MA 203 (Probability)

Probability and Random processes for Electrical engineering
— Alberto Leon-Garcia.

• Random experiment: experiments which have more than one possible outcomes

Eg 1) Toss a coin and note head or tail.
2) Throw a dice and note the number on its face.
3) Select a ball from an urn containing 10 balls numbered ~~fo~~ from one to ten

• Sample space: collection of all possible outcomes of random exp

1) $S = \{H, T\}$
2) $S = \{1, 2, 3, 4, 5, 6\}$
3) $S = \{1, 2, \cdots 10\}$

• Event: A subset of sample space.

1) $A_1 = \{H\}$, $A_2 = \{T\}$, $A_3 = S = \{H, T\}$, $A_4 = \phi$
↳ certain event ↳ Impossible event

2) Power set of S, $P(S)$ = all subsets of S.

$\# P(S) = 2^6$
↳ no of elements

• Probability :— It is a law or rule for assigning numbers to the events associated with a random exp

$P(\{H\}) = 1/2$ $P(\{T\}) = 1/2$

$P(\{H, T\}) = 1$ $P(\phi) = 0$

⇒ Random exp: Toss a coin till first head appears and note the number of tosses required.

$S = \{1, 2, 3, \dots\}$

⟶ countably infinite sample space.
can make one to one correspondence with the set of natural numbers

⟶ uncountably infinite sample space
cannot make " " " " "

$P(\{1\}) = \frac{1}{2}$ $\quad P(\{j\}) = \left(\frac{1}{2}\right)^j$

$P(\{2\}) = \left(\frac{1}{2}\right)^2$

$S = \{1\} \cup \{2\} \cup \{3\} \dots \cup \{j\} \cup \dots$

$P(S) = 1 = \sum_{j=1}^{\infty} P(\{j\})$

$= \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = \frac{1/2}{1 - 1/2} = 1$ ⟶ infinite geometric series

$P(\phi) = 0$

⟶ uncountably infinite sample space.

* Pick a real number from the interval $(0,1)$

Sample space, $S = (0, 1)$

$P(\{0.5\}) = 0$

$P(\{x\}) = 0, \quad x \in (0, 1)$

$(0,1) \neq \bigcup_{j=1}^{\infty} \{x_j\}$ ⇒ because it is uncountably infinite.

consider $(a,b) \subset (0,1)$ $\quad 0 \leq a < b \leq 1$

$P(a,b) = b - a$

$P(0,1) = 1 - 0 = 1$

$P(\{x\}) = P(x,x) = 0$

→ Uncountably infinite sample space.

R.E. Pick a real number from the interval $(0,1)$

$S = (0,1)$, $[P(\{x\}) = 0 , \quad x \in (0,1)]$

$(a,b) \subset (0,1)$

↳ The ~~number~~ selected number x belongs to (a,b)

$x \in (a,b)$ on $a < x < b$

$P(a,b) = b - a$, $P(0,1) = 1 - 0$

~~$S = \{1, 2, \ldots, 6\}$~~ $\quad P(\{0.5\}) = 0$.

~~$A = 3, 6 \}$~~ $\quad P(0.3, 0.6) = 0.3 \Rightarrow (b - a)$

RE. Pick a number from the interval (a,b)

$S = (a,b)$

$(c,d) \in (a,b)$ $\quad a < c < d < b$

$P(c,d) = \dfrac{d-c}{b-a} \approx$ length of (c,d) wrt (a,b)

$\left\{ P(a,b) = \dfrac{b-a}{b-a} = 1 \right.$

~~[illegible]~~

$S = \{1, 2 \cdots 6\}$ $\#S = 6$

$m(S) = 6$

A = set of even numbers

$= \{2, 4, 6\}$

$$P(A) = \frac{m(A)}{m(S)} = \frac{3}{6} = \frac{1}{2}$$

→ In case of uncountably infinite space the measure is taken by the length.

→ 2D UISP the size concept is area

→ 3D " " " " is volume.

⇒ sample space S or Ω ; ⇒ set of event A or E ; ⇒ Probability law P

$P(S) \to$ powerset of S

⇒ σ-algebra on S → Denote it by $\mathcal{F}$

It is a set of subsets of S satisfying the following axiom

i) $S \in \mathcal{F}$

ii) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

iii) if $A_1, A_2, A_3 \cdots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_i \in \mathcal{F}$

- $S = \{H, T\}$

$\mathcal{F} = P(S) = \{\phi, S, \{H\}, \{T\}\}$

$\mathcal{F} = \{\phi, S\}$

$\mathcal{F} = \{\phi, S, \{H\}\}$ ⊗ 2^{nd} axiom is not satisfied.

(1) **Note**:— If S is a sample space then

$\mathcal{F}_0 = \{\phi, S\}$ is a smallest σ-algebra on S

(2) Power set of S is the largest σ-algebra on S.

R.E. $S = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F}_0 = \{\phi, S\}$, $\# P(S) = 2^6$ elements → largest σ-algebra.

$\mathcal{F}_1 = \{\phi, S, \{1\}, \{2, 3, 4, 5, 6\}\}$ ✓

$\mathcal{F}_2 = \{\phi, S, \{2\}, \{1, 3, 4, 5, 6\}\}$

~~$\mathcal{F} = \{\phi, S, \{1\}, \{2, 3, 4, 5, 6\}\}$~~,

σ-algebra containg $\{1\}, \{2\}$

$\mathcal{F} = \{\phi, S, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{3, 4, 5, 6\}\}$

8/8 **Probability.**

→ Let $P : \mathcal{F} \to [0, 1]$ and satifying the following axioms:

1) $P(S) = 1$, $P(\phi) = 0$

2) If $\{A_i\}_{i=1}^{\infty} \in F$ then $A_i \cap A_j = \phi$, $\forall\, i \neq j$

the $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

mutually exclusive

$P(A \cup B) = P(A) + P(B) + P(A \cap B)$

$A \cap B = \phi$ for mutuall exclu-sive

$P(A \cap B) = \phi$

$\Rightarrow P(A \cup B) = P(A) + P(B)$

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

- Let $\{A_i\}_{i=1}^{\infty}$ be a countable sequence such that

$A_1 = A$, $A_2 = B$, $A_i = \phi$, $\forall$ $i \geq 3$

and $A \cap B = \phi$

$A \cup B \cup \phi \cup \phi \cdots \phi = A \cup B$

---

consequence of the axiom.
we can prove that

P.I. If $A \in F$ then $A^c \in F$ ~~and probability of A^c~~ is

and $P(A^c) = 1 - P(A)$

Let $A_1 = A$, $A_2 = A^c$, $A_i = \phi$, $\forall$ $i \geq 3$

$A_1 \cap A_2 = \phi$

Apply 2nd Axiom.

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$P(\underbrace{A \cup A^c}_{S}) = P(A_1) + P(A_2)$$

$$P(S) = P(A_1) + P(A^c)$$

$$1 = P(A) + P(A^c)$$ by 1 axiom $P(S) = 1$

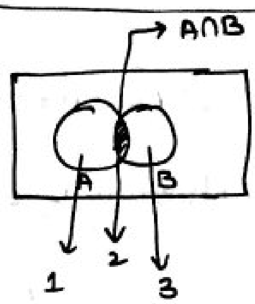
$$\Rightarrow P(A^c) = 1 - P(A)$$

(1) $P(A^c) = 1 - P(A)$, $A \in F$

(2) $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$, $A_i \cap A_j = \phi$ $\forall\ i \neq j$

} inferences

Venn Diagram.



★ for proving.
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$

$= A_1 \cup A_2 \cup A_3$ → all mutually exclusive

$A \setminus B = A \setminus (A \cap B)$

→ $P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$

→ $P(A \setminus B) = P(A) - P(A \cap B)$

→ $P(A \cup B) = P(A) - P(\cancel{A \cap B}) + P(\cancel{A \cap B}) + P(B) - P(A \cap B)$

→ $\underline{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

$A = (A \setminus B) \cup (A \cap B)$

$(A \setminus B) \cap (A \cap B) = \phi$ (disjoint)

→ $P(A) = P(A \setminus B) + P(A \cap B)$

III^{ly} $B = (B \setminus A) \cup (A \cap B)$

→ $P(B) = P(B/A) + P(A \cap B)$

→ computing probability using counting methods.

(1) choose k objects (SAMPLING OF OBJECTS) (only for finite S)
i^{th} object has n_i possible choices
$i = 1, 2 \cdots k$
(1^{st} obj has n_1 possible choices
2^{nd} obj has n_2 possible choices)

number of ways of choosing k objects

→ $n_1 \times n_2 \times \cdots n_k$.

Ex # $\{(x_1, x_2, \cdots x_k) : x_i$ has n_i possible choices $i = 1, 2 \cdots k\}$

$\rightarrow (n_1 n_2 \cdots n_k)$

(ii) Sampling with replacement and with ordering

→ No. of ways of choosing k objects from n distinct objects

(The object after selection is being replaced)

$n_1 = n_2 = \cdots = n.$

$\rightarrow n^k$

(iii) Sampling with out replacement and with ordering

$$= n(n-1)(n-2) \cdots (n-(k-1))$$

$$= \frac{n!}{(n-k)!} = {}^nP_k$$

(iii)

Corrolary Permutation of n distinct objects

$k = n \qquad {}^nP_n = n!$

(vii) Sampling without replacement and without ordering

$${}^nC_k = \frac{{}^nP_k}{k!} = \frac{n!}{k!(n-k)!}$$

1/8 Counting methods

→ Sampling with replacement and with ordering

n^k ways

→ Sampling without replacing and with ordering

$$n(n-1)(n-2) \ldots (n-(k-1)) = \frac{n!}{(n-k)!} = {}^nP_k.$$

→ Sampling without replacing and without ordering.

$${}^nC_k = \frac{n!}{k!(n-k)!} = {}^nC_{n-k}.$$

Binomial expansion.

$$(x_1+x_2)^n = \sum_{k=0}^{n} \binom{n}{k} x_1^k x_2^{n-k}$$

Ex: A fair coin is tossed 100 times. What is the probability of obtaining 20 H

$$P(20H) \Rightarrow P(k=20) \qquad \cancel{100/4\,20} = \underbrace{\binom{100}{20}}_{\text{no of ways in which 20H can occur in 100 tosses}} \underbrace{\left(\frac{1}{2}\right)^{20}\left(\frac{1}{2}\right)^{80}}_{\text{Sample space}}$$

Ex: A fair dice is tossed 10 times. What is prob of obtaining 2 ones, 3 twos, 2 threes, 1 four, 1 five, 1 six

$$\frac{10!}{\lfloor 2 \lfloor 3 \lfloor 2 \lfloor 1 \lfloor 1 \lfloor 1} \underbrace{\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^3\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)}_{\text{sample space}}$$

* Multinomial coefficient

Let n objects is divided into M classes, let x_i denote the num of objects of class i. Then number of ways of doing this partition is

$$\frac{n!}{x_1!\,x_2!\,x_3!\cdots x_m!}\ , \quad x_1+x_2+x_3\cdots+x_M=n$$

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(4) → Sampling with replacement and without ordering

Let x_i be the number of times object i is chosen

$$i=1,2,\ldots n$$

$$x_i \in \{0,1,2\cdots k\}$$

$$x_1+x_2+\cdots x_n=k.$$

$\binom{n+k-1}{k}$ ways → no of solutions $\quad {}^{n+k-1}C_k$.

Ex: $n=4 \quad k=5$

1	2	3	4
x	x	xx	x

$\times \downarrow 5$

$| \rightarrow (n-1)$

$= 4-1$

$= 3$

∴ from n-1 lines and k crosses we have to choose k crosses or n-1 lines

Ex: A batch of 50 items contain 10 defective items.

Suppose 10 items are selected at random. what is the prob. that 5 items are defective

$$\frac{{}^{10}C_5 \; {}^{40}C_5}{{}^{50}C_{10}}$$

40 non-def → 5, 10 def → 5

⊛ Stirling's formula.

$$n! \approx \sqrt{2\pi} \; n^{n+1/2} \; e^{-n} \quad \text{for large } n.$$

⊛ Conditional probability.

Let A and B be two events then conditional probability of A when event B has occured is defined as

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) > 0$$

after vertical line, that event has occured

Ex: Dice throw exp

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{2\}$ $B = \{\text{even number}\}$ $B = \{2, 4, 6\}$

what is the prob of A when B has occured.

$P(A|B) = ?$

$$P(A|B) = \frac{P(\{2\}\cap\{2,4,6\})}{P(\{2,4,6\})} = \frac{1/6}{3/6} = \frac{1}{3}$$

$\cancel{\frac{1}{3}}$ $C = \{3\}$

$P(C|B) = 0$

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* Conditional probability Let A and B are two events

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

lly, $P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$

Independent events : Let A and B be two events then A and B are said to be independent if

$\Rightarrow P(A \cap B) = P(A) \, P(B)$

Mutually exclusive events $A \cap B = \phi$

$P(A \cap B) = 0$

Toss a coin two time

$S = \{HH, HT, TH, TT\}$

$A = \{HH\}$ $B = \{TT\}$

$P(A) = \frac{1}{4}$, $P(B) = \frac{1}{4}$, $P(A \cap B) = 0$

$P(A \cap B) \neq P(A) P(B)$

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$\rightarrow$ $P(A|B) = P(A|B)\ P(B)$,

$\rightarrow$ $P(A \cap B) = P(B|A)\ P(A)$

If A and B are independent events

$$P(A)\ P(B) = P(A|B)\ P(B)$$

$$\Longrightarrow P(A) = P(A|B), \quad P(B) > 0$$

~~P(B)~~

$$P(A)\ P(B) = P(B|A)\ P(A)$$

$$\Longrightarrow P(B) = P(B|A), \quad P(A) > 0$$

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* <u>Independent of more than two event</u>

Let $A_1, A_2, \ldots A_n$ be n events

$\rightarrow$ If $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\ P(A_2) \cdots P(A_n)$

$\rightarrow$ then $A_1\ A_2\ A_3 \ldots A_n$ are said to be independent events.

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* <u>Independence of countable sequence of event</u>

The countable sequence $\{A_n\}_{n=1}^{\infty}$ is said to be independent if any finite sub-sequence of events ~~is~~ is independents

* <u>Partition S</u>: Let $\{B_n\}_{n=1}^{\infty}$ be a countable sequence of mutually exclusive events such that $S = \bigcup_{n=1}^{\infty} B_n$

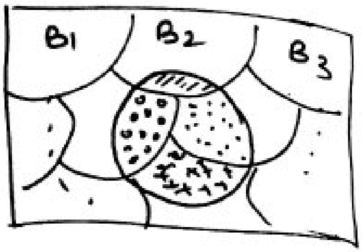
Then $\{B_n\}_{n=1}^{\infty}$ is called a partition of S

Law of Total probability:

Let S be the sample space of a random experiment and $\{B_n\}_{n=1}^{\infty}$ be a partition of S. Let A be any event, then

$P(\cancel{A})$

$$A = \bigcup_{n=1}^{\infty} (A \cap B_n)$$



$$P(A) = \sum_{n=1}^{\infty} P(A \cap B_n)$$

$$= \sum_{n=1}^{\infty} P(A|B_n)\, P(B_n)$$

(weighted avg of conditional probability of A)

Note:- $P(A)$ is a weighted avg of all conditional probability of A in the partition $\{B_n\}$

Bayes' theorem:-

Let $\{B_n\}_{n=1}^{\infty}$ be a partition of sample space S. Let $\{P(B_n)\}$ be an apriori estimate on the probability of $\{B_n\}$. On the occurrence of an event A, we may obtain a posteriori estimate of prob. of $\{B$

$$P(B_n|A) = \frac{P(B_n \cap A)}{P(A)} = \frac{P(A|B_n)\, P(B_n)}{\sum_{n=1}^{\infty} P(A|B_n)\, P(B_n)}$$

$$= \frac{P(A|B_n)\, P(B_n)}{P(A)}$$

* Random variable :- (cont)

3 coin toss : $S = \{HHH, HHT, \ldots TTT\}$

$X : S \longrightarrow R$

Associating numerical value to events to perform diff mathematical op.

$S\{\omega\}$

$X(HHH) = 3$ $X(HHT) = 2$

16/8 **Probability space:**

Let S be the sample space, F a σ-algebra on S and P be probability measure on F. Then the triplet (S, F, P) is called probability space.

Borel σ-algebra:

This is the σ-algebra on $\mathbb{R}$ generated by all closed intervals on $\mathbb{R}$

$$G = \{[a,b] \in \mathbb{R} : a < b\}$$

$$\boxed{B(\mathbb{R}) = \{[a,b], (c,d), (-\infty,a), (b,\infty), (-\infty,a], [b,\infty), \bigcup_{i=1}^{\infty}[a_i, b_i], \bigcup_{i=1}^{\infty}(c_i, d_i), \bigcap_{i=1}^{\infty}[a_i b_i], \bigcap_{i=1}^{\infty}(c,d)\}}$$

If $[1,2] \in G$

$[1,2]^c = (-\infty, -1) \cup (2, \infty)$

$\rightarrow$ $A \subset S$, $B \subset S$

$$\boxed{(A^c \cup B^c)^c = A \cap B}$$

$$\boxed{(A \cup B)^c = A^c \cap B^c}$$

$$F_{\{A\}} = \{\phi, S, A, A^c\}$$

$$F_{\{A,B\}} = \{\phi, S, A, B, A^c, B^c, A \cup B, A \cap B, A^c \cup B^c\}$$

ex

* Borel σ-algebra on $\mathbb{R}$ is denoted by $B(\mathbb{R})$

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func mapping from some set to real no.

Random variable: A real valued function X on S is said to be a random variable if for every Borel set $B \in B(\mathbb{R})$

$$X^{-1}(B) = \{X \in B\} = \{w \in S : X(w) \in B\} \in F$$

$X : S \rightarrow \mathbb{R}$, $F = P(S)$

$X : (S, F) \rightarrow (\mathbb{R}, B(\mathbb{R}))$

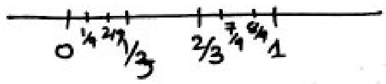
<u>Notation:</u> $X : (S, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

<u>Borel set examples</u> :-

- $\mathcal{B}(\mathbb{R}) = \{ [a,b], (c,d), (-\infty, a), (b, \infty),$
$(-\infty, a], [b, \infty), \bigcup_{i=1}^{\infty} [a_i, b_i], \bigcup_{i=1}^{\infty} (c_i, d_i)$
$\bigcap_{i=1}^{\infty} [a_i, b_i], \bigcap_{i=1}^{\infty} (c_i, d_i) \}$

- $[1,2], (3,4), [1,2] \cup (3,4)$

- $A_0 = [0,1]$



$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$A_2 = [0, 1/9] \cup [2/9, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{9}{9}]$

$\vdots$

$\lim_{n \to \infty} |A_n| = 0.$ (Length is 0)

(closed intervals but they do not have length)

Eg: $S = \{HH, HT, TH, TT\}$

$\mathcal{F} = P(S)$

$X : (S, P(S)) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$X(\omega)$ = no of H in ω

$X(HH) = 2$; $X(HT) = X(TH) = 1$; $X(TT) = 0$.

$\rightarrow B = [3,4]$, $X^{-1}([3,4]) = \{\omega \in S.\ X(\omega) \in [3,4]\} = \phi \in P\{S\}$

$\rightarrow B = [0,3]$, $X^{-1}([0,3]) = S \in P(S)$

$B = [0,1] \cdot \quad X^{-1}([0,1]) = \{TT, HT, TH\} \in P(S)$

If $X : (S, F_0) \longrightarrow (\mathbb{R}, B(\mathbb{R}))$, $F_0 = (\phi, S)$

$$Y^{-1}([0,1]) = \{TT, HT, TH\} \notin F_0$$

<u>Y is not a random variable.</u>

*

Let : $F_{\{H,H\}} = \{\phi, S, \{HH\}, \{\cancel{HH,T}\ HT, TT, TH\}\}$

$\rightarrow Z : (S, F_{\{H,H\}}) \longrightarrow (\mathbb{R}, B(\mathbb{R}))$

$\cancel{Z^{-1}([1,2]) =}$

$$Z^{-1}([1, 1.5]) = \{HT, TH\} \notin F_{\{H,H\}}$$

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* Distribution of random variable :

~~Let $X : \overbrace{(S, F, P)}^{\text{probability space}} \longrightarrow (B, \mathbb{R}, \mu_x)$ be a random variable defined on $(S, F, P$~~

prob. of S

prob. measure

Let $X : \overbrace{(S, F, P)}^{\text{prob space}} \longrightarrow (\mathbb{R}, B(\mathbb{R}), \mu_x)$ be a random variable defined on

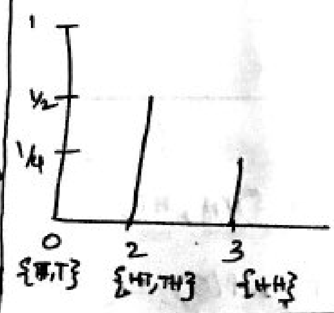
prob. of S

prob. distribution

(S, F, P)

If $X = [0, 1, 2]$

for prev prob



$\cancel{\mu_x(x)}$

Here μ_x is the probability distribution ~~of~~ of X and is defined

by $\mu_x(B) = P(X \in B) = P(X^{-1}(B))$

$B \in B(\mathbb{R})$

20/8 **Probability Distribution of a random variable.**

$$X : (S, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_x)$$

$$\mu_x(B) = P(X^{-1}(B)) \ , \forall \ B \in \mathcal{B}(\mathbb{R})$$

$$X^{-1}(B) = \{\omega \in S \ . \ X(\omega) \in B\} \in \mathcal{F}$$

EX: A coin is tossed 3 times and sequence of H and T is noted

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTT, TTH\}$$

$$X : (S, P(S), P) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_x)$$

$X(\omega) =$ no of H in ω

Range of $X = \{0, 1, 2, 3\}$

$\rightarrow$ $B = (-1, 0)$, $P(X^{-1}((-1,0))) = P(\phi) = 0$

· $B_1 = [0, 3]$ · $B_2 = [0, 4]$ · $B_3 = [0, \infty]$ · $B_4 = [0, 2]$

$\rightarrow$ $\mu_x(B_1) = P(X^{-1}(B_1)) = 1$.

lllly $P(X^{-1}(B_2)) = 1$

$P(X^{-1}(B_3)) = 1$

$\rightarrow$ $\mu_x(B_4) = P(X^{-1}(B_4))$

$= \cancel{P(\{}$ $= P(S \mid \{HHH\})$

$= \frac{7}{8}$

Cumulative Distribution function (cdf)

The cdf of a random variable X is defined as

any real no.

$$F_X(x) = P(X \leq x), \quad x \in \mathbb{R}$$

$$\{X \leq x\} = \{\omega \in S : X(\omega) \leq x\}$$

$$= X^{-1}((-\infty, x])$$

↳ Belongs to $B(\mathbb{R})$

* for the prev example

$$\rightarrow F_X(1) = P(-\infty < X \leq 1) = P(X^{-1}(-\infty, 1])$$

$$= P(X=0) + P(X=1)$$

$$= P(TTT) + P(HTT, THT, TTH)$$

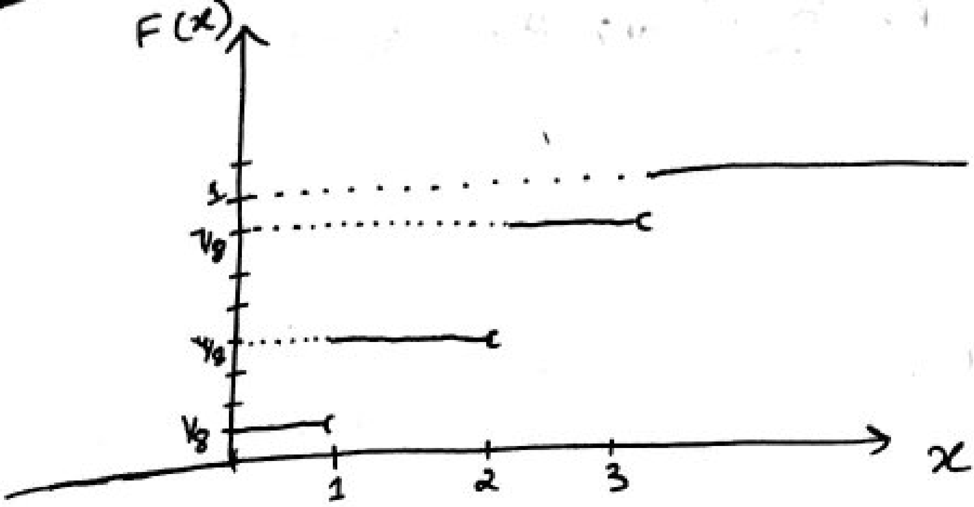
$$= \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$$

$$\rightarrow F_X(2) = P(-\infty < X \leq 2) = P(X^{-1}(-\infty, 2])$$

$$= P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

$\rightarrow F_X(x) = 1, \; x \geq 3 \quad ; \quad F_X(-1) = 0$



Jump ~~cont~~ discontinuous

properties

i) F_X is non negative func

ii) F_X is non decreasing

iii) $F_X(-\infty) = 0$

iv) $F_X(\infty) = 1$

v) F_X is right continuous , i.e $F_X(x) = F_X(x+0)$

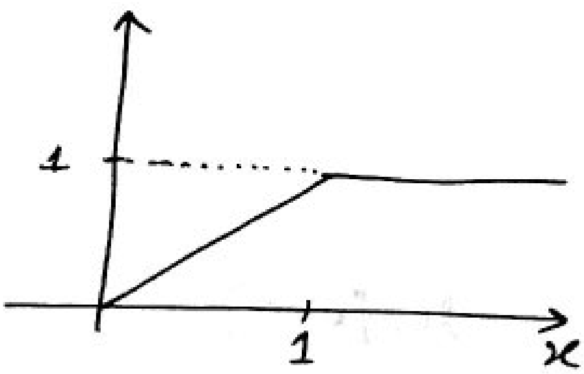
Ex: $S = [0, 1]$

$$X : (S, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$$

$$X(\omega) = \omega \;,\; \omega \in S$$

$$F_X(-1) = P(X \leq -1) = 0$$

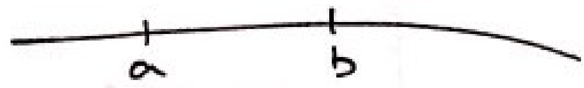
$$F_X(1/2) = \frac{1}{2} \Rightarrow \frac{1/2 - 0}{1 - 0} = 1/2$$



~~dis~~continuous

Let us consider the Borel set $B = (a, b]$

$\mu_X((a,b])$



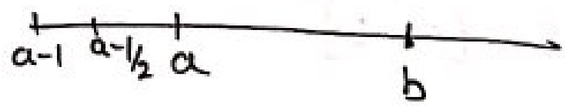
$(a,b] = (-\infty, b] - (-\infty, a]$

$\rightarrow \mu_X((a,b]) = P(X^{-1}(a,b])$

$= P(X^{-1}(-\infty,b]) - P(X^{-1}(-\infty,a])$

$= F_X(b) - F_X(a)$

$\mu_X[a,b] = F_X(b) - F_X(a-0)$



$[a,b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b]$

$\hookrightarrow$ (It will shrink to $[a,b]$)

$[a,b] \subset (a-\frac{1}{n}, b]$

$\forall n = 1, 2, 3 \ldots$

$\mu_X[a,b]$

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MA 203

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$X : (S, F, P) \longrightarrow (\mathbb{R}, B(\mathbb{R}), \mu_x)$

$\rightarrow$ prob. distribution $\mu_X(B) = P(X^{-1}(B))$, $B \in B(\mathbb{R})$

$\rightarrow$ cdf $F_X(x) = P(X \leq x)$, $x \in \mathbb{R}$

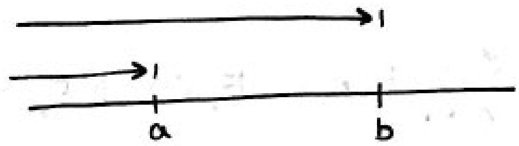
$$= \mu_X((-\infty, x])$$

$$\mu_X((a,b]) = P(a < X \leq b)$$

$$= P(-\infty < X \leq b) - P(-\infty < X \leq a)$$

$$\boxed{X^{-1}(B) = \{X \in B\} = \{\omega \in S : X(\omega) \in B\}}$$

$\rightarrow \mu_X((a,b]) = F_X(b) - F_X(a)$



$\rightarrow \mu_X([a,b]) =$

$$\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right]$$

$$= \lim_{m \to \infty} \left[\bigcap_{n=1}^{m} \left(a - \frac{1}{n}, b\right] \right]$$

$$= \lim_{m \to \infty} A_m$$

$$= \lim_{m \to \infty} \left(a - \frac{1}{m}, b\right]$$

$$= [a,b]$$

$$[a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right]$$

$$A_n = \left(a - \frac{1}{n}, b\right] , n \in \mathbb{N}$$

$$[a,b] \subset A_n \quad \forall n \in \mathbb{N}$$

$$A_{n+1} \subset A_n \quad \forall n \in \mathbb{N}$$

(monotone seq of intervals)

$$\therefore \quad \mu_X([a,b]) = \mu_X\left(\bigcap_{n=1}^{\infty}\left(a-\tfrac{1}{n}, b\right]\right)$$

$$= \lim_{n\to\infty} \mu_X\left(a-\tfrac{1}{n}, b\right]$$

$$= \lim_{n\to\infty} \left[F_X(b) - F_X\left(a-\tfrac{1}{n}\right)\right]$$

$$= F_X(b) - F_X(a-0)$$

↳ (left hand lim)

$\lim\limits_{x\to a} F_X(x) = F_X(a) \Rightarrow F_X$ is cont at $x = a$

$F_X(a+0) = F_X(a)$

(right hand continuity)

$\lim\limits_{x\to a+0} F_X(x) = F_X(a)$ $\qquad$ $\lim\limits_{x\to a-0} F_X(x) \neq F_X(a)$ $\quad$ } right cont

$F_X(a-0) = F_X(a)$

(left hand continuity)

$$\rightarrow \quad \mu_X(\{a\}) = \mu_X\left(\bigcap_{n=1}^{\infty}\left(a-\tfrac{1}{n}, a\right)\right)$$

$$= F_X(a) - F_X(a-0)$$

* Discrete Random variable -

A random variable X is called discrete if the range of X is finite or countable

---

Eg. A coin is tossed 3 times and the sequence of H and T is noted $S = \{HHH, HHT, HTT, HTH, THH, TTH, THT, TTT\}$

$X(\omega) =$ no of H in ω

Range of X, $S_X = \{0, 1, 2, 3\}$

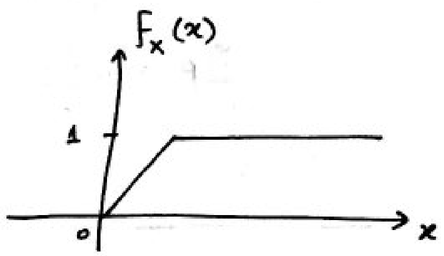
---

* Continuous Random variable.

A random variable is said to be continuous if the cdf of X is continuous.

Eg. $X(\omega) = \omega, \quad \omega \in [0, 1]$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0, 1) \\ 1, & x \geq 1 \end{cases}$$



→ Probability mass function (pmf)

Let X be a discrete r.v

$$X : S \longrightarrow \mathbb{R}$$

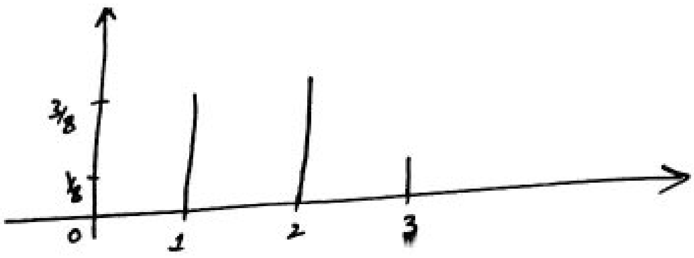
Then $S_X = \{x_1, x_2, \ldots, x_n\}$

$p_1 = \cancel{P(X)} \; P(X = x_1) = P(X \leq x_1) = P(\{\omega \in S : X(\omega) = x_1\})$

pmf of X is $\{p_K = P(X = x_K) : K = 1, 2 \cdots n\}$

for prev. prob Pmf of X : $\{P(X=0), P(\ldots$

$$= \left\{ \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8} \right\}$$



$\rightarrow$ Uniform discrete random variable.

All the outcomes are equally likely and $P(X = x_k) = \frac{1}{N}$,

where $N = \# S_X$

If a sample is countably infinite it cannot have uniform distribution

$$\sum_{n=1}^{\infty} p = p \sum_{n=1}^{\infty} 1 \neq 1$$

$$P(X = x_k = k) = \left(\frac{1}{2}\right)^k \quad \forall \ k \in \mathbb{N}$$

$$S = \mathbb{N} \ , \quad X : S \rightarrow \mathbb{R}$$

$X(1) = 1$ $\qquad$ $X(\omega) = \omega$

$X(k) = k$

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1/2}{1/2 - 1} = 1$$

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23/8 Discrete R.V X range of X is S_X

pmf of X $\{P_k = P(X = x_k) : x_k \in S_X\}$

i) $P_k \geq 0 \quad \forall k$

ii) $\sum_k P_k = 1$

iii) $P(X \in B) = \sum_{x_k \in B} P_k$, $B \in \mathcal{B}(\mathbb{R})$

cdf of X $\quad F_X(x) = P(X \leq x) = \sum_{\{x_k \leq x\}} P_k$

$$= \sum_{x_k \in S_X} P_k \, u(x - x_k)$$

Where $u(x)$ is the unit ~~s~~ step function

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Eg: Let X denote the number of heads in 3 coin tosses

Find the cdf of X

$S_X = \{0, 1, 2, 3\}$ $\quad \{P_k\} = \left\{\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right\}$

$$F_X(x) = \frac{1}{8} u(x) + \frac{3}{8} u(x-1) + \frac{3}{8} u(x-2) + \frac{1}{8} u(x-3)$$

$F_X(2.5) = P_1 + P_2 + P_3$

$= P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{8} u(2.5) + \frac{3}{8} u(2.5-1) + \frac{3}{8} u(2.5-2) + \boxed{\frac{1}{8} u(2.5-3)}$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + 0$

$\Rightarrow P(X = x_k) = F_X(x_k) - F_X(x_k - 0)$

Special types of discrete R.V

(1) Uniform ~~r.v~~ discrete r.v : $X(N)$ ↱ no of possible outcomes / Range of x

$$S_x = \{1, 2, \ldots N\}$$

$$pmf = \{P_k = \frac{1}{N} ; k = 1, 2 \ldots N\}$$

(2) Bernoulli r.v :

$$X(p)$$

P = prob of success

$$S_x = \{0, 1\} , \quad pmf \ \{p, 1-p\}$$

(3) Binomial r.v :

$$X(N, p)$$

It is a sequence of N (independent) Bernoulli r.v with parameter p

$$pmf : \left\{ P_k = \binom{N}{k} p^k (1-p)^{N-k} : k = 0, 1, \ldots, N \right\}$$

$$S_x = \{0, 1, 2, \ldots N\}$$

(4) Geometric r.v : $X(p)$

Let x denote the number of tosses required to get first Head

$$S_x = \{1, 2, 3 \ldots\} = \mathbb{N}$$

$$pmf = \{ P_k = (1-p)^{k-1} p \ . \ k = 1, 2 \cdots \}$$

first k-1 are failures and the last is success

$$\sum_{k=1}^{\infty} p_k = 1$$

→ Continuous R. V.

X is said to be a continuous random variable if cdf of X is continu- -ous everywhere and there exist a function $f_X(x)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad , \quad x \in \mathbb{R}$$

$f_X(x)$ is called probability density function of X

//-------

$$P(x < X \le x+h) = F_X(x+h) - F_X(x)$$

$$= \left(\frac{F_X(x+h) - F_X(x)}{h}\right) h$$

If we derivate it

$$\approx f_X(x)h \qquad (\text{from})$$

$$\therefore \quad f_X(x) \approx \frac{P(x < X \le x+h)}{h} \approx f_X(x)h$$

Note 1) $\frac{d}{dx} F_X(x) = f_X(x)$

→ Properties of probability density function (pdf)

i) $f_X(x) \ge 0 \qquad x \in \mathbb{R}$

ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

iii) $P(X \in [a,b]) = F_X(b) - F_X(a-0)$

$$= F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx$$

Eg: Let $g(x) = \begin{cases} c e^{-\alpha x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$, $c > 0$

$\alpha > 0$ is constant

find c such that $g(x)$ is pdf of some r.v.

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{0} 0 + \int_{0}^{\infty} c e^{-\alpha x}\,dx = \frac{c}{-\alpha} e^{-\alpha x}\Big|_0^\infty$$

$$= \frac{c}{\alpha}$$

1 (2nd property)

for this to be pdf $\frac{c}{\alpha} = 1$

$$\Rightarrow c = \alpha$$

cdf $F_X(x) = \int_{-\infty}^{x} g(t)\,dt = \begin{cases} 0, & x < 0 \\ \int_0^x \alpha e^{-\alpha t}\,dt, & x \geq 0 \end{cases}$

<u>Special type of continuous R.V</u>

1) Uniform r.v

$X(\omega) = \omega$, $\omega \in [0,1]$

cdf $F_X(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x \geq 1 \end{cases}$

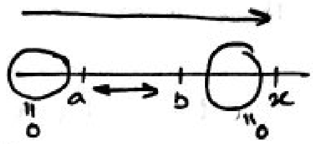
$f_X(x) = \begin{cases} 0, & x < 0 \\ 1, & x \in [0,1] \\ 0, & x \geq 1 \end{cases}$

1) Uniform random variable

Let $S = [0,1]$ then

cdf $F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$

pdf $f_X(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x \le 1 \\ 0, & x \ge 1 \end{cases}$

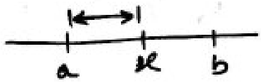


$\{X \le x\} = \{X \le a\} \cup \{a < X \le$ $\cup \{b < X \le x\}$

Sample space, $S = [a,b]$

$X : [a,b] \longrightarrow \mathbb{R}$

cdf $F_X(x) = P(X \le x)$, $x \in \mathbb{R}$



For $x < a$, $F_X(x) = 0$

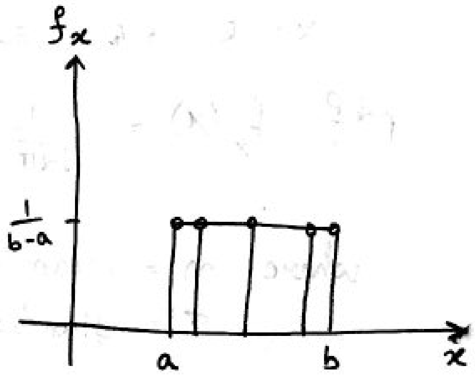
For $x \in [a,b]$, $\{X \le x\} = \{X < a\} \cup \{a \le X \le x\}\ \cancel{\cup\{b < X \le x\}}$

$$F_X(x) = P(a < X \le x) = \frac{x-a}{b-a}$$

For $x \ge b = 1$

$F_X(x) = \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a < x \le b \\ 1, & x > b \end{cases}$

pdf $f_X(x) = \begin{cases} 0, & x \le a \\ \frac{1}{b-a}, & a < x \le b \\ 0, & x > b \end{cases}$



$x_0 \in [a,b]$

$$P\{X = x_0\} = F_X(x_0) - F_X(x_0 - 0)$$

$$= \lim_{h \to 0} P(x_0 < X \le x_0 + h)$$

$$= \lim_{h \to 0} \int_{x_0}^{x_0+h} f_X(x)\,dx = 0$$

2) Exponential Random Variable :

$S = [0, \infty)$, $X : [0, \infty) \longrightarrow \mathbb{R}$

$P(\{X > x\}) = e^{-\lambda x}$, $x > 0$

Here $\lambda > 0$ is a constant parameter of X

cdf $F_X(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$

$x \in \mathbb{R}$

$x = -0.5$, $F_X(-0.5) = 0$

$x = 0.5$, $F_X(0.5) = 1 - e^{-0.5\lambda}$

pdf $f_X(x) = \begin{cases} 0 & , x < 0 \\ \lambda e^{-\lambda x} & , x \geq 0 \end{cases}$

3) Normal random variable or Gaussian R.V

$X : \mathbb{R} \longrightarrow \mathbb{R}$, sample space $= \mathbb{R}$

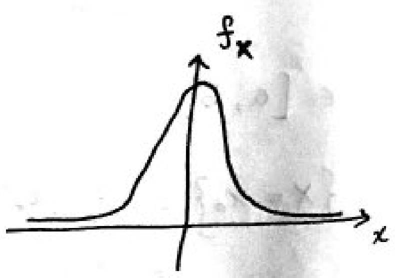
pdf $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2}$, $x \in \mathbb{R}$

where m = mean of $X > 0$

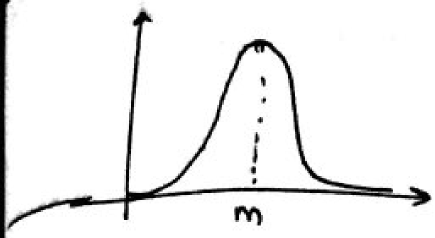
σ = standard deviation of $X > 0$

cdf : $F_X(x) = P(X \leq x)$

$= \int_{-\infty}^{x} f_X(t)\,dt$



$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-(t-m)^2/2\sigma^2}\,dt$



→ **conditional cdf**

Let c be an event which has occured

Then,

$$F_X(x|c) = P(X \le x \mid c) \quad , \quad P(X \in c) > 0$$

$$= \cancel{\frac{P(\{X \le x\} \cap \{X \in c\})}{P(X \in c)}} \qquad \frac{P(\{X \le x\} \cap \{X \in c\})}{P(X \in c)}$$

Let $\{B_i\}_{i=1}^{\infty}$ be a partition of sample space S

Then $F_X(x) = \sum_{i=1}^{\infty} F_X(x|B_i)\, P(X \in B_i)$

→ **conditional pdf**

$$f_X(x|c) = \frac{d}{dx} F_X(x|c) \quad , \quad P(X \in c) > 0$$

→ **conditional pmf**

$$P_X(x_i|c) = \frac{P_X(\{x_i\} \cap c)}{P_X(X \in c)} \quad , \quad P(X \in c) > 0$$

Expectation (Mean) $X : (S, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_x)$

$$E(X) = \int_S X(\omega)\, dP(\omega) \quad , \quad \omega \in S$$

$$= \sum_{x_i \in S} x_i P_X (X = x_i) \quad \text{--- Discrete}$$

$$= \int_R x f_X(x)\, dx \quad \text{--- Continuous}$$

cont . . .

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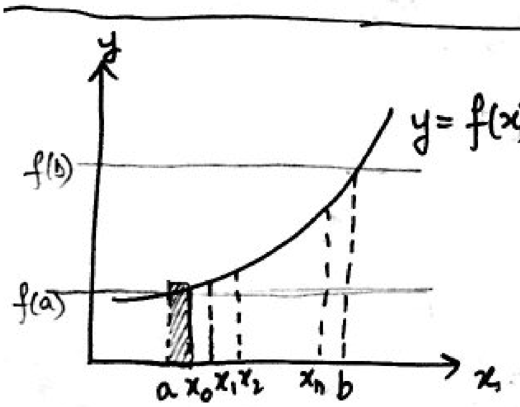
Expectation: Let $X : S \to R$ be a random variable. Then expectation of X is

$$E(X) = \int_S X(\omega)\, dP(\omega)$$

finite decimal interval. $f_X(x)\,dx = P(X \in (x, x+dx))$

$$= \int_S x\, f_X(x)\, dx \quad , \text{ if } X \text{ is continuous r.v}$$

$$= \sum_{x_i \in S} x_i\, P_X(X = x_i) \quad , \text{ if } X \text{ is discrete r.v}$$



$$\int_a^b f(x)\, dx \qquad f : [a,b] \to R$$

$$a = x_0 < x_1 < \cdots < x_n = b$$

Lower Riemann sum

$$LR = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$$

$$UR = \sum_{i=0}^{n-1} f(x_{i+1})(x_{i+1} - x_i)$$

If LR = UR the integral is defined.

If we partition the range we take the corresponding interval in the domain the sub intervals are borel sets

Lebesgue integration.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases} \quad , x \in [0,1]$$

$[0,1]$: $x_0 = 0$, $x_1 = x_0 + \underbrace{0.01}_{= h}$, ... $x_i = x_0 + ih$

$x_{100} = 1$

LR = 0 ; UR = 0 , $\int_0^1 f(x)dx$ does not exist

→ $S = [0,1] = A \cup B$

$A = \{x \in S : x \text{ is rational}\}$

$B = \{x \in S : x \text{ is irrational}\}$

$A \cap B = \phi$, $P(S) = P(A) + P(B) = 1$

$P(A) = \sum_{i=1}^{\infty} P(\{x_i\}) = 0$; $P(B) = 1$

prob of finding singleton is 0

rationals are countable
irrationals are uncountable

Range of $f = \{0, 1\} = y_0 = 0 < y_1 = 1$

$f^{-1}(\{y_0 = 0\}) = A$, $f^{-1}(\{y_1 = R\}) = B$

$$\int_0^1 f(x)dx = y_0 \times \text{length}(A) + y_1 \times \text{length}(B)$$
$$= 1$$

Range of $f = \{y_0, y_1, \ldots\ldots y_n\}$

$$\int_S f(x)dx = \sum_i y_i \times f^{-1}(\{y_i\})$$

Range of $f = [m_f, M_f]$

$y_0 = m_f < y_1 < \cdots\cdots < y_n = M_f$

lower Lebesque sum $= \sum_{i=0}^{n-1} y_i \, l\left(f^{-1}([y_i, y_{i+1}))\right)$, $n \longrightarrow \infty$

$f_X(x)\,dx$

$= P(X \in (x, x+dx])$

$\cancel{f = f^+ - f^- = \max(f, 0)}$

$\cancel{f(x) = x \ , \ x \in [-1, 1]}$

$f = f^+ - f^-$, $f^+ = \max(f, 0)$

$f^{-1} = \max(-f, 0)$

$f(x) = x \ , \ x \in [-1, 1]$

$$f^+(x) = \begin{cases} x, & x \in [0,1] \\ 0, & x \in [-1, 0] \end{cases} \quad , \quad f^-(x) = \begin{cases} 0, & x \in [0,1] \\ -x, & x \in [-1, 0] \end{cases}$$

$$\int_a^b f(x)\,dx = \int_a^b f^+(x)\,dx - \int_a^b f^-(x)\,dx$$

if $\int_a^b f^+(x)\,dx = \infty$, $\int_a^b f^-(x)\,dx < \infty$

$\Rightarrow \int_a^b f(x)\,dx = \infty$

If $\int_a^b f^+(x)\,dx = \infty$ and $\int_a^b f^-(x)\,dx = \infty$

then $\int_a^b f(x)\,dx$ is not defined

$\int_a^b |f|\,dx < \infty$

$|f| = f^+ + f^-$

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Expectation : $E(X) = \int_S X(\omega)\, dP(\omega)$

$$= \begin{cases} \int_{\mathbb{R}} x f_X(x)\,dx, & X \text{ is cont} \\ \sum_{x_i \in S_X} x_i P_X(X = x_i); & X \text{ is discont} \end{cases}$$

Variance $\quad Var(x) = E(X - E(x))^2$

i) Uniform discrete random variable

Range of X, $S_X = \{1, 2, \dots N\}$

$P_X(X = i) = \frac{1}{N}$, $x_i = i$; $\quad i = 1, 2, \dots N$

$$E(x) = \sum_{i=1}^{N} i\,\frac{1}{N} = \frac{1}{N}\,\frac{(N)(N+1)}{2} = \frac{N+1}{2}$$

$S_X = \{0, 1, 2 \dots N\}$

$$E(x) = \sum_{i=0}^{n} i\,P(X = x_i) = \sum_{i=0}^{N} i\,\frac{1}{N+1} = \frac{1}{N+1}\,\frac{(N)(N+1)}{2} = \frac{N}{2}$$

ii) Bernoulli r.v $\quad S_X = \{0, 1\}$

$P(X = 1) = P$, $P(X = 0) = 1 - P$

$E(x) = P$

iii) Geometric r.v

$$S_X = \{1, 2, \dots\} \; ; \quad P(X=i) = (1-p)^{i-1} p$$

$$i = 1, 2, 3 \dots$$

$$E(X) = \sum_{i=1}^{\infty} i (1-p)^{i-1} p$$

$$= p(1-(1-p))^{-2}$$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$(1-x)^{-1} = \sum_{i=0}^{\infty} x^i \quad , \; |x| < 1$$

$$(1-x)^{-2} = \sum_{i=1}^{\infty} i \, x^{i-1}$$

iv) Poisson r.v

$$S_X = \{0, 1, 2 \dots\}$$

poisson's const

$$P(X=i) = \frac{\alpha^i}{i!} e^{-\alpha} \quad , \; \alpha > 0 \quad , \; i = 0, 1, \dots$$

to check if it is pmf

$$\sum_{i=0}^{\infty} P(X=i) = e^{-\alpha} \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} = e^{-\alpha} e^{\alpha} = 1$$

$\sum_{i=0}^{\infty} P(X=i)$ should be $= 1$

find.

$$\rightarrow \quad E(X) = \sum_{i=0}^{\infty} i \, e^{-\alpha} \frac{\alpha^i}{i!} = \alpha$$

Uniform continuous r.v : $S = [a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} \, , & x \in [a, b] \\ 0 \, , & x \notin [a, b] \end{cases}$$

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$$E(x) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

Exponential r.v :

$$S = \mathbb{R}$$

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x > 0 \end{cases}$$

as $x \to \infty$ this will become 0

$$E(x) = \int_0^{\infty} x \lambda e^{-\lambda x}\, dx = \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-\lambda x}}{\lambda} dx \right]$$

$$= \frac{1}{\lambda}$$

Gaussian r.v : $\quad S = \mathbb{R}$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}, \quad x \in \mathbb{R}$$

$$E(x) = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{2\pi}}$$

Let $y = \frac{x-m}{\sqrt{2\sigma^2}}$, $dy = \frac{dx}{\sqrt{2\sigma^2}}$

$$\Rightarrow E(x) = \int_{-\infty}^{\infty} \frac{x-m}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx + \int_{-\infty}^{\infty} \frac{m}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx$$

$= 0$ $\quad \int_{-a}^{a} f(x) = \begin{cases} 0, & \text{odd } f^n \\ 2\int_0^a f(x), & \text{even} \end{cases}$

$$\Downarrow$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx = 1$$

$$= m \, ;$$

Expectation is a linear function

$$E(aX+bY) = aE(X) + bE(Y) \quad , \text{ for}$$

any $a, b \in \mathbb{R}$ any two r.v X and Y

Integral is linear operator

we will use this to get simpler eq^n for variance

$$\mathrm{Var}(X) = E\left(X^2 - 2E(X)X + [E(X)]^2\right)$$

$$= E(X^2) - 2[E(X)]^2 + [E(X)]^2$$

$$= E(X^2) - [E(X)]^2$$

↳ mean.

→ **Independent Random Variable.**

$X_1 : (S, \mathcal{F}, P_1) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_{X_1})$

$X_2 : (S, \mathcal{F}, P_2) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_{X_2})$

$P(A \cap B \cap C) = P(A)P(B)P(C)$

$P(A \cap B) = P(A)P(B)$

$P(B \cap C) = P(B)P(C)$

$P(C \cap A) = P(C)P(A)$

$\sigma(X_1) = \sigma\left[\{X_1^{-1}([a,b]) : [a,b] \in \mathbb{R}\}\right]$

$\sigma(X_2) =$

X_1 and X_2 are said to be independent

if $P(A \cap B) = P(A)P(B)$, $\forall A \in \sigma(X_1)$ and $B \in \sigma(X_2)$

$\rightarrow p_1 = P(X=1)$

$\rightarrow p_2 = \frac{p_1}{2} = P(X=2)$

$\rightarrow p_3 = \frac{p_1}{3} = P(X=3)$

$\rightarrow p_4 = \frac{p_1}{4} = P(X=4)$

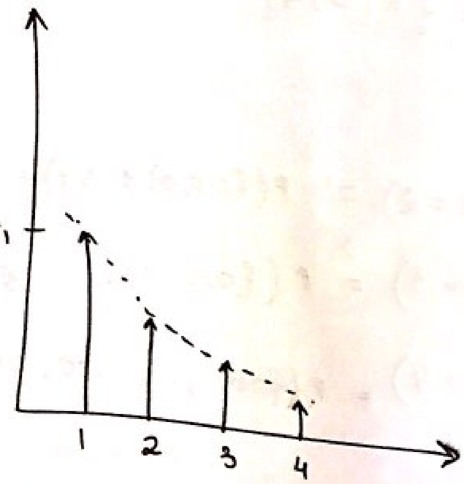
$$\Rightarrow \sum_{k=1}^{4} p_k = 1$$

$$\Rightarrow p_1\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = 1$$

$$\Rightarrow p_1\left(\frac{24+12+8+6}{24}\right) = 1$$

$$\Rightarrow p_1 = \frac{24}{50}$$

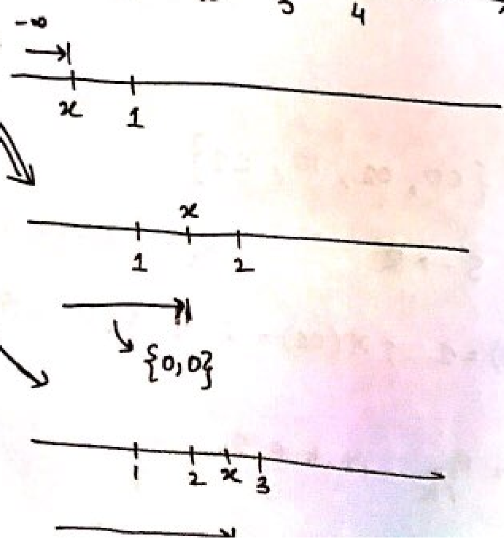
pmf: $\left\{p_1, \frac{p_1}{2}, \frac{p_1}{3}, \frac{p_1}{4}\right\}$



cdf:

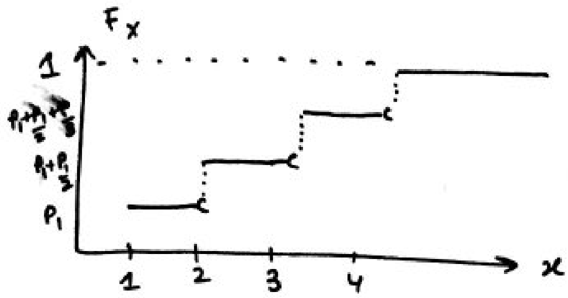
$$\boxed{F_X(x) = P(X \le x),\ x \in R}$$

$$= \begin{cases} 0 & , x < 1 \\ p_1 & , 1 \le x < 2 \\ p_1 + \frac{p_1}{2} & , 2 \le x < 3 \\ p_1 + \frac{p_1}{2} + \frac{p_1}{3} & , 3 \le x < 4 \\ \cancel{p_1 + \frac{p_1}{2} + p_1} = \underline{1} & , x \le 4 \end{cases}$$



$$= \begin{cases} 0, & x < 1 \\ P_1, & 1 \le x \le 2 \\ P_1 + \frac{P_1}{2}, & 2 \le x < 3 \\ P + \frac{P_1}{2} + \frac{P_1}{3}, & 3 \le x < 4 \\ 1, & x \le 4 \end{cases}$$

plot of cdf



b) $P_{k+1} = P_k/2 \qquad k = 1, 2, 3$

$P_2 = P_1/2$

$P_3 = P_2/2 = \frac{P_1}{4}$

$P_4 = P_3/2 = \frac{P_1}{8}$

to find P_1 add all probabilities and make it equal to 1

d) $S_X = \mathbb{N} \longrightarrow$ is it possible

$X : S \longrightarrow \mathbb{N}$ where $\mathbb{N}$ is range of X

$\downarrow$ 4 element

Range is collection of images so it can have only 4 values if it has more than 4 values it breakes down

e) $P(\{X \le 1\}) = F_X(1) - \overset{0}{F_X(-\infty)}$

F_X at $-\infty$ is 0

F_X at $+\infty$ is 1

$= P_1$

$P(\{X < 2.5\}) = \boxed{P_1 + P_2}$

interval $(-\infty, 2.5) \Rightarrow \cancel{(-\infty, 1) \cup [1, 2.5)}$

$\underset{0}{(-\infty, 1)} \cup \underset{F_X(1) = P_1}{[1,2)} \cup \underset{P_2}{[2, 2.5)}$

$$(-\infty, 2.5) = (-\infty, 1] \cup (1, 2] \cup (2, 2.5)$$

$$P\left(X \in (a,b]\right) = F_X(b) - F_X(a)$$

$$(a,b] = (-\infty, b] - (-\infty, a]$$

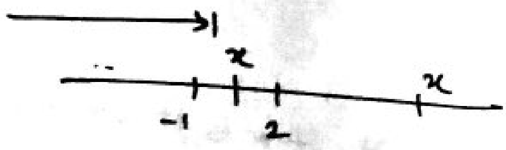
$$P(a,b] = P(-\infty, b] - P(-\infty, a]$$

$\rightarrow$ $\{0.5 < X \leq 2.5\}$

$$= (0.5, 1] \cup (1, 2] \cup (2, 2.5)$$

(13) X is uniform in $[-1, 2]$

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & , x < -1 \\ \dfrac{x-(-1)}{2-(-1)} & , -1 \leq x < 2 \\ 1 & , x > 2 \end{cases}$$



MA-2

$$E(X) = \int_S X(\omega)\, dP(\omega) =$$ written before

$$var(x) = E(X - E(X))^2 = E(X^2) - ((E(X))^2$$

→ n^{th} moment of X

mean is first moment
variance is expressed in 2nd moment

$$E(X^n) = \int_S X^n(\omega)\, dP(\omega)$$

1) Bernoulli r.v $X(p)$, $S_X = \{0, 1\}$

$$P(X=0) = 1-p \quad , \quad P(X=1) = p$$

$$E(X) = p \quad , \Rightarrow Var(X) = E(X^2) + E(X)^2$$

$$= 0^2 P(X=0) + 1^2 P(X=1) - p^2$$

$$= p(1-p)$$

---

2) Binomial r.v $X(N, p)$

$$S_X = \{0, 1, 2, \dots N\} \quad ; \quad P(X=i) = \binom{N}{i} p^i (1-p)^{N-i}$$

$$E(X) = \sum_{i=0}^{N} i \binom{N}{i} p^i (1-p)^{N-i} = Np$$ (sum of n bernoulli r.v)

$$X = \sum_{i=1}^{N} X_i \quad \text{where } X_i(p) \sim \text{Bernoulli}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$\Rightarrow \boxed{E(X^2) = \sum i^2 \binom{N}{i} p^i (1-p)^{N-i}}$$

$$= \sum_{i=0}^{N} i^2 \frac{N!}{i!(N-i)!} p^i (1-p)^{N-i}$$

$$= \sum_{i=1}^{N} i^2 \frac{N!}{i!\,(N-i)!} p^i (1-p)^{N-i}$$

$$= \sum_{i=1}^{N} i \frac{N!}{(i-1)!\,(N-i)!} p^i (1-p)^{N-i}$$

$$= \sum_{i=0}^{N-1} \frac{(i+1)\; N\,(N-1)!}{i!\;(N-i-1)!} p^{i+1} (1-p)^{N-1-i}$$

$$= Np \sum_{i=0}^{N-1} \frac{(i+1)\;(N-1)!}{i!\;(N-1-i)!} p^{i} (1-p)^{N-1-i}$$

$$= Np \left[\sum_{i=0}^{N-1} i \frac{(N-1)!}{i!\,(N-1-i)!} p^i (1-p)^{N-1-i} + \sum_{i=0}^{N-1} \frac{(N-i)!}{i!\,(N-1-i)!} p^i (1-p)^{N-} \right.$$

Binomial r.v. with parameters $N-1$ & p

$$E(x^2) = Np\left[(N-1)p + 1 \right]$$

sum of binomial r.v

$$Var(X) = Np\left[(N-1)p + 1\right] - N^2p^2$$

$$= Np(1-p)$$

variation does not agree with linear

Let X_1, X_2 be two r.v and a, b be two real constants

Then $Y = aX_1 + bX_2$ is also a random variable

$$E(Y) = E(aX_1 + bX_2) = aE(X_1) + bE(X_2)$$

$$E(Y^2) = \int_S Y(\omega)^2 \, dP(\omega)$$

$$E(Y^2) = \int_S \left[a^2 X_1^2(\omega) + 2ab X_1(\omega) X_2(\omega) + b^2 X_2^2(\omega)\right] dP(\omega)$$

$$E(Y^2) = a^2 \int_S X_1^2(\omega) dP(\omega) + b^2 \int_S X_2^2(\omega) dP(\omega) + 2ab \int_S X_1(\omega) X_2(\omega) dP(\omega)$$

$\cancel{Var(Y) = E(Y^2) - [E(Y)]^2}$

$$E(Y^2) = a^2 E(X_1^2) + b^2 E(X_2^2) + 2ab \int_S X_1(\omega) X_2(\omega) dP(\omega)$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= a^2 E(X_1^2) + b^2 E(X_2^2) + 2ab \int_S X_1(\omega) X_2(\omega) dP(\omega)$$

$$- \left[a^2 \cancel{Var} E(X_1)^2 + b^2 E(X_2)^2 + 2ab E(X_1) E(X_2)\right]$$

$$= a^2 Var(X_1) + b^2 Var(X_2) + 2ab \underbrace{\left[\int X_1(\omega) X_2(\omega) dP(\omega) + E(X_1) E(X_2)\right]}_{\text{covariance}}$$

Covariance of two random variable.

Let X_1, X_2 be two r.v with finite expectation, i.e $E(X_1)$ and $E(X_2) < \infty$

~~with the~~ $$Cov(X_1, X_2) = E\left[(X_1 - E(X_1))(X_2 - E(X_2))\right]$$

When $X_1 = X_2$
$Cov(X_1, X_2) = Var(X_1)$

$$= E\left[X_1 X_2 - X_1 E(X_2) - X_2 E(X_1) + E(X_1) E(X_2)\right]$$

$$= E(X_1 X_2) - E(X_1) E(X_2)$$

→ Moment generating function and moment

Theorem:

Moment generating function (mgf)

Let X be a random variable then mgf of X is

$$M_X(t) = E(e^{tX}) \quad , \ t \in \mathbb{R}$$

$$= E\left[1 + tx + \frac{t^2x^2}{2!} + \cdots\cdots\right]$$

using linearity prop

$$= 1 + tE(x) + \frac{t^2}{2!}E(X^2) + \cdots$$

for 5^{th} moment differentiate 5 times and put $t=0$

* Moment theorem

$$E(X^n) = \left[\frac{d^n}{dt^n} M_X(t)\Big|_{t=0}\right]$$

* Probability generating function (pgf): (in case of discrete r.v)

$$G_X(z) = E(z^X) \quad , \ z \in \mathbb{R}$$

$$= M_X(\log z) \qquad z = e^t$$

$$\log z = t$$

$$M_X(t) = G_X(e^t)$$

10) $P(\text{error}) = p = 10^{-6}$

10,000 bits = n

N = no. of errors

$P[N=0] = e^{-\lambda} \dfrac{\lambda^0}{0!}$ $\quad \lambda = np$

$= {}^nC_0\, p^0 (1-p)^n$

$P[N \le 3] = \sum_{k=0}^{3} e^{-\lambda} \dfrac{\lambda^k}{k!} \approx \sum_{k=0}^{3} \binom{n}{k} p^k (1-p)^{n-k}$

11 Y = diff of no. of head & tails in 3 tosses

$Y = \{-1, 1, 3, -3\}$

$P(Y=-1) = ({}^3C_1)\left(\frac{1}{2}\right)^3$

$P(Y=1) = ({}^3C_1)\left(\frac{1}{2}\right)^3$

$P(Y=-3) = \left(\frac{1}{2}\right)^3$ $\quad P(Y=3) = \left(\frac{1}{2}\right)^3$

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mgf: $M_X(t) = E(e^{tX})$, $t \in \mathbb{R}$

pgf: $G_X(z) = E(z^x)$, $z \in \mathbb{R}$

Moment theorem

$$E(x^n) = \frac{d^n}{dt^n} M_X(t)\Big|_{t=0}$$

i) Bernoulli r.v: $X(p)$

$$M_X(t) = E(e^{tX}) = e^{t\times 0}(1-p) + e^{t\cdot 1} p$$

$$= \underbrace{(1-p) + pe^t}_{\Downarrow}$$

moment generating f^n of Bernoulli r.v

pgf: $G_X(z) = (1-p) + pz$

Characteristic function

$$\chi_X(t) = E(e^{itx}), \quad t \in \mathbb{R}, \quad i = \sqrt{-1}$$

$$\chi_X(t) = M_X(it) = (1-p) + pe^{it}$$

ii) Binomial r.v $X(N, p)$

$S_X = \{0, 1, \dots N\}$

$S_{e^{tX}} = \{e^0, e^t, e^{2t} \dots e^{Nt}\}$

$$M_X(t) = E(e^{tx})$$

$$= \sum_{j=0}^{N} e^{tj} \binom{N}{j} p^j (1-p)^{N-j}$$

$$= \sum_{j=0}^{N} \binom{N}{j} (pe^t)^j (1-p)^{N-j}$$

$$= \left[(1-p) + pe^t\right]^N$$

$\longrightarrow$ $G_X(z) = \left[(1-p) + p z\right]^N$

$\Longrightarrow$ $G_X(z) = \left[(1-p) + pz\right]^N$

$\Longrightarrow$ $\chi_X(t) = \left[(1-p) + pe^{it}\right]^N$

iii) <u>Geometric r.v</u>

$\longrightarrow$ $M_X(t) = E(e^{tX})$

$$= \sum_{j=1}^{\infty} e^{tj} (1-p)^{j-1} p$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ |x|<1$$

$$= \frac{p}{1-p} \sum_{j=1}^{\infty} \{(1-p)e^t\}^j$$

$$= \frac{p}{1-p}\left[\sum_{j=0}^{\infty} \{(1-p)e^t\}^j - 1\right]$$

$$= \frac{p}{1-p}\left[\frac{1}{1-e^t(1-p)} - 1\right], \quad e^t(1-p) < 1$$

$$t < -\log(1-p)$$

$$= \frac{pe^t}{1-e^t(1-p)}, \quad t < -\log(1-p)$$

$\longrightarrow$ $G_X(z) = \dfrac{pz}{1-z(1-p)}, \ |z| < \dfrac{1}{1-p}$

$\longrightarrow$ $\chi_X(t) = \dfrac{pe^{it}}{1-(1-p)e^{it}}, \ |(1-p)e^{it}| = (1-p) < 1$

iv) Poisson r.v

$$M_X(t) = E(e^{tX})$$

$$= \sum_{j=0}^{\infty} e^{tj} \frac{e^{-\alpha}\alpha^j}{j!}$$

$$= e^{-\alpha} \sum_{j=0}^{\infty} \frac{(\alpha e^t)^j}{j!}$$

$$= e^{-\alpha} \sum_{j=0}^{\infty} \frac{(\alpha e^t)^j}{j!} = e^{-\alpha} e^{\alpha e^t}$$

$$= \cancel{e^{\alpha(e^t-1)}} = e^{\alpha(e^t-1)}$$

$$G_X(z) = e^{\alpha(z-1)}$$

$$\chi_X(t) = e^{\alpha(e^{it}-1)}$$

continuous r.v $\quad X \sim U([a,b])$

$$M_X(t) = E(e^{tX}) = \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)}$$

(pgf find only when prob are discrete)

$$\chi_X(t) = \frac{e^{ibt} - e^{iat}}{it(b-a)}$$

v) Exponential r.v :

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \int_0^{\infty} \lambda e^{-(\lambda - t)x} dx = \cancel{\int \frac{\lambda}{t-\lambda} e^{-(\lambda-x)}}$$

$$= \frac{\lambda}{t-\lambda} \; e^{-(\lambda-t)x} \Big|_0^\infty$$

$$M_X(t) = \frac{\lambda}{\lambda-t} \; , \; t < \lambda$$

$$\chi_X(t) = \frac{\lambda}{\lambda-it}$$

$$e^{-(\lambda-it)x}$$

$$e^{-\lambda x} \; e^{itx} \Big|_0^\infty$$

$$\chi_X(t) = E(itX)$$

$$= E\left(1 + itx + \frac{i^2t^2x^2}{2!} + \cdots\right)$$

$$= 1 + it\,E(x) + \frac{i^2t^2}{2!}E(x^2) + \cdots + \frac{i^mt^m}{m!}E(x^m)$$

Moment theorem

$$E(x^n) = \frac{1}{i^n} \frac{d^n}{dt^n} \chi_X(t) \Big|_{t=0}$$

$$\chi_X(t) = E(\cos(tX) + i\sin(tX))$$

$$= E(\cos(tx)) + i\,E(\sin(tx))$$

Normal r.v $\quad M_X(t) = ?$

$M_X(t) = E(e^{tX}) =$

Suppose X is cont r.v

$$E(e^{tx}) = \int_{\mathbb{R}} e^{tx} f_X(x)\, dx$$

→ Laplace transform of f is defined as

$$L[f](s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$M_X(-t) = \int_{\mathbb{R}} e^{-tx} f_X(x)\, dx \Rightarrow \text{two sided laplace transform}$$

$$\rightarrow \chi_X(t) = \cancel{E(e^{itx})}\ E(e^{itx}) = \int_{\mathbb{R}} e^{itx} f_X(x)\, dx.$$

$\chi_X(t)$ = Fourier transform of $f_X(x)$

$f_X(t)$ = inverse Fourier transform of $\chi_X(t)$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \chi_X(t)\, dt$$

Function of Random variable.

Let $X: S \longrightarrow \mathbb{R}$ be a random variable and $g: \mathbb{R} \to \mathbb{R}$ be a Borel measurable function then $Y = g(X)$ is also a random variable

$g^{-1}(B)$ is borel set

for every borel set B

Find cdf pdf of Y in terms of cdf and pdf of X

$Y = aX + b$

Function of Random variable

$Y = g(X)$, X is r.v

Q) Find cdf and pdf of Y in terms of cdf and pdf of X

Ex : $Y = aX + b$

cdf Y $\quad F_Y(y) = P(Y \le y)$

$$= P(aX + b \le y)$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right), & a > 0 \\ P\left(X \ge \frac{y-b}{a}\right), & a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0 \\ 1 - F_X\left(\frac{y-b}{a} - 0\right), & a < 0, \text{ if X is continuous} \end{cases}$$

~~pdf of~~

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right), & a < 0, \text{ if X is continuous} \end{cases}$$

pdf of Y $\quad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$X \sim N(m, \sigma^2)$, $Y = aX + b$, $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$

$$f_Y(y) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{y-b}{a} - m\right)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 a^2}} e^{-(y-(am+b))^2/2a^2\sigma^2},$$

$$Y \sim N(am+b, a^2\sigma^2)$$

Normal random variable.

Ex $\quad Y = X^2$

cdf $\quad F_Y(y) = P(Y \le y)$

$$= P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= \begin{cases} 0, & y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y} - 0), & y \ge 0 \end{cases}$$

pdf $\quad f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), & y \ge 0. \\ 0, & y < 0 \end{cases}$

$Y = g(X)$

Ex

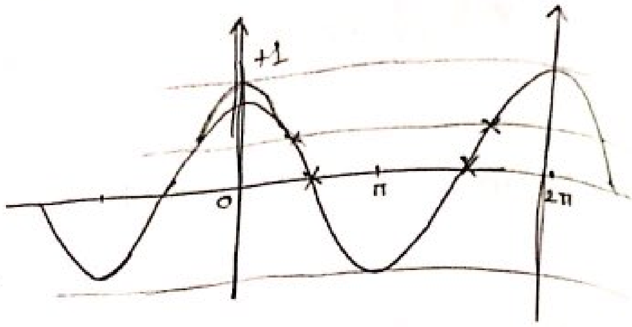
$Y = \cos(X)$

$\cancel{x = \cos}$

$x = \cos^{-1} y$

$y^2 = 1 - \sin^2 x$

$F_Y(y) = P(\cos(X) \leq y)$.



Let X be uniform on $(0, 2\pi]$

$y = \cos(x)$, find roots

$x_0 = \cos^{-1}(y)$, $x_1 = 2\pi - \cos^{-1}(y)$

$\cancel{\frac{dy}{dx} =}$

$$\frac{dy}{dx} = -\sin(x)$$

$$\left|\frac{dy}{dx}\right| = \left|-\sin\left(\cos^{-1}(y)\right)\right| = \sqrt{1-y^2}$$

$$\frac{dy}{dx} = 2x.$$

$$\left.\frac{dy}{dx}\right|_{x=\sqrt{y}} = 2\sqrt{y}$$

$$\left.\frac{dy}{dx}\right|_{x=-\sqrt{y}} = -2\sqrt{y}$$

$$f_Y(y) = f_X(\cos^{-1} y)\left|\frac{dx}{dy}\right|_{x=\cos^{-1} y} + f_X(2\pi - \cos^{-1} y)\left|\frac{dx}{dy}\right|_{x=x_1}$$

$$f_Y(y) = \begin{cases} \dfrac{f_X(\cos^{-1} y)}{\sqrt{1-y^2}} + \dfrac{f_X(2\pi - \cos^{-1} y)}{\sqrt{1-y^2}}, & y \in (-1, 1) \\ 0, & y \notin (-1, 1) \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{2\pi}, & x \in (0, 2\pi] \\ 0, & x \notin (0, 2\pi] \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}}, & y \in (-1, 1) \\ 0, & y \notin (-1, 1) \end{cases}$$

cdf

$$F_Y(y) = \begin{cases} 0 \; ; & y < -1 \\ \int_{-1}^{y} f_Y(t)\,dt \, , & -1 \le y < 1 \\ 1 \, , & y \ge 1 \end{cases}$$

Three coin toss exp

$X =$ No of H

$S_X = \{0, 1, 2, 3, 4\}$

$$F_X(x) = \begin{cases} 0 \, , & x < 0 \\ 1/8 \, , & 0 \le x < 1 \\ \frac{4}{8} \, , & 1 \le x < 2 \\ \frac{7}{8} \, , & 2 \le x < 3 \\ 1 \, , & x \ge 3 \end{cases}$$

$$F_X(x) = \frac{1}{8} u(x) + \frac{3}{8} u(x-1) + \frac{3}{8} u(x-2) + \frac{1}{8} u(x-3)$$

$$f_X(x) = \frac{1}{8} \delta(x) + \frac{3}{8} \delta(x-1) + \frac{3}{8} \delta(x-2) + \frac{1}{8} \delta(x-3).$$

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19-9-18

Function of random variable

$Y = g(X)$, g is Borel measurable function.

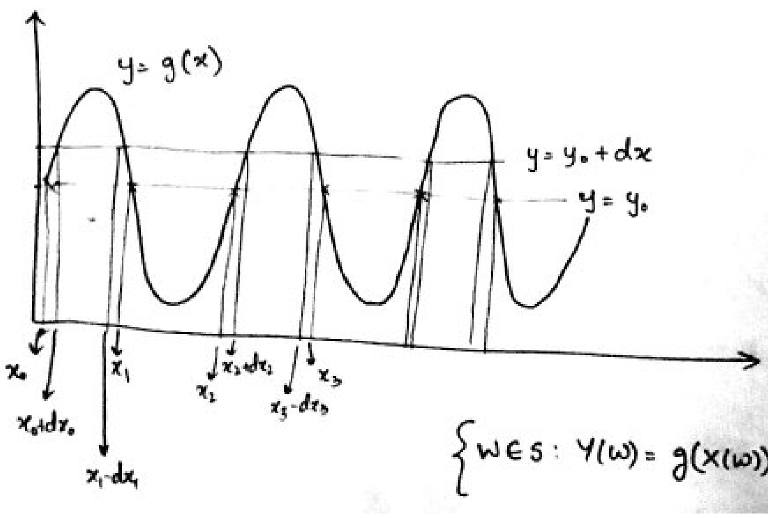
Find cdf of Y in terms of cdf of X

Eg: $Y = aX + b$, $F_Y(y) = \begin{cases} F_X\left(\frac{y-b}{a}\right), & a>0 \\ 1 - F_X\left(\frac{y-b}{a} - 0\right), & a<0 \end{cases}$

$Y = X^2$

$$F_Y(y) = P(Y \leq y)$$
$$= P(X^2 \leq y)$$
$$= P(-\sqrt{y} < X \leq \sqrt{y})$$
$$= \begin{cases} 0, & y<0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}), & y \geq 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 0, & y<0 \\ \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), & y \geq 0 \end{cases}$$



$\{ \omega \in S : Y(\omega) = g(X(\omega)) \in (y_0, y_0 + dy) \}$

$\Rightarrow P(y_0 < Y < y_0 + dy)$

$$\{y_0 < Y < y_0 + dy\} = \{x_0 < X < x_0 + dx_0\} \cup \{x_1 - dx_1 < X < x_1\}$$
$$\cup \{x_2 < X < x_2 + dx_2\} \cup \{x_3 - dx_3 < X < x_3\}$$

$$\Rightarrow P(y_0 < Y < y_0 + dy) = P(x_0 < X < x_0 + dx_0) + P(x_1 - dx_1 < X < x_1)$$
$$+ P(x_2 < X < x_2 + dx_2) + P(x_3 - dx_3 < X < x_3)$$

$$\rightarrow f_Y(y)\,dy = f_X(x_0)\,dx_0 + f_X(x_1)\,dx_1 + f_X(x_3)\,dx_3$$

$$\rightarrow f_Y(y) = \Sigma f_X(x_i) \left|\frac{dx}{dy}\right|_{x = x_1}, \; y = g(x_i)$$

$\because dx_0 \cdot dx_1 \; dx_2 \cdots$ are infintesmall
we can replace by 1 derivative

$$y = x^2, \quad \frac{dy}{dx} = 2x, \quad \frac{dx}{dy} = \frac{1}{2x}$$

$\downarrow$

roots $\quad x_0 = \sqrt{y} \quad x_1 = -\sqrt{y},$

instead of $\frac{-1}{2\sqrt{y}}$
absolute value

$$\Rightarrow f_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \times \frac{1}{2\sqrt{y}}$$

$$x = \frac{y-b}{a}$$

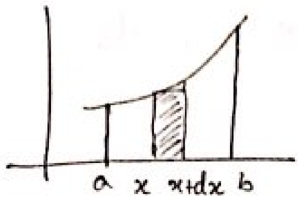
$$\frac{dy}{dx} = a$$

$$\left|\frac{dx}{dy}\right|_{x = \frac{y-b}{a}} = |a|^{-1}$$

Explanations

$P(x < X < x+dx)$

$$P(a < X < b) = \int_a^b f_X(x)\,dx$$



$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\frac{1}{|a|}$$

---

* Markov inequality.

Let X be a non negative random variable. Such that $E(X) = m < \infty$. Then for any $a > 0$

$$P(X \geq a) \leq \frac{E(X)}{a}$$

proof: $E(X) = \int_0^\infty x f_X(x)\,dx$

$$= \int_0^a x f_X(x)\,dx + \int_a^\infty x f_X(x)\,dx$$

$$\geq \int_a^\infty x f_X(x)\,dx$$

$$\geq \int_a^\infty a f_X(x)\,dx$$

$$= a\,P(X \geq a)$$

$$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}$$

* Chebyshev inequality

Let X be a random variable such that $E(x) = m < \infty$ and $\text{var}(x) = \sigma^2 < \infty$. Then $P(|X-m| \geq a) \leq \frac{\text{Var}(x)}{a^2}$, $a>0$

proof

Let $Y^2 = (X-m)^2 \geq 0$ non-negative r.v

$$E(Y^2) = E(X-m)^2 = \text{Var}(X) < \infty$$

$$P(Y^2 \geq a^2) \leq \frac{E(Y^2)}{a^2}$$

$$\Rightarrow P((X-m)^2 \geq a^2) \leq \frac{\text{Var}(X)}{a^2}$$

$$\Rightarrow P(|X-m| \geq a) \leq \frac{\text{Var}(X)}{a^2} \;;\; a>0$$

$$E(x) = \int_{\mathbb{R}} x f_X(x)\,dx$$

* ~~Chernoff bound~~.

Let $A - [a, \infty)$, $\{X \geq a\}$

$$I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

$$E(I_A(X)) = \int_A 1 \cdot f_X(x)\,dx + \int_{\mathbb{R}/A} 0 \cdot f_X(x)\,dx$$

$$P(X \geq a) = E(I_{\{x \geq a\}}) \leq E(e^{s(X-a)})$$

$$\boxed{P(X \geq a) \leq e^{-as} E(e^{sX})} \quad s>0$$

Chernoff bound.

Chernoff bound for Normal r.v.

Let $X \sim N(m, \sigma^2)$

$$E(e^{sX}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{sx} \, e^{-(x-m)^2/2\sigma^2} \, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x^2 - 2mx + m^2 - 2\sigma^2 sx)/2\sigma^2} \, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-[(x-(m+\sigma^2 s))^2 - \sigma^4 s^2 - 2m\sigma^2 s]/2\sigma^2} \, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{(ms + \frac{1}{2}\sigma^2 s^2)} \int_{-\infty}^{\infty} e^{-(x-(m+\sigma^2 s))^2/2\sigma^2} \, dx$$

$$= \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \; e^{(ms + \frac{1}{2}\sigma^2 s^2)}$$

$$P(X \geq a) \leq \left(e^{-as} \, e^{ms + \frac{1}{2}\sigma^2 s^2} \right), \quad s > 0$$

Let $f(s) = e^{(m-a)s + \frac{1}{2}\sigma^2 s^2}$

$$f'(s) = \left[(m-a) + \sigma^2 s \right] e^{(m-a)s + \frac{1}{2}\sigma^2 s^2} = 0$$

$$\Rightarrow s = \frac{a-m}{\gamma^2}$$

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$$P(x \geq a) = E\left(I_{\{x \geq a\}}\right)$$

$$\leq E\left(e^{s(x-a)}\right)$$

$$I_{\{x \geq a\}} \leq e^{s(x-a)} \quad , s > 0$$

$x \geq a$

$$\boxed{P(x \geq a) \leq e^{-as} E(e^{sx})} \; , \; s > 0$$

$$P(x \geq a) \leq e^{-(a-m)\frac{(a-m)}{\sigma^2} + \frac{1}{2}\frac{\sigma^2 (a-m)}{\sigma^4}}$$

$$= e^{-\frac{(a-m)^2}{2\sigma^2}}$$

* Vector random variable.

Let S be the sample space. Then

$$\vec{X} = (X_1, X_2) : S \to \mathbb{R}^2$$

is said to be a vector r.v if X_1 and X_2 are random variables on S

$$X_1 : (S, \mathcal{F}, P) \to (\mathbb{R}, B(\mathbb{R}), \mu_X)$$

$$\vec{X} : (S, \mathcal{F}, P) \to (\mathbb{R}^2, B(\mathbb{R}^2), \mu_{\vec{X}})$$

$B(\mathbb{R}^2) \to$ Borel σ-algebra on $\mathbb{R}^2$

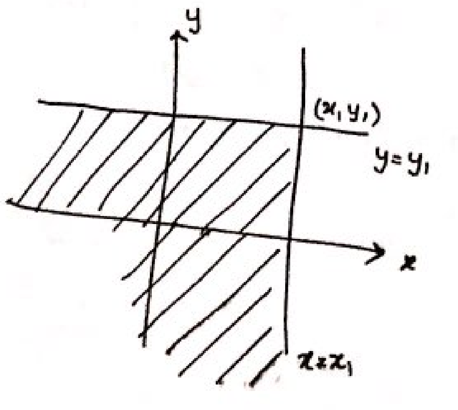
$$\mu_{\vec{X}}(B) = P(\vec{X} \in B), \quad B \in B(\mathbb{R}^2)$$

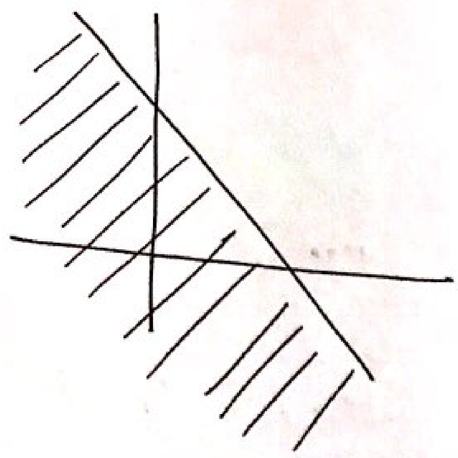
In $\mathbb{R}$ $\quad (-\infty, x], \ (a, b]$

In $\mathbb{R}^2$ $\quad \{(x,y) \in \mathbb{R}^2 : -\infty < x \le x_1, \ -\infty < y \le y_1\}$

$= (-\infty, x_1] \times (-\infty, y_1] \to$ semi-infinite rectangle

$(a, b] \times (c, d]$





$$\mu_{\vec{X}}\left[(-\infty, x_1] \times (-\infty, y_1]\right] = P(X_1 \le x_1 \ , \ X_2 \le y_1)$$

$$= P(\{X_1 \le x_1\} \cap \{X_2 \le y_1\})$$

$$\mu_{\vec{X}}\left[(a,b] \times (c,d]\right] = P(X_1 \in (a,b] \ , X_2 \in (c,d])$$

<u>cdf</u> $\quad F_{\vec{X}}(x,y) = P(X_1 \le x, \ X_2 \le y)$.

joint cdf of $\vec{X}$

properties of joint cdf

i) $F_{\vec{X}}$ is non-negative

ii) $F_{\vec{X}}$ is non-decreasing in each variable

$$F_{\vec{X}}(x_1, y_1) \le F_{\vec{X}}(x_2, y_2) \ , \ x_1 \le x_2$$
$$y_1 \le y_2$$

iii) $F_{\vec{X}}(-\infty, y_1) = F_{\vec{X}}(x_1 \ , -\infty) = 0$

iv) $F_{\vec{X}}(-\infty, \infty) = 1$

v) $F_{\vec{X}}(x_1, \infty) = F_{X_1}(x_1) \rightarrow$ marginal cdf of X_1

$F_{\vec{X}}(\infty, y_1) = F_{X_2}(y_1) \rightarrow$ marginal ~~cdf~~ of X_2

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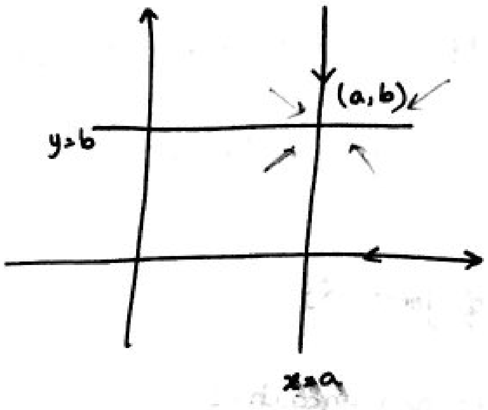
$$\vec{X} = (X_1, X_2), \quad S \to \mathbb{R}^2$$

$$F_{\vec{X}}(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2), \quad (x_1, x_2) \in \mathbb{R}^2$$

vi) cdf is continuous from north side and east side

$$\lim_{x \to a+} F_{\vec{X}}(x, y) = F_{\vec{X}}(a, y)$$

$$\lim_{y \to b^+} F_{\vec{X}}(x, y) = F_{\vec{X}}(x, b)$$

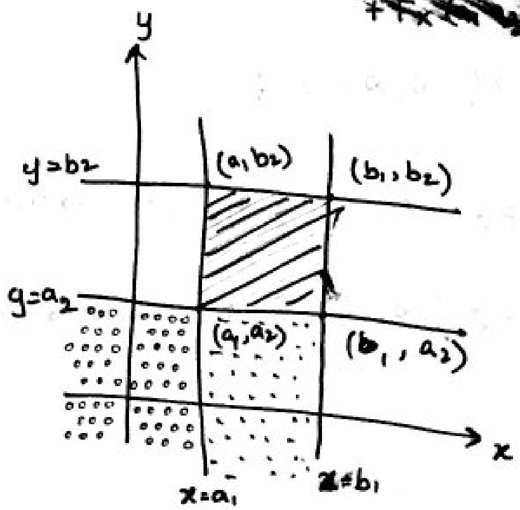


If a, b are cont together then it can be done in 4 ways as shown in diag.

vii) $P(a_1 \le X_1 \le b_1, \; a_2 \le X_2 \le b_2) = F_x(b_1, b_2) - F_x(a_1, b_2) \cancel{- F_x(b_1, a_2) + F_x(a_1, a_2)}$

$P(a_1 \le X_1 \le b_1, \; a_2 \le X_2 \le b_2)$

$$= F_{\vec{X}}(b_1, b_2) - F_x(a_1, b_2) - F_x(b_1, a_2) + F_x(a_1, a_2)$$



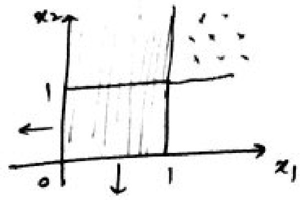
$$\boxed{P(a < X < b) = F_X(b) - F_X(a)}$$

$$F_{\vec{X}}(b_1, b_2) = P(A) + F_{\vec{X}}(a_1, a_2)$$
$$+ F_{\vec{X}}(b_1, a_2) - F_{\vec{X}}(a_1, a_2)$$
$$+ F_{\vec{X}}(a_1, b_2) - F_{\vec{X}}(a_1, a_2)$$

$\rightarrow P(A) = F_{\vec{X}}(b_1, b_2) - F_{\vec{X}}(b_1, a_2) - F_{\vec{X}}(a_1, b_2) + F_{\vec{X}}(a_1, a_2)$

Let X_1 is uniform on $[0, 1]$ and X_2 is uniform on $[0, 1]$

Find joint cdf of $\vec{X} = (X_1, X_2)$



$$F_{\vec{X}}(x_1, x_2) = P(X_1 \le x_1,\ X_2 \le x_2)$$

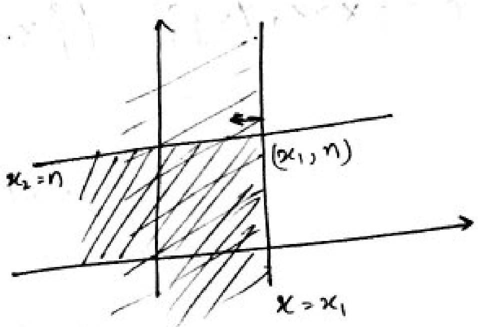
$$= \begin{cases} 0, & \{-\infty < x_1 < 0\} \cup \{-\infty < x_2 < 0\} \\ x_1 x_2, & \{0 \le x_1 < 1\} \cup \{0 \le x_2 < 1\} \\ x_1, & \{0 \le x_1 < 1\} \cup \{x_2 > 1\} \\ x_2, & \{x_1 > 1\} \cap \{0 \le x_2 < 1\} \\ 1, & \{x_1 \ge 1\} \cap \{x_2 \ge 1\} \end{cases}$$

$F_{\vec{X}}(x_1, \infty) = F_{X_1}(x_1) \longrightarrow$ marginal cdf

$$F_{\vec{X}}(x_1, \infty) = \lim_{n \to \infty} P(X_1 \le x_1,\ X_2 \le n)$$

$$= P(X_1 \le x_1)$$

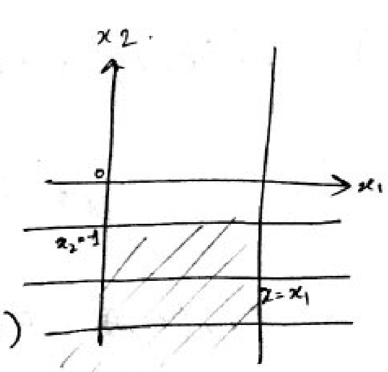
$$= F_{X_1}(x_1)$$



$F_{\vec{X}}(\infty, x_2) = F_{X_2}(x_2)$

$$F_{\vec{X}}(x_1, -\infty) = \lim_{n \to \infty} P(X_1 \le x_1,\ X_2 \le -n)$$

$$\Rightarrow P(\phi) = 0$$



$F_X(-\infty, x_2) = 0$

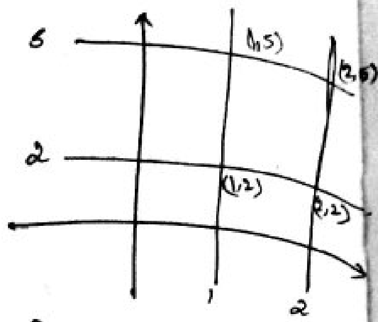
$$F_{\vec{X}}(\infty, \infty) = \lim_{m \to \infty} \lim_{n \to \infty} P(X_1 \le m,\ X_2 \le n)$$

$$= P(\mathbb{R}^2)$$

$$= 1$$

Let

$$F_{\vec{X}}(x,y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta y}) & , x \geq 0 , y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$



$A = \{X \leq 1 , Y \leq 1\}$, $P(A) = F_{X,Y}(1,1)$

$$= (1-e^{-\alpha})(1-e^{-\beta})$$

$B = \{1 < X \leq 2 , 2 < Y \leq 5\}$

$$P(B) = F_{\vec{X}}(2,5) - F_{\vec{X}}(1,5) - F_{\vec{X}}(2,2) + F_{X}(1,2)$$

$C = \{X > x , Y > y\}$

$$C^c = \{X > x , Y > y\}^c = (\{X > x\} \cap \{Y > y\})^c$$

$$= \{X > x\}^c \cup \{Y > y\}^c$$

$$= \{X \leq x\} \cup \{Y \leq y\}$$

$$P(C^c) = P(\{X \leq x\} \cup \{Y \leq y\})$$

$$= P(\{X \leq x\}) + P(\{Y \leq y\}) - P(\{X \leq x , Y \leq y\})$$

$$= F_{X,Y}(x,\infty) + F_{X,Y}(\infty , y) - F_{X,Y}(x,y)$$

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$$F(x,y) = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta x}) & , x \geq 0, \quad y \geq 0. \\ 0 & \text{otherwise} \quad , \alpha > 0, \quad \beta > 0 \end{cases}$$

Pair of discrete random variable (X, Y)

Joint pmf of (x, y) is defined as

$$\{ P(X = x_i , Y = y_i) : (x_i , y_i) \in S_{x,y} \}$$

Jointly continuous random variable.

A pair of random variable (X, Y) is said to be jointly continuous, if there exist a non-negative function $f_{X,Y}(x,y)$ such that

$$P((X,Y) \in B) = \int_B f_{X,Y}(x,y)\, dx\, dy \quad , \forall \; B \in \mathcal{B}(\mathbb{R}^2)$$

Joint pdf :- The non-negative function $f_{XY}(x,y)$ is called the joint pdf of (X, Y)

1) $\int_{\mathbb{R}^2} f_{XY}(x,y)\, dx\, dy = 1.$

2) $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$

3) $F_{XY}(x,y) = F(X \leq x, \; Y \leq y)$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x', y')\, dx'\, dy'$$

4) $F_X(x) = F_{XY}(x, \infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x', y')\, dx'\, dy'$

Diff wrt x

$$f_X(x) = \frac{d}{dx} F_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy.$$

5) $f_Y(y) = \frac{d}{dy} F_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx.$

Ex:

$$\frac{\partial F_{X,Y}}{\partial x}(x, y) = \begin{cases} \alpha e^{-\alpha x}(1 - e^{-\beta y}), & x \geq 0, \ y \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x\, \partial y} = \begin{cases} \alpha\beta e^{-\alpha x} e^{-\beta y}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \begin{cases} \int_0^{\infty} \alpha\beta e^{-\alpha x} e^{-\beta y}\, dy, & x \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \beta e^{-\beta y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

$$F_{XY}(x, y) = F_X(x)\, F_Y(y)$$

$\Rightarrow$ X and Y are independent random variable

Ex consider the function

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)}$$

$$(x, y) \in \mathbb{R}^2$$

Here $\rho \in [-1, 1]$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_{-\infty}^{\infty} \frac{dy}{2\pi\sqrt{1-\rho^2}}\, e^{-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)}$$

$$y^2 - 2\rho xy = (y - \rho x)^2 - \rho^2 x^2$$

$$\Rightarrow f_X(x) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-x^2/2(1-\rho^2)} \int_{-\infty}^{\infty} e^{-((y-\rho x)^2 - \rho^2 x^2)/2(1-\rho^2)}\, dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-[x^2-\rho^2x^2]/2(1-\rho^2)} \int_{-\infty}^{\infty} e^{-(y-\rho x)^2/2(1-\rho^2)} dy$$

$$= \left[\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right] \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-(y-\rho x)^2/2(1-\rho^2)} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sim N(0,1)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \sim N(0,1)$$

$$f_X(x)\, f_Y(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

Here ρ is correlation coefficient of (X,Y)

if $\rho = 0$ then (X,Y) are independent r.v

For $\rho \neq 0$; (X,Y) are dependent r.v.

<u>Independent r.v</u>

$$X : (S_1, \mathcal{F}_1, P_1) \longrightarrow (\mathbb{R}, B(\mathbb{R}), \mu_1)$$

$$Y : (S_2, \mathcal{F}_2, P_2) \longrightarrow (\mathbb{R}, B(\mathbb{R}), \mu_2)$$

$$P(X \in A_1, Y \in A_2) = P(X \in A_1)\, P(Y \in A_2)$$

$$X \in A_1 \in \mathcal{F}_1 \quad , \quad X \in A_2 \in \mathcal{F}_2$$

X, Y are ind r.v

joint cdf

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= P(X \le x)\, P(Y \le y)$$

$$= F_X(x)\, F_Y(y)$$

Diff wrt x and y

$$\frac{\partial F_{X,Y}}{\partial x} = \left[\frac{d}{dx} F_X(x)\right] F_Y(y)$$

$$\frac{\partial^2 F}{\partial x\, \partial y} = \left[\frac{d}{dx} F_X(x)\right]\left[\frac{d}{dy} F_Y(y)\right]$$

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y), \quad (x,y) \in \mathbb{R}^2$$

jk^{th} joint moment of (X,Y)

$$E(X^j Y^k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x,y)\, dx\, dy$$

For $j=1$ $k=0$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx$$

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$j = 1,\ k = 1$

$$E(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

correlation of X, Y

Tut sheet 4

(Q) a) X is uniform in $[0, b]$

$$F_X(x) = \begin{cases} 0 & , x < 0 \\ \frac{x}{b} & , x \in [0,b) \\ 1 & , x \geq b \end{cases}$$

$$P(X \geq c) = 1 - P(X < c)$$
$$= 1 - F_X(c)$$

$$= \begin{cases} 1 & , c < 0 \\ 1 - \frac{c}{b} & , c \in [0,b) \\ 0 & , c \geq b \end{cases}$$

Markov inequality: $P(X \geq c) \leq \frac{E(X)}{c} = \frac{b}{2c}$, $c > 0$

$$1 - \frac{c}{b} \leq \frac{b}{2c} \quad , c \in [0,b)$$

$$0 \leq \frac{b}{2c} \quad , c \geq b \longrightarrow \text{trivial}$$

$y = |X|$, X is continuous r.v

i) $F_Y(y) = P(Y \leq y)$

$= P(|X| \leq y)$

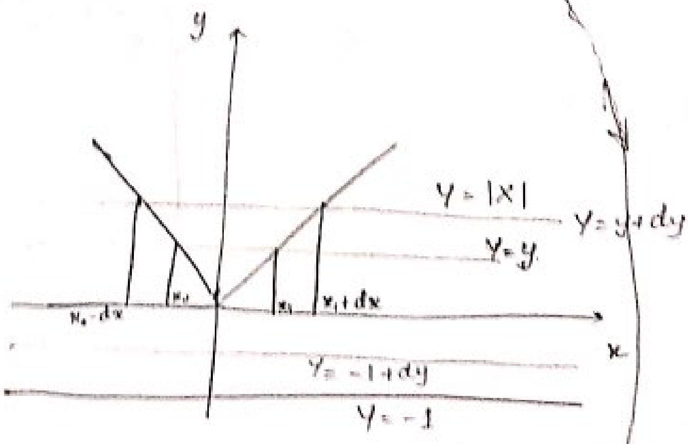
$= \begin{cases} P(-y \leq X \leq y) & , y > 0 \\ 0 & , y \leq 0 \end{cases}$

$P(a < X < b) = \int_a^b f_X(x)\,dx$

$= \begin{cases} F_X(y) - F_X(-y) & , y > 0 \\ 0 & , y \leq 0 \end{cases}$

$f_Y(y) = \begin{cases} f_X(y) + f_X(-y) & , y > 0 \\ 0 & , y \leq 0 \end{cases}$

b) $\{y < Y < y + dy\}$



$= \{x_0 - dx < X < x_0\} \cup \{x_1 < X < x_1 + dx\}$

$P(y < Y < y + dy) = P(x_0 - dx < X < x_0) + P(x_1 < X < x_1 + dx)$

$\Rightarrow f_Y(y)\,dy = f_X(x_0)\,dx + f_X(x_1)\,dx \qquad \int_y^{y+dy} f_Y(y')\,dy'$

$\Rightarrow f_Y(y) = f_X(x_0) \left| \frac{dx}{dy} \right|_{x = x_0} + f_X(x_1) \left| \frac{dx}{dy} \right|_{x = x_1}$

$y = |x|$

$\Rightarrow x_0 = y \,,\ x_1 = -y \,,\ \dfrac{dy}{dx} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

$\boxed{\text{eq}^n\ |x| = -1 \text{ has no sol}^n}$

$$f_y(y) = \begin{cases} f_x(-y) + f_x(y) \,, & y > 0 \\ 0 \,, & y \le 0 \end{cases}$$

<u>cdf</u>

$$F_x(y) = \int_{-\infty}^{y} f_y(y')\,dy' = \int_{-\infty}^{0} f_y(y')\,dy' + \int_{0}^{y} f_y(y')\,dy'$$

$$= \begin{cases} \int_0^y f_x(y')\,dy' + \int_0^y f_x(-y')\,dy' \,, & y > 0 \\ 0 \,, & y \le 0 \end{cases}$$

$$\boxed{f_x(x) = \frac{d}{dx} F_x(x)}$$

$$F_x(y) = \begin{cases} F_x(y) - F_x(0) + \int_0^{-y} f_x(y')\,dy' & \\ F_x(y) - F_x(-y) \,, & y > 0 \\ 0 \,, & y \le 0 \end{cases}$$

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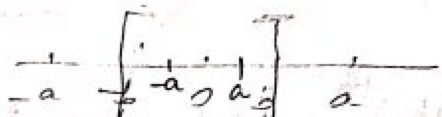
1) $P(|x-m|>a) \le \dfrac{\sigma^2}{a^2}$, $a>0$

2) $X \sim U[-b,b]$, $m=0$.



$$P(|X|>a) = 1-P(|x|\le a) = 1-P(-a\le X \le a)$$

$$= 1-\left[F_X(a)-F_X(-a)\right]$$

$$F_X(x) = \begin{cases} 0, & x<-b \\ \dfrac{x+b}{2b}, & x\in[-b,b) \\ 1, & x\ge b \end{cases} \qquad = \begin{cases} 1-1=0, & a>b \\ 1-\left[\dfrac{a+b}{2b}-\dfrac{b-a}{2b}\right], & a<b \end{cases}$$

26-9.

$(jk)^{th}$ joint moment

$$E(X^j Y^k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x,y)\,dx\,dy$$

$$= \sum_{x_n}\sum_{y_m} x_n^j y_m^k \, P_{X,Y}(X = x_n, Y = y_m)$$

$j = 0$, $k = 1$

$E(Y)$

$j = 1$, $k = 0$, $E(X)$

$j = 1$, $k = 1$, $E(X,Y) \rightarrow$ correlation of X, Y

* Suppose X and Y are independent r.v

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \left[\int_{-\infty}^{\infty} x\, f_X(x)\,dx\right]\left[\int_{-\infty}^{\infty} y\, f_Y(y)\,dy\right]$$

$$= E(X)\,E(Y)$$

$E(X^j Y^k) = E(X^j)\, E(Y^k)$, X, Y are ind. r.v

$E(g_1(X)\, g_2(Y)) = E(g_1(X))\, E(g_2(Y))$

$(jk)^{th}$ joint central moment (centered around the expectation)

$E\left[(X-E(X))^j \,(Y-E(Y))^k\right]$

For $j=1$ & $k=1$

$$\operatorname{cov}(X,Y) = E\left[(X-E(X))(Y-E(Y))\right]$$

$$= E\left[XY - \underbrace{E(X)Y}_{E(X)E(Y)} - \underbrace{X E(Y)}_{E(X)E(Y)} + E(X)E(Y)\right]$$

$$= E(XY) - E(X)E(Y)$$

If X and Y are independent r.v then $\operatorname{cov}(X,Y) = 0$

but if $\operatorname{cov}(X,Y) = 0 \not\Rightarrow$ X & Y are independent

Example. $X = \cos\theta \quad Y = \sin\theta \qquad \theta$ is uniform on $(0, 2\pi)$

$$E(X) = \int_0^{2\pi} \cos\theta \,\frac{1}{2\pi}\, d\theta = 0,$$

$$E(Y) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta \, d\theta = 0,$$

$$E(XY) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \sin\theta \, d\theta$$

$$= 0.$$

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y) = 0$$

$X = \cos\theta \qquad Y = \sin\theta = \sqrt{1-\cos^2\theta}$

$\Rightarrow Y = \sqrt{1-X^2}$

Though X & Y are dependent their cov is 0

X & Y are dependent random variable

* <u>Correlation</u> <u>coefficient</u> of <u>2 random</u> <u>variables</u>

$$\rho_{X,Y} = \frac{\text{covariance}(X,Y)}{\sigma_X \sigma_Y} = \frac{\text{cov}(X,Y)}{\sigma_X , \sigma_Y}$$

$$\rightarrow \quad -1 \le \rho_{X,Y} \le 1$$

$$E\left(\frac{X-E(X)}{\sigma_X} \pm \frac{Y-E(Y)}{\sigma_Y}\right)^2 \ge 0$$

$$\rightarrow E\left(\frac{(X-E(X))^2}{\sigma_X^2} + \frac{(Y-E(Y))^2}{\sigma_Y^2} \pm \frac{2(X-E(X))(Y-E(Y))}{\sigma_X \sigma_Y}\right)$$

$$\Rightarrow \quad 1 + 1 \pm 2\,\frac{cov(X,Y)}{\sigma_X \sigma_Y} \geq 0$$

$$\Rightarrow \quad 1 \pm \rho_{X,Y} \geq 0$$

$$\Rightarrow \quad \boxed{-1 \leq \rho_{X,Y} \leq 1}$$

* $\rho_{X,Y} = 0 \Rightarrow$ X and Y are ind r.v

$$X,\ Y = aX + b \qquad , \quad E(Y) = aE(X) + b$$

$$cov(X,Y) = E\big((X - E(X))(Y - E(Y))\big)$$

$$= E\big[(X - E(X))(Y - E(Y))\big]$$

$$= E\big[(X - E(X))(aX + \cancel{b} - aE(X) - \cancel{b})\big]$$

$$= E\big[(X - E(X))(a(X - E(X)))\big]$$

$$= a\,E(X - E(X))^2$$

$$= a\,\sigma_X^2$$

$$\sigma_Y^2 = var(Y) = a^2 \sigma_X^2$$

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{a\,\sigma_X^2}{\sigma_X (\pm a)\,\sigma_X} = \begin{cases} +1, & a > 0 \\ -1, & a < 0 \end{cases}$$

* Conditional probability

$$P(Y=y \mid X=x) = \frac{P(Y=y,\ X=x)}{P(X=x)}$$

y is varying in all values of y

$$P(X=x) > 0$$

for discrete r.v

$$P(Y \in A) = \sum_{\text{all } x_k} \sum_{y_j \in A} P_{x,y}(X=x_k,\ Y=y_j)$$

$$= \sum_{\text{all } x_k} \sum_{y_j \in A} P_y(Y=y_j \mid X=x_k)\, P_X(X=x_k)$$

$$= \sum_{\text{all } x_k} P_X(X=x_k) \left(\sum_{y_j \in A} P_y(Y=y_j \mid X=x_k) \right)$$

$$= \sum_{\text{all } x_k} P_X(X=x_k)\, P_y(Y \in A \mid X=x_k)$$

$$= \sum_{\text{all } x_k} P_Y(Y \in A \mid X=x_k)\, P_X(X=x_k)$$

for continuous r.v.

~~P(Y∈A) =~~ $\cancel{\int f_y(y|x) f_x(}$

$$P(Y \in A) = \int_{-\infty}^{\infty} \left[\int_{y \in A} f_y(y|x)\,dy \right] f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} f_Y (Y \in A \mid x) \; f_X(x) \, dx$$

conditional Expectation:

$$Z = \frac{Y}{X}$$

$$E(Y) = E\big(E(Y \mid X)\big)$$

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y \, f_Y(y \mid x) \, dy = g(x)$$

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx$$

law of total expectation

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27-9

Ex 1 Let X be input signal

$$X = \begin{cases} +1 \\ -1 \end{cases}$$

discrete

Output signal $\quad Y = X + N$, where $N \sim N(0,1)$

continuous

Find ~~pdf~~ conditional pdf of Y given $X = +1$ or $X = -1$

conditional cdf of Y

$$F_Y(y|X=+1) = P(Y \le y | X = +1)$$

$$= P(X+N \le y | X=+1)$$

$$= P(1+N \le y | X = +1)$$

$$= P(N \le y-1 | X = +1)$$

$$= P(N \le y-1) \qquad P(A|B) = P(A) \quad \because A \,\&\, B \text{ are independent}$$

$$F_Y(y|X=+1) = \int_{-\infty}^{y-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \,, \quad \cancel{x+1 = t}, \ \cancel{dx = dt}, \ \cancel{x = y-1}, \ \cancel{y-1+1 = t}$$

conditional pdf of Y

$$f_Y(y|X=+1) = \frac{d}{dy} F_Y(y|X=+1)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} \sim N(1,1)$$

$\longrightarrow$ $F_Y(y|X=-1) = P(Y \le y | X = -1)$

$$= P(X+N \le y | X = -1)$$

$$= P(-1+N \le y | X = -1)$$

$= P(N \leq y+1 \mid X = -1)$

$= P(N \leq y+1)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y+1} e^{-x^2/2}\, dx$$

$$f_Y(y \mid X = -1) = \frac{1}{\sqrt{2\pi}} e^{-(y+1)^2/2} \quad \sim N(-1, 1)$$

compute $\quad P(X = +1 \mid Y > 0)$, $\quad P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Bayes Therem $\quad P(X = +1 \mid Y > 0) = \frac{P(Y > 0 \mid X = +1)\, P(X = +1)}{P(Y > 0)}$

Law of total probability

$$P(Y>0) = P(Y>0 \mid X = +1)\, P(X = +1) + P(Y>0 \mid X = -1) \times P(X = -1)$$

$$P(Y>0 \mid X = +1) = \int_0^{\infty} f_Y(y \mid X = +1)\, dy$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2}\, dy \quad , \quad y - 1 = t$$

$$= \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$P(Y>0 \mid X = -1) = \int_0^{\infty} f_Y(y \mid X = -1)\, dy$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y+1)^2/2}\, dy \quad y + 1 = t$$

$$= \int_1^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

Find $E(Y)$

$$E(Y) = E(E(Y|X))$$

$$E(Y|X=+1) = 1 = \int_{-\infty}^{\infty} y f_y(y|X=+1)\, dy$$

$$E(Y|X=-1) = -1 =$$

$$E(Y) = E(g(x)) \neq \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= E(Y|X=+1)\, P(X=+1) + E(Y|X=-1)\, P(X=-1)$$

$$= P(X=+1) - P(X=-1)$$

$$E(g(x)) = \sum_{i=1}^{2} g(X=x_i)\, P(X=x_i)$$

$Y = g(x)$ $\qquad$ $f_y(y) \neq f_X(x)$

$$E(Y) = \int_{-\infty}^{\infty} y f_y(y) \qquad f_y(y) = \sum_{\text{all } x_k} f_X\, x_k \left|\frac{dx}{dy}\right|_{x=x_k}$$

x_k are roots of $y=g(x)$
i.e $y = g(x_k)$ for all k

$$\int_{-\infty}^{\infty} g(x) f_{g(x)}(x)\, dx$$

x_k are roots of $y=g(x)$
i.e $y = g(x_k)$ for all x_k

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compute $P(X=+1 \mid Y=y)$

$$P(X=+1 \mid Y=y) = \frac{P(X=+1, Y=y)}{P(Y=y)} \quad \times \qquad P(Y=y)=0$$

$$P(X=+1 \mid Y=y) = \lim_{h\to 0} P(X=+1 \mid y < Y \leq y+h)$$

$$P(X=+1 \mid y < Y \leq y+h) = \frac{P(y < Y \leq y+h \mid X=+1)\, P(X=+1)}{P(y < Y \leq y+h)}$$

$$= \frac{\int_y^{y+h} f_Y(y' \mid X=+1)\, dy'\; P(X=+1)}{\int_y^{y+h} f_Y(y')\, dy'}$$

$$= \frac{\cancel{K} f_Y(y \mid X=+1)\, P(X=+1)}{\cancel{K} f_Y(y)}$$

$$P(X=+1 \mid y < Y \leq y+h) = \frac{f_Y(y \mid X=+1)\, P(X=+1)}{f_Y(y)}$$

$$= \frac{f_Y(y \mid X=+1)\, P(X=+1)}{f_Y(y)}$$

$$P(X=+1 \mid Y=y) = \frac{f_Y(y \mid X=+1)\, P(X=+1)}{f_Y(y)}$$

$$\left[\left[f_Y(y) = f_Y(y \mid X=+1)\, P(X=+1) + f_Y(y \mid X=-1)\, P(X=-1) \right]\right]$$

$\downarrow$

In this case Y is continuous, X is discrete

If both X and Y are continuous

$$f_Y(y) = \int_{-\infty}^{\infty} f_Y(y|x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$f_X(x,y) = f_Y(y|x)\, f_X(x)$$

$$f_X(x,y) = f_X(x|y)\, f_Y(y)$$

1/10/18

Function of random vector.

Let $\vec{X} = (X,Y)$ be a random vector. Consider the function

$$Z = F(\vec{X}) = F(X,Y)$$

Find cdf and pdf of Z in terms of joint cdf and joint pdf of (X,Y)

By definition

$$F_Z(z) = P(Z \leq z) \quad , \quad z \in \mathbb{R}$$

$$= P(F(X,Y) \leq z)$$

$$= P((X,Y) \in R_z)$$

where $R_z = \{(x,y) \in \mathbb{R}^2 \mid F(x,y) \leq z\}$

Then $F_Z(z) = \iint_{R_z} f_{X,Y}(x',y')\, dx'\, dy'$

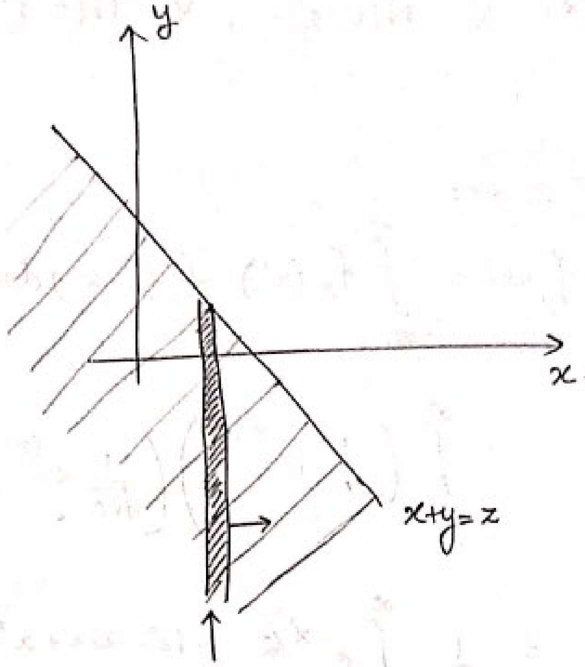
pdf $\quad f_Z(z) = \frac{d}{dz} F_Z(z)$

let $Z = X + Y$.

$F_Z(z) = P(Z \le z)$

$= P(X+Y \le z)$

$= \iint_{R_z} f_{X,Y}(x', y')\, dx'dy'$



$R_z = \{(x,y) \in \mathbb{R}^2 : x+y \le z\}$

$$\iint_{R_z} f_{X,Y}(x', y')\, dx'\, dy'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x'} f_{X,Y}(x', y')\, dx'dy'$$

Now, $f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x', z-x')\, dx'$

$$\left.\begin{aligned} f_{X,Y}(x,y) &= f_X(x|y)\, f_Y(y) \\ &\text{or } f_Y(y|x)\, f_X(x) \end{aligned}\right\} X, Y \text{ are dependent r.v}$$

If X and Y are independent r.v then

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x')\; f_Y(z-x')\, dx' = (f_X * f_Y)(z)$$

$$= \text{convolution}(f_X, f_Y)$$

Ex:- $X \sim N(0,1)$, $Y \sim N(0,1)$, $\rho_{x,y} = 0$

$\Rightarrow$ X and Y are ind

$Z = X+Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x') \, f_Y(z-x') \, dx'$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-x'^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-(z-x')^2/2}\right) dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x'^2/2} \, e^{-(z^2-2zx'+x'^2)/2} \, dx'$$

$$= \frac{e^{-z^2/2}}{2\pi} \int_{-\infty}^{\infty} e^{-(2x'^2-2zx')/2} \, dx'$$

$$= \frac{e^{-z^2/2}}{2\pi} \int_{-\infty}^{\infty} e^{-(x'^2-zx')} \, dx'$$

$$= \frac{e^{-z^2/2}}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(x'-\frac{z}{2}\right)^2 - \frac{z^2}{4}\right]} \, dx'$$

$$= \frac{e^{-z^2/2}}{2\pi} \int_{-\infty}^{\infty} e^{-(x'-z/2)^2} \, dx' \; e^{+z^2/4}$$

$\sigma_Z^2 = \sigma_x^2 + \sigma_y^2$, $\sigma_Z^2 = 2$, $\sigma_Z = \sqrt{2}$

$Z \sim N(0,2)$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/4}$$

Ex: Let $Z = \frac{X}{Y}$, where X and Y are ind exp r.v with mean 1

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$$F_Z(z|y) = P(Z \leq z|y)$$

$$= P\left(\frac{X}{Y} \leq z|y\right)$$

$$= P\left(\frac{X}{y} \leq z|y\right)$$

$$= \begin{cases} P(X \leq yz|y), & y > 0 \\ P(X \geq yz|y), & y < 0 \end{cases}$$

$$= \begin{cases} P(X \leq yz), & y > 0 \\ P(X \geq yz), & y < 0 \end{cases}$$

$$F_Z(z|y) = \begin{cases} F_X(yz), & y > 0 \\ 1 - F_X(yz), & y < 0 \end{cases}$$

$$f_Z(z|y) = \frac{d}{dz} F_Z(z|y) = |y| f_X(y,z)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y) f_Y(y)\, dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_X(yz) f_Y(y)\, dy$$

$$= \int_0^{\infty} y\, e^{-yz} e^{-y}\, dy$$

$$= \int_0^{\infty} y\, e^{-y(z+1)}\, dy$$

$$= \frac{1}{(1+z)^2}$$

Transformation of R.V.

$$(X,Y) \longmapsto (z_1, z_2)$$

Let $z_1 = g_1(X,Y)$ and $z_2 = g_2(X,Y)$

Find joint cdf and joint pdf of (Z_1, Z_2)

$$F_{Z_1,Z_2}(z_1,z_2) = P(Z_1 \leq z_1 \;\; Z_2 \leq z_2)$$

$$= P(g_1(X,Y) \leq z \;, g_2(X,Y) \leq z)$$

$$= \iint_{R_z} f_{X,Y}(x',y')\, dx'\, dy'$$

$$R_z = \{(x,y) \in \mathbb{R}^2 : g_1(x,y) \leq z \text{ and } g_2(x,y) \leq z\}$$

Tutorial

Let $X \sim N(0,1)$, $Y \sim N(0,1)$ are independent r.v

$Z = X+Y$, $E(Z) = E(X)+E(Y) = 0+0 = 0$

$Var(Z) = E(Z-E(Z))^2 = E(Z^2) = E(X^2+2XY+Y^2)$

$= E(X^2)+E(Y^2)+E(2XY)$

$= Var(X)+Var(Y)$

$[\because E(XY) = E(X)E(Y) = 0]$

$\sigma_Z^2 = 2$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x') f_Y(z-x')\,dx'$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-x'^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-(z-x')^2/2}\right) dx'$$

$$= \cancel{\frac{e^{-z^2/2}}{2\pi}}\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-[x'^2+z^2+x'^2-2zx']/2}\,dx'$$

$$= \frac{e^{-z^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-(x'^2-zx')/2(1/2)}\,dx'$$

$$= \frac{e^{-z^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-\left[\left(x'-\frac{z}{2}\right)^2-\frac{z^2}{4}\right]/2(1/2)}\,dx'$$

$$= \sqrt{\frac{1}{2}}\ \frac{e^{-z^2/4}}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-\left(x'-\frac{z}{2}\right)^2/2\left(\frac{1}{2}\right)}\,dx' \times \frac{1}{\sqrt{2\pi}\sqrt{1/2}}\right]$$

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$$= \frac{e^{-z^2/2(2)}}{\sqrt{2\pi}\sqrt{2}} \sim N(0,2)$$

$$X \sim N(m, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2}$$

$\Rightarrow$ $Z = X + Y$, $X \sim N(0,1)$, $Y \sim N(0,1)$

$\rho_{X,Y} = \rho \neq 0 \Rightarrow$ X and Y are dependent

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}\, e^{-\left[\left(\frac{x-m_1}{\sigma_X}\right)^2 - 2\rho_{X,Y}\left(\frac{x-m_1}{\sigma_X}\right)\left(\frac{y-m_2}{\sigma_Y}\right) + \left(\frac{y-m_2}{\sigma_Y}\right)^2\right]/2(1-\rho_{XY}^2)}$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-[x^2 - 2\rho xy + y^2]/2(1-\rho^2)}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x', z-x')\,dx' = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-[x'^2 - 2\rho x'(z-x') + (z-x')^2]/2(1-\rho^2)}$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-[x'^2 + 2\rho x'^2 + x'^2 - 2\rho x'z - 2zx' + z^2]/2(1-\rho^2)}$$

$$= \frac{e^{-z^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-[2(\rho+1)x'^2 - 2x'z(\rho+1)]/2(1-\rho^2)} \, dx'$$

$$= \frac{e^{-z^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-[x'^2 - x'z]/2\left(\frac{1-\rho}{2}\right)} \, dx'$$

$$= \frac{e^{z^2/4(1-\rho)} \, e^{-z^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-(x'-\frac{z}{2})^2/2\left(\frac{1-\rho}{2}\right)} \, dx'$$

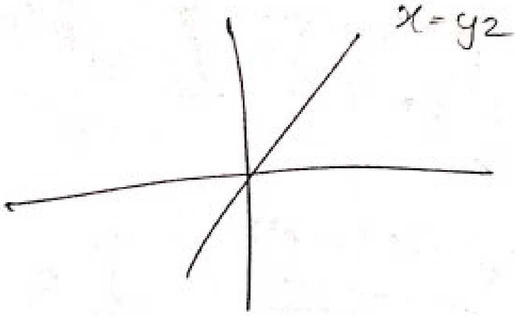
$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{1-\rho}{2}}} \int_{-\infty}^{\infty} e^{-(x'-\frac{z}{2})^2/2\left(\frac{1-\rho}{2}\right)} \, dx'$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{1-\rho}{2}}} \int_{-\infty}^{\infty} e^{-(x'-\frac{z}{2})^2/2\left(\frac{1-\rho}{2}\right)} \, dx' = 1$$

$-x')^2]/2(1-\rho^2)$

$-\rho^2)$

$Z = \frac{X}{Y}$



$F_Z(z) = P(Z \leq z)$

$= P\left(\frac{X}{Y} \leq z\right)$

$\left\{\frac{x}{y} \leq z\right\} = \left\{x \leq yz : y > 0\right\}$

$\cup \left\{x \geq yz ; y < 0\right\}$

$= \begin{cases} P(X < yz), & y > 0 \\ P(X \geq yz), & y < 0 \end{cases} = P(X \leq yz)\, I_{\{y>0\}}$

$+ P(X \geq yz)\, I_{\{y<0\}}$

Indication f^n of set A

$I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$

$Z = \frac{X}{Y}$

cond cdf $F_Z(z/y) \longrightarrow f_Z(z/y)$

$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z/y)\, f_Y(y)\, dy$

$P(\{Z \in A\}) = \int_A f_Z(z)\, dz$

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Transformation of random variable

$$Z_1 = g_1(X,Y) \qquad Z_2 = g_2(X,Y)$$

$$F_{Z_1,Z_2}(z_1,z_2) = P(Z_1 \le z_1 \,,\; Z_2 \le z_2)$$

$$= \int_{R_{z_1 z_2}} f_{X,Y}(x',y')\,dx'dy'$$

where $R_{z_1,z_2} = \left\{ (x,y) \in \mathbb{R}^2 : g_1(x,y) \le z_1 \text{ and } g_2(x,y) \le z_2 \right\}$

Ex: $Z_1 = \min(X,Y)$ and $Z_2 = \max(X,Y)$

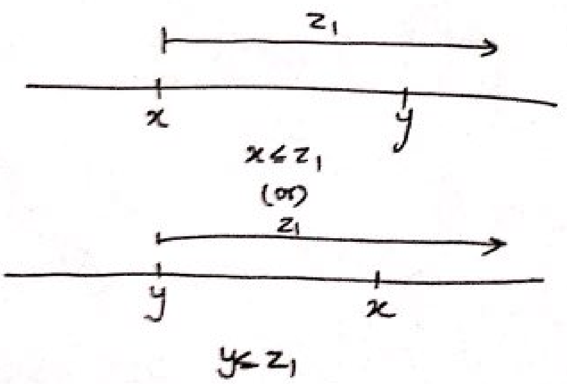
$$F_{Z_1,Z_2}(z_1,z_2) = P(Z_1 \le z_1 \,,\; Z_2 \le z_2)$$

$$\{Z_1 \le z_1\} = \{\min(X,Y) \le z_1\} = \{X \le z_1\} \cup \{Y \le z_1\}$$

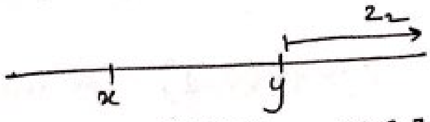
$$\{Z_2 \le z_2\} = \{\max(X,Y) \le z_2\} = \{X \le z_2\} \cap \{Y \le z_2\}$$

$\min(x,y) \le z_1$

~~Z~~

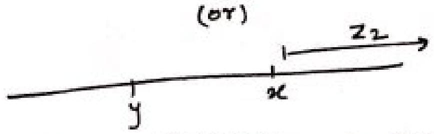


$max(x,y) \leq z_2$



$x \leq y \leq z_2$ $\quad$ $x \leq z_2$ and $y \leq z_2$

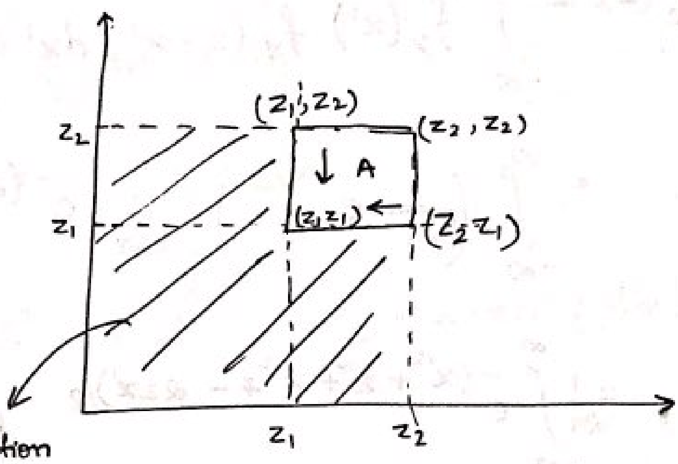
(or)



$y < x \leq z_2$, $x \leq z_2$ and $y \leq z_2$

case I $\quad z_1 \leq z_2$

case II $\quad z_1 \geq z_2$



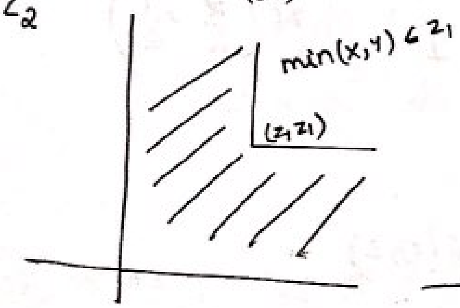
$$F_{z_1 z_2}(z_1, z_2) = F_{x,y}(z_2\, z_2) - P(A)$$

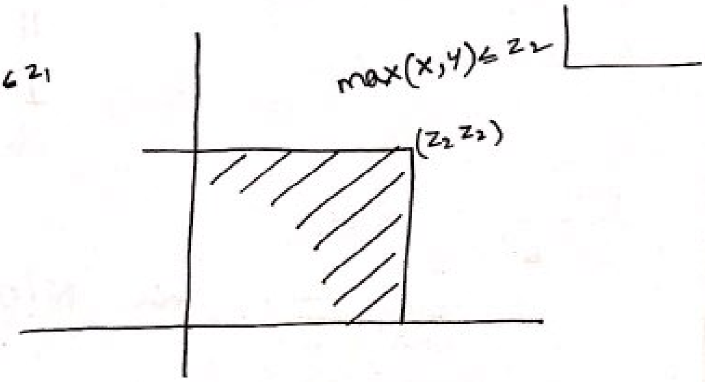
$$= \cancel{F_{xy}(z_2, z_2)} - \left\{ \cancel{F_{xy}(z_2, z_2)} + F_{x,y}(z_1, z_1) - F_{xy}(z_1, z_2) - F_{xy}(z_2, z_1) \right.$$

$$= F_{xy}(z_1, z_2) + F_{xy}(z_2, z_1) - F_{x,y}(z_1, z_1)$$

$z_1 \geq z_2$

(or)





$$F_{z_1 z_2}(z_1\, z_2) = F_{x,y}(z_2, z_2)$$

$X \sim N(0,1)$, $Y \sim N(0,1)$

$$\rho_{x,y} = 0$$

$Z = X + Y$, X and Y are independent r.v

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x') \, f_Y(z - x') \, dx'$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-x'^2/2} \right) \left(\frac{1}{\sqrt{2\pi}} e^{-(z-x')^2/2} \right) dx'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x'^2 + z^2 + x'^2 - 2zx')/2} \, dx'$$

$$= \frac{e^{-z^2/2}}{2\pi} \int_{-\infty}^{\infty} e^{-(x'^2 - zx')/2\left(\frac{1}{2}\right)} \, dx'$$

$$= \frac{e^{-z^2/2}}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(x' - \frac{z}{2}\right)^2 - \frac{z^2}{4}\right]/2\left(\frac{1}{2}\right)} \, dx'$$

$$= \frac{e^{-z^2/4}}{\sqrt{2}\sqrt{2\pi}} \underbrace{\left[\frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{z}{2}\right)^2/2\left(\frac{1}{2}\right)} \, dx' \right]}_{= 1}$$

$\sim N\left(\frac{z}{2}, \frac{1}{2}\right)$

$$= \frac{e^{-z^2/2(2)}}{\sqrt{2\pi}\sqrt{2}} \sim N(0,2)$$

Ex $X \sim N(0,1)$ $Y \sim N(0,1)$ X and Y are ~~not~~ ind

$R = (X^2 + Y^2)^{1/2}$, $\Theta = \tan^{-1}(Y/X)$

Find joint cdf of (R, Θ)

$$F_{R,\Theta}(r, \theta) = \int_{R_{r\theta}} f_{X,Y}(x', y')\,dx'dy'$$

$$R_{r\theta} = \left\{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^{1/2} \le r ,\ 0 \le \tan^{-1}(y/x) \le \theta\right\}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$F_{R\Theta}(r,\theta) = \int_0^\theta \int_0^r \frac{1}{2\pi} e^{-r'^2/2} \, r' dr' d\theta'$$

$$= \left(\frac{1}{2\pi}\theta\right) \int_0^r r' e^{-r'^2/2} \, dr'$$

$$\frac{r'^2}{2} = t$$

$$r' dr' = dt$$

$$= \left(\frac{1\theta}{2\pi}\right) \int_0^{r^2/2} e^{-t} \, dt$$

$$F_{R\Theta}(r,\theta) = \left(\frac{1}{2\pi}\theta\right)\left(1 - e^{-r^2/2}\right)$$

$$f_{R\Theta}(r,\theta) = \frac{1}{2\pi} r e^{-r^2/2}$$

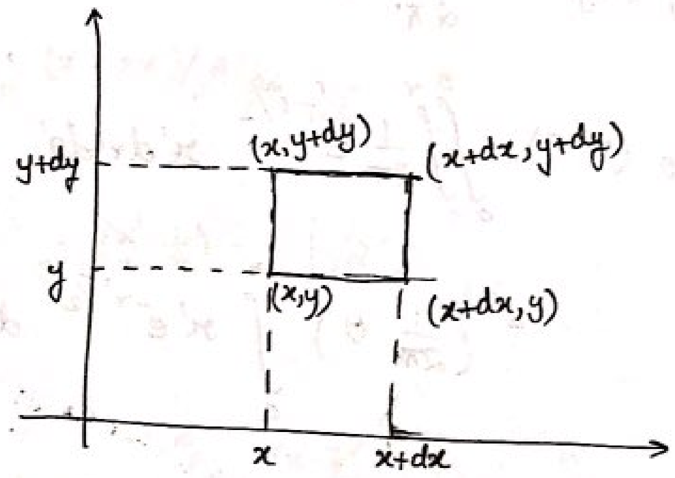
Θ is uniform r.v on $(0, 2\pi)$

R is Rayleigh random variable

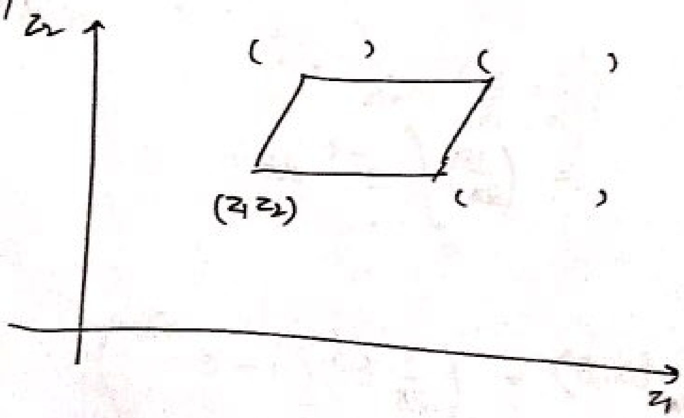
* **Linear transformation of Random variables**

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix} = A^{-1} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

$$f_{XY}(x,y)\, dx\, dy \approx \int_x^{x+dx} \int_y^{y+dy} f_{XY}(x', y')\, dx'\, dy'$$



$f_{Z_1 Z_2}(z_1\, z_2) |dP|$



$dP = |A|\, dx\, dy$

$$f_{z_1 z_2}(z_1\ z_2) = \frac{f_{X,Y}(x,y)(dx\,dy)}{|A|\,dx\,dy}$$

$$f_{z_1,z_2}(z_1 z_2) = \frac{f_{XY}-\left(A^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)}{|A|}$$

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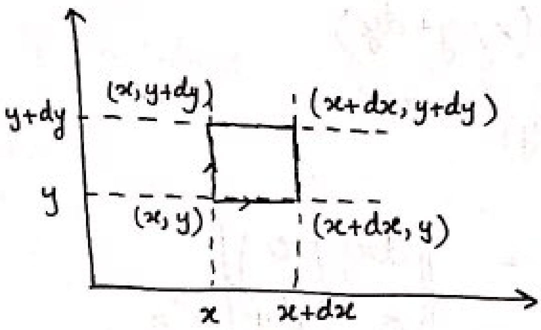
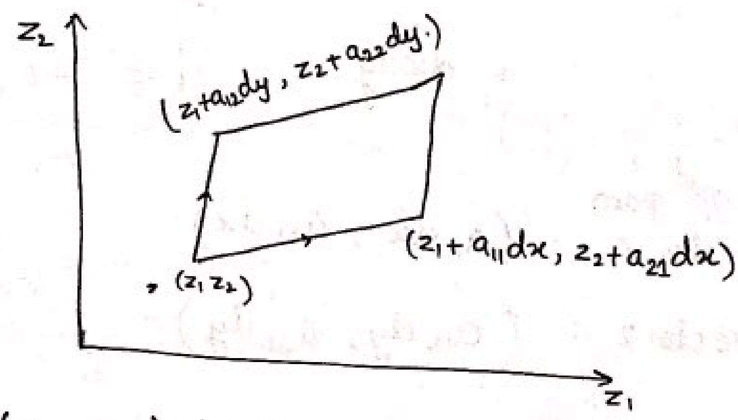
joint pdf of linear transformation.

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \Rightarrow \vec{Z} = A\vec{X}$$



$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$



$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x+dx \\ y \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} dx \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} a_{11}dx \\ a_{21}dx \end{pmatrix} = \begin{pmatrix} z_1 + a_{11}dx \\ z_2 + a_{21}dx \end{pmatrix}$$

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$(x, y+dy) \longmapsto (z_1 + a_{12}\,dy,\ z_2 + a_{22}\,dy)$

$(x+dx, y+dy) \longmapsto (z_1 + a_{11}\,dx + a_{12}\,dy,$
$z_2 + a_{21}\,dx + a_{22}\,dy)$

$$f_{X,Y}(x,y)\,dx\,dy = f_{Z_1 Z_2}(z_1, z_2)\,|dP|$$

(*) $|dP|$ = area of parallelogram. (It is the crossproduct of 2 sides)

$(x,y) \longrightarrow (x+dx, y)$

vector $(dx, 0)$

$(x,y) \longrightarrow (x, y+dy)$

vector $(0, dy)$

Area of rect $= \left| \begin{pmatrix} dx \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ dy \end{pmatrix} \right|$

thats what dxdy repts

$= dx\,dy$

for $\|^{el}$gram

$\longrightarrow$ vector 1 : $(a_{11}\,dx,\ a_{21}\,dx)$

$\longrightarrow$ vector 2 : $(a_{12}\,dy,\ a_{22}\,dy)$

$$\text{Area} = \begin{vmatrix} a_{11}\,dx & a_{12}\,dy \\ a_{21}\,dx & a_{22}\,dy \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})\,dx\,dy$$

$$= |A|\,dx\,dy$$

$$f_{z_1,z_2}(z_1 z_2) = \frac{f_{X,Y}(x,y)\,dx\,dy}{|A|\,dx\,dy}$$

[we want pdf f_Z in terms of z_1, z_2 so we want to remove x, y terms]

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad = \boxed{\frac{f_{X,Y}(A^{-1}z)}{|A|}}$$

$$f_{\underline{Z}}(\underline{z}) = \frac{f_{\underline{X}}(A^{-1}\underline{z})}{|A|}$$

<u>Ex:</u>

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

where $X \sim N(0,1)$, $Y \sim N(0,1)$, $\rho_{XY} = \rho \neq 0$

Find $f_{Z_1 Z_2}(z_1 z_2)$?

$|A| = \frac{1}{\sqrt{2}}(1+1) = \sqrt{2}$, $\quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$f_{Z_1 Z_2}(z_1, z_2) = \frac{f_{XY}\left(A^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)}{|A|} \qquad A^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{z_1 - z_2}{2} \\ \frac{z_1 + z_2}{2} \end{pmatrix}$$

$$= \frac{f_{X,Y}\left(\frac{z_1 - z_2}{2}, \frac{z_1 + z_2}{2}\right)}{\sqrt{2}}$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)}$$

$$\left[\left(\frac{z_1 - z_2}{2}\right)^2 - 2\rho\left(\frac{z_1 - z_2}{2}\right)\left(\frac{z_1 + z_2}{2}\right) + \left(\frac{z_1 + z_2}{2}\right)^2\right] \times \frac{1}{2(1-\rho^2)}$$

$$= \left[z_1^2 + z_2^2 - \cancel{2z_1 z_2} - 2\rho z_1^2 + 2\rho z_2^2 + z_1^2 + z_2^2 + \cancel{2z_1 z_2} \right] \times \frac{1}{8(1-\rho^2)}$$

$$= 2\left[z_1^2(1-\rho) + z_2^2(1-\rho) \right] \times \frac{1}{8(1-\rho^2)}$$

$$f_{z_1 z_2}(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sqrt{2}}\; e^{-\left[\frac{z_1^2}{4(1+\rho)} + \frac{z_2^2}{4(1-\rho)}\right]}$$

$z_1\ z_2$ are independent normal r.v

mean is 0

~~variance~~

$$\sigma_{z_1}^2 = 2(1+\rho)$$

$$\sigma_{z_2}^2 = 2(1-\rho)$$

$$z_1 = g_1(X, Y)$$

need not be linear

$$z_2 = g_2(X, Y)$$

$$F_{z_1 z_2}(z_1, z_2) = \int_{R_{z_1 z_2}} f_{X,Y}(x', y')\, dx'\, dy'$$

$$R_{z_1 z_2} = \left\{ (x,y) \in \mathbb{R}^2 : g_1(x,y) \le z_1 \text{ and } g_2(x,y) \le z_2 \right\}$$

$$f_{z_1 z_2}(z_1, z_2)\,|dP| = f_{x,y}(x,y)\, dx\, dy.$$

$|dP|$ given as jacobian of the transformation.

$$|dP| = \left| \frac{\partial(z_1 z_2)}{\partial(x,y)} \right| dx\, dy$$

$$\left| \frac{\partial(z_1 z_2)}{\partial(x,y)} \right| = \begin{vmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial y} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix}$$

$$g_1(x,y) = a_{11}x + a_{12}y.$$
$$g_2(x,y) = a_{21}x + a_{22}y$$

[In lt jacobian is matrix of the transformation
↓
linear trasformation]

Ex $z_1 = X + Y$

$z_2 = \dfrac{X}{X+Y} = \dfrac{X}{z_1}$ $\Rightarrow X = z_1 z_2$
$\Rightarrow Y = z_1 - X = z_1 - z_1 z_2$

$f_{z_1 z_2}(z_1, z_2) = ?$

for finding $|dP|$ =

$$\left| \frac{\partial(z_1; z_2)}{\partial(x,y)} \right| = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix}$$

$$\frac{\partial z_2}{\partial x} = \frac{1}{x+y} - \frac{x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$\frac{\partial z_2}{\partial y} = \frac{-x}{(x+y)^2}$$

$$= -\frac{x}{(x+y)^2} - \frac{y}{(x+y)^2} = \frac{-1}{x+y} = \left| \frac{-1}{z_1} \right| = \frac{1}{z_1}$$

$$f_{Z_1 Z_2}(z_1, z_2) = \frac{f_{X,Y}(z_1 z_2,\ z_1 - z_1 z_2)}{|z_1|}$$

Joint pdf of linear transformation.

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In the prev ex

$$\vec{Z} = A\vec{X}$$

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \vec{X} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$f_{\vec{Z}}(z_1, z_2) = \frac{f_{X,Y}(A^{-1}\vec{Z})}{|A|}$$

$X \sim N(0,1) \quad Y \sim N(0,1), \quad \rho_{x,y} = \rho \neq 0$

$$|A| = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$$

$$A^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad A^{-1}\vec{Z} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

$$= \frac{Z_1 - Z_2}{\sqrt{2}}$$

$$\frac{z_1 + z_2}{\sqrt{2}}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)}$$

$$f_{\vec{Z}}(z_1, z_2) = f_{XY}\left(\frac{z_1 - z_2}{\sqrt{2}}, \frac{z_1 + z_2}{\sqrt{2}}\right)$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(\left(\frac{z_1-z_2}{\sqrt{2}}\right)^2 - 2\rho\left(\frac{z_1-z_2}{\sqrt{2}}\right)\left(\frac{z_1+z_2}{\sqrt{2}}\right) + \left(\frac{z_1+z_2}{2}\right)^2\right) / 2(1-\rho^2)}$$

$$\frac{1}{2(1-\rho^2)} \times \frac{1}{2}\left[z_1^2 + z_2^2 - 2z_1z_2 - 2\rho z_1^2 + 2\rho z_2^2 + z_1^2 + z_2^2 + 2z_1z_2\right]$$

$$= \frac{1}{2(1-\rho^2)} \left[z_1^2(1-\rho) + z_2^2(1+\rho)\right]$$

$$f_{\vec{Z}}(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left[\frac{z_1^2}{2(1+\rho)} + \frac{z_2^2}{2(1-\rho)}\right]}$$

$z_1 \sim N(0, 1+\rho)$ $\qquad$ $z_2 \sim N(0, 1-\rho)$

$\rho_{z_1, z_2} = 0$.

Random Vectors: $\quad \vec{X} = (X_1, X_2, \ldots X_n)$

where X_i is a random variable for $i = 1, 2, \ldots n$

Let A be an n-dimensional event in product form.

$$P(\vec{X} \in A) = P(X_1 \in A_1, X_2 \in A_2, \ldots X_n \in A_n)$$

where $A = A_1 \times A_2 \times A_3 \times \cdots A_n$

joint cdf of $\vec{X}$

$$F_{\vec{X}}(x_1, x_2, \ldots x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots X_n \le x_n)$$

$$(x_1, x_2, \ldots x_n) \in \mathbb{R}^n$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f_{\vec{X}}(s_1, s_2, \ldots s_n)\, ds_1\, ds_2 \ldots ds_n$$

If $\vec{X}$ is jointly continuous random vec

where $f_{\vec{X}}(x_1, x_2, \ldots x_n)$ is joint pdf of $\vec{X}$

$$f_{\vec{X}}(x_1, x_2 \ldots x_n) = \frac{\partial^n F_{\vec{X}}(x_1, x_2 \ldots x_n)}{\partial x_1\, \partial x_2 \ldots \partial x_n}$$

marginal cdf, pdf

$$F_{X_1}(x_1) = F_{\vec{X}}(x_1, \infty, \ldots \infty)$$

$$F_{X_1, X_2}(x_1, x_2) = F_{\vec{X}}(x_1, x_2, \infty \ldots \infty)$$

$\vdots$

$$F_{X_1 X_2 \ldots X_{n-1}}(x_1, x_2 \ldots x_{n-1}) = F_X(x_1, x_2, x_3 \ldots x_{n-1}, \infty)$$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\vec{X}}(x_1, x_2, \ldots x_n)\, dx_2 \ldots dx_n$$

$(n-1)$ integrals.

$$f_{X_1, X_2}(x_1, x_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\vec{X}}(x_1, x_2, \ldots x_n)\, dx_3 \ldots dx_n$$

Conditional cdf and conditional pdf:

$\Rightarrow F_{X_1 X_2}(x_1, x_2) = F_{X_1}(x_1|x_2)\, F_{X_2}(x_2)$

$F_{\underline{X}}(x_1, x_2, \dots x_n) = F_{X_1}(x_1|x_2, \dots x_n)\, F_{X_2 \dots X_n}(x_2, \dots x_n)$

$= F_{X_1}(x_1|x_2 \dots x_n)\, F_{X_2}(x_2|x_3, \dots, x_n) \dots$

$\dots F_{X_{n-1}}(x_{n-1}|x_n)\, F_{X_n}(x_n)$

Ex: Let (X_1, X_2, X_3) be jointly Normal random ~~variable~~ vector such that

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \frac{1}{2\pi\sqrt{\pi}} e^{-\left(x_1^2 + x_2^2 - \sqrt{2}x_1 x_2 + \frac{1}{2}x_3^2\right)}$$

Find marginal pdf $f_{X_1 X_3}(x_1, x_3)$ and conditional pdf $f_{X_2}(x_2|x_1, x_3)$

$$f_{X_1 X_3}(x_1, x_3) = \int_{-\infty}^{\infty} f_{X_1 X_2 X_3}(x_1, x_2, x_3)\, dx_2$$

$$= \frac{1}{2\pi\sqrt{\pi}} \left(\int_{-\infty}^{\infty} e^{-(x_2^2 - \sqrt{2}x_1 x_2)} dx_2 \right) e^{-x_1^2} e^{-x_3^2/2}$$

$$= \left(\frac{e^{-x_1^2/2}}{\sqrt{2\pi}} \right) \left(\frac{e^{-x_3^2/2}}{\sqrt{2\pi}} \right) \left(\frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\left(x_2 - \frac{x_1}{\sqrt{2}}\right)^2 / 2\left(\frac{1}{2}\right)} dx_2 \right)$$

$$= \left(\frac{e^{-x_1^2/2}}{\sqrt{2\pi}}\right)\left(\frac{e^{-x_3^2/2}}{\sqrt{2\pi}}\right)$$

$X_1 \sim N(0,1) \quad X_3 \sim N(0,1) \quad \rho_{x_1 x_3} = 0$

$$f_{X_2}(x_2 \mid x_1, x_3) = \frac{f_{X_1, X_2 X_3}(x_1, x_2, x_3)}{f_{X_1, X_3}(x_1, x_3)}$$

$$= \frac{1}{2\pi\sqrt{\pi}} e^{-(x_1^2 + x_2^2 - \sqrt{2}x_1x_2 + \frac{1}{2}x_3^2)} \times \frac{}{e^{-(}}$$

$$= \cancel{\frac{1}{\sqrt{\pi}} e^{-(x_2^2 - \sqrt{2}x_1x_2 -}}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\left(\frac{x_1^2}{2} + x_2^2 - \sqrt{2}x_1x_2\right)}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\left[\left(x_2 - \frac{\sqrt{2}x_1}{2}\right)^2 - \frac{x_1^2}{2} + \frac{x_1^2}{2}\right]}$$

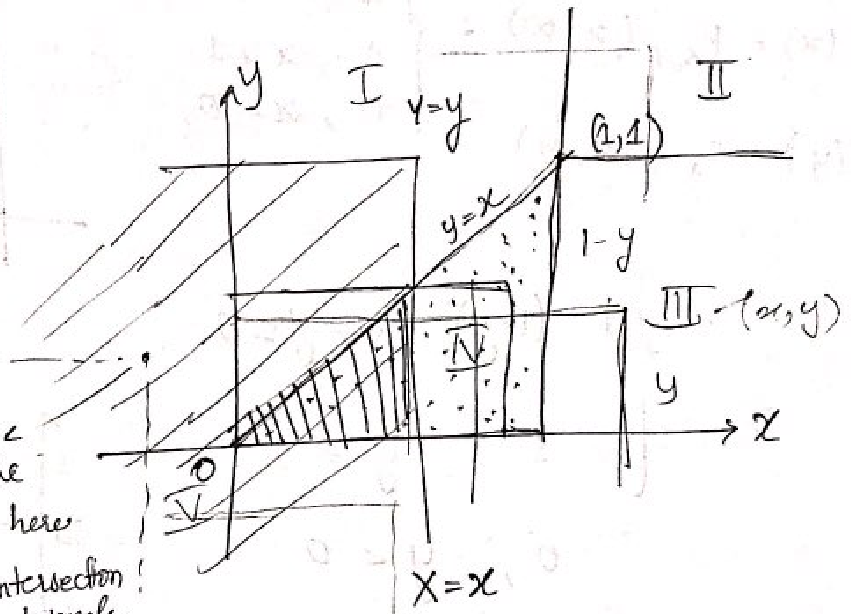
$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}} e^{-\left(x_2 - \frac{x_1}{\sqrt{2}}\right)^2 / 2\left(\frac{1}{2}\right)}$$

$$X_2 \mid X_1, X_3 \sim N\left(\frac{x_1}{\sqrt{2}}, \frac{1}{2}\right)$$

Tut - 5.

(2) $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\} = S$, sample space

$$F_{X,Y}(x,y) = P(X \le x,\ Y \le y),\quad (x,y) \in \mathbb{R}^2$$



If the point is here there is no intersection with the triangle.

$$= \begin{cases} 0, & \{x<0\} \cup \{y<0\} \\ \dfrac{\frac{1}{2}x^2}{1/2}, & (x,y) \in \mathrm{I} \\ \dfrac{xy - \frac{1}{2}y^2}{1/2} & (x,y) \in \mathrm{IV} \\ \dfrac{\frac{1}{2} - \frac{1}{2}(1-y)^2}{1/2} & (x,y) \in \mathrm{III} \\ 1, & (x,y) \in \mathrm{II} \\ 0, & (x,y) \in \mathrm{V} \end{cases}$$

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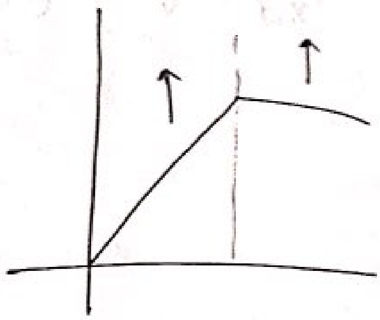
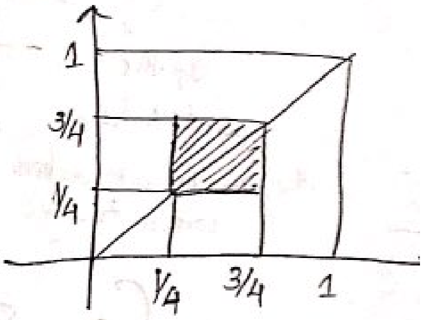
$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{1/2}, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} x^2, & 0 \le x < 1 \\ 1, & x > 1 \\ 0, & x < 0 \end{cases}$$



$$F_Y(y) = F_{X,Y}(\infty, y)$$

$$= \begin{cases} 1-(1-y)^2, & 0 \le y \le 1 \\ 1, & y \ge 1 \\ 0, & y < 0 \end{cases}$$



$$A = \left\{ X \le \frac{1}{2}, \; Y \le \frac{3}{4} \right\}$$

$$P(A) = F_{XY}\left(\frac{1}{2}, \frac{3}{4}\right) = F_{XY}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$P(B) = F_{XY}\left(\frac{3}{4}, \frac{3}{4}\right)$$

$$+ F_{XY}\left(\frac{1}{4} \, \frac{1}{4}\right) - F_{XY}\left(\frac{3}{4} \, \frac{1}{4}\right)$$

$$- F_{XY}\left(\frac{1}{4} \, \frac{3}{4}\right)$$

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(i) $F_{XY}(x,y) \geq 0 \quad \forall\, x,y \in \mathbb{R}^2$

(ii) $\dfrac{\partial F_{XY}}{\partial x}(x,y) = \begin{cases} \frac{2}{x^2+y^2}, & x>1,\ y>1 \\ 0, & \text{otherwise} \end{cases} \geq 0$

(iii) $\dfrac{\partial F_{XY}}{\partial y}(x,y) = \begin{cases} \frac{2}{x^2+y^2} & x>1\ \ y>1 \\ 0, & \text{otherwise} \end{cases}$

(iv) $F_{XY}(x,-\infty) = 0 = F_{XY}(-\infty, y)$

v) $F_{XY}(\infty,\infty) = 1$

(6) $X = \begin{cases} +1 \\ -1 \end{cases} \quad P(X=+1) = p,\ P(X=-1) = 1-p$

output signal: $Y = X + N$, $N \sim (0, 0.05 = \sigma^2)$

1) Find $P(X=j,\ Y \leq y) = P(Y \leq y \mid X=j)\, P(X=j)$

$$P(Y \leq y \mid X=j) = P(X+N \leq y \mid X=j)$$

$$= P(X+N \leq y \mid X=j)$$

$$= P(N \leq y-j \mid X=j$$

$$= P(N \leq y-j) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{y-1} e^{-x^2/2\sigma^2}\,dx$$

$$P(X=+1,\ Y \leq y) = P(N \leq y-1)\,P(X=+1) = \frac{p}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{y-1} e^{-x^2/2\sigma^2}\,dx$$

$$P(X=-1, Y\le y) = P(N \le y+1)\, P(X=-1)$$

$$= \frac{(1-p)}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{y+1} e^{-x^2/2\sigma^2}\, dx.$$

$$P(X=+1) = \lim_{y\to\infty} P(X=+1, Y\le y) = p$$

$$P(X=-1) = \lim_{y\to\infty} P(X=-1, Y\le y) = 1-p$$

$$P(Y\le y) = P(X=+1, Y\le y) + P(X=-1, Y\le y)$$

1/10

Independent Random vector

Random vector $\vec{X}$ is said to be independent

if $F_{\vec{X}}(x_1, x_2, \cdots x_n) = F_{X_1}(x_1)\, F_{X_2}(x_2) \cdots \cdots F_{X_n}(x_n)$

where $\vec{X} = (X_1, X_2, \cdots X_n)$

iid random vector : iid stands for independent and identically distributed random variables

Ex: Let $\vec{X} = (X_1, X_2, \cdots X_n)$ be an iid random vector

Find the cdf of

$V = \max(X_1, X_2, \cdots X_n)$

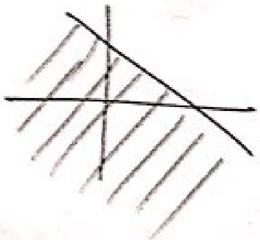
and $W = \min(X_1, X_2 \cdots X_n)$

$$F_V(v) = P(V \le v)$$
$$= P(\max(X_1, X_2, \cdots X_n) \le v)$$
$$= P(X_1 \le v, X_2 \le v, \ldots X_n \le v)$$
$$= P(X_1 \le v)\, P(X_2 \le v) \cdots P(X_n \le v)$$
$$= [F_X(v)]^n$$

$$F_W(\omega) = P(W \le \omega)$$
$$= P(\min(X_1, X_2, \cdots X_n) \le \omega)$$
$$= 1 - P(\min(X_1, X_2, \cdots X_n) > \omega)$$
$$= 1 - P(X_1 > \omega, X_2 > \omega, \cdots X_n > \omega)$$
$$= 1 - P(X_1 > \omega)\, P(X_2 > \omega) \cdots P(X_n > \omega)$$
$$= 1 - (1 - P(X_1 \le \omega))(1 - P(X_2 \le \omega)) \cdots\cdots (1 - P(X_n \le \omega))$$
$$= 1 - (1 - F_X(\omega))^n$$

Let $Z = X + Y$ $\quad F_Z(z)$

$$= P(Z \le z)$$
$$= P(X + Y \le z)$$



Eg! Let $Z = X_1 + X_2 + X_3$

Find the pdf of Z

Let $Z_1 = X_1$

$Z_2 = X_1 + X_2$

$Z_3 = X_1 + X_2 + X_3$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$f_{\vec{Z}}(z_1, z_2, z_3) = \frac{f_{\vec{X}}(A^{-1}\vec{z})}{|A|}$$

$|A| = 1$

$x_1 = z_1$

$x_2 = z_2 - z_1$

$x_3 = z_3 - z_2$

$$f_{\vec{Z}}(z_1, z_2, z_3) = f_{\vec{X}}(z_1, z_2 - z_1, z_3 - z_2)$$

$$f_{Z_3}(z_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\vec{Z}}(z_1\ z_2\ z_3)\, dz_1\, dz_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\vec{X}}(z_1, z_2 - z_1, z_3 - z_2)\, dz_1\, dz_2$$

$$f_{\vec{Z}}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, y - x)\, dx$$

If X_1, X_2, X_3 are independent r.v.

$$f_Z(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1}(z_1) f_{X_2}(z_2 - z_1) f_{X_3}(z_3 - z_2)\, dz_1\, dz_2$$

If $z_1 = g_1(x, y)$ $z_2 = g_2(x, y)$

$$f_{z_1 z_2}(z_1, z_2) = \frac{f_{x,y}(h_1(z_1, z_2), h_2(z_1, z_2))}{|J(x, y)|}$$

where $x = h_1(z_1, z_2)$ $y = h_2(z_1, z_2)$

and $J(x, y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix}$

Let $\vec{X} = (X_1, X_2, \cdots X_n)$ is transformed to

$\vec{Z} = (Z_1, Z_2 \cdots Z_n)$ by

$Z_1 = g_1(X_1, X_2, \cdots X_n)$

$Z_2 = g_2(X_1, X_2, \cdots X_n)$

$\vdots$

$Z_n = g_n(X_1, X_2, \cdots X_n)$

then $f_{\vec{Z}}(z_1, z_2, \cdots z_n) =$

$$= \frac{f_{\vec{X}}(h_1(\vec{Z}), h_2(\vec{Z}) \cdots \cdots h_n(\vec{Z}))}{|J(x_1, x_2, x_3 \cdots x_n)|}$$

where $x_i = h_i(\vec{z})$ inverse of g.

$i = 1, 2, \dots, n$

and $J(x_1, x_2, \dots, x_n)$

$$= \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}$$

$Z = X_1 + X_2 + X_3$

$\vec{Z} = A\vec{X}$

$Z_1 = 1 X_1 + 0 \cdot X_2 + 0 X_3$

$Z_2 = 1 \cdot X_1 + 1 \cdot X_2 + 0 \cdot X_3$

$Z_3 = 1 \cdot X_1 + 1 \cdot X_2 + 1 \cdot X_3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Mean of $\vec{X} = (X_1, X_2, \dots X_n)$ (or expectation)

$E(\vec{X}) = \left(E(X_1), E(X_2), \dots E(X_n)\right)$

correlation matrix of $\vec{X}$

$$R_{\vec{X}} = \begin{bmatrix} E(X_1^2) & E(X_1X_2) & \cdots & E(X_1X_n) \\ E(X_2X_1) & & & \\ \vdots & & & \\ E(X_nX_1) & \cdots & & E(X_n^2) \end{bmatrix}_{n \times n}$$

$R_{\vec{X}}$ is symmetric

$R_{\vec{X}} = E(\vec{X}\vec{X}^T)$, $\vec{X}$ is the column vector

11/10 cov

Covariance matrix

$$K_{\vec{X}} = E\left((\vec{X}-E(\vec{X}))((\vec{X}-E(\vec{X}))^T\right)$$

$$= E\left((\vec{X}\vec{X}^T - \vec{X}\,E(\vec{X})^T - E(\vec{X})\,\vec{X}^T + E(\vec{X})\,E(\vec{X})^T\right)$$

$$= R_{\vec{X}} - E(\vec{X})\,E(\vec{X})^T$$

$$\text{Cov}(X,Y) = E(X,Y) - E(X)E(Y)$$

det

Covariance matrix of $\vec{X} = (x_1, x_2, \cdots x_n)$

$$K_{\vec{X}} = E\left[(X - m_{\vec{X}})(X - m_{\vec{X}})^T\right] \quad , \quad m_{\vec{X}} = E(\vec{X})$$

$$= E(\vec{X}\vec{X}^T) - m_{\vec{X}} m_{\vec{X}}^T$$

$$= R_{\vec{X}} - m_{\vec{X}} m_{\vec{X}}^T$$

let $\vec{X} = (X_1, X_2)$

$$K_{\vec{X}} = \begin{bmatrix} var(X_1) & cov(X_1 X_2) \\ cov(X_2 X_1) & var(X_2) \end{bmatrix}$$

cross correlation matrix of 2 random vectors

$\vec{X} = (X_1, X_2, \cdots, X_n)$ and $Y = (Y_1, Y_2, \cdots Y_n)$

$$R_{\vec{X}\vec{Y}} = E(XY^T) = \begin{bmatrix} E(X_1 Y_1) & E(X_1 Y_2) & \cdots & E(X_1 Y_n) \\ E(X_2 Y_1) & E(X_2 Y_2) & \cdots & E(X_2 Y_n) \\ \vdots & & & \\ E(X_n Y_1) & \cdots & \cdots & E(X_n Y_n) \end{bmatrix}$$

cross-covariance of $\vec{X}$ and $\vec{Y}$

$$K_{\vec{X}\vec{Y}} = E\left[(\vec{X} - m_{\vec{X}})(\vec{Y} - m_{\vec{Y}})^T\right]$$

$$K_{\vec{X}\vec{Y}} = R_{\vec{X}\vec{Y}} - m_{\vec{X}} m_{\vec{Y}}^T$$

$$K_{\vec{X}\vec{Y}} \neq K_{\vec{Y}\vec{X}}$$

Consider a linear transformation of random vector $\vec{X}$

$$\vec{Y} = A\vec{X}$$

$$m_{\vec{Y}} = E(\vec{Y}) = E(A\vec{X}) = AE(\vec{X}) = Am_{\vec{X}}$$

$$K_{\vec{Y}} = E\left[(\vec{Y} - m_{\vec{Y}})(\vec{Y} - m_{\vec{Y}})^T\right]$$

$$= E\left[(A\vec{X} - Am_{\vec{X}})(A\vec{X} - Am_{\vec{X}})^T\right]$$

$$= E\left[A(\vec{X} - m_{\vec{X}})(\vec{X} - m_{\vec{X}})^T A^T\right]$$

$$= AE\left[(\vec{X} - m_{\vec{X}})(\vec{X} - m_{\vec{X}})^T\right] A^T$$

$$K_{\vec{Y}} = AK_{\vec{X}}A^T$$

$$R_{\vec{X}\vec{Y}} = E(XY^T) = E(X(AX)^T) = E(XX^TA^T)$$

$$= E(XX^T)A^T$$

$$= R_{\vec{X}}A^T$$

$$\rightarrow K_{\vec{X}\vec{Y}} = R_{\vec{X}\vec{Y}} - m_{\vec{X}}m_{\vec{Y}}^T = R_{\vec{X}}A^T - m_{\vec{X}}(Am_{\vec{X}})^T$$

$$= (R_{\vec{X}} - m_{\vec{X}}m_{\vec{X}}^T)A^T$$

$$= K_{\vec{X}}A^T$$

Ex: $f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \frac{1}{2\pi\sqrt{\pi}} e^{-\left(x_1^2 + x_2^2 - \sqrt{2}\, x_1 x_2 + \frac{1}{2} x_3^2\right)}$

$m_{\underline{X}} = (0, 0, 0) \quad = \frac{e^{-x_3^2/2}}{\sqrt{2\pi}} \quad \frac{e^{-\left(x_1^2 + x_2^2 - 2\frac{1}{\sqrt{2}} x_1 x_2\right)}}{2\pi\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}}$

$$K_{\underline{X}} = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{cov}(X_1, X_2) & \mathrm{cov}(X_1, X_3) \\ \mathrm{cov}(X_1, X_2) & \mathrm{var}(X_2) & \mathrm{cov}(X_2, X_3) \\ \mathrm{cov}(X_1, X_3) & \mathrm{cov}(X_2, X_3) & \mathrm{var}(X_3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{\sigma_1 \sigma_2 2\pi \sqrt{1 - \rho_{X_1 X_2}^2}} \; e^{-\left\{\frac{\left[\left(\frac{x_1 - m_1}{\sigma_1}\right)^2 - 2\rho_{X_1 X_2}\left(\frac{x_1 - m_1}{\sigma_1}\right)\left(\frac{x_2 - m_2}{\sigma_2}\right) + \left(\frac{x_2 - m_2}{\sigma_2}\right)^2\right]}{2\left(1 - \rho_{X_1 X_2}^2\right)}\right\}}$$

$K_X = I K_{\underline{X}} I'$

$$\rightarrow \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Row trans}} K_{\underline{X}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{column trans}}$$

$\vdots$

$K_{\underline{Y}} = A K_{\underline{X}} A^T$

Let $\vec{X}$ be jointly gaussian random vector

$\vec{X} = (x_1, x_2 \cdots x_n)$

$$f_{\vec{X}}(\vec{x}) = \frac{1}{\left(\sqrt{2\pi}\right)^n |K_{\vec{X}}|^{1/2}} e^{-\frac{1}{2}\left[(\vec{x}-m_{\vec{x}}) K_{\vec{X}}^{-1} (\vec{x}-m_{\vec{x}})^T\right]}$$

where $\vec{x} = (x_1, x_2, \ldots\ldots x_n)_{n\times 1}$

For $\vec{X} = (X_1, X_2)$ $\quad \rho_{\vec{X}_1\vec{X}_2} = \dfrac{\operatorname{cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}}$

$$K_{\vec{X}} = \begin{bmatrix} \sigma_{X_1}^2 & \rho_{\vec{X}_1\vec{X}_2}\sigma_{X_1}\sigma_{X_2} \\ \rho_{X_1X_2}\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix}$$

$$K_{\vec{X}}^{-1} = \frac{1}{\sigma_{X_1}^2\sigma_{X_2}^2\left(1-\rho_{X_1X_2}^2\right)} \begin{bmatrix} \sigma_{X_2}^2 & -\rho_{\vec{X}_1\vec{X}_2}\sigma_{X_1}\sigma_{X_2} \\ -\rho_{X_1X_2}\sigma_{X_1}\sigma_{X_2} & \sigma_{X_1}^2 \end{bmatrix}$$

Joint charateristic function of $\vec{X}$

$$\phi_{\vec{X}}(\omega_1, \omega_2, \cdots \omega_n) = E\left(e^{i(\omega_1X_1 + \omega_2X_2 + \cdots \omega_nX_n)}\right)$$

$$= \phi_{\vec{X}}(\vec{\omega}) = E\left(e^{i\vec{\omega}\vec{X}}\right),$$ $\quad \vec{\omega} = (\omega_1, \omega_2 \cdots \omega_n)$, $\vec{X} = (X_1, X_2 \cdots X_n)$

Let $n=2$

$$\phi_{X_1X_2}(\omega_1, \omega_2) = E\left(e^{i(\omega_1X_1 + \omega_2X_2)}\right)$$

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marginal characteristic function

$$\phi_{x_1}(\omega_1) = \phi_{x_1,x_2}(\omega_1, 0) \;;\; \phi_{x_2}(\omega_2) = \phi_{x_1,x_2}(0, \omega_2)$$

If x_1 and x_2 are independent

$$\phi_{x_1,x_2}(\omega_1,\omega_2) = E\left(e^{i(\omega_1 x_1 + \omega_2 x_2)}\right)$$

$$= E\left(e^{i\omega_1 x_1}\, e^{i\omega_2 x_2}\right)$$

$$= E\left(e^{i\omega_1 x_1}\right) E\left(e^{i\omega_2 x_2}\right)$$

$$= \phi_{x_1}(\omega_1)\, \phi_{x_2}(\omega_2)$$

Let $Z = aX + bY$

$$\phi_Z(\omega) = E\left(e^{i\omega z}\right) = E\left(e^{i\omega(aX+bY)}\right)$$

$$= \phi_{X,Y}(\omega a, \omega b)$$

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