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MA 204: Tutorial Sheet 1

Prob 1) Expand the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

in a series by using the exponential series and integrating. Obtain the Taylor series of $\operatorname{erf}(x)$ about zero directly. Are the two series the same? Evaluate $\operatorname{erf}(1)$ by adding four terms of the series and compare with the value $\operatorname{erf}(1) \approx 0.8427$, which is correct to four decimal places.
Hint: Recall from the Fundamental Theorem of Calculus that

$$\frac{d}{dx} \int_0^x f(t)dt = f(x).$$

Prob 2) What is the least number of terms required to obtain π correct up to four decimal places, using the series

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right].$$

Prob 3) What are the condition numbers of the following functions? Where are they large?

(i) $(x-1)^a$, where $a > 0$. (ii) $x^{-1}e^x$. (iii) $\cos^{-1} x$.

Prob 4) We consider a classic example given by Wilkinson. Let

$$f(x) = (x-1)(x-2)...(x-20) \text{ and } g(x) = x^{19}.$$

The roots of f are obviously the integers 1,2,3,...,20. How is the root $r = 20$ affected by perturbing f to $f + \epsilon g$?

Prob 5) Let the Bisection algorithm is applied to a continuous function f on an interval $[a, b]$ to solve $f(x) = 0$, where $f(a)f(b) < 0$. Denote the successive intervals that arise in the Bisection method by $[a_0, b_0], [a_1, b_1], ..., [a_n, b_n]$ and so on with $a = a_0$ and $b = b_0$. Show that

a) $a_0 \leq a_1 \leq a_2 \leq .....$ and $b_0 \geq b_1 \geq b_2 \geq ......$

b) $b_n - a_n = 2^{-n}(b_0 - a_0)$.

c) After $n-$steps, an approximate root will have been computed with error at most $(b_0 - a_0)/2^{(n+1)}$.

Further, if $a = 0.1$ and $b = 1.0$, how many steps of the Bisection method are required to determine the root with an error of at most $\frac{1}{2} \times 10^{-8}$.

Prob 6) Using Bisection method, find where the graphs of $y = 3x$ and $y = e^x$ intersect by finding roots of $e^x - 3x = 0$ correct to four decimal digits.

Prob 7) Verify that when Newton's Method is used to compute $\sqrt{N}$ (by solving the equation $x^2 = N$), the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right).$$

Perform three iterations of this scheme for computing $\sqrt{2}$, starting with $x_0 = 1$, and of the Bisection method for $\sqrt{2}$, starting with interval $[1, 2]$. How many iterations are needed for each method in order to obtain 10^{-6} accuracy?

Lab Exercises

Ex 1) Write codes for solving the problems in Problems 6 and 7.

Ex 2) Write a program to solve for a root of the equation $e^{-x^2} = \cos x + 1$ on $[0, 4]$. What happens in Newton's method if we start with $x_0 = 0$ or with $x_0 = 1$?

Ex 3) (**Circuit Problem**) A simple circuit with resistance R, capacitance C in series with a battery of voltage V is given by

$$Q = CV\left(1 - e^{-T/(RC)}\right),$$

where Q is the charge of the capacitor and T is the time needed to obtain the charge. We wish to solve for the unknown C. For example, using Bisection method, solve this exercise

$$f(x) = 10x\left[1 - e^{-0.004/(2000x)}\right] - 0.00001.$$

Plot the curve.

Ex 4) In celestial mechanics, **Kepler's Equation** is important. It reads

$$x = y - \epsilon \sin y,$$

in which x is a planet's mean anomaly, y its eccentric anomaly, and ϵ the eccentricity of its orbit. Taking $\epsilon = 0.9$, construct a table of y for 30 equally spaced values of x in the interval $0 \leq x \leq \pi$. Use Newton's Method to obtain each value of y. The y corresponding to an x can be used as the starting point for the iteration when x is changed slightly.