11/02/19

Gauss elimination

for a linear system,

$$Ax = b, \quad |A| \neq 0, \quad A \in \mathbb{R}^{n \times n}$$

If $a_{11} \neq 0$

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} \cdot a_{ij}$$

$$a_{21} = a_{21} - a_{21} \cdot \frac{a_{21}}{a_{11}}$$

$$a_{ij} = a_{ij} - \frac{a_{ik}}{a_{ii}} a_{kj}$$

$$b_j^{(2)} = b_j - \frac{a_{j1}}{a_{11}} \cdot b_1$$

we update $(n-1) \times n$ entries, including the b-vector. $\quad i, j = 2, \dots n$

If $a_{22} \neq 0$

$$a_{ij}^{(2)} = a_{ij} - \frac{a_{i2}}{a_{22}} \cdot a_{2j}$$

$$b_j^{(2)} = b_j^{(2)} - \frac{a_{i2}}{a_{22}} \cdot b_2$$

$i, j = 3, \dots, n.$

We update $\overset{\text{cols.}}{(n-1)} \times \overset{\text{rows}}{(n-2)}$ elements

In general

$$a_{ij}^{(k)} = a_{ij}^{(k)} - \frac{a_{ik}}{a_{kk}} \cdot a_{kj}$$

$$b_j^{(k)} = b_j - \frac{a_{ik}}{a_{kk}} \cdot b_k, \quad i, j = k+1, \dots n$$

and we update :

$$n(n-1) + (n-1)(n-2) + \dots + 1$$

entries in total.

Floating point operations per second

1 flop = 1 (add/sub) + 1 (mul/div)

To update each entry in GE:

Step1 needs $(n)(n-1)$ flops + n divisions

as each op. is

$\underbrace{1 \text{ sub} + 1 \text{mul}}_{1 \text{ flop}} + \underline{1 \text{ div}}$

Step2 needs $(n-1)(n-2)$ flops + $(n-1)$ divisions

Total

$$\underbrace{[(n)(n-1) + (n-1)(n-2) + \cdots + 1]}_{\text{flops}} + \underbrace{[n + (n-1) + \cdots + 1]}_{\text{divisions}}$$

$$= [(n-1)(n-1+1) + (n-2)(n-2+1) + \cdots] + [n + n-1 + \cdots + 1]$$
$$+ [(n-1) + \cdots + 1]$$

$$= [(n-1)^2 + (n-2)^2 + \cdots + 1] + [(n-1) + \cdots] + [n + (n-1) \ldots]$$

$$= \frac{n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n+1)}{2}.$$

$1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n.$

$\approx \underline{O(n^3)}$

Notice

$O(n^3) \ll O(n!)$

Thus more efficient than Cramer's rule.

BACKWORD SUBSTITUTION

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ & & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\cancel{\sum_{i=1}^{n} a_{ij}\, x_j = b_j}$$

$$\cancel{a\, x_j = b_j}$$

$$\sum_{i=1}^{n} a_{ij} \cdot x_j = b_j$$

$$a_{ii} x_j + \sum_{k=i+1}^{n} a_{kj} \cdot x_j = b_j$$

$$\boxed{x_i = -\frac{1}{a_{ii}} \left(\sum_{k=i+1}^{n} a_{ik} \cdot x_k + b_j \right)}$$

1 div in step 1

$\underbrace{1 \text{ mul} + 1 \text{ sub}}$ + 1 div in step 2.

$\underbrace{2 \text{ mul} + 2 \text{ sub}}$ + 1 div in step 3.

⋮

$\frac{n(n+1)}{2} + n$ flops $\approx$ $\underline{O(n^2)}$

$\therefore$ for GE + Back Sub

complexity $O(n^3) + O(n^2)$

$\approx O(n^3)$

NOTE

verify Only when the leading minors are non-singular, GE works.

$\square$

LU decomposition

$Ax = b.$

$= (LU)(x) = b$

letting $Ux = y$.

$\Rightarrow \boxed{Ly = b}$

$\Rightarrow y = L^{-1}b$

and $Ux = L^{-1}b.$

$x = U^{-1}L^{-1}b$

We assume $diag(L) \mid diag(U) = 1$ as

1.) n^2 variables

2.) non-singular : guaranteed.

Proof. Crout is unique, and in general too.

i.e. Given $A = LU$, L, U are unique.

$$A = LU$$

Let $A = L_2U_2 = L_1U_1$

where $\mathrm{diag}(U) = 1$.

unit upper $\triangle$ matrix

$\Rightarrow \det U = 1$

Consider

$\Rightarrow L_2^{-1} L_1 U_1 = U_2$.

$\Rightarrow L_2^{-1} L_1 = U_2 U_1^{-1}$

Note inv(U), U being $\triangle$ is is also $\triangle$ and so for lower.

$\Rightarrow$ $L_2^{-1}L_1 = I$ and $U_2U_1^{-1} = I$. | usually diagonal, but unit $\triangle \Rightarrow I$.

$\rightarrow L_2 = L_1$, $U_2 = U_1$

□

Sufficient conditions on existence of LU decompositions

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = LU$.

1 - 6

= -7

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{bmatrix}$$

$u_{11} = 1$, $u_{12} = 2$, $u_{13} = 3$.

$l_{21} = 2 = l_{31}$.

$v_{22} =$

$$2 \cdot 2 + v_{22} = 4$$

$$u_{22} = 0$$

$$u_{33} = 3/2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$\det A = 5$ $\det L = 1$ $\det U = 0$

$$5 \neq 1 \cdot 0$$

Thus, non-singularity of A is insufficient.

SUFFICIENT CONDN.

A should have all principal minors be non-singular to have an LU decomposition

In the above ex. $|\text{minor}(a_{33})| = 0$.

Now

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$ (after row ops.)

at $\text{minor}(a_{11}) \neq \text{minor}(a_{22}) \neq \text{minor}(a_{33}) \neq 0$

NOTE

Let $D = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix}$. $D^{-1} = \begin{bmatrix} 1/u_{11} & 0 \\ 0 & 1/u_{22} \end{bmatrix}$

$u_{11} \times u_{22} \neq 0$.

$D^{-1}U = \begin{bmatrix} 1 & u_{12}/u_{11} \\ 0 & 1 \end{bmatrix}$ for $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$

Then $V = D^{-1}U$ is a unit upper Δ. matrix.

$\therefore$ $U = DV$

$\Rightarrow$ $A = LU = LDV$

Case I

Now, if $A = A^T$

$LDV = V^T D^T L^T = V^T D L^T$ ($D^T = D$, diagonal)

$\Rightarrow$ $L = V^T$, $V = L^T$

$\Rightarrow$ $\boxed{A = LDL^T}$

Case II

A is positive definite i.e. $x^T A x > 0 \ \forall x$.

$Ax = \lambda x$.

$\Rightarrow$ $x^T A x = \lambda x^T x$.

$\Rightarrow$ $\lambda_i > 0$

Define $\sqrt{D} = \text{diag}(\sqrt{u_{11}}, \sqrt{u_{22}})$. (P. def is needed here)

$A = (L\sqrt{D})(\sqrt{D}L^T) = G$ $L\sqrt{D}.\sqrt{D}L^T$

$= \cancel{LF} = GG^T$

$G = L\sqrt{D}$

$G^T = \sqrt{D}^T L^T = \sqrt{D} L^T$

$\Rightarrow$ $\boxed{A = GG^T}$

Partial pivoting

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 3 & 3 & 4 & | & 20 \\ 2 & 1 & 3 & | & 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 4 & | & 20 \\ 1 & 1 & 1 & | & 6 \\ 2 & 1 & 3 & | & 13 \end{bmatrix}$$

$R_2 \to R_2 - R_1/3$

$R_3 \to R_2 - \frac{2}{3}R_1$

$$\begin{bmatrix} 3 & 3 & 4 & | & 20 \\ 0 & 0 & -1/3 & | & -2/3 \\ 0 & -1 & 1/3 & | & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 4 & | & 20 \\ 0 & -1 & 1/3 & | & -1/3 \\ 0 & 0 & -1/3 & | & -2/3 \end{bmatrix}$$

$R_2 \to R_2 \times -1$

$R_3 \to R_3 \times -3$

$$= \begin{bmatrix} 3 & 3 & 4 & | & 20 \\ 0 & 1 & -1/3 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

$z = \cancel{2/3}$ 2

$y = \cancel{1/9}$ 1

$x = \cancel{17/3}$ 3

$\frac{1}{3} \cdot \frac{2}{9}$

$3x + \cancel{\frac{1}{3}} + \frac{8}{3} = 20$

$x = 17/3$

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complete pivoting

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

make 4 the pivot : it is the largest entry ever.

$C_1 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

$$R_I = \begin{bmatrix} 4 & 3 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 6 \\ 13 \end{bmatrix}$$

$R_2 \to R_2 - 1/4\, R_1$

$R_3 \to R_1 - 3/4\, R_1$

$$= \begin{bmatrix} 4 & 3 & 3 \\ 0 & 1/4 & 1/4 \\ 0 & \boxed{5/4} & -1/4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 3 \\ 0 & -5/4 & 1/4 \\ 0 & 1/4 & 1/4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ -2 \\ 1 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{5} R_2$

$$= \begin{bmatrix} 4 & 3 & 3 \\ 0 & -5/4 & 1/4 \\ 0 & 0 & \cancel{3/10}\ 1/5 \end{bmatrix}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 20 \\ -2 \\ 3/5 \end{bmatrix}$$

$x_1 = 3$

$x_2 = 1$

$x_3 = 2/.$

Now, to solve,

$Ax = B$

Direct Methods → Gauss, LU, GG^T, Thomas' Algo for tridiagonal systems.

Indirect ⇒ Gauss - Seidel.

Vector norm

Let $(V, +, \cdot)$ be a vector space $\forall x \in V$,

$\|\cdot\| : V \to \mathbb{R}^+ \cup \{0\}$ s.t.

1.) $\|x\| > 0 \quad \forall x \neq 0$

2.) $\|x\| = 0 \quad \forall x = 0$

3.) $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$.

4.) $\|x + y\| \leq \|x\| + \|y\|$. $\forall x, y \in V$.

Eg. → $\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$ (Euclidean norm)

→ $\|x\|_\infty = \max_i |x_i| = \max \{|x_1|, \ldots, |x_n|\}$

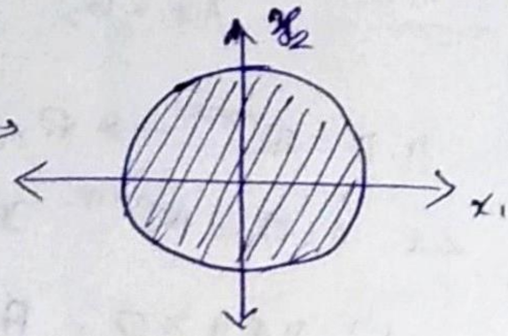
(infinite norm)

→ $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

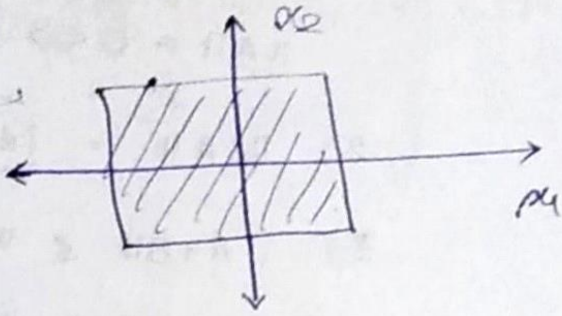
Q. find the points $x = (x_1, x_2) \in \mathbb{R}^2$ s.t.

$D_1 = \{x \mid \|x\|_2 \leq 1\}$

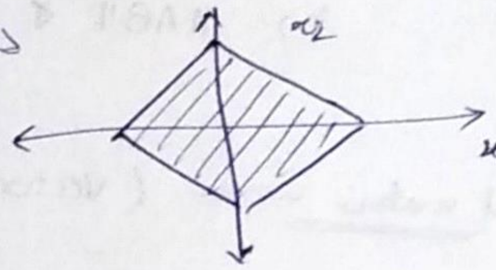
$D_2 = \{x_1^2 + x_2^2 \leq 1\} \longrightarrow$



$D_\infty = \{x \mid \|x\|_\infty \leq 1\}$



$D_1 = \{x \mid \|x\|_1 \leq 1\}$



Q. $x = (4, 4, -4, 4)$ $\quad$ $4, 1, -9, -1$

$v = (0, 5, 5, 5)$

$w = (6, 0, 0, 0)$

	$\|x\|_1$	$\|x\|_2$	$\|x\|_\infty$
x	16	8	4
v	15	$5\sqrt{3}$	5
w	6	6	6
$x - v$	15	$3\sqrt{11}$	9

Matrix norm

for

$$Ax = b, \quad |A| \neq 0, \quad A \in \mathbb{R}^{n \times n}$$

$$\|\cdot\| : \mathbb{R}^{n \times n} \to \mathbb{R}_{\geq 0}$$

Let $A, B \in \mathbb{R}^{n \times n}$. Then

1.) $\|A\| \geq 0, \quad A \neq \bar{0}$

$\|A\| = 0 \Leftrightarrow A = \bar{0}$

2.) $\|\alpha A\| = |\alpha| \|A\|$

3.) $\|A + B\| \leq \|A\| + \|B\|$

4.) $\|AB\| \leq \underline{\|A\| \|B\|}$ (consistency)

Induced matrix norm (vector norm).

Let $\|\cdot\|_v$ be a vector norm.

$$\Rightarrow \quad \|A\| = \max_{\|x\|_v \neq 0} \left(\frac{\|Ax\|_v}{\|x\|_v} \right)$$

$\boxed{\text{Consistency}}$

for any x,

$$\|A\| \geq \frac{\|Ax\|_v}{\|x\|_v}$$

and

$$\|Ax\|_v \leq [\|A\|][\|x\|_v]$$

NOTE

$$\|ABx\|_v \leq \|AB\|_v \|x\|_v \leq \|A\|_v \|B\|_v \cdot \|x\|_v x.$$

Then

$$\|AB\| = \max_{\|x\|_v \neq 0} \frac{\|ABx\|_v}{\|x\|_v}$$

eg $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

$$\|A\|_1 = \max_{j \in 1 \dots n} \sum_{i=1}^{N} |a_{ij}| \quad \text{[Column sum norm]}$$

$$= \max\{ |1| + |4| , |-2| + |3| \} = \underline{5}$$

$$\|A\|_2 = \max_{i \in 1 \dots n} \sum_{j=1}^{N} |a_{ij}| = m\{ |1| + |-2|, |4| + |3| \}$$

$$= \underline{\underline{7}}$$

Tutorial

20/02/19

1) Thomas' Algorithm

$$\begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & b_3 & \\ & & & \ddots & \\ & & & c_{n-1} & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

useful in solving parabolic PDE's

$$R_2 \rightarrow R_2 - R_1 . (c_1/a_1)$$

$$R_3 \rightarrow R_3 - R_2 (c_2/a_2)$$

$$\vdots$$

$$R_n \rightarrow R_n - R_{n-1} \left(\frac{c_{n-1}}{a_{n-1}}\right)$$

$$x_n = \frac{d_n}{a_n}$$

$$x_n = \frac{d_n - b_n x_{n+1}}{a_n}$$

$$x_3 = \frac{d_3 - \frac{b_3}{m_{34}} \cdot x_4}{m_{33}\ a_3}$$

---

1) <u>Gaussian</u> elimination

$$R_i \rightarrow R_i - \frac{a_{ij}}{a_{jj}} R_j$$

21/02/19

Spectral radius

For a square matrix :

$\cancel{\max(\lambda) \in \sigma(A)}$

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

where $\sigma(A)$ is the set of all eigenvalues of A.

Now.

$$\|A\|_2 = \sqrt{\rho(A^T A)}$$

2
17
13
51
17
221

Eg

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$

and
$$A^T A = \begin{bmatrix} 17 & 10 \\ 10 & 13 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$= \begin{vmatrix} 17-\lambda & 10 \\ 10 & 13-\lambda \end{vmatrix} = 0$$

$$= (17-\lambda)(13-\lambda) - 100 = 0$$

$$= 17 \cdot 13 - 30\lambda + \lambda^2 - 100 = 0$$

$$= \lambda^2 - 30\lambda + 121 = 0$$

$$\lambda = \frac{30 \pm \sqrt{30^2 - 4(121)}}{2}$$

$$= \frac{30 \pm \sqrt{900 - 484}}{2}$$

$\cancel{\lambda_{max}}$ =

$$\lambda = (25.2, 4.8)$$

$$\rho(A^T A) = 25.2$$

$$\therefore \|A\|_2 = \sqrt{25.2}$$

FROBENIUS NORM

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}$$

Eg

$$\|A\|_F = \sqrt{1^2+2^2+3^2+4^2}$$

$$= \sqrt{1+4+9+16} = \underline{\sqrt{30}}$$

In general,

$$\|A\|_F \leq \|A\|_2.$$

Prove using Cauchy-Schwarz inequality

Hint $\quad \|Ax\|_2 = \|A\|_F \cdot \|x\|_2.$

ITERATIVE METHODS

~~#A~~

Let $Ax = b$, $A \in \mathbb{R}^{n \times n}$

Define the splitting matrix $Q \in \mathbb{R}^{n \times n}$ s.t $|Q| \neq 0$. ~~Q being a split~~

$$Ax = b.$$

$$\Rightarrow \quad Qx + 0 = Qx + b - Ax.$$

$$(or) \quad Qx = (Q - A)x + b. \qquad \text{--- (1)}$$

Define an iterative scheme:

$$Qx^{(k+1)} = (Q-A)\,x^{(k)} + b. \qquad \text{--- (2)}$$

We can rewrite Q as:

$$Q^{-1}(Qx) = Q^{-1}(Q-A) + Q^{-1}b.$$

$$\Rightarrow x = (I - Q^{-1}A)x + Q^{-1}b. \qquad \text{--- (3)}$$

and so:

$$x^{(k+1)} = (I - Q^{-1}A)x^{(k)} - Q^{-1}b. \qquad \text{--- (4)}$$

$$(4) - (3) \Rightarrow (x^{(n)} - x) = H(x^{(n)} - x)$$

where $H = I - Q^{-1}A$.

$$\Rightarrow$$

$$\| x^{(n)} - x \| = \|H\| \, \|x^0 - x\|$$

$$\leq \|H\| \, \|x^0 - x\|$$

for

$n=0$:

$$\|x^1 - x\| \leq \|H\| \cdot \|x^0 - x\|$$

$n=1$

$$\|x^2 - x\| \leq \|H\| \cdot \|x^1 - x\| \leq \|H^2\|^2 \|x^0 - x\|$$

in general,

$$\|x^{(n+1)} - x\| \leq \|H\|^{n+1} \|x^{(0)} - x\|$$

Similar to $|\phi'(a)| \leq 1$,

we have to say

$$\boxed{\|H\| \leq 1}$$

for convergence.

Note that the choice of a affects the rate of convergence.

and that the norms used here are general i.e. use any convenient norm.

Formally

$$\|H\| < 1 \Rightarrow \lim_{n \to \infty} \|x^{(n+1)} - x\| = 0$$

$\square$.

STOPPING CRITERIA

Consider $\|x^{(n+1)} - x\| \leq \|H\| \cdot \|x^{(n)} - x\|$.

Prove that $\|x^{(n+1)} - x\| \leq \dfrac{\|H\|}{1 - \|H\|} \|x^{(n+1)} - x^{(n)}\|$

$$\cancel{\|x^{(n+1)} - x^{(n)} + x^{(n)} - x\|}$$

$$\cancel{\leq \|x^{(n+1)} - x^{(n)}\| + \|x^{(n)} - x\|}$$

$$\|x^{(n)} - x^{(n+1)} + x^{(n+1)} - x\|$$

$$\leq \|x^{(n)} - x^{(n+1)}\| + \|x^{(n+1)} - x\|$$

$$\therefore \quad \|x^{(n)} - x\| \leq \|H\|\left(\|x^{(n+1)} - x^{(n)}\| + \|x^{(n)} - x\|\right)$$

$$[1 - \|H\|]\,\|x^n - x\| \leq \|H\| \, [\|x^{(n+1)} - x^n\|]$$

$$\therefore \quad \boxed{\|x^n - x\| \leq \frac{\|H\|}{1 - \|H\|}\|x^{(n+1)} - x^{(n)}\|}$$

Relative error is obtained by dividing by $\|x^{(n)}\|$

$\square$