16.5 Relation with Transfer Function

Consider,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Br(t)\\ C(t) &= Dx(t) + Er(t)\end{aligned}$$

Assuming the initial condition to be zero,

$$\begin{aligned}sX(s) &= AX(s) + BR(s)\\ C(s) &= DX(s) + ER(s)\\ (sI - A)X(s) &= BR(s)\\ R(s) &= B^{-1}(SI - A)X(s)\end{aligned}$$

Now, the transfer function is defined as the matrix $G(s)$ such that

$$\begin{aligned}G(s)R(s) &= C(s) = Dx(s) + ER(s)\\ [G(s) - E]R(s) &= DX(s)\\ [G(s) - E]B^{-1}(SI - A)X(s) &= DX(s)\\ G(s) - E &= D[B^{-1}(SI - A)]^{-1}\\ G(s) - E &= D(SI - A)^{-1}B\\ G(s) &= D(sI - A)^{-1}B + E\end{aligned}$$

Existence of $G(s)$ requires $(sI - A)$ to be a non-singular matrix.

How is the characteristic equation of a system related to the eigenvalues and eigenvectors of A?

$$\begin{aligned}G(s) &= D[(sI - A)^{-1}]B + E\\ &= D[\frac{\mathrm{adj}(sI - A)}{|sI - A|}]B + E\\ &= \frac{D[\mathrm{adj}(sI - A)]B + E|sI - A|}{|sI - A|}\end{aligned}$$

We know that if we set the denominator of $G(s) = 0$, then we get the characteristic equation of $G(s)$. So,

$$|sI - A| = 0$$

is the characteristic equation of $G(s)$, and its roots are the poles of $G(s)$. Also,

$$|sI = A| = 0$$

gives the eigenvalues of A. so, the eigenvalues of A are the poles of $G(s)$.

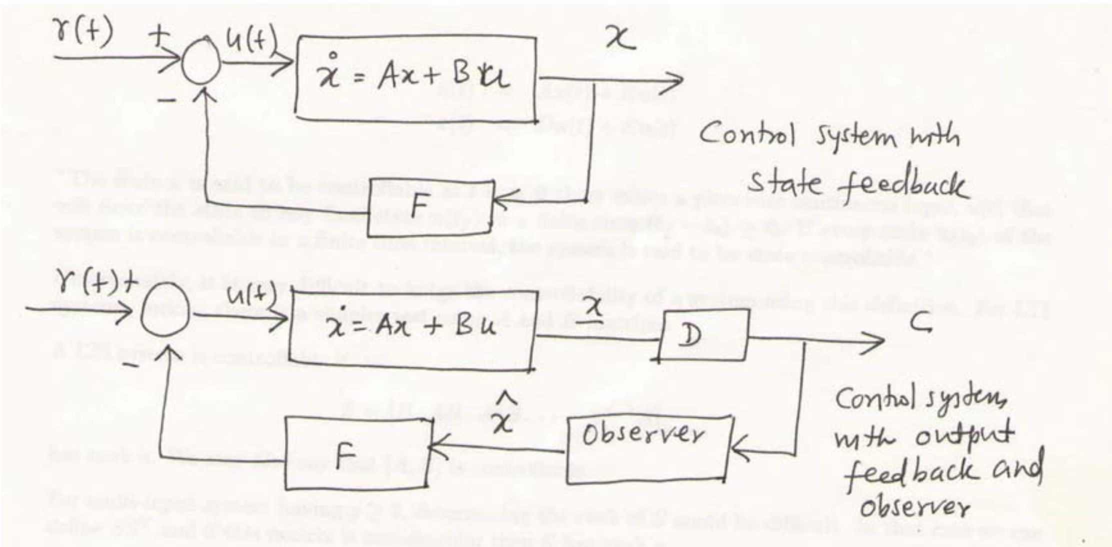
Note: The stability of a LTI system depends only on the eigenvalues of the matrix A.

16.6 Controllability and Observability of a Linear System

These are very important notions. Controllability and observability are basic notion that govern the existence of a solution to an optimal control problem.

We will see that the notion of controllability is closely related to the existence of solutions of state feedback for the purpose of placing the eigenvalues of the systems arbitrarily.

The concept of observability relates to the condition of observing or estimating the state variable from the output variable, which are generally measurable.



$$\dot{\mathbf{x}} = A\mathbf{x}(t) + B\mathbf{u}(t)$$

Feedback the state variables through a constant matrix F.

$$\begin{aligned}
\mathbf{u}(t) &= \mathbf{r}(t) - F\mathbf{x}(t) \\
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B[\mathbf{r}(t) - \mathbf{x}(t)] \\
\dot{\mathbf{x}}(t) &= (A - BF)\mathbf{x}(t) + B\mathbf{r}(t)
\end{aligned}$$

The design objective is to find the feedback matrix F such that the eigenvalues of $(A - BF)$, of the closed loop system, are at certain prescribed values.

This is also the pole placement design problem through state feedback.

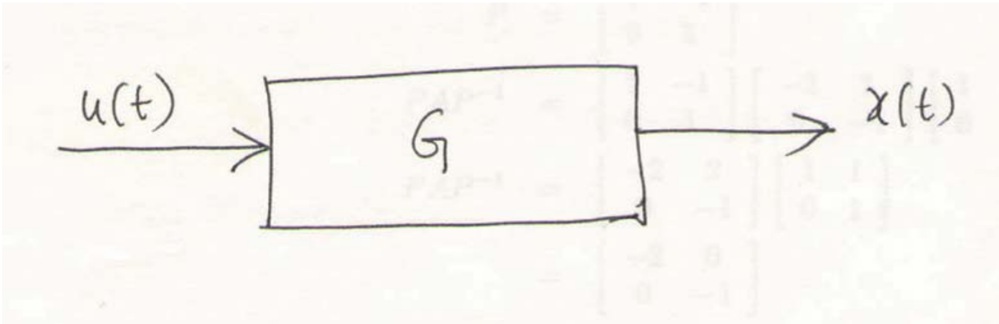
Note that ”pole”s are the poles of the closed loop transfer function matrix and is the same as the eigenvalue of $A - BF$.

Why do we need the observer?

Because we don't have the states. We have only the outputs. So we need to construct the states from the outputs. This is what the observer does.

What says that we can design such an observer? The condition that guarantees it is called the observability of the system.

16.6.1 Formal Definition of State Controllability



$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{c}(t) &= D\mathbf{x}(t) + E\mathbf{u}(t) \end{aligned}$$

"The state $\mathbf{x}$ is said to be controllable at $t = t_0$ if there exists a piecewise continuous input $\mathbf{u}(t)$ that will drive the state to any final state $\mathbf{x}(t_f)$ in a finite time $(t_f - t_0) \geq 0$. If every state $\mathbf{x}(t_0)$ of the system is controllable in a finite time interval, the system is said to be state controllable."

Unfortunately, it is very difficult to judge the controllability of a system using this definition. For LTI systems, luckily, there is a simpler test using A and B matrices.

A LTI system is controllable if

$$S = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n \times np}$$

has rank n. We may also say that $[A, B]$ is controllable.

For multi-input system having $p \geq 2$, determining the rank of S could be difficult. In that case we can define SS^T and if this matrix is non-singular then S has rank n.

An Example: Controllability

Let

$$\begin{aligned} A &= \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ S &= \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$|S| = 0$ Singular, and so, uncontrollable

Is there any other way of doing this?

Consider a similarity transformation P that diagonalizes A.

$$\dot{x} = Ax + Bu$$

Let

$$\begin{aligned} y &= Px \\ \dot{y} &= P\dot{x} \\ P^{-1}\dot{y} &= \dot{x} \\ P^{-1}\dot{y} &= AP^{-1}y + Bu \\ \dot{y} &= PAP^{-1}y + PBu \end{aligned}$$

Consider

$$\begin{aligned} P &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ PAP^{-1} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ PAP^{-1} &= \begin{bmatrix} -2 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \\ PB &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \end{aligned}$$

So, $u(t)$ can affect only y_1, but not y_2 and so the system is "uncontrollable".

Another Example: Shows that the way we define the state variables affects the controllability of the system

Consider a linear system

$$\frac{d^2c(t)}{dt^2} + 2\frac{dc(t)}{dt} + c(t) = \frac{du(t)}{dt} + u(t)$$

Let

$$\begin{aligned} x_1 &= C \\ x_2 &= \frac{dc(t)}{dt} - u \\ \dot{x_1}(t) &= x_2(t) + u \end{aligned}$$

$$
\begin{aligned}
\dot{x_2}(t) &= \frac{d^2c(t)}{dt^2} - \frac{du(t)}{dt} = -2\frac{dc(t)}{dt} - c(t) + u(t) \\
&= -2(x(t) + u) - x_1(t) + u(t) \\
&= -2x_2(t) - x_1(t) - u(t) \\
\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
S &= \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}
$$

Now, $|S| = 0 \rightarrow$, and so S is singular and hence the system is uncontrollable.

But another way to express the same system is through the following equations (we omit details).

$$
\begin{aligned}
\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
c &= x_1 + x_2 \\
S &= \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \\
|S| &= 1 \neq 0 \ \ \text{[nonsingular]}
\end{aligned}
$$

and so the system is completely state controllable.

16.6.2 Observability of Linear system

In general, a system is completely observable if every state variable of the system affects some of the outputs.

Definition: Given a LTI system

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\
\mathbf{c}t) &= D\mathbf{x}(t) + E\mathbf{u}(t)
\end{aligned}
$$

the state $\mathbf{x}(t_o)$is said to be observable if given any input $\mathbf{u}(t)$, there exists a finite time $t_f \geq t_0$ such that the knowledge of $\mathbf{u}(t)$ for $t_0 \leq t \leq t_f$, knowledge of the matrices A, B, D, E, and the output $\mathbf{c}(t)$ for $t_0 \leq t \leq t_f$, are sufficient to determine $\mathbf{x}(t_0)$. If every state of the system is observable for a finite t_f, we say that the system is completely observable, or simply observable.

Again, the above is a difficult condition to verify, and for LTI system we have a simpler test that depends on A and D. Let,

$$
V = \begin{bmatrix} D^T & A^T D^T & (A^T)^2 D^T & \cdots & (A^T)^{n-1} D^T \end{bmatrix}
$$

where, 'T' in the superscript stands for 'transpose'. If V had rank n then the system is completely observable. We may say that the pair $[A, D]$ is observable.

PROBLEM SET 10

1. Write the state and output equations corresponding to the following differential equations. Find the state transition matrix. Also, find the state transition equation when there is an unit step input. Assume some arbitrary initial conditions for the states.

   (a)

$$\frac{d^2c(t)}{dt^2} + 10\frac{dc(t)}{dt} + 3c(t) = r(t)$$

   (b)

$$\frac{d^3c(t)}{dt^3} + 3\frac{d^2c(t)}{dt^2} + 7\frac{dc(t)}{dt} = 2\frac{dr(t)}{dt} + r(t)$$

2. Suppose,

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   Check if this can be converted to the phase variable canonical form. If yes, then find the appropriate transformation and convert this system to a phase variable canonical form.

3. Do the same as above for the following system of equations:

$$\dot{x}_1 = x_1 + 2x_2 + 2u$$
$$\dot{x}_2 = x_1 + x_2 + 2u$$

4. Determine the controllability of each of the systems given above.

16.6.3 Some Invariance Theorems

1. If $[A, B]$ is controllable then $[P^{-1}AP, P^{-1}B]$ is also controllable if P is a nonsingular matrix.

   So, a nonsingular transformation of the state variable does not change the controllability of the system.

2. If $[A, D]$ is observable then $[P^{-1}AP, DP]$ is also observable is P is a non-singular matrix.

   So, any nonsingular transformation of the state variable does not change the observability of the system.

3. Controllability of closed-loop system with state feedback.

   Let

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

   be completely controllable. Then any close loop system obtained through state feedback.

$$\mathbf{u}(t) = \mathbf{r}(t) - F\mathbf{x}(t)$$

   yields a state equation

$$\dot{\mathbf{x}}(t) = (A - BF)\mathbf{x}(t) + B\mathbf{r}(t)$$

   which is also controllable. So, if $[A, B]$ is controllable then $[A - BF, B]$ is also controllable. On the other hand if, [A,B] is uncontrollable then there exists no F so that $[A - BF, B]$ is controllable, or $[A - BF, B]$ is uncontrollable for all F.

   Thus, an uncontrollable system cannot be made controllable by state feedback.

4. Observability of closed-loop system with state feedback.

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{c}(t) &= D\mathbf{x}(t) \end{aligned}$$

   If this system is both controllable and observable, state feedback can destroy observability.

   So, observability of open loop system and the corresponding closed loop system are unrelated.

Example 7: How state feedback can destroy observability

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}: \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

The system is both controllable and observable, since

$$\begin{aligned} S &= \begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix}; \quad |S| = -6 \\ V &= \begin{bmatrix} 1 & -4 \\ 2 & -5 \end{bmatrix}; \quad |V| = 3 \end{aligned}$$

Let the state feedback be defined as

$$\begin{aligned}
u(t) &= r(t) - Fx(t) \\
F &= [f_1 \quad f_2] \\
\dot{x}(t) &= (A - BF)x(t) + Br(t) \\
A - BG &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} = \begin{bmatrix} -f_1 & 1 - f_2 \\ -2 - f_1 & -3 - f_2 \end{bmatrix} \\
V &= \begin{bmatrix} D^T & (A - BF)^T D^T \end{bmatrix} = \begin{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} -f_1 & -2 - f_1 \\ 1 - f_2 & -3 - f_1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -3f_1 - 4 \\ 2 & -3f_2 - 5 \end{bmatrix} \\
|V| &= 6f_1 - 3f_2 + 3 = 0
\end{aligned}$$

If $f_1 = 1, f_2 = 3$, the system becomes unobservable.

16.6.4 Relation with Transfer Functions

How do we relate the notions of controllability and observability with transfer functions? That is, can we have systems that are uncontrollable/unobservavble when expressed in the form of transfer functions?

The answer lies in the effect of pole-zero cancellation.

"If the input-output transfer function of a linear system has pole-zero cancellation, the system will be either not state controllable, or unobservable, depending on how the state variables are defined. If the input-output transfer function of a linear system does not have pole-zero cancellation, the system can always be represented by dynamic equations as a completely controllable and observable system".

An Example

Consider

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
D &= \begin{bmatrix} 1 & 0 \end{bmatrix}
\end{aligned}$$

This system is both uncontrollable and unobservable.

This system corresponds to

$$\frac{C(s)}{U(s)} = \frac{s+1}{s^2 + 2s + 1} = \frac{(s+1)}{(s+1)^2} = \frac{1}{s+1}$$

which shows pole-zero cancellation.

16.6.5 Some comments

Controllability and observability are concepts distinctive to the state-space approach. They were introduced by Kalman and has interesting ramifications in classical control design and points out the pitfall of the logic “Why don’t we simply cancel the RHS poles by using some RHS zeros?”