Journal Problem

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Problem: Prove $\int_0^1 \frac{x\ln(1+x)}{1+x^2}dx = \frac{\pi^2}{96} + \frac{(\ln 2)^2}{8}$

Solution:(Joshua Steier)

First, introduce a variable t. This leads to the following declaration: $F(t) = \int_0^1 \frac{x\ln(1+xt)}{1+x^2}dx$.

Next, differentiate $F(t)$.

This leads to $F'(t) = \int_0^1 \frac{x^2}{(1+x^2)(1+xt)}dx$. Note that the x's will disappear, in favor of the parameter t.

Then, do a partial fraction decomposition which yields:
$F'(t) = \frac{\ln(1+t)}{t(1+t^2)} + \frac{t}{1+t^2}\frac{\ln(2)}{2} - \frac{1}{1+t^2}\frac{\pi}{4}$

Do another partial fraction decomposition which leads to the following: $F'(t) = \frac{\ln(1+t)}{t} - \frac{\ln(1+t)}{1+t^2} + \frac{t}{1+t^2}\frac{\ln(2)}{2} - \frac{1}{1+t^2}\frac{\pi}{4}$

Now, integrating by both sides from 0 to 1, this yields:
$\frac{\pi^2}{96} + \frac{(\ln^2)^2}{8}$