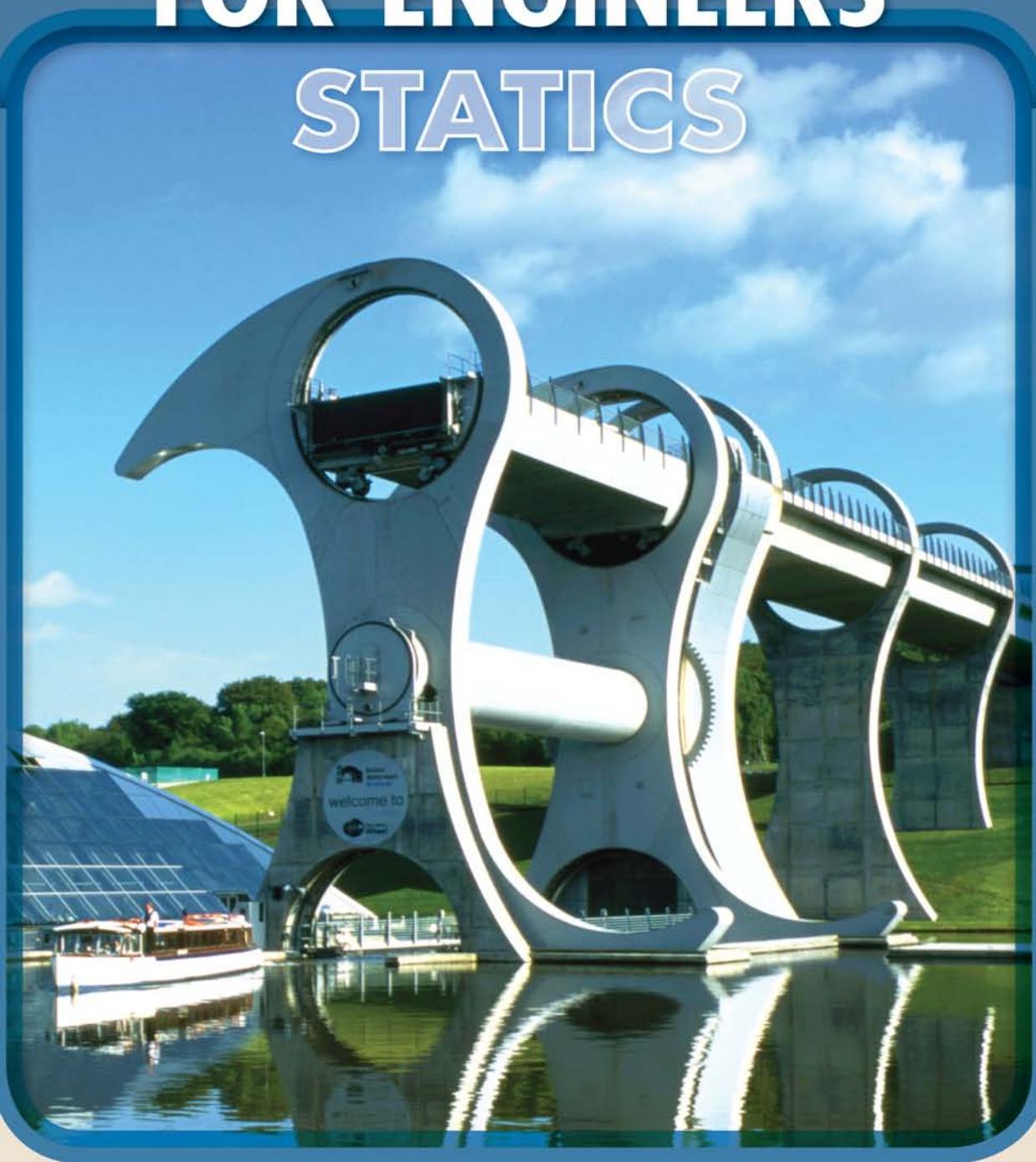


# VECTOR MECHANICS FOR ENGINEERS

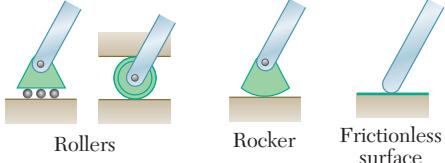
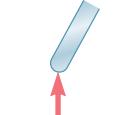
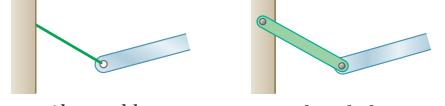
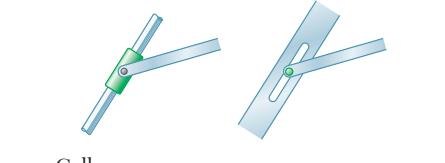
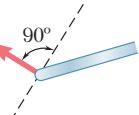
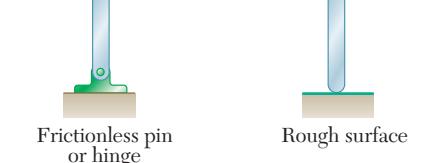
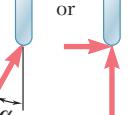
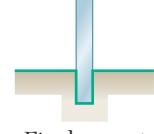
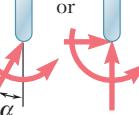
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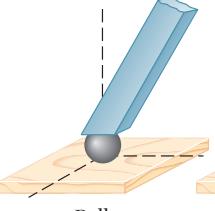
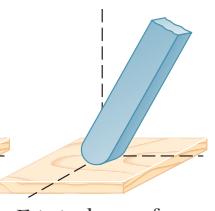
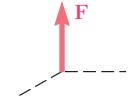
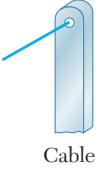
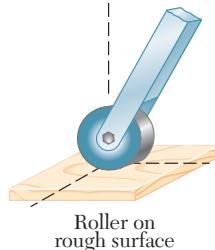
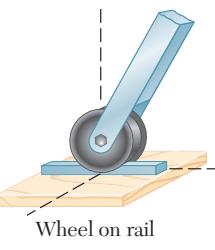
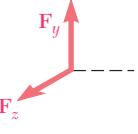
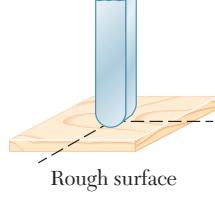
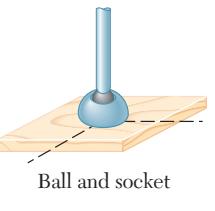
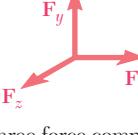
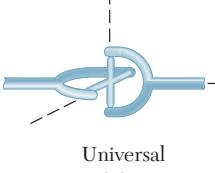
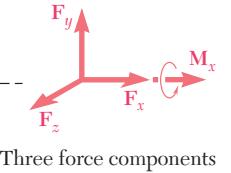
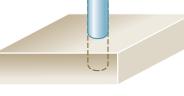
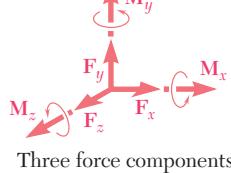
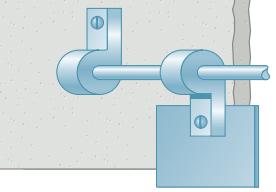
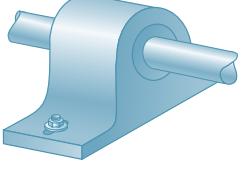
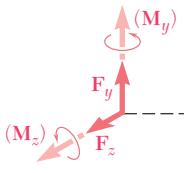
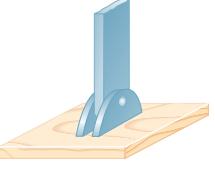
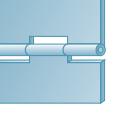
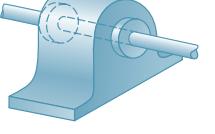
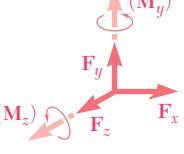
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## Reactions at Supports and Connections for a Two-Dimensional Structure

Support or Connection	Reaction	Number of Unknowns
 Rollers      Rocker      Frictionless surface	 Force with known line of action	1
 Short cable      Short link	 Force with known line of action	1
 Collar on frictionless rod      Frictionless pin in slot	 Force with known line of action	1
 Frictionless pin or hinge      Rough surface	 Force of unknown direction	2
 Fixed support	 Force and couple	3

The first step in the solution of any problem concerning the equilibrium of a rigid body is to construct an appropriate free-body diagram of the body. As part of that process, it is necessary to show on the diagram the reactions through which the ground and other bodies oppose a possible motion of the body. The figures on this and the facing page summarize the possible reactions exerted on two- and three-dimensional bodies.

## Reactions at Supports and Connections for a Three-Dimensional Structure

				Force with known line of action (one unknown)	Force with known line of action (one unknown)	
				Two force components		
				Three force components		
				Three force components and one couple	Three force components and three couples	
				Hinge and bearing supporting radial load only	Two force components (and two couples; see page 191)	
				Pin and bracket	Hinge and bearing supporting axial thrust and radial load	Three force components (and two couples; see page 191)



# **VECTOR MECHANICS FOR ENGINEERS**

**Statics**



NINTH EDITION

# VECTOR MECHANICS FOR ENGINEERS

## Statics

**Ferdinand P. Beer**

Late of Lehigh University

**E. Russell Johnston, Jr.**

University of Connecticut

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U.S. Coast Guard Academy

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# About the Authors

As publishers of the books by Ferd Beer and Russ Johnston, we are often asked how they happened to write their books together with one of them at Lehigh and the other at the University of Connecticut.

The answer to this question is simple. Russ Johnston's first teaching appointment was in the Department of Civil Engineering and Mechanics at Lehigh University. There he met Ferd Beer, who had joined that department two years earlier and was in charge of the courses in mechanics.

Ferd was delighted to discover that the young man who had been hired chiefly to teach graduate structural engineering courses was not only willing but eager to help him reorganize the mechanics courses. Both believed that these courses should be taught from a few basic principles and that the various concepts involved would be best understood and remembered by the students if they were presented to them in a graphic way. Together they wrote lecture notes in statics and dynamics, to which they later added problems they felt would appeal to future engineers, and soon they produced the manuscript of the first edition of *Mechanics for Engineers* that was published in June 1956.

The second edition of *Mechanics for Engineers* and the first edition of *Vector Mechanics for Engineers* found Russ Johnston at Worcester Polytechnic Institute and the next editions at the University of Connecticut. In the meantime, both Ferd and Russ assumed administrative responsibilities in their departments, and both were involved in research, consulting, and supervising graduate students—Ferd in the area of stochastic processes and random vibrations and Russ in the area of elastic stability and structural analysis and design. However, their interest in improving the teaching of the basic mechanics courses had not subsided, and they both taught sections of these courses as they kept revising their texts and began writing the manuscript of the first edition of their *Mechanics of Materials* text.

Their collaboration spanned more than half a century and many successful revisions of all of their textbooks, and Ferd's and Russ's contributions to engineering education have earned them a number of honors and awards. They were presented with the Western Electric Fund Award for excellence in the instruction of engineering students by their respective regional sections of the American Society for Engineering Education, and they both received the Distinguished Educator Award from the Mechanics Division of the same society. Starting in 2001, the New Mechanics Educator Award of the Mechanics Division has been named in honor of the Beer and Johnston author team.

**Ferdinand P. Beer.** Born in France and educated in France and Switzerland, Ferd received an M.S. degree from the Sorbonne and an Sc.D. degree in theoretical mechanics from the University of Geneva.

He came to the United States after serving in the French army during the early part of World War II and taught for four years at Williams College in the Williams-MIT joint arts and engineering program. Following his service at Williams College, Ferd joined the faculty of Lehigh University where he taught for thirty-seven years. He held several positions, including University Distinguished Professor and chairman of the Department of Mechanical Engineering and Mechanics, and in 1995 Ferd was awarded an honorary Doctor of Engineering degree by Lehigh University.

**E. Russell Johnston, Jr.** Born in Philadelphia, Russ holds a B.S. degree in civil engineering from the University of Delaware and an Sc.D. degree in the field of structural engineering from the Massachusetts Institute of Technology. He taught at Lehigh University and Worcester Polytechnic Institute before joining the faculty of the University of Connecticut where he held the position of chairman of the Department of Civil Engineering and taught for twenty-six years. In 1991 Russ received the Outstanding Civil Engineer Award from the Connecticut Section of the American Society of Civil Engineers.

**David F. Mazurek.** David holds a B.S. degree in ocean engineering and an M.S. degree in civil engineering from the Florida Institute of Technology and a Ph.D. degree in civil engineering from the University of Connecticut. He was employed by the Electric Boat Division of General Dynamics Corporation and taught at Lafayette College prior to joining the U.S. Coast Guard Academy, where he has been since 1990. He has served on the American Railway Engineering and Maintenance of Way Association's Committee 15—Steel Structures for the past eighteen years. His professional interests include bridge engineering, tall towers, structural forensics, and blast-resistant design.

**Elliot R. Eisenberg.** Elliot holds a B.S. degree in engineering and an M.E. degree, both from Cornell University. He has focused his scholarly activities on professional service and teaching, and he was recognized for this work in 1992 when the American Society of Mechanical Engineers awarded him the Ben C. Sparks Medal for his contributions to mechanical engineering and mechanical engineering technology education and for service to the American Society for Engineering Education. Elliot taught for thirty-two years, including twenty-nine years at Penn State where he was recognized with awards for both teaching and advising.

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# Preface

## OBJECTIVES

The main objective of a first course in mechanics should be to develop in the engineering student the ability to analyze any problem in a simple and logical manner and to apply to its solution a few, well-understood, basic principles. It is hoped that this text, designed for the first course in statics offered in the sophomore year, and the volume that follows, *Vector Mechanics for Engineers: Dynamics*, will help the instructor achieve this goal.<sup>†</sup>

## GENERAL APPROACH

Vector analysis is introduced early in the text and is used in the presentation and discussion of the fundamental principles of mechanics. Vector methods are also used to solve many problems, particularly three-dimensional problems where these techniques result in a simpler and more concise solution. The emphasis in this text, however, remains on the correct understanding of the principles of mechanics and on their application to the solution of engineering problems, and vector analysis is presented chiefly as a convenient tool.<sup>‡</sup>

**Practical Applications Are Introduced Early.** One of the characteristics of the approach used in these volumes is that mechanics of *particles* is clearly separated from the mechanics of *rigid bodies*. This approach makes it possible to consider simple practical applications at an early stage and to postpone the introduction of the more difficult concepts. For example:

- In *Statics*, statics of particles is treated first (Chap. 2); after the rules of addition and subtraction of vectors are introduced, the principle of equilibrium of a particle is immediately applied to practical situations involving only concurrent forces. The statics of rigid bodies is considered in Chaps. 3 and 4. In Chap. 3, the vector and scalar products of two vectors are introduced and used to define the moment of a force about a point and about an axis. The presentation of these new concepts is followed by a thorough and rigorous discussion of equivalent systems of forces leading, in Chap. 4, to many practical applications involving the equilibrium of rigid bodies under general force systems.
- In *Dynamics*, the same division is observed. The basic concepts of force, mass, and acceleration, of work and energy, and

<sup>†</sup>This text is available in a single volume, *Vector Mechanics for Engineers: Statics and Dynamics*, ninth edition.

<sup>‡</sup>In a parallel text, *Mechanics for Engineers: Statics*, fifth edition, the use of vector algebra is limited to the addition and subtraction of vectors.

of impulse and momentum are introduced and first applied to problems involving only particles. Thus, students can familiarize themselves with the three basic methods used in dynamics and learn their respective advantages before facing the difficulties associated with the motion of rigid bodies.

**New Concepts Are Introduced in Simple Terms.** Since this text is designed for the first course in statics, new concepts are presented in simple terms and every step is explained in detail. On the other hand, by discussing the broader aspects of the problems considered, and by stressing methods of general applicability, a definite maturity of approach is achieved. For example, the concepts of partial constraints and statical indeterminacy are introduced early and are used throughout.

**Fundamental Principles Are Placed in the Context of Simple Applications.** The fact that mechanics is essentially a *deductive* science based on a few fundamental principles is stressed. Derivations have been presented in their logical sequence and with all the rigor warranted at this level. However, the learning process being largely *inductive*, simple applications are considered first. For example:

- The statics of particles precedes the statics of rigid bodies, and problems involving internal forces are postponed until Chap. 6.
- In Chap. 4, equilibrium problems involving only coplanar forces are considered first and solved by ordinary algebra, while problems involving three-dimensional forces and requiring the full use of vector algebra are discussed in the second part of the chapter.

**Free-Body Diagrams Are Used Both to Solve Equilibrium Problems and to Express the Equivalence of Force Systems.**

Free-body diagrams are introduced early, and their importance is emphasized throughout the text. They are used not only to solve equilibrium problems but also to express the equivalence of two systems of forces or, more generally, of two systems of vectors. The advantage of this approach becomes apparent in the study of the dynamics of rigid bodies, where it is used to solve three-dimensional as well as two-dimensional problems. By placing the emphasis on “free-body-diagram equations” rather than on the standard algebraic equations of motion, a more intuitive and more complete understanding of the fundamental principles of dynamics can be achieved. This approach, which was first introduced in 1962 in the first edition of *Vector Mechanics for Engineers*, has now gained wide acceptance among mechanics teachers in this country. It is, therefore, used in preference to the method of dynamic equilibrium and to the equations of motion in the solution of all sample problems in this book.

**A Four-Color Presentation Uses Color to Distinguish Vectors.**

Color has been used, not only to enhance the quality of the illustrations, but also to help students distinguish among the various types of vectors they will encounter. While there was no intention to “color code” this text, the same color is used in any given chapter to represent vectors of the same type. Throughout *Statics*, for example, red

is used exclusively to represent forces and couples, while position vectors are shown in blue and dimensions in black. This makes it easier for the students to identify the forces acting on a given particle or rigid body and to follow the discussion of sample problems and other examples given in the text.

### A Careful Balance Between SI and U.S. Customary Units Is Consistently Maintained.

Because of the current trend in the American government and industry to adopt the international system of units (SI metric units), the SI units most frequently used in mechanics are introduced in Chap. 1 and are used throughout the text. Approximately half of the sample problems and 60 percent of the homework problems are stated in these units, while the remainder are in U.S. customary units. The authors believe that this approach will best serve the need of students, who, as engineers, will have to be conversant with both systems of units.

It also should be recognized that using both SI and U.S. customary units entails more than the use of conversion factors. Since the SI system of units is an absolute system based on the units of time, length, and mass, whereas the U.S. customary system is a gravitational system based on the units of time, length, and force, different approaches are required for the solution of many problems. For example, when SI units are used, a body is generally specified by its mass expressed in kilograms; in most problems of statics it will be necessary to determine the weight of the body in newtons, and an additional calculation will be required for this purpose. On the other hand, when U.S. customary units are used, a body is specified by its weight in pounds and, in dynamics problems, an additional calculation will be required to determine its mass in slugs (or  $\text{lb} \cdot \text{s}^2/\text{ft}$ ). The authors, therefore, believe that problem assignments should include both systems of units.

The *Instructor's and Solutions Manual* provides six different lists of assignments so that an equal number of problems stated in SI units and in U.S. customary units can be selected. If so desired, two complete lists of assignments can also be selected with up to 75 percent of the problems stated in SI units.

### Optional Sections Offer Advanced or Specialty Topics.

A large number of optional sections have been included. These sections are indicated by asterisks and thus are easily distinguished from those which form the core of the basic statics course. They may be omitted without prejudice to the understanding of the rest of the text.

Among the topics covered in these additional sections are the reduction of a system of forces to a wrench, applications to hydrostatics, shear and bending-moment diagrams for beams, equilibrium of cables, products of inertia and Mohr's circle, the determination of the principal axes and the mass moments of inertia of a body of arbitrary shape, and the method of virtual work. The sections on beams are especially useful when the course in statics is immediately followed by a course in mechanics of materials, while the sections on the inertia properties of three-dimensional bodies are primarily intended for students who will later study in dynamics the three-dimensional motion of rigid bodies.

The material presented in the text and most of the problems require no previous mathematical knowledge beyond algebra, trigonometry, and elementary calculus; all the elements of vector algebra necessary to the understanding of the text are carefully presented in Chaps. 2 and 3. In general, a greater emphasis is placed on the correct understanding of the basic mathematical concepts involved than on the nimble manipulation of mathematical formulas. In this connection, it should be mentioned that the determination of the centroids of composite areas precedes the calculation of centroids by integration, thus making it possible to establish the concept of the moment of an area firmly before introducing the use of integration.

## CHAPTER ORGANIZATION AND PEDAGOGICAL FEATURES

**Chapter Introduction.** Each chapter begins with an introductory section setting the purpose and goals of the chapter and describing in simple terms the material to be covered and its application to the solution of engineering problems. Chapter outlines provide students with a preview of chapter topics.

**Chapter Lessons.** The body of the text is divided into units, each consisting of one or several theory sections, one or several sample problems, and a large number of problems to be assigned. Each unit corresponds to a well-defined topic and generally can be covered in one lesson. In a number of cases, however, the instructor will find it desirable to devote more than one lesson to a given topic. The *Instructor's and Solutions Manual* contains suggestions on the coverage of each lesson.

**Sample Problems.** The sample problems are set up in much the same form that students will use when solving the assigned problems. They thus serve the double purpose of amplifying the text and demonstrating the type of neat, orderly work that students should cultivate in their own solutions.

**Solving Problems on Your Own.** A section entitled *Solving Problems on Your Own* is included for each lesson, between the sample problems and the problems to be assigned. The purpose of these sections is to help students organize in their own minds the preceding theory of the text and the solution methods of the sample problems, so that they can more successfully solve the homework problems. Also included in these sections are specific suggestions and strategies that will enable students to more efficiently attack any assigned problems.

**Homework Problem Sets.** Most of the problems are of a practical nature and should appeal to engineering students. They are primarily designed, however, to illustrate the material presented in the text and to help students understand the principles of mechanics. The problems are grouped according to the portions of material they illustrate and are arranged in order of increasing difficulty. Problems requiring special attention are indicated by asterisks. Answers to 70 percent of the problems are given at the end of the book. Problems for

which the answers are given are set in straight type in the text, while problems for which no answer is given are set in italic.

**Chapter Review and Summary.** Each chapter ends with a review and summary of the material covered in that chapter. Marginal notes are used to help students organize their review work, and cross-references have been included to help them find the portions of material requiring their special attention.

**Review Problems.** A set of review problems is included at the end of each chapter. These problems provide students further opportunity to apply the most important concepts introduced in the chapter.

**Computer Problems.** Each chapter includes a set of problems designed to be solved with computational software. Many of these problems are relevant to the design process. In statics, for example, they may involve the analysis of a structure for various configurations and loadings of the structure or the determination of the equilibrium positions of a mechanism which may require an iterative method of solution. Developing the algorithm required to solve a given mechanics problem will benefit students in two different ways: (1) it will help them gain a better understanding of the mechanics principles involved; (2) it will provide them with an opportunity to apply their computer skills to the solution of a meaningful engineering problem.

## SUPPLEMENTS

An extensive supplements package for both instructors and students is available with the text.

**Instructor's and Solutions Manual.** The *Instructor's and Solutions Manual* that accompanies the ninth edition features typeset, one-per-page solutions to the homework problems. This manual also features a number of tables designed to assist instructors in creating a schedule of assignments for their courses. The various topics covered in the text are listed in Table I, and a suggested number of periods to be spent on each topic is indicated. Table II provides a brief description of all groups of problems and a classification of the problems in each group according to the units used. Sample lesson schedules are shown in Tables III, IV, and V.

## McGRAW-HILL CONNECT ENGINEERING

McGraw-Hill Connect Engineering is a web-based assignment and assessment platform that gives students the means to better connect with their coursework, their instructors, and the important concepts that they will need to know for success now and in the future. With Connect Engineering, instructors can deliver assignments, quizzes, and tests easily online. Students can practice important skills at their own pace and on their own schedule.

Connect Engineering for *Vector Mechanics for Engineers* is available at [www.mhhe.com/beerjohnston](http://www.mhhe.com/beerjohnston) and includes algorithmic problems from the text, Lecture PowerPoints, an image bank, and animations.

**Hands-on Mechanics.** Hands-on Mechanics is a website designed for instructors who are interested in incorporating three-dimensional, hands-on teaching aids into their lectures. Developed through a partnership between the McGraw-Hill Engineering Team and the Department of Civil and Mechanical Engineering at the United States Military Academy at West Point, this website not only provides detailed instructions for how to build 3-D teaching tools using materials found in any lab or local hardware store but also provides a community where educators can share ideas, trade best practices, and submit their own demonstrations for posting on the site. Visit [www.handsonmechanics.com](http://www.handsonmechanics.com).

## ELECTRONIC TEXTBOOK OPTIONS

Ebooks are an innovative way for students to save money and create a greener environment at the same time. An ebook can save students about half the cost of a traditional textbook and offers unique features like a powerful search engine, highlighting, and the ability to share notes with classmates using ebooks.

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E. Russell Johnston, Jr.  
David F. Mazurek  
Elliot R. Eisenberg

# List of Symbols

$a$	Constant; radius; distance
<b>A, B, C, ...</b>	Reactions at supports and connections
$A, B, C, \dots$	Points
$A$	Area
$b$	Width; distance
$c$	Constant
$C$	Centroid
$d$	Distance
$e$	Base of natural logarithms
<b>F</b>	Force; friction force
$g$	Acceleration of gravity
$G$	Center of gravity; constant of gravitation
$h$	Height; sag of cable
<b>i, j, k</b>	Unit vectors along coordinate axes
$I, I_x, \dots$	Moments of inertia
$\bar{I}$	Centroidal moment of inertia
$I_{xy}, \dots$	Products of inertia
$J$	Polar moment of inertia
$k$	Spring constant
$k_x, k_y, k_O$	Radii of gyration
$\bar{k}$	Centroidal radius of gyration
$l$	Length
$L$	Length; span
$m$	Mass
<b>M</b>	Couple; moment
$\mathbf{M}_O$	Moment about point $O$
$\mathbf{M}_O^R$	Moment resultant about point $O$
$M$	Magnitude of couple or moment; mass of earth
$M_{OL}$	Moment about axis $OL$
<b>N</b>	Normal component of reaction
$O$	Origin of coordinates
$p$	Pressure
<b>P</b>	Force; vector
<b>Q</b>	Force; vector
<b>r</b>	Position vector
$r$	Radius; distance; polar coordinate
<b>R</b>	Resultant force; resultant vector; reaction
$R$	Radius of earth
<b>s</b>	Position vector
$s$	Length of arc; length of cable
<b>S</b>	Force; vector
$t$	Thickness
<b>T</b>	Force
$T$	Tension
<b>U</b>	Work

<b>V</b>	Vector product; shearing force
<i>V</i>	Volume; potential energy; shear
<i>w</i>	Load per unit length
<b>W, W</b>	Weight; load
<i>x, y, z</i>	Rectangular coordinates; distances
$\bar{x}, \bar{y}, \bar{z}$	Rectangular coordinates of centroid or center of gravity
$\alpha, \beta, \gamma$	Angles
$\gamma$	Specific weight
$\delta$	Elongation
$\delta\mathbf{r}$	Virtual displacement
$\delta U$	Virtual work
$\lambda$	Unit vector along a line
$\eta$	Efficiency
$\theta$	Angular coordinate; angle; polar coordinate
$\mu$	Coefficient of friction
$\rho$	Density
$\phi$	Angle of friction; angle



# **VECTOR MECHANICS FOR ENGINEERS**

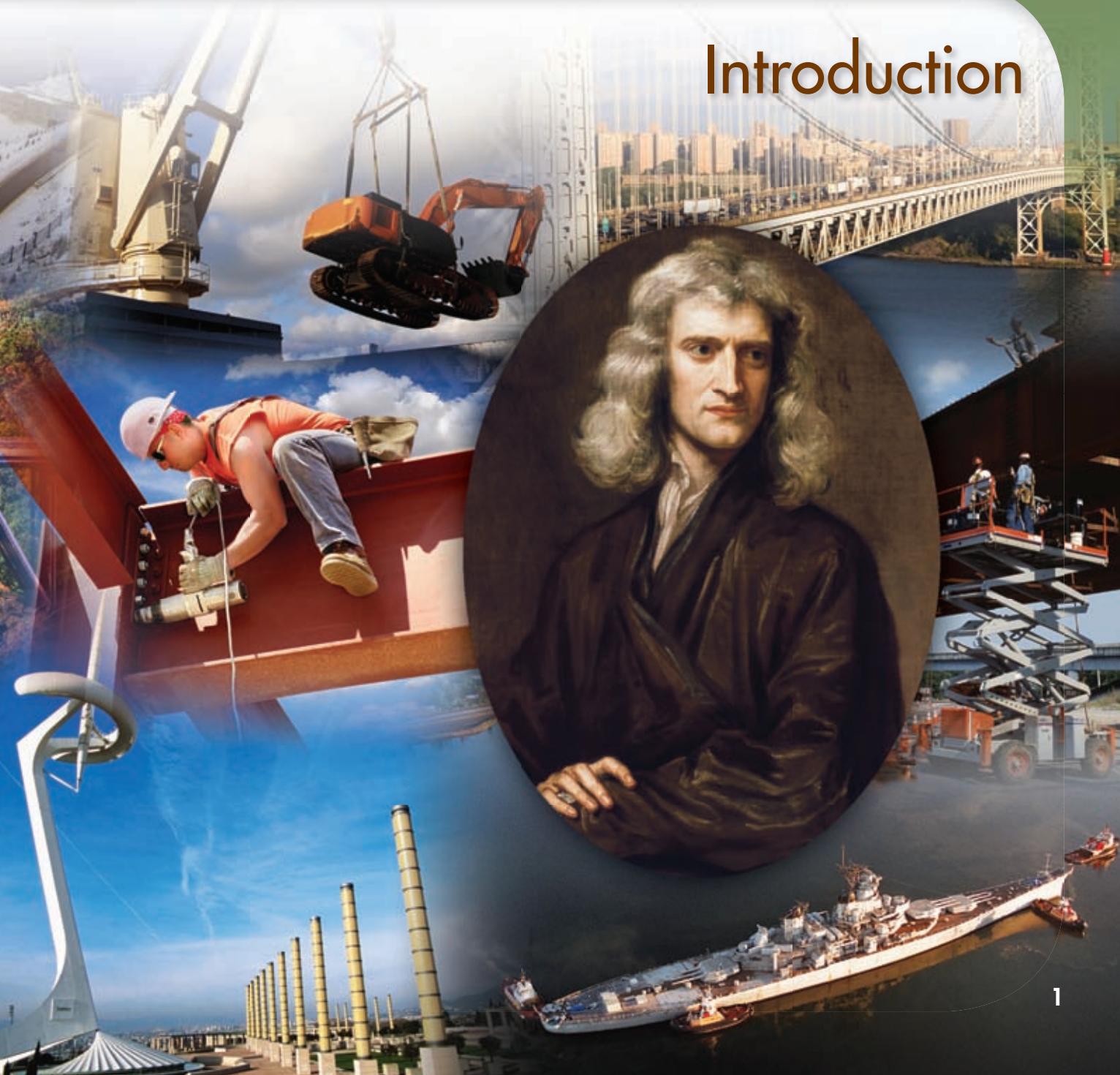
**Statics**

**In the latter part of the seventeenth century, Sir Isaac Newton stated the fundamental principles of mechanics, which are the foundation of much of today's engineering.**



# CHAPTER

## Introduction



## Chapter 1 Introduction

- 1.1 What Is Mechanics?
- 1.2 Fundamental Concepts and Principles
- 1.3 Systems of Units
- 1.4 Conversion from One System of Units to Another
- 1.5 Method of Problem Solution
- 1.6 Numerical Accuracy

### 1.1 WHAT IS MECHANICS?

Mechanics can be defined as that science which describes and predicts the conditions of rest or motion of bodies under the action of forces. It is divided into three parts: mechanics of *rigid bodies*, mechanics of *deformable bodies*, and mechanics of *fluids*.

The mechanics of rigid bodies is subdivided into *statics* and *dynamics*, the former dealing with bodies at rest, the latter with bodies in motion. In this part of the study of mechanics, bodies are assumed to be perfectly rigid. Actual structures and machines, however, are never absolutely rigid and deform under the loads to which they are subjected. But these deformations are usually small and do not appreciably affect the conditions of equilibrium or motion of the structure under consideration. They are important, though, as far as the resistance of the structure to failure is concerned and are studied in mechanics of materials, which is a part of the mechanics of deformable bodies. The third division of mechanics, the mechanics of fluids, is subdivided into the study of *incompressible fluids* and of *compressible fluids*. An important subdivision of the study of incompressible fluids is *hydraulics*, which deals with problems involving water.

Mechanics is a physical science, since it deals with the study of physical phenomena. However, some associate mechanics with mathematics, while many consider it as an engineering subject. Both these views are justified in part. Mechanics is the foundation of most engineering sciences and is an indispensable prerequisite to their study. However, it does not have the *empiricism* found in some engineering sciences, i.e., it does not rely on experience or observation alone; by its rigor and the emphasis it places on deductive reasoning it resembles mathematics. But, again, it is not an *abstract* or even a *pure* science; mechanics is an *applied* science. The purpose of mechanics is to explain and predict physical phenomena and thus to lay the foundations for engineering applications.

### 1.2 FUNDAMENTAL CONCEPTS AND PRINCIPLES

Although the study of mechanics goes back to the time of Aristotle (384–322 B.C.) and Archimedes (287–212 B.C.), one has to wait until Newton (1642–1727) to find a satisfactory formulation of its fundamental principles. These principles were later expressed in a modified form by d'Alembert, Lagrange, and Hamilton. Their validity remained unchallenged, however, until Einstein formulated his *theory of relativity* (1905). While its limitations have now been recognized, *newtonian mechanics* still remains the basis of today's engineering sciences.

The basic concepts used in mechanics are *space*, *time*, *mass*, and *force*. These concepts cannot be truly defined; they should be accepted on the basis of our intuition and experience and used as a mental frame of reference for our study of mechanics.

The concept of *space* is associated with the notion of the position of a point *P*. The position of *P* can be defined by three lengths measured from a certain reference point, or *origin*, in three given directions. These lengths are known as the *coordinates* of *P*.

To define an event, it is not sufficient to indicate its position in space. The *time* of the event should also be given.

The concept of *mass* is used to characterize and compare bodies on the basis of certain fundamental mechanical experiments. Two bodies of the same mass, for example, will be attracted by the earth in the same manner; they will also offer the same resistance to a change in translational motion.

A *force* represents the action of one body on another. It can be exerted by actual contact or at a distance, as in the case of gravitational forces and magnetic forces. A force is characterized by its *point of application*, its *magnitude*, and its *direction*; a force is represented by a *vector* (Sec. 2.3).

In newtonian mechanics, space, time, and mass are absolute concepts, independent of each other. (This is not true in *relativistic mechanics*, where the time of an event depends upon its position, and where the mass of a body varies with its velocity.) On the other hand, the concept of force is not independent of the other three. Indeed, one of the fundamental principles of newtonian mechanics listed below indicates that the resultant force acting on a body is related to the mass of the body and to the manner in which its velocity varies with time.

You will study the conditions of rest or motion of particles and rigid bodies in terms of the four basic concepts we have introduced. By *particle* we mean a very small amount of matter which may be assumed to occupy a single point in space. A *rigid body* is a combination of a large number of particles occupying fixed positions with respect to each other. The study of the mechanics of particles is obviously a prerequisite to that of rigid bodies. Besides, the results obtained for a particle can be used directly in a large number of problems dealing with the conditions of rest or motion of actual bodies.

The study of elementary mechanics rests on six fundamental principles based on experimental evidence.

**The Parallelogram Law for the Addition of Forces.** This states that two forces acting on a particle may be replaced by a single force, called their *resultant*, obtained by drawing the diagonal of the parallelogram which has sides equal to the given forces (Sec. 2.2).

**The Principle of Transmissibility.** This states that the conditions of equilibrium or of motion of a rigid body will remain unchanged if a force acting at a given point of the rigid body is replaced by a force of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action (Sec. 3.3).

**Newton's Three Fundamental Laws.** Formulated by Sir Isaac Newton in the latter part of the seventeenth century, these laws can be stated as follows:

**FIRST LAW.** If the resultant force acting on a particle is zero, the particle will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion) (Sec. 2.10).

**SECOND LAW.** If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.

As you will see in Sec. 12.2, this law can be stated as

$$\mathbf{F} = m\mathbf{a} \quad (1.1)$$

where  $\mathbf{F}$ ,  $m$ , and  $\mathbf{a}$  represent, respectively, the resultant force acting on the particle, the mass of the particle, and the acceleration of the particle, expressed in a consistent system of units.

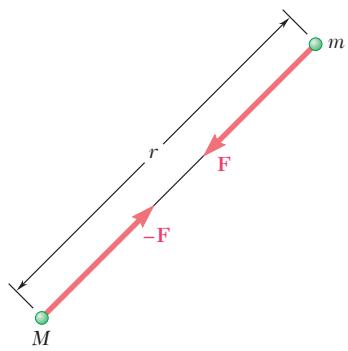


Fig. 1.1

**THIRD LAW.** The forces of action and reaction between bodies in contact have the same magnitude, same line of action, and opposite sense (Sec. 6.1).

**Newton's Law of Gravitation.** This states that two particles of mass  $M$  and  $m$  are mutually attracted with equal and opposite forces  $\mathbf{F}$  and  $-\mathbf{F}$  (Fig. 1.1) of magnitude  $F$  given by the formula

$$F = G \frac{Mm}{r^2} \quad (1.2)$$

where  $r$  = distance between the two particles

$G$  = universal constant called the *constant of gravitation*

Newton's law of gravitation introduces the idea of an action exerted at a distance and extends the range of application of Newton's third law: the action  $\mathbf{F}$  and the reaction  $-\mathbf{F}$  in Fig. 1.1 are equal and opposite, and they have the same line of action.

A particular case of great importance is that of the attraction of the earth on a particle located on its surface. The force  $\mathbf{F}$  exerted by the earth on the particle is then defined as the *weight*  $\mathbf{W}$  of the particle. Taking  $M$  equal to the mass of the earth,  $m$  equal to the mass of the particle, and  $r$  equal to the radius  $R$  of the earth, and introducing the constant

$$g = \frac{GM}{R^2} \quad (1.3)$$

the magnitude  $W$  of the weight of a particle of mass  $m$  may be expressed as†

$$W = mg \quad (1.4)$$

The value of  $R$  in formula (1.3) depends upon the elevation of the point considered; it also depends upon its latitude, since the earth is not truly spherical. The value of  $g$  therefore varies with the position of the point considered. As long as the point actually remains on the surface of the earth, it is sufficiently accurate in most engineering computations to assume that  $g$  equals  $9.81 \text{ m/s}^2$  or  $32.2 \text{ ft/s}^2$ .



**Photo 1.1** When in earth orbit, people and objects are said to be *weightless* even though the gravitational force acting is approximately 90% of that experienced on the surface of the earth. This apparent contradiction will be resolved in Chapter 12 when we apply Newton's second law to the motion of particles.

†A more accurate definition of the weight  $\mathbf{W}$  should take into account the rotation of the earth.

The principles we have just listed will be introduced in the course of our study of mechanics as they are needed. The study of the statics of particles carried out in Chap. 2, will be based on the parallelogram law of addition and on Newton's first law alone. The principle of transmissibility will be introduced in Chap. 3 as we begin the study of the statics of rigid bodies, and Newton's third law in Chap. 6 as we analyze the forces exerted on each other by the various members forming a structure. In the study of dynamics, Newton's second law and Newton's law of gravitation will be introduced. It will then be shown that Newton's first law is a particular case of Newton's second law (Sec. 12.2) and that the principle of transmissibility could be derived from the other principles and thus eliminated (Sec. 16.5). In the meantime, however, Newton's first and third laws, the parallelogram law of addition, and the principle of transmissibility will provide us with the necessary and sufficient foundation for the entire study of the statics of particles, rigid bodies, and systems of rigid bodies.

As noted earlier, the six fundamental principles listed above are based on experimental evidence. Except for Newton's first law and the principle of transmissibility, they are independent principles which cannot be derived mathematically from each other or from any other elementary physical principle. On these principles rests most of the intricate structure of newtonian mechanics. For more than two centuries a tremendous number of problems dealing with the conditions of rest and motion of rigid bodies, deformable bodies, and fluids have been solved by applying these fundamental principles. Many of the solutions obtained could be checked experimentally, thus providing a further verification of the principles from which they were derived. It is only in the twentieth century that Newton's mechanics was found at fault, in the study of the motion of atoms and in the study of the motion of certain planets, where it must be supplemented by the theory of relativity. But on the human or engineering scale, where velocities are small compared with the speed of light, Newton's mechanics has yet to be disproved.

### 1.3 SYSTEMS OF UNITS

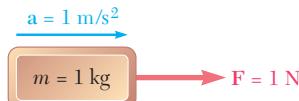
With the four fundamental concepts introduced in the preceding section are associated the so-called *kinetic units*, i.e., the units of *length*, *time*, *mass*, and *force*. These units cannot be chosen independently if Eq. (1.1) is to be satisfied. Three of the units may be defined arbitrarily; they are then referred to as *basic units*. The fourth unit, however, must be chosen in accordance with Eq. (1.1) and is referred to as a *derived unit*. Kinetic units selected in this way are said to form a *consistent system of units*.

**International System of Units (SI Units†).** In this system, which will be in universal use after the United States has completed its conversion to SI units, the base units are the units of length, mass, and time, and they are called, respectively, the *meter* (m), the *kilogram* (kg), and the *second* (s). All three are arbitrarily defined. The second,

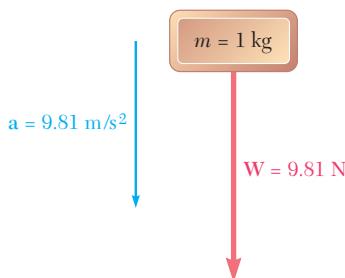
†SI stands for *Système International d'Unités* (French).

which was originally chosen to represent 1/86 400 of the mean solar day, is now defined as the duration of 9 192 631 770 cycles of the radiation corresponding to the transition between two levels of the fundamental state of the cesium-133 atom. The meter, originally defined as one ten-millionth of the distance from the equator to either pole, is now defined as 1 650 763.73 wavelengths of the orange-red light corresponding to a certain transition in an atom of krypton-86. The kilogram, which is approximately equal to the mass of 0.001 m<sup>3</sup> of water, is defined as the mass of a platinum-iridium standard kept at the International Bureau of Weights and Measures at Sèvres, near Paris, France. The unit of force is a derived unit. It is called the *newton* (N) and is defined as the force which gives an acceleration of 1 m/s<sup>2</sup> to a mass of 1 kg (Fig. 1.2). From Eq. (1.1) we write

$$1 \text{ N} = (1 \text{ kg})(1 \text{ m/s}^2) = 1 \text{ kg} \cdot \text{m/s}^2 \quad (1.5)$$



**Fig. 1.2**



**Fig. 1.3**

The SI units are said to form an *absolute* system of units. This means that the three base units chosen are independent of the location where measurements are made. The meter, the kilogram, and the second may be used anywhere on the earth; they may even be used on another planet. They will always have the same significance.

The *weight* of a body, or the *force of gravity* exerted on that body, should, like any other force, be expressed in newtons. From Eq. (1.4) it follows that the weight of a body of mass 1 kg (Fig. 1.3) is

$$\begin{aligned} W &= mg \\ &= (1 \text{ kg})(9.81 \text{ m/s}^2) \\ &= 9.81 \text{ N} \end{aligned}$$

Multiples and submultiples of the fundamental SI units may be obtained through the use of the prefixes defined in Table 1.1. The multiples and submultiples of the units of length, mass, and force most frequently used in engineering are, respectively, the *kilometer* (km) and the *millimeter* (mm); the *megagram*† (Mg) and the *gram* (g); and the *kilonewton* (kN). According to Table 1.1, we have

$$\begin{array}{ll} 1 \text{ km} = 1000 \text{ m} & 1 \text{ mm} = 0.001 \text{ m} \\ 1 \text{ Mg} = 1000 \text{ kg} & 1 \text{ g} = 0.001 \text{ kg} \\ & 1 \text{ kN} = 1000 \text{ N} \end{array}$$

The conversion of these units into meters, kilograms, and newtons, respectively, can be effected by simply moving the decimal point three places to the right or to the left. For example, to convert 3.82 km into meters, one moves the decimal point three places to the right:

$$3.82 \text{ km} = 3820 \text{ m}$$

Similarly, 47.2 mm is converted into meters by moving the decimal point three places to the left:

$$47.2 \text{ mm} = 0.0472 \text{ m}$$

†Also known as a *metric ton*.

**TABLE 1.1 SI Prefixes**

Multiplication Factor	Prefix†	Symbol
$1\ 000\ 000\ 000\ 000 = 10^{12}$	tera	T
$1\ 000\ 000\ 000 = 10^9$	giga	G
$1\ 000\ 000 = 10^6$	mega	M
$1\ 000 = 10^3$	kilo	k
$100 = 10^2$	hecto‡	h
$10 = 10^1$	deka‡	da
$0.1 = 10^{-1}$	deci‡	d
$0.01 = 10^{-2}$	centi‡	c
$0.001 = 10^{-3}$	milli	m
$0.000\ 001 = 10^{-6}$	micro	$\mu$
$0.000\ 000\ 001 = 10^{-9}$	nano	n
$0.000\ 000\ 000\ 001 = 10^{-12}$	pico	p
$0.000\ 000\ 000\ 000\ 001 = 10^{-15}$	femto	f
$0.000\ 000\ 000\ 000\ 000\ 001 = 10^{-18}$	atto	a

†The first syllable of every prefix is accented so that the prefix will retain its identity. Thus, the preferred pronunciation of kilometer places the accent on the first syllable, not the second.

‡The use of these prefixes should be avoided, except for the measurement of areas and volumes and for the nontechnical use of centimeter, as for body and clothing measurements.

Using scientific notation, one may also write

$$\begin{aligned}3.82 \text{ km} &= 3.82 \times 10^3 \text{ m} \\47.2 \text{ mm} &= 47.2 \times 10^{-3} \text{ m}\end{aligned}$$

The multiples of the unit of time are the *minute* (min) and the *hour* (h). Since  $1 \text{ min} = 60 \text{ s}$  and  $1 \text{ h} = 60 \text{ min} = 3600 \text{ s}$ , these multiples cannot be converted as readily as the others.

By using the appropriate multiple or submultiple of a given unit, one can avoid writing very large or very small numbers. For example, one usually writes 427.2 km rather than 427 200 m, and 2.16 mm rather than 0.002 16 m.†

**Units of Area and Volume.** The unit of area is the *square meter* ( $\text{m}^2$ ), which represents the area of a square of side 1 m; the unit of volume is the *cubic meter* ( $\text{m}^3$ ), equal to the volume of a cube of side 1 m. In order to avoid exceedingly small or large numerical values in the computation of areas and volumes, one uses systems of subunits obtained by respectively squaring and cubing not only the millimeter but also two intermediate submultiples of the meter, namely, the *decimeter* (dm) and the *centimeter* (cm). Since, by definition,

$$\begin{aligned}1 \text{ dm} &= 0.1 \text{ m} = 10^{-1} \text{ m} \\1 \text{ cm} &= 0.01 \text{ m} = 10^{-2} \text{ m} \\1 \text{ mm} &= 0.001 \text{ m} = 10^{-3} \text{ m}\end{aligned}$$

†It should be noted that when more than four digits are used on either side of the decimal point to express a quantity in SI units—as in 427 200 m or 0.002 16 m—spaces, never commas, should be used to separate the digits into groups of three. This is to avoid confusion with the comma used in place of a decimal point, which is the convention in many countries.

the submultiples of the unit of area are

$$\begin{aligned}1 \text{ dm}^2 &= (1 \text{ dm})^2 = (10^{-1} \text{ m})^2 = 10^{-2} \text{ m}^2 \\1 \text{ cm}^2 &= (1 \text{ cm})^2 = (10^{-2} \text{ m})^2 = 10^{-4} \text{ m}^2 \\1 \text{ mm}^2 &= (1 \text{ mm})^2 = (10^{-3} \text{ m})^2 = 10^{-6} \text{ m}^2\end{aligned}$$

and the submultiples of the unit of volume are

$$\begin{aligned}1 \text{ dm}^3 &= (1 \text{ dm})^3 = (10^{-1} \text{ m})^3 = 10^{-3} \text{ m}^3 \\1 \text{ cm}^3 &= (1 \text{ cm})^3 = (10^{-2} \text{ m})^3 = 10^{-6} \text{ m}^3 \\1 \text{ mm}^3 &= (1 \text{ mm})^3 = (10^{-3} \text{ m})^3 = 10^{-9} \text{ m}^3\end{aligned}$$

It should be noted that when the volume of a liquid is being measured, the cubic decimeter ( $\text{dm}^3$ ) is usually referred to as a *liter* (L).

Other derived SI units used to measure the moment of a force, the work of a force, etc., are shown in Table 1.2. While these units will be introduced in later chapters as they are needed, we should note an important rule at this time: When a derived unit is obtained by dividing a base unit by another base unit, a prefix may be used in the numerator of the derived unit but not in its denominator. For example, the constant  $k$  of a spring which stretches 20 mm under a load of 100 N will be expressed as

$$k = \frac{100 \text{ N}}{20 \text{ mm}} = \frac{100 \text{ N}}{0.020 \text{ m}} = 5000 \text{ N/m} \quad \text{or} \quad k = 5 \text{ kN/m}$$

but never as  $k = 5 \text{ N/mm}$ .

**TABLE 1.2 Principal SI Units Used in Mechanics**

Quantity	Unit	Symbol	Formula
Acceleration	Meter per second squared	...	$\text{m/s}^2$
Angle	Radian	rad	†
Angular acceleration	Radian per second squared	...	$\text{rad/s}^2$
Angular velocity	Radian per second	...	$\text{rad/s}$
Area	Square meter	...	$\text{m}^2$
Density	Kilogram per cubic meter	...	$\text{kg/m}^3$
Energy	Joule	J	$\text{N} \cdot \text{m}$
Force	Newton	N	$\text{kg} \cdot \text{m/s}^2$
Frequency	Hertz	Hz	$\text{s}^{-1}$
Impulse	Newton-second	...	$\text{kg} \cdot \text{m/s}$
Length	Meter	m	†
Mass	Kilogram	kg	†
Moment of a force	Newton-meter	...	$\text{N} \cdot \text{m}$
Power	Watt	W	$\text{J/s}$
Pressure	Pascal	Pa	$\text{N/m}^2$
Stress	Pascal	Pa	$\text{N/m}^2$
Time	Second	s	†
Velocity	Meter per second	...	$\text{m/s}$
Volume	Cubic meter	...	$\text{m}^3$
Solids		L	$10^{-3} \text{ m}^3$
Liquids	Liter	J	$\text{N} \cdot \text{m}$
Work	Joule		

†Supplementary unit (1 revolution =  $2\pi$  rad =  $360^\circ$ ).

‡Base unit.

**U.S. Customary Units.** Most practicing American engineers still commonly use a system in which the base units are the units of length, force, and time. These units are, respectively, the *foot* (ft), the *pound* (lb), and the *second* (s). The second is the same as the corresponding SI unit. The foot is defined as 0.3048 m. The pound is defined as the *weight* of a platinum standard, called the *standard pound*, which is kept at the National Institute of Standards and Technology outside Washington, the mass of which is 0.453 592 43 kg. Since the weight of a body depends upon the earth's gravitational attraction, which varies with location, it is specified that the standard pound should be placed at sea level and at a latitude of 45° to properly define a force of 1 lb. Clearly the U.S. customary units do not form an absolute system of units. Because of their dependence upon the gravitational attraction of the earth, they form a *gravitational* system of units.

While the standard pound also serves as the unit of mass in commercial transactions in the United States, it cannot be so used in engineering computations, since such a unit would not be consistent with the base units defined in the preceding paragraph. Indeed, when acted upon by a force of 1 lb, that is, when subjected to the force of gravity, the standard pound receives the acceleration of gravity,  $g = 32.2 \text{ ft/s}^2$  (Fig. 1.4), not the unit acceleration required by Eq. (1.1). The unit of mass consistent with the foot, the pound, and the second is the mass which receives an acceleration of  $1 \text{ ft/s}^2$  when a force of 1 lb is applied to it (Fig. 1.5). This unit, sometimes called a *slug*, can be derived from the equation  $F = ma$  after substituting 1 lb and  $1 \text{ ft/s}^2$  for  $F$  and  $a$ , respectively. We write

$$F = ma \quad 1 \text{ lb} = (1 \text{ slug})(1 \text{ ft/s}^2)$$

and obtain

$$1 \text{ slug} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} \quad (1.6)$$

Comparing Figs. 1.4 and 1.5, we conclude that the slug is a mass 32.2 times larger than the mass of the standard pound.

The fact that in the U.S. customary system of units bodies are characterized by their weight in pounds rather than by their mass in slugs will be a convenience in the study of statics, where one constantly deals with weights and other forces and only seldom with masses. However, in the study of dynamics, where forces, masses, and accelerations are involved, the mass  $m$  of a body will be expressed in slugs when its weight  $W$  is given in pounds. Recalling Eq. (1.4), we write

$$m = \frac{W}{g} \quad (1.7)$$

where  $g$  is the acceleration of gravity ( $g = 32.2 \text{ ft/s}^2$ ).

Other U.S. customary units frequently encountered in engineering problems are the *mile* (mi), equal to 5280 ft; the *inch* (in.), equal to  $\frac{1}{12}$  ft; and the *kilopound* (kip), equal to a force of 1000 lb. The *ton* is often used to represent a mass of 2000 lb but, like the pound, must be converted into slugs in engineering computations.

The conversion into feet, pounds, and seconds of quantities expressed in other U.S. customary units is generally more involved and

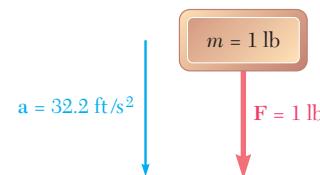


Fig. 1.4

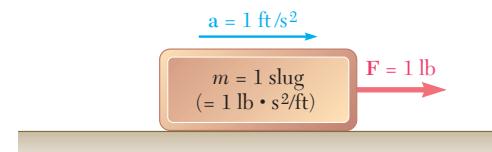


Fig. 1.5

requires greater attention than the corresponding operation in SI units. If, for example, the magnitude of a velocity is given as  $v = 30 \text{ mi/h}$ , we convert it to ft/s as follows. First we write

$$v = 30 \frac{\text{mi}}{\text{h}}$$

Since we want to get rid of the unit miles and introduce instead the unit feet, we should multiply the right-hand member of the equation by an expression containing miles in the denominator and feet in the numerator. But, since we do not want to change the value of the right-hand member, the expression used should have a value equal to unity. The quotient  $(5280 \text{ ft})/(1 \text{ mi})$  is such an expression. Operating in a similar way to transform the unit hour into seconds, we write

$$v = \left(30 \frac{\text{mi}}{\text{h}}\right) \left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) \left(\frac{1 \text{ h}}{3600 \text{ s}}\right)$$

Carrying out the numerical computations and canceling out units which appear in both the numerator and the denominator, we obtain

$$v = 44 \frac{\text{ft}}{\text{s}} = 44 \text{ ft/s}$$

## 1.4 CONVERSION FROM ONE SYSTEM OF UNITS TO ANOTHER

There are many instances when an engineer wishes to convert into SI units a numerical result obtained in U.S. customary units or vice versa. Because the unit of time is the same in both systems, only two kinetic base units need be converted. Thus, since all other kinetic units can be derived from these base units, only two conversion factors need be remembered.

**Units of Length.** By definition the U.S. customary unit of length is

$$1 \text{ ft} = 0.3048 \text{ m} \quad (1.8)$$

It follows that

$$1 \text{ mi} = 5280 \text{ ft} = 5280(0.3048 \text{ m}) = 1609 \text{ m}$$

or

$$1 \text{ mi} = 1.609 \text{ km} \quad (1.9)$$

Also

$$1 \text{ in.} = \frac{1}{12} \text{ ft} = \frac{1}{12}(0.3048 \text{ m}) = 0.0254 \text{ m}$$

or

$$1 \text{ in.} = 25.4 \text{ mm} \quad (1.10)$$

**Units of Force.** Recalling that the U.S. customary unit of force (pound) is defined as the weight of the standard pound (of mass 0.4536 kg) at sea level and at a latitude of  $45^\circ$  (where  $g = 9.807 \text{ m/s}^2$ ) and using Eq. (1.4), we write

$$\begin{aligned} W &= mg \\ 1 \text{ lb} &= (0.4536 \text{ kg})(9.807 \text{ m/s}^2) = 4.448 \text{ kg} \cdot \text{m/s}^2 \end{aligned}$$

or, recalling Eq. (1.5),

$$1 \text{ lb} = 4.448 \text{ N} \quad (1.11)$$

**Units of Mass.** The U.S. customary unit of mass (slug) is a derived unit. Thus, using Eqs. (1.6), (1.8), and (1.11), we write

$$1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = \frac{4.448 \text{ N}}{0.3048 \text{ m/s}^2} = 14.59 \text{ N} \cdot \text{s}^2/\text{m}$$

and, recalling Eq. (1.5),

$$1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} = 14.59 \text{ kg} \quad (1.12)$$

Although it cannot be used as a consistent unit of mass, we recall that the mass of the standard pound is, by definition,

$$1 \text{ pound mass} = 0.4536 \text{ kg} \quad (1.13)$$

This constant may be used to determine the *mass* in SI units (kilograms) of a body which has been characterized by its *weight* in U.S. customary units (pounds).

To convert a derived U.S. customary unit into SI units, one simply multiplies or divides by the appropriate conversion factors. For example, to convert the moment of a force which was found to be  $M = 47 \text{ lb} \cdot \text{in.}$  into SI units, we use formulas (1.10) and (1.11) and write

$$\begin{aligned} M &= 47 \text{ lb} \cdot \text{in.} = 47(4.448 \text{ N})(25.4 \text{ mm}) \\ &= 5310 \text{ N} \cdot \text{mm} = 5.31 \text{ N} \cdot \text{m} \end{aligned}$$

The conversion factors given in this section may also be used to convert a numerical result obtained in SI units into U.S. customary units. For example, if the moment of a force was found to be  $M = 40 \text{ N} \cdot \text{m}$ , we write, following the procedure used in the last paragraph of Sec. 1.3,

$$M = 40 \text{ N} \cdot \text{m} = (40 \text{ N} \cdot \text{m}) \left( \frac{1 \text{ lb}}{4.448 \text{ N}} \right) \left( \frac{1 \text{ ft}}{0.3048 \text{ m}} \right)$$

Carrying out the numerical computations and canceling out units which appear in both the numerator and the denominator, we obtain

$$M = 29.5 \text{ lb} \cdot \text{ft}$$

The U.S. customary units most frequently used in mechanics are listed in Table 1.3 with their SI equivalents.

## 1.5 METHOD OF PROBLEM SOLUTION

You should approach a problem in mechanics as you would approach an actual engineering situation. By drawing on your own experience and intuition, you will find it easier to understand and formulate the problem. Once the problem has been clearly stated, however, there is

**TABLE 1.3 U.S. Customary Units and Their SI Equivalents**

Quantity	U.S. Customary Unit	SI Equivalent
Acceleration	$\text{ft}/\text{s}^2$	$0.3048 \text{ m}/\text{s}^2$
	$\text{in.}/\text{s}^2$	$0.0254 \text{ m}/\text{s}^2$
Area	$\text{ft}^2$	$0.0929 \text{ m}^2$
	$\text{in.}^2$	$645.2 \text{ mm}^2$
Energy	$\text{ft} \cdot \text{lb}$	$1.356 \text{ J}$
Force	kip	$4.448 \text{ kN}$
	lb	$4.448 \text{ N}$
	oz	$0.2780 \text{ N}$
Impulse	$\text{lb} \cdot \text{s}$	$4.448 \text{ N} \cdot \text{s}$
Length	ft	$0.3048 \text{ m}$
	in.	$25.40 \text{ mm}$
	mi	$1.609 \text{ km}$
Mass	oz mass	$28.35 \text{ g}$
	lb mass	$0.4536 \text{ kg}$
	slug	$14.59 \text{ kg}$
	ton	$907.2 \text{ kg}$
Moment of a force	$\text{lb} \cdot \text{ft}$	$1.356 \text{ N} \cdot \text{m}$
	$\text{lb} \cdot \text{in.}$	$0.1130 \text{ N} \cdot \text{m}$
Moment of inertia		
Of an area	$\text{in.}^4$	$0.4162 \times 10^6 \text{ mm}^4$
Of a mass	$\text{lb} \cdot \text{ft} \cdot \text{s}^2$	$1.356 \text{ kg} \cdot \text{m}^2$
Momentum	$\text{lb} \cdot \text{s}$	$4.448 \text{ kg} \cdot \text{m/s}$
Power	$\text{ft} \cdot \text{lb/s}$	$1.356 \text{ W}$
	hp	$745.7 \text{ W}$
Pressure or stress	$\text{lb}/\text{ft}^2$	$47.88 \text{ Pa}$
	$\text{lb}/\text{in.}^2$ (psi)	$6.895 \text{ kPa}$
Velocity	ft/s	$0.3048 \text{ m/s}$
	in./s	$0.0254 \text{ m/s}$
	mi/h (mph)	$0.4470 \text{ m/s}$
	mi/h (mph)	$1.609 \text{ km/h}$
Volume	$\text{ft}^3$	$0.02832 \text{ m}^3$
	$\text{in.}^3$	$16.39 \text{ cm}^3$
Liquids	gal	$3.785 \text{ L}$
	qt	$0.9464 \text{ L}$
Work	$\text{ft} \cdot \text{lb}$	$1.356 \text{ J}$

no place in its solution for your particular fancy. *The solution must be based on the six fundamental principles stated in Sec. 1.2 or on theorems derived from them.* Every step taken must be justified on that basis. Strict rules must be followed, which lead to the solution in an almost automatic fashion, leaving no room for your intuition or “feeling.” After an answer has been obtained, it should be checked. Here again, you may call upon your common sense and personal experience. If not completely satisfied with the result obtained, you should carefully check your formulation of the problem, the validity of the methods used for its solution, and the accuracy of your computations.

The statement of a problem should be clear and precise. It should contain the given data and indicate what information is required. A neat drawing showing all quantities involved should be included. Separate diagrams should be drawn for all bodies involved, indicating clearly the forces acting on each body. These diagrams are known as *free-body diagrams* and are described in detail in Secs. 2.11 and 4.2.

The fundamental principles of mechanics listed in Sec. 1.2 will be used to write equations expressing the conditions of rest or motion of the bodies considered. Each equation should be clearly related to one of the free-body diagrams. You will then proceed to solve the problem, observing strictly the usual rules of algebra and recording neatly the various steps taken.

After the answer has been obtained, it should be *carefully checked*. Mistakes in *reasoning* can often be detected by checking the units. For example, to determine the moment of a force of 50 N about a point 0.60 m from its line of action, we would have written (Sec. 3.12)

$$M = Fd = (50 \text{ N})(0.60 \text{ m}) = 30 \text{ N} \cdot \text{m}$$

The unit N · m obtained by multiplying newtons by meters is the correct unit for the moment of a force; if another unit had been obtained, we would have known that some mistake had been made.

Errors in *computation* will usually be found by substituting the numerical values obtained into an equation which has not yet been used and verifying that the equation is satisfied. The importance of correct computations in engineering cannot be overemphasized.

## 1.6 NUMERICAL ACCURACY

The accuracy of the solution of a problem depends upon two items: (1) the accuracy of the given data and (2) the accuracy of the computations performed.

The solution cannot be more accurate than the less accurate of these two items. For example, if the loading of a bridge is known to be 75,000 lb with a possible error of 100 lb either way, the relative error which measures the degree of accuracy of the data is

$$\frac{100 \text{ lb}}{75,000 \text{ lb}} = 0.0013 = 0.13 \text{ percent}$$

In computing the reaction at one of the bridge supports, it would then be meaningless to record it as 14,322 lb. The accuracy of the solution cannot be greater than 0.13 percent, no matter how accurate the computations are, and the possible error in the answer may be as large as  $(0.13/100)(14,322 \text{ lb}) \approx 20 \text{ lb}$ . The answer should be properly recorded as  $14,320 \pm 20 \text{ lb}$ .

In engineering problems, the data are seldom known with an accuracy greater than 0.2 percent. It is therefore seldom justified to write the answers to such problems with an accuracy greater than 0.2 percent. A practical rule is to use 4 figures to record numbers beginning with a “1” and 3 figures in all other cases. Unless otherwise indicated, the data given in a problem should be assumed known with a comparable degree of accuracy. A force of 40 lb, for example, should be read 40.0 lb, and a force of 15 lb should be read 15.00 lb.

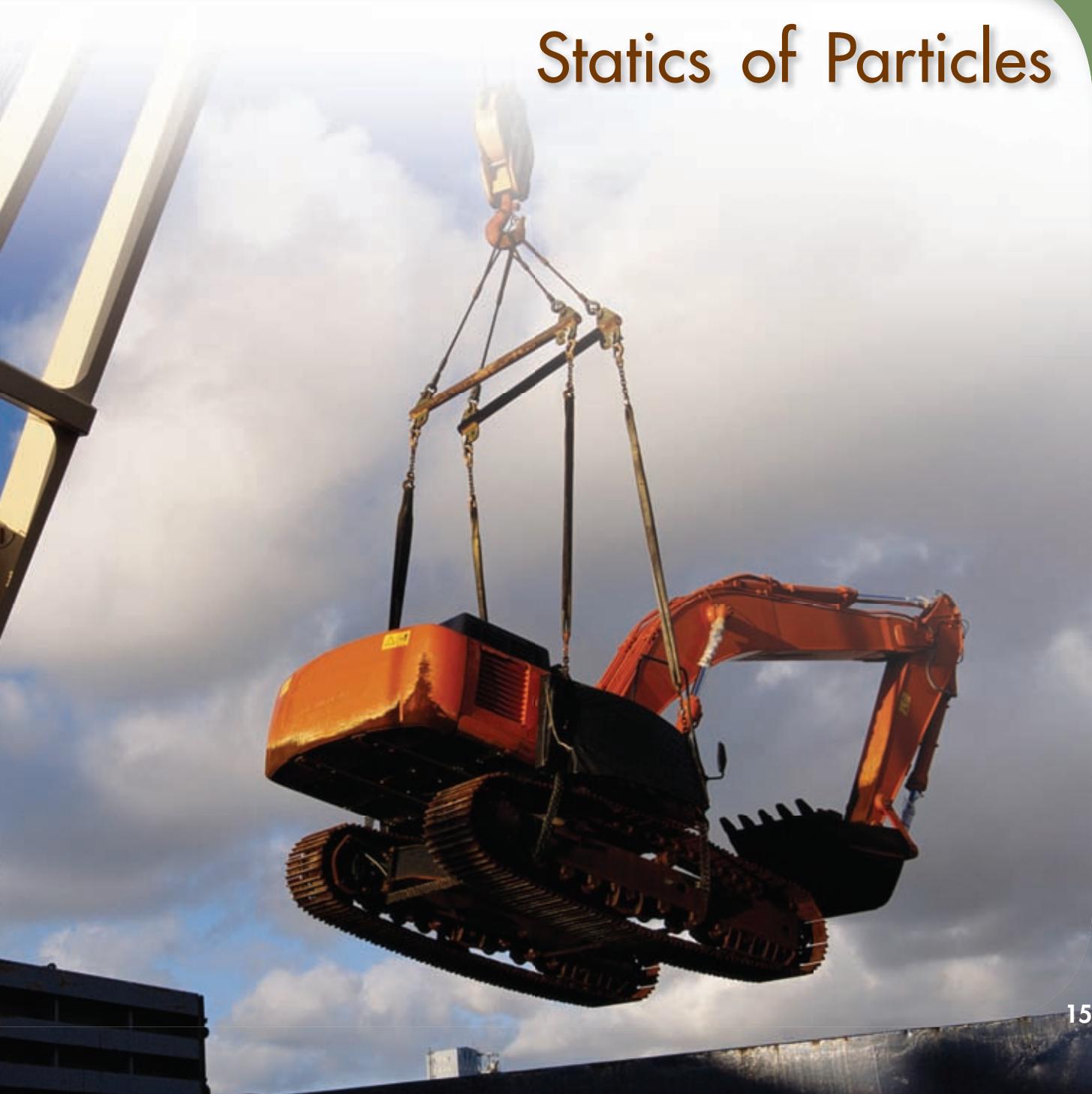
Pocket electronic calculators are widely used by practicing engineers and engineering students. The speed and accuracy of these calculators facilitate the numerical computations in the solution of many problems. However, students should not record more significant figures than can be justified merely because they are easily obtained. As noted above, an accuracy greater than 0.2 percent is seldom necessary or meaningful in the solution of practical engineering problems.

**Many engineering problems can be solved by considering the equilibrium of a “particle.” In the case of this excavator, which is being loaded onto a ship, a relation between the tensions in the various cables involved can be obtained by considering the equilibrium of the hook to which the cables are attached.**



CHAPTER  
2

# Statics of Particles



## Chapter 2 Statics of Particles

- 2.1** Introduction
- 2.2** Force on a Particle. Resultant of Two Forces
- 2.3** Vectors
- 2.4** Addition of Vectors
- 2.5** Resultant of Several Concurrent Forces
- 2.6** Resolution of a Force into Components
- 2.7** Rectangular Components of a Force. Unit Vectors
- 2.8** Addition of Forces by Summing X and Y Components
- 2.9** Equilibrium of a Particle
- 2.10** Newton's First Law of Motion
- 2.11** Problems Involving the Equilibrium of a Particle. Free-Body Diagrams
- 2.12** Rectangular Components of a Force in Space
- 2.13** Force Defined by Its Magnitude and Two Points on Its Line of Action
- 2.14** Addition of Concurrent Forces in Space
- 2.15** Equilibrium of a Particle in Space

## 2.1 INTRODUCTION

In this chapter you will study the effect of forces acting on particles. First you will learn how to replace two or more forces acting on a given particle by a single force having the same effect as the original forces. This single equivalent force is the *resultant* of the original forces acting on the particle. Later the relations which exist among the various forces acting on a particle in a state of *equilibrium* will be derived and used to determine some of the forces acting on the particle.

The use of the word "particle" does not imply that our study will be limited to that of small corpuscles. What it means is that the size and shape of the bodies under consideration will not significantly affect the solution of the problems treated in this chapter and that all the forces acting on a given body will be assumed to be applied at the same point. Since such an assumption is verified in many practical applications, you will be able to solve a number of engineering problems in this chapter.

The first part of the chapter is devoted to the study of forces contained in a single plane, and the second part to the analysis of forces in three-dimensional space.

## FORCES IN A PLANE

### 2.2 FORCE ON A PARTICLE. RESULTANT OF TWO FORCES

A force represents the action of one body on another and is generally characterized by its *point of application*, its **magnitude**, and its **direction**. Forces acting on a given particle, however, have the same point of application. Each force considered in this chapter will thus be completely defined by its magnitude and direction.

The magnitude of a force is characterized by a certain number of units. As indicated in Chap. 1, the SI units used by engineers to measure the magnitude of a force are the newton (N) and its multiple the kilonewton (kN), equal to 1000 N, while the U.S. customary units used for the same purpose are the pound (lb) and its multiple the kilopound (kip), equal to 1000 lb. The direction of a force is defined by the *line of action* and the *sense* of the force. The line of action is the infinite straight line along which the force acts; it is characterized by the angle it forms with some fixed axis (Fig. 2.1). The force itself is represented by a segment of

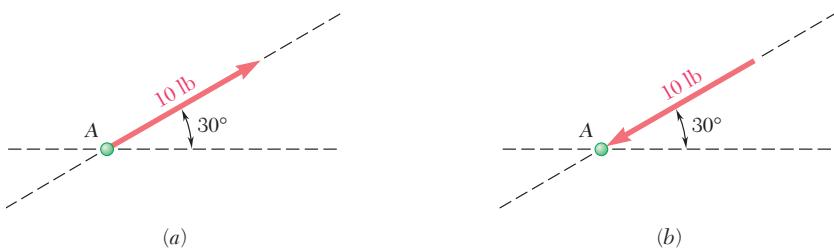


Fig. 2.1

that line; through the use of an appropriate scale, the length of this segment may be chosen to represent the magnitude of the force. Finally, the sense of the force should be indicated by an arrowhead. It is important in defining a force to indicate its sense. Two forces having the same magnitude and the same line of action but different sense, such as the forces shown in Fig. 2.1a and b, will have directly opposite effects on a particle.

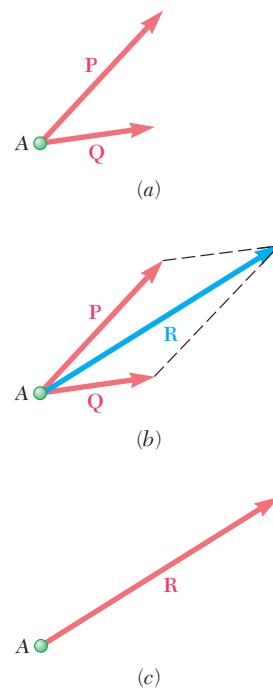
Experimental evidence shows that two forces **P** and **Q** acting on a particle A (Fig. 2.2a) can be replaced by a single force **R** which has the same effect on the particle (Fig. 2.2c). This force is called the *resultant* of the forces **P** and **Q** and can be obtained, as shown in Fig. 2.2b, by constructing a parallelogram, using **P** and **Q** as two adjacent sides of the parallelogram. *The diagonal that passes through A represents the resultant.* This method for finding the resultant is known as the *parallelogram law* for the addition of two forces. This law is based on experimental evidence; it cannot be proved or derived mathematically.

## 2.3 VECTORS

It appears from the above that forces do not obey the rules of addition defined in ordinary arithmetic or algebra. For example, two forces acting at a right angle to each other, one of 4 lb and the other of 3 lb, add up to a force of 5 lb, *not* to a force of 7 lb. Forces are not the only quantities which follow the parallelogram law of addition. As you will see later, *displacements, velocities, accelerations, and momenta* are other examples of physical quantities possessing magnitude and direction that are added according to the parallelogram law. All these quantities can be represented mathematically by *vectors*, while those physical quantities which have magnitude but not direction, such as *volume, mass, or energy*, are represented by plain numbers or *scalars*.

Vectors are defined as *mathematical expressions possessing magnitude and direction, which add according to the parallelogram law*. Vectors are represented by arrows in the illustrations and will be distinguished from scalar quantities in this text through the use of boldface type (**P**). In longhand writing, a vector may be denoted by drawing a short arrow above the letter used to represent it ( $\overrightarrow{P}$ ) or by underlining the letter ( $\underline{P}$ ). The last method may be preferred since underlining can also be used on a typewriter or computer. The magnitude of a vector defines the length of the arrow used to represent the vector. In this text, italic type will be used to denote the magnitude of a vector. Thus, the magnitude of the vector **P** will be denoted by  $P$ .

A vector used to represent a force acting on a given particle has a well-defined point of application, namely, the particle itself. Such a vector is said to be a *fixed, or bound, vector* and cannot be moved without modifying the conditions of the problem. Other physical quantities, however, such as couples (see Chap. 3), are represented by vectors which may be freely moved in space; these



**Fig. 2.2**

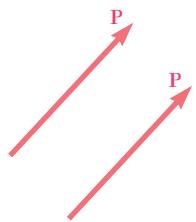


Fig. 2.4

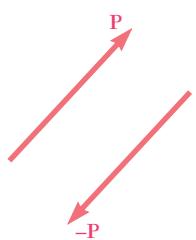


Fig. 2.5

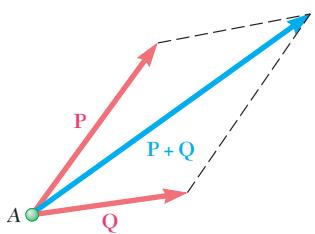


Fig. 2.6

vectors are called *free vectors*. Still other physical quantities, such as forces acting on a rigid body (see Chap. 3), are represented by vectors which can be moved, or slid, along their lines of action; they are known as *sliding vectors*.†

Two vectors which have the same magnitude and the same direction are said to be *equal*, whether or not they also have the same point of application (Fig. 2.4); equal vectors may be denoted by the same letter.

The *negative vector* of a given vector  $\mathbf{P}$  is defined as a vector having the same magnitude as  $\mathbf{P}$  and a direction opposite to that of  $\mathbf{P}$  (Fig. 2.5); the negative of the vector  $\mathbf{P}$  is denoted by  $-\mathbf{P}$ . The vectors  $\mathbf{P}$  and  $-\mathbf{P}$  are commonly referred to as *equal and opposite vectors*. Clearly, we have

$$\mathbf{P} + (-\mathbf{P}) = 0$$

## 2.4 ADDITION OF VECTORS

We saw in the preceding section that, by definition, vectors add according to the parallelogram law. Thus, the sum of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  is obtained by attaching the two vectors to the same point  $A$  and constructing a parallelogram, using  $\mathbf{P}$  and  $\mathbf{Q}$  as two sides of the parallelogram (Fig. 2.6). The diagonal that passes through  $A$  represents the sum of the vectors  $\mathbf{P}$  and  $\mathbf{Q}$ , and this sum is denoted by  $\mathbf{P} + \mathbf{Q}$ . The fact that the sign  $+$  is used to denote both vector and scalar addition should not cause any confusion if vector and scalar quantities are always carefully distinguished. Thus, we should note that the magnitude of the vector  $\mathbf{P} + \mathbf{Q}$  is *not*, in general, equal to the sum  $P + Q$  of the magnitudes of the vectors  $\mathbf{P}$  and  $\mathbf{Q}$ .

Since the parallelogram constructed on the vectors  $\mathbf{P}$  and  $\mathbf{Q}$  does not depend upon the order in which  $\mathbf{P}$  and  $\mathbf{Q}$  are selected, we conclude that the addition of two vectors is *commutative*, and we write

$$\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P} \quad (2.1)$$

†Some expressions have magnitude and direction, but do not add according to the parallelogram law. While these expressions may be represented by arrows, they *cannot* be considered as vectors.

A group of such expressions is the finite rotations of a rigid body. Place a closed book on a table in front of you, so that it lies in the usual fashion, with its front cover up and its binding to the left. Now rotate it through  $180^\circ$  about an axis parallel to the binding (Fig. 2.3a); this rotation may be represented by an arrow of length equal to 180 units and oriented as shown. Picking up the book as it lies in its new position, rotate

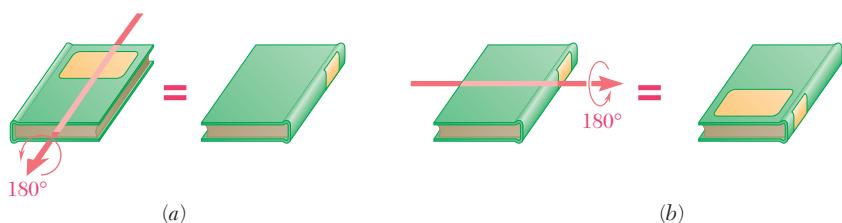


Fig. 2.3 Finite rotations of a rigid body

From the parallelogram law, we can derive an alternative method for determining the sum of two vectors. This method, known as the *triangle rule*, is derived as follows. Consider Fig. 2.6, where the sum of the vectors  $\mathbf{P}$  and  $\mathbf{Q}$  has been determined by the parallelogram law. Since the side of the parallelogram opposite  $\mathbf{Q}$  is equal to  $\mathbf{Q}$  in magnitude and direction, we could draw only half of the parallelogram (Fig. 2.7a). The sum of the two vectors can thus be found by arranging  $\mathbf{P}$  and  $\mathbf{Q}$  in tip-to-tail fashion and then connecting the tail of  $\mathbf{P}$  with the tip of  $\mathbf{Q}$ . In Fig. 2.7b, the other half of the parallelogram is considered, and the same result is obtained. This confirms the fact that vector addition is commutative.

The *subtraction* of a vector is defined as the addition of the corresponding negative vector. Thus, the vector  $\mathbf{P} - \mathbf{Q}$  representing the difference between the vectors  $\mathbf{P}$  and  $\mathbf{Q}$  is obtained by adding to  $\mathbf{P}$  the negative vector  $-\mathbf{Q}$  (Fig. 2.8). We write

$$\mathbf{P} - \mathbf{Q} = \mathbf{P} + (-\mathbf{Q}) \quad (2.2)$$

Here again we should observe that, while the same sign is used to denote both vector and scalar subtraction, confusion will be avoided if care is taken to distinguish between vector and scalar quantities.

We will now consider the *sum of three or more vectors*. The sum of three vectors  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{S}$  will, *by definition*, be obtained by first adding the vectors  $\mathbf{P}$  and  $\mathbf{Q}$  and then adding the vector  $\mathbf{S}$  to the vector  $\mathbf{P} + \mathbf{Q}$ . We thus write

$$\mathbf{P} + \mathbf{Q} + \mathbf{S} = (\mathbf{P} + \mathbf{Q}) + \mathbf{S} \quad (2.3)$$

Similarly, the sum of four vectors will be obtained by adding the fourth vector to the sum of the first three. It follows that the sum of any number of vectors can be obtained by applying repeatedly the parallelogram law to successive pairs of vectors until all the given vectors are replaced by a single vector.

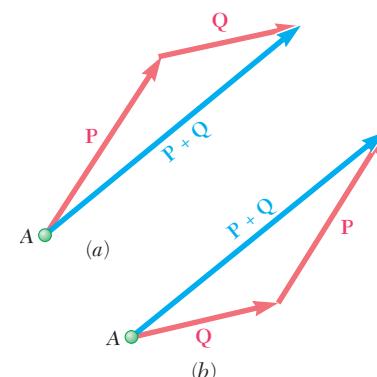


Fig. 2.7

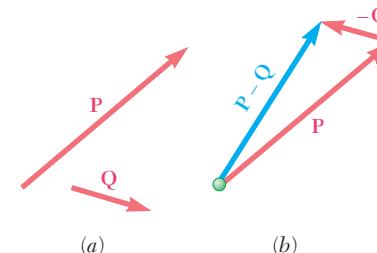
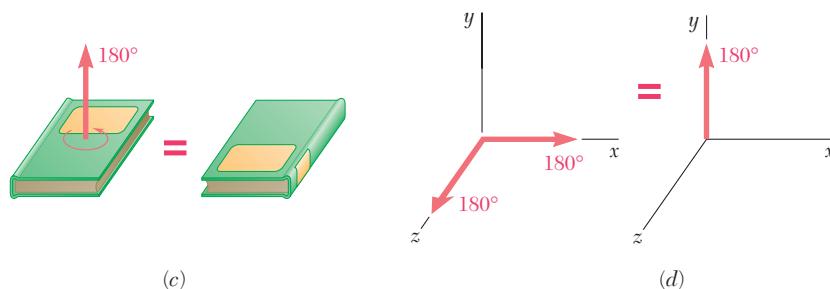


Fig. 2.8

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it now through  $180^\circ$  about a horizontal axis perpendicular to the binding (Fig. 2.3b); this second rotation may be represented by an arrow  $180$  units long and oriented as shown. But the book could have been placed in this final position through a single  $180^\circ$  rotation about a vertical axis (Fig. 2.3c). We conclude that the sum of the two  $180^\circ$  rotations represented by arrows directed respectively along the  $z$  and  $x$  axes is a  $180^\circ$  rotation represented by an arrow directed along the  $y$  axis (Fig. 2.3d). Clearly, the finite rotations of a rigid body *do not* obey the parallelogram law of addition; therefore, they *cannot* be represented by vectors.



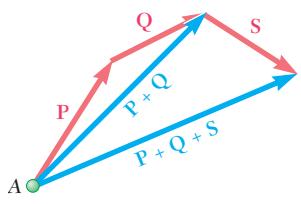


Fig. 2.9

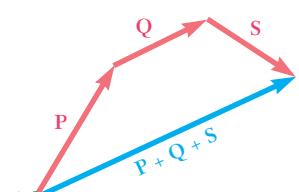


Fig. 2.10

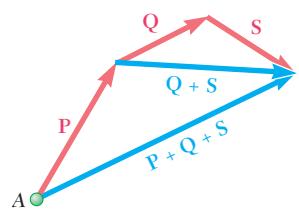


Fig. 2.11

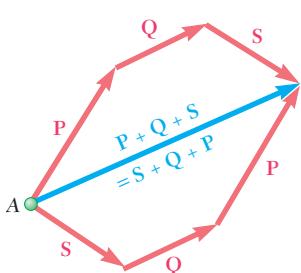


Fig. 2.12

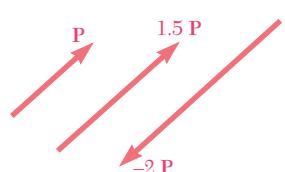


Fig. 2.13

If the given vectors are *coplanar*, i.e., if they are contained in the same plane, their sum can be easily obtained graphically. For this case, the repeated application of the triangle rule is preferred to the application of the parallelogram law. In Fig. 2.9 the sum of three vectors  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{S}$  was obtained in that manner. The triangle rule was first applied to obtain the sum  $\mathbf{P} + \mathbf{Q}$  of the vectors  $\mathbf{P}$  and  $\mathbf{Q}$ ; it was applied again to obtain the sum of the vectors  $\mathbf{P} + \mathbf{Q}$  and  $\mathbf{S}$ . The determination of the vector  $\mathbf{P} + \mathbf{Q}$ , however, could have been omitted and the sum of the three vectors could have been obtained directly, as shown in Fig. 2.10, by *arranging the given vectors in tip-to-tail fashion and connecting the tail of the first vector with the tip of the last one*. This is known as the *polygon rule* for the addition of vectors.

We observe that the result obtained would have been unchanged if, as shown in Fig. 2.11, the vectors  $\mathbf{Q}$  and  $\mathbf{S}$  had been replaced by their sum  $\mathbf{Q} + \mathbf{S}$ . We may thus write

$$\mathbf{P} + \mathbf{Q} + \mathbf{S} = (\mathbf{P} + \mathbf{Q}) + \mathbf{S} = \mathbf{P} + (\mathbf{Q} + \mathbf{S}) \quad (2.4)$$

which expresses the fact that vector addition is *associative*. Recalling that vector addition has also been shown, in the case of two vectors, to be commutative, we write

$$\begin{aligned} \mathbf{P} + \mathbf{Q} + \mathbf{S} &= (\mathbf{P} + \mathbf{Q}) + \mathbf{S} = \mathbf{S} + (\mathbf{P} + \mathbf{Q}) \\ &= \mathbf{S} + (\mathbf{Q} + \mathbf{P}) = \mathbf{S} + \mathbf{Q} + \mathbf{P} \end{aligned} \quad (2.5)$$

This expression, as well as others which may be obtained in the same way, shows that the order in which several vectors are added together is immaterial (Fig. 2.12).

**Product of a Scalar and a Vector.** Since it is convenient to denote the sum  $\mathbf{P} + \mathbf{P}$  by  $2\mathbf{P}$ , the sum  $\mathbf{P} + \mathbf{P} + \mathbf{P}$  by  $3\mathbf{P}$ , and, in general, the sum of  $n$  equal vectors  $\mathbf{P}$  by the product  $n\mathbf{P}$ , we will define the product  $n\mathbf{P}$  of a positive integer  $n$  and a vector  $\mathbf{P}$  as a vector having the same direction as  $\mathbf{P}$  and the magnitude  $nP$ . Extending this definition to include all scalars, and recalling the definition of a negative vector given in Sec. 2.3, we define the product  $k\mathbf{P}$  of a scalar  $k$  and a vector  $\mathbf{P}$  as a vector having the same direction as  $\mathbf{P}$  (if  $k$  is positive), or a direction opposite to that of  $\mathbf{P}$  (if  $k$  is negative), and a magnitude equal to the product of  $P$  and of the absolute value of  $k$  (Fig. 2.13).

## 2.5 RESULTANT OF SEVERAL CONCURRENT FORCES

Consider a particle A acted upon by several coplanar forces, i.e., by several forces contained in the same plane (Fig. 2.14a). Since the forces considered here all pass through A, they are also said to be *concurrent*. The vectors representing the forces acting on A may be added by the polygon rule (Fig. 2.14b). Since the use of the polygon rule is equivalent to the repeated application of the parallelogram law, the vector  $\mathbf{R}$  thus obtained represents the resultant of the given concurrent forces, i.e., the single force which has the same effect on

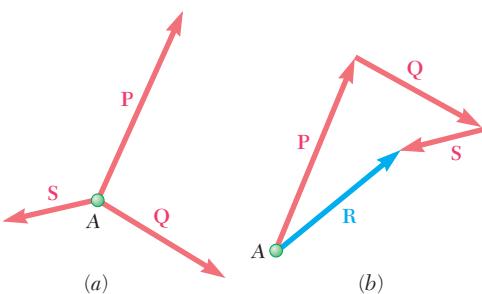


Fig. 2.14

the particle  $A$  as the given forces. As indicated above, the order in which the vectors  $P$ ,  $Q$ , and  $S$  representing the given forces are added together is immaterial.

## 2.6 RESOLUTION OF A FORCE INTO COMPONENTS

We have seen that two or more forces acting on a particle may be replaced by a single force which has the same effect on the particle. Conversely, a single force  $F$  acting on a particle may be replaced by two or more forces which, together, have the same effect on the particle. These forces are called the *components* of the original force  $F$ , and the process of substituting them for  $F$  is called *resolving the force  $F$  into components*.

Clearly, for each force  $F$  there exist an infinite number of possible sets of components. Sets of *two components*  $P$  and  $Q$  are the most important as far as practical applications are concerned. But, even then, the number of ways in which a given force  $F$  may be resolved into two components is unlimited (Fig. 2.15). Two cases are of particular interest:

- 1. One of the Two Components,  $P$ , Is Known.** The second component,  $Q$ , is obtained by applying the triangle rule and joining the tip of  $P$  to the tip of  $F$  (Fig. 2.16); the magnitude and direction of  $Q$  are determined graphically or by trigonometry. Once  $Q$  has been determined, both components  $P$  and  $Q$  should be applied at  $A$ .
- 2. The Line of Action of Each Component Is Known.** The magnitude and sense of the components are obtained by applying the parallelogram law and drawing lines, through the tip of  $F$ , parallel to the given lines of action (Fig. 2.17). This process leads to two well-defined components,  $P$  and  $Q$ , which can be determined graphically or computed trigonometrically by applying the law of sines.

Many other cases can be encountered; for example, the direction of one component may be known, while the magnitude of the other component is to be as small as possible (see Sample Prob. 2.2). In all cases the appropriate triangle or parallelogram which satisfies the given conditions is drawn.

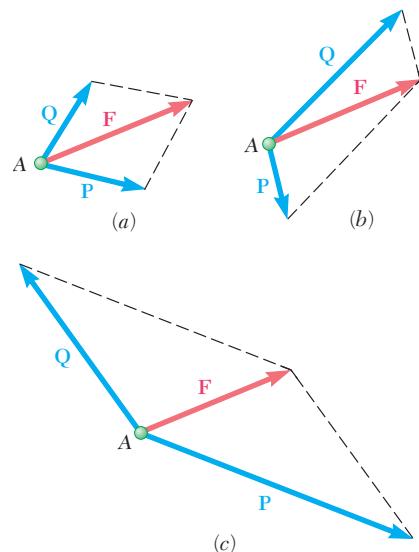


Fig. 2.15

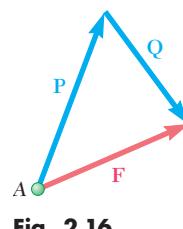


Fig. 2.16

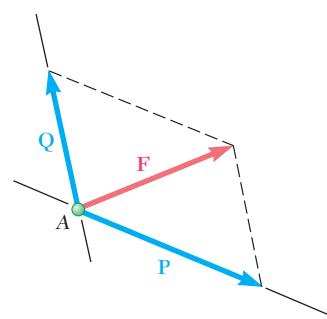
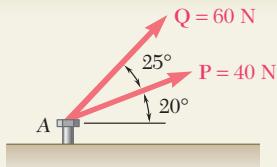


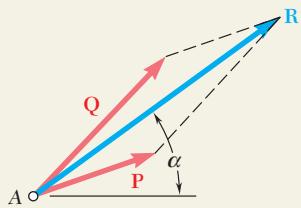
Fig. 2.17



## SAMPLE PROBLEM 2.1

The two forces **P** and **Q** act on a bolt **A**. Determine their resultant.

### SOLUTION

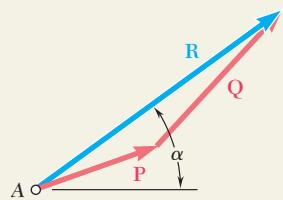


**Graphical Solution.** A parallelogram with sides equal to **P** and **Q** is drawn to scale. The magnitude and direction of the resultant are measured and found to be

$$R = 98 \text{ N} \quad \alpha = 35^\circ \quad \mathbf{R} = 98 \text{ N} \angle 35^\circ$$

The triangle rule may also be used. Forces **P** and **Q** are drawn in tip-to-tail fashion. Again the magnitude and direction of the resultant are measured.

$$R = 98 \text{ N} \quad \alpha = 35^\circ \quad \mathbf{R} = 98 \text{ N} \angle 35^\circ$$



**Trigonometric Solution.** The triangle rule is again used; two sides and the included angle are known. We apply the law of cosines.

$$\begin{aligned} R^2 &= P^2 + Q^2 - 2PQ \cos B \\ R^2 &= (40 \text{ N})^2 + (60 \text{ N})^2 - 2(40 \text{ N})(60 \text{ N}) \cos 155^\circ \\ R &= 97.73 \text{ N} \end{aligned}$$

Now, applying the law of sines, we write

$$\frac{\sin A}{Q} = \frac{\sin B}{R} \quad \frac{\sin A}{60 \text{ N}} = \frac{\sin 155^\circ}{97.73 \text{ N}} \quad (1)$$

Solving Eq. (1) for  $\sin A$ , we have

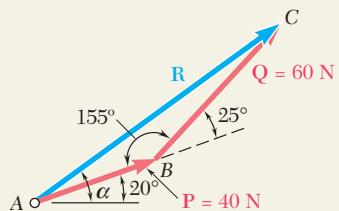
$$\sin A = \frac{(60 \text{ N}) \sin 155^\circ}{97.73 \text{ N}}$$

Using a calculator, we first compute the quotient, then its arc sine, and obtain

$$A = 15.04^\circ \quad \alpha = 20^\circ + A = 35.04^\circ$$

We use 3 significant figures to record the answer (cf. Sec. 1.6):

$$\mathbf{R} = 97.7 \text{ N} \angle 35.0^\circ$$



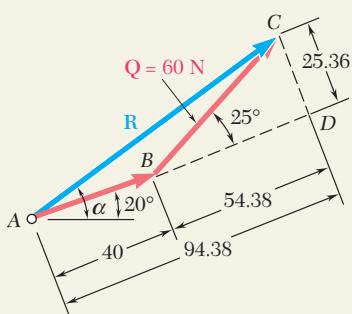
**Alternative Trigonometric Solution.** We construct the right triangle **BCD** and compute

$$\begin{aligned} CD &= (60 \text{ N}) \sin 25^\circ = 25.36 \text{ N} \\ BD &= (60 \text{ N}) \cos 25^\circ = 54.38 \text{ N} \end{aligned}$$

Then, using triangle **ACD**, we obtain

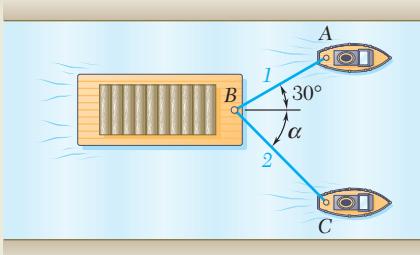
$$\tan A = \frac{25.36 \text{ N}}{94.38 \text{ N}} \quad A = 15.04^\circ$$

$$R = \frac{25.36}{\sin A} \quad R = 97.73 \text{ N}$$



Again,

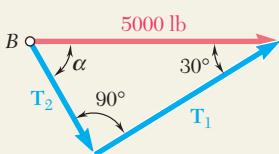
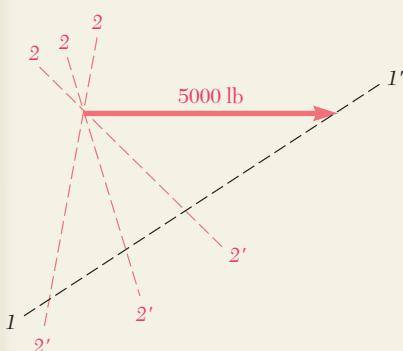
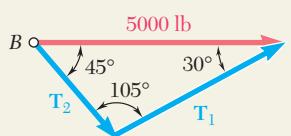
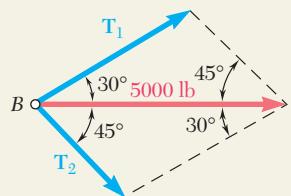
$$\alpha = 20^\circ + A = 35.04^\circ \quad \mathbf{R} = 97.7 \text{ N} \angle 35.0^\circ$$



## SAMPLE PROBLEM 2.2

A barge is pulled by two tugboats. If the resultant of the forces exerted by the tugboats is a 5000-lb force directed along the axis of the barge, determine (a) the tension in each of the ropes knowing that  $\alpha = 45^\circ$ , (b) the value of  $\alpha$  for which the tension in rope 2 is minimum.

## SOLUTION



**a. Tension for  $\alpha = 45^\circ$ . Graphical Solution.** The parallelogram law is used; the diagonal (resultant) is known to be equal to 5000 lb and to be directed to the right. The sides are drawn parallel to the ropes. If the drawing is done to scale, we measure

$$T_1 = 3700 \text{ lb} \quad T_2 = 2600 \text{ lb}$$

**Trigonometric Solution.** The triangle rule can be used. We note that the triangle shown represents half of the parallelogram shown above. Using the law of sines, we write

$$\frac{T_1}{\sin 45^\circ} = \frac{T_2}{\sin 30^\circ} = \frac{5000 \text{ lb}}{\sin 105^\circ}$$

With a calculator, we first compute and store the value of the last quotient. Multiplying this value successively by  $\sin 45^\circ$  and  $\sin 30^\circ$ , we obtain

$$T_1 = 3660 \text{ lb} \quad T_2 = 2590 \text{ lb}$$

**b. Value of  $\alpha$  for Minimum  $T_2$ .** To determine the value of  $\alpha$  for which the tension in rope 2 is minimum, the triangle rule is again used. In the sketch shown, line I-I' is the known direction of  $\mathbf{T}_1$ . Several possible directions of  $\mathbf{T}_2$  are shown by the lines 2-2'. We note that the minimum value of  $T_2$  occurs when  $\mathbf{T}_1$  and  $\mathbf{T}_2$  are perpendicular. The minimum value of  $T_2$  is

$$T_2 = (5000 \text{ lb}) \sin 30^\circ = 2500 \text{ lb}$$

Corresponding values of  $T_1$  and  $\alpha$  are

$$T_1 = (5000 \text{ lb}) \cos 30^\circ = 4330 \text{ lb}$$

$$\alpha = 90^\circ - 30^\circ$$

$$\alpha = 60^\circ$$

# SOLVING PROBLEMS ON YOUR OWN

The preceding sections were devoted to the *parallelogram law* for the addition of vectors and to its applications.

Two sample problems were presented. In Sample Prob. 2.1, the parallelogram law was used to determine the resultant of two forces of known magnitude and direction. In Sample Prob. 2.2, it was used to resolve a given force into two components of known direction.

You will now be asked to solve problems on your own. Some may resemble one of the sample problems; others may not. What all problems and sample problems in this section have in common is that they can be solved by the direct application of the parallelogram law.

Your solution of a given problem should consist of the following steps:

**1. Identify which of the forces are the applied forces and which is the resultant.** It is often helpful to write the vector equation which shows how the forces are related. For example, in Sample Prob. 2.1 we would have

$$\mathbf{R} = \mathbf{P} + \mathbf{Q}$$

You may want to keep that relation in mind as you formulate the next part of your solution.

**2. Draw a parallelogram with the applied forces as two adjacent sides and the resultant as the included diagonal (Fig. 2.2).** Alternatively, you can *use the triangle rule*, with the applied forces drawn in tip-to-tail fashion and the resultant extending from the tail of the first vector to the tip of the second (Fig. 2.7).

**3. Indicate all dimensions.** Using one of the triangles of the parallelogram, or the triangle constructed according to the triangle rule, indicate all dimensions—whether sides or angles—and determine the unknown dimensions either graphically or by trigonometry. If you use trigonometry, remember that the law of cosines should be applied first if two sides and the included angle are known [Sample Prob. 2.1], and the law of sines should be applied first if one side and all angles are known [Sample Prob. 2.2].

If you have had prior exposure to mechanics, you might be tempted to ignore the solution techniques of this lesson in favor of resolving the forces into rectangular components. While this latter method is important and will be considered in the next section, use of the parallelogram law simplifies the solution of many problems and should be mastered at this time.

# PROBLEMS<sup>†</sup>

- 2.1** Two forces **P** and **Q** are applied as shown at point A of a hook support. Knowing that  $P = 75 \text{ N}$  and  $Q = 125 \text{ N}$ , determine graphically the magnitude and direction of their resultant using (a) the parallelogram law, (b) the triangle rule.

- 2.2** Two forces **P** and **Q** are applied as shown at point A of a hook support. Knowing that  $P = 60 \text{ lb}$  and  $Q = 25 \text{ lb}$ , determine graphically the magnitude and direction of their resultant using (a) the parallelogram law, (b) the triangle rule.

- 2.3** The cable stays  $AB$  and  $AD$  help support pole  $AC$ . Knowing that the tension is  $120 \text{ lb}$  in  $AB$  and  $40 \text{ lb}$  in  $AD$ , determine graphically the magnitude and direction of the resultant of the forces exerted by the stays at  $A$  using (a) the parallelogram law, (b) the triangle rule.

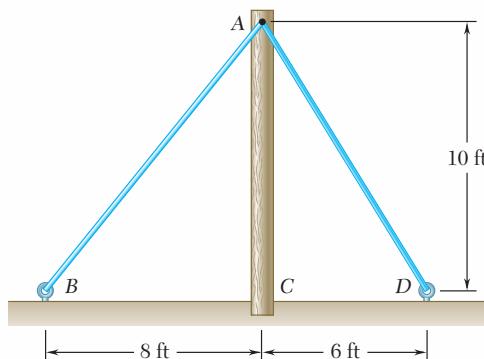


Fig. P2.3

- 2.4** Two forces are applied at point  $B$  of beam  $AB$ . Determine graphically the magnitude and direction of their resultant using (a) the parallelogram law, (b) the triangle rule.

- 2.5** The 300-lb force is to be resolved into components along lines  $a-a'$  and  $b-b'$ . (a) Determine the angle  $\alpha$  by trigonometry knowing that the component along line  $a-a'$  is to be 240 lb. (b) What is the corresponding value of the component along  $b-b'$ ?

- 2.6** The 300-lb force is to be resolved into components along lines  $a-a'$  and  $b-b'$ . (a) Determine the angle  $\alpha$  by trigonometry knowing that the component along line  $b-b'$  is to be 120 lb. (b) What is the corresponding value of the component along  $a-a'$ ?

- 2.7** Two forces are applied as shown to a hook support. Knowing that the magnitude of **P** is  $35 \text{ N}$ , determine by trigonometry (a) the required angle  $\alpha$  if the resultant **R** of the two forces applied to the support is to be horizontal, (b) the corresponding magnitude of **R**.

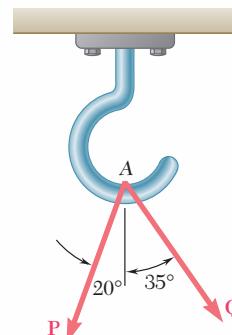


Fig. P2.1 and P2.2

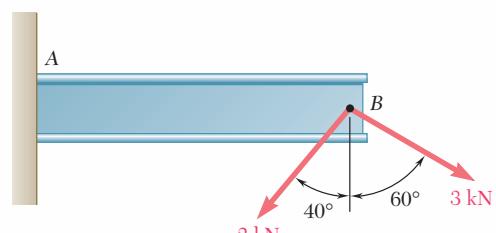


Fig. P2.4

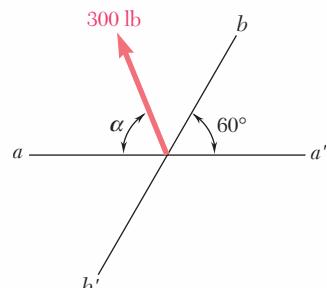


Fig. P2.5 and P2.6

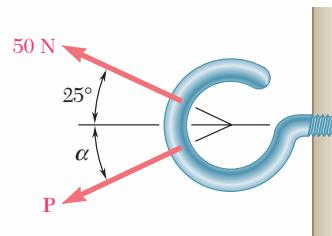


Fig. P2.7

<sup>†</sup>Answers to all problems set in straight type (such as 2.1) are given at the end of the book. Answers to problems with a number set in italic type (such as 2.4) are not given.

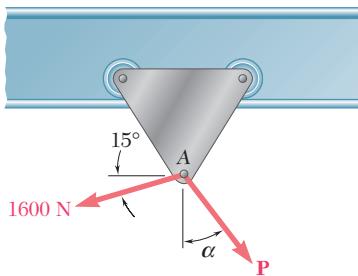


Fig. P2.9 and P2.10

- 2.8** For the hook support of Prob. 2.1, knowing that the magnitude of  $\mathbf{P}$  is 75 N, determine by trigonometry (a) the required magnitude of the force  $\mathbf{Q}$  if the resultant  $\mathbf{R}$  of the two forces applied at A is to be vertical, (b) the corresponding magnitude of  $\mathbf{R}$ .

- 2.9** A trolley that moves along a horizontal beam is acted upon by two forces as shown. (a) Knowing that  $\alpha = 25^\circ$ , determine by trigonometry the magnitude of the force  $\mathbf{P}$  so that the resultant force exerted on the trolley is vertical. (b) What is the corresponding magnitude of the resultant?

- 2.10** A trolley that moves along a horizontal beam is acted upon by two forces as shown. Determine by trigonometry the magnitude and direction of the force  $\mathbf{P}$  so that the resultant is a vertical force of 2500 N.

- 2.11** A steel tank is to be positioned in an excavation. Knowing that  $\alpha = 20^\circ$ , determine by trigonometry (a) the required magnitude of the force  $\mathbf{P}$  if the resultant  $\mathbf{R}$  of the two forces applied at A is to be vertical, (b) the corresponding magnitude of  $\mathbf{R}$ .

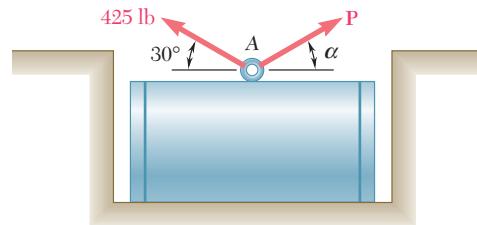


Fig. P2.11 and P2.12

- 2.12** A steel tank is to be positioned in an excavation. Knowing that the magnitude of  $\mathbf{P}$  is 500 lb, determine by trigonometry (a) the required angle  $\alpha$  if the resultant  $\mathbf{R}$  of the two forces applied at A is to be vertical, (b) the corresponding magnitude of  $\mathbf{R}$ .

- 2.13** For the hook support of Prob. 2.7, determine by trigonometry (a) the magnitude and direction of the smallest force  $\mathbf{P}$  for which the resultant  $\mathbf{R}$  of the two forces applied to the support is horizontal, (b) the corresponding magnitude of  $\mathbf{R}$ .

- 2.14** For the steel tank of Prob. 2.11, determine by trigonometry (a) the magnitude and direction of the smallest force  $\mathbf{P}$  for which the resultant  $\mathbf{R}$  of the two forces applied at A is vertical, (b) the corresponding magnitude of  $\mathbf{R}$ .

- 2.15** Solve Prob. 2.2 by trigonometry.

- 2.16** Solve Prob. 2.3 by trigonometry.

- 2.17** Solve Prob. 2.4 by trigonometry.

- 2.18** Two structural members A and B are bolted to a bracket as shown. Knowing that both members are in compression and that the force is 15 kN in member A and 10 kN in member B, determine by trigonometry the magnitude and direction of the resultant of the forces applied to the bracket by members A and B.

- 2.19** Two structural members A and B are bolted to a bracket as shown. Knowing that both members are in compression and that the force is 10 kN in member A and 15 kN in member B, determine by trigonometry the magnitude and direction of the resultant of the forces applied to the bracket by members A and B.

- 2.20** For the hook support of Prob. 2.7, knowing that  $P = 75$  N and  $\alpha = 50^\circ$ , determine by trigonometry the magnitude and direction of the resultant of the two forces applied to the support.

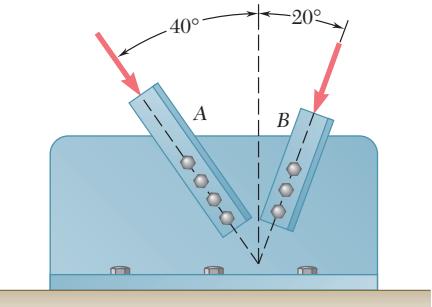


Fig. P2.18 and P2.19

## 2.7 RECTANGULAR COMPONENTS OF A FORCE. UNIT VECTORS†

In many problems it will be found desirable to resolve a force into two components which are perpendicular to each other. In Fig. 2.18, the force  $\mathbf{F}$  has been resolved into a component  $\mathbf{F}_x$  along the  $x$  axis and a component  $\mathbf{F}_y$  along the  $y$  axis. The parallelogram drawn to obtain the two components is a *rectangle*, and  $\mathbf{F}_x$  and  $\mathbf{F}_y$  are called *rectangular components*.

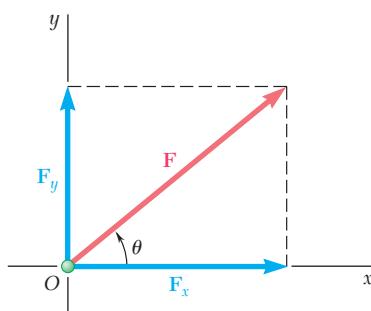


Fig. 2.18

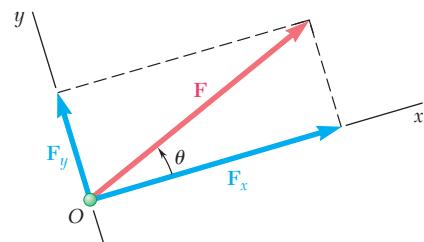


Fig. 2.19

The  $x$  and  $y$  axes are usually chosen horizontal and vertical, respectively, as in Fig. 2.18; they may, however, be chosen in any two perpendicular directions, as shown in Fig. 2.19. In determining the rectangular components of a force, the student should think of the construction lines shown in Figs. 2.18 and 2.19 as being *parallel* to the  $x$  and  $y$  axes, rather than *perpendicular* to these axes. This practice will help avoid mistakes in determining *oblique* components as in Sec. 2.6.

†The properties established in Secs. 2.7 and 2.8 may be readily extended to the rectangular components of any vector quantity.

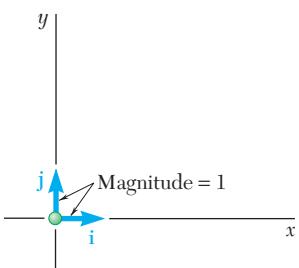


Fig. 2.20

Two vectors of unit magnitude, directed respectively along the positive  $x$  and  $y$  axes, will be introduced at this point. These vectors are called **unit vectors** and are denoted by  $\mathbf{i}$  and  $\mathbf{j}$ , respectively (Fig. 2.20). Recalling the definition of the product of a scalar and a vector given in Sec. 2.4, we note that the rectangular components  $\mathbf{F}_x$  and  $\mathbf{F}_y$  of a force  $\mathbf{F}$  may be obtained by multiplying respectively the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$  by appropriate scalars (Fig. 2.21). We write

$$\mathbf{F}_x = F_x \mathbf{i} \quad \mathbf{F}_y = F_y \mathbf{j} \quad (2.6)$$

and

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} \quad (2.7)$$

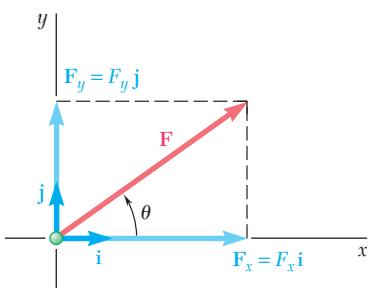


Fig. 2.21

While the scalars  $F_x$  and  $F_y$  may be positive or negative, depending upon the sense of  $\mathbf{F}_x$  and of  $\mathbf{F}_y$ , their absolute values are respectively equal to the magnitudes of the component forces  $\mathbf{F}_x$  and  $\mathbf{F}_y$ . The scalars  $F_x$  and  $F_y$  are called the *scalar components* of the force  $\mathbf{F}$ , while the actual component forces  $\mathbf{F}_x$  and  $\mathbf{F}_y$  should be referred to as the *vector components* of  $\mathbf{F}$ . However, when there exists no possibility of confusion, the vector as well as the scalar components of  $\mathbf{F}$  may be referred to simply as the *components* of  $\mathbf{F}$ . We note that the scalar component  $F_x$  is positive when the vector component  $\mathbf{F}_x$  has the same sense as the unit vector  $\mathbf{i}$  (i.e., the same sense as the positive  $x$  axis) and is negative when  $\mathbf{F}_x$  has the opposite sense. A similar conclusion may be drawn regarding the sign of the scalar component  $F_y$ .

Denoting by  $F$  the magnitude of the force  $\mathbf{F}$  and by  $\theta$  the angle between  $\mathbf{F}$  and the  $x$  axis, measured counterclockwise from the positive  $x$  axis (Fig. 2.21), we may express the scalar components of  $\mathbf{F}$  as follows:

$$F_x = F \cos \theta \quad F_y = F \sin \theta \quad (2.8)$$

We note that the relations obtained hold for any value of the angle  $\theta$  from  $0^\circ$  to  $360^\circ$  and that they define the signs as well as the absolute values of the scalar components  $F_x$  and  $F_y$ .

**EXAMPLE 1.** A force of 800 N is exerted on a bolt  $A$  as shown in Fig. 2.22a. Determine the horizontal and vertical components of the force.

In order to obtain the correct sign for the scalar components  $F_x$  and  $F_y$ , the value  $180^\circ - 35^\circ = 145^\circ$  should be substituted for  $\theta$  in Eqs. (2.8). However, it will be found more practical to determine by inspection the signs of  $F_x$  and  $F_y$  (Fig. 2.22b) and to use the trigonometric functions of the angle  $\alpha = 35^\circ$ . We write, therefore,

$$F_x = -F \cos \alpha = -(800 \text{ N}) \cos 35^\circ = -655 \text{ N}$$

$$F_y = +F \sin \alpha = +(800 \text{ N}) \sin 35^\circ = +459 \text{ N}$$

The vector components of  $\mathbf{F}$  are thus

$$\mathbf{F}_x = -(655 \text{ N})\mathbf{i} \quad \mathbf{F}_y = +(459 \text{ N})\mathbf{j}$$

and we may write  $\mathbf{F}$  in the form

$$\mathbf{F} = -(655 \text{ N})\mathbf{i} + (459 \text{ N})\mathbf{j} \blacksquare$$

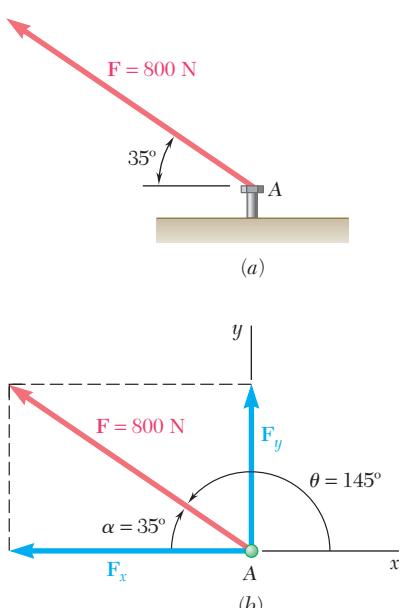


Fig. 2.22

**EXAMPLE 2.** A man pulls with a force of 300 N on a rope attached to a building, as shown in Fig. 2.23a. What are the horizontal and vertical components of the force exerted by the rope at point A?

It is seen from Fig. 2.23b that

$$F_x = +(300 \text{ N}) \cos \alpha \quad F_y = -(300 \text{ N}) \sin \alpha$$

Observing that  $AB = 10 \text{ m}$ , we find from Fig. 2.23a

$$\cos \alpha = \frac{8 \text{ m}}{AB} = \frac{8 \text{ m}}{10 \text{ m}} = \frac{4}{5} \quad \sin \alpha = \frac{6 \text{ m}}{AB} = \frac{6 \text{ m}}{10 \text{ m}} = \frac{3}{5}$$

We thus obtain

$$F_x = +(300 \text{ N}) \frac{4}{5} = +240 \text{ N} \quad F_y = -(300 \text{ N}) \frac{3}{5} = -180 \text{ N}$$

and write

$$\mathbf{F} = (240 \text{ N})\mathbf{i} - (180 \text{ N})\mathbf{j}$$

When a force  $\mathbf{F}$  is defined by its rectangular components  $F_x$  and  $F_y$  (see Fig. 2.21), the angle  $\theta$  defining its direction can be obtained by writing

$$\tan \theta = \frac{F_y}{F_x} \quad (2.9)$$

The magnitude  $F$  of the force can be obtained by applying the Pythagorean theorem and writing

$$F = \sqrt{F_x^2 + F_y^2} \quad (2.10)$$

or by solving for  $F$  one of the Eqs. (2.8).

**EXAMPLE 3.** A force  $\mathbf{F} = (700 \text{ lb})\mathbf{i} + (1500 \text{ lb})\mathbf{j}$  is applied to a bolt A. Determine the magnitude of the force and the angle  $\theta$  it forms with the horizontal.

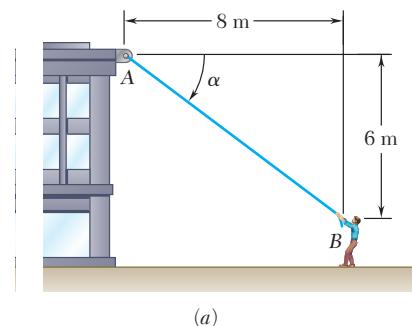
First we draw a diagram showing the two rectangular components of the force and the angle  $\theta$  (Fig. 2.24). From Eq. (2.9), we write

$$\tan \theta = \frac{F_y}{F_x} = \frac{1500 \text{ lb}}{700 \text{ lb}}$$

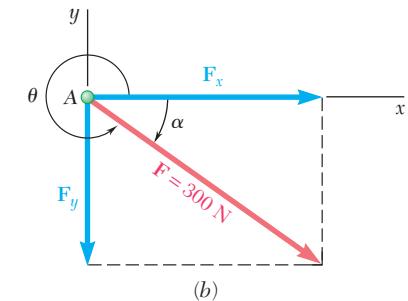
Using a calculator,<sup>†</sup> we enter 1500 lb and divide by 700 lb; computing the arc tangent of the quotient, we obtain  $\theta = 65.0^\circ$ . Solving the second of Eqs. (2.8) for  $F$ , we have

$$F = \frac{F_y}{\sin \theta} = \frac{1500 \text{ lb}}{\sin 65.0^\circ} = 1655 \text{ lb}$$

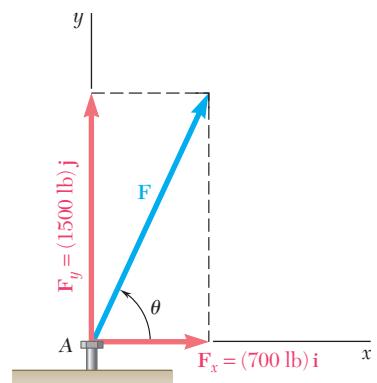
The last calculation is facilitated if the value of  $F_y$  is stored when originally entered; it may then be recalled to be divided by  $\sin \theta$ .



(a)



(b)

**Fig. 2.23****Fig. 2.24**

<sup>†</sup>It is assumed that the calculator used has keys for the computation of trigonometric and inverse trigonometric functions. Some calculators also have keys for the direct conversion of rectangular coordinates into polar coordinates, and vice versa. Such calculators eliminate the need for the computation of trigonometric functions in Examples 1, 2, and 3 and in problems of the same type.

## 2.8 ADDITION OF FORCES BY SUMMING X AND Y COMPONENTS

It was seen in Sec. 2.2 that forces should be added according to the parallelogram law. From this law, two other methods, more readily applicable to the *graphical* solution of problems, were derived in Secs. 2.4 and 2.5: the triangle rule for the addition of two forces and the polygon rule for the addition of three or more forces. It was also seen that the force triangle used to define the resultant of two forces could be used to obtain a *trigonometric* solution.

When three or more forces are to be added, no practical trigonometric solution can be obtained from the force polygon which defines the resultant of the forces. In this case, an *analytic* solution of the problem can be obtained by resolving each force into two rectangular components. Consider, for instance, three forces  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{S}$  acting on a particle  $A$  (Fig. 2.25a). Their resultant  $\mathbf{R}$  is defined by the relation

$$\mathbf{R} = \mathbf{P} + \mathbf{Q} + \mathbf{S} \quad (2.11)$$

Resolving each force into its rectangular components, we write

$$\begin{aligned} R_x\mathbf{i} + R_y\mathbf{j} &= P_x\mathbf{i} + P_y\mathbf{j} + Q_x\mathbf{i} + Q_y\mathbf{j} + S_x\mathbf{i} + S_y\mathbf{j} \\ &= (P_x + Q_x + S_x)\mathbf{i} + (P_y + Q_y + S_y)\mathbf{j} \end{aligned}$$

from which it follows that

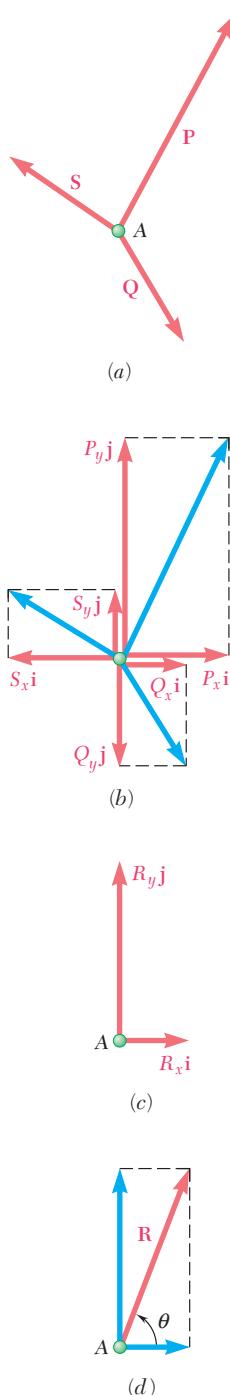
$$R_x = P_x + Q_x + S_x \quad R_y = P_y + Q_y + S_y \quad (2.12)$$

or, for short,

$$R_x = \sum F_x \quad R_y = \sum F_y \quad (2.13)$$

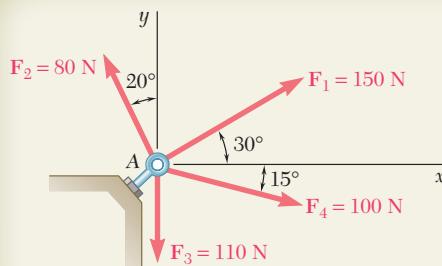
We thus conclude that the scalar components  $R_x$  and  $R_y$  of the resultant  $\mathbf{R}$  of several forces acting on a particle are obtained by adding algebraically the corresponding scalar components of the given forces.<sup>†</sup>

In practice, the determination of the resultant  $\mathbf{R}$  is carried out in three steps as illustrated in Fig. 2.25. First the given forces shown in Fig. 2.25a are resolved into their  $x$  and  $y$  components (Fig. 2.25b). Adding these components, we obtain the  $x$  and  $y$  components of  $\mathbf{R}$  (Fig. 2.25c). Finally, the resultant  $\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$  is determined by applying the parallelogram law (Fig. 2.25d). The procedure just described will be carried out most efficiently if the computations are arranged in a table. While it is the only practical analytic method for adding three or more forces, it is also often preferred to the trigonometric solution in the case of the addition of two forces.



**Fig. 2.25**

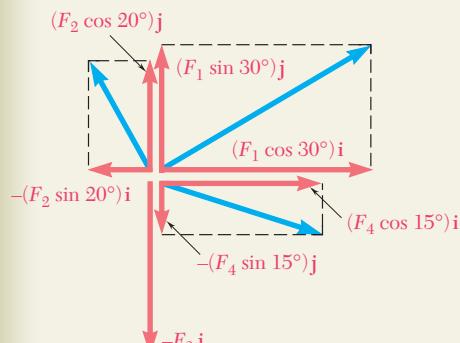
<sup>†</sup>Clearly, this result also applies to the addition of other vector quantities, such as velocities, accelerations, or momenta.



### SAMPLE PROBLEM 2.3

Four forces act on bolt A as shown. Determine the resultant of the forces on the bolt.

### SOLUTION



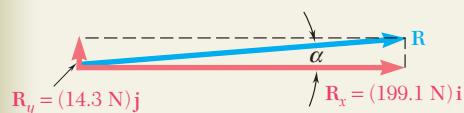
The  $x$  and  $y$  components of each force are determined by trigonometry as shown and are entered in the table below. According to the convention adopted in Sec. 2.7, the scalar number representing a force component is positive if the force component has the same sense as the corresponding coordinate axis. Thus,  $x$  components acting to the right and  $y$  components acting upward are represented by positive numbers.

Force	Magnitude, N	$x$ Component, N	$y$ Component, N
$\mathbf{F}_1$	150	+129.9	+75.0
$\mathbf{F}_2$	80	-27.4	+75.2
$\mathbf{F}_3$	110	0	-110.0
$\mathbf{F}_4$	100	+96.6	-25.9
		$R_x = +199.1$	$R_y = +14.3$

Thus, the resultant  $\mathbf{R}$  of the four forces is

$$\mathbf{R} = R_x \mathbf{i} + R_y \mathbf{j} \quad \mathbf{R} = (199.1 \text{ N})\mathbf{i} + (14.3 \text{ N})\mathbf{j}$$

The magnitude and direction of the resultant may now be determined. From the triangle shown, we have



$$\tan \alpha = \frac{R_y}{R_x} = \frac{14.3 \text{ N}}{199.1 \text{ N}} \quad \alpha = 4.1^\circ$$

$$R = \frac{14.3 \text{ N}}{\sin \alpha} = 199.6 \text{ N} \quad \mathbf{R} = 199.6 \text{ N} \angle 4.1^\circ$$

With a calculator, the last computation may be facilitated if the value of  $R_y$  is stored when originally entered; it may then be recalled to be divided by  $\sin \alpha$ . (Also see the footnote on p. 29.)

# SOLVING PROBLEMS ON YOUR OWN

You saw in the preceding lesson that the resultant of two forces may be determined either graphically or from the trigonometry of an oblique triangle.

**A. When three or more forces are involved,** the determination of their resultant  $\mathbf{R}$  is best carried out by first resolving each force into *rectangular components*. Two cases may be encountered, depending upon the way in which each of the given forces is defined:

**Case 1. The force  $\mathbf{F}$  is defined by its magnitude  $F$  and the angle  $\alpha$  it forms with the  $x$  axis.** The  $x$  and  $y$  components of the force can be obtained by multiplying  $F$  by  $\cos \alpha$  and  $\sin \alpha$ , respectively [Example 1].

**Case 2. The force  $\mathbf{F}$  is defined by its magnitude  $F$  and the coordinates of two points  $A$  and  $B$  on its line of action** (Fig. 2.23). The angle  $\alpha$  that  $\mathbf{F}$  forms with the  $x$  axis may first be determined by trigonometry. However, the components of  $\mathbf{F}$  may also be obtained directly from proportions among the various dimensions involved, without actually determining  $\alpha$  [Example 2].

**B. Rectangular components of the resultant.** The components  $R_x$  and  $R_y$  of the resultant can be obtained by adding algebraically the corresponding components of the given forces [Sample Prob. 2.3].

You can express the resultant in *vectorial form* using the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ , which are directed along the  $x$  and  $y$  axes, respectively:

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$

Alternatively, you can determine the *magnitude and direction* of the resultant by solving the right triangle of sides  $R_x$  and  $R_y$  for  $R$  and for the angle that  $\mathbf{R}$  forms with the  $x$  axis.

# PROBLEMS

**2.21 and 2.22** Determine the  $x$  and  $y$  components of each of the forces shown.

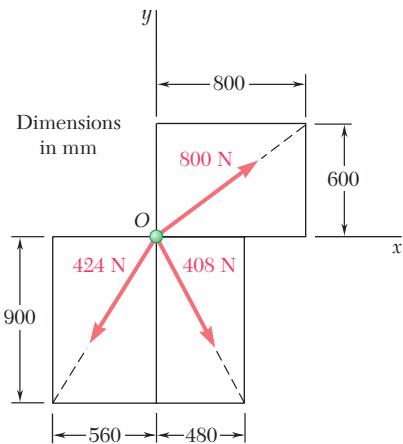


Fig. P2.21

**2.23 and 2.24** Determine the  $x$  and  $y$  components of each of the forces shown.

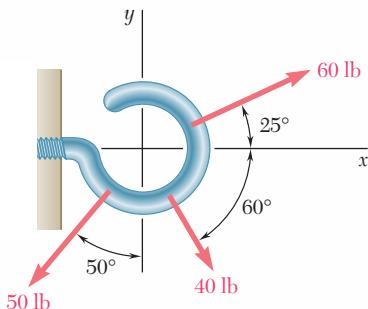


Fig. P2.23

**2.25** Member  $BD$  exerts on member  $ABC$  a force  $\mathbf{P}$  directed along line  $BD$ . Knowing that  $\mathbf{P}$  must have a 300-lb horizontal component, determine (a) the magnitude of the force  $\mathbf{P}$ , (b) its vertical component.

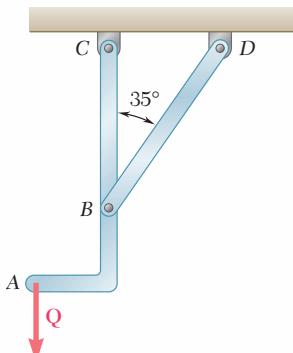


Fig. P2.25

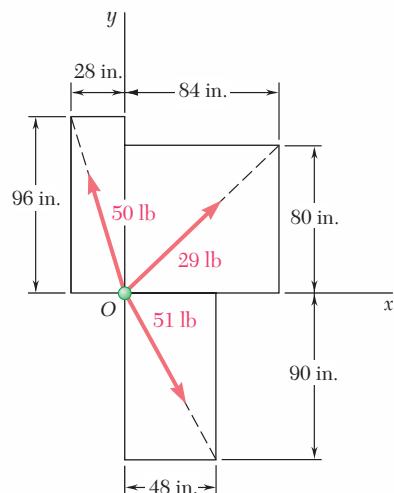


Fig. P2.22

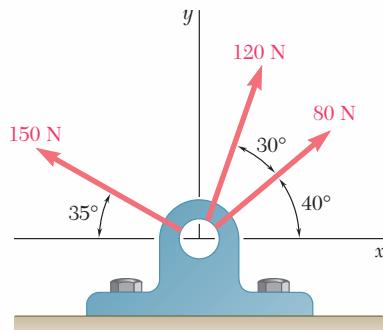


Fig. P2.24

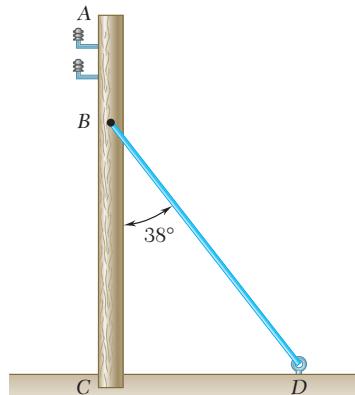


Fig. P2.27 and P2.28

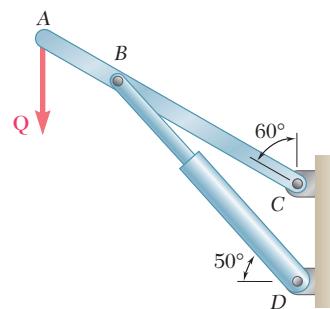


Fig. P2.26

- 2.26** The hydraulic cylinder  $BD$  exerts on member  $ABC$  a force  $\mathbf{P}$  directed along line  $BD$ . Knowing that  $\mathbf{P}$  must have a 750-N component perpendicular to member  $ABC$ , determine (a) the magnitude of the force  $\mathbf{P}$ , (b) its component parallel to  $ABC$ .

- 2.27** The guy wire  $BD$  exerts on the telephone pole  $AC$  a force  $\mathbf{P}$  directed along  $BD$ . Knowing that  $\mathbf{P}$  must have a 120-N component perpendicular to the pole  $AC$ , determine (a) the magnitude of the force  $\mathbf{P}$ , (b) its component along line  $AC$ .

- 2.28** The guy wire  $BD$  exerts on the telephone pole  $AC$  a force  $\mathbf{P}$  directed along  $BD$ . Knowing that  $\mathbf{P}$  has a 180-N component along line  $AC$ , determine (a) the magnitude of the force  $\mathbf{P}$ , (b) its component in a direction perpendicular to  $AC$ .

- 2.29** Member  $CB$  of the vise shown exerts on block  $B$  a force  $\mathbf{P}$  directed along line  $CB$ . Knowing that  $\mathbf{P}$  must have a 1200-N horizontal component, determine (a) the magnitude of the force  $\mathbf{P}$ , (b) its vertical component.

- 2.30** Cable  $AC$  exerts on beam  $AB$  a force  $\mathbf{P}$  directed along line  $AC$ . Knowing that  $\mathbf{P}$  must have a 350-lb vertical component, determine (a) the magnitude of the force  $\mathbf{P}$ , (b) its horizontal component.

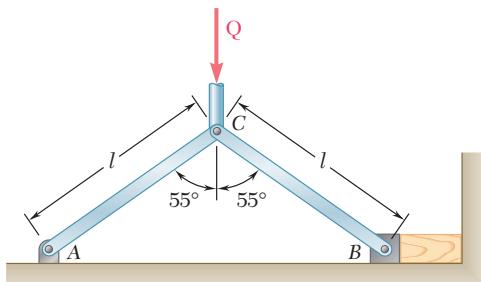


Fig. P2.29

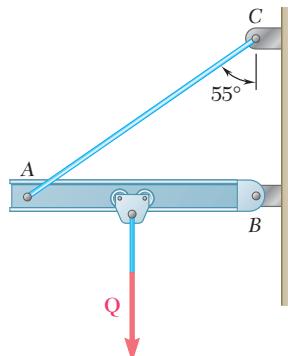


Fig. P2.30

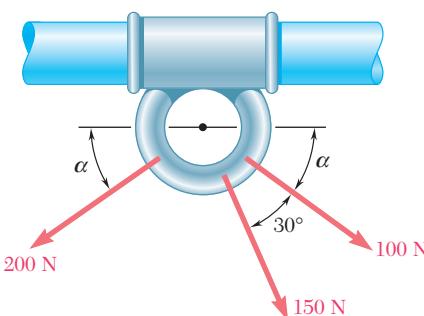


Fig. P2.35

- 2.31** Determine the resultant of the three forces of Prob. 2.22.

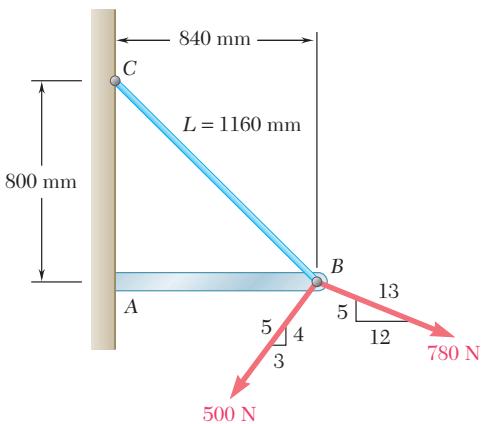
- 2.32** Determine the resultant of the three forces of Prob. 2.24.

- 2.33** Determine the resultant of the three forces of Prob. 2.23.

- 2.34** Determine the resultant of the three forces of Prob. 2.21.

- 2.35** Knowing that  $\alpha = 35^\circ$ , determine the resultant of the three forces shown.

- 2.36** Knowing that the tension in cable  $BC$  is 725 N, determine the resultant of the three forces exerted at point  $B$  of beam  $AB$ .



**Fig. P2.36**

- 2.37** Knowing that  $\alpha = 40^\circ$ , determine the resultant of the three forces shown.

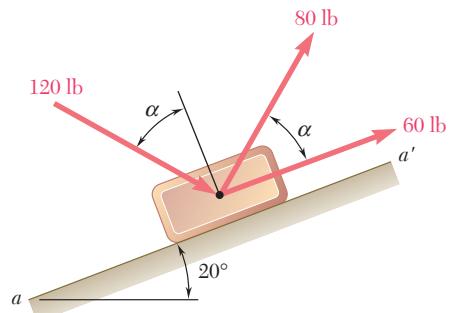
- 2.38** Knowing that  $\alpha = 75^\circ$ , determine the resultant of the three forces shown.

- 2.39** For the collar of Prob. 2.35, determine (a) the required value of  $\alpha$  if the resultant of the three forces shown is to be vertical, (b) the corresponding magnitude of the resultant.

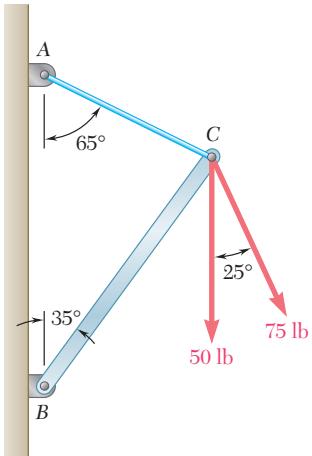
- 2.40** For the beam of Prob. 2.36, determine (a) the required tension in cable  $BC$  if the resultant of the three forces exerted at point  $B$  is to be vertical, (b) the corresponding magnitude of the resultant.

- 2.41** Determine (a) the required tension in cable  $AC$ , knowing that the resultant of the three forces exerted at point  $C$  of boom  $BC$  must be directed along  $BC$ , (b) the corresponding magnitude of the resultant.

- 2.42** For the block of Probs. 2.37 and 2.38, determine (a) the required value of  $\alpha$  if the resultant of the three forces shown is to be parallel to the incline, (b) the corresponding magnitude of the resultant.



**Fig. P2.37 and P2.38**

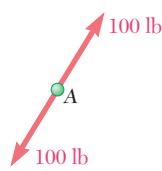


**Fig. P2.41**

## 2.9 EQUILIBRIUM OF A PARTICLE

In the preceding sections, we discussed the methods for determining the resultant of several forces acting on a particle. Although it has not occurred in any of the problems considered so far, it is quite possible for the resultant to be zero. In such a case, the net effect of the given forces is zero, and the particle is said to be in equilibrium. We thus have the following definition: *When the resultant of all the forces acting on a particle is zero, the particle is in equilibrium.*

A particle which is acted upon by two forces will be in equilibrium if the two forces have the same magnitude and the same line of action but opposite sense. The resultant of the two forces is then zero. Such a case is shown in Fig. 2.26.



**Fig. 2.26**

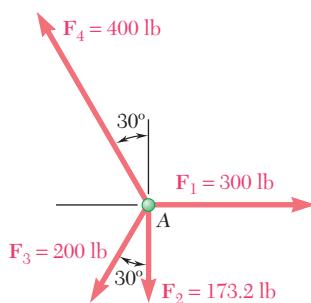


Fig. 2.27

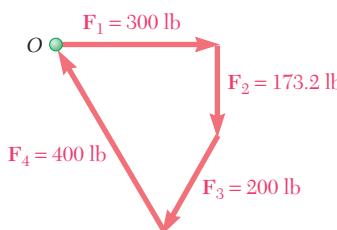


Fig. 2.28

Another case of equilibrium of a particle is represented in Fig. 2.27, where four forces are shown acting on A. In Fig. 2.28, the resultant of the given forces is determined by the polygon rule. Starting from point  $O$  with  $\mathbf{F}_1$  and arranging the forces in tip-to-tail fashion, we find that the tip of  $\mathbf{F}_4$  coincides with the starting point  $O$ . Thus the resultant  $\mathbf{R}$  of the given system of forces is zero, and the particle is in equilibrium.

The closed polygon drawn in Fig. 2.28 provides a *graphical* expression of the equilibrium of A. To express *algebraically* the conditions for the equilibrium of a particle, we write

$$\mathbf{R} = \sum \mathbf{F} = 0 \quad (2.14)$$

Resolving each force  $\mathbf{F}$  into rectangular components, we have

$$\Sigma(F_x \mathbf{i} + F_y \mathbf{j}) = 0 \quad \text{or} \quad (\Sigma F_x) \mathbf{i} + (\Sigma F_y) \mathbf{j} = 0$$

We conclude that the necessary and sufficient conditions for the equilibrium of a particle are

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad (2.15)$$

Returning to the particle shown in Fig. 2.27, we check that the equilibrium conditions are satisfied. We write

$$\begin{aligned}\Sigma F_x &= 300 \text{ lb} - (200 \text{ lb}) \sin 30^\circ - (400 \text{ lb}) \sin 30^\circ \\ &= 300 \text{ lb} - 100 \text{ lb} - 200 \text{ lb} = 0 \\ \Sigma F_y &= -173.2 \text{ lb} - (200 \text{ lb}) \cos 30^\circ + (400 \text{ lb}) \cos 30^\circ \\ &= -173.2 \text{ lb} - 173.2 \text{ lb} + 346.4 \text{ lb} = 0\end{aligned}$$

## 2.10 NEWTON'S FIRST LAW OF MOTION

In the latter part of the seventeenth century, Sir Isaac Newton formulated three fundamental laws upon which the science of mechanics is based. The first of these laws can be stated as follows:

*If the resultant force acting on a particle is zero, the particle will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion).*

From this law and from the definition of equilibrium given in Sec. 2.9, it is seen that a particle in equilibrium either is at rest or is moving in a straight line with constant speed. In the following section, various problems concerning the equilibrium of a particle will be considered.

## 2.11 PROBLEMS INVOLVING THE EQUILIBRIUM OF A PARTICLE. FREE-BODY DIAGRAMS

In practice, a problem in engineering mechanics is derived from an actual physical situation. A sketch showing the physical conditions of the problem is known as a *space diagram*.

The methods of analysis discussed in the preceding sections apply to a system of forces acting on a particle. A large number of problems involving actual structures, however, can be reduced to problems concerning the equilibrium of a particle. This is done by

choosing a significant particle and drawing a separate diagram showing this particle and all the forces acting on it. Such a diagram is called a *free-body diagram*.

As an example, consider the 75-kg crate shown in the space diagram of Fig. 2.29a. This crate was lying between two buildings, and it is now being lifted onto a truck, which will remove it. The crate is supported by a vertical cable, which is joined at A to two ropes which pass over pulleys attached to the buildings at B and C. It is desired to determine the tension in each of the ropes AB and AC.

In order to solve this problem, a free-body diagram showing a particle in equilibrium must be drawn. Since we are interested in the rope tensions, the free-body diagram should include at least one of these tensions or, if possible, both tensions. Point A is seen to be a good free body for this problem. The free-body diagram of point A is shown in Fig. 2.29b. It shows point A and the forces exerted on A by the vertical cable and the two ropes. The force exerted by the cable is directed downward, and its magnitude is equal to the weight W of the crate. Recalling Eq. (1.4), we write

$$W = mg = (75 \text{ kg})(9.81 \text{ m/s}^2) = 736 \text{ N}$$

and indicate this value in the free-body diagram. The forces exerted by the two ropes are not known. Since they are respectively equal in magnitude to the tensions in rope AB and rope AC, we denote them by  $\mathbf{T}_{AB}$  and  $\mathbf{T}_{AC}$  and draw them away from A in the directions shown in the space diagram. No other detail is included in the free-body diagram.

Since point A is in equilibrium, the three forces acting on it must form a closed triangle when drawn in tip-to-tail fashion. This *force triangle* has been drawn in Fig. 2.29c. The values  $T_{AB}$  and  $T_{AC}$  of the tension in the ropes may be found graphically if the triangle is drawn to scale, or they may be found by trigonometry. If the latter method of solution is chosen, we use the law of sines and write

$$\frac{T_{AB}}{\sin 60^\circ} = \frac{T_{AC}}{\sin 40^\circ} = \frac{736 \text{ N}}{\sin 80^\circ}$$

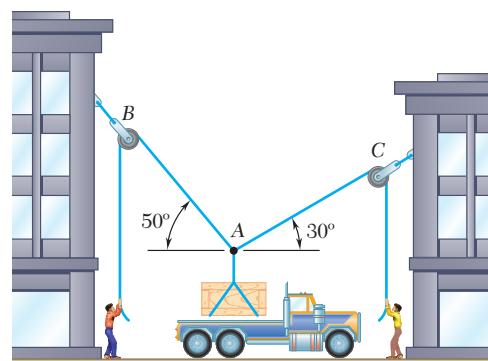
$$T_{AB} = 647 \text{ N} \quad T_{AC} = 480 \text{ N}$$

When a particle is in *equilibrium under three forces*, the problem can be solved by drawing a force triangle. When a particle is in *equilibrium under more than three forces*, the problem can be solved graphically by drawing a force polygon. If an analytic solution is desired, the *equations of equilibrium* given in Sec. 2.9 should be solved:

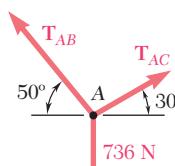
$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad (2.15)$$

These equations can be solved for no more than *two unknowns*; similarly, the force triangle used in the case of equilibrium under three forces can be solved for two unknowns.

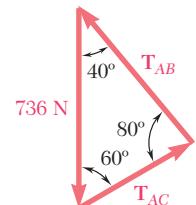
The more common types of problems are those in which the two unknowns represent (1) the two components (or the magnitude and direction) of a single force, (2) the magnitudes of two forces, each of known direction. Problems involving the determination of the maximum or minimum value of the magnitude of a force are also encountered (see Probs. 2.57 through 2.61).



(a) Space diagram



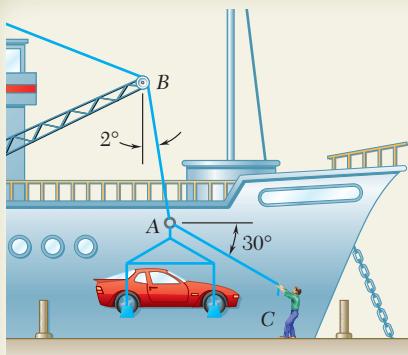
(b) Free-body diagram



(c) Force triangle



**Photo 2.1** As illustrated in the above example, it is possible to determine the tensions in the cables supporting the shaft shown by treating the hook as a particle and then applying the equations of equilibrium to the forces acting on the hook.



## SAMPLE PROBLEM 2.4

In a ship-unloading operation, a 3500-lb automobile is supported by a cable. A rope is tied to the cable at A and pulled in order to center the automobile over its intended position. The angle between the cable and the vertical is 2°, while the angle between the rope and the horizontal is 30°. What is the tension in the rope?

## SOLUTION

**Free-Body Diagram.** Point A is chosen as a free body, and the complete free-body diagram is drawn.  $T_{AB}$  is the tension in the cable  $AB$ , and  $T_{AC}$  is the tension in the rope.

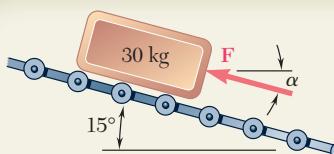
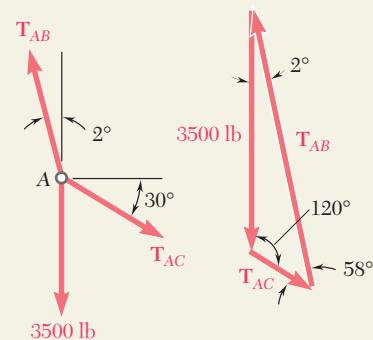
**Equilibrium Condition.** Since only three forces act on the free body, we draw a force triangle to express that it is in equilibrium. Using the law of sines, we write

$$\frac{T_{AB}}{\sin 120^\circ} = \frac{T_{AC}}{\sin 2^\circ} = \frac{3500 \text{ lb}}{\sin 58^\circ}$$

With a calculator, we first compute and store the value of the last quotient. Multiplying this value successively by  $\sin 120^\circ$  and  $\sin 2^\circ$ , we obtain

$$T_{AB} = 3570 \text{ lb}$$

$$T_{AC} = 144 \text{ lb}$$



## SAMPLE PROBLEM 2.5

Determine the magnitude and direction of the smallest force  $F$  which will maintain the package shown in equilibrium. Note that the force exerted by the rollers on the package is perpendicular to the incline.

## SOLUTION

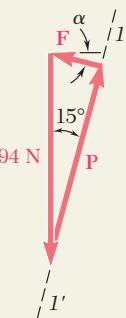
**Free-Body Diagram.** We choose the package as a free body, assuming that it can be treated as a particle. We draw the corresponding free-body diagram.

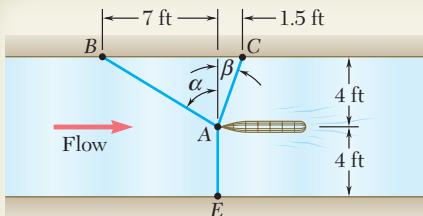
**Equilibrium Condition.** Since only three forces act on the free body, we draw a force triangle to express that it is in equilibrium. Line  $l-l'$  represents the known direction of  $P$ . In order to obtain the minimum value of the force  $F$ , we choose the direction of  $F$  perpendicular to that of  $P$ . From the geometry of the triangle obtained, we find

$$F = (294 \text{ N}) \sin 15^\circ = 76.1 \text{ N}$$

$$\alpha = 15^\circ$$

$$F = 76.1 \text{ N} \angle 15^\circ$$

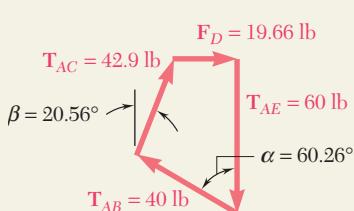
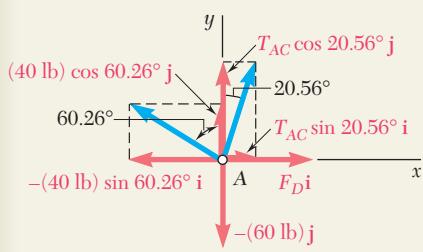
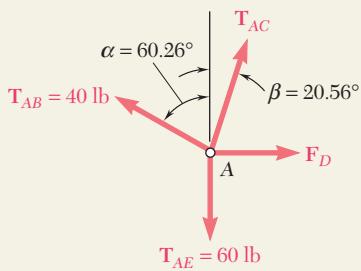




## SAMPLE PROBLEM 2.6

As part of the design of a new sailboat, it is desired to determine the drag force which may be expected at a given speed. To do so, a model of the proposed hull is placed in a test channel and three cables are used to keep its bow on the centerline of the channel. Dynamometer readings indicate that for a given speed, the tension is 40 lb in cable AB and 60 lb in cable AE. Determine the drag force exerted on the hull and the tension in cable AC.

## SOLUTION



**Determination of the Angles.** First, the angles  $\alpha$  and  $\beta$  defining the direction of cables AB and AC are determined. We write

$$\begin{aligned}\tan \alpha &= \frac{7 \text{ ft}}{4 \text{ ft}} = 1.75 & \tan \beta &= \frac{1.5 \text{ ft}}{4 \text{ ft}} = 0.375 \\ \alpha &= 60.26^\circ & \beta &= 20.56^\circ\end{aligned}$$

**Free-Body Diagram.** Choosing the hull as a free body, we draw the free-body diagram shown. It includes the forces exerted by the three cables on the hull, as well as the drag force  $\mathbf{F}_D$  exerted by the flow.

**Equilibrium Condition.** We express that the hull is in equilibrium by writing that the resultant of all forces is zero:

$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AE} + \mathbf{F}_D = 0 \quad (1)$$

Since more than three forces are involved, we resolve the forces into  $x$  and  $y$  components:

$$\begin{aligned}\mathbf{T}_{AB} &= -(40 \text{ lb}) \sin 60.26^\circ \mathbf{i} + (40 \text{ lb}) \cos 60.26^\circ \mathbf{j} \\ &= -(34.73 \text{ lb}) \mathbf{i} + (19.84 \text{ lb}) \mathbf{j} \\ \mathbf{T}_{AC} &= T_{AC} \sin 20.56^\circ \mathbf{i} + T_{AC} \cos 20.56^\circ \mathbf{j} \\ &= 0.3512 T_{AC} \mathbf{i} + 0.9363 T_{AC} \mathbf{j} \\ \mathbf{T}_{AE} &= -(60 \text{ lb}) \mathbf{j} \\ \mathbf{F}_D &= F_D \mathbf{i}\end{aligned}$$

Substituting the expressions obtained into Eq. (1) and factoring the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ , we have

$$(-34.73 \text{ lb} + 0.3512 T_{AC} + F_D) \mathbf{i} + (19.84 \text{ lb} + 0.9363 T_{AC} - 60 \text{ lb}) \mathbf{j} = 0$$

This equation will be satisfied if, and only if, the coefficients of  $\mathbf{i}$  and  $\mathbf{j}$  are equal to zero. We thus obtain the following two equilibrium equations, which express, respectively, that the sum of the  $x$  components and the sum of the  $y$  components of the given forces must be zero.

$$(\sum F_x = 0): \quad -34.73 \text{ lb} + 0.3512 T_{AC} + F_D = 0 \quad (2)$$

$$(\sum F_y = 0): \quad 19.84 \text{ lb} + 0.9363 T_{AC} - 60 \text{ lb} = 0 \quad (3)$$

From Eq. (3) we find

$$T_{AC} = +42.9 \text{ lb} \quad \blacktriangleleft$$

and, substituting this value into Eq. (2),

$$F_D = +19.66 \text{ lb} \quad \blacktriangleleft$$

In drawing the free-body diagram, we assumed a sense for each unknown force. A positive sign in the answer indicates that the assumed sense is correct. The complete force polygon may be drawn to check the results.

# SOLVING PROBLEMS ON YOUR OWN

When a particle is in *equilibrium*, the resultant of the forces acting on the particle must be zero. Expressing this fact in the case of a particle under *coplanar forces* will provide you with two relations among these forces. As you saw in the preceding sample problems, these relations may be used to determine two unknowns—such as the magnitude and direction of one force or the magnitudes of two forces.

**Drawing a free-body diagram is the first step** in the solution of a problem involving the equilibrium of a particle. This diagram shows the particle and all the forces acting on it. Indicate in your free-body diagram the magnitudes of known forces, as well as any angle or dimensions that define the direction of a force. Any unknown magnitude or angle should be denoted by an appropriate symbol. Nothing else should be included in the free-body diagram.

*Drawing a clear and accurate free-body diagram is a must in the solution of any equilibrium problem.* Skipping this step might save you pencil and paper, but is very likely to lead you to a wrong solution.

**Case 1. If only three forces are involved** in the free-body diagram, the rest of the solution is best carried out by drawing these forces in tip-to-tail fashion to form a *force triangle*. This triangle can be solved graphically or by trigonometry for no more than two unknowns [Sample Probs. 2.4 and 2.5].

**Case 2. If more than three forces are involved,** it is to your advantage to use an *analytic solution*. You select  $x$  and  $y$  axes and resolve each of the forces shown in the free-body diagram into  $x$  and  $y$  components. Expressing that the sum of the  $x$  components and the sum of the  $y$  components of all the forces are both zero, you will obtain two equations which you can solve for no more than two unknowns [Sample Prob. 2.6].

It is strongly recommended that when using an analytic solution the equations of equilibrium be written in the same form as Eqs. (2) and (3) of Sample Prob. 2.6. The practice adopted by some students of initially placing the unknowns on the left side of the equation and the known quantities on the right side may lead to confusion in assigning the appropriate sign to each term.

We have noted that regardless of the method used to solve a two-dimensional equilibrium problem we can determine at most two unknowns. If a two-dimensional problem involves more than two unknowns, one or more additional relations must be obtained from the information contained in the statement of the problem.

# PROBLEMS

- 2.43** Two cables are tied together at  $C$  and are loaded as shown. Knowing that  $\alpha = 20^\circ$ , determine the tension (a) in cable  $AC$ , (b) in cable  $BC$ .

- 2.44** Two cables are tied together at  $C$  and are loaded as shown. Determine the tension (a) in cable  $AC$ , (b) in cable  $BC$ .

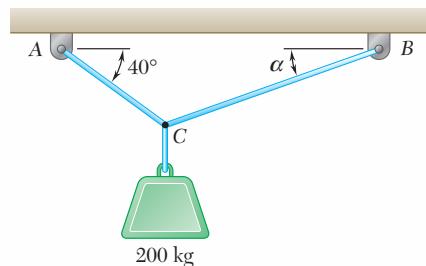


Fig. P2.43

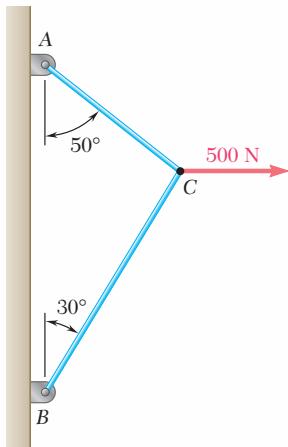


Fig. P2.44

- 2.45** Two cables are tied together at  $C$  and are loaded as shown. Knowing that  $P = 500 \text{ N}$  and  $\alpha = 60^\circ$ , determine the tension in (a) in cable  $AC$ , (b) in cable  $BC$ .

- 2.46** Two cables are tied together at  $C$  and are loaded as shown. Determine the tension (a) in cable  $AC$ , (b) in cable  $BC$ .

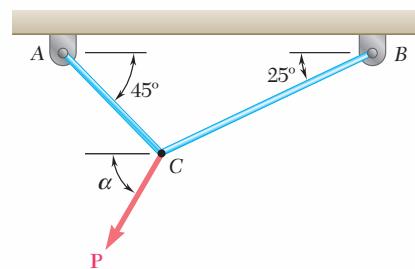


Fig. P2.45

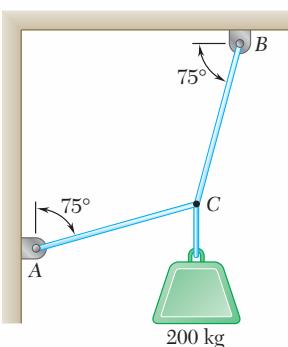


Fig. P2.46

- 2.47** Knowing that  $\alpha = 20^\circ$ , determine the tension (a) in cable AC, (b) in rope BC.

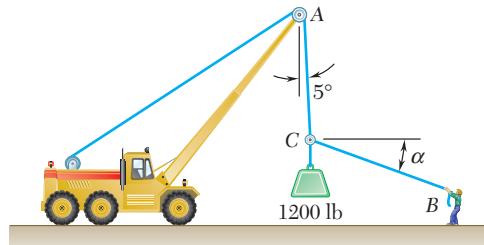


Fig. P2.47

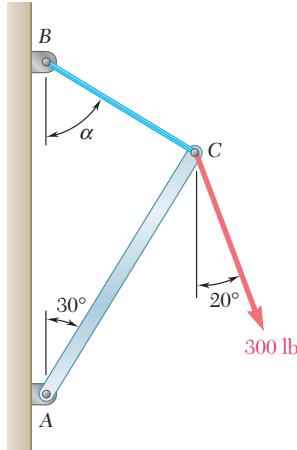


Fig. P2.48

- 2.48** Knowing that  $\alpha = 55^\circ$  and that boom AC exerts on pin C a force directed along line AC, determine (a) the magnitude of that force, (b) the tension in cable BC.

- 2.49** Two forces **P** and **Q** are applied as shown to an aircraft connection. Knowing that the connection is in equilibrium and that  $P = 500$  lb and  $Q = 650$  lb, determine the magnitudes of the forces exerted on the rods A and B.

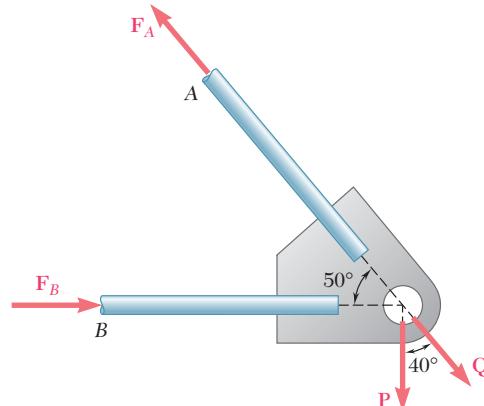


Fig. P2.49 and P2.50

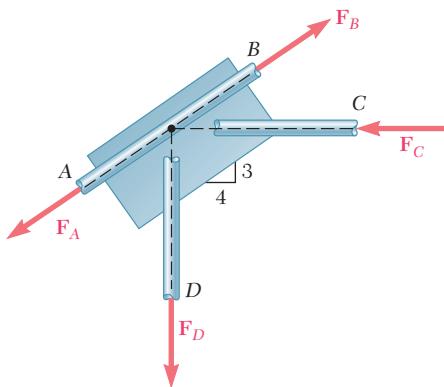


Fig. P2.51 and P2.52

- 2.50** Two forces **P** and **Q** are applied as shown to an aircraft connection. Knowing that the connection is in equilibrium and that the magnitudes of the forces exerted on rods A and B are  $F_A = 750$  lb and  $F_B = 400$  lb, determine the magnitudes of **P** and **Q**.

- 2.51** A welded connection is in equilibrium under the action of the four forces shown. Knowing that  $F_A = 8$  kN and  $F_B = 16$  kN, determine the magnitudes of the other two forces.

- 2.52** A welded connection is in equilibrium under the action of the four forces shown. Knowing that  $F_A = 5$  kN and  $F_D = 6$  kN, determine the magnitudes of the other two forces.

- 2.53** Two cables tied together at  $C$  are loaded as shown. Knowing that  $Q = 60$  lb, determine the tension (a) in cable  $AC$ , (b) in cable  $BC$ .

- 2.54** Two cables tied together at  $C$  are loaded as shown. Determine the range of values of  $Q$  for which the tension will not exceed 60 lb in either cable.

- 2.55** A sailor is being rescued using a boatswain's chair that is suspended from a pulley that can roll freely on the support cable  $ACB$  and is pulled at a constant speed by cable  $CD$ . Knowing that  $\alpha = 30^\circ$  and  $\beta = 10^\circ$  and that the combined weight of the boatswain's chair and the sailor is 900 N, determine the tension (a) in the support cable  $ACB$ , (b) in the traction cable  $CD$ .

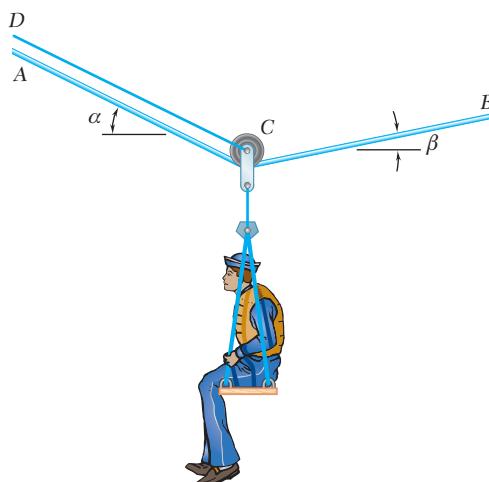


Fig. P2.55 and P2.56

- 2.56** A sailor is being rescued using a boatswain's chair that is suspended from a pulley that can roll freely on the support cable  $ACB$  and is pulled at a constant speed by cable  $CD$ . Knowing that  $\alpha = 25^\circ$  and  $\beta = 15^\circ$  and that the tension in cable  $CD$  is 80 N, determine (a) the combined weight of the boatswain's chair and the sailor, (b) the tension in the support cable  $ACB$ .

- 2.57** For the cables of Prob. 2.45, it is known that the maximum allowable tension is 600 N in cable  $AC$  and 750 N in cable  $BC$ . Determine (a) the maximum force  $\mathbf{P}$  that can be applied at  $C$ , (b) the corresponding value of  $\alpha$ .

- 2.58** For the situation described in Fig. P2.47, determine (a) the value of  $\alpha$  for which the tension in rope  $BC$  is as small as possible, (b) the corresponding value of the tension.

- 2.59** For the structure and loading of Prob. 2.48, determine (a) the value of  $\alpha$  for which the tension in cable  $BC$  is as small as possible, (b) the corresponding value of the tension.

- 2.60** Knowing that portions  $AC$  and  $BC$  of cable  $ACB$  must be equal, determine the shortest length of cable that can be used to support the load shown if the tension in the cable is not to exceed 870 N.

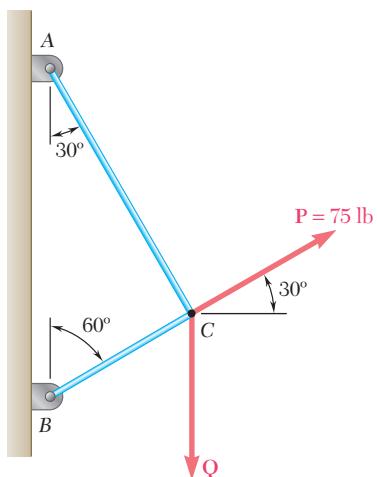


Fig. P2.53 and P2.54

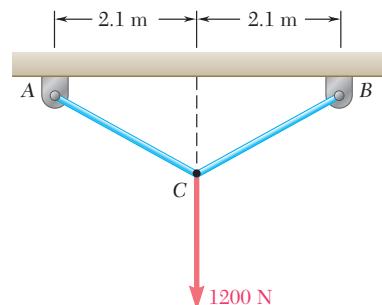


Fig. P2.60

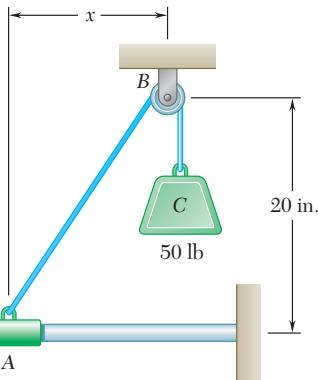


Fig. P2.63 and P2.64

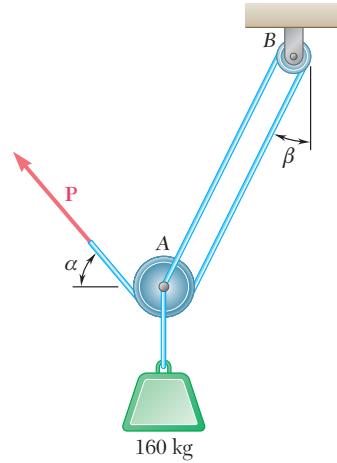


Fig. P2.65 and P2.66

- 2.61** Two cables tied together at *C* are loaded as shown. Knowing that the maximum allowable tension in each cable is 800 N, determine (a) the magnitude of the largest force **P** that can be applied at *C*, (b) the corresponding value of  $\alpha$ .

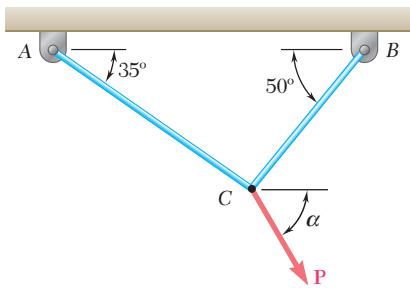


Fig. P2.61 and P2.62

- 2.62** Two cables tied together at *C* are loaded as shown. Knowing that the maximum allowable tension is 1200 N in cable *AC* and 600 N in cable *BC*, determine (a) the magnitude of the largest force **P** that can be applied at *C*, (b) the corresponding value of  $\alpha$ .

- 2.63** Collar *A* is connected as shown to a 50-lb load and can slide on a frictionless horizontal rod. Determine the magnitude of the force **P** required to maintain the equilibrium of the collar when (a)  $x = 4.5$  in., (b)  $x = 15$  in.

- 2.64** Collar *A* is connected as shown to a 50-lb load and can slide on a frictionless horizontal rod. Determine the distance  $x$  for which the collar is in equilibrium when  $P = 48$  lb.

- 2.65** A 160-kg load is supported by the rope-and-pulley arrangement shown. Knowing that  $\beta = 20^\circ$ , determine the magnitude and direction of the force **P** that must be exerted on the free end of the rope to maintain equilibrium. (Hint: The tension in the rope is the same on each side of a simple pulley. This can be proved by the methods of Chap. 4.)

- 2.66** A 160-kg load is supported by the rope-and-pulley arrangement shown. Knowing that  $\alpha = 40^\circ$ , determine (a) the angle  $\beta$ , (b) the magnitude of the force **P** that must be exerted on the free end of the rope to maintain equilibrium. (See the hint for Prob. 2.65.)

- 2.67** A 600-lb crate is supported by several rope-and-pulley arrangements as shown. Determine for each arrangement the tension in the rope. (See the hint for Prob. 2.65.)

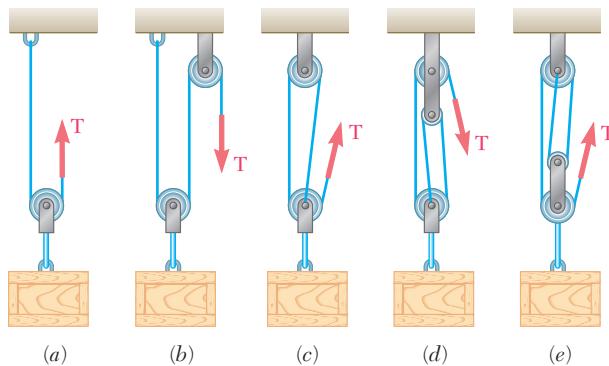


Fig. P2.67

**2.68** Solve parts *b* and *d* of Prob. 2.67, assuming that the free end of the rope is attached to the crate.

**2.69** A load **Q** is applied to the pulley *C*, which can roll on the cable *ACB*. The pulley is held in the position shown by a second cable *CAD*, which passes over the pulley *A* and supports a load **P**. Knowing that  $P = 750 \text{ N}$ , determine (*a*) the tension in cable *ACB*, (*b*) the magnitude of load **Q**.

**2.70** An 1800-N load **Q** is applied to the pulley *C*, which can roll on the cable *ACB*. The pulley is held in the position shown by a second cable *CAD*, which passes over the pulley *A* and supports a load **P**. Determine (*a*) the tension in cable *ACB*, (*b*) the magnitude of load **P**.

2.12 Rectangular Components of a Force in Space

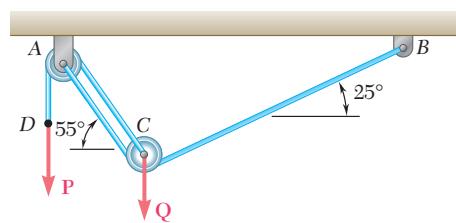


Fig. P2.69 and P2.70

## FORCES IN SPACE

### 2.12 RECTANGULAR COMPONENTS OF A FORCE IN SPACE

The problems considered in the first part of this chapter involved only two dimensions; they could be formulated and solved in a single plane. In this section and in the remaining sections of the chapter, we will discuss problems involving the three dimensions of space.

Consider a force **F** acting at the origin *O* of the system of rectangular coordinates *x*, *y*, *z*. To define the direction of **F**, we draw the vertical plane *OBAC* containing **F** (Fig. 2.30*a*). This plane passes through the vertical *y* axis; its orientation is defined by the angle  $\phi$  it forms with the *xy* plane. The direction of **F** within the plane is defined by the angle  $\theta_y$  that **F** forms with the *y* axis. The force **F** may be resolved into a vertical component **F<sub>y</sub>** and a horizontal component **F<sub>h</sub>**; this operation, shown in Fig. 2.30*b*, is carried out in plane *OBAC* according to the rules developed in the first part of the chapter. The corresponding scalar components are

$$F_y = F \cos \theta_y \quad F_h = F \sin \theta_y \quad (2.16)$$

But **F<sub>h</sub>** may be resolved into two rectangular components **F<sub>x</sub>** and **F<sub>z</sub>** along the *x* and *z* axes, respectively. This operation, shown in Fig. 2.30*c*, is carried out in the *xz* plane. We obtain the following expressions for the corresponding scalar components:

$$\begin{aligned} F_x &= F_h \cos \phi = F \sin \theta_y \cos \phi \\ F_z &= F_h \sin \phi = F \sin \theta_y \sin \phi \end{aligned} \quad (2.17)$$

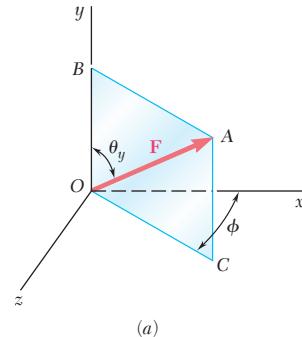
The given force **F** has thus been resolved into three rectangular vector components **F<sub>x</sub>**, **F<sub>y</sub>**, **F<sub>z</sub>**, which are directed along the three coordinate axes.

Applying the Pythagorean theorem to the triangles *OAB* and *ODC* of Fig. 2.30, we write

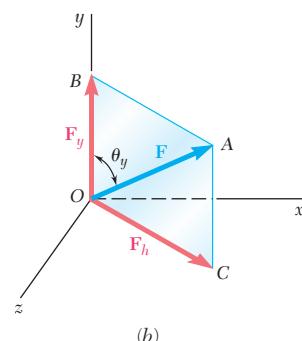
$$\begin{aligned} F^2 &= (OA)^2 = (OB)^2 + (BA)^2 = F_y^2 + F_h^2 \\ F_h^2 &= (OC)^2 = (OD)^2 + (DC)^2 = F_x^2 + F_z^2 \end{aligned}$$

Eliminating **F<sub>h</sub>** from these two equations and solving for **F**, we obtain the following relation between the magnitude of **F** and its rectangular scalar components:

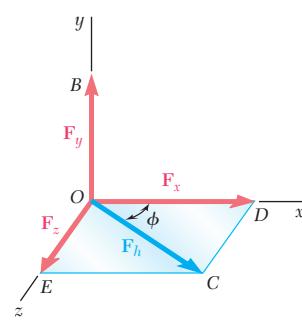
$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \quad (2.18)$$



(a)



(b)



(c)

Fig. 2.30

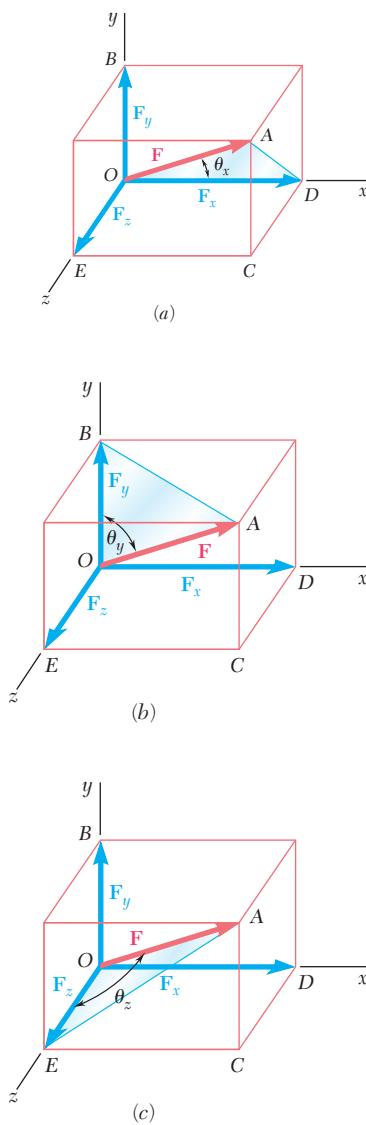


Fig. 2.31

The relationship existing between the force  $\mathbf{F}$  and its three components  $\mathbf{F}_x$ ,  $\mathbf{F}_y$ ,  $\mathbf{F}_z$  is more easily visualized if a “box” having  $\mathbf{F}_x$ ,  $\mathbf{F}_y$ ,  $\mathbf{F}_z$  for edges is drawn as shown in Fig. 2.31. The force  $\mathbf{F}$  is then represented by the diagonal  $OA$  of this box. Figure 2.31b shows the right triangle  $OAB$  used to derive the first of the formulas (2.16):  $F_y = F \cos \theta_y$ . In Fig. 2.31a and c, two other right triangles have also been drawn:  $OAD$  and  $OAE$ . These triangles are seen to occupy in the box positions comparable with that of triangle  $OAB$ . Denoting by  $\theta_x$  and  $\theta_z$ , respectively, the angles that  $\mathbf{F}$  forms with the  $x$  and  $z$  axes, we can derive two formulas similar to  $F_y = F \cos \theta_y$ . We thus write

$$F_x = F \cos \theta_x \quad F_y = F \cos \theta_y \quad F_z = F \cos \theta_z \quad (2.19)$$

The three angles  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  define the direction of the force  $\mathbf{F}$ ; they are more commonly used for this purpose than the angles  $\theta_y$  and  $\phi$  introduced at the beginning of this section. The cosines of  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  are known as the *direction cosines* of the force  $\mathbf{F}$ .

Introducing the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ , directed respectively along the  $x$ ,  $y$ , and  $z$  axes (Fig. 2.32), we can express  $\mathbf{F}$  in the form

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \quad (2.20)$$

where the scalar components  $F_x$ ,  $F_y$ ,  $F_z$  are defined by the relations (2.19).

**EXAMPLE 1.** A force of 500 N forms angles of  $60^\circ$ ,  $45^\circ$ , and  $120^\circ$ , respectively, with the  $x$ ,  $y$ , and  $z$  axes. Find the components  $F_x$ ,  $F_y$ , and  $F_z$  of the force.

Substituting  $F = 500$  N,  $\theta_x = 60^\circ$ ,  $\theta_y = 45^\circ$ ,  $\theta_z = 120^\circ$  into formulas (2.19), we write

$$\begin{aligned} F_x &= (500 \text{ N}) \cos 60^\circ = +250 \text{ N} \\ F_y &= (500 \text{ N}) \cos 45^\circ = +354 \text{ N} \\ F_z &= (500 \text{ N}) \cos 120^\circ = -250 \text{ N} \end{aligned}$$

Carrying into Eq. (2.20) the values obtained for the scalar components of  $\mathbf{F}$ , we have

$$\mathbf{F} = (250 \text{ N})\mathbf{i} + (354 \text{ N})\mathbf{j} - (250 \text{ N})\mathbf{k}$$

As in the case of two-dimensional problems, a plus sign indicates that the component has the same sense as the corresponding axis, and a minus sign indicates that it has the opposite sense. ■

The angle a force  $\mathbf{F}$  forms with an axis should be measured from the positive side of the axis and will always be between 0 and  $180^\circ$ . An angle  $\theta_x$  smaller than  $90^\circ$  (acute) indicates that  $\mathbf{F}$  (assumed attached to  $O$ ) is on the same side of the  $yz$  plane as the positive  $x$  axis;  $\cos \theta_x$  and  $F_x$  will then be positive. An angle  $\theta_x$  larger than  $90^\circ$  (obtuse) indicates that  $\mathbf{F}$  is on the other side of the  $yz$  plane;  $\cos \theta_x$  and  $F_x$  will then be negative. In Example 1 the angles  $\theta_x$  and  $\theta_y$  are acute, while  $\theta_z$  is obtuse; consequently,  $F_x$  and  $F_y$  are positive, while  $F_z$  is negative.

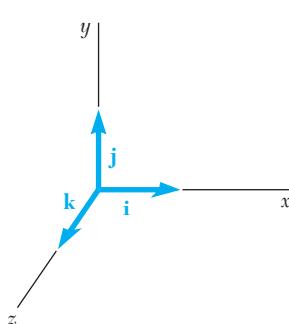


Fig. 2.32

Substituting into (2.20) the expressions obtained for  $F_x$ ,  $F_y$ ,  $F_z$  in (2.19), we write

$$\mathbf{F} = F(\cos \theta_x \mathbf{i} + \cos \theta_y \mathbf{j} + \cos \theta_z \mathbf{k}) \quad (2.21)$$

which shows that the force  $\mathbf{F}$  can be expressed as the product of the scalar  $F$  and the vector

$$\boldsymbol{\lambda} = \cos \theta_x \mathbf{i} + \cos \theta_y \mathbf{j} + \cos \theta_z \mathbf{k} \quad (2.22)$$

Clearly, the vector  $\boldsymbol{\lambda}$  is a vector whose magnitude is equal to 1 and whose direction is the same as that of  $\mathbf{F}$  (Fig. 2.33). The vector  $\boldsymbol{\lambda}$  is referred to as the *unit vector* along the line of action of  $\mathbf{F}$ . It follows from (2.22) that the components of the unit vector  $\boldsymbol{\lambda}$  are respectively equal to the direction cosines of the line of action of  $\mathbf{F}$ :

$$\lambda_x = \cos \theta_x \quad \lambda_y = \cos \theta_y \quad \lambda_z = \cos \theta_z \quad (2.23)$$

We should observe that the values of the three angles  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  are not independent. Recalling that the sum of the squares of the components of a vector is equal to the square of its magnitude, we write

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1$$

or, substituting for  $\lambda_x$ ,  $\lambda_y$ ,  $\lambda_z$  from (2.23),

$$\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1 \quad (2.24)$$

In Example 1, for instance, once the values  $\theta_x = 60^\circ$  and  $\theta_y = 45^\circ$  have been selected, the value of  $\theta_z$  *must* be equal to  $60^\circ$  or  $120^\circ$  in order to satisfy identity (2.24).

When the components  $F_x$ ,  $F_y$ ,  $F_z$  of a force  $\mathbf{F}$  are given, the magnitude  $F$  of the force is obtained from (2.18).† The relations (2.19) can then be solved for the direction cosines,

$$\cos \theta_x = \frac{F_x}{F} \quad \cos \theta_y = \frac{F_y}{F} \quad \cos \theta_z = \frac{F_z}{F} \quad (2.25)$$

and the angles  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  characterizing the direction of  $\mathbf{F}$  can be found.

**EXAMPLE 2.** A force  $\mathbf{F}$  has the components  $F_x = 20$  lb,  $F_y = -30$  lb,  $F_z = 60$  lb. Determine its magnitude  $F$  and the angles  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  it forms with the coordinate axes.

From formula (2.18) we obtain†

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2 + F_z^2} \\ &= \sqrt{(20 \text{ lb})^2 + (-30 \text{ lb})^2 + (60 \text{ lb})^2} \\ &= \sqrt{4900} \text{ lb} = 70 \text{ lb} \end{aligned}$$

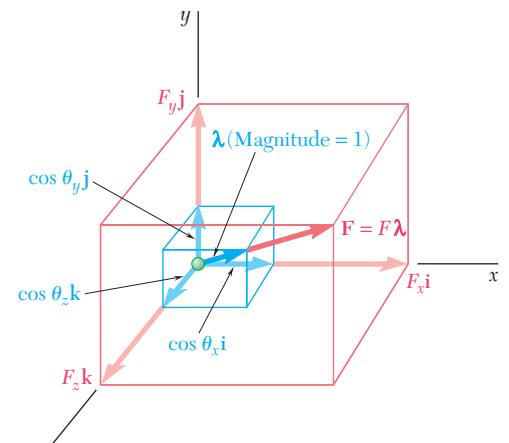


Fig. 2.33

†With a calculator programmed to convert rectangular coordinates into polar coordinates, the following procedure will be found more expeditious for computing  $F$ : First determine  $F_h$  from its two rectangular components  $F_x$  and  $F_z$  (Fig. 2.30c), then determine  $F$  from its two rectangular components  $F_h$  and  $F_y$  (Fig. 2.30b). The actual order in which the three components  $F_x$ ,  $F_y$ ,  $F_z$  are entered is immaterial.

Substituting the values of the components and magnitude of  $\mathbf{F}$  into Eqs. (2.25), we write

$$\cos \theta_x = \frac{F_x}{F} = \frac{20 \text{ lb}}{70 \text{ lb}} \quad \cos \theta_y = \frac{F_y}{F} = \frac{-30 \text{ lb}}{70 \text{ lb}} \quad \cos \theta_z = \frac{F_z}{F} = \frac{60 \text{ lb}}{70 \text{ lb}}$$

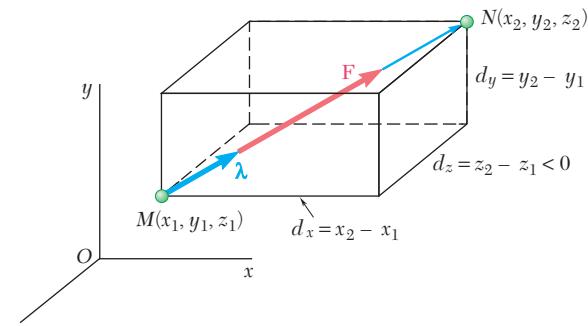
Calculating successively each quotient and its arc cosine, we obtain

$$\theta_x = 73.4^\circ \quad \theta_y = 115.4^\circ \quad \theta_z = 31.0^\circ$$

These computations can be carried out easily with a calculator. ■

## 2.13 FORCE DEFINED BY ITS MAGNITUDE AND TWO POINTS ON ITS LINE OF ACTION

In many applications, the direction of a force  $\mathbf{F}$  is defined by the coordinates of two points,  $M(x_1, y_1, z_1)$  and  $N(x_2, y_2, z_2)$ , located on its line of action (Fig. 2.34). Consider the vector  $\overrightarrow{MN}$  joining  $M$  and  $N$



**Fig. 2.34**

and of the same sense as  $\mathbf{F}$ . Denoting its scalar components by  $d_x$ ,  $d_y$ ,  $d_z$ , respectively, we write

$$\overrightarrow{MN} = d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k} \quad (2.26)$$

The unit vector  $\lambda$  along the line of action of  $\mathbf{F}$  (i.e., along the line  $MN$ ) may be obtained by dividing the vector  $\overrightarrow{MN}$  by its magnitude  $MN$ . Substituting for  $\overrightarrow{MN}$  from (2.26) and observing that  $MN$  is equal to the distance  $d$  from  $M$  to  $N$ , we write

$$\lambda = \frac{\overrightarrow{MN}}{MN} = \frac{1}{d} (d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k}) \quad (2.27)$$

Recalling that  $\mathbf{F}$  is equal to the product of  $F$  and  $\lambda$ , we have

$$\mathbf{F} = F \lambda = \frac{F}{d} (d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k}) \quad (2.28)$$

from which it follows that the scalar components of  $\mathbf{F}$  are, respectively,

$$F_x = \frac{Fd_x}{d} \quad F_y = \frac{Fd_y}{d} \quad F_z = \frac{Fd_z}{d} \quad (2.29)$$

The relations (2.29) considerably simplify the determination of the components of a force  $\mathbf{F}$  of given magnitude  $F$  when the line of action of  $\mathbf{F}$  is defined by two points  $M$  and  $N$ . Subtracting the coordinates of  $M$  from those of  $N$ , we first determine the components of the vector  $\overrightarrow{MN}$  and the distance  $d$  from  $M$  to  $N$ :

$$\begin{aligned} d_x &= x_2 - x_1 & d_y &= y_2 - y_1 & d_z &= z_2 - z_1 \\ d &= \sqrt{d_x^2 + d_y^2 + d_z^2} \end{aligned}$$

Substituting for  $F$  and for  $d_x, d_y, d_z$ , and  $d$  into the relations (2.29), we obtain the components  $F_x, F_y, F_z$  of the force.

The angles  $\theta_x, \theta_y, \theta_z$  that  $\mathbf{F}$  forms with the coordinate axes can then be obtained from Eqs. (2.25). Comparing Eqs. (2.22) and (2.27), we can also write

$$\cos \theta_x = \frac{d_x}{d} \quad \cos \theta_y = \frac{d_y}{d} \quad \cos \theta_z = \frac{d_z}{d} \quad (2.30)$$

and determine the angles  $\theta_x, \theta_y, \theta_z$  directly from the components and magnitude of the vector  $\overrightarrow{MN}$ .

## 2.14 ADDITION OF CONCURRENT FORCES IN SPACE

The resultant  $\mathbf{R}$  of two or more forces in space will be determined by summing their rectangular components. Graphical or trigonometric methods are generally not practical in the case of forces in space.

The method followed here is similar to that used in Sec. 2.8 with coplanar forces. Setting

$$\mathbf{R} = \Sigma \mathbf{F}$$

we resolve each force into its rectangular components and write

$$\begin{aligned} R_x \mathbf{i} + R_y \mathbf{j} + R_z \mathbf{k} &= \Sigma(F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}) \\ &= (\Sigma F_x) \mathbf{i} + (\Sigma F_y) \mathbf{j} + (\Sigma F_z) \mathbf{k} \end{aligned}$$

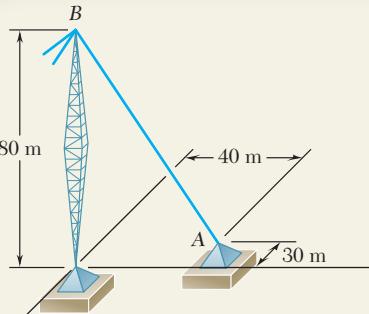
from which it follows that

$$R_x = \Sigma F_x \quad R_y = \Sigma F_y \quad R_z = \Sigma F_z \quad (2.31)$$

The magnitude of the resultant and the angles  $\theta_x, \theta_y, \theta_z$  that the resultant forms with the coordinate axes are obtained using the method discussed in Sec. 2.12. We write

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \quad (2.32)$$

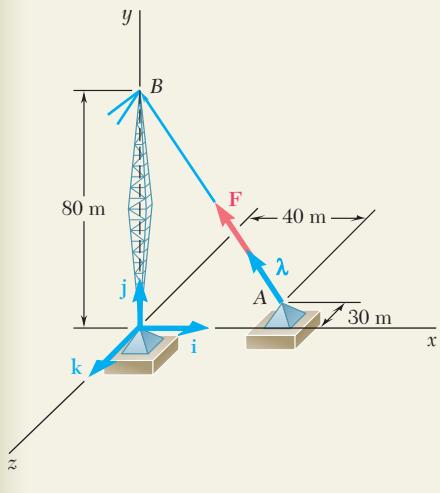
$$\cos \theta_x = \frac{R_x}{R} \quad \cos \theta_y = \frac{R_y}{R} \quad \cos \theta_z = \frac{R_z}{R} \quad (2.33)$$



## SAMPLE PROBLEM 2.7

A tower guy wire is anchored by means of a bolt at A. The tension in the wire is 2500 N. Determine (a) the components  $F_x$ ,  $F_y$ ,  $F_z$  of the force acting on the bolt, (b) the angles  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  defining the direction of the force.

## SOLUTION



**a. Components of the Force.** The line of action of the force acting on the bolt passes through A and B, and the force is directed from A to B. The components of the vector  $\overrightarrow{AB}$ , which has the same direction as the force, are

$$d_x = -40 \text{ m} \quad d_y = +80 \text{ m} \quad d_z = +30 \text{ m}$$

The total distance from A to B is

$$AB = d = \sqrt{d_x^2 + d_y^2 + d_z^2} = 94.3 \text{ m}$$

Denoting by  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  the unit vectors along the coordinate axes, we have

$$\overrightarrow{AB} = -(40 \text{ m})\mathbf{i} + (80 \text{ m})\mathbf{j} + (30 \text{ m})\mathbf{k}$$

Introducing the unit vector  $\boldsymbol{\lambda} = \overrightarrow{AB}/AB$ , we write

$$\mathbf{F} = F\boldsymbol{\lambda} = F \frac{\overrightarrow{AB}}{AB} = \frac{2500 \text{ N}}{94.3 \text{ m}} \overrightarrow{AB}$$

Substituting the expression found for  $\overrightarrow{AB}$ , we obtain

$$\mathbf{F} = \frac{2500 \text{ N}}{94.3 \text{ m}} [-(40 \text{ m})\mathbf{i} + (80 \text{ m})\mathbf{j} + (30 \text{ m})\mathbf{k}]$$

$$\mathbf{F} = -(1060 \text{ N})\mathbf{i} + (2120 \text{ N})\mathbf{j} + (795 \text{ N})\mathbf{k}$$

The components of  $\mathbf{F}$ , therefore, are

$$F_x = -1060 \text{ N} \quad F_y = +2120 \text{ N} \quad F_z = +795 \text{ N}$$

**b. Direction of the Force.** Using Eqs. (2.25), we write

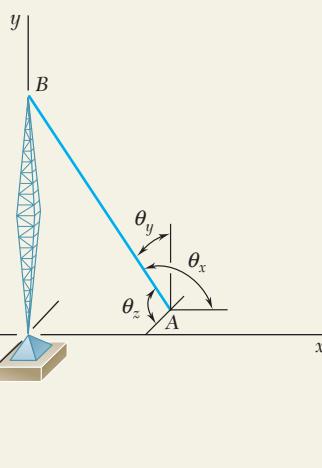
$$\cos \theta_x = \frac{F_x}{F} = \frac{-1060 \text{ N}}{2500 \text{ N}} \quad \cos \theta_y = \frac{F_y}{F} = \frac{+2120 \text{ N}}{2500 \text{ N}}$$

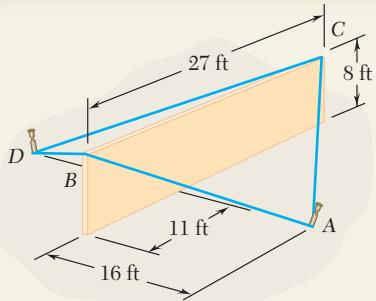
$$\cos \theta_z = \frac{F_z}{F} = \frac{+795 \text{ N}}{2500 \text{ N}}$$

Calculating successively each quotient and its arc cosine, we obtain

$$\theta_x = 115.1^\circ \quad \theta_y = 32.0^\circ \quad \theta_z = 71.5^\circ$$

(Note. This result could have been obtained by using the components and magnitude of the vector  $\overrightarrow{AB}$  rather than those of the force  $\mathbf{F}$ .)





## SAMPLE PROBLEM 2.8

A wall section of precast concrete is temporarily held by the cables shown. Knowing that the tension is 840 lb in cable  $AB$  and 1200 lb in cable  $AC$ , determine the magnitude and direction of the resultant of the forces exerted by cables  $AB$  and  $AC$  on stake  $A$ .

### SOLUTION

**Components of the Forces.** The force exerted by each cable on stake  $A$  will be resolved into  $x$ ,  $y$ , and  $z$  components. We first determine the components and magnitude of the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$ , measuring them from  $A$  toward the wall section. Denoting by  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  the unit vectors along the coordinate axes, we write

$$\begin{aligned}\overrightarrow{AB} &= -(16 \text{ ft})\mathbf{i} + (8 \text{ ft})\mathbf{j} + (11 \text{ ft})\mathbf{k} & AB = 21 \text{ ft} \\ \overrightarrow{AC} &= -(16 \text{ ft})\mathbf{i} + (8 \text{ ft})\mathbf{j} - (16 \text{ ft})\mathbf{k} & AC = 24 \text{ ft}\end{aligned}$$

Denoting by  $\lambda_{AB}$  the unit vector along  $AB$ , we have

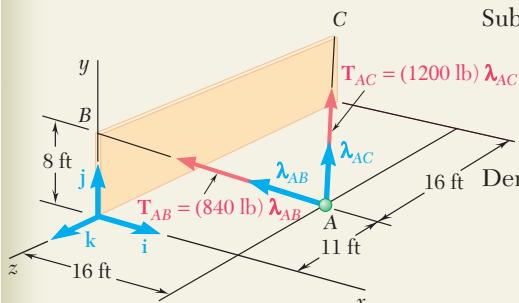
$$\mathbf{T}_{AB} = T_{AB}\lambda_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = \frac{840 \text{ lb}}{21 \text{ ft}}\overrightarrow{AB}$$

Substituting the expression found for  $\overrightarrow{AB}$ , we obtain

$$\begin{aligned}\mathbf{T}_{AB} &= \frac{840 \text{ lb}}{21 \text{ ft}}[-(16 \text{ ft})\mathbf{i} + (8 \text{ ft})\mathbf{j} + (11 \text{ ft})\mathbf{k}] \\ \mathbf{T}_{AB} &= -(640 \text{ lb})\mathbf{i} + (320 \text{ lb})\mathbf{j} + (440 \text{ lb})\mathbf{k}\end{aligned}$$

Denoting by  $\lambda_{AC}$  the unit vector along  $AC$ , we obtain in a similar way

$$\begin{aligned}\mathbf{T}_{AC} &= T_{AC}\lambda_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = \frac{1200 \text{ lb}}{24 \text{ ft}}\overrightarrow{AC} \\ \mathbf{T}_{AC} &= -(800 \text{ lb})\mathbf{i} + (400 \text{ lb})\mathbf{j} - (800 \text{ lb})\mathbf{k}\end{aligned}$$



**Resultant of the Forces.** The resultant  $\mathbf{R}$  of the forces exerted by the two cables is

$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} = -(1440 \text{ lb})\mathbf{i} + (720 \text{ lb})\mathbf{j} - (360 \text{ lb})\mathbf{k}$$

The magnitude and direction of the resultant are now determined:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{(-1440)^2 + (720)^2 + (-360)^2} \quad R = 1650 \text{ lb}$$

From Eqs. (2.33) we obtain

$$\begin{aligned}\cos \theta_x &= \frac{R_x}{R} = \frac{-1440 \text{ lb}}{1650 \text{ lb}} & \cos \theta_y = \frac{R_y}{R} = \frac{+720 \text{ lb}}{1650 \text{ lb}} \\ \cos \theta_z &= \frac{R_z}{R} = \frac{-360 \text{ lb}}{1650 \text{ lb}}\end{aligned}$$

Calculating successively each quotient and its arc cosine, we have

$$\theta_x = 150.8^\circ \quad \theta_y = 64.1^\circ \quad \theta_z = 102.6^\circ$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson we saw that *forces in space* may be defined by their magnitude and direction or by the three rectangular components  $F_x$ ,  $F_y$ , and  $F_z$ .

**A. When a force is defined by its magnitude and direction,** its rectangular components  $F_x$ ,  $F_y$ , and  $F_z$  may be found as follows:

**Case 1.** If the direction of the force  $\mathbf{F}$  is defined by the angles  $\theta_y$  and  $\phi$  shown in Fig. 2.30, projections of  $\mathbf{F}$  through these angles or their complements will yield the components of  $\mathbf{F}$  [Eqs. (2.17)]. Note that the  $x$  and  $z$  components of  $\mathbf{F}$  are found by first projecting  $\mathbf{F}$  onto the horizontal plane; the projection  $\mathbf{F}_h$  obtained in this way is then resolved into the components  $\mathbf{F}_x$  and  $\mathbf{F}_z$  (Fig. 2.30c).

**Case 2.** If the direction of the force  $\mathbf{F}$  is defined by the angles  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  that  $\mathbf{F}$  forms with the coordinate axes, each component can be obtained by multiplying the magnitude  $F$  of the force by the cosine of the corresponding angle [Example 1]:

$$F_x = F \cos \theta_x \quad F_y = F \cos \theta_y \quad F_z = F \cos \theta_z$$

**Case 3.** If the direction of the force  $\mathbf{F}$  is defined by two points  $M$  and  $N$  located on its line of action (Fig. 2.34), you will first express the vector  $\overrightarrow{MN}$  drawn from  $M$  to  $N$  in terms of its components  $d_x$ ,  $d_y$ ,  $d_z$  and the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ :

$$\overrightarrow{MN} = d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k}$$

Next, you will determine the unit vector  $\boldsymbol{\lambda}$  along the line of action of  $\mathbf{F}$  by dividing the vector  $\overrightarrow{MN}$  by its magnitude  $MN$ . Multiplying  $\boldsymbol{\lambda}$  by the magnitude of  $\mathbf{F}$ , you will obtain the desired expression for  $\mathbf{F}$  in terms of its rectangular components [Sample Prob. 2.7]:

$$\mathbf{F} = F \boldsymbol{\lambda} = \frac{F}{d} (d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k})$$

It is advantageous to use a consistent and meaningful system of notation when determining the rectangular components of a force. The method used in this text is illustrated in Sample Prob. 2.8 where, for example, the force  $\mathbf{T}_{AB}$  acts from stake  $A$  toward point  $B$ . Note that the subscripts have been ordered to agree with the direction of the force. It is recommended that you adopt the same notation, as it will help you identify point 1 (the first subscript) and point 2 (the second subscript).

When forming the vector defining the line of action of a force, you may think of its scalar components as the number of steps you must take in each coordinate direction to go from point 1 to point 2. It is essential that you always remember to assign the correct sign to each of the components.

**B. When a force is defined by its rectangular components  $F_x$ ,  $F_y$ ,  $F_z$** , you can obtain its magnitude  $F$  by writing

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

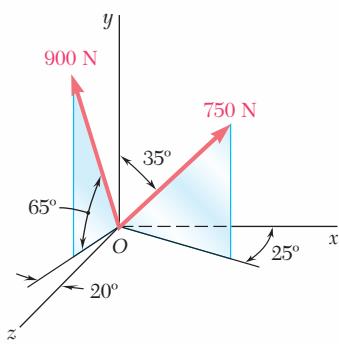
You can determine the direction cosines of the line of action of  $\mathbf{F}$  by dividing the components of the force by  $F$ :

$$\cos \theta_x = \frac{F_x}{F} \quad \cos \theta_y = \frac{F_y}{F} \quad \cos \theta_z = \frac{F_z}{F}$$

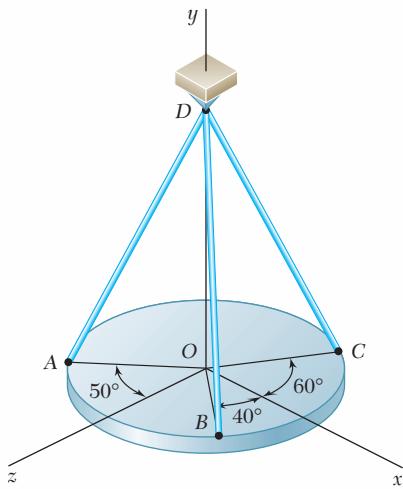
From the direction cosines you can obtain the angles  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  that  $\mathbf{F}$  forms with the coordinate axes [Example 2].

**C. To determine the resultant  $R$  of two or more forces** in three-dimensional space, first determine the rectangular components of each force by one of the procedures described above. Adding these components will yield the components  $R_x$ ,  $R_y$ ,  $R_z$  of the resultant. The magnitude and direction of the resultant may then be obtained as indicated above for a force  $\mathbf{F}$  [Sample Prob. 2.8].

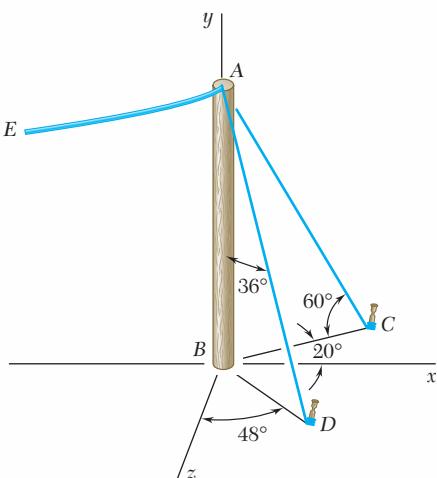
# PROBLEMS



**Fig. P2.71 and P2.72**



**Fig. P2.73, P2.74, P2.75, and P2.76**



**Fig. P2.77 and P2.78**

**2.71** Determine (a) the  $x$ ,  $y$ , and  $z$  components of the 750-N force, (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force forms with the coordinate axes.

**2.72** Determine (a) the  $x$ ,  $y$ , and  $z$  components of the 900-N force, (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force forms with the coordinate axes.

**2.73** A horizontal circular plate is suspended as shown from three wires that are attached to a support at  $D$  and form  $30^\circ$  angles with the vertical. Knowing that the  $x$  component of the force exerted by wire  $AD$  on the plate is 110.3 N, determine (a) the tension in wire  $AD$ , (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force exerted at  $A$  forms with the coordinate axes.

**2.74** A horizontal circular plate is suspended as shown from three wires that are attached to a support at  $D$  and form  $30^\circ$  angles with the vertical. Knowing that the  $z$  component of the force exerted by wire  $BD$  on the plate is  $-32.14$  N, determine (a) the tension in wire  $BD$ , (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force exerted at  $B$  forms with the coordinate axes.

**2.75** A horizontal circular plate is suspended as shown from three wires that are attached to a support at  $D$  and form  $30^\circ$  angles with the vertical. Knowing that the tension in wire  $CD$  is 60 lb, determine (a) the components of the force exerted by this wire on the plate, (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force forms with the coordinate axes.

**2.76** A horizontal circular plate is suspended as shown from three wires that are attached to a support at  $D$  and form  $30^\circ$  angles with the vertical. Knowing that the  $x$  component of the force exerted by wire  $CD$  on the plate is  $-20$  lb, determine (a) the tension in wire  $CD$ , (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force exerted at  $C$  forms with the coordinate axes.

**2.77** The end of the coaxial cable  $AE$  is attached to the pole  $AB$ , which is strengthened by the guy wires  $AC$  and  $AD$ . Knowing that the tension in wire  $AC$  is 120 lb, determine (a) the components of the force exerted by this wire on the pole, (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force forms with the coordinate axes.

**2.78** The end of the coaxial cable  $AE$  is attached to the pole  $AB$ , which is strengthened by the guy wires  $AC$  and  $AD$ . Knowing that the tension in wire  $AD$  is 85 lb, determine (a) the components of the force exerted by this wire on the pole, (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  that the force forms with the coordinate axes.

**2.79** Determine the magnitude and direction of the force  $\mathbf{F} = (320 \text{ N})\mathbf{i} + (400 \text{ N})\mathbf{j} - (250 \text{ N})\mathbf{k}$ .

- 2.80** Determine the magnitude and direction of the force  $\mathbf{F} = (240 \text{ N})\mathbf{i} - (270 \text{ N})\mathbf{j} + (680 \text{ N})\mathbf{k}$ .

- 2.81** A force acts at the origin of a coordinate system in a direction defined by the angles  $\theta_x = 70.9^\circ$  and  $\theta_y = 144.9^\circ$ . Knowing that the  $z$  component of the force is  $-52 \text{ lb}$ , determine (a) the angle  $\theta_z$ , (b) the other components and the magnitude of the force.

- 2.82** A force acts at the origin of a coordinate system in a direction defined by the angles  $\theta_y = 55^\circ$  and  $\theta_z = 45^\circ$ . Knowing that the  $x$  component of the force is  $-500 \text{ lb}$ , determine (a) the angle  $\theta_x$ , (b) the other components and the magnitude of the force.

- 2.83** A force  $\mathbf{F}$  of magnitude  $210 \text{ N}$  acts at the origin of a coordinate system. Knowing that  $F_x = 80 \text{ N}$ ,  $\theta_z = 151.2^\circ$ , and  $F_y < 0$ , determine (a) the components  $F_y$  and  $F_z$ , (b) the angles  $\theta_x$  and  $\theta_y$ .

- 2.84** A force  $\mathbf{F}$  of magnitude  $230 \text{ N}$  acts at the origin of a coordinate system. Knowing that  $\theta_x = 32.5^\circ$ ,  $F_y = -60 \text{ N}$ , and  $F_z > 0$ , determine (a) the components  $F_x$  and  $F_z$ , (b) the angles  $\theta_y$  and  $\theta_z$ .

- 2.85** A transmission tower is held by three guy wires anchored by bolts at  $B$ ,  $C$ , and  $D$ . If the tension in wire  $AB$  is  $525 \text{ lb}$ , determine the components of the force exerted by the wire on the bolt at  $B$ .

- 2.86** A transmission tower is held by three guy wires anchored by bolts at  $B$ ,  $C$ , and  $D$ . If the tension in wire  $AD$  is  $315 \text{ lb}$ , determine the components of the force exerted by the wire on the bolt at  $D$ .

- 2.87** A frame  $ABC$  is supported in part by cable  $DBE$  that passes through a frictionless ring at  $B$ . Knowing that the tension in the cable is  $385 \text{ N}$ , determine the components of the force exerted by the cable on the support at  $D$ .

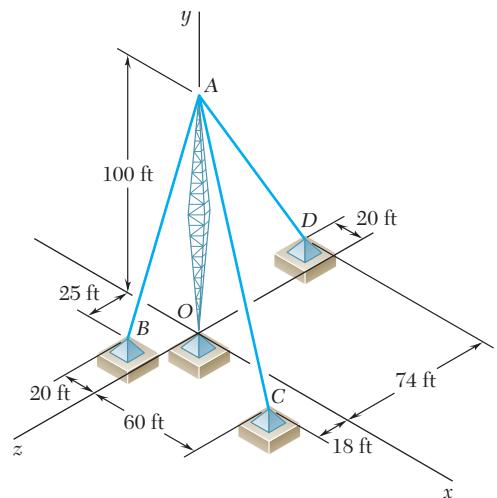


Fig. P2.85 and P2.86

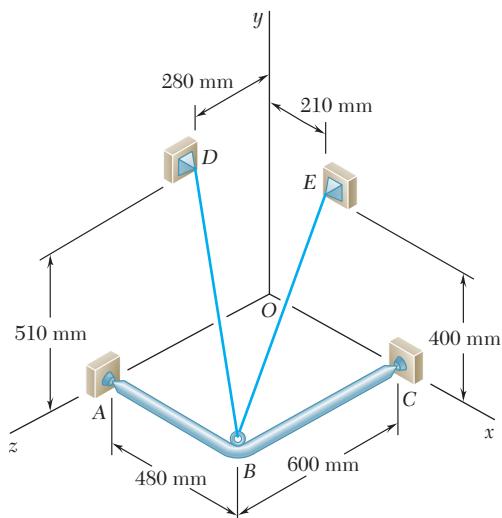
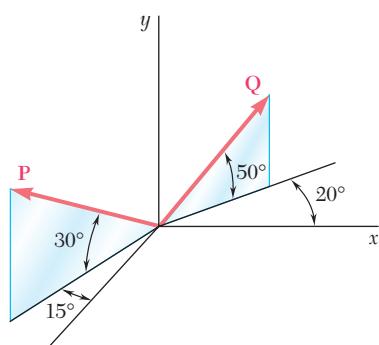


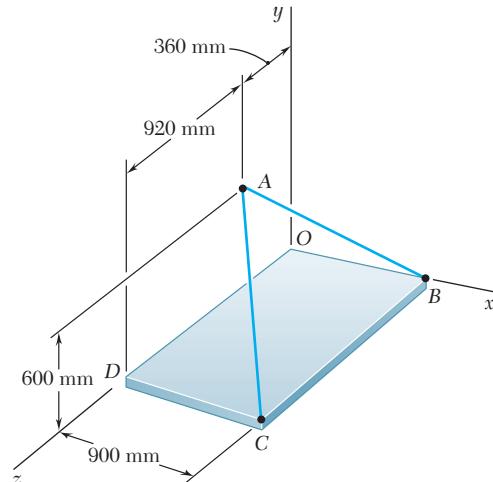
Fig. P2.87

- 2.88** For the frame and cable of Prob. 2.87, determine the components of the force exerted by the cable on the support at *E*.

- 2.89** Knowing that the tension in cable *AB* is 1425 N, determine the components of the force exerted on the plate at *B*.



**Fig. P2.91 and P2.92**



**Fig. P2.89 and P2.90**

- 2.90** Knowing that the tension in cable *AC* is 2130 N, determine the components of the force exerted on the plate at *C*.

- 2.91** Find the magnitude and direction of the resultant of the two forces shown knowing that  $P = 300$  N and  $Q = 400$  N.

- 2.92** Find the magnitude and direction of the resultant of the two forces shown knowing that  $P = 400$  N and  $Q = 300$  N.

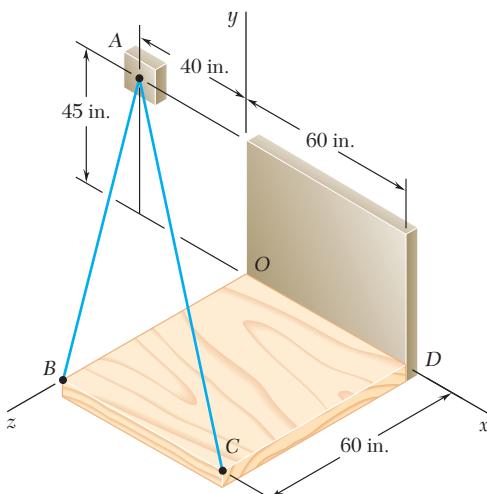
- 2.93** Knowing that the tension is 425 lb in cable *AB* and 510 lb in cable *AC*, determine the magnitude and direction of the resultant of the forces exerted at *A* by the two cables.

- 2.94** Knowing that the tension is 510 lb in cable *AB* and 425 lb in cable *AC*, determine the magnitude and direction of the resultant of the forces exerted at *A* by the two cables.

- 2.95** For the frame of Prob. 2.87, determine the magnitude and direction of the resultant of the forces exerted by the cable at *B* knowing that the tension in the cable is 385 N.

- 2.96** For the cables of Prob. 2.89, knowing that the tension is 1425 N in cable *AB* and 2130 N in cable *AC*, determine the magnitude and direction of the resultant of the forces exerted at *A* by the two cables.

**Fig. P2.93 and P2.94**



- 2.97** The end of the coaxial cable AE is attached to the pole AB, which is strengthened by the guy wires AC and AD. Knowing that the tension in AC is 150 lb and that the resultant of the forces exerted at A by wires AC and AD must be contained in the  $xy$  plane, determine (a) the tension in AD, (b) the magnitude and direction of the resultant of the two forces.

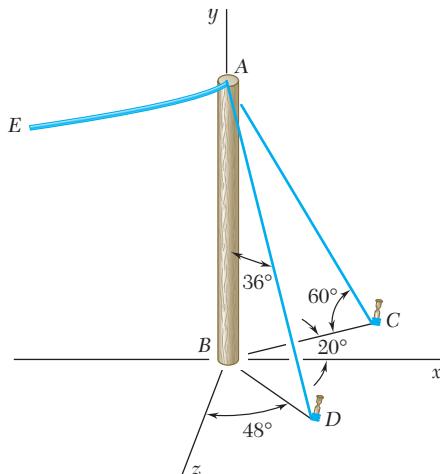


Fig. P2.97 and P2.98

- 2.98** The end of the coaxial cable AE is attached to the pole AB, which is strengthened by the guy wires AC and AD. Knowing that the tension in AD is 125 lb and that the resultant of the forces exerted at A by wires AC and AD must be contained in the  $xy$  plane, determine (a) the tension in AC, (b) the magnitude and direction of the resultant of the two forces.

## 2.15 EQUILIBRIUM OF A PARTICLE IN SPACE

According to the definition given in Sec. 2.9, a particle A is in equilibrium if the resultant of all the forces acting on A is zero. The components  $R_x$ ,  $R_y$ ,  $R_z$  of the resultant are given by the relations (2.31); expressing that the components of the resultant are zero, we write

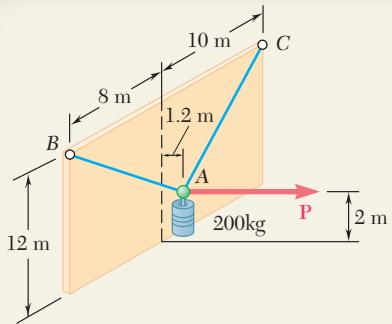
$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0 \quad (2.34)$$

Equations (2.34) represent the necessary and sufficient conditions for the equilibrium of a particle in space. They can be used to solve problems dealing with the equilibrium of a particle involving no more than three unknowns.

To solve such problems, you first should draw a free-body diagram showing the particle in equilibrium and *all* the forces acting on it. You can then write the equations of equilibrium (2.34) and solve them for three unknowns. In the more common types of problems, these unknowns will represent (1) the three components of a single force or (2) the magnitude of three forces, each of known direction.



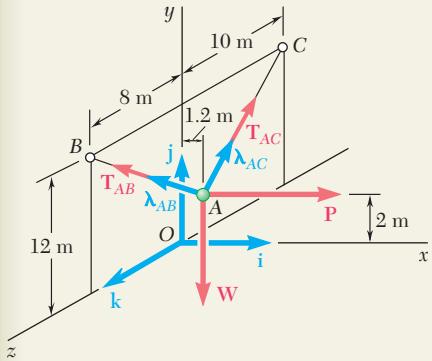
**Photo 2.2** While the tension in the four cables supporting the car cannot be found using the three equations of (2.34), a relation between the tensions can be obtained by considering the equilibrium of the hook.



## SAMPLE PROBLEM 2.9

A 200-kg cylinder is hung by means of two cables  $AB$  and  $AC$ , which are attached to the top of a vertical wall. A horizontal force  $\mathbf{P}$  perpendicular to the wall holds the cylinder in the position shown. Determine the magnitude of  $\mathbf{P}$  and the tension in each cable.

## SOLUTION



**Free-body Diagram.** Point  $A$  is chosen as a free body; this point is subjected to four forces, three of which are of unknown magnitude.

Introducing the unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , we resolve each force into rectangular components.

$$\mathbf{P} = P\mathbf{i} \quad (1)$$

$$\mathbf{W} = -mg\mathbf{j} = -(200 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(1962 \text{ N})\mathbf{j}$$

In the case of  $\mathbf{T}_{AB}$  and  $\mathbf{T}_{AC}$ , it is necessary first to determine the components and magnitudes of the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$ . Denoting by  $\lambda_{AB}$  the unit vector along  $AB$ , we write

$$\overrightarrow{AB} = -(1.2 \text{ m})\mathbf{i} + (10 \text{ m})\mathbf{j} + (8 \text{ m})\mathbf{k} \quad AB = 12.862 \text{ m}$$

$$\lambda_{AB} = \frac{\overrightarrow{AB}}{AB} = -0.09330\mathbf{i} + 0.7775\mathbf{j} + 0.6220\mathbf{k}$$

$$\mathbf{T}_{AB} = T_{AB}\lambda_{AB} = -0.09330T_{AB}\mathbf{i} + 0.7775T_{AB}\mathbf{j} + 0.6220T_{AB}\mathbf{k} \quad (2)$$

Denoting by  $\lambda_{AC}$  the unit vector along  $AC$ , we write in a similar way

$$\overrightarrow{AC} = -(1.2 \text{ m})\mathbf{i} + (10 \text{ m})\mathbf{j} - (10 \text{ m})\mathbf{k} \quad AC = 14.193 \text{ m}$$

$$\lambda_{AC} = \frac{\overrightarrow{AC}}{AC} = -0.08455\mathbf{i} + 0.7046\mathbf{j} - 0.7046\mathbf{k}$$

$$\mathbf{T}_{AC} = T_{AC}\lambda_{AC} = -0.08455T_{AC}\mathbf{i} + 0.7046T_{AC}\mathbf{j} - 0.7046T_{AC}\mathbf{k} \quad (3)$$

**Equilibrium Condition.** Since  $A$  is in equilibrium, we must have

$$\Sigma \mathbf{F} = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{P} + \mathbf{W} = 0$$

or, substituting from (1), (2), (3) for the forces and factoring  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ,

$$(-0.09330T_{AB} - 0.08455T_{AC} + P)\mathbf{i} + (0.7775T_{AB} + 0.7046T_{AC} - 1962 \text{ N})\mathbf{j} + (0.6220T_{AB} - 0.7046T_{AC})\mathbf{k} = 0$$

Setting the coefficients of  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  equal to zero, we write three scalar equations, which express that the sums of the  $x$ ,  $y$ , and  $z$  components of the forces are respectively equal to zero.

$$(\Sigma F_x = 0:) \quad -0.09330T_{AB} - 0.08455T_{AC} + P = 0$$

$$(\Sigma F_y = 0:) \quad +0.7775T_{AB} + 0.7046T_{AC} - 1962 \text{ N} = 0$$

$$(\Sigma F_z = 0:) \quad +0.6220T_{AB} - 0.7046T_{AC} = 0$$

Solving these equations, we obtain

$$P = 235 \text{ N} \quad T_{AB} = 1402 \text{ N} \quad T_{AC} = 1238 \text{ N} \quad \blacktriangleleft$$

# SOLVING PROBLEMS ON YOUR OWN

We saw earlier that when a particle is in *equilibrium*, the resultant of the forces acting on the particle must be zero. Expressing this fact in the case of the equilibrium of a *particle in three-dimensional space* will provide you with three relations among the forces acting on the particle. These relations may be used to determine three unknowns—usually the magnitudes of three forces.

Your solution will consist of the following steps:

**1. Draw a free-body diagram of the particle.** This diagram shows the particle and all the forces acting on it. Indicate on the diagram the magnitudes of known forces, as well as any angles or dimensions that define the direction of a force. Any unknown magnitude or angle should be denoted by an appropriate symbol. Nothing else should be included in your free-body diagram.

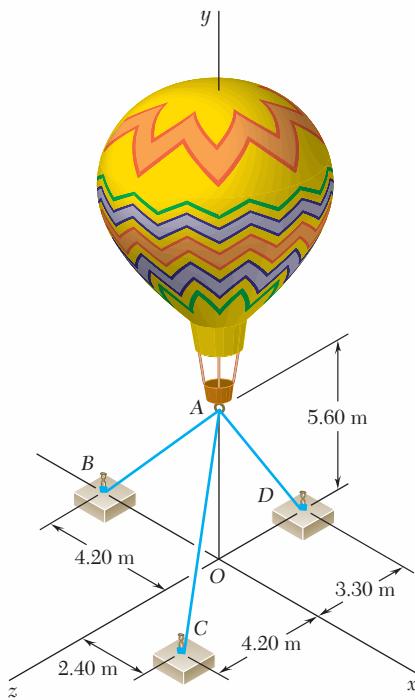
**2. Resolve each of the forces into rectangular components.** Following the method used in the preceding lesson, you will determine for each force  $\mathbf{F}$  the unit vector  $\boldsymbol{\lambda}$  defining the direction of that force and express  $\mathbf{F}$  as the product of its magnitude  $F$  and the unit vector  $\boldsymbol{\lambda}$ . You will obtain an expression of the form

$$\mathbf{F} = F\boldsymbol{\lambda} = \frac{F}{d} (d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k})$$

where  $d$ ,  $d_x$ ,  $d_y$ , and  $d_z$  are dimensions obtained from the free-body diagram of the particle. If a force is known in magnitude as well as in direction, then  $F$  is known and the expression obtained for  $\mathbf{F}$  is well defined; otherwise  $F$  is one of the three unknowns that should be determined.

**3. Set the resultant, or sum, of the forces exerted on the particle equal to zero.** You will obtain a vectorial equation consisting of terms containing the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , or  $\mathbf{k}$ . You will group the terms containing the same unit vector and factor that vector. For the vectorial equation to be satisfied, the coefficient of each of the unit vectors must be set equal to zero. This will yield three scalar equations that you can solve for no more than three unknowns [Sample Prob. 2.9].

# PROBLEMS



**Fig. P2.99, P2.100, P2.101, and P2.102**

**2.99** Three cables are used to tether a balloon as shown. Determine the vertical force  $\mathbf{P}$  exerted by the balloon at A knowing that the tension in cable AB is 259 N.

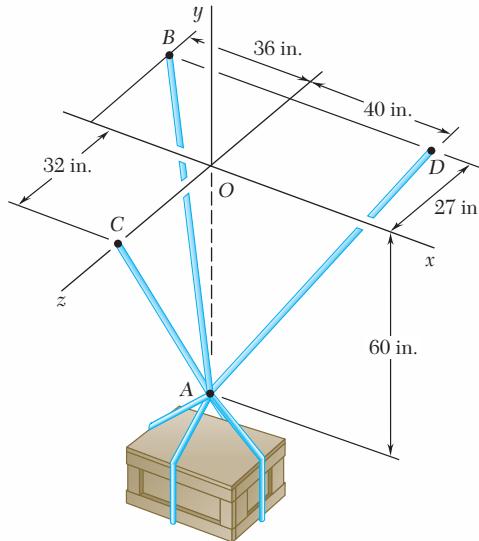
**2.100** Three cables are used to tether a balloon as shown. Determine the vertical force  $\mathbf{P}$  exerted by the balloon at A knowing that the tension in cable AC is 444 N.

**2.101** Three cables are used to tether a balloon as shown. Determine the vertical force  $\mathbf{P}$  exerted by the balloon at A knowing that the tension in cable AD is 481 N.

**2.102** Three cables are used to tether a balloon as shown. Knowing that the balloon exerts an 800-N vertical force at A, determine the tension in each cable.

**2.103** A crate is supported by three cables as shown. Determine the weight of the crate knowing that the tension in cable AB is 750 lb.

**2.104** A crate is supported by three cables as shown. Determine the weight of the crate knowing that the tension in cable AD is 616 lb.



**Fig. P2.103, P2.104, P2.105, and P2.106**

**2.105** A crate is supported by three cables as shown. Determine the weight of the crate knowing that the tension in cable AC is 544 lb.

**2.106** A 1600-lb crate is supported by three cables as shown. Determine the tension in each cable.

- 2.107** Three cables are connected at A, where the forces **P** and **Q** are applied as shown. Knowing that  $Q = 0$ , find the value of  $P$  for which the tension in cable  $AD$  is 305 N.

- 2.108** Three cables are connected at A, where the forces **P** and **Q** are applied as shown. Knowing that  $P = 1200$  N, determine the values of  $Q$  for which cable  $AD$  is taut.

- 2.109** A transmission tower is held by three guy wires attached to a pin at A and anchored by bolts at B, C, and D. If the tension in wire  $AB$  is 630 lb, determine the vertical force **P** exerted by the tower on the pin at A.

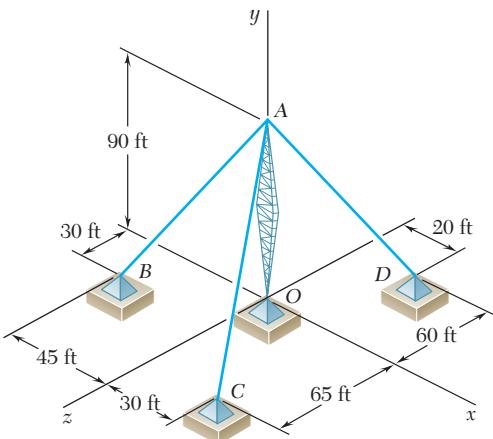


Fig. P2.109, and P2.110

- 2.110** A transmission tower is held by three guy wires attached to a pin at A and anchored by bolts at B, C, and D. If the tension in wire  $AC$  is 920 lb, determine the vertical force **P** exerted by the tower on the pin at A.

- 2.111** A rectangular plate is supported by three cables as shown. Knowing that the tension in cable  $AC$  is 60 N, determine the weight of the plate.

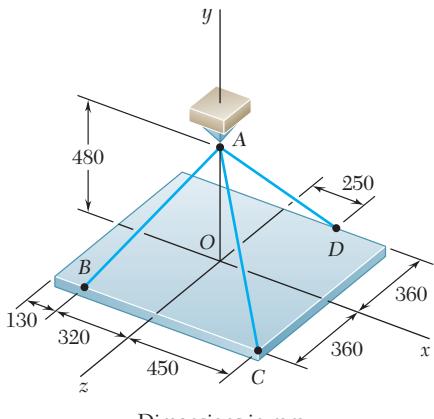


Fig. P2.111 and P2.112

- 2.112** A rectangular plate is supported by three cables as shown. Knowing that the tension in cable  $AD$  is 520 N, determine the weight of the plate.

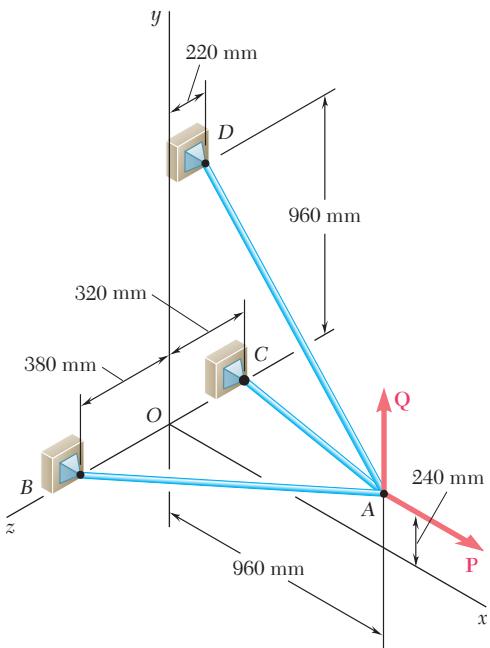


Fig. P2.107 and P2.108

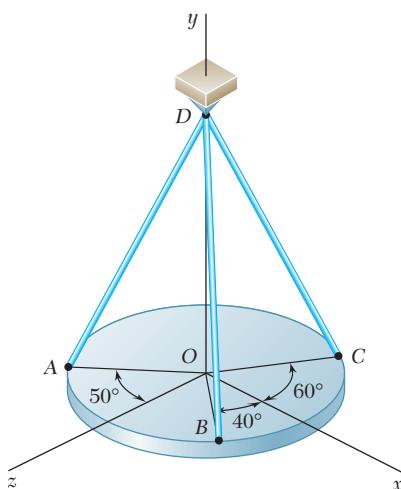


Fig. P2.114

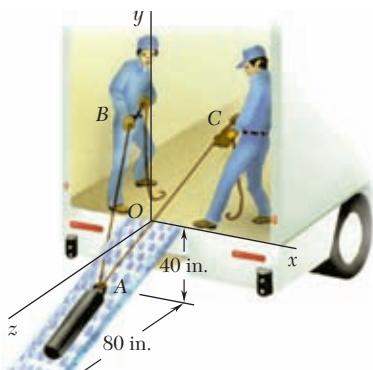


Fig. P2.119

- 2.113** For the transmission tower of Probs. 2.109 and 2.110, determine the tension in each guy wire knowing that the tower exerts on the pin at *A* an upward vertical force of 2100 lb.

- 2.114** A horizontal circular plate weighing 60 lb is suspended as shown from three wires that are attached to a support at *D* and form  $30^\circ$  angles with the vertical. Determine the tension in each wire.

- 2.115** For the rectangular plate of Probs. 2.111 and 2.112, determine the tension in each of the three cables knowing that the weight of the plate is 792 N.

- 2.116** For the cable system of Probs. 2.107 and 2.108, determine the tension in each cable knowing that  $P = 2880$  N and  $Q = 0$ .

- 2.117** For the cable system of Probs. 2.107 and 2.108, determine the tension in each cable knowing that  $P = 2880$  N and  $Q = 576$  N.

- 2.118** For the cable system of Probs. 2.107 and 2.108, determine the tension in each cable knowing that  $P = 2880$  N and  $Q = -576$  N ( $\mathbf{Q}$  is directed downward).

- 2.119** Using two ropes and a roller chute, two workers are unloading a 200-lb cast-iron counterweight from a truck. Knowing that at the instant shown the counterweight is kept from moving and that the positions of points *A*, *B*, and *C* are, respectively,  $A(0, -20 \text{ in.}, 40 \text{ in.})$ ,  $B(-40 \text{ in.}, 50 \text{ in.}, 0)$ , and  $C(45 \text{ in.}, 40 \text{ in.}, 0)$ , and assuming that no friction exists between the counterweight and the chute, determine the tension in each rope. (Hint: Since there is no friction, the force exerted by the chute on the counterweight must be perpendicular to the chute.)

- 2.120** Solve Prob. 2.119 assuming that a third worker is exerting a force  $\mathbf{P} = -(40 \text{ lb})\mathbf{i}$  on the counterweight.

- 2.121** A container of weight  $W$  is suspended from ring *A*. Cable *BAC* passes through the ring and is attached to fixed supports at *B* and *C*. Two forces  $\mathbf{P} = P\mathbf{i}$  and  $\mathbf{Q} = Q\mathbf{k}$  are applied to the ring to maintain the container in the position shown. Knowing that  $W = 376$  N, determine  $P$  and  $Q$ . (Hint: The tension is the same in both portions of cable *BAC*.)

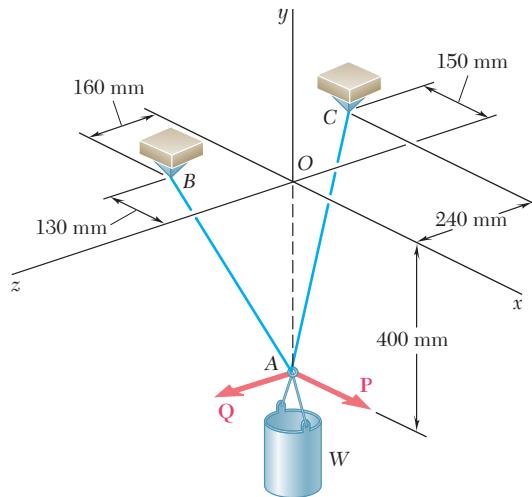


Fig. P2.121

**2.122** For the system of Prob. 2.121, determine  $W$  and  $Q$  knowing that  $P = 164$  N.

**2.123** A container of weight  $W$  is suspended from ring  $A$ , to which cables  $AC$  and  $AE$  are attached. A force  $\mathbf{P}$  is applied to the end  $F$  of a third cable that passes over a pulley at  $B$  and through ring  $A$  and that is attached to a support at  $D$ . Knowing that  $W = 1000$  N, determine the magnitude of  $\mathbf{P}$ . (*Hint:* The tension is the same in all portions of cable  $FBAD$ .)

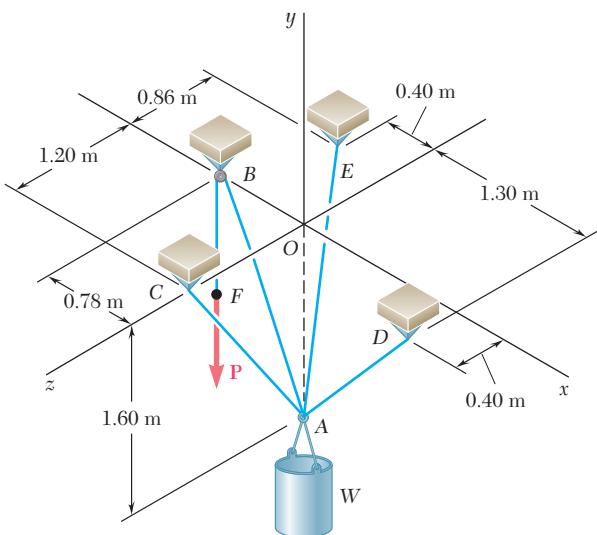


Fig. P2.123

**2.124** Knowing that the tension in cable  $AC$  of the system described in Prob. 2.123 is 150 N, determine (a) the magnitude of the force  $\mathbf{P}$ , (b) the weight  $W$  of the container.

**2.125** Collars  $A$  and  $B$  are connected by a 25-in.-long wire and can slide freely on frictionless rods. If a 60-lb force  $\mathbf{Q}$  is applied to collar  $B$  as shown, determine (a) the tension in the wire when  $x = 9$  in., (b) the corresponding magnitude of the force  $\mathbf{P}$  required to maintain the equilibrium of the system.

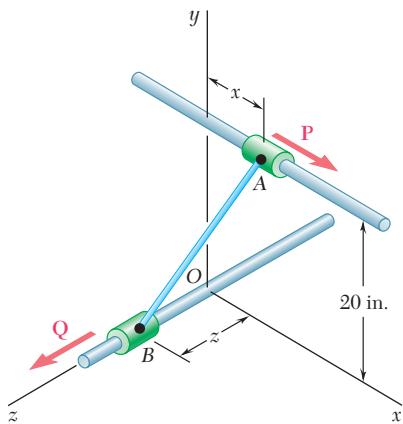


Fig. P2.125 and P2.126

**2.126** Collars  $A$  and  $B$  are connected by a 25-in.-long wire and can slide freely on frictionless rods. Determine the distances  $x$  and  $z$  for which the equilibrium of the system is maintained when  $P = 120$  lb and  $Q = 60$  lb.

# REVIEW AND SUMMARY

In this chapter we have studied the effect of forces on particles, i.e., on bodies of such shape and size that all forces acting on them may be assumed applied at the same point.

## Resultant of two forces

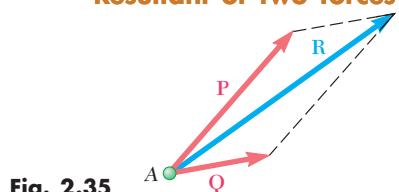


Fig. 2.35

## Components of a force

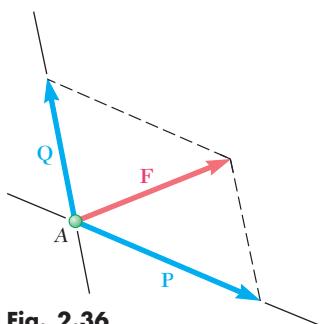


Fig. 2.36

## Rectangular components Unit vectors

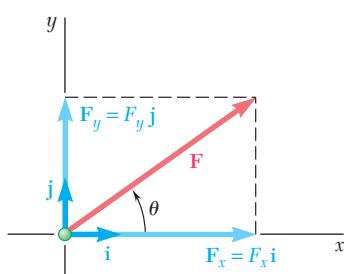


Fig. 2.37

Forces are *vector quantities*; they are characterized by a *point of application*, a *magnitude*, and a *direction*, and they add according to the *parallelogram law* (Fig. 2.35). The magnitude and direction of the resultant **R** of two forces **P** and **Q** can be determined either graphically or by trigonometry, using successively the law of cosines and the law of sines [Sample Prob. 2.1].

Any given force acting on a particle can be resolved into two or more *components*, i.e., it can be replaced by two or more forces which have the same effect on the particle. A force **F** can be resolved into two components **P** and **Q** by drawing a parallelogram which has **F** for its diagonal; the components **P** and **Q** are then represented by the two adjacent sides of the parallelogram (Fig. 2.36) and can be determined either graphically or by trigonometry [Sec. 2.6].

A force **F** is said to have been resolved into two *rectangular components* if its components **F<sub>x</sub>** and **F<sub>y</sub>** are perpendicular to each other and are directed along the coordinate axes (Fig. 2.37). Introducing the *unit vectors* **i** and **j** along the *x* and *y* axes, respectively, we write [Sec. 2.7]

$$\mathbf{F}_x = F_x \mathbf{i} \quad \mathbf{F}_y = F_y \mathbf{j} \quad (2.6)$$

and

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} \quad (2.7)$$

where  $F_x$  and  $F_y$  are the *scalar components* of **F**. These components, which can be positive or negative, are defined by the relations

$$F_x = F \cos \theta \quad F_y = F \sin \theta \quad (2.8)$$

When the rectangular components  $F_x$  and  $F_y$  of a force **F** are given, the angle  $\theta$  defining the direction of the force can be obtained by writing

$$\tan \theta = \frac{F_y}{F_x} \quad (2.9)$$

The magnitude  $F$  of the force can then be obtained by solving one of the equations (2.8) for  $F$  or by applying the Pythagorean theorem and writing

$$F = \sqrt{F_x^2 + F_y^2} \quad (2.10)$$

When *three or more coplanar forces* act on a particle, the rectangular components of their resultant  $\mathbf{R}$  can be obtained by adding algebraically the corresponding components of the given forces [Sec. 2.8]. We have

$$R_x = \sum F_x \quad R_y = \sum F_y \quad (2.13)$$

The magnitude and direction of  $\mathbf{R}$  can then be determined from relations similar to Eqs. (2.9) and (2.10) [Sample Prob. 2.3].

A force  $\mathbf{F}$  in *three-dimensional space* can be resolved into rectangular components  $\mathbf{F}_x$ ,  $\mathbf{F}_y$ , and  $\mathbf{F}_z$  [Sec. 2.12]. Denoting by  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$ , respectively, the angles that  $\mathbf{F}$  forms with the  $x$ ,  $y$ , and  $z$  axes (Fig. 2.38), we have

$$F_x = F \cos \theta_x \quad F_y = F \cos \theta_y \quad F_z = F \cos \theta_z \quad (2.19)$$

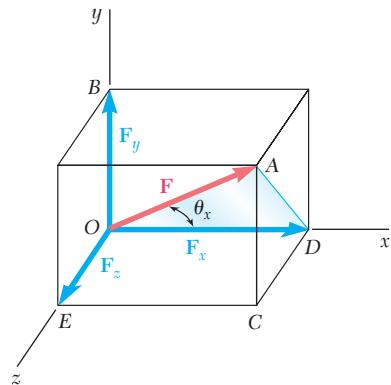
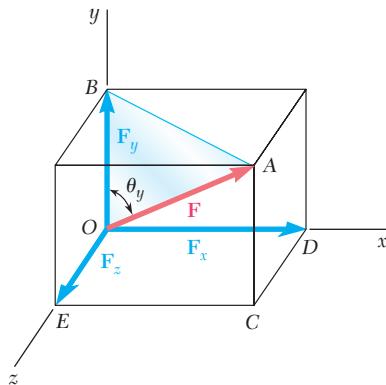
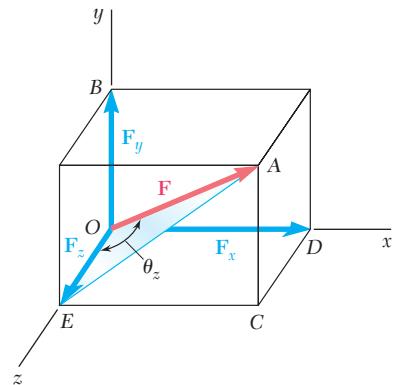


Fig. 2.38



(b)



(c)

The cosines of  $\theta_x$ ,  $\theta_y$ ,  $\theta_z$  are known as the *direction cosines* of the force  $\mathbf{F}$ . Introducing the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  along the coordinate axes, we write

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \quad (2.20)$$

or

$$\mathbf{F} = F(\cos \theta_x \mathbf{i} + \cos \theta_y \mathbf{j} + \cos \theta_z \mathbf{k}) \quad (2.21)$$

which shows (Fig. 2.39) that  $\mathbf{F}$  is the product of its magnitude  $F$  and the unit vector

$$\boldsymbol{\lambda} = \cos \theta_x \mathbf{i} + \cos \theta_y \mathbf{j} + \cos \theta_z \mathbf{k}$$

Since the magnitude of  $\boldsymbol{\lambda}$  is equal to unity, we must have

$$\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1 \quad (2.24)$$

When the rectangular components  $F_x$ ,  $F_y$ ,  $F_z$  of a force  $\mathbf{F}$  are given, the magnitude  $F$  of the force is found by writing

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \quad (2.18)$$

and the direction cosines of  $\mathbf{F}$  are obtained from Eqs. (2.19). We have

$$\cos \theta_x = \frac{F_x}{F} \quad \cos \theta_y = \frac{F_y}{F} \quad \cos \theta_z = \frac{F_z}{F} \quad (2.25)$$

## Resultant of several coplanar forces

### Forces in space

### Direction cosines

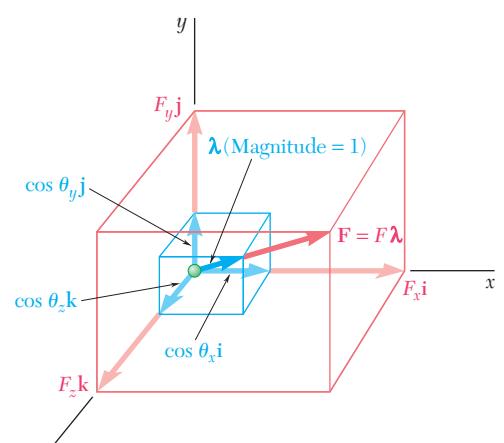


Fig. 2.39

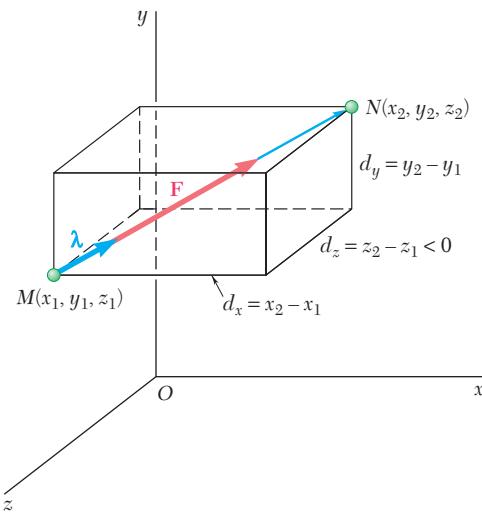


Fig. 2.40

### Resultant of forces in space

When a force  $\mathbf{F}$  is defined in three-dimensional space by its magnitude  $F$  and two points  $M$  and  $N$  on its line of action [Sec. 2.13], its rectangular components can be obtained as follows. We first express the vector  $\overrightarrow{MN}$  joining points  $M$  and  $N$  in terms of its components  $d_x$ ,  $d_y$ , and  $d_z$  (Fig. 2.40); we write

$$\overrightarrow{MN} = d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k} \quad (2.26)$$

We next determine the unit vector  $\lambda$  along the line of action of  $\mathbf{F}$  by dividing  $\overrightarrow{MN}$  by its magnitude  $MN = d$ :

$$\lambda = \frac{\overrightarrow{MN}}{MN} = \frac{1}{d}(d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k}) \quad (2.27)$$

Recalling that  $\mathbf{F}$  is equal to the product of  $F$  and  $\lambda$ , we have

$$\mathbf{F} = F\lambda = \frac{F}{d}(d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k}) \quad (2.28)$$

from which it follows [Sample Probs. 2.7 and 2.8] that the scalar components of  $\mathbf{F}$  are, respectively,

$$F_x = \frac{Fd_x}{d} \quad F_y = \frac{Fd_y}{d} \quad F_z = \frac{Fd_z}{d} \quad (2.29)$$

When *two or more forces* act on a particle in *three-dimensional space*, the rectangular components of their resultant  $\mathbf{R}$  can be obtained by adding algebraically the corresponding components of the given forces [Sec. 2.14]. We have

$$R_x = \sum F_x \quad R_y = \sum F_y \quad R_z = \sum F_z \quad (2.31)$$

The magnitude and direction of  $\mathbf{R}$  can then be determined from relations similar to Eqs. (2.18) and (2.25) [Sample Prob. 2.8].

A particle is said to be in *equilibrium* when the resultant of all the forces acting on it is zero [Sec. 2.9]. The particle will then remain at rest (if originally at rest) or move with constant speed in a straight line (if originally in motion) [Sec. 2.10].

### Free-body diagram

To solve a problem involving a particle in equilibrium, one first should draw a *free-body diagram* of the particle showing all the forces acting on it [Sec. 2.11]. If *only three coplanar forces* act on the particle, a *force triangle* may be drawn to express that the particle is in equilibrium. Using graphical methods of trigonometry, this triangle can be solved for no more than two unknowns [Sample Prob. 2.4]. If *more than three coplanar forces* are involved, the equations of equilibrium

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad (2.15)$$

should be used. These equations can be solved for no more than two unknowns [Sample Prob. 2.6].

### Equilibrium in space

When a particle is in *equilibrium in three-dimensional space* [Sec. 2.15], the three equations of equilibrium

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0 \quad (2.34)$$

should be used. These equations can be solved for no more than three unknowns [Sample Prob. 2.9].

# REVIEW PROBLEMS

- 2.127** The direction of the 75-lb forces may vary, but the angle between the forces is always  $50^\circ$ . Determine the value of  $\alpha$  for which the resultant of the forces acting at A is directed horizontally to the left.

- 2.128** A stake is being pulled out of the ground by means of two ropes as shown. Knowing the magnitude and direction of the force exerted on one rope, determine the magnitude and direction of the force  $P$  that should be exerted on the other rope if the resultant of these two forces is to be a 40-lb vertical force.

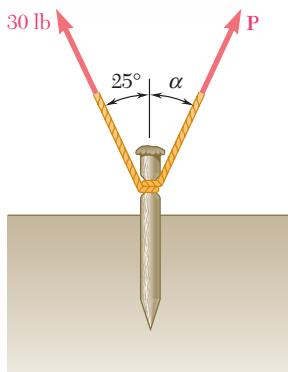


Fig. P2.128

- 2.129** Member BD exerts on member ABC a force  $P$  directed along line BD. Knowing that  $P$  must have a 240-lb vertical component, determine (a) the magnitude of the force  $P$ , (b) its horizontal component.

- 2.130** Two cables are tied together at C and loaded as shown. Determine the tension (a) in cable AC, (b) in cable BC.

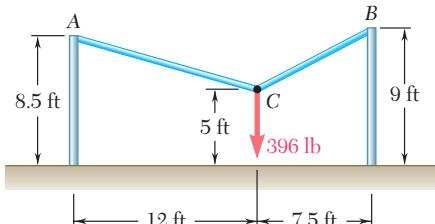


Fig. P2.130

- 2.131** Two cables are tied together at C and loaded as shown. Knowing that  $P = 360 \text{ N}$ , determine the tension (a) in cable AC, (b) in cable BC.

- 2.132** Two cables are tied together at C and loaded as shown. Determine the range of values of  $P$  for which both cables remain taut.

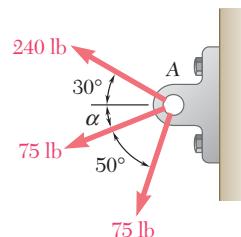


Fig. P2.127

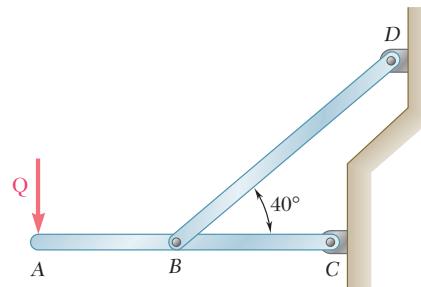


Fig. P2.129

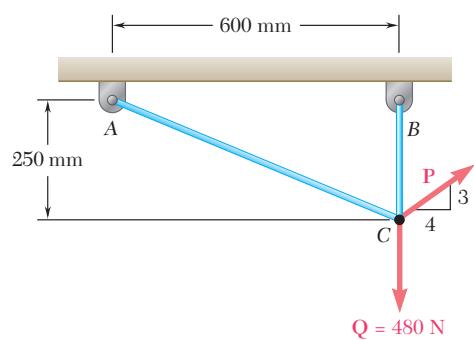
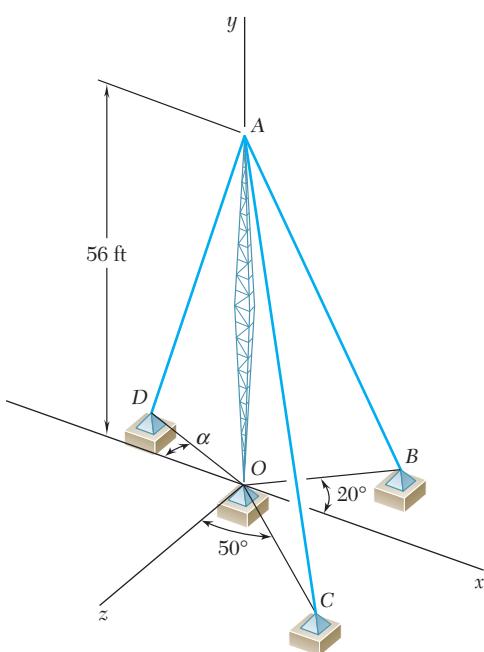


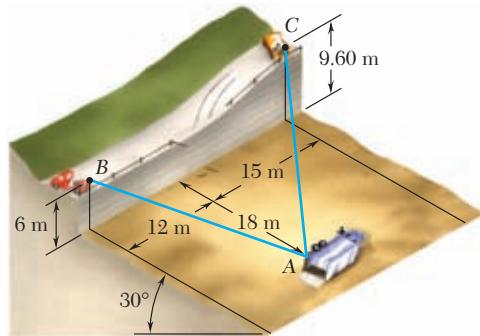
Fig. P2.131 and P2.132

**Fig. P2.134**

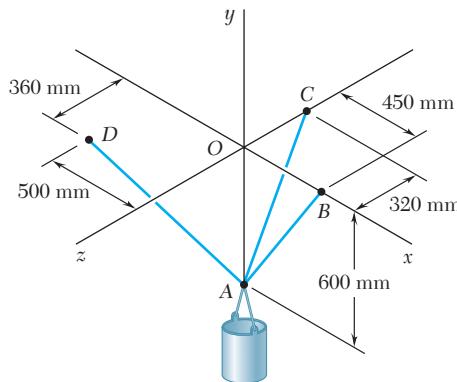
- 2.133** A force acts at the origin of a coordinate system in a direction defined by the angles  $\theta_x = 69.3^\circ$  and  $\theta_z = 57.9^\circ$ . Knowing that the  $y$  component of the force is  $-174$  lb, determine (a) the angle  $\theta_y$ , (b) the other components and the magnitude of the force.

- 2.134** Cable  $AB$  is 65 ft long, and the tension in that cable is 3900 lb. Determine (a) the  $x$ ,  $y$ , and  $z$  components of the force exerted by the cable on the anchor  $B$ , (b) the angles  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  defining the direction of that force.

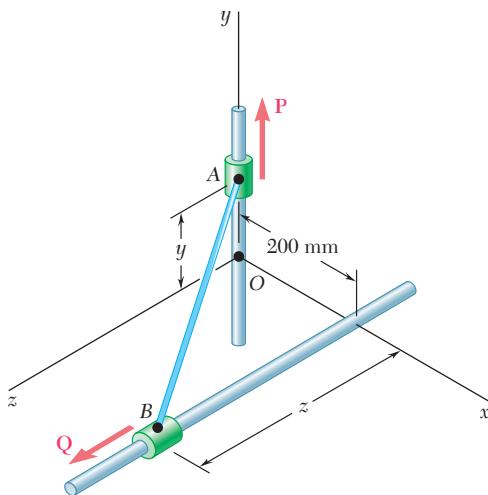
- 2.135** In order to move a wrecked truck, two cables are attached at  $A$  and pulled by winches  $B$  and  $C$  as shown. Knowing that the tension is 10 kN in cable  $AB$  and 7.5 kN in cable  $AC$ , determine the magnitude and direction of the resultant of the forces exerted at  $A$  by the two cables.

**Fig. P2.135**

- 2.136** A container of weight  $W = 1165$  N is supported by three cables as shown. Determine the tension in each cable.

**Fig. P2.136**

- 2.137** Collars A and B are connected by a 525-mm-long wire and can slide freely on frictionless rods. If a force  $\mathbf{P} = (341 \text{ N})\mathbf{j}$  is applied to collar A, determine (a) the tension in the wire when  $y = 155 \text{ mm}$ , (b) the magnitude of the force  $\mathbf{Q}$  required to maintain the equilibrium of the system.

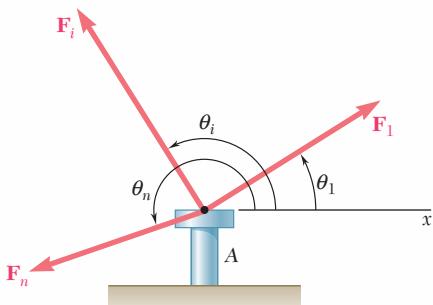


**Fig. P2.137**

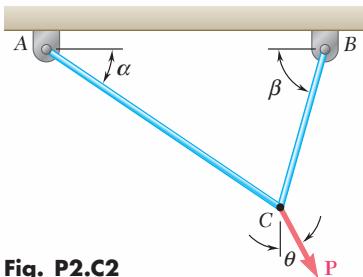
- 2.138** Solve Prob. 2.137 assuming that  $y = 275 \text{ mm}$ .

# COMPUTER PROBLEMS

**2.C1** Using computational software, determine the magnitude and direction of the resultant of  $n$  coplanar forces applied at a point A. Use this program to solve Probs. 2.32, 2.33, 2.35, and 2.38.



**Fig. P2.C1**

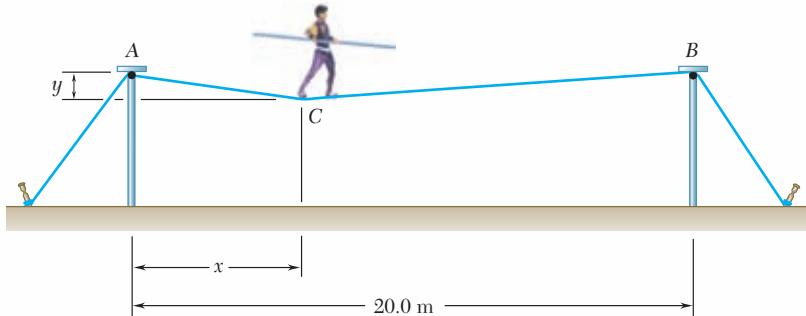


**Fig. P2.C2**

**2.C2** A load  $\mathbf{P}$  is supported by two cables as shown. Using computational software, determine the tension in each cable for any given value of  $P$  and for values of  $\theta$  ranging from  $\theta_1 = \beta - 90^\circ$  to  $\theta_2 = 90^\circ - \alpha$ , using given increments  $\Delta\theta$ . Use this program to determine for the following three sets of numerical values (a) the tension in each cable for values of  $\theta$  ranging from  $\theta_1$  to  $\theta_2$ , (b) the value of  $\theta$  for which the tension in the two cables is as small as possible, (c) the corresponding value of the tension:

- (1)  $\alpha = 35^\circ$ ,  $\beta = 75^\circ$ ,  $P = 400$  lb,  $\Delta\theta = 5^\circ$
- (2)  $\alpha = 50^\circ$ ,  $\beta = 30^\circ$ ,  $P = 600$  lb,  $\Delta\theta = 10^\circ$
- (3)  $\alpha = 40^\circ$ ,  $\beta = 60^\circ$ ,  $P = 250$  lb,  $\Delta\theta = 5^\circ$

**2.C3** An acrobat is walking on a tightrope of length  $L = 20.1$  m attached to supports A and B at a distance of 20.0 m from each other. The combined weight of the acrobat and his balancing pole is 800 N, and the friction between his shoes and the rope is large enough to prevent him from slipping. Neglecting the weight of the rope and any elastic deformation, use computational software to calculate the deflection  $y$  and the tension in portions AC and BC of the rope for values of  $x$  from 0.5 m to 10.0 m using 0.5-m increments. From the data obtained, determine (a) the maximum deflection of the rope, (b) the maximum tension in the rope, (c) the smallest values of the tension in portions AC and BC of the rope.



**Fig. P2.C3**

**2.C4** Write a computer program that can be used to determine the magnitude and direction of the resultant of  $n$  forces  $\mathbf{F}_i$ , where  $i = 1, 2, \dots, n$ , that are applied at point  $A_0$  of coordinates  $x_0, y_0$ , and  $z_0$ , knowing that the line of action of  $\mathbf{F}_i$  passes through point  $A_i$  of coordinates  $x_i, y_i$ , and  $z_i$ . Use this program to solve Probs. 2.93, 2.94, 2.95, and 2.135.

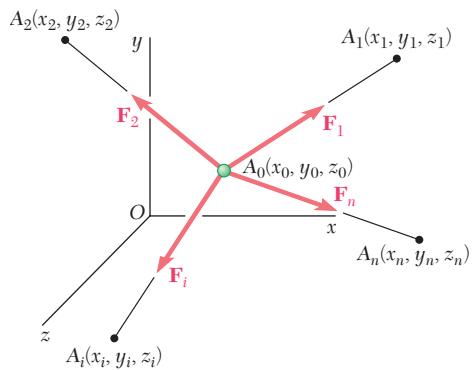


Fig. P2.C4

**2.C5** Three cables are attached at points  $A_1, A_2$ , and  $A_3$ , respectively, and are connected at point  $A_0$ , to which a given load  $\mathbf{P}$  is applied as shown. Write a computer program that can be used to determine the tension in each of the cables. Use this program to solve Probs. 2.102, 2.106, 2.107, and 2.115.

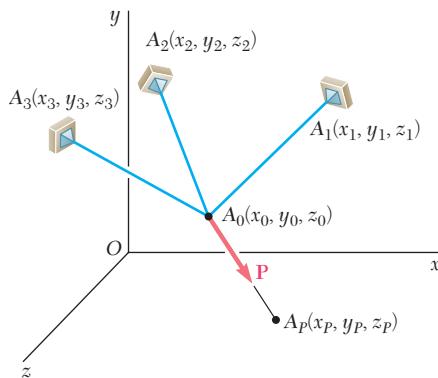
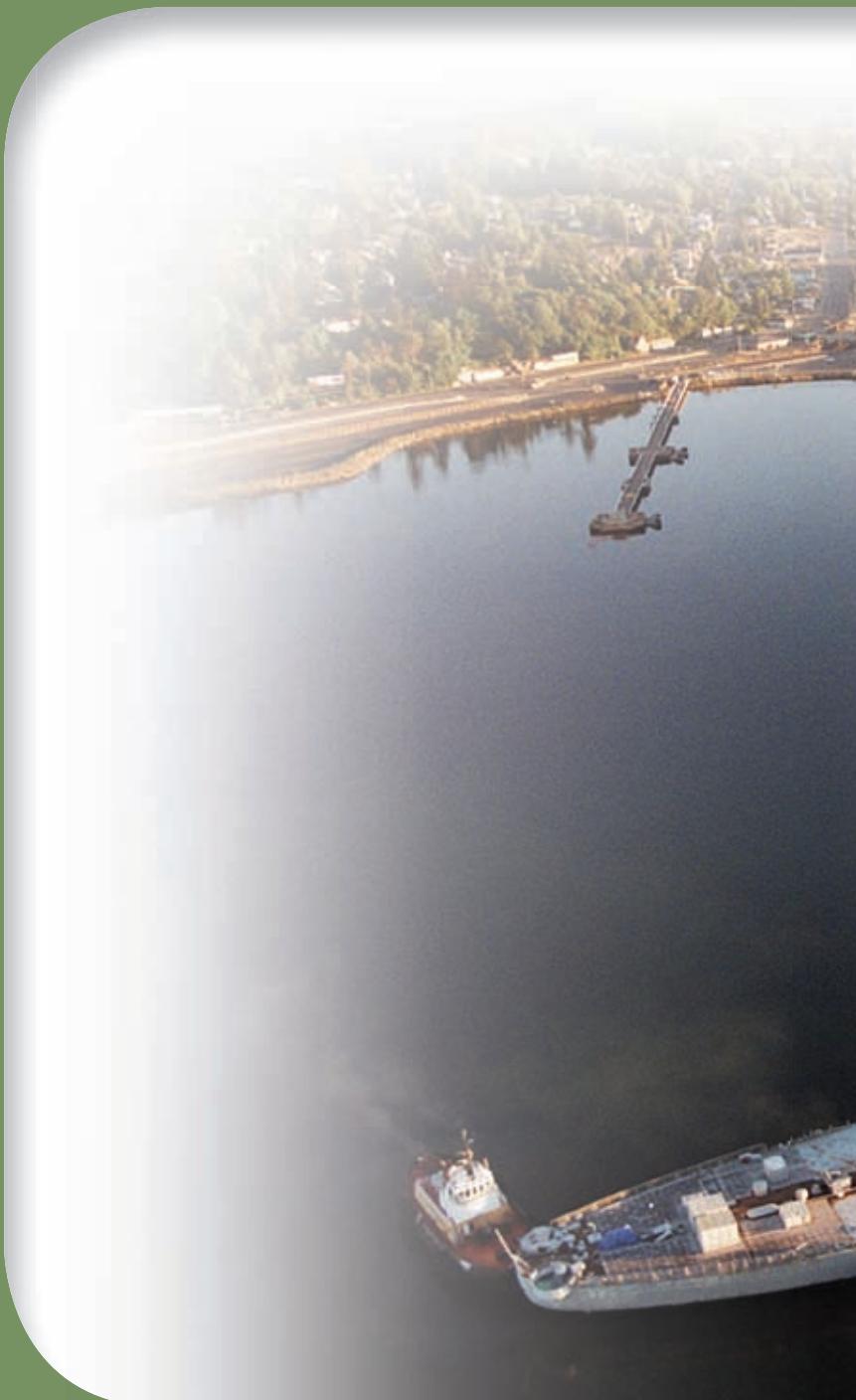


Fig. P2.C5

The battleship USS *New Jersey* is maneuvered by four tugboats at Bremerton Naval Shipyard. It will be shown in this chapter that the forces exerted on the ship by the tugboats could be replaced by an equivalent force exerted by a single, more powerful, tugboat.



# Rigid Bodies: Equivalent Systems of Forces



## Chapter 3 Rigid Bodies: Equivalent Systems of Forces

- 3.1** Introduction
- 3.2** External and Internal Forces
- 3.3** Principle of Transmissibility.  
Equivalent Forces
- 3.4** Vector Product of Two Vectors
- 3.5** Vector Products Expressed in  
Terms of Rectangular  
Components
- 3.6** Moment of a Force about a Point
- 3.7** Varignon's Theorem
- 3.8** Rectangular Components of the  
Moment of a Force
- 3.9** Scalar Product of Two Vectors
- 3.10** Mixed Triple Product of Three  
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- 3.11** Moment of a Force about a  
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- 3.12** Moment of a Couple
- 3.13** Equivalent Couples
- 3.14** Addition of Couples
- 3.15** Couples Can Be Represented  
by Vectors
- 3.16** Resolution of a Given Force into  
a Force at  $O$  and a Couple
- 3.17** Reduction of a System of Forces  
to One Force and One Couple
- 3.18** Equivalent Systems of Forces
- 3.19** Equipollent Systems of Vectors
- 3.20** Further Reduction of a System  
of Forces
- 3.21** Reduction of a System of Forces  
to a Wrench

### 3.1 INTRODUCTION

In the preceding chapter it was assumed that each of the bodies considered could be treated as a single particle. Such a view, however, is not always possible, and a body, in general, should be treated as a combination of a large number of particles. The size of the body will have to be taken into consideration, as well as the fact that forces will act on different particles and thus will have different points of application.

Most of the bodies considered in elementary mechanics are assumed to be *rigid*, a *rigid body* being defined as one which does not deform. Actual structures and machines, however, are never absolutely rigid and deform under the loads to which they are subjected. But these deformations are usually small and do not appreciably affect the conditions of equilibrium or motion of the structure under consideration. They are important, though, as far as the resistance of the structure to failure is concerned and are considered in the study of mechanics of materials.

In this chapter you will study the effect of forces exerted on a rigid body, and you will learn how to replace a given system of forces by a simpler equivalent system. This analysis will rest on the fundamental assumption that the effect of a given force on a rigid body remains unchanged if that force is moved along its line of action (*principle of transmissibility*). It follows that forces acting on a rigid body can be represented by *sliding vectors*, as indicated earlier in Sec. 2.3.

Two important concepts associated with the effect of a force on a rigid body are the *moment of a force about a point* (Sec. 3.6) and the *moment of a force about an axis* (Sec. 3.11). Since the determination of these quantities involves the computation of vector products and scalar products of two vectors, the fundamentals of vector algebra will be introduced in this chapter and applied to the solution of problems involving forces acting on rigid bodies.

Another concept introduced in this chapter is that of a *couple*, i.e., the combination of two forces which have the same magnitude, parallel lines of action, and opposite sense (Sec. 3.12). As you will see, any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple. This basic system is called a *force-couple system*. In the case of concurrent, coplanar, or parallel forces, the equivalent force-couple system can be further reduced to a single force, called the *resultant* of the system, or to a single couple, called the *resultant couple* of the system.

### 3.2 EXTERNAL AND INTERNAL FORCES

Forces acting on rigid bodies can be separated into two groups: (1) *external forces* and (2) *internal forces*.

1. The *external forces* represent the action of other bodies on the rigid body under consideration. They are entirely responsible for the external behavior of the rigid body. They will either cause it to move or ensure that it remains at rest. We shall be concerned only with external forces in this chapter and in Chaps. 4 and 5.

2. The *internal forces* are the forces which hold together the particles forming the rigid body. If the rigid body is structurally composed of several parts, the forces holding the component parts together are also defined as internal forces. Internal forces will be considered in Chaps. 6 and 7.

As an example of external forces, let us consider the forces acting on a disabled truck that three people are pulling forward by means of a rope attached to the front bumper (Fig. 3.1). The external forces acting on the truck are shown in a *free-body diagram* (Fig. 3.2). Let us first consider the *weight* of the truck. Although it embodies the effect of the earth's pull on each of the particles forming the truck, the weight can be represented by the single force  $\mathbf{W}$ . The *point of application* of this force, i.e., the point at which the force acts, is defined as the *center of gravity* of the truck. It will be seen in Chap. 5 how centers of gravity can be determined. The weight  $\mathbf{W}$  tends to make the truck move vertically downward. In fact, it would actually cause the truck to move downward, i.e., to fall, if it were not for the presence of the ground. The ground opposes the downward motion of the truck by means of the reactions  $\mathbf{R}_1$  and  $\mathbf{R}_2$ . These forces are exerted by the ground *on* the truck and must therefore be included among the external forces acting on the truck.

The people pulling on the rope exert the force  $\mathbf{F}$ . The point of application of  $\mathbf{F}$  is on the front bumper. The force  $\mathbf{F}$  tends to make the truck move forward in a straight line and does actually make it move, since no external force opposes this motion. (Rolling resistance has been neglected here for simplicity.) This forward motion of the truck, during which each straight line keeps its original orientation (the floor of the truck remains horizontal, and the walls remain vertical), is known as a *translation*. Other forces might cause the truck to move differently. For example, the force exerted by a jack placed under the front axle would cause the truck to pivot about its rear axle. Such a motion is a *rotation*. It can be concluded, therefore, that each of the *external forces* acting on a *rigid body* can, if unopposed, impart to the rigid body a motion of translation or rotation, or both.

### 3.3 PRINCIPLE OF TRANSMISSIBILITY. EQUIVALENT FORCES

The *principle of transmissibility* states that the conditions of equilibrium or motion of a rigid body will remain unchanged if a force  $\mathbf{F}$  acting at a given point of the rigid body is replaced by a force  $\mathbf{F}'$  of the same magnitude and same direction, but acting at a different point, *provided that the two forces have the same line of action* (Fig. 3.3). The two forces  $\mathbf{F}$  and  $\mathbf{F}'$  have the same effect on the rigid body and are said to be *equivalent*. This principle, which states that the action of a force may be *transmitted* along its line of action, is based on experimental evidence. It *cannot* be derived from the properties established so far in this text and must therefore be accepted as an experimental law. However, as you will see in Sec. 16.5, the principle of transmissibility can be derived from the study of the dynamics of rigid bodies, but this study requires the introduction of Newton's

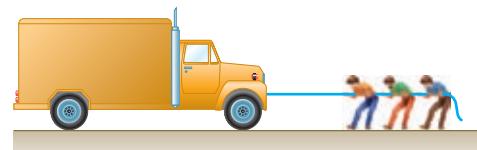


Fig. 3.1

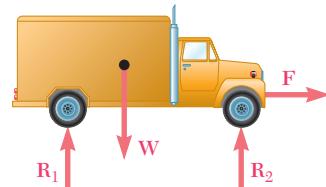


Fig. 3.2

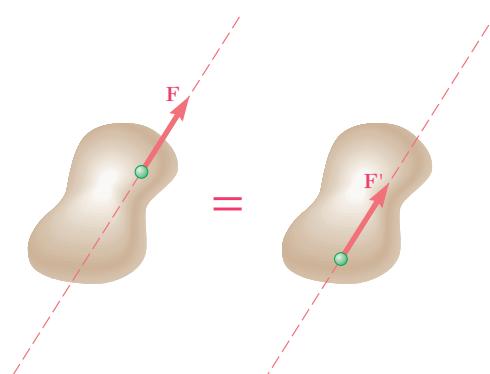
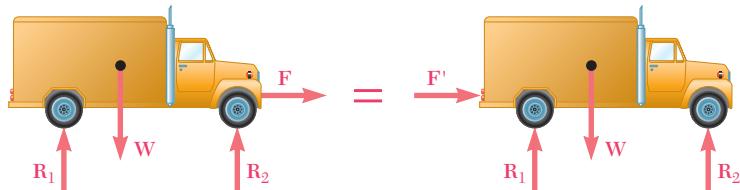


Fig. 3.3

second and third laws and of a number of other concepts as well. Therefore, our study of the statics of rigid bodies will be based on the three principles introduced so far, i.e., the parallelogram law of addition, Newton's first law, and the principle of transmissibility.

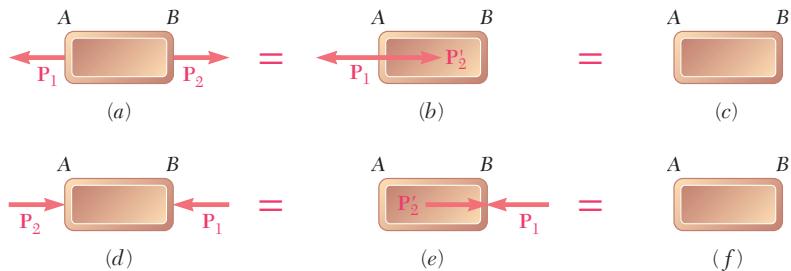
It was indicated in Chap. 2 that the forces acting on a particle could be represented by vectors. These vectors had a well-defined point of application, namely, the particle itself, and were therefore fixed, or bound, vectors. In the case of forces acting on a rigid body, however, the point of application of the force does not matter, as long as the line of action remains unchanged. Thus, forces acting on a rigid body must be represented by a different kind of vector, known as a *sliding vector*, since forces may be allowed to slide along their lines of action. We should note that all the properties which will be derived in the following sections for the forces acting on a rigid body will be valid more generally for any system of sliding vectors. In order to keep our presentation more intuitive, however, we will carry it out in terms of physical forces rather than in terms of mathematical sliding vectors.



**Fig. 3.4**

Returning to the example of the truck, we first observe that the line of action of the force  $\mathbf{F}$  is a horizontal line passing through both the front and the rear bumpers of the truck (Fig. 3.4). Using the principle of transmissibility, we can therefore replace  $\mathbf{F}$  by an *equivalent force*  $\mathbf{F}'$  acting on the rear bumper. In other words, the conditions of motion are unaffected, and all the other external forces acting on the truck ( $\mathbf{W}$ ,  $\mathbf{R}_1$ ,  $\mathbf{R}_2$ ) remain unchanged if the people push on the rear bumper instead of pulling on the front bumper.

The principle of transmissibility and the concept of equivalent forces have limitations, however. Consider, for example, a short bar  $AB$  acted upon by equal and opposite axial forces  $\mathbf{P}_1$  and  $\mathbf{P}_2$ , as shown in Fig. 3.5a. According to the principle of transmissibility, the force  $\mathbf{P}_2$  can be replaced by a force  $\mathbf{P}'_2$  having the same magnitude, the same direction, and the same line of action but acting at  $A$  instead of  $B$  (Fig. 3.5b). The forces  $\mathbf{P}_1$  and  $\mathbf{P}'_2$  acting on the same particle



**Fig. 3.5**

can be added according to the rules of Chap. 2, and, as these forces are equal and opposite, their sum is equal to zero. Thus, in terms of the external behavior of the bar, the original system of forces shown in Fig. 3.5a is equivalent to no force at all (Fig. 3.5c).

Consider now the two equal and opposite forces  $\mathbf{P}_1$  and  $\mathbf{P}_2$  acting on the bar  $AB$  as shown in Fig. 3.5d. The force  $\mathbf{P}_2$  can be replaced by a force  $\mathbf{P}'_2$  having the same magnitude, the same direction, and the same line of action but acting at  $B$  instead of at  $A$  (Fig. 3.5e). The forces  $\mathbf{P}_1$  and  $\mathbf{P}'_2$  can then be added, and their sum is again zero (Fig. 3.5f). From the point of view of the mechanics of rigid bodies, the systems shown in Fig. 3.5a and d are thus equivalent. But the *internal forces* and *deformations* produced by the two systems are clearly different. The bar of Fig. 3.5a is in *tension* and, if not absolutely rigid, will increase in length slightly; the bar of Fig. 3.5d is in *compression* and, if not absolutely rigid, will decrease in length slightly. Thus, while the principle of transmissibility may be used freely to determine the conditions of motion or equilibrium of rigid bodies and to compute the external forces acting on these bodies, it should be avoided, or at least used with care, in determining internal forces and deformations.

### 3.4 VECTOR PRODUCT OF TWO VECTORS

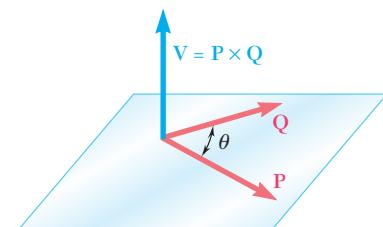
In order to gain a better understanding of the effect of a force on a rigid body, a new concept, the concept of *a moment of a force about a point*, will be introduced at this time. This concept will be more clearly understood, and applied more effectively, if we first add to the mathematical tools at our disposal the *vector product* of two vectors.

The vector product of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  is defined as the vector  $\mathbf{V}$  which satisfies the following conditions.

1. The line of action of  $\mathbf{V}$  is perpendicular to the plane containing  $\mathbf{P}$  and  $\mathbf{Q}$  (Fig. 3.6a).
2. The magnitude of  $\mathbf{V}$  is the product of the magnitudes of  $\mathbf{P}$  and  $\mathbf{Q}$  and of the sine of the angle  $\theta$  formed by  $\mathbf{P}$  and  $\mathbf{Q}$  (the measure of which will always be  $180^\circ$  or less); we thus have

$$V = PQ \sin \theta \quad (3.1)$$

3. The direction of  $\mathbf{V}$  is obtained from the *right-hand rule*. Close your right hand and hold it so that your fingers are curled in the same sense as the rotation through  $\theta$  which brings the vector  $\mathbf{P}$  in line with the vector  $\mathbf{Q}$ ; your thumb will then indicate the direction of the vector  $\mathbf{V}$  (Fig. 3.6b). Note that if  $\mathbf{P}$  and  $\mathbf{Q}$  do not have a common point of application, they should first be redrawn from the same point. The three vectors  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{V}$ —taken in that order—are said to form a *right-handed triad*.†



(a)



(b)

**Fig. 3.6**

†We should note that the  $x$ ,  $y$ , and  $z$  axes used in Chap. 2 form a right-handed system of orthogonal axes and that the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  defined in Sec. 2.12 form a right-handed orthogonal triad.

As stated above, the vector  $\mathbf{V}$  satisfying these three conditions (which define it uniquely) is referred to as the vector product of  $\mathbf{P}$  and  $\mathbf{Q}$ ; it is represented by the mathematical expression

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} \quad (3.2)$$

Because of the notation used, the vector product of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  is also referred to as the *cross product* of  $\mathbf{P}$  and  $\mathbf{Q}$ .

It follows from Eq. (3.1) that, when two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  have either the same direction or opposite directions, their vector product is zero. In the general case when the angle  $\theta$  formed by the two vectors is neither  $0^\circ$  nor  $180^\circ$ , Eq. (3.1) can be given a simple geometric interpretation: The magnitude  $V$  of the vector product of  $\mathbf{P}$  and  $\mathbf{Q}$  is equal to the area of the parallelogram which has  $\mathbf{P}$  and  $\mathbf{Q}$  for sides (Fig. 3.7). The vector product  $\mathbf{P} \times \mathbf{Q}$  will therefore remain unchanged if we replace  $\mathbf{Q}$  by a vector  $\mathbf{Q}'$  which is coplanar with  $\mathbf{P}$  and  $\mathbf{Q}$  and such that the line joining the tips of  $\mathbf{Q}$  and  $\mathbf{Q}'$  is parallel to  $\mathbf{P}$ . We write

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = \mathbf{P} \times \mathbf{Q}' \quad (3.3)$$

From the third condition used to define the vector product  $\mathbf{V}$  of  $\mathbf{P}$  and  $\mathbf{Q}$ , namely, the condition stating that  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{V}$  must form a right-handed triad, it follows that vector products *are not commutative*, i.e.,  $\mathbf{Q} \times \mathbf{P}$  is not equal to  $\mathbf{P} \times \mathbf{Q}$ . Indeed, we can easily check that  $\mathbf{Q} \times \mathbf{P}$  is represented by the vector  $-\mathbf{V}$ , which is equal and opposite to  $\mathbf{V}$ . We thus write

$$\mathbf{Q} \times \mathbf{P} = -(\mathbf{P} \times \mathbf{Q}) \quad (3.4)$$

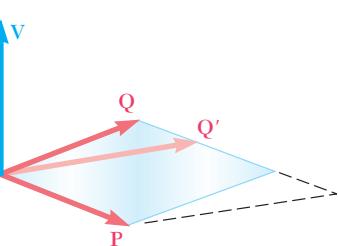


Fig. 3.7

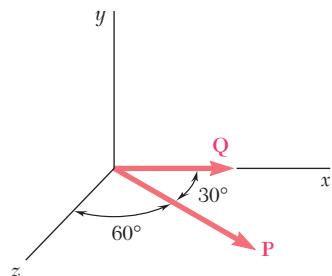


Fig. 3.8

**EXAMPLE** Let us compute the vector product  $\mathbf{V} = \mathbf{P} \times \mathbf{Q}$  where the vector  $\mathbf{P}$  is of magnitude 6 and lies in the  $zx$  plane at an angle of  $30^\circ$  with the  $x$  axis, and where the vector  $\mathbf{Q}$  is of magnitude 4 and lies along the  $x$  axis (Fig. 3.8).

It follows immediately from the definition of the vector product that the vector  $\mathbf{V}$  must lie along the  $y$  axis, have the magnitude

$$V = PQ \sin \theta = (6)(4) \sin 30^\circ = 12$$

and be directed upward. ■

We saw that the commutative property does not apply to vector products. We may wonder whether the *distributive* property holds, i.e., whether the relation

$$\mathbf{P} \times (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \times \mathbf{Q}_1 + \mathbf{P} \times \mathbf{Q}_2 \quad (3.5)$$

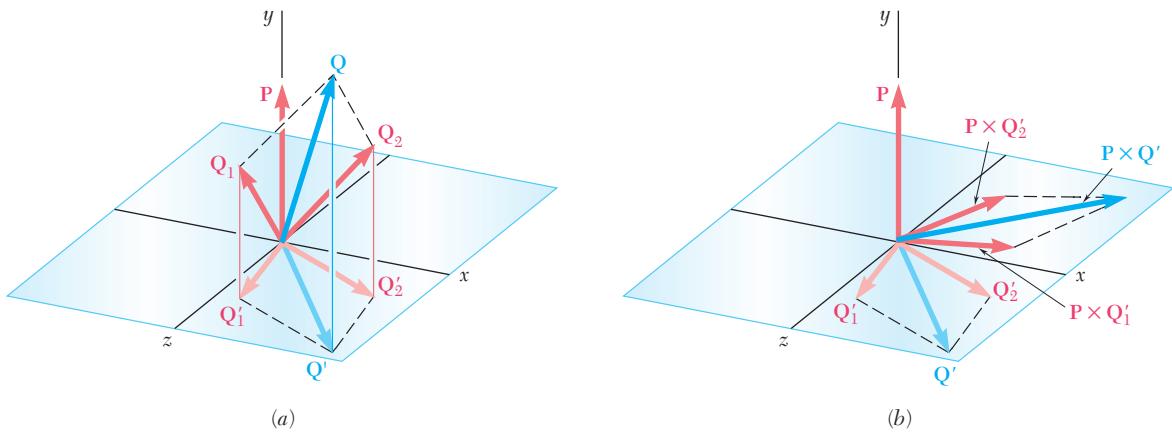
is valid. The answer is *yes*. Many readers are probably willing to accept without formal proof an answer which they intuitively feel is correct. However, since the entire structure of both vector algebra and statics depends upon the relation (3.5), we should take time out to derive it.

We can, without any loss of generality, assume that  $\mathbf{P}$  is directed along the  $y$  axis (Fig. 3.9a). Denoting by  $\mathbf{Q}$  the sum of  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$ , we drop perpendiculars from the tips of  $\mathbf{Q}$ ,  $\mathbf{Q}_1$ , and  $\mathbf{Q}_2$  onto the  $zx$  plane, defining in this way the vectors  $\mathbf{Q}'$ ,  $\mathbf{Q}'_1$ , and  $\mathbf{Q}'_2$ . These vectors will be referred to, respectively, as the *projections* of  $\mathbf{Q}$ ,  $\mathbf{Q}_1$ , and  $\mathbf{Q}_2$  on the  $zx$  plane. Recalling the property expressed by Eq. (3.3), we

note that the left-hand member of Eq. (3.5) can be replaced by  $\mathbf{P} \times \mathbf{Q}'$  and that, similarly, the vector products  $\mathbf{P} \times \mathbf{Q}_1$  and  $\mathbf{P} \times \mathbf{Q}_2$  can respectively be replaced by  $\mathbf{P} \times \mathbf{Q}'_1$  and  $\mathbf{P} \times \mathbf{Q}'_2$ . Thus, the relation to be proved can be written in the form

$$\mathbf{P} \times \mathbf{Q}' = \mathbf{P} \times \mathbf{Q}'_1 + \mathbf{P} \times \mathbf{Q}'_2 \quad (3.5')$$

We now observe that  $\mathbf{P} \times \mathbf{Q}'$  can be obtained from  $\mathbf{Q}'$  by multiplying this vector by the scalar  $P$  and rotating it counterclockwise through  $90^\circ$  in the  $zx$  plane (Fig. 3.9b); the other two vector



**Fig. 3.9**

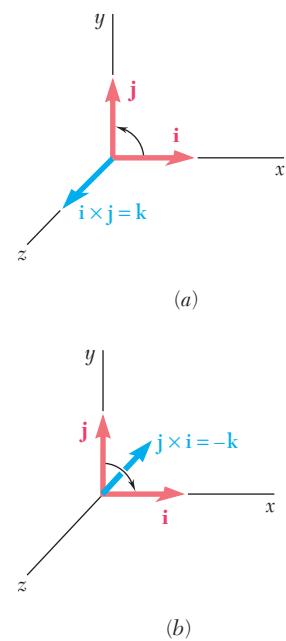
products in (3.5') can be obtained in the same manner from  $\mathbf{Q}'_1$  and  $\mathbf{Q}'_2$ , respectively. Now, since the projection of a parallelogram onto an arbitrary plane is a parallelogram, the projection  $\mathbf{Q}'$  of the sum  $\mathbf{Q}$  of  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  must be the sum of the projections  $\mathbf{Q}'_1$  and  $\mathbf{Q}'_2$  of  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  on the same plane (Fig. 3.9a). This relation between the vectors  $\mathbf{Q}'$ ,  $\mathbf{Q}'_1$ , and  $\mathbf{Q}'_2$  will still hold after the three vectors have been multiplied by the scalar  $P$  and rotated through  $90^\circ$  (Fig. 3.9b). Thus, the relation (3.5') has been proved, and we can now be sure that the distributive property holds for vector products.

A third property, the associative property, does not apply to vector products; we have in general

$$(\mathbf{P} \times \mathbf{Q}) \times \mathbf{S} \neq \mathbf{P} \times (\mathbf{Q} \times \mathbf{S}) \quad (3.6)$$

### 3.5 VECTOR PRODUCTS EXPRESSED IN TERMS OF RECTANGULAR COMPONENTS

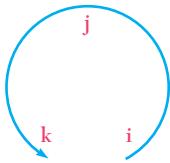
Let us now determine the vector product of any two of the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ , which were defined in Chap. 2. Consider first the product  $\mathbf{i} \times \mathbf{j}$  (Fig. 3.10a). Since both vectors have a magnitude equal to 1 and since they are at a right angle to each other, their vector product will also be a unit vector. This unit vector must be  $\mathbf{k}$ , since the vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are mutually perpendicular and form a right-handed triad. On the other hand, it follows from the right-hand rule given on page 77 that the product  $\mathbf{j} \times \mathbf{i}$  will be equal to  $-\mathbf{k}$  (Fig. 3.10b). Finally, it should be observed that the vector product



**Fig. 3.10**

of a unit vector with itself, such as  $\mathbf{i} \times \mathbf{i}$ , is equal to zero, since both vectors have the same direction. The vector products of the various possible pairs of unit vectors are

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} = -\mathbf{j} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{array} \quad (3.7)$$



**Fig. 3.11**

By arranging in a circle and in counterclockwise order the three letters representing the unit vectors (Fig. 3.11), we can simplify the determination of the sign of the vector product of two unit vectors: The product of two unit vectors will be positive if they follow each other in counterclockwise order and will be negative if they follow each other in clockwise order.

We can now easily express the vector product  $\mathbf{V}$  of two given vectors  $\mathbf{P}$  and  $\mathbf{Q}$  in terms of the rectangular components of these vectors. Resolving  $\mathbf{P}$  and  $\mathbf{Q}$  into components, we first write

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = (P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k}) \times (Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k})$$

Making use of the distributive property, we express  $\mathbf{V}$  as the sum of vector products, such as  $P_x \mathbf{i} \times Q_y \mathbf{j}$ . Observing that each of the expressions obtained is equal to the vector product of two unit vectors, such as  $\mathbf{i} \times \mathbf{j}$ , multiplied by the product of two scalars, such as  $P_x Q_y$ , and recalling the identities (3.7), we obtain, after factoring out  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ ,

$$\mathbf{V} = (P_y Q_z - P_z Q_y) \mathbf{i} + (P_z Q_x - P_x Q_z) \mathbf{j} + (P_x Q_y - P_y Q_x) \mathbf{k} \quad (3.8)$$

The rectangular components of the vector product  $\mathbf{V}$  are thus found to be

$$\begin{aligned} V_x &= P_y Q_z - P_z Q_y \\ V_y &= P_z Q_x - P_x Q_z \\ V_z &= P_x Q_y - P_y Q_x \end{aligned} \quad (3.9)$$

Returning to Eq. (3.8), we observe that its right-hand member represents the expansion of a determinant. The vector product  $\mathbf{V}$  can thus be expressed in the following form, which is more easily memorized:<sup>†</sup>

$$\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad (3.10)$$

<sup>†</sup>Any determinant consisting of three rows and three columns can be evaluated by repeating the first and second columns and forming products along each diagonal line. The sum of the products obtained along the red lines is then subtracted from the sum of the products obtained along the black lines.



## 3.6 MOMENT OF A FORCE ABOUT A POINT

Let us now consider a force  $\mathbf{F}$  acting on a rigid body (Fig. 3.12a). As we know, the force  $\mathbf{F}$  is represented by a vector which defines its magnitude and direction. However, the effect of the force on the rigid body depends also upon its point of application  $A$ . The position of  $A$  can be conveniently defined by the vector  $\mathbf{r}$  which joins the fixed reference point  $O$  with  $A$ ; this vector is known as the *position vector* of  $A$ .<sup>f</sup> The position vector  $\mathbf{r}$  and the force  $\mathbf{F}$  define the plane shown in Fig. 3.12a.

We will define the *moment of  $\mathbf{F}$  about  $O$*  as the vector product of  $\mathbf{r}$  and  $\mathbf{F}$ :

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \quad (3.11)$$

According to the definition of the vector product given in Sec. 3.4, the moment  $\mathbf{M}_O$  must be perpendicular to the plane containing  $O$  and the force  $\mathbf{F}$ . The sense of  $\mathbf{M}_O$  is defined by the sense of the rotation which will bring the vector  $\mathbf{r}$  in line with the vector  $\mathbf{F}$ ; this rotation will be observed as *countrerclockwise* by an observer located at the tip of  $\mathbf{M}_O$ . Another way of defining the sense of  $\mathbf{M}_O$  is furnished by a variation of the right-hand rule: Close your right hand and hold it so that your fingers are curled in the sense of the rotation that  $\mathbf{F}$  would impart to the rigid body about a fixed axis directed along the line of action of  $\mathbf{M}_O$ ; your thumb will indicate the sense of the moment  $\mathbf{M}_O$  (Fig. 3.12b).

Finally, denoting by  $\theta$  the angle between the lines of action of the position vector  $\mathbf{r}$  and the force  $\mathbf{F}$ , we find that the magnitude of the moment of  $\mathbf{F}$  about  $O$  is

$$M_O = rF \sin \theta = Fd \quad (3.12)$$

where  $d$  represents the perpendicular distance from  $O$  to the line of action of  $\mathbf{F}$ . Since the tendency of a force  $\mathbf{F}$  to make a rigid body rotate about a fixed axis perpendicular to the force depends upon the distance of  $\mathbf{F}$  from that axis as well as upon the magnitude of  $\mathbf{F}$ , we note that *the magnitude of  $\mathbf{M}_O$  measures the tendency of the force  $\mathbf{F}$  to make the rigid body rotate about a fixed axis directed along  $\mathbf{M}_O$* .

In the SI system of units, where a force is expressed in newtons (N) and a distance in meters (m), the moment of a force is expressed in newton-meters (N · m). In the U.S. customary system of units, where a force is expressed in pounds and a distance in feet or inches, the moment of a force is expressed in lb · ft or lb · in.

We can observe that although the moment  $\mathbf{M}_O$  of a force about a point depends upon the magnitude, the line of action, and the sense of the force, it does *not* depend upon the actual position of the point of application of the force along its line of action. Conversely, the moment  $\mathbf{M}_O$  of a force  $\mathbf{F}$  does not characterize the position of the point of application of  $\mathbf{F}$ .

<sup>f</sup>We can easily verify that position vectors obey the law of vector addition and, thus, are truly vectors. Consider, for example, the position vectors  $\mathbf{r}$  and  $\mathbf{r}'$  of  $A$  with respect to two reference points  $O$  and  $O'$  and the position vector  $\mathbf{s}$  of  $O$  with respect to  $O'$  (Fig. 3.40a, Sec. 3.16). We verify that the position vector  $\mathbf{r}' = \overrightarrow{O'A}$  can be obtained from the position vectors  $\mathbf{s} = \overrightarrow{O'O}$  and  $\mathbf{r} = \overrightarrow{OA}$  by applying the triangle rule for the addition of vectors.

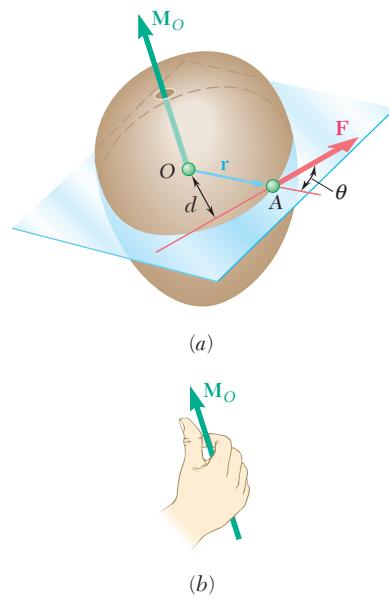


Fig. 3.12

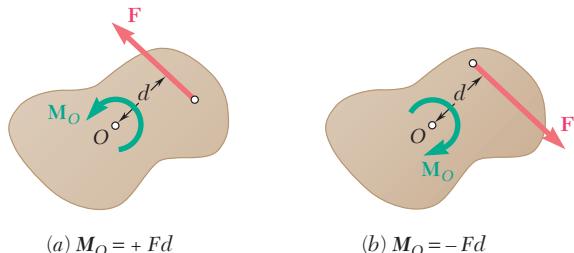
However, as it will be seen presently, the moment  $\mathbf{M}_O$  of a force  $\mathbf{F}$  of given magnitude and direction *completely defines the line of action of  $\mathbf{F}$* . Indeed, the line of action of  $\mathbf{F}$  must lie in a plane through  $O$  perpendicular to the moment  $\mathbf{M}_O$ ; its distance  $d$  from  $O$  must be equal to the quotient  $M_O/F$  of the magnitudes of  $\mathbf{M}_O$  and  $\mathbf{F}$ ; and the sense of  $\mathbf{M}_O$  determines whether the line of action of  $\mathbf{F}$  is to be drawn on one side or the other of the point  $O$ .

We recall from Sec. 3.3 that the principle of transmissibility states that two forces  $\mathbf{F}$  and  $\mathbf{F}'$  are equivalent (i.e., have the same effect on a rigid body) if they have the same magnitude, same direction, and same line of action. This principle can now be restated as follows: *Two forces  $\mathbf{F}$  and  $\mathbf{F}'$  are equivalent if, and only if, they are equal* (i.e., have the same magnitude and same direction) *and have equal moments about a given point  $O$* . The necessary and sufficient conditions for two forces  $\mathbf{F}$  and  $\mathbf{F}'$  to be equivalent are thus

$$\mathbf{F} = \mathbf{F}' \quad \text{and} \quad \mathbf{M}_O = \mathbf{M}'_O \quad (3.13)$$

We should observe that it follows from this statement that if the relations (3.13) hold for a given point  $O$ , they will hold for any other point.

**Problems Involving Only Two Dimensions.** Many applications deal with two-dimensional structures, i.e., structures which have length and breadth but only negligible depth and which are subjected to forces contained in the plane of the structure. Two-dimensional structures and the forces acting on them can be readily represented on a sheet of paper or on a blackboard. Their analysis is therefore considerably simpler than that of three-dimensional structures and forces.



**Fig. 3.13**

Consider, for example, a rigid slab acted upon by a force  $\mathbf{F}$  (Fig. 3.13). The moment of  $\mathbf{F}$  about a point  $O$  chosen in the plane of the figure is represented by a vector  $\mathbf{M}_O$  perpendicular to that plane and of magnitude  $Fd$ . In the case of Fig. 3.13a the vector  $\mathbf{M}_O$  points *out of* the paper, while in the case of Fig. 3.13b it points *into* the paper. As we look at the figure, we observe in the first case that  $\mathbf{F}$  tends to rotate the slab counterclockwise and in the second case that it tends to rotate the slab clockwise. Therefore, it is natural to refer to the sense of the moment of  $\mathbf{F}$  about  $O$  in Fig. 3.13a as counterclockwise  $\uparrow$ , and in Fig. 3.13b as clockwise  $\downarrow$ .

Since the moment of a force  $\mathbf{F}$  acting in the plane of the figure must be perpendicular to that plane, we need only specify the *magnitude* and the *sense* of the moment of  $\mathbf{F}$  about  $O$ . This can be done by assigning to the magnitude  $M_O$  of the moment a positive or negative sign according to whether the vector  $\mathbf{M}_O$  points out of or into the paper.

### 3.7 VARIGNON'S THEOREM

The distributive property of vector products can be used to determine the moment of the resultant of several *concurrent forces*. If several forces  $\mathbf{F}_1, \mathbf{F}_2, \dots$  are applied at the same point  $A$  (Fig. 3.14), and if we denote by  $\mathbf{r}$  the position vector of  $A$ , it follows immediately from Eq. (3.5) of Sec. 3.4 that

$$\mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \dots) = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \dots \quad (3.14)$$

In words, *the moment about a given point  $O$  of the resultant of several concurrent forces is equal to the sum of the moments of the various forces about the same point  $O$* . This property, which was originally established by the French mathematician Varignon (1654–1722) long before the introduction of vector algebra, is known as *Varignon's theorem*.

The relation (3.14) makes it possible to replace the direct determination of the moment of a force  $\mathbf{F}$  by the determination of the moments of two or more component forces. As you will see in the next section,  $\mathbf{F}$  will generally be resolved into components parallel to the coordinate axes. However, it may be more expeditious in some instances to resolve  $\mathbf{F}$  into components which are not parallel to the coordinate axes (see Sample Prob. 3.3).

### 3.8 RECTANGULAR COMPONENTS OF THE MOMENT OF A FORCE

In general, the determination of the moment of a force in space will be considerably simplified if the force and the position vector of its point of application are resolved into rectangular  $x$ ,  $y$ , and  $z$  components. Consider, for example, the moment  $\mathbf{M}_O$  about  $O$  of a force  $\mathbf{F}$  whose components are  $F_x$ ,  $F_y$ , and  $F_z$  and which is applied at a point  $A$  of coordinates  $x$ ,  $y$ , and  $z$  (Fig. 3.15). Observing that the components of the position vector  $\mathbf{r}$  are respectively equal to the coordinates  $x$ ,  $y$ , and  $z$  of the point  $A$ , we write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (3.15)$$

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \quad (3.16)$$

Substituting for  $\mathbf{r}$  and  $\mathbf{F}$  from (3.15) and (3.16) into

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \quad (3.11)$$

and recalling the results obtained in Sec. 3.5, we write the moment  $\mathbf{M}_O$  of  $\mathbf{F}$  about  $O$  in the form

$$\mathbf{M}_O = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} \quad (3.17)$$

where the components  $M_x$ ,  $M_y$ , and  $M_z$  are defined by the relations

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \quad (3.18)$$

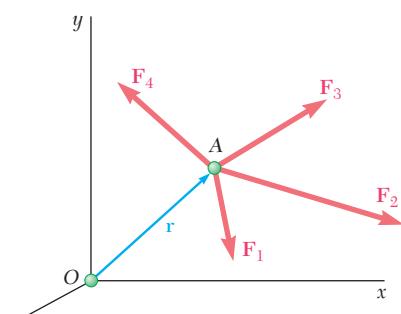


Fig. 3.14

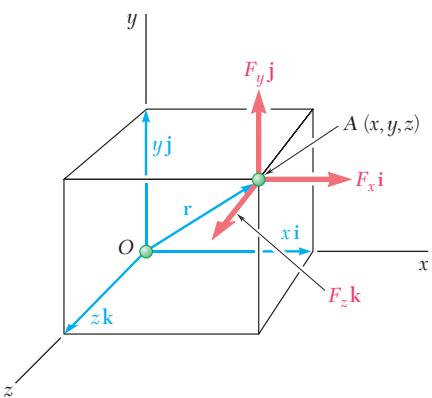


Fig. 3.15

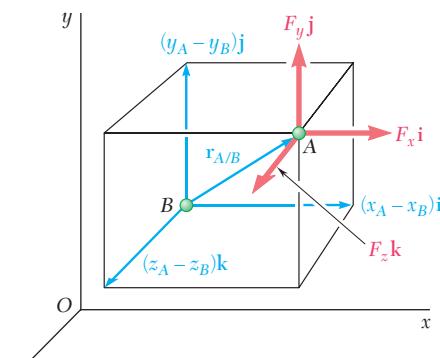


Fig. 3.16

As you will see in Sec. 3.11, the scalar components  $M_x$ ,  $M_y$ , and  $M_z$  of the moment  $\mathbf{M}_O$  measure the tendency of the force  $\mathbf{F}$  to impart to a rigid body a motion of rotation about the  $x$ ,  $y$ , and  $z$  axes, respectively. Substituting from (3.18) into (3.17), we can also write  $\mathbf{M}_O$  in the form of the determinant

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \quad (3.19)$$

To compute the moment  $\mathbf{M}_B$  about an arbitrary point  $B$  of a force  $\mathbf{F}$  applied at  $A$  (Fig. 3.16), we must replace the position vector  $\mathbf{r}$  in Eq. (3.11) by a vector drawn from  $B$  to  $A$ . This vector is the *position vector of A relative to B* and will be denoted by  $\mathbf{r}_{A/B}$ . Observing that  $\mathbf{r}_{A/B}$  can be obtained by subtracting  $\mathbf{r}_B$  from  $\mathbf{r}_A$ , we write

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} \quad (3.20)$$

or, using the determinant form,

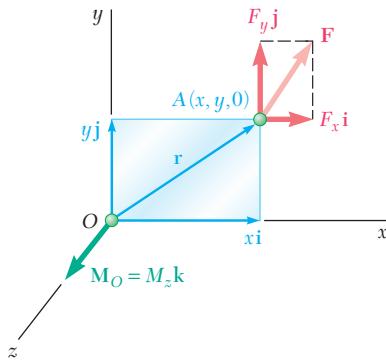


Fig. 3.17

$$\mathbf{M}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \quad (3.21)$$

where  $x_{A/B}$ ,  $y_{A/B}$ , and  $z_{A/B}$  denote the components of the vector  $\mathbf{r}_{A/B}$ :

$$x_{A/B} = x_A - x_B \quad y_{A/B} = y_A - y_B \quad z_{A/B} = z_A - z_B$$

In the case of *problems involving only two dimensions*, the force  $\mathbf{F}$  can be assumed to lie in the  $xy$  plane (Fig. 3.17). Setting  $z = 0$  and  $F_z = 0$  in Eq. (3.19), we obtain

$$\mathbf{M}_O = (xF_y - yF_x)\mathbf{k}$$

We verify that the moment of  $\mathbf{F}$  about  $O$  is perpendicular to the plane of the figure and that it is completely defined by the scalar

$$M_O = M_z = xF_y - yF_x \quad (3.22)$$

As noted earlier, a positive value for  $M_O$  indicates that the vector  $\mathbf{M}_O$  points out of the paper (the force  $\mathbf{F}$  tends to rotate the body counter-clockwise about  $O$ ), and a negative value indicates that the vector  $\mathbf{M}_O$  points into the paper (the force  $\mathbf{F}$  tends to rotate the body clockwise about  $O$ ).

To compute the moment about  $B(x_B, y_B)$  of a force lying in the  $xy$  plane and applied at  $A(x_A, y_A)$  (Fig. 3.18), we set  $z_{A/B} = 0$  and  $F_z = 0$  in the relations (3.21) and note that the vector  $\mathbf{M}_B$  is perpendicular to the  $xy$  plane and is defined in magnitude and sense by the scalar

$$M_B = (x_A - x_B)F_y - (y_A - y_B)F_x \quad (3.23)$$

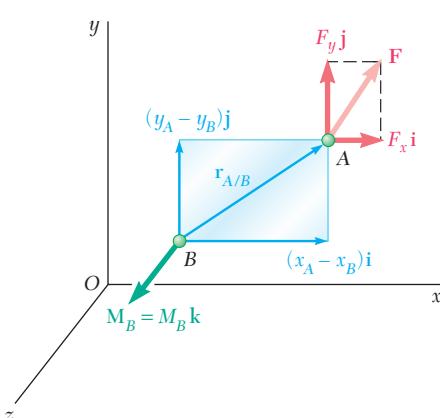
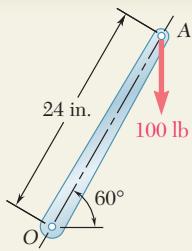


Fig. 3.18



## SAMPLE PROBLEM 3.1

A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at  $O$ . Determine (a) the moment of the 100-lb force about  $O$ ; (b) the horizontal force applied at  $A$  which creates the same moment about  $O$ ; (c) the smallest force applied at  $A$  which creates the same moment about  $O$ ; (d) how far from the shaft a 240-lb vertical force must act to create the same moment about  $O$ ; (e) whether any one of the forces obtained in parts b, c, and d is equivalent to the original force.

## SOLUTION

**a. Moment about  $O$ .** The perpendicular distance from  $O$  to the line of action of the 100-lb force is

$$d = (24 \text{ in.}) \cos 60^\circ = 12 \text{ in.}$$

The magnitude of the moment about  $O$  of the 100-lb force is

$$M_O = Fd = (100 \text{ lb})(12 \text{ in.}) = 1200 \text{ lb} \cdot \text{in.}$$

Since the force tends to rotate the lever clockwise about  $O$ , the moment will be represented by a vector  $\mathbf{M}_O$  perpendicular to the plane of the figure and pointing *into* the paper. We express this fact by writing

$$\mathbf{M}_O = 1200 \text{ lb} \cdot \text{in.} \downarrow$$

**b. Horizontal Force.** In this case, we have

$$d = (24 \text{ in.}) \sin 60^\circ = 20.8 \text{ in.}$$

Since the moment about  $O$  must be 1200 lb · in., we write

$$M_O = Fd$$

$$1200 \text{ lb} \cdot \text{in.} = F(20.8 \text{ in.})$$

$$F = 57.7 \text{ lb}$$

$$\mathbf{F} = 57.7 \text{ lb} \rightarrow$$

**c. Smallest Force.** Since  $M_O = Fd$ , the smallest value of  $F$  occurs when  $d$  is maximum. We choose the force perpendicular to  $OA$  and note that  $d = 24 \text{ in.}$ ; thus

$$M_O = Fd$$

$$1200 \text{ lb} \cdot \text{in.} = F(24 \text{ in.})$$

$$F = 50 \text{ lb}$$

$$\mathbf{F} = 50 \text{ lb} \nwarrow 30^\circ$$

**d. 240-lb Vertical Force.** In this case  $M_O = Fd$  yields

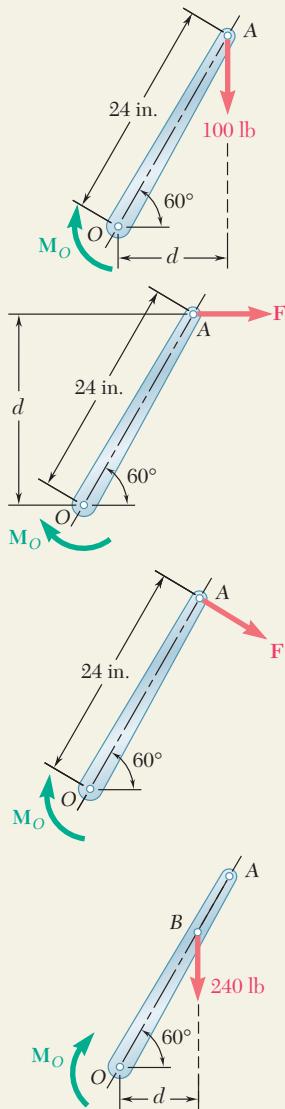
$$1200 \text{ lb} \cdot \text{in.} = (240 \text{ lb})d \quad d = 5 \text{ in.}$$

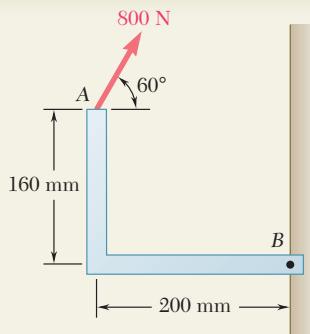
but

$$OB \cos 60^\circ = d$$

$$OB = 10 \text{ in.}$$

**e.** None of the forces considered in parts b, c, and d is equivalent to the original 100-lb force. Although they have the same moment about  $O$ , they have different  $x$  and  $y$  components. In other words, although each force tends to rotate the shaft in the same manner, each causes the lever to pull on the shaft in a different way.





## SAMPLE PROBLEM 3.2

A force of 800 N acts on a bracket as shown. Determine the moment of the force about *B*.

### SOLUTION

The moment  $\mathbf{M}_B$  of the force  $\mathbf{F}$  about *B* is obtained by forming the vector product

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F}$$

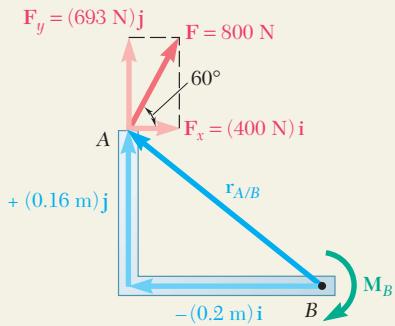
where  $\mathbf{r}_{A/B}$  is the vector drawn from *B* to *A*. Resolving  $\mathbf{r}_{A/B}$  and  $\mathbf{F}$  into rectangular components, we have

$$\begin{aligned}\mathbf{r}_{A/B} &= -(0.2 \text{ m})\mathbf{i} + (0.16 \text{ m})\mathbf{j} \\ \mathbf{F} &= (800 \text{ N}) \cos 60^\circ \mathbf{i} + (800 \text{ N}) \sin 60^\circ \mathbf{j} \\ &= (400 \text{ N})\mathbf{i} + (693 \text{ N})\mathbf{j}\end{aligned}$$

Recalling the relations (3.7) for the cross products of unit vectors (Sec. 3.5), we obtain

$$\begin{aligned}\mathbf{M}_B &= \mathbf{r}_{A/B} \times \mathbf{F} = [-(0.2 \text{ m})\mathbf{i} + (0.16 \text{ m})\mathbf{j}] \times [(400 \text{ N})\mathbf{i} + (693 \text{ N})\mathbf{j}] \\ &= -(138.6 \text{ N} \cdot \text{m})\mathbf{k} - (64.0 \text{ N} \cdot \text{m})\mathbf{k} \\ &= -(202.6 \text{ N} \cdot \text{m})\mathbf{k} \quad \mathbf{M}_B = 203 \text{ N} \cdot \text{m} \downarrow\end{aligned}$$

The moment  $\mathbf{M}_B$  is a vector perpendicular to the plane of the figure and pointing *into* the paper.



## SAMPLE PROBLEM 3.3

A 30-lb force acts on the end of the 3-ft lever as shown. Determine the moment of the force about *O*.

### SOLUTION

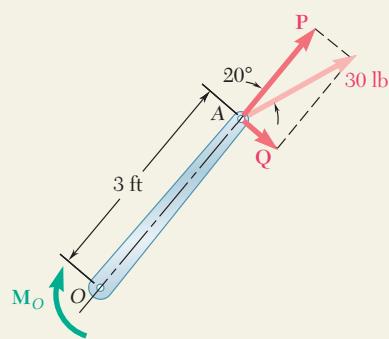
The force is replaced by two components, one component  $\mathbf{P}$  in the direction of  $OA$  and one component  $\mathbf{Q}$  perpendicular to  $OA$ . Since *O* is on the line of action of  $\mathbf{P}$ , the moment of  $\mathbf{P}$  about *O* is zero and the moment of the 30-lb force reduces to the moment of  $\mathbf{Q}$ , which is clockwise and, thus, is represented by a negative scalar.

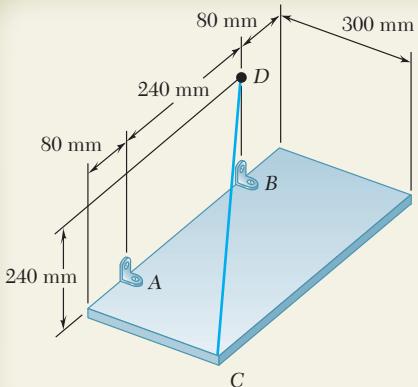
$$Q = (30 \text{ lb}) \sin 20^\circ = 10.26 \text{ lb}$$

$$M_O = -Q(3 \text{ ft}) = -(10.26 \text{ lb})(3 \text{ ft}) = -30.8 \text{ lb} \cdot \text{ft}$$

Since the value obtained for the scalar  $M_O$  is negative, the moment  $\mathbf{M}_O$  points *into* the paper. We write

$$\mathbf{M}_O = 30.8 \text{ lb} \cdot \text{ft} \downarrow$$





## SAMPLE PROBLEM 3.4

A rectangular plate is supported by brackets at *A* and *B* and by a wire *CD*. Knowing that the tension in the wire is 200 N, determine the moment about *A* of the force exerted by the wire on point *C*.

## SOLUTION

The moment  $\mathbf{M}_A$  about *A* of the force  $\mathbf{F}$  exerted by the wire on point *C* is obtained by forming the vector product

$$\mathbf{M}_A = \mathbf{r}_{C/A} \times \mathbf{F} \quad (1)$$

where  $\mathbf{r}_{C/A}$  is the vector drawn from *A* to *C*,

$$\mathbf{r}_{C/A} = \overrightarrow{AC} = (0.3 \text{ m})\mathbf{i} + (0.08 \text{ m})\mathbf{k} \quad (2)$$

and  $\mathbf{F}$  is the 200-N force directed along *CD*. Introducing the unit vector  $\lambda = \overrightarrow{CD}/CD$ , we write

$$\mathbf{F} = F\lambda = (200 \text{ N}) \frac{\overrightarrow{CD}}{CD} \quad (3)$$

Resolving the vector  $\overrightarrow{CD}$  into rectangular components, we have

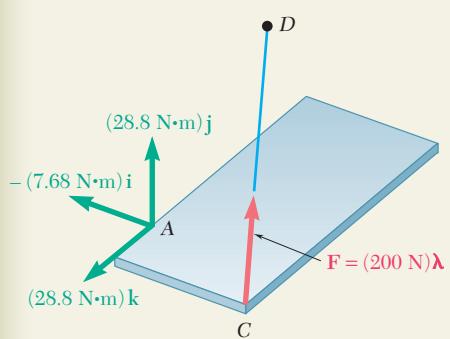
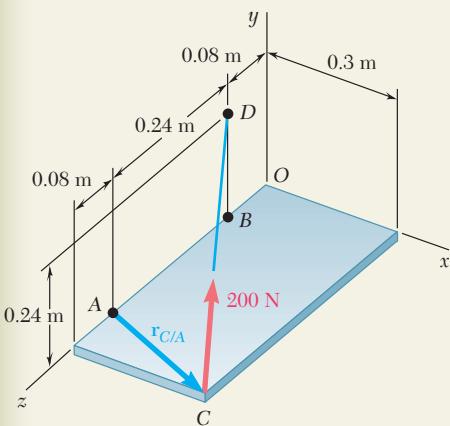
$$\overrightarrow{CD} = -(0.3 \text{ m})\mathbf{i} + (0.24 \text{ m})\mathbf{j} - (0.32 \text{ m})\mathbf{k} \quad CD = 0.50 \text{ m}$$

Substituting into (3), we obtain

$$\begin{aligned} \mathbf{F} &= \frac{200 \text{ N}}{0.50 \text{ m}} [-(0.3 \text{ m})\mathbf{i} + (0.24 \text{ m})\mathbf{j} - (0.32 \text{ m})\mathbf{k}] \\ &= -(120 \text{ N})\mathbf{i} + (96 \text{ N})\mathbf{j} - (128 \text{ N})\mathbf{k} \end{aligned} \quad (4)$$

Substituting for  $\mathbf{r}_{C/A}$  and  $\mathbf{F}$  from (2) and (4) into (1) and recalling the relations (3.7) of Sec. 3.5, we obtain

$$\begin{aligned} \mathbf{M}_A &= \mathbf{r}_{C/A} \times \mathbf{F} = (0.3\mathbf{i} + 0.08\mathbf{k}) \times (-120\mathbf{i} + 96\mathbf{j} - 128\mathbf{k}) \\ &= (0.3)(96)\mathbf{k} + (0.3)(-128)(-\mathbf{j}) + (0.08)(-120)\mathbf{j} + (0.08)(96)(-\mathbf{i}) \\ \mathbf{M}_A &= -(7.68 \text{ N} \cdot \text{m})\mathbf{i} + (28.8 \text{ N} \cdot \text{m})\mathbf{j} + (28.8 \text{ N} \cdot \text{m})\mathbf{k} \end{aligned} \quad \blacktriangleleft$$



**Alternative Solution.** As indicated in Sec. 3.8, the moment  $\mathbf{M}_A$  can be expressed in the form of a determinant:

$$\mathbf{M}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_C - x_A & y_C - y_A & z_C - z_A \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\mathbf{M}_A = -(7.68 \text{ N} \cdot \text{m})\mathbf{i} + (28.8 \text{ N} \cdot \text{m})\mathbf{j} + (28.8 \text{ N} \cdot \text{m})\mathbf{k} \quad \blacktriangleleft$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced the *vector product* or *cross product* of two vectors. In the following problems, you may want to use the vector product to compute the *moment of a force about a point* and also to determine the *perpendicular distance* from a point to a line.

We defined the moment of the force  $\mathbf{F}$  about the point  $O$  of a rigid body as

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \quad (3.11)$$

where  $\mathbf{r}$  is the position vector from  $O$  to *any point* on the line of action of  $\mathbf{F}$ . Since the vector product is not commutative, it is absolutely necessary when computing such a product that you place the vectors in the proper order and that each vector have the correct sense. The moment  $\mathbf{M}_O$  is important because its magnitude is a measure of the tendency of the force  $\mathbf{F}$  to cause the rigid body to rotate about an axis directed along  $\mathbf{M}_O$ .

**1. Computing the moment  $M_O$  of a force in two dimensions.** You can use one of the following procedures:

- a. Use Eq. (3.12),  $M_O = Fd$ , which expresses the magnitude of the moment as the product of the magnitude of  $\mathbf{F}$  and the *perpendicular distance*  $d$  from  $O$  to the line of action of  $\mathbf{F}$  [Sample Prob. 3.1].
- b. Express  $\mathbf{r}$  and  $\mathbf{F}$  in component form and formally evaluate the vector product  $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$  [Sample Prob. 3.2].
- c. Resolve  $\mathbf{F}$  into components respectively parallel and perpendicular to the position vector  $\mathbf{r}$ . Only the perpendicular component contributes to the moment of  $\mathbf{F}$  [Sample Prob. 3.3].
- d. Use Eq. (3.22),  $M_O = M_z = xF_y - yF_x$ . When applying this method, the simplest approach is to treat the scalar components of  $\mathbf{r}$  and  $\mathbf{F}$  as positive and then to assign, by observation, the proper sign to the moment produced by each force component. For example, applying this method to solve Sample Prob. 3.2, we observe that both force components tend to produce a clockwise rotation about  $B$ . Therefore, the moment of each force about  $B$  should be represented by a negative scalar. We then have for the total moment

$$M_B = -(0.16 \text{ m})(400 \text{ N}) - (0.20 \text{ m})(693 \text{ N}) = -202.6 \text{ N} \cdot \text{m}$$

**2. Computing the moment  $M_O$  of a force  $\mathbf{F}$  in three dimensions.** Following the method of Sample Prob. 3.4, the first step in the process is to select the most convenient (simplest) position vector  $\mathbf{r}$ . You should next express  $\mathbf{F}$  in terms of its rectangular components. The final step is to evaluate the vector product  $\mathbf{r} \times \mathbf{F}$  to determine the moment. In most three-dimensional problems you will find it easiest to calculate the vector product using a determinant.

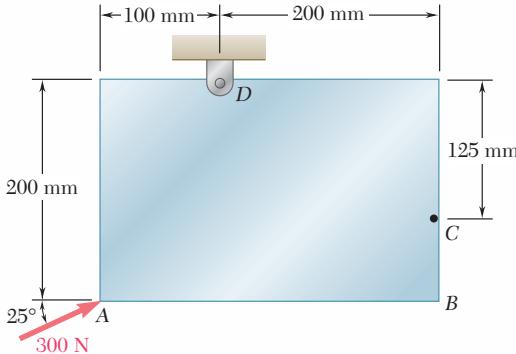
**3. Determining the perpendicular distance  $d$  from a point  $A$  to a given line.** First assume that a force  $\mathbf{F}$  of known magnitude  $F$  lies along the given line. Next determine its moment about  $A$  by forming the vector product  $\mathbf{M}_A = \mathbf{r} \times \mathbf{F}$ , and calculate this product as indicated above. Then compute its magnitude  $M_A$ . Finally, substitute the values of  $F$  and  $M_A$  into the equation  $M_A = Fd$  and solve for  $d$ .

# PROBLEMS

- 3.1** A foot valve for a pneumatic system is hinged at *B*. Knowing that  $\alpha = 28^\circ$ , determine the moment of the 16-N force about point *B* by resolving the force into horizontal and vertical components.

- 3.2** A foot valve for a pneumatic system is hinged at *B*. Knowing that  $\alpha = 28^\circ$ , determine the moment of the 16-N force about point *B* by resolving the force into components along *ABC* and in a direction perpendicular to *ABC*.

- 3.3** A 300-N force is applied at *A* as shown. Determine (a) the moment of the 300-N force about *D*, (b) the smallest force applied at *B* that creates the same moment about *D*.



**Fig. P3.3 and P3.4**

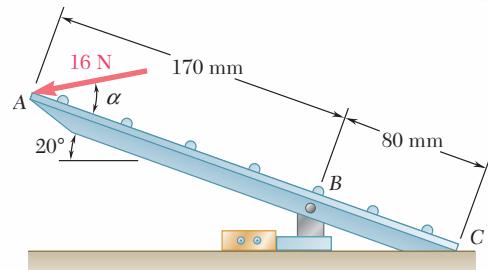
- 3.4** A 300-N force is applied at *A* as shown. Determine (a) the moment of the 300-N force about *D*, (b) the magnitude and sense of the horizontal force applied at *C* that creates the same moment about *D*, (c) the smallest force applied at *C* that creates the same moment about *D*.

- 3.5** An 8-lb force **P** is applied to a shift lever. Determine the moment of **P** about *B* when  $\alpha$  is equal to  $25^\circ$ .

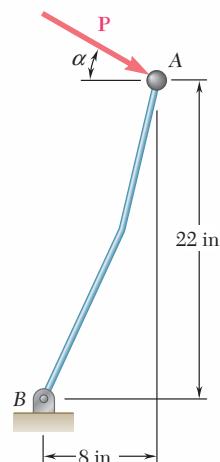
- 3.6** For the shift lever shown, determine the magnitude and the direction of the smallest force **P** that has a 210-lb · in. clockwise moment about *B*.

- 3.7** An 11-lb force **P** is applied to a shift lever. The moment of **P** about *B* is clockwise and has a magnitude of 250 lb · in. Determine the value of  $\alpha$ .

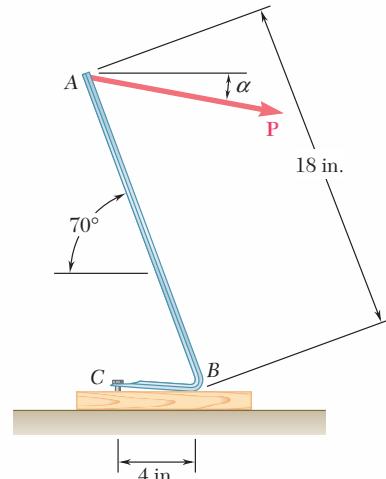
- 3.8** It is known that a vertical force of 200 lb is required to remove the nail at *C* from the board. As the nail first starts moving, determine (a) the moment about *B* of the force exerted on the nail, (b) the magnitude of the force **P** that creates the same moment about *B* if  $\alpha = 10^\circ$ , (c) the smallest force **P** that creates the same moment about *B*.



**Fig. P3.1 and P3.2**



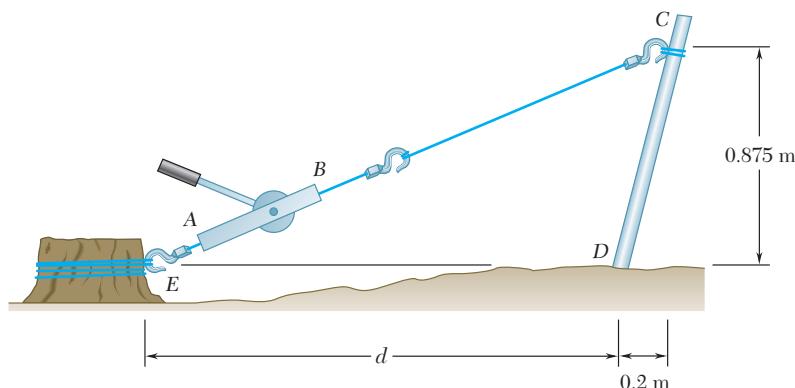
**Fig. P3.5, P3.6, and P3.7**



**Fig. P3.8**

- 3.9** A winch puller  $AB$  is used to straighten a fence post. Knowing that the tension in cable  $BC$  is 1040 N and length  $d$  is 1.90 m, determine the moment about  $D$  of the force exerted by the cable at  $C$  by resolving that force into horizontal and vertical components applied (a) at point  $C$ , (b) at point  $E$ .

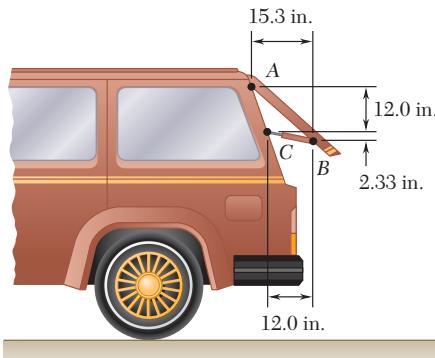
- 3.10** It is known that a force with a moment of 960 N · m about  $D$  is required to straighten the fence post  $CD$ . If  $d = 2.80$  m, determine the tension that must be developed in the cable of winch puller  $AB$  to create the required moment about point  $D$ .



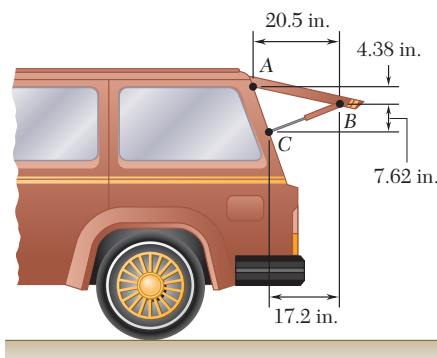
**Fig. P3.9, P3.10, and P3.11**

- 3.11** It is known that a force with a moment of 960 N · m about  $D$  is required to straighten the fence post  $CD$ . If the capacity of winch puller  $AB$  is 2400 N, determine the minimum value of distance  $d$  to create the specified moment about point  $D$ .

- 3.12 and 3.13** The tailgate of a car is supported by the hydraulic lift  $BC$ . If the lift exerts a 125-lb force directed along its centerline on the ball and socket at  $B$ , determine the moment of the force about  $A$ .



**Fig. P3.12**



**Fig. P3.13**

- 3.14** A mechanic uses a piece of pipe  $AB$  as a lever when tightening an alternator belt. When he pushes down at  $A$ , a force of 485 N is exerted on the alternator at  $B$ . Determine the moment of that force about bolt  $C$  if its line of action passes through  $O$ .

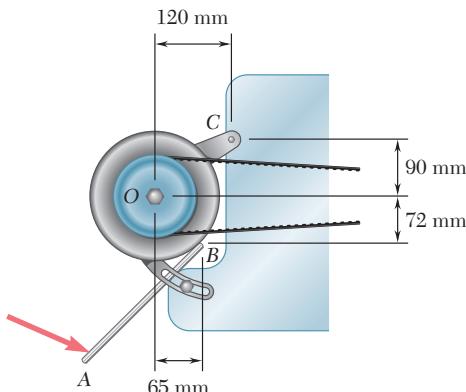


Fig. P3.14

- 3.15** Form the vector products  $\mathbf{B} \times \mathbf{C}$  and  $\mathbf{B}' \times \mathbf{C}$ , where  $B = B'$ , and use the results obtained to prove the identity

$$\sin \alpha \cos \beta = \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta).$$

- 3.16** A line passes through the points  $(20 \text{ m}, 16 \text{ m})$  and  $(-1 \text{ m}, -4 \text{ m})$ . Determine the perpendicular distance  $d$  from the line to the origin  $O$  of the system of coordinates.

- 3.17** The vectors  $\mathbf{P}$  and  $\mathbf{Q}$  are two adjacent sides of a parallelogram. Determine the area of the parallelogram when (a)  $\mathbf{P} = -7\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$  and  $\mathbf{Q} = 2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ , (b)  $\mathbf{P} = 6\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$  and  $\mathbf{Q} = -2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ .

- 3.18** A plane contains the vectors  $\mathbf{A}$  and  $\mathbf{B}$ . Determine the unit vector normal to the plane when  $\mathbf{A}$  and  $\mathbf{B}$  are equal to, respectively, (a)  $\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$  and  $4\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$ , (b)  $3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$  and  $-2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ .

- 3.19** Determine the moment about the origin  $O$  of the force  $\mathbf{F} = 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$  that acts at a point  $A$ . Assume that the position vector of  $A$  is (a)  $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ , (b)  $\mathbf{r} = 2\mathbf{i} + 2.5\mathbf{j} - 1.5\mathbf{k}$ , (c)  $\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ .

- 3.20** Determine the moment about the origin  $O$  of the force  $\mathbf{F} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$  that acts at a point  $A$ . Assume that the position vector of  $A$  is (a)  $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ , (b)  $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ , (c)  $\mathbf{r} = -4\mathbf{i} + 10\mathbf{k}$ .

- 3.21** A 200-N force is applied as shown to the bracket  $ABC$ . Determine the moment of the force about  $A$ .

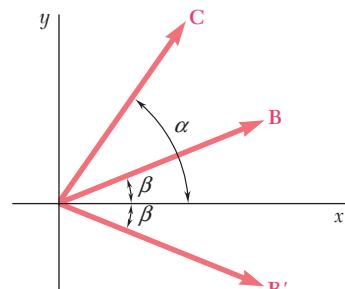


Fig. P3.15

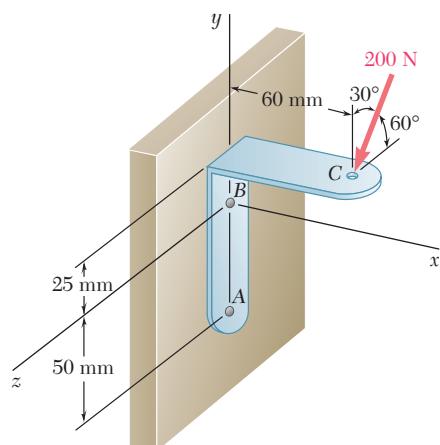


Fig. P3.21

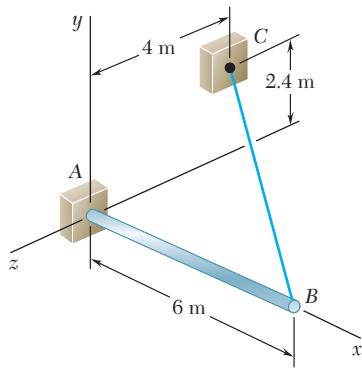


Fig. P3.23

- 3.22** Before the trunk of a large tree is felled, cables  $AB$  and  $BC$  are attached as shown. Knowing that the tensions in cables  $AB$  and  $BC$  are 555 N and 660 N, respectively, determine the moment about  $O$  of the resultant force exerted on the tree by the cables at  $B$ .

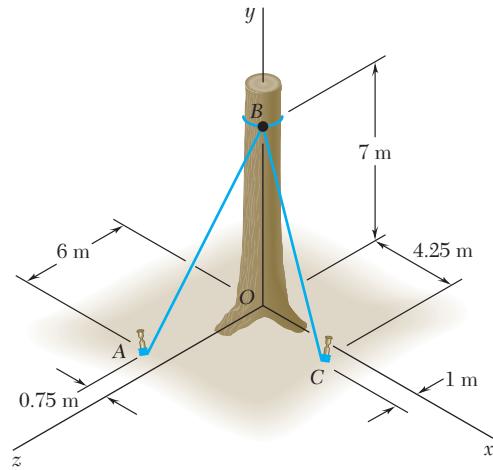


Fig. P3.22

- 3.23** The 6-m boom  $AB$  has a fixed end  $A$ . A steel cable is stretched from the free end  $B$  of the boom to a point  $C$  located on the vertical wall. If the tension in the cable is 2.5 kN, determine the moment about  $A$  of the force exerted by the cable at  $B$ .

- 3.24** A wooden board  $AB$ , which is used as a temporary prop to support a small roof, exerts at point  $A$  of the roof a 57-lb force directed along  $BA$ . Determine the moment about  $C$  of that force.

- 3.25** The ramp  $ABCD$  is supported by cables at corners  $C$  and  $D$ . The tension in each of the cables is 810 N. Determine the moment about  $A$  of the force exerted by (a) the cable at  $D$ , (b) the cable at  $C$ .

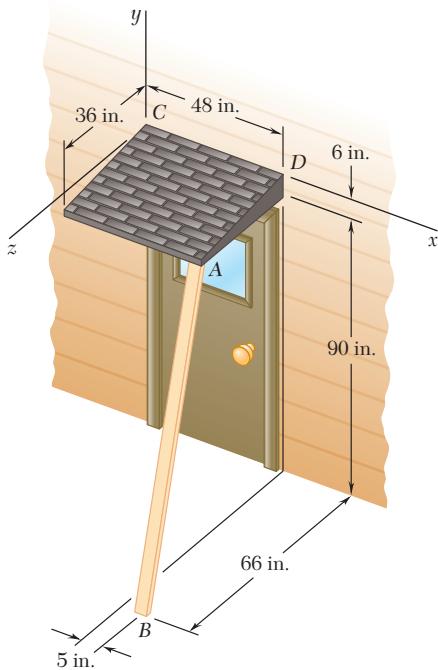


Fig. P3.24

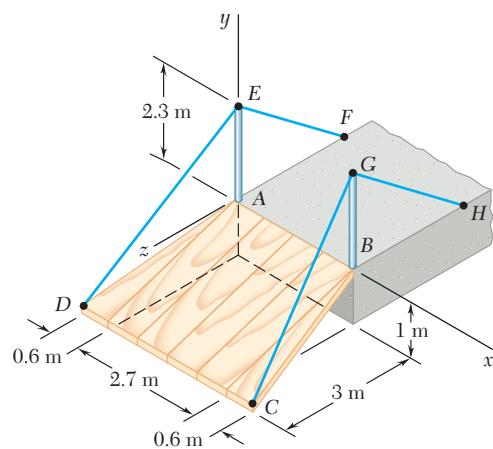


Fig. P3.25

- 3.26** A small boat hangs from two davits, one of which is shown in the figure. The tension in line  $ABAD$  is 82 lb. Determine the moment about  $C$  of the resultant force  $\mathbf{R}_A$  exerted on the davit at  $A$ .

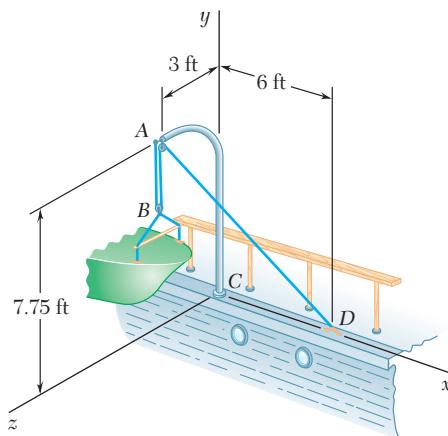


Fig. P3.26

- 3.27** In Prob. 3.22, determine the perpendicular distance from point  $O$  to cable  $AB$ .
- 3.28** In Prob. 3.22, determine the perpendicular distance from point  $O$  to cable  $BC$ .
- 3.29** In Prob. 3.24, determine the perpendicular distance from point  $D$  to a line drawn through points  $A$  and  $B$ .
- 3.30** In Prob. 3.24, determine the perpendicular distance from point  $C$  to a line drawn through points  $A$  and  $B$ .
- 3.31** In Prob. 3.25, determine the perpendicular distance from point  $A$  to portion  $DE$  of cable  $DEF$ .
- 3.32** In Prob. 3.25, determine the perpendicular distance from point  $A$  to a line drawn through points  $C$  and  $G$ .
- 3.33** In Prob. 3.26, determine the perpendicular distance from point  $C$  to portion  $AD$  of the line  $ABAD$ .
- 3.34** Determine the value of  $a$  that minimizes the perpendicular distance from point  $C$  to a section of pipeline that passes through points  $A$  and  $B$ .

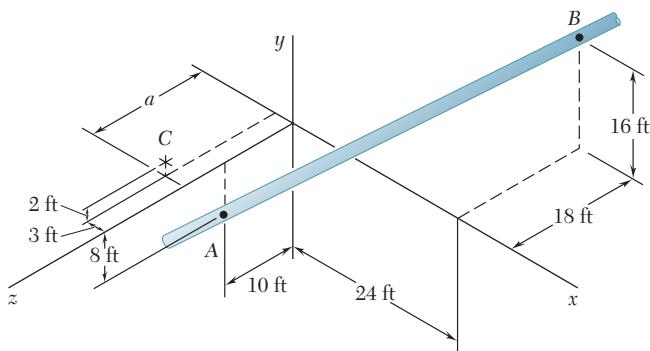


Fig. P3.34

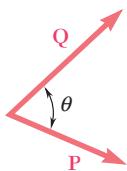


Fig. 3.19

### 3.9 SCALAR PRODUCT OF TWO VECTORS

The *scalar product* of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  is defined as the product of the magnitudes of  $\mathbf{P}$  and  $\mathbf{Q}$  and of the cosine of the angle  $\theta$  formed by  $\mathbf{P}$  and  $\mathbf{Q}$  (Fig. 3.19). The scalar product of  $\mathbf{P}$  and  $\mathbf{Q}$  is denoted by  $\mathbf{P} \cdot \mathbf{Q}$ . We write therefore

$$\mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta \quad (3.24)$$

Note that the expression just defined is not a vector but a *scalar*, which explains the name *scalar product*; because of the notation used,  $\mathbf{P} \cdot \mathbf{Q}$  is also referred to as the *dot product* of the vectors  $\mathbf{P}$  and  $\mathbf{Q}$ .

It follows from its very definition that the scalar product of two vectors is *commutative*, i.e., that

$$\mathbf{P} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{P} \quad (3.25)$$

To prove that the scalar product is also *distributive*, we must prove the relation

$$\mathbf{P} \cdot (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P} \cdot \mathbf{Q}_2 \quad (3.26)$$

We can, without any loss of generality, assume that  $\mathbf{P}$  is directed along the  $y$  axis (Fig. 3.20). Denoting by  $\mathbf{Q}$  the sum of  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$  and by  $\theta_y$  the angle  $\mathbf{Q}$  forms with the  $y$  axis, we express the left-hand member of (3.26) as follows:

$$\mathbf{P} \cdot (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta_y = PQ_y \quad (3.27)$$

where  $Q_y$  is the  $y$  component of  $\mathbf{Q}$ . We can, in a similar way, express the right-hand member of (3.26) as

$$\mathbf{P} \cdot \mathbf{Q}_1 + \mathbf{P} \cdot \mathbf{Q}_2 = P(Q_1)_y + P(Q_2)_y \quad (3.28)$$

Since  $\mathbf{Q}$  is the sum of  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$ , its  $y$  component must be equal to the sum of the  $y$  components of  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$ . Thus, the expressions obtained in (3.27) and (3.28) are equal, and the relation (3.26) has been proved.

As far as the third property—the associative property—is concerned, we note that this property cannot apply to scalar products. Indeed,  $(\mathbf{P} \cdot \mathbf{Q}) \cdot \mathbf{S}$  has no meaning, since  $\mathbf{P} \cdot \mathbf{Q}$  is not a vector but a scalar.

The scalar product of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  can be expressed in terms of their rectangular components. Resolving  $\mathbf{P}$  and  $\mathbf{Q}$  into components, we first write

$$\mathbf{P} \cdot \mathbf{Q} = (P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k}) \cdot (Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k})$$

Making use of the distributive property, we express  $\mathbf{P} \cdot \mathbf{Q}$  as the sum of scalar products, such as  $P_x \mathbf{i} \cdot Q_x \mathbf{i}$  and  $P_x \mathbf{i} \cdot Q_y \mathbf{j}$ . However, from the

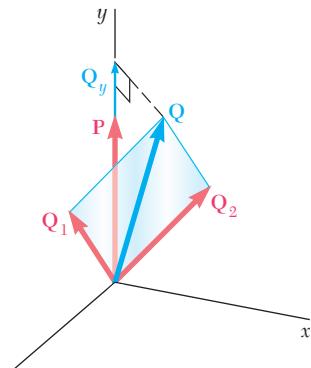


Fig. 3.20

definition of the scalar product it follows that the scalar products of the unit vectors are either zero or one.

$$\begin{aligned}\mathbf{i} \cdot \mathbf{i} &= 1 & \mathbf{j} \cdot \mathbf{j} &= 1 & \mathbf{k} \cdot \mathbf{k} &= 1 \\ \mathbf{i} \cdot \mathbf{j} &= 0 & \mathbf{j} \cdot \mathbf{k} &= 0 & \mathbf{k} \cdot \mathbf{i} &= 0\end{aligned}\quad (3.29)$$

Thus, the expression obtained for  $\mathbf{P} \cdot \mathbf{Q}$  reduces to

$$\mathbf{P} \cdot \mathbf{Q} = P_x Q_x + P_y Q_y + P_z Q_z \quad (3.30)$$

In the particular case when  $\mathbf{P}$  and  $\mathbf{Q}$  are equal, we note that

$$\mathbf{P} \cdot \mathbf{P} = P_x^2 + P_y^2 + P_z^2 = P^2 \quad (3.31)$$

## Applications

- Angle formed by two given vectors. Let two vectors be given in terms of their components:

$$\begin{aligned}\mathbf{P} &= P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k} \\ \mathbf{Q} &= Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k}\end{aligned}$$

To determine the angle formed by the two vectors, we equate the expressions obtained in (3.24) and (3.30) for their scalar product and write

$$PQ \cos \theta = P_x Q_x + P_y Q_y + P_z Q_z$$

Solving for  $\cos \theta$ , we have

$$\cos \theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ} \quad (3.32)$$

- Projection of a vector on a given axis. Consider a vector  $\mathbf{P}$  forming an angle  $\theta$  with an axis, or directed line,  $OL$  (Fig. 3.21). The *projection of  $\mathbf{P}$  on the axis  $OL$*  is defined as the scalar

$$P_{OL} = P \cos \theta \quad (3.33)$$

We note that the projection  $P_{OL}$  is equal in absolute value to the length of the segment  $OA$ ; it will be positive if  $OA$  has the same sense as the axis  $OL$ , that is, if  $\theta$  is acute, and negative otherwise. If  $\mathbf{P}$  and  $OL$  are at a right angle, the projection of  $\mathbf{P}$  on  $OL$  is zero.

Consider now a vector  $\mathbf{Q}$  directed along  $OL$  and of the same sense as  $OL$  (Fig. 3.22). The scalar product of  $\mathbf{P}$  and  $\mathbf{Q}$  can be expressed as

$$\mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta = P_{OL} Q \quad (3.34)$$

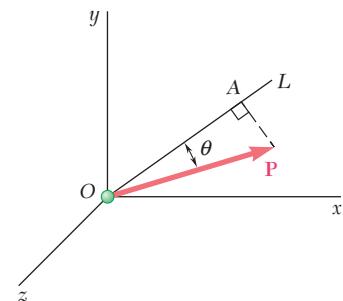


Fig. 3.21

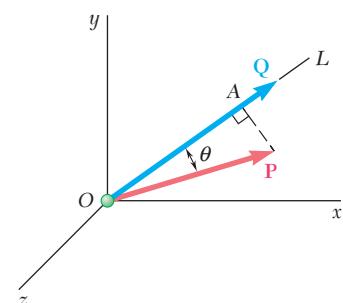


Fig. 3.22

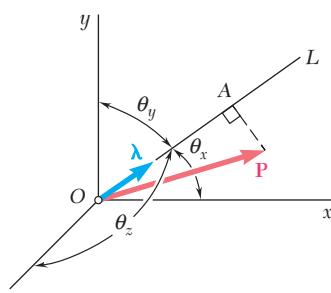


Fig. 3.23

from which it follows that

$$P_{OL} = \frac{\mathbf{P} \cdot \mathbf{Q}}{Q} = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{Q} \quad (3.35)$$

In the particular case when the vector selected along  $OL$  is the unit vector  $\lambda$  (Fig. 3.23), we write

$$P_{OL} = \mathbf{P} \cdot \lambda \quad (3.36)$$

Resolving  $\mathbf{P}$  and  $\lambda$  into rectangular components and recalling from Sec. 2.12 that the components of  $\lambda$  along the coordinate axes are respectively equal to the direction cosines of  $OL$ , we express the projection of  $\mathbf{P}$  on  $OL$  as

$$P_{OL} = P_x \cos \theta_x + P_y \cos \theta_y + P_z \cos \theta_z \quad (3.37)$$

where  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  denote the angles that the axis  $OL$  forms with the coordinate axes.

### 3.10 MIXED TRIPLE PRODUCT OF THREE VECTORS

We define the *mixed triple product* of the three vectors  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  as the scalar expression

$$\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) \quad (3.38)$$

obtained by forming the scalar product of  $\mathbf{S}$  with the vector product of  $\mathbf{P}$  and  $\mathbf{Q}$ .†

A simple geometrical interpretation can be given for the mixed triple product of  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  (Fig. 3.24). We first recall from Sec. 3.4 that the vector  $\mathbf{P} \times \mathbf{Q}$  is perpendicular to the plane containing  $\mathbf{P}$  and  $\mathbf{Q}$  and that its magnitude is equal to the area of the parallelogram which has  $\mathbf{P}$  and  $\mathbf{Q}$  for sides. On the other hand, Eq. (3.34) indicates that the scalar product of  $\mathbf{S}$  and  $\mathbf{P} \times \mathbf{Q}$  can be obtained by multiplying the magnitude of  $\mathbf{P} \times \mathbf{Q}$  (i.e., the area of the parallelogram defined by  $\mathbf{P}$  and  $\mathbf{Q}$ ) by the projection of  $\mathbf{S}$  on the vector  $\mathbf{P} \times \mathbf{Q}$  (i.e., by the projection of  $\mathbf{S}$  on the normal to the plane containing the parallelogram). The mixed triple product is thus equal, in absolute value, to the volume of the parallelepiped having the vectors  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  for sides (Fig. 3.25). We note that the sign of the mixed triple product will be positive if  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  form a right-handed triad and negative if they form a left-handed triad [that is,  $\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q})$  will be negative if the rotation which brings  $\mathbf{P}$  into line with  $\mathbf{Q}$  is observed as clockwise from the

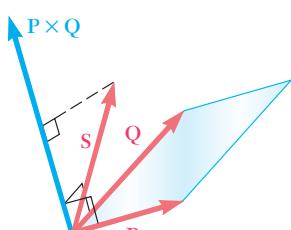


Fig. 3.24

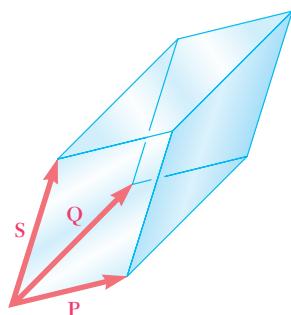


Fig. 3.25

†Another kind of triple product will be introduced later (Chap. 15): the *vector triple product*  $\mathbf{S} \times (\mathbf{P} \times \mathbf{Q})$ .

tip of  $\mathbf{S}$ ]. The mixed triple product will be zero if  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  are coplanar.

Since the parallelepiped defined in the preceding paragraph is independent of the order in which the three vectors are taken, the six mixed triple products which can be formed with  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  will all have the same absolute value, although not the same sign. It is easily shown that

$$\begin{aligned}\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) &= \mathbf{P} \cdot (\mathbf{Q} \times \mathbf{S}) = \mathbf{Q} \cdot (\mathbf{S} \times \mathbf{P}) \\ &= -\mathbf{S} \cdot (\mathbf{Q} \times \mathbf{P}) = -\mathbf{P} \cdot (\mathbf{S} \times \mathbf{Q}) = -\mathbf{Q} \cdot (\mathbf{P} \times \mathbf{S})\end{aligned}\quad (3.39)$$

Arranging in a circle and in counterclockwise order the letters representing the three vectors (Fig. 3.26), we observe that the sign of the mixed triple product remains unchanged if the vectors are permuted in such a way that they are still read in counterclockwise order. Such a permutation is said to be a *circular permutation*. It also follows from Eq. (3.39) and from the commutative property of scalar products that the mixed triple product of  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  can be defined equally well as  $\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q})$  or  $(\mathbf{S} \times \mathbf{P}) \cdot \mathbf{Q}$ .

The mixed triple product of the vectors  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  can be expressed in terms of the rectangular components of these vectors. Denoting  $\mathbf{P} \times \mathbf{Q}$  by  $\mathbf{V}$  and using formula (3.30) to express the scalar product of  $\mathbf{S}$  and  $\mathbf{V}$ , we write

$$\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) = \mathbf{S} \cdot \mathbf{V} = S_x V_x + S_y V_y + S_z V_z$$

Substituting from the relations (3.9) for the components of  $\mathbf{V}$ , we obtain

$$\begin{aligned}\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) &= S_x(P_y Q_z - P_z Q_y) + S_y(P_z Q_x - P_x Q_z) \\ &\quad + S_z(P_x Q_y - P_y Q_x)\end{aligned}\quad (3.40)$$

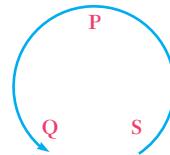
This expression can be written in a more compact form if we observe that it represents the expansion of a determinant:

$$\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad (3.41)$$

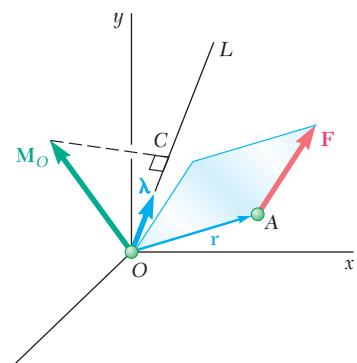
By applying the rules governing the permutation of rows in a determinant, we could easily verify the relations (3.39) which were derived earlier from geometrical considerations.

## 3.11 MOMENT OF A FORCE ABOUT A GIVEN AXIS

Now that we have further increased our knowledge of vector algebra, we can introduce a new concept, the concept of *moment of a force about an axis*. Consider again a force  $\mathbf{F}$  acting on a rigid body and the moment  $\mathbf{M}_O$  of that force about  $O$  (Fig. 3.27). Let  $OL$  be



**Fig. 3.26**



**Fig. 3.27**

an axis through  $O$ ; we define the moment  $M_{OL}$  of  $\mathbf{F}$  about  $OL$  as the projection  $OC$  of the moment  $\mathbf{M}_O$  onto the axis  $OL$ . Denoting by  $\lambda$  the unit vector along  $OL$  and recalling from Secs. 3.9 and 3.6, respectively, the expressions (3.36) and (3.11) obtained for the projection of a vector on a given axis and for the moment  $\mathbf{M}_O$  of a force  $\mathbf{F}$ , we write

$$M_{OL} = \lambda \cdot \mathbf{M}_O = \lambda \cdot (\mathbf{r} \times \mathbf{F}) \quad (3.42)$$

which shows that the moment  $M_{OL}$  of  $\mathbf{F}$  about the axis  $OL$  is the scalar obtained by forming the mixed triple product of  $\lambda$ ,  $\mathbf{r}$ , and  $\mathbf{F}$ . Expressing  $M_{OL}$  in the form of a determinant, we write

$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \quad (3.43)$$

where  $\lambda_x, \lambda_y, \lambda_z$  = direction cosines of axis  $OL$

$x, y, z$  = coordinates of point of application of  $\mathbf{F}$

$F_x, F_y, F_z$  = components of force  $\mathbf{F}$

The physical significance of the moment  $M_{OL}$  of a force  $\mathbf{F}$  about a fixed axis  $OL$  becomes more apparent if we resolve  $\mathbf{F}$  into two rectangular components  $\mathbf{F}_1$  and  $\mathbf{F}_2$ , with  $\mathbf{F}_1$  parallel to  $OL$  and  $\mathbf{F}_2$  lying in a plane  $P$  perpendicular to  $OL$  (Fig. 3.28). Resolving  $\mathbf{r}$  similarly into two components  $\mathbf{r}_1$  and  $\mathbf{r}_2$  and substituting for  $\mathbf{F}$  and  $\mathbf{r}$  into (3.42), we write

$$\begin{aligned} M_{OL} &= \lambda \cdot [(\mathbf{r}_1 + \mathbf{r}_2) \times (\mathbf{F}_1 + \mathbf{F}_2)] \\ &= \lambda \cdot (\mathbf{r}_1 \times \mathbf{F}_1) + \lambda \cdot (\mathbf{r}_1 \times \mathbf{F}_2) + \lambda \cdot (\mathbf{r}_2 \times \mathbf{F}_1) + \lambda \cdot (\mathbf{r}_2 \times \mathbf{F}_2) \end{aligned}$$

Noting that all of the mixed triple products except the last one are equal to zero, since they involve vectors which are coplanar when drawn from a common origin (Sec. 3.10), we have

$$M_{OL} = \lambda \cdot (\mathbf{r}_2 \times \mathbf{F}_2) \quad (3.44)$$

The vector product  $\mathbf{r}_2 \times \mathbf{F}_2$  is perpendicular to the plane  $P$  and represents the moment of the component  $\mathbf{F}_2$  of  $\mathbf{F}$  about the point  $Q$  where  $OL$  intersects  $P$ . Therefore, the scalar  $M_{OL}$ , which will be positive if  $\mathbf{r}_2 \times \mathbf{F}_2$  and  $OL$  have the same sense and negative otherwise, measures the tendency of  $\mathbf{F}_2$  to make the rigid body rotate about the fixed axis  $OL$ . Since the other component  $\mathbf{F}_1$  of  $\mathbf{F}$  does not tend to make the body rotate about  $OL$ , we conclude that *the moment  $M_{OL}$  of  $\mathbf{F}$  about  $OL$  measures the tendency of the force  $\mathbf{F}$  to impart to the rigid body a motion of rotation about the fixed axis  $OL$ .*

It follows from the definition of the moment of a force about an axis that the moment of  $\mathbf{F}$  about a coordinate axis is equal to the component of  $\mathbf{M}_O$  along that axis. Substituting successively each

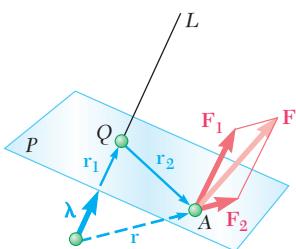


Fig. 3.28

of the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  for  $\boldsymbol{\lambda}$  in (3.42), we observe that the expressions thus obtained for the *moments of  $\mathbf{F}$  about the coordinate axes* are respectively equal to the expressions obtained in Sec. 3.8 for the components of the moment  $\mathbf{M}_O$  of  $\mathbf{F}$  about  $O$ :

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \quad (3.18)$$

We observe that just as the components  $F_x$ ,  $F_y$ , and  $F_z$  of a force  $\mathbf{F}$  acting on a rigid body measure, respectively, the tendency of  $\mathbf{F}$  to move the rigid body in the  $x$ ,  $y$ , and  $z$  directions, the moments  $M_x$ ,  $M_y$ , and  $M_z$  of  $\mathbf{F}$  about the coordinate axes measure the tendency of  $\mathbf{F}$  to impart to the rigid body a motion of rotation about the  $x$ ,  $y$ , and  $z$  axes, respectively.

More generally, the moment of a force  $\mathbf{F}$  applied at  $A$  about an axis which does not pass through the origin is obtained by choosing an arbitrary point  $B$  on the axis (Fig. 3.29) and determining the projection on the axis  $BL$  of the moment  $\mathbf{M}_B$  of  $\mathbf{F}$  about  $B$ . We write

$$M_{BL} = \boldsymbol{\lambda} \cdot \mathbf{M}_B = \boldsymbol{\lambda} \cdot (\mathbf{r}_{A/B} \times \mathbf{F}) \quad (3.45)$$

where  $\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B$  represents the vector drawn from  $B$  to  $A$ . Expressing  $M_{BL}$  in the form of a determinant, we have

$$M_{BL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \quad (3.46)$$

where  $\lambda_x, \lambda_y, \lambda_z$  = direction cosines of axis  $BL$

$$x_{A/B} = x_A - x_B \quad y_{A/B} = y_A - y_B \quad z_{A/B} = z_A - z_B$$

$F_x, F_y, F_z$  = components of force  $\mathbf{F}$

It should be noted that the result obtained is independent of the choice of the point  $B$  on the given axis. Indeed, denoting by  $M_{CL}$  the result obtained with a different point  $C$ , we have

$$\begin{aligned} M_{CL} &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_A - \mathbf{r}_C) \times \mathbf{F}] \\ &= \boldsymbol{\lambda} \cdot [(\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}] + \boldsymbol{\lambda} \cdot [(\mathbf{r}_B - \mathbf{r}_C) \times \mathbf{F}] \end{aligned}$$

But, since the vectors  $\boldsymbol{\lambda}$  and  $\mathbf{r}_B - \mathbf{r}_C$  lie in the same line, the volume of the parallelepiped having the vectors  $\boldsymbol{\lambda}$ ,  $\mathbf{r}_B - \mathbf{r}_C$ , and  $\mathbf{F}$  for sides is zero, as is the mixed triple product of these three vectors (Sec. 3.10). The expression obtained for  $M_{CL}$  thus reduces to its first term, which is the expression used earlier to define  $M_{BL}$ . In addition, it follows from Sec. 3.6 that, when computing the moment of  $\mathbf{F}$  about the given axis,  $A$  can be any point on the line of action of  $\mathbf{F}$ .

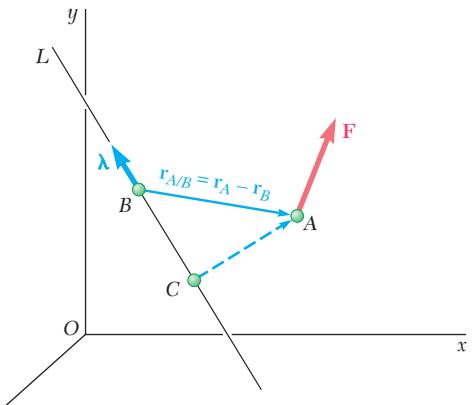
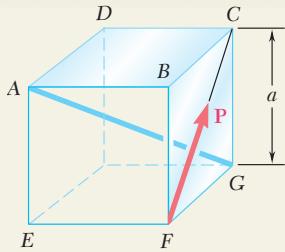


Fig. 3.29



## SAMPLE PROBLEM 3.5

A cube of side  $a$  is acted upon by a force  $\mathbf{P}$  as shown. Determine the moment of  $\mathbf{P}$  (a) about  $A$ , (b) about the edge  $AB$ , (c) about the diagonal  $AG$  of the cube, (d). Using the result of part c, determine the perpendicular distance between  $AG$  and  $FC$ .

### SOLUTION

**a. Moment about A.** Choosing  $x$ ,  $y$ , and  $z$  axes as shown, we resolve into rectangular components the force  $\mathbf{P}$  and the vector  $\mathbf{r}_{F/A} = \overrightarrow{AF}$  drawn from  $A$  to the point of application  $F$  of  $\mathbf{P}$ .

$$\begin{aligned}\mathbf{r}_{F/A} &= ai - aj = a(\mathbf{i} - \mathbf{j}) \\ \mathbf{P} &= (P/\sqrt{2})\mathbf{j} - (P/\sqrt{2})\mathbf{k} = (P/\sqrt{2})(\mathbf{j} - \mathbf{k})\end{aligned}$$

The moment of  $\mathbf{P}$  about  $A$  is

$$\begin{aligned}\mathbf{M}_A &= \mathbf{r}_{F/A} \times \mathbf{P} = a(\mathbf{i} - \mathbf{j}) \times (P/\sqrt{2})(\mathbf{j} - \mathbf{k}) \\ \mathbf{M}_A &= (ap/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k})\end{aligned}$$

**b. Moment about AB.** Projecting  $\mathbf{M}_A$  on  $AB$ , we write

$$\begin{aligned}M_{AB} &= \mathbf{i} \cdot \mathbf{M}_A = \mathbf{i} \cdot (ap/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ M_{AB} &= ap/\sqrt{2}\end{aligned}$$

We verify that, since  $AB$  is parallel to the  $x$  axis,  $M_{AB}$  is also the  $x$  component of the moment  $\mathbf{M}_A$ .

**c. Moment about Diagonal AG.** The moment of  $\mathbf{P}$  about  $AG$  is obtained by projecting  $\mathbf{M}_A$  on  $AG$ . Denoting by  $\lambda$  the unit vector along  $AG$ , we have

$$\lambda = \frac{\overrightarrow{AG}}{AG} = \frac{ai - aj - ak}{a\sqrt{3}} = (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k})$$

$$\begin{aligned}M_{AG} &= \lambda \cdot \mathbf{M}_A = (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (ap/\sqrt{2})(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ M_{AG} &= (ap/\sqrt{6})(1 - 1 - 1) \quad M_{AG} = -ap/\sqrt{6}\end{aligned}$$

**Alternative Method.** The moment of  $\mathbf{P}$  about  $AG$  can also be expressed in the form of a determinant:

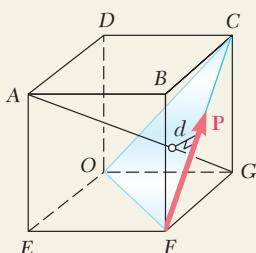
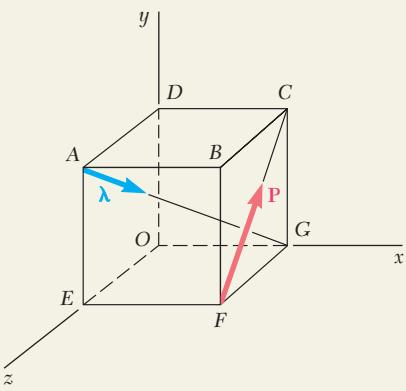
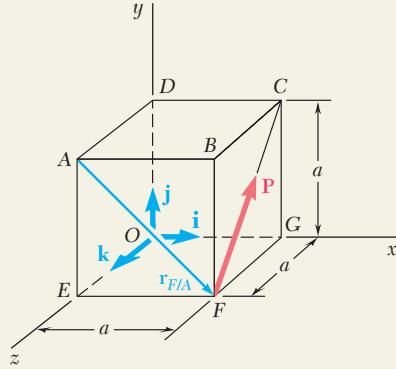
$$M_{AG} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{F/A} & y_{F/A} & z_{F/A} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ a & -a & 0 \\ 0 & P/\sqrt{2} & -P/\sqrt{2} \end{vmatrix} = -ap/\sqrt{6}$$

**d. Perpendicular Distance between AG and FC.** We first observe that  $\mathbf{P}$  is perpendicular to the diagonal  $AG$ . This can be checked by forming the scalar product  $\mathbf{P} \cdot \lambda$  and verifying that it is zero:

$$\mathbf{P} \cdot \lambda = (P/\sqrt{2})(\mathbf{j} - \mathbf{k}) \cdot (1/\sqrt{3})(\mathbf{i} - \mathbf{j} - \mathbf{k}) = (P\sqrt{6})(0 - 1 + 1) = 0$$

The moment  $M_{AG}$  can then be expressed as  $-Pd$ , where  $d$  is the perpendicular distance from  $AG$  to  $FC$ . (The negative sign is used since the rotation imparted to the cube by  $\mathbf{P}$  appears as clockwise to an observer at  $G$ .) Recalling the value found for  $M_{AG}$  in part c,

$$M_{AG} = -Pd = -ap/\sqrt{6} \quad d = a/\sqrt{6}$$



# SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson you will apply the *scalar product* or *dot product* of two vectors to determine the *angle formed by two given vectors* and the *projection of a force on a given axis*. You will also use the *mixed triple product* of three vectors to find the *moment of a force about a given axis* and the *perpendicular distance between two lines*.

**1. Calculating the angle formed by two given vectors.** First express the vectors in terms of their components and determine the magnitudes of the two vectors. The cosine of the desired angle is then obtained by dividing the scalar product of the two vectors by the product of their magnitudes [Eq. (3.32)].

**2. Computing the projection of a vector  $\mathbf{P}$  on a given axis  $OL$ .** In general, begin by expressing  $\mathbf{P}$  and the unit vector  $\lambda$ , that defines the direction of the axis, in component form. Take care that  $\lambda$  has the correct sense (that is,  $\lambda$  is directed from  $O$  to  $L$ ). The required projection is then equal to the scalar product  $\mathbf{P} \cdot \lambda$ . However, if you know the angle  $\theta$  formed by  $\mathbf{P}$  and  $\lambda$ , the projection is also given by  $P \cos \theta$ .

**3. Determining the moment  $M_{OL}$  of a force about a given axis  $OL$ .** We defined  $M_{OL}$  as

$$M_{OL} = \lambda \cdot \mathbf{M}_O = \lambda \cdot (\mathbf{r} \times \mathbf{F}) \quad (3.42)$$

where  $\lambda$  is the unit vector along  $OL$  and  $\mathbf{r}$  is a position vector *from any point* on the line  $OL$  *to any point* on the line of action of  $\mathbf{F}$ . As was the case for the moment of a force about a point, choosing the most convenient position vector will simplify your calculations. Also, recall the warning of the previous lesson: the vectors  $\mathbf{r}$  and  $\mathbf{F}$  must have the correct sense, and they must be placed in the proper order. The procedure you should follow when computing the moment of a force about an axis is illustrated in part *c* of Sample Prob. 3.5. The two essential steps in this procedure are to first express  $\lambda$ ,  $\mathbf{r}$ , and  $\mathbf{F}$  in terms of their rectangular components and to then evaluate the mixed triple product  $\lambda \cdot (\mathbf{r} \times \mathbf{F})$  to determine the moment about the axis. In most three-dimensional problems the most convenient way to compute the mixed triple product is by using a determinant.

As noted in the text, when  $\lambda$  is directed along one of the coordinate axes,  $M_{OL}$  is equal to the scalar component of  $\mathbf{M}_O$  along that axis.

(continued)

**4. Determining the perpendicular distance between two lines.** You should remember that it is the perpendicular component  $\mathbf{F}_2$  of the force  $\mathbf{F}$  that tends to make a body rotate about a given axis  $OL$  (Fig. 3.28). It then follows that

$$M_{OL} = F_2 d$$

where  $M_{OL}$  is the moment of  $\mathbf{F}$  about axis  $OL$  and  $d$  is the perpendicular distance between  $OL$  and the line of action of  $\mathbf{F}$ . This last equation gives us a simple technique for determining  $d$ . First assume that a force  $\mathbf{F}$  of known magnitude  $F$  lies along one of the given lines and that the unit vector  $\lambda$  lies along the other line. Next compute the moment  $M_{OL}$  of the force  $\mathbf{F}$  about the second line using the method discussed above. The magnitude of the parallel component,  $F_1$ , of  $\mathbf{F}$  is obtained using the scalar product:

$$F_1 = \mathbf{F} \cdot \lambda$$

The value of  $F_2$  is then determined from

$$F_2 = \sqrt{F^2 - F_1^2}$$

Finally, substitute the values of  $M_{OL}$  and  $F_2$  into the equation  $M_{OL} = F_2 d$  and solve for  $d$ .

You should now realize that the calculation of the perpendicular distance in part d of Sample Prob. 3.5 was simplified by  $\mathbf{P}$  being perpendicular to the diagonal  $AG$ . In general, the two given lines will not be perpendicular, so that the technique just outlined will have to be used when determining the perpendicular distance between them.

# PROBLEMS

- 3.35** Given the vectors  $\mathbf{P} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{Q} = 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ , and  $\mathbf{S} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ , compute the scalar products  $\mathbf{P} \cdot \mathbf{Q}$ ,  $\mathbf{P} \cdot \mathbf{S}$ , and  $\mathbf{Q} \cdot \mathbf{S}$ .

- 3.36** Form the scalar products  $\mathbf{B} \cdot \mathbf{C}$  and  $\mathbf{B}' \cdot \mathbf{C}$ , where  $B = B'$ , and use the results obtained to prove the identity

$$\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta).$$

- 3.37** Section  $AB$  of a pipeline lies in the  $yz$  plane and forms an angle of  $37^\circ$  with the  $z$  axis. Branch lines  $CD$  and  $EF$  join  $AB$  as shown. Determine the angle formed by pipes  $AB$  and  $CD$ .

- 3.38** Section  $AB$  of a pipeline lies in the  $yz$  plane and forms an angle of  $37^\circ$  with the  $z$  axis. Branch lines  $CD$  and  $EF$  join  $AB$  as shown. Determine the angle formed by pipes  $AB$  and  $EF$ .

- 3.39** Consider the volleyball net shown. Determine the angle formed by guy wires  $AB$  and  $AC$ .

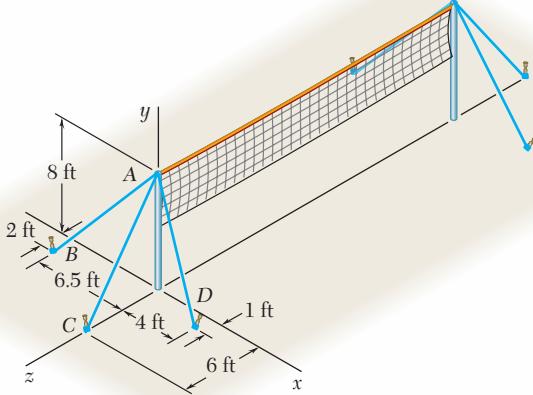


Fig. P3.39 and P3.40

- 3.40** Consider the volleyball net shown. Determine the angle formed by guy wires  $AC$  and  $AD$ .

- 3.41** Knowing that the tension in cable  $AC$  is 1260 N, determine (a) the angle between cable  $AC$  and the boom  $AB$ , (b) the projection on  $AB$  of the force exerted by cable  $AC$  at point  $A$ .

- 3.42** Knowing that the tension in cable  $AD$  is 405 N, determine (a) the angle between cable  $AD$  and the boom  $AB$ , (b) the projection on  $AB$  of the force exerted by cable  $AD$  at point  $A$ .

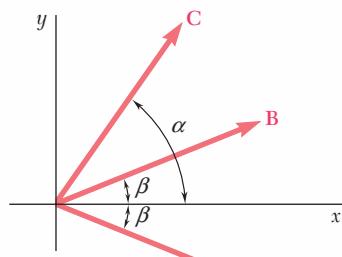


Fig. P3.36

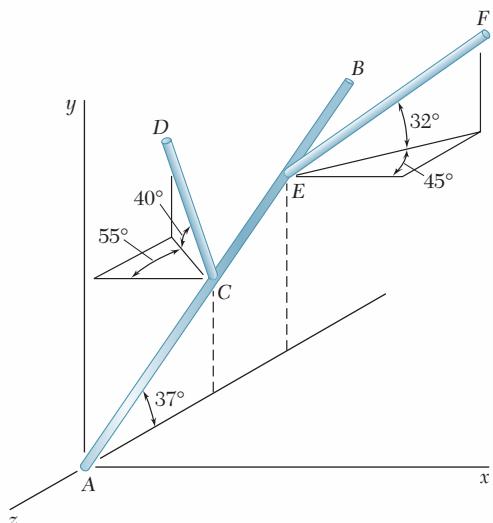


Fig. P3.37 and P3.38

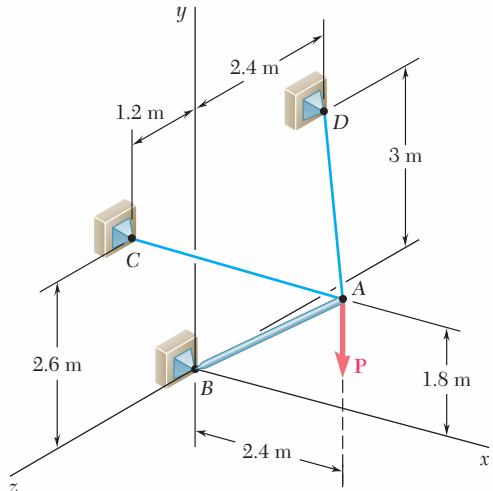


Fig. P3.41 and P3.42

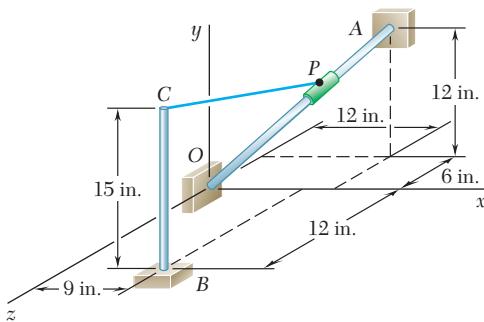


Fig. P3.43 and P3.44

- 3.43** Slider  $P$  can move along rod  $OA$ . An elastic cord  $PC$  is attached to the slider and to the vertical member  $BC$ . Knowing that the distance from  $O$  to  $P$  is 6 in. and that the tension in the cord is 3 lb, determine (a) the angle between the elastic cord and the rod  $OA$ , (b) the projection on  $OA$  of the force exerted by cord  $PC$  at point  $P$ .

- 3.44** Slider  $P$  can move along rod  $OA$ . An elastic cord  $PC$  is attached to the slider and to the vertical member  $BC$ . Determine the distance from  $O$  to  $P$  for which cord  $PC$  and rod  $OA$  are perpendicular.

- 3.45** Determine the volume of the parallelepiped of Fig. 3.25 when (a)  $\mathbf{P} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{Q} = -2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ , and  $\mathbf{S} = 7\mathbf{i} + \mathbf{j} - \mathbf{k}$ , (b)  $\mathbf{P} = 5\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ ,  $\mathbf{Q} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ , and  $\mathbf{S} = -3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ .

- 3.46** Given the vectors  $\mathbf{P} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{Q} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ , and  $\mathbf{S} = S_x\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ , determine the value of  $S_x$  for which the three vectors are coplanar.

- 3.47** The  $0.61 \times 1.00 of a storage bin is hinged along side  $AB$  and is held open by looping cord  $DEC$  over a frictionless hook at  $E$ . If the tension in the cord is 66 N, determine the moment about each of the coordinate axes of the force exerted by the cord at  $D$ .$

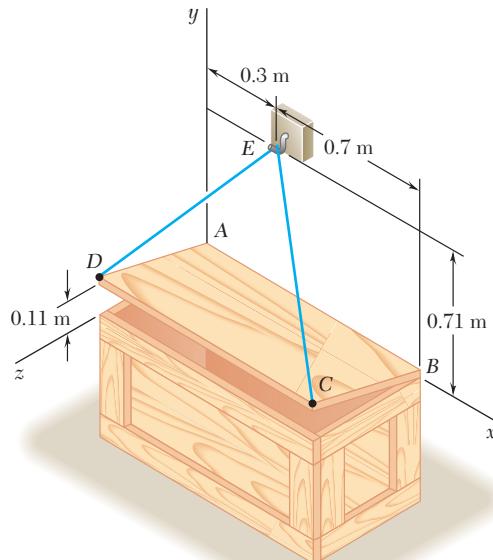


Fig. P3.47 and P3.48

- 3.48** The  $0.61 \times 1.00 of a storage bin is hinged along side  $AB$  and is held open by looping cord  $DEC$  over a frictionless hook at  $E$ . If the tension in the cord is 66 N, determine the moment about each of the coordinate axes of the force exerted by the cord at  $C$ .$

- 3.49** To lift a heavy crate, a man uses a block and tackle attached to the bottom of an I-beam at hook *B*. Knowing that the moments about the *y* and the *z* axes of the force exerted at *B* by portion *AB* of the rope are, respectively,  $120 \text{ N} \cdot \text{m}$  and  $-460 \text{ N} \cdot \text{m}$ , determine the distance *a*.

- 3.50** To lift a heavy crate, a man uses a block and tackle attached to the bottom of an I-beam at hook *B*. Knowing that the man applies a  $195\text{-N}$  force to end *A* of the rope and that the moment of that force about the *y* axis is  $132 \text{ N} \cdot \text{m}$ , determine the distance *a*.

- 3.51** A small boat hangs from two davits, one of which is shown in the figure. It is known that the moment about the *z* axis of the resultant force  $\mathbf{R}_A$  exerted on the davit at *A* must not exceed  $279 \text{ lb} \cdot \text{ft}$  in absolute value. Determine the largest allowable tension in line *ABAD* when  $x = 6 \text{ ft}$ .

- 3.52** For the davit of Prob. 3.51, determine the largest allowable distance *x* when the tension in line *ABAD* is  $60 \text{ lb}$ .

- 3.53** To loosen a frozen valve, a force  $\mathbf{F}$  of magnitude  $70 \text{ lb}$  is applied to the handle of the valve. Knowing that  $\theta = 25^\circ$ ,  $M_x = -61 \text{ lb} \cdot \text{ft}$ , and  $M_z = -43 \text{ lb} \cdot \text{ft}$ , determine  $\phi$  and  $d$ .

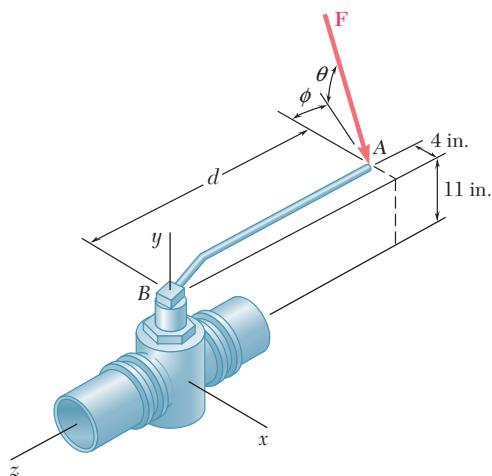


Fig. P3.53 and P3.54

- 3.54** When a force  $\mathbf{F}$  is applied to the handle of the valve shown, its moments about the *x* and *z* axes are, respectively,  $M_x = -77 \text{ lb} \cdot \text{ft}$  and  $M_z = -81 \text{ lb} \cdot \text{ft}$ . For  $d = 27 \text{ in.}$ , determine the moment  $M_y$  of  $\mathbf{F}$  about the *y* axis.

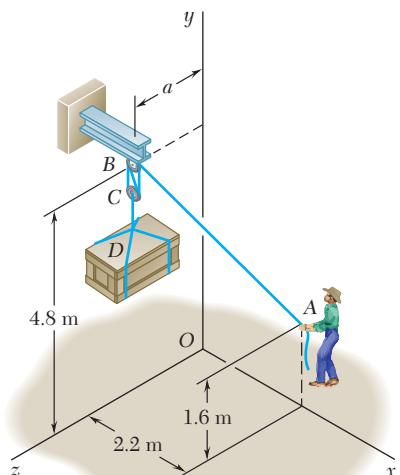


Fig. P3.49 and P3.50

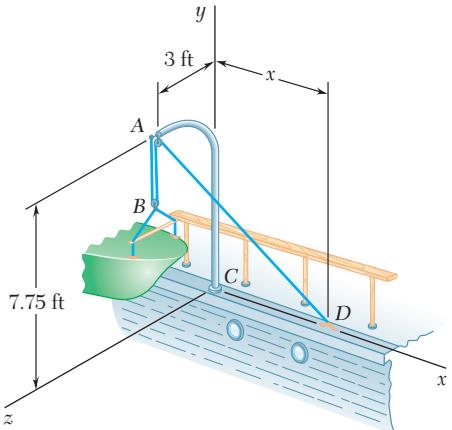
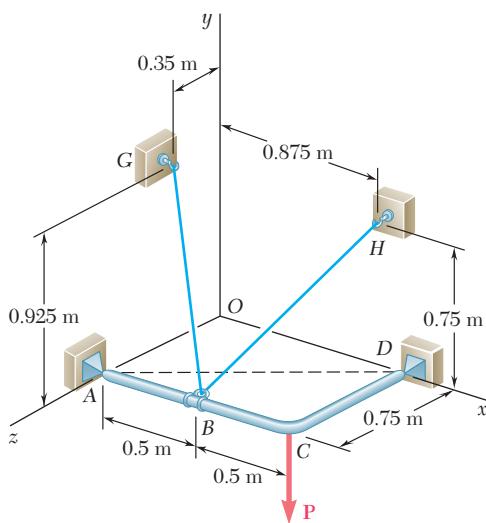


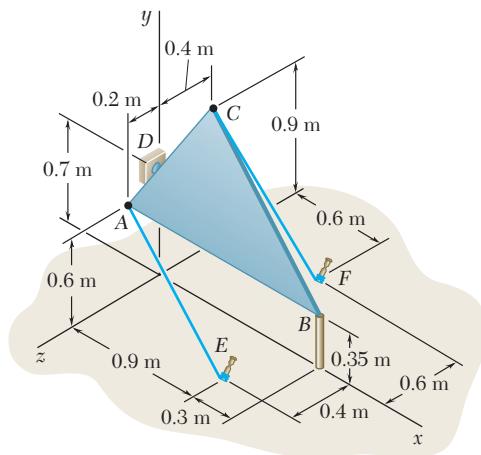
Fig. P3.51

**Fig. P3.55**

- 3.55** The frame  $ACD$  is hinged at  $A$  and  $D$  and is supported by a cable that passes through a ring at  $B$  and is attached to hooks at  $G$  and  $H$ . Knowing that the tension in the cable is 450 N, determine the moment about the diagonal  $AD$  of the force exerted on the frame by portion  $BH$  of the cable.

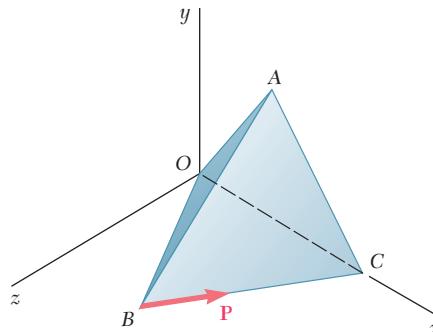
- 3.56** In Prob. 3.55, determine the moment about the diagonal  $AD$  of the force exerted on the frame by portion  $BG$  of the cable.

- 3.57** The triangular plate  $ABC$  is supported by ball-and-socket joints at  $B$  and  $D$  and is held in the position shown by cables  $AE$  and  $CF$ . If the force exerted by cable  $AE$  at  $A$  is 55 N, determine the moment of that force about the line joining points  $D$  and  $B$ .

**Fig. P3.57 and P3.58**

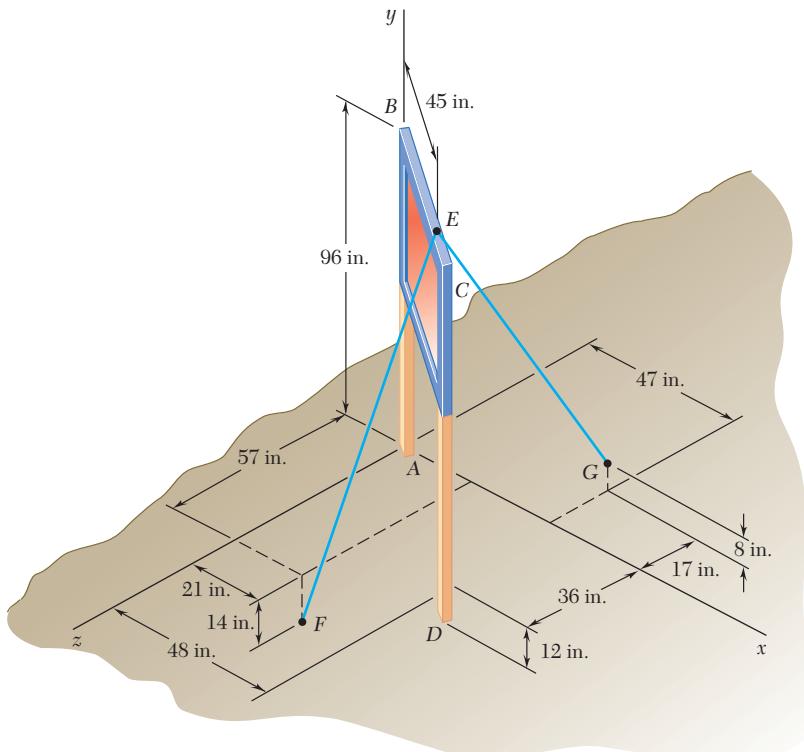
- 3.58** The triangular plate  $ABC$  is supported by ball-and-socket joints at  $B$  and  $D$  and is held in the position shown by cables  $AE$  and  $CF$ . If the force exerted by cable  $CF$  at  $C$  is 33 N, determine the moment of that force about the line joining points  $D$  and  $B$ .

- 3.59** A regular tetrahedron has six edges of length  $a$ . A force  $\mathbf{P}$  is directed as shown along edge  $BC$ . Determine the moment of  $\mathbf{P}$  about edge  $OA$ .

**Fig. P3.59 and P3.60**

- 3.60** A regular tetrahedron has six edges of length  $a$ . (a) Show that two opposite edges, such as  $OA$  and  $BC$ , are perpendicular to each other. (b) Use this property and the result obtained in Prob. 3.59 to determine the perpendicular distance between edges  $OA$  and  $BC$ .

- 3.61** A sign erected on uneven ground is guyed by cables  $EF$  and  $EG$ . If the force exerted by cable  $EF$  at  $E$  is 46 lb, determine the moment of that force about the line joining points  $A$  and  $D$ .



**Fig. P3.61 and P3.62**

- 3.62** A sign erected on uneven ground is guyed by cables  $EF$  and  $EG$ . If the force exerted by cable  $EG$  at  $E$  is 54 lb, determine the moment of that force about the line joining points  $A$  and  $D$ .

- 3.63** Two forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  in space have the same magnitude  $F$ . Prove that the moment of  $\mathbf{F}_1$  about the line of action of  $\mathbf{F}_2$  is equal to the moment of  $\mathbf{F}_2$  about the line of action of  $\mathbf{F}_1$ .

- \*3.64** In Prob. 3.55, determine the perpendicular distance between portion  $BH$  of the cable and the diagonal  $AD$ .

- \*3.65** In Prob. 3.56, determine the perpendicular distance between portion  $BG$  of the cable and the diagonal  $AD$ .

- \*3.66** In Prob. 3.57, determine the perpendicular distance between cable  $AE$  and the line joining points  $D$  and  $B$ .

- \*3.67** In Prob. 3.58, determine the perpendicular distance between cable  $CF$  and the line joining points  $D$  and  $B$ .

- \*3.68** In Prob. 3.61, determine the perpendicular distance between cable  $EF$  and the line joining points  $A$  and  $D$ .

- \*3.69** In Prob. 3.62, determine the perpendicular distance between cable  $EG$  and the line joining points  $A$  and  $D$ .

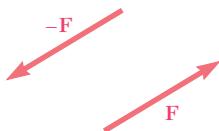


Fig. 3.30

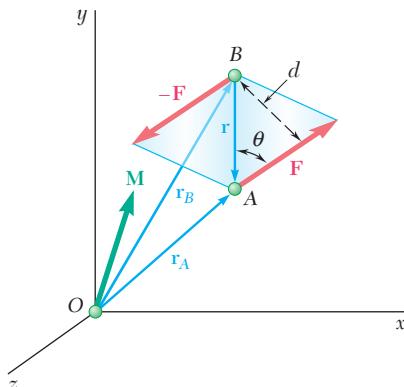


Fig. 3.31

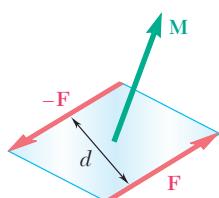


Fig. 3.32



**Photo 3.1** The parallel upward and downward forces of equal magnitude exerted on the arms of the lug nut wrench are an example of a couple.

## 3.12 MOMENT OF A COUPLE

Two forces  $\mathbf{F}$  and  $-\mathbf{F}$  having the same magnitude, parallel lines of action, and opposite sense are said to form a couple (Fig. 3.30). Clearly, the sum of the components of the two forces in any direction is zero. The sum of the moments of the two forces about a given point, however, is not zero. While the two forces will not translate the body on which they act, they will tend to make it rotate.

Denoting by  $\mathbf{r}_A$  and  $\mathbf{r}_B$ , respectively, the position vectors of the points of application of  $\mathbf{F}$  and  $-\mathbf{F}$  (Fig. 3.31), we find that the sum of the moments of the two forces about  $O$  is

$$\mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

Setting  $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$ , where  $\mathbf{r}$  is the vector joining the points of application of the two forces, we conclude that the sum of the moments of  $\mathbf{F}$  and  $-\mathbf{F}$  about  $O$  is represented by the vector

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \quad (3.47)$$

The vector  $\mathbf{M}$  is called the *moment of the couple*; it is a vector perpendicular to the plane containing the two forces, and its magnitude is

$$M = rF \sin \theta = Fd \quad (3.48)$$

where  $d$  is the perpendicular distance between the lines of action of  $\mathbf{F}$  and  $-\mathbf{F}$ . The sense of  $\mathbf{M}$  is defined by the right-hand rule.

Since the vector  $\mathbf{r}$  in (3.47) is independent of the choice of the origin  $O$  of the coordinate axes, we note that the same result would have been obtained if the moments of  $\mathbf{F}$  and  $-\mathbf{F}$  had been computed about a different point  $O'$ . Thus, the moment  $\mathbf{M}$  of a couple is a *free vector* (Sec. 2.3) which can be applied at any point (Fig. 3.32).

From the definition of the moment of a couple, it also follows that two couples, one consisting of the forces  $\mathbf{F}_1$  and  $-\mathbf{F}_1$ , the other of the forces  $\mathbf{F}_2$  and  $-\mathbf{F}_2$  (Fig. 3.33), will have equal moments if

$$F_1 d_1 = F_2 d_2 \quad (3.49)$$

and if the two couples lie in parallel planes (or in the same plane) and have the same sense.

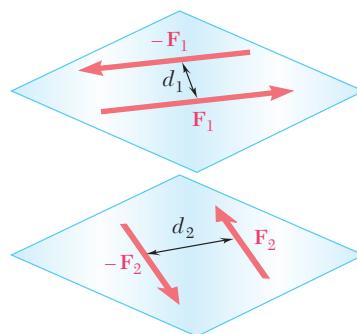
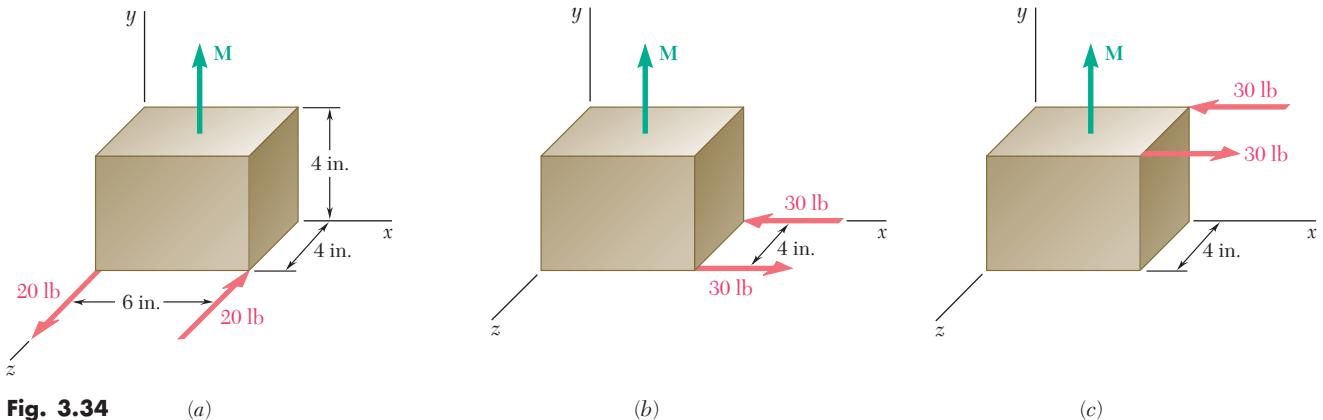


Fig. 3.33

### 3.13 EQUIVALENT COUPLES

Figure 3.34 shows three couples which act successively on the same rectangular box. As seen in the preceding section, the only motion a couple can impart to a rigid body is a rotation. Since each of the three couples shown has the same moment  $\mathbf{M}$  (same direction and same magnitude  $M = 120 \text{ lb} \cdot \text{in.}$ ), we can expect the three couples to have the same effect on the box.



**Fig. 3.34**

(a)

(b)

(c)

As reasonable as this conclusion appears, we should not accept it hastily. While intuitive feeling is of great help in the study of mechanics, it should not be accepted as a substitute for logical reasoning. Before stating that two systems (or groups) of forces have the same effect on a rigid body, we should prove that fact on the basis of the experimental evidence introduced so far. This evidence consists of the parallelogram law for the addition of two forces (Sec. 2.2) and the principle of transmissibility (Sec. 3.3). Therefore, we will state that *two systems of forces are equivalent* (i.e., they have the same effect on a rigid body) if we can transform one of them into the other by means of one or several of the following operations: (1) replacing two forces acting on the same particle by their resultant; (2) resolving a force into two components; (3) canceling two equal and opposite forces acting on the same particle; (4) attaching to the same particle two equal and opposite forces; (5) moving a force along its line of action. Each of these operations is easily justified on the basis of the parallelogram law or the principle of transmissibility.

Let us now prove that *two couples having the same moment  $\mathbf{M}$  are equivalent*. First consider two couples contained in the same plane, and assume that this plane coincides with the plane of the figure (Fig. 3.35). The first couple consists of the forces  $\mathbf{F}_1$  and  $-\mathbf{F}_1$  of magnitude  $F_1$ , which are located at a distance  $d_1$  from each other (Fig. 3.35a), and the second couple consists of the forces  $\mathbf{F}_2$  and  $-\mathbf{F}_2$  of magnitude  $F_2$ , which are located at a distance  $d_2$  from each other (Fig. 3.35d). Since the two couples have the same moment  $\mathbf{M}$ , which is perpendicular to the plane of the figure, they must have the same sense (assumed here to be counterclockwise), and the relation

$$F_1 d_1 = F_2 d_2 \quad (3.49)$$

must be satisfied. To prove that they are equivalent, we shall show that the first couple can be transformed into the second by means of the operations listed above.

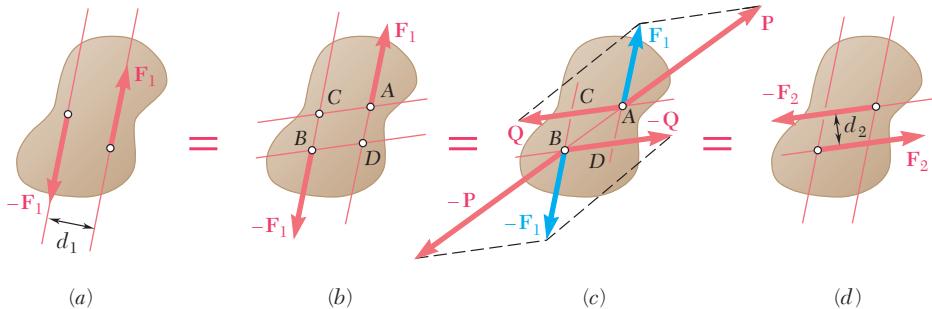


Fig. 3.35

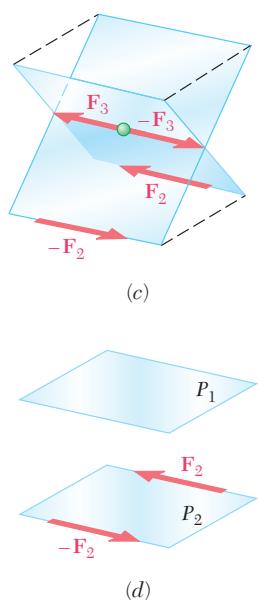
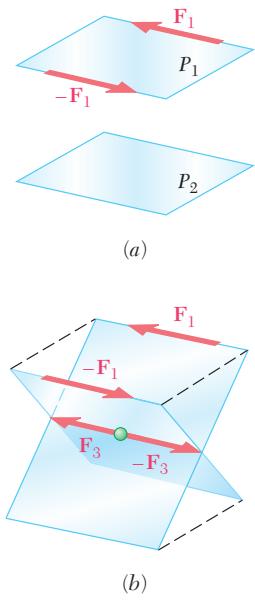


Fig. 3.36

Denoting by  $A$ ,  $B$ ,  $C$ , and  $D$  the points of intersection of the lines of action of the two couples, we first slide the forces  $\mathbf{F}_1$  and  $-\mathbf{F}_1$  until they are attached, respectively, at  $A$  and  $B$ , as shown in Fig. 3.35b. The force  $\mathbf{F}_1$  is then resolved into a component  $\mathbf{P}$  along line  $AB$  and a component  $\mathbf{Q}$  along  $AC$  (Fig. 3.35c); similarly, the force  $-\mathbf{F}_1$  is resolved into  $-\mathbf{P}$  along  $AB$  and  $-\mathbf{Q}$  along  $BD$ . The forces  $\mathbf{P}$  and  $-\mathbf{P}$  have the same magnitude, the same line of action, and opposite sense; they can be moved along their common line of action until they are applied at the same point and may then be canceled. Thus the couple formed by  $\mathbf{F}_1$  and  $-\mathbf{F}_1$  reduces to a couple consisting of  $\mathbf{Q}$  and  $-\mathbf{Q}$ .

We will now show that the forces  $\mathbf{Q}$  and  $-\mathbf{Q}$  are respectively equal to the forces  $-\mathbf{F}_2$  and  $\mathbf{F}_2$ . The moment of the couple formed by  $\mathbf{Q}$  and  $-\mathbf{Q}$  can be obtained by computing the moment of  $\mathbf{Q}$  about  $B$ ; similarly, the moment of the couple formed by  $\mathbf{F}_1$  and  $-\mathbf{F}_1$  is the moment of  $\mathbf{F}_1$  about  $B$ . But, by Varignon's theorem, the moment of  $\mathbf{F}_1$  is equal to the sum of the moments of its components  $\mathbf{P}$  and  $\mathbf{Q}$ . Since the moment of  $\mathbf{P}$  about  $B$  is zero, the moment of the couple formed by  $\mathbf{Q}$  and  $-\mathbf{Q}$  must be equal to the moment of the couple formed by  $\mathbf{F}_1$  and  $-\mathbf{F}_1$ . Recalling (3.49), we write

$$Qd_2 = F_1d_1 = F_2d_2 \quad \text{and} \quad Q = F_2$$

Thus the forces  $\mathbf{Q}$  and  $-\mathbf{Q}$  are respectively equal to the forces  $-\mathbf{F}_2$  and  $\mathbf{F}_2$ , and the couple of Fig. 3.35a is equivalent to the couple of Fig. 3.35d.

Next consider two couples contained in parallel planes  $P_1$  and  $P_2$ ; we will prove that they are equivalent if they have the same moment. In view of the foregoing, we can assume that the couples consist of forces of the same magnitude  $F$  acting along parallel lines (Fig. 3.36a and d). We propose to show that the couple contained in plane  $P_1$  can be transformed into the couple contained in plane  $P_2$  by means of the standard operations listed above.

Let us consider the two planes defined respectively by the lines of action of  $\mathbf{F}_1$  and  $-\mathbf{F}_2$  and by those of  $-\mathbf{F}_1$  and  $\mathbf{F}_2$  (Fig. 3.36b). At a point on their line of intersection we attach two forces  $\mathbf{F}_3$  and  $-\mathbf{F}_3$ , respectively equal to  $\mathbf{F}_1$  and  $-\mathbf{F}_1$ . The couple formed by  $\mathbf{F}_1$  and  $-\mathbf{F}_3$  can be replaced by a couple consisting of  $\mathbf{F}_3$  and  $-\mathbf{F}_2$  (Fig. 3.36c), since both couples clearly have the same moment and are contained in the same plane. Similarly, the couple formed by  $-\mathbf{F}_1$  and  $\mathbf{F}_3$  can be replaced by a couple consisting of  $-\mathbf{F}_3$  and  $\mathbf{F}_2$ . Canceling the two equal and opposite forces  $\mathbf{F}_3$  and  $-\mathbf{F}_3$ , we obtain the desired couple in plane  $P_2$  (Fig. 3.36d). Thus, we conclude that two couples having

the same moment  $\mathbf{M}$  are equivalent, whether they are contained in the same plane or in parallel planes.

The property we have just established is very important for the correct understanding of the mechanics of rigid bodies. It indicates that when a couple acts on a rigid body, it does not matter where the two forces forming the couple act or what magnitude and direction they have. The only thing which counts is the *moment* of the couple (magnitude and direction). Couples with the same moment will have the same effect on the rigid body.

### 3.14 ADDITION OF COUPLES

Consider two intersecting planes  $P_1$  and  $P_2$  and two couples acting respectively in  $P_1$  and  $P_2$ . We can, without any loss of generality, assume that the couple in  $P_1$  consists of two forces  $\mathbf{F}_1$  and  $-\mathbf{F}_1$  perpendicular to the line of intersection of the two planes and acting respectively at  $A$  and  $B$  (Fig. 3.37a). Similarly, we assume that the couple in  $P_2$  consists of two forces  $\mathbf{F}_2$  and  $-\mathbf{F}_2$  perpendicular to  $AB$  and acting respectively at  $A$  and  $B$ . It is clear that the resultant  $\mathbf{R}$  of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  and the resultant  $-\mathbf{R}$  of  $-\mathbf{F}_1$  and  $-\mathbf{F}_2$  form a couple. Denoting by  $\mathbf{r}$  the vector joining  $B$  to  $A$  and recalling the definition of the moment of a couple (Sec. 3.12), we express the moment  $\mathbf{M}$  of the resulting couple as follows:

$$\mathbf{M} = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2)$$

and, by Varignon's theorem,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

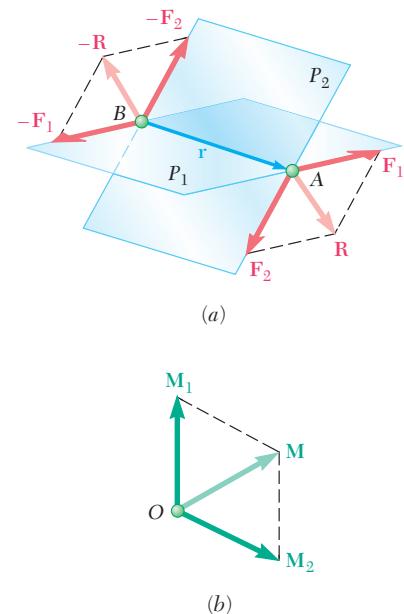
But the first term in the expression obtained represents the moment  $\mathbf{M}_1$  of the couple in  $P_1$ , and the second term represents the moment  $\mathbf{M}_2$  of the couple in  $P_2$ . We have

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 \quad (3.50)$$

and we conclude that the sum of two couples of moments  $\mathbf{M}_1$  and  $\mathbf{M}_2$  is a couple of moment  $\mathbf{M}$  equal to the vector sum of  $\mathbf{M}_1$  and  $\mathbf{M}_2$  (Fig. 3.37b).

### 3.15 COUPLES CAN BE REPRESENTED BY VECTORS

As we saw in Sec. 3.13, couples which have the same moment, whether they act in the same plane or in parallel planes, are equivalent. There is therefore no need to draw the actual forces forming a given couple in order to define its effect on a rigid body (Fig. 3.38a). It is sufficient to draw an arrow equal in magnitude and direction to the moment  $\mathbf{M}$  of the couple (Fig. 3.38b). On the other hand, we saw in Sec. 3.14 that the sum of two couples is itself a couple and that the moment  $\mathbf{M}$  of the resultant couple can be obtained by forming the vector sum of the moments  $\mathbf{M}_1$  and  $\mathbf{M}_2$  of the given couples. Thus, couples obey the law of addition of vectors, and the arrow used in Fig. 3.38b to represent the couple defined in Fig. 3.38a can truly be considered a vector.



**Fig. 3.37**

The vector representing a couple is called a *couple vector*. Note that, in Fig. 3.38, a red arrow is used to distinguish the couple vector, which represents the couple itself, from the moment of the couple, which was represented by a green arrow in earlier figures. Also note that the symbol  $\uparrow$  is added to this red arrow to avoid any confusion with vectors representing forces. A couple vector, like the moment of a couple, is a free vector. Its point of application, therefore, can be chosen at the origin of the system of coordinates, if so desired (Fig. 3.38c). Furthermore, the couple vector  $\mathbf{M}$  can be resolved into component vectors  $\mathbf{M}_x$ ,  $\mathbf{M}_y$ , and  $\mathbf{M}_z$ , which are directed along the coordinate axes (Fig. 3.38d). These component vectors represent couples acting, respectively, in the  $yz$ ,  $zx$ , and  $xy$  planes.

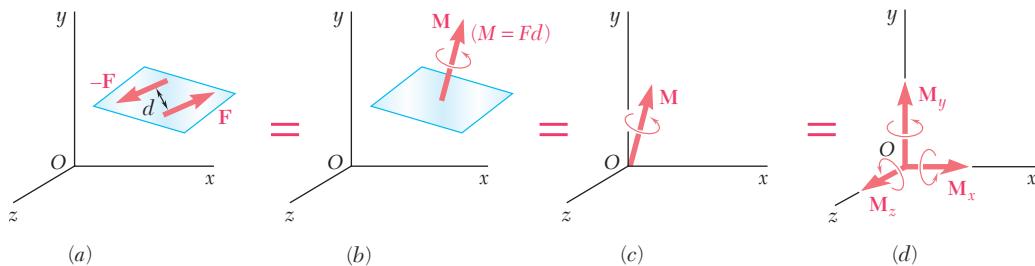


Fig. 3.38

### 3.16 RESOLUTION OF A GIVEN FORCE INTO A FORCE AT O AND A COUPLE

Consider a force  $\mathbf{F}$  acting on a rigid body at a point  $A$  defined by the position vector  $\mathbf{r}$  (Fig. 3.39a). Suppose that for some reason we would rather have the force act at point  $O$ . While we can move  $\mathbf{F}$  along its line of action (principle of transmissibility), we cannot move it to a point  $O$  which does not lie on the original line of action without modifying the action of  $\mathbf{F}$  on the rigid body.

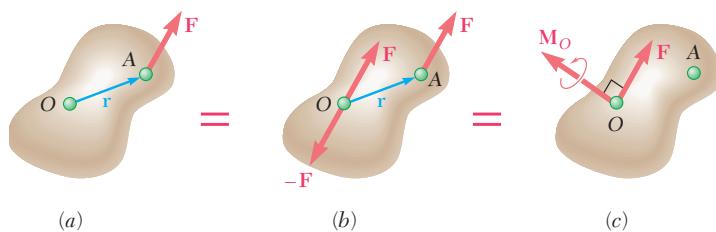


Fig. 3.39

We can, however, attach two forces at point  $O$ , one equal to  $\mathbf{F}$  and the other equal to  $-\mathbf{F}$ , without modifying the action of the original force on the rigid body (Fig. 3.39b). As a result of this transformation, a force  $\mathbf{F}$  is now applied at  $O$ ; the other two forces form a couple of moment  $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$ . Thus, *any force  $\mathbf{F}$  acting on a rigid body can be moved to an arbitrary point  $O$  provided that a couple is added whose moment is equal to the moment of  $\mathbf{F}$  about  $O$* . The

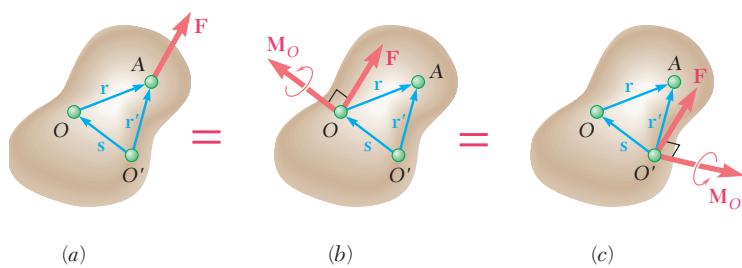
couple tends to impart to the rigid body the same rotational motion about  $O$  that the force  $\mathbf{F}$  tended to produce before it was transferred to  $O$ . The couple is represented by a couple vector  $\mathbf{M}_O$  perpendicular to the plane containing  $\mathbf{r}$  and  $\mathbf{F}$ . Since  $\mathbf{M}_O$  is a free vector, it may be applied anywhere; for convenience, however, the couple vector is usually attached at  $O$ , together with  $\mathbf{F}$ , and the combination obtained is referred to as a *force-couple system* (Fig. 3.39c).

If the force  $\mathbf{F}$  had been moved from  $A$  to a different point  $O'$  (Fig. 3.40a and c), the moment  $\mathbf{M}_{O'} = \mathbf{r}' \times \mathbf{F}$  of  $\mathbf{F}$  about  $O'$  should have been computed, and a new force-couple system, consisting of  $\mathbf{F}$  and of the couple vector  $\mathbf{M}_{O'}$ , would have been attached at  $O'$ . The relation existing between the moments of  $\mathbf{F}$  about  $O$  and  $O'$  is obtained by writing

$$\mathbf{M}_{O'} = \mathbf{r}' \times \mathbf{F} = (\mathbf{r} + \mathbf{s}) \times \mathbf{F} = \mathbf{r} \times \mathbf{F} + \mathbf{s} \times \mathbf{F}$$

$$\mathbf{M}_{O'} = \mathbf{M}_O + \mathbf{s} \times \mathbf{F} \quad (3.51)$$

where  $\mathbf{s}$  is the vector joining  $O'$  to  $O$ . Thus, the moment  $\mathbf{M}_{O'}$  of  $\mathbf{F}$  about  $O'$  is obtained by adding to the moment  $\mathbf{M}_O$  of  $\mathbf{F}$  about  $O$  the vector product  $\mathbf{s} \times \mathbf{F}$  representing the moment about  $O'$  of the force  $\mathbf{F}$  applied at  $O$ .



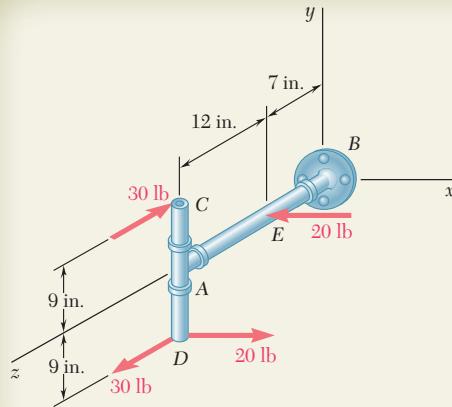
**Fig. 3.40**

This result could also have been established by observing that, in order to transfer to  $O'$  the force-couple system attached at  $O$  (Fig. 3.40b and c), the couple vector  $\mathbf{M}_O$  can be freely moved to  $O'$ ; to move the force  $\mathbf{F}$  from  $O$  to  $O'$ , however, it is necessary to add to  $\mathbf{F}$  a couple vector whose moment is equal to the moment about  $O'$  of the force  $\mathbf{F}$  applied at  $O$ . Thus, the couple vector  $\mathbf{M}_{O'}$  must be the sum of  $\mathbf{M}_O$  and the vector  $\mathbf{s} \times \mathbf{F}$ .

As noted above, the force-couple system obtained by transferring a force  $\mathbf{F}$  from a point  $A$  to a point  $O$  consists of  $\mathbf{F}$  and a couple vector  $\mathbf{M}_O$  perpendicular to  $\mathbf{F}$ . Conversely, any force-couple system consisting of a force  $\mathbf{F}$  and a couple vector  $\mathbf{M}_O$  which are *mutually perpendicular* can be replaced by a single equivalent force. This is done by moving the force  $\mathbf{F}$  in the plane perpendicular to  $\mathbf{M}_O$  until its moment about  $O$  is equal to the moment of the couple to be eliminated.

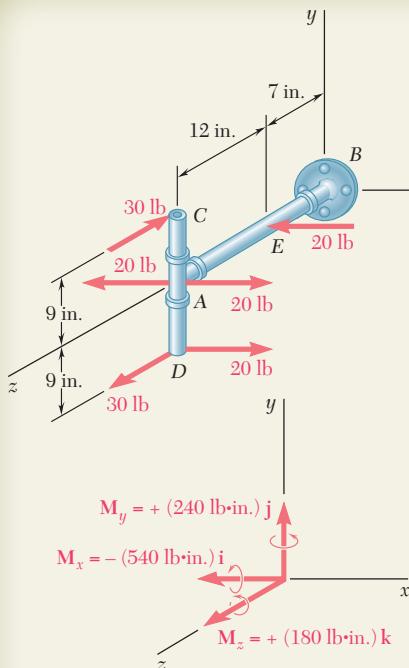


**Photo 3.2** The force exerted by each hand on the wrench could be replaced with an equivalent force-couple system acting on the nut.



### SAMPLE PROBLEM 3.6

Determine the components of the single couple equivalent to the two couples shown.



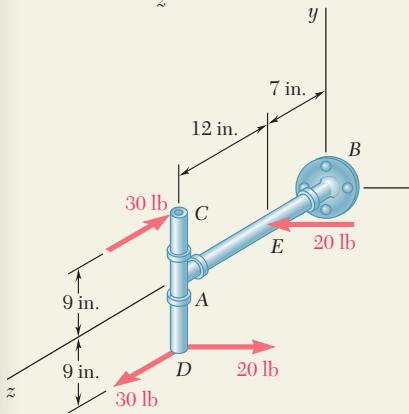
### SOLUTION

Our computations will be simplified if we attach two equal and opposite 20-lb forces at A. This enables us to replace the original 20-lb-force couple by two new 20-lb-force couples, one of which lies in the  $zx$  plane and the other in a plane parallel to the  $xy$  plane. The three couples shown in the adjoining sketch can be represented by three couple vectors  $\mathbf{M}_x$ ,  $\mathbf{M}_y$ , and  $\mathbf{M}_z$  directed along the coordinate axes. The corresponding moments are

$$\begin{aligned} M_x &= -(30 \text{ lb})(18 \text{ in.}) = -540 \text{ lb} \cdot \text{in.} \\ M_y &= +(20 \text{ lb})(12 \text{ in.}) = +240 \text{ lb} \cdot \text{in.} \\ M_z &= +(20 \text{ lb})(9 \text{ in.}) = +180 \text{ lb} \cdot \text{in.} \end{aligned}$$

These three moments represent the components of the single couple  $\mathbf{M}$  equivalent to the two given couples. We write

$$\mathbf{M} = -(540 \text{ lb} \cdot \text{in.})\mathbf{i} + (240 \text{ lb} \cdot \text{in.})\mathbf{j} + (180 \text{ lb} \cdot \text{in.})\mathbf{k}$$

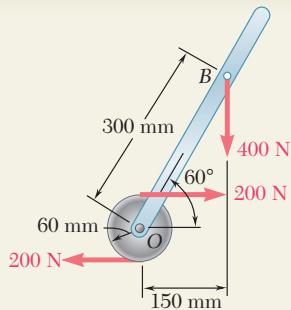


**Alternative Solution.** The components of the equivalent single couple  $\mathbf{M}$  can also be obtained by computing the sum of the moments of the four given forces about an arbitrary point. Selecting point D, we write

$$\mathbf{M} = \mathbf{M}_D = (18 \text{ in.})\mathbf{j} \times (-30 \text{ lb})\mathbf{k} + [(9 \text{ in.})\mathbf{j} - (12 \text{ in.})\mathbf{k}] \times (-20 \text{ lb})\mathbf{i}$$

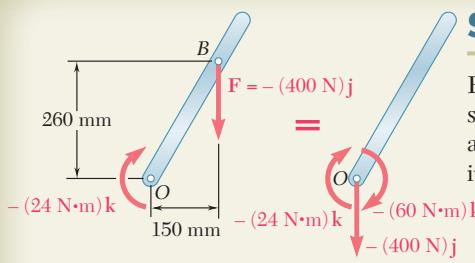
and, after computing the various cross products,

$$\mathbf{M} = -(540 \text{ lb} \cdot \text{in.})\mathbf{i} + (240 \text{ lb} \cdot \text{in.})\mathbf{j} + (180 \text{ lb} \cdot \text{in.})\mathbf{k}$$



## SAMPLE PROBLEM 3.7

Replace the couple and force shown by an equivalent single force applied to the lever. Determine the distance from the shaft to the point of application of this equivalent force.



## SOLUTION

First the given force and couple are replaced by an equivalent force-couple system at  $O$ . We move the force  $\mathbf{F} = -(400 \text{ N})\mathbf{j}$  to  $O$  and at the same time add a couple of moment  $\mathbf{M}_O$  equal to the moment about  $O$  of the force in its original position.

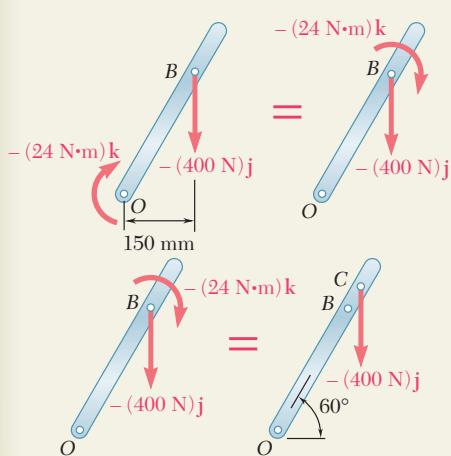
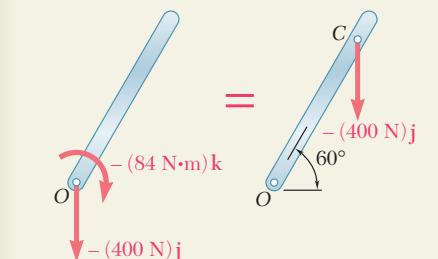
$$\begin{aligned}\mathbf{M}_O &= \overrightarrow{OB} \times \mathbf{F} = [(0.150 \text{ m})\mathbf{i} + (0.260 \text{ m})\mathbf{j}] \times (-400 \text{ N})\mathbf{j} \\ &= -(60 \text{ N} \cdot \text{m})\mathbf{k}\end{aligned}$$

This couple is added to the couple of moment  $-(24 \text{ N} \cdot \text{m})\mathbf{k}$  formed by the two 200-N forces, and a couple of moment  $-(84 \text{ N} \cdot \text{m})\mathbf{k}$  is obtained. This last couple can be eliminated by applying  $\mathbf{F}$  at a point  $C$  chosen in such a way that

$$\begin{aligned}-(84 \text{ N} \cdot \text{m})\mathbf{k} &= \overrightarrow{OC} \times \mathbf{F} \\ &= [(OC) \cos 60^\circ \mathbf{i} + (OC) \sin 60^\circ \mathbf{j}] \times (-400 \text{ N})\mathbf{j} \\ &= -(OC) \cos 60^\circ (400 \text{ N})\mathbf{k}\end{aligned}$$

We conclude that

$$(OC) \cos 60^\circ = 0.210 \text{ m} = 210 \text{ mm} \quad OC = 420 \text{ mm} \quad \blacktriangleleft$$



**Alternative Solution.** Since the effect of a couple does not depend on its location, the couple of moment  $-(24 \text{ N} \cdot \text{m})\mathbf{k}$  can be moved to  $B$ ; we thus obtain a force-couple system at  $B$ . The couple can now be eliminated by applying  $\mathbf{F}$  at a point  $C$  chosen in such a way that

$$\begin{aligned}-(24 \text{ N} \cdot \text{m})\mathbf{k} &= \overrightarrow{BC} \times \mathbf{F} \\ &= -(BC) \cos 60^\circ (400 \text{ N})\mathbf{k}\end{aligned}$$

We conclude that

$$\begin{aligned}(BC) \cos 60^\circ &= 0.060 \text{ m} = 60 \text{ mm} \quad BC = 120 \text{ mm} \\ OC &= OB + BC = 300 \text{ mm} + 120 \text{ mm} \quad OC = 420 \text{ mm} \quad \blacktriangleleft\end{aligned}$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson we discussed the properties of *couples*. To solve the problems which follow, you will need to remember that the net effect of a couple is to produce a moment  $\mathbf{M}$ . Since this moment is independent of the point about which it is computed,  $\mathbf{M}$  is a *free vector* and thus remains unchanged as it is moved from point to point. Also, two couples are *equivalent* (that is, they have the same effect on a given rigid body) if they produce the same moment.

When determining the moment of a couple, all previous techniques for computing moments apply. Also, since the moment of a couple is a free vector, it should be computed relative to the most convenient point.

Because the only effect of a couple is to produce a moment, it is possible to represent a couple with a vector, the *couple vector*, which is equal to the moment of the couple. The couple vector is a free vector and will be represented by a special symbol,  $\mathcal{M}$ , to distinguish it from force vectors.

In solving the problems in this lesson, you will be called upon to perform the following operations:

**1. Adding two or more couples.** This results in a new couple, the moment of which is obtained by adding vectorially the moments of the given couples [Sample Prob. 3.6].

**2. Replacing a force with an equivalent force-couple system at a specified point.** As explained in Sec. 3.16, the force of the force-couple system is equal to the original force, while the required couple vector is equal to the moment of the original force about the given point. In addition, it is important to observe that the force and the couple vector are perpendicular to each other. Conversely, it follows that a force-couple system can be reduced to a single force only if the force and couple vector are mutually perpendicular (see the next paragraph).

**3. Replacing a force-couple system (with  $\mathbf{F}$  perpendicular to  $\mathbf{M}$ ) with a single equivalent force.** Note that the requirement that  $\mathbf{F}$  and  $\mathbf{M}$  be mutually perpendicular will be satisfied in all two-dimensional problems. The single equivalent force is equal to  $\mathbf{F}$  and is applied in such a way that its moment about the original point of application is equal to  $\mathbf{M}$  [Sample Prob. 3.7].

# PROBLEMS

- 3.70** Two parallel 60-N forces are applied to a lever as shown. Determine the moment of the couple formed by the two forces (a) by resolving each force into horizontal and vertical components and adding the moments of the two resulting couples, (b) by using the perpendicular distance between the two forces, (c) by summing the moments of the two forces about point A.

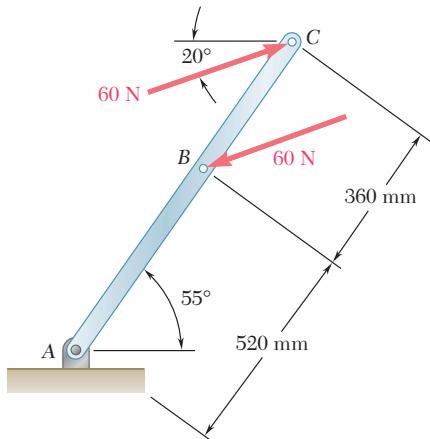


Fig. P3.70

- 3.71** A plate in the shape of a parallelogram is acted upon by two couples. Determine (a) the moment of the couple formed by the two 21-lb forces, (b) the perpendicular distance between the 12-lb forces if the resultant of the two couples is zero, (c) the value of  $\alpha$  if the resultant couple is 72 lb · in. clockwise and  $d$  is 42 in.

- 3.72** A couple  $\mathbf{M}$  of magnitude 18 N · m is applied to the handle of a screwdriver to tighten a screw into a block of wood. Determine the magnitudes of the two smallest horizontal forces that are equivalent to  $\mathbf{M}$  if they are applied (a) at corners A and D, (b) at corners B and C, (c) anywhere on the block.

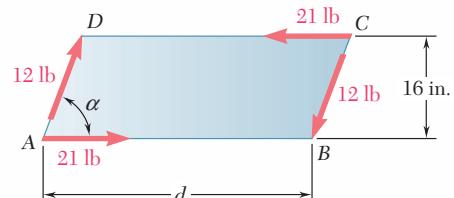


Fig. P3.71

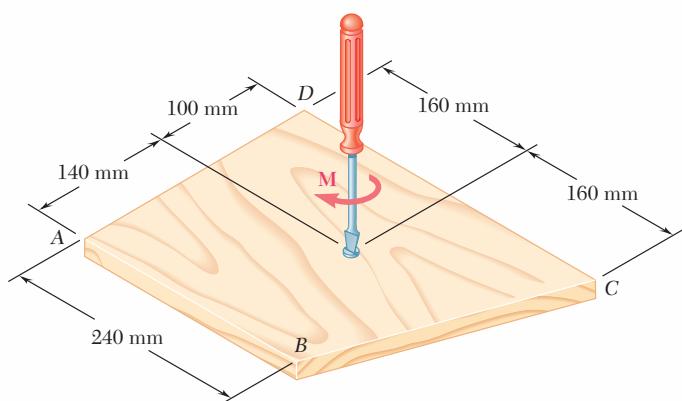


Fig. P3.72

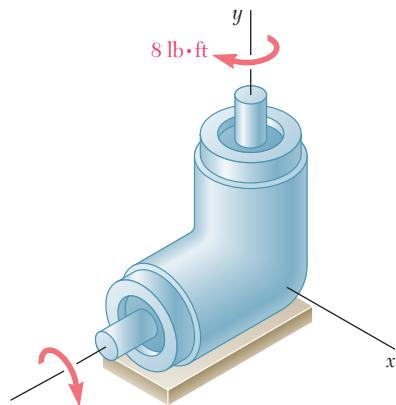


Fig. P3.75

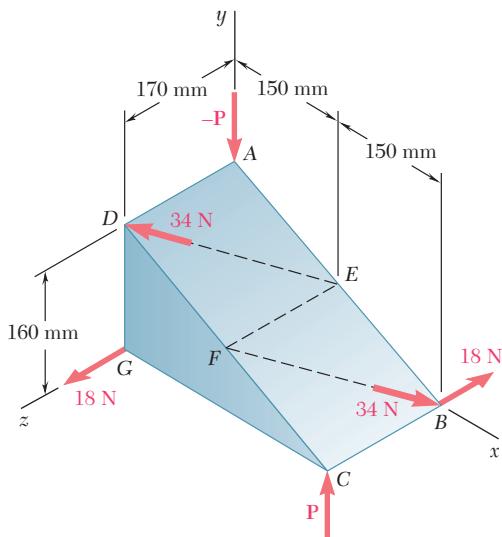


Fig. P3.76 and P3.79

- 3.73** Four 1-in.-diameter pegs are attached to a board as shown. Two strings are passed around the pegs and pulled with the forces indicated. (a) Determine the resultant couple acting on the board. (b) If only one string is used, around which pegs should it pass and in what directions should it be pulled to create the same couple with the minimum tension in the string? (c) What is the value of that minimum tension?

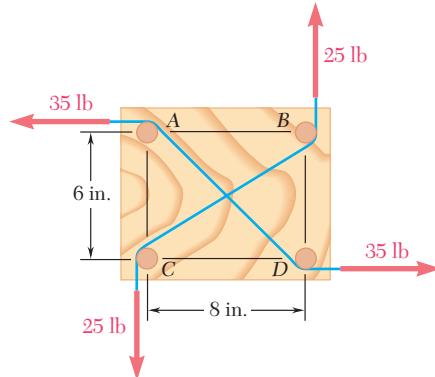


Fig. P3.73 and P3.74

- 3.74** Four pegs of the same diameter are attached to a board as shown. Two strings are passed around the pegs and pulled with the forces indicated. Determine the diameter of the pegs knowing that the resultant couple applied to the board is 485 lb · in. counterclockwise.

- 3.75** The shafts of an angle drive are acted upon by the two couples shown. Replace the two couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

- 3.76 and 3.77** If  $P = 0$ , replace the two remaining couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

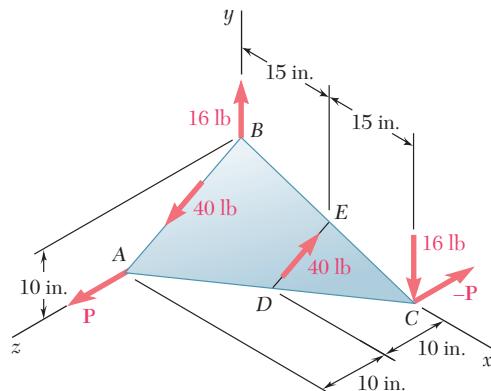


Fig. P3.77 and P3.78

- 3.78** If  $P = 20$  lb, replace the three couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

- 3.79** If  $P = 20$  N, replace the three couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

- 3.80** Shafts A and B connect the gear box to the wheel assemblies of a tractor, and shaft C connects it to the engine. Shafts A and B lie in the vertical  $yz$  plane, while shaft C is directed along the  $x$  axis. Replace the couples applied to the shafts with a single equivalent couple, specifying its magnitude and the direction of its axis.

- 3.81** The tension in the cable attached to the end C of an adjustable boom ABC is 560 lb. Replace the force exerted by the cable at C with an equivalent force-couple system (a) at A, (b) at B.

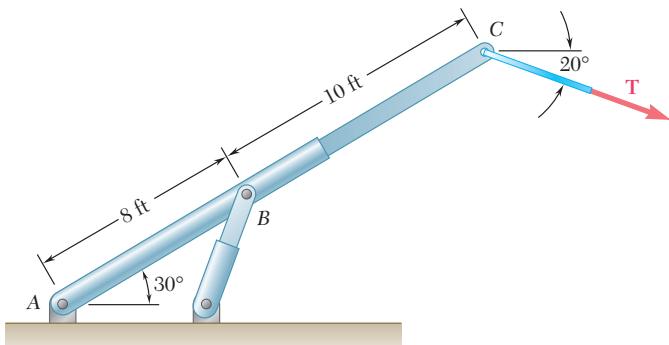


Fig. P3.81

- 3.82** A 160-lb force  $\mathbf{P}$  is applied at point A of a structural member. Replace  $\mathbf{P}$  with (a) an equivalent force-couple system at C, (b) an equivalent system consisting of a vertical force at B and a second force at D.

- 3.83** The 80-N horizontal force  $\mathbf{P}$  acts on a bell crank as shown. (a) Replace  $\mathbf{P}$  with an equivalent force-couple system at B. (b) Find the two vertical forces at C and D that are equivalent to the couple found in part a.

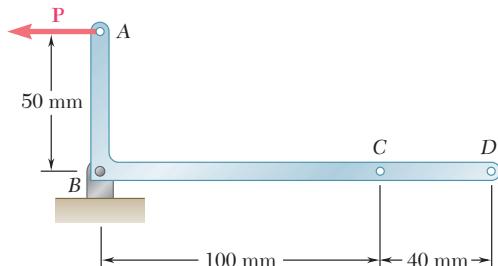


Fig. P3.83

- 3.84** A dirigible is tethered by a cable attached to its cabin at B. If the tension in the cable is 1040 N, replace the force exerted by the cable at B with an equivalent system formed by two parallel forces applied at A and C.

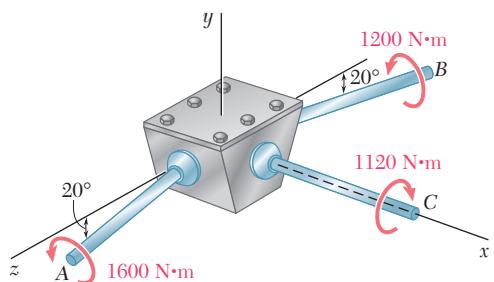


Fig. P3.80

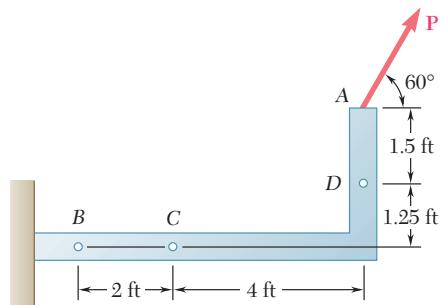


Fig. P3.82

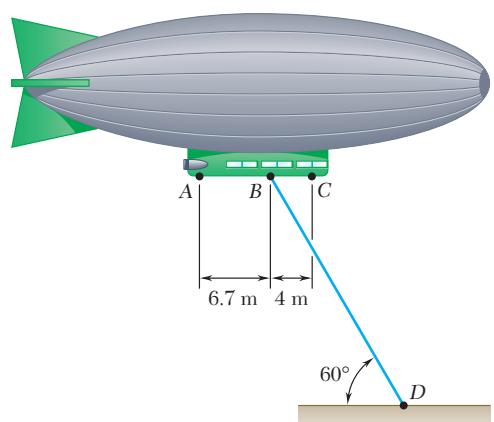


Fig. P3.84

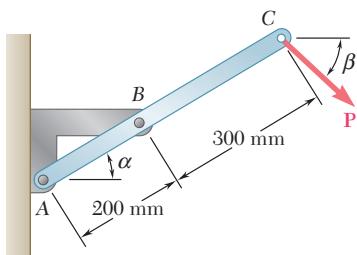


Fig. P3.85

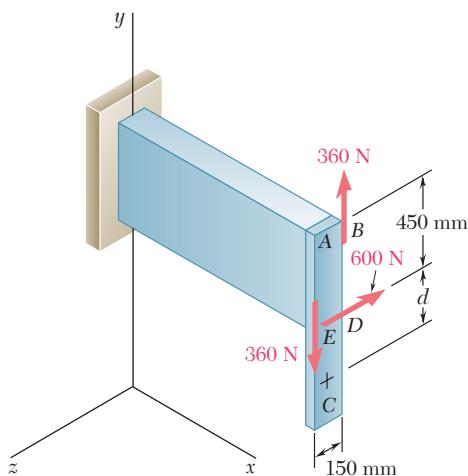


Fig. P3.87

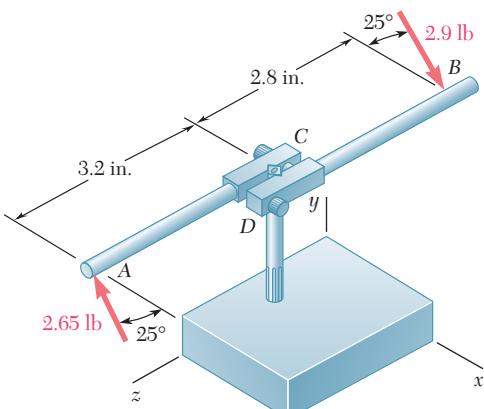


Fig. P3.89

- 3.85** The force  $\mathbf{P}$  has a magnitude of 250 N and is applied at the end  $C$  of a 500-mm rod  $AC$  attached to a bracket at  $A$  and  $B$ . Assuming  $\alpha = 30^\circ$  and  $\beta = 60^\circ$ , replace  $\mathbf{P}$  with (a) an equivalent force-couple system at  $B$ , (b) an equivalent system formed by two parallel forces applied at  $A$  and  $B$ .

**3.86** Solve Prob. 3.85, assuming  $\alpha = \beta = 25^\circ$ .

- 3.87** A force and a couple are applied as shown to the end of a cantilever beam. (a) Replace this system with a single force  $\mathbf{F}$  applied at point  $C$ , and determine the distance  $d$  from  $C$  to a line drawn through points  $D$  and  $E$ . (b) Solve part *a* if the directions of the two 360-N forces are reversed.

- 3.88** The shearing forces exerted on the cross section of a steel channel can be represented by a 900-N vertical force and two 250-N horizontal forces as shown. Replace this force and couple with a single force  $\mathbf{F}$  applied at point  $C$ , and determine the distance  $x$  from  $C$  to line  $BD$ . (Point  $C$  is defined as the *shear center* of the section.)

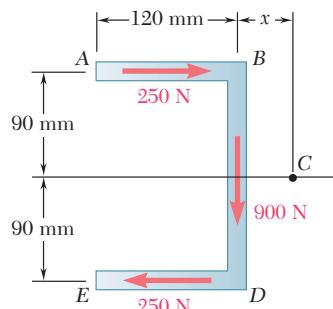


Fig. P3.88

- 3.89** While tapping a hole, a machinist applies the horizontal forces shown to the handle of the tap wrench. Show that these forces are equivalent to a single force, and specify, if possible, the point of application of the single force on the handle.

- 3.90** Three control rods attached to a lever  $ABC$  exert on it the forces shown. (a) Replace the three forces with an equivalent force-couple system at  $B$ . (b) Determine the single force that is equivalent to the force-couple system obtained in part *a*, and specify its point of application on the lever.

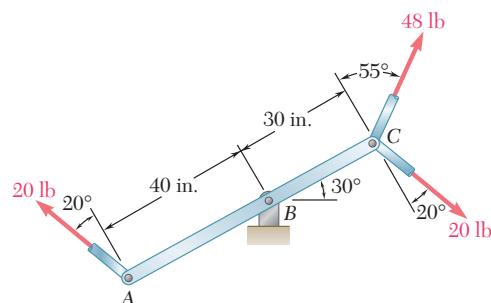


Fig. P3.90

- 3.91** A hexagonal plate is acted upon by the force **P** and the couple shown. Determine the magnitude and the direction of the smallest force **P** for which this system can be replaced with a single force at *E*.

- 3.92** A rectangular plate is acted upon by the force and couple shown. This system is to be replaced with a single equivalent force. (a) For  $\alpha = 40^\circ$ , specify the magnitude and the line of action of the equivalent force. (b) Specify the value of  $\alpha$  if the line of action of the equivalent force is to intersect line *CD* 300 mm to the right of *D*.

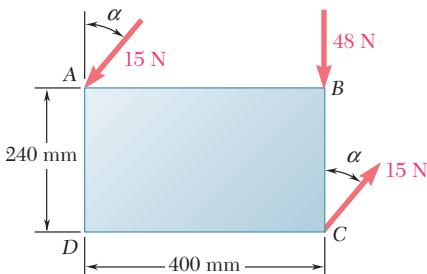


Fig. P3.92

- 3.93** An eccentric, compressive 1220-N force **P** is applied to the end of a cantilever beam. Replace **P** with an equivalent force-couple system at *G*.

- 3.94** To keep a door closed, a wooden stick is wedged between the floor and the doorknob. The stick exerts at *B* a 175-N force directed along line *AB*. Replace that force with an equivalent force-couple system at *C*.

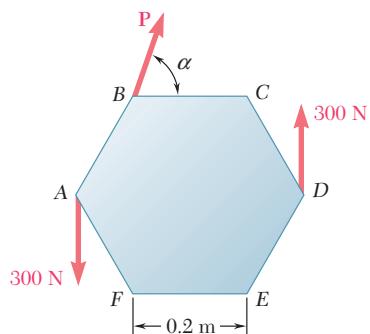


Fig. P3.91

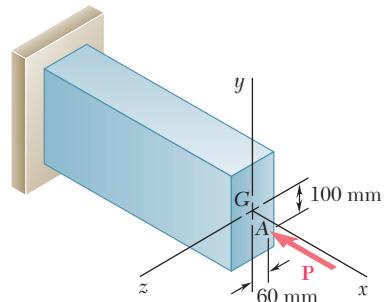


Fig. P3.93

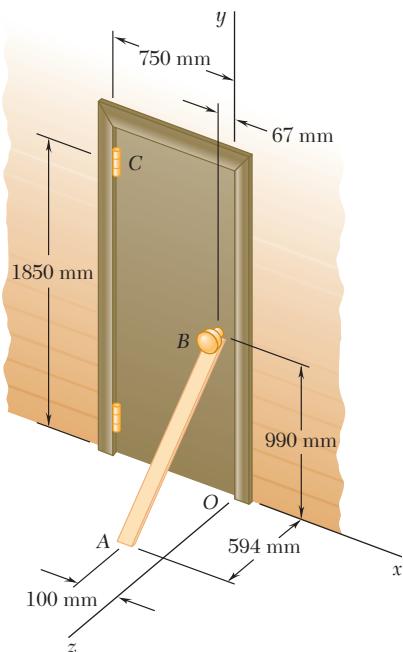
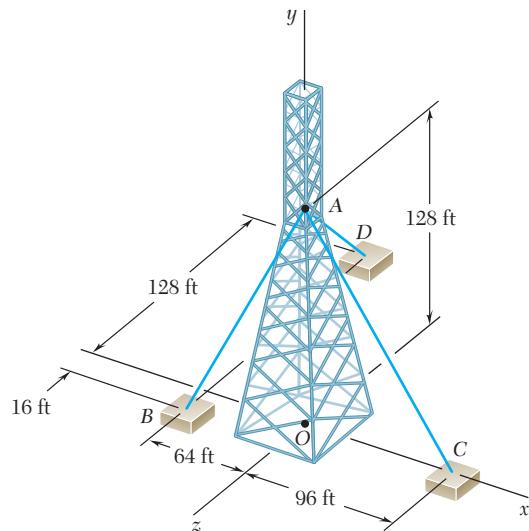


Fig. P3.94

- 3.95** An antenna is guyed by three cables as shown. Knowing that the tension in cable  $AB$  is 288 lb, replace the force exerted at  $A$  by cable  $AB$  with an equivalent force-couple system at the center  $O$  of the base of the antenna.

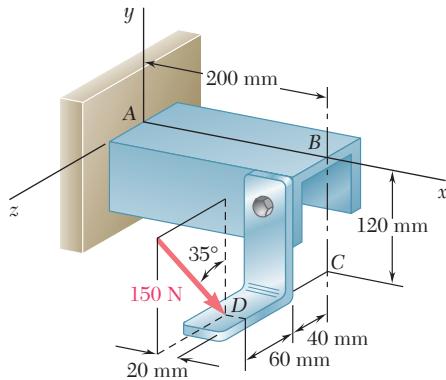


**Fig. P3.95 and P3.96**

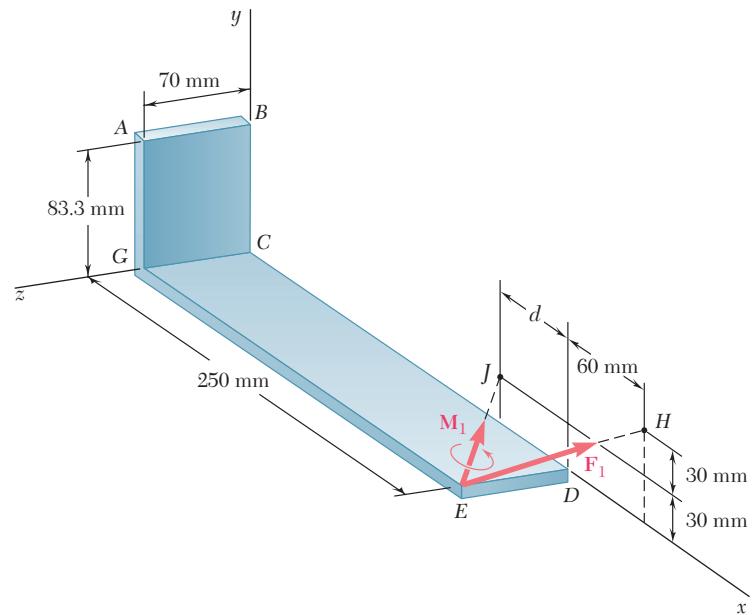
- 3.96** An antenna is guyed by three cables as shown. Knowing that the tension in cable  $AD$  is 270 lb, replace the force exerted at  $A$  by cable  $AD$  with an equivalent force-couple system at the center  $O$  of the base of the antenna.

- 3.97** Replace the 150-N force with an equivalent force-couple system at  $A$ .

- 3.98** A 77-N force  $\mathbf{F}_1$  and a 31-N · m couple  $\mathbf{M}_1$  are applied to corner  $E$  of the bent plate shown. If  $\mathbf{F}_1$  and  $\mathbf{M}_1$  are to be replaced with an equivalent force-couple system ( $\mathbf{F}_2$ ,  $\mathbf{M}_2$ ) at corner  $B$  and if  $(M_2)_z = 0$ , determine (a) the distance  $d$ , (b)  $\mathbf{F}_2$  and  $\mathbf{M}_2$ .



**Fig. P3.97**



**Fig. P3.98**

- 3.99** A 46-lb force  $\mathbf{F}$  and a 2120-lb · in. couple  $\mathbf{M}$  are applied to corner A of the block shown. Replace the given force-couple system with an equivalent force-couple system at corner H.

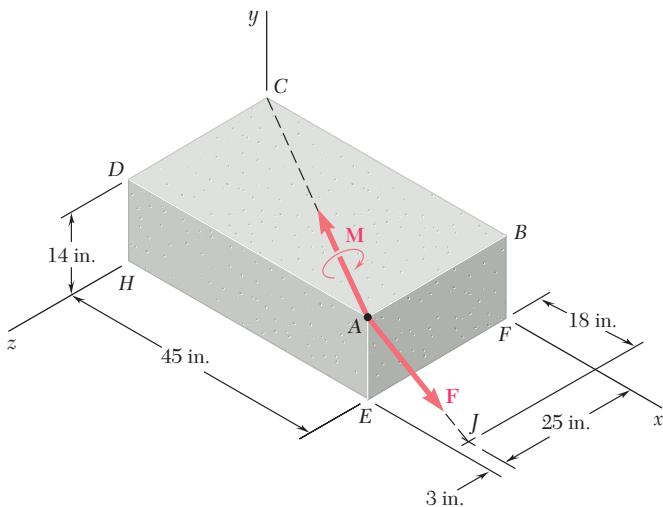


Fig. P3.99

- 3.100** The handpiece for a miniature industrial grinder weighs 0.6 lb, and its center of gravity is located on the  $y$  axis. The head of the handpiece is offset in the  $xz$  plane in such a way that line BC forms an angle of  $25^\circ$  with the  $x$  direction. Show that the weight of the handpiece and the two couples  $\mathbf{M}_1$  and  $\mathbf{M}_2$  can be replaced with a single equivalent force. Further, assuming that  $M_1 = 0.68$  lb · in. and  $M_2 = 0.65$  lb · in., determine (a) the magnitude and the direction of the equivalent force, (b) the point where its line of action intersects the  $xz$  plane.

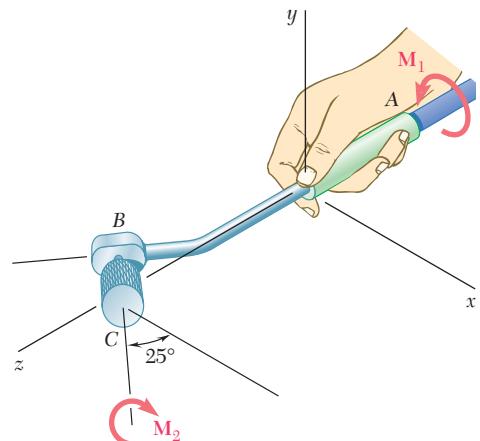


Fig. P3.100

### 3.17 REDUCTION OF A SYSTEM OF FORCES TO ONE FORCE AND ONE COUPLE

Consider a system of forces  $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \dots$ , acting on a rigid body at the points  $A_1, A_2, A_3, \dots$ , defined by the position vectors  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots$  etc. (Fig. 3.41a). As seen in the preceding section,  $\mathbf{F}_1$  can be moved from  $A_1$  to a given point  $O$  if a couple of moment  $\mathbf{M}_1$  equal to the moment  $\mathbf{r}_1 \times \mathbf{F}_1$  of  $\mathbf{F}_1$  about  $O$  is added to the original system of forces. Repeating this procedure with  $\mathbf{F}_2, \mathbf{F}_3, \dots$ , we obtain the

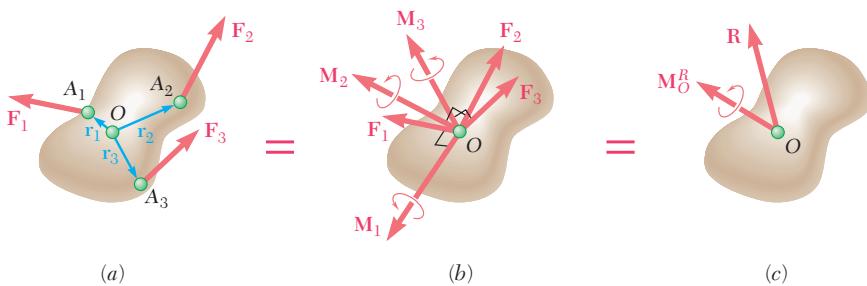


Fig. 3.41

system shown in Fig. 3.41b, which consists of the original forces, now acting at  $O$ , and the added couple vectors. Since the forces are now concurrent, they can be added vectorially and replaced by their resultant  $\mathbf{R}$ . Similarly, the couple vectors  $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \dots$ , can be added vectorially and replaced by a single couple vector  $\mathbf{M}_O^R$ . Any system of forces, however complex, can thus be reduced to an *equivalent force-couple system acting at a given point O* (Fig. 3.41c). We should note that while each of the couple vectors  $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \dots$ , in Fig. 3.41b is perpendicular to its corresponding force, the resultant force  $\mathbf{R}$  and the resultant couple vector  $\mathbf{M}_O^R$  in Fig. 3.41c will not, in general, be perpendicular to each other.

The equivalent force-couple system is defined by the equations

$$\mathbf{R} = \sum \mathbf{F} \quad \mathbf{M}_O^R = \sum \mathbf{M}_O = \sum (\mathbf{r} \times \mathbf{F}) \quad (3.52)$$

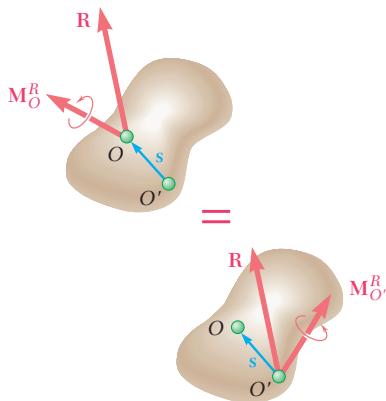


Fig. 3.42

which express that the force  $\mathbf{R}$  is obtained by adding all the forces of the system, while the moment of the resultant couple vector  $\mathbf{M}_O^R$ , called the *moment resultant* of the system, is obtained by adding the moments about  $O$  of all the forces of the system.

Once a given system of forces has been reduced to a force and a couple at a point  $O$ , it can easily be reduced to a force and a couple at another point  $O'$ . While the resultant force  $\mathbf{R}$  will remain unchanged, the new moment resultant  $\mathbf{M}_{O'}^R$  will be equal to the sum of  $\mathbf{M}_O^R$  and the moment about  $O'$  of the force  $\mathbf{R}$  attached at  $O$  (Fig. 3.42). We have

$$\mathbf{M}_{O'}^R = \mathbf{M}_O^R + \mathbf{s} \times \mathbf{R} \quad (3.53)$$

In practice, the reduction of a given system of forces to a single force  $\mathbf{R}$  at  $O$  and a couple vector  $\mathbf{M}_O^R$  will be carried out in terms of components. Resolving each position vector  $\mathbf{r}$  and each force  $\mathbf{F}$  of the system into rectangular components, we write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (3.54)$$

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \quad (3.55)$$

Substituting for  $\mathbf{r}$  and  $\mathbf{F}$  in (3.52) and factoring out the unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , we obtain  $\mathbf{R}$  and  $\mathbf{M}_O^R$  in the form

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} \quad \mathbf{M}_O^R = M_x^R\mathbf{i} + M_y^R\mathbf{j} + M_z^R\mathbf{k} \quad (3.56)$$

The components  $R_x, R_y, R_z$  represent, respectively, the sums of the  $x$ ,  $y$ , and  $z$  components of the given forces and measure the tendency of the system to impart to the rigid body a motion of translation in the  $x$ ,  $y$ , or  $z$  direction. Similarly, the components  $M_x^R, M_y^R, M_z^R$  represent, respectively, the sum of the moments of the given forces about the  $x$ ,  $y$ , and  $z$  axes and measure the tendency of the system to impart to the rigid body a motion of rotation about the  $x$ ,  $y$ , or  $z$  axis.

If the magnitude and direction of the force  $\mathbf{R}$  are desired, they can be obtained from the components  $R_x, R_y, R_z$  by means of the relations (2.18) and (2.19) of Sec. 2.12; similar computations will yield the magnitude and direction of the couple vector  $\mathbf{M}_O^R$ .

## 3.18 EQUIVALENT SYSTEMS OF FORCES

We saw in the preceding section that any system of forces acting on a rigid body can be reduced to a force-couple system at a given point  $O$ . This equivalent force-couple system characterizes completely the effect of the given force system on the rigid body. *Two systems of forces are equivalent, therefore, if they can be reduced to the same force-couple system at a given point  $O$ .* Recalling that the force-couple system at  $O$  is defined by the relations (3.52), we state that *two systems of forces,  $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \dots$ , and  $\mathbf{F}'_1, \mathbf{F}'_2, \mathbf{F}'_3, \dots$ , which act on the same rigid body are equivalent if, and only if, the sums of the forces and the sums of the moments about a given point  $O$  of the forces of the two systems are, respectively, equal.* Expressed mathematically, the necessary and sufficient conditions for the two systems of forces to be equivalent are

$$\Sigma \mathbf{F} = \Sigma \mathbf{F}' \quad \text{and} \quad \Sigma \mathbf{M}_O = \Sigma \mathbf{M}'_O \quad (3.57)$$

Note that to prove that two systems of forces are equivalent, the second of the relations (3.57) must be established with respect to *only one point  $O$* . It will hold, however, with respect to *any point* if the two systems are equivalent.

Resolving the forces and moments in (3.57) into their rectangular components, we can express the necessary and sufficient conditions for the equivalence of two systems of forces acting on a rigid body as follows:

$$\begin{aligned} \Sigma F_x &= \Sigma F'_x & \Sigma F_y &= \Sigma F'_y & \Sigma F_z &= \Sigma F'_z \\ \Sigma M_x &= \Sigma M'_x & \Sigma M_y &= \Sigma M'_y & \Sigma M_z &= \Sigma M'_z \end{aligned} \quad (3.58)$$

These equations have a simple physical significance. They express that two systems of forces are equivalent if they tend to impart to the rigid body (1) the same translation in the  $x$ ,  $y$ , and  $z$  directions, respectively, and (2) the same rotation about the  $x$ ,  $y$ , and  $z$  axes, respectively.

## 3.19 EQUIPOLLENT SYSTEMS OF VECTORS

In general, when two systems of vectors satisfy Eqs. (3.57) or (3.58), i.e., when their resultants and their moment resultants about an arbitrary point  $O$  are respectively equal, the two systems are said to be *equipollent*. The result established in the preceding section can thus be restated as follows: *If two systems of forces acting on a rigid body are equipollent, they are also equivalent.*

It is important to note that this statement does not apply to *any* system of vectors. Consider, for example, a system of forces acting on a set of independent particles which do *not* form a rigid body. A different system of forces acting on the same particles may happen to be equipollent to the first one; i.e., it may have the same resultant and the same moment resultant. Yet, since different forces will now act on the various particles, their effects on these particles will be different; the two systems of forces, while equipollent, are *not equivalent*.



**Photo 3.3** The forces exerted by the children upon the wagon can be replaced with an equivalent force-couple system when analyzing the motion of the wagon.

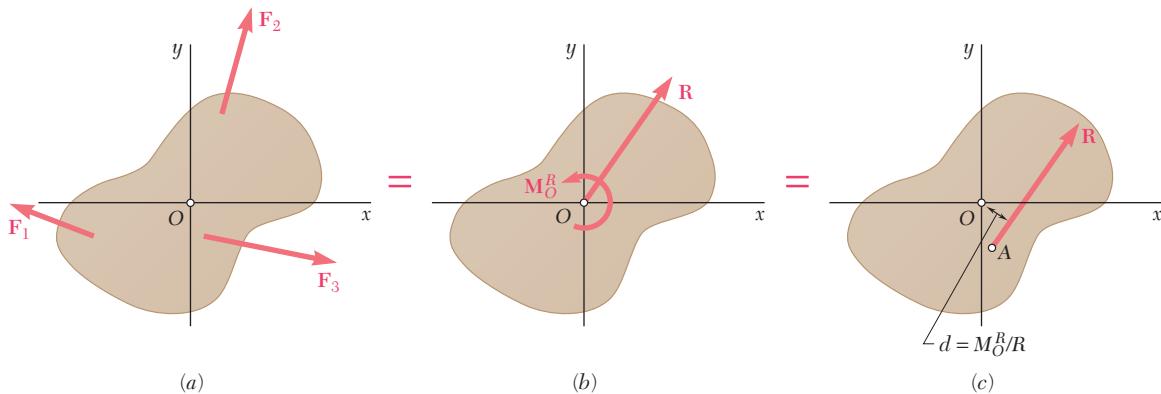
## 3.20 FURTHER REDUCTION OF A SYSTEM OF FORCES

We saw in Sec. 3.17 that any given system of forces acting on a rigid body can be reduced to an equivalent force-couple system at  $O$  consisting of a force  $\mathbf{R}$  equal to the sum of the forces of the system and a couple vector  $\mathbf{M}_O^R$  of moment equal to the moment resultant of the system.

When  $\mathbf{R} = 0$ , the force-couple system reduces to the couple vector  $\mathbf{M}_O^R$ . The given system of forces can then be reduced to a single couple, called the *resultant couple* of the system.

Let us now investigate the conditions under which a given system of forces can be reduced to a single force. It follows from Sec. 3.16 that the force-couple system at  $O$  can be replaced by a single force  $\mathbf{R}$  acting along a new line of action if  $\mathbf{R}$  and  $\mathbf{M}_O^R$  are mutually perpendicular. The systems of forces which can be reduced to a single force, or *resultant*, are therefore the systems for which the force  $\mathbf{R}$  and the couple vector  $\mathbf{M}_O^R$  are mutually perpendicular. While this condition is generally not satisfied by systems of forces in space, it will be satisfied by systems consisting of (1) concurrent forces, (2) coplanar forces, or (3) parallel forces. These three cases will be discussed separately.

1. *Concurrent forces* are applied at the same point and can therefore be added directly to obtain their resultant  $\mathbf{R}$ . Thus, they always reduce to a single force. Concurrent forces were discussed in detail in Chap. 2.
2. *Coplanar forces* act in the same plane, which may be assumed to be the plane of the figure (Fig. 3.43a). The sum  $\mathbf{R}$  of the forces of the system will also lie in the plane of the figure, while the moment of each force about  $O$ , and thus the moment resultant  $\mathbf{M}_O^R$ , will be perpendicular to that plane. The force-couple system at  $O$  consists, therefore, of a force  $\mathbf{R}$  and a couple vector  $\mathbf{M}_O^R$  which are mutually perpendicular (Fig. 3.43b).† They can be reduced to a single force  $\mathbf{R}$  by moving  $\mathbf{R}$  in the plane of the figure until its moment about  $O$  becomes equal to  $\mathbf{M}_O^R$ . The distance from  $O$  to the line of action of  $\mathbf{R}$  is  $d = M_O^R/R$  (Fig. 3.43c).

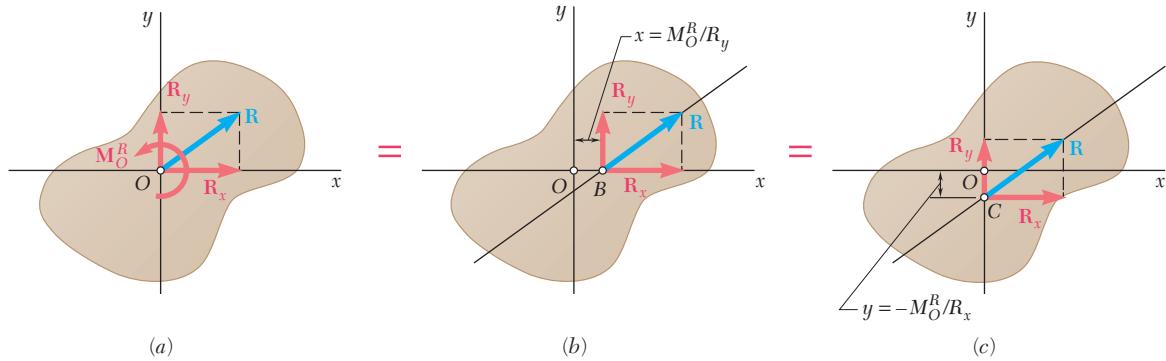


**Fig. 3.43**

†Since the couple vector  $\mathbf{M}_O^R$  is perpendicular to the plane of the figure, it has been represented by the symbol  $\uparrow$ . A counterclockwise couple  $\uparrow$  represents a vector pointing out of the paper, and a clockwise couple  $\downarrow$  represents a vector pointing into the paper.

As noted in Sec. 3.17, the reduction of a system of forces is considerably simplified if the forces are resolved into rectangular components. The force-couple system at  $O$  is then characterized by the components (Fig. 3.44a)

$$R_x = \Sigma F_x \quad R_y = \Sigma F_y \quad M_O^R = M_O^R = \Sigma M_O \quad (3.59)$$



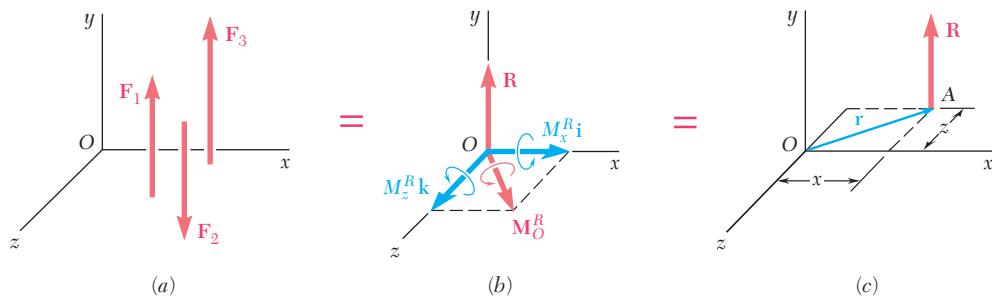
**Fig. 3.44**

To reduce the system to a single force  $\mathbf{R}$ , we express that the moment of  $\mathbf{R}$  about  $O$  must be equal to  $\mathbf{M}_O^R$ . Denoting by  $x$  and  $y$  the coordinates of the point of application of the resultant and recalling formula (3.22) of Sec. 3.8, we write

$$xR_y - yR_x = M_O^R$$

which represents the equation of the line of action of  $\mathbf{R}$ . We can also determine directly the  $x$  and  $y$  intercepts of the line of action of the resultant by noting that  $\mathbf{M}_O^R$  must be equal to the moment about  $O$  of the  $y$  component of  $\mathbf{R}$  when  $\mathbf{R}$  is attached at  $B$  (Fig. 3.44b) and to the moment of its  $x$  component when  $\mathbf{R}$  is attached at  $C$  (Fig. 3.44c).

3. *Parallel forces* have parallel lines of action and may or may not have the same sense. Assuming here that the forces are parallel to the  $y$  axis (Fig. 3.45a), we note that their sum  $\mathbf{R}$  will also be parallel to the  $y$  axis. On the other hand, since the moment of a given force must be perpendicular to that force, the moment about  $O$  of each force of the system, and thus the moment resultant  $\mathbf{M}_O^R$ , will lie in the  $zx$  plane. The force-couple system at  $O$  consists, therefore,



**Fig. 3.45**



**Photo 3.4** The parallel wind forces acting on the highway signs can be reduced to a single equivalent force. Determining this force can simplify the calculation of the forces acting on the supports of the frame to which the signs are attached.

of a force  $\mathbf{R}$  and a couple vector  $\mathbf{M}_O^R$  which are mutually perpendicular (Fig. 3.45b). They can be reduced to a single force  $\mathbf{R}$  (Fig. 3.45c) or, if  $\mathbf{R} = 0$ , to a single couple of moment  $\mathbf{M}_O^R$ .

In practice, the force-couple system at  $O$  will be characterized by the components

$$R_y = \sum F_y \quad M_x^R = \sum M_x \quad M_z^R = \sum M_z \quad (3.60)$$

The reduction of the system to a single force can be carried out by moving  $\mathbf{R}$  to a new point of application  $A(x, 0, z)$  chosen so that the moment of  $\mathbf{R}$  about  $O$  is equal to  $\mathbf{M}_O^R$ . We write

$$\begin{aligned} \mathbf{r} \times \mathbf{R} &= \mathbf{M}_O^R \\ (x\mathbf{i} + z\mathbf{k}) \times R_y\mathbf{j} &= M_x^R\mathbf{i} + M_z^R\mathbf{k} \end{aligned}$$

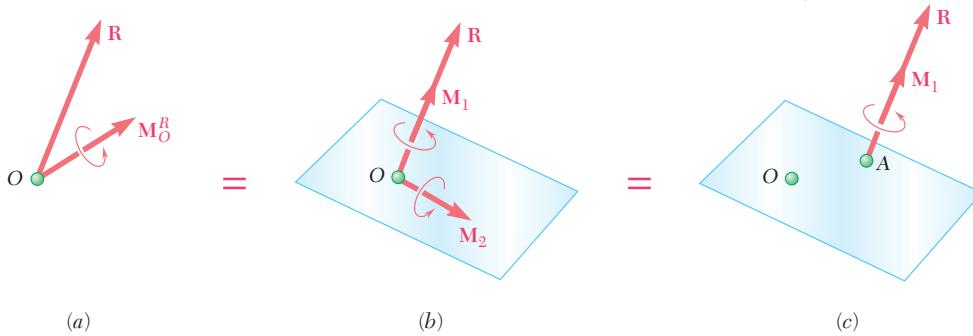
By computing the vector products and equating the coefficients of the corresponding unit vectors in both members of the equation, we obtain two scalar equations which define the coordinates of  $A$ :

$$-zR_y = M_x^R \quad xR_y = M_z^R$$

These equations express that the moments of  $\mathbf{R}$  about the  $x$  and  $z$  axes must, respectively, be equal to  $M_x^R$  and  $M_z^R$ .

### \*3.21 REDUCTION OF A SYSTEM OF FORCES TO A WRENCH

In the general case of a system of forces in space, the equivalent force-couple system at  $O$  consists of a force  $\mathbf{R}$  and a couple vector  $\mathbf{M}_O^R$  which are not perpendicular, and neither of which is zero (Fig. 3.46a). Thus, the system of forces *cannot* be reduced to a single force or to a single couple. The couple vector, however, can be replaced by two other couple vectors obtained by resolving  $\mathbf{M}_O^R$  into a component  $\mathbf{M}_1$  along  $\mathbf{R}$  and a component  $\mathbf{M}_2$  in a plane perpendicular to  $\mathbf{R}$  (Fig. 3.46b). The couple vector  $\mathbf{M}_2$  and the force  $\mathbf{R}$  can then be replaced by a single force  $\mathbf{R}$  acting along a new line of action. The original system of forces thus reduces to  $\mathbf{R}$  and to the couple vector  $\mathbf{M}_1$  (Fig. 3.46c), i.e., to  $\mathbf{R}$  and a couple acting in the plane perpendicular to  $\mathbf{R}$ . This particular force-couple system is called a *wrench* because the resulting combination of push and twist is the same as that which would be caused by an actual wrench. The line of action of  $\mathbf{R}$  is known as the *axis of the wrench*, and the ratio  $p = M_1/R$  is called the *pitch*



**Fig. 3.46**

of the wrench. A wrench, therefore, consists of two collinear vectors, namely, a force  $\mathbf{R}$  and a couple vector

$$\mathbf{M}_1 = p\mathbf{R} \quad (3.61)$$

Recalling the expression (3.35) obtained in Sec. 3.9 for the projection of a vector on the line of action of another vector, we note that the projection of  $\mathbf{M}_O^R$  on the line of action of  $\mathbf{R}$  is

$$M_1 = \frac{\mathbf{R} \cdot \mathbf{M}_O^R}{R}$$

Thus, the pitch of the wrench can be expressed as†

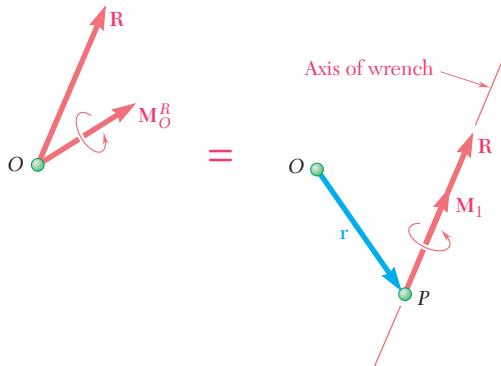
$$p = \frac{M_1}{R} = \frac{\mathbf{R} \cdot \mathbf{M}_O^R}{R^2} \quad (3.62)$$

To define the axis of the wrench, we can write a relation involving the position vector  $\mathbf{r}$  of an arbitrary point  $P$  located on that axis. Attaching the resultant force  $\mathbf{R}$  and couple vector  $\mathbf{M}_1$  at  $P$  (Fig. 3.47) and expressing that the moment about  $O$  of this force-couple system is equal to the moment resultant  $\mathbf{M}_O^R$  of the original force system, we write

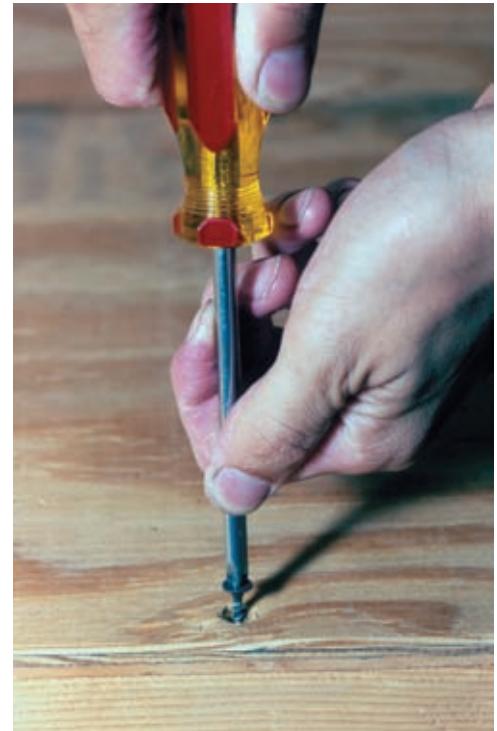
$$\mathbf{M}_1 + \mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R \quad (3.63)$$

or, recalling Eq. (3.61),

$$p\mathbf{R} + \mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R \quad (3.64)$$



**Fig. 3.47**



**Photo 3.5** The pushing-turning action associated with the tightening of a screw illustrates the collinear lines of action of the force and couple vector that constitute a wrench.

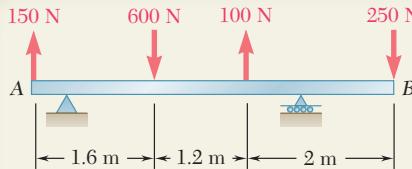
†The expressions obtained for the projection of the couple vector on the line of action of  $\mathbf{R}$  and for the pitch of the wrench are independent of the choice of point  $O$ . Using the relation (3.53) of Sec. 3.17, we note that if a different point  $O'$  had been used, the numerator in (3.62) would have been

$$\mathbf{R} \cdot \mathbf{M}_{O'}^R = \mathbf{R} \cdot (\mathbf{M}_O^R + \mathbf{s} \times \mathbf{R}) = \mathbf{R} \cdot \mathbf{M}_O^R + \mathbf{R} \cdot (\mathbf{s} \times \mathbf{R})$$

Since the mixed triple product  $\mathbf{R} \cdot (\mathbf{s} \times \mathbf{R})$  is identically equal to zero, we have

$$\mathbf{R} \cdot \mathbf{M}_{O'}^R = \mathbf{R} \cdot \mathbf{M}_O^R$$

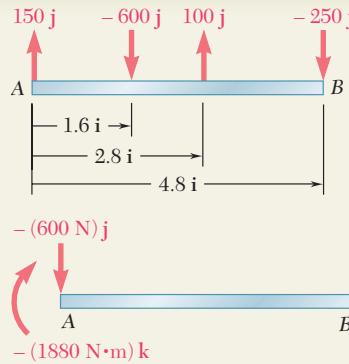
Thus, the scalar product  $\mathbf{R} \cdot \mathbf{M}_O^R$  is independent of the choice of point  $O$ .



## SAMPLE PROBLEM 3.8

A 4.80-m-long beam is subjected to the forces shown. Reduce the given system of forces to (a) an equivalent force-couple system at A, (b) an equivalent force-couple system at B, (c) a single force or resultant.

Note. Since the reactions at the supports are not included in the given system of forces, the given system will not maintain the beam in equilibrium.



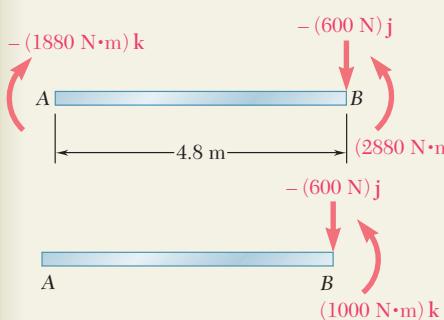
## SOLUTION

**a. Force-Couple System at A.** The force-couple system at A equivalent to the given system of forces consists of a force  $\mathbf{R}$  and a couple  $\mathbf{M}_A^R$  defined as follows:

$$\begin{aligned}\mathbf{R} &= \Sigma \mathbf{F} \\ &= (150 \text{ N})\mathbf{j} - (600 \text{ N})\mathbf{j} + (100 \text{ N})\mathbf{j} - (250 \text{ N})\mathbf{j} = -(600 \text{ N})\mathbf{j} \\ \mathbf{M}_A^R &= \Sigma (\mathbf{r} \times \mathbf{F}) \\ &= (1.6\mathbf{i}) \times (-600\mathbf{j}) + (2.8\mathbf{i}) \times (100\mathbf{j}) + (4.8\mathbf{i}) \times (-250\mathbf{j}) \\ &= -(1880 \text{ N} \cdot \text{m})\mathbf{k}\end{aligned}$$

The equivalent force-couple system at A is thus

$$\mathbf{R} = 600 \text{ N} \downarrow \quad \mathbf{M}_A^R = 1880 \text{ N} \cdot \text{m} \downarrow$$



**b. Force-Couple System at B.** We propose to find a force-couple system at B equivalent to the force-couple system at A determined in part a. The force  $\mathbf{R}$  is unchanged, but a new couple  $\mathbf{M}_B^R$  must be determined, the moment of which is equal to the moment about B of the force-couple system determined in part a. Thus, we have

$$\begin{aligned}\mathbf{M}_B^R &= \mathbf{M}_A^R + \overrightarrow{BA} \times \mathbf{R} \\ &= -(1880 \text{ N} \cdot \text{m})\mathbf{k} + (-4.8\mathbf{i}) \times (-600 \text{ N})\mathbf{j} \\ &= -(1880 \text{ N} \cdot \text{m})\mathbf{k} + (2880 \text{ N} \cdot \text{m})\mathbf{k} = +(1000 \text{ N} \cdot \text{m})\mathbf{k}\end{aligned}$$

The equivalent force-couple system at B is thus

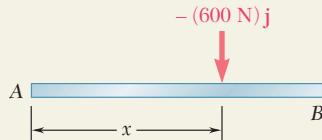
$$\mathbf{R} = 600 \text{ N} \downarrow \quad \mathbf{M}_B^R = 1000 \text{ N} \cdot \text{m} \uparrow$$

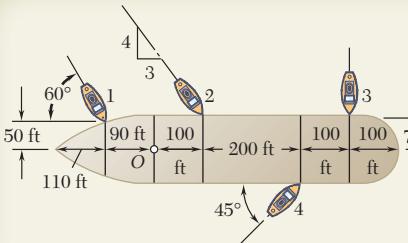
**c. Single Force or Resultant.** The resultant of the given system of forces is equal to  $\mathbf{R}$ , and its point of application must be such that the moment of  $\mathbf{R}$  about A is equal to  $\mathbf{M}_A^R$ . We write

$$\begin{aligned}\mathbf{r} \times \mathbf{R} &= \mathbf{M}_A^R \\ x\mathbf{i} \times (-600 \text{ N})\mathbf{j} &= -(1880 \text{ N} \cdot \text{m})\mathbf{k} \\ -x(600 \text{ N})\mathbf{k} &= -(1880 \text{ N} \cdot \text{m})\mathbf{k}\end{aligned}$$

and conclude that  $x = 3.13 \text{ m}$ . Thus, the single force equivalent to the given system is defined as

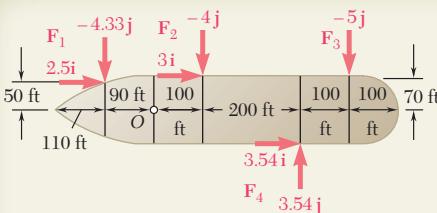
$$\mathbf{R} = 600 \text{ N} \downarrow \quad x = 3.13 \text{ m}$$





## SAMPLE PROBLEM 3.9

Four tugboats are used to bring an ocean liner to its pier. Each tugboat exerts a 5000-lb force in the direction shown. Determine (a) the equivalent force-couple system at the foremast  $O$ , (b) the point on the hull where a single, more powerful tugboat should push to produce the same effect as the original four tugboats.



## SOLUTION

**a. Force-Couple System at  $O$ .** Each of the given forces is resolved into components in the diagram shown (kip units are used). The force-couple system at  $O$  equivalent to the given system of forces consists of a force  $\mathbf{R}$  and a couple  $\mathbf{M}_O^R$  defined as follows:

$$\begin{aligned}\mathbf{R} &= \sum \mathbf{F} \\ &= (2.50\mathbf{i} - 4.33\mathbf{j}) + (3.00\mathbf{i} - 4.00\mathbf{j}) + (-5.00\mathbf{j}) + (3.54\mathbf{i} + 3.54\mathbf{j}) \\ &= 9.04\mathbf{i} - 9.79\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{M}_O^R &= \sum (\mathbf{r} \times \mathbf{F}) \\ &= (-90\mathbf{i} + 50\mathbf{j}) \times (2.50\mathbf{i} - 4.33\mathbf{j}) \\ &\quad + (100\mathbf{i} + 70\mathbf{j}) \times (3.00\mathbf{i} - 4.00\mathbf{j}) \\ &\quad + (400\mathbf{i} + 70\mathbf{j}) \times (-5.00\mathbf{j}) \\ &\quad + (300\mathbf{i} - 70\mathbf{j}) \times (3.54\mathbf{i} + 3.54\mathbf{j}) \\ &= (390 - 125 - 400 - 210 - 2000 + 1062 + 248)\mathbf{k} \\ &= -1035\mathbf{k}\end{aligned}$$

The equivalent force-couple system at  $O$  is thus

$$\mathbf{R} = (9.04 \text{ kips})\mathbf{i} - (9.79 \text{ kips})\mathbf{j} \quad \mathbf{M}_O^R = -(1035 \text{ kip} \cdot \text{ft})\mathbf{k}$$

or  $\mathbf{R} = 13.33 \text{ kips} \angle 47.3^\circ \quad \mathbf{M}_O^R = 1035 \text{ kip} \cdot \text{ft} \downarrow$

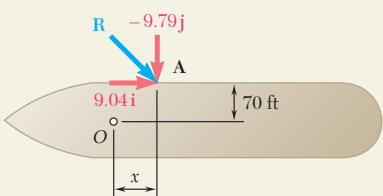
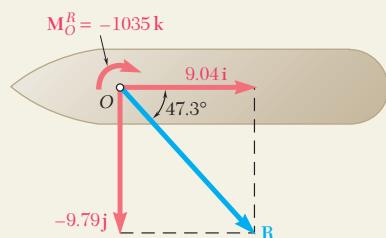
*Remark.* Since all the forces are contained in the plane of the figure, we could have expected the sum of their moments to be perpendicular to that plane. Note that the moment of each force component could have been obtained directly from the diagram by first forming the product of its magnitude and perpendicular distance to  $O$  and then assigning to this product a positive or a negative sign depending upon the sense of the moment.

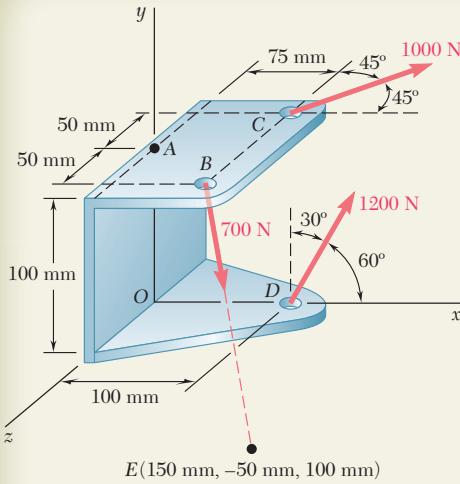
**b. Single Tugboat.** The force exerted by a single tugboat must be equal to  $\mathbf{R}$ , and its point of application  $A$  must be such that the moment of  $\mathbf{R}$  about  $O$  is equal to  $\mathbf{M}_O^R$ . Observing that the position vector of  $A$  is

$$\mathbf{r} = x\mathbf{i} + 70\mathbf{j}$$

we write

$$\begin{aligned}\mathbf{r} \times \mathbf{R} &= \mathbf{M}_O^R \\ (\mathbf{x}\mathbf{i} + 70\mathbf{j}) \times (9.04\mathbf{i} - 9.79\mathbf{j}) &= -1035\mathbf{k} \\ -x(9.79)\mathbf{k} - 633\mathbf{k} &= -1035\mathbf{k} \quad x = 41.1 \text{ ft} \quad \blacktriangleleft\end{aligned}$$





## SAMPLE PROBLEM 3.10

Three cables are attached to a bracket as shown. Replace the forces exerted by the cables with an equivalent force-couple system at A.

### SOLUTION

We first determine the relative position vectors drawn from point A to the points of application of the various forces and resolve the forces into rectangular components. Observing that  $\mathbf{F}_B = (700 \text{ N})\lambda_{BE}$  where

$$\lambda_{BE} = \frac{\overrightarrow{BE}}{BE} = \frac{75\mathbf{i} - 150\mathbf{j} + 50\mathbf{k}}{175}$$

we have, using meters and newtons,

$$\begin{aligned}\mathbf{r}_{B/A} &= \overrightarrow{AB} = 0.075\mathbf{i} + 0.050\mathbf{k} & \mathbf{F}_B &= 300\mathbf{i} - 600\mathbf{j} + 200\mathbf{k} \\ \mathbf{r}_{C/A} &= \overrightarrow{AC} = 0.075\mathbf{i} - 0.050\mathbf{k} & \mathbf{F}_C &= 707\mathbf{i} - 707\mathbf{k} \\ \mathbf{r}_{D/A} &= \overrightarrow{AD} = 0.100\mathbf{i} - 0.100\mathbf{j} & \mathbf{F}_D &= 600\mathbf{i} + 1039\mathbf{j}\end{aligned}$$

The force-couple system at A equivalent to the given forces consists of a force  $\mathbf{R} = \Sigma \mathbf{F}$  and a couple  $\mathbf{M}_A^R = \Sigma (\mathbf{r} \times \mathbf{F})$ . The force  $\mathbf{R}$  is readily obtained by adding respectively the  $x$ ,  $y$ , and  $z$  components of the forces:

$$\mathbf{R} = \Sigma \mathbf{F} = (1607 \text{ N})\mathbf{i} + (439 \text{ N})\mathbf{j} - (507 \text{ N})\mathbf{k}$$

The computation of  $\mathbf{M}_A^R$  will be facilitated if we express the moments of the forces in the form of determinants (Sec. 3.8):

$$\begin{aligned}\mathbf{r}_{B/A} \times \mathbf{F}_B &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\mathbf{i} - 45\mathbf{k} \\ \mathbf{r}_{C/A} \times \mathbf{F}_C &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\mathbf{j} \\ \mathbf{r}_{D/A} \times \mathbf{F}_D &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\mathbf{k}\end{aligned}$$

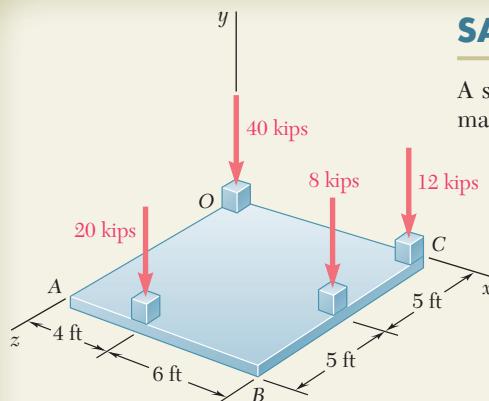
Adding the expressions obtained, we have

$$\mathbf{M}_A^R = \Sigma (\mathbf{r} \times \mathbf{F}) = (30 \text{ N} \cdot \text{m})\mathbf{i} + (17.68 \text{ N} \cdot \text{m})\mathbf{j} + (118.9 \text{ N} \cdot \text{m})\mathbf{k}$$

The rectangular components of the force  $\mathbf{R}$  and the couple  $\mathbf{M}_A^R$  are shown in the adjoining sketch.

## SAMPLE PROBLEM 3.11

A square foundation mat supports the four columns shown. Determine the magnitude and point of application of the resultant of the four loads.

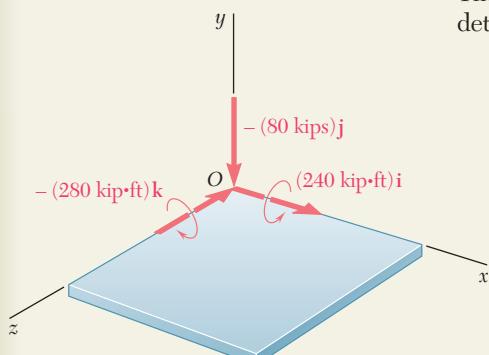


## SOLUTION

We first reduce the given system of forces to a force-couple system at the origin  $O$  of the coordinate system. This force-couple system consists of a force  $\mathbf{R}$  and a couple vector  $\mathbf{M}_O^R$  defined as follows:

$$\mathbf{R} = \sum \mathbf{F} \quad \mathbf{M}_O^R = \sum (\mathbf{r} \times \mathbf{F})$$

The position vectors of the points of application of the various forces are determined, and the computations are arranged in tabular form.



$\mathbf{r}, \text{ft}$	$\mathbf{F}, \text{kips}$	$\mathbf{r} \times \mathbf{F}, \text{kip} \cdot \text{ft}$
0	$-40\mathbf{j}$	0
$10\mathbf{i}$	$-12\mathbf{j}$	$-120\mathbf{k}$
$10\mathbf{i} + 5\mathbf{k}$	$-8\mathbf{j}$	$40\mathbf{i} - 80\mathbf{k}$
$4\mathbf{i} + 10\mathbf{k}$	$-20\mathbf{j}$	$200\mathbf{i} - 80\mathbf{k}$
	$\mathbf{R} = -80\mathbf{j}$	$\mathbf{M}_O^R = 240\mathbf{i} - 280\mathbf{k}$

Since the force  $\mathbf{R}$  and the couple vector  $\mathbf{M}_O^R$  are mutually perpendicular, the force-couple system obtained can be reduced further to a single force  $\mathbf{R}$ . The new point of application of  $\mathbf{R}$  will be selected in the plane of the mat and in such a way that the moment of  $\mathbf{R}$  about  $O$  will be equal to  $\mathbf{M}_O^R$ . Denoting by  $\mathbf{r}$  the position vector of the desired point of application, and by  $x$  and  $z$  its coordinates, we write

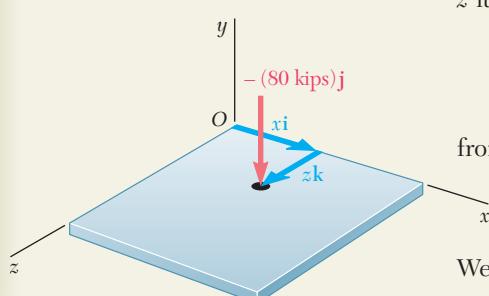
$$\begin{aligned} \mathbf{r} \times \mathbf{R} &= \mathbf{M}_O^R \\ (x\mathbf{i} + z\mathbf{k}) \times (-80\mathbf{j}) &= 240\mathbf{i} - 280\mathbf{k} \\ -80x\mathbf{k} + 80z\mathbf{i} &= 240\mathbf{i} - 280\mathbf{k} \end{aligned}$$

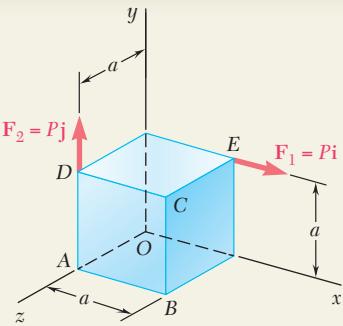
from which it follows that

$$\begin{aligned} -80x &= -280 & 80z &= 240 \\ x &= 3.50 \text{ ft} & z &= 3.00 \text{ ft} \end{aligned}$$

We conclude that the resultant of the given system of forces is

$$\mathbf{R} = 80 \text{ kips} \downarrow \quad \text{at } x = 3.50 \text{ ft}, z = 3.00 \text{ ft}$$





## SAMPLE PROBLEM 3.12

Two forces of the same magnitude  $P$  act on a cube of side  $a$  as shown. Replace the two forces by an equivalent wrench, and determine (a) the magnitude and direction of the resultant force  $\mathbf{R}$ , (b) the pitch of the wrench, (c) the point where the axis of the wrench intersects the  $yz$  plane.

### SOLUTION

**Equivalent Force-Couple System at  $O$ .** We first determine the equivalent force-couple system at the origin  $O$ . We observe that the position vectors of the points of application  $E$  and  $D$  of the two given forces are  $\mathbf{r}_E = a\mathbf{i} + a\mathbf{j}$  and  $\mathbf{r}_D = a\mathbf{j} + a\mathbf{k}$ . The resultant  $\mathbf{R}$  of the two forces and their moment resultant  $\mathbf{M}_O^R$  about  $O$  are

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = P\mathbf{i} + P\mathbf{j} = P(\mathbf{i} + \mathbf{j}) \quad (1)$$

$$\begin{aligned} \mathbf{M}_O^R &= \mathbf{r}_E \times \mathbf{F}_1 + \mathbf{r}_D \times \mathbf{F}_2 = (a\mathbf{i} + a\mathbf{j}) \times P\mathbf{i} + (a\mathbf{j} + a\mathbf{k}) \times P\mathbf{j} \\ &= -Pak - Pai = -Pa(\mathbf{i} + \mathbf{k}) \end{aligned} \quad (2)$$

**a. Resultant Force  $\mathbf{R}$ .** It follows from Eq. (1) and the adjoining sketch that the resultant force  $\mathbf{R}$  has the magnitude  $R = P\sqrt{2}$ , lies in the  $xy$  plane, and forms angles of  $45^\circ$  with the  $x$  and  $y$  axes. Thus

$$R = P\sqrt{2} \quad \theta_x = \theta_y = 45^\circ \quad \theta_z = 90^\circ \quad \blacktriangleleft$$

**b. Pitch of Wrench.** Recalling formula (3.62) of Sec. 3.21 and Eqs. (1) and (2) above, we write

$$p = \frac{\mathbf{R} \cdot \mathbf{M}_O^R}{R^2} = \frac{P(\mathbf{i} + \mathbf{j}) \cdot (-Pa)(\mathbf{i} + \mathbf{k})}{(P\sqrt{2})^2} = \frac{-P^2a(1 + 0 + 0)}{2P^2} \quad p = -\frac{a}{2} \quad \blacktriangleleft$$

**c. Axis of Wrench.** It follows from the above and from Eq. (3.61) that the wrench consists of the force  $\mathbf{R}$  found in (1) and the couple vector

$$\mathbf{M}_1 = p\mathbf{R} = -\frac{a}{2}P(\mathbf{i} + \mathbf{j}) = -\frac{Pa}{2}(\mathbf{i} + \mathbf{j}) \quad (3)$$

To find the point where the axis of the wrench intersects the  $yz$  plane, we express that the moment of the wrench about  $O$  is equal to the moment resultant  $\mathbf{M}_O^R$  of the original system:

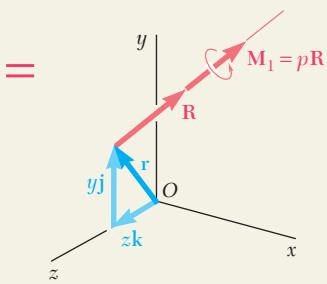
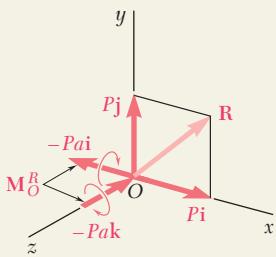
$$\mathbf{M}_1 + \mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R$$

or, noting that  $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$  and substituting for  $\mathbf{R}$ ,  $\mathbf{M}_O^R$ , and  $\mathbf{M}_1$  from Eqs. (1), (2), and (3),

$$\begin{aligned} -\frac{Pa}{2}(\mathbf{i} + \mathbf{j}) + (y\mathbf{j} + z\mathbf{k}) \times P(\mathbf{i} + \mathbf{j}) &= -Pa(\mathbf{i} + \mathbf{k}) \\ -\frac{Pa}{2}\mathbf{i} - \frac{Pa}{2}\mathbf{j} - Py\mathbf{k} + Pz\mathbf{j} - Pz\mathbf{i} &= -Pa\mathbf{i} - Pa\mathbf{k} \end{aligned}$$

Equating the coefficients of  $\mathbf{k}$ , and then the coefficients of  $\mathbf{j}$ , we find

$$y = a \quad z = a/2 \quad \blacktriangleleft$$



# SOLVING PROBLEMS ON YOUR OWN

This lesson was devoted to the reduction and simplification of force systems. In solving the problems which follow, you will be asked to perform the operations discussed below.

**1. Reducing a force system to a force and a couple at a given point A.** The force is the *resultant*  $\mathbf{R}$  of the system and is obtained by adding the various forces; the moment of the couple is the *moment resultant* of the system and is obtained by adding the moments about  $A$  of the various forces. We have

$$\mathbf{R} = \Sigma \mathbf{F} \quad \mathbf{M}_A^R = \Sigma (\mathbf{r} \times \mathbf{F})$$

where the position vector  $\mathbf{r}$  is drawn from  $A$  to *any point* on the line of action of  $\mathbf{F}$ .

**2. Moving a force-couple system from point A to point B.** If you wish to reduce a given force system to a force-couple system at point  $B$  after you have reduced it to a force-couple system at point  $A$ , you need not recompute the moments of the forces about  $B$ . The resultant  $\mathbf{R}$  remains unchanged, and the new moment resultant  $\mathbf{M}_B^R$  can be obtained by adding to  $\mathbf{M}_A^R$  the moment about  $B$  of the force  $\mathbf{R}$  applied at  $A$  [Sample Prob. 3.8]. Denoting by  $\mathbf{s}$  the vector drawn from  $B$  to  $A$ , you can write

$$\mathbf{M}_B^R = \mathbf{M}_A^R + \mathbf{s} \times \mathbf{R}$$

**3. Checking whether two force systems are equivalent.** First reduce each force system to a force-couple system *at the same, but arbitrary, point A* (as explained in paragraph 1). The two systems are equivalent (that is, they have the same effect on the given rigid body) if the two force-couple systems you have obtained are identical, that is, if

$$\Sigma \mathbf{F} = \Sigma \mathbf{F}' \quad \text{and} \quad \Sigma \mathbf{M}_A = \Sigma \mathbf{M}'_A$$

You should recognize that if the first of these equations is not satisfied, that is, if the two systems do not have the same resultant  $\mathbf{R}$ , the two systems cannot be equivalent and there is then no need to check whether or not the second equation is satisfied.

**4. Reducing a given force system to a single force.** First reduce the given system to a force-couple system consisting of the resultant  $\mathbf{R}$  and the couple vector  $\mathbf{M}_A^R$  at some convenient point  $A$  (as explained in paragraph 1). You will recall from the previous lesson that further reduction to a single force is possible *only if the*

(continued)

force  $\mathbf{R}$  and the couple vector  $\mathbf{M}_A^R$  are mutually perpendicular. This will certainly be the case for systems of forces which are either *concurrent*, *coplanar*, or *parallel*. The required single force can then be obtained by moving  $\mathbf{R}$  until its moment about  $A$  is equal to  $\mathbf{M}_A^R$ , as you did in several problems of the preceding lesson. More formally, you can write that the position vector  $\mathbf{r}$  drawn from  $A$  to any point on the line of action of the single force  $\mathbf{R}$  must satisfy the equation

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_A^R$$

This procedure was used in Sample Probs. 3.8, 3.9, and 3.11.

**5. Reducing a given force system to a wrench.** If the given system is comprised of forces which are not concurrent, coplanar, or parallel, the equivalent force-couple system at a point  $A$  will consist of a force  $\mathbf{R}$  and a couple vector  $\mathbf{M}_A^R$  which, in general, *are not mutually perpendicular*. (To check whether  $\mathbf{R}$  and  $\mathbf{M}_A^R$  are mutually perpendicular, form their scalar product. If this product is zero, they are mutually perpendicular; otherwise, they are not.) If  $\mathbf{R}$  and  $\mathbf{M}_A^R$  are not mutually perpendicular, the force-couple system (and thus the given system of forces) *cannot be reduced to a single force*. However, the system can be reduced to a *wrench*—the combination of a force  $\mathbf{R}$  and a couple vector  $\mathbf{M}_1$  directed along a common line of action called the *axis of the wrench* (Fig. 3.47). The ratio  $p = M_1/R$  is called the *pitch* of the wrench.

To reduce a given force system to a wrench, you should follow these steps:

- a. Reduce the given system to an equivalent force-couple system  $(\mathbf{R}, \mathbf{M}_O^R)$ , typically located at the origin  $O$ .
- b. Determine the pitch  $p$  from Eq. (3.62)

$$p = \frac{M_1}{R} = \frac{\mathbf{R} \cdot \mathbf{M}_O^R}{R^2} \quad (3.62)$$

and the couple vector from  $\mathbf{M}_1 = p\mathbf{R}$ .

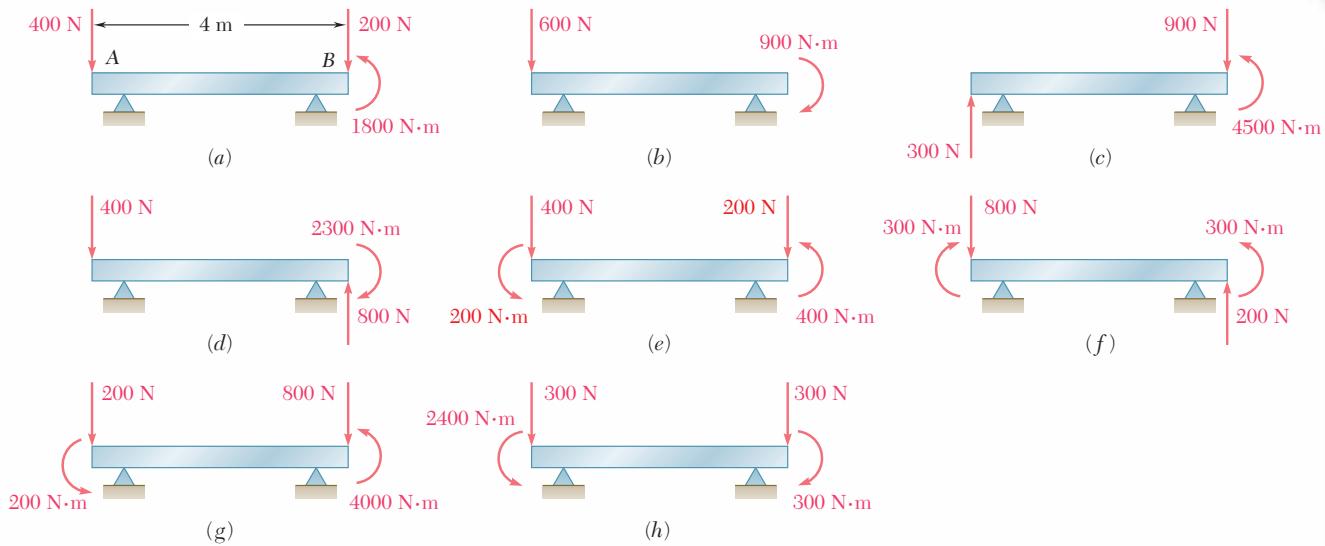
- c. Express that the moment about  $O$  of the wrench is equal to the moment resultant  $\mathbf{M}_O^R$  of the force-couple system at  $O$ :

$$\mathbf{M}_1 + \mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R \quad (3.63)$$

This equation allows you to determine the point where the line of action of the wrench intersects a specified plane, since the position vector  $\mathbf{r}$  is directed from  $O$  to that point.

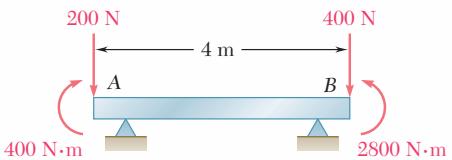
These steps are illustrated in Sample Prob. 3.12. Although the determination of a wrench and the point where its axis intersects a plane may appear difficult, the process is simply the application of several of the ideas and techniques developed in this chapter. Thus, once you have mastered the wrench, you can feel confident that you understand much of Chap. 3.

# PROBLEMS

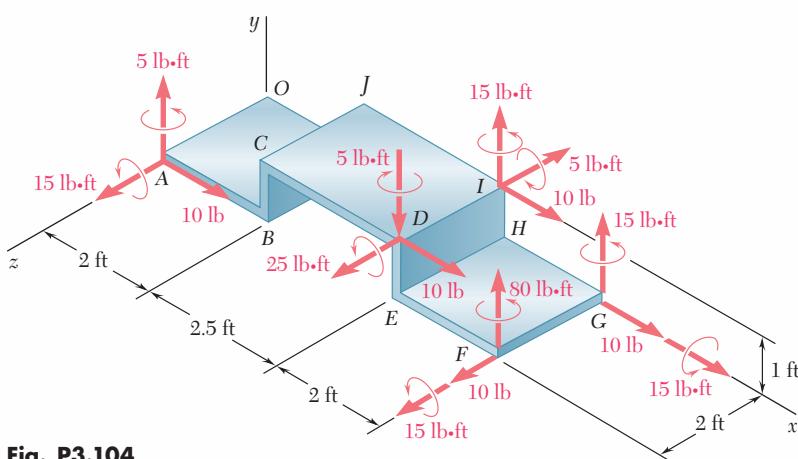


**Fig. P3.101**

- 3.101** A 4-m-long beam is subjected to a variety of loadings. (a) Replace each loading with an equivalent force-couple system at end A of the beam. (b) Which of the loadings are equivalent?
- 3.102** A 4-m-long beam is loaded as shown. Determine the loading of Prob. 3.101 which is equivalent to this loading.
- 3.103** Determine the single equivalent force and the distance from point A to its line of action for the beam and loading of (a) Prob. 3.101b, (b) Prob. 3.101d, (c) Prob. 3.101e.
- 3.104** Five separate force-couple systems act at the corners of a piece of sheet metal, which has been bent into the shape shown. Determine which of these systems is equivalent to a force  $\mathbf{F} = (10 \text{ lb})\mathbf{i}$  and a couple of moment  $\mathbf{M} = (15 \text{ lb} \cdot \text{ft})\mathbf{j} + (15 \text{ lb} \cdot \text{ft})\mathbf{k}$  located at the origin.



**Fig. P3.102**



**Fig. P3.104**

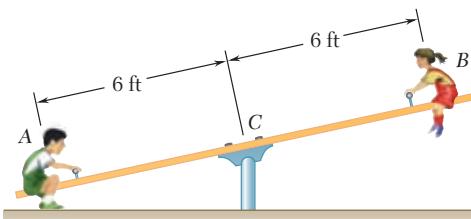


Fig. P3.105

- 3.105** The weights of two children sitting at ends *A* and *B* of a seesaw are 84 lb and 64 lb, respectively. Where should a third child sit so that the resultant of the weights of the three children will pass through *C* if she weighs (a) 60 lb, (b) 52 lb.

- 3.106** Three stage lights are mounted on a pipe as shown. The lights at *A* and *B* each weigh 4.1 lb, while the one at *C* weighs 3.5 lb. (a) If  $d = 25$  in., determine the distance from *D* to the line of action of the resultant of the weights of the three lights. (b) Determine the value of  $d$  so that the resultant of the weights passes through the midpoint of the pipe.

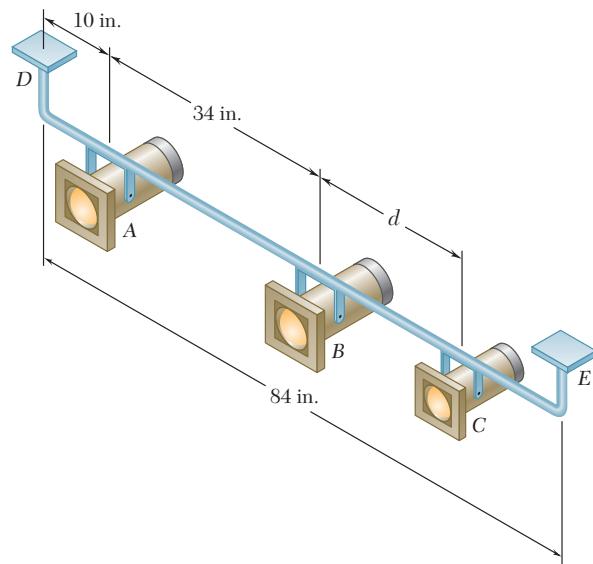


Fig. P3.106

- 3.107** A beam supports three loads of given magnitude and a fourth load whose magnitude is a function of position. If  $b = 1.5$  m and the loads are to be replaced with a single equivalent force, determine (a) the value of  $a$  so that the distance from support *A* to the line of action of the equivalent force is maximum, (b) the magnitude of the equivalent force and its point of application on the beam.

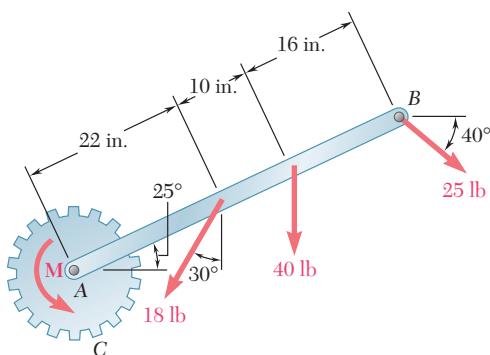


Fig. P3.108

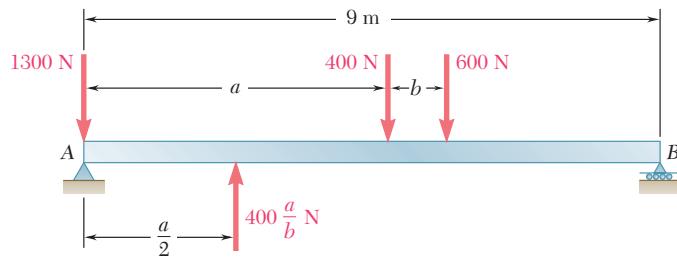


Fig. P3.107

- 3.108** Gear *C* is rigidly attached to arm *AB*. If the forces and couple shown can be reduced to a single equivalent force at *A*, determine the equivalent force and the magnitude of the couple *M*.

- 3.109** A couple of magnitude  $M = 54 \text{ lb} \cdot \text{in}$ . and the three forces shown are applied to an angle bracket. (a) Find the resultant of this system of forces. (b) Locate the points where the line of action of the resultant intersects line  $AB$  and line  $BC$ .

- 3.110** A couple  $\mathbf{M}$  and the three forces shown are applied to an angle bracket. Find the moment of the couple if the line of action of the resultant of the force system is to pass through (a) point  $A$ , (b) point  $B$ , (c) point  $C$ .

- 3.111** Four forces act on a  $700 \times 375\text{-mm}$  plate as shown. (a) Find the resultant of these forces. (b) Locate the two points where the line of action of the resultant intersects the edge of the plate.

- 3.112** Solve Prob. 3.111, assuming that the 760-N force is directed to the right.

- 3.113** A truss supports the loading shown. Determine the equivalent force acting on the truss and the point of intersection of its line of action with a line drawn through points  $A$  and  $G$ .

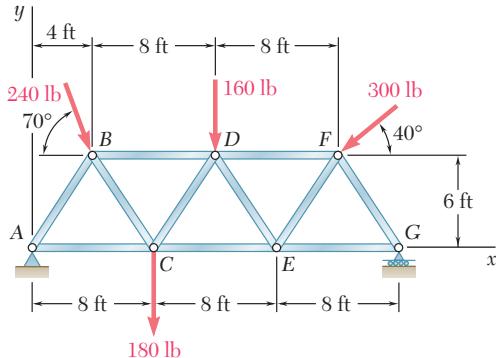


Fig. P3.113

- 3.114** Pulleys  $A$  and  $B$  are mounted on bracket  $CDEF$ . The tension on each side of the two belts is as shown. Replace the four forces with a single equivalent force, and determine where its line of action intersects the bottom edge of the bracket.

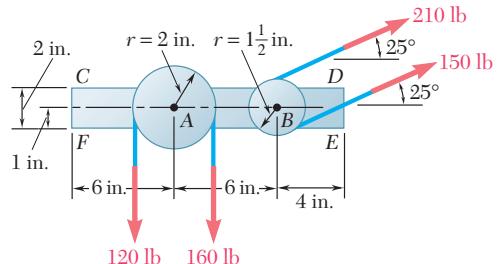


Fig. P3.114

- 3.115** A machine component is subjected to the forces and couples shown. The component is to be held in place by a single rivet that can resist a force but not a couple. For  $P = 0$ , determine the location of the rivet hole if it is to be located (a) on line  $FG$ , (b) on line  $GH$ .

- 3.116** Solve Prob. 3.115, assuming that  $P = 60 \text{ N}$ .

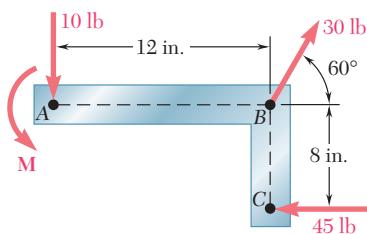


Fig. P3.109 and P3.110

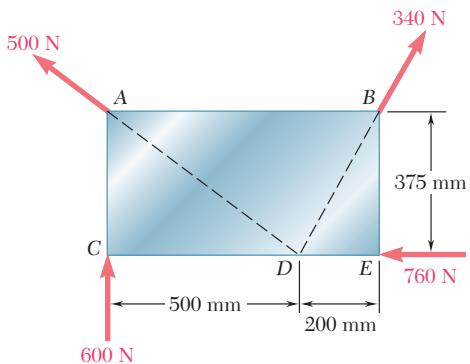


Fig. P3.111

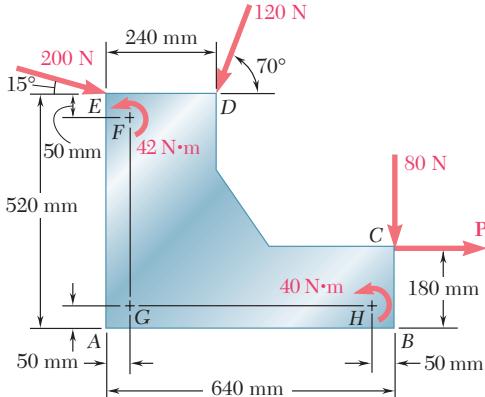


Fig. P3.115

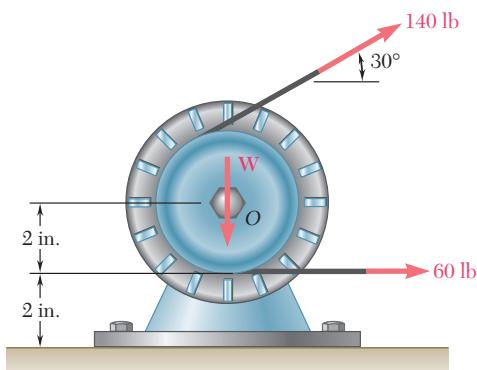


Fig. P3.117

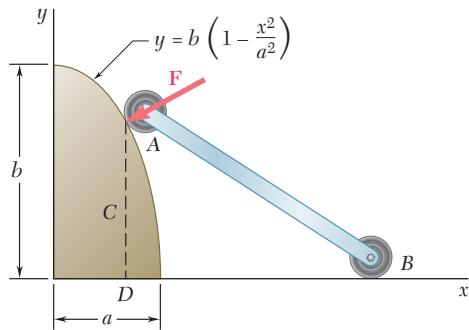


Fig. P3.118

- 3.117** A 32-lb motor is mounted on the floor. Find the resultant of the weight and the forces exerted on the belt, and determine where the line of action of the resultant intersects the floor.

- 3.118** As follower *AB* rolls along the surface of member *C*, it exerts a constant force **F** perpendicular to the surface. (a) Replace **F** with an equivalent force-couple system at the point *D* obtained by drawing the perpendicular from the point of contact to the *x* axis. (b) For *a* = 1 m and *b* = 2 m, determine the value of *x* for which the moment of the equivalent force-couple system at *D* is maximum.

- 3.119** Four forces are applied to the machine component *ABDE* as shown. Replace these forces by an equivalent force-couple system at *A*.

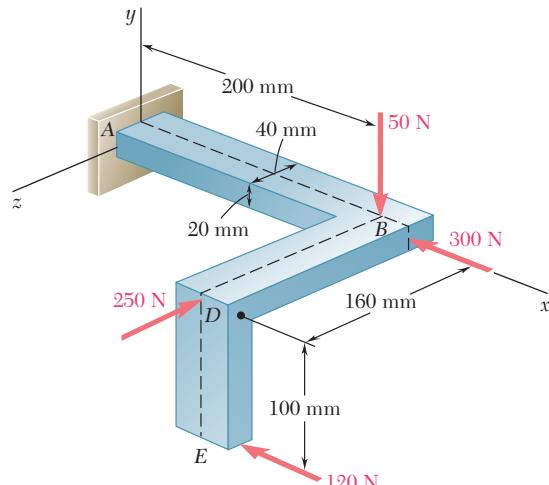


Fig. P3.119

- 3.120** Two 150-mm-diameter pulleys are mounted on line shaft *AD*. The belts at *B* and *C* lie in vertical planes parallel to the *yz* plane. Replace the belt forces shown with an equivalent force-couple system at *A*.

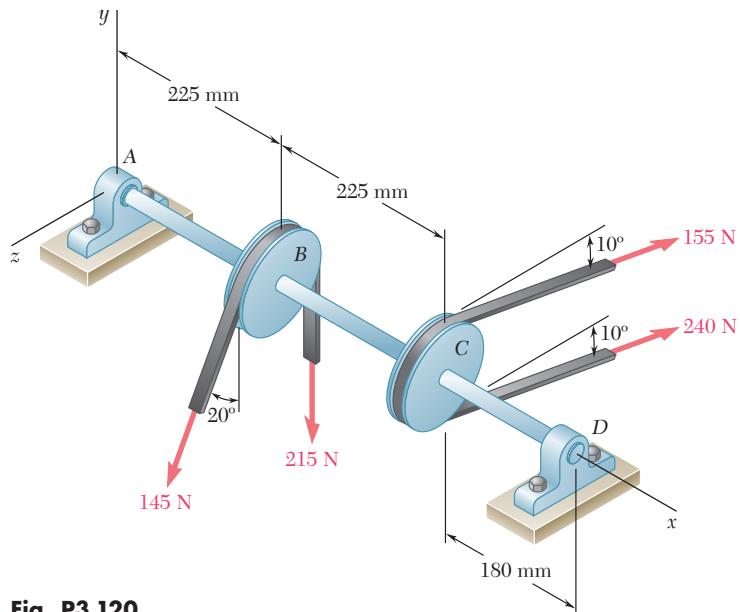


Fig. P3.120

- 3.121** While using a pencil sharpener, a student applies the forces and couple shown. (a) Determine the forces exerted at *B* and *C* knowing that these forces and the couple are equivalent to a force-couple system at *A* consisting of the force  $\mathbf{R} = (2.6 \text{ lb})\mathbf{i} + R_y\mathbf{j} - (0.7 \text{ lb})\mathbf{k}$  and the couple  $\mathbf{M}_A^R = M_x\mathbf{i} + (1.0 \text{ lb} \cdot \text{ft})\mathbf{j} - (0.72 \text{ lb} \cdot \text{ft})\mathbf{k}$ . (b) Find the corresponding values of  $R_y$  and  $M_x$ .

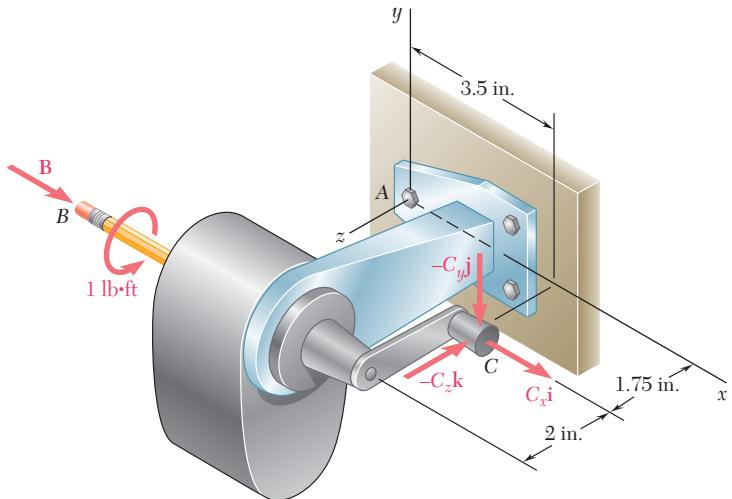


Fig. P3.121

- 3.122** A mechanic uses a crowfoot wrench to loosen a bolt at *C*. The mechanic holds the socket wrench handle at points *A* and *B* and applies forces at these points. Knowing that these forces are equivalent to a force-couple system at *C* consisting of the force  $\mathbf{C} = -(8 \text{ lb})\mathbf{i} + (4 \text{ lb})\mathbf{k}$  and the couple  $\mathbf{M}_C = (360 \text{ lb} \cdot \text{in.})\mathbf{i}$ , determine the forces applied at *A* and at *B* when  $A_z = 2 \text{ lb}$ .

- 3.123** As an adjustable brace *BC* is used to bring a wall into plumb, the force-couple system shown is exerted on the wall. Replace this force-couple system with an equivalent force-couple system at *A* if  $R = 21.2 \text{ lb}$  and  $M = 13.25 \text{ lb} \cdot \text{ft}$ .

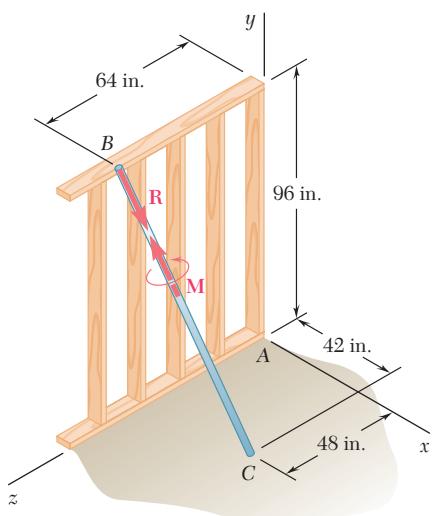


Fig. P3.123

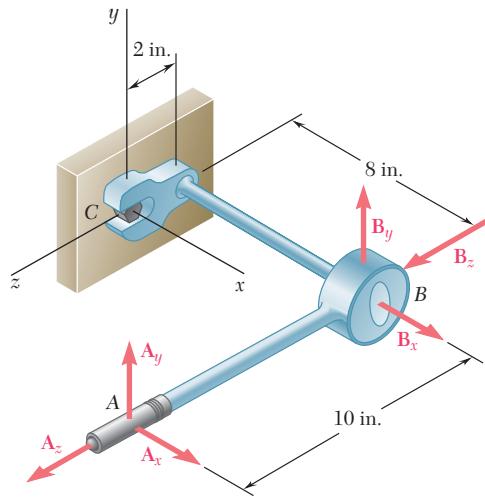


Fig. P3.122

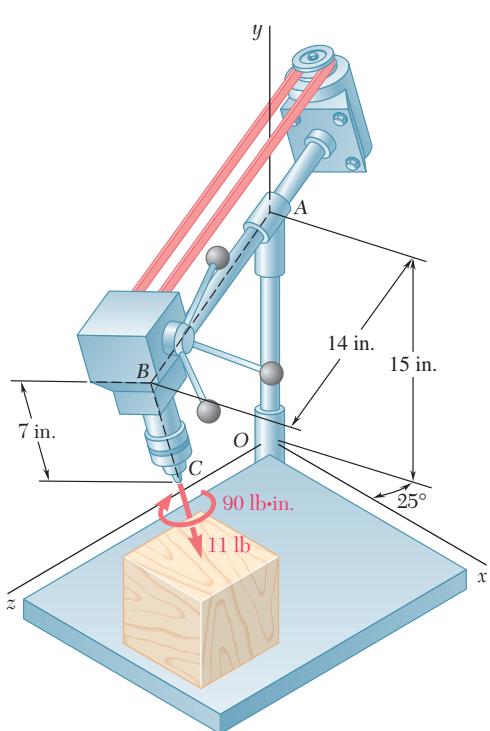


Fig. P3.126

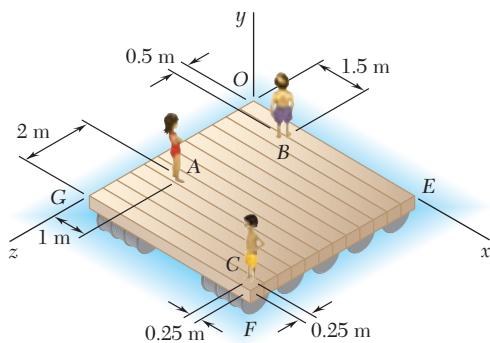


Fig. P3.127 and P3.128

- 3.124** A mechanic replaces a car's exhaust system by firmly clamping the catalytic converter  $FG$  to its mounting brackets  $H$  and  $I$  and then loosely assembling the mufflers and the exhaust pipes. To position the tailpipe  $AB$ , he pushes in and up at  $A$  while pulling down at  $B$ . (a) Replace the given force system with an equivalent force-couple system at  $D$ . (b) Determine whether pipe  $CD$  tends to rotate clockwise or counterclockwise relative to muffler  $DE$ , as viewed by the mechanic.

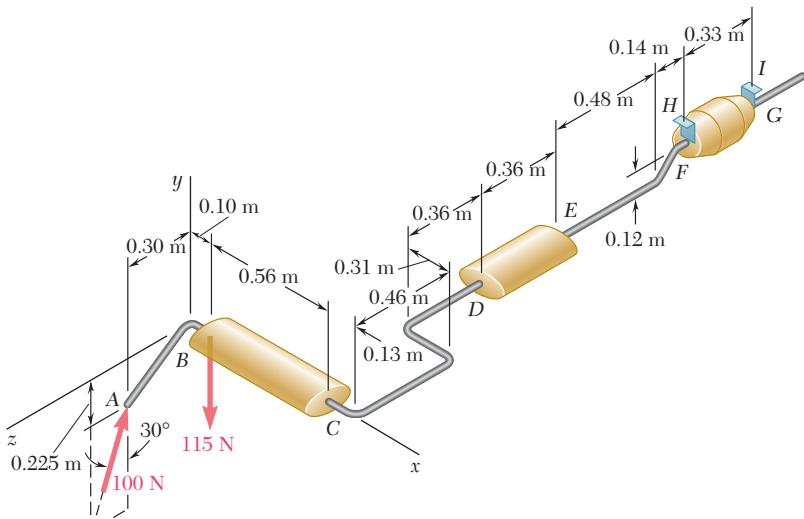


Fig. P3.124

- 3.125** For the exhaust system of Prob. 3.124, (a) replace the given force system with an equivalent force-couple system at  $F$ , where the exhaust pipe is connected to the catalytic converter, (b) determine whether pipe  $EF$  tends to rotate clockwise or counterclockwise, as viewed by the mechanic.

- 3.126** The head-and-motor assembly of a radial drill press was originally positioned with arm  $AB$  parallel to the  $z$  axis and the axis of the chuck and bit parallel to the  $y$  axis. The assembly was then rotated  $25^\circ$  about the  $y$  axis and  $20^\circ$  about the centerline of the horizontal arm  $AB$ , bringing it into the position shown. The drilling process was started by switching on the motor and rotating the handle to bring the bit into contact with the workpiece. Replace the force and couple exerted by the drill press with an equivalent force-couple system at the center  $O$  of the base of the vertical column.

- 3.127** Three children are standing on a  $5 \times 5$ -m raft. If the weights of the children at points  $A$ ,  $B$ , and  $C$  are 375 N, 260 N, and 400 N, respectively, determine the magnitude and the point of application of the resultant of the three weights.

- 3.128** Three children are standing on a  $5 \times 5$ -m raft. The weights of the children at points  $A$ ,  $B$ , and  $C$  are 375 N, 260 N, and 400 N, respectively. If a fourth child of weight 425 N climbs onto the raft, determine where she should stand if the other children remain in the positions shown and the line of action of the resultant of the four weights is to pass through the center of the raft.

- 3.129** Four signs are mounted on a frame spanning a highway, and the magnitudes of the horizontal wind forces acting on the signs are as shown. Determine the magnitude and the point of application of the resultant of the four wind forces when  $a = 1$  ft and  $b = 12$  ft.

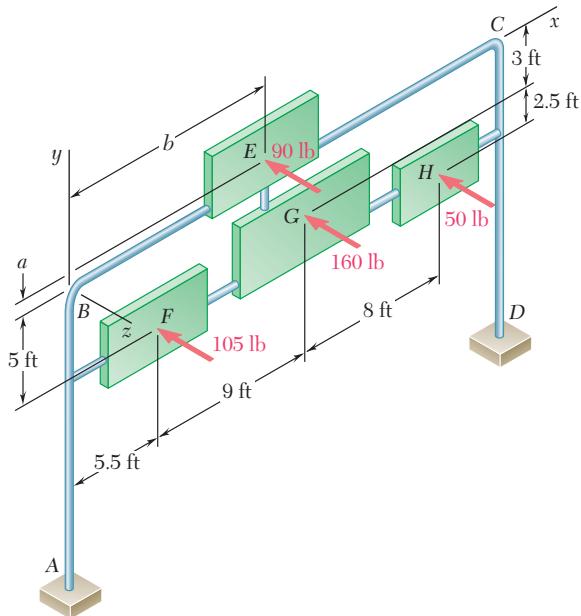


Fig. P3.129 and P3.130

- 3.130** Four signs are mounted on a frame spanning a highway, and the magnitudes of the horizontal wind forces acting on the signs are as shown. Determine  $a$  and  $b$  so that the point of application of the resultant of the four forces is at  $G$ .

- \*3.131** A group of students loads a  $2 \times 3.3$ -m flatbed trailer with two  $0.66 \times 0.66 \times 0.66$ -m boxes and one  $0.66 \times 0.66 \times 1.2$ -m box. Each of the boxes at the rear of the trailer is positioned so that it is aligned with both the back and a side of the trailer. Determine the smallest load the students should place in a second  $0.66 \times 0.66 \times 1.2$ -m box and where on the trailer they should secure it, without any part of the box overhanging the sides of the trailer, if each box is uniformly loaded and the line of action of the resultant of the weights of the four boxes is to pass through the point of intersection of the centerlines of the trailer and the axle. (Hint: Keep in mind that the box may be placed either on its side or on its end.)

- \*3.132** Solve Prob. 3.131 if the students want to place as much weight as possible in the fourth box and at least one side of the box must coincide with a side of the trailer.

- \*3.133** Three forces of the same magnitude  $P$  act on a cube of side  $a$  as shown. Replace the three forces by an equivalent wrench and determine (a) the magnitude and direction of the resultant force  $\mathbf{R}$ , (b) the pitch of the wrench, (c) the axis of the wrench.

- \*3.134** A piece of sheet metal is bent into the shape shown and is acted upon by three forces. If the forces have the same magnitude  $P$ , replace them with an equivalent wrench and determine (a) the magnitude and the direction of the resultant force  $\mathbf{R}$ , (b) the pitch of the wrench, (c) the axis of the wrench.

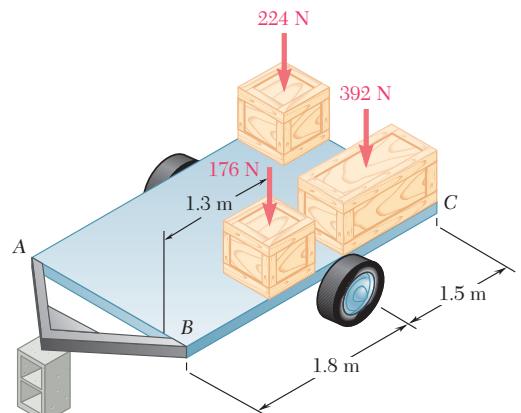


Fig. P3.131

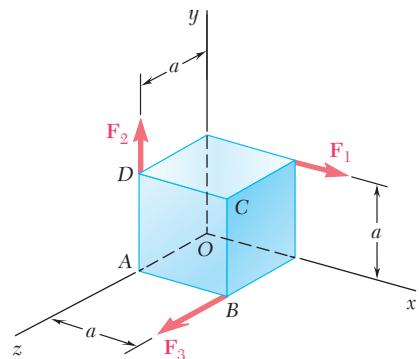


Fig. P3.133

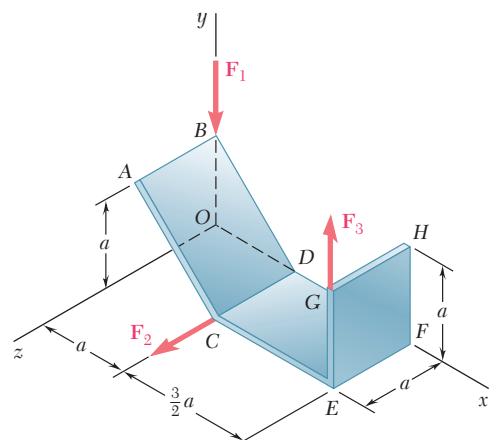


Fig. P3.134

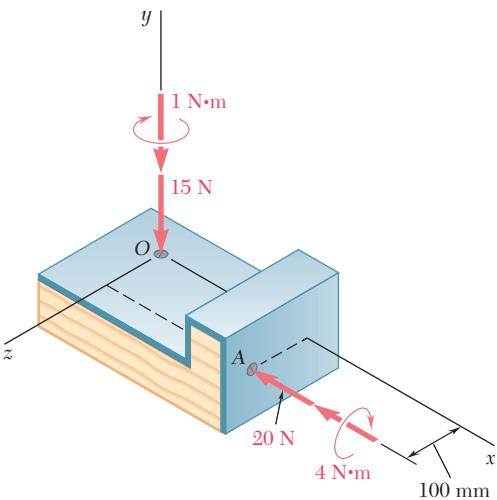


Fig. P3.135

**\*3.135 and \*3.136** The forces and couples shown are applied to two screws as a piece of sheet metal is fastened to a block of wood. Reduce the forces and the couples to an equivalent wrench and determine (a) the resultant force  $\mathbf{R}$ , (b) the pitch of the wrench, (c) the point where the axis of the wrench intersects the  $xz$  plane.

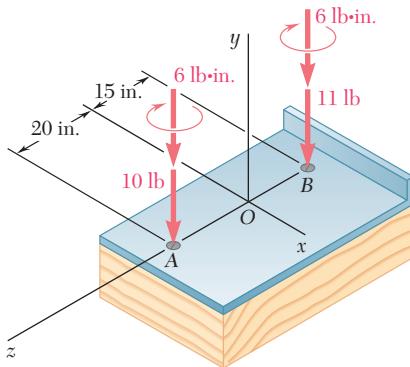


Fig. P3.136

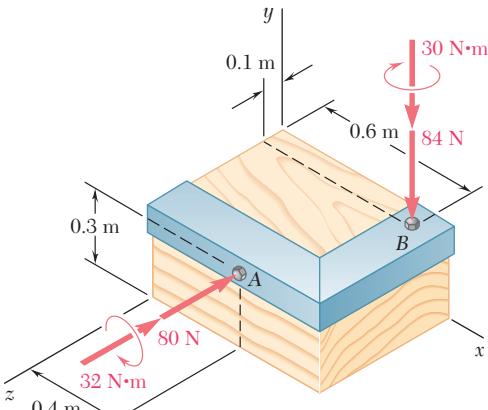


Fig. P3.137

**\*3.137 and \*3.138** Two bolts at  $A$  and  $B$  are tightened by applying the forces and couples shown. Replace the two wrenches with a single equivalent wrench and determine (a) the resultant  $\mathbf{R}$ , (b) the pitch of the single equivalent wrench, (c) the point where the axis of the wrench intersects the  $xz$  plane.

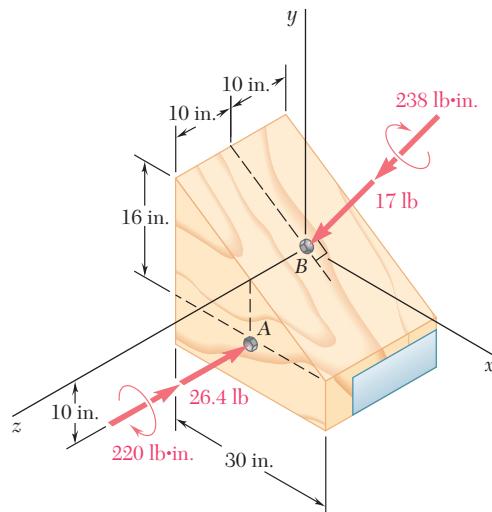


Fig. P3.138

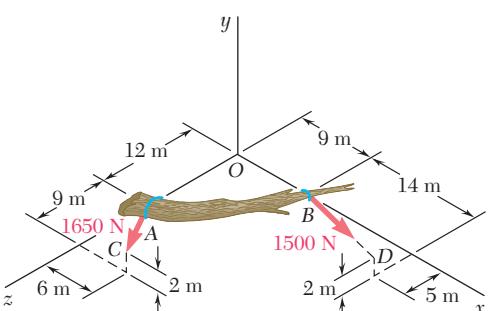


Fig. P3.139

**\*3.139** Two ropes attached at  $A$  and  $B$  are used to move the trunk of a fallen tree. Replace the forces exerted by the ropes with an equivalent wrench and determine (a) the resultant force  $\mathbf{R}$ , (b) the pitch of the wrench, (c) the point where the axis of the wrench intersects the  $yz$  plane.

- \*3.140** A flagpole is guyed by three cables. If the tensions in the cables have the same magnitude  $P$ , replace the forces exerted on the pole with an equivalent wrench and determine (a) the resultant force  $\mathbf{R}$ , (b) the pitch of the wrench, (c) the point where the axis of the wrench intersects the  $xz$  plane.

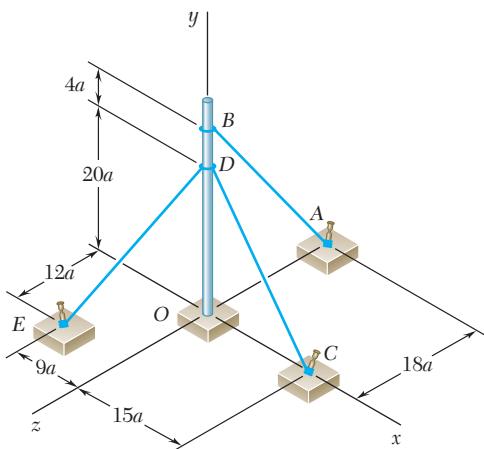


Fig. P3.140

- \*3.141 and \*3.142** Determine whether the force-and-couple system shown can be reduced to a single equivalent force  $\mathbf{R}$ . If it can, determine  $\mathbf{R}$  and the point where the line of action of  $\mathbf{R}$  intersects the  $yz$  plane. If it cannot be so reduced, replace the given system with an equivalent wrench and determine its resultant, its pitch, and the point where its axis intersects the  $yz$  plane.

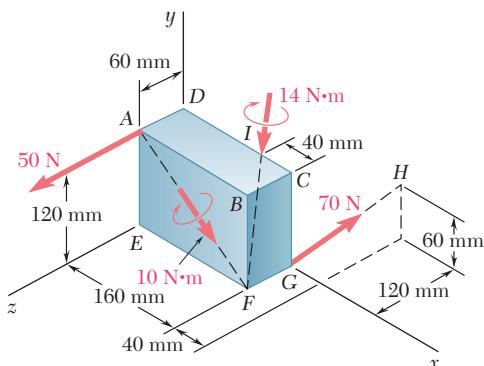


Fig. P3.141

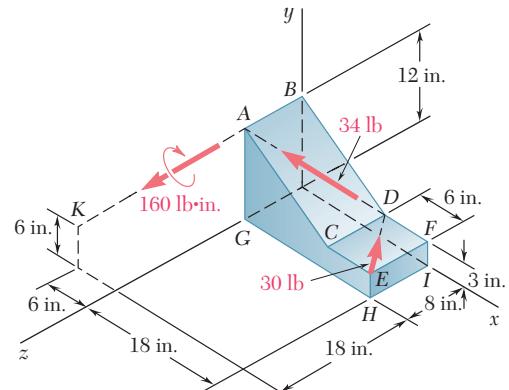


Fig. P3.142

- \*3.143** Replace the wrench shown with an equivalent system consisting of two forces perpendicular to the  $y$  axis and applied respectively at  $A$  and  $B$ .

- \*3.144** Show that, in general, a wrench can be replaced with two forces chosen in such a way that one force passes through a given point while the other force lies in a given plane.

- \*3.145** Show that a wrench can be replaced with two perpendicular forces, one of which is applied at a given point.

- \*3.146** Show that a wrench can be replaced with two forces, one of which has a prescribed line of action.

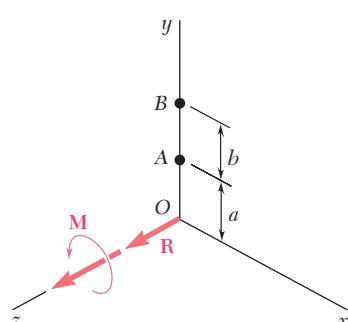


Fig. P3.143

# REVIEW AND SUMMARY

## Principle of transmissibility

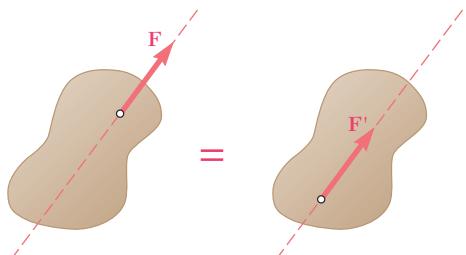


Fig. 3.48

In this chapter we studied the effect of forces exerted on a rigid body. We first learned to distinguish between *external* and *internal* forces [Sec. 3.2] and saw that, according to the *principle of transmissibility*, the effect of an external force on a rigid body remains unchanged if that force is moved along its line of action [Sec. 3.3]. In other words, two forces  $\mathbf{F}$  and  $\mathbf{F}'$  acting on a rigid body at two different points have the same effect on that body if they have the same magnitude, same direction, and same one of action (Fig. 3.48). Two such forces are said to be *equivalent*.

Before proceeding with the discussion of *equivalent systems of forces*, we introduced the concept of the *vector product of two vectors* [Sec. 3.4]. The vector product

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q}$$

of the vectors  $\mathbf{P}$  and  $\mathbf{Q}$  was defined as a vector perpendicular to the plane containing  $\mathbf{P}$  and  $\mathbf{Q}$  (Fig. 3.49), of magnitude

$$V = PQ \sin \theta \quad (3.1)$$

and directed in such a way that a person located at the tip of  $\mathbf{V}$  will observe as counterclockwise the rotation through  $\theta$  which brings the vector  $\mathbf{P}$  in line with the vector  $\mathbf{Q}$ . The three vectors  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{V}$ —taken in that order—are said to form a *right-handed triad*. It follows that the vector products  $\mathbf{Q} \times \mathbf{P}$  and  $\mathbf{P} \times \mathbf{Q}$  are represented by equal and opposite vectors. We have

$$\mathbf{Q} \times \mathbf{P} = -(\mathbf{P} \times \mathbf{Q}) \quad (3.4)$$

It also follows from the definition of the vector product of two vectors that the vector products of the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are

$$\mathbf{i} \times \mathbf{i} = 0 \quad \mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

and so on. The sign of the vector product of two unit vectors can be obtained by arranging in a circle and in counterclockwise order the three letters representing the unit vectors (Fig. 3.50): The vector product of two unit vectors will be positive if they follow each other in counterclockwise order and negative if they follow each other in clockwise order.

The *rectangular components of the vector product*  $\mathbf{V}$  of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  were expressed [Sec. 3.5] as

$$\begin{aligned} V_x &= P_y Q_z - P_z Q_y \\ V_y &= P_z Q_x - P_x Q_z \\ V_z &= P_x Q_y - P_y Q_x \end{aligned} \quad (3.9)$$

## Rectangular components of vector product

Fig. 3.49

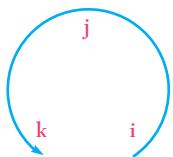
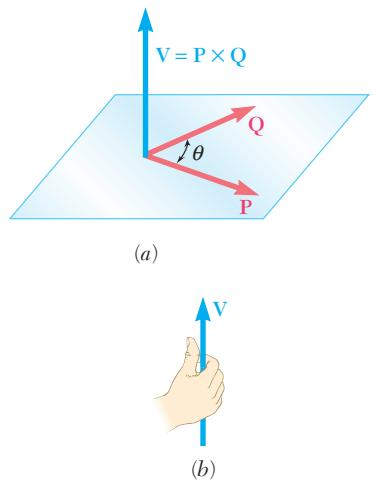


Fig. 3.50

Using a determinant, we also wrote

$$\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad (3.10)$$

The *moment of a force  $\mathbf{F}$  about a point  $O$*  was defined [Sec. 3.6] as the vector product

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \quad (3.11)$$

where  $\mathbf{r}$  is the *position vector* drawn from  $O$  to the point of application  $A$  of the force  $\mathbf{F}$  (Fig. 3.51). Denoting by  $\theta$  the angle between the lines of action of  $\mathbf{r}$  and  $\mathbf{F}$ , we found that the magnitude of the moment of  $\mathbf{F}$  about  $O$  can be expressed as

$$M_O = rF \sin \theta = Fd \quad (3.12)$$

where  $d$  represents the perpendicular distance from  $O$  to the line of action of  $\mathbf{F}$ .

The *rectangular components of the moment  $\mathbf{M}_O$  of a force  $\mathbf{F}$*  were expressed [Sec. 3.8] as

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \quad (3.18)$$

where  $x, y, z$  are the components of the position vector  $\mathbf{r}$  (Fig. 3.52). Using a determinant form, we also wrote

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \quad (3.19)$$

In the more general case of the moment about an arbitrary point  $B$  of a force  $\mathbf{F}$  applied at  $A$ , we had

$$\mathbf{M}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \quad (3.21)$$

where  $x_{A/B}$ ,  $y_{A/B}$ , and  $z_{A/B}$  denote the components of the vector  $\mathbf{r}_{A/B}$ :

$$x_{A/B} = x_A - x_B \quad y_{A/B} = y_A - y_B \quad z_{A/B} = z_A - z_B$$

In the case of *problems involving only two dimensions*, the force  $\mathbf{F}$  can be assumed to lie in the  $xy$  plane. Its moment  $\mathbf{M}_B$  about a point  $B$  in the same plane is perpendicular to that plane (Fig. 3.53) and is completely defined by the scalar

$$M_B = (x_A - x_B)F_y - (y_A - y_B)F_x \quad (3.23)$$

Various methods for the computation of the moment of a force about a point were illustrated in Sample Probs. 3.1 through 3.4.

The *scalar product* of two vectors  $\mathbf{P}$  and  $\mathbf{Q}$  [Sec. 3.9] was denoted by  $\mathbf{P} \cdot \mathbf{Q}$  and was defined as the scalar quantity

$$\mathbf{P} \cdot \mathbf{Q} = PQ \cos \theta \quad (3.24)$$

### Moment of a force about a point

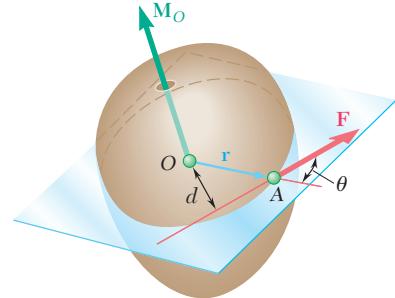


Fig. 3.51

### Rectangular components of moment

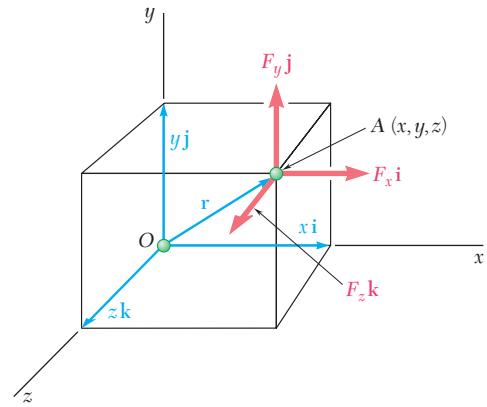


Fig. 3.52

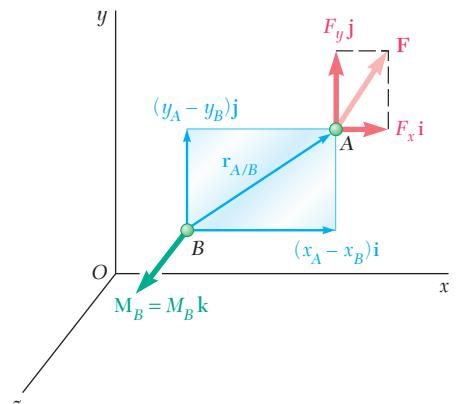


Fig. 3.53

### Scalar product of two vectors

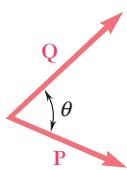


Fig. 3.54

where  $\theta$  is the angle between  $\mathbf{P}$  and  $\mathbf{Q}$  (Fig. 3.54). By expressing the scalar product of  $\mathbf{P}$  and  $\mathbf{Q}$  in terms of the rectangular components of the two vectors, we determined that

$$\mathbf{P} \cdot \mathbf{Q} = P_x Q_x + P_y Q_y + P_z Q_z \quad (3.30)$$

The *projection of a vector  $\mathbf{P}$  on an axis  $OL$*  (Fig. 3.55) can be obtained by forming the scalar product of  $\mathbf{P}$  and the unit vector  $\lambda$  along  $OL$ . We have

$$P_{OL} = \mathbf{P} \cdot \lambda \quad (3.36)$$

or, using rectangular components,

$$P_{OL} = P_x \cos \theta_x + P_y \cos \theta_y + P_z \cos \theta_z \quad (3.37)$$

where  $\theta_x$ ,  $\theta_y$ , and  $\theta_z$  denote the angles that the axis  $OL$  forms with the coordinate axes.

The *mixed triple product* of the three vectors  $\mathbf{S}$ ,  $\mathbf{P}$ , and  $\mathbf{Q}$  was defined as the scalar expression

$$\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) \quad (3.38)$$

obtained by forming the scalar product of  $\mathbf{S}$  with the vector product of  $\mathbf{P}$  and  $\mathbf{Q}$  [Sec. 3.10]. It was shown that

$$\mathbf{S} \cdot (\mathbf{P} \times \mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad (3.41)$$

where the elements of the determinant are the rectangular components of the three vectors.

### Moment of a force about an axis

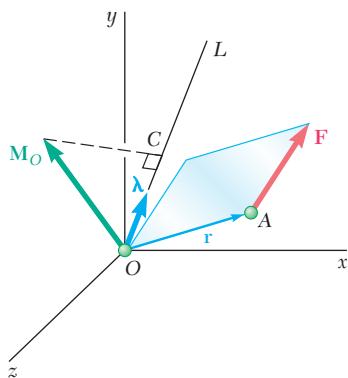


Fig. 3.56

The *moment of a force  $\mathbf{F}$  about an axis  $OL$*  [Sec. 3.11] was defined as the projection  $OC$  on  $OL$  of the moment  $\mathbf{M}_O$  of the force  $\mathbf{F}$  (Fig. 3.56), i.e., as the mixed triple product of the unit vector  $\lambda$ , the position vector  $\mathbf{r}$ , and the force  $\mathbf{F}$ :

$$M_{OL} = \lambda \cdot \mathbf{M}_O = \lambda \cdot (\mathbf{r} \times \mathbf{F}) \quad (3.42)$$

Using the determinant form for the mixed triple product, we have

$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \quad (3.43)$$

where  $\lambda_x, \lambda_y, \lambda_z$  = direction cosines of axis  $OL$

$x, y, z$  = components of  $\mathbf{r}$

$F_x, F_y, F_z$  = components of  $\mathbf{F}$

An example of the determination of the moment of a force about a skew axis was given in Sample Prob. 3.5.

Two forces  $\mathbf{F}$  and  $-\mathbf{F}$  having the same magnitude, parallel lines of action, and opposite sense are said to form a couple [Sec. 3.12]. It was shown that the moment of a couple is independent of the point about which it is computed; it is a vector  $\mathbf{M}$  perpendicular to the plane of the couple and equal in magnitude to the product of the common magnitude  $F$  of the forces and the perpendicular distance  $d$  between their lines of action (Fig. 3.57).

Two couples having the same moment  $\mathbf{M}$  are *equivalent*, i.e., they have the same effect on a given rigid body [Sec. 3.13]. The sum of two couples is itself a couple [Sec. 3.14], and the moment  $\mathbf{M}$  of the resultant couple can be obtained by adding vectorially the moments  $\mathbf{M}_1$  and  $\mathbf{M}_2$  of the original couples [Sample Prob. 3.6]. It follows that a couple can be represented by a vector, called a *couple vector*, equal in magnitude and direction to the moment  $\mathbf{M}$  of the couple [Sec. 3.15]. A couple vector is a *free vector* which can be attached to the origin  $O$  if so desired and resolved into components (Fig. 3.58).

## Couples

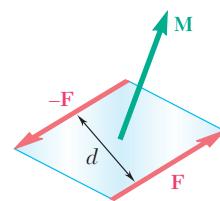


Fig. 3.57

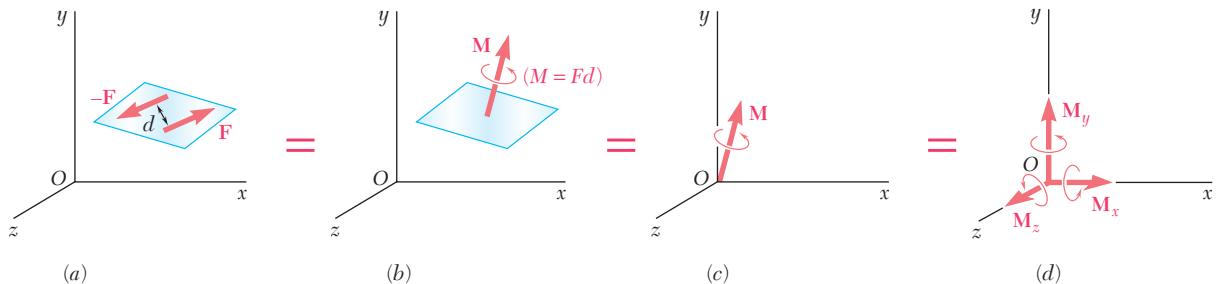


Fig. 3.58

Any force  $\mathbf{F}$  acting at a point  $A$  of a rigid body can be replaced by a *force-couple system* at an arbitrary point  $O$ , consisting of the force  $\mathbf{F}$  applied at  $O$  and a couple of moment  $\mathbf{M}_O$  equal to the moment about  $O$  of the force  $\mathbf{F}$  in its original position [Sec. 3.16]; it should be noted that the force  $\mathbf{F}$  and the couple vector  $\mathbf{M}_O$  are always perpendicular to each other (Fig. 3.59).

## Force-couple system

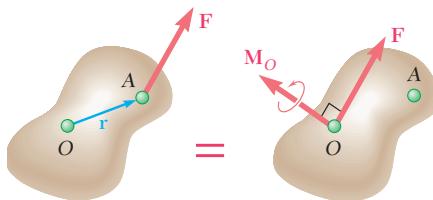


Fig. 3.59

It follows [Sec. 3.17] that *any system of forces can be reduced to a force-couple system at a given point  $O$*  by first replacing each of the forces of the system by an equivalent force-couple system at  $O$ .

## Reduction of a system of forces to a force-couple system

(Fig. 3.60) and then adding all the forces and all the couples determined in this manner to obtain a resultant force  $\mathbf{R}$  and a resultant couple vector  $\mathbf{M}_O^R$  [Sample Probs. 3.8 through 3.11]. Note that, in general, the resultant  $\mathbf{R}$  and the couple vector  $\mathbf{M}_O^R$  will not be perpendicular to each other.

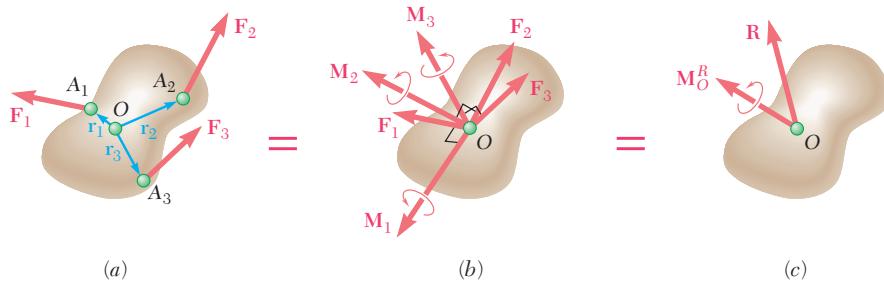


Fig. 3.60

### Equivalent systems of forces

We concluded from the above [Sec. 3.18] that, as far as rigid bodies are concerned, *two systems of forces,  $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \dots$  and  $\mathbf{F}'_1, \mathbf{F}'_2, \mathbf{F}'_3, \dots$ , are equivalent if, and only if,*

$$\Sigma \mathbf{F} = \Sigma \mathbf{F}' \quad \text{and} \quad \Sigma \mathbf{M}_O = \Sigma \mathbf{M}'_O \quad (3.57)$$

### Further reduction of a system of forces

If the resultant force  $\mathbf{R}$  and the resultant couple vector  $\mathbf{M}_O^R$  are perpendicular to each other, the force-couple system at  $O$  can be further reduced to a single resultant force [Sec. 3.20]. This will be the case for systems consisting either of (a) concurrent forces (cf. Chap. 2), (b) coplanar forces [Sample Probs. 3.8 and 3.9], or (c) parallel forces [Sample Prob. 3.11]. If the resultant  $\mathbf{R}$  and the couple vector  $\mathbf{M}_O^R$  are *not* perpendicular to each other, the system *cannot* be reduced to a single force. It can, however, be reduced to a special type of force-couple system called a *wrench*, consisting of the resultant  $\mathbf{R}$  and a couple vector  $\mathbf{M}_1$  directed along  $\mathbf{R}$  [Sec. 3.21 and Sample Prob. 3.12].

# REVIEW PROBLEMS

- 3.147** A crate of mass 80 kg is held in the position shown. Determine (a) the moment produced by the weight  $\mathbf{W}$  of the crate about  $E$ , (b) the smallest force applied at  $B$  that creates a moment of equal magnitude and opposite sense about  $E$ .

- 3.148** It is known that the connecting rod  $AB$  exerts on the crank  $BC$  a 1.5-kN force directed down and to the left along the centerline of  $AB$ . Determine the moment of the force about  $C$ .

- 3.149** A 6-ft-long fishing rod  $AB$  is securely anchored in the sand of a beach. After a fish takes the bait, the resulting force in the line is 6 lb. Determine the moment about  $A$  of the force exerted by the line at  $B$ .

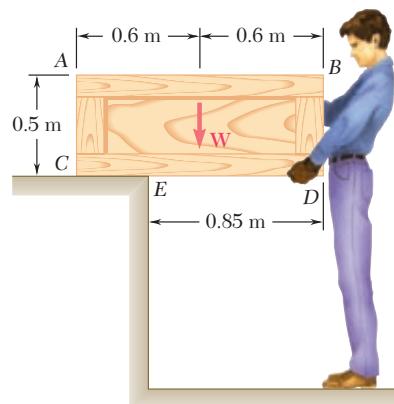


Fig. P3.147

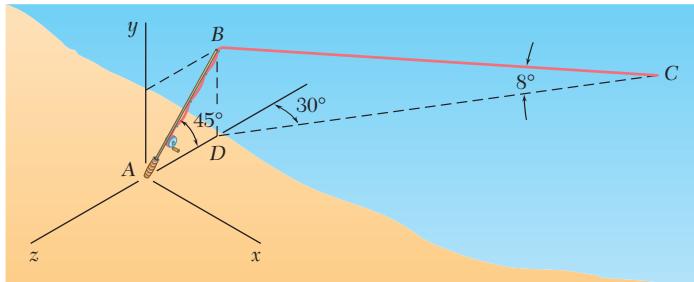


Fig. P3.149

- 3.150** Ropes  $AB$  and  $BC$  are two of the ropes used to support a tent. The two ropes are attached to a stake at  $B$ . If the tension in rope  $AB$  is 540 N, determine (a) the angle between rope  $AB$  and the stake, (b) the projection on the stake of the force exerted by rope  $AB$  at point  $B$ .

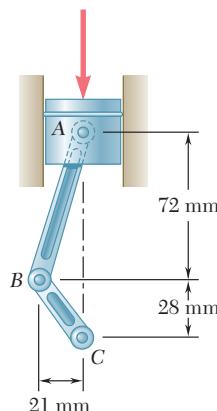


Fig. P3.148

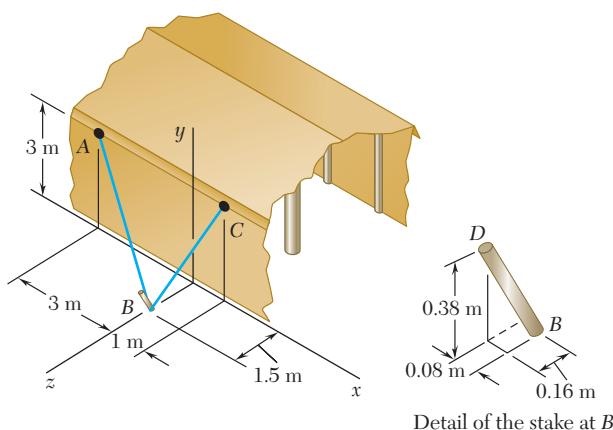
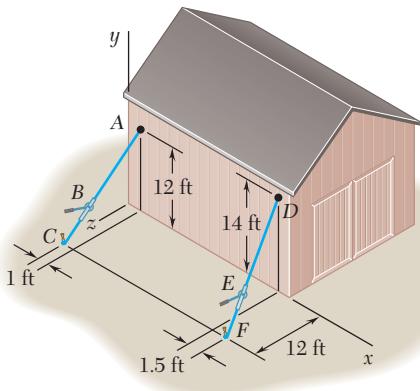


Fig. P3.150

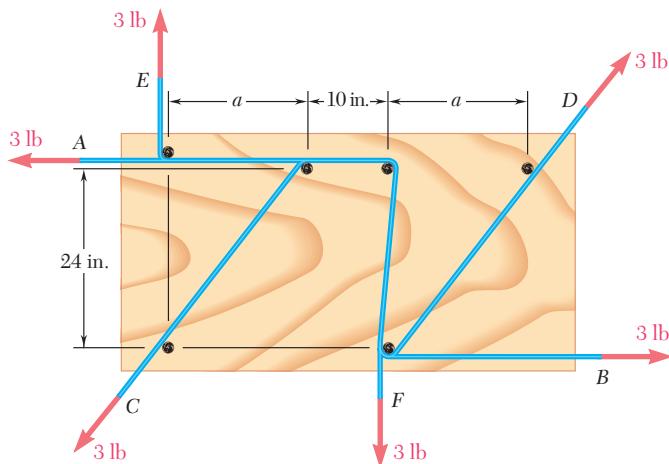


**Fig. P3.151**

**3.151** A farmer uses cables and winch pullers  $B$  and  $E$  to plumb one side of a small barn. If it is known that the sum of the moments about the  $x$  axis of the forces exerted by the cables on the barn at points  $A$  and  $D$  is equal to  $4728 \text{ lb} \cdot \text{ft}$ , determine the magnitude of  $\mathbf{T}_{DE}$  when  $T_{AB} = 255 \text{ lb}$ .

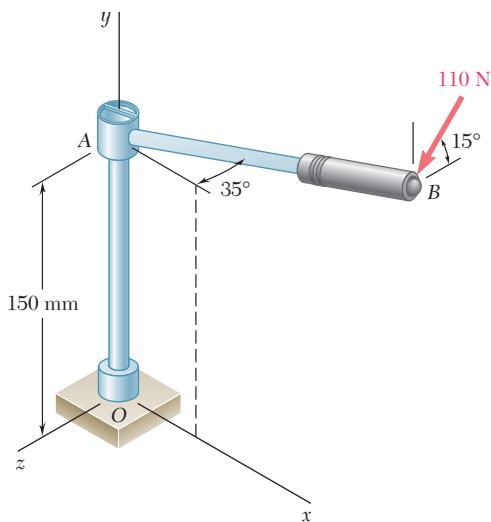
**3.152** Solve Prob. 3.151 when the tension in cable  $AB$  is 306 lb.

**3.153** A wiring harness is made by routing either two or three wires around 2-in.-diameter pegs mounted on a sheet of plywood. If the force in each wire is 3 lb, determine the resultant couple acting on the plywood when  $a = 18$  in. and (a) only wires  $AB$  and  $CD$  are in place, (b) all three wires are in place.

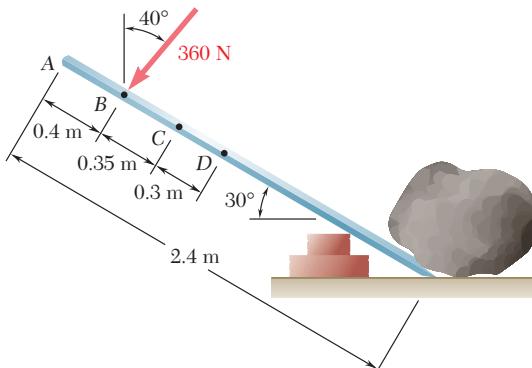


**Fig. P3.153**

**3.154** A worker tries to move a rock by applying a 360-N force to a steel bar as shown. (a) Replace that force with an equivalent force-couple system at  $D$ . (b) Two workers attempt to move the same rock by applying a vertical force at  $A$  and another force at  $D$ . Determine these two forces if they are to be equivalent to the single force of part *a*.



**Fig. P3.155**

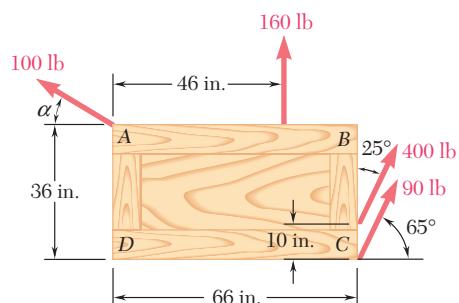
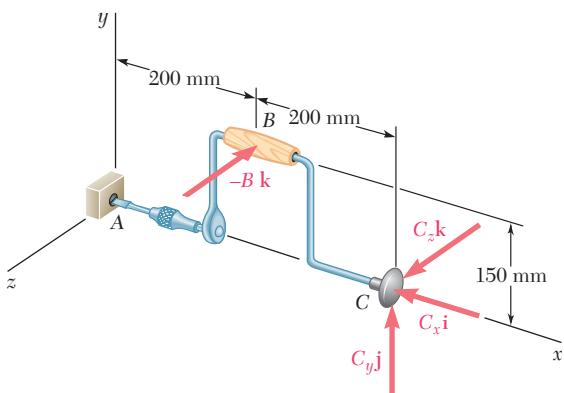


**Fig. P3.154**

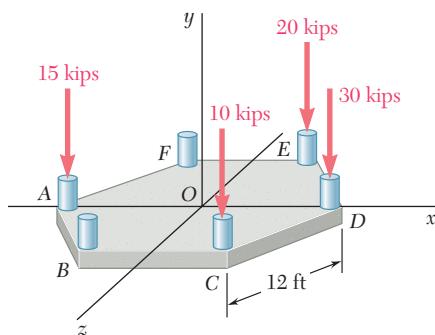
**3.155** A 110-N force acting in a vertical plane parallel to the  $yz$  plane is applied to the 220-mm-long horizontal handle  $AB$  of a socket wrench. Replace the force with an equivalent force-couple system at the origin  $O$  of the coordinate system.

- 3.156** Four ropes are attached to a crate and exert the forces shown. If the forces are to be replaced with a single equivalent force applied at a point on line  $AB$ , determine (a) the equivalent force and the distance from  $A$  to the point of application of the force when  $\alpha = 30^\circ$ , (b) the value of  $\alpha$  so that the single equivalent force is applied at point  $B$ .

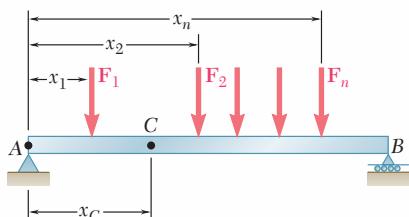
- 3.157** A blade held in a brace is used to tighten a screw at  $A$ . (a) Determine the forces exerted at  $B$  and  $C$ , knowing that these forces are equivalent to a force-couple system at  $A$  consisting of  $\mathbf{R} = -(30 \text{ N})\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$  and  $\mathbf{M}_A^R = -(12 \text{ N} \cdot \text{m})\mathbf{i}$ . (b) Find the corresponding values of  $R_y$  and  $R_z$ . (c) What is the orientation of the slot in the head of the screw for which the blade is least likely to slip when the brace is in the position shown?

**Fig. P3.156****Fig. P3.157**

- 3.158** A concrete foundation mat in the shape of a regular hexagon of side 12 ft supports four column loads as shown. Determine the magnitudes of the additional loads that must be applied at  $B$  and  $F$  if the resultant of all six loads is to pass through the center of the mat.

**Fig. P3.158**

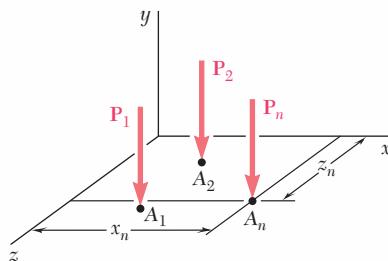
# COMPUTER PROBLEMS



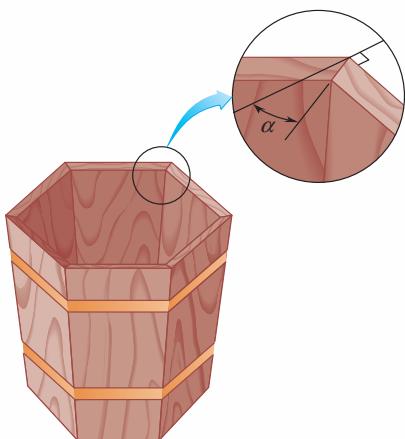
**Fig. P3.C1**

**3.C1** A beam  $AB$  is subjected to several vertical forces as shown. Using computational software, determine the magnitude of the resultant of the forces and the distance  $x_C$  to point  $C$ , the point where the line of action of the resultant intersects  $AB$ . Use this program to solve (a) Sample Prob. 3.8c, (b) Prob. 3.106a.

**3.C2** Using computational software, determine the magnitude and the point of application of the resultant of the vertical forces  $P_1, P_2, \dots, P_n$  that act at points  $A_1, A_2, \dots, A_n$  that are located in the  $xz$  plane. Use this program to solve (a) Sample Prob. 3.11, (b) Prob. 3.127, (c) Prob. 3.129.



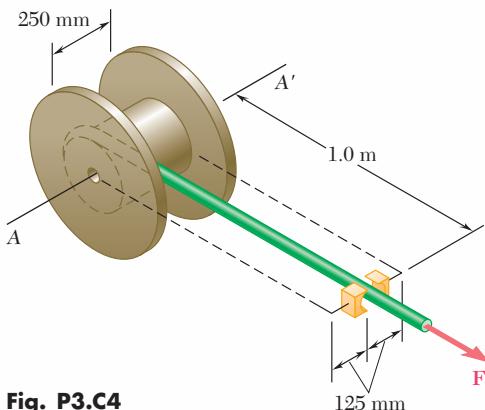
**Fig. P3.C2**



**Fig. P3.C3**

**3.C3** A friend asks for your help in designing flower planter boxes. The boxes are to have 4, 5, 6, or 8 sides, which are to tilt outward at  $10^\circ, 20^\circ$ , or  $30^\circ$ . Using computational software, determine the bevel angle  $\alpha$  for each of the twelve planter designs. (*Hint:* The bevel angle is equal to one-half of the angle formed by the inward normals of two adjacent sides.)

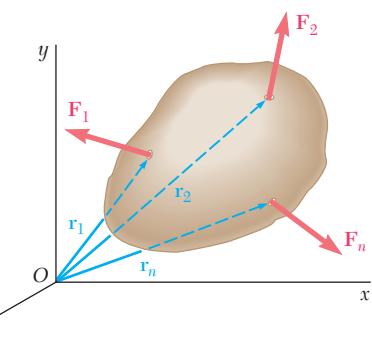
**3.C4** The manufacturer of a spool for hoses wants to determine the moment of the force  $\mathbf{F}$  about the axis  $AA'$ . The magnitude of the force, in newtons, is defined by the relation  $F = 300(1 - x/L)$ , where  $x$  is the length of hose wound on the 0.6-m-diameter drum and  $L$  is the total length of the hose. Using computational software, calculate the required moment for a hose 30 m long and 50 mm in diameter. Beginning with  $x = 0$ , compute the moment after every revolution of the drum until the hose is wound on the drum.



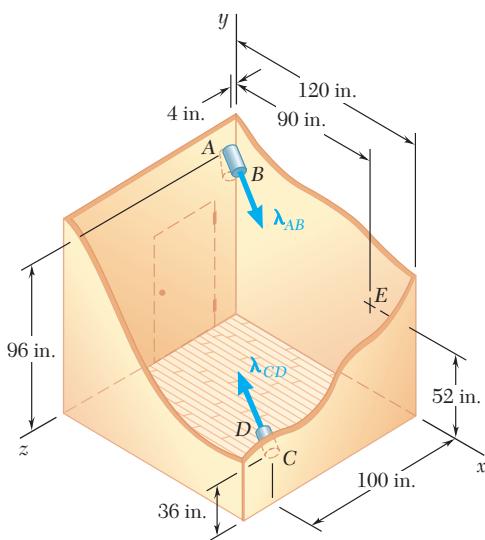
**Fig. P3.C4**

**3.C5** A body is acted upon by a system of  $n$  forces. Using computational software, calculate the equivalent force-couple system at the origin of the coordinate axes and to determine, if the equivalent force and the equivalent couple are orthogonal, the magnitude and the point of application in the  $xz$  plane of the resultant of the original force system. Use this program to solve (a) Prob. 3.113, (b) Prob. 3.120, (c) Prob. 3.127.

**3.C6** Two cylindrical ducts,  $AB$  and  $CD$ , enter a room through two parallel walls. The centerlines of the ducts are parallel to each other but are not perpendicular to the walls. The ducts are to be connected by two flexible elbows and a straight center portion. Using computational software, determine the lengths of  $AB$  and  $CD$  that minimize the distance between the axis of the straight portion and a thermometer mounted on the wall at  $E$ . Assume that the elbows are of negligible length and that  $AB$  and  $CD$  have centerlines defined by  $\lambda_{AB} = (7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})/9$  and  $\lambda_{CD} = (-7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})/9$  and can vary in length from 9 in. to 36 in.

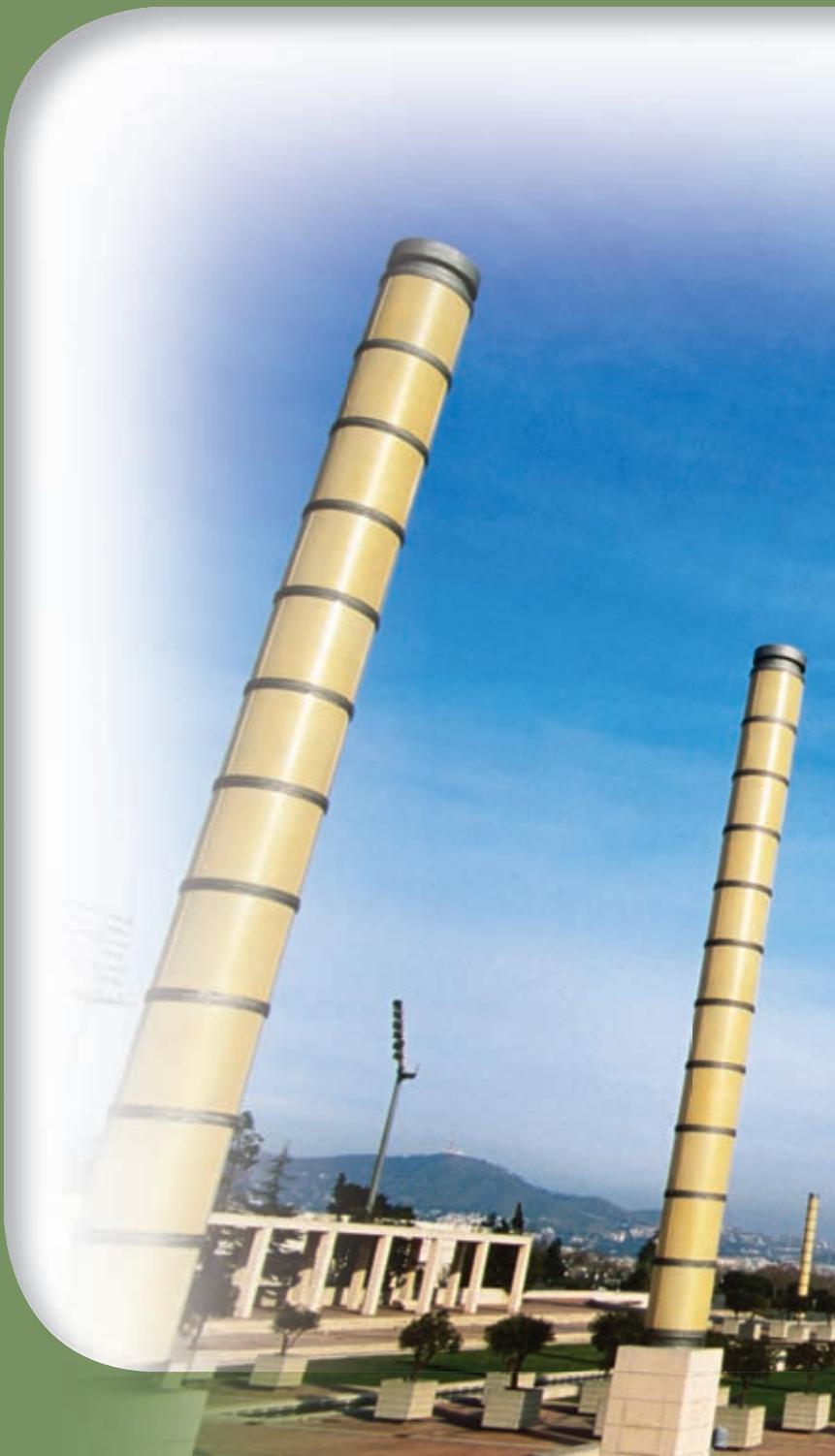


**Fig. P3.C5**



**Fig. P3.C6**

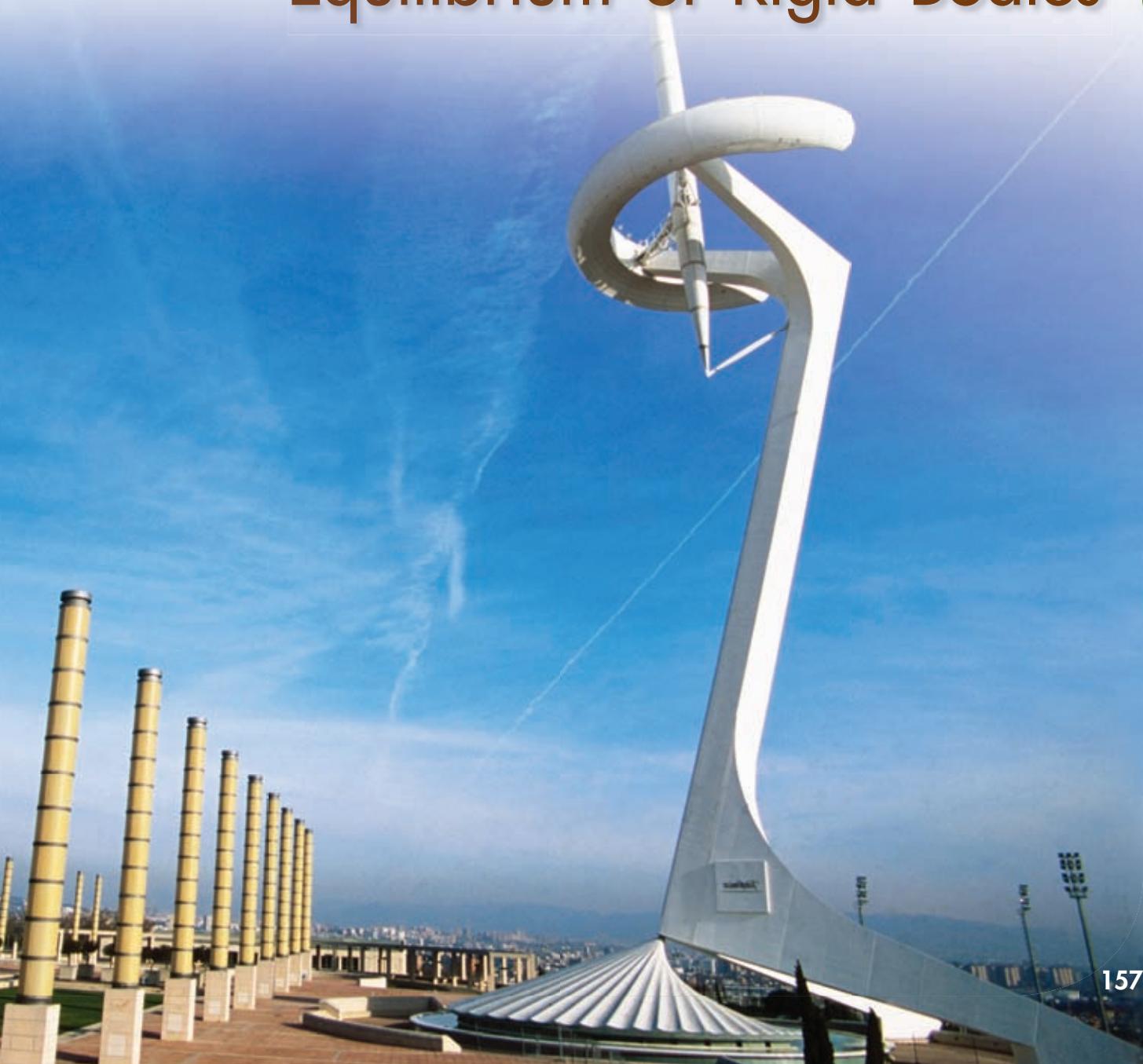
**This telecommunications tower,  
constructed in the heart of the  
Barcelona Olympic complex to  
broadcast the 1992 games, was  
designed to remain in equilibrium  
under the vertical force of gravity and  
the lateral forces exerted by wind.**



# CHAPTER

# 4

# Equilibrium of Rigid Bodies



## Chapter 4 Equilibrium of Rigid Bodies

- 4.1 Introduction
- 4.2 Free-Body Diagram
- 4.3 Reactions at Supports and Connections for a Two-Dimensional Structure
- 4.4 Equilibrium of a Rigid Body in Two Dimensions
- 4.5 Statically Indeterminate Reactions. Partial Constraints
- 4.6 Equilibrium of a Two-Force Body
- 4.7 Equilibrium of a Three-Force Body
- 4.8 Equilibrium of a Rigid Body in Three Dimensions
- 4.9 Reactions at Supports and Connections for a Three-Dimensional Structure

### 4.1 INTRODUCTION

We saw in the preceding chapter that the external forces acting on a rigid body can be reduced to a force-couple system at some arbitrary point  $O$ . When the force and the couple are both equal to zero, the external forces form a system equivalent to zero, and the rigid body is said to be in *equilibrium*.

The necessary and sufficient conditions for the equilibrium of a rigid body, therefore, can be obtained by setting  $\mathbf{R}$  and  $\mathbf{M}_O^R$  equal to zero in the relations (3.52) of Sec. 3.17:

$$\Sigma \mathbf{F} = 0 \quad \Sigma \mathbf{M}_O = \Sigma (\mathbf{r} \times \mathbf{F}) = 0 \quad (4.1)$$

Resolving each force and each moment into its rectangular components, we can express the necessary and sufficient conditions for the equilibrium of a rigid body with the following six scalar equations:

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0 \quad (4.2)$$

$$\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0 \quad (4.3)$$

The equations obtained can be used to determine unknown forces applied to the rigid body or unknown reactions exerted on it by its supports. We note that Eqs. (4.2) express the fact that the components of the external forces in the  $x$ ,  $y$ , and  $z$  directions are balanced; Eqs. (4.3) express the fact that the moments of the external forces about the  $x$ ,  $y$ , and  $z$  axes are balanced. Therefore, for a rigid body in equilibrium, the system of the external forces will impart no translational or rotational motion to the body considered.

In order to write the equations of equilibrium for a rigid body, it is essential to first identify all of the forces acting on that body and then to draw the corresponding *free-body diagram*. In this chapter we first consider the equilibrium of *two-dimensional structures* subjected to forces contained in their planes and learn how to draw their free-body diagrams. In addition to the forces *applied* to a structure, the *reactions* exerted on the structure by its supports will be considered. A specific reaction will be associated with each type of support. You will learn how to determine whether the structure is properly supported, so that you can know in advance whether the equations of equilibrium can be solved for the unknown forces and reactions.

Later in the chapter, the equilibrium of three-dimensional structures will be considered, and the same kind of analysis will be given to these structures and their supports.

## 4.2 FREE-BODY DIAGRAM

In solving a problem concerning the equilibrium of a rigid body, it is essential to consider *all* of the forces acting on the body; it is equally important to exclude any force which is not directly applied to the body. Omitting a force or adding an extraneous one would destroy the conditions of equilibrium. Therefore, the first step in the solution of the problem should be to draw a *free-body diagram* of the rigid body under consideration. Free-body diagrams have already been used on many occasions in Chap. 2. However, in view of their importance to the solution of equilibrium problems, we summarize here the various steps which must be followed in drawing a free-body diagram.

1. A clear decision should be made regarding the choice of the free body to be used. This body is then detached from the ground and is separated from all other bodies. The contour of the body thus isolated is sketched.
2. All external forces should be indicated on the free-body diagram. These forces represent the actions exerted *on* the free body *by* the ground and *by* the bodies which have been detached; they should be applied at the various points where the free body was supported by the ground or was connected to the other bodies. The *weight* of the free body should also be included among the external forces, since it represents the attraction exerted by the earth on the various particles forming the free body. As will be seen in Chap. 5, the weight should be applied at the center of gravity of the body. When the free body is made of several parts, the forces the various parts exert on each other should *not* be included among the external forces. These forces are internal forces as far as the free body is concerned.
3. The magnitudes and directions of the *known external forces* should be clearly marked on the free-body diagram. When indicating the directions of these forces, it must be remembered that the forces shown on the free-body diagram must be those which are exerted *on*, and not *by*, the free body. Known external forces generally include the *weight* of the free body and *forces applied* for a given purpose.
4. *Unknown external forces* usually consist of the *reactions*, through which the ground and other bodies oppose a possible motion of the free body. The reactions constrain the free body to remain in the same position, and, for that reason, are sometimes called *constraining forces*. Reactions are exerted at the points where the free body is *supported by* or *connected to* other bodies and should be clearly indicated. Reactions are discussed in detail in Secs. 4.3 and 4.8.
5. The free-body diagram should also include dimensions, since these may be needed in the computation of moments of forces. Any other detail, however, should be omitted.



**Photo 4.1** A free-body diagram of the tractor shown would include all of the external forces acting on the tractor: the weight of the tractor, the weight of the load in the bucket, and the forces exerted by the ground on the tires.



**Photo 4.2** In Chap. 6, we will discuss how to determine the internal forces in structures made of several connected pieces, such as the forces in the members that support the bucket of the tractor of Photo 4.1.

## EQUILIBRIUM IN TWO DIMENSIONS

### 4.3 REACTIONS AT SUPPORTS AND CONNECTIONS FOR A TWO-DIMENSIONAL STRUCTURE

In the first part of this chapter, the equilibrium of a two-dimensional structure is considered; i.e., it is assumed that the structure being analyzed and the forces applied to it are contained in the same plane. Clearly, the reactions needed to maintain the structure in the same position will also be contained in this plane.

The reactions exerted on a two-dimensional structure can be divided into three groups corresponding to three types of *supports*, or *connections*:



**Photo 4.3** As the link of the awning window opening mechanism is extended, the force it exerts on the slider results in a normal force being applied to the rod, which causes the window to open.



**Photo 4.4** The abutment-mounted rocker bearing shown is used to support the roadway of a bridge.



**Photo 4.5** Shown is the rocker expansion bearing of a plate girder bridge. The convex surface of the rocker allows the support of the girder to move horizontally.

#### 1. Reactions Equivalent to a Force with Known Line of Action.

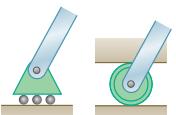
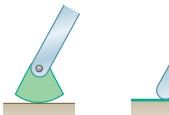
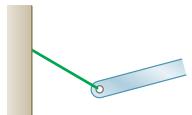
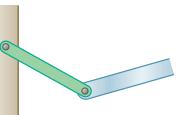
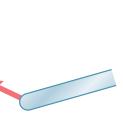
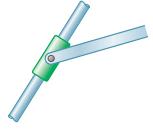
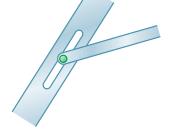
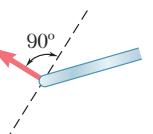
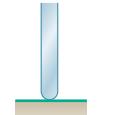
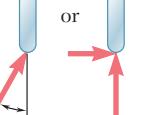
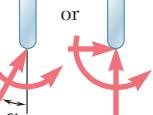
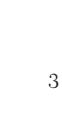
Supports and connections causing reactions of this type include *rollers*, *rockers*, *frictionless surfaces*, *short links and cables*, *collars on frictionless rods*, and *frictionless pins in slots*. Each of these supports and connections can prevent motion in one direction only. They are shown in Fig. 4.1, together with the reactions they produce. Each of these reactions involves *one unknown*, namely, the magnitude of the reaction; this magnitude should be denoted by an appropriate letter. The line of action of the reaction is known and should be indicated clearly in the free-body diagram. The sense of the reaction must be as shown in Fig. 4.1 for the cases of a frictionless surface (toward the free body) or a cable (away from the free body). The reaction can be directed either way in the case of double-track rollers, links, collars on rods, and pins in slots. Single-track rollers and rockers are generally assumed to be reversible, and thus the corresponding reactions can also be directed either way.

#### 2. Reactions Equivalent to a Force of Unknown Direction and Magnitude.

Supports and connections causing reactions of this type include *frictionless pins in fitted holes*, *hinges*, and *rough surfaces*. They can prevent translation of the free body in all directions, but they cannot prevent the body from rotating about the connection. Reactions of this group involve *two unknowns* and are usually represented by their *x* and *y* components. In the case of a rough surface, the component normal to the surface must be directed away from the surface.

#### 3. Reactions Equivalent to a Force and a Couple.

These reactions are caused by *fixed supports*, which oppose any motion of the free body and thus constrain it completely. Fixed supports actually produce forces over the entire surface of contact; these forces, however, form a system which can be reduced to a force and a couple. Reactions of this group involve *three unknowns*, consisting usually of the two components of the force and the moment of the couple.

Support or Connection	Reaction	Number of Unknowns
Rollers  Rocker  Frictionless surface 	Force with known line of action 	1
Short cable  Short link 	Force with known line of action 	1
Collar on frictionless rod  Frictionless pin in slot 	Force with known line of action 	1
Frictionless pin or hinge  Rough surface 	Force of unknown direction  or 	2
Fixed support 	Force and couple  or 	3

**Fig. 4.1** Reactions at supports and connections.

When the sense of an unknown force or couple is not readily apparent, no attempt should be made to determine it. Instead, the sense of the force or couple should be arbitrarily assumed; the sign of the answer obtained will indicate whether the assumption is correct or not.

## 4.4 EQUILIBRIUM OF A RIGID BODY IN TWO DIMENSIONS

The conditions stated in Sec. 4.1 for the equilibrium of a rigid body become considerably simpler for the case of a two-dimensional structure. Choosing the  $x$  and  $y$  axes to be in the plane of the structure, we have

$$F_z = 0 \quad M_x = M_y = 0 \quad M_z = M_O$$

for each of the forces applied to the structure. Thus, the six equations of equilibrium derived in Sec. 4.1 reduce to

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M_O = 0 \quad (4.4)$$

and to three trivial identities,  $0 = 0$ . Since  $\Sigma M_O = 0$  must be satisfied regardless of the choice of the origin  $O$ , we can write the equations of equilibrium for a two-dimensional structure in the more general form

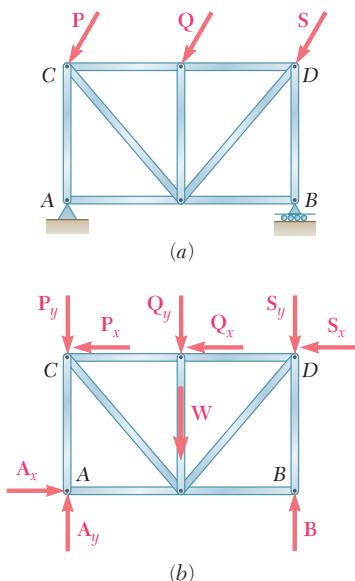
$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M_A = 0 \quad (4.5)$$

where  $A$  is any point in the plane of the structure. The three equations obtained can be solved for no more than *three unknowns*.

We saw in the preceding section that unknown forces include reactions and that the number of unknowns corresponding to a given reaction depends upon the type of support or connection causing that reaction. Referring to Sec. 4.3, we observe that the equilibrium equations (4.5) can be used to determine the reactions associated with two rollers and one cable, one fixed support, or one roller and one pin in a fitted hole, etc.

Consider Fig. 4.2a, in which the truss shown is subjected to the given forces  $\mathbf{P}$ ,  $\mathbf{Q}$ , and  $\mathbf{S}$ . The truss is held in place by a pin at  $A$  and a roller at  $B$ . The pin prevents point  $A$  from moving by exerting on the truss a force which can be resolved into the components  $\mathbf{A}_x$  and  $\mathbf{A}_y$ ; the roller keeps the truss from rotating about  $A$  by exerting the vertical force  $\mathbf{B}$ . The free-body diagram of the truss is shown in Fig. 4.2b; it includes the reactions  $\mathbf{A}_x$ ,  $\mathbf{A}_y$ , and  $\mathbf{B}$  as well as the applied forces  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{S}$  and the weight  $\mathbf{W}$  of the truss. Expressing that the sum of the moments about  $A$  of all of the forces shown in Fig. 4.2b is zero, we write the equation  $\Sigma M_A = 0$ , which can be used to determine the magnitude  $B$  since it does not contain  $A_x$  or  $A_y$ . Next, expressing that the sum of the  $x$  components and the sum of the  $y$  components of the forces are zero, we write the equations  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$ , from which we can obtain the components  $A_x$  and  $A_y$ , respectively.

An additional equation could be obtained by expressing that the sum of the moments of the external forces about a point other than  $A$  is zero. We could write, for instance,  $\Sigma M_B = 0$ . Such a statement, however, does not contain any new information, since it has already been established that the system of the forces shown in Fig. 4.2b is equivalent to zero. The additional equation is not independent and cannot be used to determine a fourth unknown. It will be useful,



**Fig. 4.2**

however, for checking the solution obtained from the original three equations of equilibrium.

While the three equations of equilibrium cannot be *augmented* by additional equations, any of them can be *replaced* by another equation. Therefore, an alternative system of equations of equilibrium is

$$\Sigma F_x = 0 \quad \Sigma M_A = 0 \quad \Sigma M_B = 0 \quad (4.6)$$

where the second point about which the moments are summed (in this case, point *B*) cannot lie on the line parallel to the *y* axis that passes through point *A* (Fig. 4.2*b*). These equations are sufficient conditions for the equilibrium of the truss. The first two equations indicate that the external forces must reduce to a single vertical force at *A*. Since the third equation requires that the moment of this force be zero about a point *B* which is not on its line of action, the force must be zero, and the rigid body is in equilibrium.

A third possible set of equations of equilibrium is

$$\Sigma M_A = 0 \quad \Sigma M_B = 0 \quad \Sigma M_C = 0 \quad (4.7)$$

where the points *A*, *B*, and *C* do not lie in a straight line (Fig. 4.2*b*). The first equation requires that the external forces reduce to a single force at *A*; the second equation requires that this force pass through *B*; and the third equation requires that it pass through *C*. Since the points *A*, *B*, *C* do not lie in a straight line, the force must be zero, and the rigid body is in equilibrium.

The equation  $\Sigma M_A = 0$ , which expresses that the sum of the moments of the forces about pin *A* is zero, possesses a more definite physical meaning than either of the other two equations (4.7). These two equations express a similar idea of balance, but with respect to points about which the rigid body is not actually hinged. They are, however, as useful as the first equation, and our choice of equilibrium equations should not be unduly influenced by the physical meaning of these equations. Indeed, it will be desirable in practice to choose equations of equilibrium containing only one unknown, since this eliminates the necessity of solving simultaneous equations. Equations containing only one unknown can be obtained by summing moments about the point of intersection of the lines of action of two unknown forces or, if these forces are parallel, by summing components in a direction perpendicular to their common direction. For example, in Fig. 4.3, in which the truss shown is held by rollers at *A* and *B* and a short link at *D*, the reactions at *A* and *B* can be eliminated by summing *x* components. The reactions at *A* and *D* will be eliminated by summing moments about *C*, and the reactions at *B* and *D* by summing moments about *D*. The equations obtained are

$$\Sigma F_x = 0 \quad \Sigma M_C = 0 \quad \Sigma M_D = 0$$

Each of these equations contains only one unknown.

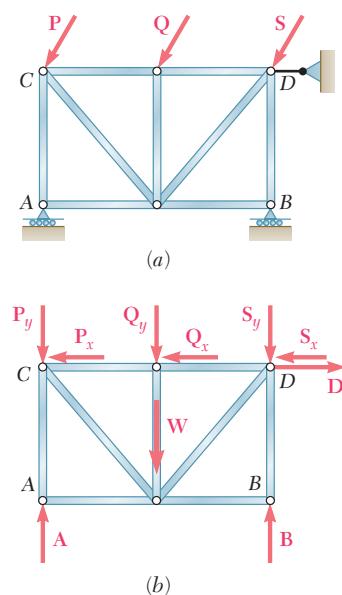


Fig. 4.3

## 4.5 STATICALLY INDETERMINATE REACTIONS. PARTIAL CONSTRAINTS

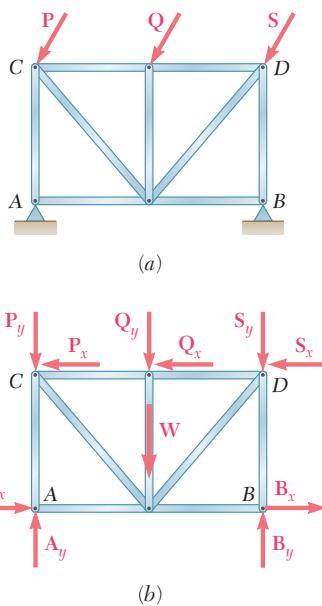
In the two examples considered in the preceding section (Figs. 4.2 and 4.3), the types of supports used were such that the rigid body could not possibly move under the given loads or under any other loading conditions. In such cases, the rigid body is said to be *completely constrained*. We also recall that the reactions corresponding to these supports involved *three unknowns* and could be determined by solving the three equations of equilibrium. When such a situation exists, the reactions are said to be *statically determinate*.

Consider Fig. 4.4a, in which the truss shown is held by pins at A and B. These supports provide more constraints than are necessary to keep the truss from moving under the given loads or under any other loading conditions. We also note from the free-body diagram of Fig. 4.4b that the corresponding reactions involve *four unknowns*. Since, as was pointed out in Sec. 4.4, only three independent equilibrium equations are available, there are *more unknowns than equations*; thus, all of the unknowns cannot be determined. While the equations  $\sum M_A = 0$  and  $\sum M_B = 0$  yield the vertical components  $B_y$  and  $A_y$ , respectively, the equation  $\sum F_x = 0$  gives only the sum  $A_x + B_x$  of the horizontal components of the reactions at A and B. The components  $A_x$  and  $B_x$  are said to be *statically indeterminate*. They could be determined by considering the deformations produced in the truss by the given loading, but this method is beyond the scope of statics and belongs to the study of mechanics of materials.

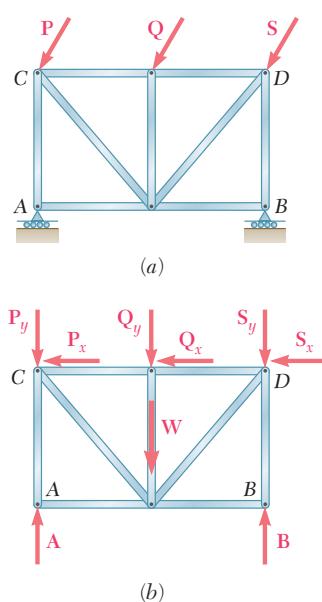
The supports used to hold the truss shown in Fig. 4.5a consist of rollers at A and B. Clearly, the constraints provided by these supports are not sufficient to keep the truss from moving. While any vertical motion is prevented, the truss is free to move horizontally. The truss is said to be *partially constrained*.<sup>†</sup> Turning our attention to Fig. 4.5b, we note that the reactions at A and B involve only *two unknowns*. Since three equations of equilibrium must still be satisfied, there are *fewer unknowns than equations*, and, in general, one of the equilibrium equations will not be satisfied. While the equations  $\sum M_A = 0$  and  $\sum M_B = 0$  can be satisfied by a proper choice of reactions at A and B, the equation  $\sum F_x = 0$  will not be satisfied unless the sum of the horizontal components of the applied forces happens to be zero. We thus observe that the equilibrium of the truss of Fig. 4.5 cannot be maintained under general loading conditions.

It appears from the above that if a rigid body is to be completely constrained and if the reactions at its supports are to be statically determinate, *there must be as many unknowns as there are equations of equilibrium*. When this condition is *not* satisfied, we can be certain that either the rigid body is not completely constrained or that the reactions at its supports are not statically determinate; it is also possible that the rigid body is not completely constrained and that the reactions are statically indeterminate.

We should note, however, that, while *necessary*, the above condition is *not sufficient*. In other words, the fact that the number of



**Fig. 4.4** Statically indeterminate reactions.



**Fig. 4.5** Partial constraints.

<sup>†</sup>Partially constrained bodies are often referred to as *unstable*. However, to avoid confusion between this type of instability, due to insufficient constraints, and the type of instability considered in Chap. 10, which relates to the behavior of a rigid body when its equilibrium is disturbed, we shall restrict the use of the words *stable* and *unstable* to the latter case.

unknowns is equal to the number of equations is no guarantee that the body is completely constrained or that the reactions at its supports are statically determinate. Consider Fig. 4.6a, in which the truss shown is held by rollers at A, B, and E. While there are three unknown reactions, **A**, **B**, and **E** (Fig. 4.6b), the equation  $\sum F_x = 0$  will not be satisfied unless the sum of the horizontal components of the applied forces happens to be zero. Although there are a sufficient number of constraints, these constraints are not properly arranged, and the truss is free to move horizontally. We say that the truss is *improperly constrained*. Since only two equilibrium equations are left for determining three unknowns, the reactions will be statically indeterminate. Thus, improper constraints also produce static indeterminacy.

Another example of improper constraints—and of static indeterminacy—is provided by the truss shown in Fig. 4.7. This truss is held by a pin at A and by rollers at B and C, which altogether involve four unknowns. Since only three independent equilibrium equations are available, the reactions at the supports are statically indeterminate. On the other hand, we note that the equation  $\sum M_A = 0$  cannot be satisfied under general loading conditions, since the lines of action of the reactions **B** and **C** pass through A. We conclude that the truss can rotate about A and that it is improperly constrained.<sup>†</sup>

The examples of Figs. 4.6 and 4.7 lead us to conclude that *a rigid body is improperly constrained whenever the supports, even though they may provide a sufficient number of reactions, are arranged in such a way that the reactions must be either concurrent or parallel.*<sup>‡</sup>

In summary, to be sure that a two-dimensional rigid body is completely constrained and that the reactions at its supports are statically determinate, we should verify that the reactions involve three—and only three—unknowns and that the supports are arranged in such a way that they do not require the reactions to be either concurrent or parallel.

Supports involving statically indeterminate reactions should be used with care in the *design* of structures and only with a full knowledge of the problems they may cause. On the other hand, the *analysis* of structures possessing statically indeterminate reactions often can be partially carried out by the methods of statics. In the case of the truss of Fig. 4.4, for example, the vertical components of the reactions at A and B were obtained from the equilibrium equations.

For obvious reasons, supports producing partial or improper constraints should be avoided in the design of stationary structures. However, a partially or improperly constrained structure will not necessarily collapse; under particular loading conditions, equilibrium can be maintained. For example, the trusses of Figs. 4.5 and 4.6 will be in equilibrium if the applied forces **P**, **Q**, and **S** are vertical. Besides, structures which are designed to move *should* be only partially constrained. A railroad car, for instance, would be of little use if it were completely constrained by having its brakes applied permanently.

<sup>†</sup>Rotation of the truss about A requires some “play” in the supports at B and C. In practice such play will always exist. In addition, we note that if the play is kept small, the displacements of the rollers B and C and, thus, the distances from A to the lines of action of the reactions **B** and **C** will also be small. The equation  $\sum M_A = 0$  then requires that **B** and **C** be very large, a situation which can result in the failure of the supports at B and C.

<sup>‡</sup>Because this situation arises from an inadequate arrangement or *geometry* of the supports, it is often referred to as *geometric instability*.

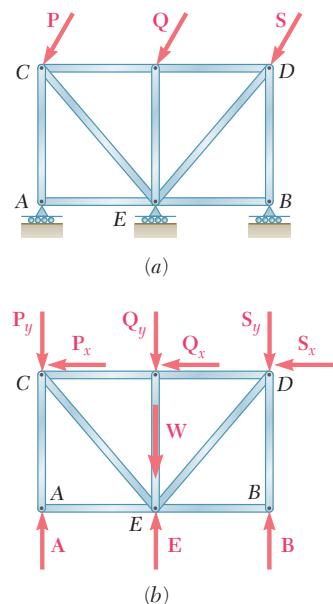


Fig. 4.6 Improper constraints.

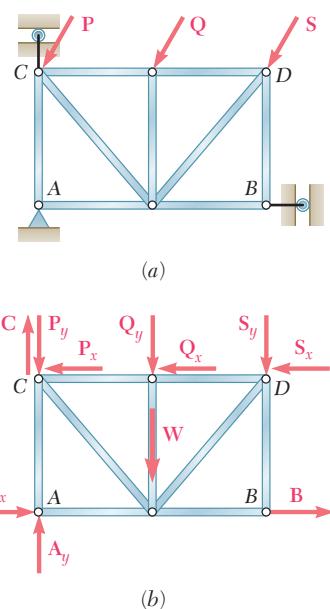
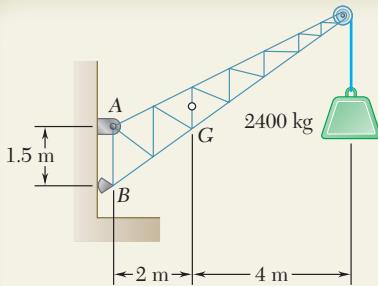
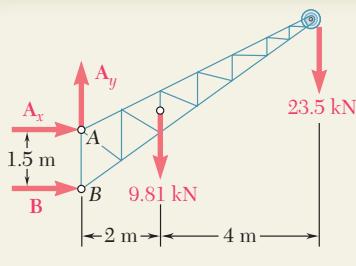


Fig. 4.7 Improper constraints.



## SAMPLE PROBLEM 4.1

A fixed crane has a mass of 1000 kg and is used to lift a 2400-kg crate. It is held in place by a pin at A and a rocker at B. The center of gravity of the crane is located at G. Determine the components of the reactions at A and B.



## SOLUTION

**Free-Body Diagram.** A free-body diagram of the crane is drawn. By multiplying the masses of the crane and of the crate by  $g = 9.81 \text{ m/s}^2$ , we obtain the corresponding weights, that is, 9810 N or 9.81 kN, and 23 500 N or 23.5 kN. The reaction at pin A is a force of unknown direction; it is represented by its components  $A_x$  and  $A_y$ . The reaction at the rocker B is perpendicular to the rocker surface; thus, it is horizontal. We assume that  $A_x$ ,  $A_y$ , and  $B$  act in the directions shown.

**Determination of  $B$ .** We express that the sum of the moments of all external forces about point A is zero. The equation obtained will contain neither  $A_x$  nor  $A_y$ , since the moments of  $A_x$  and  $A_y$  about A are zero. Multiplying the magnitude of each force by its perpendicular distance from A, we write

$$+\uparrow\sum M_A = 0: \quad +B(1.5 \text{ m}) - (9.81 \text{ kN})(2 \text{ m}) - (23.5 \text{ kN})(6 \text{ m}) = 0 \\ B = +107.1 \text{ kN} \quad \mathbf{B = 107.1 \text{ kN}} \rightarrow$$

Since the result is positive, the reaction is directed as assumed.

**Determination of  $A_x$ .** The magnitude of  $A_x$  is determined by expressing that the sum of the horizontal components of all external forces is zero.

$$\pm\rightarrow\sum F_x = 0: \quad A_x + B = 0 \\ A_x + 107.1 \text{ kN} = 0 \\ A_x = -107.1 \text{ kN} \quad \mathbf{A_x = 107.1 \text{ kN}} \leftarrow$$

Since the result is negative, the sense of  $A_x$  is opposite to that assumed originally.

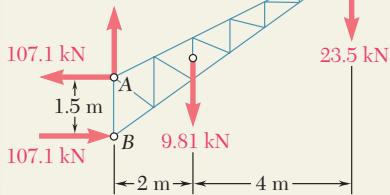
**Determination of  $A_y$ .** The sum of the vertical components must also equal zero.

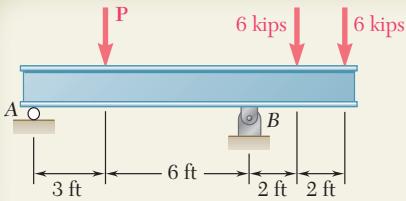
$$+\uparrow\sum F_y = 0: \quad A_y - 9.81 \text{ kN} - 23.5 \text{ kN} = 0 \\ A_y = +33.3 \text{ kN} \quad \mathbf{A_y = 33.3 \text{ kN}} \uparrow$$

Adding vectorially the components  $A_x$  and  $A_y$ , we find that the reaction at A is 112.2 kN  $\angle 17.3^\circ$ .

**Check.** The values obtained for the reactions can be checked by recalling that the sum of the moments of all of the external forces about any point must be zero. For example, considering point B, we write

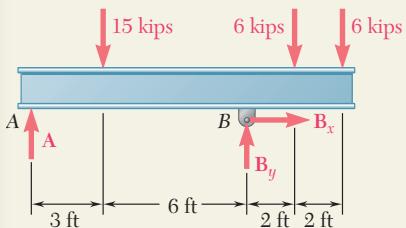
$$+\uparrow\sum M_B = -(9.81 \text{ kN})(2 \text{ m}) - (23.5 \text{ kN})(6 \text{ m}) + (107.1 \text{ kN})(1.5 \text{ m}) = 0$$





## SAMPLE PROBLEM 4.2

Three loads are applied to a beam as shown. The beam is supported by a roller at A and by a pin at B. Neglecting the weight of the beam, determine the reactions at A and B when  $P = 15$  kips.



## SOLUTION

**Free-Body Diagram.** A free-body diagram of the beam is drawn. The reaction at A is vertical and is denoted by  $\mathbf{A}$ . The reaction at B is represented by components  $\mathbf{B}_x$  and  $\mathbf{B}_y$ . Each component is assumed to act in the direction shown.

**Equilibrium Equations.** We write the following three equilibrium equations and solve for the reactions indicated:

$$\rightarrow \sum F_x = 0: \quad B_x = 0 \quad \mathbf{B}_x = 0$$

$$+\uparrow \sum M_A = 0: \quad -(15 \text{ kips})(3 \text{ ft}) + B_y(9 \text{ ft}) - (6 \text{ kips})(11 \text{ ft}) - (6 \text{ kips})(13 \text{ ft}) = 0 \\ B_y = +21.0 \text{ kips} \quad \mathbf{B}_y = 21.0 \text{ kips} \uparrow$$

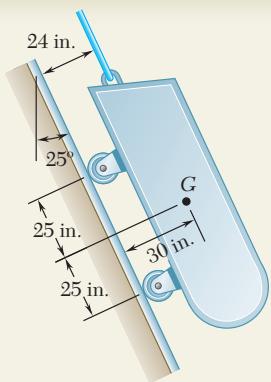
$$+\uparrow \sum M_B = 0: \quad -A(9 \text{ ft}) + (15 \text{ kips})(6 \text{ ft}) - (6 \text{ kips})(2 \text{ ft}) - (6 \text{ kips})(4 \text{ ft}) = 0 \\ A = +6.00 \text{ kips} \quad \mathbf{A} = 6.00 \text{ kips} \uparrow$$

**Check.** The results are checked by adding the vertical components of all of the external forces:

$$+\uparrow \sum F_y = +6.00 \text{ kips} - 15 \text{ kips} + 21.0 \text{ kips} - 6 \text{ kips} - 6 \text{ kips} = 0$$

**Remark.** In this problem the reactions at both A and B are vertical; however, these reactions are vertical for different reasons. At A, the beam is supported by a roller; hence the reaction cannot have any horizontal component. At B, the horizontal component of the reaction is zero because it must satisfy the equilibrium equation  $\sum F_x = 0$  and because none of the other forces acting on the beam has a horizontal component.

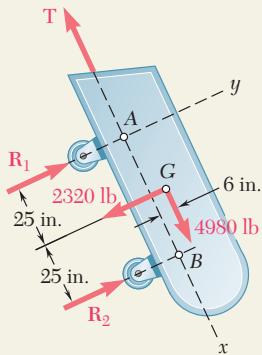
We could have noticed at first glance that the reaction at B was vertical and dispensed with the horizontal component  $\mathbf{B}_x$ . This, however, is a bad practice. In following it, we would run the risk of forgetting the component  $\mathbf{B}_x$  when the loading conditions require such a component (i.e., when a horizontal load is included). Also, the component  $\mathbf{B}_x$  was found to be zero by using and solving an equilibrium equation,  $\sum F_x = 0$ . By setting  $\mathbf{B}_x$  equal to zero immediately, we might not realize that we actually make use of this equation and thus might lose track of the number of equations available for solving the problem.



## SAMPLE PROBLEM 4.3

A loading car is at rest on a track forming an angle of  $25^\circ$  with the vertical. The gross weight of the car and its load is 5500 lb, and it is applied at a point 30 in. from the track, halfway between the two axles. The car is held by a cable attached 24 in. from the track. Determine the tension in the cable and the reaction at each pair of wheels.

## SOLUTION



**Free-Body Diagram.** A free-body diagram of the car is drawn. The reaction at each wheel is perpendicular to the track, and the tension force  $\mathbf{T}$  is parallel to the track. For convenience, we choose the  $x$  axis parallel to the track and the  $y$  axis perpendicular to the track. The 5500-lb weight is then resolved into  $x$  and  $y$  components.

$$W_x = +(5500 \text{ lb}) \cos 25^\circ = +4980 \text{ lb}$$

$$W_y = -(5500 \text{ lb}) \sin 25^\circ = -2320 \text{ lb}$$

**Equilibrium Equations.** We take moments about  $A$  to eliminate  $\mathbf{T}$  and  $\mathbf{R}_1$  from the computation.

$$+\gamma\sum M_A = 0: \quad -(2320 \text{ lb})(25 \text{ in.}) - (4980 \text{ lb})(6 \text{ in.}) + R_2(50 \text{ in.}) = 0$$

$$R_2 = +1758 \text{ lb} \quad \mathbf{R}_2 = 1758 \text{ lb} \nearrow$$

Now, taking moments about  $B$  to eliminate  $\mathbf{T}$  and  $\mathbf{R}_2$  from the computation, we write

$$+\gamma\sum M_B = 0: \quad (2320 \text{ lb})(25 \text{ in.}) - (4980 \text{ lb})(6 \text{ in.}) - R_1(50 \text{ in.}) = 0$$

$$R_1 = +562 \text{ lb} \quad \mathbf{R}_1 = +562 \text{ lb} \nearrow$$

The value of  $T$  is found by writing

$$\nabla + \sum F_x = 0: \quad +4980 \text{ lb} - T = 0$$

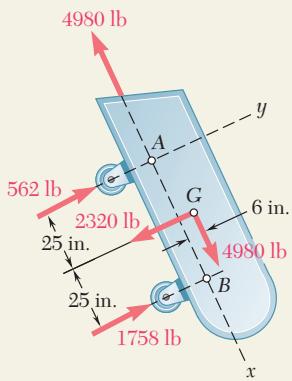
$$T = +4980 \text{ lb} \quad \mathbf{T} = 4980 \text{ lb} \nwarrow$$

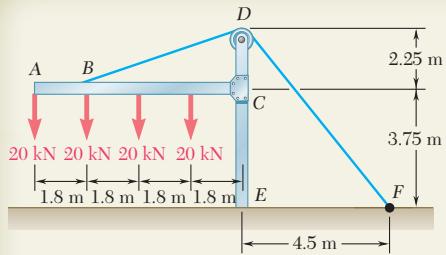
The computed values of the reactions are shown in the adjacent sketch.

**Check.** The computations are verified by writing

$$\nearrow + \sum F_y = +562 \text{ lb} + 1758 \text{ lb} - 2320 \text{ lb} = 0$$

The solution could also have been checked by computing moments about any point other than  $A$  or  $B$ .





## SAMPLE PROBLEM 4.4

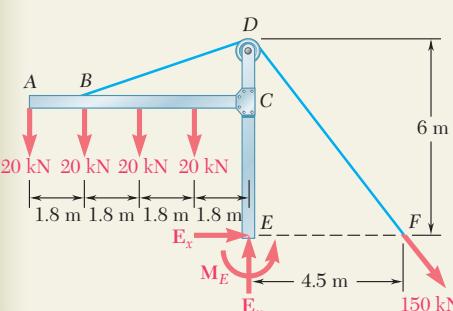
The frame shown supports part of the roof of a small building. Knowing that the tension in the cable is 150 kN, determine the reaction at the fixed end  $E$ .

### SOLUTION

**Free-Body Diagram.** A free-body diagram of the frame and of the cable  $BDF$  is drawn. The reaction at the fixed end  $E$  is represented by the force components  $\mathbf{E}_x$  and  $\mathbf{E}_y$  and the couple  $\mathbf{M}_E$ . The other forces acting on the free body are the four 20-kN loads and the 150-kN force exerted at end  $F$  of the cable.

**Equilibrium Equations.** Noting that  $DF = \sqrt{(4.5 \text{ m})^2 + (6 \text{ m})^2} = 7.5 \text{ m}$ , we write

$$\begin{aligned} \rightarrow \sum F_x &= 0: & E_x + \frac{4.5}{7.5}(150 \text{ kN}) &= 0 \\ && E_x = -90.0 \text{ kN} & \mathbf{E}_x = 90.0 \text{ kN} \leftarrow \\ + \uparrow \sum F_y &= 0: & E_y - 4(20 \text{ kN}) - \frac{6}{7.5}(150 \text{ kN}) &= 0 \\ && E_y = +200 \text{ kN} & \mathbf{E}_y = 200 \text{ kN} \uparrow \\ + \uparrow \sum M_E &= 0: & (20 \text{ kN})(7.2 \text{ m}) + (20 \text{ kN})(5.4 \text{ m}) + (20 \text{ kN})(3.6 \text{ m}) \\ && + (20 \text{ kN})(1.8 \text{ m}) - \frac{6}{7.5}(150 \text{ kN})(4.5 \text{ m}) + M_E &= 0 \\ && M_E = +180.0 \text{ kN} \cdot \text{m} & \mathbf{M}_E = 180.0 \text{ kN} \cdot \text{m} \uparrow \end{aligned}$$



## SAMPLE PROBLEM 4.5

A 400-lb weight is attached at  $A$  to the lever shown. The constant of the spring  $BC$  is  $k = 250 \text{ lb/in.}$ , and the spring is unstretched when  $\theta = 0$ . Determine the position of equilibrium.

### SOLUTION

**Free-Body Diagram.** We draw a free-body diagram of the lever and cylinder. Denoting by  $s$  the deflection of the spring from its undeformed position, and noting that  $s = r\theta$ , we have  $F = ks = kr\theta$ .

**Equilibrium Equation.** Summing the moments of  $\mathbf{W}$  and  $\mathbf{F}$  about  $O$ , we write

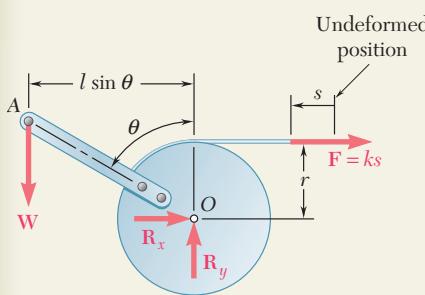
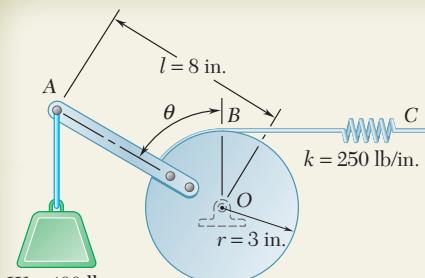
$$+\uparrow \sum M_O = 0: \quad Wl \sin \theta - r(kr\theta) = 0 \quad \sin \theta = \frac{kr^2}{Wl} \theta$$

Substituting the given data, we obtain

$$\sin \theta = \frac{(250 \text{ lb/in.})(3 \text{ in.})^2}{(400 \text{ lb})(8 \text{ in.})} \theta \quad \sin \theta = 0.703 \theta$$

Solving by trial and error, we find

$$\theta = 0 \quad \theta = 80.3^\circ$$



# SOLVING PROBLEMS ON YOUR OWN

You saw that the external forces acting on a rigid body in equilibrium form a system equivalent to zero. To solve an equilibrium problem your first task is to draw a neat, reasonably large *free-body diagram* on which you will show all external forces. Both known and unknown forces must be included.

**For a two-dimensional rigid body,** the reactions at the supports can involve one, two, or three unknowns depending on the type of support (Fig. 4.1). For the successful solution of a problem, a correct free-body diagram is essential. Never proceed with the solution of a problem until you are sure that your free-body diagram includes all loads, all reactions, and the weight of the body (if appropriate).

**1. You can write three equilibrium equations** and solve them for *three unknowns*. The three equations might be

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M_O = 0$$

However, there are usually several sets of equations that you can write, such as

$$\Sigma F_x = 0 \quad \Sigma M_A = 0 \quad \Sigma M_B = 0$$

where point *B* is chosen in such a way that the line *AB* is not parallel to the *y* axis, or

$$\Sigma M_A = 0 \quad \Sigma M_B = 0 \quad \Sigma M_C = 0$$

where the points *A*, *B*, and *C* do not lie in a straight line.

**2. To simplify your solution,** it may be helpful to use one of the following solution techniques if applicable.

a. **By summing moments about the point of intersection** of the lines of action of two unknown forces, you will obtain an equation in a single unknown.

b. **By summing components in a direction perpendicular to two unknown parallel forces,** you will obtain an equation in a single unknown.

**3. After drawing your free-body diagram,** you may find that one of the following special situations exists.

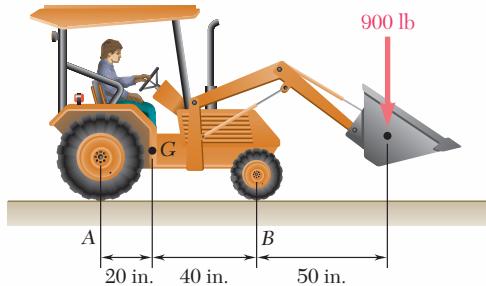
a. **The reactions involve fewer than three unknowns;** the body is said to be *partially constrained* and motion of the body is possible.

b. **The reactions involve more than three unknowns;** the reactions are said to be *statically indeterminate*. While you may be able to calculate one or two reactions, you cannot determine all of the reactions.

c. **The reactions pass through a single point or are parallel;** the body is said to be *improperly constrained* and motion can occur under a general loading condition.

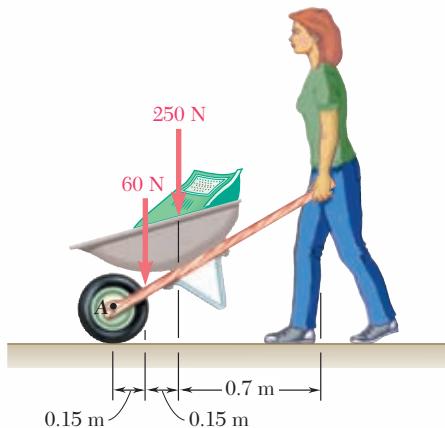
# PROBLEMS

- 4.1** A 2100-lb tractor is used to lift 900 lb of gravel. Determine the reaction at each of the two (a) rear wheels A, (b) front wheels B.



**Fig. P4.1**

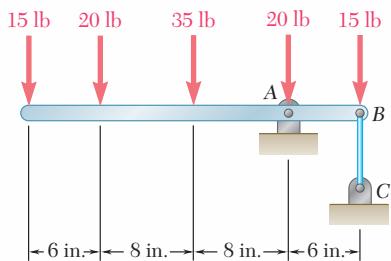
- 4.2** A gardener uses a 60-N wheelbarrow to transport a 250-N bag of fertilizer. What force must she exert on each handle?



**Fig. P4.2**

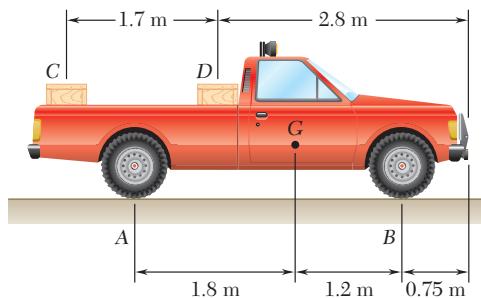
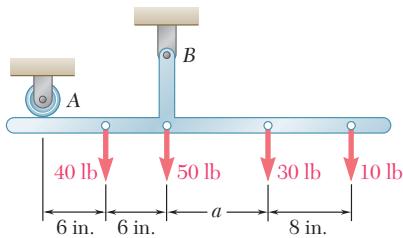
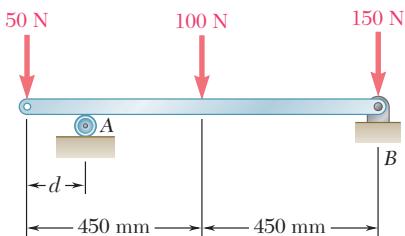
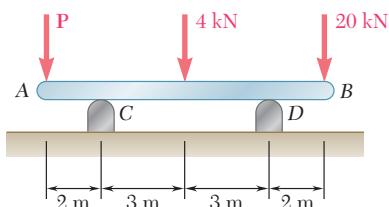
- 4.3** The gardener of Prob. 4.2 wishes to transport a second 250-N bag of fertilizer at the same time as the first one. Determine the maximum allowable horizontal distance from the axle A of the wheelbarrow to the center of gravity of the second bag if she can hold only 75 N with each arm.

- 4.4** For the beam and loading shown, determine (a) the reaction at A, (b) the tension in cable BC.



**Fig. P4.4**

- 4.5** Two crates, each of mass 350 kg, are placed as shown in the bed of a 1400-kg pickup truck. Determine the reactions at each of the two (a) rear wheels A, (b) front wheels B.

**Fig. P4.5****Fig. P4.7****Fig. P4.9****Fig. P4.12 and P4.13**

- 4.6** Solve Prob. 4.5, assuming that crate D is removed and that the position of crate C is unchanged.

- 4.7** A T-shaped bracket supports the four loads shown. Determine the reactions at A and B (a) if  $a = 10$  in., (b) if  $a = 7$  in.

- 4.8** For the bracket and loading of Prob. 4.7, determine the smallest distance  $a$  if the bracket is not to move.

- 4.9** The maximum allowable value of each of the reactions is 180 N. Neglecting the weight of the beam, determine the range of the distance  $d$  for which the beam is safe.

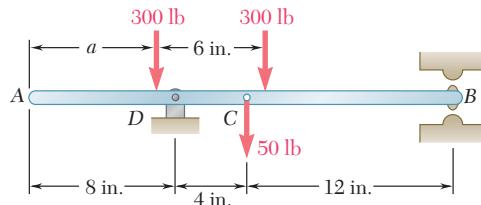
- 4.10** Solve Prob. 4.9 if the 50-N load is replaced by an 80-N load.

- 4.11** For the beam of Sample Prob. 4.2, determine the range of values of  $P$  for which the beam will be safe, knowing that the maximum allowable value of each of the reactions is 30 kips and that the reaction at A must be directed upward.

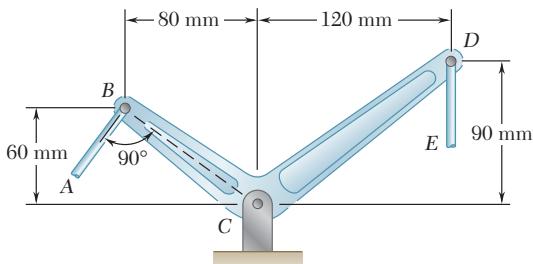
- 4.12** The 10-m beam AB rests upon, but is not attached to, supports at C and D. Neglecting the weight of the beam, determine the range of values of  $P$  for which the beam will remain in equilibrium.

- 4.13** The maximum allowable value of each of the reactions is 50 kN, and each reaction must be directed upward. Neglecting the weight of the beam, determine the range of values of  $P$  for which the beam is safe.

- 4.14** For the beam and loading shown, determine the range of the distance  $a$  for which the reaction at B does not exceed 100 lb downward or 200 lb upward.

**Fig. P4.14**

- 4.15** Two links  $AB$  and  $DE$  are connected by a bell crank as shown. Knowing that the tension in link  $AB$  is 720 N, determine (a) the tension in link  $DE$ , (b) the reaction at  $C$ .



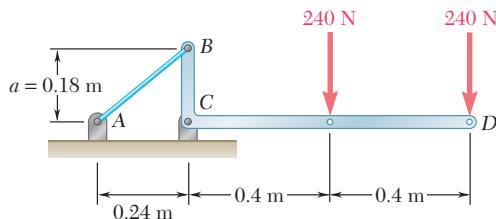
**Fig. P4.15 and P4.16**

- 4.16** Two links  $AB$  and  $DE$  are connected by a bell crank as shown. Determine the maximum force that can be safely exerted by link  $AB$  on the bell crank if the maximum allowable value for the reaction at  $C$  is 1600 N.

- 4.17** The required tension in cable  $AB$  is 200 lb. Determine (a) the vertical force  $P$  that must be applied to the pedal, (b) the corresponding reaction at  $C$ .

- 4.18** Determine the maximum tension that can be developed in cable  $AB$  if the maximum allowable value of the reaction at  $C$  is 250 lb.

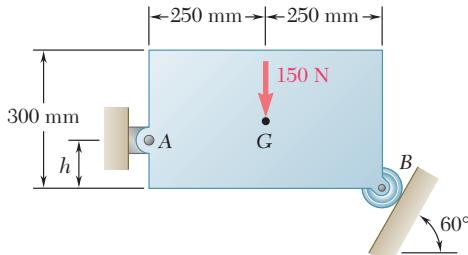
- 4.19** The bracket  $BCD$  is hinged at  $C$  and attached to a control cable at  $B$ . For the loading shown, determine (a) the tension in the cable, (b) the reaction at  $C$ .



**Fig. P4.19**

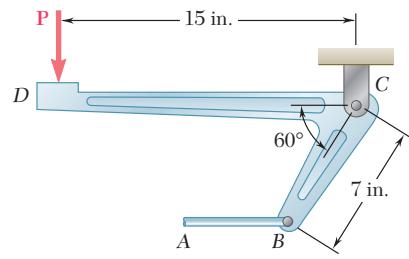
- 4.20** Solve Prob. 4.19, assuming that  $a = 0.32$  m.

- 4.21** Determine the reactions at  $A$  and  $B$  when (a)  $h = 0$ , (b)  $h = 200$  mm.

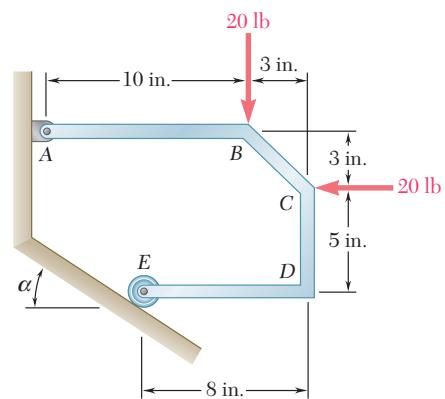


**Fig. P4.21**

- 4.22** For the frame and loading shown, determine the reactions at  $A$  and  $E$  when (a)  $\alpha = 30^\circ$ , (b)  $\alpha = 45^\circ$ .



**Fig. P4.17 and P4.18**



**Fig. P4.22**

**4.23 and 4.24** For each of the plates and loadings shown, determine the reactions at *A* and *B*.

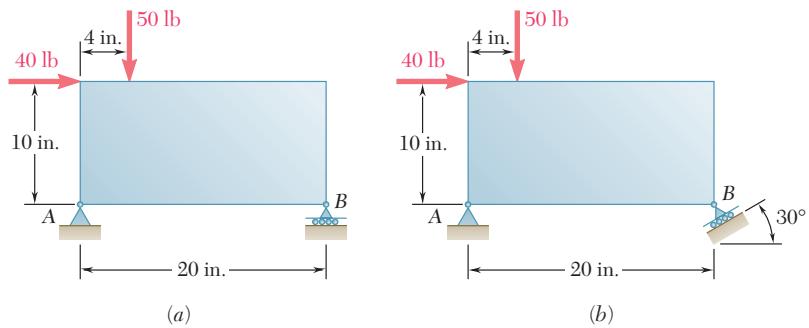


Fig. P4.23

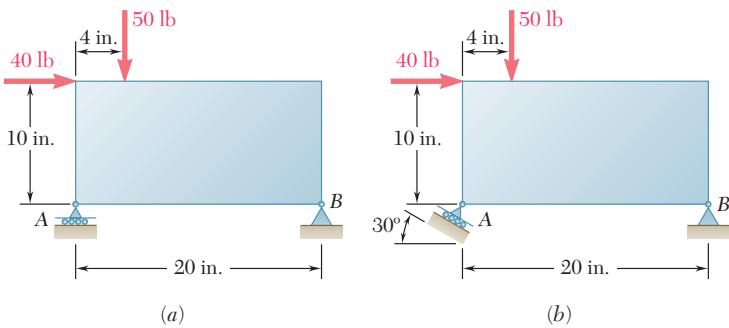


Fig. P4.24

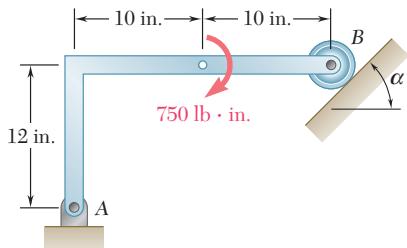


Fig. P4.25

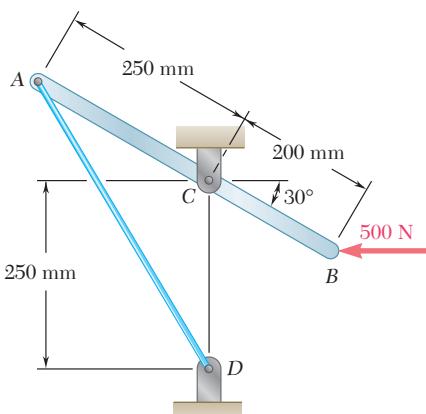


Fig. P4.28

**4.25** Determine the reactions at *A* and *B* when (a)  $\alpha = 0$ , (b)  $\alpha = 90^\circ$ , (c)  $\alpha = 30^\circ$ .

**4.26** A rod *AB*, hinged at *A* and attached at *B* to cable *BD*, supports the loads shown. Knowing that  $d = 200$  mm, determine (a) the tension in cable *BD*, (b) the reaction at *A*.

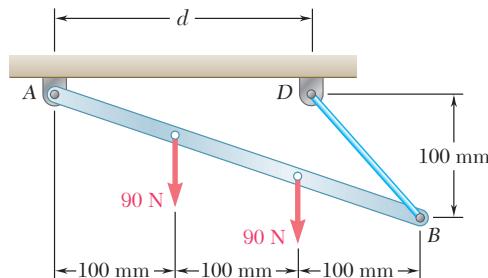


Fig. P4.26 and P4.27

**4.27** A rod *AB*, hinged at *A* and attached at *B* to cable *BD*, supports the loads shown. Knowing that  $d = 150$  mm, determine (a) the tension in cable *BD*, (b) the reaction at *A*.

**4.28** A lever *AB* is hinged at *C* and attached to a control cable at *A*. If the lever is subjected to a 500-N horizontal force at *B*, determine (a) the tension in the cable, (b) the reaction at *C*.

- 4.29** A force  $\mathbf{P}$  of magnitude 280 lb is applied to member  $ABCD$ , which is supported by a frictionless pin at  $A$  and by the cable  $CED$ . Since the cable passes over a small pulley at  $E$ , the tension may be assumed to be the same in portions  $CE$  and  $ED$  of the cable. For the case when  $a = 3$  in., determine (a) the tension in the cable, (b) the reaction at  $A$ .

- 4.30** Neglecting friction, determine the tension in cable  $ABD$  and the reaction at support  $C$ .

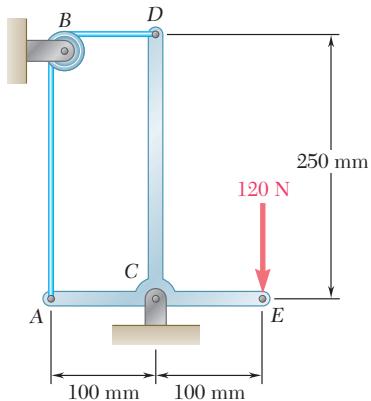


Fig. P4.30

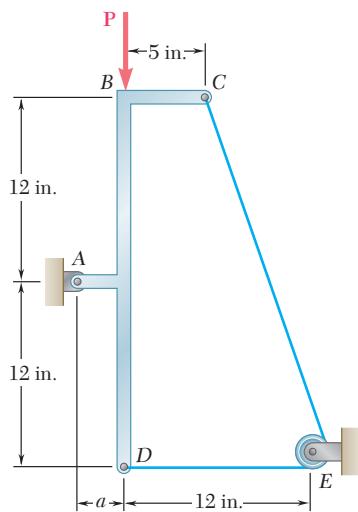


Fig. P4.29

- 4.31** Rod  $ABC$  is bent in the shape of an arc of circle of radius  $R$ . Knowing that  $\theta = 30^\circ$ , determine the reaction (a) at  $B$ , (b) at  $C$ .

- 4.32** Rod  $ABC$  is bent in the shape of an arc of circle of radius  $R$ . Knowing that  $\theta = 60^\circ$ , determine the reaction (a) at  $B$ , (b) at  $C$ .

- 4.33** Neglecting friction, determine the tension in cable  $ABD$  and the reaction at  $C$  when  $\theta = 60^\circ$ .

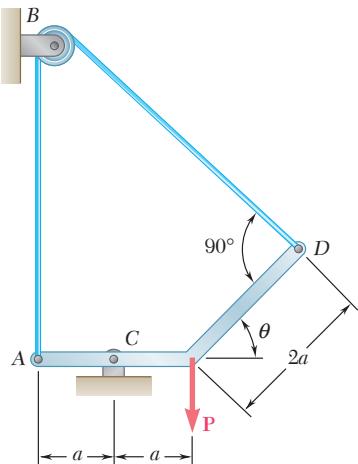


Fig. P4.33 and P4.34

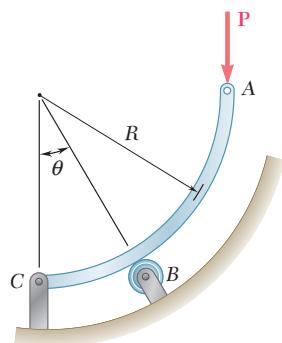


Fig. P4.31 and P4.32

- 4.34** Neglecting friction, determine the tension in cable  $ABD$  and the reaction at  $C$  when  $\theta = 45^\circ$ .

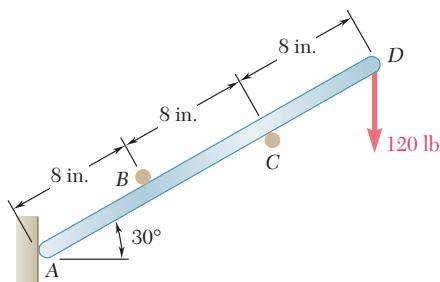


Fig. P4.35

- 4.35** A light rod  $AD$  is supported by frictionless pegs at  $B$  and  $C$  and rests against a frictionless wall at  $A$ . A vertical 120-lb force is applied at  $D$ . Determine the reactions at  $A$ ,  $B$ , and  $C$ .

- 4.36** A light bar  $AD$  is suspended from a cable  $BE$  and supports a 50-lb block at  $C$ . The ends  $A$  and  $D$  of the bar are in contact with frictionless vertical walls. Determine the tension in cable  $BE$  and the reactions at  $A$  and  $D$ .

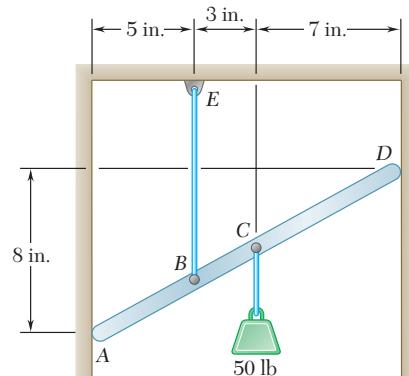


Fig. P4.36

- 4.37** Bar  $AC$  supports two 400-N loads as shown. Rollers at  $A$  and  $C$  rest against frictionless surfaces and a cable  $BD$  is attached at  $B$ . Determine (a) the tension in cable  $BD$ , (b) the reaction at  $A$ , (c) the reaction at  $C$ .

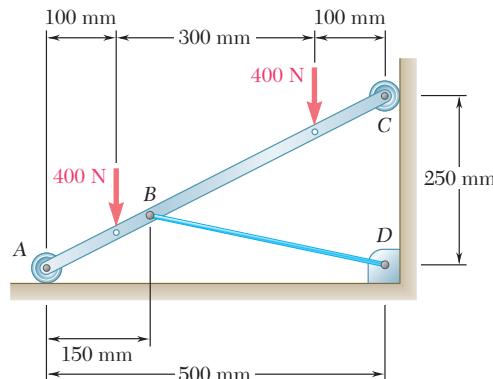


Fig. P4.37

- 4.38** Determine the tension in each cable and the reaction at  $D$ .

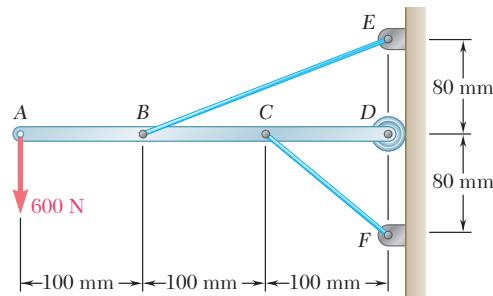


Fig. P4.38

- 4.39** Two slots have been cut in plate *DEF*, and the plate has been placed so that the slots fit two fixed, frictionless pins *A* and *B*. Knowing that  $P = 15 \text{ lb}$ , determine (a) the force each pin exerts on the plate, (b) the reaction at *F*.

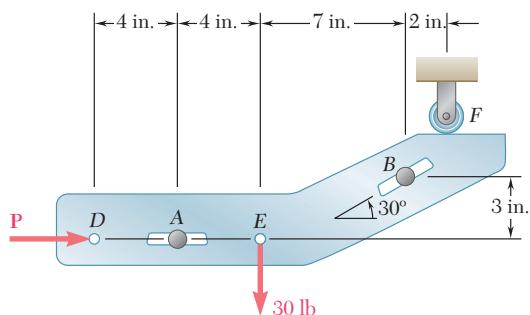


Fig. P4.39

- 4.40** For the plate of Prob. 4.39 the reaction at *F* must be directed downward, and its maximum allowable value is 20 lb. Neglecting friction at the pins, determine the required range of values of  $P$ .

- 4.41** Bar *AD* is attached at *A* and *C* to collars that can move freely on the rods shown. If the cord *BE* is vertical ( $\alpha = 0$ ), determine the tension in the cord and the reactions at *A* and *C*.

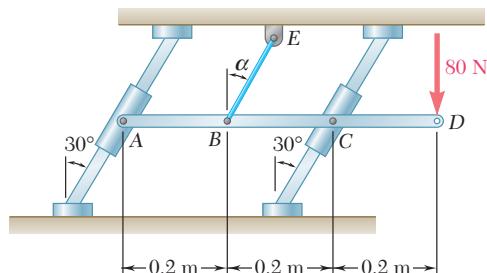


Fig. P4.41

- 4.42** Solve Prob. 4.41 if the cord *BE* is parallel to the rods ( $\alpha = 30^\circ$ ).

- 4.43** An 8-kg mass can be supported in the three different ways shown. Knowing that the pulleys have a 100-mm radius, determine the reaction at *A* in each case.

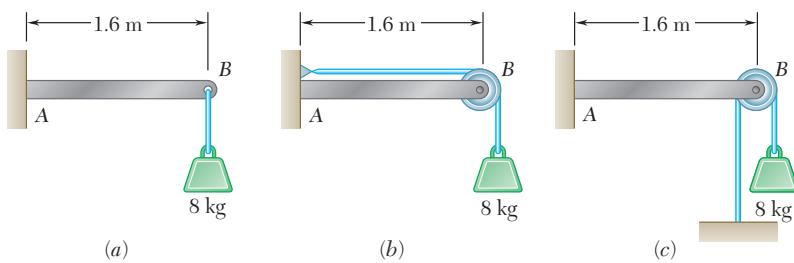
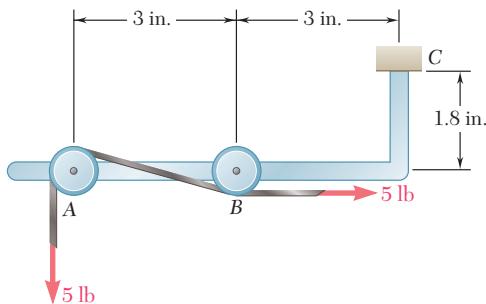


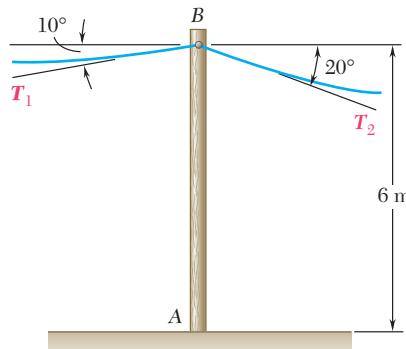
Fig. P4.43

- 4.44** A tension of 5 lb is maintained in a tape as it passes through the support system shown. Knowing that the radius of each pulley is 0.4 in., determine the reaction at *C*.

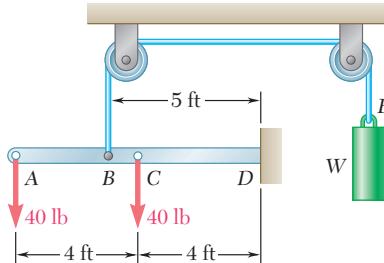
**Fig. P4.44**

- 4.45** Solve Prob. 4.44, assuming that 0.6-in.-radius pulleys are used.

- 4.46** A 6-m telephone pole weighing 1600 N is used to support the ends of two wires. The wires form the angles shown with the horizontal and the tensions in the wires are, respectively,  $T_1 = 600$  N and  $T_2 = 375$  N. Determine the reaction at the fixed end *A*.

**Fig. P4.46**

- 4.47** Beam *AD* carries the two 40-lb loads shown. The beam is held by a fixed support at *D* and by the cable *BE* that is attached to the counterweight *W*. Determine the reaction at *D* when (a)  $W = 100$  lb, (b)  $W = 90$  lb.

**Fig. P4.47 and P4.48**

- 4.48** For the beam and loading shown, determine the range of values of *W* for which the magnitude of the couple at *D* does not exceed 40 lb · ft.

- 4.49** Knowing that the tension in wire  $BD$  is 1300 N, determine the reaction at the fixed support  $C$  of the frame shown.

- 4.50** Determine the range of allowable values of the tension in wire  $BD$  if the magnitude of the couple at the fixed support  $C$  is not to exceed 100 N · m.

- 4.51** A vertical load  $\mathbf{P}$  is applied at end  $B$  of rod  $BC$ . (a) Neglecting the weight of the rod, express the angle  $\theta$  corresponding to the equilibrium position in terms of  $P$ ,  $l$ , and the counterweight  $W$ . (b) Determine the value of  $\theta$  corresponding to equilibrium if  $P = 2W$ .

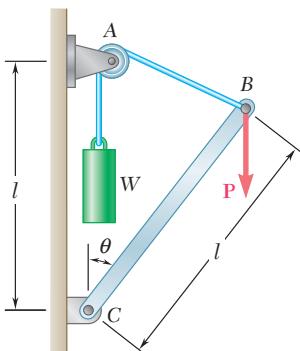


Fig. P4.51

- 4.52** A slender rod  $AB$ , of weight  $W$ , is attached to blocks  $A$  and  $B$ , which move freely in the guides shown. The blocks are connected by an elastic cord that passes over a pulley at  $C$ . (a) Express the tension in the cord in terms of  $W$  and  $\theta$ . (b) Determine the value of  $\theta$  for which the tension in the cord is equal to  $3W$ .

- 4.53** Rod  $AB$  is acted upon by a couple  $\mathbf{M}$  and two forces, each of magnitude  $P$ . (a) Derive an equation in  $\theta$ ,  $P$ ,  $M$ , and  $l$  that must be satisfied when the rod is in equilibrium. (b) Determine the value of  $\theta$  corresponding to equilibrium when  $M = 150$  N · m,  $P = 200$  N, and  $l = 600$  mm.

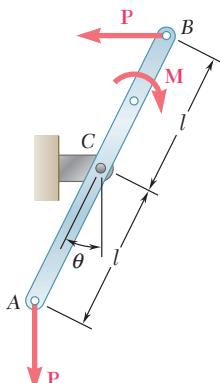


Fig. P4.53

- 4.54** Rod  $AB$  is attached to a collar at  $A$  and rests against a small roller at  $C$ . (a) Neglecting the weight of rod  $AB$ , derive an equation in  $P$ ,  $Q$ ,  $a$ ,  $l$ , and  $\theta$  that must be satisfied when the rod is in equilibrium. (b) Determine the value of  $\theta$  corresponding to equilibrium when  $P = 16$  lb,  $Q = 12$  lb,  $l = 20$  in., and  $a = 5$  in.

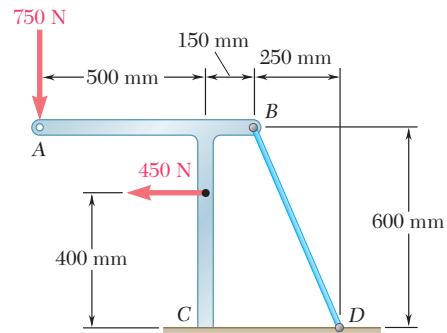


Fig. P4.49 and P4.50

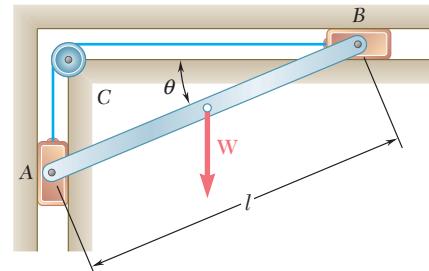


Fig. P4.52

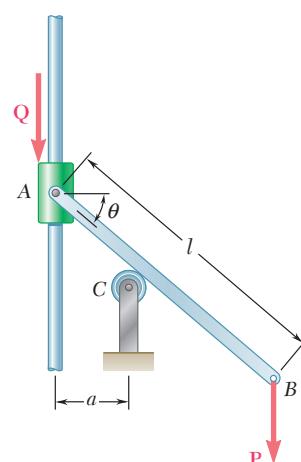


Fig. P4.54

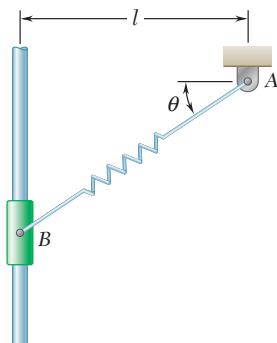


Fig. P4.55

- 4.55** A collar *B* of weight *W* can move freely along the vertical rod shown. The constant of the spring is *k*, and the spring is unstretched when  $\theta = 0$ . (a) Derive an equation in  $\theta$ , *W*, *k*, and *l* that must be satisfied when the collar is in equilibrium. (b) Knowing that *W* = 300 N, *l* = 500 mm, and *k* = 800 N/m, determine the value of  $\theta$  corresponding to equilibrium.

- 4.56** A vertical load **P** is applied at end *B* of rod *BC*. The constant of the spring is *k*, and the spring is unstretched when  $\theta = 90^\circ$ . (a) Neglecting the weight of the rod, express the angle  $\theta$  corresponding to equilibrium in terms of *P*, *k*, and *l*. (b) Determine the value of  $\theta$  corresponding to equilibrium when *P* =  $\frac{1}{4}kl$ .

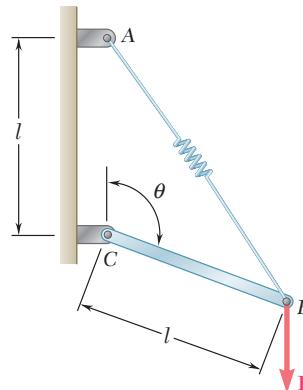


Fig. P4.56

- 4.57** Solve Sample Prob. 4.5, assuming that the spring is unstretched when  $\theta = 90^\circ$ .

- 4.58** A slender rod *AB*, of weight *W*, is attached to blocks *A* and *B* that move freely in the guides shown. The constant of the spring is *k*, and the spring is unstretched when  $\theta = 0$ . (a) Neglecting the weight of the blocks, derive an equation in *W*, *k*, *l*, and  $\theta$  that must be satisfied when the rod is in equilibrium. (b) Determine the value of  $\theta$  when *W* = 75 lb, *l* = 30 in., and *k* = 3 lb/in.

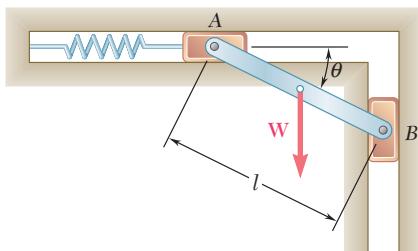


Fig. P4.58

- 4.59** Eight identical 500 × 750-mm rectangular plates, each of mass *m* = 40 kg, are held in a vertical plane as shown. All connections consist of frictionless pins, rollers, or short links. In each case, determine whether (a) the plate is completely, partially, or improperly constrained, (b) the reactions are statically determinate or indeterminate, (c) the equilibrium of the plate is maintained in the position shown. Also, wherever possible, compute the reactions.

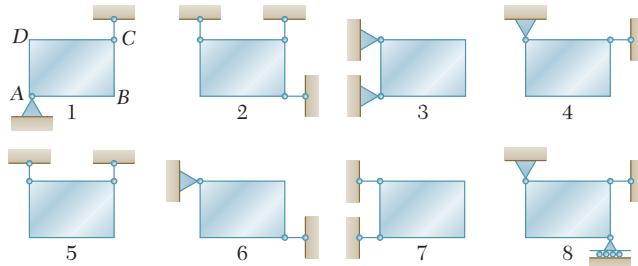
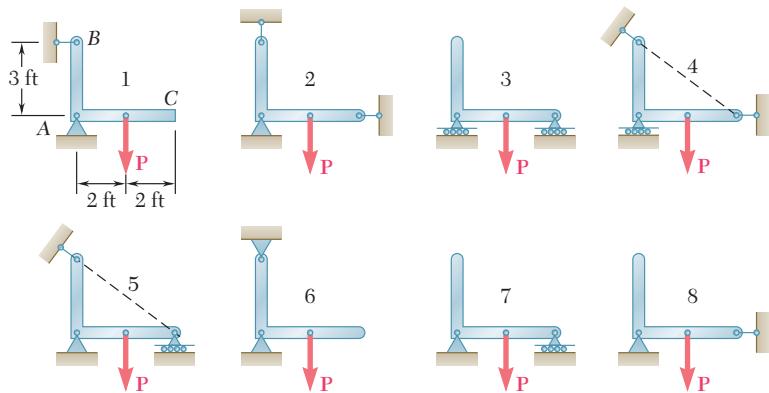


Fig. P4.59

- 4.60** The bracket ABC can be supported in the eight different ways shown. All connections consist of smooth pins, rollers, or short links. For each case, answer the questions listed in Prob. 4.59, and, wherever possible, compute the reactions, assuming that the magnitude of the force  $\mathbf{P}$  is 100 lb.

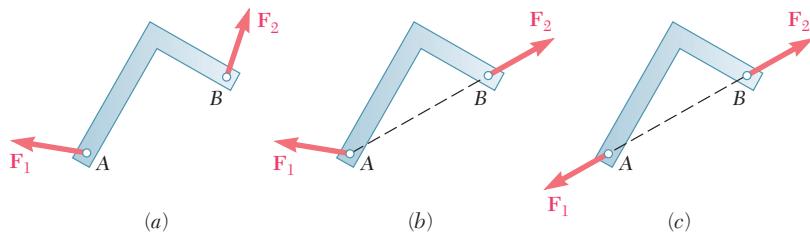


**Fig. P4.60**

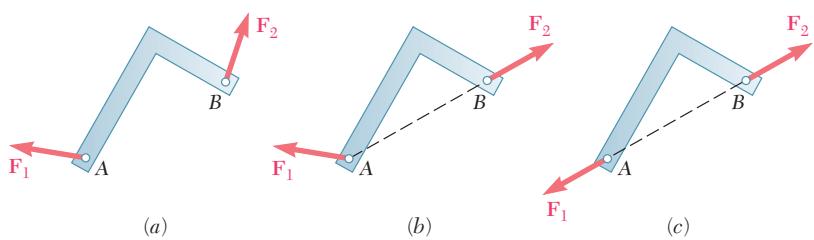
## 4.6 EQUILIBRIUM OF A TWO-FORCE BODY

A particular case of equilibrium which is of considerable interest is that of a rigid body subjected to two forces. Such a body is commonly called a *two-force body*. It will be shown that *if a two-force body is in equilibrium, the two forces must have the same magnitude, the same line of action, and opposite sense*.

Consider a corner plate subjected to two forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  acting at A and B, respectively (Fig. 4.8a). If the plate is to be in equilibrium, the sum of the moments of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  about any axis must be zero. First, we sum moments about A. Since the moment of  $\mathbf{F}_1$  is obviously zero, the moment of  $\mathbf{F}_2$  must also be zero and the line of action of  $\mathbf{F}_2$  must pass through A (Fig. 4.8b). Summing moments about B, we prove similarly that the line of action of  $\mathbf{F}_1$  must pass through B (Fig. 4.8c). Therefore, both forces have the same line of action (line AB). From either of the equations  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$  it is seen that they must also have the same magnitude but opposite sense.



**Fig. 4.8**

**Fig. 4.8 (repeated)**

If several forces act at two points  $A$  and  $B$ , the forces acting at  $A$  can be replaced by their resultant  $\mathbf{F}_1$  and those acting at  $B$  can be replaced by their resultant  $\mathbf{F}_2$ . Thus a two-force body can be more generally defined as *a rigid body subjected to forces acting at only two points*. The resultants  $\mathbf{F}_1$  and  $\mathbf{F}_2$  then must have the same line of action, the same magnitude, and opposite sense (Fig. 4.8).

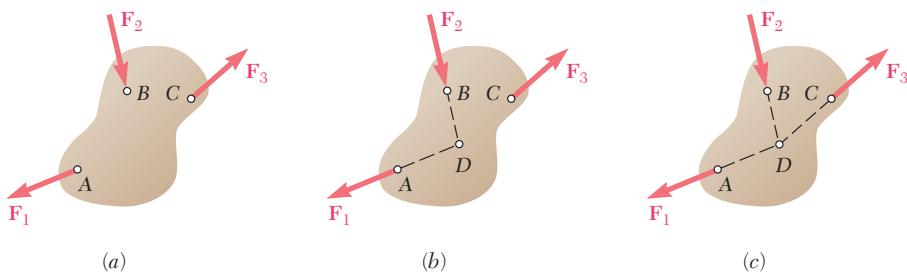
In the study of structures, frames, and machines, you will see how the recognition of two-force bodies simplifies the solution of certain problems.

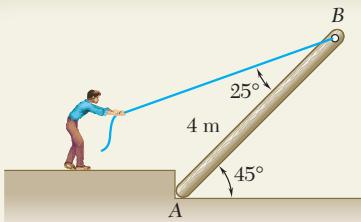
## 4.7 EQUILIBRIUM OF A THREE-FORCE BODY

Another case of equilibrium that is of great interest is that of a *three-force body*, i.e., a rigid body subjected to three forces or, more generally, *a rigid body subjected to forces acting at only three points*. Consider a rigid body subjected to a system of forces which can be reduced to three forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ , and  $\mathbf{F}_3$  acting at  $A$ ,  $B$ , and  $C$ , respectively (Fig. 4.9a). It will be shown that if the body is in equilibrium, *the lines of action of the three forces must be either concurrent or parallel*.

Since the rigid body is in equilibrium, the sum of the moments of  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ , and  $\mathbf{F}_3$  about any axis must be zero. Assuming that the lines of action of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  intersect and denoting their point of intersection by  $D$ , we sum moments about  $D$  (Fig. 4.9b). Since the moments of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  about  $D$  are zero, the moment of  $\mathbf{F}_3$  about  $D$  must also be zero, and the line of action of  $\mathbf{F}_3$  must pass through  $D$  (Fig. 4.9c). Therefore, the three lines of action are concurrent. The only exception occurs when none of the lines intersect; the lines of action are then parallel.

Although problems concerning three-force bodies can be solved by the general methods of Secs. 4.3 to 4.5, the property just established can be used to solve them either graphically or mathematically from simple trigonometric or geometric relations.

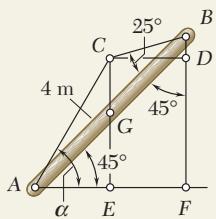
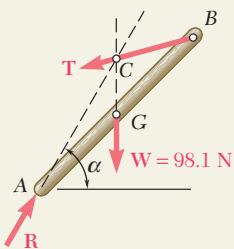
**Fig. 4.9**



## SAMPLE PROBLEM 4.6

A man raises a 10-kg joist, of length 4 m, by pulling on a rope. Find the tension  $T$  in the rope and the reaction at  $A$ .

## SOLUTION



**Three-Force Body.** Since the joist is a three-force body, the forces acting on it must be concurrent. The reaction  $\mathbf{R}$ , therefore, will pass through the point of intersection  $C$  of the lines of action of the weight  $\mathbf{W}$  and the tension force  $\mathbf{T}$ . This fact will be used to determine the angle  $\alpha$  that  $\mathbf{R}$  forms with the horizontal.

Drawing the vertical  $BF$  through  $B$  and the horizontal  $CD$  through  $C$ , we note that

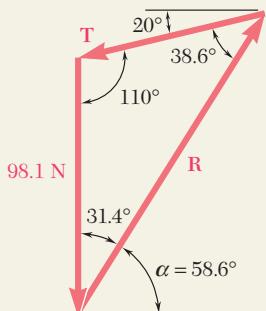
$$\begin{aligned} AF &= BF = (AB) \cos 45^\circ = (4 \text{ m}) \cos 45^\circ = 2.828 \text{ m} \\ CD &= EF = AE = \frac{1}{2}(AF) = 1.414 \text{ m} \\ BD &= (CD) \cot(45^\circ + 25^\circ) = (1.414 \text{ m}) \tan 20^\circ = 0.515 \text{ m} \\ CE &= DF = BF - BD = 2.828 \text{ m} - 0.515 \text{ m} = 2.313 \text{ m} \end{aligned}$$

We write

$$\tan \alpha = \frac{CE}{AE} = \frac{2.313 \text{ m}}{1.414 \text{ m}} = 1.636$$

$$\alpha = 58.6^\circ$$

We now know the direction of all the forces acting on the joist.



**Force Triangle.** A force triangle is drawn as shown, and its interior angles are computed from the known directions of the forces. Using the law of sines, we write

$$\frac{T}{\sin 31.4^\circ} = \frac{R}{\sin 110^\circ} = \frac{98.1 \text{ N}}{\sin 38.6^\circ}$$

$$T = 81.9 \text{ N}$$

$$R = 147.8 \text{ N} \angle 58.6^\circ$$

# SOLVING PROBLEMS ON YOUR OWN

The preceding sections covered two particular cases of equilibrium of a rigid body.

**1. A two-force body is a body subjected to forces at only two points.** The resultants of the forces acting at each of these points must have the *same magnitude, the same line of action, and opposite sense*. This property will allow you to simplify the solutions of some problems by replacing the two unknown components of a reaction by a single force of unknown magnitude but of *known direction*.

**2. A three-force body is subjected to forces at only three points.** The resultants of the forces acting at each of these points must be *concurrent or parallel*. To solve a problem involving a three-force body with concurrent forces, draw your free-body diagram showing that these three forces pass through the same point. The use of simple geometry may then allow you to complete the solution by using a force triangle [Sample Prob. 4.6].

Although the principle noted above for the solution of problems involving three-force bodies is easily understood, it can be difficult to sketch the needed geometric constructions. If you encounter difficulty, first draw a reasonably large free-body diagram and then seek a relation between known or easily calculated lengths and a dimension that involves an unknown. This was done in Sample Prob. 4.6, where the easily calculated dimensions  $AE$  and  $CE$  were used to determine the angle  $\alpha$ .

# PROBLEMS

**4.61** Determine the reactions at *A* and *B* when  $a = 180$  mm.

**4.62** For the bracket and loading shown, determine the range of values of the distance *a* for which the magnitude of the reaction at *B* does not exceed 600 N.

**4.63** Using the method of Sec. 4.7, solve Prob. 4.17.

**4.64** Using the method of Sec. 4.7, solve Prob. 4.18.

**4.65** The spanner shown is used to rotate a shaft. A pin fits in a hole at *A*, while a flat, frictionless surface rests against the shaft at *B*. If a 60-lb force **P** is exerted on the spanner at *D*, find the reactions at *A* and *B*.

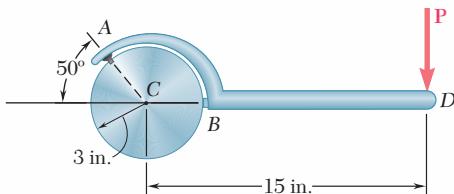


Fig. P4.65

**4.66** Determine the reactions at *B* and *D* when  $b = 60$  mm.

**4.67** Determine the reactions at *B* and *D* when  $b = 120$  mm.

**4.68** Determine the reactions at *B* and *C* when  $a = 1.5$  in.

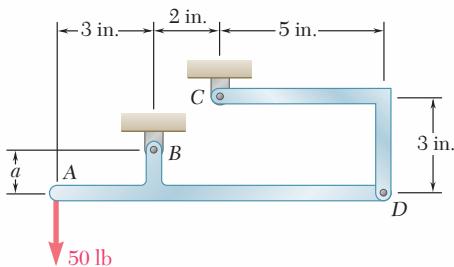


Fig. P4.68

**4.69** A 50-kg crate is attached to the trolley-beam system shown. Knowing that  $a = 1.5$  m, determine (a) the tension in cable *CD*, (b) the reaction at *B*.

**4.70** Solve Prob. 4.69, assuming that  $a = 3$  m.

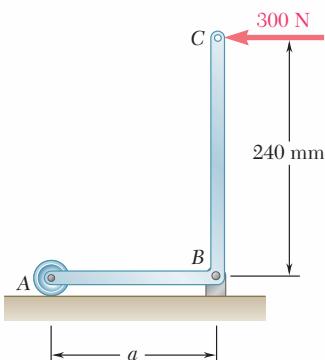


Fig. P4.61 and P4.62

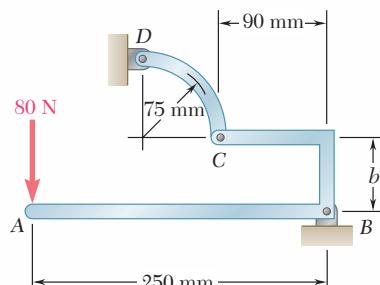


Fig. P4.66 and P4.67

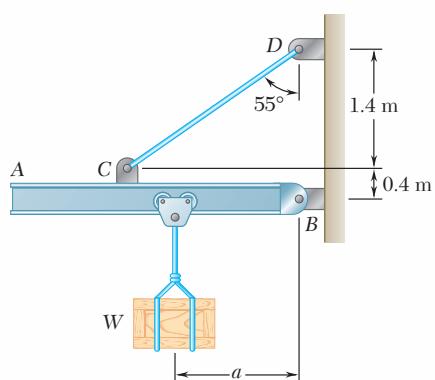


Fig. P4.69

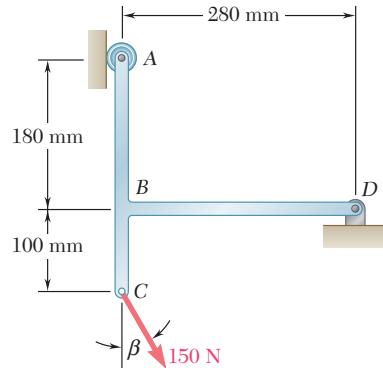


Fig. P4.72 and P4.73

- 4.71** One end of rod  $AB$  rests in the corner  $A$  and the other end is attached to cord  $BD$ . If the rod supports a 40-lb load at its midpoint  $C$ , find the reaction at  $A$  and the tension in the cord.

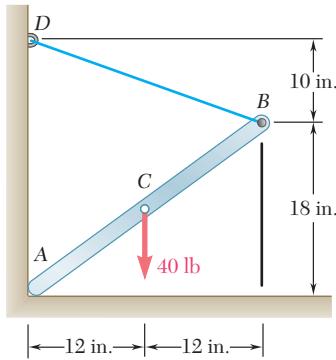


Fig. P4.71

- 4.72** Determine the reactions at  $A$  and  $D$  when  $\beta = 30^\circ$ .

- 4.73** Determine the reactions at  $A$  and  $D$  when  $\beta = 60^\circ$ .

- 4.74** A 40-lb roller, of diameter 8 in., which is to be used on a tile floor, is resting directly on the subflooring as shown. Knowing that the thickness of each tile is 0.3 in., determine the force  $P$  required to move the roller onto the tiles if the roller is (a) pushed to the left, (b) pulled to the right.

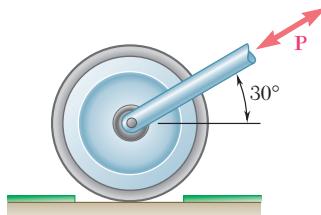


Fig. P4.74

- 4.75 and 4.76** Member  $ABC$  is supported by a pin and bracket at  $B$  and by an inextensible cord attached at  $A$  and  $C$  and passing over a frictionless pulley at  $D$ . The tension may be assumed to be the same in portions  $AD$  and  $CD$  of the cord. For the loading shown and neglecting the size of the pulley, determine the tension in the cord and the reaction at  $B$ .

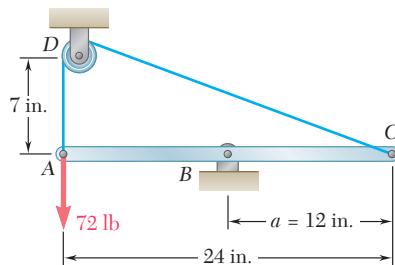


Fig. P4.75

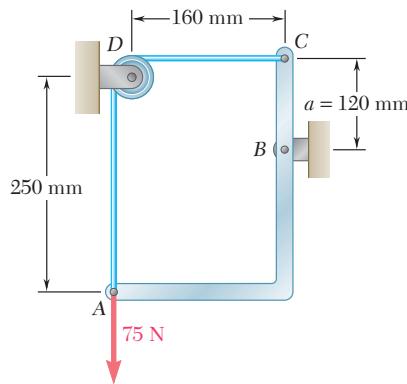


Fig. P4.76

- 4.77** Rod  $AB$  is supported by a pin and bracket at  $A$  and rests against a frictionless peg at  $C$ . Determine the reactions at  $A$  and  $C$  when a 170-N vertical force is applied at  $B$ .

- 4.78** Solve Prob. 4.77, assuming that the 170-N force applied at  $B$  is horizontal and directed to the left.

- 4.79** Using the method of Sec. 4.7, solve Prob. 4.21.

- 4.80** Using the method of Sec. 4.7, solve Prob. 4.28.

- 4.81** Knowing that  $\theta = 30^\circ$ , determine the reaction (a) at  $B$ , (b) at  $C$ .

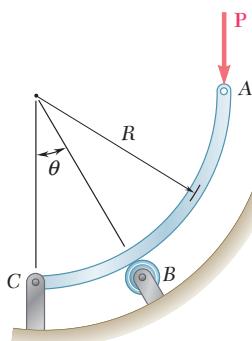


Fig. P4.81 and P4.82

- 4.82** Knowing that  $\theta = 60^\circ$ , determine the reaction (a) at  $B$ , (b) at  $C$ .

- 4.83** Rod  $AB$  is bent into the shape of an arc of circle and is lodged between two pegs  $D$  and  $E$ . It supports a load  $P$  at end  $B$ . Neglecting friction and the weight of the rod, determine the distance  $c$  corresponding to equilibrium when  $a = 20$  mm and  $R = 100$  mm.

- 4.84** A slender rod of length  $L$  is attached to collars that can slide freely along the guides shown. Knowing that the rod is in equilibrium, derive an expression for the angle  $\theta$  in terms of the angle  $\beta$ .

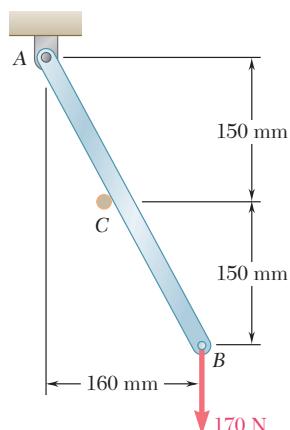


Fig. P4.77

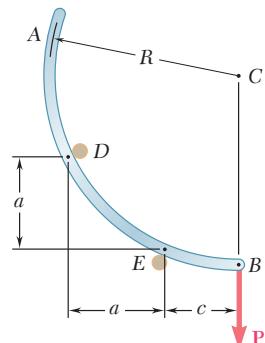


Fig. P4.83

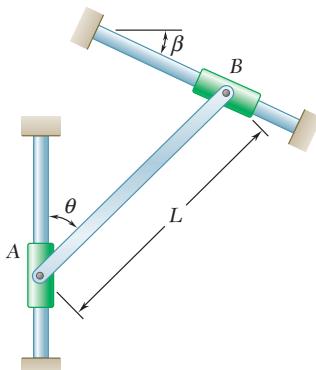
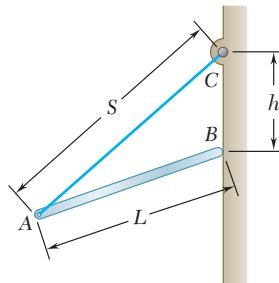
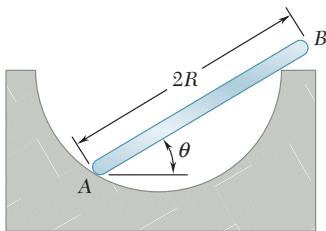


Fig. P4.84 and P4.85

- 4.85** An 8-kg slender rod of length  $L$  is attached to collars that can slide freely along the guides shown. Knowing that the rod is in equilibrium and that  $\beta = 30^\circ$ , determine (a) the angle  $\theta$  that the rod forms with the vertical, (b) the reactions at  $A$  and  $B$ .

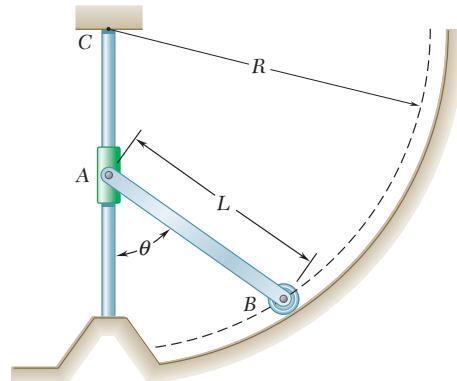
- 4.86** A slender uniform rod of length  $L$  is held in equilibrium as shown, with one end against a frictionless wall and the other end attached to a cord of length  $S$ . Derive an expression for the distance  $h$  in terms of  $L$  and  $S$ . Show that this position of equilibrium does not exist if  $S > 2L$ .

**Fig. P4.86 and P4.87****Fig. P4.88**

- 4.87** A slender uniform rod of length  $L = 20$  in. is held in equilibrium as shown, with one end against a frictionless wall and the other end attached to a cord of length  $S = 30$  in. Knowing that the weight of the rod is 10 lb, determine (a) the distance  $h$ , (b) the tension in the cord, (c) the reaction at  $B$ .

- 4.88** A uniform rod  $AB$  of length  $2R$  rests inside a hemispherical bowl of radius  $R$  as shown. Neglecting friction, determine the angle  $\theta$  corresponding to equilibrium.

- 4.89** A slender rod of length  $L$  and weight  $W$  is attached to a collar at  $A$  and is fitted with a small wheel at  $B$ . Knowing that the wheel rolls freely along a cylindrical surface of radius  $R$ , and neglecting friction, derive an equation in  $\theta$ ,  $L$ , and  $R$  that must be satisfied when the rod is in equilibrium.

**Fig. P4.89**

- 4.90** Knowing that for the rod of Prob. 4.89,  $L = 15$  in.,  $R = 20$  in., and  $W = 10$  lb, determine (a) the angle  $\theta$  corresponding to equilibrium, (b) the reactions at  $A$  and  $B$ .

## 4.8 EQUILIBRIUM OF A RIGID BODY IN THREE DIMENSIONS

We saw in Sec. 4.1 that six scalar equations are required to express the conditions for the equilibrium of a rigid body in the general three-dimensional case:

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0 \quad (4.2)$$

$$\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0 \quad (4.3)$$

These equations can be solved for no more than *six unknowns*, which generally will represent reactions at supports or connections.

In most problems the scalar equations (4.2) and (4.3) will be more conveniently obtained if we first express in vector form the conditions for the equilibrium of the rigid body considered. We write

$$\Sigma \mathbf{F} = 0 \quad \Sigma \mathbf{M}_O = \Sigma (\mathbf{r} \times \mathbf{F}) = 0 \quad (4.1)$$

and express the forces  $\mathbf{F}$  and position vectors  $\mathbf{r}$  in terms of scalar components and unit vectors. Next, we compute all vector products, either by direct calculation or by means of determinants (see Sec. 3.8). We observe that as many as three unknown reaction components may be eliminated from these computations through a judicious choice of the point  $O$ . By equating to zero the coefficients of the unit vectors in each of the two relations (4.1), we obtain the desired scalar equations.<sup>†</sup>

## 4.9 REACTIONS AT SUPPORTS AND CONNECTIONS FOR A THREE-DIMENSIONAL STRUCTURE

The reactions on a three-dimensional structure range from the single force of known direction exerted by a frictionless surface to the force-couple system exerted by a fixed support. Consequently, in problems involving the equilibrium of a three-dimensional structure, there can be between one and six unknowns associated with the reaction at each support or connection. Various types of supports and

<sup>†</sup>In some problems, it will be found convenient to eliminate the reactions at two points  $A$  and  $B$  from the solution by writing the equilibrium equation  $\Sigma M_{AB} = 0$ , which involves the determination of the moments of the forces about the axis  $AB$  joining points  $A$  and  $B$  (see Sample Prob. 4.10).



**Photo 4.6** Universal joints, easily seen on the drive shafts of rear-wheel-drive cars and trucks, allow rotational motion to be transferred between two noncollinear shafts.

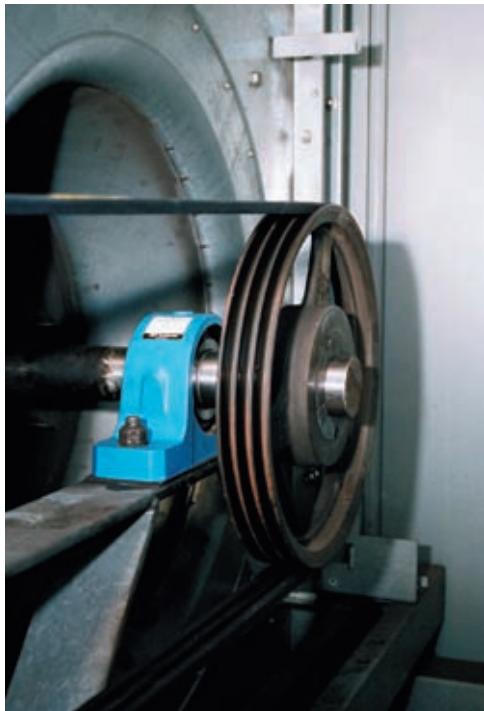
connections are shown in Fig. 4.10 with their corresponding reactions. A simple way of determining the type of reaction corresponding to a given support or connection and the number of unknowns involved is to find which of the six fundamental motions (translation in the  $x$ ,  $y$ , and  $z$  directions, rotation about the  $x$ ,  $y$ , and  $z$  axes) are allowed and which motions are prevented.

Ball supports, frictionless surfaces, and cables, for example, prevent translation in one direction only and thus exert a single force whose line of action is known; each of these supports involves one unknown, namely, the magnitude of the reaction. Rollers on rough surfaces and wheels on rails prevent translation in two directions; the corresponding reactions consist of two unknown force components. Rough surfaces in direct contact and ball-and-socket supports prevent translation in three directions; these supports involve three unknown force components.

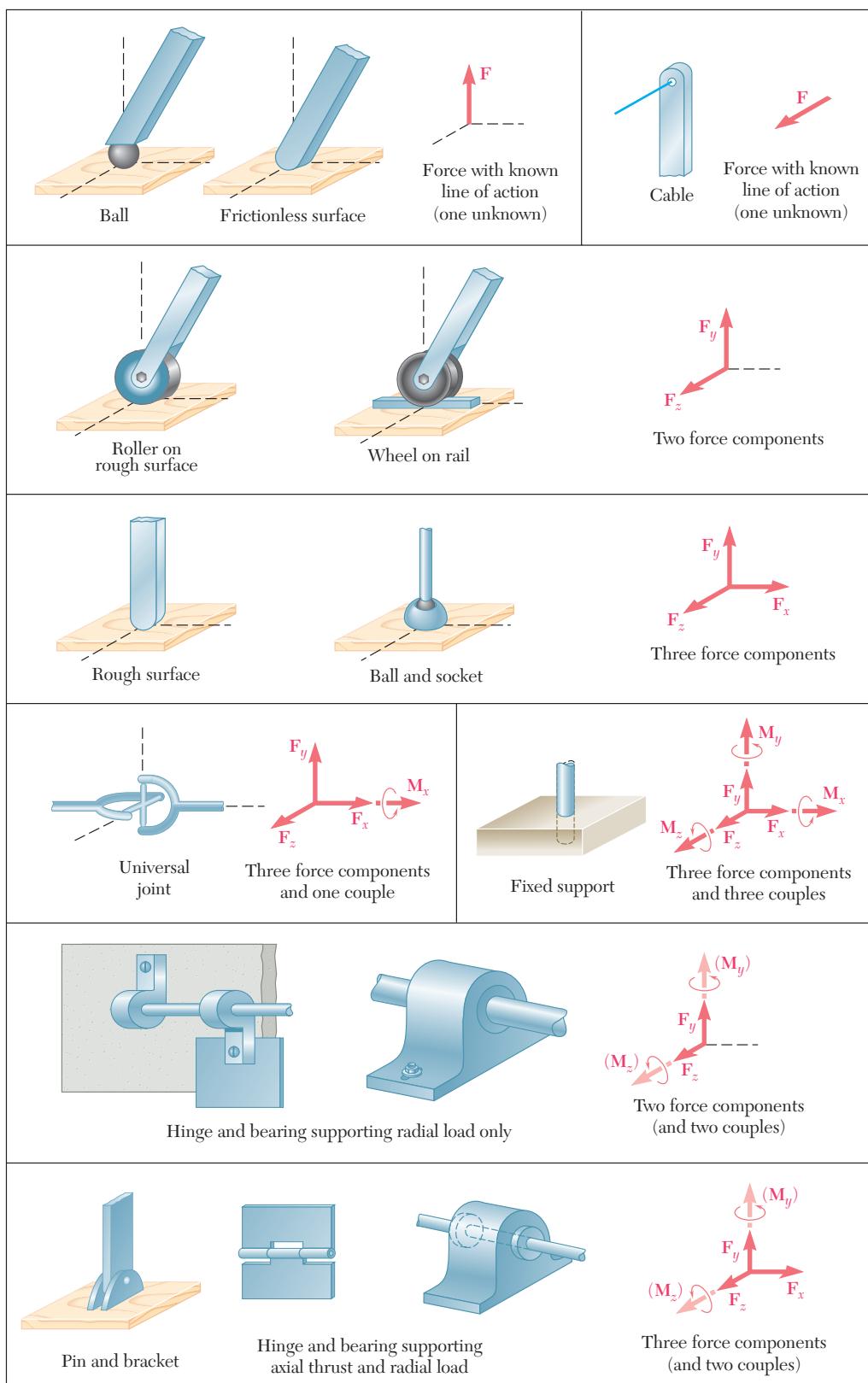
Some supports and connections can prevent rotation as well as translation; the corresponding reactions include couples as well as forces. For example, the reaction at a fixed support, which prevents any motion (rotation as well as translation), consists of three unknown forces and three unknown couples. A universal joint, which is designed to allow rotation about two axes, will exert a reaction consisting of three unknown force components and one unknown couple.

Other supports and connections are primarily intended to prevent translation; their design, however, is such that they also prevent some rotations. The corresponding reactions consist essentially of force components but *may* also include couples. One group of supports of this type includes hinges and bearings designed to support radial loads only (for example, journal bearings, roller bearings). The corresponding reactions consist of two force components but may also include two couples. Another group includes pin-and-bracket supports, hinges, and bearings designed to support an axial thrust as well as a radial load (for example, ball bearings). The corresponding reactions consist of three force components but may include two couples. However, these supports will not exert any appreciable couples under normal conditions of use. Therefore, *only* force components should be included in their analysis *unless* it is found that couples are necessary to maintain the equilibrium of the rigid body, or unless the support is known to have been specifically designed to exert a couple (see Probs. 4.119 through 4.122).

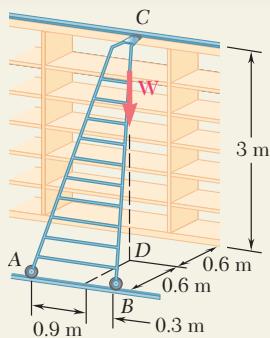
If the reactions involve more than six unknowns, there are more unknowns than equations, and some of the reactions are *statically indeterminate*. If the reactions involve fewer than six unknowns, there are more equations than unknowns, and some of the equations of equilibrium cannot be satisfied under general loading conditions; the rigid body is only *partially constrained*. Under the particular loading conditions corresponding to a given problem, however, the extra equations often reduce to trivial identities, such as  $0 = 0$ , and can be disregarded; although only partially constrained, the rigid body remains in equilibrium (see Sample Probs. 4.7 and 4.8). Even with six or more unknowns, it is possible that some equations of equilibrium will not be satisfied. This can occur when the reactions associated with the given supports either are parallel or intersect the same line; the rigid body is then *improperly constrained*.



**Photo 4.7** The pillow block bearing shown supports the shaft of a fan used in an industrial facility.



**Fig. 4.10** Reactions at supports and connections.



## SAMPLE PROBLEM 4.7

A 20-kg ladder used to reach high shelves in a storeroom is supported by two flanged wheels *A* and *B* mounted on a rail and by an unflanged wheel *C* resting against a rail fixed to the wall. An 80-kg man stands on the ladder and leans to the right. The line of action of the combined weight **W** of the man and ladder intersects the floor at point *D*. Determine the reactions at *A*, *B*, and *C*.

## SOLUTION

**Free-Body Diagram.** A free-body diagram of the ladder is drawn. The forces involved are the combined weight of the man and ladder,

$$\mathbf{W} = -mg\mathbf{j} = -(80 \text{ kg} + 20 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(981 \text{ N})\mathbf{j}$$

and five unknown reaction components, two at each flanged wheel and one at the unflanged wheel. The ladder is thus only partially constrained; it is free to roll along the rails. It is, however, in equilibrium under the given load since the equation  $\sum F_x = 0$  is satisfied.

**Equilibrium Equations.** We express that the forces acting on the ladder form a system equivalent to zero:

$$\begin{aligned} \sum \mathbf{F} = 0: \quad & A_y\mathbf{j} + A_z\mathbf{k} + B_y\mathbf{j} + B_z\mathbf{k} - (981 \text{ N})\mathbf{j} + C\mathbf{k} = 0 \\ & (A_y + B_y - 981 \text{ N})\mathbf{j} + (A_z + B_z + C)\mathbf{k} = 0 \end{aligned} \quad (1)$$

$$\sum \mathbf{M}_A = \sum (\mathbf{r} \times \mathbf{F}) = 0: \quad 1.2\mathbf{i} \times (B_y\mathbf{j} + B_z\mathbf{k}) + (0.9\mathbf{i} - 0.6\mathbf{k}) \times (-981\mathbf{j}) + (0.6\mathbf{i} + 3\mathbf{j} - 1.2\mathbf{k}) \times C\mathbf{k} = 0$$

Computing the vector products, we have†

$$\begin{aligned} 1.2B_y\mathbf{k} - 1.2B_z\mathbf{j} - 882.9\mathbf{k} - 588.6\mathbf{i} - 0.6C\mathbf{j} + 3C\mathbf{i} = 0 \\ (3C - 588.6)\mathbf{i} - (1.2B_z + 0.6C)\mathbf{j} + (1.2B_y - 882.9)\mathbf{k} = 0 \end{aligned} \quad (2)$$

Setting the coefficients of **i**, **j**, **k** equal to zero in Eq. (2), we obtain the following three scalar equations, which express that the sum of the moments about each coordinate axis must be zero:

$$\begin{array}{ll} 3C - 588.6 = 0 & C = +196.2 \text{ N} \\ 1.2B_z + 0.6C = 0 & B_z = -98.1 \text{ N} \\ 1.2B_y - 882.9 = 0 & B_y = +736 \text{ N} \end{array}$$

The reactions at *B* and *C* are therefore

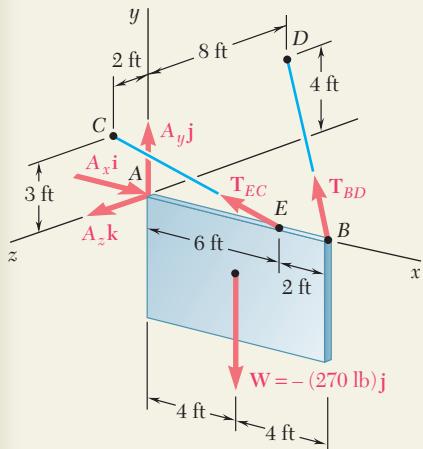
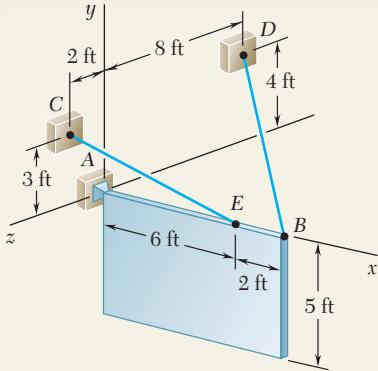
$$\mathbf{B} = +(736 \text{ N})\mathbf{j} - (98.1 \text{ N})\mathbf{k} \quad \mathbf{C} = +(196.2 \text{ N})\mathbf{k}$$

Setting the coefficients of **j** and **k** equal to zero in Eq. (1), we obtain two scalar equations expressing that the sums of the components in the **y** and **z** directions are zero. Substituting for  $B_y$ ,  $B_z$ , and  $C$  the values obtained above, we write

$$\begin{array}{lll} A_y + B_y - 981 = 0 & A_y + 736 - 981 = 0 & A_y = +245 \text{ N} \\ A_z + B_z + C = 0 & A_z - 98.1 + 196.2 = 0 & A_z = -98.1 \text{ N} \end{array}$$

We conclude that the reaction at *A* is  $\mathbf{A} = +(245 \text{ N})\mathbf{j} - (98.1 \text{ N})\mathbf{k}$

†The moments in this sample problem and in Sample Probs. 4.8 and 4.9 can also be expressed in the form of determinants (see Sample Prob. 3.10).



## SAMPLE PROBLEM 4.8

A  $5 \times 8$ -ft sign of uniform density weighs 270 lb and is supported by a ball-and-socket joint at A and by two cables. Determine the tension in each cable and the reaction at A.

### SOLUTION

**Free-Body Diagram.** A free-body diagram of the sign is drawn. The forces acting on the free body are the weight  $\mathbf{W} = -(270 \text{ lb})\mathbf{j}$  and the reactions at A, B, and E. The reaction at A is a force of unknown direction and is represented by three unknown components. Since the directions of the forces exerted by the cables are known, these forces involve only one unknown each, namely, the magnitudes  $T_{BD}$  and  $T_{EC}$ . Since there are only five unknowns, the sign is partially constrained. It can rotate freely about the x axis; it is, however, in equilibrium under the given loading, since the equation  $\sum M_x = 0$  is satisfied.

The components of the forces  $\mathbf{T}_{BD}$  and  $\mathbf{T}_{EC}$  can be expressed in terms of the unknown magnitudes  $T_{BD}$  and  $T_{EC}$  by writing

$$\begin{aligned}\overrightarrow{BD} &= -(8 \text{ ft})\mathbf{i} + (4 \text{ ft})\mathbf{j} - (8 \text{ ft})\mathbf{k} & BD = 12 \text{ ft} \\ \overrightarrow{EC} &= -(6 \text{ ft})\mathbf{i} + (3 \text{ ft})\mathbf{j} + (2 \text{ ft})\mathbf{k} & EC = 7 \text{ ft} \\ \mathbf{T}_{BD} &= T_{BD} \left( \frac{\overrightarrow{BD}}{BD} \right) = T_{BD} \left( -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \right) \\ \mathbf{T}_{EC} &= T_{EC} \left( \frac{\overrightarrow{EC}}{EC} \right) = T_{EC} \left( -\frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} - \frac{2}{7}\mathbf{k} \right)\end{aligned}$$

**Equilibrium Equations.** We express that the forces acting on the sign form a system equivalent to zero:

$$\begin{aligned}\Sigma \mathbf{F} = 0: \quad A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} + \mathbf{T}_{BD} + \mathbf{T}_{EC} - (270 \text{ lb})\mathbf{j} &= 0 \\ (A_x - \frac{2}{3}T_{BD} - \frac{6}{7}T_{EC})\mathbf{i} + (A_y + \frac{1}{3}T_{BD} + \frac{3}{7}T_{EC} - 270 \text{ lb})\mathbf{j} &+ (A_z - \frac{2}{3}T_{BD} + \frac{2}{7}T_{EC})\mathbf{k} = 0 \quad (1)\end{aligned}$$

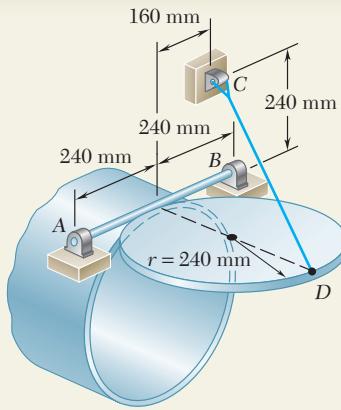
$$\begin{aligned}\Sigma \mathbf{M}_A = \Sigma (\mathbf{r} \times \mathbf{F}) = 0: \quad & (8 \text{ ft})\mathbf{i} \times T_{BD} \left( -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \right) + (6 \text{ ft})\mathbf{i} \times T_{EC} \left( -\frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k} \right) \\ & + (4 \text{ ft})\mathbf{i} \times (-270 \text{ lb})\mathbf{j} = 0 \\ (2.667T_{BD} + 2.571T_{EC} - 1080 \text{ lb})\mathbf{k} + (5.333T_{BD} - 1.714T_{EC})\mathbf{j} &= 0 \quad (2)\end{aligned}$$

Setting the coefficients of  $\mathbf{j}$  and  $\mathbf{k}$  equal to zero in Eq. (2), we obtain two scalar equations which can be solved for  $T_{BD}$  and  $T_{EC}$ :

$$T_{BD} = 101.3 \text{ lb} \quad T_{EC} = 315 \text{ lb} \quad \blacktriangleleft$$

Setting the coefficients of  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  equal to zero in Eq. (1), we obtain three more equations, which yield the components of  $\mathbf{A}$ . We have

$$\mathbf{A} = +(338 \text{ lb})\mathbf{i} + (101.2 \text{ lb})\mathbf{j} - (22.5 \text{ lb})\mathbf{k} \quad \blacktriangleleft$$



## SAMPLE PROBLEM 4.9

A uniform pipe cover of radius  $r = 240$  mm and mass 30 kg is held in a horizontal position by the cable  $CD$ . Assuming that the bearing at  $B$  does not exert any axial thrust, determine the tension in the cable and the reactions at  $A$  and  $B$ .

### SOLUTION

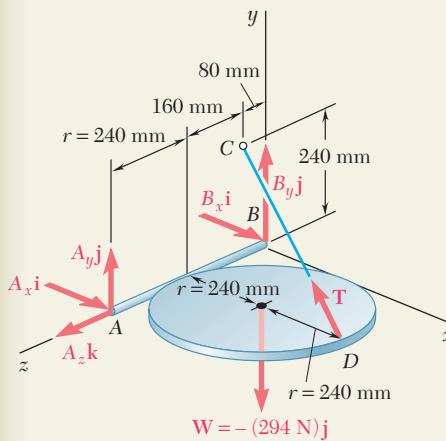
**Free-Body Diagram.** A free-body diagram is drawn with the coordinate axes shown. The forces acting on the free body are the weight of the cover,

$$\mathbf{W} = -mg\mathbf{j} = -(30 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(294 \text{ N})\mathbf{j}$$

and reactions involving six unknowns, namely, the magnitude of the force  $\mathbf{T}$  exerted by the cable, three force components at hinge  $A$ , and two at hinge  $B$ . The components of  $\mathbf{T}$  are expressed in terms of the unknown magnitude  $T$  by resolving the vector  $\overrightarrow{DC}$  into rectangular components and writing

$$\overrightarrow{DC} = -(480 \text{ mm})\mathbf{i} + (240 \text{ mm})\mathbf{j} - (160 \text{ mm})\mathbf{k} \quad DC = 560 \text{ mm}$$

$$\mathbf{T} = T \frac{\overrightarrow{DC}}{DC} = -\frac{6}{7}T\mathbf{i} + \frac{3}{7}T\mathbf{j} - \frac{2}{7}T\mathbf{k}$$



**Equilibrium Equations.** We express that the forces acting on the pipe cover form a system equivalent to zero:

$$\begin{aligned} \Sigma \mathbf{F} = 0: \quad & A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} + B_x\mathbf{i} + B_y\mathbf{j} + \mathbf{T} - (294 \text{ N})\mathbf{j} = 0 \\ & (A_x + B_x - \frac{6}{7}T)\mathbf{i} + (A_y + B_y + \frac{3}{7}T - 294 \text{ N})\mathbf{j} + (A_z - \frac{2}{7}T)\mathbf{k} = 0 \quad (1) \end{aligned}$$

$$\begin{aligned} \Sigma \mathbf{M}_B = \Sigma (\mathbf{r} \times \mathbf{F}) = 0: \quad & 2r\mathbf{k} \times (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \\ & + (2r\mathbf{i} + r\mathbf{k}) \times (-\frac{6}{7}T\mathbf{i} + \frac{3}{7}T\mathbf{j} - \frac{2}{7}T\mathbf{k}) \\ & + (\mathbf{r}_i + r\mathbf{k}) \times (-294 \text{ N})\mathbf{j} = 0 \\ & (-2A_y - \frac{3}{7}T + 294 \text{ N})r\mathbf{i} + (2A_x - \frac{2}{7}T)r\mathbf{j} + (\frac{6}{7}T - 294 \text{ N})r\mathbf{k} = 0 \quad (2) \end{aligned}$$

Setting the coefficients of the unit vectors equal to zero in Eq. (2), we write three scalar equations, which yield

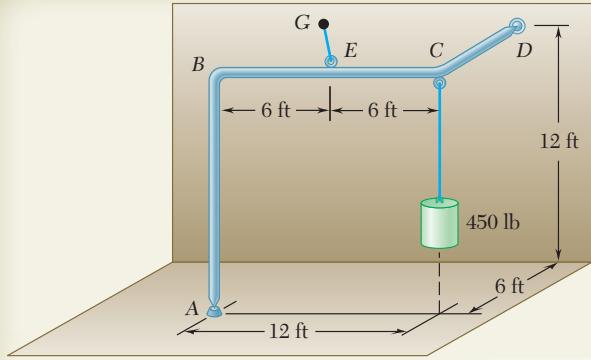
$$A_x = +49.0 \text{ N} \quad A_y = +73.5 \text{ N} \quad T = 343 \text{ N} \quad \blacktriangleleft$$

Setting the coefficients of the unit vectors equal to zero in Eq. (1), we obtain three more scalar equations. After substituting the values of  $T$ ,  $A_x$ , and  $A_y$  into these equations, we obtain

$$A_z = +98.0 \text{ N} \quad B_x = +245 \text{ N} \quad B_y = +73.5 \text{ N}$$

The reactions at  $A$  and  $B$  are therefore

$$\begin{aligned} \mathbf{A} &= +(49.0 \text{ N})\mathbf{i} + (73.5 \text{ N})\mathbf{j} + (98.0 \text{ N})\mathbf{k} \\ \mathbf{B} &= +(245 \text{ N})\mathbf{i} + (73.5 \text{ N})\mathbf{j} \quad \blacktriangleright \end{aligned}$$

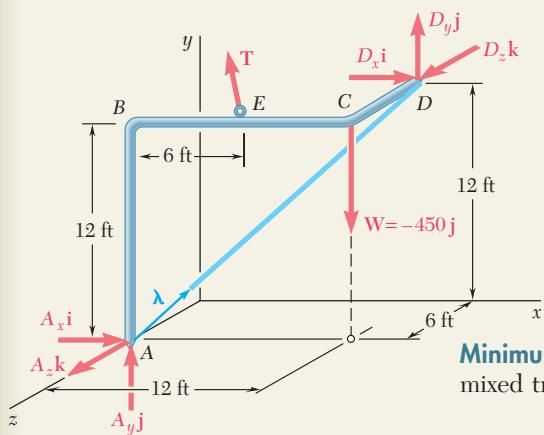


## SAMPLE PROBLEM 4.10

A 450-lb load hangs from the corner *C* of a rigid piece of pipe *ABCD* which has been bent as shown. The pipe is supported by the ball-and-socket joints *A* and *D*, which are fastened, respectively, to the floor and to a vertical wall, and by a cable attached at the midpoint *E* of the portion *BC* of the pipe and at a point *G* on the wall. Determine (a) where *G* should be located if the tension in the cable is to be minimum, (b) the corresponding minimum value of the tension.

## SOLUTION

**Free-Body Diagram.** The free-body diagram of the pipe includes the load  $\mathbf{W} = (-450 \text{ lb})\mathbf{j}$ , the reactions at *A* and *D*, and the force  $\mathbf{T}$  exerted by the cable. To eliminate the reactions at *A* and *D* from the computations, we express that the sum of the moments of the forces about *AD* is zero. Denoting by  $\lambda$  the unit vector along *AD*, we write



$$\sum M_{AD} = 0: \lambda \cdot (\overrightarrow{AE} \times \mathbf{T}) + \lambda \cdot (\overrightarrow{AC} \times \mathbf{W}) = 0 \quad (1)$$

The second term in Eq. (1) can be computed as follows:

$$\begin{aligned} \overrightarrow{AC} \times \mathbf{W} &= (12\mathbf{i} + 12\mathbf{j}) \times (-450\mathbf{j}) = -5400\mathbf{k} \\ \lambda &= \frac{\overrightarrow{AD}}{AD} = \frac{12\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}}{\sqrt{18}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \end{aligned}$$

$$\lambda \cdot (\overrightarrow{AC} \times \mathbf{W}) = (\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}) \cdot (-5400\mathbf{k}) = +1800$$

Substituting the value obtained into Eq. (1), we write

$$\lambda \cdot (\overrightarrow{AE} \times \mathbf{T}) = -1800 \text{ lb} \cdot \text{ft} \quad (2)$$

**Minimum Value of Tension.** Recalling the commutative property for mixed triple products, we rewrite Eq. (2) in the form

$$\mathbf{T} \cdot (\lambda \times \overrightarrow{AE}) = -1800 \text{ lb} \cdot \text{ft} \quad (3)$$

which shows that the projection of  $\mathbf{T}$  on the vector  $\lambda \times \overrightarrow{AE}$  is a constant. It follows that  $\mathbf{T}$  is minimum when parallel to the vector

$$\lambda \times \overrightarrow{AE} = (\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}) \times (6\mathbf{i} + 12\mathbf{j}) = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

Since the corresponding unit vector is  $\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ , we write

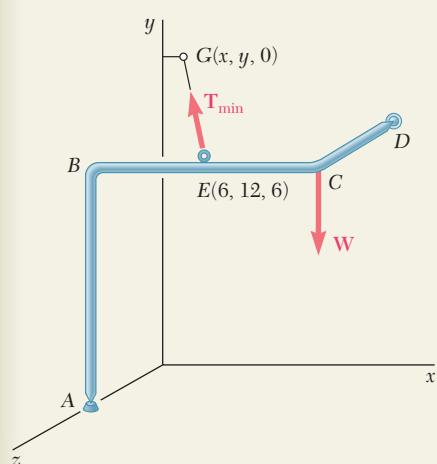
$$\mathbf{T}_{\min} = T(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}) \quad (4)$$

Substituting for  $\mathbf{T}$  and  $\lambda \times \overrightarrow{AE}$  in Eq. (3) and computing the dot products, we obtain  $6T = -1800$  and, thus,  $T = -300$ . Carrying this value into (4), we obtain

$$\mathbf{T}_{\min} = -200\mathbf{i} + 100\mathbf{j} - 200\mathbf{k} \quad \mathbf{T}_{\min} = 300 \text{ lb} \quad \blacktriangleleft$$

**Location of *G*.** Since the vector  $\overrightarrow{EG}$  and the force  $\mathbf{T}_{\min}$  have the same direction, their components must be proportional. Denoting the coordinates of *G* by *x*, *y*, 0, we write

$$\frac{x - 6}{-200} = \frac{y - 12}{+100} = \frac{0 - 6}{-200} \quad x = 0 \quad y = 15 \text{ ft} \quad \blacktriangleleft$$



# SOLVING PROBLEMS ON YOUR OWN

The equilibrium of a *three-dimensional body* was considered in the sections you just completed. It is again most important that you draw a complete *free-body diagram* as the first step of your solution.

**1. As you draw the free-body diagram, pay particular attention to the reactions at the supports.** The number of unknowns at a support can range from one to six (Fig. 4.10). To decide whether an unknown reaction or reaction component exists at a support, ask yourself whether the support prevents motion of the body in a certain direction or about a certain axis.

a. **If motion is prevented in a certain direction,** include in your free-body diagram an unknown *reaction* or *reaction component* that acts in the *same direction*.

b. **If a support prevents rotation about a certain axis,** include in your free-body diagram a *couple* of unknown magnitude that acts about the *same axis*.

**2. The external forces acting on a three-dimensional body form a system equivalent to zero.** Writing  $\Sigma F = 0$  and  $\Sigma M_A = 0$  about an appropriate point A, and setting the coefficients of  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  in both equations equal to zero will provide you with six scalar equations. In general, these equations will contain six unknowns and may be solved for these unknowns.

**3. After completing your free-body diagram, you may want to seek equations involving as few unknowns as possible.** The following strategies may help you.

a. By summing moments about a ball-and-socket support or a hinge, you will obtain equations from which three unknown reaction components have been eliminated [Sample Probs. 4.8 and 4.9].

b. If you can draw an axis through the points of application of all but one of the unknown reactions, summing moments about that axis will yield an equation in a single unknown [Sample Prob. 4.10].

**4. After drawing your free-body diagram, you may find that one of the following situations exists.**

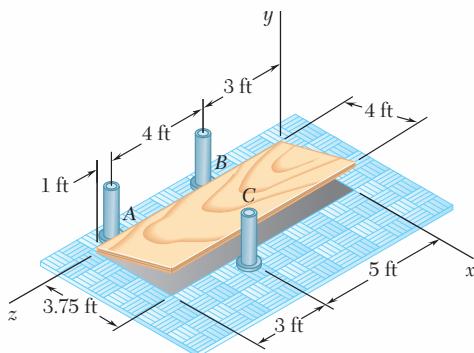
a. **The reactions involve fewer than six unknowns;** the body is said to be *partially constrained* and motion of the body is possible. However, you may be able to determine the reactions for a given loading condition [Sample Prob. 4.7].

b. **The reactions involve more than six unknowns;** the reactions are said to be *statically indeterminate*. Although you may be able to calculate one or two reactions, you cannot determine all of the reactions [Sample Prob. 4.10].

c. **The reactions are parallel or intersect the same line;** the body is said to be *improperly constrained*, and motion can occur under a general loading condition.

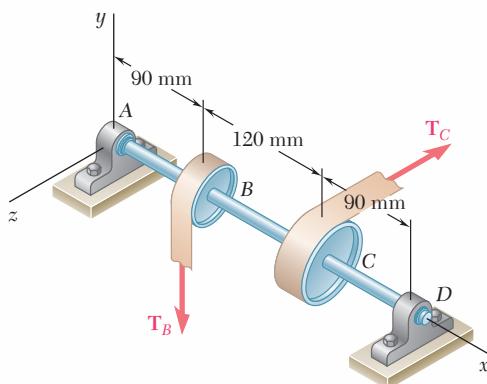
# PROBLEMS

- 4.91** A  $4 \times 8$ -ft sheet of plywood weighing 34 lb has been temporarily placed among three pipe supports. The lower edge of the sheet rests on small collars at *A* and *B* and its upper edge leans against pipe *C*. Neglecting friction at all surfaces, determine the reactions at *A*, *B*, and *C*.



**Fig. P4.91**

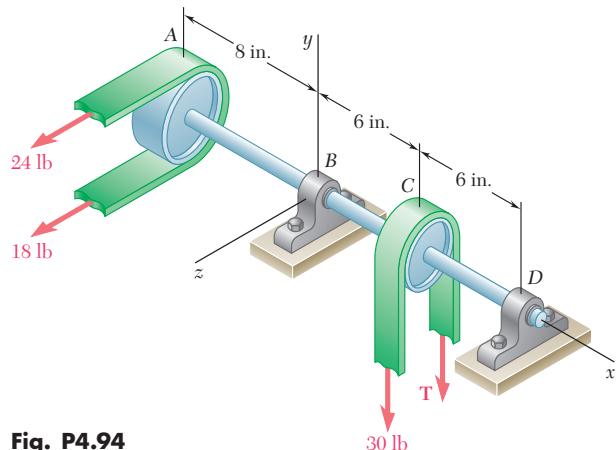
- 4.92** Two tape spools are attached to an axle supported by bearings at *A* and *D*. The radius of spool *B* is 30 mm and the radius of spool *C* is 40 mm. Knowing that  $T_B = 80$  N and that the system rotates at a constant rate, determine the reactions at *A* and *D*. Assume that the bearing at *A* does not exert any axial thrust and neglect the weights of the spools and axle.



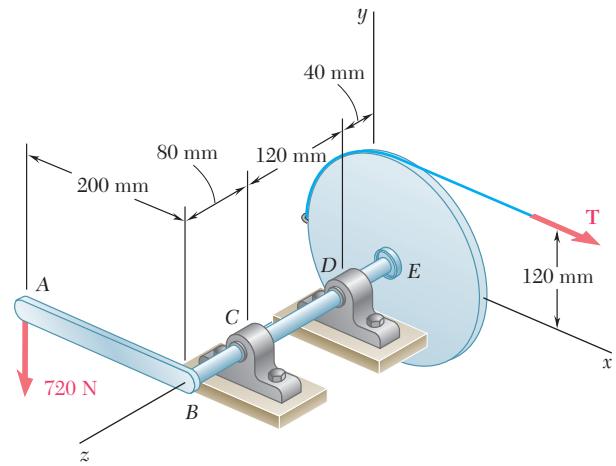
**Fig. P4.92**

- 4.93** Solve Prob. 4.92, assuming that the spool *C* is replaced by a spool of radius 50 mm.

- 4.94** Two transmission belts pass over sheaves welded to an axle supported by bearings at *B* and *D*. The sheave at *A* has a radius of 2.5 in., and the sheave at *C* has a radius of 2 in. Knowing that the system rotates at a constant rate, determine (a) the tension *T*, (b) the reactions at *B* and *D*. Assume that the bearing at *D* does not exert any axial thrust and neglect the weights of the sheaves and axle.

**Fig. P4.94**

- 4.95** A 200-mm lever and a 240-mm-diameter pulley are welded to the axle *BE* that is supported by bearings at *C* and *D*. If a 720-N vertical load is applied at *A* when the lever is horizontal, determine (a) the tension in the cord, (b) the reactions at *C* and *D*. Assume that the bearing at *D* does not exert any axial thrust.

**Fig. P4.95**

- 4.96** Solve Prob. 4.95, assuming that the axle has been rotated clockwise in its bearings by  $30^\circ$  and that the 720-N load remains vertical.

- 4.97** An opening in a floor is covered by a  $1 \times 1.2\text{-m}$  sheet of plywood of mass 18 kg. The sheet is hinged at *A* and *B* and is maintained in a position slightly above the floor by a small block *C*. Determine the vertical component of the reaction (a) at *A*, (b) at *B*, (c) at *C*.

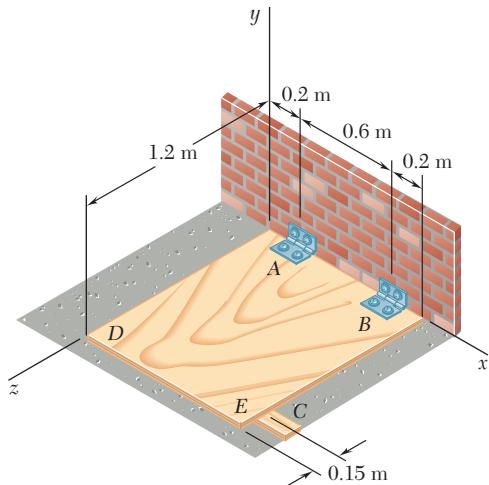


Fig. P4.97

- 4.98** Solve Prob. 4.97, assuming that the small block *C* is moved and placed under edge *DE* at a point 0.15 m from corner *E*.

- 4.99** The rectangular plate shown weighs 80 lb and is supported by three vertical wires. Determine the tension in each wire.

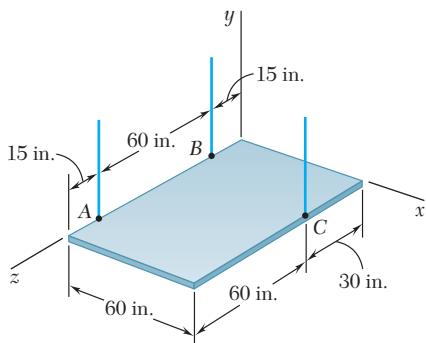
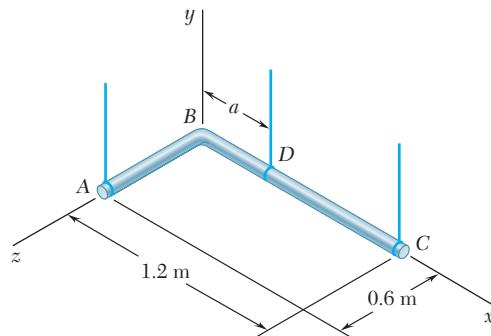


Fig. P4.99 and P4.100

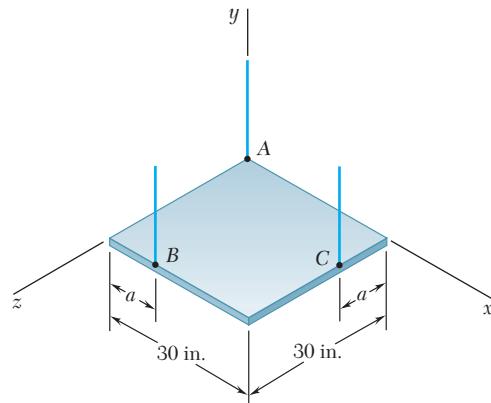
- 4.100** The rectangular plate shown weighs 80 lb and is supported by three vertical wires. Determine the weight and location of the lightest block that should be placed on the plate if the tensions in the three wires are to be equal.

- 4.101** Two steel pipes  $AB$  and  $BC$ , each having a mass per unit length of  $8 \text{ kg/m}$ , are welded together at  $B$  and supported by three wires. Knowing that  $a = 0.4 \text{ m}$ , determine the tension in each wire.

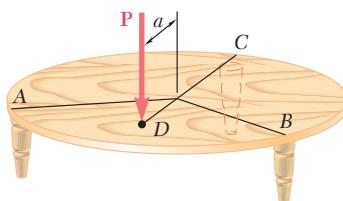
**Fig. P4.101**

- 4.102** For the pipe assembly of Prob. 4.101, determine (a) the largest permissible value of  $a$  if the assembly is not to tip, (b) the corresponding tension in each wire.

- 4.103** The 24-lb square plate shown is supported by three vertical wires. Determine (a) the tension in each wire when  $a = 10 \text{ in.}$ , (b) the value of  $a$  for which the tension in each wire is 8 lb.

**Fig. P4.103**

- 4.104** The table shown weighs 30 lb and has a diameter of 4 ft. It is supported by three legs equally spaced around the edge. A vertical load  $\mathbf{P}$  of magnitude 100 lb is applied to the top of the table at  $D$ . Determine the maximum value of  $a$  if the table is not to tip over. Show, on a sketch, the area of the table over which  $\mathbf{P}$  can act without tipping the table.

**Fig. P4.104**

- 4.105** A 10-ft boom is acted upon by the 840-lb force shown. Determine the tension in each cable and the reaction at the ball-and-socket joint at *A*.

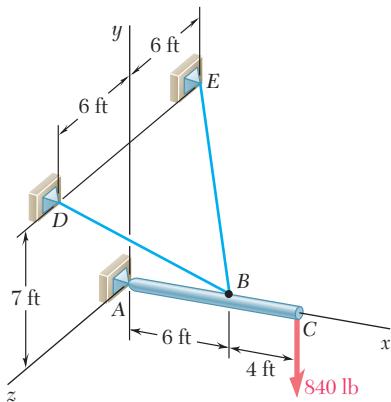


Fig. P4.105

- 4.106** A 2.4-m boom is held by a ball-and-socket joint at *C* and by two cables *AD* and *AE*. Determine the tension in each cable and the reaction at *C*.

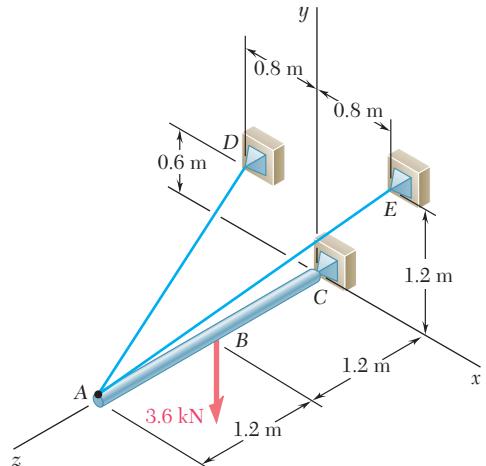


Fig. P4.106

- 4.107** Solve Prob. 4.106, assuming that the 3.6-kN load is applied at point *A*.

- 4.108** A 600-lb crate hangs from a cable that passes over a pulley *B* and is attached to a support at *H*. The 200-lb boom *AB* is supported by a ball-and-socket joint at *A* and by two cables *DE* and *DF*. The center of gravity of the boom is located at *G*. Determine (a) the tension in cables *DE* and *DF*, (b) the reaction at *A*.

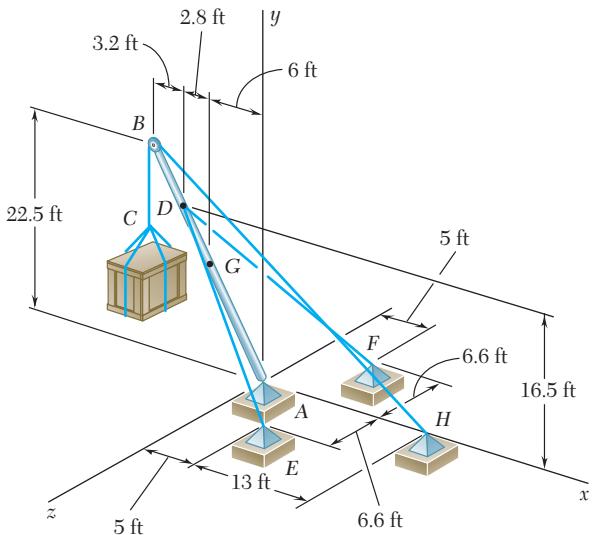


Fig. P4.108

- 4.109** A 3-m pole is supported by a ball-and-socket joint at *A* and by the cables *CD* and *CE*. Knowing that the 5-kN force acts vertically downward ( $\phi = 0$ ), determine (a) the tension in cables *CD* and *CE*, (b) the reaction at *A*.

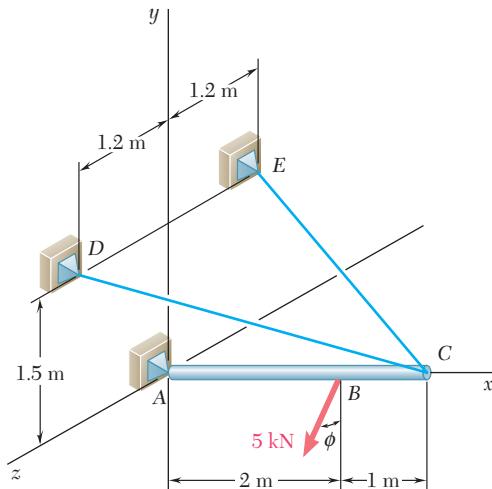


Fig. P4.109 and P4.110

- 4.110** A 3-m pole is supported by a ball-and-socket joint at *A* and by the cables *CD* and *CE*. Knowing that the line of action of the 5-kN force forms an angle  $\phi = 30^\circ$  with the vertical *xy* plane, determine (a) the tension in cables *CD* and *CE*, (b) the reaction at *A*.

- 4.111** A 48-in. boom is held by a ball-and-socket joint at *C* and by two cables *BF* and *DAE*; cable *DAE* passes around a frictionless pulley at *A*. For the loading shown, determine the tension in each cable and the reaction at *C*.

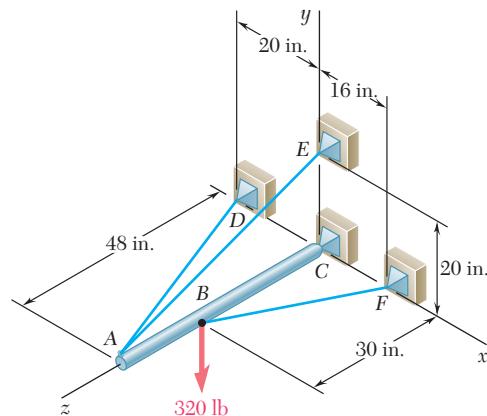
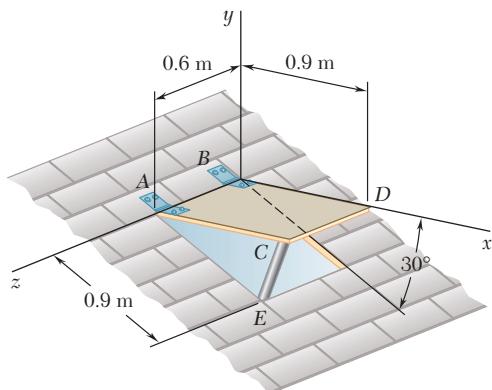


Fig. P4.111

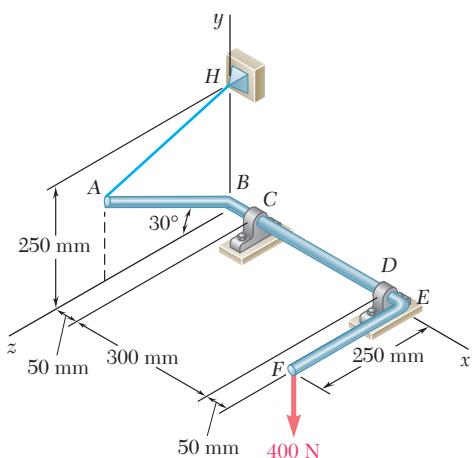
**4.112** Solve Prob. 4.111, assuming that the 320-lb load is applied at A.

**4.113** A 20-kg cover for a roof opening is hinged at corners A and B. The roof forms an angle of  $30^\circ$  with the horizontal, and the cover is maintained in a horizontal position by the brace CE. Determine (a) the magnitude of the force exerted by the brace, (b) the reactions at the hinges. Assume that the hinge at A does not exert any axial thrust.



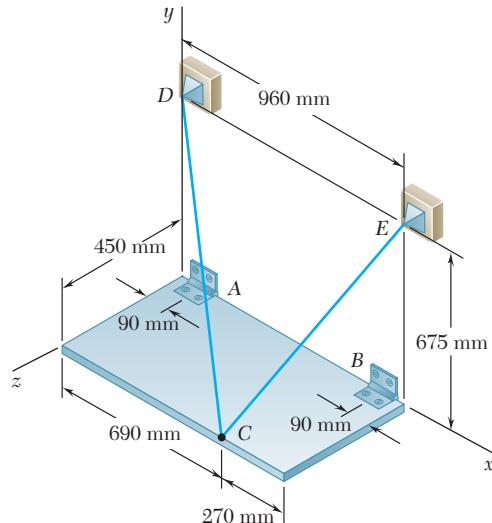
**Fig. P4.113**

**4.114** The bent rod ABEF is supported by bearings at C and D and by wire AH. Knowing that portion AB of the rod is 250 mm long, determine (a) the tension in wire AH, (b) the reactions at C and D. Assume that the bearing at D does not exert any axial thrust.



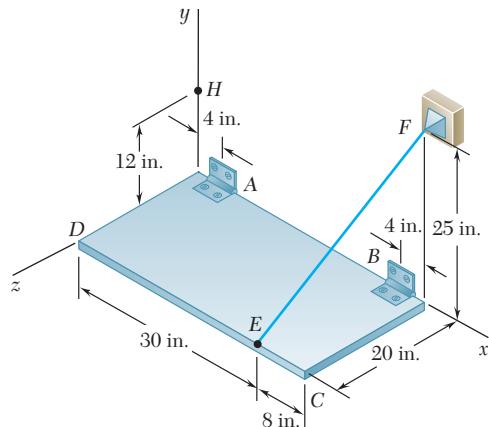
**Fig. P4.114**

- 4.115** A 100-kg uniform rectangular plate is supported in the position shown by hinges *A* and *B* and by cable *DCE* that passes over a frictionless hook at *C*. Assuming that the tension is the same in both parts of the cable, determine (a) the tension in the cable, (b) the reactions at *A* and *B*. Assume that the hinge at *B* does not exert any axial thrust.

**Fig. P4.115**

- 4.116** Solve Prob. 4.115, assuming that cable *DCE* is replaced by a cable attached to point *E* and hook *C*.

- 4.117** The rectangular plate shown weighs 75 lb and is held in the position shown by hinges at *A* and *B* and by cable *EF*. Assuming that the hinge at *B* does not exert any axial thrust, determine (a) the tension in the cable, (b) the reactions at *A* and *B*.

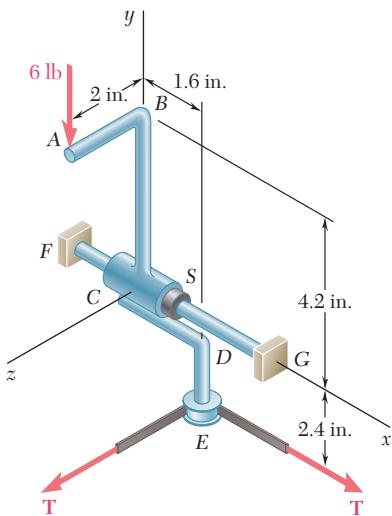
**Fig. P4.117**

- 4.118** Solve Prob. 4.117, assuming that cable *EF* is replaced by a cable attached at points *E* and *H*.

- 4.119** Solve Prob. 4.114, assuming that the bearing at  $D$  is removed and that the bearing at  $C$  can exert couples about axes parallel to the  $y$  and  $z$  axes.

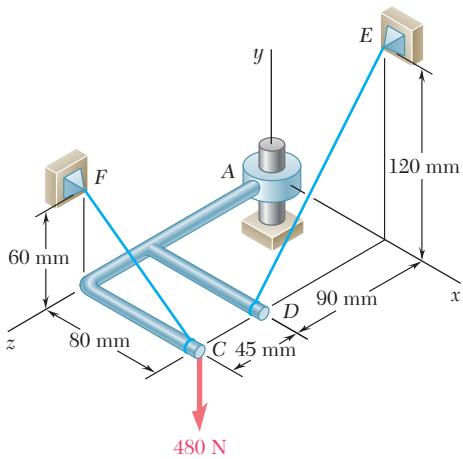
- 4.120** Solve Prob. 4.117, assuming that the hinge at  $B$  is removed and that the hinge at  $A$  can exert couples about axes parallel to the  $y$  and  $z$  axes.

- 4.121** The assembly shown is used to control the tension  $T$  in a tape that passes around a frictionless spool at  $E$ . Collar  $C$  is welded to rods  $ABC$  and  $CDE$ . It can rotate about shaft  $FG$  but its motion along the shaft is prevented by a washer  $S$ . For the loading shown, determine (a) the tension  $T$  in the tape, (b) the reaction at  $C$ .



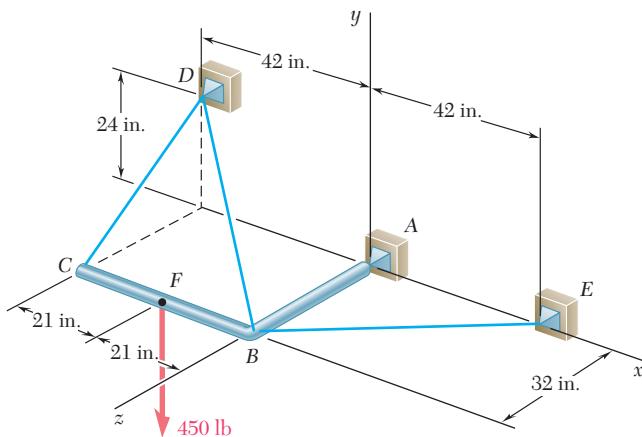
**Fig. P4.121**

- 4.122** The assembly shown is welded to collar A that fits on the vertical pin shown. The pin can exert couples about the  $x$  and  $z$  axes but does not prevent motion about or along the  $y$  axis. For the loading shown, determine the tension in each cable and the reaction at A.



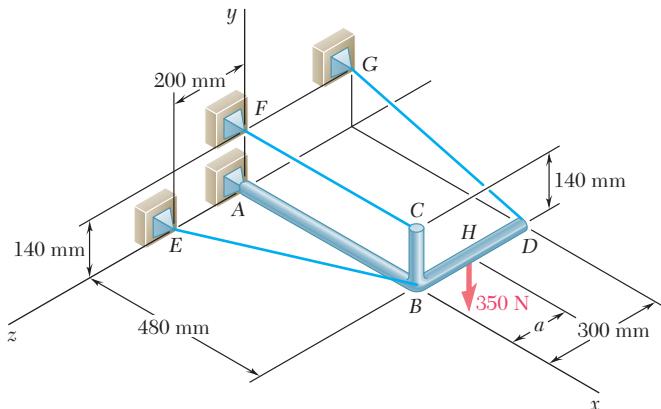
**Fig. P4.122**

- 4.123** The rigid L-shaped member  $ABC$  is supported by a ball-and-socket joint at  $A$  and by three cables. If a 450-lb load is applied at  $F$ , determine the tension in each cable.

**Fig. P4.123**

- 4.124** Solve Prob. 4.123, assuming that the 450-lb load is applied at  $C$ .

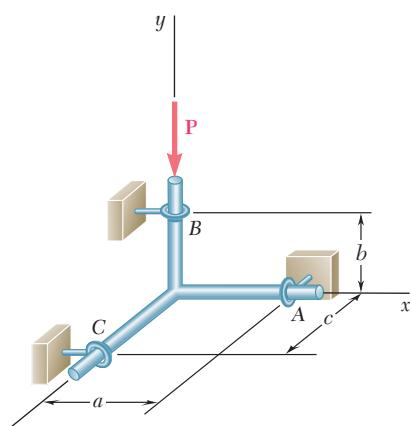
- 4.125** Frame  $ABCD$  is supported by a ball-and-socket joint at  $A$  and by three cables. For  $a = 150$  mm, determine the tension in each cable and the reaction at  $A$ .

**Fig. P4.125 and P4.126**

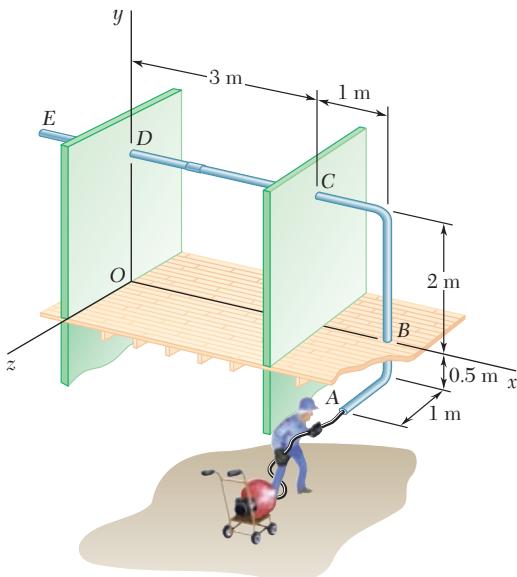
- 4.126** Frame  $ABCD$  is supported by a ball-and-socket joint at  $A$  and by three cables. Knowing that the 350-N load is applied at  $D$  ( $a = 300$  mm), determine the tension in each cable and the reaction at  $A$ .

- 4.127** Three rods are welded together to form a “corner” that is supported by three eyebolts. Neglecting friction, determine the reactions at  $A$ ,  $B$ , and  $C$  when  $P = 240$  lb,  $a = 12$  in.,  $b = 8$  in., and  $c = 10$  in.

- 4.128** Solve Prob. 4.127, assuming that the force  $\mathbf{P}$  is removed and is replaced by a couple  $\mathbf{M} = +(600 \text{ lb} \cdot \text{in.})\mathbf{j}$  acting at  $B$ .

**Fig. P4.127**

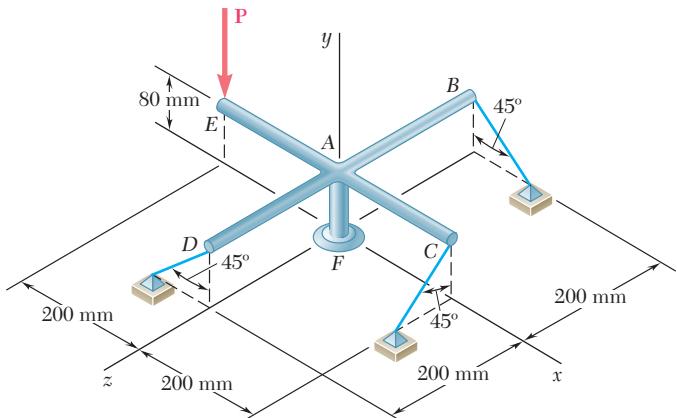
- 4.129** In order to clean the clogged drainpipe  $AE$ , a plumber has disconnected both ends of the pipe and inserted a power snake through the opening at  $A$ . The cutting head of the snake is connected by a heavy cable to an electric motor that rotates at a constant speed as the plumber forces the cable into the pipe. The forces exerted by the plumber and the motor on the end of the cable can be represented by the wrench  $\mathbf{F} = -(48 \text{ N})\mathbf{k}$ ,  $\mathbf{M} = -(90 \text{ N} \cdot \text{m})\mathbf{k}$ . Determine the additional reactions at  $B$ ,  $C$ , and  $D$  caused by the cleaning operation. Assume that the reaction at each support consists of two force components perpendicular to the pipe.



**Fig. P4.129**

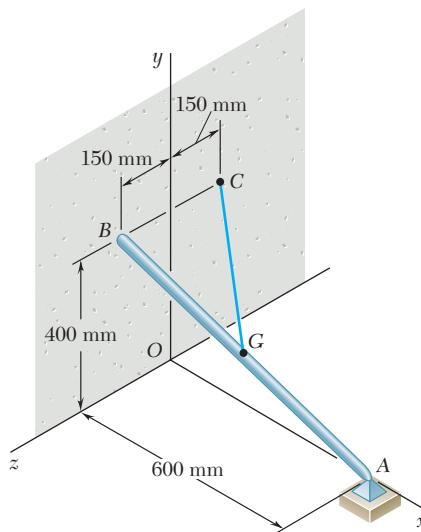
- 4.130** Solve Prob. 4.129, assuming that the plumber exerts a force  $\mathbf{F} = -(48 \text{ N})\mathbf{k}$  and that the motor is turned off ( $\mathbf{M} = 0$ ).

- 4.131** The assembly shown consists of an 80-mm rod  $AF$  that is welded to a cross consisting of four 200-mm arms. The assembly is supported by a ball-and-socket joint at  $F$  and by three short links, each of which forms an angle of  $45^\circ$  with the vertical. For the loading shown, determine (a) the tension in each link, (b) the reaction at  $F$ .

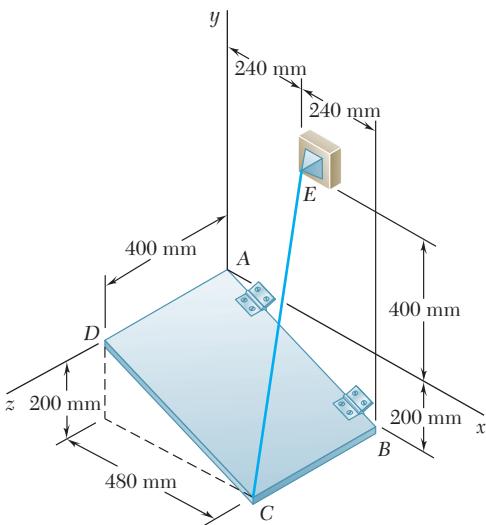
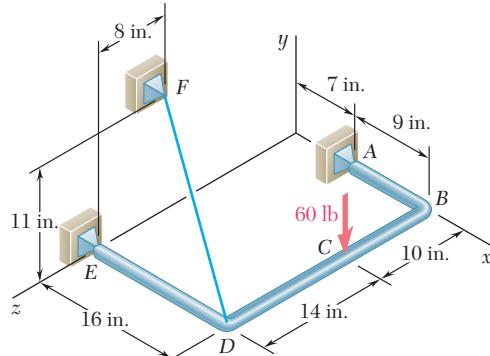


**Fig. P4.131**

- 4.132** The uniform 10-kg rod  $AB$  is supported by a ball-and-socket joint at  $A$  and by the cord  $CG$  that is attached to the midpoint  $G$  of the rod. Knowing that the rod leans against a frictionless vertical wall at  $B$ , determine (a) the tension in the cord, (b) the reactions at  $A$  and  $B$ .

**Fig. P4.132**

- 4.133** The bent rod  $ABDE$  is supported by ball-and-socket joints at  $A$  and  $E$  and by the cable  $DF$ . If a 60-lb load is applied at  $C$  as shown, determine the tension in the cable.

**Fig. P4.135****Fig. P4.133**

- 4.134** Solve Prob. 4.133, assuming that cable  $DF$  is replaced by a cable connecting  $B$  and  $F$ .

- 4.135** The 50-kg plate  $ABCD$  is supported by hinges along edge  $AB$  and by wire  $CE$ . Knowing that the plate is uniform, determine the tension in the wire.

- 4.136** Solve Prob. 4.135, assuming that wire  $CE$  is replaced by a wire connecting  $E$  and  $D$ .

- 4.137** Two rectangular plates are welded together to form the assembly shown. The assembly is supported by ball-and-socket joints at *B* and *D* and by a ball on a horizontal surface at *C*. For the loading shown, determine the reaction at *C*.

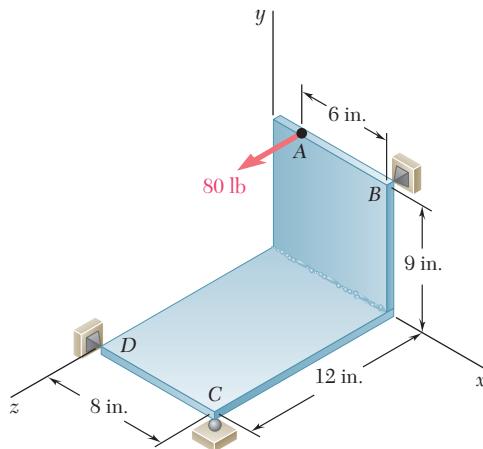


Fig. P4.137

- 4.138** Two  $2 \times 4$ -ft plywood panels, each of weight 12 lb, are nailed together as shown. The panels are supported by ball-and-socket joints at *A* and *F* and by the wire *BH*. Determine (a) the location of *H* in the *xy* plane if the tension in the wire is to be minimum, (b) the corresponding minimum tension.

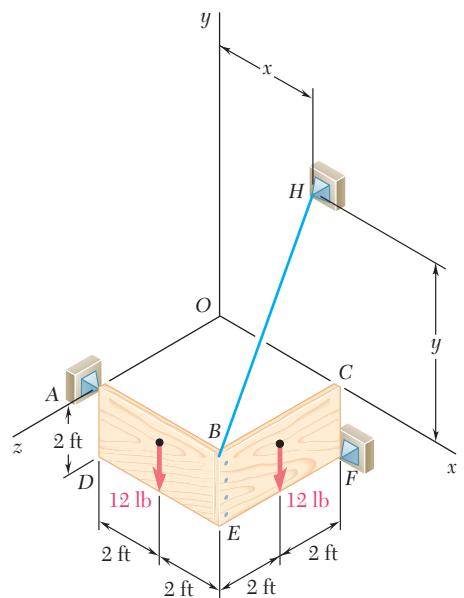


Fig. P4.138

- 4.139** Solve Prob. 4.138, subject to the restriction that *H* must lie on the *y* axis.

- 4.140** The pipe *ACDE* is supported by ball-and-socket joints at *A* and *E* and by the wire *DF*. Determine the tension in the wire when a 640-N load is applied at *B* as shown.

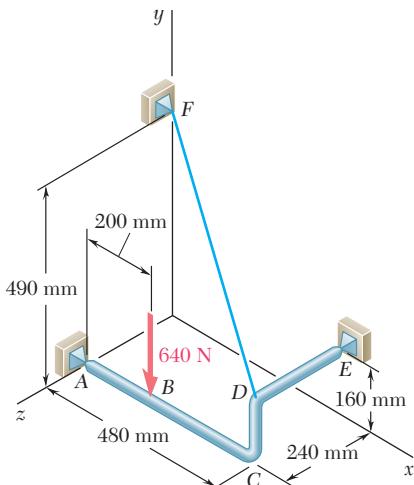


Fig. P4.140

- 4.141** Solve Prob. 4.140, assuming that wire *DF* is replaced by a wire connecting *C* and *F*.

# REVIEW AND SUMMARY

## Equilibrium equations

This chapter was devoted to the study of the *equilibrium of rigid bodies*, i.e., to the situation when the external forces acting on a rigid body form a system equivalent to zero [Sec. 4.1]. We then have

$$\Sigma \mathbf{F} = 0 \quad \Sigma \mathbf{M}_O = \Sigma (\mathbf{r} \times \mathbf{F}) = 0 \quad (4.1)$$

Resolving each force and each moment into its rectangular components, we can express the necessary and sufficient conditions for the equilibrium of a rigid body with the following six scalar equations:

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0 \quad (4.2)$$

$$\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0 \quad (4.3)$$

These equations can be used to determine unknown forces applied to the rigid body or unknown reactions exerted by its supports.

## Free-body diagram

When solving a problem involving the equilibrium of a rigid body, it is essential to consider *all* of the forces acting on the body. Therefore, the first step in the solution of the problem should be to draw a *free-body diagram* showing the body under consideration and all of the unknown as well as known forces acting on it [Sec. 4.2].

## Equilibrium of a two-dimensional structure

In the first part of the chapter, we considered the *equilibrium of a two-dimensional structure*; i.e., we assumed that the structure considered and the forces applied to it were contained in the same plane. We saw that each of the reactions exerted on the structure by its supports could involve one, two, or three unknowns, depending upon the type of support [Sec. 4.3].

In the case of a two-dimensional structure, Eqs. (4.1), or Eqs. (4.2) and (4.3), reduce to *three equilibrium equations*, namely

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M_A = 0 \quad (4.5)$$

where  $A$  is an arbitrary point in the plane of the structure [Sec. 4.4]. These equations can be used to solve for three unknowns. While the three equilibrium equations (4.5) cannot be *augmented* with additional equations, any of them can be *replaced* by another equation. Therefore, we can write alternative sets of equilibrium equations, such as

$$\Sigma F_x = 0 \quad \Sigma M_A = 0 \quad \Sigma M_B = 0 \quad (4.6)$$

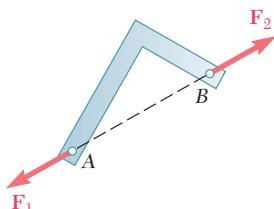
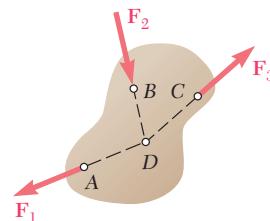
where point  $B$  is chosen in such a way that the line  $AB$  is not parallel to the  $y$  axis, or

$$\Sigma M_A = 0 \quad \Sigma M_B = 0 \quad \Sigma M_C = 0 \quad (4.7)$$

where the points  $A$ ,  $B$ , and  $C$  do not lie in a straight line.

Since any set of equilibrium equations can be solved for only three unknowns, the reactions at the supports of a rigid two-dimensional structure cannot be completely determined if they involve *more than three unknowns*; they are said to be *statically indeterminate* [Sec. 4.5]. On the other hand, if the reactions involve *fewer than three unknowns*, equilibrium will not be maintained under general loading conditions; the structure is said to be *partially constrained*. The fact that the reactions involve exactly three unknowns is no guarantee that the equilibrium equations can be solved for all three unknowns. If the supports are arranged in such a way that the reactions are *either concurrent or parallel*, the reactions are statically indeterminate, and the structure is said to be *improperly constrained*.

Two particular cases of equilibrium of a rigid body were given special attention. In Sec. 4.6, a *two-force body* was defined as a rigid body subjected to forces at only two points, and it was shown that the resultants  $\mathbf{F}_1$  and  $\mathbf{F}_2$  of these forces must have the *same magnitude, the same line of action, and opposite sense* (Fig. 4.11), a property which will simplify the solution of certain problems in later chapters. In Sec. 4.7, a *three-force body* was defined as a rigid body subjected to forces at only three points, and it was shown that the resultants  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ , and  $\mathbf{F}_3$  of these forces must be *either concurrent* (Fig. 4.12) or *parallel*. This property provides us with an alternative approach to the solution of problems involving a three-force body [Sample Prob. 4.6].

**Fig. 4.11****Fig. 4.12**

In the second part of the chapter, we considered the *equilibrium of a three-dimensional body* and saw that each of the reactions exerted on the body by its supports could involve between one and six unknowns, depending upon the type of support [Sec. 4.8].

In the general case of the equilibrium of a three-dimensional body, all of the six scalar equilibrium equations (4.2) and (4.3) listed at the beginning of this review should be used and solved for *six unknowns* [Sec. 4.9]. In most problems, however, these equations will be more conveniently obtained if we first write

$$\Sigma \mathbf{F} = 0 \quad \Sigma \mathbf{M}_O = \Sigma (\mathbf{r} \times \mathbf{F}) = 0 \quad (4.1)$$

and express the forces  $\mathbf{F}$  and position vectors  $\mathbf{r}$  in terms of scalar components and unit vectors. The vector products can then be computed either directly or by means of determinants, and the desired scalar equations obtained by equating to zero the coefficients of the unit vectors [Sample Probs. 4.7 through 4.9].

## Statical indeterminacy

### Partial constraints

### Improper constraints

### Two-force body

### Three-force body

## Equilibrium of a three-dimensional body

We noted that as many as three unknown reaction components may be eliminated from the computation of  $\Sigma \mathbf{M}_O$  in the second of the relations (4.1) through a judicious choice of point  $O$ . Also, the reactions at two points  $A$  and  $B$  can be eliminated from the solution of some problems by writing the equation  $\Sigma M_{AB} = 0$ , which involves the computation of the moments of the forces about an axis  $AB$  joining points  $A$  and  $B$  [Sample Prob. 4.10].

If the reactions involve more than six unknowns, some of the reactions are *statically indeterminate*; if they involve fewer than six unknowns, the rigid body is only *partially constrained*. Even with six or more unknowns, the rigid body will be *improperly constrained* if the reactions associated with the given supports either are parallel or intersect the same line.

# REVIEW PROBLEMS

- 4.142** A hand truck is used to move two kegs, each of mass 40 kg. Neglecting the mass of the hand truck, determine (a) the vertical force  $\mathbf{P}$  that should be applied to the handle to maintain equilibrium when  $\alpha = 35^\circ$ , (b) the corresponding reaction at each of the two wheels.

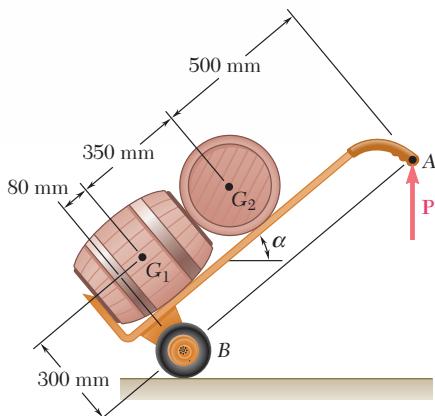


Fig. P4.142

- 4.143** Determine the reactions at A and C when (a)  $\alpha = 0$ , (b)  $\alpha = 30^\circ$ .

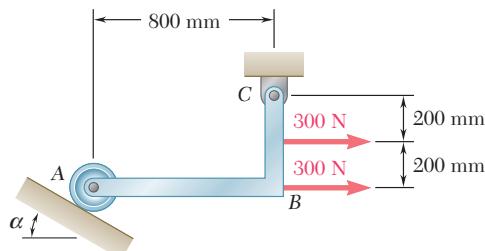


Fig. P4.143

- 4.144** A lever AB is hinged at C and attached to a control cable at A. If the lever is subjected to a 75-lb vertical force at B, determine (a) the tension in the cable, (b) the reaction at C.

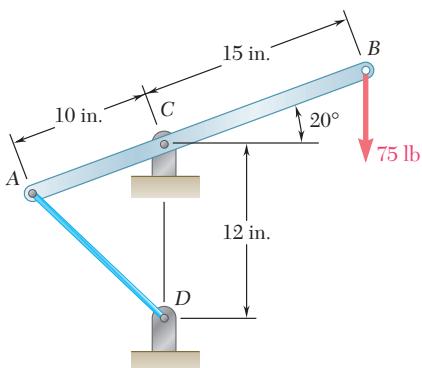


Fig. P4.144

- 4.145** Neglecting friction and the radius of the pulley, determine (a) the tension in cable ADB, (b) the reaction at C.

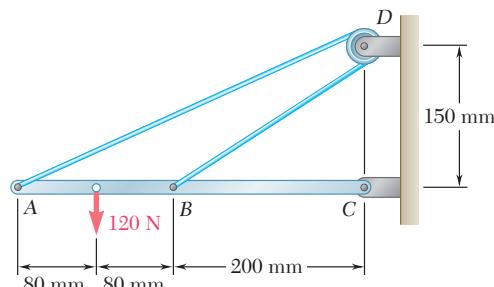


Fig. P4.145

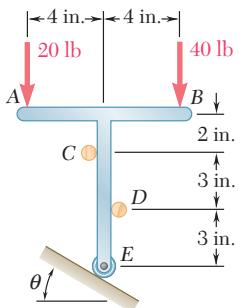
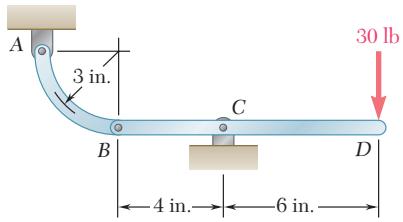
Fig. P4.146  
and P4.147

Fig. P4.148

- 4.146** The T-shaped bracket shown is supported by a small wheel at *E* and pegs at *C* and *D*. Neglecting the effect of friction, determine the reactions at *C*, *D*, and *E* when  $\theta = 30^\circ$ .

- 4.147** The T-shaped bracket shown is supported by a small wheel at *E* and pegs at *C* and *D*. Neglecting the effect of friction, determine (a) the smallest value of  $\theta$  for which the equilibrium of the bracket is maintained, (b) the corresponding reactions at *C*, *D*, and *E*.

- 4.148** For the frame and loading shown, determine the reactions at *A* and *C*.

- 4.149** Determine the reactions at *A* and *B* when  $\beta = 50^\circ$ .

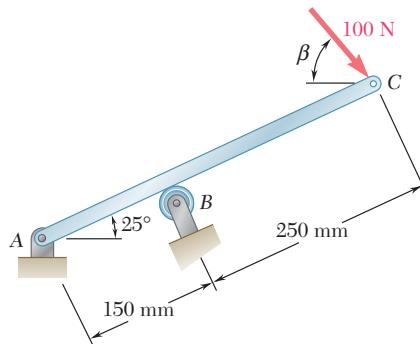


Fig. P4.149

- 4.150** The 6-m pole *ABC* is acted upon by a 455-N force as shown. The pole is held by a ball-and-socket joint at *A* and by two cables *BD* and *BE*. For  $a = 3$  m, determine the tension in each cable and the reaction at *A*.

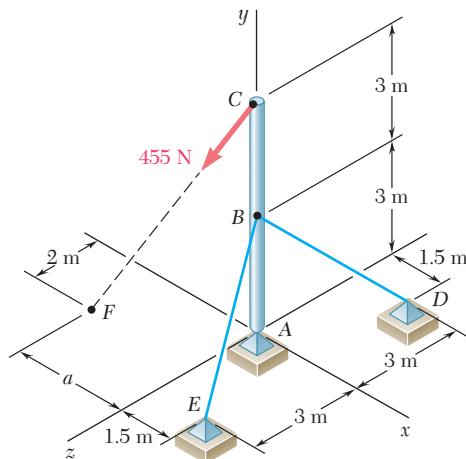
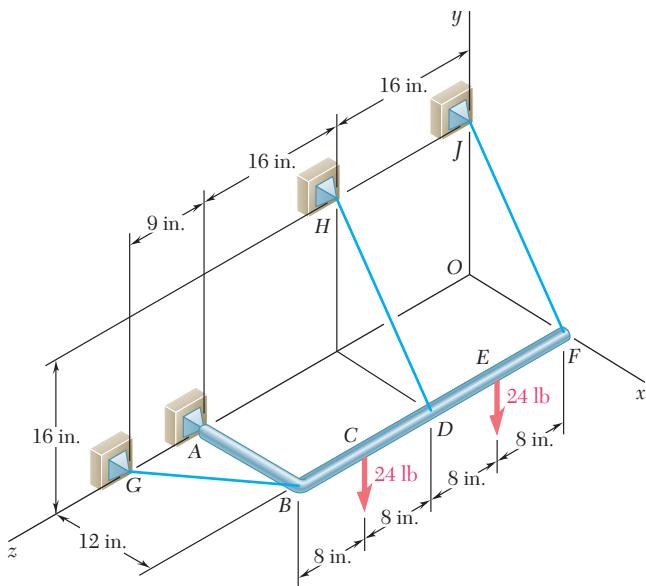


Fig. P4.150

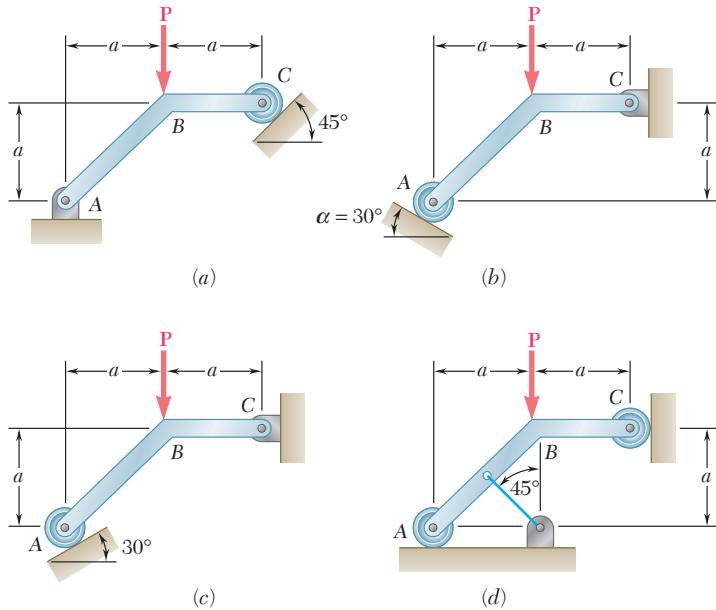
- 4.151** Solve Prob. 4.150 for  $a = 1.5$  m.

- 4.152** The rigid L-shaped member  $ABF$  is supported by a ball-and-socket joint at  $A$  and by three cables. For the loading shown, determine the tension in each cable and the reaction at  $A$ .



**Fig. P4.152**

- 4.153** A force  $\mathbf{P}$  is applied to a bent rod  $ABC$ , which may be supported in four different ways as shown. In each case, if possible, determine the reactions at the supports.



**Fig. P4.153**

# COMPUTER PROBLEMS

**4.C1** The position of the L-shaped rod shown is controlled by a cable attached at  $B$ . Knowing that the rod supports a load of magnitude  $P = 50$  lb, use computational software to calculate the tension  $T$  in the cable for values of  $\theta$  from 0 to  $120^\circ$  using  $10^\circ$  increments. Using appropriate smaller increments, calculate the maximum tension  $T$  and the corresponding value of  $\theta$ .

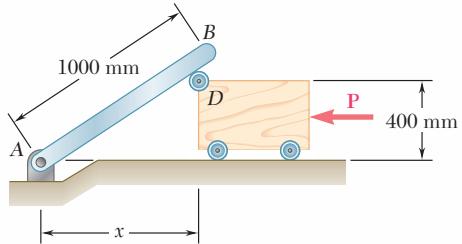


Fig. P4.C2

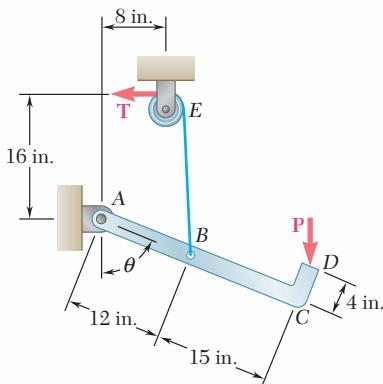


Fig. P4.C1

**4.C2** The position of the 10-kg rod  $AB$  is controlled by the block shown, which is slowly moved to the left by the force  $\mathbf{P}$ . Neglecting the effect of friction, use computational software to calculate the magnitude  $P$  of the force for values of  $x$  decreasing from 750 mm to 0 using 50-mm increments. Using appropriate smaller increments, determine the maximum value of  $P$  and the corresponding value of  $x$ .

**4.C3 and 4.C4** The constant of spring  $AB$  is  $k$ , and the spring is unstretched when  $\theta = 0$ . Knowing that  $R = 10$  in.,  $a = 20$  in., and  $k = 5$  lb/in., use computational software to calculate the weight  $W$  corresponding to equilibrium for values of  $\theta$  from 0 to  $90^\circ$  using  $10^\circ$  increments. Using appropriate smaller increments, determine the value of  $\theta$  corresponding to equilibrium when  $W = 5$  lb.

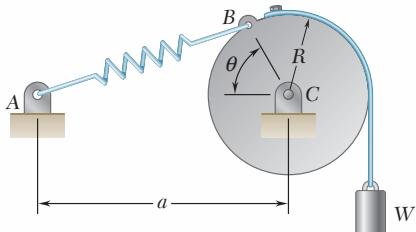


Fig. P4.C3

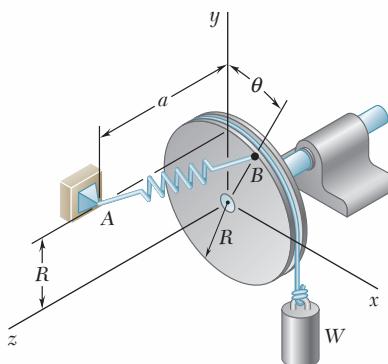
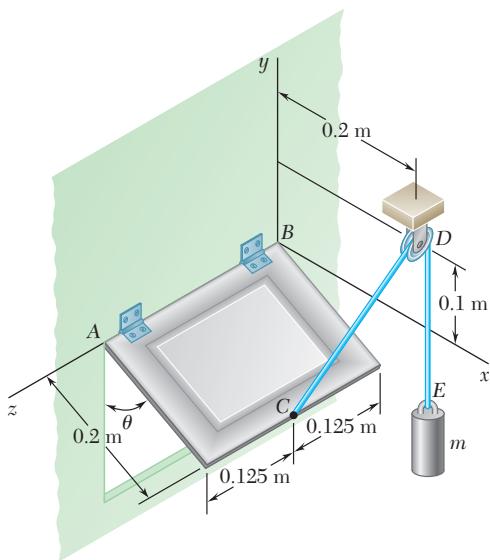


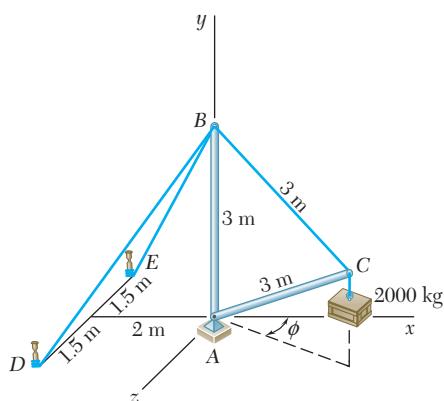
Fig. P4.C4

**4.C5** A  $200 \times 250$ -mm panel of mass 20 kg is supported by hinges along edge AB. Cable CDE is attached to the panel at C, passes over a small pulley at D, and supports a cylinder of mass  $m$ . Neglecting the effect of friction, use computational software to calculate the mass of the cylinder corresponding to equilibrium for values of  $\theta$  from 0 to  $90^\circ$  using  $10^\circ$  increments. Using appropriate smaller increments, determine the value of  $\theta$  corresponding to  $m = 10$  kg.



**Fig. P4.C5**

**4.C6** The derrick shown supports a 2000-kg crate. It is held by a ball-and-socket joint at A and by two cables attached at D and E. Knowing that the derrick stands in a vertical plane forming an angle  $\phi$  with the  $xy$  plane, use computational software to calculate the tension in each cable for values of  $\phi$  from 0 to  $60^\circ$  using  $5^\circ$  increments. Using appropriate smaller increments, determine the value of  $\phi$  for which the tension in cable BE is maximum.



**Fig. P4.C6**

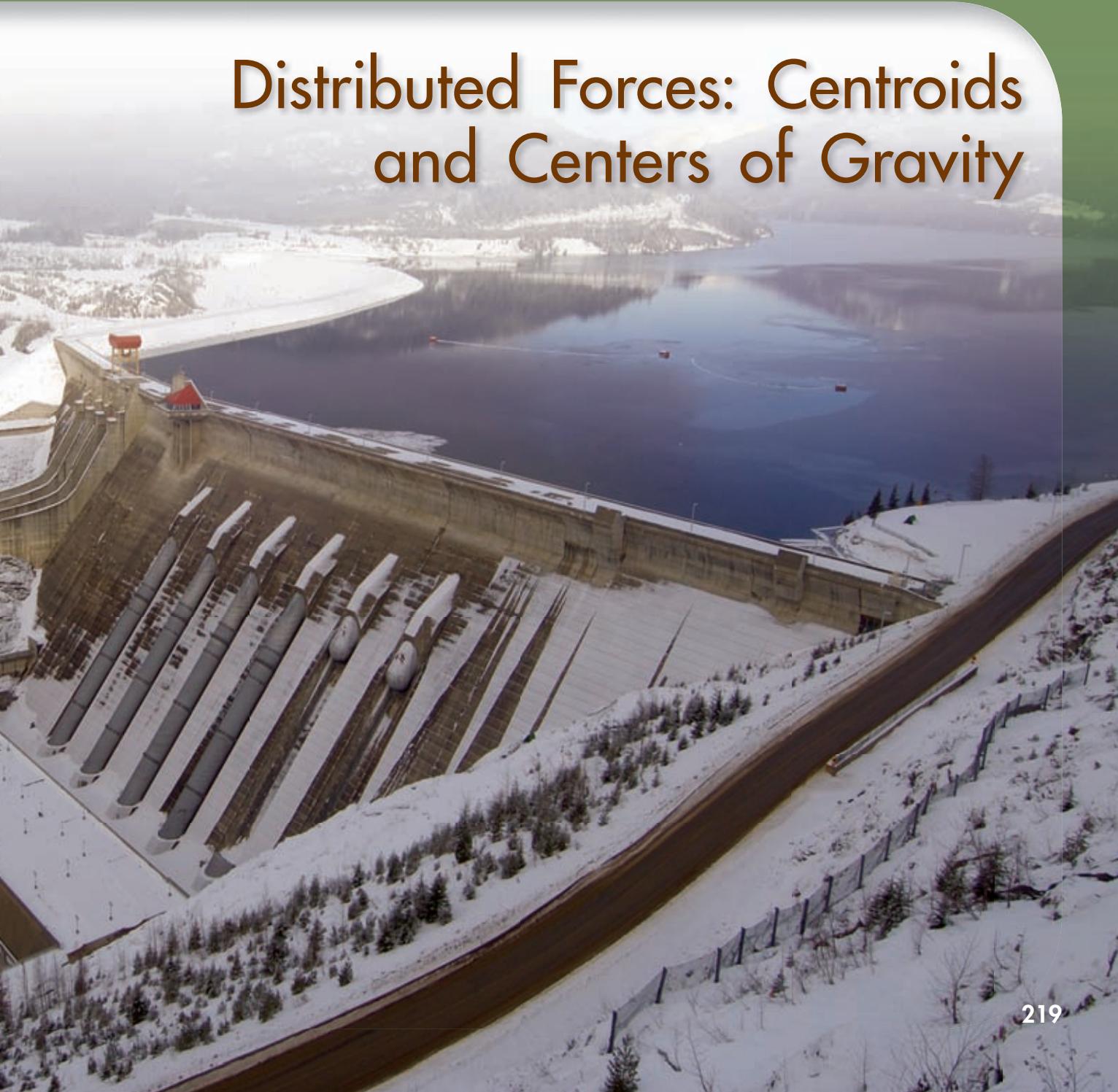
**The Revelstoke Dam, located on the Columbia River in British Columbia, is subjected to three different kinds of distributed forces: the weights of its constituent elements, the pressure forces exerted by the water of its submerged face, and the pressure forces exerted by the ground on its base.**



# 5

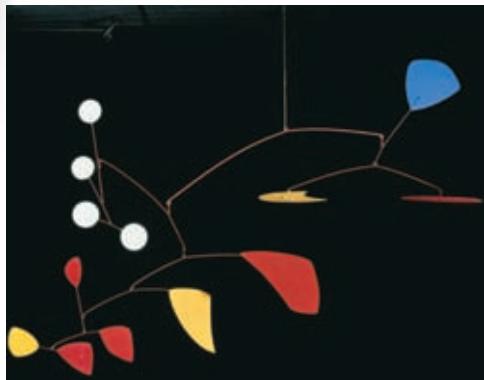
CHAPTER

## Distributed Forces: Centroids and Centers of Gravity



## Chapter 5 Distributed Forces: Centroids and Centers of Gravity

- 5.1** Introduction
- 5.2** Center of Gravity of a Two-Dimensional Body
- 5.3** Centroids of Areas and Lines
- 5.4** First Moments of Areas and Lines
- 5.5** Composite Plates and Wires
- 5.6** Determination of Centroids by Integration
- 5.7** Theorems of Pappus-Guldinus
- 5.8** Distributed Loads on Beams
- 5.9** Forces on Submerged Surfaces
- 5.10** Center of Gravity of a Three-Dimensional Body. Centroid of a Volume
- 5.11** Composite Bodies
- 5.12** Determination of Centroids of Volumes by Integration



**Photo 5.1** The precise balancing of the components of a mobile requires an understanding of centers of gravity and centroids, the main topics of this chapter.

### 5.1 INTRODUCTION

We have assumed so far that the attraction exerted by the earth on a rigid body could be represented by a single force  $\mathbf{W}$ . This force, called the force of gravity or the weight of the body, was to be applied at the *center of gravity* of the body (Sec. 3.2). Actually, the earth exerts a force on each of the particles forming the body. The action of the earth on a rigid body should thus be represented by a large number of small forces distributed over the entire body. You will learn in this chapter, however, that all of these small forces can be replaced by a single equivalent force  $\mathbf{W}$ . You will also learn how to determine the center of gravity, i.e., the point of application of the resultant  $\mathbf{W}$ , for bodies of various shapes.

In the first part of the chapter, two-dimensional bodies, such as flat plates and wires contained in a given plane, are considered. Two concepts closely associated with the determination of the center of gravity of a plate or a wire are introduced: the concept of the *centroid* of an area or a line and the concept of the *first moment* of an area or a line with respect to a given axis.

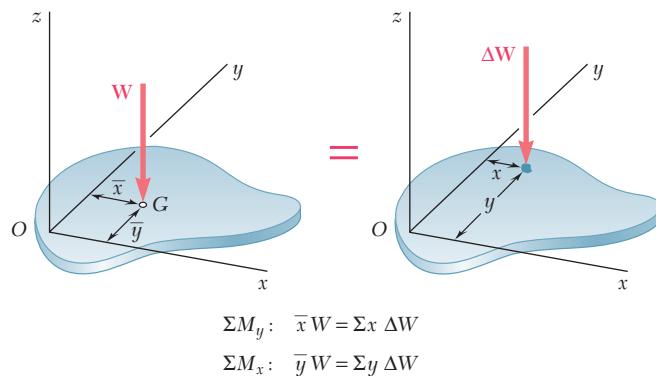
You will also learn that the computation of the area of a surface of revolution or of the volume of a body of revolution is directly related to the determination of the centroid of the line or area used to generate that surface or body of revolution (Theorems of Pappus-Guldinus). And, as is shown in Secs. 5.8 and 5.9, the determination of the centroid of an area simplifies the analysis of beams subjected to distributed loads and the computation of the forces exerted on submerged rectangular surfaces, such as hydraulic gates and portions of dams.

In the last part of the chapter, you will learn how to determine the center of gravity of a three-dimensional body as well as the centroid of a volume and the first moments of that volume with respect to the coordinate planes.

### AREAS AND LINES

## 5.2 CENTER OF GRAVITY OF A TWO-DIMENSIONAL BODY

Let us first consider a flat horizontal plate (Fig. 5.1). We can divide the plate into  $n$  small elements. The coordinates of the first element



**Fig. 5.1** Center of gravity of a plate.

are denoted by  $x_1$  and  $y_1$ , those of the second element by  $x_2$  and  $y_2$ , etc. The forces exerted by the earth on the elements of plate will be denoted, respectively, by  $\Delta\mathbf{W}_1$ ,  $\Delta\mathbf{W}_2$ , ...,  $\Delta\mathbf{W}_n$ . These forces or weights are directed toward the center of the earth; however, for all practical purposes they can be assumed to be parallel. Their resultant is therefore a single force in the same direction. The magnitude  $W$  of this force is obtained by adding the magnitudes of the elemental weights.

$$\Sigma F_z: \quad W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_n$$

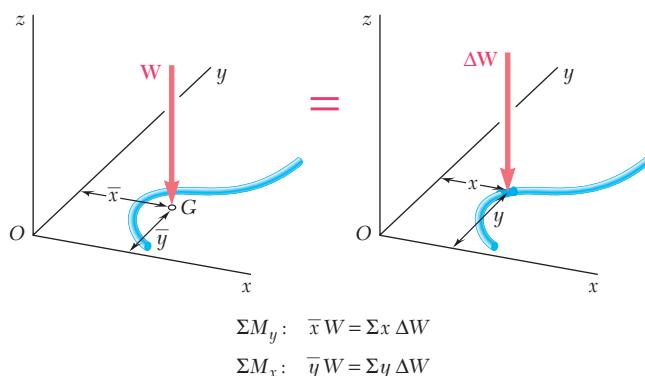
To obtain the coordinates  $\bar{x}$  and  $\bar{y}$  of the point  $G$  where the resultant  $\mathbf{W}$  should be applied, we write that the moments of  $\mathbf{W}$  about the  $y$  and  $x$  axes are equal to the sum of the corresponding moments of the elemental weights,

$$\begin{aligned} \Sigma M_y: \quad \bar{x}W &= x_1 \Delta W_1 + x_2 \Delta W_2 + \cdots + x_n \Delta W_n \\ \Sigma M_x: \quad \bar{y}W &= y_1 \Delta W_1 + y_2 \Delta W_2 + \cdots + y_n \Delta W_n \end{aligned} \quad (5.1)$$

If we now increase the number of elements into which the plate is divided and simultaneously decrease the size of each element, we obtain in the limit the following expressions:

$$W = \int dW \quad \bar{x}W = \int x dW \quad \bar{y}W = \int y dW \quad (5.2)$$

These equations define the weight  $\mathbf{W}$  and the coordinates  $\bar{x}$  and  $\bar{y}$  of the center of gravity  $G$  of a flat plate. The same equations can be derived for a wire lying in the  $xy$  plane (Fig. 5.2). We note that the center of gravity  $G$  of a wire is usually not located on the wire.



**Fig. 5.2** Center of gravity of a wire.

### 5.3 CENTROIDS OF AREAS AND LINES

In the case of a flat homogeneous plate of uniform thickness, the magnitude  $\Delta W$  of the weight of an element of the plate can be expressed as

$$\Delta W = \gamma t \Delta A$$

where  $\gamma$  = specific weight (weight per unit volume) of the material

$t$  = thickness of the plate

$\Delta A$  = area of the element

Similarly, we can express the magnitude  $W$  of the weight of the entire plate as

$$W = \gamma t A$$

where  $A$  is the total area of the plate.

If U.S. customary units are used, the specific weight  $\gamma$  should be expressed in  $\text{lb}/\text{ft}^3$ , the thickness  $t$  in feet, and the areas  $\Delta A$  and  $A$  in square feet. We observe that  $\Delta W$  and  $W$  will then be expressed in pounds. If SI units are used,  $\gamma$  should be expressed in  $\text{N}/\text{m}^3$ ,  $t$  in meters, and the areas  $\Delta A$  and  $A$  in square meters; the weights  $\Delta W$  and  $W$  will then be expressed in newtons.<sup>†</sup>

Substituting for  $\Delta W$  and  $W$  in the moment equations (5.1) and dividing throughout by  $\gamma t$ , we obtain

$$\begin{aligned}\Sigma M_y: \quad \bar{x}A &= x_1 \Delta A_1 + x_2 \Delta A_2 + \cdots + x_n \Delta A_n \\ \Sigma M_x: \quad \bar{y}A &= y_1 \Delta A_1 + y_2 \Delta A_2 + \cdots + y_n \Delta A_n\end{aligned}$$

If we increase the number of elements into which the area  $A$  is divided and simultaneously decrease the size of each element, we obtain in the limit

$$\bar{x}A = \int x dA \quad \bar{y}A = \int y dA \quad (5.3)$$

These equations define the coordinates  $\bar{x}$  and  $\bar{y}$  of the center of gravity of a homogeneous plate. The point whose coordinates are  $\bar{x}$  and  $\bar{y}$  is also known as the *centroid C of the area A* of the plate (Fig. 5.3). If the plate is not homogeneous, these equations cannot be used to determine the center of gravity of the plate; they still define, however, the centroid of the area.

In the case of a homogeneous wire of uniform cross section, the magnitude  $\Delta W$  of the weight of an element of wire can be expressed as

$$\Delta W = \gamma a \Delta L$$

where  $\gamma$  = specific weight of the material

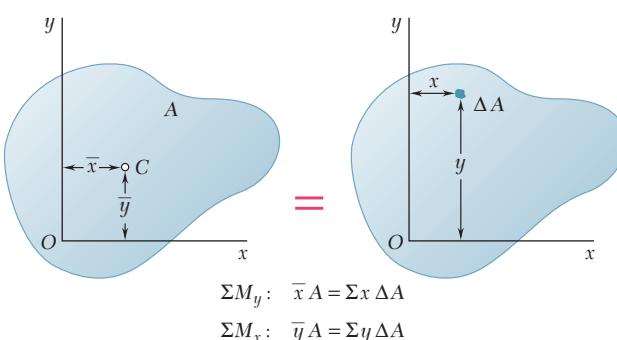
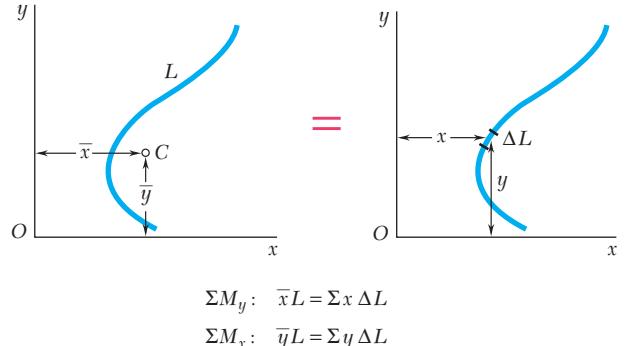
$a$  = cross-sectional area of the wire

$\Delta L$  = length of the element

<sup>†</sup>It should be noted that in the SI system of units a given material is generally characterized by its density  $\rho$  (mass per unit volume) rather than by its specific weight  $\gamma$ . The specific weight of the material can then be obtained from the relation

$$\gamma = \rho g$$

where  $g = 9.81 \text{ m/s}^2$ . Since  $\rho$  is expressed in  $\text{kg}/\text{m}^3$ , we observe that  $\gamma$  will be expressed in  $(\text{kg}/\text{m}^3)(\text{m/s}^2)$ , that is, in  $\text{N}/\text{m}^3$ .

**Fig. 5.3** Centroid of an area.**Fig. 5.4** Centroid of a line.

The center of gravity of the wire then coincides with the *centroid*  $C$  of the line  $L$  defining the shape of the wire (Fig. 5.4). The coordinates  $\bar{x}$  and  $\bar{y}$  of the centroid of the line  $L$  are obtained from the equations

$$\bar{x} L = \int x dL \quad \bar{y} L = \int y dL \quad (5.4)$$

## 5.4 FIRST MOMENTS OF AREAS AND LINES

The integral  $\int x dA$  in Eqs. (5.3) of the preceding section is known as the *first moment of the area  $A$  with respect to the  $y$  axis* and is denoted by  $Q_y$ . Similarly, the integral  $\int y dA$  defines the *first moment of  $A$  with respect to the  $x$  axis* and is denoted by  $Q_x$ . We write

$$Q_y = \int x dA \quad Q_x = \int y dA \quad (5.5)$$

Comparing Eqs. (5.3) with Eqs. (5.5), we note that the first moments of the area  $A$  can be expressed as the products of the area and the coordinates of its centroid:

$$Q_y = \bar{x} A \quad Q_x = \bar{y} A \quad (5.6)$$

It follows from Eqs. (5.6) that the coordinates of the centroid of an area can be obtained by dividing the first moments of that area by the area itself. The first moments of the area are also useful in mechanics of materials for determining the shearing stresses in beams under transverse loadings. Finally, we observe from Eqs. (5.6) that if the centroid of an area is located on a coordinate axis, the first moment of the area with respect to that axis is zero. Conversely, if the first moment of an area with respect to a coordinate axis is zero, then the centroid of the area is located on that axis.

Relations similar to Eqs. (5.5) and (5.6) can be used to define the first moments of a line with respect to the coordinate axes and

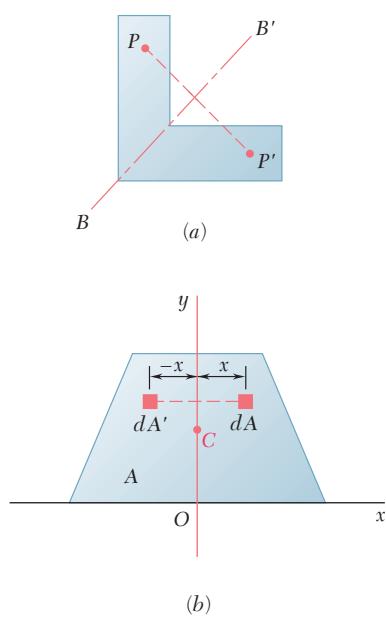


Fig. 5.5

to express these moments as the products of the length  $L$  of the line and the coordinates  $\bar{x}$  and  $\bar{y}$  of its centroid.

An area  $A$  is said to be *symmetric with respect to an axis  $BB'$*  if for every point  $P$  of the area there exists a point  $P'$  of the same area such that the line  $PP'$  is perpendicular to  $BB'$  and is divided into two equal parts by that axis (Fig. 5.5a). A line  $L$  is said to be symmetric with respect to an axis  $BB'$  if it satisfies similar conditions. When an area  $A$  or a line  $L$  possesses an axis of symmetry  $BB'$ , its first moment with respect to  $BB'$  is zero, and its centroid is located on that axis. For example, in the case of the area  $A$  of Fig. 5.5b, which is symmetric with respect to the  $y$  axis, we observe that for every element of area  $dA$  of abscissa  $x$  there exists an element  $dA'$  of equal area and with abscissa  $-x$ . It follows that the integral in the first of Eqs. (5.5) is zero and, thus, that  $Q_y = 0$ . It also follows from the first of the relations (5.3) that  $\bar{x} = 0$ . Thus, if an area  $A$  or a line  $L$  possesses an axis of symmetry, its centroid  $C$  is located on that axis.

We further note that if an area or line possesses two axes of symmetry, its centroid  $C$  must be located at the intersection of the two axes (Fig. 5.6). This property enables us to determine immediately the centroid of areas such as circles, ellipses, squares, rectangles, equilateral triangles, or other symmetric figures as well as the centroid of lines in the shape of the circumference of a circle, the perimeter of a square, etc.

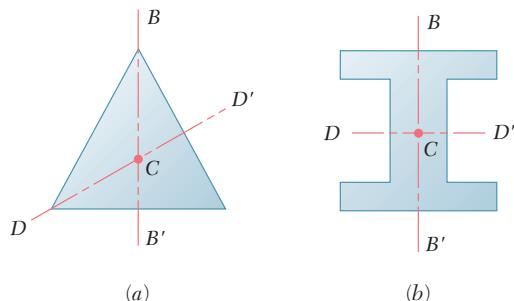


Fig. 5.6

An area  $A$  is said to be *symmetric with respect to a center  $O$*  if for every element of area  $dA$  of coordinates  $x$  and  $y$  there exists an element  $dA'$  of equal area with coordinates  $-x$  and  $-y$  (Fig. 5.7). It then follows that the integrals in Eqs. (5.5) are both zero and that  $Q_x = Q_y = 0$ . It also follows from Eqs. (5.3) that  $\bar{x} = \bar{y} = 0$ , that is, that the centroid of the area coincides with its center of symmetry  $O$ . Similarly, if a line possesses a center of symmetry  $O$ , the centroid of the line will coincide with the center  $O$ .

It should be noted that a figure possessing a center of symmetry does not necessarily possess an axis of symmetry (Fig. 5.7), while a figure possessing two axes of symmetry does not necessarily possess a center of symmetry (Fig. 5.6a). However, if a figure possesses two axes of symmetry at a right angle to each other, the point of intersection of these axes is a center of symmetry (Fig. 5.6b).

Determining the centroids of unsymmetrical areas and lines and of areas and lines possessing only one axis of symmetry will be discussed in Secs. 5.6 and 5.7. Centroids of common shapes of areas and lines are shown in Fig. 5.8A and B.

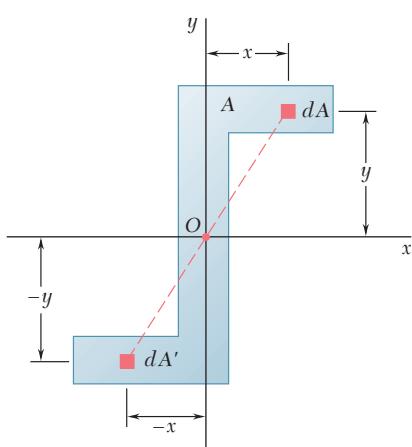
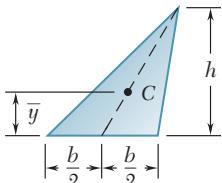
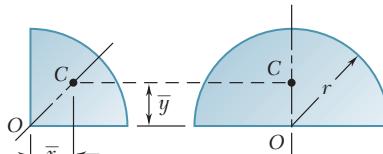
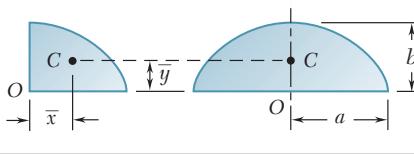
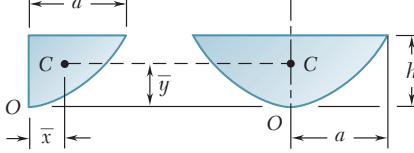
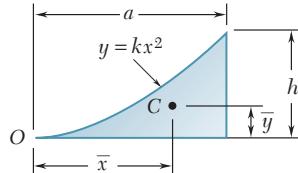
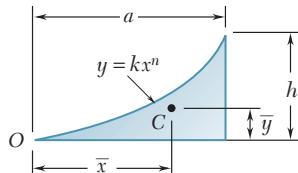
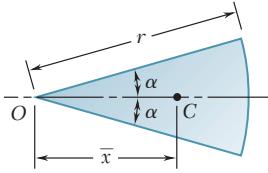


Fig. 5.7

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	$\alpha r^2$

**Fig. 5.8A** Centroids of common shapes of areas.

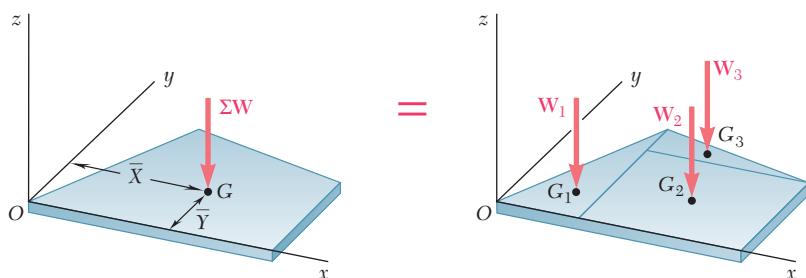
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	$\pi r$
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

**Fig. 5.8B** Centroids of common shapes of lines.

## 5.5 COMPOSITE PLATES AND WIRES

In many instances, a flat plate can be divided into rectangles, triangles, or the other common shapes shown in Fig. 5.8A. The abscissa  $\bar{X}$  of its center of gravity  $G$  can be determined from the abscissas  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$  of the centers of gravity of the various parts by expressing that the moment of the weight of the whole plate about the  $y$  axis is equal to the sum of the moments of the weights of the various parts about the same axis (Fig. 5.9). The ordinate  $\bar{Y}$  of the center of gravity of the plate is found in a similar way by equating moments about the  $x$  axis. We write

$$\begin{aligned} \Sigma M_y: \quad \bar{X}(W_1 + W_2 + \dots + W_n) &= \bar{x}_1 W_1 + \bar{x}_2 W_2 + \dots + \bar{x}_n W_n \\ \Sigma M_x: \quad \bar{Y}(W_1 + W_2 + \dots + W_n) &= \bar{y}_1 W_1 + \bar{y}_2 W_2 + \dots + \bar{y}_n W_n \end{aligned}$$



$$\Sigma M_y: \quad \bar{X} \Sigma W = \Sigma \bar{x} W$$

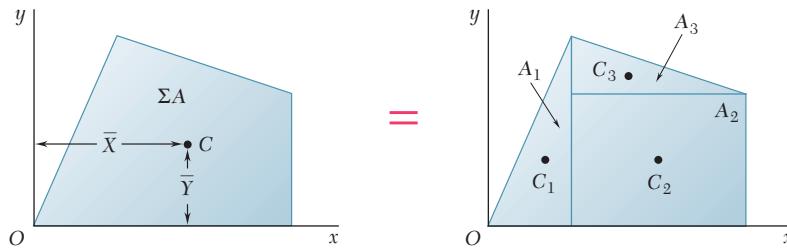
$$\Sigma M_x: \quad \bar{Y} \Sigma W = \Sigma \bar{y} W$$

**Fig. 5.9** Center of gravity of a composite plate.

or, for short,

$$\bar{X}\Sigma A = \Sigma \bar{x}A \quad \bar{Y}\Sigma A = \Sigma \bar{y}A \quad (5.7)$$

These equations can be solved for the coordinates  $\bar{X}$  and  $\bar{Y}$  of the center of gravity of the plate.



$$Q_y = \bar{X}\Sigma A = \Sigma \bar{x}A$$

$$Q_x = \bar{Y}\Sigma A = \Sigma \bar{y}A$$

**Fig. 5.10** Centroid of a composite area.

If the plate is homogeneous and of uniform thickness, the center of gravity coincides with the centroid  $C$  of its area. The abscissa  $\bar{X}$  of the centroid of the area can be determined by noting that the first moment  $Q_y$  of the composite area with respect to the  $y$  axis can be expressed both as the product of  $\bar{X}$  and the total area and as the sum of the first moments of the elementary areas with respect to the  $y$  axis (Fig. 5.10). The ordinate  $\bar{Y}$  of the centroid is found in a similar way by considering the first moment  $Q_x$  of the composite area. We have

$$Q_y = \bar{X}(A_1 + A_2 + \dots + A_n) = \bar{x}_1 A_1 + \bar{x}_2 A_2 + \dots + \bar{x}_n A_n$$

$$Q_x = \bar{Y}(A_1 + A_2 + \dots + A_n) = \bar{y}_1 A_1 + \bar{y}_2 A_2 + \dots + \bar{y}_n A_n$$

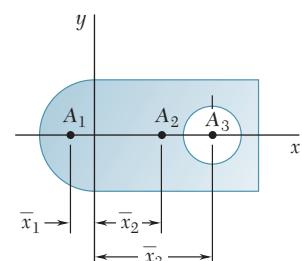
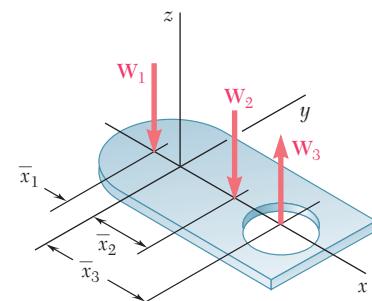
or, for short,

$$Q_y = \bar{X}\Sigma A = \Sigma \bar{x}A \quad Q_x = \bar{Y}\Sigma A = \Sigma \bar{y}A \quad (5.8)$$

These equations yield the first moments of the composite area, or they can be used to obtain the coordinates  $\bar{X}$  and  $\bar{Y}$  of its centroid.

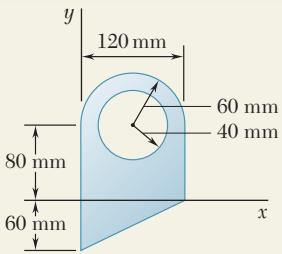
Care should be taken to assign the appropriate sign to the moment of each area. First moments of areas, like moments of forces, can be positive or negative. For example, an area whose centroid is located to the left of the  $y$  axis will have a negative first moment with respect to that axis. Also, the area of a hole should be assigned a negative sign (Fig. 5.11).

Similarly, it is possible in many cases to determine the center of gravity of a composite wire or the centroid of a composite line by dividing the wire or line into simpler elements (see Sample Prob. 5.2).



	$\bar{x}$	$A$	$\bar{x}A$
$A_1$ Semicircle	-	+	-
$A_2$ Full rectangle	+	+	+
$A_3$ Circular hole	+	-	-

**Fig. 5.11**

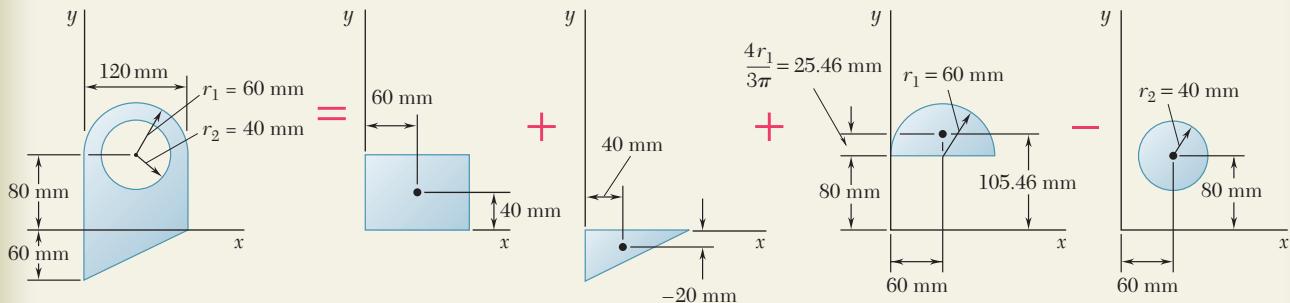


## SAMPLE PROBLEM 5.1

For the plane area shown, determine (a) the first moments with respect to the  $x$  and  $y$  axes, (b) the location of the centroid.

## SOLUTION

**Components of Area.** The area is obtained by adding a rectangle, a triangle, and a semicircle and by then subtracting a circle. Using the coordinate axes shown, the area and the coordinates of the centroid of each of the component areas are determined and entered in the table below. The area of the circle is indicated as negative, since it is to be subtracted from the other areas. We note that the coordinate  $\bar{y}$  of the centroid of the triangle is negative for the axes shown. The first moments of the component areas with respect to the coordinate axes are computed and entered in the table.



Component	$A, \text{mm}^2$	$\bar{x}, \text{mm}$	$\bar{y}, \text{mm}$	$\bar{x}A, \text{mm}^3$	$\bar{y}A, \text{mm}^3$
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	-20	$+144 \times 10^3$	$-72 \times 10^3$
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	$-301.6 \times 10^3$	$-402.2 \times 10^3$
	$\Sigma A = 13.828 \times 10^3$			$\Sigma \bar{x}A = +757.7 \times 10^3$	$\Sigma \bar{y}A = +506.2 \times 10^3$

**a. First Moments of the Area.** Using Eqs. (5.8), we write

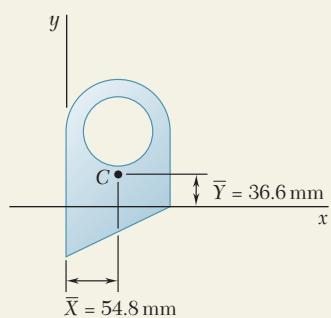
$$Q_x = \Sigma \bar{y}A = 506.2 \times 10^3 \text{ mm}^3 \quad Q_x = 506 \times 10^3 \text{ mm}^3 \quad \blacktriangleright$$

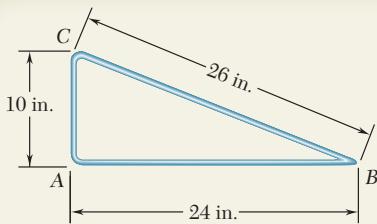
$$Q_y = \Sigma \bar{x}A = 757.7 \times 10^3 \text{ mm}^3 \quad Q_y = 758 \times 10^3 \text{ mm}^3 \quad \blacktriangleright$$

**b. Location of Centroid.** Substituting the values given in the table into the equations defining the centroid of a composite area, we obtain

$$\bar{X} \Sigma A = \Sigma \bar{x}A: \quad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3 \quad \bar{X} = 54.8 \text{ mm} \quad \blacktriangleright$$

$$\bar{Y} \Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3 \quad \bar{Y} = 36.6 \text{ mm} \quad \blacktriangleright$$

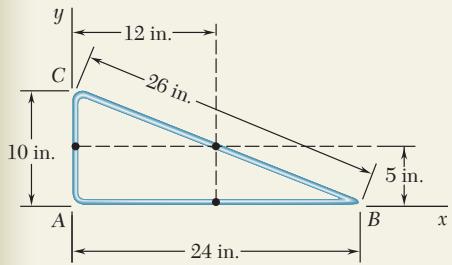




## SAMPLE PROBLEM 5.2

The figure shown is made from a piece of thin, homogeneous wire. Determine the location of its center of gravity.

## SOLUTION



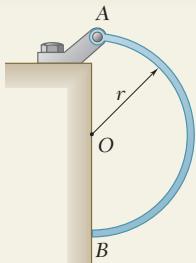
Since the figure is formed of homogeneous wire, its center of gravity coincides with the centroid of the corresponding line. Therefore, that centroid will be determined. Choosing the coordinate axes shown, with origin at A, we determine the coordinates of the centroid of each line segment and compute the first moments with respect to the coordinate axes.

Segment	$L$ , in.	$\bar{x}$ , in.	$\bar{y}$ , in.	$\bar{x}L$ , in $^2$	$\bar{y}L$ , in $^2$
AB	24	12	0	288	0
BC	26	12	5	312	130
CA	10	0	5	0	50
$\Sigma L = 60$				$\Sigma \bar{x}L = 600$	$\Sigma \bar{y}L = 180$

Substituting the values obtained from the table into the equations defining the centroid of a composite line, we obtain

$$\begin{aligned}\bar{X}\Sigma L &= \Sigma \bar{x}L: & \bar{X}(60 \text{ in.}) &= 600 \text{ in}^2 \\ \bar{Y}\Sigma L &= \Sigma \bar{y}L: & \bar{Y}(60 \text{ in.}) &= 180 \text{ in}^2\end{aligned}$$

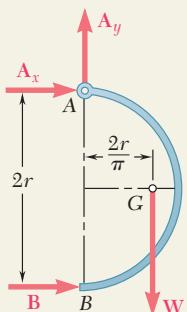
$$\begin{aligned}\bar{X} &= 10 \text{ in.} \\ \bar{Y} &= 3 \text{ in.}\end{aligned}$$



## SAMPLE PROBLEM 5.3

A uniform semicircular rod of weight  $W$  and radius  $r$  is attached to a pin at  $A$  and rests against a frictionless surface at  $B$ . Determine the reactions at  $A$  and  $B$ .

## SOLUTION



**Free-Body Diagram.** A free-body diagram of the rod is drawn. The forces acting on the rod are its weight  $\mathbf{W}$ , which is applied at the center of gravity  $G$  (whose position is obtained from Fig. 5.8B); a reaction at  $A$ , represented by its components  $\mathbf{A}_x$  and  $\mathbf{A}_y$ ; and a horizontal reaction at  $B$ .

### Equilibrium Equations

$$+\nabla \sum M_A = 0: \quad B(2r) - W\left(\frac{2r}{\pi}\right) = 0$$

$$B = +\frac{W}{\pi} \quad \mathbf{B} = \frac{W}{\pi} \rightarrow \blacktriangleleft$$

$$+\rightarrow \sum F_x = 0: \quad A_x + B = 0$$

$$A_x = -B = -\frac{W}{\pi} \quad \mathbf{A}_x = \frac{W}{\pi} \leftarrow$$

$$+\uparrow \sum F_y = 0: \quad A_y - W = 0 \quad \mathbf{A}_y = W \uparrow$$

Adding the two components of the reaction at  $A$ :

$$A = \left[ W^2 + \left(\frac{W}{\pi}\right)^2 \right]^{1/2}$$

$$A = W \left( 1 + \frac{1}{\pi^2} \right)^{1/2} \blacktriangleright$$

$$\tan \alpha = \frac{W}{W/\pi} = \pi$$

$$\alpha = \tan^{-1} \pi \blacktriangleright$$

The answers can also be expressed as follows:

$$\mathbf{A} = 1.049W \angle 72.3^\circ$$

$$\mathbf{B} = 0.318W \rightarrow \blacktriangleright$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson we developed the general equations for locating the centers of gravity of two-dimensional bodies and wires [Eqs. (5.2)] and the centroids of plane areas [Eqs. (5.3)] and lines [Eqs. (5.4)]. In the following problems, you will have to locate the centroids of composite areas and lines or determine the first moments of the area for composite plates [Eqs. (5.8)].

**1. Locating the centroids of composite areas and lines.** Sample Problems 5.1 and 5.2 illustrate the procedure you should follow when solving problems of this type. There are, however, several points that should be emphasized.

- a. The first step in your solution should be to decide how to construct the given area or line from the common shapes of Fig. 5.8. You should recognize that for plane areas it is often possible to construct a particular shape in more than one way. Also, showing the different components (as is done in Sample Prob. 5.1) will help you to correctly establish their centroids and areas or lengths. Do not forget that you can subtract areas as well as add them to obtain a desired shape.
- b. We strongly recommend that for each problem you construct a table containing the areas or lengths and the respective coordinates of the centroids. It is essential for you to remember that areas which are “removed” (for example, holes) are treated as negative. Also, the sign of negative coordinates must be included. Therefore, you should always carefully note the location of the origin of the coordinate axes.
- c. When possible, use symmetry [Sec. 5.4] to help you determine the location of a centroid.
- d. In the formulas for the circular sector and for the arc of a circle in Fig. 5.8, the angle  $\alpha$  must always be expressed in radians.

**2. Calculating the first moments of an area.** The procedures for locating the centroid of an area and for determining the first moments of an area are similar; however, for the latter it is not necessary to compute the total area. Also, as noted in Sec. 5.4, you should recognize that the first moment of an area relative to a centroidal axis is zero.

**3. Solving problems involving the center of gravity.** The bodies considered in the following problems are homogeneous; thus, their centers of gravity and centroids coincide. In addition, when a body that is suspended from a single pin is in equilibrium, the pin and the body's center of gravity must lie on the same vertical line.

It may appear that many of the problems in this lesson have little to do with the study of mechanics. However, being able to locate the centroid of composite shapes will be essential in several topics that you will soon encounter.

# PROBLEMS

**5.1 through 5.9** Locate the centroid of the plane area shown.

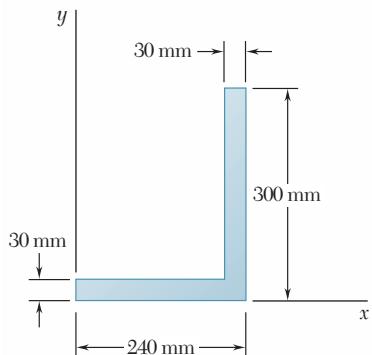


Fig. P5.1

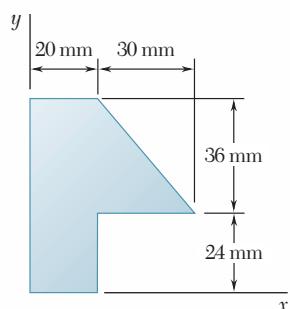


Fig. P5.2

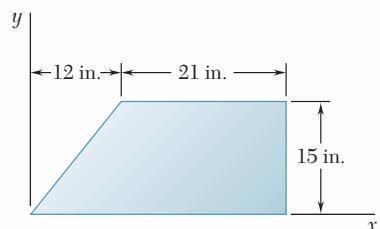


Fig. P5.3

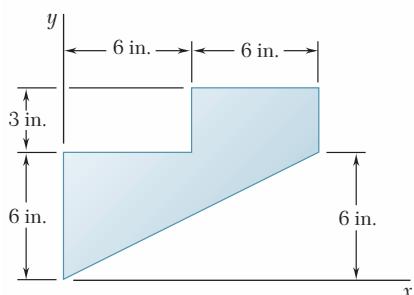


Fig. P5.4

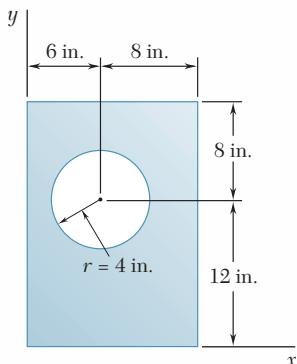


Fig. P5.5

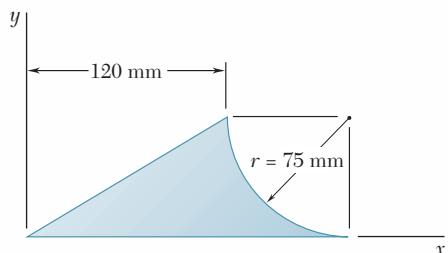


Fig. P5.6

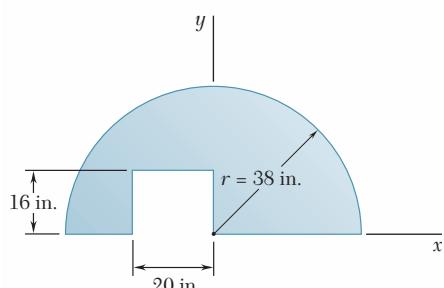


Fig. P5.7

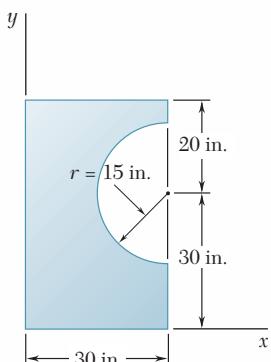


Fig. P5.8

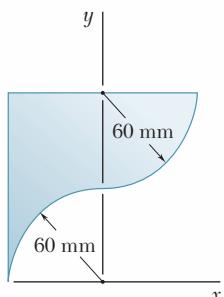


Fig. P5.9

**5.10 through 5.15** Locate the centroid of the plane area shown.

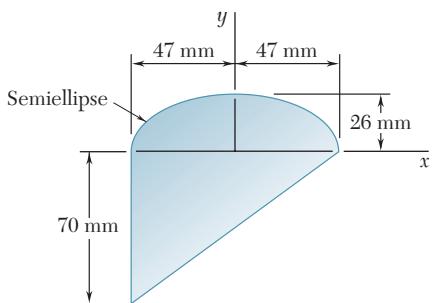


Fig. P5.10

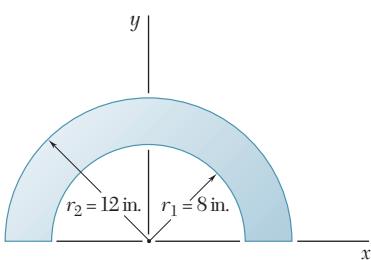


Fig. P5.11

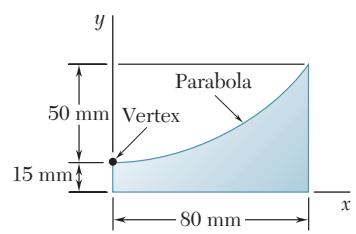


Fig. P5.12

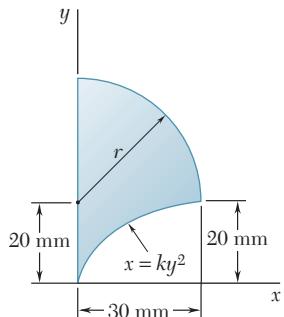


Fig. P5.13

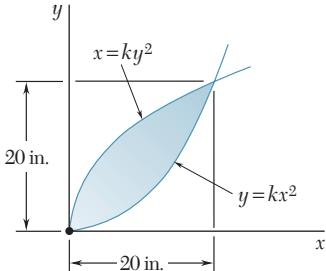


Fig. P5.14

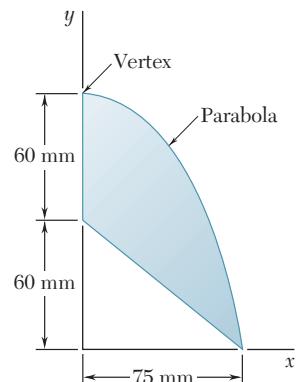


Fig. P5.15

**5.16** Determine the  $y$  coordinate of the centroid of the shaded area in terms of  $r_1$ ,  $r_2$ , and  $\alpha$ .

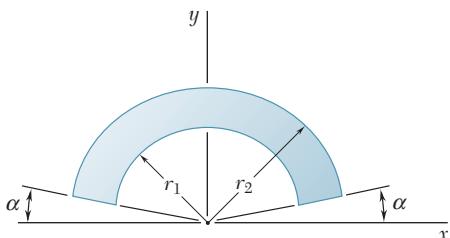


Fig. P5.16 and P5.17

**5.17** Show that as  $r_1$  approaches  $r_2$ , the location of the centroid approaches that for an arc of circle of radius  $(r_1 + r_2)/2$ .

**5.18** For the area shown, determine the ratio  $a/b$  for which  $\bar{x} = \bar{y}$ .

**5.19** For the semiannular area of Prob. 5.11, determine the ratio  $r_2/r_1$  so that  $\bar{y} = 3r_1/4$ .

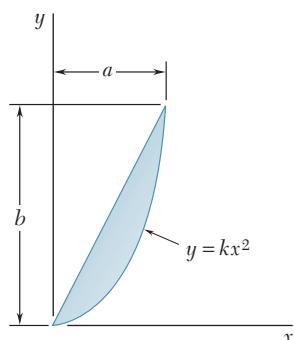
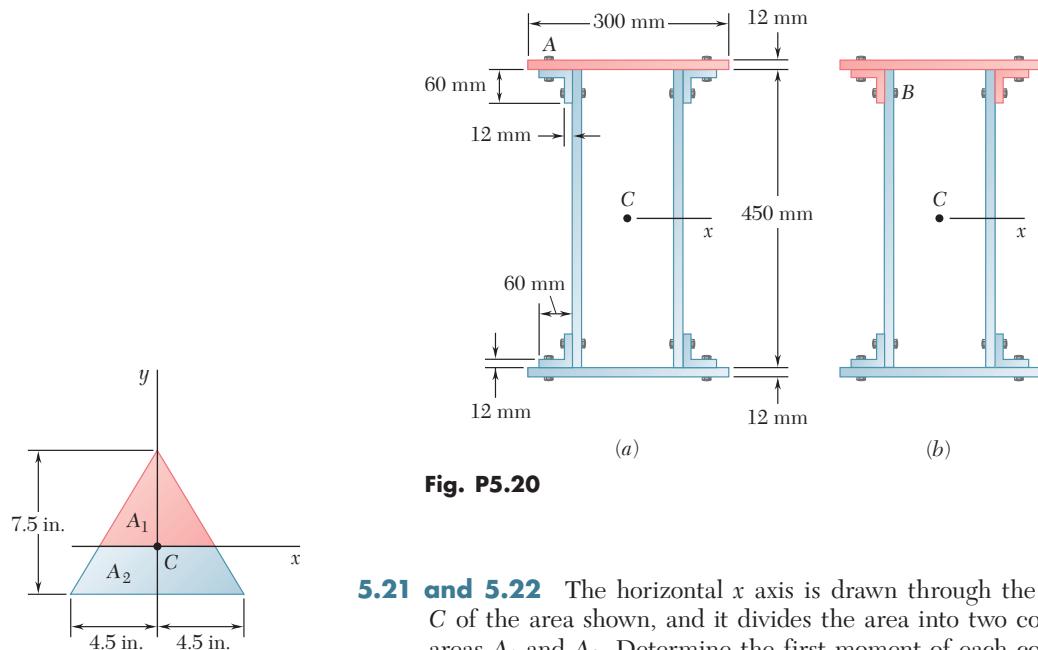
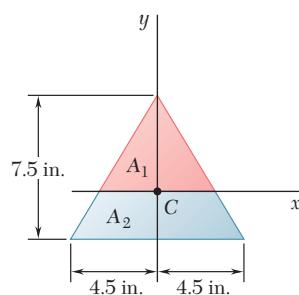
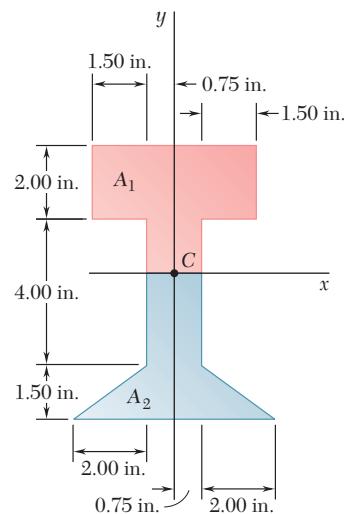


Fig. P5.18

- 5.20** A composite beam is constructed by bolting four plates to four  $60 \times 60 \times 12$ -mm angles as shown. The bolts are equally spaced along the beam, and the beam supports a vertical load. As proved in mechanics of materials, the shearing forces exerted on the bolts at *A* and *B* are proportional to the first moments with respect to the centroidal *x* axis of the red shaded areas shown, respectively, in parts *a* and *b* of the figure. Knowing that the force exerted on the bolt at *A* is 280 N, determine the force exerted on the bolt at *B*.

**Fig. P5.20****Fig. P5.21**

- 5.21 and 5.22** The horizontal *x* axis is drawn through the centroid *C* of the area shown, and it divides the area into two component areas *A*<sub>1</sub> and *A*<sub>2</sub>. Determine the first moment of each component area with respect to the *x* axis, and explain the results obtained.

**Fig. P5.22**

- 5.23** The first moment of the shaded area with respect to the  $x$  axis is denoted by  $Q_x$ . (a) Express  $Q_x$  in terms of  $b$ ,  $c$ , and the distance  $y$  from the base of the shaded area to the  $x$  axis. (b) For what value of  $y$  is  $Q_x$  maximum, and what is that maximum value?

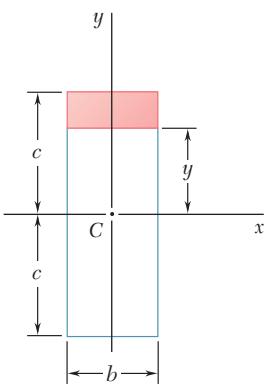


Fig. P5.23

- 5.24 through 5.27** A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

**5.24** Fig. P5.1.

**5.25** Fig. P5.2.

**5.26** Fig. P5.3.

**5.27** Fig. P5.7.

- 5.28** A uniform circular rod of weight 8 lb and radius 10 in. is attached to a pin at  $C$  and to the cable  $AB$ . Determine (a) the tension in the cable, (b) the reaction at  $C$ .

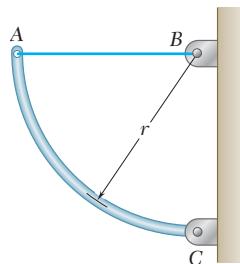


Fig. P5.28

- 5.29** Member  $ABCDE$  is a component of a mobile and is formed from a single piece of aluminum tubing. Knowing that the member is supported at  $C$  and that  $l = 2$  m, determine the distance  $d$  so that portion  $BCD$  of the member is horizontal.

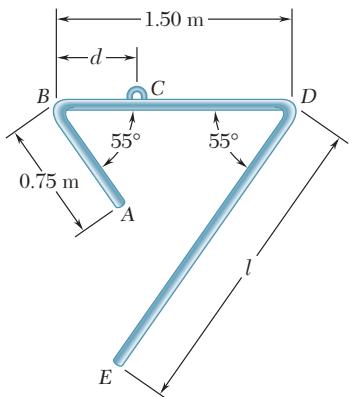


Fig. P5.29 and P5.30

- 5.30** Member  $ABCDE$  is a component of a mobile and is formed from a single piece of aluminum tubing. Knowing that the member is supported at  $C$  and that  $d$  is 0.50 m, determine the length  $l$  of arm  $DE$  so that this portion of the member is horizontal.

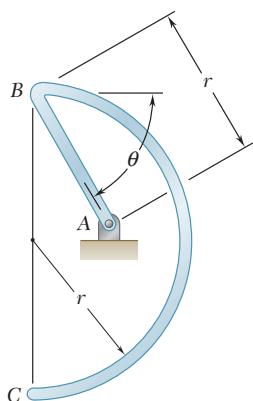


Fig. P5.31

- 5.31** The homogeneous wire  $ABC$  is bent into a semicircular arc and a straight section as shown and is attached to a hinge at  $A$ . Determine the value of  $\theta$  for which the wire is in equilibrium for the indicated position.

- 5.32** Determine the distance  $h$  for which the centroid of the shaded area is as far above line  $BB'$  as possible when (a)  $k = 0.10$ , (b)  $k = 0.80$ .

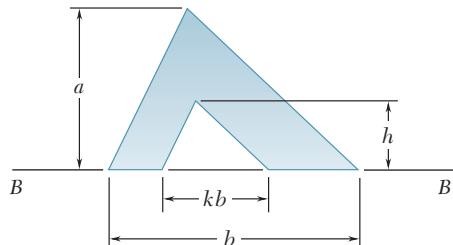


Fig. P5.32 and P5.33

- 5.33** Knowing that the distance  $h$  has been selected to maximize the distance  $\bar{y}$  from line  $BB'$  to the centroid of the shaded area, show that  $\bar{y} = 2h/3$ .

## 5.6 DETERMINATION OF CENTROIDS BY INTEGRATION

The centroid of an area bounded by analytical curves (i.e., curves defined by algebraic equations) is usually determined by evaluating the integrals in Eqs. (5.3) of Sec. 5.3:

$$\bar{x}A = \int x \, dA \quad \bar{y}A = \int y \, dA \quad (5.3)$$

If the element of area  $dA$  is a small rectangle of sides  $dx$  and  $dy$ , the evaluation of each of these integrals requires a *double integration* with respect to  $x$  and  $y$ . A double integration is also necessary if polar coordinates are used for which  $dA$  is a small element of sides  $dr$  and  $r \, d\theta$ .

In most cases, however, it is possible to determine the coordinates of the centroid of an area by performing a single integration. This is achieved by choosing  $dA$  to be a thin rectangle or strip or a thin sector or pie-shaped element (Fig. 5.12); the centroid of the thin rectangle is located at its center, and the centroid of the thin sector is located at a distance  $\frac{2}{3}r$  from its vertex (as it is for a triangle). The coordinates of the centroid of the area under consideration are then obtained by expressing that the first moment of the entire area with respect to each of the coordinate axes is equal to the sum (or integral) of the corresponding moments of the elements of area.

Denoting by  $\bar{x}_{el}$  and  $\bar{y}_{el}$  the coordinates of the centroid of the element  $dA$ , we write

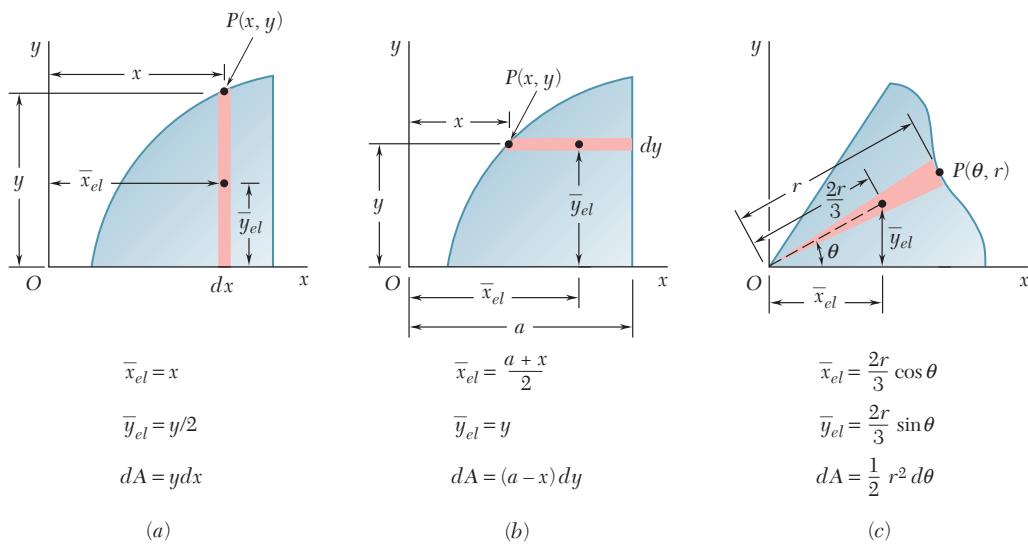
$$\begin{aligned} Q_y &= \bar{x}A = \int \bar{x}_{el} dA \\ Q_x &= \bar{y}A = \int \bar{y}_{el} dA \end{aligned} \quad (5.9)$$

If the area  $A$  is not already known, it can also be computed from these elements.

The coordinates  $\bar{x}_{el}$  and  $\bar{y}_{el}$  of the centroid of the element of area  $dA$  should be expressed in terms of the coordinates of a point located on the curve bounding the area under consideration. Also, the area of the element  $dA$  should be expressed in terms of the coordinates of that point and the appropriate differentials. This has been done in Fig. 5.12 for three common types of elements; the pie-shaped element of part *c* should be used when the equation of the curve bounding the area is given in polar coordinates. The appropriate expressions should be substituted into formulas (5.9), and the equation of the bounding curve should be used to express one of the coordinates in terms of the other. The integration is thus reduced to a single integration. Once the area has been determined and the integrals in Eqs. (5.9) have been evaluated, these equations can be solved for the coordinates  $\bar{x}$  and  $\bar{y}$  of the centroid of the area.

When a line is defined by an algebraic equation, its centroid can be determined by evaluating the integrals in Eqs. (5.4) of Sec. 5.3:

$$\bar{x}L = \int x dL \quad \bar{y}L = \int y dL \quad (5.4)$$



**Fig. 5.12** Centroids and areas of differential elements.

The differential length  $dL$  should be replaced by one of the following expressions, depending upon which coordinate,  $x$ ,  $y$ , or  $\theta$ , is chosen as the independent variable in the equation used to define the line (these expressions can be derived using the Pythagorean theorem):

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

After the equation of the line has been used to express one of the coordinates in terms of the other, the integration can be performed, and Eqs. (5.4) can be solved for the coordinates  $\bar{x}$  and  $\bar{y}$  of the centroid of the line.

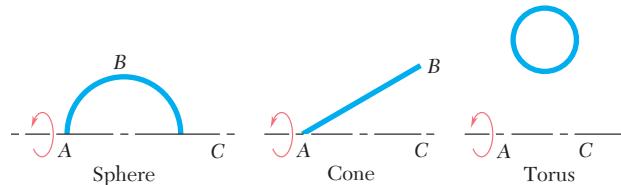


**Photo 5.2** The storage tanks shown are all bodies of revolution. Thus, their surface areas and volumes can be determined using the theorems of Pappus-Guldinus.

## 5.7 THEOREMS OF PAPPUS-GULDINUS

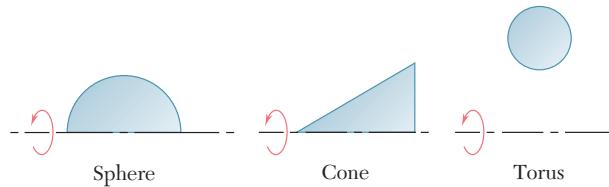
These theorems, which were first formulated by the Greek geometer Pappus during the third century A.D. and later restated by the Swiss mathematician Guldinus, or Guldin, (1577–1643) deal with surfaces and bodies of revolution.

A *surface of revolution* is a surface which can be generated by rotating a plane curve about a fixed axis. For example (Fig. 5.13), the



**Fig. 5.13**

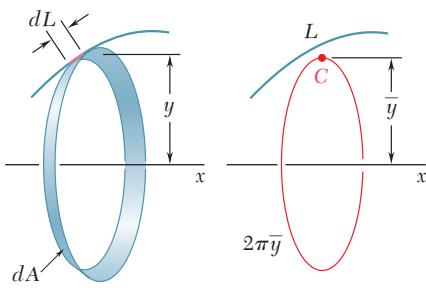
surface of a sphere can be obtained by rotating a semicircular arc  $ABC$  about the diameter  $AC$ , the surface of a cone can be produced by rotating a straight line  $AB$  about an axis  $AC$ , and the surface of a torus or ring can be generated by rotating the circumference of a circle about a nonintersecting axis. A *body of revolution* is a body which can be generated by rotating a plane area about a fixed axis. As shown in Fig. 5.14, a sphere, a cone, and a torus can each be generated by rotating the appropriate shape about the indicated axis.



**Fig. 5.14**

**THEOREM I.** *The area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid of the curve while the surface is being generated.*

**Proof.** Consider an element  $dL$  of the line  $L$  (Fig. 5.15), which is revolved about the  $x$  axis. The area  $dA$  generated by the element

**Fig. 5.15**

$dL$  is equal to  $2\pi y \, dL$ . Thus, the entire area generated by  $L$  is  $A = \int 2\pi y \, dL$ . Recalling that we found in Sec. 5.3 that the integral  $\int y \, dL$  is equal to  $\bar{y}L$ , we therefore have

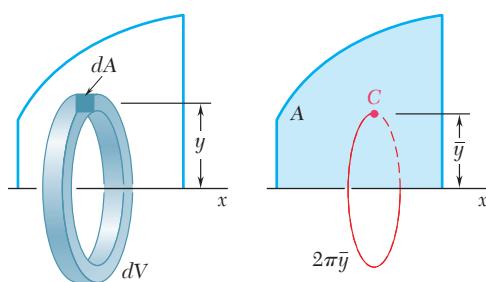
$$A = 2\pi \bar{y}L \quad (5.10)$$

where  $2\pi \bar{y}$  is the distance traveled by the centroid of  $L$  (Fig. 5.15). It should be noted that the generating curve must not cross the axis about which it is rotated; if it did, the two sections on either side of the axis would generate areas having opposite signs, and the theorem would not apply.

**THEOREM II.** *The volume of a body of revolution is equal to the generating area times the distance traveled by the centroid of the area while the body is being generated.*

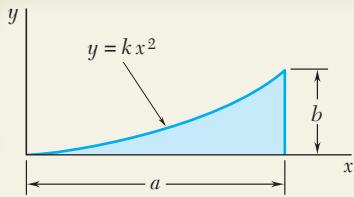
**Proof.** Consider an element  $dA$  of the area  $A$  which is revolved about the  $x$  axis (Fig. 5.16). The volume  $dV$  generated by the element  $dA$  is equal to  $2\pi y \, dA$ . Thus, the entire volume generated by  $A$  is  $V = \int 2\pi y \, dA$ , and since the integral  $\int y \, dA$  is equal to  $\bar{y}A$  (Sec. 5.3), we have

$$V = 2\pi \bar{y}A \quad (5.11)$$

**Fig. 5.16**

where  $2\pi \bar{y}$  is the distance traveled by the centroid of  $A$ . Again, it should be noted that the theorem does not apply if the axis of rotation intersects the generating area.

The theorems of Pappus-Guldinus offer a simple way to compute the areas of surfaces of revolution and the volumes of bodies of revolution. Conversely, they can also be used to determine the centroid of a plane curve when the area of the surface generated by the curve is known or to determine the centroid of a plane area when the volume of the body generated by the area is known (see Sample Prob. 5.8).



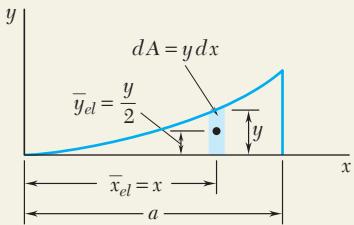
## SAMPLE PROBLEM 5.4

Determine by direct integration the location of the centroid of a parabolic spandrel.

### SOLUTION

**Determination of the Constant  $k$ .** The value of  $k$  is determined by substituting  $x = a$  and  $y = b$  into the given equation. We have  $b = ka^2$  or  $k = b/a^2$ . The equation of the curve is thus

$$y = \frac{b}{a^2}x^2 \quad \text{or} \quad x = \frac{a}{b^{1/2}}y^{1/2}$$



**Vertical Differential Element.** We choose the differential element shown and find the total area of the figure.

$$A = \int dA = \int y dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[ \frac{b}{a^2} \frac{x^3}{3} \right]_0^a = \frac{ab}{3}$$

The first moment of the differential element with respect to the  $y$  axis is  $\bar{x}_{el} dy$ ; hence, the first moment of the entire area with respect to this axis is

$$Q_y = \int \bar{x}_{el} dA = \int xy dx = \int_0^a x \left( \frac{b}{a^2}x^2 \right) dx = \left[ \frac{b}{a^2} \frac{x^4}{4} \right]_0^a = \frac{a^2 b}{4}$$

Since  $Q_y = \bar{x}A$ , we have

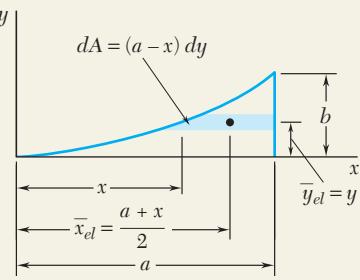
$$\bar{x}A = \int \bar{x}_{el} dA \quad \bar{x} \frac{ab}{3} = \frac{a^2 b}{4} \quad \bar{x} = \frac{3}{4}a \quad \blacktriangleleft$$

Likewise, the first moment of the differential element with respect to the  $x$  axis is  $\bar{y}_{el} dy$ , and the first moment of the entire area is

$$Q_x = \int \bar{y}_{el} dA = \int \frac{y}{2}y dx = \int_0^a \frac{1}{2} \left( \frac{b}{a^2}x^2 \right)^2 dx = \left[ \frac{b^2}{2a^4} \frac{x^5}{5} \right]_0^a = \frac{ab^2}{10}$$

Since  $Q_x = \bar{y}A$ , we have

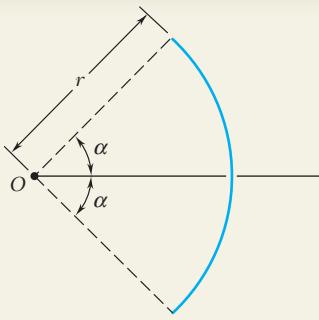
$$\bar{y}A = \int \bar{y}_{el} dA \quad \bar{y} \frac{ab}{3} = \frac{ab^2}{10} \quad \bar{y} = \frac{3}{10}b \quad \blacktriangleleft$$



**Horizontal Differential Element.** The same results can be obtained by considering a horizontal element. The first moments of the area are

$$\begin{aligned} Q_y &= \int \bar{x}_{el} dA = \int \frac{a+x}{2}(a-x) dy = \int_0^b \frac{a^2 - x^2}{2} dy \\ &= \frac{1}{2} \int_0^b \left( a^2 - \frac{a^2}{b}y \right) dy = \frac{a^2 b}{4} \\ Q_x &= \int \bar{y}_{el} dA = \int y(a-x) dy = \int y \left( a - \frac{a}{b^{1/2}}y^{1/2} \right) dy \\ &= \int_0^b \left( ay - \frac{a}{b^{1/2}}y^{3/2} \right) dy = \frac{ab^2}{10} \end{aligned}$$

To determine  $\bar{x}$  and  $\bar{y}$ , the expressions obtained are again substituted into the equations defining the centroid of the area.



## SAMPLE PROBLEM 5.5

Determine the location of the centroid of the arc of circle shown.

### SOLUTION

Since the arc is symmetrical with respect to the  $x$  axis,  $\bar{y} = 0$ . A differential element is chosen as shown, and the length of the arc is determined by integration.

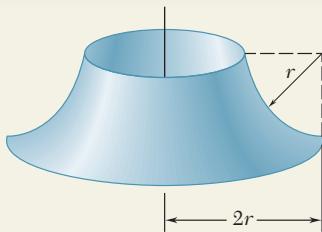
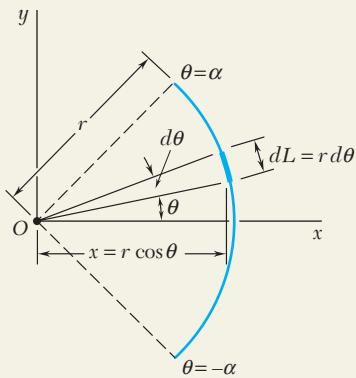
$$L = \int dL = \int_{-\alpha}^{\alpha} r d\theta = r \int_{-\alpha}^{\alpha} d\theta = 2r\alpha$$

The first moment of the arc with respect to the  $y$  axis is

$$\begin{aligned} Q_y &= \int x dL = \int_{-\alpha}^{\alpha} (r \cos \theta)(r d\theta) = r^2 \int_{-\alpha}^{\alpha} \cos \theta d\theta \\ &= r^2 [\sin \theta]_{-\alpha}^{\alpha} = 2r^2 \sin \alpha \end{aligned}$$

Since  $Q_y = \bar{x}L$ , we write

$$\bar{x}(2r\alpha) = 2r^2 \sin \alpha \quad \bar{x} = \frac{r \sin \alpha}{\alpha} \quad \blacktriangleleft$$

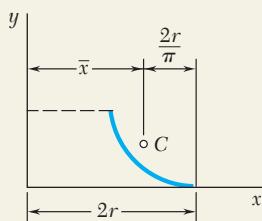


## SAMPLE PROBLEM 5.6

Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.

### SOLUTION

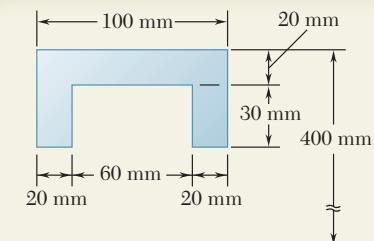
According to Theorem I of Pappus-Guldinus, the area generated is equal to the product of the length of the arc and the distance traveled by its centroid. Referring to Fig. 5.8B, we have



$$\bar{x} = 2r - \frac{2r}{\pi} = 2r \left(1 - \frac{1}{\pi}\right)$$

$$A = 2\pi \bar{x} L = 2\pi \left[2r \left(1 - \frac{1}{\pi}\right)\right] \left(\frac{\pi r}{2}\right)$$

$$A = 2\pi r^2 (\pi - 1) \quad \blacktriangleleft$$

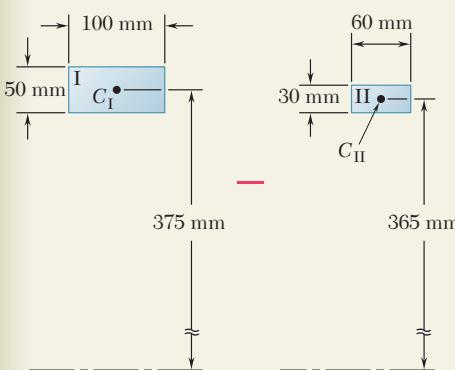


## SAMPLE PROBLEM 5.7

The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is  $\rho = 7.85 \times 10^3 \text{ kg/m}^3$ , determine the mass and the weight of the rim.

## SOLUTION

The volume of the rim can be found by applying Theorem II of Pappus-Guldinus, which states that the volume equals the product of the given cross-sectional area and the distance traveled by its centroid in one complete revolution. However, the volume can be more easily determined if we observe that the cross section can be formed from rectangle I, whose area is positive, and rectangle II, whose area is negative.



	Area, $\text{mm}^2$	$\bar{y}$ , mm	Distance Traveled by C, mm	Volume, $\text{mm}^3$
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	-1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
Volume of rim = $7.65 \times 10^6$				

Since  $1 \text{ mm} = 10^{-3} \text{ m}$ , we have  $1 \text{ mm}^3 = (10^{-3} \text{ m})^3 = 10^{-9} \text{ m}^3$ , and we obtain  $V = 7.65 \times 10^6 \text{ mm}^3 = (7.65 \times 10^6)(10^{-9} \text{ m}^3) = 7.65 \times 10^{-3} \text{ m}^3$ .

$$\begin{aligned} m &= \rho V = (7.85 \times 10^3 \text{ kg/m}^3)(7.65 \times 10^{-3} \text{ m}^3) & m &= 60.0 \text{ kg} \\ W &= mg = (60.0 \text{ kg})(9.81 \text{ m/s}^2) = 589 \text{ kg} \cdot \text{m/s}^2 & W &= 589 \text{ N} \end{aligned}$$

## SAMPLE PROBLEM 5.8

Using the theorems of Pappus-Guldinus, determine (a) the centroid of a semicircular area, (b) the centroid of a semicircular arc. We recall that the volume and the surface area of a sphere are  $\frac{4}{3}\pi r^3$  and  $4\pi r^2$ , respectively.

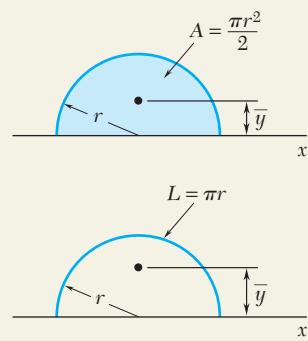
## SOLUTION

The volume of a sphere is equal to the product of the area of a semicircle and the distance traveled by the centroid of the semicircle in one revolution about the x axis.

$$V = 2\pi\bar{y}A \quad \frac{4}{3}\pi r^3 = 2\pi\bar{y}(\frac{1}{2}\pi r^2) \quad \bar{y} = \frac{4r}{3\pi}$$

Likewise, the area of a sphere is equal to the product of the length of the generating semicircle and the distance traveled by its centroid in one revolution.

$$A = 2\pi\bar{y}L \quad 4\pi r^2 = 2\pi\bar{y}(\pi r) \quad \bar{y} = \frac{2r}{\pi}$$



# SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will use the equations

$$\bar{x}A = \int x \, dA \quad \bar{y}A = \int y \, dA \quad (5.3)$$

$$\bar{x}L = \int x \, dL \quad \bar{y}L = \int y \, dL \quad (5.4)$$

to locate the centroids of plane areas and lines, respectively. You will also apply the theorems of Pappus-Guldinus (Sec. 5.7) to determine the areas of surfaces of revolution and the volumes of bodies of revolution.

**1. Determining by direct integration the centroids of areas and lines.** When solving problems of this type, you should follow the method of solution shown in Sample Probs. 5.4 and 5.5: compute  $A$  or  $L$ , determine the first moments of the area or the line, and solve Eqs. (5.3) or (5.4) for the coordinates of the centroid. In addition, you should pay particular attention to the following points.

a. Begin your solution by carefully defining or determining each term in the applicable integral formulas. We strongly encourage you to show on your sketch of the given area or line your choice for  $dA$  or  $dL$  and the distances to its centroid.

b. As explained in Sec. 5.6, the  $x$  and the  $y$  in the above equations represent the *coordinates of the centroid* of the differential elements  $dA$  and  $dL$ . It is important to recognize that the coordinates of the centroid of  $dA$  are not equal to the coordinates of a point located on the curve bounding the area under consideration. You should carefully study Fig. 5.12 until you fully understand this important point.

c. To possibly simplify or minimize your computations, always examine the shape of the given area or line before defining the differential element that you will use. For example, sometimes it may be preferable to use horizontal rectangular elements instead of vertical ones. Also, it will usually be advantageous to use polar coordinates when a line or an area has circular symmetry.

d. Although most of the integrations in this lesson are straightforward, at times it may be necessary to use more advanced techniques, such as trigonometric substitution or integration by parts. Of course, using a table of integrals is the fastest method to evaluate difficult integrals.

**2. Applying the theorems of Pappus-Guldinus.** As shown in Sample Probs. 5.6 through 5.8, these simple, yet very useful theorems allow you to apply your knowledge of centroids to the computation of areas and volumes. Although the theorems refer to the distance traveled by the centroid and to the length of the generating curve or to the generating area, the resulting equations [Eqs. (5.10) and (5.11)] contain the products of these quantities, which are simply the first moments of a line ( $\bar{y}L$ ) and an area ( $\bar{y}A$ ), respectively. Thus, for those problems for which the generating line or area consists of more than one common shape, you need only determine  $\bar{y}L$  or  $\bar{y}A$ ; you do not have to calculate the length of the generating curve or the generating area.

# PROBLEMS

**5.34 through 5.36** Determine by direct integration the centroid of the area shown. Express your answer in terms of  $a$  and  $h$ .

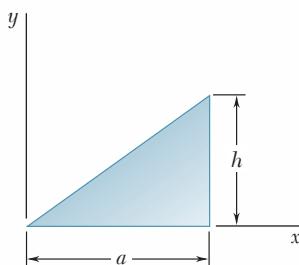


Fig. P5.34

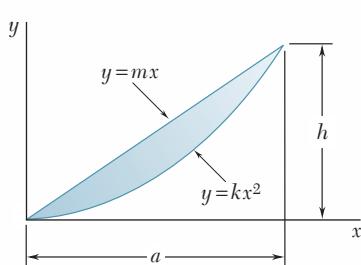


Fig. P5.35

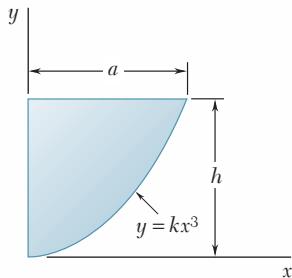


Fig. P5.36

**5.37 through 5.39** Determine by direct integration the centroid of the area shown.

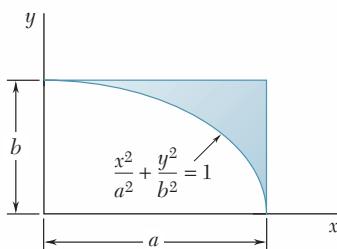


Fig. P5.37

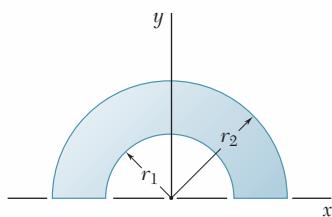


Fig. P5.38

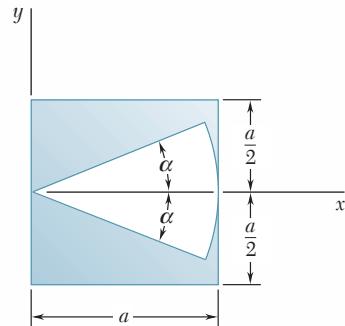


Fig. P5.39

**5.40 and 5.41** Determine by direct integration the centroid of the area shown. Express your answer in terms of  $a$  and  $b$ .

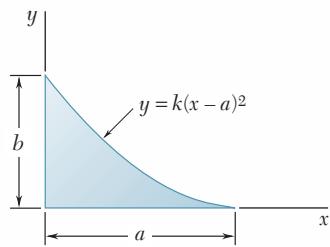


Fig. P5.40

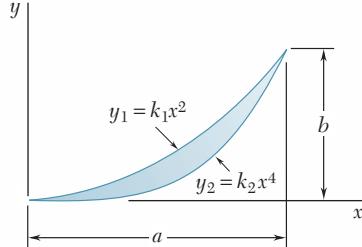
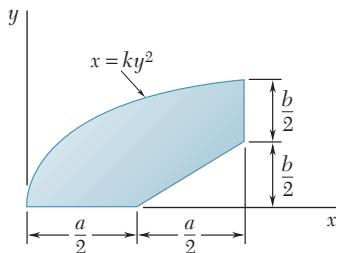


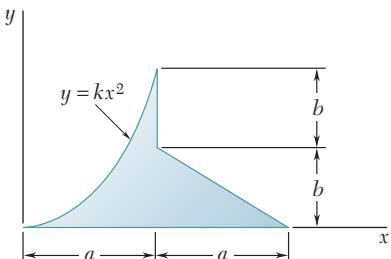
Fig. P5.41

**5.42** Determine by direct integration the centroid of the area shown.

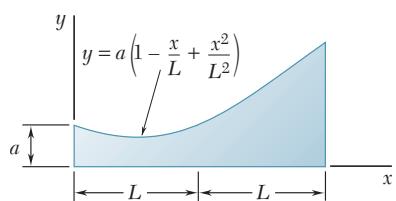
**5.43 and 5.44** Determine by direct integration the centroid of the area shown. Express your answer in terms of  $a$  and  $b$ .



**Fig. P5.43**

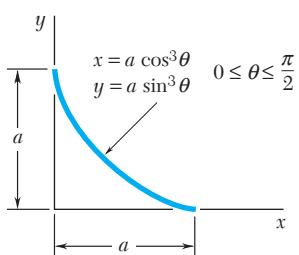


**Fig. P5.44**

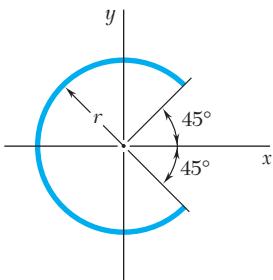


**Fig. P5.42**

**5.45 and 5.46** A homogeneous wire is bent into the shape shown. Determine by direct integration the  $x$  coordinate of its centroid.



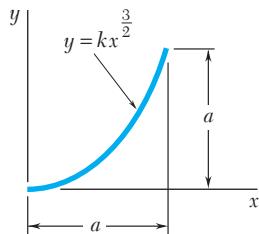
**Fig. P5.45**



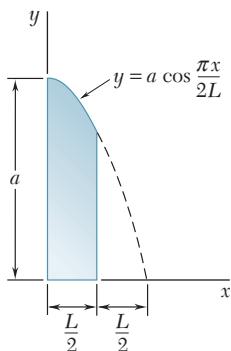
**Fig. P5.46**

**\*5.47** A homogeneous wire is bent into the shape shown. Determine by direct integration the  $x$  coordinate of its centroid. Express your answer in terms of  $a$ .

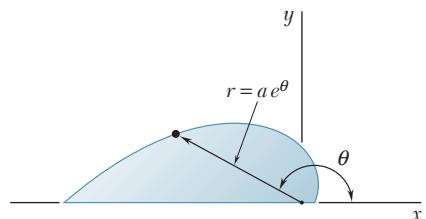
**\*5.48 and \*5.49** Determine by direct integration the centroid of the area shown.



**Fig. P5.47**



**Fig. P5.48**



**Fig. P5.49**

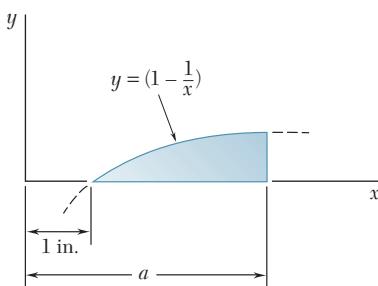


Fig. P5.50 and P5.51

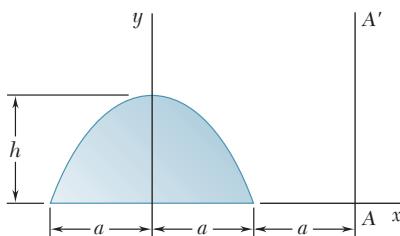


Fig. P5.55

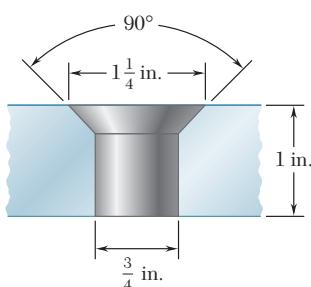


Fig. P5.58

**5.50** Determine the centroid of the area shown when  $a = 2$  in.

**5.51** Determine the value of  $a$  for which the ratio  $\bar{x}/\bar{y}$  is 9.

**5.52** Determine the volume and the surface area of the solid obtained by rotating the area of Prob. 5.1 about (a) the line  $x = 240$  mm, (b) the  $y$  axis.

**5.53** Determine the volume and the surface area of the solid obtained by rotating the area of Prob. 5.2 about (a) the line  $y = 60$  mm, (b) the  $y$  axis.

**5.54** Determine the volume and the surface area of the solid obtained by rotating the area of Prob. 5.8 about (a) the  $x$  axis, (b) the  $y$  axis.

**5.55** Determine the volume of the solid generated by rotating the parabolic area shown about (a) the  $x$  axis, (b) the axis  $AA'$ .

**5.56** Determine the volume and the surface area of the chain link shown, which is made from a 6-mm-diameter bar, if  $R = 10$  mm and  $L = 30$  mm.

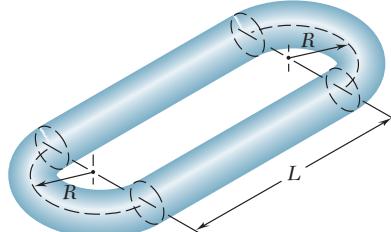


Fig. P5.56

**5.57** Verify that the expressions for the volumes of the first four shapes in Fig. 5.21 on page 260 are correct.

**5.58** A  $\frac{3}{4}$ -in.-diameter hole is drilled in a piece of 1-in.-thick steel; the hole is then countersunk as shown. Determine the volume of steel removed during the countersinking process.

**5.59** Determine the capacity, in liters, of the punch bowl shown if  $R = 250$  mm.

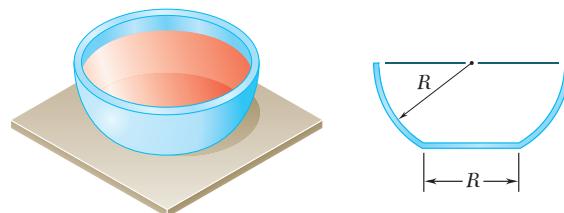


Fig. P5.59

- 5.60** Three different drive belt profiles are to be studied. If at any given time each belt makes contact with one-half of the circumference of its pulley, determine the *contact area* between the belt and the pulley for each design.

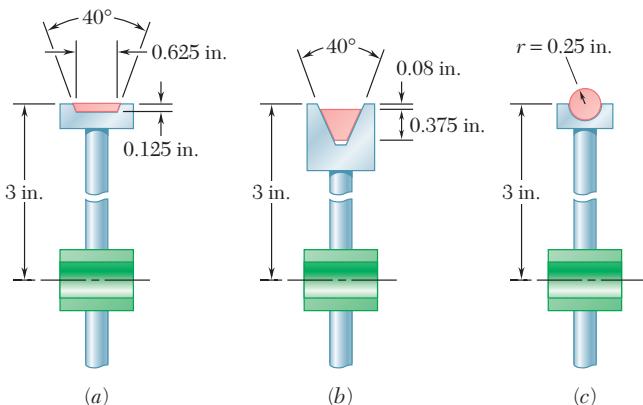


Fig. P5.60

- 5.61** The aluminum shade for the small high-intensity lamp shown has a uniform thickness of 1 mm. Knowing that the density of aluminum is  $2800 \text{ kg/m}^3$ , determine the mass of the shade.

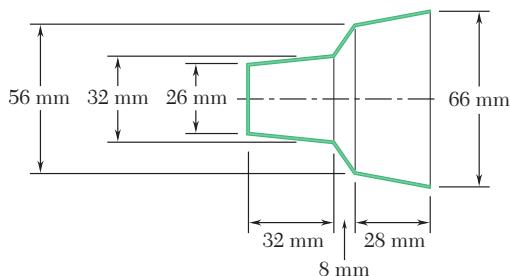
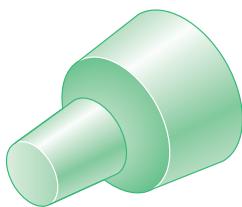


Fig. P5.61

- 5.62** The escutcheon (a decorative plate placed on a pipe where the pipe exits from a wall) shown is cast from brass. Knowing that the density of brass is  $8470 \text{ kg/m}^3$ , determine the mass of the escutcheon.

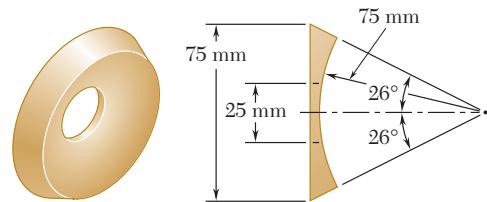


Fig. P5.62

- 5.63** A manufacturer is planning to produce 20,000 wooden pegs having the shape shown. Determine how many gallons of paint should be ordered, knowing that each peg will be given two coats of paint and that one gallon of paint covers  $100 \text{ ft}^2$ .

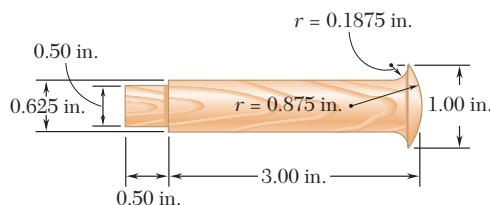


Fig. P5.63 and P5.64

- 5.64** The wooden peg shown is turned from a dowel 1 in. in diameter and 4 in. long. Determine the percentage of the initial volume of the dowel that becomes waste.

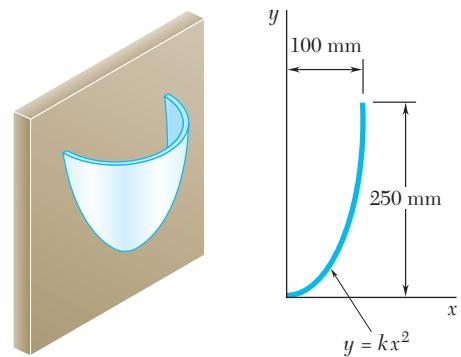


Fig. P5.65

- \*5.65** The shade for a wall-mounted light is formed from a thin sheet of translucent plastic. Determine the surface area of the outside of the shade, knowing that it has the parabolic cross section shown.

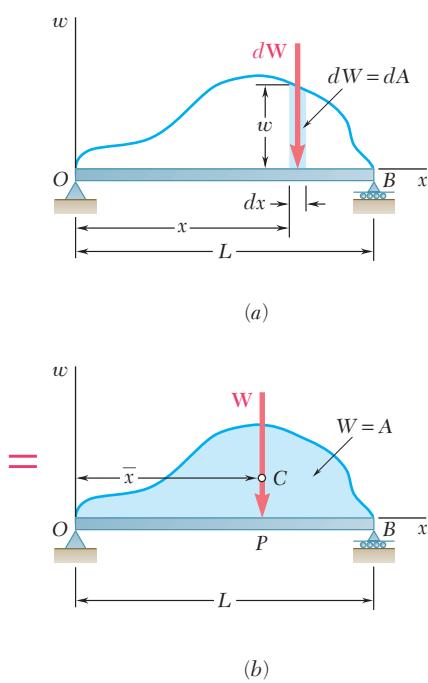


Fig. 5.17



**Photo 5.3** The roofs of the buildings shown must be able to support not only the total weight of the snow but also the nonsymmetric distributed loads resulting from drifting of the snow.

## \*5.8 DISTRIBUTED LOADS ON BEAMS

The concept of the centroid of an area can be used to solve other problems besides those dealing with the weights of flat plates. Consider, for example, a beam supporting a *distributed load*; this load may consist of the weight of materials supported directly or indirectly by the beam, or it may be caused by wind or hydrostatic pressure. The distributed load can be represented by plotting the load  $w$  supported per unit length (Fig. 5.17); this load is expressed in N/m or in lb/ft. The magnitude of the force exerted on an element of beam of length  $dx$  is  $dW = w dx$ , and the total load supported by the beam is

$$W = \int_0^L w dx$$

We observe that the product  $w dx$  is equal in magnitude to the element of area  $dA$  shown in Fig. 5.17a. The load  $W$  is thus equal in magnitude to the total area  $A$  under the load curve:

$$W = \int dA = A$$

We now determine where a *single concentrated load*  $\mathbf{W}$ , of the same magnitude  $W$  as the total distributed load, should be applied on the beam if it is to produce the same reactions at the supports (Fig. 5.17b). However, this concentrated load  $\mathbf{W}$ , which represents the resultant of the given distributed loading, is equivalent to the loading only when considering the free-body diagram of the entire beam. The point of application  $P$  of the equivalent concentrated load  $\mathbf{W}$  is obtained by expressing that the moment of  $\mathbf{W}$  about point  $O$  is equal to the sum of the moments of the elemental loads  $d\mathbf{W}$  about  $O$ :

$$(OP)W = \int x dW$$

or, since  $dW = w dx = dA$  and  $W = A$ ,

$$(OP)A = \int_0^L x dA \quad (5.12)$$

Since the integral represents the first moment with respect to the  $w$  axis of the area under the load curve, it can be replaced by the product  $\bar{x}A$ . We therefore have  $OP = \bar{x}$ , where  $\bar{x}$  is the distance from the  $w$  axis to the centroid  $C$  of the area  $A$  (this is *not* the centroid of the beam).

A *distributed load on a beam can thus be replaced by a concentrated load; the magnitude of this single load is equal to the area under the load curve, and its line of action passes through the centroid of that area*. It should be noted, however, that the concentrated load is equivalent to the given loading only as far as external forces are concerned. It can be used to determine reactions but should not be used to compute internal forces and deflections.

## \*5.9 FORCES ON SUBMERGED SURFACES

The approach used in the preceding section can be used to determine the resultant of the hydrostatic pressure forces exerted on a *rectangular surface* submerged in a liquid. Consider the rectangular plate shown in Fig. 5.18, which is of length  $L$  and width  $b$ , where  $b$  is measured perpendicular to the plane of the figure. As noted in Sec. 5.8, the load exerted on an element of the plate of length  $dx$  is  $w dx$ , where  $w$  is the load per unit length. However, this load can also be expressed as  $p dA = pb dx$ , where  $p$  is the gage pressure in the liquid† and  $b$  is the width of the plate; thus,  $w = bp$ . Since the gage pressure in a liquid is  $p = \gamma h$ , where  $\gamma$  is the specific weight of the liquid and  $h$  is the vertical distance from the free surface, it follows that

$$w = bp = b\gamma h \quad (5.13)$$

which shows that the load per unit length  $w$  is proportional to  $h$  and, thus, varies linearly with  $x$ .

Recalling the results of Sec. 5.8, we observe that the resultant  $\mathbf{R}$  of the hydrostatic forces exerted on one side of the plate is equal in magnitude to the trapezoidal area under the load curve and that its line of action passes through the centroid  $C$  of that area. The point  $P$  of the plate where  $\mathbf{R}$  is applied is known as the *center of pressure*.‡

Next, we consider the forces exerted by a liquid on a curved surface of constant width (Fig. 5.19a). Since the determination of the resultant  $\mathbf{R}$  of these forces by direct integration would not be easy, we consider the free body obtained by detaching the volume of liquid  $ABD$  bounded by the curved surface  $AB$  and by the two plane surfaces  $AD$  and  $DB$  shown in Fig. 5.19b. The forces acting on the free body  $ABD$  are the weight  $\mathbf{W}$  of the detached volume of liquid, the resultant  $\mathbf{R}_1$  of the forces exerted on  $AD$ , the resultant  $\mathbf{R}_2$  of the forces exerted on  $BD$ , and the resultant  $-\mathbf{R}$  of the forces exerted by the curved surface on the liquid. The resultant  $-\mathbf{R}$  is equal and opposite to, and has the same line of action as, the resultant  $\mathbf{R}$  of the forces exerted by the liquid on the curved surface. The forces  $\mathbf{W}$ ,  $\mathbf{R}_1$ , and  $\mathbf{R}_2$  can be determined by standard methods; after their values have been found, the force  $-\mathbf{R}$  is obtained by solving the equations of equilibrium for the free body of Fig. 5.19b. The resultant  $\mathbf{R}$  of the hydrostatic forces exerted on the curved surface is then obtained by reversing the sense of  $-\mathbf{R}$ .

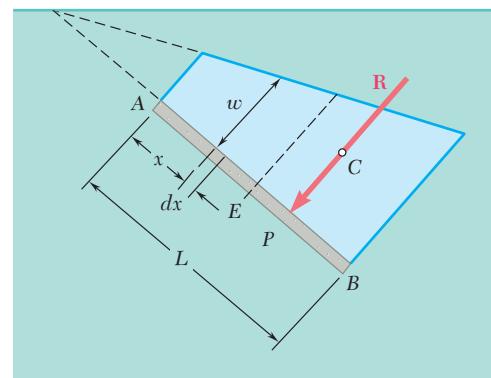
The methods outlined in this section can be used to determine the resultant of the hydrostatic forces exerted on the surfaces of dams and rectangular gates and vanes. The resultants of forces on submerged surfaces of variable width will be determined in Chap. 9.

†The pressure  $p$ , which represents a load per unit area, is expressed in  $\text{N/m}^2$  or in  $\text{lb/ft}^2$ . The derived SI unit  $\text{N/m}^2$  is called a *pascal* (Pa).

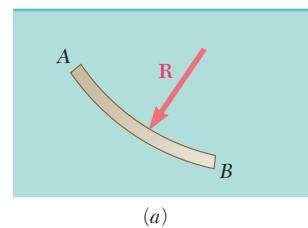
‡Noting that the area under the load curve is equal to  $w_E L$ , where  $w_E$  is the load per unit length at the center  $E$  of the plate, and recalling Eq. (5.13), we can write

$$R = w_E L = (bp_E)L = p_E(bL) = p_E A$$

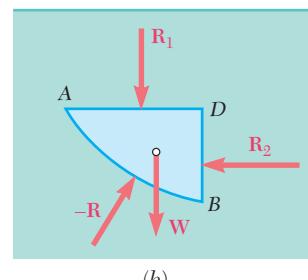
where  $A$  denotes the area of the *plate*. Thus, the magnitude of  $\mathbf{R}$  can be obtained by multiplying the area of the plate by the pressure at its center  $E$ . The resultant  $\mathbf{R}$ , however, should be applied at  $P$ , not at  $E$ .



**Fig. 5.18**

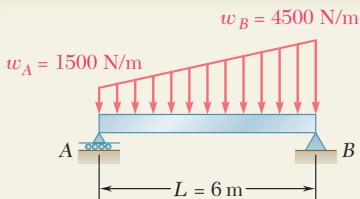


(a)



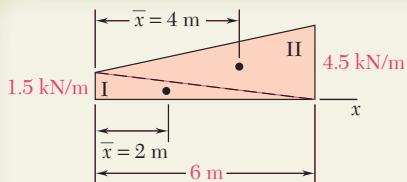
(b)

**Fig. 5.19**



## SAMPLE PROBLEM 5.9

A beam supports a distributed load as shown. (a) Determine the equivalent concentrated load. (b) Determine the reactions at the supports.



## SOLUTION

**a. Equivalent Concentrated Load.** The magnitude of the resultant of the load is equal to the area under the load curve, and the line of action of the resultant passes through the centroid of the same area. We divide the area under the load curve into two triangles and construct the table below. To simplify the computations and tabulation, the given loads per unit length have been converted into kN/m.

Component	$A, \text{kN}$	$\bar{x}, \text{m}$	$\bar{x}A, \text{kN} \cdot \text{m}$
Triangle I	4.5	2	9
	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$

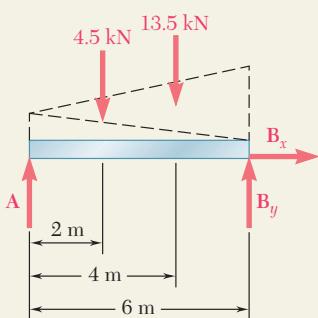
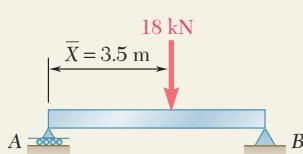
$$\text{Thus, } \bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(18 \text{ kN}) = 63 \text{ kN} \cdot \text{m} \quad \bar{X} = 3.5 \text{ m}$$

The equivalent concentrated load is

$$\mathbf{W} = 18 \text{ kN} \downarrow$$

and its line of action is located at a distance

$$\bar{X} = 3.5 \text{ m to the right of A}$$



**b. Reactions.** The reaction at A is vertical and is denoted by  $\mathbf{A}$ ; the reaction at B is represented by its components  $\mathbf{B}_x$  and  $\mathbf{B}_y$ . The given load can be considered to be the sum of two triangular loads as shown. The resultant of each triangular load is equal to the area of the triangle and acts at its centroid. We write the following equilibrium equations for the free body shown:

$$+\sum F_x = 0:$$

$$\mathbf{B}_x = 0$$

$$+\gamma \sum M_A = 0: \quad -(4.5 \text{ kN})(2 \text{ m}) - (13.5 \text{ kN})(4 \text{ m}) + B_y(6 \text{ m}) = 0$$

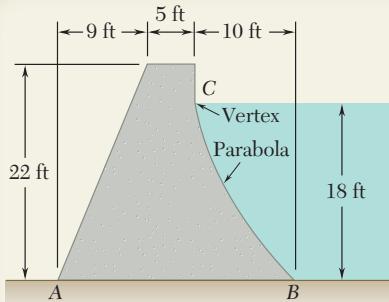
$$\mathbf{B}_y = 10.5 \text{ kN} \uparrow$$

$$+\gamma \sum M_B = 0: \quad +(4.5 \text{ kN})(4 \text{ m}) + (13.5 \text{ kN})(2 \text{ m}) - A(6 \text{ m}) = 0$$

$$\mathbf{A} = 7.5 \text{ kN} \uparrow$$

**Alternative Solution.** The given distributed load can be replaced by its resultant, which was found in part a. The reactions can be determined by writing the equilibrium equations  $\sum F_x = 0$ ,  $\sum M_A = 0$ , and  $\sum M_B = 0$ . We again obtain

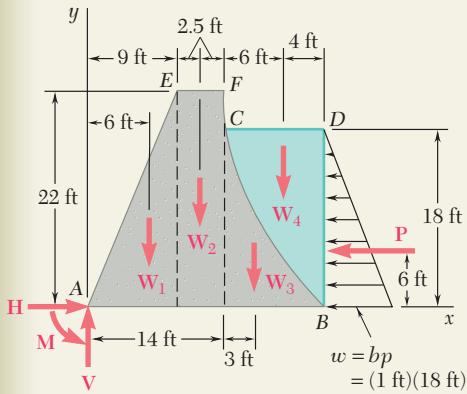
$$\mathbf{B}_x = 0 \quad \mathbf{B}_y = 10.5 \text{ kN} \uparrow \quad \mathbf{A} = 7.5 \text{ kN} \uparrow$$



## SAMPLE PROBLEM 5.10

The cross section of a concrete dam is as shown. Consider a 1-ft-thick section of the dam, and determine (a) the resultant of the reaction forces exerted by the ground on the base  $AB$  of the dam, (b) the resultant of the pressure forces exerted by the water on the face  $BC$  of the dam. The specific weights of concrete and water are  $150 \text{ lb}/\text{ft}^3$  and  $62.4 \text{ lb}/\text{ft}^3$ , respectively.

## SOLUTION



**a. Ground Reaction.** We choose as a free body the 1-ft-thick section  $AECFCDDB$  of the dam and water. The reaction forces exerted by the ground on the base  $AB$  are represented by an equivalent force-couple system at  $A$ . Other forces acting on the free body are the weight of the dam, represented by the weights of its components  $\mathbf{W}_1$ ,  $\mathbf{W}_2$ , and  $\mathbf{W}_3$ ; the weight of the water  $\mathbf{W}_4$ ; and the resultant  $\mathbf{P}$  of the pressure forces exerted on section  $BD$  by the water to the right of section  $BD$ . We have

$$\begin{aligned} W_1 &= \frac{1}{2}(9 \text{ ft})(22 \text{ ft})(1 \text{ ft})(150 \text{ lb}/\text{ft}^3) = 14,850 \text{ lb} \\ W_2 &= (5 \text{ ft})(22 \text{ ft})(1 \text{ ft})(150 \text{ lb}/\text{ft}^3) = 16,500 \text{ lb} \\ W_3 &= \frac{1}{3}(10 \text{ ft})(18 \text{ ft})(1 \text{ ft})(150 \text{ lb}/\text{ft}^3) = 9000 \text{ lb} \\ W_4 &= \frac{2}{3}(10 \text{ ft})(18 \text{ ft})(1 \text{ ft})(62.4 \text{ lb}/\text{ft}^3) = 7488 \text{ lb} \\ w &= bp = (1 \text{ ft})(18 \text{ ft})(62.4 \text{ lb}/\text{ft}^3) \\ P &= \frac{1}{2}(18 \text{ ft})(1 \text{ ft})(18 \text{ ft})(62.4 \text{ lb}/\text{ft}^3) = 10,109 \text{ lb} \end{aligned}$$

### Equilibrium Equations

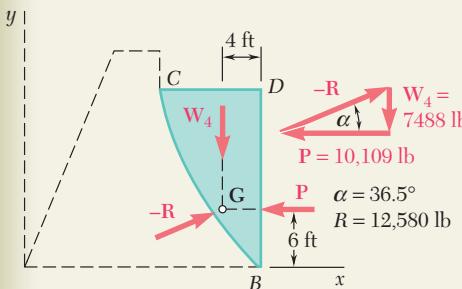
$$\begin{aligned} \stackrel{+}{\rightarrow} \sum F_x &= 0: \quad H - 10,109 \text{ lb} = 0 & H &= 10,110 \text{ lb} \rightarrow \\ +\uparrow \sum F_y &= 0: \quad V - 14,850 \text{ lb} - 16,500 \text{ lb} - 9000 \text{ lb} - 7488 \text{ lb} = 0 & V &= 47,840 \text{ lb} \uparrow \\ +\uparrow \sum M_A &= 0: \quad -(14,850 \text{ lb})(6 \text{ ft}) - (16,500 \text{ lb})(11.5 \text{ ft}) \\ &\quad - (9000 \text{ lb})(17 \text{ ft}) - (7488 \text{ lb})(20 \text{ ft}) + (10,109 \text{ lb})(6 \text{ ft}) + M = 0 & M &= 520,960 \text{ lb} \cdot \text{ft} \downarrow \end{aligned}$$

We can replace the force-couple system obtained by a single force acting at a distance  $d$  to the right of  $A$ , where

$$d = \frac{520,960 \text{ lb} \cdot \text{ft}}{47,840 \text{ lb}} = 10.89 \text{ ft}$$

**b. Resultant  $\mathbf{R}$  of Water Forces.** The parabolic section of water  $BCD$  is chosen as a free body. The forces involved are the resultant  $-\mathbf{R}$  of the forces exerted by the dam on the water, the weight  $\mathbf{W}_4$ , and the force  $\mathbf{P}$ . Since these forces must be concurrent,  $-\mathbf{R}$  passes through the point of intersection  $G$  of  $\mathbf{W}_4$  and  $\mathbf{P}$ . A force triangle is drawn from which the magnitude and direction of  $-\mathbf{R}$  are determined. The resultant  $\mathbf{R}$  of the forces exerted by the water on the face  $BC$  is equal and opposite:

$$\mathbf{R} = 12,580 \text{ lb} \angle 36.5^\circ \quad \blacktriangleleft$$



# SOLVING PROBLEMS ON YOUR OWN

The problems in this lesson involve two common and very important types of loading: distributed loads on beams and forces on submerged surfaces of constant width. As we discussed in Secs. 5.8 and 5.9 and illustrated in Sample Probs. 5.9 and 5.10, determining the single equivalent force for each of these loadings requires a knowledge of centroids.

**1. Analyzing beams subjected to distributed loads.** In Sec. 5.8, we showed that a distributed load on a beam can be replaced by a single equivalent force. The magnitude of this force is equal to the area under the distributed load curve and its line of action passes through the centroid of that area. Thus, you should begin your solution by replacing the various distributed loads on a given beam by their respective single equivalent forces. The reactions at the supports of the beam can then be determined by using the methods of Chap. 4.

When possible, complex distributed loads should be divided into the common-shape areas shown in Fig. 5.8A [Sample Prob. 5.9]. Each of these areas can then be replaced by a single equivalent force. If required, the system of equivalent forces can be reduced further to a single equivalent force. As you study Sample Prob. 5.9, note how we have used the analogy between force and area and the techniques for locating the centroid of a composite area to analyze a beam subjected to a distributed load.

**2. Solving problems involving forces on submerged bodies.** The following points and techniques should be remembered when solving problems of this type.

a. The pressure  $p$  at a depth  $h$  below the free surface of a liquid is equal to  $\gamma h$  or  $\rho gh$ , where  $\gamma$  and  $\rho$  are the specific weight and the density of the liquid, respectively. The load per unit length  $w$  acting on a submerged surface of constant width  $b$  is then

$$w = bp = b\gamma h = b\rho gh$$

b. The line of action of the resultant force  $\mathbf{R}$  acting on a submerged plane surface is perpendicular to the surface.

c. For a vertical or inclined plane rectangular surface of width  $b$ , the loading on the surface can be represented by a linearly distributed load which is trapezoidal in shape (Fig. 5.18). Further, the magnitude of  $\mathbf{R}$  is given by

$$R = \gamma h_E A$$

where  $h_E$  is the vertical distance to the center of the surface and  $A$  is the area of the surface.

**d.** The load curve will be triangular (rather than trapezoidal) when the top edge of a plane rectangular surface coincides with the free surface of the liquid, since the pressure of the liquid at the free surface is zero. For this case, the line of action of  $\mathbf{R}$  is easily determined, for it passes through the centroid of a *triangular* distributed load.

**e.** For the general case, rather than analyzing a trapezoid, we suggest that you use the method indicated in part *b* of Sample Prob. 5.9. First divide the trapezoidal distributed load into two triangles, and then compute the magnitude of the resultant of each triangular load. (The magnitude is equal to the area of the triangle times the width of the plate.) Note that the line of action of each resultant force passes through the centroid of the corresponding triangle and that the sum of these forces is equivalent to  $\mathbf{R}$ . Thus, rather than using  $\mathbf{R}$ , you can use the two equivalent resultant forces, whose points of application are easily calculated. Of course, the equation given for  $R$  in paragraph *c* should be used when only the magnitude of  $\mathbf{R}$  is needed.

**f.** When the submerged surface of constant width is curved, the resultant force acting on the surface is obtained by considering the equilibrium of the volume of liquid bounded by the curved surface and by horizontal and vertical planes (Fig. 5.19). Observe that the force  $\mathbf{R}_1$  of Fig. 5.19 is equal to the weight of the liquid lying above the plane  $AD$ . The method of solution for problems involving curved surfaces is shown in part *b* of Sample Prob. 5.10.

In subsequent mechanics courses (in particular, mechanics of materials and fluid mechanics), you will have ample opportunity to use the ideas introduced in this lesson.

# PROBLEMS

**5.66 and 5.67** For the beam and loading shown, determine (a) the magnitude and location of the resultant of the distributed load, (b) the reactions at the beam supports.

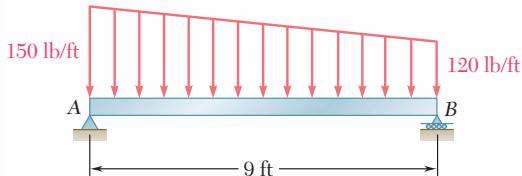


Fig. P5.66

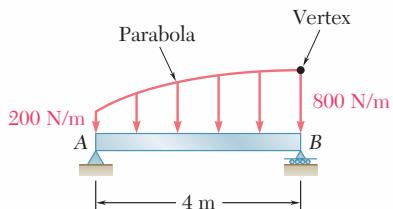


Fig. P5.67

**5.68 through 5.73** Determine the reactions at the beam supports for the given loading.

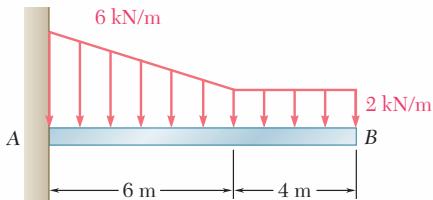


Fig. P5.68

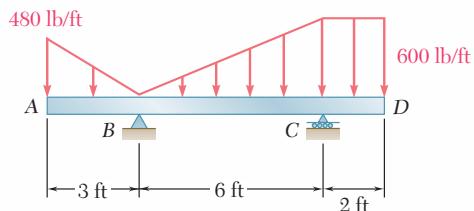


Fig. P5.69

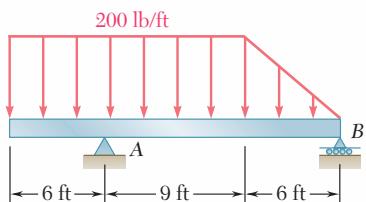


Fig. P5.70

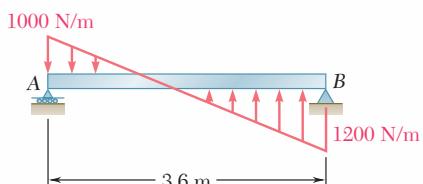


Fig. P5.71

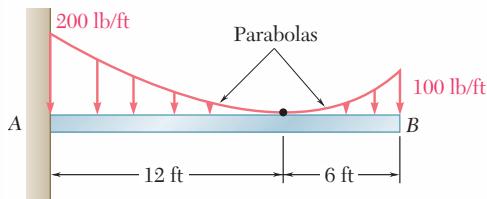


Fig. P5.72

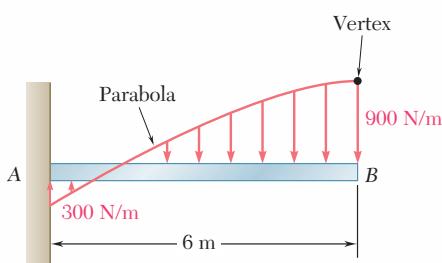


Fig. P5.73

**5.74** Determine (a) the distance  $a$  so that the vertical reactions at supports  $A$  and  $B$  are equal, (b) the corresponding reactions at the supports.

**5.75** Determine (a) the distance  $a$  so that the reaction at support  $B$  is minimum, (b) the corresponding reactions at the supports.

**5.76** Determine the reactions at the beam supports for the given loading when  $w_0 = 150 \text{ lb/ft}$ .

**5.77** Determine (a) the distributed load  $w_0$  at the end  $D$  of the beam  $ABCD$  for which the reaction at  $B$  is zero, (b) the corresponding reaction at  $C$ .

**5.78** The beam  $AB$  supports two concentrated loads and rests on soil that exerts a linearly distributed upward load as shown. Determine the values of  $w_A$  and  $w_B$  corresponding to equilibrium.

**5.79** For the beam and loading of Prob. 5.78, determine (a) the distance  $a$  for which  $w_A = 20 \text{ kN/m}$ , (b) the corresponding value of  $w_B$ .

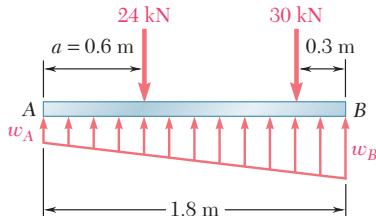


Fig. P5.78

In the following problems, use  $\gamma = 62.4 \text{ lb/ft}^3$  for the specific weight of fresh water and  $\gamma_c = 150 \text{ lb/ft}^3$  for the specific weight of concrete if U.S. customary units are used. With SI units, use  $\rho = 10^3 \text{ kg/m}^3$  for the density of fresh water and  $\rho_c = 2.40 \times 10^3 \text{ kg/m}^3$  for the density of concrete. (See the footnote on page 222 for how to determine the specific weight of a material given its density.)

**5.80** The cross section of a concrete dam is as shown. For a 1-m-wide dam section determine (a) the resultant of the reaction forces exerted by the ground on the base  $AB$  of the dam, (b) the point of application of the resultant of part a, (c) the resultant of the pressure forces exerted by the water on the face  $BC$  of the dam.

**5.81** The cross section of a concrete dam is as shown. For a 1-ft-wide dam section determine (a) the resultant of the reaction forces exerted by the ground on the base  $AB$  of the dam, (b) the point of application of the resultant of part a, (c) the resultant of the pressure forces exerted by the water on the face  $BC$  of the dam.

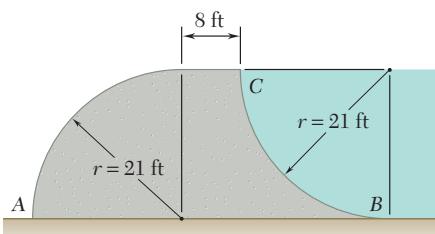


Fig. P5.81

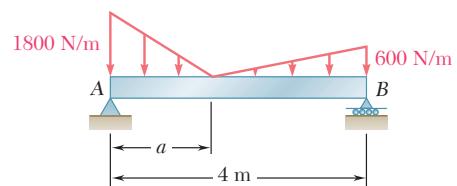


Fig. P5.74 and P5.75

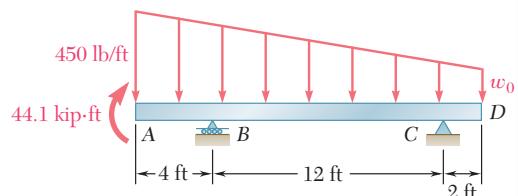


Fig. P5.76 and P5.77

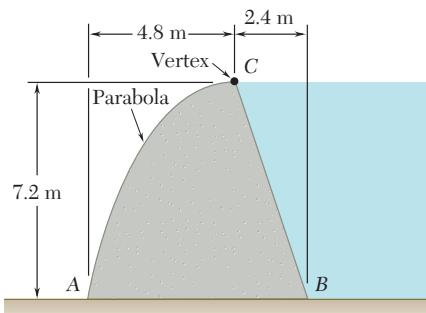


Fig. P5.80

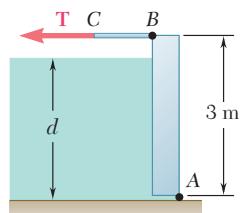


Fig. P5.82 and P5.83

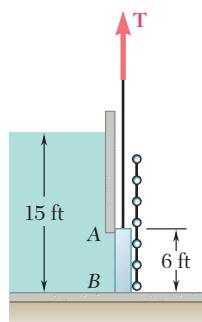


Fig. P5.84

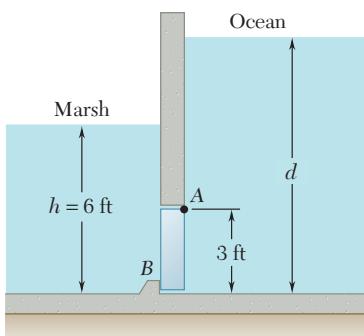


Fig. P5.85

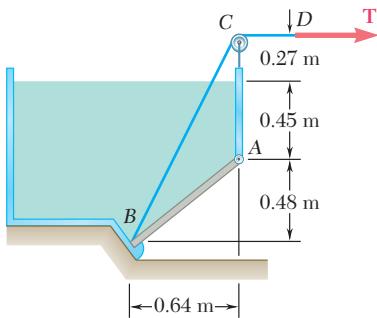


Fig. P5.88 and P5.89

- 5.82** The  $3 \times 4\text{-m}$  side  $AB$  of a tank is hinged at its bottom  $A$  and is held in place by a thin rod  $BC$ . The maximum tensile force the rod can withstand without breaking is 200 kN, and the design specifications require the force in the rod not to exceed 20 percent of this value. If the tank is slowly filled with water, determine the maximum allowable depth of water  $d$  in the tank.

- 5.83** The  $3 \times 4\text{-m}$  side of an open tank is hinged at its bottom  $A$  and is held in place by a thin rod  $BC$ . The tank is to be filled with glycerine, whose density is  $1263 \text{ kg/m}^3$ . Determine the force  $\mathbf{T}$  in the rod and the reactions at the hinge after the tank is filled to a depth of 2.9 m.

- 5.84** The friction force between a  $6 \times 6\text{-ft}$  square sluice gate  $AB$  and its guides is equal to 10 percent of the resultant of the pressure forces exerted by the water on the face of the gate. Determine the initial force needed to lift the gate if it weighs 1000 lb.

- 5.85** A freshwater marsh is drained to the ocean through an automatic tide gate that is 4 ft wide and 3 ft high. The gate is held by hinges located along its top edge at  $A$  and bears on a sill at  $B$ . If the water level in the marsh is  $h = 6$  ft, determine the ocean level  $d$  for which the gate will open. (Specific weight of salt water =  $64 \text{ lb/ft}^3$ .)

- 5.86** The dam for a lake is designed to withstand the additional force caused by silt that has settled on the lake bottom. Assuming that silt is equivalent to a liquid of density  $\rho_s = 1.76 \times 10^3 \text{ kg/m}^3$  and considering a 1-m-wide section of dam, determine the percentage increase in the force acting on the dam face for a silt accumulation of depth 2 m.

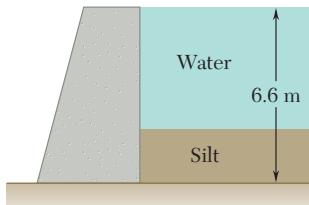


Fig. P5.86 and P5.87

- 5.87** The base of a dam for a lake is designed to resist up to 120 percent of the horizontal force of the water. After construction, it is found that silt (that is equivalent to a liquid of density  $\rho_s = 1.76 \times 10^3 \text{ kg/m}^3$ ) is settling on the lake bottom at the rate of 12 mm/year. Considering a 1-m-wide section of dam, determine the number of years until the dam becomes unsafe.

- 5.88** A  $0.5 \times 0.8\text{-m}$  gate  $AB$  is located at the bottom of a tank filled with water. The gate is hinged along its top edge  $A$  and rests on a frictionless stop at  $B$ . Determine the reactions at  $A$  and  $B$  when cable  $BCD$  is slack.

- 5.89** A  $0.5 \times 0.8\text{-m}$  gate  $AB$  is located at the bottom of a tank filled with water. The gate is hinged along its top edge  $A$  and rests on a frictionless stop at  $B$ . Determine the minimum tension required in cable  $BCD$  to open the gate.

- 5.90** A long trough is supported by a continuous hinge along its lower edge and by a series of horizontal cables attached to its upper edge. Determine the tension in each of the cables, at a time when the trough is completely full of water.

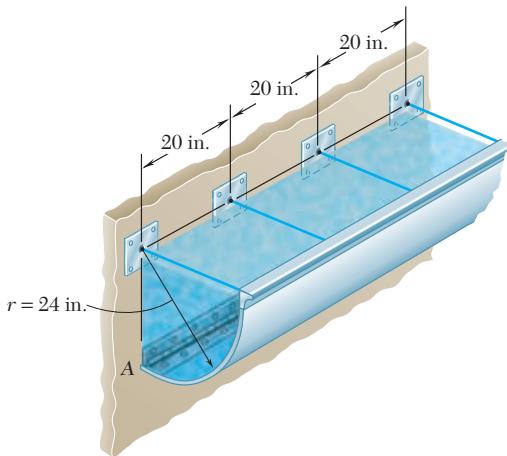


Fig. P5.90

- 5.91** A  $4 \times 2$ -ft gate is hinged at A and is held in position by rod CD. End D rests against a spring whose constant is 828 lb/ft. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod CD on the gate remains horizontal, determine the minimum depth of water  $d$  for which the bottom B of the gate will move to the end of the cylindrical portion of the floor.

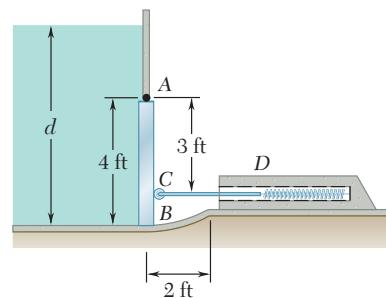


Fig. P5.91

- 5.92** Solve Prob. 5.91 if the gate weighs 1000 lb.

- 5.93** A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at A and rests on a frictionless support at B. The pin is located at a distance  $h = 0.10$  m below the center of gravity C of the gate. Determine the depth of water  $d$  for which the gate will open.

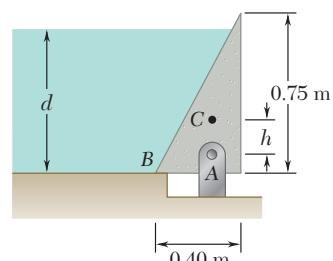


Fig. P5.93 and P5.94

- 5.94** A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at A and rests on a frictionless support at B. Determine the distance  $h$  if the gate is to open when  $d = 0.75$  m.

- 5.95** A 55-gallon 23-in.-diameter drum is placed on its side to act as a dam in a 30-in.-wide freshwater channel. Knowing that the drum is anchored to the sides of the channel, determine the resultant of the pressure forces acting on the drum.



Fig. P5.95

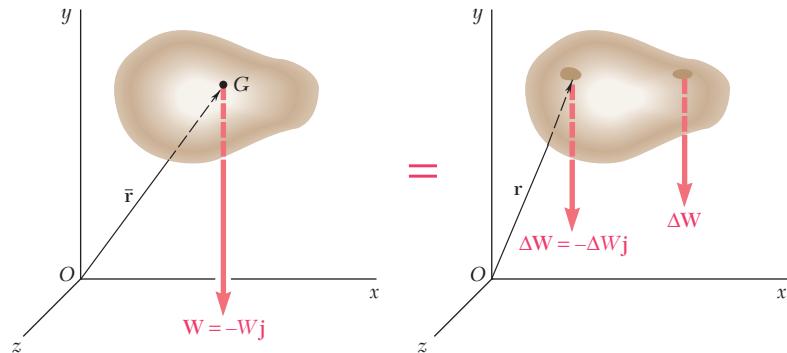


**Photo 5.4** To predict the flight characteristics of the modified Boeing 747 when used to transport a space shuttle, the center of gravity of each craft had to be determined.

## VOLUMES

### 5.10 CENTER OF GRAVITY OF A THREE-DIMENSIONAL BODY. CENTROID OF A VOLUME

The *center of gravity*  $G$  of a three-dimensional body is obtained by dividing the body into small elements and by then expressing that the weight  $\mathbf{W}$  of the body acting at  $G$  is equivalent to the system of distributed forces  $\Delta\mathbf{W}$  representing the weights of the small elements. Choosing the  $y$  axis to be vertical with positive sense upward (Fig. 5.20) and denoting by  $\bar{\mathbf{r}}$  the position vector of  $G$ , we write that



**Fig. 5.20**

$\mathbf{W}$  is equal to the sum of the elemental weights  $\Delta\mathbf{W}$  and that its moment about  $O$  is equal to the sum of the moments about  $O$  of the elemental weights:

$$\begin{aligned} \Sigma\mathbf{F}: \quad & -W\mathbf{j} = \Sigma(-\Delta W\mathbf{j}) \\ \Sigma\mathbf{M}_O: \quad & \bar{\mathbf{r}} \times (-W\mathbf{j}) = \Sigma[\mathbf{r} \times (-\Delta W\mathbf{j})] \end{aligned} \quad (5.14)$$

Rewriting the last equation in the form

$$\bar{\mathbf{r}}\mathbf{W} \times (-\mathbf{j}) = (\Sigma\mathbf{r} \Delta W) \times (-\mathbf{j}) \quad (5.15)$$

we observe that the weight  $\mathbf{W}$  of the body is equivalent to the system of the elemental weights  $\Delta\mathbf{W}$  if the following conditions are satisfied:

$$\mathbf{W} = \Sigma \Delta\mathbf{W} \quad \bar{\mathbf{r}}\mathbf{W} = \Sigma \mathbf{r} \Delta W$$

Increasing the number of elements and simultaneously decreasing the size of each element, we obtain in the limit

$$\mathbf{W} = \int d\mathbf{W} \quad \bar{\mathbf{r}}\mathbf{W} = \int \mathbf{r} d\mathbf{W} \quad (5.16)$$

We note that the relations obtained are independent of the orientation of the body. For example, if the body and the coordinate axes were rotated so that the  $z$  axis pointed upward, the unit vector  $-\mathbf{j}$  would be replaced by  $-\mathbf{k}$  in Eqs. (5.14) and (5.15), but the relations (5.16) would remain unchanged. Resolving the vectors  $\bar{\mathbf{r}}$  and  $\mathbf{r}$  into rectangular components, we note that the second of the relations (5.16) is equivalent to the three scalar equations

$$\bar{x}\mathbf{W} = \int x d\mathbf{W} \quad \bar{y}\mathbf{W} = \int y d\mathbf{W} \quad \bar{z}\mathbf{W} = \int z d\mathbf{W} \quad (5.17)$$

If the body is made of a homogeneous material of specific weight  $\gamma$ , the magnitude  $dW$  of the weight of an infinitesimal element can be expressed in terms of the volume  $dV$  of the element, and the magnitude  $W$  of the total weight can be expressed in terms of the total volume  $V$ . We write

$$dW = \gamma dV \quad W = \gamma V$$

Substituting for  $dW$  and  $W$  in the second of the relations (5.16), we write

$$\bar{\mathbf{r}}V = \int \mathbf{r} dV \quad (5.18)$$

or, in scalar form,

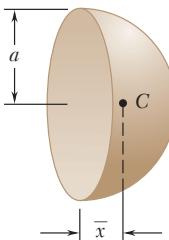
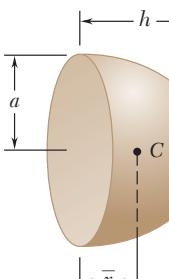
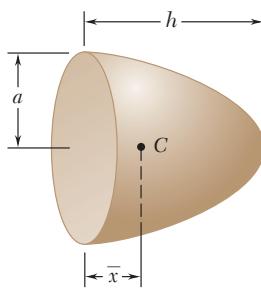
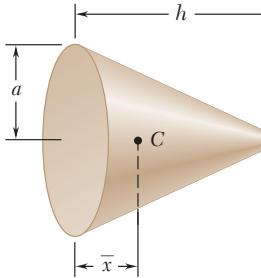
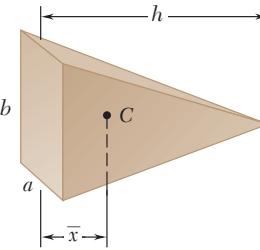
$$\bar{x}V = \int x dV \quad \bar{y}V = \int y dV \quad \bar{z}V = \int z dV \quad (5.19)$$

The point whose coordinates are  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$  is also known as the *centroid*  $C$  of the volume  $V$  of the body. If the body is not homogeneous, Eqs. (5.19) cannot be used to determine the center of gravity of the body; however, Eqs. (5.19) still define the centroid of the volume.

The integral  $\int x dV$  is known as the *first moment of the volume with respect to the  $yz$  plane*. Similarly, the integrals  $\int y dV$  and  $\int z dV$  define the first moments of the volume with respect to the  $zx$  plane and the  $xy$  plane, respectively. It is seen from Eqs. (5.19) that if the centroid of a volume is located in a coordinate plane, the first moment of the volume with respect to that plane is zero.

A volume is said to be symmetrical with respect to a given plane if for every point  $P$  of the volume there exists a point  $P'$  of the same volume, such that the line  $PP'$  is perpendicular to the given plane and is bisected by that plane. The plane is said to be a *plane of symmetry* for the given volume. When a volume  $V$  possesses a plane of symmetry, the first moment of  $V$  with respect to that plane is zero, and the centroid of the volume is located in the plane of symmetry. When a volume possesses two planes of symmetry, the centroid of the volume is located on the line of intersection of the two planes. Finally, when a volume possesses three planes of symmetry which intersect at a well-defined point (i.e., not along a common line), the point of intersection of the three planes coincides with the centroid of the volume. This property enables us to determine immediately the locations of the centroids of spheres, ellipsoids, cubes, rectangular parallelepipeds, etc.

The centroids of unsymmetrical volumes or of volumes possessing only one or two planes of symmetry should be determined by integration (Sec. 5.12). The centroids of several common volumes are shown in Fig. 5.21. It should be observed that in general the centroid of a volume of revolution does not coincide with the centroid of its cross section. Thus, the centroid of a hemisphere is different from that of a semicircular area, and the centroid of a cone is different from that of a triangle.

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

**Fig. 5.21** Centroids of common shapes and volumes.

## 5.11 COMPOSITE BODIES

If a body can be divided into several of the common shapes shown in Fig. 5.21, its center of gravity  $G$  can be determined by expressing that the moment about  $O$  of its total weight is equal to the sum of the moments about  $O$  of the weights of the various component parts. Proceeding as in Sec. 5.10, we obtain the following equations defining the coordinates  $\bar{X}$ ,  $\bar{Y}$ ,  $\bar{Z}$  of the center of gravity  $G$ .

$$\bar{X}\Sigma W = \Sigma \bar{x}W \quad \bar{Y}\Sigma W = \Sigma \bar{y}W \quad \bar{Z}\Sigma W = \Sigma \bar{z}W \quad (5.20)$$

If the body is made of a homogeneous material, its center of gravity coincides with the centroid of its volume, and we obtain:

$$\bar{X}\Sigma V = \Sigma \bar{x}V \quad \bar{Y}\Sigma V = \Sigma \bar{y}V \quad \bar{Z}\Sigma V = \Sigma \bar{z}V \quad (5.21)$$

## 5.12 DETERMINATION OF CENTROIDS OF VOLUMES BY INTEGRATION

The centroid of a volume bounded by analytical surfaces can be determined by evaluating the integrals given in Sec. 5.10:

$$\bar{x}V = \int x dV \quad \bar{y}V = \int y dV \quad \bar{z}V = \int z dV \quad (5.22)$$

If the element of volume  $dV$  is chosen to be equal to a small cube of sides  $dx$ ,  $dy$ , and  $dz$ , the evaluation of each of these integrals requires a *triple integration*. However, it is possible to determine the coordinates of the centroid of most volumes by *double integration* if  $dV$  is chosen to be equal to the volume of a thin filament (Fig. 5.22). The coordinates of the centroid of the volume are then obtained by rewriting Eqs. (5.22) as

$$\bar{x}V = \int \bar{x}_{el} dV \quad \bar{y}V = \int \bar{y}_{el} dV \quad \bar{z}V = \int \bar{z}_{el} dV \quad (5.23)$$

and by then substituting the expressions given in Fig. 5.22 for the volume  $dV$  and the coordinates  $\bar{x}_{el}$ ,  $\bar{y}_{el}$ ,  $\bar{z}_{el}$ . By using the equation of the surface to express  $z$  in terms of  $x$  and  $y$ , the integration is reduced to a double integration in  $x$  and  $y$ .

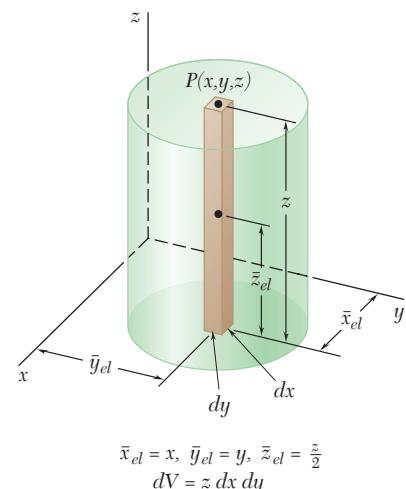
If the volume under consideration possesses *two planes of symmetry*, its centroid must be located on the line of intersection of the two planes. Choosing the  $x$  axis to lie along this line, we have

$$\bar{y} = \bar{z} = 0$$

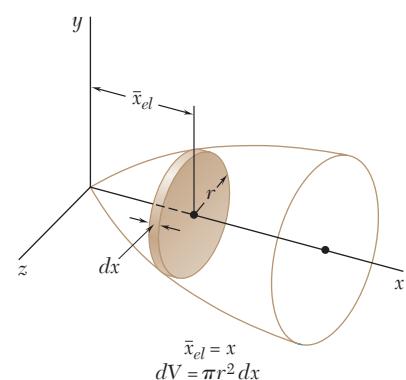
and the only coordinate to determine is  $\bar{x}$ . This can be done with a *single integration* by dividing the given volume into thin slabs parallel to the  $yz$  plane and expressing  $dV$  in terms of  $x$  and  $dx$  in the equation

$$\bar{x}V = \int \bar{x}_{el} dV \quad (5.24)$$

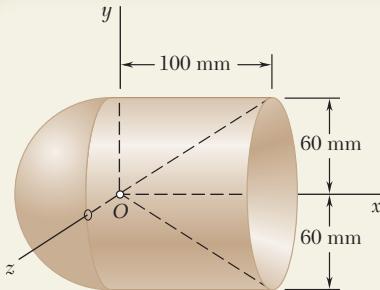
For a body of revolution, the slabs are circular and their volume is given in Fig. 5.23.



**Fig. 5.22** Determination of the centroid of a volume by double integration.



**Fig. 5.23** Determination of the centroid of a body of revolution.

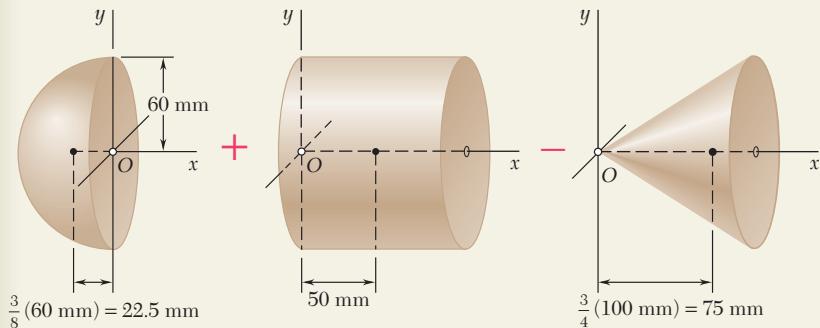


## SAMPLE PROBLEM 5.11

Determine the location of the center of gravity of the homogeneous body of revolution shown, which was obtained by joining a hemisphere and a cylinder and carving out a cone.

## SOLUTION

Because of symmetry, the center of gravity lies on the  $x$  axis. As shown in the figure below, the body can be obtained by adding a hemisphere to a cylinder and then subtracting a cone. The volume and the abscissa of the centroid of each of these components are obtained from Fig. 5.21 and are entered in the table below. The total volume of the body and the first moment of its volume with respect to the  $yz$  plane are then determined.



Component	Volume, $\text{mm}^3$	$\bar{x}$ , mm	$\bar{x}V$ , $\text{mm}^4$
Hemisphere	$\frac{1}{2} \frac{4\pi}{3} (60)^3 = 0.4524 \times 10^6$	-22.5	$-10.18 \times 10^6$
Cylinder	$\pi(60)^2(100) = 1.1310 \times 10^6$	+50	$+56.55 \times 10^6$
Cone	$-\frac{\pi}{3} (60)^2(100) = -0.3770 \times 10^6$	+75	$-28.28 \times 10^6$
$\Sigma V = 1.206 \times 10^6$			$\Sigma \bar{x}V = +18.09 \times 10^6$

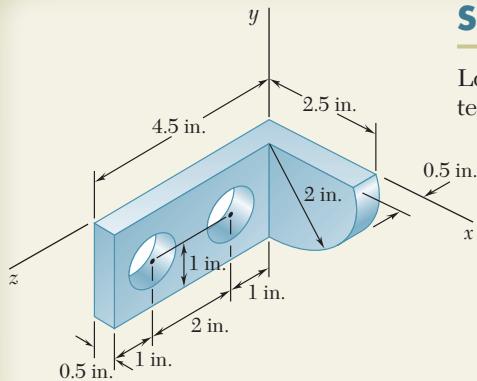
Thus,

$$\bar{X}\Sigma V = \Sigma \bar{x}V; \quad \bar{X}(1.206 \times 10^6 \text{ mm}^3) = 18.09 \times 10^6 \text{ mm}^4$$

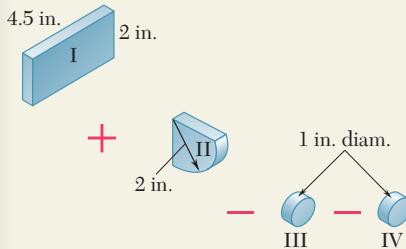
$$\bar{X} = 15 \text{ mm}$$

## SAMPLE PROBLEM 5.12

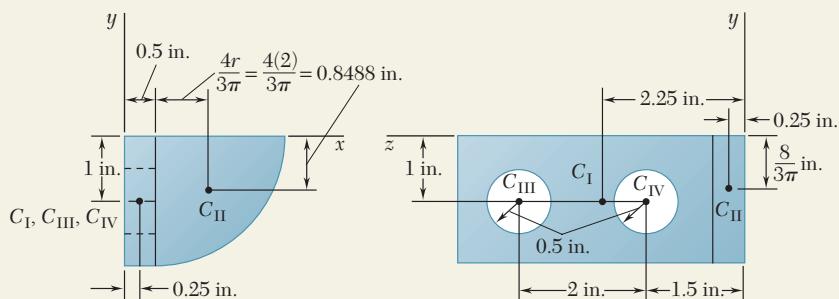
Locate the center of gravity of the steel machine element shown. The diameter of each hole is 1 in.



## SOLUTION



The machine element can be obtained by adding a rectangular parallelepiped (I) to a quarter cylinder (II) and then subtracting two 1-in.-diameter cylinders (III and IV). The volume and the coordinates of the centroid of each component are determined and are entered in the table below. Using the data in the table, we then determine the total volume and the moments of the volume with respect to each of the coordinate planes.

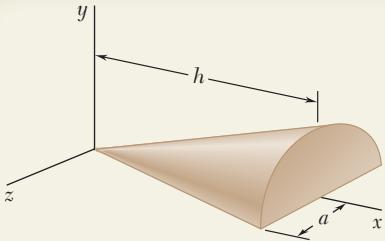


	$V, \text{ in}^3$	$\bar{x}, \text{ in.}$	$\bar{y}, \text{ in.}$	$\bar{z}, \text{ in.}$	$\bar{x}V, \text{ in}^4$	$\bar{y}V, \text{ in}^4$	$\bar{z}V, \text{ in}^4$
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma \bar{x}V = 3.048$	$\Sigma \bar{y}V = -5.047$	$\Sigma \bar{z}V = 8.555$

Thus,

$$\begin{aligned} \bar{X}\Sigma V &= \Sigma \bar{x}V: & \bar{X}(5.286 \text{ in}^3) &= 3.048 \text{ in}^4 \\ \bar{Y}\Sigma V &= \Sigma \bar{y}V: & \bar{Y}(5.286 \text{ in}^3) &= -5.047 \text{ in}^4 \\ \bar{Z}\Sigma V &= \Sigma \bar{z}V: & \bar{Z}(5.286 \text{ in}^3) &= 8.555 \text{ in}^4 \end{aligned}$$

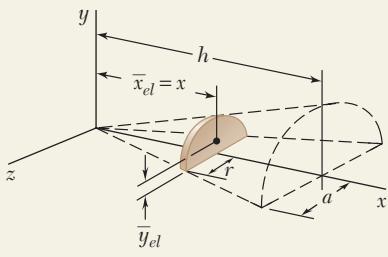
$$\begin{aligned} \bar{X} &= 0.577 \text{ in.} \\ \bar{Y} &= -0.955 \text{ in.} \\ \bar{Z} &= 1.618 \text{ in.} \end{aligned}$$



### SAMPLE PROBLEM 5.13

Determine the location of the centroid of the half right circular cone shown.

### SOLUTION



Since the  $xy$  plane is a plane of symmetry, the centroid lies in this plane and  $\bar{z} = 0$ . A slab of thickness  $dx$  is chosen as a differential element. The volume of this element is

$$dV = \frac{1}{2}\pi r^2 dx$$

The coordinates  $\bar{x}_{el}$  and  $\bar{y}_{el}$  of the centroid of the element are obtained from Fig. 5.8 (semicircular area).

$$\bar{x}_{el} = x \quad \bar{y}_{el} = \frac{4r}{3\pi}$$

We observe that  $r$  is proportional to  $x$  and write

$$\frac{r}{x} = \frac{a}{h} \quad r = \frac{a}{h}x$$

The volume of the body is

$$V = \int dV = \int_0^h \frac{1}{2}\pi r^2 dx = \int_0^h \frac{1}{2}\pi \left(\frac{a}{h}x\right)^2 dx = \frac{\pi a^2 h}{6}$$

The moment of the differential element with respect to the  $yz$  plane is  $\bar{x}_{el} dV$ ; the total moment of the body with respect to this plane is

$$\int \bar{x}_{el} dV = \int_0^h x \left(\frac{1}{2}\pi r^2\right) dx = \int_0^h x \left(\frac{1}{2}\pi\right) \left(\frac{a}{h}x\right)^2 dx = \frac{\pi a^2 h^2}{8}$$

Thus,

$$\bar{x}V = \int \bar{x}_{el} dV \quad \bar{x} \frac{\pi a^2 h^2}{6} = \frac{\pi a^2 h^2}{8} \quad \bar{x} = \frac{3}{4}h \quad \blacktriangleleft$$

Likewise, the moment of the differential element with respect to the  $zx$  plane is  $\bar{y}_{el} dV$ ; the total moment is

$$\int \bar{y}_{el} dV = \int_0^h \frac{4r}{3\pi} \left(\frac{1}{2}\pi r^2\right) dx = \frac{2}{3} \int_0^h \left(\frac{a}{h}x\right)^3 dx = \frac{a^3 h}{6}$$

Thus,

$$\bar{y}V = \int \bar{y}_{el} dV \quad \bar{y} \frac{\pi a^2 h^2}{6} = \frac{a^3 h}{6} \quad \bar{y} = \frac{a}{\pi} \quad \blacktriangleleft$$

# SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will be asked to locate the centers of gravity of three-dimensional bodies or the centroids of their volumes. All of the techniques we previously discussed for two-dimensional bodies—using symmetry, dividing the body into common shapes, choosing the most efficient differential element, etc.—may also be applied to the general three-dimensional case.

**1. Locating the centers of gravity of composite bodies.** In general, Eqs. (5.20) must be used:

$$\bar{X}\Sigma W = \Sigma \bar{x}W \quad \bar{Y}\Sigma W = \Sigma \bar{y}W \quad \bar{Z}\Sigma W = \Sigma \bar{z}W \quad (5.20)$$

However, for the case of a *homogeneous body*, the center of gravity of the body coincides with the *centroid of its volume*. Therefore, for this special case, the center of gravity of the body can also be located using Eqs. (5.21):

$$\bar{X}\Sigma V = \Sigma \bar{x}V \quad \bar{Y}\Sigma V = \Sigma \bar{y}V \quad \bar{Z}\Sigma V = \Sigma \bar{z}V \quad (5.21)$$

You should realize that these equations are simply an extension of the equations used for the two-dimensional problems considered earlier in the chapter. As the solutions of Sample Probs. 5.11 and 5.12 illustrate, the methods of solution for two- and three-dimensional problems are identical. Thus, we once again strongly encourage you to construct appropriate diagrams and tables when analyzing composite bodies. Also, as you study Sample Prob. 5.12, observe how the  $x$  and  $y$  coordinates of the centroid of the quarter cylinder were obtained using the equations for the centroid of a quarter circle.

We note that *two special cases* of interest occur when the given body consists of either uniform wires or uniform plates made of the same material.

a. For a body made of *several wire elements* of the *same uniform cross section*, the cross-sectional area  $A$  of the wire elements will factor out of Eqs. (5.21) when  $V$  is replaced with the product  $AL$ , where  $L$  is the length of a given element. Equations (5.21) thus reduce in this case to

$$\bar{X}\Sigma L = \Sigma \bar{x}L \quad \bar{Y}\Sigma L = \Sigma \bar{y}L \quad \bar{Z}\Sigma L = \Sigma \bar{z}L$$

b. For a body made of *several plates* of the *same uniform thickness*, the thickness  $t$  of the plates will factor out of Eqs. (5.21) when  $V$  is replaced with the product  $tA$ , where  $A$  is the area of a given plate. Equations (5.21) thus reduce in this case to

$$\bar{X}\Sigma A = \Sigma \bar{x}A \quad \bar{Y}\Sigma A = \Sigma \bar{y}A \quad \bar{Z}\Sigma A = \Sigma \bar{z}A$$

**2. Locating the centroids of volumes by direct integration.** As explained in Sec. 5.12, evaluating the integrals of Eqs. (5.22) can be simplified by choosing either a thin filament (Fig. 5.22) or a thin slab (Fig. 5.23) for the element of volume  $dV$ . Thus, you should begin your solution by identifying, if possible, the  $dV$  which produces the single or double integrals that are the easiest to compute. For bodies of revolution, this may be a thin slab (as in Sample Prob. 5.13) or a thin cylindrical shell. However, it is important to remember that the relationship that you establish among the variables (like the relationship between  $r$  and  $x$  in Sample Prob. 5.13) will directly affect the complexity of the integrals you will have to compute. Finally, we again remind you that  $\bar{x}_{el}$ ,  $\bar{y}_{el}$ , and  $\bar{z}_{el}$  in Eqs. (5.23) are the coordinates of the centroid of  $dV$ .

# PROBLEMS

- 5.96** Determine the location of the centroid of the composite body shown when (a)  $h = 2b$ , (b)  $h = 2.5b$ .

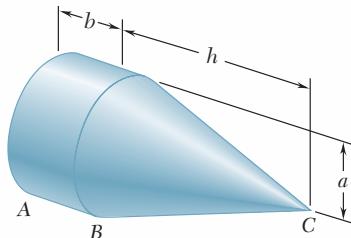


Fig. P5.96

- 5.97** Determine the  $y$  coordinate of the centroid of the body shown.

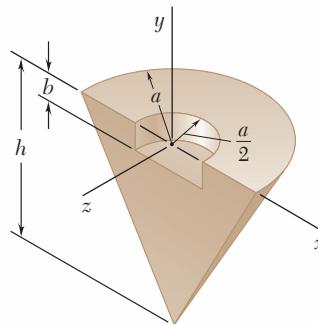


Fig. P5.97 and P5.98

- 5.98** Determine the  $z$  coordinate of the centroid of the body shown.  
(Hint: Use the result of Sample Prob. 5.13.)

- 5.99** The composite body shown is formed by removing a semiellipsoid of revolution of semimajor axis  $h$  and semiminor axis  $a/2$  from a hemisphere of radius  $a$ . Determine (a) the  $y$  coordinate of the centroid when  $h = a/2$ , (b) the ratio  $h/a$  for which  $\bar{y} = -0.4a$ .

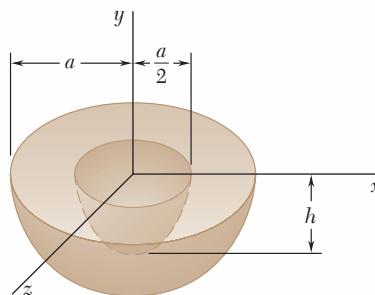


Fig. P5.99

**5.100** For the stop bracket shown, locate the  $x$  coordinate of the center of gravity.

**5.101** For the stop bracket shown, locate the  $z$  coordinate of the center of gravity.

**5.102 and 5.103** For the machine element shown, locate the  $y$  coordinate of the center of gravity.

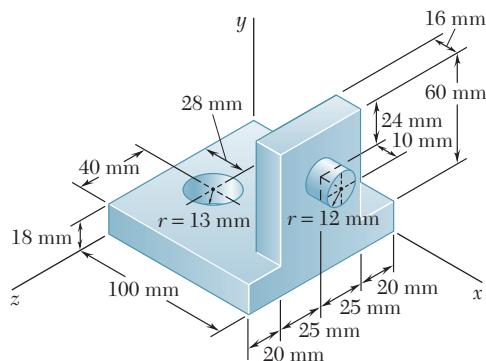


Fig. P5.102 and P5.105

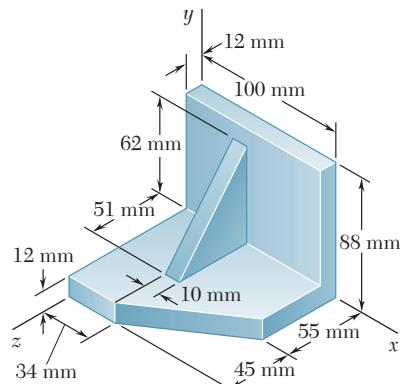


Fig. P5.100 and P5.101

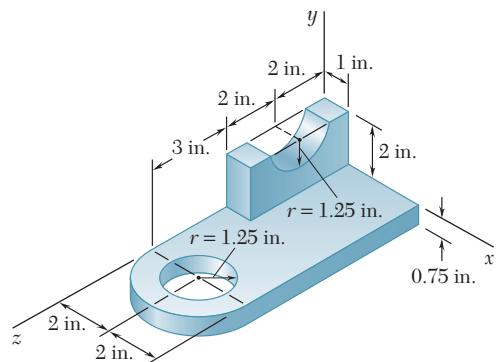


Fig. P5.103 and P5.104

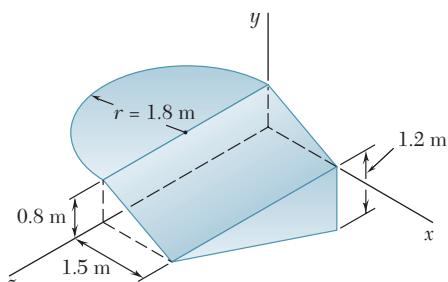


Fig. P5.106

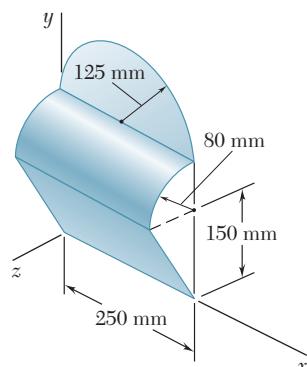
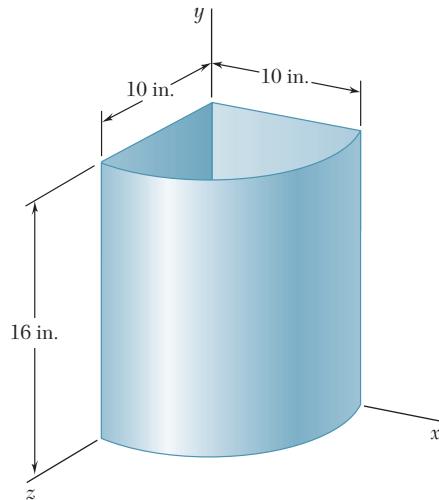


Fig. P5.107

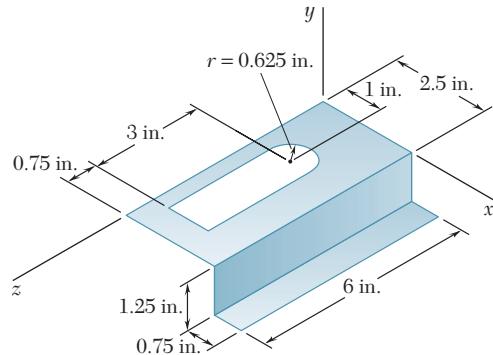
**5.106 and 5.107** Locate the center of gravity of the sheet-metal form shown.

- 5.108** A wastebasket, designed to fit in the corner of a room, is 16 in. high and has a base in the shape of a quarter circle of radius 10 in. Locate the center of gravity of the wastebasket, knowing that it is made of sheet metal of uniform thickness.



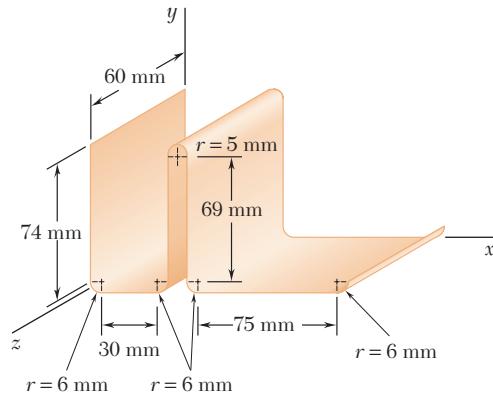
**Fig. P5.108**

- 5.109** A mounting bracket for electronic components is formed from sheet metal of uniform thickness. Locate the center of gravity of the bracket.



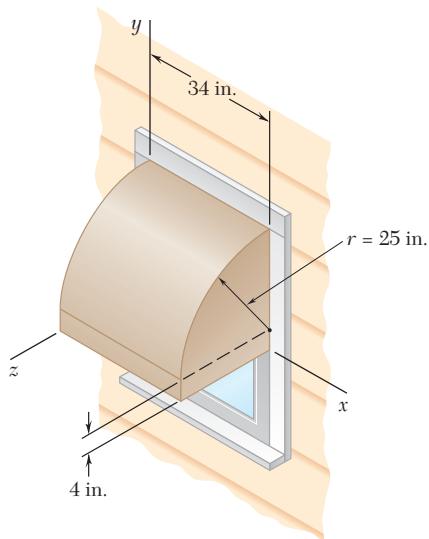
**Fig. P5.109**

- 5.110** A thin sheet of plastic of uniform thickness is bent to form a desk organizer. Locate the center of gravity of the organizer.



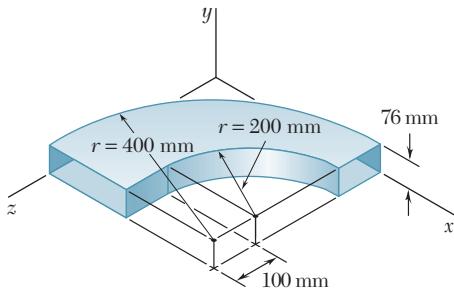
**Fig. P5.110**

- 5.111** A window awning is fabricated from sheet metal of uniform thickness. Locate the center of gravity of the awning.



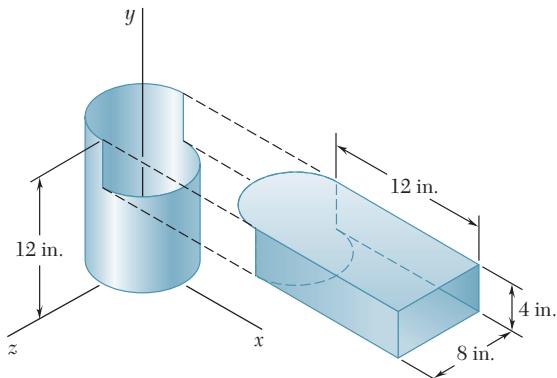
**Fig. P5.111**

- 5.112** An elbow for the duct of a ventilating system is made of sheet metal of uniform thickness. Locate the center of gravity of the elbow.



**Fig. P5.112**

- 5.113** An 8-in.-diameter cylindrical duct and a  $4 \times 8$ -in. rectangular duct are to be joined as indicated. Knowing that the ducts were fabricated from the same sheet metal, which is of uniform thickness, locate the center of gravity of the assembly.



**Fig. P5.113**

- 5.114** A thin steel wire of uniform cross section is bent into the shape shown. Locate its center of gravity.

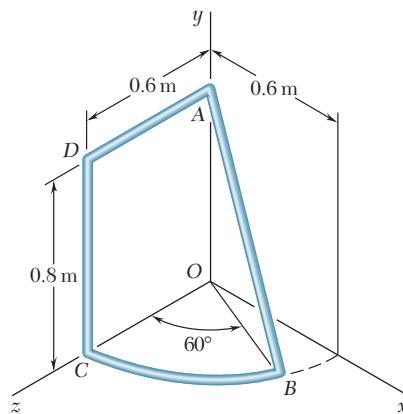


Fig. P5.114

- 5.115 and 5.116** Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

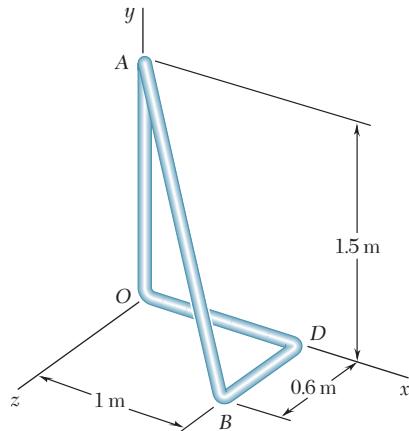


Fig. P5.115

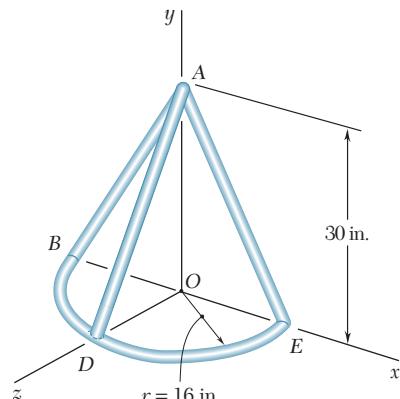


Fig. P5.116

- 5.117** The frame of a greenhouse is constructed from uniform aluminum channels. Locate the center of gravity of the portion of the frame shown.

- 5.118** A scratch awl has a plastic handle and a steel blade and shank. Knowing that the density of plastic is  $1030 \text{ kg/m}^3$  and of steel is  $7860 \text{ kg/m}^3$ , locate the center of gravity of the awl.

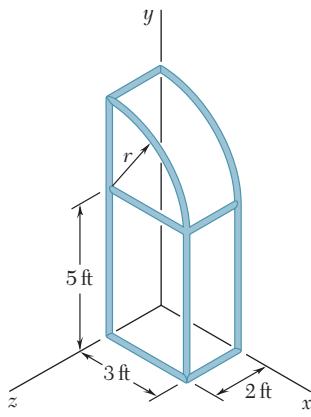


Fig. P5.117

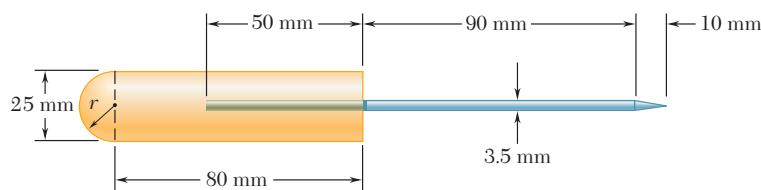


Fig. P5.118

- 5.119** A bronze bushing is mounted inside a steel sleeve. Knowing that the specific weight of bronze is  $0.318 \text{ lb/in}^3$  and of steel is  $0.284 \text{ lb/in}^3$ , determine the location of the center of gravity of the assembly.

- 5.120** A brass collar, of length 2.5 in., is mounted on an aluminum rod of length 4 in. Locate the center of gravity of the composite body. (Specific weights: brass =  $0.306 \text{ lb/in}^3$ , aluminum =  $0.101 \text{ lb/in}^3$ .)

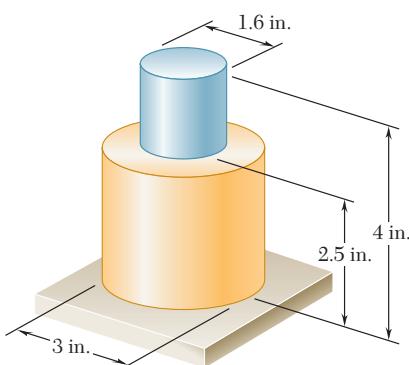


Fig. P5.120

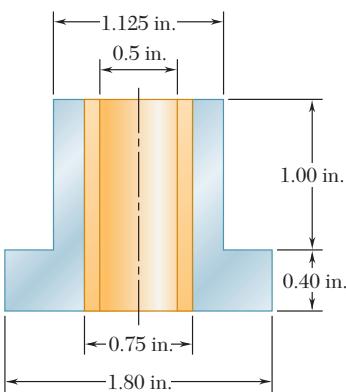


Fig. P5.119

- 5.121** The three legs of a small glass-topped table are equally spaced and are made of steel tubing, which has an outside diameter of 24 mm and a cross-sectional area of  $150 \text{ mm}^2$ . The diameter and the thickness of the table top are 600 mm and 10 mm, respectively. Knowing that the density of steel is  $7860 \text{ kg/m}^3$  and of glass is  $2190 \text{ kg/m}^3$ , locate the center of gravity of the table.

- 5.122 through 5.124** Determine by direct integration the values of  $\bar{x}$  for the two volumes obtained by passing a vertical cutting plane through the given shape of Fig. 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

**5.122** A hemisphere

**5.123** A semiellipsoid of revolution

**5.124** A paraboloid of revolution.

- 5.125 and 5.126** Locate the centroid of the volume obtained by rotating the shaded area about the  $x$  axis.

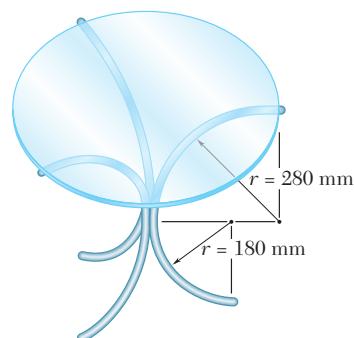


Fig. P5.121

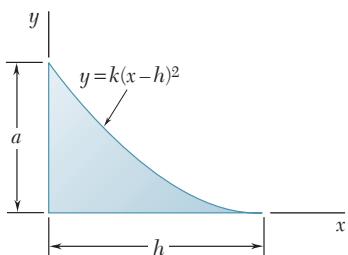


Fig. P5.125

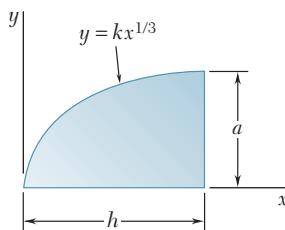


Fig. P5.126

- 5.127** Locate the centroid of the volume obtained by rotating the shaded area about the line  $x = h$ .

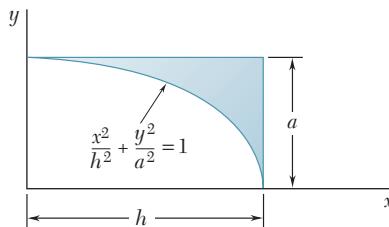


Fig. P5.127

- \*5.128** Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the  $x$  axis.

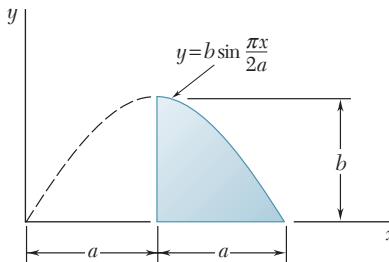


Fig. P5.128 and P5.129

- \*5.129** Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the  $y$  axis. (Hint: Use a thin cylindrical shell of radius  $r$  and thickness  $dr$  as the element of volume.)

- \*5.130** Show that for a regular pyramid of height  $h$  and  $n$  sides ( $n = 3, 4, \dots$ ) the centroid of the volume of the pyramid is located at a distance  $h/4$  above the base.

- 5.131** Determine by direct integration the location of the centroid of one-half of a thin, uniform hemispherical shell of radius  $R$ .

- 5.132** The sides and the base of a punch bowl are of uniform thickness  $t$ . If  $t \ll R$  and  $R = 250$  mm, determine the location of the center of gravity of (a) the bowl, (b) the punch.

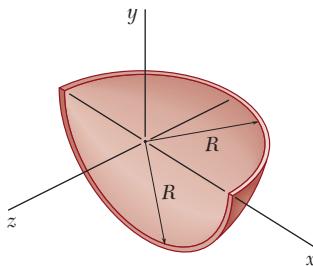


Fig. P5.131

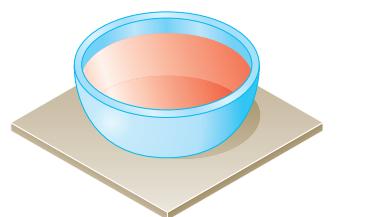


Fig. P5.132

- 5.133** After grading a lot, a builder places four stakes to designate the corners of the slab for a house. To provide a firm, level base for the slab, the builder places a minimum of 3 in. of gravel beneath the slab. Determine the volume of gravel needed and the  $x$  coordinate of the centroid of the volume of the gravel. (*Hint:* The bottom surface of the gravel is an oblique plane, which can be represented by the equation  $y = a + bx + cz$ .)

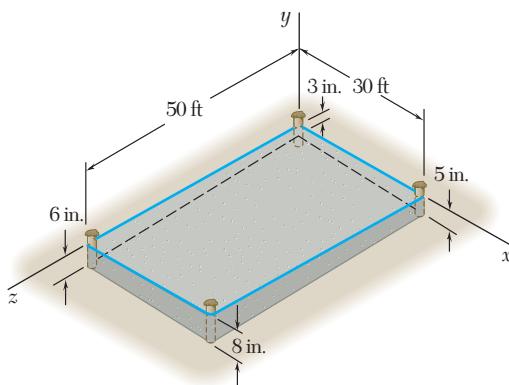


Fig. P5.133

- 5.134** Determine by direct integration the location of the centroid of the volume between the  $xz$  plane and the portion shown of the surface  $y = 16h(ax - x^2)(bz - z^2)/a^2b^2$ .

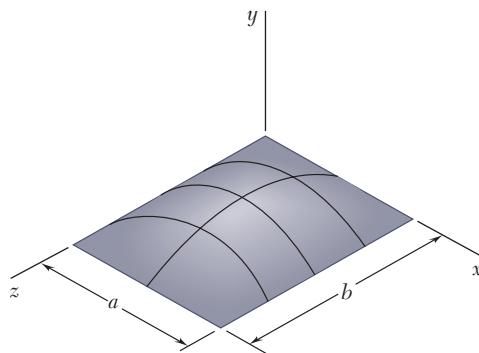


Fig. P5.134

- 5.135** Locate the centroid of the section shown, which was cut from a thin circular pipe by two oblique planes.

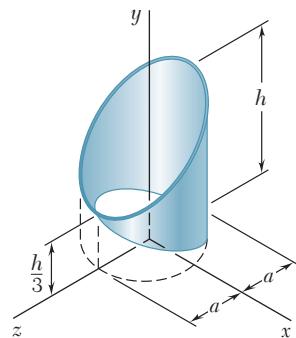


Fig. P5.135

- \*5.136** Locate the centroid of the section shown, which was cut from an elliptical cylinder by an oblique plane.

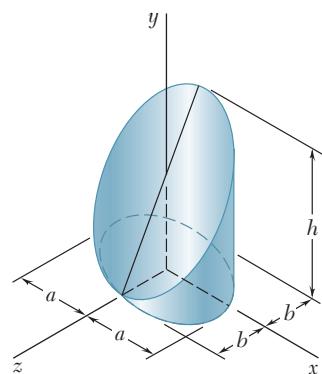


Fig. P5.136

# REVIEW AND SUMMARY

This chapter was devoted chiefly to the determination of the *center of gravity* of a rigid body, i.e., to the determination of the point  $G$  where a single force  $\mathbf{W}$ , called the *weight* of the body, can be applied to represent the effect of the earth's attraction on the body.

## Center of gravity of a two-dimensional body

In the first part of the chapter, we considered *two-dimensional bodies*, such as flat plates and wires contained in the  $xy$  plane. By adding force components in the vertical  $z$  direction and moments about the horizontal  $y$  and  $x$  axes [Sec. 5.2], we derived the relations

$$W = \int dW \quad \bar{x}W = \int x \, dW \quad \bar{y}W = \int y \, dW \quad (5.2)$$

which define the weight of the body and the coordinates  $\bar{x}$  and  $\bar{y}$  of its center of gravity.

## Centroid of an area or line

In the case of a *homogeneous flat plate of uniform thickness* [Sec. 5.3], the center of gravity  $G$  of the plate coincides with the *centroid*  $C$  of the area  $A$  of the plate, the coordinates of which are defined by the relations

$$\bar{x}A = \int x \, dA \quad \bar{y}A = \int y \, dA \quad (5.3)$$

Similarly, the determination of the center of gravity of a *homogeneous wire of uniform cross section* contained in a plane reduces to the determination of the *centroid*  $C$  of the line  $L$  representing the wire; we have

$$\bar{x}L = \int x \, dL \quad \bar{y}L = \int y \, dL \quad (5.4)$$

## First moments

The integrals in Eqs. (5.3) are referred to as the *first moments* of the area  $A$  with respect to the  $y$  and  $x$  axes and are denoted by  $Q_y$  and  $Q_x$ , respectively [Sec. 5.4]. We have

$$Q_y = \bar{x}A \quad Q_x = \bar{y}A \quad (5.6)$$

The first moments of a line can be defined in a similar way.

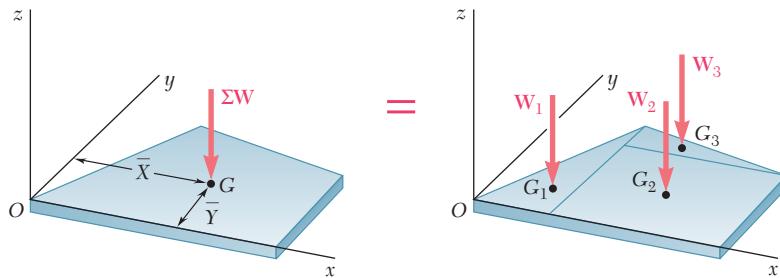
## Properties of symmetry

The determination of the centroid  $C$  of an area or line is simplified when the area or line possesses certain *properties of symmetry*. If the area or line is symmetric with respect to an axis, its centroid  $C$

lies on that axis; if it is symmetric with respect to two axes,  $C$  is located at the intersection of the two axes; if it is symmetric with respect to a center  $O$ ,  $C$  coincides with  $O$ .

The *areas and the centroids of various common shapes* are tabulated in Fig. 5.8. When a flat plate can be divided into several of these shapes, the coordinates  $\bar{X}$  and  $\bar{Y}$  of its center of gravity  $G$  can be determined from the coordinates  $\bar{x}_1, \bar{x}_2, \dots$  and  $\bar{y}_1, \bar{y}_2, \dots$  of the centers of gravity  $G_1, G_2, \dots$  of the various parts [Sec. 5.5]. Equating moments about the  $y$  and  $x$  axes, respectively (Fig. 5.24), we have

$$\bar{X}\Sigma W = \Sigma \bar{x}W \quad \bar{Y}\Sigma W = \Sigma \bar{y}W \quad (5.7)$$



**Fig. 5.24**

If the plate is homogeneous and of uniform thickness, its center of gravity coincides with the centroid  $C$  of the area of the plate, and Eqs. (5.7) reduce to

$$Q_y = \bar{X}\Sigma A = \Sigma \bar{x}A \quad Q_x = \bar{Y}\Sigma A = \Sigma \bar{y}A \quad (5.8)$$

These equations yield the first moments of the composite area, or they can be solved for the coordinates  $\bar{X}$  and  $\bar{Y}$  of its centroid [Sample Prob. 5.1]. The determination of the center of gravity of a composite wire is carried out in a similar fashion [Sample Prob. 5.2].

When an area is bounded by analytical curves, the coordinates of its centroid can be determined by *integration* [Sec. 5.6]. This can be done by evaluating either the double integrals in Eqs. (5.3) or a *single integral* which uses one of the thin rectangular or pie-shaped elements of area shown in Fig. 5.12. Denoting by  $\bar{x}_{el}$  and  $\bar{y}_{el}$  the coordinates of the centroid of the element  $dA$ , we have

$$Q_y = \bar{x}A = \int \bar{x}_{el} dA \quad Q_x = \bar{y}A = \int \bar{y}_{el} dA \quad (5.9)$$

It is advantageous to use the same element of area to compute both of the first moments  $Q_y$  and  $Q_x$ ; the same element can also be used to determine the area  $A$  [Sample Prob. 5.4].

### Center of gravity of a composite body

### Determination of centroid by integration

### Theorems of Pappus-Guldinus

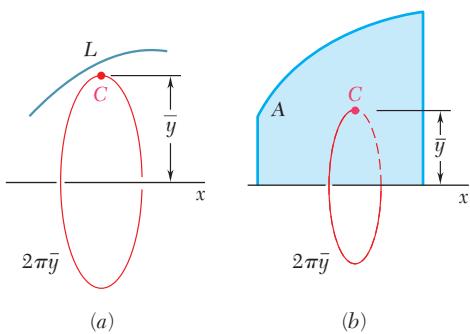


Fig. 5.25

### Distributed loads

The concept of centroid of an area can also be used to solve problems other than those dealing with the weight of flat plates. For example, to determine the reactions at the supports of a beam [Sec. 5.8], we can replace a *distributed load*  $w$  by a concentrated load  $\mathbf{W}$  equal in magnitude to the area  $A$  under the load curve and passing through the centroid  $C$  of that area (Fig. 5.26). The same approach can be used to determine the resultant of the hydrostatic forces exerted on a *rectangular plate submerged in a liquid* [Sec. 5.9].

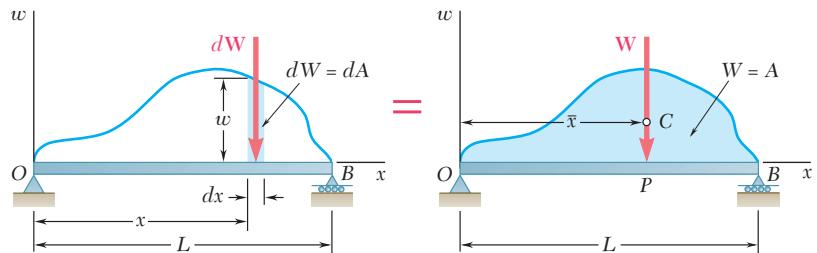


Fig. 5.26

### Center of gravity of a three-dimensional body

The last part of the chapter was devoted to the determination of the *center of gravity  $G$  of a three-dimensional body*. The coordinates  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$  of  $G$  were defined by the relations

$$\bar{x}W = \int x \, dW \quad \bar{y}W = \int y \, dW \quad \bar{z}W = \int z \, dW \quad (5.17)$$

### Centroid of a volume

In the case of a *homogeneous body*, the center of gravity  $G$  coincides with the *centroid  $C$  of the volume  $V$*  of the body; the coordinates of  $C$  are defined by the relations

$$\bar{x}V = \int x \, dV \quad \bar{y}V = \int y \, dV \quad \bar{z}V = \int z \, dV \quad (5.19)$$

If the volume possesses a *plane of symmetry*, its centroid  $C$  will lie in that plane; if it possesses two planes of symmetry,  $C$  will be located on the line of intersection of the two planes; if it possesses three planes of symmetry which intersect at only one point,  $C$  will coincide with that point [Sec. 5.10].

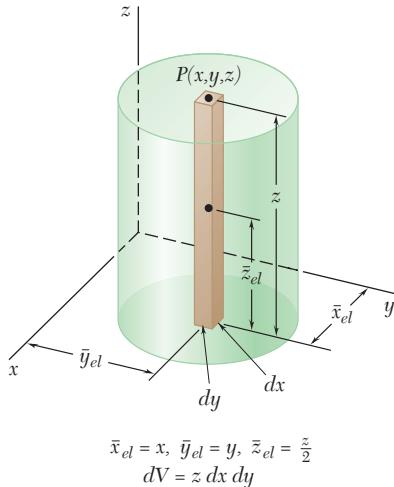
The *volumes and centroids of various common three-dimensional shapes* are tabulated in Fig. 5.21. When a body can be divided into several of these shapes, the coordinates  $\bar{X}$ ,  $\bar{Y}$ ,  $\bar{Z}$  of its center of gravity  $G$  can be determined from the corresponding coordinates of the centers of gravity of its various parts [Sec. 5.11]. We have

$$\bar{X}\Sigma W = \Sigma \bar{x}W \quad \bar{Y}\Sigma W = \Sigma \bar{y}W \quad \bar{Z}\Sigma W = \Sigma \bar{z}W \quad (5.20)$$

If the body is made of a homogeneous material, its center of gravity coincides with the centroid  $C$  of its volume, and we write [Sample Probs. 5.11 and 5.12]

$$\bar{X}\Sigma V = \Sigma \bar{x}V \quad \bar{Y}\Sigma V = \Sigma \bar{y}V \quad \bar{Z}\Sigma V = \Sigma \bar{z}V \quad (5.21)$$

When a volume is bounded by analytical surfaces, the coordinates of its centroid can be determined by *integration* [Sec. 5.12]. To avoid the computation of the triple integrals in Eqs. (5.19), we can use elements of volume in the shape of thin filaments, as shown in Fig. 5.27.



**Fig. 5.27**

Denoting by  $\bar{x}_{el}$ ,  $\bar{y}_{el}$ , and  $\bar{z}_{el}$  the coordinates of the centroid of the element  $dV$ , we rewrite Eqs. (5.19) as

$$\bar{x}V = \int \bar{x}_{el} dV \quad \bar{y}V = \int \bar{y}_{el} dV \quad \bar{z}V = \int \bar{z}_{el} dV \quad (5.23)$$

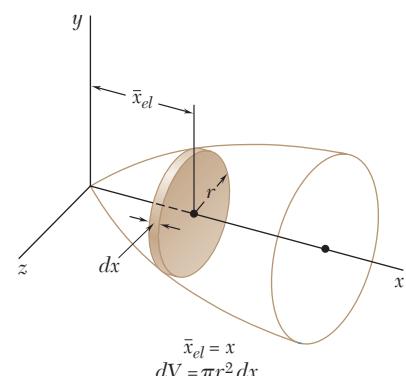
which involve only double integrals. If the volume possesses *two planes of symmetry*, its centroid  $C$  is located on their line of intersection. Choosing the  $x$  axis to lie along that line and dividing the volume into thin slabs parallel to the  $yz$  plane, we can determine  $C$  from the relation

$$\bar{x}V = \int \bar{x}_{el} dV \quad (5.24)$$

with a *single integration* [Sample Prob. 5.13]. For a body of revolution, these slabs are circular and their volume is given in Fig. 5.28.

## Center of gravity of a composite body

### Determination of centroid by integration



**Fig. 5.28**

# REVIEW PROBLEMS

**5.137 and 5.138** Locate the centroid of the plane area shown.

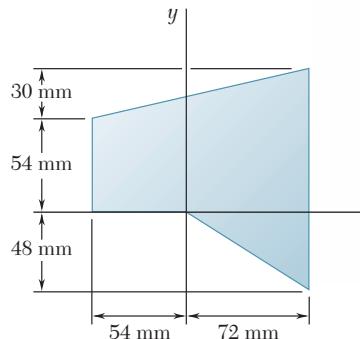


Fig. P5.137

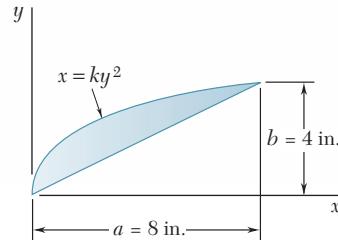


Fig. P5.138

**5.139** The frame for a sign is fabricated from thin, flat steel bar stock of mass per unit length  $4.73 \text{ kg/m}$ . The frame is supported by a pin at  $C$  and by a cable  $AB$ . Determine (a) the tension in the cable, (b) the reaction at  $C$ .

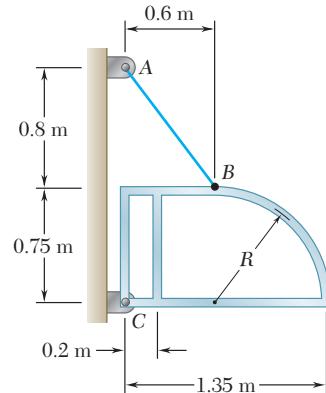


Fig. P5.139

**5.140** Determine by direct integration the centroid of the area shown. Express your answer in terms of  $a$  and  $h$ .

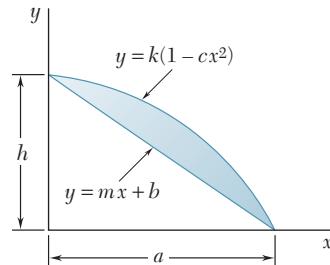
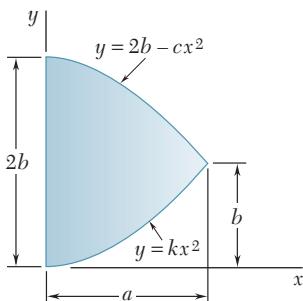
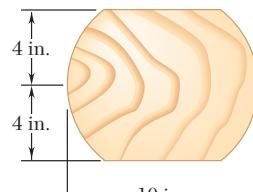


Fig. P5.140

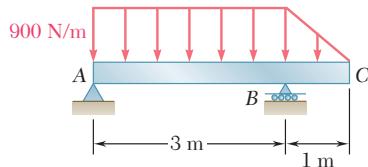
- 5.141** Determine by direct integration the centroid of the area shown. Express your answer in terms of  $a$  and  $b$ .

**Fig. P5.141**

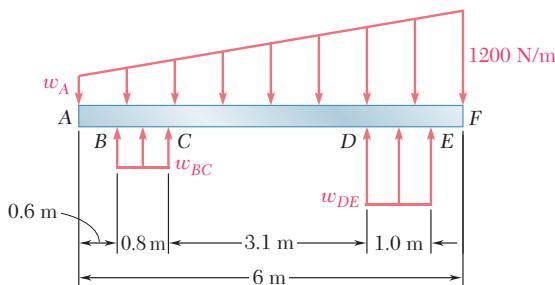
- 5.142** Knowing that two equal caps have been removed from a 10-in.-diameter wooden sphere, determine the total surface area of the remaining portion.

**Fig. P5.142**

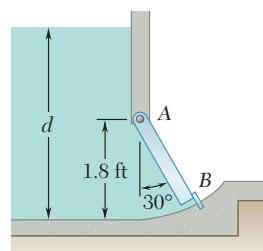
- 5.143** Determine the reactions at the beam supports for the given loading.

**Fig. P5.143**

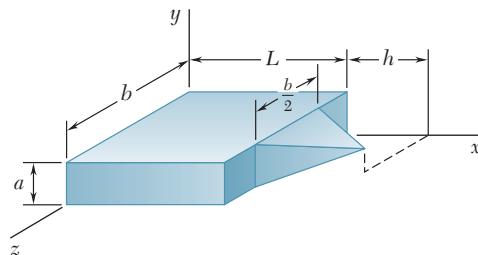
- 5.144** A beam is subjected to a linearly distributed downward load and rests on two wide supports  $BC$  and  $DE$ , which exert uniformly distributed upward loads as shown. Determine the values of  $w_{BC}$  and  $w_{DE}$  corresponding to equilibrium when  $w_A = 600 \text{ N/m}$ .

**Fig. P5.144**

- 5.145** The square gate  $AB$  is held in the position shown by hinges along its top edge  $A$  and by a shear pin at  $B$ . For a depth of water  $d = 3.5 \text{ ft}$ , determine the force exerted on the gate by the shear pin.

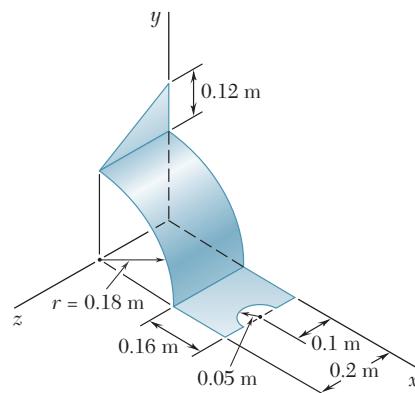
**Fig. P5.145**

- 5.146** Consider the composite body shown. Determine (a) the value of  $\bar{x}$  when  $h = L/2$ , (b) the ratio  $h/L$  for which  $\bar{x} = L$ .



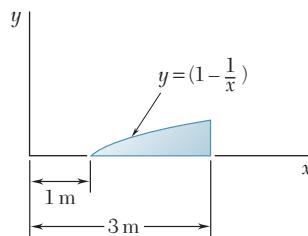
**Fig. P5.146**

- 5.147** Locate the center of gravity of the sheet-metal form shown.



**Fig. P5.147**

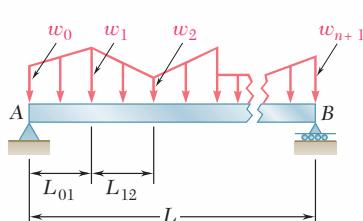
- 5.148** Locate the centroid of the volume obtained by rotating the shaded area about the  $x$  axis.



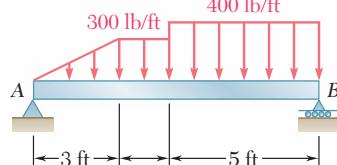
**Fig. P5.148**

# COMPUTER PROBLEMS

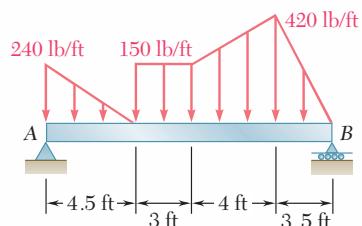
**5.C1** A beam is to carry a series of uniform and uniformly varying distributed loads as shown in part *a* of the figure. Divide the area under each portion of the load curve into two triangles (see Sample Prob. 5.9), and then use computational software to calculate the reactions at *A* and *B*. Use this program to calculate the reactions at the supports for the beams shown in parts *b* and *c* of the figure.



(a)



(b)



(c)

Fig. P5.C1

**5.C2** The three-dimensional structure shown is fabricated from five thin steel rods of equal diameter. Using computational software, calculate the coordinates of the center of gravity of the structure. Use this program to locate the center of gravity when (a)  $h = 12$  m,  $R = 5$  m,  $\alpha = 90^\circ$ ; (b)  $h = 570$  mm,  $R = 760$  mm,  $\alpha = 30^\circ$ ; (c)  $h = 21$  m,  $R = 20$  m,  $\alpha = 135^\circ$ .

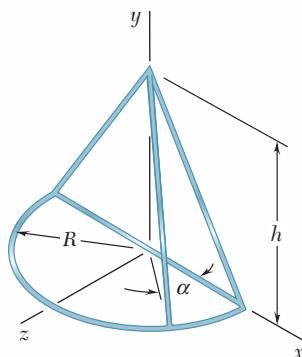


Fig. P5.C2

**5.C3** An open tank is to be slowly filled with water. (The density of water is  $10^3 \text{ kg/m}^3$ .) Using computational software, determine the resultant of the pressure forces exerted by the water on a 1-m-wide section of side ABC of the tank. Determine the resultant of the pressure forces for values of  $d$  from 0 to 3 m using 0.25-m increments.

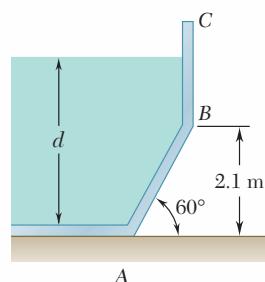


Fig. P5.C3

**5.C4** Approximate the curve shown using 10 straight-line segments, and then use computational software, to determine the location of the centroid of the line. Use this program to determine the location of the centroid when (a)  $a = 1$  in.,  $L = 11$  in.,  $h = 2$  in.; (b)  $a = 2$  in.,  $L = 17$  in.,  $h = 4$  in.; (c)  $a = 5$  in.,  $L = 12$  in.,  $h = 1$  in.

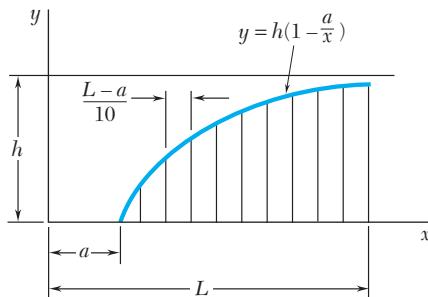


Fig. P5.C4

**5.C5** Approximate the general spandrel shown using a series of  $n$  rectangles, each of width  $\Delta a$  and of the form  $bcc'b'$ , and then use computational software to calculate the coordinates of the centroid of the area. Use this program to locate the centroid when (a)  $m = 2$ ,  $a = 80$  mm,  $h = 80$  mm; (b)  $m = 2$ ,  $a = 80$  mm,  $h = 500$  mm; (c)  $m = 5$ ,  $a = 80$  mm,  $h = 80$  mm; (d)  $m = 5$ ,  $a = 80$  mm,  $h = 500$  mm. In each case, compare the answers obtained to the exact values of  $\bar{x}$  and  $\bar{y}$  computed from the formulas given in Fig. 5.8A and determine the percentage error.

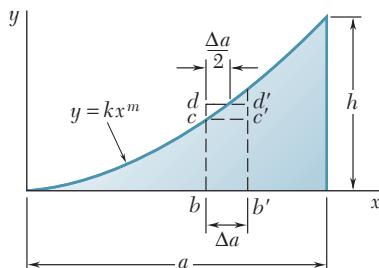


Fig. P5.C5

**5.C6** Solve Prob. 5.C5, using rectangles of the form  $bdd'b'$ .

**\*5.C7** A farmer asks a group of engineering students to determine the volume of water in a small pond. Using cord, the students first establish a  $2 \times 2$ -ft grid across the pond and then record the depth of the water, in feet, at each intersection point of the grid (see the accompanying table). Using computational software, determine (a) the volume of water in the

pond, (b) the location of the center of gravity of the water. Approximate the depth of each  $2 \times 2$ -ft element of water using the average of the water depths at the four corners of the element.

		Cord									
		1	2	3	4	5	6	7	8	9	10
Cord	1	...	...	...	...	0	0	0	...	...	...
	2	...	...	0	0	0	1	0	0	0	...
	3	...	0	0	1	3	3	3	1	0	0
	4	0	0	1	3	6	6	6	3	1	0
	5	0	1	3	6	8	8	6	3	1	0
	6	0	1	3	6	8	7	7	3	0	0
	7	0	3	4	6	6	6	4	1	0	...
	8	0	3	3	3	3	3	1	0	0	...
	9	0	0	0	1	1	0	0	0	...	...
	10	...	...	0	0	0	0	...	...	...	...

Trusses, such as this Pratt-style cantilever arch bridge in New York State, provide both a practical and an economical solution to many engineering problems.



CHAPTER

# Analysis of Structures

## Chapter 6 Analysis of Structures

- 6.1 Introduction
- 6.2 Definition of a Truss
- 6.3 Simple Trusses
- 6.4 Analysis of Trusses by the Method of Joints
- 6.5 Joints under Special Loading Conditions
- 6.6 Space Trusses
- 6.7 Analysis of Trusses by the Method of Sections
- 6.8 Trusses Made of Several Simple Trusses
- 6.9 Structures Containing Multiforce Members
- 6.10 Analysis of a Frame
- 6.11 Frames Which Cease to Be Rigid When Detached from Their Supports
- 6.12 Machines

### 6.1 INTRODUCTION

The problems considered in the preceding chapters concerned the equilibrium of a single rigid body, and all forces involved were external to the rigid body. We now consider problems dealing with the equilibrium of structures made of several connected parts. These problems call for the determination not only of the external forces acting on the structure but also of the forces which hold together the various parts of the structure. From the point of view of the structure as a whole, these forces are *internal forces*.

Consider, for example, the crane shown in Fig. 6.1a, which carries a load  $W$ . The crane consists of three beams  $AD$ ,  $CF$ , and  $BE$  connected by frictionless pins; it is supported by a pin at  $A$  and by a cable  $DG$ . The free-body diagram of the crane has been drawn in Fig. 6.1b. The external forces, which are shown in the diagram, include the weight  $W$ , the two components  $A_x$  and  $A_y$  of the reaction at  $A$ , and the force  $T$  exerted by the cable at  $D$ . The internal forces holding the various parts of the crane together do not appear in the diagram. If, however, the crane is dismembered and if a free-body diagram is drawn for each of its component parts, the forces holding the three beams together will also be represented, since these forces are external forces from the point of view of each component part (Fig. 6.1c).

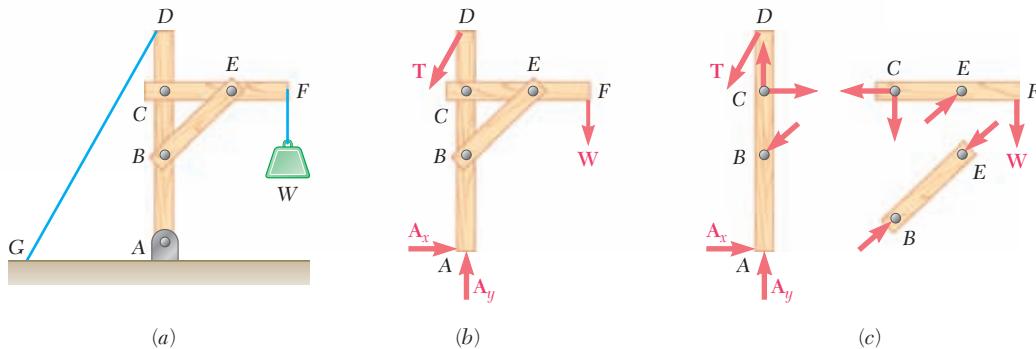


Fig. 6.1

It will be noted that the force exerted at  $B$  by member  $BE$  on member  $AD$  has been represented as equal and opposite to the force exerted at the same point by member  $AD$  on member  $BE$ ; the force exerted at  $E$  by  $BE$  on  $CF$  is shown equal and opposite to the force exerted by  $CF$  on  $BE$ ; and the components of the force exerted at  $C$  by  $CF$  on  $AD$  are shown equal and opposite to the components of the force exerted by  $AD$  on  $CF$ . This is in conformity with Newton's third law, which states that *the forces of action and reaction between bodies in contact have the same magnitude, same line of action, and opposite sense*. As pointed out in Chap. 1, this law, which is based on experimental evidence, is one of the six fundamental principles of elementary mechanics, and its application is essential to the solution of problems involving connected bodies.

In this chapter, three broad categories of engineering structures will be considered:

- 1.** *Trusses*, which are designed to support loads and are usually stationary, fully constrained structures. Trusses consist exclusively of straight members connected at joints located at the ends of each member. Members of a truss, therefore, are *two-force members*, i.e., members acted upon by two equal and opposite forces directed along the member.
- 2.** *Frames*, which are also designed to support loads and are also usually stationary, fully constrained structures. However, like the crane of Fig. 6.1, frames always contain at least one *multipforce member*, i.e., a member acted upon by three or more forces which, in general, are not directed along the member.
- 3.** *Machines*, which are designed to transmit and modify forces and are structures containing moving parts. Machines, like frames, always contain at least one multipforce member.



**Photo 6.1** Shown is a pin-jointed connection on the approach span to the San Francisco–Oakland Bay Bridge.

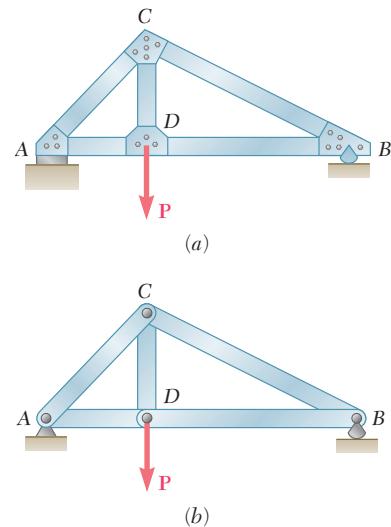
## TRUSSES

### 6.2 DEFINITION OF A TRUSS

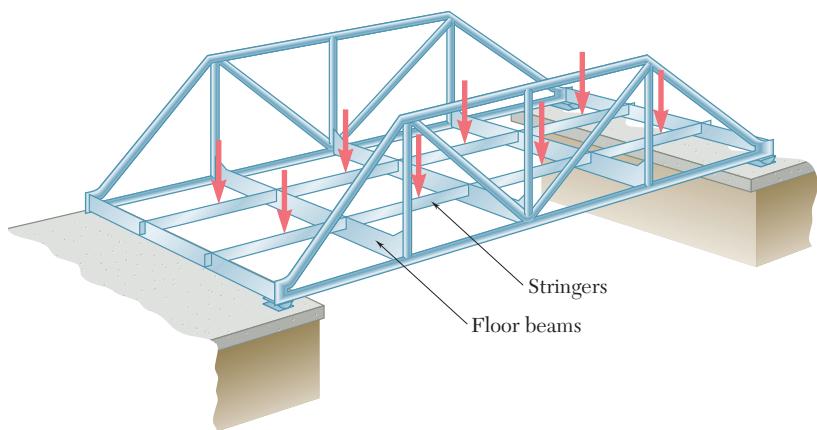
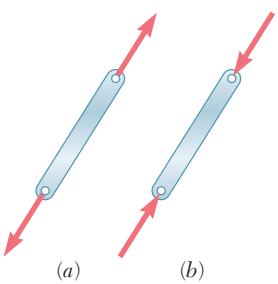
The truss is one of the major types of engineering structures. It provides both a practical and an economical solution to many engineering situations, especially in the design of bridges and buildings. A typical truss is shown in Fig. 6.2a. A truss consists of straight members connected at joints. Truss members are connected at their extremities only; thus no member is continuous through a joint. In Fig. 6.2a, for example, there is no member  $AB$ ; there are instead two distinct members  $AD$  and  $DB$ . Most actual structures are made of several trusses joined together to form a space framework. Each truss is designed to carry those loads which act in its plane and thus may be treated as a two-dimensional structure.

In general, the members of a truss are slender and can support little lateral load; all loads, therefore, must be applied to the various joints, and not to the members themselves. When a concentrated load is to be applied between two joints, or when a distributed load is to be supported by the truss, as in the case of a bridge truss, a floor system must be provided which, through the use of stringers and floor beams, transmits the load to the joints (Fig. 6.3).

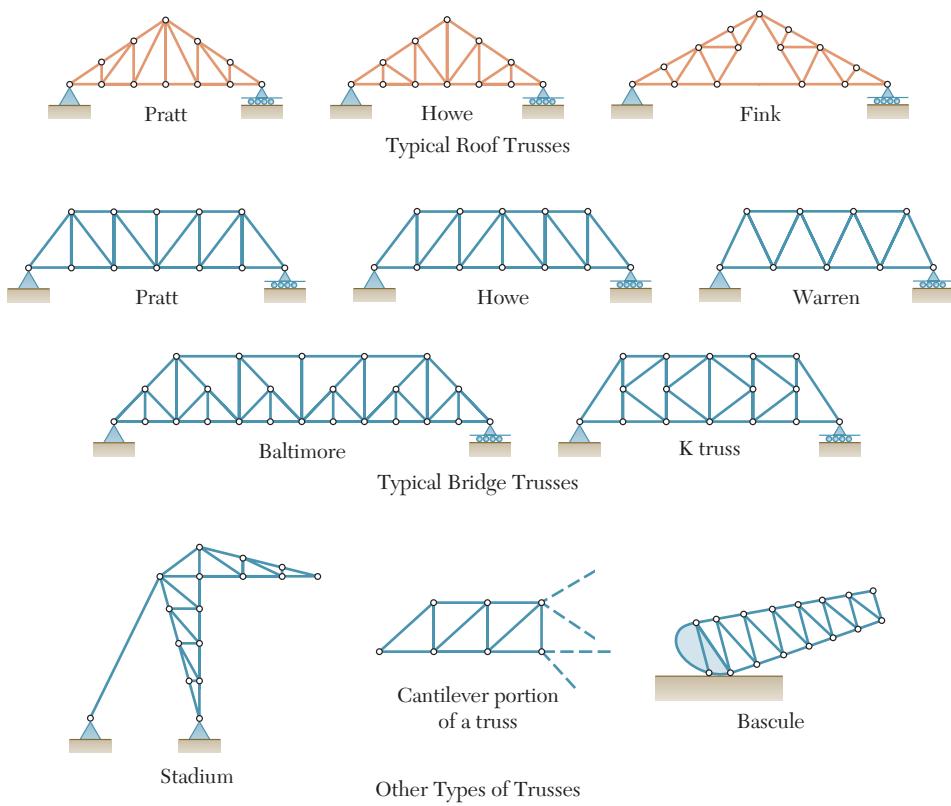
The weights of the members of the truss are also assumed to be applied to the joints, half of the weight of each member being applied to each of the two joints the member connects. Although the members are actually joined together by means of welded, bolted, or riveted connections, it is customary to assume that the members are pinned together; therefore, the forces acting at each end of a member reduce to a single force and no couple. Thus, the only forces assumed to be applied to a truss member are a single



**Fig. 6.2**

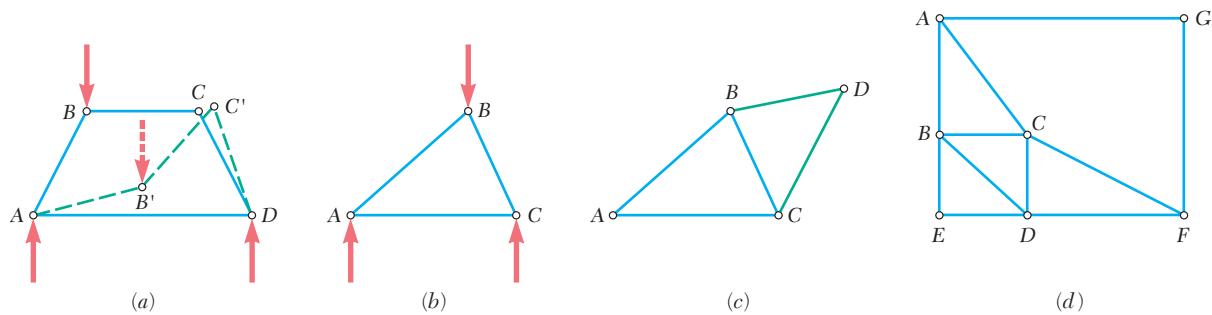
**Fig. 6.3****Fig. 6.4**

force at each end of the member. Each member can then be treated as a two-force member, and the entire truss can be considered as a group of pins and two-force members (Fig. 6.2b). An individual member can be acted upon as shown in either of the two sketches of Fig. 6.4. In Fig. 6.4a, the forces tend to pull the member apart, and the member is in tension; in Fig. 6.4b, the forces tend to compress the member, and the member is in compression. A number of typical trusses are shown in Fig. 6.5.

**Fig. 6.5**

## 6.3 SIMPLE TRUSSES

Consider the truss of Fig. 6.6a, which is made of four members connected by pins at A, B, C, and D. If a load is applied at B, the truss will greatly deform, completely losing its original shape. In contrast, the truss of Fig. 6.6b, which is made of three members connected by pins at A, B, and C, will deform only slightly under a load applied at B. The only possible deformation for this truss is one involving small changes in the length of its members. The truss of Fig. 6.6b is said to be a *rigid truss*, the term rigid being used here to indicate that the truss *will not collapse*.



**Fig. 6.6**

As shown in Fig. 6.6c, a larger rigid truss can be obtained by adding two members  $BD$  and  $CD$  to the basic triangular truss of Fig. 6.6b. This procedure can be repeated as many times as desired, and the resulting truss will be rigid if each time two new members are added, they are attached to two existing joints and connected at a new joint.<sup>†</sup> A truss which can be constructed in this manner is called a *simple truss*.

It should be noted that a simple truss is not necessarily made only of triangles. The truss of Fig. 6.6d, for example, is a simple truss which was constructed from triangle  $ABC$  by adding successively the joints  $D$ ,  $E$ ,  $F$ , and  $G$ . On the other hand, rigid trusses are not always simple trusses, even when they appear to be made of triangles. The Fink and Baltimore trusses shown in Fig. 6.5, for instance, are not simple trusses, since they cannot be constructed from a single triangle in the manner described above. All the other trusses shown in Fig. 6.5 are simple trusses, as may be easily checked. (For the K truss, start with one of the central triangles.)

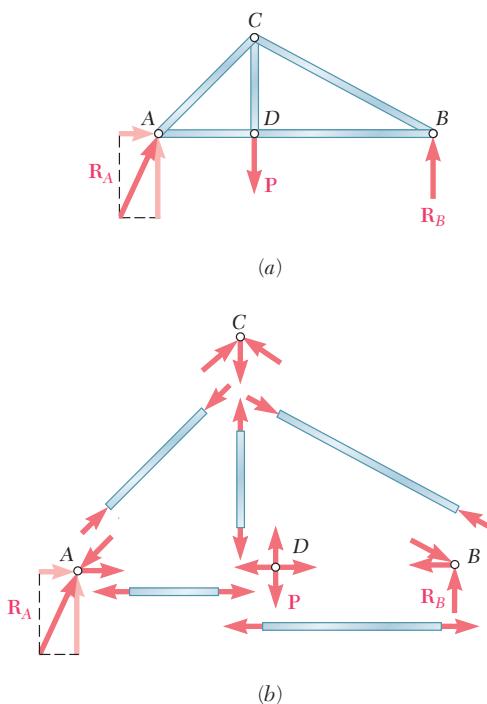
Returning to Fig. 6.6, we note that the basic triangular truss of Fig. 6.6b has three members and three joints. The truss of Fig. 6.6c has two more members and one more joint, i.e., five members and four joints altogether. Observing that every time two new members are added, the number of joints is increased by one, we find that in a simple truss the total number of members is  $m = 2n - 3$ , where  $n$  is the total number of joints.



**Photo 6.2** Two K-trusses were used as the main components of the movable bridge shown which moved above a large stockpile of ore. The bucket below the trusses picked up ore and redeposited it until the ore was thoroughly mixed. The ore was then sent to the mill for processing into steel.

<sup>†</sup>The three joints must not be in a straight line.

## 6.4 ANALYSIS OF TRUSSES BY THE METHOD OF JOINTS



**Fig. 6.7**



**Photo 6.3** Because roof trusses, such as those shown, require support only at their ends, it is possible to construct buildings with large, unobstructed floor areas.

We saw in Sec. 6.2 that a truss can be considered as a group of pins and two-force members. The truss of Fig. 6.2, whose free-body diagram is shown in Fig. 6.7a, can thus be dismembered, and a free-body diagram can be drawn for each pin and each member (Fig. 6.7b). Each member is acted upon by two forces, one at each end; these forces have the same magnitude, same line of action, and opposite sense (Sec. 4.6). Furthermore, Newton's third law indicates that the forces of action and reaction between a member and a pin are equal and opposite. Therefore, the forces exerted by a member on the two pins it connects must be directed along that member and be equal and opposite. The common magnitude of the forces exerted by a member on the two pins it connects is commonly referred to as the *force in the member* considered, even though this quantity is actually a scalar. Since the lines of action of all the internal forces in a truss are known, the analysis of a truss reduces to computing the forces in its various members and to determining whether each of its members is in tension or in compression.

Since the entire truss is in equilibrium, each pin must be in equilibrium. The fact that a pin is in equilibrium can be expressed by drawing its free-body diagram and writing two equilibrium equations (Sec. 2.9). If the truss contains  $n$  pins, there will, therefore, be  $2n$  equations available, which can be solved for  $2n$  unknowns. In the case of a simple truss, we have  $m = 2n - 3$ , that is,  $2n = m + 3$ , and the number of unknowns which can be determined from the free-body diagrams of the pins is thus  $m + 3$ . This means that the forces in all the members, the two components of the reaction  $\mathbf{R}_A$ , and the reaction  $\mathbf{R}_B$  can be found by considering the free-body diagrams of the pins.

The fact that the entire truss is a rigid body in equilibrium can be used to write three more equations involving the forces shown in the free-body diagram of Fig. 6.7a. Since they do not contain any new information, these equations are not independent of the equations associated with the free-body diagrams of the pins. Nevertheless, they can be used to determine the components of the reactions at the supports. The arrangement of pins and members in a simple truss is such that it will then always be possible to find a joint involving only two unknown forces. These forces can be determined by the methods of Sec. 2.11 and their values transferred to the adjacent joints and treated as known quantities at these joints. This procedure can be repeated until all unknown forces have been determined.

As an example, the truss of Fig. 6.7 will be analyzed by considering the equilibrium of each pin successively, starting with a joint at which only two forces are unknown. In the truss considered, all pins are subjected to at least three unknown forces. Therefore, the reactions at the supports must first be determined by considering the entire truss as a free body and using the equations of equilibrium of a rigid body. We find in this way that  $\mathbf{R}_A$  is vertical and determine the magnitudes of  $\mathbf{R}_A$  and  $\mathbf{R}_B$ .

The number of unknown forces at joint A is thus reduced to two, and these forces can be determined by considering the equilibrium of pin A. The reaction  $\mathbf{R}_A$  and the forces  $\mathbf{F}_{AC}$  and  $\mathbf{F}_{AD}$  exerted

on pin A by members AC and AD, respectively, must form a force triangle. First we draw  $\mathbf{R}_A$  (Fig. 6.8); noting that  $\mathbf{F}_{AC}$  and  $\mathbf{F}_{AD}$  are directed along AC and AD, respectively, we complete the triangle and determine the magnitude and sense of  $\mathbf{F}_{AC}$  and  $\mathbf{F}_{AD}$ . The magnitudes  $F_{AC}$  and  $F_{AD}$  represent the forces in members AC and AD. Since  $\mathbf{F}_{AC}$  is directed down and to the left, that is, *toward* joint A, member AC pushes on pin A and is in compression. Since  $\mathbf{F}_{AD}$  is directed *away* from joint A, member AD pulls on pin A and is in tension.

	Free-body diagram	Force polygon
Joint A		
Joint D		
Joint C		
Joint B		

Fig. 6.8

We can now proceed to joint D, where only two forces,  $\mathbf{F}_{DC}$  and  $\mathbf{F}_{DB}$ , are still unknown. The other forces are the load  $\mathbf{P}$ , which is given, and the force  $\mathbf{F}_{DA}$  exerted on the pin by member AD. As indicated above, this force is equal and opposite to the force  $\mathbf{F}_{AD}$  exerted by the same member on pin A. We can draw the force polygon corresponding to joint D, as shown in Fig. 6.8, and determine the forces

$\mathbf{F}_{DC}$  and  $\mathbf{F}_{DB}$  from that polygon. However, when more than three forces are involved, it is usually more convenient to solve the equations of equilibrium  $\sum F_x = 0$  and  $\sum F_y = 0$  for the two unknown forces. Since both of these forces are found to be directed away from joint  $D$ , members  $DC$  and  $DB$  pull on the pin and are in tension.

Next, joint  $C$  is considered; its free-body diagram is shown in Fig. 6.8. It is noted that both  $\mathbf{F}_{CD}$  and  $\mathbf{F}_{CA}$  are known from the analysis of the preceding joints and that only  $\mathbf{F}_{CB}$  is unknown. Since the equilibrium of each pin provides sufficient information to determine two unknowns, a check of our analysis is obtained at this joint. The force triangle is drawn, and the magnitude and sense of  $\mathbf{F}_{CB}$  are determined. Since  $\mathbf{F}_{CB}$  is directed toward joint  $C$ , member  $CB$  pushes on pin  $C$  and is in compression. The check is obtained by verifying that the force  $\mathbf{F}_{CB}$  and member  $CB$  are parallel.

At joint  $B$ , all of the forces are known. Since the corresponding pin is in equilibrium, the force triangle must close and an additional check of the analysis is obtained.

It should be noted that the force polygons shown in Fig. 6.8 are not unique. Each of them could be replaced by an alternative configuration. For example, the force triangle corresponding to joint  $A$  could be drawn as shown in Fig. 6.9. The triangle actually shown in Fig. 6.8 was obtained by drawing the three forces  $\mathbf{R}_A$ ,  $\mathbf{F}_{AC}$ , and  $\mathbf{F}_{AD}$  in tip-to-tail fashion in the order in which their lines of action are encountered when moving clockwise around joint  $A$ . The other force polygons in Fig. 6.8, having been drawn in the same way, can be made to fit into a single diagram, as shown in Fig. 6.10. Such a diagram, known as *Maxwell's diagram*, greatly facilitates the graphical analysis of truss problems.

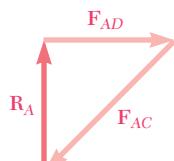


Fig. 6.9

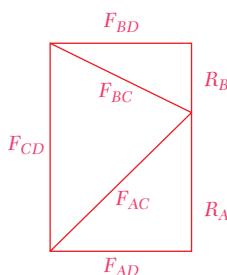


Fig. 6.10

## \*6.5 JOINTS UNDER SPECIAL LOADING CONDITIONS

Consider Fig. 6.11a, in which the joint shown connects four members lying in two intersecting straight lines. The free-body diagram of Fig. 6.11b shows that pin  $A$  is subjected to two pairs of directly opposite forces. The corresponding force polygon, therefore, must be a parallelogram (Fig. 6.11c), and *the forces in opposite members must be equal*.

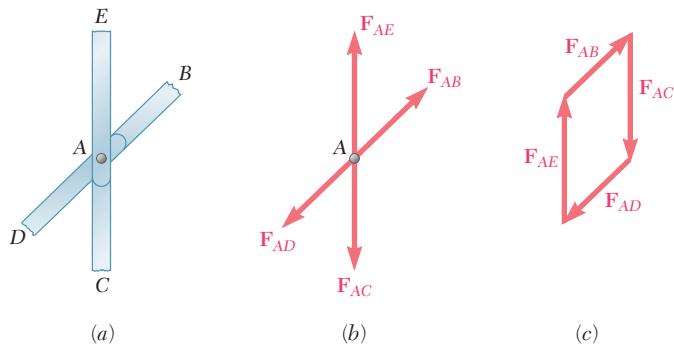
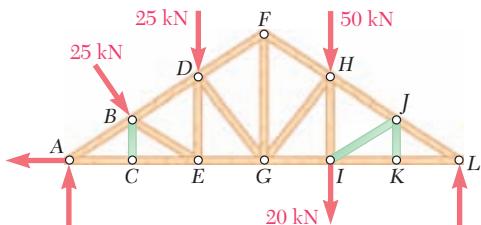


Fig. 6.11

Consider next Fig. 6.12a, in which the joint shown connects three members and supports a load  $\mathbf{P}$ . Two of the members lie in the same line, and the load  $\mathbf{P}$  acts along the third member. The free-body diagram of pin A and the corresponding force polygon will be as shown in Fig. 6.11b and c, with  $\mathbf{F}_{AE}$  replaced by the load  $\mathbf{P}$ . Thus, *the forces in the two opposite members must be equal, and the force in the other member must equal  $P$* . A particular case of special interest is shown in Fig. 6.12b. Since, in this case, no external load is applied to the joint, we have  $P = 0$ , and the force in member AC is zero. Member AC is said to be a *zero-force member*.

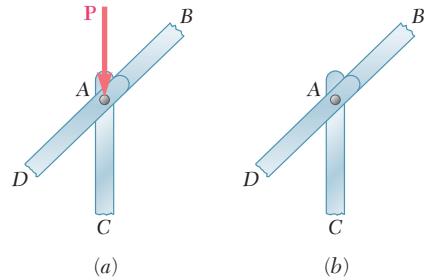
Consider now a joint connecting two members only. From Sec. 2.9, we know that a particle which is acted upon by two forces will be in equilibrium if the two forces have the same magnitude, same line of action, and opposite sense. In the case of the joint of Fig. 6.13a, which connects two members  $AB$  and  $AD$  lying in the same line, *the forces in the two members must be equal* for pin A to be in equilibrium. In the case of the joint of Fig. 6.13b, pin A cannot be in equilibrium unless the forces in both members are zero. Members connected as shown in Fig. 6.13b, therefore, must be *zero-force members*.

Spotting the joints which are under the special loading conditions listed above will expedite the analysis of a truss. Consider, for example, a Howe truss loaded as shown in Fig. 6.14. All of the members represented by green lines will be recognized as zero-force members. Joint C connects three members, two of which lie in the same line, and is not subjected to any external load; member  $BC$  is thus a zero-force member. Applying the same reasoning to joint K, we find that member  $JK$  is also a zero-force member. But joint J is now in the same situation as joints C and K, and member  $IJ$  must be a zero-force member. The examination of joints C, J, and K also shows that the forces in members  $AC$  and  $CE$  are equal, that the forces in members  $HJ$  and  $JL$  are equal, and that the forces in members  $IK$  and  $KL$  are equal. Turning our attention to joint I, where the 20-kN load and member  $HI$  are collinear, we note that the force in member  $HI$  is 20 kN (tension) and that the forces in members  $GI$  and  $IK$  are equal. Hence, the forces in members  $GI$ ,  $IK$ , and  $KL$  are equal.

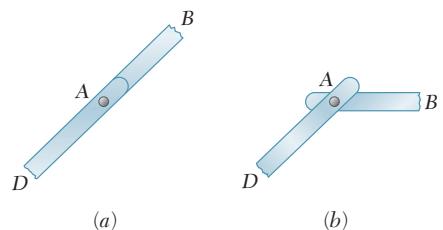


**Fig. 6.14**

Note that the conditions described above do not apply to joints *B* and *D* in Fig. 6.14, and it would be wrong to assume that the force in member *DE* is 25 kN or that the forces in members *AB* and *BD* are equal. The forces in these members and in all remaining members should be found by carrying out the analysis of joints *A*, *B*, *D*, *E*, *F*, *G*, *H*, and *L* in the usual manner. Thus, until you have become thoroughly familiar with the conditions under which the rules established in this



**Fig. 6.12**



**Fig. 6.13**



**Photo 6.4** Three-dimensional or space trusses are used for broadcast and power transmission line towers, roof framing, and spacecraft applications, such as components of the *International Space Station*.

section can be applied, you should draw the free-body diagrams of all pins and write the corresponding equilibrium equations (or draw the corresponding force polygons) whether or not the joints being considered are under one of the special loading conditions described above.

A final remark concerning zero-force members: These members are not useless. For example, although the zero-force members of Fig. 6.14 do not carry any loads under the loading conditions shown, the same members would probably carry loads if the loading conditions were changed. Besides, even in the case considered, these members are needed to support the weight of the truss and to maintain the truss in the desired shape.

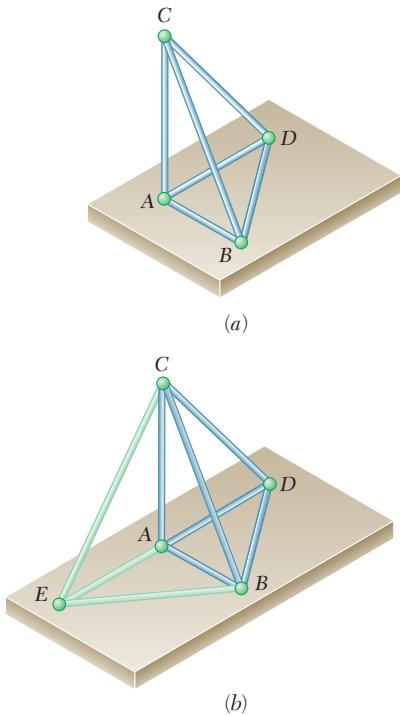
## \*6.6 SPACE TRUSSES

When several straight members are joined together at their extremities to form a three-dimensional configuration, the structure obtained is called a *space truss*.

We recall from Sec. 6.3 that the most elementary two-dimensional rigid truss consisted of three members joined at their extremities to form the sides of a triangle; by adding two members at a time to this basic configuration, and connecting them at a new joint, it was possible to obtain a larger rigid structure which was defined as a simple truss. Similarly, the most elementary rigid space truss consists of six members joined at their extremities to form the edges of a tetrahedron  $ABCD$  (Fig. 6.15a). By adding three members at a time to this basic configuration, such as  $AE$ ,  $BE$ , and  $CE$ , attaching them to three existing joints, and connecting them at a new joint,† we can obtain a larger rigid structure which is defined as a *simple space truss* (Fig. 6.15b). Observing that the basic tetrahedron has six members and four joints and that every time three members are added, the number of joints is increased by one, we conclude that in a simple space truss the total number of members is  $m = 3n - 6$ , where  $n$  is the total number of joints.

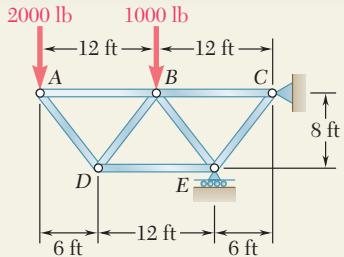
If a space truss is to be completely constrained and if the reactions at its supports are to be statically determinate, the supports should consist of a combination of balls, rollers, and balls and sockets which provides six unknown reactions (see Sec. 4.9). These unknown reactions may be readily determined by solving the six equations expressing that the three-dimensional truss is in equilibrium.

Although the members of a space truss are actually joined together by means of bolted or welded connections, it is assumed that each joint consists of a ball-and-socket connection. Thus, no couple will be applied to the members of the truss, and each member can be treated as a two-force member. The conditions of equilibrium for each joint will be expressed by the three equations  $\sum F_x = 0$ ,  $\sum F_y = 0$ , and  $\sum F_z = 0$ . In the case of a simple space truss containing  $n$  joints, writing the conditions of equilibrium for each joint will thus yield  $3n$  equations. Since  $m = 3n - 6$ , these equations suffice to determine all unknown forces (forces in  $m$  members and six reactions at the supports). However, to avoid the necessity of solving simultaneous equations, care should be taken to select joints in such an order that no selected joint will involve more than three unknown forces.



**Fig. 6.15**

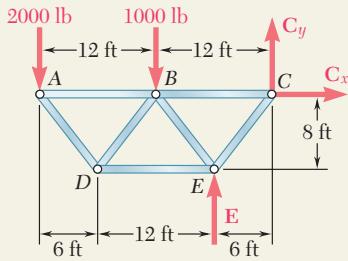
†The four joints must not lie in a plane.



## SAMPLE PROBLEM 6.1

Using the method of joints, determine the force in each member of the truss shown.

## SOLUTION

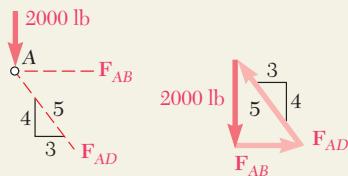


**Free-Body: Entire Truss.** A free-body diagram of the entire truss is drawn; external forces acting on this free body consist of the applied loads and the reactions at  $C$  and  $E$ . We write the following equilibrium equations.

$$+\uparrow \sum M_C = 0: \quad (2000 \text{ lb})(24 \text{ ft}) + (1000 \text{ lb})(12 \text{ ft}) - E(6 \text{ ft}) = 0 \\ E = +10,000 \text{ lb} \quad \mathbf{E} = 10,000 \text{ lb} \uparrow$$

$$+\rightarrow \sum F_x = 0: \quad \mathbf{C}_x = 0$$

$$+\uparrow \sum F_y = 0: \quad -2000 \text{ lb} - 1000 \text{ lb} + 10,000 \text{ lb} + C_y = 0 \\ C_y = -7000 \text{ lb} \quad \mathbf{C}_y = 7000 \text{ lb} \downarrow$$

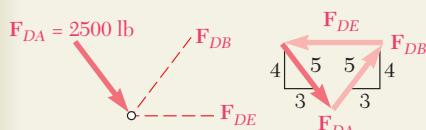


**Free-Body: Joint A.** This joint is subjected to only two unknown forces, namely, the forces exerted by members  $AB$  and  $AD$ . A force triangle is used to determine  $\mathbf{F}_{AB}$  and  $\mathbf{F}_{AD}$ . We note that member  $AB$  pulls on the joint and thus is in tension and that member  $AD$  pushes on the joint and thus is in compression. The magnitudes of the two forces are obtained from the proportion

$$\frac{2000 \text{ lb}}{4} = \frac{F_{AB}}{3} = \frac{F_{AD}}{5}$$

$$F_{AB} = 1500 \text{ lb } T$$

$$F_{AD} = 2500 \text{ lb } C$$



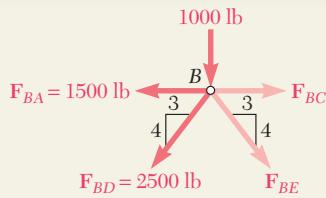
**Free-Body: Joint D.** Since the force exerted by member  $AD$  has been determined, only two unknown forces are now involved at this joint. Again, a force triangle is used to determine the unknown forces in members  $DB$  and  $DE$ .

$$F_{DB} = F_{DA}$$

$$F_{DE} = 2(\frac{3}{5})F_{DA}$$

$$F_{DB} = 2500 \text{ lb } T$$

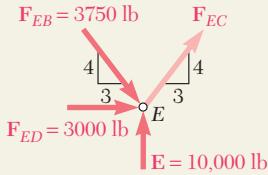
$$F_{DE} = 3000 \text{ lb } C$$



**Free-Body: Joint B.** Since more than three forces act at this joint, we determine the two unknown forces  $\mathbf{F}_{BC}$  and  $\mathbf{F}_{BE}$  by solving the equilibrium equations  $\sum F_x = 0$  and  $\sum F_y = 0$ . We arbitrarily assume that both unknown forces act away from the joint, i.e., that the members are in tension. The positive value obtained for  $F_{BC}$  indicates that our assumption was correct; member  $BC$  is in tension. The negative value of  $F_{BE}$  indicates that our assumption was wrong; member  $BE$  is in compression.

$$+\uparrow \sum F_y = 0: -1000 - \frac{4}{5}(2500) - \frac{4}{5}F_{BE} = 0 \\ F_{BE} = -3750 \text{ lb} \quad F_{BE} = 3750 \text{ lb } C \quad \blacktriangleleft$$

$$\stackrel{+}{\rightarrow} \sum F_x = 0: F_{BC} - 1500 - \frac{3}{5}(2500) - \frac{3}{5}(3750) = 0 \\ F_{BC} = +5250 \text{ lb} \quad F_{BC} = 5250 \text{ lb } T \quad \blacktriangleleft$$

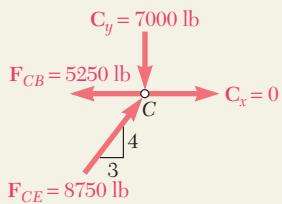


**Free-Body: Joint E.** The unknown force  $\mathbf{F}_{EC}$  is assumed to act away from the joint. Summing  $x$  components, we write

$$\stackrel{+}{\rightarrow} \sum F_x = 0: \frac{3}{5}F_{EC} + 3000 + \frac{3}{5}(3750) = 0 \\ F_{EC} = -8750 \text{ lb} \quad F_{EC} = 8750 \text{ lb } C \quad \blacktriangleleft$$

Summing  $y$  components, we obtain a check of our computations:

$$+\uparrow \sum F_y = 10,000 - \frac{4}{5}(3750) - \frac{4}{5}(8750) \\ = 10,000 - 3000 - 7000 = 0 \quad (\text{checks})$$



**Free-Body: Joint C.** Using the computed values of  $\mathbf{F}_{CB}$  and  $\mathbf{F}_{CE}$ , we can determine the reactions  $\mathbf{C}_x$  and  $\mathbf{C}_y$  by considering the equilibrium of this joint. Since these reactions have already been determined from the equilibrium of the entire truss, we will obtain two checks of our computations. We can also simply use the computed values of all forces acting on the joint (forces in members and reactions) and check that the joint is in equilibrium:

$$\stackrel{+}{\rightarrow} \sum F_x = -5250 + \frac{3}{5}(8750) = -5250 + 5250 = 0 \quad (\text{checks}) \\ +\uparrow \sum F_y = -7000 + \frac{4}{5}(8750) = -7000 + 7000 = 0 \quad (\text{checks})$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to use the *method of joints* to determine the forces in the members of a *simple truss*, that is, a truss that can be constructed from a basic triangular truss by adding to it two new members at a time and connecting them at a new joint.

Your solution will consist of the following steps:

1. **Draw a free-body diagram of the entire truss,** and use this diagram to determine the reactions at the supports.
2. **Locate a joint connecting only two members, and draw the free-body diagram of its pin.** Use this free-body diagram to determine the unknown force in each of the two members. If only three forces are involved (the two unknown forces and a known one), you will probably find it more convenient to draw and solve the corresponding force triangle. If more than three forces are involved, you should write and solve the equilibrium equations for the pin,  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$ , assuming that the members are in tension. A positive answer means that the member is in tension, a negative answer that the member is in compression. Once the forces have been found, enter their values on a sketch of the truss, with *T* for tension and *C* for compression.
3. **Next, locate a joint where the forces in only two of the connected members are still unknown.** Draw the free-body diagram of the pin and use it as indicated above to determine the two unknown forces.
4. **Repeat this procedure until the forces in all the members of the truss have been found.** Since you previously used the three equilibrium equations associated with the free-body diagram of the entire truss to determine the reactions at the supports, you will end up with three extra equations. These equations can be used to check your computations.
5. **Note that the choice of the first joint is not unique.** Once you have determined the reactions at the supports of the truss, you can choose either of two joints as a starting point for your analysis. In Sample Prob. 6.1, we started at joint *A* and proceeded through joints *D*, *B*, *E*, and *C*, but we could also have started at joint *C* and proceeded through joints *E*, *B*, *D*, and *A*. On the other hand, having selected a first joint, you may in some cases reach a point in your analysis beyond which you cannot proceed. You must then start again from another joint to complete your solution.

Keep in mind that the analysis of a *simple truss* can always be carried out by the method of joints. Also remember that it is helpful to outline your solution *before* starting any computations.

# PROBLEMS

**6.1 through 6.8** Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

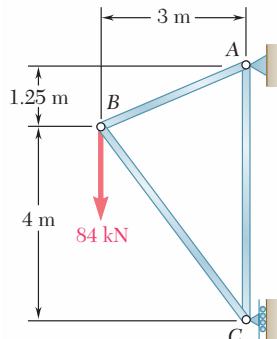


Fig. P6.1

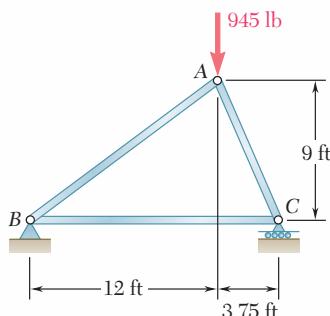


Fig. P6.2

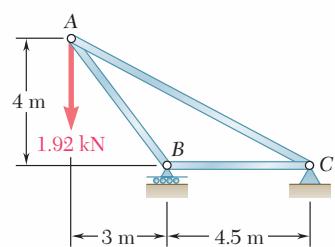


Fig. P6.3

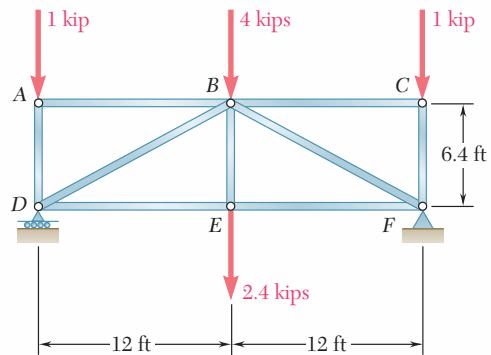


Fig. P6.4

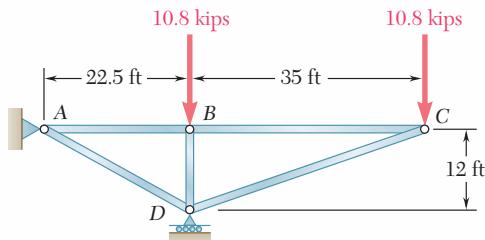


Fig. P6.5

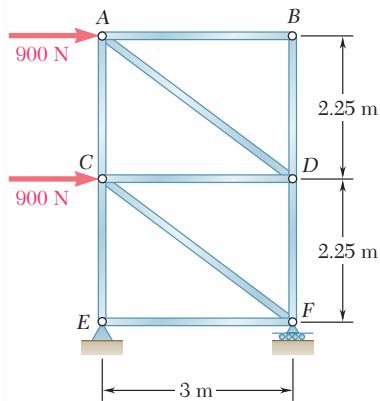


Fig. P6.6

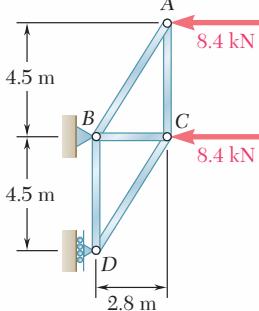


Fig. P6.7

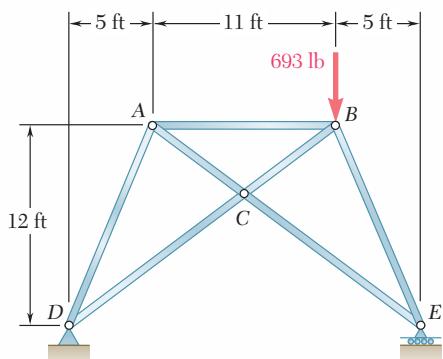


Fig. P6.8

- 6.9** Determine the force in each member of the Pratt roof truss shown. State whether each member is in tension or compression.

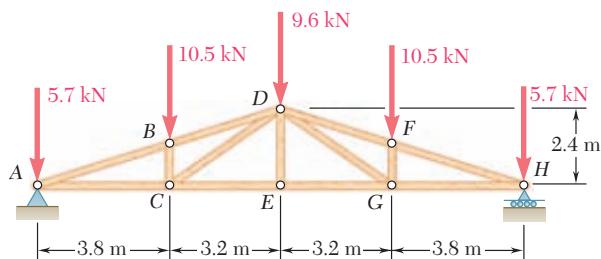


Fig. P6.9

- 6.10** Determine the force in each member of the fan roof truss shown. State whether each member is in tension or compression.

- 6.11** Determine the force in each member of the Howe roof truss shown. State whether each member is in tension or compression.

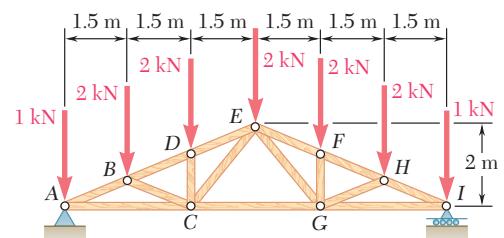


Fig. P6.10

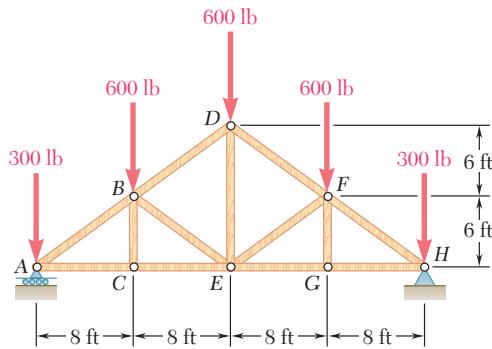


Fig. P6.11

- 6.12** Determine the force in each member of the Gambrel roof truss shown. State whether each member is in tension or compression.

- 6.13** Determine the force in each member of the truss shown.

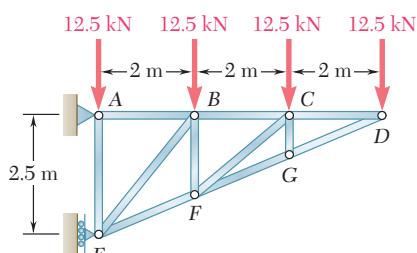


Fig. P6.13

- 6.14** Determine the force in each member of the roof truss shown. State whether each member is in tension or compression.

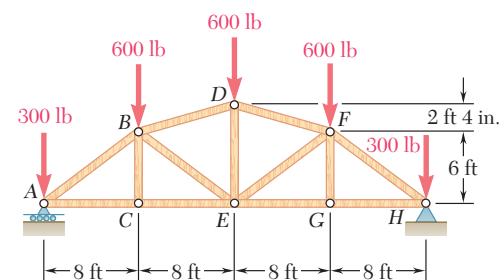


Fig. P6.12

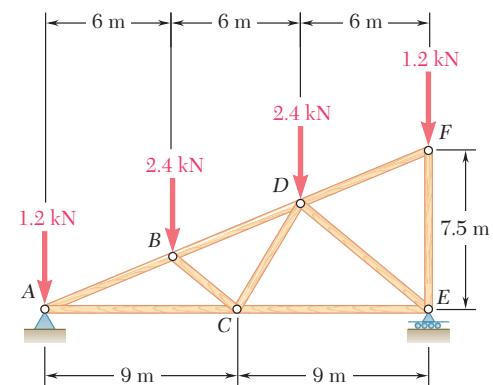


Fig. P6.14

- 6.15** Determine the force in each member of the Warren bridge truss shown. State whether each member is in tension or compression.

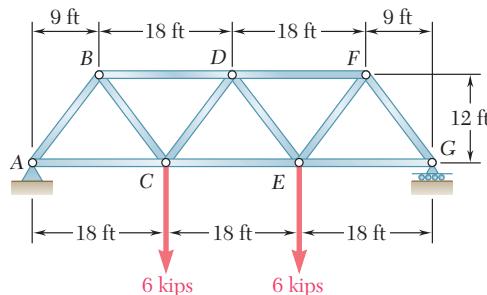


Fig. P6.15

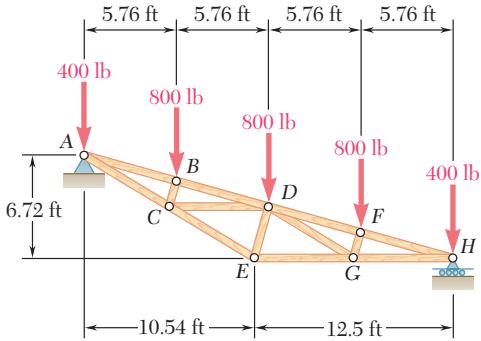


Fig. P6.17 and P6.18

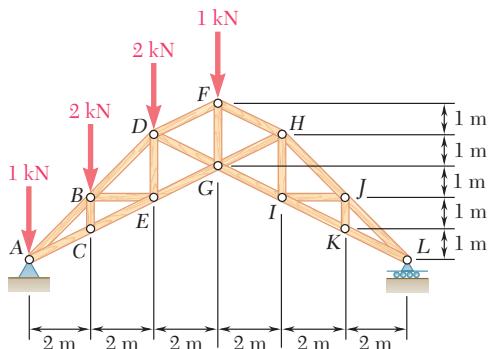


Fig. P6.19 and P6.20

- 6.16** Solve Prob. 6.15 assuming that the load applied at *E* has been removed.

- 6.17** Determine the force in member *DE* and in each of the members located to the left of *DE* for the inverted Howe roof truss shown. State whether each member is in tension or compression.

- 6.18** Determine the force in each of the members located to the right of *DE* for the inverted Howe roof truss shown. State whether each member is in tension or compression.

- 6.19** Determine the force in each of the members located to the left of *FG* for the scissors roof truss shown. State whether each member is in tension or compression.

- 6.20** Determine the force in member *FG* and in each of the members located to the right of *FG* for the scissors roof truss shown. State whether each member is in tension or compression.

- 6.21** Determine the force in each of the members located to the left of line *FGH* for the studio roof truss shown. State whether each member is in tension or compression.

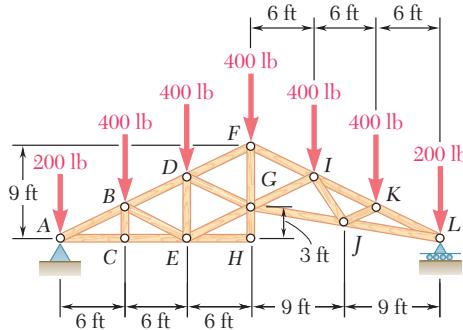


Fig. P6.21 and P6.22

- 6.22** Determine the force in member *FG* and in each of the members located to the right of *FG* for the studio roof truss shown. State whether each member is in tension or compression.

- 6.23** Determine the force in each of the members connecting joints A through F of the vaulted roof truss shown. State whether each member is in tension or compression.

- 6.24** The portion of truss shown represents the upper part of a power transmission line tower. For the given loading, determine the force in each of the members located above HJ. State whether each member is in tension or compression.

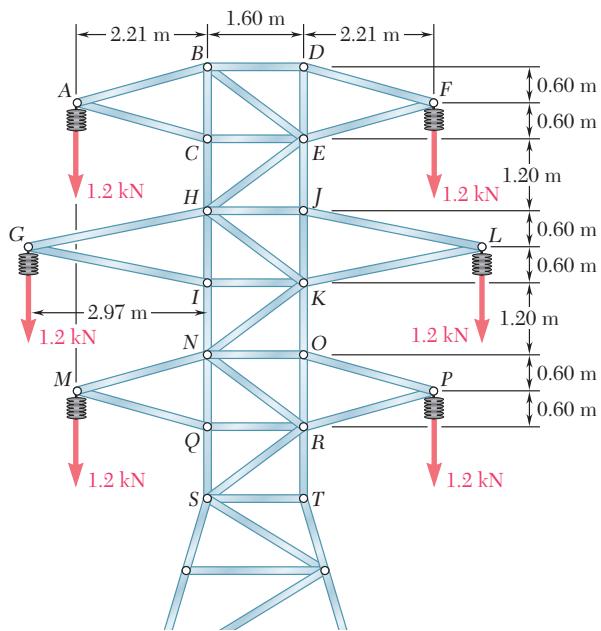


Fig. P6.24

- 6.25** For the tower and loading of Prob. 6.24 and knowing that  $F_{CH} = F_{EJ} = 1.2 \text{ kN}$  and  $F_{EH} = 0$ , determine the force in member HJ and in each of the members located between HJ and NO. State whether each member is in tension or compression.

- 6.26** Solve Prob. 6.24 assuming that the cables hanging from the right side of the tower have fallen to the ground.

- 6.27** Determine the force in each member of the truss shown. State whether each member is in tension or compression.

- 6.28** Determine the force in each member of the truss shown. State whether each member is in tension or compression.

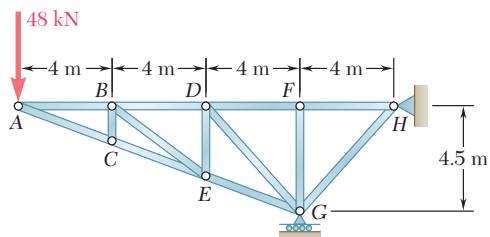


Fig. P6.28

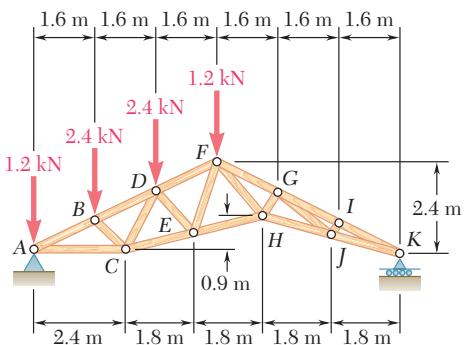


Fig. P6.23

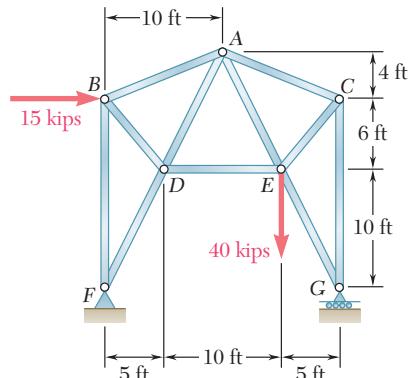


Fig. P6.27

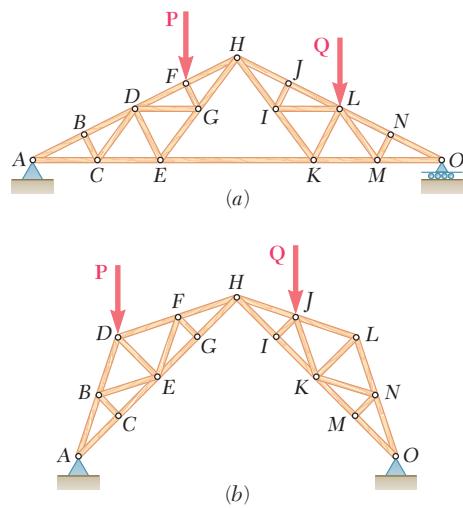


Fig. P6.31

**6.29** Determine whether the trusses of Probs. 6.31a, 6.32a, and 6.33a are simple trusses.

**6.30** Determine whether the trusses of Probs. 6.31b, 6.32b, and 6.33b are simple trusses.

**6.31** For the given loading, determine the zero-force members in each of the two trusses shown.

**6.32** For the given loading, determine the zero-force members in each of the two trusses shown.

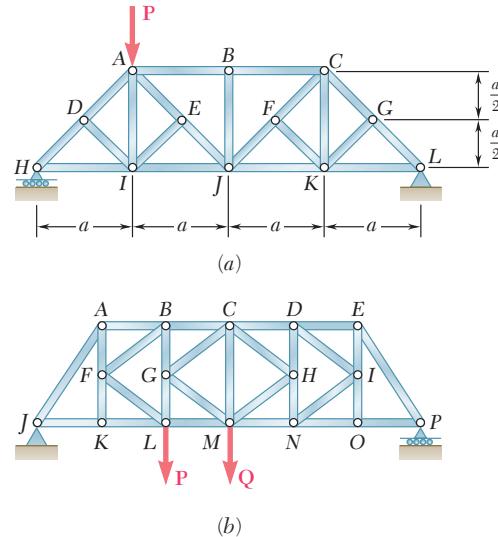


Fig. P6.32

**6.33** For the given loading, determine the zero-force members in each of the two trusses shown.

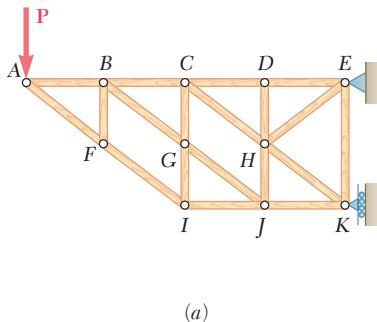
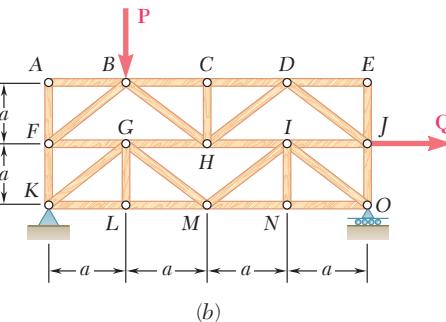


Fig. P6.33



**6.34** Determine the zero-force members in the truss of (a) Prob. 6.23, (b) Prob. 6.28.

- \*6.35** The truss shown consists of six members and is supported by a short link at *A*, two short links at *B*, and a ball and socket at *D*. Determine the force in each of the members for the given loading.

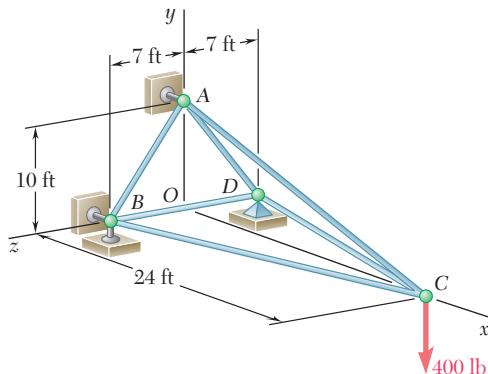


Fig. P6.35

- \*6.36** The truss shown consists of six members and is supported by a ball and socket at *B*, a short link at *C*, and two short links at *D*. Determine the force in each of the members for  $\mathbf{P} = (-2184 \text{ N})\mathbf{j}$  and  $\mathbf{Q} = 0$ .

- \*6.37** The truss shown consists of six members and is supported by a ball and socket at *B*, a short link at *C*, and two short links at *D*. Determine the force in each of the members for  $\mathbf{P} = 0$  and  $\mathbf{Q} = (2968 \text{ N})\mathbf{i}$ .

- \*6.38** The truss shown consists of nine members and is supported by a ball and socket at *A*, two short links at *B*, and a short link at *C*. Determine the force in each of the members for the given loading.

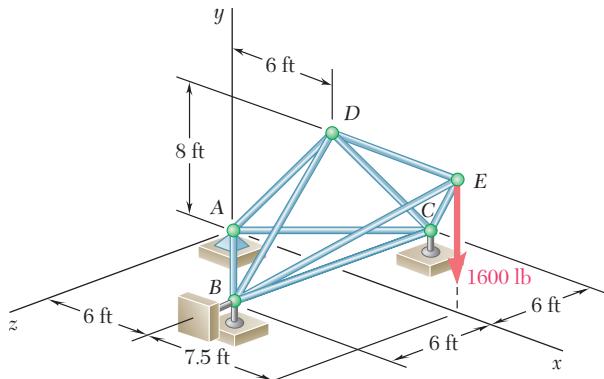


Fig. P6.38

- \*6.39** The truss shown consists of nine members and is supported by a ball and socket at *B*, a short link at *C*, and two short links at *D*. (a) Check that this truss is a simple truss, that it is completely constrained, and that the reactions at its supports are statically determinate. (b) Determine the force in each member for  $\mathbf{P} = (-1200 \text{ N})\mathbf{j}$  and  $\mathbf{Q} = 0$ .

- \*6.40** Solve Prob. 6.39 for  $\mathbf{P} = 0$  and  $\mathbf{Q} = (-900 \text{ N})\mathbf{k}$ .

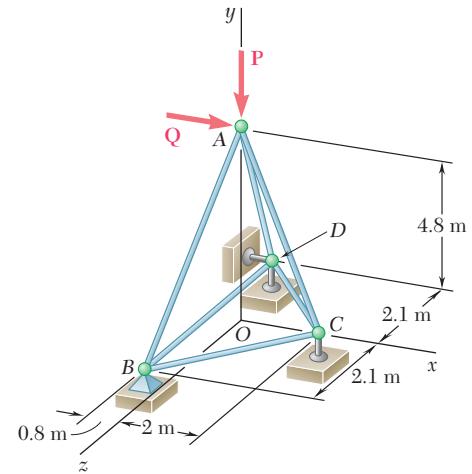


Fig. P6.36 and P6.37

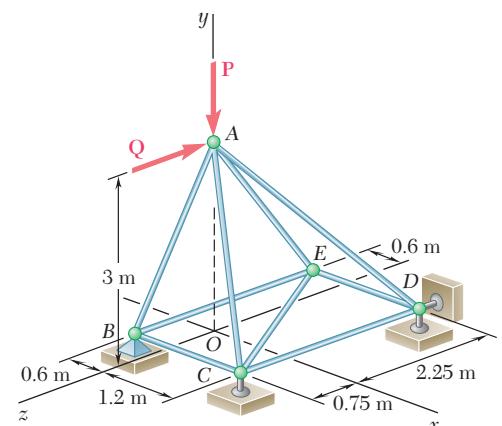


Fig. P6.39

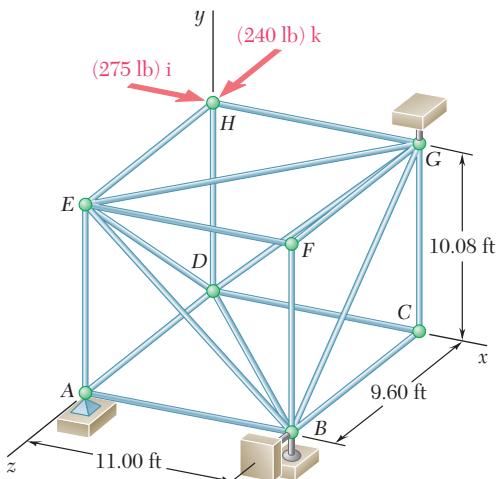


Fig. P6.41 and P6.42

- \*6.41** The truss shown consists of 18 members and is supported by a ball and socket at *A*, two short links at *B*, and one short link at *G*. (a) Check that this truss is a simple truss, that it is completely constrained, and that the reactions at its supports are statically determinate. (b) For the given loading, determine the force in each of the six members joined at *E*.

- \*6.42** The truss shown consists of 18 members and is supported by a ball and socket at *A*, two short links at *B*, and one short link at *G*. (a) Check that this truss is a simple truss, that it is completely constrained, and that the reactions at its supports are statically determinate. (b) For the given loading, determine the force in each of the six members joined at *G*.

## 6.7 ANALYSIS OF TRUSSES BY THE METHOD OF SECTIONS

The method of joints is most effective when the forces in all the members of a truss are to be determined. If, however, the force in only one member or the forces in a very few members are desired, another method, the method of sections, is more efficient.

Assume, for example, that we want to determine the force in member *BD* of the truss shown in Fig. 6.16a. To do this, we must determine the force with which member *BD* acts on either joint *B* or joint *D*. If we were to use the method of joints, we would choose either joint *B* or joint *D* as a free body. However, we can also choose as a free body a larger portion of the truss, composed of several joints and members, provided that the desired force is one of the external forces acting on that portion. If, in addition, the portion of the truss is chosen so that there is a total of only three unknown forces acting upon it, the desired force can be obtained by solving the equations of equilibrium for this portion of the truss. In practice, the portion of the truss to be utilized is obtained by *passing a section* through three members of the truss, one of which is the desired member, i.e., by drawing a line which divides the truss into two completely separate parts but does not intersect more than three members. Either of the two portions of the truss obtained after the intersected members have been removed can then be used as a free body.<sup>†</sup>

In Fig. 6.16a, the section *nn* has been passed through members *BD*, *BE*, and *CE*, and the portion *ABC* of the truss is chosen as the free body (Fig. 6.16b). The forces acting on the free body are the loads  $\mathbf{P}_1$  and  $\mathbf{P}_2$  at points *A* and *B* and the three unknown forces  $\mathbf{F}_{BD}$ ,  $\mathbf{F}_{BE}$ , and  $\mathbf{F}_{CE}$ . Since it is not known whether the members removed were in tension or compression, the three forces have been arbitrarily drawn away from the free body as if the members were in tension.

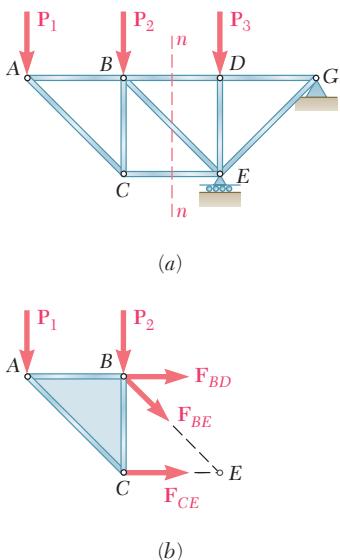


Fig. 6.16

<sup>†</sup>In the analysis of certain trusses, sections are passed which intersect more than three members; the forces in one, or possibly two, of the intersected members may be obtained if equilibrium equations can be found, each of which involves only one unknown (see Probs. 6.61 through 6.64).

The fact that the rigid body  $ABC$  is in equilibrium can be expressed by writing three equations which can be solved for the three unknown forces. If only the force  $\mathbf{F}_{BD}$  is desired, we need write only one equation, provided that the equation does not contain the other unknowns. Thus the equation  $\sum M_E = 0$  yields the value of the magnitude  $F_{BD}$  of the force  $\mathbf{F}_{BD}$  (Fig. 6.16b). A positive sign in the answer will indicate that our original assumption regarding the sense of  $\mathbf{F}_{BD}$  was correct and that member  $BD$  is in tension; a negative sign will indicate that our assumption was incorrect and that  $BD$  is in compression.

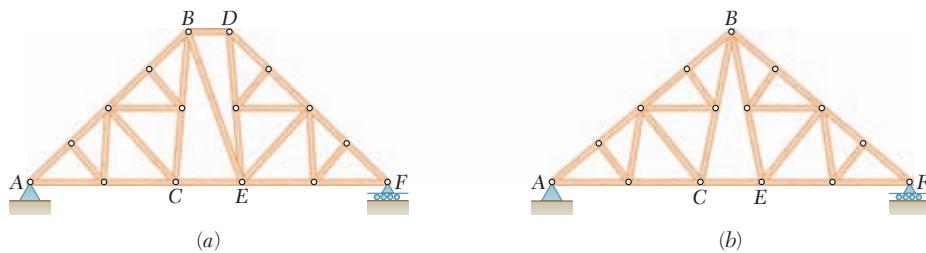
On the other hand, if only the force  $\mathbf{F}_{CE}$  is desired, an equation which does not involve  $\mathbf{F}_{BD}$  or  $\mathbf{F}_{BE}$  should be written; the appropriate equation is  $\sum M_B = 0$ . Again a positive sign for the magnitude  $F_{CE}$  of the desired force indicates a correct assumption, that is, tension; and a negative sign indicates an incorrect assumption, that is, compression.

If only the force  $\mathbf{F}_{BE}$  is desired, the appropriate equation is  $\sum F_y = 0$ . Whether the member is in tension or compression is again determined from the sign of the answer.

When the force in only one member is determined, no independent check of the computation is available. However, when all the unknown forces acting on the free body are determined, the computations can be checked by writing an additional equation. For instance, if  $\mathbf{F}_{BD}$ ,  $\mathbf{F}_{BE}$ , and  $\mathbf{F}_{CE}$  are determined as indicated above, the computation can be checked by verifying that  $\sum F_x = 0$ .

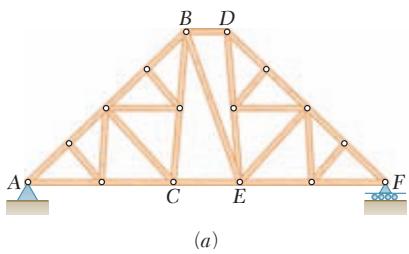
## \*6.8 TRUSSES MADE OF SEVERAL SIMPLE TRUSSES

Consider two simple trusses  $ABC$  and  $DEF$ . If they are connected by three bars  $BD$ ,  $BE$ , and  $CE$  as shown in Fig. 6.17a, they will form together a rigid truss  $ABDF$ . The trusses  $ABC$  and  $DEF$  can also be combined into a single rigid truss by joining joints  $B$  and  $D$  into a single joint  $B$  and by connecting joints  $C$  and  $E$  by a bar  $CE$  (Fig. 6.17b). The truss thus obtained is known as a *Fink truss*. It should be noted that the trusses of Fig. 6.17a and b are *not* simple trusses; they cannot be constructed from a triangular truss by adding successive pairs of members as prescribed in Sec. 6.3. They are rigid trusses, however, as we can check by comparing the systems of connections used to hold the simple trusses  $ABC$  and  $DEF$  together (three bars in Fig. 6.17a, one pin and one bar in Fig. 6.17b) with the systems of supports discussed in Secs. 4.4 and 4.5. Trusses made of several simple trusses rigidly connected are known as *compound trusses*.

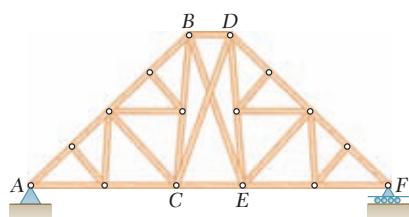


**Fig. 6.17**

In a compound truss the number of members  $m$  and the number of joints  $n$  are still related by the formula  $m = 2n - 3$ . This can be verified by observing that, if a compound truss is supported by a frictionless pin and a roller (involving three unknown reactions), the total number of unknowns is  $m + 3$ , and this number must be equal to the number  $2n$  of equations obtained by expressing that the  $n$  pins are in equilibrium; it follows that  $m = 2n - 3$ . Compound trusses supported by a pin and a roller, or by an equivalent system of supports, are *statically determinate, rigid, and completely constrained*. This means that all of the unknown reactions and the forces in all the members can be determined by the methods of statics, and that the truss will neither collapse nor move. The forces in the members, however, cannot all be determined by the method of joints, except by solving a large number of simultaneous equations. In the case of the compound truss of Fig. 6.17a, for example, it is more efficient to pass a section through members  $BD$ ,  $BE$ , and  $CE$  to determine the forces in these members.



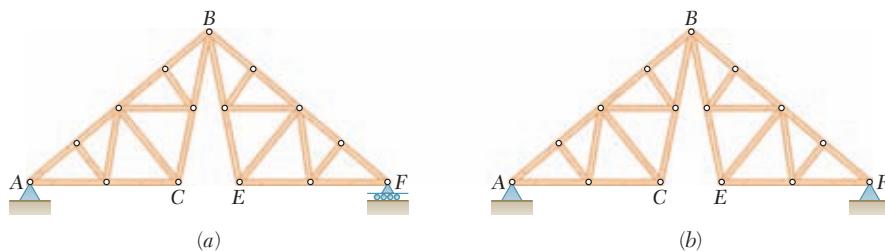
**Fig. 6.17** (repeated)



**Fig. 6.18**

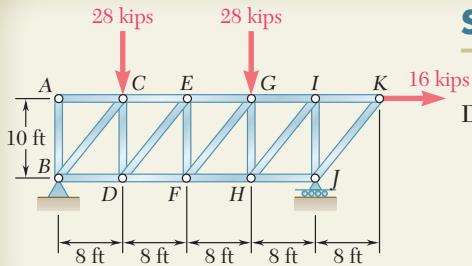
Suppose, now, that the simple trusses  $ABC$  and  $DEF$  are connected by four bars  $BD$ ,  $BE$ ,  $CD$ , or  $CE$  (Fig. 6.18). The number of members  $m$  is now larger than  $2n - 3$ ; the truss obtained is *overrigid*, and one of the four members  $BD$ ,  $BE$ ,  $CD$ , or  $CE$  is said to be *redundant*. If the truss is supported by a pin at  $A$  and a roller at  $F$ , the total number of unknowns is  $m + 3$ . Since  $m > 2n - 3$ , the number  $m + 3$  of unknowns is now larger than the number  $2n$  of available independent equations; the truss is *statically indeterminate*.

Finally, let us assume that the two simple trusses  $ABC$  and  $DEF$  are joined by a pin as shown in Fig. 6.19a. The number of members  $m$  is smaller than  $2n - 3$ . If the truss is supported by a pin at  $A$  and a roller at  $F$ , the total number of unknowns is  $m + 3$ . Since  $m < 2n - 3$ , the number  $m + 3$  of unknowns is now smaller than the number  $2n$  of equilibrium equations which should be satisfied; the truss is *non-rigid* and will collapse under its own weight. However, if two pins are used to support it, the truss becomes *rigid* and will not collapse (Fig. 6.19b). We note that the total number of unknowns is now  $m + 4$  and is equal to the number  $2n$  of equations. More generally, if the reactions at the supports involve  $r$  unknowns, the condition for a compound truss to be statically determinate, rigid, and completely constrained is  $m + r = 2n$ . However, while necessary this condition is not sufficient for the equilibrium of a structure which ceases to be rigid when detached from its supports (see Sec. 6.11).



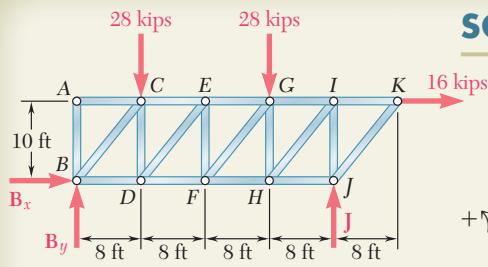
**Fig. 6.19**

## SAMPLE PROBLEM 6.2



Determine the force in members *EF* and *GI* of the truss shown.

## SOLUTION



**Free-Body: Entire Truss.** A free-body diagram of the entire truss is drawn; external forces acting on this free body consist of the applied loads and the reactions at *B* and *J*. We write the following equilibrium equations.

$$+\uparrow\sum M_B = 0:$$

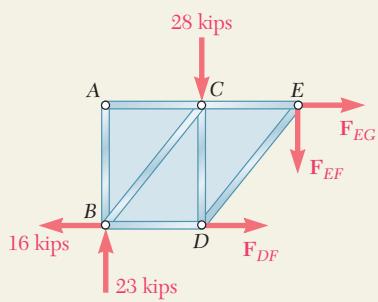
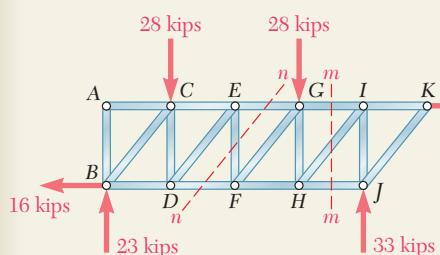
$$-(28 \text{ kips})(8 \text{ ft}) - (28 \text{ kips})(24 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) + J(32 \text{ ft}) = 0 \\ J = +33 \text{ kips} \quad \mathbf{J} = 33 \text{ kips} \uparrow$$

$$+\sum F_x = 0: \quad B_x + 16 \text{ kips} = 0$$

$$B_x = -16 \text{ kips} \quad \mathbf{B}_x = 16 \text{ kips} \leftarrow$$

$$+\uparrow\sum M_J = 0:$$

$$(28 \text{ kips})(24 \text{ ft}) + (28 \text{ kips})(8 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) - B_y(32 \text{ ft}) = 0 \\ B_y = +23 \text{ kips} \quad \mathbf{B}_y = 23 \text{ kips} \uparrow$$

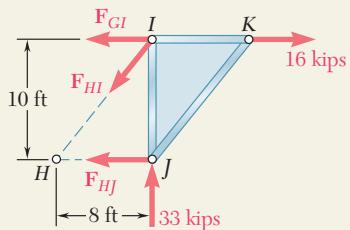


**Force in Member *EF*.** Section *nn* is passed through the truss so that it intersects member *EF* and only two additional members. After the intersected members have been removed, the left-hand portion of the truss is chosen as a free body. Three unknowns are involved; to eliminate the two horizontal forces, we write

$$+\uparrow\sum F_y = 0: \quad +23 \text{ kips} - 28 \text{ kips} - F_{EF} = 0 \\ F_{EF} = -5 \text{ kips}$$

The sense of  $F_{EF}$  was chosen assuming member *EF* to be in tension; the negative sign obtained indicates that the member is in compression.

$$F_{EF} = 5 \text{ kips} \mathbf{C}$$

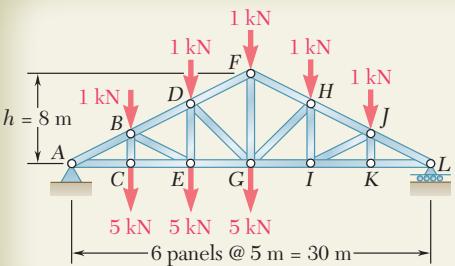


**Force in Member *GI*.** Section *mm* is passed through the truss so that it intersects member *GI* and only two additional members. After the intersected members have been removed, we choose the right-hand portion of the truss as a free body. Three unknown forces are again involved; to eliminate the two forces passing through point *H*, we write

$$+\uparrow\sum M_H = 0: \quad (33 \text{ kips})(8 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) + F_{GI}(10 \text{ ft}) = 0$$

$$F_{GI} = -10.4 \text{ kips} \quad \mathbf{F}_{GI} = 10.4 \text{ kips} \mathbf{C}$$

## SAMPLE PROBLEM 6.3



Determine the force in members  $FH$ ,  $GH$ , and  $GI$  of the roof truss shown.

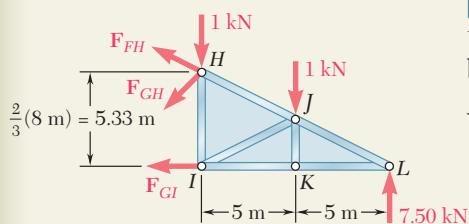
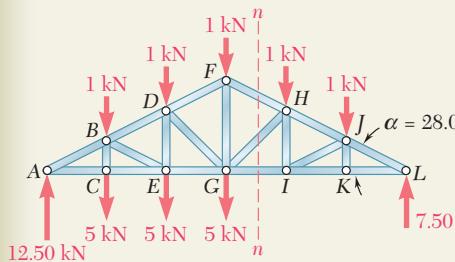
## SOLUTION

**Free Body: Entire Truss.** From the free-body diagram of the entire truss, we find the reactions at  $A$  and  $L$ :

$$A = 12.50 \text{ kN} \uparrow \quad L = 7.50 \text{ kN} \uparrow$$

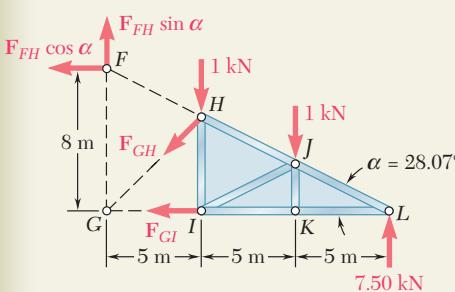
We note that

$$\tan \alpha = \frac{FG}{GL} = \frac{8 \text{ m}}{15 \text{ m}} = 0.5333 \quad \alpha = 28.07^\circ$$



**Force in Member  $GI$ .** Section  $nn$  is passed through the truss as shown. Using the portion  $HLI$  of the truss as a free body, the value of  $F_{GI}$  is obtained by writing

$$+\uparrow\sum M_H = 0: \quad (7.50 \text{ kN})(10 \text{ m}) - (1 \text{ kN})(5 \text{ m}) - F_{GI}(5.33 \text{ m}) = 0 \\ F_{GI} = +13.13 \text{ kN} \quad F_{GI} = 13.13 \text{ kN T} \quad \blacktriangleleft$$



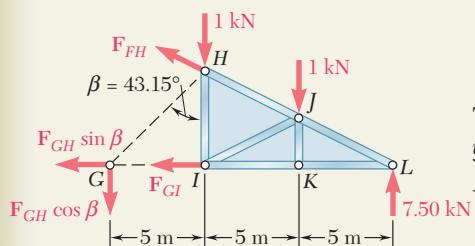
**Force in Member  $FH$ .** The value of  $F_{FH}$  is obtained from the equation  $\sum M_G = 0$ . We move  $\mathbf{F}_{FH}$  along its line of action until it acts at point  $F$ , where it is resolved into its  $x$  and  $y$  components. The moment of  $\mathbf{F}_{FH}$  with respect to point  $G$  is now equal to  $(F_{FH} \cos \alpha)(8 \text{ m})$ .

$$+\uparrow\sum M_G = 0: \quad (7.50 \text{ kN})(15 \text{ m}) - (1 \text{ kN})(10 \text{ m}) - (1 \text{ kN})(5 \text{ m}) + (F_{FH} \cos \alpha)(8 \text{ m}) = 0 \\ F_{FH} = -13.81 \text{ kN} \quad F_{FH} = 13.81 \text{ kN C} \quad \blacktriangleleft$$

**Force in Member  $GH$ .** We first note that

$$\tan \beta = \frac{GI}{HI} = \frac{5 \text{ m}}{\frac{2}{3}(8 \text{ m})} = 0.9375 \quad \beta = 43.15^\circ$$

The value of  $F_{GH}$  is then determined by resolving the force  $\mathbf{F}_{GH}$  into  $x$  and  $y$  components at point  $G$  and solving the equation  $\sum M_L = 0$ .



$$+\uparrow\sum M_L = 0: \quad (1 \text{ kN})(10 \text{ m}) + (1 \text{ kN})(5 \text{ m}) + (F_{GH} \cos \beta)(15 \text{ m}) = 0 \\ F_{GH} = -1.371 \text{ kN} \quad F_{GH} = 1.371 \text{ kN C} \quad \blacktriangleleft$$

# SOLVING PROBLEMS ON YOUR OWN

The *method of joints* that you studied earlier is usually the best method to use when the forces *in all the members* of a simple truss are to be found. However, the *method of sections*, which was covered in this lesson, is more effective when the force *in only one member* or the forces *in a very few members* of a simple truss are desired. The *method of sections* must also be used when the truss is *not a simple truss*.

**A. To determine the force in a given truss member** by the *method of sections*, you should follow these steps:

- 1. Draw a free-body diagram of the entire truss,** and use this diagram to determine the reactions at the supports.
- 2. Pass a section through three members of the truss,** one of which is the desired member. After you have removed these members, you will obtain two separate portions of truss.
- 3. Select one of the two portions of truss you have obtained, and draw its free-body diagram.** This diagram should include the external forces applied to the selected portion as well as the forces exerted on it by the intersected members before these members were removed.
- 4. You can now write three equilibrium equations** which can be solved for the forces in the three intersected members.
- 5. An alternative approach is to write a single equation,** which can be solved for the force in the desired member. To do so, first observe whether the forces exerted by the other two members on the free body are parallel or whether their lines of action intersect.
  - a. If these forces are parallel,** they can be eliminated by writing an equilibrium equation involving *components in a direction perpendicular* to these two forces.
  - b. If their lines of action intersect at a point H,** they can be eliminated by writing an equilibrium equation involving *moments about H*.
- 6. Keep in mind that the section you use must intersect three members only.** This is because the equilibrium equations in step 4 can be solved for three unknowns only. However, you can pass a section through more than three members to find the force in one of those members if you can write an equilibrium equation containing only that force as an unknown. Such special situations are found in Probs. 6.61 through 6.64.

(continued)

## B. About completely constrained and determinate trusses:

**1. First note that any simple truss which is simply supported** is a completely constrained and determinate truss.

**2. To determine whether any other truss is or is not completely constrained and determinate,** you first count the number  $m$  of its members, the number  $n$  of its joints, and the number  $r$  of the reaction components at its supports. You then compare the sum  $m + r$  representing the number of unknowns and the product  $2n$  representing the number of available independent equilibrium equations.

**a. If  $m + r < 2n$ ,** there are fewer unknowns than equations. Thus, some of the equations cannot be satisfied; the truss is only *partially constrained*.

**b. If  $m + r > 2n$ ,** there are more unknowns than equations. Thus, some of the unknowns cannot be determined; the truss is *indeterminate*.

**c. If  $m + r = 2n$ ,** there are as many unknowns as there are equations. This, however, does not mean that all the unknowns can be determined and that all the equations can be satisfied. To find out whether the truss is *completely* or *improperly constrained*, you should try to determine the reactions at its supports and the forces in its members. If all can be found, the truss is *completely constrained and determinate*.

# PROBLEMS

**6.43** A Warren bridge truss is loaded as shown. Determine the force in members  $CE$ ,  $DE$ , and  $DF$ .

**6.44** A Warren bridge truss is loaded as shown. Determine the force in members  $EG$ ,  $FG$ , and  $FH$ .

**6.45** Determine the force in members  $BD$  and  $DE$  of the truss shown.

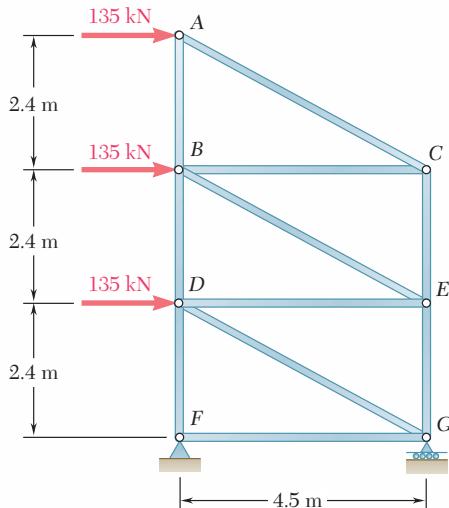


Fig. P6.45 and P6.46

**6.46** Determine the force in members  $DG$  and  $EG$  of the truss shown.

**6.47** A floor truss is loaded as shown. Determine the force in members  $CF$ ,  $EF$ , and  $EG$ .

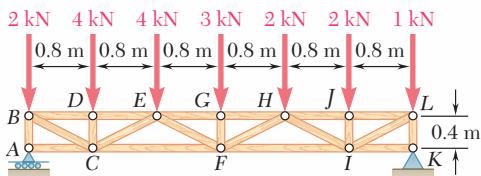


Fig. P6.47 and P6.48

**6.48** A floor truss is loaded as shown. Determine the force in members  $FI$ ,  $HI$ , and  $HJ$ .

**6.49** A pitched flat roof truss is loaded as shown. Determine the force in members  $CE$ ,  $DE$ , and  $DF$ .

**6.50** A pitched flat roof truss is loaded as shown. Determine the force in members  $EG$ ,  $GH$ , and  $HJ$ .

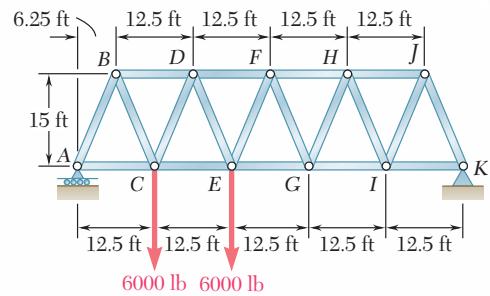


Fig. P6.43 and P6.44

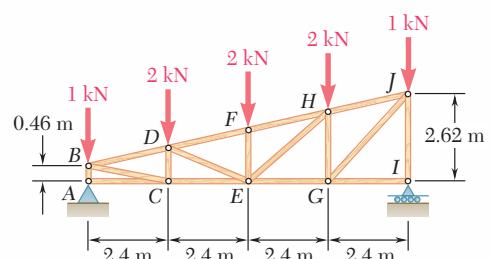
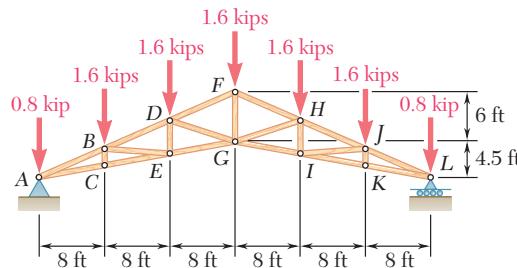


Fig. P6.49 and P6.50

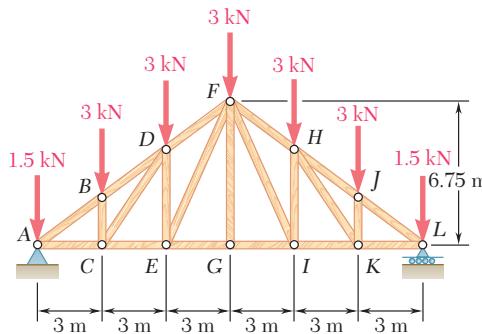
- 6.51** A Howe scissors roof truss is loaded as shown. Determine the force in members  $DF$ ,  $DG$ , and  $EG$ .



**Fig. P6.51 and P6.52**

- 6.52** A Howe scissors roof truss is loaded as shown. Determine the force in members  $GI$ ,  $HI$ , and  $HJ$ .

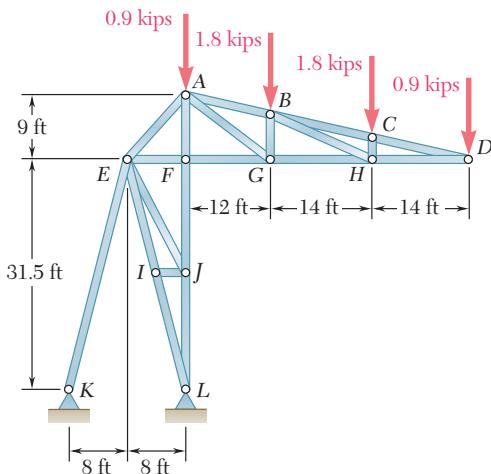
- 6.53** A Pratt roof truss is loaded as shown. Determine the force in members  $CE$ ,  $DE$ , and  $DF$ .



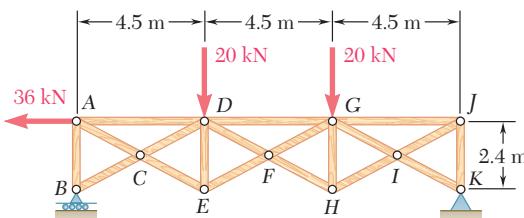
**Fig. P6.53 and P6.54**

- 6.54** A Pratt roof truss is loaded as shown. Determine the force in members  $FH$ ,  $FI$ , and  $GI$ .

- 6.55** Determine the force in members  $AD$ ,  $CD$ , and  $CE$  of the truss shown.



**Fig. P6.57 and P6.58**



**Fig. P6.55 and P6.56**

- 6.56** Determine the force in members  $DG$ ,  $FG$ , and  $FH$  of the truss shown.

- 6.57** A stadium roof truss is loaded as shown. Determine the force in members  $AB$ ,  $AG$ , and  $FG$ .

- 6.58** A stadium roof truss is loaded as shown. Determine the force in members  $AE$ ,  $EF$ , and  $FJ$ .

- 6.59** A Polynesian, or duopitch, roof truss is loaded as shown. Determine the force in members  $DF$ ,  $EF$ , and  $EG$ .

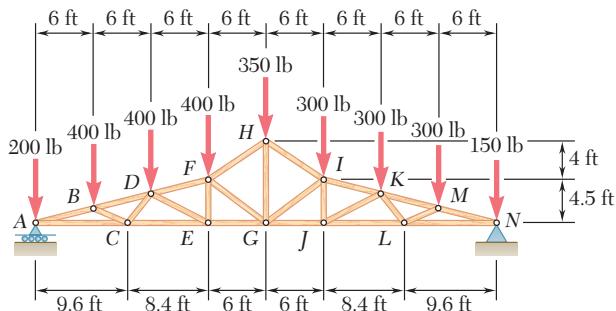


Fig. P6.59 and P6.60

- 6.60** A Polynesian, or duopitch, roof truss is loaded as shown. Determine the force in members  $HI$ ,  $GI$ , and  $GJ$ .

- 6.61** Determine the force in members  $AF$  and  $EJ$  of the truss shown when  $P = Q = 1.2$  kN. (Hint: Use section  $aa$ .)

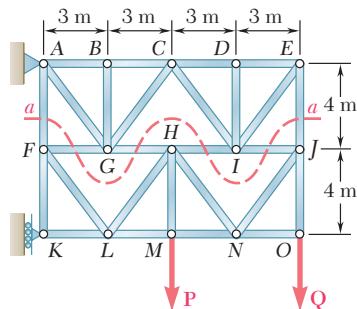


Fig. P6.61 and P6.62

- 6.62** Determine the force in members  $AF$  and  $EJ$  of the truss shown when  $P = 1.2$  kN and  $Q = 0$ . (Hint: Use section  $aa$ .)

- 6.63** Determine the force in members  $EH$  and  $GI$  of the truss shown. (Hint: Use section  $aa$ .)

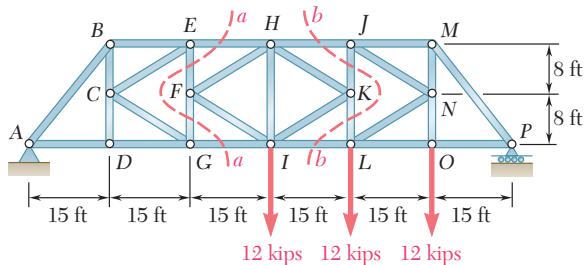


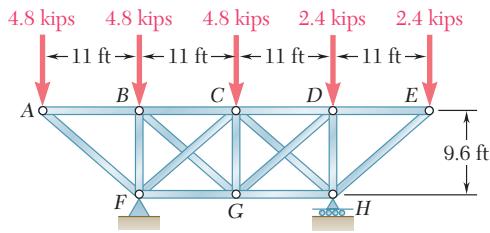
Fig. P6.63 and P6.64

- 6.64** Determine the force in members  $HJ$  and  $IL$  of the truss shown. (Hint: Use section  $bb$ .)

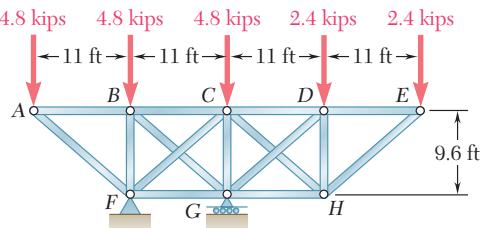
**6.65 and 6.66** The diagonal members in the center panels of the power transmission line tower shown are very slender and can act only in tension; such members are known as *counters*. For the given loading, determine (a) which of the two counters listed below is acting, (b) the force in that counter.

### 6.65 Counters *CJ* and *HE*.

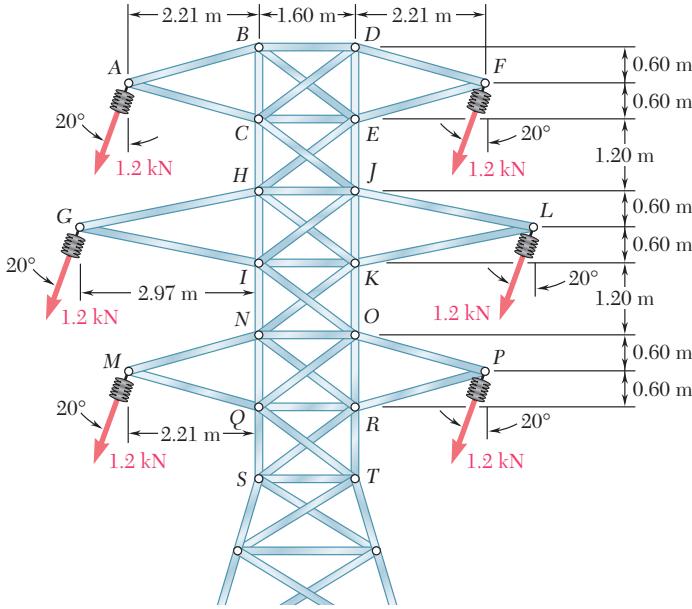
### **6.66** Counters $IO$ and $KN$ .



**Fig. P6.67**



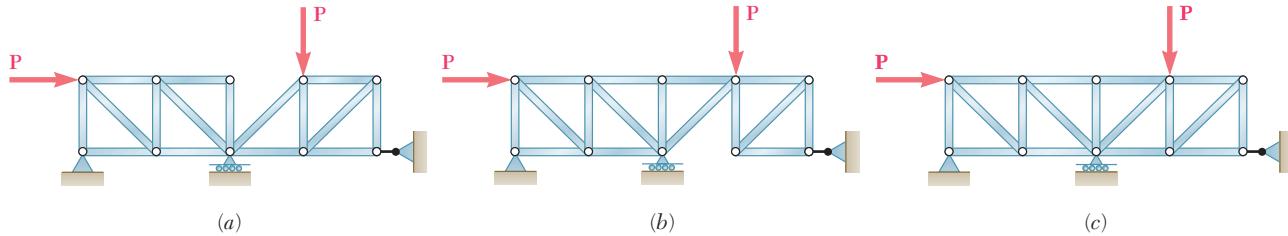
**Fig. P6.68**



### **Fig. P6.65 and P6.66**

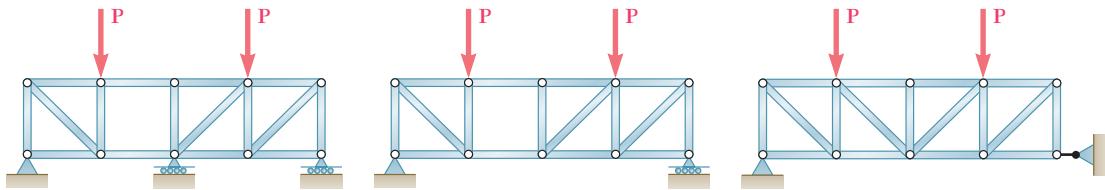
**6.67 and 6.68** The diagonal members in the center panels of the truss shown are very slender and can act only in tension; such members are known as *counters*. Determine the forces in the counters that are acting under the given loading.

**6.69** Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



**Fig. P6.69**

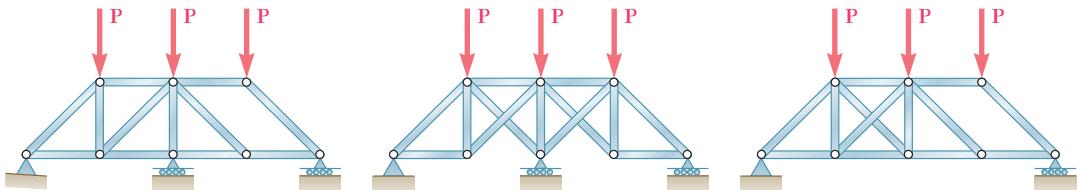
**6.70 through 6.74** Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



**Fig. P6.70** (a)

(b)

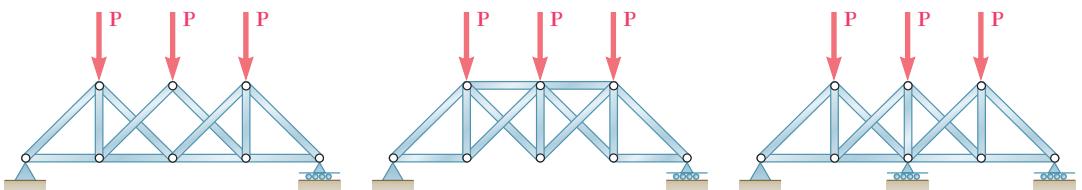
(c)



**Fig. P6.71** (a)

(b)

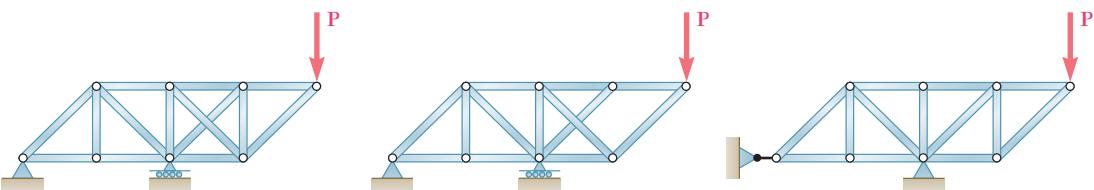
(c)



**Fig. P6.72** (a)

(b)

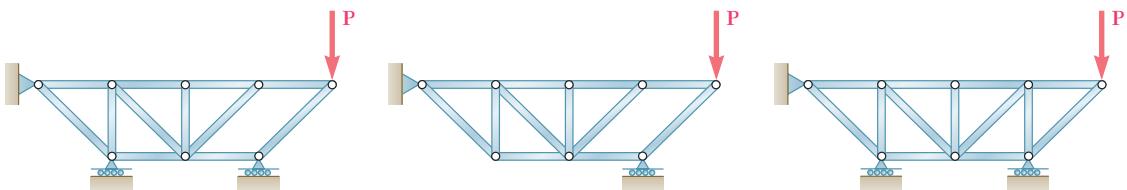
(c)



**Fig. P6.73** (a)

(b)

(c)



**Fig. P6.74** (a)

(b)

(c)

## FRAMES AND MACHINES

### 6.9 STRUCTURES CONTAINING MULTIFORCE MEMBERS

Under trusses, we have considered structures consisting entirely of pins and straight two-force members. The forces acting on the two-force members were known to be directed along the members themselves. We now consider structures in which at least one of the members is a *multipforce* member, i.e., a member acted upon by three or more forces. These forces will generally not be directed along the members on which they act; their direction is unknown, and they should be represented therefore by two unknown components.

Frames and machines are structures containing multipforce members. *Frames* are designed to support loads and are usually stationary, fully constrained structures. *Machines* are designed to transmit and modify forces; they may or may not be stationary and will always contain moving parts.

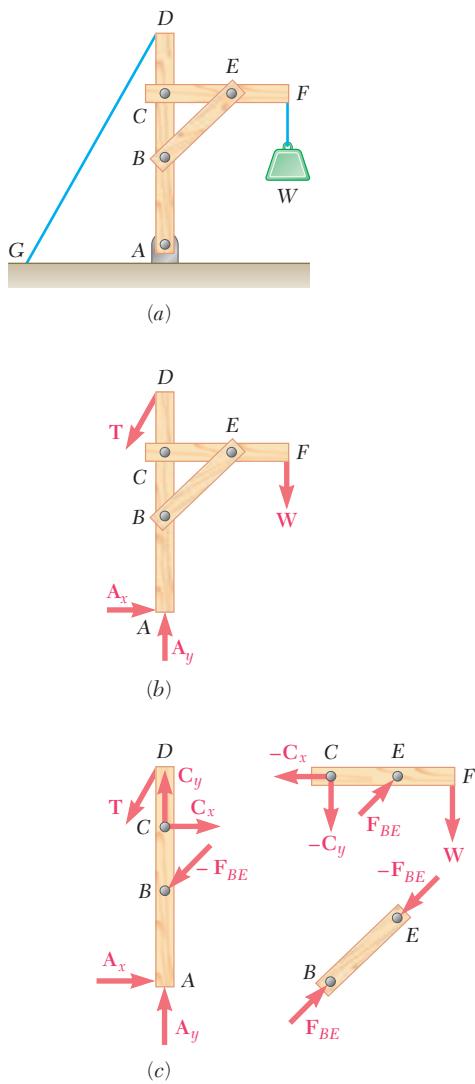
### 6.10 ANALYSIS OF A FRAME

As a first example of analysis of a frame, the crane described in Sec. 6.1, which carries a given load  $W$  (Fig. 6.20a), will again be considered. The free-body diagram of the entire frame is shown in Fig. 6.20b. This diagram can be used to determine the external forces acting on the frame. Summing moments about  $A$ , we first determine the force  $T$  exerted by the cable; summing  $x$  and  $y$  components, we then determine the components  $A_x$  and  $A_y$  of the reaction at the pin  $A$ .

In order to determine the internal forces holding the various parts of a frame together, we must dismember the frame and draw a free-body diagram for each of its component parts (Fig. 6.20c). First, the two-force members should be considered. In this frame, member  $BE$  is the only two-force member. The forces acting at each end of this member must have the same magnitude, same line of action, and opposite sense (Sec. 4.6). They are therefore directed along  $BE$  and will be denoted, respectively, by  $\mathbf{F}_{BE}$  and  $-\mathbf{F}_{BE}$ . Their sense will be arbitrarily assumed as shown in Fig. 6.20c; later the sign obtained for the common magnitude  $F_{BE}$  of the two forces will confirm or deny this assumption.

Next, we consider the multipforce members, i.e., the members which are acted upon by three or more forces. According to Newton's third law, the force exerted at  $B$  by member  $BE$  on member  $AD$  must be equal and opposite to the force  $\mathbf{F}_{BE}$  exerted by  $AD$  on  $BE$ . Similarly, the force exerted at  $E$  by member  $BE$  on member  $CF$  must be equal and opposite to the force  $-\mathbf{F}_{BE}$  exerted by  $CF$  on  $BE$ . Thus the forces that the two-force member  $BE$  exerts on  $AD$  and  $CF$  are, respectively, equal to  $-\mathbf{F}_{BE}$  and  $\mathbf{F}_{BE}$ ; they have the same magnitude  $F_{BE}$  and opposite sense, and should be directed as shown in Fig. 6.20c.

At  $C$  two multipforce members are connected. Since neither the direction nor the magnitude of the forces acting at  $C$  is known, these forces will be represented by their  $x$  and  $y$  components. The components  $\mathbf{C}_x$  and  $\mathbf{C}_y$  of the force acting on member  $AD$  will be



**Fig. 6.20**

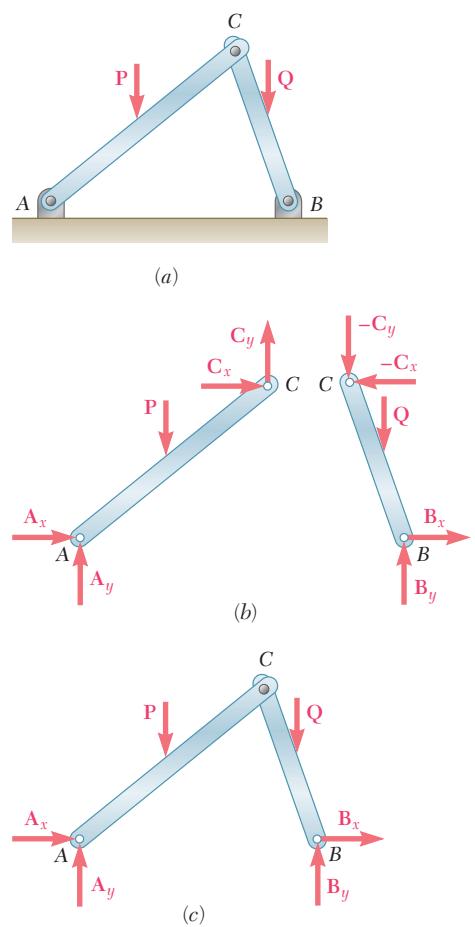
arbitrarily directed to the right and upward. Since, according to Newton's third law, the forces exerted by member *CF* on *AD* and by member *AD* on *CF* are equal and opposite, the components of the force acting on member *CF* must be directed to the left and downward; they will be denoted, respectively, by  $-C_x$  and  $-C_y$ . Whether the force  $C_x$  is actually directed to the right and the force  $-C_x$  is actually directed to the left will be determined later from the sign of their common magnitude  $C_x$ , a plus sign indicating that the assumption made was correct, and a minus sign that it was wrong. The free-body diagrams of the multiforce members are completed by showing the external forces acting at *A*, *D*, and *F*.†

The internal forces can now be determined by considering the free-body diagram of either of the two multiforce members. Choosing the free-body diagram of *CF*, for example, we write the equations  $\Sigma M_C = 0$ ,  $\Sigma E = 0$ , and  $\Sigma F_x = 0$ , which yield the values of the magnitudes  $F_{BE}$ ,  $C_y$ , and  $C_x$ , respectively. These values can be checked by verifying that member *AD* is also in equilibrium.

It should be noted that the pins in Fig. 6.20 were assumed to form an integral part of one of the two members they connected and so it was not necessary to show their free-body diagram. This assumption can always be used to simplify the analysis of frames and machines. When a pin connects three or more members, however, or when a pin connects a support and two or more members, or when a load is applied to a pin, a clear decision must be made in choosing the member to which the pin will be assumed to belong. (If multiforce members are involved, the pin should be attached to one of these members.) The various forces exerted on the pin should then be clearly identified. This is illustrated in Sample Prob. 6.6.

## 6.11 FRAMES WHICH CEASE TO BE RIGID WHEN DETACHED FROM THEIR SUPPORTS

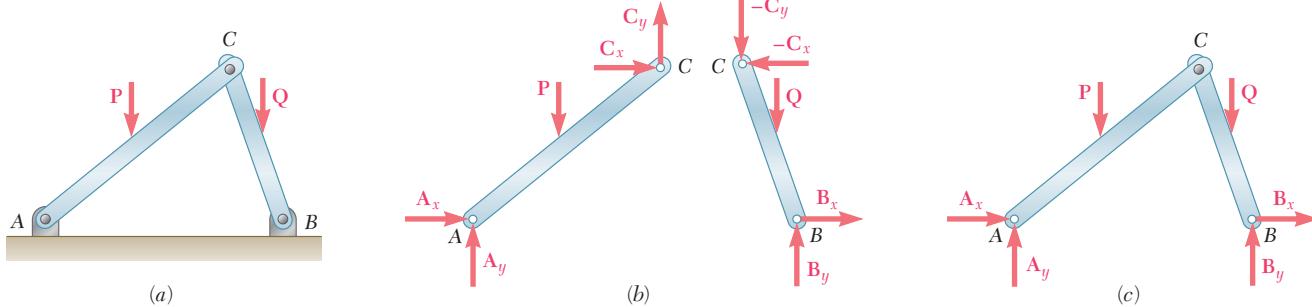
The crane analyzed in Sec. 6.10 was so constructed that it could keep the same shape without the help of its supports; it was therefore considered as a rigid body. Many frames, however, will collapse if detached from their supports; such frames cannot be considered as rigid bodies. Consider, for example, the frame shown in Fig. 6.21a, which consists of two members *AC* and *CB* carrying loads **P** and **Q** at their midpoints; the members are supported by pins at *A* and *B* and are connected by a pin at *C*. If detached from its supports, this frame will not maintain its shape; it should therefore be considered as made of two distinct rigid parts *AC* and *CB*.



**Fig. 6.21**

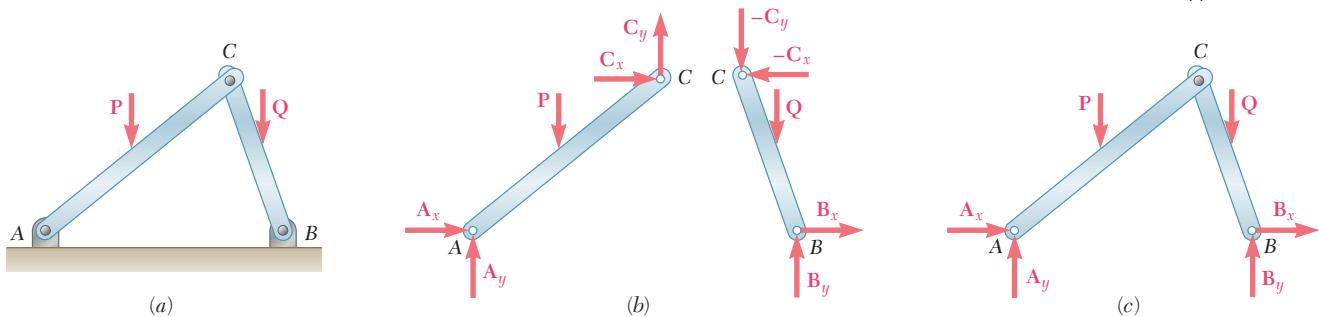
†It is not strictly necessary to use a minus sign to distinguish the force exerted by one member on another from the equal and opposite force exerted by the second member on the first, since the two forces belong to different free-body diagrams and thus cannot easily be confused. In the Sample Problems, the same symbol is used to represent equal and opposite forces which are applied to different free bodies. It should be noted that, under these conditions, the sign obtained for a given force component will not directly relate the sense of that component to the sense of the corresponding coordinate axis. Rather, a positive sign will indicate that the sense assumed for that component in the free-body diagram is correct, and a negative sign will indicate that it is wrong.

The equations  $\Sigma F_x = 0$ ,  $\Sigma F_y = 0$ ,  $\Sigma M = 0$  (about any given point) express the conditions for the *equilibrium of a rigid body* (Chap. 4); we should use them, therefore, in connection with the free-body diagrams of rigid bodies, namely, the free-body diagrams of members  $AC$  and  $CB$  (Fig. 6.21b). Since these members are multi-force members, and since pins are used at the supports and at the connection, the reactions at  $A$  and  $B$  and the forces at  $C$  will each be represented by two components. In accordance with Newton's third law, the components of the force exerted by  $CB$  on  $AC$  and the components of the force exerted by  $AC$  on  $CB$  will be represented by vectors of the same magnitude and opposite sense; thus, if the first pair of components consists of  $\mathbf{C}_x$  and  $\mathbf{C}_y$ , the second pair will be represented by  $-\mathbf{C}_x$  and  $-\mathbf{C}_y$ . We note that four unknown force components act on free body  $AC$ , while only three independent equations can be used to express that the body is in equilibrium; similarly, four unknowns, but only three equations, are associated with  $CB$ . However, only six different unknowns are involved in the analysis of the two members, and altogether six equations are available to express that the members are in equilibrium. Writing  $\Sigma M_A = 0$  for free body  $AC$  and  $\Sigma M_B = 0$  for  $CB$ , we obtain two simultaneous equations which may be solved for the common magnitude  $C_x$  of the components  $\mathbf{C}_x$  and  $-\mathbf{C}_x$ , and for the common magnitude  $C_y$  of the components  $\mathbf{C}_y$  and  $-\mathbf{C}_y$ . We then write  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$  for each of the two free bodies, obtaining, successively, the magnitudes  $A_x$ ,  $A_y$ ,  $B_x$ , and  $B_y$ .



**Fig. 6.21** (repeated)

It can now be observed that since the equations of equilibrium  $\Sigma F_x = 0$ ,  $\Sigma F_y = 0$ , and  $\Sigma M = 0$  (about any given point) are satisfied by the forces acting on free body  $AC$ , and since they are also satisfied by the forces acting on free body  $CB$ , they must be satisfied when the forces acting on the two free bodies are considered simultaneously. Since the internal forces at  $C$  cancel each other, we find that the equations of equilibrium must be satisfied by the external forces shown on the free-body diagram of the frame  $ACB$  itself (Fig. 6.21c), although the frame is not a rigid body. These equations can be used to determine some of the components of the reactions at  $A$  and  $B$ . We will also find, however, that *the reactions cannot be completely determined from the free-body diagram of the whole frame*. It is thus necessary to

**Fig. 6.21** (repeated)

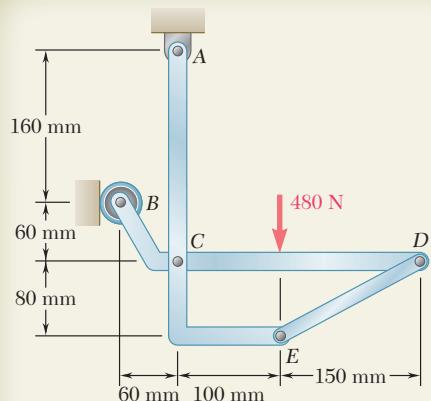
dismember the frame and to consider the free-body diagrams of its component parts (Fig. 6.21b), even when we are interested in determining external reactions only. This is because the equilibrium equations obtained for free body *ACB* are *necessary conditions* for the equilibrium of a nonrigid structure, *but are not sufficient conditions*.

The method of solution outlined in the second paragraph of this section involved simultaneous equations. A more efficient method is now presented, which utilizes the free body *ACB* as well as the free bodies *AC* and *CB*. Writing  $\sum M_A = 0$  and  $\sum M_B = 0$  for free body *ACB*, we obtain  $B_y$  and  $A_y$ . Writing  $\sum M_C = 0$ ,  $\sum F_x = 0$ , and  $\sum F_y = 0$  for free body *AC*, we obtain, successively,  $A_x$ ,  $C_x$ , and  $C_y$ . Finally, writing  $\sum F_x = 0$  for *ACB*, we obtain  $B_x$ .

We noted above that the analysis of the frame of Fig. 6.21 involves six unknown force components and six independent equilibrium equations. (The equilibrium equations for the whole frame were obtained from the original six equations and, therefore, are not independent.) Moreover, we checked that all unknowns could be actually determined and that all equations could be satisfied. The frame considered is *statically determinate and rigid*.† In general, to determine whether a structure is statically determinate and rigid, we should draw a free-body diagram for each of its component parts and count the reactions and internal forces involved. We should also determine the number of independent equilibrium equations (excluding equations expressing the equilibrium of the whole structure or of groups of component parts already analyzed). If there are more unknowns than equations, the structure is *statically indeterminate*. If there are fewer unknowns than equations, the structure is *nonrigid*. If there are as many unknowns as equations, and if all unknowns can be determined and all equations satisfied under general loading conditions, the structure is *statically determinate and rigid*. If, however, due to an *improper arrangement* of members and supports, all unknowns cannot be determined and all equations cannot be satisfied, the structure is *statically indeterminate and nonrigid*.

†The word “rigid” is used here to indicate that the frame will maintain its shape as long as it remains attached to its supports.

## SAMPLE PROBLEM 6.4

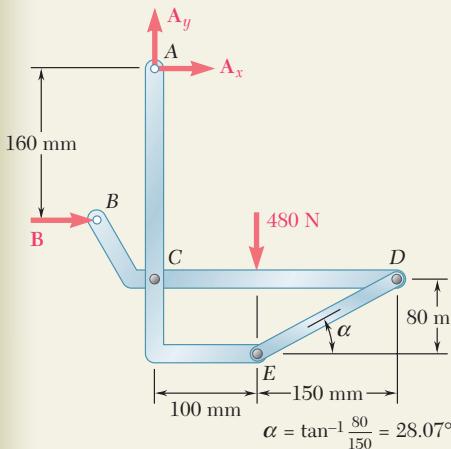


In the frame shown, members *ACE* and *BCD* are connected by a pin at *C* and by the link *DE*. For the loading shown, determine the force in link *DE* and the components of the force exerted at *C* on member *BCD*.

### SOLUTION

**Free Body: Entire Frame.** Since the external reactions involve only three unknowns, we compute the reactions by considering the free-body diagram of the entire frame.

$$\begin{aligned} +\uparrow \sum F_y &= 0: & A_y - 480 \text{ N} &= 0 & A_y &= +480 \text{ N} & A_y &= 480 \text{ N} \uparrow \\ +\nabla \sum M_A &= 0: & -(480 \text{ N})(100 \text{ mm}) + B(160 \text{ mm}) &= 0 & B &= +300 \text{ N} & B &= 300 \text{ N} \rightarrow \\ +\rightarrow \sum F_x &= 0: & B + A_x &= 0 & 300 \text{ N} + A_x &= 0 & A_x &= -300 \text{ N} & A_x &= 300 \text{ N} \leftarrow \end{aligned}$$



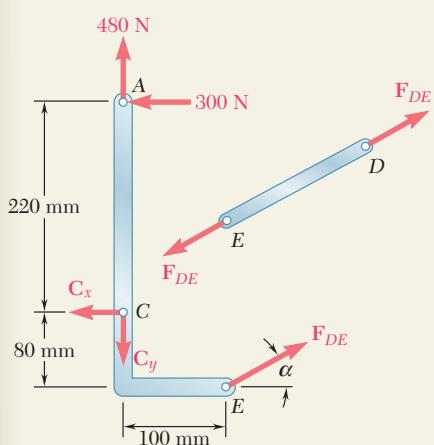
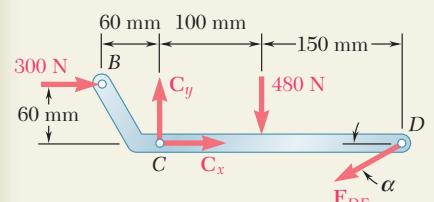
**Members.** We now dismember the frame. Since only two members are connected at *C*, the components of the unknown forces acting on *ACE* and *BCD* are, respectively, equal and opposite and are assumed directed as shown. We assume that link *DE* is in tension and exerts equal and opposite forces at *D* and *E*, directed as shown.

**Free Body: Member *BCD*.** Using the free body *BCD*, we write

$$\begin{aligned} +\downarrow \sum M_C &= 0: & (F_{DE} \sin \alpha)(250 \text{ mm}) + (300 \text{ N})(80 \text{ mm}) + (480 \text{ N})(100 \text{ mm}) &= 0 & F_{DE} &= 561 \text{ N} & F_{DE} &= 561 \text{ N} \text{ C} \\ +\rightarrow \sum F_x &= 0: & C_x - F_{DE} \cos \alpha + 300 \text{ N} &= 0 & C_x &= -795 \text{ N} \\ +\uparrow \sum F_y &= 0: & C_y - F_{DE} \sin \alpha - 480 \text{ N} &= 0 & C_y &= +216 \text{ N} & C_y &= 216 \text{ N} \uparrow \end{aligned}$$

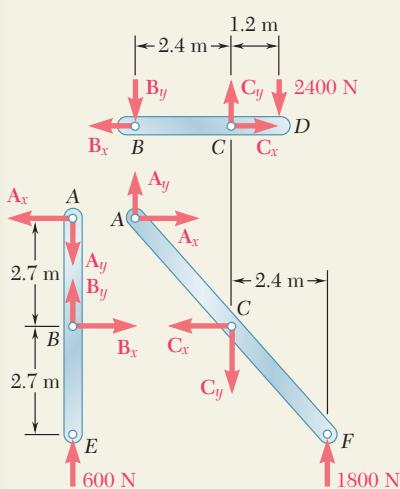
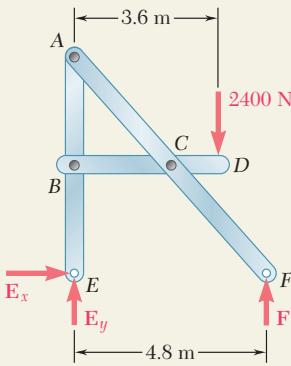
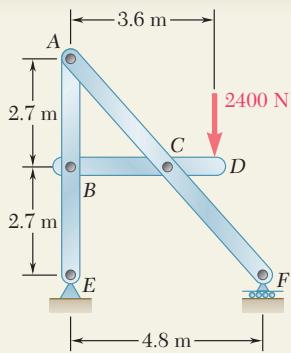
From the signs obtained for  $C_x$  and  $C_y$  we conclude that the force components  $\mathbf{C}_x$  and  $\mathbf{C}_y$  exerted on member *BCD* are directed, respectively, to the left and up. We have

$$\mathbf{C}_x = 795 \text{ N} \leftarrow, \mathbf{C}_y = 216 \text{ N} \uparrow$$



**Free Body: Member *ACE* (Check).** The computations are checked by considering the free body *ACE*. For example,

$$\begin{aligned} +\nabla \sum M_A &= (F_{DE} \cos \alpha)(300 \text{ mm}) + (F_{DE} \sin \alpha)(100 \text{ mm}) - C_x(220 \text{ mm}) \\ &= (-561 \cos \alpha)(300) + (-561 \sin \alpha)(100) - (-795)(220) = 0 \end{aligned}$$



## SAMPLE PROBLEM 6.5

Determine the components of the forces acting on each member of the frame shown.

### SOLUTION

**Free Body: Entire Frame.** Since the external reactions involve only three unknowns, we compute the reactions by considering the free-body diagram of the entire frame.

$$+\uparrow \sum M_E = 0: -(2400 \text{ N})(3.6 \text{ m}) + F(4.8 \text{ m}) = 0$$

$$F = +1800 \text{ N}$$

$$\mathbf{F} = 1800 \text{ N} \uparrow$$

$$+\uparrow \sum F_y = 0: -2400 \text{ N} + 1800 \text{ N} + E_y = 0$$

$$E_y = +600 \text{ N}$$

$$\mathbf{E}_y = 600 \text{ N} \uparrow$$

$$\pm \sum F_x = 0:$$

$$\mathbf{E}_x = 0$$

**Members.** The frame is now dismembered; since only two members are connected at each joint, equal and opposite components are shown on each member at each joint.

#### Free Body: Member BCD

$$+\uparrow \sum M_B = 0: -(2400 \text{ N})(3.6 \text{ m}) + C_y(2.4 \text{ m}) = 0 \quad C_y = +3600 \text{ N}$$

$$\mathbf{C}_y = 3600 \text{ N} \uparrow$$

$$+\uparrow \sum M_C = 0: -(2400 \text{ N})(1.2 \text{ m}) + B_y(2.4 \text{ m}) = 0 \quad B_y = +1200 \text{ N}$$

$$\mathbf{B}_y = 1200 \text{ N} \uparrow$$

$$\pm \sum F_x = 0: -B_x + C_x = 0$$

We note that neither  $B_x$  nor  $C_x$  can be obtained by considering only member BCD. The positive values obtained for  $B_y$  and  $C_y$  indicate that the force components  $\mathbf{B}_y$  and  $\mathbf{C}_y$  are directed as assumed.

#### Free Body: Member ABE

$$+\uparrow \sum M_A = 0: B_x(2.7 \text{ m}) = 0$$

$$\mathbf{B}_x = 0$$

$$\pm \sum F_x = 0: +B_x - A_x = 0$$

$$\mathbf{A}_x = 0$$

$$+\uparrow \sum F_y = 0: -A_y + B_y + 600 \text{ N} = 0$$

$$-A_y + 1200 \text{ N} + 600 \text{ N} = 0 \quad \mathbf{A}_y = +1800 \text{ N}$$

$$-A_y + 1200 \text{ N} + 600 \text{ N} = 0 \quad \mathbf{A}_y = +1800 \text{ N}$$

#### Free Body: Member BCD.

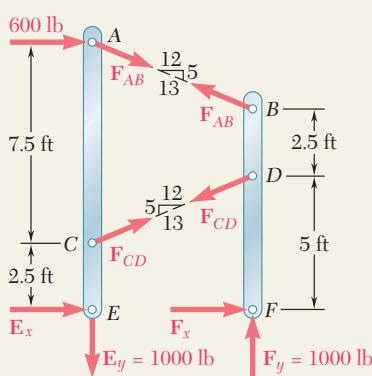
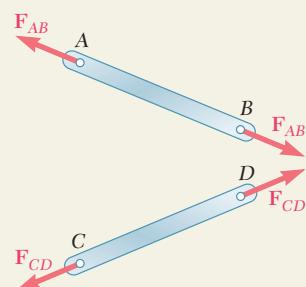
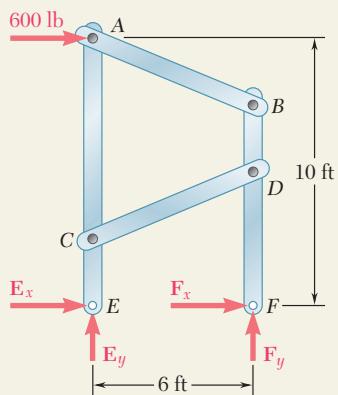
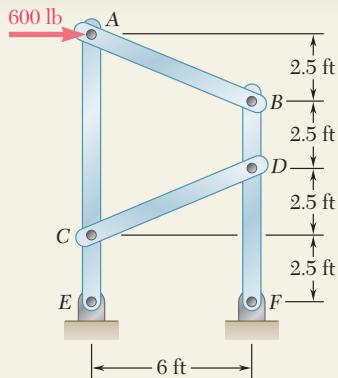
Returning now to member BCD, we write

$$\pm \sum F_x = 0: -B_x + C_x = 0 \quad 0 + C_x = 0$$

$$\mathbf{C}_x = 0$$

**Free Body: Member ACF (Check).** All unknown components have now been found; to check the results, we verify that member ACF is in equilibrium.

$$+\uparrow \sum M_C = (1800 \text{ N})(2.4 \text{ m}) - A_y(2.4 \text{ m}) - A_x(2.7 \text{ m}) \\ = (1800 \text{ N})(2.4 \text{ m}) - (1800 \text{ N})(2.4 \text{ m}) - 0 = 0 \quad (\text{checks})$$



## SAMPLE PROBLEM 6.6

A 600-lb horizontal force is applied to pin A of the frame shown. Determine the forces acting on the two vertical members of the frame.

### SOLUTION

**Free Body: Entire Frame.** The entire frame is chosen as a free body; although the reactions involve four unknowns,  $\mathbf{E}_y$  and  $\mathbf{F}_y$  may be determined by writing

$$+\uparrow\sum M_E = 0: \quad -(600 \text{ lb})(10 \text{ ft}) + F_y(6 \text{ ft}) = 0 \\ F_y = +1000 \text{ lb} \quad \mathbf{F}_y = 1000 \text{ lb}\uparrow$$

$$+\uparrow\sum F_y = 0: \quad E_y + F_y = 0 \\ E_y = -1000 \text{ lb} \quad \mathbf{E}_y = 1000 \text{ lb}\downarrow$$

**Members.** The equations of equilibrium of the entire frame are not sufficient to determine  $\mathbf{E}_x$  and  $\mathbf{F}_x$ . The free-body diagrams of the various members must now be considered in order to proceed with the solution. In dismembering the frame we will assume that pin A is attached to the multiforce member ACE and, thus, that the 600-lb force is applied to that member. We also note that AB and CD are two-force members.

#### Free Body: Member ACE

$$+\uparrow\sum F_y = 0: \quad -\frac{5}{13}F_{AB} + \frac{5}{13}F_{CD} - 1000 \text{ lb} = 0$$

$$+\uparrow\sum M_E = 0: \quad -(600 \text{ lb})(10 \text{ ft}) - (\frac{12}{13}F_{AB})(10 \text{ ft}) - (\frac{12}{13}F_{CD})(2.5 \text{ ft}) = 0$$

Solving these equations simultaneously, we find

$$F_{AB} = -1040 \text{ lb} \quad F_{CD} = +1560 \text{ lb}$$

The signs obtained indicate that the sense assumed for  $F_{CD}$  was correct and the sense for  $F_{AB}$  incorrect. Summing now  $x$  components,

$$+\rightarrow\sum F_x = 0: \quad 600 \text{ lb} + \frac{12}{13}(-1040 \text{ lb}) + \frac{12}{13}(+1560 \text{ lb}) + E_x = 0 \\ E_x = -1080 \text{ lb} \quad \mathbf{E}_x = 1080 \text{ lb}\leftarrow$$

**Free Body: Entire Frame.** Since  $\mathbf{E}_x$  has been determined, we can return to the free-body diagram of the entire frame and write

$$+\rightarrow\sum F_x = 0: \quad 600 \text{ lb} - 1080 \text{ lb} + F_x = 0 \\ F_x = +480 \text{ lb} \quad \mathbf{F}_x = 480 \text{ lb}\rightarrow$$

**Free Body: Member BDF (Check).** We can check our computations by verifying that the equation  $\sum M_B = 0$  is satisfied by the forces acting on member BDF.

$$+\uparrow\sum M_B = -(\frac{12}{13}F_{CD})(2.5 \text{ ft}) + (F_x)(7.5 \text{ ft}) \\ = -\frac{12}{13}(1560 \text{ lb})(2.5 \text{ ft}) + (480 \text{ lb})(7.5 \text{ ft}) \\ = -3600 \text{ lb}\cdot\text{ft} + 3600 \text{ lb}\cdot\text{ft} = 0 \quad (\text{checks})$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to analyze frames containing one or more multiforce members. In the problems that follow you will be asked to determine the external reactions exerted on the frame and the internal forces that hold together the members of the frame.

In solving problems involving frames containing one or more multiforce members, follow these steps:

**1. Draw a free-body diagram of the entire frame.** Use this free-body diagram to calculate, to the extent possible, the reactions at the supports. (In Sample Prob. 6.6 only two of the four reaction components could be found from the free body of the entire frame.)

**2. Dismember the frame, and draw a free-body diagram of each member.**

**3. Considering first the two-force members,** apply equal and opposite forces to each two-force member at the points where it is connected to another member. If the two-force member is a straight member, these forces will be directed along the axis of the member. If you cannot tell at this point whether the member is in tension or compression, just *assume* that the member is in tension and *direct both of the forces away from the member*. Since these forces have the same unknown magnitude, give them both the *same name* and, to avoid any confusion later, *do not use a plus sign or a minus sign*.

**4. Next, consider the multiforce members.** For each of these members, show all the forces acting on the member, including *applied loads, reactions, and internal forces at connections*. The magnitude and direction of any reaction or reaction component found earlier from the free-body diagram of the entire frame should be clearly indicated.

**a. Where a multiforce member is connected to a two-force member,** apply to the multiforce member a force *equal and opposite* to the force drawn on the free-body diagram of the two-force member, *giving it the same name*.

**b. Where a multiforce member is connected to another multiforce member,** use *horizontal and vertical components* to represent the internal forces at that point, since neither the direction nor the magnitude of these forces is known. The direction you choose for each of the two force components exerted on the first multiforce member is arbitrary, but *you must apply equal and opposite force components of the same name* to the other multiforce member. Again, *do not use a plus sign or a minus sign*.

(continued)

**5. The internal forces may now be determined**, as well as any *reactions* that you have not already found.

a. **The free-body diagram** of each of the multiforce members can provide you with *three equilibrium equations*.

b. **To simplify your solution**, you should seek a way to write an equation involving a single unknown. If you can locate *a point where all but one of the unknown force components intersect*, you will obtain an equation in a single unknown by summing moments about that point. *If all unknown forces except one are parallel*, you will obtain an equation in a single unknown by summing force components in a direction perpendicular to the parallel forces.

c. **Since you arbitrarily chose the direction of each of the unknown forces**, you cannot determine until the solution is completed whether your guess was correct. To do that, consider the *sign* of the value found for each of the unknowns: a *positive* sign means that the direction you selected was *correct*; a *negative* sign means that the direction is *opposite* to the direction you assumed.

**6. To be more effective and efficient** as you proceed through your solution, observe the following rules:

a. **If an equation involving only one unknown can be found**, write that equation and *solve it for that unknown*. Immediately *replace* that unknown wherever it appears on other free-body diagrams *by the value you have found*. Repeat this process by seeking equilibrium equations involving only one unknown until you have found all of the internal forces and unknown reactions.

b. **If an equation involving only one unknown cannot be found**, you may have to *solve a pair of simultaneous equations*. Before doing so, check that you have shown the values of all of the reactions that were obtained from the free-body diagram of the entire frame.

c. **The total number of equations** of equilibrium for the entire frame and for the individual members *will be larger than the number of unknown forces and reactions*. After you have found all the reactions and all the internal forces, you can use the remaining equations to check the accuracy of your computations.

# PROBLEMS

- 6.75** For the frame and loading shown, determine the force acting on member  $ABC$  (a) at  $B$ , (b) at  $C$ .

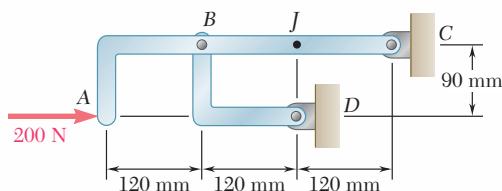


Fig. P6.75

- 6.76** Determine the force in member  $BD$  and the components of the reaction at  $C$ .

- 6.77** Rod  $CD$  is fitted with a collar at  $D$  that can be moved along rod  $AB$ , which is bent in the shape of an arc of circle. For the position when  $\theta = 30^\circ$ , determine (a) the force in rod  $CD$ , (b) the reaction at  $B$ .

- 6.78** Solve Prob. 6.77 when  $\theta = 150^\circ$ .

- 6.79** Determine the components of all forces acting on member  $ABCD$  when  $\theta = 0$ .

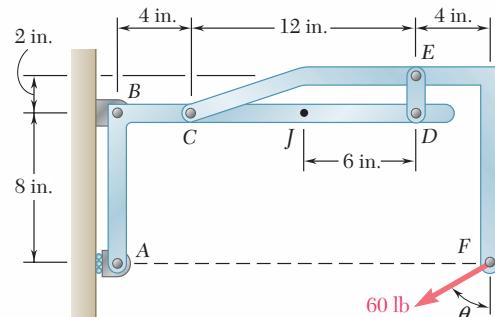


Fig. P6.79 and P6.80

- 6.80** Determine the components of all forces acting on member  $ABCD$  when  $\theta = 90^\circ$ .

- 6.81** For the frame and loading shown, determine the components of all forces acting on member  $ABC$ .

- 6.82** Solve Prob. 6.81 assuming that the 18-kN load is replaced by a clockwise couple of magnitude  $72 \text{ kN} \cdot \text{m}$  applied to member  $CDEF$  at point  $D$ .

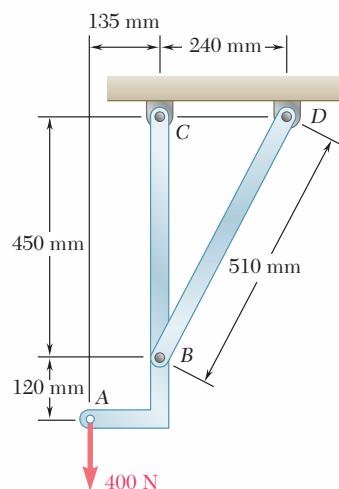


Fig. P6.76

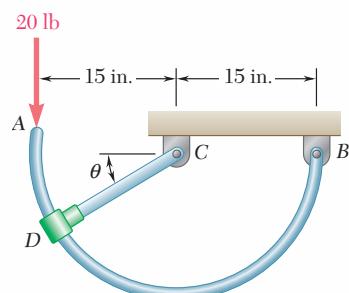


Fig. P6.77

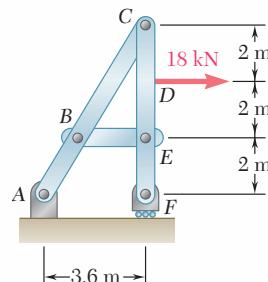


Fig. P6.81

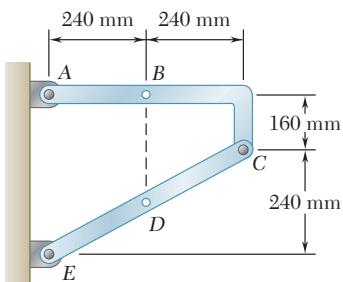


Fig. P6.83 and P6.85

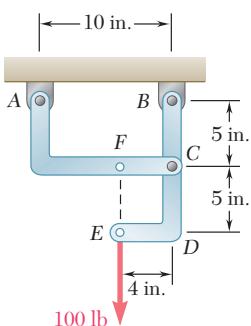


Fig. P6.87

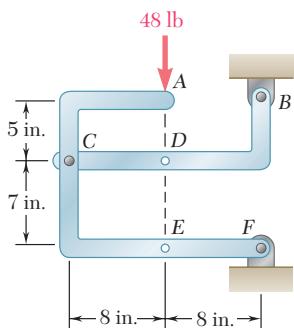


Fig. P6.88 and P6.89

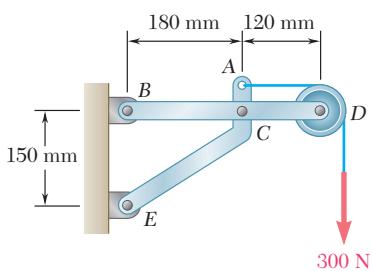


Fig. P6.91

- 6.83 and 6.84** Determine the components of the reactions at A and E if a 750-N force directed vertically downward is applied (a) at B, (b) at D.

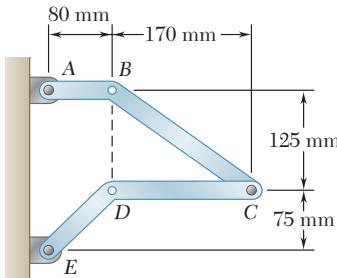


Fig. P6.84 and P6.86

- 6.85 and 6.86** Determine the components of the reactions at A and E if the frame is loaded by a clockwise couple of magnitude 36 N · m applied (a) at B, (b) at D.

- 6.87** Determine the components of the reactions at A and B, (a) if the 500-N load is applied as shown, (b) if the 500-N load is moved along its line of action and is applied at point F.

- 6.88** The 48-lb load can be moved along the line of action shown and applied at A, D, or E. Determine the components of the reactions at B and F if the 48-lb load is applied (a) at A, (b) at D, (c) at E.

- 6.89** The 48-lb load is removed and a 288-lb · in. clockwise couple is applied successively at A, D, and E. Determine the components of the reactions at B and F if the couple is applied (a) at A, (b) at D, (c) at E.

- 6.90** (a) Show that when a frame supports a pulley at A, an equivalent loading of the frame and of each of its component parts can be obtained by removing the pulley and applying at A two forces equal and parallel to the forces that the cable exerts on the pulley. (b) Show that if one end of the cable is attached to the frame at a point B, a force of magnitude equal to the tension in the cable should also be applied at B.

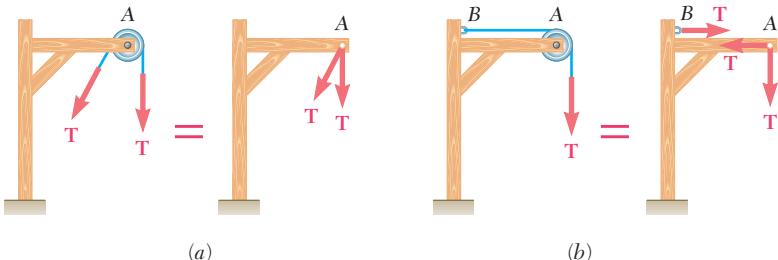


Fig. P6.90

- 6.91** Knowing that the pulley has a radius of 50 mm, determine the components of the reactions at B and E.

- 6.92** Knowing that each pulley has a radius of 250 mm, determine the components of the reactions at *D* and *E*.

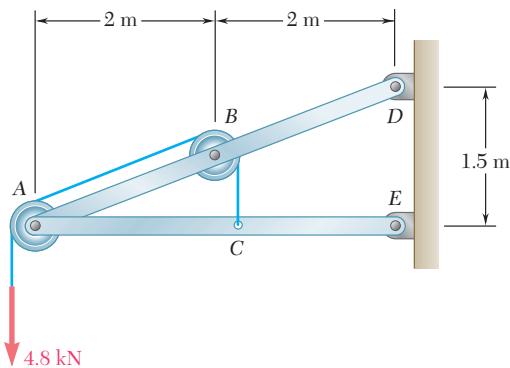


Fig. P6.92

- 6.93** Two 9-in.-diameter pipes (pipe 1 and pipe 2) are supported every 7.5 ft by a small frame like that shown. Knowing that the combined weight of each pipe and its contents is 30 lb/ft and assuming frictionless surfaces, determine the components of the reactions at *A* and *G*.

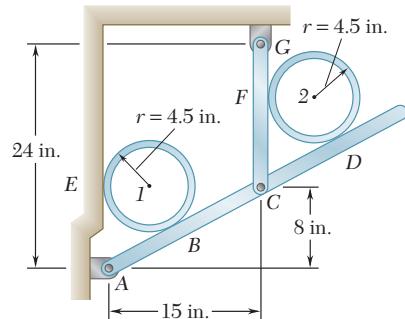


Fig. P6.93

- 6.94** Solve Prob. 6.93 assuming that pipe 1 is removed and that only pipe 2 is supported by the frames.

- 6.95** A trailer weighing 2400 lb is attached to a 2900-lb pickup truck by a ball-and-socket truck hitch at *D*. Determine (a) the reactions at each of the six wheels when the truck and trailer are at rest, (b) the additional load on each of the truck wheels due to the trailer.

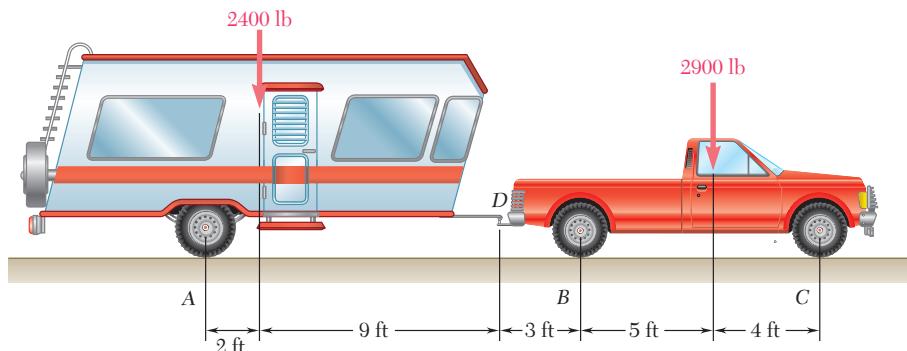


Fig. P6.95

- 6.96** In order to obtain a better weight distribution over the four wheels of the pickup truck of Prob. 6.95, a compensating hitch of the type shown is used to attach the trailer to the truck. The hitch consists of two bar springs (only one is shown in the figure) that fit into bearings inside a support rigidly attached to the truck. The springs are also connected by chains to the trailer frame, and specially designed hooks make it possible to place both chains in tension. (a) Determine the tension *T* required in each of the two chains if the additional load due to the trailer is to be evenly distributed over the four wheels of the truck. (b) What are the resulting reactions at each of the six wheels of the trailer-truck combination?

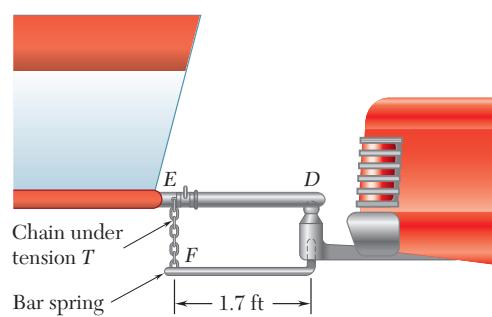
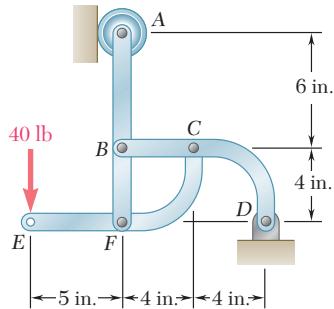
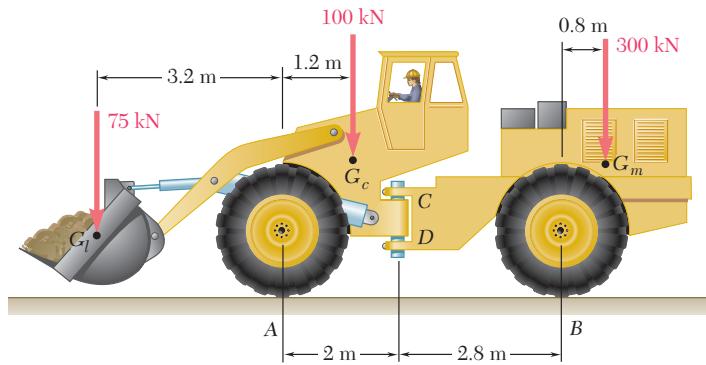


Fig. P6.96

**6.97** The cab and motor units of the front-end loader shown are connected by a vertical pin located 2 m behind the cab wheels. The distance from  $C$  to  $D$  is 1 m. The center of gravity of the 300-kN motor unit is located at  $G_m$ , while the centers of gravity of the 100-kN cab and 75-kN load are located, respectively, at  $G_c$  and  $G_l$ . Knowing that the machine is at rest with its brakes released, determine (a) the reactions at each of the four wheels, (b) the forces exerted on the motor unit at  $C$  and  $D$ .



**Fig. P6.99**



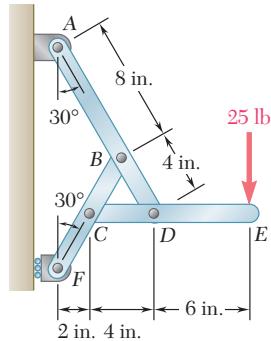
**Fig. P6.97**

**6.98** Solve Prob. 6.97 assuming that the 75-kN load has been removed.

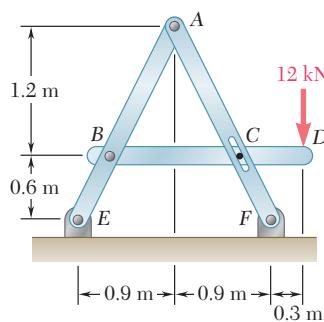
**6.99** For the frame and loading shown, determine the components of the forces acting on member *CFE* at *C* and *F*.

**6.100** For the frame and loading shown, determine the components of the forces acting on member *CDE* at *C* and *D*.

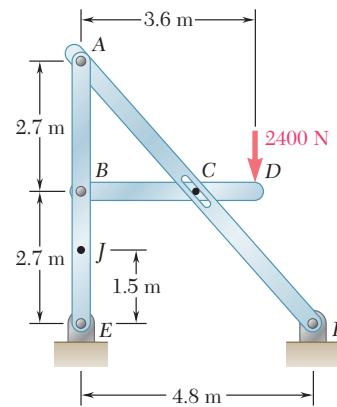
**6.101 and 6.102** For the frame and loading shown, determine the components of all forces acting on member *ABE*.



**Fig. P6.100**



**Fig. P6.101**



**Fig. P6.102**

- 6.103** Knowing that  $P = 15$  lb and  $Q = 65$  lb, determine the components of the forces exerted (a) on member  $BCDF$  at  $C$  and  $D$ , (b) on member  $ACEG$  at  $E$ .

- 6.104** Knowing that  $P = 25$  lb and  $Q = 55$  lb, determine the components of the forces exerted (a) on member  $BCDF$  at  $C$  and  $D$ , (b) on member  $ACEG$  at  $E$ .

- 6.105** For the frame and loading shown, determine the components of the forces acting on member  $DABC$  at  $B$  and  $D$ .

- 6.106** Solve Prob. 6.105 assuming that the 6-kN load has been removed.

- 6.107** The axis of the three-hinge arch  $ABC$  is a parabola with vertex at  $B$ . Knowing that  $P = 112$  kN and  $Q = 140$  kN, determine (a) the components of the reaction at  $A$ , (b) the components of the force exerted at  $B$  on segment  $AB$ .

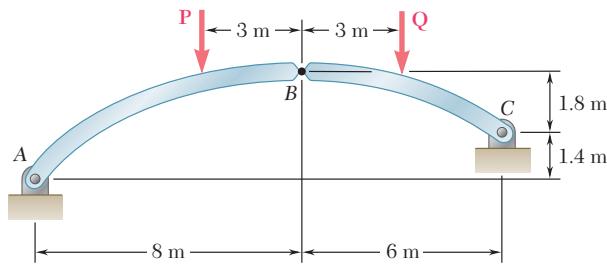


Fig. P6.107 and P6.108

- 6.108** The axis of the three-hinge arch  $ABC$  is a parabola with vertex at  $B$ . Knowing that  $P = 140$  kN and  $Q = 112$  kN, determine (a) the components of the reaction at  $A$ , (b) the components of the force exerted at  $B$  on segment  $AB$ .

- 6.109** Knowing that the surfaces at  $A$  and  $D$  are frictionless, determine the forces exerted at  $B$  and  $C$  on member  $BCE$ .

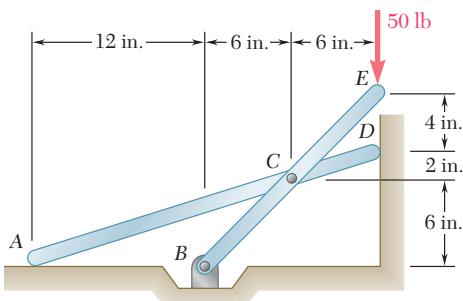


Fig. P6.109

- 6.110** For the frame and loading shown, determine (a) the reaction at  $C$ , (b) the force in member  $AD$ .

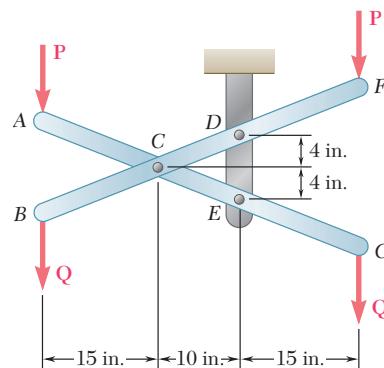


Fig. P6.103 and P6.104

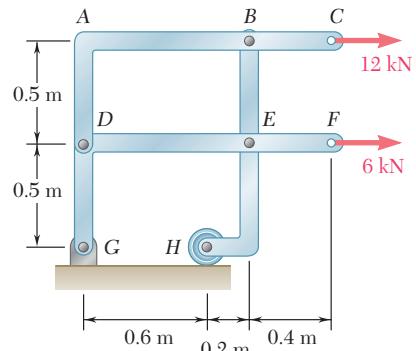


Fig. P6.105

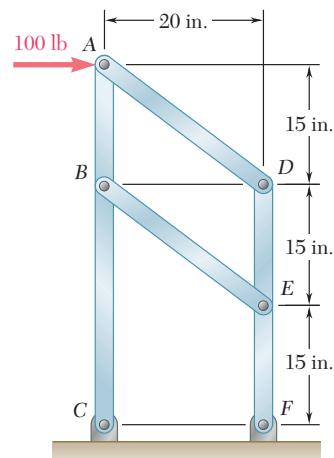


Fig. P6.110

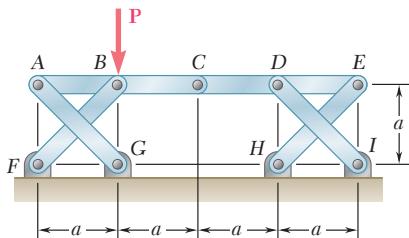


Fig. P6.111

**6.111, 6.112, and 6.113** Members ABC and CDE are pin-connected at C and supported by four links. For the loading shown, determine the force in each link.

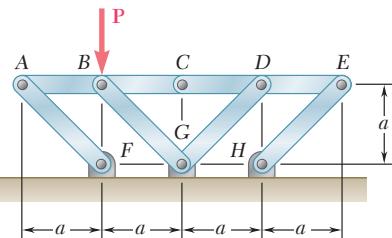


Fig. P6.112

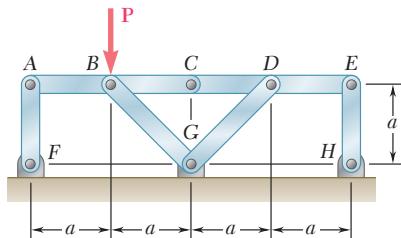


Fig. P6.113

**6.114** Members ABC and CDE are pin-connected at C and supported by the four links AF, BG, DG, and EH. For the loading shown, determine the force in each link.

**6.115** Solve Prob. 6.113 assuming that the force **P** is replaced by a clockwise couple of moment **M<sub>0</sub>** applied to member CDE at D.

**6.116** Solve Prob. 6.114 assuming that the force **P** is replaced by a clockwise couple of moment **M<sub>0</sub>** applied to member CDE at D.

**6.117** Four beams, each of length  $3a$ , are held together by single nails at A, B, C, and D. Each beam is attached to a support located at a distance  $a$  from an end of the beam as shown. Assuming that only vertical forces are exerted at the connections, determine the vertical reactions at E, F, G, and H.

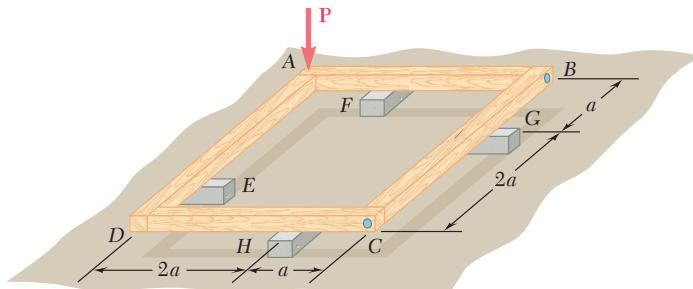


Fig. P6.117

**6.118** Four beams, each of length  $2a$ , are nailed together at their midpoints to form the support system shown. Assuming that only vertical forces are exerted at the connections, determine the vertical reactions at A, D, E, and H.

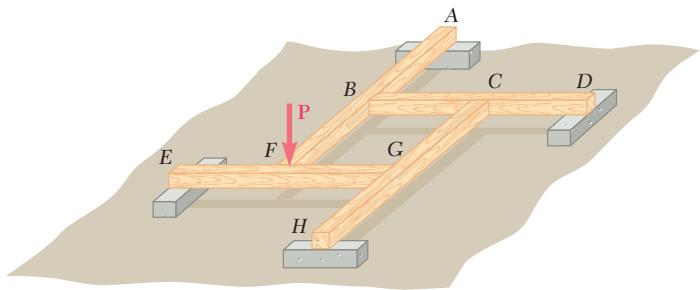


Fig. P6.118

**6.119 through 6.121** Each of the frames shown consists of two L-shaped members connected by two rigid links. For each frame, determine the reactions at the supports and indicate whether the frame is rigid.

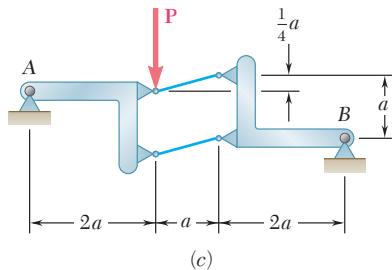
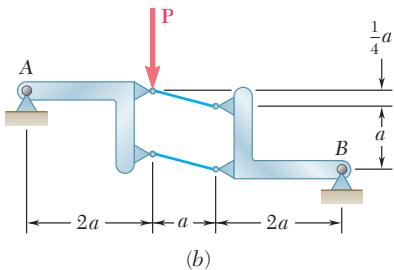
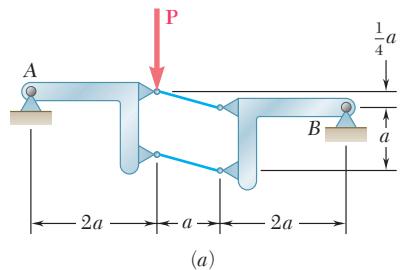


Fig. P6.119

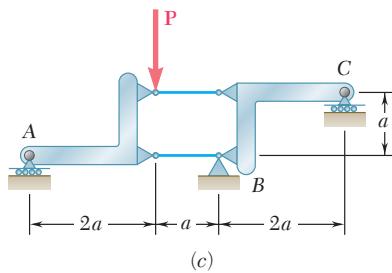
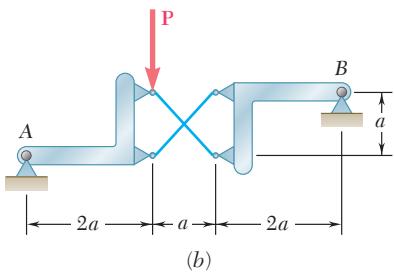
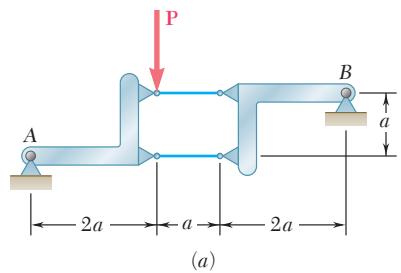


Fig. P6.120

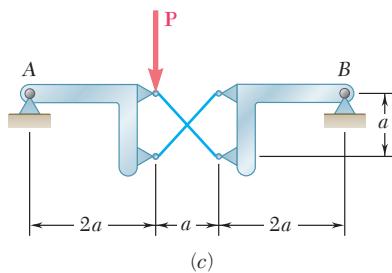
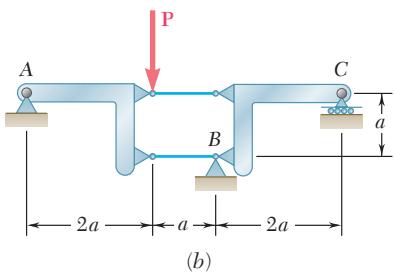
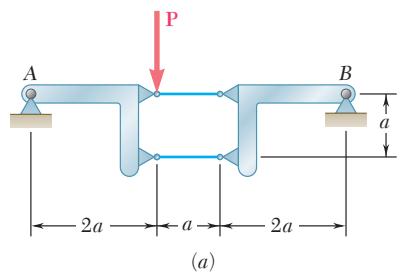


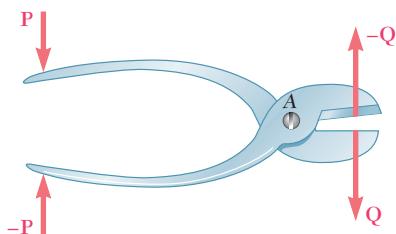
Fig. P6.121

## 6.12 MACHINES

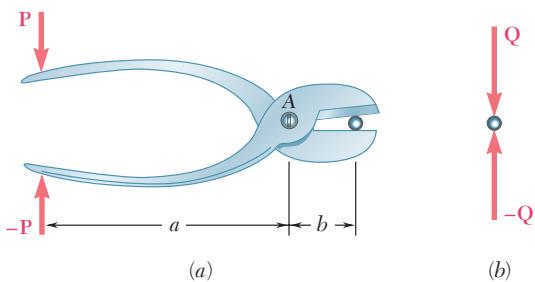
Machines are structures designed to transmit and modify forces. Whether they are simple tools or include complicated mechanisms, their main purpose is to transform *input forces* into *output forces*. Consider, for example, a pair of cutting pliers used to cut a wire (Fig. 6.22a). If we apply two equal and opposite forces  $\mathbf{P}$  and  $-\mathbf{P}$  on their handles, they will exert two equal and opposite forces  $\mathbf{Q}$  and  $-\mathbf{Q}$  on the wire (Fig. 6.22b).



**Photo 6.5** The lamp shown can be placed in many positions. By considering various free bodies, the force in the springs and the internal forces at the joints can be determined.

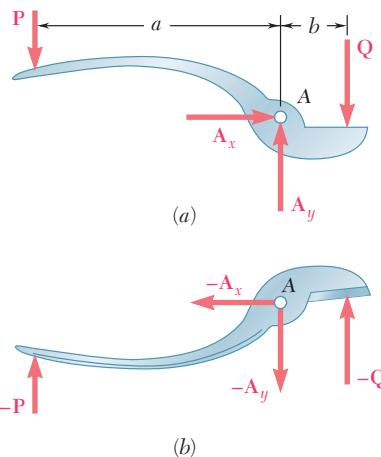


**Fig. 6.23**



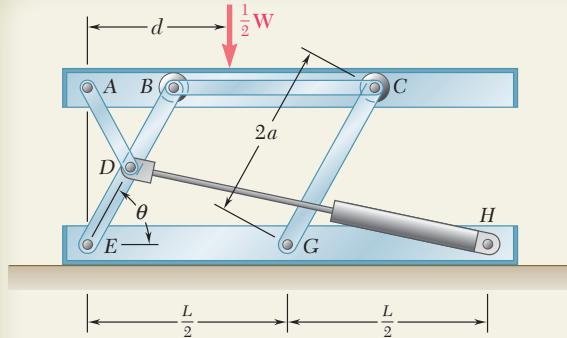
**Fig. 6.22**

To determine the magnitude  $Q$  of the output forces when the magnitude  $P$  of the input forces is known (or, conversely, to determine  $P$  when  $Q$  is known), we draw a free-body diagram of the pliers *alone*, showing the input forces  $\mathbf{P}$  and  $-\mathbf{P}$  and the *reactions*  $-\mathbf{Q}$  and  $\mathbf{Q}$  that the wire exerts on the pliers (Fig. 6.23). However, since a pair of pliers forms a nonrigid structure, we must use one of the component parts as a free body in order to determine the unknown forces. Considering Fig. 6.24a, for example, and taking moments about  $A$ , we obtain the relation  $Pa = Qb$ , which defines the magnitude  $Q$  in terms of  $P$  or  $P$  in terms of  $Q$ . The same free-body diagram can be used to determine the components of the internal force at  $A$ ; we find  $A_x = 0$  and  $A_y = P + Q$ .



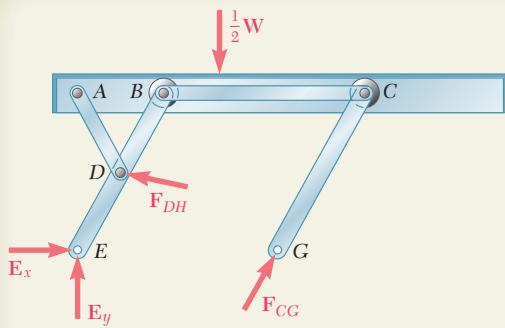
**Fig. 6.24**

In the case of more complicated machines, it generally will be necessary to use several free-body diagrams and, possibly, to solve simultaneous equations involving various internal forces. The free bodies should be chosen to include the input forces and the reactions to the output forces, and the total number of unknown force components involved should not exceed the number of available independent equations. It is advisable, before attempting to solve a problem, to determine whether the structure considered is determinate. There is no point, however, in discussing the rigidity of a machine, since a machine includes moving parts and thus *must* be nonrigid.



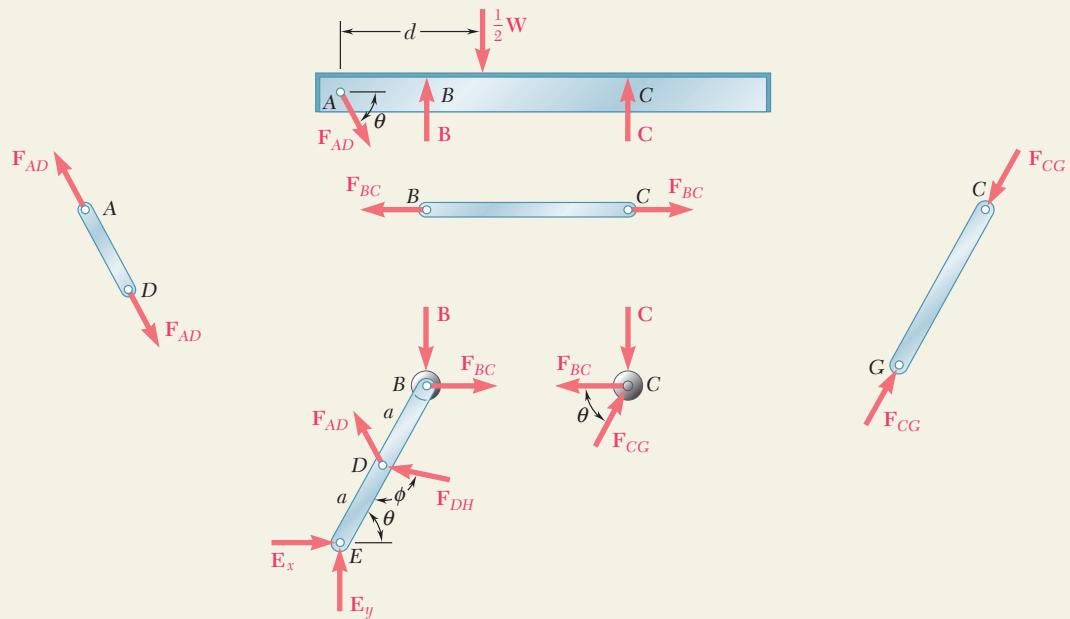
## SAMPLE PROBLEM 6.7

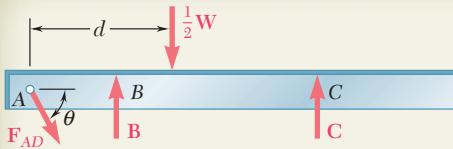
A hydraulic-lift table is used to raise a 1000-kg crate. It consists of a platform and two identical linkages on which hydraulic cylinders exert equal forces. (Only one linkage and one cylinder are shown.) Members  $EDB$  and  $CG$  are each of length  $2a$ , and member  $AD$  is pinned to the midpoint of  $EDB$ . If the crate is placed on the table, so that half of its weight is supported by the system shown, determine the force exerted by each cylinder in raising the crate for  $\theta = 60^\circ$ ,  $a = 0.70$  m, and  $L = 3.20$  m. Show that the result obtained is independent of the distance  $d$ .



## SOLUTION

The machine considered consists of the platform and of the linkage. Its free-body diagram includes an input force  $\mathbf{F}_{DH}$  exerted by the cylinder, the weight  $\frac{1}{2}\mathbf{W}$ , equal and opposite to the output force, and reactions at  $E$  and  $G$  that we assume to be directed as shown. Since more than three unknowns are involved, this diagram will not be used. The mechanism is dismembered and a free-body diagram is drawn for each of its component parts. We note that  $AD$ ,  $BC$ , and  $CG$  are two-force members. We already assumed member  $CG$  to be in compression; we now assume that  $AD$  and  $BC$  are in tension and direct as shown the forces exerted on them. Equal and opposite vectors will be used to represent the forces exerted by the two-force members on the platform, on member  $BDE$ , and on roller  $C$ .

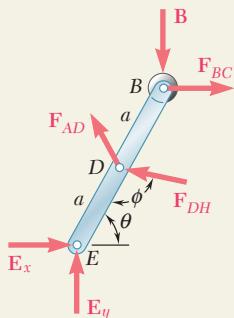




**Free Body: Platform ABC.**

$$\begin{aligned}\stackrel{+}{\rightarrow} \sum F_x &= 0: & F_{AD} \cos \theta &= 0 & F_{AD} &= 0 \\ +\uparrow \sum F_y &= 0: & B + C - \frac{1}{2}W &= 0 & B + C &= \frac{1}{2}W\end{aligned}\quad (1)$$

**Free Body: Roller C.** We draw a force triangle and obtain  $F_{BC} = C \cot \theta$ .



**Free Body: Member BDE.** Recalling that  $F_{AD} = 0$ ,

$$\begin{aligned}+\uparrow \sum M_E &= 0: & F_{DH} \cos(\phi - 90^\circ)a - B(2a \cos \theta) - F_{BC}(2a \sin \theta) &= 0 \\ F_{DH} \sin \phi - B(2a \cos \theta) - (C \cot \theta)(2a \sin \theta) &= 0 \\ F_{DH} \sin \phi - 2(B + C) \cos \theta &= 0\end{aligned}$$

Recalling Eq. (1), we have

$$F_{DH} = W \frac{\cos \theta}{\sin \phi} \quad (2)$$

and we observe that *the result obtained is independent of d*. ◀

Applying first the law of sines to triangle EDH, we write

$$\frac{\sin \phi}{EH} = \frac{\sin \theta}{DH} \quad \sin \phi = \frac{EH}{DH} \sin \theta \quad (3)$$

Using now the law of cosines, we have

$$\begin{aligned}(DH)^2 &= a^2 + L^2 - 2aL \cos \theta \\ &= (0.70)^2 + (3.20)^2 - 2(0.70)(3.20) \cos 60^\circ \\ (DH)^2 &= 8.49 \quad DH = 2.91 \text{ m}\end{aligned}$$

We also note that

$$W = mg = (1000 \text{ kg})(9.81 \text{ m/s}^2) = 9810 \text{ N} = 9.81 \text{ kN}$$

Substituting for  $\sin \phi$  from (3) into (2) and using the numerical data, we write

$$F_{DH} = W \frac{DH}{EH} \cot \theta = (9.81 \text{ kN}) \frac{2.91 \text{ m}}{3.20 \text{ m}} \cot 60^\circ$$

$$F_{DH} = 5.15 \text{ kN} \quad \text{◀}$$

# SOLVING PROBLEMS ON YOUR OWN

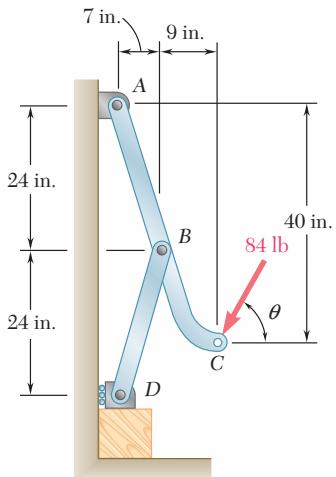
This lesson was devoted to the analysis of *machines*. Since machines are designed to transmit or modify forces, they always contain moving parts. However, the machines considered here will always be at rest, and you will be working with the set of *forces required to maintain the equilibrium of the machine*.

Known forces that act on a machine are called *input forces*. A *machine transforms the input forces into output forces*, such as the cutting forces applied by the pliers of Fig. 6.22. You will determine the output forces by finding the forces equal and opposite to the output forces that should be applied to the machine to maintain its equilibrium.

In the preceding lesson you analyzed frames; you will now use almost the same procedure to analyze machines:

- 1. Draw a free-body diagram of the whole machine,** and use it to determine as many as possible of the unknown forces exerted on the machine.
- 2. Dismember the machine, and draw a free-body diagram of each member.**
- 3. Considering first the two-force members,** apply equal and opposite forces to each two-force member at the points where it is connected to another member. If you cannot tell at this point whether the member is in tension or in compression just *assume* that the member is in tension and *direct both of the forces away from the member*. Since these forces have the same unknown magnitude, *give them both the same name*.
- 4. Next consider the multiforce members.** For each of these members, show all the forces acting on the member, including applied loads and forces, reactions, and internal forces at connections.
  - a. Where a multiforce member is connected to a two-force member,** apply to the multiforce member a force *equal and opposite* to the force drawn on the free-body diagram of the two-force member, *giving it the same name*.
  - b. Where a multiforce member is connected to another multiforce member,** use *horizontal and vertical components* to represent the internal forces at that point. The directions you choose for each of the two force components exerted on the first multiforce member are arbitrary, but *you must apply equal and opposite force components of the same name* to the other multiforce member.
- 5. Equilibrium equations can be written** after you have completed the various free-body diagrams.
  - a. To simplify your solution,** you should, whenever possible, write and solve equilibrium equations involving single unknowns.
  - b. Since you arbitrarily chose the direction of each of the unknown forces,** you must determine at the end of the solution whether your guess was correct. To that effect, *consider the sign* of the value found for each of the unknowns. A *positive* sign indicates that your guess was correct, and a *negative* sign indicates that it was not.
- 6. Finally, you should check your solution** by substituting the results obtained into an equilibrium equation that you have not previously used.

# PROBLEMS

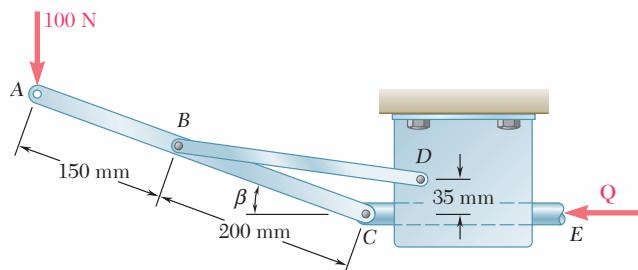


**Fig. P6.122**

- 6.122** An 84-lb force is applied to the toggle vise at *C*. Knowing that  $\theta = 90^\circ$ , determine (a) the vertical force exerted on the block at *D*, (b) the force exerted on member *ABC* at *B*.

- 6.123** Solve Prob. 6.122 when  $\theta = 0$ .

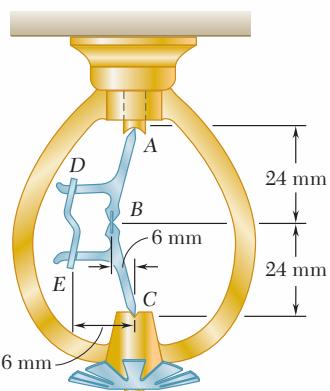
- 6.124** The control rod *CE* passes through a horizontal hole in the body of the toggle system shown. Knowing that link *BD* is 250 mm long, determine the force *Q* required to hold the system in equilibrium when  $\beta = 20^\circ$ .



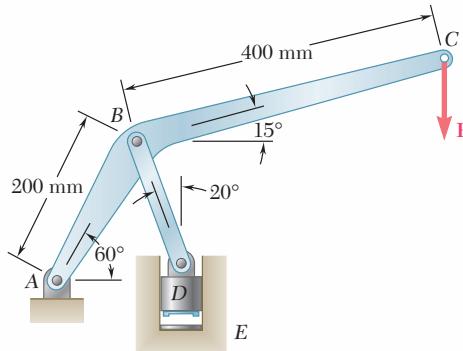
**Fig. P6.124**

- 6.125** Solve Prob. 6.124 when (a)  $\beta = 0$ , (b)  $\beta = 6^\circ$ .

- 6.126** The press shown is used to emboss a small seal at *E*. Knowing that *P* = 250 N, determine (a) the vertical component of the force exerted on the seal, (b) the reaction at *A*.



**Fig. P6.128**

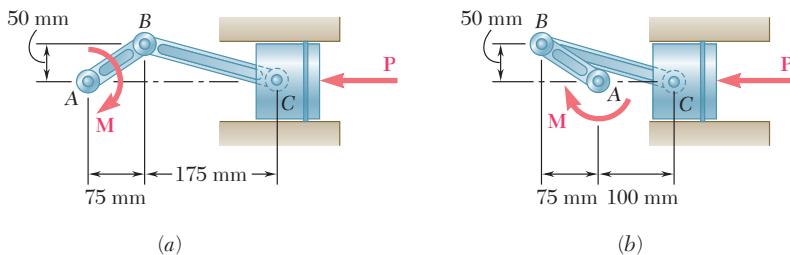


**Fig. P6.126 and P6.127**

- 6.127** The press shown is used to emboss a small seal at *E*. Knowing that the vertical component of the force exerted on the seal must be 900 N, determine (a) the required vertical force *P*, (b) the corresponding reaction at *A*.

- 6.128** Water pressure in the supply system exerts a downward force of 135 N on the vertical plug at *A*. Determine the tension in the fusible link *DE* and the force exerted on member *BCE* at *B*.

- 6.129** A couple  $\mathbf{M}$  of magnitude  $1.5 \text{ kN} \cdot \text{m}$  is applied to the crank of the engine system shown. For each of the two positions shown, determine the force  $\mathbf{P}$  required to hold the system in equilibrium.



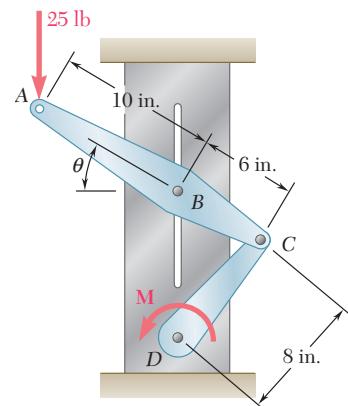
**Fig. P6.129 and P6.130**

- 6.130** A force  $\mathbf{P}$  of magnitude  $16 \text{ kN}$  is applied to the piston of the engine system shown. For each of the two positions shown, determine the couple  $\mathbf{M}$  required to hold the system in equilibrium.

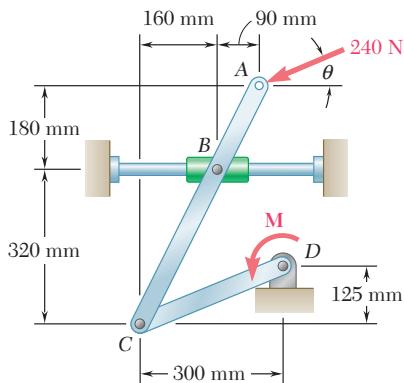
- 6.131** The pin at  $B$  is attached to member  $ABC$  and can slide freely along the slot cut in the fixed plate. Neglecting the effect of friction, determine the couple  $\mathbf{M}$  required to hold the system in equilibrium when  $\theta = 30^\circ$ .

- 6.132** The pin at  $B$  is attached to member  $ABC$  and can slide freely along the slot cut in the fixed plate. Neglecting the effect of friction, determine the couple  $\mathbf{M}$  required to hold the system in equilibrium when  $\theta = 60^\circ$ .

- 6.133** Arm  $ABC$  is connected by pins to a collar at  $B$  and to crank  $CD$ . Neglecting the effect of friction, determine the couple  $\mathbf{M}$  required to hold the system in equilibrium when  $\theta = 0$ .



**Fig. P6.131 and P6.132**



**Fig. P6.133 and P6.134**

- 6.134** Arm  $ABC$  is connected by pins to a collar at  $B$  and to crank  $CD$ . Neglecting the effect of friction, determine the couple  $\mathbf{M}$  required to hold the system in equilibrium when  $\theta = 90^\circ$ .

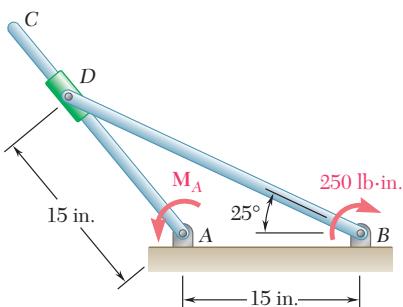


Fig. P6.135

**6.135 and 6.136** Two rods are connected by a slider block as shown. Neglecting the effect of friction, determine the couple  $\mathbf{M}_A$  required to hold the system in equilibrium.

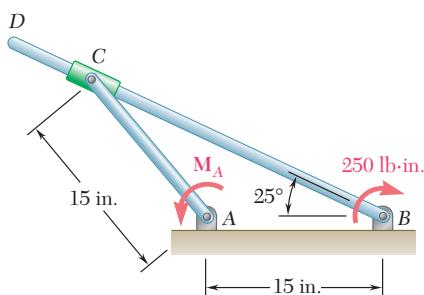


Fig. P6.136

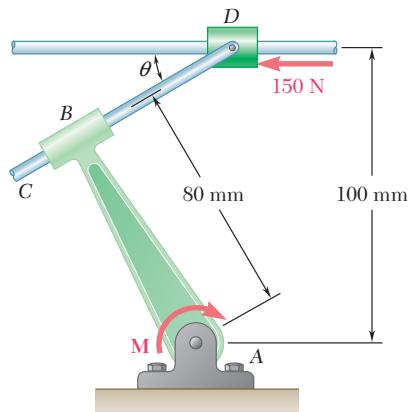


Fig. P6.138

**6.137 and 6.138** Rod CD is attached to the collar D and passes through a collar welded to end B of lever AB. Neglecting the effect of friction, determine the couple  $\mathbf{M}$  required to hold the system in equilibrium when  $\theta = 30^\circ$ .

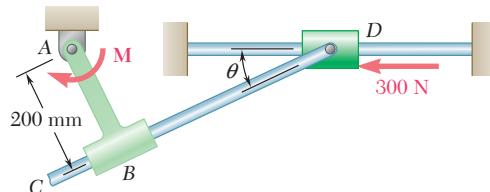


Fig. P6.137

**6.139** Two hydraulic cylinders control the position of the robotic arm ABC. Knowing that in the position shown the cylinders are parallel, determine the force exerted by each cylinder when  $P = 160 \text{ N}$  and  $Q = 80 \text{ N}$ .

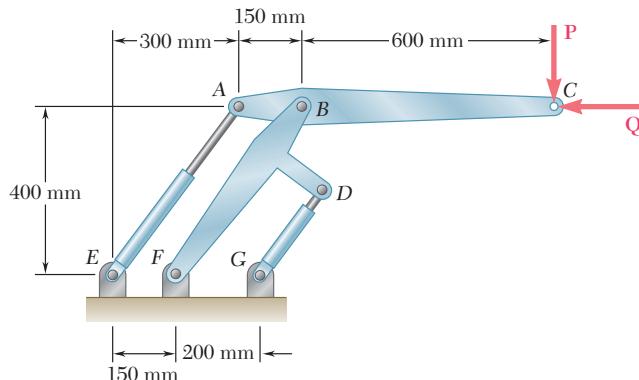


Fig. P6.139 and P6.140

**6.140** Two hydraulic cylinders control the position of the robotic arm ABC. In the position shown, the cylinders are parallel and both are in tension. Knowing that  $F_{AE} = 600 \text{ N}$  and  $F_{DG} = 50 \text{ N}$ , determine the forces  $\mathbf{P}$  and  $\mathbf{Q}$  applied at C to arm ABC.

- 6.141** A log weighing 800 lb is lifted by a pair of tongs as shown. Determine the forces exerted at *E* and *F* on tong *DEF*.

- 6.142** A 39-ft length of railroad rail of weight 44 lb/ft is lifted by the tongs shown. Determine the forces exerted at *D* and *F* on tong *BDF*.

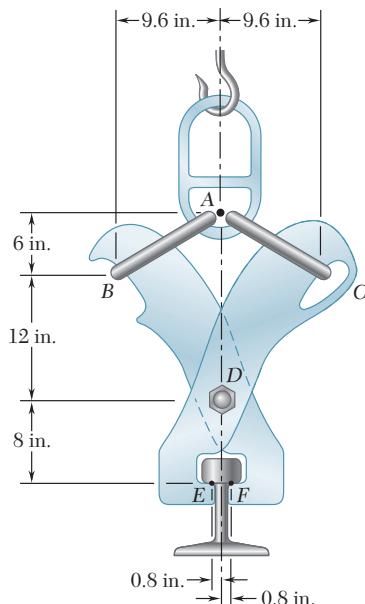


Fig. P6.142

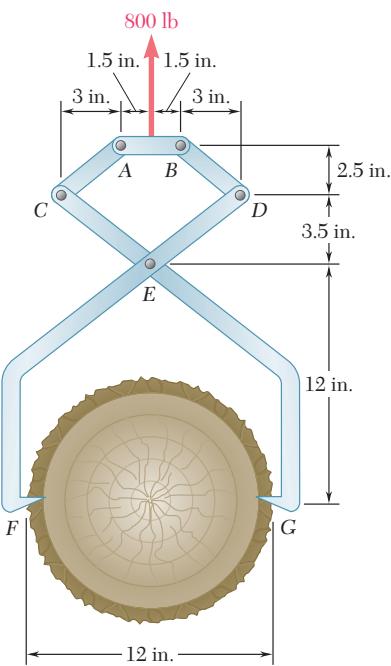


Fig. P6.141

- 6.143** The tongs shown are used to apply a total upward force of 45 kN on a pipe cap. Determine the forces exerted at *D* and *F* on tong *ADF*.

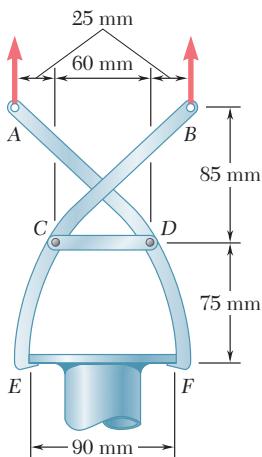


Fig. P6.143

- 6.144** If the toggle shown is added to the tongs of Prob. 6.143 and a single vertical force is applied at *G*, determine the forces exerted at *D* and *F* on tong *ADF*.

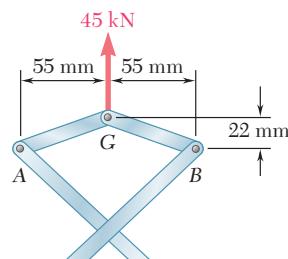


Fig. P6.144

- 6.145** The pliers shown are used to grip a 0.3-in.-diameter rod. Knowing that two 60-lb forces are applied to the handles, determine (a) the magnitude of the forces exerted on the rod, (b) the force exerted by the pin at A on portion AB of the pliers.

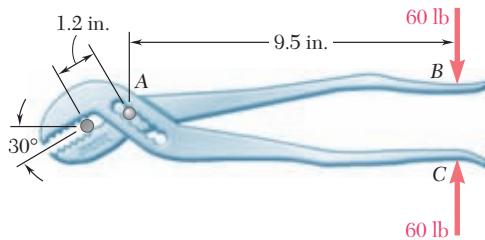


Fig. P6.145

- 6.146** In using the bolt cutter shown, a worker applies two 300-N forces to the handles. Determine the magnitude of the forces exerted by the cutter on the bolt.

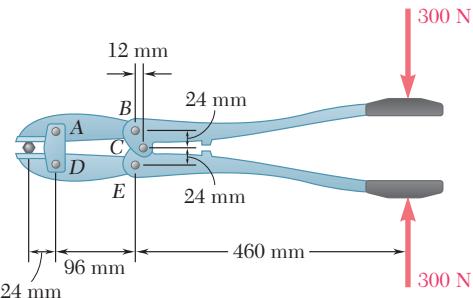


Fig. P6.146

- 6.147** Determine the magnitude of the gripping forces exerted along line aa on the nut when two 50-lb forces are applied to the handles as shown. Assume that pins A and D slide freely in slots cut in the jaws.

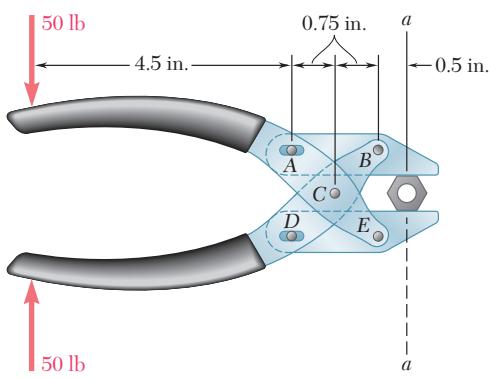


Fig. P6.147

- 6.148** Determine the magnitude of the gripping forces produced when two 300-N forces are applied as shown.

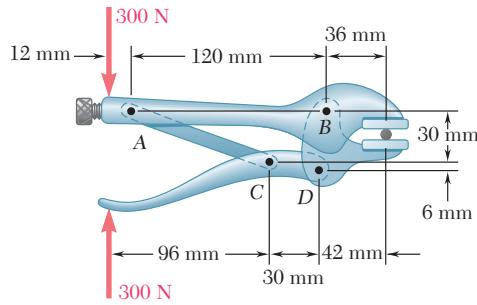


Fig. P6.148

- 6.149** Knowing that the frame shown has a sag at B of  $a = 1$  in., determine the force  $\mathbf{P}$  required to maintain equilibrium in the position shown.

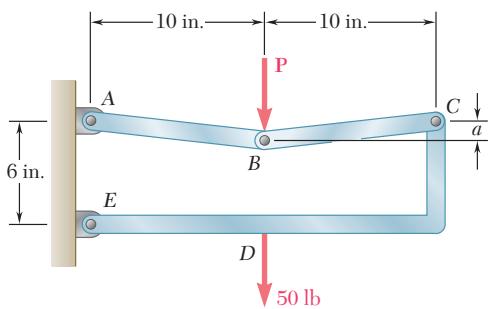


Fig. P6.149 and P6.150

- 6.150** Knowing that the frame shown has a sag at B of  $a = 0.5$  in., determine the force  $\mathbf{P}$  required to maintain equilibrium in the position shown.

- 6.151** The garden shears shown consist of two blades and two handles. The two handles are connected by pin *C* and the two blades are connected by pin *D*. The left blade and the right handle are connected by pin *A*; the right blade and the left handle are connected by pin *B*. Determine the magnitude of the forces exerted on the small branch at *E* when two 80-N forces are applied to the handles as shown.

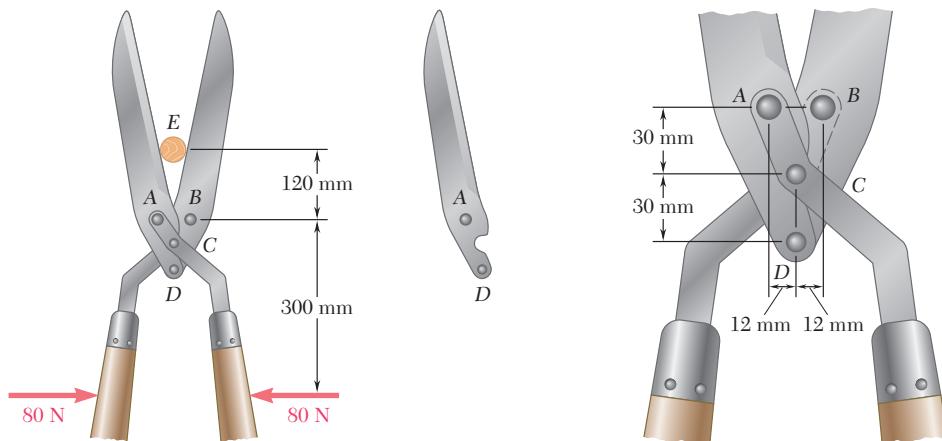


Fig. P6.151

- 6.152** The telescoping arm *ABC* is used to provide an elevated platform for construction workers. The workers and the platform together have a mass of 200 kg and have a combined center of gravity located directly above *C*. For the position when  $\theta = 20^\circ$ , determine (a) the force exerted at *B* by the single hydraulic cylinder *BD*, (b) the force exerted on the supporting carriage at *A*.

- 6.153** The telescoping arm *ABC* can be lowered until end *C* is close to the ground, so that workers can easily board the platform. For the position when  $\theta = -20^\circ$ , determine (a) the force exerted at *B* by the single hydraulic cylinder *BD*, (b) the force exerted on the supporting carriage at *A*.

- 6.154** The position of member *ABC* is controlled by the hydraulic cylinder *CD*. Knowing that  $\theta = 30^\circ$ , determine for the loading shown (a) the force exerted by the hydraulic cylinder on pin *C*, (b) the reaction at *B*.

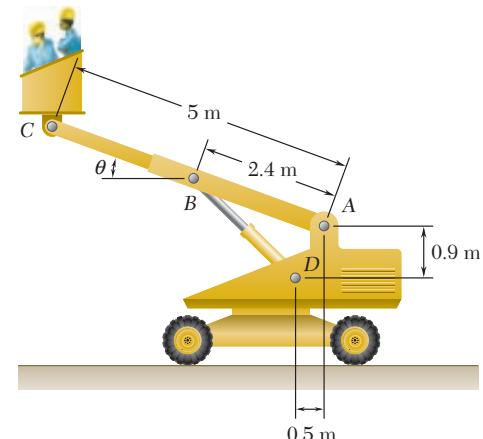


Fig. P6.152 and P6.153

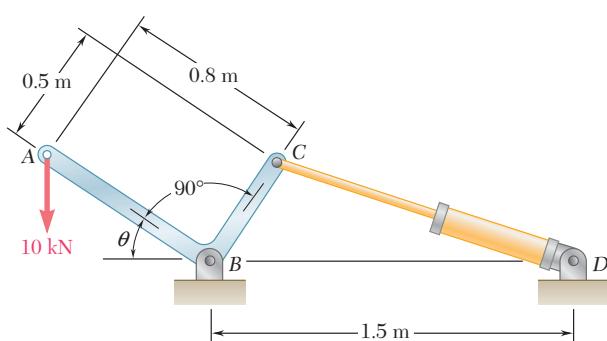
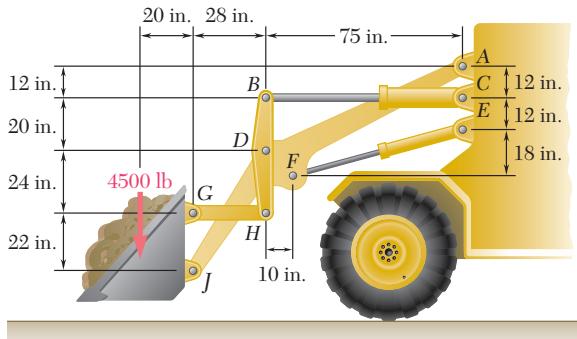


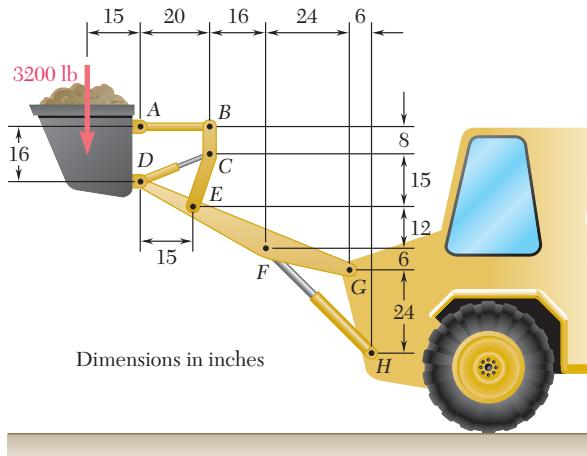
Fig. P6.154

**6.155** The motion of the bucket of the front-end loader shown is controlled by two arms and a linkage that are pin-connected at  $D$ . The arms are located symmetrically with respect to the central, vertical, and longitudinal plane of the loader; one arm  $AFJ$  and its control cylinder  $EF$  are shown. The single linkage  $GHDB$  and its control cylinder  $BC$  are located in the plane of symmetry. For the position and loading shown, determine the force exerted (a) by cylinder  $BC$ , (b) by cylinder  $EF$ .



**Fig. P6.155**

**6.156** The bucket of the front-end loader shown carries a 3200-lb load. The motion of the bucket is controlled by two identical mechanisms, only one of which is shown. Knowing that the mechanism shown supports one-half of the 3200-lb load, determine the force exerted (a) by cylinder  $CD$ , (b) by cylinder  $FH$ .



**Fig. P6.156**

- 6.157** The motion of the backhoe bucket shown is controlled by the hydraulic cylinders  $AD$ ,  $CG$ , and  $EF$ . As a result of an attempt to dislodge a portion of a slab, a 2-kip force  $\mathbf{P}$  is exerted on the bucket teeth at  $J$ . Knowing that  $\theta = 45^\circ$ , determine the force exerted by each cylinder.

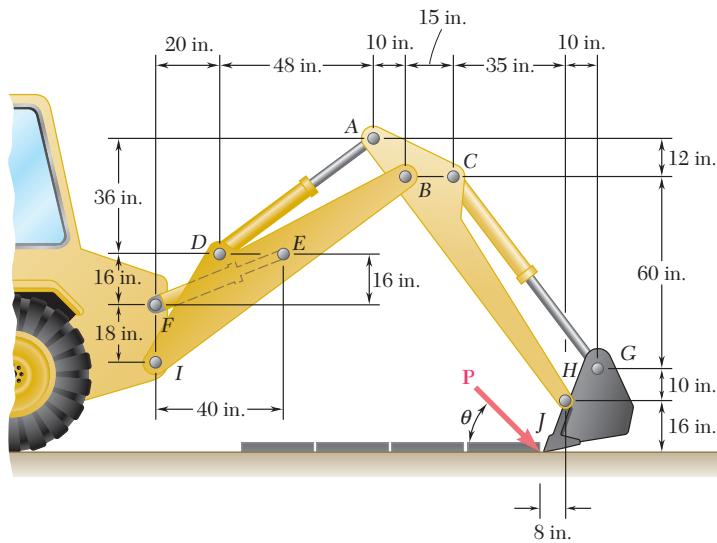


Fig. P6.157

- 6.158** Solve Prob. 6.157 assuming that the 2-kip force  $\mathbf{P}$  acts horizontally to the right ( $\theta = 0$ ).

- 6.159** The gears  $D$  and  $G$  are rigidly attached to shafts that are held by frictionless bearings. If  $r_D = 90 \text{ mm}$  and  $r_G = 30 \text{ mm}$ , determine (a) the couple  $\mathbf{M}_0$  that must be applied for equilibrium, (b) the reactions at  $A$  and  $B$ .

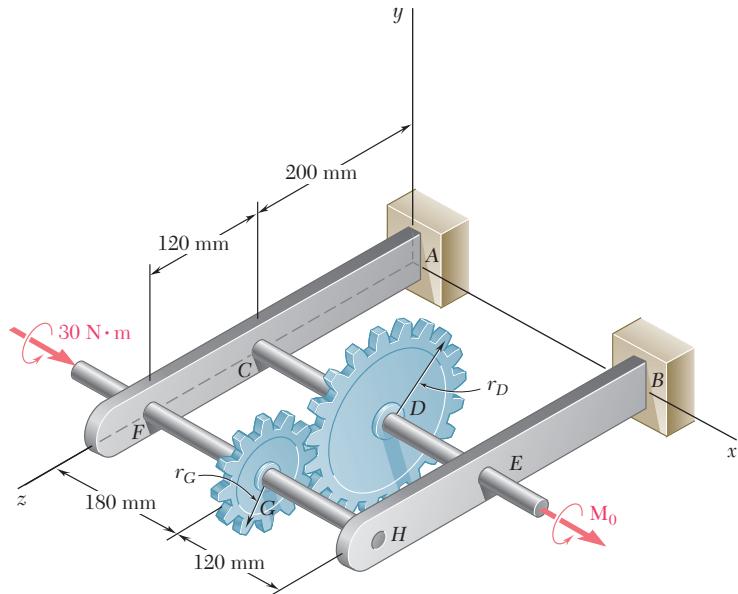


Fig. P6.159

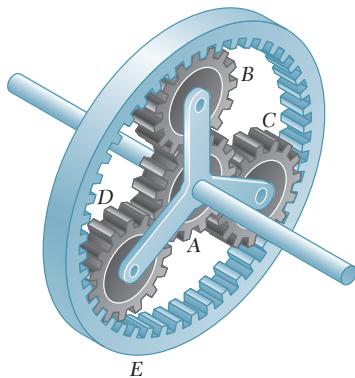


Fig. P6.160

- 6.160** In the planetary gear system shown, the radius of the central gear  $A$  is  $a = 18$  mm, the radius of each planetary gear is  $b$ , and the radius of the outer gear  $E$  is  $(a + 2b)$ . A clockwise couple of magnitude  $M_A = 10$  N · m is applied to the central gear  $A$  and a counterclockwise couple of magnitude  $M_S = 50$  N · m is applied to the spider  $BCD$ . If the system is to be in equilibrium, determine (a) the required radius  $b$  of the planetary gears, (b) the magnitude  $M_E$  of the couple that must be applied to the outer gear  $E$ .

- \*6.161** Two shafts  $AC$  and  $CF$ , which lie in the vertical  $xy$  plane, are connected by a universal joint at  $C$ . The bearings at  $B$  and  $D$  do not exert any axial force. A couple of magnitude  $500$  lb · in. (clockwise when viewed from the positive  $x$  axis) is applied to shaft  $CF$  at  $F$ . At a time when the arm of the crosspiece attached to shaft  $CF$  is horizontal, determine (a) the magnitude of the couple that must be applied to shaft  $AC$  at  $A$  to maintain equilibrium, (b) the reactions at  $B$ ,  $D$ , and  $E$ . (Hint: The sum of the couples exerted on the crosspiece must be zero.)

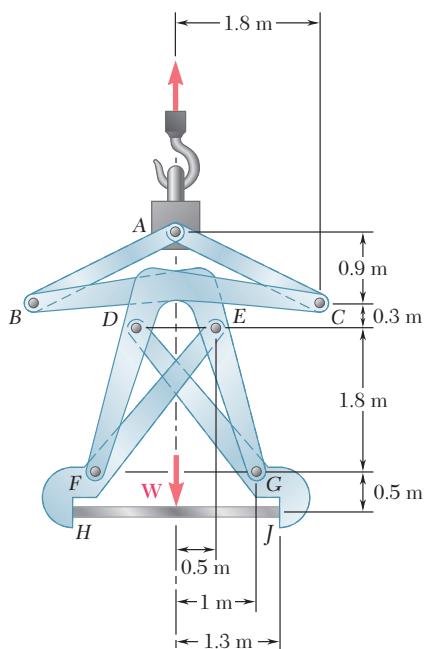


Fig. P6.163

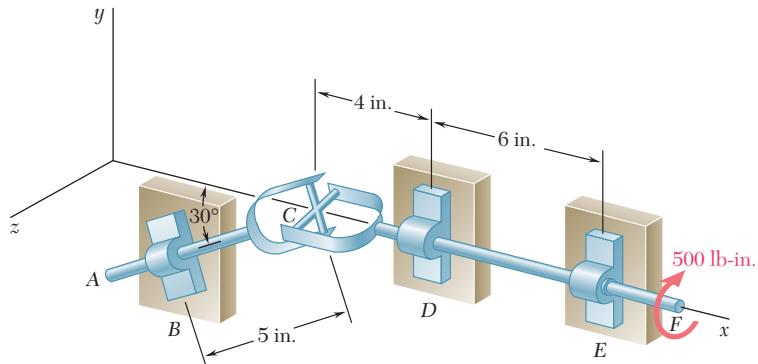


Fig. P6.161

- \*6.162** Solve Prob. 6.161 assuming that the arm of the crosspiece attached to shaft  $CF$  is vertical.

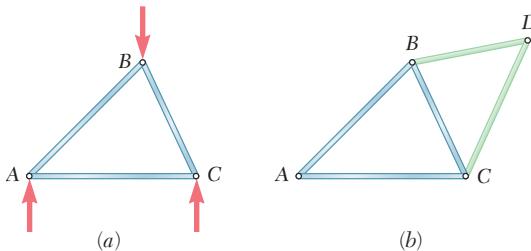
- \*6.163** The large mechanical tongs shown are used to grab and lift a thick 7500-kg steel slab  $HJ$ . Knowing that slipping does not occur between the tong grips and the slab at  $H$  and  $J$ , determine the components of all forces acting on member  $EFH$ . (Hint: Consider the symmetry of the tongs to establish relationships between the components of the force acting at  $E$  on  $EFH$  and the components of the force acting at  $D$  on  $CDF$ .)

# REVIEW AND SUMMARY

In this chapter you learned to determine the *internal forces* holding together the various parts of a structure.

The first half of the chapter was devoted to the analysis of *trusses*, i.e., to the analysis of structures consisting of *straight members connected at their extremities only*. The members being slender and unable to support lateral loads, all the loads must be applied at the joints; a truss may thus be assumed to consist of *pins and two-force members* [Sec. 6.2].

A truss is said to be *rigid* if it is designed in such a way that it will not greatly deform or collapse under a small load. A triangular truss consisting of three members connected at three joints is clearly a rigid truss (Fig. 6.25a) and so will be the truss obtained by adding two new members to the first one and connecting them at a new joint (Fig. 6.25b). Trusses obtained by repeating this procedure are called *simple trusses*. We may check that in a simple truss the total number of members is  $m = 2n - 3$ , where  $n$  is the total number of joints [Sec. 6.3].



**Fig. 6.25**

The forces in the various members of a simple truss can be determined by the *method of joints* [Sec. 6.4]. First, the reactions at the supports can be obtained by considering the entire truss as a free body. The free-body diagram of each pin is then drawn, showing the forces exerted on the pin by the members or supports it connects. Since the members are straight two-force members, the force exerted by a member on the pin is directed along that member, and only the magnitude of the force is unknown. It is always possible in the case of a simple truss to draw the free-body diagrams of the pins in such an order that only two unknown forces are included in each diagram. These forces can be obtained from the corresponding two equilibrium equations or—if only three forces are involved—from the corresponding force triangle. If the force exerted by a member on a pin is directed toward that pin, the member is in *compression*;

## Analysis of trusses

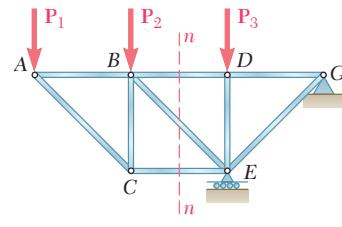
## Simple trusses

## Method of joints

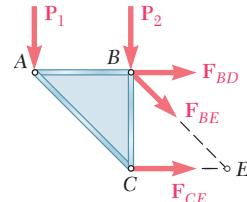
if it is directed away from the pin, the member is in *tension* [Sample Prob. 6.1]. The analysis of a truss is sometimes expedited by first recognizing *joints under special loading conditions* [Sec. 6.5]. The method of joints can also be extended to the analysis of three-dimensional or *space trusses* [Sec. 6.6].

### Method of sections

The *method of sections* is usually preferred to the method of joints when the force in only one member—or very few members—of a truss is desired [Sec. 6.7]. To determine the force in member *BD* of the truss of Fig. 6.26a, for example, we *pass a section* through members *BD*, *BE*, and *CE*, remove these members, and use the portion *ABC* of the truss as a free body (Fig. 6.26b). Writing  $\sum M_E = 0$ , we determine the magnitude of the force  $\mathbf{F}_{BD}$ , which represents the force in member *BD*. A positive sign indicates that the member is in *tension*; a negative sign indicates that it is in *compression* [Sample Probs. 6.2 and 6.3].



(a)



(b)

**Fig. 6.26**

### Compound trusses

The method of sections is particularly useful in the analysis of *compound trusses*, i.e., trusses which cannot be constructed from the basic triangular truss of Fig. 6.25a but which can be obtained by rigidly connecting several simple trusses [Sec. 6.8]. If the component trusses have been properly connected (e.g., one pin and one link, or three nonconcurrent and nonparallel links) and if the resulting structure is properly supported (e.g., one pin and one roller), the compound truss is *statically determinate, rigid, and completely constrained*. The following necessary—but not sufficient—condition is then satisfied:  $m + r = 2n$ , where  $m$  is the number of members,  $r$  is the number of unknowns representing the reactions at the supports, and  $n$  is the number of joints.

The second part of the chapter was devoted to the analysis of *frames and machines*. Frames and machines are structures which contain *multiforce members*, i.e., members acted upon by three or more forces. Frames are designed to support loads and are usually stationary, fully constrained structures. Machines are designed to transmit or modify forces and always contain moving parts [Sec. 6.9].

To *analyze a frame*, we first consider the *entire frame as a free body* and write three equilibrium equations [Sec. 6.10]. If the frame remains rigid when detached from its supports, the reactions involve only three unknowns and may be determined from these equations [Sample Probs. 6.4 and 6.5]. On the other hand, if the frame ceases to be rigid when detached from its supports, the reactions involve more than three unknowns and cannot be completely determined from the equilibrium equations of the frame [Sec. 6.11; Sample Prob. 6.6].

We then *dismember the frame* and identify the various members as either two-force members or multiforce members; pins are assumed to form an integral part of one of the members they connect. We draw the free-body diagram of each of the multiforce members, noting that when two multiforce members are connected to the same two-force member, they are acted upon by that member with *equal and opposite forces of unknown magnitude but known direction*. When two multiforce members are connected by a pin, they exert on each other *equal and opposite forces of unknown direction*, which should be represented by *two unknown components*. The equilibrium equations obtained from the free-body diagrams of the multiforce members can then be solved for the various internal forces [Sample Probs. 6.4 and 6.5]. The equilibrium equations can also be used to complete the determination of the reactions at the supports [Sample Prob. 6.6]. Actually, if the frame is *statically determinate and rigid*, the free-body diagrams of the multiforce members could provide as many equations as there are unknown forces (including the reactions) [Sec. 6.11]. However, as suggested above, it is advisable to first consider the free-body diagram of the entire frame to minimize the number of equations that must be solved simultaneously.

To *analyze a machine*, we dismember it and, following the same procedure as for a frame, draw the free-body diagram of each of the multiforce members. The corresponding equilibrium equations yield the *output forces* exerted by the machine in terms of the *input forces* applied to it, as well as the *internal forces* at the various connections [Sec. 6.12; Sample Prob. 6.7].

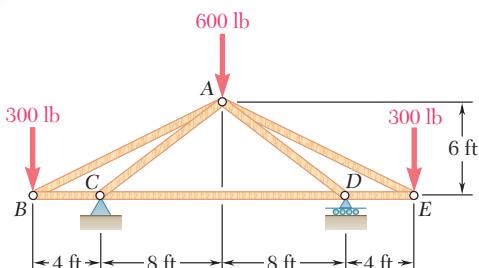
## Frames and machines

### Analysis of a frame

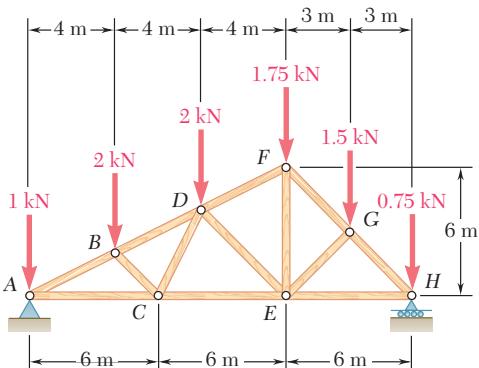
### Multiforce members

### Analysis of a machine

# REVIEW PROBLEMS



**Fig. P6.164**

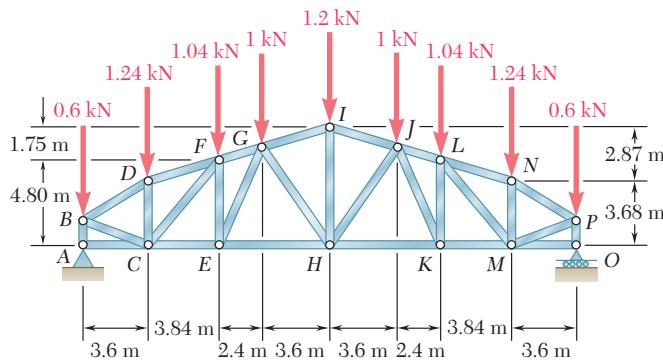


**Fig. P6.165**

- 6.164** Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

- 6.165** Using the method of joints, determine the force in each member of the double-pitch roof truss shown. State whether each member is in tension or compression.

- 6.166** The truss shown was designed to support the roof of a food market. For the given loading, determine the force in members *FG*, *EG*, and *EH*.



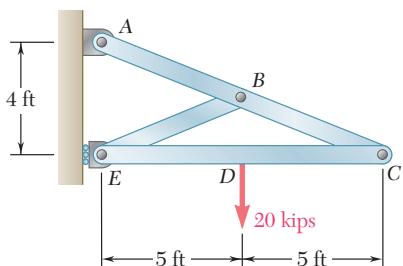
**Fig. P6.166 and P6.167**

- 6.167** The truss shown was designed to support the roof of a food market. For the given loading, determine the force in members *KM*, *LM*, and *LN*.

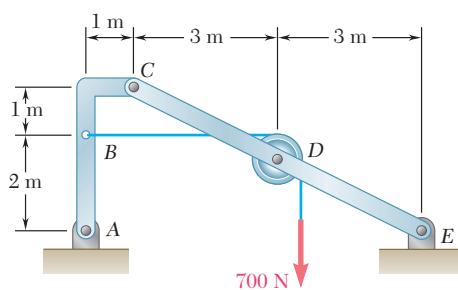
- 6.168** For the frame and loading shown, determine the components of all forces acting on member *ABC*.

- 6.169** Solve Prob. 6.168 assuming that the 20-kip load is replaced by a clockwise couple of magnitude 100 kip · ft applied to member *EDC* at point *D*.

- 6.170** Knowing that the pulley has a radius of 0.5 m, determine the components of the reactions at *A* and *E*.



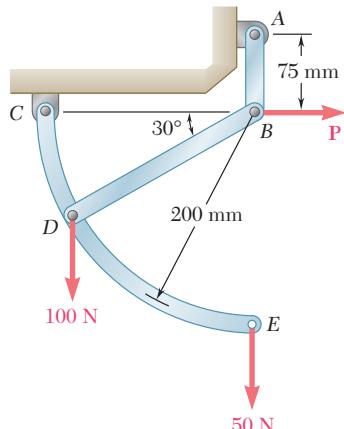
**Fig. P6.168**



**Fig. P6.170**

- 6.171** For the frame and loading shown, determine the reactions at *A*, *B*, *D*, and *E*. Assume that the surface at each support is frictionless.

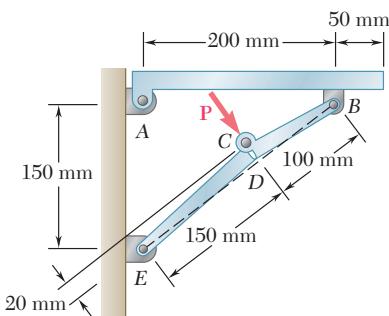
**6.172** For the system and loading shown, determine (a) the force **P** required for equilibrium, (b) the corresponding force in member *BD*, (c) the corresponding reaction at *C*.



**Fig. P6.172**

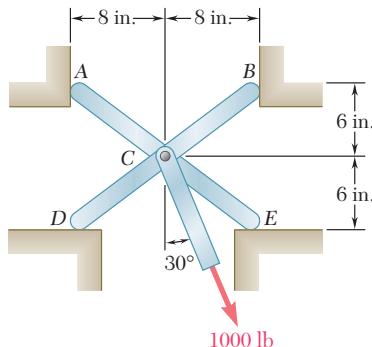
- 6.173** A small barrel weighing 60 lb is lifted by a pair of tongs as shown. Knowing that  $a = 5$  in., determine the forces exerted at  $B$  and  $D$  on tong  $ABD$ .

- 6.174** A 20-kg shelf is held horizontally by a self-locking brace that consists of two parts  $EDC$  and  $CDB$  hinged at  $C$  and bearing against each other at  $D$ . Determine the force  $\mathbf{P}$  required to release the brace.

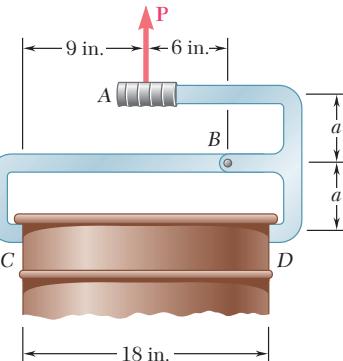


**Fig. P6.174**

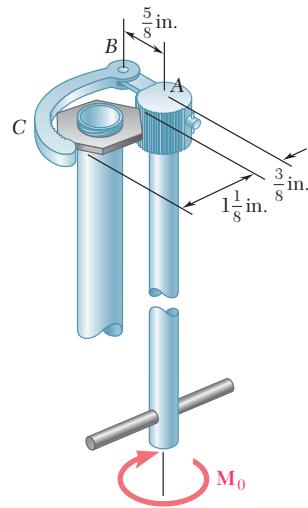
- 6.175** The specialized plumbing wrench shown is used in confined areas (e.g., under a basin or sink). It consists essentially of a jaw  $BC$  pinned at  $B$  to a long rod. Knowing that the forces exerted on the nut are equivalent to a clockwise (when viewed from above) couple of magnitude  $135 \text{ lb} \cdot \text{in.}$ , determine (a) the magnitude of the force exerted by pin  $B$  on jaw  $BC$ , (b) the couple  $\mathbf{M}_0$  that is applied to the wrench.



**Fig. P6.171**

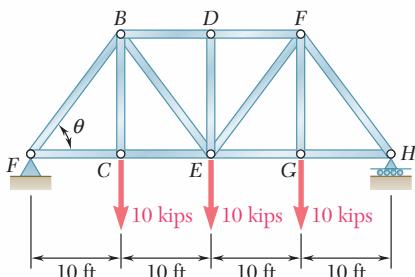


**Fig. P6.173**



**Fig. P6.175**

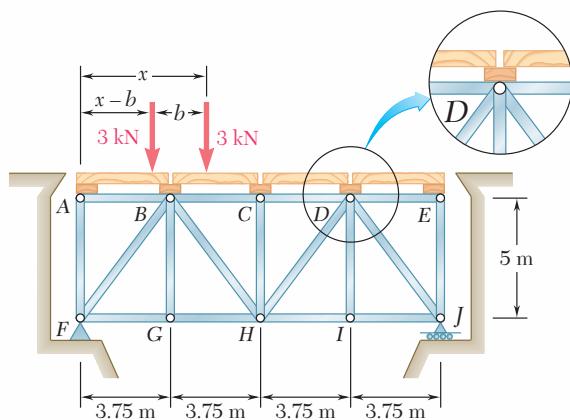
# COMPUTER PROBLEMS



**Fig. P6.C1**

**6.C1** A Pratt steel truss is to be designed to support three 10-kip loads as shown. The length of the truss is to be 40 ft. The height of the truss and thus the angle  $\theta$ , as well as the cross-sectional areas of the various members, are to be selected to obtain the most economical design. Specifically, the cross-sectional area of each member is to be chosen so that the stress (force divided by area) in that member is equal to 20 kips/in<sup>2</sup>, the allowable stress for the steel used; the total weight of the steel, and thus its cost, must be as small as possible. (a) Knowing that the specific weight of the steel used is 0.284 lb/in<sup>3</sup>, use computational software to calculate the weight of the truss and the cross-sectional area of each load-bearing member located to the left of  $DE$  for values of  $\theta$  from  $20^\circ$  to  $80^\circ$  using  $5^\circ$  increments. (b) Using appropriate smaller increments, determine the optimum value of  $\theta$  and the corresponding values of the weight of the truss and of the cross-sectional areas of the various members. Ignore the weight of any zero-force member in your computations.

**6.C2** The floor of a bridge will rest on stringers that will be simply supported by transverse floor beams, as in Fig. 6.3. The ends of the beams will be connected to the upper joints of two trusses, one of which is shown in Fig. P6.C2. As part of the design of the bridge, it is desired to simulate the effect on this truss of driving a 12-kN truck over the bridge. Knowing that the distance between the truck's axles is  $b = 2.25$  m and assuming that the weight of the truck is equally distributed over its four wheels, use computational software to calculate the forces created by the truck in members  $BH$  and  $GH$  for values of  $x$  from 0 to 17.25 m using 0.75-m increments. From the results obtained, determine (a) the maximum tensile force in  $BH$ , (b) the maximum compressive force in  $BH$ , (c) the maximum tensile force in  $GH$ . Indicate in each case the corresponding value of  $x$ . (Note: The increments have been selected so that the desired values are among those that will be tabulated.)



**Fig. P6.C2**

**6.C3** In the mechanism shown the position of boom  $AC$  is controlled by arm  $BD$ . For the loading shown, use computational software to determine the couple  $\mathbf{M}$  required to hold the system in equilibrium for values of  $\theta$  from  $-30^\circ$  to  $90^\circ$  using  $10^\circ$  increments. Also, for the same values of  $\theta$ , determine the reaction at  $A$ . As a part of the design process of the mechanism, use appropriate smaller increments and determine (a) the value of  $\theta$  for which  $M$  is maximum and the corresponding value of  $M$ , (b) the value of  $\theta$  for which the reaction at  $A$  is maximum and the corresponding magnitude of this reaction.

**6.C4** The design of a robotic system calls for the two-rod mechanism shown. Rods  $AC$  and  $BD$  are connected by a slider block  $D$  as shown. Neglecting the effect of friction, use computational software to determine the couple  $\mathbf{M}_A$  required to hold the rods in equilibrium for values of  $\theta$  from  $0$  to  $120^\circ$  using  $10^\circ$  increments. For the same values of  $\theta$ , determine the magnitude of the force  $\mathbf{F}$  exerted by rod  $AC$  on the slider block.

**6.C5** The compound-lever pruning shears shown can be adjusted by placing pin  $A$  at various ratchet positions on blade  $ACE$ . Knowing that the length  $AB$  is 0.85 in., use computational software to determine the magnitude of the vertical forces applied to the small branch for values of  $d$  from 0.4 in. to 0.6 in. using 0.025 in. increments. As a part of the design of the shears, use appropriate smaller increments and determine the smallest allowable value of  $d$  if the force in link  $AB$  is not to exceed 500 lb.

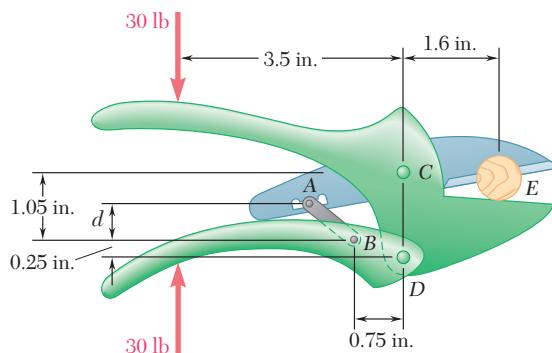


Fig. P6.C5

**6.C6** Rod  $CD$  is attached to collar  $D$  and passes through a collar welded to end  $B$  of lever  $AB$ . As an initial step in the design of lever  $AB$ , use computational software to calculate the magnitude  $M$  of the couple required to hold the system in equilibrium for values of  $\theta$  from  $15^\circ$  to  $90^\circ$  using  $5^\circ$  increments. Using appropriate smaller increments, determine the value of  $\theta$  for which  $M$  is minimum and the corresponding value of  $M$ .

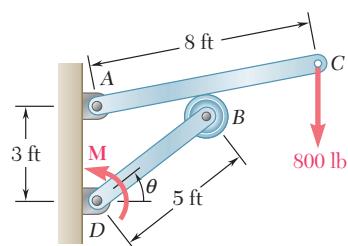


Fig. P6.C3

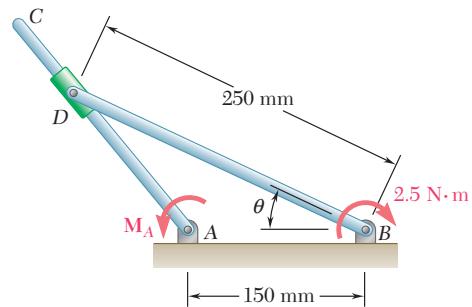


Fig. P6.C4

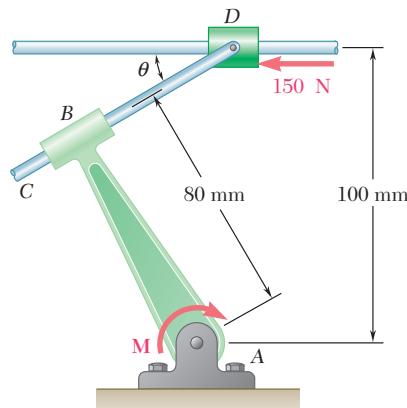
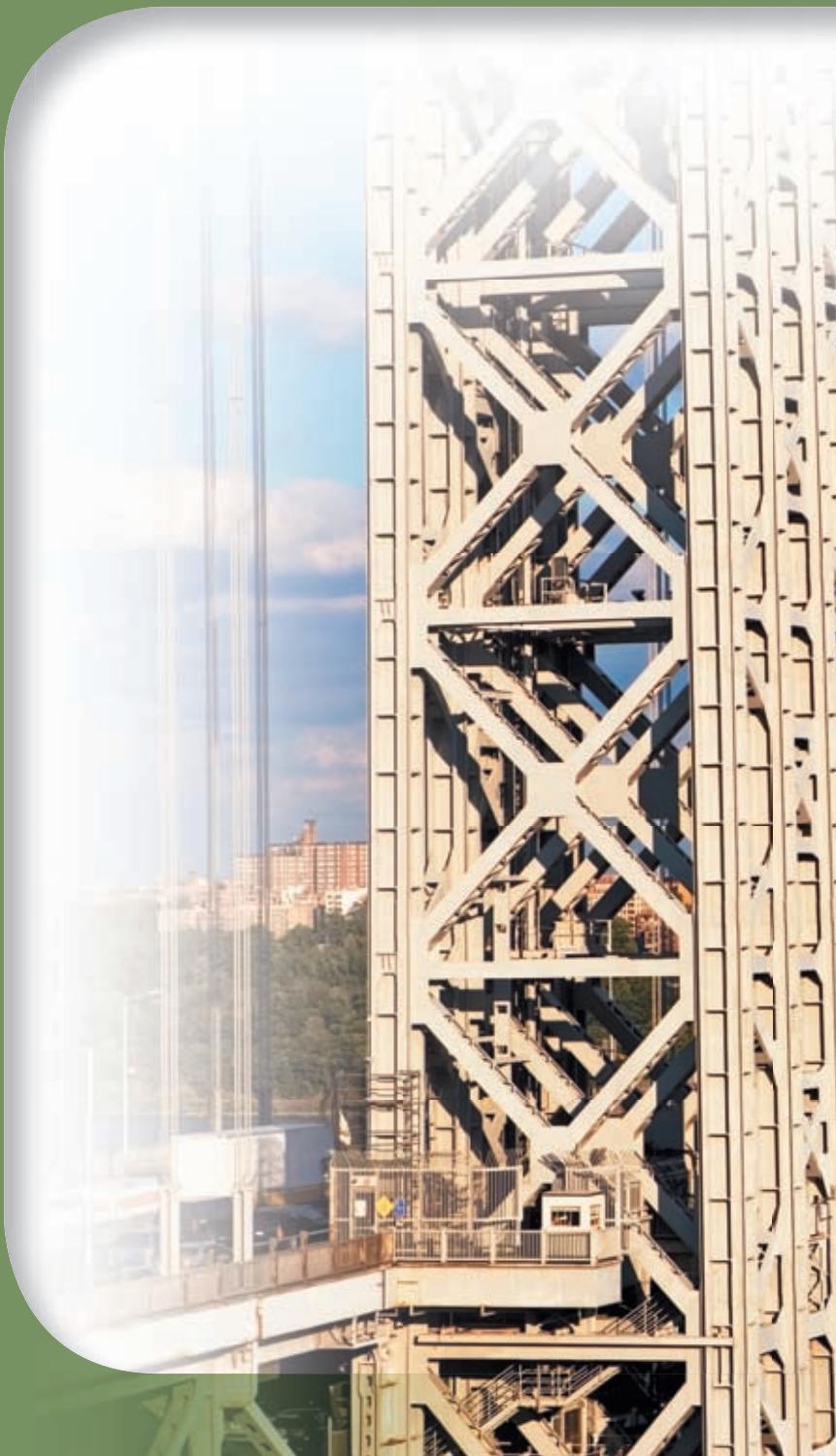


Fig. P6.C6

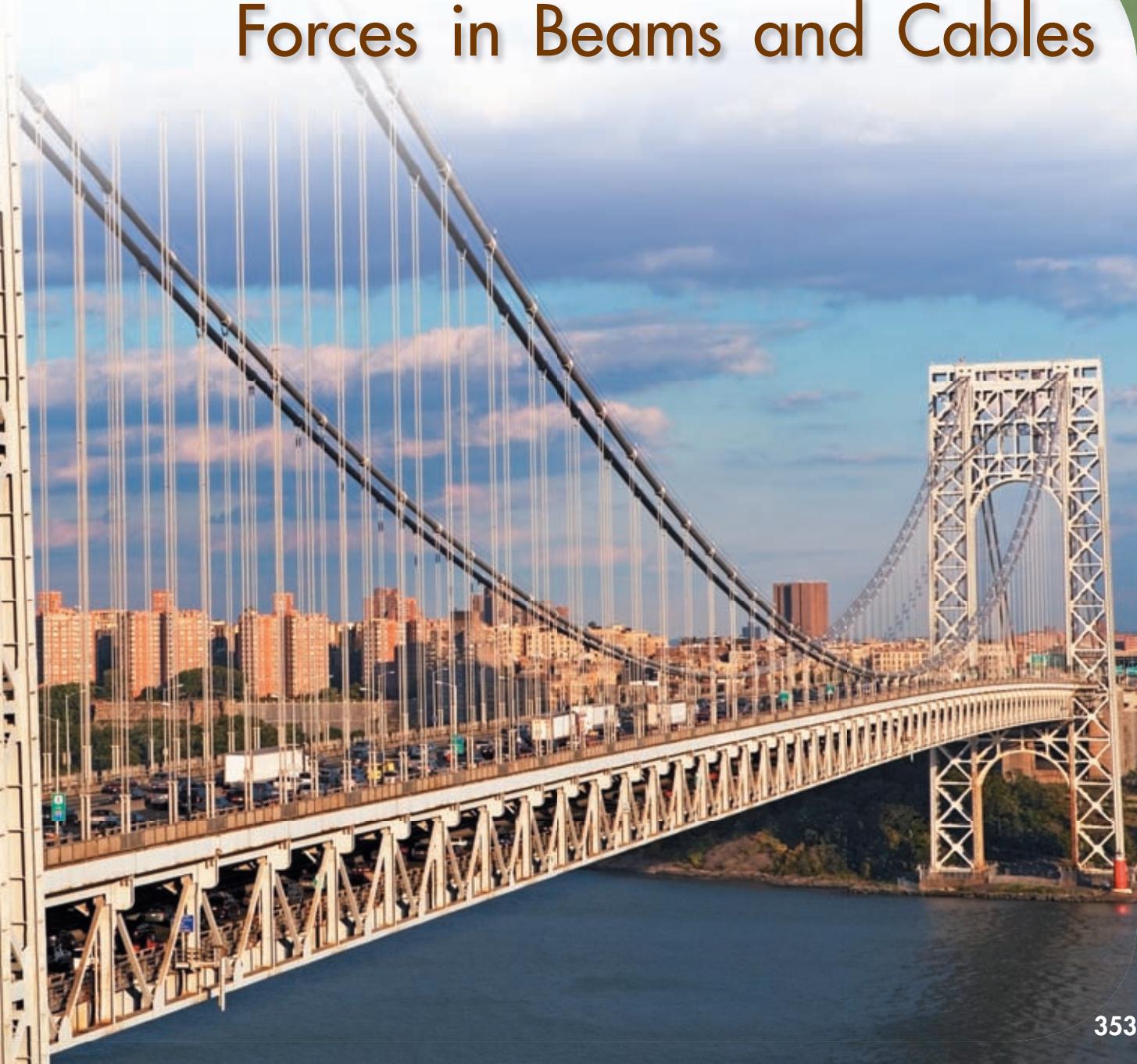
**The George Washington Bridge** connects Manhattan, New York, and Fort Lee, New Jersey. This suspension bridge carries traffic on two levels over roadways that are supported by a system of beams. Trusses are used both to connect these roadways to complete the overall bridge span as well as to form the towers. The bridge span itself is supported by the cable system.



# CHAPTER

7

# Forces in Beams and Cables



## Chapter 7 Forces in Beams and Cables

- 7.1** Introduction
- 7.2** Internal Forces in Members
- 7.3** Various Types of Loading and Support
- 7.4** Shear and Bending Moment in a Beam
- 7.5** Shear and Bending-Moment Diagrams
- 7.6** Relations among Load, Shear, and Bending Moment
- 7.7** Cables with Concentrated Loads
- 7.8** Cables with Distributed Loads
- 7.9** Parabolic Cable
- 7.10** Catenary

### \*7.1 INTRODUCTION

In preceding chapters, two basic problems involving structures were considered: (1) determining the external forces acting on a structure (Chap. 4) and (2) determining the forces which hold together the various members forming a structure (Chap. 6). The problem of determining the internal forces which hold together the various parts of a given member will now be considered.

We will first analyze the internal forces in the members of a frame, such as the crane considered in Secs. 6.1 and 6.10, noting that whereas the internal forces in a straight two-force member can produce only *tension* or *compression* in that member, the internal forces in any other type of member usually produce *shear* and *bending* as well.

Most of this chapter will be devoted to the analysis of the internal forces in two important types of engineering structures, namely,

1. *Beams*, which are usually long, straight prismatic members designed to support loads applied at various points along the member.
2. *Cables*, which are flexible members capable of withstanding only tension, designed to support either concentrated or distributed loads. Cables are used in many engineering applications, such as suspension bridges and transmission lines.

### \*7.2 INTERNAL FORCES IN MEMBERS

Let us first consider a *straight two-force member AB* (Fig. 7.1a). From Sec. 4.6, we know that the forces  $\mathbf{F}$  and  $-\mathbf{F}$  acting at A and B, respectively, must be directed along AB in opposite sense and have the same magnitude  $F$ . Now, let us cut the member at C. To maintain the equilibrium of the free bodies  $AC$  and  $CB$  thus obtained, we must apply to  $AC$  a force  $-\mathbf{F}$  equal and opposite to  $\mathbf{F}$ , and to  $CB$  a force  $\mathbf{F}$  equal and opposite to  $-\mathbf{F}$  (Fig. 7.1b). These new forces are directed along AB in opposite sense and have the same magnitude  $F$ . Since the two parts  $AC$  and  $CB$  were in equilibrium before the member was cut, *internal forces* equivalent to these new forces must have existed in the member itself. We conclude that in the case of a straight two-force member, the internal forces that the two portions of the member exert on each other are equivalent to *axial forces*. The common magnitude  $F$  of these forces does not depend upon the location of the section C and is referred to as the *force in member AB*. In the case considered, the member is in tension and will elongate under the action of the internal forces. In the case represented in Fig. 7.2, the member is in compression and will decrease in length under the action of the internal forces.

Next, let us consider a *multipurpose member*. Take, for instance, member  $AD$  of the crane analyzed in Sec. 6.10. This crane is shown again in Fig. 7.3a, and the free-body diagram of member  $AD$  is drawn in Fig. 7.3b. We now cut member  $AD$  at  $J$  and draw a free-body diagram for each of the portions  $JD$  and  $AJ$  of the member

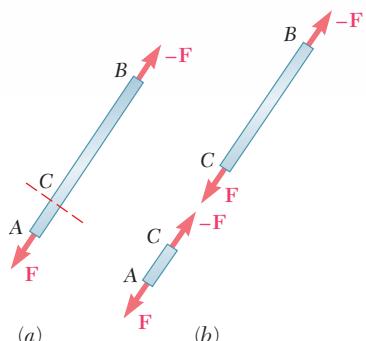


Fig. 7.1

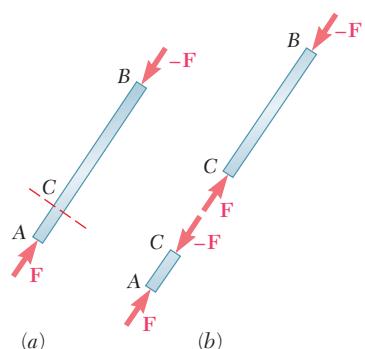


Fig. 7.2

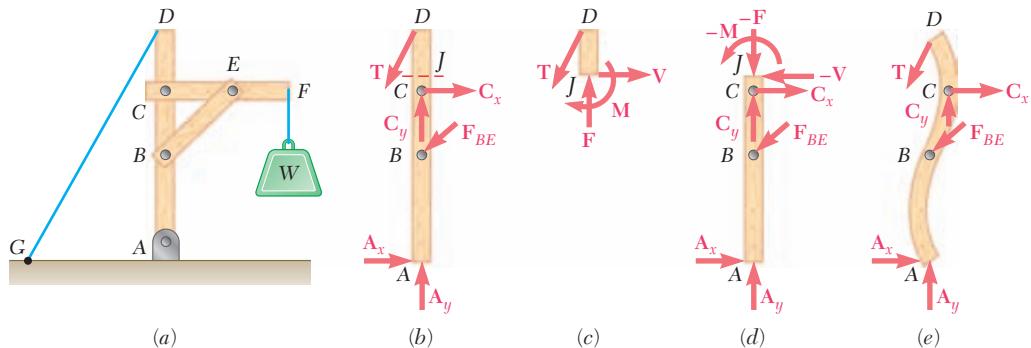


Fig. 7.3

(Fig. 7.3c and d). Considering the free body  $JD$ , we find that its equilibrium will be maintained if we apply at  $J$  a force  $\mathbf{F}$  to balance the vertical component of  $\mathbf{T}$ , a force  $\mathbf{V}$  to balance the horizontal component of  $\mathbf{T}$ , and a couple  $\mathbf{M}$  to balance the moment of  $\mathbf{T}$  about  $J$ . Again we conclude that internal forces must have existed at  $J$  before member  $AD$  was cut. The internal forces acting on the portion  $JD$  of member  $AD$  are equivalent to the force-couple system shown in Fig. 7.3c. According to Newton's third law, the internal forces acting on  $AJ$  must be equivalent to an equal and opposite force-couple system, as shown in Fig. 7.3d. It is clear that the action of the internal forces in member  $AD$  is not limited to producing tension or compression as in the case of straight two-force members; the internal forces also produce shear and bending. The force  $\mathbf{F}$  is an *axial force*; the force  $\mathbf{V}$  is called a *shearing force*; and the moment  $\mathbf{M}$  of the couple is known as the *bending moment at  $J$* . We note that when determining internal forces in a member, we should clearly indicate on which portion of the member the forces are supposed to act. The deformation which will occur in member  $AD$  is sketched in Fig. 7.3e. The actual analysis of such a deformation is part of the study of mechanics of materials.

It should be noted that in a *two-force member which is not straight*, the internal forces are also equivalent to a force-couple system. This is shown in Fig. 7.4, where the two-force member  $ABC$  has been cut at  $D$ .



**Photo 7.1** The design of the shaft of a circular saw must account for the internal forces resulting from the forces applied to the teeth of the blade. At a given point in the shaft, these internal forces are equivalent to a force-couple system consisting of axial and shearing forces and a couple representing the bending and torsional moments.

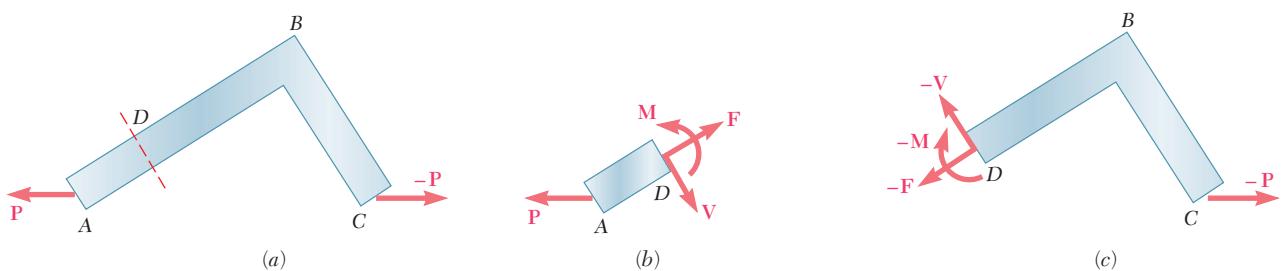
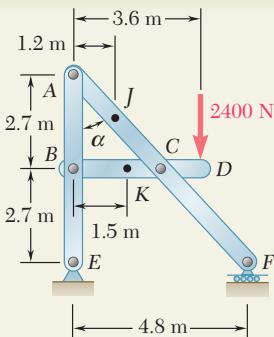
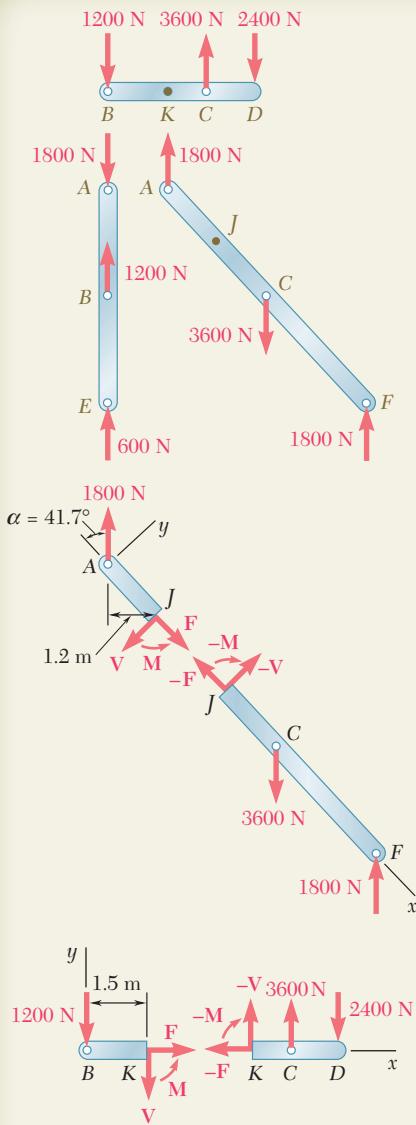


Fig. 7.4



## SAMPLE PROBLEM 7.1

In the frame shown, determine the internal forces (a) in member  $ACF$  at point  $J$ , (b) in member  $BCD$  at point  $K$ . This frame has been previously considered in Sample Prob. 6.5.



## SOLUTION

**Reactions and Forces at Connections.** The reactions and the forces acting on each member of the frame are determined; this has been previously done in Sample Prob. 6.5, and the results are repeated here.

**a. Internal Forces at  $J$ .** Member  $ACF$  is cut at point  $J$ , and the two parts shown are obtained. The internal forces at  $J$  are represented by an equivalent force-couple system and can be determined by considering the *free body*  $AJ$ , we write

$$\begin{aligned} +\uparrow \sum M_J &= 0: & -(1800 \text{ N})(1.2 \text{ m}) + M = 0 \\ && M = +2160 \text{ N} \cdot \text{m} \uparrow & \mathbf{M} = 2160 \text{ N} \cdot \text{m} \uparrow \\ +\downarrow \sum F_x &= 0: & F - (1800 \text{ N}) \cos 41.7^\circ = 0 \\ && F = +1344 \text{ N} & \mathbf{F} = 1344 \text{ N} \downarrow \\ +\nearrow \sum F_y &= 0: & -V + (1800 \text{ N}) \sin 41.7^\circ = 0 \\ && V = +1197 \text{ N} & \mathbf{V} = 1197 \text{ N} \leftarrow \end{aligned}$$

The internal forces at  $J$  are therefore equivalent to a couple  $\mathbf{M}$ , an axial force  $\mathbf{F}$ , and a shearing force  $\mathbf{V}$ . The internal force-couple system acting on part  $JCF$  is equal and opposite.

**b. Internal Forces at  $K$ .** We cut member  $BCD$  at  $K$  and obtain the two parts shown. Considering the *free body*  $BK$ , we write

$$\begin{aligned} +\uparrow \sum M_K &= 0: & (1200 \text{ N})(1.5 \text{ m}) + M = 0 \\ && M = -1800 \text{ N} \cdot \text{m} & \mathbf{M} = 1800 \text{ N} \cdot \text{m} \downarrow \\ +\rightarrow \sum F_x &= 0: & F = 0 & \mathbf{F} = 0 \\ +\uparrow \sum F_y &= 0: & -1200 \text{ N} - V = 0 \\ && V = -1200 \text{ N} & \mathbf{V} = 1200 \text{ N} \uparrow \end{aligned}$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to determine the internal forces in the member of a frame. The internal forces at a given point in a *straight two-force member* reduce to an axial force, but in all other cases, they are equivalent to a *force-couple system* consisting of an *axial force F*, a *shearing force V*, and a couple *M* representing the *bending moment* at that point.

To determine the internal forces at a given point *J* of the member of a frame, you should take the following steps.

- 1. Draw a free-body diagram of the entire frame,** and use it to determine as many of the reactions at the supports as you can.
- 2. Dismember the frame, and draw a free-body diagram of each of its members.** Write as many equilibrium equations as are necessary to find all the forces acting on the member on which point *J* is located.
- 3. Cut the member at point *J*, and draw a free-body diagram of each of the two portions** of the member that you have obtained, applying to each portion at point *J* the force components and couple representing the internal forces exerted by the other portion. Note that these force components and couples are equal in magnitude and opposite in sense.
- 4. Select one of the two free-body diagrams** you have drawn and use it to write three equilibrium equations for the corresponding portion of member.
  - a. Summing moments about *J*** and equating them to zero will yield the bending moment at point *J*.
  - b. Summing components in directions parallel and perpendicular** to the member at *J* and equating them to zero will yield, respectively, the axial and shearing force.
- 5. When recording your answers, be sure to specify the portion of the member** you have used, since the forces and couples acting on the two portions have opposite senses.

Since the solutions of the problems in this lesson require the determination of the forces exerted on each other by the various members of a frame, be sure to review the methods used in Chap. 6 to solve this type of problem. When frames involve pulleys and cables, for instance, remember that the forces exerted by a pulley on the member of the frame to which it is attached have the same magnitude and direction as the forces exerted by the cable on the pulley [Prob. 6.90].

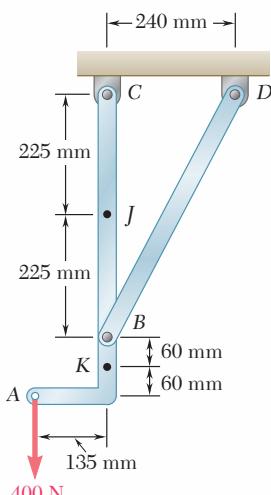
# PROBLEMS

**7.1 and 7.2** Determine the internal forces (axial force, shearing force, and bending moment) at point J of the structure indicated.

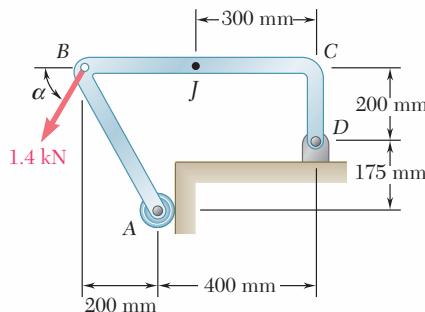
### 7.1 Frame and loading of Prob. 6.79

## 7.2 Frame and loading of Prob. 6.80

**7.3** Determine the internal forces at point  $J$  when  $\alpha = 90^\circ$ .



**Fig. P7.5 and P7.6**



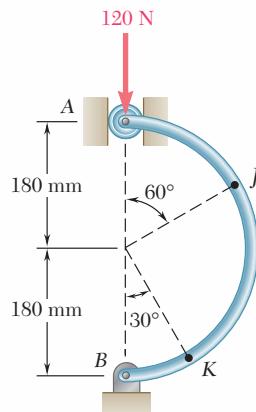
**Fig. P7.3 and P7.4**

**7.4** Determine the internal forces at point  $J$  when  $\alpha = 0$ .

**7.5** Determine the internal forces at point  $J$  of the structure shown.

**7.6** Determine the internal forces at point  $K$  of the structure shown.

**7.7** A semicircular rod is loaded as shown. Determine the internal forces at point  $J$ .



**Fig. P7.7 and P7.8**

**7.8** A semicircular rod is loaded as shown. Determine the internal forces at point  $K$ .

- 7.9** An archer aiming at a target is pulling with a 45-lb force on the bowstring. Assuming that the shape of the bow can be approximated by a parabola, determine the internal forces at point *J*.

- 7.10** For the bow of Prob. 7.9, determine the magnitude and location of the maximum (a) axial force, (b) shearing force, (c) bending moment.

- 7.11** Two members, each consisting of a straight and a quarter-circular portion of rod, are connected as shown and support a 75-lb load at A. Determine the internal forces at point *J*.

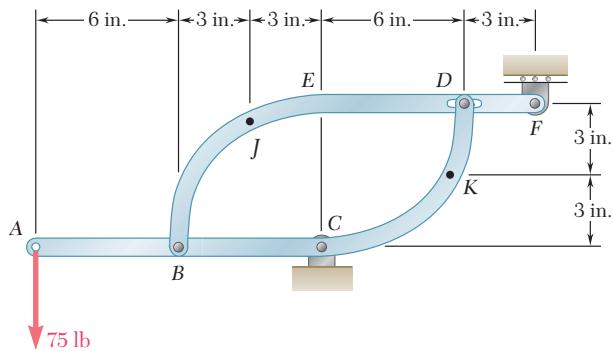


Fig. P7.11 and P7.12

- 7.12** Two members, each consisting of a straight and a quarter-circular portion of rod, are connected as shown and support a 75-lb load at A. Determine the internal forces at point *K*.

- 7.13** A semicircular rod is loaded as shown. Determine the internal forces at point *J* knowing that  $\theta = 30^\circ$ .

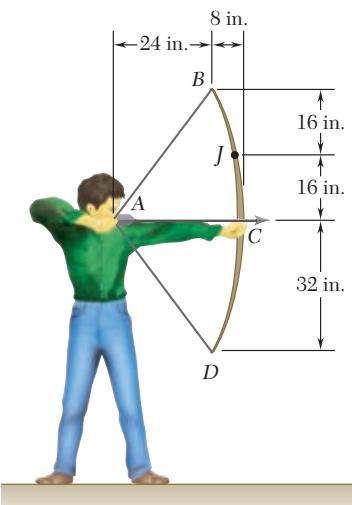


Fig. P7.9

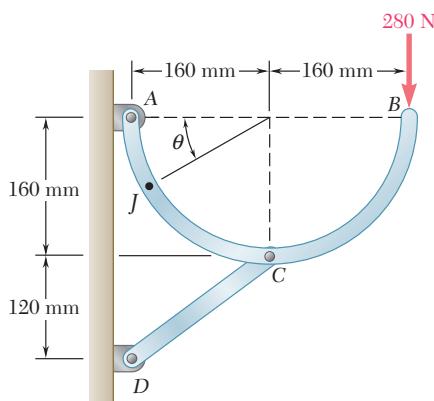


Fig. P7.13 and P7.14

- 7.14** A semicircular rod is loaded as shown. Determine the magnitude and location of the maximum bending moment in the rod.

- 7.15** Knowing that the radius of each pulley is 150 mm, that  $\alpha = 20^\circ$ , and neglecting friction, determine the internal forces at (a) point J, (b) point K.

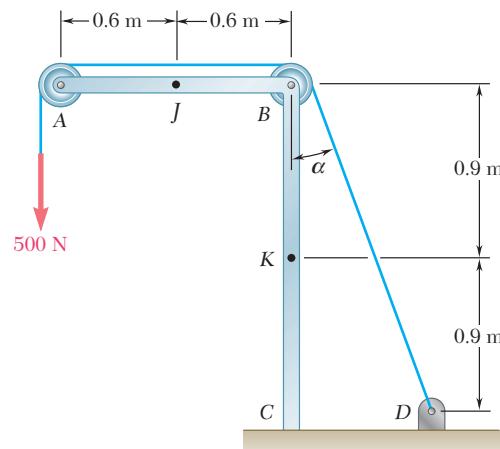


Fig. P7.15 and P7.16

- 7.16** Knowing that the radius of each pulley is 150 mm, that  $\alpha = 30^\circ$ , and neglecting friction, determine the internal forces at (a) point J, (b) point K.

- 7.17** Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at point J of the frame shown.

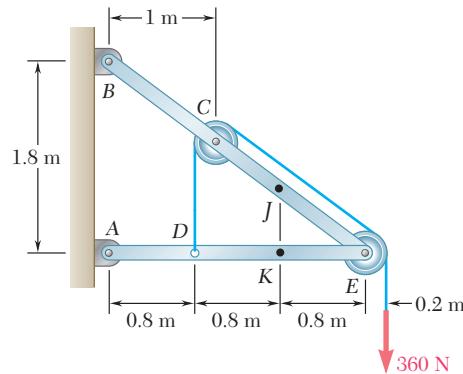


Fig. P7.17 and P7.18

- 7.18** Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at point K of the frame shown.

- 7.19** A 5-in.-diameter pipe is supported every 9 ft by a small frame consisting of two members as shown. Knowing that the combined weight of the pipe and its contents is 10 lb/ft and neglecting the effect of friction, determine the magnitude and location of the maximum bending moment in member AC.

- 7.20** For the frame of Prob. 7.19, determine the magnitude and location of the maximum bending moment in member BC.

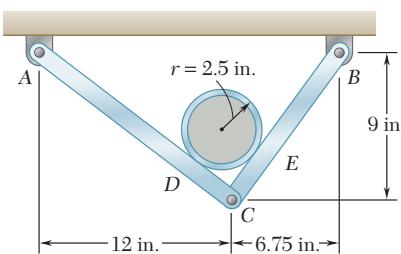


Fig. P7.19

- 7.21 and 7.22** A force  $\mathbf{P}$  is applied to a bent rod that is supported by a roller and a pin and bracket. For each of the three cases shown, determine the internal forces at point  $J$ .

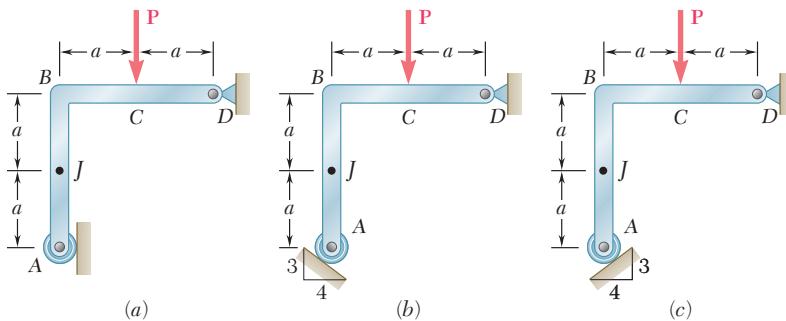


Fig. P7.21

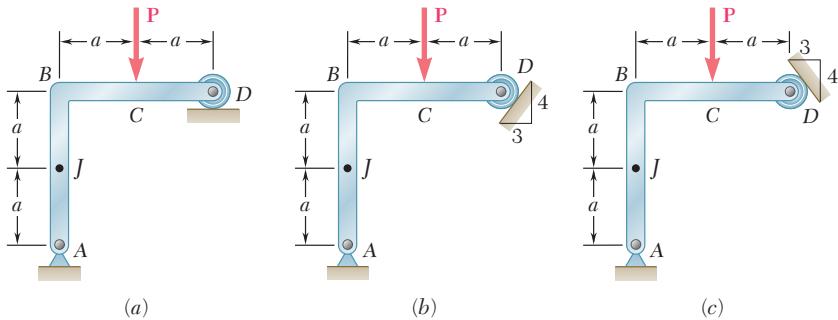


Fig. P7.22

- 7.23** A semicircular rod of weight  $W$  and uniform cross section is supported as shown. Determine the bending moment at point  $J$  when  $\theta = 60^\circ$ .

- 7.24** A semicircular rod of weight  $W$  and uniform cross section is supported as shown. Determine the bending moment at point  $J$  when  $\theta = 150^\circ$ .

- 7.25 and 7.26** A quarter-circular rod of weight  $W$  and uniform cross section is supported as shown. Determine the bending moment at point  $J$  when  $\theta = 30^\circ$ .

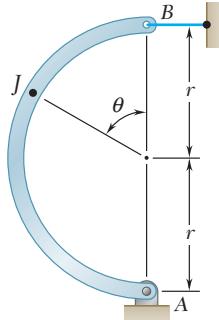


Fig. P7.23 and P7.24

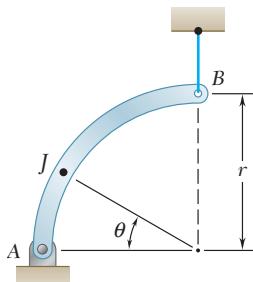


Fig. P7.25

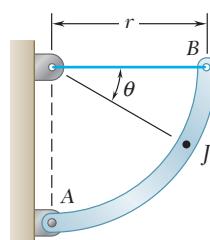


Fig. P7.26

- 7.27** For the rod of Prob. 7.25, determine the magnitude and location of the maximum bending moment.

- 7.28** For the rod of Prob. 7.26, determine the magnitude and location of the maximum bending moment.

## BEAMS

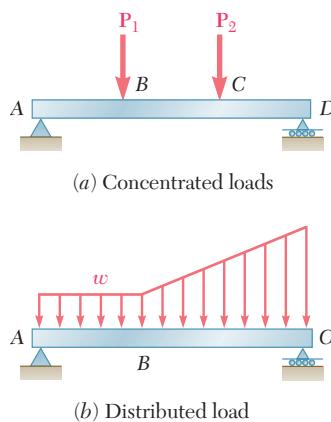
### \*7.3 VARIOUS TYPES OF LOADING AND SUPPORT

A structural member designed to support loads applied at various points along the member is known as a *beam*. In most cases, the loads are perpendicular to the axis of the beam and will cause only shear and bending in the beam. When the loads are not at a right angle to the beam, they will also produce axial forces in the beam.

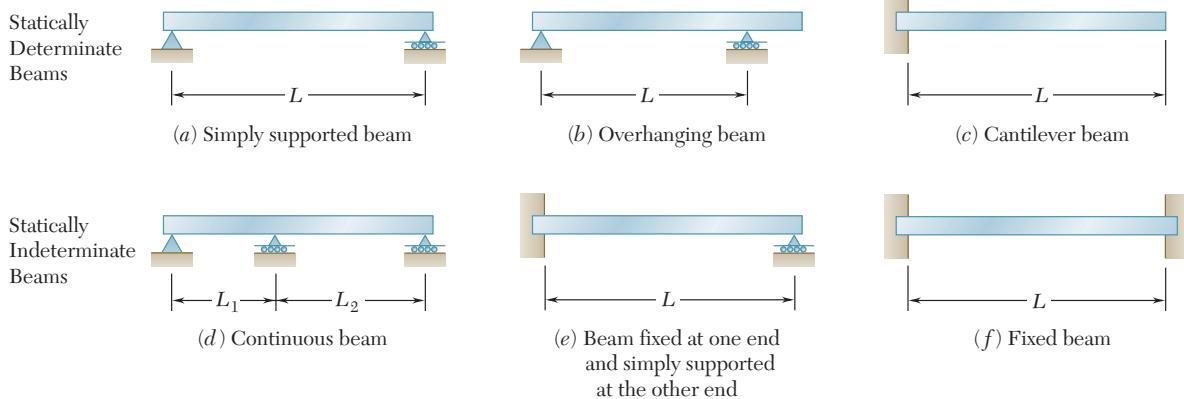
Beams are usually long, straight prismatic bars. Designing a beam for the most effective support of the applied loads is a two-part process: (1) determining the shearing forces and bending moments produced by the loads and (2) selecting the cross section best suited to resist the shearing forces and bending moments determined in the first part. Here we are concerned with the first part of the problem of beam design. The second part belongs to the study of mechanics of materials.

A beam can be subjected to *concentrated loads*  $\mathbf{P}_1$ ,  $\mathbf{P}_2$ , . . . , expressed in newtons, pounds, or their multiples kilonewtons and kips (Fig. 7.5a), to a *distributed load*  $w$ , expressed in N/m, kN/m, lb/ft, or kips/ft (Fig. 7.5b), or to a combination of both. When the load  $w$  per unit length has a constant value over part of the beam (as between A and B in Fig. 7.5b), the load is said to be *uniformly distributed* over that part of the beam. The determination of the reactions at the supports is considerably simplified if distributed loads are replaced by equivalent concentrated loads, as explained in Sec. 5.8. This substitution, however, should not be performed, or at least should be performed with care, when internal forces are being computed (see Sample Prob. 7.3).

Beams are classified according to the way in which they are supported. Several types of beams frequently used are shown in Fig. 7.6. The distance  $L$  between supports is called the *span*. It should be noted that the reactions will be determinate if the supports involve only three unknowns. If more unknowns are involved, the



**Fig. 7.5**



**Fig. 7.6**

reactions will be statically indeterminate and the methods of statics will not be sufficient to determine the reactions; the properties of the beam with regard to its resistance to bending must then be taken into consideration. Beams supported by two rollers are not shown here; they are only partially constrained and will move under certain loadings.

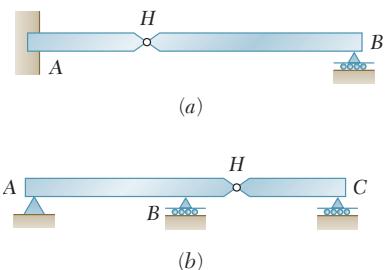
Sometimes two or more beams are connected by hinges to form a single continuous structure. Two examples of beams hinged at a point  $H$  are shown in Fig. 7.7. It will be noted that the reactions at the supports involve four unknowns and cannot be determined from the free-body diagram of the two-beam system. They can be determined, however, by considering the free-body diagram of each beam separately; six unknowns are involved (including two force components at the hinge), and six equations are available.

## \*7.4 SHEAR AND BENDING MOMENT IN A BEAM

Consider a beam  $AB$  subjected to various concentrated and distributed loads (Fig. 7.8a). We propose to determine the shearing force and bending moment at any point of the beam. In the example considered here, the beam is simply supported, but the method used could be applied to any type of statically determinate beam.

First we determine the reactions at  $A$  and  $B$  by choosing the entire beam as a free body (Fig. 7.8b); writing  $\Sigma M_A = 0$  and  $\Sigma M_B = 0$ , we obtain, respectively,  $\mathbf{R}_B$  and  $\mathbf{R}_A$ .

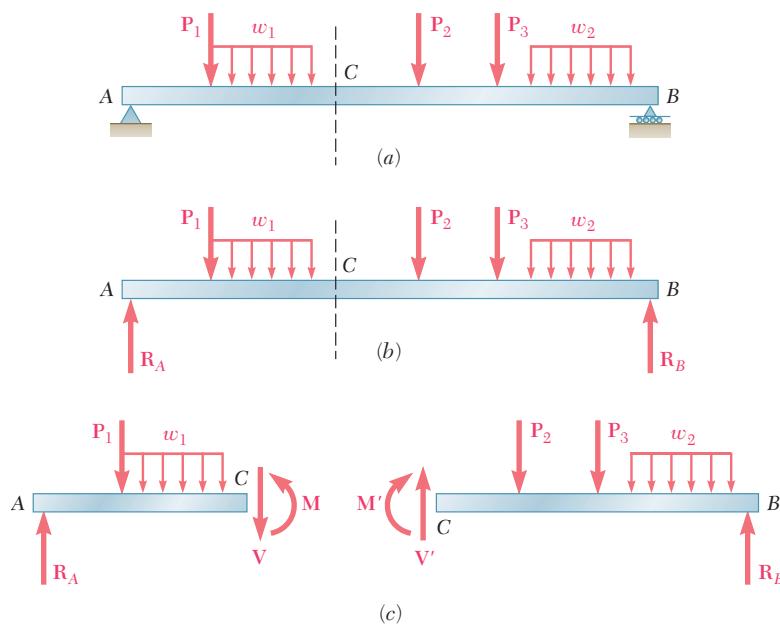
To determine the internal forces at  $C$ , we cut the beam at  $C$  and draw the free-body diagrams of the portions  $AC$  and  $CB$  of the



**Fig. 7.7**



**Photo 7.2** The internal forces in the beams of the overpass shown vary as the truck crosses the overpass.



**Fig. 7.8**

beam (Fig. 7.8c). Using the free-body diagram of  $AC$ , we can determine the shearing force  $\mathbf{V}$  at  $C$  by equating to zero the sum of the vertical components of all forces acting on  $AC$ . Similarly, the bending moment  $\mathbf{M}$  at  $C$  can be found by equating to zero the sum of the moments about  $C$  of all forces and couples acting on  $AC$ . Alternatively, we could use the free-body diagram of  $CB^{\dagger}$  and determine the shearing force  $\mathbf{V}'$  and the bending moment  $\mathbf{M}'$  by equating to zero the sum of the vertical components and the sum of the moments about  $C$  of all forces and couples acting on  $CB$ . While this choice of free bodies may facilitate the computation of the numerical values of the shearing force and bending moment, it makes it necessary to indicate on which portion of the beam the internal forces considered are acting. If the shearing force and bending moment are to be computed at every point of the beam and efficiently recorded, we must find a way to avoid having to specify every time which portion of the beam is used as a free body. We shall adopt, therefore, the following conventions:

In determining the shearing force in a beam, *it will always be assumed* that the internal forces  $\mathbf{V}$  and  $\mathbf{V}'$  are directed as shown in Fig. 7.8c. A positive value obtained for their common magnitude  $V$  will indicate that this assumption was correct and that the shearing forces are actually directed as shown. A negative value obtained for  $V$  will indicate that the assumption was wrong and that the shearing forces are directed in the opposite way. Thus, only the magnitude  $V$ , together with a plus or minus sign, needs to be recorded to define completely the shearing forces at a given point of the beam. The scalar  $V$  is commonly referred to as the *shear* at the given point of the beam.

Similarly, *it will always be assumed* that the internal couples  $\mathbf{M}$  and  $\mathbf{M}'$  are directed as shown in Fig. 7.8c. A positive value obtained for their magnitude  $M$ , commonly referred to as the bending moment, will indicate that this assumption was correct, and a negative value will indicate that it was wrong. Summarizing the sign conventions we have presented, we state:

*The shear  $V$  and the bending moment  $M$  at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown in Fig. 7.9a.*

These conventions can be more easily remembered if we note that:

1. *The shear at  $C$  is positive when the **external** forces (loads and reactions) acting on the beam tend to shear off the beam at  $C$  as indicated in Fig. 7.9b.*
2. *The bending moment at  $C$  is positive when the **external** forces acting on the beam tend to bend the beam at  $C$  as indicated in Fig. 7.9c.*

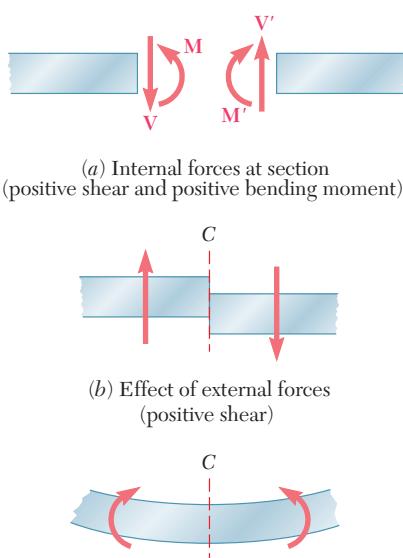


Fig. 7.9

<sup>†</sup>The force and couple representing the internal forces acting on  $CB$  will now be denoted by  $\mathbf{V}'$  and  $\mathbf{M}'$ , rather than by  $-\mathbf{V}$  and  $-\mathbf{M}$  as done earlier, in order to avoid confusion when applying the sign convention which we are about to introduce.

It may also help to note that the situation described in Fig. 7.9, in which the values of the shear and of the bending moment are positive, is precisely the situation which occurs in the left half of a simply supported beam carrying a single concentrated load at its midpoint. This particular example is fully discussed in the following section.

## \*7.5 SHEAR AND BENDING-MOMENT DIAGRAMS

Now that shear and bending moment have been clearly defined in sense as well as in magnitude, we can easily record their values at any point of a beam by plotting these values against the distance  $x$  measured from one end of the beam. The graphs obtained in this way are called, respectively, the *shear diagram* and the *bending-moment diagram*. As an example, consider a simply supported beam  $AB$  of span  $L$  subjected to a single concentrated load  $P$  applied at its midpoint  $D$  (Fig. 7.10a). We first determine the reactions at the supports from the free-body diagram of the entire beam (Fig. 7.10b); we find that the magnitude of each reaction is equal to  $P/2$ .

Next we cut the beam at a point  $C$  between  $A$  and  $D$  and draw the free-body diagrams of  $AC$  and  $CB$  (Fig. 7.10c). Assuming that shear and bending moment are positive, we direct the internal forces  $V$  and  $V'$  and the internal couples  $M$  and  $M'$  as indicated in Fig. 7.9a. Considering the free body  $AC$  and writing that the sum of the vertical components and the sum of the moments about  $C$  of the forces acting on the free body are zero, we find  $V = +P/2$  and  $M = +Px/2$ . Both shear and bending moment are therefore positive; this can be checked by observing that the reaction at  $A$  tends to shear off and to bend the beam at  $C$  as indicated in Fig. 7.9b and c. We can plot  $V$  and  $M$  between  $A$  and  $D$  (Fig. 7.10e and f); the shear has a constant value  $V = P/2$ , while the bending moment increases linearly from  $M = 0$  at  $x = 0$  to  $M = PL/4$  at  $x = L/2$ .

Cutting, now, the beam at a point  $E$  between  $D$  and  $B$  and considering the free body  $EB$  (Fig. 7.10d), we write that the sum of the vertical components and the sum of the moments about  $E$  of the forces acting on the free body are zero. We obtain  $V = -P/2$  and  $M = P(L - x)/2$ . The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at  $B$  bends the beam at  $E$  as indicated in Fig. 7.9c but tends to shear it off in a manner opposite to that shown in Fig. 7.9b. We can complete, now, the shear and bending-moment diagrams of Fig. 7.10e and f; the shear has a constant value  $V = -P/2$  between  $D$  and  $B$ , while the bending moment decreases linearly from  $M = PL/4$  at  $x = L/2$  to  $M = 0$  at  $x = L$ .

It should be noted that when a beam is subjected to concentrated loads only, the shear is of constant value between loads and the bending moment varies linearly between loads, but when a beam is subjected to distributed loads, the shear and bending moment vary quite differently (see Sample Prob. 7.3).

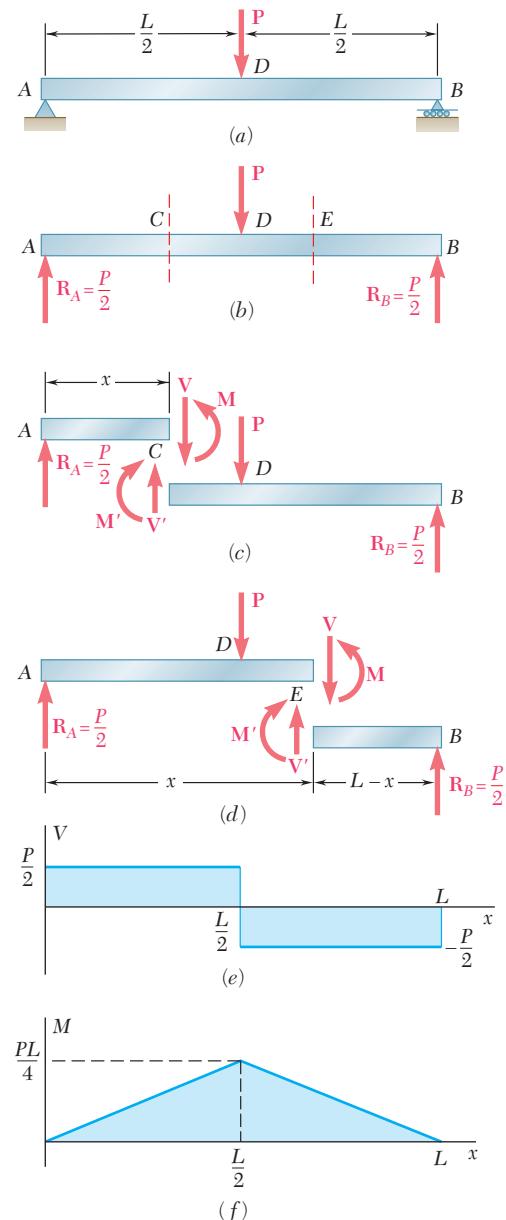
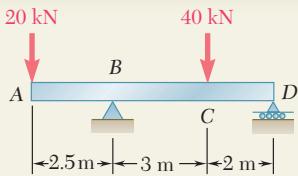


Fig. 7.10



## SAMPLE PROBLEM 7.2

Draw the shear and bending-moment diagrams for the beam and loading shown.

## SOLUTION

**Free-Body: Entire Beam.** From the free-body diagram of the entire beam, we find the reactions at *B* and *D*:

$$\mathbf{R}_B = 46 \text{ kN} \uparrow \quad \mathbf{R}_D = 14 \text{ kN} \uparrow$$

**Shear and Bending Moment.** We first determine the internal forces just to the right of the 20-kN load at *A*. Considering the stub of beam to the left of section 1 as a free body and assuming *V* and *M* to be positive (according to the standard convention), we write

$$\begin{aligned} +\uparrow \sum F_y &= 0: & -20 \text{ kN} - V_1 &= 0 & V_1 &= -20 \text{ kN} \\ +\uparrow \sum M_1 &= 0: & (20 \text{ kN})(0 \text{ m}) + M_1 &= 0 & M_1 &= 0 \end{aligned}$$

We next consider as a free body the portion of the beam to the left of section 2 and write

$$\begin{aligned} +\uparrow \sum F_y &= 0: & -20 \text{ kN} - V_2 &= 0 & V_2 &= -20 \text{ kN} \\ +\uparrow \sum M_2 &= 0: & (20 \text{ kN})(2.5 \text{ m}) + M_2 &= 0 & M_2 &= -50 \text{ kN} \cdot \text{m} \end{aligned}$$

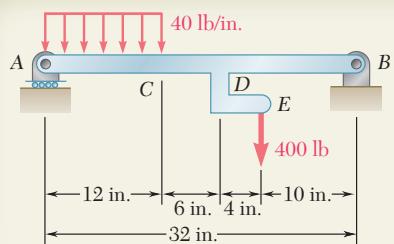
The shear and bending moment at sections 3, 4, 5, and 6 are determined in a similar way from the free-body diagrams shown. We obtain

$$\begin{array}{ll} V_3 = +26 \text{ kN} & M_3 = -50 \text{ kN} \cdot \text{m} \\ V_4 = +26 \text{ kN} & M_4 = +28 \text{ kN} \cdot \text{m} \\ V_5 = -14 \text{ kN} & M_5 = +28 \text{ kN} \cdot \text{m} \\ V_6 = -14 \text{ kN} & M_6 = 0 \end{array}$$

For several of the latter sections, the results are more easily obtained by considering as a free body the portion of the beam to the right of the section. For example, considering the portion of the beam to the right of section 4, we write

$$\begin{aligned} +\uparrow \sum F_y &= 0: & V_4 - 40 \text{ kN} + 14 \text{ kN} &= 0 & V_4 &= +26 \text{ kN} \\ +\uparrow \sum M_4 &= 0: & -M_4 + (14 \text{ kN})(2 \text{ m}) &= 0 & M_4 &= +28 \text{ kN} \cdot \text{m} \end{aligned}$$

**Shear and Bending-Moment Diagrams.** We can now plot the six points shown on the shear and bending-moment diagrams. As indicated in Sec. 7.5, the shear is of constant value between concentrated loads, and the bending moment varies linearly; we therefore obtain the shear and bending-moment diagrams shown.



## SAMPLE PROBLEM 7.3

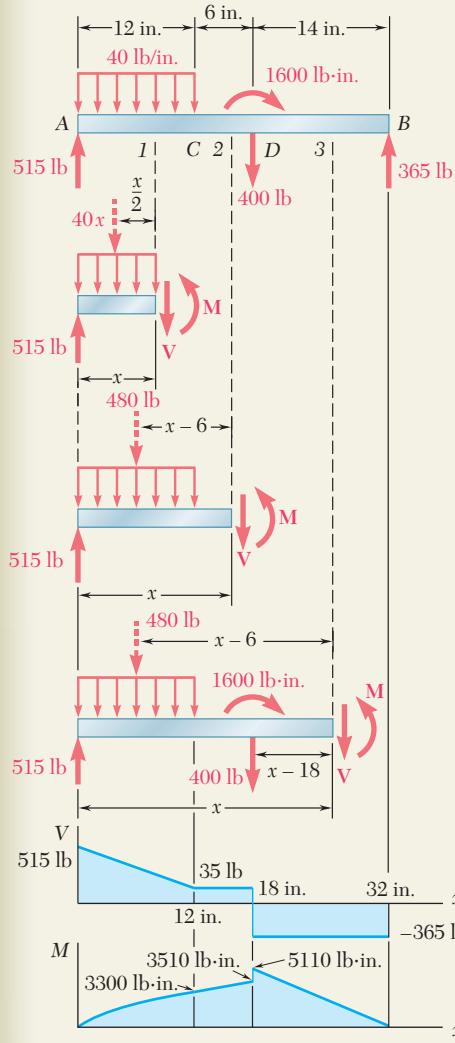
Draw the shear and bending-moment diagrams for the beam AB. The distributed load of 40 lb/in. extends over 12 in. of the beam, from A to C, and the 400-lb load is applied at E.

## SOLUTION

**Free-Body: Entire Beam.** The reactions are determined by considering the entire beam as a free body.

$$\begin{aligned} +\uparrow \sum M_A = 0: \quad & B_y(32 \text{ in.}) - (480 \text{ lb})(6 \text{ in.}) - (400 \text{ lb})(22 \text{ in.}) = 0 \\ & B_y = +365 \text{ lb} \quad \mathbf{B}_y = 365 \text{ lb} \uparrow \\ +\uparrow \sum M_B = 0: \quad & (480 \text{ lb})(26 \text{ in.}) + (400 \text{ lb})(10 \text{ in.}) - A(32 \text{ in.}) = 0 \\ & A = +515 \text{ lb} \quad \mathbf{A} = 515 \text{ lb} \uparrow \\ \rightarrow \sum F_x = 0: \quad & B_x = 0 \quad \mathbf{B}_x = 0 \end{aligned}$$

The 400-lb load is now replaced by an equivalent force-couple system acting on the beam at point D.



**Shear and Bending Moment. From A to C.** We determine the internal forces at a distance  $x$  from point A by considering the portion of the beam to the left of section 1. That part of the distributed load acting on the free body is replaced by its resultant, and we write

$$\begin{aligned} +\uparrow \sum F_y = 0: \quad & 515 - 40x - V = 0 \quad V = 515 - 40x \\ +\uparrow \sum M_1 = 0: \quad & -515x + 40x(\frac{1}{2}x) + M = 0 \quad M = 515x - 20x^2 \end{aligned}$$

Since the free-body diagram shown can be used for all values of  $x$  smaller than 12 in., the expressions obtained for  $V$  and  $M$  are valid throughout the region  $0 < x < 12$  in.

**From C to D.** Considering the portion of the beam to the left of section 2 and again replacing the distributed load by its resultant, we obtain

$$\begin{aligned} +\uparrow \sum F_y = 0: \quad & 515 - 480 - V = 0 \quad V = 35 \text{ lb} \\ +\uparrow \sum M_2 = 0: \quad & -515x + 480(x - 6) + M = 0 \quad M = (2880 + 35x) \text{ lb} \cdot \text{in.} \end{aligned}$$

These expressions are valid in the region  $12 \text{ in.} < x < 18 \text{ in.}$

**From D to B.** Using the portion of the beam to the left of section 3, we obtain for the region  $18 \text{ in.} < x < 32 \text{ in.}$

$$\begin{aligned} +\uparrow \sum F_y = 0: \quad & 515 - 480 - 400 - V = 0 \quad V = -365 \text{ lb} \\ +\uparrow \sum M_3 = 0: \quad & -515x + 480(x - 6) - 1600 + 400(x - 18) + M = 0 \\ & M = (11,680 - 365x) \text{ lb} \cdot \text{in.} \end{aligned}$$

**Shear and Bending-Moment Diagrams.** The shear and bending-moment diagrams for the entire beam can now be plotted. We note that the couple of moment 1600 lb · in. applied at point D introduces a discontinuity into the bending-moment diagram.

# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to determine the shear  $V$  and the bending moment  $M$  at any point in a beam. You also learned to draw the *shear diagram* and the *bending-moment diagram* for the beam by plotting, respectively,  $V$  and  $M$  against the distance  $x$  measured along the beam.

**A. Determining the shear and bending moment in a beam.** To determine the shear  $V$  and the bending moment  $M$  at a given point  $C$  of a beam, you should take the following steps.

**1. Draw a free-body diagram of the entire beam,** and use it to determine the reactions at the beam supports.

**2. Cut the beam at point  $C$ ,** and, using the original loading, select one of the two portions of the beam you have obtained.

**3. Draw the free-body diagram of the portion of the beam you have selected,** showing:

**a. The loads and the reaction** exerted on that portion of the beam, replacing each distributed load by an equivalent concentrated load as explained earlier in Sec. 5.8.

**b. The shearing force and the bending couple representing the internal forces at  $C$ .** To facilitate recording the shear  $V$  and the bending moment  $M$  after they have been determined, follow the convention indicated in Figs. 7.8 and 7.9. Thus, if you are using the portion of the beam located to the *left of  $C$* , apply at  $C$  a *shearing force  $\mathbf{V}$  directed downward* and a *bending couple  $\mathbf{M}$  directed counterclockwise*. If you are using the portion of the beam located to the *right of  $C$* , apply at  $C$  a *shearing force  $\mathbf{V}'$  directed upward* and a *bending couple  $\mathbf{M}'$  directed clockwise* [Sample Prob. 7.2].

**4. Write the equilibrium equations for the portion of the beam you have selected.** Solve the equation  $\sum F_y = 0$  for  $V$  and the equation  $\sum M_C = 0$  for  $M$ .

**5. Record the values of  $V$  and  $M$  with the sign obtained for each of them.** A positive sign for  $V$  means that the shearing forces exerted at  $C$  on each of the two portions of the beam are directed as shown in Figs. 7.8 and 7.9; a negative sign means that they have the opposite sense. Similarly, a positive sign for  $M$  means that the bending couples at  $C$  are directed as shown in these figures, and a negative sign means that they have the opposite sense. In addition, a positive sign for  $M$  means that the concavity of the beam at  $C$  is directed upward, and a negative sign means that it is directed downward.

**B. Drawing the shear and bending-moment diagrams for a beam.** These diagrams are obtained by plotting, respectively,  $V$  and  $M$  against the distance  $x$  measured along the beam. However, in most cases the values of  $V$  and  $M$  need to be computed only at a few points.

**1. For a beam supporting only concentrated loads,** we note [Sample Prob. 7.2] that

a. **The shear diagram consists of segments of horizontal lines.** Thus, to draw the shear diagram of the beam you will need to compute  $V$  only just to the left or just to the right of the points where the loads or the reactions are applied.

b. **The bending-moment diagram consists of segments of oblique straight lines.** Thus, to draw the bending-moment diagram of the beam you will need to compute  $M$  only at the points where the loads or the reactions are applied.

**2. For a beam supporting uniformly distributed loads,** we note [Sample Prob. 7.3] that under each of the distributed loads:

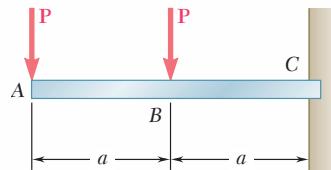
a. **The shear diagram consists of a segment of an oblique straight line.** Thus, you will need to compute  $V$  only where the distributed load begins and where it ends.

b. **The bending-moment diagram consists of an arc of parabola.** In most cases you will need to compute  $M$  only where the distributed load begins and where it ends.

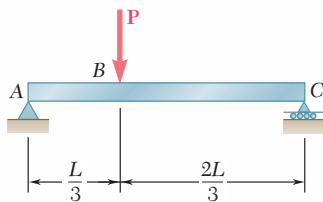
**3. For a beam with a more complicated loading,** it is necessary to consider the free-body diagram of a portion of the beam of arbitrary length  $x$  and determine  $V$  and  $M$  as functions of  $x$ . This procedure may have to be repeated several times, since  $V$  and  $M$  are often represented by different functions in various parts of the beam [Sample Prob. 7.3].

**4. When a couple is applied to a beam,** the shear has the same value on both sides of the point of application of the couple, but the bending-moment diagram will show a discontinuity at that point, rising or falling by an amount equal to the magnitude of the couple. Note that a couple can either be applied directly to the beam, or result from the application of a load on a curved member rigidly attached to the beam [Sample Prob. 7.3].

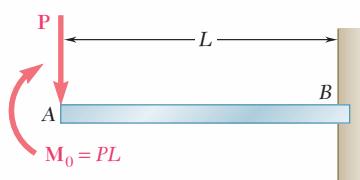
# PROBLEMS



**Fig. P7.29**

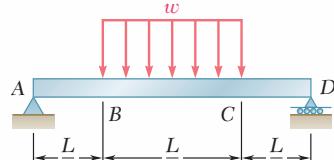


**Fig. P7.30**

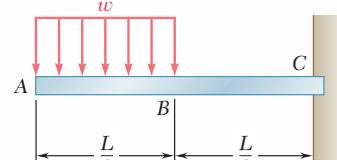


**Fig. P7.33**

**7.29 through 7.32** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

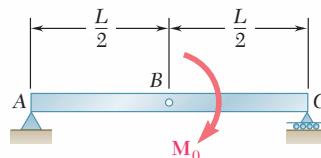


**Fig. P7.31**

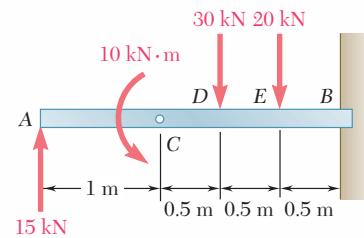


**Fig. P7.32**

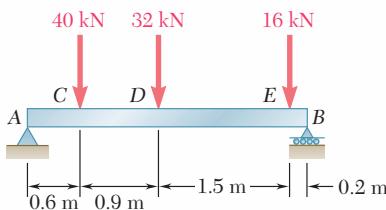
**7.33 and 7.34** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.



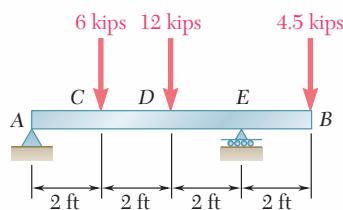
**Fig. P7.34**



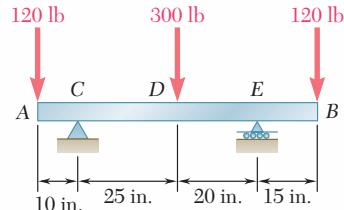
**Fig. P7.35**



**Fig. P7.36**



**Fig. P7.37**



**Fig. P7.38**

**7.37 and 7.38** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

- 7.39 through 7.42** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

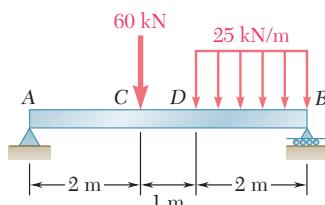


Fig. P7.39

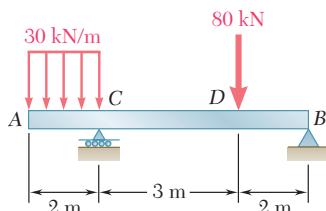


Fig. P7.40

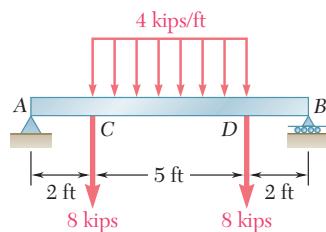


Fig. P7.41

- 7.43** Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed and knowing that  $a = 0.3$  m, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

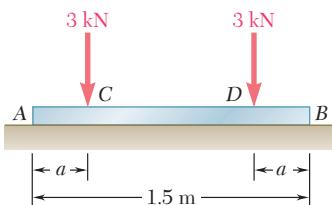


Fig. P7.43

- 7.44** Solve Prob. 7.43 knowing that  $a = 0.5$  m.

- 7.45 and 7.46** Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

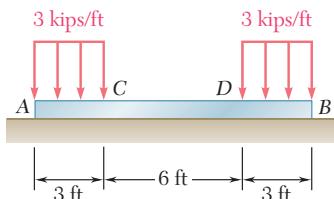


Fig. P7.45

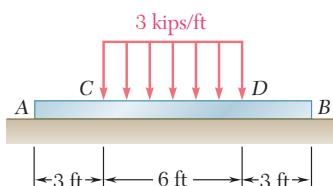


Fig. P7.46

- 7.47** Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed and knowing that  $P = wa$ , (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

- 7.48** Solve Prob. 7.47 knowing that  $P = 3wa$ .

- 7.49** Draw the shear and bending-moment diagrams for the beam *AB*, and determine the shear and bending moment (a) just to the left of *C*, (b) just to the right of *C*.

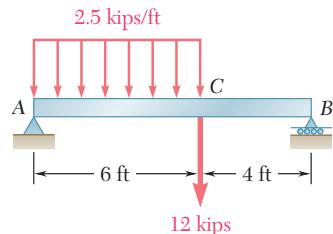


Fig. P7.42

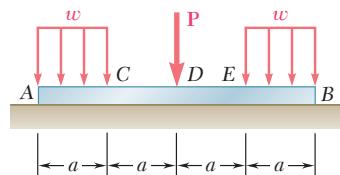


Fig. P7.47

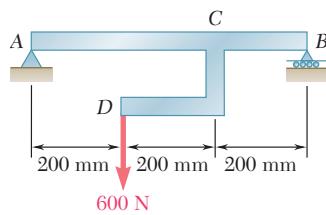


Fig. P7.49

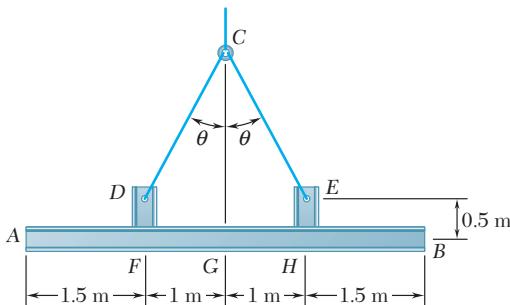


Fig. P7.50

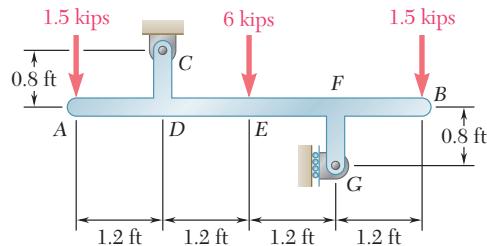


Fig. P7.52

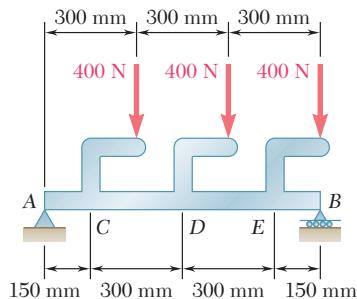


Fig. P7.54

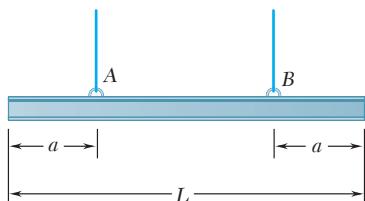


Fig. P7.58

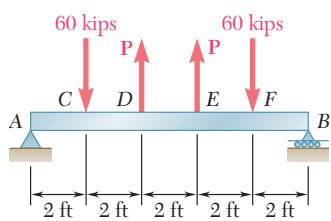


Fig. P7.59

- 7.50** Two small channel sections *DF* and *EH* have been welded to the uniform beam *AB* of weight  $W = 3 \text{ kN}$  to form the rigid structural member shown. This member is being lifted by two cables attached at *D* and *E*. Knowing that  $\theta = 30^\circ$  and neglecting the weight of the channel sections, (a) draw the shear and bending-moment diagrams for beam *AB*, (b) determine the maximum absolute values of the shear and bending moment in the beam.

- 7.51** Solve Prob. 7.50 when  $\theta = 60^\circ$ .

- 7.52 through 7.54** Draw the shear and bending-moment diagrams for the beam *AB*, and determine the maximum absolute values of the shear and bending moment.

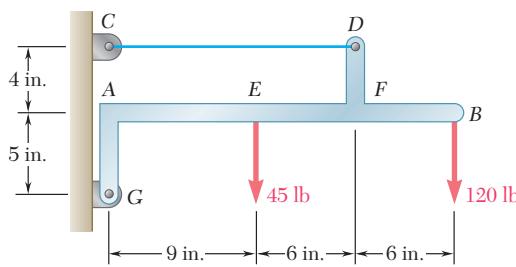


Fig. P7.53

- 7.55** For the structural member of Prob. 7.50, determine (a) the angle  $\theta$  for which the maximum absolute value of the bending moment in beam *AB* is as small as possible, (b) the corresponding value of  $|M|_{\max}$ . (Hint: Draw the bending-moment diagram and then equate the absolute values of the largest positive and negative bending moments obtained.)

- 7.56** For the beam of Prob. 7.43, determine (a) the distance  $a$  for which the maximum absolute value of the bending moment in the beam is as small as possible, (b) the corresponding value of  $|M|_{\max}$ . (See hint for Prob. 7.55.)

- 7.57** For the beam of Prob. 7.47, determine (a) the ratio  $k = P/wa$  for which the maximum absolute value of the bending moment in the beam is as small as possible, (b) the corresponding value of  $|M|_{\max}$ . (See hint for Prob. 7.55.)

- 7.58** A uniform beam is to be picked up by crane cables attached at *A* and *B*. Determine the distance  $a$  from the ends of the beam to the points where the cables should be attached if the maximum absolute value of the bending moment in the beam is to be as small as possible. (Hint: Draw the bending-moment diagram in terms of  $a$ ,  $L$ , and the weight  $w$  per unit length, and then equate the absolute values of the largest positive and negative bending moments obtained.)

- 7.59** For the beam shown, determine (a) the magnitude  $P$  of the two upward forces for which the maximum absolute value of the bending moment in the beam is as small as possible, (b) the corresponding value of  $|M|_{\max}$ . (See hint for Prob. 7.55.)

- 7.60** Knowing that  $P = Q = 150$  lb, determine (a) the distance  $a$  for which the maximum absolute value of the bending moment in beam  $AB$  is as small as possible, (b) the corresponding value of  $|M|_{\max}$ . (See hint for Prob. 7.55.)

- 7.61** Solve Prob. 7.60 assuming that  $P = 300$  lb and  $Q = 150$  lb.

- \*7.62** In order to reduce the bending moment in the cantilever beam  $AB$ , a cable and counterweight are permanently attached at end  $B$ . Determine the magnitude of the counterweight for which the maximum absolute value of the bending moment in the beam is as small as possible and the corresponding value of  $|M|_{\max}$ . Consider (a) the case when the distributed load is permanently applied to the beam, (b) the more general case when the distributed load may either be applied or removed.

## \*7.6 RELATIONS AMONG LOAD, SHEAR, AND BENDING MOMENT

When a beam carries more than two or three concentrated loads, or when it carries distributed loads, the method outlined in Sec. 7.5 for plotting shear and bending moment is likely to be quite cumbersome. The construction of the shear diagram and, especially, of the bending-moment diagram will be greatly facilitated if certain relations existing among load, shear, and bending moment are taken into consideration.

Let us consider a simply supported beam  $AB$  carrying a distributed load  $w$  per unit length (Fig. 7.11a), and let  $C$  and  $C'$  be two points of the beam at a distance  $\Delta x$  from each other. The shear and bending moment at  $C$  will be denoted by  $V$  and  $M$ , respectively, and will be assumed positive; the shear and bending moment at  $C'$  will be denoted by  $V + \Delta V$  and  $M + \Delta M$ .

Let us now detach the portion of beam  $CC'$  and draw its free-body diagram (Fig. 7.11b). The forces exerted on the free body include a load of magnitude  $w \Delta x$  and internal forces and couples at  $C$  and  $C'$ . Since shear and bending moment have been assumed positive, the forces and couples will be directed as shown in the figure.

**Relations between Load and Shear.** We write that the sum of the vertical components of the forces acting on the free body  $CC'$  is zero:

$$V - (V + \Delta V) - w \Delta x = 0 \\ \Delta V = -w \Delta x$$

Dividing both members of the equation by  $\Delta x$  and then letting  $\Delta x$  approach zero, we obtain

$$\frac{dV}{dx} = -w \quad (7.1)$$

Formula (7.1) indicates that for a beam loaded as shown in Fig. 7.11a, the slope  $dV/dx$  of the shear curve is negative; the numerical value of the slope at any point is equal to the load per unit length at that point.

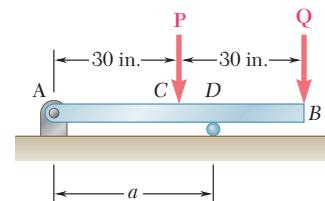


Fig. P7.60

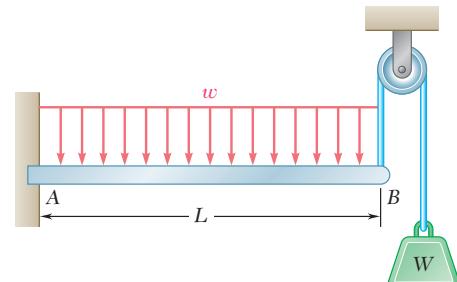
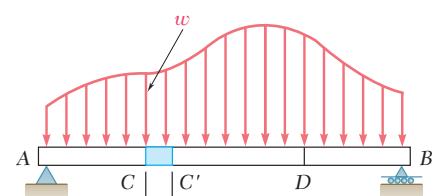
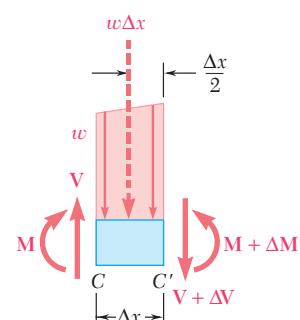


Fig. P7.62



(a)



(b)

Fig. 7.11

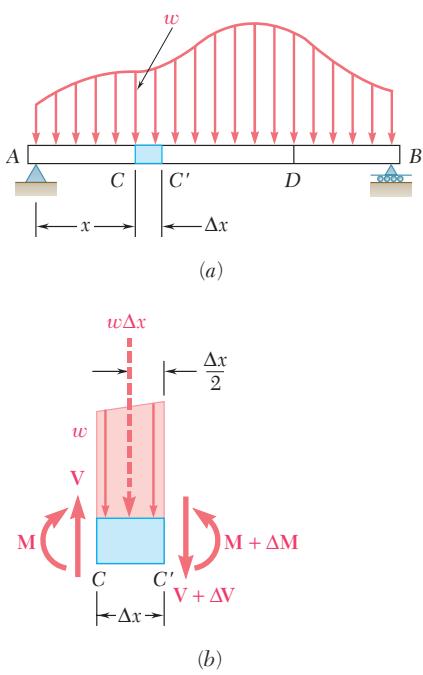


Fig. 7.11 (repeated)

Integrating (7.1) between points  $C$  and  $D$ , we obtain

$$V_D - V_C = - \int_{x_C}^{x_D} w \, dx \quad (7.2)$$

$$V_D - V_C = -( \text{area under load curve between } C \text{ and } D ) \quad (7.2')$$

Note that this result could also have been obtained by considering the equilibrium of the portion of beam  $CD$ , since the area under the load curve represents the total load applied between  $C$  and  $D$ .

It should be observed that formula (7.1) is *not valid* at a point where a concentrated load is applied; the shear curve is discontinuous at such a point, as seen in Sec. 7.5. Similarly, formulas (7.2) and (7.2') cease to be valid when concentrated loads are applied between  $C$  and  $D$ , since they do not take into account the sudden change in shear caused by a concentrated load. Formulas (7.2) and (7.2'), therefore, should be applied only between successive concentrated loads.

**Relations between Shear and Bending Moment.** Returning to the free-body diagram of Fig. 7.11b, and writing now that the sum of the moments about  $C'$  is zero, we obtain

$$(M + \Delta M) - M - V \Delta x + w \Delta x \frac{\Delta x}{2} = 0 \\ \Delta M = V \Delta x - \frac{1}{2} w (\Delta x)^2$$

Dividing both members of the equation by  $\Delta x$  and then letting  $\Delta x$  approach zero, we obtain

$$\frac{dM}{dx} = V \quad (7.3)$$

Formula (7.3) indicates that the slope  $dM/dx$  of the bending-moment curve is equal to the value of the shear. This is true at any point where the shear has a well-defined value, i.e., at any point where no concentrated load is applied. Formula (7.3) also shows that the shear is zero at points where the bending moment is maximum. This property facilitates the determination of the points where the beam is likely to fail under bending.

Integrating (7.3) between points  $C$  and  $D$ , we obtain

$$M_D - M_C = \int_{x_C}^{x_D} V \, dx \quad (7.4)$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \quad (7.4')$$

Note that the area under the shear curve should be considered positive where the shear is positive and negative where the shear is negative. Formulas (7.4) and (7.4') are valid even when concentrated loads are applied between  $C$  and  $D$ , as long as the shear curve has been correctly drawn. The formulas cease to be valid, however, if a couple is applied at a point between  $C$  and  $D$ , since they do not take into account the sudden change in bending moment caused by a couple (see Sample Prob. 7.7).

**EXAMPLE** Let us consider a simply supported beam  $AB$  of span  $L$  carrying a uniformly distributed load  $w$  (Fig. 7.12a). From the free-body diagram of the entire beam we determine the magnitude of the reactions at the supports:  $R_A = R_B = wL/2$  (Fig. 7.12b). Next, we draw the shear diagram. Close to the end  $A$  of the beam, the shear is equal to  $R_A$ , that is, to  $wL/2$ , as we can check by considering a very small portion of the beam as a free body. Using formula (7.2), we can then determine the shear  $V$  at any distance  $x$  from  $A$ . We write

$$V - V_A = - \int_0^x w \, dx = -wx$$

$$V = V_A - wx = \frac{wL}{2} - wx = w\left(\frac{L}{2} - x\right)$$

The shear curve is thus an oblique straight line which crosses the  $x$  axis at  $x = L/2$  (Fig. 7.12c). Considering, now, the bending moment, we first observe that  $M_A = 0$ . The value  $M$  of the bending moment at any distance  $x$  from  $A$  can then be obtained from formula (7.4); we have

$$M - M_A = \int_0^x V \, dx$$

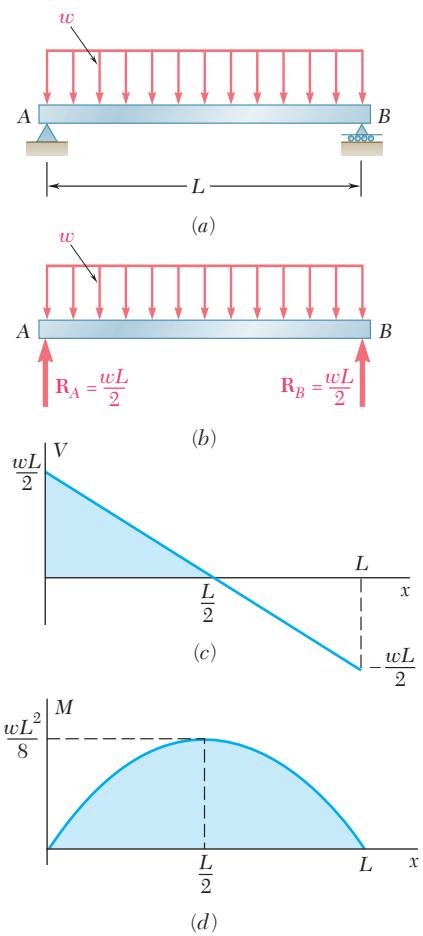
$$M = \int_0^x w\left(\frac{L}{2} - x\right) dx = \frac{w}{2}(Lx - x^2)$$

The bending-moment curve is a parabola. The maximum value of the bending moment occurs when  $x = L/2$ , since  $V$  (and thus  $dM/dx$ ) is zero for that value of  $x$ . Substituting  $x = L/2$  in the last equation, we obtain  $M_{\max} = wL^2/8$ . ■

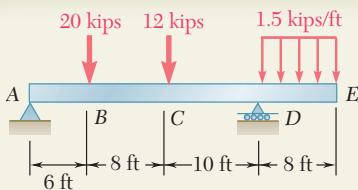
In most engineering applications, the value of the bending moment needs to be known only at a few specific points. Once the shear diagram has been drawn, and after  $M$  has been determined at one of the ends of the beam, the value of the bending moment can then be obtained at any given point by computing the area under the shear curve and using formula (7.4'). For instance, since  $M_A = 0$  for the beam of Fig. 7.12, the maximum value of the bending moment for that beam can be obtained simply by measuring the area of the shaded triangle in the shear diagram:

$$M_{\max} = \frac{1}{2} \frac{L}{2} \frac{wL}{2} = \frac{wL^2}{8}$$

In this example, the load curve is a horizontal straight line, the shear curve is an oblique straight line, and the bending-moment curve is a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree), and the bending-moment curve would have been a cubic (third degree). The shear and bending-moment curves will always be, respectively, one and two degrees higher than the load curve. Thus, once a few values of the shear and bending moment have been computed, we should be able to sketch the shear and bending-moment diagrams without actually determining the functions  $V(x)$  and  $M(x)$ . The sketches obtained will be more accurate if we make use of the fact that at any point where the curves are continuous, the slope of the shear curve is equal to  $-w$  and the slope of the bending-moment curve is equal to  $V$ .



**Fig. 7.12**



## SAMPLE PROBLEM 7.4

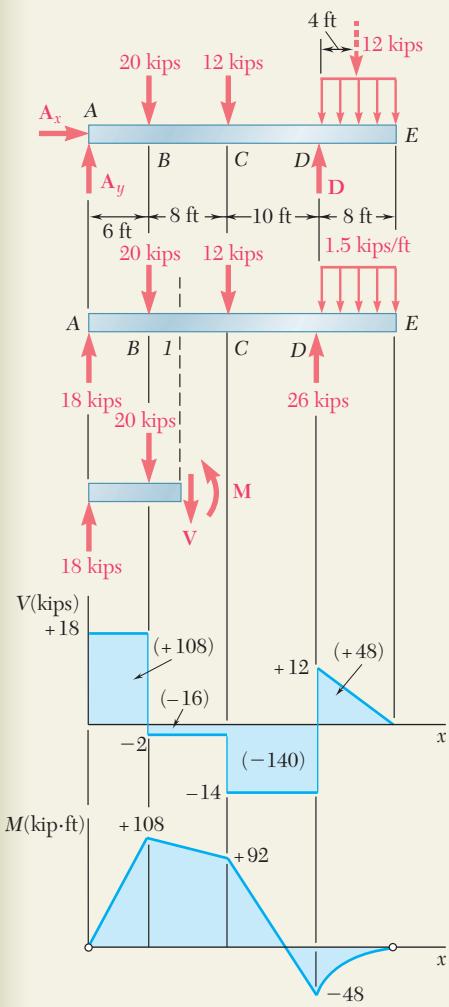
Draw the shear and bending-moment diagrams for the beam and loading shown.

### SOLUTION

**Free-Body: Entire Beam.** Considering the entire beam as a free body, we determine the reactions:

$$\begin{aligned}
 +\sum \text{M}_A &= 0: & D(24 \text{ ft}) - (20 \text{ kips})(6 \text{ ft}) - (12 \text{ kips})(14 \text{ ft}) - (12 \text{ kips})(28 \text{ ft}) &= 0 \\
 && D = +26 \text{ kips} & \mathbf{D} = 26 \text{ kips} \uparrow \\
 +\uparrow \sum F_y &= 0: & A_y - 20 \text{ kips} - 12 \text{ kips} + 26 \text{ kips} - 12 \text{ kips} &= 0 \\
 && A_y = +18 \text{ kips} & \mathbf{A}_y = 18 \text{ kips} \uparrow \\
 \rightarrow \sum F_x &= 0: & A_x = 0 & \mathbf{A}_x = 0
 \end{aligned}$$

We also note that at both A and E the bending moment is zero; thus two points (indicated by small circles) are obtained on the bending-moment diagram.



**Shear Diagram.** Since  $dV/dx = -w$ , we find that between concentrated loads and reactions the slope of the shear diagram is zero (i.e., the shear is constant). The shear at any point is determined by dividing the beam into two parts and considering either part as a free body. For example, using the portion of beam to the left of section 1, we obtain the shear between B and C:

$$+\uparrow \sum F_y = 0: \quad +18 \text{ kips} - 20 \text{ kips} - V = 0 \quad V = -2 \text{ kips}$$

We also find that the shear is +12 kips just to the right of D and zero at end E. Since the slope  $dV/dx = -w$  is constant between D and E, the shear diagram between these two points is a straight line.

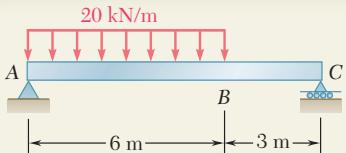
**Bending-Moment Diagram.** We recall that the area under the shear curve between two points is equal to the change in bending moment between the same two points. For convenience, the area of each portion of the shear diagram is computed and is indicated on the diagram. Since the bending moment  $M_A$  at the left end is known to be zero, we write

$$\begin{array}{ll}
 M_B - M_A = +108 & M_B = +108 \text{ kip} \cdot \text{ft} \\
 M_C - M_B = -16 & M_C = +92 \text{ kip} \cdot \text{ft} \\
 M_D - M_C = -140 & M_D = -48 \text{ kip} \cdot \text{ft} \\
 M_E - M_D = +48 & M_E = 0
 \end{array}$$

Since  $M_E$  is known to be zero, a check of the computations is obtained.

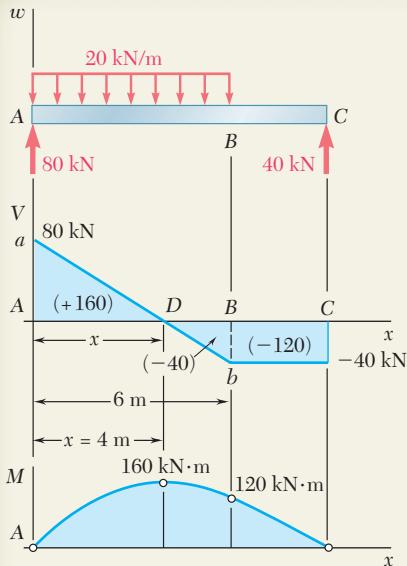
Between the concentrated loads and reactions the shear is constant; thus the slope  $dM/dx$  is constant, and the bending-moment diagram is drawn by connecting the known points with straight lines. Between D and E, where the shear diagram is an oblique straight line, the bending-moment diagram is a parabola.

From the V and M diagrams we note that  $V_{\max} = 18 \text{ kips}$  and  $M_{\max} = 108 \text{ kip} \cdot \text{ft}$ .



## SAMPLE PROBLEM 7.5

Draw the shear and bending-moment diagrams for the beam and loading shown and determine the location and magnitude of the maximum bending moment.



## SOLUTION

**Free-Body: Entire Beam.** Considering the entire beam as a free body, we obtain the reactions

$$R_A = 80 \text{ kN} \uparrow \quad R_C = 40 \text{ kN} \uparrow$$

**Shear Diagram.** The shear just to the right of A is  $V_A = +80 \text{ kN}$ . Since the change in shear between two points is equal to *minus* the area under the load curve between the same two points, we obtain  $V_B$  by writing

$$\begin{aligned} V_B - V_A &= -(20 \text{ kN/m})(6 \text{ m}) = -120 \text{ kN} \\ V_B &= -120 + V_A = -120 + 80 = -40 \text{ kN} \end{aligned}$$

Since the slope  $dV/dx = -w$  is constant between A and B, the shear diagram between these two points is represented by a straight line. Between B and C, the area under the load curve is zero; therefore,

$$V_C - V_B = 0 \quad V_C = V_B = -40 \text{ kN}$$

and the shear is constant between B and C.

**Bending-Moment Diagram.** We note that the bending moment at each end of the beam is zero. In order to determine the maximum bending moment, we locate the section D of the beam where  $V = 0$ . We write

$$\begin{aligned} V_D - V_A &= -wx \\ 0 - 80 \text{ kN} &= -(20 \text{ kN/m})x \end{aligned}$$

and, solving for  $x$ :

$x = 4 \text{ m}$

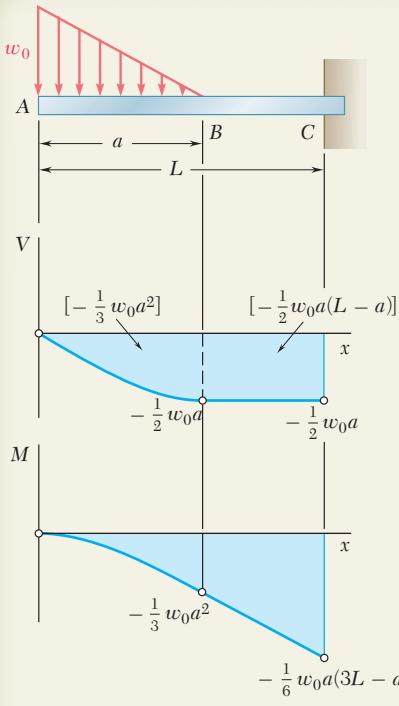
The maximum bending moment occurs at point D, where we have  $dM/dx = V = 0$ . The areas of the various portions of the shear diagram are computed and are given (in parentheses) on the diagram. Since the area of the shear diagram between two points is equal to the change in bending moment between the same two points, we write

$$\begin{array}{ll} M_D - M_A = +160 \text{ kN} \cdot \text{m} & M_D = +160 \text{ kN} \cdot \text{m} \\ M_B - M_D = -40 \text{ kN} \cdot \text{m} & M_B = +120 \text{ kN} \cdot \text{m} \\ M_C - M_B = -120 \text{ kN} \cdot \text{m} & M_C = 0 \end{array}$$

The bending-moment diagram consists of an arc of parabola followed by a segment of straight line; the slope of the parabola at A is equal to the value of  $V$  at that point.

The maximum bending moment is

$$M_{\max} = M_D = +160 \text{ kN} \cdot \text{m}$$



## SAMPLE PROBLEM 7.6

Sketch the shear and bending-moment diagrams for the cantilever beam shown.

### SOLUTION

**Shear Diagram.** At the free end of the beam, we find  $V_A = 0$ . Between A and B, the area under the load curve is  $\frac{1}{2}w_0a$ ; we find  $V_B$  by writing

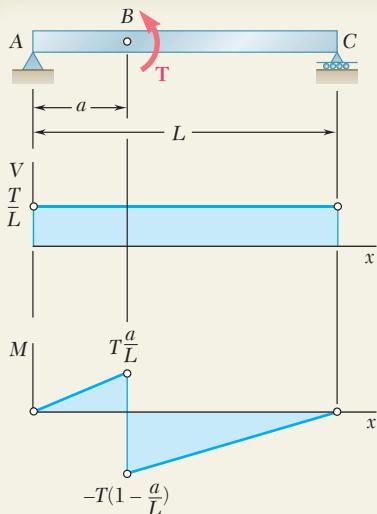
$$V_B - V_A = -\frac{1}{2}w_0a \quad V_B = -\frac{1}{2}w_0a$$

Between B and C, the beam is not loaded; thus  $V_C = V_B$ . At A, we have  $w = w_0$ , and, according to Eq. (7.1), the slope of the shear curve is  $dV/dx = -w_0$ , while at B the slope is  $dV/dx = 0$ . Between A and B, the loading decreases linearly, and the shear diagram is parabolic. Between B and C,  $w = 0$ , and the shear diagram is a horizontal line.

**Bending-Moment Diagram.** We note that  $M_A = 0$  at the free end of the beam. We compute the area under the shear curve and write

$$\begin{aligned} M_B - M_A &= -\frac{1}{3}w_0a^2 & M_B &= -\frac{1}{3}w_0a^2 \\ M_C - M_B &= -\frac{1}{2}w_0a(L-a) & \\ M_C &= -\frac{1}{6}w_0a(3L-a) \end{aligned}$$

The sketch of the bending-moment diagram is completed by recalling that  $dM/dx = V$ . We find that between A and B the diagram is represented by a cubic curve with zero slope at A, and between B and C the diagram is represented by a straight line.



## SAMPLE PROBLEM 7.7

The simple beam AC is loaded by a couple of magnitude  $T$  applied at point B. Draw the shear and bending-moment diagrams for the beam.

### SOLUTION

**Free Body: Entire Beam.** The entire beam is taken as a free body, and we obtain

$$\mathbf{R}_A = \frac{T}{L} \uparrow \quad \mathbf{R}_C = \frac{T}{L} \downarrow$$

**Shear and Bending-Moment Diagrams.** The shear at any section is constant and equal to  $T/L$ . Since a couple is applied at B, the bending-moment diagram is discontinuous at B; the bending moment decreases suddenly by an amount equal to  $T$ .

# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned how to use the relations existing among load, shear, and bending moment to simplify the drawing of the shear and bending-moment diagrams. These relations are

$$\frac{dV}{dx} = -w \quad (7.1)$$

$$\frac{dM}{dx} = V \quad (7.3)$$

$$V_D - V_C = -( \text{area under load curve between } C \text{ and } D) \quad (7.2')$$

$$M_D - M_C = (\text{area under shear curve between } C \text{ and } D) \quad (7.4')$$

Taking into account these relations, you can use the following procedure to draw the shear and bending-moment diagrams for a beam.

**1. Draw a free-body diagram of the entire beam,** and use it to determine the reactions at the beam supports.

**2. Draw the shear diagram.** This can be done as in the preceding lesson by cutting the beam at various points and considering the free-body diagram of one of the two portions of the beam that you have obtained [Sample Prob. 7.3]. You can, however, consider one of the following alternative procedures.

a. **The shear  $V$  at any point of the beam is the sum of the reactions and loads to the left of that point;** an upward force is counted as positive, and a downward force is counted as negative.

b. **For a beam carrying a distributed load,** you can start from a point where you know  $V$  and use Eq. (7.2') repeatedly to find  $V$  at all the other points of interest.

**3. Draw the bending-moment diagram,** using the following procedure.

a. **Compute the area under each portion of the shear curve,** assigning a positive sign to areas located above the  $x$  axis and a negative sign to areas located below the  $x$  axis.

b. **Apply Eq. (7.4') repeatedly** [Sample Probs. 7.4 and 7.5], starting from the left end of the beam, where  $M = 0$  (except if a couple is applied at that end, or if the beam is a cantilever beam with a fixed left end).

c. **Where a couple is applied to the beam,** be careful to show a discontinuity in the bending-moment diagram by *increasing* the value of  $M$  at that point by an amount equal to the magnitude of the couple if the couple is *clockwise*, or *decreasing* the value of  $M$  by that amount if the couple is *counterclockwise* [Sample Prob. 7.7].

(continued)

**4. Determine the location and magnitude of  $|M|_{max}$ .** The maximum absolute value of the bending moment occurs at one of the points where  $dM/dx = 0$ , that is, according to Eq. (7.3), at a point where  $V$  is equal to zero or changes sign. You should, therefore:

a. **Determine from the shear diagram the value of  $|M|$  where  $V$  changes sign;** this will occur under the concentrated loads [Sample Prob. 7.4].

b. **Determine the points where  $V = 0$  and the corresponding values of  $|M|$ ;** this will occur under a distributed load. To find the distance  $x$  between point  $C$ , where the distributed load starts, and point  $D$ , where the shear is zero, use Eq. (7.2'); for  $V_C$  use the known value of the shear at point  $C$ , for  $V_D$  use zero, and express the area under the load curve as a function of  $x$  [Sample Prob. 7.5].

**5. You can improve the quality of your drawings** by keeping in mind that at any given point, according to Eqs. (7.1) and (7.3), the slope of the  $V$  curve is equal to  $-w$  and the slope of the  $M$  curve is equal to  $V$ .

**6. Finally, for beams supporting a distributed load expressed as a function  $w(x)$ ,** remember that the shear  $V$  can be obtained by integrating the function  $-w(x)$ , and the bending moment  $M$  can be obtained by integrating  $V(x)$  [Eqs. (7.3) and (7.4)].

# PROBLEMS

**7.63** Using the method of Sec. 7.6, solve Prob. 7.29.

**7.64** Using the method of Sec. 7.6, solve Prob. 7.30.

**7.65** Using the method of Sec. 7.6, solve Prob. 7.31.

**7.66** Using the method of Sec. 7.6, solve Prob. 7.32.

**7.67** Using the method of Sec. 7.6, solve Prob. 7.33.

**7.68** Using the method of Sec. 7.6, solve Prob. 7.34.

**7.69 and 7.70** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

**7.71** Using the method of Sec. 7.6, solve Prob. 7.39.

**7.72** Using the method of Sec. 7.6, solve Prob. 7.40.

**7.73** Using the method of Sec. 7.6, solve Prob. 7.41.

**7.74** Using the method of Sec. 7.6, solve Prob. 7.42.

**7.75 and 7.76** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

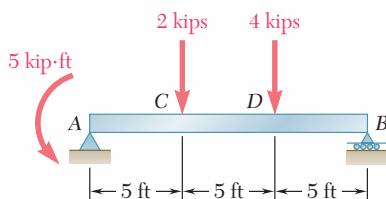


Fig. P7.75

**7.77 and 7.78** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the magnitude and location of the maximum bending moment.

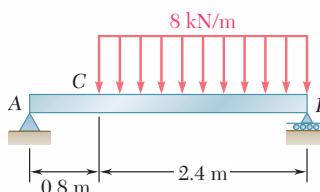


Fig. P7.77

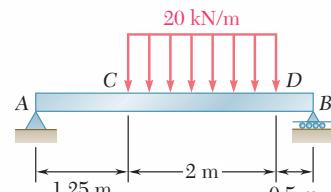


Fig. P7.78

**7.79 and 7.80** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the magnitude and location of the maximum absolute value of the bending moment.

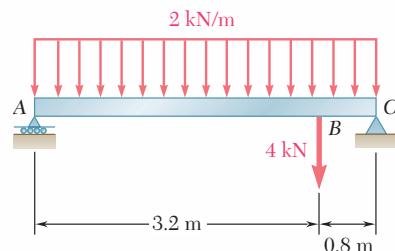


Fig. P7.69

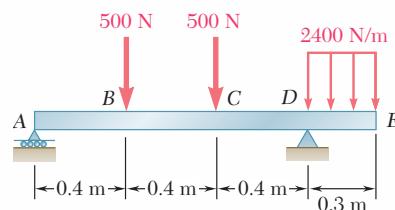


Fig. P7.70

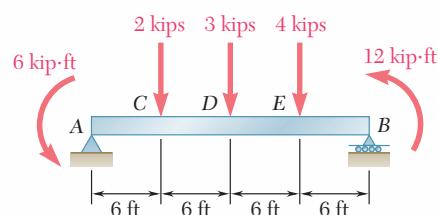


Fig. P7.76

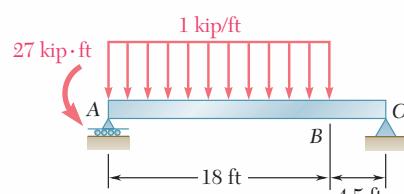


Fig. P7.79

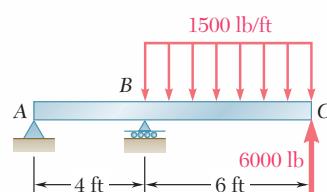


Fig. P7.80

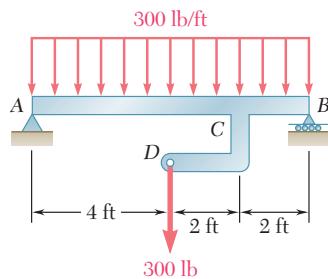


Fig. P7.81

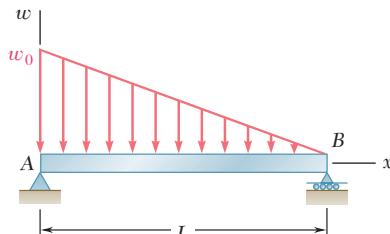


Fig. P7.85

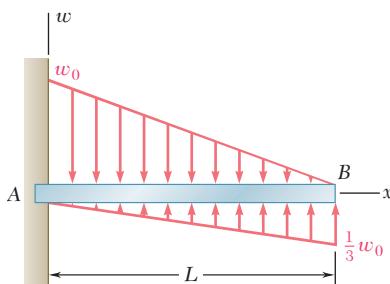


Fig. P7.86

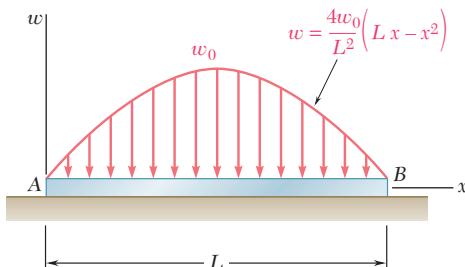


Fig. P7.88

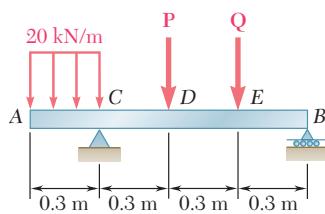


Fig. P7.89

- 7.81** (a) Draw the shear and bending-moment diagrams for beam  $AB$ , (b) determine the magnitude and location of the maximum absolute value of the bending moment.

- 7.82** Solve Prob. 7.81 assuming that the 300-lb force applied at  $D$  is directed upward.

- 7.83** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the magnitude and location of the maximum absolute value of the bending moment.

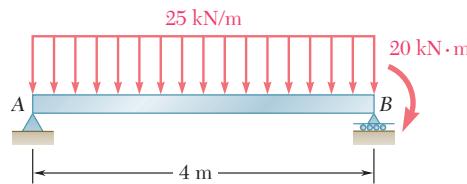


Fig. P7.83

- 7.84** Solve Prob. 7.83 assuming that the 20-kN · m couple applied at  $B$  is counterclockwise.

- 7.85 and 7.86** For the beam and loading shown, (a) write the equations of the shear and bending-moment curves, (b) determine the magnitude and location of the maximum bending moment.

- 7.87** For the beam and loading shown, (a) write the equations of the shear and bending-moment curves, (b) determine the magnitude and location of the maximum bending moment.

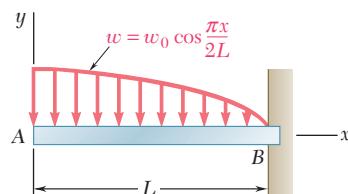


Fig. P7.87

- 7.88** The beam  $AB$ , which lies on the ground, supports the parabolic load shown. Assuming the upward reaction of the ground to be uniformly distributed, (a) write the equations of the shear and bending-moment curves, (b) determine the maximum bending moment.

- 7.89** The beam  $AB$  is subjected to the uniformly distributed load shown and to two unknown forces  $P$  and  $Q$ . Knowing that it has been experimentally determined that the bending moment is  $+800 \text{ N} \cdot \text{m}$  at  $D$  and  $+1300 \text{ N} \cdot \text{m}$  at  $E$ , (a) determine  $P$  and  $Q$ , (b) draw the shear and bending-moment diagrams for the beam.

- 7.90** Solve Prob. 7.89 assuming that the bending moment was found to be  $+650 \text{ N} \cdot \text{m}$  at  $D$  and  $+1450 \text{ N} \cdot \text{m}$  at  $E$ .

- \*7.91** The beam  $AB$  is subjected to the uniformly distributed load shown and to two unknown forces  $\mathbf{P}$  and  $\mathbf{Q}$ . Knowing that it has been experimentally determined that the bending moment is  $+6.10 \text{ kip} \cdot \text{ft}$  at  $D$  and  $+5.50 \text{ kip} \cdot \text{ft}$  at  $E$ , (a) determine  $\mathbf{P}$  and  $\mathbf{Q}$ , (b) draw the shear and bending-moment diagrams for the beam.

- \*7.92** Solve Prob. 7.91 assuming that the bending moment was found to be  $+5.96 \text{ kip} \cdot \text{ft}$  at  $D$  and  $+6.84 \text{ kip} \cdot \text{ft}$  at  $E$ .

## CABLES

### \*7.7 CABLES WITH CONCENTRATED LOADS

Cables are used in many engineering applications, such as suspension bridges, transmission lines, aerial tramways, guy wires for high towers, etc. Cables may be divided into two categories, according to their loading: (1) cables supporting concentrated loads, (2) cables supporting distributed loads. In this section, cables of the first category are examined.

Consider a cable attached to two fixed points  $A$  and  $B$  and supporting  $n$  vertical concentrated loads  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$  (Fig. 7.13a). We assume that the cable is *flexible*, i.e., that its resistance to bending is small and can be neglected. We further assume that the *weight of the cable is negligible* compared with the loads supported by the cable. Any portion of cable between successive loads can therefore be considered as a two-force member, and the internal forces at any point in the cable reduce to a *force of tension directed along the cable*.

We assume that each of the loads lies in a given vertical line, i.e., that the horizontal distance from support  $A$  to each of the loads is known; we also assume that the horizontal and vertical distances between the supports are known. We propose to determine the shape of the cable, i.e., the vertical distance from support  $A$  to each of the points  $C_1, C_2, \dots, C_n$ , and also the tension  $T$  in each portion of the cable.

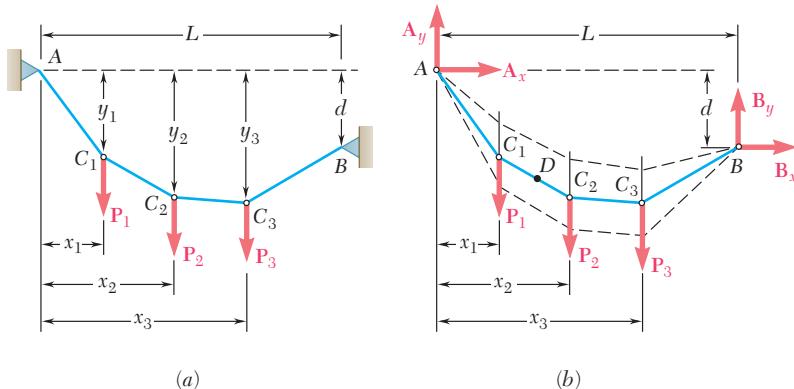


Fig. 7.13

We first draw the free-body diagram of the entire cable (Fig. 7.13b). Since the slope of the portions of cable attached at  $A$  and  $B$  is not known, the reactions at  $A$  and  $B$  must be represented by two components each. Thus, four unknowns are involved, and the three equations of equilibrium are not sufficient to determine the reactions at  $A$  and  $B$ .† We must

†Clearly, the cable is not a rigid body; the equilibrium equations represent, therefore, necessary but not sufficient conditions (see Sec. 6.11).

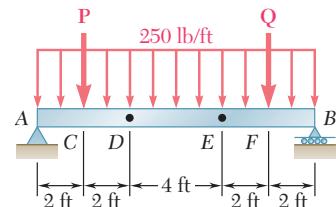
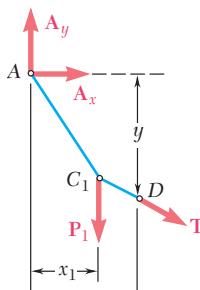


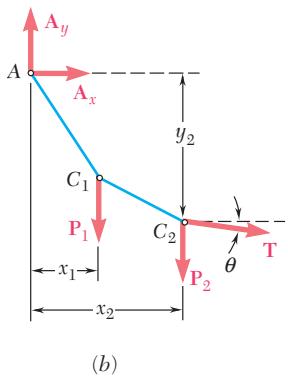
Fig. P7.91



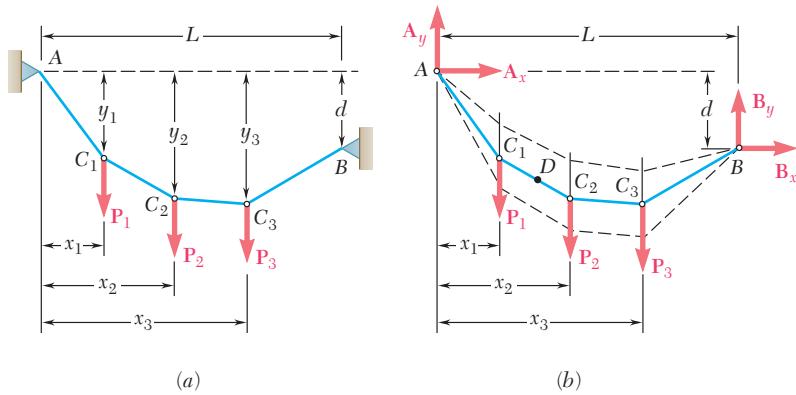
Photo 7.3 Since the weight of the cable of the chairlift shown is negligible compared to the weights of the chairs and skiers, the methods of this section can be used to determine the force at any point in the cable.



(a)



(b)

**Fig. 7.14****Fig. 7.13 (repeated)**

therefore obtain an additional equation by considering the equilibrium of a portion of the cable. This is possible if we know the coordinates  $x$  and  $y$  of a point  $D$  of the cable. Drawing the free-body diagram of the portion of cable  $AD$  (Fig. 7.14a) and writing  $\sum M_D = 0$ , we obtain an additional relation between the scalar components  $A_x$  and  $A_y$  and can determine the reactions at  $A$  and  $B$ . The problem would remain indeterminate, however, if we did not know the coordinates of  $D$ , unless some other relation between  $A_x$  and  $A_y$  (or between  $B_x$  and  $B_y$ ) were given. The cable might hang in any of various possible ways, as indicated by the dashed lines in Fig. 7.13b.

Once  $A_x$  and  $A_y$  have been determined, the vertical distance from  $A$  to any point of the cable can easily be found. Considering point  $C_2$ , for example, we draw the free-body diagram of the portion of cable  $AC_2$  (Fig. 7.14b). Writing  $\sum M_{C_2} = 0$ , we obtain an equation which can be solved for  $y_2$ . Writing  $\sum F_x = 0$  and  $\sum F_y = 0$ , we obtain the components of the force  $\mathbf{T}$  representing the tension in the portion of cable to the right of  $C_2$ . We observe that  $T \cos \theta = -A_x$ ; *the horizontal component of the tension force is the same at any point of the cable*. It follows that the tension  $T$  is maximum when  $\cos \theta$  is minimum, i.e., in the portion of cable which has the largest angle of inclination  $\theta$ . Clearly, this portion of cable must be adjacent to one of the two supports of the cable.

## \*7.8 CABLES WITH DISTRIBUTED LOADS

Consider a cable attached to two fixed points  $A$  and  $B$  and carrying a *distributed load* (Fig. 7.15a). We saw in the preceding section that for a cable supporting concentrated loads, the internal force at any point is a force of tension directed along the cable. In the case of a cable carrying a distributed load, the cable hangs in the shape of a curve, and the internal force at a point  $D$  is a force of tension  $\mathbf{T}$  *directed along the tangent to the curve*. In this section, you will learn to determine the tension at any point of a cable supporting a given distributed load. In the following sections, the shape of the cable will be determined for two particular types of distributed loads.

Considering the most general case of distributed load, we draw the free-body diagram of the portion of cable extending from the lowest point  $C$  to a given point  $D$  of the cable (Fig. 7.15b). The

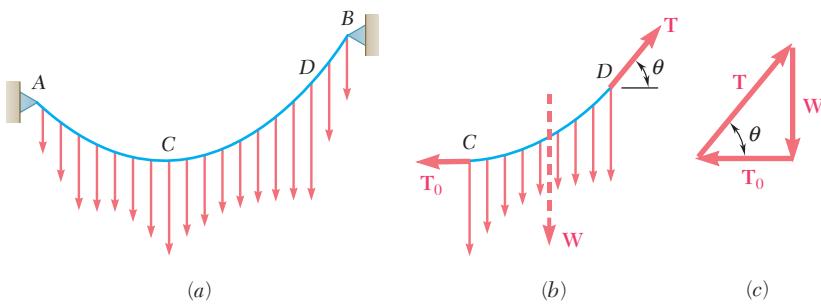


Fig. 7.15

forces acting on the free body are the tension force  $\mathbf{T}_0$  at  $C$ , which is horizontal, the tension force  $\mathbf{T}$  at  $D$ , directed along the tangent to the cable at  $D$ , and the resultant  $\mathbf{W}$  of the distributed load supported by the portion of cable  $CD$ . Drawing the corresponding force triangle (Fig. 7.15c), we obtain the following relations:

$$T \cos \theta = T_0 \quad T \sin \theta = W \quad (7.5)$$

$$T = \sqrt{T_0^2 + W^2} \quad \tan \theta = \frac{W}{T_0} \quad (7.6)$$

From the relations (7.5), it appears that the horizontal component of the tension force  $\mathbf{T}$  is the same at any point and that the vertical component of  $\mathbf{T}$  is equal to the magnitude  $W$  of the load measured from the lowest point. Relations (7.6) show that the tension  $T$  is minimum at the lowest point and maximum at one of the two points of support.

## \*7.9 PARABOLIC CABLE

Let us assume, now, that the cable  $AB$  carries a load *uniformly distributed along the horizontal* (Fig. 7.16a). Cables of suspension bridges may be assumed loaded in this way, since the weight of the cables is small compared with the weight of the roadway. We denote by  $w$  the load per unit length (*measured horizontally*) and express it in N/m or in lb/ft. Choosing coordinate axes with origin at the lowest point  $C$  of the cable, we find that the magnitude  $W$  of the total load carried by the portion of cable extending from  $C$  to the point  $D$  of coordinates  $x$  and  $y$  is  $W = wx$ . The relations (7.6) defining the magnitude and direction of the tension force at  $D$  become

$$T = \sqrt{T_0^2 + w^2 x^2} \quad \tan \theta = \frac{wx}{T_0} \quad (7.7)$$

Moreover, the distance from  $D$  to the line of action of the resultant  $\mathbf{W}$  is equal to half the horizontal distance from  $C$  to  $D$  (Fig. 7.16b). Summing moments about  $D$ , we write

$$+\uparrow \sum M_D = 0: \quad wx \frac{x}{2} - T_0 y = 0$$

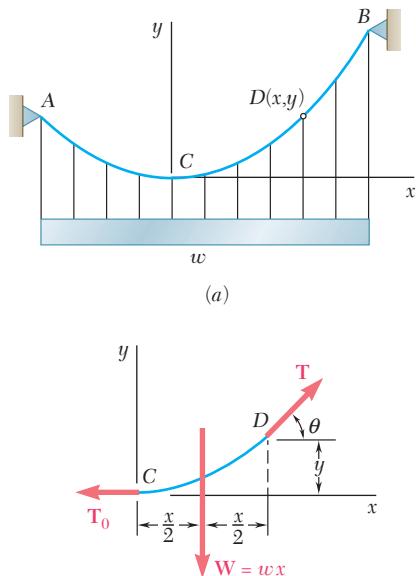


Fig. 7.16

and, solving for  $y$ ,

$$y = \frac{wx^2}{2T_0} \quad (7.8)$$

This is the equation of a *parabola* with a vertical axis and its vertex at the origin of coordinates. The curve formed by cables loaded uniformly along the horizontal is thus a parabola.<sup>†</sup>

When the supports  $A$  and  $B$  of the cable have the same elevation, the distance  $L$  between the supports is called the *span* of the cable and the vertical distance  $h$  from the supports to the lowest point is called the *sag* of the cable (Fig. 7.17a). If the span and sag of a cable are known, and if the load  $w$  per unit horizontal length is given, the minimum tension  $T_0$  may be found by substituting  $x = L/2$  and  $y = h$  in Eq. (7.8). Equations (7.7) will then yield the tension and the slope at any point of the cable and Eq. (7.8) will define the shape of the cable.

When the supports have different elevations, the position of the lowest point of the cable is not known and the coordinates  $x_A, y_A$  and  $x_B, y_B$  of the supports must be determined. To this effect, we express that the coordinates of  $A$  and  $B$  satisfy Eq. (7.8) and that  $x_B - x_A = L$  and  $y_B - y_A = d$ , where  $L$  and  $d$  denote, respectively, the horizontal and vertical distances between the two supports (Fig. 7.17b and c).

The length of the cable from its lowest point  $C$  to its support  $B$  can be obtained from the formula

$$s_B = \int_0^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (7.9)$$

Differentiating (7.8), we obtain the derivative  $dy/dx = wx/T_0$ ; substituting into (7.9) and using the binomial theorem to expand the radical in an infinite series, we have

$$\begin{aligned} s_B &= \int_0^{x_B} \sqrt{1 + \frac{w^2 x^2}{T_0^2}} dx = \int_0^{x_B} \left(1 + \frac{w^2 x^2}{2T_0^2} - \frac{w^4 x^4}{8T_0^4} + \dots\right) dx \\ s_B &= x_B \left(1 + \frac{w^2 x_B^2}{6T_0^2} - \frac{w^4 x_B^4}{40T_0^4} + \dots\right) \end{aligned}$$

and, since  $wx_B^2/2T_0 = y_B$ ,

$$s_B = x_B \left[1 + \frac{2}{3} \left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5} \left(\frac{y_B}{x_B}\right)^4 + \dots\right] \quad (7.10)$$

The series converges for values of the ratio  $y_B/x_B$  less than 0.5; in most cases, this ratio is much smaller, and only the first two terms of the series need be computed.

<sup>†</sup>Cables hanging under their own weight are not loaded uniformly along the horizontal, and they do not form a parabola. The error introduced by assuming a parabolic shape for cables hanging under their weight, however, is small when the cable is sufficiently taut. A complete discussion of cables hanging under their own weight is given in the next section.

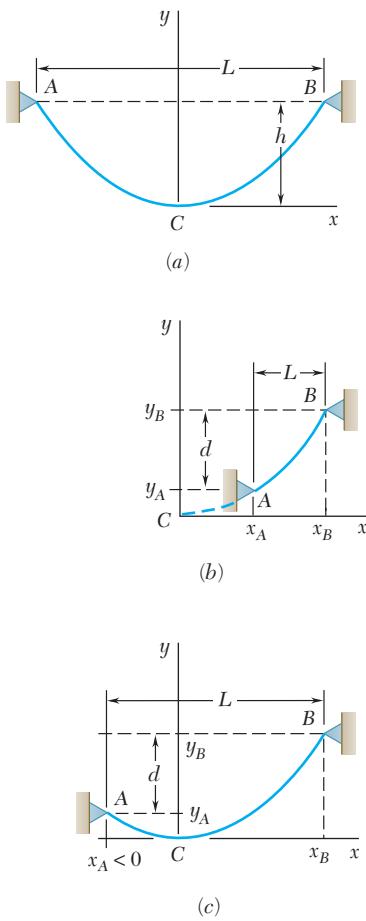
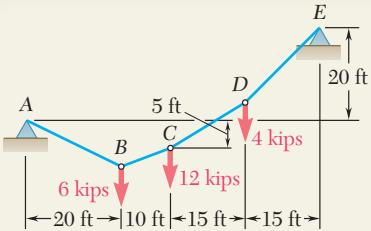


Fig. 7.17



## SAMPLE PROBLEM 7.8

The cable  $AE$  supports three vertical loads from the points indicated. If point  $C$  is 5 ft below the left support, determine (a) the elevation of points  $B$  and  $D$ , (b) the maximum slope and the maximum tension in the cable.

## SOLUTION

**Reactions at Supports.** The reaction components  $A_x$  and  $A_y$  are determined as follows:

### Free Body: Entire Cable

$$+\uparrow \sum M_E = 0: A_x(20 \text{ ft}) - A_y(60 \text{ ft}) + (6 \text{ kips})(40 \text{ ft}) + (12 \text{ kips})(30 \text{ ft}) + (4 \text{ kips})(15 \text{ ft}) = 0 \\ 20A_x - 60A_y + 660 = 0$$

### Free Body: ABC

$$+\uparrow \sum M_C = 0: -A_x(5 \text{ ft}) - A_y(30 \text{ ft}) + (6 \text{ kips})(10 \text{ ft}) = 0 \\ -5A_x - 30A_y + 60 = 0$$

Solving the two equations simultaneously, we obtain

$$\begin{aligned} A_x &= -18 \text{ kips} & A_x &= 18 \text{ kips} \leftarrow \\ A_y &= +5 \text{ kips} & A_y &= 5 \text{ kips} \uparrow \end{aligned}$$

### a. Elevation of Points B and D.

**Free Body: AB** Considering the portion of cable  $AB$  as a free body, we write

$$+\uparrow \sum M_B = 0: (18 \text{ kips})y_B - (5 \text{ kips})(20 \text{ ft}) = 0$$

$$y_B = 5.56 \text{ ft below } A$$

**Free Body: ABCD** Using the portion of cable  $ABCD$  as a free body, we write

$$+\uparrow \sum M_D = 0:$$

$$-(18 \text{ kips})y_D - (5 \text{ kips})(45 \text{ ft}) + (6 \text{ kips})(25 \text{ ft}) + (12 \text{ kips})(15 \text{ ft}) = 0$$

$$y_D = 5.83 \text{ ft above } A$$

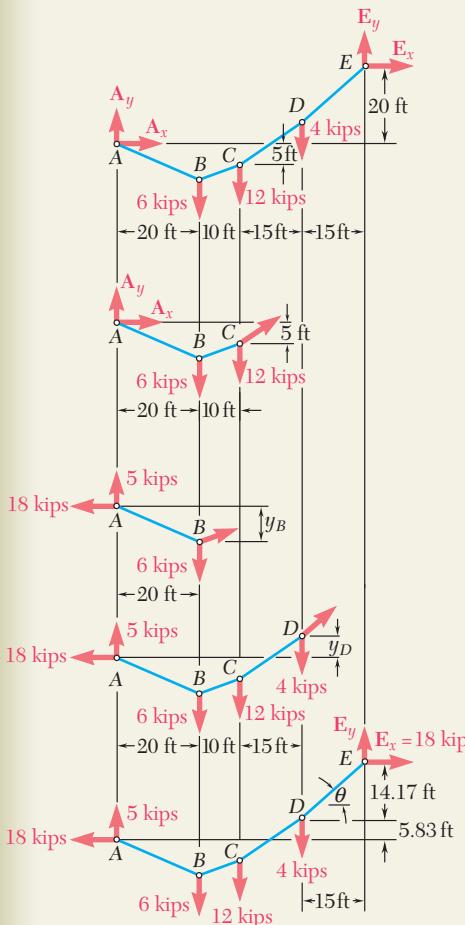
**b. Maximum Slope and Maximum Tension.** We observe that the maximum slope occurs in portion  $DE$ . Since the horizontal component of the tension is constant and equal to 18 kips, we write

$$\tan \theta = \frac{14.17}{15 \text{ ft}}$$

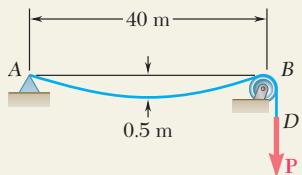
$$\theta = 43.4^\circ$$

$$T_{\max} = \frac{18 \text{ kips}}{\cos \theta}$$

$$T_{\max} = 24.8 \text{ kips}$$

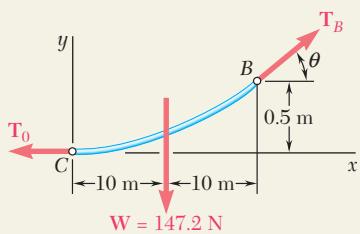


## SAMPLE PROBLEM 7.9



A light cable is attached to a support at A, passes over a small pulley at B, and supports a load  $\mathbf{P}$ . Knowing that the sag of the cable is 0.5 m and that the mass per unit length of the cable is 0.75 kg/m, determine (a) the magnitude of the load  $\mathbf{P}$ , (b) the slope of the cable at B, (c) the total length of the cable from A to B. Since the ratio of the sag to the span is small, assume the cable to be parabolic. Also, neglect the weight of the portion of cable from B to D.

## SOLUTION



**a. Load  $\mathbf{P}$ .** We denote by  $C$  the lowest point of the cable and draw the free-body diagram of the portion  $CB$  of cable. Assuming the load to be uniformly distributed along the horizontal, we write

$$w = (0.75 \text{ kg/m})(9.81 \text{ m/s}^2) = 7.36 \text{ N/m}$$

The total load for the portion  $CB$  of cable is

$$W = wx_B = (7.36 \text{ N/m})(20 \text{ m}) = 147.2 \text{ N}$$

and is applied halfway between  $C$  and  $B$ . Summing moments about  $B$ , we write

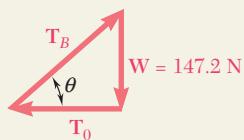
$$+\gamma \sum M_B = 0: \quad (147.2 \text{ N})(10 \text{ m}) - T_0(0.5 \text{ m}) = 0 \quad T_0 = 2944 \text{ N}$$

From the force triangle we obtain

$$\begin{aligned} T_B &= \sqrt{T_0^2 + W^2} \\ &= \sqrt{(2944 \text{ N})^2 + (147.2 \text{ N})^2} = 2948 \text{ N} \end{aligned}$$

Since the tension on each side of the pulley is the same, we find

$$P = T_B = 2948 \text{ N} \quad \blacktriangleleft$$



**b. Slope of Cable at  $B$ .** We also obtain from the force triangle

$$\tan \theta = \frac{W}{T_0} = \frac{147.2 \text{ N}}{2944 \text{ N}} = 0.05$$

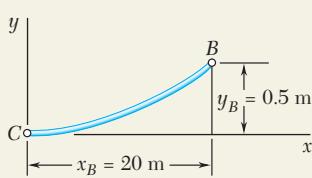
$$\theta = 2.9^\circ \quad \blacktriangleleft$$

**c. Length of Cable.** Applying Eq. (7.10) between  $C$  and  $B$ , we write

$$\begin{aligned} s_B &= x_B \left[ 1 + \frac{2}{3} \left( \frac{y_B}{x_B} \right)^2 + \dots \right] \\ &= (20 \text{ m}) \left[ 1 + \frac{2}{3} \left( \frac{0.5 \text{ m}}{20 \text{ m}} \right)^2 + \dots \right] = 20.00833 \text{ m} \end{aligned}$$

The total length of the cable between A and B is twice this value,

$$\text{Length} = 2s_B = 40.0167 \text{ m} \quad \blacktriangleleft$$



# SOLVING PROBLEMS ON YOUR OWN

In the problems of this section you will apply the equations of equilibrium to cables that lie in a vertical plane. We assume that a cable cannot resist bending, so that the force of tension in the cable is always directed along the cable.

**A. In the first part of this lesson we considered cables subjected to concentrated loads.** Since the weight of the cable is neglected, the cable is straight between loads.

Your solution will consist of the following steps:

**1. Draw a free-body diagram of the entire cable** showing the loads and the horizontal and vertical components of the reaction at each support. Use this free-body diagram to write the corresponding equilibrium equations.

**2. You will be confronted with four unknown components and only three equations of equilibrium** (see Fig. 7.13). You must therefore find an additional piece of information, such as the *position* of a point on the cable or the *slope* of the cable at a given point.

**3. After you have identified the point of the cable where the additional information exists,** cut the cable at that point, and draw a free-body diagram of one of the two portions of the cable you have obtained.

**a. If you know the position** of the point where you have cut the cable, writing  $\Sigma M = 0$  about that point for the new free body will yield the additional equation required to solve for the four unknown components of the reactions. [Sample Prob. 7.8].

**b. If you know the slope** of the portion of the cable you have cut, writing  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$  for the new free body will yield two equilibrium equations which, together with the original three, can be solved for the four reaction components and for the tension in the cable where it has been cut.

**4. To find the elevation of a given point of the cable and the slope and tension at that point** once the reactions at the supports have been found, you should cut the cable at that point and draw a free-body diagram of one of the two portions of the cable you have obtained. Writing  $\Sigma M = 0$  about the given point yields its elevation. Writing  $\Sigma F_x = 0$  and  $\Sigma F_y = 0$  yields the components of the tension force, from which its magnitude and direction can easily be found.

(continued)

**5. For a cable supporting vertical loads only,** you will observe that *the horizontal component of the tension force is the same at any point*. It follows that, for such a cable, the *maximum tension occurs in the steepest portion of the cable*.

**B. In the second portion of this lesson we considered cables carrying a load uniformly distributed along the horizontal.** The shape of the cable is then parabolic.

Your solution will use one or more of the following concepts:

**1. Placing the origin of coordinates at the lowest point of the cable** and directing the  $x$  and  $y$  axes to the right and upward, respectively, we find that *the equation of the parabola is*

$$y = \frac{wx^2}{2T_0} \quad (7.8)$$

The minimum cable tension occurs at the origin, where the cable is horizontal, and the maximum tension is at the support where the slope is maximum.

**2. If the supports of the cable have the same elevation,** the sag  $h$  of the cable is the vertical distance from the lowest point of the cable to the horizontal line joining the supports. To solve a problem involving such a parabolic cable, you should write Eq. (7.8) for one of the supports; this equation can be solved for one unknown.

**3. If the supports of the cable have different elevations,** you will have to write Eq. (7.8) for each of the supports (see Fig. 7.17).

**4. To find the length of the cable** from the lowest point to one of the supports, you can use Eq. (7.10). In most cases, you will need to compute only the first two terms of the series.

# PROBLEMS

- 7.93** Two loads are suspended as shown from the cable *ABCD*. Knowing that  $h_B = 1.8 \text{ m}$ , determine (a) the distance  $h_C$ , (b) the components of the reaction at *D*, (c) the maximum tension in the cable.

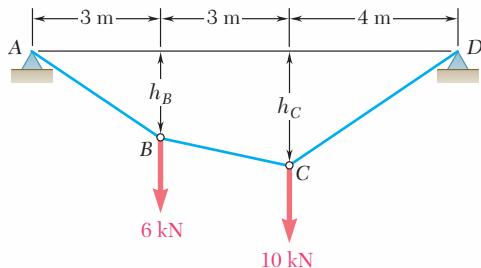


Fig. P7.93 and P7.94

- 7.94** Knowing that the maximum tension in cable *ABCD* is 15 kN, determine (a) the distance  $h_B$ , (b) the distance  $h_C$ .

- 7.95** If  $d_C = 8 \text{ ft}$ , determine (a) the reaction at *A*, (b) the reaction at *E*.

- 7.96** If  $d_C = 4.5 \text{ ft}$ , determine (a) the reaction at *A*, (b) the reaction at *E*.

- 7.97** Knowing that  $d_C = 3 \text{ m}$ , determine (a) the distances  $d_B$  and  $d_D$ , (b) the reaction at *E*.

- 7.98** Determine (a) distance  $d_C$  for which portion *DE* of the cable is horizontal, (b) the corresponding reactions at *A* and *E*.

- 7.99** If  $d_C = 15 \text{ ft}$ , determine (a) the distances  $d_B$  and  $d_D$ , (b) the maximum tension in the cable.

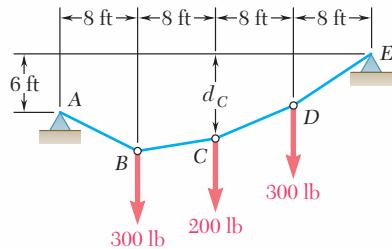


Fig. P7.95 and P7.96

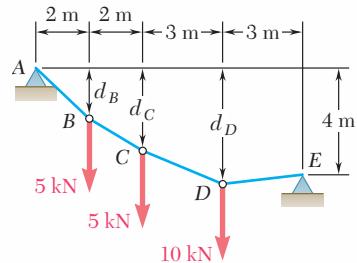


Fig. P7.97 and P7.98

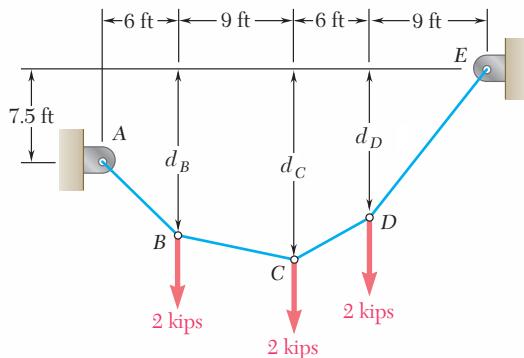


Fig. P7.99 and P7.100

- 7.100** Determine (a) the distance  $d_C$  for which portion *BC* of the cable is horizontal, (b) the corresponding components of the reaction at *E*.

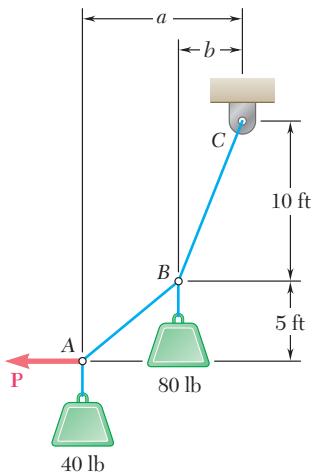


Fig. P7.101 and P7.102

- 7.101** Cable  $ABC$  supports two loads as shown. Knowing that  $b = 4$  ft, determine (a) the required magnitude of the horizontal force  $\mathbf{P}$ , (b) the corresponding distance  $a$ .

- 7.102** Cable  $ABC$  supports two loads as shown. Determine the distances  $a$  and  $b$  when a horizontal force  $\mathbf{P}$  of magnitude 60 lb is applied at  $A$ .

- 7.103** Knowing that  $m_B = 70$  kg and  $m_C = 25$  kg, determine the magnitude of the force  $\mathbf{P}$  required to maintain equilibrium.

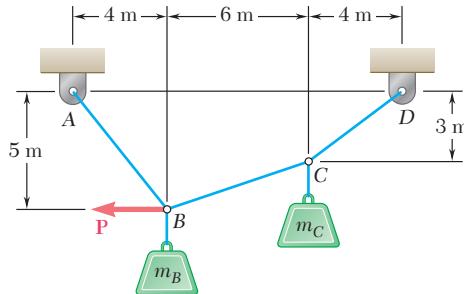


Fig. P7.103 and P7.104

- 7.104** Knowing that  $m_B = 18$  kg and  $m_C = 10$  kg, determine the magnitude of the force  $\mathbf{P}$  required to maintain equilibrium.

- 7.105** If  $a = 3$  m, determine the magnitudes of  $\mathbf{P}$  and  $\mathbf{Q}$  required to maintain the cable in the shape shown.

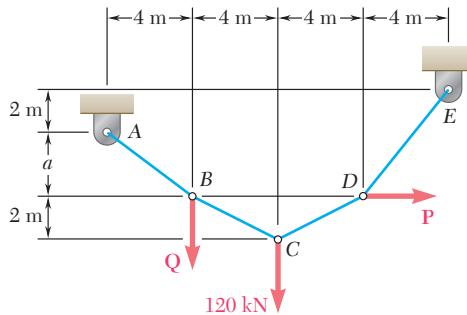


Fig. P7.105 and P7.106

- 7.106** If  $a = 4$  m, determine the magnitudes of  $\mathbf{P}$  and  $\mathbf{Q}$  required to maintain the cable in the shape shown.

- 7.107** A wire having a mass per unit length of 0.65 kg/m is suspended from two supports at the same elevation that are 120 m apart. If the sag is 30 m, determine (a) the total length of the wire, (b) the maximum tension in the wire.

- 7.108** Two cables of the same gauge are attached to a transmission tower at  $B$ . Since the tower is slender, the horizontal component of the resultant of the forces exerted by the cables at  $B$  is to be zero. Knowing that the mass per unit length of the cables is 0.4 kg/m, determine (a) the required sag  $h$ , (b) the maximum tension in each cable.

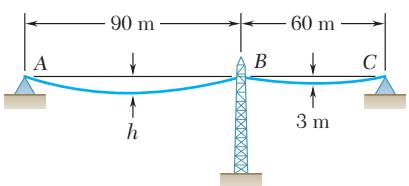


Fig. P7.108

- 7.109** Each cable of the Golden Gate Bridge supports a load  $w = 11.1$  kips/ft along the horizontal. Knowing that the span  $L$  is 4150 ft and that the sag  $h$  is 464 ft, determine (a) the maximum tension in each cable, (b) the length of each cable.

- 7.110** The center span of the George Washington Bridge, as originally constructed, consisted of a uniform roadway suspended from four cables. The uniform load supported by each cable was  $w = 9.75$  kips/ft along the horizontal. Knowing that the span  $L$  is 3500 ft and that the sag  $h$  is 316 ft, determine for the original configuration (a) the maximum tension in each cable, (b) the length of each cable.

- 7.111** The total mass of cable  $AC$  is 25 kg. Assuming that the mass of the cable is distributed uniformly along the horizontal, determine the sag  $h$  and the slope of the cable at  $A$  and  $C$ .

- 7.112** A 50.5-m length of wire having a mass per unit length of 0.75 kg/m is used to span a horizontal distance of 50 m. Determine (a) the approximate sag of the wire, (b) the maximum tension in the wire. [Hint: Use only the first two terms of Eq. (7.10).]

- 7.113** A cable of length  $L + \Delta$  is suspended between two points that are at the same elevation and a distance  $L$  apart. (a) Assuming that  $\Delta$  is small compared to  $L$  and that the cable is parabolic, determine the approximate sag in terms of  $L$  and  $\Delta$ . (b) If  $L = 100$  ft and  $\Delta = 4$  ft, determine the approximate sag. [Hint: Use only the first two terms of Eq. (7.10).]

- 7.114** The center span of the Verrazano-Narrows Bridge consists of two uniform roadways suspended from four cables. The design of the bridge allows for the effect of extreme temperature changes that cause the sag of the center span to vary from  $h_w = 386$  ft in winter to  $h_s = 394$  ft in summer. Knowing that the span is  $L = 4260$  ft, determine the change in length of the cables due to extreme temperature changes.

- 7.115** Each cable of the side spans of the Golden Gate Bridge supports a load  $w = 10.2$  kips/ft along the horizontal. Knowing that for the side spans the maximum vertical distance  $h$  from each cable to the chord  $AB$  is 30 ft and occurs at midspan, determine (a) the maximum tension in each cable, (b) the slope at  $B$ .

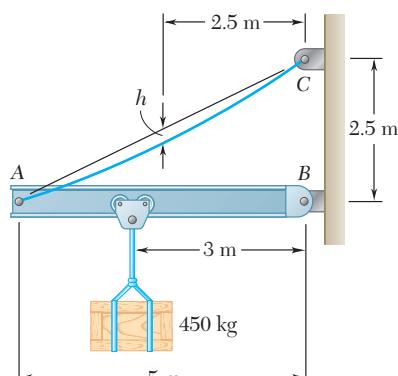


Fig. P7.111

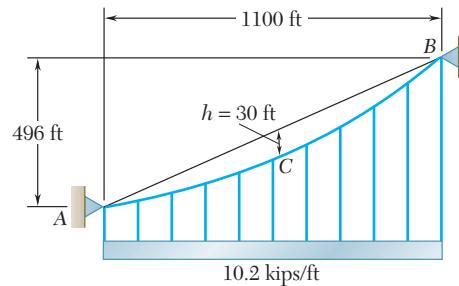
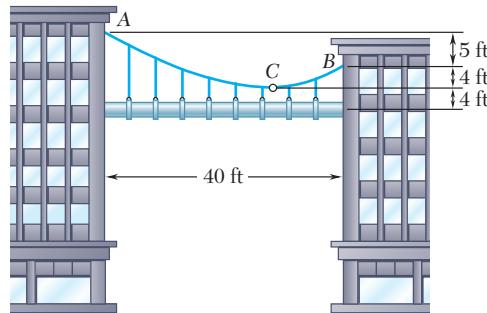
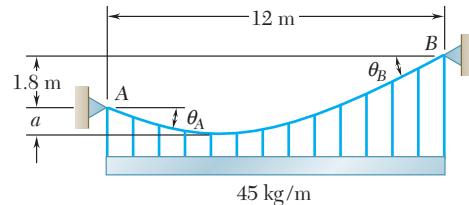
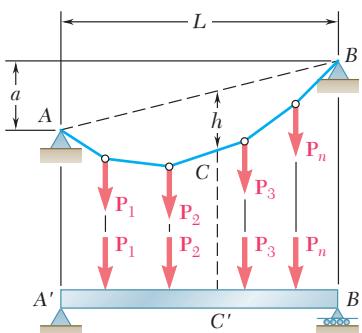


Fig. P7.115

- 7.116** A steam pipe weighing 45 lb/ft that passes between two buildings 40 ft apart is supported by a system of cables as shown. Assuming that the weight of the cable system is equivalent to a uniformly distributed loading of 5 lb/ft, determine (a) the location of the lowest point  $C$  of the cable, (b) the maximum tension in the cable.

**Fig. P7.116**

- 7.117** Cable  $AB$  supports a load uniformly distributed along the horizontal as shown. Knowing that at  $B$  the cable forms an angle  $\theta_B = 35^\circ$  with the horizontal, determine (a) the maximum tension in the cable, (b) the vertical distance  $a$  from  $A$  to the lowest point of the cable.

**Fig. P7.117 and P7.118****Fig. P7.119**

- 7.118** Cable  $AB$  supports a load uniformly distributed along the horizontal as shown. Knowing that the lowest point of the cable is located at a distance  $a = 0.6$  m below  $A$ , determine (a) the maximum tension in the cable, (b) the angle  $\theta_B$  that the cable forms with the horizontal at  $B$ .

- \*7.119** A cable  $AB$  of span  $L$  and a simple beam  $A'B'$  of the same span are subjected to identical vertical loadings as shown. Show that the magnitude of the bending moment at a point  $C'$  in the beam is equal to the product  $T_0 h$ , where  $T_0$  is the magnitude of the horizontal component of the tension force in the cable and  $h$  is the vertical distance between point  $C$  and the chord joining the points of support  $A$  and  $B$ .

- 7.120 through 7.123** Making use of the property established in Prob. 7.119, solve the problem indicated by first solving the corresponding beam problem.

**7.120** Prob. 7.94a.

**7.121** Prob. 7.97a.

**7.122** Prob. 7.99a.

**7.123** Prob. 7.100a.

**\*7.124** Show that the curve assumed by a cable that carries a distributed load  $w(x)$  is defined by the differential equation  $d^2y/dx^2 = w(x)/T_0$ , where  $T_0$  is the tension at the lowest point.

**\*7.125** Using the property indicated in Prob. 7.124, determine the curve assumed by a cable of span  $L$  and sag  $h$  carrying a distributed load  $w = w_0 \cos(\pi x/L)$ , where  $x$  is measured from mid-span. Also determine the maximum and minimum values of the tension in the cable.

**\*7.126** If the weight per unit length of the cable  $AB$  is  $w_0/\cos^2 \theta$ , prove that the curve formed by the cable is a circular arc. (*Hint:* Use the property indicated in Prob. 7.124.)

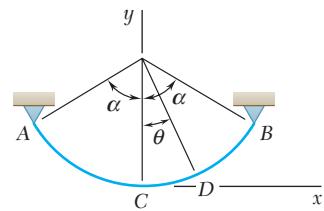


Fig. P7.126

## 7.10 CATENARY

Let us now consider a cable  $AB$  carrying a load *uniformly distributed along the cable itself* (Fig. 7.18a). Cables hanging under their own weight are loaded in this way. We denote by  $w$  the load per unit length (*measured along the cable*) and express it in N/m or in lb/ft. The magnitude  $W$  of the total load carried by a portion of cable of length  $s$  extending from the lowest point  $C$  to a point  $D$  is  $W = ws$ . Substituting this value for  $W$  in formula (7.6), we obtain the tension at  $D$ :

$$T = \sqrt{T_0^2 + w^2 s^2}$$

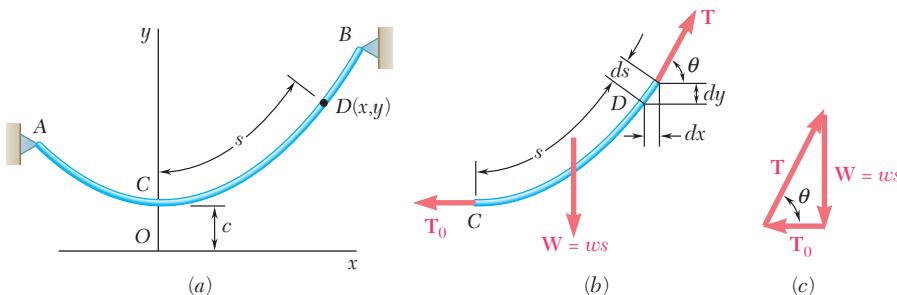


Fig. 7.18

In order to simplify the subsequent computations, we introduce the constant  $c = T_0/w$ . We thus write

$$T_0 = wc \quad W = ws \quad T = w\sqrt{c^2 + s^2} \quad (7.11)$$

The free-body diagram of the portion of cable  $CD$  is shown in Fig. 7.18b. This diagram, however, cannot be used to obtain directly the equation of the curve assumed by the cable, since we do not know the horizontal distance from  $D$  to the line of action of the resultant  $\mathbf{W}$  of the load. To obtain this equation, we first write that the horizontal projection of a small element of cable of length  $ds$  is



**Photo 7.4** The forces on the supports and the internal forces in the cables of the power line shown are discussed in this section.

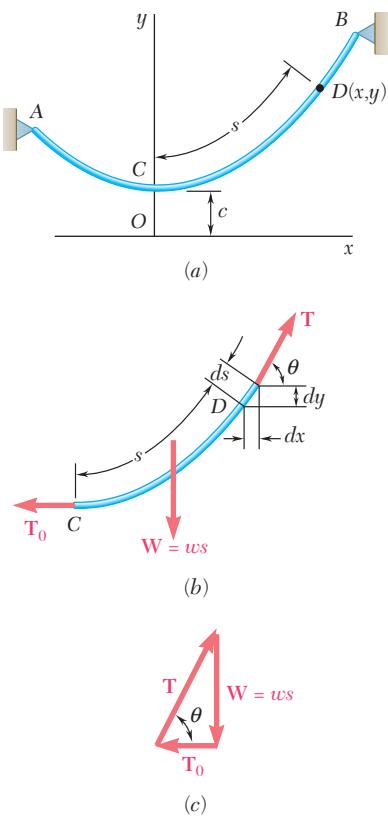


Fig. 7.18 (continued)

$dx = ds \cos \theta$ . Observing from Fig. 7.18c that  $\cos \theta = T_0/T$  and using (7.11), we write

$$dx = ds \cos \theta = \frac{T_0}{T} ds = \frac{wc ds}{w\sqrt{c^2 + s^2}} = \frac{ds}{\sqrt{1 + s^2/c^2}}$$

Selecting the origin  $O$  of the coordinates at a distance  $c$  directly below  $C$  (Fig. 7.18a) and integrating from  $C(0, c)$  to  $D(x, y)$ , we obtain†

$$x = \int_0^s \frac{ds}{\sqrt{1 + s^2/c^2}} = c \left[ \sinh^{-1} \frac{s}{c} \right]_0^s = c \sinh^{-1} \frac{s}{c}$$

This equation, which relates the length  $s$  of the portion of cable  $CD$  and the horizontal distance  $x$ , can be written in the form

$$s = c \sinh \frac{x}{c} \quad (7.15)$$

The relation between the coordinates  $x$  and  $y$  can now be obtained by writing  $dy = dx \tan \theta$ . Observing from Fig. 7.18c that  $\tan \theta = W/T_0$  and using (7.11) and (7.15), we write

$$dy = dx \tan \theta = \frac{W}{T_0} dx = \frac{s}{c} dx = \sinh \frac{x}{c} dx$$

Integrating from  $C(0, c)$  to  $D(x, y)$  and using (7.12) and (7.13), we obtain

$$\begin{aligned} y - c &= \int_0^x \sinh \frac{x}{c} dx = c \left[ \cosh \frac{x}{c} \right]_0^x = c \left( \cosh \frac{x}{c} - 1 \right) \\ y - c &= c \cosh \frac{x}{c} - c \end{aligned}$$

†This integral can be found in all standard integral tables. The function

$$z = \sinh^{-1} u$$

(read “arc hyperbolic sine  $u$ ”) is the *inverse* of the function  $u = \sinh z$  (read “hyperbolic sine  $z$ ”). This function and the function  $v = \cosh z$  (read “hyperbolic cosine  $z$ ”) are defined as follows:

$$u = \sinh z = \frac{1}{2}(e^z - e^{-z}) \quad v = \cosh z = \frac{1}{2}(e^z + e^{-z})$$

Numerical values of the functions  $\sinh z$  and  $\cosh z$  are found in *tables of hyperbolic functions*. They may also be computed on most calculators either directly or from the above definitions. The student is referred to any calculus text for a complete description of the properties of these functions. In this section, we use only the following properties, which are easily derived from the above definitions:

$$\frac{d \sinh z}{dz} = \cosh z \quad \frac{d \cosh z}{dz} = \sinh z \quad (7.12)$$

$$\sinh 0 = 0 \quad \cosh 0 = 1 \quad (7.13)$$

$$\cosh^2 z - \sinh^2 z = 1 \quad (7.14)$$

which reduces to

$$y = c \cosh \frac{x}{c} \quad (7.16)$$

This is the equation of a *catenary* with vertical axis. The ordinate  $c$  of the lowest point  $C$  is called the *parameter* of the catenary. Squaring both sides of Eqs. (7.15) and (7.16), subtracting, and taking (7.14) into account, we obtain the following relation between  $y$  and  $s$ :

$$y^2 - s^2 = c^2 \quad (7.17)$$

Solving (7.17) for  $s^2$  and carrying into the last of the relations (7.11), we write these relations as follows:

$$T_0 = wc \quad W = ws \quad T = wy \quad (7.18)$$

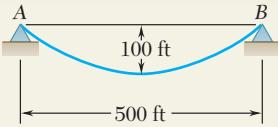
The last relation indicates that the tension at any point  $D$  of the cable is proportional to the vertical distance from  $D$  to the horizontal line representing the  $x$  axis.

When the supports  $A$  and  $B$  of the cable have the same elevation, the distance  $L$  between the supports is called the *span* of the cable and the vertical distance  $h$  from the supports to the lowest point  $C$  is called the *sag* of the cable. These definitions are the same as those given in the case of parabolic cables, but it should be noted that because of our choice of coordinate axes, the sag  $h$  is now

$$h = y_A - c \quad (7.19)$$

It should also be observed that certain catenary problems involve transcendental equations which must be solved by successive approximations (see Sample Prob. 7.10). When the cable is fairly taut, however, the load can be assumed uniformly distributed *along the horizontal* and the catenary can be replaced by a parabola. This greatly simplifies the solution of the problem, and the error introduced is small.

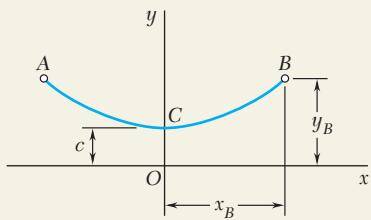
When the supports  $A$  and  $B$  have different elevations, the position of the lowest point of the cable is not known. The problem can then be solved in a manner similar to that indicated for parabolic cables, by expressing that the cable must pass through the supports and that  $x_B - x_A = L$  and  $y_B - y_A = d$ , where  $L$  and  $d$  denote, respectively, the horizontal and vertical distances between the two supports.



## SAMPLE PROBLEM 7.10

A uniform cable weighing 3 lb/ft is suspended between two points A and B as shown. Determine (a) the maximum and minimum values of the tension in the cable, (b) the length of the cable.

## SOLUTION



**Equation of Cable.** The origin of coordinates is placed at a distance  $c$  below the lowest point of the cable. The equation of the cable is given by Eq. (7.16),

$$y = c \cosh \frac{x}{c}$$

The coordinates of point B are

$$x_B = 250 \text{ ft} \quad y_B = 100 + c$$

Substituting these coordinates into the equation of the cable, we obtain

$$\begin{aligned} 100 + c &= c \cosh \frac{250}{c} \\ \frac{100}{c} + 1 &= \cosh \frac{250}{c} \end{aligned}$$

The value of  $c$  is determined by assuming successive trial values, as shown in the following table:

$c$	$\frac{250}{c}$	$\frac{100}{c}$	$\frac{100}{c} + 1$	$\cosh \frac{250}{c}$
300	0.833	0.333	1.333	1.367
350	0.714	0.286	1.286	1.266
330	0.758	0.303	1.303	1.301
328	0.762	0.305	1.305	1.305

Taking  $c = 328$ , we have

$$y_B = 100 + c = 428 \text{ ft}$$

**a. Maximum and Minimum Values of the Tension.** Using Eqs. (7.18), we obtain

$$\begin{aligned} T_{\min} &= T_0 = wc = (3 \text{ lb/ft})(328 \text{ ft}) & T_{\min} &= 984 \text{ lb} \\ T_{\max} &= T_B = wy_B = (3 \text{ lb/ft})(428 \text{ ft}) & T_{\max} &= 1284 \text{ lb} \end{aligned}$$

**b. Length of Cable.** One-half the length of the cable is found by solving Eq. (7.17):

$$y_B^2 - s_{CB}^2 = c^2 \quad s_{CB}^2 = y_B^2 - c^2 = (428)^2 - (328)^2 \quad s_{CB} = 275 \text{ ft}$$

The total length of the cable is therefore

$$s_{AB} = 2s_{CB} = 2(275 \text{ ft}) \quad s_{AB} = 550 \text{ ft}$$

# SOLVING PROBLEMS ON YOUR OWN

In the last section of this chapter you learned to solve problems involving a *cable carrying a load uniformly distributed along the cable*. The shape assumed by the cable is a catenary and is defined by the equation:

$$y = c \cosh \frac{x}{c} \quad (7.16)$$

**1. You should keep in mind that the origin of coordinates for a catenary is located at a distance  $c$  directly below the lowest point of the catenary.** The length of the cable from the origin to any point is expressed as

$$s = c \sinh \frac{x}{c} \quad (7.15)$$

**2. You should first identify all of the known and unknown quantities.** Then consider each of the equations listed in the text (Eqs. 7.15 through 7.19), and solve an equation that contains only one unknown. Substitute the value found into another equation, and solve that equation for another unknown.

**3. If the sag  $h$  is given,** use Eq. (7.19) to replace  $y$  by  $h + c$  in Eq. (7.16) if  $x$  is known [Sample Prob. 7.10], or in Eq. (7.17) if  $s$  is known, and solve the equation obtained for the constant  $c$ .

**4. Many of the problems that you will encounter will involve the solution by trial and error** of an equation involving a hyperbolic sine or cosine. You can make your work easier by keeping track of your calculations in a table, as in Sample Prob. 7.10, or by applying a numerical methods approach using a computer or calculator.

# PROBLEMS

- 7.127** A 30-m cable is strung as shown between two buildings. The maximum tension is found to be 500 N, and the lowest point of the cable is observed to be 4 m above the ground. Determine (a) the horizontal distance between the buildings, (b) the total mass of the cable.

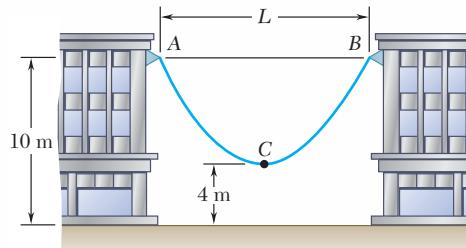


Fig. P7.127

- 7.128** A 200-ft steel surveying tape weighs 4 lb. If the tape is stretched between two points at the same elevation and pulled until the tension at each end is 16 lb, determine the horizontal distance between the ends of the tape. Neglect the elongation of the tape due to the tension.

- 7.129** A 200-m-long aerial tramway cable having a mass per unit length of 3.5 kg/m is suspended between two points at the same elevation. Knowing that the sag is 50 m, find (a) the horizontal distance between the supports, (b) the maximum tension in the cable.

- 7.130** An electric transmission cable of length 400 ft weighing 2.5 lb/ft is suspended between two points at the same elevation. Knowing that the sag is 100 ft, determine the horizontal distance between the supports and the maximum tension.

- 7.131** A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at A and to a collar at B. Neglecting the effect of friction, determine (a) the force  $\mathbf{P}$  for which  $h = 8 \text{ m}$ , (b) the corresponding span  $L$ .

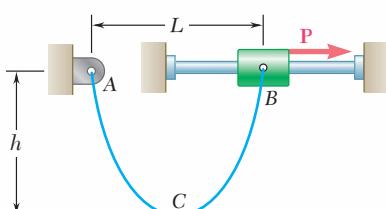


Fig. P7.131, P7.132, and P7.133

- 7.132** A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at A and to a collar at B. Knowing that the magnitude of the horizontal force applied to the collar is  $P = 20 \text{ N}$ , determine (a) the sag  $h$ , (b) the span  $L$ .

- 7.133** A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at A and to a collar at B. Neglecting the effect of friction, determine (a) the sag  $h$  for which  $L = 15 \text{ m}$ , (b) the corresponding force  $\mathbf{P}$ .

- 7.134** Determine the sag of a 30-ft chain that is attached to two points at the same elevation that are 20 ft apart.

- 7.135** A 90-m wire is suspended between two points at the same elevation that are 60 m apart. Knowing that the maximum tension is 300 N, determine (a) the sag of the wire, (b) the total mass of the wire.

- 7.136** A counterweight  $D$  is attached to a cable that passes over a small pulley at  $A$  and is attached to a support at  $B$ . Knowing that  $L = 45$  ft and  $h = 15$  ft, determine (a) the length of the cable from  $A$  to  $B$ , (b) the weight per unit length of the cable. Neglect the weight of the cable from  $A$  to  $D$ .

- 7.137** A uniform cord 50 in. long passes over a pulley at  $B$  and is attached to a pin support at  $A$ . Knowing that  $L = 20$  in. and neglecting the effect of friction, determine the smaller of the two values of  $h$  for which the cord is in equilibrium.

- 7.138** A cable weighing 2 lb/ft is suspended between two points at the same elevation that are 160 ft apart. Determine the smallest allowable sag of the cable if the maximum tension is not to exceed 400 lb.

- 7.139** A motor  $M$  is used to slowly reel in the cable shown. Knowing that the mass per unit length of the cable is 0.4 kg/m, determine the maximum tension in the cable when  $h = 5$  m.

- 7.140** A motor  $M$  is used to slowly reel in the cable shown. Knowing that the mass per unit length of the cable is 0.4 kg/m, determine the maximum tension in the cable when  $h = 3$  m.

- 7.141** A uniform cable weighing 3 lb/ft is held in the position shown by a horizontal force  $\mathbf{P}$  applied at  $B$ . Knowing that  $P = 180$  lb and  $\theta_A = 60^\circ$ , determine (a) the location of point  $B$ , (b) the length of the cable.

- 7.142** A uniform cable weighing 3 lb/ft is held in the position shown by a horizontal force  $\mathbf{P}$  applied at  $B$ . Knowing that  $P = 150$  lb and  $\theta_A = 60^\circ$ , determine (a) the location of point  $B$ , (b) the length of the cable.

- 7.143** To the left of point  $B$  the long cable  $ABDE$  rests on the rough horizontal surface shown. Knowing that the mass per unit length of the cable is 2 kg/m, determine the force  $\mathbf{F}$  when  $a = 3.6$  m.

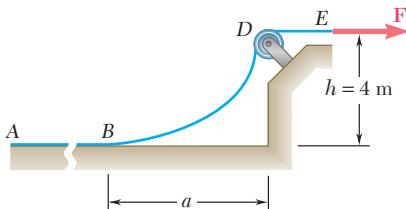


Fig. P7.143 and P7.144

- 7.144** To the left of point  $B$  the long cable  $ABDE$  rests on the rough horizontal surface shown. Knowing that the mass per unit length of the cable is 2 kg/m, determine the force  $\mathbf{F}$  when  $a = 6$  m.

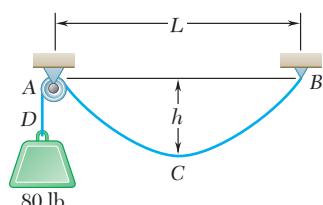


Fig. P7.136

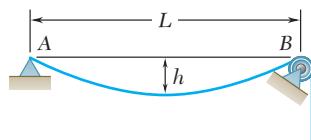


Fig. P7.137

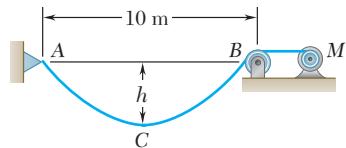


Fig. P7.139 and P7.140

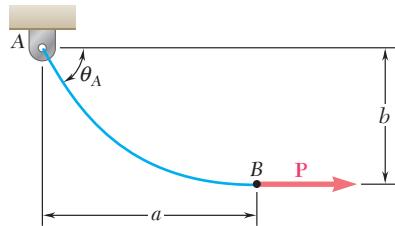


Fig. P7.141 and P7.142

- 7.145** The cable  $ACB$  has a mass per unit length of  $0.45 \text{ kg/m}$ . Knowing that the lowest point of the cable is located at a distance  $a = 0.6 \text{ m}$  below the support  $A$ , determine (a) the location of the lowest point  $C$ , (b) the maximum tension in the cable.

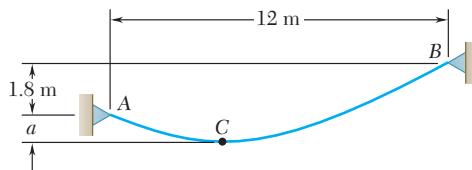


Fig. P7.145 and P7.146

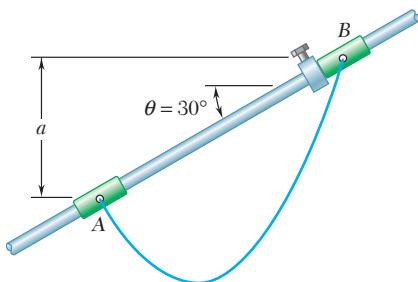


Fig. P7.147

- 7.146** The cable  $ACB$  has a mass per unit length of  $0.45 \text{ kg/m}$ . Knowing that the lowest point of the cable is located at a distance  $a = 2 \text{ m}$  below the support  $A$ , determine (a) the location of the lowest point  $C$ , (b) the maximum tension in the cable.

- \*7.147** The 10-ft cable  $AB$  is attached to two collars as shown. The collar at  $A$  can slide freely along the rod; a stop attached to the rod prevents the collar at  $B$  from moving on the rod. Neglecting the effect of friction and the weight of the collars, determine the distance  $a$ .

- \*7.148** Solve Prob. 7.147 assuming that the angle  $\theta$  formed by the rod and the horizontal is  $45^\circ$ .

- 7.149** Denoting by  $\theta$  the angle formed by a uniform cable and the horizontal, show that at any point (a)  $s = c \tan \theta$ , (b)  $y = c \sec \theta$ .

- \*7.150** (a) Determine the maximum allowable horizontal span for a uniform cable of weight  $w$  per unit length if the tension in the cable is not to exceed a given value  $T_m$ . (b) Using the result of part a, determine the maximum span of a steel wire for which  $w = 0.25 \text{ lb/ft}$  and  $T_m = 8000 \text{ lb}$ .

- \*7.151** A cable has a mass per unit length of  $3 \text{ kg/m}$  and is supported as shown. Knowing that the span  $L$  is  $6 \text{ m}$ , determine the two values of the sag  $h$  for which the maximum tension is  $350 \text{ N}$ .

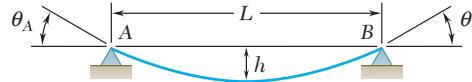


Fig. P7.151, P7.152, and P7.153

- \*7.152** Determine the sag-to-span ratio for which the maximum tension in the cable is equal to the total weight of the entire cable  $AB$ .

- \*7.153** A cable of weight  $w$  per unit length is suspended between two points at the same elevation that are a distance  $L$  apart. Determine (a) the sag-to-span ratio for which the maximum tension is as small as possible, (b) the corresponding values of  $\theta_B$  and  $T_m$ .

# REVIEW AND SUMMARY

In this chapter you learned to determine the internal forces which hold together the various parts of a given member in a structure.

Considering first a *straight two-force member AB* [Sec. 7.2], we recall that such a member is subjected at A and B to equal and opposite forces  $\mathbf{F}$  and  $-\mathbf{F}$  directed along AB (Fig. 7.19a). Cutting member AB at C and drawing the free-body diagram of portion AC, we conclude that the internal forces which existed at C in member AB are equivalent to an *axial force*  $-\mathbf{F}$  equal and opposite to  $\mathbf{F}$  (Fig. 7.19b). We note that in the case of a two-force member which is not straight, the internal forces reduce to a force-couple system and not to a single force.

## Forces in straight two-force members

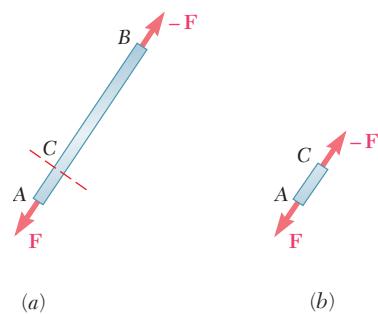


Fig. 7.19

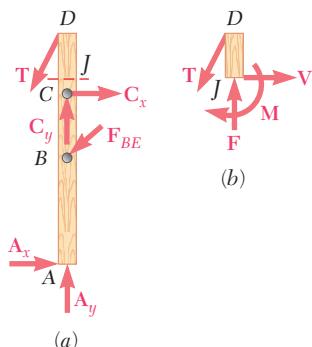


Fig. 7.20

Considering next a *multiforce member AD* (Fig. 7.20a), cutting it at J, and drawing the free-body diagram of portion JD, we conclude that the internal forces at J are equivalent to a force-couple system consisting of the *axial force*  $\mathbf{F}$ , the *shearing force*  $\mathbf{V}$ , and a couple  $\mathbf{M}$  (Fig. 7.20b). The magnitude of the shearing force measures the *shear* at point J, and the moment of the couple is referred to as the *bending moment* at J. Since an equal and opposite force-couple system would have been obtained by considering the free-body diagram of portion AJ, it is necessary to specify which portion of member AD was used when recording the answers [Sample Prob. 7.1].

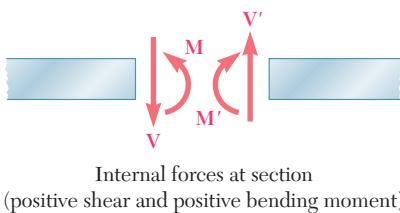
## Forces in multiforce members

Most of the chapter was devoted to the analysis of the internal forces in two important types of engineering structures: *beams* and *cables*. *Beams* are usually long, straight prismatic members designed to support loads applied at various points along the member. In general the loads are perpendicular to the axis of the beam and produce only *shear and bending* in the beam. The loads may be either *concentrated*

## Forces in beams

at specific points, or *distributed* along the entire length or a portion of the beam. The beam itself may be supported in various ways; since only statically determinate beams are considered in this text, we limited our analysis to that of *simply supported beams*, *overhanging beams*, and *cantilever beams* [Sec. 7.3].

### Shear and bending moment in a beam



**Fig. 7.21**

To obtain the *shear*  $V$  and *bending moment*  $M$  at a given point  $C$  of a beam, we first determine the reactions at the supports by considering the entire beam as a free body. We then cut the beam at  $C$  and use the free-body diagram of one of the two portions obtained in this fashion to determine  $V$  and  $M$ . In order to avoid any confusion regarding the sense of the shearing force  $\mathbf{V}$  and couple  $\mathbf{M}$  (which act in opposite directions on the two portions of the beam), the sign convention illustrated in Fig. 7.21 was adopted [Sec. 7.4]. Once the values of the shear and bending moment have been determined at a few selected points of the beam, it is usually possible to draw a *shear diagram* and a *bending-moment diagram* representing, respectively, the shear and bending moment at any point of the beam [Sec. 7.5]. When a beam is subjected to concentrated loads only, the shear is of constant value between loads and the bending moment varies linearly between loads [Sample Prob. 7.2]. On the other hand, when a beam is subjected to distributed loads, the shear and bending moment vary quite differently [Sample Prob. 7.3].

The construction of the shear and bending-moment diagrams is facilitated if the following relations are taken into account. Denoting by  $w$  the distributed load per unit length (assumed positive if directed downward), we have [Sec. 7.5]:

$$\frac{dV}{dx} = -w \quad (7.1)$$

$$\frac{dM}{dx} = V \quad (7.3)$$

or, in integrated form,

$$V_D - V_C = -( \text{area under load curve between } C \text{ and } D ) \quad (7.2')$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \quad (7.4')$$

Equation (7.2') makes it possible to draw the shear diagram of a beam from the curve representing the distributed load on that beam and the value of  $V$  at one end of the beam. Similarly, Eq. (7.4') makes it possible to draw the bending-moment diagram from the shear diagram and the value of  $M$  at one end of the beam. However, concentrated loads introduce discontinuities in the shear diagram and concentrated couples in the bending-moment diagram, none of which are accounted for in these equations [Sample Probs. 7.4 and 7.7]. Finally, we note from Eq. (7.3) that the points of the beam where the bending moment is maximum or minimum are also the points where the shear is zero [Sample Prob. 7.5].

### Relations among load, shear, and bending moment

### Cables with concentrated loads

The second half of the chapter was devoted to the analysis of *flexible cables*. We first considered a cable of negligible weight supporting *concentrated loads* [Sec. 7.7]. Using the entire cable  $AB$  as a free

body (Fig. 7.22), we noted that the three available equilibrium equations were not sufficient to determine the four unknowns representing the reactions at the supports *A* and *B*. However, if the coordinates of a point *D* of the cable are known, an additional equation can be obtained by considering the free-body diagram of the portion *AD* or *DB* of the cable. Once the reactions at the supports have been determined, the elevation of any point of the cable and the tension in any portion of the cable can be found from the appropriate free-body diagram [Sample Prob. 7.8]. It was noted that the horizontal component of the force **T** representing the tension is the same at any point of the cable.

We next considered cables carrying *distributed loads* [Sec. 7.8]. Using as a free body a portion of cable *CD* extending from the lowest point *C* to an arbitrary point *D* of the cable (Fig. 7.23), we observed that the horizontal component of the tension force **T** at *D* is constant and equal to the tension  $T_0$  at *C*, while its vertical component is equal to the weight *W* of the portion of cable *CD*. The magnitude and direction of **T** were obtained from the force triangle:

$$T = \sqrt{T_0^2 + W^2} \quad \tan \theta = \frac{W}{T_0} \quad (7.6)$$

In the case of a load *uniformly distributed along the horizontal*—as in a suspension bridge (Fig. 7.24)—the load supported by portion *CD* is  $W = wx$ , where *w* is the constant load per unit horizontal length [Sec. 7.9]. We also found that the curve formed by the cable is a *parabola* of equation

$$y = \frac{wx^2}{2T_0} \quad (7.8)$$

and that the length of the cable can be found by using the expansion in series given in Eq. (7.10) [Sample Prob. 7.9].

In the case of a load *uniformly distributed along the cable itself*—e.g., a cable hanging under its own weight (Fig. 7.25)—the load supported by portion *CD* is  $W = ws$ , where *s* is the length measured along the cable and *w* is the constant load per unit length [Sec. 7.10]. Choosing the origin *O* of the coordinate axes at a distance  $c = T_0/w$  below *C*, we derived the relations

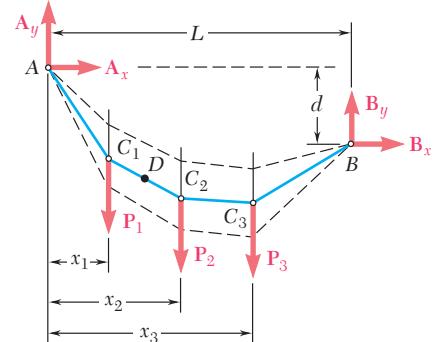
$$s = c \sinh \frac{x}{c} \quad (7.15)$$

$$y = c \cosh \frac{x}{c} \quad (7.16)$$

$$y^2 - s^2 = c^2 \quad (7.17)$$

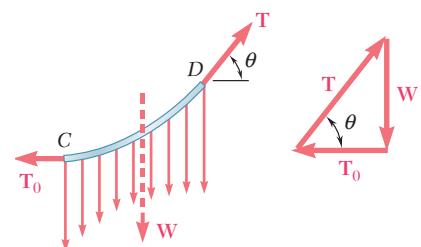
$$T_0 = wc \quad W = ws \quad T = wy \quad (7.18)$$

which can be used to solve problems involving cables hanging under their own weight [Sample Prob. 7.10]. Equation (7.16), which defines the shape of the cable, is the equation of a *catenary*.



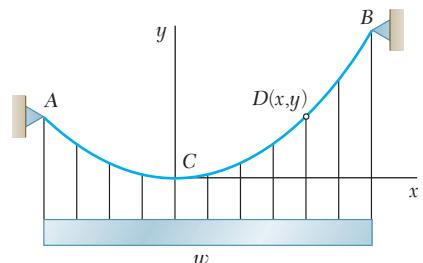
**Fig. 7.22**

### Cables with distributed loads



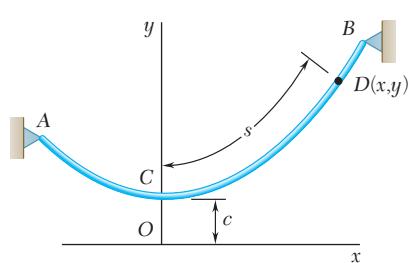
**Fig. 7.23**

### Parabolic cable



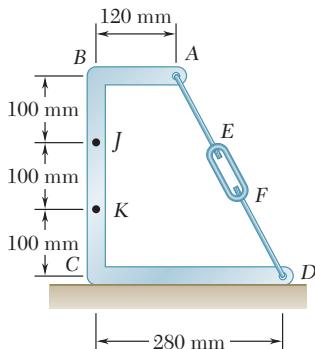
**Fig. 7.24**

### Catenary



**Fig. 7.25**

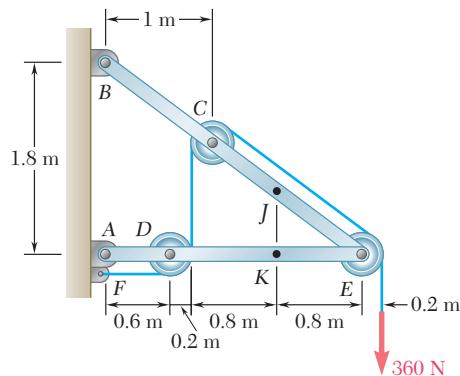
# REVIEW PROBLEMS



**Fig. P7.154**

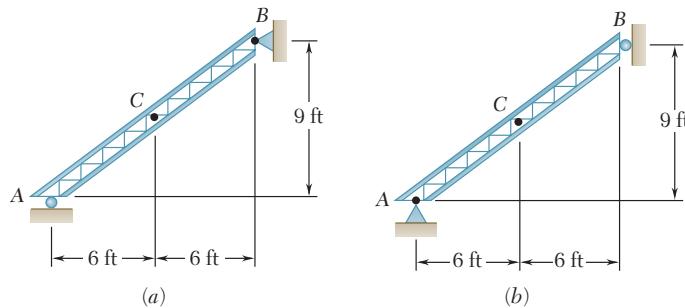
- 7.154** It has been experimentally determined that the bending moment at point *K* of the frame shown is  $300 \text{ N} \cdot \text{m}$ . Determine (a) the tension in rods *AE* and *FD*, (b) the corresponding internal forces at point *J*.

- 7.155** Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at point *J* of the frame shown.

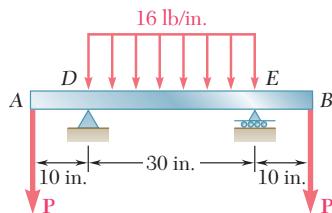


**Fig. P7.155**

- 7.156** A steel channel of weight per unit length  $w = 20 \text{ lb/ft}$  forms one side of a flight of stairs. Determine the internal forces at the center *C* of the channel due to its own weight for each of the support conditions shown.



**Fig. P7.156**



**Fig. P7.157 and P7.158**

- 7.157** For the beam shown, determine (a) the magnitude *P* of the two concentrated loads for which the maximum absolute value of the bending moment is as small as possible, (b) the corresponding value of  $|M|_{\max}$ .

- 7.158** Knowing that the magnitude of the concentrated loads *P* is 75 lb, (a) draw the shear and bending-moment diagrams for beam *AB*, (b) determine the maximum absolute values of the shear and bending moment.

- 7.159** For the beam and loading shown, (a) draw the shear and bending-moment diagrams, (b) determine the maximum absolute values of the shear and bending moment.

- 7.160** For the beam shown, draw the shear and bending-moment diagrams, and determine the magnitude and location of the maximum absolute value of the bending moment, knowing that (a)  $P = 6$  kips, (b)  $P = 3$  kips.

- 7.161** For the beam and loading shown, (a) write the equations of the shear and bending-moment curves, (b) determine the maximum bending moment.

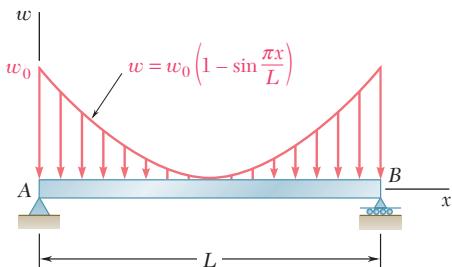


Fig. P7.161

- 7.162** An oil pipeline is supported at 6-ft intervals by vertical hangers attached to the cable shown. Due to the combined weight of the pipe and its contents the tension in each hanger is 400 lb. Knowing that  $d_C = 12$  ft, determine (a) the maximum tension in the cable, (b) the distance  $d_D$ .

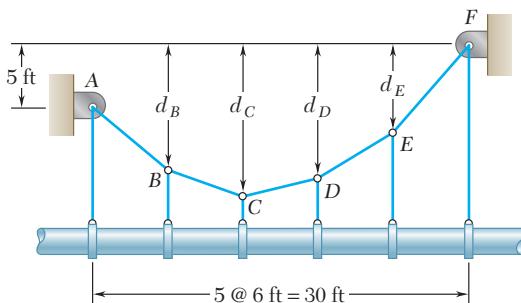


Fig. P7.162

- 7.163** Solve Prob. 7.162 assuming that  $d_C = 9$  ft.

- 7.164** A transmission cable having a mass per unit length of 0.8 kg/m is strung between two insulators at the same elevation that are 75 m apart. Knowing that the sag of the cable is 2 m, determine (a) the maximum tension in the cable, (b) the length of the cable.

- 7.165** Cable ACB supports a load uniformly distributed along the horizontal as shown. The lowest point C is located 9 m to the right of A. Determine (a) the vertical distance  $a$ , (b) the length of the cable, (c) the components of the reaction at A.

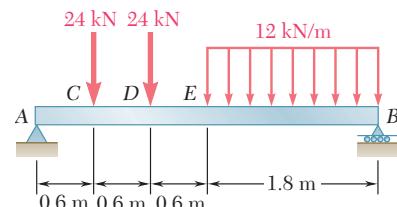


Fig. P7.159

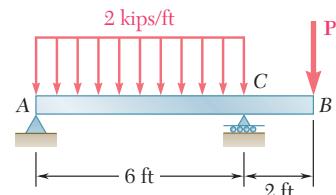


Fig. P7.160

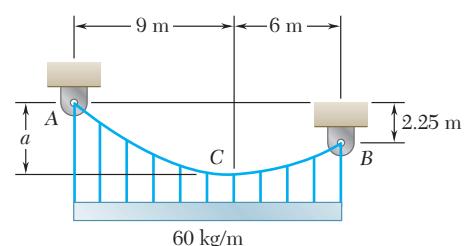
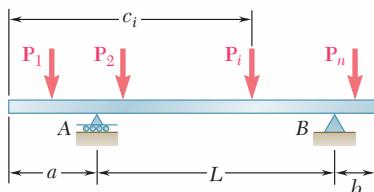


Fig. P7.165

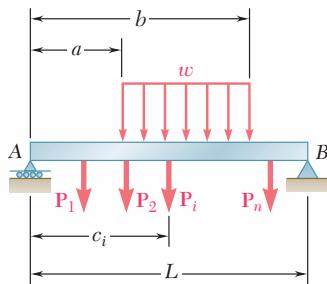
# COMPUTER PROBLEMS



**Fig. P7.C1**

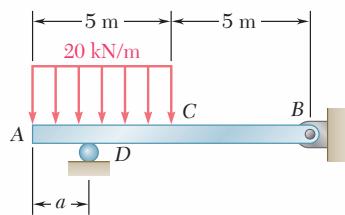
**7.C1** An overhanging beam is to be designed to support several concentrated loads. One of the first steps in the design of the beam is to determine the values of the bending moment that can be expected at the supports A and B and under each of the concentrated loads. Using computational software, calculate those values for the arbitrary beam and loading shown. Use this program for the beam and loading of (a) Prob. 7.36, (b) Prob. 7.37, (c) Prob. 7.38.

**7.C2** Several concentrated loads and a uniformly distributed load are to be applied to a simply supported beam AB. As a first step in the design of the beam, use computational software to calculate the shear and bending moment in the beam for the arbitrary loading shown using given increments  $\Delta x$ . Use this program for the beam of (a) Prob. 7.39, with  $\Delta x = 0.25$  m; (b) Prob. 7.41, with  $\Delta x = 0.5$  ft; (c) Prob. 7.42, with  $\Delta x = 0.5$  ft.



**Fig. P7.C2**

**7.C3** A beam AB hinged at B and supported by a roller at D is to be designed to carry a load uniformly distributed from its end A to its midpoint C with maximum efficiency. As part of the design process, use computational software to determine the distance  $a$  from end A to the point D where the roller should be placed to minimize the absolute value of the bending moment  $M$  in the beam. (Note: A short preliminary analysis will show that the roller should be placed under the load and that the largest negative value of  $M$  will occur at D, while its largest positive value will occur somewhere between D and C. Also see the hint for Prob. 7.55.)



**Fig. P7.C3**

**7.C4** The floor of a bridge will consist of narrow planks resting on two simply supported beams, one of which is shown in the figure. As part of the design of the bridge, it is desired to simulate the effect that driving a 3000-lb truck over the bridge will have on this beam. The distance between the truck's axles is 6 ft, and it is assumed that the weight of the truck is equally distributed over its four wheels. (a) Using computational software, calculate the magnitude and location of the maximum bending moment in the beam for values of  $x$  from  $-3$  ft to  $10$  ft using  $0.5$ -ft increments. (b) Using smaller increments if necessary, determine the largest value of the bending moment that occurs in the beam as the truck is driven over the bridge and determine the corresponding value of  $x$ .

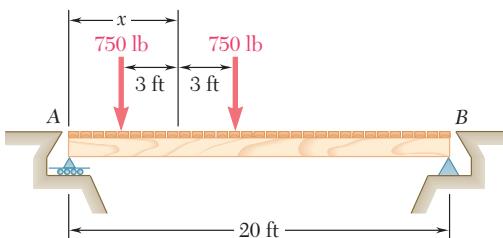


Fig. P7.C4

**\*7.C5** Using computational software, plot the shear and bending-moment diagrams for the beam of Prob. 7.C1. Using this program and a plotting increment  $\Delta x \leq L/100$ , plot the V and M diagrams for the beam and loading of (a) Prob. 7.36, (b) Prob. 7.37, (c) Prob. 7.38.

**\*7.C6** Using computational software, plot the shear and bending-moment diagrams for the beam of Prob. 7.C2. Using this program and a plotting increment  $\Delta x \leq L/100$ , plot the V and M diagrams for the beam and loading of (a) Prob. 7.39, (b) Prob. 7.41, (c) Prob. 7.42.

**7.C7** Using computational software, design cable supports to calculate the horizontal and vertical components of the reaction at the support  $A_n$  from values of the loads  $P_1, P_2, \dots, P_{n-1}$ , the horizontal distances  $d_1, d_2, \dots, d_n$ , and the two vertical distances  $h_0$  and  $h_k$ . Use this program to solve Probs. 7.95b, 7.96b, and 7.97b.

**7.C8** A typical transmission-line installation consists of a cable of length  $s_{AB}$  and weight  $w$  per unit length suspended as shown between two points at the same elevation. Using computational software, develop a table that can be used in the design of future installations. The table should present the dimensionless quantities  $h/L$ ,  $s_{AB}/L$ ,  $T_0/wL$ , and  $T_{max}/wL$  for values of  $c/L$  from 0.2 to 0.5 using 0.025 increments and from 0.5 to 4 using 0.5 increments.

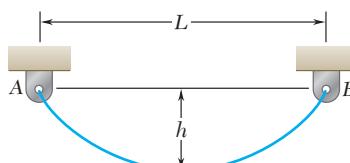


Fig. P7.C8

**7.C9** Using computational software, solve Prob. 7.132 for values of  $P$  from 0 to 50 N using 5-N increments.

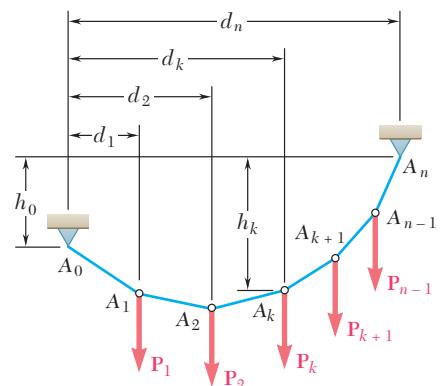


Fig. P7.C7

**The tractive force that a railroad locomotive can develop depends upon the frictional resistance between the drive wheels and the rails. When the potential exists for wheel slip to occur, such as when a train travels upgrade over wet rails, sand is deposited on top of the railhead to increase this friction.**





# CHAPTER

# 8

## Friction

## Chapter 8 Friction

- 8.1** Introduction
- 8.2** The Laws of Dry Friction.  
Coefficients of Friction
- 8.3** Angles of Friction
- 8.4** Problems Involving Dry Friction
- 8.5** Wedges
- 8.6** Square-Threaded Screws
- 8.7** Journal Bearings. Axle Friction
- 8.8** Thrust Bearings. Disk Friction
- 8.9** Wheel Friction. Rolling Resistance
- 8.10** Belt Friction

## 8.1 INTRODUCTION

In the preceding chapters, it was assumed that surfaces in contact were either *frictionless* or *rough*. If they were frictionless, the force each surface exerted on the other was normal to the surfaces and the two surfaces could move freely with respect to each other. If they were rough, it was assumed that tangential forces could develop to prevent the motion of one surface with respect to the other.

This view was a simplified one. Actually, no perfectly frictionless surface exists. When two surfaces are in contact, tangential forces, called *friction forces*, will always develop if one attempts to move one surface with respect to the other. On the other hand, these friction forces are limited in magnitude and will not prevent motion if sufficiently large forces are applied. The distinction between frictionless and rough surfaces is thus a matter of degree. This will be seen more clearly in the present chapter, which is devoted to the study of friction and of its applications to common engineering situations.

There are two types of friction: *dry friction*, sometimes called *Coulomb friction*, and *fluid friction*. Fluid friction develops between layers of fluid moving at different velocities. Fluid friction is of great importance in problems involving the flow of fluids through pipes and orifices or dealing with bodies immersed in moving fluids. It is also basic in the analysis of the motion of *lubricated mechanisms*. Such problems are considered in texts on fluid mechanics. The present study is limited to dry friction, i.e., to problems involving rigid bodies which are in contact along *nonlubricated* surfaces.

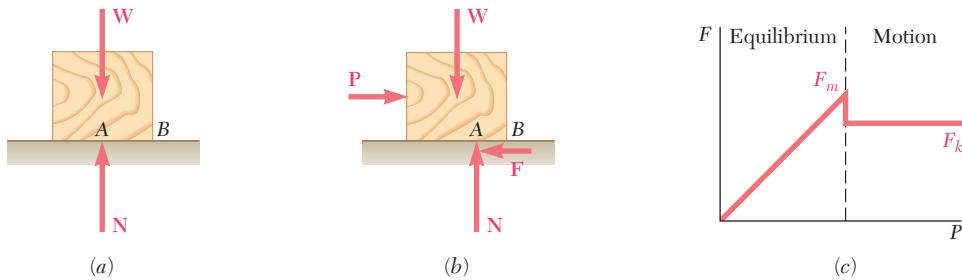
In the first part of this chapter, the equilibrium of various rigid bodies and structures, assuming dry friction at the surfaces of contact, is analyzed. Later a number of specific engineering applications where dry friction plays an important role are considered: wedges, square-threaded screws, journal bearings, thrust bearings, rolling resistance, and belt friction.

## 8.2 THE LAWS OF DRY FRICTION. COEFFICIENTS OF FRICTION

The laws of dry friction are exemplified by the following experiment. A block of weight **W** is placed on a horizontal plane surface (Fig. 8.1a). The forces acting on the block are its weight **W** and the reaction of the surface. Since the weight has no horizontal component, the reaction of the surface also has no horizontal component; the reaction is therefore *normal* to the surface and is represented by **N** in Fig. 8.1a. Suppose, now, that a horizontal force **P** is applied to the block (Fig. 8.1b). If **P** is small, the block will not move; some other horizontal force must therefore exist, which balances **P**. This other force is the *static-friction force* **F**, which is actually the resultant of a great number of forces acting over the entire surface of contact between the block and the plane. The nature of these forces is not known exactly, but it is generally assumed that these forces are due

to the irregularities of the surfaces in contact and, to a certain extent, to molecular attraction.

If the force  $\mathbf{P}$  is increased, the friction force  $\mathbf{F}$  also increases, continuing to oppose  $\mathbf{P}$ , until its magnitude reaches a certain *maximum value*  $F_m$  (Fig. 8.1c). If  $\mathbf{P}$  is further increased, the friction force



**Fig. 8.1**

cannot balance it any more and the block starts sliding.<sup>†</sup> As soon as the block has been set in motion, the magnitude of  $\mathbf{F}$  drops from  $F_m$  to a lower value  $F_k$ . This is because there is less interpenetration between the irregularities of the surfaces in contact when these surfaces move with respect to each other. From then on, the block keeps sliding with increasing velocity while the friction force, denoted by  $\mathbf{F}_k$  and called the *kinetic-friction force*, remains approximately constant.

Experimental evidence shows that the maximum value  $F_m$  of the static-friction force is proportional to the normal component  $N$  of the reaction of the surface. We have

$$F_m = \mu_s N \quad (8.1)$$

where  $\mu_s$  is a constant called the *coefficient of static friction*. Similarly, the magnitude  $F_k$  of the kinetic-friction force may be put in the form

$$F_k = \mu_k N \quad (8.2)$$

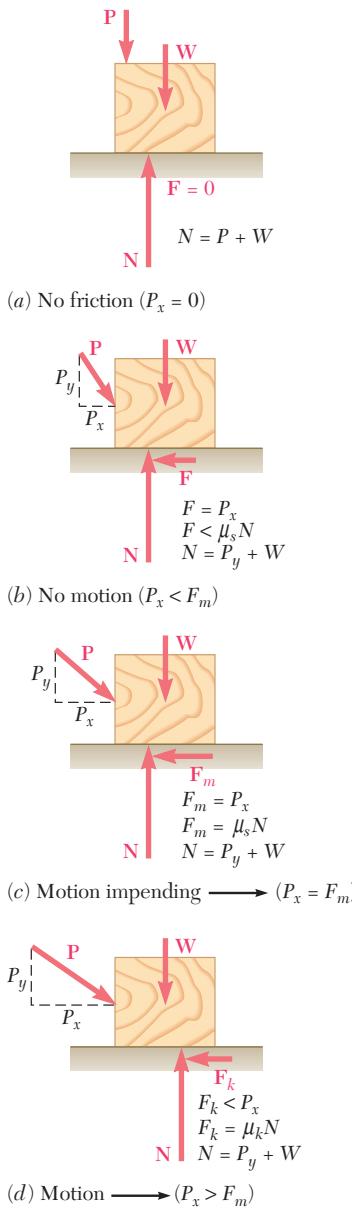
where  $\mu_k$  is a constant called the *coefficient of kinetic friction*. The coefficients of friction  $\mu_s$  and  $\mu_k$  do not depend upon the area of

<sup>†</sup>It should be noted that, as the magnitude  $F$  of the friction force increases from 0 to  $F_m$ , the point of application A of the resultant  $\mathbf{N}$  of the normal forces of contact moves to the right, so that the couples formed, respectively, by  $\mathbf{P}$  and  $\mathbf{F}$  and by  $\mathbf{W}$  and  $\mathbf{N}$  remain balanced. If  $\mathbf{N}$  reaches B before  $F$  reaches its maximum value  $F_m$ , the block will tip about B before it can start sliding (see Probs. 8.15 and 8.16).

the surfaces in contact. Both coefficients, however, depend strongly on the *nature* of the surfaces in contact. Since they also depend upon the exact condition of the surfaces, their value is seldom known with an accuracy greater than 5 percent. Approximate values of coefficients of static friction for various dry surfaces are given in Table 8.1. The corresponding values of the coefficient of kinetic friction would be about 25 percent smaller. Since coefficients of friction are dimensionless quantities, the values given in Table 8.1 can be used with both SI units and U.S. customary units.

**TABLE 8.1 Approximate Values of Coefficient of Static Friction for Dry Surfaces**

Metal on metal	0.15–0.60
Metal on wood	0.20–0.60
Metal on stone	0.30–0.70
Metal on leather	0.30–0.60
Wood on wood	0.25–0.50
Wood on leather	0.25–0.50
Stone on stone	0.40–0.70
Earth on earth	0.20–1.00
Rubber on concrete	0.60–0.90



**Fig. 8.2**

From the description given above, it appears that four different situations can occur when a rigid body is in contact with a horizontal surface:

1. The forces applied to the body do not tend to move it along the surface of contact; there is no friction force (Fig. 8.2a).
2. The applied forces tend to move the body along the surface of contact but are not large enough to set it in motion. The friction force  $\mathbf{F}$  which has developed can be found by solving the equations of equilibrium for the body. Since there is no evidence that  $\mathbf{F}$  has reached its maximum value, the equation  $F_m = \mu_s N$  cannot be used to determine the friction force (Fig. 8.2b).
3. The applied forces are such that the body is just about to slide. We say that *motion is impending*. The friction force  $\mathbf{F}$  has reached its maximum value  $F_m$  and, together with the normal force  $\mathbf{N}$ , balances the applied forces. Both the equations of equilibrium and the equation  $F_m = \mu_s N$  can be used. We also note that the friction force has a sense opposite to the sense of impending motion (Fig. 8.2c).
4. The body is sliding under the action of the applied forces, and the equations of equilibrium do not apply any more. However,  $\mathbf{F}$  is now equal to  $\mathbf{F}_k$  and the equation  $F_k = \mu_k N$  may be used. The sense of  $\mathbf{F}_k$  is opposite to the sense of motion (Fig. 8.2d).

### 8.3 ANGLES OF FRICTION

It is sometimes convenient to replace the normal force  $\mathbf{N}$  and the friction force  $\mathbf{F}$  by their resultant  $\mathbf{R}$ . Let us consider again a block of weight  $\mathbf{W}$  resting on a horizontal plane surface. If no horizontal force is applied to the block, the resultant  $\mathbf{R}$  reduces to the normal force  $\mathbf{N}$  (Fig. 8.3a). However, if the applied force  $\mathbf{P}$  has a horizontal component  $\mathbf{P}_x$  which tends to move the block, the force  $\mathbf{R}$  will have a horizontal component  $\mathbf{F}$  and, thus, will form an angle  $\phi$  with the normal to the surface (Fig. 8.3b). If  $\mathbf{P}_x$  is increased until motion becomes impending, the angle between  $\mathbf{R}$  and the vertical grows and reaches a maximum value (Fig. 8.3c). This value is called the *angle of static friction* and is denoted by  $\phi_s$ . From the geometry of Fig. 8.3c, we note that

$$\tan \phi_s = \frac{F_m}{N} = \frac{\mu_s N}{N}$$

$$\tan \phi_s = \mu_s \quad (8.3)$$

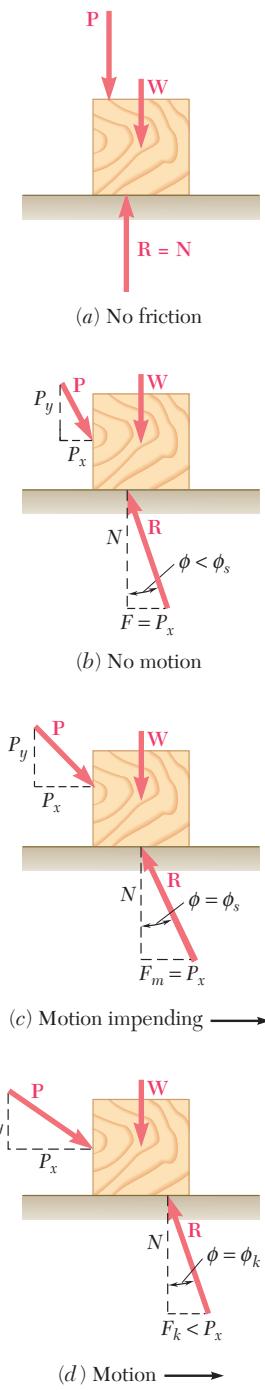
If motion actually takes place, the magnitude of the friction force drops to  $F_k$ ; similarly, the angle  $\phi$  between  $\mathbf{R}$  and  $\mathbf{N}$  drops to a lower value  $\phi_k$ , called the *angle of kinetic friction* (Fig. 8.3d). From the geometry of Fig. 8.3d, we write

$$\tan \phi_k = \frac{F_k}{N} = \frac{\mu_k N}{N}$$

$$\tan \phi_k = \mu_k \quad (8.4)$$

Another example will show how the angle of friction can be used to advantage in the analysis of certain types of problems. Consider a block resting on a board and subjected to no other force than its weight  $\mathbf{W}$  and the reaction  $\mathbf{R}$  of the board. The board can be given any desired inclination. If the board is horizontal, the force  $\mathbf{R}$  exerted by the board on the block is perpendicular to the board and balances the weight  $\mathbf{W}$  (Fig. 8.4a). If the board is given a small angle of inclination  $\theta$ , the force  $\mathbf{R}$  will deviate from the perpendicular to the board by the angle  $\theta$  and will keep balancing  $\mathbf{W}$  (Fig. 8.4b); it will then have a normal component  $\mathbf{N}$  of magnitude  $N = W \cos \theta$  and a tangential component  $\mathbf{F}$  of magnitude  $F = W \sin \theta$ .

If we keep increasing the angle of inclination, motion will soon become impending. At that time, the angle between  $\mathbf{R}$  and the normal will have reached its maximum value  $\phi_s$  (Fig. 8.4c). The value of the angle of inclination corresponding to impending motion is called the *angle of repose*. Clearly, the angle of repose is equal to the angle of static friction  $\phi_s$ . If the angle of inclination  $\theta$  is further increased, motion starts and the angle between  $\mathbf{R}$  and the normal drops to the lower value  $\phi_k$  (Fig. 8.4d). The reaction  $\mathbf{R}$  is not vertical any more, and the forces acting on the block are unbalanced.



**Fig. 8.3**

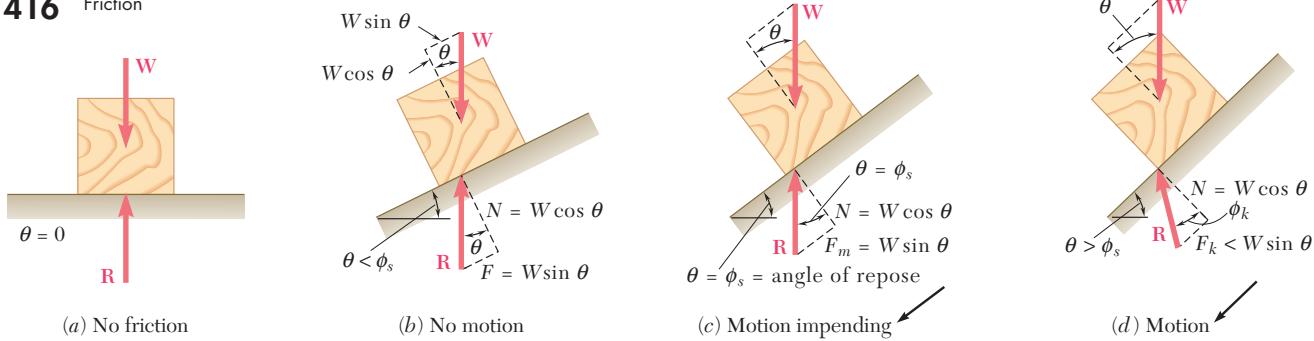


Fig. 8.4



**Photo 8.1** The coefficient of static friction between a package and the inclined conveyor belt must be sufficiently large to enable the package to be transported without slipping.

## 8.4 PROBLEMS INVOLVING DRY FRICTION

Problems involving dry friction are found in many engineering applications. Some deal with simple situations such as the block sliding on a plane described in the preceding sections. Others involve more complicated situations as in Sample Prob. 8.3; many deal with the stability of rigid bodies in accelerated motion and will be studied in dynamics. Also, a number of common machines and mechanisms can be analyzed by applying the laws of dry friction. These include wedges, screws, journal and thrust bearings, and belt transmissions. They will be studied in the following sections.

The *methods* which should be used to solve problems involving dry friction are the same that were used in the preceding chapters. If a problem involves only a motion of translation, with no possible rotation, the body under consideration can usually be treated as a particle, and the methods of Chap. 2 used. If the problem involves a possible rotation, the body must be considered as a rigid body, and the methods of Chap. 4 should be used. If the structure considered is made of several parts, the principle of action and reaction must be used as was done in Chap. 6.

If the body considered is acted upon by more than three forces (including the reactions at the surfaces of contact), the reaction at each surface will be represented by its components **N** and **F** and the problem will be solved from the equations of equilibrium. If only three forces act on the body under consideration, it may be more convenient to represent each reaction by the single force **R** and to solve the problem by drawing a force triangle.

Most problems involving friction fall into one of the following *three groups*: In the *first group* of problems, all applied forces are given and the coefficients of friction are known; we are to determine whether the body considered will remain at rest or slide. The friction force **F required to maintain equilibrium** is unknown (its magnitude is *not* equal to  $\mu_s N$ ) and should be determined, together with the normal force **N**, by drawing a free-body diagram and *solving the equations of equilibrium* (Fig. 8.5a). The value found for the magnitude **F** of the friction force is then compared with the maximum value  $F_m = \mu_s N$ . If **F** is smaller than or equal to  $F_m$ , the body remains at rest. If the value found for **F** is larger than  $F_m$ , equilibrium cannot

be maintained and motion takes place; the actual magnitude of the friction force is then  $F_k = \mu_k N$ .

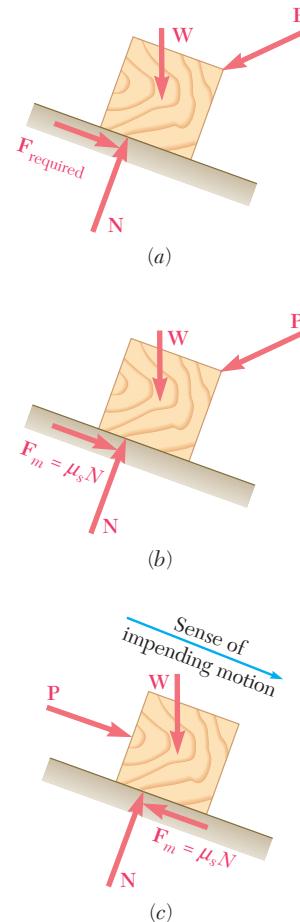
In problems of the *second group*, all applied forces are given and the motion is known to be impending; we are to determine the value of the coefficient of static friction. Here again, we determine the friction force and the normal force by drawing a free-body diagram and solving the equations of equilibrium (Fig. 8.5b). Since we know that the value found for  $F$  is the maximum value  $F_m$ , the coefficient of friction may be found by writing and solving the equation  $F_m = \mu_s N$ .

In problems of the *third group*, the coefficient of static friction is given, and it is known that the motion is impending in a given direction; we are to determine the magnitude or the direction of one of the applied forces. The friction force should be shown in the free-body diagram with a *sense opposite to that of the impending motion* and with a magnitude  $F_m = \mu_s N$  (Fig. 8.5c). The equations of equilibrium can then be written, and the desired force determined.

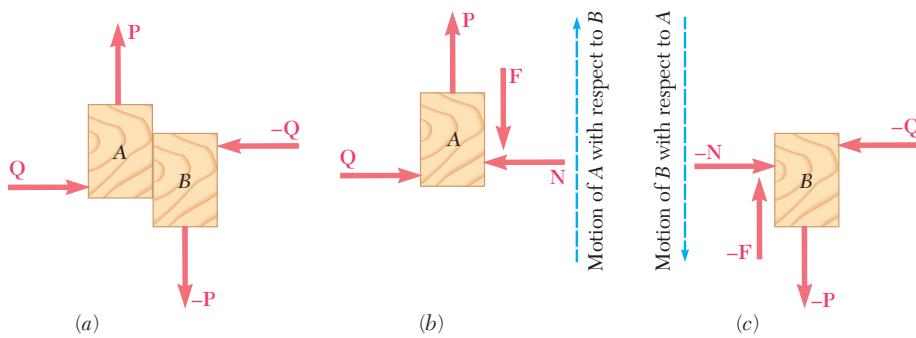
As noted above, when only three forces are involved it may be more convenient to represent the reaction of the surface by a single force **R** and to solve the problem by drawing a force triangle. Such a solution is used in Sample Prob. 8.2.

When two bodies *A* and *B* are in contact (Fig. 8.6a), the forces of friction exerted, respectively, by *A* on *B* and by *B* on *A* are equal and opposite (Newton's third law). In drawing the free-body diagram of one of the bodies, it is important to include the appropriate friction force with its correct sense. The following rule should then be observed: *The sense of the friction force acting on A is opposite to that of the motion (or impending motion) of A as observed from B* (Fig. 8.6b).† The sense of the friction force acting on *B* is determined in a similar way (Fig. 8.6c). Note that the motion of *A* as observed from *B* is a *relative motion*. For example, if body *A* is fixed and body *B* moves, body *A* will have a relative motion with respect to *B*. Also, if both *B* and *A* are moving down but *B* is moving faster than *A*, body *A* will be observed, from *B*, to be moving up.

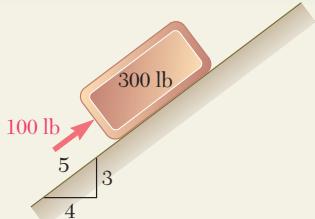
†It is therefore *the same as that of the motion of B as observed from A*.



**Fig. 8.5**



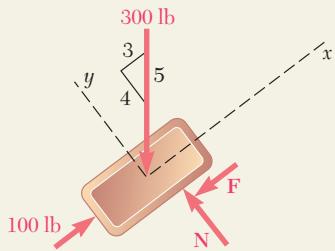
**Fig. 8.6**



## SAMPLE PROBLEM 8.1

A 100-lb force acts as shown on a 300-lb block placed on an inclined plane. The coefficients of friction between the block and the plane are  $\mu_s = 0.25$  and  $\mu_k = 0.20$ . Determine whether the block is in equilibrium, and find the value of the friction force.

## SOLUTION



**Force Required for Equilibrium.** We first determine the value of the friction force required to maintain equilibrium. Assuming that  $\mathbf{F}$  is directed down and to the left, we draw the free-body diagram of the block and write

$$+\nearrow \sum F_x = 0: \quad 100 \text{ lb} - \frac{3}{5}(300 \text{ lb}) - F = 0 \\ F = -80 \text{ lb} \quad \mathbf{F} = 80 \text{ lb} \nearrow$$

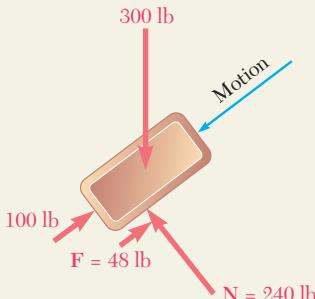
$$+\nwarrow \sum F_y = 0: \quad N - \frac{4}{5}(300 \text{ lb}) = 0 \\ N = +240 \text{ lb} \quad \mathbf{N} = 240 \text{ lb} \nwarrow$$

The force  $\mathbf{F}$  required to maintain equilibrium is an 80-lb force directed up and to the right; the tendency of the block is thus to move down the plane.

**Maximum Friction Force.** The magnitude of the maximum friction force which may be developed is

$$F_m = \mu_s N \quad F_m = 0.25(240 \text{ lb}) = 60 \text{ lb}$$

Since the value of the force required to maintain equilibrium (80 lb) is larger than the maximum value which may be obtained (60 lb), equilibrium will not be maintained and *the block will slide down the plane*.



**Actual Value of Friction Force.** The magnitude of the actual friction force is obtained as follows:

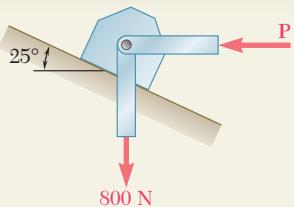
$$F_{\text{actual}} = F_k = \mu_k N \\ = 0.20(240 \text{ lb}) = 48 \text{ lb}$$

The sense of this force is opposite to the sense of motion; the force is thus directed up and to the right:

$$\mathbf{F}_{\text{actual}} = 48 \text{ lb} \nearrow$$

It should be noted that the forces acting on the block are not balanced; the resultant is

$$\frac{3}{5}(300 \text{ lb}) - 100 \text{ lb} - 48 \text{ lb} = 32 \text{ lb} \nwarrow$$



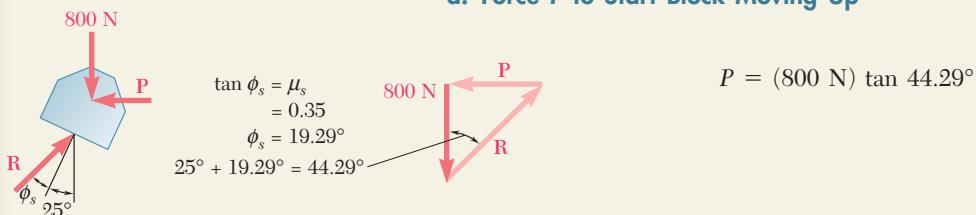
## SAMPLE PROBLEM 8.2

A support block is acted upon by two forces as shown. Knowing that the coefficients of friction between the block and the incline are  $\mu_s = 0.35$  and  $\mu_k = 0.25$ , determine the force  $P$  required (a) to start the block moving up the incline, (b) to keep it moving up, (c) to prevent it from sliding down.

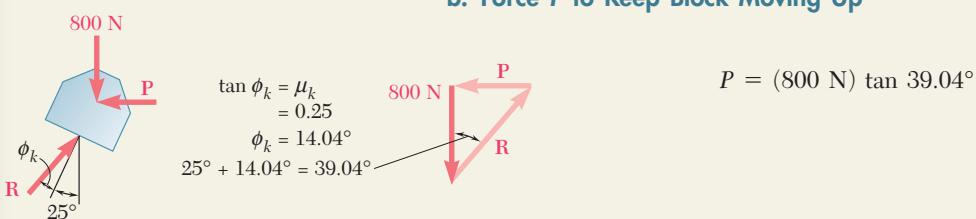
## SOLUTION

**Free-Body Diagram.** For each part of the problem we draw a free-body diagram of the block and a force triangle including the 800-N vertical force, the horizontal force  $P$ , and the force  $R$  exerted on the block by the incline. The direction of  $R$  must be determined in each separate case. We note that since  $P$  is perpendicular to the 800-N force, the force triangle is a right triangle, which can easily be solved for  $P$ . In most other problems, however, the force triangle will be an oblique triangle and should be solved by applying the law of sines.

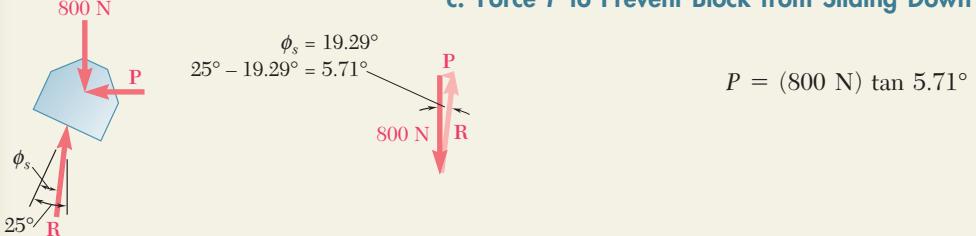
### a. Force $P$ to Start Block Moving Up

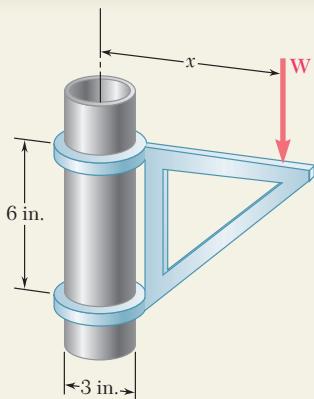


### b. Force $P$ to Keep Block Moving Up



### c. Force $P$ to Prevent Block from Sliding Down

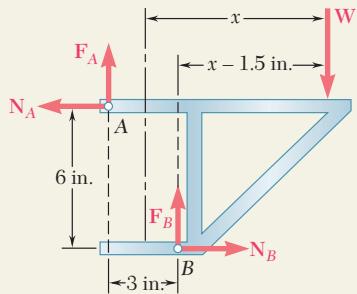




## SAMPLE PROBLEM 8.3

The movable bracket shown may be placed at any height on the 3-in.-diameter pipe. If the coefficient of static friction between the pipe and bracket is 0.25, determine the minimum distance  $x$  at which the load  $\mathbf{W}$  can be supported. Neglect the weight of the bracket.

## SOLUTION



**Free-Body Diagram.** We draw the free-body diagram of the bracket. When  $\mathbf{W}$  is placed at the minimum distance  $x$  from the axis of the pipe, the bracket is just about to slip, and the forces of friction at  $A$  and  $B$  have reached their maximum values:

$$F_A = \mu_s N_A = 0.25 N_A$$

$$F_B = \mu_s N_B = 0.25 N_B$$

### Equilibrium Equations

$$\stackrel{\perp}{\rightarrow} \sum F_x = 0: \quad N_B - N_A = 0$$

$$N_B = N_A$$

$$\stackrel{+}{\uparrow} \sum F_y = 0: \quad F_A + F_B - W = 0$$

$$0.25N_A + 0.25N_B = W$$

And, since  $N_B$  has been found equal to  $N_A$ ,

$$0.50N_A = W$$

$$N_A = 2W$$

$$\stackrel{+\nabla}{\rightarrow} \sum M_B = 0: \quad N_A(6 \text{ in.}) - F_A(3 \text{ in.}) - W(x - 1.5 \text{ in.}) = 0$$

$$6N_A - 3(0.25N_A) - Wx + 1.5W = 0$$

$$6(2W) - 0.75(2W) - Wx + 1.5W = 0$$

Dividing through by  $W$  and solving for  $x$ ,

$$x = 12 \text{ in.}$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson you studied and applied the *laws of dry friction*. Previously you had encountered only (a) frictionless surfaces that could move freely with respect to each other, (b) rough surfaces that allowed no motion relative to each other.

**A. In solving problems involving dry friction,** you should keep the following in mind.

**1. The reaction  $\mathbf{R}$  exerted by a surface on a free body** can be resolved into a component  $\mathbf{N}$  and a tangential component  $\mathbf{F}$ . The tangential component is known as the *friction force*. When a body is in contact with a fixed surface the direction of the friction force  $\mathbf{F}$  is opposite to that of the actual or impending motion of the body.

a. **No motion will occur** as long as  $F$  does not exceed the maximum value  $F_m = \mu_s N$ , where  $\mu_s$  is the *coefficient of static friction*.

b. **Motion will occur** if a value of  $F$  larger than  $F_m$  is required to maintain equilibrium. As motion takes place, the actual value of  $F$  drops to  $F_k = \mu_k N$ , where  $\mu_k$  is the *coefficient of kinetic friction* [Sample Prob. 8.1].

**2. When only three forces are involved** an alternative approach to the analysis of friction may be preferred [Sample Prob. 8.2]. The reaction  $\mathbf{R}$  is defined by its magnitude  $R$  and the angle  $\phi$  it forms with the normal to the surface. No motion will occur as long as  $\phi$  does not exceed the maximum value  $\phi_s$ , where  $\tan \phi_s = \mu_s$ . Motion will occur if a value of  $\phi$  larger than  $\phi_s$  is required to maintain equilibrium, and the actual value of  $\phi$  will drop to  $\phi_k$ , where  $\tan \phi_k = \mu_k$ .

**3. When two bodies are in contact** the sense of the actual or impending relative motion at the point of contact must be determined. On each of the two bodies a friction force  $\mathbf{F}$  should be shown in a direction opposite to that of the actual or impending motion of the body as seen from the other body.

(continued)

**B. Methods of solution.** The first step in your solution is to *draw a free-body diagram* of the body under consideration, resolving the force exerted on each surface where friction exists into a normal component  $\mathbf{N}$  and a friction force  $\mathbf{F}$ . If several bodies are involved, draw a free-body diagram of each of them, labeling and directing the forces at each surface of contact as you learned to do when analyzing frames in Chap. 6.

The problem you have to solve may fall in one of the following three categories:

**1. All the applied forces and the coefficients of friction are known, and you must determine whether equilibrium is maintained.** Note that in this situation the friction force is unknown and *cannot be assumed to be equal to  $\mu_s N$* .

a. **Write the equations of equilibrium to determine  $\mathbf{N}$  and  $\mathbf{F}$ .**

b. **Calculate the maximum allowable friction force,  $F_m = \mu_s N$ .** If  $F \leq F_m$ , equilibrium is maintained. If  $F > F_m$ , motion occurs, and the magnitude of the friction force is  $F_k = \mu_k N$  [Sample Prob. 8.1].

**2. All the applied forces are known, and you must find the smallest allowable value of  $\mu_s$  for which equilibrium is maintained.** You will assume that motion is impending and determine the corresponding value of  $\mu_s$ .

a. **Write the equations of equilibrium to determine  $\mathbf{N}$  and  $\mathbf{F}$ .**

b. **Since motion is impending,  $F = F_m$ .** Substitute the values found for  $N$  and  $F$  into the equation  $F_m = \mu_s N$  and solve for  $\mu_s$ .

**3. The motion of the body is impending and  $\mu_s$  is known; you must find some unknown quantity,** such as a distance, an angle, the magnitude of a force, or the direction of a force.

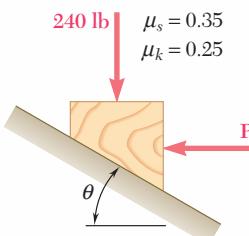
a. **Assume a possible motion of the body** and, on the free-body diagram, draw the friction force in a direction opposite to that of the assumed motion.

b. **Since motion is impending,  $F = F_m = \mu_s N$ .** Substituting for  $\mu_s$  its known value, you can express  $F$  in terms of  $N$  on the free-body diagram, thus eliminating one unknown.

c. **Write and solve the equilibrium equations for the unknown you seek** [Sample Prob. 8.3].

# PROBLEMS

- 8.1** Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when  $\theta = 25^\circ$  and  $P = 150$  lb.



**Fig. P8.1 and P8.2**

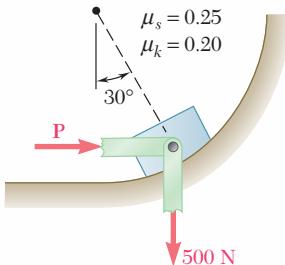
- 8.2** Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when  $\theta = 30^\circ$  and  $P = 30$  lb.

- 8.3** Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when  $\theta = 40^\circ$  and  $P = 400$  N.

- 8.4** Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when  $\theta = 35^\circ$  and  $P = 200$  N.

- 8.5** Knowing that  $\theta = 45^\circ$ , determine the range of values of  $P$  for which equilibrium is maintained.

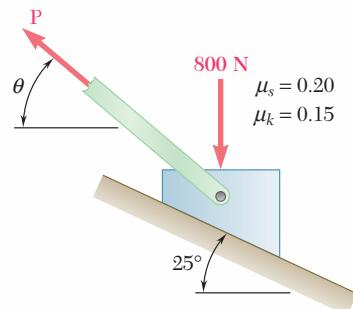
- 8.6** Determine the range of values of  $P$  for which equilibrium of the block shown is maintained.



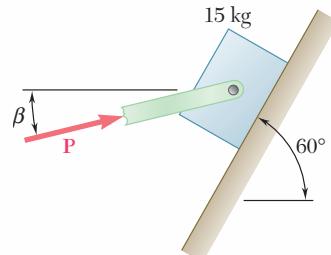
**Fig. P8.6**

- 8.7** Knowing that the coefficient of friction between the 15-kg block and the incline is  $\mu_s = 0.25$ , determine (a) the smallest value of  $P$  required to maintain the block in equilibrium, (b) the corresponding value of  $\beta$ .

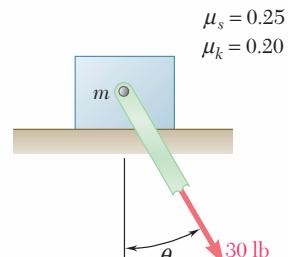
- 8.8** Considering only values of  $\theta$  less than  $90^\circ$ , determine the smallest value of  $\theta$  required to start the block moving to the right when (a)  $W = 75$  lb, (b)  $W = 100$  lb.



**Fig. P8.3, P8.4, and P8.5**



**Fig. P8.7**



**Fig. P8.8**

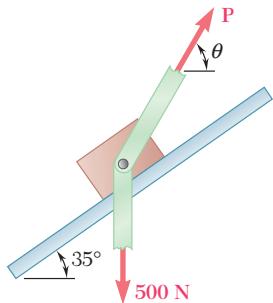


Fig. P8.9

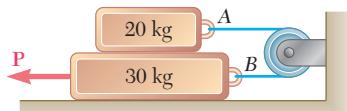


Fig. P8.11

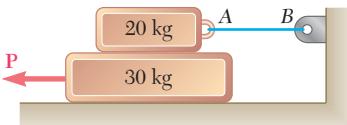


Fig. P8.12

- 8.9** The coefficients of friction between the block and the rail are  $\mu_s = 0.30$  and  $\mu_k = 0.25$ . Knowing that  $\theta = 65^\circ$ , determine the smallest value of  $P$  required (a) to start the block moving up the rail, (b) to keep it from moving down.

- 8.10** The 80-lb block is attached to link AB and rests on a moving belt. Knowing that  $\mu_s = 0.25$  and  $\mu_k = 0.20$ , determine the magnitude of the horizontal force  $P$  that should be applied to the belt to maintain its motion (a) to the right, (b) to the left.

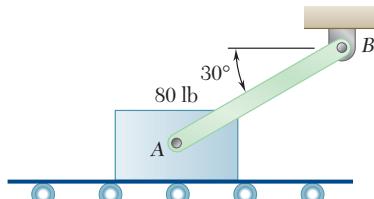


Fig. P8.10

- 8.11 and 8.12** The coefficients of friction are  $\mu_s = 0.40$  and  $\mu_k = 0.30$  between all surfaces of contact. Determine the smallest force  $P$  required to start the 30-kg block moving if cable AB (a) is attached as shown, (b) is removed.

- 8.13** Three 4-kg packages A, B, and C are placed on a conveyor belt that is at rest. Between the belt and both packages A and C the coefficients of friction are  $\mu_s = 0.30$  and  $\mu_k = 0.20$ ; between package B and the belt the coefficients are  $\mu_s = 0.10$  and  $\mu_k = 0.08$ . The packages are placed on the belt so that they are in contact with each other and at rest. Determine which, if any, of the packages will move and the friction force acting on each package.

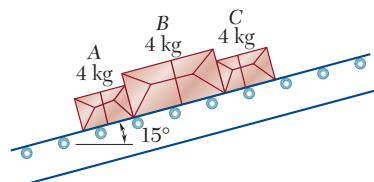


Fig. P8.13

- 8.14** Solve Prob. 8.13 assuming that package B is placed to the right of both packages A and C.

- 8.15** A 120-lb cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.30. If  $h = 32$  in., determine the magnitude of the force  $P$  required to move the cabinet to the right (a) if all casters are locked, (b) if the casters at B are locked and the casters at A are free to rotate, (c) if the casters at A are locked and the casters at B are free to rotate.

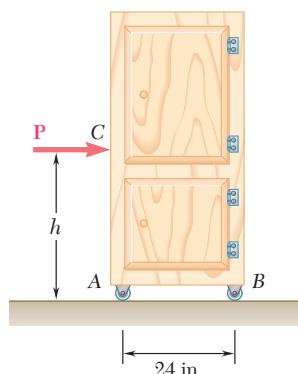


Fig. P8.15 and P8.16

- 8.16** A 120-lb cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.30. Assuming that the casters at both A and B are locked, determine (a) the force  $P$  required to move the cabinet to the right, (b) the largest allowable value of  $h$  if the cabinet is not to tip over.

- 8.17** The cylinder shown is of weight  $W$  and radius  $r$ , and the coefficient of static friction  $\mu_s$  is the same at  $A$  and  $B$ . Determine the magnitude of the largest couple  $\mathbf{M}$  that can be applied to the cylinder if it is not to rotate.

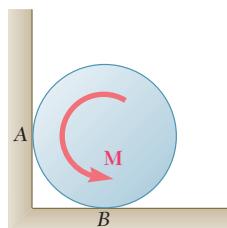


Fig. P8.17 and P8.18

- 8.18** The cylinder shown is of weight  $W$  and radius  $r$ . Express in terms of  $W$  and  $r$  the magnitude of the largest couple  $\mathbf{M}$  that can be applied to the cylinder if it is not to rotate, assuming the coefficient of static friction to be (a) zero at  $A$  and 0.30 at  $B$ , (b) 0.25 at  $A$  and 0.30 at  $B$ .

- 8.19** The hydraulic cylinder shown exerts a force of 3 kN directed to the right on point  $B$  and to the left on point  $E$ . Determine the magnitude of the couple  $\mathbf{M}$  required to rotate the drum clockwise at a constant speed.

- 8.20** A couple  $\mathbf{M}$  of magnitude 100 N · m is applied to the drum as shown. Determine the smallest force that must be exerted by the hydraulic cylinder on joints  $B$  and  $E$  if the drum is not to rotate.

- 8.21** A 6.5-m ladder  $AB$  leans against a wall as shown. Assuming that the coefficient of static friction  $\mu_s$  is zero at  $B$ , determine the smallest value of  $\mu_s$  at  $A$  for which equilibrium is maintained.

- 8.22** A 6.5-m ladder  $AB$  leans against a wall as shown. Assuming that the coefficient of static friction  $\mu_s$  is the same at  $A$  and  $B$ , determine the smallest value of  $\mu_s$  for which equilibrium is maintained.

- 8.23 and 8.24** End  $A$  of a slender, uniform rod of length  $L$  and weight  $W$  bears on a surface as shown, while end  $B$  is supported by a cord  $BC$ . Knowing that the coefficients of friction are  $\mu_s = 0.40$  and  $\mu_k = 0.30$ , determine (a) the largest value of  $\theta$  for which motion is impending, (b) the corresponding value of the tension in the cord.

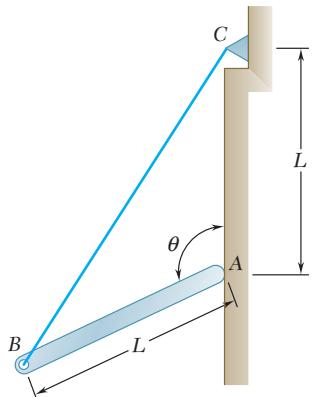


Fig. P8.23

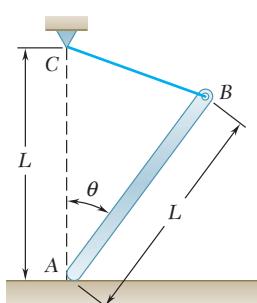


Fig. P8.24

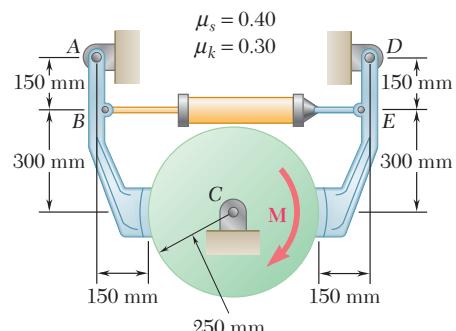


Fig. P8.19 and P8.20

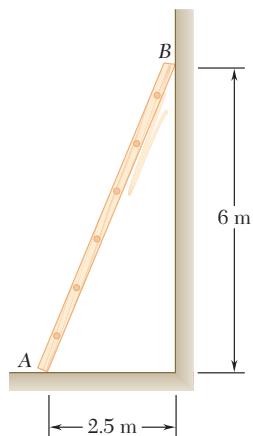


Fig. P8.21 and P8.22

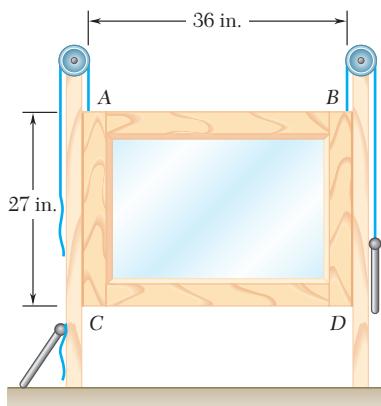


Fig. P8.25

- 8.25** A window sash weighing 10 lb is normally supported by two 5-lb sash weights. Knowing that the window remains open after one sash cord has broken, determine the smallest possible value of the coefficient of static friction. (Assume that the sash is slightly smaller than the frame and will bind only at points A and D.)

- 8.26** A 500-N concrete block is to be lifted by the pair of tongs shown. Determine the smallest allowable value of the coefficient of static friction between the block and the tongs at F and G.

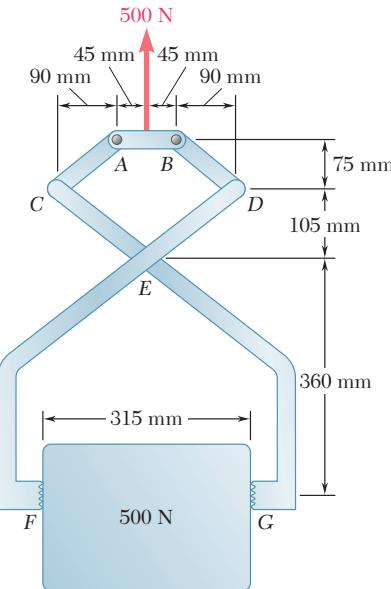


Fig. P8.26

- 8.27** The press shown is used to emboss a small seal at E. Knowing that the coefficient of static friction between the vertical guide and the embossing die D is 0.30, determine the force exerted by the die on the seal.

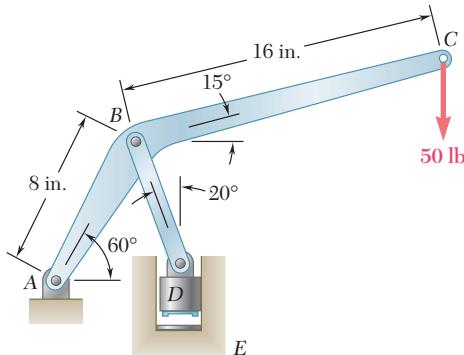


Fig. P8.27

- 8.28** The 100-mm-radius cam shown is used to control the motion of the plate CD. Knowing that the coefficient of static friction between the cam and the plate is 0.45 and neglecting friction at the roller supports, determine (a) the force **P** required to maintain the motion of the plate, knowing that the plate is 20 mm thick, (b) the largest thickness of the plate for which the mechanism is self locking (i.e., for which the plate cannot be moved however large the force **P** may be).

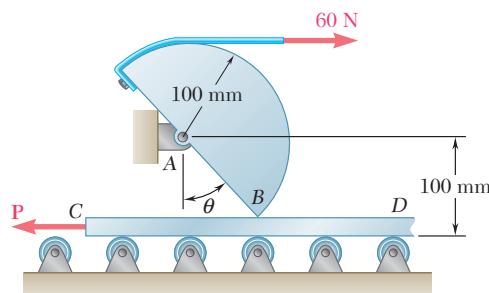


Fig. P8.28

- 8.29** A slender rod of length  $L$  is lodged between peg  $C$  and the vertical wall and supports a load  $\mathbf{P}$  at end  $A$ . Knowing that the coefficient of static friction is 0.20 at both  $B$  and  $C$ , find the range of values of the ratio  $L/a$  for which equilibrium is maintained.

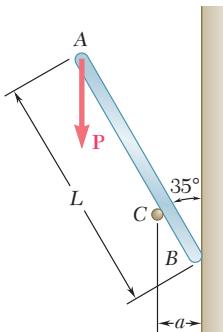


Fig. P8.29

- 8.30** The 50-lb plate  $ABCD$  is attached at  $A$  and  $D$  to collars that can slide on the vertical rod. Knowing that the coefficient of static friction is 0.40 between both collars and the rod, determine whether the plate is in equilibrium in the position shown when the magnitude of the vertical force applied at  $E$  is (a)  $P = 0$ , (b)  $P = 20$  lb.

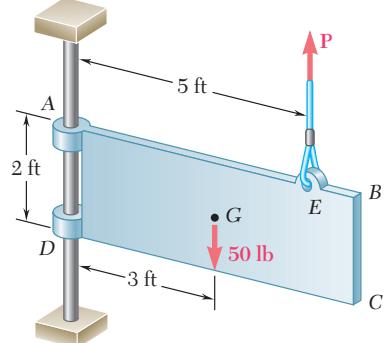


Fig. P8.30

- 8.31** In Prob. 8.30, determine the range of values of the magnitude  $P$  of the vertical force applied at  $E$  for which the plate will move downward.

- 8.32** A pipe of diameter 60 mm is gripped by the stillson wrench shown. Portions  $AB$  and  $DE$  of the wrench are rigidly attached to each other, and portion  $CF$  is connected by a pin at  $D$ . If the wrench is to grip the pipe and be self-locking, determine the required minimum coefficients of friction at  $A$  and  $C$ .

- 8.33** Solve Prob. 8.32 assuming that the diameter of the pipe is 30 mm.

- 8.34** A 10-ft beam, weighing 1200 lb, is to be moved to the left onto the platform. A horizontal force  $\mathbf{P}$  is applied to the dolly, which is mounted on frictionless wheels. The coefficients of friction between all surfaces are  $\mu_s = 0.30$  and  $\mu_k = 0.25$ , and initially  $x = 2$  ft. Knowing that the top surface of the dolly is slightly higher than the platform, determine the force  $\mathbf{P}$  required to start moving the beam. (Hint: The beam is supported at  $A$  and  $D$ .)

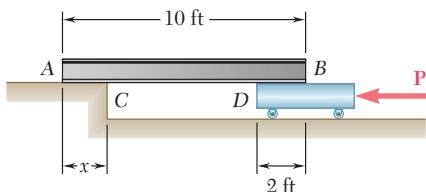


Fig. P8.34

- 8.35** (a) Show that the beam of Prob. 8.34 cannot be moved if the top surface of the dolly is slightly lower than the platform. (b) Show that the beam can be moved if two 175-lb workers stand on the beam at  $B$  and determine how far to the left the beam can be moved.

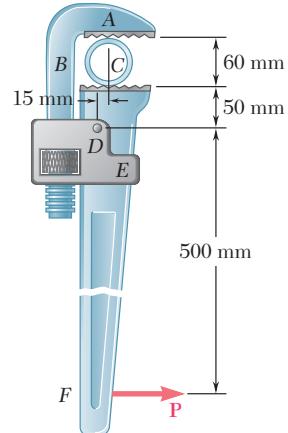


Fig. P8.32

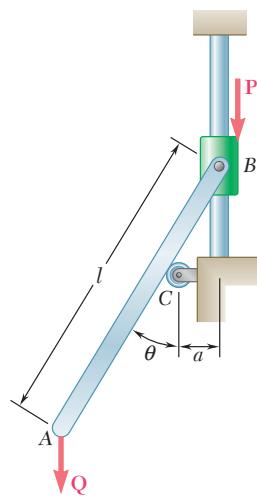


Fig. P8.38

- 8.36** Knowing that the coefficient of static friction between the collar and the rod is 0.35, determine the range of values of  $P$  for which equilibrium is maintained when  $\theta = 50^\circ$  and  $M = 20 \text{ N} \cdot \text{m}$ .

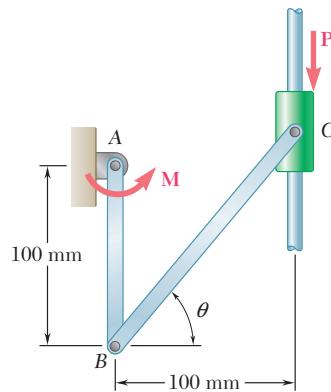


Fig. P8.36 and P8.37

- 8.37** Knowing that the coefficient of static friction between the collar and the rod is 0.40, determine the range of values of  $M$  for which equilibrium is maintained when  $\theta = 60^\circ$  and  $P = 200 \text{ N}$ .

- 8.38** The slender rod  $AB$  of length  $l = 600 \text{ mm}$  is attached to a collar at  $B$  and rests on a small wheel located at a horizontal distance  $a = 80 \text{ mm}$  from the vertical rod on which the collar slides. Knowing that the coefficient of static friction between the collar and the vertical rod is 0.25 and neglecting the radius of the wheel, determine the range of values of  $P$  for which equilibrium is maintained when  $Q = 100 \text{ N}$  and  $\theta = 30^\circ$ .

- 8.39** Two 10-lb blocks  $A$  and  $B$  are connected by a slender rod of negligible weight. The coefficient of static friction is 0.30 between all surfaces of contact, and the rod forms an angle  $\theta = 30^\circ$  with the vertical. (a) Show that the system is in equilibrium when  $P = 0$ . (b) Determine the largest value of  $P$  for which equilibrium is maintained.

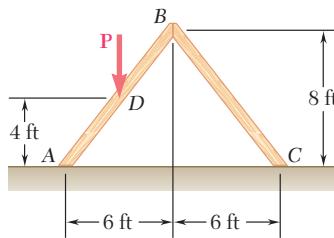


Fig. P8.40

$W = 10 \text{ lb}$

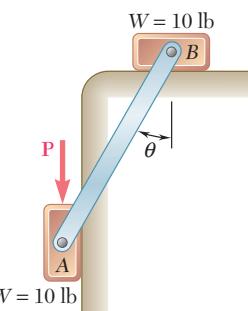


Fig. P8.39

- 8.40** Two identical uniform boards, each of weight 40 lb, are temporarily leaned against each other as shown. Knowing that the coefficient of static friction between all surfaces is 0.40, determine (a) the largest magnitude of the force  $P$  for which equilibrium will be maintained, (b) the surface at which motion will impend.

- 8.41** Two identical 5-ft-long rods connected by a pin at *B* are placed between two walls and a horizontal surface as shown. Denoting by  $\mu_s$  the coefficient of static friction at *A*, *B*, and *C*, determine the smallest value of  $\mu_s$  for which equilibrium is maintained.

- 8.42** Two 8-kg blocks *A* and *B* resting on shelves are connected by a rod of negligible mass. Knowing that the magnitude of a horizontal force  $\mathbf{P}$  applied at *C* is slowly increased from zero, determine the value of  $P$  for which motion occurs, and what that motion is, when the coefficient of static friction between all surfaces is (a)  $\mu_s = 0.40$ , (b)  $\mu_s = 0.50$ .

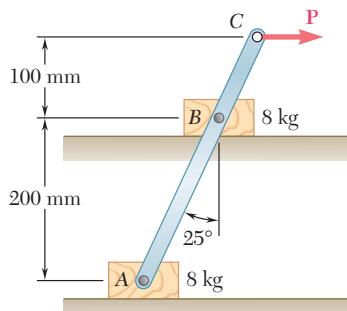


Fig. P8.42

- 8.43** A slender steel rod of length 225 mm is placed inside a pipe as shown. Knowing that the coefficient of static friction between the rod and the pipe is 0.20, determine the largest value of  $\theta$  for which the rod will not fall into the pipe.

- 8.44** In Prob. 8.43, determine the smallest value of  $\theta$  for which the rod will not fall out of the pipe.

- 8.45** Two slender rods of negligible weight are pin-connected at *C* and attached to blocks *A* and *B*, each of weight  $W$ . Knowing that  $\theta = 80^\circ$  and that the coefficient of static friction between the blocks and the horizontal surface is 0.30, determine the largest value of  $P$  for which equilibrium is maintained.

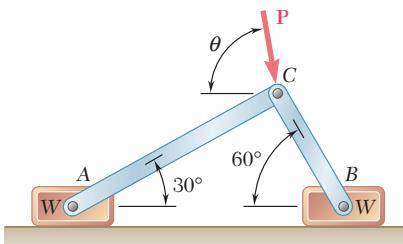


Fig. P8.45

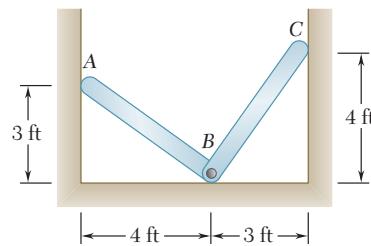


Fig. P8.41

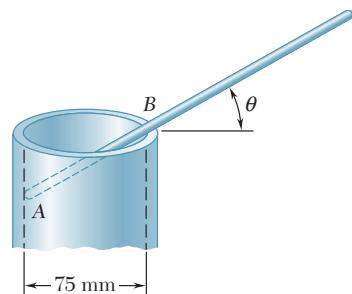


Fig. P8.43

## 8.5 WEDGES

Wedges are simple machines used to raise large stone blocks and other heavy loads. These loads can be raised by applying to the wedge a force usually considerably smaller than the weight of the

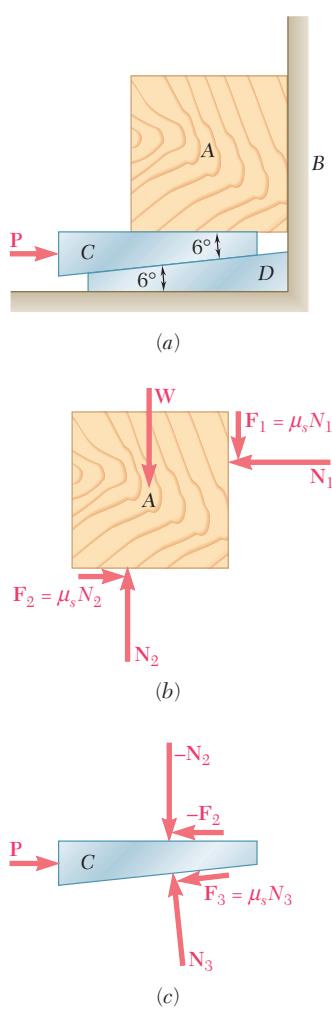


Fig. 8.7

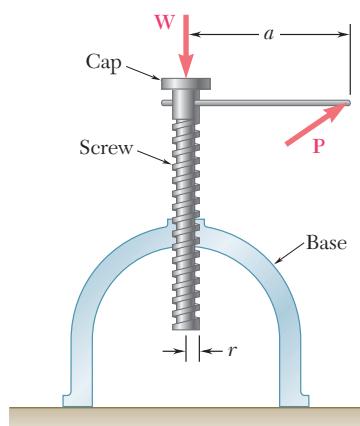


Fig. 8.8

load. In addition, because of the friction between the surfaces in contact, a properly shaped wedge will remain in place after being forced under the load. Wedges can thus be used advantageously to make small adjustments in the position of heavy pieces of machinery.

Consider the block *A* shown in Fig. 8.7*a*. This block rests against a vertical wall *B* and is to be raised slightly by forcing a wedge *C* between block *A* and a second wedge *D*. We want to find the minimum value of the force **P** which must be applied to the wedge *C* to move the block. It will be assumed that the weight **W** of the block is known, either given in pounds or determined in newtons from the mass of the block expressed in kilograms.

The free-body diagrams of block *A* and of wedge *C* have been drawn in Fig. 8.7*b* and *c*. The forces acting on the block include its weight and the normal and friction forces at the surfaces of contact with wall *B* and wedge *C*. The magnitudes of the friction forces **F**<sub>1</sub> and **F**<sub>2</sub> are equal, respectively, to  $\mu_s N_1$  and  $\mu_s N_2$  since the motion of the block must be started. It is important to show the friction forces with their correct sense. Since the block will move upward, the force **F**<sub>1</sub> exerted by the wall on the block must be directed downward. On the other hand, since the wedge *C* moves to the right, the relative motion of *A* with respect to *C* is to the left and the force **F**<sub>2</sub> exerted by *C* on *A* must be directed to the right.

Considering now the free body *C* in Fig. 8.7*c*, we note that the forces acting on *C* include the applied force **P** and the normal and friction forces at the surfaces of contact with *A* and *D*. The weight of the wedge is small compared with the other forces involved and can be neglected. The forces exerted by *A* on *C* are equal and opposite to the forces **N**<sub>2</sub> and **F**<sub>2</sub> exerted by *C* on *A* and are denoted, respectively, by **-N**<sub>2</sub> and **-F**<sub>2</sub>; the friction force **-F**<sub>2</sub> must therefore be directed to the left. We check that the force **F**<sub>3</sub> exerted by *D* is also directed to the left.

The total number of unknowns involved in the two free-body diagrams can be reduced to four if the friction forces are expressed in terms of the normal forces. Expressing that block *A* and wedge *C* are in equilibrium will provide four equations which can be solved to obtain the magnitude of **P**. It should be noted that in the example considered here, it will be more convenient to replace each pair of normal and friction forces by their resultant. Each free body is then subjected to only three forces, and the problem can be solved by drawing the corresponding force triangles (see Sample Prob. 8.4).

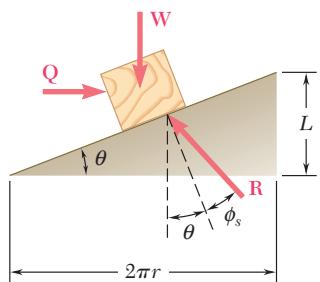
## 8.6 SQUARE-THREADED SCREWS

Square-threaded screws are frequently used in jacks, presses, and other mechanisms. Their analysis is similar to the analysis of a block sliding along an inclined plane.

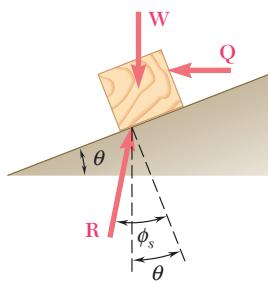
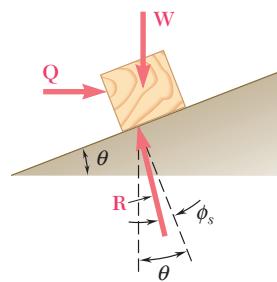
Consider the jack shown in Fig. 8.8. The screw carries a load **W** and is supported by the base of the jack. Contact between screw and base takes place along a portion of their threads. By applying a force **P** on the handle, the screw can be made to turn and to raise the load **W**.

The thread of the base has been unwrapped and shown as a straight line in Fig. 8.9a. The correct slope was obtained by plotting horizontally the product  $2\pi r$ , where  $r$  is the mean radius of the thread, and vertically the *lead*  $L$  of the screw, i.e., the distance through which the screw advances in one turn. The angle  $\theta$  this line forms with the horizontal is the *lead angle*. Since the force of friction between two surfaces in contact does not depend upon the area of contact, a much smaller than actual area of contact between the two threads can be assumed, and the screw can be represented by the block shown in Fig. 8.9a. It should be noted, however, that in this analysis of the jack, the friction between cap and screw is neglected.

The free-body diagram of the block should include the load  $\mathbf{W}$ , the reaction  $\mathbf{R}$  of the base thread, and a horizontal force  $\mathbf{Q}$  having the same effect as the force  $\mathbf{P}$  exerted on the handle. The force  $\mathbf{Q}$  should have the same moment as  $\mathbf{P}$  about the axis of the screw and its magnitude should thus be  $Q = Pa/r$ . The force  $\mathbf{Q}$ , and thus the force  $\mathbf{P}$  required to raise the load  $\mathbf{W}$ , can be obtained from the free-body diagram shown in Fig. 8.9a. The friction angle is taken equal to  $\phi_s$  since the load will presumably be raised through a succession of short strokes. In mechanisms providing for the continuous rotation of a screw, it may be desirable to distinguish between the force required to start motion (using  $\phi_s$ ) and that required to maintain motion (using  $\phi_k$ ).



(a) Impending motion upward

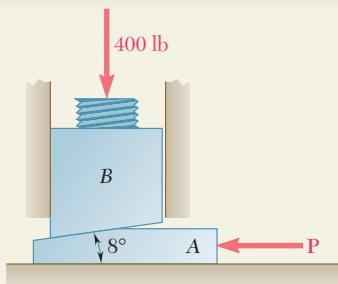
(b) Impending motion downward with  $\phi_s > \theta$ (c) Impending motion downward with  $\phi_s < \theta$ **Fig. 8.9** Block-and-incline analysis of a screw.

If the friction angle  $\phi_s$  is larger than the lead angle  $\theta$ , the screw is said to be *self-locking*; it will remain in place under the load. To lower the load, we must then apply the force shown in Fig. 8.9b. If  $\phi_s$  is smaller than  $\theta$ , the screw will unwind under the load; it is then necessary to apply the force shown in Fig. 8.9c to maintain equilibrium.

The lead of a screw should not be confused with its *pitch*. The lead was defined as the distance through which the screw advances in one turn; the pitch is the distance measured between two consecutive threads. While lead and pitch are equal in the case of *single-threaded* screws, they are different in the case of *multiple-threaded* screws, i.e., screws having several independent threads. It is easily verified that for double-threaded screws, the lead is twice as large as the pitch; for triple-threaded screws, it is three times as large as the pitch; etc.



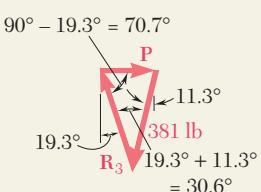
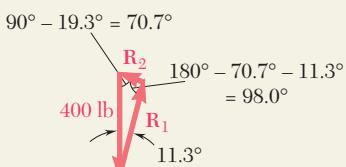
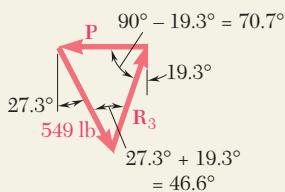
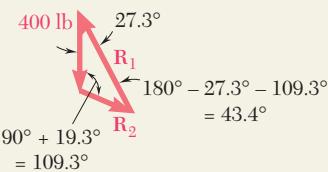
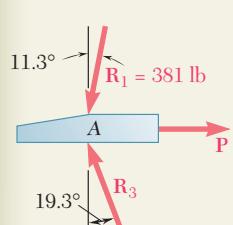
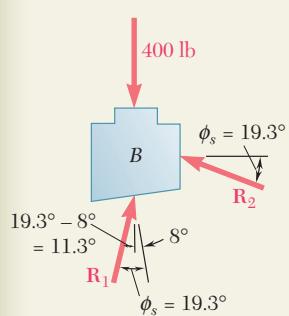
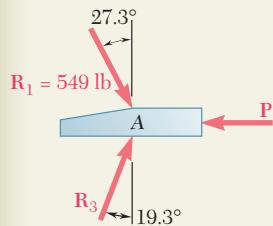
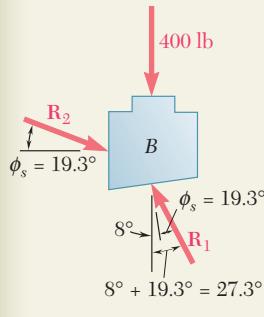
**Photo 8.2** Wedges are used as shown to split tree trunks because the normal forces exerted by the wedges on the wood are much larger than the forces required to insert the wedges.



## SAMPLE PROBLEM 8.4

The position of the machine block  $B$  is adjusted by moving the wedge  $A$ . Knowing that the coefficient of static friction is 0.35 between all surfaces of contact, determine the force  $\mathbf{P}$  required (a) to raise block  $B$ , (b) to lower block  $B$ .

## SOLUTION



$$\phi_s = \tan^{-1} 0.35 = 19.3^\circ$$

### a. Force $\mathbf{P}$ to Raise Block

#### Free Body: Block B

$$\frac{R_1}{\sin 109.3^\circ} = \frac{400 \text{ lb}}{\sin 43.4^\circ}$$

$$R_1 = 549 \text{ lb}$$

#### Free Body: Wedge A

$$\frac{P}{\sin 46.6^\circ} = \frac{549 \text{ lb}}{\sin 70.7^\circ}$$

$$P = 423 \text{ lb} \quad \mathbf{P} = 423 \text{ lb} \leftarrow$$

### b. Force $\mathbf{P}$ to Lower Block

#### Free Body: Block B

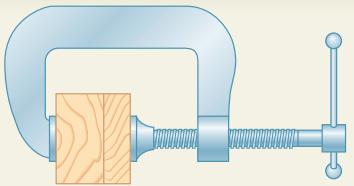
$$\frac{R_1}{\sin 70.7^\circ} = \frac{400 \text{ lb}}{\sin 98.0^\circ}$$

$$R_1 = 381 \text{ lb}$$

#### Free Body: Wedge A

$$\frac{P}{\sin 30.6^\circ} = \frac{381 \text{ lb}}{\sin 70.7^\circ}$$

$$P = 206 \text{ lb} \quad \mathbf{P} = 206 \text{ lb} \rightarrow$$



## SAMPLE PROBLEM 8.5

A clamp is used to hold two pieces of wood together as shown. The clamp has a double square thread of mean diameter equal to 10 mm with a pitch of 2 mm. The coefficient of friction between threads is  $\mu_s = 0.30$ . If a maximum couple of 40 N · m is applied in tightening the clamp, determine (a) the force exerted on the pieces of wood, (b) the couple required to loosen the clamp.

## SOLUTION

**a. Force Exerted by Clamp.** The mean radius of the screw is  $r = 5$  mm. Since the screw is double-threaded, the lead  $L$  is equal to twice the pitch:  $L = 2(2 \text{ mm}) = 4 \text{ mm}$ . The lead angle  $\theta$  and the friction angle  $\phi_s$  are obtained by writing

$$\tan \theta = \frac{L}{2\pi r} = \frac{4 \text{ mm}}{10\pi \text{ mm}} = 0.1273 \quad \theta = 7.3^\circ$$

$$\tan \phi_s = \mu_s = 0.30 \quad \phi_s = 16.7^\circ$$

The force  $\mathbf{Q}$  which should be applied to the block representing the screw is obtained by expressing that its moment  $Qr$  about the axis of the screw is equal to the applied couple.

$$Q(5 \text{ mm}) = 40 \text{ N} \cdot \text{m}$$

$$Q = \frac{40 \text{ N} \cdot \text{m}}{5 \text{ mm}} = \frac{40 \text{ N} \cdot \text{m}}{5 \times 10^{-3} \text{ m}} = 8000 \text{ N} = 8 \text{ kN}$$

The free-body diagram and the corresponding force triangle can now be drawn for the block; the magnitude of the force  $\mathbf{W}$  exerted on the pieces of wood is obtained by solving the triangle.

$$W = \frac{Q}{\tan(\theta + \phi_s)} = \frac{8 \text{ kN}}{\tan 24.0^\circ}$$

$$W = 17.97 \text{ kN} \quad \blacktriangleleft$$

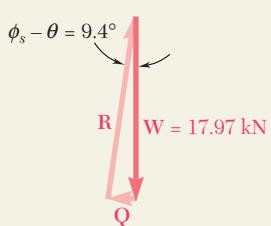
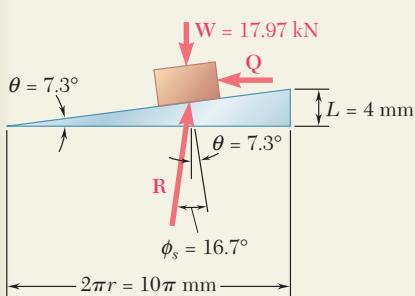
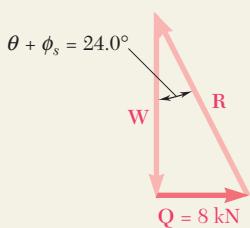
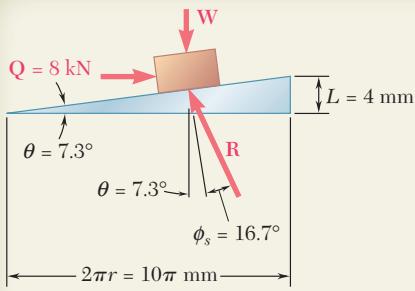
**b. Couple Required to Loosen Clamp.** The force  $\mathbf{Q}$  required to loosen the clamp and the corresponding couple are obtained from the free-body diagram and force triangle shown.

$$Q = W \tan(\phi_s - \theta) = (17.97 \text{ kN}) \tan 9.4^\circ$$

$$= 2.975 \text{ kN}$$

$$\begin{aligned} \text{Couple} &= Qr = (2.975 \text{ kN})(5 \text{ mm}) \\ &= (2.975 \times 10^3 \text{ N})(5 \times 10^{-3} \text{ m}) = 14.87 \text{ N} \cdot \text{m} \end{aligned}$$

$$\text{Couple} = 14.87 \text{ N} \cdot \text{m} \quad \blacktriangleleft$$



# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to apply the laws of friction to the solution of problems involving *wedges* and *square-threaded screws*.

**1. Wedges.** Keep the following in mind when solving a problem involving a wedge:

a. **First draw a free-body diagram of the wedge and of all the other bodies involved.** Carefully note the sense of the relative motion of all surfaces of contact and show each friction force acting in a *direction opposite* to the direction of that relative motion.

b. **Show the maximum static friction force  $F_m$**  at each surface if the wedge is to be inserted or removed, *since motion will be impending in each of these cases*.

c. **The reaction R and the angle of friction,** rather than the normal force and the friction force, can be used in many applications. You can then draw one or more force triangles and determine the unknown quantities either graphically or by trigonometry [Sample Prob. 8.4].

**2. Square-Threaded Screws.** The analysis of a square-threaded screw is equivalent to the analysis of a block sliding on an incline. To draw the appropriate incline, you should unwrap the thread of the screw and represent it by a straight line [Sample Prob. 8.5]. When solving a problem involving a square-threaded screw, keep the following in mind:

a. **Do not confuse the pitch of a screw with the lead of a screw.** The *pitch* of a screw is the distance between two consecutive threads, while the *lead* of a screw is the distance the screw advances in one full turn. The lead and the pitch are equal only in single-threaded screws. In a double-threaded screw, the lead is twice the pitch.

b. **The couple required to tighten a screw is different from the couple required to loosen it.** Also, screws used in jacks and clamps are usually *self-locking*; that is, the screw will remain stationary as long as no couple is applied to it, and a couple must be applied to the screw to loosen it [Sample Prob. 8.5].

# PROBLEMS

**8.46** The machine part *ABC* is supported by a frictionless hinge at *B* and a  $10^\circ$  wedge at *C*. Knowing that the coefficient of static friction is 0.20 at both surfaces of the wedge, determine (a) the force **P** required to move the wedge to the left, (b) the components of the corresponding reaction at *B*.

**8.47** Solve Prob. 8.46 assuming that the wedge is to be moved to the right.

**8.48 and 8.49** Two  $8^\circ$  wedges of negligible weight are used to move and position the 800-kg block. Knowing that the coefficient of static friction is 0.30 at all surfaces of contact, determine the smallest force **P** that should be applied as shown to one of the wedges.

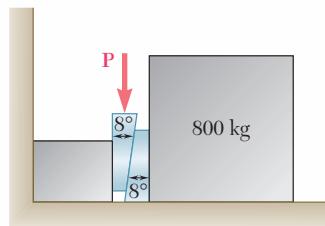


Fig. P8.48

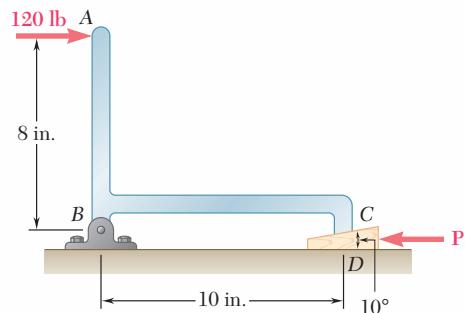


Fig. P8.46

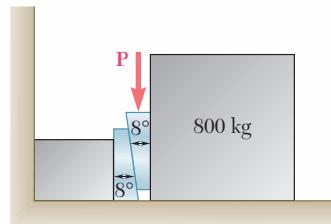


Fig. P8.49

**8.50 and 8.51** The elevation of the end of the steel beam supported by a concrete floor is adjusted by means of the steel wedges *E* and *F*. The base plate *CD* has been welded to the lower flange of the beam, and the end reaction of the beam is known to be 100 kN. The coefficient of static friction is 0.30 between two steel surfaces and 0.60 between steel and concrete. If the horizontal motion of the beam is prevented by the force **Q**, determine (a) the force **P** required to raise the beam, (b) the corresponding force **Q**.

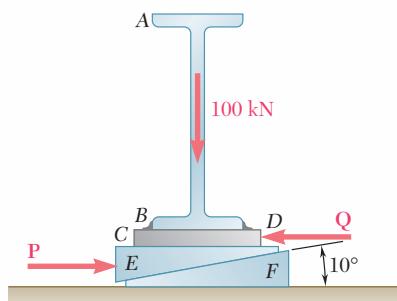


Fig. P8.50

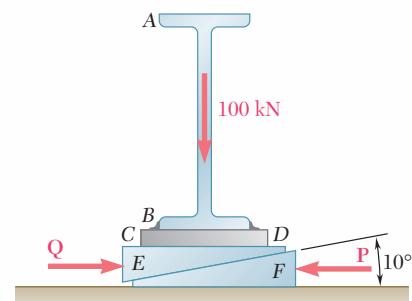


Fig. P8.51

- 8.52** A wedge *A* of negligible weight is to be driven between two 100-lb plates *B* and *C*. The coefficient of static friction between all surfaces of contact is 0.35. Determine the magnitude of the force **P** required to start moving the wedge (*a*) if the plates are equally free to move, (*b*) if plate *C* is securely bolted to the surface.

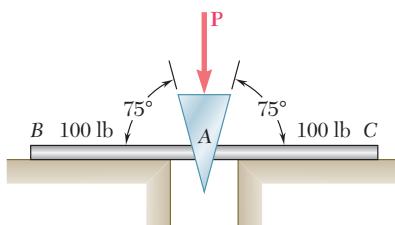


Fig. P8.52

- 8.53** Block *A* supports a pipe column and rests as shown on wedge *B*. Knowing that the coefficient of static friction at all surfaces of contact is 0.25 and that  $\theta = 45^\circ$ , determine the smallest force **P** required to raise block *A*.

- 8.54** Block *A* supports a pipe column and rests as shown on wedge *B*. Knowing that the coefficient of static friction at all surfaces of contact is 0.25 and that  $\theta = 45^\circ$ , determine the smallest force **P** for which equilibrium is maintained.

- 8.55** Block *A* supports a pipe column and rests as shown on wedge *B*. The coefficient of static friction at all surfaces of contact is 0.25. If **P** = 0, determine (*a*) the angle  $\theta$  for which sliding is impending, (*b*) the corresponding force exerted on the block by the vertical wall.

- 8.56** A  $12^\circ$  wedge is used to spread a split ring. The coefficient of static friction between the wedge and the ring is 0.30. Knowing that a force **P** of magnitude 25 lb was required to insert the wedge, determine the magnitude of the forces exerted on the ring by the wedge after insertion.

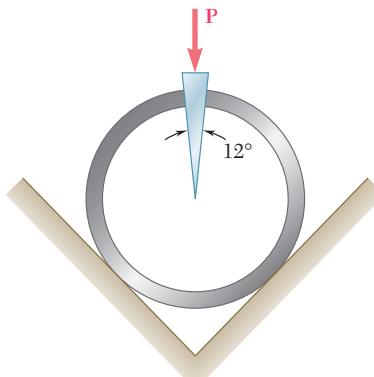


Fig. P8.56

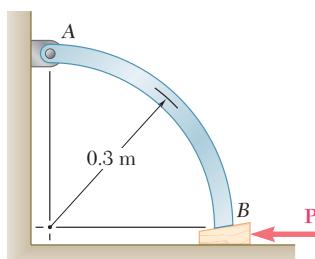


Fig. P8.57

- 8.57** A  $10^\circ$  wedge is to be forced under end *B* of the 5-kg rod *AB*. Knowing that the coefficient of static friction is 0.40 between the wedge and the rod and 0.20 between the wedge and the floor, determine the smallest force **P** required to raise end *B* of the rod.

- 8.58** A  $10^\circ$  wedge is used to split a section of a log. The coefficient of static friction between the wedge and the log is 0.35. Knowing that a force  $\mathbf{P}$  of magnitude 600 lb was required to insert the wedge, determine the magnitude of the forces exerted on the wood by the wedge after insertion.

- 8.59** A conical wedge is placed between two horizontal plates that are then slowly moved toward each other. Indicate what will happen to the wedge (a) if  $\mu_s = 0.20$ , (b) if  $\mu_s = 0.30$ .

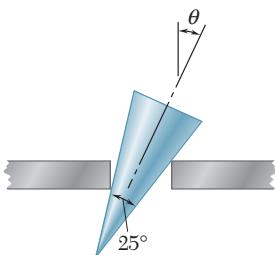


Fig. P8.59

- 8.60** A  $15^\circ$  wedge is forced under a 50-kg pipe as shown. The coefficient of static friction at all surfaces is 0.20. (a) Show that slipping will occur between the pipe and the vertical wall. (b) Determine the force  $\mathbf{P}$  required to move the wedge.

- 8.61** A  $15^\circ$  wedge is forced under a 50-kg pipe as shown. Knowing that the coefficient of static friction at both surfaces of the wedge is 0.20, determine the largest coefficient of static friction between the pipe and the vertical wall for which slipping will occur at A.

- 8.62** An  $8^\circ$  wedge is to be forced under a machine base at B. Knowing that the coefficient of static friction at all surfaces of contact is 0.15, (a) determine the force  $\mathbf{P}$  required to move the wedge, (b) indicate whether the machine base will slide on the floor.

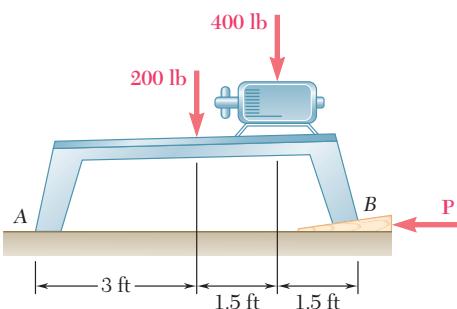


Fig. P8.62

- 8.63** Solve Prob. 8.62 assuming that the wedge is to be forced under the machine base at A instead of B.

- \*8.64** A 200-N block rests as shown on a wedge of negligible weight. The coefficient of static friction  $\mu_s$  is the same at both surfaces of the wedge, and friction between the block and the vertical wall may be neglected. For  $P = 100$  N, determine the value of  $\mu_s$  for which motion is impending. (Hint: Solve the equation obtained by trial and error.)

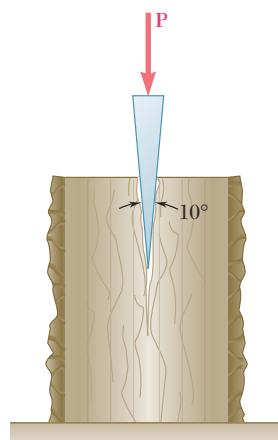


Fig. P8.58

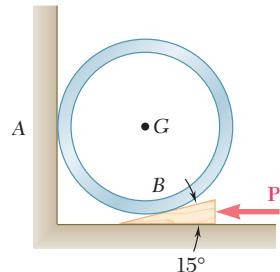


Fig. P8.60 and P8.61

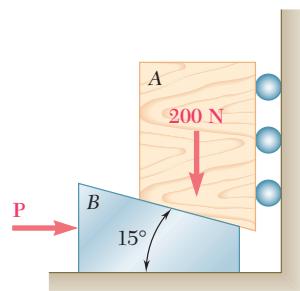


Fig. P8.64

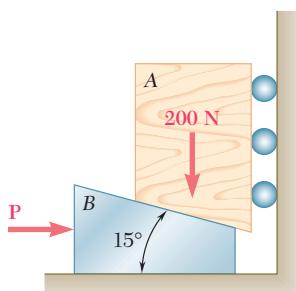


Fig. P8.64 (repeated)

**\*8.65** Solve Prob. 8.64 assuming that the rollers are removed and that  $\mu_s$  is the coefficient of friction at all surfaces of contact.

**8.66** Derive the following formulas relating the load  $\mathbf{W}$  and the force  $\mathbf{P}$  exerted on the handle of the jack discussed in Sec. 8.6. (a)  $P = (Wr/a) \tan(\theta + \phi_s)$ , to raise the load; (b)  $P = (Wr/a) \tan(\phi_s - \theta)$ , to lower the load if the screw is self-locking; (c)  $P = (Wr/a) \tan(\theta - \phi_s)$ , to hold the load if the screw is not self-locking.

**8.67** The square-threaded worm gear shown has a mean radius of 1.5 in. and a lead of 0.375 in. The large gear is subjected to a constant clockwise couple of 7.2 kip · in. Knowing that the coefficient of static friction between the two gears is 0.12, determine the couple that must be applied to shaft AB in order to rotate the large gear counterclockwise. Neglect friction in the bearings at A, B, and C.

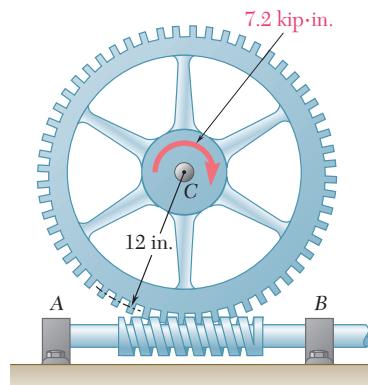


Fig. P8.67

**8.68** In Prob. 8.67, determine the couple that must be applied to shaft AB in order to rotate the large gear clockwise.

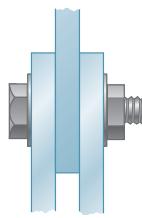


Fig. P8.69

**8.69** High-strength bolts are used in the construction of many steel structures. For a 24-mm-nominal-diameter bolt the required minimum bolt tension is 210 kN. Assuming the coefficient of friction to be 0.40, determine the required couple that should be applied to the bolt and nut. The mean diameter of the thread is 22.6 mm, and the lead is 3 mm. Neglect friction between the nut and washer, and assume the bolt to be square-threaded.

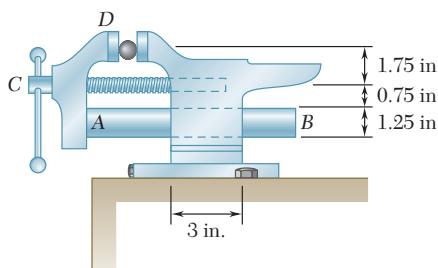
**8.70** The ends of two fixed rods A and B are each made in the form of a single-threaded screw of mean radius 6 mm and pitch 2 mm. Rod A has a right-handed thread and rod B has a left-handed thread. The coefficient of static friction between the rods and the threaded sleeve is 0.12. Determine the magnitude of the couple that must be applied to the sleeve in order to draw the rods closer together.



Fig. P8.70

- 8.71** Assuming that in Prob. 8.70 a right-handed thread is used on *both* rods A and B, determine the magnitude of the couple that must be applied to the sleeve in order to rotate it.

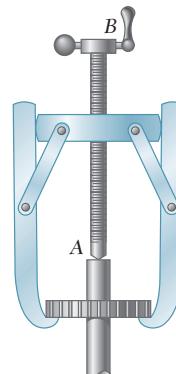
- 8.72** In the machinist's vise shown, the movable jaw D is rigidly attached to the tongue AB that fits loosely into the fixed base of the vise. The screw is single-threaded into the fixed base and has a mean diameter of 0.75 in. and a pitch of 0.25 in. The coefficient of static friction is 0.25 between the threads and also between the tongue and the body. Neglecting bearing friction between the screw and the movable head, determine the couple that must be applied to the handle in order to produce a clamping force of 1 kip.



**Fig. P8.72**

- 8.73** In Prob. 8.72, a clamping force of 1 kip was obtained by tightening the vise. Determine the couple that must be applied to the screw to loosen the vise.

- 8.74** In the gear-pulling assembly shown the square-threaded screw AB has a mean radius of 15 mm and a lead of 4 mm. Knowing that the coefficient of static friction is 0.10, determine the couple that must be applied to the screw in order to produce a force of 3 kN on the gear. Neglect friction at end A of the screw.



**Fig. P8.74**

## \*8.7 JOURNAL BEARINGS. AXLE FRICTION

Journal bearings are used to provide lateral support to rotating shafts and axles. Thrust bearings, which will be studied in the next section, are used to provide axial support to shafts and axles. If the journal bearing is fully lubricated, the frictional resistance depends upon the speed of rotation, the clearance between axle and bearing, and the viscosity of the lubricant. As indicated in Sec. 8.1, such problems are studied in fluid mechanics. The methods of this chapter, however, can be applied to the study of axle friction when the bearing is not lubricated or only partially lubricated. It can then be assumed that the axle and the bearing are in direct contact along a single straight line.

Consider two wheels, each of weight **W**, rigidly mounted on an axle supported symmetrically by two journal bearings (Fig. 8.10a). If the wheels rotate, we find that to keep them rotating at constant speed, it is necessary to apply to each of them a couple **M**. The free-body

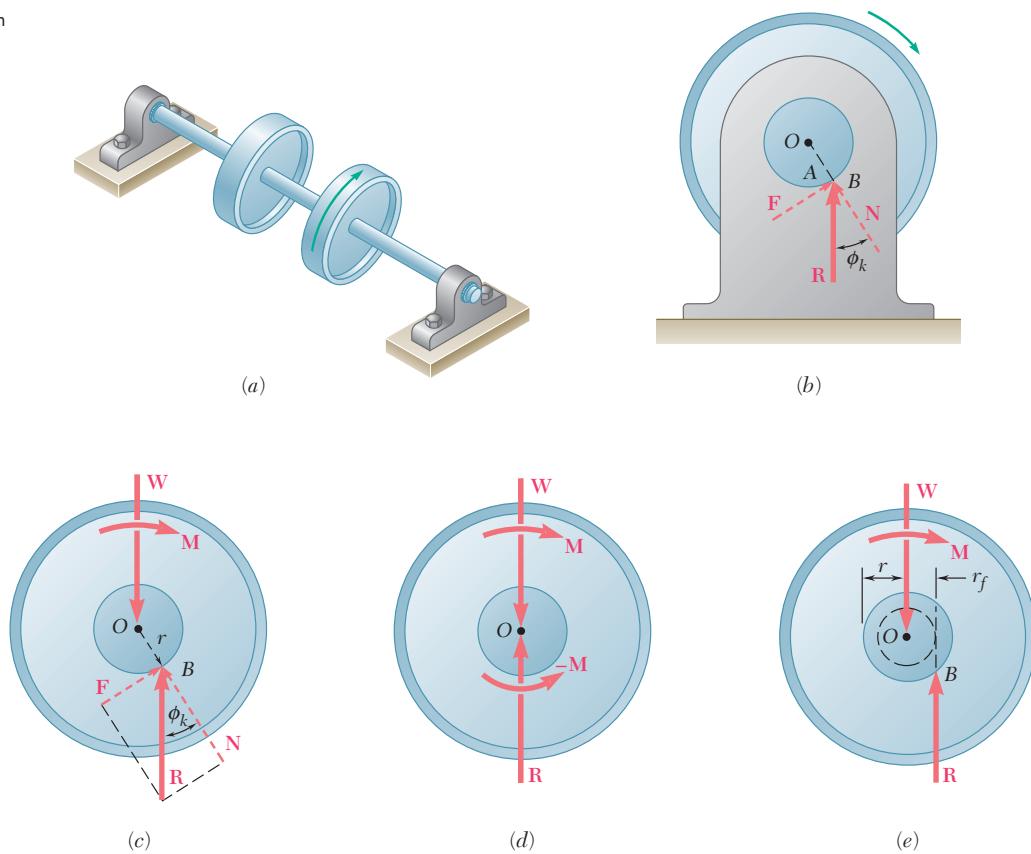


Fig. 8.10

diagram in Fig. 8.10c represents one of the wheels and the corresponding half axle in projection on a plane perpendicular to the axle. The forces acting on the free body include the weight  $\mathbf{W}$  of the wheel, the couple  $\mathbf{M}$  required to maintain its motion, and a force  $\mathbf{R}$  representing the reaction of the bearing. This force is vertical, equal, and opposite to  $\mathbf{W}$  but does not pass through the center  $O$  of the axle;  $\mathbf{R}$  is located to the right of  $O$  at a distance such that its moment about  $O$  balances the moment  $\mathbf{M}$  of the couple. Therefore, contact between the axle and bearing does not take place at the lowest point  $A$  when the axle rotates. It takes place at point  $B$  (Fig. 8.10b) or, rather, along a straight line intersecting the plane of the figure at  $B$ . Physically, this is explained by the fact that when the wheels are set in motion, the axle “climbs” in the bearings until slippage occurs. After sliding back slightly, the axle settles more or less in the position shown. This position is such that the angle between the reaction  $\mathbf{R}$  and the normal to the surface of the bearing is equal to the angle of kinetic friction  $\phi_k$ . The distance from  $O$  to the line of action of  $\mathbf{R}$  is thus  $r \sin \phi_k$ , where  $r$  is the radius of the axle. Writing that  $\sum M_O = 0$  for the forces acting on the free body considered, we obtain the magnitude of the couple  $\mathbf{M}$  required to overcome the frictional resistance of one of the bearings:

$$M = Rr \sin \phi_k \quad (8.5)$$

Observing that, for small values of the angle of friction,  $\sin \phi_k$  can be replaced by  $\tan \phi_k$ , that is, by  $\mu_k$ , we write the approximate formula

$$M \approx Rr\mu_k \quad (8.6)$$

In the solution of certain problems, it may be more convenient to let the line of action of  $\mathbf{R}$  pass through  $O$ , as it does when the axle does not rotate. A couple  $-\mathbf{M}$  of the same magnitude as the couple  $\mathbf{M}$  but of opposite sense must then be added to the reaction  $\mathbf{R}$  (Fig. 8.10d). This couple represents the frictional resistance of the bearing.

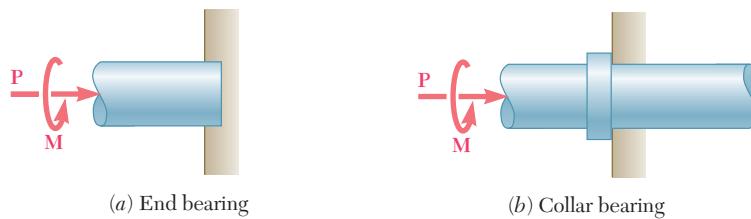
In case a graphical solution is preferred, the line of action of  $\mathbf{R}$  can be readily drawn (Fig. 8.10e) if we note that it must be tangent to a circle centered at  $O$  and of radius

$$r_f = r \sin \phi_k \approx r\mu_k \quad (8.7)$$

This circle is called the *circle of friction* of the axle and bearing and is independent of the loading conditions of the axle.

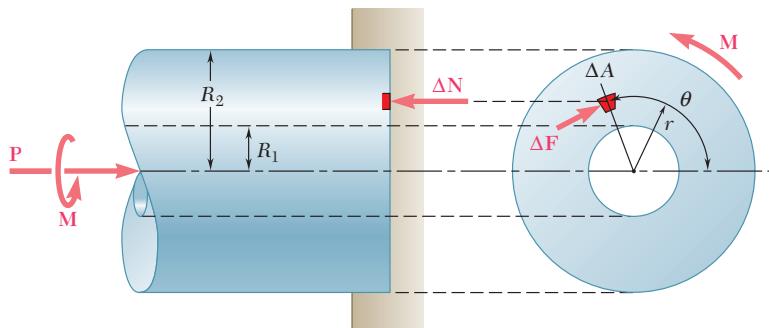
## \*8.8 THRUST BEARINGS. DISK FRICTION

Two types of thrust bearings are used to provide axial support to rotating shafts and axles: (1) *end bearings* and (2) *collar bearings* (Fig. 8.11). In the case of collar bearings, friction forces develop between the two ring-shaped areas which are in contact. In the case of end bearings, friction takes place over full circular areas, or over ring-shaped areas when the end of the shaft is hollow. Friction between circular areas, called *disk friction*, also occurs in other mechanisms, such as *disk clutches*.



**Fig. 8.11** Thrust bearings.

To obtain a formula which is valid in the most general case of disk friction, let us consider a rotating hollow shaft. A couple  $\mathbf{M}$  keeps the shaft rotating at constant speed while a force  $\mathbf{P}$  maintains it in contact with a fixed bearing (Fig. 8.12). Contact between the shaft and



**Fig. 8.12**

the bearing takes place over a ring-shaped area of inner radius  $R_1$  and outer radius  $R_2$ . Assuming that the pressure between the two surfaces in contact is uniform, we find that the magnitude of the normal force  $\Delta N$  exerted on an element of area  $\Delta A$  is  $\Delta N = P \Delta A/A$ , where  $A = \pi(R_2^2 - R_1^2)$ , and that the magnitude of the friction force  $\Delta F$  acting on  $\Delta A$  is  $\Delta F = \mu_k \Delta N$ . Denoting by  $r$  the distance from the axis of the shaft to the element of area  $\Delta A$ , we express the magnitude  $\Delta M$  of the moment of  $\Delta F$  about the axis of the shaft as follows:

$$\Delta M = r \Delta F = \frac{r \mu_k P \Delta A}{\pi(R_2^2 - R_1^2)}$$

The equilibrium of the shaft requires that the moment  $\mathbf{M}$  of the couple applied to the shaft be equal in magnitude to the sum of the moments of the friction forces  $\Delta F$ . Replacing  $\Delta A$  by the infinitesimal element  $dA = r d\theta dr$  used with polar coordinates, and integrating over the area of contact, we thus obtain the following expression for the magnitude of the couple  $\mathbf{M}$  required to overcome the frictional resistance of the bearing:

$$\begin{aligned} M &= \frac{\mu_k P}{\pi(R_2^2 - R_1^2)} \int_0^{2\pi} \int_{R_1}^{R_2} r^2 dr d\theta \\ &= \frac{\mu_k P}{\pi(R_2^2 - R_1^2)} \int_0^{2\pi} \frac{1}{3}(R_2^3 - R_1^3) d\theta \\ M &= \frac{2}{3}\mu_k P \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2} \end{aligned} \quad (8.8)$$

When contact takes place over a full circle of radius  $R$ , formula (8.8) reduces to

$$M = \frac{2}{3}\mu_k PR \quad (8.9)$$

The value of  $M$  is then the same as would be obtained if contact between shaft and bearing took place at a single point located at a distance  $2R/3$  from the axis of the shaft.

The largest couple which can be transmitted by a disk clutch without causing slippage is given by a formula similar to (8.9), where  $\mu_k$  has been replaced by the coefficient of static friction  $\mu_s$ .

## \*8.9 WHEEL FRICTION. ROLLING RESISTANCE

The wheel is one of the most important inventions of our civilization. Its use makes it possible to move heavy loads with relatively little effort. Because the point of the wheel in contact with the ground at any given instant has no relative motion with respect to the ground, the wheel eliminates the large friction forces which would arise if the load were in direct contact with the ground. However, some resistance to the wheel's motion exists. This resistance has two distinct causes. It is due (1) to a combined effect of axle friction and friction at the rim and (2) to the fact that the wheel and the ground

deform, with the result that contact between wheel and ground takes place over a certain area, rather than at a single point.

To understand better the first cause of resistance to the motion of a wheel, let us consider a railroad car supported by eight wheels mounted on axles and bearings. The car is assumed to be moving to the right at constant speed along a straight horizontal track. The free-body diagram of one of the wheels is shown in Fig. 8.13a. The forces acting on the free body include the load **W** supported by the wheel and the normal reaction **N** of the track. Since **W** is drawn through the center *O* of the axle, the frictional resistance of the bearing should be represented by a counterclockwise couple **M** (see Sec. 8.7). To keep the free body in equilibrium, we must add two equal and opposite forces **P** and **F**, forming a clockwise couple of moment **-M**. The force **F** is the friction force exerted by the track on the wheel, and **P** represents the force which should be applied to the wheel to keep it rolling at constant speed. Note that the forces **P** and **F** would not exist if there were no friction between wheel and track. The couple **M** representing the axle friction would then be zero; the wheel would slide on the track without turning in its bearing.

The couple **M** and the forces **P** and **F** also reduce to zero when there is no axle friction. For example, a wheel which is not held in bearings and rolls freely and at constant speed on horizontal ground (Fig. 8.13b) will be subjected to only two forces: its own weight **W** and the normal reaction **N** of the ground. Regardless of the value of the coefficient of friction between wheel and ground no friction force will act on the wheel. A wheel rolling freely on horizontal ground should thus keep rolling indefinitely.

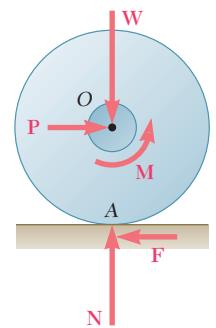
Experience, however, indicates that the wheel will slow down and eventually come to rest. This is due to the second type of resistance mentioned at the beginning of this section, known as the *rolling resistance*. Under the load **W**, both the wheel and the ground deform slightly, causing the contact between wheel and ground to take place over a certain area. Experimental evidence shows that the resultant of the forces exerted by the ground on the wheel over this area is a force **R** applied at a point *B*, which is not located directly under the center *O* of the wheel, but slightly in front of it (Fig. 8.13c). To balance the moment of **W** about *B* and to keep the wheel rolling at constant speed, it is necessary to apply a horizontal force **P** at the center of the wheel. Writing  $\Sigma M_B = 0$ , we obtain

$$Pr = Wb \quad (8.10)$$

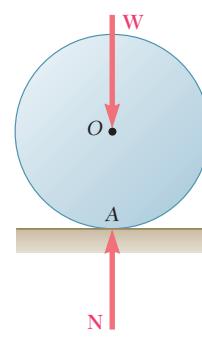
where *r* = radius of wheel

*b* = horizontal distance between *O* and *B*

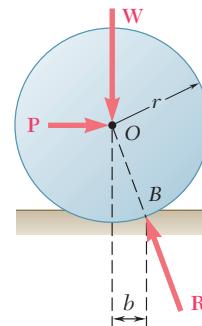
The distance *b* is commonly called the *coefficient of rolling resistance*. It should be noted that *b* is not a dimensionless coefficient since it represents a length; *b* is usually expressed in inches or in millimeters. The value of *b* depends upon several parameters in a manner which has not yet been clearly established. Values of the coefficient of rolling resistance vary from about 0.01 in. or 0.25 mm for a steel wheel on a steel rail to 5.0 in. or 125 mm for the same wheel on soft ground.



(a) Effect of axle friction



(b) Free wheel



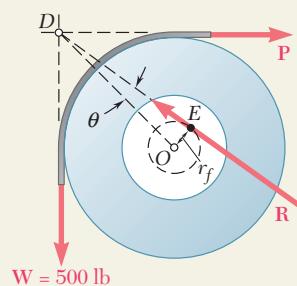
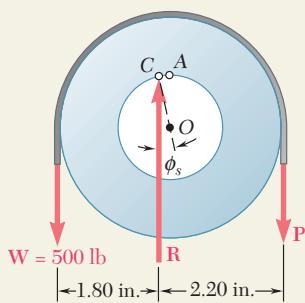
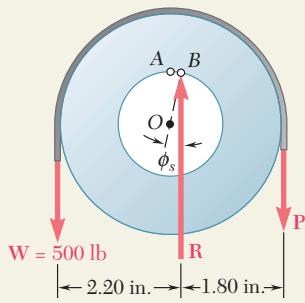
(c) Rolling resistance

**Fig. 8.13**

## SAMPLE PROBLEM 8.6

A pulley of diameter 4 in. can rotate about a fixed shaft of diameter 2 in. The coefficient of static friction between the pulley and shaft is 0.20. Determine (a) the smallest vertical force  $\mathbf{P}$  required to start raising a 500-lb load, (b) the smallest vertical force  $\mathbf{P}$  required to hold the load, (c) the smallest horizontal force  $\mathbf{P}$  required to start raising the same load.

### SOLUTION



**a. Vertical Force  $\mathbf{P}$  Required to Start Raising the Load.** When the forces in both parts of the rope are equal, contact between the pulley and shaft takes place at A. When  $\mathbf{P}$  is increased, the pulley rolls around the shaft slightly and contact takes place at B. The free-body diagram of the pulley when motion is impending is drawn. The perpendicular distance from the center  $O$  of the pulley to the line of action of  $\mathbf{R}$  is

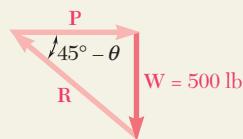
$$r_f = r \sin \phi_s \approx r\mu_s \quad r_f \approx (1 \text{ in.})0.20 = 0.20 \text{ in.}$$

Summing moments about B, we write

$$+\uparrow \sum M_B = 0: \quad (2.20 \text{ in.})(500 \text{ lb}) - (1.80 \text{ in.})P = 0 \\ P = 611 \text{ lb} \quad \mathbf{P} = 611 \text{ lb} \downarrow$$

**b. Vertical Force  $\mathbf{P}$  to Hold the Load.** As the force  $\mathbf{P}$  is decreased, the pulley rolls around the shaft and contact takes place at C. Considering the pulley as a free body and summing moments about C, we write

$$+\uparrow \sum M_C = 0: \quad (1.80 \text{ in.})(500 \text{ lb}) - (2.20 \text{ in.})P = 0 \\ P = 409 \text{ lb} \quad \mathbf{P} = 409 \text{ lb} \downarrow$$



**c. Horizontal Force  $\mathbf{P}$  to Start Raising the Load.** Since the three forces  $\mathbf{W}$ ,  $\mathbf{P}$ , and  $\mathbf{R}$  are not parallel, they must be concurrent. The direction of  $\mathbf{R}$  is thus determined from the fact that its line of action must pass through the point of intersection D of  $\mathbf{W}$  and  $\mathbf{P}$ , and must be tangent to the circle of friction. Recalling that the radius of the circle of friction is  $r_f = 0.20$  in., we write

$$\sin \theta = \frac{OE}{OD} = \frac{0.20 \text{ in.}}{(2 \text{ in.})\sqrt{2}} = 0.0707 \quad \theta = 4.1^\circ$$

From the force triangle, we obtain

$$P = W \cot (45^\circ - \theta) = (500 \text{ lb}) \cot 40.9^\circ \\ = 577 \text{ lb} \quad \mathbf{P} = 577 \text{ lb} \rightarrow$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned about several additional engineering applications of the laws of friction.

**1. Journal bearings and axle friction.** In journal bearings, the *reaction does not pass through the center of the shaft or axle* which is being supported. The distance from the center of the shaft or axle to the line of action of the reaction (Fig. 8.10) is defined by the equation.

$$r_f = r \sin \phi_k \approx r\mu_k$$

if motion is actually taking place, and by the equation

$$r_f = r \sin \phi_s \approx r\mu_s$$

if the motion is impending.

Once you have determined the line of action of the reaction, you can draw a *free-body diagram* and use the corresponding equations of equilibrium to complete your solution [Sample Prob. 8.6]. In some problems, it is useful to observe that the line of action of the reaction must be tangent to a circle of radius  $r_f \approx r\mu_k$ , or  $r_f \approx r\mu_s$ , known as the *circle of friction* [Sample Prob. 8.6, part c].

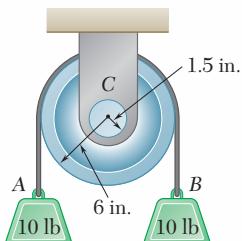
**2. Thrust bearings and disk friction.** In a *thrust bearing* the magnitude of the couple required to overcome frictional resistance is equal to the sum of the moments of the *kinetic friction forces* exerted on the elements of the end of the shaft [Eqs. (8.8) and (8.9)].

An example of disk friction is the *disk clutch*. It is analyzed in the same way as a thrust bearing, except that to determine the largest couple that can be transmitted, you must compute the sum of the moments of the *maximum static friction forces* exerted on the disk.

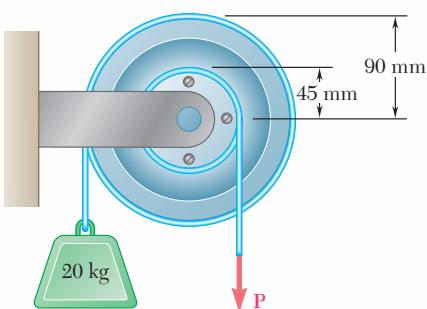
**3. Wheel friction and rolling resistance.** You saw that the rolling resistance of a wheel is caused by deformations of both the wheel and the ground. The line of action of the reaction **R** of the ground on the wheel intersects the ground at a horizontal distance  $b$  from the center of the wheel. The distance  $b$  is known as the *coefficient of rolling resistance* and is expressed in inches or millimeters.

**4. In problems involving both rolling resistance and axle friction,** your free-body diagram should show that the line of action of the reaction **R** of the ground on the wheel is tangent to the friction circle of the axle and intersects the ground at a horizontal distance from the center of the wheel equal to the coefficient of rolling resistance.

# PROBLEMS



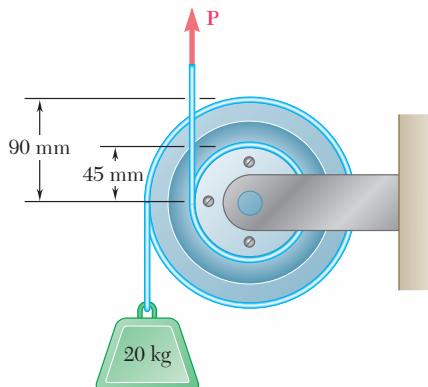
**Fig. P8.75**



**Fig. P8.76 and P8.78**

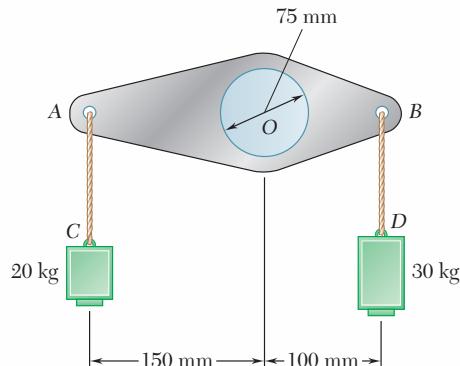
**8.75** A 6-in.-radius pulley of weight 5 lb is attached to a 1.5-in.-radius shaft that fits loosely in a fixed bearing. It is observed that the pulley will just start rotating if a 0.5-lb weight is added to block A. Determine the coefficient of static friction between the shaft and the bearing.

**8.76 and 8.77** The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the force **P** required to start raising the load.



**Fig. P8.77 and P8.79**

**8.78 and 8.79** The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the smallest force **P** required to maintain equilibrium.



**Fig. P8.80**

**8.80** A lever of negligible weight is loosely fitted onto a 75-mm-diameter fixed shaft. It is observed that the lever will just start rotating if a 3-kg mass is added at C. Determine the coefficient of static friction between the shaft and the lever.

- 8.81** The block and tackle shown are used to raise a 150-lb load. Each of the 3-in.-diameter pulleys rotates on a 0.5-in.-diameter axle. Knowing that the coefficient of static friction is 0.20, determine the tension in each portion of the rope as the load is slowly raised.

- 8.82** The block and tackle shown are used to lower a 150-lb load. Each of the 3-in.-diameter pulleys rotates on a 0.5-in.-diameter axle. Knowing that the coefficient of static friction is 0.20, determine the tension in each portion of the rope as the load is slowly lowered.

- 8.83** A loaded railroad car has a mass of 30 Mg and is supported by eight 800-mm-diameter wheels with 125-mm-diameter axles. Knowing that the coefficients of friction are  $\mu_s = 0.020$  and  $\mu_k = 0.015$ , determine the horizontal force required (a) to start the car moving, (b) to keep the car moving at a constant speed. Neglect rolling resistance between the wheels and the track.

- 8.84 and 8.85** A lever  $AB$  of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force  $P$  required to start the lever rotating counterclockwise.

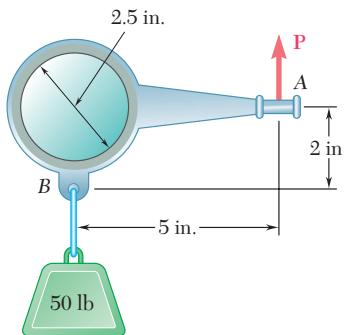


Fig. P8.84 and P8.86

- 8.86 and 8.87** A lever  $AB$  of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force  $P$  required to start the lever rotating clockwise.

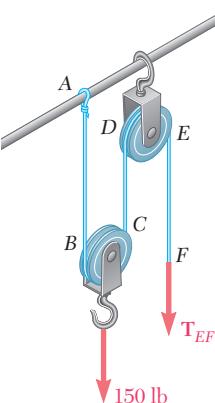


Fig. P8.81 and P8.82

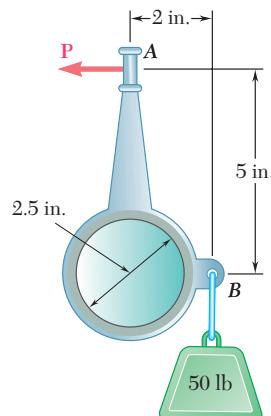


Fig. P8.85 and P8.87

- 8.88** The link arrangement shown is frequently used in highway bridge construction to allow for expansion due to changes in temperature. At each of the 60-mm-diameter pins  $A$  and  $B$  the coefficient of static friction is 0.20. Knowing that the vertical component of the force exerted by  $BC$  on the link is 200 kN, determine (a) the horizontal force that should be exerted on beam  $BC$  to just move the link, (b) the angle that the resulting force exerted by beam  $BC$  on the link will form with the vertical.

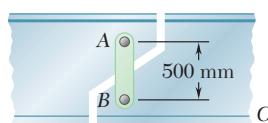


Fig. P8.88

- 8.89** A scooter is to be designed to roll down a 2 percent slope at a constant speed. Assuming that the coefficient of kinetic friction between the 25-mm-diameter axles and the bearings is 0.10, determine the required diameter of the wheels. Neglect the rolling resistance between the wheels and the ground.

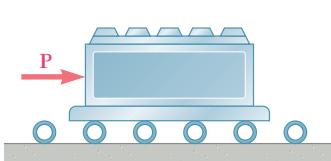


Fig. P8.96

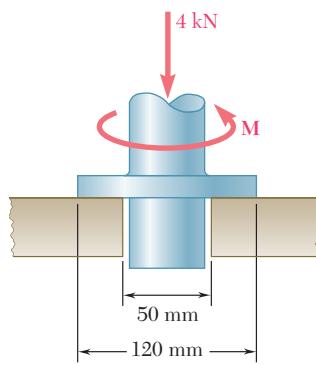


Fig. P8.91

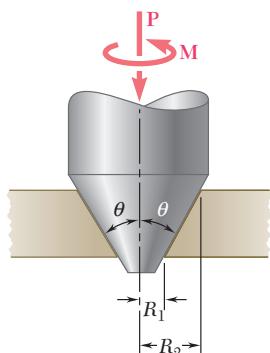


Fig. P8.94

- 8.90** A 50-lb electric floor polisher is operated on a surface for which the coefficient of kinetic friction is 0.25. Assuming that the normal force per unit area between the disk and the floor is uniformly distributed, determine the magnitude  $Q$  of the horizontal forces required to prevent motion of the machine.

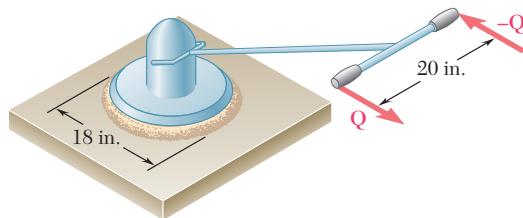


Fig. P8.90

- 8.91** Knowing that a couple of magnitude  $30 \text{ N} \cdot \text{m}$  is required to start the vertical shaft rotating, determine the coefficient of static friction between the annular surfaces of contact.

- \*8.92** The frictional resistance of a thrust bearing decreases as the shaft and bearing surfaces wear out. It is generally assumed that the wear is directly proportional to the distance traveled by any given point of the shaft and thus to the distance  $r$  from the point to the axis of the shaft. Assuming, then, that the normal force per unit area is inversely proportional to  $r$ , show that the magnitude  $M$  of the couple required to overcome the frictional resistance of a worn-out end bearing (with contact over the full circular area) is equal to 75 percent of the value given by Eq. (8.9) for a new bearing.

- \*8.93** Assuming that bearings wear out as indicated in Prob. 8.92, show that the magnitude  $M$  of the couple required to overcome the frictional resistance of a worn-out collar bearing is

$$M = \frac{1}{2}\mu_k P(R_1 + R_2)$$

where  $P$  = magnitude of the total axial force

$R_1, R_2$  = inner and outer radii of collar

- \*8.94** Assuming that the pressure between the surfaces of contact is uniform, show that the magnitude  $M$  of the couple required to overcome frictional resistance for the conical bearing shown is

$$M = \frac{2}{3} \frac{\mu_k P}{\sin \theta} \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$$

- 8.95** Solve Prob. 8.90 assuming that the normal force per unit area between the disk and the floor varies linearly from a maximum at the center to zero at the circumference of the disk.

- 8.96** A 900-kg machine base is rolled along a concrete floor using a series of steel pipes with outside diameters of 100 mm. Knowing that the coefficient of rolling resistance is 0.5 mm between the pipes and the base and 1.25 mm between the pipes and the concrete floor, determine the magnitude of the force  $P$  required to slowly move the base along the floor.

**8.97** Knowing that a 6-in.-diameter disk rolls at a constant velocity down a 2 percent incline, determine the coefficient of rolling resistance between the disk and the incline.

**8.98** Determine the horizontal force required to move a 2500-lb automobile with 23-in.-diameter tires along a horizontal road at a constant speed. Neglect all forms of friction except rolling resistance, and assume the coefficient of rolling resistance to be 0.05 in.

**8.99** Solve Prob. 8.83 including the effect of a coefficient of rolling resistance of 0.5 mm.

**8.100** Solve Prob. 8.89 including the effect of a coefficient of rolling resistance of 1.75 mm.

## 8.10 BELT FRICTION

Consider a flat belt passing over a fixed cylindrical drum (Fig. 8.14a). We propose to determine the relation existing between the values  $T_1$  and  $T_2$  of the tension in the two parts of the belt when the belt is just about to slide toward the right.

Let us detach from the belt a small element  $PP'$  subtending an angle  $\Delta\theta$ . Denoting by  $T$  the tension at  $P$  and by  $T + \Delta T$  the tension at  $P'$ , we draw the free-body diagram of the element of the belt (Fig. 8.14b). Besides the two forces of tension, the forces acting on the free body are the normal component  $\Delta N$  of the reaction of the drum and the friction force  $\Delta F$ . Since motion is assumed to be impending, we have  $\Delta F = \mu_s \Delta N$ . It should be noted that if  $\Delta\theta$  is made to approach zero, the magnitudes  $\Delta N$  and  $\Delta F$ , and the difference  $\Delta T$  between the tension at  $P$  and the tension at  $P'$ , will also approach zero; the value  $T$  of the tension at  $P$ , however, will remain unchanged. This observation helps in understanding our choice of notations.

Choosing the coordinate axes shown in Fig. 8.14b, we write the equations of equilibrium for the element  $PP'$ :

$$\Sigma F_x = 0: \quad (T + \Delta T) \cos \frac{\Delta\theta}{2} - T \cos \frac{\Delta\theta}{2} - \mu_s \Delta N = 0 \quad (8.11)$$

$$\Sigma F_y = 0: \quad \Delta N - (T + \Delta T) \sin \frac{\Delta\theta}{2} - T \sin \frac{\Delta\theta}{2} = 0 \quad (8.12)$$

Solving Eq. (8.12) for  $\Delta N$  and substituting into (8.11), we obtain after reductions

$$\Delta T \cos \frac{\Delta\theta}{2} - \mu_s (2T + \Delta T) \sin \frac{\Delta\theta}{2} = 0$$

Both terms are now divided by  $\Delta\theta$ . For the first term, this is done simply by dividing  $\Delta T$  by  $\Delta\theta$ . The division of the second term is

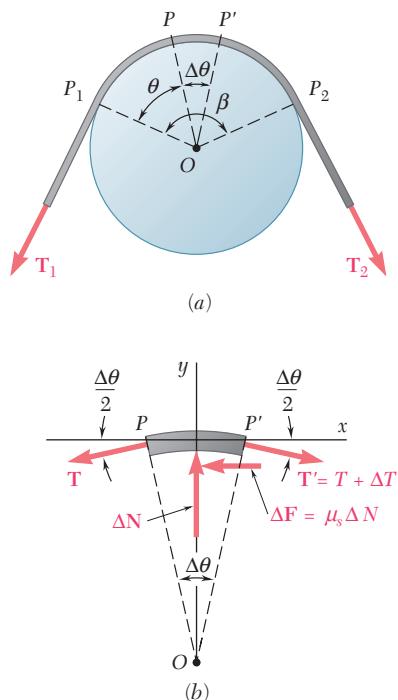


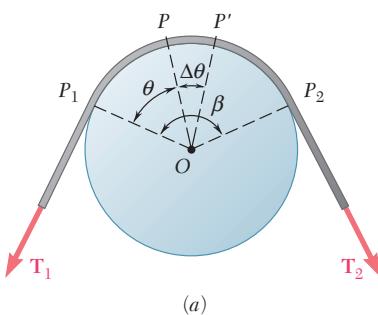
Fig. 8.14

carried out by dividing the terms in the parentheses by 2 and the sine by  $\Delta\theta/2$ . We write

$$\frac{\Delta T}{\Delta\theta} \cos \frac{\Delta\theta}{2} - \mu_s \left( T + \frac{\Delta T}{2} \right) \frac{\sin(\Delta\theta/2)}{\Delta\theta/2} = 0$$

If we now let  $\Delta\theta$  approach 0, the cosine approaches 1 and  $\Delta T/2$  approaches zero, as noted above. The quotient of  $\sin(\Delta\theta/2)$  over  $\Delta\theta/2$  approaches 1, according to a lemma derived in all calculus textbooks. Since the limit of  $\Delta T/\Delta\theta$  is by definition equal to the derivative  $dT/d\theta$ , we write

$$\frac{dT}{d\theta} - \mu_s T = 0 \quad \frac{dT}{T} = \mu_s d\theta$$



**Fig. 8.14a** (repeated)

Both members of the last equation (Fig. 8.14a) will now be integrated from  $P_1$  to  $P_2$ . At  $P_1$ , we have  $\theta = 0$  and  $T = T_1$ ; at  $P_2$ , we have  $\theta = \beta$  and  $T = T_2$ . Integrating between these limits, we write

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^\beta \mu_s d\theta$$

$$\ln T_2 - \ln T_1 = \mu_s \beta$$

or, noting that the left-hand member is equal to the natural logarithm of the quotient of  $T_2$  and  $T_1$ ,

$$\ln \frac{T_2}{T_1} = \mu_s \beta \quad (8.13)$$

This relation can also be written in the form

$$\frac{T_2}{T_1} = e^{\mu_s \beta} \quad (8.14)$$



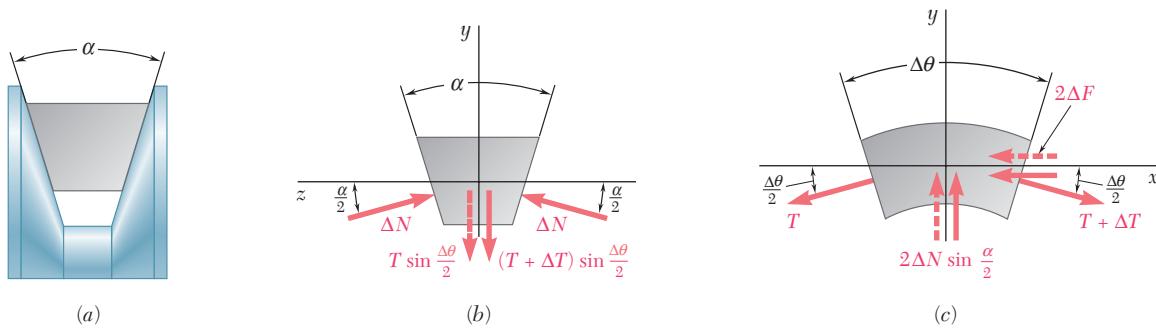
**Photo 8.3** By wrapping the rope around the bollard, the force exerted by the worker to control the rope is much smaller than the tension in the taut portion of the rope.

The formulas we have derived apply equally well to problems involving flat belts passing over fixed cylindrical drums and to problems involving ropes wrapped around a post or capstan. They can also be used to solve problems involving band brakes. In such problems, it is the drum which is about to rotate, while the band remains fixed. The formulas can also be applied to problems involving belt drives. In these problems, both the pulley and the belt rotate; our concern is then to find whether the belt will slip, i.e., whether it will move *with respect* to the pulley.

Formulas (8.13) and (8.14) should be used only if the belt, rope, or brake is *about to slip*. Formula (8.14) will be used if  $T_1$  or  $T_2$  is desired; formula (8.13) will be preferred if either  $\mu_s$  or the angle of contact  $\beta$  is desired. We should note that  $T_2$  is always larger than  $T_1$ ;  $T_2$  therefore represents the tension in that part of the belt or rope which *pulls*, while  $T_1$  is the tension in the part which *resists*. We should also observe that the angle of contact  $\beta$  must be expressed in *radians*. The angle  $\beta$  may be larger than  $2\pi$ ; for example, if a rope is wrapped  $n$  times around a post,  $\beta$  is equal to  $2\pi n$ .

If the belt, rope, or brake is actually slipping, formulas similar to (8.13) and (8.14), but involving the coefficient of kinetic friction  $\mu_k$ , should be used. If the belt, rope, or brake is not slipping and is not about to slip, none of these formulas can be used.

The belts used in belt drives are often V-shaped. In the V belt shown in Fig. 8.15a contact between belt and pulley takes place



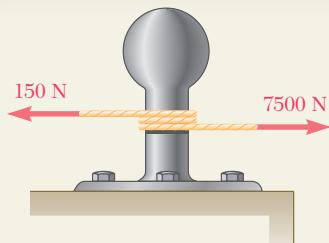
**Fig. 8.15**

along the sides of the groove. The relation existing between the values  $T_1$  and  $T_2$  of the tension in the two parts of the belt when the belt is just about to slip can again be obtained by drawing the free-body diagram of an element of belt (Fig. 8.15b and c). Equations similar to (8.11) and (8.12) are derived, but the magnitude of the total friction force acting on the element is now  $2 \Delta F$ , and the sum of the  $y$  components of the normal forces is  $2 \Delta N \sin (\alpha/2)$ . Proceeding as above, we obtain

$$\ln \frac{T_2}{T_1} = \frac{\mu_s \beta}{\sin(\alpha/2)} \quad (8.15)$$

or,

$$\frac{T_2}{T_1} = e^{\mu_s \beta / \sin(\alpha/2)} \quad (8.16)$$



## SAMPLE PROBLEM 8.7

A hawser thrown from a ship to a pier is wrapped two full turns around a bollard. The tension in the hawser is 7500 N; by exerting a force of 150 N on its free end, a dockworker can just keep the hawser from slipping. (a) Determine the coefficient of friction between the hawser and the bollard. (b) Determine the tension in the hawser that could be resisted by the 150-N force if the hawser were wrapped three full turns around the bollard.

## SOLUTION

**a. Coefficient of Friction.** Since slipping of the hawser is impending, we use Eq. (8.13):

$$\ln \frac{T_2}{T_1} = \mu_s \beta$$

Since the hawser is wrapped two full turns around the bollard, we have

$$\begin{aligned}\beta &= 2(2\pi \text{ rad}) = 12.57 \text{ rad} \\ T_1 &= 150 \text{ N} \quad T_2 = 7500 \text{ N}\end{aligned}$$

Therefore,

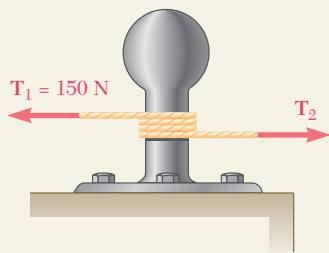
$$\begin{aligned}\mu_s \beta &= \ln \frac{T_2}{T_1} \\ \mu_s(12.57 \text{ rad}) &= \ln \frac{7500 \text{ N}}{150 \text{ N}} = \ln 50 = 3.91 \\ \mu_s &= 0.311 \quad \mu_s = 0.311\end{aligned}$$

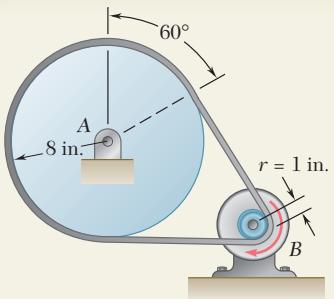
**b. Hawser Wrapped Three Turns around Bollard.** Using the value of  $\mu_s$  obtained in part *a*, we now have

$$\begin{aligned}\beta &= 3(2\pi \text{ rad}) = 18.85 \text{ rad} \\ T_1 &= 150 \text{ N} \quad \mu_s = 0.311\end{aligned}$$

Substituting these values into Eq. (8.14), we obtain

$$\begin{aligned}\frac{T_2}{T_1} &= e^{\mu_s \beta} \\ \frac{T_2}{150 \text{ N}} &= e^{(0.311)(18.85)} = e^{5.862} = 351.5 \\ T_2 &= 52725 \text{ N} \\ T_2 &= 52.7 \text{ kN}\end{aligned}$$

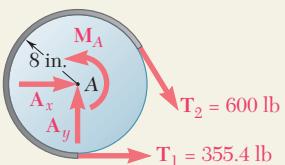
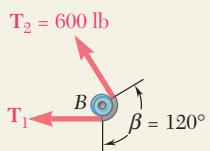
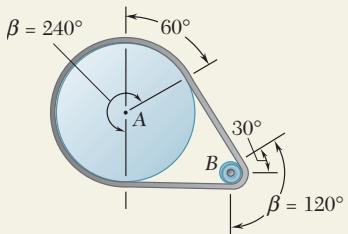




## SAMPLE PROBLEM 8.8

A flat belt connects pulley A, which drives a machine tool, to pulley B, which is attached to the shaft of an electric motor. The coefficients of friction are  $\mu_s = 0.25$  and  $\mu_k = 0.20$  between both pulleys and the belt. Knowing that the maximum allowable tension in the belt is 600 lb, determine the largest torque which can be exerted by the belt on pulley A.

## SOLUTION



Since the resistance to slippage depends upon the angle of contact  $\beta$  between pulley and belt, as well as upon the coefficient of static friction  $\mu_s$ , and since  $\mu_s$  is the same for both pulleys, slippage will occur first on pulley B, for which  $\beta$  is smaller.

**Pulley B.** Using Eq. (8.14) with  $T_2 = 600$  lb,  $\mu_s = 0.25$ , and  $\beta = 120^\circ = 2\pi/3$  rad, we write

$$\frac{T_2}{T_1} = e^{\mu_s \beta} \quad \frac{600 \text{ lb}}{T_1} = e^{0.25(2\pi/3)} = 1.688$$

$$T_1 = \frac{600 \text{ lb}}{1.688} = 355.4 \text{ lb}$$

**Pulley A.** We draw the free-body diagram of pulley A. The couple  $M_A$  is applied to the pulley by the machine tool to which it is attached and is equal and opposite to the torque exerted by the belt. We write

$$+\uparrow \sum M_A = 0: \quad M_A - (600 \text{ lb})(8 \text{ in.}) + (355.4 \text{ lb})(8 \text{ in.}) = 0$$

$$M_A = 1957 \text{ lb} \cdot \text{in.} \quad M_A = 163.1 \text{ lb} \cdot \text{ft} \quad \blacktriangleleft$$

**Note.** We may check that the belt does not slip on pulley A by computing the value of  $\mu_s$  required to prevent slipping at A and verifying that it is smaller than the actual value of  $\mu_s$ . From Eq. (8.13) we have

$$\mu_s \beta = \ln \frac{T_2}{T_1} = \ln \frac{600 \text{ lb}}{355.4 \text{ lb}} = 0.524$$

and, since  $\beta = 240^\circ = 4\pi/3$  rad,

$$\frac{4\pi}{3} \mu_s = 0.524 \quad \mu_s = 0.125 < 0.25$$

# SOLVING PROBLEMS ON YOUR OWN

In the preceding section you learned about *belt friction*. The problems you will have to solve include belts passing over fixed drums, band brakes in which the drum rotates while the band remains fixed, and belt drives.

**1. Problems involving belt friction** fall into one of the following two categories:

a. **Problems in which slipping is impending.** One of the following formulas, involving the *coefficient of static friction*  $\mu_s$ , may then be used,

$$\ln \frac{T_2}{T_1} = \mu_s \beta \quad (8.13)$$

or

$$\frac{T_2}{T_1} = e^{\mu_s \beta} \quad (8.14)$$

b. **Problems in which slipping is occurring.** The formulas to be used can be obtained from Eqs. (8.13) and (8.14) by replacing  $\mu_s$  with the *coefficient of kinetic friction*  $\mu_k$ .

**2. As you start solving a belt-friction problem,** be sure to remember the following:

a. **The angle  $\beta$  must be expressed in radians.** In a belt-and-drum problem, this is the angle subtending the arc of the drum on which the belt is wrapped.

b. **The larger tension is always denoted by  $T_2$  and the smaller tension is denoted by  $T_1$ .**

c. **The larger tension occurs at the end of the belt which is in the direction of the motion,** or impending motion, of the belt relative to the drum.

**3. In each of the problems you will be asked to solve,** three of the four quantities  $T_1$ ,  $T_2$ ,  $\beta$ , and  $\mu_s$  (or  $\mu_k$ ) will either be given or readily found, and you will then solve the appropriate equation for the fourth quantity. Here are two kinds of problems that you will encounter:

a. **Find  $\mu_s$  between belt and drum, knowing that slipping is impending.** From the given data, determine  $T_1$ ,  $T_2$ , and  $\beta$ ; substitute these values into Eq. (8.13) and solve for  $\mu_s$  [Sample Prob. 8.7, part a]. Follow the same procedure to find the *smallest value* of  $\mu_s$  for which slipping will not occur.

b. **Find the magnitude of a force or couple applied to the belt or drum, knowing that slipping is impending.** The given data should include  $\mu_s$  and  $\beta$ . If it also includes  $T_1$  or  $T_2$ , use Eq. (8.14) to find the other tension. If neither  $T_1$  nor  $T_2$  is known but some other data is given, use the free-body diagram of the belt-drum system to write an equilibrium equation that you will solve simultaneously with Eq. (8.14) for  $T_1$  and  $T_2$ . You will then be able to find the magnitude of the specified force or couple from the free-body diagram of the system. Follow the same procedure to determine the *largest value* of a force or couple which can be applied to the belt or drum if no slipping is to occur [Sample Prob. 8.8].

# PROBLEMS

- 8.101** A hawser is wrapped two full turns around a bollard. By exerting an 80-lb force on the free end of the hawser, a dockworker can resist a force of 5000 lb on the other end of the hawser. Determine (a) the coefficient of static friction between the hawser and the bollard, (b) the number of times the hawser should be wrapped around the bollard if a 20,000-lb force is to be resisted by the same 80-lb force.

- 8.102** A rope  $ABCD$  is looped over two pipes as shown. Knowing that the coefficient of static friction is 0.25, determine (a) the smallest value of the mass  $m$  for which equilibrium is possible, (b) the corresponding tension in portion  $BC$  of the rope.

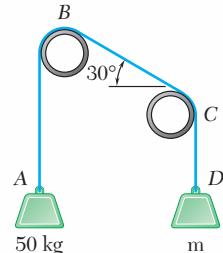


Fig. P8.102 and P8.103

- 8.103** A rope  $ABCD$  is looped over two pipes as shown. Knowing that the coefficient of static friction is 0.25, determine (a) the largest value of the mass  $m$  for which equilibrium is possible, (b) the corresponding tension in portion  $BC$  of the rope.

- 8.104** A 300-lb block is supported by a rope that is wrapped  $1\frac{1}{2}$  times around a horizontal rod. Knowing that the coefficient of static friction between the rope and the rod is 0.15, determine the range of values of  $P$  for which equilibrium is maintained.

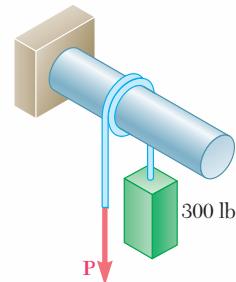


Fig. P8.104

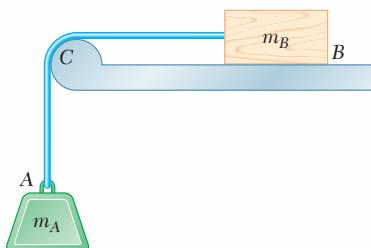


Fig. P8.105 and P8.106

- 8.106** The coefficient of static friction  $\mu_s$  is the same between block  $B$  and the horizontal surface and between the rope and support  $C$ . Knowing that  $m_A = m_B$ , determine the smallest value of  $\mu_s$  for which equilibrium is maintained.

- 8.107** A flat belt is used to transmit a couple from drum  $B$  to drum  $A$ . Knowing that the coefficient of static friction is 0.40 and that the allowable belt tension is 450 N, determine the largest couple that can be exerted on drum  $A$ .

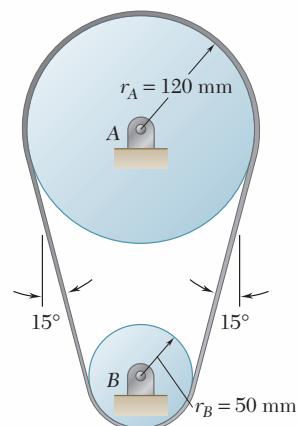
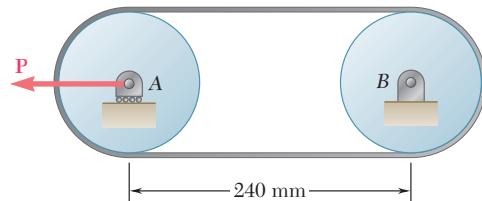


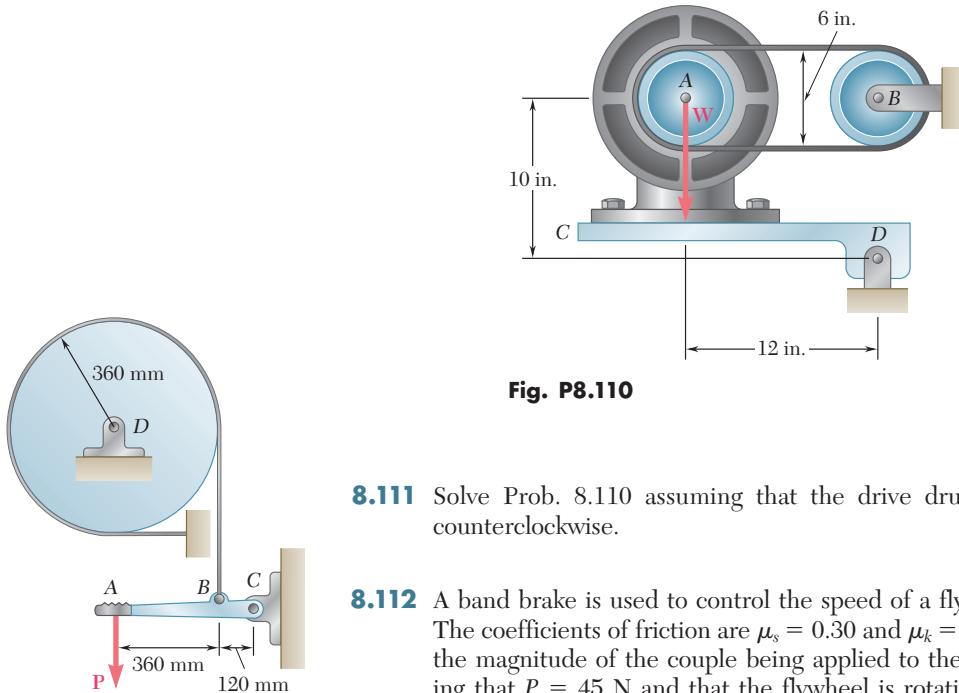
Fig. P8.107

- 8.108** A flat belt is used to transmit a couple from pulley A to pulley B. The radius of each pulley is 60 mm, and a force of magnitude  $P = 900 \text{ N}$  is applied as shown to the axle of pulley A. Knowing that the coefficient of static friction is 0.35, determine (a) the largest couple that can be transmitted, (b) the corresponding maximum value of the tension in the belt.

**Fig. P8.108**

- 8.109** Solve Prob. 8.108 assuming that the belt is looped around the pulleys in a figure eight.

- 8.110** In the pivoted motor mount shown the weight  $\mathbf{W}$  of the 175-lb motor is used to maintain tension in the drive belt. Knowing that the coefficient of static friction between the flat belt and drums A and B is 0.40, and neglecting the weight of platform CD, determine the largest couple that can be transmitted to drum B when the drive drum A is rotating clockwise.

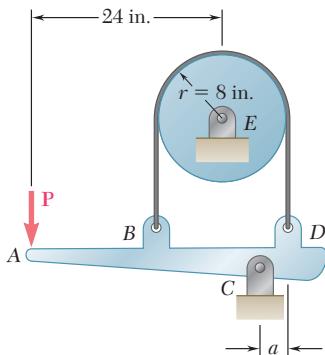
**Fig. P8.112**

- 8.111** Solve Prob. 8.110 assuming that the drive drum A is rotating counterclockwise.

- 8.112** A band brake is used to control the speed of a flywheel as shown. The coefficients of friction are  $\mu_s = 0.30$  and  $\mu_k = 0.25$ . Determine the magnitude of the couple being applied to the flywheel, knowing that  $P = 45 \text{ N}$  and that the flywheel is rotating counterclockwise at a constant speed.

- 8.113** The speed of the brake drum shown is controlled by a belt attached to the control bar  $AD$ . A force  $\mathbf{P}$  of magnitude 25 lb is applied to the control bar at  $A$ . Determine the magnitude of the couple being applied to the drum, knowing that the coefficient of kinetic friction between the belt and the drum is 0.25, that  $a = 4$  in., and that the drum is rotating at a constant speed (a) counterclockwise, (b) clockwise.

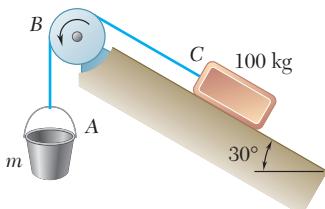
- 8.114** Knowing that  $a = 4$  in., determine the maximum value of the coefficient of static friction for which the brake is not self-locking when the drum rotates counterclockwise.



**Fig. P8.113, P8.114,  
and P8.115**

- 8.115** Knowing that the coefficient of static friction is 0.30 and that the brake drum is rotating counterclockwise, determine the minimum value of  $a$  for which the brake is not self-locking.

- 8.116** Bucket  $A$  and block  $C$  are connected by a cable that passes over drum  $B$ . Knowing that drum  $B$  rotates slowly counterclockwise and that the coefficients of friction at all surfaces are  $\mu_s = 0.35$  and  $\mu_k = 0.25$ , determine the smallest combined mass  $m$  of the bucket and its contents for which block  $C$  will (a) remain at rest, (b) start moving up the incline, (c) continue moving up the incline at a constant speed.



**Fig. P8.116**

- 8.117** Solve Prob. 8.116 assuming that drum  $B$  is frozen and cannot rotate.

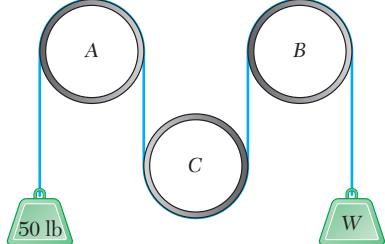


Fig. P8.118 and P8.119

**8.118 and 8.120** A cable is placed around three parallel pipes. Knowing that the coefficients of friction are  $\mu_s = 0.25$  and  $\mu_k = 0.20$ , determine (a) the smallest weight  $W$  for which equilibrium is maintained, (b) the largest weight  $W$  that can be raised if pipe  $B$  is slowly rotated counterclockwise while pipes  $A$  and  $C$  remain fixed.

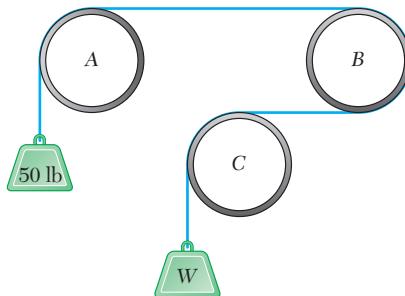


Fig. P8.120 and P8.121

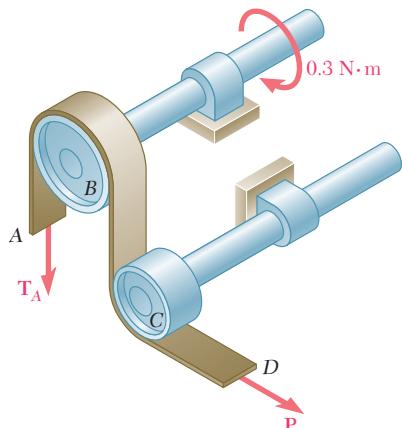


Fig. P8.122

**8.119 and 8.121** A cable is placed around three parallel pipes. Two of the pipes are fixed and do not rotate; the third pipe is slowly rotated. Knowing that the coefficients of friction are  $\mu_s = 0.25$  and  $\mu_k = 0.20$ , determine the largest weight  $W$  that can be raised (a) if only pipe  $A$  is rotated counterclockwise, (b) if only pipe  $C$  is rotated clockwise.

**8.122** A recording tape passes over the 20-mm-radius drive drum  $B$  and under the idler drum  $C$ . Knowing that the coefficients of friction between the tape and the drums are  $\mu_s = 0.40$  and  $\mu_k = 0.30$  and that drum  $C$  is free to rotate, determine the smallest allowable value of  $P$  if slipping of the tape on drum  $B$  is not to occur.

**8.123** Solve Prob. 8.122 assuming that the idler drum  $C$  is frozen and cannot rotate.

**8.124** The 10-lb bar  $AE$  is suspended by a cable that passes over a 5-in.-radius drum. Vertical motion of end  $E$  of the bar is prevented by the two stops shown. Knowing that  $\mu_s = 0.30$  between the cable and the drum, determine (a) the largest counterclockwise couple  $M_0$  that can be applied to the drum if slipping is not to occur, (b) the corresponding force exerted on end  $E$  of the bar.

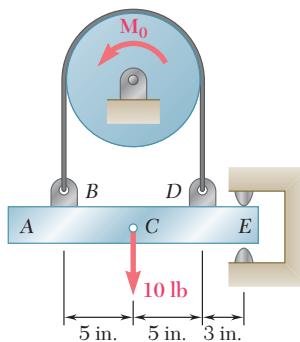


Fig. P8.124

**8.125** Solve Prob. 8.124 assuming that a clockwise couple  $\mathbf{M}_0$  is applied to the drum.

**8.126** The strap wrench shown is used to grip the pipe firmly without marring the external surface of the pipe. Knowing that the coefficient of static friction is the same for all surfaces of contact, determine the smallest value of  $\mu_s$  for which the wrench will be self-locking when  $a = 200$  mm,  $r = 30$  mm, and  $\theta = 65^\circ$ .

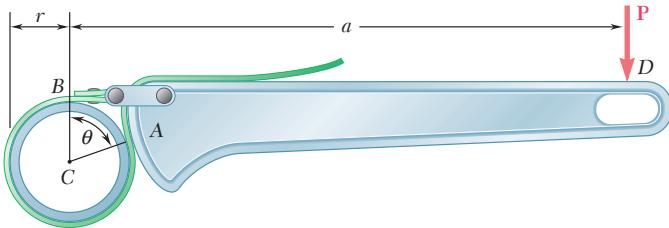


Fig. P8.126

**8.127** Solve Prob. 8.126 assuming that  $\theta = 75^\circ$ .

**8.128** Prove that Eqs. (8.13) and (8.14) are valid for any shape of surface provided that the coefficient of friction is the same at all points of contact.

**8.129** Complete the derivation of Eq. (8.15), which relates the tension in both parts of a V belt.

**8.130** Solve Prob. 8.107 assuming that the flat belt and drums are replaced by a V belt and V pulleys with  $\alpha = 36^\circ$ . (The angle  $\alpha$  is as shown in Fig. 8.15a.)

**8.131** Solve Prob. 8.108 assuming that the flat belt and pulleys are replaced by a V belt and V pulleys with  $\alpha = 36^\circ$ . (The angle  $\alpha$  is as shown in Fig. 8.15a.)

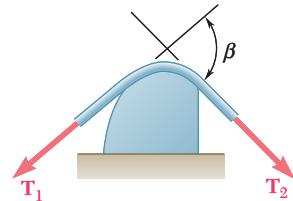
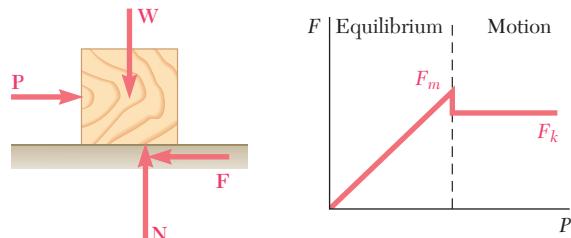


Fig. P8.128

# REVIEW AND SUMMARY

This chapter was devoted to the study of *dry friction*, i.e., to problems involving rigid bodies which are in contact along *nonlubricated surfaces*.



**Fig. 8.16**

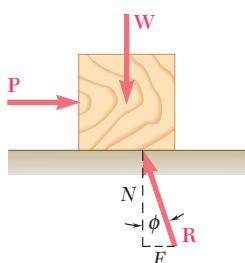
## Static and kinetic friction

Applying a horizontal force  $\mathbf{P}$  to a block resting on a horizontal surface [Sec. 8.2], we note that the block at first does not move. This shows that a *friction force*  $\mathbf{F}$  must have developed to balance  $\mathbf{P}$  (Fig. 8.16). As the magnitude of  $\mathbf{P}$  is increased, the magnitude of  $\mathbf{F}$  also increases until it reaches a maximum value  $F_m$ . If  $\mathbf{P}$  is further increased, the block starts sliding and the magnitude of  $\mathbf{F}$  drops from  $F_m$  to a lower value  $F_k$ . Experimental evidence shows that  $F_m$  and  $F_k$  are proportional to the normal component  $N$  of the reaction of the surface. We have

$$F_m = \mu_s N \quad F_k = \mu_k N \quad (8.1, 8.2)$$

where  $\mu_s$  and  $\mu_k$  are called, respectively, the *coefficient of static friction* and the *coefficient of kinetic friction*. These coefficients depend on the nature and the condition of the surfaces in contact. Approximate values of the coefficients of static friction were given in Table 8.1.

## Angles of friction



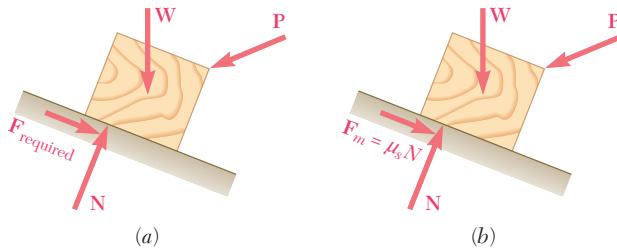
**Fig. 8.17**

It is sometimes convenient to replace the normal force  $\mathbf{N}$  and the friction force  $\mathbf{F}$  by their resultant  $\mathbf{R}$  (Fig. 8.17). As the friction force increases and reaches its maximum value  $F_m = \mu_s N$ , the angle  $\phi$  that  $\mathbf{R}$  forms with the normal to the surface increases and reaches a maximum value  $\phi_s$ , called the *angle of static friction*. If motion actually takes place, the magnitude of  $\mathbf{F}$  drops to  $F_k$ ; similarly the angle  $\phi$  drops to a lower value  $\phi_k$ , called the *angle of kinetic friction*. As shown in Sec. 8.3, we have

$$\tan \phi_s = \mu_s \quad \tan \phi_k = \mu_k \quad (8.3, 8.4)$$

When solving equilibrium problems involving friction, we should keep in mind that the magnitude  $F$  of the friction force is equal to  $F_m = \mu_s N$  only if the body is about to slide [Sec. 8.4]. If motion is not impending,  $F$  and  $N$  should be considered as independent unknowns to be determined from the equilibrium equations (Fig. 8.18a). We

### Problems involving friction



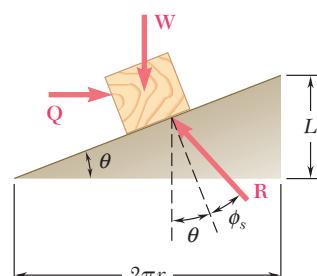
**Fig. 8.18**

should also check that the value of  $F$  required to maintain equilibrium is not larger than  $F_m$ ; if it were, the body would move and the magnitude of the friction force would be  $F_k = \mu_k N$  [Sample Prob. 8.1]. On the other hand, if motion is known to be impending,  $F$  has reached its maximum value  $F_m = \mu_s N$  (Fig. 8.18b), and this expression may be substituted for  $F$  in the equilibrium equations [Sample Prob. 8.3]. When only three forces are involved in a free-body diagram, including the reaction **R** of the surface in contact with the body, it is usually more convenient to solve the problem by drawing a force triangle [Sample Prob. 8.2].

When a problem involves the analysis of the forces exerted on each other by *two bodies A and B*, it is important to show the friction forces with their correct sense. The correct sense for the friction force exerted by *B* on *A*, for instance, is opposite to that of the *relative motion* (or impending motion) of *A* with respect to *B* [Fig. 8.6].

In the second part of the chapter we considered a number of specific engineering applications where dry friction plays an important role. In the case of *wedges*, which are simple machines used to raise heavy loads [Sec. 8.5], two or more free-body diagrams were drawn and care was taken to show each friction force with its correct sense [Sample Prob. 8.4]. The analysis of *square-threaded screws*, which are frequently used in jacks, presses, and other mechanisms, was reduced to the analysis of a block sliding on an incline by unwrapping the thread of the screw and showing it as a straight line [Sec. 8.6]. This is done again in Fig. 8.19, where  $r$  denotes the *mean radius* of the thread,  $L$  is the *lead* of the screw, i.e., the distance through which the screw advances in one turn,  $W$  is the load, and  $Qr$  is equal to the couple exerted on the screw. It was noted that in the case of multiple-threaded screws the lead  $L$  of the screw is *not* equal to its pitch, which is the distance measured between two consecutive threads.

### Wedges and screws



**Fig. 8.19**

Other engineering applications considered in this chapter were *journal bearings* and *axle friction* [Sec. 8.7], *thrust bearings* and *disk friction* [Sec. 8.8], *wheel friction* and *rolling resistance* [Sec. 8.9], and *belt friction* [Sec. 8.10].

### Belt friction

In solving a problem involving a *flat belt* passing over a fixed cylinder, it is important to first determine the direction in which the belt slips or is about to slip. If the drum is rotating, the motion or impending motion of the belt should be determined *relative* to the rotating drum. For instance, if the belt shown in Fig. 8.20 is about to slip to

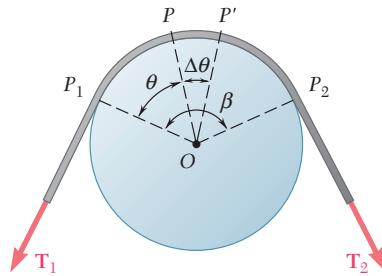


Fig. 8.20

the right relative to the drum, the friction forces exerted by the drum on the belt will be directed to the left and the tension will be larger in the right-hand portion of the belt than in the left-hand portion. Denoting the larger tension by  $T_2$ , the smaller tension by  $T_1$ , the coefficient of static friction by  $\mu_s$ , and the angle (in radians) subtended by the belt by  $\beta$ , we derived in Sec. 8.10 the formulas

$$\ln \frac{T_2}{T_1} = \mu_s \beta \quad (8.13)$$

$$\frac{T_2}{T_1} = e^{\mu_s \beta} \quad (8.14)$$

which were used in solving Sample Probs. 8.7 and 8.8. If the belt actually slips on the drum, the coefficient of static friction  $\mu_s$  should be replaced by the coefficient of kinetic friction  $\mu_k$  in both of these formulas.

# REVIEW PROBLEMS

- 8.132** Knowing that the coefficient of friction between the 25-kg block and the incline is  $\mu_s = 0.25$ , determine (a) the smallest value of  $P$  required to start the block moving up the incline, (b) the corresponding value of  $\beta$ .

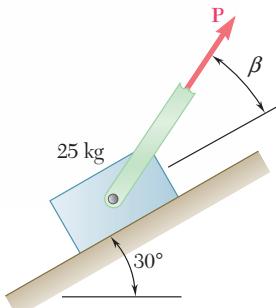


Fig. P8.132

- 8.133** The 20-lb block  $A$  and the 30-lb block  $B$  are supported by an incline that is held in the position shown. Knowing that the coefficient of static friction is 0.15 between all surfaces of contact, determine the value of  $\theta$  for which motion is impending.

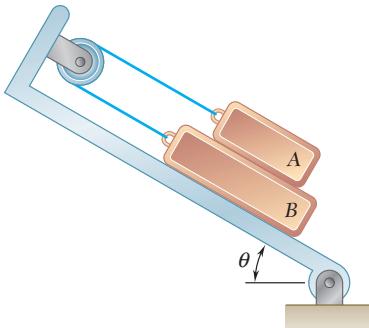


Fig. P8.133

- 8.134** A worker slowly moves a 50-kg crate to the left along a loading dock by applying a force  $P$  at corner  $B$  as shown. Knowing that the crate starts to tip about the edge  $E$  of the loading dock when  $a = 200$  mm, determine (a) the coefficient of kinetic friction between the crate and the loading dock, (b) the corresponding magnitude  $P$  of the force.

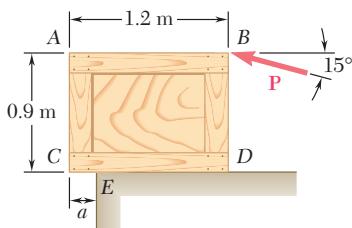
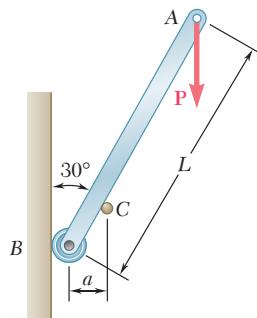
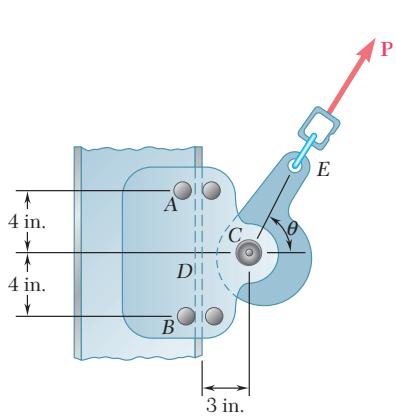
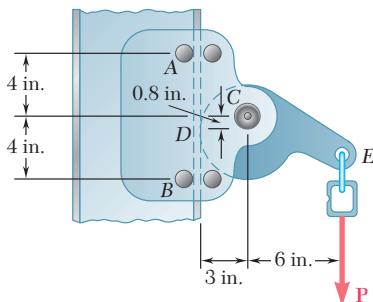


Fig. P8.134

- 8.135** A slender rod of length  $L$  is lodged between peg  $C$  and the vertical wall and supports a load  $\mathbf{P}$  at end  $A$ . Knowing that the coefficient of static friction between the peg and the rod is 0.15 and neglecting friction at the roller, determine the range of values of the ratio  $L/a$  for which equilibrium is maintained.

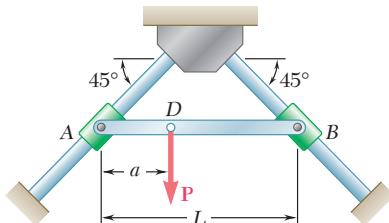
**Fig. P8.135**

- 8.136** A safety device used by workers climbing ladders fixed to high structures consists of a rail attached to the ladder and a sleeve that can slide on the flange of the rail. A chain connects the worker's belt to the end of an eccentric cam that can be rotated about an axle attached to the sleeve at  $C$ . Determine the smallest allowable common value of the coefficient of static friction between the flange of the rail, the pins at  $A$  and  $B$ , and the eccentric cam if the sleeve is not to slide down when the chain is pulled vertically downward.

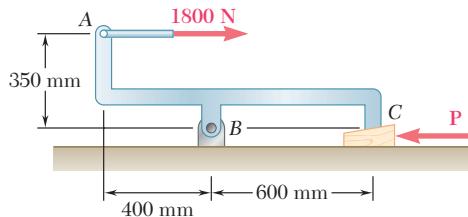
**Fig. P8.137****Fig. P8.136**

- 8.137** To be of practical use, the safety sleeve described in the preceding problem must be free to slide along the rail when pulled upward. Determine the largest allowable value of the coefficient of static friction between the flange of the rail and the pins at  $A$  and  $B$  if the sleeve is to be free to slide when pulled as shown in the figure, assuming (a)  $\theta = 60^\circ$ , (b)  $\theta = 50^\circ$ , (c)  $\theta = 40^\circ$ .

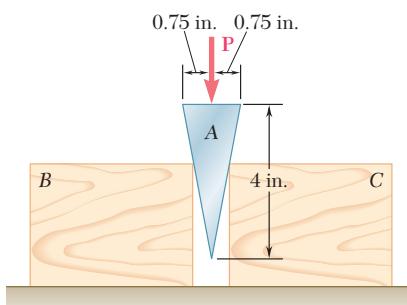
- 8.138** Bar  $AB$  is attached to collars that can slide on the inclined rods shown. A force  $\mathbf{P}$  is applied at point  $D$  located at a distance  $a$  from end  $A$ . Knowing that the coefficient of static friction  $\mu_s$  between each collar and the rod upon which it slides is 0.30 and neglecting the weights of the bar and of the collars, determine the smallest value of the ratio  $a/L$  for which equilibrium is maintained.

**Fig. P8.138**

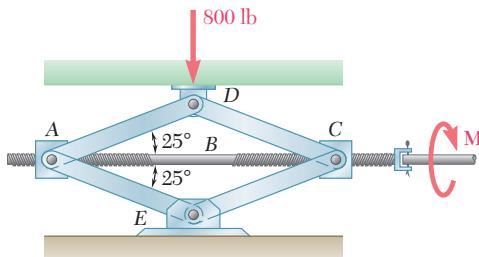
- 8.139** The machine part  $ABC$  is supported by a frictionless hinge at  $B$  and a  $10^\circ$  wedge at  $C$ . Knowing that the coefficient of static friction at both surfaces of the wedge is 0.20, determine (a) the force  $\mathbf{P}$  required to move the wedge, (b) the components of the corresponding reaction at  $B$ .

**Fig. P8.139**

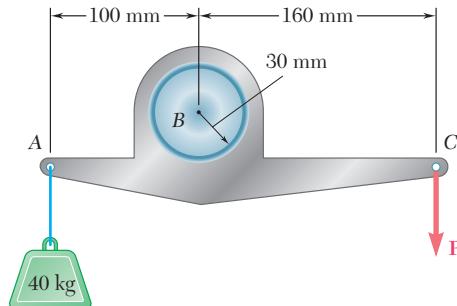
- 8.140** A wedge  $A$  of negligible weight is to be driven between two 100-lb blocks  $B$  and  $C$  resting on a horizontal surface. Knowing that the coefficient of static friction at all surfaces of contact is 0.35, determine the smallest force  $\mathbf{P}$  required to start moving the wedge (a) if the blocks are equally free to move, (b) if block  $C$  is securely bolted to the horizontal surface.

**Fig. P8.140**

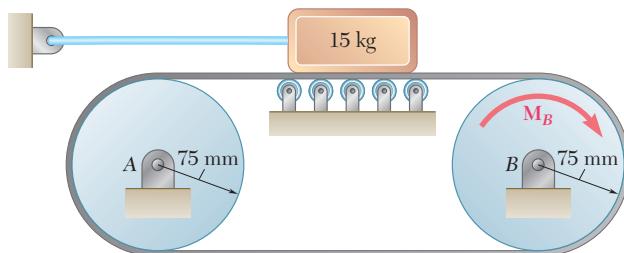
- 8.141** The position of the automobile jack shown is controlled by a screw  $ABC$  that is single-threaded at each end (right-handed thread at  $A$ , left-handed thread at  $C$ ). Each thread has a pitch of 0.1 in. and a mean diameter of 0.375 in. If the coefficient of static friction is 0.15, determine the magnitude of the couple  $\mathbf{M}$  that must be applied to raise the automobile.

**Fig. P8.141**

- 8.142** A lever of negligible weight is loosely fitted onto a 30-mm-radius fixed shaft as shown. Knowing that a force  $\mathbf{P}$  of magnitude 275 N will just start the lever rotating clockwise, determine (a) the coefficient of static friction between the shaft and the lever, (b) the smallest force  $\mathbf{P}$  for which the lever does not start rotating counterclockwise.

**Fig. P8.142**

- 8.143** A couple  $\mathbf{M}_B$  is applied to the drive drum  $B$  to maintain a constant speed in the polishing belt shown. Knowing that  $\mu_k = 0.45$  between the belt and the 15-kg block being polished and  $\mu_s = 0.30$  between the belt and the drive drum  $B$ , determine (a) the couple  $\mathbf{M}_B$ , (b) the minimum tension in the lower portion of the belt if no slipping is to occur between the belt and the drive drum.

**Fig. P8.143**

# COMPUTER PROBLEMS

**8.C1** The position of the 10-kg rod *AB* is controlled by the 2-kg block shown, which is slowly moved to the left by the force **P**. Knowing that the coefficient of kinetic friction between all surfaces of contact is 0.25, use computational software to calculate the magnitude *P* of the force for values of *x* from 900 to 100 mm, using 50-mm decrements. Using appropriate smaller decrements, determine the maximum value of *P* and the corresponding value of *x*.

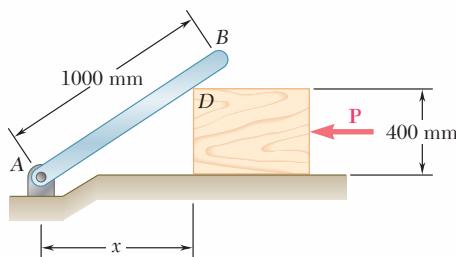


Fig. P8.C1

**8.C2** Blocks *A* and *B* are supported by an incline that is held in the position shown. Knowing that block *A* weighs 20 lb and that the coefficient of static friction between all surfaces of contact is 0.15, use computational software to calculate the value of  $\theta$  for which motion is impending for weights of block *B* from 0 to 100 lb, using 10-lb increments.

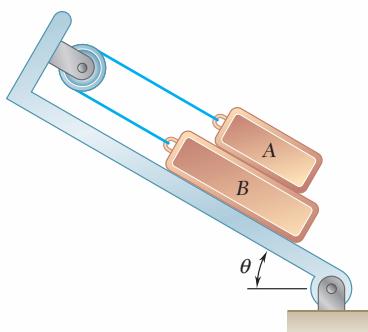


Fig. P8.C2

**8.C3** A 300-g cylinder *C* rests on cylinder *D* as shown. Knowing that the coefficient of static friction  $\mu_s$  is the same at *A* and *B*, use computational software to determine, for values of  $\mu_s$  from 0 to 0.40 and using 0.05 increments, the largest counterclockwise couple **M** that can be applied to cylinder *D* if it is not to rotate.

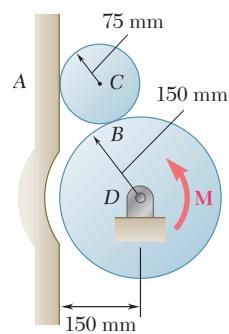


Fig. P8.C3

**8.C4** Two rods are connected by a slider block  $D$  and are held in equilibrium by the couple  $\mathbf{M}_A$  as shown. Knowing that the coefficient of static friction between rod  $AC$  and the slider block is 0.40, use computational software to determine, for values of  $\theta$  from 0 to  $120^\circ$  and using  $10^\circ$  increments, the range of values of  $M_A$  for which equilibrium is maintained.

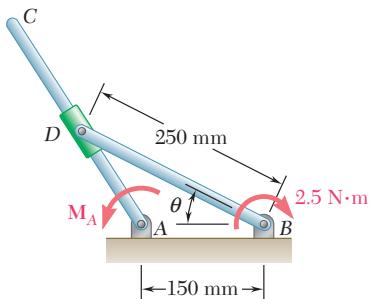


Fig. P8.C4

**8.C5** The 10-lb block  $A$  is slowly moved up the circular cylindrical surface by a cable that passes over a small fixed cylindrical drum at  $B$ . The coefficient of kinetic friction is known to be 0.30 between the block and the surface and between the cable and the drum. Using computational software, calculate the force  $\mathbf{P}$  required to maintain the motion for values of  $\theta$  from 0 to  $90^\circ$ , using  $10^\circ$  increments. For the same values of  $\theta$  calculate the magnitude of the reaction between the block and the surface. [Note that the angle of contact between the cable and the fixed drum is  $\beta = \pi - (\theta/2)$ .]

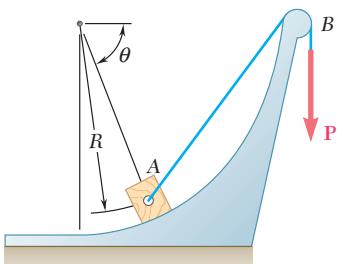


Fig. P8.C5

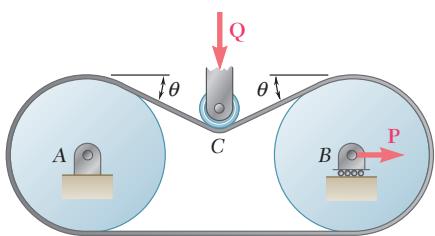


Fig. P8.C6

**8.C6** A flat belt is used to transmit a couple from drum  $A$  to drum  $B$ . The radius of each drum is 80 mm, and the system is fitted with an idler wheel  $C$  that is used to increase the contact between the belt and the drums. The allowable belt tension is 200 N, and the coefficient of static friction between the belt and the drums is 0.30. Using computational software, calculate the largest couple that can be transmitted for values of  $\theta$  from 0 to  $30^\circ$ , using  $5^\circ$  increments.

**8.C7** Two collars A and B that slide on vertical rods with negligible friction are connected by a 30-in. cord that passes over a fixed shaft at C. The coefficient of static friction between the cord and the fixed shaft is 0.30. Knowing that the weight of collar B is 8 lb, use computational software to determine, for values of  $\theta$  from 0 to  $60^\circ$  and using  $10^\circ$  increments, the largest and smallest weight of collar A for which equilibrium is maintained.

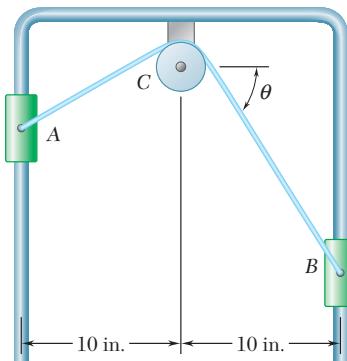


Fig. P8.C7

**8.C8** The end B of a uniform beam of length  $L$  is being pulled by a stationary crane. Initially the beam lies on the ground with end A directly below pulley C. As the cable is slowly pulled in, the beam first slides to the left with  $\theta = 0$  until it has moved through a distance  $x_0$ . In a second phase, end B is raised, while end A keeps sliding to the left until  $x$  reaches its maximum value  $x_m$  and  $\theta$  the corresponding value  $\theta_1$ . The beam then rotates about A' while  $\theta$  keeps increasing. As  $\theta$  reaches the value  $\theta_2$ , end A starts sliding to the right and keeps sliding in an irregular manner until B reaches C. Knowing that the coefficients of friction between the beam and the ground are  $\mu_s = 0.50$  and  $\mu_k = 0.40$ , (a) use computational software to compute  $x$  for any value of  $\theta$  while the beam is sliding to the left and use this program to determine  $x_0$ ,  $x_m$ , and  $\theta_1$ , (b) modify the program to compute for any  $\theta$  the value of  $x$  for which sliding would be impending to the right and use this new program to determine the value  $\theta_2$  of  $\theta$  corresponding to  $x = x_m$ .

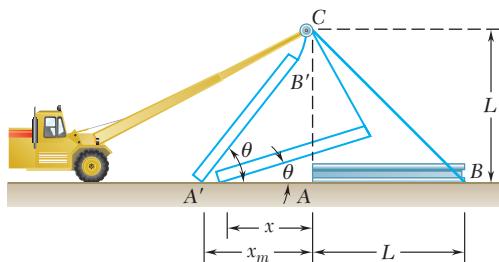


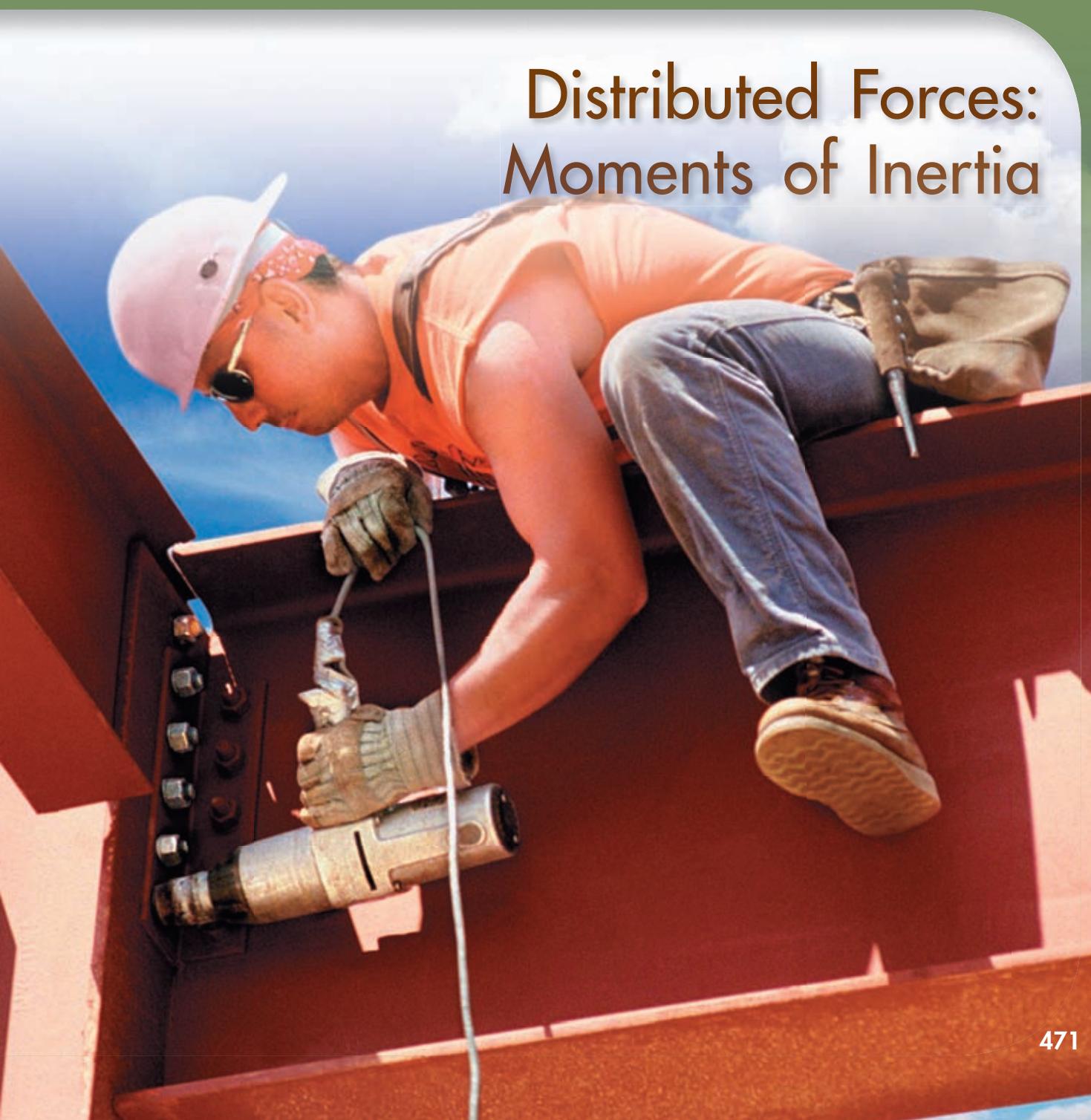
Fig. P8.C8

The strength of structural members used in the construction of buildings depends to a large extent on the properties of their cross sections. This includes the second moments of area, or moments of inertia, of these cross sections.

CHAPTER

9

# Distributed Forces: Moments of Inertia



## Chapter 9 Distributed Forces: Moments of Inertia

- 9.1 Introduction
- 9.2 Second Moment, or Moment of Inertia, of an Area
- 9.3 Determination of the Moment of Inertia of an Area by Integration
- 9.4 Polar Moment of Inertia
- 9.5 Radius of Gyration of an Area
- 9.6 Parallel-Axis Theorem
- 9.7 Moments of Inertia of Composite Areas
- 9.8 Product of Inertia
- 9.9 Principal Axes and Principal Moments of Inertia
- 9.10 Mohr's Circle for Moments and Products of Inertia
- 9.11 Moment of Inertia of a Mass
- 9.12 Parallel-Axis Theorem
- 9.13 Moments of Inertia of Thin Plates
- 9.14 Determination of the Moment of Inertia of a Three-Dimensional Body by Integration
- 9.15 Moments of Inertia of Composite Bodies
- 9.16 Moment of Inertia of a Body with Respect to an Arbitrary Axis through O. Mass Products of Inertia
- 9.17 Ellipsoid of Inertia. Principal Axes of Inertia
- 9.18 Determination of the Principal Axes and Principal Moments of Inertia of a Body of Arbitrary Shape

## 9.1 INTRODUCTION

In Chap. 5, we analyzed various systems of forces distributed over an area or volume. The three main types of forces considered were (1) weights of homogeneous plates of uniform thickness (Secs. 5.3 through 5.6), (2) distributed loads on beams (Sec. 5.8) and hydrostatic forces (Sec. 5.9), and (3) weights of homogeneous three-dimensional bodies (Secs. 5.10 and 5.11). In the case of homogeneous plates, the magnitude  $\Delta W$  of the weight of an element of a plate was proportional to the area  $\Delta A$  of the element. For distributed loads on beams, the magnitude  $\Delta W$  of each elemental weight was represented by an element of area  $\Delta A = \Delta W$  under the load curve; in the case of hydrostatic forces on submerged rectangular surfaces, a similar procedure was followed. In the case of homogeneous three-dimensional bodies, the magnitude  $\Delta W$  of the weight of an element of the body was proportional to the volume  $\Delta V$  of the element. Thus, in all cases considered in Chap. 5, the distributed forces were proportional to the elemental areas or volumes associated with them. The resultant of these forces, therefore, could be obtained by summing the corresponding areas or volumes, and the moment of the resultant about any given axis could be determined by computing the first moments of the areas or volumes about that axis.

In the first part of this chapter, we consider distributed forces  $\Delta \mathbf{F}$  whose magnitudes depend not only upon the elements of area  $\Delta A$  on which these forces act but also upon the distance from  $\Delta A$  to some given axis. More precisely, the magnitude of the force per unit area  $\Delta F / \Delta A$  is assumed to vary linearly with the distance to the axis. As indicated in the next section, forces of this type are found in the study of the bending of beams and in problems involving submerged non-rectangular surfaces. Assuming that the elemental forces involved are distributed over an area  $A$  and vary linearly with the distance  $y$  to the  $x$  axis, it will be shown that while the magnitude of their resultant  $\mathbf{R}$  depends upon the first moment  $Q_x = \int y dA$  of the area  $A$ , the location of the point where  $\mathbf{R}$  is applied depends upon the *second moment, or moment of inertia*,  $I_x = \int y^2 dA$  of the same area with respect to the  $x$  axis. You will learn to compute the moments of inertia of various areas with respect to given  $x$  and  $y$  axes. Also introduced in the first part of this chapter is the *polar moment of inertia*  $J_O = \int r^2 dA$  of an area, where  $r$  is the distance from the element of area  $dA$  to the point  $O$ . To facilitate your computations, a relation will be established between the moment of inertia  $I_x$  of an area  $A$  with respect to a given  $x$  axis and the moment of inertia  $I_{x'}$  of the same area with respect to the parallel centroidal  $x'$  axis (parallel-axis theorem). You will also study the transformation of the moments of inertia of a given area when the coordinate axes are rotated (Secs. 9.9 and 9.10).

In the second part of the chapter, you will learn how to determine the moments of inertia of various *masses* with respect to a given axis. As you will see in Sec. 9.11, the moment of inertia of a given mass about an axis  $AA'$  is defined as  $I = \int r^2 dm$ , where  $r$  is the distance from the axis  $AA'$  to the element of mass  $dm$ . Moments of inertia of masses are encountered in dynamics in problems involving the rotation of a rigid body about an axis. To facilitate the computation

of mass moments of inertia, the parallel-axis theorem will be introduced (Sec. 9.12). Finally, you will learn to analyze the transformation of moments of inertia of masses when the coordinate axes are rotated (Sects. 9.16 through 9.18).

## MOMENTS OF INERTIA OF AREAS

### 9.2 SECOND MOMENT, OR MOMENT OF INERTIA, OF AN AREA

In the first part of this chapter, we consider distributed forces  $\Delta F$  whose magnitudes  $\Delta F$  are proportional to the elements of area  $\Delta A$  on which the forces act and at the same time vary linearly with the distance from  $\Delta A$  to a given axis.

Consider, for example, a beam of uniform cross section which is subjected to two equal and opposite couples applied at each end of the beam. Such a beam is said to be in *pure bending*, and it is shown in mechanics of materials that the internal forces in any section of the beam are distributed forces whose magnitudes  $\Delta F = ky \Delta A$  vary linearly with the distance  $y$  between the element of area  $\Delta A$  and an axis passing through the centroid of the section. This axis, represented by the  $x$  axis in Fig. 9.1, is known as the *neutral axis* of the section. The forces on one side of the neutral axis are forces of compression, while those on the other side are forces of tension; on the neutral axis itself the forces are zero.

The magnitude of the resultant  $R$  of the elemental forces  $\Delta F$  which act over the entire section is

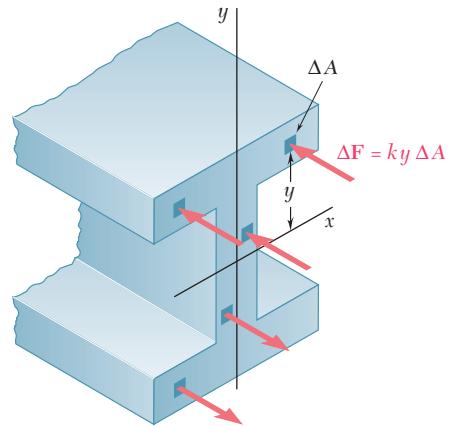
$$R = \int k y dA = k \int y dA$$

The last integral obtained is recognized as the *first moment*  $Q_x$  of the section about the  $x$  axis; it is equal to  $\bar{y}A$  and is thus equal to zero, since the centroid of the section is located on the  $x$  axis. The system of the forces  $\Delta F$  thus reduces to a couple. The magnitude  $M$  of this couple (bending moment) must be equal to the sum of the moments  $\Delta M_x = y \Delta F = ky^2 \Delta A$  of the elemental forces. Integrating over the entire section, we obtain

$$M = \int k y^2 dA = k \int y^2 dA$$

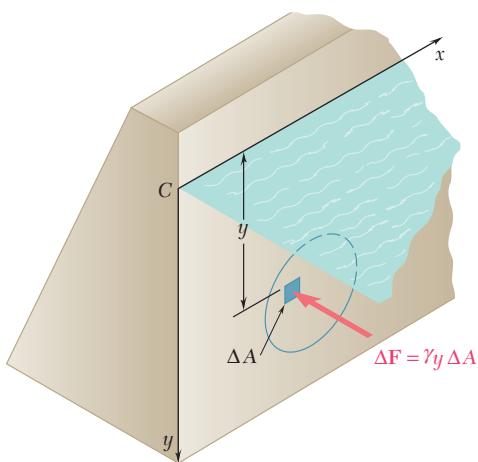
The last integral is known as the *second moment*, or *moment of inertia*,† of the beam section with respect to the  $x$  axis and is denoted by  $I_x$ . It is obtained by multiplying each element of area  $dA$  by the *square of its distance* from the  $x$  axis and integrating over the beam section. Since each product  $y^2 dA$  is positive, regardless of the sign of  $y$ , or zero (if  $y$  is zero), the integral  $I_x$  will always be positive.

Another example of a second moment, or moment of inertia, of an area is provided by the following problem from hydrostatics: A



**Fig. 9.1**

†The term *second moment* is more proper than the term moment of inertia, since, logically, the latter should be used only to denote integrals of mass (see Sec. 9.11). In engineering practice, however, moment of inertia is used in connection with areas as well as masses.

**Fig. 9.2**

vertical circular gate used to close the outlet of a large reservoir is submerged under water as shown in Fig. 9.2. What is the resultant of the forces exerted by the water on the gate, and what is the moment of the resultant about the line of intersection of the plane of the gate and the water surface ( $x$  axis)?

If the gate were rectangular, the resultant of the forces of pressure could be determined from the pressure curve, as was done in Sec. 5.9. Since the gate is circular, however, a more general method must be used. Denoting by  $y$  the depth of an element of area  $\Delta A$  and by  $\gamma$  the specific weight of water, the pressure at the element is  $p = \gamma y$ , and the magnitude of the elemental force exerted on  $\Delta A$  is  $\Delta F = p \Delta A = \gamma y \Delta A$ . The magnitude of the resultant of the elemental forces is thus

$$R = \int \gamma y dA = \gamma \int y dA$$

and can be obtained by computing the first moment  $Q_x = \int y dA$  of the area of the gate with respect to the  $x$  axis. The moment  $M_x$  of the resultant must be equal to the sum of the moments  $\Delta M_x = y \Delta F = \gamma y^2 \Delta A$  of the elemental forces. Integrating over the area of the gate, we have

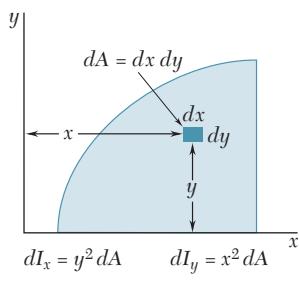
$$M_x = \int \gamma y^2 dA = \gamma \int y^2 dA$$

Here again, the integral obtained represents the second moment, or moment of inertia,  $I_x$  of the area with respect to the  $x$  axis.

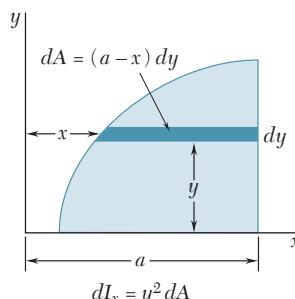
### 9.3 DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

We defined in the preceding section the second moment, or moment of inertia, of an area  $A$  with respect to the  $x$  axis. Defining in a similar way the moment of inertia  $I_y$  of the area  $A$  with respect to the  $y$  axis, we write (Fig. 9.3a)

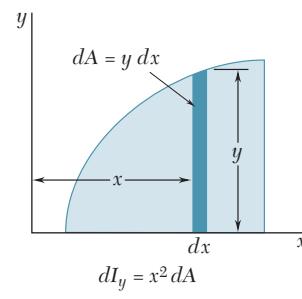
$$I_x = \int y^2 dA \quad I_y = \int x^2 dA \quad (9.1)$$



(a)



(b)



(c)

**Fig. 9.3**

These integrals, known as the *rectangular moments of inertia* of the area  $A$ , can be more easily evaluated if we choose  $dA$  to be a thin strip parallel to one of the coordinate axes. To compute  $I_x$ , the strip is chosen parallel to the  $x$  axis, so that all of the points of the strip are at the same distance  $y$  from the  $x$  axis (Fig. 9.3b); the moment of inertia  $dI_x$  of the strip is then obtained by multiplying the area  $dA$  of the strip by  $y^2$ . To compute  $I_y$ , the strip is chosen parallel to the  $y$  axis so that all of the points of the strip are at the same distance  $x$  from the  $y$  axis (Fig. 9.3c); the moment of inertia  $dI_y$  of the strip is  $x^2 dA$ .

**Moment of Inertia of a Rectangular Area.** As an example, let us determine the moment of inertia of a rectangle with respect to its base (Fig. 9.4). Dividing the rectangle into strips parallel to the  $x$  axis, we obtain

$$dA = b dy \quad dI_x = y^2 b dy$$

$$I_x = \int_0^h by^2 dy = \frac{1}{3}bh^3 \quad (9.2)$$

**Computing  $I_x$  and  $I_y$  Using the Same Elemental Strips.** The formula just derived can be used to determine the moment of inertia  $dI_x$  with respect to the  $x$  axis of a rectangular strip which is parallel to the  $y$  axis, such as the strip shown in Fig. 9.3c. Setting  $b = dx$  and  $h = y$  in formula (9.2), we write

$$dI_x = \frac{1}{3}y^3 dx$$

On the other hand, we have

$$dI_y = x^2 dA = x^2 y dx$$

The same element can thus be used to compute the moments of inertia  $I_x$  and  $I_y$  of a given area (Fig. 9.5).

## 9.4 POLAR MOMENT OF INERTIA

An integral of great importance in problems concerning the torsion of cylindrical shafts and in problems dealing with the rotation of slabs is

$$J_O = \int r^2 dA \quad (9.3)$$

where  $r$  is the distance from  $O$  to the element of area  $dA$  (Fig. 9.6). This integral is the *polar moment of inertia* of the area  $A$  with respect to the “pole”  $O$ .

The polar moment of inertia of a given area can be computed from the rectangular moments of inertia  $I_x$  and  $I_y$  of the area if these quantities are already known. Indeed, noting that  $r^2 = x^2 + y^2$ , we write

$$J_O = \int r^2 dA = \int (x^2 + y^2) dA = \int y^2 dA + \int x^2 dA$$

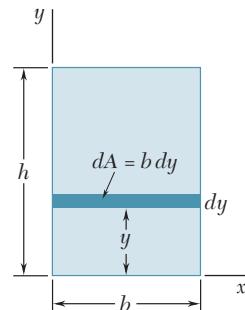


Fig. 9.4

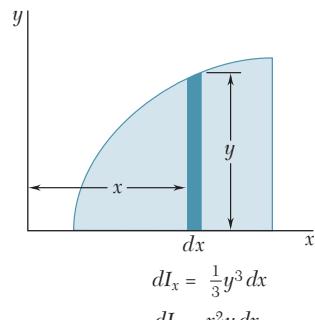


Fig. 9.5

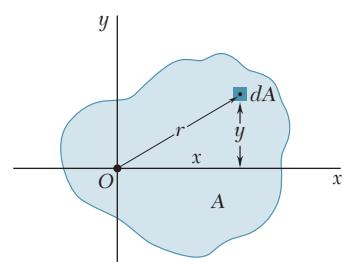
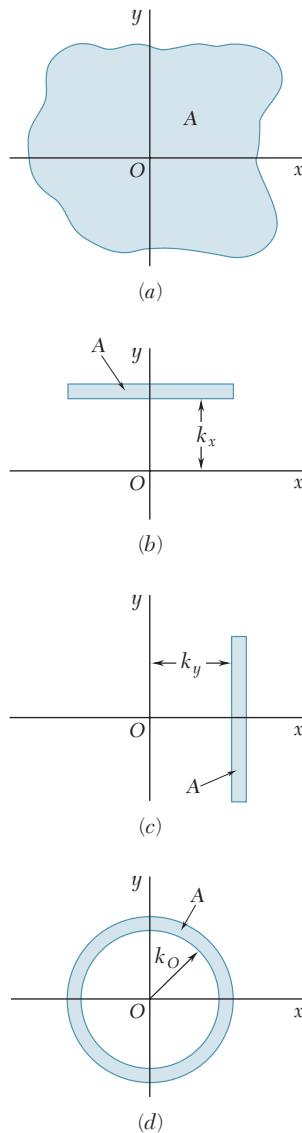


Fig. 9.6

that is,

$$J_O = I_x + I_y \quad (9.4)$$



**Fig. 9.7**

## 9.5 RADIUS OF GYRATION OF AN AREA

Consider an area  $A$  which has a moment of inertia  $I_x$  with respect to the  $x$  axis (Fig. 9.7a). Let us imagine that we concentrate this area into a thin strip parallel to the  $x$  axis (Fig. 9.7b). If the area  $A$ , thus concentrated, is to have the same moment of inertia with respect to the  $x$  axis, the strip should be placed at a distance  $k_x$  from the  $x$  axis, where  $k_x$  is defined by the relation

$$I_x = k_x^2 A$$

Solving for  $k_x$ , we write

$$k_x = \sqrt{\frac{I_x}{A}} \quad (9.5)$$

The distance  $k_x$  is referred to as the *radius of gyration* of the area with respect to the  $x$  axis. In a similar way, we can define the radii of gyration  $k_y$  and  $k_O$  (Fig. 9.7c and d); we write

$$I_y = k_y^2 A \quad k_y = \sqrt{\frac{I_y}{A}} \quad (9.6)$$

$$J_O = k_O^2 A \quad k_O = \sqrt{\frac{J_O}{A}} \quad (9.7)$$

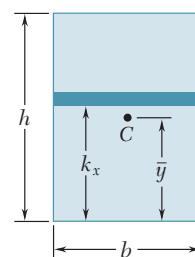
If we rewrite Eq. (9.4) in terms of the radii of gyration, we find that

$$k_O^2 = k_x^2 + k_y^2 \quad (9.8)$$

**EXAMPLE** For the rectangle shown in Fig. 9.8, let us compute the radius of gyration  $k_x$  with respect to its base. Using formulas (9.5) and (9.2), we write

$$k_x^2 = \frac{I_x}{A} = \frac{\frac{1}{3}bh^3}{bh} = \frac{h^2}{3} \quad k_x = \frac{h}{\sqrt{3}}$$

The radius of gyration  $k_x$  of the rectangle is shown in Fig. 9.8. It should not be confused with the ordinate  $\bar{y} = h/2$  of the centroid of the area. While  $k_x$  depends upon the *second moment*, or moment of inertia, of the area, the ordinate  $\bar{y}$  is related to the *first moment* of the area. ■

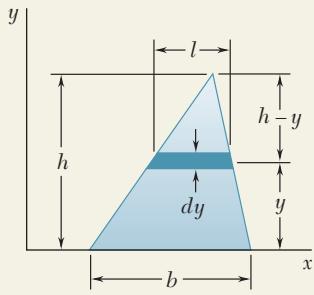


**Fig. 9.8**

## SAMPLE PROBLEM 9.1

Determine the moment of inertia of a triangle with respect to its base.

### SOLUTION



A triangle of base  $b$  and height  $h$  is drawn; the  $x$  axis is chosen to coincide with the base. A differential strip parallel to the  $x$  axis is chosen to be  $dA$ . Since all portions of the strip are at the same distance from the  $x$  axis, we write

$$dI_x = y^2 dA \quad dA = l dy$$

Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \quad l = b \frac{h-y}{h} \quad dA = b \frac{h-y}{h} dy$$

Integrating  $dI_x$  from  $y = 0$  to  $y = h$ , we obtain

$$\begin{aligned} I_x &= \int y^2 dA = \int_0^h y^2 b \frac{h-y}{h} dy = \frac{b}{h} \int_0^h (hy^2 - y^3) dy \\ &= \frac{b}{h} \left[ h \frac{y^3}{3} - \frac{y^4}{4} \right]_0^h \quad I_x = \frac{bh^3}{12} \end{aligned}$$

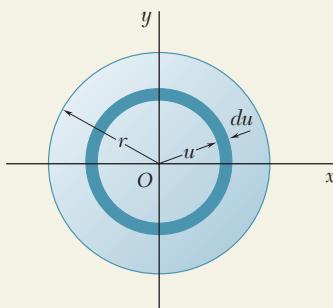
## SAMPLE PROBLEM 9.2

- (a) Determine the centroidal polar moment of inertia of a circular area by direct integration. (b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.

### SOLUTION

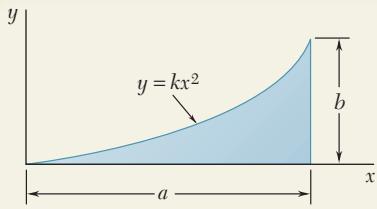
**a. Polar Moment of Inertia.** An annular differential element of area is chosen to be  $dA$ . Since all portions of the differential area are at the same distance from the origin, we write

$$\begin{aligned} dJ_O &= u^2 dA \quad dA = 2\pi u du \\ J_O &= \int dJ_O = \int_0^r u^2 (2\pi u du) = 2\pi \int_0^r u^3 du \\ J_O &= \frac{\pi}{2} r^4 \end{aligned}$$



**b. Moment of Inertia with Respect to a Diameter.** Because of the symmetry of the circular area, we have  $I_x = I_y$ . We then write

$$J_O = I_x + I_y = 2I_x \quad \frac{\pi}{2} r^4 = 2I_x \quad I_{\text{diameter}} = I_x = \frac{\pi}{4} r^4$$



## SAMPLE PROBLEM 9.3

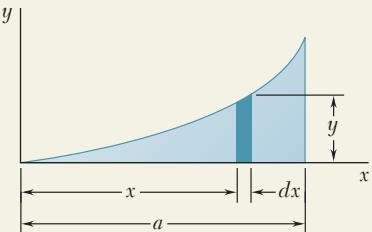
(a) Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes. (Properties of this area were considered in Sample Prob. 5.4.) (b) Using the results of part a, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.

## SOLUTION

Referring to Sample Prob. 5.4, we obtain the following expressions for the equation of the curve and the total area:

$$y = \frac{b}{a^2}x^2 \quad A = \frac{1}{3}ab$$

**Moment of Inertia  $I_x$ .** A vertical differential element of area is chosen to be  $dA$ . Since all portions of this element are *not* at the same distance from the  $x$  axis, we must treat the element as a thin rectangle. The moment of inertia of the element with respect to the  $x$  axis is then



$$\begin{aligned} dI_x &= \frac{1}{3}y^3 dx = \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{b^3}{a^6}x^6 dx \\ I_x &= \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a^6}x^6 dx = \left[\frac{1}{3}\frac{b^3}{a^6}\frac{x^7}{7}\right]_0^a \\ I_x &= \frac{ab^3}{21} \end{aligned}$$

**Moment of Inertia  $I_y$ .** The same vertical differential element of area is used. Since all portions of the element are at the same distance from the  $y$  axis, we write

$$\begin{aligned} dI_y &= x^2 dA = x^2(y dx) = x^2\left(\frac{b}{a^2}x^2\right)dx = \frac{b}{a^2}x^4 dx \\ I_y &= \int dI_y = \int_0^a \frac{b}{a^2}x^4 dx = \left[\frac{b}{a^2}\frac{x^5}{5}\right]_0^a \\ I_y &= \frac{a^3 b}{5} \end{aligned}$$

**Radii of Gyration  $k_x$  and  $k_y$ .** We have, by definition,

$$k_x^2 = \frac{I_x}{A} = \frac{ab^3/21}{ab/3} = \frac{b^2}{7} \quad k_x = \sqrt{\frac{1}{7}b}$$

and

$$k_y^2 = \frac{I_y}{A} = \frac{a^3 b/5}{ab/3} = \frac{3}{5}a^2 \quad k_y = \sqrt{\frac{3}{5}}a$$

# SOLVING PROBLEMS ON YOUR OWN

The purpose of this lesson was to introduce the *rectangular and polar moments of inertia of areas* and the corresponding *radii of gyration*. Although the problems you are about to solve may appear to be more appropriate for a calculus class than for one in mechanics, we hope that our introductory comments have convinced you of the relevance of the moments of inertia to your study of a variety of engineering topics.

**1. Calculating the rectangular moments of inertia  $I_x$  and  $I_y$ .** We defined these quantities as

$$I_x = \int y^2 dA \quad I_y = \int x^2 dA \quad (9.1)$$

where  $dA$  is a differential element of area  $dx dy$ . The moments of inertia are *the second moments of the area*; it is for that reason that  $I_x$ , for example, depends on the perpendicular distance  $y$  to the area  $dA$ . As you study Sec. 9.3, you should recognize the importance of carefully defining the shape and the orientation of  $dA$ . Further, you should note the following points.

a. **The moments of inertia of most areas can be obtained by means of a single integration.** The expressions given in Figs. 9.3b and c and Fig. 9.5 can be used to calculate  $I_x$  and  $I_y$ . Regardless of whether you use a single or a double integration, be sure to show on your sketch the element  $dA$  that you have chosen.

b. **The moment of inertia of an area is always positive,** regardless of the location of the area with respect to the coordinate axes. This is because it is obtained by integrating the product of  $dA$  and the *square* of distance. (Note how this differs from the results for the first moment of the area.) Only when an area is *removed* (as in the case for a hole) will its moment of inertia be entered in your computations with a minus sign.

c. **As a partial check of your work,** observe that the moments of inertia are equal to an area times the square of a length. Thus, every term in an expression for a moment of inertia must be a *length to the fourth power*.

**2. Computing the polar moment of inertia  $J_O$ .** We defined  $J_O$  as

$$J_O = \int r^2 dA \quad (9.3)$$

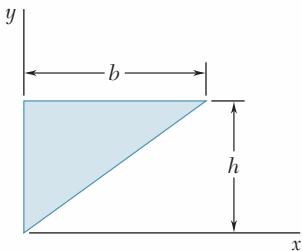
where  $r^2 = x^2 + y^2$ . If the given area has circular symmetry (as in Sample Prob. 9.2), it is possible to express  $dA$  as a function of  $r$  and to compute  $J_O$  with a single integration. When the area lacks circular symmetry, it is usually easier first to calculate  $I_x$  and  $I_y$  and then to determine  $J_O$  from

$$J_O = I_x + I_y \quad (9.4)$$

Lastly, if the equation of the curve that bounds the given area is expressed in polar coordinates, then  $dA = r dr d\theta$  and a double integration is required to compute the integral for  $J_O$  [see Prob. 9.27].

**3. Determining the radii of gyration  $k_x$  and  $k_y$  and the polar radius of gyration  $k_O$ .** These quantities were defined in Sec. 9.5, and you should realize that they can be determined only after the area and the appropriate moments of inertia have been computed. It is important to remember that  $k_x$  is measured in the  $y$  direction, while  $k_y$  is measured in the  $x$  direction; you should carefully study Sec. 9.5 until you understand this point.

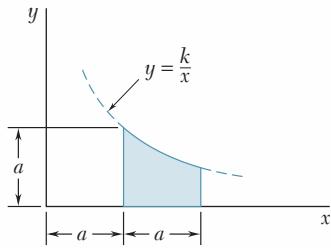
# PROBLEMS



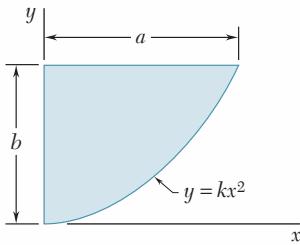
**Fig. P9.1 and P9.5**

**9.1 through 9.4** Determine by direct integration the moment of inertia of the shaded area with respect to the  $y$  axis.

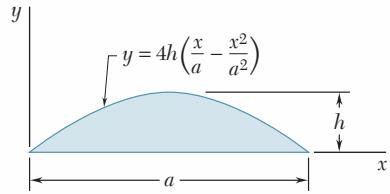
**9.5 through 9.8** Determine by direct integration the moment of inertia of the shaded area with respect to the  $x$  axis.



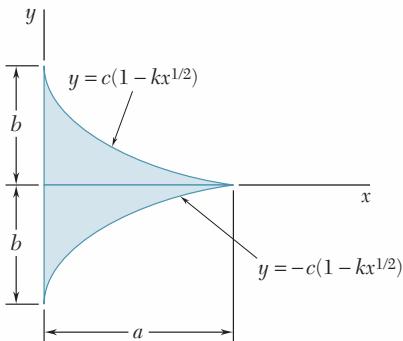
**Fig. P9.2 and P9.6**



**Fig. P9.3 and P9.7**



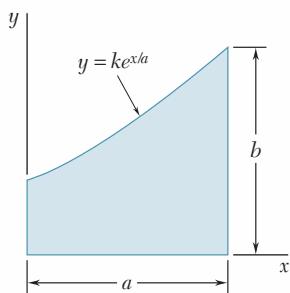
**Fig. P9.4 and P9.8**



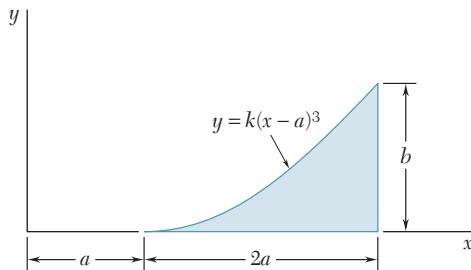
**Fig. P9.9 and P9.12**

**9.9 through 9.11** Determine by direct integration the moment of inertia of the shaded area with respect to the  $x$  axis.

**9.12 through 9.14** Determine by direct integration the moment of inertia of the shaded area with respect to the  $y$  axis.



**Fig. P9.10 and P9.13**



**Fig. P9.11 and P9.14**

- 9.15 and 9.16** Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the  $x$  axis.

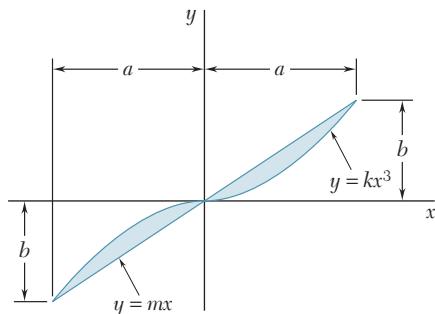


Fig. P9.15 and P9.17

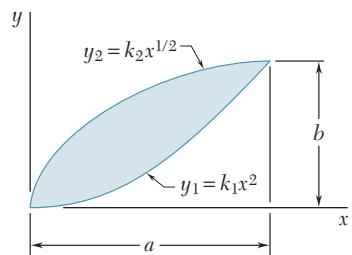


Fig. P9.16 and P9.18

- 9.17 and 9.18** Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the  $y$  axis.

- 9.19** Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the  $x$  axis.

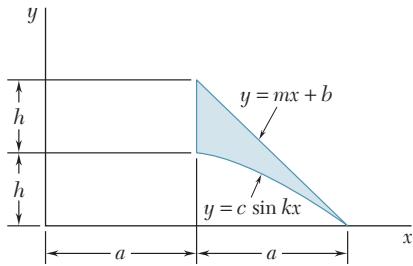


Fig. P9.19 and P9.20

- 9.20** Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the  $y$  axis.

- 9.21 and 9.22** Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to point  $P$ .

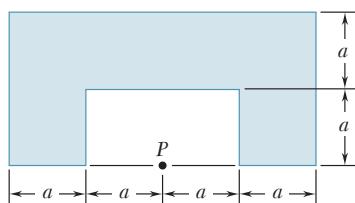


Fig. P9.21

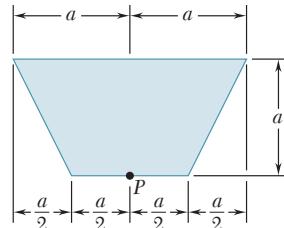


Fig. P9.22

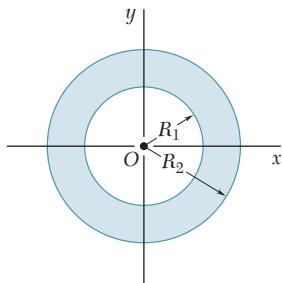


Fig. P9.23 and P9.24

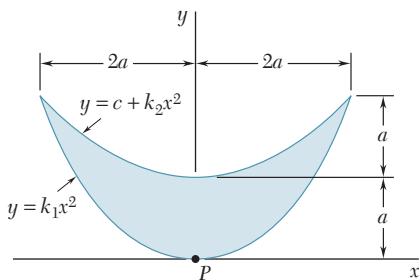


Fig. P9.25

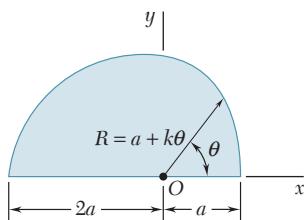


Fig. P9.27

- 9.23** (a) Determine by direct integration the polar moment of inertia of the annular area shown with respect to point  $O$ . (b) Using the result of part *a*, determine the moment of inertia of the given area with respect to the  $x$  axis.

- 9.24** (a) Show that the polar radius of gyration  $k_O$  of the annular area shown is approximately equal to the mean radius  $R_m = (R_1 + R_2)/2$  for small values of the thickness  $t = R_2 - R_1$ . (b) Determine the percentage error introduced by using  $R_m$  in place of  $k_O$  for the following values of  $t/R_m$ :  $1$ ,  $\frac{1}{2}$ , and  $\frac{1}{10}$ .

- 9.25 and 9.26** Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to point  $P$ .

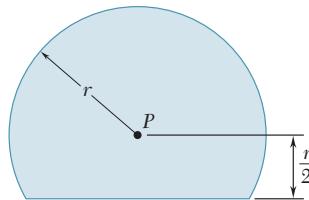


Fig. P9.26

- 9.27** Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to point  $O$ .

- 9.28** Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to point  $O$ .

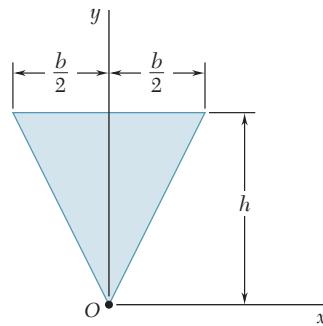


Fig. P9.28

- \*9.29** Using the polar moment of inertia of the isosceles triangle of Prob. 9.28, show that the centroidal polar moment of inertia of a circular area of radius  $r$  is  $\pi r^4/2$ . (Hint: As a circular area is divided into an increasing number of equal circular sectors, what is the approximate shape of each circular sector?)

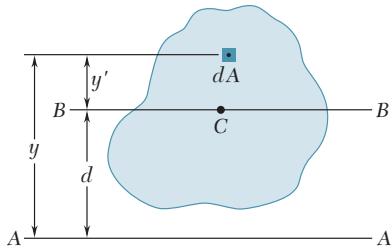
- \*9.30** Prove that the centroidal polar moment of inertia of a given area  $A$  cannot be smaller than  $A^2/2\pi$ . (Hint: Compare the moment of inertia of the given area with the moment of inertia of a circle that has the same area and the same centroid.)

## 9.6 PARALLEL-AXIS THEOREM

Consider the moment of inertia  $I$  of an area  $A$  with respect to an axis  $AA'$  (Fig. 9.9). Denoting by  $y$  the distance from an element of area  $dA$  to  $AA'$ , we write

$$I = \int y^2 dA$$

Let us now draw through the centroid  $C$  of the area an axis  $BB'$  parallel to  $AA'$ ; this axis is called a *centroidal axis*. Denoting by  $y'$



**Fig. 9.9**

the distance from the element  $dA$  to  $BB'$ , we write  $y = y' + d$ , where  $d$  is the distance between the axes  $AA'$  and  $BB'$ . Substituting for  $y$  in the above integral, we write

$$\begin{aligned} I &= \int y^2 dA = \int (y' + d)^2 dA \\ &= \int y'^2 dA + 2d \int y' dA + d^2 \int dA \end{aligned}$$

The first integral represents the moment of inertia  $\bar{I}$  of the area with respect to the centroidal axis  $BB'$ . The second integral represents the first moment of the area with respect to  $BB'$ ; since the centroid  $C$  of the area is located on that axis, the second integral must be zero. Finally, we observe that the last integral is equal to the total area  $A$ . Therefore, we have

$$I = \bar{I} + Ad^2 \quad (9.9)$$

This formula expresses that the moment of inertia  $I$  of an area with respect to any given axis  $AA'$  is equal to the moment of inertia  $\bar{I}$  of the area with respect to a centroidal axis  $BB'$  parallel to  $AA'$  plus the product of the area  $A$  and the square of the distance  $d$  between the two axes. This theorem is known as the *parallel-axis theorem*. Substituting  $k^2A$  for  $I$  and  $\bar{k}^2A$  for  $\bar{I}$ , the theorem can also be expressed as

$$k^2 = \bar{k}^2 + d^2 \quad (9.10)$$

A similar theorem can be used to relate the polar moment of inertia  $J_O$  of an area about a point  $O$  to the polar moment of inertia  $\bar{J}_C$  of the same area about its centroid  $C$ . Denoting by  $d$  the distance between  $O$  and  $C$ , we write

$$J_O = \bar{J}_C + Ad^2 \quad \text{or} \quad k_O^2 = \bar{k}_C^2 + d^2 \quad (9.11)$$

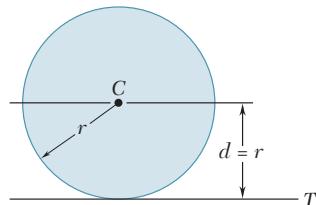


Fig. 9.10

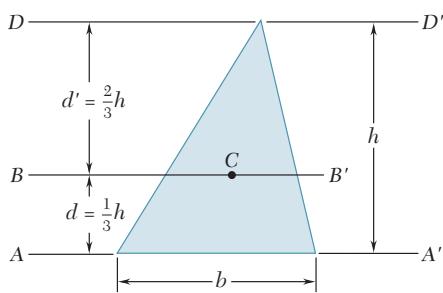


Fig. 9.11

**EXAMPLE 1** As an application of the parallel-axis theorem, let us determine the moment of inertia  $I_T$  of a circular area with respect to a line tangent to the circle (Fig. 9.10). We found in Sample Prob. 9.2 that the moment of inertia of a circular area about a centroidal axis is  $\bar{I} = \frac{1}{4}\pi r^4$ . We can write, therefore,

$$I_T = \bar{I} + Ad^2 = \frac{1}{4}\pi r^4 + (\pi r^2)r^2 = \frac{5}{4}\pi r^4 \blacksquare$$

**EXAMPLE 2** The parallel-axis theorem can also be used to determine the centroidal moment of inertia of an area when the moment of inertia of the area with respect to a parallel axis is known. Consider, for instance, a triangular area (Fig. 9.11). We found in Sample Prob. 9.1 that the moment of inertia of a triangle with respect to its base  $AA'$  is equal to  $\frac{1}{12}bh^3$ . Using the parallel-axis theorem, we write

$$\begin{aligned} I_{AA'} &= \bar{I}_{BB'} + Ad^2 \\ \bar{I}_{BB'} &= I_{AA'} - Ad^2 = \frac{1}{12}bh^3 - \frac{1}{2}bh\left(\frac{1}{3}h\right)^2 = \frac{1}{36}bh^3 \end{aligned}$$

It should be observed that the product  $Ad^2$  was *subtracted* from the given moment of inertia in order to obtain the centroidal moment of inertia of the triangle. Note that this product is *added* when transferring *from* a centroidal axis to a parallel axis, but it should be *subtracted* when transferring *to* a centroidal axis. In other words, the moment of inertia of an area is always smaller with respect to a centroidal axis than with respect to any parallel axis.

Returning to Fig. 9.11, we observe that the moment of inertia of the triangle with respect to the line  $DD'$  (which is drawn through a vertex) can be obtained by writing

$$I_{DD'} = \bar{I}_{BB'} + Ad'^2 = \frac{1}{36}bh^3 + \frac{1}{2}bh\left(\frac{2}{3}h\right)^2 = \frac{1}{4}bh^3$$

Note that  $I_{DD'}$  could not have been obtained directly from  $I_{AA'}$ . The parallel-axis theorem can be applied only if one of the two parallel axes passes through the centroid of the area. ■

## 9.7 MOMENTS OF INERTIA OF COMPOSITE AREAS

Consider a composite area  $A$  made of several component areas  $A_1, A_2, A_3, \dots$ . Since the integral representing the moment of inertia of  $A$  can be subdivided into integrals evaluated over  $A_1, A_2, A_3, \dots$ , the moment of inertia of  $A$  with respect to a given axis is obtained by adding the moments of inertia of the areas  $A_1, A_2, A_3, \dots$ , with respect to the same axis. The moment of inertia of an area consisting of several of the common shapes shown in Fig. 9.12 can thus be obtained by using the formulas given in that figure. Before adding the moments of inertia of the component areas, however, the parallel-axis theorem may have to be used to transfer each moment of inertia to the desired axis. This is shown in Sample Probs. 9.4 and 9.5.

The properties of the cross sections of various structural shapes are given in Fig. 9.13. As noted in Sec. 9.2, the moment of inertia of a beam section about its neutral axis is closely related to the computation of the bending moment in that section of the beam. The



**Photo 9.1** Figure 9.13 tabulates data for a small sample of the rolled-steel shapes that are readily available. Shown above are two examples of wide-flange shapes that are commonly used in the construction of buildings.

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

**Fig. 9.12** Moments of inertia of common geometric shapes.

determination of moments of inertia is thus a prerequisite to the analysis and design of structural members.

It should be noted that the radius of gyration of a composite area is *not* equal to the sum of the radii of gyration of the component areas. In order to determine the radius of gyration of a composite area, it is first necessary to compute the moment of inertia of the area.

	Designation	Area in <sup>2</sup>	Depth in.	Width in.	Axis X-X			Axis Y-Y		
					$\bar{I}_x$ , in <sup>4</sup>	$\bar{k}_x$ , in.	$\bar{y}$ , in.	$\bar{I}_y$ , in <sup>4</sup>	$\bar{k}_y$ , in.	$\bar{x}$ , in.
W Shapes (Wide-Flange Shapes)	W18 × 76†	22.3	18.2	11.0	1330	7.73		152	2.61	
	W16 × 57	16.8	16.4	7.12	758	6.72		43.1	1.60	
	W14 × 38	11.2	14.1	6.77	385	5.87		26.7	1.55	
	W8 × 31	9.12	8.00	8.00	110	3.47		37.1	2.02	
S Shapes (American Standard Shapes)	S18 × 54.7†	16.0	18.0	6.00	801	7.07		20.7	1.14	
	S12 × 31.8	9.31	12.0	5.00	217	4.83		9.33	1.00	
	S10 × 25.4	7.45	10.0	4.66	123	4.07		6.73	0.950	
	S6 × 12.5	3.66	6.00	3.33	22.0	2.45		1.80	0.702	
C Shapes (American Standard Channels)	C12 × 20.7†	6.08	12.0	2.94	129	4.61		3.86	0.797	0.698
	C10 × 15.3	4.48	10.0	2.60	67.3	3.87		2.27	0.711	0.634
	C8 × 11.5	3.37	8.00	2.26	32.5	3.11		1.31	0.623	0.572
	C6 × 8.2	2.39	6.00	1.92	13.1	2.34		0.687	0.536	0.512
Angles	L6 × 6 × 1‡	11.0			35.4	1.79	1.86	35.4	1.79	1.86
	L4 × 4 × $\frac{1}{2}$	3.75			5.52	1.21	1.18	5.52	1.21	1.18
	L3 × 3 × $\frac{1}{4}$	1.44			1.23	0.926	0.836	1.23	0.926	0.836
	L6 × 4 × $\frac{1}{2}$	4.75			17.3	1.91	1.98	6.22	1.14	0.981
	L5 × 3 × $\frac{1}{2}$	3.75			9.43	1.58	1.74	2.55	0.824	0.746
	L3 × 2 × $\frac{1}{4}$	1.19			1.09	0.953	0.980	0.390	0.569	0.487

**Fig. 9.13A** Properties of rolled-steel shapes (U.S. customary units).\*

\*Courtesy of the American Institute of Steel Construction, Chicago, Illinois

†Nominal depth in inches and weight in pounds per foot

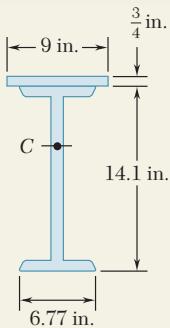
‡Depth, width, and thickness in inches

	Designation	Area mm <sup>2</sup>	Depth mm	Width mm	Axis X-X			Axis Y-Y		
					$\bar{I}_x$ $10^6 \text{ mm}^4$	$\bar{k}_x$ mm	$\bar{y}$ mm	$\bar{I}_y$ $10^6 \text{ mm}^4$	$\bar{k}_y$ mm	$\bar{x}$ mm
W Shapes (Wide-Flange Shapes)	W460 × 113†	14400	462	279	554	196		63.3	66.3	
	W410 × 85	10800	417	181	316	171		17.9	40.6	
	W360 × 57.8	7230	358	172	160	149		11.1	39.4	
	W200 × 46.1	5880	203	203	45.8	88.1		15.4	51.3	
S Shapes (American Standard Shapes)	S460 × 81.4‡	10300	457	152	333	180		8.62	29.0	
	S310 × 47.3	6010	305	127	90.3	123		3.88	25.4	
	S250 × 37.8	4810	254	118	51.2	103		2.80	24.1	
	S150 × 18.6	2360	152	84.6	9.16	62.2		0.749	17.8	
C Shapes (American Standard Channels)	C310 × 30.8†	3920	305	74.7	53.7	117		1.61	20.2	17.7
	C250 × 22.8	2890	254	66.0	28.0	98.3		0.945	18.1	16.1
	C200 × 17.1	2170	203	57.4	13.5	79.0		0.545	15.8	14.5
	C150 × 12.2	1540	152	48.8	5.45	59.4		0.286	13.6	13.0
Angles	L152 × 152 × 25.4‡	7100			14.7	45.5	47.2	14.7	45.5	47.2
	L102 × 102 × 12.7	2420			2.30	30.7	30.0	2.30	30.7	30.0
	L76 × 76 × 6.4	929			0.512	23.5	21.2	0.512	23.5	21.2
	L152 × 102 × 12.7	3060			7.20	48.5	50.3	2.59	29.0	24.9
	L127 × 76 × 12.7	2420			3.93	40.1	44.2	1.06	20.9	18.9
	L76 × 51 × 6.4	768			0.454	24.2	24.9	0.162	14.5	12.4

**Fig. 9.13B** Properties of rolled-steel shapes (SI units).

†Nominal depth in millimeters and mass in kilograms per meter

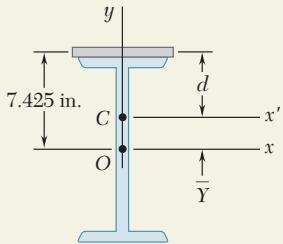
‡Depth, width, and thickness in millimeters



## SAMPLE PROBLEM 9.4

The strength of a W14 × 38 rolled-steel beam is increased by attaching a 9 ×  $\frac{3}{4}$ -in. plate to its upper flange as shown. Determine the moment of inertia and the radius of gyration of the composite section with respect to an axis which is parallel to the plate and passes through the centroid C of the section.

## SOLUTION



The origin O of the coordinates is placed at the centroid of the wide-flange shape, and the distance  $\bar{Y}$  to the centroid of the composite section is computed using the methods of Chap. 5. The area of the wide-flange shape is found by referring to Fig. 9.13A. The area and the  $y$  coordinate of the centroid of the plate are

$$A = (9 \text{ in.})(0.75 \text{ in.}) = 6.75 \text{ in}^2$$

$$\bar{y} = \frac{1}{2}(14.1 \text{ in.}) + \frac{1}{2}(0.75 \text{ in.}) = 7.425 \text{ in.}$$

Section	Area, in <sup>2</sup>	$\bar{y}$ , in.	$\bar{y}A$ , in <sup>3</sup>
Plate	6.75	7.425	50.12
Wide-flange shape	11.2	0	0
	$\Sigma A = 17.95$		$\Sigma \bar{y}A = 50.12$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A \quad \bar{Y}(17.95) = 50.12 \quad \bar{Y} = 2.792 \text{ in.}$$

**Moment of Inertia.** The parallel-axis theorem is used to determine the moments of inertia of the wide-flange shape and the plate with respect to the  $x'$  axis. This axis is a centroidal axis for the composite section but *not* for either of the elements considered separately. The value of  $\bar{I}_x$  for the wide-flange shape is obtained from Fig. 9.13A.

For the wide-flange shape,

$$I_{x'} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.2)(2.792)^2 = 472.3 \text{ in}^4$$

For the plate,

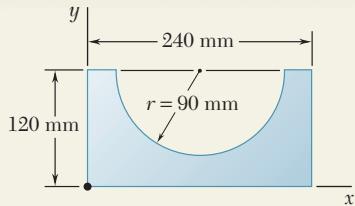
$$I_{x'} = \bar{I}_x + Ad^2 = (\frac{1}{12})(9)(\frac{3}{4})^3 + (6.75)(7.425 - 2.792)^2 = 145.2 \text{ in}^4$$

For the composite area,

$$I_{x'} = 472.3 + 145.2 = 617.5 \text{ in}^4 \quad I_{x'} = 618 \text{ in}^4 \quad \blacktriangleleft$$

**Radius of Gyration.** We have

$$k_{x'}^2 = \frac{I_{x'}}{A} = \frac{617.5 \text{ in}^4}{17.95 \text{ in}^2} \quad k_{x'} = 5.87 \text{ in.} \quad \blacktriangleleft$$

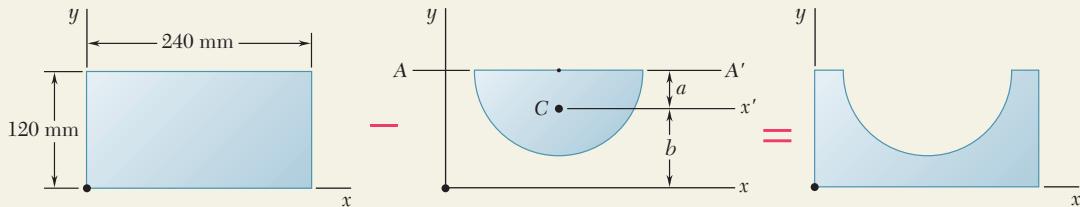


## SAMPLE PROBLEM 9.5

Determine the moment of inertia of the shaded area with respect to the  $x$  axis.

### SOLUTION

The given area can be obtained by subtracting a half circle from a rectangle. The moments of inertia of the rectangle and the half circle will be computed separately.



**Moment of Inertia of Rectangle.** Referring to Fig. 9.12, we obtain

$$I_x = \frac{1}{3}bh^3 = \frac{1}{3}(240 \text{ mm})(120 \text{ mm})^3 = 138.2 \times 10^6 \text{ mm}^4$$

**Moment of Inertia of Half Circle.** Referring to Fig. 5.8, we determine the location of the centroid  $C$  of the half circle with respect to diameter  $AA'$ .

$$a = \frac{4r}{3\pi} = \frac{(4)(90 \text{ mm})}{3\pi} = 38.2 \text{ mm}$$

The distance  $b$  from the centroid  $C$  to the  $x$  axis is

$$b = 120 \text{ mm} - a = 120 \text{ mm} - 38.2 \text{ mm} = 81.8 \text{ mm}$$

Referring now to Fig. 9.12, we compute the moment of inertia of the half circle with respect to diameter  $AA'$ ; we also compute the area of the half circle.

$$\begin{aligned} I_{AA'} &= \frac{1}{8}\pi r^4 = \frac{1}{8}\pi(90 \text{ mm})^4 = 25.76 \times 10^6 \text{ mm}^4 \\ A &= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(90 \text{ mm})^2 = 12.72 \times 10^3 \text{ mm}^2 \end{aligned}$$

Using the parallel-axis theorem, we obtain the value of  $\bar{I}_{x'}$ :

$$\begin{aligned} I_{AA'} &= \bar{I}_{x'} + Aa^2 \\ 25.76 \times 10^6 \text{ mm}^4 &= \bar{I}_{x'} + (12.72 \times 10^3 \text{ mm}^2)(38.2 \text{ mm})^2 \\ \bar{I}_{x'} &= 7.20 \times 10^6 \text{ mm}^4 \end{aligned}$$

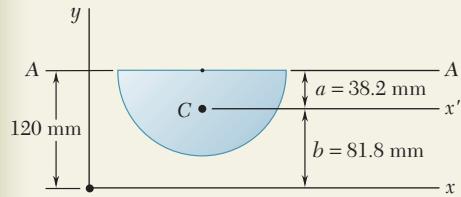
Again using the parallel-axis theorem, we obtain the value of  $I_x$ :

$$\begin{aligned} I_x &= \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 \text{ mm}^4 + (12.72 \times 10^3 \text{ mm}^2)(81.8 \text{ mm})^2 \\ &= 92.3 \times 10^6 \text{ mm}^4 \end{aligned}$$

**Moment of Inertia of Given Area.** Subtracting the moment of inertia of the half circle from that of the rectangle, we obtain

$$I_x = 138.2 \times 10^6 \text{ mm}^4 - 92.3 \times 10^6 \text{ mm}^4$$

$$I_x = 45.9 \times 10^6 \text{ mm}^4$$



# SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced the *parallel-axis theorem* and illustrated how it can be used to simplify the computation of moments and polar moments of inertia of composite areas. The areas that you will consider in the following problems will consist of common shapes and rolled-steel shapes. You will also use the parallel-axis theorem to locate the point of application (the center of pressure) of the resultant of the hydrostatic forces acting on a submerged plane area.

**1. Applying the parallel-axis theorem.** In Sec. 9.6 we derived the parallel-axis theorem

$$I = \bar{I} + Ad^2 \quad (9.9)$$

which states that the moment of inertia  $I$  of an area  $A$  with respect to a given axis is equal to the sum of the moment of inertia  $\bar{I}$  of that area with respect to a *parallel centroidal axis* and the product  $Ad^2$ , where  $d$  is the distance between the two axes. It is important that you remember the following points as you use the parallel-axis theorem.

a. **The centroidal moment of inertia  $\bar{I}$  of an area  $A$  can be obtained by subtracting the product  $Ad^2$  from the moment of inertia  $I$  of the area with respect to a parallel axis.** It follows that the moment of inertia  $\bar{I}$  is *smaller* than the moment of inertia  $I$  of the same area with respect to any parallel axis.

b. **The parallel-axis theorem can be applied only if one of the two axes involved is a centroidal axis.** Therefore, as we noted in Example 2, to compute the moment of inertia of an area with respect to a *noncentroidal axis* when the moment of inertia of the area is known with respect to *another noncentroidal axis*, it is necessary to *first compute* the moment of inertia of the area with respect to a *centroidal axis parallel to the two given axes*.

**2. Computing the moments and polar moments of inertia of composite areas.** Sample Probs. 9.4 and 9.5 illustrate the steps you should follow to solve problems of this type. As with all composite-area problems, you should show on your sketch the common shapes or rolled-steel shapes that constitute the various elements of the given area, as well as the distances between the centroidal axes of the elements and the axes about which the moments of inertia are to be computed. In addition, it is important that the following points be noted.

a. **The moment of inertia of an area is always positive,** regardless of the location of the axis with respect to which it is computed. As pointed out in the comments for the preceding lesson, it is only when an area is *removed* (as in the case of a hole) that its moment of inertia should be entered in your computations with a minus sign.

**b. The moments of inertia of a semiellipse and a quarter ellipse** can be determined by dividing the moment of inertia of an ellipse by 2 and 4, respectively. It should be noted, however, that the moments of inertia obtained in this manner are *with respect to the axes of symmetry of the ellipse*. To obtain the *centroidal* moments of inertia of these shapes, the parallel-axis theorem should be used. Note that this remark also applies to a semicircle and to a quarter circle and that the expressions given for these shapes in Fig. 9.12 are *not* centroidal moments of inertia.

**c. To calculate the polar moment of inertia** of a composite area, you can use either the expressions given in Fig. 9.12 for  $J_O$  or the relationship

$$J_O = I_x + I_y \quad (9.4)$$

depending on the shape of the given area.

**d. Before computing the centroidal moments of inertia** of a given area, you may find it necessary to first locate the centroid of the area using the methods of Chapter 5.

### 3. Locating the point of application of the resultant of a system of hydrostatic forces.

In Sec. 9.2 we found that

$$\begin{aligned} R &= \gamma \int y \, dA = \gamma \bar{y} A \\ M_x &= \gamma \int y^2 \, dA = \gamma I_x \end{aligned}$$

where  $\bar{y}$  is the distance from the  $x$  axis to the centroid of the submerged plane area. Since  $\mathbf{R}$  is equivalent to the system of elemental hydrostatic forces, it follows that

$$\Sigma M_x: \quad y_p R = M_x$$

where  $y_p$  is the depth of the point of application of  $\mathbf{R}$ . Then

$$y_p(\gamma \bar{y} A) = \gamma I_x \quad \text{or} \quad y_p = \frac{I_x}{\bar{y} A}$$

In closing, we encourage you to carefully study the notation used in Fig. 9.13 for the rolled-steel shapes, as you will likely encounter it again in subsequent engineering courses.

# PROBLEMS

**9.31 and 9.32** Determine the moment of inertia and the radius of gyration of the shaded area with respect to the  $x$  axis.

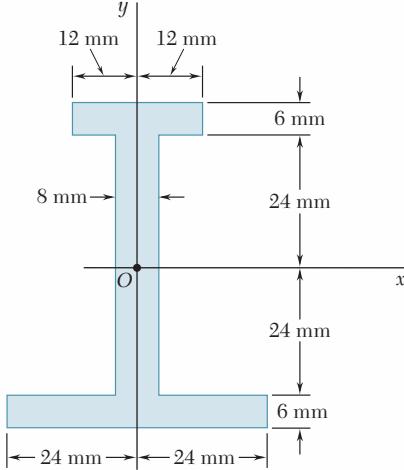


Fig. P9.31 and P9.33

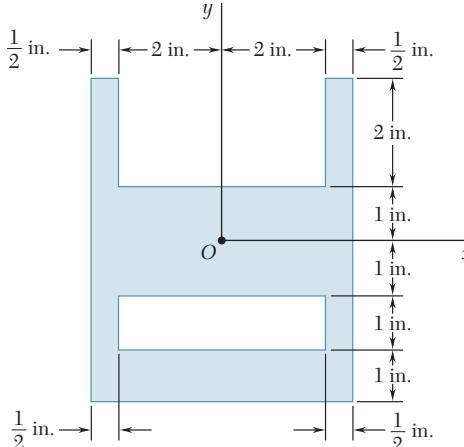


Fig. P9.32 and P9.34

**9.33 and 9.34** Determine the moment of inertia and the radius of gyration of the shaded area with respect to the  $y$  axis.

**9.35 and 9.36** Determine the moments of inertia of the shaded area shown with respect to the  $x$  and  $y$  axes when  $a = 20 \text{ mm}$ .

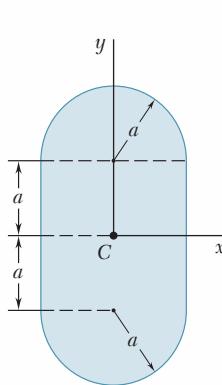


Fig. P9.35

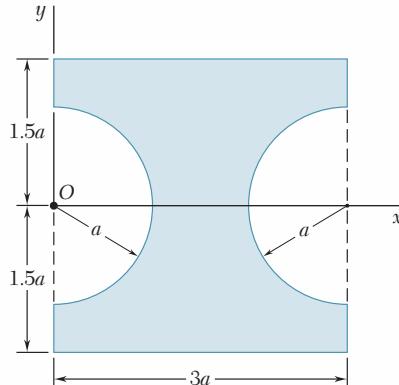


Fig. P9.36

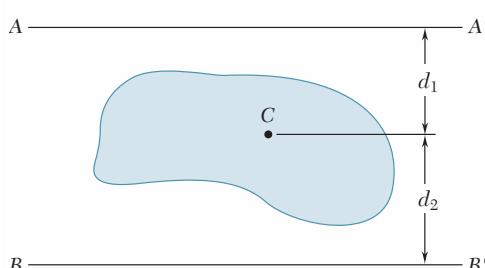


Fig. P9.37 and P9.38

**9.37** For the  $4000\text{-mm}^2$  shaded area shown, determine the distance  $d_2$  and the moment of inertia with respect to the centroidal axis parallel to  $AA'$  knowing that the moments of inertia with respect to  $AA'$  and  $BB'$  are  $12 \times 10^6 \text{ mm}^4$  and  $23.9 \times 10^6 \text{ mm}^4$ , respectively, and that  $d_1 = 25 \text{ mm}$ .

**9.38** Determine for the shaded region the area and the moment of inertia with respect to the centroidal axis parallel to  $BB'$ , knowing that  $d_1 = 25 \text{ mm}$  and  $d_2 = 15 \text{ mm}$  and that the moments of inertia with respect to  $AA'$  and  $BB'$  are  $7.84 \times 10^6 \text{ mm}^4$  and  $5.20 \times 10^6 \text{ mm}^4$ , respectively.

- 9.39** The shaded area is equal to 50 in.<sup>2</sup>. Determine its centroidal moments of inertia  $\bar{I}_x$  and  $\bar{I}_y$ , knowing that  $\bar{I}_y = 2\bar{I}_x$  and that the polar moment of inertia of the area about point A is  $J_A = 2250 \text{ in}^4$ .

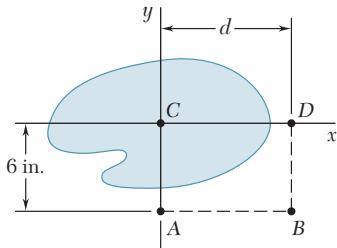


Fig. P9.39 and P9.40

- 9.40** The polar moments of inertia of the shaded area with respect to points A, B, and D are, respectively,  $J_A = 2880 \text{ in}^4$ ,  $J_B = 6720 \text{ in}^4$ , and  $J_D = 4560 \text{ in}^4$ . Determine the shaded area, its centroidal moment of inertia  $\bar{J}_C$ , and the distance d from C to D.

- 9.41 through 9.44** Determine the moments of inertia  $\bar{I}_x$  and  $\bar{I}_y$  of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

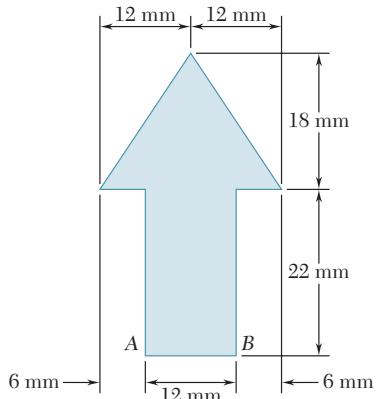


Fig. P9.42

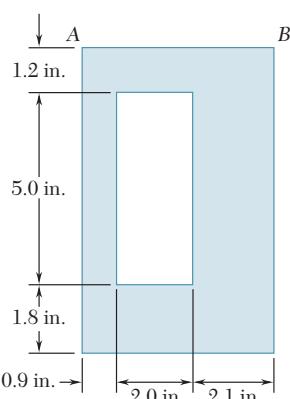


Fig. P9.43

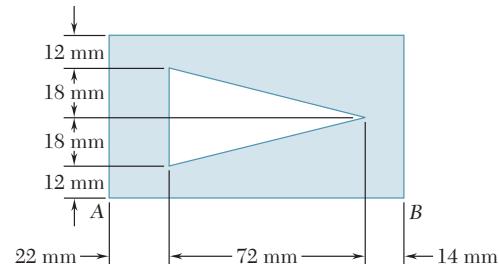


Fig. P9.41

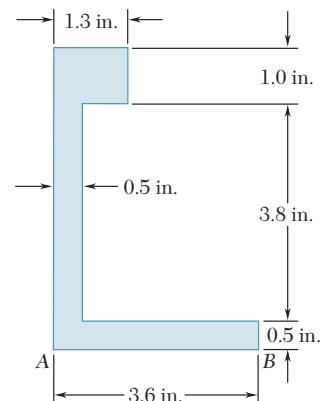


Fig. P9.44

- 9.45 and 9.46** Determine the polar moment of inertia of the area shown with respect to (a) point O, (b) the centroid of the area.

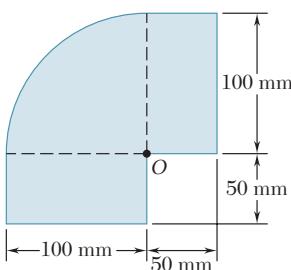


Fig. P9.45

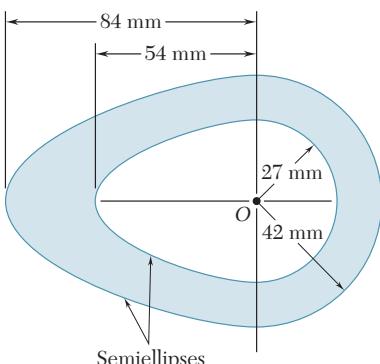


Fig. P9.46

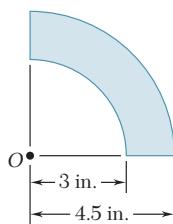


Fig. P9.47

**9.47 and 9.48** Determine the polar moment of inertia of the area shown with respect to (a) point  $O$ , (b) the centroid of the area.

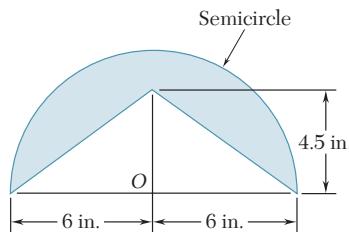


Fig. P9.48

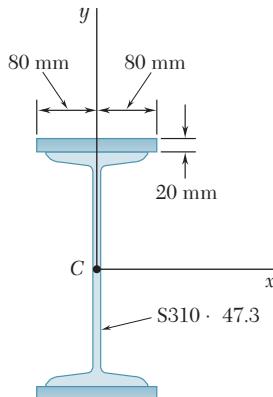


Fig. P9.49

**9.49** Two 20-mm steel plates are welded to a rolled S section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal  $x$  and  $y$  axes.

**9.50** Two channels are welded to a rolled W section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal  $x$  and  $y$  axes.

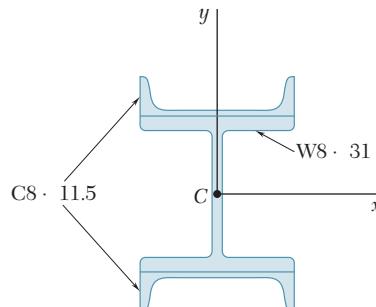


Fig. P9.50

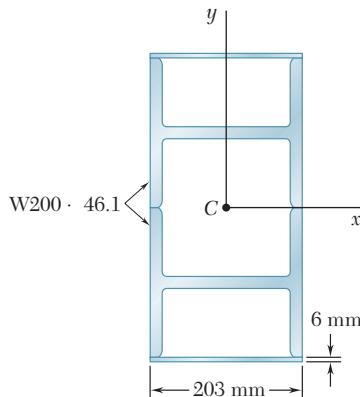


Fig. P9.51

**9.51** To form a reinforced box section, two rolled W sections and two plates are welded together. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal axes shown.

**9.52** Two channels are welded to a  $d \times 12$ -in. steel plate as shown. Determine the width  $d$  for which the ratio  $\bar{I}_x/\bar{I}_y$  of the centroidal moments of inertia of the section is 16.

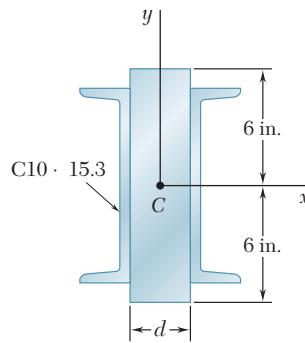


Fig. P9.52

- 9.53** Two L $76 \times 76 \times 6.4$ -mm angles are welded to a C $250 \times 22.8$  channel. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the web of the channel.

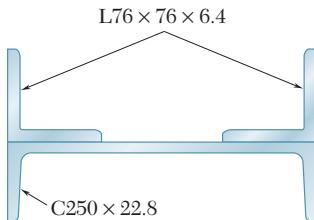


Fig. P9.53

- 9.54** Two L $4 \times 4 \times \frac{1}{2}$ -in. angles are welded to a steel plate as shown. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the plate.

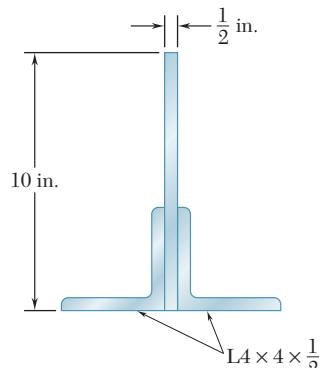


Fig. P9.54

- 9.55** The strength of the rolled W section shown is increased by welding a channel to its upper flange. Determine the moments of inertia of the combined section with respect to its centroidal  $x$  and  $y$  axes.

- 9.56** Two L $5 \times 3 \times \frac{1}{2}$ -in. angles are welded to a  $\frac{1}{2}$ -in. steel plate. Determine the distance  $b$  and the centroidal moments of inertia  $\bar{I}_x$  and  $\bar{I}_y$  of the combined section, knowing that  $\bar{I}_y = 4\bar{I}_x$ .

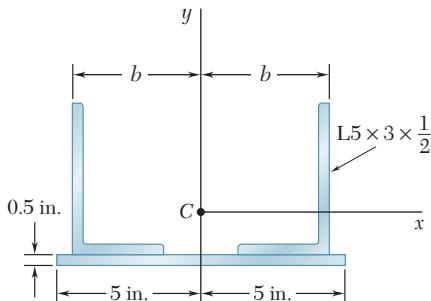


Fig. P9.56

- 9.57 and 9.58** The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to Sec. 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

- 9.59 and \*9.60** The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to Sec. 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

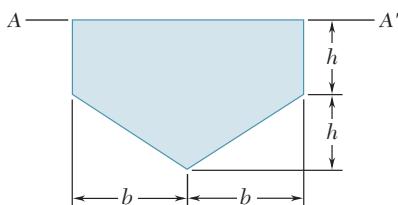


Fig. P9.59

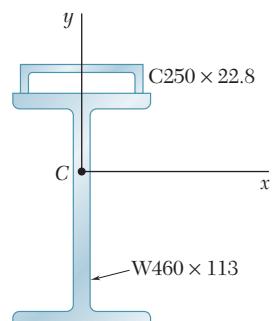


Fig. P9.57

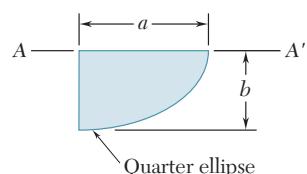


Fig. P9.58

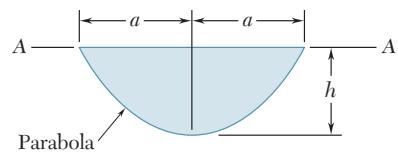
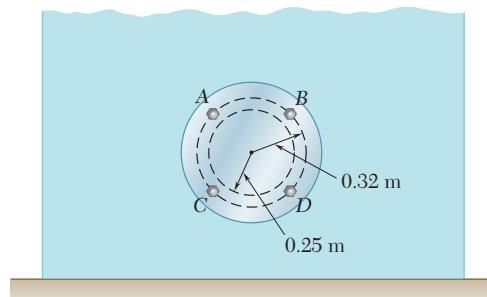
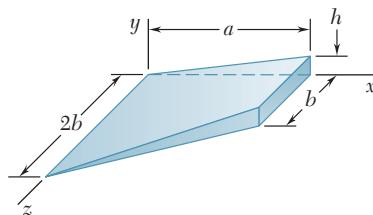
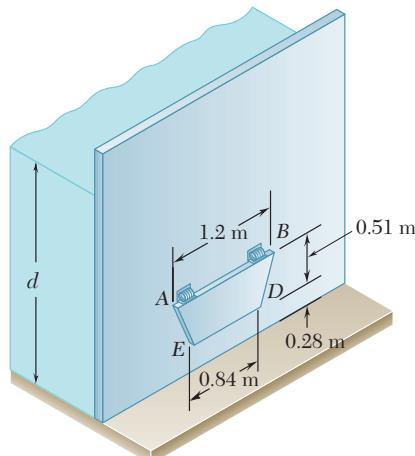
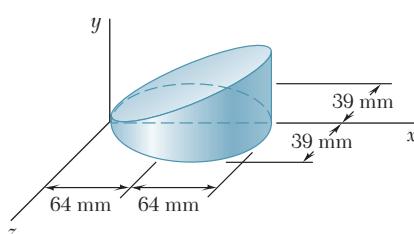


Fig. P9.60

- 9.61** The cover for a 0.5-m-diameter access hole in a water storage tank is attached to the tank with four equally spaced bolts as shown. Determine the additional force on each bolt due to the water pressure when the center of the cover is located 1.4 m below the water surface.

**Fig. P9.61**

- 9.62** A vertical trapezoidal gate that is used as an automatic valve is held shut by two springs attached to hinges located along edge  $AB$ . Knowing that each spring exerts a couple of magnitude 1470 N · m, determine the depth  $d$  of water for which the gate will open.

**Fig. P9.62****Fig. P9.62****Fig. P9.63**

- \*9.63** Determine the  $x$  coordinate of the centroid of the volume shown. (*Hint:* The height  $y$  of the volume is proportional to the  $x$  coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

- \*9.64** Determine the  $x$  coordinate of the centroid of the volume shown; this volume was obtained by intersecting an elliptic cylinder with an oblique plane. (See hint of Prob. 9.63.)

- \*9.65** Show that the system of hydrostatic forces acting on a submerged plane area  $A$  can be reduced to a force  $\mathbf{P}$  at the centroid  $C$  of the area and two couples. The force  $\mathbf{P}$  is perpendicular to the area and is of magnitude  $P = \gamma A \bar{y} \sin \theta$ , where  $\gamma$  is the specific weight of the liquid, and the couples are  $\mathbf{M}_{x'} = (\gamma \bar{I}_{x'} \sin \theta) \mathbf{i}$  and  $\mathbf{M}_{y'} = (\gamma \bar{I}_{x'y'} \sin \theta) \mathbf{j}$ , where  $\bar{I}_{x'y'} = \int x'y' dA$  (see Sec. 9.8). Note that the couples are independent of the depth at which the area is submerged.

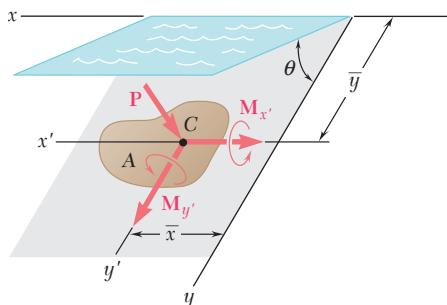


Fig. P9.65

- \*9.66** Show that the resultant of the hydrostatic forces acting on a submerged plane area  $A$  is a force  $\mathbf{P}$  perpendicular to the area and of magnitude  $P = \gamma A \bar{y} \sin \theta = \bar{p} A$ , where  $\gamma$  is the specific weight of the liquid and  $\bar{p}$  is the pressure at the centroid  $C$  of the area. Show that  $\mathbf{P}$  is applied at a point  $C_p$ , called the center of pressure, whose coordinates are  $x_p = I_{xy}/A \bar{y}$  and  $y_p = I_x/A \bar{y}$ , where  $I_{xy} = \int xy dA$  (see Sec. 9.8). Show also that the difference of ordinates  $y_p - \bar{y}$  is equal to  $\bar{k}_x^2/\bar{y}$  and thus depends upon the depth at which the area is submerged.

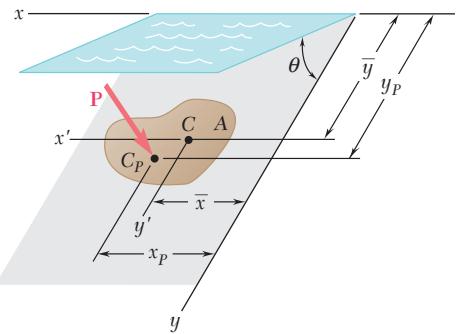


Fig. P9.66

## 9.8 PRODUCT OF INERTIA

The integral

$$I_{xy} = \int xy dA \quad (9.12)$$

which is obtained by multiplying each element  $dA$  of an area  $A$  by its coordinates  $x$  and  $y$  and integrating over the area (Fig. 9.14), is known as the *product of inertia* of the area  $A$  with respect to the  $x$  and  $y$  axes. Unlike the moments of inertia  $I_x$  and  $I_y$ , the product of inertia  $I_{xy}$  can be positive, negative, or zero.

When one or both of the  $x$  and  $y$  axes are axes of symmetry for the area  $A$ , the product of inertia  $I_{xy}$  is zero. Consider, for example, the channel section shown in Fig. 9.15. Since this section is symmetrical with respect to the  $x$  axis, we can associate with each element  $dA$  of coordinates  $x$  and  $y$  an element  $dA'$  of coordinates  $x$  and  $-y$ . Clearly, the contributions to  $I_{xy}$  of any pair of elements chosen in this way cancel out, and the integral (9.12) reduces to zero.

A parallel-axis theorem similar to the one established in Sec. 9.6 for moments of inertia can be derived for products of inertia. Consider an area  $A$  and a system of rectangular coordinates  $x$  and  $y$

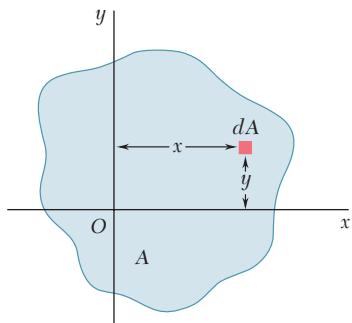


Fig. 9.14

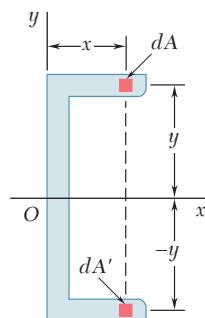


Fig. 9.15

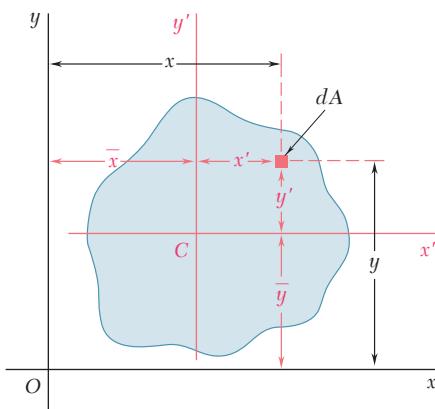


Fig. 9.16

(Fig. 9.16). Through the centroid  $C$  of the area, of coordinates  $\bar{x}$  and  $\bar{y}$ , we draw two *centroidal axes*  $x'$  and  $y'$  which are parallel, respectively, to the  $x$  and  $y$  axes. Denoting by  $x$  and  $y$  the coordinates of an element of area  $dA$  with respect to the original axes, and by  $x'$  and  $y'$  the coordinates of the same element with respect to the centroidal axes, we write  $x = x' + \bar{x}$  and  $y = y' + \bar{y}$ . Substituting into (9.12), we obtain the following expression for the product of inertia  $I_{xy}$ :

$$\begin{aligned} I_{xy} &= \int xy \, dA = \int (x' + \bar{x})(y' + \bar{y}) \, dA \\ &= \int x'y' \, dA + \bar{y} \int x' \, dA + \bar{x} \int y' \, dA + \bar{x}\bar{y} \int \, dA \end{aligned}$$

The first integral represents the product of inertia  $\bar{I}_{x'y'}$  of the area  $A$  with respect to the centroidal axes  $x'$  and  $y'$ . The next two integrals represent first moments of the area with respect to the centroidal axes; they reduce to zero, since the centroid  $C$  is located on these axes. Finally, we observe that the last integral is equal to the total area  $A$ . Therefore, we have

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \quad (9.13)$$

## \*9.9 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA

Consider the area  $A$  and the coordinate axes  $x$  and  $y$  (Fig. 9.17). Assuming that the moments and product of inertia

$$I_x = \int y^2 \, dA \quad I_y = \int x^2 \, dA \quad I_{xy} = \int xy \, dA \quad (9.14)$$

of the area  $A$  are known, we propose to determine the moments and product of inertia  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  of  $A$  with respect to new axes  $x'$  and  $y'$  which are obtained by rotating the original axes about the origin through an angle  $\theta$ .

We first note the following relations between the coordinates  $x'$ ,  $y'$  and  $x$ ,  $y$  of an element of area  $dA$ :

$$x' = x \cos \theta + y \sin \theta \quad y' = y \cos \theta - x \sin \theta$$

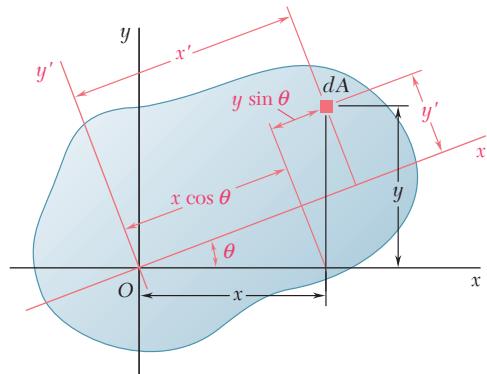


Fig. 9.17

Substituting for  $y'$  in the expression for  $I_{x'}$ , we write

$$\begin{aligned} I_{x'} &= \int (y')^2 dA = \int (y \cos \theta - x \sin \theta)^2 dA \\ &= \cos^2 \theta \int y^2 dA - 2 \sin \theta \cos \theta \int xy dA + \sin^2 \theta \int x^2 dA \end{aligned}$$

Using the relations (9.14), we write

$$I_{x'} = I_x \cos^2 \theta - 2I_{xy} \sin \theta \cos \theta + I_y \sin^2 \theta \quad (9.15)$$

Similarly, we obtain for  $I_{y'}$  and  $I_{x'y'}$  the expressions

$$I_{y'} = I_x \sin^2 \theta + 2I_{xy} \sin \theta \cos \theta + I_y \cos^2 \theta \quad (9.16)$$

$$I_{x'y'} = (I_x - I_y) \sin \theta \cos \theta + I_{xy}(\cos^2 \theta - \sin^2 \theta) \quad (9.17)$$

Recalling the trigonometric relations

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

and

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

we can write (9.15), (9.16), and (9.17) as follows:

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad (9.18)$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad (9.19)$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \quad (9.20)$$

Adding (9.18) and (9.19) we observe that

$$I_{x'} + I_{y'} = I_x + I_y \quad (9.21)$$

This result could have been anticipated, since both members of (9.21) are equal to the polar moment of inertia  $J_O$ .

Equations (9.18) and (9.20) are the parametric equations of a circle. This means that if we choose a set of rectangular axes and plot a point  $M$  of abscissa  $I_{x'}$  and ordinate  $I_{x'y'}$  for any given value of the parameter  $\theta$ , all of the points thus obtained will lie on a circle. To establish this property, we eliminate  $\theta$  from Eqs. (9.18) and (9.20); this is done by transposing  $(I_x + I_y)/2$  in Eq. (9.18), squaring both members of Eqs. (9.18) and (9.20), and adding. We write

$$\left( I_{x'} - \frac{I_x + I_y}{2} \right)^2 + I_{x'y'}^2 = \left( \frac{I_x - I_y}{2} \right)^2 + I_{xy}^2 \quad (9.22)$$

Setting

$$I_{\text{ave}} = \frac{I_x + I_y}{2} \quad \text{and} \quad R = \sqrt{\left( \frac{I_x - I_y}{2} \right)^2 + I_{xy}^2} \quad (9.23)$$

we write the identity (9.22) in the form

$$(I_{x'} - I_{\text{ave}})^2 + I_{x'y'}^2 = R^2 \quad (9.24)$$

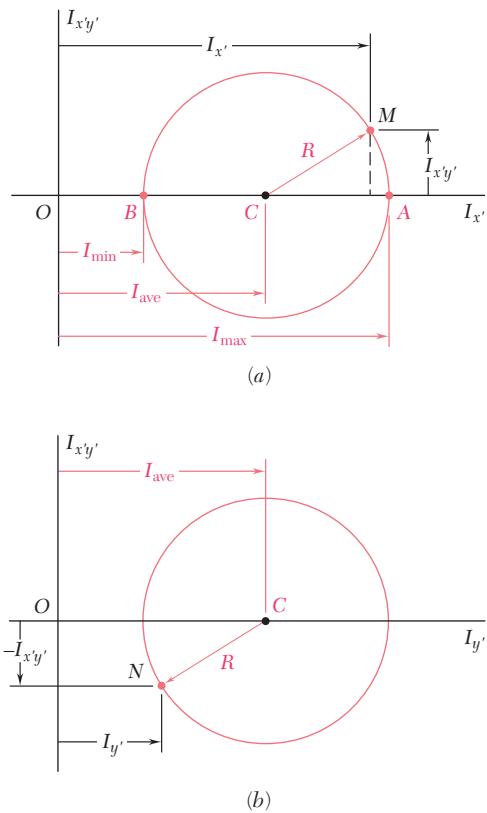


Fig. 9.18

which is the equation of a circle of radius  $R$  centered at the point  $C$  whose  $x$  and  $y$  coordinates are  $I_{\text{ave}}$  and 0, respectively (Fig. 9.18a). We observe that Eqs. (9.19) and (9.20) are the parametric equations of the same circle. Furthermore, because of the symmetry of the circle about the horizontal axis, the same result would have been obtained if instead of plotting  $M$ , we had plotted a point  $N$  of coordinates  $I_{y'}$  and  $-I_{x'y'}$  (Fig. 9.18b). This property will be used in Sec. 9.10.

The two points  $A$  and  $B$  where the above circle intersects the horizontal axis (Fig. 9.18a) are of special interest: Point  $A$  corresponds to the maximum value of the moment of inertia  $I_{x'}$ , while point  $B$  corresponds to its minimum value. In addition, both points correspond to a zero value of the product of inertia  $I_{x'y'}$ . Thus, the values  $\theta_m$  of the parameter  $\theta$  which correspond to the points  $A$  and  $B$  can be obtained by setting  $I_{x'y'} = 0$  in Eq. (9.20). We obtain†

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} \quad (9.25)$$

This equation defines two values  $2\theta_m$  which are  $180^\circ$  apart and thus two values  $\theta_m$  which are  $90^\circ$  apart. One of these values corresponds to point  $A$  in Fig. 9.18a and to an axis through  $O$  in Fig. 9.17 with respect to which the moment of inertia of the given area is maximum; the other value corresponds to point  $B$  and to an axis through  $O$  with respect to which the moment of inertia of the area is minimum. The two axes thus defined, which are perpendicular to each other, are called the *principal axes of the area about  $O$* , and the corresponding values  $I_{\max}$  and  $I_{\min}$  of the moment of inertia are called the *principal moments of inertia of the area about  $O$* . Since the two values  $\theta_m$  defined by Eq. (9.25) were obtained by setting  $I_{x'y'} = 0$  in Eq. (9.20), it is clear that the product of inertia of the given area with respect to its principal axes is zero.

We observe from Fig. 9.18a that

$$I_{\max} = I_{\text{ave}} + R \quad I_{\min} = I_{\text{ave}} - R \quad (9.26)$$

Using the values for  $I_{\text{ave}}$  and  $R$  from formulas (9.23), we write

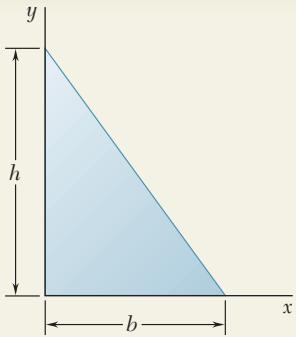
$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad (9.27)$$

Unless it is possible to tell by inspection which of the two principal axes corresponds to  $I_{\max}$  and which corresponds to  $I_{\min}$ , it is necessary to substitute one of the values of  $\theta_m$  into Eq. (9.18) in order to determine which of the two corresponds to the maximum value of the moment of inertia of the area about  $O$ .

Referring to Sec. 9.8, we note that if an area possesses an axis of symmetry through a point  $O$ , this axis must be a principal axis of the area about  $O$ . On the other hand, a principal axis does not need to be an axis of symmetry; whether or not an area possesses any axes of symmetry, it will have two principal axes of inertia about any point  $O$ .

The properties we have established hold for any point  $O$  located inside or outside the given area. If the point  $O$  is chosen to coincide with the centroid of the area, any axis through  $O$  is a centroidal axis; the two principal axes of the area about its centroid are referred to as the *principal centroidal axes of the area*.

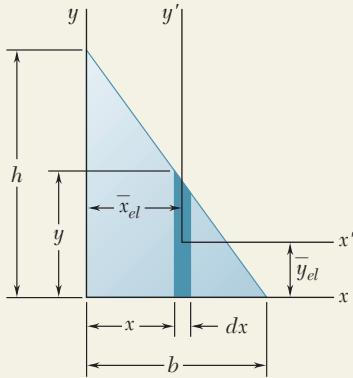
†This relation can also be obtained by differentiating  $I_{x'}$  in Eq. (9.18) and setting  $dI_{x'}/d\theta = 0$ .



## SAMPLE PROBLEM 9.6

Determine the product of inertia of the right triangle shown (a) with respect to the  $x$  and  $y$  axes and (b) with respect to centroidal axes parallel to the  $x$  and  $y$  axes.

## SOLUTION



**a. Product of Inertia  $I_{xy}$ .** A vertical rectangular strip is chosen as the differential element of area. Using the parallel-axis theorem, we write

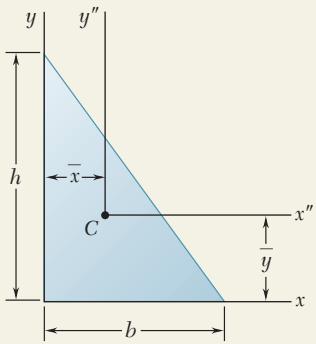
$$dI_{xy} = dI_{x'y'} + \bar{x}_{el}\bar{y}_{el} dA$$

Since the element is symmetrical with respect to the  $x'$  and  $y'$  axes, we note that  $dI_{x'y'} = 0$ . From the geometry of the triangle, we obtain

$$\begin{aligned} y &= h\left(1 - \frac{x}{b}\right) & dA &= y dx = h\left(1 - \frac{x}{b}\right) dx \\ \bar{x}_{el} &= x & \bar{y}_{el} &= \frac{1}{2}y = \frac{1}{2}h\left(1 - \frac{x}{b}\right) \end{aligned}$$

Integrating  $dI_{xy}$  from  $x = 0$  to  $x = b$ , we obtain

$$\begin{aligned} I_{xy} &= \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el} dA = \int_0^b x\left(\frac{1}{2}h\right)^2 \left(1 - \frac{x}{b}\right)^2 dx \\ &= h^2 \int_0^b \left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right) dx = h^2 \left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2}\right]_0^b \\ &= \frac{1}{24}b^2h^2 \end{aligned}$$

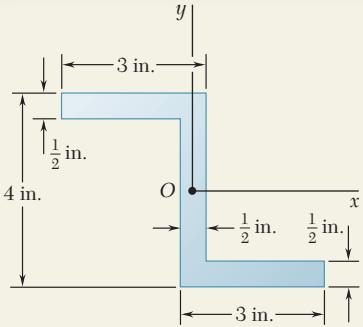


**b. Product of Inertia  $\bar{I}_{x''y''}$ .** The coordinates of the centroid of the triangle relative to the  $x$  and  $y$  axes are

$$\bar{x} = \frac{1}{3}b \quad \bar{y} = \frac{1}{3}h$$

Using the expression for  $I_{xy}$  obtained in part a, we apply the parallel-axis theorem and write

$$\begin{aligned} I_{xy} &= \bar{I}_{x''y''} + \bar{x}\bar{y}A \\ \frac{1}{24}b^2h^2 &= \bar{I}_{x''y''} + (\frac{1}{3}b)(\frac{1}{3}h)(\frac{1}{2}bh) \\ \bar{I}_{x''y''} &= \frac{1}{24}b^2h^2 - \frac{1}{18}b^2h^2 \\ \bar{I}_{x''y''} &= -\frac{1}{72}b^2h^2 \end{aligned}$$



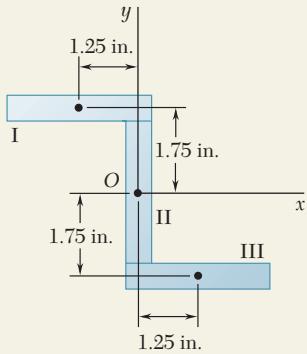
## SAMPLE PROBLEM 9.7

For the section shown, the moments of inertia with respect to the  $x$  and  $y$  axes have been computed and are known to be

$$I_x = 10.38 \text{ in}^4 \quad I_y = 6.97 \text{ in}^4$$

Determine (a) the orientation of the principal axes of the section about  $O$ , (b) the values of the principal moments of inertia of the section about  $O$ .

## SOLUTION



We first compute the product of inertia with respect to the  $x$  and  $y$  axes. The area is divided into three rectangles as shown. We note that the product of inertia  $\bar{I}_{x'y'}$  with respect to centroidal axes parallel to the  $x$  and  $y$  axes is zero for each rectangle. Using the parallel-axis theorem  $I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$ , we find that  $I_{xy}$  reduces to  $\bar{x}\bar{y}A$  for each rectangle.

Rectangle	Area, in <sup>2</sup>	$\bar{x}$ , in.	$\bar{y}$ , in.	$\bar{x}\bar{y}A$ , in <sup>4</sup>
I	1.5	-1.25	+1.75	-3.28
II	1.5	0	0	0
III	1.5	+1.25	-1.75	-3.28
				$\Sigma\bar{x}\bar{y}A = -6.56$

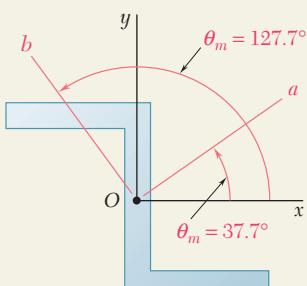
$$I_{xy} = \Sigma\bar{x}\bar{y}A = -6.56 \text{ in}^4$$

**a. Principal Axes.** Since the magnitudes of  $I_x$ ,  $I_y$ , and  $I_{xy}$  are known, Eq. (9.25) is used to determine the values of  $\theta_m$ :

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-6.56)}{10.38 - 6.97} = +3.85$$

$$2\theta_m = 75.4^\circ \text{ and } 255.4^\circ$$

$$\theta_m = 37.7^\circ \quad \text{and} \quad \theta_m = 127.7^\circ$$



**b. Principal Moments of Inertia.** Using Eq. (9.27), we write

$$\begin{aligned} I_{\max,\min} &= \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \\ &= \frac{10.38 + 6.97}{2} \pm \sqrt{\left(\frac{10.38 - 6.97}{2}\right)^2 + (-6.56)^2} \\ &I_{\max} = 15.45 \text{ in}^4 \quad I_{\min} = 1.897 \text{ in}^4 \end{aligned}$$

Noting that the elements of the area of the section are more closely distributed about the  $b$  axis than about the  $a$  axis, we conclude that  $I_a = I_{\max} = 15.45 \text{ in}^4$  and  $I_b = I_{\min} = 1.897 \text{ in}^4$ . This conclusion can be verified by substituting  $\theta = 37.7^\circ$  into Eqs. (9.18) and (9.19).

# SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will continue your work with *moments of inertia* and will utilize various techniques for computing *products of inertia*. Although the problems are generally straightforward, several items are worth noting.

**1. Calculating the product of inertia  $I_{xy}$  by integration.** We defined this quantity as

$$I_{xy} = \int xy \, dA \quad (9.12)$$

and stated that its value can be positive, negative, or zero. The product of inertia can be computed directly from the above equation using double integration, or it can be determined using single integration as shown in Sample Prob. 9.6. When applying the latter technique and using the parallel-axis theorem, it is important to remember that  $\bar{x}_{el}$  and  $\bar{y}_{el}$  in the equation

$$dI_{xy} = dI_{x'y'} + \bar{x}_{el}\bar{y}_{el} \, dA$$

are the coordinates of the centroid of the element of area  $dA$ . Thus, if  $dA$  is not in the first quadrant, one or both of these coordinates will be negative.

**2. Calculating the products of inertia of composite areas.** They can easily be computed from the products of inertia of their component parts by using the parallel-axis theorem

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \quad (9.13)$$

The proper technique to use for problems of this type is illustrated in Sample Probs. 9.6 and 9.7. In addition to the usual rules for composite-area problems, it is essential that you remember the following points.

a. **If either of the centroidal axes of a component area is an axis of symmetry for that area, the product of inertia  $\bar{I}_{x'y'}$  for that area is zero.** Thus,  $\bar{I}_{x'y'}$  is zero for component areas such as circles, semicircles, rectangles, and isosceles triangles which possess an axis of symmetry parallel to one of the coordinate axes.

b. **Pay careful attention to the signs of the coordinates  $\bar{x}$  and  $\bar{y}$**  of each component area when you use the parallel-axis theorem [Sample Prob. 9.7].

**3. Determining the moments of inertia and the product of inertia for rotated coordinate axes.** In Sec. 9.9 we derived Eqs. (9.18), (9.19), and (9.20), from which the moments of inertia and the product of inertia can be computed for coordinate axes which have been rotated about the origin  $O$ . To apply these equations, you must know a set of values  $I_x$ ,  $I_y$ , and  $I_{xy}$  for a given orientation of the axes, and you must remember that  $\theta$  is positive for counterclockwise rotations of the axes and negative for clockwise rotations of the axes.

**4. Computing the principal moments of inertia.** We showed in Sec. 9.9 that there is a particular orientation of the coordinate axes for which the moments of inertia attain their maximum and minimum values,  $I_{\max}$  and  $I_{\min}$ , and for which the product of inertia is zero. Equation (9.27) can be used to compute these values, known as the *principal moments of inertia* of the area about  $O$ . The corresponding axes are referred to as the *principal axes* of the area about  $O$ , and their orientation is defined by Eq. (9.25). To determine which of the principal axes corresponds to  $I_{\max}$  and which corresponds to  $I_{\min}$ , you can either follow the procedure outlined in the text after Eq. (9.27) or observe about which of the two principal axes the area is more closely distributed; that axis corresponds to  $I_{\min}$  [Sample Prob. 9.7].

# PROBLEMS

**9.67 through 9.70** Determine by direct integration the product of inertia of the given area with respect to the  $x$  and  $y$  axes.

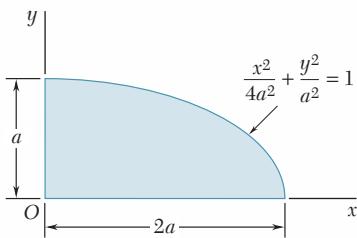


Fig. P9.67

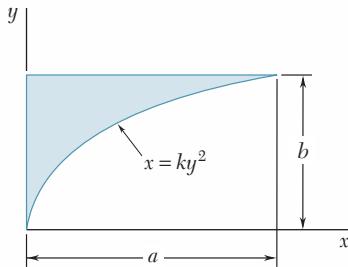


Fig. P9.68

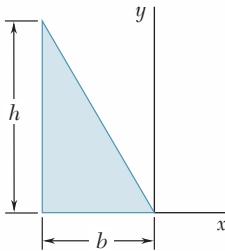


Fig. P9.69

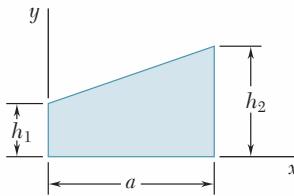


Fig. P9.70

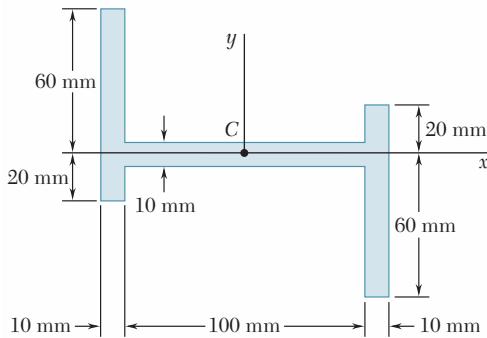


Fig. P9.71

**9.71 through 9.74** Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal  $x$  and  $y$  axes.

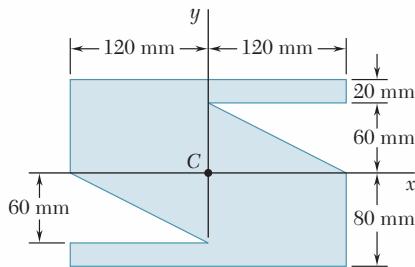


Fig. P9.72

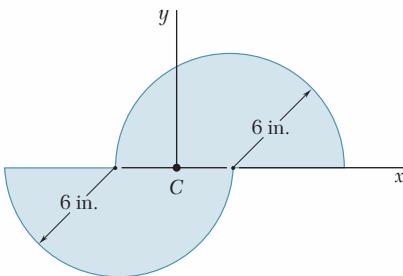


Fig. P9.73

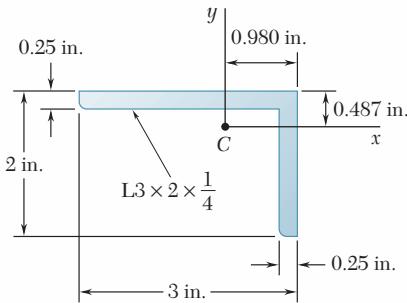


Fig. P9.74

- 9.75 through 9.78** Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal  $x$  and  $y$  axes.

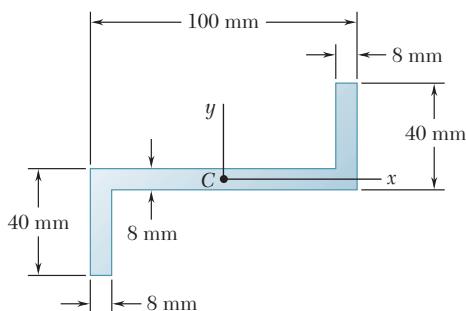


Fig. P9.75

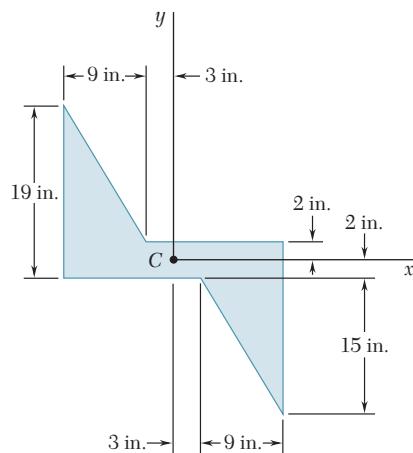


Fig. P9.76

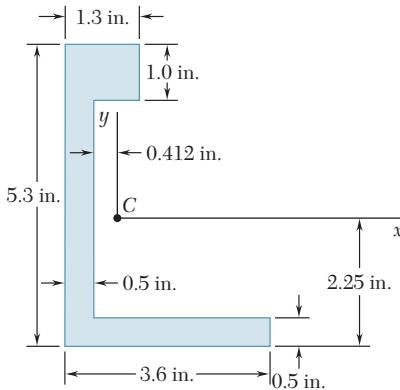


Fig. P9.77

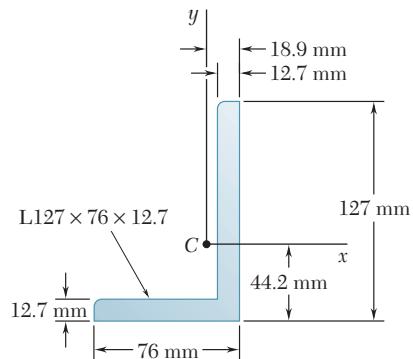


Fig. P9.78

- 9.79** Determine for the quarter ellipse of Prob. 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the  $x$  and  $y$  axes about  $O$  (a) through  $45^\circ$  counterclockwise, (b) through  $30^\circ$  clockwise.

- 9.80** Determine the moments of inertia and the product of inertia of the area of Prob. 9.72 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $30^\circ$  counterclockwise.

- 9.81** Determine the moments of inertia and the product of inertia of the area of Prob. 9.73 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $60^\circ$  counterclockwise.

- 9.82** Determine the moments of inertia and the product of inertia of the area of Prob. 9.75 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $45^\circ$  clockwise.

- 9.83** Determine the moments of inertia and the product of inertia of the L3  $\times$  2  $\times$   $\frac{1}{4}$ -in. angle cross section of Prob. 9.74 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $30^\circ$  clockwise.

**9.84** Determine the moments of inertia and the product of inertia of the L127 × 76 × 12.7-mm angle cross section of Prob. 9.78 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes 45° counterclockwise.

**9.85** For the quarter ellipse of Prob. 9.67, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

**9.86 through 9.88** For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

**9.86** Area of Prob. 9.72

**9.87** Area of Prob. 9.73

**9.88** Area of Prob. 9.75

**9.89 and 9.90** For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

**9.89** The L3 × 2 ×  $\frac{1}{4}$ -in. angle cross section of Prob. 9.74

**9.90** The L127 × 76 × 12.7-mm angle cross section of Prob. 9.78

## \*9.10 MOHR'S CIRCLE FOR MOMENTS AND PRODUCTS OF INERTIA

The circle used in the preceding section to illustrate the relations existing between the moments and products of inertia of a given area with respect to axes passing through a fixed point  $O$  was first introduced by the German engineer Otto Mohr (1835–1918) and is known as *Mohr's circle*. It will be shown that if the moments and product of inertia of an area  $A$  are known with respect to two rectangular  $x$  and  $y$  axes which pass through a point  $O$ , Mohr's circle can be used to graphically determine (a) the principal axes and principal moments of inertia of the area about  $O$  and (b) the moments and product of inertia of the area with respect to any other pair of rectangular axes  $x'$  and  $y'$  through  $O$ .

Consider a given area  $A$  and two rectangular coordinate axes  $x$  and  $y$  (Fig. 9.19a). Assuming that the moments of inertia  $I_x$  and  $I_y$  and the product of inertia  $I_{xy}$  are known, we will represent them on a diagram by plotting a point  $X$  of coordinates  $I_x$  and  $I_{xy}$  and a point  $Y$  of coordinates  $I_y$  and  $-I_{xy}$  (Fig. 9.19b). If  $I_{xy}$  is positive, as assumed in Fig. 9.19a, point  $X$  is located above the horizontal axis and point  $Y$  is located below, as shown in Fig. 9.19b. If  $I_{xy}$  is negative,  $X$  is located below the horizontal axis and  $Y$  is located above. Joining  $X$  and  $Y$  with a straight line, we denote by  $C$  the point of intersection of line  $XY$  with the horizontal axis and draw the circle of center  $C$  and diameter  $XY$ . Noting that the abscissa of  $C$  and the radius of the circle are respectively equal to the quantities  $I_{ave}$  and  $R$  defined by the formula (9.23), we conclude that the circle obtained is Mohr's circle for the given area about point  $O$ . Thus, the abscissas of the points  $A$  and  $B$  where the circle intersects the horizontal axis represent respectively the principal moments of inertia  $I_{max}$  and  $I_{min}$  of the area.

We also note that, since  $\tan(XCA) = 2I_{xy}/(I_x - I_y)$ , the angle  $XCA$  is equal in magnitude to one of the angles  $2\theta_m$  which satisfy Eq. (9.25);

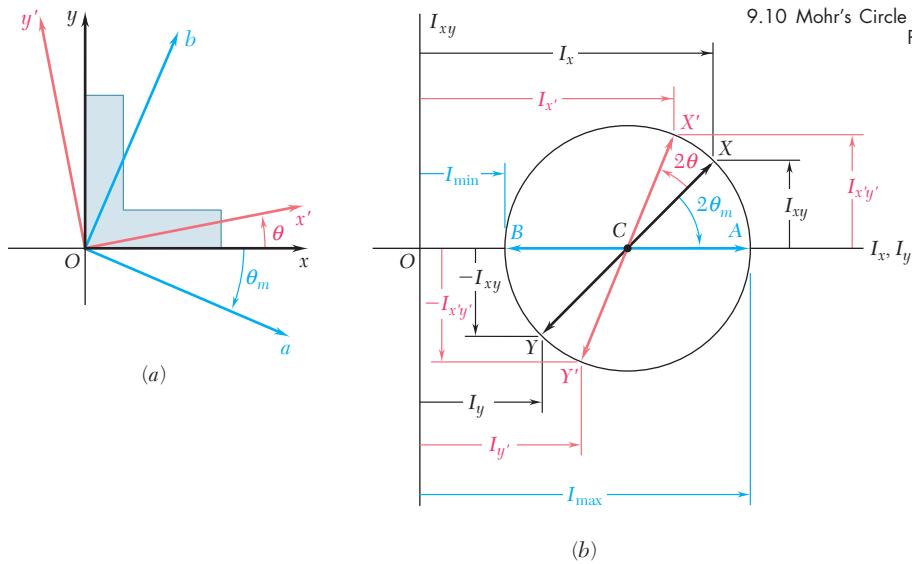
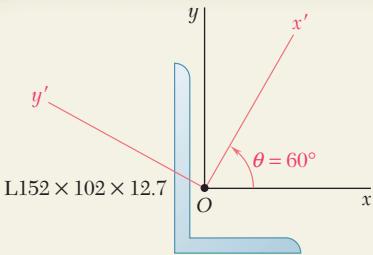


Fig. 9.19

thus, the angle  $\theta_m$ , which defines in Fig. 9.19a the principal axis  $Oa$  corresponding to point  $A$  in Fig. 9.19b, is equal to half of the angle  $XCA$  of Mohr's circle. We further observe that if  $I_x > I_y$  and  $I_{xy} > 0$ , as in the case considered here, the rotation which brings  $CX$  into  $CA$  is clockwise. Also, under these conditions, the angle  $\theta_m$  obtained from Eq. (9.25), which defines the principal axis  $Oa$  in Fig. 9.19a, is negative; thus, the rotation which brings  $Ox$  into  $Oa$  is also clockwise. We conclude that the senses of rotation in both parts of Fig. 9.19 are the same. If a clockwise rotation through  $2\theta_m$  is required to bring  $CX$  into  $CA$  on Mohr's circle, a clockwise rotation through  $\theta_m$  will bring  $Ox$  into the corresponding principal axis  $Oa$  in Fig. 9.19a.

Since Mohr's circle is uniquely defined, the same circle can be obtained by considering the moments and product of inertia of the area  $A$  with respect to the rectangular axes  $x'$  and  $y'$  (Fig. 9.19a). The point  $X'$  of coordinates  $I_{x'}$  and  $I_{x'y'}$  and the point  $Y'$  of coordinates  $I_{y'}$  and  $-I_{x'y'}$  are thus located on Mohr's circle, and the angle  $X'CA$  in Fig. 9.19b must be equal to twice the angle  $x'Oa$  in Fig. 9.19a. Since, as noted above, the angle  $XCA$  is twice the angle  $xOa$ , it follows that the angle  $XCX'$  in Fig. 9.19b is twice the angle  $xOx'$  in Fig. 9.19a. The diameter  $X'Y'$ , which defines the moments and product of inertia  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  of the given area with respect to rectangular axes  $x'$  and  $y'$  forming an angle  $\theta$  with the  $x$  and  $y$  axes can be obtained by rotating through an angle  $2\theta$  the diameter  $XY$  which corresponds to the moments and product of inertia  $I_x$ ,  $I_y$ , and  $I_{xy}$ . We note that the rotation which brings the diameter  $XY$  into the diameter  $X'Y'$  in Fig. 9.19b has the same sense as the rotation which brings the  $x$  and  $y$  axes into the  $x'$  and  $y'$  axes in Fig. 9.19a.

It should be noted that the use of Mohr's circle is not limited to graphical solutions, i.e., to solutions based on the careful drawing and measuring of the various parameters involved. By merely sketching Mohr's circle and using trigonometry, one can easily derive the various relations required for a numerical solution of a given problem (see Sample Prob. 9.8).

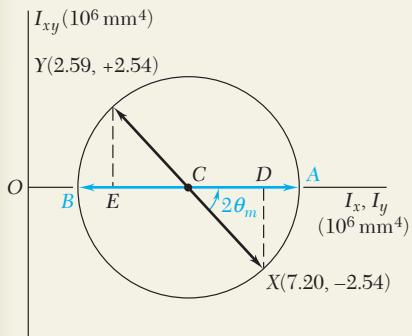


## SAMPLE PROBLEM 9.8

For the section shown, the moments and product of inertia with respect to the  $x$  and  $y$  axes are known to be

$$I_x = 7.20 \times 10^6 \text{ mm}^4 \quad I_y = 2.59 \times 10^6 \text{ mm}^4 \quad I_{xy} = -2.54 \times 10^6 \text{ mm}^4$$

Using Mohr's circle, determine (a) the principal axes of the section about  $O$ , (b) the values of the principal moments of inertia of the section about  $O$ , (c) the moments and product of inertia of the section with respect to the  $x'$  and  $y'$  axes which form an angle of  $60^\circ$  with the  $x$  and  $y$  axes.



## SOLUTION

**Drawing Mohr's Circle.** We first plot point  $X$  of coordinates  $I_x = 7.20$ ,  $I_{xy} = -2.54$ , and point  $Y$  of coordinates  $I_y = 2.59$ ,  $-I_{xy} = +2.54$ . Joining  $X$  and  $Y$  with a straight line, we define the center  $C$  of Mohr's circle. The abscissa of  $C$ , which represents  $I_{ave}$ , and the radius  $R$  of the circle can be measured directly or calculated as follows:

$$\begin{aligned} I_{ave} &= OC = \frac{1}{2}(I_x + I_y) = \frac{1}{2}(7.20 \times 10^6 + 2.59 \times 10^6) = 4.895 \times 10^6 \text{ mm}^4 \\ CD &= \frac{1}{2}(I_x - I_y) = \frac{1}{2}(7.20 \times 10^6 - 2.59 \times 10^6) = 2.305 \times 10^6 \text{ mm}^4 \\ R &= \sqrt{(CD)^2 + (DX)^2} = \sqrt{(2.305 \times 10^6)^2 + (2.54 \times 10^6)^2} \\ &= 3.430 \times 10^6 \text{ mm}^4 \end{aligned}$$

**a. Principal Axes.** The principal axes of the section correspond to points  $A$  and  $B$  on Mohr's circle, and the angle through which we should rotate  $CX$  to bring it into  $CA$  defines  $2\theta_m$ . We have

$$\tan 2\theta_m = \frac{DX}{CD} = \frac{2.54}{2.305} = 1.102 \quad 2\theta_m = 47.8^\circ \quad \theta_m = 23.9^\circ$$

Thus, the principal axis  $OA$  corresponding to the maximum value of the moment of inertia is obtained by rotating the  $x$  axis through  $23.9^\circ$  counterclockwise; the principal axis  $Ob$  corresponding to the minimum value of the moment of inertia can be obtained by rotating the  $y$  axis through the same angle.

**b. Principal Moments of Inertia.** The principal moments of inertia are represented by the abscissas of  $A$  and  $B$ . We have

$$\begin{aligned} I_{max} &= OA = OC + CA = I_{ave} + R = (4.895 + 3.430)10^6 \text{ mm}^4 \\ &\quad I_{max} = 8.33 \times 10^6 \text{ mm}^4 \\ I_{min} &= OB = OC - BC = I_{ave} - R = (4.895 - 3.430)10^6 \text{ mm}^4 \\ &\quad I_{min} = 1.47 \times 10^6 \text{ mm}^4 \end{aligned}$$

### c. Moments and Product of Inertia with Respect to the $x'$ and $y'$ Axes

On Mohr's circle, the points  $X'$  and  $Y'$ , which correspond to the  $x'$  and  $y'$  axes, are obtained by rotating  $CX$  and  $CY$  through an angle  $2\theta = 2(60^\circ) = 120^\circ$  counterclockwise. The coordinates of  $X'$  and  $Y'$  yield the desired moments and product of inertia. Noting that the angle that  $CX'$  forms with the horizontal axis is  $\phi = 120^\circ - 47.8^\circ = 72.2^\circ$ , we write

$$I_{x'} = OF = OC + CF = 4.895 \times 10^6 \text{ mm}^4 + (3.430 \times 10^6 \text{ mm}^4) \cos 72.2^\circ \quad I_{x'} = 5.94 \times 10^6 \text{ mm}^4$$

$$I_{y'} = OG = OC - GC = 4.895 \times 10^6 \text{ mm}^4 - (3.430 \times 10^6 \text{ mm}^4) \cos 72.2^\circ \quad I_{y'} = 3.85 \times 10^6 \text{ mm}^4$$

$$I_{x'y'} = FX' = (3.430 \times 10^6 \text{ mm}^4) \sin 72.2^\circ \quad I_{x'y'} = 3.27 \times 10^6 \text{ mm}^4$$

# SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will use *Mohr's circle* to determine the moments and products of inertia of a given area for different orientations of the coordinate axes. Although in some cases using Mohr's circle may not be as direct as substituting into the appropriate equations [Eqs. (9.18) through (9.20)], this method of solution has the advantage of providing a visual representation of the relationships among the various variables. Further, Mohr's circle shows all of the values of the moments and products of inertia which are possible for a given problem.

**Using Mohr's circle.** The underlying theory was presented in Sec. 9.9, and we discussed the application of this method in Sec. 9.10 and in Sample Prob. 9.8. In the same problem, we presented the steps you should follow to determine the *principal axes*, the *principal moments of inertia*, and the *moments and product of inertia with respect to a specified orientation of the coordinates axes*. When you use Mohr's circle to solve problems, it is important that you remember the following points.

**a. Mohr's circle is completely defined by the quantities  $R$  and  $I_{ave}$ ,** which represent, respectively, the radius of the circle and the distance from the origin  $O$  to the center  $C$  of the circle. These quantities can be obtained from Eqs. (9.23) if the moments and product of inertia are known for a given orientation of the axes. However, Mohr's circle can be defined by other combinations of known values [Probs. 9.103, 9.106, and 9.107]. For these cases, it may be necessary to first make one or more assumptions, such as choosing an arbitrary location for the center when  $I_{ave}$  is unknown, assigning relative magnitudes to the moments of inertia (for example,  $I_x > I_y$ ), or selecting the sign of the product of inertia.

**b. Point X of coordinates  $(I_x, I_{xy})$  and point Y of coordinates  $(I_y, -I_{xy})$**  are both located on Mohr's circle and are diametrically opposite.

**c. Since moments of inertia must be positive,** the entire Mohr's circle must lie to the right of the  $I_{xy}$  axis; it follows that  $I_{ave} > R$  for all cases.

**d. As the coordinate axes are rotated through an angle  $\theta$ ,** the associated rotation of the diameter of Mohr's circle is equal to  $2\theta$  and is in the same sense (clockwise or counterclockwise). We strongly suggest that the known points on the circumference of the circle be labeled with the appropriate capital letter, as was done in Fig. 9.19b and for the Mohr circles of Sample Prob. 9.8. This will enable you to determine, for each value of  $\theta$ , the sign of the corresponding product of inertia and to determine which moment of inertia is associated with each of the coordinate axes [Sample Prob. 9.8, parts *a* and *c*].

Although we have introduced Mohr's circle within the specific context of the study of moments and products of inertia, the Mohr circle technique is also applicable to the solution of analogous but physically different problems in mechanics of materials. This multiple use of a specific technique is not unique, and as you pursue your engineering studies, you will encounter several methods of solution which can be applied to a variety of problems.

# PROBLEMS

- 9.91** Using Mohr's circle, determine for the quarter ellipse of Prob. 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the  $x$  and  $y$  axes about  $O$  (a) through  $45^\circ$  counterclockwise, (b) through  $30^\circ$  clockwise.
- 9.92** Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Prob. 9.72 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $30^\circ$  counterclockwise.
- 9.93** Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Prob. 9.73 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $60^\circ$  counterclockwise.
- 9.94** Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Prob. 9.75 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $45^\circ$  clockwise.
- 9.95** Using Mohr's circle, determine the moments of inertia and the product of inertia of the  $L3 \times 2 \times \frac{1}{4}$ -in. angle cross section of Prob. 9.74 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $30^\circ$  clockwise.
- 9.96** Using Mohr's circle, determine the moments of inertia and the product of inertia of the  $L127 \times 76 \times 12.7$ -mm angle cross section of Prob. 9.78 with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $45^\circ$  counterclockwise.
- 9.97** For the quarter ellipse of Prob. 9.67, use Mohr's circle to determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.
- 9.98 through 9.102** Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.
- 9.98** Area of Prob. 9.72
- 9.99** Area of Prob. 9.76
- 9.100** Area of Prob. 9.73
- 9.101** Area of Prob. 9.74
- 9.102** Area of Prob. 9.77
- (The moments of inertia  $\bar{I}_x$  and  $\bar{I}_y$  of the area of Prob. 9.102 were determined in Prob. 9.44.)
- 9.103** The moments and product of inertia of an  $L4 \times 3 \times \frac{1}{4}$ -in. angle cross section with respect to two rectangular axes  $x$  and  $y$  through  $C$  are, respectively,  $\bar{I}_x = 1.33 \text{ in}^4$ ,  $\bar{I}_y = 2.75 \text{ in}^4$ , and  $\bar{I}_{xy} < 0$ , with the minimum value of the moment of inertia of the area with respect to any axis through  $C$  being  $\bar{I}_{\min} = 0.692 \text{ in}^4$ . Using Mohr's circle, determine (a) the product of inertia  $\bar{I}_{xy}$  of the area, (b) the orientation of the principal axes, (c) the value of  $\bar{I}_{\max}$ .

- 9.104 and 9.105** Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Fig. 9.13.)

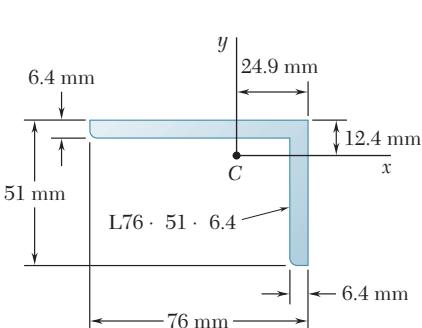


Fig. P9.104

- \*9.106** For a given area the moments of inertia with respect to two rectangular centroidal  $x$  and  $y$  axes are  $\bar{I}_x = 1200 \text{ in}^4$  and  $\bar{I}_y = 300 \text{ in}^4$ , respectively. Knowing that after rotating the  $x$  and  $y$  axes about the centroid  $30^\circ$  counterclockwise, the moment of inertia relative to the rotated  $x$  axis is  $1450 \text{ in}^4$ , use Mohr's circle to determine (a) the orientation of the principal axes, (b) the principal centroidal moments of inertia.

- 9.107** It is known that for a given area  $\bar{I}_y = 48 \times 10^6 \text{ mm}^4$  and  $\bar{I}_{xy} = -20 \times 10^6 \text{ mm}^4$ , where the  $x$  and  $y$  axes are rectangular centroidal axes. If the axis corresponding to the maximum product of inertia is obtained by rotating the  $x$  axis  $67.5^\circ$  counterclockwise about  $C$ , use Mohr's circle to determine (a) the moment of inertia  $\bar{I}_x$  of the area, (b) the principal centroidal moments of inertia.

- 9.108** Using Mohr's circle, show that for any regular polygon (such as a pentagon) (a) the moment of inertia with respect to every axis through the centroid is the same, (b) the product of inertia with respect to every pair of rectangular axes through the centroid is zero.

- 9.109** Using Mohr's circle, prove that the expression  $I_x I_{y'} - I_{x'y'}^2$  is independent of the orientation of the  $x'$  and  $y'$  axes, where  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  represent the moments and product of inertia, respectively, of a given area with respect to a pair of rectangular axes  $x'$  and  $y'$  through a given point  $O$ . Also show that the given expression is equal to the square of the length of the tangent drawn from the origin of the coordinate system to Mohr's circle.

- 9.110** Using the invariance property established in the preceding problem, express the product of inertia  $I_{xy}$  of an area  $A$  with respect to a pair of rectangular axes through  $O$  in terms of the moments of inertia  $I_x$  and  $I_y$  of  $A$  and the principal moments of inertia  $I_{\min}$  and  $I_{\max}$  of  $A$  about  $O$ . Use the formula obtained to calculate the product of inertia  $I_{xy}$  of the  $L3 \times 2 \times \frac{1}{4}$ -in. angle cross section shown in Fig. 9.13A, knowing that its maximum moment of inertia is  $1.257 \text{ in}^4$ .

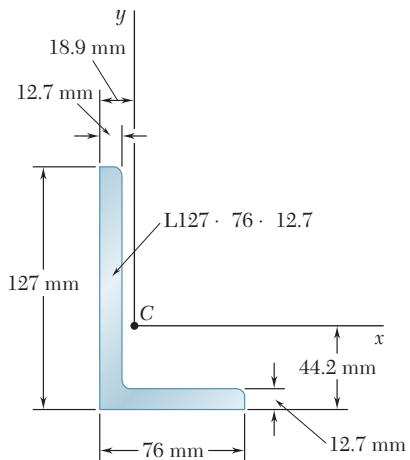
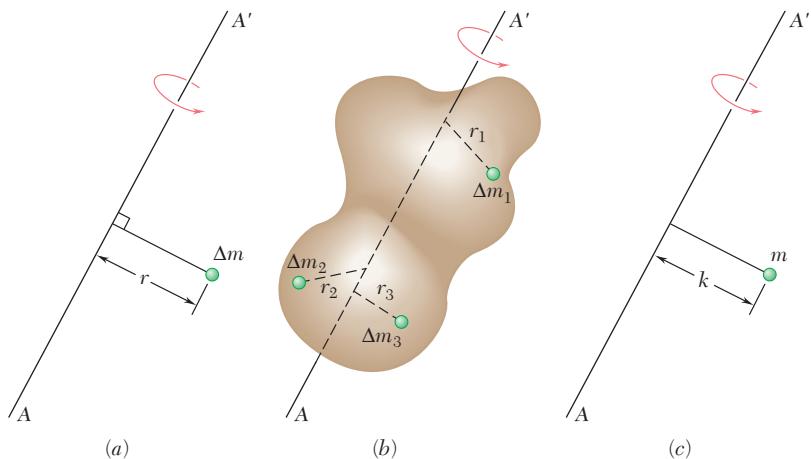


Fig. P9.105

## MOMENTS OF INERTIA OF MASSES

### 9.11 MOMENT OF INERTIA OF A MASS

Consider a small mass  $\Delta m$  mounted on a rod of negligible mass which can rotate freely about an axis  $AA'$  (Fig. 9.20a). If a couple is applied to the system, the rod and mass, assumed to be initially at rest, will start rotating about  $AA'$ . The details of this motion will be studied later in dynamics. At present, we wish only to indicate that the time required for the system to reach a given speed of rotation is proportional to the mass  $\Delta m$  and to the square of the distance  $r$ . The product  $r^2 \Delta m$  provides, therefore, a measure of the *inertia* of the system, i.e., a measure of the resistance the system offers when we try to set it in motion. For this reason, the product  $r^2 \Delta m$  is called the *moment of inertia* of the mass  $\Delta m$  with respect to the axis  $AA'$ .



**Fig. 9.20**

Consider now a body of mass  $m$  which is to be rotated about an axis  $AA'$  (Fig. 9.20b). Dividing the body into elements of mass  $\Delta m_1$ ,  $\Delta m_2$ , etc., we find that the body's resistance to being rotated is measured by the sum  $r_1^2 \Delta m_1 + r_2^2 \Delta m_2 + \dots$ . This sum defines, therefore, the moment of inertia of the body with respect to the axis  $AA'$ . Increasing the number of elements, we find that the moment of inertia is equal, in the limit, to the integral

$$I = \int r^2 dm \quad (9.28)$$

The *radius of gyration*  $k$  of the body with respect to the axis  $AA'$  is defined by the relation

$$I = k^2 m \quad \text{or} \quad k = \sqrt{\frac{I}{m}} \quad (9.29)$$

The radius of gyration  $k$  represents, therefore, the distance at which the entire mass of the body should be concentrated if its moment of inertia with respect to  $AA'$  is to remain unchanged (Fig. 9.20c). Whether it is kept in its original shape (Fig. 9.20b) or whether it is concentrated as shown in Fig. 9.20c, the mass  $m$  will react in the same way to a rotation, or *gyration*, about  $AA'$ .

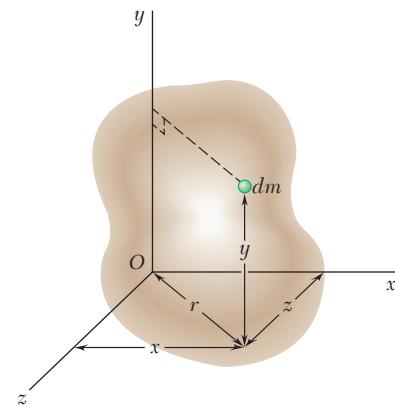
If SI units are used, the radius of gyration  $k$  is expressed in meters and the mass  $m$  in kilograms, and thus the unit used for the moment of inertia of a mass is  $\text{kg} \cdot \text{m}^2$ . If U.S. customary units are used, the radius of gyration is expressed in feet and the mass in slugs (i.e., in  $\text{lb} \cdot \text{s}^2/\text{ft}$ ), and thus the derived unit used for the moment of inertia of a mass is  $\text{lb} \cdot \text{ft} \cdot \text{s}^2$ .†

The moment of inertia of a body with respect to a coordinate axis can easily be expressed in terms of the coordinates  $x$ ,  $y$ ,  $z$  of the element of mass  $dm$  (Fig. 9.21). Noting, for example, that the square of the distance  $r$  from the element  $dm$  to the  $y$  axis is  $z^2 + x^2$ , we express the moment of inertia of the body with respect to the  $y$  axis as

$$I_y = \int r^2 dm = \int (z^2 + x^2) dm$$

Similar expressions can be obtained for the moments of inertia with respect to the  $x$  and  $z$  axes. We write

$$\begin{aligned} I_x &= \int (y^2 + z^2) dm \\ I_y &= \int (z^2 + x^2) dm \\ I_z &= \int (x^2 + y^2) dm \end{aligned} \quad (9.30)$$



**Fig. 9.21**



†It should be kept in mind when converting the moment of inertia of a mass from U.S. customary units to SI units that the base unit *pound* used in the derived unit  $\text{lb} \cdot \text{ft} \cdot \text{s}^2$  is a unit of force (not of mass) and should therefore be converted into newtons. We have

$$1 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 = (4.45 \text{ N})(0.3048 \text{ m})(1 \text{ s})^2 = 1.356 \text{ N} \cdot \text{m} \cdot \text{s}^2$$

or, since  $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$ ,

$$1 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 = 1.356 \text{ kg} \cdot \text{m}^2$$

**Photo 9.2** As you will discuss in your dynamics course, the rotational behavior of the camshaft shown is dependent upon the mass moment of inertia of the camshaft with respect to its axis of rotation.

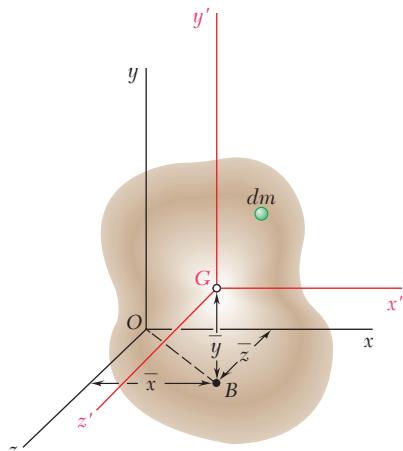


Fig. 9.22

## 9.12 PARALLEL-AXIS THEOREM

Consider a body of mass  $m$ . Let  $Oxyz$  be a system of rectangular coordinates whose origin is at the arbitrary point  $O$ , and  $Gx'y'z'$  a system of parallel *centroidal axes*, i.e., a system whose origin is at the center of gravity  $G$  of the body† and whose axes  $x', y', z'$  are parallel to the  $x, y$ , and  $z$  axes, respectively (Fig. 9.22). Denoting by  $\bar{x}, \bar{y}, \bar{z}$  the coordinates of  $G$  with respect to  $Oxyz$ , we write the following relations between the coordinates  $x, y, z$  of the element  $dm$  with respect to  $Oxyz$  and its coordinates  $x', y', z'$  with respect to the centroidal axes  $Gx'y'z'$ :

$$x = x' + \bar{x} \quad y = y' + \bar{y} \quad z = z' + \bar{z} \quad (9.31)$$

Referring to Eqs. (9.30), we can express the moment of inertia of the body with respect to the  $x$  axis as follows:

$$\begin{aligned} I_x &= \int (y^2 + z^2) dm = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2] dm \\ &= \int (y'^2 + z'^2) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + (\bar{y}^2 + \bar{z}^2) \int dm \end{aligned}$$

The first integral in this expression represents the moment of inertia  $\bar{I}_{x'}$  of the body with respect to the centroidal axis  $x'$ ; the second and third integrals represent the first moment of the body with respect to the  $z'x'$  and  $x'y'$  planes, respectively, and, since both planes contain  $G$ , the two integrals are zero; the last integral is equal to the total mass  $m$  of the body. We write, therefore,

$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2) \quad (9.32)$$

and, similarly,

$$I_y = \bar{I}_{y'} + m(\bar{z}^2 + \bar{x}^2) \quad I_z = \bar{I}_{z'} + m(\bar{x}^2 + \bar{y}^2) \quad (9.32')$$

We easily verify from Fig. 9.22 that the sum  $\bar{z}^2 + \bar{x}^2$  represents the square of the distance  $OB$  between the  $y$  and  $y'$  axes. Similarly,  $\bar{y}^2 + \bar{z}^2$  and  $\bar{x}^2 + \bar{y}^2$  represent the squares of the distance between the  $x$  and  $x'$  axes and the  $z$  and  $z'$  axes, respectively. Denoting by  $d$  the distance between an arbitrary axis  $AA'$  and a parallel centroidal axis  $BB'$  (Fig. 9.23), we can, therefore, write the following general relation between the moment of inertia  $I$  of the body with respect to  $AA'$  and its moment of inertia  $\bar{I}$  with respect to  $BB'$ :

$$I = \bar{I} + md^2 \quad (9.33)$$

Expressing the moments of inertia in terms of the corresponding radii of gyration, we can also write

$$k^2 = \bar{k}^2 + d^2 \quad (9.34)$$

where  $k$  and  $\bar{k}$  represent the radii of gyration of the body about  $AA'$  and  $BB'$ , respectively.

†Note that the term *centroidal* is used here to define an axis passing through the center of gravity  $G$  of the body, whether or not  $G$  coincides with the centroid of the volume of the body.

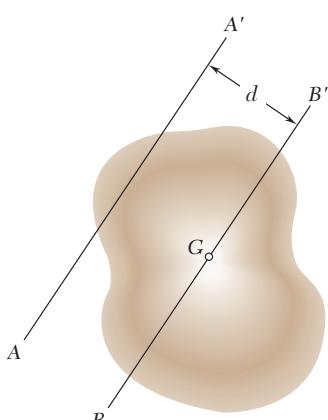


Fig. 9.23

## 9.13 MOMENTS OF INERTIA OF THIN PLATES

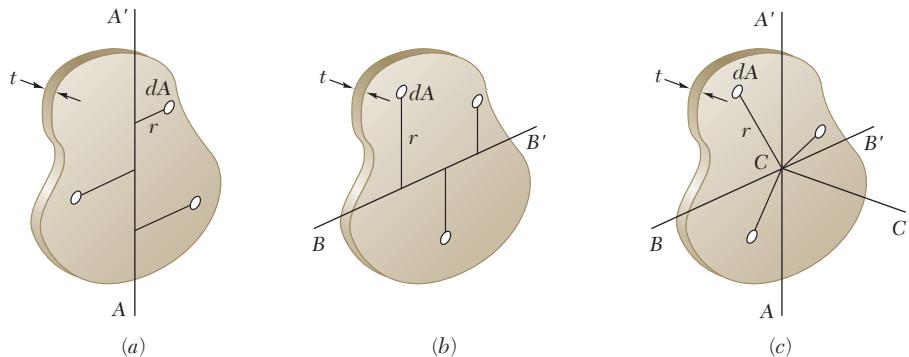
Consider a thin plate of uniform thickness  $t$ , which is made of a homogeneous material of density  $\rho$  (density = mass per unit volume). The mass moment of inertia of the plate with respect to an axis  $AA'$  contained in the plane of the plate (Fig. 9.24a) is

$$I_{AA', \text{mass}} = \int r^2 dm$$

Since  $dm = \rho t dA$ , we write

$$I_{AA', \text{mass}} = \rho t \int r^2 dA$$

But  $r$  represents the distance of the element of area  $dA$  to the axis



**Fig. 9.24**

$AA'$ ; the integral is therefore equal to the moment of inertia of the area of the plate with respect to  $AA'$ . We have

$$I_{AA', \text{mass}} = \rho t I_{AA', \text{area}} \quad (9.35)$$

Similarly, for an axis  $BB'$  which is contained in the plane of the plate and is perpendicular to  $AA'$  (Fig. 9.24b), we have

$$I_{BB', \text{mass}} = \rho t I_{BB', \text{area}} \quad (9.36)$$

Considering now the axis  $CC'$  which is *perpendicular* to the plate and passes through the point of intersection  $C$  of  $AA'$  and  $BB'$  (Fig. 9.24c), we write

$$I_{CC', \text{mass}} = \rho t J_{C, \text{area}} \quad (9.37)$$

where  $J_C$  is the *polar* moment of inertia of the area of the plate with respect to point  $C$ .

Recalling the relation  $J_C = I_{AA'} + I_{BB'}$  which exists between polar and rectangular moments of inertia of an area, we write the following relation between the mass moments of inertia of a thin plate:

$$I_{CC'} = I_{AA'} + I_{BB'} \quad (9.38)$$

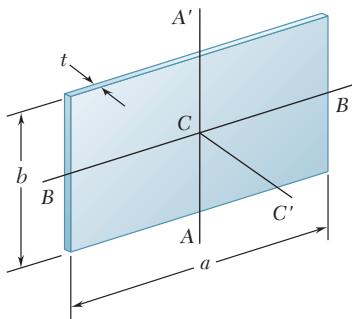


Fig. 9.25

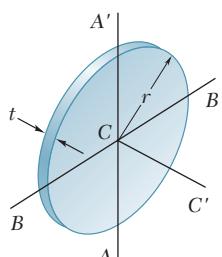


Fig. 9.26

**Rectangular Plate.** In the case of a rectangular plate of sides  $a$  and  $b$  (Fig. 9.25), we obtain the following mass moments of inertia with respect to axes through the center of gravity of the plate:

$$I_{AA',\text{mass}} = \rho t I_{AA',\text{area}} = \rho t \left(\frac{1}{12}a^3b\right)$$

$$I_{BB',\text{mass}} = \rho t I_{BB',\text{area}} = \rho t \left(\frac{1}{12}ab^3\right)$$

Observing that the product  $\rho abt$  is equal to the mass  $m$  of the plate, we write the mass moments of inertia of a thin rectangular plate as follows:

$$I_{AA'} = \frac{1}{12}ma^2 \quad I_{BB'} = \frac{1}{12}mb^2 \quad (9.39)$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{12}m(a^2 + b^2) \quad (9.40)$$

**Circular Plate.** In the case of a circular plate, or disk, of radius  $r$  (Fig. 9.26), we write

$$I_{AA',\text{mass}} = \rho t I_{AA',\text{area}} = \rho t \left(\frac{1}{4}\pi r^4\right)$$

Observing that the product  $\rho\pi r^2t$  is equal to the mass  $m$  of the plate and that  $I_{AA'} = I_{BB'}$ , we write the mass moments of inertia of a circular plate as follows:

$$I_{AA'} = I_{BB'} = \frac{1}{4}mr^2 \quad (9.41)$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{2}mr^2 \quad (9.42)$$

## 9.14 DETERMINATION OF THE MOMENT OF INERTIA OF A THREE-DIMENSIONAL BODY BY INTEGRATION

The moment of inertia of a three-dimensional body is obtained by evaluating the integral  $I = \int r^2 dm$ . If the body is made of a homogeneous material of density  $\rho$ , the element of mass  $dm$  is equal to  $\rho dV$  and we can write  $I = \rho \int r^2 dV$ . This integral depends only upon the shape of the body. Thus, in order to compute the moment of inertia of a three-dimensional body, it will generally be necessary to perform a triple, or at least a double, integration.

However, if the body possesses two planes of symmetry, it is usually possible to determine the body's moment of inertia with a single integration by choosing as the element of mass  $dm$  a thin slab which is perpendicular to the planes of symmetry. In the case of bodies of revolution, for example, the element of mass would be a thin disk (Fig. 9.27). Using formula (9.42), the moment of inertia of the disk with respect to the axis of revolution can be expressed as indicated in Fig. 9.27. Its moment of inertia with respect to each of the other two coordinate axes is obtained by using formula (9.41) and the parallel-axis theorem. Integration of the expression obtained yields the desired moment of inertia of the body.

## 9.15 MOMENTS OF INERTIA OF COMPOSITE BODIES

The moments of inertia of a few common shapes are shown in Fig. 9.28. For a body consisting of several of these simple shapes, the moment of inertia of the body with respect to a given axis can be obtained by first computing the moments of inertia of its component parts about the desired axis and then adding them together. As was the case for areas, the radius of gyration of a composite body *cannot* be obtained by adding the radii of gyration of its component parts.

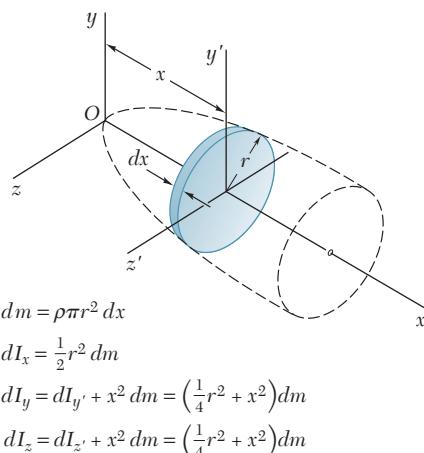
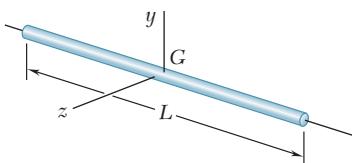
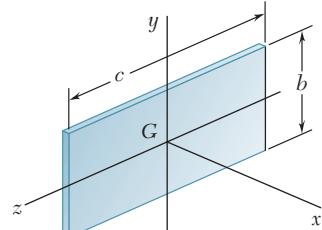
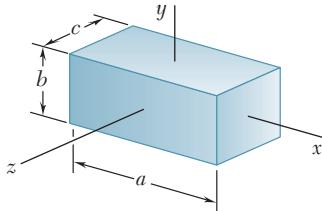
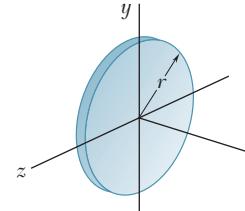
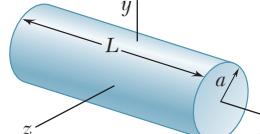
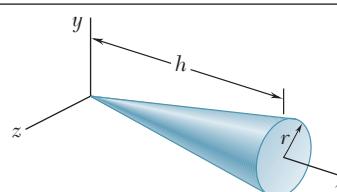
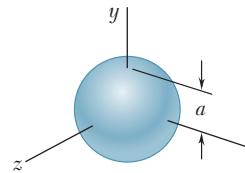
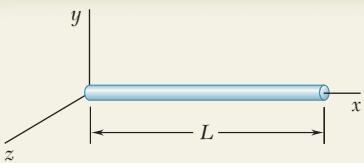


Fig. 9.27 Determination of the moment of inertia of a body of revolution.

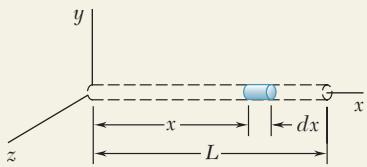
Slender rod		$I_y = I_z = \frac{1}{12} m L^2$
Thin rectangular plate		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} m c^2$ $I_z = \frac{1}{12} m b^2$
Rectangular prism		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} m(c^2 + a^2)$ $I_z = \frac{1}{12} m(a^2 + b^2)$
Thin disk		$I_x = \frac{1}{2} m r^2$ $I_y = I_z = \frac{1}{4} m r^2$
Circular cylinder		$I_x = \frac{1}{2} m a^2$ $I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$
Circular cone		$I_x = \frac{3}{10} m a^2$ $I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$
Sphere		$I_x = I_y = I_z = \frac{2}{5} m a^2$

**Fig. 9.28** Mass moments of inertia of common geometric shapes.



## SAMPLE PROBLEM 9.9

Determine the moment of inertia of a slender rod of length  $L$  and mass  $m$  with respect to an axis which is perpendicular to the rod and passes through one end of the rod.

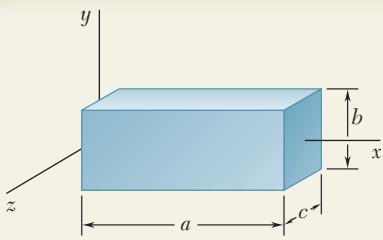


## SOLUTION

Choosing the differential element of mass shown, we write

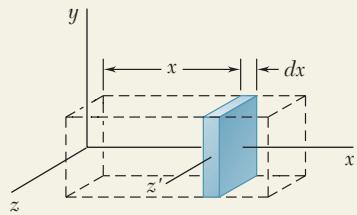
$$dm = \frac{m}{L}dx$$

$$I_y = \int x^2 dm = \int_0^L x^2 \frac{m}{L}dx = \left[ \frac{m}{L} \frac{x^3}{3} \right]_0^L \quad I_y = \frac{1}{3}mL^2 \quad \blacktriangleleft$$



## SAMPLE PROBLEM 9.10

For the homogeneous rectangular prism shown, determine the moment of inertia with respect to the  $z$  axis.



## SOLUTION

We choose as the differential element of mass the thin slab shown; thus

$$dm = \rho bc dx$$

Referring to Sec. 9.13, we find that the moment of inertia of the element with respect to the  $z'$  axis is

$$dI_{z'} = \frac{1}{12}b^2 dm$$

Applying the parallel-axis theorem, we obtain the mass moment of inertia of the slab with respect to the  $z$  axis.

$$dI_z = dI_{z'} + x^2 dm = \frac{1}{12}b^2 dm + x^2 dm = (\frac{1}{12}b^2 + x^2)\rho bc dx$$

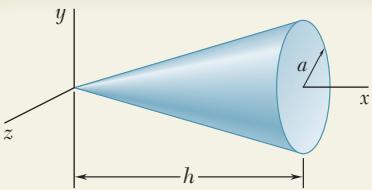
Integrating from  $x = 0$  to  $x = a$ , we obtain

$$I_z = \int dI_z = \int_0^a (\frac{1}{12}b^2 + x^2)\rho bc dx = \rho abc(\frac{1}{12}b^2 + \frac{1}{3}a^2)$$

Since the total mass of the prism is  $m = \rho abc$ , we can write

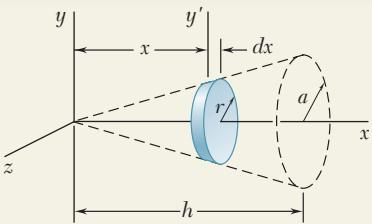
$$I_z = m(\frac{1}{12}b^2 + \frac{1}{3}a^2) \quad I_z = \frac{1}{12}m(4a^2 + b^2) \quad \blacktriangleleft$$

We note that if the prism is thin,  $b$  is small compared to  $a$ , and the expression for  $I_z$  reduces to  $\frac{1}{3}ma^2$ , which is the result obtained in Sample Prob. 9.9 when  $L = a$ .



## SAMPLE PROBLEM 9.11

Determine the moment of inertia of a right circular cone with respect to (a) its longitudinal axis, (b) an axis through the apex of the cone and perpendicular to its longitudinal axis, (c) an axis through the centroid of the cone and perpendicular to its longitudinal axis.



## SOLUTION

We choose the differential element of mass shown.

$$r = a \frac{x}{h} \quad dm = \rho \pi r^2 dx = \rho \pi \frac{a^2}{h^2} x^2 dx$$

**a. Moment of Inertia  $I_x$ .** Using the expression derived in Sec. 9.13 for a thin disk, we compute the mass moment of inertia of the differential element with respect to the  $x$  axis.

$$dI_x = \frac{1}{2} r^2 dm = \frac{1}{2} \left( a \frac{x}{h} \right)^2 \left( \rho \pi \frac{a^2}{h^2} x^2 dx \right) = \frac{1}{2} \rho \pi \frac{a^4}{h^4} x^4 dx$$

Integrating from  $x = 0$  to  $x = h$ , we obtain

$$I_x = \int dI_x = \int_0^h \frac{1}{2} \rho \pi \frac{a^4}{h^4} x^4 dx = \frac{1}{2} \rho \pi \frac{a^4}{h^4} \frac{h^5}{5} = \frac{1}{10} \rho \pi a^4 h$$

Since the total mass of the cone is  $m = \frac{1}{3} \rho \pi a^2 h$ , we can write

$$I_x = \frac{1}{10} \rho \pi a^4 h = \frac{3}{10} a^2 \left( \frac{1}{3} \rho \pi a^2 h \right) = \frac{3}{10} m a^2 \quad I_x = \frac{3}{10} m a^2 \quad \blacktriangleleft$$

**b. Moment of Inertia  $I_y$ .** The same differential element is used. Applying the parallel-axis theorem and using the expression derived in Sec. 9.13 for a thin disk, we write

$$dI_y = dI_{y'} + x^2 dm = \frac{1}{4} r^2 dm + x^2 dm = (\frac{1}{4} r^2 + x^2) dm$$

Substituting the expressions for  $r$  and  $dm$  into the equation, we obtain

$$\begin{aligned} dI_y &= \left( \frac{1}{4} \frac{a^2}{h^2} x^2 + x^2 \right) \left( \rho \pi \frac{a^2}{h^2} x^2 dx \right) = \rho \pi \frac{a^2}{h^2} \left( \frac{a^2}{4h^2} + 1 \right) x^4 dx \\ I_y &= \int dI_y = \int_0^h \rho \pi \frac{a^2}{h^2} \left( \frac{a^2}{4h^2} + 1 \right) x^4 dx = \rho \pi \frac{a^2}{h^2} \left( \frac{a^2}{4h^2} + 1 \right) \frac{h^5}{5} \end{aligned}$$

Introducing the total mass of the cone  $m$ , we rewrite  $I_y$  as follows:

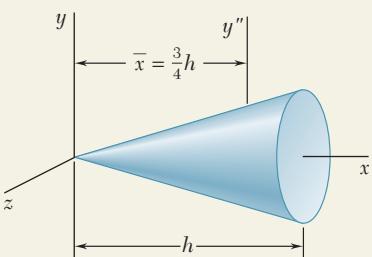
$$I_y = \frac{3}{5} (\frac{1}{4} a^2 + h^2) \frac{1}{3} \rho \pi a^2 h \quad I_y = \frac{3}{5} m (\frac{1}{4} a^2 + h^2) \quad \blacktriangleleft$$

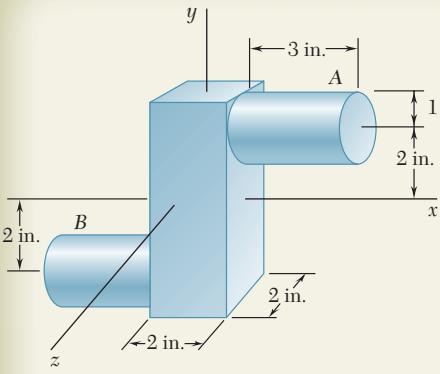
**c. Moment of Inertia  $\bar{I}_{y''}$ .** We apply the parallel-axis theorem and write

$$I_y = \bar{I}_{y''} + m \bar{x}^2$$

Solving for  $\bar{I}_{y''}$  and recalling that  $\bar{x} = \frac{3}{4}h$ , we have

$$\begin{aligned} \bar{I}_{y''} &= I_y - m \bar{x}^2 = \frac{3}{5} m (\frac{1}{4} a^2 + h^2) - m (\frac{3}{4} h)^2 \\ \bar{I}_{y''} &= \frac{3}{20} m (a^2 + \frac{1}{4} h^2) \quad \blacktriangleleft \end{aligned}$$





## SAMPLE PROBLEM 9.12

A steel forging consists of a  $6 \times 2 \times 2$ -in. rectangular prism and two cylinders of diameter 2 in. and length 3 in. as shown. Determine the moments of inertia of the forging with respect to the coordinate axes, knowing that the specific weight of steel is  $490 \text{ lb}/\text{ft}^3$ .

## SOLUTION

### Computation of Masses

#### Prism

$$V = (2 \text{ in.})(2 \text{ in.})(6 \text{ in.}) = 24 \text{ in}^3$$

$$W = \frac{(24 \text{ in}^3)(490 \text{ lb}/\text{ft}^3)}{1728 \text{ in}^3/\text{ft}^3} = 6.81 \text{ lb}$$

$$m = \frac{6.81 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.211 \text{ lb} \cdot \text{s}^2/\text{ft}$$

#### Each Cylinder

$$V = \pi(1 \text{ in.})^2(3 \text{ in.}) = 9.42 \text{ in}^3$$

$$W = \frac{(9.42 \text{ in}^3)(490 \text{ lb}/\text{ft}^3)}{1728 \text{ in}^3/\text{ft}^3} = 2.67 \text{ lb}$$

$$m = \frac{2.67 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.0829 \text{ lb} \cdot \text{s}^2/\text{ft}$$

**Moments of Inertia.** The moments of inertia of each component are computed from Fig. 9.28, using the parallel-axis theorem when necessary. Note that all lengths should be expressed in feet.

#### Prism

$$I_x = I_z = \frac{1}{12}(0.211 \text{ lb} \cdot \text{s}^2/\text{ft})[(\frac{6}{12} \text{ ft})^2 + (\frac{2}{12} \text{ ft})^2] = 4.88 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = \frac{1}{12}(0.211 \text{ lb} \cdot \text{s}^2/\text{ft})[(\frac{2}{12} \text{ ft})^2 + (\frac{2}{12} \text{ ft})^2] = 0.977 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

#### Each Cylinder

$$I_x = \frac{1}{2}ma^2 + m\bar{y}^2 = \frac{1}{2}(0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})(\frac{1}{12} \text{ ft})^2 + (0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})(\frac{2}{12} \text{ ft})^2 = 2.59 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = \frac{1}{12}m(3a^2 + L^2) = m\bar{x}^2 = \frac{1}{12}(0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})[3(\frac{1}{12} \text{ ft})^2 + (\frac{3}{12} \text{ ft})^2]$$

$$+ (0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})(\frac{2.5}{12} \text{ ft})^2 = 4.17 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_z = \frac{1}{12}m(3a^2 + L^2) + m(\bar{x}^2 + \bar{y}^2) = \frac{1}{12}(0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})[3(\frac{1}{12} \text{ ft})^2 + (\frac{3}{12} \text{ ft})^2]$$

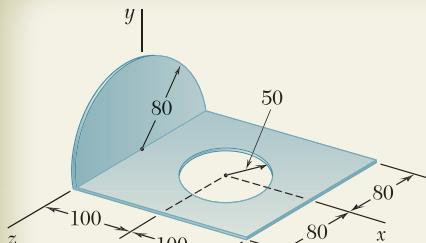
$$+ (0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})[(\frac{2.5}{12} \text{ ft})^2 + (\frac{2}{12} \text{ ft})^2] = 6.48 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

**Entire Body.** Adding the values obtained,

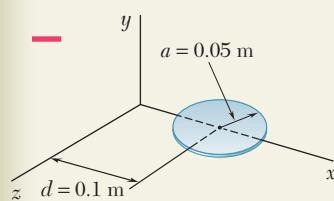
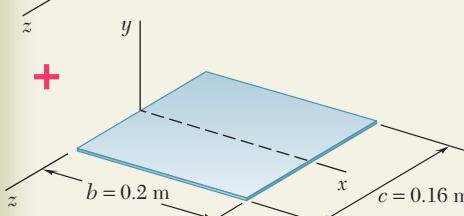
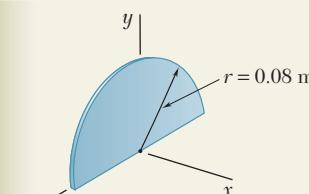
$$I_x = 4.88 \times 10^{-3} + 2(2.59 \times 10^{-3}) \quad I_x = 10.06 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = 0.977 \times 10^{-3} + 2(4.17 \times 10^{-3}) \quad I_y = 9.32 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_z = 4.88 \times 10^{-3} + 2(6.48 \times 10^{-3}) \quad I_z = 17.84 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$



Dimensions in mm



## SAMPLE PROBLEM 9.13

A thin steel plate which is 4 mm thick is cut and bent to form the machine part shown. Knowing that the density of steel is  $7850 \text{ kg/m}^3$ , determine the moments of inertia of the machine part with respect to the coordinate axes.

### SOLUTION

We observe that the machine part consists of a semicircular plate and a rectangular plate from which a circular plate has been removed.

#### Computation of Masses. Semicircular Plate

$$V_1 = \frac{1}{2}\pi r^2 t = \frac{1}{2}\pi(0.08 \text{ m})^2(0.004 \text{ m}) = 40.21 \times 10^{-6} \text{ m}^3$$

$$m_1 = \rho V_1 = (7.85 \times 10^3 \text{ kg/m}^3)(40.21 \times 10^{-6} \text{ m}^3) = 0.3156 \text{ kg}$$

#### Rectangular Plate

$$V_2 = (0.200 \text{ m})(0.160 \text{ m})(0.004 \text{ m}) = 128 \times 10^{-6} \text{ m}^3$$

$$m_2 = \rho V_2 = (7.85 \times 10^3 \text{ kg/m}^3)(128 \times 10^{-6} \text{ m}^3) = 1.005 \text{ kg}$$

#### Circular Plate

$$V_3 = \pi a^2 t = \pi(0.050 \text{ m})^2(0.004 \text{ m}) = 31.42 \times 10^{-6} \text{ m}^3$$

$$m_3 = \rho V_3 = (7.85 \times 10^3 \text{ kg/m}^3)(31.42 \times 10^{-6} \text{ m}^3) = 0.2466 \text{ kg}$$

**Moments of Inertia.** Using the method presented in Sec. 9.13, we compute the moments of inertia of each component.

**Semicircular Plate.** From Fig. 9.28, we observe that for a circular plate of mass  $m$  and radius  $r$

$$I_x = \frac{1}{2}mr^2 \quad I_y = I_z = \frac{1}{4}mr^2$$

Because of symmetry, we note that for a semicircular plate

$$I_x = \frac{1}{2}(\frac{1}{2}mr^2) \quad I_y = I_z = \frac{1}{2}(\frac{1}{4}mr^2)$$

Since the mass of the semicircular plate is  $m_1 = \frac{1}{2}m$ , we have

$$I_x = \frac{1}{2}m_1r^2 = \frac{1}{2}(0.3156 \text{ kg})(0.08 \text{ m})^2 = 1.010 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = I_z = \frac{1}{4}(\frac{1}{2}mr^2) = \frac{1}{4}m_1r^2 = \frac{1}{4}(0.3156 \text{ kg})(0.08 \text{ m})^2 = 0.505 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

#### Rectangular Plate

$$I_x = \frac{1}{12}m_2c^2 = \frac{1}{12}(1.005 \text{ kg})(0.16 \text{ m})^2 = 2.144 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = \frac{1}{3}m_2b^2 = \frac{1}{3}(1.005 \text{ kg})(0.2 \text{ m})^2 = 13.400 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = I_x + I_z = (2.144 + 13.400)(10^{-3}) = 15.544 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

#### Circular Plate

$$I_x = \frac{1}{4}m_3a^2 = \frac{1}{4}(0.2466 \text{ kg})(0.05 \text{ m})^2 = 0.154 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = \frac{1}{2}m_3a^2 + m_3d^2$$

$$= \frac{1}{2}(0.2466 \text{ kg})(0.05 \text{ m})^2 + (0.2466 \text{ kg})(0.1 \text{ m})^2 = 2.774 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = \frac{1}{4}m_3a^2 + m_3d^2 = \frac{1}{4}(0.2466 \text{ kg})(0.05 \text{ m})^2 + (0.2466 \text{ kg})(0.1 \text{ m})^2$$

$$= 2.620 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

#### Entire Machine Part

$$I_x = (1.010 + 2.144 - 0.154)(10^{-3}) \text{ kg} \cdot \text{m}^2 \quad I_x = 3.00 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = (0.505 + 15.544 - 2.774)(10^{-3}) \text{ kg} \cdot \text{m}^2 \quad I_y = 13.28 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_z = (0.505 + 13.400 - 2.620)(10^{-3}) \text{ kg} \cdot \text{m}^2 \quad I_z = 11.29 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced the *mass moment of inertia* and the *radius of gyration* of a three-dimensional body with respect to a given axis [Eqs. (9.28) and (9.29)]. We also derived a *parallel-axis theorem* for use with mass moments of inertia and discussed the computation of the mass moments of inertia of thin plates and three-dimensional bodies.

**1. Computing mass moments of inertia.** The mass moment of inertia  $I$  of a body with respect to a given axis can be calculated directly from the definition given in Eq. (9.28) for simple shapes [Sample Prob. 9.9]. In most cases, however, it is necessary to divide the body into thin slabs, compute the moment of inertia of a typical slab with respect to the given axis—using the parallel-axis theorem if necessary—and integrate the expression obtained.

**2. Applying the parallel-axis theorem.** In Sec. 9.12 we derived the parallel-axis theorem for mass moments of inertia

$$I = \bar{I} + md^2 \quad (9.33)$$

which states that the moment of inertia  $I$  of a body of mass  $m$  with respect to a given axis is equal to the sum of the moment of inertia  $\bar{I}$  of that body with respect to a *parallel centroidal axis* and the product  $md^2$ , where  $d$  is the distance between the two axes. When the moment of inertia of a three-dimensional body is calculated with respect to one of the coordinate axes,  $d^2$  can be replaced by the sum of the squares of distances measured along the other two coordinate axes [Eqs. (9.32) and (9.32')].

**3. Avoiding unit-related errors.** To avoid errors, it is essential that you be consistent in your use of units. Thus, all lengths should be expressed in meters or feet, as appropriate, and for problems using U.S. customary units, masses should be given in  $\text{lb} \cdot \text{s}^2/\text{ft}$ . In addition, we strongly recommend that you include units as you perform your calculations [Sample Probs. 9.12 and 9.13].

**4. Calculating the mass moment of inertia of thin plates.** We showed in Sec. 9.13 that the mass moment of inertia of a thin plate with respect to a given axis can be obtained by multiplying the corresponding moment of inertia of the area of the plate by the density  $\rho$  and the thickness  $t$  of the plate [Eqs. (9.35) through (9.37)]. Note that since the axis  $CC'$  in Fig. 9.24c is *perpendicular to the plate*,  $I_{CC',\text{mass}}$  is associated with the *polar* moment of inertia  $J_{C,\text{area}}$ .

Instead of calculating directly the moment of inertia of a thin plate with respect to a specified axis, you may sometimes find it convenient to first compute its moment of inertia with respect to an axis parallel to the specified axis and then apply the parallel-axis theorem. Further, to determine the moment of inertia of a thin plate with respect to an axis perpendicular to the plate, you may wish to first determine its moments of inertia with respect to two perpendicular in-plane axes and then use Eq. (9.38). Finally, remember that the mass of a plate of area  $A$ , thickness  $t$ , and density  $\rho$  is  $m = \rho t A$ .

**5. Determining the moment of inertia of a body by direct single integration.** We discussed in Sec. 9.14 and illustrated in Sample Probs. 9.10 and 9.11 how single integration can be used to compute the moment of inertia of a body that can be divided into a series of thin, parallel slabs. For such cases, you will often need to express the mass of the body in terms of the body's density and dimensions. Assuming that the body has been divided, as in the sample problems, into thin slabs perpendicular to the  $x$  axis, you will need to express the dimensions of each slab as functions of the variable  $x$ .

a. **In the special case of a body of revolution,** the elemental slab is a thin disk, and the equations given in Fig. 9.27 should be used to determine the moments of inertia of the body [Sample Prob. 9.11].

b. **In the general case, when the body is not of revolution,** the differential element is not a disk, but a thin slab of a different shape, and the equations of Fig. 9.27 cannot be used. See, for example, Sample Prob. 9.10, where the element was a thin, rectangular slab. For more complex configurations, you may want to use one or more of the following equations, which are based on Eqs. (9.32) and (9.32') of Sec. 9.12.

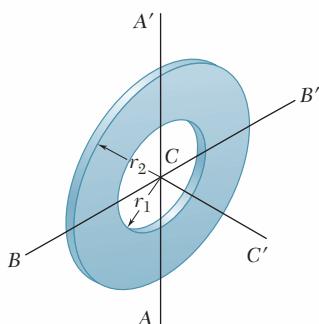
$$\begin{aligned} dI_x &= dI_{x'} + (\bar{y}_{el}^2 + \bar{z}_{el}^2) dm \\ dI_y &= dI_{y'} + (\bar{z}_{el}^2 + \bar{x}_{el}^2) dm \\ dI_z &= dI_{z'} + (\bar{x}_{el}^2 + \bar{y}_{el}^2) dm \end{aligned}$$

where the primes denote the centroidal axes of each elemental slab, and where  $\bar{x}_{el}$ ,  $\bar{y}_{el}$ , and  $\bar{z}_{el}$  represent the coordinates of its centroid. The centroidal moments of inertia of the slab are determined in the manner described earlier for a thin plate: Referring to Fig. 9.12 on page 485, calculate the corresponding moments of inertia of the area of the slab and multiply the result by the density  $\rho$  and the thickness  $t$  of the slab. Also, assuming that the body has been divided into thin slabs perpendicular to the  $x$  axis, remember that you can obtain  $dI_{x'}$  by adding  $dI_{y'}$  and  $dI_{z'}$  instead of computing it directly. Finally, using the geometry of the body, express the result obtained in terms of the single variable  $x$  and integrate in  $x$ .

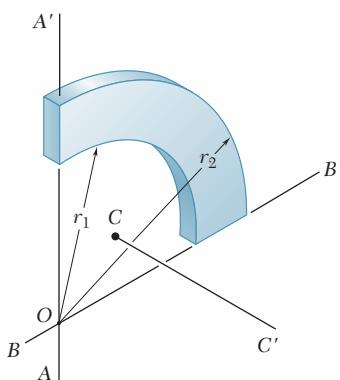
**6. Computing the moment of inertia of a composite body.** As stated in Sec. 9.15, the moment of inertia of a composite body with respect to a specified axis is equal to the sum of the moments of its components with respect to that axis. Sample Probs. 9.12 and 9.13 illustrate the appropriate method of solution. You must also remember that the moment of inertia of a component will be negative only if the component is *removed* (as in the case of a hole).

Although the composite-body problems in this lesson are relatively straightforward, you will have to work carefully to avoid computational errors. In addition, if some of the moments of inertia that you need are not given in Fig. 9.28, you will have to derive your own formulas, using the techniques of this lesson.

# PROBLEMS



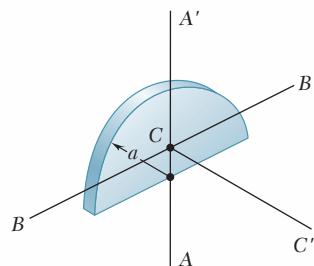
**Fig. P9.111**



**Fig. P9.113**

- 9.111** Determine the mass moment of inertia of a ring of mass  $m$ , cut from a thin uniform plate, with respect to (a) the axis  $AA'$ , (b) the centroidal axis  $CC'$  that is perpendicular to the plane of the ring.

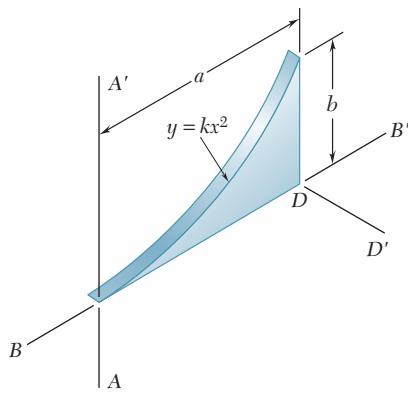
- 9.112** A thin semicircular plate has a radius  $a$  and a mass  $m$ . Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis  $BB'$ , (b) the centroidal axis  $CC'$  that is perpendicular to the plate.



**Fig. P9.112**

- 9.113** The quarter ring shown has a mass  $m$  and was cut from a thin, uniform plate. Knowing that  $r_1 = \frac{3}{4}r_2$ , determine the mass moment of inertia of the quarter ring with respect to (a) the axis  $AA'$ , (b) the centroidal axis  $CC'$  that is perpendicular to the plane of the quarter ring.

- 9.114** The parabolic spandrel shown was cut from a thin, uniform plate. Denoting the mass of the spandrel by  $m$ , determine its mass moment of inertia with respect to (a) the axis  $BB'$ , (b) the axis  $DD'$  that is perpendicular to the spandrel. (*Hint:* See Sample Prob. 9.3.)



**Fig. P9.114**

- 9.115** A thin plate of mass  $m$  was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (a) the  $x$  axis, (b) the axis  $BB'$ , which is perpendicular to the plate.

- 9.116** A thin plate of mass  $m$  was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (a) the  $y$  axis, (b) the axis  $AA'$ , which is perpendicular to the plate.

- 9.117** A thin plate of mass  $m$  has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (a) the  $x$  axis, (b) the  $y$  axis.

- 9.118** A thin plate of mass  $m$  has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis  $CC'$  that is perpendicular to the plate, (b) the axis  $AA'$  that is parallel to the  $x$  axis and is located at a distance  $1.5a$  from the plate.

- 9.119** The area shown is revolved about the  $x$  axis to form a homogeneous solid of revolution of mass  $m$ . Using direct integration, express the mass moment of inertia of the solid with respect to the  $x$  axis in terms of  $m$  and  $h$ .

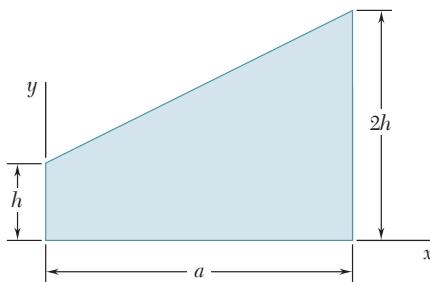


Fig. P9.119

- 9.120** Determine by direct integration the mass moment of inertia with respect to the  $y$  axis of the right circular cylinder shown, assuming that it has a uniform density and a mass  $m$ .

- 9.121** The area shown is revolved about the  $x$  axis to form a homogeneous solid of revolution of mass  $m$ . Determine by direct integration the mass moment of inertia of the solid with respect to (a) the  $x$  axis, (b) the  $y$  axis. Express your answers in terms of  $m$  and the dimensions of the solid.

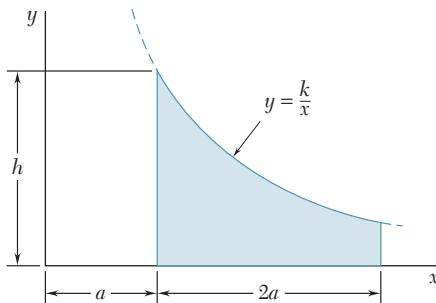


Fig. P9.121

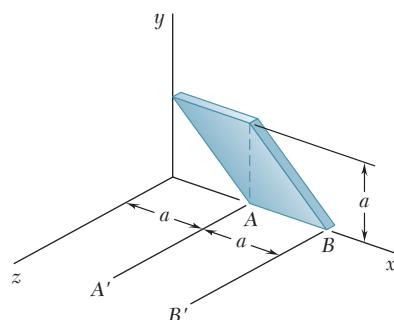


Fig. P9.115 and P9.116

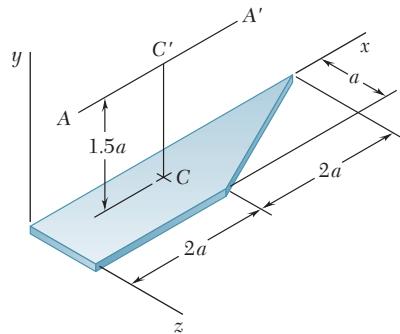


Fig. P9.117 and P9.118

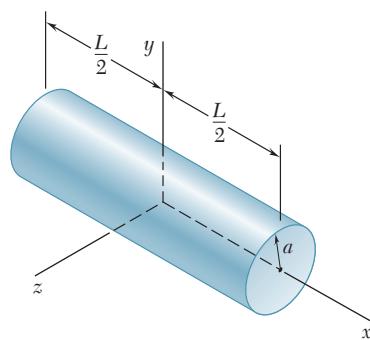


Fig. P9.120

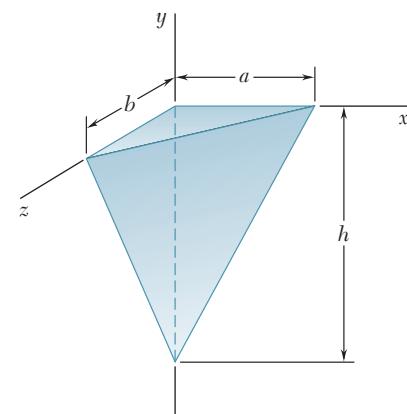


Fig. P9.122 and P9.123

- 9.122** Determine by direct integration the mass moment of inertia with respect to the  $x$  axis of the tetrahedron shown, assuming that it has a uniform density and a mass  $m$ .

- 9.123** Determine by direct integration the mass moment of inertia with respect to the  $y$  axis of the tetrahedron shown, assuming that it has a uniform density and a mass  $m$ .

- \*9.124** Determine by direct integration the mass moment of inertia with respect to the  $z$  axis of the semiellipsoid shown, assuming that it has a uniform density and a mass  $m$ .

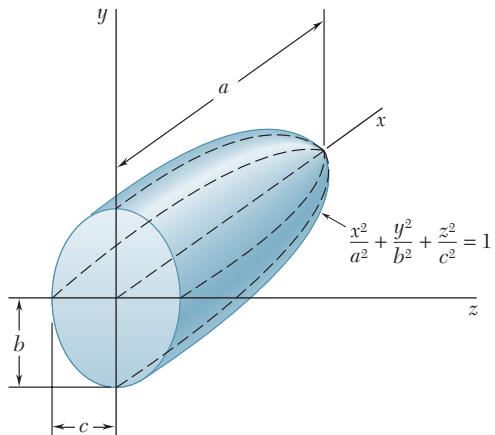


Fig. P9.124

- \*9.125** A thin steel wire is bent into the shape shown. Denoting the mass per unit length of the wire by  $m'$ , determine by direct integration the mass moment of inertia of the wire with respect to each of the coordinate axes.

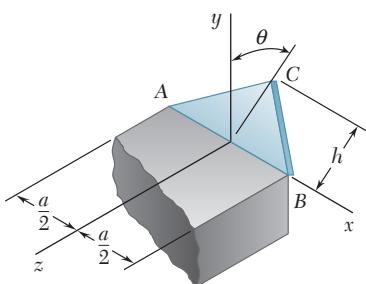


Fig. P9.126

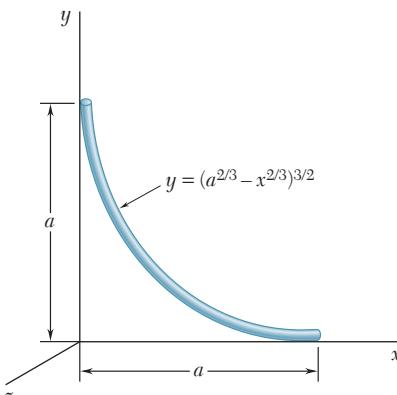


Fig. P9.125

- 9.126** A thin triangular plate of mass  $m$  is welded along its base  $AB$  to a block as shown. Knowing that the plate forms an angle  $\theta$  with the  $y$  axis, determine by direct integration the mass moment of inertia of the plate with respect to (a) the  $x$  axis, (b) the  $y$  axis, (c) the  $z$  axis.

- 9.127** Shown is the cross section of a molded flat-belt pulley. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA'. (The density of brass is  $8650 \text{ kg/m}^3$  and the density of the fiber-reinforced polycarbonate used is  $1250 \text{ kg/m}^3$ .)

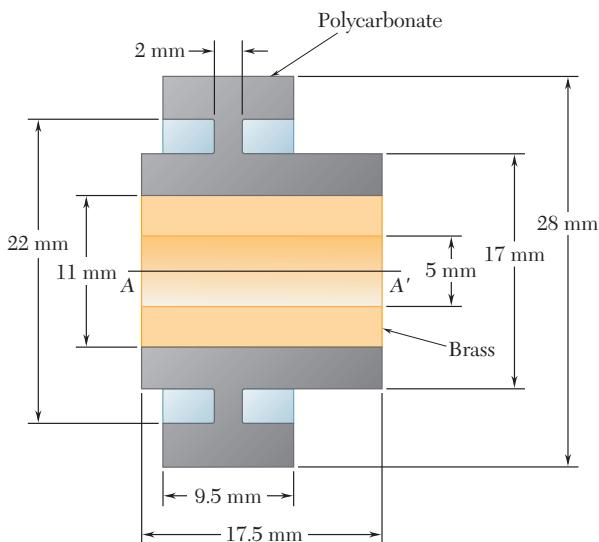


Fig. P9.127

- 9.128** Shown is the cross section of an idler roller. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA'. (The specific weight of bronze is  $0.310 \text{ lb/in}^3$ ; of aluminum,  $0.100 \text{ lb/in}^3$ ; and of neoprene,  $0.0452 \text{ lb/in}^3$ .)

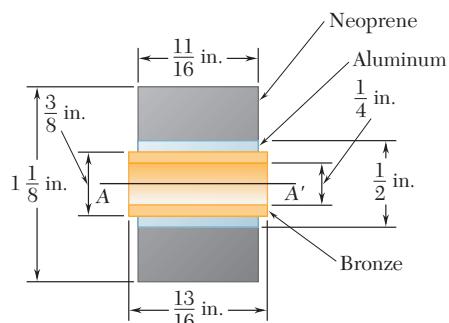


Fig. P9.128

- 9.129** Knowing that the thin cylindrical shell shown is of mass  $m$ , thickness  $t$ , and height  $h$ , determine the mass moment of inertia of the shell with respect to the  $x$  axis. (Hint: Consider the shell as formed by removing a cylinder of radius  $a$  and height  $h$  from a cylinder of radius  $a + t$  and height  $h$ ; then neglect terms containing  $t^2$  and  $t^3$  and keep those terms containing  $t$ .)

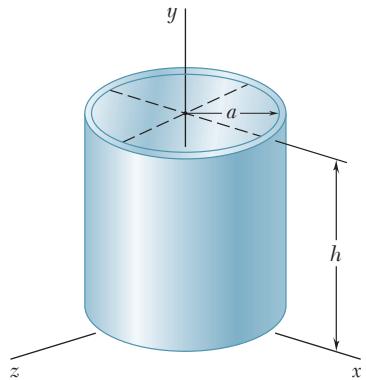


Fig. P9.129

- 9.130** The machine part shown is formed by machining a conical surface into a circular cylinder. For  $b = \frac{1}{2}h$ , determine the mass moment of inertia and the radius of gyration of the machine part with respect to the  $y$  axis.

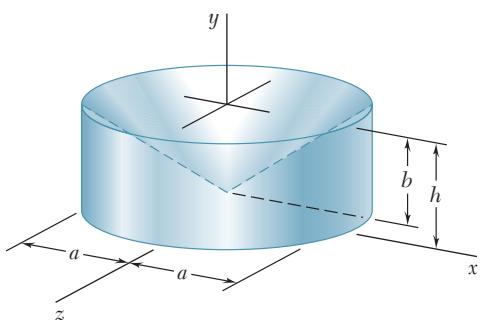
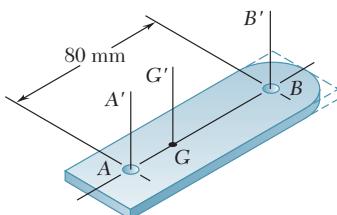
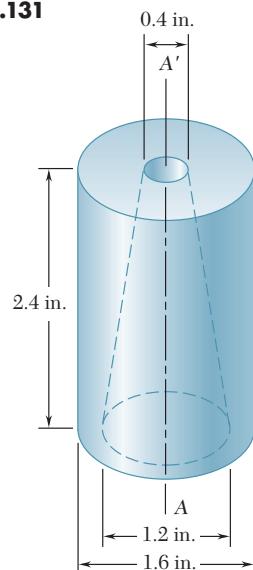


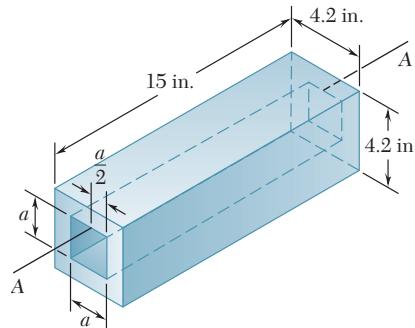
Fig. P9.130

**Fig. P9.131****Fig. P9.132**

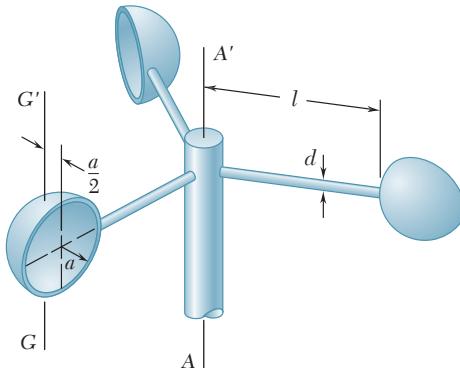
- 9.131** After a period of use, one of the blades of a shredder has been worn to the shape shown and is of mass 0.18 kg. Knowing that the mass moments of inertia of the blade with respect to the  $AA'$  and  $BB'$  axes are  $0.320 \text{ g} \cdot \text{m}^2$  and  $0.680 \text{ g} \cdot \text{m}^2$ , respectively, determine (a) the location of the centroidal axis  $GG'$ , (b) the radius of gyration with respect to axis  $GG'$ .

- 9.132** Determine the mass moment of inertia of the 0.9-lb machine component shown with respect to the axis  $AA'$ .

- 9.133** A square hole is centered in and extends through the aluminum machine component shown. Determine (a) the value of  $a$  for which the mass moment of inertia of the component with respect to the axis  $AA'$ , which bisects the top surface of the hole, is maximum, (b) the corresponding values of the mass moment of inertia and the radius of gyration with respect to the axis  $AA'$ . (The specific weight of aluminum is  $0.100 \text{ lb/in}^3$ .)

**Fig. P9.133**

- 9.134** The cups and the arms of an anemometer are fabricated from a material of density  $\rho$ . Knowing that the mass moment of inertia of a thin, hemispherical shell of mass  $m$  and thickness  $t$  with respect to its centroidal axis  $GG'$  is  $5ma^2/12$ , determine (a) the mass moment of inertia of the anemometer with respect to the axis  $AA'$ , (b) the ratio of  $a$  to  $l$  for which the centroidal moment of inertia of the cups is equal to 1 percent of the moment of inertia of the cups with respect to the axis  $AA'$ .

**Fig. P9.134**

- 9.135 and 9.136** A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is  $7850 \text{ kg/m}^3$ , determine the mass moment of inertia of the component with respect to each of the coordinate axes.

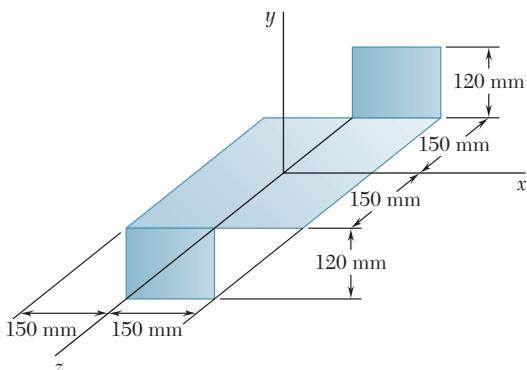


Fig. P9.135

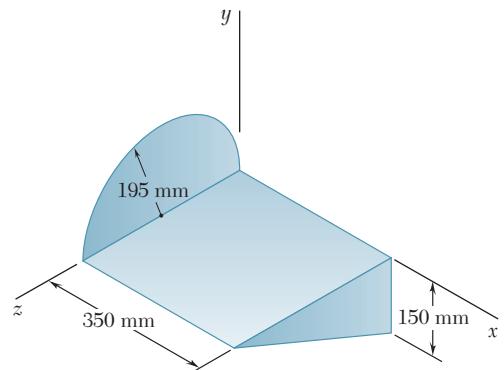


Fig. P9.136

- 9.137** The cover for an electronic device is formed from sheet aluminum that is 0.05 in. thick. Determine the mass moment of inertia of the cover with respect to each of the coordinate axes. (The specific weight of aluminum is  $0.100 \text{ lb/in}^3$ .)

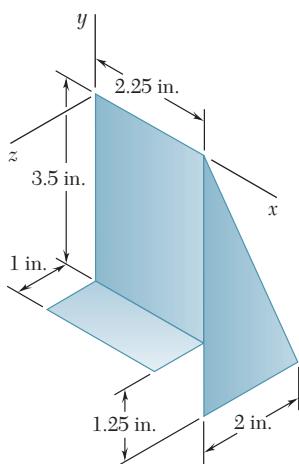


Fig. P9.138

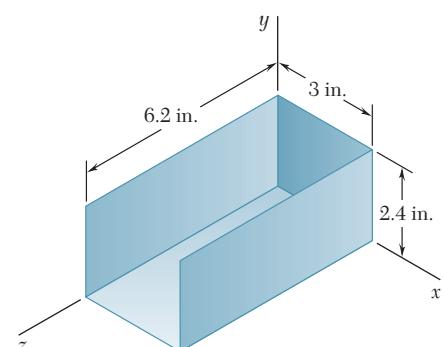


Fig. P9.137

- 9.138** A framing anchor is formed of 0.05-in.-thick galvanized steel. Determine the mass moment of inertia of the anchor with respect to each of the coordinate axes. (The specific weight of galvanized steel is  $470 \text{ lb/ft}^3$ .)

- 9.139** A subassembly for a model airplane is fabricated from three pieces of 1.5-mm plywood. Neglecting the mass of the adhesive used to assemble the three pieces, determine the mass moment of inertia of the subassembly with respect to each of the coordinate axes. (The density of the plywood is  $780 \text{ kg/m}^3$ .)

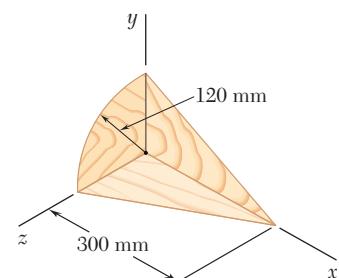
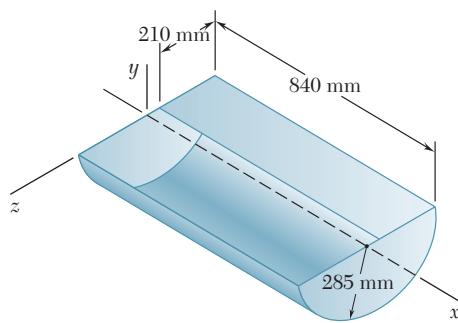
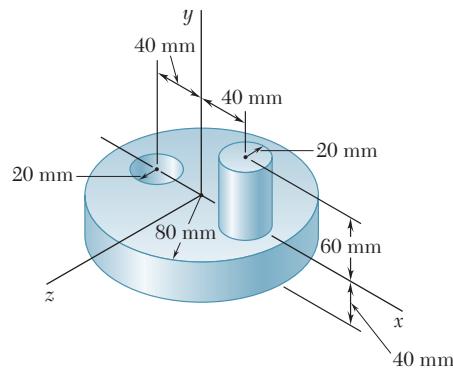


Fig. P9.139

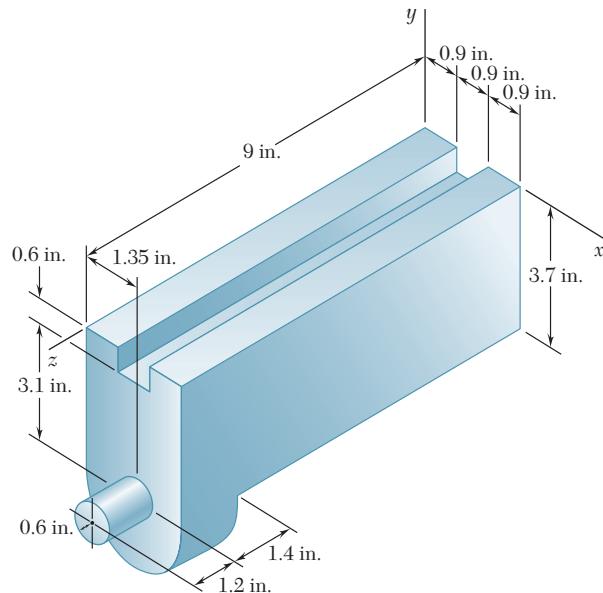
**Fig. P9.140**

**\*9.140** A farmer constructs a trough by welding a rectangular piece of 2-mm-thick sheet steel to half of a steel drum. Knowing that the density of steel is  $7850 \text{ kg/m}^3$  and that the thickness of the walls of the drum is 1.8 mm, determine the mass moment of inertia of the trough with respect to each of the coordinate axes. Neglect the mass of the welds.

**9.141** The machine element shown is fabricated from steel. Determine the mass moment of inertia of the assembly with respect to (a) the  $x$  axis, (b) the  $y$  axis, (c) the  $z$  axis. (The density of steel is  $7850 \text{ kg/m}^3$ .)

**Fig. P9.141**

**9.142** Determine the mass moment of inertia of the steel machine element shown with respect to the  $y$  axis. (The specific weight of steel is  $490 \text{ lb/ft}^3$ .)

**Fig. P9.142 and P9.143**

**9.143** Determine the mass moment of inertia of the steel machine element shown with respect to the  $z$  axis. (The specific weight of steel is  $490 \text{ lb/ft}^3$ .)

- 9.144** Determine the mass moment of inertia and the radius of gyration of the steel machine element shown with respect to the  $x$  axis. (The density of steel is  $7850 \text{ kg/m}^3$ .)

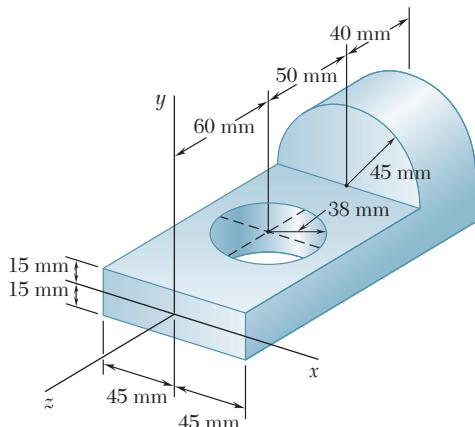


Fig. P9.144

- 9.145** Determine the mass moment of inertia of the steel fixture shown with respect to (a) the  $x$  axis, (b) the  $y$  axis, (c) the  $z$  axis. (The density of steel is  $7850 \text{ kg/m}^3$ .)

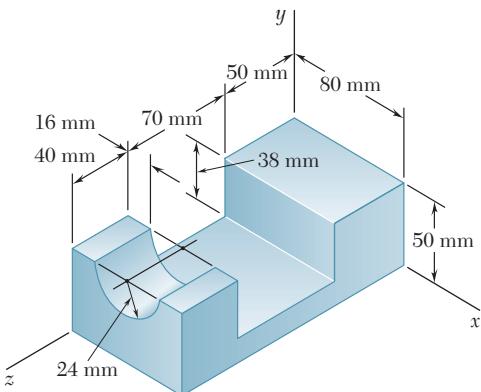


Fig. P9.145

- 9.146** Aluminum wire with a weight per unit length of  $0.033 \text{ lb/ft}$  is used to form the circle and the straight members of the figure shown. Determine the mass moment of inertia of the assembly with respect to each of the coordinate axes.

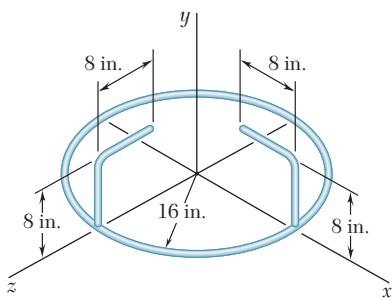


Fig. P9.146

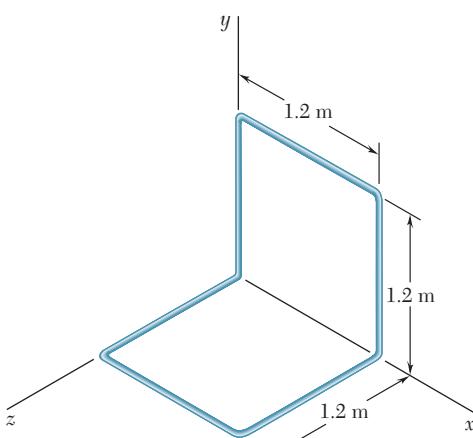


Fig. P9.148

- 9.147** The figure shown is formed of  $\frac{1}{8}$ -in.-diameter steel wire. Knowing that the specific weight of the steel is  $490 \text{ lb/ft}^3$ , determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

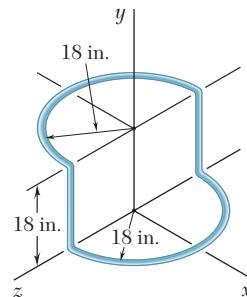


Fig. P9.147

- 9.148** A homogeneous wire with a mass per unit length of  $0.056 \text{ kg/m}$  is used to form the figure shown. Determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

## \*9.16 MOMENT OF INERTIA OF A BODY WITH RESPECT TO AN ARBITRARY AXIS THROUGH O. MASS PRODUCTS OF INERTIA

In this section you will see how the moment of inertia of a body can be determined with respect to an arbitrary axis  $OL$  through the origin (Fig. 9.29) if its moments of inertia with respect to the three coordinate axes, as well as certain other quantities to be defined below, have already been determined.

The moment of inertia  $I_{OL}$  of the body with respect to  $OL$  is equal to  $\int p^2 dm$ , where  $p$  denotes the perpendicular distance from the element of mass  $dm$  to the axis  $OL$ . If we denote by  $\lambda$  the unit vector along  $OL$  and by  $\mathbf{r}$  the position vector of the element  $dm$ , we observe that the perpendicular distance  $p$  is equal to  $r \sin \theta$ , which is the magnitude of the vector product  $\lambda \times \mathbf{r}$ . We therefore write

$$I_{OL} = \int p^2 dm = \int |\lambda \times \mathbf{r}|^2 dm \quad (9.43)$$

Expressing  $|\lambda \times \mathbf{r}|^2$  in terms of the rectangular components of the vector product, we have

$$I_{OL} = \int [(\lambda_x y - \lambda_y x)^2 + (\lambda_y z - \lambda_z y)^2 + (\lambda_z x - \lambda_x z)^2] dm$$

where the components  $\lambda_x, \lambda_y, \lambda_z$  of the unit vector  $\lambda$  represent the direction cosines of the axis  $OL$  and the components  $x, y, z$  of  $\mathbf{r}$  represent the coordinates of the element of mass  $dm$ . Expanding the squares and rearranging the terms, we write

$$\begin{aligned} I_{OL} &= \lambda_x^2 \int (y^2 + z^2) dm + \lambda_y^2 \int (z^2 + x^2) dm + \lambda_z^2 \int (x^2 + y^2) dm \\ &\quad - 2\lambda_x \lambda_y \int xy dm - 2\lambda_y \lambda_z \int yz dm - 2\lambda_z \lambda_x \int zx dm \end{aligned} \quad (9.44)$$

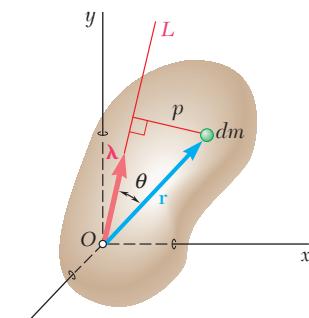


Fig. 9.29

Referring to Eqs. (9.30), we note that the first three integrals in (9.44) represent, respectively, the moments of inertia  $I_x$ ,  $I_y$ , and  $I_z$  of the body with respect to the coordinate axes. The last three integrals in (9.44), which involve products of coordinates, are called the *products of inertia* of the body with respect to the  $x$  and  $y$  axes, the  $y$  and  $z$  axes, and the  $z$  and  $x$  axes, respectively. We write

$$I_{xy} = \int xy \, dm \quad I_{yz} = \int yz \, dm \quad I_{zx} = \int zx \, dm \quad (9.45)$$

Rewriting Eq. (9.44) in terms of the integrals defined in Eqs. (9.30) and (9.45), we have

$$I_{OL} = I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x \quad (9.46)$$

We note that the definition of the products of inertia of a mass given in Eqs. (9.45) is an extension of the definition of the product of inertia of an area (Sec. 9.8). Mass products of inertia reduce to zero under the same conditions of symmetry as do products of inertia of areas, and the parallel-axis theorem for mass products of inertia is expressed by relations similar to the formula derived for the product of inertia of an area. Substituting the expressions for  $x$ ,  $y$ , and  $z$  given in Eqs. (9.31) into Eqs. (9.45), we find that

$$\begin{aligned} I_{xy} &= \bar{I}_{x'y'} + m\bar{x}\bar{y} \\ I_{yz} &= \bar{I}_{y'z'} + m\bar{y}\bar{z} \\ I_{zx} &= \bar{I}_{z'x'} + m\bar{z}\bar{x} \end{aligned} \quad (9.47)$$

where  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$  are the coordinates of the center of gravity  $G$  of the body and  $\bar{I}_{x'y'}$ ,  $\bar{I}_{y'z'}$ ,  $\bar{I}_{z'x'}$  denote the products of inertia of the body with respect to the centroidal axes  $x'$ ,  $y'$ ,  $z'$  (Fig. 9.22).

## \*9.17 ELLIPSOID OF INERTIA. PRINCIPAL AXES OF INERTIA

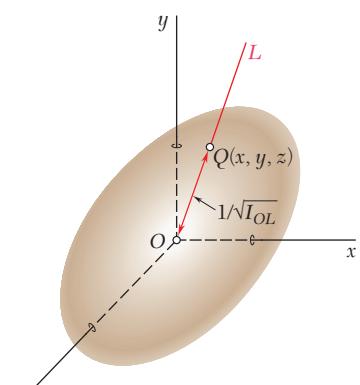
Let us assume that the moment of inertia of the body considered in the preceding section has been determined with respect to a large number of axes  $OL$  through the fixed point  $O$  and that a point  $Q$  has been plotted on each axis  $OL$  at a distance  $OQ = 1/\sqrt{I_{OL}}$  from  $O$ . The locus of the points  $Q$  thus obtained forms a surface (Fig. 9.30). The equation of that surface can be obtained by substituting  $1/(OQ)^2$  for  $I_{OL}$  in (9.46) and then multiplying both sides of the equation by  $(OQ)^2$ . Observing that

$$(OQ)\lambda_x = x \quad (OQ)\lambda_y = y \quad (OQ)\lambda_z = z$$

where  $x$ ,  $y$ ,  $z$  denote the rectangular coordinates of  $Q$ , we write

$$I_x x^2 + I_y y^2 + I_z z^2 - 2I_{xy}xy - 2I_{yz}yz - 2I_{zx}zx = 1 \quad (9.48)$$

The equation obtained is the equation of a *quadric surface*. Since the moment of inertia  $I_{OL}$  is different from zero for every axis  $OL$ , no point  $Q$  can be at an infinite distance from  $O$ . Thus, the quadric surface obtained is an *ellipsoid*. This ellipsoid, which defines the



**Fig. 9.30**

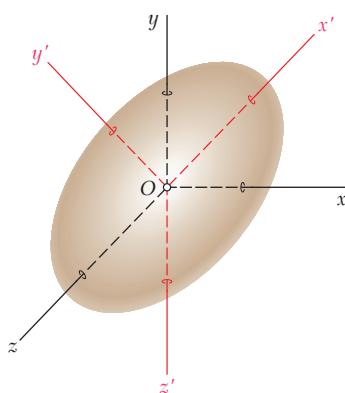


Fig. 9.31

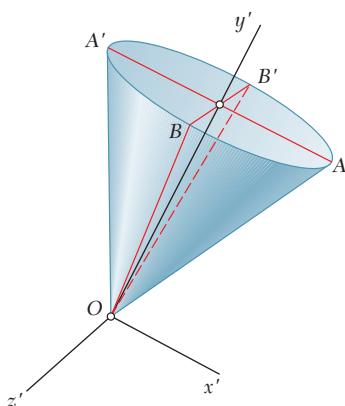


Fig. 9.32

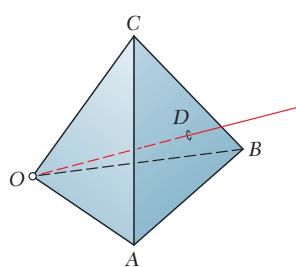


Fig. 9.33

moment of inertia of the body with respect to any axis through  $O$ , is known as the *ellipsoid of inertia* of the body at  $O$ .

We observe that if the axes in Fig. 9.30 are rotated, the coefficients of the equation defining the ellipsoid change, since they are equal to the moments and products of inertia of the body with respect to the rotated coordinate axes. However, the *ellipsoid itself remains unaffected*, since its shape depends only upon the distribution of mass in the given body. Suppose that we choose as coordinate axes the principal axes  $x'$ ,  $y'$ ,  $z'$  of the ellipsoid of inertia (Fig. 9.31). The equation of the ellipsoid with respect to these coordinate axes is known to be of the form

$$I_x x'^2 + I_y y'^2 + I_z z'^2 = 1 \quad (9.49)$$

which does not contain any products of the coordinates. Comparing Eqs. (9.48) and (9.49), we observe that the products of inertia of the body with respect to the  $x'$ ,  $y'$ ,  $z'$  axes must be zero. The  $x'$ ,  $y'$ ,  $z'$  axes are known as the *principal axes of inertia* of the body at  $O$ , and the coefficients  $I_{x'}$ ,  $I_{y'}$ ,  $I_{z'}$  are referred to as the *principal moments of inertia* of the body at  $O$ . Note that, given a body of arbitrary shape and a point  $O$ , it is always possible to find axes which are the principal axes of inertia of the body at  $O$ , that is, axes with respect to which the products of inertia of the body are zero. Indeed, whatever the shape of the body, the moments and products of inertia of the body with respect to  $x$ ,  $y$ , and  $z$  axes through  $O$  will define an ellipsoid, and this ellipsoid will have principal axes which, by definition, are the principal axes of inertia of the body at  $O$ .

If the principal axes of inertia  $x'$ ,  $y'$ ,  $z'$  are used as coordinate axes, the expression obtained in Eq. (9.46) for the moment of inertia of a body with respect to an arbitrary axis through  $O$  reduces to

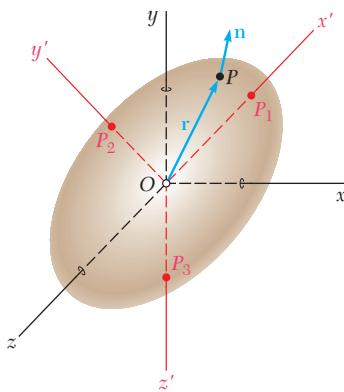
$$I_{OL} = I_{x'} \lambda_{x'}^2 + I_{y'} \lambda_{y'}^2 + I_{z'} \lambda_{z'}^2 \quad (9.50)$$

The determination of the principal axes of inertia of a body of arbitrary shape is somewhat involved and will be discussed in the next section. There are many cases, however, where these axes can be spotted immediately. Consider, for instance, the homogeneous cone of elliptical base shown in Fig. 9.32; this cone possesses two mutually perpendicular planes of symmetry  $OAA'$  and  $OBP'$ . From the definition (9.45), we observe that if the  $x'y'$  and  $y'z'$  planes are chosen to coincide with the two planes of symmetry, all of the products of inertia are zero. The  $x'$ ,  $y'$ , and  $z'$  axes thus selected are therefore the principal axes of inertia of the cone at  $O$ . In the case of the homogeneous regular tetrahedron  $OABC$  shown in Fig. 9.33, the line joining the corner  $O$  to the center  $D$  of the opposite face is a principal axis of inertia at  $O$ , and any line through  $O$  perpendicular to  $OD$  is also a principal axis of inertia at  $O$ . This property is apparent if we observe that rotating the tetrahedron through  $120^\circ$  about  $OD$  leaves its shape and mass distribution unchanged. It follows that the ellipsoid of inertia at  $O$  also remains unchanged under this rotation. The ellipsoid, therefore, is a body of revolution whose axis of revolution is  $OD$ , and the line  $OD$ , as well as any perpendicular line through  $O$ , must be a principal axis of the ellipsoid.

## \*9.18 DETERMINATION OF THE PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA OF A BODY OF ARBITRARY SHAPE

The method of analysis described in this section should be used when the body under consideration has no obvious property of symmetry.

Consider the ellipsoid of inertia of the body at a given point  $O$  (Fig. 9.34); let  $\mathbf{r}$  be the radius vector of a point  $P$  on the surface of the ellipsoid and let  $\mathbf{n}$  be the unit vector along the normal to that surface at  $P$ . We observe that the only points where  $\mathbf{r}$  and  $\mathbf{n}$  are collinear are the points  $P_1$ ,  $P_2$ , and  $P_3$ , where the principal axes intersect the visible portion of the surface of the ellipsoid, and the corresponding points on the other side of the ellipsoid.



**Fig. 9.34**

We now recall from calculus that the direction of the normal to a surface of equation  $f(x, y, z) = 0$  at a point  $P(x, y, z)$  is defined by the gradient  $\nabla f$  of the function  $f$  at that point. To obtain the points where the principal axes intersect the surface of the ellipsoid of inertia, we must therefore write that  $\mathbf{r}$  and  $\nabla f$  are collinear,

$$\nabla f = (2K)\mathbf{r} \quad (9.51)$$

where  $K$  is a constant,  $\mathbf{r} = xi + yj + zk$ , and

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Recalling Eq. (9.48), we note that the function  $f(x, y, z)$  corresponding to the ellipsoid of inertia is

$$f(x, y, z) = I_x x^2 + I_y y^2 + I_z z^2 - 2I_{xy}xy - 2I_{yz}yz - 2I_{zx}zx - 1$$

Substituting for  $\mathbf{r}$  and  $\nabla f$  into Eq. (9.51) and equating the coefficients of the unit vectors, we write

$$\begin{aligned} I_x x - I_{xy}y - I_{zx}z &= Kx \\ -I_{xy}x + I_y y - I_{yz}z &= Ky \\ -I_{zx}x - I_{yz}y + I_z z &= Kz \end{aligned} \quad (9.52)$$

Dividing each term by the distance  $r$  from  $O$  to  $P$ , we obtain similar equations involving the direction cosines  $\lambda_x$ ,  $\lambda_y$ , and  $\lambda_z$ :

$$\begin{aligned} I_x \lambda_x - I_{xy} \lambda_y - I_{zx} \lambda_z &= K \lambda_x \\ -I_{xy} \lambda_x + I_y \lambda_y - I_{yz} \lambda_z &= K \lambda_y \\ -I_{zx} \lambda_x - I_{yz} \lambda_y + I_z \lambda_z &= K \lambda_z \end{aligned} \quad (9.53)$$

Transposing the right-hand members leads to the following homogeneous linear equations:

$$\begin{aligned} (I_x - K) \lambda_x - I_{xy} \lambda_y - I_{zx} \lambda_z &= 0 \\ -I_{xy} \lambda_x + (I_y - K) \lambda_y - I_{yz} \lambda_z &= 0 \\ -I_{zx} \lambda_x - I_{yz} \lambda_y + (I_z - K) \lambda_z &= 0 \end{aligned} \quad (9.54)$$

For this system of equations to have a solution different from  $\lambda_x = \lambda_y = \lambda_z = 0$ , its discriminant must be zero:

$$\begin{vmatrix} I_x - K & -I_{xy} & -I_{zx} \\ -I_{xy} & I_y - K & -I_{yz} \\ -I_{zx} & -I_{yz} & I_z - K \end{vmatrix} = 0 \quad (9.55)$$

Expanding this determinant and changing signs, we write

$$K^3 - (I_x + I_y + I_z)K^2 + (I_x I_y + I_y I_z + I_z I_x - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)K - (I_x I_y I_z - I_x I_{yz}^2 - I_y I_{zx}^2 - I_z I_{xy}^2 - 2I_{xy} I_{yz} I_{zx}) = 0 \quad (9.56)$$

This is a cubic equation in  $K$ , which yields three real, positive roots  $K_1$ ,  $K_2$ , and  $K_3$ .

To obtain the direction cosines of the principal axis corresponding to the root  $K_1$  we substitute  $K_1$  for  $K$  in Eqs. (9.54). Since these equations are now linearly dependent, only two of them may be used to determine  $\lambda_x$ ,  $\lambda_y$ , and  $\lambda_z$ . An additional equation may be obtained, however, by recalling from Sec. 2.12 that the direction cosines must satisfy the relation

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \quad (9.57)$$

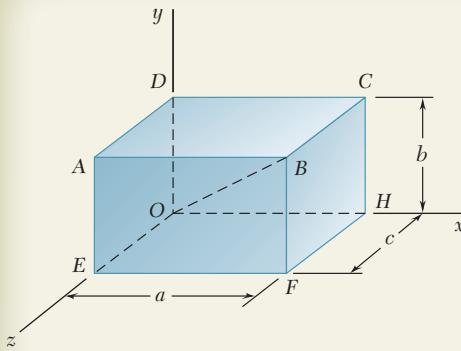
Repeating this procedure with  $K_2$  and  $K_3$ , we obtain the direction cosines of the other two principal axes.

We will now show that the roots  $K_1$ ,  $K_2$ , and  $K_3$  of Eq. (9.56) are the principal moments of inertia of the given body. Let us substitute for  $K$  in Eqs. (9.53) the root  $K_1$ , and for  $\lambda_x$ ,  $\lambda_y$ , and  $\lambda_z$  the corresponding values  $(\lambda_x)_1$ ,  $(\lambda_y)_1$ , and  $(\lambda_z)_1$  of the direction cosines; the three equations will be satisfied. We now multiply by  $(\lambda_x)_1$ ,  $(\lambda_y)_1$ , and  $(\lambda_z)_1$ , respectively, each term in the first, second, and third equation and add the equations obtained in this way. We write

$$\begin{aligned} I_x^2 (\lambda_x)_1^2 + I_y^2 (\lambda_y)_1^2 + I_z^2 (\lambda_z)_1^2 - 2I_{xy} (\lambda_x)_1 (\lambda_y)_1 \\ - 2I_{yz} (\lambda_y)_1 (\lambda_z)_1 - 2I_{zx} (\lambda_z)_1 (\lambda_x)_1 = K_1 [(\lambda_x)_1^2 + (\lambda_y)_1^2 + (\lambda_z)_1^2] \end{aligned}$$

Recalling Eq. (9.46), we observe that the left-hand member of this equation represents the moment of inertia of the body with respect to the principal axis corresponding to  $K_1$ ; it is thus the principal moment of inertia corresponding to that root. On the other hand, recalling Eq. (9.57), we note that the right-hand member reduces to  $K_1$ . Thus  $K_1$  itself is the principal moment of inertia. We can show in the same fashion that  $K_2$  and  $K_3$  are the other two principal moments of inertia of the body.

## SAMPLE PROBLEM 9.14



Consider a rectangular prism of mass  $m$  and sides  $a, b, c$ . Determine (a) the moments and products of inertia of the prism with respect to the coordinate axes shown, (b) its moment of inertia with respect to the diagonal  $OB$ .

### SOLUTION

**a. Moments and Products of Inertia with Respect to the Coordinate Axes.** **Moments of Inertia.** Introducing the centroidal axes  $x', y', z'$ , with respect to which the moments of inertia are given in Fig. 9.28, we apply the parallel-axis theorem:

$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2) = \frac{1}{12}m(b^2 + c^2) + m(\frac{1}{4}b^2 + \frac{1}{4}c^2)$$

$$I_x = \frac{1}{3}m(b^2 + c^2) \quad \blacksquare$$

Similarly,

$$I_y = \frac{1}{3}m(c^2 + a^2) \quad I_z = \frac{1}{3}m(a^2 + b^2) \quad \blacksquare$$

**Products of Inertia.** Because of symmetry, the products of inertia with respect to the centroidal axes  $x', y', z'$  are zero, and these axes are principal axes of inertia. Using the parallel-axis theorem, we have

$$I_{xy} = \bar{I}_{x'y'} + m\bar{x}\bar{y} = 0 + m(\frac{1}{2}a)(\frac{1}{2}b) \quad I_{xy} = \frac{1}{4}mab \quad \blacksquare$$

Similarly,

$$I_{yz} = \frac{1}{4}mbc \quad I_{zx} = \frac{1}{4}mca \quad \blacksquare$$

**b. Moment of Inertia with Respect to  $OB$ .** We recall Eq. (9.46):

$$I_{OB} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

where the direction cosines of  $OB$  are

$$\lambda_x = \cos \theta_x = \frac{OH}{OB} = \frac{a}{(a^2 + b^2 + c^2)^{1/2}}$$

$$\lambda_y = \frac{b}{(a^2 + b^2 + c^2)^{1/2}} \quad \lambda_z = \frac{c}{(a^2 + b^2 + c^2)^{1/2}}$$

Substituting the values obtained for the moments and products of inertia and for the direction cosines into the equation for  $I_{OB}$ , we have

$$I_{OB} = \frac{1}{a^2 + b^2 + c^2} \left[ \frac{1}{3}m(b^2 + c^2)a^2 + \frac{1}{3}m(c^2 + a^2)b^2 + \frac{1}{3}m(a^2 + b^2)c^2 - \frac{1}{2}ma^2b^2 - \frac{1}{2}mb^2c^2 - \frac{1}{2}mc^2a^2 \right]$$

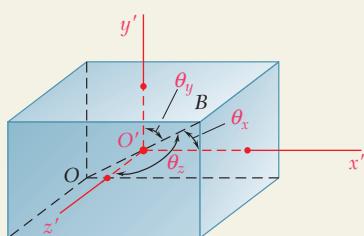
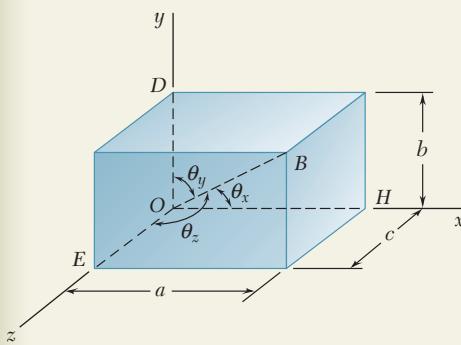
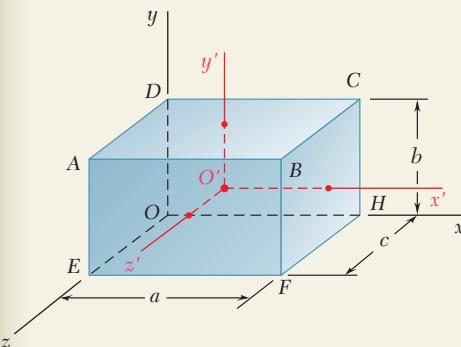
$$I_{OB} = \frac{m a^2b^2 + b^2c^2 + c^2a^2}{6 a^2 + b^2 + c^2} \quad \blacksquare$$

**Alternative Solution.** The moment of inertia  $I_{OB}$  can be obtained directly from the principal moments of inertia  $\bar{I}_x, \bar{I}_y, \bar{I}_z$ , since the line  $OB$  passes through the centroid  $O'$ . Since the  $x', y', z'$  axes are principal axes of inertia, we use Eq. (9.50) to write

$$I_{OB} = \bar{I}_x\lambda_x^2 + \bar{I}_y\lambda_y^2 + \bar{I}_z\lambda_z^2$$

$$= \frac{1}{a^2 + b^2 + c^2} \left[ \frac{m}{12}(b^2 + c^2)a^2 + \frac{m}{12}(c^2 + a^2)b^2 + \frac{m}{12}(a^2 + b^2)c^2 \right]$$

$$I_{OB} = \frac{m a^2b^2 + b^2c^2 + c^2a^2}{6 a^2 + b^2 + c^2} \quad \blacksquare$$



## SAMPLE PROBLEM 9.15

If  $a = 3c$  and  $b = 2c$  for the rectangular prism of Sample Prob. 9.14, determine (a) the principal moments of inertia at the origin  $O$ , (b) the principal axes of inertia at  $O$ .

### SOLUTION

**a. Principal Moments of Inertia at the Origin  $O$ .** Substituting  $a = 3c$  and  $b = 2c$  into the solution to Sample Prob. 9.14, we have

$$\begin{aligned} I_x &= \frac{5}{3}mc^2 & I_y &= \frac{10}{3}mc^2 & I_z &= \frac{13}{3}mc^2 \\ I_{xy} &= \frac{3}{2}mc^2 & I_{yz} &= \frac{1}{2}mc^2 & I_{zx} &= \frac{3}{4}mc^2 \end{aligned}$$

Substituting the values of the moments and products of inertia into Eq. (9.56) and collecting terms yields

$$K^3 - \left(\frac{28}{3}mc^2\right)K^2 + \left(\frac{3479}{144}m^2c^4\right)K - \frac{589}{54}m^3c^6 = 0$$

We then solve for the roots of this equation; from the discussion in Sec. 9.18, it follows that these roots are the principal moments of inertia of the body at the origin.

$$\begin{aligned} K_1 &= 0.568867mc^2 & K_2 &= 4.20885mc^2 & K_3 &= 4.55562mc^2 \\ K_1 &= 0.569mc^2 & K_2 &= 4.21mc^2 & K_3 &= 4.56mc^2 \end{aligned}$$

**b. Principal Axes of Inertia at  $O$ .** To determine the direction of a principal axis of inertia, we first substitute the corresponding value of  $K$  into two of the equations (9.54); the resulting equations together with Eq. (9.57) constitute a system of three equations from which the direction cosines of the corresponding principal axis can be determined. Thus, we have for the first principal moment of inertia  $K_1$ :

$$\begin{aligned} \left(\frac{5}{3} - 0.568867\right)mc^2(\lambda_x)_1 - \frac{3}{2}mc^2(\lambda_y)_1 - \frac{3}{4}mc^2(\lambda_z)_1 &= 0 \\ -\frac{3}{2}mc^2(\lambda_x)_1 + \left(\frac{10}{3} - 0.568867\right)mc^2(\lambda_y)_1 - \frac{1}{2}mc^2(\lambda_z)_1 &= 0 \\ (\lambda_x)_1^2 + (\lambda_y)_1^2 + (\lambda_z)_1^2 &= 1 \end{aligned}$$

Solving yields

$$(\lambda_x)_1 = 0.836600 \quad (\lambda_y)_1 = 0.496001 \quad (\lambda_z)_1 = 0.232557$$

The angles that the first principal axis of inertia forms with the coordinate axes are then

$$(\theta_x)_1 = 33.2^\circ \quad (\theta_y)_1 = 60.3^\circ \quad (\theta_z)_1 = 76.6^\circ$$

Using the same set of equations successively with  $K_2$  and  $K_3$ , we find that the angles associated with the second and third principal moments of inertia at the origin are, respectively,

$$(\theta_x)_2 = 57.8^\circ \quad (\theta_y)_2 = 146.6^\circ \quad (\theta_z)_2 = 98.0^\circ$$

and

$$(\theta_x)_3 = 82.8^\circ \quad (\theta_y)_3 = 76.1^\circ \quad (\theta_z)_3 = 164.3^\circ$$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson we defined the *mass products of inertia*  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of a body and showed you how to determine the moments of inertia of that body with respect to an arbitrary axis passing through the origin  $O$ . You also learned how to determine at the origin  $O$  the *principal axes of inertia* of a body and the corresponding *principal moments of inertia*.

**1. Determining the mass products of inertia of a composite body.** The mass products of inertia of a composite body with respect to the coordinate axes can be expressed as the sums of the products of inertia of its component parts with respect to those axes. For each component part, we can use the parallel-axis theorem and write Eqs. (9.47)

$$I_{xy} = \bar{I}_{x'y'} + m\bar{x}\bar{y} \quad I_{yz} = \bar{I}_{y'z'} + m\bar{y}\bar{z} \quad I_{zx} = \bar{I}_{z'x'} + m\bar{z}\bar{x}$$

where the primes denote the centroidal axes of each component part and where  $\bar{x}$ ,  $\bar{y}$ , and  $\bar{z}$  represent the coordinates of its center of gravity. Keep in mind that the mass products of inertia can be positive, negative, or zero, and be sure to take into account the signs of  $\bar{x}$ ,  $\bar{y}$ , and  $\bar{z}$ .

**a. From the properties of symmetry of a component part,** you can deduce that two or all three of its centroidal mass products of inertia are zero. For instance, you can verify that for a thin plate parallel to the  $xy$  plane; a wire lying in a plane parallel to the  $xy$  plane; a body with a plane of symmetry parallel to the  $xy$  plane; and a body with an axis of symmetry parallel to the  $z$  axis, *the products of inertia  $\bar{I}_{y'z'}$  and  $\bar{I}_{z'x'}$  are zero*.

For rectangular, circular, or semicircular plates with axes of symmetry parallel to the coordinate axes; straight wires parallel to a coordinate axis; circular and semicircular wires with axes of symmetry parallel to the coordinate axes; and rectangular prisms with axes of symmetry parallel to the coordinate axes, *the products of inertia  $\bar{I}_{x'y'}$ ,  $\bar{I}_{y'z'}$ , and  $\bar{I}_{z'x'}$  are all zero*.

**b. Mass products of inertia which are different from zero** can be computed from Eqs. (9.45). Although, in general, a triple integration is required to determine a mass product of inertia, a single integration can be used if the given body can be divided into a series of thin, parallel slabs. The computations are then similar to those discussed in the previous lesson for moments of inertia.

(continued)

**2. Computing the moment of inertia of a body with respect to an arbitrary axis  $OL$ .** An expression for the moment of inertia  $I_{OL}$  was derived in Sec. 9.16 and is given in Eq. (9.46). Before computing  $I_{OL}$ , you must first determine the mass moments and products of inertia of the body with respect to the given coordinate axes as well as the direction cosines of the unit vector  $\lambda$  along  $OL$ .

**3. Calculating the principal moments of inertia of a body and determining its principal axes of inertia.** You saw in Sec. 9.17 that it is always possible to find an orientation of the coordinate axes for which the mass products of inertia are zero. These axes are referred to as the *principal axes of inertia* and the corresponding moments of inertia are known as the *principal moments of inertia* of the body. In many cases, the principal axes of inertia of a body can be determined from its properties of symmetry. The procedure required to determine the principal moments and principal axes of inertia of a body with no obvious property of symmetry was discussed in Sec. 9.18 and was illustrated in Sample Prob. 9.15. It consists of the following steps.

**a. Expand the determinant in Eq. (9.55) and solve the resulting cubic equation.** The solution can be obtained by trial and error or, preferably, with an advanced scientific calculator or with the appropriate computer software. The roots  $K_1$ ,  $K_2$ , and  $K_3$  of this equation are the principal moments of inertia of the body.

**b. To determine the direction of the principal axis corresponding to  $K_1$ ,** substitute this value for  $K$  in two of the equations (9.54) and solve these equations together with Eq. (9.57) for the direction cosines of the principal axis corresponding to  $K_1$ .

**c. Repeat this procedure with  $K_2$  and  $K_3$**  to determine the directions of the other two principal axes. As a check of your computations, you may wish to verify that the scalar product of any two of the unit vectors along the three axes you have obtained is zero and, thus, that these axes are perpendicular to each other.

# PROBLEMS

- 9.149** Determine the mass products of inertia  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of the steel fixture shown. (The density of steel is  $7850 \text{ kg/m}^3$ .)

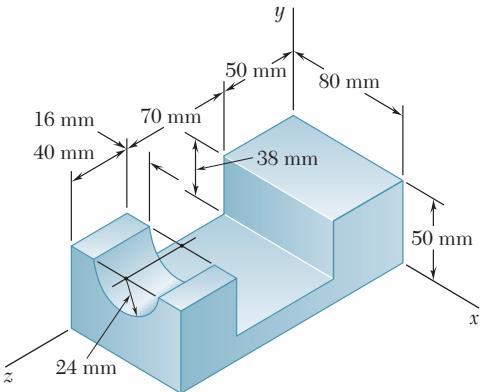


Fig. P9.149

- 9.150** Determine the mass products of inertia  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of the steel machine element shown. (The density of steel is  $7850 \text{ kg/m}^3$ .)

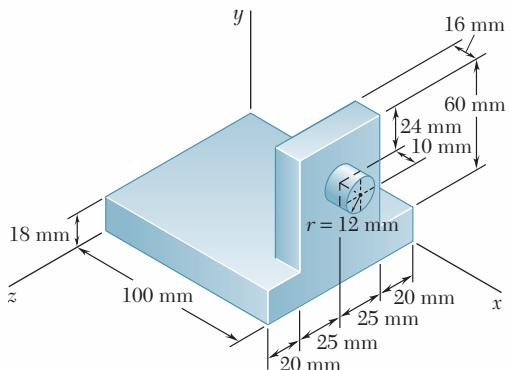


Fig. P9.150

- 9.151 and 9.152** Determine the mass products of inertia  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of the cast aluminum machine component shown. (The specific weight of aluminum is  $0.100 \text{ lb/in.}^3$ )

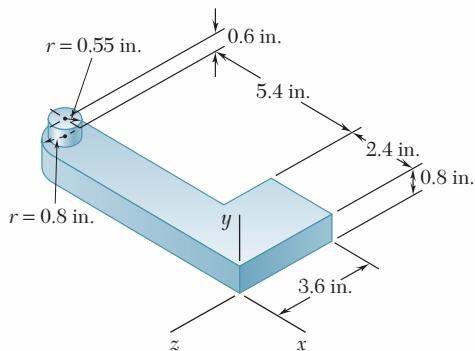


Fig. P9.151

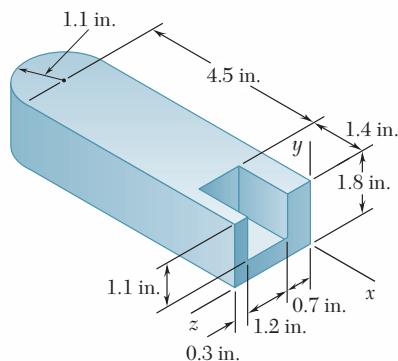


Fig. P9.152

**9.153 through 9.156** A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is  $7850 \text{ kg/m}^3$ , determine the mass products of inertia  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of the component.

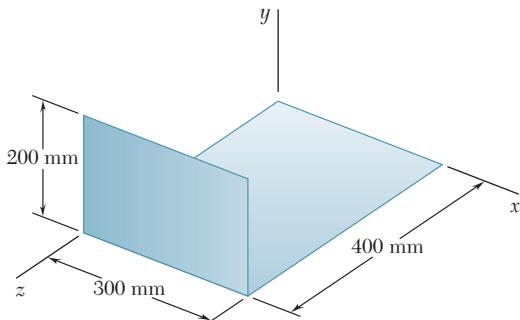


Fig. P9.153

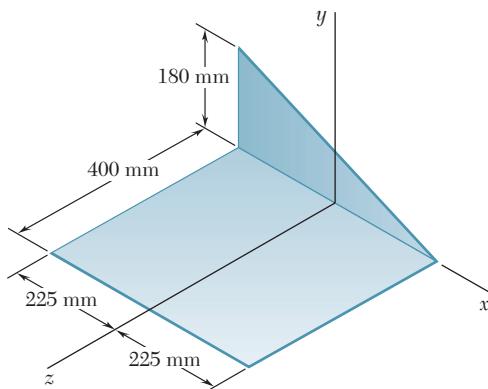


Fig. P9.154

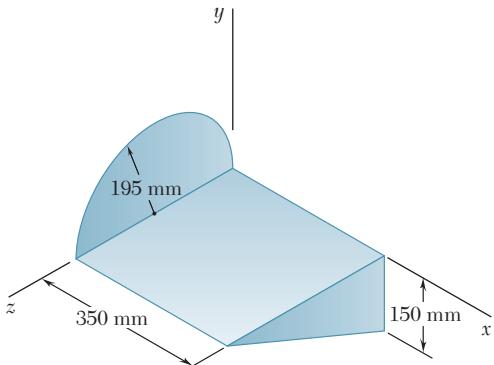


Fig. P9.155

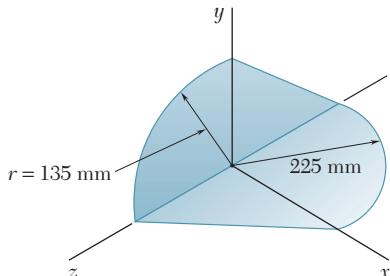


Fig. P9.156

**9.157 and 9.158** Brass wire with a weight per unit length  $w$  is used to form the figure shown. Determine the mass products of inertia  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of the wire figure.

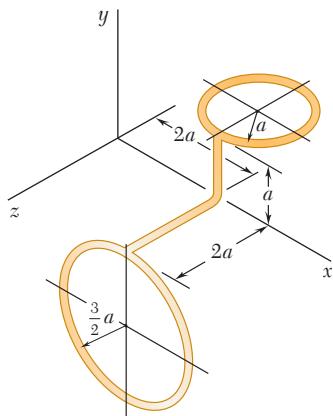


Fig. P9.157

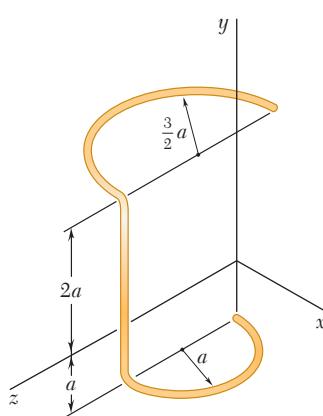


Fig. P9.158

- 9.159** The figure shown is formed of 1.5-mm-diameter aluminum wire. Knowing that the density of aluminum is  $2800 \text{ kg/m}^3$ , determine the mass products of inertia  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of the wire figure.

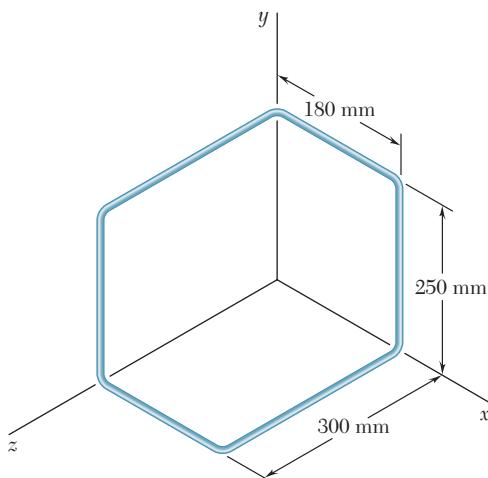


Fig. P9.159

- 9.160** Thin aluminum wire of uniform diameter is used to form the figure shown. Denoting by  $m'$  the mass per unit length of the wire, determine the mass products of inertia  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$  of the wire figure.

- 9.161** Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

- 9.162** For the homogeneous tetrahedron of mass  $m$  shown, (a) determine by direct integration the mass product of inertia  $I_{zx}$ , (b) deduce  $I_{yz}$  and  $I_{xy}$  from the result obtained in part a.

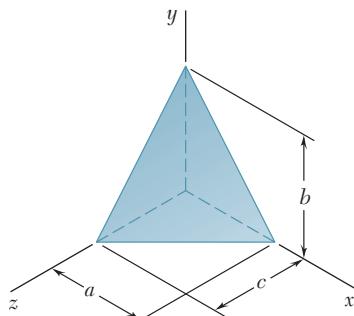


Fig. P9.162

- 9.163** The homogeneous circular cylinder shown has a mass  $m$ . Determine the mass moment of inertia of the cylinder with respect to the line joining the origin  $O$  and point  $A$  that is located on the perimeter of the top surface of the cylinder.

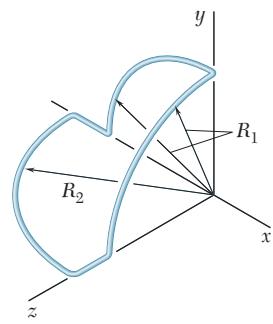


Fig. P9.160

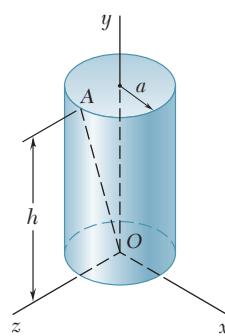
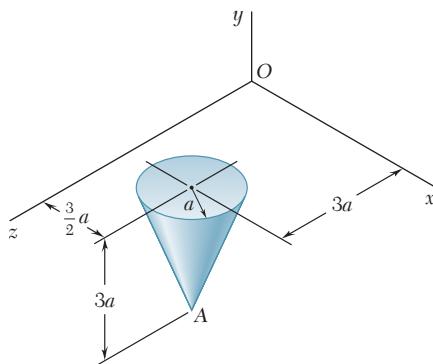
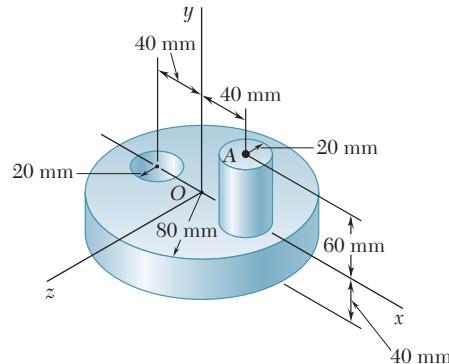


Fig. P9.163

- 9.164** The homogeneous circular cone shown has a mass  $m$ . Determine the mass moment of inertia of the cone with respect to the line joining the origin  $O$  and point  $A$ .

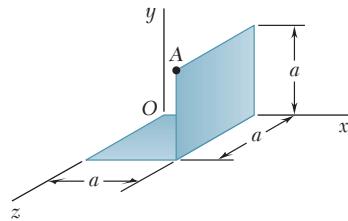
**Fig. P9.164**

- 9.165** Shown is the machine element of Prob. 9.141. Determine its mass moment of inertia with respect to the line joining the origin  $O$  and point  $A$ .

**Fig. P9.165**

- 9.166** Determine the mass moment of inertia of the steel fixture of Probs. 9.145 and 9.149 with respect to the axis through the origin that forms equal angles with the  $x$ ,  $y$ , and  $z$  axes.

- 9.167** The thin bent plate shown is of uniform density and weight  $W$ . Determine its mass moment of inertia with respect to the line joining the origin  $O$  and point  $A$ .

**Fig. P9.167**

- 9.168** A piece of sheet steel of thickness  $t$  and specific weight  $\gamma$  is cut and bent into the machine component shown. Determine the mass moment of inertia of the component with respect to the line joining the origin  $O$  and point  $A$ .

- 9.169** Determine the mass moment of inertia of the machine component of Probs. 9.136 and 9.155 with respect to the axis through the origin characterized by the unit vector  $\lambda = (-4\mathbf{i} + 8\mathbf{j} + \mathbf{k})/9$ .

- 9.170 through 9.172** For the wire figure of the problem indicated, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector  $\lambda = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$ .

**9.170** Prob. 9.148

**9.171** Prob. 9.147

**9.172** Prob. 9.146

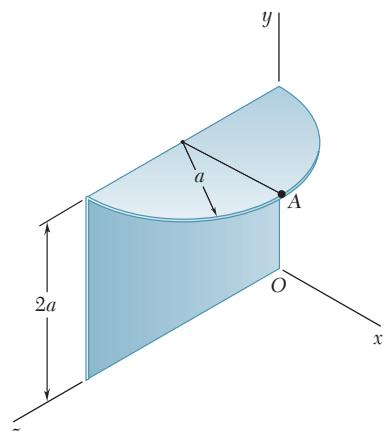


Fig. P9.168

- 9.173** For the rectangular prism shown, determine the values of the ratios  $b/a$  and  $c/a$  so that the ellipsoid of inertia of the prism is a sphere when computed (a) at point  $A$ , (b) at point  $B$ .

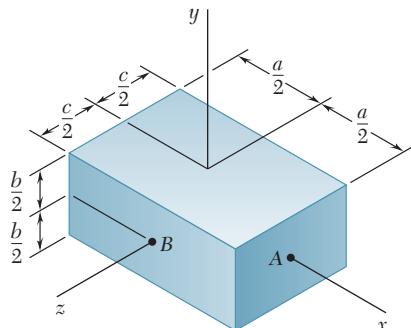


Fig. P9.173

- 9.174** For the right circular cone of Sample Prob. 9.11, determine the value of the ratio  $a/h$  for which the ellipsoid of inertia of the cone is a sphere when computed (a) at the apex of the cone, (b) at the center of the base of the cone.

- 9.175** For the homogeneous circular cylinder shown, of radius  $a$  and length  $L$ , determine the value of the ratio  $a/L$  for which the ellipsoid of inertia of the cylinder is a sphere when computed (a) at the centroid of the cylinder, (b) at point  $A$ .

- 9.176** Given an arbitrary body and three rectangular axes  $x$ ,  $y$ , and  $z$ , prove that the mass moment of inertia of the body with respect to any one of the three axes cannot be larger than the sum of the mass moments of inertia of the body with respect to the other two axes. That is, prove that the inequality  $I_x \leq I_y + I_z$  and the two similar inequalities are satisfied. Further, prove that  $I_y \geq \frac{1}{2}I_x$  if the body is a homogeneous solid of revolution, where  $x$  is the axis of revolution and  $y$  is a transverse axis.

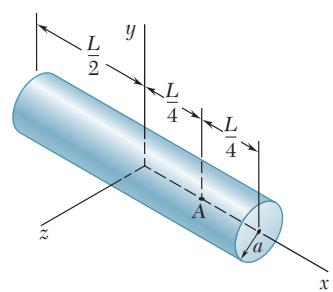


Fig. P9.175

**9.177** Consider a cube of mass  $m$  and side  $a$ . (a) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the moment of inertia of the cube with respect to one of its diagonals. (b) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

**9.178** Given a homogeneous body of mass  $m$  and of arbitrary shape and three rectangular axes  $x$ ,  $y$ , and  $z$  with origin at  $O$ , prove that the sum  $I_x + I_y + I_z$  of the mass moments of inertia of the body cannot be smaller than the similar sum computed for a sphere of the same mass and the same material centered at  $O$ . Further, using the result of Prob. 9.176, prove that if the body is a solid of revolution, where  $x$  is the axis of revolution, its mass moment of inertia  $I_y$  about a transverse axis  $y$  cannot be smaller than  $3ma^2/10$ , where  $a$  is the radius of the sphere of the same mass and the same material.

**\*9.179** The homogeneous circular cylinder shown has a mass  $m$ , and the diameter  $OB$  of its top surface forms  $45^\circ$  angles with the  $x$  and  $z$  axes. (a) Determine the principal mass moments of inertia of the cylinder at the origin  $O$ . (b) Compute the angles that the principal axes of inertia at  $O$  form with the coordinate axes. (c) Sketch the cylinder, and show the orientation of the principal axes of inertia relative to the  $x$ ,  $y$ , and  $z$  axes.

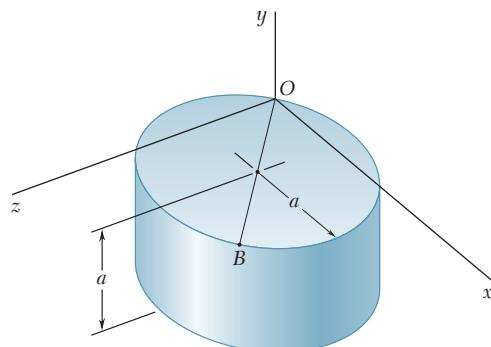


Fig. P9.179

**9.180 through 9.184** For the component described in the problem indicated, determine (a) the principal mass moments of inertia at the origin, (b) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the  $x$ ,  $y$ , and  $z$  axes.

**\*9.180** Prob. 9.165

**\*9.181** Probs. 9.145 and 9.149

**\*9.182** Prob. 9.167

**\*9.183** Prob. 9.168

**\*9.184** Probs. 9.148 and 9.170

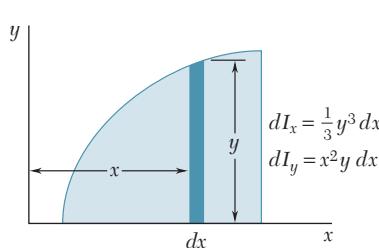
# REVIEW AND SUMMARY

In the first half of this chapter, we discussed the determination of the resultant  $\mathbf{R}$  of forces  $\Delta\mathbf{F}$  distributed over a plane area  $A$  when the magnitudes of these forces are proportional to both the areas  $\Delta A$  of the elements on which they act and the distances  $y$  from these elements to a given  $x$  axis; we thus had  $\Delta F = ky \Delta A$ . We found that the magnitude of the resultant  $\mathbf{R}$  is proportional to the first moment  $Q_x = \int y dA$  of the area  $A$ , while the moment of  $\mathbf{R}$  about the  $x$  axis is proportional to the *second moment*, or *moment of inertia*,  $I_x = \int y^2 dA$  of  $A$  with respect to the same axis [Sec. 9.2].

The *rectangular moments of inertia*  $I_x$  and  $I_y$  of an area [Sec. 9.3] were obtained by evaluating the integrals

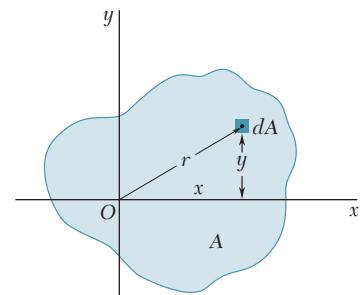
$$I_x = \int y^2 dA \quad I_y = \int x^2 dA \quad (9.1)$$

These computations can be reduced to single integrations by choosing  $dA$  to be a thin strip parallel to one of the coordinate axes. We also recall that it is possible to compute  $I_x$  and  $I_y$  from the same elemental strip (Fig. 9.35) using the formula for the moment of inertia of a rectangular area [Sample Prob. 9.3].



**Fig. 9.35**

## Rectangular moments of inertia



**Fig. 9.36**

The *polar moment of inertia* of an area  $A$  with respect to the pole  $O$  [Sec. 9.4] was defined as

$$J_O = \int r^2 dA \quad (9.3)$$

where  $r$  is the distance from  $O$  to the element of area  $dA$  (Fig. 9.36). Observing that  $r^2 = x^2 + y^2$ , we established the relation

$$J_O = I_x + I_y \quad (9.4)$$

## Polar moment of inertia

### Radius of gyration

The *radius of gyration of an area A with respect to the x axis* [Sec. 9.5] was defined as the distance  $k_x$ , where  $I_x = k_x^2 A$ . With similar definitions for the radii of gyration of A with respect to the y axis and with respect to O, we had

$$k_x = \sqrt{\frac{I_x}{A}} \quad k_y = \sqrt{\frac{I_y}{A}} \quad k_O = \sqrt{\frac{J_O}{A}} \quad (9.5-9.7)$$

### Parallel-axis theorem

The *parallel-axis theorem* was presented in Sec. 9.6. It states that the moment of inertia  $I$  of an area with respect to any given axis AA' (Fig. 9.37) is equal to the moment of inertia  $\bar{I}$  of the area with respect to the centroidal axis BB' that is parallel to AA' plus the product of the area A and the square of the distance  $d$  between the two axes:

$$I = \bar{I} + Ad^2 \quad (9.9)$$

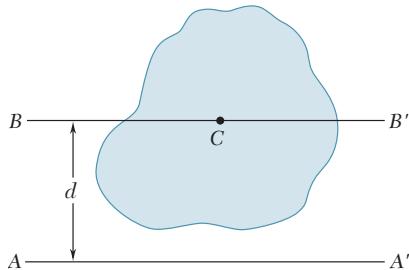


Fig. 9.37

This formula can also be used to determine the moment of inertia  $\bar{I}$  of an area with respect to a centroidal axis BB' when its moment of inertia  $I$  with respect to a parallel axis AA' is known. In this case, however, the product  $Ad^2$  should be *subtracted* from the known moment of inertia  $I$ .

A similar relation holds between the polar moment of inertia  $J_O$  of an area about a point O and the polar moment of inertia  $\bar{J}_C$  of the same area about its centroid C. Letting  $d$  be the distance between O and C, we have

$$J_O = \bar{J}_C + Ad^2 \quad (9.11)$$

### Composite areas

The parallel-axis theorem can be used very effectively to compute the *moment of inertia of a composite area* with respect to a given axis [Sec. 9.7]. Considering each component area separately, we first compute the moment of inertia of each area with respect to its centroidal axis, using the data provided in Figs. 9.12 and 9.13 whenever possible. The parallel-axis theorem is then applied to determine the moment of inertia of each component area with respect to the desired axis, and the various values obtained are added [Sample Probs. 9.4 and 9.5].

### Product of inertia

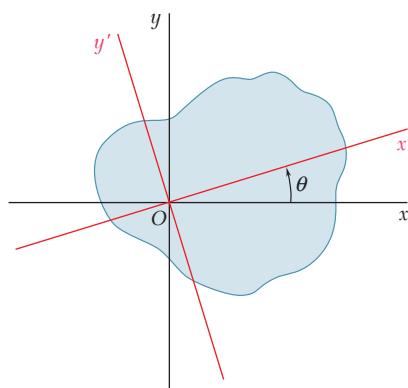
Sections 9.8 through 9.10 were devoted to the transformation of the moments of inertia of an area *under a rotation of the coordinate axes*. First, we defined the *product of inertia of an area A* as

$$I_{xy} = \int xy \, dA \quad (9.12)$$

and showed that  $I_{xy} = 0$  if the area A is symmetrical with respect to either or both of the coordinate axes. We also derived the *parallel-axis theorem for products of inertia*. We had

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \quad (9.13)$$

where  $\bar{I}_{x'y'}$  is the product of inertia of the area with respect to the centroidal axes  $x'$  and  $y'$  which are parallel to the  $x$  and  $y$  axis and  $\bar{x}$  and  $\bar{y}$  are the coordinates of the centroid of the area [Sec. 9.8].

**Fig. 9.38**

In Sec. 9.9 we determined the moments and product of inertia  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  of an area with respect to  $x'$  and  $y'$  axes obtained by rotating the original  $x$  and  $y$  coordinate axes through an angle  $\theta$  counterclockwise (Fig. 9.38). We expressed  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  in terms of the moments and product of inertia  $I_x$ ,  $I_y$ , and  $I_{xy}$  computed with respect to the original  $x$  and  $y$  axes. We had

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad (9.18)$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad (9.19)$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \quad (9.20)$$

### Rotation of axes

The *principal axes of the area about O* were defined as the two axes perpendicular to each other, with respect to which the moments of inertia of the area are maximum and minimum. The corresponding values of  $\theta$ , denoted by  $\theta_m$ , were obtained from the formula

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} \quad (9.25)$$

The corresponding maximum and minimum values of  $I$  are called the *principal moments of inertia* of the area about  $O$ ; we had

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad (9.27)$$

### Principal axes

### Principal moments of inertia

We also noted that the corresponding value of the product of inertia is zero.

The transformation of the moments and product of inertia of an area under a rotation of axes can be represented graphically by drawing *Mohr's circle* [Sec. 9.10]. Given the moments and product of inertia  $I_x$ ,  $I_y$ , and  $I_{xy}$  of the area with respect to the  $x$  and  $y$  coordinate axes, we

### Mohr's circle

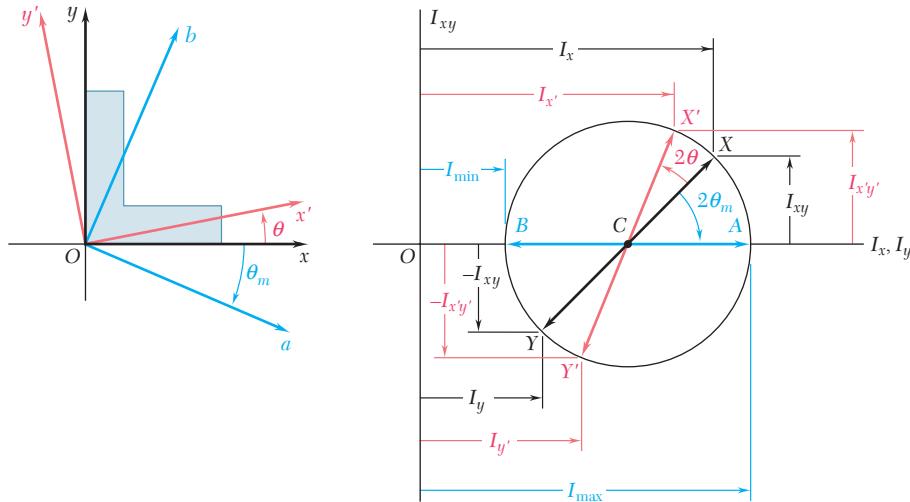


Fig. 9.39

plot points  $X$  ( $I_x, I_{xy}$ ) and  $Y$  ( $I_y, -I_{xy}$ ) and draw the line joining these two points (Fig. 9.39). This line is a diameter of Mohr's circle and thus defines this circle. As the coordinate axes are rotated through  $\theta$ , the diameter rotates through *twice that angle*, and the coordinates of  $X'$  and  $Y'$  yield the new values  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  of the moments and product of inertia of the area. Also, the angle  $\theta_m$  and the coordinates of points  $A$  and  $B$  define the principal axes  $a$  and  $b$  and the principal moments of inertia of the area [Sample Prob. 9.8].

### Moments of inertia of masses

The second half of the chapter was devoted to the determination of *moments of inertia of masses*, which are encountered in dynamics in problems involving the rotation of a rigid body about an axis. The mass moment of inertia of a body with respect to an axis  $AA'$  (Fig. 9.40) was defined as

$$I = \int r^2 dm \quad (9.28)$$

where  $r$  is the distance from  $AA'$  to the element of mass [Sec. 9.11]. The *radius of gyration* of the body was defined as

$$k = \sqrt{\frac{I}{m}} \quad (9.29)$$

The moments of inertia of a body with respect to the coordinate axes were expressed as

$$\begin{aligned} I_x &= \int (y^2 + z^2) dm \\ I_y &= \int (z^2 + x^2) dm \\ I_z &= \int (x^2 + y^2) dm \end{aligned} \quad (9.30)$$

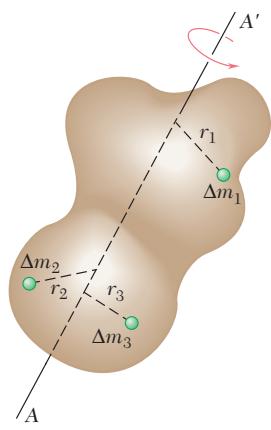
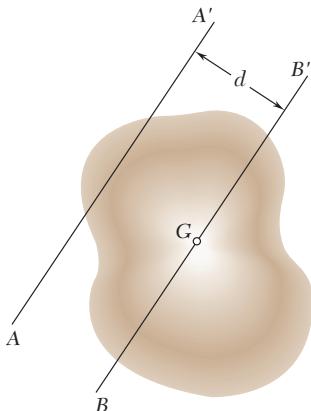


Fig. 9.40

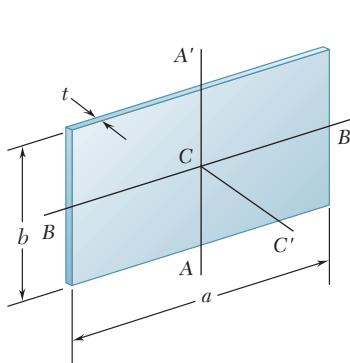
We saw that the *parallel-axis theorem* also applies to mass moments of inertia [Sec. 9.12]. Thus, the moment of inertia  $I$  of a body with respect to an arbitrary axis  $AA'$  (Fig. 9.41) can be expressed as

$$I = \bar{I} + md^2 \quad (9.33)$$

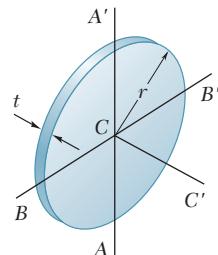
where  $\bar{I}$  is the moment of inertia of the body with respect to the centroidal axis  $BB'$  which is parallel to the axis  $AA'$ ,  $m$  is the mass of the body, and  $d$  is the distance between the two axes.



**Fig. 9.41**



**Fig. 9.42**



**Fig. 9.43**

The moments of inertia of *thin plates* can be readily obtained from the moments of inertia of their areas [Sec. 9.13]. We found that for a *rectangular plate* the moments of inertia with respect to the axes shown (Fig. 9.42) are

$$I_{AA'} = \frac{1}{12}ma^2 \quad I_{BB'} = \frac{1}{12}mb^2 \quad (9.39)$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{12}m(a^2 + b^2) \quad (9.40)$$

while for a *circular plate* (Fig. 9.43) they are

$$I_{AA'} = I_{BB'} = \frac{1}{4}mr^2 \quad (9.41)$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{2}mr^2 \quad (9.42)$$

When a body possesses *two planes of symmetry*, it is usually possible to use a single integration to determine its moment of inertia with respect to a given axis by selecting the element of mass  $dm$  to be a thin plate [Sample Probs. 9.10 and 9.11]. On the other hand, when a body consists of *several common geometric shapes*, its moment of inertia with respect to a given axis can be obtained by using the formulas given in Fig. 9.28 together with the parallel-axis theorem [Sample Probs. 9.12 and 9.13].

In the last portion of the chapter, we learned to determine the moment of inertia of a body *with respect to an arbitrary axis OL* which is drawn through the origin O [Sec. 9.16]. Denoting by  $\lambda_x, \lambda_y$ ,

### Moments of inertia of thin plates

### Composite bodies

### Moment of inertia with respect to an arbitrary axis

$\lambda_z$  the components of the unit vector  $\lambda$  along  $OL$  (Fig. 9.44) and introducing the *products of inertia*

$$I_{xy} = \int xy \, dm \quad I_{yz} = \int yz \, dm \quad I_{zx} = \int zx \, dm \quad (9.45)$$

we found that the moment of inertia of the body with respect to  $OL$  could be expressed as

$$I_{OL} = I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x \quad (9.46)$$

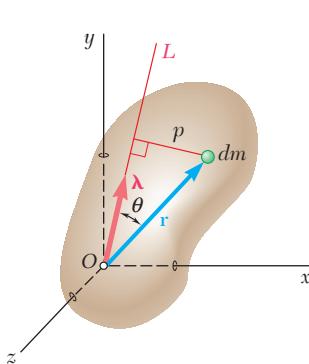


Fig. 9.44

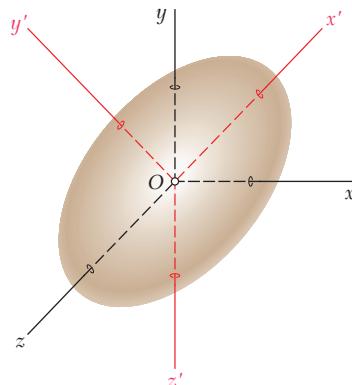


Fig. 9.45

### Ellipsoid of inertia

#### Principal axes of inertia Principal moments of inertia

By plotting a point  $Q$  along each axis  $OL$  at a distance  $OQ = 1/\sqrt{I_{OL}}$  from  $O$  [Sec. 9.17], we obtained the surface of an ellipsoid, known as the *ellipsoid of inertia* of the body at point  $O$ . The principal axes  $x'$ ,  $y'$ ,  $z'$  of this ellipsoid (Fig. 9.45) are the *principal axes of inertia* of the body; that is, the products of inertia  $I_{x'y'}$ ,  $I_{y'z'}$ ,  $I_{z'x'}$  of the body with respect to these axes are all zero. There are many situations when the principal axes of inertia of a body can be deduced from properties of symmetry of the body. Choosing these axes to be the coordinate axes, we can then express  $I_{OL}$  as

$$I_{OL} = I_{x'} \lambda_{x'}^2 + I_{y'} \lambda_{y'}^2 + I_{z'} \lambda_{z'}^2 \quad (9.50)$$

where  $I_{x'}$ ,  $I_{y'}$ ,  $I_{z'}$  are the *principal moments of inertia* of the body at  $O$ .

When the principal axes of inertia cannot be obtained by observation [Sec. 9.17], it is necessary to solve the cubic equation

$$K^3 - (I_x + I_y + I_z)K^2 + (I_x I_y + I_y I_z + I_z I_x - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)K - (I_x I_y I_z - I_x I_{yz}^2 - I_y I_{zx}^2 - I_z I_{xy}^2 - 2I_{xy} I_{yz} I_{zx}) = 0 \quad (9.56)$$

We found [Sec. 9.18] that the roots  $K_1$ ,  $K_2$ , and  $K_3$  of this equation are the principal moments of inertia of the given body. The direction cosines  $(\lambda_x)_1$ ,  $(\lambda_y)_1$ , and  $(\lambda_z)_1$  of the principal axis corresponding to the principal moment of inertia  $K_1$  are then determined by substituting  $K_1$  into Eqs. (9.54) and solving two of these equations and Eq. (9.57) simultaneously. The same procedure is then repeated using  $K_2$  and  $K_3$  to determine the direction cosines of the other two principal axes [Sample Prob. 9.15].

# REVIEW PROBLEMS

**9.185** Determine by direct integration the moments of inertia of the shaded area with respect to the  $x$  and  $y$  axes.

**9.186** Determine the moments of inertia and the radii of gyration of the shaded area shown with respect to the  $x$  and  $y$  axes.

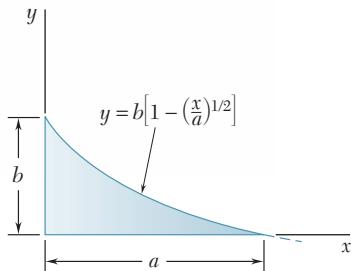


Fig. P9.186

**9.187** Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the  $y$  axis.

**9.188** Determine the moments of inertia of the shaded area shown with respect to the  $x$  and  $y$  axes.

**9.189** Determine the polar moment of inertia of the area shown with respect to (a) point  $O$ , (b) the centroid of the area.

**9.190** To form an unsymmetrical girder, two L76 × 76 × 6.4-mm angles and two L152 × 102 × 12.7-mm angles are welded to a 16-mm steel plate as shown. Determine the moments of inertia of the combined section with respect to its centroidal  $x$  and  $y$  axes.

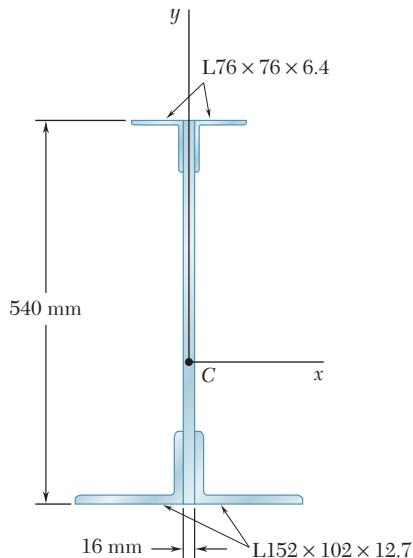


Fig. P9.190

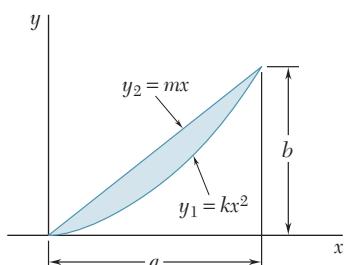


Fig. P9.185

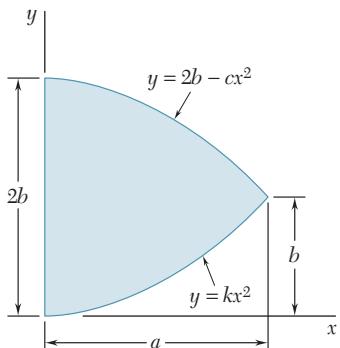


Fig. P9.187

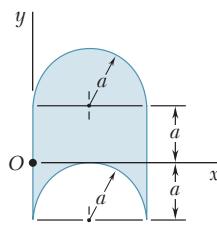


Fig. P9.188

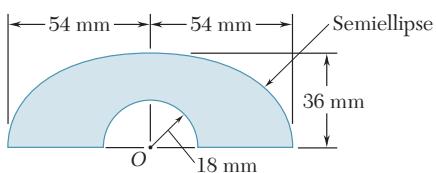


Fig. P9.189

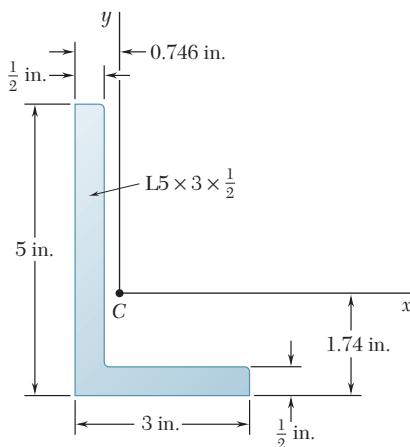


Fig. P9.191 and P9.192

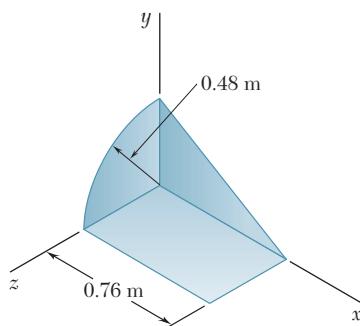


Fig. P9.195

- 9.191** Using the parallel-axis theorem, determine the product of inertia of the  $L5 \times 3 \times \frac{1}{2}$ -in. angle cross section shown with respect to the centroidal  $x$  and  $y$  axes.

- 9.192** For the  $L5 \times 3 \times \frac{1}{2}$ -in. angle cross section shown, use Mohr's circle to determine (a) the moments of inertia and the product of inertia with respect to new centroidal axes obtained by rotating the  $x$  and  $y$  axes  $30^\circ$  clockwise, (b) the orientation of the principal axes through the centroid and the corresponding values of the moments of inertia.

- 9.193** A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by  $m$ , determine its mass moment of inertia with respect to (a) the  $x$  axis, (b) the  $y$  axis.

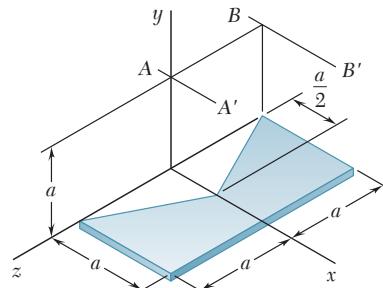


Fig. P9.193 and P9.194

- 9.194** A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by  $m$ , determine its mass moment of inertia with respect to (a) the axis  $AA'$ , (b) the axis  $BB'$ , where the  $AA'$  and  $BB'$  axes are parallel to the  $x$  axis and lie in a plane parallel to and at a distance  $a$  above the  $xz$  plane.

- 9.195** A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is  $7850 \text{ kg/m}^3$ , determine the mass moment of inertia of the component with respect to each of the coordinate axes.

- 9.196** Determine the mass moments of inertia and the radii of gyration of the steel machine element shown with respect to the  $x$  and  $y$  axes. (The density of steel is  $7850 \text{ kg/m}^3$ .)

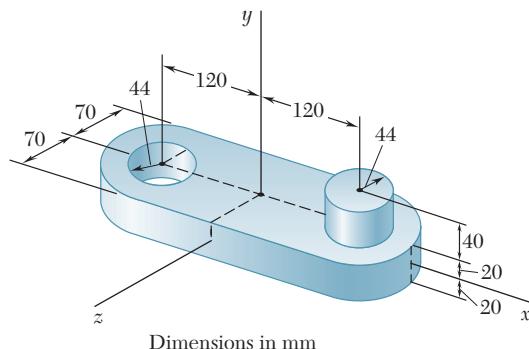


Fig. P9.196

# COMPUTER PROBLEMS

**9.C1** Use computational software that, for an area with known moments and product of inertia  $I_x$ ,  $I_y$ , and  $I_{xy}$ , can be used to calculate the moments and product of inertia  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  of the area with respect to axes  $x'$  and  $y'$  obtained by rotating the original axes counterclockwise through an angle  $\theta$ . Use this program to compute  $I_{x'}$ ,  $I_{y'}$ , and  $I_{x'y'}$  for the section of Sample Prob. 9.7 for values of  $\theta$  from 0 to  $90^\circ$  using  $5^\circ$  increments.

**9.C2** Use computational software that, for an area with known moments and product of inertia  $I_x$ ,  $I_y$ , and  $I_{xy}$ , can be used to calculate the orientation of the principal axes of the area and the corresponding values of the principal moments of inertia. Use this program to solve (a) Prob. 9.89, (b) Sample Prob. 9.7.

**9.C3** Many cross sections can be approximated by a series of rectangles as shown. Using computational software, calculate the moments of inertia and the radii of gyration of cross sections of this type with respect to horizontal and vertical centroidal axes. Apply this program to the cross sections shown in (a) Figs. P9.31 and P9.33, (b) Figs. P9.32 and P9.34, (c) Fig. P9.43, (d) Fig. P9.44.

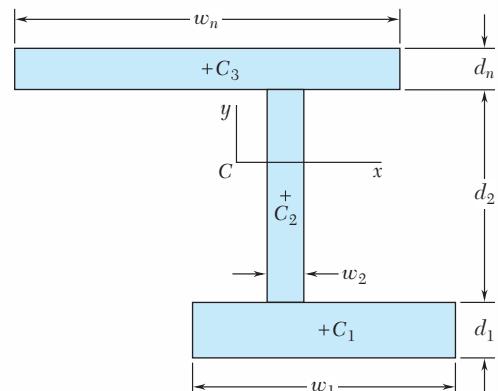
**9.C4** Many cross sections can be approximated by a series of rectangles as shown. Using computational software, calculate the products of inertia of cross sections of this type with respect to horizontal and vertical centroidal axes. Use this program to solve (a) Prob. P9.71, (b) Prob. P9.75, (c) Prob. 9.77.

**9.C5** The area shown is revolved about the  $x$  axis to form a homogeneous solid of mass  $m$ . Approximate the area using a series of 400 rectangles of the form  $bcc'b'$ , each of width  $\Delta l$ , and then use computational software to determine the mass moment of inertia of the solid with respect to the  $x$  axis. Use this program to solve part a of (a) Sample Prob. 9.11, (b) Prob. 9.121, assuming that in these problems  $m = 2 \text{ kg}$ ,  $a = 100 \text{ mm}$ , and  $h = 400 \text{ mm}$ .

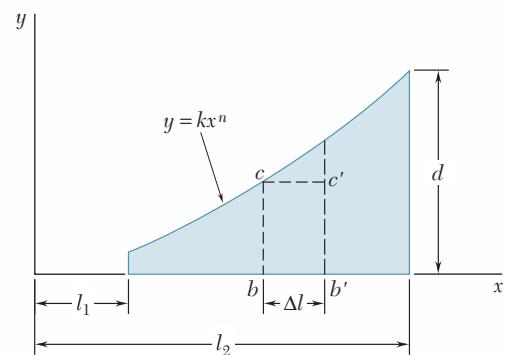
**9.C6** A homogeneous wire with a weight per unit length of 0.04 lb/ft is used to form the figure shown. Approximate the figure using 10 straight line segments, and then use computational software to determine the mass moment of inertia  $I_x$  of the wire with respect to the  $x$  axis. Use this program to determine  $I_x$  when (a)  $a = 1 \text{ in.}$ ,  $L = 11 \text{ in.}$ ,  $h = 4 \text{ in.}$ , (b)  $a = 2 \text{ in.}$ ,  $L = 17 \text{ in.}$ ,  $h = 10 \text{ in.}$ , (c)  $a = 5 \text{ in.}$ ,  $L = 25 \text{ in.}$ ,  $h = 6 \text{ in.}$

**\*9.C7** Use computational software that, for a body with known mass moments and products of inertia  $I_x$ ,  $I_y$ ,  $I_z$ ,  $I_{xy}$ ,  $I_{yz}$ , and  $I_{zx}$ , can be used to calculate the principal mass moments of inertia  $K_1$ ,  $K_2$ , and  $K_3$  of the body at the origin. Use this program to solve part a of (a) Prob. 9.180, (b) Prob. 9.181, (c) Prob. 9.184.

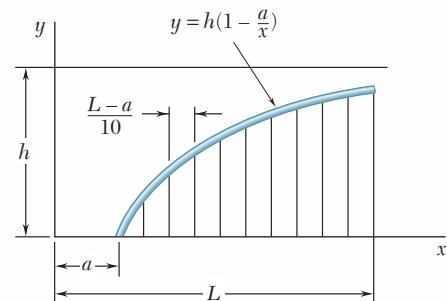
**\*9.C8** Using the software from Prob. 9.C7, include the computation of the angles that the principal axes of inertia at the origin form with the coordinate axes. Use this program to solve (a) Prob. 9.180, (b) Prob. 9.181, (c) Prob. 9.184.



**Fig. P9.C3 and P9.C4**



**Fig. P9.C5**



**Fig. P9.C6**

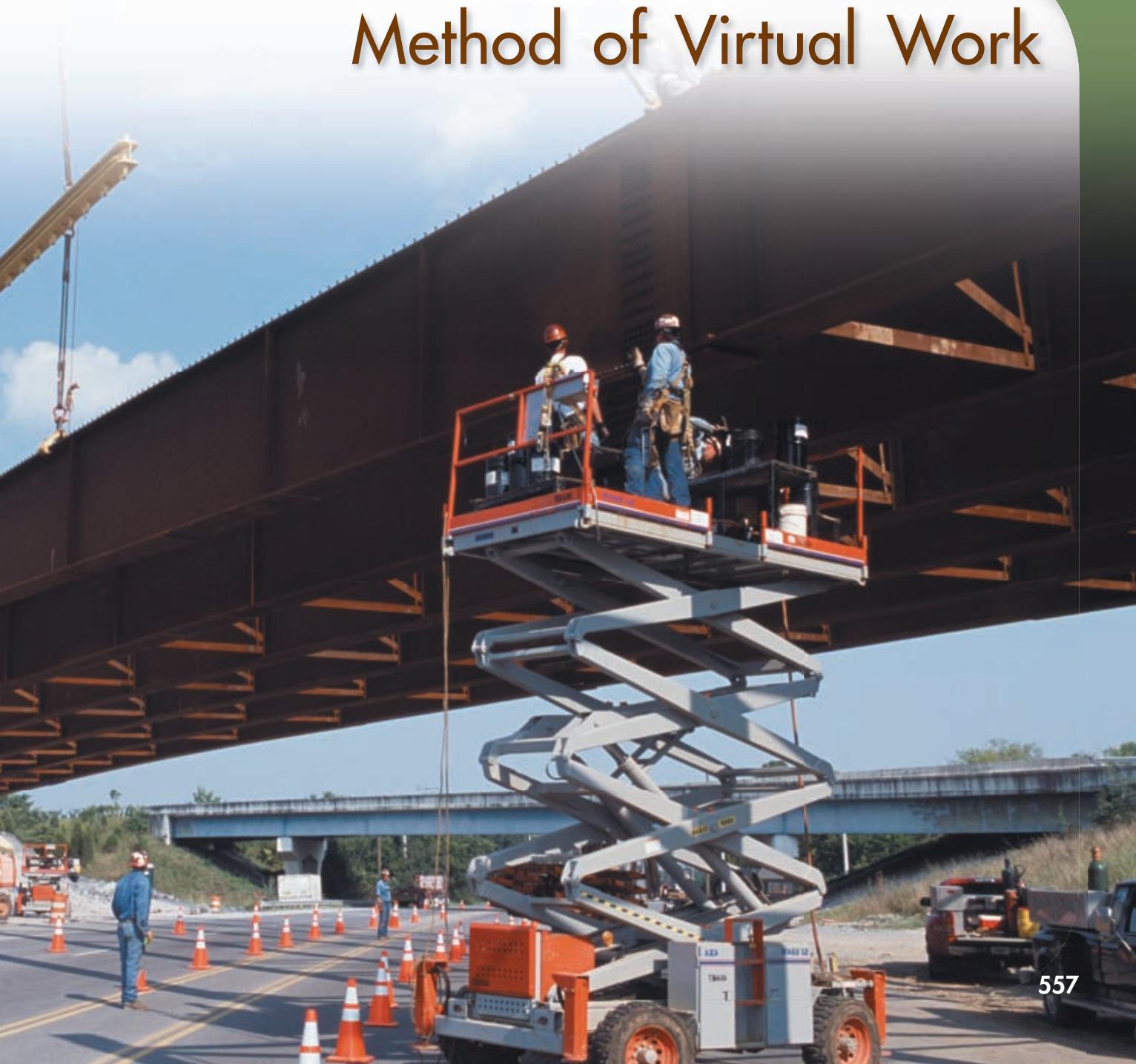
The method of virtual work is particularly effective when a simple relation can be found among the displacements of the points of application of the various forces involved. This is the case for the scissor lift platform being used by workers to gain access to a highway bridge under construction.



# 10

CHAPTER

## Method of Virtual Work



## Chapter 10 Method of Virtual Work

- 10.1 Introduction
- 10.2 Work of a Force
- 10.3 Principle of Virtual Work
- 10.4 Applications of the Principle of Virtual Work
- 10.5 Real Machines. Mechanical Efficiency
- 10.6 Work of a Force during a Finite Displacement
- 10.7 Potential Energy
- 10.8 Potential Energy and Equilibrium
- 10.9 Stability of Equilibrium

### \*10.1 INTRODUCTION

In the preceding chapters, problems involving the equilibrium of rigid bodies were solved by expressing that the external forces acting on the bodies were balanced. The equations of equilibrium  $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum M_A = 0$  were written and solved for the desired unknowns. A different method, which will prove more effective for solving certain types of equilibrium problems, will now be considered. This method is based on the *principle of virtual work* and was first formally used by the Swiss mathematician Jean Bernoulli in the eighteenth century.

As you will see in Sec. 10.3, the principle of virtual work states that if a particle or rigid body, or, more generally, a system of connected rigid bodies, which is in equilibrium under various external forces, is given an arbitrary displacement from that position of equilibrium, the total work done by the external forces during the displacement is zero. This principle is particularly effective when applied to the solution of problems involving the equilibrium of machines or mechanisms consisting of several connected members.

In the second part of the chapter, the method of virtual work will be applied in an alternative form based on the concept of *potential energy*. It will be shown in Sec. 10.8 that if a particle, rigid body, or system of rigid bodies is in equilibrium, then the derivative of its potential energy with respect to a variable defining its position must be zero.

In this chapter, you will also learn to evaluate the mechanical efficiency of a machine (Sec. 10.5) and to determine whether a given position of equilibrium is stable, unstable, or neutral (Sec. 10.9).

### \*10.2 WORK OF A FORCE

Let us first define the terms *displacement* and *work* as they are used in mechanics. Consider a particle which moves from a point  $A$  to a neighboring point  $A'$  (Fig. 10.1). If  $\mathbf{r}$  denotes the position vector corresponding to point  $A$ , the small vector joining  $A$  and  $A'$  may be denoted by the differential  $d\mathbf{r}$ ; the vector  $d\mathbf{r}$  is called the *displacement* of the particle. Now let us assume that a force  $\mathbf{F}$  is acting on the particle. The *work of the force  $\mathbf{F}$  corresponding to the displacement  $d\mathbf{r}$*  is defined as the quantity

$$dU = \mathbf{F} \cdot d\mathbf{r} \quad (10.1)$$

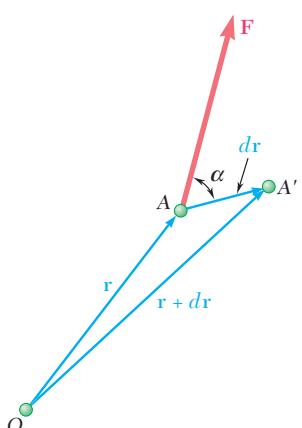


Fig. 10.1

obtained by forming the scalar product of the force  $\mathbf{F}$  and the displacement  $d\mathbf{r}$ . Denoting respectively by  $F$  and  $ds$  the magnitudes of the force and of the displacement, and by  $\alpha$  the angle formed by  $\mathbf{F}$  and  $d\mathbf{r}$ , and recalling the definition of the scalar product of two vectors (Sec. 3.9), we write

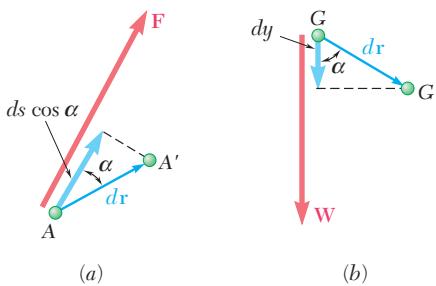
$$dU = F ds \cos \alpha \quad (10.1')$$

Being a *scalar quantity*, work has a magnitude and a sign, but no direction. We also note that work should be expressed in units obtained

by multiplying units of length by units of force. Thus, if U.S. customary units are used, work should be expressed in  $\text{ft} \cdot \text{lb}$  or  $\text{in} \cdot \text{lb}$ . If SI units are used, work should be expressed in  $\text{N} \cdot \text{m}$ . The unit of work  $\text{N} \cdot \text{m}$  is called a *joule* (J).†

It follows from (10.1') that the work  $dU$  is positive if the angle  $\alpha$  is acute and negative if  $\alpha$  is obtuse. Three particular cases are of special interest. If the force  $\mathbf{F}$  has the same direction as  $d\mathbf{r}$ , the work  $dU$  reduces to  $F ds$ . If  $\mathbf{F}$  has a direction opposite to that of  $d\mathbf{r}$ , the work is  $dU = -F ds$ . Finally, if  $\mathbf{F}$  is perpendicular to  $d\mathbf{r}$ , the work  $dU$  is zero.

The work  $dU$  of a force  $\mathbf{F}$  during a displacement  $d\mathbf{r}$  can also be considered as the product of  $F$  and the component  $ds \cos \alpha$  of the displacement  $d\mathbf{r}$  along  $\mathbf{F}$  (Fig. 10.2a). This view is particularly



**Fig. 10.2**

useful in the computation of the work done by the weight  $\mathbf{W}$  of a body (Fig. 10.2b). The work of  $\mathbf{W}$  is equal to the product of  $W$  and the vertical displacement  $dy$  of the center of gravity  $G$  of the body. If the displacement is downward, the work is positive; if it is upward, the work is negative.

A number of forces frequently encountered in statics *do no work*: forces applied to fixed points ( $ds = 0$ ) or acting in a direction perpendicular to the displacement ( $\cos \alpha = 0$ ). Among these forces are the reaction at a frictionless pin when the body supported rotates about the pin; the reaction at a frictionless surface when the body in contact moves along the surface; the reaction at a roller moving along its track; the weight of a body when its center of gravity moves horizontally; and the friction force acting on a wheel rolling without slipping (since at any instant the point of contact does not move). Examples of forces which *do work* are the weight of a body (except in the case considered above), the friction force acting on a body sliding on a rough surface, and most forces applied on a moving body.



**Photo 10.1** The forces exerted by the hydraulic cylinders to position the bucket lift shown can be effectively determined using the method of virtual work since a simple relation exists among the displacements of the points of application of the forces acting on the members of the lift.

†The joule is the SI unit of *energy*, whether in mechanical form (work, potential energy, kinetic energy) or in chemical, electrical, or thermal form. We should note that even though  $\text{N} \cdot \text{m} = \text{J}$ , the moment of a force must be expressed in  $\text{N} \cdot \text{m}$ , and not in joules, since the moment of a force is not a form of energy.

In certain cases, the sum of the work done by several forces is zero. Consider, for example, two rigid bodies  $AC$  and  $BC$  connected at  $C$  by a *frictionless pin* (Fig. 10.3a). Among the forces acting on  $AC$  is the force  $\mathbf{F}$  exerted at  $C$  by  $BC$ . In general, the work of this

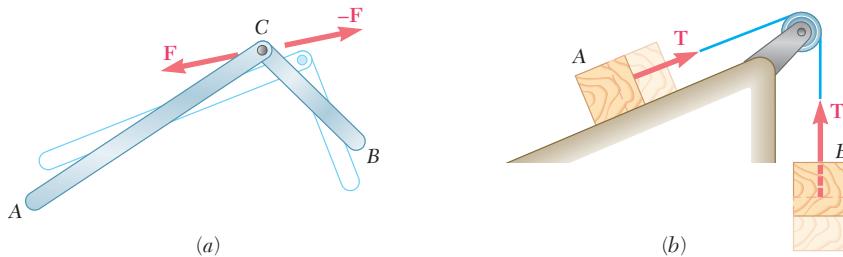


Fig. 10.3

force will not be zero, but it will be equal in magnitude and opposite in sign to the work of the force  $-\mathbf{F}$  exerted by  $AC$  on  $BC$ , since these forces are equal and opposite and are applied to the same particle. Thus, when the total work done by all the forces acting on  $AB$  and  $BC$  is considered, the work of the two internal forces at  $C$  cancels out. A similar result is obtained if we consider a system consisting of two blocks connected by an *inextensible cord*  $AB$  (Fig. 10.3b). The work of the tension force  $\mathbf{T}$  at  $A$  is equal in magnitude to the work of the tension force  $\mathbf{T}'$  at  $B$ , since these forces have the same magnitude and the points  $A$  and  $B$  move through the same distance; but in one case the work is positive, and in the other it is negative. Thus, the work of the internal forces again cancels out.

It can be shown that the total work of the internal forces holding together the particles of a rigid body is zero. Consider two particles  $A$  and  $B$  of a rigid body and the two equal and opposite forces  $\mathbf{F}$  and  $-\mathbf{F}$  they exert on each other (Fig. 10.4). While, in general,

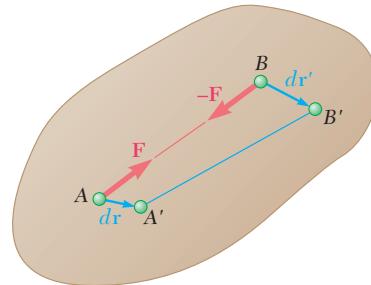


Fig. 10.4

small displacements  $d\mathbf{r}$  and  $d\mathbf{r}'$  of the two particles are different, the components of these displacements along  $AB$  must be equal; otherwise, the particles would not remain at the same distance from each other, and the body would not be rigid. Therefore, the work of  $\mathbf{F}$  is equal in magnitude and opposite in sign to the work of  $-\mathbf{F}$ , and their sum is zero.

In computing the work of the external forces acting on a rigid body, it is often convenient to determine the work of a couple without considering separately the work of each of the two forces forming the couple. Consider the two forces  $\mathbf{F}$  and  $-\mathbf{F}$  forming a couple of

moment  $\mathbf{M}$  and acting on a rigid body (Fig. 10.5). Any small displacement of the rigid body bringing  $A$  and  $B$ , respectively, into  $A'$  and  $B''$  can be divided into two parts, one in which points  $A$  and  $B$  undergo equal displacements  $d\mathbf{r}_1$ , the other in which  $A'$  remains fixed while  $B'$  moves into  $B''$  through a displacement  $d\mathbf{r}_2$  of magnitude  $ds_2 = r d\theta$ . In the first part of the motion, the work of  $\mathbf{F}$  is equal in magnitude and opposite in sign to the work of  $-\mathbf{F}$ , and their sum is zero. In the second part of the motion, only force  $\mathbf{F}$  works, and its work is  $dU = F ds_2 = Fr d\theta$ . But the product  $Fr$  is equal to the magnitude  $M$  of the moment of the couple. Thus, the work of a couple of moment  $\mathbf{M}$  acting on a rigid body is

$$dU = M d\theta \quad (10.2)$$

where  $d\theta$  is the small angle expressed in radians through which the body rotates. We again note that work should be expressed in units obtained by multiplying units of force by units of length.

### \*10.3 PRINCIPLE OF VIRTUAL WORK

Consider a particle acted upon by several forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$  (Fig. 10.6). We can imagine that the particle undergoes a small displacement from  $A$  to  $A'$ . This displacement is possible, but it will not necessarily take place. The forces may be balanced and the particle at rest, or the particle may move under the action of the given forces in a direction different from that of  $AA'$ . Since the displacement considered does not actually occur, it is called a *virtual displacement* and is denoted by  $\delta\mathbf{r}$ . The symbol  $\delta\mathbf{r}$  represents a differential of the first order; it is used to distinguish the virtual displacement from the displacement  $d\mathbf{r}$  which would take place under actual motion. As you will see, virtual displacements can be used to determine whether the conditions of equilibrium of a particle are satisfied.

The work of each of the forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$  during the virtual displacement  $\delta\mathbf{r}$  is called *virtual work*. The virtual work of all the forces acting on the particle of Fig. 10.6 is

$$\begin{aligned} \delta U &= \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \dots + \mathbf{F}_n \cdot \delta\mathbf{r} \\ &= (\mathbf{F}_1 + \mathbf{F}_2 + \dots + \mathbf{F}_n) \cdot \delta\mathbf{r} \end{aligned}$$

or

$$\delta U = \mathbf{R} \cdot \delta\mathbf{r} \quad (10.3)$$

where  $\mathbf{R}$  is the resultant of the given forces. Thus, the total virtual work of the forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$  is equal to the virtual work of their resultant  $\mathbf{R}$ .

The principle of virtual work for a particle states that *if a particle is in equilibrium, the total virtual work of the forces acting on the particle is zero for any virtual displacement of the particle*. This condition is necessary: if the particle is in equilibrium, the resultant  $\mathbf{R}$  of the forces is zero, and it follows from (10.3) that the total virtual work  $\delta U$  is zero. The condition is also sufficient: if the total virtual work  $\delta U$  is zero for any virtual displacement, the scalar product  $\mathbf{R} \cdot \delta\mathbf{r}$  is zero for any  $\delta\mathbf{r}$ , and the resultant  $\mathbf{R}$  must be zero.

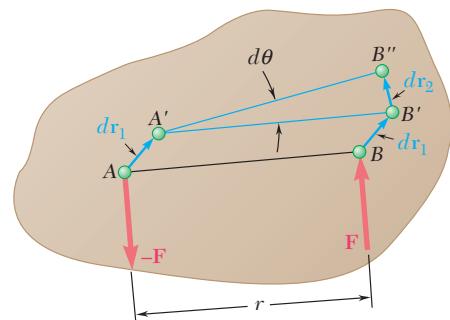


Fig. 10.5

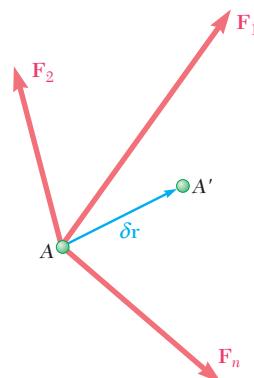


Fig. 10.6

In the case of a rigid body, the principle of virtual work states that *if a rigid body is in equilibrium, the total virtual work of the external forces acting on the rigid body is zero for any virtual displacement of the body*. The condition is necessary: if the body is in equilibrium, all the particles forming the body are in equilibrium and the total virtual work of the forces acting on all the particles must be zero; but we have seen in the preceding section that the total work of the internal forces is zero; the total work of the external forces must therefore also be zero. The condition can also be proved to be sufficient.

The principle of virtual work can be extended to the case of a *system of connected rigid bodies*. If the system remains connected during the virtual displacement, *only the work of the forces external to the system need be considered*, since the total work of the internal forces at the various connections is zero.

## \*10.4 APPLICATIONS OF THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work is particularly effective when applied to the solution of problems involving machines or mechanisms consisting of several connected rigid bodies. Consider, for instance, the toggle vise  $ACB$  of Fig. 10.7a, used to compress a wooden block. We

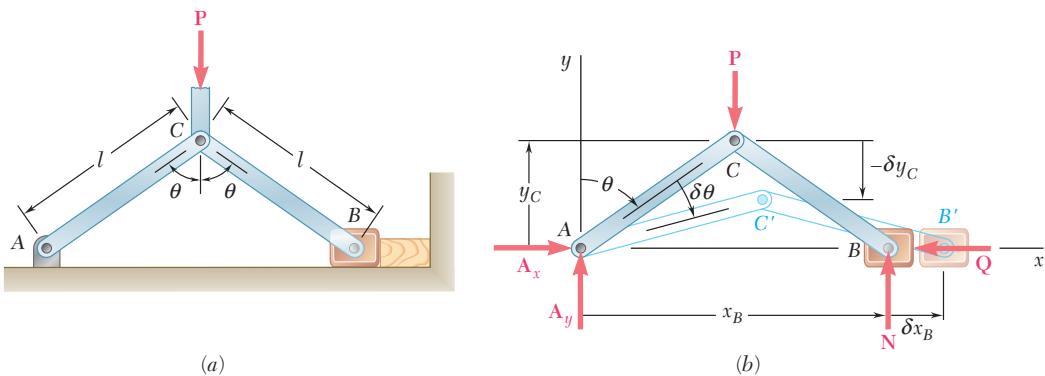


Fig. 10.7

wish to determine the force exerted by the vise on the block when a given force  $\mathbf{P}$  is applied at  $C$ , assuming that there is no friction. Denoting by  $\mathbf{Q}$  the reaction of the block on the vise, we draw the free-body diagram of the vise and consider the virtual displacement obtained by giving a positive increment  $\delta\theta$  to the angle  $\theta$  (Fig. 10.7b). Choosing a system of coordinate axes with origin at  $A$ , we note that  $x_B$  increases while  $y_C$  decreases. This is indicated in the figure, where a positive increment  $\delta x_B$  and a negative increment  $-\delta y_C$  are shown. The reactions  $\mathbf{A}_x$ ,  $\mathbf{A}_y$ , and  $\mathbf{N}$  will do no work during the virtual displacement considered, and we need only compute the work of  $\mathbf{P}$  and  $\mathbf{Q}$ . Since  $\mathbf{Q}$  and  $\delta x_B$  have opposite senses, the virtual work of  $\mathbf{Q}$  is  $\delta U_Q = -Q \delta x_B$ . Since  $\mathbf{P}$  and the increment shown ( $-\delta y_C$ ) have the same sense, the virtual work of  $\mathbf{P}$  is  $\delta U_P = +P(-\delta y_C) = -P \delta y_C$ . The minus signs obtained could have been predicted by simply noting that the forces  $\mathbf{Q}$  and  $\mathbf{P}$  are directed opposite to the positive

$x$  and  $y$  axes, respectively. Expressing the coordinates  $x_B$  and  $y_C$  in terms of the angle  $\theta$  and differentiating, we obtain

$$\begin{aligned}x_B &= 2l \sin \theta & y_C &= l \cos \theta \\ \delta x_B &= 2l \cos \theta \delta\theta & \delta y_C &= -l \sin \theta \delta\theta\end{aligned}\quad (10.4)$$

The total virtual work of the forces  $\mathbf{Q}$  and  $\mathbf{P}$  is thus

$$\begin{aligned}\delta U &= \delta U_Q + \delta U_P = -Q \delta x_B - P \delta y_C \\ &= -2Ql \cos \theta \delta\theta + Pl \sin \theta \delta\theta\end{aligned}$$

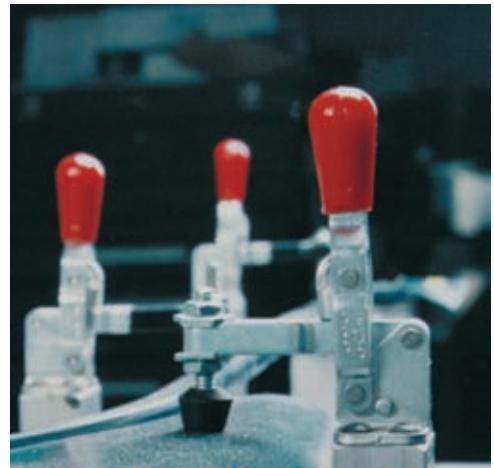
Making  $\delta U = 0$ , we obtain

$$2Ql \cos \theta \delta\theta = Pl \sin \theta \delta\theta \quad (10.5)$$

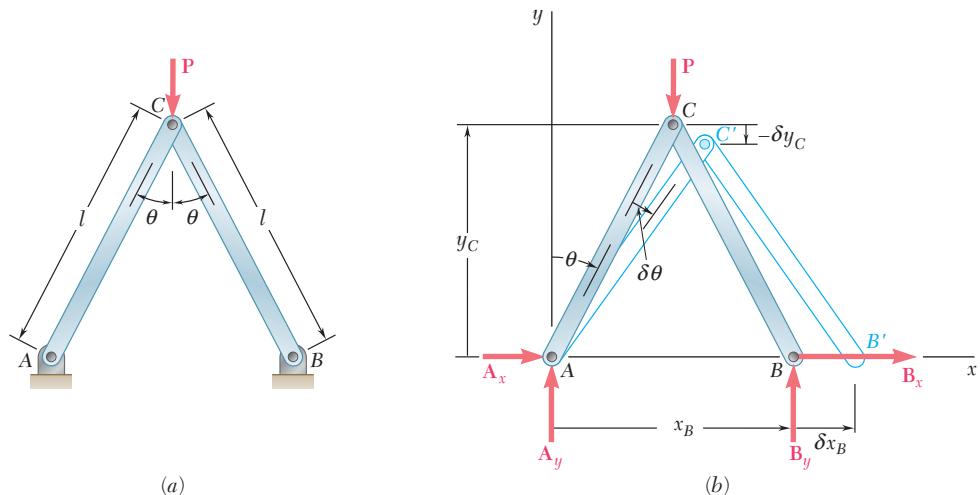
$$Q = \frac{1}{2}P \tan \theta \quad (10.6)$$

The superiority of the method of virtual work over the conventional equilibrium equations in the problem considered here is clear: by using the method of virtual work, we were able to eliminate all unknown reactions, while the equation  $\sum M_A = 0$  would have eliminated only two of the unknown reactions. This property of the method of virtual work can be used in solving many problems involving machines and mechanisms. *If the virtual displacement considered is consistent with the constraints imposed by the supports and connections, all reactions and internal forces are eliminated and only the work of the loads, applied forces, and friction forces need be considered.*

The method of virtual work can also be used to solve problems involving completely constrained structures, although the virtual displacements considered will never actually take place. Consider, for example, the frame  $ACB$  shown in Fig. 10.8a. If point  $A$  is kept fixed, while  $B$  is given a horizontal virtual displacement (Fig. 10.8b), we need consider only the work of  $\mathbf{P}$  and  $\mathbf{B}_x$ . We can thus determine



**Photo 10.2** The clamping force of the toggle clamp shown can be expressed as a function of the force applied to the handle by first establishing the geometric relations among the members of the clamp and then applying the method of virtual work.



**Fig. 10.8**

the reaction component  $\mathbf{B}_x$  in the same way as the force  $\mathbf{Q}$  of the preceding example (Fig. 10.7b); we have

$$B_x = -\frac{1}{2}P \tan \theta$$

Keeping  $B$  fixed and giving to  $A$  a horizontal virtual displacement, we can similarly determine the reaction component  $\mathbf{A}_x$ . The components  $\mathbf{A}_y$  and  $\mathbf{B}_y$  can be determined by rotating the frame  $ACB$  as a rigid body about  $B$  and  $A$ , respectively.

The method of virtual work can also be used to determine the configuration of a system in equilibrium under given forces. For example, the value of the angle  $\theta$  for which the linkage of Fig. 10.7 is in equilibrium under two given forces  $\mathbf{P}$  and  $\mathbf{Q}$  can be obtained by solving Eq. (10.6) for  $\tan \theta$ .

It should be noted, however, that the attractiveness of the method of virtual work depends to a large extent upon the existence of simple geometric relations between the various virtual displacements involved in the solution of a given problem. When no such simple relations exist, it is usually advisable to revert to the conventional method of Chap. 6.

## \*10.5 REAL MACHINES. MECHANICAL EFFICIENCY

In analyzing the toggle vise in the preceding section, we assumed that no friction forces were involved. Thus, the virtual work consisted only of the work of the applied force  $\mathbf{P}$  and of the reaction  $\mathbf{Q}$ . But the work of the reaction  $\mathbf{Q}$  is equal in magnitude and opposite in sign to the work of the force exerted by the vise on the block. Equation (10.5), therefore, expresses that the *output work*  $2Ql \cos \theta \delta\theta$  is equal to the *input work*  $P l \sin \theta \delta\theta$ . A machine in which input and output work are equal is said to be an “ideal” machine. In a “real” machine, friction forces will always do some work, and the output work will be smaller than the input work.

Consider, for example, the toggle vise of Fig. 10.7a, and assume now that a friction force  $\mathbf{F}$  develops between the sliding block  $B$  and the horizontal plane (Fig. 10.9). Using the conventional methods of statics and summing moments about  $A$ , we find  $N = P/2$ . Denoting by  $\mu$  the coefficient of friction between block  $B$  and the horizontal

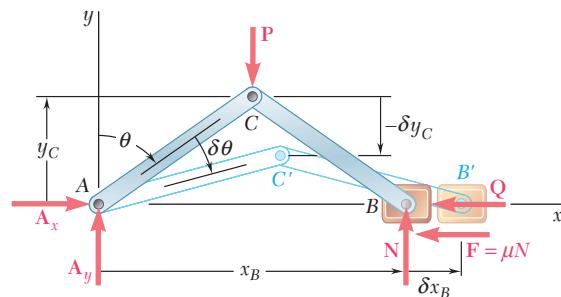


Fig. 10.9

plane, we have  $F = \mu N = \mu P/2$ . Recalling formulas (10.4), we find that the total virtual work of the forces  $\mathbf{Q}$ ,  $\mathbf{P}$ , and  $\mathbf{F}$  during the virtual displacement shown in Fig. 10.9 is

$$\begin{aligned}\delta U &= -Q \delta x_B - P \delta y_C - F \delta x_B \\ &= -2Ql \cos \theta \delta\theta + Pl \sin \theta \delta\theta - \mu Pl \cos \theta \delta\theta\end{aligned}$$

Making  $\delta U = 0$ , we obtain

$$2Ql \cos \theta \delta\theta = Pl \sin \theta \delta\theta - \mu Pl \cos \theta \delta\theta \quad (10.7)$$

which expresses that the output work is equal to the input work minus the work of the friction force. Solving for  $Q$ , we have

$$Q = \frac{1}{2}P(\tan \theta - \mu) \quad (10.8)$$

We note that  $Q = 0$  when  $\tan \theta = \mu$ , that is, when  $\theta$  is equal to the angle of friction  $\phi$ , and that  $Q < 0$  when  $\theta < \phi$ . The toggle vise may thus be used only for values of  $\theta$  larger than the angle of friction.

The *mechanical efficiency* of a machine is defined as the ratio

$$\eta = \frac{\text{output work}}{\text{input work}} \quad (10.9)$$

Clearly, the mechanical efficiency of an ideal machine is  $\eta = 1$ , since input and output work are then equal, while the mechanical efficiency of a real machine will always be less than 1.

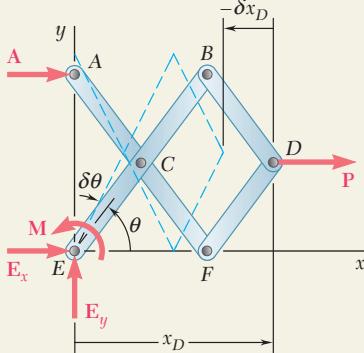
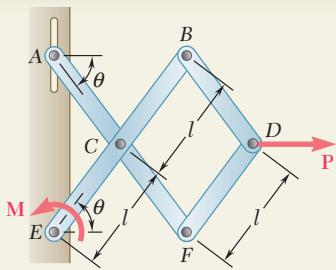
In the case of the toggle vise we have just analyzed, we write

$$\eta = \frac{\text{output work}}{\text{input work}} = \frac{2Ql \cos \theta \delta\theta}{Pl \sin \theta \delta\theta}$$

Substituting from (10.8) for  $Q$ , we obtain

$$\eta = \frac{P(\tan \theta - \mu)l \cos \theta \delta\theta}{Pl \sin \theta \delta\theta} = 1 - \mu \cot \theta \quad (10.10)$$

We check that in the absence of friction forces, we would have  $\mu = 0$  and  $\eta = 1$ . In the general case, when  $\mu$  is different from zero, the efficiency  $\eta$  becomes zero for  $\mu \cot \theta = 1$ , that is, for  $\tan \theta = \mu$ , or  $\theta = \tan^{-1} \mu = \phi$ . We note again that the toggle vise can be used only for values of  $\theta$  larger than the angle of friction  $\phi$ .



## SAMPLE PROBLEM 10.1

Using the method of virtual work, determine the magnitude of the couple  $\mathbf{M}$  required to maintain the equilibrium of the mechanism shown.

### SOLUTION

Choosing a coordinate system with origin at  $E$ , we write

$$x_D = 3l \cos \theta \quad \delta x_D = -3l \sin \theta \delta\theta$$

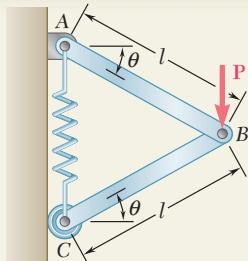
**Principle of Virtual Work.** Since the reactions  $\mathbf{A}$ ,  $\mathbf{E}_x$ , and  $\mathbf{E}_y$  will do no work during the virtual displacement, the total virtual work done by  $\mathbf{M}$  and  $\mathbf{P}$  must be zero. Noting that  $\mathbf{P}$  acts in the positive  $x$  direction and  $\mathbf{M}$  acts in the positive  $\theta$  direction, we write

$$\delta U = 0:$$

$$+M \delta\theta + P \delta x_D = 0$$

$$+M \delta\theta + P(-3l \sin \theta \delta\theta) = 0$$

$$M = 3Pl \sin \theta \quad \blacktriangleleft$$



## SAMPLE PROBLEM 10.2

Determine the expressions for  $\theta$  and for the tension in the spring which correspond to the equilibrium position of the mechanism. The unstretched length of the spring is  $h$ , and the constant of the spring is  $k$ . Neglect the weight of the mechanism.

### SOLUTION

With the coordinate system shown

$$y_B = l \sin \theta \quad y_C = 2l \sin \theta \\ \delta y_B = l \cos \theta \delta\theta \quad \delta y_C = 2l \cos \theta \delta\theta$$

The elongation of the spring is  $s = y_C - h = 2l \sin \theta - h$

The magnitude of the force exerted at  $C$  by the spring is

$$F = ks = k(2l \sin \theta - h) \quad (1)$$

**Principle of Virtual Work.** Since the reactions  $\mathbf{A}_x$ ,  $\mathbf{A}_y$ , and  $\mathbf{C}$  do no work, the total virtual work done by  $\mathbf{P}$  and  $\mathbf{F}$  must be zero.

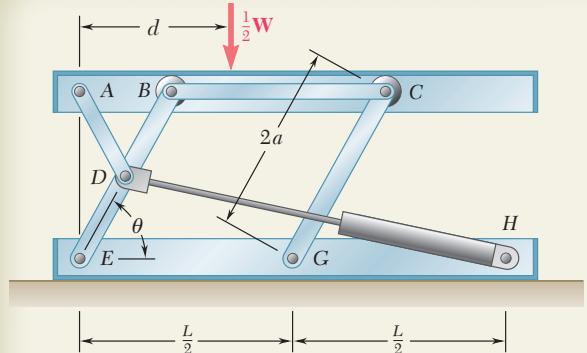
$$\delta U = 0: \quad P \delta y_B - F \delta y_C = 0$$

$$P(l \cos \theta \delta\theta) - k(2l \sin \theta - h)(2l \cos \theta \delta\theta) = 0$$

$$\sin \theta = \frac{P + 2kh}{4kl} \quad \blacktriangleleft$$

Substituting this expression into (1), we obtain

$$F = \frac{1}{2}P \quad \blacktriangleleft$$



### SAMPLE PROBLEM 10.3

A hydraulic-lift table is used to raise a 1000-kg crate. It consists of a platform and of two identical linkages on which hydraulic cylinders exert equal forces. (Only one linkage and one cylinder are shown.) Members  $EDB$  and  $CG$  are each of length  $2a$ , and member  $AD$  is pinned to the midpoint of  $EDB$ . If the crate is placed on the table, so that half of its weight is supported by the system shown, determine the force exerted by each cylinder in raising the crate for  $\theta = 60^\circ$ ,  $a = 0.70$  m, and  $L = 3.20$  m. This mechanism has been previously considered in Sample Prob. 6.7.

### SOLUTION

The machine considered consists of the platform and of the linkage, with an input force  $\mathbf{F}_{DH}$  exerted by the cylinder and an output force equal and opposite to  $\frac{1}{2}\mathbf{W}$ .

**Principle of Virtual Work.** We first observe that the reactions at  $E$  and  $G$  do no work. Denoting by  $y$  the elevation of the platform above the base, and by  $s$  the length  $DH$  of the cylinder-and-piston assembly, we write

$$\delta U = 0: \quad -\frac{1}{2}W \delta y + F_{DH} \delta s = 0 \quad (1)$$

The vertical displacement  $\delta y$  of the platform is expressed in terms of the angular displacement  $\delta\theta$  of  $EDB$  as follows:

$$y = (EB) \sin \theta = 2a \sin \theta \\ \delta y = 2a \cos \theta \delta\theta$$

To express  $\delta s$  similarly in terms of  $\delta\theta$ , we first note that by the law of cosines,

$$s^2 = a^2 + L^2 - 2aL \cos \theta$$

Differentiating,

$$2s \delta s = -2aL(-\sin \theta) \delta\theta \\ \delta s = \frac{aL \sin \theta}{s} \delta\theta$$

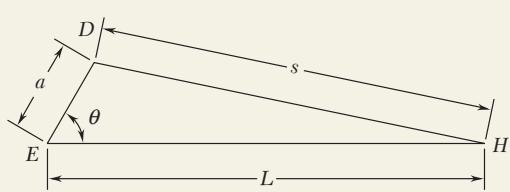
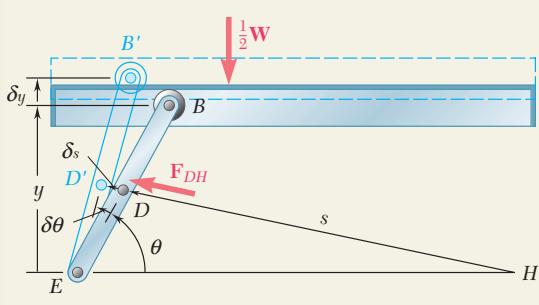
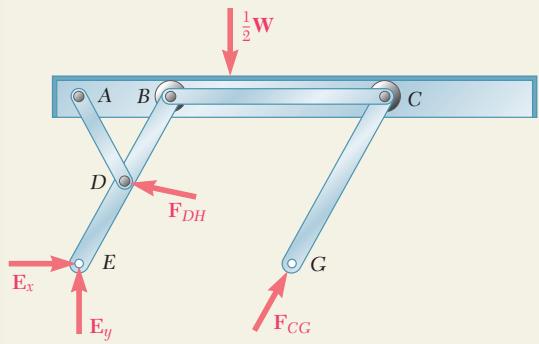
Substituting for  $\delta y$  and  $\delta s$  into (1), we write

$$(-\frac{1}{2}W)2a \cos \theta \delta\theta + F_{DH} \frac{aL \sin \theta}{s} \delta\theta = 0 \\ F_{DH} = W \frac{s}{L} \cot \theta$$

With the given numerical data, we have

$$W = mg = (1000 \text{ kg})(9.81 \text{ m/s}^2) = 9810 \text{ N} = 9.81 \text{ kN} \\ s^2 = a^2 + L^2 - 2aL \cos \theta \\ = (0.70)^2 + (3.20)^2 - 2(0.70)(3.20) \cos 60^\circ = 8.49 \\ s = 2.91 \text{ m} \\ F_{DH} = W \frac{s}{L} \cot \theta = (9.81 \text{ kN}) \frac{2.91 \text{ m}}{3.20 \text{ m}} \cot 60^\circ$$

$$F_{DH} = 5.15 \text{ kN}$$



# SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to use the *method of virtual work*, which is a different way of solving problems involving the equilibrium of rigid bodies.

The work done by a force during a displacement of its point of application or by a couple during a rotation is found by using Eqs. (10.1) and (10.2), respectively:

$$dU = F ds \cos \alpha \quad (10.1)$$

$$dU = M d\theta \quad (10.2)$$

**Principle of virtual work.** In its more general and more useful form, this principle can be stated as follows: *If a system of connected rigid bodies is in equilibrium, the total virtual work of the external forces applied to the system is zero for any virtual displacement of the system.*

As you apply the principle of virtual work, keep in mind the following:

**1. Virtual displacement.** A machine or mechanism in equilibrium has no tendency to move. However, *we can cause, or imagine, a small displacement*. Since it does not actually occur, such a displacement is called a *virtual displacement*.

**2. Virtual work.** The work done by a force or couple during a virtual displacement is called *virtual work*.

**3. You need consider only the forces which do work** during the virtual displacement.

**4. Forces which do no work** during a virtual displacement that is consistent with the constraints imposed on the system are:

- a. Reactions at supports
- b. Internal forces at connections
- c. Forces exerted by inextensible cords and cables

None of these forces need be considered when you use the method of virtual work.

**5. Be sure to express the various virtual displacements** involved in your computations in terms of a *single virtual displacement*. This is done in each of the three preceding sample problems, where the virtual displacements are all expressed in terms of  $\delta\theta$ .

**6. Remember that the method of virtual work is effective only in those cases** where the geometry of the system makes it relatively easy to relate the displacements involved.

# PROBLEMS

**10.1** Determine the vertical force  $\mathbf{P}$  that must be applied at  $C$  to maintain the equilibrium of the linkage.

**10.2** Determine the horizontal force  $\mathbf{P}$  that must be applied at  $A$  to maintain the equilibrium of the linkage.

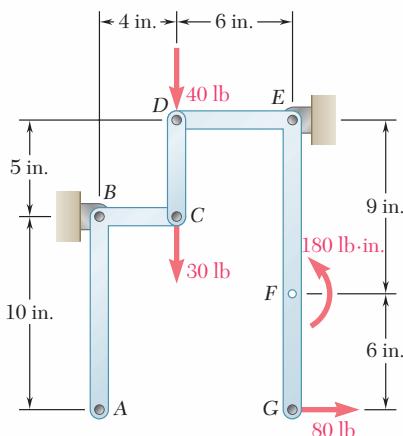


Fig. P10.2 and P10.4

**10.3 and 10.4** Determine the couple  $\mathbf{M}$  that must be applied to member  $ABC$  to maintain the equilibrium of the linkage.

**10.5** Knowing that the maximum friction force exerted by the bottle on the cork is 60 lb, determine (a) the force  $\mathbf{P}$  that must be applied to the corkscrew to open the bottle, (b) the maximum force exerted by the base of the corkscrew on the top of the bottle.

**10.6** The two-bar linkage shown is supported by a pin and bracket at  $B$  and a collar at  $D$  that slides freely on a vertical rod. Determine the force  $\mathbf{P}$  required to maintain the equilibrium of the linkage.

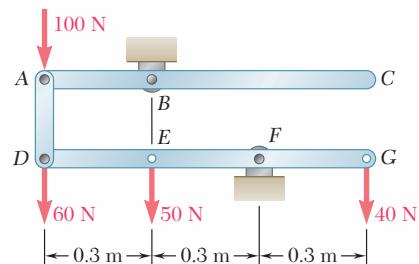


Fig. P10.1 and P10.3

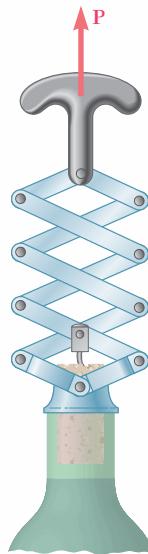


Fig. P10.5

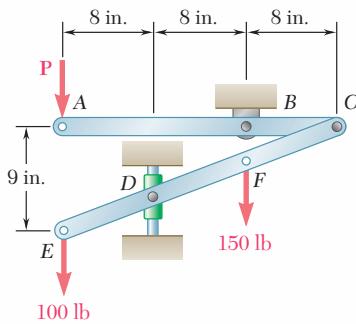


Fig. P10.6

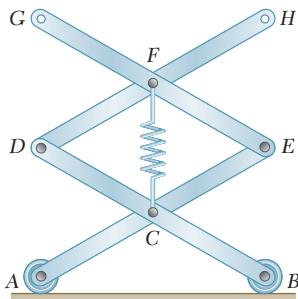


Fig. P10.7 and P10.8

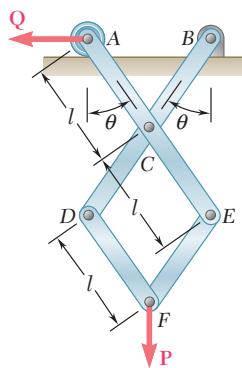


Fig. P10.11

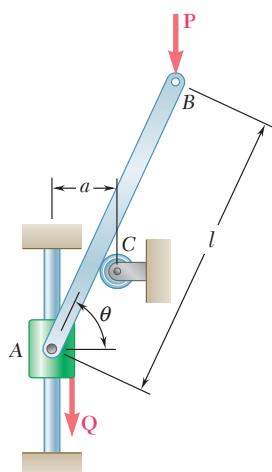


Fig. P10.12

- 10.7** A spring of constant 15 kN/m connects points C and F of the linkage shown. Neglecting the weight of the spring and linkage, determine the force in the spring and the vertical motion of point G when a vertical downward 120-N force is applied (a) at point C, (b) at points C and H.

- 10.8** A spring of constant 15 kN/m connects points C and F of the linkage shown. Neglecting the weight of the spring and linkage, determine the force in the spring and the vertical motion of point G when a vertical downward 120-N force is applied (a) at point E, (b) at points E and F.

- 10.9** Knowing that the line of action of the force **Q** passes through point C, derive an expression for the magnitude of **Q** required to maintain equilibrium.

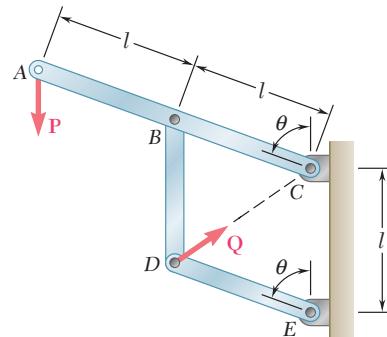


Fig. P10.9

- 10.10** Solve Prob. 10.9 assuming that the force **P** applied at point A acts horizontally to the left.

- 10.11** The mechanism shown is acted upon by the force **P**; derive an expression for the magnitude of the force **Q** required to maintain equilibrium.

- 10.12 and 10.13** The slender rod AB is attached to a collar A and rests on a small wheel at C. Neglecting the radius of the wheel and the effect of friction, derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the rod.

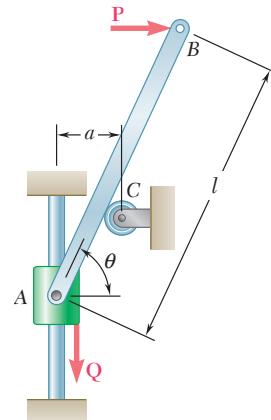


Fig. P10.13

- 10.14** Derive an expression for the magnitude of the force  $\mathbf{Q}$  required to maintain the equilibrium of the mechanism shown.

- 10.15** A uniform rod  $AB$  of length  $l$  and weight  $W$  is suspended from two cords  $AC$  and  $BC$  of equal length. Derive an expression for the magnitude of the couple  $\mathbf{M}$  required to maintain equilibrium of the rod in the position shown.

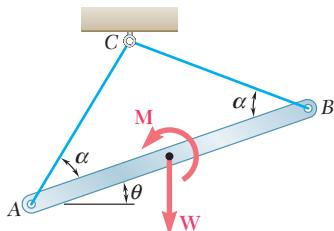


Fig. P10.15

- 10.16 and 10.17** Derive an expression for the magnitude of the couple  $\mathbf{M}$  required to maintain the equilibrium of the linkage shown.

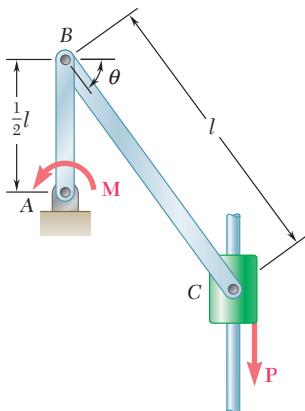


Fig. P10.16

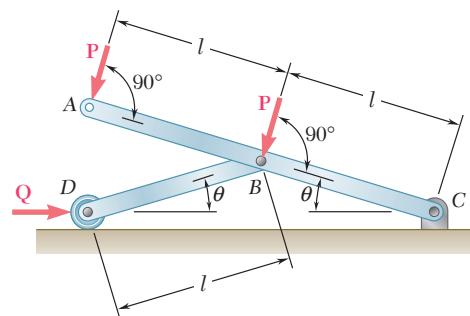


Fig. P10.14

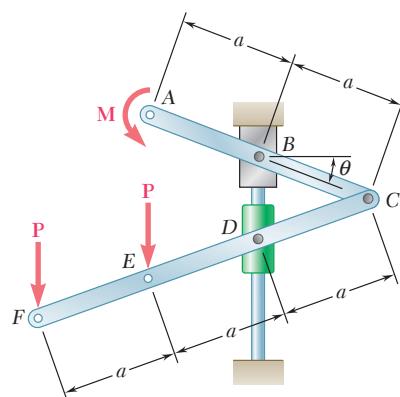


Fig. P10.17

- 10.18** The pin at  $C$  is attached to member  $BCD$  and can slide along a slot cut in the fixed plate shown. Neglecting the effect of friction, derive an expression for the magnitude of the couple  $\mathbf{M}$  required to maintain equilibrium when the force  $\mathbf{P}$  that acts at  $D$  is directed (a) as shown, (b) vertically downward, (c) horizontally to the right.

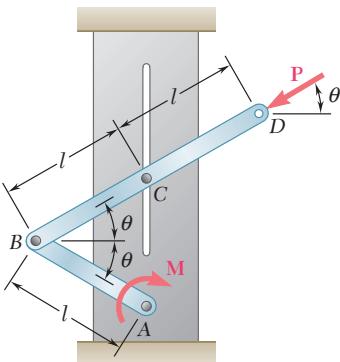


Fig. P10.18

- 10.19** A 4-kN force  $\mathbf{P}$  is applied as shown to the piston of the engine system. Knowing that  $AB = 50$  mm and  $BC = 200$  mm, determine the couple  $\mathbf{M}$  required to maintain the equilibrium of the system when (a)  $\theta = 30^\circ$ , (b)  $\theta = 150^\circ$ .

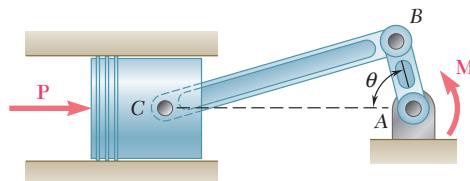


Fig. P10.19 and P10.20

- 10.20** A couple  $\mathbf{M}$  of magnitude 100 N · m is applied as shown to the crank of the engine system. Knowing that  $AB = 50$  mm and  $BC = 200$  mm, determine the force  $\mathbf{P}$  required to maintain the equilibrium of the system when (a)  $\theta = 60^\circ$ , (b)  $\theta = 120^\circ$ .

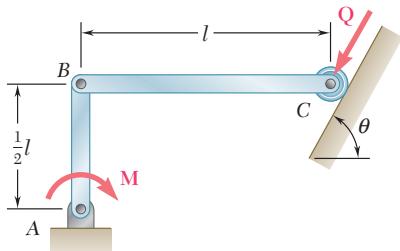


Fig. P10.21 and P10.22

- 10.21** For the linkage shown, determine the couple  $\mathbf{M}$  required for equilibrium when  $l = 1.8$  ft,  $Q = 40$  lb, and  $\theta = 65^\circ$ .

- 10.22** For the linkage shown, determine the force  $\mathbf{Q}$  required for equilibrium when  $l = 18$  in.,  $M = 600$  lb · in., and  $\theta = 70^\circ$ .

- 10.23** Determine the value of  $\theta$  corresponding to the equilibrium position of the mechanism of Prob. 10.11 when  $P = 45$  lb and  $Q = 160$  lb.

- 10.24** Determine the value of  $\theta$  corresponding to the equilibrium position of the mechanism of Prob. 10.9 when  $P = 80$  N and  $Q = 100$  N.

- 10.25** Rod  $AB$  is attached to a block at  $A$  that can slide freely in the vertical slot shown. Neglecting the effect of friction and the weights of the rods, determine the value of  $\theta$  corresponding to equilibrium.

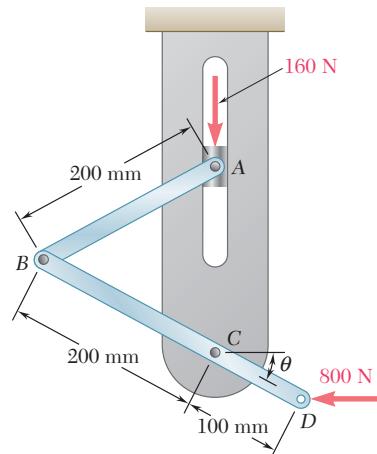


Fig. P10.25

- 10.26** Solve Prob. 10.25 assuming that the 800-N force is replaced by a 24-N · m clockwise couple applied at *D*.

- 10.27** Determine the value of  $\theta$  corresponding to the equilibrium position of the rod of Prob. 10.12 when  $l = 30$  in.,  $a = 5$  in.,  $P = 25$  lb, and  $Q = 40$  lb.

- 10.28** Determine the values of  $\theta$  corresponding to the equilibrium position of the rod of Prob. 10.13 when  $l = 600$  mm,  $a = 100$  mm,  $P = 50$  N, and  $Q = 90$  N.

- 10.29** Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is  $k$ , and the spring is unstretched when  $\theta = 30^\circ$ . For the loading shown, derive an equation in  $P$ ,  $\theta$ ,  $l$ , and  $k$  that must be satisfied when the system is in equilibrium.

- 10.30** Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is 1.5 lb/in., and the spring is unstretched when  $\theta = 30^\circ$ . Knowing that  $l = 10$  in. and neglecting the weight of the rods, determine the value of  $\theta$  corresponding to equilibrium when  $P = 40$  lb.

- 10.31** Solve Prob. 10.30 assuming that force **P** is moved to *C* and acts vertically downward.

- 10.32** Rod *ABC* is attached to blocks *A* and *B* that can move freely in the guides shown. The constant of the spring attached at *A* is  $k = 3$  kN/m, and the spring is unstretched when the rod is vertical. For the loading shown, determine the value of  $\theta$  corresponding to equilibrium.

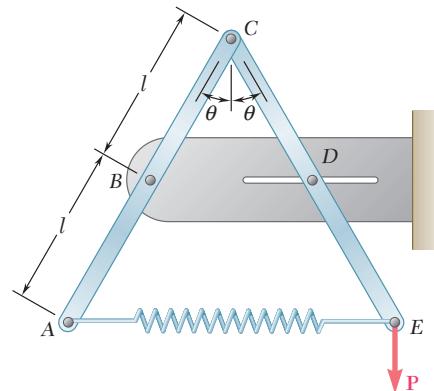


Fig. P10.29 and P10.30

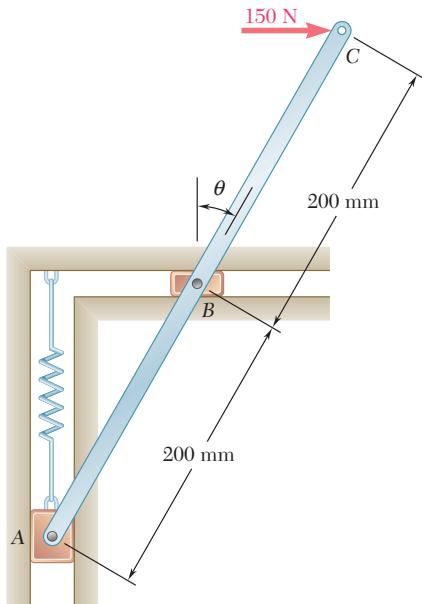


Fig. P10.32

- 10.33** A load  $\mathbf{W}$  of magnitude 600 N is applied to the linkage at  $B$ . The constant of the spring is  $k = 2.5 \text{ kN/m}$ , and the spring is unstretched when  $AB$  and  $BC$  are horizontal. Neglecting the weight of the linkage and knowing that  $l = 300 \text{ mm}$ , determine the value of  $\theta$  corresponding to equilibrium.

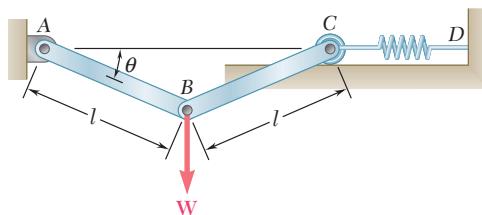


Fig. P10.33 and P10.34

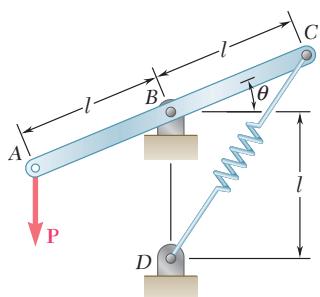


Fig. P10.35 and P10.36

- 10.34** A vertical load  $\mathbf{W}$  is applied to the linkage at  $B$ . The constant of the spring is  $k$ , and the spring is unstretched when  $AB$  and  $BC$  are horizontal. Neglecting the weight of the linkage, derive an equation in  $\theta$ ,  $W$ ,  $l$ , and  $k$  that must be satisfied when the linkage is in equilibrium.

- 10.35 and 10.36** Knowing that the constant of spring  $CD$  is  $k$  and that the spring is unstretched when rod  $ABC$  is horizontal, determine the value of  $\theta$  corresponding to equilibrium for the data indicated.

$$\mathbf{10.35} \quad P = 300 \text{ N}, l = 400 \text{ mm}, k = 5 \text{ kN/m}.$$

$$\mathbf{10.36} \quad P = 75 \text{ lb}, l = 15 \text{ in.}, k = 20 \text{ lb/in.}$$

- 10.37** A load  $\mathbf{W}$  of magnitude 72 lb is applied to the mechanism at  $C$ . Neglecting the weight of the mechanism, determine the value of  $\theta$  corresponding to equilibrium. The constant of the spring is  $k = 20 \text{ lb/in.}$ , and the spring is unstretched when  $\theta = 0$ .

- 10.38** A force  $\mathbf{P}$  of magnitude 240 N is applied to end  $E$  of cable  $CDE$ , which passes under pulley  $D$  and is attached to the mechanism at  $C$ . Neglecting the weight of the mechanism and the radius of the pulley, determine the value of  $\theta$  corresponding to equilibrium. The constant of the spring is  $k = 4 \text{ kN/m}$ , and the spring is unstretched when  $\theta = 90^\circ$ .

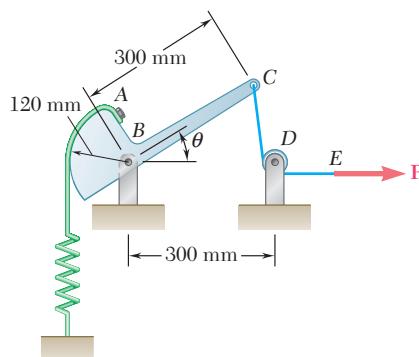


Fig. P10.38

- 10.39** The lever  $AB$  is attached to the horizontal shaft  $BC$  that passes through a bearing and is welded to a fixed support at  $C$ . The torsional spring constant of the shaft  $BC$  is  $K$ ; that is, a couple of magnitude  $K$  is required to rotate end  $B$  through 1 rad. Knowing that the shaft is untwisted when  $AB$  is horizontal, determine the value of  $\theta$  corresponding to the position of equilibrium when  $P = 100 \text{ N}$ ,  $l = 250 \text{ mm}$ , and  $K = 12.5 \text{ N} \cdot \text{m}/\text{rad}$ .

- 10.40** Solve Prob. 10.39 assuming that  $P = 350 \text{ N}$ ,  $l = 250 \text{ mm}$ , and  $K = 12.5 \text{ N} \cdot \text{m}/\text{rad}$ . Obtain answers in each of the following quadrants:  $0 < \theta < 90^\circ$ ,  $270^\circ < \theta < 360^\circ$ ,  $360^\circ < \theta < 450^\circ$ .

- 10.41** The position of boom  $ABC$  is controlled by the hydraulic cylinder  $BD$ . For the loading shown, determine the force exerted by the hydraulic cylinder on pin  $B$  when  $\theta = 65^\circ$ .

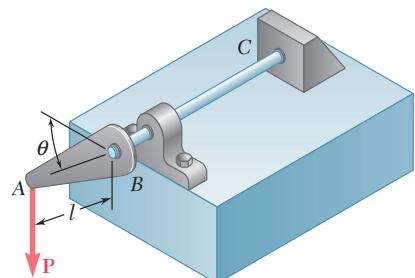


Fig. P10.39

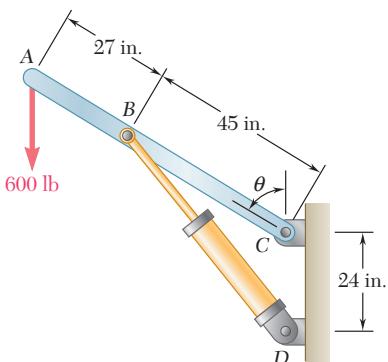


Fig. P10.41 and P10.42

- 10.42** The position of boom  $ABC$  is controlled by the hydraulic cylinder  $BD$ . For the loading shown, (a) express the force exerted by the hydraulic cylinder on pin  $B$  as a function of the length  $BD$ , (b) determine the smallest possible value of the angle  $\theta$  if the maximum force that the cylinder can exert on pin  $B$  is 2.5 kips.

- 10.43** The position of member  $ABC$  is controlled by the hydraulic cylinder  $CD$ . For the loading shown, determine the force exerted by the hydraulic cylinder on pin  $C$  when  $\theta = 55^\circ$ .

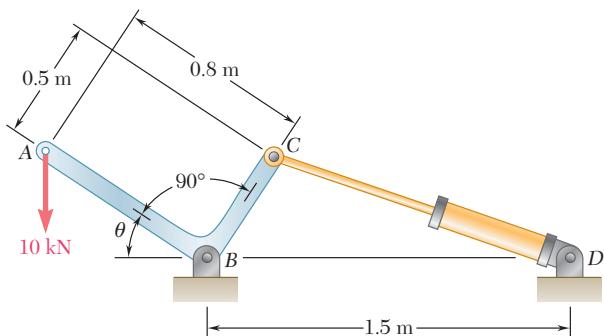
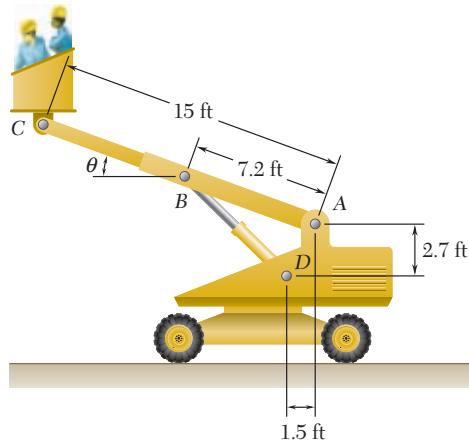


Fig. P10.43 and P10.44

- 10.44** The position of member  $ABC$  is controlled by the hydraulic cylinder  $CD$ . Determine the angle  $\theta$  knowing that the hydraulic cylinder exerts a 15-kN force on pin  $C$ .

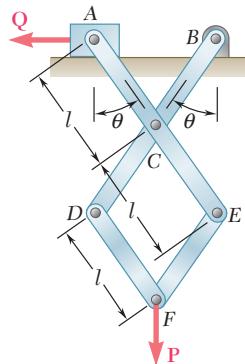
- 10.45** The telescoping arm  $ABC$  is used to provide an elevated platform for construction workers. The workers and the platform together weigh 500 lb and their combined center of gravity is located directly above  $C$ . For the position when  $\theta = 20^\circ$ , determine the force exerted on pin  $B$  by the single hydraulic cylinder  $BD$ .

**Fig. P10.45**

- 10.46** Solve Prob. 10.45 assuming that the workers are lowered to a point near the ground so that  $\theta = -20^\circ$ .

- 10.47** A block of weight  $W$  is pulled up a plane forming an angle  $\alpha$  with the horizontal by a force  $\mathbf{P}$  directed along the plane. If  $\mu$  is the coefficient of friction between the block and the plane, derive an expression for the mechanical efficiency of the system. Show that the mechanical efficiency cannot exceed  $\frac{1}{2}$  if the block is to remain in place when the force  $\mathbf{P}$  is removed.

- 10.48** Denoting by  $\mu_s$  the coefficient of static friction between the block attached to rod  $ACE$  and the horizontal surface, derive expressions in terms of  $P$ ,  $\mu_s$ , and  $\theta$  for the largest and smallest magnitude of the force  $\mathbf{Q}$  for which equilibrium is maintained.

**Fig. P10.48 and P10.49**

- 10.49** Knowing that the coefficient of static friction between the block attached to rod  $ACE$  and the horizontal surface is 0.15, determine the magnitude of the largest and smallest force  $\mathbf{Q}$  for which equilibrium is maintained when  $\theta = 30^\circ$ ,  $l = 0.2$  m, and  $P = 40$  N.

- 10.50** Denoting by  $\mu_s$  the coefficient of static friction between collar  $C$  and the vertical rod, derive an expression for the magnitude of the largest couple  $\mathbf{M}$  for which equilibrium is maintained in the position shown. Explain what happens if  $\mu_s \geq \tan \theta$ .

- 10.51** Knowing that the coefficient of static friction between collar  $C$  and the vertical rod is 0.40, determine the magnitude of the largest and smallest couple  $\mathbf{M}$  for which equilibrium is maintained in the position shown, when  $\theta = 35^\circ$ ,  $l = 600 \text{ mm}$ , and  $P = 300 \text{ N}$ .

- 10.52** Derive an expression for the mechanical efficiency of the jack discussed in Sec. 8.6. Show that if the jack is to be self-locking, the mechanical efficiency cannot exceed  $\frac{1}{2}$ .

- 10.53** Using the method of virtual work, determine the reaction at  $E$ .

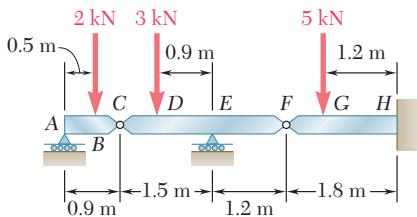


Fig. P10.53 and P10.54

- 10.54** Using the method of virtual work, determine separately the force and couple representing the reaction at  $H$ .

- 10.55** Referring to Prob. 10.43 and using the value found for the force exerted by the hydraulic cylinder  $CD$ , determine the change in the length of  $CD$  required to raise the 10-kN load by 15 mm.

- 10.56** Referring to Prob. 10.45 and using the value found for the force exerted by the hydraulic cylinder  $BD$ , determine the change in the length of  $BD$  required to raise the platform attached at  $C$  by 2.5 in.

- 10.57** Determine the vertical movement of joint  $D$  if the length of member  $BF$  is increased by 1.5 in. (Hint: Apply a vertical load at joint  $D$ , and, using the methods of Chap. 6, compute the force exerted by member  $BF$  on joints  $B$  and  $F$ . Then apply the method of virtual work for a virtual displacement resulting in the specified increase in length of member  $BF$ . This method should be used only for small changes in the lengths of members.)

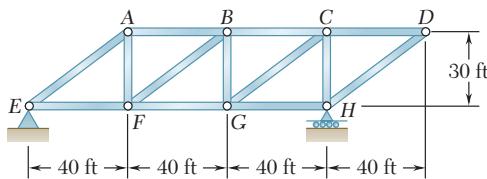


Fig. P10.57 and P10.58

- 10.58** Determine the horizontal movement of joint  $D$  if the length of member  $BF$  is increased by 1.5 in. (See the hint for Prob. 10.57.)

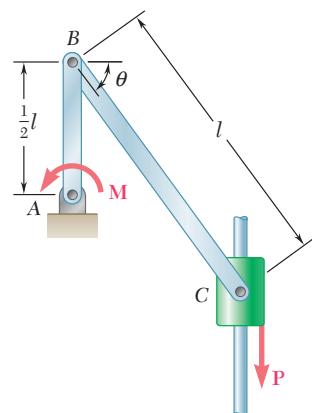


Fig. P10.50 and P10.51

## \*10.6 WORK OF A FORCE DURING A FINITE DISPLACEMENT

Consider a force  $\mathbf{F}$  acting on a particle. The work of  $\mathbf{F}$  corresponding to an infinitesimal displacement  $d\mathbf{r}$  of the particle was defined in Sec. 10.2 as

$$dU = \mathbf{F} \cdot d\mathbf{r} \quad (10.1)$$

The work of  $\mathbf{F}$  corresponding to a finite displacement of the particle from  $A_1$  to  $A_2$  (Fig. 10.10a) is denoted by  $U_{1 \rightarrow 2}$  and is obtained by integrating (10.1) along the curve described by the particle:

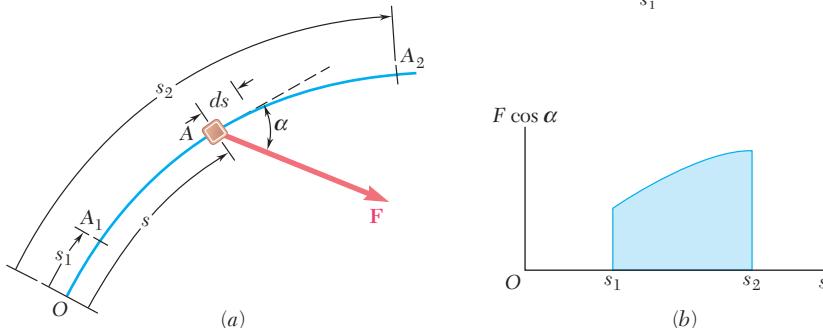
$$U_{1 \rightarrow 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \quad (10.11)$$

Using the alternative expression

$$dU = F ds \cos \alpha \quad (10.1')$$

given in Sec. 10.2 for the elementary work  $dU$ , we can also express the work  $U_{1 \rightarrow 2}$  as

$$U_{1 \rightarrow 2} = \int_{s_1}^{s_2} (F \cos \alpha) ds \quad (10.11')$$



**Fig. 10.10**

where the variable of integration  $s$  measures the distance along the path traveled by the particle. The work  $U_{1 \rightarrow 2}$  is represented by the area under the curve obtained by plotting  $F \cos \alpha$  against  $s$  (Fig. 10.10b). In the case of a force  $\mathbf{F}$  of constant magnitude acting in the direction of motion, formula (10.11') yields  $U_{1 \rightarrow 2} = F(s_2 - s_1)$ .

Recalling from Sec. 10.2 that the work of a couple of moment  $\mathbf{M}$  during an infinitesimal rotation  $d\theta$  of a rigid body is

$$dU = M d\theta \quad (10.2)$$

we express as follows the work of the couple during a finite rotation of the body:

$$U_{1 \rightarrow 2} = \int_{\theta_1}^{\theta_2} M d\theta \quad (10.12)$$

In the case of a constant couple, formula (10.12) yields

$$U_{1 \rightarrow 2} = M(\theta_2 - \theta_1)$$

**Work of a Weight.** It was stated in Sec. 10.2 that the work of the weight  $\mathbf{W}$  of a body during an infinitesimal displacement of the body is equal to the product of  $W$  and the vertical displacement of the center of gravity of the body. With the  $y$  axis pointing upward, the work of  $\mathbf{W}$  during a finite displacement of the body (Fig. 10.11) is obtained by writing

$$dU = -W dy$$

Integrating from  $A_1$  to  $A_2$ , we have

$$U_{1 \rightarrow 2} = - \int_{y_1}^{y_2} W dy = Wy_1 - Wy_2 \quad (10.13)$$

or

$$U_{1 \rightarrow 2} = -W(y_2 - y_1) = -W \Delta y \quad (10.13')$$

where  $\Delta y$  is the vertical displacement from  $A_1$  to  $A_2$ . The work of the weight  $\mathbf{W}$  is thus equal to *the product of  $W$  and the vertical displacement of the center of gravity of the body*. The work is *positive* when  $\Delta y < 0$ , that is, *when the body moves down*.

**Work of the Force Exerted by a Spring.** Consider a body  $A$  attached to a fixed point  $B$  by a spring; it is assumed that the spring is undeformed when the body is at  $A_0$  (Fig. 10.12a). Experimental evidence shows that the magnitude of the force  $\mathbf{F}$  exerted by the spring on a body  $A$  is proportional to the deflection  $x$  of the spring measured from the position  $A_0$ . We have

$$F = kx \quad (10.14)$$

where  $k$  is the *spring constant*, expressed in N/m if SI units are used and expressed in lb/ft or lb/in. if U.S. customary units are used. The work of the force  $\mathbf{F}$  exerted by the spring during a finite displacement of the body from  $A_1(x = x_1)$  to  $A_2(x = x_2)$  is obtained by writing

$$dU = -F dx = -kx dx$$

$$U_{1 \rightarrow 2} = - \int_{x_1}^{x_2} kx dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \quad (10.15)$$

Care should be taken to express  $k$  and  $x$  in consistent units. For example, if U.S. customary units are used,  $k$  should be expressed in lb/ft and  $x$  expressed in feet, or  $k$  in lb/in. and  $x$  in inches; in the first case, the work is obtained in ft · lb; in the second case, in in · lb. We note that the work of the force  $\mathbf{F}$  exerted by the spring on the body is *positive* when  $x_2 < x_1$ , that is, *when the spring is returning to its undeformed position*.

Since Eq. (10.14) is the equation of a straight line of slope  $k$  passing through the origin, the work  $U_{1 \rightarrow 2}$  of  $\mathbf{F}$  during the displacement from  $A_1$  to  $A_2$  can be obtained by evaluating the area of the trapezoid shown in Fig. 10.12b. This is done by computing the values  $F_1$  and  $F_2$  and multiplying the base  $\Delta x$  of the trapezoid by its mean height  $\frac{1}{2}(F_1 + F_2)$ . Since the work of the force  $\mathbf{F}$  exerted by the spring is positive for a negative value of  $\Delta x$ , we write

$$U_{1 \rightarrow 2} = -\frac{1}{2}(F_1 + F_2) \Delta x \quad (10.16)$$

Formula (10.16) is usually more convenient to use than (10.15) and affords fewer chances of confusing the units involved.

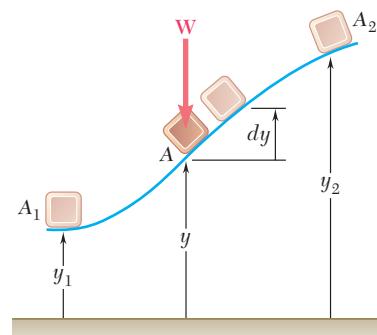


Fig. 10.11

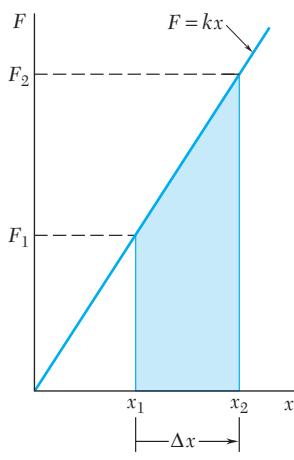
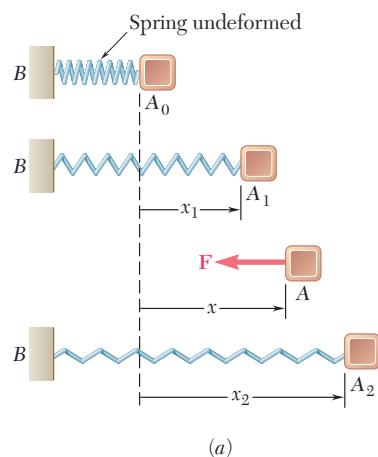


Fig. 10.12

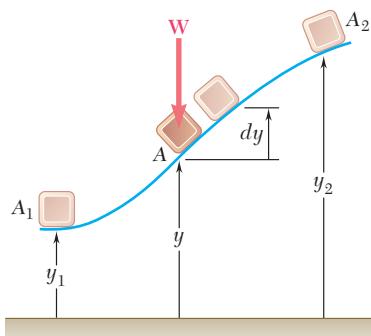


Fig. 10.11 (repeated)

## \*10.7 POTENTIAL ENERGY

Considering again the body of Fig. 10.11, we note from Eq. (10.13) that the work of the weight  $\mathbf{W}$  during a finite displacement is obtained by subtracting the value of the function  $W_y$  corresponding to the second position of the body from its value corresponding to the first position. The work of  $\mathbf{W}$  is thus independent of the actual path followed; it depends only upon the initial and final values of the function  $W_y$ . This function is called the *potential energy* of the body with respect to the *force of gravity*  $\mathbf{W}$  and is denoted by  $V_g$ . We write

$$U_{1 \rightarrow 2} = (V_g)_1 - (V_g)_2 \quad \text{with } V_g = W_y \quad (10.17)$$

We note that if  $(V_g)_2 > (V_g)_1$ , that is, if the potential energy increases during the displacement (as in the case considered here), the work  $U_{1 \rightarrow 2}$  is negative. If, on the other hand, the work of  $\mathbf{W}$  is positive, the potential energy decreases. Therefore, the potential energy  $V_g$  of the body provides a measure of the work which can be done by its weight  $\mathbf{W}$ . Since only the change in potential energy, and not the actual value of  $V_g$ , is involved in formula (10.17), an arbitrary constant can be added to the expression obtained for  $V_g$ . In other words, the level from which the elevation  $y$  is measured can be chosen arbitrarily. Note that potential energy is expressed in the same units as work, i.e., in joules (J) if SI units are used† and in ft · lb or in · lb if U.S. customary units are used.

Considering now the body of Fig. 10.12a, we note from Eq. (10.15) that the work of the elastic force  $\mathbf{F}$  is obtained by subtracting the value of the function  $\frac{1}{2}kx^2$  corresponding to the second position of the body from its value corresponding to the first position. This function is denoted by  $V_e$  and is called the *potential energy* of the body with respect to the *elastic force*  $\mathbf{F}$ . We write

$$U_{1 \rightarrow 2} = (V_e)_1 - (V_e)_2 \quad \text{with } V_e = \frac{1}{2}kx^2 \quad (10.18)$$

and observe that during the displacement considered, the work of the force  $\mathbf{F}$  exerted by the spring on the body is negative and the potential energy  $V_e$  increases. We should note that the expression obtained for  $V_e$  is valid only if the deflection of the spring is measured from its undeformed position.

The concept of potential energy can be used when forces other than gravity forces and elastic forces are involved. It remains valid as long as the elementary work  $dU$  of the force considered is an *exact differential*. It is then possible to find a function  $V$ , called potential energy, such that

$$dU = -dV \quad (10.19)$$

Integrating (10.19) over a finite displacement, we obtain the general formula

$$U_{1 \rightarrow 2} = V_1 - V_2 \quad (10.20)$$

which expresses that *the work of the force is independent of the path followed and is equal to minus the change in potential energy*. A force which satisfies Eq. (10.20) is said to be a *conservative force*.‡

†See footnote, page 559.

‡A detailed discussion of conservative forces is given in Sec. 13.7 of *Dynamics*.

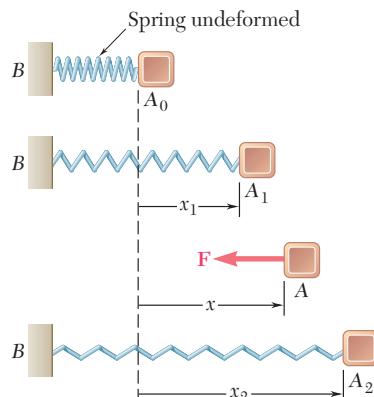


Fig. 10.12a (repeated)

## \*10.8 POTENTIAL ENERGY AND EQUILIBRIUM

The application of the principle of virtual work is considerably simplified when the potential energy of a system is known. In the case of a virtual displacement, formula (10.19) becomes  $\delta U = -\delta V$ . Moreover, if the position of the system is defined by a single independent variable  $\theta$ , we can write  $\delta V = (dV/d\theta) \delta\theta$ . Since  $\delta\theta$  must be different from zero, the condition  $\delta U = 0$  for the equilibrium of the system becomes

$$\frac{dV}{d\theta} = 0 \quad (10.21)$$

In terms of potential energy, therefore, the principle of virtual work states that *if a system is in equilibrium, the derivative of its total potential energy is zero*. If the position of the system depends upon several independent variables (the system is then said to possess *several degrees of freedom*), the partial derivatives of  $V$  with respect to each of the independent variables must be zero.

Consider, for example, a structure made of two members  $AC$  and  $CB$  and carrying a load  $W$  at  $C$ . The structure is supported by a pin at  $A$  and a roller at  $B$ , and a spring  $BD$  connects  $B$  to a fixed point  $D$  (Fig. 10.13a). The constant of the spring is  $k$ , and it is assumed that the natural length of the spring is equal to  $AD$  and thus that the spring is undeformed when  $B$  coincides with  $A$ . Neglecting the friction forces and the weight of the members, we find that the only forces which work during a displacement of the structure are the weight  $\mathbf{W}$  and the force  $\mathbf{F}$  exerted by the spring at point  $B$  (Fig. 10.13b). The total potential energy of the system will thus be obtained by adding the potential energy  $V_g$  corresponding to the gravity force  $\mathbf{W}$  and the potential energy  $V_e$  corresponding to the elastic force  $\mathbf{F}$ .

Choosing a coordinate system with origin at  $A$  and noting that the deflection of the spring, measured from its undeformed position, is  $AB = x_B$ , we write

$$V_e = \frac{1}{2}kx_B^2 \quad V_g = W y_C$$

Expressing the coordinates  $x_B$  and  $y_C$  in terms of the angle  $\theta$ , we have

$$\begin{aligned} x_B &= 2l \sin \theta & y_C &= l \cos \theta \\ V_e &= \frac{1}{2}k(2l \sin \theta)^2 & V_g &= W(l \cos \theta) \\ V &= V_e + V_g = 2kl^2 \sin^2 \theta + Wl \cos \theta \end{aligned} \quad (10.22)$$

The positions of equilibrium of the system are obtained by equating to zero the derivative of the potential energy  $V$ . We write

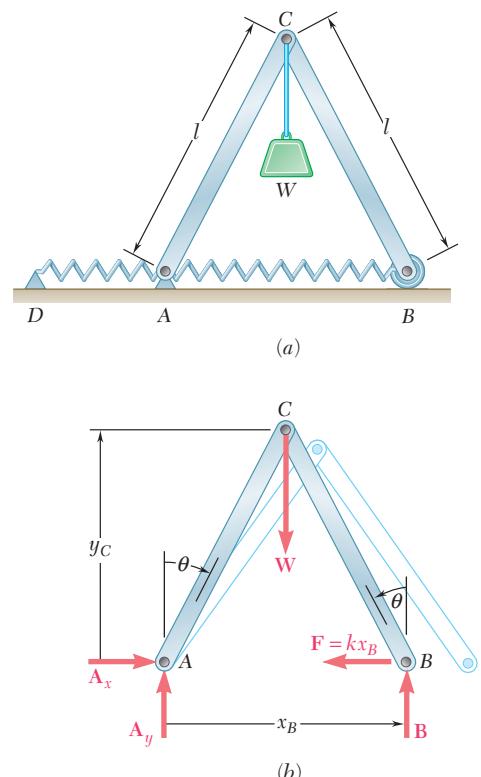
$$\frac{dV}{d\theta} = 4kl^2 \sin \theta \cos \theta - Wl \sin \theta = 0$$

or, factoring  $l \sin \theta$ ,

$$\frac{dV}{d\theta} = l \sin \theta (4kl \cos \theta - W) = 0$$

There are therefore two positions of equilibrium, corresponding to the values  $\theta = 0$  and  $\theta = \cos^{-1}(W/4kl)$ , respectively.<sup>†</sup>

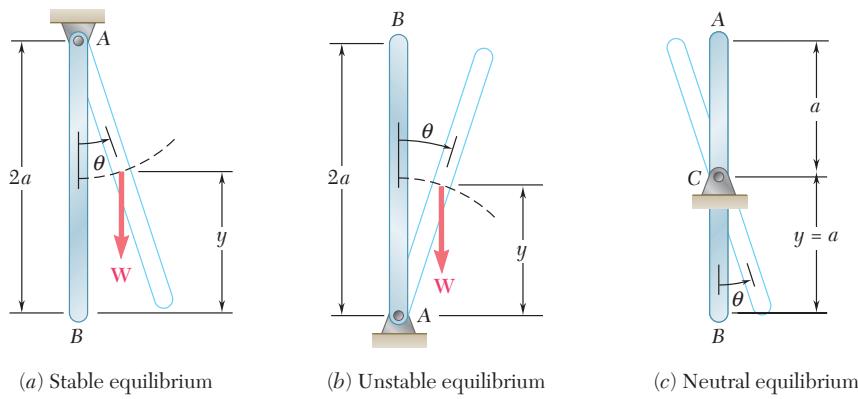
<sup>†</sup>The second position does not exist if  $W > 4kl$ .



**Fig. 10.13**

## \*10.9 STABILITY OF EQUILIBRIUM

Consider the three uniform rods of length  $2a$  and weight  $\mathbf{W}$  shown in Fig. 10.14. While each rod is in equilibrium, there is an important difference between the three cases considered. Suppose that each rod is slightly disturbed from its position of equilibrium and then released: rod  $a$  will move back toward its original position, rod  $b$  will keep moving away from its original position, and rod  $c$  will remain in its new position. In case  $a$ , the equilibrium of the rod is said to be *stable*; in case  $b$ , it is said to be *unstable*; and, in case  $c$ , it is said to be *neutral*.



**Fig. 10.14**

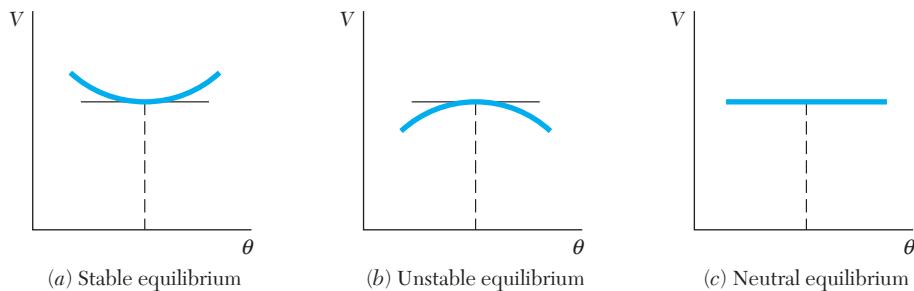
Recalling from Sec. 10.7 that the potential energy  $V_g$  with respect to gravity is equal to  $Wy$ , where  $y$  is the elevation of the point of application of  $\mathbf{W}$  measured from an arbitrary level, we observe that the potential energy of rod  $a$  is minimum in the position of equilibrium considered, that the potential energy of rod  $b$  is maximum, and that the potential energy of rod  $c$  is constant. Equilibrium is thus *stable*, *unstable*, or *neutral* according to whether the potential energy is *minimum*, *maximum*, or *constant* (Fig. 10.15).

That the result obtained is quite general can be seen as follows: We first observe that a force always tends to do positive work and thus to decrease the potential energy of the system on which it is applied. Therefore, when a system is disturbed from its position of equilibrium, the forces acting on the system will tend to bring it back to its original position if  $V$  is minimum (Fig. 10.15a) and to move it farther away if  $V$  is maximum (Fig. 10.15b). If  $V$  is constant (Fig. 10.15c), the forces will not tend to move the system either way.

Recalling from calculus that a function is minimum or maximum according to whether its second derivative is positive or negative, we can summarize the conditions for the equilibrium of a system

with one degree of freedom (i.e., a system the position of which is defined by a single independent variable  $\theta$ ) as follows:

$$\begin{aligned} \frac{dV}{d\theta} = 0 & \quad \frac{d^2V}{d\theta^2} > 0: \text{stable equilibrium} \\ \frac{dV}{d\theta} = 0 & \quad \frac{d^2V}{d\theta^2} < 0: \text{unstable equilibrium} \end{aligned} \quad (10.23)$$

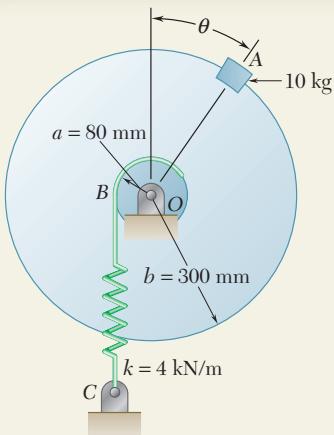


**Fig. 10.15**

If both the first and the second derivatives of  $V$  are zero, it is necessary to examine derivatives of a higher order to determine whether the equilibrium is stable, unstable, or neutral. The equilibrium will be neutral if all derivatives are zero, since the potential energy  $V$  is then a constant. The equilibrium will be stable if the first derivative found to be different from zero is of even order and positive. In all other cases the equilibrium will be unstable.

If the system considered possesses *several degrees of freedom*, the potential energy  $V$  depends upon several variables, and it is thus necessary to apply the theory of functions of several variables to determine whether  $V$  is minimum. It can be verified that a system with 2 degrees of freedom will be stable, and the corresponding potential energy  $V(\theta_1, \theta_2)$  will be minimum, if the following relations are satisfied simultaneously:

$$\begin{aligned} \frac{\partial V}{\partial \theta_1} = \frac{\partial V}{\partial \theta_2} &= 0 \\ \left( \frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} \right)^2 - \frac{\partial^2 V}{\partial \theta_1^2} \frac{\partial^2 V}{\partial \theta_2^2} &< 0 \\ \frac{\partial^2 V}{\partial \theta_1^2} > 0 & \quad \text{or} \quad \frac{\partial^2 V}{\partial \theta_2^2} > 0 \end{aligned} \quad (10.24)$$



## SAMPLE PROBLEM 10.4

A 10-kg block is attached to the rim of a 300-mm-radius disk as shown. Knowing that spring BC is unstretched when  $\theta = 0$ , determine the position or positions of equilibrium, and state in each case whether the equilibrium is stable, unstable, or neutral.

### SOLUTION

**Potential Energy.** Denoting by  $s$  the deflection of the spring from its undeformed position and placing the origin of coordinates at  $O$ , we obtain

$$V_e = \frac{1}{2}ks^2 \quad V_g = W_y = mgy$$

Measuring  $\theta$  in radians, we have

$$s = a\theta \quad y = b \cos \theta$$

Substituting for  $s$  and  $y$  in the expressions for  $V_e$  and  $V_g$ , we write

$$\begin{aligned} V_e &= \frac{1}{2}ka^2\theta^2 & V_g &= mgb \cos \theta \\ V &= V_e + V_g = \frac{1}{2}ka^2\theta^2 + mgb \cos \theta \end{aligned}$$

**Positions of Equilibrium.** Setting  $dV/d\theta = 0$ , we write

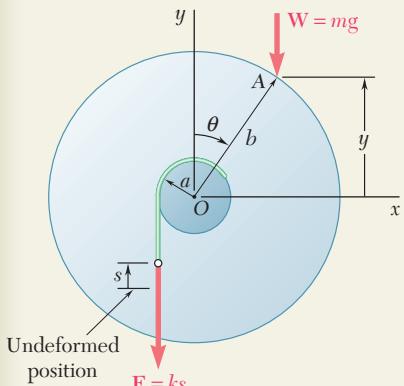
$$\begin{aligned} \frac{dV}{d\theta} &= ka^2\theta - mgb \sin \theta = 0 \\ \sin \theta &= \frac{ka^2}{mgb} \theta \end{aligned}$$

Substituting  $a = 0.08$  m,  $b = 0.3$  m,  $k = 4$  kN/m, and  $m = 10$  kg, we obtain

$$\begin{aligned} \sin \theta &= \frac{(4 \text{ kN/m})(0.08 \text{ m})^2}{(10 \text{ kg})(9.81 \text{ m/s}^2)(0.3 \text{ m})} \theta \\ \sin \theta &= 0.8699 \theta \end{aligned}$$

where  $\theta$  is expressed in radians. Solving by trial and error for  $\theta$ , we find

$$\begin{aligned} \theta &= 0 \quad \text{and} \quad \theta = 0.902 \text{ rad} \\ \theta &= 0 \quad \text{and} \quad \theta = 51.7^\circ \end{aligned}$$



**Stability of Equilibrium.** The second derivative of the potential energy  $V$  with respect to  $\theta$  is

$$\begin{aligned} \frac{d^2V}{d\theta^2} &= ka^2 - mgb \cos \theta \\ &= (4 \text{ kN/m})(0.08 \text{ m})^2 - (10 \text{ kg})(9.81 \text{ m/s}^2)(0.3 \text{ m}) \cos \theta \\ &= 25.6 - 29.43 \cos \theta \end{aligned}$$

$$\text{For } \theta = 0: \quad \frac{d^2V}{d\theta^2} = 25.6 - 29.43 \cos 0^\circ = -3.83 < 0$$

The equilibrium is unstable for  $\theta = 0$

$$\text{For } \theta = 51.7^\circ: \quad \frac{d^2V}{d\theta^2} = 25.6 - 29.43 \cos 51.7^\circ = +7.36 > 0$$

The equilibrium is stable for  $\theta = 51.7^\circ$

# SOLVING PROBLEMS ON YOUR OWN

In this lesson we defined the *work of a force during a finite displacement* and the *potential energy* of a rigid body or a system of rigid bodies. You learned to use the concept of potential energy to determine the *equilibrium position* of a rigid body or a system of rigid bodies.

**1. The potential energy  $V$  of a system** is the sum of the potential energies associated with the various forces acting on the system that *do work* as the system moves. In the problems of this lesson you will determine the following:

a. **Potential energy of a weight.** This is the potential energy due to *gravity*,  $V_g = Wy$ , where  $y$  is the elevation of the weight  $W$  measured from some arbitrary reference level. Note that the potential energy  $V_g$  may be used with any vertical force  $\mathbf{P}$  of constant magnitude directed downward; we write  $V_g = Py$ .

b. **Potential energy of a spring.** This is the potential energy due to the *elastic* force exerted by a spring,  $V_e = \frac{1}{2}kx^2$ , where  $k$  is the constant of the spring and  $x$  is the deformation of the spring *measured from its unstretched position*.

Reactions at fixed supports, internal forces at connections, forces exerted by inextensible cords and cables, and other forces which do no work do not contribute to the potential energy of the system.

**2. Express all distances and angles in terms of a single variable**, such as an angle  $\theta$ , when computing the potential energy  $V$  of a system. This is necessary, since the determination of the equilibrium position of the system requires the computation of the derivative  $dV/d\theta$ .

**3. When a system is in equilibrium, the first derivative of its potential energy is zero.** Therefore:

a. **To determine a position of equilibrium of a system**, once its potential energy  $V$  has been expressed in terms of the single variable  $\theta$ , compute its derivative and solve the equation  $dV/d\theta = 0$  for  $\theta$ .

b. **To determine the force or couple required to maintain a system in a given position of equilibrium**, substitute the known value of  $\theta$  in the equation  $dV/d\theta = 0$  and solve this equation for the desired force or couple.

**4. Stability of equilibrium.** The following rules generally apply:

a. **Stable equilibrium** occurs when the potential energy of the system is *minimum*, that is, when  $dV/d\theta = 0$  and  $d^2V/d\theta^2 > 0$  (Figs. 10.14a and 10.15a).

b. **Unstable equilibrium** occurs when the potential energy of the system is *maximum*, that is, when  $dV/d\theta = 0$  and  $d^2V/d\theta^2 < 0$  (Figs. 10.14b and 10.15b).

c. **Neutral equilibrium** occurs when the potential energy of the system is *constant*;  $dV/d\theta$ ,  $d^2V/d\theta^2$ , and all the successive derivatives of  $V$  are then equal to zero (Figs. 10.14c and 10.15c).

See page 583 for a discussion of the case when  $dV/d\theta$ ,  $d^2V/d\theta^2$  but *not all* of the successive derivatives of  $V$  are equal to zero.

# PROBLEMS

**10.59** Using the method of Sec. 10.8, solve Prob. 10.29.

**10.60** Using the method of Sec. 10.8, solve Prob. 10.30.

**10.61** Using the method of Sec. 10.8, solve Prob. 10.33.

**10.62** Using the method of Sec. 10.8, solve Prob. 10.34.

**10.63** Using the method of Sec. 10.8, solve Prob. 10.35.

**10.64** Using the method of Sec. 10.8, solve Prob. 10.36.

**10.65** Using the method of Sec. 10.8, solve Prob. 10.31.

**10.66** Using the method of Sec. 10.8, solve Prob. 10.38.

**10.67** Show that the equilibrium is neutral in Prob. 10.1.

**10.68** Show that the equilibrium is neutral in Prob. 10.6.

**10.69** Two uniform rods, each of mass  $m$  and length  $l$ , are attached to drums that are connected by a belt as shown. Assuming that no slipping occurs between the belt and the drums, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

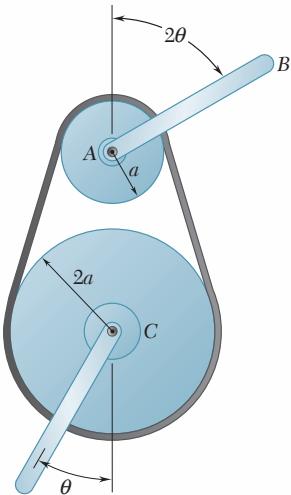


Fig. P10.69

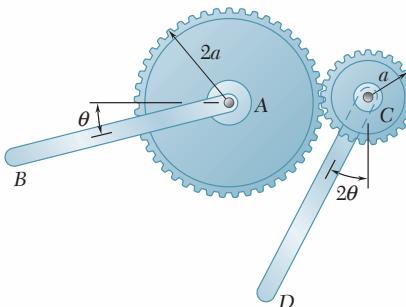
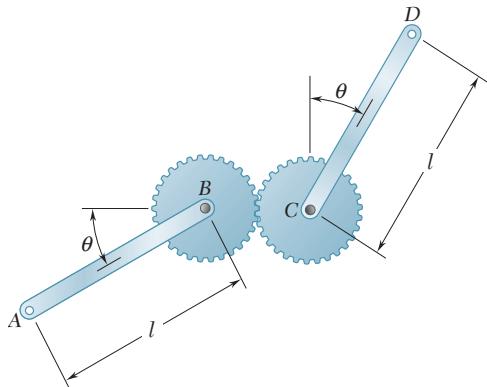


Fig. P10.70

**10.70** Two uniform rods  $AB$  and  $CD$ , of the same length  $l$ , are attached to gears as shown. Knowing that rod  $AB$  weighs 3 lb and that rod  $CD$  weighs 2 lb, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

- 10.71** Two uniform rods, each of mass  $m$ , are attached to gears of equal radii as shown. Determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.



**Fig. P10.71 and P10.72**

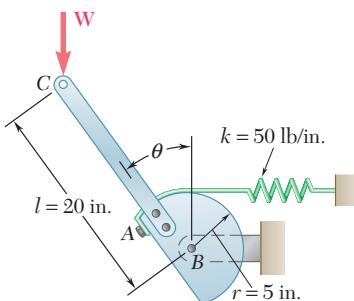
- 10.72** Two uniform rods,  $AB$  and  $CD$ , are attached to gears of equal radii as shown. Knowing that  $W_{AB} = 8 \text{ lb}$  and  $W_{CD} = 4 \text{ lb}$ , determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

- 10.73** Using the method of Sec. 10.8, solve Prob. 10.39. Determine whether the equilibrium is stable, unstable, or neutral. (*Hint:* The potential energy corresponding to the couple exerted by a torsion spring is  $\frac{1}{2}K\theta^2$ , where  $K$  is the torsional spring constant and  $\theta$  is the angle of twist.)

- 10.74** In Prob. 10.40, determine whether each of the positions of equilibrium is stable, unstable, or neutral. (See hint for Prob. 10.73.)

- 10.75** A load  $\mathbf{W}$  of magnitude 100 lb is applied to the mechanism at  $C$ . Knowing that the spring is unstretched when  $\theta = 15^\circ$ , determine that value of  $\theta$  corresponding to equilibrium and check that the equilibrium is stable.

- 10.76** A load  $\mathbf{W}$  of magnitude 100 lb is applied to the mechanism at  $C$ . Knowing that the spring is unstretched when  $\theta = 30^\circ$ , determine that value of  $\theta$  corresponding to equilibrium and check that the equilibrium is stable.



**Fig. P10.75 and P10.76**

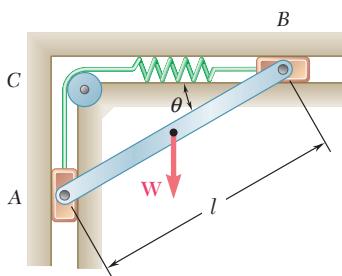


Fig. P10.77 and P10.78

- 10.77** A slender rod  $AB$ , of weight  $W$ , is attached to two blocks  $A$  and  $B$  that can move freely in the guides shown. The constant of the spring is  $k$ , and the spring is unstretched when  $AB$  is horizontal. Neglecting the weight of the blocks, derive an equation in  $\theta$ ,  $W$ ,  $l$ , and  $k$  that must be satisfied when the rod is in equilibrium.

- 10.78** A slender rod  $AB$ , of weight  $W$ , is attached to two blocks  $A$  and  $B$  that can move freely in the guides shown. Knowing that the spring is unstretched when  $AB$  is horizontal, determine three values of  $\theta$  corresponding to equilibrium when  $W = 300$  lb,  $l = 16$  in., and  $k = 75$  lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

- 10.79** A slender rod  $AB$ , of weight  $W$ , is attached to two blocks  $A$  and  $B$  that can move freely in the guides shown. Knowing that the spring is unstretched when  $y = 0$ , determine the value of  $y$  corresponding to equilibrium when  $W = 80$  N,  $l = 500$  mm, and  $k = 600$  N/m.

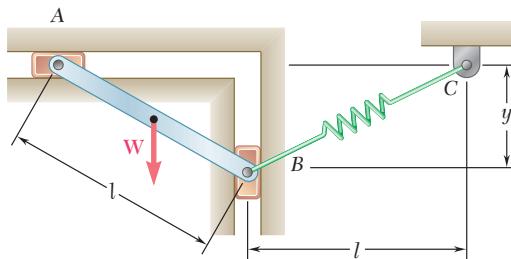


Fig. P10.79

- 10.80** Knowing that both springs are unstretched when  $y = 0$ , determine the value of  $y$  corresponding to equilibrium when  $W = 80$  N,  $l = 500$  mm, and  $k = 600$  N/m.

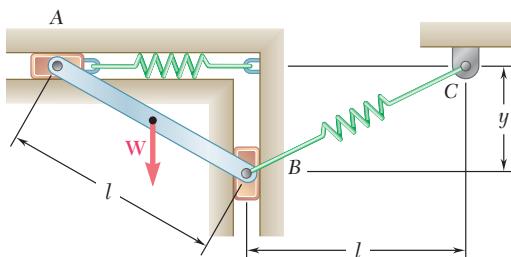


Fig. P10.80

- 10.81** A spring  $AB$  of constant  $k$  is attached to two identical gears as shown. Knowing that the spring is undeformed when  $\theta = 0$ , determine two values of the angle  $\theta$  corresponding to equilibrium when  $P = 30$  lb,  $a = 4$  in.,  $b = 3$  in.,  $r = 6$  in., and  $k = 5$  lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

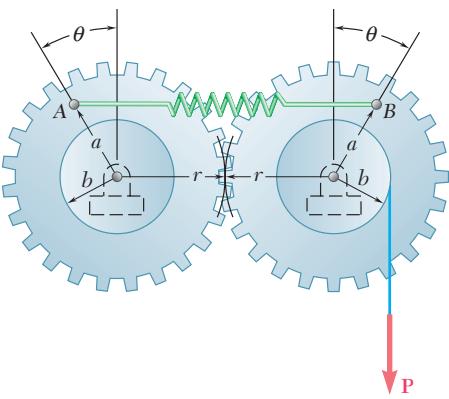


Fig. P10.81 and P10.82

- 10.82** A spring  $AB$  of constant  $k$  is attached to two identical gears as shown. Knowing that the spring is undeformed when  $\theta = 0$ , and given that  $a = 60$  mm,  $b = 45$  mm,  $r = 90$  mm, and  $k = 6$  kN/m, determine (a) the range of values of  $P$  for which a position of equilibrium exists, (b) two values of  $\theta$  corresponding to equilibrium if the value of  $P$  is equal to half the upper limit of the range found in part *a*.

- 10.83** A slender rod  $AB$  is attached to two collars  $A$  and  $B$  that can move freely along the guide rods shown. Knowing that  $\beta = 30^\circ$  and  $P = Q = 400 \text{ N}$ , determine the value of the angle  $\theta$  corresponding to equilibrium.

- 10.84** A slender rod  $AB$  is attached to two collars  $A$  and  $B$  that can move freely along the guide rods shown. Knowing that  $\beta = 30^\circ$ ,  $P = 100 \text{ N}$ , and  $Q = 25 \text{ N}$ , determine the value of the angle  $\theta$  corresponding to equilibrium.

- 10.85 and 10.86** Collar  $A$  can slide freely on the semicircular rod shown. Knowing that the constant of the spring is  $k$  and that the unstretched length of the spring is equal to the radius  $r$ , determine the value of  $\theta$  corresponding to equilibrium when  $W = 50 \text{ lb}$ ,  $r = 9 \text{ in.}$ , and  $k = 15 \text{ lb/in.}$

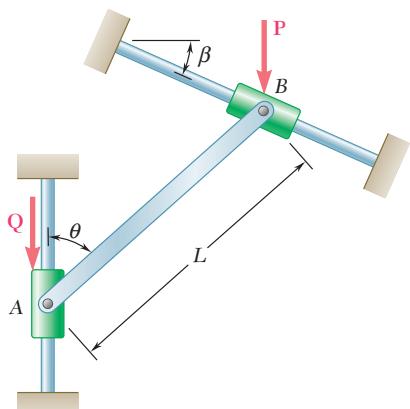


Fig. P10.83 and P10.84

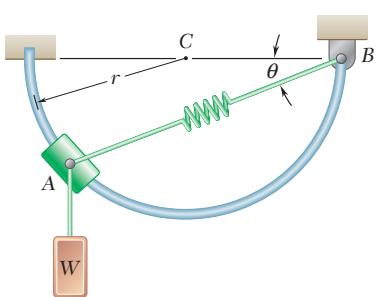


Fig. P10.85

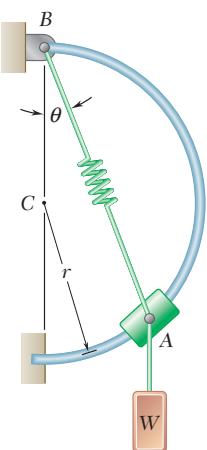


Fig. P10.86

- 10.87 and 10.88** Cart  $B$ , which weighs  $75 \text{ kN}$ , rolls along a sloping track that forms an angle  $\beta$  with the horizontal. The spring constant is  $5 \text{ kN/m}$ , and the spring is unstretched when  $x = 0$ . Determine the distance  $x$  corresponding to equilibrium for the angle  $\beta$  indicated.

**10.87** Angle  $\beta = 30^\circ$

**10.88** Angle  $\beta = 60^\circ$

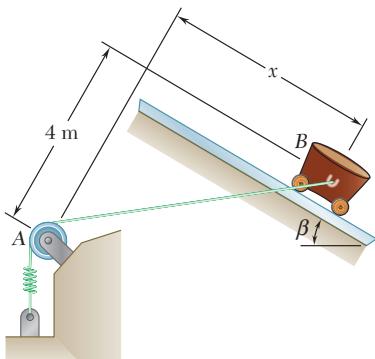


Fig. P10.87 and P10.88

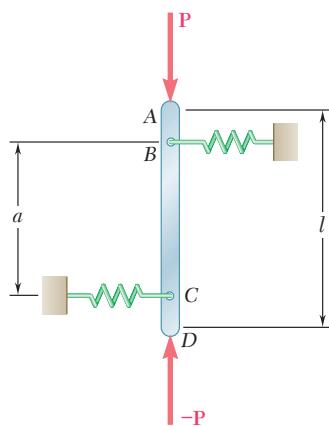


Fig. P10.89

- 10.89** A vertical bar  $AD$  is attached to two springs of constant  $k$  and is in equilibrium in the position shown. Determine the range of values of the magnitude  $P$  of two equal and opposite vertical forces  $\mathbf{P}$  and  $-\mathbf{P}$  for which the equilibrium position is stable if (a)  $AB = CD$ , (b)  $AB = 2CD$ .

- 10.90** Rod  $AB$  is attached to a hinge at  $A$  and to two springs, each of constant  $k$ . If  $h = 25$  in.,  $d = 12$  in., and  $W = 80$  lb, determine the range of values of  $k$  for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

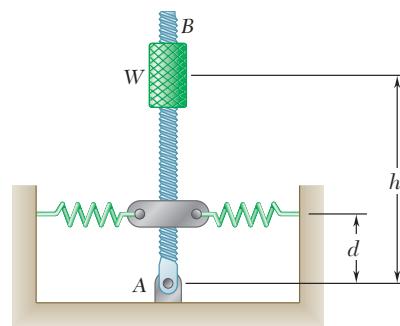


Fig. P10.90 and P10.91

- 10.91** Rod  $AB$  is attached to a hinge at  $A$  and to two springs, each of constant  $k$ . If  $h = 45$  in.,  $k = 6$  lb/in., and  $W = 60$  lb, determine the smallest distance  $d$  for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

- 10.92 and 10.93** Two bars are attached to a single spring of constant  $k$  that is unstretched when the bars are vertical. Determine the range of values of  $P$  for which the equilibrium of the system is stable in the position shown.

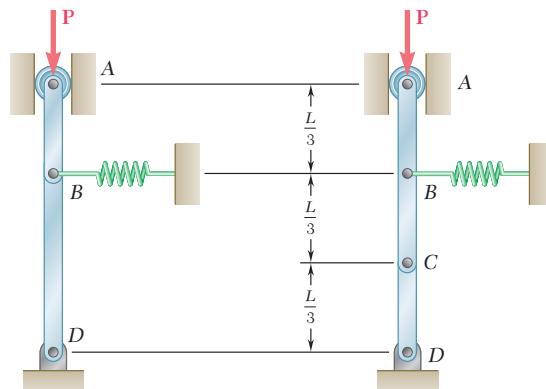


Fig. P10.92 and P10.93

- 10.94** Two bars  $AB$  and  $BC$  are attached to a single spring of constant  $k$  that is unstretched when the bars are vertical. Determine the range of values of  $P$  for which the equilibrium of the system is stable in the position shown.

- 10.95** The horizontal bar  $BEH$  is connected to three vertical bars. The collar at  $E$  can slide freely on bar  $DF$ . Determine the range of values of  $Q$  for which the equilibrium of the system is stable in the position shown when  $a = 24$  in.,  $b = 20$  in., and  $P = 150$  lb.

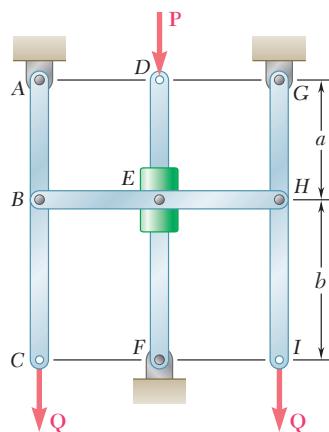


Fig. P10.95 and P10.96

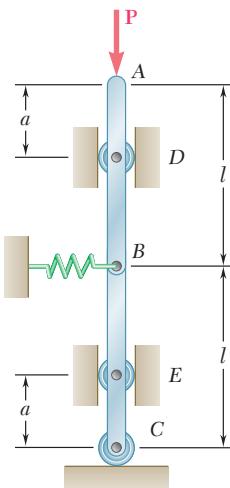


Fig. P10.94

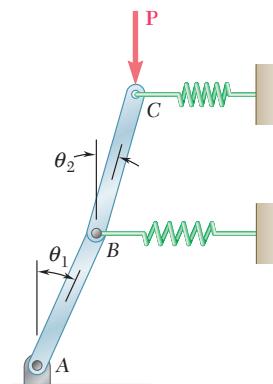


Fig. P10.97

- 10.96** The horizontal bar  $BEH$  is connected to three vertical bars. The collar at  $E$  can slide freely on bar  $DF$ . Determine the range of values of  $P$  for which the equilibrium of the system is stable in the position shown when  $a = 150$  mm,  $b = 200$  mm, and  $Q = 45$  N.

- \*10.97** Bars  $AB$  and  $BC$ , each of length  $l$  and of negligible weight, are attached to two springs, each of constant  $k$ . The springs are undeformed, and the system is in equilibrium when  $\theta_1 = \theta_2 = 0$ . Determine the range of values of  $P$  for which the equilibrium position is stable.

- \*10.98** Solve Prob. 10.97 knowing that  $l = 800$  mm and  $k = 2.5$  kN/m.

- \*10.99** Two rods of negligible weight are attached to drums of radius  $r$  that are connected by a belt and spring of constant  $k$ . Knowing that the spring is undeformed when the rods are vertical, determine the range of values of  $P$  for which the equilibrium position  $\theta_1 = \theta_2 = 0$  is stable.

- \*10.100** Solve Prob. 10.99 knowing that  $k = 20$  lb/in.,  $r = 3$  in.,  $l = 6$  in., and (a)  $W = 15$  lb, (b)  $W = 60$  lb.

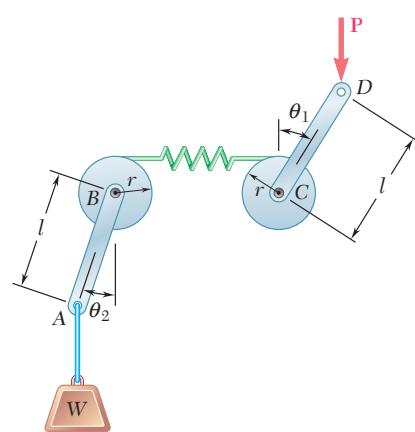


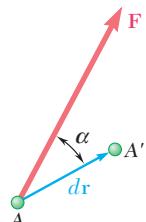
Fig. P10.99

# REVIEW AND SUMMARY

## Work of a force

The first part of this chapter was devoted to the *principle of virtual work* and to its direct application to the solution of equilibrium problems. We first defined the *work of a force  $\mathbf{F}$  corresponding to the small displacement  $d\mathbf{r}$*  [Sec. 10.2] as the quantity

$$dU = \mathbf{F} \cdot d\mathbf{r} \quad (10.1)$$



**Fig. 10.16**

obtained by forming the scalar product of the force  $\mathbf{F}$  and the displacement  $d\mathbf{r}$  (Fig. 10.16). Denoting respectively by  $F$  and  $ds$  the magnitudes of the force and of the displacement, and by  $\alpha$  the angle formed by  $\mathbf{F}$  and  $d\mathbf{r}$ , we wrote

$$dU = F ds \cos \alpha \quad (10.1')$$

The work  $dU$  is positive if  $\alpha < 90^\circ$ , zero if  $\alpha = 90^\circ$ , and negative if  $\alpha > 90^\circ$ . We also found that the *work of a couple of moment  $\mathbf{M}$*  acting on a rigid body is

$$dU = M d\theta \quad (10.2)$$

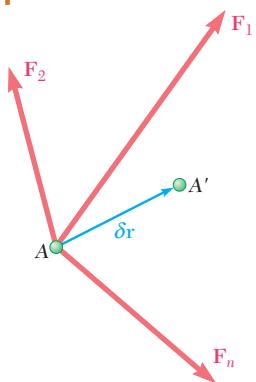
where  $d\theta$  is the small angle expressed in radians through which the body rotates.

## Virtual displacement

Considering a particle located at  $A$  and acted upon by several forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$  [Sec. 10.3], we imagined that the particle moved to a new position  $A'$  (Fig. 10.17). Since this displacement did not actually take place, it was referred to as a *virtual displacement* and denoted by  $\delta\mathbf{r}$ , while the corresponding work of the forces was called *virtual work* and denoted by  $\delta U$ . We had

$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \dots + \mathbf{F}_n \cdot \delta\mathbf{r}$$

## Principle of virtual work



**Fig. 10.17**

The *principle of virtual work* states that if a particle is in equilibrium, the total virtual work  $\delta U$  of the forces acting on the particle is zero for any virtual displacement of the particle.

The principle of virtual work can be extended to the case of rigid bodies and systems of rigid bodies. Since it involves *only forces which do work*, its application provides a useful alternative to the use of the equilibrium equations in the solution of many engineering problems. It is particularly effective in the case of machines and mechanisms consisting of connected rigid bodies, since the work of the reactions at the supports is zero and the work of the internal forces at the pin connections cancels out [Sec. 10.4; Sample Probs. 10.1, 10.2, and 10.3].

In the case of *real machines*, however [Sec. 10.5], the work of the friction forces should be taken into account, with the result that the *output work will be less than the input work*. Defining the *mechanical efficiency* of a machine as the ratio

$$\eta = \frac{\text{output work}}{\text{input work}} \quad (10.9)$$

we also noted that for an ideal machine (no friction)  $\eta = 1$ , while for a real machine  $\eta < 1$ .

In the second part of the chapter we considered the *work of forces corresponding to finite displacements* of their points of application. The work  $U_{1 \rightarrow 2}$  of the force  $\mathbf{F}$  corresponding to a displacement of the particle A from  $A_1$  to  $A_2$  (Fig. 10.18) was obtained by integrating the right-hand member of Eq. (10.1) or (10.1') along the curve described by the particle [Sec. 10.6]:

$$U_{1 \rightarrow 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \quad (10.11)$$

or

$$U_{1 \rightarrow 2} = \int_{s_1}^{s_2} (F \cos \alpha) ds \quad (10.11')$$

Similarly, the work of a couple of moment  $\mathbf{M}$  corresponding to a finite rotation from  $\theta_1$  to  $\theta_2$  of a rigid body was expressed as

$$U_{1 \rightarrow 2} = \int_{\theta_1}^{\theta_2} M d\theta \quad (10.12)$$

The *work of the weight  $\mathbf{W}$  of a body* as its center of gravity moves from the elevation  $y_1$  to  $y_2$  (Fig. 10.19) can be obtained by making  $F = W$  and  $\alpha = 180^\circ$  in Eq. (10.11'):

$$U_{1 \rightarrow 2} = - \int_{y_1}^{y_2} W dy = Wy_1 - Wy_2 \quad (10.13)$$

The work of  $\mathbf{W}$  is therefore positive *when the elevation  $y$  decreases*.

## Mechanical efficiency

## Work of a force over a finite displacement

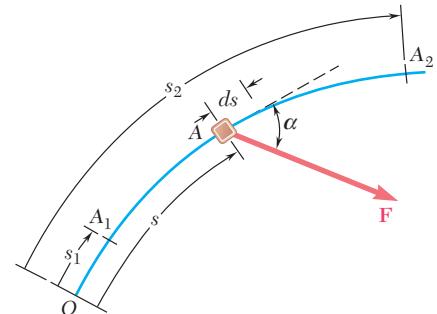


Fig. 10.18

## Work of a weight

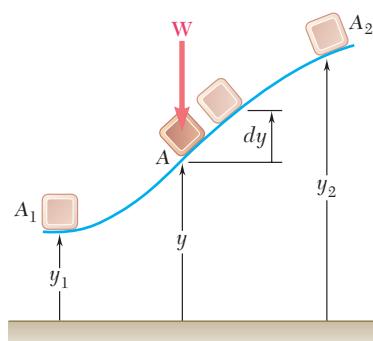


Fig. 10.19

## Work of the force exerted by a spring

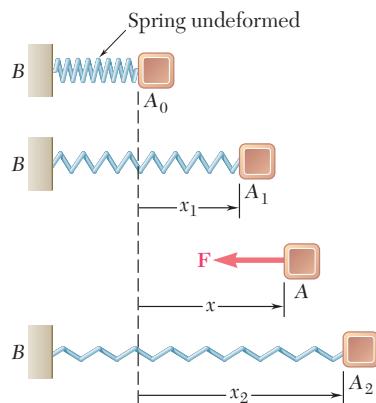


Fig. 10.20

### Potential energy

The *work of the force  $\mathbf{F}$  exerted by a spring* on a body  $A$  as the spring is stretched from  $x_1$  to  $x_2$  (Fig. 10.20) can be obtained by making  $F = kx$ , where  $k$  is the constant of the spring, and  $\alpha = 180^\circ$  in Eq. (10.11'):

$$U_{1 \rightarrow 2} = - \int_{x_1}^{x_2} kx \, dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \quad (10.15)$$

The work of  $\mathbf{F}$  is therefore positive *when the spring is returning to its undeformed position*.

When the work of a force  $\mathbf{F}$  is independent of the path actually followed between  $A_1$  and  $A_2$ , the force is said to be a *conservative force*, and its work can be expressed as

$$U_{1 \rightarrow 2} = V_1 - V_2 \quad (10.20)$$

where  $V$  is the *potential energy* associated with  $\mathbf{F}$ , and  $V_1$  and  $V_2$  represent the values of  $V$  at  $A_1$  and  $A_2$ , respectively [Sec. 10.7]. The potential energies associated, respectively, with the *force of gravity*  $\mathbf{W}$  and the *elastic force*  $\mathbf{F}$  exerted by a spring were found to be

$$V_g = Wy \quad \text{and} \quad V_e = \frac{1}{2}kx^2 \quad (10.17, 10.18)$$

## Alternative expression for the principle of virtual work

When the position of a mechanical system depends upon a single independent variable  $\theta$ , the potential energy of the system is a function  $V(\theta)$  of that variable, and it follows from Eq. (10.20) that  $\delta U = -\delta V = -(dV/d\theta) \delta\theta$ . The condition  $\delta U = 0$  required by the principle of virtual work for the equilibrium of the system can thus be replaced by the condition

$$\frac{dV}{d\theta} = 0 \quad (10.21)$$

When all the forces involved are conservative, it may be preferable to use Eq. (10.21) rather than apply the principle of virtual work directly [Sec. 10.8; Sample Prob. 10.4].

## Stability of equilibrium

This approach presents another advantage, since it is possible to determine from the sign of the second derivative of  $V$  whether the equilibrium of the system is *stable*, *unstable*, or *neutral* [Sec. 10.9]. If  $d^2V/d\theta^2 > 0$ ,  $V$  is *minimum* and the equilibrium is *stable*; if  $d^2V/d\theta^2 < 0$ ,  $V$  is *maximum* and the equilibrium is *unstable*; if  $d^2V/d\theta^2 = 0$ , it is necessary to examine derivatives of a higher order.

# REVIEW PROBLEMS

**10.101** Determine the vertical force  $\mathbf{P}$  that must be applied at  $G$  to maintain the equilibrium of the linkage.

**10.102** Determine the couple  $\mathbf{M}$  that must be applied to member  $DEFG$  to maintain the equilibrium of the linkage.

**10.103** Derive an expression for the magnitude of the couple  $\mathbf{M}$  required to maintain the equilibrium of the linkage shown.

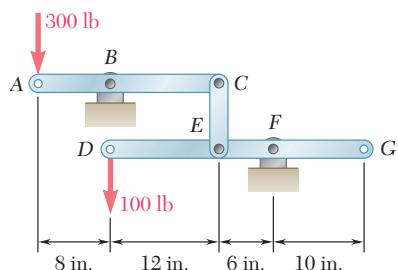


Fig. P10.101 and P10.102

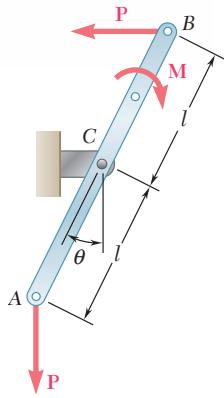


Fig. P10.103

**10.104** Collars  $A$  and  $B$  are connected by the wire  $AB$  and can slide freely on the rods shown. Knowing that the length of the wire is 440 mm and that the weight  $W$  of collar  $A$  is 90 N, determine the magnitude of the force  $\mathbf{P}$  required to maintain equilibrium of the system when (a)  $c = 80$  mm, (b)  $c = 280$  mm.

**10.105** Collar  $B$  can slide along rod  $AC$  and is attached by a pin to a block that can slide in the vertical slot shown. Derive an expression for the magnitude of the couple  $\mathbf{M}$  required to maintain equilibrium.

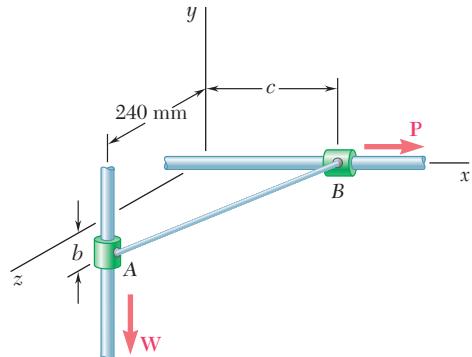


Fig. P10.104

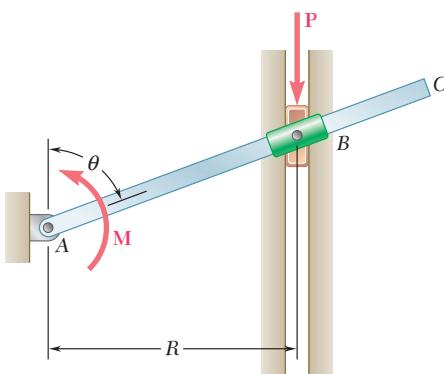


Fig. P10.105

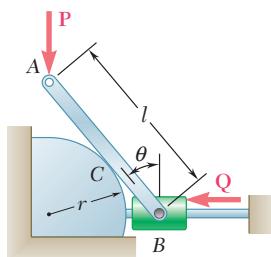


Fig. P10.106

- 10.106** A slender rod of length  $l$  is attached to a collar at  $B$  and rests on a portion of a circular cylinder of radius  $r$ . Neglecting the effect of friction, determine the value of  $\theta$  corresponding to the equilibrium position of the mechanism when  $l = 200$  mm,  $r = 60$  mm,  $P = 40$  N, and  $Q = 80$  N.

- 10.107** A horizontal force  $\mathbf{P}$  of magnitude 40 lb is applied to the mechanism at  $C$ . The constant of the spring is  $k = 9$  lb/in., and the spring is unstretched when  $\theta = 0$ . Neglecting the weight of the mechanism, determine the value of  $\theta$  corresponding to equilibrium.

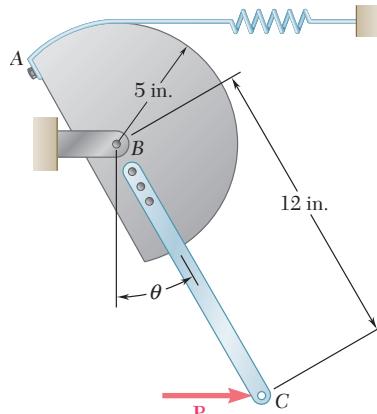


Fig. P10.107

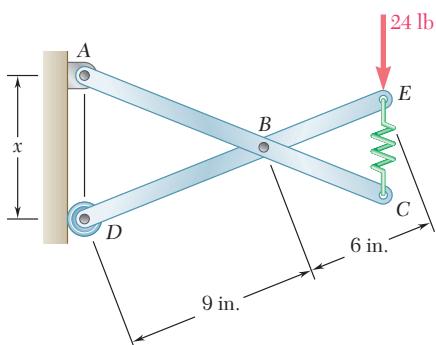


Fig. P10.108

- 10.108** Two identical rods  $ABC$  and  $DBE$  are connected by a pin at  $B$  and by a spring  $CE$ . Knowing that the spring is 4 in. long when unstretched and that the constant of the spring is 8 lb/in., determine the distance  $x$  corresponding to equilibrium when a 24-lb load is applied at  $E$  as shown.

- 10.109** Solve Prob. 10.108 assuming that the 24-lb load is applied at  $C$  instead of  $E$ .

- 10.110** Two uniform rods, each of mass  $m$  and length  $l$ , are attached to gears as shown. For the range  $0 \leq \theta \leq 180^\circ$ , determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

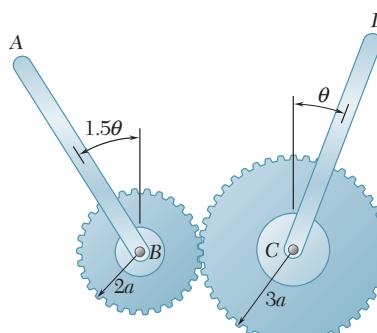
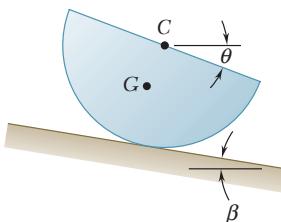


Fig. P10.110

- 10.111** A homogeneous hemisphere of radius  $r$  is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine the angle  $\theta$  corresponding to equilibrium when  $\beta = 10^\circ$ .



**Fig. P10.111 and P10.112**

- 10.112** A homogeneous hemisphere of radius  $r$  is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine (a) the largest angle  $\beta$  for which a position of equilibrium exists, (b) the angle  $\theta$  corresponding to equilibrium when the angle  $\beta$  is equal to half the value found in part a.

# COMPUTER PROBLEMS

**10.C1** A couple  $\mathbf{M}$  is applied to crank  $AB$  in order to maintain the equilibrium of the engine system shown when a force  $\mathbf{P}$  is applied to the piston. Knowing that  $b = 2.4$  in. and  $l = 7.5$  in., use computational software to calculate the ratio  $M/P$  for values of  $\theta$  from 0 to  $180^\circ$  using  $10^\circ$  increments. Using appropriate smaller increments, determine the value of  $\theta$  for which the ratio  $M/P$  is maximum, and the corresponding value of  $M/P$ .

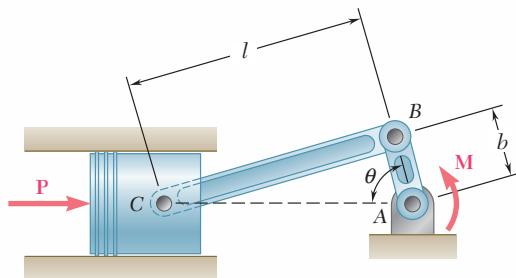


Fig. P10.C1

**10.C2** Knowing that  $a = 500$  mm,  $b = 150$  mm,  $L = 500$  mm, and  $P = 100$  N, use computational software to calculate the force in member  $BD$  for values of  $\theta$  from  $30^\circ$  to  $150^\circ$  using  $10^\circ$  increments. Using appropriate smaller increments, determine the range of values of  $\theta$  for which the absolute value of the force in member  $BD$  is less than 400 N.

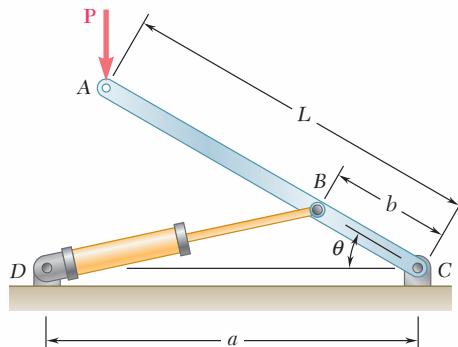


Fig. P10.C2

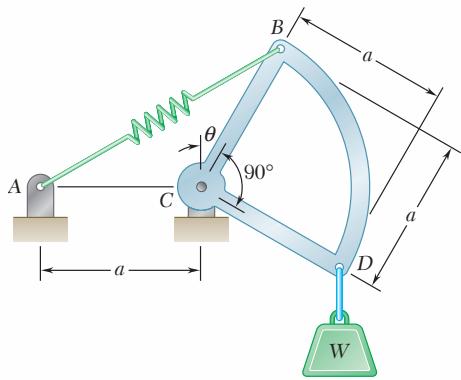


Fig. P10.C4

**10.C3** Solve Prob. 10.C2 assuming that the force  $\mathbf{P}$  applied at  $A$  is directed horizontally to the right.

**10.C4** The constant of spring  $AB$  is  $k$ , and the spring is unstretched when  $\theta = 0$ . (a) Neglecting the weight of the member  $BCD$ , use computational software to calculate the potential energy of the system and its derivative  $dV/d\theta$ . (b) For  $W = 150$  lb,  $a = 10$  in., and  $k = 75$  lb/in., calculate and plot the potential energy versus  $\theta$  for values of  $\theta$  from 0 to  $165^\circ$  using  $15^\circ$  increments. (c) Using appropriate smaller increments, determine the values of  $\theta$  for which the system is in equilibrium and state in each case whether the equilibrium is stable, unstable, or neutral.

**10.C5** Two rods,  $AC$  and  $DE$ , each of length  $L$ , are connected by a collar that is attached to rod  $AC$  at its midpoint  $B$ . (a) Using computational software, calculate the potential energy  $V$  of the system and its derivative  $dV/d\theta$ . (b) For  $W = 75$  N,  $P = 200$  N, and  $L = 500$  mm, calculate  $V$  and  $dV/d\theta$  for values of  $\theta$  from 0 to  $70^\circ$  using  $5^\circ$  increments. (c) Using appropriate smaller increments, determine the values of  $\theta$  for which the system is in equilibrium and state in each case whether the equilibrium is stable, unstable, or neutral.

**10.C6** A slender rod  $ABC$  is attached to blocks  $A$  and  $B$  that can move freely in the guides shown. The constant of the spring is  $k$ , and the spring is unstretched when the rod is vertical. (a) Neglecting the weights of the rod and of the blocks, use computational software to calculate the potential energy  $V$  of the system and its derivative  $dV/d\theta$ . (b) For  $P = 150$  N,  $l = 200$  mm, and  $k = 3$  kN/m, calculate and plot the potential energy versus  $\theta$  for values of  $\theta$  from 0 to  $75^\circ$  using  $5^\circ$  increments. (c) Using appropriate smaller increments, determine any positions of equilibrium in the range  $0 \leq \theta \leq 75^\circ$  and state in each case whether the equilibrium is stable, unstable, or neutral.

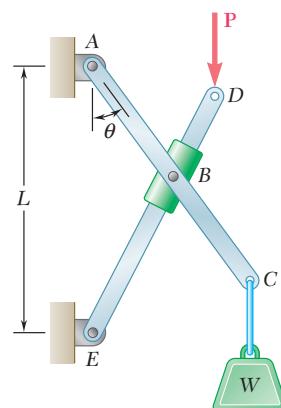


Fig. P10.C5

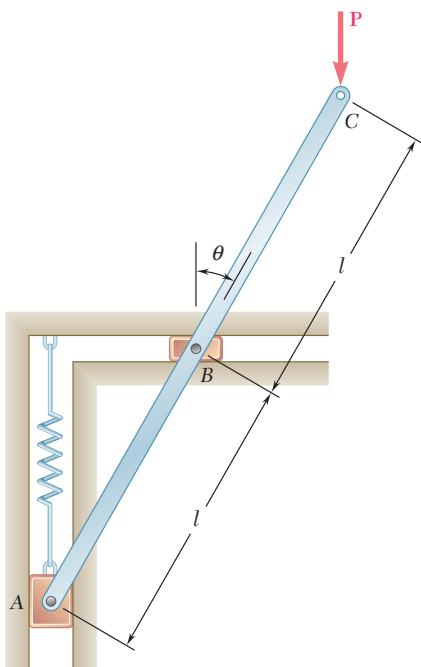


Fig. P10.C6

**10.C7** Solve Prob. 10.C6 assuming that the force  $\mathbf{P}$  applied at  $C$  is directed horizontally to the right.



# Appendix

## Fundamentals of Engineering Examination

Engineers are required to be licensed when their work directly affects the public health, safety, and welfare. The intent is to ensure that engineers have met minimum qualifications involving competence, ability, experience, and character. The licensing process involves an initial exam, called the *Fundamentals of Engineering Examination*; professional experience; and a second exam, called the *Principles and Practice of Engineering*. Those who successfully complete these requirements are licensed as a *Professional Engineer*. The exams are developed under the auspices of the *National Council of Examiners for Engineering and Surveying*.

The first exam, the *Fundamentals of Engineering Examination*, can be taken just before or after graduation from a four-year accredited engineering program. The exam stresses subject material in a typical undergraduate engineering program, including statics. The topics included in the exam cover much of the material in this book. The following is a list of the main topic areas, with references to the appropriate sections in this book. Also included are problems that can be solved to review this material.

### **Concurrent Force Systems (2.2–2.9; 2.12–2.14)**

Problems: 2.33, 2.35, 2.36, 2.37, 2.73, 2.83, 2.92, 2.93, 2.97

### **Vector Forces (3.4–3.11)**

Problems: 3.16, 3.18, 3.25, 3.31, 3.38, 3.40

### **Equilibrium in Two Dimensions (2.11; 4.1–4.7)**

Problems: 4.5, 4.13, 4.14, 4.17, 4.29, 4.38, 4.66, 4.75

### **Equilibrium in Three Dimensions (2.15; 4.8–4.9)**

Problems: 4.101, 4.104, 4.103, 4.106, 4.115, 4.117, 4.127, 4.132, 4.140

### **Centroid of an Area (5.2–5.7)**

Problems: 5.6, 5.18, 5.29, 5.35, 5.40, 5.56, 5.58, 5.99, 5.103, 5.104, 5.125

**Analysis of Trusses (6.2–6.7)**

Problems: 6.3, 6.4, 6.33, 6.43, 6.44, 6.56

**Equilibrium of Two-Dimensional Frames (6.9–6.11)**

Problems: 6.76, 6.80, 6.87, 6.91, 6.92

**Shear and Bending Moment (7.3–7.6)**

Problems: 7.22, 7.25, 7.31, 7.36, 7.45, 7.49, 7.70, 7.83

**Friction (8.2–8.5; 8.10)**

Problems: 8.11, 8.15, 8.21, 8.30, 8.50, 8.53, 8.101, 8.104, 8.105

**Moments of Inertia (9.2–9.10)**

Problems: 9.5, 9.31, 9.32, 9.33, 9.77, 9.78, 9.84, 9.89, 9.101, 9.103

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# Answers to Problems

Answers to problems with a number set in straight type are given on this and the following pages. Answers to problems set in italic are not listed.

## CHAPTER 2

- 2.1** 179 N  $\nwarrow$  75.1°.  
**2.2** 77.1 lb  $\swarrow$  85.4°.  
**2.3** 139.1 lb  $\nearrow$  67.0°.  
**2.5** (a) 76.1°. (b) 336 lb.  
**2.7** (a) 37.1°. (b) 73.2 N.  
**2.8** (a) 44.7 N. (b) 107.1 N.  
**2.9** (a) 3660 N. (b) 3730 N.  
**2.10** 2600 N  $\nwarrow$  53.5°.  
**2.11** (a) 392 lb. (b) 346 lb.  
**2.13** (a) 21.1 N ↓. (b) 45.3 N.  
**2.14** (a) 368 lb →. (b) 213 lb.  
**2.15** 77.1 lb  $\swarrow$  85.4°.  
**2.16** 139.1 lb  $\nearrow$  67.0°.  
**2.17** 3.30 kN  $\nwarrow$  66.6°.  
**2.19** 21.8 kN  $\nwarrow$  86.6°.  
**2.21** (800 N) 640 N, 480 N; (424 N) -224 N, -360 N; (408 N) 192.0 N, -360 N.  
**2.22** (29 lb) 21.0 lb, 20.0 lb; (50 lb) -14.00 lb, 48.0 lb; (51 lb) 24.0 lb, -45.0 lb.  
**2.23** (40 lb) 20.0 lb, -34.6 lb; (50 lb) -38.3 lb, -32.1 lb; (60 lb) 54.4 lb, 25.4 lb.  
**2.25** (a) 523 lb. (b) 428 lb.  
**2.26** (a) 2190 N. (b) 2060 N.  
**2.27** (a) 194.9 N. (b) 153.6 N.  
**2.30** (a) 610 lb. (b) 500 lb.  
**2.31** 38.6 lb  $\angle$  36.6°.  
**2.32** 251 N  $\Delta$  85.3°.  
**2.34** 654 N  $\nwarrow$  21.5°.  
**2.35** 309 N  $\nearrow$  86.6°.  
**2.36** 226 N  $\nearrow$  62.3°.  
**2.37** 203 lb  $\angle$  8.46°.  
**2.39** (a) 21.7°. (b) 229 N.  
**2.40** (a) 580 N. (b) 300 N.  
**2.42** (a) 56.3°. (b) 204 lb.  
**2.43** (a) 2.13 kN. (b) 1.735 kN.  
**2.45** (a) 305 N. (b) 514 N.  
**2.47** (a) 1244 lb. (b) 115.4 lb.  
**2.48** (a) 172.7 lb. (b) 231 lb.  
**2.49**  $F_A = 1303$  lb;  $F_B = 420$  lb.  
**2.51**  $F_C = 6.40$  kN;  $F_D = 4.80$  kN.  
**2.52**  $F_B = 15.00$  kN;  $F_C = 8.00$  kN.  
**2.53** (a) 52.0 lb. (b) 45.0 lb.  
**2.55** (a) 1213 N. (b) 166.3 N.  
**2.56** (a) 863 N. (b) 1216 N.  
**2.57** (a) 784 N. (b) 71.0°.  
**2.59** (a) 60.0°. (b) 230 lb.  
**2.60** 5.80 m.  
**2.61** (a) 1081 N. (b) 82.5°.  
**2.62** (a) 1294 N. (b) 62.5°.  
**2.63** (a) 10.98 lb. (b) 30.0 lb.
- 2.65** (a) 602 N  $\Delta$  46.8°. (b) 1365 N  $\nearrow$  46.8°.  
**2.67** (a) 300 lb. (b) 300 lb. (c) 200 lb. (d) 200 lb. (e) 150.0 lb.  
**2.68** (b) 200 lb. (d) 150.0 lb.  
**2.69** (a) 1293 N. (b) 2220 N.  
**2.71** (a) +390 N, +614 N, +181.8 N. (b) 58.7°, 35.0°, 76.0°.  
**2.72** (a) -130.1 N, +816 N, +357 N. (b) 98.3°, 25.0°, 66.6°.  
**2.73** (a) 288 N. (b) 67.5°, 30.0°, 108.7°.  
**2.74** (a) 100.0 N. (b) 112.5°, 30.0°, 108.7°.  
**2.76** (a) 80.0 lb. (b) 104.5°, 30.0°, 64.3°.  
**2.77** (a) +56.4 lb, -103.9 lb, -20.5 lb. (b) 62.0°, 150.0°, 99.8°.  
**2.79**  $F = 570$  N;  $\theta_x = 55.8^\circ$ ,  $\theta_y = 45.4^\circ$ ,  $\theta_z = 116.0^\circ$ .  
**2.81** (a) 118.2°. (b)  $F_x = 36.0$  lb,  $F_y = -90.0$  lb;  $F = 110.0$  lb.  
**2.82** (a) 114.4°. (b)  $F_y = 694$  lb,  $F_z = 855$  lb;  $F = 1209$  lb.  
**2.84** (a)  $F_x = 194.0$  N,  $F_z = 108.0$  N. (b)  $\theta_y = 105.1^\circ$ ,  $\theta_z = 62.0^\circ$ .  
**2.85** +100.0 lb, +500 lb, -125.0 lb.  
**2.86** +50.0 lb, +250 lb, +185.0 lb.  
**2.87** +240 N, -255 N, +160.0 N.  
**2.89** -1125 N, +750 N, +450 N.  
**2.91** 515 N;  $\theta_x = 70.2^\circ$ ,  $\theta_y = 27.6^\circ$ ,  $\theta_z = 71.5^\circ$ .  
**2.92** 515 N;  $\theta_x = 79.8^\circ$ ,  $\theta_y = 33.4^\circ$ ,  $\theta_z = 58.6^\circ$ .  
**2.94** 913 lb;  $\theta_x = 50.6^\circ$ ,  $\theta_y = 117.6^\circ$ ,  $\theta_z = 51.8^\circ$ .  
**2.95** 748 N;  $\theta_x = 120.1^\circ$ ,  $\theta_y = 52.5^\circ$ ,  $\theta_z = 128.0^\circ$ .  
**2.96** 3120 N;  $\theta_x = 37.4^\circ$ ,  $\theta_y = 122.0^\circ$ ,  $\theta_z = 72.6^\circ$ .  
**2.97** (a) 65.2 lb. (b) 208 lb;  $\theta_x = 61.6^\circ$ ,  $\theta_y = 151.6^\circ$ ,  $\theta_z = 90.0^\circ$ .  
**2.99** 1031 N ↑.  
**2.101** 926 N ↑.  
**2.103** 2100 lb.  
**2.104** 1868 lb.  
**2.105** 1049 lb.  
**2.107** 960 N.  
**2.108**  $0 \leq Q < 300$  N.  
**2.109** 1572 lb.  
**2.111** 845 N.  
**2.112** 768 N.  
**2.113**  $T_{AB} = 842$  lb;  $T_{AC} = 624$  lb;  $T_{AD} = 1088$  lb.  
**2.114**  $T_{AD} = 29.5$  lb;  $T_{BD} = 10.25$  lb;  $T_{CD} = 29.5$  lb.  
**2.115**  $T_{AB} = 510$  N;  $T_{AC} = 56.2$  N;  $T_{AD} = 536$  N.  
**2.116**  $T_{AB} = 1340$  N;  $T_{AC} = 1025$  N;  $T_{AD} = 915$  N.  
**2.117**  $T_{AB} = 1431$  N;  $T_{AC} = 1560$  N;  $T_{AD} = 183.0$  N.  
**2.118**  $T_{AB} = 1249$  N;  $T_{AC} = 490$  N;  $T_{AD} = 1647$  N.  
**2.121**  $P = 131.2$  N;  $Q = 29.6$  N.  
**2.123** 378 N.  
**2.125** (a) 125.0 lb. (b) 45.0 lb.  
**2.126**  $x = 13.42$  in.,  $z = 6.71$  in.  
**2.127** 37.0°.  
**2.130** (a) 500 lb. (b) 544 lb.  
**2.131** (a) 312 N. (b) 144 N.  
**2.133** (a) 140.3°. (b)  $F_x = 79.9$  lb,  $F_z = 120.1$  lb;  $F = 226$  lb.  
**2.134** (a) -1861 lb, +3360 lb, +677 lb. (b) 118.5°, 30.5°, 80.0°.  
**2.135** 15.13 kN;  $\theta_x = 133.4^\circ$ ,  $\theta_y = 43.6^\circ$ ,  $\theta_z = 86.6^\circ$ .  
**2.136**  $T_{AB} = 500$  N;  $T_{AC} = 459$  N;  $T_{AD} = 516$  N.

- 2.137** (a) 1155 N. (b) 1012 N.  
**2.C2** (1) (b) 20°; (c) 244 lb. (2) (b) -10°; (c) 467 lb. (3) (b) 10°; (c) 163.2 lb.  
**2.C3** (a) 1.001 m. (b) 4.01 kN. (c) 1.426 kN; 1.194 kN.

## CHAPTER 3

- 3.1** 1.277 N · m ↴.  
**3.2** 1.277 N · m ↴.  
**3.3** (a) 41.7 N · m ↴. (b) 147.4 N ↗ 45.0°.  
**3.4** (a) 41.7 N · m ↴. (b) 176.8 N ↗ 58.0°.  
**3.5** 186.6 lb · in. ↓.  
**3.7** 6.12° or 33.8°.  
**3.9** (a) 760 N · m ↴. (b) 760 N · m ↴.  
**3.10** 1224 N.  
**3.12** 116.2 lb · ft ↴.  
**3.13** 128.2 lb · ft ↴.  
**3.16** 2.21 m.  
**3.17** (a) 41.0. (b) 26.9.  
**3.19** (a) -11i + 22j + 22k. (b) 0. (c) -45i + 30j - 10k.  
**3.21** (7.50 N · m)i - (6.00 N · m)j - (10.39 N · m)k.  
**3.22** (3080 N · m)i - (2070 N · m)k.  
**3.24** -(153.0 lb · ft)i + (63.0 lb · ft)j + (215 lb · ft)k.  
**3.26** (492 lb · ft)i + (144.0 lb · ft)j - (372 lb · ft)k.  
**3.27** 4.58 m.  
**3.28** 3.70 m.  
**3.30** 57.0 in.  
**3.31** 1.564 m.  
**3.32** 3.29 m.  
**3.33** 4.86 ft.  
**3.35**  $\mathbf{P} \cdot \mathbf{Q} = 1$ ;  $\mathbf{P} \cdot \mathbf{S} = -11$ ;  $\mathbf{Q} \cdot \mathbf{S} = 10$ .  
**3.37** 27.4°.  
**3.39** 43.6°.  
**3.40** 38.9°.  
**3.41** (a) 59.0°. (b) 648 N.  
**3.43** (a) 71.1°. (b) 0.973 lb.  
**3.44** 12.00 in.  
**3.45** (a) 67. (b) 111.  
**3.46** 7.  
**3.47**  $M_x = -31.2$  N · m;  $M_y = 13.20$  N · m;  $M_z = -2.42$  N · m.  
**3.48**  $M_x = -25.6$  N · m;  $M_y = 10.80$  N · m;  $M_z = 40.6$  N · m.  
**3.49** 1.252 m.  
**3.50** 1.256 m.  
**3.51** 61.5 lb.  
**3.53**  $\phi = 24.6^\circ$ ;  $d = 34.6$  in.  
**3.55** -90.0 N · m.  
**3.56** -111.0 N · m.  
**3.57** 2.28 N · m.  
**3.58** -9.50 N · m.  
**3.59**  $aP/\sqrt{2}$ .  
**3.61** 1359 lb · in.  
**3.65** 0.249 m.  
**3.66** 0.1198 m.  
**3.68** 30.4 in.  
**3.69** 43.5 in.  
**3.70** (a) 12.39 N · m ↓. (b) 12.39 N · m ↓. (c) 12.39 N · m ↓.  
**3.71** (a) 336 lb · in. ↴. (b) 28.0 in. (c) 54.0°.  
**3.72** (a) 75.0 N. (b) 71.2 N. (c) 45.0 N.  
**3.75**  $M = 10.00$  lb · ft;  $\theta_x = 90.0^\circ$ ,  $\theta_y = 143.1^\circ$ ,  $\theta_z = 126.9^\circ$ .  
**3.76**  $M = 9.21$  N · m;  $\theta_x = 77.9^\circ$ ,  $\theta_y = 12.05^\circ$ ,  $\theta_z = 90.0^\circ$ .  
**3.77**  $M = 604$  lb · in.;  $\theta_x = 72.8^\circ$ ,  $\theta_y = 27.3^\circ$ ,  $\theta_z = 110.5^\circ$ .  
**3.78**  $M = 1170$  lb · in.;  $\theta_x = 81.2^\circ$ ,  $\theta_y = 13.70^\circ$ ,  $\theta_z = 100.4^\circ$ .
- 3.79**  $M = 10.92$  N · m;  $\theta_x = 97.8^\circ$ ,  $\theta_y = 34.5^\circ$ ,  $\theta_z = 56.7^\circ$ .  
**3.80**  $M = 2860$  N · m;  $\theta_x = 113.0^\circ$ ,  $\theta_y = 92.7^\circ$ ,  $\theta_z = 23.2^\circ$ .  
**3.81** (a)  $\mathbf{F} = 560$  lb ↘ 20.0°;  $\mathbf{M} = 7720$  lb · ft ↓.  
(b)  $\mathbf{F} = 560$  lb ↘ 20.0°;  $\mathbf{M} = 4290$  lb · ft ↓.  
**3.82** (a)  $\mathbf{F} = 160.0$  lb ↗ 60.0°;  $\mathbf{M} = 334$  lb · ft ↗.  
(b)  $\mathbf{F}_B = 20.0$  lb ↑;  $\mathbf{F}_D = 143.0$  lb ↗ 56.0°.  
**3.83** (a)  $\mathbf{F}_B = 80.0$  N ←;  $\mathbf{M}_B = 4.00$  N · m ↴.  
(b)  $\mathbf{F}_C = 100.0$  N ↓;  $\mathbf{F}_D = 100.0$  N ↑.  
**3.85** (a)  $\mathbf{F}_B = 250$  N ↘ 60.0°;  $\mathbf{M}_B = 75.0$  N · m ↓.  
(b)  $\mathbf{F}_A = 375$  N ↙ 60.0°;  $\mathbf{F}_B = 625$  N ↘ 60.0°.  
**3.87** (a)  $\mathbf{F} = -(600 \text{ N})\mathbf{k}$ ;  $d = 90.0\text{mm}$  below  $ED$ .  
(b)  $\mathbf{F} = -(600 \text{ N})\mathbf{k}$ ;  $d = 90.0\text{mm}$  above  $ED$ .  
**3.88**  $\mathbf{F} = 900$  N ↓;  $x = 50.0$  mm.  
**3.89** (0.227 lb)i + (0.1057 lb)j; 63.6 in. to the right of  $B$ .  
**3.90** (a)  $\mathbf{F} = 48.0$  lb ↗ 65.0°;  $\mathbf{M} = 490$  lb · in. ↓.  
(b)  $\mathbf{F} = 48.0$  lb ↗ 65.0°; 17.78 in. to the left of  $B$ .  
**3.93**  $\mathbf{F} = -(1220 \text{ N})\mathbf{i}$ ;  $\mathbf{M} = (73.2 \text{ N} \cdot \text{m})\mathbf{j} - (122.0 \text{ N} \cdot \text{m})\mathbf{k}$ .  
**3.94**  $\mathbf{F}_C = (5.00 \text{ N})\mathbf{i} + (150.0 \text{ N})\mathbf{j} - (90.0 \text{ N})\mathbf{k}$ ;  
 $\mathbf{M}_C = (77.4 \text{ N} \cdot \text{m})\mathbf{i} + (61.5 \text{ N} \cdot \text{m})\mathbf{j} + (106.8 \text{ N} \cdot \text{m})\mathbf{k}$ .  
**3.95**  $\mathbf{F} = -(128.0 \text{ lb})\mathbf{i} - (256 \text{ lb})\mathbf{j} + (32.0 \text{ lb})\mathbf{k}$ ;  
 $\mathbf{M} = (4.10 \text{ kip} \cdot \text{ft})\mathbf{i} + (16.38 \text{ kip} \cdot \text{ft})\mathbf{k}$ .  
**3.97**  $\mathbf{F} = -(122.9 \text{ N})\mathbf{j} - (86.0 \text{ N})\mathbf{k}$ ;  
 $\mathbf{M} = (22.6 \text{ N} \cdot \text{m})\mathbf{i} + (15.49 \text{ N} \cdot \text{m})\mathbf{j} - (22.1 \text{ N} \cdot \text{m})\mathbf{k}$ .  
**3.98** (a) 135.0 mm. (b)  $\mathbf{F}_2 = (42.0 \text{ N})\mathbf{i} + (42.0 \text{ N})\mathbf{j} - (49.0 \text{ N})\mathbf{k}$ ;  
 $\mathbf{M}_2 = -(25.9 \text{ N} \cdot \text{m})\mathbf{i} + (21.2 \text{ N} \cdot \text{m})\mathbf{j}$ .  
**3.99**  $\mathbf{F} = (36.0 \text{ lb})\mathbf{i} - (28.0 \text{ lb})\mathbf{j} - (6.00 \text{ lb})\mathbf{k}$ ;  
 $\mathbf{M} = -(157.0 \text{ lb} \cdot \text{ft})\mathbf{i} + (22.5 \text{ lb} \cdot \text{ft})\mathbf{j} - (240 \text{ lb} \cdot \text{ft})\mathbf{k}$ .  
**3.101** (a) Loading a:  $\mathbf{R} = 600$  N ↓;  $\mathbf{M} = 1000$  N · m ↴.  
Loading b:  $\mathbf{R} = 600$  N ↓;  $\mathbf{M} = 900$  N · m ↓.  
Loading c:  $\mathbf{R} = 600$  N ↓;  $\mathbf{M} = 900$  N · m ↴.  
Loading d:  $\mathbf{R} = 400$  N ↑;  $\mathbf{M} = 900$  N · m ↴.  
Loading e:  $\mathbf{R} = 600$  N ↓;  $\mathbf{M} = 200$  N · m ↓.  
Loading f:  $\mathbf{R} = 600$  N ↓;  $\mathbf{M} = 800$  N · m ↴.  
Loading g:  $\mathbf{R} = 1000$  N ↓;  $\mathbf{M} = 1000$  N · m ↴.  
Loading h:  $\mathbf{R} = 600$  N ↓;  $\mathbf{M} = 900$  N · m ↴.  
(b) Loadings c and h.  
**3.102** Loading f.  
**3.104** Force–couple system at  $D$ .  
**3.105** (a) 2.00 ft to the right of  $C$ . (b) 2.31 ft to the right of  $C$ .  
**3.106** (a) 39.6 in. to the right of  $D$ . (b) 33.1 in.  
**3.108**  $\mathbf{R} = 72.4$  lb ↘ 81.9°;  $\mathbf{M} = 206$  lb · ft.  
**3.109** (a) 34.0 lb ↙ 28.0°. (b) AB: 11.64 in. to the left of  $B$ ;  
BC: 6.20 in. below  $B$ .  
**3.110** (a) 48.2 lb · in. ↴. (b) 240 lb · in. ↴. (c) 0.  
**3.111** (a) 1562 N ↙ 50.2°. (b) 250 mm to the right of  $C$  and  
300 mm above  $C$ .  
**3.112** (a) 1308 N ↗ 66.6°. (b) 412 mm to the right of  $A$  and  
250 mm to the right of  $C$ .  
**3.113** 773 lb ↗ 79.0°; 9.54 ft to the right of  $A$ .  
**3.115** (a) 0.365 m above  $G$ . (b) 0.227 m to the right of  $G$ .  
**3.116** (a) 0.299 m above  $G$ . (b) 0.259 m to the right of  $G$ .  
**3.118** (a)  $\mathbf{R} = F \bar{\ell} \tan^{-1}(a^2/2bx)$ ;  
 $\mathbf{M} = 2Fb^2(x - x^3/a^2)/\sqrt{a^4 + 4b^2x^2}$  ↴. (b) 0.369 m.  
**3.119**  $\mathbf{R} = -(420 \text{ N})\mathbf{i} - (50.0 \text{ N})\mathbf{j} - (250 \text{ N})\mathbf{k}$ ;  
 $\mathbf{M} = (30.8 \text{ N} \cdot \text{m})\mathbf{j} - (22.0 \text{ N} \cdot \text{m})\mathbf{k}$ .  
**3.120**  $\mathbf{R} = -(420 \text{ N})\mathbf{j} - (339 \text{ N})\mathbf{k}$ ;  
 $\mathbf{M} = (1.125 \text{ N} \cdot \text{m})\mathbf{i} + (163.9 \text{ N} \cdot \text{m})\mathbf{j} - (109.9 \text{ N} \cdot \text{m})\mathbf{k}$ .  
**3.121** (a)  $\mathbf{B} = (2.50 \text{ lb})\mathbf{i}$ ;  
 $\mathbf{C} = (0.1000 \text{ lb})\mathbf{i} - (2.47 \text{ lb})\mathbf{j} - (0.700 \text{ lb})\mathbf{k}$ .  
(b)  $R_y = -2.47$  lb;  $M_x = 1.360$  lb · ft.  
**3.122**  $\mathbf{A} = (1.600 \text{ lb})\mathbf{i} - (36.0 \text{ lb})\mathbf{j} + (2.00 \text{ lb})\mathbf{k}$ ;  
 $\mathbf{B} = -(9.60 \text{ lb})\mathbf{i} + (36.0 \text{ lb})\mathbf{j} + (2.00 \text{ lb})\mathbf{k}$ .

- 3.124** (a)  $\mathbf{R} = -(28.4 \text{ N})\mathbf{j} - (50.0 \text{ N})\mathbf{k}$ ;  $\mathbf{M} = (8.56 \text{ N} \cdot \text{m})\mathbf{i} - (24.0 \text{ N} \cdot \text{m})\mathbf{j} + (2.13 \text{ N} \cdot \text{m})\mathbf{k}$ .  
 (b) Counterclockwise.
- 3.125** (a)  $\mathbf{R} = -(28.4 \text{ N})\mathbf{j} - (50.0 \text{ N})\mathbf{k}$ ;  
 $\mathbf{M} = (42.4 \text{ N} \cdot \text{m})\mathbf{i} - (24.0 \text{ N} \cdot \text{m})\mathbf{j} + (2.13 \text{ N} \cdot \text{m})\mathbf{k}$ .  
 (b) Counterclockwise.
- 3.127** 1035 N; 2.57 m from OG and 3.05 m from OE.
- 3.128** 2.32 m from OG and 1.165 m from OE.
- 3.129** 405 lb; 12.60 ft to the right of AB and 2.94 ft below BC.
- 3.130**  $a = 0.722 \text{ ft}$ ;  $b = 20.6 \text{ ft}$ .
- 3.133** (a)  $P\sqrt{3}$ ;  $\theta_x = \theta_y = \theta_z = 54.7^\circ$ . (b)  $-a$ . (c) Axis of the wrench is diagonal OA.
- 3.134** (a)  $P$ ;  $\theta_x = 90.0^\circ$ ,  $\theta_y = 90.0^\circ$ ,  $\theta_z = 0$ . (b)  $5a/2$ . (c) Axis of wrench is parallel to the z axis at  $x = a$ ,  $y = -a$ .
- 3.136** (a)  $-(21.0 \text{ lb})\mathbf{j}$ . (b) 0.571 in. (c) Axis of wrench is parallel to the y axis at  $x = 0$ ,  $z = 1.667 \text{ in}$ .
- 3.137** (a)  $-(84.0 \text{ N})\mathbf{j} - (80.0 \text{ N})\mathbf{k}$ . (b) 0.477 m.  
 (c)  $x = 0.526 \text{ m}$ ,  $z = -0.1857 \text{ m}$ .
- 3.140** (a)  $3P(2\mathbf{i} - 20\mathbf{j} - \mathbf{k})/25$ . (b)  $-0.0988a$ .  
 (c)  $x = 2.00a$ ,  $z = -1.990a$ .
- 3.141**  $\mathbf{R} = (20.0 \text{ N})\mathbf{i} + (30.0 \text{ N})\mathbf{j} - (10.0 \text{ N})\mathbf{k}$ ;  
 $y = -0.540 \text{ m}$ ,  $z = -0.420 \text{ m}$ .
- 3.143**  $\mathbf{F}_A = (M/b)\mathbf{i} + R[1 + (a/b)]\mathbf{k}$ ;  $\mathbf{F}_B = -(M/b)\mathbf{i} - (aR/b)\mathbf{k}$ .
- 3.147** (a) 196.2 N · m ↴. (b) 199.0 N ↘ 59.5°.
- 3.148** 42.0 N · m ↗.
- 3.149**  $-(25.4 \text{ lb} \cdot \text{ft})\mathbf{i} - (12.60 \text{ lb} \cdot \text{ft})\mathbf{j} - (12.60 \text{ lb} \cdot \text{ft})\mathbf{k}$ .
- 3.151** 283 lb.
- 3.153** (a) 151.2 lb · in. ↗. (b) 67.2 lb · in. ↗.
- 3.155**  $\mathbf{F} = -(28.5 \text{ N})\mathbf{j} + (106.3 \text{ N})\mathbf{k}$ ;  
 $\mathbf{M} = (12.35 \text{ N} \cdot \text{m})\mathbf{i} - (19.16 \text{ N} \cdot \text{m})\mathbf{j} - (5.13 \text{ N} \cdot \text{m})\mathbf{k}$ .
- 3.156** (a) 665 lb ↙ 79.6°; 64.9 in. to the right of A. (b) 22.9°.
- 3.157** (a)  $\mathbf{F}_B = -(80.0 \text{ N})\mathbf{k}$ ;  $\mathbf{F}_C = -(30.0 \text{ N})\mathbf{i} + (40.0 \text{ N})\mathbf{k}$ .  
 (b)  $R_y = 0$ ;  $R_z = -40.0 \text{ N}$ . (c) When the slot is vertical.
- 3.C3** 4 sides:  $\beta = 10^\circ$ ,  $\alpha = 44.1^\circ$ ;  
 $\beta = 20^\circ$ ,  $\alpha = 41.6^\circ$ ;  
 $\beta = 30^\circ$ ,  $\alpha = 37.8^\circ$ .
- 3.C4**  $\theta = 0 \text{ rev}$ :  $M = 97.0 \text{ N} \cdot \text{m}$ ;  
 $\theta = 6 \text{ rev}$ :  $M = 63.3 \text{ N} \cdot \text{m}$ ;  
 $\theta = 12 \text{ rev}$ :  $M = 9.17 \text{ N} \cdot \text{m}$ .
- 3.C6**  $d_{AB} = 36.0 \text{ in.}$ ;  $d_{CD} = 9.00 \text{ in.}$ ;  $d_{\min} = 58.3 \text{ in.}$
- 4.24** (a)  $\mathbf{A} = 20.0 \text{ lb} \uparrow$ ;  $\mathbf{B} = 50.0 \text{ lb} \nwarrow 36.9^\circ$ .  
 (b)  $\mathbf{A} = 23.1 \text{ lb} \angle 60.0^\circ$ ;  $\mathbf{B} = 59.6 \text{ lb} \nwarrow 30.2^\circ$ .
- 4.26** (a) 190.9 N. (b) 142.3 N ↙ 18.43°.
- 4.27** (a) 324 N. (b) 270 N →.
- 4.28** (a) 400 N. (b)  $\mathbf{C} = 458 \text{ N} \angle 49.1^\circ$ .
- 4.29** (a) 875 lb (b) 1584 lb ↘ 45.0°.
- 4.30**  $T = 80.0 \text{ N}$ ;  $\mathbf{C} = 89.4 \text{ N} \angle 26.6^\circ$ .
- 4.33**  $T = 2P/3$ ;  $\mathbf{C} = 0.577 P \rightarrow$ .
- 4.34**  $T = 0.586 P$ ;  $\mathbf{C} = 0.414 P \rightarrow$ .
- 4.35**  $\mathbf{A} = 69.3 \text{ lb} \rightarrow$ ;  $\mathbf{B} = 34.6 \text{ lb} \nwarrow 60.0^\circ$ ;  $\mathbf{C} = 173.2 \text{ lb} \nwarrow 60.0^\circ$ .
- 4.36**  $T_{BE} = 50.0 \text{ lb}$ ;  $\mathbf{A} = 18.75 \text{ lb} \rightarrow$ ;  $\mathbf{D} = 18.75 \text{ lb} \leftarrow$ .
- 4.37** (a) 1432 N. (b) 1100 N ↑. (c) 1400 N ←.
- 4.38**  $T_{BE} = 3230 \text{ N}$ ;  $T_{CF} = 960 \text{ N}$ ;  $\mathbf{D} = 3750 \text{ N} \leftarrow$ .
- 4.41**  $T = 80.0 \text{ N}$ ;  $\mathbf{A} = 160.0 \text{ N} \nwarrow 30.0^\circ$ ;  $\mathbf{C} = 160.0 \text{ N} \nwarrow 30.0^\circ$ .
- 4.42**  $T = 69.3 \text{ N}$ ;  $\mathbf{A} = 140.0 \text{ N} \nwarrow 30.0^\circ$ ;  $\mathbf{C} = 180.0 \text{ N} \nwarrow 30.0^\circ$ .
- 4.43** (a)  $\mathbf{A} = 78.5 \text{ N}$ ;  $\mathbf{M}_A = 125.6 \text{ N} \cdot \text{m} \uparrow$ .  
 (b)  $\mathbf{A} = 111.0 \text{ N} \angle 45.0^\circ$ ;  $\mathbf{M}_A = 125.6 \text{ N} \cdot \text{m} \uparrow$ .  
 (c)  $\mathbf{A} = 157.0 \text{ N} \uparrow$ ;  $\mathbf{M}_A = 251 \text{ N} \cdot \text{m} \uparrow$ .
- 4.44**  $\mathbf{C} = 7.07 \text{ lb} \nwarrow 45.0^\circ$ ;  $\mathbf{M}_C = 43.0 \text{ lb} \cdot \text{in.} \downarrow$ .
- 4.46**  $\mathbf{A} = 1848 \text{ N} \angle 82.6^\circ$ ;  $\mathbf{M}_A = 1431 \text{ N} \cdot \text{m} \downarrow$ .
- 4.47** (a)  $\mathbf{D} = 20.0 \text{ lb} \downarrow$ ;  $\mathbf{M}_D = 20.0 \text{ lb} \cdot \text{ft} \uparrow$ .  
 (b)  $\mathbf{D} = 10.00 \text{ lb} \downarrow$ ;  $\mathbf{M}_D = 30.0 \text{ lb} \cdot \text{ft} \downarrow$ .
- 4.49**  $\mathbf{C} = 1951 \text{ N} \nwarrow 88.5^\circ$ ;  $\mathbf{M}_C = 75.0 \text{ N} \cdot \text{m} \downarrow$ .
- 4.50**  $1.232 \text{ kN} \leq T \leq 1.774 \text{ kN}$ .
- 4.51** (a)  $\theta = 2 \sin^{-1}(W/2P)$ . (b)  $\theta = 29.0^\circ$ .
- 4.52** (a)  $T = \frac{1}{2}W/(1 - \tan \theta)$ . (b)  $\theta = 39.8^\circ$ .
- 4.53** (a)  $\sin \theta + \cos \theta = M/Pl$ . (b)  $17.11^\circ$  and  $72.9^\circ$ .
- 4.54** (a)  $\cos^3 \theta = a(P + Q)/Pl$ . (b)  $40.6^\circ$ .
- 4.57** 141.1°
- 4.58** (a)  $(1 - \cos \theta) \tan \theta = W/2kl$ . (b) 49.7°.
- 4.59** (1) completely constrained; determinate;  $\mathbf{A} = \mathbf{C} = 196.2 \text{ N} \uparrow$ .  
 (2) completely constrained; determinate;  $\mathbf{B} = 0$ ,  $\mathbf{C} = \mathbf{D} = 196.2 \text{ N} \uparrow$ .  
 (3) completely constrained; indeterminate;  $\mathbf{A}_x = 294 \text{ N} \rightarrow$ ,  $\mathbf{D}_x = 294 \text{ N} \leftarrow$ .  
 (4) improperly constrained; indeterminate; no equilibrium.  
 (5) partially constrained; determinate; equilibrium;  $\mathbf{C} = \mathbf{D} = 196.2 \text{ N} \uparrow$ .  
 (6) completely constrained; determinate;  $\mathbf{B} = 294 \text{ N} \rightarrow$ ,  $\mathbf{D} = 491 \text{ N} \nwarrow 53.1^\circ$ .  
 (7) partially constrained; no equilibrium.  
 (8) completely constrained; indeterminate;  $\mathbf{B} = 196.2 \text{ N} \uparrow$ ,  $\mathbf{D}_y = 196.2 \text{ N} \uparrow$ .
- 4.61**  $\mathbf{A} = 400 \text{ N} \uparrow$ ;  $\mathbf{B} = 500 \text{ N} \nwarrow 53.1^\circ$ .
- 4.62**  $a \geq 138.6 \text{ mm}$ .
- 4.66**  $\mathbf{B} = 888 \nwarrow 41.3^\circ$ ;  $\mathbf{D} = 943 \text{ N} \nwarrow 45.0^\circ$ .
- 4.67**  $\mathbf{B} = 1001 \text{ N} \nwarrow 48.2^\circ$ ;  $\mathbf{D} = 943 \text{ N} \nwarrow 45.0^\circ$ .
- 4.69** (a) 499 N. (b) 457 N ↘ 26.6°.
- 4.70** (a) 998 N. (b) 822 N ↗ 5.72°.
- 4.71**  $\mathbf{A} = 37.1 \text{ lb} \angle 62.4^\circ$ ;  $T = 18.57 \text{ lb}$ .
- 4.74** (a)  $24.9 \text{ lb} \nwarrow 30.0^\circ$ . (b)  $15.34 \text{ lb} \angle 30.0^\circ$ .
- 4.75**  $T = 100.0 \text{ lb}$ ;  $\mathbf{B} = 111.1 \text{ lb} \nwarrow 30.3^\circ$ .
- 4.77**  $\mathbf{A} = 170.0 \text{ N} \nwarrow 33.9^\circ$ ;  $\mathbf{C} = 160.0 \text{ N} \angle 28.1^\circ$ .
- 4.80** (a)  $F_{AD} = 400 \text{ N}$ . (b)  $\mathbf{C} = 458 \text{ N} \angle 49.1^\circ$ .
- 4.81** (a)  $2P \nwarrow 60.0^\circ$ . (b)  $1.239 P \nwarrow 36.2^\circ$ .
- 4.82** (a)  $1.155 P \nwarrow 30.0^\circ$ . (b)  $1.086 P \angle 22.9^\circ$ .
- 4.83** 60.0 mm.
- 4.84**  $\tan \theta = 2 \tan \beta$ .
- 4.85** (a)  $49.1^\circ$ . (b)  $\mathbf{A} = 45.3 \text{ N} \leftarrow$ ;  $\mathbf{B} = 90.6 \text{ N} \angle 60.0^\circ$ .
- 4.87** (a) 12.91 in. (b) 11.62 lb. (c) 5.92 lb.
- 4.88** 32.5°.
- 4.90** (a)  $59.4^\circ$ . (b)  $\mathbf{A} = 8.45 \text{ lb} \rightarrow$ ;  $\mathbf{B} = 13.09 \text{ lb} \nwarrow 49.8^\circ$ .

## CHAPTER 4

- 4.1** (a) 325 lb ↑. (b) 1175 lb ↑.
- 4.2** 42.0 N ↑.
- 4.3** 0.264 m.
- 4.4** (a) 245 lb. ↑. (b) 140.0 lb.
- 4.5** (a) 6.07 kN ↑. (b) 4.23 kN ↑.
- 4.6** (a) 4.89 kN ↑. (b) 3.69 kN ↑.
- 4.9**  $150.0 \text{ mm} \leq d \leq 400 \text{ mm}$ .
- 4.11** 6.00 kips  $\leq P \leq 42.0 \text{ kips}$ .
- 4.12**  $3.50 \text{ kN} \leq P \leq 86.0 \text{ kN}$ .
- 4.14** 2.00 in.  $\leq a \leq 10.00 \text{ in}$ .
- 4.15** (a)  $F_{DE} = 600 \text{ N}$ . (b)  $\mathbf{C} = 1253 \text{ N} \angle 69.8^\circ$ .
- 4.17** (a) 80.0 lb ↓. (b) 216 lb ↙ 22.0°.
- 4.18** 232 lb.
- 4.19** (a) 2.00 kN. (b) 2.32 kN ↙ 46.4°.
- 4.21** (a)  $\mathbf{A} = 150.0 \text{ N} \angle 30.0^\circ$ ;  $\mathbf{B} = 150.0 \text{ N} \nwarrow 30.0^\circ$ .  
 (b)  $\mathbf{A} = 433 \text{ N} \nwarrow 12.55^\circ$ ;  $\mathbf{B} = 488 \nwarrow 30.0^\circ$ .
- 4.23** (a)  $\mathbf{A} = 44.7 \text{ lb} \nwarrow 26.6^\circ$ ;  $\mathbf{B} = 30.0 \text{ lb} \uparrow$ .  
 (b)  $\mathbf{A} = 30.2 \text{ lb} \nwarrow 41.4^\circ$ ;  $\mathbf{B} = 34.6 \text{ lb} \nwarrow 60.0^\circ$ .

- 4.91**  $\mathbf{A} = (22.9 \text{ lb})\mathbf{i} + (8.50 \text{ lb})\mathbf{j}$ ;  $\mathbf{B} = (22.9 \text{ lb})\mathbf{i} + (25.5 \text{ lb})\mathbf{j}$ ;  $\mathbf{C} = -(45.8 \text{ lb})\mathbf{i}$ .
- 4.92**  $\mathbf{A} = (56.0 \text{ N})\mathbf{j} + (18.00 \text{ N})\mathbf{k}$ ;  $\mathbf{D} = (24.0 \text{ N})\mathbf{j} + (42.0 \text{ N})\mathbf{k}$ .
- 4.93**  $\mathbf{A} = (56.0 \text{ N})\mathbf{j} + (14.40 \text{ N})\mathbf{k}$ ;  $\mathbf{D} = (24.0 \text{ N})\mathbf{j} + (33.6 \text{ N})\mathbf{k}$ .
- 4.94** (a) 37.5 lb. (b)  $\mathbf{B} = (33.8 \text{ lb})\mathbf{j} - (70.0 \text{ lb})\mathbf{k}$ ;  $\mathbf{D} = (33.8 \text{ lb})\mathbf{j} + (28.0 \text{ lb})\mathbf{k}$ .
- 4.97** (a) 121.9 N. (b) -46.2 N. (c) 100.9 N.
- 4.98** (a) 95.6 N. (b) -7.36 N. (c) 88.3 N.
- 4.99**  $T_A = 30.0 \text{ lb}$ ;  $T_B = 10.00 \text{ lb}$ ;  $T_C = 40.0 \text{ lb}$ .
- 4.100**  $(W_D)_{\min} = 40.0 \text{ lb}$ ;  $X = 0 \text{ in.}$ ;  $Z = 30.0 \text{ in.}$
- 4.101**  $T_A = 23.5 \text{ N}$ ;  $T_C = 11.77 \text{ N}$ ;  $T_D = 105.9 \text{ N}$ .
- 4.102** (a) 0.480 m. (b)  $T_A = 23.5 \text{ N}$ ;  $T_C = 0$ ;  $T_D = 117.7 \text{ N}$ .
- 4.105**  $T_{BD} = T_{BE} = 1100 \text{ lb}$ ;  $\mathbf{A} = (1200 \text{ lb})\mathbf{i} - (560 \text{ lb})\mathbf{j}$ .
- 4.106**  $T_{AD} = 2.60 \text{ kN}$ ;  $T_{AE} = 2.80 \text{ kN}$ ;  $\mathbf{C} = (1.800 \text{ kN})\mathbf{j} + (4.80 \text{ kN})\mathbf{k}$ .
- 4.107**  $T_{AD} = 5.20 \text{ kN}$ ;  $T_{AE} = 5.60 \text{ kN}$ ;  $\mathbf{C} = (9.60 \text{ kN})\mathbf{k}$ .
- 4.108** (a)  $T_{DE} = T_{DF} = 262 \text{ lb}$ . (b)  $\mathbf{A} = -(801 \text{ lb})\mathbf{i} + (1544 \text{ lb})\mathbf{j}$ .
- 4.109** (a)  $T_{CD} = T_{CE} = 3.96 \text{ kN}$ . (b)  $\mathbf{A} = (6.67 \text{ kN})\mathbf{i} + (1.667 \text{ kN})\mathbf{j}$ .
- 4.110** (a)  $T_{CD} = 0.954 \text{ kN}$ ;  $T_{CE} = 5.90 \text{ kN}$ .  
(b)  $\mathbf{A} = (5.77 \text{ kN})\mathbf{i} + (1.443 \text{ kN})\mathbf{j} - (0.833 \text{ kN})\mathbf{k}$ .
- 4.113** (a) 101.6 N. (b)  $\mathbf{A} = -(26.3 \text{ N})\mathbf{i}$ ;  $\mathbf{B} = (98.1 \text{ N})\mathbf{j}$ .
- 4.114** (a) 462 N. (b)  $\mathbf{C} = -(336 \text{ N})\mathbf{j} + (467 \text{ N})\mathbf{k}$ ;  $\mathbf{D} = (505 \text{ N})\mathbf{j} - (66.7 \text{ N})\mathbf{k}$ .
- 4.117** (a) 49.5 lb. (b)  $\mathbf{A} = -(12.00 \text{ lb})\mathbf{i} + (22.5 \text{ lb})\mathbf{j} - (4.00 \text{ lb})\mathbf{k}$ ;  $\mathbf{B} = (15.00 \text{ lb})\mathbf{j} + (34.0 \text{ lb})\mathbf{k}$ .
- 4.118** (a) 118.8 lb. (b)  $\mathbf{A} = (93.8 \text{ lb})\mathbf{i} + (22.5 \text{ lb})\mathbf{j} + (70.8 \text{ lb})\mathbf{k}$ ;  $\mathbf{B} = (15.00 \text{ lb})\mathbf{j} - (8.33 \text{ lb})\mathbf{k}$ .
- 4.119** (a) 462 N. (b)  $\mathbf{C} = (169.1 \text{ N})\mathbf{j} + (400 \text{ N})\mathbf{k}$ ;  $\mathbf{M}_C = (20.0 \text{ N} \cdot \text{m})\mathbf{j} + (151.5 \text{ N} \cdot \text{m})\mathbf{k}$ .
- 4.120** (a) 49.5 lb. (b)  $\mathbf{A} = -(12.00 \text{ lb})\mathbf{i} + (37.5 \text{ lb})\mathbf{j} + (30.0 \text{ lb})\mathbf{k}$ ;  $\mathbf{M}_A = -(1020 \text{ lb} \cdot \text{in.})\mathbf{j} + (450 \text{ lb} \cdot \text{in.})\mathbf{k}$ .
- 4.121** (a) 5.00 lb. (b)  $\mathbf{C} = -(5.00 \text{ lb})\mathbf{i} + (6.00 \text{ lb})\mathbf{j} - (5.00 \text{ lb})\mathbf{k}$ ;  $\mathbf{M}_C = (8.00 \text{ lb} \cdot \text{in.})\mathbf{j} - (12.00 \text{ lb} \cdot \text{in.})\mathbf{k}$ .
- 4.122**  $T_{CF} = 200 \text{ N}$ ;  $T_{DE} = 450 \text{ N}$ ;  $\mathbf{A} = (160.0 \text{ N})\mathbf{i} + (270 \text{ N})\mathbf{k}$ ;  $\mathbf{M}_A = -(16.20 \text{ N} \cdot \text{m})\mathbf{i}$ .
- 4.125**  $T_{BE} = 975 \text{ N}$ ;  $T_{CF} = 600 \text{ N}$ ;  $T_{DG} = 625 \text{ N}$ ;  $\mathbf{A} = (2100 \text{ N})\mathbf{i} + (175.0 \text{ N})\mathbf{j} - (375 \text{ N})\mathbf{k}$ .
- 4.126**  $T_{BE} = 1950 \text{ N}$ ;  $T_{CF} = 0$ ;  $T_{DG} = 1250 \text{ N}$ ;  $\mathbf{A} = (3000 \text{ N})\mathbf{i} - (750 \text{ N})\mathbf{k}$ .
- 4.127**  $\mathbf{A} = (120.0 \text{ lb})\mathbf{j} - (150.0 \text{ lb})\mathbf{k}$ ;  $\mathbf{B} = (180.0 \text{ lb})\mathbf{i} + (150.0 \text{ lb})\mathbf{k}$ ;  $\mathbf{C} = -(180.0 \text{ lb})\mathbf{i} + (120.0 \text{ lb})\mathbf{j}$ .
- 4.128**  $\mathbf{A} = (20.0 \text{ lb})\mathbf{j} + (25.0 \text{ lb})\mathbf{k}$ ;  $\mathbf{B} = (30.0 \text{ lb})\mathbf{i} - (25.0 \text{ lb})\mathbf{k}$ ;  $\mathbf{C} = -(30.0 \text{ lb})\mathbf{i} - (20.0 \text{ lb})\mathbf{j}$ .
- 4.129**  $\mathbf{B} = (60.0 \text{ N})\mathbf{k}$ ;  $\mathbf{C} = (30.0 \text{ N})\mathbf{j} - (16.00 \text{ N})\mathbf{k}$ ;  $\mathbf{D} = -(30.0 \text{ N})\mathbf{j} + (4.00 \text{ N})\mathbf{k}$ .
- 4.130**  $\mathbf{B} = (60.0 \text{ N})\mathbf{k}$ ;  $\mathbf{C} = -(16.00 \text{ N})\mathbf{k}$ ;  $\mathbf{D} = (4.00 \text{ N})\mathbf{k}$ .
- 4.133** 85.3 lb.
- 4.134** 181.7 lb.
- 4.135** 373 N
- 4.136** 301 N
- 4.137** (45.0 lb) $\mathbf{j}$
- 4.138** (a)  $x = 4.00 \text{ ft}$ ;  $y = 8.00 \text{ ft}$ . (b) 10.73 lb.
- 4.139** (a)  $x = 0 \text{ ft}$ ;  $y = 16.00 \text{ ft}$ . (b) 11.31 lb.
- 4.142** (a) 37.9 N  $\uparrow$ . (b) 373 N  $\uparrow$ .
- 4.143** (a)  $\mathbf{A} = 225 \text{ N} \uparrow$ ;  $\mathbf{C} = 641 \text{ N} \nearrow 20.6^\circ$ .  
(b)  $\mathbf{A} = 365 \text{ N} \angle 60.0^\circ$ ;  $\mathbf{B} = 844 \text{ N} \nearrow 22.0^\circ$ .
- 4.145** (a) 130.0 N. (b) 224 N  $\nearrow 2.05^\circ$ .
- 4.146**  $\mathbf{C} = 7.97 \text{ lb} \rightarrow$ ;  $\mathbf{D} = 42.6 \text{ lb} \leftarrow$ ;  $\mathbf{E} = 69.3 \text{ lb} \angle 60.0^\circ$ .
- 4.148**  $\mathbf{A} = 63.6 \text{ lb} \nwarrow 45.0^\circ$ ;  $\mathbf{C} = 87.5 \text{ lb} \angle 59.0^\circ$ .
- 4.150**  $T_{BD} = 780 \text{ N}$ ;  $T_{BE} = 390 \text{ N}$ ;  $\mathbf{A} = -(195.0 \text{ N})\mathbf{i} + (1170 \text{ N})\mathbf{j} + (130.0 \text{ N})\mathbf{k}$ .
- 4.152**  $T_{FJ} = 0$ ;  $T_{DH} = 60.0 \text{ lb}$ ;  $T_{BG} = 80.0 \text{ lb}$ ;  $\mathbf{A} = (100.0 \text{ lb})\mathbf{i} - (48.0 \text{ lb})\mathbf{k}$ .
- 4.153** (a)  $\mathbf{A} = 0.745 P \angle 63.4^\circ$ ;  $\mathbf{C} = 0.471 P \angle 45.0^\circ$ .  
(b)  $\mathbf{A} = 0.812 P \angle 60.0^\circ$ ;  $\mathbf{C} = 0.503 P \nearrow 36.2^\circ$ .  
(c)  $\mathbf{A} = 0.448 P \angle 60.0^\circ$ ;  $\mathbf{C} = 0.652 P \angle 69.9^\circ$ .  
(d) improperly constrained; no equilibrium.
- 4.C1**  $\theta = 20^\circ$ :  $T = 114.8 \text{ lb}$ ;  $\theta = 70^\circ$ :  $T = 127.7 \text{ lb}$ ;  $T_{\max} = 132.2 \text{ lb}$  at  $\theta = 50.4^\circ$ .
- 4.C2**  $x = 600 \text{ mm}$ :  $P = 31.4 \text{ N}$ ;  $x = 150 \text{ mm}$ :  $P = 37.7 \text{ N}$ ;  $P_{\max} = 47.2 \text{ N}$  at  $x = 283 \text{ mm}$ .
- 4.C3**  $\theta = 30^\circ$ :  $W = 9.66 \text{ lb}$ ;  $\theta = 60^\circ$ :  $W = 36.6 \text{ lb}$ ;  $W = 5 \text{ lb}$  at  $\theta = 22.9^\circ$  [Also at  $\theta = 175.7^\circ$ ].
- 4.C4**  $\theta = 30^\circ$ :  $W = 0.80 \text{ lb}$ ;  $\theta = 60^\circ$ :  $W = 4.57 \text{ lb}$ ;  $W = 5 \text{ lb}$  at  $\theta = 62.6^\circ$  [Also at  $\theta = 159.6^\circ$ ].
- 4.C5**  $\theta = 30^\circ$ :  $m = 7.09 \text{ kg}$ ;  $\theta = 60^\circ$ :  $m = 11.02 \text{ kg}$ . When  $m = 10 \text{ kg}$ ,  $\theta = 51.0^\circ$ .
- 4.C6**  $\theta = 15^\circ$ :  $T_{BD} = 10.30 \text{ kN}$ ,  $T_{BE} = 21.7 \text{ kN}$ ;  $\theta = 30^\circ$ :  $T_{BD} = 5.69 \text{ kN}$ ,  $T_{BE} = 24.4 \text{ kN}$ ;  $T_{\max} = 26.5 \text{ kN}$  at  $\theta = 36.9^\circ$ .

## CHAPTER 5

- 5.1**  $\bar{X} = 175.6 \text{ mm}$ ,  $\bar{Y} = 94.4 \text{ mm}$ .
- 5.2**  $\bar{X} = 16.21 \text{ mm}$ ,  $Y = 31.9 \text{ mm}$ .
- 5.3**  $\bar{X} = 19.28 \text{ in.}$ ,  $\bar{Y} = 6.94 \text{ in.}$
- 5.4**  $\bar{X} = 5.67 \text{ in.}$ ,  $\bar{Y} = 5.17 \text{ in.}$
- 5.5**  $\bar{X} = 7.22 \text{ in.}$ ,  $Y = 9.56 \text{ in.}$
- 5.6**  $\bar{X} = 92.0 \text{ mm}$ ,  $\bar{Y} = 23.3 \text{ mm}$ .
- 5.9**  $\bar{X} = -10.00 \text{ mm}$ ,  $\bar{Y} = 87.5 \text{ mm}$ .
- 5.10**  $\bar{X} = -9.89 \text{ mm}$ ,  $\bar{Y} = -10.67 \text{ mm}$
- 5.11**  $\bar{X} = 0$ ,  $\bar{Y} = 6.45 \text{ in.}$
- 5.12**  $\bar{X} = 50.5 \text{ mm}$ ,  $\bar{Y} = 19.34 \text{ mm}$ .
- 5.14**  $\bar{X} = \bar{Y} = 9.00 \text{ in.}$
- 5.16**  $\bar{Y} = \frac{2}{3} \left( \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} \right) \left( \frac{2 \cos \alpha}{\pi - 2\alpha} \right)$
- 5.17**  $\bar{Y} = \frac{r_1 + r_2}{\pi - 2\alpha} \cos \alpha$
- 5.18**  $a/b = 4/5$
- 5.20** 459 N
- 5.21**  $(Q_x)_1 = 25.0 \text{ in}^3$ ;  $(Q_x)_2 = -25.0 \text{ in}^3$ .
- 5.22**  $(Q_x)_1 = 23.3 \text{ in}^3$ ;  $(Q_x)_2 = -23.3 \text{ in}^3$ .
- 5.24**  $\bar{X} = 172.5 \text{ mm}$ ,  $\bar{Y} = 97.5 \text{ mm}$
- 5.26**  $\bar{X} = 18.45 \text{ in.}$ ,  $\bar{Y} = 6.48 \text{ in.}$
- 5.28** (a)  $T = 5.09 \text{ lb}$ . (b)  $\mathbf{C} = 9.48 \text{ lb} \angle 57.5^\circ$ .
- 5.29** 0.739 m
- 5.30**  $L = 0.204 \text{ m}$  or  $0.943 \text{ m}$
- 5.32** (a)  $h = 0.513a$ . (b)  $h = 0.691a$
- 5.34**  $\bar{x} = 2/3a$ ,  $\bar{y} = \frac{1}{3}h$ .
- 5.35**  $\bar{x} = a/2$ ,  $\bar{y} = 2h/5$ .
- 5.37**  $\bar{x} = 2a/3(4 - \pi)$ ,  $\bar{y} = 2b/3(4 - \pi)$
- 5.39**  $\bar{x} = a(3 - 4 \sin \alpha)/6(1 - \alpha)$ ,  $\bar{y} = 0$ .
- 5.40**  $\bar{x} = a/4$ ,  $\bar{y} = 3b/10$ .
- 5.41**  $\bar{x} = 5a/8$ ,  $\bar{y} = b/3$ .
- 5.42**  $\bar{x} = 5L/4$ ,  $\bar{y} = 33a/40$ .
- 5.44**  $\bar{x} = a$ ,  $\bar{y} = 17b/35$ .
- 5.45**  $2/5a$
- 5.46**  $-2\sqrt{2}r/3\pi$ .
- 5.48**  $\bar{x} = 0.236L$ ,  $\bar{y} = 0.454a$ .
- 5.49**  $\bar{x} = -9.27a$ ,  $\bar{y} = 3.09a$ .
- 5.50**  $\bar{x} = 1.629 \text{ in.}$ ,  $\bar{y} = 0.1853 \text{ in.}$
- 5.51**  $a = 1.901 \text{ in.}$  or  $3.74 \text{ in.}$

- 5.52** (a)  $V = 6.19 \cdot 10^6 \text{ mm}^3$ ;  $A = 458 \cdot 10^3 \text{ mm}^2$ .  
(b)  $V = 16.88 \cdot 10^6 \text{ mm}^3$ ;  $A = 1.171 \cdot 10^6 \text{ mm}^2$ .
- 5.53** (a)  $V = 308 \cdot 10^3 \text{ mm}^3$ ;  $A = 38.2 \cdot 10^3 \text{ mm}^2$ .  
(b)  $V = 177.2 \cdot 10^3 \text{ mm}^3$ ;  $A = 22.4 \cdot 10^3 \text{ mm}^2$ .
- 5.54** (a)  $V = 169.0 \cdot 10^3 \text{ in}^3$ ;  $A = 28.4 \cdot 10^3 \text{ in}^2$ .  
(b)  $V = 88.9 \cdot 10^3 \text{ in}^3$ ;  $A = 15.48 \cdot 10^3 \text{ in}^2$ .
- 5.56**  $V = 3470 \text{ mm}^3$ ;  $A = 2320 \text{ mm}^2$ .
- 5.58**  $V = 0.0900 \text{ in}^3$ .
- 5.59**  $V = 31.9 \text{ liters}$ .
- 5.61**  $m = 0.0305 \text{ kg}$ .
- 5.63** 22.0 gallons
- 5.64** 66.5%
- 5.66** (a)  $\mathbf{R} = 1215 \text{ lb } \downarrow$ ,  $\bar{x} = 4.33 \text{ ft}$ .  
(b)  $\mathbf{A} = 630 \text{ lb } \uparrow$ ;  $\mathbf{B} = 585 \text{ lb } \uparrow$
- 5.67** (a)  $\mathbf{R} = 2400 \text{ N } \downarrow$ , 2.33 m to the right of A.  
(b)  $\mathbf{A} = 1000 \text{ N } \uparrow$ ;  $\mathbf{B} = 1400 \text{ N } \uparrow$ .
- 5.68**  $\mathbf{A} = 32.0 \text{ kN}$ ;  $\mathbf{M}_A = 124.0 \text{ kN} \cdot \text{m } \uparrow$ .
- 5.69**  $\mathbf{B} = 1360 \text{ lb } \uparrow$ ;  $\mathbf{C} = 2360 \text{ lb } \uparrow$ .
- 5.71**  $\mathbf{A} = 480 \text{ N } \uparrow$ ;  $\mathbf{B} = 840 \text{ N } \downarrow$ .
- 5.73**  $\mathbf{A} = 3000 \text{ N } \uparrow$ ;  $\mathbf{M}_A = 12.60 \text{ kN} \cdot \text{m } \uparrow$ .
- 5.74** (a)  $a = 0.536 \text{ m}$ . (b)  $\mathbf{A} = \mathbf{B} = 761 \text{ N } \uparrow$ .
- 5.76**  $\mathbf{B} = 150.0 \text{ lb } \uparrow$ ;  $\mathbf{C} = 5250 \text{ lb } \uparrow$ .
- 5.77** (a)  $w_0 = 100.0 \text{ lb/ft}$ . (b)  $\mathbf{C} = 4950 \text{ lb } \uparrow$ .
- 5.78**  $w_A = 10.00 \text{ kN/m}$ ;  $w_B = 50.0 \text{ kN/m}$ .
- 5.80** (a)  $\mathbf{H} = 254 \text{ kN } \rightarrow$ ;  $\mathbf{V} = 831 \text{ kN } \uparrow$ .  
(b)  $x = 3.25 \text{ m}$  to the right of A.  
(c)  $\mathbf{R} = 268 \text{ kN } \angle 18.4^\circ$ .
- 5.81** (a)  $\mathbf{H} = 13.76 \text{ kips } \rightarrow$ ;  $\mathbf{V} = 113.0 \text{ kips } \uparrow$ .  
(b) 22.4 ft to the right of A.  
(c)  $\mathbf{R} = 25.6 \text{ kips } \angle 57.5^\circ$ .
- 5.82**  $d = 2.64 \text{ m}$ .
- 5.83**  $\mathbf{T} = 67.2 \text{ kN } \leftarrow$ ;  $\mathbf{A} = 141.2 \text{ kN } \leftarrow$ .
- 5.84**  $\mathbf{T} = 3.70 \text{ kips } \uparrow$ .
- 5.85**  $d = 5.88 \text{ ft}$ .
- 5.88**  $\mathbf{A} = 1197 \text{ N } \angle 53.1^\circ$ ;  $\mathbf{B} = 1511 \text{ N } \angle 53.1^\circ$ .
- 5.89**  $T = 3570 \text{ N}$ .
- 5.90**  $T = 208 \text{ lb}$ .
- 5.91**  $d = 6.00 \text{ ft}$ .
- 5.93**  $d = 0.683 \text{ m}$ .
- 5.94**  $h = 0.0711 \text{ m}$ .
- 5.96** (a)  $b/10$  to the left of base of cone.  
(b) 0.01136b to the right of base of cone.
- 5.97**  $\bar{Y} = -(2h^2 - 3b^2)/(2(4h - 3b))$ .
- 5.98**  $\bar{Z} = -a(4h - 2b)/\pi(4h - 3b)$ .
- 5.99** (a)  $\bar{Y} = -0.402a$ . (b)  $h/a = 2/5$  or  $2/3$ .
- 5.100**  $\bar{X} = 46.8 \text{ mm}$ .
- 5.101**  $\bar{Z} = 26.2 \text{ mm}$ .
- 5.103**  $\bar{Y} = -0.1403 \text{ in}$ .
- 5.104**  $\bar{Z} = 3.47 \text{ in}$ .
- 5.106**  $\bar{X} = 0.295 \text{ m}$ ,  $\bar{Y} = 0.423 \text{ m}$ ,  $\bar{Z} = 1.703 \text{ m}$ .
- 5.107**  $\bar{X} = 125.0 \text{ mm}$ ,  $\bar{Y} = 167.0 \text{ mm}$ ,  $\bar{Z} = 33.5 \text{ mm}$ .
- 5.108**  $\bar{X} = \bar{Z} = 4.21 \text{ in}$ ,  $\bar{Y} = 7.03 \text{ in}$ .
- 5.110**  $\bar{X} = 46.5 \text{ mm}$ ,  $\bar{Y} = 27.2 \text{ mm}$ ,  $\bar{Z} = 30.0 \text{ mm}$ .
- 5.111**  $\bar{X} = 17.00 \text{ in}$ ,  $\bar{Y} = 15.68 \text{ in}$ ,  $\bar{Z} = 14.16 \text{ in}$ .
- 5.112**  $\bar{X} = 180.2 \text{ mm}$ ,  $\bar{Y} = 38.0 \text{ mm}$ ,  $\bar{Z} = 193.5 \text{ mm}$ .
- 5.114**  $\bar{X} = 0.1452 \text{ m}$ ,  $\bar{Y} = 0.396 \text{ m}$ ,  $\bar{Z} = 0.370 \text{ m}$ .
- 5.115**  $\bar{X} = 0.410 \text{ m}$ ,  $\bar{Y} = 0.510 \text{ m}$ ,  $\bar{Z} = 0.1500 \text{ m}$ .
- 5.117**  $\bar{X} = 1.750 \text{ ft}$ ,  $\bar{Y} = 4.14 \text{ ft}$ ,  $\bar{Z} = 1.355 \text{ ft}$ .
- 5.118**  $\bar{X} = 61.6 \text{ mm}$  from the end of the handle.
- 5.119**  $\bar{Y} = 0.526 \text{ in}$ . above the base.
- 5.121**  $\bar{Y} = 421 \text{ mm}$ . above the floor.
- 5.122**  $(\bar{x})_1 = 21a/88$ ;  $(\bar{x})_2 = 27a/40$ .
- 5.123**  $(\bar{x})_1 = 21h/88$ ;  $(\bar{x})_2 = 27h/40$ .
- 5.124**  $(\bar{x})_1 = 2h/9$ ;  $(\bar{x})_2 = 2h/3$ .
- 5.125**  $\bar{x} = h/6$ ;  $\bar{y} = \bar{z} = 0$ .
- 5.128**  $\bar{x} = 1.297a$ ,  $\bar{y} = \bar{z} = 0$ .
- 5.129**  $\bar{x} = \bar{z} = 0$ ,  $\bar{y} = 0.374b$ .
- 5.132** (a)  $\bar{x} = \bar{z} = 0$ ,  $\bar{y} = -121.9 \text{ mm}$ .  
(b)  $\bar{x} = \bar{z} = 0$ ,  $\bar{y} = -90.2 \text{ mm}$ .
- 5.133**  $V = 688 \text{ ft}^3$ ;  $\bar{x} = 15.91 \text{ ft}$ .
- 5.134**  $\bar{x} = a/2$ ,  $\bar{y} = 8h/25$ ,  $\bar{z} = b/2$ .
- 5.136**  $\bar{x} = 0$ ,  $\bar{y} = 5h/16$ ,  $\bar{z} = -b/4$ .
- 5.137**  $\bar{X} = 19.27 \text{ mm}$ ,  $\bar{Y} = 26.6 \text{ mm}$ .
- 5.138**  $\bar{X} = 3.20 \text{ in}$ ,  $\bar{Y} = 2.00 \text{ in}$ .
- 5.140**  $\bar{x} = a/2$ ,  $\bar{y} = 3h/5$ .
- 5.141**  $\bar{x} = 3a/8$ ,  $\bar{y} = b$ .
- 5.143**  $\mathbf{A} = 1300 \text{ N } \uparrow$ ;  $\mathbf{B} = 1850 \text{ N } \uparrow$ .
- 5.146** (a)  $\bar{x} = 0.548L$ . (b)  $h/L = 2\sqrt{3}$ .
- 5.147**  $\bar{X} = 0.1402 \text{ m}$ ,  $\bar{Y} = 0.0944 \text{ m}$ ,  $\bar{Z} = 0.0959 \text{ m}$ .
- 5.148**  $\bar{x} = 2.34 \text{ m}$ ,  $\bar{y} = \bar{z} = 0$ .
- 5.C1** (a)  $\mathbf{A} = 1220 \text{ lb } \uparrow$ ;  $\mathbf{B} = 1830 \text{ lb } \uparrow$ .  
(c)  $\mathbf{A} = 1265 \text{ lb } \uparrow$ ;  $\mathbf{B} = 1601 \text{ lb } \uparrow$ .
- 5.C2** (a)  $\bar{X} = 0$ ,  $\bar{Y} = 0.278 \text{ m}$ ,  $\bar{Z} = 0.0878 \text{ m}$ .  
(b)  $\bar{X} = 0.0487 \text{ mm}$ ,  $\bar{Y} = 0.1265 \text{ mm}$ ,  $\bar{Z} = 0.0997 \text{ mm}$ .  
(c)  $\bar{X} = -0.0372 \text{ m}$ ,  $\bar{Y} = 0.1659 \text{ m}$ ,  $\bar{Z} = 0.1043 \text{ m}$ .
- 5.C3**  $d = 1.00 \text{ m}$ :  $\mathbf{F} = 5.66 \text{ kN } \angle 30^\circ$ .  
 $d = 3.00 \text{ m}$ :  $\mathbf{F} = 49.9 \text{ kN } \angle 27.7^\circ$ .
- 5.C4** (a)  $\bar{X} = 5.80 \text{ in}$ ,  $\bar{Y} = 1.492 \text{ in}$ . (b)  $\bar{X} = 9.11 \text{ in}$ ,  $\bar{Y} = 2.78 \text{ in}$ .  
(c)  $\bar{X} = 8.49 \text{ in}$ ,  $\bar{Y} = 0.375 \text{ in}$ .
- 5.C5** With  $n = 40$ : (a)  $\bar{X} = 60.2 \text{ mm}$ ,  $\bar{Y} = 23.4 \text{ mm}$ .  
(b)  $\bar{X} = 60.2 \text{ mm}$ ,  $\bar{Y} = 146.2 \text{ mm}$ .  
(c)  $\bar{X} = 68.7 \text{ mm}$ ,  $\bar{Y} = 20.4 \text{ mm}$ .  
(d)  $\bar{X} = 68.7 \text{ mm}$ ,  $\bar{Y} = 127.8 \text{ mm}$ .
- 5.C6** With  $n = 40$ : (a)  $\bar{X} = 60.0 \text{ mm}$ ,  $\bar{Y} = 24.0 \text{ mm}$ .  
(b)  $\bar{X} = 60.0 \text{ mm}$ ,  $\bar{Y} = 150.0 \text{ mm}$ .  
(c)  $\bar{X} = 68.6 \text{ mm}$ ,  $\bar{Y} = 21.8 \text{ mm}$ .  
(d)  $\bar{X} = 68.6 \text{ mm}$ ,  $\bar{Y} = 136.1 \text{ mm}$ .
- 5.C7** (a)  $V = 628 \text{ ft}^3$ .  
(b)  $\bar{X} = 8.65 \text{ ft}$ ,  $\bar{Y} = -4.53 \text{ ft}$ ,  $\bar{Z} = 9.27 \text{ ft}$ .

## CHAPTER 6

- 6.1**  $F_{AB} = 52.0 \text{ kN } T$ ;  $F_{AC} = 64.0 \text{ kN } T$ ;  $F_{BC} = 80.0 \text{ kN } C$ .
- 6.2**  $F_{AB} = 375 \text{ lb } C$ ;  $F_{AC} = 780 \text{ lb } C$ ;  $F_{BC} = 300 \text{ lb } T$ .
- 6.3**  $F_{AB} = 4.00 \text{ kN } C$ ;  $F_{AC} = 2.72 \text{ kN } T$ ;  $F_{BC} = 2.40 \text{ kN } C$ .
- 6.5**  $F_{AB} = F_{BC} = 31.5 \text{ kips } T$ ;  $F_{AD} = 35.7 \text{ kips } C$ ;  
 $F_{BD} = 10.80 \text{ kips } C$ ;  $F_{CD} = 33.3 \text{ kips } C$ .
- 6.6**  $F_{AB} = F_{BD} = 0$ ;  $F_{AC} = 675 \text{ N } T$ ;  $F_{AD} = 1125 \text{ N } C$ ;  
 $F_{CD} = 900 \text{ N } T$ ;  $F_{CE} = 2025 \text{ N } T$ ;  $F_{CF} = 2250 \text{ N } C$ ;  
 $F_{DF} = 675 \text{ N } C$ ;  $F_{EF} = 1800 \text{ N } T$ .
- 6.7**  $F_{AB} = 15.90 \text{ kN } C$ ;  $F_{AC} = 13.50 \text{ kN } T$ ;  $F_{BC} = 16.80 \text{ kN } C$ ;  
 $F_{BD} = 13.50 \text{ kN } C$ ;  $F_{CD} = 15.90 \text{ kN } T$ .
- 6.9**  $F_{AB} = 47.2 \text{ kN } C$ ;  $F_{AC} = 44.6 \text{ kN } T$ ;  $F_{BC} = 10.50 \text{ kN } C$ ;  
 $F_{BD} = 47.2 \text{ kN } C$ ;  $F_{CD} = 17.50 \text{ kN } T$ ;  $F_{CE} = 30.6 \text{ kN } T$ ;  
 $F_{DE} = 0$ .
- 6.10**  $F_{AB} = F_{HI} = 12.31 \text{ kN } C$ ;  $F_{AC} = F_{GI} = 11.25 \text{ kN } T$ ;  $F_{BC} = F_{GH} = 2.46 \text{ kN } C$ ;  $F_{BD} = F_{DE} = F_{EF} = F_{FH} = 9.85 \text{ kN } C$ ;  
 $F_{CD} = F_{FG} = 2.00 \text{ kN } C$ ;  $F_{CE} = F_{EG} = 3.75 \text{ kN } T$ ;  
 $F_{CG} = 6.75 \text{ kN } T$ .
- 6.11**  $F_{AB} = F_{FH} = 1500 \text{ lb } C$ ;  $F_{AC} = F_{CE} = F_{EG} = F_{GH} = 1200 \text{ lb } T$ ;  $F_{BC} = F_{FG} = 0$ ;  $F_{BD} = F_{DF} = 1000 \text{ lb } C$ ;  
 $F_{BE} = F_{EF} = 500 \text{ lb } C$ ;  $F_{DE} = 600 \text{ lb } T$ .

- 6.12**  $F_{AB} = F_{FH} = 1500 \text{ lb}$  C;  $F_{AC} = F_{CE} = F_{EG} = F_{GH} = 1200 \text{ lb}$  T;  $F_{BC} = F_{FG} = 0$ ;  $F_{BD} = F_{DF} = 1200 \text{ lb}$  C;  $F_{BE} = F_{EF} = 60.0 \text{ lb}$  C;  $F_{DE} = 72.0 \text{ lb}$  T.
- 6.15**  $F_{AB} = 7.50 \text{ kips}$  C;  $F_{AC} = 4.50 \text{ kips}$  T;  $F_{BC} = 7.50 \text{ kips}$  T;  $F_{BD} = 9.00 \text{ kips}$  C;  $F_{CD} = 0$ ;  $F_{CE} = 9.00 \text{ kips}$  T.
- 6.16**  $F_{AB} = 5.00 \text{ kips}$  C;  $F_{AC} = 3.00 \text{ kips}$  T;  $F_{BC} = 5.00 \text{ kips}$  T;  $F_{BD} = 6.00 \text{ kips}$  C;  $F_{CD} = F_{EF} = 2.50 \text{ kips}$  T;  $F_{CE} = 4.50 \text{ kips}$  T;  $F_{DE} = F_{FG} = 2.50 \text{ kips}$  C;  $F_{DF} = 3.00 \text{ kips}$  C;  $F_{EG} = 1.500 \text{ kips}$  T.
- 6.19**  $F_{AB} = 9.90 \text{ kN}$  C;  $F_{AC} = 7.83 \text{ kN}$  T;  $F_{BC} = 0$ ;  $F_{BD} = 7.07 \text{ kN}$  C;  $F_{BE} = 2.00 \text{ kN}$  C;  $F_{CE} = 7.83 \text{ kN}$  T;  $F_{DE} = 1.000 \text{ kN}$  T;  $F_{DF} = 5.03 \text{ kN}$  C;  $F_{DG} = 0.559 \text{ kN}$  C;  $F_{EG} = 5.59 \text{ kN}$  T.
- 6.20**  $F_{FG} = 3.50 \text{ kN}$  T;  $F_{FH} = 5.03 \text{ kN}$  C;  $F_{CH} = 1.677 \text{ kN}$  T;  $F_{GI} = F_{IK} = F_{KL} = 3.35 \text{ kN}$  T;  $F_{HI} = F_{IJ} = F_{JK} = 0$ ;  $F_{HJ} = F_{JL} = 4.42 \text{ kN}$  C.
- 6.21**  $F_{AB} = 2240 \text{ lb}$  C;  $F_{AC} = F_{CE} = 2000 \text{ lb}$  T;  $F_{BC} = F_{EH} = 0$ ;  $F_{BD} = 1789 \text{ lb}$  C;  $F_{BE} = 447 \text{ lb}$  C;  $F_{DE} = 600 \text{ lb}$  C;  $F_{DF} = 2010 \text{ lb}$  C;  $F_{DG} = 224 \text{ lb}$  T;  $F_{EG} = 1789 \text{ lb}$  T.
- 6.22**  $F_{FC} = 1400 \text{ lb}$  T;  $F_{FI} = 2010 \text{ lb}$  C;  $F_{GI} = 671 \text{ lb}$  C;  $F_{GJ} = 2430 \text{ lb}$  T;  $F_{HJ} = 361 \text{ lb}$  T;  $F_{IK} = 2910 \text{ lb}$  C;  $F_{JK} = 447 \text{ lb}$  C;  $F_{JL} = 3040 \text{ lb}$  T;  $F_{KL} = 3350 \text{ lb}$  C.
- 6.23**  $F_{AB} = 9.39 \text{ kN}$  C;  $F_{AC} = 8.40 \text{ kN}$  T;  $F_{BC} = 2.26 \text{ kN}$  C;  $F_{BD} = 7.60 \text{ kN}$  C;  $F_{CD} = 0.128 \text{ kN}$  C;  $F_{CE} = 7.07 \text{ kN}$  T;  $F_{DE} = 2.14 \text{ kN}$  C;  $F_{DF} = 6.10 \text{ kN}$  C;  $F_{EF} = 2.23 \text{ kN}$  T.
- 6.24**  $F_{AB} = F_{DF} = 2.29 \text{ kN}$  T;  $F_{AC} = F_{EF} = 2.29 \text{ kN}$  C;  $F_{BC} = F_{DE} = 0.600 \text{ kN}$  C;  $F_{BD} = 2.21 \text{ kN}$  T;  $F_{BE} = F_{EH} = 0$ ;  $F_{CE} = 2.21 \text{ kN}$  C;  $F_{CH} = F_{EJ} = 1.200 \text{ kN}$  C.
- 6.27**  $F_{AB} = 31.0 \text{ kips}$  C;  $F_{AC} = 28.3 \text{ kips}$  C;  $F_{AD} = 15.09 \text{ kips}$  T;  $F_{AE} = 9.50 \text{ kips}$  T;  $F_{BD} = 21.5 \text{ kips}$  T;  $F_{BF} = 28.0 \text{ kips}$  C;  $F_{CE} = 41.0 \text{ kips}$  T;  $F_{CG} = 42.0 \text{ kips}$  C;  $F_{DE} = 22.0 \text{ kips}$  T;  $F_{DF} = 33.5 \text{ kips}$  T;  $F_{EG} = 0$ .
- 6.28**  $F_{AB} = 128.0 \text{ kN}$  T;  $F_{AC} = 136.7 \text{ kN}$  C;  $F_{BD} = F_{DF} = F_{FH} = 128.0 \text{ kN}$  T;  $F_{CE} = F_{EG} = 136.7 \text{ kN}$  C;  $F_{GH} = 192.7 \text{ kN}$  C.
- 6.29** Truss of Prob. 6.33a is the only simple truss.
- 6.30** Truss of Prob. 6.32b is the only simple truss.
- 6.31** (a) BC, CD, IJ, IL, LM, MN. (b) BC, BE, DE, EF, FG, IJ, KN, MN.
- 6.32** (a) AI, BJ, CK, DI, EI, FK, GK. (b) FK, IO.
- 6.35**  $F_{AB} = F_{AD} = 244 \text{ lb}$  C;  $F_{AC} = 1040 \text{ lb}$  T;  $F_{BC} = F_{CD} = 500 \text{ lb}$  C;  $F_{BD} = 280 \text{ lb}$  T.
- 6.36**  $F_{AB} = F_{AD} = 861 \text{ N}$  C;  $F_{AC} = 676 \text{ N}$  C;  $F_{BC} = F_{CD} = 162.5 \text{ N}$  T;  $F_{BD} = 244 \text{ N}$  T.
- 6.37**  $F_{AB} = F_{AD} = 2810 \text{ N}$  T;  $F_{AC} = 5510 \text{ N}$  C;  $F_{BC} = F_{CD} = 1325 \text{ N}$  T;  $F_{BD} = 1908 \text{ N}$  C.
- 6.38**  $F_{AB} = F_{AC} = 1061 \text{ lb}$  C;  $F_{AD} = 2500 \text{ lb}$  T;  $F_{BC} = 2100 \text{ lb}$  T;  $F_{BD} = F_{CD} = 1250 \text{ lb}$  C;  $F_{BE} = F_{CE} = 1250 \text{ lb}$  C;  $F_{DE} = 1500 \text{ lb}$  T.
- 6.39**  $F_{AB} = 840 \text{ N}$  C;  $F_{AC} = 110.6 \text{ N}$  C;  $F_{AD} = 394 \text{ N}$  C;  $F_{AE} = 0$ ;  $F_{BC} = 160.0 \text{ N}$  T;  $F_{BE} = 200 \text{ N}$  T;  $F_{CD} = 225 \text{ N}$  T;  $F_{CE} = 233 \text{ N}$  C;  $F_{DE} = 120.0 \text{ N}$  T.
- 6.40**  $F_{AB} = 0$ ;  $F_{AC} = 995 \text{ N}$  T;  $F_{AD} = 1181 \text{ N}$  C;  $F_{AE} = F_{BC} = 0$ ;  $F_{BE} = 600 \text{ N}$  T;  $F_{CD} = 375 \text{ N}$  T;  $F_{CE} = 700 \text{ N}$  C;  $F_{DE} = 360 \text{ N}$  T.
- 6.43**  $F_{CE} = 8000 \text{ lb}$  T;  $F_{DE} = 2600 \text{ lb}$  T;  $F_{DF} = 9000 \text{ lb}$  C
- 6.44**  $F_{EG} = 7500 \text{ lb}$  T;  $F_{FG} = 3900 \text{ lb}$  C;  $F_{FH} = 6000 \text{ lb}$  C
- 6.45**  $F_{BD} = 216 \text{ kN}$  T;  $F_{DE} = 270 \text{ kN}$  T.
- 6.46**  $F_{DG} = 459 \text{ kN}$  C;  $F_{EG} = 216 \text{ kN}$  C.
- 6.49**  $F_{CE} = 7.20 \text{ kN}$  T;  $F_{DE} = 1.047 \text{ kN}$  C;  $F_{DF} = 6.39 \text{ kN}$  C.
- 6.50**  $F_{EG} = 3.46 \text{ kN}$  T;  $F_{GH} = 3.78 \text{ kN}$  C;  $F_{HJ} = 3.55 \text{ kN}$  C.
- 6.51**  $F_{DF} = 10.48 \text{ kips}$  C;  $F_{DG} = 3.35 \text{ kips}$  C;  $F_{EG} = 13.02 \text{ kips}$  T.
- 6.52**  $F_{OI} = 13.02 \text{ kips}$  T;  $F_{HI} = 0.800 \text{ kips}$  T;  $F_{HJ} = 13.97 \text{ kips}$  C.
- 6.53**  $F_{CE} = 8.00 \text{ kN}$  T;  $F_{DE} = 4.50 \text{ kN}$  C;  $F_{DF} = 10.00 \text{ kN}$  C.
- 6.54**  $F_{FH} = 10.00 \text{ kN}$  C;  $F_{FI} = 4.92 \text{ kN}$  T;  $F_{CI} = 6.00 \text{ kN}$  T.
- 6.55**  $F_{AD} = 13.5 \text{ kN}$  C;  $F_{CD} = 0$ ;  $F_{CE} = 56.1 \text{ kN}$  T.
- 6.56**  $F_{DG} = 75.0 \text{ kN}$  C;  $F_{FG} = 56.1 \text{ kN}$  T;  $F_{FH} = 69.7 \text{ kN}$  T.
- 6.57**  $F_{AB} = 8.20 \text{ kips}$  T;  $F_{AG} = 4.50 \text{ kips}$  T;  $F_{FG} = 11.60 \text{ kips}$  C.
- 6.58**  $F_{AE} = 17.46 \text{ kips}$  T;  $F_{EF} = 11.60 \text{ kips}$  C;  $F_{FJ} = 18.45 \text{ kips}$  C.
- 6.61**  $F_{AF} = 1.500 \text{ kN}$  T;  $F_{EJ} = 0.900 \text{ kN}$  T.
- 6.62**  $F_{AF} = 0.900 \text{ kN}$  T;  $F_{EJ} = 0.300 \text{ kN}$  T.
- 6.65** (a) CJ. (b) 1.026 kN T.
- 6.66** (a) IO. (b) 2.05 kN T.
- 6.67**  $F_{BG} = 5.48 \text{ kips}$  T;  $F_{DC} = 1.825 \text{ kips}$  T.
- 6.68**  $F_{CF} = 3.65 \text{ kips}$  T;  $F_{CH} = 7.30 \text{ kips}$  T.
- 6.69** (a) improperly constrained. (b) completely constrained, determinate. (c) completely constrained, indeterminate.
- 6.70** (a) completely constrained, determinate. (b) partially constrained. (c) improperly constrained.
- 6.71** (a) completely constrained, determinate. (b) completely constrained, indeterminate. (c) improperly constrained.
- 6.72** (a) partially constrained. (b) completely constrained, determinate. (c) completely constrained, indeterminate.
- 6.75** (a) 125 N ↘ 36.9°. (b) 125 N ↗ 36.9°.
- 6.76**  $F_{BD} = 255 \text{ N}$  C;  $\mathbf{C}_x = 120.0 \text{ N} \rightarrow$ ;  $\mathbf{C}_y = 625 \text{ N} \uparrow$ .
- 6.77** (a) 80.0 lb T. (b) 72.1 lb ↗ 16.1°.
- 6.78** (a) 80.0 lb T. (b) 72.1 lb ↘ 16.1°.
- 6.81**  $\mathbf{A}_x = 18.00 \text{ kN} \leftarrow$ ;  $\mathbf{A}_y = 20.0 \text{ kN} \downarrow$ ;  $\mathbf{B} = 9.00 \text{ kN} \rightarrow$ ;  $\mathbf{C}_x = 9.00 \text{ kN} \rightarrow$ ;  $\mathbf{C}_y = 20.0 \text{ kN} \uparrow$ .
- 6.82**  $\mathbf{A} = 20.0 \text{ kN} \downarrow$ ;  $\mathbf{B} = 18.00 \text{ kN} \leftarrow$ ;  $\mathbf{C}_x = 18.00 \text{ kN} \rightarrow$ ;  $\mathbf{C}_y = 20.0 \text{ kN} \uparrow$ .
- 6.83** (a)  $\mathbf{A}_x = 450 \text{ N} \leftarrow$ ,  $\mathbf{A}_y = 525 \text{ N} \uparrow$ ;  $\mathbf{E}_x = 450 \text{ N} \rightarrow$ ,  $\mathbf{E}_y = 225 \text{ N} \uparrow$ ; (b)  $\mathbf{A}_x = 450 \text{ N} \leftarrow$ ,  $\mathbf{A}_y = 150.0 \text{ N} \uparrow$ ;  $\mathbf{E}_x = 450 \text{ N} \rightarrow$ ,  $\mathbf{E}_y = 600 \text{ N} \uparrow$ .
- 6.84** (a)  $\mathbf{A}_x = 300 \text{ N} \leftarrow$ ,  $\mathbf{A}_y = 660 \text{ N} \uparrow$ ;  $\mathbf{E}_x = 300 \text{ N} \rightarrow$ ,  $\mathbf{E}_y = 90.0 \text{ N} \uparrow$ . (b)  $\mathbf{A}_x = 300 \text{ N} \leftarrow$ ,  $\mathbf{A}_y = 150.0 \text{ N} \uparrow$ ;  $\mathbf{E}_x = 300 \text{ N} \rightarrow$ ,  $\mathbf{E}_y = 600 \text{ N} \uparrow$ .
- 6.87** (a)  $\mathbf{A}_x = 80.0 \text{ lb} \leftarrow$ ,  $\mathbf{A}_y = 40.0 \text{ lb} \uparrow$ ;  $\mathbf{B}_x = 80.0 \text{ lb} \rightarrow$ ,  $\mathbf{B}_y = 60.0 \text{ lb} \uparrow$ . (b)  $\mathbf{A}_x = 0$ ,  $\mathbf{A}_y = 40.0 \text{ lb} \uparrow$ ;  $\mathbf{B}_x = 0$ ,  $\mathbf{B}_y = 60.0 \text{ lb} \uparrow$ .
- 6.88** (a) and (c)  $\mathbf{B}_x = 32.0 \text{ lb} \rightarrow$ ,  $\mathbf{B}_y = 10.00 \text{ lb} \uparrow$ ;  $\mathbf{F}_x = 32.0 \text{ lb} \leftarrow$ ,  $\mathbf{F}_y = 38.0 \text{ lb} \uparrow$ . (b)  $\mathbf{B}_x = 32.0 \text{ lb} \rightarrow$ ,  $\mathbf{B}_y = 34.0 \text{ lb} \uparrow$ ,  $\mathbf{F}_x = 32.0 \text{ lb} \leftarrow$ ,  $\mathbf{F}_y = 14.00 \text{ lb} \uparrow$ .
- 6.89** (a) and (c)  $\mathbf{B}_x = 24.0 \text{ lb} \leftarrow$ ,  $\mathbf{B}_y = 7.5 \text{ lb} \downarrow$ ;  $\mathbf{F}_x = 24.0 \text{ lb} \rightarrow$ ,  $\mathbf{F}_y = 7.50 \text{ lb} \uparrow$ . (b)  $\mathbf{B}_x = 24.0 \text{ lb} \leftarrow$ ,  $\mathbf{B}_y = 10.50 \text{ lb} \uparrow$ ;  $\mathbf{F}_x = 24.0 \text{ lb} \rightarrow$ ,  $\mathbf{F}_y = 10.50 \text{ lb} \downarrow$ .
- 6.91**  $\mathbf{B}_x = 700 \text{ N} \leftarrow$ ,  $\mathbf{B}_y = 200 \text{ N} \downarrow$ ;  $\mathbf{E}_x = 700 \text{ N} \rightarrow$ ,  $\mathbf{E}_y = 500 \text{ N} \uparrow$ .
- 6.92**  $\mathbf{D}_x = 13.60 \text{ kN} \rightarrow$ ,  $\mathbf{D}_y = 7.50 \text{ kN} \uparrow$ ;  $\mathbf{E}_x = 13.60 \text{ kN} \leftarrow$ ,  $\mathbf{E}_y = 2.70 \text{ kN} \downarrow$ .
- 6.93**  $\mathbf{A}_x = 176.3 \text{ lb} \leftarrow$ ,  $\mathbf{A}_y = 60.0 \text{ lb} \downarrow$ ;  $\mathbf{G}_x = 56.3 \text{ lb} \rightarrow$ ,  $\mathbf{G}_y = 510 \text{ lb} \uparrow$ .
- 6.94**  $\mathbf{A}_x = 56.3 \text{ lb} \leftarrow$ ,  $\mathbf{A}_y = 157.5 \text{ lb} \downarrow$ ;  $\mathbf{G}_x = 56.3 \text{ lb} \rightarrow$ ,  $\mathbf{G}_y = 383 \text{ lb} \uparrow$ .
- 6.95** (a)  $\mathbf{A} = 982 \text{ lb} \uparrow$ ;  $\mathbf{B} = 935 \text{ lb} \uparrow$ ;  $\mathbf{C} = 733 \text{ lb} \uparrow$ . (b)  $\Delta B = +291 \text{ lb}$ ;  $\Delta C = -72.7 \text{ lb}$ .
- 6.96** (a) = 572 lb. (b)  $\mathbf{A} = 1070 \text{ lb} \uparrow$ ;  $\mathbf{B} = 709 \text{ lb} \uparrow$ ;  $\mathbf{C} = 870 \text{ lb} \uparrow$ .
- 6.99**  $\mathbf{C}_x = 78.0 \text{ lb} \rightarrow$ ,  $\mathbf{C}_y = 28.0 \text{ lb} \uparrow$ ;  $\mathbf{F}_x = 78.0 \text{ lb} \leftarrow$ ,  $\mathbf{F}_y = 12.00 \text{ lb} \uparrow$ .
- 6.100**  $\mathbf{C}_x = 21.7 \text{ lb} \rightarrow$ ,  $\mathbf{C}_y = 37.5 \text{ lb} \downarrow$ ;  $\mathbf{D}_x = 21.7 \text{ lb} \leftarrow$ ,  $\mathbf{D}_y = 62.5 \text{ lb} \uparrow$ .
- 6.101**  $\mathbf{A}_x = 13.00 \text{ kN} \leftarrow$ ,  $\mathbf{A}_y = 4.00 \text{ kN} \downarrow$ ;  $\mathbf{B}_x = 36.0 \text{ kN} \rightarrow$ ,  $\mathbf{B}_y = 6.00 \text{ kN} \uparrow$ ;  $\mathbf{E}_x = 23.0 \text{ kN} \leftarrow$ ,  $\mathbf{E}_y = 2.00 \text{ kN} \downarrow$ .
- 6.102**  $\mathbf{A}_x = 2025 \text{ N} \leftarrow$ ,  $\mathbf{A}_y = 1800 \text{ N} \downarrow$ ;  $\mathbf{B}_x = 4050 \text{ N} \rightarrow$ ,  $\mathbf{B}_y = 1200 \text{ N} \uparrow$ ;  $\mathbf{E}_x = 2025 \text{ N} \leftarrow$ ,  $\mathbf{E}_y = 600 \text{ N} \uparrow$ .

- 6.103** (a)  $\mathbf{C}_x = 100.0 \text{ lb } \leftarrow$ ,  $\mathbf{C}_y = 100.0 \text{ lb } \uparrow$ ;  $\mathbf{D}_x = 100.0 \text{ lb } \rightarrow$ ,  $\mathbf{D}_y = 20.0 \text{ lb } \downarrow$ . (b)  $\mathbf{E}_x = 100.0 \text{ lb } \leftarrow$ ,  $\mathbf{E}_y = 180.0 \text{ lb } \uparrow$ .
- 6.104** (a)  $\mathbf{C}_x = 100.0 \text{ lb } \leftarrow$ ,  $\mathbf{C}_y = 60.0 \text{ lb } \uparrow$ ;  $\mathbf{D}_x = 100.0 \text{ lb } \rightarrow$ ,  $\mathbf{D}_y = 20.0 \text{ lb } \uparrow$ . (b)  $\mathbf{E}_x = 100.0 \text{ lb } \leftarrow$ ,  $\mathbf{E}_y = 140.0 \text{ lb } \uparrow$ .
- 6.107** (a)  $\mathbf{A}_x = 200 \text{ kN } \rightarrow$ ,  $\mathbf{A}_y = 122 \text{ kN } \uparrow$ . (b)  $\mathbf{B}_x = 200 \text{ kN } \leftarrow$ ,  $\mathbf{B}_y = 10.00 \text{ kN } \downarrow$ .
- 6.108** (a)  $\mathbf{A}_x = 205 \text{ kN } \rightarrow$ ,  $\mathbf{A}_y = 134.5 \text{ kN } \uparrow$ . (b)  $\mathbf{B}_x = 205 \text{ kN } \leftarrow$ ,  $\mathbf{B}_y = 5.50 \text{ kN } \uparrow$ .
- 6.109**  $\mathbf{B} = 98.5 \text{ lb } \angle 24.0^\circ$ ;  $\mathbf{C} = 90.6 \text{ lb } \angle 6.3^\circ$ .
- 6.110** (a)  $301 \text{ lb } \angle 48.4^\circ$ . (b)  $375 \text{ lb } T$ .
- 6.111**  $F_{AG} = \sqrt{2} P/6 C$ ;  $F_{BF} = 2\sqrt{2} P/3 C$ ;  $F_{DI} = \sqrt{2} P/3 C$ ;  $F_{EH} = \sqrt{2} P/6 T$ .
- 6.113**  $F_{AF} = P/4 C$ ;  $F_{BG} = F_{DG} = P/\sqrt{2} C$ ;  $F_{EH} = P/4 T$ .
- 6.115**  $F_{AF} = M_0/4a C$ ;  $F_{BG} = F_{DG} = M_0/\sqrt{2}a T$ ;  $F_{EH} = 3M_0/4a C$ .
- 6.116**  $F_{AF} = \sqrt{2} M_0/3a C$ ;  $F_{BG} = M_0/a T$ ;  $F_{DG} = M_0/a C$ ;  $F_{EH} = 2\sqrt{2} M_0/3a T$ .
- 6.117**  $\mathbf{E} = P/5 \downarrow$ ;  $\mathbf{F} = 8P/5 \uparrow$ ;  $\mathbf{G} = 4P/5 \downarrow$ ;  $\mathbf{H} = 2P/5 \uparrow$ .
- 6.118**  $\mathbf{A} = P/15 \uparrow$ ;  $\mathbf{D} = 2P/15 \uparrow$ ;  $\mathbf{E} = 8P/15 \uparrow$ ;  $\mathbf{H} = 4P/15 \uparrow$ .
- 6.119** (a)  $\mathbf{A} = 2.06P \angle 14.0^\circ$ ;  $\mathbf{B} = 2.06P \angle 14.0^\circ$ ; frame is rigid.  
(b) Frame is not rigid. (c)  $\mathbf{A} = 1.25P \angle 36.9^\circ$ ;  $\mathbf{B} = 1.031P \angle 14.0^\circ$ ; frame is rigid.
- 6.122** (a)  $(F_{BD})_y = 96.0 \text{ lb } \downarrow$  (b)  $F_{BD} = 100.0 \text{ lb } \angle 73.7^\circ$ .
- 6.123** (a)  $(F_{BD})_y = 240 \text{ lb } \downarrow$  (b)  $F_{BD} = 250 \text{ lb } \angle 73.7^\circ$ .
- 6.126** (a)  $746 \text{ N } \downarrow$ . (b)  $565 \text{ N } \angle 61.3^\circ$ .
- 6.127** (a)  $302 \text{ N } \downarrow$ . (b)  $682 \text{ N } \angle 61.3^\circ$ .
- 6.128**  $T_{DE} = 81.0 \text{ N}$ ;  $\mathbf{B} = 216 \text{ N } \downarrow$ .
- 6.129** (a)  $21.0 \text{ kN } \leftarrow$ . (b)  $52.5 \text{ kN } \leftarrow$ .
- 6.130** (a)  $1143 \text{ N } \cdot \text{m } \downarrow$ . (b)  $457 \text{ N } \cdot \text{m } \downarrow$ .
- 6.131**  $832 \text{ lb } \cdot \text{in. } \uparrow$ .
- 6.132**  $360 \text{ lb } \cdot \text{in. } \uparrow$ .
- 6.133**  $195.0 \text{ kN } \cdot \text{m } \downarrow$ .
- 6.134**  $40.5 \text{ kN } \cdot \text{m } \uparrow$ .
- 6.137**  $208 \text{ N } \cdot \text{m } \downarrow$ .
- 6.138**  $18.43 \text{ N } \cdot \text{m } \downarrow$ .
- 6.139**  $F_{AE} = 800 \text{ N } T$ ;  $F_{DG} = 100.0 \text{ N } C$ .
- 6.140**  $\mathbf{P} = 120.0 \text{ N } \downarrow$ ;  $\mathbf{Q} = 110.0 \text{ N } \leftarrow$ .
- 6.142**  $\mathbf{F} = 3290 \text{ lb } \angle 15.12^\circ$ ;  $\mathbf{D} = 4450 \text{ lb } \leftarrow$ .
- 6.143**  $\mathbf{D} = 30.0 \text{ kN } \leftarrow$ ;  $\mathbf{F} = 37.5 \text{ kN } \angle 36.9^\circ$ .
- 6.144**  $\mathbf{D} = 150.0 \text{ kN } \leftarrow$ ;  $\mathbf{F} = 96.4 \text{ kN } \angle 13.50^\circ$ .
- 6.145** (a)  $475 \text{ lb}$ . (b)  $528 \text{ lb } \angle 63.3^\circ$ .
- 6.146**  $44.8 \text{ kN}$ .
- 6.148**  $8.45 \text{ kN}$ .
- 6.149**  $25.0 \text{ lb } \downarrow$
- 6.150**  $10.00 \text{ lb } \downarrow$ .
- 6.151**  $240 \text{ N}$ .
- 6.154** (a)  $14.11 \text{ kN } \angle 19.10^\circ$ . (b)  $19.79 \text{ kN } \angle 47.6^\circ$ .
- 6.155** (a)  $4.91 \text{ kips } C$ . (b)  $10.69 \text{ kips } C$ .
- 6.156** (a)  $2.86 \text{ kips } C$ . (b)  $9.43 \text{ kips } C$ .
- 6.159** (a)  $M_0 = (90.0 \text{ N } \cdot \text{m})\mathbf{i}$ . (b)  $\mathbf{A} = 0$ ;  $\mathbf{B} = 0$ ;  $\mathbf{M}_B = -(72.0 \text{ N } \cdot \text{m})\mathbf{i}$
- 6.160** (a)  $27.0 \text{ mm}$ . (b)  $40.0 \text{ N } \cdot \text{m } \downarrow$ .
- 6.163**  $\mathbf{E}_x = 100.0 \text{ kN } \rightarrow$ ,  $\mathbf{E}_y = 154.9 \text{ kN } \uparrow$ ;  $\mathbf{F}_x = 26.5 \text{ kN } \rightarrow$ ,  $\mathbf{F}_y = 118.1 \text{ kN } \downarrow$ ;  $\mathbf{H}_x = 126.5 \text{ kN } \leftarrow$ ,  $\mathbf{H}_y = 36.8 \text{ kN } \downarrow$ .
- 6.164**  $F_{AB} = F_{AE} = 671 \text{ lb } T$ ;  $F_{AC} = F_{AD} = 1000 \text{ lb } C$ ;  $F_{BC} = F_{DE} = 600 \text{ lb } C$ ;  $F_{CD} = 200 \text{ lb } T$ .
- 6.166**  $F_{FG} = 5.23 \text{ kN } C$ ;  $F_{EG} = 0.1476 \text{ kN } C$ ;  $F_{EH} = 5.08 \text{ kN } T$ .
- 6.167**  $F_{KM} = 5.02 \text{ kN } T$ ;  $F_{LM} = 1.963 \text{ kN } C$ ;  $F_{LN} = 3.95 \text{ kN } C$ .
- 6.168**  $\mathbf{A}_x = 25.0 \text{ kips } \leftarrow$ ,  $\mathbf{A}_y = 20.0 \text{ kips } \uparrow$ ;  $\mathbf{B}_x = 25.0 \text{ kips } \leftarrow$ ,  $\mathbf{B}_y = 190.0 \text{ kips } \downarrow$ ;  $\mathbf{C}_x = 50.0 \text{ kips } \rightarrow$ ,  $\mathbf{C}_y = 10.00 \text{ kips } \downarrow$ .
- 6.170**  $\mathbf{A}_x = 150.0 \text{ N } \leftarrow$ ,  $\mathbf{A}_y = 250 \text{ N } \uparrow$ ;  $\mathbf{E}_x = 150.0 \text{ N } \rightarrow$ ,  $\mathbf{E}_y = 450 \text{ N } \uparrow$ .
- 6.171**  $\mathbf{A} = 327 \text{ lb } \rightarrow$ ;  $\mathbf{B} = 827 \text{ lb } \leftarrow$ ;  $\mathbf{D} = 621 \text{ lb } \uparrow$ ;  $\mathbf{E} = 246 \text{ lb } \uparrow$ .

**6.172** (a)  $\mathbf{P} = 109.8 \text{ N } \rightarrow$ . (b)  $126.8 \text{ N } T$ , (c)  $139.8 \text{ N } \angle 38.3^\circ$ .

**6.175** (a)  $312 \text{ lb}$ . (b)  $135.0 \text{ lb } \cdot \text{in. } \downarrow$ .

- 6.C1** (a)  $\theta = 30^\circ$ ;  $W = 472 \text{ lb}$ ,  $A_{AB} = 1.500 \text{ in}^2$ ,  $A_{AC} = A_{CE} = 1.299 \text{ in}^2$ ,  $A_{BC} = A_{BE} = 0.500 \text{ in}^2$ ,  $A_{BD} = 1.732 \text{ in}^2$ .  
(b)  $\theta_{\text{opt}} = 56.8^\circ$ ;  $W = 312 \text{ lb}$ ,  $A_{AB} = 0.896 \text{ in}^2$ ,  $A_{AC} = 0.491 \text{ in}^2$ ,  $A_{BC} = 0.500 \text{ in}^2$ ,  $A_{BE} = 0.299 \text{ in}^2$ ,  $A_{BD} = 0.655 \text{ in}^2$ .

- 6.C2** (a) For  $x = 9.75 \text{ m}$ ,  $F_{BH} = 3.19 \text{ kN } T$ . (b) For  $x = 3.75 \text{ m}$ ,  $F_{BH} = 1.313 \text{ kN } C$ . (c) For  $x = 6 \text{ m}$ ,  $F_{CH} = 3.04 \text{ kN } T$ .

**6.C3**  $\theta = 30^\circ$ ;  $\mathbf{M} = 5860 \text{ lb } \cdot \text{ft } \uparrow$ ;  $\mathbf{A} = 670 \text{ lb } \angle 75.5^\circ$ .

- (a)  $M_{\text{max}} = 8680 \text{ lb } \cdot \text{ft}$  when  $\theta = 65.9^\circ$ .  
(b)  $A_{\text{max}} = 1436 \text{ lb}$  when  $\theta = 68.5^\circ$ .

- 6.C4**  $\theta = 30^\circ$ ;  $\mathbf{M}_A = 1.669 \text{ N } \cdot \text{m } \uparrow$ ,  $F = 11.79 \text{ N}$ .  $\theta = 80^\circ$ ;  $\mathbf{M}_A = 3.21 \text{ N } \cdot \text{m } \uparrow$ ,  $F = 11.98 \text{ N}$ .

- 6.C5**  $d = 0.40 \text{ in.}$ :  $634 \text{ lb } C$ ;  $d = 0.55 \text{ in.}$ :  $286 \text{ lb } C$ ;  $d = 0.473 \text{ in.}$ :  $F_{AB} = 500 \text{ lb } C$ .

- 6.C6**  $\theta = 20^\circ$ ;  $M = 31.8 \text{ N } \cdot \text{m}$ ;  $\theta = 75^\circ$ ;  $M = 12.75 \text{ N } \cdot \text{m}$ ;  $\theta = 60.0^\circ$ ;  $M_{\text{min}} = 12.00 \text{ N } \cdot \text{m}$ .

## CHAPTER 7

**7.1**  $\mathbf{F} = 0$ ;  $\mathbf{V} = 80.0 \text{ lb } \uparrow$ ;  $\mathbf{M} = 480 \text{ lb } \cdot \text{in. } \uparrow$ .

**7.2**  $\mathbf{F} = 0$ ;  $\mathbf{V} = 40.0 \text{ lb } \uparrow$ ;  $\mathbf{M} = 240 \text{ lb } \cdot \text{in. } \uparrow$ .

**7.3**  $\mathbf{F} = 4.80 \text{ kN } \leftarrow$ ;  $\mathbf{V} = 1.400 \text{ kN } \downarrow$ ;  $\mathbf{M} = 1.380 \text{ kN } \cdot \text{m } \downarrow$ .

**7.4**  $\mathbf{F} = 3.00 \text{ kN } \leftarrow$ ;  $\mathbf{V} = 0$ ;  $\mathbf{M} = 0.600 \text{ kN } \cdot \text{m } \downarrow$ .

**7.7** (On AJ)  $\mathbf{F} = 103.9 \text{ N } \nwarrow$ ;  $\mathbf{V} = 60.0 \text{ N } \nearrow$ ;  $\mathbf{M} = 18.71 \text{ N } \cdot \text{m } \downarrow$ .

**7.8** (On BK)  $\mathbf{F} = 60.0 \text{ N } \swarrow$ ;  $\mathbf{V} = 103.9 \text{ N } \searrow$ ;  $\mathbf{M} = 10.80 \text{ N } \cdot \text{m } \uparrow$ .

**7.9** (On CJ)  $\mathbf{F} = 23.6 \text{ lb } \searrow$ ;  $\mathbf{V} = 29.1 \text{ lb } \swarrow$ ;  $\mathbf{M} = 540 \text{ lb } \cdot \text{in. } \uparrow$ .

**7.10** (a)  $30.0 \text{ lb}$  at  $C$ . (b)  $33.5 \text{ lb}$  at  $B$  and  $D$ . (c)  $960 \text{ lb } \cdot \text{in.}$  at  $C$ .

**7.13** (On AJ)  $\mathbf{F} = 194.6 \text{ N } \angle 60^\circ$ ;  $\mathbf{V} = 257 \text{ N } \angle 30^\circ$ ;  $\mathbf{M} = 24.7 \text{ N } \cdot \text{m } \downarrow$ .

**7.14**  $45.2 \text{ N } \cdot \text{m}$  for  $\theta = 82.9^\circ$ .

**7.15** (a)  $\mathbf{F} = 500 \text{ N } \leftarrow$ ;  $\mathbf{V} = 500 \text{ N } \uparrow$ ;  $\mathbf{M} = 300 \text{ N } \cdot \text{m } \downarrow$ .  
(b)  $\mathbf{F} = 970 \text{ N } \uparrow$ ;  $\mathbf{V} = 171.0 \text{ N } \leftarrow$ ;  $\mathbf{M} = 446 \text{ N } \cdot \text{m } \downarrow$ .

**7.16** (a)  $\mathbf{F} = 500 \text{ N } \leftarrow$ ;  $\mathbf{V} = 500 \text{ N } \uparrow$ ;  $\mathbf{M} = 300 \text{ N } \cdot \text{m } \downarrow$ .  
(b)  $\mathbf{F} = 933 \text{ N } \uparrow$ ;  $\mathbf{V} = 250 \text{ N } \leftarrow$ ;  $\mathbf{M} = 375 \text{ N } \cdot \text{m } \downarrow$ .

**7.17** (On BJ)  $\mathbf{F} = 200 \text{ N } \searrow$ ;  $\mathbf{V} = 120.0 \text{ N } \nearrow$ ;  $\mathbf{M} = 120.0 \text{ N } \cdot \text{m } \uparrow$ .

**7.18** (On AK)  $\mathbf{F} = 520 \text{ N } \leftarrow$ ;  $\mathbf{V} = 120.0 \text{ N } \downarrow$ ;  $\mathbf{M} = 96.0 \text{ N } \cdot \text{m } \downarrow$ .

**7.19**  $150.0 \text{ lb } \cdot \text{in.}$  at  $D$ .

**7.20**  $105.0 \text{ lb } \cdot \text{in.}$  at  $E$ .

**7.23** (On BJ)  $0.289 \text{ Wr } \uparrow$ .

**7.24** (On BJ)  $0.417 \text{ Wr } \uparrow$ .

**7.27**  $0.1009 \text{ Wr}$  for  $\theta = 57.3^\circ$ .

**7.28**  $0.357 \text{ Wr}$  for  $\theta = 49.3^\circ$ .

**7.29** (b)  $|V|_{\text{max}} = 2P$ ;  $|M|_{\text{max}} = 3Pa$ .

**7.30** (b)  $|V|_{\text{max}} = 2P/3$ ;  $|M|_{\text{max}} = 2PL/9$ .

**7.31** (b)  $wL/4$ ;  $3wL^2/32$ .

**7.32** (b)  $wL/2$ ;  $3wL^2/8$ .

**7.35** (b)  $|V|_{\text{max}} = 35.0 \text{ kN}$ ;  $|M|_{\text{max}} = 12.50 \text{ kN } \cdot \text{m}$ .

**7.36** (b)  $|V|_{\text{max}} = 50.5 \text{ kN}$ ;  $|M|_{\text{max}} = 39.8 \text{ kN } \cdot \text{m}$ .

**7.39** (b)  $|V|_{\text{max}} = 64.0 \text{ kN}$ ;  $|M|_{\text{max}} = 92.0 \text{ kN } \cdot \text{m}$ .

**7.40** (b)  $|V|_{\text{max}} = 60.0 \text{ kN}$ ;  $|M|_{\text{max}} = 72.0 \text{ kN } \cdot \text{m}$ .

**7.41** (b)  $|V|_{\text{max}} = 18.00 \text{ kips}$ ;  $|M|_{\text{max}} = 48.5 \text{ kip } \cdot \text{ft}$ .

**7.42** (b)  $|V|_{\text{max}} = 15.30 \text{ kips}$ ;  $|M|_{\text{max}} = 46.8 \text{ kip } \cdot \text{ft}$ .

**7.43** (b)  $|V|_{\text{max}} = 1.800 \text{ kN}$ ;  $|M|_{\text{max}} = 0.225 \text{ kN } \cdot \text{m}$ .

**7.44** (b)  $|V|_{\text{max}} = 2.00 \text{ kN}$ ;  $|M|_{\text{max}} = 0.500 \text{ kN } \cdot \text{m}$ .

**7.45** (a)  $M \leq 0$  everywhere.

(b)  $|V|_{\text{max}} = 4.50 \text{ kips}$ ;  $|M|_{\text{max}} = 13.50 \text{ kip } \cdot \text{ft}$ .

**7.46** (a)  $M \geq 0$  everywhere.

(b)  $|V|_{\text{max}} = 4.50 \text{ kips}$ ;  $|M|_{\text{max}} = 13.50 \text{ kip } \cdot \text{ft}$ .

- 7.49** (a) +400 N; +160.0 N · m. (b) -200N; +40.0 N · m.
- 7.52**  $|V|_{\max} = 7.50$  kips;  $|M|_{\max} = 7.20$  kip · ft.
- 7.53**  $|V|_{\max} = 165$  lb;  $|M|_{\max} = 1625$  lb · in.
- 7.54**  $|V|_{\max} = 800$  N;  $|M|_{\max} = 180.0$  N · m.
- 7.55** (a)  $54.5^\circ$ . (b)  $675$  N · m.
- 7.56** (a)  $0.311$  m. (b)  $193.0$  N · m.
- 7.57** (a)  $1.236$ . (b)  $0.1180 w a^2$ .
- 7.58**  $a = 0.207 L$
- 7.59** (a)  $40.0$  kips, (b)  $40.0$  kip · ft.
- 7.62** (a)  $0.414 wL$ ;  $0.0858 wL^2$ . (b)  $0.250 wL$ ;  $0.250 wL^2$ .
- 7.63**  $|V|_{\max} = 2P$ ;  $|M|_{\max} = 3Pa$
- 7.69**  $|V|_{\max} = 7.20$  kN;  $|M|_{\max} = 5.76$  kN · m.
- 7.70**  $|V|_{\max} = 720$  N;  $|M|_{\max} = 164.0$  N · m.
- 7.72**  $|V|_{\max} = 60.0$  kN;  $|M|_{\max} = 72.0$  kN · m.
- 7.77** (b)  $9.00$  kN · m,  $1.700$  m from A.
- 7.78** (b)  $26.4$  kN · m,  $2.05$  m from A.
- 7.79** (b)  $45.0$  kip · ft,  $12.00$  ft from A.
- 7.80** (b)  $12.00$  kip · ft,  $6.00$  ft from A.
- 7.83** (b)  $40.5$  kN · m,  $1.800$  m from A.
- 7.84** (b)  $60.5$  kN · m,  $2.20$  m from A.
- 7.85** (a)  $V = (w_0/6L)(3x^2 - 6Lx + 2L^2)$ ;  
 $M = (w_0/6L)(x^3 - 3Lx^2 + 2L^2x)$ .  
(b)  $0.0642w_0L^2$ , at  $x = 0.423L$ .
- 7.86** (a)  $V = (w_0/3L)(2x^2 - 3Lx + L^2)$ ;  
 $M = (w_0/18L)(4x^3 - 9Lx^2 + 6L^2x - L^3)$ .  
(b)  $w_0L^2/72$ , at  $x = L/2$ .
- 7.89** (a)  $\mathbf{P} = 4.00$  kN ↓;  $\mathbf{Q} = 6.00$  kN ↓. (b)  $M_C = -900$  N · m.
- 7.90** (a)  $\mathbf{P} = 2.50$  kN ↓;  $\mathbf{Q} = 7.50$  kN ↓. (b)  $M_C = -900$  N · m.
- 7.91** (a)  $\mathbf{P} = 1.350$  kips ↓;  $\mathbf{Q} = 0.450$  kips ↓.  
(b)  $V_{\max} = 2.70$  kips at A;  $M_{\max} = 6.345$  kip · ft,  
 $5.40$  ft from A.
- 7.92** (a)  $\mathbf{P} = 0.540$  kips ↓;  $\mathbf{Q} = 1.860$  kips ↓.  
(b)  $|V|_{\max} = 3.14$  kips at B;  $M_{\max} = 6.997$  kip · ft,  
 $6.88$  ft from A.
- 7.93** (a)  $2.28$  m. (b)  $\mathbf{D}_x = 13.67$  kN →;  $\mathbf{D}_y = 7.80$  kN ↑.  
(c)  $15.94$  kN.
- 7.94** (a)  $1.959$  m. (b)  $2.48$  m.
- 7.95** (a)  $838$  lb ↘  $17.4^\circ$ . (b)  $971$  lb ↙  $34.5^\circ$ .
- 7.96** (a)  $2670$  lb ↗  $21.0^\circ$ . (b)  $2810$  lb ↖  $18.6^\circ$ .
- 7.97** (a)  $d_B = 1.733$  m;  $d_D = 4.20$  m. (b)  $21.5$  kN ↖  $3.8^\circ$ .
- 7.98** (a)  $2.8$  m. (b)  $\mathbf{A} = 32.0$  kN ↘  $38.7^\circ$ ;  $\mathbf{E} = 25.0$  kN →.
- 7.101** (a)  $48.0$  lb. (b)  $10.00$  ft.
- 7.102** (a)  $12.50$  ft. (b)  $5.00$  ft.
- 7.103**  $196.2$  N.
- 7.104**  $157.0$  N.
- 7.107** (a)  $138.1$  m. (b)  $602$  N.
- 7.108** (a)  $6.75$  m. (b)  $T_{AB} = 615$  N;  $T_{BC} = 600$  N.
- 7.109** (a)  $56,400$  kips. (b)  $4280$  ft.
- 7.110** (a)  $50,200$  kips. (b)  $3580$  ft.
- 7.113** (a)  $\sqrt{3L\Delta/8}$ . (b)  $12.25$  ft.
- 7.114**  $3.75$  ft.
- 7.115** (a)  $58940$  kips. (b)  $29.2^\circ$ .
- 7.116** (a)  $16.00$  ft to the left of B. (b)  $2000$  lb.
- 7.117** (a)  $5880$  N. (b)  $0.873$  m.
- 7.118** (a)  $6860$  N. (b)  $31.0^\circ$ .
- 7.125**  $y = h[1 - \cos(\pi x/L)]$ ;  $T_0 = w_0L^2/h\pi^2$ ;  
 $T_{\max} = (w_0L/\pi)\sqrt{(L^2/h^2\pi^2) + 1}$ .
- 7.127** (a)  $26.7$  m. (b)  $70.3$  kg.
- 7.128**  $199.5$  ft
- 7.129** (a)  $164.8$  m. (b)  $4290$  N.
- 7.130**  $330$  ft;  $625$  lb.
- 7.133** (a)  $5.89$  m. (b)  $10.89$  N →.
- 7.134**  $10.05$  ft.
- 7.135** (a)  $30.2$  m. (b)  $56.6$  kg.
- 7.136** (a)  $56.3$  ft. (b)  $2.36$  lb/ft.
- 7.139**  $31.8$  N.
- 7.140**  $29.8$  N.
- 7.141** (a)  $a = 79.0$  ft;  $b = 60.0$  ft. (b)  $103.9$  ft.
- 7.142** (a)  $a = 65.8$  ft;  $b = 50.0$  ft. (b)  $= 86.6$  ft.
- 7.143**  $119.1$  N →.
- 7.144**  $177.6$  N →.
- 7.147**  $3.50$  ft.
- 7.148**  $5.71$  ft.
- 7.151**  $0.394$  m and  $10.97$  m.
- 7.152**  $0.1408$ .
- 7.153** (a)  $0.338$ . (b)  $56.5^\circ$ ;  $0.755 wL$ .
- 7.154** (a)  $1500$  N. (b) (On ABJ)  $\mathbf{F} = 1324$  N ↑;  
 $\mathbf{V} = 706$  N ←;  $\mathbf{M} = 229$  N · m ↗.
- 7.155** (On BJ)  $\mathbf{F} = 250$  N ↘;  $\mathbf{V} = 120.0$  N ↗;  $\mathbf{M} = 120.0$  N · m ↗.
- 7.156** (a) (On AC)  $\mathbf{F} = \mathbf{V} = 0$ ;  $\mathbf{M} = 450$  lb · ft ↗.  
(b) (On AC)  $\mathbf{F} = 250$  lb ↗;  $\mathbf{V} = 0$ ;  $\mathbf{M} = 450$  lb · ft ↗.
- 7.157** (a)  $90.0$  lb. (b)  $900$  lb · in.
- 7.159** (b)  $41.4$  kN;  $35.3$  kN · m.
- 7.160** (a)  $12.00$  kip · ft, at C. (b)  $6.25$  kip · ft,  $2.50$  ft from A.
- 7.162** (a)  $1229$  lb. (b)  $11.00$  ft.
- 7.164** (a)  $2770$  N. (b)  $75.14$  m.
- 7.C1** (a)  $M_D = +39.8$  kN · m. (b)  $M_D = +14.00$  kip · ft.  
(c)  $M_D = +1800$  lb · in.
- 7.C3**  $a = 1.923$  m;  $M_{\max} = 37.0$  kN · m at  $4.64$  m from A.
- 7.C4** (b)  $M_{\max} = 5.42$  kip · ft when  $x = 8.5$  ft and  $11.5$  ft.
- 7.C8**  $c/L = 0.300$ ;  $h/L = 0.5225$ ;  $s_{AB}/L = 1.532$ ;  
 $T_0/wL = 0.300$ ;  $T_{\max}/wL = 0.823$ .

## CHAPTER 8

- 8.1** Equilibrium;  $\mathbf{F} = 34.5$  lb ↘.
- 8.2** Block moves;  $\mathbf{F} = 55.7$  lb ↗.
- 8.3** Equilibrium;  $\mathbf{F} = 48.3$  N ↗.
- 8.4** Block moves;  $\mathbf{F} = 103.5$  N ↗.
- 8.5**  $225$  N ≤  $P$  ≤  $479$  N.
- 8.6**  $143.0$  N ≤  $P$  ≤  $483$  N.
- 8.7** (a)  $105.8$  N. (b)  $46.0^\circ$ .
- 8.9** (a)  $403$  N. (b)  $229$  N.
- 8.11** (a)  $353$  N ←. (b)  $196.2$  N ←.
- 8.12** (a)  $275$  N ←. (b)  $196.2$  N ←.
- 8.15** (a)  $36.0$  lb →. (b)  $30.0$  lb. (c)  $12.86$  lb →.
- 8.16** (a)  $36.0$  lb →. (b)  $40.0$  in.
- 8.17**  $M = Wr\mu_s(1 + \mu_s)/(1 + \mu_s^2)$ .
- 8.18** (a)  $0.300$  Wr. (b)  $0.349$  Wr.
- 8.19**  $151.5$  N · m.
- 8.20**  $1.473$  kN.
- 8.21**  $0.208$ .
- 8.23** (a)  $136.4^\circ$ . (b)  $0.928$  W.
- 8.25**  $0.750$ .
- 8.26**  $0.860$ .
- 8.27**  $132.9$  lb ↓.
- 8.28** (a)  $112.5$  N. (b)  $8.81$  mm.
- 8.29**  $3.46 \leq \frac{L}{a} \leq 13.63$ .
- 8.30** (a) Plate in equilibrium. (b) Plate moves downward.
- 8.31**  $10.00$  lb <  $P$  <  $36.7$  lb.
- 8.34**  $135.0$  lb.
- 8.36**  $168.4$  N ≤  $P$  ≤  $308$  N.
- 8.37**  $9.38$  N · m ≤  $M$  ≤  $15.01$  N · m.

- 8.38**  $-46.8 \text{ N} \leq P \leq 34.3 \text{ N}$ .
- 8.39** (b) 2.69 lb.
- 8.41** 0.0949.
- 8.42** (a) System slides;  $P = 62.8 \text{ N}$ .  
(b) System rotates about  $B$ ;  $P = 73.2 \text{ N}$ .
- 8.43**  $35.8^\circ$ .
- 8.44**  $20.5^\circ$ .
- 8.45** 1.225 W.
- 8.46** (a)  $\mathbf{P} = 56.6 \text{ lb} \leftarrow$ . (b)  $\mathbf{B}_x = 82.6 \text{ lb} \leftarrow$ ;  $\mathbf{B}_y = 96.0 \text{ lb} \downarrow$ .
- 8.47** (a)  $\mathbf{P} = 21.4 \text{ lb} \rightarrow$ . (b)  $\mathbf{B}_x = 122.2 \text{ lb} \leftarrow$ ;  $\mathbf{B}_y = 96.0 \text{ lb} \downarrow$ .
- 8.48**  $\mathbf{P} = 2080 \text{ N} \downarrow$ .
- 8.49**  $\mathbf{P} = 1966 \text{ N} \downarrow$ .
- 8.52** (a) 62.7 lb. (b) 62.7 lb.
- 8.53** 9.86 kN  $\leftarrow$ .
- 8.54** 913 N  $\leftarrow$ .
- 8.55** (a)  $28.1^\circ$ . (b)  $728 \text{ N} \not\propto 14.0^\circ$ .
- 8.56** 29.9 lb.
- 8.57** 67.4 N.
- 8.60** (b) 283 N  $\leftarrow$ .
- 8.61** 0.442.
- 8.62** (a) 90.0 lb. (b) Base moves.
- 8.63** (a) 89.4 lb. (b) Base does not move.
- 8.64** 0.1103.
- 8.65** 0.1013.
- 8.69**  $1068 \text{ N} \cdot \text{m}$ .
- 8.70**  $4.18 \text{ N} \cdot \text{m}$ .
- 8.72**  $169.7 \text{ lb} \cdot \text{in}$ .
- 8.73**  $32.7 \text{ lb} \cdot \text{in}$ .
- 8.75** 0.098.
- 8.76** 450 N.
- 8.77** 412 N.
- 8.78** 344 N.
- 8.79** 376 N.
- 8.80** 0.226.
- 8.82**  $T_{AB} = 77.5 \text{ lb}$ ;  $T_{CD} = 72.5 \text{ lb}$ ;  $T_{EF} = 67.8 \text{ lb}$ .
- 8.85**  $22.0 \text{ lb} \leftarrow$ .
- 8.86**  $1.948 \text{ lb} \downarrow$ .
- 8.87**  $18.01 \text{ lb} \leftarrow$ .
- 8.88** (a) 4.80 kN. (b)  $1.375^\circ$ .
- 8.90** 3.75 lb.
- 8.91** 0.1670.
- 8.96** 154.4 N.
- 8.97** 0.0600 in.
- 8.98** 10.87 lb
- 8.99** (a) 1.288 kN. (b) 1.058 kN.
- 8.100** 300 mm.
- 8.101** (a) 0.329. (b) 2.67 turns.
- 8.102** (a) 22.8 kg. (b) 291 N.
- 8.103** (a) 109.7 kg. (b) 828 N.
- 8.104**  $73.0 \text{ lb} \leq P \leq 1233 \text{ lb}$ .
- 8.107**  $35.1 \text{ N} \cdot \text{m}$
- 8.108** (a)  $27.0 \text{ N} \cdot \text{m}$  (b) 675 N.
- 8.109** (a) 39.0 N  $\cdot$  m. (b) 844 N.
- 8.110** 421 lb  $\cdot$  in.
- 8.111** 301 lb  $\cdot$  in.
- 8.112** 44.9 N  $\cdot$  m  $\uparrow$ .
- 8.115** 4.49 in.
- 8.116** (a) 11.66 kg. (b) 38.6 kg. (c) 34.4 kg.
- 8.117** (a) 9.46 kg. (b) 167.2 kg. (c) 121.0 kg.
- 8.120** (a) 10.39 lb. (b) 58.5 lb.
- 8.121** (a) and (b) 28.9 lb.
- 8.122** 5.97 N.
- 8.123** 9.56 N.
- 8.124** (a)  $30.3 \text{ lb} \cdot \text{in} \uparrow$ . (b)  $3.78 \text{ lb} \downarrow$ .
- 8.125** (a)  $17.23 \text{ lb} \cdot \text{in} \downarrow$ . (b)  $2.15 \text{ lb} \uparrow$ .
- 8.126** 0.350.
- 8.131** (a)  $51.0 \text{ N} \cdot \text{m}$ . (b) 875 N.
- 8.132** (a) 170.5 N. (b)  $14.04^\circ$ .
- 8.133**  $53.5^\circ$
- 8.135**  $6.35 \leq L/a \leq 10.81$ .
- 8.136** 0.0533.
- 8.138** 0.225.
- 8.139** (a)  $620 \text{ N} \leftarrow$ . (b)  $\mathbf{B}_x = 1390 \text{ N} \leftarrow$ ;  $\mathbf{B}_y = 1050 \text{ N} \downarrow$ .
- 8.140** (a) and (b) 50.4 lb.
- 8.142** (a) 0.238. (b)  $218 \text{ N} \downarrow$ .
- 8.C1**  $x = 500 \text{ mm}$ ;  $63.3 \text{ N}$ ;  $P_{\max} = 67.8 \text{ N}$  at  $x = 355 \text{ mm}$ .
- 8.C2**  $W_B = 10 \text{ lb}$ ;  $\theta = 46.4^\circ$ ;  $W_B = 70 \text{ lb}$ ;  $\theta = 21.3^\circ$ .
- 8.C3**  $\mu_A = 0.25$ ;  $M = 0.0603 \text{ N} \cdot \text{m}$ .
- 8.C4**  $\theta = 30^\circ$ ;  $1.336 \text{ N} \cdot \text{m} \leq M_A \leq 2.23 \text{ N} \cdot \text{m}$ .
- 8.C5**  $\theta = 60^\circ$ ;  $\mathbf{P} = 16.40 \text{ lb} \downarrow$ ;  $R = 5.14 \text{ lb}$ .
- 8.C6**  $\theta = 20^\circ$ ;  $10.39 \text{ N} \cdot \text{m}$ .
- 8.C7**  $\theta = 20^\circ$ ;  $30.3 \text{ lb}$ ;  $13.25 \text{ lb}$ .
- 8.C8** (a)  $x_0 = 0.600L$ ;  $x_m = 0.604L$ ;  $\theta_1 = 5.06^\circ$ . (b)  $\theta_2 = 55.4^\circ$ .

## CHAPTER 9

- 9.1**  $b^3h/12$ .
- 9.2**  $3a^4/2$ .
- 9.3**  $2a^3b/15$ .
- 9.4**  $ha^3/5$
- 9.6**  $a^4/8$ .
- 9.7**  $2ab^3/7$ .
- 9.9**  $ab^3/15$ .
- 9.10**  $0.1056 ab^3$ .
- 9.11**  $ab^3/15$ .
- 9.12**  $2a^3b/21$ .
- 9.15**  $ab^3/10$ ;  $b/\sqrt{5}$ .
- 9.16**  $3ab^3/35$ ;  $b\sqrt{9/35}$ .
- 9.17**  $a^3b/6$ ;  $a/\sqrt{3}$ .
- 9.18**  $3a^3b/35$ ;  $b\sqrt{9/35}$ .
- 9.21**  $20a^4$ ;  $1.826a$ .
- 9.22**  $43a^4/48$ ;  $0.773a$
- 9.23**  $(\pi/2)(R_2^4 - R_1^4)$ ;  $(\pi/4)(R_2^4 - R_1^4)$
- 9.24** (b) for  $t/R_m = 1$ ,  $-10.56\%$ ; for  $t/R_m = \frac{1}{2}$ ,  $-2.99\%$ ,  
for  $t/R_m = \frac{1}{10}$ ,  $-0.1250\%$ .
- 9.25**  $64a^4/15$ ;  $1.265a$ .
- 9.28**  $bh(12h^2 + b^2)/48$ ;  $\sqrt{(12h^2 + b^2)/24}$
- 9.31**  $390 \times 10^3 \text{ mm}^4$ ;  $21.9 \text{ mm}$ .
- 9.32**  $46.0 \text{ in}^4$ ;  $1.599 \text{ in}$ .
- 9.33**  $64.3 \times 10^3 \text{ mm}^4$ ;  $8.87 \text{ mm}$ .
- 9.34**  $46.5 \text{ in}^4$ ;  $1.607 \text{ in}$ .
- 9.37**  $I = 9.50 \cdot 10^6 \text{ mm}^4$ ;  $d_2 = 60.0 \text{ mm}$ .
- 9.38**  $A = 6600 \text{ mm}^2$ ;  $\bar{I} = 3.72 \cdot 10^6 \text{ mm}^4$ .
- 9.39**  $\bar{I}_x = 150.0 \text{ in}^4$ ;  $\bar{I}_y = 300 \text{ in}^4$ .
- 9.41**  $\bar{I}_x = 1.874 \cdot 10^6 \text{ mm}^4$ ;  $\bar{I}_y = 5.82 \cdot 10^6 \text{ mm}^4$ .
- 9.42**  $\bar{I}_x = 48.9 \cdot 10^3 \text{ mm}^4$ ;  $\bar{I}_y = 8.35 \cdot 10^3 \text{ mm}^4$ .
- 9.43**  $\bar{I}_x = 191.3 \text{ in}^4$ ;  $\bar{I}_y = 75.2 \text{ in}^4$ .
- 9.44**  $\bar{I}_x = 18.13 \text{ in}^4$ ;  $\bar{I}_y = 4.51 \text{ in}^4$ .
- 9.45** (a)  $80.9 \cdot 10^6 \text{ mm}^4$ . (b)  $57.4 \cdot 10^6 \text{ mm}^4$ .
- 9.46** (a)  $12.16 \cdot 10^6 \text{ mm}^4$ . (b)  $9.73 \cdot 10^6 \text{ mm}^4$ .
- 9.49**  $\bar{I}_x = 260 \cdot 10^6 \text{ mm}^4$ ;  $\bar{k}_x = 144.6 \text{ mm}$ ;  $\bar{I}_y = 17.53 \cdot 10^6 \text{ mm}^4$ ;  
 $\bar{k}_y = 37.6 \text{ mm}$ .
- 9.50**  $\bar{I}_x = 254 \text{ in}^4$ ;  $\bar{k}_x = 4.00 \text{ in}$ ;  $\bar{I}_y = 102.1 \text{ in}^4$ ;  $\bar{k}_y = 2.54 \text{ in}$ .

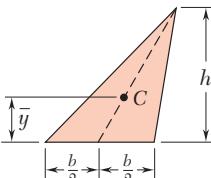
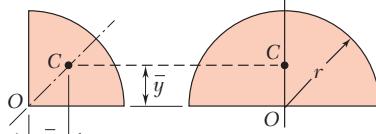
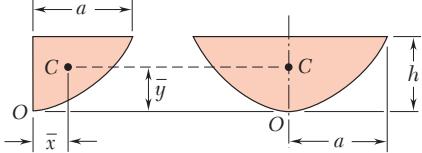
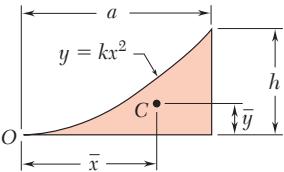
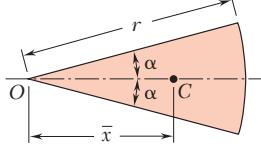
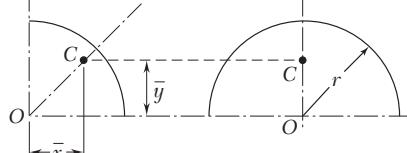
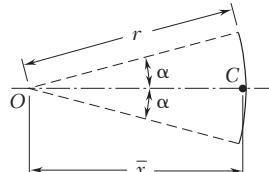
- 9.51**  $\bar{I}_x = 255 \cdot 10^6 \text{ mm}^4$ ;  $\bar{k}_x = 134.1 \text{ mm}$ ;  $\bar{I}_y = 100.0 \cdot 10^6 \text{ mm}^4$ ;  $\bar{k}_y = 83.9 \text{ mm}$ .
- 9.52** 1.077 in.
- 9.53**  $\bar{I}_x = 3.55 \cdot 10^6 \text{ mm}^4$ ;  $\bar{I}_y = 49.8 \cdot 10^6 \text{ mm}^4$ .
- 9.55**  $\bar{I}_x = 745 \cdot 10^6 \text{ mm}^4$ ;  $\bar{I}_y = 91.3 \cdot 10^6 \text{ mm}^4$ .
- 9.57**  $3\pi r/16$ .
- 9.58**  $3\pi b/16$ .
- 9.59**  $15h/14$ .
- 9.60**  $4h/7$ .
- 9.63**  $5a/8$ .
- 9.64** 80.0 mm.
- 9.67**  $a^4/2$ .
- 9.68**  $a^2b^2/12$ .
- 9.69**  $-b^2h^2/8$ .
- 9.71**  $-1.760 \cdot 10^6 \text{ mm}^4$ .
- 9.72**  $-21.6 \cdot 10^6 \text{ mm}^4$ .
- 9.74**  $-0.380 \text{ in}^4$ .
- 9.75**  $471 \cdot 10^3 \text{ mm}^4$ .
- 9.76**  $-9010 \text{ in}^4$ .
- 9.78**  $1.165 \cdot 10^6 \text{ mm}^4$ .
- 9.79** (a)  $\bar{I}_{x'} = 0.482a^4$ ;  $\bar{I}_{y'} = 1.482a^4$ ;  $\bar{I}_{x'y'} = -0.589a^4$ .  
(b)  $\bar{I}_{x'} = 1.120a^4$ ;  $\bar{I}_{y'} = 0.843a^4$ ;  $\bar{I}_{x'y'} = 0.760a^4$ .
- 9.80**  $\bar{I}_{x'} = 103.5 \cdot 10^6 \text{ mm}^4$ ;  $\bar{I}_{y'} = 97.9 \cdot 10^6 \text{ mm}^4$ ;  $\bar{I}_{x'y'} = -38.3 \cdot 10^6 \text{ mm}^4$ .
- 9.81**  $\bar{I}_{x'} = 1033 \text{ in}^4$ ;  $\bar{I}_{y'} = 2020 \text{ in}^4$ ;  $\bar{I}_{x'y'} = -873 \text{ in}^4$ .
- 9.83**  $\bar{I}_{x'} = 0.236 \text{ in}^4$ ;  $\bar{I}_{y'} = 1.244 \text{ in}^4$ ;  $\bar{I}_{x'y'} = 0.1132 \text{ in}^4$ .
- 9.85**  $20.2^\circ$ ;  $1.754a^4$ ,  $0.209a^4$ .
- 9.86**  $-17.11^\circ$ ;  $139.1 \cdot 10^6 \text{ mm}^4$ ,  $62.3 \cdot 10^6 \text{ mm}^4$ .
- 9.87**  $29.7^\circ$ ;  $2530 \text{ in}^4$ ,  $524 \text{ in}^4$ .
- 9.89**  $-23.7^\circ$  and  $66.3^\circ$ ;  $1.257 \text{ in}^4$ ,  $0.224 \text{ in}^4$ .
- 9.91** (a)  $\bar{I}_{x'} = 0.482a^4$ ;  $\bar{I}_{y'} = 1.482a^4$ ;  $\bar{I}_{x'y'} = -0.589a^4$ .  
(b)  $\bar{I}_{x'} = 1.120a^4$ ;  $\bar{I}_{y'} = 0.843a^4$ ;  $\bar{I}_{x'y'} = 0.760a^4$ .
- 9.92**  $\bar{I}_{x'} = 103.5 \cdot 10^6 \text{ mm}^4$ ;  $\bar{I}_{y'} = 97.9 \cdot 10^6 \text{ mm}^4$ ;  $\bar{I}_{x'y'} = -38.3 \cdot 10^6 \text{ mm}^4$ .
- 9.93**  $\bar{I}_{x'} = 1033 \text{ in}^4$ ;  $\bar{I}_{y'} = 2020 \text{ in}^4$ ;  $\bar{I}_{x'y'} = -873 \text{ in}^4$ .
- 9.95**  $\bar{I}_{x'} = 0.236 \text{ in}^4$ ;  $\bar{I}_{y'} = 1.244 \text{ in}^4$ ;  $\bar{I}_{x'y'} = 0.1132 \text{ in}^4$ .
- 9.97**  $20.2^\circ$ ;  $1.754a^4$ ,  $0.209a^4$ .
- 9.98**  $-17.11^\circ$ ;  $139.1 \cdot 10^6 \text{ mm}^4$ ,  $62.3 \cdot 10^6 \text{ mm}^4$ .
- 9.99**  $-33.4^\circ$ ;  $22.1 \cdot 10^3 \text{ in}^4$ ,  $2490 \text{ in}^4$ .
- 9.100**  $29.7^\circ$ ;  $2530 \text{ in}^4$ ,  $524 \text{ in}^4$ .
- 9.103** (a)  $-1.146 \text{ in}^4$ . (b)  $29.1^\circ$  clockwise. (c)  $3.39 \text{ in}^4$ .
- 9.104**  $23.8^\circ$  clockwise;  $0.524 \cdot 10^6 \text{ mm}^4$ ,  $0.0917 \cdot 10^6 \text{ mm}^4$ .
- 9.105**  $19.54^\circ$  counterclockwise;  $4.34 \cdot 10^6 \text{ mm}^4$ ,  $0.647 \cdot 10^6 \text{ mm}^4$ .
- 9.106** (a)  $25.3^\circ$ . (b)  $1459 \text{ in}^4$ ,  $40.5 \text{ in}^4$ .
- 9.107** (a)  $88.0 \cdot 10^6 \text{ mm}^4$ . (b)  $96.3 \cdot 10^6 \text{ mm}^4$ ,  $39.7 \cdot 10^6 \text{ mm}^4$ .
- 9.111** (a)  $m(r_1^2 + r_2^2)/4$ . (b)  $m(r_1^2 + r_2^2)/2$ .
- 9.112** (a)  $0.0699 ma^2$ . (b)  $0.320 ma^2$ .
- 9.113** (a)  $25 mr_2^2/64$ . (b)  $0.1522 mr_2^2$ .
- 9.114** (a)  $mb^2/7$ . (b)  $m(7a^2 + 10b^2)/70$ .
- 9.115** (a)  $ma^2/3$ . (b)  $3ma^2/2$ .
- 9.116** (a)  $7ma^2/6$ . (b)  $ma^2/2$ .
- 9.119**  $1.329 \text{ mh}^2$ .
- 9.120**  $m(3a^2 + L^2)/12$ .
- 9.121** (a)  $0.241 mh^2$ . (b)  $m(3a^2 + 0.1204 h^2)$ .
- 9.122**  $m(b^2 + h^2)/10$ .
- 9.124**  $m(a^2 + b^2)/5$ .
- 9.125**  $I_x = I_y = ma^2/4$ ;  $I_z = ma^2/2$ .
- 9.127**  $837 \times 10^{-9} \text{ kg} \cdot \text{m}^2$ ;  $6.92 \text{ mm}$ .
- 9.128**  $1.160 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $0.341 \text{ in}$ .
- 9.129**  $m(3a^2 + 2h^2)/6$ .
- 9.131** (a) 27.5 mm to the right of A. (b) 32.0 mm.
- 9.133** (a) 2.30 in. (b)  $20.6 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $2.27 \text{ in}$ .
- 9.134** (a)  $\pi\rho l^2 \left[ 6a^2 t \left( \frac{5a^2}{3l^2} + \frac{2a}{l} + 1 \right) + \frac{d^2 l^2}{4} \right]$ . (b) 0.1851.
- 9.135**  $I_x = 26 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_y = 38.2 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_z = 17.55 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.136**  $I_x = 175.5 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_y = 309 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_z = 154.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.137**  $I_x = 745 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_y = 896 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_z = 304 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .
- 9.138**  $I_x = 344 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_y = 132.1 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_z = 453 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .
- 9.141** (a)  $13.99 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ . (b)  $20.6 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .  
(c)  $14.30 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.142**  $0.1785 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .
- 9.144**  $I_x = 38.1 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $k_x = 110.7 \text{ mm}$ .
- 9.145** (a)  $26.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ . (b)  $31.2 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .  
(c)  $8.58 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.147**  $I_x = 0.0392 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_y = 0.0363 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_z = 0.0304 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .
- 9.148**  $I_x = 0.323 \text{ kg} \cdot \text{m}^2$ ;  $I_y = I_z = 0.419 \text{ kg} \cdot \text{m}^2$ .
- 9.149**  $I_{xy} = 2.50 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_{yz} = 4.06 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_{zx} = 8.81 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.150**  $I_{xy} = 2.44 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_{yz} = 1.415 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_{zx} = 4.59 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.151**  $I_{xy} = -538 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_{yz} = -171.4 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_{zx} = 1120 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .
- 9.152**  $I_{xy} = -1.726 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_{yz} = 0.507 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ;  $I_{zx} = -2.12 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .
- 9.155**  $I_{xy} = -8.04 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_{yz} = 12.90 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $I_{zx} = 94.0 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.156**  $I_{xy} = 0$ ;  $I_{yz} = 48.3 \times 10^{-6} \text{ kg} \cdot \text{m}^2$ ;  $I_{zx} = -4.43 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.157**  $I_{xy} = wa^3(1 - 5\pi)/g$ ;  $I_{yz} = -11\pi wa^3/g$ ;  $I_{zx} = 4wa^3(1 + 2\pi)g$ .
- 9.158**  $I_{xy} = -11wa^3/g$ ;  $I_{yz} = wa^3(\pi + 6)/2g$ ;  $I_{zx} = -wa^3/4g$ .
- 9.159**  $I_{xy} = 47.9 \times 10^{-6} \text{ kg} \cdot \text{m}^2$ ;  $I_{yz} = 102.1 \times 10^{-6} \text{ kg} \cdot \text{m}^2$ ;  $I_{zx} = 64.1 \times 10^{-6} \text{ kg} \cdot \text{m}^2$ .
- 9.160**  $I_{xy} = -m'R_1^3/2$ ;  $I_{yz} = m'R_1^3/2$ ;  $I_{zx} = -m'R_2^3/2$ .
- 9.162** (a)  $mac/20$ . (b)  $I_{xy} = mab/20$ ;  $I_{yz} = mbc/20$ .
- 9.165**  $18.17 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.166**  $11.81 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.167**  $5 Wa^2/18g$ .
- 9.168**  $4.41\gamma ta^4/g$ .
- 9.169**  $281 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.170**  $0.354 \text{ kg} \cdot \text{m}^2$ .
- 9.173** (a)  $b/a = 2$ ;  $c/a = 2$ . (b)  $b/a = 1$ ;  $c/a = 0.5$ .
- 9.174** (a) 2. (b)  $\sqrt{2/3}$ .
- 9.175** (a)  $1/\sqrt{3}$ . (b)  $\sqrt{7/12}$ .
- 9.179** (a)  $K_1 = 0.363ma^2$ ;  $K_2 = 1.583ma^2$ ;  $K_3 = 1.720ma^2$ .  
(b)  $(\theta_x)_1 = (\theta_z)_1 = 49.7^\circ$ ;  $(\theta_y)_1 = 113.7^\circ$ .  
 $(\theta_x)_2 = 45^\circ$ ;  $(\theta_y)_2 = 90^\circ$ ;  $(\theta_z)_2 = 135^\circ$ .  
 $(\theta_x)_3 = (\theta_z)_3 = 73.5^\circ$ ;  $(\theta_y)_3 = 23.7^\circ$ .
- 9.180** (a)  $K_1 = 14.30 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $K_2 = 13.96 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ;  $K_3 = 20.6 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .  
(b)  $(\theta_x)_1 = (\theta_y)_1 = 90.0^\circ$ ;  $(\theta_z)_1 = 0^\circ$ .  
 $(\theta_x)_2 = 3.42^\circ$ ;  $(\theta_y)_2 = 86.6^\circ$ ;  $(\theta_z)_2 = 90.0^\circ$ .  
 $(\theta_x)_3 = 93.4^\circ$ ;  $(\theta_y)_3 = 3.4^\circ$ ;  $(\theta_z)_3 = 90.0^\circ$ .
- 9.182** (a)  $K_1 = 0.1639Wa^2/g$ ;  $K_2 = 1.054Wa^2/g$ ;  $K_3 = 1.115Wa^2/g$ .  
(b)  $(\theta_x)_1 = 36.7^\circ$ ;  $(\theta_y)_1 = 71.6^\circ$ ;  $(\theta_z)_1 = 59.5^\circ$ .  
 $(\theta_x)_2 = 74.9^\circ$ ;  $(\theta_y)_2 = 54.5^\circ$ ;  $(\theta_z)_2 = 140.5^\circ$ .  
 $(\theta_x)_3 = 57.5^\circ$ ;  $(\theta_y)_3 = 138.8^\circ$ ;  $(\theta_z)_3 = 112.4^\circ$ .

- 9.183** (a)  $K_1 = 2.26\gamma ta^4/g$ ;  $K_2 = 17.27\gamma ta^4/g$ ;  $K_3 = 19.08\gamma ta^4/g$ .  
 (b)  $(\theta_x)_1 = 85.0^\circ$ ,  $(\theta_y)_1 = 36.8^\circ$ ,  $(\theta_z)_1 = 53.7^\circ$ ;  
 $(\theta_x)_2 = 81.7^\circ$ ,  $(\theta_y)_2 = 54.7^\circ$ ,  $(\theta_z)_2 = 143.4^\circ$ ;  
 $(\theta_x)_3 = 9.70^\circ$ ,  $(\theta_y)_3 = 99.0^\circ$ ,  $(\theta_z)_3 = 86.3^\circ$ .
- 9.185**  $I_x = ab^3/28$ ;  $I_y = a^3b/20$ .
- 9.187**  $4a^3b/15$ ;  $a/\sqrt{5}$ .
- 9.188**  $I_x = 4a^4$ ;  $I_y = 16a^4/3$ .
- 9.189** (a)  $3.13 \times 10^6 \text{ mm}^4$ . (b)  $2.41 \times 10^6 \text{ mm}^4$ .
- 9.190**  $I_x = 634 \times 10^6 \text{ mm}^4$ ;  $I_y = 38.0 \times 10^6 \text{ mm}^4$ .
- 9.191**  $I_{xy} = -2.81 \text{ in}^4$ .
- 9.193** (a)  $7 \text{ ma}^2/18$ . (b)  $0.819 \text{ ma}^2$ .
- 9.195**  $I_x = 0.877 \text{ kg} \cdot \text{m}^2$ ;  $I_y = 1.982 \text{ kg} \cdot \text{m}^2$ ;  $I_z = 1.652 \text{ kg} \cdot \text{m}^2$ .
- 9.C1**  $\theta = 20^\circ$ :  $I_x' = 14.20 \text{ in}^4$ ,  $I_y' = 3.15 \text{ in}^4$ ,  $I_{x'y'} = -3.93 \text{ in}^4$ .
- 9.C3** (a)  $\bar{I}_{x'} = 371 \times 10^3 \text{ mm}^4$ ,  $\bar{I}_{y'} = 64.3 \times 10^3 \text{ mm}^4$ ;  
 $\bar{k}_{x'} = 21.3 \text{ mm}$ ,  $\bar{k}_{y'} = 8.87 \text{ mm}$ . (b)  $\bar{I}_{x'} = 40.4 \text{ in}^4$ ,  
 $\bar{I}_{y'} = 46.5 \text{ in}^4$ ;  $\bar{k}_{x'} = 1.499 \text{ in}$ ,  $\bar{k}_{y'} = 1.607 \text{ in}$ .  
(c)  $\bar{k}_x = 2.53 \text{ in}$ ,  $\bar{k}_y = 1.583 \text{ in}$ . (d)  $\bar{k}_x = 1.904 \text{ in}$ ,  
 $\bar{k}_y = 0.950 \text{ in}$ .
- 9.C5** (a)  $5.99 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ . (b)  $77.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ .
- 9.C6** (a)  $74.0 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ . (b)  $645 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .  
(c)  $208 \times 10^{-6} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ .

## CHAPTER 10

- 10.1**  $82.5 \text{ N} \downarrow$ .
- 10.2**  $120 \text{ lb} \rightarrow$ .
- 10.3**  $49.5 \text{ N} \cdot \text{m} \downarrow$ .
- 10.4**  $1200 \text{ lb} \cdot \text{in.} \uparrow$ .
- 10.7** (a)  $60.0 \text{ N C}$ ,  $8.00 \text{ mm} \downarrow$ . (b)  $300 \text{ N C}$ ,  $40.0 \text{ mm} \downarrow$ .
- 10.8** (a)  $120.0 \text{ N C}$ ,  $16.00 \text{ mm} \downarrow$ . (b)  $300 \text{ N C}$ ,  $40.0 \text{ mm} \downarrow$ .
- 10.9**  $Q = 2P \sin \theta / \cos \theta/2$ .
- 10.10**  $Q = 2P \cos \theta / \cos \theta/2$ .
- 10.11**  $Q = (3P/2) \tan \theta$
- 10.12**  $Q = P[(l/a)\cos^3 \theta - 1]$ .
- 10.15**  $M = \frac{1}{2}Wl \tan \alpha \sin \theta$ .
- 10.16**  $M = Pl/2 \tan \theta$ .
- 10.17**  $M = 7Pa \cos \theta$ .
- 10.18** (a)  $M = Pl \sin 2\theta$ . (b)  $M = 3Pl \cos \theta$ . (c)  $M = Pl \sin \theta$ .
- 10.21**  $85.2 \text{ lb} \cdot \text{ft} \downarrow$ .
- 10.22**  $22.8 \text{ lb} \nearrow 70.0^\circ$ .
- 10.24**  $36.4^\circ$ .
- 10.25**  $38.7^\circ$ .
- 10.26**  $68.0^\circ$ .
- 10.28**  $19.81^\circ$  and  $51.9^\circ$ .
- 10.30**  $25.0^\circ$ .
- 10.31**  $39.7^\circ$  and  $69.0^\circ$ .
- 10.32**  $52.2^\circ$ .
- 10.33**  $40.2^\circ$ .
- 10.35**  $22.6^\circ$ .
- 10.36**  $51.1^\circ$ .
- 10.37**  $52.4^\circ$ .
- 10.38**  $19.40^\circ$ .
- 10.39**  $59.0^\circ$ .
- 10.40**  $78.7^\circ$ ,  $324^\circ$ ,  $379^\circ$ .
- 10.43**  $12.03 \text{ kN} \searrow$ .
- 10.44**  $20.4^\circ$ .
- 10.45**  $2370 \text{ lb} \nwarrow$ .
- 10.46**  $2550 \text{ lb} \nwarrow$ .
- 10.47**  $\eta = 1/(1 + \mu \cot \alpha)$
- 10.49**  $37.6 \text{ N}$ ,  $31.6 \text{ N}$ .
- 10.51**  $300 \text{ N} \cdot \text{m}$ ,  $81.8 \text{ N} \cdot \text{m}$ .
- 10.52**  $\eta = \tan \theta / \tan(\theta + \phi_s)$ .
- 10.53**  $7.75 \text{ kN} \uparrow$ .
- 10.54**  $H = 1.361 \text{ kN} \uparrow$ ;  $M_H = 550 \text{ N} \cdot \text{m} \uparrow$ .
- 10.57**  $0.833 \text{ in.} \downarrow$ .
- 10.58**  $0.625 \text{ in.} \rightarrow$ .
- 10.66**  $19.40^\circ$ .
- 10.67** Equilibrium is neutral.
- 10.69**  $\theta = 0$  and  $\theta = 180.0^\circ$ , unstable;  
 $\theta = 75.5^\circ$  and  $\theta = 284^\circ$ , stable.
- 10.70**  $\theta = 90.0^\circ$  and  $\theta = 270^\circ$  unstable;  
 $\theta = 22.0^\circ$  and  $\theta = 158.0^\circ$ , stable.
- 10.71**  $\theta = -45.0^\circ$ , unstable;  $\theta = 135.0^\circ$ , stable.
- 10.72**  $\theta = -63.4^\circ$ , unstable;  $\theta = 116.6^\circ$ , stable.
- 10.73**  $59.0^\circ$  stable.
- 10.74**  $78.7^\circ$ , stable;  $324^\circ$ , unstable;  $379^\circ$ , stable.
- 10.78**  $9.39^\circ$  and  $90.0^\circ$ , stable;  $34.2^\circ$ , unstable.
- 10.79**  $357 \text{ mm}$ .
- 10.80**  $252 \text{ mm}$ .
- 10.81**  $17.11^\circ$ , stable;  $72.9^\circ$ , unstable.
- 10.83**  $49.1^\circ$ .
- 10.85**  $54.8^\circ$ .
- 10.86**  $37.4^\circ$ .
- 10.88**  $16.88 \text{ m}$ .
- 10.90**  $k > 6.94 \text{ lb/in.}$
- 10.91**  $15.00 \text{ in.}$
- 10.92**  $P < 2kL/9$ .
- 10.93**  $P < kL/18$ .
- 10.94**  $P < k(l - a)^2/2l$ .
- 10.96**  $P < 160.0 \text{ N}$ .
- 10.98**  $P < 764 \text{ N}$ .
- 10.100** (a)  $P < 10.00 \text{ lb}$ . (b)  $P < 20.0 \text{ lb}$ .
- 10.101**  $60.0 \text{ lb} \downarrow$ .
- 10.102**  $600 \text{ lb} \cdot \text{in.} \downarrow$ .
- 10.104** (a)  $20.0 \text{ N}$ . (b)  $105.0 \text{ N}$ .
- 10.106**  $39.2^\circ$ .
- 10.107**  $60.4^\circ$ .
- 10.108**  $7.13 \text{ in.}$
- 10.110** (a)  $0$ , unstable. (b)  $137.8^\circ$ , stable.
- 10.112** (a)  $22.0^\circ$ . (b)  $30.6^\circ$ .
- 10.C1**  $\theta = 60^\circ$ :  $2.42 \text{ in.}$ ;  $\theta = 120^\circ$ :  $1.732 \text{ in.}$ ;  
 $(M/P)_{\max} = 2.52 \text{ in. at } \theta = 73.7^\circ$ .
- 10.C2**  $\theta = 60^\circ$ :  $171.1 \text{ N C}$ . For  $32.5^\circ \leq \theta \leq 134.3^\circ$ ,  $|F| \leq 400 \text{ N}$ .
- 10.C3**  $\theta = 60^\circ$ :  $296 \text{ N T}$ . For  $\theta \leq 125.7^\circ$ ,  $|F| \leq 400 \text{ N}$ .
- 10.C4** (b)  $\theta = 60^\circ$ , datum at C:  $V = -294 \text{ in} \cdot \text{lb}$ .  
(c)  $34.2^\circ$ , stable;  $90^\circ$ , unstable;  $145.8^\circ$ , stable
- 10.C5** (b)  $\theta = 50^\circ$ , datum at E:  $V = 100.5 \text{ J}$ .  $dV/d\theta = 22.9 \text{ J}$ .  
(c)  $\theta = 0$ , unstable;  $30.4^\circ$ .
- 10.C6** (b)  $\theta = 60^\circ$ , datum at B:  $30.0 \text{ J}$ .  
(c)  $\theta = 0$ , unstable;  $41.4^\circ$ , stable.
- 10.C7** (b)  $\theta = 60^\circ$ , datum at  $\theta = 0$ :  $-37.0 \text{ J}$ . (c)  $52.2^\circ$ , stable.

## Centroids of Common Shapes of Areas and Lines

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	$\alpha r^2$
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	$\pi r$
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

## Moments of Inertia of Common Geometric Shapes

Rectangle

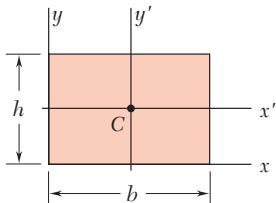
$$\bar{I}_{x'} = \frac{1}{12}bh^3$$

$$\bar{I}_{y'} = \frac{1}{12}b^3h$$

$$I_x = \frac{1}{3}bh^3$$

$$I_y = \frac{1}{3}b^3h$$

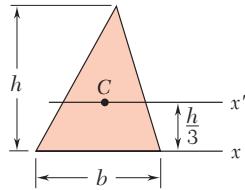
$$J_C = \frac{1}{12}bh(b^2 + h^2)$$



Triangle

$$\bar{I}_{x'} = \frac{1}{36}bh^3$$

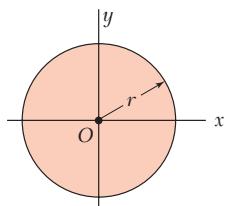
$$I_x = \frac{1}{12}bh^3$$



Circle

$$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$$

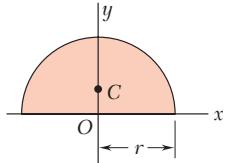
$$J_O = \frac{1}{2}\pi r^4$$



Semicircle

$$I_x = I_y = \frac{1}{8}\pi r^4$$

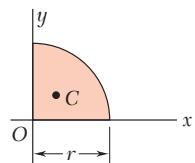
$$J_O = \frac{1}{4}\pi r^4$$



Quarter circle

$$I_x = I_y = \frac{1}{16}\pi r^4$$

$$J_O = \frac{1}{8}\pi r^4$$

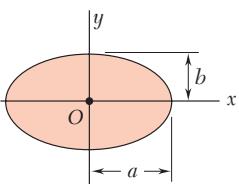


Ellipse

$$\bar{I}_x = \frac{1}{4}\pi ab^3$$

$$\bar{I}_y = \frac{1}{4}\pi a^3b$$

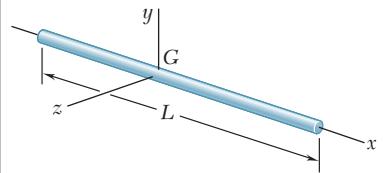
$$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$$



## Mass Moments of Inertia of Common Geometric Shapes

Slender rod

$$I_y = I_z = \frac{1}{12}mL^2$$

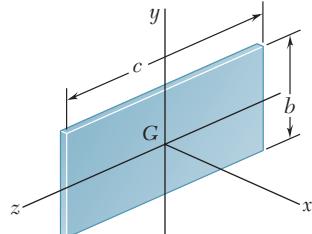


Thin rectangular plate

$$I_x = \frac{1}{12}m(b^2 + c^2)$$

$$I_y = \frac{1}{12}mc^2$$

$$I_z = \frac{1}{12}mb^2$$

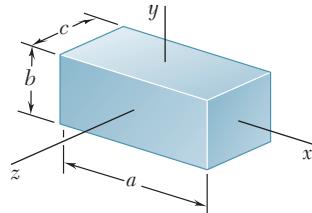


Rectangular prism

$$I_x = \frac{1}{12}m(b^2 + c^2)$$

$$I_y = \frac{1}{12}m(c^2 + a^2)$$

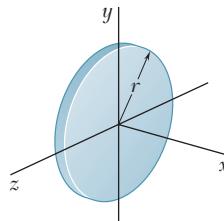
$$I_z = \frac{1}{12}m(a^2 + b^2)$$



Thin disk

$$I_x = \frac{1}{2}mr^2$$

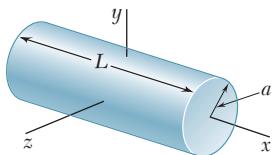
$$I_y = I_z = \frac{1}{4}mr^2$$



Circular cylinder

$$I_x = \frac{1}{2}ma^2$$

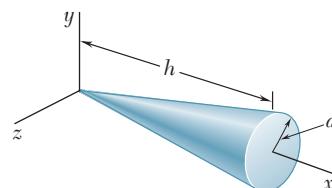
$$I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$$



Circular cone

$$I_x = \frac{3}{10}ma^2$$

$$I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$$



Sphere

$$I_x = I_y = I_z = \frac{2}{5}ma^2$$

