Mahindra
École Centrale

Engineering Mathematics-1
Problem Sheet-3
Topics: Applications of Differential Calculus of Multi Variables and Line Integrals

Assignment Problems

1. Let $f(x,y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$ whenever $x^2y^2 + (x-y)^2 \neq 0$. Show that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] = \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right] = 0$ but that $f(x,y)$ does not tend to a limit as $(x,y) \to (0,0)$.

2. Let $f(x) = \begin{cases} x\sin\dfrac{1}{y} & \text{if } \; y \neq 0 \\ 0 & \text{if } \; y = 0 \end{cases}$
   Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ but that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$.

3. If $(x,y) \neq (0,0)$, let $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the line $y = mx$. Is it possible to define $f(0,0)$ so as to make f continuous at $(0,0)$?

4. A scalar field f is defined on $\mathbb{R}^n$ by the equation $f(\mathbf{x}) = \|\mathbf{x}\|^4$. Compute $f'(\mathbf{x};\mathbf{y})$ for arbitrary $\mathbf{x}$ and $\mathbf{y}$.

5. Compute the first order partial derivatives of the given scalar field.
   (i) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0).$
   (ii) $f(\mathbf{x}) = \mathbf{a}.\mathbf{x}$, $\mathbf{a}$ fixed. Here f is defined on $\mathbb{R}^n$.

6. Let $f(x,y) = \dfrac{1}{y}\cos x^2, \quad y \neq 0$. Verify that the mixed partials $D_1(D_2 f)$ and $D_2(D_1 f)$ are equal.

7. Evaluate the directional derivative of the scalar field $f(x,y,z) = x^2 + 2y^2 + 3z^2$ at $(1,1,0)$ in the direction of $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

8. Find the values of the constants a, b and c such that the directional derivative of $f(x,y,z) = axy^2 + byz + cz^2x^3$ at the point $(1,2,-1)$ has a maximum value of 64 in a direction parallel to the z-axis.

9. In $\mathbb{R}^3$ let $\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $r(x,y,z) = \|\mathbf{r}(x,y,z)\|$. Show that $\nabla(r^n) = nr^{n-2}\mathbf{r}$ if n is a positive integer.

10. The equations $u = f(x,y)$, $x = X(t)$, $y = Y(t)$ define u as a function of t, say $u = F(t)$. Compute $F'(t)$ and $F''(t)$ in terms of t for the following functions:
    (i) $f(x,y) = x^2 + y^2$, $X(t) = t$, $Y(t) = t^2$.

(ii) $f(x,y) = e^{xy}\cos(xy^2)$, $X(t) = \cos t$, $Y(t) = \sin t$.

11. Find a constant c such that at any point of intersection of the two spheres $(x-c)^2+y^2+z^2 = 3$ and $x^2 + (y-1)^2 + z^2 = 1$ the corresponding tangent planes will be perpendicular to each other.

12. Locate and classify the stationary points (if any) of the following surfaces.

    (i) $z = x^3 + y^3 - 3xy$.

    (ii) $z = \sin x \cosh y$.

13. Determine all the relative and absolute extreme values and the saddle points for the function $f(x,y) = xy(1-x^2-y^2)$ on the square $0 \le x \le 1$, $0 \le y \le 1$.

14. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 = 8$.

15. Find the extreme values of the scalar field $f(x,y,z) = x-2y+2z$ on the sphere $x^2+y^2+z^2 = 1$.

16. If a, b, and c are positive numbers, find the maximum value of $f(x,y,z) = x^a y^b z^c$ subject to the side condition $x+y+z = 1$.

17. Use the method of Lagrage's multipliers to find the greatest and least distances of a point on the ellipse $x^2 + 4y^2 = 4$ from the straight line $x + y = 4$.

18. Calculate the line integral of the vector field $\mathbf{f}$ along the path described.

    (i) $\mathbf{f}(x,y,z) = (y^2 - z^2)\mathbf{i} + 2yz\mathbf{j} - x^2\mathbf{k}$, along the path described by $\alpha(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$.

    (ii) $\mathbf{f}(x,y,z) = 2xy\mathbf{i} + (x^2+z)\mathbf{j} + y\mathbf{k}$, from $(1,0,2)$ to $(3,4,1)$ along a line segment.

19. Compute the value of the given line integral.

    (i) $\displaystyle\int_C (x^2-2xy)dx+(y^2-2xy)dy$, where C is a path from $(-2,4)$ to $(1,1)$ along the parabola $y = x^2$.

    (ii) $\displaystyle\int_C \frac{(x+y)dx-(x-y)dy}{x^2+y^2}$, where C is the circle $x^2+y^2 = a^2$, traversed once in a counter-clockwise direction.

20. Let $\mathbf{f}$ be vector field defined as below. In each case determine whether or not $\mathbf{f}$ is the gradient of a scalar field. When $\mathbf{f}$ is a gradient, find a corresponding potential function ϕ.

    (i) $\mathbf{f}(x,y) = (2xe^y + y)\mathbf{i} + (x^2e^y + x - 2y)\mathbf{j}$.

    (ii) $\mathbf{f}(x,y,z) = 2xy^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$.