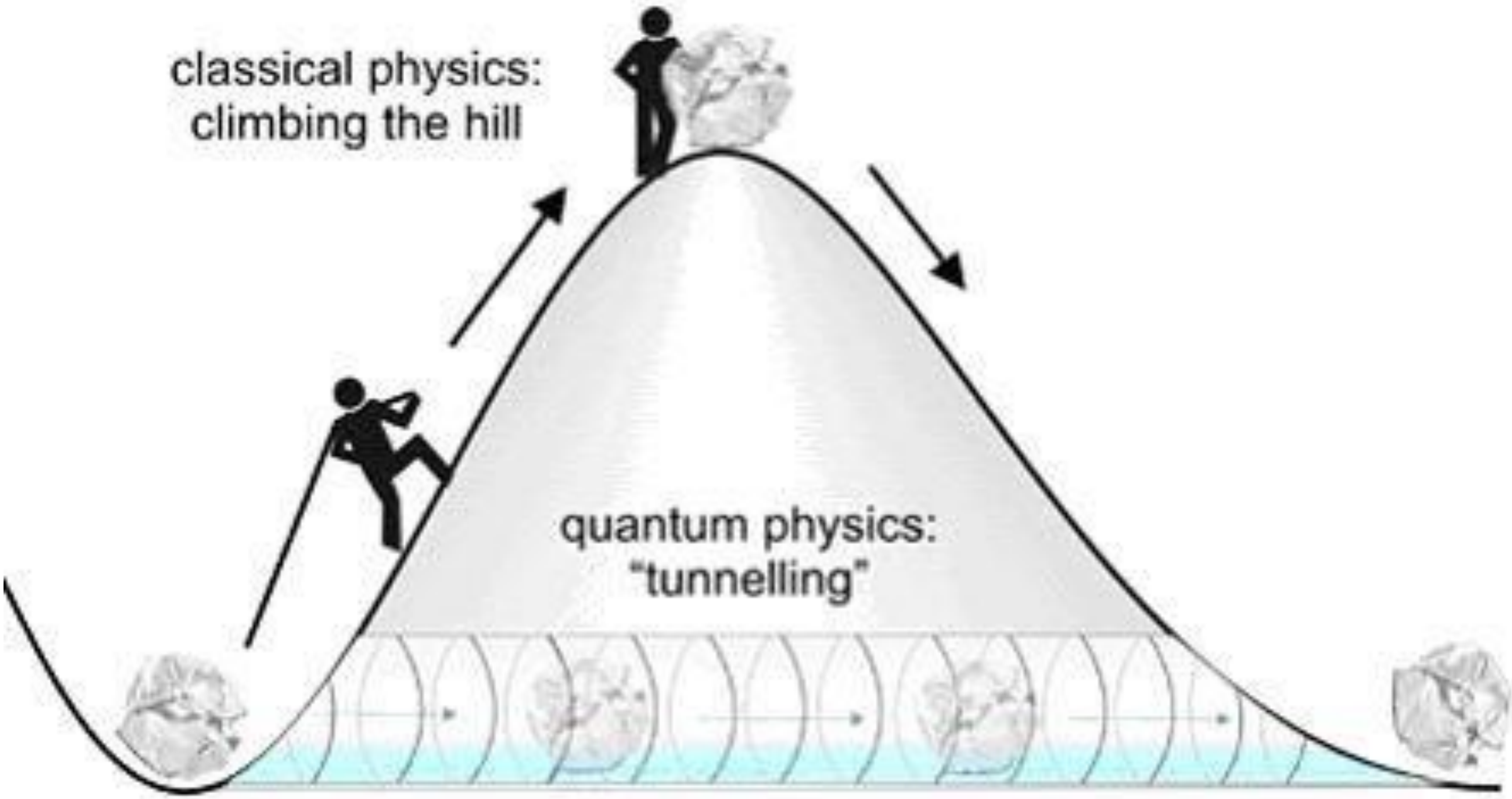
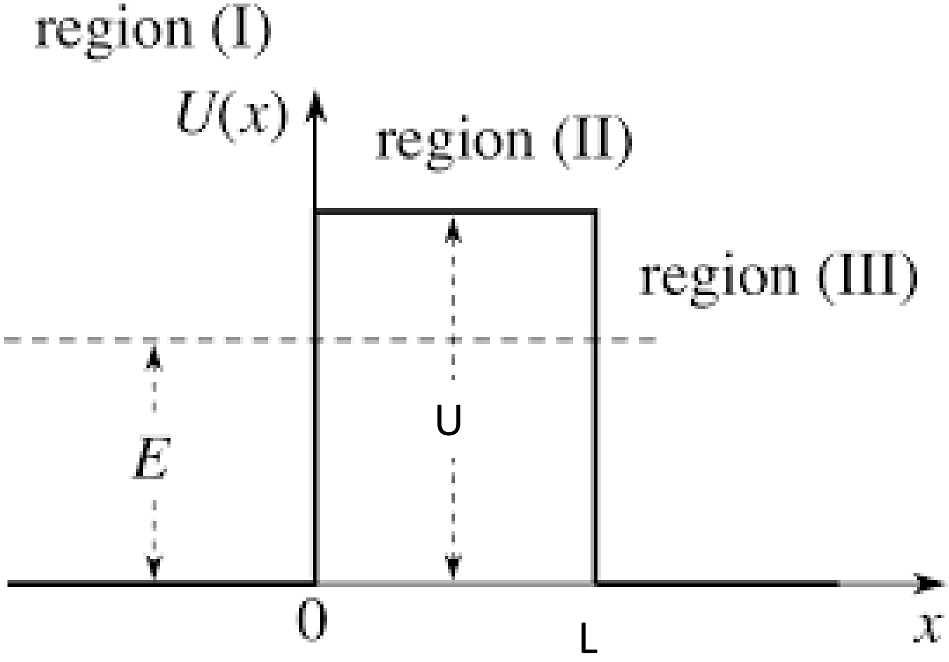
# Modern Physics

## Lecture 20

# Tunnel Effect

- Tunneling is a quantum mechanical phenomenon with no analog in classical physics
- Occurs when an electron passes through a potential barrier without having enough energy to do so


classical physics:
climbing the hill
quantum physics:
"tunnelling"




The potential energy has a constant value $U$ in the region of width $L$ and zero in all other regions

This a called a **square barrier**

$U$ is called the barrier **height**

Start with TISE

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial^2 x}+V(x)\psi(x)=E\psi(x)$$

# Defining potential barrier

Region I

*For* $x < 0$; $V(x) = 0$

Region II

For $0 < x < L$; $V(x) = U$

Region III

*For* $x > 0$; $V(x) = 0$

**In region I**

$$\frac{d^2\psi_I}{dx^2} + \frac{2mE}{\hbar^2}\psi_I = 0$$

**In region II**

$$\frac{d^2\psi_{II}}{dx^2} - \frac{2m}{\hbar^2}(U - E)\psi_{II} = 0$$

**In region III**

$$\frac{d^2\psi_{III}}{dx^2} + \frac{2mE}{\hbar^2}\psi_{III} = 0$$

# Solutions

Region I

$$\psi_I = Ae^{ik_1x} + Be^{-ik_1x}$$

$$\psi_I = \psi_{I+} + \psi_{I-}$$

where $\psi_{1+} = Ae^{ik_1x}$ and $\psi_{1-} = Be^{-ik_1x}$

Region III

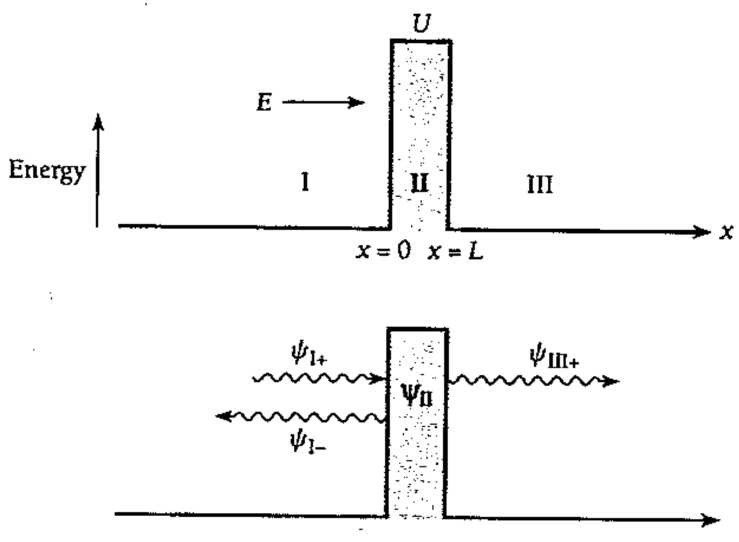
$$\psi_{III} = Fe^{ik_1x} + Ge^{-ik_1x}$$

Since there is no reflected wave

$$G = 0$$

Therefore

$$\psi_{III} = Fe^{ik_1x}$$

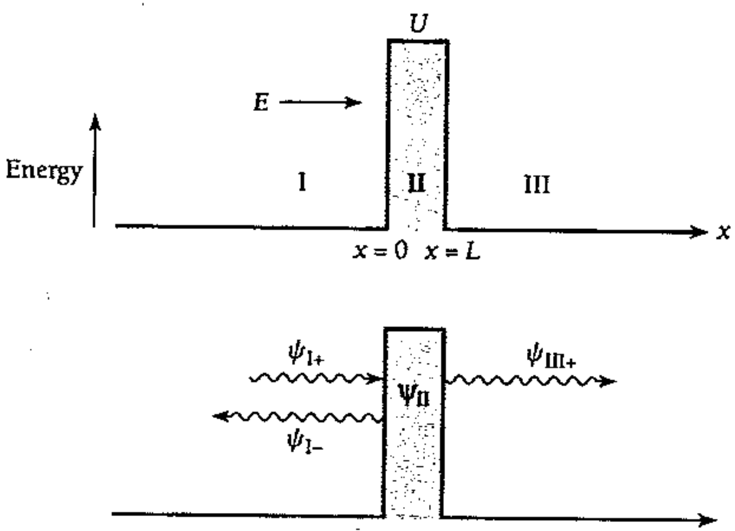


# Region II

$$\psi_{II} = Ce^{-k_2 x} + De^{k_2 x}$$

Where $$k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$$

Boundary conditions:




$\boxed{\text{At x = 0}}$

$$\psi_I = \psi_{II}$$

$$\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}$$

$\boxed{\text{At x = L}}$

$$\psi_{II} = \psi_{III}$$

$$\frac{d\psi_{II}}{dx} = \frac{d\psi_{III}}{dx}$$

# Using boundary conditions

At x = 0

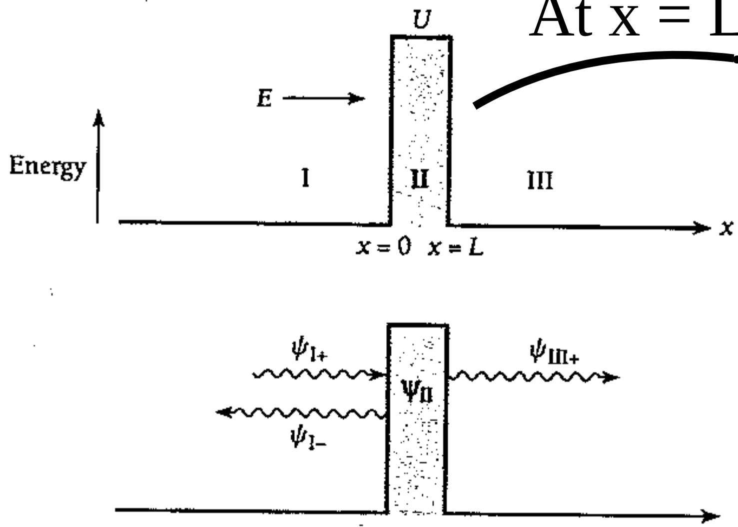
$$A + B = C + D$$

$$ik_1 A - ik_1 B = -k_2 C + k_2 D$$

At x = L

$$Ce^{-k_2 L} + De^{k_2 L} = Fe^{ik_1 L}$$

$$-k_2 Ce^{-k_2 L} + k_2 De^{k_2 L} = ik_1 Fe^{ik_1 L}$$




4 equations 5 unknowns (A, B, C, D, F)

# But we can find reflectance (R) and transmittance (T)

Transmittance $$T = \frac{|F|^2}{|A|^2} = \frac{F^* F}{A^* A}$$

Reflectance $$R = \frac{|B|^2}{|A|^2} = \frac{B^* B}{A^* A}$$

$$T = \left[\frac{16}{4 + (k_2 / k_1)^2}\right] e^{-2k_2 L}$$

$$\left(\frac{k_2}{k_1}\right)^2 = \frac{2m(U - E)/\hbar^2}{2mE/\hbar^2}$$

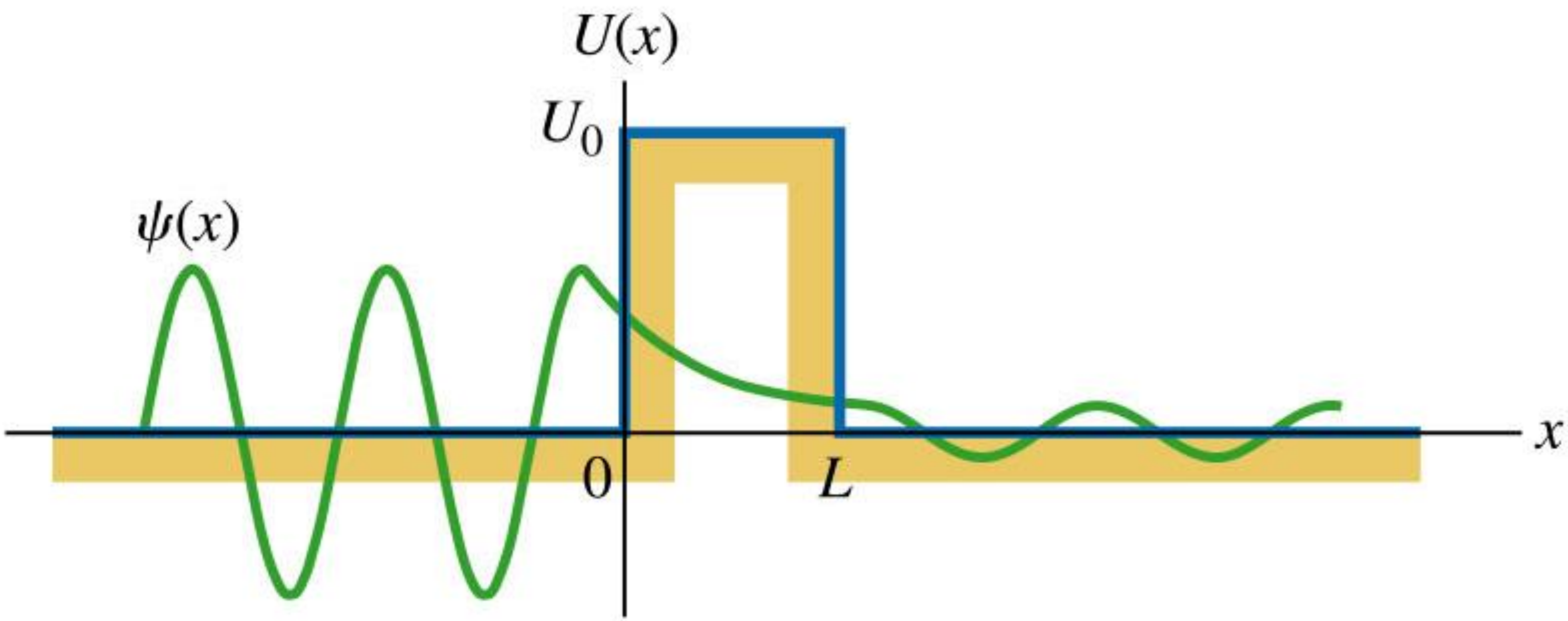
$$T = \left[\frac{16}{4 + \frac{U}{E} - 1}\right] e^{-2k_2 L} \cong e^{-2k_2 L}$$

$$\left(\frac{k_2}{k_1}\right)^2 = \frac{U}{E} - 1$$

$$\psi_{1+} = Ae^{ik_1 x}$$ Incident wave

$$\psi_{III} = Fe^{ik_1 x}$$ Transmitted wave

$$T = e^{-2k_2 L}$$ Tunnel coefficient



$$T = e^{-2k_2 L}$$

Tunnel coefficient (T) vs Barrier Length (L)

Tunnel coefficient (T) vs Particle Energy (E)

Electrons with energies of 1.0 eV and 2.0 eV are incident on a barrier 10.0 eV high and 0.50 nm wide. (a) Find their respective transmission probabilities (b) How are these affected if the barrier is doubled in width?

Part a

For 1 eV electrons

$$k_2 = \frac{\sqrt{2m(U-E)}}{\hbar}$$

$$k_2 = \frac{\sqrt{2\times 9.1\times 10^{-31}(10-1)\times 1.6\times 10^{-19}}}{1.054\times 10^{-34}}$$

$$k_2 = 1.6\times 10^{10}\ m^{-1}$$

Now L = 0.5 nm, Therefore

$$2k_2L = 2\times 1.6\times 10^{10}\times 5.0\times 10^{-10}$$

$$2k_2L = 16$$

$$T = e^{-2k_2 L} = e^{-16}$$

$$T = 1.1 \times 10^{-7}$$

This means 1 electron out of 9090000 electron can tunnel through the potential barrier

For 2 eV electron

$$T = 2.4 \times 10^{-7}$$

This means 1 electron out of 4166666 electron can tunnel through the potential barrier

Part b

If the barrier width is doubled

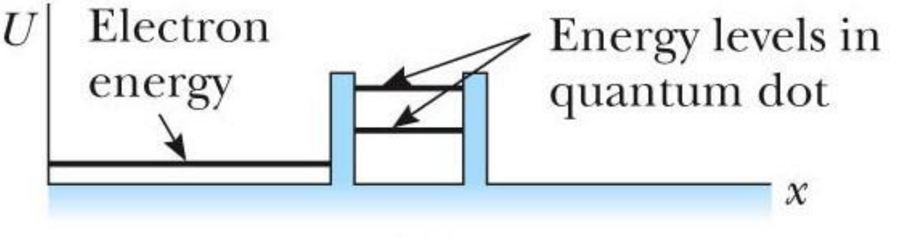
For 1 eV electrons $T = 1.3 \times 10^{-14}$

For 2 eV electrons $T = 5.1 \times 10^{-14}$

# Applications of Tunnel Effect

# Resonant Tunneling Diodes

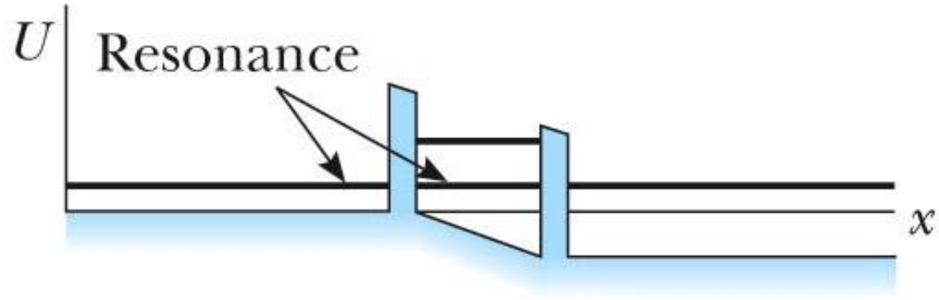
- Figure b shows the potential barriers and the energy levels in the quantum well
- The electron with the energy shown encounters the first barrier, it has no energy levels available on the right side of the barrier
- This greatly reduces the probability of tunneling



(b)

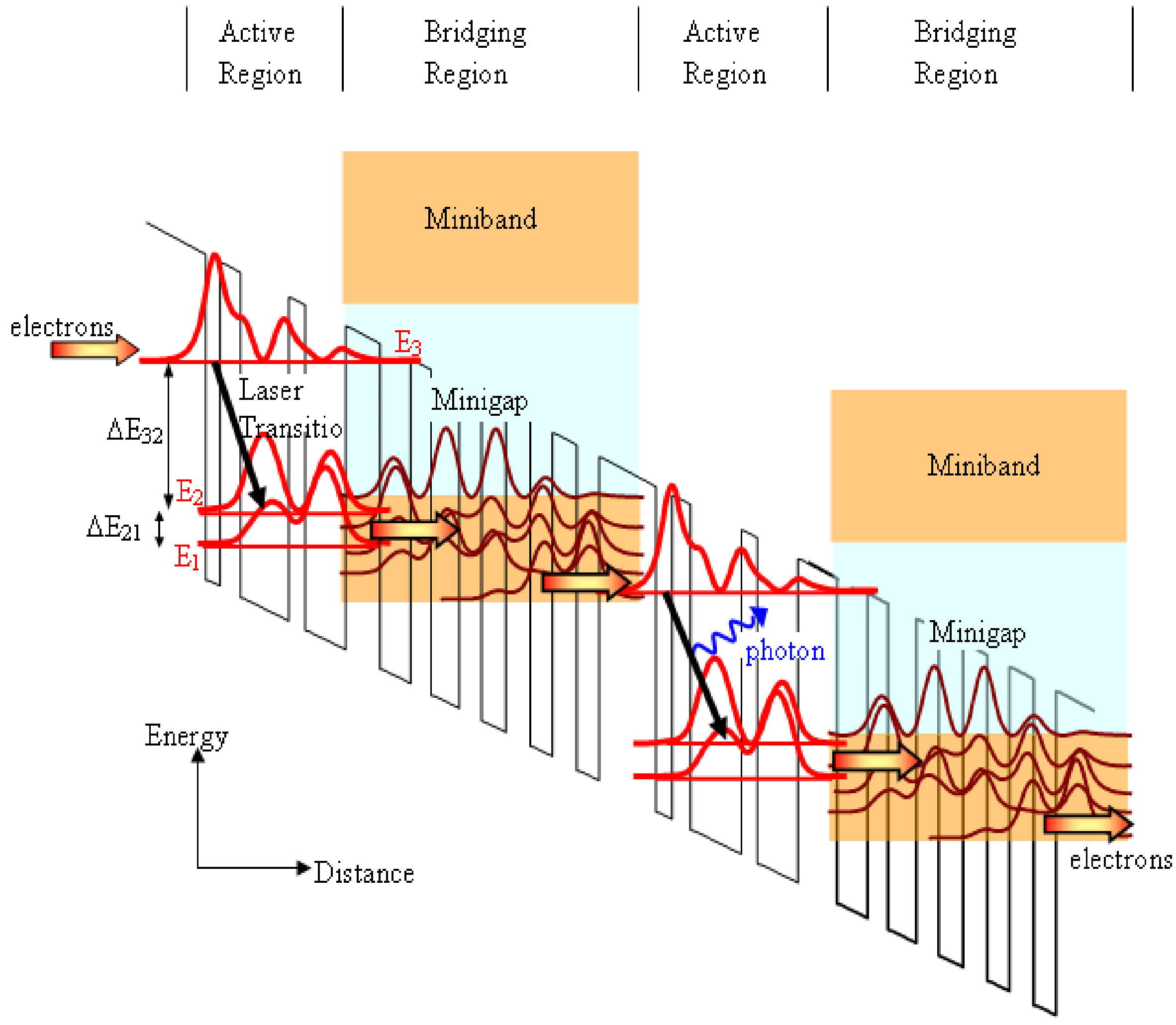
# Resonant Tunneling Diodes

- Applying a voltage decreases the potential with position
- The deformation of the potential barrier results in lowering the energy level in the quantum well
- The resonance of energies gives the device its name



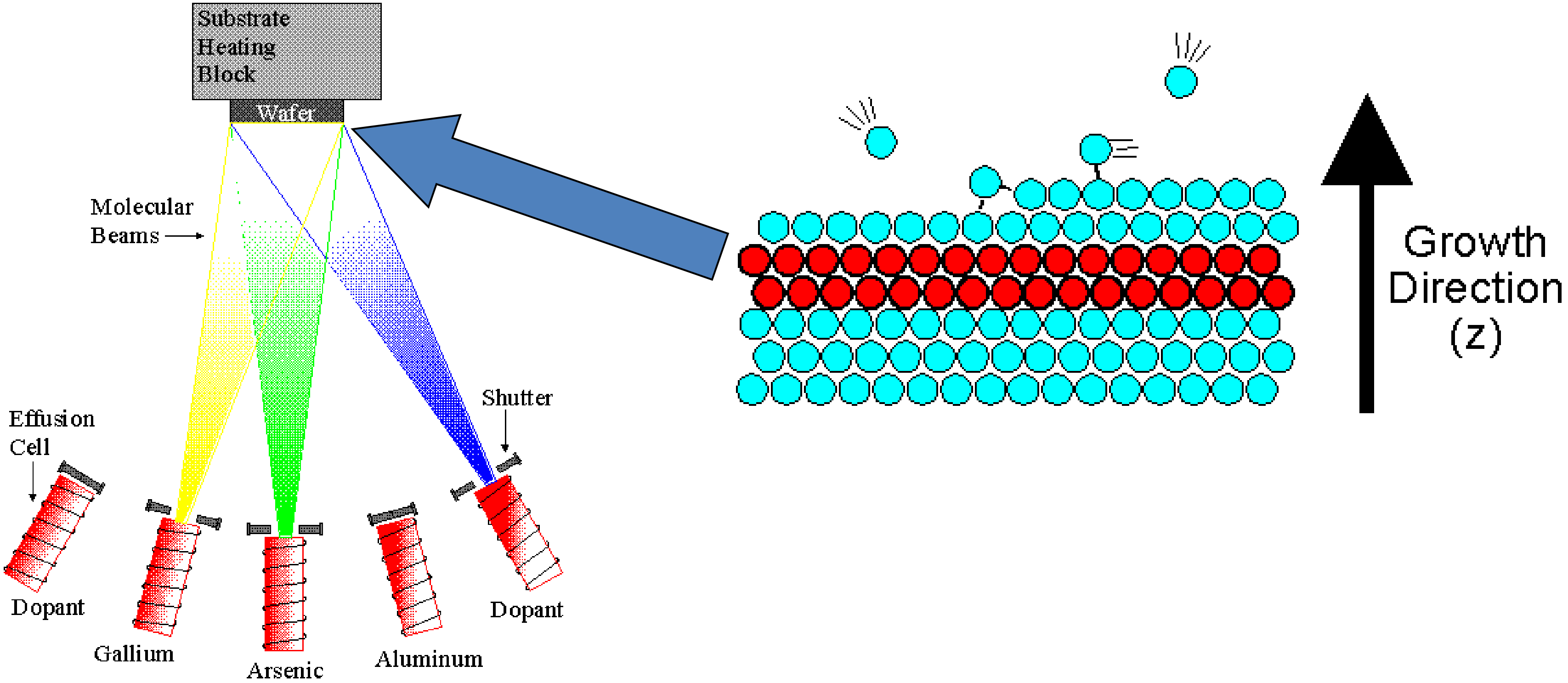

(c)

# Quantum Cascade Laser: Engineering with electron wavefunctions

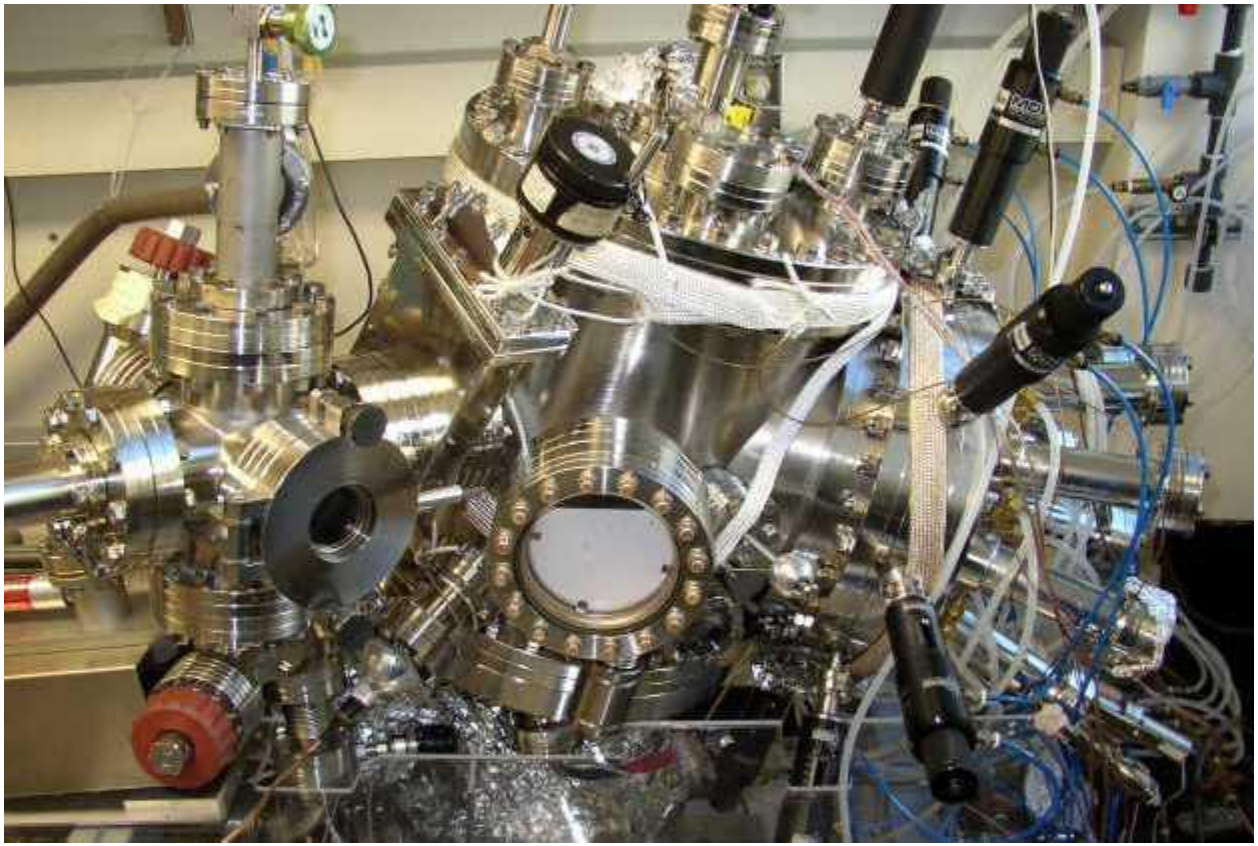


# Molecular Beam Epitaxy (MBE)

Man-made potential wells for Quantum mechanical engineering

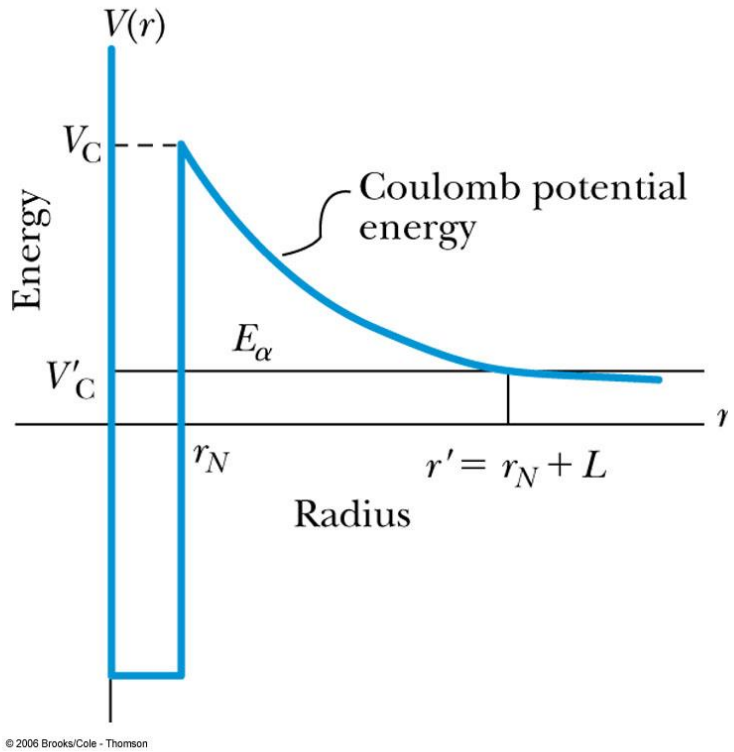


# Molecular Beam Epitaxy: Man-made potential wells for Quantum mechanical engineering



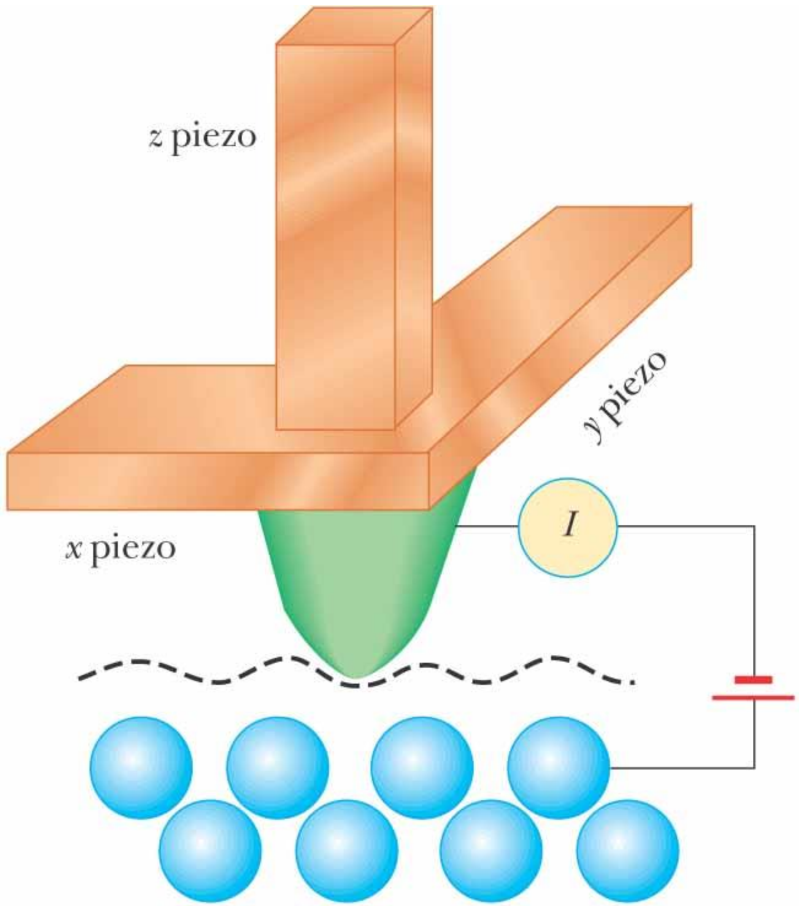
# Alpha-Particle Decay

- The phenomenon of tunneling explains the alpha-particle decay of heavy, radioactive nuclei.
- Inside the nucleus, an alpha particle feels the strong, short-range attractive nuclear force as well as the repulsive Coulomb force.
- The nuclear force dominates inside the nuclear radius where the potential is approximately a square well.
- The Coulomb force dominates outside the nuclear radius.
- The potential barrier at the nuclear radius is several times greater than the energy of an alpha particle.
- According to quantum mechanics, however, the alpha particle can “tunnel” through the barrier. Hence this is observed as radioactive decay.



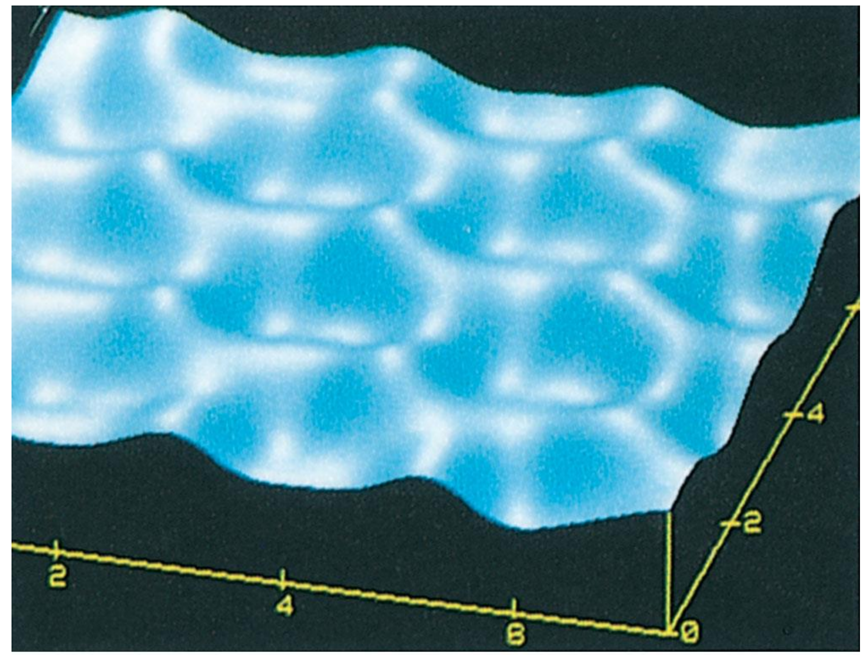
# Applications of Tunneling – Scanning Tunneling Microscope

- An electrically conducting probe with a very sharp edge is brought near the surface to be studied
- The empty space between the tip and the surface represents the “barrier”
- The tip and the surface are two walls of the “potential barrier”



# Scanning Tunneling Microscope

- The STM allows highly detailed images of surfaces with resolutions comparable to the size of a single atom
- At right is the surface of graphite “viewed” with the STM






- The STM is very sensitive to the distance from the tip to the surface
  - This is the thickness of the barrier

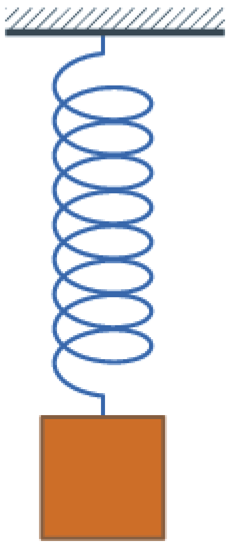
# Simple Harmonic Oscillator

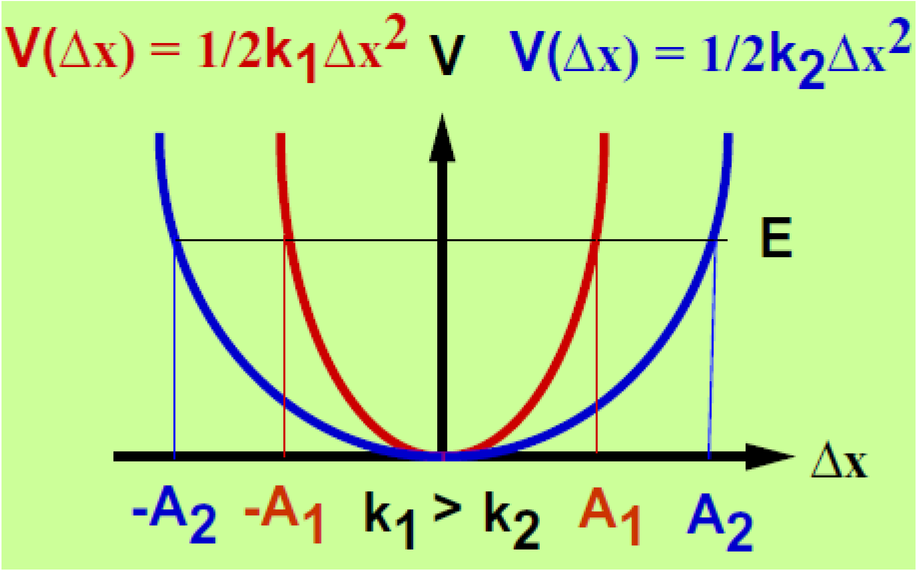
In general, according to Hooke's Law:

**F = -k x**

i.e. the force proportional to displacement and pointing in opposite direction and where k is the force constant and x is the displacement.

Corresponding Potential Energy $\int dU = \int -F dx = \frac{1}{2}kx^2$






➢The **parabolic potential energy V = ½ *kx*²** *of a harmonic oscillator, where x is* the displacement from equilibrium.

➢The **narrowness of the curve depends on the force constant *k***: *the larger the* value of *k*, *the narrower the well.*

# Time Independent Schrodinger's Equation

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\psi = 0$$ Using SHM potential

We make the following substitutions,

$$y = \sqrt{\frac{2\pi m\nu}{\hbar}}x$$ and $$\alpha = \frac{2E}{\hbar}\sqrt{\frac{m}{k}} = \frac{2E}{h\nu}$$

$$\frac{d^2\psi}{dy^2} + \left(\alpha - y^2\right)\psi = 0$$

After solving Energy levels are given by,

$$\alpha = 2n + 1; n = 0,1,2,3........$$

This means,

$$E_n = (n + \frac{1}{2})h\nu$$

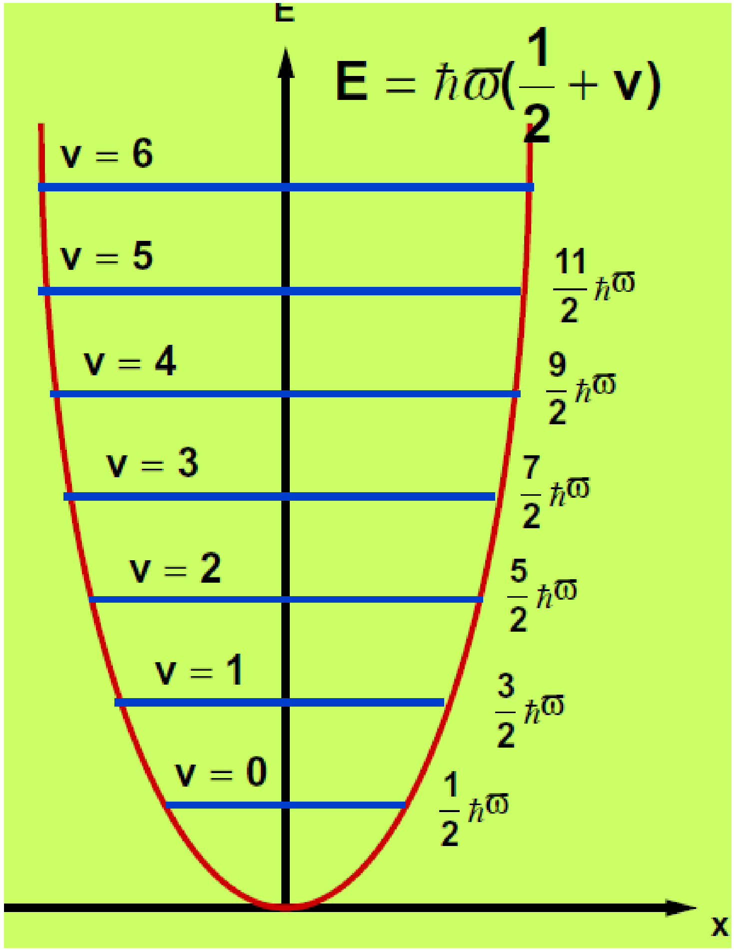
If $n = 0$

Ground state energy will be

$$E_0 = \frac{1}{2}h\nu$$

This is the minimum energy of Simple Harmonic Oscillator

- Spacing between allowed energy levels for the harmonic oscillator is constant, whereas for the particle in a box, the spacing between levels rises as the quantum number increases.
- Lowest level $v = 0$ is possible since E will not be zero.



$$\frac{d^2\psi}{dy^2} + \left(\alpha - y^2\right)\psi = 0$$

Solutions are the following functions,

$$\psi_n = \left(\frac{2m\nu}{\hbar}\right)^{1/4} (2^n n!)^{-1/2} H_n(y) e^{-y^2/2}$$

$H_n(y)$ *is Hermite Polynomial*

# Hermite polynomials

| $v$ | $H_v$ |
|---|---|
| 0 | $1$ |
| 1 | $2y$ |
| 2 | $4y^2 - 2$ |
| 3 | $8y^3 - 12y$ |
| 4 | $16y^4 - 48y^2 + 12$ |
| 5 | $32y^5 - 160y^3 + 120y$ |
| 6 | $64y^6 - 48y^4 + 72y^2 - 120$ |

# Wave functions will look like