MA 203: Tutorial Sheet 1: Probability

1. Two dice are tossed and the magnitude of the difference in the number of dots facing up in the two dice is noted.
   (a) Find the sample space.
   (b) Find the set A corresponding to the event "magnitude of difference is 3."

2. A binary communication system transmits a signal X that is either a $+2$ voltage signal or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal. (a) Find the sample space. (b) Find the set of outcomes corresponding to the event "transmitted signal was definitely $+2$." (c) Describe in words the event corresponding to the outcome $Y = 0$.

3. The sample space of an experiment is the real line. Let the events A and B correspond to the following subsets of the real line: $A = (-\infty, r]$ and $B = (-\infty, s]$, where $r \leq s$. Find an expression for the event $C = (r, s]$ in terms of A and B. Show that $B = A \cup C$ and $A \cap C = \emptyset$.

4. A random experiment has sample space $S = \{a, b, c, d\}$. Suppose that $P(\{a, b\}) = 3/8$, $P(\{b, c\}) = 6/8$, $P(\{d\}) = 1/8$, and $P(\{c, d\}) = 5/8$. Use the axioms of probability to find the probabilities of the elementary events.

5. A die is tossed and the number of dots facing up is noted.
   (a) Find the probability of the elementary events under the assumption that all faces of the die are equally likely to be facing up after a toss.
   (b) Find the probability of the events: $A = \{more\ than\ 3\ dots\}$; $B = \{odd\ number\ of\ dots\}$.
   (c) Find the probability of $A \cup B$, $A \cap B$, A^c.

6. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$. (a) Find the probabilities of A, B, $A \cap B$, and $A \cap C$. (b) Find the probabilities of $A \cup B$, $A \cup C$, and $A \cup B \cup C$, first, by directly evaluating the sets and then their probabilities, and second, by using the appropriate axioms or corollaries.

7. The lifetime of a device behaves according to the probability law $P[(t, \infty)] = 1/t$ for $t > 1$. Let A be the event "lifetime is greater than 4," and B the event "lifetime is greater than 8."
   (a) Find the probability of $A \cap B$ and $A \cup B$.
   (b) Find the probability of the event "lifetime is greater than 6 but less than or equal to 12."

8. Consider an experiment for which the sample space is the real line. A probability law assigns probabilities to subsets of the form $(-\infty, r]$.
   (a) Show that we must have $P[(-\infty, r]] \leq P[(-\infty, s]]$ when $r < s$.
   (b) Find an expression for $P[(r, s]]$ in terms of $P[(-\infty, r]]$ and $P[(-\infty, s]]$.
   (c) Find an expression for $P[(s, \infty)]$.

9. Two numbers (x, y) are selected at random from the interval $[0, 1]$.
(a) Find the probability that the pair of numbers are inside the unit circle.
(b) Find the probability that $y > 2x$.

10. A biased coin is tossed repeatedly until heads has come up three times. Find the probability that k tosses are required.

11. Ten passengers get on an airport shuttle at the airport. The shuttle has a route that includes 5 hotels, and each passenger gets off the shuttle at his/her hotel. The driver records how many passengers leave the shuttle at each hotel. How many different possibilities exist?

12. A student needs eight chips of a certain type to build a circuit. It is known that 5% of these chips are defective. How many chips should he buy for there to be a greater than 90% probability of having enough chips for the circuit?

13. A multiple choice test has 10 questions with 4 choices each. How many ways are there to answer the test? What is the probability that two papers have the same answers?

14. Let U be selected at random from the unit interval. Let $A = \{0 < U < 1/2\}$, $B = \{1/4 < U < 3/4\}$, and $C = \{1/2 < U < 1\}$. Are any of these events independent?

15. An experiment consists of picking one of two urns at random and then selecting a ball from the urn and noting its color (black or white). Let A be the event "urn 1 is selected" and B the event "a black ball is observed." Under what conditions are A and B independent?

16. Let A be any subset of S . Show that the class of sets $\{\emptyset, A, A^c, S\}$ is a σ-algebra.

17. Let f be a mapping from a sample space S to a finite set $S' = \{y_1, y_2, \cdots, y_k\}$.
(a) Show that the set of inverse images $A_k = f^{-1}(\{y_k\})$ forms a partition of S.
(b) Show that any event B of S' can be related to a union of A_k's.

18. Find the countable union of the following sequences of events:
(a) $A_n = [a + 1/n, b - 1/n]$.
(b) $B_n = (-n, b - 1/n]$.
(c) $C_n = [a + 1/n, b)$.

19. Find the countable intersection of the following sequences of events:
(a) $A_n = (a - 1/n, b + 1/n)$.
(b) $B_n = [a, b + 1/n)$.
(c) $C_n = (a - 1/n, b]$.