(t)$ sampled $\frac{1}{2B}$

quantized and represented using N bits

$f(t)$ has "T" duration

$\therefore$ total # of bits : $2BTN$

Say $F(\omega)$ has smaller essential B.W.
better to sample $F(\omega)$ and this information sent.
Transform coding

Cosine transform : $\int_{-\infty}^{\infty} f(t) \cos(\omega t)\, dt$

MPEG

Sampling rate for $F(\omega)$: $\frac{1}{2T}$

easiest to implement among other transforms : Harr, Hadamard ...

2-dimensional Fourier Transform

$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, e^{-j(\omega_x x + \omega_y y)}\, dx\, dy$$

Optics :

Diffraction Integral { Rayleigh - Sommerfield
Fraunhofer
→ Fresnel

Fourier Optics : In far field : double Fourier Transform

Optical signal Processing to implement double Fourier Transform which occurs a natural phenomenon in Fresnel region.

Laplace Transform

$$F(s) \triangleq \int_0^{\infty} f(t)\, e^{-st}\, dt$$ One-sided Laplace Transf.

$$f(t) = \begin{cases} \text{exists} & t>0 \\ 0 & t<0 \end{cases}$$ f(t) is causal

$s = \sigma + j\omega$ (complex)

for $\sigma = 0$ and f(t) is causal L.T. becomes Fourier Transform

by considering $s = \sigma + j\omega$ increasing # of signals for which a transformation exists.

$$|F(s)| = \left| \int_0^{\infty} f(t)\, e^{-st}\, dt \right| \leq \int_0^{\infty} |f(t)\, e^{-st}|\, dt$$
$$= \int_0^{\infty} |f(t)\, e^{-\sigma t}|\, dt$$

∴ more functions can have L.T. compared to F.T.

Consider:
$$f(t) = \begin{cases} e^{-at} & 0<t<\infty \\ 0 & t<0 \end{cases}$$

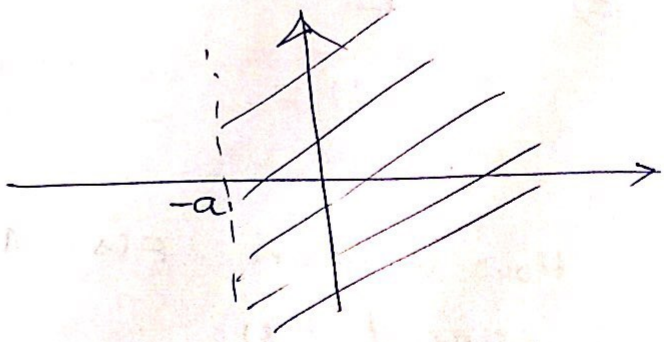
$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-at}\, e^{-st}\, dt = \int_0^{\infty} e^{-(s+a)t}\, dt$$

integral exists for $Re(s+a) > 0 \Rightarrow Re\{s\} > -a$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s+a} \; ; \; Re\{s\} > -a$$

R.O.C.

∴ Laplace Transform of f(t) has to be accompanied by R.O.C.



- $\lim_{s \to \infty} F(s) = 0$

- linearity: $\alpha f_1(t) + \beta f_2(t) \leftrightarrow \alpha F_1(s) + \beta F_2(s)$

- $\dfrac{dF(s)}{ds} \longleftrightarrow -t\, f(t)$; $t^2 f(t) \longleftrightarrow \dfrac{d^2F}{ds^2}$

  $t^n f(t) \longleftrightarrow (-1)^n \dfrac{d^nF}{ds^n}$

- $\dfrac{df}{dt} \longleftrightarrow sF(s) - f(0)$

  $\int_0^\infty \dfrac{df}{dt} e^{-st}\, dt$

  $= \int_0^\infty e^{-st}\, df$

- $\dfrac{d^2f}{dt^2} = \dfrac{d}{dt}\left(\dfrac{df}{dt}\right) \longleftrightarrow s\left[sF(s) - f(0)\right] - f'(0)$

  $= s^2F(s) - sf(0) - f'(0)$

$\therefore$ Easy to solve const. coeff. O.D.E.

$\mathcal{L}\left\{ \dfrac{d^2f}{dt^2} + a\dfrac{df}{dt} + bf = g \right\}$

$\Rightarrow s^2F(s) - sf(0) - f'(0) + a\left[sF(s) - f(0)\right] + bF(s) = G(s)$

$\Rightarrow F(s)\left[s^2 + as + b\right] = G(s) + sf(0) + f'(0) + af(0)$

$\therefore F(s) = \dfrac{G(s) + sf(0) + f'(0) + af(0)}{s^2 + as + b}$

How to invert $F(s)$ and get $f(t)$

Given: R.O.C. & $F(s)$

- Initial value Theorem:

Given $F(s)$; find $f(0)$:

$$\lim_{s\to\infty} \int_0^\infty \frac{df}{dt} e^{-st} dt = \lim_{s\to\infty} sF(s) - f(0) = 0$$

$$\therefore f(0) = \lim_{s\to\infty} sF(s)$$

- Final value theorem:

Given $F(s)$; find $f(\infty)$:

$$\lim_{s\to 0} \int_0^\infty \frac{df}{dt} e^{-st} dt = \lim_{s\to 0} sF(s) - f(0)$$

$$f(\infty) - \cancel{f(0)} = \lim_{s\to 0} sF(s) - \cancel{f(0)}$$

say $F(s_p) \to \infty$ $s = s_p$ is a pole

say $F(s_z) = 0$ $s = s_z$ is a zero

for poles in RHP ; $f(\infty) \to \infty$

" " in LHP : $f(\infty) \to 0$

" " on $j\omega$-axis : $f(\infty)$ does not exist
oscillatory function

$\therefore \lim_{s\to 0} sF(s) = f(\infty)$ only when $F(s)$ has a pole @ $s=0$

14 Sept.

$f(t)$ is a causal function $\quad f(t) = 0 \quad t < 0$

laplace transform: $\quad F(s) \triangleq \int_0^{\infty} f(t)\, e^{-st}\, dt$

s : complex variable

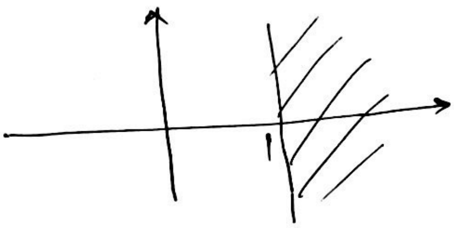
$s = \sigma + j\omega$

must also specify ROC along w/ the laplace transform $F(s)$ to obtain $f(t)$ uniquely.

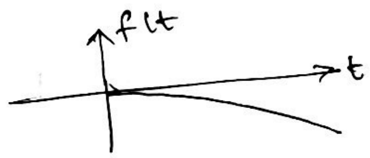
Examples:

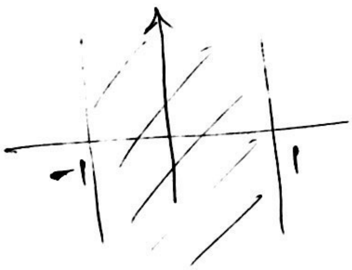
Given : $\quad F(s) = \dfrac{2}{1-s^2}$

with: ROC :

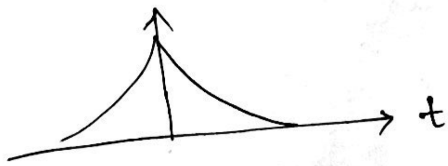


then $f(t) = e^{t} - e^{-t} \quad t > 0$

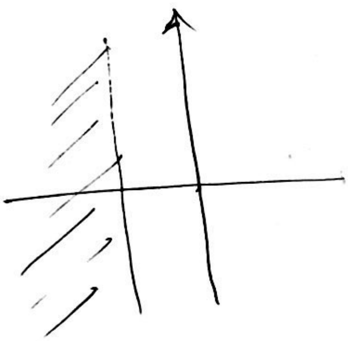




then $f(t) = e^{-|t|}$



then $f(t) = e^{t} - e^{-t} \quad t < 0$



Inverse laplace transform given :

F(s) and ROC

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$$

c : boundary along ROC.

$$F(s) \triangleq \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds = \int_{c-j\infty}^{c+j\infty} \int_0^\infty f(x)\, e^{-sx}\, dx\, e^{st}\, ds$$

$$= - \int_0^\infty dx\, f(x) \int_{c-j\infty}^{c+j\infty} e^{s(t-x)}\, ds$$

consider : $\displaystyle\int_{c-j\infty}^{c+j\infty} \frac{e^{as}}{s}\, ds = g(a)$

pole @ s=0

e^{as} is analytic in whole of s-plane

Cauchy Integral theorem :

$$\oint \frac{G(s)}{s-s_j}\, ds = \begin{cases} 0 & s_j \text{ outside domain of integration} \\ -2\pi j\, F(s_j) & \text{" inside "} \end{cases}$$

$$\frac{d}{dt} \frac{e^{s(t-x)}}{s} = e^{s(t-x)} \Rightarrow \int_{c-j\infty}^{c+j\infty} e^{s(t-x)}\, ds = \frac{d}{dt} \int_{c-j\infty}^{c+j\infty} \frac{e^{s(t-x)}}{s}\, ds$$

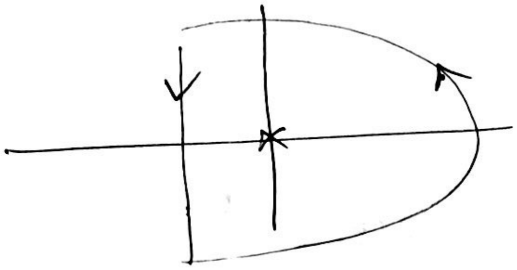
given a value of "t" : $(t-x) > 0$ (or) $(t-x) < 0$

$$2\pi j \, e^{a(0)} = 2\pi j \cdot 1.$$

$$\oint = \int_{-j\infty}^{+j\infty} + \oint$$

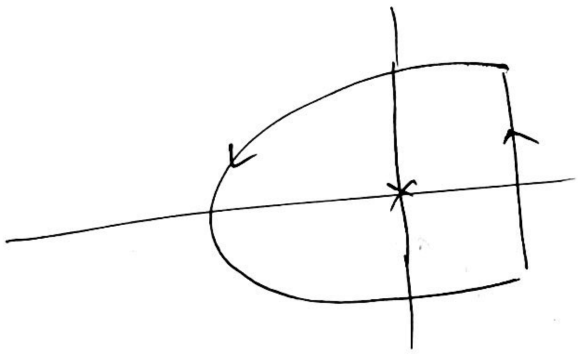
Region of Convergence : valid for : $a < 0$

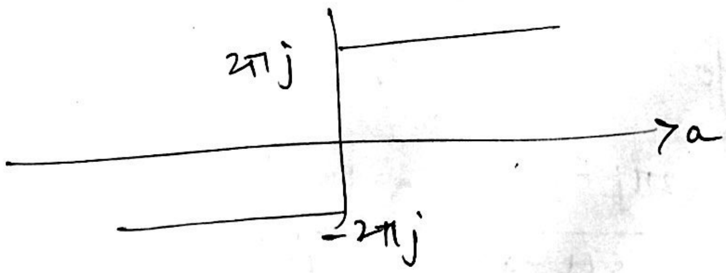
so that $\oint \to 0$



R.O.C. valid for $a > 0$

so that $\oint \to 0$





derivative is Impulse fn. @ $a = 0$

Fourier Stieltjes

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$$

$$f(t) \longrightarrow \boxed{\text{D.E.}} \longrightarrow g(t)$$

$$G(s) = F(s)\, H(s)$$

say $F(s) = \dfrac{P(s)}{Q(s)}$

For L.T.I. systems $H(s)$ is ratio of 2 polynomials

$\therefore$ $G(s)$ is also ratio of 2 polynomials

Given a polynomial

$F(s) =$ Taylor series form) express as ratio of 2 polynomials (Pade's method)

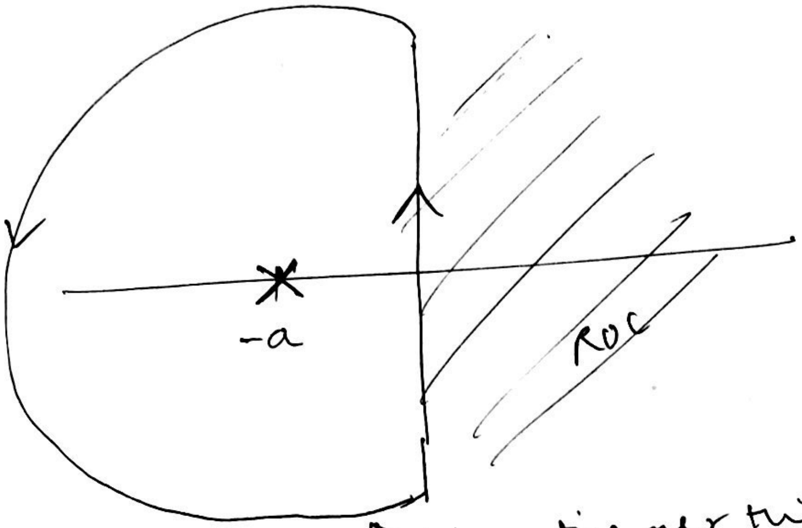
$$\therefore F(s) = \frac{(s-s_1)(s-s_3)\cdots}{(s-s_0)(s-s_2)\cdots}$$

s_{odd} : zeros

s_{even} : poles

$$\therefore f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{(s-s_1)(s-s_3)\cdots}{(s-s_0)(s-s_2)\cdots}\, e^{st}\, ds$$

$$= \sum_k (s-s_k)\, F(s) \Big|_{s=s_k}$$



integrating over this to include pole @ $s=-a$

$$F(s) = \frac{1}{s+a}$$

$$\frac{1}{2\pi j}\int_{-a-j\infty}^{-a+j\infty} \frac{1}{s+a} e^{st}\, ds = e^{-at}$$

---

say $F(s) = \frac{P(s)}{Q(s)} = \frac{A}{s-s_0} + \frac{B}{s-s_2} + \cdots$

$$\therefore f(t) = A e^{s_0 t} + B e^{s_2 t} + \cdots$$

higher order poles expanded as Laurent series

Rice university notes for Laplace transforms!

when there are const. coeff D.E.s ...

polynomials have real coeff.s

$\therefore$ poles when complex are always accompanied by their complex conjugates.

18 sept.

$$F(s) = \frac{P(s)}{Q(s)}$$ (ratio of 2 polynomials)

Continuous time single valued functions $f(t)$ $-\infty < t < \infty$

$f(nT)$: samples spaced by T

$f[n]$: sequence of samples without explicitly refering to "T"

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \sum_{n=0}^{\infty} f(t)\, u_0(t - nT)\, e^{-st}\, dt$$

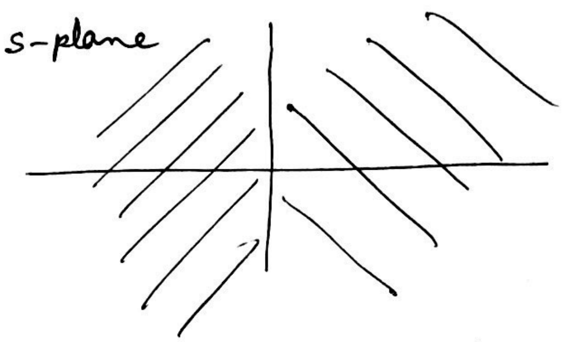
$$= \sum_{n=0}^{\infty} f(nT)\, e^{-snT}$$

Considering $T = 1$:

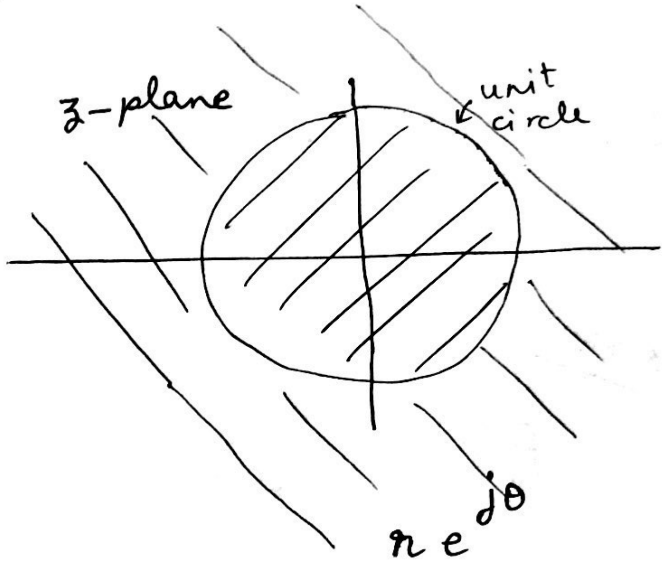
$$\sum_{n=0}^{\infty} f[n]\, e^{-sn}$$

considering $z = e^{-s}$: $\sum_{n=0}^{\infty} f[n]\, z^{n}$

$$e^{s} = z \Rightarrow e^{\sigma + j\omega} = \cancel{z} \rightarrow r e^{j\theta}$$



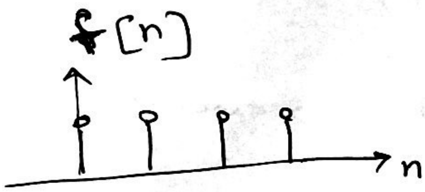
$$e^{\sigma} \; e^{j\omega}$$



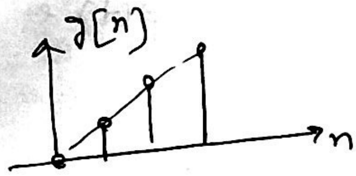
$$r e^{j\theta}$$

Kroneker Delta $\delta[n]$ to describe discrete sequences :

$$f[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



$$g[n] = \sum_{k=0}^{\infty} \cancel{k} \, \delta[n-k] \; n$$



$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) \, e^{st} \, ds$$

$$f(nT) = \frac{1}{2\pi j} \cancel{Z} \int_{c-j\infty}^{c+j\infty} F(s) \, e^{snT} \, ds$$

$$f(nT) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{snT}\, e^{-s}\, dz$$

$$= \frac{1}{2\pi j} \oint F(z)\, z^{n-1}\, dz \qquad (z = e^{s})$$

over the curve bounding ROC.

$x[n] = Ce^{an}$ Exponential function

$x[n] = x[n+N]$ periodic function

Consider: $e^{j\omega_0 n} = e^{j\omega_0(n+N)} = e^{j\omega_0 n} e^{j\omega_0 N}$

when $\omega_0 N = 2k\pi$ $k = 1, 2 \ldots$ then $e^{j\omega_0 N}$ is periodic w/ N

$\therefore e^{j2n}$ is not periodic

However, $e^{j\pi n}$ is periodic

Even : $x[n] = x[-n]$

odd : $x[n] = -x[-n]$

Then $x[n] = x_e[n] + x_o[n]$

$x[-n] = x_e[n] - x_o[n]$

$\Rightarrow$ $x_e[n] = \dfrac{x[n]+x[-n]}{2}$

$x_o[n] = \dfrac{x[n]-x[-n]}{2}$

Consider:

$$x[n] = \begin{array}{ccccccccc} x(0) & x(1) & x(2) & \ldots & & & & & \\ \end{array}$$

$$x[n] = [1 \quad 0 \quad 0 \quad \tfrac{1}{2} \quad 0 \quad 0 \quad \tfrac{1}{4} \quad 0 \ldots]$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-3k]$$

$$X(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-3k}$$ ROC: $|z| > 1$

$$\sum_{n=0}^{\infty} x[n]\, z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

say $x[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

19 Sept. Discrete signals

$x[n]$ n : integers $-\infty < n < \infty$

- $x[n] = \begin{cases} 1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \ \cdots & n = 0, \cdots \infty \\ 0 & n < 0 \end{cases}$
(Causal)

- $x[n] = \sum_{k=0}^{\infty} x(k)\ \delta[n-k]$

- $x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n = 0, \cdots \infty \\ 0 & n < 0 \end{cases}$

Impulses are orthogonal functions
sampled signal using impulses is an orthogonal representation

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Average power : $\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

Time limited function to 2N+1 sample values the $\lim_{N\to\infty}$ is not required.

$x[n]$ $\quad$ $y[n]$

Correlation function:

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n+l]\, y^*[n]$$

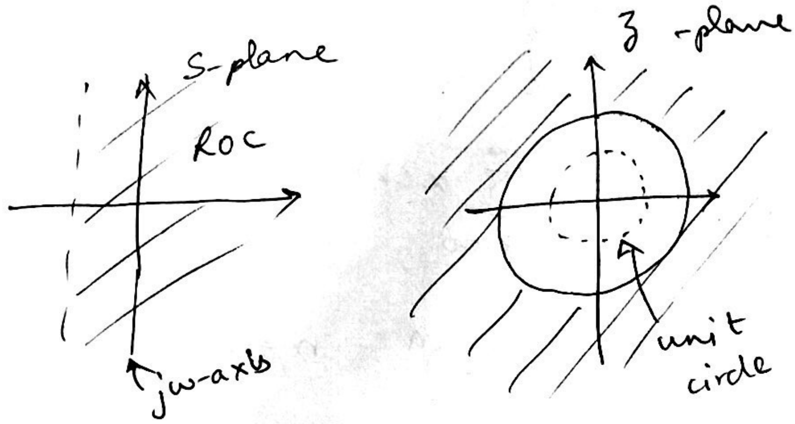
when $x = y$ Autocorrelation function $r_{xx}[l]$

$$\rho_{xy}(l) = \frac{r_{xy}[l]}{\sqrt{E_x}\sqrt{E_y}} \qquad |\rho_{xy}[l]| \le 1$$

(proved using Schwartz Inequality)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$z = e^s$$



say: $x[n] = 1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \cdots$

$$\therefore X(z) = 1 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} + \cdots = \frac{1}{1 - \frac{1}{2z}} = \frac{z}{z - \frac{1}{2}}$$

series converges when $\left|\frac{1}{2z}\right| < 1 \Rightarrow |z| > \frac{1}{2}$ (ROC)

$x[n] \longleftrightarrow X(z)$ uniqueness guarenteed by ROC.

Scanned with CamScanner

$z = e^s$ is not a unique transformation without specifying a valid "cut" say $-\pi$ to π

$$X(z) = \frac{P(z)}{Q(z)}$$

$$\dot{x} + ax = y$$

$$\frac{x[n+1) - x[n]}{1} + a\,x[n] = y[n]$$

O.D.E. w/ real const. coeff.s $\Rightarrow X(z) = \dfrac{P(z)}{Q(z)}$

P, Q are polynomials in 'z'

- brute force method to get $x[n]$ from $X(z)$

Expand $X(z)$ as Taylor series in z^{-1} and coeff.s give $x[n]$
(or long division P/Q)

- $X(z) = \dfrac{\prod_i (z - z_i)}{\prod_j (z - z_j)}$ z_i s : zeros

z_j s : poles

$$= \frac{A}{\left(1 - \frac{z_1}{z}\right)} + \frac{B}{\left(1 - \frac{z_2}{z}\right)} + \cdots$$

& we know series corresponding to $\dfrac{1}{\left(1 - \frac{z_j}{z}\right)}$ (Z-transform ref. table)

- $x[n] = \dfrac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$

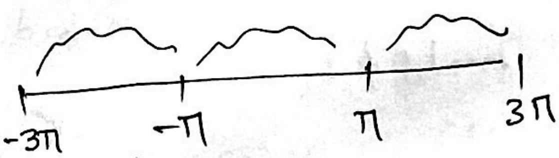
boundary of ROC. circle w/ poles inside

Ref. for Laplace & Z-transforms:

Baranuick (Rice University)

Given $X(z)$: freq. response :

$$X(e^{j\omega}) = \frac{P(e^{j\omega})}{Q(e^{j\omega})} \qquad z = e^{s} \text{ w/ } s = j\omega$$



Riemann Surface : suface formed by helix of unit radius

$|e^{j\theta}|$: unity

$T=1$ is considered for discrete sequences to connect w/ perspective of sampled signals (w/ period : T)

Random Signals : $x[n]$ for a given 'n' is r.v.

$p_{X,n}(x, n)$ probability density function

$P_{X,n}(x, n)$ " distribution "

$x[m]$ $m \neq n$ is same r.v. w/ same p.d.f.

then x is stationary

Most of the time we choose p.d.f. as Gaussian.

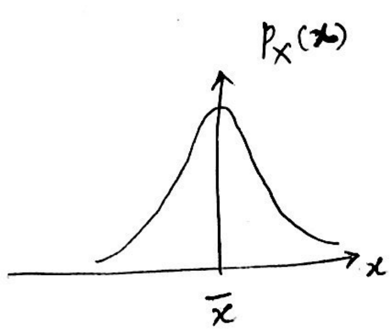
21 sept.

$$p_{X,n}(x,n) = f(x)$$

$$\int_{-\infty}^{x} p_X(x)\, dx = P\{X < x\} = P_X(x)$$

- $f(x)$ is +ve $\forall x$
- $\int_{-\infty}^{\infty} p_X(x)\, dx = 1$

Gaussian p.D.F. $p_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$



Mean, $\bar{x} = \int_{-\infty}^{\infty} x\, p_X(x)\, dx$

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n\, p_X(x)\, dx \quad (n^{th} \text{ moment})$$

Central limit theorem (C.L.T.)

$$X = \sum_{i=1}^{N} a_i X_i \qquad N: \text{very large}$$

X_i s are iid r.v.s

when N is very large irrespective of p.d.f. of X_i s

X's p.d.f. is Gaussian

2^{nd} moment : power : $\int_{-\infty}^{\infty} x^2 p_X(x)\, dx$

Correlation function ρ_{ij}

$x[i]$ $\quad$ $x[j]$ $\quad$ (or) $\quad$ $x(i)$ $\quad$ $x(j)$

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i \cancel{(i)}\, x_j \cancel{(j)}\; p_{X_i X_j}(x_i, x_j)\, dx_i\, dx_j$$

say $j = i + k$

Normalization of ρ_{ij} w/ $\sqrt{\int x_i^2 dx_i \int x_j^2 dy_i}$

say $k = 0$:

$\rho_{ii} =$

$$p_{X_1, X_2 \cdots}(x_1, x_2 \cdots)$$

Wide sense stationary (WSS)

$$E(X(i)) = \text{constant} \quad \forall i$$

$$E(X(i)\, X(j)) = f(i-j)$$

Time average = Statistical average

Ergodocity

25 sept.

For WSS

$$R_X(n) = R_X(n_1 - n_2) = R_X(x(n_1)\, x(n_2))$$

Z-transform of $R_X(n)$; $S_X(z) = \sum_{n=-\infty}^{\infty} R_X(n)\, z^{-n}$

$$S_X(e^{jw}) = \sum_{n=-\infty}^{\infty} R_X(n)\, e^{-jwn}$$

Discrete Power Spectrum

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A WSS process does not need to have same p.d.f. $\forall$ n or t

only requires : (1) $\int x(t)\, f_X(x)\, dx$ = Constant

(2)

generally , $R_X(t_1-t_2) \downarrow\downarrow$ as $(t_1-t_2) \uparrow\uparrow$

$R_X(t_1-t_2)$ is an even function in (t_1-t_2)

$R_X(\tau)$ could be negative

$R_X(0)$ is power

$R_X(\tau) \longleftrightarrow S_X(\omega)$ $\qquad$ $R_X(\tau)$ is even

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega)\, e^{j\omega\tau}\, d\omega$$

$\because \int_{-\infty}^{\infty} R_X(\tau) \underbrace{\sin\omega\tau}_{\text{even} * \text{odd} = \text{odd}}\, d\tau = 0$ $\qquad$ $\therefore S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) \cos\omega\tau\, d\tau$

$\downarrow$

Power spectral density

$\boxed{\text{Weiner Khinchin Theorem}}$

fluctuations in value of r.v. (s)

To measure $R_X(\tau)$ when ergodic:

- measure time function
- shift
- integrate

$S_X(\omega)$ is an even function : $\boxed{S_X(\omega) = S_X(-\omega)}$

Total power : $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$

$S_X(\omega)\, \Delta\omega$ is always +ve

$\therefore \boxed{S_X(\omega) \geq 0 \quad \forall\, \omega}$

$x(t)$: random process

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \phi_n(t)$$

$x_n = \int \cancel{E}\left[x(t)\, \phi_n(t)\right] dt$

when $x(t)$ is stationary

$E[x_n] = \int E[x(t)]\, \phi_n(t)\, dt$

x_n s are r.v.s

a set of $\phi_n(t)$ s (deterministic) leads to uncorrelated x_n s

[KL expansion]

(independent ?)

uncorrelated r.v.s x_n, x_m : $E[x_n x_m] = 0$

Indepedent r.v.s : $p_{X_n X_m}(x_n, x_m) = p_{X_n}(x_n)\, p_{X_m}(x_m)$

when $p_{X_n X_m}$ is Gaussian and x_n, x_m are uncorrelated then x_n, x_m also become independent.

26/09/'18

random process $x(t) = \sum_{n=-\infty}^{\infty} x_n \, \phi_n(t)$; $x_n = \int_{-\infty}^{\infty} x(t)\, \phi_n(t)\, dt$

r.v., deterministic

$x(t)$ is w.s.s.

$R_x(i,j) = R_x(i-j)$

Find $\{\phi_n(t)\}$ Kahruen-Loeve (KL) set such that x_n s are uncorrelated

when $x(t)$ is a Gaussian random process :

then x_n s are Gaussian

then when x_n s are uncorrelated : x_n s are also independent (i.e. Joint p.d.f = product of marginal p.d.f.s)

$$E[x_i \, x_j] = E\left[\int_{-\infty}^{\infty} x(t)\, \phi_i(t)\, dt \int_{-\infty}^{\infty} x(u)\, \phi_j(u)\, du \right]$$

$$= \iint_{-\infty}^{\infty} E[x(t)\, x(u)]\, \phi_i(t)\, \phi_j(u)\, dt\, du$$

$$= \iint_{-\infty}^{\infty} R_x(t-u)\, \phi_i(t)\, \phi_j(u)\, dt\, du$$

(for x_i , x_j to be uncorrelated : $E[x_i x_j] = \lambda_j \delta_{ij}$)

$$= \int_{-\infty}^{\infty} \phi_i(t) \int_{-\infty}^{\infty} R_x(t-u)\, \phi_j(u)\, du \;\; dt$$

say, $$\boxed{\int_{-\infty}^{\infty} R_X(t-u)\, \phi_j(u)\, du = \lambda_j\, \phi_j(t)} \quad (1)$$

then:

$$= \int_{-\infty}^{\infty} \phi_i(t)\ \lambda_j\, \phi_j(t)\ dt$$

$$= \lambda_j\, \delta_{ij}$$

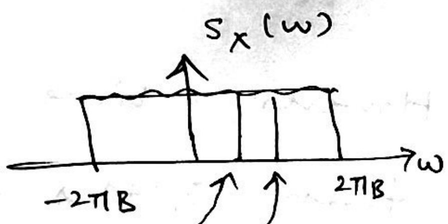
$\therefore$ Need to solve eq. (1) (Freedholm integral equation of first kind)

to obtain $\{\phi_n(t)\}$ set such that x_ns are uncorrelated

---

$x(t)$ is bandlimited

$\therefore$ $S_X(\omega)$ is bandlimited



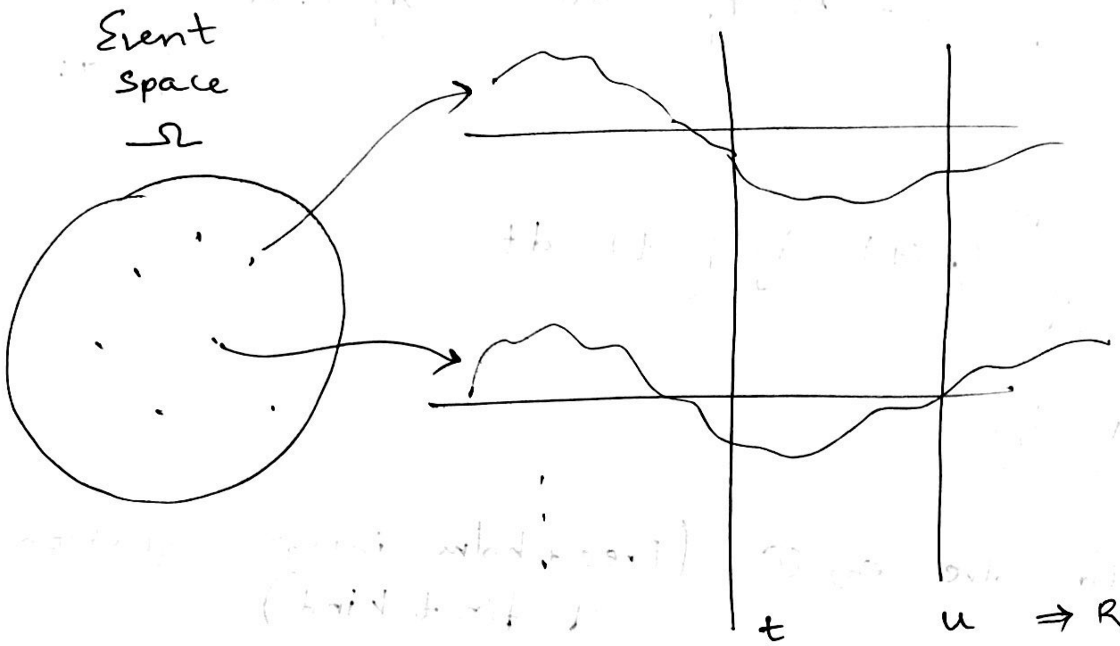
$A_1 \sin(\omega_1 t + \theta)$

$A_2 \sin(\omega_2 t + \theta)$

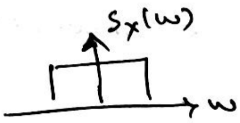
$\vdots$

$A_n \sin(\omega_n t + \theta)$

deterministic (A_n) Random (ω_n, θ)



$\Rightarrow R_x(t-u) \Rightarrow S_x(\omega)$

Say 

$\Rightarrow$ $R_x(\tau)$ is sinc( )

Then $\{\phi_n(t)\}$ are sinc( ) functions but $\int_{-\infty}^{\infty}$ is required

i.e. not time limited

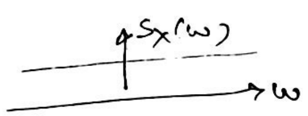
However, time limited r.p. & band limited r.p.

then $\phi_n(t)$ s are prolate spheroidal waveforms

(Slepian from Bell Labs)

Say when $R_x(t-u) = \lambda u_0(t-u)$ white process

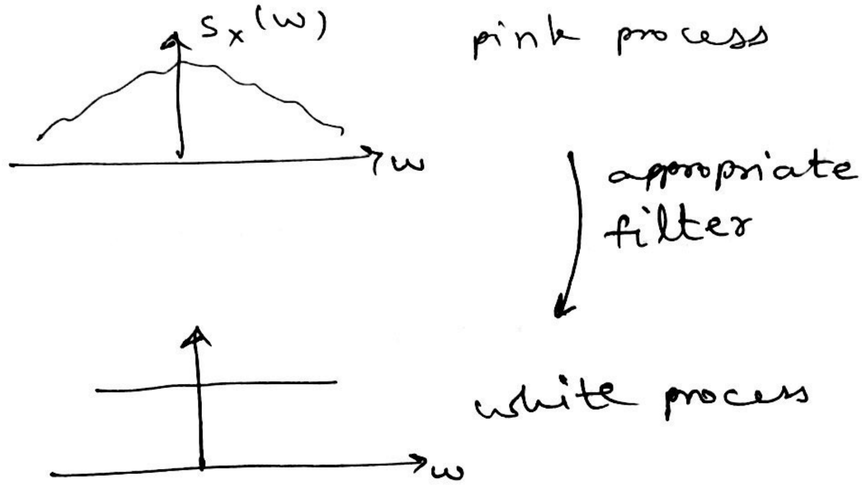
(otherwise non-white / pink process)



$\therefore$ for white processes !

$$\int_{-\infty}^{\infty} \lambda u_0(t-u)\, \phi_j(u)\, du = \lambda_j\, \phi_j(t)$$

& irrespective of $\{\phi_n(t)\}$ (whichever kind) above eq. is satisfied

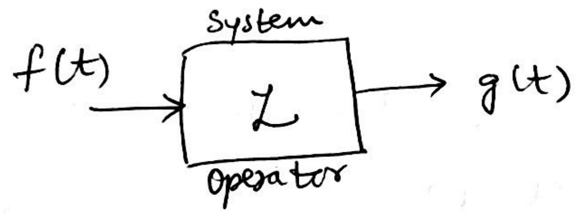


28 Sept.

Systems

Guillemin
(Network Theory)
↓
Weiner
(System Theory)

Zadeh
(State-space
Fuzzy)



$$\mathcal{L}\left[f(\cdot)\right] = g(\cdot)$$

- Linear / non-linear system / operator
- Time-invariant / Time varying "
- Deterministic / Random "

Solutions to non-linear systems problems posed in general sense do not have closed form solutions

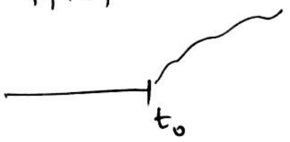
specific problems (specific i/p s & systems)
may be solved using computers

Linear : $a f_1(t) + b f_2(t) \rightarrow a g_1(t) + b g_2(t)$

a, b may be irrational #s in general

Time Invariant :

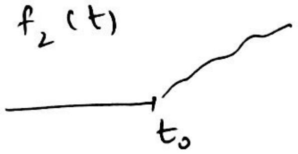
$f_1(t)$



Response is same for both cases

for $t < t_0$

$f_2(t)$



Impulse Response: $u_0(t) \rightarrow \boxed{\mathcal{L}} \rightarrow h(t)$

(LTI systems)

$u_0(t-\tau) \longrightarrow h(t-\tau)$ ($\because$ Time Invariant)

$f(\tau)\, u_0(t-\tau) \longrightarrow f(\tau)\, h(t-\tau)$

$\int_{-\infty}^{\infty} f(\tau)\, u_0(t-\tau)\, d\tau \longrightarrow \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$ ($\because$ linear)

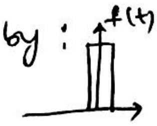
$\therefore \quad f(t) \rightarrow \boxed{\mathcal{L}} \rightarrow g(t) = f(t) * h(t)$

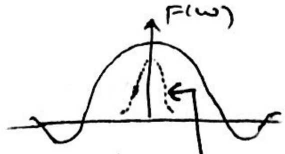
$G(\omega) = H(\omega)\, F(\omega)$

Apply Impulse and find h(t)

Apply $\sin \omega t \longrightarrow H(\omega)$

[ideal impulse generator approximated by :





as long as $H(\omega)$ is narrow enough

h(t) : Impulse Response

$\updownarrow$

$H(j\omega)$: Transfer function

$$\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau$$

when $h(t) \neq 0$ $t<0$

then to compute $g(t_0)$, we need $f(t)$ for $t > t_0$

Causality also demands poles in LHP for $H(s)$

$(h(t) = 0$ for $t < 0)$

$H(s)$ converges in RHP, poles in LHP $\Rightarrow$ $h(t) = 0$, $t<0$

$$\int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau$$

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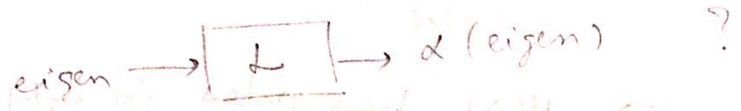
Causality also demands poles in LHP for $H(s)$

$(h(t)=0$ for $t<0)$

$H(s)$ converges in RHP, poles in LHP ? $\Rightarrow$ $h(t)=0$, $t<0$



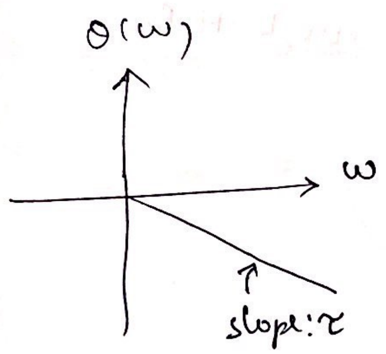
9 oct. Continuous systems

$h(t)$: Impulse response $\longleftrightarrow$ $H(s)$: Transfer function

$g(t) = h(t) * f(t)$

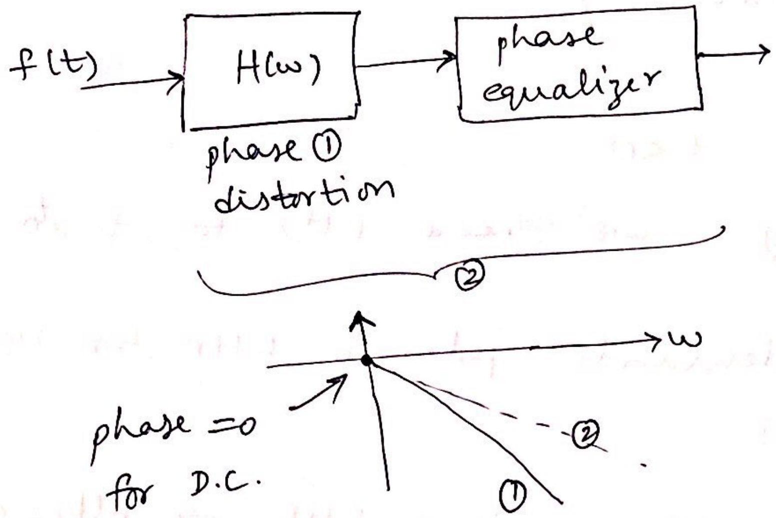
$G(s) = H(s) F(s)$

$H(j\omega)$: frequency response $= \underbrace{|H(j\omega)|}_{\text{amplitude response}} e^{j\overbrace{\theta(\omega)}^{\text{phase response}}}$



$$G(j\omega) = 1 . e^{-j\omega\tau} F(j\omega)$$

$$\therefore g(t) = f(t-\tau)$$



$h(t)$: causal $\Rightarrow$ $H(s)$ has poles in L.H.P.

$\Downarrow$

$h(t) = 0$ for $t < 0$

$h(t)$: stable $\Rightarrow$ $H(s)$ has poles in L.H.P.

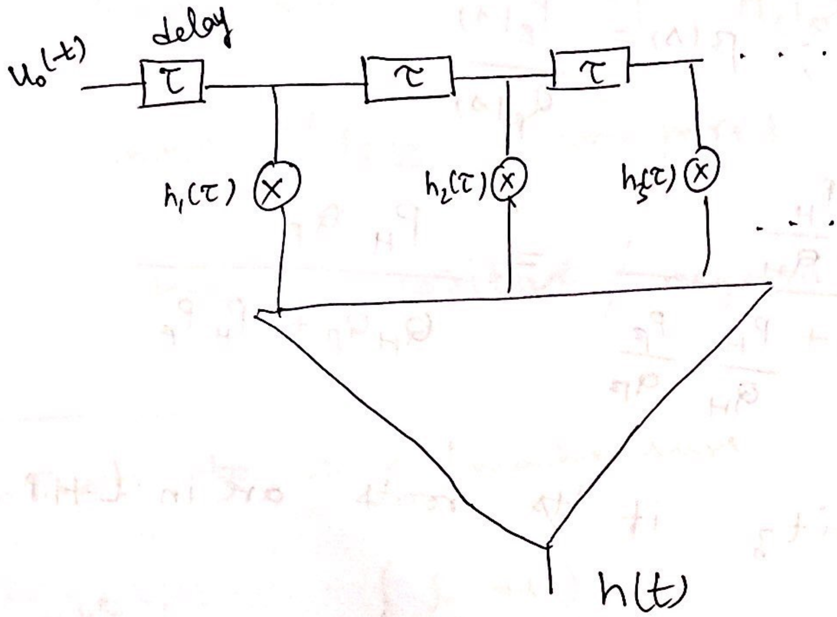
$\Downarrow$

$h(t) \to 0$ as $t \to \infty$

$$H(s) = \frac{P(s)}{Q(s)} = \sum \frac{k_i}{s+s_i} \longleftrightarrow h(t) = \sum k_i e^{s_i t}$$

$h(t)$: passive $\Rightarrow$ $H(s)$ has poles in L.H.P.

Transversal Filter



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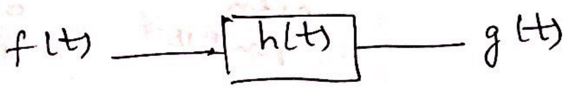
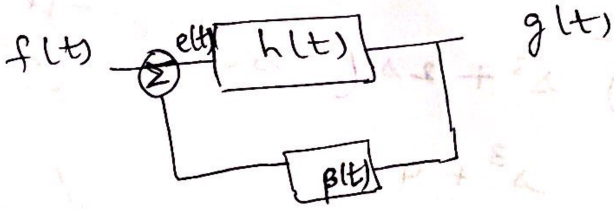
open loop system :



$H(s)$: open loop transfer function

closed loop system :
(feedback)

1940s
Black
Bell labs



$$G(s) = H(s)\,E(s) = H(s)\left(F(s) - \beta(s)\,G(s)\right)$$

$$\therefore\ G(s) = \left(\frac{H(s)}{\underbrace{1+\beta(s)\,H(s)}}\right) F(s)$$

closed loop transfer function

Even if $H(s)$ is stable, $\dfrac{H(s)}{1+\beta(s)H(s)}$ may not be stable

$$H(s) = \frac{P_H(s)}{Q_H(s)} \quad ; \quad \beta(s) = \frac{P_\beta(s)}{Q_\beta(s)}$$

$$\therefore H_{\text{closed loop}}(s) = \frac{\frac{P_H}{Q_H}}{1+\frac{P_H}{Q_H}\frac{P_\beta}{Q_\beta}} = \frac{P_H Q_\beta}{Q_H Q_\beta + P_H P_\beta}$$

$Q(s)$ is Hurwitz if its roots are in L.H.P.

Routh - Hurwitz $\longleftrightarrow$ Hurwitz test

$$s^4 + s^3 + 3s^2 + 2s + 4 = \underset{\text{even powers}}{m(s)} + \underset{\text{odd powers}}{n(s)}$$

$$
\begin{array}{l}
s^3+2s \,)\, s^4 + 3s^2 + 4 \,(\, s \\
\quad\quad\quad\;\; \underline{s^4 + 2s^2} \\
\quad\quad\quad\quad s^2+4 \,)\, s^3 + 2s \,(\, s \\
\quad\quad\quad\quad\quad\quad\quad\;\; \underline{s^3 + 4s} \\
\quad\quad\quad\quad\quad\quad\quad\; -2s \,)\, s^2 + 4 \,(\, -\tfrac{1}{2}s \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; \underline{s^2} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; 4 \,)\, -2s \,(\, -\tfrac{1}{2}s \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; \underline{2s} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

both +ve & -ve coeff.s

$\therefore$ pole(s) in R.H.P.

Gain associated w/ $H_{closed\ loop}(s)$ is

more / less compared to $H(s)$

then positive / negative feedback

Test to see / make sure $\frac{P_H}{Q_H} \frac{P_\beta}{Q_\beta}$ is not

-ve. (Bode)

$H(s) = K \frac{P_H}{Q_H}$

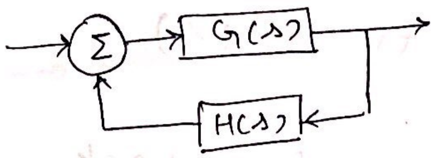
movement of poles with variations in K

Root locus

Nyquist plot

11 Oct.

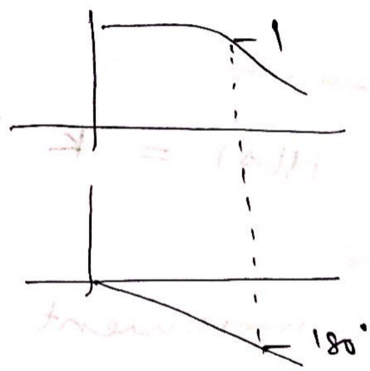
Classical control theory



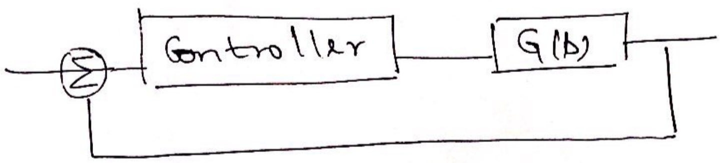
closed loop Gain : $\dfrac{G(s)}{1+G(s)H(s)}$

- Most times $H(s)$ is constant (resistive feedback)
- Even when $G(s), H(s)$ are stable, still need to check whether $1+G(s)H(s)$ is stable.
- Hurwitz test (similar to Routz-Hurwitz test)
- Bode plot : $G(s)H(s)\big|_{s=j\omega} \overset{?}{=} -1$

In such a case :
since $G(s)$ is generally fixed
$H(s)$ is designed to avoid it



Another system form :



Gain margin

$|GH| \geq 1 \pm \Delta$

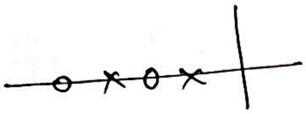
Phase margin

$\angle GH \geq 180° \pm \Delta$

$$G(s)H(s)\Big|_{s=j\omega} = \underset{\text{loop gain}}{L(s)}\Big|_{s=j\omega} = \frac{P(s)}{Q(s)}\Big|_{s=j\omega}$$

$$\frac{P(s)}{Q(s)} = \frac{(s-s_{z_1})(s-s_{z_2})\cdots}{(s-s_{p_1})(s-s_{p_2})\cdots}$$

say s_{z_i} , s_{p_i} are on −ve real axis

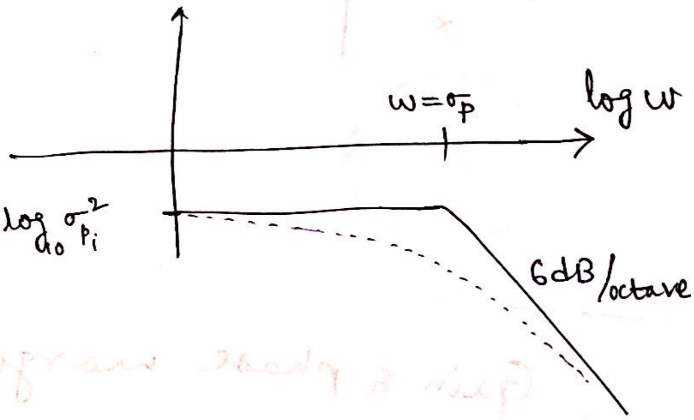


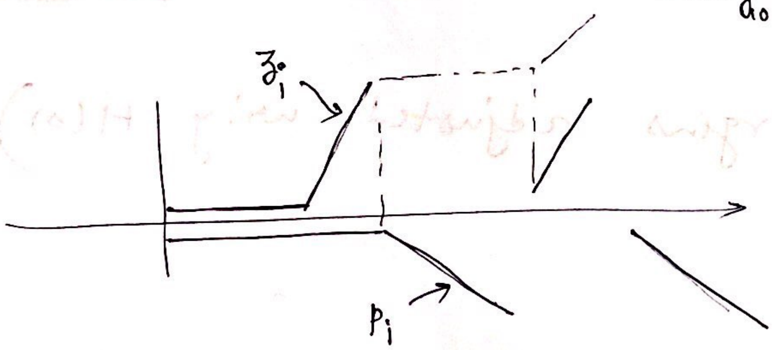
$$20 \log_{10} \left| \frac{s - s_{z_1}}{s - s_{p_i}} \right|_{s=j\omega}$$

say $Re\{s_{p_i}\} = -\sigma_{p_i}$

$$-20 \log_{10} | j\omega + \sigma_{p_i} | = -20 \log_{10} (\omega^2 + \sigma_{p_i}^2)^{1/2}$$

$$= -10 \log_{10} (\omega^2 + \sigma_{p_i}^2)$$





Similar plot for phase:

small "ω" $\phi \approx 0°$

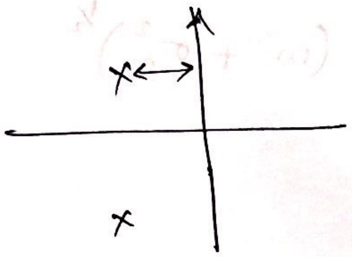
$\omega \approx \sigma_{p_i}$ $\phi \approx 45°$

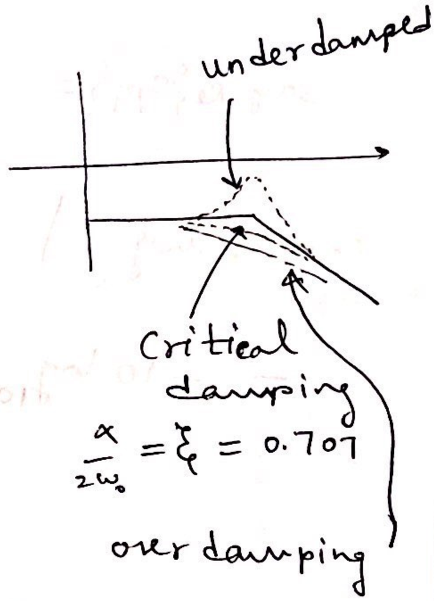
large "ω" $\phi \approx 90°$

$s^2 + as + b$

$|-\omega^2 + j\omega a + b|$

$\sqrt{(b-\omega^2)^2 + \omega^2 a^2}$





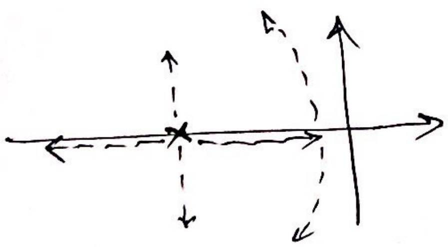
(Gain & phase margins adjusted using $H(s)$)

<u>Root Locus</u>

$G(s) = \cancel{k} \; g(s)$

close loop gain: $\dfrac{k\,g(s)}{1 + k\,g(s)H(s)}$

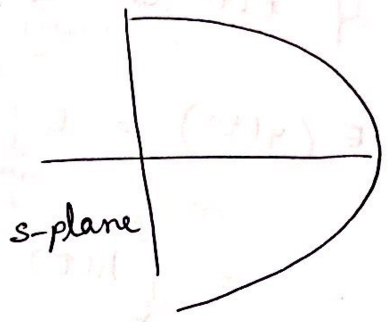
w/ fluctuation in physical parameters) say gain "k", is the system going to remain stable? Root locus answers this

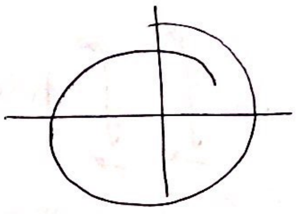


Nyquist plot

Q. Does $(1+G(s)H(s))$ have roots in R.H.P.

Consider a contour in R.H.P. enclosing whole of RHP.





12 Oct

deterministic

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t) = h(t) * x(t)$$

If $x(t)$ is w.s.s., is $y(t)$ w.s.s.?

$$E(y(t)) = E\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, E(x(t-\tau))\, d\tau = \bar{x} \int_{-\infty}^{\infty} h(\tau)\, d\tau \quad ✓$$

$$E(y(t)\, y(t+\tau)) = E\left[\int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha \int_{-\infty}^{\infty} h(\beta)\, x(t+\tau-\beta)\, d\beta\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)\, h(\beta)\, R_{xx}(\tau+\alpha-\beta)\, d\alpha\, d\beta = R_{yy}(\tau) \quad ✓$$

$\therefore$ $y(t)$ is also w.s.s.

& if $x(t)$ is zero mean then $y(t)$ is also zero mean

also if $x(t)$ is Gaussian then $y(t)$ is also Gaussian

($\because$ sum of Gaussian processes is also Gaussian and integration is like summation (in limit))

$$\therefore E(y(t)\, y(t+\tau)) = \frac{1}{2\pi} \iiint_{-\infty}^{\infty} h(\alpha)\, h(\beta)\, S_{xx}(\omega)\, e^{j\omega(\tau+\alpha-\beta)}\, d\omega\, d\alpha\, d\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(-\omega)\, H(\omega)\, S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\therefore \quad S_{yy}(\omega) = S_{xx}(\omega)\, |H(j\omega)|^2$$

$$\left(\because \quad H(-\omega) = H^*(\omega) \quad \text{for real } h(t)\right)$$

$$\therefore \quad R_{yy}(\tau) = R_{xx}(\tau) * \mathcal{F}^{-1}\{|H(j\omega)|^2\}$$

If $x(t)$ is w.s.s. and also a white process:

i.e. $R_{xx}(\tau) = K\, u_0(\tau)$

$$S_{xx}(\omega) = K$$

then $S_{yy}(\omega) = K\,|H(j\omega)|^2$

$\therefore$ Any random process of a given $R_{yy}(\tau)$ or $S_{yy}(\omega)$ may be generated from a w.s.s.

Wiener

$$S_{xx}(\omega) \quad \xrightarrow[\text{white process}]{\text{w.s.s.}} \boxed{} \longrightarrow |H(j\omega)|^2 = S_{yy}(\omega)$$

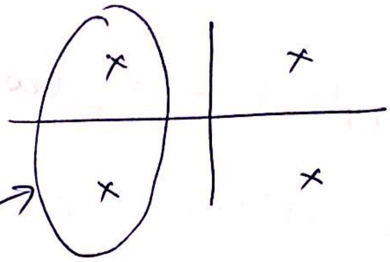
$|H(j\omega)|^2$ needs to be realizable

$$H(s) = \frac{P(s)}{Q(s)} \quad \Rightarrow \quad |H(j\omega)|^2 = \left.\frac{P(s)\,P(-s)}{Q(s)\,Q(-s)}\right|_{s^2=-\omega^2}$$

$S_{yy}(\omega)$ is always +ve and <u>even</u>

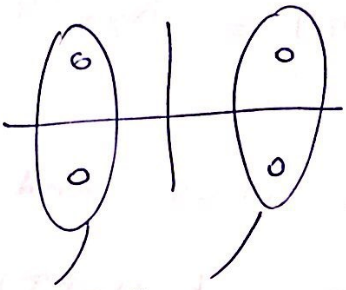
$$\therefore S_{yy}(\omega) = \frac{A(\omega^2)}{B(\omega^2)} = \frac{A(-s^2)}{B(-s^2)}$$

Quadrental symmetry



for $Q(s)$

A



or

for $P(s)$

If we pick $P(s)$ roots also in L.H.P.

then such $\frac{P}{Q}$ is called minimum phase T.F.

$\therefore$ $P(s)$ in R.H.P. has +ve phase adding to phase of $Q(s)$

Conversely pink noise may be converted to white noise by an appropriate $H(j\omega)$

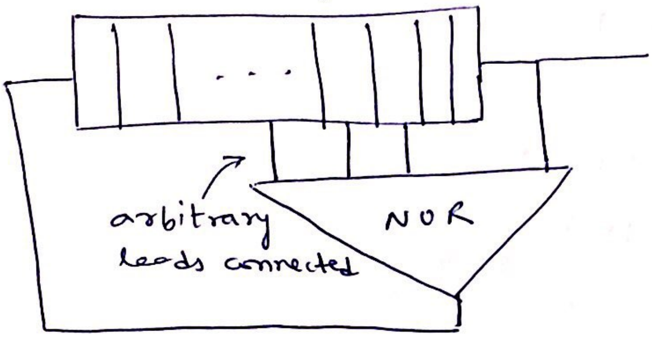
Generation of white noise :

most random processes in nature are white

- voltage across an unbiased resistor
- o/p of a transistor without any i/p signal
- Generate a deterministic signal which satisfies properties of a white W.S.S.

i.e. Pseudo Noise (PN) sequence

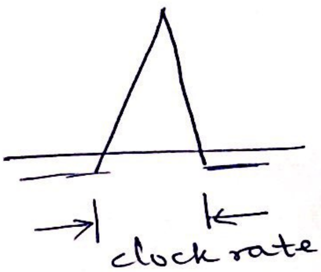
shift register (of length "n")

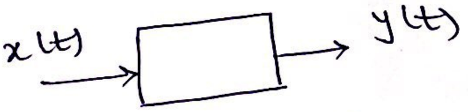


PN (or) shift register sequence

repetition rate of sequence is: $2^n - 1$

Autocorrelation of PN sequence





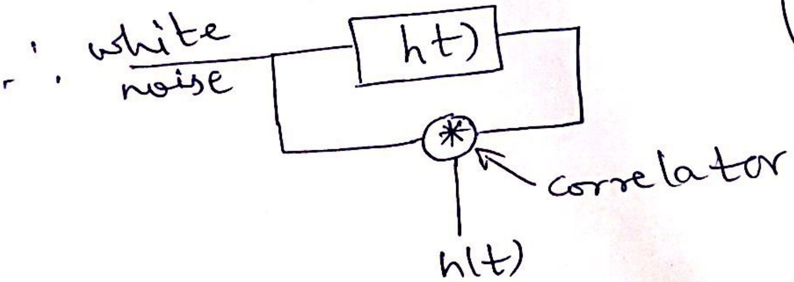
$$R_{YX} = E\left(X(t+\tau) \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha \right)$$

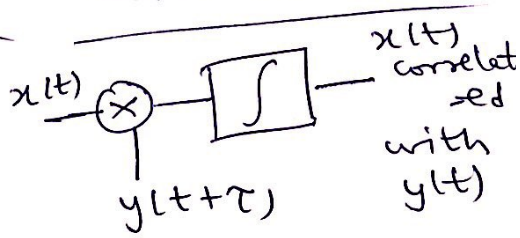
$$= \int_{-\infty}^{\infty} R_{XX}(\tau+\alpha)\, h(\alpha)\, d\alpha$$

$x(t)$ is white noise

$\therefore R_{YX}(\tau) = h(\tau)$

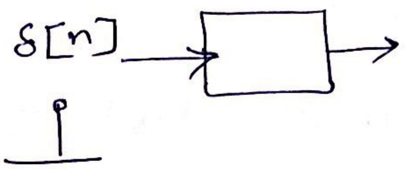
when $R_{XX}(\tau)$ is delta function





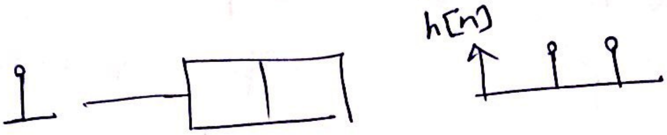
$\therefore$ any unknown $h(t)$ may be found by inputting white noise to it & using a correlator

Discrete systems :



Impulse response of discrete system
h[n]

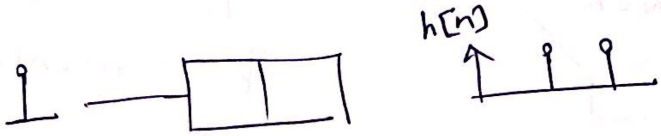
Consider a shift register :



Discrete systems :

$\delta[n] \longrightarrow \boxed{} \longrightarrow$ Impulse response of discrete system $h[n]$

Consider a shift register :



17 Oct.

$$x[n] \longrightarrow \underset{\text{LTI}}{\boxed{h[n]}} \longrightarrow y[n]$$

$\delta[n]$ $\qquad$ $h[n]$ $\qquad$ Impulse response

$\delta[n-k]$ $\qquad$ $h[n-k]$

$\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$ $\qquad$ $\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

$x[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Causal LTI system : $h[k] = 0$ for $k<0$

say $x[n]$ is also causal : $x[n] = 0$ for $n<0$

$$\therefore\ y[n] = \sum_{k=0}^{n} h[k]\,x[n-k]$$

Convolution

linear convolution

cyclic convolution (periodic) i.e. for periodic signals.

For BIBO stable systems

$h[n]$:

Infinite Impulse Response (IIR) system

$n \in (-\infty, \infty)$

Finite Impulse Response (FIR) system

$n \in [k_1, k_2]$

k_1, k_2 : finite

$$|y[n]| = \left|\sum_k h[k]\, x[n-k]\right| \leq \left|\sum h[k]\right| \left|\sum x[n-k]\right|$$

bounded

$\therefore \left|\sum h[k]\right| < \infty$ is required

IIR

AR (Auto Regressor) systems

FIR

MA (Moving Average) systems

For MA systems : $y[n] = a_0 x[n] + a_1 x[n-1] + \cdots$

For AR systems : $b_0 y[n] + b_1 y[n-1] + \cdots = x[n]$

ARMA system : $a_0 x[n] + a_1 x[n-1] + \cdots = b_0 y[n] + b_1 y[n-1] + \cdots$

Z-transform :

$$a_0 X(z) + a_1 X(z) z^{-1} + \cdots + a_m X(z) z^{-m}$$
$$= b_0 Y(z) + b_1 Y(z) z^{-1} + \cdots + b_n Y(z) z^{-n}$$

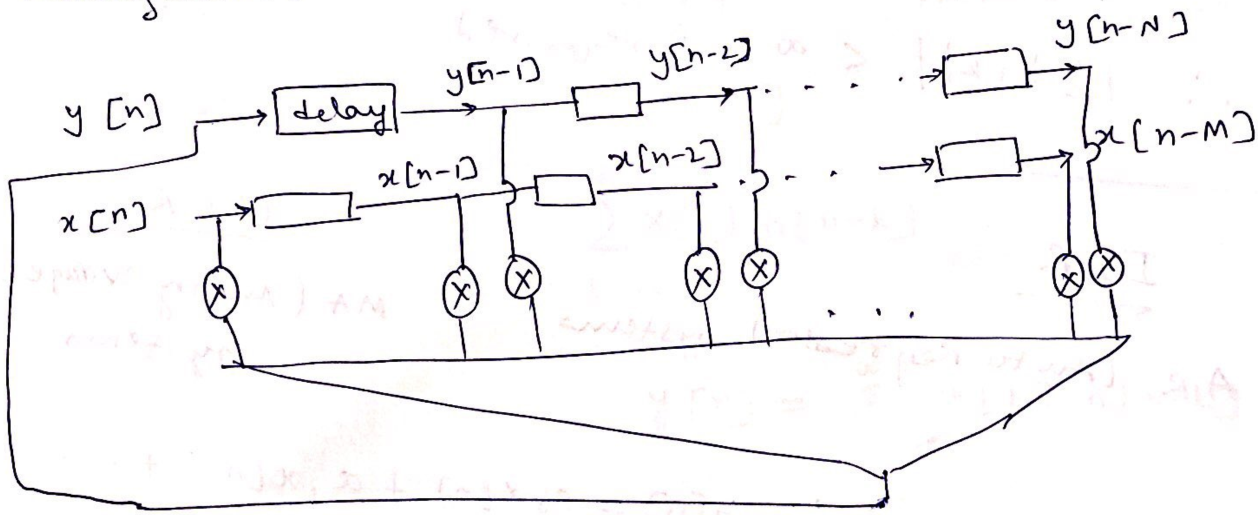
$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}}{b_0 + b_1 z^{-1} + \cdots + a_N z^{-N}}$$ Discrete Transfer function

Say: $H(z) = \frac{1}{1 - 1/z}$ AR system

zeros from MA (numerator)

poles from AR (denominator)

Realization:



$H(z) \rightarrow H(e^{j\omega})$ repeats Freq. response

$x[n] \rightarrow \boxed{LTI} \rightarrow y[n]$

WSS process

WSS ?

$E(y[n]) = \text{constant}$

$E\{Y[n+m]\,Y[m]\})$

$= E\left(\sum_k h[k]\, x[n+m-k] \sum_j h[j]\, x[n+j] \right)$

$= R_{YY}(m)$

(Fokker - Plank technique exists to find joint probability density of $Y[n]$ from joint pdf of $x[n]$)

$R_{YY}(m)$ need not be +ve $\forall$ m

$R_{YY}(0)$ is finite ($\because$ power is finite)
& positive ?

$R_{YY}(m) \leq R_{YY}(0)$ (from Schwartz inequality)

$S_{YY}(w)$ is always positive

$\int S_{YY}(w)\, dw = R_{YY}(0)$

30 Oct.

State Space

(LTI system)

linear system

TI system

(A, B are not functions of time)

(C, D also time variant)

State eq.: $\underset{n\times1}{\dot{X}} = \underset{n\times n}{A}\,\underset{n\times1}{X} + \underset{n\times p}{B}\,\underset{p\times1}{U}$

matrix Ricatti eq.

Output eq.: $\underset{r\times1}{Y} = \underset{r\times n}{C}\,\underset{n\times1}{X} + \underset{r\times p}{D}\,\underset{p\times1}{U}$

Def. of state variables:

A min. set of variables (X) in a system given at time "t_0" $X(t_0)$ and input $u(t)$, $t > t_0$ sufficient to find the complete dynamics of the system uniquely for $t \geq t_0$

For a non-LTI system:

state eq.: $\dot{X} = f(X, u, t)$

so possibly time-varying

may be non linear function

Consider an LTI system: $\ddot{y} + a\dot{y} + by = u$

$y = x_1$; $x_2 = \dot{x}_1 = \dot{y}$

$$\therefore \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_2 - b x_1 + u \end{cases}$$

$$x(t) = \underbrace{\overbrace{e^{At} x(t_0)}^{\text{zero-i/p-response}}}_{\substack{\text{response}\\ \text{due to}\\ \text{i.c.s}}} + \underbrace{\overbrace{\int_{t_0}^{t} e^{A(t-\tau)} B\, U(\tau)\, d\tau}^{\text{zero-state-response}}}_{\substack{\text{convolution integral}\\ \text{response due to i/ps}}}$$

Superposition

State Transition matrix $\phi(t)$: e^{At}

$$e^{At} = I + \frac{A}{1!}t + \frac{A^2}{2!}t^2 + \dots + \frac{A^n}{n!}t^n + \dots$$

To find e^{At} :

$x \rightarrow MZ$

linear transformation matrix

$\therefore M\dot{Z} = AMZ + BU$

$\dot{Z} = M^{-1}AMZ + M^{-1}BU$

$M^{-1}AM = D \Rightarrow A = MDM^{-1}$

$\therefore e^{At} = M e^{Dt} M^{-1}$

Another approach to find e^{At} :

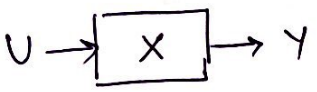
consider : $\dot{X} = AX$

$|\lambda I - A| = 0$ (characteristic eq. : $Q(\lambda) = 0$)

By Cayley-Hamilton theorem : $Q(A) = 0$

(Kalman)

<u>Observability of System</u>



Can $x(t)$ be found uniquely from "$Y(t)$" ?
(with $U = 0$)

i.e. given $Y(t)$, can we find $x(t_0)$?

$\because$ $x(t)$ may be found from $x(t_0)$ (& $v(t)$)

<u>31 Oct.</u>

$x(t_0)$

$y(t) = \sum k_i e^{s_i t}$

when pole-zero cancellation certain s_i's may not show up in $y(t)$. Then all states are not observable thro' $y(t)$.

Controllability

State-feedback

complete control over placement of poles ∵ full-state is known

output feedback

Zero-input response: $e^{At} x(t_0)$

$$\int_{t_0}^{t} e^{A(t-\tau)} x(t_0)\, d\tau = \int_{t_0}^{t} e^{At} x(\cancel{t-t_0})\, d\tau$$

t_0

$$\dot{x} = Ax + BU$$

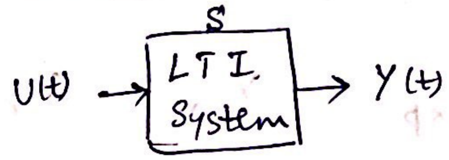
Laplace transform

$$sX(s) - X(0) = A\,X(s) + B\,U(s)$$

$$\therefore X(s) = (sI - A)^{-1}\left(X(0) + B\,U(s)\right)$$

$$\therefore e^{At} = \mathcal{L}^{-1}\left((sI-A)^{-1}\right)$$

02 Nov.



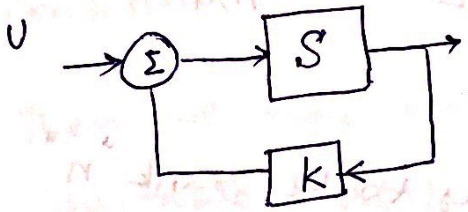
$\dot{X} = AX + BU$

$Y = CX + DU$

$|SI - A| = 0$: characteristic function $\Phi(s)$

roots : S_i : poles : eigen values

State feedback :



Observability :

$Y \to X$

depends on matrices A, B, C, D

Controllability

Given X, can system be stabilized thro' K.

∵ unlike in most circuits systems in general are not amenable to modifying "S". ∴ Observability, Controllability conditions are useful. } Desoer

---

Knowing $Y(t) \longrightarrow$ find $\underset{\sim}{X}(t) \longrightarrow$ then $\underset{\sim}{X}(t_0)$

$$Y(t_0) = C X(t_0)$$

$$Y^{(1)}(t_0) = C X^{(1)}(t_0) = C A X(t_0)$$

$$Y^{(2)}(t_0) = C X^{(2)}(t_0) = C A^2 X(t_0)$$

$$\vdots$$

$$Y^{(n-1)}(t_0) = C X^{(n-1)}(t_0) = C A^{n-1} X(t_0)$$

$X_{n\times 1}$ $\quad A_{n\times n}$ $\quad B_{n\times p}$ $\quad Y_{r\times 1}$ $\quad C_{r\times n}$

$$\begin{pmatrix} Y(t_0) \\ Y^{(1)}(t_0) \\ \vdots \\ Y^{(n-1)}(t_0) \end{pmatrix}_{rn\times 1} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{rn\times n}}_{\text{This matrix has to be rank "n"}} \begin{bmatrix} X(t_0) \end{bmatrix}_{n\times 1}$$

Kalman Observability :

Controllability

$$X(t) = e^{A(t-t_0)} X(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$x(t_0)$ $\quad$ $X(t_1)$

does $u(t)$ exist to go from $X(t_0)$ to $X(t_1)$

$$e^{-At} X(t) = e^{-At_0} X(t_0) + \int_{t_0}^{t} e^{-A\tau} B\, u(\tau)\, d\tau$$

By Cayley-Hamilton thr. :

$$e^{-At} X(t) = \left(\sum_{i=0}^{n-1} \alpha_i(\tau) A^i\right) X(t)$$

$$\therefore\ e^{-At} X(t) = e^{-At_0} X(t_0) + \sum_{i=0}^{n-1} A^i \int_{t_0}^{t} \alpha_i(\tau) B\, u(\tau)\, d\tau$$

say scalars

$$= e^{-At_0} X(t_0) + \sum_{i=0}^{n-1} A^i B \int_{t_0}^{t} \alpha_i(\tau)\, u(\tau)\, d\tau$$

$$\begin{bmatrix} \\ \\ \vdots & \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} B & \vdots & AB & \vdots & A^2B & \vdots & \cdots & A^{n-1}B \end{bmatrix}}_{n \times n} \begin{bmatrix} \int_{t_0}^{t} \alpha_0(\tau)\, u(\tau)\, d\tau \\ \vdots \\ \int_{t_0}^{t} \alpha_{n-1}(\tau)\, u(\tau)\, d\tau \end{bmatrix}$$

Controllability condition { This matrix has to be rank "n"

Then we may invert ↑this matrix to find a suitable $u(t)$ to control $x(t)$

Example:

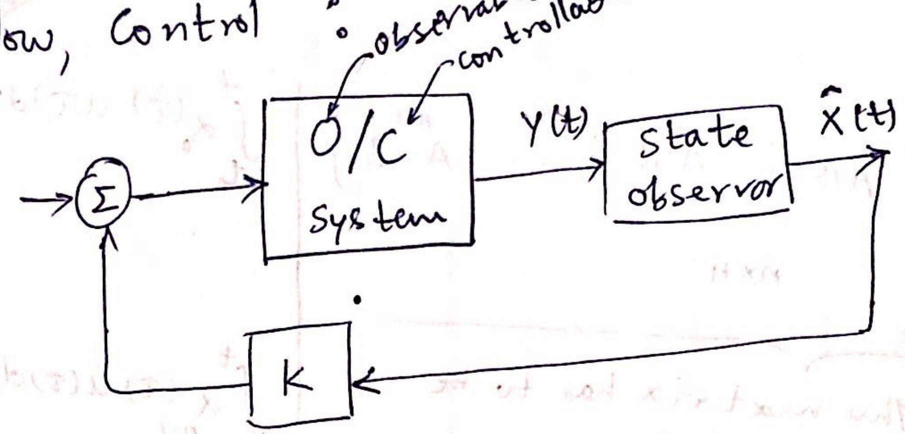
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u$$

$$y = \underset{C}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} C \\ \hline CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \hline 1 & 1 \end{bmatrix} \quad \therefore \det \begin{bmatrix} C \\ \hline CA \end{bmatrix} \neq 0 \Rightarrow \text{observable}$$

$$\begin{bmatrix} B & \vdots & AB \end{bmatrix} = \begin{bmatrix} 0 & \vdots & 1 \\ 1 & \vdots & -1 \end{bmatrix} \quad \therefore \det \begin{bmatrix} B & \vdots & AB \end{bmatrix} \neq 0 \Rightarrow \text{controllable}$$

Now, Control :



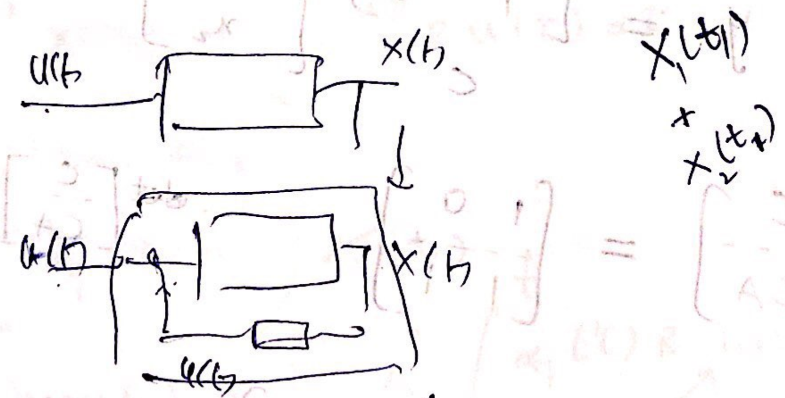
New system's poles : $|sI - \hat{A}| = 0$

($u(t)$ affects $Y(t)$ & ~~not~~ $X(t)$ but not A, B)

For real systems : we have to further estimate A, B before observability, controllability.

Adaptive ~~System~~ control, stochastic control, Optimal control,

performance measure



$\hat{x} = Ax_0 + Bu$

$\hat{x} = Ax + B[u + Kx(t)]$

$=$

13 Nov.

ARMA :

$$\sum_{k=0}^{N} b_k \, y(n-k) = \sum_{k=0}^{M} a_k \, x(n-k)$$

AR MA

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}$$

denominator order

$(M \leq N)$

numerator order

M=N : State-space representation for discrete system

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \\ \vdots \\ x_N(n+1) \end{bmatrix} = A \begin{bmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_N(n) \end{bmatrix} + B \begin{bmatrix} u_1(n) \\ \vdots \\ u_r(n) \end{bmatrix}$$

$$Y(n) = C\, X(n) + D\, U(n)$$

Consider : $u(n)$ is $|x|$

$[x(n)]$ may be replaced by $[x'(n)]$ where invertible

(A) $[A']$ are obtained thro' a linear transformation

$\therefore$ $[A]$ and $[A']$ have N same eigen values.

Control Canonical form:

$$\begin{bmatrix} X_1(n+1) \\ \vdots \\ X_N(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & & & \ddots & 1 \\ -b_N & \cdots & & & -b_1 \end{bmatrix} \begin{bmatrix} X_1(n) \\ \vdots \\ X_N(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{N\times 1} u(n)$$

consider $[u(n)]$ is an input vector ~~1×1~~ of size 1×1

$$H(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}} \qquad M \leq N$$

$$\frac{\overbrace{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}^{N(z)}}{\underbrace{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}_{D(z)}} = \frac{Y(z)}{U(z)}$$

$$W(z) = \frac{U(z)}{D(z)}$$

$$Y(z) = N(z)\, W(z)$$

$$\therefore u(n) = z^{-1}\left[D(z)\; W(z) \right]$$

$$u(n) = b_0 W(n) + b_1 W(n-1) + \cdots + b_N W(n-N) \qquad \because U(z) = D(z)W(z)$$

$$Y(n) = a_0 W(n) + a_1 W(n-1) + \cdots + a_N W(n-N) \qquad \because Y(z) = N(z)W(z)$$

by choosing

$$W(n) = X_N(n+1)$$

$$W(n-1) = X_N(n)$$

$$W(n-2) = X_{N-1}(n)$$

$$\vdots$$

$$W(n-N) = X_1(n)$$

Given any controllable "A"
$\exists$ a linear invertible transformation "T"
that gives us an "A'" which is of
control canonical form.

example : $A = \begin{bmatrix} 0 & 1 \\ -b_2 & -b_1 \end{bmatrix}$

$$|zI - A| = \begin{vmatrix} z & -1 \\ b_2 & z+b_1 \end{vmatrix} = \underbrace{z(z+b_1) + b_2}_{D(z)}$$

roots of $D(z)$ inside unit circle $\Rightarrow$ system stable

$$y(n) = \left[(a_N - a_0 b_N) \;\; (a_{N-1} - a_0 b_{N-1}) \cdots (a_1 - a_0 b_1)\right] \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} + a_0 u(n)$$

① Stability
③ Controllability
② observability

① $|zI - A| = 0$ roots, z_is inside unit circle

② $\Theta = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix}$ ~~$Np \times N$~~ $Np \times N$

Θ must be of rank N
p : # of observations $[Y]_{p \times 1}$

③ $G = [B \vdots AB \vdots \cdots A^{N-1}B]_{N \times Nr}$ is rank N

r : # of inputs $[U]_{r \times 1}$

---

example : $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\frac{|zI - A|}{z^2} = \frac{z^2 + 3z + 2}{z^2} \Rightarrow z_i^{-1}s = -1, -\frac{1}{2}$$

$z_i s : -1, -2$

$\Theta = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$ rank 2 (observable)

$G' = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ 4 (controllable)

$$X(n+1) = AX(n) + BU(n)$$

$Z\{\}$ on both sides :

$$zX(z) - zX(0) = AX(z) + BU(z)$$

$$(zI - A)X(z) = zX(0) + BU(z)$$

$$X(z) = (zI-A)^{-1} zX(0) + (zI-A)^{-1}BU(z)$$

$$\therefore X(n) = Z^{-1}\{(zI-A)^{-1}z\} X(0) + Z^{-1}\{\quad\}$$

$\therefore\ Z^{-1}\{(zI-A)^{-1}z\}$ is the state transition matrix for discrete systems

$$x(n+1) = A\,x(n) + B\,u(n)$$

$$x(1) = Ax(0) + B\,u(0)$$

$$x(2) = A\,x(1) + B\,u(1)$$

$$= A^2x(0) + AB\,u(0) + B\,u(1)$$

$$x(3) = A^3x(0) + A^2B\,u(0) + AB\,u(1) + B\,u(2)$$

$$\therefore\ Z^{-1}\{(zI-A)^{-1}z\} = A^N$$

$\because M \le N$, without loss of generality consider $M=N$

$$\frac{Y(z)}{U(z)} = \frac{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}}$$

let : $$\frac{W(z)}{U(z)} = \frac{1}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}}$$

$$\therefore W(z) + b_1 z^{-1} W(z) + \cdots + b_N z^{-N} W(z) = U(z)$$

$$W(z) = -b_N z^{-N} W(z) - b_{N-1} z^{-(N-1)} W(z) - \cdots - b_2 z^{-2} W(z) - b_1 z^{-1} W(z) + U(z)$$

$$W(n) = -b_N W(n-N) - b_{N-1} W(n-(N-1)) - \cdots - b_2 W(n-2) - b_1 W(n-1) + u(n)$$

$x_N(n+1) = x(n)$

$x_N(n) = x(n-1)$

$x_{N-1}(n) = x(n-2)$

$x_{N-1}(n+1) = x(n-1) = x_N(n)$

$\vdots$

$x_{N-(N-2)}(n) = x(n-(N-1)) \Rightarrow x_2(n) = x(n-(N-1))$

$x_2(n+1) = x(n-(N-2)) = x_3(n)$

$x_{N-(N-1)}(n) = x(n-N) \Rightarrow x_1(n) = x(n-N)$

$x_1(n+1) = x(n-(N-1)) = x_2(n)$

$$w(n) = -b_N\, w(n-N) - b_{N-1}\, w(n-(N-1)) \quad \cdots$$

$$\cdots - b_2\, w(n-2) - b_1\, w(n-1) + u(n)$$

$$x_1(n) = w(n-N) \qquad x_1(n+1) = x_2(n)$$

$$x_2(n) = w(n-(N-1)) \qquad x_2(n+1) = x_3(n)$$

$$x_3(n) = w(n-(N-2)) \qquad x_3(n+1) = x_4(n)$$

$$\vdots \qquad \vdots$$

$$x_{N-2+1}(n) = w(n-(2)) \qquad x_{N-1}(n+1) = x_N(n)$$

$$x_{N-1+1}(n) = w(n-(1)) \qquad x_N(n+1) = w(n)$$

$$Y(z) = W(z)\left(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}\right)$$

$$y(n) = a_0\, w(n) + a_1\, w(n-1) + \cdots + a_N\, w(n-N)$$

$$= a_0\, w(n) + a_1\, x_N(n) + \cdots + a_N\, x_1(n)$$

$$= a_0\left[-b_N x_1(n) - b_{N-1} x_2(n) - \cdots - b_2 x_{N-1}(n) - b_1 x_N(n) + u(n)\right]$$

$$+ a_1 x_N(n) + \cdots + a_N x_1(n)$$

$$= (a_N - a_0 b_N)\, x_1(n) + (a_{N-1} - a_0 b_{N-1})\, x_2(n) + \cdots$$

$$\cdots + (a_1 - a_0 b_1)\, x_N(n) + a_0\, u(n)$$

$$\underset{N\times 1}{x(n+1)} = A \underset{N\times 1}{x(n)} + B\, u(n)_{p\times 1}$$

$$\underset{r\times 1}{y(n)} = C \underset{N\times 1}{x(n)} + D \underset{p\times 1}{u(n)}$$

$A_{N\times N}$ $B_{N\times r}$ $C_{r\times N}$ $D_{r\times p}$

$$z X(z) - z x_0 = A X(z) + B U(z)$$

$$Y(z) = C X(z) + D U(z)$$

$$(zI - A) X(z) = z x_0 + B U(z)$$

$$\therefore X(z) = (zI-A)^{-1} z x_0 + (zI-A)^{-1} B U(z)$$

set i.c., $x_0 = 0$ for transfer fn.

$$Y(z) = C \left\{ (zI-A)^{-1} B U(z) \right\} + D U(z)$$

$$\therefore \boxed{\frac{Y(z)}{U(z)} = C (zI-A)^{-1} B + D}$$

Nov. 14.

Detection & Estimation

Signal + "Noise"

↑ cannot eliminate completely due to randomness

$r(t) = s(t) + n(t)$

only when $n_c(t)$ is coherent with $n(t)$

then $n(t) - n_c(t)$ reduces noise

if they're not then noise actually increases

Consider :

binary signal → $\boxed{\text{Communication system}}$ → $r = s + n$

↑ consider zero-mean Gaussian

1 0

$\underbrace{s_1(t) \quad s_0(t)}$

amplitude, or phase or freq. change between $s_1(t)$ & $s_0(t)$

$r = s + n$

(r.v.) (deterministic) (r.v.)

0 or 1

r	s	Cost	Probability
0	0	C_{00}	P_{00}
1	0	C_{10}	P_{10}
0	1	C_{01}	P_{01}
1	1	C_{11}	P_{11}

Minimize the cost function :

$$C_{00} P_{00} + C_{10} P_{10} + C_{01} P_{01} + C_{11} P_{11}$$

$$P_{rx,tx} = P_{rx/tx} \, P_{tx}$$

$\therefore$ cost function $= C_{00} P_{0/0} P_0 + C_{10} P_{1/0} P_0 + C_{01} P_{0/1} P_1 + C_{11} P_{1/1} P_1$

Hypothesis testing

we know two hypothesis exist apriori

$s = 1 \; : \; H_1$

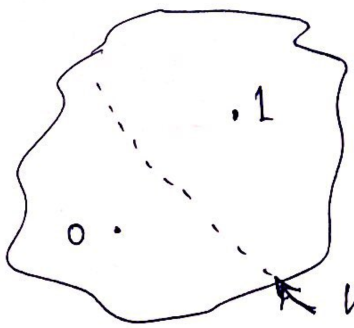
$s = 0 \; : \; H_0$

$\therefore$ cost (C) $= C_{00} P_{H_0/H_0} P_0 + C_{10} P_{H_1/H_0} P_0 + C_{01} P_{H_0/H_1} P_1 + C_{11} P_{H_1/H_1} P_1$

say $C_{00} = C_{11} = 0$, C_{10} & C_{01} are the numbers

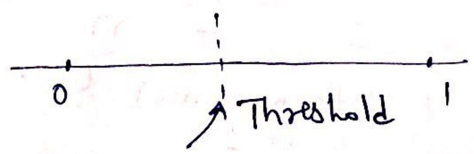
$\therefore$ minimize C : $\quad C_{10} \underbrace{P_{H_1/H_0}}_{\text{(false alarm)}} P_0 + C_{01} \underbrace{P_{H_0/H_1}}_{\text{(miss)}} P_1$

Prob./Event space :



how to demarkate prob. space

Actually a 1-dimensional problem :



1 : $r >$ threshold (γ)

0 : $r <$ threshold (γ)

$$C = C_{10}\left\{\int_{>\gamma} p_{r/H_0}\,dr\right\} P_0 + C_{01}\left\{\int_{<\gamma} p_{r/H_1}\,dr\right\}$$

$$= C_{10}\left\{\int_{\gamma}^{\infty} p_{r/H_0}\,dr\right\} P_0 + C_{01}\left\{\int_{-\infty}^{\gamma} p_{r/H_1}\,dr\right\} P_1$$

for Gaussian:

H_1 : $r = 1 + n$ $\qquad p_{r/H_1} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r-1)^2}{2\sigma^2}\right)$

H_0 : $r = n$ $\qquad p_{r/H_0} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right)$

$$\therefore C = C_{10} P_0 \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr$$

$$+ C_{01} P_1 \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r-1)^2}{2\sigma^2}\right) dr$$

say : $C_{10} = C_{01}$ and $P_0 = P_1$

$$\therefore C \propto \int_{\gamma}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\text{constant}} \exp\left(\frac{-r^2}{2\sigma^2}\right) dr + \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(r-1)^2}{2\sigma^2}\right) dr$$

$$\therefore C \propto 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\gamma} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr + \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\gamma} \exp\left(-\frac{(r-1)^2}{2\sigma^2}\right) dr$$

$$\because \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(r-1)^2}{2\sigma^2}\right) dr + \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-r^2}{2\sigma^2}\right) dr = 1$$

$$\therefore C = 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\int_{-\infty}^{\gamma} \left(e^{-r^2/2\sigma^2} - e^{-(r-1)^2/2\sigma^2} \right) dr}_{\text{maximize}}$$

$\uparrow$

minimize

$\because$ both integrals are +ve :

$$\int_{-\infty}^{\gamma} e^{-r^2/2\sigma^2} dr > \int_{-\infty}^{\gamma} e^{-(r-1)^2/2\sigma^2} dr$$

$$\therefore \quad P_{r/H_0} > P_{r/H_1}$$

$$\frac{P_{r/H_0}}{P_{r/H_1}} \underset{H_1}{\overset{H_0}{\gtrless}} \text{(some threshold)}$$

only for Gaussian

Communication System

Message signal : $s(t)$
(baseband)

modulation →

Analog

$$f(t) \begin{cases} s(t)\,A\cos(\omega_c t+\phi) \\ \text{(or)} \\ A\cos(\cancel{\omega_c}\, t+\phi) \\ \text{(or)} \\ A\cos(\omega_c t+\cancel{\phi}) \end{cases}$$

$\omega_c^{\pm\Delta\omega}$ changes around ω_c w.r.t. $s(t)$

At receiver : $f(t) + n(t)$

Discrete

Sample $s(t)$
↓ quantize
↓ digitize

if # of bits used to represent a quantized level ↑ then bit rate ↑ and bit duration ↓ thereby BW ↑

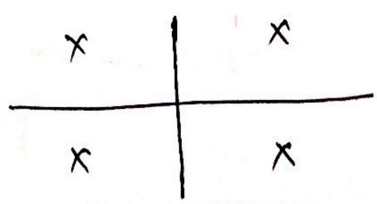
on-off signals : $f(t)$ is present for 1
0 for 0

Higher (2) order system :

Band

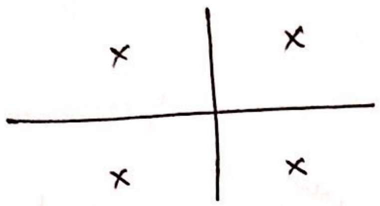
0 0
0 1
1 0
1 1

Band :
1 $\sin(\omega_c t)$
+
0 $\cos(\omega_c t)$

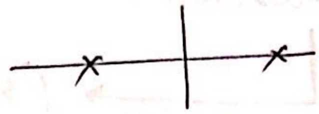


Trade-Off :

BW $\downarrow$
prob. error $\uparrow$
more susceptible to noise

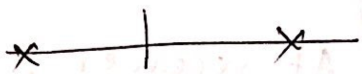


prob. error $\downarrow$
less susceptible to noise
BW $\uparrow$



Now consider a simple system :

$$r = \underbrace{s}_{1\ 0} + n$$



(or) $r(t) = \underbrace{s(t)}_{s_0(t)\ \ s_1(t)} + n(t)$ ← random process

represent $r(t)$ as orthogonal represention in terms of $\phi_0(t) = s_0(t)$, $\phi_1(t) = s_1(t)$, . . .

project $r(t)$ on $s_0(t)$ and $r(t)$ on $s_1(t)$

$\therefore$ $r_0 = s_0 + n_0$

$r_1 = s_1 + n_1$

$r_2 = 0 + n_2$

$r_3 = 0 + n_3$

⋮

$n_0, n_1, n_2, \dots$ may be correlated

$\because$ since $\phi_0(t), \phi_1(t) \dots$ are not obtained thro' K-L expansion

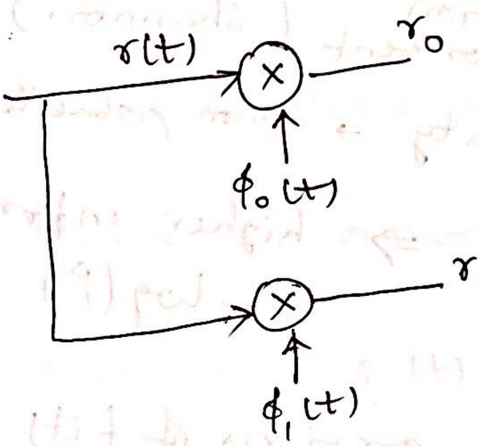
Say, white noise then

$n_0, n_1 \ldots$ are uncorrelated

(if they are correlated (pink noise) then convert to white noise using an appropriate filter)

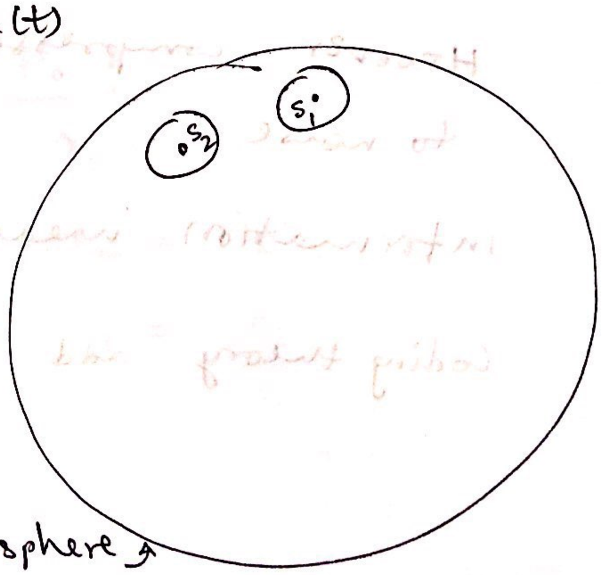
(use $s_0(t) \longrightarrow \boxed{} \longrightarrow s_0'(t)$

$s_1(t) \longrightarrow \boxed{} \longrightarrow s_1'(t)$)



r_0, r_1 are random numbers based on which we need to decide whether $r(t)$ is $s_0(t)$ or $s_1(t)$

Consider : $s_i(t) + n(t)$



Say same r.m.s. value for all $s_i(t)$ s

$n(t)$'s r.m.s. value decides radius of smaller spheres

How many smaller spheres can we pack in an n-dimensional sphere without smaller spheres overlapping

As no. of dimensions "n" increases surface area of sphere → volume of sphere.

∴ shannon's channel capacity: $C = B \ln\left(1+\frac{S}{N}\right)$

∃ a modulation scheme that gives max. data rate (error-free) given by channel capacity "c", for any given B.W. available & SNR.

---

bit represents information content (shannon)

information ⇒ uncertainity → apriori probabilities

low prob. of event occuring ⇒ higher infor. $-\log(P)$

∴ first remove ~~redu~~redundant portions of f(t) i.e. compress f(t)

However compressed f(t) is more susceptible to noise.

information measure vs. value measure

Coding theory add bits to f(t) (for error correcting codes)

20 Nov.

Detection :

$r(t) = s(t) + n(t)$ $\qquad 0 < t < T$

Question: Is $r(t)$ due to $s_1(t) = 1$
(or)
$s_0(t) = 0$

$H_1 : r(t) = s(t) + n(t)$
$H_0 : r(t) = 0 + n(t)$

} OOK
on-off keying

Consider : $s_1(t) = \sqrt{E}\, s(t)$

Normalized waveform : $\int_0^T s(t)\, dt = 1$

Taking an orthonormal expansion:

$\phi_i(t)$: $\phi_1(t) = s(t)$
$\phi_2(t)$
$\vdots$

<u>H_1</u> :

$r_1 = \sqrt{E} + n_1$

$r_2 = n_2$

$r_3 = n_3$

$\vdots$

<u>H_0</u> :

$r_1 = n_1$

$r_2 = n_2$

$r_3 = n_3$

$\vdots$

Scanned with CamScanner

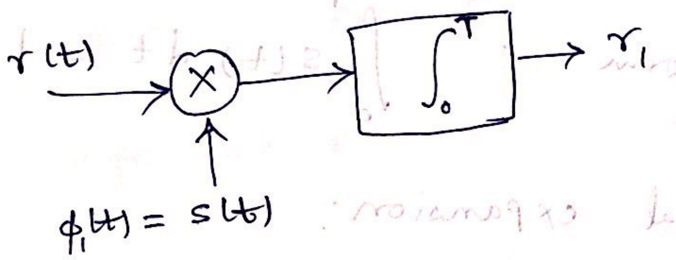
for H_1 & H_0 mentioned earlier to be valid:

→ noise may be white and then $\phi_1, \phi_2 \ldots$ may be any orthonormal set.

→ noise is not white and then $\phi_1, \phi_2 \ldots$ need to be from K-L expansion

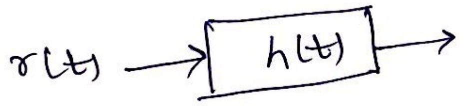
$\therefore$ $r_2, r_3 \ldots$ are irrelevant:

$\underline{H_1}$: $r_1 = \sqrt{E} + n_1$; $\underline{H_0}$: $r_1 = n_1$



Correlation receiver

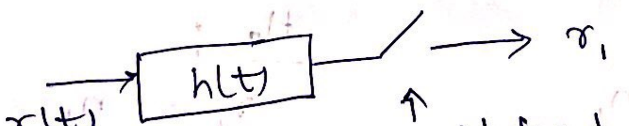
$\therefore$ Impulse response?



$r_1 = \int_0^T r(t)\, h(t-\tau)\, d\tau$

then : $h(t) = s(T-t)$

$\therefore$



switched at $t = nT$, $n = 1, 2 \ldots$

Matched filter

20-NOV-18

Detection $r(t) = s(t) + n(t), 0<t<T$

Q : k this $r(t)$ due to $s_1(t)$ — 1

or $s_0(t)$ — 0

$H_1 : r(t)$: $s(t) + n(t)$

$H_0 : r(t)$: $0 + n(t)$

Now, $s_1(t) = \sqrt{E}\, s(t)$ Normalized waveform $\int_0^T s^2(t)\,dt = 1$

Taking an orthonormal expansion

$\phi_i(t) \Rightarrow \phi_1(t) = s(t)$

$\phi_2(t) =$

Expanding $s_1(t)$ in terms of these components

H_1		H_0
$r_1 = \sqrt{E} + n_1$		$r_1 = n_1$
$r_2 = \quad + n_2$		$r_2 = n_2$

If $n(t)$ is Gauss white, any orthonormal expansion will give rise to uncorrelated $n_1, n_2, n_3, \ldots$

$r_3 = \quad + n_3$

$\vdots$

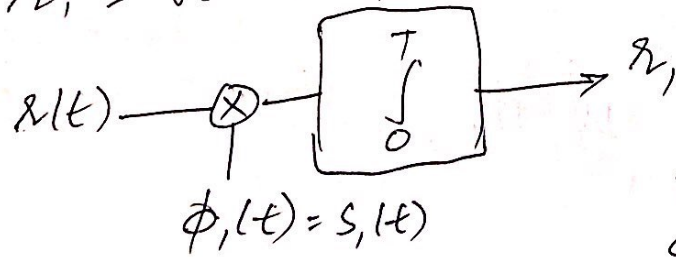
The signal info is contained only in r_1

The others are called irrelevant info.

So

H_1	H_0
$r_1 = \sqrt{E} + n_1$	$r_1 = n_1$

The decision whether $r = 1$ or 0 is based only on this info



This is called the correlation receiver

We want to replace it by $r(t) \rightarrow \boxed{h} \rightarrow r_1$

what should be the impulse resp. of this?

$$r(t) \longrightarrow \boxed{h(t)} \longrightarrow \int_0^T r(\tau)\, h(t-\tau)\, d\tau$$

If $h(t) = S(T-t)$

If we put a switch here $\quad t = T$

which is actuated at $t = T$

to get exactly this

This is called a <u>Matched filter</u> which

is more convenient if it is analog.

The ~~first~~ first step in communications is

to do a <u>matched filtering</u>.

For white noise it turns out that $\sigma_{n_i}^2$ is

the height of the spectrum $\quad N_0/2$

in communications, we

call this height $\frac{N_0}{2}$ where

2 comes in due to double sided.

So n is a Gauss random variable

$$n = N\left(0, \frac{N_0}{2}\right)$$

Given the r we need to decide if H_1 is true

or H_0 is true. How to do this? Minimize

a cost function

$$\text{Cost} = \cancel{C_{00} P_0 P_r(H_0|H_0)} + \underbrace{C_{10} P_0 P_r[H_1/H_0]}_{\text{false alarm (radar language)}}$$

(transmitted / interpreted)

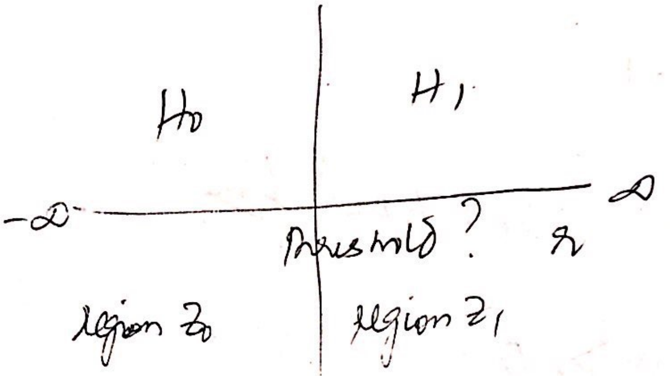
$$+ \underbrace{C_{01} P_1 P_r[H_0|H_1]}_{\text{miss (radar language)}} + \cancel{C_{11} P_1 P_r[H_1/H_1]}$$

probability of detection = 1 − miss = P_D ← prob. of detection

In communication, both, the false alarm & miss have equal cost

$\therefore$ cost = $P_0 \, P_r[H_1/H_0] + P_1 \, P_r[H_0/H_1]$

r is Gaussian random i.e. $-\infty < r < \infty$



The threshold has to be chosen judiciously so that the cost is minimized.

$$\text{Cost} = P_0 \int_{Z_1} p(r|H_0)\,dr + P_1 \int_{Z_0} p(r|H_1)\,dr$$

$$= P_0 - P_0 \int_{Z_0} p(r|H_0)\,dr + P_1 \int_{Z_0} p(r|H_1)\,dr$$

[$\because Z_0 + Z_1$ is the total sample space]

$$= P_0 - \cancel{P_0}\int_{Z_0} \underbrace{\left[P_0\, p(r|H_0) - P_1\, p(r|H_1)\right] dr}_{\Rightarrow \text{ maximize this}}$$

The decision rule is

If $P_0\, p(r|H_0) > P_1\, p(r|H_1)$ interpreted as r in Z_0

If $P_1\, p(r|H_1) > P_0\, p(r|H_0)$ interpreted as r in Z_1

$$\frac{\cancel{P_1}\, p(r|H_1)}{\cancel{P_0}\, p(r|H_0)} \gtrless \frac{P_0}{P_1} = \eta$$

aposteriori probabilities

apriori probabilities.

In our case $\dfrac{p(r|H_1)}{p(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$

~~Gaussian~~ ~~distributions~~ prob. dens $\Rightarrow$ $\ln \underbrace{\dfrac{p(r|H_1)}{p(r|H_0)}}$ $\gtrless \ln 1$

given a name $\Gamma(r)$ - the likelihood fn.

$$\ln p(r|H_1) - \ln p(r|H_0) \gtrless 0$$

or $\ln \Gamma(r) \underset{H_0}{\overset{H_1}{\gtrless}} 0$

Now $p(r|H_1) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(r-\sqrt{E})^2/2\sigma^2}$

$p(r|H_0) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-r^2/2\sigma^2}$

where $\sigma^2 = \dfrac{N_0}{2}$.

$$\therefore \ln \Gamma(r) = \frac{-(r-\sqrt{E})^2}{2\sigma^2} + \frac{r^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\therefore -\cancel{r^2} - E + 2r\sqrt{E} + \cancel{r^2} \underset{H_0}{\overset{H_1}{\gtrless}} 0 \Rightarrow \boxed{r \gtrless \frac{\sqrt{E}}{2}}$$

So put the threshold here (decision rule)

$$\therefore \text{Error} = \frac{1}{2}\int_{\sqrt{E}/2}^{\infty} p(r|H_0)\,dr + \frac{1}{2}\int_{-\infty}^{\sqrt{E}/2} p(r|H_1)\,dr$$

but, these are known

$$= \frac{1}{2}\int_{\sqrt{E}/2}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-r^2/N_0}\,dr + \frac{1}{2}\int_{-\infty}^{\sqrt{E}/2} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-(r-\sqrt{E})^2/N_0}\,dr.$$

change variable

$r - \sqrt{E} = x$

$dr = dx$

$\sqrt{\frac{E}{2}} = \frac{r-x}{2}$

$$= \qquad\qquad + \int$$

$$= \int_{\sqrt{E}/2}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-r^2/N_0}\,dr$$

$$\frac{r}{\sqrt{N_0}} = \frac{z}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2\pi \frac{N_0}{2}}}\int_{\sqrt{E/2N_0}}^{\infty} e^{-z^2/2}\,dz\,\sqrt{\frac{N_0}{2}}$$

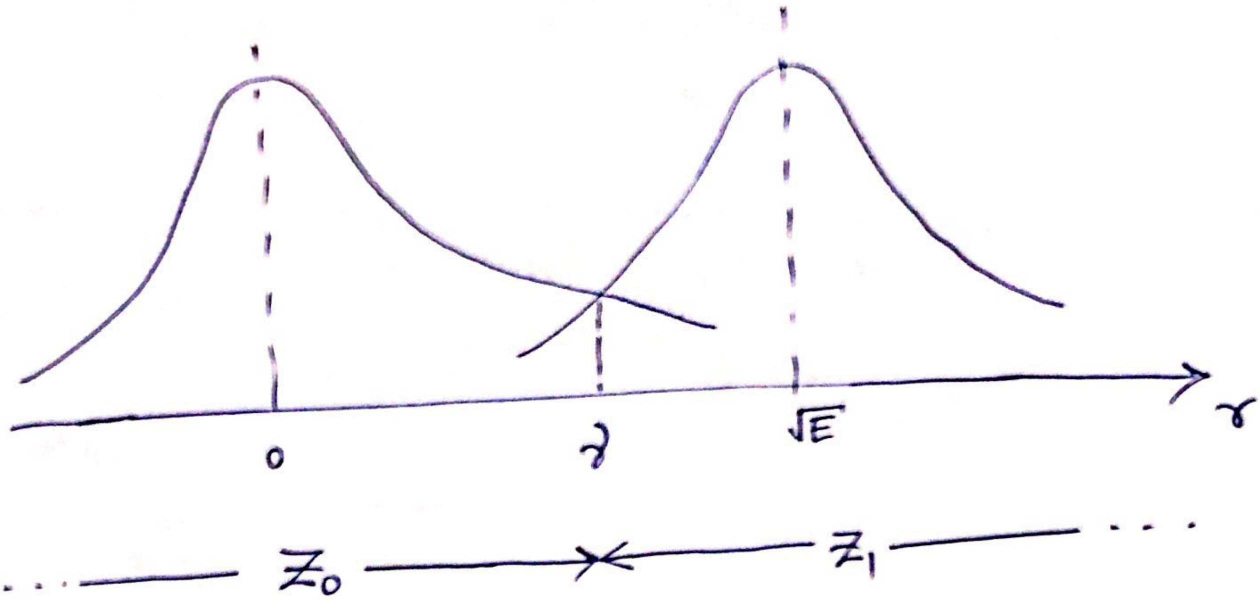
the error fn.

S/N ratio (SNR)

received ↑

So, if SNR is ↑ ⇒ error ↓ i.e. transmit more energy ⇒ radar uses MW of power.

Original work done by <u>Marcum</u> in <u>Rand Labs</u> ⇒



0
γ
√E
r
Z0
Z1

21 Nov.

Estimation.

$r(t) = A\,s(t) + n(t) \qquad 0 < t < T$

s(t) is normalized : $\int_0^T s^2(t)\,dt = 1$

n(t) is zero-mean white Gaussian noise

with $S_n(\omega) = \frac{N_0}{2}$

Choosing an orthonormal set ... :

$r = A + n$

In detection, it was checked whether

A was present or not

in estimation, we estimate "A"

Estimate of A, $\hat{A}$ is obtained by:

$\hat{A}$ # maximizes $p(A/r)$

Baye's rule : $p(A/r) = \frac{p(A, r)}{p(r)}$ ← joint prob.

$= \frac{p(r/A)\;p(A)}{p(r)}$

$\frac{\partial}{\partial A}\left[p(r/A)\,p(A)\right] \overset{\hat{A}}{=} 0$

∵ function are Gaussian : convenient to take ln.

$\frac{\partial}{\partial A}\left[\ln(p(r/A))\right] + \frac{\partial}{\partial A}\left[\ln(p(A))\right] \overset{\hat{A}}{=} 0$

$$p(r/A) = \frac{1}{2\pi\sigma_n^2} e^{-(r-A)^2/2\sigma_n^2}$$

noise

Assume "A" is a zero-mean Gaussian

$$p(A) = \frac{1}{\sqrt{2\pi\sigma_a^2}} e^{-A^2/2\sigma_a^2}$$

r.v. "A"

21-11-18

Estimation

$r(t) = A\,s(t) + n(t)$, $0 < t < T$

assume $s(t)$ is normalized $\ni \int_0^T s^2(t)\,dt = 1$

zero mean Gauss White

spectral density $\frac{N_0}{2} = S_n(\omega)$

Choosing orthonormal expansions, etc:

$r = A + n$ (using match filter implementation)

detection $\Rightarrow$ is A present?

estimation $\Rightarrow$ what is the value of A?

$p(A/r)$: the a posteriori probability

given r

A that maximizes $p(A/r) = \hat{A}$ ← estimated value

Baye's rule:

$p(A/r) = \frac{p(A, r)}{p(r)}$ ← joint probability; strictly $p_A(a, r)$

$= \frac{p(r/A)\,p(A)}{p(r)}$

$p(r)$ independent of A, per say

So, maximizing $\Rightarrow$

$\frac{\partial}{\partial A}\left[p(r/A)\,p(A)\right] \overset{\hat{A}}{=} 0$

$\hat{A}$ is taken as the estimate of A.

sometimes, the log is taken $\because$ the fns are Gaussian

$$\frac{\partial}{\partial A}\left\{\ln p(r/A)\right] + \frac{\partial}{\partial A}\left[\ln p(A)\right] \overset{\hat{A}}{=} 0$$

but $p(r/A) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \; e^{-(r-A)^2/2\sigma_n^2}$

$$p(A) = \frac{1}{\sqrt{2\pi\sigma_a^2}} \, e^{-A^2/2\sigma_a^2}$$

$$\frac{\partial}{\partial A}\left[-\frac{(r-A)^2}{2\sigma_n^2}\right] + \frac{\partial}{\partial A}\left[\frac{-A^2}{2\sigma_a^2}\right] = 0$$

$$\frac{2(r-A)}{2\sigma_n^2} - \frac{2A}{2\sigma_a^2} = 0 \quad \Rightarrow \boxed{\hat{A} = \frac{\sigma_a^2}{(\sigma_a^2+\sigma_n^2)} r}$$

i.e. we take r and weigh it down

σ_a^2/σ_n^2 is the S/N ratio.

MAP estimation (maximum a priori probability)

If A is known, can use Maximum likelihood (ML) estimation.

If noise is high, the best estimate is to take the mean value of the noise.

- A extension of this problem could be $A = A(t)$ which is a random process $\Rightarrow$ a modulation problem.
- Another extension: $s(t)$ itself is random a famous problem solved by Norbert Weiner in 1940 resulting in winning the WW II: to predict trajectory from the ~~per~~ previous trajectory. This is the foundation of communication theory.

The ~~British~~ Americans didn't have a high power source. ~~In America~~ In Birmingham, they developed the magnetron and smuggled it to the US. The magnetron attached to ~~a~~ a radar & it worked.

$s(t)$ is a random process immersed in $n(t)$ which is another random process. In radar, the power returned is very small. Send MW & get mW or μW back, which is ~~a~~ immersed in noise. From this extract $s(t)$ with minimum mean square error.

Radiation Laboratory (of MIT) series books in 28 volumes
→ renamed RLE (Research labs for electronics)

In the 1960s Kalman modelled the same problem in state space — the Kalman filter.

Weiner's $s(t)$ is a stationary random process but Kalman can be used for a non-stationary random process too.

Can have nonlinear modulation schemes where $s = s(t, a(t))$.
random modulation

Another scenario is as follows: $w(t)\, s(t, a(t))$
another random process

e.g. in a fading channel!

We said $r(t) = A\,s(t) + n(t)$

If $s(t)$ is known: $r = A + n$

in the sense that it is being sent by the transmitter.

$s(t)$ may be a sinusoid and we don't know its phase. If it differs from the local $s(t)$, it won't work. So, synchronize them.

— feedback equalization

— diversity

What we did not learn

- Time varying systems
- Nonlinear systems

Inserts:

A in $A\,s(t)$ is random : MAP estimation

A in $A\,s(t)$ is deterministic : ML estimation

Extension to commn. systems: $A\,s(t)$
message

∴ need to estimate message (A) at receiver

(or) like in radar: $A\,s(t)$ is a random process embedded in noise, $n(t)$, another " "

& need to estimate $A\,s(t)$ (trajectory of enemy plane)
(courtesy of Norbert Wiener)