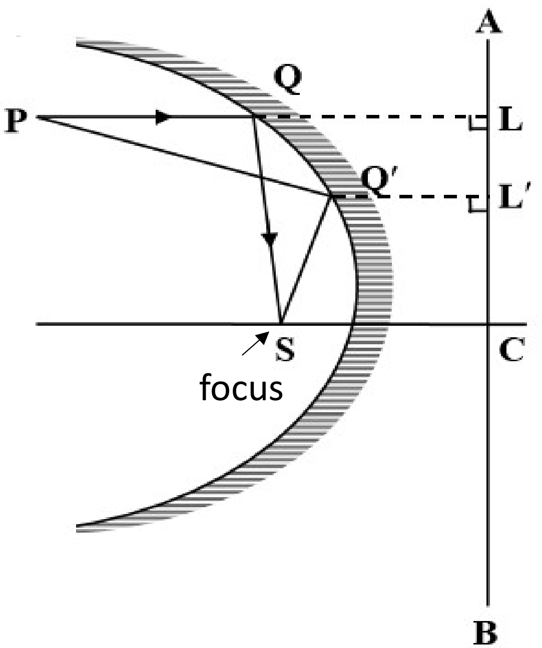
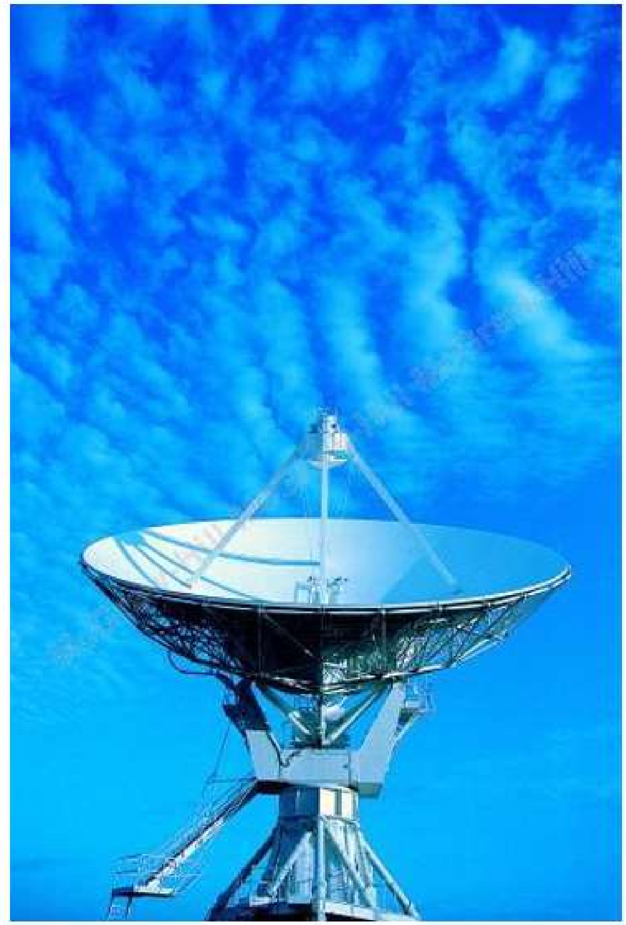
PH203: Optics

Lecture #3

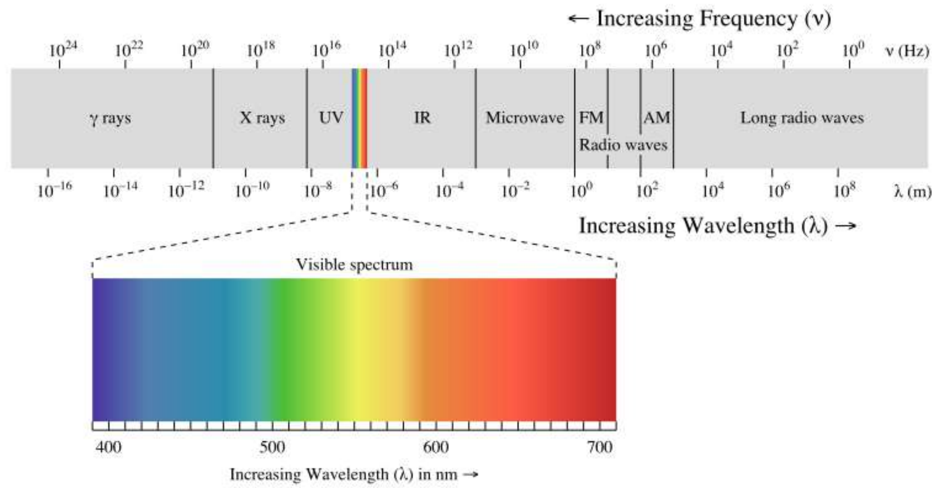
02.11.2018

From Fermat's principle set of rays parallel to the axis of a paraboloid mirror will pass through its focus





A paraboloidal satellite dish



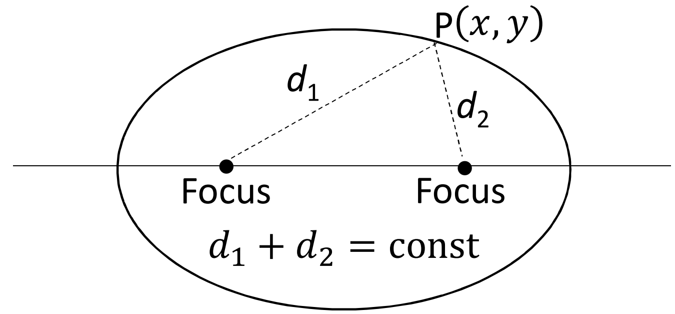
One of the 30 paraboloidal dishes each of 45 meter diameter
fully steerable Giant Metrewave Radio Telescope (GMRT) @ Pune

Radio waves from interstellar space is recorded

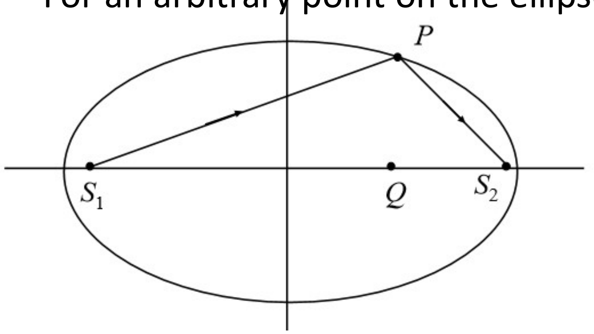
Radio wave forms part of EM spectrum

Problem: Consider an elliptical reflector having foci at S_1 and S_2. All the rays emanating from S_1 will pass through S_2 after undergoing reflection

An ellipse is the set of all points in a plane, the sum of whose distances from two distinct fixed points (foci) is constant



For an arbitrary point on the ellipse,



$$S_1P + S_2P = const$$

$\Rightarrow$ All rays from S_1 will pass through S_2

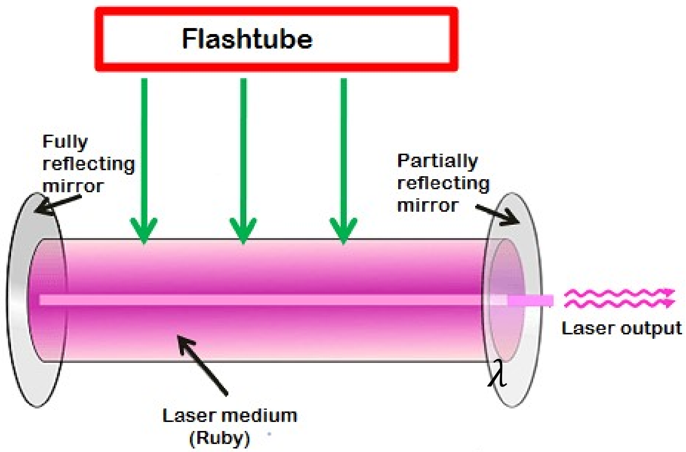
Since optical path that connects S_1 and P will be a st line, and likewise for point P to S_2

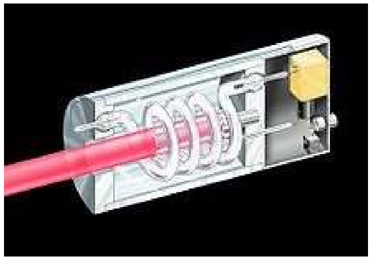
$\Rightarrow$ Both together will make ray path between S_1 and S_2 via reflection at the ellipsoid will be a minimum

Same will be true for any other reflection point

This is an example where time taken by a ray connecting the points S_1 and S_2 via reflection at any point P on the ellipse is stationary

This property is often used in constructing lasers like a Ruby laser that emits light at $\lambda = 694.3 \text{ nm}$





1st Ruby laser

https://en.wikipedia.org/wiki/Ruby_laser

In a ruby laser the laser rod and the flash lamp coincide with the focal lines of a cylindrical reflector with elliptical cross section

http://www.physics-and-radio-electronics.com/physics/laser/rubylaserdefinitionconstructionworking.html

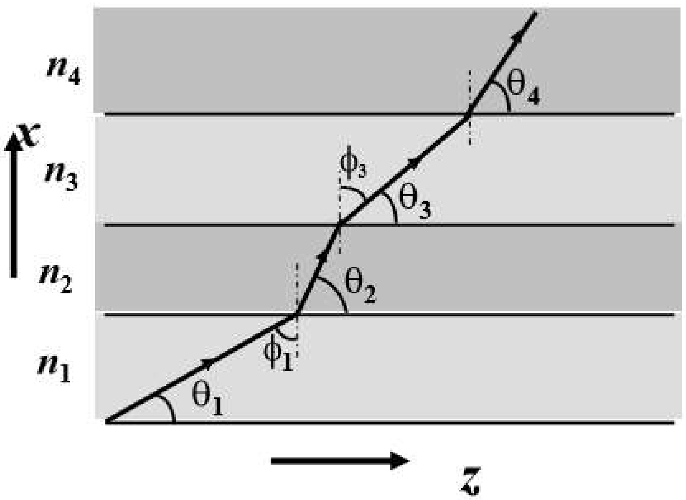
Flash lamp is used to pump the ground state atoms (which absorb the pump light) to a higher energy metastable state and stimulated to emit characteristic radiation corresponding to the energy difference between the metastable state and ground state thereby yielding light amplification

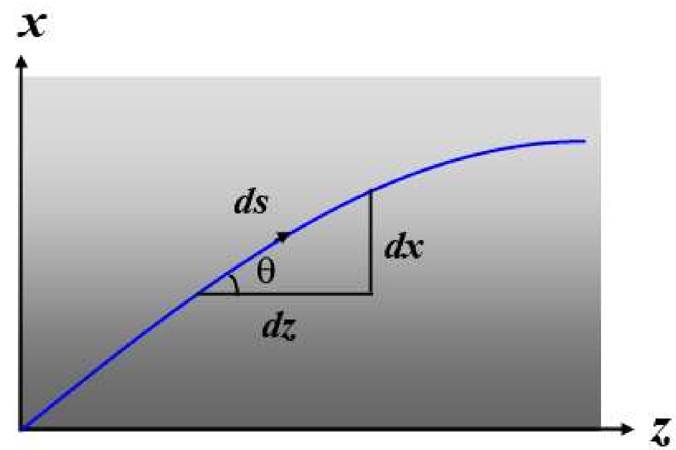
By means of a pair of mirrors at the end of the Ruby rod feedback is provided to this amplified light, which yields the laser light

Ray paths in an inhomogeneous medium:

Let R. I. : *n* (*x*) is increasing continuously e.g. hot air near the ground on a hot day

It can be approximated as a limiting case of a continuous set of thin slabs of media of slightly higher refractive indices in subsequent layers





At each interface, apply law of refraction: $n_1 \sin\phi_1 = n_2 \sin\phi_2 = n_3 \sin\phi_3 = \cdots$

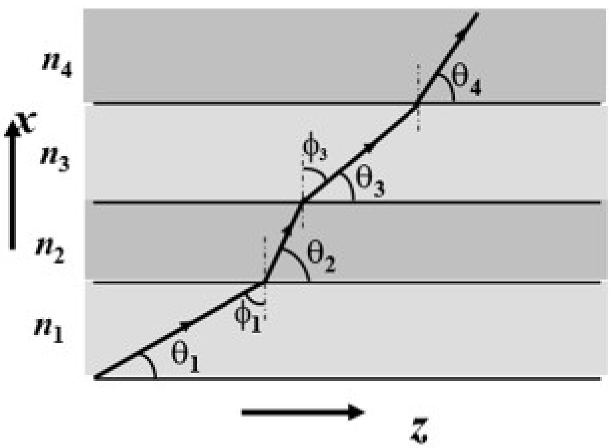
$$\Rightarrow \quad n_1 \sin\left(\frac{\pi}{2} - \theta_1\right) = n_2 sin\left(\frac{\pi}{2} - \theta_2\right) = n_3 sin\left(\frac{\pi}{2} - \theta_3\right) = \ldots.$$

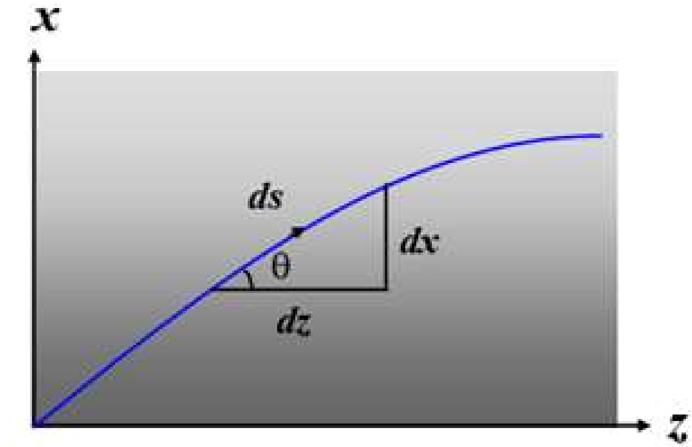
In the limiting case of continuous variation in R. I. :

$$\Rightarrow \quad n(x)\sin\phi(x) = n(x)\cos\theta(x) = \widetilde{\beta} \quad \text{(a constant)}$$

$$\Rightarrow \quad n(x)\cos\theta(x) = n_1 \cos\theta_1 = \widetilde{\beta}$$

$\Rightarrow$ As the R. I. changes the ray bends in a way to maintain the product $n(x)\cos\theta(x)$ constant





Ray equation

From the figure

$$(ds)^2 = (dx)^2 + (dz)^2 \quad \Rightarrow \quad \left(\frac{ds}{dz}\right)^2 = \left(\frac{dx}{dz}\right)^2 + 1$$

Again

$$\frac{dz}{ds} = \cos\theta = \frac{\widetilde{\beta}}{n(x)}$$

Hence

$$\left(\frac{dx}{dz}\right)^2 = \frac{n^2(x)}{\widetilde{\beta}^2} - 1$$ **: Ray equation**

By further differentiation

$$\left(\frac{dx}{dz}\right)^2 = \frac{n^2(x)}{\tilde{\beta}^2} - 1 \quad \rightarrow \quad 2\cancel{\left(\frac{dx}{dz}\right)}\frac{d^2x}{dz^2} = \frac{1}{\tilde{\beta}^2}\frac{d}{dz}n^2(x)$$

$$= \frac{1}{\tilde{\beta}^2}\frac{dn^2(x)}{dx}\cancel{\frac{dx}{dz}}$$

$$\frac{d^2x}{dz^2} = \frac{1}{2\tilde{\beta}^2}\frac{dn^2(x)}{dx}:$$ **alternate form of ray equation**

If we consider light propagation in a uniform medium of ri : $n(x) = n$

$\Rightarrow$ RHS will be 0 because $n(x) = n$ is independent of spatial coordinate x

$$\frac{d^2x}{dz^2} = 0 \quad \Rightarrow \frac{d}{dz}\left(\frac{dx}{dz}\right) = 0 \Rightarrow \left(\frac{dx}{dz}\right) = \text{const} \quad \Rightarrow \quad x = Az + \text{const}$$ This is eq to a straight line

Phenomenon of mirage

On a hot day near the ground R. I. of air near the ground

$$n(x) \approx n_0 + k\,x \qquad 0 < x < \text{few metres} \qquad \uparrow x$$

n_0 → Just above the ground

where $k = 1.234 \times 10^{-5}\, m^{-1}$

From ray eq.

$$\frac{d^2x}{dz^2} = \frac{1}{2\tilde{\beta}^2}\,\frac{dn^2}{dx} = \frac{1}{\cancel{2}\tilde{\beta}^2}[\cancel{2}n_0k + \cancel{2}k^2x] = \frac{n_0k}{\tilde{\beta}^2} + \frac{k^2x}{\tilde{\beta}^2} = \frac{k}{\tilde{\beta}^2}[n_0 + kx]; \qquad X \equiv x + \frac{n_0}{k}$$

$$\frac{d^2X}{dz^2} = \kappa^2 X(z); \qquad \kappa = \frac{k}{\tilde{\beta}}$$

Hence the ray path: $X(z) = x(z) + \frac{n_0}{k} = C_1 e^{\kappa z} + C_2 e^{-\kappa z};$

$C_{1,2}$ are determined from boundary condition:

At z = 0, and assume a ray at $x = x_1$ is launched at an angle θ_1

$$\Rightarrow x = x_1; \quad \left.\frac{dx}{dz}\right|_{z=0} = \tan\theta_1 \text{ and using } \tilde{\beta} = n_1 \cos\theta_1$$

We will get

$$C_1 = \frac{1}{2}\left[x_1 + \frac{1}{k}(n_0 + n_1 \sin\theta_1)\right]$$

$$C_2 = \frac{1}{2}\left[x_1 + \frac{1}{k}(n_0 - n_1 \sin\theta_1)\right]$$

$$x(z=0) = -\frac{n_0}{k} + C_1 e^{\kappa z} + C_2 e^{-\kappa z}$$

Thus, $x_1 = -\frac{n_0}{k} + C_1 + C_2$ (1)

Again

$$\left.\frac{dx}{dz}\right|_{z=0} = \tan\theta_1 = C_1\kappa - C_2\kappa$$

$$\tan\theta_1 = \frac{\sin \quad _1}{\cos\theta_1} = \frac{n_1 \sin\theta_1}{\tilde{\beta}} = C_1\kappa - C_2\kappa \quad \dots (2)$$

Multiply (1) by κ and to (2) and solve for C_1

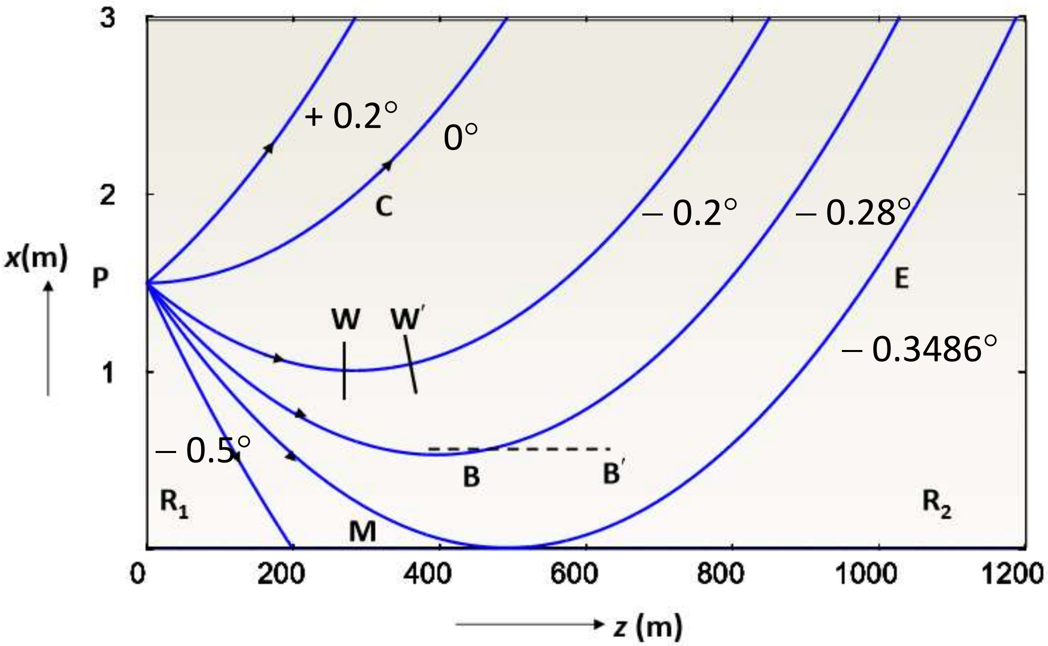
$$\Rightarrow C_1 = \frac{1}{2}\left[x_1 + \frac{n_0}{k} + \frac{n_1 \text{ si} \quad _1}{k}\right]$$

Assume $x_1 = 1.5\, m$ at which $n_1 = 1.00026$ and $k = 1.234 \times 10^{-5}\, m^{-1}$

Corresponding ray paths for different θ's are shown in the figure

$\Rightarrow$ each ray has a specific $\tilde{\beta} = n_1 \cos\theta_1$

Consider a ray path, which becomes horizontal at x = 0, which corresponds to θ_1 = - 0.28°



Assume r.i. = n_e at the eye level

Let θ_e: angle the eye makes with the horizontal at x_1

$$\Rightarrow \quad \tilde{\beta} = n_0 \cos 0^{\mathrm{o}} = n_e \cos \theta_e$$

Just above the ground

For

$$\theta_e \ll 1, \cos\theta_e \approx 1 - \frac{1}{2}\theta_e^2$$

$$\Rightarrow \theta_e \cong \sqrt{2\left(1 - \frac{n_0}{n_e}\right)}$$

At const air pressure, $n_{0,e}$ are related to temperatures of air at the ground and eye levels as

$$T_0(n_0 - 1) = T_e(n_e - 1) \quad \Rightarrow \quad 1 - \frac{n_0 - 1}{n_e - 1} = 1 - \frac{T_e}{T_0}$$

LHS: $\dfrac{n_e - n_0}{n_e - 1} = n_e \dfrac{\left(1 - \frac{n_0}{n_e}\right)}{n_e - 1}$ Thus, $\left(1 - \frac{n_0}{n_e}\right) = \frac{n_e - 1}{n_e}\left(1 - \frac{T_e}{T_0}\right) = \left(1 - \frac{1}{n_e}\right)\left(1 - \frac{T_e}{T_0}\right)$

$$\Rightarrow \theta_e = \sqrt{2\left(1 - \frac{1}{n_e}\right)\left(1 - \frac{T_e}{T_0}\right)}$$

Typically on a hot day, $T_0 \sim 323\ K$ At a height of 1.5 m from the ground, $T_e \sim 303\ K$

One gets $\theta_e = 5.67 \times 10^{-3} radians = 0.00567 \times \frac{180}{\pi} \approx 0.325°$

Thus only PME ray will reach eye and hence to one's eye, ray from $x = x_1$ will appear to come from point below the ground level i.e. $x < 0 \Rightarrow$ point *P* will appear as mirage

No other ray from *P* reaches eye!

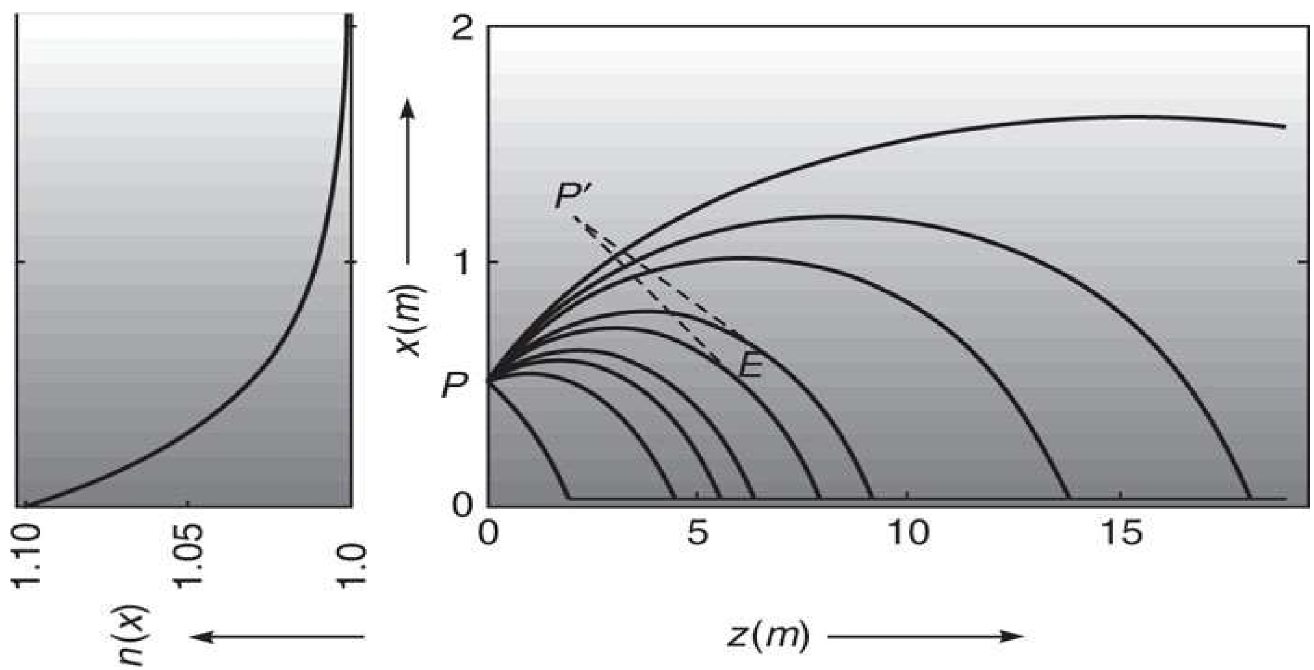
In the region R_2 no ray from *P* will reach and hence that region will appear as the shadow region

Temperatures nearer the sea/ocean are cooler than in the atmosphere above it

$$n^2(x) = n_0^2 + n_2^2 e^{-\alpha x}$$

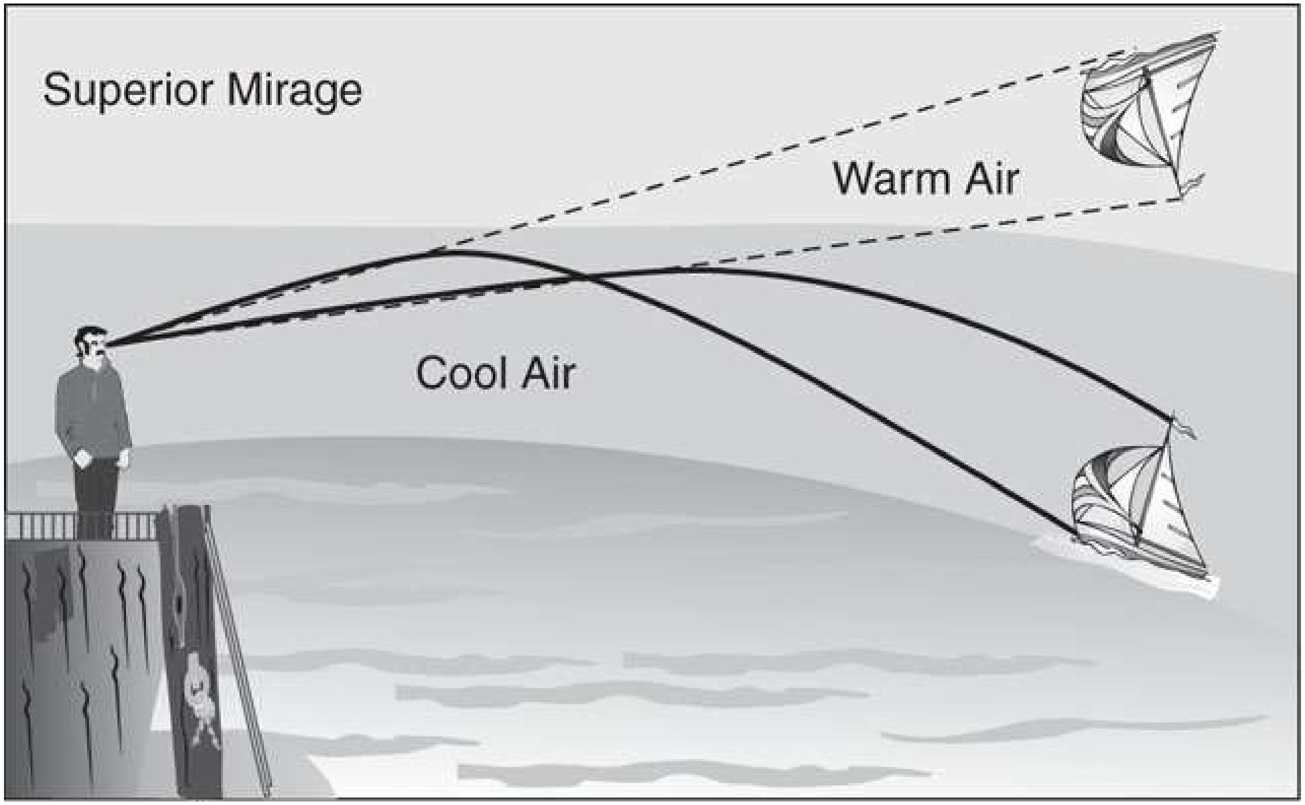
where $n_0 = 1.00023, n_2 = 0.45836, \alpha = 2.303^{-1}$

$\Rightarrow$ R.I. is max at $x = 0$ and decreases with height



Height of P = 0.5 m

For eye at E, received rays appear to come from P'


Superior Mirage
Warm Air
Cool Air

Ray paths in a graded index medium:

$$n^2(x) = n_1^2 - \gamma^2 x^2$$

From Ray eq:

$$\left(\frac{dx}{dz}\right)^2 = \frac{n^2(x)}{\tilde{\beta}^2} - 1$$

$$\int \frac{dx}{\sqrt{n^2(x) - \tilde{\beta}^2}} = \pm\frac{1}{\tilde{\beta}}\int dz \qquad \Rightarrow \qquad \int \frac{dx}{\sqrt{n_1^2 - \gamma^2 x^2 - \tilde{\beta}^2}} = \pm\frac{1}{\tilde{\beta}}\int dz$$

$$\int \frac{dx}{\gamma\sqrt{\frac{n_1^2 - \tilde{\beta}^2}{\gamma^2} - x^2}} = \pm\frac{1}{\tilde{\beta}}\int dz \quad \Rightarrow \quad \int \frac{dx}{\sqrt{\frac{n_1^2 - \tilde{\beta}^2}{\gamma^2} - x^2}} = \pm\underbrace{\left(\frac{\gamma}{\tilde{\beta}}\right)}_{\equiv\ \Gamma}\int dz \quad \Rightarrow \quad \int \frac{dx}{\sqrt{x_0^2 - x^2}} = \pm\Gamma\int dz$$

where

$$x_0 = \frac{\sqrt{n_1^2 - \tilde{\beta}^2}}{\gamma}$$

Let $\quad x = x_0 \sin\theta \quad \Rightarrow dx = x_0 \cos\theta \, d\theta \qquad \Rightarrow$ Integral becomes $\displaystyle\int \frac{x_0\cos\theta\, d\theta}{x_0\cos\theta} = \pm\Gamma\int dz$

$$\Rightarrow \quad \theta = sin^{-1}\left(\frac{x}{x_0}\right) = \pm\,\Gamma(z - z_0) \quad \Rightarrow x = \pm\, x_0 \sin[\Gamma(z - z_0)]$$