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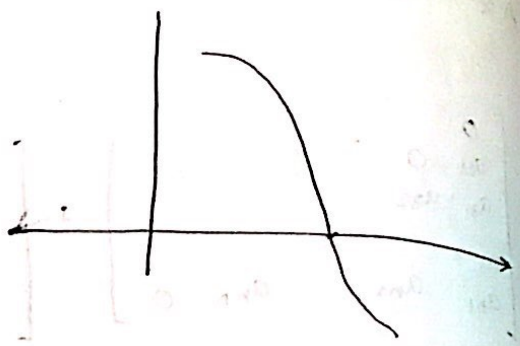
Recap

Root finding



Let

$f : [a,b] \to \mathbb{R}$ be continuous

$f(a)f(b) < 0$

Generate a sequence of numbers using iterative methods

$$x_0, x_1, x_2, x_3 \dots$$

$\lim_{n\to\infty} x_n = r$, r is a root

order of convergence

$$\lim_{x\to\infty} \frac{|e_{k+1}|}{|e_k|^p} = M \qquad M \neq 0, M \neq \infty$$

if it satifies these conditions the p is the order of convergence

Ex: $x_n = \frac{1}{n}$, $n = 1, 2, 3$

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\dots\right\} \quad , \quad r = 0$$

$e_n = x_n - r$; $e_{n+1} = x_{n+1} - r$

$e_n = \frac{1}{n}$ $\qquad e_{n+1} = \frac{1}{n+1}$

$$\Rightarrow \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \infty$$

Ex: $x_0 = \frac{1}{3}$, $x_{n+1} = x_n^2$, $n = 1, 2, 3, 4 \dots$

clearly $r = 0$

$e_n = x_n$; $e_{n+1} = x_{n+1} - r = x_n^2$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = 1 \neq 0$$

order of conv = 2

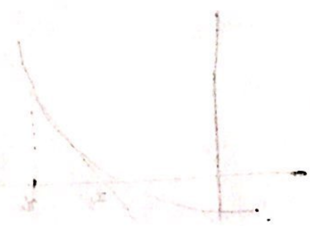
<u>Iterative</u> <u>method</u>.

$$x_{n+1} = \phi(x_n) \quad , \quad n = 0, 1, 2, \dots$$

<u>NR</u>

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$



<u>sufficient conditions</u>

$$|\phi'(x)| < 1$$

<u>Fixed point</u> $\quad \phi(r) = r$

Let $x_{n+1} - r = \phi(x_n) - \phi(r)$

$= \phi'(\xi)(x_n - r)$ where ξ lies bet^n x_n & r

$|x_{n+1} - r| = |\phi'(\xi)|^{n+1} (x_0 - r)$ $\quad$ ξ lies in the small intervals of the sequence

Case(i)

$$\phi'(x) \neq 0 \quad , \forall \; x \in [a,b]$$

$$x_{n+1} - r = \phi(x_n) - \phi(r) \qquad e_n = x_n - r$$

$$x_n = r + e_n$$

$$= \phi(r + e_n) - \phi(r)$$

$$= \left[\cancel{\phi(r)} + e_n \phi'(r) + \frac{e_n^2}{2!}\phi''(r) + \cdots\right] - \cancel{\phi(r)}$$

$$= e_n \phi'(r) + \frac{e_n^2}{2!}\phi''(r) + \frac{e_n^3}{3!}\phi'''(r)$$

$$\phi'''(r) \neq 0$$

case(ii) : If $\phi'(r) = 0$, $\phi''(r) = 0$, then the order of fixed point iterative scheme is $p = 3$

Stopping criteria

$$\text{Relative error} = \frac{|x_{n+1} - x_n|}{|x_n|} < \epsilon$$

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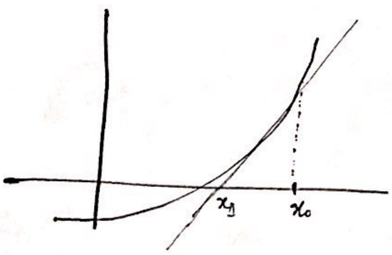

$$f(x) = 0$$

x_n

$$x_{n+1} = x_n + h$$

$$f(x_n + h) = f(x_n) + h f'(x_n) + \cancel{\frac{h^2}{2!} f''(x_n)} + \cancel{\frac{h^3}{3!} f'''(x_n)}$$

$$\Rightarrow f(x_n) + h f'(x_n) = 0 \qquad h = -\frac{f(x_n)}{f'(x_n)}$$

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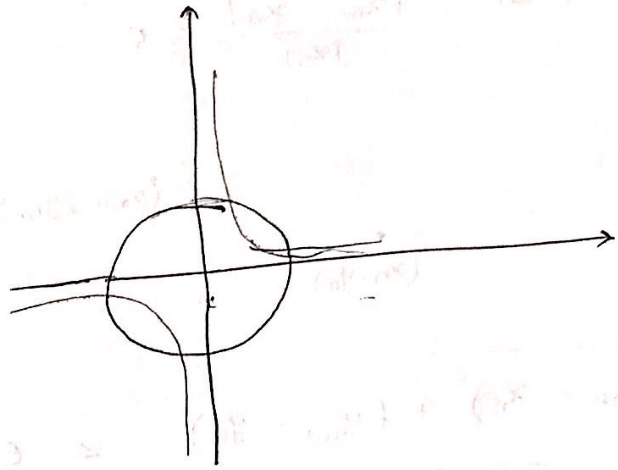
Ex : $\left.\begin{aligned} x^2+y^2 &= 1 \\ xy &= 1 \end{aligned}\right\}$



2D

$\left.\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}\right]$

$(x_0, y_0) \longrightarrow (x_1, y_1)$

$x_1 = x_0 + h$

$y_1 = y_0 + h$

$\cancel{f(x_0+h,\ y_0}+k)^{\text{zero}}$

$$= f(x_0, y_0) + h \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} + k \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} + \cdots\cdots$$

$\cancel{g(x_0+h},\ y_0+k)^{\text{zero}}$

$$= g(x_0, y_0) + h \left.\frac{\partial g}{\partial x}\right|_{(x_0,y_0)} + k \left.\frac{\partial g}{\partial y}\right|_{(x_0,y_0)} + \cdots\cdots$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x_0,y_0)} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -f(x_0,y_0) \\ -g(x_0,y_0) \end{bmatrix}$$

$\boxed{JH = F}$

$x_1 = x_0 + h \qquad y_1 = y_0 + k$

$x_2 = x_1 + h \qquad y_2 = y_1 + k$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J^{-1} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$

Relative error $= \dfrac{|x_{n+1} - x_n|}{|x_n|} < \epsilon$

(x_{n+1}, y_{n+1})

(x_n, y_n)

$$\frac{\sqrt{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2}}{\sqrt{x_n^2 + y_n^2}} < \epsilon$$

Solving linear system of equations

Eg $ax + by = k_1 \Rightarrow f(x,y) = ax + by - k_1 = 0$

$cx + dy = k_2 \Rightarrow g(x,y) = cx + dy - k_2 = 0$

(x_0, y_0) $\quad J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |J| = ad - bc \neq 0$

Gauss elimination

$Ax = b$ $\qquad |A| \neq 0$

$x_i = \dfrac{\det(A_i)}{\det A}$ $\qquad A_i$ th column is replaced by RHS

If $n = 100$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & & & & \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$a_{22}^{(1)} = a_{22} - \frac{a_{12}}{a_{11}} a_{21}$$

$$a_{32}^{(1)} = a_{32} - \frac{a_{21}}{a_{11}} a_{13}$$

$$\vdots$$

$$a_{n2} = a_{n2} - \frac{a_{n-1\,2}}{a_{11}} a_{11}$$

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$Ax = b \qquad |A| \neq 0$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

if $a_{11} \neq 0$

step 1 $\quad a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \qquad i = 2, 3, \dots n \quad j = 2, 3, \dots n$

$$b_j = b_j - \frac{a_{i1}}{a_{11}} b_1 \qquad i = 2, \dots n \quad j = 2, \dots n$$

no of iteration $\boxed{(n-1)n}$

Step② $a_{22} \neq 0$

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}} a_{2j}^{(1)} \qquad i = 3, 4, \ldots n. \quad j = 3\ 4, \ldots n$$

$$b_j^{(2)} = b_j^{(1)} - \frac{a_{i2}}{a_{22}} b_2^{(2)} \qquad \boxed{(n-2)(n-1)}$$

* Floating point operations (Flop)

1 flop = 1 (add/sub) + 1 (mul/div)

$Ax = b$

$Ux = \tilde{b}$

To update each element

$$\underbrace{1 \text{ sub} + 1 \text{ mul}}_{1 \text{ flop}} + 1 \text{ div}$$

Step 1 $\quad n(n-1) + n$ div

Step 2 $\quad (n-1)(n-2) + (n-1)$ div.

Total $\quad \underbrace{[n(n-1) + (n-1)(n-2) + \cdots + 1]}_{\text{flop}} + \underbrace{[n + (n-1) + \cdots + 1]}_{\text{div}}$

$$= [(n-1)(n-1+1) + (n-2)(n-2+1) + \cdots\ 1]$$
$$[n + (n-1) + \cdots\ 1]$$

$$= [(n-1)^2 + (n-2)^2 + \cdots + 1]$$
$$+ [(n-1) + (n-2) + \cdots\ 1]$$
$$+ [n + (n-1) + \cdots + 1]$$

$$= \frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n(2n^2-3n+1)}{6} + \frac{n(n-1)}{2} + \frac{n(n+1)}{2}$$

$$= O(n^3)$$

Backward Sub

If $a_{ii} \neq 0$, $i = 1, 2, \ldots, n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 0 & a_{22} & \cdots & & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & a_{n-1,n-1} & a_{n-1,n} \\ 0 & & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}\, x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{1}{a_{n-1,n-2}} \left(b_{n-2} - a_{n-2,n}\, x_n - a_{n-2,n-1}\, x_{n-1} \right)$$

2 mul + 2 sub + 1 div.

for $j = n : -1 : 1$

$$x_j = \frac{1}{a_{jj}} \left(b_j - \sum_{k=j+1}^{n} a_{jk}\, x_k \right)$$

end

LU decomposition

$Ax = b \qquad |A| \neq 0, \quad A \in \mathbb{R}^{n \times n}$

$LUx = b$ —— ①

let $Ux = y$ —— ②

① $Ly = b$ —— ③

$A = LU$ is unique ?

Let A has crouts decomposition, then prove that L & U are unique

Let $A = L_1 U_1 = L_2 U_2$

where U is an upper triangular matrix

$\longrightarrow |U| = 1$

consider $L_1 U_1 = L_2 U_2$

$L_2^{-1}(L_1 U_1) = L_2^{-1}(L_2 U_2)$

$\rightarrow (L_2^{-1} L_1) U_1 = I U_2$

$\Rightarrow L_2^{-1} L_1 = U_2 U_1^{-1} = I$

H·W

Note: Inverse of upper/lower triangular matrix is an upper/lower triangular

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Recap. $A = LU$ —— (1)

For uniqueness Let L be an unit lower triangular matrix

Let $L_1U_1 = L_2U_2$

$\Rightarrow L_2^{-1}L_1 = U_2U_1^{-1} = L$

L^{-1} is a lower triangle ??

$\Rightarrow L_2 = L_1$ and $U_2 = U_1$

$\therefore$ LU decomposition is unique.

Sufficient condition on existance of LU decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \quad : \quad |A| = -5 \neq 0$$

→ Test condition
If matrix is non singular then it is LU decomposable.

$A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = 1 ; u_{12} = 2 ; u_{13} = 3$

$l_{21}u_{11} = 2 \Rightarrow l_{21} = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & \frac{15-6}{2} \end{bmatrix}$$

$A = L\,U$

$|A| = |L||U|$

$-5 = 1 \times 0$?

* A has LU decomposition if all the principal minors are non singular

$\Rightarrow |a_{11}| \neq 0$,

$\Rightarrow \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$,

$\Rightarrow |A| \neq 0$.

In the case of example.

$\rightarrow |a_{11}| \neq 0$

$\rightarrow \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$\rightarrow |A| = -5 \neq 0$

$\therefore$ since one of the principle minors is zero LU decomp not applicable

we can ~~change~~ interchange rows

$\Longrightarrow A = \begin{vmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Note: Let $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$: $|U| = U_{11} U_{22} \neq 0$

Define $D = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix}$: $D^{-1} = \begin{bmatrix} \frac{1}{u_{11}} & 0 \\ 0 & \frac{1}{u_{22}} \end{bmatrix}$

$$D^{-1}U = \begin{bmatrix} 1 & u_{12}/u_{22} \\ 0 & 1 \end{bmatrix} \quad ?$$

Let $V = D^{-1}U$ is a unit upper triangular

$\longrightarrow \quad U = DV$

consider $A = LU$

$$\boxed{A = LDV}$$

case (i) If $A = A^T$

$$LDV = (LDV)^T$$
$$= V^T D L^T$$

$\Longrightarrow \quad L = V^T$ and $V = L^T$

$\therefore \quad A = LDL^T$

case (ii) A is positive definite $x^T A x > 0, \ \forall x$

$\Rightarrow A x = \lambda x$

$\Rightarrow x^T A x = \lambda x^T x^*$

$\longrightarrow \quad \lambda_i > 0$, for $i = 1, 2, \cdots n$

Define

$$\sqrt{D} = \begin{bmatrix} \sqrt{u_{11}} & 0 \\ 0 & \sqrt{u_{22}} \end{bmatrix}$$

$$\Rightarrow (\sqrt{D})(\sqrt{D}) = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix} = D$$

$$A = LDL^T$$
$$= L(\sqrt{D}\sqrt{D})L^T$$
$$= (L\sqrt{D})(\sqrt{D}L^T)$$
$$\boxed{A = GG^T}$$

Let
$$G = L\sqrt{D}$$
$$G^T = \sqrt{D}L^T$$

Define

$$G = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

Partial pivoting

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 13 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1/3$; $R_3 \rightarrow R_3 - \frac{2R_1}{3}$

$$\begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & -1/3 \\ 0 & -1 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -2/3 \\ -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 4 \\ 0 & -1 & 1/3 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ -1/3 \\ -2/3 \end{bmatrix} \uparrow$$

$x_1 \; x_2 \; x_3$

$= (3 \quad 1 \quad 2)$

complete pivoting

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 3x_2 + 4x_3 &= 20 \\ 2x_1 + x_2 + 3x_3 &= 13 \end{aligned}$$

$C_1 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 4 & 3 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 13 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - \frac{R_1}{4}$, $R_3 \longrightarrow R_3 - \frac{3}{4}R_1$

$$\begin{bmatrix} 4 & 3 & 3 \\ 0 & 1/4 & 1/4 \\ 0 & \frac{-5}{4} & -1/4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \\ -2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 4 & 3 & 3 \\ 0 & -5/4 & -1/4 \\ 0 & 1/4 & 1/4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ -2 \\ 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + \frac{1}{5} R_2$

$$\begin{bmatrix} 4 & 3 & 3 \\ 0 & -5/4 & -1/4 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ -2 \\ 3/5 \end{bmatrix} \Rightarrow \begin{array}{l} x_3 = 2 \\ x_2 = 1 \\ x_1 = 3 \end{array}$$

$Ax = b \qquad |A| \neq 0$

Direct methods — GEM, LU, GG^T → Thomas algorithm

Iterative methods

<u>Vector norm</u> : Let $(V, +, \cdot)$ be a vector space let $x \in V$

$$\|\cdot\| : V \rightarrow \mathbb{R}^+ \cup \{0\}$$

a) $\|x\| > 0$ if $x \neq 0$

$\|x\| = 0 \iff x = 0$

b) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$

c) $\|x+y\| \leq \|x\| + \|y\|$, $\forall \, x, y \in V$

Ex: Euclidean norm

~~$\|x\|_2 = |x_1| + |x_2| + \cdots + |x_n|$~~

~~$= \sum_{i=1}^{n} |x_i|$~~

Ex: Euclidean norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \quad \text{where, } x = (x_1, x_2, \ldots x_n)$$

Ex Max norm

$$\|x\|_\infty = \max_{1 \le i \le n} \|x_i\| = \max\{|x_1|, |x_2|, \ldots |x_n|\}$$

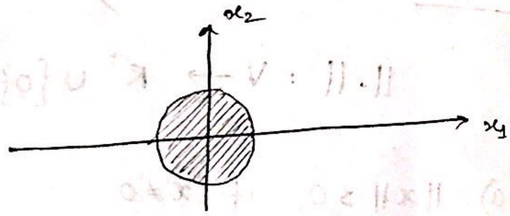
one-norm:

$$\|x_1\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

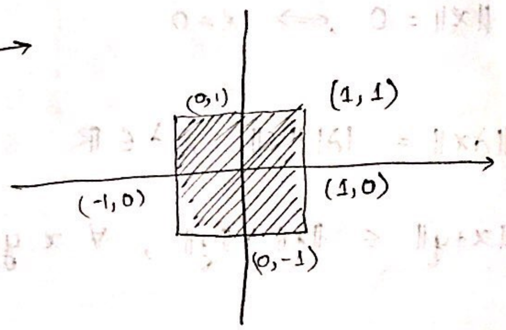
$$= \sum_{i=1}^{n} |x_i|$$

Ex: Find the points $x = (x_1, x_2) \in \mathbb{R}^2$ such that

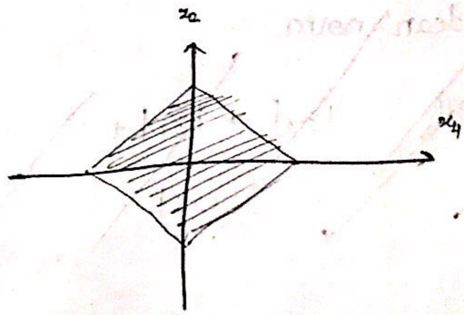
$D_1 = \{x / \|x\|_2 \le 1\}$



$D_2 = \{x / \|x\|_\infty \le 1\}$ $\longrightarrow$



$D_3 = \{x / \|x\|_1 \le 1\}$ $\longrightarrow$



Scanned with CamScanner

$x = (4, 4, -4, 4)$ $\quad v = (0, 5, 5, 5)$ $\quad w = (6, 0, 0, 0)$

	$\lVert\cdot\rVert_1$	$\lVert\cdot\rVert_2$	$\lVert\cdot\rVert_\infty$
x	16	8	4
v	15	$5\sqrt{3}$	5
w	6	6	6
$x-v$	15	$3\sqrt{11}$	9

<u>Matrix norm</u> : Let $Ax = b$ $\quad |A| \neq 0$, $A \in \mathbb{R}^{n \times n}$

$$\lVert\cdot\rVert : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^+ \cup \{0\}$$

Let $A, B \in \mathbb{R}^{n \times n}$, $\alpha \in \mathbb{R}$

a) $\lVert A \rVert \geq 0$, $\forall A \in \mathbb{R}^{n+n}$

$\lVert A \rVert = 0 \iff A = 0$ (yero matrix)

b) $\lVert \alpha A \rVert = |\alpha| \, \lVert A \rVert$

c) $\lVert A + B \rVert \leq \lVert A \rVert + \lVert B \rVert$

d) $\lVert AB \rVert \leq \lVert A \rVert \, \lVert B \rVert$ (consistency property)

<u>Induced matrix norm</u> (vector norm)

Let $\lVert\cdot\rVert_v$ be a vector norm

$$\lVert A \rVert = \max_{\lVert x \rVert_v \neq 0} \frac{\lVert Ax \rVert_v}{\lVert x \rVert_v}$$

Consistency property.

$$\|A\| \geq \frac{\|Ax\|_v}{\|x\|_v}$$

$$\Rightarrow \boxed{\|Ax\|_v \leq \|A\| \, \|x\|_v}$$

Note:

$$\|(AB)x\|_v \leq \|AB\| \, \|x\|_v$$
$$\leq \|A\| \, \|B\| \, \|x\|_v$$

$$\|AB\| = \max_{\|x\|_v \neq 0} \frac{\|ABx\|_v}{\|x\|_v}$$

Ex: $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad \text{(column sum norm)}$$

$$\|A\|_1 = \max\{|1| + |4| \, , \, |-2| + |3|\}$$
$$= 5$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \quad \text{(Row sum)}$$

$$\|A\|_\infty = \max\{|1| + |-2| \, , \, |4| + |3|\} = 7$$

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spectral radius of A :

$$\rho(A) = \max|\lambda|$$

$$\lambda \in \sigma(A)$$

where $\sigma(A)$ = set of all eigen values of A

$$\|A\|_2 = \sqrt{\rho(A^T A)}$$

Ex :— $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$ $\quad A^T A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 17 & 10 \\ 10 & 13 \end{bmatrix}$$

Eigen values

$$Ax = \lambda x$$

$$\begin{bmatrix} 17-\lambda & 10 \\ 10 & 13-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 30\lambda + 121 = 0$$

$$\lambda = \frac{30 \pm \sqrt{30^2 - 4 \times 1 \times 121}}{2}$$

$$(\lambda_1, \lambda_2) = (25.2, 4.8)$$

$$\rho(A^T A) = 25.2$$

$$\|A\|_2 = \sqrt{25.2}$$

Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)}$$

$A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$ = ~~30~~ $\quad \|A\|_F = \sqrt{1+4+6+9} = \sqrt{30}$

$$\|A\|_2 \le \|A\|_F$$

Using cauchy-shwartz inequality prove that:

$$\|A\|_2 \le \|A\|_F \|x\|_2 \qquad \forall\, x \in \mathbb{R}^n$$

$$\Longleftrightarrow \boxed{\frac{\|Ax\|_2}{\|x\|_2} \le \|A\|_F}$$

<u>Iterative methods</u>

$f(x)=0,\ x\in[a,b]$
$x_{n+1} = \phi(x_n)$
$n=0,1,\dots$

Let $Ax=b \qquad \|A\| \neq 0 \qquad A \in \mathbb{R}^{n\times n}$

Let $Q \in \mathbb{R}^{n\times n}$, $|Q| \neq 0$, Q is a splitting matrix

A is exact value

$$0 = -Ax+b$$

$$Qx + 0 = Qx - Ax + b$$

$$Qx = (Q-A)x + b \longrightarrow (1)$$

Define an iterative scheme

$$Qx^{(n+1)} = (Q-A)x^{(n)} + b \longrightarrow (2)$$

$$n = 0, 1, 2 \dots$$

$$Q^{-1}\left(Qx^{(n+1)}\right) = Q^{-1}(Q-A)x^{(n)} + Q^{-1}b$$

$$x = (I - Q^{-1}A)x + Q^{-1}b \text{ --- } (3)$$

$$x^{(n+1)} = (I - Q^{-1}A)x^{(n)} + Q^{-1}b \text{ --- } (4)$$

Let $H = I - Q^{-1}A$

$$\Rightarrow x^{(n+1)} - x^{(n)} = H(x^{(n)} - x)$$

$$\|x^{(n+1)} - x\| = \|H(x^{(n)} - x)\|$$

$$\le \|H\|\,\|x^{(n)} - x\|, \quad x^{(0)} \text{ is known}$$

$n=0$ $\| x^{(1)} - x \| \leq \|H\| \; \| x^{(0)} - x \|$

$n=1$ $\| x^{(2)} - x \| \leq \|H\| \; \| x^{(1)} - x \|$

$\leq (\|H\|)^2 \; \| x^{(0)} - x \|$

⋮

$\| x^{(n+1)} - x \| < (\|H\|)^{n+1} \; \| x^{(0)} - x \|$

size should be less than 1 $\qquad$ $\|H\| < 1$ for convergence

$\boxed{\text{Iterative process converges if } \|H\| < 1}$

$$\lim_{n\to\infty} \| x^{(n+1)} - x \| = 0$$

stopping criteria.

consider

$\| x^{(n+1)} - x \| < \|H\| \; \| x^{(n)} - x \| \quad \text{——— (1)}$.

prove that

$$\| x^{(n+1)} - x \| \leq \frac{\|H\|}{1 - \|H\|} \; \| x^{(n+1)} - x^{(n)} \|$$

from (1)

$\| x^{n+1} - x \| \leq \|H\| \; \| x^{(n+1)} - x + x^{(n)} - x^{(n+1)} \|$

$\| x^{n+1} - x \| (1 - \|H\|) \leq \|H\| \; \| x^{n+1} - x^{n} \|$

$\Rightarrow \| x^{n+1} - x \| \leq \dfrac{\|H\|}{(1 - \|H\|)} \left(\| x^{(n+1)} - x^{(n)} \| \right)$

→ Gauss Jacobi Iterative method.

Let $A = L + D + U$

Ex: $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

for jacobi method $\boxed{Q = D}$.

Jacobi iteration method.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3$$

Initial vector $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$

$$x_1 = \frac{1}{a_{11}}\left(d_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

$$x_2 = \frac{1}{a_{22}}\left(d_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right)$$

$$x_3 = \frac{1}{a_{33}}\left(d_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{n+1} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(n)} + \begin{bmatrix} \frac{d_1}{a_{11}} \\ \frac{d_2}{a_{22}} \\ \frac{d_3}{a_{33}} \end{bmatrix}$$

$x^{(n+1)} = Hx^{(n)} + C$, $x^{(0)}$ is known

Max eigen value of H - spectral radius

Sufficient for convergence is $\|H\|_\infty < 1$

Diagonal entries should be large $\|H\|_\infty < 1$ Then it converg

column sum

$$\|H\|_1 = \max\left\{\left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{31}}{a_{33}}\right|, \left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{32}}{a_{33}}\right|, \left|\frac{a_{13}}{a_{11}}\right| + \left|\frac{a_{23}}{a_{22}}\right|\right\}$$

$$\|H\|_\infty = \max\left\{\left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{13}}{a_{11}}\right|, \left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{23}}{a_{22}}\right|, \left|\frac{a_{31}}{a_{33}}\right| + \left|\frac{a_{32}}{a_{33}}\right|\right\}$$

→ Diagonal dominant matrix

Diagnal dominant matrix

$$|a_{ii}| >> \sum_{\substack{j=1 \\ (j \neq i)}}^{n} a_{ij}$$

Ex:

$5x_1 + 2x_2 - 10x_3 = -3$ —— ③

$10x_1 + 4x_2 - 2x_3 = 12$ —— ①

$x_1 - 10x_2 - x_3 = -10$ —— ②

order – according to diagonal entries

Gauss Seidal method.

$$x^0 = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$$

$$x_1^{(1)} = \frac{1}{a_{11}} (d_1 - a_{12} x_2^{(0)} - a_{13} x_3^{(0)})$$

$$x_2^{(1)} = \frac{1}{a_{22}} (d_2 - a_{21} x_1^{(1)} - a_{23} x_3^{(0)})$$

$$x_3^{(1)} = \frac{1}{a_{33}} (d_3 - a_{31} x_1^{(1)} - a_{32} x_2^{(1)})$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{22}} & 1 & 0 \\ \frac{a_{31}}{a_{33}} & \frac{a_{32}}{a_{33}} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(n+1)} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ 0 & 0 & -\frac{a_{23}}{a_{22}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{(n)} + \begin{bmatrix} \frac{d_1}{a_{11}} \\ \frac{d_2}{a_{22}} \\ \frac{d_3}{a_{33}} \end{bmatrix}$$

$$x^{(n+1)} = Hx^{(n)} + C \qquad A = L + D + U$$

$$\boxed{H = -(D+L)^{-1}U}$$

For convergence $\|H\| < 1 \Rightarrow |a_{ii}| \gg \sum_{j=1}^{n} |a_{ij}|$ $(j \neq i)$

(twice faster ?)

Note:

$Ax = b$, $|A| \neq 0$

✓ $\boxed{b} \longrightarrow \boxed{A} \longrightarrow x$

✓ $\boxed{b+\tilde{b}} \longrightarrow \boxed{A} \longrightarrow x+\tilde{x}$

$\Rightarrow$ stable.

$\boxed{b+\tilde{b}} \longrightarrow \boxed{A+\tilde{A}} \longrightarrow x+\tilde{x}$

Theorem In $Ax = b$, $|A| \neq 0$

$$\frac{1}{K(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|x - x^2\|}{\|x\|} \leq K(A) \frac{\|r\|}{\|b\|}$$

where $r = b - \tilde{b} = \delta b$

$e = x - \tilde{x} = \delta x$

condition no.

$K(A) = \|A\| \|A^{-1}\|$

27/1 **Theorem:** Let $Ax = b$ $|A| \neq 0$, then

$$\frac{1}{K(A)} \frac{\|r\|}{\|b\|} \leq \underbrace{\frac{\|x - \tilde{x}\|}{\|x\|}}_{\text{relative error}} \leq k(A) \underbrace{\frac{\|r\|}{\|b\|}}_{\text{residual error}}$$

where $r = b - A\tilde{x} = b - \tilde{b}$ | $K(A) = \|A\| \cdot \|A^{-1}\|$

$e = x - \tilde{x}$

proof

$b = Ax$

$\|b\| = \|Ax\| \leq \|A\| \cdot \|x\|$ $\quad (x \neq 0)$

$\rightarrow \frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$ .... ①

Also, $A\tilde{x} = b + \delta b$

$A(x + \delta x) = b + \delta b$

$Ax + A\delta x = b + \delta b$

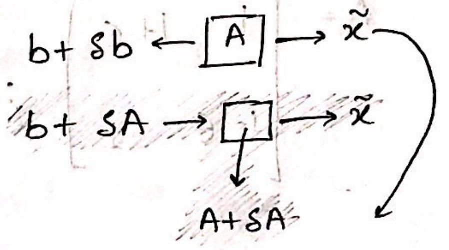
$\cancel{b} + A\delta x = \cancel{b} + \delta b$

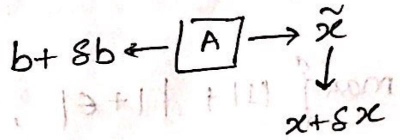
$\Rightarrow A\delta x = \delta b$

$\delta x = A^{-1}\delta b$

$\|\delta x\| = \|A^{-1}\delta b\|$

$\leq \|A^{-1}\| \|\delta b\|$ —— ②





$\tilde{x} = x + \delta x$

$\delta x = \tilde{x} - x$

$\tilde{b} = b + \delta b$

$\tilde{b} - b = \delta b$

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|b - \tilde{b}\|}{\|b\|} \text{ .... ③}$$

consider, $Ax = b \Rightarrow x = A^{-1}b$

$$\Rightarrow \|x\| \le \|A^{-1}\| \|b\|$$

$$\Rightarrow \frac{1}{\|A^{-1}\| \|b\|} \le \frac{1}{\|x\|} \quad \cdots\cdots \text{(4)}$$

$$\delta b = A\delta x$$

$$\|\delta b\| \le \|A\| \|\delta x\| \qquad \frac{\|\delta b\|}{\|A\|} \le \|\delta x\| \quad \cdots\cdots \text{(5)}$$

from (4) & (5)

$$\frac{1}{\|A\| \|A^{-1}\|} \frac{\|b - \tilde{b}\|}{\|b\|} \le \frac{\|x - \tilde{x}\|}{\|x\|} \quad \cdots\cdots \text{(6)}$$

Ex:

$$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix} : A^{-1} = \frac{1}{\epsilon^2}\begin{bmatrix} 1 & -1-\epsilon \\ -1+\epsilon & 1 \end{bmatrix} ; \epsilon > 0$$

$$\|A\|_\infty = \max\{|1| + |1+\epsilon|, |1-\epsilon| + |1|\} = 2+\epsilon$$

$$\|A^{-1}\|_\infty = \max\left\{\frac{|1| + |-1 \mp \epsilon|}{\epsilon^2}, \frac{|-1+\epsilon| + |1|}{\epsilon^2}\right\} = \frac{2+\epsilon}{\epsilon^2}$$

$$K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = \frac{(2+\epsilon)^2}{\epsilon^2}$$

case (i) : If $\epsilon = 0.01$

$$K_\infty(A) = \frac{(2.01)^2}{(0.01)^2} = 40401$$

$$\Rightarrow \frac{\|x - \tilde{x}\|}{\|x\|} \le 40401 \frac{\|b - \tilde{b}\|}{\|b\|}$$

case (ii) : If $\epsilon = 10$

$$K_\infty(A) = 1.44$$

$$\frac{\|x - \tilde{x}\|}{\|x\|} \le 1.44 \frac{\|b - \tilde{b}\|}{\|b\|}$$

If condition no. is high sol does not exist.

Ex

$$x_1 + (1+\epsilon)x_2 = 3 \longrightarrow m_1 = \frac{-1}{1+\epsilon}$$

$$(1-\epsilon)x_1 + x_2 = 5 \longrightarrow m_2 = \frac{\epsilon - 1}{1}$$

If $\epsilon = 0.01 \longrightarrow m_1 = -0.99$; $m_2 = -0.99$

If $\epsilon = 10 \longrightarrow m_1 = -0.09$ $m_2 = 9$

★ Power method to find numerically largest eigen value and the corresponding eigen vector.

Let $Ax = b$, $|A| \neq 0$ ..... ① : $x \in \mathbb{R}^n$

Let $x_1, x_2, x_3, \ldots, x_n$ be linearly independent eigen vectors corresponding to $\lambda_1, \lambda_2, \ldots \lambda_n$

Let λ_1 is a numerically largest eigen value

Let $x = c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$, $c_i \in \mathbb{R}$, $i = 1, \ldots n$

Let $Ax_i = \lambda_i x_i$, $i = 1, 2, \ldots n$, $x_i \neq 0$

$$Ax = A(c_1x_1 + c_2x_2 + \cdots + c_nx_n)$$

$$= c_1Ax_1 + c_2Ax_2 + \cdots + c_nAx_n$$

$$Ax = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n \text{ ——— ①}$$

$$A(Ax) = A(c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n)$$

$$A^2x = c_1\lambda_1^2x_1 + c_2\lambda_2^2x_2 + \cdots c_n\lambda_n^2x_n$$

$$\boxed{A^mx = c_1\lambda_1^mx_1 + c_2\lambda_2^mx_2 + \cdots + c_n\lambda_n^mx_n}$$

suppose

$x, Ax, A^2x, A^3x, \ldots$

$1, \lambda, \lambda^2, \ldots\ldots$

18/2

Matrix eigen value problem (power method)

Let $A \in \mathbb{R}^{n\times n}$, $\exists$ a basis such that $\Rightarrow$ $x_1, x_2, \dots x_n$

$$Ax_i = \lambda x_i \quad \cdots\cdots \textcircled{1}$$

$i = 1, 2, 3 \dots$

$\lambda_1, \lambda_2, \dots \lambda_n$

$|\lambda_1| > |\lambda_j|$

$j = 2, 3, \dots n$

Let

$$x^{(0)} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$Ax^{(0)} = c_1 A x_1 + c_2 A x_2 + \cdots c_n A x_n$$

$$x^{(1)} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots c_n \lambda_n x_n$$

$$\Longrightarrow Ax^{(1)} = x^{(2)} = c_1 \lambda_1^2 x_1 + c_2 \lambda_2^2 x_2 + \cdots + c_n \lambda_n^2 x_n$$

$\vdots$

$$Ax^{(m-1)} = x^{(m)} = c_1 \lambda_1^m x_1 + c_2 \lambda_2^m x_2 + \cdots + c_n \lambda_n^m x_n$$

$$= \lambda_1^m \Big[c_1 x_1 + \underbrace{c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^m x_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^m x_n}_{e^{(m)}} \Big]$$

$$\lim_{m\to\infty} x^{(m)} = \lambda_1^m c_1 x_1$$

Let $x^{(m)} = \lambda_1^m \left[c_1 x_1 + e^{(m)} \right]$ $\quad\Big|\quad x, x_1 \in \mathbb{R}^n$

$$x^{(m+1)} = \lambda_1^{m+1} \left[c_1 x_1 + e^{(m+1)} \right]$$

Define $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear function.

$$\phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y), \quad \alpha, \beta \in \mathbb{R}$$
$$x, y \in \mathbb{R}^n$$

~~Let $\phi(x) = \|x\|_\infty$~~

Let $\phi(z) = z_j$, $j = 1, 2, \dots n$

$z = (z_1, z_2, \dots, z_n)$

$$\phi(x^{(m)}) = \phi\left[\lambda_1^m (c_1 x_1 + e^{(m)})\right]$$

$$= \lambda_1^m \left[\phi (c_1 x_1 + e^{(m)})\right]$$

$$\phi(x^{(m+1)}) = \lambda_1^{m+1}\left[\phi(c_1 x_1 + e^{(m+1)})\right]$$

$$\lim_{m\to\infty} \frac{\phi(x^{(m+1)})}{\phi(x^{(m)})} = \lim_{m\to\infty} \lambda_1 \left[\frac{\phi(c_1 x_1 + e^{(m+1)})}{\phi(c_1 x_1 + e^{(m)})}\right] = \lambda_1$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, Let $x^{(0)} = (1, 1)$

$$Ax^{(0)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 5\underbrace{\begin{bmatrix} 3/5 \\ 1 \end{bmatrix}}_{x^{(1)}}$$

$$Ax^{(1)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} 13/5 \\ 19/5 \end{bmatrix} = \frac{19}{5}\begin{bmatrix} 13/19 \\ 1 \end{bmatrix}$$

$$Ax^{(2)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 13/19 \\ 1 \end{bmatrix} = \begin{bmatrix} 51/19 \\ 77/19 \end{bmatrix} = \frac{77}{19}\begin{bmatrix} 51/77 \\ 1 \end{bmatrix}$$

$$Ax^{(3)} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 51/77 \\ 1 \end{bmatrix} = \begin{bmatrix} 205/77 \\ 307/77 \end{bmatrix} = 3.98\begin{bmatrix} 0.667 \\ 1 \end{bmatrix}$$

Exact eigen vector : $x = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}}_{x} = \begin{bmatrix} 8/3 \\ 4 \end{bmatrix} = \underset{\lambda}{4}\underset{x}{\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}}$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-3x_1 + 2x_2 = 0$$

$$3x_1 = 2x_2$$

$$x_1 = 2/3\, x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

$$\boxed{Ax = \lambda x}$$

Note $x = (1, -1)$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}\underset{x}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underset{\lambda}{(-1)}\underset{x}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$\boxed{\left|\lambda^{(n+1)} - \lambda^{(n)}\right| < 10^{-4}}$$

$$\boxed{\begin{aligned} A^{-1}x^{(0)} &= x^{(1)} \\ x^{(0)} &= Ax^{(1)} \end{aligned}}$$

Note: To find the numerically smallest eigen value.

$$x^{(0)} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$A^{-1}x^{(0)} = c_1\frac{x_1}{\lambda_1} + c_2\frac{x_2}{\lambda_2} + \cdots + c_n\frac{x_n}{\lambda_n}$$

5/3

Weierstrass approximation theorem

Let $f:[a,b] \to R$ be continuous. Given $\epsilon > 0$ there exists n (depends only on ϵ) such that

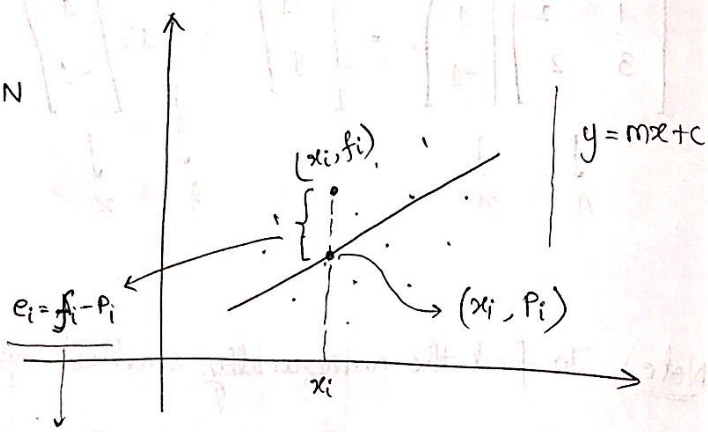
$$|f(x) - P_n(x)| < \epsilon, \quad \text{for all } x \in [a,b]$$

where P_n is a polynomial of degree $\leq n$

Any cont f^n on a bounded interval can be uniformly approximated by polynomials

$\{x_i, f(x_i)\}_{i=1,2,\cdots N}$

$P(x) = c_1 + c_2 x$



$$e_i = f_i - P_i$$

Error,

$$E_i = f(x_i) - p(x_i), \quad i = 1, 2, \cdots N$$

$$I(c_1, c_2) = \sum_{i=1}^{N} \left[E(x_i)\right]^2$$

$$= \sum_{i=1}^{N} \left[f(x_i) - P(x_i)\right]^2$$

$$= \sum_{i=1}^{N} \left[f(x_i) - (c_1 + c_2 x_i)\right]^2.$$

$$\frac{\partial I}{\partial c_1} = 0$$

$$\Rightarrow \sum_{i=1}^{N} 2\left\{f(x_i) - (c_1 + c_2 x_i)\right\} \times (-1) = 0$$

$$\frac{\partial I}{\partial c_2} = 0 \Rightarrow \sum_{i=1}^{N} 2[f(x_i) - (c_1 + c_2 x)] \times (-x_i) = 0$$

$$\sum_{i=1}^{N} f(x_i) = N c_1 + c_2 \sum_{i=1}^{N} x_i$$

$$\sum_{i=1}^{N} x_i f(x_i) = c_1 \sum_{i=1}^{N} x_i + c_2 \sum_{i=1}^{N} x_i^2$$

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} f(x_i) \\ \sum_{i=1}^{N} x_i f(x_i) \end{bmatrix}$$

Note: $p(x) = c_1 + c_2 x + c_3 x^2$
$x \in [a,b]$

Note: $p(x) = c_1 + c_2 x + c_3 x^2 \quad x \in [a,b]$

$$\begin{bmatrix} N & \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 \\ \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} f(x_i) \\ \sum_{i=1}^{N} x_i f(x_i) \\ \sum_{i=1}^{N} x_i^2 f(x_i) \end{bmatrix}$$

Ex: Find a least square straight line and quadratic polynomial to the following data

x	-0.5	1	1.5	2	2.5
$f(x)$	0.75	3	4.75	7	9.75

case (i) $p(x) = c_1 + c_2 x$

$$\begin{bmatrix} 5 & 6.5 \\ 6.5 & 13.75 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 25.25 \\ 48.125 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 1.2972. \\ c_2 = 2.8868 \end{matrix}$$

case(ii) $p(x) = c_1 + c_2 x + c_3 x^2$

$$\begin{bmatrix} 5 & 6.5 & 13.75 \\ 6.5 & 13.75 & 27.87 \\ 13.75 & 27.87 & 61.1875 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 25.25 \\ 48.125 \\ 102.8125 \end{bmatrix}$$

Let $f:[a,b] \longrightarrow \mathbb{R}$ be continuous

case (i) $p(x) = c_1 + x c_2$, $x \in [a,b]$

$E(x) = f(x) - p(x)$, $x \in [a,b]$

$\not\exists$

$$I(c_1, c_2) = \int_a^b [f(x) - p(x)]^2 \, dx$$

$$= \int_a^b \{ f(x) - (c_1 + c_2 x)]^2 \, dx$$

$$\frac{\partial I}{\partial c_1} = \int_a^b 2 [f(x) - (c_1 + c_2 x)] \times (-1) \, dx$$

$$\frac{\partial I}{\partial c_2} = 0 = \int_a^b 2 [f(x) - (c_1 + c_2 x)] \times (-x) \, dx$$

$$\begin{bmatrix} b-a & \int_a^b x \, dx \\ \int_a^b x \, dx & \int_a^b x^2 \, dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_a^b f(x) \, dx \\ \int_a^b x f(x) \, dx \end{bmatrix}$$

Note: $p(x) = c_1 + c_2 x + c_3 x^2 \, , \; x \in [a, b]$

$$\begin{bmatrix} b-a & \int_a^b x \, dx & \int_a^b x^2 \, dx \\ \int_a^b x \, dx & \int_a^b x^2 \, dx & \int_a^b x^3 \, dx \\ \int x^2 \, dx & \int x^3 \, dx & \int x^4 \, dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \int_a^b f(x) \, dx \\ \int_a^b x f(x) \, dx \\ \int_a^b x^2 f(x) \, dx \end{bmatrix}$$

Exe: $f(x) = x^3$, $x \in [0,1]$

a) find $p(x) = c_1 + c_2 x$

b) find $p(x) = c_1 + c_2 x + c_3 x^2$

Sol:
$$\begin{bmatrix} \int_0^1 dx & \int_0^1 x\,dx \\ \int_0^1 x\,dx & \int_0^1 x^2\,dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x)\,dx \\ \int_0^1 x f(x)\,dx \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/5 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = -\frac{2}{10} \\ c_2 = \frac{9}{10} \end{array}$$

$$\Longrightarrow p(x) = c_1 + c_2 x = -\frac{2}{10} + \frac{9x}{10} = \frac{1}{10}(9x-2)$$

Note $f(0) = 0$ $p(0) = -\frac{2}{10}$ | $f\left(\frac{1}{2}\right) = \frac{1}{8}$

$f(1) = 1$ $p(1) = \frac{7}{10}$ | $p\left(\frac{1}{2}\right) = \cancel{\frac{16}{10}}\ \frac{5}{20}$

b) $p(x) = c_1 + c_2 x + c_3 x^2$, $x \in [0,1]$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \end{bmatrix}$$

$c_1 = 0.05$, $c_2 = -0.6$, $c_3 = 1.5$

$p(x) = 0.05 - 0.6x + 1.5x^2$, $x \in [0, 1]$

$f(\frac{1}{2}) = \frac{1}{8} = 0.125$ | $f(0) = 0$ | $f(1) = 1$

$p(\frac{1}{2}) = 0.125$ | $p(0) = 0.05$ | $P(1) =$?

$H = \frac{1}{i+j-1}$, $i, j = 1, 2, \ldots N$

$\text{cond}(H_3) = \|H_3\| \, \|H_3^{-1}\| = 748$

$\rightarrow \frac{\|x - \tilde{x}\|}{\|x\|} \leq 748 \frac{\|b - \tilde{b}\|}{\|b\|}$

$I = AA^{-1}$

$\|I\| \leq \underbrace{\|A\| \, \|A^{-1}\|}$

$1 \leq k(A)$

orthonormal basis

$\{\phi_1, \phi_2, \ldots \phi_n\} \in \{-1, 1\}$ $x \in \mathbb{R}$

$<\phi_i, \phi_j> = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

$<\phi_i, \phi_j> = \int_{-1}^{1} w(x) \, \phi_i(x) \, \phi_j(x) \, dx$, $w(x) > 0$

Legendre polynomials

$\phi_0 \quad \phi_1 \quad \phi_2$

$\{ 1 \quad t \quad \frac{1}{3}(3t^2 - 1) \ldots \}$ $w(t) = 1$

$\int_{-1}^{1} \phi_i(t) \, \phi_j(t) \, dt = \begin{cases} \frac{2}{2i+1}, & i = j \\ 0, & i \neq j \end{cases}$

✓ $i=j=0$ $\int_{-1}^{1} \phi_0^2(t)\,dt = \int_{-1}^{1} 1\,dt = \cancel{\frac{2}{3}}\ 2$

✓ $i=j=1$ $\int_{-1}^{1} \phi_2^2(t)\,dt = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$

Ex $f(x) = x^3,\ x \in [0,1]$ | $f(x) = ,\ x \in [a,b]$

$[a,b] \rightleftharpoons [-1,1]$

$x = c_1 t + c_2$

If $x = a \to t = -1 \Rightarrow a = -c_1 + c_2$

$x = b \Rightarrow t = 1 \Longrightarrow b = c_1 + c_2$

$c_1 = \frac{b-a}{2}$

$a + b = 2c_2$

$c_2 = \frac{1}{2}(a+b)$

$x = \left(\frac{b-a}{2}\right)t + \frac{1}{2}(a+b)$

$f(x) = x^3,\ x \in [0,1]$

$x = \frac{1}{2}t + \frac{1}{2}$

$g(t) = f\left(\frac{1}{2}t + \frac{1}{2}\right) = \frac{1}{8}(t+1)^3,\quad t \in [-1,1]$

Let $p(t) = c_1 \phi_0(t) + c_2 \phi_1(t) + c_3 \phi_2(t)$

$$E(x) = g(t) - p(t) \quad , \quad x \in [-1, 1]$$

$$I(c_1, c_2) = \int_{-1}^{1} [g(t) - p(t)]^2 \, dt$$

$$\int_{-1}^{1} [g(t) - c_1 \phi_0(t) - c_2 \phi_1(t) - c_3 \phi_2(c)]^2 \, dt$$

$$\left.\begin{array}{l} \frac{\partial I}{\partial c_1} = 0 \\ \\ \frac{\partial I}{\partial c_2} = 0 \\ \\ \frac{\partial I}{\partial c_3} = 0 \end{array}\right\} \begin{bmatrix} \int_{-1}^{1} \phi_0^2 dt & 0 & 0 \\ 0 & \int_{-1}^{1} \phi_1^2 dt & 0 \\ 0 & 0 & \int_{-1}^{1} \phi_2^2 dt \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} g(t)\phi_0(t) dt \\ \int_{-1}^{1} g(t)\phi_1(t) dt \\ \int_{-1}^{1} g(t)\phi_2(t) dt \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/10 \\ 1/5 \end{bmatrix}$$

$$\int_{-1}^{1} g(t)\phi_0(t) = \int_{-1}^{1} \frac{1}{8} (t+1)^3 dt =$$

$$c_1 = \frac{1}{4} \quad ; \quad c_2 = \frac{9}{20} \quad , \quad c_3 = \frac{1}{\cancel{2}4}$$

$$p(t) = \frac{1}{4} + \frac{9}{20} t + \frac{1}{\cancel{2}4} \times \frac{1}{2} (3t^2 - 1)$$

since $t = 2x - 1$

$$p(x) = \frac{1}{4} + \frac{9}{20} (2x - 1) + \frac{1}{8} [3(2x-1)^2 - 1]$$

$$= \frac{1}{4} + \frac{9}{20} (2x - 1) + \frac{1}{8} [3(4x^2 + 1 - 4x) - 1]$$

$$= \cancel{\frac{3}{20}} + \cancel{\frac{-21}{10}x} + \cancel{3x^2}$$

$$= \frac{3}{10} + \frac{-21}{10}x + 3x^2$$

$$p(x) = 0.05 - 0.6x + 1.5x^2$$

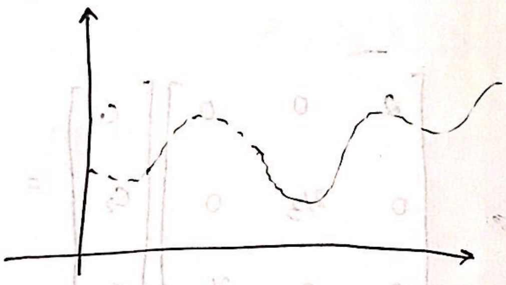
$$\int_{-1}^{1} \phi_i(x)\,\phi_j(x)\,dx = \begin{cases} \frac{2}{2i+1}, & i \neq j \\ 0, & i \neq j \end{cases}$$

~~7/3~~

Interpolation

Let

$\{x_i, f(x_i)\}$, $i = 0, 1, 2, \dots n$



Let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots c_n x^n$$

Interpolation conditions $f(x_i) = p(x_i)$, $i = 0, 1, 2 \dots n$

At $\underline{x = x_0}$ $\quad p(x_0) = f(x_0)$

$$c_0 + c_1 x_0 + c_2 x_0^2 + \dots + c_n x_0^n = f(x_0)$$

$\underline{x = x_1}$

$$c_0 + c_1 x_1 + c_2 x_1^2 + \dots + c_n x_1^n = f(x_1)$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \\ f(x_n) \end{bmatrix}$$

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \end{bmatrix}$$

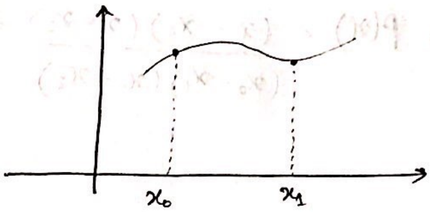
$$x_1 - x_0 \neq 0$$

the points $\{x_0, x_1, \dots, x_n\}$ are distinct

Ex: $V_3 = \begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix}$, $K(V_3) = 21 \times 17$

Lagrange's form

$\{(x_0, f(x_0)), (x_1, f(x_1))\}$



$p(x) = c_0 + c_1 x$

$$\left.\begin{array}{l} p(x_0) = f(x_0) \Rightarrow c_0 + c_1 x_0 = f(x_0) \\ p(x_1) = f(x_1) \Rightarrow c_0 + c_1 x_0 = f(x_1) \end{array}\right\} \quad c_0 = \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1} \quad c_1 = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$p(x) = c_0 + c_1 x$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1} + \frac{[f(x_0) - f(x_1)]}{x_0 - x_1} \times x$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1} + \frac{x f(x_0) - x f(x_1)}{x_0 - x_1}$$

$$p(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x_0 - x}{x_0 - x_1}\right) f(x_1)$$

$$P(x) \Longrightarrow \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1)$$

Note: $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

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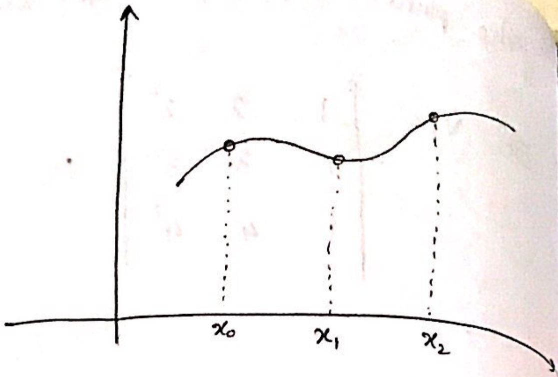
$l_0(x_0) = 1 \qquad l_1(x_1) = 1$



$$\rightarrow P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2) f(x_1)}{(x_1-x_0)(x_1-x_2)}$$

$$+ \frac{(x-x_0)(x-x_1) f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

In general

$$p(x) = \sum_{i=0}^{n} l_i(x) f(x_i), \qquad l_i(x) = \prod_{\substack{j=0 \\ (j \neq i)}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Note : $\sum_{i=0}^{n} l_i(x) = 1$

Ex: Find an interpolating polynomial for following data using legrangian method.

$$\left.\begin{array}{llll} x & : & 1 & 2 \\ f(x) & : & 2 & 5 \end{array}\right\} \quad (1, 2) \quad (2, 5)$$

x_0, f_0 $\quad$ x_1, f_1

$$p(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$\frac{x-2}{-1} \times 2 + \frac{x-1}{1} \times 5 = 3x-1$$

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Ex-2

$x =$	1	2	4
$f(x) =$	2	5	17

$$p(x) = \frac{(x-2)(x-4)}{-1 \times -3} \times 2 + \frac{(x-1)(x-4)}{1 \times -2} \times 5 + \frac{(x-1)(x-2)}{3 \times 2} \times 17$$

$$p(x) = \frac{x^2 - 6x + 8}{3} \times 2 + \frac{x^2 - 5x + 4}{-2} \times 5 + \frac{x^2 - 3x + 2}{6} \times 17$$

$$p(x) = x^2 + 1$$

problem sheet - 5
syllabus to minor II.