$$y' = f(x,y)$$

$$y(0) = y_0$$

$$\rightarrow y(t) = y_0 + \int_0^t f(x,y)\,dx$$

$$\rightarrow y_1 = y_0 + \int_0^t f(x,y_0)\,dx$$

$$\rightarrow y_1 = y_0 + \int_0^t f(x,y_0)\,dx$$

$$\rightarrow y_2 = y_0 + \int_0^t f(x,y_1)\,dx$$

$$\rightarrow y_3 = y_0 + \int_0^t f(x,y_2)\,dx$$

$$\vdots$$

$$y_n = y_0 + \int_0^t f(x,y_{n-1})\,dx$$

$$\left\{ \begin{array}{l} y_n \longrightarrow y \\ \text{as } n \longrightarrow \infty \end{array} \right.$$

Error $\qquad x \nearrow$ exact value $\qquad x_A \nearrow$ Approx value of x

- Error $= |x - x_A|$

- Relative error $= \dfrac{|x - x_A|}{|x|}$

- significant figures : 1, 2, ... 9
  0 is also significant fig except when used to fix the decimal point or when it is used in place of unknown or discarded digits

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	sig fig
4309	4
0.004309	4
1.0409	5

$430\underline{900}$

cannot be decided normally coz we dont know if they are used in place of discarded or unknown digits

4.309×10^5 —— 4

4.30900×10^5 —— 6

→ Binary to decimal

$$(111.011)_2 = 1\times2^2 + 1\times2^1 + 1\times2^0 + 0\times2^{-1} + 1\times2^{-2} + 1\times2^{-3}$$

$$= 7.375$$

→ Decimal to binary

$$(58)_{10} = 111010$$

$$\begin{array}{r|l}
2 & 58 - 0 \\
2 & 29 - 1 \\
2 & 14 - 0 \\
2 & 7 - 1 \\
2 & 3 - 1 \\
 & 1
\end{array}$$

$$\underbrace{(0.859375)}_{N_k}{}_{10}$$

$$b_k = \begin{cases} 1 & \text{if } 2N_k \geq 1 \\ 0 & \text{if } 2N_k < 1 \end{cases}$$

K	b_{-K}	N_K
0		0.859375×2
	1	0.718750×2
	1	0.437500×2
	0	0.875000×2
	1	0.750000×2
	1	0.50000×2
	1	0

→ product - b_K

$(0.110111)_2$

$(4309.38)_{10}$

$= 4 \times 10^3 + 3 \times 10^2 + 0 \times 10 + 9 \times 10^0 + 3 \times 10^{-1} + 8 \times 10^{-2}$

ITERATIVE METHODS.

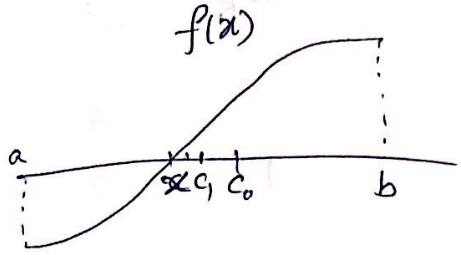
1) Bisection Method.

2) Regular Falsi / Secant Method.

3) Newton's Method.

Bolzano's thm

$f(x) = x^3 - 3x + 1$



Bisection method.

$f(a) \not> 0$
$f(b) > 0$ then there exist $x = 0$
$f(x) = 0$

it will have root $[a, b]$

$c_0 = \frac{a+b}{2}$

$f(a) \not< 0$
$f(c_0) > 0$ then we can shorten our interval

$[a, c_0]$

$c_1 = \frac{a + c_0}{2}$

$f(a) < 0$
$f(c_1) > 0$ then we shorten our interval.

$[a, c_1]$

⋮

$c_2 = \frac{a + c_1}{2}$

$f(c_2) < 0$

$f(a) < 0$

$\left.\begin{array}{l} f(c_2) < 0 \\ f(c_1) > 0 \end{array}\right\}$ then by bolzanos thm we choose this

new interval $[c_2, c_1]$

$c_3 = \frac{c_2 + c_1}{2}$

we can stop when $|c_3 - x| < \epsilon$

$|c_n - c_{n+1}| < \epsilon \}$ → stopping criteria.

for equations which are above the axis we cant use bisection method.

Ex → tangential eq

→ Ex : $x^3 - 3x + 1$.

$f(0) > 0$

$f(1) < 0$

f has root in $[0, 1]$

→ $\cancel{\frac{0+1}{2} = \frac{1}{2}}$

$\cancel{f(\frac{1}{2})}$

If we know $c_0 = 0$. length of interval w are tak as er

$|c_n - c_{n+1}|$

n	c_n	$f(c_n)$	error
0	0.5	-0.375	0.5
$[0, 0.5]$			
1	0.25	0.266	0.25
$[0.25, 0.5]$			
2	0.375	-7.23×10^{-2}	0.125
$[0.25, 0.375]$			
3			

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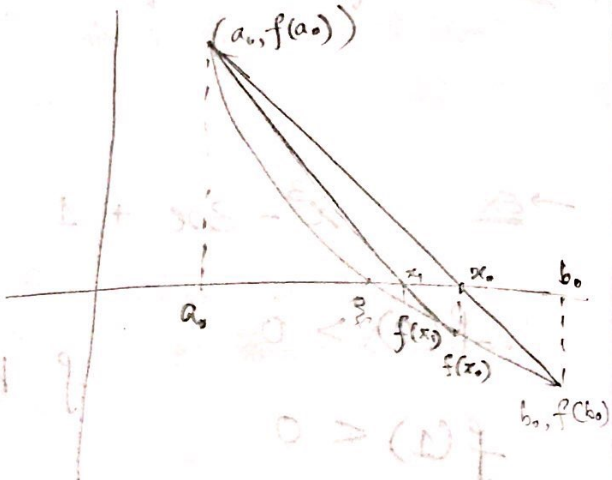
* Regularfalsi (~~sheo~~ for shear convergence)

* Secant / Regularfalsi Method



$f(a_0)f(b_0) < 0$

draw secant bet^n a_0 b_0

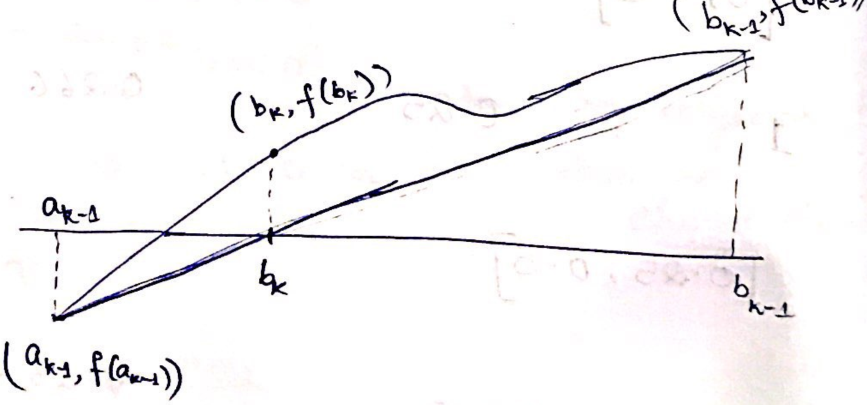
$[a_0 \; x_0]$

as $f(a_0)f(x_0) < 0$ → if it is >0
change it to $[x_0, b_0]$

now draw secant bet^n a_0 x_1

$[a_0 \; x_1]$ as $f(a_0)f(x_1) < 0$

x_1
$\vdots$
x_n



$$\frac{b_k - a_{k-1}}{-f(a_{k-1})} = \frac{b_{k-1} - b_k}{f(b_{k-1})}$$ [similar Δ^{les}]

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$$\frac{b_k - a_{k-1}}{-f(a_{k-1})} = \frac{b_{k-1} - b_k}{f(b_{k-1})}$$

$b_k = c$

$b_{k-1} = b$

$a_{k-1} = a$

$$\frac{b-c}{f(b)} = \frac{c-a}{-f(a)}$$

$$\Rightarrow \quad c = b - f(b)\,\frac{a-b}{f(a)-f(b)}$$

$$= \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$\Rightarrow \quad \boxed{x_{n+1} = x_n - f(x_n)\,\frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)}}$$

→ iteration method for secant/regula falsi

If $f(x_{n+1})\,f(x_n) < 0$

$[x_n , x_{n+1}]$

else $f(x_{n+1})\,f(x_{n-1}) < 0$

$[x_{n-1} , x_{n+1}]$

→ regula falsi cond.

Ex

* $f(x) = x^3 - 5x + 1 = 0 \qquad [0,1]$

$x_0 = 0 \quad , \quad x_1 = 1$

$f(x_0) = 1 \qquad f(x_1) = -3$

$$\rightarrow x_2 = x_1 - f(x_1)\,\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 0.25$$

$f_2 = f(x_2) = -0.234375 < 0$

By secant method $\Rightarrow$ new interval $[x_1, x_2] = [a, b]$

By regular falsi $\Rightarrow$ $[x_0, x_2] = [a, b]$

(In regular falsi we keep checking Bolzano's thm)

<u>by secant method</u>

$$x_3 = x_2 - f(x_2) \frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

$$= 0.186441$$

$f(x_3) = 0.74276$

$$x_4 = x_3 - f(x_3) \frac{x_3 - x_2}{f(x_3) - f(x_2)} = 0.201736$$

$f(x_4) = -0.000470$

$$x_5 = x_4 - f(x_4) \frac{x_4 - x_3}{f(x_4) - f(x_3)} = 0.201640.$$

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by regular falsi

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 0.25$$

$$f(x_2) = -0.234375$$

$[x_0, x_2]$, $x_3 = x_2 - f(x_2)\dfrac{x_2 - x_0}{f(x_2) - f(x_0)} = 0.20232$

$$f(x_3) = -0.004352$$

$$f(x_3)f(x_2) > 0$$

$$\therefore \xi \in (x_0, x_3)$$

$$x_4 = x_3 - \cancel{x}\, f(x_3)\frac{x_3 - x_0}{f(x_3) - f(x_0)} = 0.201654$$

$$f(x_4) = 0.000070$$

$[x_4, x_3]$

$$\xi \in [x_4, x_3]$$

order of convergence of secant method is super linear. [1,2]

$$x_5 = x_3 - f(x_3)\frac{x_3 - x_4}{f(x_3) - f(x_4)}$$

$$(f(x_n) < \epsilon \text{ or } |x_n - x_{n+1}| < \epsilon)$$

Taylor series

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{k}(c)\ (x-c)^{k} + E_n(x)$$

$\rightarrow$ error

$$E_n(x) = \frac{1}{(n+1)!} f^{n+1}(\xi)(x-c)^{n+1}$$

Maclaurin's series $\quad c = 0$

$$f(x) = \sum \frac{1}{k!} f^{k}(0)\, x^{k} + E_n(x)$$

$$E_n(x) = \frac{1}{(n+1)!} f^{n+1}(\xi)\, x^{n+1}$$

$$f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{k}(x) + E_n(h)$$

How many terms in the series need to be used to compute $\ln 2$ with accuracy of one part of 10^{+8}

Sol^n

Let $x = 2$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots + (-2)^{n-1}\frac{1}{n} + E_n(2)$$

$$|E_n(2)| < \frac{1}{n+1}$$

$\because$ it is alternating series $\quad \left|\frac{1}{n+1}\right| < \epsilon = 10^{-8}$

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$\Rightarrow \quad n+1 \geq 10^{+8}$

Q/1.

Bisection method convergence Analysis.

Let r be the solution $f(r) = 0$ $\quad r \in [a_0, b_0]$ st $f(a_0)f(b_0) < 0$

$$c_0 = \frac{a_0 + b_0}{2}$$

$$|r - c_0| \leq \frac{b_0 - a_0}{2}$$

$$|r - c_n| \leq \frac{b_n - a_n}{2} \qquad r, c_n \in [a_n \; b_n]$$

order of convergence for BM is 1 linear method.

$$\leq \frac{b_0 - a_0}{2^{n+1}} \qquad\qquad c_n = \frac{a_n + b_n}{2}$$

$$< \epsilon$$

$$\Rightarrow \frac{b_0 - a_0}{2^{n+1}} < \epsilon$$

how many times to iterate we can calculate from

$$\left\{ n > \frac{\log(b_0 - a_0) - \log 2\epsilon}{\log 2} \right.$$

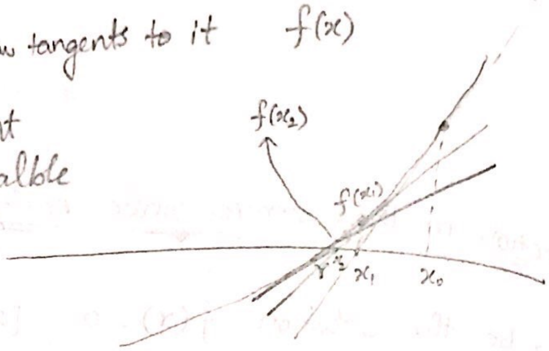
$$\frac{|x_{n+1} - r|}{|x_n - r|^{P}} < M$$

some upper bound.

order of convergence

Newton Raphson Method.

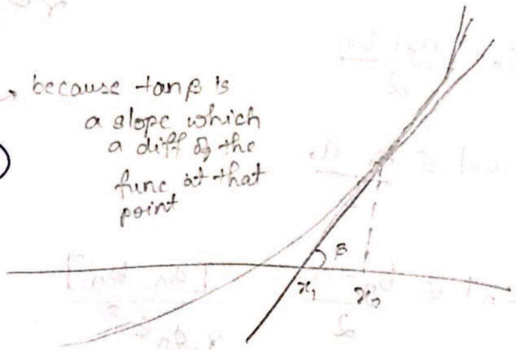
we need 1 initial point and draw tangents to it



If a curve has 2 tangents at a point it means that it is not differentiable and it ~~does~~ may not have a sol^n

$$\tan\beta = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

because $\tan\beta$ is a slope which a diff of the func at that point



$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

newton raphson is quadratic order of convergence

$$\left.x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n=1})}{x_n - x_{n-1}}}\right\} \rightarrow \text{formula for secant method}$$

we use secant method when the curve is not differentiable

Let $f(x) = 0$ $\qquad x =$ exact root

x_k approx root

$x_k + \Delta x = x$

$f(x_k + \Delta x) = 0$

$$f(x_k) + \Delta x\, f'(x_k) + \frac{(\Delta x)^2}{2} f''(x) + \cdots = 0$$

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$$f(x_k) + \Delta x\, f'(x_k) \approx 0$$

we can ignore higher orders

$$\Delta x \approx \frac{-f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

* Perform 4 iterations of N.R.M to find the smallest +ve root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

Let $x_0 = 0.5$

$$f'(x) = 3x^2 - 5$$

If convergence is slow we multiply the order of convergence m to the eq

$$x_{n+1} = x_n - \frac{M f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.176471$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.201568$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.201640$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.201640$$

* Rate of convergence.

An iterative method is said ~~to~~ be of order 'p' or has the rate of convergence 'p' if 'p' is the largest +ve real no. for which ∃ a infinite const ~~&~~ $c \neq 0$ s.t

$$|\epsilon_{k+1}| \leq c\,|\epsilon_k|^p \quad , \quad \epsilon_k = x_k - \xi$$

→ Secant method.

$$x_{k+1} = x_k - f(x_k)\,\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

~~Subtract exact root ξ from both sides~~

$$\cancel{\epsilon_{k+1}} \cancel{=} \cancel{\epsilon_k}$$

$$\underset{\downarrow}{x_k} = \overset{\nearrow \text{exact root}}{\xi} + \underset{\searrow \text{error}}{\epsilon_k}$$

K_{th} iterate of x

$$x_{k+1} = \xi + \epsilon_{k+1}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\,[f(\xi + \epsilon_k)]}{f(\epsilon_k + \xi) - f(\epsilon_{k-1} + \xi)}$$

taylor's series

$$\Rightarrow \epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[\epsilon_k f'(\xi) + \frac{(\epsilon_k)^2}{2} \times f''(\xi) + \cdots\right]}{(\epsilon_k - \epsilon_{k-1}) f'(\xi) + \frac{1}{2}(\epsilon_k^2 - \epsilon_{k-1}^2) f''(\xi) + \cdots}$$

expansion of taylor series

$$\Rightarrow \epsilon_{k+1} = \epsilon_k - \left[\epsilon_k + \frac{(\epsilon_k)^2}{2} \frac{f''(\xi)}{f'(\xi)}\right] \times \left[1 + \frac{1}{2}(\epsilon_k + \epsilon_{k-1}) \frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$\Rightarrow \epsilon_{k+1} = \frac{1}{2} \frac{f''(\xi)}{f'(\xi)} \epsilon_k \epsilon_{k-1} + O\left(\epsilon_k^2 \epsilon_{k-1} + \epsilon_k \epsilon_{k-1}^2\right)$$

10/1/3

$$\frac{|\epsilon_{k+1}|}{|\epsilon_k|^p} < C$$

for bisection rate of convergence = 1

$$\leq \frac{\frac{b_0 - a_0}{2^{n+1}}}{\left(\frac{b_0 - a_0}{2^n}\right)^1} = \frac{1}{2} C$$

Order of convergence of regular falsi

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

ξ = exact root off $\Rightarrow f(\xi) = 0$

$$x_k = \xi + \epsilon_k$$

$$x_{k+1} = \xi + \epsilon_{k+1}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) f(\xi + \epsilon_k)}{f(\xi+\epsilon_k) - f(\xi + \epsilon_{k-1})}$$

$$= \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left(\epsilon_k f'(\xi) + \frac{1}{2}\epsilon_k^2 f''(\xi) + \cdots\right)}{(\epsilon_k - \epsilon_{k-1})\left(f'(\xi)\right) + \frac{1}{2}(\epsilon_k^2 - \epsilon_{k-1}^2) f''(\xi) + \cdots}$$

$$\epsilon_{k+1} = \epsilon_k - \left[\epsilon_k + \frac{1}{2}\epsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots\right] \times \left[1 + \frac{1}{2}(\epsilon_{k-1} + \epsilon_k)\frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \epsilon_k \epsilon_{k-1} + O\left(\epsilon_k^2 \epsilon_{k-1} + \epsilon_k \epsilon_{k-1}^2\right)$$

$f = O(g)$ f goes to 0 atleast as fast as g

$f = o(g)$ f goes to 0 faster than g

we have to eliminate ϵ_{k-1} term coz there is no ϵ_{k-1} term is rate of convergence formula.

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_k\epsilon_{k-1} + O(\epsilon_k^2\epsilon_{k-1} + \epsilon_k\epsilon_{k-1}^2) \longrightarrow (*)$$

$$\cancel{\epsilon_{k+1}} = \cancel{A\epsilon}$$

Let $\epsilon_{k+1} = A\epsilon_k^p \longrightarrow (1)$

$$\epsilon_k = A\epsilon_{k-1}^p$$

$$\epsilon_{k-1} = \epsilon_k^{1/p}A^{-1/p} \longrightarrow (2)$$

substitute (1) & (2) in (*)

$$A\epsilon_k^p = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_k\epsilon_k^{1/p}A^{-1/p}$$

$$\epsilon_k^p = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_k^{1+1/p}A^{-(1+1/p)}$$

→ equating exponent of ϵ_k & coeff of ϵ_k

$$p = 1+\frac{1}{p} \Rightarrow p = \frac{1}{2}(1\pm\sqrt{5})$$ regular falsi is super linear

$$A^{-(1+\frac{1}{p})}\frac{1}{2}\frac{f''(\xi)}{f'(\xi)} = 1$$

→ big O

$f(h) = O(g(h))$ atleast as fast as g

as $h \to 0 \Rightarrow$ if $\exists$ some const C

s.t $\left|\dfrac{f(h)}{g(h)}\right| < \epsilon$ $\forall$ h sufficiently small

→ Little o

$f(h) = o(g(h))$ as $h \to 0$

if $\lim_{h \to 0} \left|\dfrac{f(h)}{g(h)}\right| = 0$

$$\text{error } f(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt \qquad \tilde{t} = \frac{t}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\tilde{t}^2/2}\,d\tilde{z}$$

16/1.

Modified Newton's Method for roots with multiplicity more than 1

→ For roots with multiplicity 1

$$x_{n+1} = x_n - \alpha \frac{f(x_n)}{f'(x_n)}$$

$f(\xi) = 0$, ξ is a exact root of multiplicity m

$$f'(\xi) = 0 = f''(\xi) = \cdots = f^{m-1}(\xi) = 0$$

$$f^{m}(\xi) \neq 0$$

→ $x_n = \xi + \epsilon_n$

$x_{n+1} = \xi + \epsilon_{n+1}$

substituting in ①

$$\epsilon_{n+1} = \epsilon_n - \alpha \frac{f(\xi + \epsilon_n)}{f'(\xi + \epsilon_{n})}$$

$$\epsilon_{n+1} = \epsilon_n - \alpha \frac{\frac{\epsilon_n^m}{m!} f^m(\xi) + \frac{\epsilon^{m+1}}{(m+1)!} f^{(m+1)}(\xi) + \cdots}{\frac{\epsilon_n^{m-1}}{(m-1)!} f^m(\xi) + \frac{\epsilon_n^m}{m!} f^{m+1}(\xi) + \cdots}$$

$$f'(\xi + \epsilon_n) = \cancel{f'(\xi)} + \cancel{\epsilon_n f''(\xi)} + \frac{\epsilon_n^2}{2!} \cancel{f'''(\xi)}$$

$$+ \frac{\epsilon_n^{m-1}}{(m-1)!} f^m(\xi) + \frac{\epsilon_n^m}{m!} f^{m+1}(\xi)$$

∵ ξ is root of multiplicity.

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$$\epsilon_{n+1} = \epsilon_n - \alpha\left(\frac{\epsilon_n}{m} + \frac{\epsilon_n^2}{(m+1)m}\frac{f^{m+1}(\xi)}{f^m(\xi)} + \cdots\right)$$

$$* \left(1 + \frac{\epsilon_n}{m}\frac{f^{m+1}(\xi)}{f^m(\xi)} + \cdots\right)^{-1}$$

$$\epsilon_{n+1} = \epsilon_n\left(1 - \frac{\alpha}{m}\right) + \alpha\left(-\frac{\epsilon_n^2}{(m+1)m} + \frac{\epsilon_n^2}{m^2}\right)\frac{f^{m+1}}{f^m}(\xi)$$

$$\epsilon_{n+1} = \epsilon_n\left(1 - \frac{\alpha}{m}\right) + \frac{\epsilon_n^2}{m^2(m+1)}\frac{f^{m+1}(\xi)}{f^m(\xi)} + \cdots$$

$$|\epsilon_{n+1}| = C|\epsilon_n|^p$$

To make order of convergence quadratic

make coeff of $\epsilon_n = 0$

$m = 1$

$\cancel{x_{n+1} = x_n - m\frac{f(x_n)}{f'(}}$

If $m > 1$

$1 - \frac{\alpha}{m} = 0$ if $m = \alpha$

* $f(x) = x^3 - 7x^2 + 16x - 12 = 0$

has double root at $x = 2$ $\quad x_0 = 1$ find root correct to 3 decimal places

using (i) N.R.M (ii) modified NRM with $m = 2$

(i) $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$x_0 = 1$

$x_1 = 1.4$, $x_2 = 1.6526$

$x_3 = 1.8$ $\cdots$ $x_{12} = 1.99953$

(ii) $$x_{k+1} = x_k - 2\frac{f(x_k)}{f'(x_k)}$$

$x_0 = 1$, $x_1 = 1.8$

$x_2 = 1.984 \cdots$

$x_3 = 1.99984$

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System of non linear equations

$f(x,y) = 0$

$g(x,y) = 0$

Let ξ, η exact root

k^{th} iteration ~~$x_k + \epsilon$~~ error

$x_k + \Delta x = \xi \Rightarrow f(x_k + \Delta x, y_k + \Delta y) = 0$

$y_k + \Delta y = \eta \Rightarrow g(x_k + \Delta x, y_k + \Delta y) = 0$

$$0 = f(x_k, y_k) + \left(\frac{\partial f(x_k, y_k)}{\partial x} \Delta x + \frac{\partial f(x_k, y_k)}{\partial y} \Delta y \right)$$

$$+ \frac{1}{2!} \left[\Delta x \frac{\partial f(x_k, y_k)}{\partial x} + \Delta y \frac{\partial f(x_k, y_k)}{\partial y} \right]^2 \rightarrow 0$$

error is small, can be ignored

$$0 = g(x_k, y_k) + \left(\Delta x \frac{\partial g}{\partial x}(x_k, y_k) + \Delta y \frac{\partial g}{\partial y}(x_k, y_k) \right)$$

$$+ \frac{1}{2!} \left[\Delta x \frac{\partial g(x_k y_k)}{\partial x} + \Delta y \frac{\partial g(x_k y_k)}{\partial y} \right]^2 + \cdots \rightarrow 0$$

$$-f(x_k\, y_k) = \frac{\partial f(x_k\, y_k)}{\partial x}\,\Delta x + \frac{\partial f(x_k, y_k)}{\partial y}\,\Delta y$$

$$-g(x_k\, y_k) = \frac{\partial g(x_k\, y_k)}{\partial x}\,\Delta x + \frac{\partial g(x_k, y_k)}{\partial y}\,\Delta y$$

$$F(x_k, y_k) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

newton · R for system of n.l eq

$$J_k = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x_k, y_k)}$$

$$F_k = \begin{bmatrix} f \\ g \end{bmatrix}_{(x_k, y_k)} \qquad \Delta X = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\Delta X = -J_k^{-1} F(x_k\, y_k)$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$= \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J_k^{-1} \begin{bmatrix} f(x_k\, y_k) \\ g(x_k\, y_k) \end{bmatrix}$$

$$f_1(x_1, x_2, x_3 \cdots x_n) = 0$$

$$f_2(x_1, x_2, \cdots \cdots x_n) = 0$$

$$\vdots$$

$$f_n(x_1, x_2 \cdots \cdots x_n) = 0$$

$$J_k = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(x_k\ y_k)}$$

The convergence of the method depends on vector $x^{(0)}$

A sufficient condition for convergence is that for each k

$$\|J_k^{-1}\| < 1$$

$$\rho(J_k^{-1}) < 1, \quad \text{iff}$$

suff cond.

(7)1 * If sin series is used in evaluating $\sin 1$ with error less than $\frac{1}{2}\times 10^{-6}$ how many terms are needed?

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \cdots\cdots$$

$\cancel{|S - S_n| \le a_{n+1}}$

$$|S - S_n| \le a_{n+1}$$

$$\frac{1}{(2n+1)!} \le \frac{1}{2}\times 10^{-6} \Rightarrow \log(2n+1)! > \log 2 + 6$$

$$\log(2n+1)! > 6.3$$

$$\log 10! \approx 6.6$$

$$\Rightarrow n > 5$$

* $$\frac{\pi^4}{90} = 1^{-4} + 2^{-4} + 3^{-4} + \cdots$$

How many terms should we take to find $\frac{\pi^4}{90}$ with error atmost $\frac{1}{2}\times 10^{-6}$

$$a_n = (n+1)^{-4} < \frac{1}{2}\times 10^{-6} \qquad n = 37$$

$$S_{37} = \sum_{K=1}^{37} K^{-4} \text{ differs from } \frac{\pi^4}{90} \text{ by } \simeq 6\times 10^{-6}$$

Another method.

$$\sum_{K=n+1}^{\infty} K^{-4} < \frac{1}{2}\times 10^{-6}$$

$$LHS < \int_n^{\infty} x^{-4}\,dx < \frac{1}{2}\times 10^{-6}$$

$$\Rightarrow -\frac{x^{-3}}{3}\Big]_n^{\infty} = \frac{1}{3n^3} < \frac{1}{2}\times 10^{-6}$$

$$\Rightarrow n \ge 88$$

* **conditioning** If it is well conditioned it is has iterations which converge

How sensitive the solution of a problem may be to small relative changes in input data

If condition number less it is well conditioned problem and it has convergence

* **MVT**

$$f(x+h) - f(x) = f'(\xi)h$$

$$\rightarrow \frac{f(x+h) - f(x)}{f(x)} = \frac{f'(\xi)h}{f(x)}$$ divide by $f(x)$

$$\approx \left(\frac{h}{x}\right)\frac{f'(x)}{f(x)} \cdot x = \left(\qquad \right)$$ multiply & divide by x and replace ξ with x

The factor $x\dfrac{f'(x)}{f(x)}$ serves as condition number

* $f(x) = \sin^{-1}x$

$$\frac{x f'(x)}{f(x)} = \frac{x}{\sin^{-1}x\,\sqrt{1-x^2}}$$

as $x \rightarrow \pm 1$ $\quad \sin^{-1}x \rightarrow \frac{\pi}{2}$

$\rightarrow -\infty$

$\therefore$ small relative error in x near $x = 1$ may lead to large relative error in $\sin^{-1}x$

Wilkinson eg :

$$f(x) = \prod_{k=1}^{20} (x-k) = (x-1)(x-2)\cdots(x-20)$$

$\boxed{f \text{ to } f+\epsilon g}$

$$g(x) = x^{20}$$

How is root 20 affected by perturbing f

$$f(x+h) - f(x) = h f'(\xi) \approx h f'(x)$$

$$\cancel{f(x)} + \epsilon g(x) - \cancel{f(x)} \approx h f'(x)$$

$$\epsilon g(x) \approx h f'(x)$$

ϵ is change in o/p

h is change in i/p

$$h \approx \frac{\epsilon g(x)}{f'(x)}$$

at $x = 20$

$$h \approx \frac{\epsilon 20^{20}}{19!} \approx \epsilon 10^{9}$$ root gets disturbed by this factor.

→ condition no. $Ax = b$

matrix ~~has~~ A has condition no

$$K(A) = \|A\| \|A^{-1}\|$$

$$\boxed{K(A) = \|A\| \|A^{-1}\|}$$

$$\bar{x}^{(k+1)} = \bar{x}^{k} - J_k^{-1} F(\bar{x}^{(k)})$$

$k = 0, 1, 2 \ldots$

$$J_k = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\bar{x}_k}$$

Take 1 step from a suitable point with N R M

$10x + \sin(x+y) = 1$ $\qquad x_0 = \frac{1}{10}$

$8y - \cos^2(\cancel{z}-y) = 1$ $\qquad y_0 = \frac{1}{4}$

$12z + \sin z = 1$ $\qquad z_0 = \frac{1}{12}$

$f_1(x,y,z) = 10x + \sin(x+y) - 1 = 0$

$f_2(x,y,z) = 8y - \cos^2(z-y) - 1 = 0$

$f_3(x,y,z) = 12z + \sin z - 1 = 0$

$$J_k = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}_{\bar{x}_k}$$

$$= \begin{bmatrix} 10 + \cos(x+y) & \cos(x+y) & 0 \\ 0 & 8 - \sin(2(z-y)) & \sin(2(z-y)) \\ 0 & 0 & 12 + \cos z \end{bmatrix}_{\bar{x}_k}$$

$$J_0 = J\left(\frac{1}{10}, \frac{1}{4}, \frac{1}{12}\right) = \begin{bmatrix} 10.9393 & 0.9393 & 0 \\ 0 & 8.327195 & -0.32 \\ 0 & 0 & 12.9 \end{bmatrix}$$

$$F_0 = \begin{bmatrix} f_1\left(\frac{1}{10}, \frac{1}{4}, \frac{1}{12}\right) \\ f_2\left(\frac{1}{10}, \frac{1}{4}, \frac{1}{12}\right) \\ f_3\left(\frac{1}{10}, \frac{1}{4}, \frac{1}{12}\right) \end{bmatrix} = \begin{bmatrix} 0.342898 \\ 0.027522 \\ 0.083237 \end{bmatrix}$$

$$\bar{x}^{k+1} = \bar{x}^{k} - J_k^{-1} F_k$$

$$x^{(1)} = x^{(0)} - J_0^{-1} F_0$$

$x_1 = 0.0689$, $y_1 = 0.246443$, $z_1 = 0.07692$

23/1 Fixed point iteration

$f(x) = 0 \longrightarrow$ (1)

$x = \phi(x)$ (2)

N.R.M

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \phi(x_n)$$

secant

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \phi(x_n, x_{n-1})$$

(1) $\longrightarrow$ $a_0x^2 + a_1x + a_2 = 0$

$a_0 \neq 0$

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_1}}{2a_0}$$

(a) $x_{k+1} = \dfrac{-a_2 + a_0 x_k^2}{a_1} = \phi(x_k)$, $k = 0, 1, 2$

(b) $x_{k+1} = \dfrac{-a_2}{a_0 x_k + a_1}$, $k = 0, 1, 2$

(c) $x_{k+1} = -\dfrac{a_2 + a_1 x_k}{a_0 x_k}$,

$$x_{k+1} = \phi(x_k)$$

Defn A seq of iterates $\{x_k\}$ is said to converge the ~~the~~ root ξ

If $\lim_{k\to\infty} |x_k - \xi| = 0$

Thm If $\phi(x)$ is a continuous fn in some interval $[a,b]$ that contains the root and $|\phi'(x)| <<< 1$ in this interval, then for any choice of $x_0 \in [a, b]$, the sequence $\{x_k\}$ determined

$$x_{k+1} = \phi(x_k) \quad , \quad k = 0, 1, 2$$

converges to root ξ of $x = \phi(x)$

$$\xi - x_{k+1} = \phi(\xi) - \phi(x_k)$$

$$\epsilon_{k+1} = \epsilon_k \, \phi'(\xi_k)$$

$$x_k < \xi_k < \xi$$

$$\epsilon_{k+1} = \epsilon_{k-1} \, \phi'(\xi_{k-1}) \, \phi'(\xi_k)$$

$x_{k-1} < \xi_{k-1} < \xi$

$$\Rightarrow = \epsilon_0 \, \phi'(\xi_k) \, \phi'(\xi_{k-1}) \cdots \phi'(\xi_0)$$

$$= \epsilon_0 \phi'(\xi_k)\phi'(\xi_{k-1}) \cdots \phi'(\xi_0)$$

$$x_0 < \xi_0 < \xi, \quad x_1 < \xi_1 < \xi, \quad \cdots \quad x_k < \xi_k < \xi$$

$$\leq \epsilon_0 c^{k+1}$$

$$|\epsilon_{k+1}| \leq |\epsilon_k| c^{k+1}$$

$$\Rightarrow k \to \infty \quad \epsilon_{k+1} \to 0 \qquad 0 < c < 1$$

Eg The eq^n $f(x) = 3x^3 + 4x^2 + 4x + 1 = 0$ has a root in $(-1, 0)$. Determine an iteration fn $\phi(x)$ s.t the sequence of iterations obtained from $x_{k+1} = \phi(x_k)$

$x_0 = -0.5$, $k = 0, 1 \ldots$

converges to the root

write the given eq as $\quad x = x + \alpha(3x^3 + 4x^2 + 4x + 1)$

$$= \phi(x)$$

α is arbitrary const to be determined

s.t $\quad |\phi'(x)| = |1 + \alpha(9x^2 + 8x + 4)| < 1$

$$\forall x \in [-1, 0]$$

$$9x^2 + 8x + 4 > 0 \quad \forall \ (-1, 0)$$

$$|\phi'(x)| < 1 \Rightarrow \alpha < 0$$

$$|\phi'(-0.5)| < 1$$

$$= \left|1 + \frac{9\alpha}{4}\right| < 1$$

$$\Rightarrow -1 < 1 + \frac{9\alpha}{4} < 1 \quad \Rightarrow \quad -\frac{8}{9} < \alpha < 0$$

Let $\alpha = -1/2$

$$x_{k+1} = x_k - \frac{1}{2}\left(3x_k^3 + 4x_k^2 + 4x_k + 1\right)$$

$$= -\frac{1}{2}\left(3x_k^3 + 4x_k^2 + 2x_k + 1\right)$$

$$= \phi(x_k)$$

$x_0 = -0.5$

$x_1 = -0.3125$, $x_2 = -0.337036$

$x_3 = -0.332723$, $x_4 = -0.333435$

$x_5 = -0.333316$

The iteration method ~~$x_{k+1} = \phi(x_{k+1})$~~ $x_{k+1} = \phi(x_k)$ is said to have p^{th} order convergence if $\phi'(\xi) = \phi''(\xi) = \dots = \phi^{p-1}(\xi) = 0$

but $\phi^p(\xi) \neq 0$

where ξ is the solution of $x = \phi(x)$

$$x_{k+1} = \phi(x_k)$$

$$x_{n+1} = \phi(x_n) = \phi(\xi) + \phi'(\xi)(x_n - \xi)$$

$$\boxed{x_n = \xi - \epsilon_n} \qquad + \frac{1}{2}\phi''(\xi)(x_n - \xi)^2$$

$$+ \dots$$

$$\phi(\xi) = \xi$$

subtract ξ from both sides of 1

$$\underbrace{x_{n+1} - \xi}_{\epsilon_{n+1}} = \cancel{\phi(\xi)} - \cancel{\xi} + \phi'(\xi)\epsilon_n + \frac{1}{2}\phi''(\xi)\epsilon$$

$$\rightarrow \epsilon_{n+1} = \phi'(\xi)\epsilon_n + \frac{1}{2}\phi''(\xi)\epsilon_n^2 \dots$$

Scanned with CamScanner

Suppose $\phi'(\xi) = 0$

$$\epsilon_{n+1} = \frac{1}{2} \phi''(\xi) \epsilon_n^2 + O(\epsilon_n^3)$$

$$\epsilon_{n+1} = C \epsilon_n^{p-2}$$

N.R.M

$$\phi(x) = x - \frac{f(x)}{f'(x)} = \xi \quad \text{at } x = \xi$$

$$\phi'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2}$$

$$= \frac{f(x) f''(x)}{(f'(x))^2} = 0 \quad \text{at } x = \xi$$

$\phi''(x)$

Scanned with CamScanner

System of linear eq^n

$$AX = B$$

$$LUX = B$$

$$m = -\frac{a_{21}}{a_{11}}$$

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} & \vdots \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

$\text{rank}(A) = \text{rank}(A|B)$

solution exists

if $\text{rank}(A) < n$

multiple solutions

if $\text{rank}(A) = n \Rightarrow$ unique solutions

$\text{rank}(A) \neq \text{rank}(A|B) \Rightarrow$ no solution

Gauss elimination

$$8x_2 + 2x_3 = -7$$

$$3x_1 + 5x_2 + 2x_3 = 8$$

$$6x_1 + 2x_2 + 8x_3 = 26$$

$$6x_1 + 2x_2 + 8x_3 = 26$$

$$3x_1 + 5x_2 + 2x_3 = 8$$

$$8x_2 + 2x_3 = -7$$

subtract $\frac{3}{6} = \frac{1}{2}$ times

The pivot eq^n from 2^{nd} row

$$6x_1 + 2x_2 + 8x_3 = 26$$
$$4x_2 - 2x_3 = -5$$
$$8x_2 + 2x_3 = -7$$

$$6x_1 + 2x_2 + 8x_3 = 26.$$
$$8x_2 + 2x_3 = -7$$
$$4x_2 - 2x_3 = -5$$

Subtract $\frac{4}{8} = \frac{1}{2}$ times

the pivot eq^n from 3^{rd} eq^n

$$6x_1 + 2x_2 + 8x_3 = 26$$
$$8x_2 + 2x_3 = -7$$
$$-3x_3 = -3/2$$

$$x_3 = 1/2$$

Back substitution

$$x_2 = \frac{1}{8}(-7 - 2x_3) = -1$$

$$x_1 = \frac{1}{6}(26 - 2x_2 - 8x_3) = 4$$

Input augumented $n \times (n+1)$

$\bar{A} = [A|B]$

For $k = 1$ to $n-1$ do $m = k$

For $j = k+1, \ldots\ldots n$ do

If $(|a_{mk}| < |a_{jk}|)$

then $m = j$

end

If $a_{mk} = 0$ break

else

rows

For $j = k+1, \ldots\ldots n$ do

$$m_{jk} := \frac{a_{jk}}{a_{kk}}$$

columns element

For $p = k+1 \ldots\ldots n+1$

$$a_{jp} := a_{jp} - m_{jk}\, a_{kp}$$

end

end

end.

$$x_n = \frac{a_{n,n+1}}{a_{nn}}$$

For $i = n-1$ to 1 do

$$x_i = \frac{1}{a_{ii}} \left(a_{i,n+1} - \sum_{j=i+1}^{n} \cancel{a_{i}a_{j}}\, a_{ij}\, x_j \right)$$

end

output $[x]$

$$0.0004x_1 + 1.402x_2 = 1.406$$

$$0.4003x_1 - 1.502x_2 = 2.501$$

$x_1 = 10$, $x_2 = 1$

$$m = \frac{0.4003}{0.0004} = 1001$$

subtract the result from 2nd eqn

$$-1405x_2 = -1404$$

$$x_2 = -\frac{1404}{-1405} = 0.9993$$

and ~~substitute~~ substitute x_2 in 1st eqn to get

$$x_1 = \frac{1}{0.0004}(1.406 - 1.402 * 0.9993)$$

$$= \frac{0.005}{0.0004} = 12.5$$

If pivoting is done

$$0.4003x_1 - 1.502x_2 = 2.501$$

$$0.0004x_1 + 1.402x_2 = 1.406$$

$$m = \frac{0.0004}{0.4003} = 0.000993$$

subtract the result from ②

$$1.404x_2 = 1.404$$

$$x_2 = 1 \rightarrow x_1 = 10$$

$AX = B$

$A = LU$

$\underset{n\times n}{L}\,\underset{n\times n}{U}\,\underset{n\times 1}{X} = B_{n\times 1}$

$\underset{n\times n}{L}\, Y_{n\times 1} = B_{n\times 1}$

Y $\qquad UX = Y$

29/1

$AX = B$

Gauss elimination method

(1) Row reduction to upper $\triangle$ form

(2) Back substitution

Gauss jordan Elimination

(1) Row reduction to diagonal matrix form

(2) Direct assignment of $\bar{x}$

$$A_{n\times n} = L_{n\times n} U_{n\times n} \qquad UX = Y$$

$$AX = B_{n\times 1}$$

$$L(U_{n\times n} X_{n\times 1}) = B_{n\times 1}$$

$$L_{n\times n} Y_{n\times 1} = B_{n\times 1}$$

$$A = \begin{bmatrix} 2 & 3 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Doolittle : $L\triangle$ diag 1

Crout : $U\triangle$ " 1

$a_{11} = u_{11} \cdot 1$

$a_{12} = u_{12} \cdot 1$

$a_{13} = u_{13} \cdot 1$

$a_{21} = L_{21} u_{11} \Rightarrow L_{21} = \frac{a_{21}}{a_{11}}$

$a_{22} = L_{21} u_{12} + 1 u_{22}$

$a_{23} = l_{21} u_{13} + u_{23}$

$u_{23} = ?$

$a_{31} = l_{31} u_{11}$

$a_{32} = l_{31} u_{12} + l_{32} u_{22}$

$a_{33} = l_{31} u_{13} + l_{32} u_{23} + l\, u_{33}$

$$AX = B = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

$$A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}}$$

$LY = B$ $\qquad$ $UX = Y$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

$$y = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$ forward substitution.

$UX = Y$

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

Backsubstitution

$UX = Y$

$$A = \begin{bmatrix} 1 & & \\ L_{21} & 1 & \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$U_{1K} = a_{1K}$

$\cancel{Y_1 = \frac{a_{j1}}{u_{11}}}$

$$L_{j1} = \frac{a_{j1}}{u_{11}}$$

$$u_{jk} = a_{jk} - \sum_{s=1}^{j-1} L_{js} u_{sk} \qquad k = j \cdots n$$

$$L_{jk} = \frac{1}{u_{kk}} \left(a_{jk} - \sum_{s=1}^{k-1} L_{js} u_{sk} \right)$$

$j = k+1 \cdots n$

Cholesky's Decomposition

A +ve definite
symmetric

$A = LL^T \qquad A = A^T$

$A = X = \lambda X \qquad A^T = A^{-1}$

A is +ve

$X^T A X > 0 \quad \forall \; x \neq 0$

$U = L^T$

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{22} & l_{13} \\ & l_{22} & l_{32} \\ & & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

$$L_{11} = \sqrt{a_{11}}$$

$$l_{j1} = \frac{a_{j1}}{L_{11}} \qquad j = 2, \dots n$$

$$L_{jj} = \sqrt{a_{jj} - \sum_{s=1}^{j-1} l_{js}^2} \quad , \quad j = 2, \dots n$$

$$l_{pj} = \frac{1}{l_{jj}} \left(a_{pj} - \sum_{s=1}^{j-1} l_{js} \, l_{ps} \right)$$

$$p = j+1, \dots n$$

$$j \geq 2$$

$4x_1 + 2x_2 + 14x_3 = 14$

$2x_1 + 17x_2 - 5x_3 = -101$

$14x_1 - 5x_2 + 83x_3 = 155$

LL^T

30/1

There is no error in LU decomposition
we get exact answers

LU: Crout's Method.

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{matrix} L_{11} & L_{11}u_{12} & L_{11}u_{13} \\ L_{21} & L_{21}u_{12} + L_{22} & L_{21}u_{13} + L_{22}u_{23} \\ L_{31} & L_{31}u_{12} + L_{32} & L_{31}u_{13} + L_{32}u_{23} + L_{33} \end{matrix}$$

1st column $\quad L_{11} = 1 \quad L_{21} = 4 \quad L_{31} = 3$

1st row $\quad u_{12} = 1 \quad u_{13} = 1$

2nd column $\quad L_{22} = -1 \quad L_{32} = 2$

2nd row $\quad L_{23} = 5 \quad L_{33} = -10$

use forward substitution

$$AX = B$$

$$L(UX) = B \Rightarrow LY = B \quad \text{where} \quad Y = UX$$

PTO

for $LY = B$

$$Y = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

$LY = B$

$\Rightarrow \quad y_1 + 0y_2 + 0y_3 = 1$

$4y_1 - y_2 + 0y_3 = 6$

$3y_1 + 2y_2 - 10y_3 = 4$

$UX = Y$

$1 = x_1 + x_2 + x_3$

$-2 = 0x_1 + x_2 + 5x_3$

$-1/2 = 0x_1 + 0x_2 + x_3$

$$X = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Cholesky's Decomposition

$$A = LL^T$$

A is symmetric $\qquad X^T A X > 0 \quad \forall X$

$$A^{-1} = (LL^T)^{-1}$$

$$A^{-1} = (L^T)^{-1} L^{-1}$$

→ positive definite matrix

$$X^T A X = X^T \lambda X$$

$$= \lambda X^2 > 0$$

$$\lambda > 0$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}}_{B}$$

$$A = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{21}L_{11} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{31}L_{11} & L_{31}L_{21} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$$

$\Rightarrow L_{11}^2 = 1$ or $L_{11} = 1$

$L_{11}L_{21} = 2 \Rightarrow L_{21} = 2$

$L_{31}L_{11} = 3 \Rightarrow L_{31} = 3$

$\rightarrow L_{21}^2 + L_{22}^2 = 8$

$\rightarrow L_{22} = 2$

$\rightarrow L_{31} L_{21} + L_{32} L_{22} = 22$

$\longrightarrow L_{32} = 8$

$L_{31}^2 + L_{32}^2 + L_{33}^2 = 82 \Rightarrow L_{33} = 3$

$A = LL^T$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

$(LL^T)X = B$

$\Rightarrow L(L^TX) = B$

$\Rightarrow LY = B$

$L^TX = Y$

~~100~~

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$L \quad Y = B$

$$Y = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

$L^T X = Y$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$x_1 = 2$

$x_2 = 3$

$x_3 = -1$

* **Gauss Seidal**

$AX = B$

$$x_1^{k+1} = -\frac{1}{a_{11}} \left(a_{12} x_2^{k} + a_{13} x_3^{k} + \cdots a_{1n} x_n^{k} \right) + \frac{b_1}{a_{11}}$$

$$x_2^{k+1} = -\frac{1}{a_{22}} \left(a_{21} x_1^{k+1} + a_{23} x_3^{k} + \cdots a_{2n} x_n^{k} \right) + \frac{b_2}{a_{22}}$$

$\vdots$

$$\boxed{x_n^{k+1} = -\frac{1}{a_{nn}} \left(a_{n1} x_1^{k+1} + \cdots a_{nn-1} x_{n-1}^{k+1} \right) + \frac{b_n}{a_{nn}}}$$

$AX = B$

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

rearrange

$$a_{11} x_1^{(k+1)} = -\sum_{i=2}^{n} a_{1i} x_i^{(k)} + b_1$$

$$a_{21} x_1^{(k+1)} + a_{22} x_2^{(k+1)} = -\sum_{i=3}^{n} a_{2i} x_i^{(k)} + b_2$$

$\vdots$

$$a_{n1} x_1^{(k+1)} + \cdots + a_{nn} x_n^{(k+1)} = b_n$$

$$(D+L) x^{(k+1)} = -U x^{(k)} + b$$

$$A = L + D + U$$

$$x^{(k+1)} = -(D+L)^{-1} U x^{(k)} + (D+L)^{-1} b$$

$$2x_1 - x_2 + 0x_3 = 7$$

$$-x_1 + 2x_2 - x_3 = 1$$

$$0x_1 - x_2 + 2x_3 = 1$$

$$X^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

31/1 Theorem: If $f \in C^2(\mathbb{R})$ is increasing, is convex and has a zero, then the zero is unique and Newton iteration will converge to it from any starting point

N.R.M

$e_n = x_n - r$

$$e_{n+1} = x_{n+1} - r = \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) - r$$

$$= e_n - \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)} \longrightarrow ②$$

$$0 = f(r) = f(x_n - e_n)$$

$$0 = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n)$$

$$r < \xi_n < x_n$$

$$\frac{1}{2} e_n^2 f''(\xi_n) = e_n f'(x_n) - f(x_n)$$

Substitute in ②

$$e_{n+1} = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} e_n^2$$

$$e_{n+1} \approx \left(\frac{1}{2} \frac{f''(r)}{f'(r)}\right) e_n^2 = C e_n^2 \longrightarrow ③$$

Pf: f is convex $\Rightarrow f''(x) > 0 \quad \forall x$ on $\mathbb{R}$

$\because f$ is increasing $\Rightarrow f'(x) > 0 \quad \forall x$ on $\mathbb{R}$

By eqn ③ $e_{n+1} > 0$

$\Rightarrow x_n > r \quad \forall n \geq 1$

$\Rightarrow \because f$ is increasing fn

$$f(x_n) > f(r) = 0$$

& eqn 2 $\Rightarrow e_{n+1} < e_n$

$[e_n], [x_n]$ are decreasing and bdd below by (0 and by r)

$$\therefore \lim_{n\to\infty} e_n = 0$$

$$\lim_{n\to\infty} x_n = r$$

$$G(x,y) = 3x^7 + 2y^5 - x^3 + y^3 - 3$$

$$G(x,y) = 0$$

x if y is given fn of x

$$y_{k+1} = y_k - \frac{G(x, y_k)}{\frac{\partial G(x, y_k)}{\partial y}}$$

$$\frac{\partial G}{\partial y}(x,y) = 10y^4 + 3y^2$$

		$x = 10$
		ϵ
$x_0 = 0$	$y_0 = 1.0$	
$x_1 = 0.1$	$y_1 = 1.0007$	
$x_2 = 0.2$	$y_2 = 1.000612$	0.89×10^{-15}
$x_{20} = 2.0$	-2.810639	-0.82×10^{-10}
$x_{80} = 8.0$	-19.92635	0.56×10^{-9}
$x_{100} = 10$		

Prove a) If U is upper Δ and invertible matrix then U^{-1} is also upper Δ

$$UU^{-1} = I$$

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \ddots & \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$U \times i^{th}$ column U^{-1} = i^{th} column of I

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i^{th} column of U^{-1}

Back substituting

$$x_n = x_{n-1} = \cdots = x_{i+1} = 0$$

$\Rightarrow U^{-1}$ is upper Δ

tri diagonal matrix

$$\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & c_3 & \\ & & \ddots & \ddots & \\ & & & & d_n \end{bmatrix}$$

$$d_2 \longleftarrow d_2 - \frac{a_1}{d_1} c_1$$

$$b_2 \longleftarrow b_2 - \frac{a_1}{d_1} b_1$$

for $i = 2$ to n

$$d_i = d_i - \frac{a_{i-1}}{d_{i-1}} c_{i-1}$$

$$b_i = b_i - \frac{a_{i-1}}{a_{i-1}} b_{i-1}$$

$$x_n \leftarrow \frac{b_n}{d_n}$$

for $i = n-1$ to 1

$$x_i = (b_i - c_i x_{i+1})/d_i$$

Tut 2 Q8

$$F(x) = x + f(x)\,g(x)$$

$$x_{n+1} = F(x_n) = x_n + f(x_n)\,g(x_n)$$

$$f(r) = 0 \quad \& \quad f'(r) \neq 0$$

$$|F'(x_n)| < 1 \qquad r \in [a,b]$$

$$\Rightarrow \quad |1 + f'(x_n)\,g(x_n) + f(x_n)\,g'(x_n)| < 1$$

$$\forall \; x_n \in [a,b]$$

Also true for $x = r$

$$-1 < 1 + f'(r)\,g(r) < 1$$

$$\Rightarrow -2 < f'(r)\,g(r) < 0$$

$$\Rightarrow -\frac{2}{f'(r)} < g(r) < 0$$

$$\epsilon_{n+1} = x_{n+1} - \xi = x_n + f(x_n)\,g(x_n) - \xi$$

$$= \epsilon_n + f(\epsilon_n + \xi)\,g(\epsilon_n + \xi)$$

$$x_{n+1} = g(x_n)$$

$$= x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow N.R.M$$

ξ root of f

$g(\xi) = \xi \rightarrow$ it fixed point of this iteration method

2nd order.

$$g'(x_n) = 1 - \frac{f''(x_n)\,f(x_n) + (f'(x_n))^2}{f'(x_n)^2}$$

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$$g'(\xi) = 1 - \frac{\cancel{(f'(\xi))^2}}{\cancel{(f'(\xi))^2}}$$

$$g'(\xi) = 0$$

$$g''(\xi) = 1$$

Q4 ~~Tut 3~~ $x_{n+1} = \frac{x_n}{2a}(3a - x_n^2)$

$a > 0$

$$x_{n+1} = \phi(x_n)$$

$$f(x) = 0$$

$$\cancel{x = 2a(3a - x^2)}$$

$$x = \frac{x}{2a}(3a - x^2)$$

$$\phi'(x) = \frac{3}{2} - \frac{3x^2}{2a}$$

$$x_{n+1} - \xi = \frac{x_n}{2a}(3a - x_n^2) - \xi$$

$$e_{n+1} = \frac{3}{2}x_n - \frac{x_n^3}{2a} - \xi$$

$$= \frac{3}{2}(x_n - \xi) - \frac{x_n^3}{2a} + \frac{1}{2}\xi$$

Tut 3 Q3

$x^2+ax+b=0$

$\alpha+\beta=-a$

$\alpha\beta=b$

(i) $x_{k+1}=-\frac{1}{x}(ax_k+b) \longrightarrow \alpha \quad |\alpha|>|\beta|$

(ii) $x_{k+1}=\frac{-b}{x_k+a} \longrightarrow \alpha \quad |\alpha|<|\beta|$

$|\phi'(x)|<1 \quad \alpha\in(c,d)$

$=\left|\frac{b}{x_k^2}\right|<1 \Rightarrow \left|\frac{\alpha\beta}{\alpha^2}\right|<1$

$\Rightarrow \frac{|\beta|}{|\alpha|}<1$

$|\phi'(x)|\le L<1$

$L\in(0,1)$

$\phi(r)=r \quad [a,b]$

Tut3 Q5

If $x_{k+1}=\phi(x_k), \quad k>>0$

$|x_{n+1}-r|\le\frac{L}{1-L}|x_{n+1}-x_n|$

$x_{n+1}=\phi(x_n)$

$r=\phi(r)$

$|x_{n+1}-r|=|\phi(x_n)-\phi(r)|$

$\approx|\phi'(\xi)|(x_n-r)$

$$|x_{n+1} - r| \le L\,|x_n - r + x_{n+1} - x_{n+1}|$$

$$|x_{n+1} - r| \le L\left(|x_n - x_{n+1}| + |x_{n+1} - r|\right)$$

$$(1-L)\,|x_{n+1} - r| \le L\,|x_n - x_{n+1}|$$

Tut 4 Q2

$$A = \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{matrix}$$

$$a_{ii} = \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1\,i-1} & a_{1\,i+1} & a_{1n} \\ a_{22} & a_{22} & a_{2\,i-1} & a_{2\,i+1} & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n\,i-1} & a_{n\,i+1} & a_{in} \end{bmatrix}$$

$$\det \sum_{i=1}^{n} a_{ij}\,(-1)^{(i+j)}\,\text{Minor}(a_{ij})$$

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$$Ax = (L+D+U)\,x = B$$

$$\begin{bmatrix} 0 & & & & \\ a_{21} & 0 & & & \\ a_{31} & a_{32} & & & \\ \vdots & & & & \\ a_{n1} & a_{n2} & & a_{nn-1} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & & \\ 0 & 0 & a_{23} & \\ 0 & 0 & 0 & a_{33} \\ \vdots & & & \\ 0 & \cdots & & a_{nn} \end{bmatrix}$$

$$(L+D+U)x = B$$

$$x^{k+1} = \frac{1}{D}\left(B - Lx^{k+1} - Ux^{k}\right)$$

$$x_1 - 0.25x_2 - 0.25x_3 = 50$$

$$-0.25x_2 + x_2 - 0.25x_4 = 50$$

$$-0.25x_1 + x_3 - 0.25x_4 = 25$$

$$-0.25x_2 - 0.25x_3 + x_4 = 25$$

$$100 = x_1^{(1)} = 0.25x_2^{(0)} + 0.25x_3^{(0)} + 50$$

$$100 = x_2^{(1)} = 0.25x_1^{(1)} + 0.25x_4^{(0)} + 50$$

$$75 = x_3^{(1)} = 0.25x_1^{(1)} + \cdots\; 0.25x_4^{(0)} + 25$$

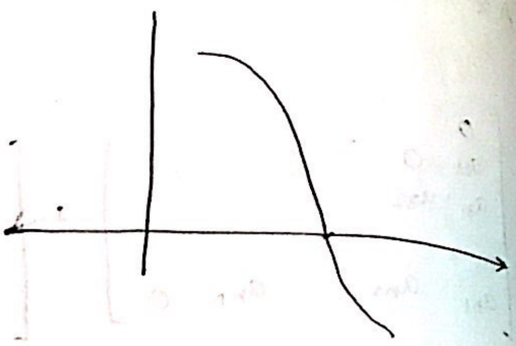
$$68.75 = x_4^{(1)} = 0.25x_2^{(1)} + 0.25x_3^{(1)} + 25$$

$$x_1^0 = x_2^0 = x_3^0 = x_4^0 = 100.$$

12/2

Recap

Root finding



Let

$f : [a,b] \to \mathbb{R}$ be continuous

$f(a) f(b) < 0$

Generate a sequence of numbers using iterative methods

$$x_0, x_1, x_2, x_3 \ldots$$

$$\lim_{n\to\infty} x_n = r, \quad r \text{ is a root}$$

order of convergence

$$\lim_{x\to\infty} \frac{|e_{k+1}|}{|e_k|^p} = M \qquad M \neq 0, \ M \neq \infty$$

if it satifies these conditions the p is the order of convergence

Ex:

$$x_n = \frac{1}{n}, \quad n = 1, 2, 3$$

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}, \quad r = 0$$

$$e_n = x_n - r \quad ; \quad e_{n+1} = x_{n+1} - r$$

$$e_n = \frac{1}{n} \qquad e_{n+1} = \frac{1}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \infty$$

Ex: $x_0 = \frac{1}{3}$, $x_{n+1} = x_n^2$, $n = 1, 2, 3, 4 \ldots\ldots$

clearly $r = 0$

$e_n = x_n$; $e_{n+1} = x_{n+1} - r = x_n^2$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = 1 \neq 0$$

order of conv $= 2$

Iterative method.

$x_{n+1} = \phi(x_n)$, $n = 0, 1, 2, \cdots$

NR

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

sufficient conditions

$$|\phi'(x)| < 1$$

Fixed point $\phi(r) = r$

Let $x_{n+1} - r = \phi(x_n) - \phi(r)$

$= \phi'(\xi)(x_n - r)$ where ξ lies bet^n x_n & r

$|x_{n+1} - r| = |\phi'(\xi)|^{n+1} (x_0 - r)$ ξ lies in the small intervals of the sequence

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case(i)

$$\phi'(x) \neq 0 \quad , \forall \; x \in [a,b]$$

$$x_{n+1} - r = \phi(x_n) - \phi(r) \qquad e_n = x_n - r$$

$$x_n = r + e_n$$

$$= \phi(r + e_n) - \phi(r)$$

$$= \left[\cancel{\phi(r)} + e_n \phi'(r) + \frac{e_n^2}{2!} \phi''(r) + \cdots\right] - \cancel{\phi(r)}$$

$$= e_n \phi'(r) + \frac{e_n^2}{2!} \phi''(r) + \frac{e_n^3}{3!} \phi'''(r)$$

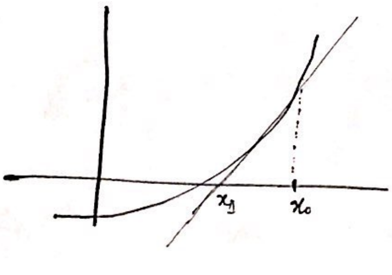
case(ii): If $\phi'(r) = 0$, $\phi''(r) = 0$, then $\phi'''(r) \neq 0$

the order of fixed point iterative scheme is $p = 3$

Stopping criteria

$$\text{Relative error} = \frac{|x_{n+1} - x_n|}{|x_n|} < \epsilon$$

N.R



$f(x) = 0$

x_n

$$x_{n+1} = x_n + h$$

$$f(x_n + h) = f(x_n) + h f'(x_n) + \cancel{\frac{h^2}{2!} f''(x_n)} + \cancel{\frac{h^3}{3!} f'''(x_n)}$$

$$\Rightarrow f(x_n) + h f'(x_n) = 0 \qquad h = -\frac{f(x_n)}{f'(x_n)}$$

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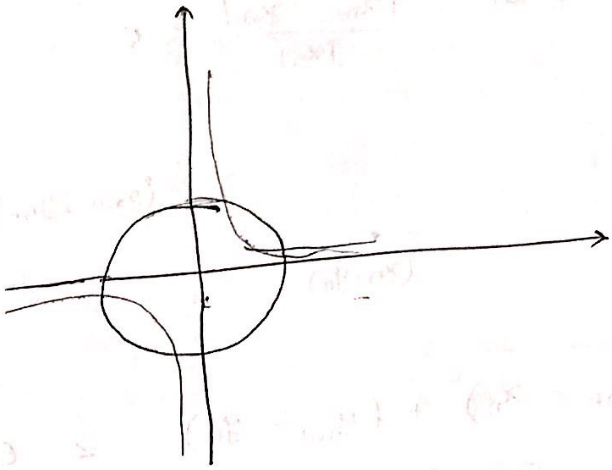
13/2 Ex: $\left.\begin{array}{l} x^2+y^2=1 \\ xy=1 \end{array}\right\}$



2D

$\left.\begin{array}{l} f(x,y)=0 \\ g(x,y)=0 \end{array}\right]$

$(x_0,y_0) \longrightarrow (x_1,y_1)$

$x_1 = x_0+h$

$y_1 = y_0+h$

$\cancel{f(x_0+h, y_0+k)}^{\text{zero}}$

$$= f(x_0,y_0) + h\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + k\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} + \cdots\cdots$$

$\cancel{g(x_0+h, y_0+k)}^{\text{zero}}$

$$= g(x_0,y_0) + h\frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} + k\frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} + \cdots\cdots$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x_0,y_0)} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -f(x_0,y_0) \\ -g(x_0,y_0) \end{bmatrix}$$

$$\boxed{JH = F}$$

$x_1 = x_0+h \qquad y_1 = y_0+k$

$x_2 = x_1+h \qquad y_2 = y_1+k$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J^{-1}\begin{bmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{bmatrix}$$

Relative error $= \frac{|x_{n+1} - x_n|}{|x_n|} < \epsilon$

(x_{n+1}, y_{n+1})

(x_n, y_n)

$$\frac{\sqrt{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2}}{\sqrt{x_n^2 + y_n^2}} < \epsilon$$

Solving linear system of equations

Eg $ax + by = k_1 \Rightarrow f(x,y) = ax + by - k_1 = 0$

$cx + dy = k_2 \Rightarrow g(x,y) = cx + dy - k_2 = 0$

(x_0, y_0) $\quad J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |J| = ad - bc \neq 0$

Gauss elimination

$Ax = b$ $\quad |A| \neq 0$

$x_i = \frac{\det(A_i)}{\det A}$ $\quad A_i$ th column is replaced by RHS

If $n = 100$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & & & & \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$a_{22}^{(1)} = a_{22} - \frac{a_{12}}{a_{11}} a_{21}$$

$$a_{32}^{(1)} = a_{32} - \frac{a_{21}}{a_{11}} a_{13}$$

$$\vdots$$

$$a_{n2} = a_{n2} - \frac{a_{n-1\,2}}{a_{11}} a_{11}$$

14/2

$Ax = b \qquad |A| \neq 0$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If $a_n \neq 0$

step 1 $\quad a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \qquad i = 2, 3, \ldots n, \quad j = 2, 3, \ldots n$

$$b_i = b_j - \frac{a_{i1}}{a_{11}} b_1 \qquad i = 2, \ldots n, \quad j = 2, \ldots n$$

no of iteration $\boxed{(n-1)n}$

Step② $a_{22} \neq 0$

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}} a_{2j}^{(1)} \qquad i = 3, 4, \dots n. \quad j = 3\ 4, \dots n$$

$$b_j^{(2)} = b_j^{(1)} - \frac{a_{i2}}{a_{22}} b_2^{(2)} \qquad \boxed{(n-2)(n-1)}$$

* Floating point operations (Flop)

1 flop = 1 (add/sub) + 1 (mul/div)

$Ax = b$

$Ux = \tilde{b}$

To update each element

$$\underbrace{1 \text{ sub} + 1 \text{ mul}}_{1 \text{ flop}} + 1 \text{ div}$$

Step 1 $\quad n(n-1) + n$ div

Step 2 $\quad (n-1)(n-2) + (n-1)$ div.

Total $\quad \underbrace{[n(n-1) + (n-1)(n-2) + \dots + 1]}_{\text{flop}} + \underbrace{[n + (n-1) + \dots + 1]}_{\text{div}}$

$$= [(n-1)(n-1+1) + (n-2)(n-2+1) + \dots 1]$$
$$[n + (n-1) + \dots 1]$$

$$= [(n-1)^2 + (n-2)^2 + \dots + 1]$$
$$+ [(n-1) + (n-2) + \dots 1]$$
$$+ [n + (n-1) + \dots + 1]$$

$$= \frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n(2n^2-3n+1)}{6} + \frac{n(n-1)}{2} + \frac{n(n+1)}{2}$$

$$= O(n^3)$$

Backward Sub

If $a_{ii} \neq 0$, $i = 1, 2, \dots, n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ \vdots & & a_{n-1,n-1} & a_{n-1,n} \\ 0 & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}\, x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{1}{a_{n-1,n-2}} \left(b_{n-2} - a_{n-2,n}\, x_n - a_{n-2,n-1}\, x_{n-1} \right)$$

2 mul + 2 sub + 1 div.

for $j = n : -1 : 1$

$$x_j = \frac{1}{a_{jj}} \left(b_j - \sum_{k=j+1}^{n} a_{jk}\, x_k \right)$$

end

LU decomposition

$Ax = b \quad |A| \neq 0, \quad A \in \mathbb{R}^{n\times n}$

$LUx = b$ —— ①

Let $Ux = y$ —— ②

① $Ly = b$ —— ③

$A = LU$ is unique ?

Let A has crouts decomposition, then prove that L & U are unique

Let $A = L_1 U_1 = L_2 U_2$

where U is an upper triangular matrix

$\longrightarrow |U| = 1$

consider $L_1 U_1 = L_2 U_2$

$L_2^{-1}(L_1 U_1) = L_2^{-1}(L_2 U_2)$

$\rightarrow (L_2^{-1} L_1) U_1 = I U_2$

$\Rightarrow L_2^{-1} L_1 = U_2 U_1^{-1} = I$

H.W

Note: Inverse of upper/lower triangular matrix is an upper/lower triangular

11/2

Recap. $A = LU$ —— ①

For uniqueness Let L be an unit lower triangular matrix

Let $L_1U_1 = L_2U_2$

$\Rightarrow L_2^{-1}L_1 = U_2U_1^{-1} = L$

L^{-1} is a lower triangle ??

$\Rightarrow L_2 = L_1$ and $U_2 = U_1$

$\therefore$ LU decomposition is unique.

Sufficient condition on existance of LU decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \; : \; |A| = -5 \neq 0$$

→ Test condition
If matrix is non singular then it is LU decomposable.

$A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = 1 ; u_{12} = 2 ; u_{13} = 3$

$l_{21} u_{11} = 2 \Rightarrow l_{21} = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & \frac{15-6}{2} \end{bmatrix}$$

$A = L\ U$

$|A| = |L||U|$

$-5 = 1 \times 0$?

* A has LU decomposition if all the principal minors are non singular

$\Rightarrow |a_{11}| \neq 0$,

$\Rightarrow \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$,

$\Rightarrow |A| \neq 0$.

In the case of example.

$\rightarrow |a_{11}| \neq 0$

$\rightarrow \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$\rightarrow |A| = -5 \neq 0$

$\therefore$ since one of the principle minors is zero LU decomp not applicable

we can ~~change~~ interchange rows

$\Longrightarrow A = \begin{vmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Note: Let $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$: $|U| = U_{11}U_{22} \neq 0$

Define $D = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix}$: $D^{-1} = \begin{bmatrix} \frac{1}{u_{11}} & 0 \\ 0 & \frac{1}{u_{22}} \end{bmatrix}$

$$D^{-1}U = \begin{bmatrix} 1 & u_{12}/u_{22} \\ 0 & 1 \end{bmatrix} \quad ?$$

Let $V = D^{-1}U$ is a unit upper triangular

$\Longrightarrow$ $U = DV$

consider $A = LU$

$$\boxed{A = LDV}$$

case (i) If $A = A^T$

$$LDV = (LDV)^T$$
$$= V^T D L^T$$

$\Longrightarrow$ $L = V^T$ and $V = L^T$

$\therefore$ $A = LDL^T$

case (ii) A is positive definite $x^TAx > 0$, $\forall x$

$\Rightarrow$ $Ax = \lambda x$

$\Rightarrow$ $x^TAx = \lambda x^Tx$

$\longrightarrow$ $\lambda_i > 0$, for $i = 1, 2, \cdots n$

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Define

$$\sqrt{D} = \begin{bmatrix} \sqrt{u_{11}} & 0 \\ 0 & \sqrt{u_{22}} \end{bmatrix}$$

$$\Rightarrow (\sqrt{D})(\sqrt{D}) = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix} = D$$

$$A = LDL^T$$
$$= L(\sqrt{D}\sqrt{D})L^T$$
$$= (L\sqrt{D})(\sqrt{D}L^T)$$
$$\boxed{A = GG^T}$$

Let

$$G = L\sqrt{D}$$
$$G^T = \sqrt{D}L^T$$

Define

$$G = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

Partial pivoting

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 13 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1/3$; $R_3 \longrightarrow R_3 - \frac{2R_1}{3}$

$$\begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & -1/3 \\ 0 & -1 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -2/3 \\ -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 4 \\ 0 & -1 & 1/3 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -1/3 \\ -2/3 \end{bmatrix}$$

$\uparrow$ $x_1 \; x_2 \; x_3$

$= (3 \quad 1 \quad 2)$

complete pivoting

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

20/2

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 3x_2 + 4x_3 &= 20 \\ 2x_1 + x_2 + 3x_3 &= 13 \end{aligned}$$

$C_1 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 4 & 3 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 13 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - \frac{R_1}{4}$, $R_3 \longrightarrow R_3 - \frac{3}{4}R_1$

$$\begin{bmatrix} 4 & 3 & 3 \\ 0 & 1/4 & 1/4 \\ 0 & \frac{-5}{4} & -1/4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \\ -2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 4 & 3 & 3 \\ 0 & -5/4 & -1/4 \\ 0 & 1/4 & 1/4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ -2 \\ 1 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{5} R_2$

$$\begin{bmatrix} 4 & 3 & 3 \\ 0 & -5/4 & -1/4 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ -2 \\ 3/5 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = 2 \\ x_2 = 1 \\ x_1 = 3 \end{matrix}$$

$Ax = b \qquad |A| \neq 0$

- Direct methods: GEM, LU, GG^T → Thomas algorithm
- Iterative methods

<u>Vector norm</u> : Let $(V, +, \cdot)$ be a vector space let $x \in V$

$$\|\cdot\| : V \to \mathbb{R}^+ \cup \{0\}$$

a) $\|x\| > 0$ if $x \neq 0$

$\|x\| = 0 \iff x = 0$

b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$

c) $\|x+y\| \leq \|x\| + \|y\|$, $\forall \, x, y \in V$

Ex: Euclidean norm

$$\cancel{\|x\|_2 = |x_1| + |x_2| + \cdots + |x_n| = \sum_{i=1}^{n} |x_i|}$$

Ex: Euclidean norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \text{ where, } x = (x_1, x_2, \dots x_n)$$

Ex Max norm

$$\|x\|_\infty = \max_{1\le i\le n} \|x_i\| = \max\{|x_1|, |x_2|, \dots |x_n|\}$$

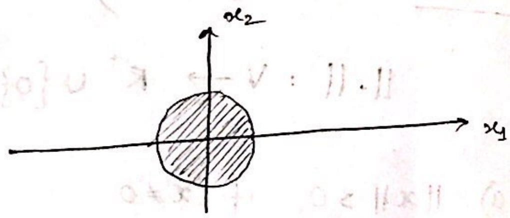
one-norm:

$$\|x_1\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

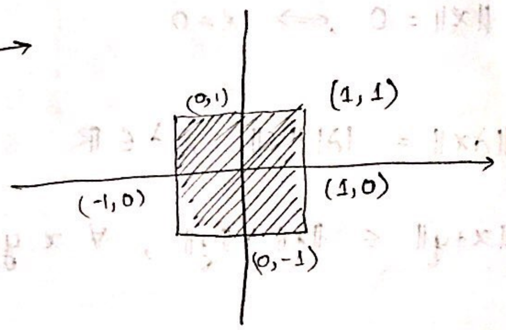
$$= \sum_{i=1}^{n} |x_i|$$

Ex: Find the points $x = (x_1, x_2) \in \mathbb{R}^2$ such that

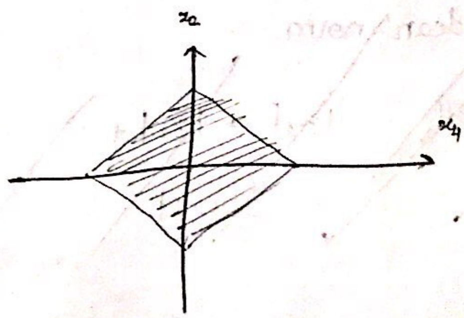
$$D_1 = \{x / \|x\|_2 \le 1\}$$



$$D_2 = \{x / \|x\|_\infty \le 1\} \longrightarrow$$



$$D_3 = \{x / \|x\|_1 \le 1\} \longrightarrow$$



$x = (4, 4, -4, 4)$ $v = (0, 5, 5, 5)$ $w = (6, 0, 0, 0)$

	$\|\cdot\|_1$	$\|\cdot\|_2$	$\|\cdot\|_\infty$
x	16	8	4
v	15	$5\sqrt{3}$	5
w	6	6	6
$x - v$	15	$3\sqrt{11}$	9

<u>Matrix norm</u> : Let $Ax = b$ $|A| \neq 0$, $A \in \mathbb{R}^{n \times n}$

$$\|\cdot\| : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^+ \cup \{0\}$$

Let $A, B \in \mathbb{R}^{n \times n}$, $\alpha \in \mathbb{R}$

a) $\|A\| \geq 0$, $\forall \; A \in \mathbb{R}^{n+n}$

$\|A\| = 0 \iff A = 0$ (zero matrix)

b) $\|\alpha A\| = |\alpha| \, \|A\|$

c) $\|A + B\| \leq \|A\| + \|B\|$

d) $\|AB\| \leq \|A\| \, \|B\|$ (consistency property)

<u>Induced matrix norm</u> (vector norm)

Let $\|\cdot\|_v$ be a vector norm

$$\|A\| = \max_{\|x\|_v \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

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Consistency property.

$$\|A\| \geq \frac{\|Ax\|_v}{\|x\|_v}$$

$$\Rightarrow \boxed{\|Ax\|_v \leq \|A\| \, \|x\|_v}$$

Note:

$$\|(AB)x\|_v \leq \|AB\| \, \|x\|_v$$

$$\leq \|A\| \, \|B\| \, \|x\|_v$$

$$\|AB\| = \max_{\|x\|_v \neq 0} \frac{\|ABx\|_v}{\|x\|_v}$$

Ex: $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad \text{(column sum norm)}$$

$$\|A\|_1 = \max\{|1| + |4|, \; |-2| + |3|\}$$

$$= 5$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \quad \text{(Row sum)}$$

$$\|A\|_\infty = \max\{|1| + |-2|, \; |4| + |3|\} = 7$$

21/2/19

spectral radius of A :

$$\rho(A) = \max |\lambda|$$

$$\lambda \in \sigma(A)$$

where $\sigma(A)$ = set of all eigen values of A

$$\|A\|_2 = \sqrt{\rho(A^T A)}$$

Ex :— $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$ $\quad A^T A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 17 & 10 \\ 10 & 13 \end{bmatrix}$$

Eigen values $\quad Ax = \lambda x$

$$\begin{bmatrix} 17-\lambda & 10 \\ 10 & 13-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 30\lambda + 121 = 0$$

$$\lambda = \frac{30 \pm \sqrt{30^2 - 4\times1\times121}}{2}$$

$$(\lambda_1, \lambda_2) = (25.2, 4.8)$$

$$\rho(A^T A) = 25.2$$

$$\|A\|_2 = \sqrt{25.2}$$

Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)}$$

$A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$ = $\cancel{\sqrt{30}}$ $\quad \|A\|_F = \sqrt{1+4+6+9} = \sqrt{30}$

$$\|A\|_2 \le \|A\|_F$$

Using cauchy-shwartz inequality prove that:

$$\|A\|_2 \leq \|A\|_F \|x\|_2 \quad \forall x \in \mathbb{R}^n$$

$$\Leftrightarrow \boxed{\frac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_F}$$

<u>Iterative methods</u>

$f(x)=0,\ x\in[a,b]$
$x_{n+1} = \phi(x_n)$
$n=0,1,\ldots$

Let $Ax=b$ $\quad \|A\| \neq 0 \quad A \in \mathbb{R}^{n\times n}$

Let $Q \in \mathbb{R}^{n\times n}$, $|Q| \neq 0$, Q is a splitting matrix

A is exact value

$$0 = -Ax+b$$

$$Qx+0 = Qx - Ax + b$$

$$Qx = (Q-A)x + b \longrightarrow (1)$$

Define an iterative scheme

$$Qx^{(n+1)} = (Q-A)x^{(n)} + b \longrightarrow (2)$$

$$n = 0, 1, 2 \ldots$$

$$Q^{-1}\left(Qx^{(n+1)}\right) = Q^{-1}(Q-A)x^{(n)} + Q^{-1}b$$

$$x = (I - Q^{-1}A)x + Q^{-1}b \text{ --- } (3)$$

$$x^{(n+1)} = (I - Q^{-1}A)x^{(n)} + Q^{-1}b \text{ --- } (4)$$

Let $H = I - Q^{-1}A$

$$\Rightarrow x^{(n+1)} - x^{(n)} = H(x^{(n)} - x)$$

$$\|x^{(n+1)} - x\| = \|H(x^{(n)} - x)\|$$

$$\leq \|H\| \|x^{(n)} - x\| \text{ , } x^{(0)} \text{ is known}$$

$n=0$ $\quad \|x^{(1)} - x\| \leq \|H\| \; \|x^{(0)} - x\|$

$n=1$ $\quad \|x^{(2)} - x\| \leq \|H\| \; \|x^{(1)} - x\|$

$\leq (\|H\|)^2 \; \|x^{(0)} - x\|$

$\vdots$

$\|x^{(n+1)} - x\| < (\|H\|)^{n+1} \; \|x^{(0)} - x\|$

size should be less than 1 $\qquad \|H\| < 1$ for convergence

$\boxed{\text{Iterative process converges if } \|H\| < 1}$

$$\lim_{n\to\infty} \|x^{(n+1)} - x\| = 0$$

stopping criteria.

consider

$$\|x^{(n+1)} - x\| < \|H\| \; \|x^{(n)} - x\| \quad \text{——— (1)}$$

prove that

$$\|x^{(n+1)} - x\| \leq \frac{\|H\|}{1 - \|H\|} \|x^{(n+1)} - x^{(n)}\|$$

from (1)

$$\|x^{n+1} - x\| \leq \|H\| \; \|x^{(n+1)} - x + x^{(n)} - x^{(n+1)}\|$$

$$\|x^{n+1} - x\| \; (1 - \|H\|) \leq \|H\| \; \|x^{n+1} - x^{n}\|$$

$$\Rightarrow \quad \|x^{n+1} - x\| \leq \frac{\|H\|}{(1 - \|H\|)} \left(\|x^{(n+1)} - x^{(n)}\|\right)$$

→ Gauss Jacobi Iterative method.

Let $A = L + D + U$

Ex: $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

for jacobi method $\boxed{Q = D}$.

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26/2

Jacobi iteration method.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3$$

Initial vector $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$

$$x_1 = \frac{1}{a_{11}}\left(d_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

$$x_2 = \frac{1}{a_{22}}\left(d_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right)$$

$$x_3 = \frac{1}{a_{33}}\left(d_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{n+1} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(n)} + \begin{bmatrix} \frac{d_1}{a_{11}} \\ \frac{d_2}{a_{22}} \\ \frac{d_3}{a_{33}} \end{bmatrix}$$

$x^{(n+1)} = Hx^{(n)} + C$, $x^{(0)}$ is known

Max eigen value of H - spectral radius

Diagonal entries should be large $\|H\|_\infty < 1$ Then it converg

Sufficient for convergence is $\|H\|_\infty < 1$

column sum

$$\|H\|_1 = \max\left\{\left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{31}}{a_{33}}\right|, \left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{32}}{a_{33}}\right|, \left|\frac{a_{13}}{a_{11}}\right| + \left|\frac{a_{23}}{a_{22}}\right|\right\}$$

$$\|H\|_\infty = \max\left\{\left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{13}}{a_{11}}\right|, \left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{23}}{a_{22}}\right|, \left|\frac{a_{31}}{a_{33}}\right| + \left|\frac{a_{32}}{a_{33}}\right|\right\}$$

→ Diagonal dominant matrix

Diagnal dominant matrix

$$|a_{ii}| >> \sum_{\substack{j=1 \\ (j \neq i)}}^{n} a_{ij}$$

Ex :

$5x_1 + 2x_2 - 10x_3 = -3$ —— ③

$10x_1 + 4x_2 - 2x_3 = 12$ —— ①

$x_1 - 10x_2 - x_3 = -10$ —— ②

order – according to diagonal entries

Gauss Seidal method

$$x^0 = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$$

$$x_1^{(1)} = \frac{1}{a_{11}} (d_1 - a_{12} x_2^{(0)} - a_{13} x_3^{(0)})$$

$$x_2^{(1)} = \frac{1}{a_{22}} (d_2 - a_{21} x_1^{(1)} - a_{23} x_3^{(0)})$$

$$x_3^{(1)} = \frac{1}{a_{33}} (d_3 - a_{31} x_1^{(1)} - a_{32} x_2^{(1)})$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{22}} & 1 & 0 \\ \frac{a_{31}}{a_{33}} & \frac{a_{32}}{a_{33}} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(n+1)} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ 0 & 0 & -\frac{a_{23}}{a_{22}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(n)} + \begin{bmatrix} \frac{d_1}{a_{11}} \\ \frac{d_2}{a_{22}} \\ \frac{d_3}{a_{33}} \end{bmatrix}$$

$$x^{(n+1)} = H x^{(n)} + c \qquad A = L + D + U$$

$$\boxed{H = -(D+L)^{-1} U}$$

For convergence $\|H\| < 1 \Rightarrow |a_{ii}| \gg \sum_{j=1}^{n} |a_{ij}|$ $(j \neq i)$

(twice faster - ?)

Note :

$Ax = b$, $|A| \neq 0$

✓ $\boxed{b} \longrightarrow \boxed{A} \longrightarrow x$

✓ $\boxed{b+\tilde{b}} \longrightarrow \boxed{A} \longrightarrow x+\tilde{x}$

$\Rightarrow$ stable .

$\boxed{b+\tilde{b}} \longrightarrow \boxed{A+\tilde{A}} \longrightarrow x+\tilde{x}$

Theorem In $Ax = b$, $|A| \neq 0$

$$\frac{1}{K(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|x - x^2\|}{\|x\|} \leq K(A) \frac{\|r\|}{\|b\|}$$

where $r = b - \tilde{b} = \delta b$

$e = x - \tilde{x} = \delta x$

condition no.

$K(A) = \|A\| \|A^{-1}\|$

27/12 **Theorem:** Let $Ax = b$, $|A| \neq 0$, then

$$\frac{1}{K(A)} \frac{\|r\|}{\|b\|} \le \underbrace{\frac{\|x - \tilde{x}\|}{\|x\|}}_{\text{relative error}} \le K(A) \frac{\|r\|}{\|b\|}$$

$\rightarrow$ residual error.

where $r = b - A\tilde{x} = b - \tilde{b}$
$e = x - \tilde{x}$

$K(A) = \|A\| \cdot \|A^{-1}\|$

proof

$b = Ax$

$\|b\| = \|Ax\| \le \|A\| \cdot \|x\|$ ($x \neq 0$)

$$\rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|} \quad \cdots\cdots\cdots \text{①}$$

Also, $A\tilde{x} = b + \delta b$

$A(x + \delta x) = b + \delta b$

$Ax + A\delta x = b + \delta b$

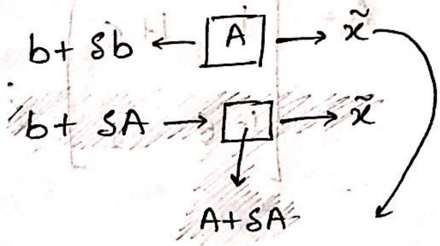
$\cancel{b} + A\delta x = \cancel{b} + \delta b$

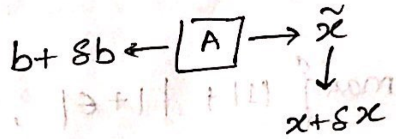
$\Rightarrow A\delta x = \delta b$

$\delta x = A^{-1}\delta b$

$\|\delta x\| = \|A^{-1}\delta b\|$

$\le \|A^{-1}\| \|\delta b\|$ —— ②





$\tilde{x} = x + \delta x$

$\delta x = \tilde{x} - x$

$\tilde{b} = b + \delta b$

$\tilde{b} - b = \delta b$

$$\frac{\|x - \tilde{x}\|}{\|x\|} \le \|A\| \|A^{-1}\| \frac{\|b - \tilde{b}\|}{\|b\|} \quad \cdots\cdots \text{③}$$

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consider, $Ax = b \Rightarrow x = A^{-1}b$

$$\Rightarrow \|x\| \le \|A^{-1}\| \, \|b\|$$

$$\Rightarrow \frac{1}{\|A^{-1}\| \, \|b\|} \le \frac{1}{\|x\|} \quad \cdots\cdots (4)$$

$$\delta b = A \delta x$$

$$\|\delta b\| \le \|A\| \, \|\delta x\| \qquad \frac{\|\delta b\|}{\|A\|} \le \|\delta x\| \quad \cdots\cdots (5)$$

from (4) & (5)

$$\frac{1}{\|A\| \, \|A^{-1}\|} \frac{\|b - \tilde{b}\|}{\|b\|} \le \frac{\|x - \tilde{x}\|}{\|x\|} \quad \cdots\cdots (6)$$

Ex:

$$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix} : A^{-1} = \frac{1}{\epsilon^2} \begin{bmatrix} 1 & -1-\epsilon \\ -1+\epsilon & 1 \end{bmatrix} ; \epsilon > 0$$

$$\|A\|_\infty = \max\{ |1| + |1+\epsilon| , |1-\epsilon| + |1| \} = 2+\epsilon$$

$$\|A^{-1}\|_\infty = \max\left\{ \frac{|1| + |-1-\epsilon|}{\epsilon^2} , \frac{|-1+\epsilon| + |1|}{\epsilon^2} \right\} = \frac{2+\epsilon}{\epsilon^2}$$

$$K_\infty(A) = \|A\|_\infty \, \|A^{-1}\|_\infty = \frac{(2+\epsilon)^2}{\epsilon^2}$$

case (i) : If $\epsilon = 0.01$

$$K_\infty(A) = \frac{(2.01)^2}{(0.01)^2} = 40401$$

$$\Rightarrow \frac{\|x - \tilde{x}\|}{\|x\|} \leq 40401 \frac{\|b - \tilde{b}\|}{\|b\|}$$

case (ii) : If $\epsilon = 10$

$$K_\infty(A) = 1.44$$

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq 1.44 \frac{\|b - \tilde{b}\|}{\|b\|}$$

If condition no. is high sol does not exist.

Ex $x_1 + (1+\epsilon)x_2 = 3 \longrightarrow m_1 = \frac{-1}{1+\epsilon}$

$(1-\epsilon)x_1 + x_2 = 5 \longrightarrow m_2 = \frac{\epsilon - 1}{1}$

If $\epsilon = 0.01$ $\longrightarrow m_1 = -0.99$; $m_2 = -0.99$

If $\epsilon = 10$ $\longrightarrow m_1 = -0.09$ $m_2 = 9$

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★ Power method to find numerically largest eigen value and the corresponding eigen vector.

Let $Ax = b$, $|A| \neq 0$ ..... (1) : $x \in \mathbb{R}^n$

Let $x_1, x_2, x_3, \ldots, x_n$ be linearly independent eigen vectors corresponding to $\lambda_1, \lambda_2, \ldots \lambda_n$

Let λ_1 is a numerically largest eigen value

Let $x = c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots + c_n x_n$, $c_i \in \mathbb{R}$, $i = 1, \ldots n$

Let $Ax_i = \lambda_i x_i$, $i = 1, 2, \ldots n$, $x_i \neq 0$

$$Ax = A(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)$$

$$= c_1 A x_1 + c_2 A x_2 + \cdots + c_n A x_n$$

$$Ax = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_n \lambda_n x_n \quad \text{——— (1)}$$

$$A(Ax) = A(c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_n \lambda_n x_n)$$

$$A^2 x = c_1 \lambda_1^2 x_1 + c_2 \lambda_2^2 x_2 + \cdots c_n \lambda_n^2 x_n$$

$$\boxed{A^m x = c_1 \lambda_1^m x_1 + c_2 \lambda_2^m x_2 + \cdots + c_n \lambda_n^m x_n}$$

suppose

$x, Ax, A^2x, A^3x, \ldots$

$1, \lambda, \lambda^2, \ldots\ldots$

18/2

Matrix eigen value problem (power method)

Let $A \in \mathbb{R}^{n\times n}$, $\exists$ a basis such that

$$Ax_i = \lambda x_i \quad \text{------} \quad (1)$$

$i = 1, 2, 3 \dots$

$x_1, x_2, \dots x_n$

$\lambda_1, \lambda_2, \dots \lambda_n$

$|\lambda_1| > |\lambda_j|$

$j = 2, 3, \dots n$

Let

$$x^{(0)} = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$Ax^{(0)} = c_1 A x_1 + c_2 A x_2 + \dots c_n A x_n$$

$$x^{(1)} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots c_n \lambda_n x_n$$

$$\Longrightarrow Ax^{(1)} = x^{(2)} = c_1 \lambda_1^2 x_1 + c_2 \lambda_2^2 x_2 + \dots + c_n \lambda_n^2 x_n$$

$\vdots$

$$Ax^{(m-1)} = x^{(m)} = c_1 \lambda_1^m x_1 + c_2 \lambda_2^m x_2 + \dots + c_n \lambda_n^m x_n$$

$$= \lambda_1^m \Big[c_1 x_1 + \underbrace{c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^m x_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^m x_n}_{e^{(m)}} \Big]$$

$$\lim_{m\to\infty} x^{(m)} = \lambda_1^m c_1 x_1$$

Let $x^{(m)} = \lambda_1^m \left[c_1 x_1 + e^{(m)} \right]$

$x^{(m+1)} = \lambda_1^{m+1} \left[c_1 x_1 + e^{(m+1)} \right]$

$x, x_1 \in \mathbb{R}^n$

Define $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear function.

$$\phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y), \quad \alpha, \beta \in \mathbb{R}$$
$$x, y \in \mathbb{R}^n$$

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~~Let $\phi(x) = \|x\|_\infty$~~

Let $\phi(z) = z_j \quad , \quad j = 1, 2, \dots n$

$$z = (z_1, z_2, \dots, z_n)$$

$$\phi(x^{(m)}) = \phi\left[\lambda_1^m (c_1 x_1 + e^{(m)})\right]$$

$$= \lambda_1^m \left[\phi(c_1 x_1 + e^{(m)})\right]$$

$$\phi(x^{(m+1)}) = \lambda_1^{m+1}\left[\phi(c_1 x_1 + e^{(m+1)})\right]$$

$$\lim_{m\to\infty} \frac{\phi(x^{(m+1)})}{\phi(x^{(m)})} = \lim_{m\to\infty} \lambda_1 \left[\frac{\phi(c_1 x_1 + e^{(m+1)})}{\phi(c_1 x_1 + e^{(m)})}\right] = \lambda_1$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, Let $x^{(0)} = (1, 1)$

$$Ax^{(0)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 5 \underbrace{\begin{bmatrix} 3/5 \\ 1 \end{bmatrix}}_{x^{(1)}}$$

$$Ax^{(1)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} 13/5 \\ 19/5 \end{bmatrix} = \frac{19}{5}\begin{bmatrix} 13/19 \\ 1 \end{bmatrix}$$

$$Ax^{(2)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 13/19 \\ 1 \end{bmatrix} = \begin{bmatrix} 51/19 \\ 77/19 \end{bmatrix} = \frac{77}{19}\begin{bmatrix} 51/77 \\ 1 \end{bmatrix}$$

$$Ax^{(3)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 51/77 \\ 1 \end{bmatrix} = \begin{bmatrix} 205/77 \\ 307/77 \end{bmatrix} = 3.98\begin{bmatrix} 0.667 \\ 1 \end{bmatrix}$$

Exact eigen vector : $x = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}\underset{x}{\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}} = \begin{bmatrix} 8/3 \\ 4 \end{bmatrix} = \underset{\lambda}{4}\underset{x}{\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}}$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-3x_1 + 2x_2 = 0$$

$$3x_1 = 2x_2$$

$$x_1 = 2/3\, x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

$$\boxed{Ax = \lambda x}$$

Note $x = (1, -1)$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}\underset{x}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underset{\lambda}{(-1)}\underset{x}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$\boxed{\left|\lambda^{(n+1)} - \lambda^{(n)}\right| < 10^{-4}}$$

Note: To find the numerically smallest eigen value.

$$\boxed{\begin{aligned} A^{-1}x^{(0)} &= x^{(1)} \\ x^{(0)} &= Ax^{(1)} \end{aligned}}$$

$$x^{(0)} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$A^{-1}x^{(0)} = c_1\frac{x_1}{\lambda_1} + c_2\frac{x_2}{\lambda_2} + \cdots + c_n\frac{x_n}{\lambda_n}$$

5/3

Weirstrass approximation theorem

Let $f : [a,b] \rightarrow R$ be continuous. Given $\epsilon > 0$ there exists n (depends only on ϵ) such that

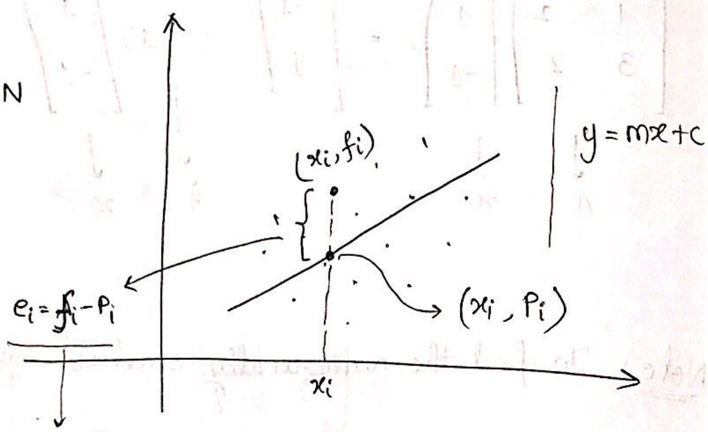
$$|f(x) - P_n(x)| < \epsilon, \text{ for all } x \in [a,b]$$

where P_n is a polynomial of degree $\leq n$

Any cont f^n on a bounded interval can be uniformly approximated by polynomials

$\{x_i, f(x_i)\}_{i=1,2,\dots N}$

$P(x) = C_1 + C_2 x$



$$e_i = f_i - P_i$$

Error,

$$E_i = f(x_i) - p(x_i), \; i = 1, 2, \dots N$$

$$I(c_1, c_2) = \sum_{i=1}^{N} [E(x_i)]^2$$

$$= \sum_{i=1}^{N} [f(x_i) - P(x_i)]^2$$

$$= \sum_{i=1}^{N} [f(x_i) - (c_1 + c_2 x_i)]^2.$$

$$\frac{\partial I}{\partial c_1} = 0$$

$$\Rightarrow \sum_{i=1}^{N} 2\left\{f(x_i) - (c_1 + c_2 x_i)\right\} \times (-1) = 0$$

$$\frac{\partial I}{\partial c_2} = 0 \Rightarrow \sum_{i=1}^{N} 2\left[f(x_i) - (c_1 + c_2 x)\right] \times (-x_i) = 0$$

$$\sum_{i=1}^{N} f(x_i) = N c_1 + c_2 \sum_{i=1}^{N} x_i$$

$$\sum_{i=1}^{N} x_i f(x_i) = c_1 \sum_{i=1}^{N} x_i + c_2 \sum_{i=1}^{N} x_i^2$$

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} f(x_i) \\ \sum_{i=1}^{N} x_i f(x_i) \end{bmatrix}$$

Note: $p(x) = c_1 + c_2 x + c_3 x^2$
$x \in [a,b]$

Note : $p(x) = c_1 + c_2 x + c_3 x^2 \quad x \in [a, b]$

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 \\ \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} f(x_i) \\ \sum_{i=1}^{N} x_i f(x_i) \\ \sum_{i=1}^{N} x_i^2 f(x_i) \end{bmatrix}$$

Ex: Find a least square straight line and quadratic polynomial to the following data

x	-0.5	1	1.5	2	2.5
$f(x)$	0.75	3	4.75	7	9.75

case (i) $p(x) = c_1 + c_2 x$

$$\begin{bmatrix} 5 & 6.5 \\ 6.5 & 13.75 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 25.25 \\ 48.125 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = 1.2972. \\ c_2 = 2.8868 \end{array}$$

case(ii) $p(x) = c_1 + c_2 x + c_3 x^2$

$$\begin{bmatrix} 5 & 6.5 & 13.75 \\ 6.5 & 13.75 & 27.87 \\ 13.75 & 27.87 & 61.1875 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 25.25 \\ 48.125 \\ 102.8125 \end{bmatrix}$$

Let $f:[a,b] \longrightarrow \mathbb{R}$ be continuous

case (i) $p(x) = c_1 + x c_2$, $x \in [a,b]$

$E(x) = f(x) - p(x)$, $x \in [a,b]$

$\mathcal{I}$

$$I(c_1, c_2) = \int_a^b [f(x) - p(x)]^2 \, dx$$

$$= \int_a^b \{ f(x) - (c_1 + c_2 x)]^2 \, dx$$

$$\frac{\partial I}{\partial c_1} = \int_a^b 2 [f(x) - (c_1 + c_2 x)] \times (-1) \, dx$$

$$\frac{\partial I}{\partial c_2} = 0 = \int_a^b 2 [f(x) - (c_1 + c_2 x)] \times (-x) \, dx$$

$$\begin{bmatrix} b-a & \int_a^b x \, dx \\ \int_a^b x \, dx & \int_a^b x^2 \, dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_a^b f(x) \, dx \\ \int_a^b x f(x) \, dx \end{bmatrix}$$

Note: $p(x) = c_1 + c_2 x + c_3 x^2 \, , \; x \in [a, b]$

$$\begin{bmatrix} b-a & \int_a^b x \, dx & \int_a^b x^2 \, dx \\ \int_a^b x \, dx & \int_a^b x^2 \, dx & \int_a^b x^3 \, dx \\ \int x^2 \, dx & \int x^3 \, dx & \int x^4 \, dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \int_a^b f(x) \, dx \\ \int_a^b x f(x) \, dx \\ \int_a^b x^2 f(x) \, dx \end{bmatrix}$$

Ex: $f(x) = x^3$, $x \in [0,1]$

a) find $p(x) = c_1 + c_2 x$

b) find $p(x) = c_1 + c_2 x + c_3 x^2$

Sol:

$$\begin{bmatrix} \int_0^1 dx & \int_0^1 x\,dx \\ \int_0^1 x\,dx & \int_0^1 x^2\,dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x)\,dx \\ \int_0^1 x f(x)\,dx \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/5 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = -\frac{2}{10} \\ c_2 = \frac{9}{10} \end{array}$$

$$\Longrightarrow p(x) = c_1 + c_2 x = -\frac{2}{10} + \frac{9x}{10} = \frac{1}{10}(9x - 2)$$

Note $f(0) = 0$ $\quad p(0) = -\frac{2}{10}$ $\quad\Big|\quad f\left(\frac{1}{2}\right) = \frac{1}{8}$

$f(1) = 1$ $\quad p(1) = \frac{7}{10}$ $\quad\Big|\quad p\left(\frac{1}{2}\right) = \cancel{\frac{16}{10}}\ \frac{5}{20}$

b) $p(x) = c_1 + c_2 x + c_3 x^2$, $x \in [0,1]$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \end{bmatrix}$$

$c_1 = 0.05$, $c_2 = -0.6$, $c_3 = 1.5$

$p(x) = 0.05 - 0.6x + 1.5x^2$, $x \in [0, 1]$

$f(\frac{1}{2}) = \frac{1}{8} = 0.125$	$f(0) = 0$	$f(1) = 1$
$p(\frac{1}{2}) = 0.125$	$p(0) = 0.05$	$P(1) =$?

$H = \frac{1}{i+j-1}$, $i, j = 1, 2, \dots N$

$\text{cond}(H_3) = \|H_3\| \, \|H_3^{-1}\| = 748$

$$\Rightarrow \frac{\|x - \tilde{x}\|}{\|x\|} \le 748 \frac{\|b - \tilde{b}\|}{\|b\|}$$

$I = AA^{-1}$

$\|I\| \le \underbrace{\|A\| \, \|A^{-1}\|}$

$1 \le k(A)$

orthonormal basis

$\{\phi_1, \phi_2, \dots \phi_n\} \in \{-1, 1\}$ $x \in \mathbb{R}$

$$\langle \phi_i, \phi_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases}$$

$$\langle \phi_i, \phi_j \rangle = \int_{-1}^{1} w(x) \, \phi_i(x) \, \phi_j(x) \, dx, \quad w(x) > 0$$

Legendre polynomials

ϕ_0 $\{ 1 \quad t \quad \frac{1}{3}(3t^2 - 1) \dots \}$ $w(t) = 1$

ϕ_1 ϕ_2

$$\int_{-1}^{1} \phi_i(t) \, \phi_j(t) \, dt = \begin{cases} \frac{2}{2i+1}, & i = j \\ 0, & i \ne j \end{cases}$$

$i=j=0$ $\quad \int_{-1}^{1} \phi_0^2(t)\,dt = \int_{-1}^{1} 1\,dt = \cancel{\frac{2}{3}}\ 2$

$i=j=1$ $\quad \int_{-1}^{1} \phi_2^2(t)\,dt = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$

Ex $\quad f(x) = x^3, \; x \in [0,1]$ | $f(x) = ,\; x \in [a,b]$

$[a,b] \rightleftharpoons [-1,1]$

$x = c_1 t + c_2$

If $x = a \to t = -1 \Rightarrow a = -c_1 + c_2$

$x = b \Rightarrow t = 1 \Longrightarrow b = c_1 + c_2$

$c_1 = \frac{b-a}{2}$ $\qquad a+b = 2c_2$

$c_2 = \frac{1}{2}(a+b)$

$x = \left(\frac{b-a}{2}\right)t + \frac{1}{2}(a+b)$

$f(x) = x^3, \; x \in [0,1]$

$x = \frac{1}{2}t + \frac{1}{2}$

$g(t) = f\left(\frac{1}{2}t + \frac{1}{2}\right) = \frac{1}{8}(t+1)^3, \quad t \in [-1,1]$

Let $p(t) = c_1 \phi_0(t) + c_2 \phi_1(t) + c_3 \phi_2(t)$

$$E(x) = g(t) - p(t) \quad , \; x \in [-1, 1]$$

$$J(c_1, c_2) = \int_{-1}^{1} [g(t) - p(t)]^2 \, dt$$

$$\int_{-1}^{1} [g(t) - c_1 \phi_0(t) - c_2 \phi_1(t) - c_3 \phi_2(t)]^2 \, dt$$

$$\left.\begin{matrix} \frac{\partial J}{\partial c_1} = 0 \\ \frac{\partial J}{\partial c_2} = 0 \\ \frac{\partial J}{\partial c_3} = 0 \end{matrix}\right\} \begin{bmatrix} \int_{-1}^{1} \phi_0^2 dt & 0 & 0 \\ 0 & \int_{-1}^{1} \phi_1^2 dt & 0 \\ 0 & 0 & \int_{-1}^{1} \phi_2^2 dt \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} g(t)\phi_0(t) dt \\ \int_{-1}^{1} g(t)\phi_1(t) dt \\ \int_{-1}^{1} g(t)\phi_2(t) dt \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/10 \\ 1/5 \end{bmatrix}$$

$$\int_{-1}^{1} g(t)\phi_0(t) = \int_{-1}^{1} \frac{1}{8} (t+1)^3 dt =$$

$$c_1 = \frac{1}{4} \; ; \quad c_2 = \frac{9}{20} \; , \quad c_3 = \frac{1}{4}$$

$$p(t) = \frac{1}{4} + \frac{9}{20} t + \frac{1}{4} \times \frac{1}{2} (3t^2 - 1)$$

since $t = 2x - 1$

$$p(x) = \frac{1}{4} + \frac{9}{20} (2x - 1) + \frac{1}{8} [3(2x-1)^2 - 1]$$

$$= \frac{1}{4} + \frac{9}{20} (2x - 1) + \frac{1}{8} [3(4x^2 + 1 - 4x) - 1]$$

$$= \cancel{\frac{3}{20}} + \cancel{\frac{-21}{10}x} + \cancel{3x^2}$$

$$= \frac{3}{10} + \frac{-21}{10}x + 3x^2$$

$$p(x) = 0.05 - 0.6x + 1.5x^2$$

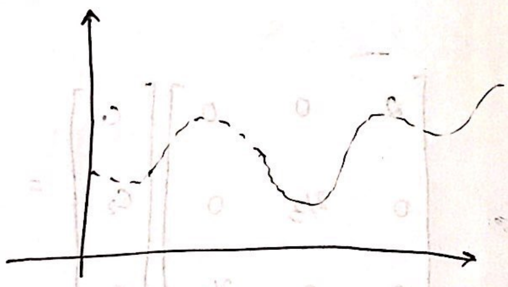
$$\int_{-1}^{1} \phi_i(x)\,\phi_j(x)\,dx = \begin{cases} \frac{2}{2i+1}, & i \neq j \\ 0, & i \neq j \end{cases}$$

~~7/3~~

Interpolation

Let

$\{x_i, f(x_i)\}$, $i = 0, 1, 2, \ldots n$



Let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n$$

Interpolation conditions $f(x_i) = p(x_i)$, $i = 0, 1, 2 \ldots n$

At $\underline{\underline{x = x_0}}$ $p(x_0) = f(x_0)$

$$c_0 + c_1 x_0 + c_2 x_0^2 + \cdots + c_n x_0^n = f(x_0)$$

$\underline{\underline{x = x_1}}$

$$c_0 + c_1 x_1 + c_2 x_1^2 + \cdots + c_n x_1^n = f(x_1)$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \\ f(x_n) \end{bmatrix}$$

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \end{bmatrix}$$

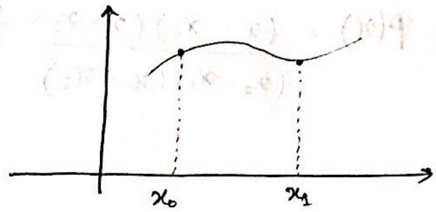
$$x_1 - x_0 \neq 0$$

The points $\{x_0, x_1, \cdots, x_n\}$ are distinct

Ex: $V_3 = \begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix}$, $K(V_3) = 21 \times 17$

Lagrange's form

$\{(x_0, f(x_0)), (x_1, f(x_1))\}$



$p(x) = c_0 + c_1 x$

$p(x_0) = f(x_0) \Rightarrow c_0 + c_1 x_0 = f(x_0)$

$p(x_1) = f(x_1) \Rightarrow c_0 + c_1 x_0 = f(x_1)$

$c_0 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1}$

$c_1 = \dfrac{f(x_0) - f(x_1)}{x_0 - x_1}$

$$p(x) = c_0 + c_1 x$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1} + \frac{[f(x_0) - f(x_1)]}{x_0 - x_1} \times x$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1} + \frac{x f(x_0) - x f(x_1)}{x_0 - x_1}$$

$$p(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x_0 - x}{x_0 - x_1}\right) f(x_1)$$

$$P(x) \Longrightarrow \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1)$$

Note: $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$

$l_0(x) = \dfrac{x - x_1}{x_0 - x_1}$ $\quad l_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

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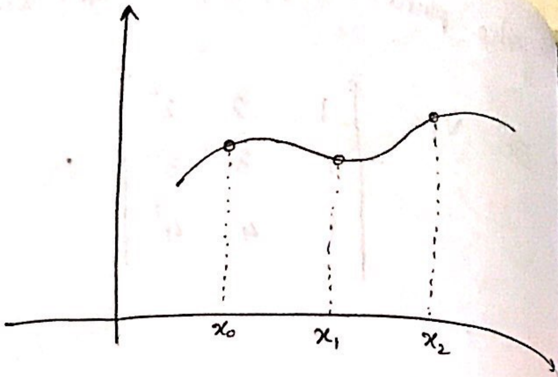
$l_0(x_0) = 1 \qquad l_1(x_1) = 1$



$$\rightarrow p(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

In general

$$p(x) = \sum_{i=0}^{n} l_i(x) f(x_i), \qquad l_i(x) = \prod_{\substack{j=0 \\ (j\neq i)}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Note : $\sum_{i=0}^{n} l_i(x) = 1$

Ex: Find an interpolating polynomial for following data using legrangian method.

x :	1	2
$f(x)$:	2	5

$(\overset{x_0}{1}, \overset{f_0}{2}) \quad (\overset{x_1}{2}, \overset{f_1}{5})$

$$p(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$\frac{x-2}{-1} \times 2 + \frac{x-1}{1} \times 5 = 3x-1$$

Ex- 2

$x =$	1	2	4
$f(x) =$	2	5	17

$$p(x) = \frac{(x-2)(x-4)}{-1 \times -3} \times 2 + \frac{(x-1)(x-4)}{1 \times -2} \times 5$$

$$+ \frac{(x-1)(x-2)}{3 \times 2} \times 17$$

$$p(x) = \frac{x^2 - 6x + 8}{3} \times 2 + \frac{x^2 - 5x + 4}{-2} \times 5$$

$$+ \frac{x^2 - 3x + 2}{6} \times 17$$

$$p(x) = x^2 + 1$$

problem sheet - 5

syllabus to minor II.

Ex - (2) $\quad x = 1 \quad 2 \quad 4$

$f(x) = 2 \quad 5 \quad 17$

$$p(x) = \frac{(x-2)(x-4)}{-1 \times -3} \times 2 + \frac{(x-1)(x-4)}{1 \times -2} \times 5 + \frac{(x-1)(x-2)}{3 \times 2} \times 17$$

$$p(x) = \frac{x^2 - 6x + 8}{3} \times 2 + \frac{x^2 - 5x + 4}{-2} \times 5 + \frac{x^2 - 3x + 2}{6} \times 17$$

$$p(x) = x^2 + 1$$

problem sheet - 5
syllabus to minor II

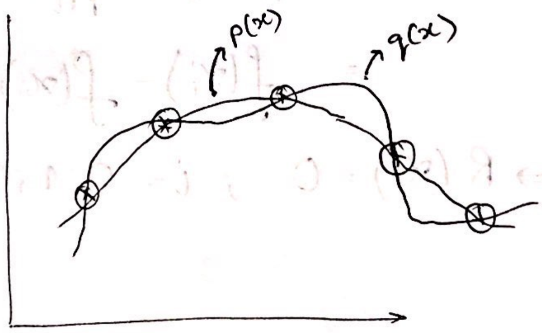
26/3

Recap

Legrange's interpolation

1D data :

$\{x_i, f(x_i)\}; \ i = 0, \dots n$



$$P_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$

where $$l_i(x) = \prod_{\substack{j=0 \\ (j \neq i)}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Note : $$l_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$l_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Existance and uniqueness of interpolation.

Existance : Let $f:[a,b]\to \mathbb{R}$ be continuous and $\{x_i, f(x_i)\}$, $i=0,1,\dots n$

~~$[x_i \to n+1$~~ $\quad [x_i \to n+1$ distinct points$]$

$$p_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$

$\longrightarrow \boxed{p(x_i) = f(x_i)}$, $i = 0,1,2,\dots n$ $\quad \because \quad l_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$\Rightarrow$ There exists an interpolating polynomial.

Uniqueness Let p_n and q_m be polynomials of degree $\leq n$, such that

$$p_n(x_i) = f(x_i) \quad \text{and} \quad q_m(x_i) = f(x_i) \quad i=0,1,\dots n$$

$\Rightarrow$ define $R(x) = p_n(x) - q_m(x)$

$\rightarrow$ Degree of $R(x)$ is $\leq n$

$\rightarrow R(x_i) = p_n(x_i) - q_m(x_i)$

$\quad\quad = f(x_i) - f(x_i) = 0$

$\Rightarrow R(x_i) = 0$, $i = 0,1,2,\dots n$

degree is $\leq n$
but there are
$n+1$ zeros
$n+1$ roots.

The above statement is true.
whenever $R(x) = 0 \ \forall x \in [a,b]$

True only for zero polynomial

$\Longrightarrow p_n(x) = q_n(x)$

$\therefore$ the interpolating polynomial is always unique.

Drawbacks of interpolating polynomial is that if a point is ~~ad~~ added then we have to do all the calculations again. but with this below method we can use prev calculation

Newton - Divided difference method:

1D data: $\{x_i, f(x_i)\}$, $i = 0, 1, 2, 3 \dots n$

$$\boxed{P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots} \quad \text{———} \quad (1)$$

$x : x_0 \quad x_1$

$f : f(x_0) \; f(x_1)$

If data set is $x : x_0 \quad x_1 \quad x_2$

$f : f(x_0) \; f(x_1) \; f(x_2)$

then $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$

$x :$	x_0	x_1	x_2	$\dots\dots$	x_n
$f(x) :$	$f(x_0)$	$f(x_1)$	$f(x_2)$		$f(x_n)$

apply the point in (1)

→ At $x = x_0$

$$P_n(x_0) = f(x_0)$$

$$a_0 = f(x_0)$$

→ At $x = x_1$

$$P(x_1) = a_0 + a_1(x_1 - x_0)$$

$$f(x_1) = f(x_0) + a_1(x_1 - x_0)$$

$$\Longrightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1] \longrightarrow \text{divided diff of the data.}$$

$p(x_i) = f(x_i)$ substitute the points and find out horsefuck

⇒ At $x = x_2$

$$\rightarrow P(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\rightarrow f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2 = \frac{1}{x_2 - x_0}\left[\frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{(f(x_0) - f(x_1))}{x_0 - x_1}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\underbrace{f[x_2, x_1]}_{\text{divided diff of } x_2\, x_1} - \underbrace{f[x_1, x_0]}_{\text{divided diff of } x_1\, x_0}\right]$$

$$= f[x_0, x_1, x_2]$$

some bitch made him solve everything fucking bastard

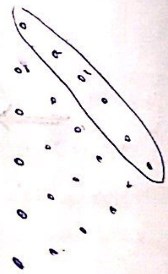
$$a_2 = \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{1}{(x_2 - x_0)(x_2 - x_1)}\left[f(x_2) - f(x_0) - \left[\frac{f(x_1) - f(x_0)}{(x_1 - x_0)}\right](x_2 - x_0)\right]$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{(f(x_0) - f(x_1))}{x_0 - x_1}\right]$$

$$= \frac{1}{(x_2 - x_0)} f[x_1, x_2] - (f[x_0, x_1])$$

$$= f[x_0, x_1, x_2]$$



Next page is fucking beauty
gotta love this bastard man

Divided difference table :

x	f	1 DD	2DD
x_0	f_0		
			$f[x_0, x_1, x_2]$
x_1	f_1	$\frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$	$\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]$
x_2	f_2	$\frac{f_2 - f_1}{x_2 - x_1} = f[x_2, x_1]$	$\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = f[x_1\, x_2\, x_3]$
x_3	f_3	$\frac{f_3 - f_2}{x_3 - x_2} = f[x_3 - x_2]$	

Ex:

x	f	1DD
1	2	
		$\frac{5-2}{2-1} = 2$
2	5	

teja ikkada.
mandi lera kathulu
leva

$p(x) = 2 + 3(x-1) = 3x - 1$ → because of uniqueness we are getting same polynomial as interpolation

god daaaaamn.

Ex:

x	f	1DD	2 DD
1	2		
		3	
2	5		1
		6	
4	4		

$p(x) = 2 + 3(x-1) + 1(x-1)(x-2)$
$= x^2 + 1$ $+a_3$

How many examples will he give

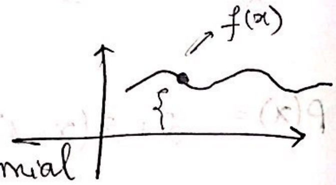
Ex:

x	f	1DD	2DD	3DD
1	2			
		3		
2	5		1	
		6		0
4	17		1	
		7		
3	10			

$$p(x) = x^2 + 1$$

why did we get the same polynomial.
lets find out in the next class

27/3 Polynomial interpolation error



Th : ~~He~~ Let f be a function on $C^{n+1}[a,b]$ and let p be a polynomial of degree n that interpolate the function f at $(n+1)$ distinct points $x_0, x_1, \ldots x_n$ in the interval $[a,b]$. To each $x \in [a,b]$ there corresponds a point

$\xi_x \in (a,b)$ s.t

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{n+1}(\xi_x) \prod_{i=0}^{n} (x - x_i) \longrightarrow (1)$$

proof: If x is one of the nodes of interpolation x_i. then (1) is true If x is any point other that node

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$(x_0, x_1, \dots x_n)$ put

$$\omega(t) = \prod_{i=0}^{n} (t - x_i) = (t - x_0) \dots (t - x_n) \longrightarrow ②$$

$$\phi = f - p - \lambda\omega \longrightarrow ③$$

where λ is a real no. such that $\phi(x) = 0$.

x is fixed point

$$\phi(x) = 0$$

$$f(x) - p(x) - \lambda\omega(x) = 0.$$

$$\lambda = \frac{f(x) - p(x)}{\omega(x)} \longrightarrow ④$$

Now $\phi \in C^{n+1}[a, b]$

ϕ vanishes at $(n+2)$ points namely $x_0 \dots\dots x_4$

Rolle's theorem

f cont $[a,b]$ $\exists$ $c \in [a, b]$

diff at $[a,b]$ $f'(c) = 0$

$f(a) = f(b)$

ϕ' has atleast $(n+1)$ distinct zeros in (a,b) (By Rolles thm)

ϕ'' has n zeros

ϕ^{n+1} has at least one zero in $[a,b]$

ξ_x

$$\phi^{(n+1)}(\xi_x) = 0$$

$$\phi^{(n+1)}(x) = f^{n+1}(x) - \underbrace{p^{n+1}(x)}_{\substack{0 \\ \text{degree "n"}}} - \lambda \underbrace{\omega^{n+1}(x)}_{\substack{\text{constant} \\ (n+1)!}}$$

$$\phi^{(n+1)}(x) = f^{(n+1)}(x) - \lambda(n+1)!$$

$$\Rightarrow 0 = \phi^{(n+1)}(\xi_x) = f^{n+1}(\xi_x) - \frac{f(x)-p(x)}{\omega(x)}(n+1)!$$

$$\Rightarrow \quad f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$\underline{\underline{n = 1}}$: $E_1 = f(x) - p(x)$

$|E_1| < (\quad)$ $\qquad$ x lies in between x_0 & x_1

$$E_1 = \frac{f''(\xi_x)}{2!}(x-x_0)(x-x_1)$$

$$|E_1| = \frac{1}{2}\left|(x-x_0)(x-x_1) f''(\xi_x)\right|$$

find max error

$$\le \frac{1}{2} \max_{x\in(x_0,x_1]} \left|(x-x_0)(x-x_1) f''(\xi_x)\right|$$

$$M_2 = \max_{x\in[x_0,x_1]} |f''(x)|$$

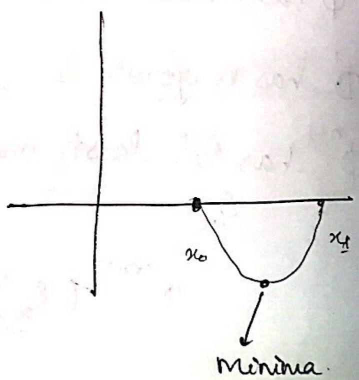
$$\le \frac{1}{2} \max_{x\in[x_0,x_1]} |(x-x_0)(x-x_1)|\, M_2$$

Let $g(x) = (x-x_0)(x-x_1)$

$$g'(x) = (x-x_1) + (x-x_0) = 0$$

$$2x = x_0 + x_1$$

$$x = \frac{x_0+x_1}{2}$$



$g''(x) = 2$

$$g''(x) > 0$$

$\Rightarrow$ minima

take mod so that we get maxima

Generalised mean value theorem.

$f[a,b] = \frac{f(a) - f(b)}{a-b} = f'(c)$, $c \in (a,b)$

Let $f : [a,b] \longrightarrow \mathbb{R}$ and

a) $f, f', \dots f^{n-1}$ are continuous on $[a,b]$

b) f^{n-1} is differentiable on (a,b)

If $x_0, x_1, \dots x_n$ are $n+1$ distinct points in $[a,b]$

then
$$f[x_0, x_1, \dots x_n] = \begin{cases} \frac{f^{(n)}(c)}{n!}, & c \in (a,b) \\ \frac{f^{(n)}(x_0)}{n!}, & \text{if } x_0 = x_1 = \dots = x_n \end{cases}$$

?

where c lies in the smallest interval which contains $x_0, x_1, \dots x_n$

Note:

x :	x_0	x_1
$f(x)$:	$f(x_0)$	$f(x_1)$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= f'(c), c \in (x_0, x_1)$$

MVT

$$f(b) - f(a) = f'(c)(b-a)$$

$$\Rightarrow \frac{f(b) - f(a)}{b-a} = f'(c), \quad c \in (a,b)$$

$$\lim_{x_1 \to x_0} f[x_0, x_1] = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

$$\rightarrow f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots$$

$$= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots$$

many centres one centre

Relation between Newton - Divided difference & taylor series

$$f(x) = f(x_0) + (x - x_0)\frac{f'(x_0)}{1!} + \frac{(x-x_0)^2 f''(x_0)}{2!} + \cdots$$

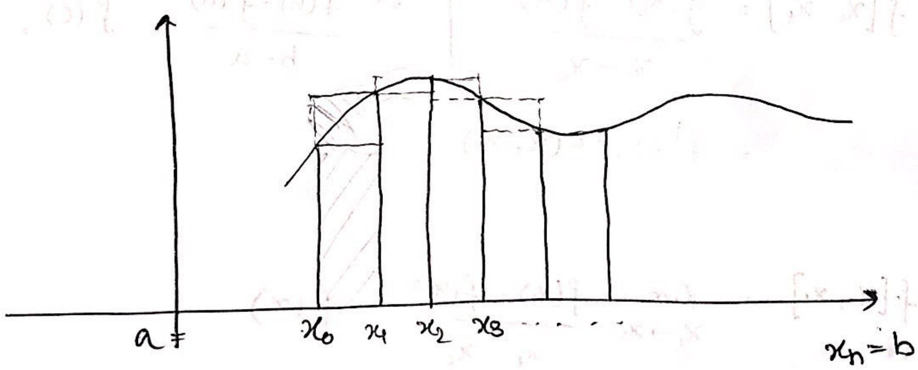
$$|f(x) - P_n(x)| = \frac{1}{(n+1)!} \prod_{i=0}^{n} (x - x_i)\, f^{(n+1)}(\xi x)$$

if all centers are same the truncation error becomes

$$|f(x) - P_n(x)| = \frac{1}{(n+1)!}(x - x_0)^{n+1} f^{(n+1)}(c)$$

where c lies between x & x_0

Numerical integration



$$L \le \int_a^b f(x)\,dx \le U$$

Let $x_{i+1} - x_i = h$; $h = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\,(x_{i+1} - x_i)$$

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$$= \cancel{\lim_{n\to\infty} \frac{(b-a)}{n}}$$

$$= \lim_{n\to\infty} (b-a)\left[\frac{f(x_0)+f(x_1)+\cdots f(x_{n-1})}{n}\right]$$

fundamental thm of calculus.

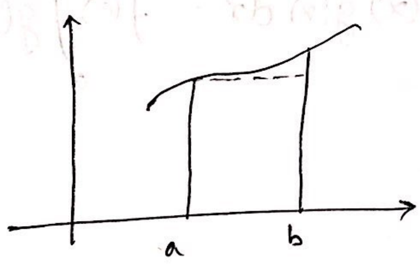
FTC : Let $f:[a,b]\to\mathbb{R}$ be continuous, then $\exists\ F:[a,b]\to\mathbb{R}$ such that $F'(x)=f(x)\ \forall x\in[a,b]$

then $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b)-F(a)$

Ex: $\int_0^1 e^{-x^2}\,dx$

Ex $\int_0^{\pi/2} \frac{dx}{\sqrt{1-a^2\sin^2 x}}$

Rectangle rule.



$x:\ a$

$f(x):\ f(a)$

Area $=(b-a)f(a)$

truncate here.

$$f(x) = f(x_0) + f[x_0,x_1](x-x_0) + \cdots$$

$$p_0(x) = f(x_0)$$

$$\int_a^b f(x)\,dx = \int_a^b f(x_0)\,dx$$

$$= f(x_0)\ x\Big|_a^b = f(x_0)(b-a)$$

$$= f(a)(b-a)$$

$$\int_a^b f(x) = f(a)(b-a)$$

<u>Error:</u> $\int_a^b f(x)\,dx - \int_a^b f(x_0)\,dx = \int_a^b f[x_0,x_1]\,(x-x_0)\,dx$

$$\int_a^b f[a,x]\,(x-a)\,dx$$

$$f[a,c].\int_a^b (x-a)\,dx$$

<u>Mean value theorem for integrals</u>

Let $f:[a,b]\to\mathbb{R}$ be continuous and let

$g:[a,b]\to\mathbb{R}$ be continuous such that either

$g(x)\geq 0$ or $g(x)\leq 0$ $\forall\ x\in[a,b]$

then $\exists$ $c\in[a,b]$, such that

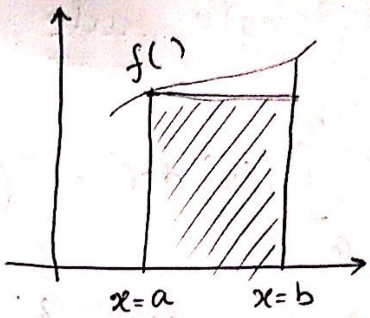
$$\int_a^b f(x)\,g(x)\,dx = f(c).\int_a^b g(x)\,dx$$

$\Leftarrow$ $g(x)$ should be always $\geq$ or ≤ 0 then only we can use

Rectangle rule

$x : a.$

$y : f(a)$



$$f(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots$$

$$f(x) = f(a) + f[a, x](x - a)$$

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$c \in (a, b)$

$$\int_a^b f(x)\,dx = \int_a^b f(a)\,dx + \int_a^b f[a, x](x-a)\,dx$$

$$= (b-a) f(a) + f[a, c] \int_a^b (x-a)\,dx, \quad \text{since } (x-a) \geq 0 \ \forall x \in (a, b)$$

$x^2 - ax$

$\frac{b^2 - a^2}{2} - a(a-b)$

$\frac{b^2 - a^2}{2} - a^2 ab$

$$\boxed{\text{MVT: } \int_a^b f(x) g(x)\,dx = f(c) \int_a^b g(x)\,dx, \quad c \in (a, b)}$$

$$\int_a^b f(x)\,dx = (b-a) f(a) + \frac{(b-a)^2}{2} \times f[a, c] \quad \left| \frac{(x-a)^2}{2} \right|_a^b$$

$$= (b-a) f(a) + \frac{f'(d)}{2} (b-a)^2, \quad d \in (a, c)$$

Note:

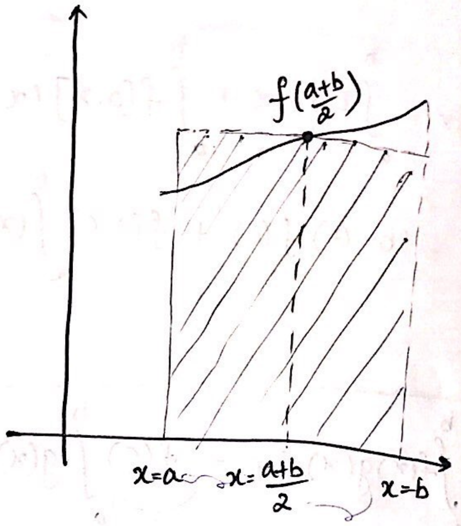
If $f'(x) = 0, \ \forall x \in (a, b)$ then the rectangle rule is exact

Rectangle rule

$$\int_a^b f(x)\,dx \simeq (b-a)\,f(a)$$

$$\text{Error} = \frac{(b-a)^2}{2} f'(d), \quad d \in [a, b)$$

Mid point rule.



$$\text{Area} = (b-a)\,f\left(\frac{a+b}{2}\right)$$

$$\rightarrow f(x) = f(x_0) + f[x_0, x]\,(x - x_0)$$

$$\rightarrow f(x) = f\left(\frac{a+b}{2}\right) + f\left[\frac{a+b}{2}, x\right]\left[x - \left(\frac{a+b}{2}\right)\right]$$

$$\rightarrow \int_a^b f(x)\,dx = \int_a^b f\left(\frac{a+b}{2}\right)dx + \underbrace{\int_a^b f\left[\frac{a+b}{2}, x\right]\left[x - \frac{a+b}{2}\right]dx}_{\text{error term.}}$$

$$= (b-a)\,f\left(\frac{a+b}{2}\right) +$$

$\left[x - \frac{a+b}{2}\right]$ is changing sign

so we can't use

MVT for integrals

In this case,

✓ $x-\left(\frac{a+b}{2}\right) \leq 0 \quad , \forall \; x \in \left[a, \frac{a+b}{2}\right]$

✓ $x-\left(\frac{a+b}{2}\right) \geq 0 \quad , \forall \; x \in \left[\frac{a+b}{2}, b\right]$

Note: $$\int_a^b \left[x-\left(\frac{a+b}{2}\right)\right] dx = \frac{1}{2}\left[x-\left(\frac{a+b}{2}\right)\right]^2 \Bigg|_a^b = 0$$

consider, is independent of x

$$f\left[c, \frac{a+b}{2}, x\right] = \frac{f\left[\frac{a+b}{2}, x\right] - f\left[c, \frac{a+b}{2}\right]}{x-c}$$

$$\Rightarrow f\left[\frac{a+b}{2}, x\right] = f\left[c, \frac{a+b}{2}\right] + f\left[c \; \frac{a+b}{2} \; x\right](x-c)$$

$\forall \; x \in [a, b]$

Error

$$\int_a^b f\left[\frac{a+b}{2}, x\right]\left[x-\frac{a+b}{2}\right] dx$$

$$= \int_a^b f\left[c, \frac{a+b}{2}\right]\left[x-\frac{a+b}{2}\right] dx$$

$$+ \int_a^b f\left[c \; \frac{a+b}{2} \; x\right](x-c)\left[x-\frac{a+b}{2}\right] dx$$

To apply MVT of integral choose $c = \frac{a+b}{2}$.

because it is true for any const c.

$\rightarrow$ If $c = \frac{a+b}{2}$

Error. $= \int_a^b f\left[\frac{a+b}{2} \; \frac{a+b}{2} \; x\right]\left[x - \left(\frac{a+b}{2}\right)\right]^2 dx$

estimated

$$f\left[\frac{a+b}{2} \; \frac{a+b}{2} \; d\right] \int_a^b \left[x - \left(\frac{a+b}{2}\right)\right]^2 dx$$

from before notes

$$= \frac{f^{(2)}(c)}{2!} \times \frac{1}{12}(b-a)^3 \, , \quad c \in (a,b)$$

Note: for what kind of func error will be zero

If $f''(x) = 0$, $\forall x \in (a,b)$

then the mid point rule is exact

Note:

$$f[x_0 \; x_1 \; x] = \begin{cases} \dfrac{f[x_1, x] - f[x_0, x_1]}{x - x_0}, & x \neq x_0 \\[2ex] \dfrac{f''(x_0)}{2!}, & x_0 = x_1 \end{cases}$$

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$$\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = \lim_{x\to x_0} \frac{f''(d_x)\cancel{(x-x_0)^2}}{2\cancel{(x-x_0)^2}}$$

mid point rule is better than rectangular rule

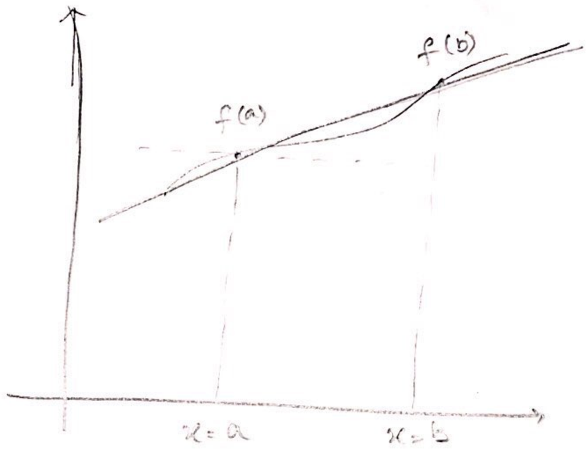
for const polynomials it is good.

verify.

→ Trapezoidal rule.

x : a b

$f(x)$: $f(a)$ $f(b)$



$$\text{Area} = (b-a)f(a) + \left(\frac{b-a}{2}\right)[f(b)-f(a)]$$

$$= \frac{(b-a)}{2}[f(a)+f(b)]$$

$$f(x) = f(a) + f[a,b](x-a) + f[a,b,x](x-a)(x-b)$$

$$\int_a^b f(x)\,dx \simeq \int_a^b f(a)\,dx + \int_a^b f[a,b](x-a)$$

$f[a,x]$

$$= (b-a)f(a) + f[a,b]\left.\frac{(x-a)^2}{2}\right|_a^b$$

$$= (b-a)f(a) + \frac{f(b)-f(a)}{(b-a)}\frac{(b-a)^2}{2}$$ → giving the area of the trapezium.

To find corres error term

$$\text{Error} = \int_a^b f[a, b]\ (x-a)(x-b)\, dx$$

3/4 Trapesoidal rule

$$\int_a^b f(x)\,dx \simeq \frac{(b-a)}{2}\left[f(a)+f(b)\right]$$

$$\text{Error} = \int_a^b f[a,b,x]\,(x-a)(x-b)\,dx$$

$$f(x) = f(a) + f[a,b]\,(x-a) + f[a,b,x]\,(x-a)(x-b)$$

Hint: $g(x) = (x-a)(x-b) \leq 0 \quad \forall\ x \in [a,b]$

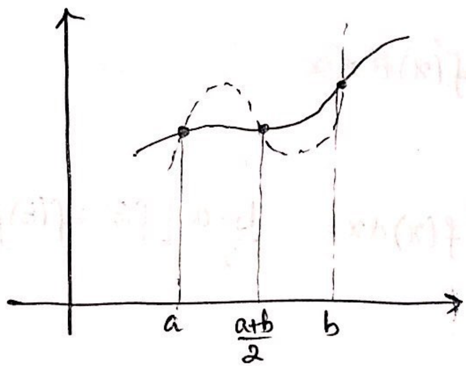
$$\text{Error} = f[a,b,c]\int_a^b (x-a)(x-b)\,dx$$

$$= -\frac{1}{12}(b-a)^3 f''(d) \quad d \in (a,b)$$

Simpson's $1/3^{rd}$ rule.

$x =$ a b $\frac{a+b}{2}$

$f(x) =$ $f(a)$ $f(b)$ $f\left(\frac{a+b}{2}\right)$



$$f(x) = f(a) + f[a,b](x-a) + f\left[a,b,\tfrac{a+b}{2}\right](x-a)(x-b)$$
$$+ f\left[a,b,\tfrac{a+b}{2},x\right](x-a)(x-b)\left[x-\tfrac{a+b}{2}\right]$$

$$\int_a^b f(x)\,dx \simeq \int_a^b f(a)\,dx + \int_a^b f[a,b]\,(x-a)\,dx$$
$$+\int_a^b f\left[a,b,\tfrac{a+b}{2}\right](x-a)(x-b)\,dx$$

$$= \frac{b-a}{2}\left[f(a)+f(b)\right] + f\left[a, b, \frac{a+b}{2}\right] \int_a^b (x-a)(x-b)\, dx$$

$$= \frac{(b-a)}{2}\left[f(a)+f(b)\right] + f\left[a, b, \frac{a+b}{2}\right] \times \frac{-1}{6}(b-a)^3$$

consider

$$f\left[a, b, \frac{a+b}{2}\right] = \frac{f\left[b, \frac{a+b}{2}\right] - f[a, b]}{\frac{a+b}{2} - a}$$

$$= \frac{\dfrac{f\left(\frac{a+b}{2}\right) - f(b)}{b - \frac{a+b}{2}} - \dfrac{(f(b) - f(a))}{b-a}}{\frac{a+b}{2} - a}$$

$$\cancel{\int_a^b f(x)\,dx} =$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right] + \frac{\dfrac{f(b) - f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}} - \dfrac{[f(b) - f(a)]}{\frac{b-a}{2} \times 2}}{\frac{b-a}{2}} \times \frac{-1}{6}(b-a)^3$$

$$= \frac{b-a}{2}\left[f(a)+f(b)\right] + \left[\frac{f(a)}{2} - f\left(\frac{a+b}{2}\right) - \frac{f(b)}{2}\right] \times \frac{-1}{6}(b-a)^3$$

$$= \cancel{\frac{b-a}{6}\left[f(a)\right.}$$

$$\int_a^b f(x)\,dx \simeq \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Error: $$\int_a^b f\left[a, b, \frac{a+b}{2}, x\right] \underbrace{(x-a)(x-b)\left[x - \frac{a+b}{2}\right]}_{g(x)} dx$$

$$g(x) \geq 0, \ \forall \ x \in \left[a, \frac{a+b}{2}\right]$$

$$g(x) \leq 0 \ \ \forall \ x \in \left[\frac{a+b}{2}, b\right]$$

Note: $$\int_a^b (x-a)(x-b)\left[x - \left(\frac{a+b}{2}\right)\right] dx = 0$$

~~H/W~~

exact for cubic

Error: $$-\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(d) \ , \ d \in (a,b)$$

$(b-a)^3$

Simpson's 3/8th rule

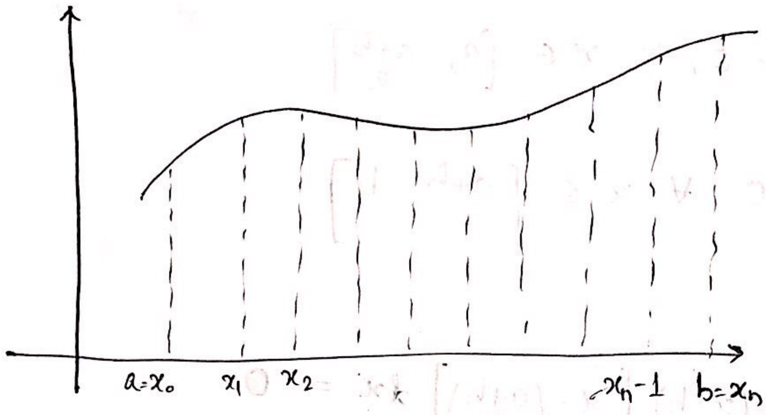
taking 4 points
3 sub intervals
for cubic it is exact

$$\int_a^b f(x)\,dx \simeq \frac{3}{8}(b-a)\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

Error: $$\frac{-3}{80}\left(\frac{b-a}{3}\right)^5 f^{(4)}(d) \ , \ d \in (a,b)$$

Ex : $\int_0^{10} e^x dx$

composite integration methods.



Rectangle rule

$$\int_a^b f(x)dx = (b-a)f(a) + \frac{(b-a)^2}{2} f'(d), \quad d \in (a,b)$$

composite rectangle rule.

$$\int_a^b f(x)dx = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (x_n - x_{n-1}) f(x_{n-1})$$

Let $x_{i+1} - x_i = h$

$$= h \sum_{i=0}^{n-1} f(x_i)$$

$$\text{Error} = \frac{(x_1 - x_0)^2}{2} f'(d_1) + \frac{(x_2 - x_1)^2}{2} f'(d_2)$$

$$+ \cdots + \frac{(x_n - x_{n-1})^2}{2} f'(d_n)$$

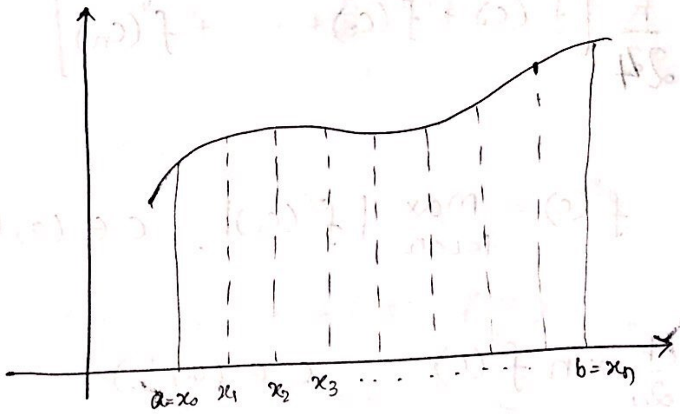
$$= \frac{h^2}{2} \sum_{i=1}^{n} f'(d_i)$$

Define $f'(d) = \max_{1 \le i \le n} |f'(d_i)|$, $d \in (a,b)$

$$\text{Error} \le \frac{h^2}{2} \times n f'(d), \quad d \in (a,b)$$

4|4 composite rectangle rule.

$$h = \frac{b-a}{n}$$



$$\int_a^b f(x)\,dx \simeq h \sum_{i=0}^{n-1} f(x_i)$$

1) $$\text{Error} = \frac{h^2}{2} \sum_{i=1}^{n} f'(d_i)$$

Define $f'(c) = \max_{1 \le i \le n} |f'(d_i)|$

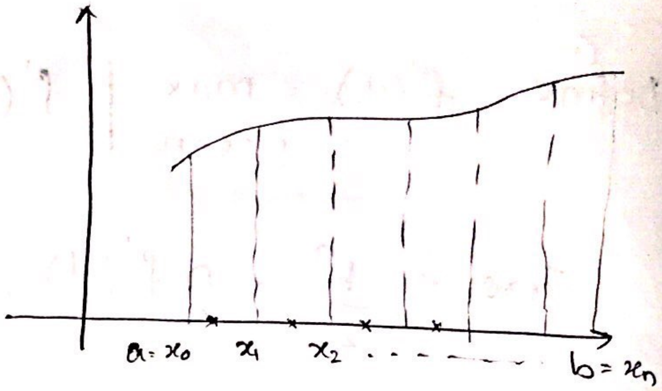
$$\text{Error} \le \frac{h^2}{2} \times n f'(c), \quad c \in (a,b)$$

$$= \frac{h^2}{2} \times \frac{b-a}{h} f'(c)$$

$$\text{Error} = \frac{h}{2}(b-a) f'(c) = O(h) \longrightarrow \text{first order}$$

- composite mid point rule.

$$\int_a^b f(x)\,dx = (b-a)\left(\frac{a+b}{2}\right)$$



$$h = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = h\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)$$

$$\text{Error} = \frac{h^3}{24}\left[f''(c_1) + f''(c_2) + \cdots + f''(c_n)\right]$$

Define $\quad f''(c) = \max_{1\le i\le n} |f''(c_i)|, \quad c \in (a,b)$

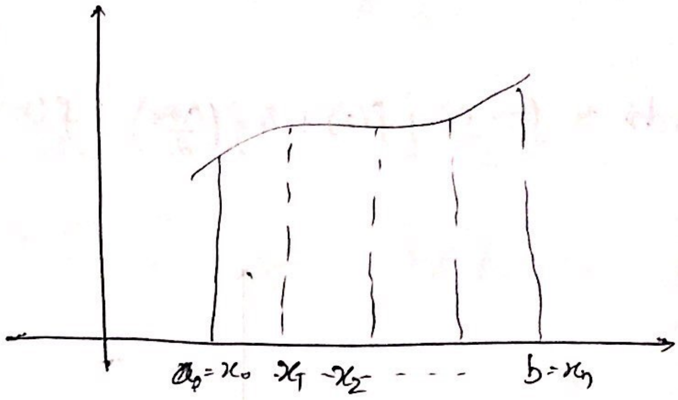
$$\text{Error} \le \frac{h^3}{24} \times n\, f''(c), \quad c \in (a,b).$$

$$= \frac{h^3}{24} \times \frac{b-a}{h} \times f''(c)$$

$$= \frac{h^2}{24} \times (b-a)\, f''(c) \longrightarrow O(h^2) \rightarrow \text{second order}$$

- composite trapezoidal rule.

$$\int_a^b f(x)dx \simeq \frac{b-a}{2}\left[f(a)+f(b)\right]$$



$$h = \frac{b-a}{n}$$

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right] + \frac{h}{2}\left[f(x_1)+f(x_2)\right] + \cdots$$

$$+ \cdots \frac{h}{2}\left[f(x_{n-1})+f(x_n)\right]$$

$$= \frac{h}{2}\left[\sum_{i=0}^{n-1}\left[f(x_i) + f(x_{i+1})\right]\right]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_n)\right] + h\left[f(x_1)+f(x_2)+f(x_3)\cdots f(x_{n-1})\right]$$

$$\text{Error} \leq \frac{h^3}{12} \times n f''(c) \quad , \quad c \in (a,b)$$

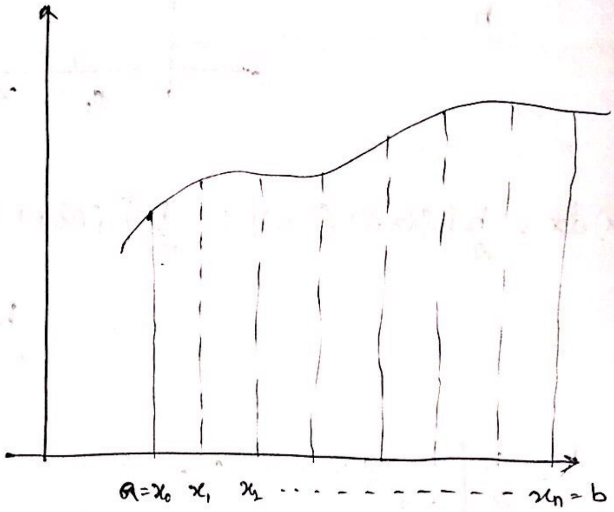
$$= \frac{h^3}{12} \times \frac{b-a}{h} \times f''(c)$$

$$= \frac{h^2}{12} \times (b-a)\, f''(c) \longrightarrow O(h^2) \rightarrow \text{second order.}$$

- composite simpson's $\frac{1}{3}^{rd}$ rule.

we should use these in exam not the basic rules

$$\int_a^b f(x)\,dx \simeq \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$



$$\int_a^b f(x)\,dx \simeq \frac{x_2 - x_0}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$+\left(\frac{x_4 - x_2}{6}\right)\left[f(x_2) + 4f(x_3) + f(x_4)\right] + \cdots$$

$$\cdots + \left(\frac{x_n - x_{n-2}}{6}\right)\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$\simeq \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$\frac{h}{3}\left[f(x_0) + f(x_n) + 4\left(f(x_1) + f(x_3) + \cdots + f(x_{n-1})\right) + 2\left(f(x_2) + f(x_4) + \cdots f(x_{n-2})\right)\right]$$

$$|Error| \le \frac{1}{90}\left(\frac{2h}{2}\right)^5 \times \frac{n}{2} f^{(4)}(c), \quad c \in (a,b).$$

$$= \frac{1}{180} h^5 \times \frac{b-a}{h} f^{(4)}(c), \quad c \in (a,b)$$

$$= \frac{1}{180} h^4 \times b-a \; f^{(4)}(c) \longrightarrow O(h^4) \rightarrow \text{fourth order.}$$

Ex: $\displaystyle\int_0^1 \frac{dx}{1+x}$

Exact value = $\ln(2) = 0.69314718$

take even subintervals is like 2n.

$n = 6$ $\quad h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

$\cancel{x} = \cancel{\frac{1}{6}}$

	x_0	x_1 / x_1	x_2 / x_2	x_3 / x_3			
x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$:	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$

Rectangle rule

$$\int_a^b f(x)dx = h \sum_{i=0}^{n-1} f(x_i)$$

$$= \cancel{\frac{1}{6}} \frac{1}{6} [\quad]$$

$$= 0.73654401$$

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9/4. $\int_0^1 \frac{dx}{1+x}$: Exact value $= \ln(2) = 0.693147$

$n=6$ $\quad h = \frac{b-a}{x} = \frac{1-0}{6} = \frac{1}{6}$

x:	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$:	1	6/7	3/4	2/3	3/5	6/11	1/2

Rectangle rule.

$$\int_0^1 \frac{dx}{1+x} = h \sum_{i=0}^{n-1} f(x_i) = 0.736544$$

$$|\text{Error}| = |\ln(2) - 0.736544| = 0.0434$$

Error bound.

$$|E| \leq \frac{h}{2}(b-a) f'(c), \quad c \in (a,b)$$

$$f(x) = \frac{1}{1+x}$$

$$\Rightarrow f'(x) = \frac{-1}{(1+x)^2}$$

$$\max_{0 \leq x \leq 1} |f'(x)| = 1$$

$$|E| \leq \frac{1}{6} \times \frac{1}{2} \times (1-0) \times 1$$

$$|E| \leq \frac{1}{12} = 0.083$$

Midpoint rule.

$$\int_0^1 \frac{1}{1+x}\,dx = h \sum_{i=0}^{n-1} f\left(\frac{x_{i+1}+x_i}{2}\right)$$

$$h = 1/3$$

$$\int_0^1 \frac{dx}{1+x} \simeq \frac{1}{3}\left[\frac{6}{7}+\frac{2}{3}+\frac{6}{11}\right] = 0.689754$$

$$|Error| = |\ln(2) - 0.689754| = 0.003392$$

$$|E| \le \frac{h^2}{24}(b-a)\,|f''(c)|, \quad c \in (a,b)$$

$$|f''(x)| = \left|\frac{2}{(1+x)^3}\right| \le 2$$

$$|E| \le \frac{\left(\frac{1}{3}\right)^2}{\cancel{24}\ 12} \times (1-0) \times 2 = 0.009259$$

Trapesoidal rule:

$$\int_a^b f(x)\,dx \simeq \frac{h}{2}\left[f(x_0)+f(x_n)+2\left(f(x_1)+f(x_2)+\dots+f(x_{n-1})\right)\right]$$

$$= \frac{(1/6)}{2}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{6}{7}+\frac{3}{4}+\frac{2}{3}+\frac{3}{5}+\frac{6}{11}\right)\right]$$

$$= 0.69476626$$

$$|Error| = |\ln(2) - 0.69476626| = 0.0016190$$

$$|E| \le \frac{h^2}{12}(b-a)\,|f''(c)|, \quad c \in (a,b)$$

$$= \frac{\left(\frac{1}{6}\right)^2}{12} \times (1-0) \times 2 = \underline{\underline{?}}$$

Simpsons $1/3^{rd}$ rule

$$\int_a^b f(x)dx \simeq \frac{h}{3}\Big[f(x_0) + f(x_n) + 4\big(f(x_1)+f(x_3)+f(x_5) + \cdots f(x_{n-1})\big) + 2\big(f(x_2)+f(x_4)+f(x_6)+ \cdots f(x_{n-2})\big)\Big]$$

$$= \frac{\left(\frac{1}{6}\right)}{3}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{6}{7}+\frac{2}{3}+\frac{6}{11}\right) + 2\left(\frac{3}{4}+\frac{3}{5}\right)\right]$$

$$= 0.69316979$$

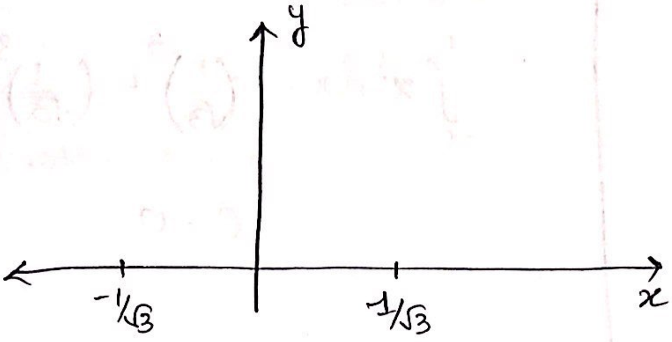
$$|\text{Error}| = |\ln(2) - 0.69316979| = 2.279 \times 10^{-5}$$

$$|E| \leq \frac{1}{180} \times h^4 \times (b-a)\, |f^{(4)}(c)|, \quad c \in (a,b)$$

order 4.

* Gauss - Legendre · quadrature

$$\int_{-1}^{1} f(x)\,dx \simeq A f\left(\frac{1}{\sqrt{3}}\right) + \cancel{B f} \quad B f\left(-\frac{1}{\sqrt{3}}\right)$$



legendre polynomials

$\Downarrow$

$$\left\{ 1,\ t,\ \frac{1}{2}(3t^2-1),\ \ldots \right\}$$

$$3t^2 - 1 = 0.$$

$$t = \pm \frac{1}{\sqrt{3}} \longrightarrow \text{roots}$$

If we keep $t=0$
then we get mid point rule

Make the above rule exact for $\{1, x, x^2, \ldots\}$

$f(x)=1$

$$\int_{-1}^{1} 1 \times dx = A f\left(\frac{1}{\sqrt{3}}\right) + B f\left(\frac{-1}{\sqrt{3}}\right)$$

$$2 = A + B \quad \text{——— } (1)$$

$f(x) \simeq x$

$$\int_{-1}^{1} x\,dx = A \times \frac{1}{\sqrt{3}} + B\left(\frac{-1}{\sqrt{3}}\right)$$

$$0 = A - B \quad \text{——— } (2)$$

$A = 1$
$B = 1$

$$\therefore \int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

Note: $f(x) = x^2$

$$\int_{-1}^{1} x^2\,dx = \frac{1}{3} + \frac{1}{3}$$

$$\frac{2}{3} = \frac{2}{3}$$

Note $f(x) = x^3$

$$\int_{-1}^{1} x^3\,dx = \left(\frac{1}{\sqrt{3}}\right)^3 + \left(\frac{-1}{\sqrt{3}}\right)^3$$

$$0 = 0$$

$$\text{Error} = \frac{1}{135} f^{(4)}(c) \quad , \quad c \in (a, b)$$

Ex: $\displaystyle\int_0^1 \frac{dx}{1+x}$

$$[0, 1] \longrightarrow [-1, 1]$$

$$x = at + b$$

If $x = 0$, $t = -1$	$0 = -a + b$	$a = 1/2$
If $x = 1$, $t = +1$	$1 = +a + b$	$b = 1/2$

$$x = \frac{1}{2}(t+1) \Rightarrow dx = \frac{1}{2}dt$$

$$\int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{\frac{1}{2}dt}{1 + \frac{1}{2}(t+1)} = \int_{-1}^{1} \frac{dt}{t+3}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{3}}\right)+3} + \frac{1}{\left(-\frac{1}{\sqrt{3}}\right)+3}$$

$$= 0.6923076923$$

$$|Error| = 8.39 \times 10^{-4}$$

10/4

Legendre polynomials on $[-1, 1]$

$\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = \frac{1}{3}(3x^2 - 1)$, $\phi_3(x) = \frac{1}{3}(5x^3 - 3x)$

and $$\int_{-1}^{1} \phi_m(x)\,\phi_n(x)\,dx = \begin{cases} \frac{2}{2n+1}, & n = m \\ 0, & n \neq m \end{cases}$$

Recurrence relation

$$(n+1)\,\phi_{n+1}(x) = (2n+1)\,x\,\phi_n(x) - n\,\phi_{n-1}(x) \quad , \; n = 1, 2, 3 \ldots$$

with $\phi_0(x) = 1$, $\phi_1(x) = x$

Gauss - legendre quadrature rule

2-point rule

$\phi_2(x) = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ | $A = B = 1$

$$\int_{-1}^{1} f(x)\,dx = A f\left(\frac{1}{\sqrt{3}}\right) + B f\left(-\frac{1}{\sqrt{3}}\right)$$

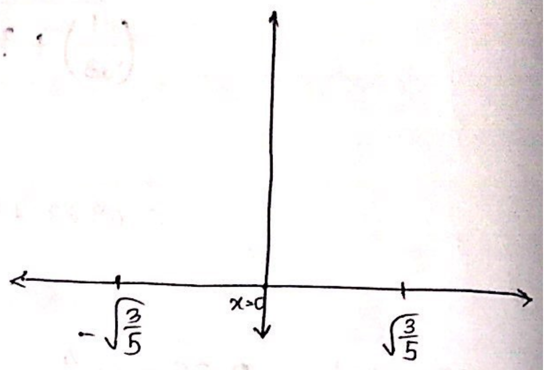
$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) + \frac{1}{135} f^{4}(c) \qquad c \in (-1, 1)$$

Exact for $\{1, x, x^2, x^3\}$

3 point rule.

$$\phi_4(x) = 0 \Rightarrow x = 0, \pm\sqrt{\frac{3}{5}}$$

$$5x^2 - 3 = 0 \Rightarrow$$



$$\int_{-1}^{1} f(x)\,dx \simeq A f\left(-\sqrt{\tfrac{3}{5}}\right) + B f(0) + C f\left(\sqrt{\tfrac{3}{5}}\right)$$

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right) + \frac{1}{15750} f^{(6)}(c)$$

$$c \in (-1, 1)$$

Exact for $\{1, x, x^2, x^3, x^4, x^5\}$

In general the n-point quadratic rule will be exact for f poly of degree $\leq 2n-1$

Ex: $$\int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{dt}{t+3}$$

$$= \frac{5}{9} \frac{1}{\left(-\sqrt{\frac{3}{5}}+3\right)} + \frac{8}{9} \times \frac{1}{3} + \frac{5}{9} \times \frac{1}{\left(-\sqrt{\frac{3}{5}}+3\right)} \quad \text{—} ?$$

$$= 0.6931216931$$

$$|\text{Error}| = |\ln(2) - 0.6931216931| = 2.5 \times 10^{-5}$$

Theorem: Prove that the coefficients in a gaussian quadrature rule are always positive.

$$\int_a^b f(x)\,dx = \int_a^b \sum_{i=0}^{n} l_i(x) f(x_i)\,dx$$

$$= \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)\,dx$$

$$= \sum_{i=0}^{n} w_i f(x_i) \quad , \text{ where } \quad w_i = \int_a^b l_i(x)\,dx.$$

prove that $w_i > 0$, for each i.

composite gaussian quadrature rule.

can go for any type of divisions.

should be better that the other case

$$\int_0^1 \frac{1}{1+x}\,dx = \int_0^{1/2} \frac{dx}{1+x} + \int_{1/2}^{1} \frac{dx}{1+x}$$

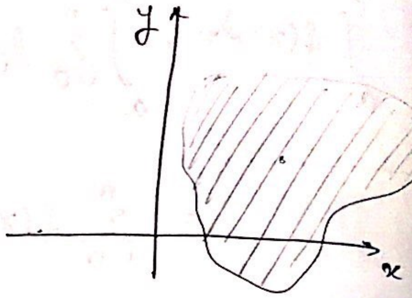
transform each of the sub interval into $(-1,1)$ and solve like before.

* Solving initial value problem

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

at a particular point x_0 y is defined.

and corresponding slope is provided



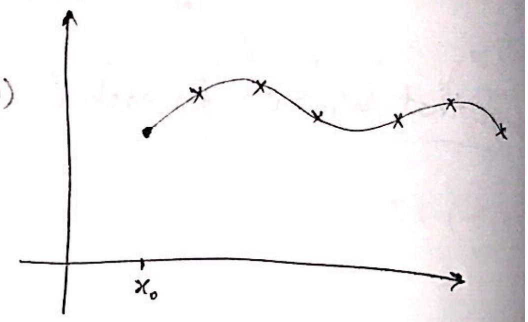
Ex:

$$\frac{d^2y}{dx^2} + \frac{g}{L}\sin y = 0 \quad \text{(SHM)}$$

If y is small

$\sin y = y$

but using n.m we can solve eq directly



Some random b.s

* Initial value problem.

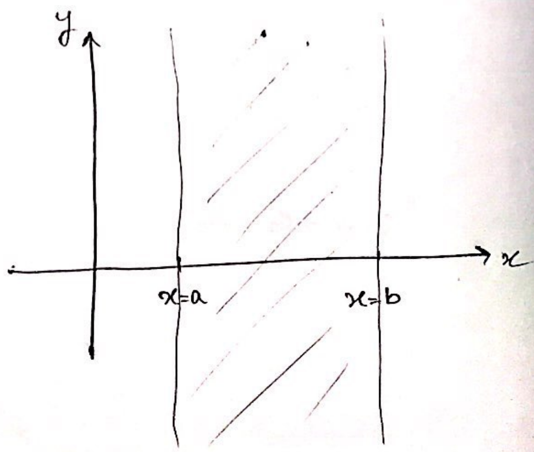
$$\frac{dy}{dx} = f(x, y(x)) \quad , \quad x \in [a,b] \subseteq \mathbb{R} \quad , \quad y \in \mathbb{R}$$

$y(x_0) = y_0 \longrightarrow$ initial condition.

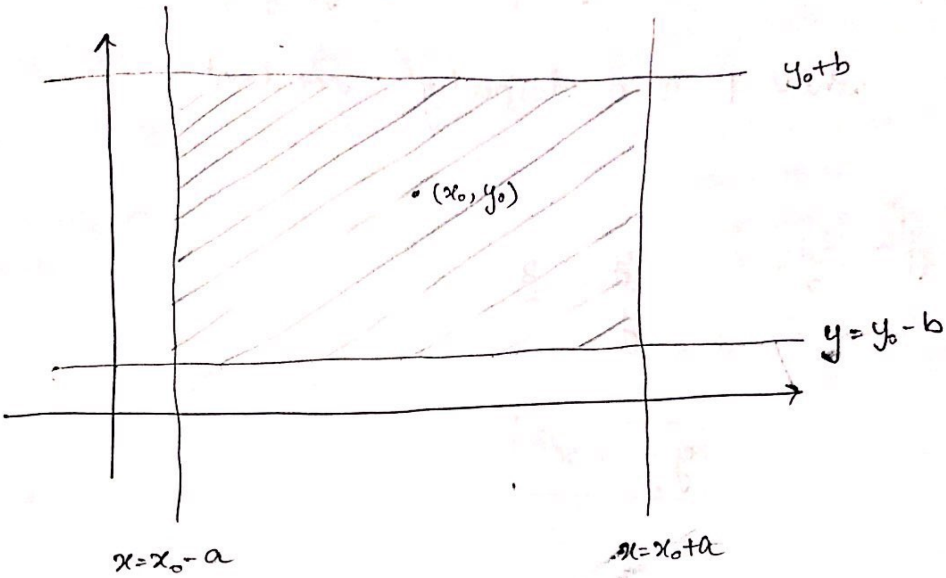
where $y : [a,b] \to \mathbb{R} \to$ unknown function.

- Let D be a rectangle
- and (x_0, y_0) is an interior point

$$D = \{(x,y) : |x - x_0| \le a \, , \; |y - y_0| \le b\}$$



$D = \{(x,y) : |x - x_0| \le a, \ |y - y_0| \le b\}$



Existance and uniqueness theorem

Existance : If $f : D \le \mathbb{R}^2 \to \mathbb{R}$ be continuous of D centered at (x_0, y_0) then the IVP

$$\frac{dy}{dx} = f(x, y(x)), \quad y(x_0) = y_0$$

has unique solution $y(x) \ \forall \ |x - x_0| \le \min\left(a, \frac{b}{m}\right)$

where $m = \max\limits_{(x,y) \in D} |f(x,y)|$

Uniqueness : If $f : D \le \mathbb{R}^2 \to \mathbb{R}$, $\frac{\partial f}{\partial y} : D \le \mathbb{R}^2 \to \mathbb{R}$ are continuous on D, then the IVP ① has a unique solution in $|x - x_0| \le \min\left(a, \frac{b}{m}\right)$

~~sufficient~~

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Sufficient condition.

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|$$

where L is a leiposi ty's constant

Ex: $y' = \frac{3y}{x} \Rightarrow \frac{y'}{y} = \frac{3}{x}$

$$\boxed{y = cx^3}$$

16/4.

Ex: $\frac{dy}{dx} = \frac{3y}{x}$

$\frac{dy}{y} = \frac{3}{x}dx$

$\ln|y| = 3\ln|x| + \ln|C|$

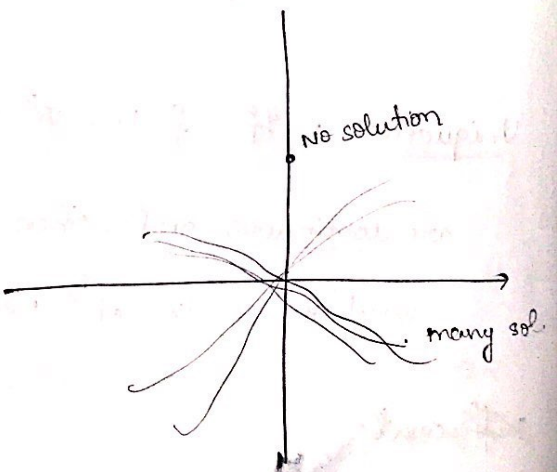
$$\boxed{y = Cx^3, \ x \in \mathbb{R}}$$

case ① $y(0) = 1$

$y = cx^3 \Rightarrow 1 = C \times 0^3$

$1 = 0$

No solution



case ② $y(0) = 0$

$y = cx^3 \Rightarrow 0 = C \times 0$

$0 = 0$

$\Rightarrow C \in \mathbb{R}$

case ③: $y(1) = 1$

$$\left.\begin{array}{l} \\ y = cx^3 \Rightarrow \quad c = 1 \end{array}\right\} \Rightarrow y = x^3 \quad \text{unique solution.}$$

now we have to fix the domain for given problem to give unique solution.

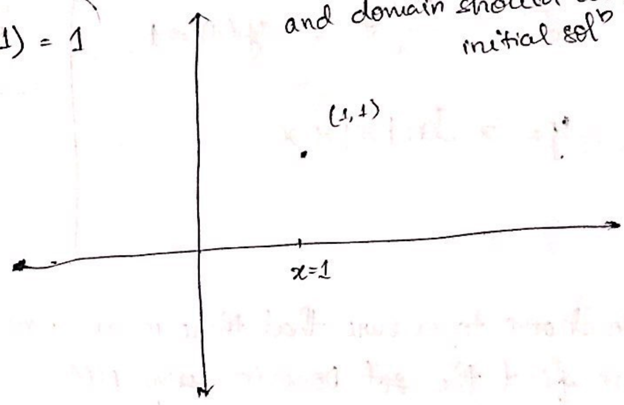
First we have to ensure that there is only one solution in the domain and then find the root using NM.

Redefine the ODE

$$\frac{dy}{dx} = \frac{3y}{x} \; ; \; y(1) = 1$$

and domain should contain initial sol^n

$$f(x, y) = \frac{3y}{x}$$



$$D = \{ (x, y) : 0 < x \leq 2 \; , \; 0 < y \leq 2 \}$$

check lipsitz condition.

$$| f(x, y_1) - f(x, y_2) | \leq L | y_1 - y_2 |$$

$$\left| \frac{3y_1}{x} - \frac{3y_2}{x} \right| \Rightarrow \frac{3}{|x|} | y_1 - y_2 |$$

$$\leq L | y_1 - y_2 |$$

now we have to find the upper bou

at $x = 0$ there is no solution so we change domain of x from $[0, 2] \to [\frac{1}{2}, 2]$

now put $x = 1/2$

$$\Rightarrow \quad L = 6$$

Ex: $\frac{dy}{dx} = 1 + \frac{y}{x} \; , \; y(1) = 1$, find domain without find actual sol^n to have unique sol^n

and domain should contain initial sol^n

$$f(x, y) = 1 + \frac{y}{x}$$

Define: $D = \{ (x, y) : \frac{1}{2} \leq x \leq 2 \; , \; 0 \leq y \leq 2 \}$

$$\left| f(x, y_1) - f(x, y_2) \right| = \left| 1 + \frac{y_1}{x} - \left(1 + \frac{y_2}{x}\right) \right|$$

$$= \frac{1}{|x|} \, |y_1 - y_2|$$

$$\leq \alpha \, |y_1 - y_2|$$

Exact solution :

$$\frac{dy}{dx} = \frac{x + y}{x} \; ; \quad y(1) = 1$$

$$y = vx$$

$$y = x \ln|x| + x$$

we have to ensure that there is an unique solution present before we find the ~~sol~~ Domain using NM

IVP

$$\frac{dy}{dx} = f(x, y(x)) \quad , \quad y(x_0) = y_0 \quad , \quad x \in I = [a, b]$$

* Taylor method

Let $y : [a, b] \to \mathbb{R}$ be differentiable $(n+1)$ times and $(n+1)^{th}$ derivative is continuous

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots\cdots$$

$$+ \frac{(x - x_0)^n}{n!} y^{(n)}(x_0) + \frac{(x - x_0)^{n+1}}{(n+1)!} y^{(n+1)}(c)$$

$$c \in (a, b)$$

$$y'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = f(x_0, y(x_0))$$

$$f \to x, y \to x$$

$$\frac{\partial f}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{dx}}_{f}$$

$$y''(x_0) = \left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left[f(x, y(x))\right]$$

$$\therefore \boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + f(x, y(x))\frac{\partial f}{\partial y}}$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + f\frac{\partial}{\partial y}$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)$$

$$T.E = \frac{(x - x_0)^{n+1}}{(n+1)!}\, y^{(n+1)}(c) \; ; \; c \in (a, b)$$

Truncation error

divide the interval into finite number of sub intervals

Let $h = \frac{b-a}{n}$

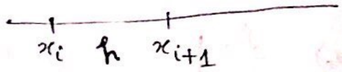
Step ① $x \in [x_0, x_1]$

$$y(x_1) \simeq y(x_0) + (x_1 - x_0)y'(x_0) + \frac{(x_1 - x_0)^2}{2!}y''(x_0)$$

and for the next step these step will be on the initial condition

$$\text{Truncation error} = \frac{(x_1 - x_0)^3}{3!}\, y^{(3)}(c) \;, \; c \in (x_0, x_1)$$

Ex: $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$, $y(1) = 1$, $x \in [1, 2]$



In general,

$h = x_{i+1} - x_i$

$$y(x_{i+1}) = y(x_i) + h y'(x_i) + \frac{h^2}{2!} y''(x_i) + \cdots\cdots + \frac{h^n}{n!} y^{(n)}(x_i) ,$$

LTE = $\dfrac{h^{n+1}}{(n+1)!} y^{n+1}(c_i)$, $c_i \in (x_i, x_{i+1})$

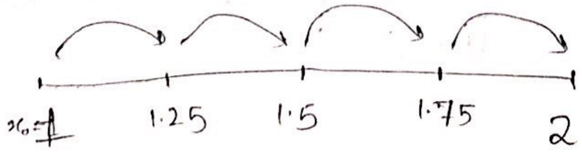
Local truncation error

Ex: $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$, $y(1) = 1$, $x \in [1, 2]$

we can take how many ever subintervals we want let $n = 4$

$h = \dfrac{b-a}{n}$

$= \dfrac{2-1}{4} = \dfrac{1}{4}$

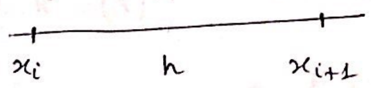


If we go directly to the end the truncation error will dominate.

Second order

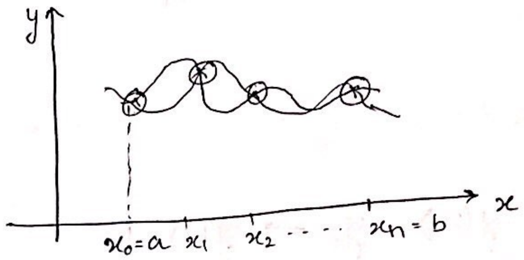
17/4 Taylor's series of order two. 2nd order.

$$\frac{dy}{dx} = f(x,y) \quad , \; y(x_0) = y_0$$



$$x \in [a,b]$$

$$h = \frac{b-a}{n}.$$



$$y(x_{i+1}) \simeq y(x_i) + h y'(x_i) + \frac{h^2}{2!} y''(x_i)$$

$$L.T.E = \frac{h^3}{3!} y'''(c_i) \quad , \quad c_i \in (x_i, x_{i+1})$$

for third order: there will be $\frac{h^3}{3!} y'''(x_i)$ term in $y(x_{i+1})$ & LTE will be the 4^{th} order term.

Total TE,

$$= \frac{h^3}{3!}\left[y'''(c_1) + y'''(c_2) + \cdots + y'''(c_n)\right]$$

$$= \frac{h^3}{3!} \times n \, y^{(3)}(c) \quad , \quad \text{Define } y''(c) = \max_{1 \le i \le n} \left| y''(c_i) \right|$$

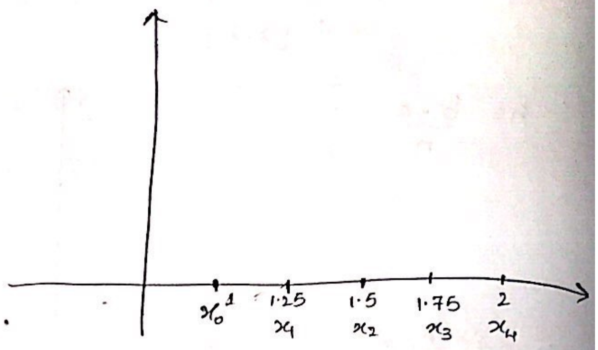
$$= \frac{h^3}{3!} \times \frac{b-a}{h} \times y^{(3)}(c) \, , \quad c \in (a,b)$$

$$\text{Total T.E} \le \frac{h^2}{3!}(b-a)\, y^{(3)}(c) \longrightarrow O(h^2)$$

Ex: $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$, $y(1) = 1$; $x \in [1,2]$

Exact solution $\quad y(x) = x \ln|x| + x$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$



Step ① $\quad y(x_0) = y_0 \rightarrow y(1) = 1.$

$$y(x_1) \simeq y(x_0) + h\, y'(x_0) + \frac{h^2}{2!} y''(x_0)$$

$$y'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = 1 + \frac{y_0}{x_0}$$

$$y''(x_0) = \left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \left.\frac{d}{dx}\left(\frac{dy}{dx}\right)\right|_{x=x_0}$$

$$= \left.\frac{d}{dx}\left(1 + \frac{y}{x}\right)\right|_{x=x_0}$$

$$= -\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x,y)$$

$$= -\frac{y}{x^2} + \frac{1}{x}\left(1 + \frac{y}{x}\right)$$

$f = 1 + \dfrac{y}{x}$

$\dfrac{\partial f}{\partial x} = -\dfrac{y}{x^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$

$$\frac{df}{dx} = \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(f(x,y)\right) = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} = \frac{1}{x}$$

$$y(x_1) \simeq y(x_0) + h \times \left(1 + \frac{y_0}{x_0}\right) + \frac{h^2}{2!} \times \frac{1}{x_0}$$

$$x_1 = x_0 + h = 1 + \frac{1}{4} = 1.25$$

$$y(1.25) \simeq 1 + 0.25 \times \left(1 + \frac{1}{1}\right) + \frac{(0.25)^2}{2} \times \frac{1}{1} = 1.5312.$$

Step② $\quad y(1.25) = 1.5312$

we have to ensure that there is only one solution using uniqueness and other thing.

$$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$$

$$y(x_2) \simeq y(x_1) + h\left(1 + \frac{y_1}{x_1}\right) + \frac{h^2}{2!} \times \frac{1}{x_1} = 2.1126$$

Step③ $\quad y(1.5) = 2.1126$

x	Exact	App.
1	1	1
1.25	1.5289	1.5312
1.5	2.1081	2.1126
1.75		
2		

Draw back of this method is that we have to find the functions derivate.

Euler's method (order 1)

$$y(x_{i+1}) = y(x_i) + h y'(x_i)$$
$$= y(x_i) + h f(x_i, y(x_i))$$

Ex: Same example from above

step ① $\quad y(1) = 1$

$y(1.25) = 1.5$ $\quad$ and we are not taking the $\frac{h^2}{2!} \times y''(x)$ term

for point wise errors is the diff bet^n App. & exact at each point

step② $\quad y(1.5) = 2.05$

* Runge - kutta method of order two

it will coincide with taylor series order 2

consider,

$$y(x_{i+1}) \simeq y(x_i) + h y'(x_i) + \frac{h^2}{2!} y''(x_i)$$

$$= y(x_i) + h f(x_i, y_i) + \frac{h^2}{2!}\left[\cancel{y}\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]_{(x_i, y_i)} + \dots \quad (1)$$

consider

$$u_{i+1} = u_i + c_1 k_1 + c_2 k_2$$

where $k_1 = hf(x_i, u_i)$

$k_2 = h f(x_i + \alpha h, u_i + \beta k_1)$

now we have to determine c_1 c_2 β so that it coincides with taylor series order 2

Aim: Determine c_1 c_2 α β so that it coincides with taylor series method of order 2

$$f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$+ h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + \dots$$

in newtons method we used this formula

something about pascal triangle

$$f(x_i + \alpha h, u_i + \beta k_1)$$

$$= f(x_i, u_i) + (\alpha h)\left.\frac{\partial f}{\partial x}\right|_{(x_i, u_i)} + (\beta k_1)\left.\frac{\partial f}{\partial y}\right|_{(x_i, u_i)}$$

$$+ (\alpha h)^2\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_i, u_i)} + 2(\alpha h)(\beta k_1)\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_i, u_i)}$$

$$+ (\beta k_1)^2\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_i, u_i)} + \dots$$

$$u_{i+1} = u_i + c_1 h f(x_i, u_i) + c_2 h \cancel{\left[f(x_i, u_i) + \alpha h \frac{\partial f}{\partial x} + \beta h f(x_i, u_i) \right]} \cancel{+ 2x} + O[h^3]$$

*

$$u_{i+1} = u_i + c_1 h f(x_i, u_i) + c_2 h \left[f(x\ u) + \alpha h \frac{\partial f}{\partial x} + \beta h f \frac{\partial f}{\partial y} \right]_{(x_i, u_i)} + O(h^3)$$

18/4

$$y(x_{i+1}) = y(x_i) + h y'(x_i) + \frac{h^2}{2!} y''(x_i) + \cdots$$

$$= y(x_i) + h f(x_i, y_i) + \frac{h^2}{2!} \left[\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right]_{(x_i, y_i)} + \cdots \quad \text{①}$$

$$u_{i+1} = u_i + c_1 k_1 + c_2 k_2 \quad \text{— (I)}$$

$$k_1 = h f(x_i, u_i)$$

$$k_2 = h f(x_i + \alpha h, u_i + \beta k_1)$$

c_1 & c_2 are weights of the slopes

$$u_{i+1} = u_i + c_1 h f(x_i, u_i) + c_2 h \left[f(x, u) + \alpha h \frac{\partial f}{\partial x} \cancel{+ \beta h} + \beta h f(x, u) \frac{\partial f}{\partial y} \right] \Bigg|_{(x_i, u_i)} + O(h^3)$$

$$= u_i + h(c_1 + c_2) f(x_i, u_i) + \frac{h^2}{2!} \left[2 c_2 \alpha \frac{\partial f}{\partial x} + 2\beta c_2 f \frac{\partial f}{\partial y} \right]_{(x_i, u_i)} + \cdots$$

②

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From ① & ②

$$c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2$$

$$2c_2\alpha = 1 \Rightarrow \alpha = \frac{1}{2c_2}$$

$$2c_2\beta = 1 \Rightarrow \beta = \frac{1}{2c_2}$$

$\boxed{\text{choose } c_2 = \frac{1}{2}.}$

$$\Rightarrow c_1 = \frac{1}{2}, \quad \alpha = 1, \quad \beta = 1$$

Runge - Kutta method of order 2

$$u_{i+1} = u_i + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = h f(x_i, u_i)$

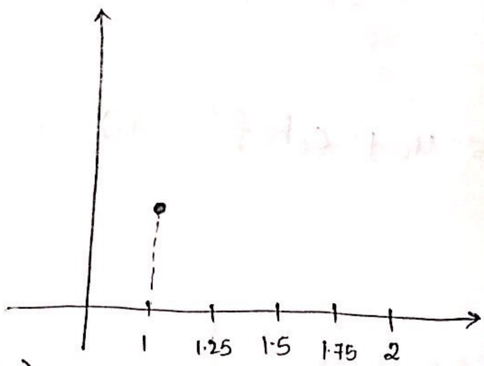
$k_2 = h f(x_i + h, u_i + k_1)$

$\cancel{c_2 = \frac{1}{4}}$
$\cancel{c_1 = 3/4}$

if the slopes are not same (weightage is not same) the the value might be less accurate

Ex: $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y(1) = 1$, $x \in [1, 2]$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$



Step① $y(1) = 1 \Rightarrow x_0 = 1, \; y_0 = 1$

Let $u_0 = y_0$

$$k_1 = h f(x_0, u_0) = h\left(1 + \frac{u_0}{x_0}\right)$$

$$= \frac{1}{2}$$

$$k_2 = h f(x_0 + h, u_0 + k_1)$$

$$= h\left(1 + \frac{u_0 + k_1}{x_0 + h}\right) = \frac{11}{20}$$

$$u_1 = u_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}\left(\frac{1}{2} + \frac{11}{20}\right) = 1.525$$

$c_1 \& c_2$ are slopes weightage
if we give equal value to slope solution will be more accurate
we solved for $c_1 \& c_2$ before
and from (I)

step② . $u(1.25) = 1.525 \Rightarrow$ (or) $u_1 = 1.525$
$x_1 = 1.25$

$$k_1 = hf(x_1, u_1) = 0.5550$$

$$k_2 = hf(x_1 + h, u + k_1) = 0.5966$$

$$u_2 = u_1 + \frac{1}{2}(k_1 + k_2)$$

$$= 1.525 + \frac{1}{2}(0.5550 + 0.5966)$$

$$= 2.1008$$

x	Exact	App.	Error
1	1	1	0
1.25	1.5289	1.525	3.4×10^{-3}
1.5	2.1081	2.1008	7.9×10^{-3}

errors will be increasing
because it is cumulated

* Runge-kutta method of order 4

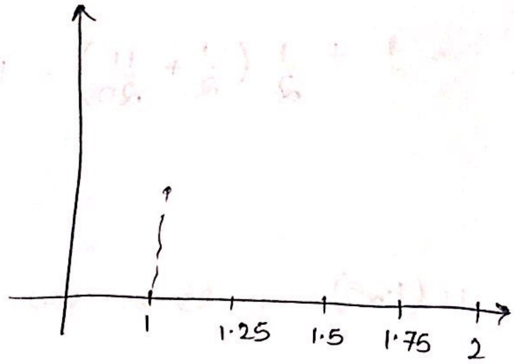
$$u_{i+1} = u_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where
$$k_1 = hf(x_i, u_i)$$
$$k_2 = hf\left(x_i + \frac{h}{2}, u_i + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_i + \frac{h}{2}, u_i + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_i + h, u_i + k_3)$$

* no proof
coz too long
yay

Ex: $\frac{dy}{dx} = 1+\frac{y}{x}$, $y(1)=1$, $x \in [1,2]$

$$h = \frac{b-a}{n} = \frac{1}{4}$$



step ①

$x_0 = 1 \quad u_0 = 1$

$$k_1 = h f(x_0, u_0) = h\left(1+\frac{u_0}{x_0}\right) = 1/2$$

$$k_2 = h f\left(x_0+\frac{h}{2},\ u_0+\frac{k_1}{2}\right) = \cancel{h\left(1+\frac{u_0+\frac{k_1}{2}}{x_0+\frac{h}{2}}\right)}$$

$$= h\left(1+\frac{u_0+\frac{k_1}{2}}{x_0+\frac{h}{2}}\right) = 0.5277$$

$k_3 = 0.5308$

$k_4 = 0.5561$

$$u_1 = u_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1.52885$$

Point wise error at $x = 1.25$ is

$$|y(1.25) - u_1| = |1.5289 - 1.52885| = 5\times10^{-5}$$

— Samapthi —

Difference Equation.

$u_{i+1} + 3u_i = 0 \quad , \quad u_0 = 1$

$u_i = \lambda^i$ $\qquad\qquad \frac{dy}{dx} + 3y = 0$

$u_{i+1} = \lambda^{i+1}$ $\qquad\qquad y = e^{mx}$

$\lambda^{i+1} + 3\lambda^i = 0$ $\qquad\qquad u_i = \lambda^i$

$\qquad\qquad u_{i+1} = \lambda^{i+1}$

$\lambda^i(\lambda+3) = 0 \Rightarrow \lambda = -3$

$u_i = c_1(-3)^i$. they are called eigen values of the function

23/4 Difference equation

$u_{i+1} + 3u_i = 0 \quad , \quad u_0 = 1$ — ①

$u_i = \lambda^i \Rightarrow u_{i+1} = \lambda^{i+1}$

Test solution

① $\Rightarrow \lambda^{i+1} + 3\lambda^i = 0$

$\lambda^i(\lambda+3) = 0 \Rightarrow \boxed{\lambda = -3}$

Sol of ① , $u_i = c_i(-3)^i \Rightarrow u_0 = c_1 \Rightarrow c_1 = 1$

$\boxed{u_i = (-3)^i}$ — ②

Stability : Small changes in the initial data leads small changes in the solution

consider, $u_{i+1} + 3u_i = 0 \quad , \quad u_0 = 1 + \epsilon$

$\tilde{u}_i = (1+\epsilon)(-3)^i$ — ③

Error $e_i = |u_i - \tilde{u}_i|$

eigen values will decide stability
modulus of eigen value should be less than 1

From ② & ③

$$\boxed{e_i = \epsilon |(-3)^i|} \qquad i = 0, 1, 2, \ldots$$

system
Let

Stability condition for difference equation.

$$\boxed{|\lambda_i| < 1 \quad , i = 1, 2, \ldots}$$

Line

Differential equations

$$\frac{dy}{dx} = \lambda y \quad , \quad y(x_0) = y_0 \quad , \lambda \in \mathbb{C} \qquad ①$$

$$y(x) = c_1 e^{\lambda x}$$

$$y(x_0) = y_0 \Rightarrow y(x_0) = c_1 e^{\lambda x_0}$$

$$\Rightarrow y_0 = c_1 e^{\lambda x_0} \Rightarrow c_1 = y_0 e^{-\lambda x_0}$$

$$y(x) = y_0 e^{\lambda(x - x_0)} \qquad \ldots\ldots \quad ②$$

consider $\frac{dy}{dx} = \lambda y \quad , \quad y(x_0) = y_0 + \epsilon$

$$\tilde{y}(x) = (y_0 + \epsilon) e^{\lambda(x - x_0)}$$

Define , $e(x) = \{y(x) - \tilde{y}(x)\}$

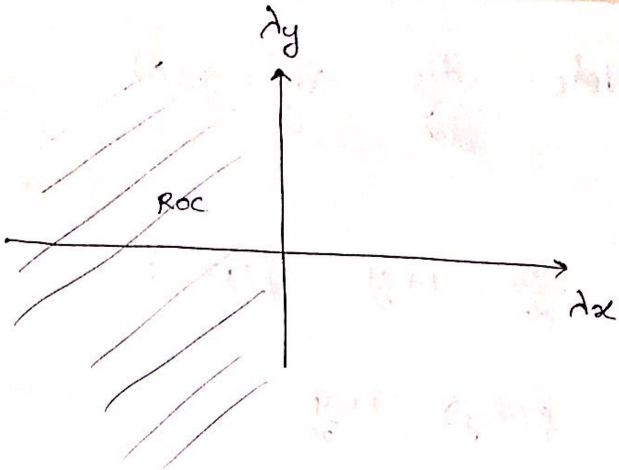
$$|e(x)| = |\epsilon \, e^{\lambda(x - x_0)}|$$

→ what is the condition on λ that this will be under co[ntrol]

Scanned with CamScanner

y system

Let $\lambda = \lambda_x + i\lambda_y$.



Linear stability analysis.

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$

sometimes we may not have no linearised func so we are linearizing it by using taylor series and then finding λ term.

consider,

$$f(x,y) = f(x_0,y_0) + (x-x_0)\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} + (y-y_0)\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} + \cdots$$

$$\Rightarrow \quad \frac{dy}{dx} = f(x_0,y_0) + (x-x_0)\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} + (y-y_0)\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} + \cdots$$

linearized equation

consider $\frac{dy}{dx} = \lambda y$, $y(x_0) = y_0$

λ

it is not going to play key role in stability

consider $\frac{dy}{dx} = \lambda y + g(x)$, $y(x_0) = y_0$

$$\boxed{\lambda = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}}$$

Note: $$\frac{d\tilde{y}}{dx} = \lambda\tilde{y} + g(x)$$

Ex: $$\frac{dy}{dx} = 1 + \frac{y}{x}, \quad y(1) = 1$$

$$f(x,y) = 1 + \frac{y}{x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \left.\frac{1}{x}\right|_{(1,1)} = 1$$

initial data

Test eq $$\frac{dy}{dx} = \lambda y, \quad y(x_0) = y_0$$

$$\frac{dy}{dx} = y, \quad y(1) = 1$$

Ex $$\frac{dy}{dx} = 100y$$

$\lambda = 100$.

then the exponential form will grow drastically so it is unstable

* Stability of Eulers method:

$$\frac{dy}{dx} = \lambda y, \quad y(x_0) = y_0 \quad \text{①}$$

λ will be in the form of $\frac{\partial f}{\partial y}$ at initial data.

$$u_{i+1} = u_i + h f(x_i, u_i), \quad u_0 = A$$

$$= u_i + h\lambda u_i$$

$$\boxed{u_{i+1} = u_i(1 + h\lambda)} \qquad \boxed{u_{i+1} = (1 + \lambda h)u_i} \quad \text{②}$$

Sol of ② $u_i = \mu^i$

$$\mu^{i+1} = (1+\lambda h)\mu^i$$

$$\mu = (1+\lambda h)$$

sol: $\boxed{u_i = c_1(1+\lambda h)^i}$

stability condition.

$\boxed{|(1+\lambda h)| < 1}$ → stability condition for euler method.

Ex: $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y(1) = 1$

$$|1+h| < 1$$

stability cond for RK-2.

RK2
$$\begin{cases} u_{i+1} = u_i + \frac{1}{2}(k_1 + k_2) \\ k_1 = hf(x_i, u_i) \quad = h\lambda u_i \\ k_2 = hf(x_i+h, u_i+k_1) \rightarrow = h\lambda(u_i+k_1) \end{cases}$$

$$= h\lambda(u_i + h\lambda u_i)$$

$$= h\lambda u_i + (h\lambda)^2 u_i$$

$$\Rightarrow u_{i+1} = u_i + \frac{1}{2}(h\lambda u_i + h\lambda u_i + (h\lambda)^2 u_i)$$

$$u_{i+1} = \left(1 + h\lambda + \frac{(h\lambda)^2}{2}\right) u_i$$

stability condition. $\left|1 + h\lambda + \frac{(h\lambda)^2}{2}\right| < 1$

stability cond.
for RK-4

$$\left| 1 + h\lambda + \frac{(h\lambda)^2}{2!} + \frac{(h\lambda)^3}{3!} + \frac{(h\lambda)^4}{4!} \right| < 1$$

we are truncating
and find the numerical
solution