CS 121: Tutorial Sheet 3: Ordinary Differentail Equations and Integral Transforms

1. Solve the following first order differential equations

   (a) $x\frac{dy}{dx} + y \; = \; x^3y^6$

   (b) $xy^2\frac{dy}{dx} + y^3 \; = \; x\cos(x)$

   (c) $(x^2y^3 + xy)\frac{dy}{dx} \; = \; 1$

   (d) $\frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1 + y^2) = 0$

   (e) $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}(1 - \frac{x}{y})dy \; = \; 0$

   (f) $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy \; = \; 0$

   (g) $y\cos(x)dx + 2\sin(x)dy \; = \; 0$

   (h) $(1 + xy)ydx + (1 - xy)xdy \; = \; 0$

   (i) $(xy - 1)dx + (x^2 - xy)dy \; = \; 0$

2. Find the curve passing through the point $(1, 0)$ and having at each of its points the slope $-\frac{x}{y}$

3. Solve the following Initial Value Problems (IVP)

   (a) $\frac{dy}{dx} = 2y + e^{2x}$, $y(0) = 3$

   (b) $\frac{dy}{dx} = 3y + 2e^{3x}$, $y(0) = 2$

   (c) $\frac{dy}{dx} = y\tan(x) + \sec x$, $y(0) = -1$

   (d) $\frac{dy}{dx} = \frac{2}{x}y + x$, $y(1) = 2$

4. Define the Wronskian $w(y_1, y_2)$ of any two differentiable functions y_1 and y_2 defined in an interval $(a, b) \subset R$. Show that $w(y_1, y_2) = 0$ if y_1 and y_2 are linearly dependent.

5. If y_1 and y_2 are any two solutions of a second order linear homogeneous ordinary differential equation which is defined in an interval $(a, b) \subset R$, then $w(y_1, y_2)$ is either identically zero or non-zero at any point of the interval (a, b).

6. Find the general solution of the following second order equations using the given known solution y_1.

   (a) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y \;=\; 0$ where $y_1(x) = x$.

   (b) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y \;=\; 0$ where $y_1(x) = x^2$.

   (c) $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y \;=\; 0$ where $y_1(x) = x$.

   (d) $x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y \;=\; 0$ where $y_1(x) = e^x$.

7. Find the general solution of each of the following equations ($D^n \equiv \frac{d^n}{dx^n}$)

   (a) $(D^4 - 81)y = 0$

   (b) $(D^3 - 4D^2 + 5D - 2)y = 0$

   (c) $(D^4 - 7D^3 + 18D^2 - 20D + 8)y = 0$

   (d) $(D^4 + 2D^3 + 3D^2 + 2D + 1)y = 0$

   (e) $(D^2 - 3D - 6)y \;=\; 3\sin 2x$

   (f) $(D^2 + 1)y \;=\; 2\cos x$

   (g) $(D^2 - 3D + 2)y \;=\; (4x+5)e^{3x}$

   (h) $(D^2 - 1)y \;=\; 3e^{2x}\cos 2x$

   (i) $(D^2 - 2D + 3)y \;=\; 3e^{-x}\cos x$

8. Solve the following using the method of variation of parameters ($D^n \equiv \frac{d^n}{dx^n}$)

   (a) $(D^2 + 1)y \;=\; cosecx$

   (b) $(D^2 - D - 6)y \;=\; e^{-x}$

   (c) $(D^2 + a^2)y \;=\; \tan ax$

   (d) $(D^2 + a^2)y \;=\; \sec 2x$

   (e) $x^2y'' - 4xy' + 6y \;=\; 21x^{-4}$

   (f) $4x^2y'' + 8xy' - 3y \;=\; 7x^2 - 15x^3$

   (g) $x^2y'' - 2xy' + 2y \;=\; x^3\cos x$

   (h) $xy'' - y' \;=\; (3+x)x^2e^x$

9. Solve the following problems using method of undetermined coefficients ($D^n \equiv \frac{d^n}{dx^n}$)

   (a) $(D^3 - 2D^2 - 5D + 6)y \;=\; 18e^x$

   (b) $(D^2 + 25)y \;=\; 50\cos 5x + 30\sin 5x$

   (c) $(D^3 - 2D^2 + 4D - 8)y \;=\; 8(x^2 + \cos 2x)$

   (d) $(D^3 + 3D^2 - 4)y \;=\; 12e^{-2x} + 9e^x$

10. Show that the general solution of $L(y) = f_1(x)+f_2(x)$, where $L(y)$ is a linear differential operator, is $y = \bar{y} + y^*$, where $\bar{y}$ is the general solution of the $L(y) = 0$ and y^* is the sum of the any particular solutions of $L(y) = f_1(x)$ and $L(y) = f_2(x)$.

11. From the definition of the Laplace transform, compute $L[f(t)]$ for

   a) $f(t) = e^t \mathrm{sin}t$,

   b) $f(t) = \begin{cases} 2 & \text{if } t \leq 1 \\ e^t & \text{if } t > 1. \end{cases}$

   c) $f(t) = \begin{cases} \mathrm{sin}wt & \text{if } 0 < t < \dfrac{\pi}{w} \\ 0 & \text{if } t \geq \dfrac{\pi}{w}. \end{cases}$

12. Let $f(t) = e^t$ on $[0, \infty)$.

   a) Show that $F(s) = L[e^t]$ converges for $\mathrm{Re}(s) > 1$.

   b) Show that $F(s) = L[e^t] \to 0$ as $\mathrm{Re}(s) \to \infty$.

13. Show that $f(t) = t^n$ has exponential order α for any $\alpha > 0, n \in \mathbb{N}$.

14. Without actually determining it, show that $f(t) = t^2 \mathrm{sinh}t$ possesses a Laplace Transform.

15. Show that $L(\sinh(wt)) = \dfrac{w}{s^2 - w^2}$.

16. Compute $L(\cos(wt))$ and $L(\sin(wt))$ from the Taylor Series representations $\cos(wt) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (wt)^{2n}}{(2n)!}$ and $\sin(wt) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (wt)^{2n+1}}{(2n+1)!}$ respectively.

17. Determine $L\left[\dfrac{1 - \cos(wt)}{t}\right]$.

18. Can $F(s) = \dfrac{s}{\mathrm{log}s}$ be the Laplace transform of some function f?. Justify.

19. Determine

   a) $L[e^{7t}\mathrm{sinh}(\sqrt{2}\,t)]$ b) $L[t^2 e^{-wt}]$ c) $L^{-1}\left[\dfrac{1}{s^2 + 2s + 5}\right]$ d) $L^{-1}\left[\dfrac{s}{(s+1)^2}\right]$.

20. Determine $L[f(t)]$ for

   a) $f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 2 \\ e^{at} & \text{if } t \geq 2. \end{cases}$ b) $f(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{2} \\ \mathrm{sin}t & \text{if } t \geq \frac{\pi}{2}. \end{cases}$

21. Find $L^{-1}\left[\dfrac{e^{-2s}}{s^3}\right]$.

22. Determine $L[t\mathrm{cosh}(wt)]$.

23. Show that $L\left(\dfrac{1 - e^{-t}}{t}\right) = \log\left(1 + \dfrac{1}{s}\right), \quad (s > 0)$.

24. Find $L^{-1}\left[\log\left(\dfrac{s^2 + a^2}{s^2 + b^2}\right)\right]$.

25. Find inverse Laplace transform of the following $F(s)$ by the partial fraction method.

a) $\dfrac{s}{2s^2+s-1}$ b) $\frac{s^2+1}{s(s-1)^3}$.

26. Find $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2-2}\right]$.

27. Determine $L[t^2\sin(wt)]$.

28. Show that $L\left[\dfrac{1-\cosh(wt)}{t}\right]=\dfrac{1}{2}\log\left(1-\dfrac{w^2}{s^2}\right)$ $(s>|w|)$.

29. Find $L^{-1}\left[\tan^{-1}\dfrac{1}{s}\right]$, $s>0$.

30. If $L^{-1}\left[\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}\right]=\dfrac{e^{-\frac{a^2}{4t}}}{\sqrt{\pi t}}$, then find $L^{-1}[e^{-a\sqrt{s}}]$.

31. Find inverse Laplace transform of $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ by partial fraction method.

32. Determine $L^{-1}\left[\dfrac{s^2}{(s^2-a^2)(s^2-b^2)(s^2-c^2)}\right]$ by the partial fraction method.