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Charles Pinter: Abstract Algebra, solutions to exercises: Rings

Problem A1:

Prove that the following are commutative rings with unity.

The ring has the addition function: $a + b = a + b - 1$

The ring has the multiplication function: $ab - (a + b) + 2$

Both these functions belong to the integers.

In order for this to be a ring, we should first check that the addition function follows an abelian group.

Let's check the conditions on the addition function to see if it is an abelian group.

First, commutativity $a + b = b + a$

$$a + b = a + b - 1$$

$$b + a = b + a - 1$$

$$a + b - 1 = b + a - 1$$

Next, check associativity

$$a + (b + c) = (a + b) + c$$

Starting on the left hand side(LHS):

$$a + (b + c) = a + b + c - 1 = a + b + c - 2$$

Now for the right hand side(RHS):

$$(a + b) + c = a + b - 1 + c$$

$$= a + b - 1 + c - 1$$

$$= a + b + c - 2$$

These two are equal, showing we have associativity.

We now must check for an identity element, and find it if it exists.

$$a + e = a$$

$$e = 1$$

Let's attempt to find inverses

$$a + a' = e$$

$$a' = 2 - a$$

Now that these conditions are checked, the addition operation follows as an abelian group.

Now, we must check if the multiplication function is associative.

$$(a * b) * c = a * (b * c)$$

Starting with the LHS:

$$a * b = ab - (a + b) + 2$$

$$(a * b) * c = abc - ac - bc - ab + a + b + c$$

Now the RHS:

$$a * (b * c)$$

$$b * c = bc - (b + c) + 2$$

$$a * (b * c) = abc - ab - ac + a - bc + b + c$$

These two are the same, thus this condition is satisfied.

Then, we must check that our functions are distributive over addition.

$$a(b + c) = ab + ca$$

$$a * (b + c) = a * b + c * a$$

LHS:

$$ab + ac - b - c + 3$$

RHS:

$$ab + ca$$

$$ab + ac - b - c + 3$$

These results are the same so this must be true.

Now, check if this is a **commutative** ring.

We must simply check if the multiplication operation is commutative.

So, $a * b = b * a$

$$a * b = ab - (a + b) + 2$$

$$b * a = ba - (b + a) + 2$$

These results are the same so this is a commutative ring.

Now we check for unity.

First I'll look for an identity element in the multiplication function if it exists.