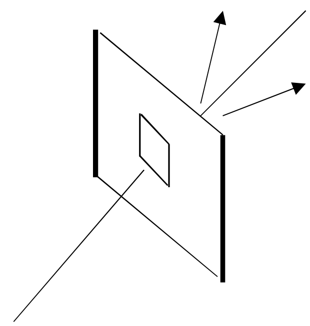
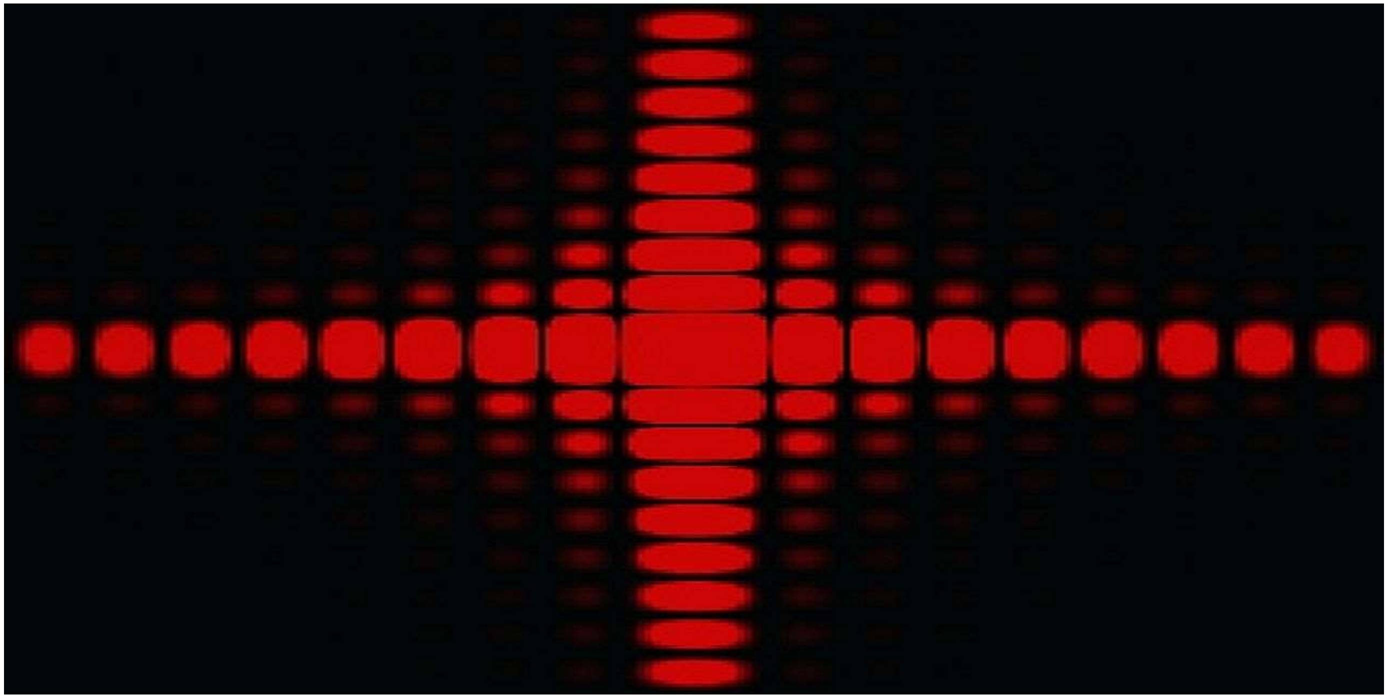
PH203: Optics

Lecture #8

22.11.2018

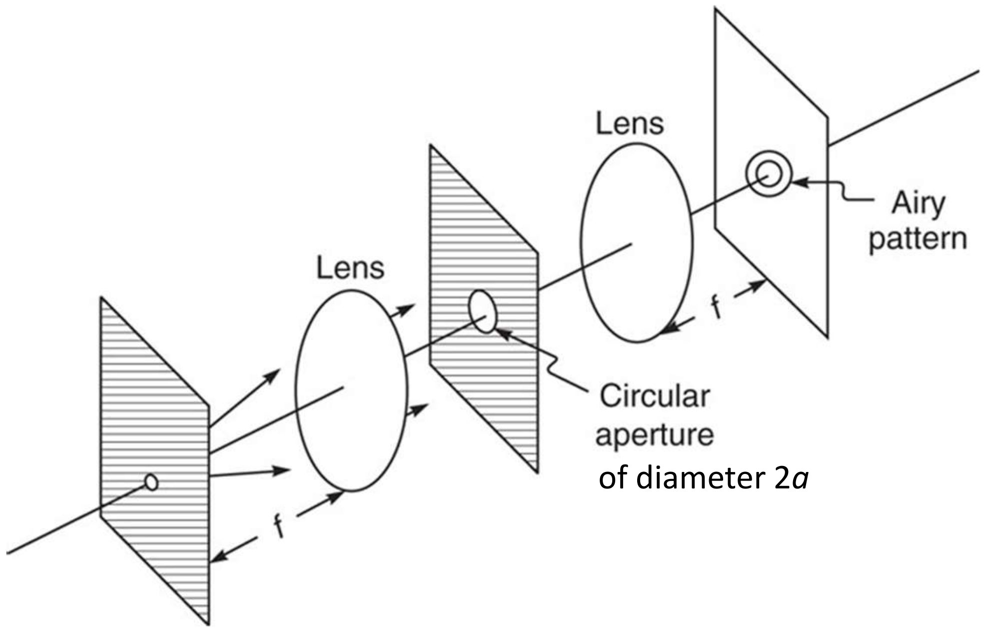
Diffraction due to a rectangular aperture

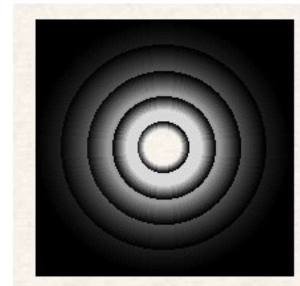




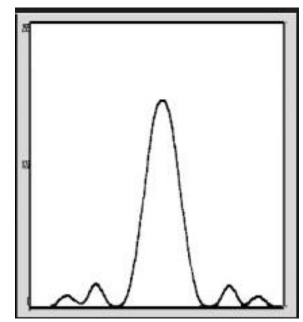
Fraunhofer diffraction pattern due to a rectangular aperture

Fraunhofer diffraction by a circular aperture:





Airy pattern



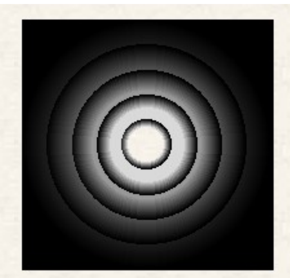
Diffraction pattern in this case involves cylindrical function of Bessel function $J_1(\nu)$, which represents solution to a second order differential eq

We quote here final result for the diffracted intensity pattern:

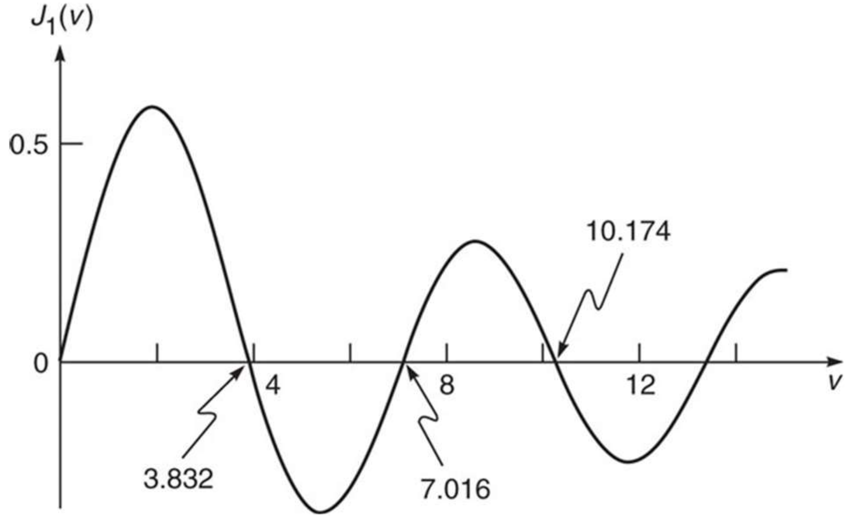
where
$$I = I_0\left[\frac{2J_1(v)}{v}\right]^2;$$ $J_1(\nu)$ is Bessel function of order 1

$$v = a\,k\sin\theta = \frac{2\pi a}{\lambda_0}\sin\theta;$$ θ is the angle of diffraction; a is the radius of the circular aperture

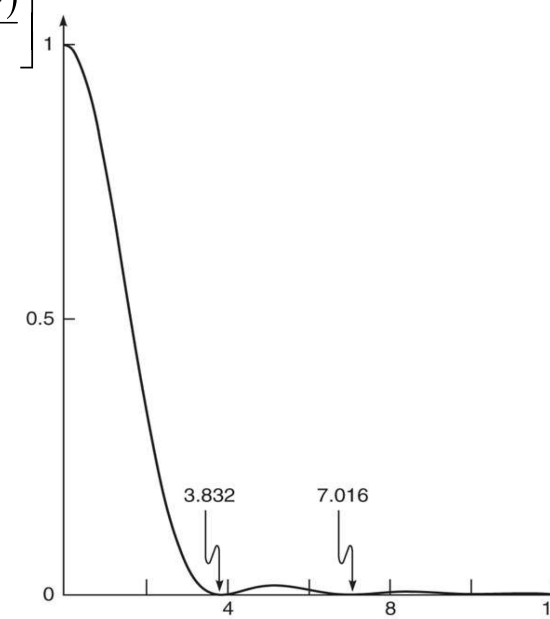
$$I = I_0\left[\frac{2J_1(v)}{v}\right]^2$$



Airy pattern



$$\frac{I}{I_0} = \left[\frac{2J_1(v)}{v}\right]^2$$



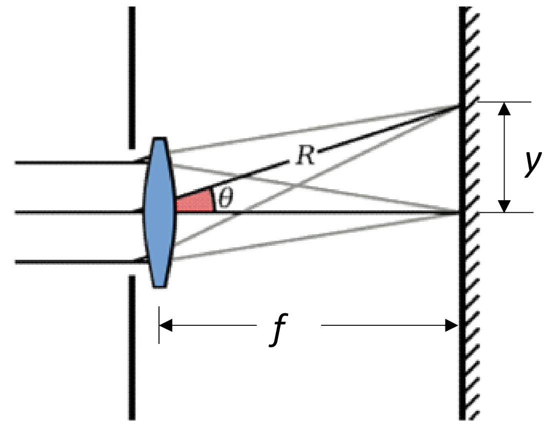
Roots of $J_1(v) = 0; \;\; v = 3.832, 7.106, 10.174, \cdots$

$\Rightarrow$ Successive dark rings in the Airy pattern appear for above values of $v = \dfrac{2\pi}{\lambda} a \sin\theta$

$$\Rightarrow \quad \sin\theta = \frac{3.832\,\lambda_0}{2\pi a}, \frac{7.016\,\lambda_0}{2\pi a}, \cdots$$

From detailed calculations , it can be shown that ~ 84% of the diffracted light is contained within the 1st dark ring!

$$\sin\theta = \frac{3.832\lambda_0}{2\pi a}, \frac{7.016\lambda_0}{2\pi a}, \ldots\ldots$$



$$\theta \approx \sin\theta \cong \tan\theta = \frac{y}{f}$$

$$\Rightarrow \quad y = f\tan\theta \approx \frac{3.832\lambda_0}{2\pi a}f, \frac{7.016\lambda_0}{2\pi a}f, \ldots\ldots$$

or

$$y \approx \underbrace{\frac{3.832\lambda_0}{\pi D}f}_{?}, \frac{7.016\lambda_0}{\pi D}f, \ldots\ldots$$

radius of the 1st dark ring in an Airy disc Likewise, 2nd term: radius of the 2nd dark ring

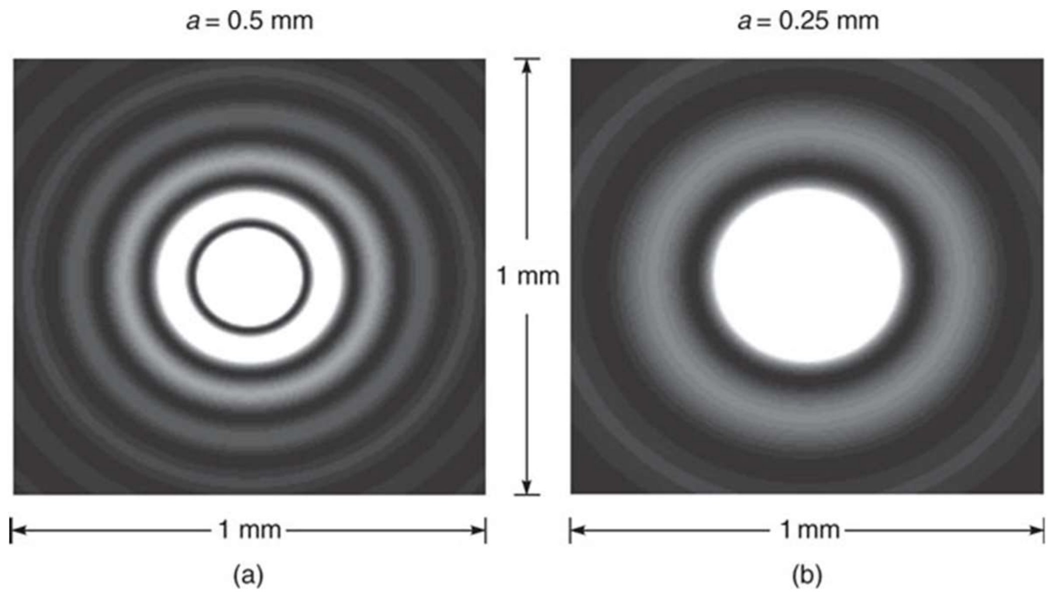
Examples

a = 0.5 mm, λ = 500 nm, and *f* = 20 cm yield $y_1 = 0.12$ mm

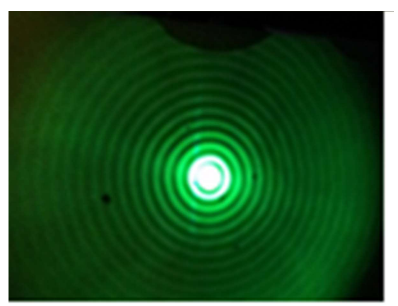
a = 0.25 mm, λ = 500 nm, and *f* = 20 cm yield $y_1 = 0.24$ mm

Important conclusion:

Smaller the diffracting aperture $\Rightarrow$ greater is the diffraction divergence



Computer generated Airy pattern at the focal plane of a lens of f = 20 cm and λ = 0.5 μm for two different apertures of different diameter



Airy disc pattern due to a pinhole taken with a green laser light

Since almost 84% of diffracted energy is contained within the 1st dark ring, it can be said that angular spread of the diffracted light is

$$\Delta\theta \approx \frac{3.832}{\pi}\frac{\lambda_0}{D} \approx 1.22\frac{\lambda_0}{D};\ \underbrace{D}_{\text{dia}} = 2\underbrace{a}_{\text{radius}} \qquad \Rightarrow \quad \Delta\theta \approx 0.61\frac{\lambda_0}{a}$$

In general,

$$\Delta\theta \approx \frac{\text{wavelength}}{\text{characteristic dimension of the aperture}}$$

Diffraction-limited spot? $\Delta\theta \times f = ?$ Spot size at the focus

For most laser beams, transverse amplitude distribution is typically Gaussian in shape: $\psi(x,y) = Ae^{-\left[\frac{x^2+y^2}{w_0^2}\right]}$;

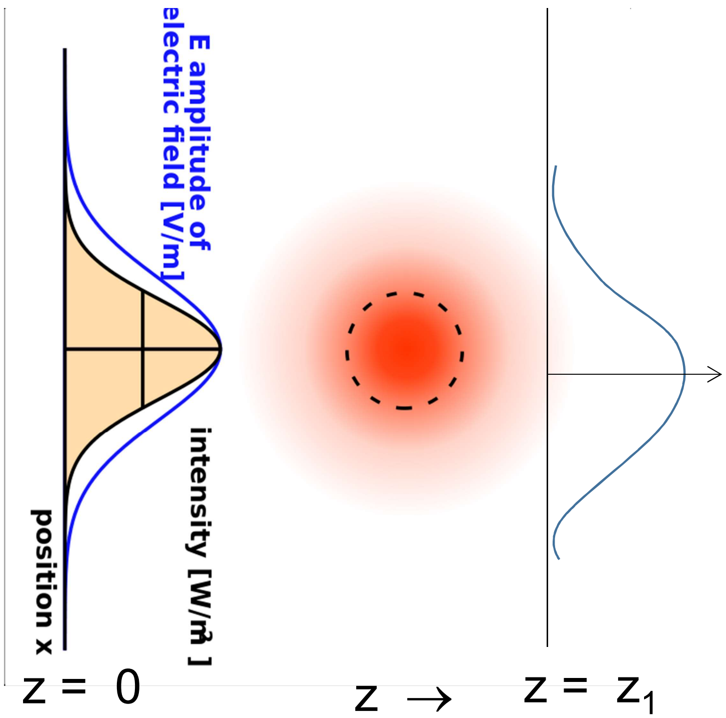
for $x^2 + y^2 = r^2 = w_0^2$,

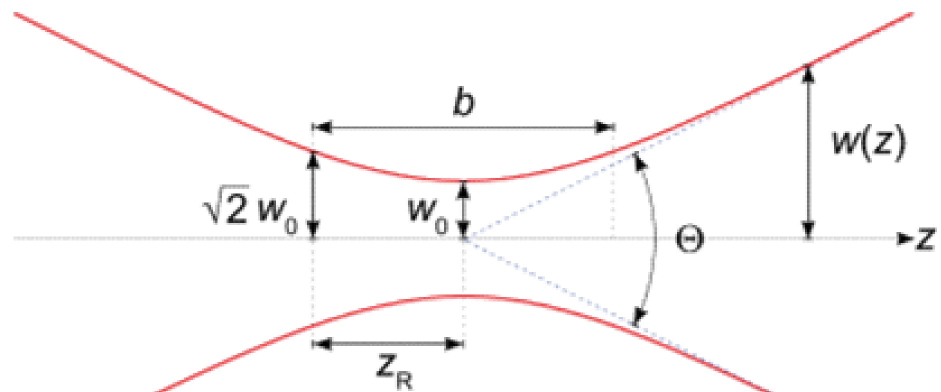
$$\psi(x,y) = Ae^{-1} = \frac{A}{e} \quad \Rightarrow \quad |\psi(x,y)|^2 = A^2 e^{-2\left(w_0^2/w_0^2\right)} = \frac{A^2}{e^2}$$

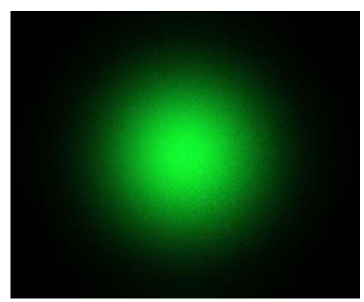
w_0 is called spot size of the beam, for which the amp falls by 1/e of its max (at (x,y) = (0,0)) and correspondingly its intensity drops by a factor $1/e^2$ of its max (at (x,y) = (0,0))

Transverse field and intensity profile of a Gaussian beam (TEM_{00} mode) from a red laser e.g. He-Ne laser

The profile its Gaussian shape with propagation except for an increase in its spot size: $w(z) \approx w_0 \left(1 + \frac{\lambda_0^2 z^2}{\pi^2 w_0^4}\right)^{1/2}$







Snapshot of intensity as recorded
from a Green laser pointer

$$w(z) \approx w_0 \left(1 + \frac{\lambda_0^2 z^2}{\pi^2 w_0^4}\right)^{1/2}$$

For large *z*

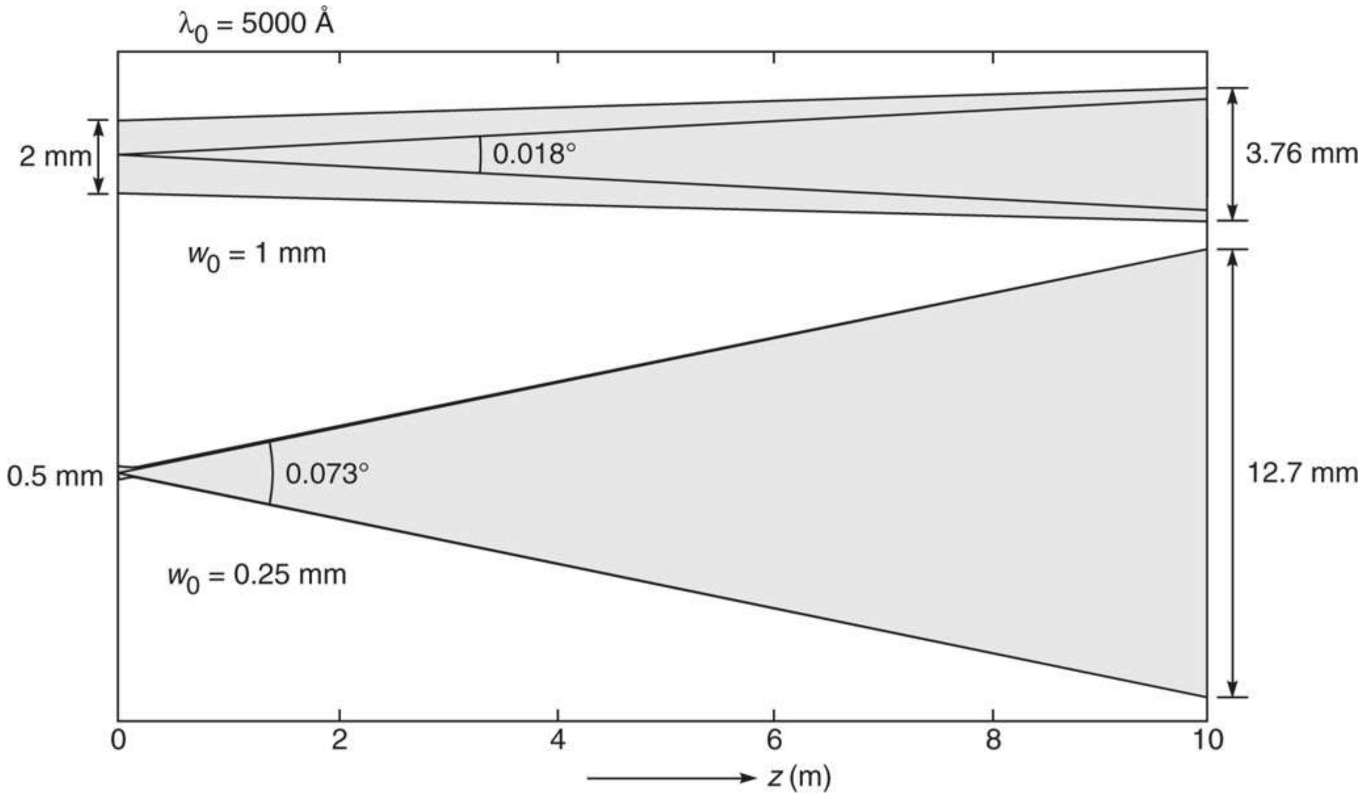
$$\underbrace{z \gg \frac{w_0^2}{\lambda_0}}$$

Condition for going from Fresnel (near field) to Fraunhofer domain

$$w(z) \approx \cancel{w}_0 \frac{\lambda_0 z}{\pi w_0^{\cancel{2}}} \approx \frac{\lambda_0}{\pi} \frac{z}{w_0}$$

$\Rightarrow$ Laser beam width increases linearly with *z* with a divergence:

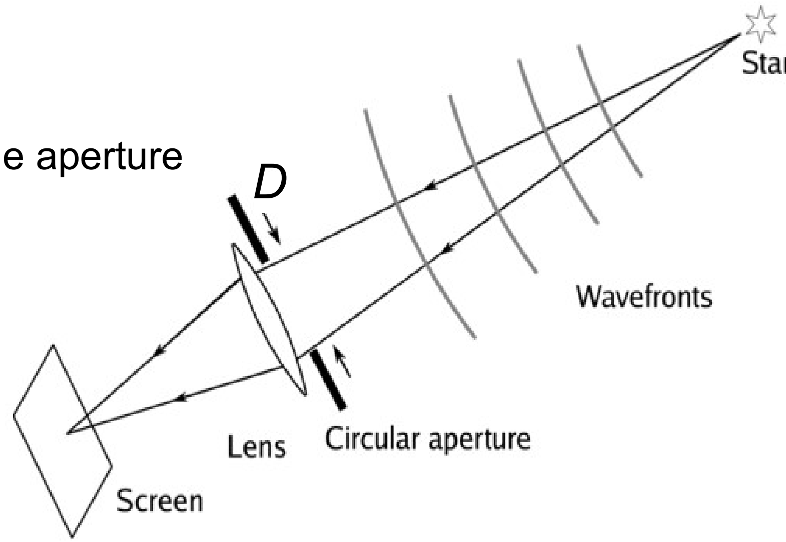
$$\tan\theta \approx \frac{w(z)}{z} \approx \frac{\lambda_0 \cancel{z}}{\cancel{z} \pi w_0} = \frac{\lambda_0}{\pi w_0}$$

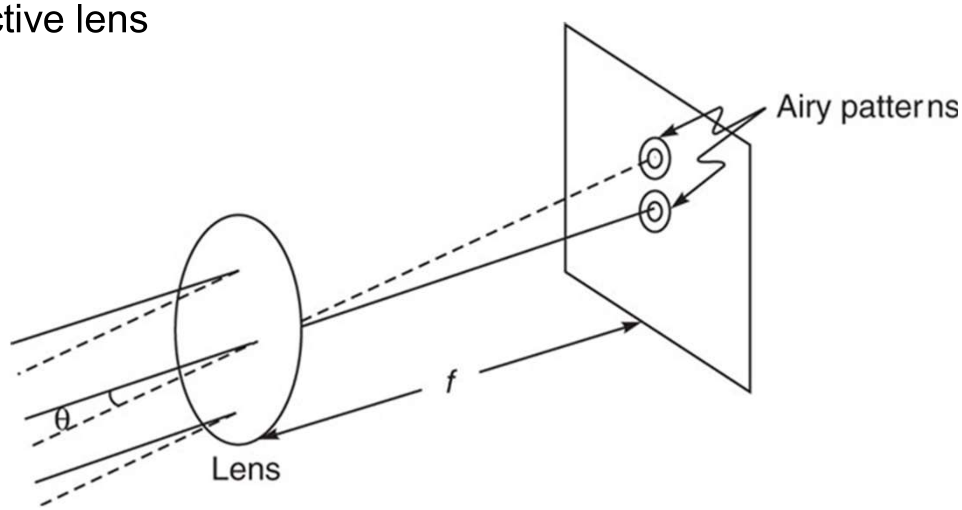


$\Rightarrow$ Smaller the aperture greater is the divergence because $\Delta\theta \sim \dfrac{\lambda}{D}$

Limit of resolution

- Let us consider resolving two stars by a telescope
- Here the circular aperture (dia D) behind the objective lens acts as the aperture
- At the focus each of these point sources (being so far away) from earth will form its own Airy pattern
- Diameter of the Airy disc will depend on λ, f, and diameter of the objective lens



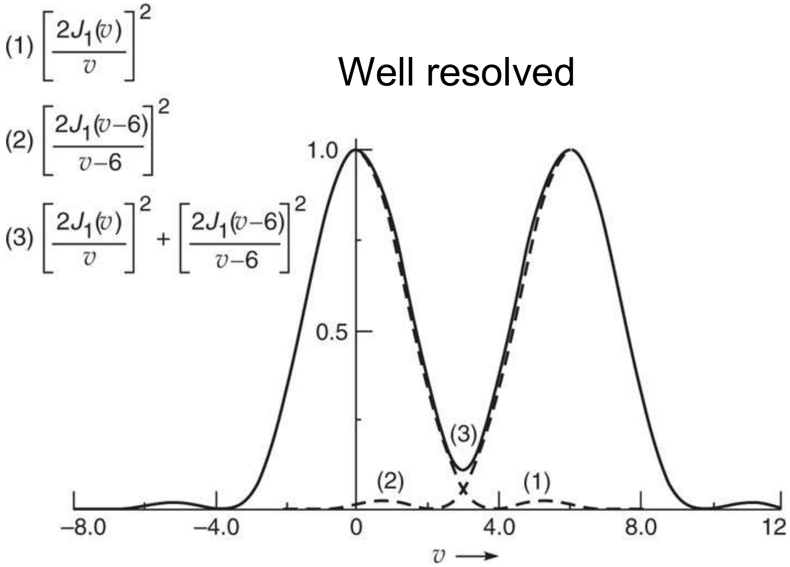


Angular spread of the 1st dark ring, which contains 84% of the diffracted energy is: $\approx 1.22\dfrac{\lambda}{D}$

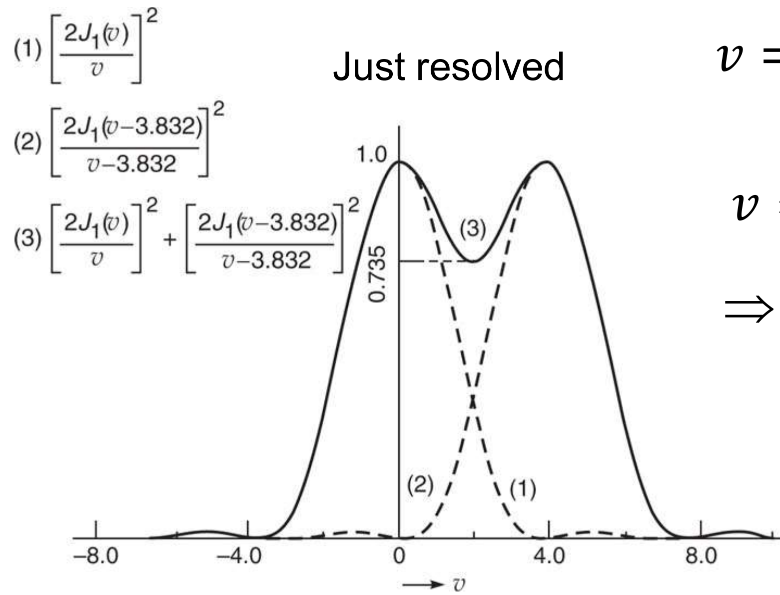
$\Rightarrow$ Larger the value of aperture, smaller will be the spread of the diffracted light

$\Rightarrow$ For higher resolution, one chooses a telescope of larger objective diameter

Well resolved



Just resolved

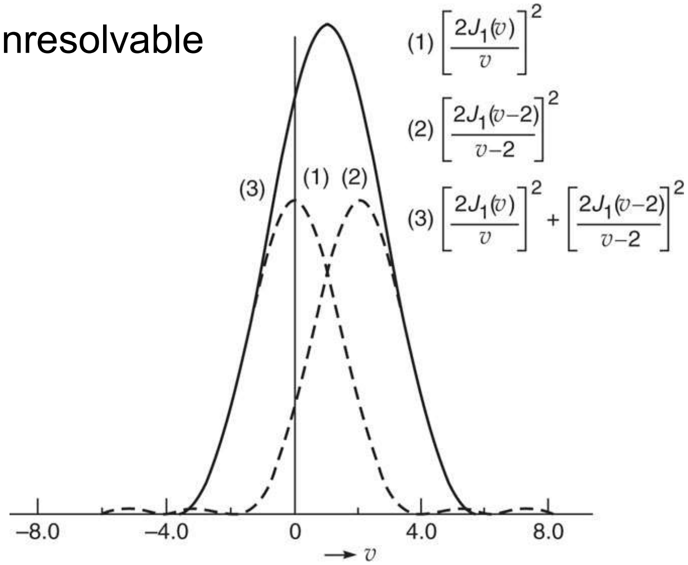


$$v = \frac{2\pi}{\lambda} a \sin\theta$$

$$v \approx \frac{\pi D}{\lambda} \times \frac{\Delta\theta}{2}$$

$$\Rightarrow \Delta\theta \sim v \frac{2}{\pi} \frac{\lambda}{D}$$

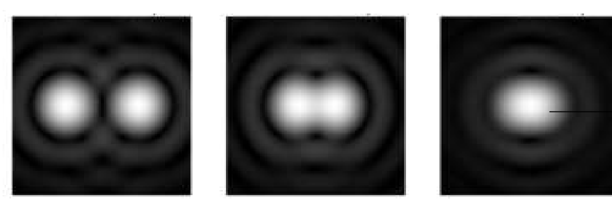
unresolvable



Just resolved

$\Delta\theta$ =

2.0 λ / D 1.0 λ / D 0.5 λ / D



unresolvable

Well resolved

Computer generated intensity patterns at the focal plane of a telescope of a pair of stars of different angular separation

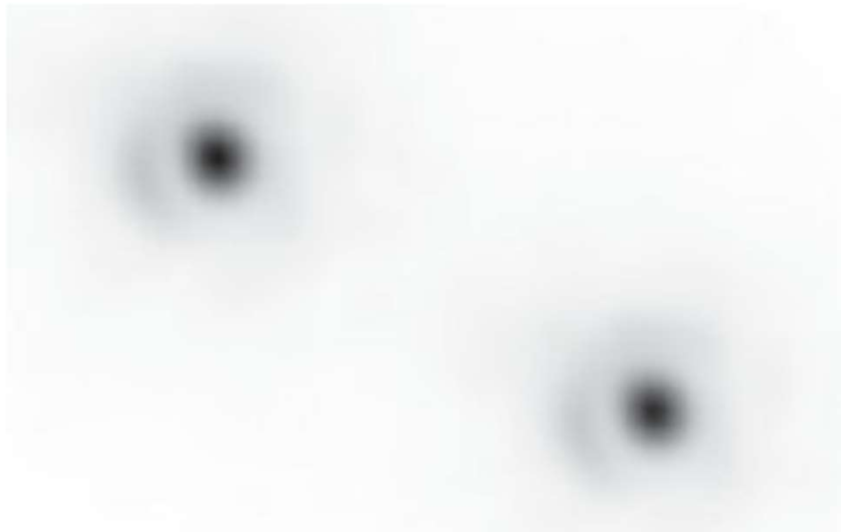


Image of a binary star Zeta Bootis (approxly 180 light years away from earth) by a 2.56 meter telescope aperture (taken by Dr. Bob Tubbs)

If angular resolution of human eye is primarily due to diffraction effects, then

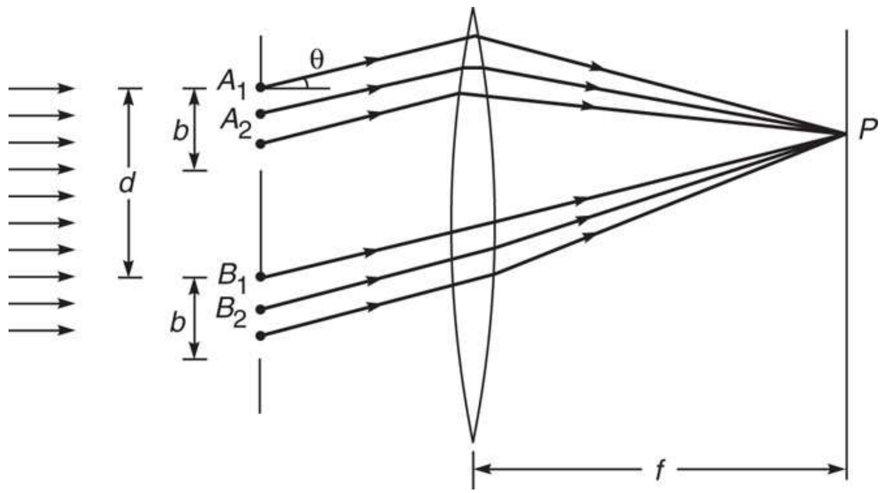
$$\Delta\theta \sim \frac{\lambda}{D} \approx \frac{6 \times 10^{-5}}{2 \times 10^{-1}} = 3 \times 10^{-4} \text{ radian}$$

Thus if we have to resolve two objects e.g. two motorbikes/vehicles at a distance of 20 m, then they must be separated by a separation of

$$\Delta\theta \times 20 = 3 \times 10^{-4} \times 20 \text{ m} = 6 \text{ mm}$$

In a Phys lab, one can qualitatively verify this by finding the distance at which a mm scale becomes blurred!

Two slit diffraction pattern:



As before assume each slit of width b consists of a large number of equally spaced (Δ) point sources and the separation between the two slits is d

Path difference between the light reaching *P* from two consecutive point sources is $\Delta \sin\theta \Rightarrow E_1 = A\frac{\sin\beta}{\beta}\cos(\omega t - \beta)$

Likewise, $E_2 = A\frac{\sin\beta}{\beta}\cos(\omega t - \beta - \Phi_1)$; $\Phi_1 = \frac{2\pi}{\lambda} d\sin\theta$

Φ_1 is the phase diff between two corresponding points in the two slits; $\beta = \frac{\pi b \sin\theta}{\lambda}$

$\Rightarrow$ pairs of points: (A_1, B_1), (A_2, B_2), …., two points in each pair is separated by *d*

At *P* resultant *E*: $E = E_1 + E_2 \ \Rightarrow\ E = A\frac{\sin\beta}{\beta}\left[\cos\left(\omega t - \beta\right) + \cos\left(\omega t - \beta - \Phi_1\right)\right]$

Using

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \quad \Rightarrow \quad E = 2A\frac{\sin\beta}{\beta}\cos\gamma\cos\left(\omega t - \beta - \frac{\Phi_1}{2}\right)$$

$$\Rightarrow E = 2A\frac{\sin\beta}{\beta}\cos\gamma\cos\left(\omega t - \frac{\beta}{2} - \frac{\Phi_1}{2}\right); \Phi_1 = \frac{2\pi}{\lambda}d\sin\theta \quad \gamma = \frac{\Phi_1}{2} = \frac{\cancel{2}\pi}{\cancel{2}\lambda}d\sin\theta = \frac{d\sin\theta}{\lambda}$$

$$\therefore\ I = I_0 \frac{\sin^2\beta}{\beta^2}\cos^2\gamma;$$ I_0 is intensity at θ = 0; 1st term represents diffraction pattern due to single slit and the 2nd term represents interference pattern due to two point sources

Positions of maxima and minima:

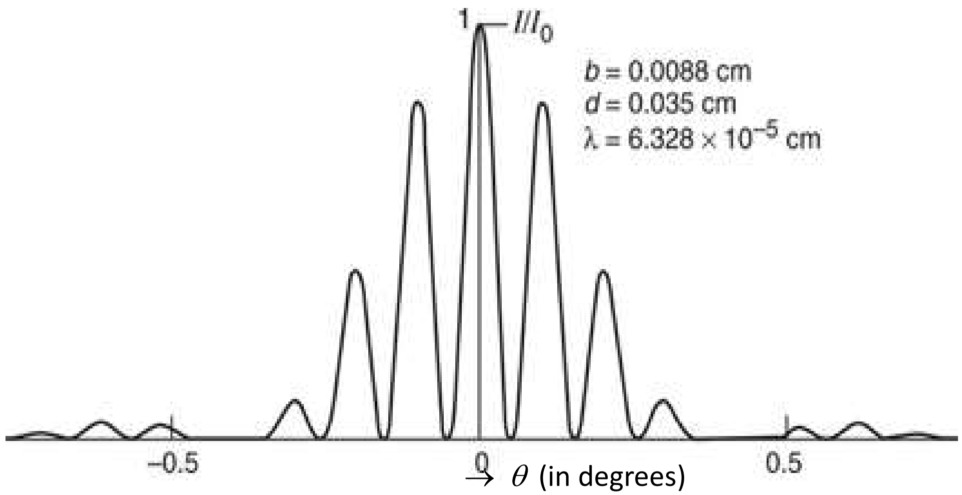
$$\left.\begin{aligned} &\beta = m\pi, m = 1,2,3, \ldots\ldots \\ \text{or} \qquad &\gamma = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots. \end{aligned}\right\} \quad \text{Intensity will be zero i.e. minima}$$

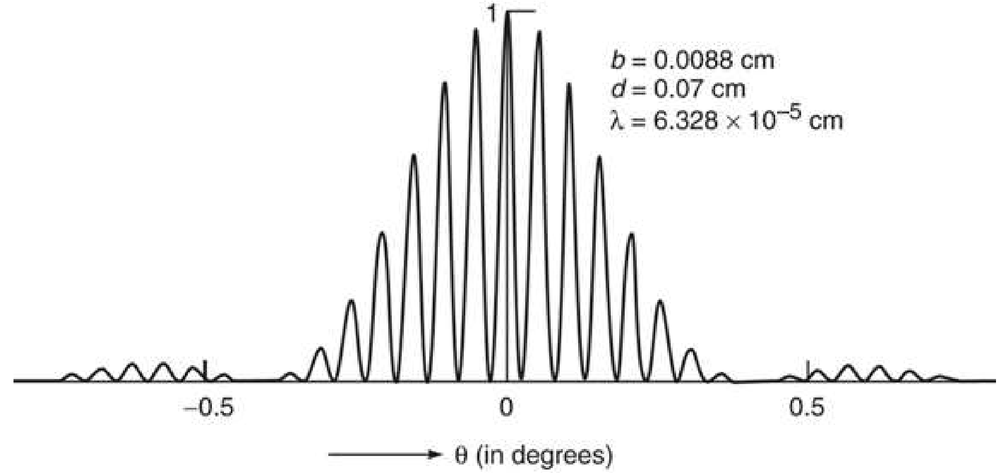
1st condition implies $b \sin\theta = m\lambda, m = 1,2,3, \ldots\ldots$

2nd condition implies $d \sin\theta = \left(p + \frac{1}{2}\right)\lambda, p = 0,1,2, \ldots.$

Maxima of interference pattern occurs whenever $\gamma = 0, \pi, 2\pi, \ldots \Rightarrow d \sin\theta = 0, \lambda, 2\lambda, \ldots.$

A maximum may not occur at all, if θ corresponds to $b \sin\theta = \lambda, 2\lambda, 3\lambda, \ldots\ldots$



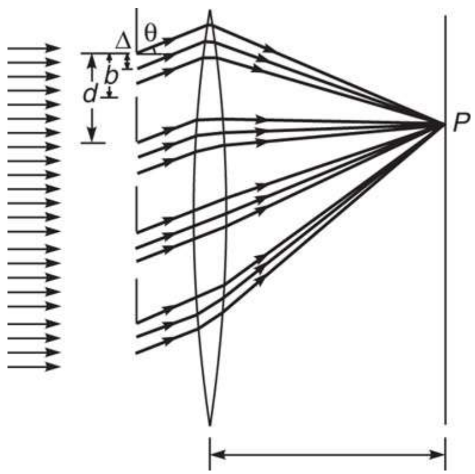


e.g. for b = 0.0088 cm, around $\theta \approx 0.41^\circ$ interference maxima are extremely weak

because

$$\theta = \sin^{-1}\left(\frac{b}{\lambda}\right) \cong 0.412^\circ$$

Diffraction due to a grating ⇒ *N*-slits:



$$E = A\frac{\sin\beta}{\beta}[\cos(\omega t-\beta)+\cos(\omega t-\beta-\Phi_1)+\cdots+\cos(\omega t-\beta-(N-1)\Phi_1)]$$

$$= A\frac{\sin\beta}{\beta}\frac{\sin N\gamma}{\sin\gamma}\cos\left(\omega t-\beta-\frac{1}{2}(N-1)\Phi_1\right)$$

where $$\gamma = \frac{\Phi_1}{2} = \frac{\pi}{\lambda}d\sin\theta$$

As seen earlier in interference,

$$\Rightarrow \quad I = I_0\frac{\sin^2\beta}{\beta^2}\frac{\sin^2 N\gamma}{\sin^2\gamma}$$

I_0 is intensity along $\theta = 0$

$\dfrac{\sin^2\beta}{\beta^2}$: Single slit diffraction

$\dfrac{\sin^2 N\gamma}{\sin^2\gamma}$: Interference between the diffracted lights from *N* equally spaced point sources

To check its correctness, we can see that

For *N* = 1, we get single slit pattern $\Rightarrow \quad I = I_0\frac{\sin^2\beta}{\beta^2}\frac{\cancel{\sin^2\gamma}}{\cancel{\sin^2\gamma}} = I_0\frac{\sin^2\beta}{\beta^2}$

For *N* = 2, we get double slit pattern

$$\Rightarrow \quad I = I_0\frac{\sin^2\beta}{\beta^2}\frac{\sin^2 2\gamma}{\sin^2\gamma} = 4I_0\frac{\sin^2\beta}{\beta^2}\frac{\cancel{\sin^2\gamma}\cos^2\gamma}{\cancel{\sin^2\gamma}} = 4I_0\frac{\sin^2\beta}{\beta^2}\cos^2\gamma$$

Positions of maxima and minima:

Maxima of interference pattern occur whenever $\gamma = m\pi; m = 0, \pi, 2\pi, ...$ $\quad d \sin\theta = 0, \lambda, 2\lambda, ...$

$$\Rightarrow \; d\sin\theta = m\lambda, m = 0,1,2, ...$$

What will be the max *m* allowed?

Since $|\sin\theta| \le 1 \quad m$ can not be greater than $\frac{d}{\lambda}$

For large *N*

$$\underset{\gamma \to m\pi}{Lt} \frac{\sin N\gamma}{\sin\gamma} = \underset{\gamma \to m\pi}{Lt} \frac{N\cos N\gamma}{\cos\gamma} = \pm N$$

$$E = N\frac{A\sin\beta}{\beta} \quad \Rightarrow \; I = I_0 N^2 \frac{\sin^2\beta}{\beta^2}$$

These maxima are called Principal maxima

where

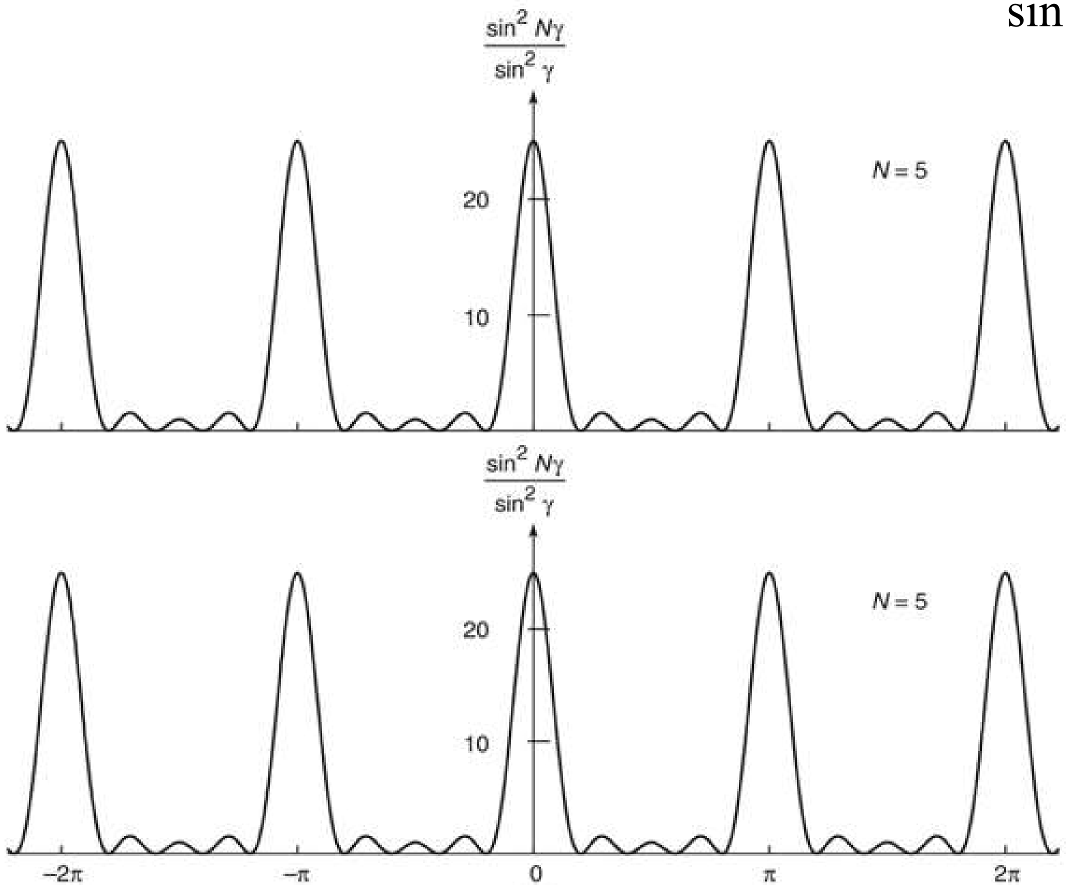
$$\beta = \frac{\pi b \sin\theta}{\lambda} = \frac{\pi b}{\lambda} \times \frac{m\lambda}{d} = \pi m\left(\frac{b}{d}\right)$$

Between two principal maxima, there could be several minima:

$$\gamma = \frac{\pi}{\lambda} d \sin\theta$$

For large N, $\dfrac{\sin^2 N\gamma}{\sin^2\gamma}$ peaks at $\gamma \approx m\pi$

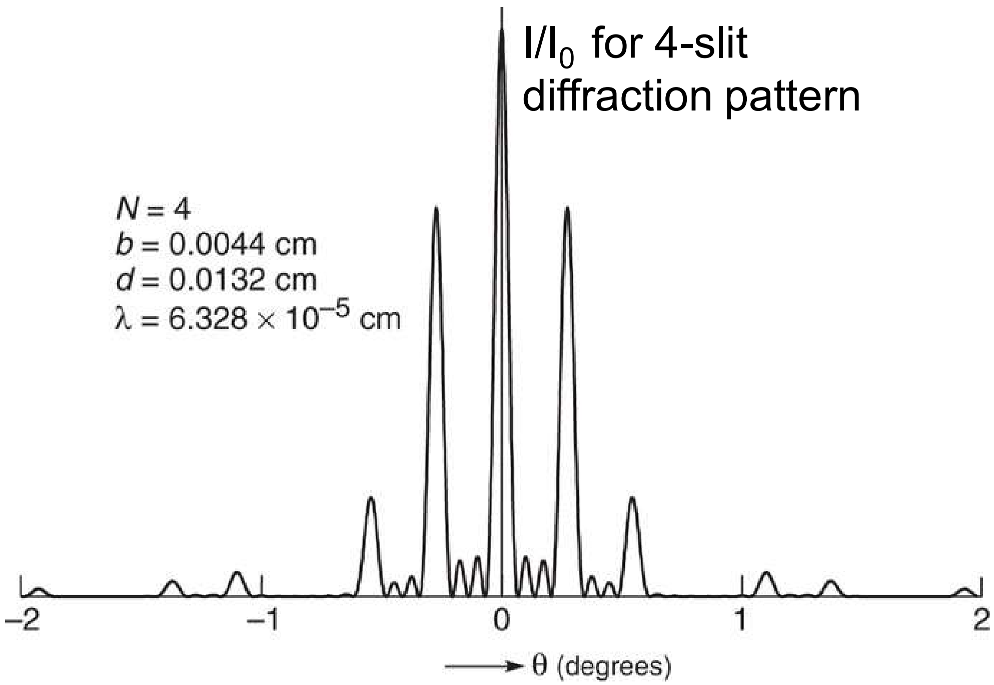
$$\Rightarrow \frac{\cancel{\pi} d \sin\theta}{\lambda} = m\cancel{\pi}$$

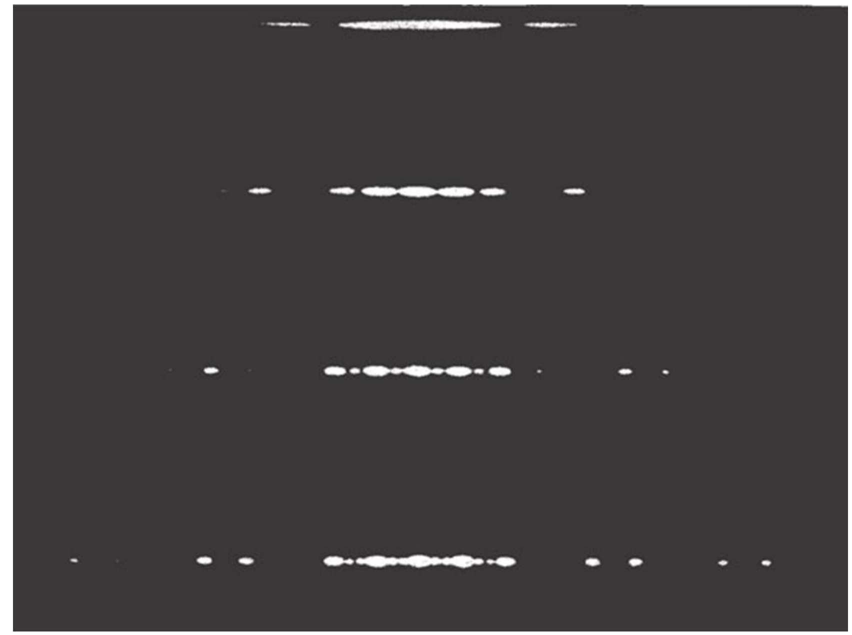


(a)

What is the difference between the two ?

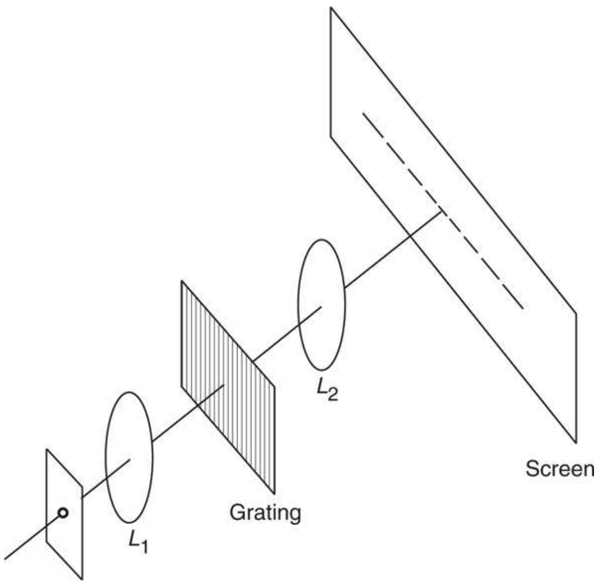
Sharper fringes with larger N (notice the difference in vertical scale)





1, 2, 3, and 4 slit diffraction patterns for *b* = 0.0044 cm, *d* = 0.0132 cm with a He-Ne laser light

Grating:



Grating equation: $d \sin\theta = m\lambda; m = 0,1,2, \dots.$

where principal maxima occurs

$\Rightarrow$

Zeroth order principal maxima occurs at $\theta = 0$ irrespective of wavelength

With white light what will be the colour of the principal maxima?

For any other order, different wavelengths of the same order will be diffracted at different angles

Hence, by measuring the diffraction angle θ, for a particular order one could find out the wavelength of that diffracted light

After differentiation we get $$\frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d\cos\theta}$$

For small θ, and for a given $\Delta\lambda$, $\Delta\theta$ is directly proportional to m

$\Delta\theta$ is inversely proportional to $d \Rightarrow$ smaller the grating element larger will be $\frac{\Delta\theta}{\Delta\lambda}$