$$\| x^{(n+1)} - x^{(n)} + x^{(n)} - x \|$$

$$\leq \| x^{(n+1)} - x^{(n)} \| + \| x^{(n)} - x \|$$

$$\| x^{(n)} - x^{(n+1)} + x^{(n+1)} - x \|$$

$$\leq \| x^{(n)} - x^{(n+1)} \| + \| x^{(n+1)} - x \|$$

$$\therefore \quad \| x^{(n)} - x \| \leq \|H\| \left(\| x^{(n+1)} - x^{(n)} \| + \| x^{(n)} - x \| \right)$$

$$[1 - \|H\|] \, \| x^{n} - x \| \leq \|H\| \, [\| x^{(n+1)} - x^{n} \|]$$

$$\therefore \quad \boxed{\| x^{n} - x \| \leq \frac{\|H\|}{1 - \|H\|} \| x^{(n+1)} - x^{(n)} \|}$$

Relative error is obtained by dividing by $\| x^{(n)} \|$

$\square$

Gauss - Jacobi

$$A = L + D + U$$

Eg $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

$D = \varphi$, the splitting matrix.

This is Gauss - Jacobi

27/02/19
Class.

Theorem

Let $Ax=b$, $|A| \neq 0$. Then,

$$\frac{1}{K(A)} \cdot \frac{\|r\|}{\|b\|} \leq \frac{\|x-\tilde{x}\|}{\|x\|} \leq K(A) \cdot \frac{\|r\|}{\|b\|}$$

where

$$r = b - A\tilde{x} = b - \tilde{b}$$
$$e = x - \tilde{x}$$
and $$K(A) = \|A\| \cdot \|A\|^{-1}$$

NOTE The term in the middle is the relative error.

This thm. is useful to determine the effect of round-off errors. As $K(A)$ is large, we see large deviations.

Proof

$$b = Ax.$$
$$\|b\| = \|Ax\| \leq \|A\| \cdot \|x\|, \quad x \neq 0.$$
$$\Rightarrow \frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|} \quad \text{--- (1)}$$

Also. $A\tilde{x} = b + \delta b.$

$$b + \delta b \leftarrow \boxed{A} \rightarrow \tilde{x} \rightarrow x + \delta x$$

$$\Rightarrow A(x + \delta x) = b + \delta b.$$

$$b + \delta b \rightarrow \boxed{A} \rightarrow \bar{x} + \delta x$$
$$\downarrow$$
$$A + \delta A$$

$$\Rightarrow \cancel{Ax} + A\delta x = \cancel{b} + \delta b$$

$$\Rightarrow A \cdot (\delta x) = \delta b,$$

As A is non-singular,

$$\boxed{\delta x = A^{-1}\,\delta b}$$

$$\Rightarrow \quad \|\delta x\| \leq \|A^{-1}\| \cdot \|b\| \qquad \text{--- (2)}$$

using $\delta x = x - \tilde{x}$ and multiplying (1), (2) :

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \cdot \|A^{-1}\| \cdot \frac{\|b - \tilde{b}\|}{\|b\|} \qquad b - \tilde{b} = \delta b$$

$$\Rightarrow \quad \frac{\|x - \tilde{x}\|}{\|x\|} \leq \left(\|A\| \cdot \|A^{-1}\|\right) \frac{\|b - \tilde{b}\|}{\|b\|} \qquad \text{--- (3).}$$

Now, we can define $\|A\| \cdot \|A^{-1}\|$ to be the condition number $\kappa(A)$. Now, to get the lower bound,

$$x = A^{-1} b. \quad \Rightarrow \quad \|x\| \leq \|A^{-1}\| \cdot \|b\|$$

$$\text{(or)} \quad \frac{1}{\|x\|} \leq \frac{1}{\|A^{-1}\| \cdot \|b\|}. \qquad \text{--- (4).}$$

consider $\delta b = A(\delta x)$

$$\|\delta b\| \leq \|A\| \cdot \|\delta x\|$$

$$\Rightarrow \quad \frac{\|\delta b\|}{\|A\|} \leq \|\delta x\| \qquad \text{--- (5).}$$

using $\|\delta x\| = \|x - \tilde{x}\|$ as before and multiplying (4), (5),

$$\frac{\|\delta x\|}{\|x\|} \geq \frac{1}{\|A\| \, \|A^{-1}\|} \cdot \frac{\|\delta b\|}{\|b\|}$$

$$\text{(or)} \quad \frac{\|x - \tilde{x}\|}{\|x\|} \geq \frac{1}{\kappa(A)} \cdot \frac{\|b - \tilde{b}\|}{\|b\|} \qquad \text{--- (6).}$$

$\square$

eg

$$A = \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}, \quad \varepsilon > 0$$

$$\|A\|_F = \sqrt{1^2 + (1+\varepsilon)^2 + (1-\varepsilon)^2 + 1^2}$$

$$= \sqrt{1^2 + 1+\varepsilon^2 + \cancel{2\varepsilon} + 1+\varepsilon^2 - \cancel{2\varepsilon} + 1^2}$$

$$= \sqrt{4 + 2\varepsilon^2}$$

$$A^{-1} = \frac{1}{\varepsilon^2}\begin{bmatrix} 1 & -1-\varepsilon \\ -1+\varepsilon & 1 \end{bmatrix}$$

$$\|A\|_\infty = \max_{\text{row}} \{ |1| + |1+\varepsilon|, |1-\varepsilon| + |1| \} = 2+\varepsilon$$

$$\|A^{-1}\|_\infty = \max_{\text{row}} \left\{ \frac{|1| + |-1-\varepsilon|}{\varepsilon^2}, \frac{|1| + |1-\varepsilon|}{\varepsilon^2} \right\} = \frac{2+\varepsilon}{\varepsilon^2}$$

Then,

$$K_\infty(A) = \frac{2+\varepsilon}{\varepsilon^2}\cdot(2+\varepsilon) = \left(\frac{2+\varepsilon}{\varepsilon}\right)^2 = \left(1+\frac{2}{\varepsilon}\right)^2$$

Case 1 Let $\varepsilon = 0.01$

$$K_\infty(A) = 40{,}401.$$

Now

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq [40{,}401] \frac{\|b - \tilde{b}\|}{\|b\|}$$

Case 2 Let $\varepsilon = 10$

$$K_\infty(A) = 1.44.$$

Now.

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq (1.44) \frac{\|b - \tilde{b}\|}{\|b\|}$$

eg

$x_1 + (1+\epsilon) x_2 = 3$

$(1-\epsilon) x_1 + x_2 = 5$

$$\begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$m_1 = \frac{-1}{1+\epsilon}$

$m_2 = \epsilon - 1$

for $f(x_1) = x_2$.

Now

Let $\epsilon = 0.01 \Rightarrow m_1, m_2 = -0.99099, -0.99$.

They will intersect at a very far away point.

Let $\epsilon = 10 \Rightarrow m_1, m_2 = -0.09, 10$

They are intersecting lines

NOTES

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POWER METHOD

- To numerically find the largest eigenvalue and its corresponding eigenvector.

Let $Ax = b$, $|A| \neq 0$. and $x_1, x_2, \ldots, x_n$ be linearly independent eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$.

This gives us a basis of n vectors.

W.L.O.G., let λ_1 be the largest eigenvalue.

Any $x \in R^n$ can be uniquely expressed as a linear combination of these eigenvectors. i.e.

$$x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$c_i \in \mathbb{R}, \; i = 1, \ldots, n.$$

and $\forall x_i$, $Ax_i = \lambda_i x_i$

$$\Rightarrow Ax = Ac_1 x_1 + \cdots + Ac_n x_n.$$

$$\Rightarrow Ax = c_1 . (\lambda_1 x_1) + \cdots + (c_n \lambda_n) x_n. \quad \text{--- (1)}$$

$$\Rightarrow A^2 x = (c_1 \lambda_1^2) x_1 + \cdots + (c_n \lambda_n^2) x_n.$$

$$\vdots$$

$$\boxed{A^m x = (c_1 \lambda_1^m) x_1 + \cdots + (c_n \lambda_n^m) x_n}$$

Thus notice:

$$\begin{array}{cccc} x, & Ax, & Ax^2, & \ldots \\ \downarrow & \downarrow & \downarrow & \\ \lambda^0 & \lambda^1 & \lambda^2 & \end{array}$$

$\therefore \quad \forall x \in \mathbb{R}^n, \quad \alpha \geq \frac{\|Ax\|_1}{\|x\|_1}$

$$\Rightarrow \sup_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\} \leq \alpha$$

$$\sup_{1 \leq j \leq n} \left\{ \frac{\|Ax\|_1}{\|x\|} \mid x \neq 0 \right\} \leq \alpha.$$

Let ~~$\alpha \leq \beta \leq \alpha$~~

$$\cancel{\sup_{1 \leq j \leq n} \left\{ \frac{\|Ax\|_1}{\|x\|_1} \mid x \neq 0 \right\} \leq \beta \leq \alpha.}$$

Let $\alpha = \sum_{i=1}^{n} |a_{ij_0}| = \|Ae_{j_0}\|_1$

$$\Rightarrow \alpha = \|Ae_{j_0}\| \leq \sup \left\{ \frac{\|Ax\|_1}{\|x\|_1} \mid x \neq 0 \right\}$$

$$\Rightarrow \alpha = \sup \left\{ \frac{\|Ax\|_1}{\|x\|_1} \mid x \neq 0 \right\}$$

$\blacksquare$

(b) P.T. $\|A\|_\infty = \sup_{1 \leq j \leq n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$ max of row sum.

We know $\|A\|_\infty = \sup \left\{ \frac{\|Ax\|_\infty}{\|x\|_\infty} \right\}$

Recall that $\|x\|_\infty = \max_{1 \leq i \leq n} \{ x_i \mid x_i \rightarrow x \}$

Then, ~~$\|A\|_\infty$~~ $= \|Ax\|_\infty = \left\{ \sum_{i=1}^{n} a_{ij} \right.$

$$\|Ax\|_\infty = \max\left\{\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \dots, \sum_{j=1}^{n} a_{nj}x_j\right\}$$

and

$$\|Ax\|_\infty = \max\left\{\sum_{j=1}^{n} a_{ij}x_j \;\middle|\; i=1,\dots n\right\}$$

$$\leq \max\left\{\sum_{j=1}^{n}|a_{ij}| \cdot \sum_{j=1}^{n}|x_j| \;\middle|\; i=1\dots n\right\}$$

$$\cancel{\leq \max\left\{\sum_{j=1}^{n}|a_{ij}| \cdot \ \middle|\; i=1\dots n,\ d=\max_i(x_i)\right\}}$$

~~Let $\sum_{j=1}^{n}|a_{ij_0}| = \alpha$, j_0 giving max.~~

$$\therefore \|Ax\|_\infty \leq \alpha \cdot \max\left\{\sum_{j=1}^{n}|x_j| \;\middle|\; i=1\dots n\right\}$$

$$\Rightarrow \|Ax\|_\infty \leq \alpha \cdot \|x\|_\infty$$

$$\therefore \boxed{\alpha \geq \frac{\|Ax\|_\infty}{\|x\|_\infty}}$$

Now, also notice that

$$\alpha = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|a_{ij}|\right) = \|A\|$$

$\max_{1\leq i\leq n}\|A^T e_i\|_1$

$$\max_{1\leq i\leq n}(A^T x_i) \leq \sup_{1\leq i\leq n}\left\{\frac{\|Ax\|_\infty}{\|x\|_\infty}\right\} \leq \|A\|_\infty.$$

Thus, $\alpha \leq \|A\|_\infty$

$$\Rightarrow \|A\|_\infty = \sup_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}.$$

$$= \beta\alpha = \cancel{\max} \sum_{j=1}^{n} |a_{ij}|$$

$$\therefore \beta \; \alpha = \frac{\|Ay\|_\infty}{\|y\|} \leq \sup \left\{ \frac{\|Ax\|_\infty}{\|x\|_\infty} \,\middle|\, x \neq 0 \right\}$$

$\square$.

(5) define the Hilbert matrix as

$$H_3 = \left[\left(\frac{1}{i+j-1} \right)_{ij} \right]_{3\times3}.$$

$$H_3 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$k(H_3) = \|H\| \cdot \|H^{-1}\| = (1.833) \cdot (408.00)$$
$$= \underline{\underline{747.5640}}.$$

Vandermonde matrix.

$$V_3 = \begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix}$$

$$k(V_3) = \|V_3\| \cdot \|V_3\|^{-1} = 29 \times 15$$
$$= \underline{\underline{435}}.$$

(6)

Consider $Ax = b$

Then, the perturbed system is

$$(A+\delta A)(x+\delta x) = (b+\delta b)$$

$$= \cancel{Ax} + A\delta x + \delta A\cdot x + \delta A\cdot\delta x = \cancel{b} + \delta b$$

$$= A\delta x + \delta A\cdot x + \delta A\cdot\delta x = \delta b$$

$$\Rightarrow \boxed{A\delta x = \delta b - \delta A x - \delta A\cdot\delta x}$$

(or) $\delta x = A^{-1}\cdot[\delta b - \delta A\cdot x - \delta A\cdot\delta x]$

$$\Rightarrow \|\delta x\| \leq \|A^{-1}\|\cdot\|\delta b - \delta A\cdot x - \delta A\cdot\delta x\|$$

Now.

$$\|\delta x\| \leq \|A^{-1}\|\cdot(\|\delta b\| + \|\delta A\|\cdot\|x\| + \|\delta A\|\cdot\|\delta x\|)$$

$$\Rightarrow [1 - \|A^{-1}\|\cdot\|\delta A\|]\|\delta x\| \leq \|A^{-1}\|(\|\delta b\| + \|\delta A\|\cdot\|x\|).$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\cdot\left(\|\delta b\|/\|x\| + \|\delta A\|\right)}{1 - \|A^{-1}\|\cdot\|\delta A\|}$$

Now. $\|x\| \geq \frac{\|b\|}{\|A\|}$ (or) $\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \left[\frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|\delta A\|}\right]\left[\frac{\|\delta b\|}{\|b\|}\|A\| + \|\delta A\|\right]$$

$$= \left[\frac{\|A^{-1}\|\|A\|}{1 - \|A^{-1}\|\cdot\|A\|\frac{\delta A}{\|A\|}}\right]\left[\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right]$$

$$\therefore \frac{\|x - \tilde{x}\|}{\|x\|} \leq \frac{K(A)}{1 - K(A)\frac{\|\delta A\|}{\|A\|}} \cdot \left[\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right]$$

$\square$

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28/02/18
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matrix eigenvalue problem

(Power method).

Let $A \in \mathbb{R}^{n \times n}$. $\exists$ a basis such that $Ax_i = \lambda x_i$ $\forall\, i = 1, \dots, n.$ (1). Then,

- $x^{(0)} = c_1 x_1 + \dots + c_n x_n.$

$x^{(1)} = Ax^{(0)} = c_1 A x_1 + \dots + c_n A x_n = c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n.$

- (or) $x^{(1)} = c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n$

- $Ax^{(1)} = x^{(2)} = c_1 \lambda_1^2 x_1 + \dots + c_n \lambda_n^2 x_n.$

$\vdots$

- $Ax^{(n-1)} = \boxed{x^{(n)} = c_1 \lambda_1^n x_1 + \dots + c_n \lambda_n^n x_n.}$

Now,

$$x^{m} = c_1 \lambda_1^m x_1 + \dots + c_n \lambda_n^m x_n.$$

$$= \lambda_1^m \left(c_1 x_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^m x_2 + c_3 \left(\frac{\lambda_3}{\lambda_1}\right)^m x_3 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^m x_n \right)$$

W.L.O.G, assume $\lambda_1 > \lambda_i$ $\forall\, i = 2, \dots, n.$

Then, as $m \to 0$, $(\lambda_i/\lambda_1) \to 0$ $\forall\, i.$

$$\therefore \boxed{\lim_{m \to \infty} x^{(m)} = \lambda_1^m c_1 x_1.}$$

and

$$\boxed{x^{(m+1)} = \lambda^{m+1}\left[c_1 x_1 + e^{(m+1)}\right]}$$

Define a linear map from $\mathbb{R}^n \to \mathbb{R}$.

i.e. $\phi : \mathbb{R}^n \to \mathbb{R}^n$ s.t $\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$.

Then,

$$\phi(x^{(m)}) = \phi(\lambda^{m}[c_1x_1 + e^{(m)}]). = \underline{\lambda^m . \phi(c_1x_1 + e^{(m)})}$$

and

$$\frac{\phi(x^{(m+1)})}{\phi(x^{(m)})} = \lambda_1 . \frac{\phi(c_1x_1 + e^{(m+1)})}{\phi(c_1x_1 + e^{m})}$$

$$\Rightarrow \lim_{m\to\infty} \frac{\phi(x^{(m+1)})}{\phi(x^{(m)})} = \lambda_1 . \lim_{m\to 0} \frac{\phi(c_1x_1 + e^{(m+1)})}{\phi(c_1x_1 + e^{(m)})} = \underline{\underline{\lambda_1}}$$

$\square$

Eg

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad x^{(0)} = (1,1).$$

$$Ax^{(0)} = x_0^{(1)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$$

$$x_2 = Ax^{(1)} = 5\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \end{bmatrix} = \frac{19}{5}\begin{bmatrix} 13/19 \\ 1 \end{bmatrix}$$

$$x_3 = Ax^{(2)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} 19/5 \begin{bmatrix} 13/19 \\ 1 \end{bmatrix} = \frac{77}{19}\begin{bmatrix} 51/77 \\ 1 \end{bmatrix}$$

$$x_4 = Ax^{(3)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\frac{77}{19}\begin{bmatrix} 51/77 \\ 1 \end{bmatrix} = \begin{bmatrix} 205/77 \\ 307/77 \end{bmatrix} = 3.98\begin{bmatrix} 0.667 \\ 1 \end{bmatrix}$$

Exact $\underline{\text{eigenvector}}$ $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

$$\therefore A\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

λ $\bar{x}$

The other eigenvector is. $(1,-1)$.

$$A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = (-1)\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\lambda_2 = -1, \quad x = (1,-1)$$

* To find the minimum, use A^{-1} instead of A.

$$x^{(0)} = c_1 x_1 + \dots + c_n x_n$$

$$x^{(1)} = A^{-1} x^{(0)} = c_1 \frac{x_1}{\lambda_1} + \dots + c_n \frac{x_n}{\lambda_n}.$$

$$\vdots$$

$$x^{(m)} = \cancel{(A)}\, A^{-1} x^{(m-1)} = \frac{c_1 x_1}{\lambda^m} + \dots + \frac{c_n x_n}{\lambda^m}$$

* Convergence is ~~a~~ sufficiently guaranteed when you scale the vectors.

* If finding A^{-1} is tricky,

$$A^{-1} x^{(0)} = x^{(1)} \Rightarrow \boxed{x^{(0)} = A x^{(1)}}$$

and this linear system has to be solved everytime, for $x^{(m)}$:

i.e. $x^{(m-1)} = A x^{(m)}$

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5 Mar 2019
Class.

Weirstrass approximation Theorem

Let $f:[a,b] \to \mathbb{R}$ be continuous. Given $\epsilon > 0$,
$\exists\, n(\epsilon)$ s.t.

$$|f(x) - p_n(x)| < \epsilon \quad \forall\, x \in [a,b]$$

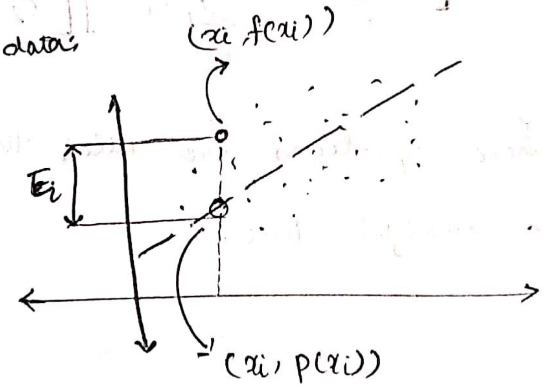
where p_n is a polynomial of degree $\leq n$.

NOTE $n : n(\epsilon)$ is a function of $\epsilon \Rightarrow$ uniform convergence.

Alternately stated,

ANY CONT. FUNCTION ON A BOUNDED INTERVAL CAN BE UNIFORMLY APPROXIMATED BY POLYNOMIALS.

To use this theorem for discrete data:



Let data be $\{(x_i, f(x_i)) \mid \mathbb{R}\}$.
$i = 1, \dots, N$.

choose a reference point and compute the error as

$$E_i = f(x_i) - p(x_i), \; i = 1, 2, \dots, N.$$

$\boxed{p(x) = c_1 + c_2 x.}$
Approximation function

Then,

$$I(c_1, c_2) = \sum_{i=1}^{N} [E(x_i)]^2$$

$$= \sum_{i=1}^{N} [f(x_i) - p(x_i)]^2 = \sum_{i=1}^{N} [f(x_i) - (c_1 + c_2 x_i)]^2$$

To get minima/maxima

$$\frac{\partial I}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial I}{\partial c_2} = 0.$$

$$\Rightarrow \sum_{i=1}^{N} 2\left[f(x_i) - (c_1 + c_2 x_i)\right](-1) = 0. \qquad \Big| \quad \frac{\partial I}{\partial c_1} = 0.$$

and $\frac{\partial I}{\partial c_2} = 0 \Rightarrow \sum_{i=1}^{N} 2\left[f(x_i) - (c_1 + c_2 x_i)\right](-x_i) = 0$

The ~~first~~ summands gives :

$$\sum_{i=N}^{N} f(x_i) = nc_1 + c_2 \sum_{i=1}^{N} x_i$$

and $\sum_{i=N}^{N} x_i f(x_i) = \sum_{i=1}^{N} x_i c_1 + c_2 \sum_{i=1}^{N} x_i^2$

$$\Rightarrow \begin{bmatrix} n & \sum_{i=1}^{N} x_i \\ \sum_i x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} f(x_i) \\ \sum_{i=1}^{N} x_i f(x_i) \end{bmatrix}$$

These equations are called the normal equations for a straight line.

NOTE For a quadratic fit : $p(x) = c_1 + c_2 x + c_3 x^2$

$$\begin{bmatrix} N & \sum \phi(x_i) & \sum \phi(x_i)^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \sum f(x_i) \\ \sum x_i f(x_i) \\ \sum x_i^2 f(x_i) \end{bmatrix}$$

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Eg Find a least square fit (straight line and quadratic) for the following data:

x	-0.5	1	1.5	2	2.5
$f(x)$	0.75	3	4.75	7	9.75

LINEAR

$$A = \begin{bmatrix} 5 & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} = \begin{bmatrix} 5 & 6.5 \\ 6.5 & 13.75 \end{bmatrix}$$

2.25
1.5

$$b = \begin{bmatrix} 25.25 \\ 48.125 \end{bmatrix}$$

$1/8$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} A^{-1} \end{bmatrix} b = \begin{bmatrix} 1.2972 \\ 2.8868 \end{bmatrix}$$

QUADRATIC

$$p(x) = c_1 + c_2 x + c_3 x^2.$$

$$\begin{bmatrix} 5 & 6.5 & 13.75 \\ 6.5 & 13.75 & 27.875 \\ 13.75 & 27.875 & 61.0625 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 25.25 \\ 48.125 \\ 102.4375 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

NORMAL EQUATIONS

FOR CONT. FUNCTIONS

Let $f : [a,b] \to \mathbb{R}$ be continuous.

For a linear fit,

$$p(x) = c_1 + c_2 x.$$

and

$$I = \int_a^b [f(x) - p(x)]^2 dx, \quad I(c_1, c_2) :\to \mathbb{R}.$$

$$\Rightarrow \frac{\partial I}{\partial c_1} = \frac{\partial}{\partial c_1} \int_a^b [f(x) - (c_1 + c_2 x)]^2 dx.$$

$$= \int_a^b \frac{\partial}{\partial c_1} [f(x) - (c_1 + c_2 x)]^2 dx = 0$$

$$\Rightarrow \int_a^b 2 \cdot [f(x) - (c_1 + c_2 x)](-1)\, dx = 0$$

Similarly.

$$\frac{\partial I}{\partial c_2}(c_1, c_2) = 0$$

$$\Rightarrow \int_a^b 2 [f(x) - (c_1 + c_2 x)](-x)\, dx$$

and so:

$$\begin{bmatrix} \int_a^b dx & \int_a^b x\,dx \\ \int_a^b x\,dx & \int_a^b x^2 dx. \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_a^b f(x)\,dx \\ \int_a^b x f(x)\,dx \end{bmatrix}$$

NOTE

for a quadratic fit,

$$p(x) = c_1 + c_2 x + c_3 x^2 \; ; \quad x \in [a,b]$$

$$\begin{bmatrix} \int_a^b dx & \int_a^b x\,dx & \int_a^b x^2\,dx \\ \int_a^b x\,dx & \int_a^b x^2\,dx & \int_a^b x^3\,dx \\ \int_a^b x^2\,dx & \int_a^b x^3\,dx & \int_a^b x^4\,dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \int_a^b f(x)\,dx \\ \int_a^b x f(x)\,dx \\ \int_a^b x^2 \cdot f(x)\,dx \end{bmatrix}$$

Ex. $f(x) = x^3 \; , \; x \in [0,1]$