

Faculty of SET / School of Computer and Mathematical Sciences

# COMP SCI 3007/7059/7659
# Artificial Intelligence
# Exact Inference

# Acknowledgement of Country

We acknowledge and pay our respects to the Kaurna people, the traditional custodians whose ancestral lands we gather on.

We acknowledge the deep feelings of attachment and relationship of the Kaurna people to the country and we respect and value their past, present and ongoing connection to the land and cultural beliefs.

# Exact Inference

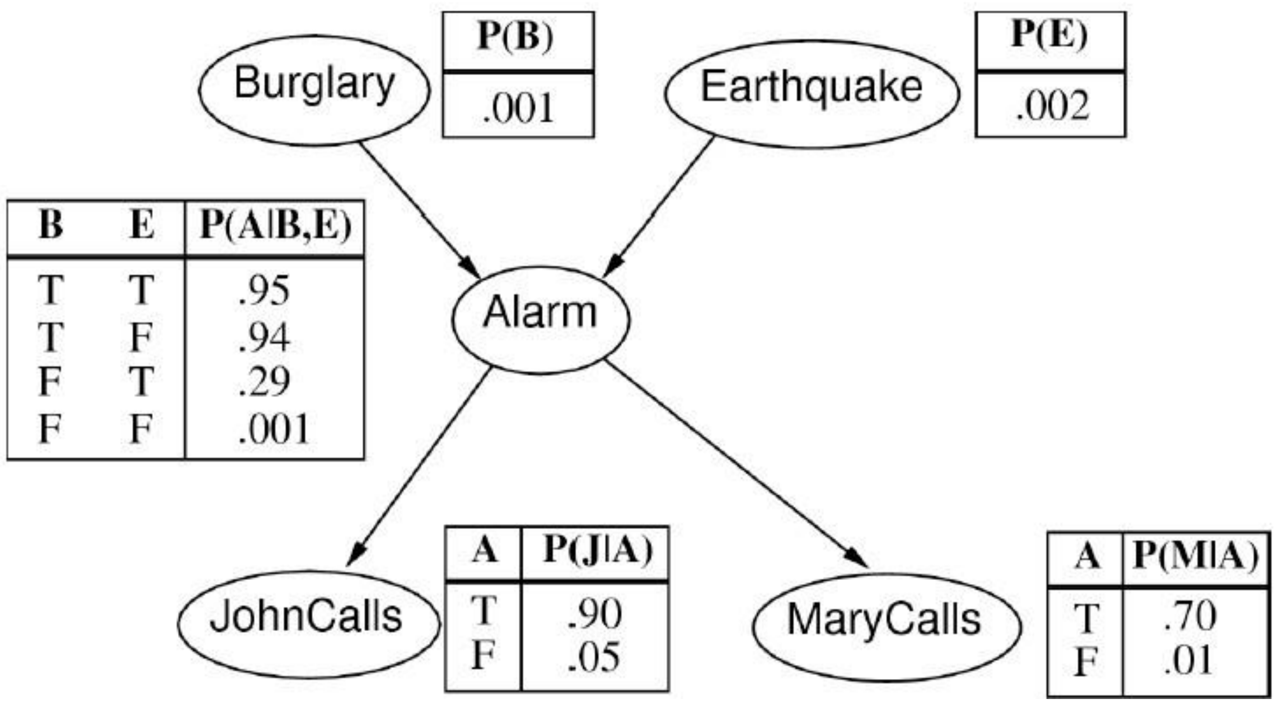
AIMA C14.4

# Outline

- Recap of Bayesian Networks
- Inference by enumeration
- Inference by variable elimination

# Recap:Inference problem

I'm at work, neighbour John called to say my alarm is ringing, but neighbour Mary didn't call. Sometimes it's set off by minor earthquakes. Is there a burglar? $P(b|j, \neg m)$




| P(B) |
|---|
| .001 |

| P(E) |
|---|
| .002 |

| B | E | P(A\|B,E) |
|---|---|---|
| T | T | .95 |
| T | F | .94 |
| F | T | .29 |
| F | F | .001 |

| A | P(J\|A) |
|---|---|
| T | .90 |
| F | .05 |

| A | P(M\|A) |
|---|---|
| T | .70 |
| F | .01 |

# Recap: Global semantic of Bayesian Networks

A Bayesian Network encodes our knowledge on how the variables interact. This is specified in terms of conditional independence assertions.

The global semantics of a network define a joint distribution of all variables as the product of local conditional distributions.

The joint distribution defined by a Bayesian Network with variables $X_1, \ldots, X_n$ is:

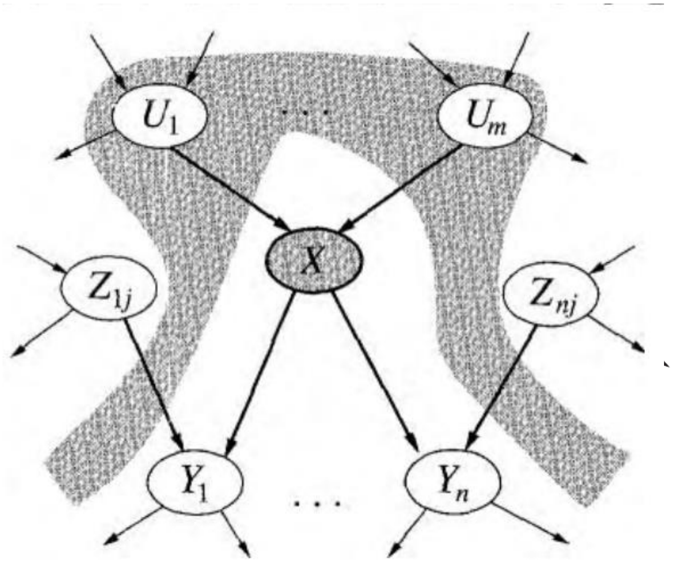
$$\begin{aligned} P(X_1, \ldots, X_n) &= P(X_1|Parents(X_1)) \times P(X_2|Parents(X_2)) \\ &\quad \times \cdots \times P(X_n|Parents(X_n)) \\ &= \prod_{i=1}^{n} P(X_i|Parents(X_i)) \end{aligned}$$

where $Parents(X_i)$ are parents of $X_i$ as specified by the particular Bayesian Network.

# Local Semantics

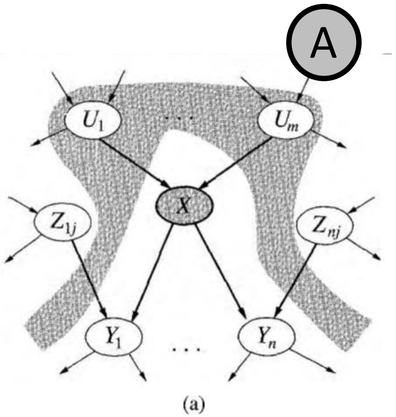
Conditional independence assumptions can simply be "read off" the network topology.

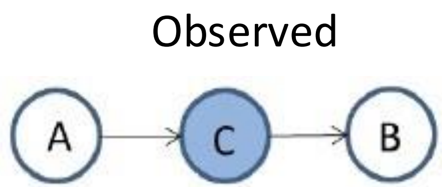
Local semantics: each node is conditionally independent of its non-descendants given its parents.




(a)

# Case 2, Chain (Head-to-Tail)




$$P(A,B|C) = P(A,B,C)/P(C)$$

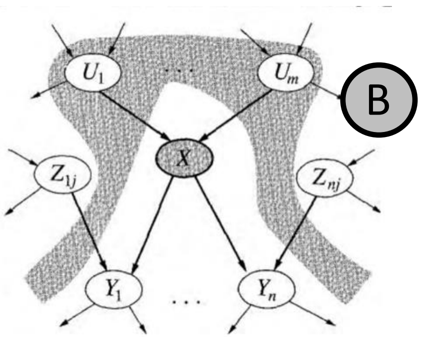
$$= P(B|C)P(C|A)P(A)/P(C)$$

$$= P(B|C)P(A|C)$$

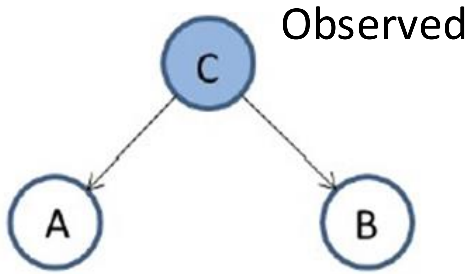
$A$ and $B$ are conditionally independent given $C$:

$$A \perp\!\!\!\perp B | C$$

# Case 1, Fork (Tail-to-Tail)




(a)




$$P(A,B|C) = P(A,B,C)/P(C)$$

Product rule

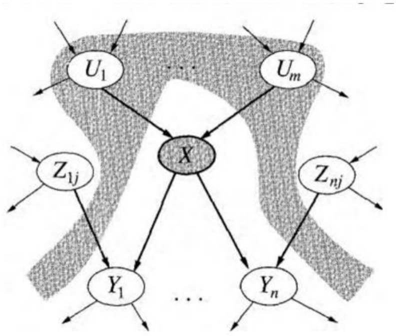
$$= P(A|C)P(B|C)P(C)/P(C)$$

$$= P(A|C)P(B|C)$$

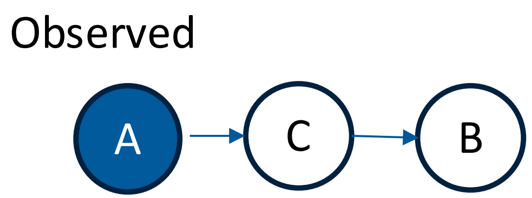
$A$ and $B$ are conditionally independent given $C$

$$A \perp\!\!\!\perp B|C$$

# Case 2.5, Chain (Head-to-Tail)




(a)




$$P(C,B|A) = P(A,B,C)/P(A)$$

$$= P(B|C)P(C|A)P(A)/P(A)$$

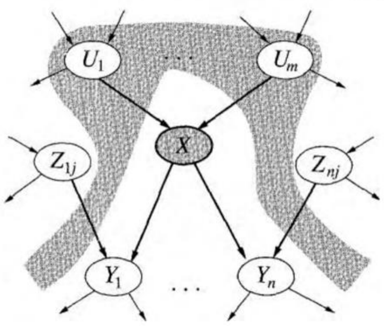
$$= P(B|C)P(C|A)$$

$C$ and $B$ are not conditionally independent given $A$

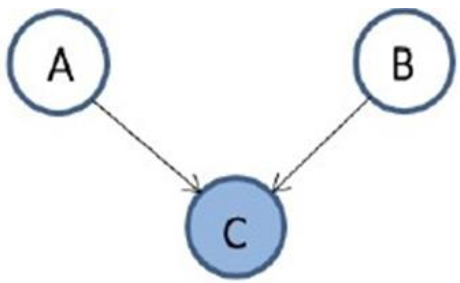
$C \not\perp B \mid A$

# Case 3, Inverted Fork

# (Head-to-Head, Collider, or V-structure)



(a)



Not observed

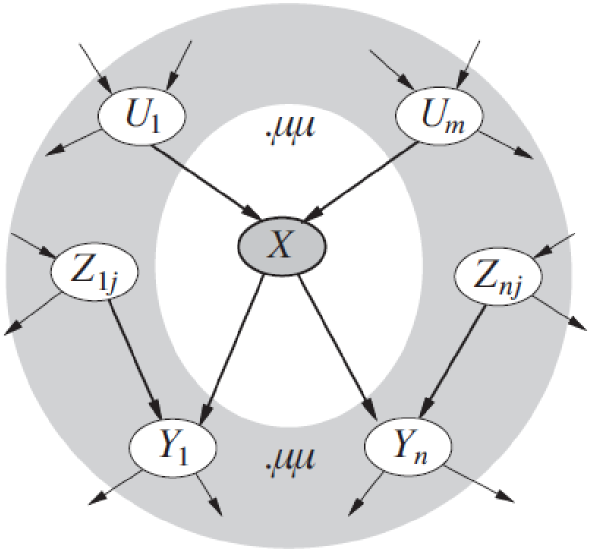
$$P(A, B, C) = P(A)P(B)P(C|A, B)$$

$$P(A, B) = \sum_{C} P(A)P(B)P(C|A, B) = P(A)P(B)$$

so $A \perp\!\!\!\perp B$

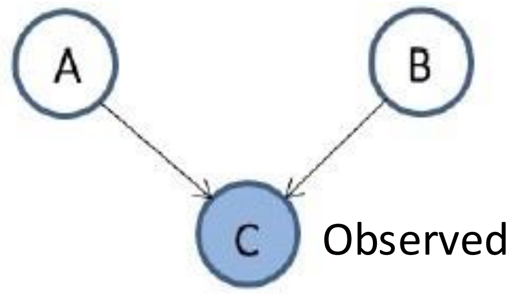
# Markov blanket

A more specific way to state the local semantics: A node is conditionally independent of all others given its parents, children, and children's <u>other</u> parents -- i.e., given the Markov blanket of the node.




Why do we need to consider the children's parents?

# Case 3, Inverted Fork (Head-to-Head)




$$P(A, B|C) = P(A, B, C)/P(C)$$
$$= P(C|A, B)P(A)P(B)/P(C)$$

so $A \not\perp\!\!\!\perp B|C$

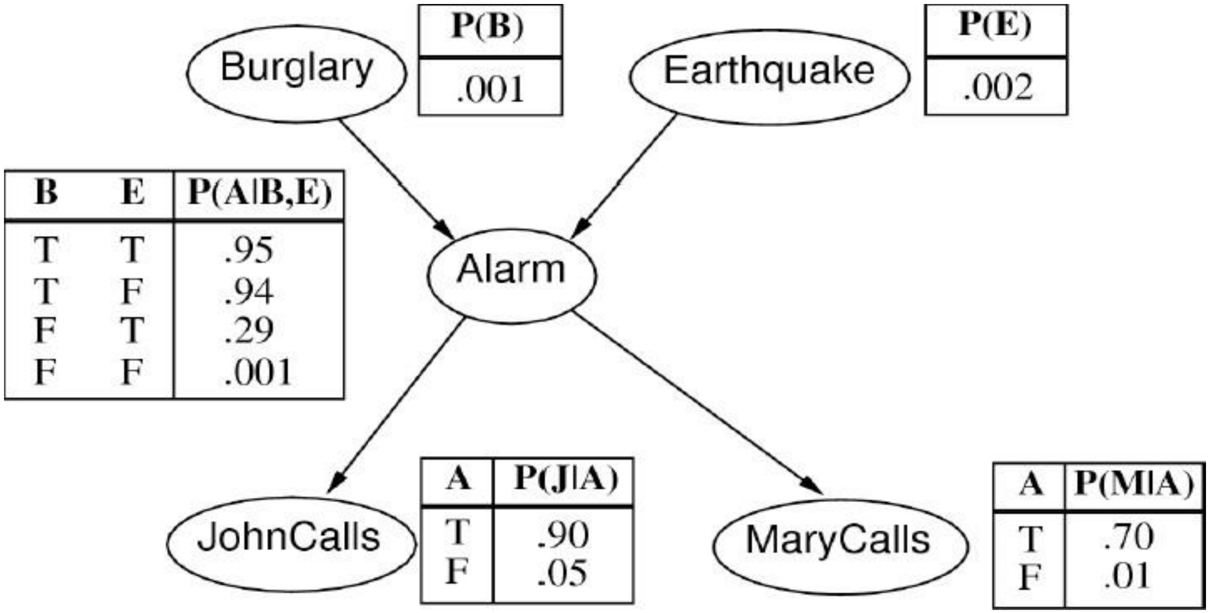
# Recap: Inference

Recall the general rule of statistical inference:

$$P(X|e) = \alpha \sum_{\forall Y} P(X, e, Y)$$

where $X$ is the query variable, $e$ the observed values for the evidence variables, and $Y$ the unobserved variables. As usual $\alpha$ is a normalisation constant that we solve for at the end.

# Performing inference



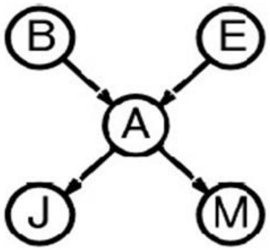

$P(B, E, A, J, M)?$ $\quad P(b|j, \neg m)?$

$$P(J, M, A, B, E) = P(J|A)P(M|A)P(A|B, E)P(B)P(E)$$

# Performing inference on Bayesian Networks

A direct application of the general rule yields

$$P(b|j, \neg m) = \alpha \sum_E \sum_A P(b, j, \neg m, E, A)$$




Observe that the summands are joint probabilities of all the variables. Hence, we introduce the global semantics of the network:

$$P(b|j, \neg m) \quad = \quad \alpha \sum_E \sum_A P(b)P(E)P(A|b, E)P(j|A)P(\neg m|A)$$

# Inference by enumeration

Expanding by enumerating the summands we obtain

$$
\begin{aligned}
P(b|j,\neg m) &= \alpha \sum_E \sum_A P(b)P(E)P(A|b,E)P(j|A)P(\neg m|A) \\
&= \alpha\,[P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a) && \text{e, a} \\
&\quad +P(b)P(e)P(\neg a|b,e)P(j|\neg a)P(\neg m|\neg a) && \text{e, ¬a} \\
&\quad +P(b)P(\neg e)P(a|b,\neg e)P(j|a)P(\neg m|a) && \text{¬e, a} \\
&\quad +P(b)P(\neg e)P(\neg a|b,\neg e)P(j|\neg a)P(\neg m|\neg a)] && \text{¬e, -a}
\end{aligned}
$$

For each of the components on the RHS, we read its value from the appropriate CPT, yielding $P(b|j,\neg m) = \alpha 0.00025677$.

Note that the result does not yet amount to a probability value as we haven't solved for $\alpha$.

# Inference by enumeration

Calculate the scaling factor

To compute $\alpha = \frac{1}{P(j,\neg m)}$ we obtain the marginal probability

$$P(j, \neg m) = \sum_B \sum_E \sum_A P(B, E, A, j, \neg m)$$

# Inference by enumeration

An alternative is to realise that $\langle P(b|j,\neg m), P(\neg b|j,\neg m)\rangle$ is a probability distribution and that $\alpha$ is a normalisation constant that ensures that the probability distribution sums to 1.

Hence, compute $P(\neg b|j,\neg m) = \alpha 0.0498$, using the global semantics and enumerating the summands as before, yielding

$$\langle P(b|j,\neg m), P(\neg b|j,\neg m)\rangle = \alpha\langle 0.00025677, 0.0498\rangle = \langle 0.0051, 0.9949\rangle$$

where $\alpha$ is solved as $\frac{1}{0.00025677+0.0498}$.

# Complexity of inference by enumeration

In the burglar-alarm example, we evaluate the expression

$$
\begin{aligned}
P(b|j,\neg m) &= \alpha \sum_E \sum_A P(b)P(E)P(A|b,E)P(j|A)P(\neg m|A) \\
&= \alpha\,[P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a) \\
&\quad +P(b)P(e)P(\neg a|b,e)P(j|\neg a)P(\neg m|\neg a) \\
&\quad +P(b)P(\neg e)P(a|b,\neg e)P(j|a)P(\neg m|a) \\
&\quad +P(b)P(\neg e)P(\neg a|b,\neg e)P(j|\neg a)P(\neg m|\neg a)]
\end{aligned}
$$

by adding up 4 terms, each obtained by multiplying 5 numbers— In total we need 16 multiplications and 3 additions (excludes the contribution due to term $\alpha$).

# Complexity of inference by enumeration

In the burglar-alarm example, we evaluate the expression

$$P(b|j, \neg m) \quad = \quad \alpha \sum_{E} \sum_{A} P(b)P(E)P(A|b, E)P(j|A)P(\neg m|A)$$

In the worst case, where we have to sum out almost all of the $n$ variables (where we assume they are all Boolean), the complexity of inference by enumeration is $\mathcal{O}(n2^n)$.

This means we will not be able to perform inference by enumeration except for the smallest networks!

# Depth-first Evaluation

An improvement can be achieved by observing that $P(b)$ is a constant that can be moved outside the summations over $E$ and $A$, while $P(e)$ can be moved outside the summation over $A$:

$$P(b|j,\neg m) = \alpha P(b) \sum_E P(E) \sum_A P(A|b,E)P(j|A)P(\neg m|A)$$

Thus we evaluate by looping through the variables in order, multiplying CPT entries as we go.

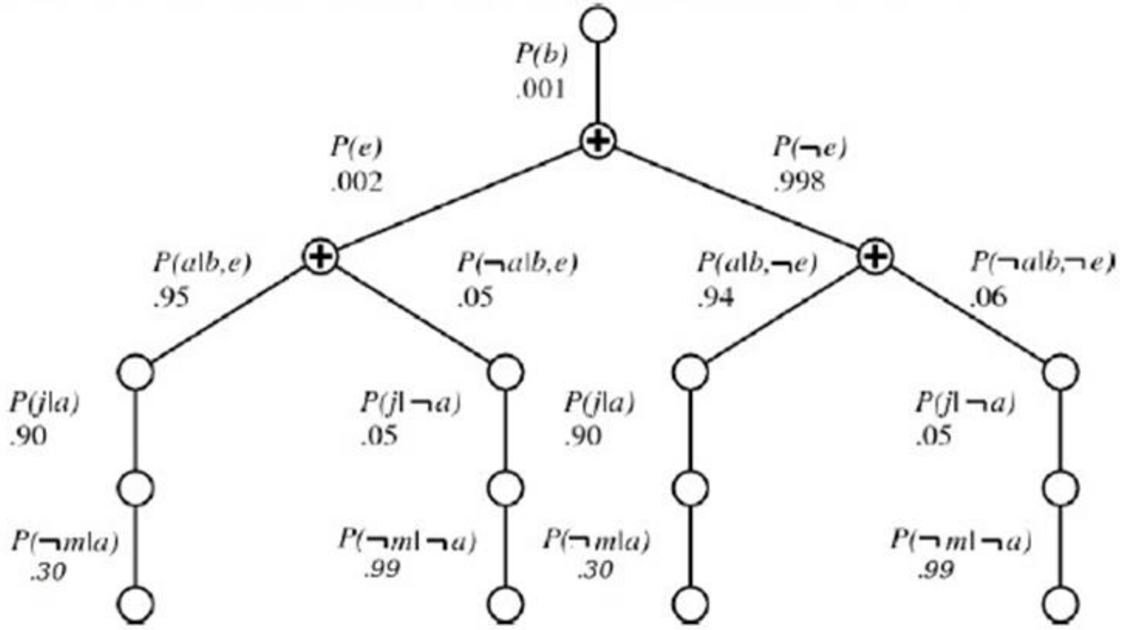
Now evaluating the expression involves 11 multiplications and 3 additions (again excluding the contribution due to term $\alpha$).

$$= \alpha P(b) \sum_E P(E)[P(a|b,E)P(j|a)P(\neg m|a) + P(\neg a|b,E)P(j|\neg a)P(\neg m|\neg a)]$$

# Depth-first Evaluation

$$P(b|j,\neg m) = \alpha P(b) \sum_E P(E) \sum_A P(A|b,E) P(j|A) P(\neg m|A)$$

The process can be illustrated as an evaluation tree.




The evaluation proceeds top-down, multiplying values along each path and summing at the “+” nodes

# Complexity of Depth-first Evaluation

$$
\begin{aligned}
P(b|j,\neg m) &= \alpha \sum_E \sum_A P(b)P(E)P(A|b,E)P(j|A)P(\neg m|A) \\
&= \alpha\,[P(b)P(e)P(a|b,e)P(j|a)P(\neg m|a) \\
&\quad +P(b)P(e)P(\neg a|b,e)P(j|\neg a)P(\neg m|\neg a) \\
&\quad +P(b)P(\neg e)P(a|b,\neg e)P(j|a)P(\neg m|a) \\
&\quad +P(b)P(\neg e)P(\neg a|b,\neg e)P(j|\neg a)P(\neg m|\neg a)]
\end{aligned}
$$

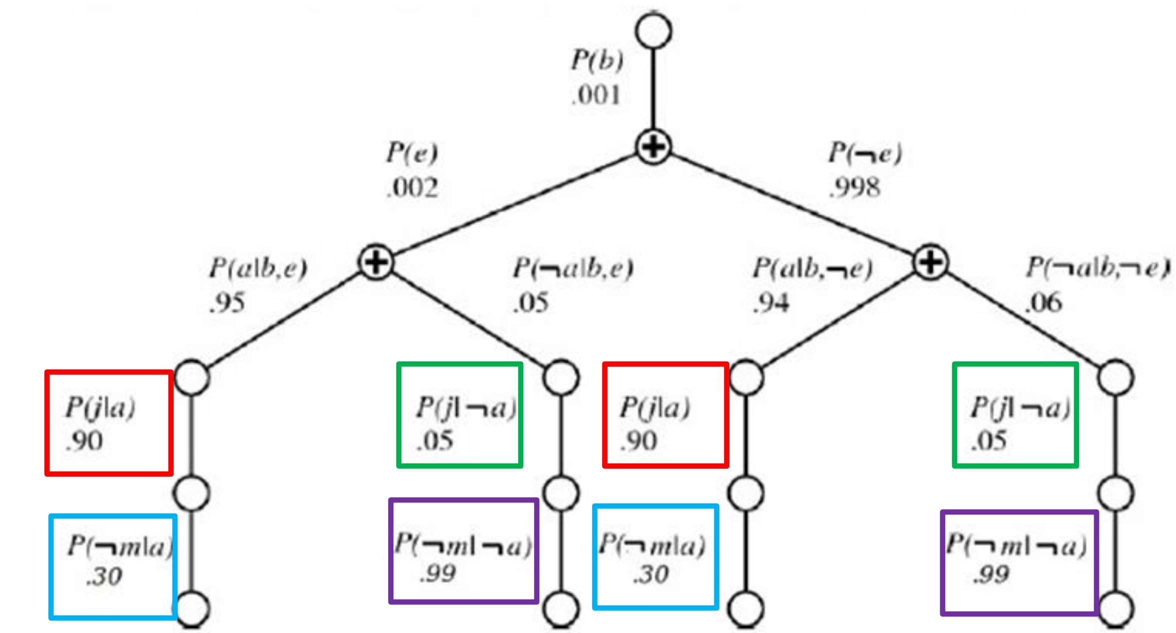
$O(n\,2^n)$

$$P(b|j,\neg m) = \alpha P(b)\sum_E P(E)\sum_A P(A|b,E)P(j|A)P(\neg m|A)$$

$O(2^n)$

# Problem of Depth-First Evaluation

$$P(b|j,\neg m) = \alpha P(b)\sum_{E} P(E)\sum_{A} P(A|b,E)P(j|A)P(\neg m|A)$$




Repeat computation!

# Problem with DF evaluation

We need to do this sum both for $E = true$ and $E = false$. For the first case:

$$P(e)\left(\begin{array}{c} P(b)\left(P(a|b,e)\boxed{P(j|a)P(m|a)} + P(\neg a|b,e)\boxed{P(j|\neg a)P(m|\neg a))} + \right. \\ P(\neg b)\left(P(a|\neg b,e)\boxed{P(j|a)P(m|a)} + P(\neg a|\neg b,e)\boxed{P(j \neg a)P(m|\neg a))}\right. \end{array}\right)$$

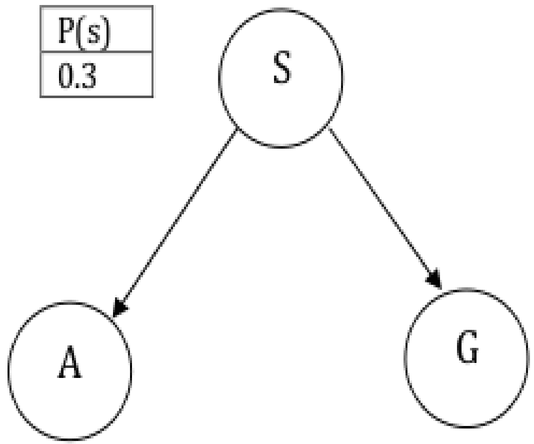
Repeat computation!

# Variable Elimination

- Recursively eliminate variables and merge terms into factors
- Store factors to avoid repeating computations

# Variable Elimination (Fork)

| P(s) |
|---|
| 0.3 |




| S | P(a\|S) |
|---|---|
| t | 0.6 |
| f | 0.3 |

| S | P(g\|S) | P(¬ g\|S) |
|---|---|---|
| t | 0.7 | 0.3 |
| f | 0.2 | 0.8 |

$$P(A) = \sum_S \sum_G P(A, S, G)$$

$$= \sum_S \sum_G P(S)P(A|S)P(G|S)$$

$$= \sum_S \left( P(S)P(A|S) \sum_G P(G|S) \right)$$

$$= \sum_S P(S)P(A|S)$$

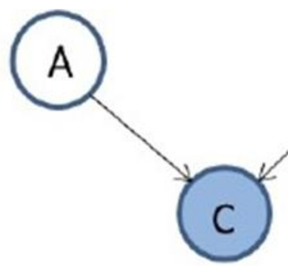
Eliminate G

P( g | s)
+ P(¬ g | s)
= 1

P( g |¬ s)
+ P(¬ g |¬ s)
= 1

# Variable Elimination (Inverted Fork)

| P(a) | P(¬ a) |
|---|---|
| z | 1-z |




| A | B | P(c) | P(¬ c) |
|---|---|---|---|
| t | t | x | 1-x |
| t | f | y | 1-y |
| f | t | m | 1-m |
| f | f | n | 1-n |

f1(B,C) = z*((B==t)*(C==c)*x +
(B==t)*(C==¬ c)*(1-x) +
(B==f)*(C==c)*y +
(B==f)*(C==¬ c)*(1-y))
+ (1-z) * (…)

| B | C | f1(B, C) |
|---|---|---|
| t | c | z*x+(1-z)*m |
| t | ¬ c | z*(1-x)+<br>(1-z)*(1-m) |
| f | c | z*y+(1-z)*n |
| f | ¬ c | z*(1-y)+<br>(1-z)*(1-n) |

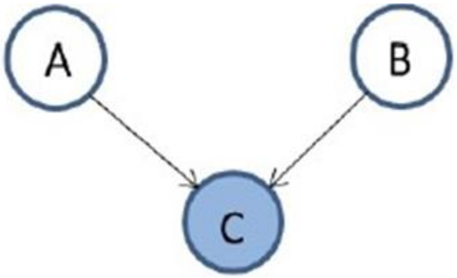
$$
\begin{aligned}
P(C) &= \sum_A \sum_B P(A,B,C) \\
&= \sum_A \sum_B P(A)P(B)P(C|A,B) \\
&= \sum_B P(B) \sum_A P(A)P(C|A,B) \\
&= \sum_B P(B) f_1(B,C) \\
&= f_2(C)
\end{aligned}
$$

Eliminate A F1(B,C) is a factor

Eliminate B

# Variable Elimination-Factor

- A factor associates a real value to each setting of its arguments.
- Factors in Bayesian Networks correspond to conditional probability distributions.
- The joint distribution is a product of factors.




$$P(A, B, C) = P(A)P(B)P(C|A, B)$$

# Variable Elimination -Factor Operation

- Let *X*, *Y* and *Z* are three random variables, and $\phi_1(X,Y)$ and $\phi_2(Y,Z)$ are two factors, their product is a new factor:

$$\psi(X,Y,Z) = \phi_1(X,Y)\phi_2(Y,Z)$$

si as in si-fi fi as in si-fi

- An Example:

$\phi_1$ has 3*2 = 6 entries

$\phi_2$ has 2*2 = 4 entries

yields:

$\psi$ has 3*2*2 = 12 entries

$\phi_1$

| *X* | *Y* | |
|---|---|---|
| $a^1$ | $b^1$ | 0.5 |
| $a^1$ | $b^2$ | 0.8 |
| $a^2$ | $b^1$ | 0.1 |
| $a^2$ | $b^2$ | 0 |
| $a^3$ | $b^1$ | 0.3 |
| $a^3$ | $b^2$ | 0.9 |

$\phi_2$

| *Y* | *Z* | |
|---|---|---|
| $b^1$ | $c^1$ | 0.5 |
| $b^1$ | $c^2$ | 0.7 |
| $b^2$ | $c^1$ | 0.1 |
| $b^2$ | $c^2$ | 0.2 |

$\psi$

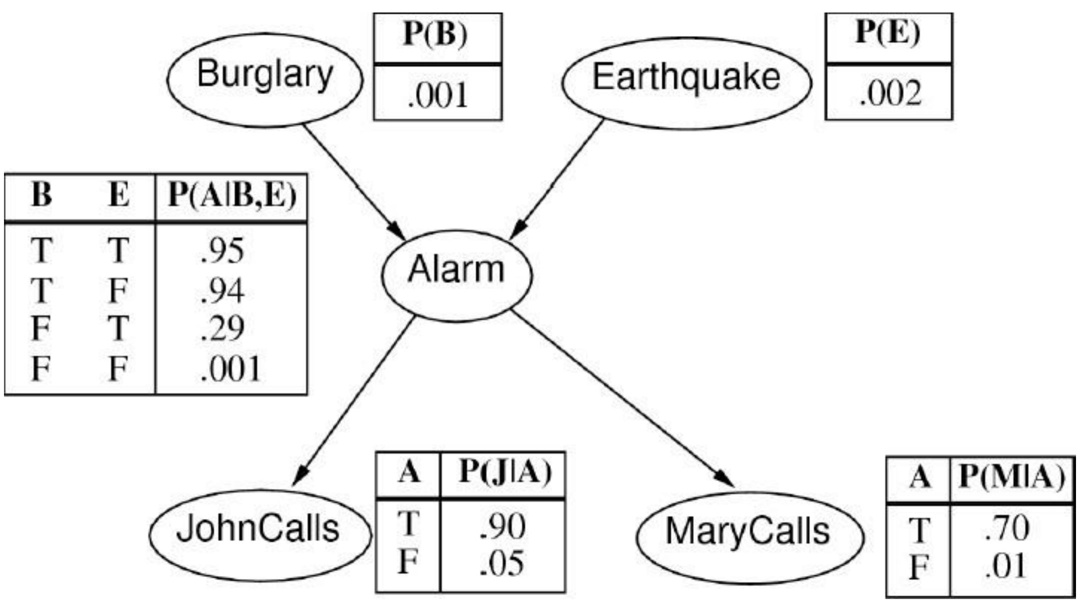
| *X* | *Y* | *Z* | |
|---|---|---|---|
| $a^1$ | $b^1$ | $c^1$ | $0.5 \cdot 0.5 = 0.25$ |
| $a^1$ | $b^1$ | $c^2$ | $0.5 \cdot 0.7 = 0.35$ |
| $a^1$ | $b^2$ | $c^1$ | $0.8 \cdot 0.1 = 0.08$ |
| $a^1$ | $b^2$ | $c^2$ | $0.8 \cdot 0.2 = 0.16$ |
| $a^2$ | $b^1$ | $c^1$ | $0.1 \cdot 0.5 = 0.05$ |
| $a^2$ | $b^1$ | $c^2$ | $0.1 \cdot 0.7 = 0.07$ |
| $a^2$ | $b^2$ | $c^1$ | $0 \cdot 0.1 = 0$ |
| $a^2$ | $b^2$ | $c^2$ | $0 \cdot 0.2 = 0$ |
| $a^3$ | $b^1$ | $c^1$ | $0.3 \cdot 0.5 = 0.15$ |
| $a^3$ | $b^1$ | $c^2$ | $0.3 \cdot 0.7 = 0.21$ |
| $a^3$ | $b^2$ | $c^1$ | $0.9 \cdot 0.1 = 0.09$ |
| $a^3$ | $b^2$ | $c^2$ | $0.9 \cdot 0.2 = 0.18$ |

# Variable Elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha \underbrace{P(E)}_{E} \sum_b \underbrace{P(b)}_{B} \sum_a \underbrace{P(a|b,E)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}$$

$$f_E(E) \qquad f_B(B) \qquad f_A(A,B,E) \qquad f_J(A) \qquad f_M(A)$$



# Variable Elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha \underbrace{P(E)}_{E} \sum_{b} \underbrace{P(b)}_{B} \sum_{a} \underbrace{P(a|b,E)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}$$

$f_E(E)$ $f_B(B)$ $f_A(A,B,E)$ $f_J(A)$ $f_M(A)$

| A | J | $f_J(A)$ |
|---|---|---|
| T | T | 0.9 |
| T | F | 0.1 |
| F | T | 0.05 |
| F | F | 0.95 |

| A | M | $f_M(A)$ |
|---|---|---|
| T | T | 0.7 |
| T | F | 0.3 |
| F | T | 0.01 |
| F | F | 0.99 |

| E | $f_E(E)$ |
|---|---|
| T | .002 |
| F | .998 |

| B | $f_B(B)$ |
|---|---|
| T | .001 |
| F | .999 |

| A | B | E | $f_A(A,B,E)$ |
|---|---|---|---|
| T | T | T | .95 |
| T | T | F | .94 |
| T | F | T | .29 |
| T | F | F | .001 |
| F | T | T | .05 |
| F | T | F | .06 |
| F | F | T | .71 |
| F | F | F | .999 |

| A | $f_J(A)$ |
|---|---|
| T | .9 |
| F | .05 |

| A | $f_M(A)$ |
|---|---|
| T | .7 |
| F | .01 |

# Variable Elimination

$$
\begin{aligned}
P(E|j,m) &= \alpha P(E,j,m) \\
&= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a) \\
&= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A)
\end{aligned}
$$

factorize

$$
\mathbf{P}(E \mid j,m) = \alpha \mathbf{f}_E(E) \times \sum_b (\mathbf{f}_B(B) \times \sum_a (\mathbf{f}_A(A,B,E) \times \mathbf{f}_J(A) \times \mathbf{f}_M(A)))
$$

# Variable Elimination

$$P(E|j,m) = \alpha P(E,j,m)$$
$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$
$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) \boxed{f_J(A) f_M(A)}$$
$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) \boxed{f_{JM}(A)}$$

factorize

factor product

| A | $f_J(A)$ |
|---|---|
| T | .9 |
| F | .05 |

| A | $f_M(A)$ |
|---|---|
| T | .7 |
| F | .01 |

=

| A | $f_{JM}(A)$ |
|---|---|
| T | .9 * .7 |
| F | .05 * .01 |

| A | J | $f_J(A)$ |
|---|---|---|
| T | T | 0.9 |
| T | F | 0.1 |
| F | T | 0.05 |
| F | F | 0.95 |

| A | M | $f_M(A)$ |
|---|---|---|
| T | T | 0.7 |
| T | F | 0.3 |
| F | T | 0.01 |
| F | F | 0.99 |

$$f_{JM}(A) = f_J(A) f_M(A)$$

# Variable Elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A)$$ factorize

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a \boxed{f_A(A,B,E) f_{JM}(A)}$$ factor product

$$= \quad \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a \boxed{f_{AJM}(A,B,E)}$$ factor product

| A | B | E | $f_A(A,B,E)$ |
|---|---|---|---|
| T | T | T | .95 |
| T | T | F | .94 |
| T | F | T | .29 |
| T | F | F | .001 |
| F | T | T | .05 |
| F | T | F | .06 |
| F | F | T | .71 |
| F | F | F | .999 |

| A | $f_{JM}(A)$ |
|---|---|
| T | .63 |
| F | .0005 |

=

| A | B | E | $f_{AJM}(A,B,E)$ |
|---|---|---|---|
| T | T | T | .95 * .63 |
| T | T | F | .94 * .63 |
| T | F | T | .29 * .63 |
| T | F | F | .001 * .63 |
| F | T | T | .05 * .0005 |
| F | T | F | .06 * .0005 |
| F | F | T | .71 * .0005 |
| F | F | F | .999 * .0005 |

$$f_{AJM}(A,B,E) = f_A(A,B,E) f_{JM}(A)$$

# Variable Elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A)$$ factorize

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_{JM}(A)$$ factor product

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \boxed{\sum_a f_{AJM}(A,B,E)}$$ factor product

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \boxed{f_{\bar{A}JM}(B,E)}$$ factor marginalization, and eliminate A

| A | B | E | $f_{AJM}(A,B,E)$ |
|---|---|---|---|
| T | T | T | .95 * .63 |
| T | T | F | .94 * .63 |
| T | F | T | .29 * .63 |
| T | F | F | .001 * .63 |
| F | T | T | .05 * .0005 |
| F | T | F | .06 * .0005 |
| F | F | T | .71 * .0005 |
| F | F | F | .999 * .0005 |

| B | E | $f_{\bar{A}JM}(B,E)$ |
|---|---|---|
| T | T | .95 * .63 + .05 * .0005 = .5985 |
| T | F | .94 * .63 + .06 * .0005 = .5922 |
| F | T | .29 * .63 + .71 * .0005 = .1830 |
| F | F | .001 * .63 + .999 * .0005 = .001129 |

$$f_{\bar{A}JM}(B,E) = \sum_a f_{AJM}(A,B,E)$$

# Variable Elimination

$$P(E|j,m) = \alpha P(E,j,m)$$

$$= \alpha P(E) \sum_b P(b) \sum_a P(a|b,E) P(j|a) P(m|a)$$

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_J(A) f_M(A)$$ factorize

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_A(A,B,E) f_{JM}(A)$$

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) \sum_a f_{AJM}(A,B,E)$$

$$= \alpha f_E(E) \textstyle\sum_b f_B(B) f_{\bar{A}JM}(B,E)$$ Eliminate A

$$= \alpha f_E(E) \textstyle\sum_b f_{B\bar{A}JM}(B,E)$$

$$= \alpha f_E(E) f_{\bar{B}\bar{A}JM}(E)$$ Eliminate B

$$= \alpha f_{E\bar{B}\bar{A}JM}(E)$$

The process of evaluation is a process of summing out variables (right to left) from pointwise products of factors to produce new factors, eventually yielding a factor that is the solution, i.e., the posterior distribution over the query variable.

It is bottom-up in the evaluation tree.

# Variable Elimination

- Recursively eliminate variables and merge terms into factors

- Store factors to avoid repeating computations

Tree-structured (Polytree)

- Complexity for Bayesian networks that are single-connected and with *n* Boolean variables are
  - linear to the size (number of CPT entries) of the network

$2^k$ CPT entries, assuming maximum k parents

$O(n2^k)$ CPT entries

# Variable elimination

Also useful for doing inference multiple times

e.g.

$$P(B|J,M) = \alpha P(B,J,M)$$
$$P(B|J) = \alpha P(B,J)$$
$$P(B|M) = \alpha P(B,M)$$

# Variable elimination

$$
\begin{aligned}
P(B|J,M) &= \alpha P(B,J,M) \\
&= \sum_a \sum_e P(B)P(e)P(a|B,e)P(J|A)P(M|a) \\
&= \alpha P(B) \sum_e P(e) \sum_a P(a|B,e)P(J|a)P(M|a) \\
&= \alpha P(B) f(J,M)
\end{aligned}
$$

| J | M | F(J,M) |
|---|---|---|
| T | T | … |
| … | … | … |
| F | F | … |

We only need to calculate *f* once

$$
\begin{aligned}
P(B|J) &= \alpha P(B,J) \\
&= \sum_m \sum_a \sum_e P(B)P(e)P(a|B,e)P(J|A)P(m|a) \\
&= \alpha P(B) \sum_m \sum_e P(e) \sum_a P(a|B,e)P(J|a)P(m|a) \\
&= \alpha P(B) \sum_m f(J,m)
\end{aligned}
$$

$$
\begin{aligned}
P(B|M) &= \alpha P(B,M) \\
&= \sum_j \sum_a \sum_e P(B)P(e)P(a|B,e)P(j|A)P(M|a) \\
&= \alpha P(B) \sum_j \sum_e P(e) \sum_a P(a|B,e)P(j|a)P(M|a) \\
&= \alpha P(B) \sum_j f(j,M)
\end{aligned}
$$