MA 203: Tutorial Sheet 4: Probability
Assignment Submission Deadline: 28/09/2018

∗ Problems to be submitted as Assignment

1. Find the cdf and pdf of $Y = \cos(X)$ where X is uniform on $(0, 2\pi]$.

2. Find the pdf of $X = -log(1 - U)$ where U is a uniform random variable in $(0, 1)$.

*3. Let $Y = |X|$ where X is an arbitrary continuous random variable.

   (a) Find the cdf of Y by finding the equivalent event of $\{Y \leq y\}$. Find the pdf of Y by differentiation of the cdf.

   (b) Find the pdf of Y by finding the equivalent event of $\{y < Y < y + dy\}$. Does the answer agree with part (a)?

   (c) What is the pdf of Y if $f_X(x)$ is an even function of x?

4. Let $Y = e^X$ where X is an arbitrary continuous random variable.

   (a) Find the cdf and pdf of Y in terms of the cdf and pdf of X.

   (b) Find the pdf of Y when X is a Gaussian random variable.

5. Compare the Chebyshev inequality and the exact probability for the event $\{|X - m| > a\}$ as a function of a for:

   *(a) X is a uniform random variable in the interval $[-b, b]$.

   (b) X is a Poission random variable with parameter α.

   (c) X is a zero-mean Gaussian random variable.

   *(d) X is a binomial random variable with $n = 10$, $p = 0.5$.

6. Suppose that light bulbs have exponentially distributed lifetimes with unknown mean $E[X]$. Suppose we measure the lifetime of n light bulbs, and we estimate the mean $E[X]$ by the arithmetic average Y of the measurements. Apply the Chebyshev inequality to the event $\{|Y - E(X)| > a\}$. What happens as $n \to \infty$?

*7. Find the Chernoff bound for the exponential random variable with $\lambda = 1$. Compare the bound to the exact value for $P(X > 5)$.

8. Let X be a Poisson random variable with parameter $\alpha = 1$. Compare the Chernoff bound and the exact value for $P(X \geq 5)$.

*9. Show that the Chernoff bound for X, a Poisson random variable with mean α, is $P[X \geq a] \leq e^{-a\log(a/\alpha)+a-\alpha}$ for $a > \alpha$.

10. Show that the Chernoff bound for X, a Gaussian random variable with mean μ and variance σ^2 is $P[X \geq a] \leq e^{-(a-\mu)/2\sigma^2}$, $a > \mu$.

*11. Let $\Phi_X(\omega)$ be the characteristic function of an exponential random variable. What random variable does $\Phi_X^n(\omega)$ correspond to?

12. The Laplace transform of the pdf of a random variable X is given by:

$$X^*(s) = \frac{ab}{(s+a)(s+b)}.$$

Find the pdf of X.