Modern Physics

Lecture 19

We have as solutions

Region I $\qquad \boxed{\psi_I(x) = Ae^{Cx}}$

Region II $\qquad \boxed{\psi_{II}(x) = F\sin(kx) + G\cos(kx)}$

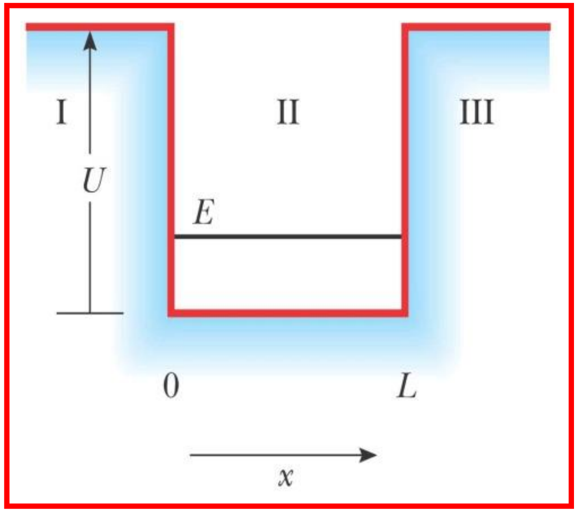
Region III $\qquad \boxed{\psi_{III}(x) = Be^{-Cx}}$

Where

$$\boxed{C^2 = \frac{2m(V-E)}{\hbar^2}}$$

and

$$\boxed{k^2 = \frac{2mE}{\hbar^2}}$$



Boundary conditions

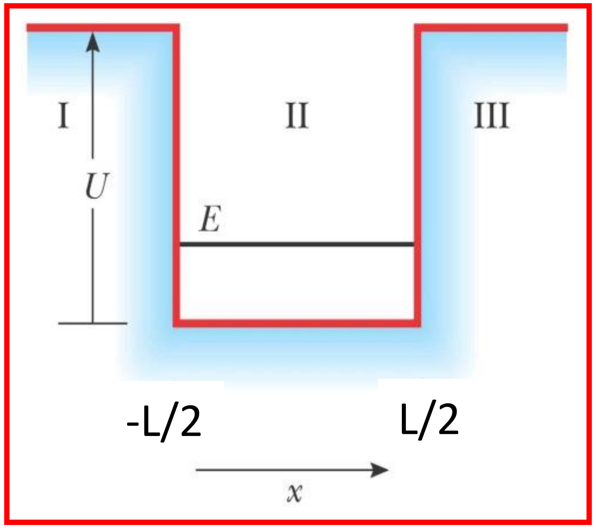
- The wave function and its first derivative must be continuous
- At x = -L/2 and at x = L/2

$$\psi_{\mathrm{I}}\left(-\frac{L}{2}\right)=\psi_{\mathrm{II}}\left(-\frac{L}{2}\right)$$

$$\frac{d}{dx}\psi_{\mathrm{I}}\left(-\frac{L}{2}\right)=\frac{d}{dx}\psi_{\mathrm{II}}\left(-\frac{L}{2}\right)$$

$$\psi_{\mathrm{II}}\left(\frac{L}{2}\right)=\psi_{\mathrm{III}}\left(\frac{L}{2}\right)$$

$$\frac{d}{dx}\psi_{\mathrm{II}}\left(\frac{L}{2}\right)=\frac{d}{dx}\psi_{\mathrm{III}}\left(\frac{L}{2}\right)$$

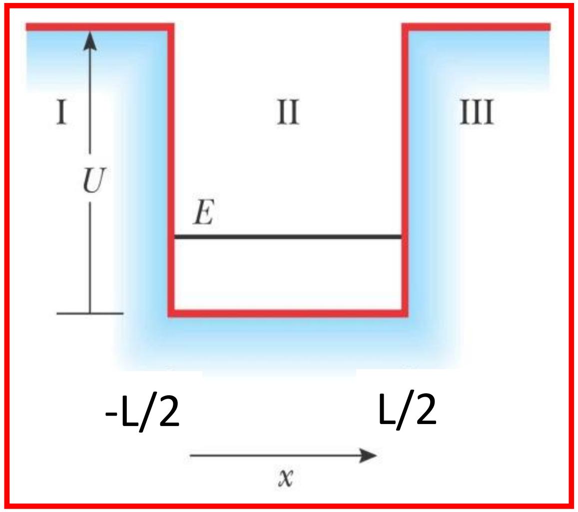


Considering even solutions inside the well

Region I $\boxed{\psi_I(x) = Ae^{Cx}}$

Region II $\boxed{\psi_{II}(x) = G\cos(kx)}$

Region III $\boxed{\psi_{III}(x) = Be^{-Cx}}$



$$\boxed{\psi_{\mathrm{I}}\left(-\frac{L}{2}\right)=\psi_{\mathrm{II}}\left(-\frac{L}{2}\right)}$$

$$\boxed{Ae^{-C\frac{L}{2}}=G\cos(k\frac{L}{2})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\boxed{\frac{d\psi_{\mathrm{I}}\left(\frac{-L}{2}\right)}{dx}=\frac{d\psi_{\mathrm{II}}\left(\frac{-L}{2}\right)}{dx}}$$

$$\boxed{-ACe^{-C\frac{L}{2}}=-Gk\sin(k\frac{L}{2})} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Dividing equations 1 and 2
We get rid of the constants

$$\boxed{C=k\tan\left(k\frac{L}{2}\right)}$$

$$\boxed{C = k\tan\left(k\frac{L}{2}\right)}$$

This contains information about the energy levels

Rewriting,

$$\boxed{\tan\left(k\frac{L}{2}\right) = \frac{C}{k}}$$

Now $\boxed{C^2 = \frac{2m(V-E)}{\hbar^2}}$ and $\boxed{k^2 = \frac{2mE}{\hbar^2}}$

Addition of these two terms, $\boxed{C^2 + k^2 = \frac{2mV}{\hbar^2}}$

Or, $\boxed{C = \sqrt{\frac{2mV}{\hbar^2} - k^2}}$

$$\boxed{C = k\sqrt{\frac{2mV}{k^2\hbar^2} - 1}}$$

After taking k outside

Or,

$$\boxed{\frac{C}{k} = \sqrt{\frac{2mV}{k^2\hbar^2} - 1}}$$

Earlier we have from BC $\boxed{\tan\left(k\frac{L}{2}\right) = \frac{C}{k}}$

Substituting $\frac{C}{k}$

$$\boxed{\tan\left(k\frac{L}{2}\right) = \sqrt{\frac{2mV}{k^2\hbar^2} - 1} = \sqrt{\frac{2m\left(\frac{L}{2}\right)^2 V}{k^2\hbar^2\left(\frac{L}{2}\right)^2} - 1}}$$

$$\boxed{\tan\left(k\frac{L}{2}\right)=\sqrt{\frac{2m\left(\frac{L}{2}\right)^2 V}{k^2\left(\frac{L}{2}\right)^2\hbar^2}-1}}$$

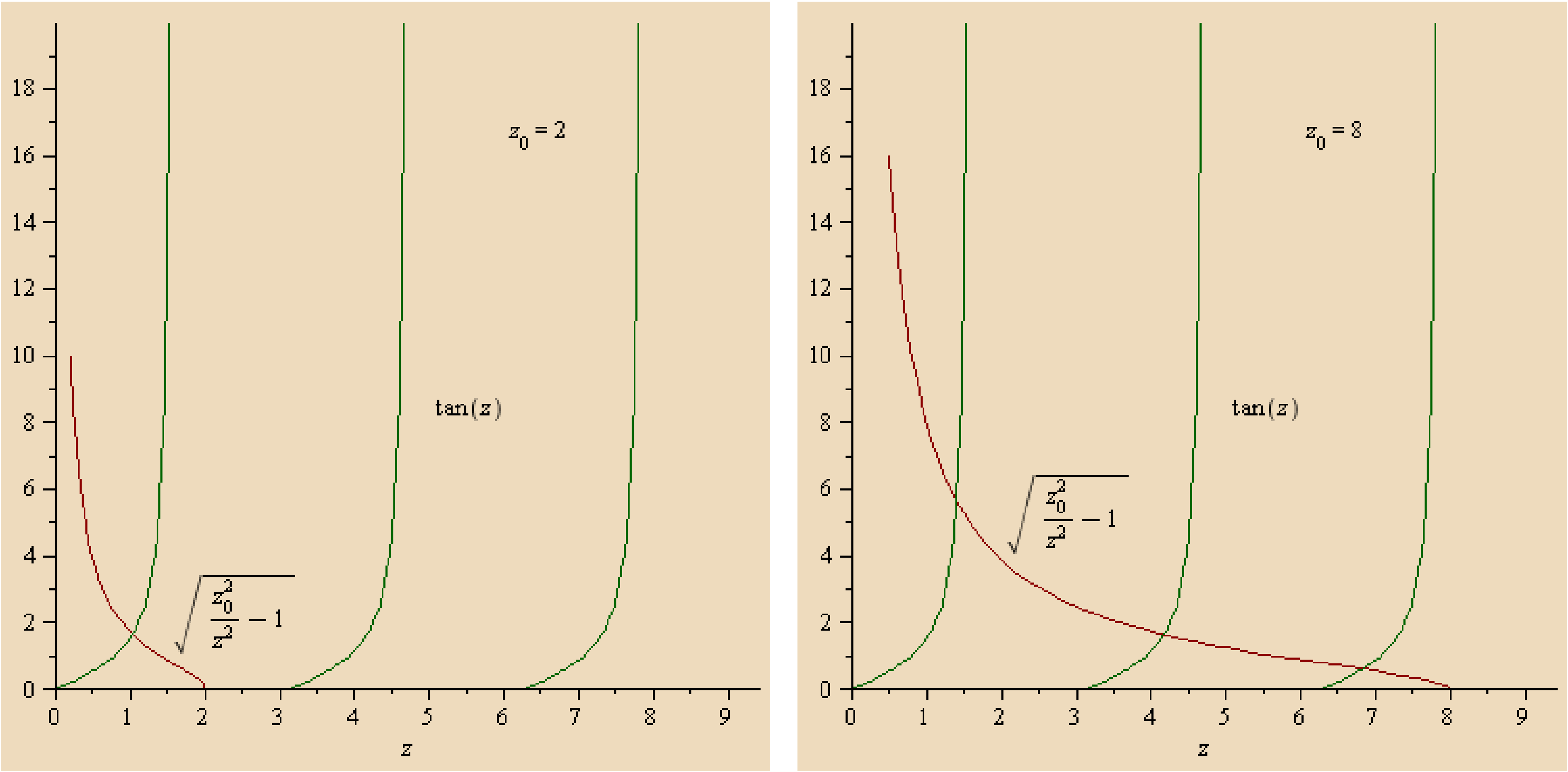
Substitute $\boxed{z=k\frac{L}{2}}$

$$\boxed{\tan(z)=\sqrt{\frac{2m\left(\frac{L}{2}\right)^2 V\big/\hbar^2}{z^2}-1}}$$

$$\boxed{\tan(z)=\sqrt{\frac{z_0^2}{z^2}-1}}$$

Where,

$$\boxed{z_0^2=2m\left(\frac{L}{2}\right)^2 V\big/\hbar^2}$$



$z_0 = 2$
$\tan(z)$
$\sqrt{\frac{z_0^2}{z^2} - 1}$
z
$z_0 = 8$
$\tan(z)$
$\sqrt{\frac{z_0^2}{z^2} - 1}$
z

- At least one bound state will be there
- V tends to infinity reproduces infinite square well situation
- Increasing well length allows more bound states inside the well

Penetration Depth

Wave function in region 3,

$$\psi_{III}(x) = Be^{-Cx}$$ where $$C = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

Penetration depth is the distance where amplitude becomes 1/e times

Therefore, $C\delta x = 1$

$$\boxed{\delta x = \frac{1}{C} = \frac{\hbar}{\sqrt{2m(V-E)}}}$$

Penetration depth

Penetration depth varies with energy level position

Classical vs. Quantum Interpretation

- According to Classical Mechanics
  - If the total energy E of the system is less than U, the particle is permanently bound in the potential well
- According to Quantum Mechanics
  - A finite probability exists that the particle can be found outside the well even if $E < U$

Application – Nanotechnology

- **Nanotechnology** refers to the design and application of devices having dimensions ranging from 1 to 100 nm
- Nanotechnology uses the idea of trapping particles in potential wells
- One area of nanotechnology of interest to researchers is the **quantum dot (QD)**
  - A quantum dot is a small region that is grown in a silicon crystal that acts as 3-D potential well