2017

Q) find values of constant a, b, c and the DD of

$f(x,y,z) = axy + byz + cz^2x^3$ at [illegible]

has a max value of 64 in a dir $\parallel$ to z-axis $(0,0,$ [illegible]

4)

DD = $\dfrac{f((1,2,-1)+t(0,0,1)) - f(1,2,-1)}{t}$

16)

DD = $\nabla f(x) \cdot \hat{u} \cos 0$ max value

64 = [illegible]

$(axy, bz, 2cx^3) \cdot (0,0,1)$

64 = $2c x^3$

64 = $(4a+2c, 4a-b, 2b-2c) \cdot (0,0,1)$

classmate

Date

Page

$2b - 2c = 64$

$4a + 3c = 0$

$4a - b = 0$

Find a, b, c

$a = 6$ $\quad b = 24$ $\quad c = -8$

Integration

16) Let $g(1) = 1$

$g'(x^2) = x^3 \quad \forall x > 0$

$g(4) = ?$

Find all points at which $f(x,y) = x^{y^2}$ is continuous

This fn is a composition of,

$x^y \circ y^2$

continuous at all real numbers except (0,0)

continuous at all real numbers.

Cont after midsem.

Recap:

f is diff'ble at $\bar{a}$ if $\exists$ a linear transfo[rmation]

$T_{\bar{a}}$ such that

$$f(\bar{a}+\bar{v}) = f(\bar{a}) + T_{\bar{a}}(\bar{v}) + \|v\| E(\bar{a},\bar{v}) \text{ for } \|\bar{v}\| \ldots$$

where $E(\bar{a},\bar{v}) \to 0$ as $\|\bar{v}\| \to$

Computing total derivative (if f is diff'ble)

$$T_{\bar{a}}(\bar{y}) = f'(\bar{a},\bar{y}) = \nabla f(\bar{a})\,\bar{y}.$$

DD of f at $\bar{a}$ in $\bar{y}$ direction is

$$\nabla f(\bar{a}) \cdot \frac{\bar{y}}{\|\bar{y}\|}$$

~~Sup~~

Sufficient Condition for Differentiability

If $z = f(x,y)$ has continuous first order partial derivatives $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$ at $\bar{a}$, then f is diff at $\bar{a}$.

The partial derivatives must exist in some ball $B(\bar{a})$

Question Show that $f(x,y) = \begin{cases} \frac{x^2-y^2}{x-y} & (x,y) \neq (1,-1) \\ 0 & (x,y) = (1,-1) \end{cases}$

is diff at $(1,-1)$

A)

$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} \Rightarrow \lim_{h\to 0} \frac{f((1,-1)+h(1,0)) - f(1,-1)}{h}$

$\Rightarrow \lim_{h\to 0} \frac{f(1+h,0) - f(1,-1)}{h}$

$\Rightarrow$ $\lim_{h\to 0} \frac{\frac{(1+h)^2}{h}}{h}$

$\Rightarrow 1$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,-1)} = \lim_{h\to 0} \frac{f((1,-1)+h(0,1)) - f(1,-1)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(0,-1) - f(1,-1)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(1+h,-1) - f(1,-1)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(1,0) - f(1,-1)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{\frac{1}{h}}$$

$$\Rightarrow \lim_{h\to 0} h$$

$$\Rightarrow 0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(1,-1+h) - f(1,-1)}{h}$$

$\Rightarrow 1$ (It is diff at $(1,-1)$)

If $\lim_{(x,y)\to(1,-1)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(1,-1)$ and

$\lim_{(x,y)\to(1,-1)} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(1,-1$ } Continuity — Find as an exercise

Question) Find an approx value of $f(2.1, 3.2)$ where,

$f(x,y) = x^y \qquad \log 2 = 0.3010$

1) If f is diff at $\bar{a}$, then:

$f(x+h, y+k) = f(x,y) + T_{\bar{a}}(\bar{v})$ as tends to 0, where

$f((x,y)+(h,k)) \Rightarrow f(\bar{a}+\bar{v})$

In this case,

$$T_{\bar{a}}(\bar{v}) = \nabla f(\bar{a}) \cdot \bar{v}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (h,k)$$

If f is totally diff then,

$$f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x}(x,y) + k\frac{\partial f}{\partial y}(x,y)$$

It is given, $f(x,y) = x^y$

$$\left.\frac{\partial f}{\partial x}\right| = y x^{y-1} \qquad \frac{\partial f}{\partial y} = x^y \log_n x$$

(partial deriv...)

$x = 2$
$h = 0.1$

$y = 3$
$k = 0.2$

For continuity,

$$\lim_{(x,y)\to(2,3)} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(2,3)$$

$$\lim_{(x,y)\to(2,3)} y^{x y-1} = \frac{\partial f}{\partial x}(2,3) \checkmark \quad \text{(because polynomial)}$$

Now,

$$\lim_{(x,y)\to(2,3)} \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(2,3) \checkmark \quad \text{(because logarithmic)}$$

$\therefore$ It is continuous

Since, partial derivatives exist and the fn is cont, it is differentiable,

Since, its differentiable,

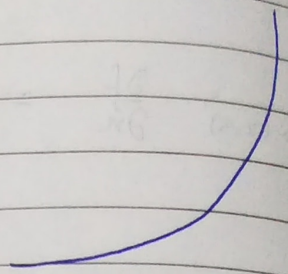
$$f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x}(x,y) + k\frac{\partial f}{\partial y}(x,y)$$

$$f(2+0.1, 3+0.2) = f(2,3) + 0.1\frac{\partial f}{\partial x}(2,3) + 0.2\frac{\partial f}{\partial y}(2,3)$$

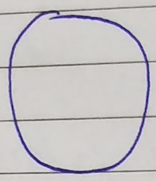
$$f(2+0.1, 3+0.2) \approx 10.3$$

Change of a Scalar Field along a Curve

$f(\bar{x})$ $\quad x \in$ Curve



Let consider a curve circle



$f(x,y)$

$C: x^2+y^2=1$ (Conv to one variable)

$x(t) = \cos t$ $\quad 0 \le t \le 2\pi$

$y(t) = \sin t$

$C: (x(t), y(t)) = \gamma(t)$

$f(x,y) = f(\gamma(t))$

Change in f along C, is $\dfrac{d}{dt} f(\gamma(t))$

$\Rightarrow \nabla f(\bar{\gamma}(t)) \cdot \bar{\gamma}'(t)$

Example 4) Find DD of Scalar field $f(x,y) = e^{xy} \cdot \cos(xy^2)$ along circle $x^2+y^2=1$ at $(0,1)$

Ans)

$\gamma(t) = (\cos t, \sin t)$ $\quad 0 \leq t \leq 2\pi$

$$\boxed{\text{DD of } f \text{ along } C \text{ is } \quad \nabla f(\bar{\gamma}(t)) \cdot \frac{\gamma'(t)}{\|\gamma'(t)\|}}$$ → unit vector stuff

We need at point $(0,1)$ i.e at $t = \pi/2$

So, at $t = \pi/2$ we need to compute,

$\nabla f(\bar{\gamma}(t)) \Rightarrow$

$\frac{\partial f}{\partial x}(\cos t, \sin t) \Rightarrow$

or

$\frac{\partial f}{\partial x}(0,1) \Rightarrow 1$ $\qquad \frac{\partial f}{\partial y}(0,1) = 0$

$\Rightarrow (1,0) + (-1,0)$

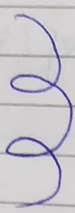
$\Rightarrow (0,0)$

Ques) Suppose partial derivatives of $f(x, y, z)$ at points on helix $x = \cos t$ $\quad y = \sin t$ $\quad z = t$ are

$f_x = \frac{\partial f}{\partial x} = \cos t \qquad f_y = \frac{\partial f}{\partial y} = \sin t \qquad f_z = \frac{\partial f}{\partial z} = t^2 + t - 2$

At what points on the helix can f take extreme values

A)



$\bar{r}(t)$ $\qquad C: (x(t), y(t), z(t))$
$\qquad\qquad (\cos t, \sin t, t)$

$g(t) = f(r(t))$

$g'(t) = 0$ (for finding critical points)

$\nabla f(\bar{r}(t)) . r'(t) = 0$

So, we need to find,

$$\left.\begin{aligned}
\frac{\partial f}{\partial x}(x(t), y(t), z(t)) &= \cos t \\
\frac{\partial f}{\partial y}(x(t), y(t), z(t)) &= \sin t \\
\frac{\partial f}{\partial z}(x(t), y(t), z(t)) &= t^2 + t - 2
\end{aligned}\right\} \text{given}$$

So,

$\nabla f(\bar{r}(t)) \cdot \bar{r}'(t)$

$(\cos t, \sin t, t^2+t-2) \cdot (-\sin t, \cos t, 1) = 0$

$g'(t) = 0, \Rightarrow t^2+t-2 = 0$

$t = 1, -2$ (critical point)

At $t = 1$

$(\cos 1, \sin 1, 1)$

At $t = 2$

$(\cos 2, -\sin 2, 2)$

$g''(t) = 2t + 1$

$g''(1) > 0$ $\qquad g''(-2) < 0$

~~So, at t = 1 and~~

Theorem

If, $f(x, y) = k$

as in $f(x, y) \Rightarrow x^2 + y^2 = 1$
or $f(\bar{\gamma}(t)) = k$

If f is const along a curve C, then ∇f is normal to the curve, because:

$$C = \bar{\gamma}(t)$$

$$f(\bar{\gamma}(t)) = k$$

$$\nabla f(\bar{\gamma}(t)) \cdot \bar{\gamma}'(t) = 0$$ (Dot product is 0)

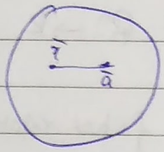
So, gradient of f is normal to tangent of curve
derivative of a curve is tangent

$e^{x-y}.(1-y) + e^{x-y}(x-1)$

$\frac{d(e^{x-y})}{dx} \Rightarrow e^{x-y}.(1-y)$

Tangent Plane

$f(x,y,z) = k$



$\underbrace{\nabla f(\bar{a}).\ (\bar{x}-\bar{a}) = 0}$

Eqn of tangent plane at $\bar{a}$

Ques) Find a pair of linear cartesian eqns for a line which is tangent to both surfaces $x^2+y^2+2z^2=4$ and $z=e^{x-y}$ at $(1,1,1)$

A)

For first surface, $f(x,y,z) = x^2+y^2+2z^2=4$

$\nabla f(1,1,1).((x,y,z)-(1,1,1)) = 0$

$(2,2,4).(x-1, y-1, z-1) = 0$

$x+y+2z = 4 \longrightarrow ①$

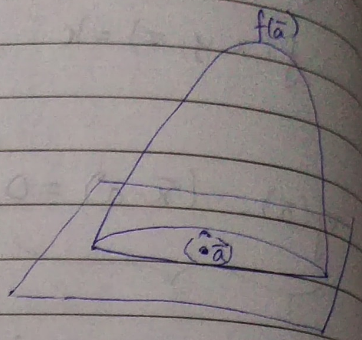
For second surface, $g(x,y,z) = e^{x-y}$ $\qquad \nabla g = (e^{x-y}, -e^{x-y}, -1$

$\nabla g(1,1,1)((x,y,z)-(1,1,1)) = 0$

$\Rightarrow x-y-z = -1 \longrightarrow ②$

These are the two lines

Application of Multi-Variable Calculus



$f: R^n \to R$

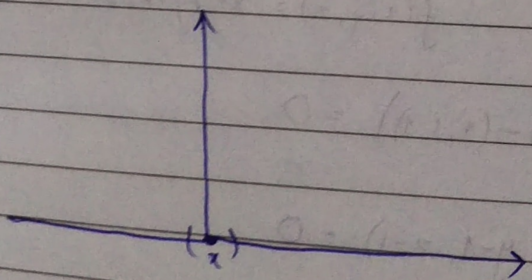
If f has relative max at $\bar{a}$ if

$f(\bar{x}) < f(\bar{a}) \quad \forall \bar{x} \in N(\bar{a})$

Saddle-point

It is a point where

$f(x) < f(0)$
$f(x) > f(0)$ in some neighbourhood of x



~~At~~ some ~~point~~ neighbourhood of ~~x~~, some values are below $f(0)$ and some above $f(0)$. This is 2D, we will deal in

Extrema of function of two variables

$z = f(x, y)$ & $F(x, y, z) = 0$

i) Assuming f is diff'ble, find ~~f~~

$\nabla f = 0$ like first derivative

This gives critical points.

ii) Find, ~~H~~

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

like second derivative

$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$

iii) Find determinant of H at each critical point

det

if $\Delta > 0$ & $f_{xx} > 0$ at a critical point $\bar{a}$ then f has rel min at $\bar{a}$

if $\Delta > 0$ & $f_{xx} < 0$ at $\bar{a}$ then f has rel max at $\bar{a}$.

if $\Delta < 0$ at $\bar{a}$ then f is a saddle point

if $\Delta = 0$ at $\bar{a}$ then f is inconclusive

Ques) Identify critical points of surface $f(x,y) = x^3 + y^3 \ldots$

A)

$\cancel{\nabla f = 3x^2 + 3y^3}$

$$\nabla f = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y$$

$$3x^2 - 3y = 0 \qquad \frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3x = 0 \qquad \frac{\partial f}{\partial y} = 0$$

So, $x = (0,0)\ (1,1) \Rightarrow$ these are critical points

So, by HHH f has extremum at $(1/2, 1/2)$

We don't know if it has rel max or rel min.

So, take points in neighborhood of $(1/2, 1/2)$

$$f(x,y) = xy$$

Let $f\left(1/2 + \varepsilon, \frac{1}{2} - \varepsilon\right)$

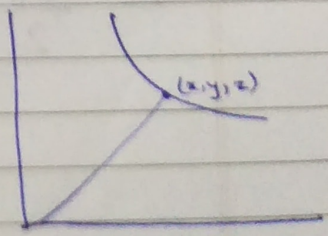
$= -ve$

The value is lesser than $f(1/2, 1/2)$

So, $f(1/2, 1/2)$ is rel max.

Q.) Find points on the surface $z^2 - xy - 1 = 0$ nearest to origin.

A)



We need to minimise the distance

Min, $f(x,y,z) = x^2 + y^2 + z^2$ — distance formula no need to take root since it will anyways maximise the root

$\rightarrow$ (1)

Subject to $g(x,y,z) = z^2 - xy - 1 = 0$

$\nabla f = \lambda \nabla g$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$

$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ $\qquad$ $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$

$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ $\qquad$ $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$ $\qquad$ $\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}$

$2x = \lambda(-y)$ $\qquad$ $2y = \lambda(-x)$ $\qquad$ $2z = \lambda(2z)$

$x = \frac{-\lambda y}{2}$ (1) $\qquad$ $y = \frac{-\lambda x}{2}$ (2) $\qquad$ z is arbitrary

$z = 0$ $\qquad$ $z \neq 0$

$\lambda = R$ $\qquad$ $\lambda = 1$

Taking $z \neq 0$

$\lambda = 1$

$x = -y/2$ $\qquad$ $y = -x/2$

$(x, y) = (0,0) \Rightarrow z = \pm 1$

$(0, 0, \pm 1)$

Taking $z = 0$

$\lambda = \mathbb{R}$

So, $g(x, y, 0) = 0 - xy - 1 = 0$

$\Rightarrow xy = -1$

Doing ① × ②

$$xy = \frac{\lambda^2 xy}{4}$$

$\lambda^2 = 4$

$\lambda = \pm 2$

So,

$x = \pm 1$ $\qquad$ $y = \mp 1$

So, points are $(1,-1,0)$ $(-1,1,0)$ $(0,0,1)$ $(0,0,-1)$