Modern Physics

Lecture 12

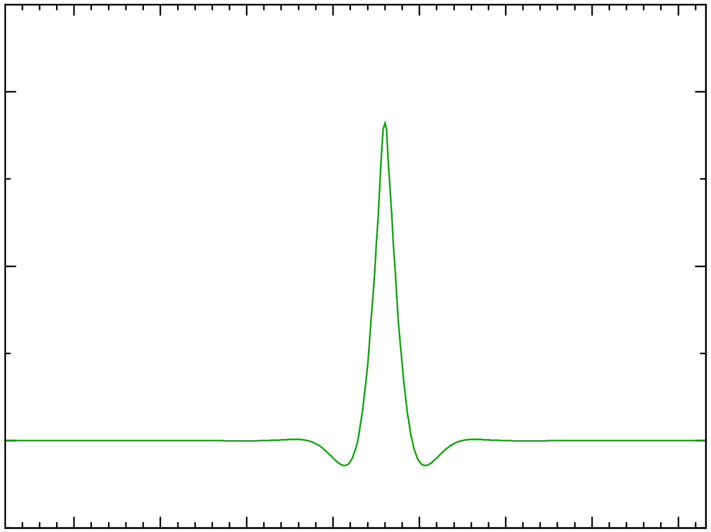
The uncertainty principle

Newtonian Physics is deterministic

Whereas Quantum Mechanics is probabilistic

Uncertainty Principle I -- derivation based on the wave properties of particles

Consider the particle represented by this wave group.



Where is the particle?

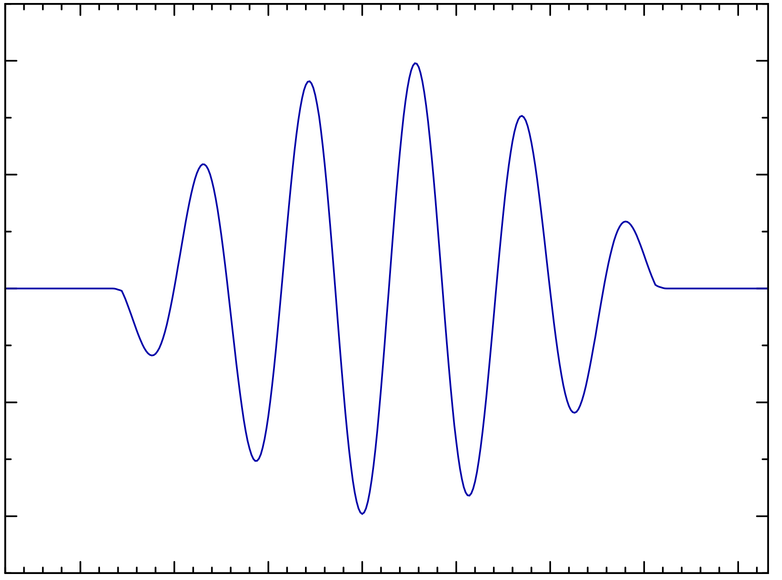
What is its wavelength?

The position is well-defined.

But the wavelength is poorly defined, and therefore there is a large uncertainty in the particle's momentum (since wavelength and momentum are connected through De Broglie relation).

"Life is uncertain. Eat dessert first."

Now consider the particle represented by this wave group.



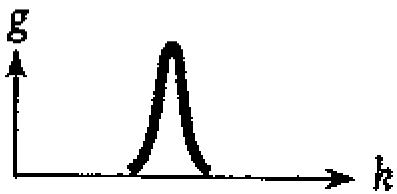
Where is the particle?

What is its wavelength?

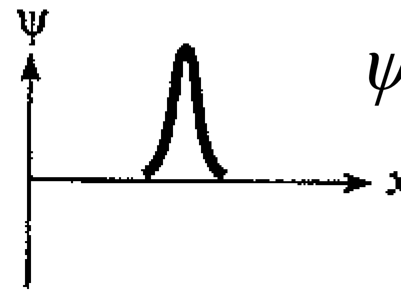
The wavelength seems to be rather well-defined, but the position is poorly defined. There is a large uncertainty in the particle's position.

To quantify the uncertainties in the wave group's position and momentum, we need to go into much more detail about Fourier transforms and representation of wave groups by summations of individual waves.

Fourier Transformation



$$g(k) = Ae^{-\alpha^2 k^2}$$



$$\psi(x) = \alpha A e^{-\frac{x^2}{4\alpha^2}}$$

In position space

In k or momentum space

Standard Gaussian is,

$$\varphi(y) = Ae^{-\left(\frac{y}{2\sigma}\right)^2}$$

By comparing the standard deviations,

$$\Delta x \Delta k \approx \frac{1}{2}$$

But wave groups are not exactly Gaussian

$$\Delta x \Delta k > \frac{1}{2}$$

Now

$$k = \frac{2\pi}{\lambda}$$

From De Broglie

$$\lambda = \frac{h}{p}$$

Substituting value of λ,

$$k = \frac{2\pi p}{h}$$

This means,

$$p = \frac{hk}{2\pi}$$

$$\Delta p = \frac{h\Delta k}{2\pi}$$

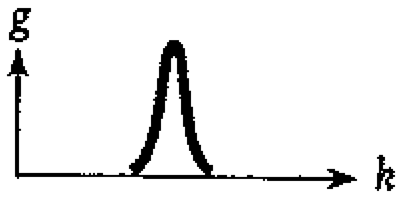
We already have from Fourier analysis,

$$\Delta x \Delta k \geq \frac{1}{2}$$

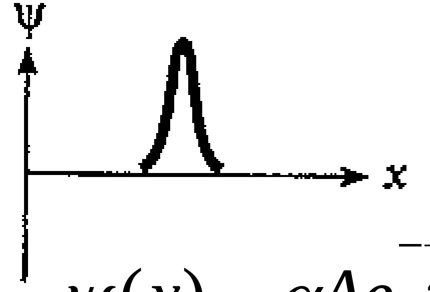
Substituting $$\frac{\Delta x 2\pi \Delta p}{h} \geq \frac{1}{2}$$

$$\Delta x \Delta p \geq \frac{h}{4\pi} \quad \rightarrow \Delta x \Delta p \geq \frac{\hbar}{2}; where\, \hbar = \frac{h}{2\pi}$$

Comparison between momentum space and position space



$$g(k) = Ae^{-\alpha^2 k^2}$$



$$\psi(x) = \alpha A e^{-\frac{x^2}{4\alpha^2}}$$

In k or momentum space

In position space

This is called Heisenberg's Uncertainty Principle (Nobel Prize 1932):

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

It is not possible to simultaneously measure, with arbitrary precision, both the position and momentum of a particle.

There are fundamental limits on how precisely we can simultaneously measure a particle's position and momentum.

These limits have nothing to do with our measurement techniques; they are built into nature.

Example 3.6

A measurement establishes the position of a proton with an accuracy of $\pm 1.00\times10^{-11}$ m. Find the uncertainty in the proton's position 1 s later. Assume v << c.

At the time of measurement, the position uncertainty is Δx_1, and

$$\Delta x_1 \Delta p_x \geq \frac{\hbar}{2}$$

$$\Delta p_x \geq \frac{\hbar}{2\Delta x_1}$$

$$\Delta p_x = \Delta(m v_x) = m\,\Delta v_x$$

Non-relativistic case, v<<c

$$\Delta v_x = \frac{\Delta p_x}{m} \geq \frac{\hbar}{2m\,\Delta x_1}$$

A time t later, the position uncertainty Δx_2 is

$$\Delta x_2 = t\,\Delta v_x \geq \frac{t\,\hbar}{2m\,\Delta x_1}$$

$$\Delta x_2 \geq \frac{(1)\left(1.054\times10^{-34}\right)}{2\left(1.67\times10^{-27}\right)\left(1.00\times10^{-11}\right)}$$

About 1/10 the size of an atom, but much larger than a nucleus.

$$\Delta x_2 \geq 3.15\times10^{3}\text{ m, or 1.96 miles.}$$

To put it bluntly, we have no clue where the proton is after 1 second.

The proton didn't spread out, because it is "somewhere," but its wave certainly did!

Size of hydrogen atom from Uncertainty principle

1 electron

H atom

1 proton

Let ***a*** be the dimension of the atom

From uncertainty principle, $$\Delta p \approx \frac{\hbar}{a}$$

Since $\Delta x = a$ is the uncertainty in electron position

Electron kinetic energy

$$E_{KE} = \frac{p^2}{2m} \approx \frac{\hbar^2}{2ma^2}$$

Electron potential energy

$$E_{PE} = -\frac{q^2}{4\pi\varepsilon_0 a}$$

Electron Total energy

$$E_T = \frac{\hbar^2}{2ma^2} - \frac{q^2}{4\pi\varepsilon_0 a}$$

System should be settling to the lowest energy state

Therefore,

$$\frac{dE_T}{da} = 0$$

$$\frac{dE_T}{da} = 0 = -\frac{\hbar^2}{ma^3} + \frac{q^2}{4\pi\varepsilon_0 a^2}$$

Therefore,

$$a = a_0 = \frac{\hbar^2}{m\dfrac{q^2}{4\pi\varepsilon_0}}$$

Substituting values as

$$\left.\begin{aligned}
\hbar &= 1.055\times10^{-34}\,Js \\
m &= 9.11\times10^{-31}kg \\
\varepsilon_0 &= 8.854\times10^{-12}CN^{-2}m^{-2} \\
q &= 1.6\times10^{-19}C
\end{aligned}\right\} \quad a = a_0 = 0.53\times10^{-10}\,m$$

$$a_0 = 0.53A^0$$

Substituting value of $a_0 = 0.53 \times 10^{-10}$ in total energy expression

$$E_T = \frac{\hbar^2}{2ma_0^2} - \frac{q^2}{4\pi\varepsilon_0 a_0}$$