MATH 55: Harvard's Honors Abstract Algebra

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Introduction

MATH 55 is an advanced undergraduate course at Harvard University. I found course notes and problem sets online, and decided to do problem sets. MATH 55 is notiroius for being "the most difficult math course at Harvard University". The course is Honors Abstract Algebra, and consists of using three books: 1). Naive Set Theory, by Halmos 2). Algebra, by Michael Artin, 3). Linear Algebra Done The Right Way by Sheldon Axler. I have not found any lectures online from MATH 55, so I may not have the full experience. However, the problem sets are important regardless. I started by reading Naive Set Theory, and finished that book before moving to Algebra. I read the first three chapters of Algebra, and did textbook problems until I felt comfortable moving to Linear Algebra Done The Right Way. The following will be answers to the PSET's for MATH 55.

First problem set- Vector Spaces: Problem 3: The problem states the following: Prove -(-v)=v for every $v \in V$.

Proof:

Suppose $v \in V$, then

-v= -1(-1v) + (-v) = 0

= (-1(-1))v + -1v

= ((-1(-1)) + -1)v = 0

0v= 0

=0, so these are additive inverses, -(-v) = v, $v \in V$

First problem set- Vector Spaces: Problem 4: Suppose $a \in F$, $v \in V$, and $av = 0$. Prove that a= 0 or v= 0.

If a is not equal to 0, then a has inverse a^{-1} such that $a^{-1} * a$ = 1.

So,

v= 1 * v= $(a^{-1} * a)$v = a^{-1}(av) = a^{-1} * 0 = 0.

First problem set- Vector Spaces: Problem 5:

For each of the following subsets of F^3, determine whether it is a subspace of F^3 :

a). $(x_1, x_2, x_3) \in F^3$: $x_1 + 2x_2 + 3x_3 = 0$;

It is a subspace of F^3 , all we need to check is additivity, closed under addition, and closed under scalar multiplication

Let's check each piece:

1). Check if $0 \in V$: (0,0,0) works

2). closed under addition: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} =$

$\begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \end{pmatrix}$ we subsitute:

$x_1 + x_1' + 2(x_2 + x_2') + 3(x_3 + x_3') = 0$
$x_1 + x_1' + 2x_2 + 2x_2' + 3x_3 + 3x_3' = 0$
$(x_1 + 2x_2 + 3x_3) + (x_1' + 2x_2' + 3x_3') = 0$

$0 + 0 = 0$ For scalar multiplication:

$k * \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = kx_1 + 2(kx_2) + 3(kx_3) = 0$ so $0 = 0$

b). $(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 43$

Check conditions: Assume $\in R^3$

0 exists, but $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ doesn't so cannot work.

c). $(x_1, x_2, x_3) \in F^3 : x_1x_2x_3 = 0$

$0 \in V$ True

and closed under addition? No, so not a supspace of F^3

d). $(x_1, x_2, x_3) \in F^3 : x_1 = 5x_3$

This is a subspace since it is closed under addition, contains the zero vector, and is closed under scalar multiplication.

First Problem Set- Vector Spaces- Problem 6. Give an example of a nonempty subset U of R^2 such that U is closed under addition and under taking additive inverses, but U is not a subspace of R^2

Denote $(x, y) \in R^2 : x, y \in Z$ by U, then U is not empty. If $(x_1, y_1) \in A$ and $(x_2, y_2) \in A$ then, $x_1, x_2, y_1, y_2 \in Z$. Hence $x_1 + x_2$ and $y_1 + y_2$ are integers. This means that $(x_1 + x_2, y_1 + y_2) = (x_1, y_1) + (x_2, y_2) \in U$ i.e. U is closed under addition. Similarly, since $(-x_1, -y_1) \in U$, it follows that U is closed under additive inverses. However, U is not closed under scalar multiplication since $(1, 1) \in A$ while (1/2)* (1,1) is not in U. Hence U is not a subspace.

First Problem Set- Vector Spaces- Problem 7: Give an example of a nonempty subset U of R^2 such that U is closed under scalar multiplication, but U is not a subspace of R^2

One such example is when x= 0 or y= 0, in this case it's closed under scalar multiplication, but not closed under addition, which means it cannot be a subspace of R^2

First Problem Set- Vector Spaces- Problem 8: Prove that the intersection of any collection of subspace of V is a subspace of V In order to satisfy subspace in V, it must have an additive identity, closed under addition, and closed under scalar multiplication.

For additivie identity: $0 \in U_i$ for every $i \in I$, hence 0 is part of the intersection.

Closed under addition: if both x and y are in the intersection, then for $i \in I$, we have $x \in U_1$ and $y \in U_i$ so $x + y \in U_i$ for U_i is closed under addition.

Closed under scalar multiplicaiton: if x is in the intersection, we have for any $k \in F$, $kx \in U_i$, is closed under scalar multplication. Therefore it's in the intersection

First Problem Set- Vector Spaces- Problem 9:

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other

Let's try to prove this by contradiction. If $U \cup W$ is a subspace of V, then $U \nsubseteq W$ and $W \nsubseteq U$. Consider $u \in U/W$ and $w \in W/U$, then $u + w \in U \cup W$ since $U \cup W$ is a subspace of V. So $u + w \in U$ or W. If $u + w \in U$, then we get $w = (u+w) - u \in U$. But this can't be true since if $u+w \in W$, then $u = (u + w) - w \in W$. So if $U \cup W$ is a subspace of V, then the subspaces must be contained in one another.

First Problem Set- Vector Spaces- Problem 10:

Suppose U is a subspace of V. What is $U + U$

Since U is a subspace of V, it must be closed under addition, by the properties of subspaces. So, if $x, y \in U$, we have $x + y \in U$. Also, if $x \in U$, then $x = x + 0 \in U + U$ It follows that $U + U = U$.

First Problem Set- Vector Spaces- Problem 11:

Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

So for $a \in U and b \in W$, because addition on V is commutative, we must have $a + b = b + a \in W + U$. This means, $U + W$ is a subset of $W + U$, and $W + U$ is a subset of $U + W$. So $U + W = W + U$.

First Problem Set- Vector Spaces- Problem 12:

Does the operaiton of addition on subspaces of V have an additive identtity? Which subspaces have additive inverses?

If U is an additive adidity, then if $W \in V$, we must have $U + W = W$. So U must be a susbet of W. So U must be zero. So zero is the additive identity here. Now if the subspace W has additive inverse, then there exists a subspace S of V such that $W + S = 0$. Which can only happen when $W = 0$

First Problem Set- Vector Spaces- Problem 13: Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$U_1 + W = U_2 + W$, then $U_1 = U_2$

One counter example could be: $U_1 = (a, b) \in R^2 : a, b \in R$, $U_2 = (a, 0) \in R^2 : a \in R$

$W = (0, b) \in R^2 : b \in R$ Then one can see that $U_1 + W = U_2 + W$, but U_1 is not equal to U_2