Mahindra
École Centrale

Engineering Mathematics-1
Problem Sheet-4
Topics: Multiple Integrals and Surface Integrals

Assignment Problems

1. Sketch the region of integration and evaluate the double integral.

   (i) $\iint_S (1+x)\sin y\,dx\,dy$, where S is the trapezoid with vertices $(0,0),(1,0),(1,2),(0,1)$.

   (ii) $\iint_S (x^2-y^2)\,dx\,dy$, where S is bounded by the curve $y=\sin x$ and the interval $[0,\pi]$.

2. Compute, by double integration, the volume of the ordinate set of f over S if: $f(x,y)=x^2+y^2$ and $S=\{(x,y):|x|\le 1,|y|\le 1\}$

3. Make a sketch of the region S and interchange the order of integration.

   (i) $\int_0^2\Big[\int_{y^2}^{2y} f(x,y)dx\Big]dy$

   (ii) $\int_1^2\Big[\int_{2-x}^{\sqrt{2x-x^2}} f(x,y)dy\Big]dx$

4. When a double integral was set up for the volume V of the solid under the surface $z=f(x,y)$ and above a region S of the xy plane, the following sum of iterated integrals was obtained:

$$V=\int_1^2\Big[\int_x^{x^2} f(x,y)dy\Big]dx+\int_2^8\Big[\int_x^8 f(x,y)dy\Big]dx.$$

   Sketch the region S and express V as an integral in which the order of integration is reversed.

5. Use Green's theorem to evaluate the line integral

$$\oint_C y^2dx+xdy$$

   when

   (i) C is the square with vertices $(\pm 2,0),(0,\pm 2)$.

   (ii) C has the vector equation $\alpha(t)=2\cos^3 t\mathbf{i}+2\sin^3 t\mathbf{j}$, $0\le t\le 2\pi$.

6. Make a sketch of the region $S=\{(x,y):0\le y\le 1-x,0\le x\le 1\}$ and express the double integral $\iint_S f(x,y)\,dx\,dy$ in polar coordinates.

7. Transform the integral to polar coordinates and compute its value, if possible.

(i) $\displaystyle\int_0^a \Big[\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dx\Big]dy$

(ii) $\displaystyle\int_0^1 \Big[\int_0^{x^2} f(x,y)dy\Big]dx$

8. If $r > 0$, let $I(r) = \int_{-r}^{r} e^{-u^2} du$.

   (i) Show that $I^2(r) = \displaystyle\iint_R e^{-(x^2+y^2)}\, dx\, dy$, where R is the square $R = [-r,r] \times [-r,r]$.

   (ii) If C_1 and C_2 are the circular disks inscribing and circumscribing R, show that

   $$\iint_{C_1} e^{-(x^2+y^2)}\, dx\, dy < I^2(r) < \iint_{C_2} e^{-(x^2+y^2)}\, dx\, dy$$

   (iii) Express the integrals over C_1 and C_2 in polar coordinates and use (ii) to deduce that $I(r) \to \sqrt{\pi}$ as $r \to \infty$.

9. Consider the mapping defined by two equations $x = u + v$, $y = v - u^2$.

   (i) A triangle T in the $uv-$ plane has vertices $(0,0),(2,0),(0,2)$. Describe, by means of a sketch, its image S in the $xy-$ plane.

   (ii) Calculate the area of S by a double integral extended over S and also by a double integral extended over T.

10. Evaluate

    $$\iiint_S dx\, dy\, dz$$

    using cylindrical coordinates, where S is the solid bounded by the three coordinate planes, the surface $z = x^2 + y^2$, and the plane $x + y = 1$.

11. Find the volume of the solid bounded by the xy-plane, the cylinder $x^2 + y^2 = 2x$, and the cone $z = \sqrt{x^2+y^2}$.

12. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

13. Let S denote the plane surface whose boundary is the triangle with vertices at $(1,0,0),(0,1,0),(0,0,1)$, and let $F(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\displaystyle\iint_S F.\,\mathbf{n}\, dS$, using:

    (i) the vector representation $\mathbf{r}(u,v) = (u+v)\mathbf{i} + (u-v)\mathbf{j} + (1-2u)\mathbf{k}$,

    (ii) an explicit representation of the form $z = f(x,y)$.

14. Integrate $f(x,y,z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.

15. Transform the surface integral $\displaystyle\iint_S (curl\ F).\,\mathbf{n}\, dS$ to line integral by using Stokes' theorem, and then evaluate the line integral.
    $F(x,y,z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$, and $\mathbf{n}$ is the unit normal with a nonnegative z-component.

16. Use Stokes' theorem to show that $\int_C (y+z)dx + (z+x)dy + (x+y)dz = 0$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$.

17. Let $\mathbf{F}(x,y,z) = y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + x^2y^2\mathbf{k}$. Show that curl $\mathbf{F}$ is not always zero, but that $F.curl\mathbf{F} = 0$. Find a scalar field μ such that $\mu\mathbf{F}$ is a gradient.

18. Let S be the surface of the unit cube, $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, and let $\mathbf{n}$ be the unit outer normal to S. If $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, use the divergence theorem to evaluate the surface integral $\iint_S F.\,\mathbf{n}\,dS$. Verify the result by evaluating surface integral directly.

19. Use the divergence theorem to find the outward flux of $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$ across the boundary of the region V: The region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

20. Prove that $\iint_S \frac{\partial f}{\partial n}\, dS = \iiint_V \nabla^2 f \, dx\, dy\, dz$.