Mahindra École Centrale, Hyderabad

ES 211 (Numerical Methods), Problem Sheet−VI

Tutorial problems

1. Experiments with a periodic process gave the following data:

t^o :	0	50	100	150	200	250	300	350
y :	0.754	1.762	2.041	1.412	0.303	−0.484	−0.380	0.520

Estimate the parameters c_0 and c_1 in the model $y = c_1 + c_0 \sin(t)$, using least square approximation.

2. Applying the method of least squares, find an equation of the form $P(x) = ae^{bx}, a, b \in \mathbb{R}$ that fits the following data:

x :	1	2	3	4
$f(x)$:	1.65	2.70	4.50	7.35

3. Obtain the least square approximation for $f(x) = \sqrt{x}, x \in [0, 1]$. (a) Using polynomials of degree one and two. (b) Using Legendre polynomials of degree one and two.

4. A function $y = f(x), x \in [a, b]$ is given at some points $x = x_0, x_1$ and x_2. Show that the Newton's divided difference formula and the corresponding Lagrange's interpolation formula are identical.

5. If $f(x) = \dfrac{1}{x}, x \in [1, 2]$, $x_0, x_1, \ldots, x_n \in [1, 2]$. Show that $f[x_0, x_1, \ldots, x_n] = \dfrac{(-1)^n}{x_0 x_1 \ldots x_n}$.

6. Prove that $f[x_0, x_1, \ldots, x_n] = \displaystyle\sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)}$, where $w'(x_i) = \displaystyle\sum_{j=0(j\neq i)}^{n} (x_i - x_j)$.

7. Let $f(x) = x^2, x \in [N, N+1], N \in \mathbb{N}$, P_1 : Linear polynomial interpolating at N and $N+1$. Then find the largest error.

8. (a) Show that the truncation error of the quadratic interpolation in a equidistant table is bounded by $\left(\dfrac{h^3}{9\sqrt{3}}\right) f'''(c), c \in (a, b)$

   (b) We want to set up an equidistant table of the function $f(x) = x^2 \ln(x), x \in [5, 10]$. The function values are rounded to 5 decimal places. Give the step size h which is to be used to yield a total error that is less than 10^{-5} on quadratic interpolation in this table.

9. A missile is launched from a ground station. The acceleration during its first 80 seconds of flight, as recorded, is given in the following table:

$t(s)$	: 0	10	20	30	40	50	60	70	80
$a(m/s^2)$	: 30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Then compute the velocity of the missile when $t = 80$ s, using composite Simpson's 1/3 rule.

10. Compute the integral $I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-(x^2/2)}\, dx$ using composite Simpson's 1/3 rule, taking h = 0.125.

11. Consider the approximation of $I = \int_1^7 \frac{1}{x}\, dx$ by (a) Composite Trapezoidal rule and (b) Composite Simpson's 1/3 rule with step size h. Determine h and the number of intervals required so that the error is less than 4×10^{-8}.

12. (**Simpson's 3/8 th rule**) Let $x_0 \in \mathbb{R}$ and $x_k = x_0 + kh, h(> 0) \in \mathbb{R}, k = 1, 2, 3$. If

$$\int_{x_0}^{x_3} f(x)\, dx \approxeq \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

Then determine the degree of the precision of this rule.

13. Construct a rule of the form

$$\int_{-1}^{1} f(x)\, dx \approxeq Af(-1/2) + Bf(0) + Cf(1/2), A, B, C \in \mathbb{R}$$

which is exact for the polynomials of degree ≤ 2.

14. Consider the IVP $\frac{dy}{dx} = y - x^2 + 1, 0 \leq x \leq 2, y(0) = 0.5$ and $D = \{(x, y) \mid 0 \leq x \leq 2, y \in \mathbb{R}\}$. Show that the IVP is well-posed.

15. Using Taylor's series method of order 2 and order 4, Runge-Kutta method of order 2 and order 4, solve the IVP $\frac{dy}{dx} = y - x^2 + 1, 0 \leq x \leq 2, y(0) = 0.5$ to find an approximate value of $y(0.8)$ by dividing the interval into 10 equal parts. Then compute the actual errors at the nodes.

16. Determine the stability region (stability means that $\lim_{i\to\infty} u_i = 0$) for the following iteration schemes, when they are applied to solve IVP $\frac{dy}{dx} = \lambda y, 0 \leq x \leq 1, y(0) = 1, \lambda \in \mathbb{C}$.

    (a) $u_{i+1} = u_i + hf(x_i, u_i), u_0 = 1, i = 0, 1, 2, \ldots, n - 1$. (Euler's method)

    (b) $u_{i+1} = u_i + hf(x_{i+1}, u_{i+1}), u_0 = 1, i = 0, 1, 2, \ldots, n - 1$. (Backward Euler's method)

    (c) $u_{i+1} = u_i + \frac{1}{2}\left[k_1 + k_2\right], k_1 = hf(x_i, u_i), k_2 = hf(x_i + h, u_i + k_1), u_0 = 1, i = 0, 1, 2, \ldots, n - 1$. (Runge-Kutta method of order 2)

17. Using Euler's method, solve the IVP (Initial Value Problem) $\frac{dy}{dx} = 100y - 101e^{-x}, 0 \leq x \leq 1, y(0) = 1$ and $D = \{(x,y) \mid 0 \leq x \leq 1, y \in \mathbb{R}\}$ to find an approximate value of $y(0.4)$ by dividing the interval into 10 equal parts. What is your conclusion ?

18. Using Runge-Kutta method of order 2, solve the following initial value problem, at $t = 0.4$ sec., with $h = 0.1$. Given that $L = 2\ ft$, $g = 32.17\ ft/sec^2$.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \theta(0) = \frac{\pi}{6}, \theta^{'}(0) = 0.$$

Then compare the angle θ obtained from the analytical solution of

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \theta(0) = \frac{\pi}{6}, \theta^{'}(0) = 0.$$

x	: 2.0	2.2	2.6
$f(x)$	: 0.69315	0.78846	0.95551

19. Solve the BVP (Boundary Value Problem)

$$\frac{d^2y}{dx^2} = y + x, y(0) = 0, y(1) = 0$$

using finite difference method with $h = \frac{1}{4}$. Then compare with the analytical solution at the nodes.

Homework Problems

1. Find the eigen value of the largest modulus, and associated eigen vector of the following matrices using power method. (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$

2. Show that the normal equations with respect to the polynomial basis in the discrete least square approximation yield a symmetric and nonsingular matrix and hence have a unique solution.

3. If $f(x_0) = f^{'}(x_0) = 0$, $f(x_1) = f^{'}(x_1) = f^{''}(x_1) = 0$. Find the polynomial of degree ≤ 5 which interpolates f at $x_0, x_0, x_1, x_1, x_1, x_2$.

4. We are given the values of a function of the variable t:

t :	0.1	0.2	0.3	0.4
$f(t)$:	0.76	0.58	0.44	0.35

Obtain a least square fit of the form $P(t) = c_0e^{-3t} + c_1e^{-2t}, c_0, c_1 \in \mathbb{R}$.

5. Find the interpolating polynomial by (a) Lagrange's formula, and (b) Newton's divided difference formula for the following data:

x:	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

(c) Estimate $f(3)$ and $f(6.5)$.

6. Find the approximate value of $I = \int_0^{\pi} \sin(x)\, dx$ using (a) Composite rectangle rule (b) Composite mid point rule (c) Composite Trapezoidal rule, and (d) Composite Simpson's 1/3 rule by dividing the range of integration into six equal parts. Then calculate the error bound and compare with the actual error.

Problems for lab

1. Write a MATLAB programs for the following problems

   (a) Homework problem 1(b) and 5.

   (b) Tutorial Problems 1, 11, 13, 15, 17, 18 and 19.