MA 203: Tutorial Sheet 5: Probability
Assignment Submission Deadline: 17/10/2018

$*$ Problems to be submitted as Assignment

*1. The pair (X, Y) has joint cdf given by:

$$F_{X,Y}(x,y) = \begin{cases} (1-1/x^2)(1-1/y^2) & \text{for } x > 1, y > 1 \\ 0 & elsewhere \end{cases}.$$

(i) Find the marginal cdf of X and of Y.
(ii) Find the probability of the following events: $\{X < 3, Y \leq 5\}$, $\{X > 4, Y > 3\}$.

2. A point (X, Y) is selected at random inside a triangle defined by $\{(x, y) : 0 \leq y \leq x \leq 1\}$. Assume the point is equally likely to fall anywhere in the triangle.
(a) Find the joint cdf of X and Y.
(b) Find the marginal cdf of X and of Y.
(c) Find the probabilities of the following events in terms of the joint cdf:
$A = \{X \leq 1/2, Y \leq 3/4\}$, $B = 1/4 < X \leq 3/4, 1/4 < Y \leq 3/4\}$.

3. Is the following a valid cdf ? Give reasons.

$$F_{X,Y}(x,y) = \begin{cases} (1-1/x^2y^2) & \text{for } x > 1, y > 1 \\ 0 & elsewhere \end{cases}.$$

4. The amplitudes of two signals X and Y have joint pdf:

$$f_{X,Y} = e^{-x/2}ye^{-y^2} \quad \text{for } x > 0, y > 0.$$

(i) Find the joint cdf and marginal pdfs.
(ii) Find $P[X^{1/2} > Y]$.

*5. The general form of the joint pdf for two jointly Gaussian random variables is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{ -\frac{\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\frac{(x-m_1)(y-m_2)}{\sigma_1\sigma_2} + \left(\frac{y-m_2}{\sigma_2}\right)^2}{2(1-\rho^2)} \right\}$$

Show that X and Y have marginal pdfs that correspond to Gaussian random variables with mean m_1 and m_2 and variances σ_1^2 and σ_2^2 respectively.

6. The input X to a communication channel is $+1$ or -1 with probability p and $1-p$, respectively. The received signal Y is the sum of X and noise N which has a Gaussian distribution with zero mean and variance $\sigma^2 = 0.25$.
(i) Find the joint probability $P[X = j, Y \leq y]$.
(ii) Find the marginal pmf of X and the marginal pdf of Y.
(iii) Suppose we are given that $Y > 0$. Which is more likely, $X = 1$ or $X = -1$?

7. Let X and Y be independent random variables. Find an expression for the probability of the following events in terms of $F_X(x)$ and $F_Y(y)$.
(i) $\{a < X \leq b\} \cap \{Y > d\}$.
(ii) $\{a < X \leq b\} \cap \{c < Y \leq d\}$.
(iii) $\{|X| < a\} \cap \{c \leq Y \leq d\}$.

8. Find $E[|X - Y|]$ if X and Y are independent exponential random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively.

9. Find $E[X^2 e^Y]$ where X and Y are independent random variables, X is a zero-mean, unit-variance Gaussian random variable, and Y is a uniform random variable in the interval $[0, 3]$.

10. Let X and Y be jointly Gaussian random variables with $E[Y] = 0$, $\sigma_1 = 1$, $\sigma_2 = 2$ and $E[X|Y] = Y/4 + 1$. Find the joint pdf of X and Y.

*11. Let X and Y be independent Gaussian random variables that are zero mean and unit variance. Let $W = X^2 + Y^2$ and let $\Theta = \tan^{-1}(Y/X)$. Find the joint pdf of W and Θ.

12. Let X and Y be independent, zero-mean, unit-variance Gaussian random variables. Let $V = aX + bY$ and $W = cX + eY$.
(i) Find the joint pdf of V and W, assuming the transformation matrix A is invertible.
(ii) Suppose$A$ is not invertible. What is the joint pdf of V and W ?

*13. Find the joint cdf of $W = \min(X, Y)$ and $Z = \max(X, Y)$ if X and Y are independent exponential random variables with the same mean μ.

14. Let X and Y be independent Gaussian random variables that are zero mean and unit variance. Find the pdf of $Z = X/Y$.