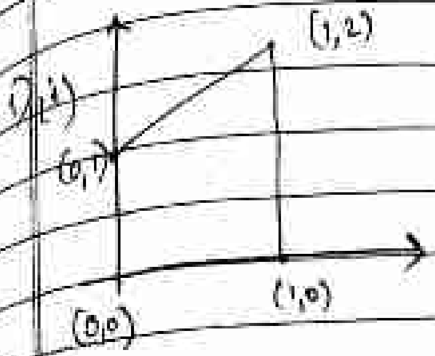
17XJ1A0543 ROCHAN A.V

MATHS ASSIGNMENT -4



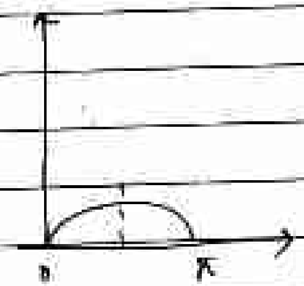
$$I = \iint (1+x)\sin y \, dx\,dy = \int_0^1 \left[\int_{y=0}^{x+1} (x+1)\sin y \, dy\right] dx$$

$$= -\int_0^1 (1+x)\left[\cos y\right]_0^{x+1} = -\int_0^1 (1+x)(\cos(x+1)-1)\,dx$$

$$= \int_0^1 1+x\,dx - \int_0^1 (1+x)\cos(1+x)\,dx$$

$$= \frac{3}{2} - 2\sin 2 + \sin 1 - \cos 2 + \cos 1$$

(ii)



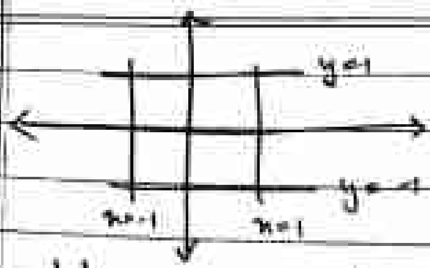
$$I = \int_0^{\pi}\int_{y=0}^{\sin x} (x^2-y^2)\,dx\,dy = \int_0^{\pi}\left[x^2y - \frac{y^3}{3}\right]_0^{\sin x} dx$$

$$= \int_0^{\pi}\left(x^2\sin x - \frac{3\sin x + \sin 3x}{12}\right)dx$$

$$= \int_0^{\pi} x^2 \sin x + 2\int x\cos x\,dx + \frac{1}{4}(2) + \frac{1}{12}\int_0^{\pi} \sin 3x$$

$$= \left[x^2\cos x + 2x\sin x - 2\cos x\right]_0^{\pi} + \frac{1}{2} + \frac{1}{18} = \boxed{\frac{\pi^2+5}{9}}$$

2)



$$\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy = \int_{-1}^{1}\left[x^2y+\frac{y^3}{3}\right]_{-1}^{1}dx = 2\int_{-1}^{1}x^2+\frac{1}{3} = \boxed{\frac{8}{3}}$$

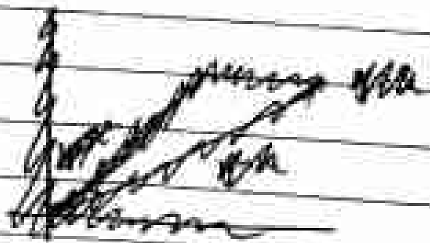
3) (i) $$\int_0^2\int_{y^2}^{2y} f(x,y)\,dy$$

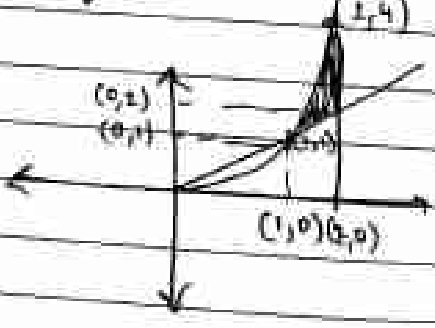
$x=y^2$
$x=2y$ intersect at $(4,2)$

$$\therefore \int_0^4\int_{y=x/2}^{\sqrt{x}} f(x,y)\,dy\,dx$$ is the answer.

(ii) $$\int_1^2\int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy\,dx = \int_0^1\int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

4)



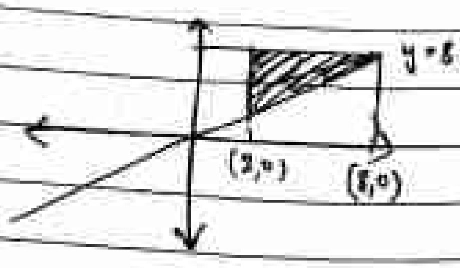


$$S_1 = \int_1^2\int_x^{x^2} f(x,y)\,dy\,dx$$

$$\left.\begin{matrix} y=x \\ y=x^2 \end{matrix}\right\} \quad \begin{matrix} x=y \\ x=\sqrt{y} \end{matrix}$$

$$S_1 = \int_1^2\int_{\sqrt{y}}^{y} f(x,y)\,dx\,dy + \int_2^4\int_{\sqrt{y}}^{2} f(x,y)\,dx\,dy$$

$$S_2 = \int_2^8\int_x^8 f(x,y)\,dy\,dx$$



$$S_2 = \int_2^{y} \int_2 f(x,y)\,dx\,dy$$

$\therefore$ Final answer $= \boxed{S_1 + S_2}$

5) (i) Given line integral $= \oint_c y^2 dx + x dy$

c is a square $(\pm 2, 0), (0, \pm 2)$

Using Green's theorem, we get $\oint y^2 dx + x dy = \int_{-2}^{2}\int_{-2}^{2} (1-2y)\,dx\,dy$

$$= \int_{-2}^{2} \left[y - y^2\right]_{-2}^{2} dx = \boxed{16}$$

(ii)

6) $S = \{(x,y),\ 0 \le y \le 1-x\ ;\ 0 \le x \le 1\}$

$\iint f(x,y)\,dx\,dy$ in polar coordinates

$$\iint f(x,y)\,dy\,dx \underset{y = Y(u,v)}{\overset{x = X(u,v)}{\Rightarrow}} \iint f(X(u,v), Y(u,v))\,|J|\,du\,dv$$

$x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$, $x+y = 1 \Rightarrow r = \dfrac{1}{\cos\theta + \sin\theta}$

$$\int_0^{\pi/2}\int_0^{1/(\sin\theta+\cos\theta)} f(\cancel{r\cos\theta}, r\sin\theta)\, r\,dr\,d\theta$$

7) (i) $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$

$\Rightarrow \iint f(x,y)\, dx\, dy \xrightarrow[y=Y(u,v)]{x=X(u,v)} \iint_T f(X(u,v), Y(u,v))\, |J|\, du\, dv$

$x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$

$\int_0^{\pi/2} \int_0^a r^2 r\, dr\, d\theta = \boxed{\frac{\pi a^4}{4}}$

(ii) $\int_0^1 \left[\int_0^{x^2} f(x,y)\, dy \right] dx$ $\quad x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$

$r = \tan\theta \sec\theta$

$\Rightarrow I = \int_0^{\pi/4} \int_0^{\tan\theta\sec\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

8) (i)

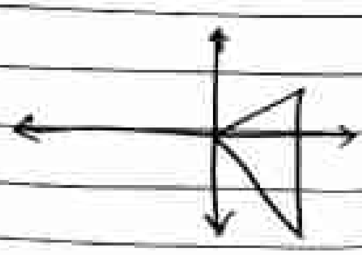
(ii)

(iii)

9) $x = u + v$, $y = v - u^2$

(i) $(0,0), (2,0), (0,2)$

$$\left.\begin{array}{l} (0,0) \longrightarrow (0,0) \\ (2,0) \longrightarrow (2,-4) \\ (0,2) \longrightarrow (2,2) \end{array}\right\}$$

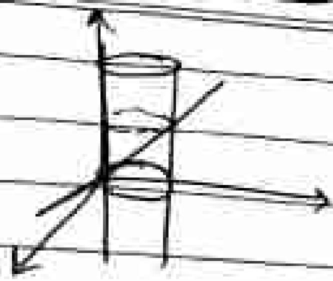


(ii) $S = \iint dx\, dy = \int_0^2 \int_0^x dy\, dx = \int_0^2 \left(x + \frac{x}{2}\right) dx = 3$

$T = \iint dv\, du = \int_0^2 \int_0^{2-v} du\, dv = \int_0^2 (2-v)\, dv = 6$

11) $x^2+y^2=2x$

$z=\sqrt{x^2+y^2}$



$$\text{vol} = \iiint dx\,dy\,dz$$

$$= \iint \left[\int_0^{\sqrt{x^2+y^2}} dz \right] dx\,dy$$

$$= \iint \sqrt{x^2+y^2}\,dx\,dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} 8\cos^3 d\theta = \boxed{\frac{32}{3}}$$

— x —

12) $x^2+y^2+z^2=a^2$, $x^2+y^2=ay$

$S = \bar{r}(u,v) = u\hat{i} + v\hat{j} + \sqrt{a^2-v^2-u^2}\,\hat{k}$

$T : u^2+v^2=a^2$

$\bar{r}_u = \hat{i} + \frac{(-u)}{\sqrt{a^2-v^2-u^2}}\hat{k}$, $\bar{r}_v = \hat{j} + \frac{(-v)}{\sqrt{a^2-v^2-u^2}}\hat{k}$

$$\bar{r}_u \times \bar{r}_v = \hat{k} + \frac{v}{\sqrt{a^2-v^2-u^2}}\hat{j} + \frac{u}{\sqrt{a^2-v^2-u^2}}\hat{i}$$

$$\|\bar{r}_u \times \bar{r}_v\| = \frac{a}{\sqrt{a^2-v^2-u^2}}$$

$$S.A = \iint 1 \cdot \|\bar{r}_u \times \bar{r}_v\| \, du\, dv = \iint \frac{a}{\sqrt{a^2 - v^2 - u^2}} \, du\, dv$$

$u = r\cos\theta$

$v = r\sin\theta$

$$\iint_R f(x,y)\, dx\, dy \xrightarrow[y = Y(u,v)]{x = X(u,v)} \iint_T f(X(u,v), Y(u,v)) |J| \, du\, dv$$

$$= \iint \frac{a}{\sqrt{a^2 - r^2}} \, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a} \frac{a}{\sqrt{a^2 - r^2}} \, r\, dr\, d\theta$$

$$= \int_0^{2\pi} a^3 \, d\theta = 2\pi a^3$$

13) $F(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

$$\iint F \cdot \hat{n}\, ds = \iint F(r(u,v)) \cdot (\bar{r}_u \times \bar{r}_v)\, du\, dv$$

(i) $r(u,v) = (u+v)\hat{i} + (u-v)\hat{j} + (1-2u)\hat{k}$

$\bar{r}_u = \hat{i} + \hat{j} - 2\hat{k}$, $\bar{r}_v = \hat{i} - \hat{j}$

$\bar{r}_u \times \bar{r}_v = -2\hat{i} - 2\hat{j} - 2\hat{k}$

$-(\bar{r}_u \times \bar{r}_v) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$$\Rightarrow \iint 2u + 2v + 2u - 2v + 2 - 4u = \iint 2\, du\, dv$$

$$\Rightarrow \int_0^{1/2}\int_{-v}^{v} 2\, du\, dv = \boxed{\frac{1}{2}}$$

(ii) $r(u,v) = u\hat{i} + v\hat{j} + (1-u-v)\hat{k}$

$\bar{r}_u = \hat{i} - \hat{k}$, $\bar{r}_v = \hat{j} - \hat{k}$

$\bar{r}_u \times \bar{r}_v = \hat{i} + \hat{j} + \hat{k}$

$$\int \hat{F}\cdot\hat{n}\,ds = \iint F\cdot(\bar{r}_u\times\bar{r}_v)\,du\,dv$$

$$= \iint\left(u\hat{i}+v\hat{j} + (1-u-v)\hat{k}\right)\cdot(1,1,1)\,du\,dv$$

$$= \iint du\,dv = \boxed{\frac{1}{2}}$$

14) $\bar{r}(u,v) = u\hat{i} + v\hat{j} + (1-v)\hat{k}$; $F = y+z$

$$\int_S F\,ds = \iint F(\bar{r}(u,v))\,\|\bar{r}_u\times\bar{r}_v\|\,dv\,du$$

$\bar{r}_u = \hat{i}$, $\bar{r}_v = \hat{j}-\hat{k}$

$\bar{r}_u\times\bar{r}_v = \hat{j}+\hat{k}$, $\|\bar{r}_u\times\bar{r}_v\| = \sqrt{2}$

$$= \int_0^1\int_0^2 \sqrt{2}\,du\,dv = 2\sqrt{2}$$

15) $\iint(\operatorname{curl}F)\cdot\hat{n}\,ds = \oint_C F\cdot d\bar{x}$

$F = y\hat{i} + z\hat{j} + x\hat{k}$, $z = 1-x^2-y^2$, $z\geq 0$

$\bar{x}(t) : \cos t\,\hat{i} + \sin t\,\hat{j}$

$$\oint F(\bar{x}(t))\cdot\bar{x}'(t)\,dt = \int(\sin t\,\hat{i} + \cos t\,\hat{k})(-\sin t\,\hat{i} + \cos t\,\hat{j})$$

$$= -\int \sin^2 t\,dt = -\pi$$

$$\oint = -\oint F\cdot d\bar{x} = -(-\pi) = \boxed{\pi}$$

16) $F(x,y,z) = (y+z,\ z+x,\ x+y)$

$$\int_C F\cdot d\bar{x} = \iint_S(\operatorname{curl}F)\cdot\hat{n}\,ds$$

$$\operatorname{curl}F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = 0 \qquad \therefore \boxed{\int_C F\cdot d\bar{x} = 0}$$

12) $F(x,y,z) = y^2z^2\hat{i} + z^2x^2\hat{j} + x^2y^2\hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^2 & z^2x^2 & x^2y^2 \end{vmatrix}$$

$$= \hat{i}\left(2x^2(y-z)\right) + \hat{j}\left(2y^2(z-x)\right) + \hat{k}\left(2z^2(x-y)\right)$$

$$F.(\nabla \times F) = 2x^2y^2z^2\left(y-z+z-x+x-y\right)$$

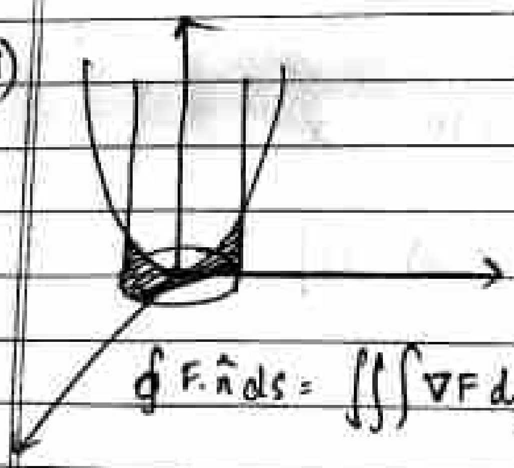
$$= \boxed{0}$$

18) $\iiint (\text{div} F)\, dx\,dy\,dz = \iint_S F.\hat{n}\, ds$

$$F = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

$$\text{div} F = \nabla . F$$
$$= 2x + 2y + 2z$$

$$\therefore \int_0^1\int_0^1\int_0^1 (2x+2y+2z)\, dx\,dy\,dz = \boxed{3}$$

19) 

$$\oint F.\hat{n}\, ds = \iiint \nabla F\, dy\,dx\,dz$$

$$F = y\hat{i} + xy\,\hat{j} - z\hat{k}\ , \quad \nabla . f = x-1$$

$I = \iiint (x-1)\, dx\,dy\,dz \longrightarrow$ cylindrical coordinates

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10) $\iiint dx\,dy\,dz$ = Volume

bounded by $z = x^2+y^2$ on top, x and y plane at bottom and $x+y=1$ plane on other side.

Since shadow region is a triangle,

$$\therefore I = \iiint_0^{x^2+y^2} dz\,dx\,dy$$

$$= \iint x^2+y^2\,dx\,dy$$

taking $x = r\cos\theta$ and $I = \iint_{x=X(u,v)}^{y=Y(u,v)} f(X(u,v), Y(u,v))\,|J|\,du\,dv$

$y = r\sin\theta$

$\therefore x+y=1$, $r = \dfrac{1}{\cos\theta+\sin\theta}$

$$\therefore I = \int_0^{\pi/2}\int_0^{1/(\cos\theta+\sin\theta)} r^3\,dr\,d\theta = \frac{1}{16}\int_0^{\pi/2}\left(\frac{1}{\cos\theta+\sin\theta}\right)^4 d\theta$$

$$= \boxed{\frac{1}{6}}$$

We get

$$\Rightarrow \int_0^{2\pi}\int_0^{4}\int_0^{x^2+y^2} (r\cos\theta \cdot 1)\,r\,dr\,d\theta\,dz$$

$$= \boxed{8\pi}$$

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20) We know that

$$\iiint \nabla F \, dx\,dy\,dz = \iint_{c} F \cdot ds$$

Assume $F = \nabla f$

$$\therefore \iiint \nabla^2 f \, dx\,dy\,dz = \iint_{s} \frac{\partial f}{\partial n} \, ds$$