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CS 121: Tutorial Sheet 2: Linear Transformations & Matrices

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be as defined below. For each of these cases, determine if T is linear. If T is linear then derive the corresponding matrix of the map. Furthermore, in this case, describe its null space and range and compute its nullity and rank.

   (a) $T(x, y) = (y, x)$

   (b) $T(x, y) = (x, x)$

   (c) $T(x, y) = (e^x, e^y)$

   (d) $T(x, y) = (x, 0)$

   (e) $T(x, y) = (x, 1)$

   (f) $T(x, y) = (2x - y, x + y)$

2. Let V be the linear space of polynomials of degree $\leq n$. For $p \in V$, $T(p) = p(x + 1)$ for all $x \in \mathbb{R}$. Is T linear? Is T an operator on V? If T is linear then derive its matrix of the linear map for $n = 2$.

3. Let V be the linear space of polynomials of degree ≤ 2 and let $T : V \to V$ be given by $T(a + bx + cx^2) = (a + 1) + (b + 1)x + (c + 1)x^2$. Is T linear? Is T an operator on V? If T is linear then derive its matrix of the linear map.

4. Let V be the linear space of polynomials of degree ≤ 3 and let $T : V \to V$ be given by $T(p) = \dfrac{\mathrm{d}^2 p}{\mathrm{d}t^2} - 2\dfrac{\mathrm{d}p}{\mathrm{d}t}$. Is T linear? Is T an operator on V? If T is linear then derive its matrix of the linear map.

5. Let V be the linear space of polynomials of degree ≤ 2 and let $T : V \to W$ be given by $T(p) = q$ where q is a polynomial such that $q(x) = \displaystyle\int_0^x p(s)\mathrm{d}s$. Is T linear? If so, for what W? If T is linear then derive its matrix of the linear map.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}$ be such that $T(av) = aT(v)$ for all $a \in \mathbb{R}$ and $v \in \mathbb{R}^2$. Show that, via an example, that T need not be linear.

7. Let $T \in \mathcal{L}(V, W)$. Prove that the null space $\mathcal{N}(T)$ and range $\mathcal{R}(T)$ are subspaces of V and W, respectively.

8. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be such that $T(x) = Ax$ where $A \in \mathbb{R}^{m\times n}$. Show that $\dim(\mathcal{R}(T))$ is the maximum number of independent columns of the matrix A.

9. Let $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ be given by $T(x) = Ax$ where $A \in \mathbb{C}^{m\times n}$. For each of the matrices below, determine the rank and nullity. Repeat the process for A^*, which is the transpose of the complex conjugate of A.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & i & 1+i \\ 2 & 2i & 2(1+i) \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 4i & 5 \\ 2 & 5i & 7 \\ 3 & 6i & 9 \end{bmatrix}$

10. Choose the basis $\{1, x, x^2, x^3\}$ in the linear space V of all real polynomials of degree ≤ 3. Let D denote the differentiation operator on V and let $T : V \to V$ be the linear transformation which maps $p(x)$ onto $xp'(x)$. Relative to the given basis, determine the matrix of the linear transformation TD. Further, let W be the image of V under TD. Find bases for V and W relative to which the matrix of TD is in diagonal form.

11. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ maps the basis vectors as follows:

$$T(1,0) = (1,0,1) \text{ and } T(0,1) = (-1,0,1).$$

(i) Determine the nullity and rank of T.

(ii) Determine the matrix of T.

(iii) Find bases $\{e_1, e_2\}$ for $\mathbb{R}^2$ and $\{w_1, w_2, w_3\}$ for $\mathbb{R}^3$ relative to which the matrix of T will be in diagonal form.

12. Use Gaussian elimination to solve the following systems of linear equations.

$$(i)\quad \begin{aligned} 2y + z &= -8 \\ x - 2y - 3z &= 0 \\ -x + y + 2z &= 3 \end{aligned} \qquad (ii)\quad \begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ x + z &= 4 \\ 2x + y + z &= 8 \end{aligned} \qquad (iii)\quad \begin{aligned} x - 2y - 6z &= 12 \\ 2x + 4y + 12z &= -17 \\ x - 4y - 12z &= 22 \end{aligned}$$

13. Compute inverses of following matrices by Gaussian elimination.

$$(i)\quad \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix} \qquad (ii)\quad \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix} \qquad (iii)\quad \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

14. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

What do you observe? Try to formulate a general principle.

15. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

What do you observe? Try to formulate a general principle.

16. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

What do you observe? Try to formulate a general principle.

17. Find the eigenvalues and eigenvectors of A, B, and $A+B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

Are eigenvalues of $A+B$ equal to sum of eigenvalues of A and B? Can this ever be true?

18. Find the eigenvalues and eigenvectors of A, B, AB, and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Observe any relation between eigenvalues of A, B, AB, and BA.

19. Let $A = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$. Using Caley-Hamilton theorem, express A^2, A^3, A^4, and A^5 in terms of A and I_2. Next, express A as $X\Lambda X^{-1}$ (where Λ is a diagonal matrix with eigenvalues of A. Then derive a general expression for A^n. Compute (if it exists) $\lim_{n\to\infty} A^n$.

20. Let $A \in \mathbb{R}^{3\times 3}$ have eigenvalues $\{0, 1, 2\}$. Determine (wherever possible) *i*) rank of A, *ii*) determinant of $A^{\mathrm{T}}A$, *iii*) eigenvalues of $A^{\mathrm{T}}A$, and *iv*) eigenvalues of $(A^2 + I)^{-1}$.

21. Is there a 2×2 real matrix A (other than I) such that $A^3 = I_2$? Can you state a general principle based on observing this problem?

22. Let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times n}$ where $m \leq n$. For every $\lambda \in \mathbb{F}$, $\det(\lambda I_n - AB) = \lambda^{n-m}\det(\lambda I_m - BA)$.

23. Let $A \in \mathbb{F}^{n\times n}$ be a *rank-one* matrix. Show that there exist $x, y \in \mathbb{F}^{n\times n}$ such that $A = xy^{\mathrm{T}}$. Determine its eigenvalues and eigenvectors (in terms of) x and y.

24. Let Q be a positive-definite matrix. Show that there exists a positive-definite matrix P such that $Q = P^2$. Such a P is called a *square root* of Q denoted by $Q^{\frac{1}{2}}$. Find square roots of *i*) I_2 and *ii*) $\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$. Are they unique?

25. Compute the exponentials of the following matrices:
i) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, *ii*) $\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$, *iii*) $\begin{bmatrix} 5 & 4 \\ -4 & 5 \end{bmatrix}$, and *iv*) $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

26. Let $a, b \in \mathbb{R}$. Compute the exponential of $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

27. Let $A \in \mathbb{F}^{n\times n}$ and $\lambda \in \mathrm{spec}(A)$. Show that $e^{\lambda} \in \mathrm{spec}(e^A)$.