MA 203: Tutorial Sheet 3: Probability

1. An information source produces binary triplets $\{000, 111, 010, 101, 001, 110, 100, 011\}$ with corresponding probabilities $\{1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16\}$. A binary code assigns a codeword of length $-\log_2 p_k$ to triplet k. Let X be the length of the string assigned to the output of the information source.
(a) Show the mapping from S to S_X the range of X.
(b) Find the probabilities for the various values of X.

2. Consider an information source that produces binary pairs that we designate as $S_X = \{1, 2, 3, 4\}$. Find and plot the pmf and cdf in the following cases:
(a) $p_k = p_1/k$ for all $k \in S_X$.
(b) $p_{k+1} = p_k/2$ for $k = 2, 3, 4$.
(c) $p_{k+1} = p_k/2^k$ for $k = 2, 3, 4$.
(d) Can the random variables in parts a, b, and c be extended to take on values in the set $\{1, 2, \cdots\}$? If yes, specify the pmf of the resulting random variables. If no, explain why not.
(e) Use the cdf to find the probability of the events: $\{X \leq 1\}$, $\{X < 2.5\}$, $\{0.5 < X \leq 2.5\}$, $\{1 < X < 4\}$.

3. Two dice are tossed and we let X be the difference in the number of dots facing up.
(a) Find and plot the pmf of X.
(b) Find the probability that $|X| \leq k$ for all k.

4. Let N be a geometric random variable with $S_N = \{1, 2, \cdots\}$. (a) Find $P[N = k | N \leq m]$.
(b) Find the probability that N is odd.

5. The number of orders waiting to be processed is given by a Poisson random variable with parameter $\alpha = \lambda/n\mu$, where λ is the average number of orders that arrive in a day, μ is the number of orders that can be processed by an employee per day, and n is the number of employees. Let $\lambda = 5$ and $\mu = 1$. Find the number of employees required so the probability that more than four orders are waiting is less than 10%. What is the probability that there are no orders waiting?

6. The number of page requests that arrive at a Web server is a Poisson random variable with an average of 6000 requests per minute.
(a) Find the probability that there are no requests in a 100-ms period. (b) Find the probability that there are between 5 and 10 requests in a 100-ms period.

7. For the Poisson random variable, show that for $\alpha < 1$, $P[N = k]$ is maximum at $k = 0$; for $\alpha > 1$, $P[N = k]$ is maximum at $[\alpha]$; and if α is a positive integer, then $P[N = k]$ is maximum at $k = \alpha$ and at $k = \alpha - 1$.

8. An LCD display has 1000×750 pixels.A display is accepted if it has 15 or fewer faulty pixels.The probability that a pixel is faulty coming out of the production line is 10^{-5}. Find the proportion of displays that are accepted.

9. Compare the Poisson approximation and the binomial probabilities for $k = 0, 1, 2, 3$ and $n = 10$, $p = 0.1$; and $n = 20$, $p = 0.005$; and $n = 100$, $p = 0.01$.

10. A binary communication channel has a probability of bit error of 10^{-6}. Suppose that transmissions occur in blocks of 10,000 bits. Let N be the number of errors introduced by the channel in a transmission block.
(a) Find $P[N = 0]$, $P[N \leq 3]$.
(b) For what value of p will the probability of 1 or more errors in a block be 99%?

11. Let Y be the difference between the number of heads and the number of tails in 3 tosses of a fair coin.
(a) Plot the cdf of the random variable Y.
(b) Express $P[|Y| < y]$ in terms of the cdf of Y.

12. Let ζ be a point selected at random from the unit interval. Consider the random variable $X = (1 - \zeta)^{-1/2}$.
(a) Sketch X as a function of ζ.
(b) Find and plot the cdf of X. (c) Find the probability of the events $\{X > 1\}$, $\{5 < X < 7\}$, $\{X \leq 20\}$.

13. The random variable X is uniformly distributed in the interval $[-1, 2]$.
(a) Find and plot the cdf of X.
(b) Use the cdf to find the probabilities of the following events: $\{X \leq 0\}$, $\{|X - 0.5| < 1\}$, $\{X > -0.5\}$.

14. The cdf of the random variable X is given by:

$$F_X(x) = \begin{cases} 0, & x < -1 \\ 0.5, & -1 \leq x \leq 0 \\ (1+x)/2, & 0 \leq x \leq 1 \\ 1, & x \geq 1. \end{cases}$$

(a) Plot the cdf and identify the type of random variable.
(b) Find $P[X \leq -1]$, $P[X = -1]$, $P[X < 0.5]$, $P[-0.5 < X < 0.5]$, $P[X > -1]$, $P[X \leq 2]$, $P[X > 3]$.

15. A random variable X has cdf:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - 0.25e^{-2x}, & x \geq 0. \end{cases}$$

(a) Plot the cdf and identify the type of random variable.
(b) Find $P[X \leq 2]$, $P[X = 0]$, $P[X < 0]$, $P[2 < X < 6]$, $P[X > 10]$.

16. The random variable X has cdf:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 0.5 + c\sin^2(\pi x/2), & 0 \leq x \leq 1 \\ 1, & x > 1. \end{cases}$$

(a) What values can c assume?
(b) Plot the cdf and find $P[X > 0]$.

17. A random variable X has pdf:

$$f_X(x) = \begin{cases} cx(1-x^2), & 0 \leq x \leq 1 \\ 0, & elsewhere. \end{cases}$$

(a) Find c and plot the pdf. Plot the cdf of X.
(b) Find $P[0 < X < 0.5]$, $P[X = 1]$, $P[0.25 < X < 0.5]$.

18. Let $Y = A\cos(\omega t) + c$ where A has mean m and variance σ^2 and ω and c are constants. Find the mean and variance of Y.

19. Let $Y = 3X + 2$.
(a) Find the mean and variance of Y in terms of the mean and variance of X.
(b) Evaluate the mean and variance of Y if X is an arbitrary Gaussian random variable.
(c) Evaluate the mean and variance of Y if $X = b\cos(2\pi U)$ where U is a uniform random variable in the unit interval.

20. Let X be a Gaussian random variable with mean 2 and variance 4.The reward in a system is given by $Y = (X)^+$. Find the pdf of Y.

21. Compare the Markov inequality and the exact probability for the event $\{X > c\}$ as a function of c for:
(a) X is a uniform random variable in the interval $[0, b]$.
(b) X is an exponential random variable with parameter λ.
(c) X is a uniform random variable in $\{1, 2, \cdots, L\}$.
(d) X is a geometric random variable.
(e) X is a binomial random variable with $n = 10$, $p = 0.5$.