Joshua Steier

Problem Statement

Let $A \in M_2(Z)$, be the set of two by two integer matrices.
Prove that $Sin(A) \in M_2(Z)$ iff A^2 is the zero matrix.

First approach: The example As an example, I took the matrix
0 1
0 0

The A^2 of this matrix results in a zero matrix so it works.

I next looked at the Sin of the matrix.

The Sin of a matrix A is given by it's Taylor series expansion:
$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1} \quad = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} \cdots$
So when we plug in the matrix above this results in:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{3!} + \frac{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{5!} - \frac{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{7!} \ldots.$$

This eventually leads to the fact that only $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ remains.

This behavior can be seen for any 2x2 matrix, where $A^2 = 0$
It's determined that the sin expansion simply converges to A, when $A^2 = 0$

Second part: Generalization I began to consider the matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Finding A^2 yields: $\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$ Next, I solved $a^2 + bc + ab + bd = 0$ and $ac + cd + bc + d^2 = 0$ Solving these yielded several cases to consider:

1). When a is not equal to 0, $b = -a$, $d = -c$. Consider the example: $A = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$, $A^2 = 0$ If we compute the sin of this matrix: sin(A) $= \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ - $\frac{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{3!}$ which continues on since $A^2 = 0$ Which just yields the original matrix, which is an integer matrix.

2). When a= 0, b= 0, d= 0 As an example, consider the following:

$\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, $A^2 = 0$, let's compute the sin(A).

Let $n \in Z$

sin(A) $= \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ - $\frac{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}{3!}$...

This continues as the previous example, and the only remaining piece is the original matrix.

In order to prove this bi-conditional statement, we must prove the following: If sin(A) yields an integer matrix A, then $A^2 = 0$.

Eigenvalues can be computed by following Schur's Lemma: which means the if there are two distinct eigenvalues, λ_1 and λ_2 then the eigenvalues are sin(λ_1), sin(λ_2).

Something interesting to note: The eigenvalues of sin(A) are transcendental whenever A has a non-zero eigenvalue, which demonstrates that if sin(A) yields an integer 2x2 matrix, then $A^2 = 0$.