Mahindra École Centrale
COLLEGE OF ENGINEERING

MA 203: Problem Sheet 6: Probability

1. Let $W = X + Y + Z$ where X, Y, and Z are zero-mean, unit-variance random variables with $COV(X, Y) = 1/2$, $COV(Y, Z) = -1/4$ and $COV(X, Z) = 1/2$. Find the mean and variance of W.

2. Let $X_1, \cdots, X_n$ be random variables with the same mean and with covariance function:
$$COV(X_i, X_j) = \begin{cases} \sigma^2 & \text{if } i = j \\ \rho\sigma^2 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$
where $\rho < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

3. Let X and Y be independent exponential random variables with parameters 1 and 4, respectively. Let $Z = X + Y$.
   (a) Find the characteristic function of Z.
   (b) Find the pdf of Z from the characteristic function found in part (a).

4. Let $Z = 3X - 7Y$ where X and Y are independent random variables.
   (a) Find the characteristic function of Z.
   (b) Find the mean and variance of Z by taking derivatives of the characteristic function found in part (a).

5. Let M_n be the sample mean of n iid random variables Find the characteristic function of M_n in terms of the characteristic function of the X_i's.

6. A random sinusoid signal $X(t) = A\sin(t)$ where A is a uniform random variable in the interval $[0, 1]$. Let $\mathbf{X} = (X(t_1), X(t_2), X(t_3))$ be samples of the signal taken at times t_1, t_2, and t_3.
   (a) Find the joint cdf of $\mathbf{X}$ in terms of the cdf of A if $t_1 = 0$, $t_2 = \pi/2$ and $t_3 = \pi$. Are $X(t_1), X(t_2), X(t_3)$ independent random variables ?
   (b) Find the joint cdf of $\mathbf{X}$ for $t_1 = \pi/6$, $t_2 = t_1 + \pi/2$ and $t_3 = t_1 + \pi$.

7. Let X and Y be zero-mean, unit-variance Gaussian random variables with correlation coefficient $1/2$. Find the joint pdf of $U = X^2$ and $V = Y^4$.

8. Let $\mathbf{X} = (X_1, X_2, X_3, X_4)$ be processed as follows:
$$Y_1 = X_1,\ Y_2 = X_1 + X_2,\ Y_3 = X_2 + X_3,\ Y_4 = X_3 + X_4.$$

(a) Find an expression for the joint pdf of $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$ in terms of the joint pdf of $\mathbf{X}$.

(b) Find the joint pdf of $\mathbf{Y}$ if $\mathbf{X}$ is a vector of independent zero-mean, unit-variance Gaussian random variables.

9. Let U_1, U_2 and U_3 be independent zero-mean, unit-variance Gaussian random variables and let $X = U_1$, $Y = U_1 + U_2$ and $Z = U_1 + U_2 + U_3$.

   (a) Find the covariance matrix of (X, Y, Z).

   (b) Find the joint pdf of (X, Y, Z).

   (c) Find the conditional pdf of Y and Z given X.

   (d) Find the conditional pdf of Z given X and Y.

10. Let $\mathbf{X} = (X_1, X_2)$ be jointly Gaussian random variables with mean vector and covariance matrix given by: $\mathbf{m_X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{K_X} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$.

    (a) Find the pdf of $\mathbf{X}$ in matrix notation.

    (b) Find the marginal pdfs of X_1 and X_2.

    (c) Find a transformation A such that the vector $\mathbf{Y} = A\mathbf{X}$ consists of independent Gaussian random variables.

    (d) Find the joint pdf of $\mathbf{Y}$.

11. Let $\mathbf{X} = (X_1, X_2, X_3)$ be the jointly Gaussian random variables with mean vector and covariance matrix given by:

$$\mathbf{m_X} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{K_X} = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}.$$

    (a) Find the pdf of $\mathbf{X}$ in matrix notation.

    (b) Find the pdf of $\mathbf{X}$ using the quadratic expression in the exponent.

    (c) Find the marginal pdfs of X_1, X_2 and X_3.

    (d) Find a transformation A such that the vector $\mathbf{Y} = A\mathbf{X}$ consists of independent Gaussian random variables.

    (e) Find the joint pdf of $\mathbf{Y}$.

12. Let $\mathbf{X} = (X_1, X_2, X_3)$ have covariance matrix:

$$\mathbf{K_X} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$$

    (i) Find the eigenvalues and eigenvectors of $\mathbf{K_X}$.

    (ii) Find the orthogonal matrix P that diagonalizes $\mathbf{K_X}$. Verify that P is orthogonal and $P^T\mathbf{K_X}P = \Lambda$, diagonal matrix.

13. Let $X = aU + bV$ and $Y = cU + dV$ where $ad - bc \neq 0$.

    (a) Find the joint characteristic function of X and Y in terms of the joint characteristic function of U and V.

    (b) Find an expression for $E[XY]$ in terms of joint moments of U and V.

14. Let X_1, X_2, X_3, X_4 be processed as follows:

$$Y_1 = X_1,\ Y_2 = X_1 + X_2,\ Y_3 = X_2 + X_3,\ Y_4 = X_3 + X_4.$$

    (a) Find an expression for the joint pdf of $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$ in terms of the joint pdf of $\mathbf{X} = (X_1, X_2, X_3, X_4)$.

    (b) Find the joint pdf of $\mathbf{Y}$ if X_1, X_2, X_3, X_4 are independent zero-mean, unit-variance Gaussian random variables.

    (c) Find the mean vector and covariance matrix $\mathbf{Y}$ of in terms of those of $\mathbf{X}$.

    (d) Find the cross-covariance matrix between $\mathbf{Y}$ and $\mathbf{X}$.

    (e) Evaluate the mean vector, covariance, and cross-covariance matrices if X_1, X_2, X_3, X_4 are independent random variables.

15. A fair coin is tossed 1000 times. Estimate the probability that the number of heads is between 400 and 600.

16. The lifetime of a cheap light bulb is an exponential random variable with mean 36 hours. Suppose that 16 light bulbs are tested and their lifetimes measured. Use the Central Limit Theorem to estimate the probability that the sum of the lifetimes is less than 600 hours.

17. Let S be the sum of 80 iid Poisson random variables with mean 0.25. Compare the exact value of $P[S = k]$ to an approximation given by the Central Limit Theorem.

18. A fair coin is tossed 100 times. Use the Chernoff bound to estimate the probability that the number of heads is greater than 90. Compare to an estimate using the Central Limit Theorem.