Assignment 3 Solution

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October 23, 2017

1

Given that

$$f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}$$

whenever $x^2y^2 + (x-y)^2 \neq 0$. Now, applying the first limit, we get

$$\lim_{y \to 0} f = \frac{x^2y^2}{x^2y^2 + (x-y)^2} = \frac{0}{0 + (x)^2} = 0$$

Now, applying the outer limit we get

$$\lim_{x \to 0} \lim_{y \to 0} f = 0$$

Similarly, we get

$$\lim_{x \to 0} f = \frac{x^2y^2}{x^2y^2 + (x-y)^2} = \frac{0}{0 + (y)^2} = 0$$

And applying the outer limit we get

$$\lim_{y \to 0} \lim_{x \to 0} f = 0$$

Hence,

$$\lim_{y \to 0} \lim_{x \to 0} f = \lim_{x \to 0} \lim_{y \to 0} f$$

Now, $\lim_{(x,y) \to (0,0)} f(x,y)$ along the $y = x$ line, we get

$$\lim_{(x,y) \to (0,0)} f = \frac{x^4}{x^4} = 1$$

Similarly, $\lim_{(x,y) \to (0,0)} f(x,y)$ along the $y = 2x$ line, we get

$$\lim_{(x,y) \to (0,0)} f = \frac{4x^4}{4x^4 + x^2} = \frac{4x^2}{4x^2 + 1} = 0$$

Hence, limit does not exits at $(x,y) \to (0,0)$.

2

See appended section.

3

Given that

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

We need to find the limit along $y = mx$ as $(x, y) \to (0, 0)$

Replacing the value of y with mx in the given function, we get

$$f = \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$

Solving the limit, we get

$$\lim_{x \to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2}$$

In order to define a $f(x, y)$ so as to make it continuous at $(0, 0)$,

$$\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

should be equal along every path $y = mx + c$ taken. Since this is not true in the given function, we cannot define $f(x, y)$ so as to make it continuous at $(0, 0)$.

4

Given a scalar field

$$f(\mathbf{x}) = ||\mathbf{x}||^4$$

Now, assume that

$$g(t) = f(\mathbf{x} + t\mathbf{y})$$

Since

$$f(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{x}) \times (\mathbf{x} \cdot \mathbf{x})$$

We get

$$g(t) = (\mathbf{x} + t\mathbf{y}) \cdot (\mathbf{x} + t\mathbf{y}) \times (\mathbf{x} + t\mathbf{y}) \cdot (\mathbf{x} + t\mathbf{y})$$

$$g(t) = (\mathbf{x} \cdot \mathbf{x} + 2t\mathbf{x} \cdot \mathbf{y} + t^2\mathbf{y} \cdot \mathbf{y}) \times (\mathbf{x} \cdot \mathbf{x} + 2t\mathbf{x} \cdot \mathbf{y} + t^2\mathbf{y} \cdot \mathbf{y})$$

$$g\prime(0) = f\prime(\mathbf{x}; \mathbf{y}) = 4||\mathbf{x}||^2(\mathbf{x} \cdot \mathbf{y})$$

5

5.1

Given that

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

Then the first order partial derivative can be calculated as

$$\frac{\partial f}{\partial x} = \frac{\partial \frac{x}{\sqrt{x^2+y^2}}}{\partial x}$$

Using division rule, we get

$$\frac{\partial f}{\partial x} = \frac{\frac{\partial}{\partial x}(x)\sqrt{x^2+y^2} - \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right)x}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$\frac{\partial f}{\partial x} = \frac{1 \cdot \sqrt{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}}x}{\left(\sqrt{x^2+y^2}\right)^2} = \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

The partial derivative with respect to y is

$$\frac{\partial f}{\partial y} = \frac{\partial \frac{x}{\sqrt{x^2+y^2}}}{\partial y}$$

$$\frac{\partial f}{\partial y} = x\frac{\partial}{\partial y}\left((x^2+y^2)^{-\frac{1}{2}}\right)$$

And using chain rule, replacing $u = x^2 + y^2$, we get

$$\frac{\partial f}{\partial y} = x\frac{\partial}{\partial u}\left(u^{-\frac{1}{2}}\right)\frac{\partial}{\partial y}\left(x^2+y^2\right)$$

$$\frac{\partial f}{\partial y} = x\left(-\frac{1}{2u^{\frac{3}{2}}}\right) \cdot 2y$$

Replacing the back the value of u, we get

$$\frac{\partial f}{\partial y} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}$$

5.2

Given

$$f(x) = \vec{a}.\vec{x}$$

$\vec{a}$ being fixed, f is defined on R^n and $\vec{a} = a_1 i + a_2 j + ...$
Then, the partial derivative in the x direction is

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f((x, y, z...) + h(1, 0, 0, 0...)) - f(x, y, z...)}{h}$$

$$= \lim_{h \to 0} \frac{\vec{a} \cdot (x + h, y, z..) - \vec{a} \cdot (x, y, z...)}{h}$$

$$= a_1$$

Then, the partial derivative in the y direction is

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f((x, y, z...) + h(0, 1, 0, 0...)) - f(x, y, z...)}{h}$$

$$= \lim_{h \to 0} \frac{\vec{a} \cdot (x, y + h, z..) - \vec{a} \cdot (x, y, z...)}{h}$$

$$= a_2$$

And so on.

6

Given the function

$$f(x, y) = \frac{1}{y} \cos x^2$$

We have

$$D_2 f = \frac{\partial \frac{1}{y} \cos x^2}{\partial y} = \frac{-1}{y^2} \cos x^2$$

$$D_1 f = \frac{\partial \frac{1}{y} \cos x^2}{\partial x} = \frac{-2x}{y} \sin x^2$$

Then, the mixed partial derivatives $D_1(D_2 f)$ and $D_2(D_1 f)$ are given by

$$D_1(D_2 f) = \frac{2x}{y^2} \sin x^2$$

$$D_2(D_1 f) = \frac{2x}{y^2} \sin x^2$$

Since $D_1(D_2 f) = D_2(D_1 f)$ for all values of (x, y)
Hence, Proved

7

Given the scalar field

$$f(x,y,z) = x^2 + 2y^2 + 3z^2$$

and the unit vector $\frac{i-j+2k}{\sqrt{6}}$

The directional derivative at $(1,1,0)$ in the direction of $\vec{v} = i - j + 2k$ equals to

$$DD = f((1,1,0);(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}})) = \lim_{h\to 0} \frac{f((1,1,0) + h(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}})) - f(1,1,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(h+\sqrt{6})^2}{6} + \frac{(-h+\sqrt{6})^2}{6} + \frac{12h^2}{6} - 3}{h}$$

Applying L' Hopitals rule, we get

$$\lim_{h\to 0} \frac{15h^2 - 2\sqrt{6}h}{6h} = \frac{-2}{\sqrt{6}}$$

8

Given the scalar field

$$f(x,y,z) = axy^2 + byz + cz^2x^3$$

has a maximum value of 64 in a direction parallel to the z -axis.

The directional derivative at $(1,2,1)$ in the direction parallel to the z axis being $\vec{v} = k$ equals to

$$\Big(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\Big) \cdot (k)$$

Replacing values, we get

$$\Big((ay^2 + 3cz^2x2)i + (2axy + bz)j + (by + 2czx^3)k\Big) \cdot \big(k\big)$$

where $(x,y,z) \to (0,0,1)$, we get

$$\Big((4a + 3c)i + (4a - b)j + (2b - 2c)k\Big) \cdot \big(k\big) = (\frac{2b - 2c}{\sqrt{1}}) = 64$$

Hence, we get

$$b - c = 32$$

If maximum occurs along a direction, then the minimum occurs along a direction perpendicular to it.

$$4a + 3c = 0$$

$$4a - b = 0$$

Solving these equations, we get

$$a = 6, b = 24, c = -8$$

9

Given

$$\mathbf{r}(x, y, z) = xi + yj + zk$$

and

$$r(x, y, z) = ||\mathbf{r}(x, y, z)|| = \sqrt{x^2 + y^2 + z^2}$$

Also

$$r^n = \sqrt[n/2]{x^2 + y^2 + z^2}$$

Computing for some value n, a positive integer, the gradient of r^n

$$\nabla(r^n) = \Big(\frac{\partial r^n}{\partial x} i + \frac{\partial r^n}{\partial y} j + \frac{\partial r^n}{\partial z} k\Big)$$

$$\nabla(r^n) = \Big(\frac{2nx \sqrt[(n-2)/2]{x^2 + y^2 + z^2}}{2} i + \frac{2ny \sqrt[(n-2)/2]{x^2 + y^2 + z^2}}{2} j + \frac{2nz \sqrt[(n-2)/2]{x^2 + y^2 + z^2}}{2} k\Big)$$

$$\nabla(r^n) = \Big(nx \sqrt[(n-2)/2]{x^2 + y^2 + z^2} i + ny \sqrt[(n-2)/2]{x^2 + y^2 + z^2} j + nz \sqrt[(n-2)/2]{x^2 + y^2 + z^2} k\Big)$$

$$\nabla(r^n) = \Big(n \sqrt[(n-2)/2]{x^2 + y^2 + z^2}\Big) \times \Big(xi + yj + zk\Big)$$

$$\nabla(r^n) = \Big(nr^{n-2}\Big) \times \Big(xi + yj + zk\Big)$$

$$\nabla(r^n) = nr^{n-2}\mathbf{r}$$

10

Given a function $u = f(x, y)$, $x = X(t)$, $y = Y(t)$ define u as a function of t, say $u = F(t)$

10.1

Given

$$f(x, y) = x^2 + y^2, X(t) = t, Y(t) = t^2$$

Replacing the given values of (x, y) as $(X(t), Y(t))$, we get

$$u(t) = t^2 + t^4$$

$F\prime(t)$ can be calculated as

$$F\prime(t) = \frac{du(t)}{dt}$$

$$F\prime(t) = \frac{d(t^2 + t^4)}{dt}$$

$$F\prime(t) = 2t + 4t^3$$

$F\prime\prime(t)$ can be calculated as

$$F\prime\prime(t) = \frac{dF\prime(t)}{dt}$$

$$F\prime\prime(t) = \frac{d(2t + 4t^3)}{dt}$$

$$F\prime\prime(t) = 2 + 12t^2$$

10.2

Given

$$f(x, y) = e^{xy} \cos(xy^2), X(t) = \cos(t), Y(t) = \sin(t)$$

Replacing the given values of (x, y) as $(X(t), Y(t))$, we get

$$u(t) = e^{\cos(t)\sin(t)} \cos(\cos(t)(\sin(t))^2)$$

$F\prime(t)$ can be calculated as

$$F\prime(t) = \frac{du(t)}{dt}$$

$$F\prime(t) = \frac{d(e^{\cos(t)\sin(t)} \cos(\cos(t)(\sin(t))^2))}{dt}$$

Have to add.

11

See appended section

12

See appended section

13

See appended section

14

See appended section

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See appended section

16

See appended section

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See appended section

18

18.1

Given

$$f(x, y, z) = (y^2 z^2)i + 2yzjx^2 k$$

along the path described by $\alpha(t) = ti + t^2 j + t^3 k$

The line integral of the vector field is

$$\int f(\alpha(t)) \cdot d\alpha = \int ((t^4 t^6)i + 2t^5 jt^2 k) \cdot (i + 2tj + 3t^2 k)dt$$

$$= \int (t^4 - t^6) + 4t^6 - 3t^4 dt$$

$$= \left(\frac{-2t^5}{5} + \frac{3t^7}{7}\right)$$

18.2

See appended section

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19.1

Given

$$\int_C (x^2 - 2xy)dx + (y^2 - 2xy)dy$$

where C is a path from $(2,4)$ to $(1,1)$ along the parabola C $y = x^2$.

The parametric equation of the curve is $x = t$ and $y = t^2$, then the line integral is

$$= \int_{-2}^{1} (t^2 - 2t^3)dt + 2(t^5 - 2t^4)dt$$

$$= \int_{-2}^{1} (t^2 - 2t^3 + 2t^5 - 4t^4)dt$$

$$= \left(\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5}\right)_{-2}^{1}$$

$$= \frac{396}{10}$$

19.2

Given

$$\int_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$$

where C is is the circle $x^2 + y^2 = a^2$ traversed once in a counter-clockwise direction.

The parametric equation of the curve is $x = a\cos t$ and $y = a\sin t$, then the line integral is

$$= \int_0^{2\pi} \frac{-a\sin t(\cos t + \sin t)dt - a\cos t(\cos t - \sin t)dt}{a^2(\cos t^2 + \sin t^2)}$$

$$= \int_0^{2\pi} \frac{-a(\sin t^2 + \cos t^2)dt}{a^2}$$

$$= \int_0^{2\pi} \frac{-dt}{a}$$

$$= \int_0^{2\pi} \frac{-dt}{a}$$

$$= \left(\frac{-t}{a}\right)_0^{2\pi}$$

$$= \frac{-2\pi}{a}$$

20

20.1

Given the vector field

$$f(x,y) = (2xe^y + y)i + (x^2e^y + x2y)j$$

We have

$$f_1(x,y) = 2xe^y + y$$

and

$$f_2(x, y) = x^2 e^y + x2y$$

Then, the partial derivatives $D_2 f_1$ and $D_1 f_2$ are given by

$$D_2 f_1 = 2xe^y + 1$$

and

$$D_1 f_2 = 2xe^y + 1$$

Since $D_2 f_1 = D_1 f_2$ for all values of (x, y), this vector field is a gradient on any open subset of R^2.

We know that

$$\frac{\partial \phi}{\partial x} = 2xe^y + 1$$

and

$$\frac{\partial \phi}{\partial y} = 2xe^y + 1$$

Using indefinite integrals and integrating the first of these equations with respect to x (holding y constant) we find

$$\phi(x, y) = \int (2xe^y + 1)dx + A(y) = x^2 e^y + x + A(y)$$

and

$$\phi(x, y) = \int (2xe^y + 1)dy + B(x) = 2xe^y + y + B(x)$$

20.2

Given the vector field

$$f(x, y, z) = 2xy^3 i + x^2 z^3 j + 3x^2 y z^2 k$$

We have

$$f_1(x, y) = 2xy^3$$
$$f_2(x, z) = x^2 z^3$$
$$f_3(x, y, z) = 3x^2 y z^2$$

Then, the partial derivatives $D_3 f_1$, $D_2 f_2$ and $D_1 f_3$ are given by

$$D_3 f_1 = 0$$

$$D_2 f_2 = 0$$

$$D_1 f_3 = 6xyz^2$$

Since $D_3 f_1 = D_2 f_2 = D_1 f_3$ only when either x, y or z is equal to 0

Hence, this vector is not a gradient of a scalar field ϕ

21 Appendix

Due to time constrains, I haven't been able to type all the solutions for the assignments in LaTeX. The answers that refer to this section are in the other document.