Faculty of SET / School of Computer and Mathematical Sciences

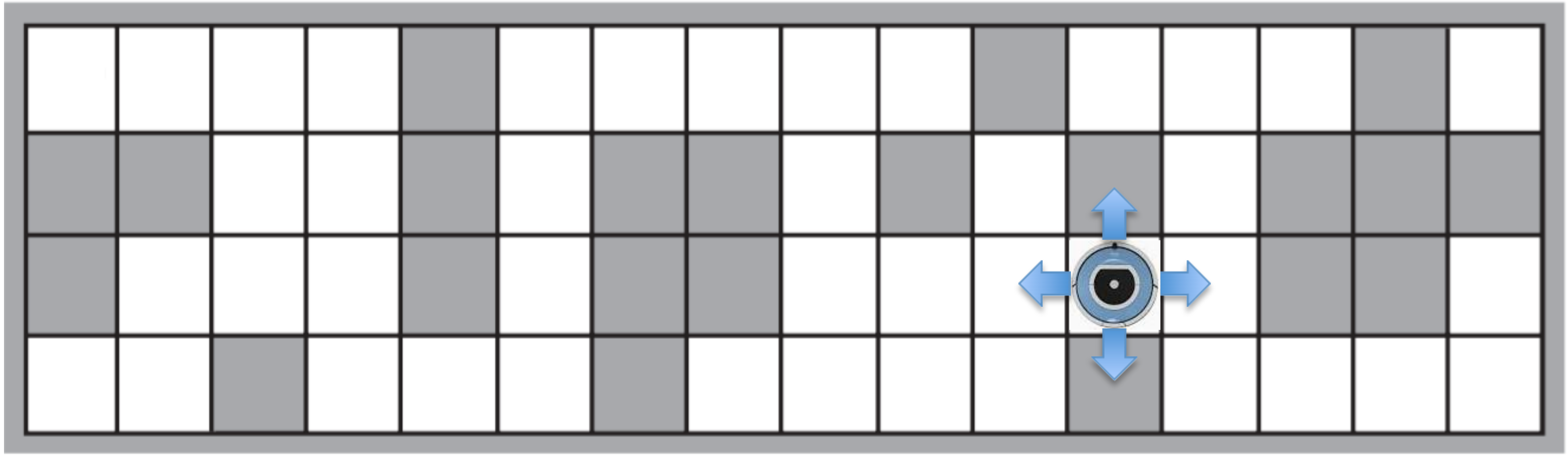
COMP SCI 3007&7059 Artificial Intelligence
Probability Reasoning Over Time 1

Acknowledgement of Country

We acknowledge and pay our respects to the Kaurna people, the traditional custodians whose ancestral lands we gather on.

We acknowledge the deep feelings of attachment and relationship of the Kaurna people to the country and we respect and value their past, present and ongoing connection to the land and cultural beliefs.

Example: robot localisation



- A robot is moving in a 2D map with obstacles. Its sensor detects the directions are blocked by obstacles at the robot's location.
- A sequence of sensor readings is observed:

Time step	1	2	3	…	t	t+1
Blocked directions	N S W	N S	N	…	S E W	S E

- At time step t+1, where is the robot?

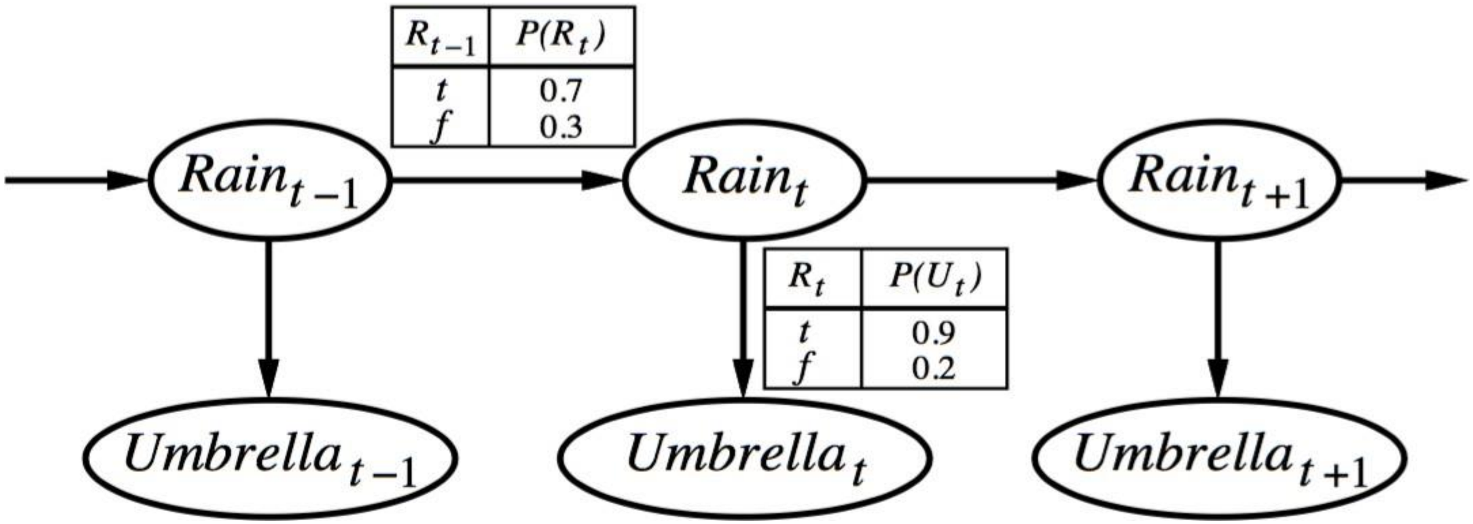
Example: is it raining outside?

- You are the security guard permanently located at a secret underground installation.
- You **cannot** see the weather outside.
- Everyday, you see the director arriving with or without an **umbrella**.
- At day t+1, the director arrived with an umbrella. Is it raining outside?

Day	1	2	3	…	t	t+1
Observed umbrella?	✓	✓	✗	…	✗	✓

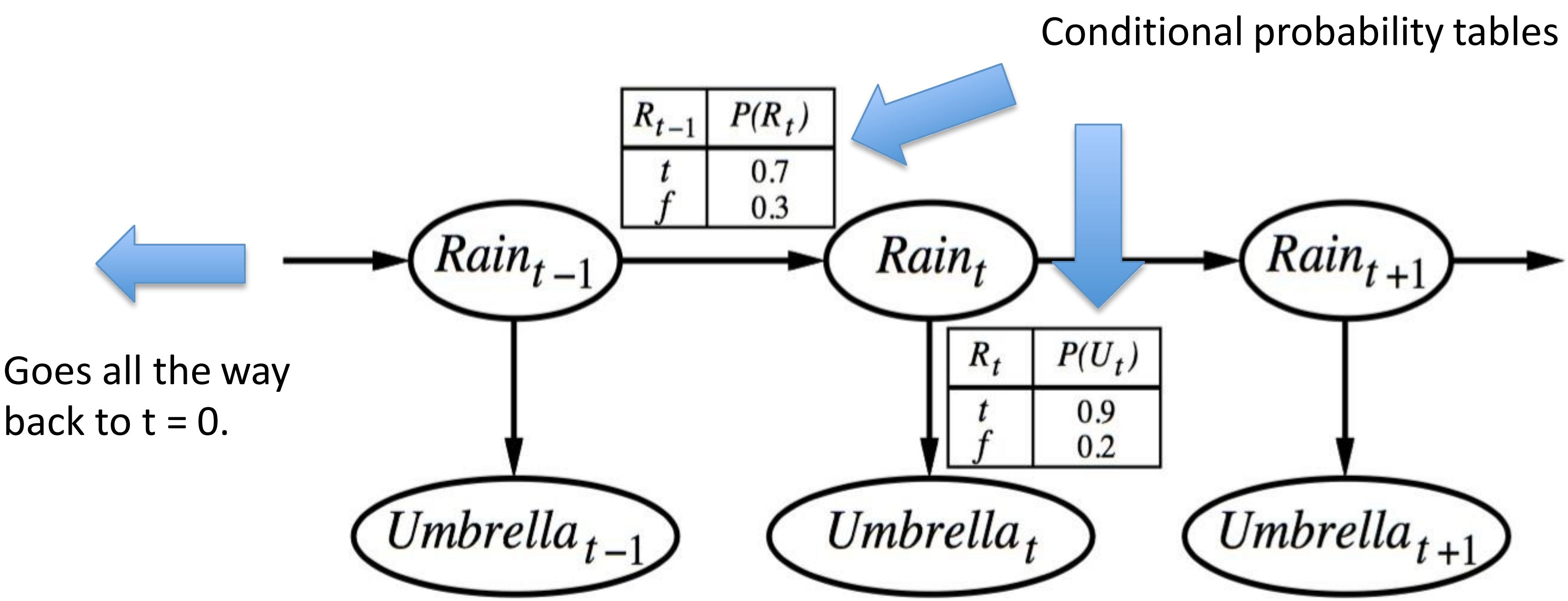
Example: is it raining outside?

- A commonly used temporal model for this kind of problem: Hidden Markov Model (HMM)



Example: is it raining outside?

- This is just a Bayesian Network with the concept of time.

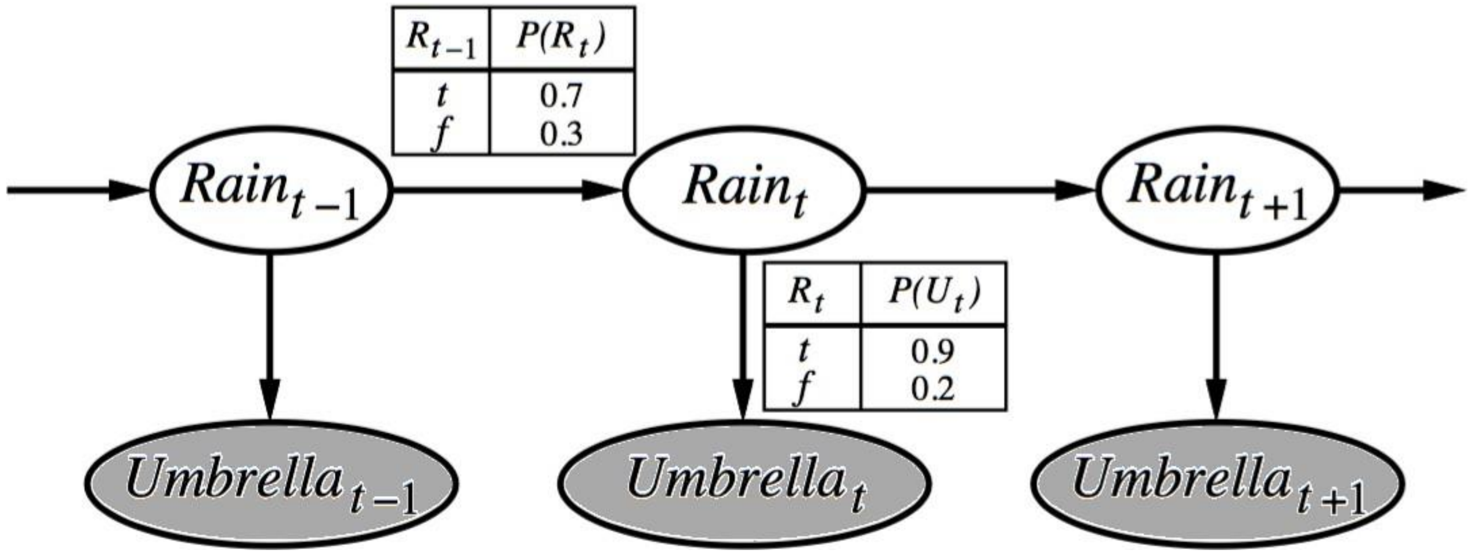


R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

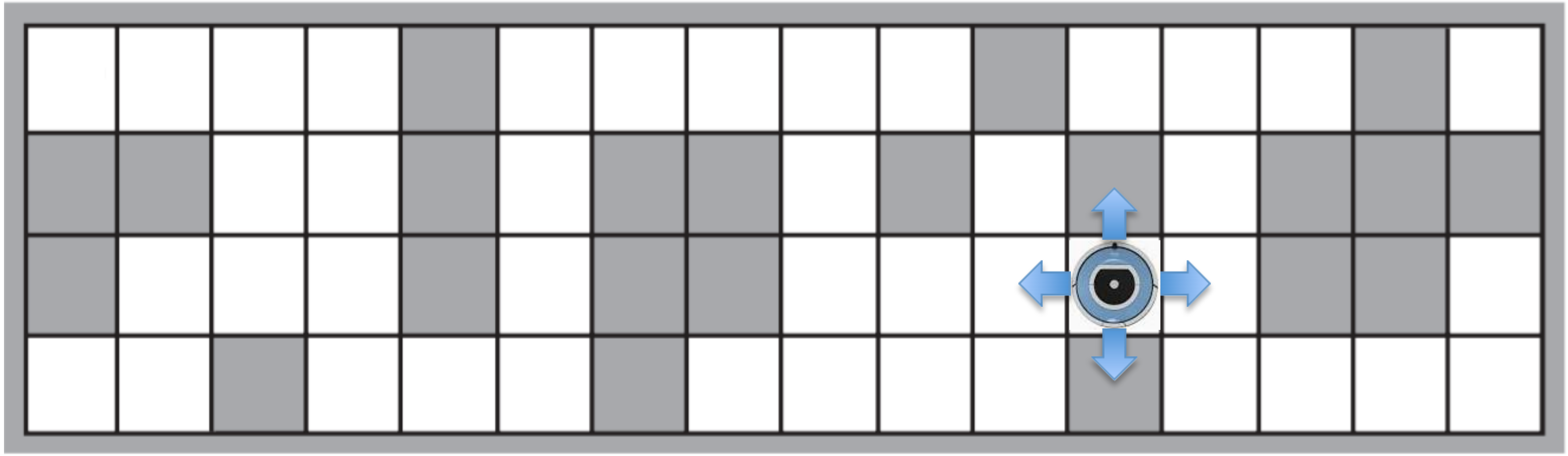
- Variables = { R_0, R_1, …, R_{t+1}, U_1, …, U_{t+1} }.

Example: is it raining outside?



- You have observed evidences(Umbrella) $\{u_1, \ldots, u_{t+1}\}$ = {true,true,false,...,false,true}.
- You want to calculate the probability
$$P(R_{t+1} \mid u_1, \ldots, u_{t+1})$$
for R_{t+1} = true and R_{t+1} = false.
- This is a special kind of probabilistic inference called filtering.

Example: robot localisation

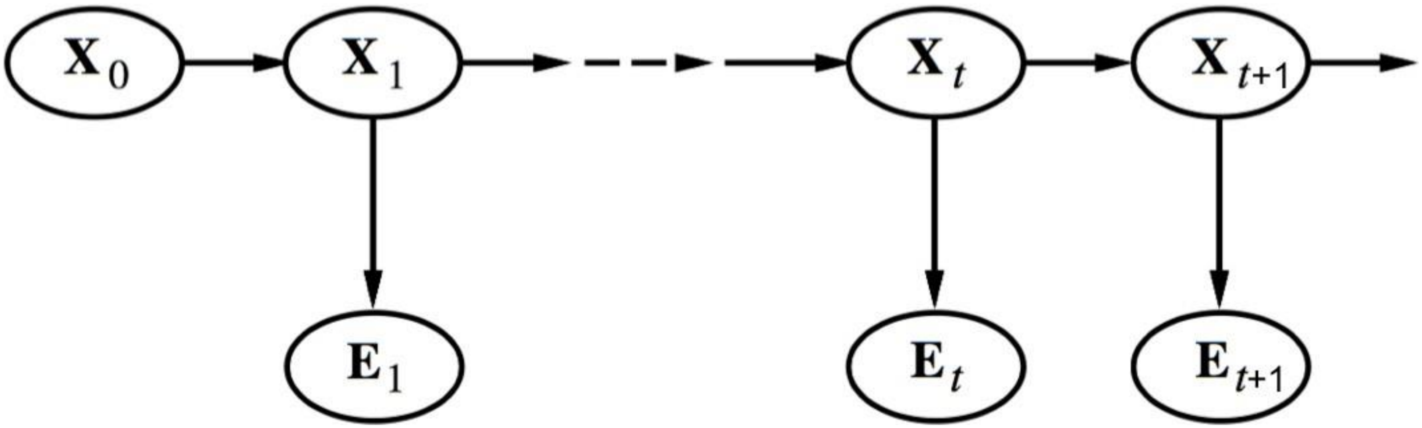


- A robot is moving in a 2D map with obstacles. Its sensor detects the directions are blocked by obstacles at the robot's location.
- A sequence of sensor readings is observed:

Time step	1	2	3	…	*t*	*t*+1
Blocked directions	N S W	N S	N	…	S E W	S E

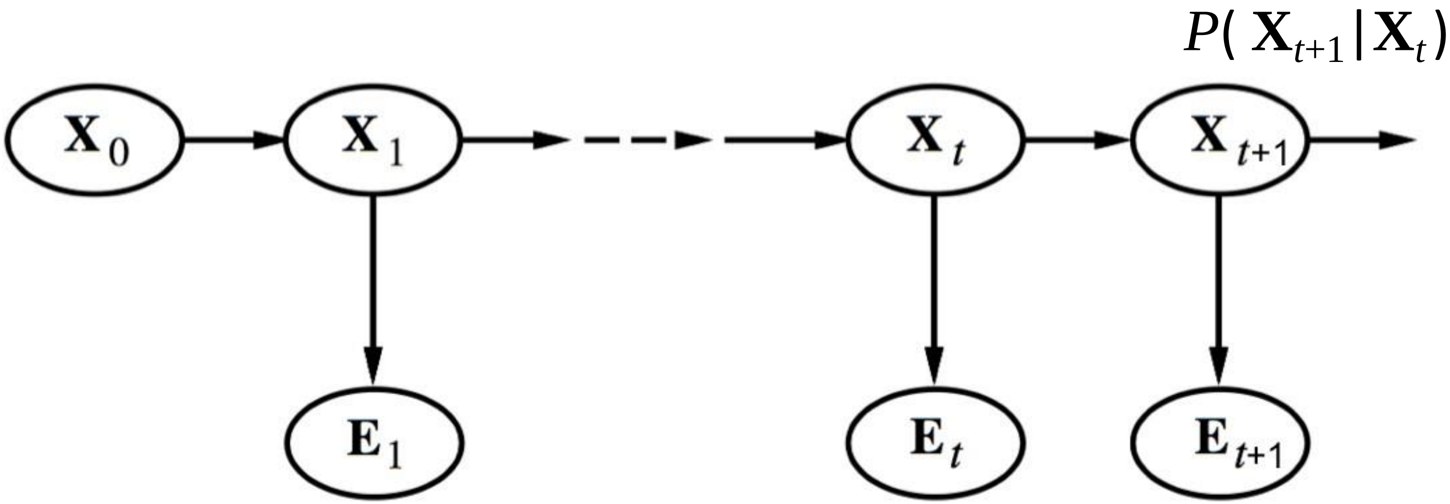
- At time step *t*+1, where is the robot?

The general case



- State variables $\{ \mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_{t+1} \}$.
- Evidence variables $\{ \mathbf{E}_1, \ldots, \mathbf{E}_{t+1} \}$.
- By convention, we assume $\mathbf{X}_t$ starts at t=0 while $\mathbf{E}_t$ starts at t=1.

The general case

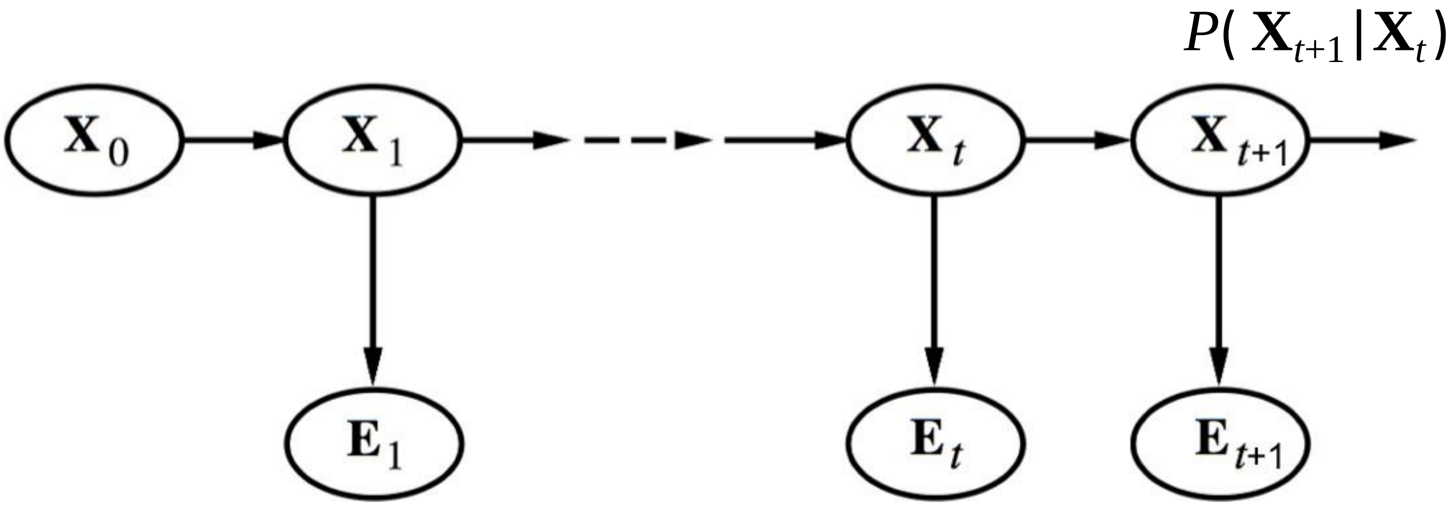


- State transition model

$$P(\mathbf{X}_{t+1} | \mathbf{X}_0, \ldots, \mathbf{X}_t) = P(\mathbf{X}_{t+1} | \mathbf{X}_t)$$

- **First order Markov assumption**: the present state depends only on the immediate previous state.

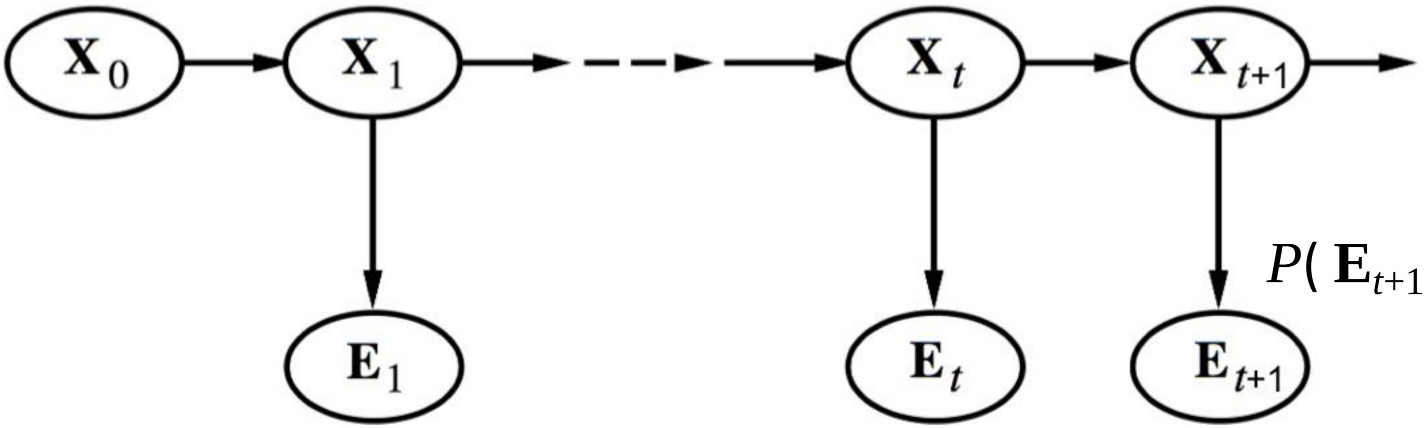
The general case



Assume the state changes are caused by a **stationary process**—that is, a process of change that is governed by laws that do not themselves change over time.

$P(\mathbf{X}_{t+1} | \mathbf{X}_t)$ is the same for all t.

The general case



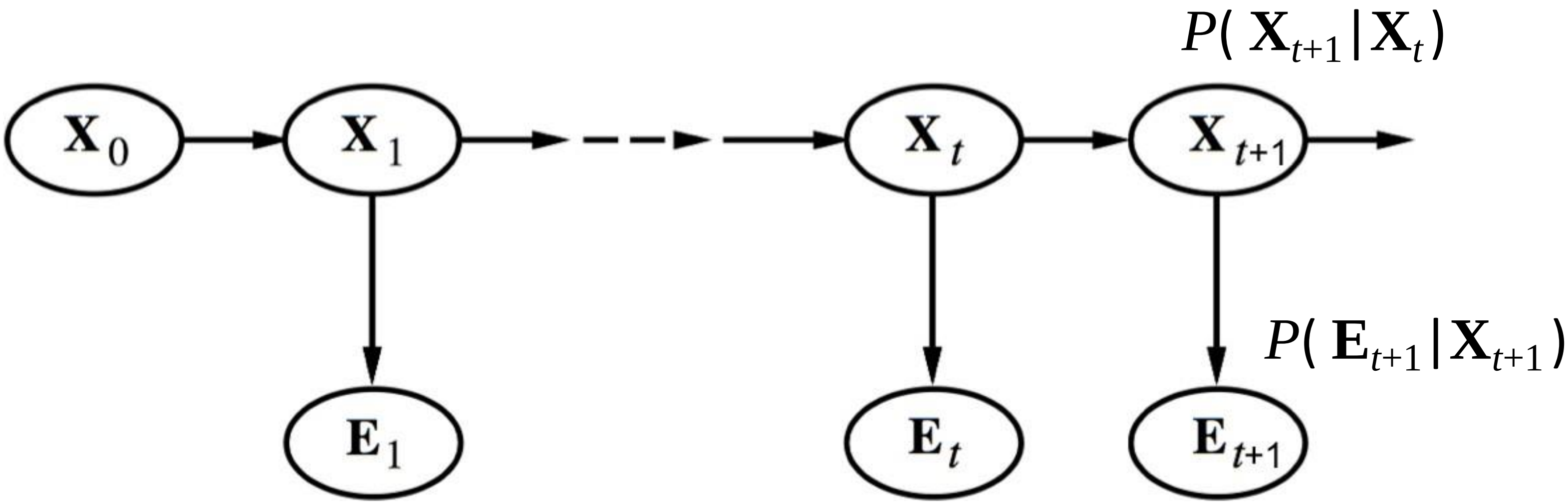
- Observation/emission/sensor model

$$P(\mathbf{E}_{t+1} | \mathbf{X}_{0:t+1}, \mathbf{E}_{0:t}) = P(\mathbf{E}_{t+1} | \mathbf{X}_{t+1})$$

- Sensor Markov assumption: the probability of observing $\mathbf{E}_t$ depends only on the state $\mathbf{X}_t$.

*Note: $\mathbf{X}_{0:t} = \mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_\mathrm{t}$

The general case



$$\mathbf{P}(\mathbf{X}_{0:t}, \mathbf{E}_{1:t}) = \mathbf{P}(\mathbf{X}_0) \prod_{i=1}^{t} \mathbf{P}(\mathbf{X}_i \mid \mathbf{X}_{i-1}) \, \mathbf{P}(\mathbf{E}_i \mid \mathbf{X}_i)$$

Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)}$$

$$= \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t}) \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)}$$

Set $A=\mathbf{X}_{t+1}$, $B = \mathbf{e}_{1:t}$, $C=\mathbf{e}_{t+1}$,

$$P(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1})$$

$$= P(A|B,C) = \frac{P(A,B,C)}{P(B,C)} = \frac{P(C|A,B)P(A|B)}{P(C|B)}$$

$$= \alpha P(C|A,B)P(A|B) = \alpha\, P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}, \mathbf{e}_{1:t})P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$\begin{aligned}
\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)} \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)} \\
&= \alpha\, \underbrace{\mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})}_{\text{Get from CPT}}\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(by the sensor Markov assumption).}
\end{aligned}$$

Sensor Markov assumption:
the probability of observing $\mathbf{E}_t$ depends only on the state $\mathbf{X}_t$

Inference: Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$\begin{aligned}
\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)} \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)} \\
&= \alpha\, \underbrace{\mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})}_{\text{Get from CPT}}\, \underbrace{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t})}_{?} \quad \text{(by the sensor Markov assumption).}
\end{aligned}$$

Filtering

$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t})$ =?

Set A=$\mathbf{X}_{t+1}$, B = $\mathbf{e}_{1:t}$, C=$\mathbf{e}_{t+1}$, D= $\mathbf{X}_t$

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) = P(A \mid B)$$

One-step prediction

$$= \frac{P(A,B)}{P(B)} = \frac{\sum_D P(A,B,D)}{P(B)} = \frac{\sum_D P(A|B,D)P(D|B)P(B)}{P(B)}$$

$$= \frac{P(B)\sum_D P(A|B,D)P(D|B)}{P(B)}$$

$$= \sum_D P(A|B,D)P(D|B)$$

$$= \sum_{\mathbf{X}_t} P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}, \mathbf{X}_t)P(\mathbf{X}_t|\mathbf{e}_{1:t})$$

Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(by the sensor Markov assumption).}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t}) \quad \text{(Markov assumption).}$$

First order Markov assumption: the present state depends only on the immediate previous state.

Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(by the sensor Markov assumption).}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

$$= \alpha\, \underbrace{\mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})}_{\text{Observation model}} \sum_{\mathbf{x}_t} \underbrace{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t)}_{\text{Transition model}} P(\mathbf{x}_t \mid \mathbf{e}_{1:t}) \quad \text{(Markov assumption).}$$

Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})} = \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(by the sensor Markov assumption).}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) \boxed{P(\mathbf{x}_t \mid \mathbf{e}_{1:t})} \quad \text{(Markov assumption).}$$

Has the same form! but at one time step before.

This process is called recursive estimation.

Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$
\begin{aligned}
\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)} \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)} \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})\, \underline{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t})} \quad \text{(by the sensor Markov assumption).} \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t \mid \mathbf{e}_{1:t}) \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \underbrace{\sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})} \quad \text{(Markov assumption).}
\end{aligned}
$$

Calculating this is called prediction.

Filtering

- We have observed $\mathbf{e}_1, \ldots, \mathbf{e}_{t+1} = \mathbf{e}_{1:t+1}$. We wish to calculate

$$\boxed{\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1})}$$

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)}$$

$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(by the sensor Markov assumption).}$$

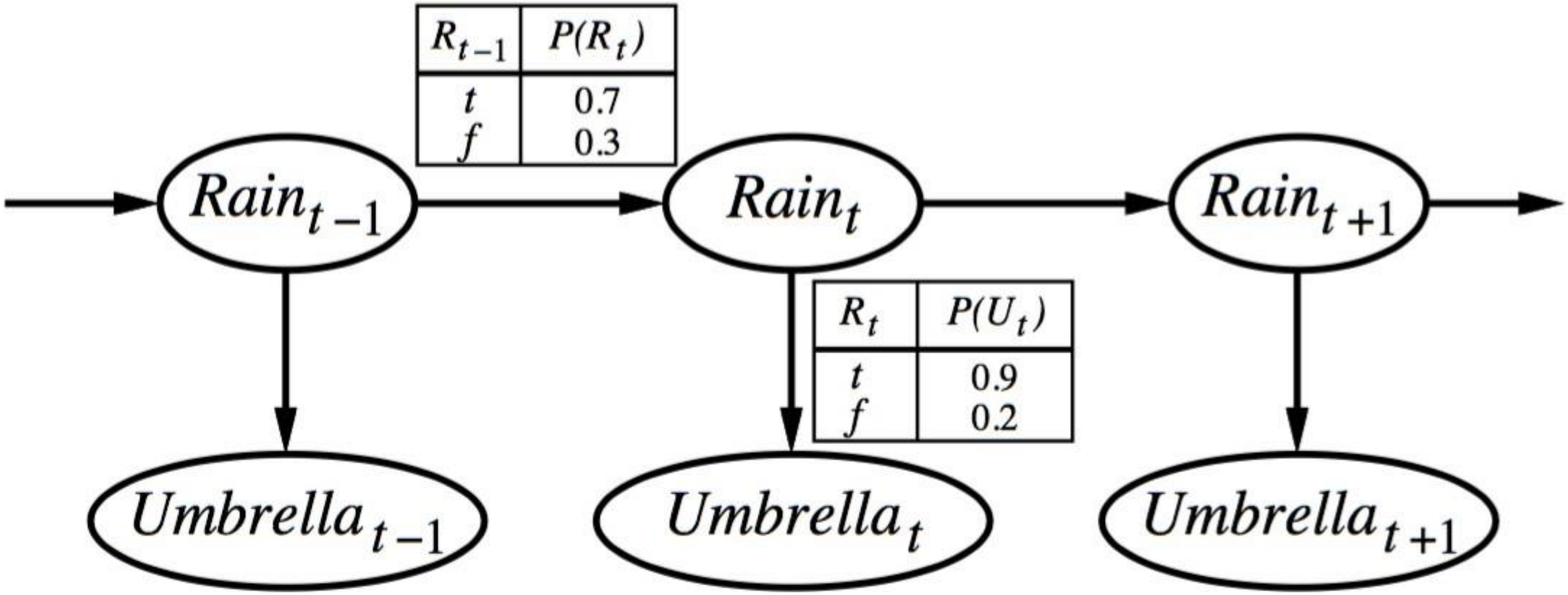
$$= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

$$= \underbrace{\alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})} \quad \text{(Markov assumption).}$$

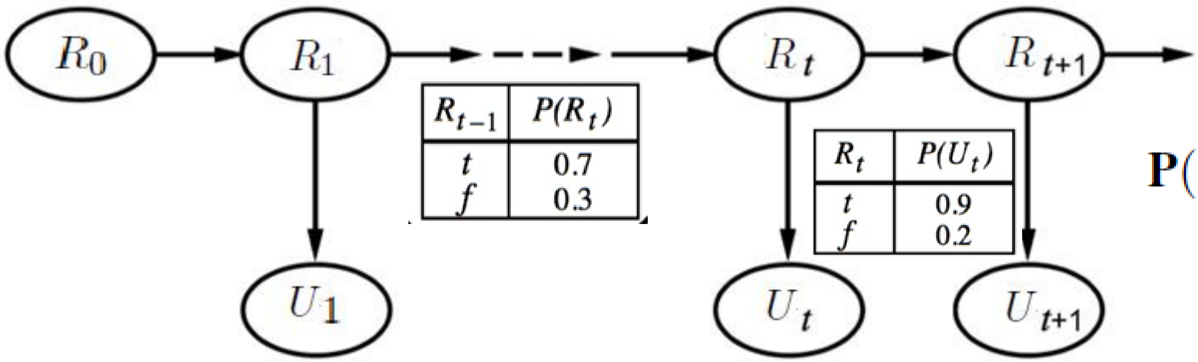
Combing the prediction with the new evidence is called update.

Example: is it raining outside?

- Form it as a first-order Markov process:



Example: is it raining outside?



R_{t-1}	$P(R_t)$
t	0.7
f	0.3

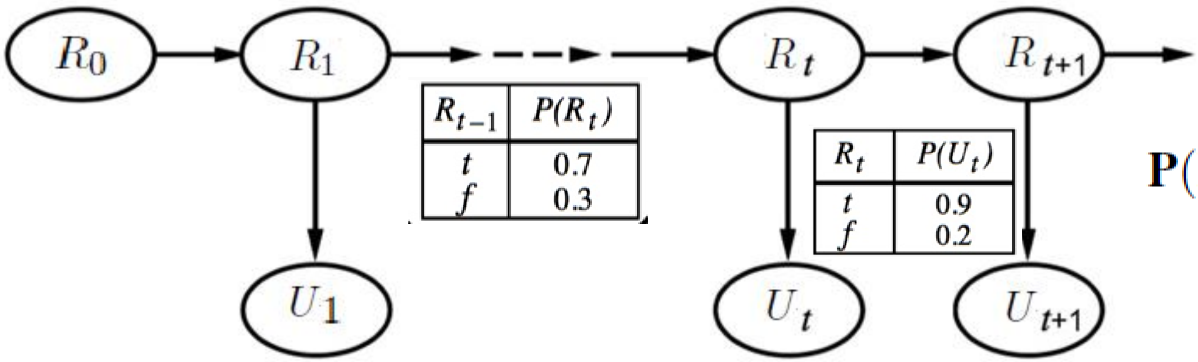
R_t	$P(U_t)$
t	0.9
f	0.2

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \alpha \, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

- On day 0, we have no observations, only the security guard's prior beliefs; let's assume that consists of $\mathbf{P}(R_0) = \langle 0.5, 0.5 \rangle$.
- On day 1, the umbrella appears, so $U_1 = true$.

$$\begin{aligned}
\mathbf{P}(R_1 \mid u_1) &= \alpha \, \mathbf{P}(u_1 \mid R_1) \sum_{r_0} \mathbf{P}(R_1 \mid r_0) P(r_0) \\
&= \alpha \, \langle \overset{r_1}{0.9}, \overset{-r_1}{0.2} \rangle \left(\langle \underset{r_1}{0.7}, \overset{1-0.7}{\underset{-r_1}{0.3}} \rangle \underset{r_0}{\times 0.5} + \langle \underset{r_1}{0.3}, \overset{1-0.3}{\underset{-r_1}{0.7}} \rangle \underset{-r_0}{\times 0.5} \right) \\
&= \alpha \, \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle \\
&= \alpha \, \langle 0.45, 0.1 \rangle \approx \langle 0.818, 0.182 \rangle
\end{aligned}$$

Example: is it raining outside?



$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

- On day 2, the umbrella appears, so $U_2 = true$.

$$\begin{aligned}
\mathbf{P}(R_2 \mid u_1, u_2) &= \alpha\, \mathbf{P}(u_2 \mid R_2) \sum_{r_1} \mathbf{P}(R_2 \mid r_1) P(r_1 \mid u_1) \\
&= \alpha \langle 0.9, 0.2 \rangle \big(\langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \big) \\
&= \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle \\
&= \alpha \langle 0.565, 0.075 \rangle \approx \langle 0.883, 0.117 \rangle
\end{aligned}$$

Can keep on going as new observations are made.

Hidden Markov Model (HMM)

- A HMM is obtained if X_t and E_t for all t are single discrete random variables.

  e.g., "is it raining outside?" is a HMM.

- In a HMM, the **transition model** can be encoded in an SxS matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}[1,1] & \mathbf{T}[1,2] & \cdots & \mathbf{T}[1,S] \\ \mathbf{T}[2,1] & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \mathbf{T}[S,1] & \cdots & \cdots & \mathbf{T}[S,S] \end{bmatrix}$$

where S is the number of possible values of X_t, and

$$\mathbf{T}[i, j] = P(X_t = j\text{-}th \mid X_{t-1} = i\text{-}th)$$

Hidden Markov Model (HMM)

- Given evidence e_t at time step t, the **observation model** can be encoded in an SxS diagonal matrix

$$\mathbf{O}_t = \begin{bmatrix} \mathbf{O}_t[1,1] & 0 & \cdots & 0 \\ 0 & \mathbf{O}_t[2,2] & \cdots & \vdots \\ \vdots & \cdots & \mathbf{O}_t[3,3] & 0 \\ 0 & \cdots & 0 & \mathbf{O}_t[S,S] \end{bmatrix}$$

where

$$\mathbf{O}_t[\mathrm{i}, \mathrm{i}] = P(e_t \mid X_t = i\text{-}th)$$

Hidden Markov Model (HMM)

- Recursive estimation (i.e., Filtering) in HMM can be computed as

$$\mathbf{f}_{t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^{T} \mathbf{f}_{t}$$

where

$\mathbf{f}_{t+1}$ is column vector form of $P(\mathrm{X}_{t+1} \mid \mathrm{e}_{1:t+1})$

$\mathbf{f}_{t}$ is column vector form of $P(\mathrm{X}_{t} \mid \mathrm{e}_{1:t})$

Hidden Markov Model (HMM)

- Recursive estimation in HMM can be computed as

$$\mathbf{f}_{t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^T \mathbf{f}_t$$

$$\begin{bmatrix} \mathbf{f}_{t+1}[1] \\ \\ \\ \mathbf{f}_{t+1}[S] \end{bmatrix} = \alpha \begin{bmatrix} \mathbf{O}_{t+1}[1,1] & 0 & & 0 \\ 0 & \mathbf{O}_{t+1}[2,2] & & \vdots \\ \vdots & & \cdots & 0 \\ 0 & \cdots & & \mathbf{O}_{t+1}[S,S] \end{bmatrix} \begin{bmatrix} \mathbf{T}[1,1] & \mathbf{T}[2,1] \cdots & \mathbf{T}[S,1] \\ \mathbf{T}[1,2] & \cdots & \vdots \\ \vdots & & \\ \mathbf{T}[1,S] & & \mathbf{T}[S,S] \end{bmatrix} \begin{bmatrix} \mathbf{f}_t[1] \\ \\ \\ \mathbf{f}_t[S] \end{bmatrix}$$

$$\mathbf{f}_{t+1}[i] = \alpha \mathbf{O}_t[\mathrm{i}, \mathrm{i}] \textstyle\sum_j \mathbf{T}[j, i] \mathbf{f}_t[j]$$

$$\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

Example: is it raining outside?

- From transition model:

$$\mathbf{T} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

R_{t-1}	$P(R_t)$
t	0.7
f	0.3

$\mathbf{T}[1,1]=P(R_t=true|\ R_{t-1}=true)$ $\mathbf{T}[1,2]=P(R_t=true|\ R_{t-1}=false)$

$\mathbf{T}[2,1]=P(R_t=false|\ R_{t-1}=true)$ $\mathbf{T}[2,2]=P(R_t=false|\ R_{t-1}=false)$

- At t=0, rain or no rain is equally likely, so

$$\mathbf{f}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Example: is it raining outside?

- From transition model:

$$\mathbf{T} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

R_{t-1}	$P(R_t)$
t	0.7
f	0.3

- At t=0, rain or no rain is equally likely, so

$$\mathbf{f}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

- At t=1, umbrella is observed, so

$$\mathbf{O}_1 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}$$

R_t	$P(U_t)$
t	0.9
f	0.2

$\mathbf{O}[1,1]=P(E_t=true|\ R_t=true)$ $\quad$ $\mathbf{O}[2,2]=P(E_t=true|\ R_t=false)$

Example: is it raining outside?

- From transition model:

$$\mathbf{T} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

R_{t-1}	$P(R_t)$
t	0.7
f	0.3

- At $t=0$, rain or no rain is equally likely, so

$$\mathbf{f}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

- At $t=1$, umbrella is observed, so

$$\mathbf{O}_1 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}$$

R_t	$P(U_t)$
t	0.9
f	0.2

- Filtering result at $t=1$ is

$$\mathbf{f}_1 = \alpha \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}^T \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \alpha \begin{bmatrix} 0.45 \\ 0.1 \end{bmatrix} \approx \begin{bmatrix} 0.818 \\ 0.182 \end{bmatrix}$$

Example: is it raining outside?

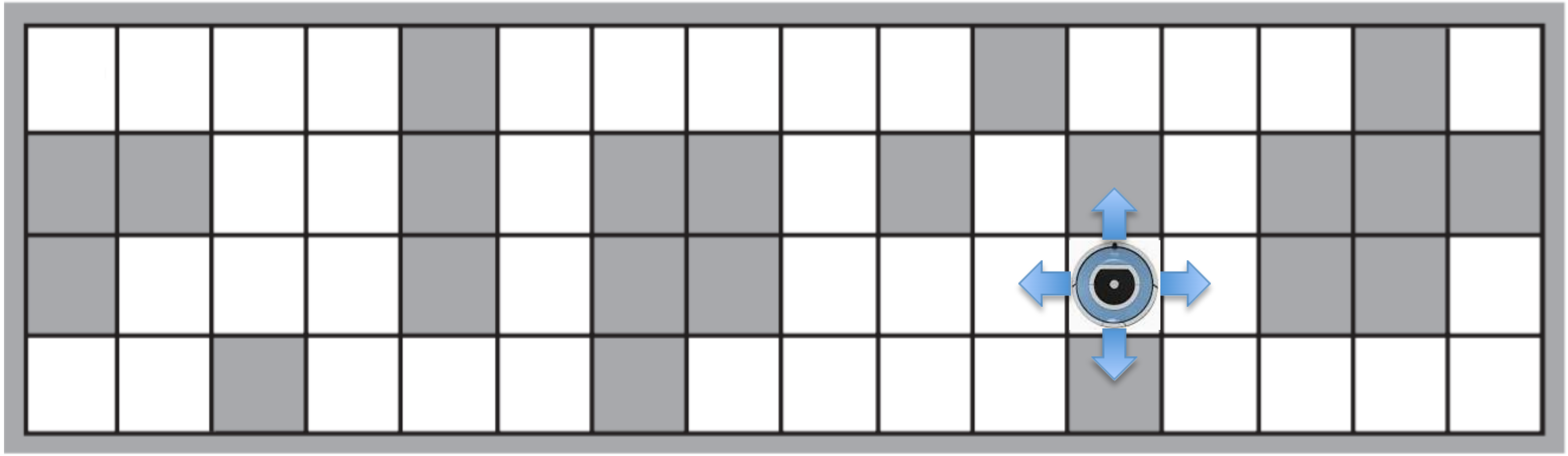
- At t=2, umbrella is observed, so

$$\mathbf{O}_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}$$

- Filtering result at t=2 is

$$\mathbf{f}_2 = \alpha \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}^T \begin{bmatrix} 0.45 \\ 0.1 \end{bmatrix} = \alpha \begin{bmatrix} 0.3105 \\ 0.041 \end{bmatrix} \approx \begin{bmatrix} 0.883 \\ 0.117 \end{bmatrix}$$

Example: robot localisation

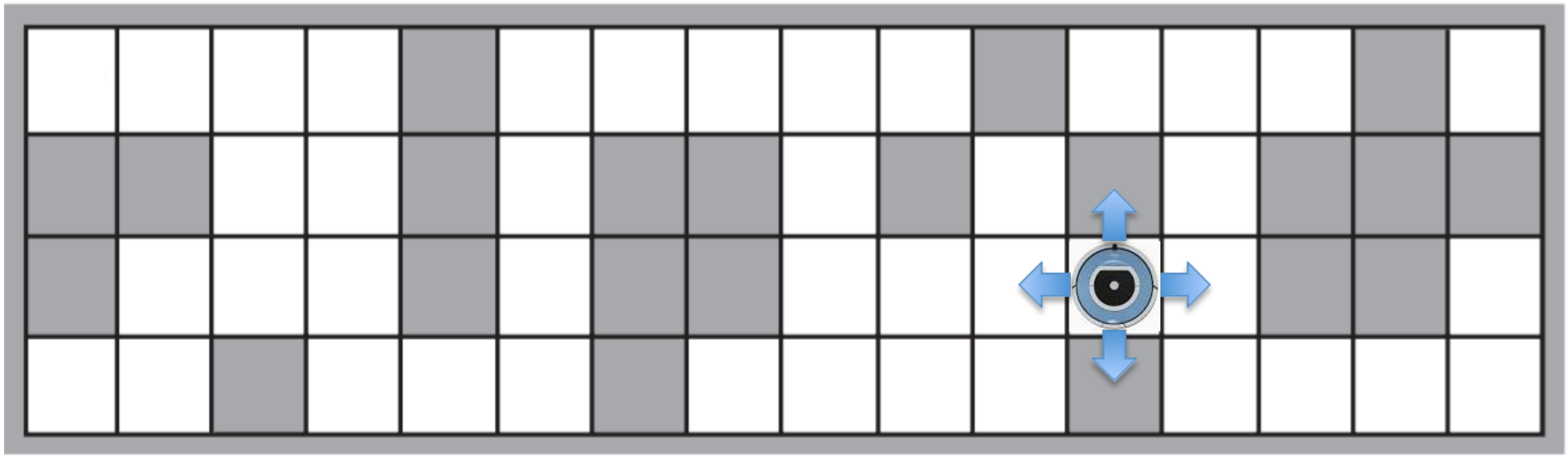


- A robot is moving in a 2D map with obstacles. Its sensor detects the directions are blocked by obstacles at the robot's location.
- A sequence of sensor readings is observed:

Time step	1	2	3	…	t	t+1
Blocked directions	N S W	N S	N	…	S E W	S E

- At time step t+1, where is the robot?

Example: robot localisation



- State variable represents the robot location:

$$X_t \in \{1, 2, \ldots, S\}$$

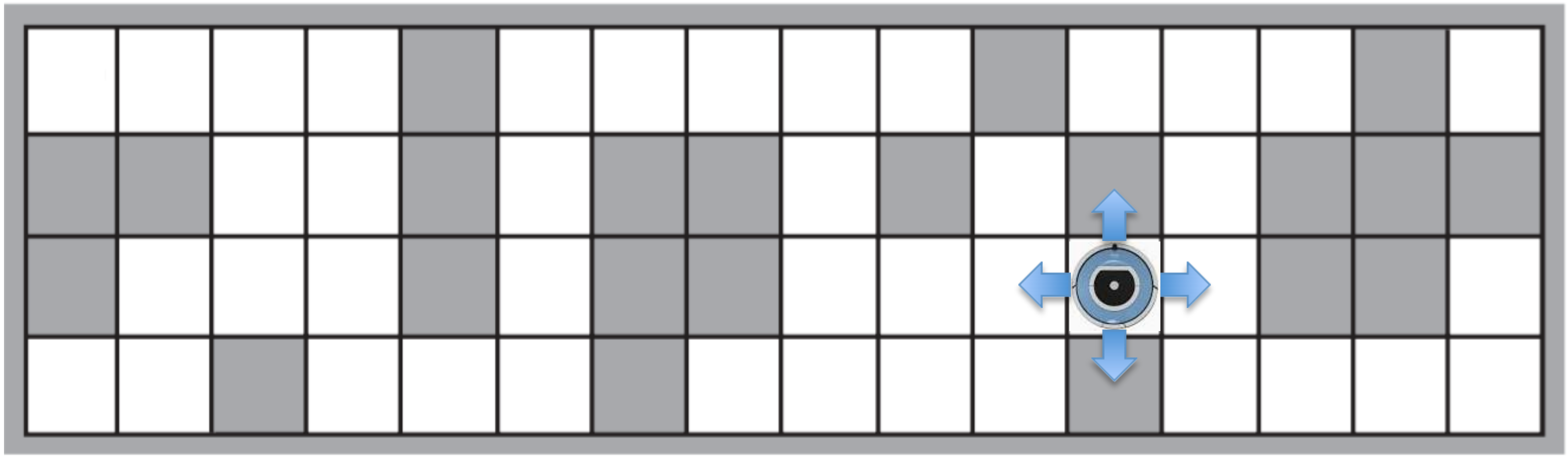
S=42=64 squares-22 blocked

- Sensor reading $E_t = e_t$: observed obstacles.

Time step	1	2	3	…	*t*	*t*+1
Blocked directions	N S W	N S	N	…	S E W	S E

E_t has 16 possible values

Example: robot localisation



- Assuming random walk, the **transition model** is

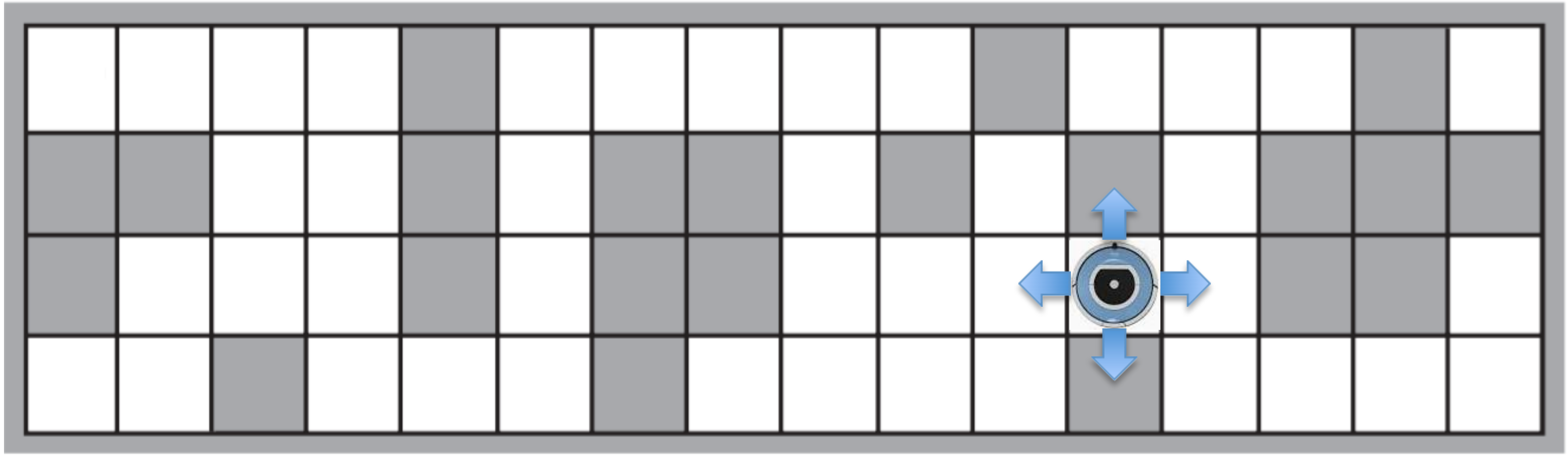
$$P(X_{t+1} = j \mid X_t = i) = \mathbf{T}_{ij} = \begin{cases} 1/N(i) & \text{if } j \in \text{NEIGHBOURS(i)} \\ 0 & \text{otherwise} \end{cases}$$

where NEIGHBOURS(i)= set of empty neighbors of cell i.

$N(i)$ = number of neighbours of cell i.

$\mathbf{T}$ has 42×42=1764 entries

Example: robot localisation



- The sensor's error rate is ϵ and error occurs independently in the four directions. This gives the **observation model**
$$P(E_t = e_t \mid X_t = i) = (1-\epsilon)^{4-d_{it}}\epsilon^{d_{it}}$$
where d_{it} is the number of directions that are wrong given location X_t = i and sensor reading $E_t = e_t$
- Example: at the robot's position in the map above, the probability of observing $e_t =$ *NSW* is $(1-\epsilon)^3\epsilon^1$

Example: robot localisation

- Assume the robot is equally likely to be at any square at $t = 0$, i.e., $\mathbf{f}_0$ is uniform: $P(X_0 = i) = 1/n$

$$\mathbf{f}_0 = \begin{bmatrix} 1/42 \\ 1/42 \\ \cdots \\ \\ \\ 1/42 \end{bmatrix}$$

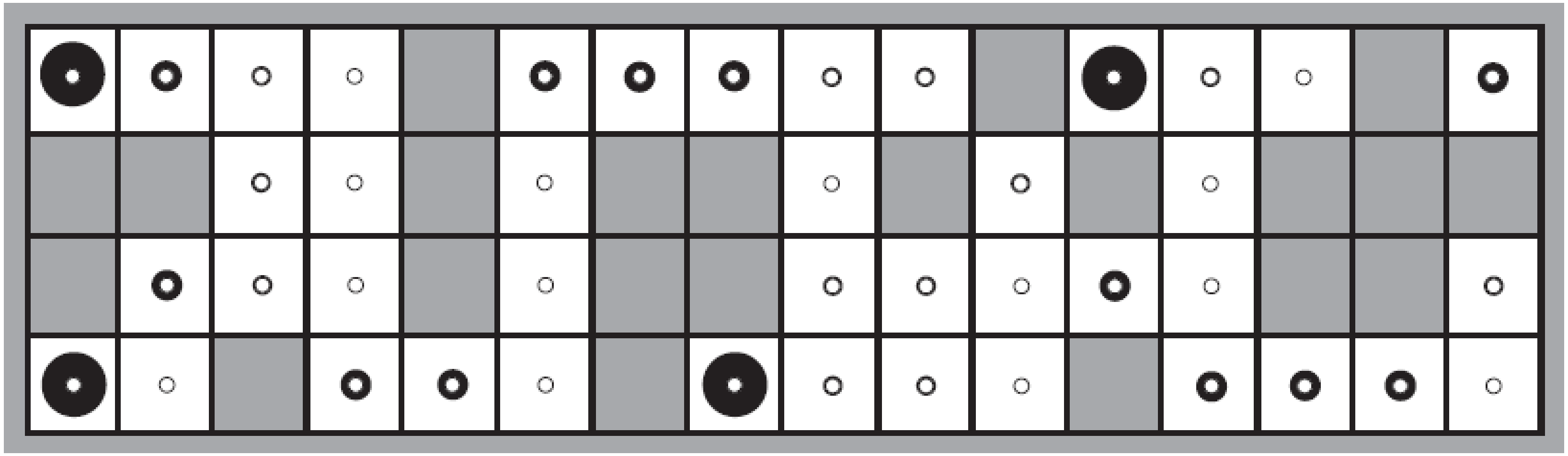
42×1

Example: robot localisation

- Assume the robot is equally likely to be at any square at $t = 0$, i.e., f_0 is uniform: $P(X_0 = i) = 1/n$

- After observing $E_1 = NSW$,

$$\mathbf{f}_1 = \boldsymbol{P}(X_1 \mid E_1 = NSW) = \alpha \mathbf{O}_1 \mathbf{T}^T \mathbf{f}_0$$

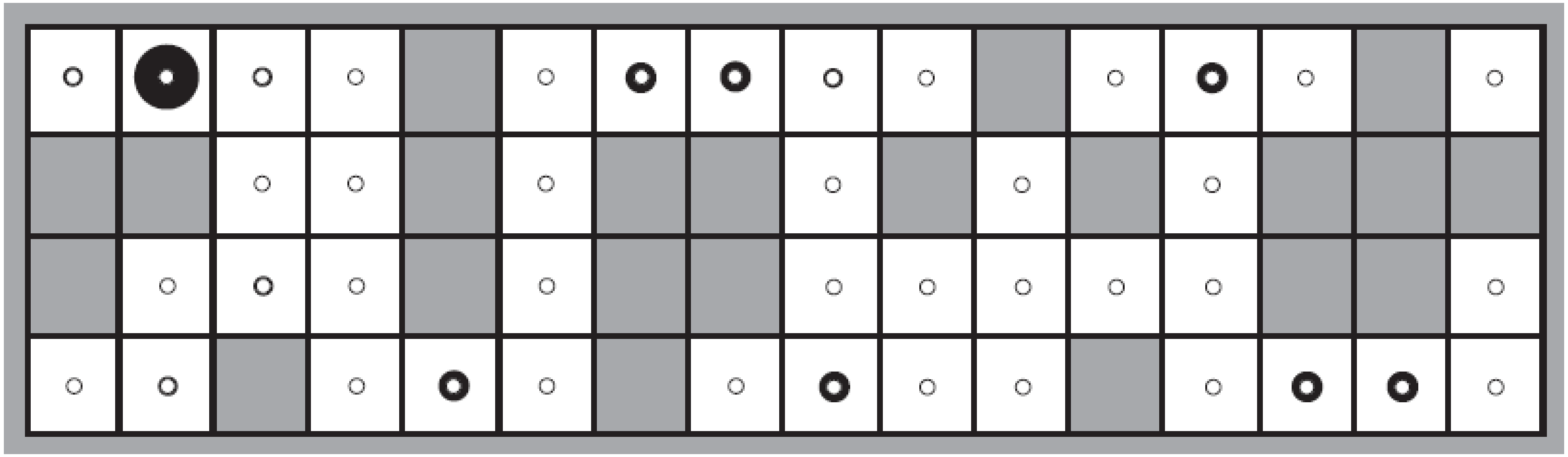


Posterior distribution over robot location after E_1= NSW $\epsilon = 0.2$

Example: robot localisation

- After observing $E_2 = NS$,

$$\mathbf{f}_2 = \boldsymbol{P}(X_2 \mid E_1 = NSW, E_2 = NS) = \alpha \mathbf{O}_2 \mathbf{T}^T \mathbf{f}_1$$



Posterior distribution over robot location after E_1= NSW, E_2= NS