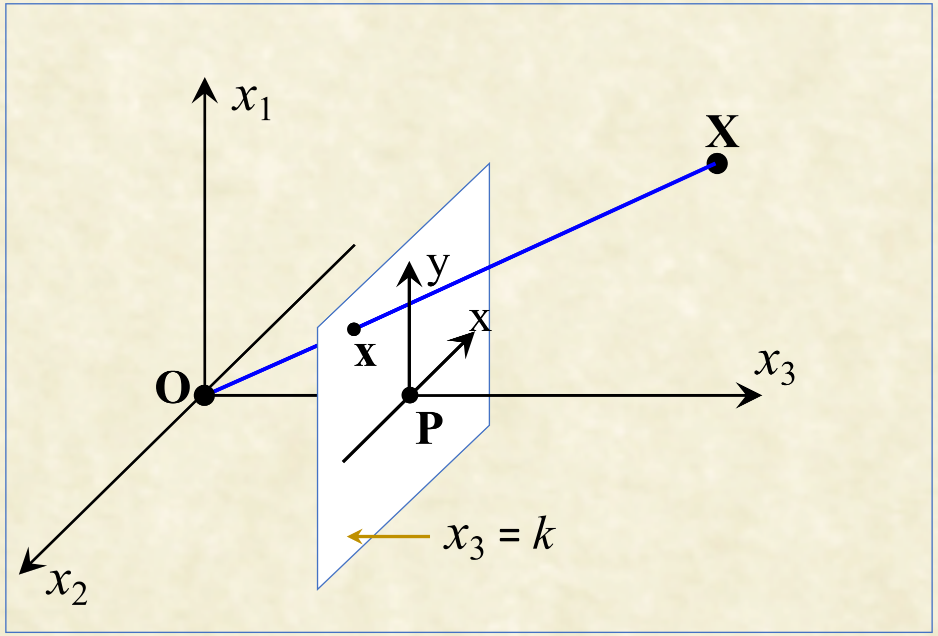
CSE 578: Computer Vision

Spring 2021: Projective Geometry and Camera Matrix



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General Camera Equation

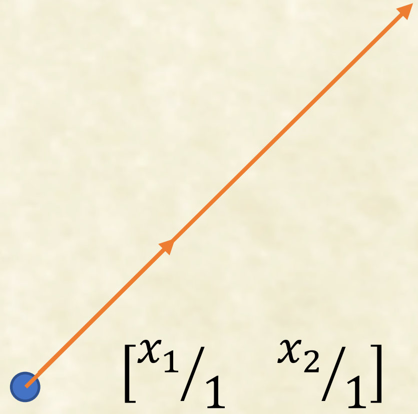
- Camera and world are related by: $$\mathbf{X_c} = \begin{bmatrix} \mathbf{R} & \mathbf{-RC} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \mathbf{X_w}$$
- 2D projection $\mathbf{x}$ of a 3D point $\mathbf{X_w}$ given by:
  - $\mathbf{x = K[\ I \mid 0\ ]\ X_c = K\ [\ R \mid -RC\ ]\ X_w}$
- $\mathbf{x = PX_w}$; camera matrix $\mathbf{P = [KR \mid -KRC] = [M \mid p_4]}$
- Common K: $$\begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$ General K: $$\begin{bmatrix} \alpha_x & s & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Points and Lines in $\mathcal{P}^2$

- Points represented by: $\mathbf{x} = [x \;\; y \;\; 1]^{\mathrm{T}}$.
- Consider the line equation: $ax + by + c = 0$.
- $[a\; b\; c][x\; y\; 1]^{\mathrm{T}} = \mathbf{l}.\mathbf{x} = \mathbf{l}^{\mathrm{T}}\,\mathbf{x} = 0$, where $\mathbf{l} = [a\; b\; c]^{\mathrm{T}}$.
- Lines are represented by 3-vectors, just like points.

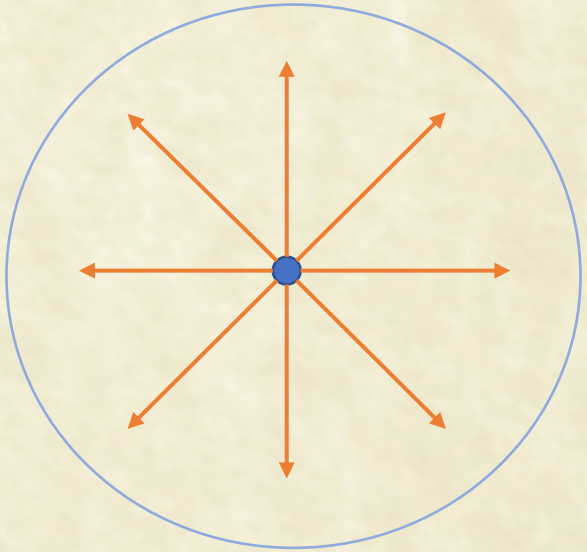
  Note: Overall scale is unimportant.
- What does $\mathbf{l}^{\mathrm{T}}\,\mathbf{x} = 0$ describe?
  - All points $\mathbf{x}$ on a fixed line $\mathbf{l}$?
  - All lines $\mathbf{l}$ passing through a fixed point, $\mathbf{x}$?

Points/Line at Infinity

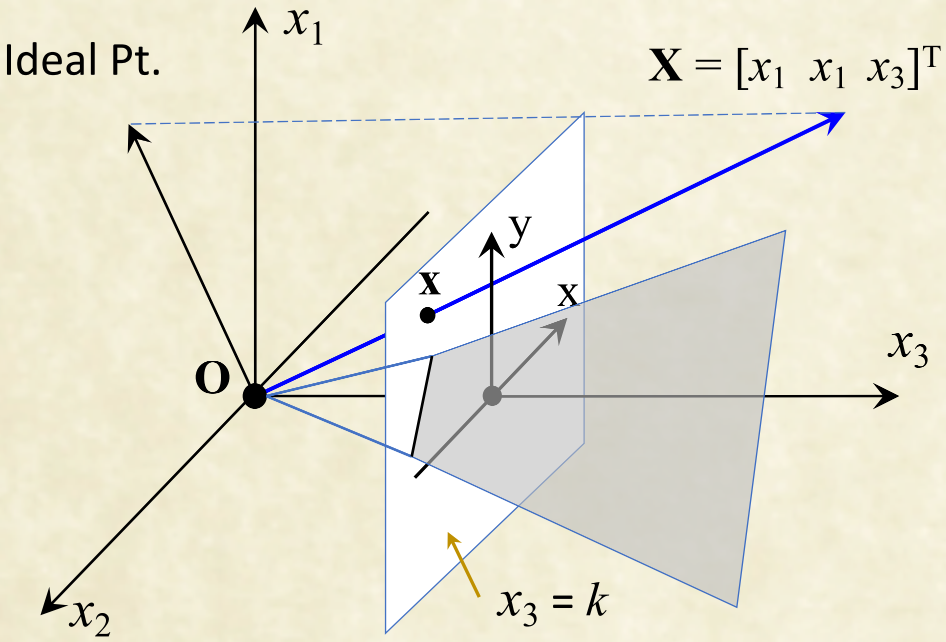


- $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^\mathrm{T}$ represents $[{}^{x_1}/_{x_3} \quad {}^{x_2}/_{x_3}]$
- What happens when $x_3 \longrightarrow 0$?
- Becomes **point at infinity**, or **vanishing point** or ideal point in the direction (x_1, x_2).
- Points at infinity can be handled like any other point in projective geometry
- $[x \quad y \quad 0]^\mathrm{T}$ are all points at infinity on the plane.
- What do they form together?
- What is the representation of $\mathbf{l}_\infty$?

$$\mathbf{l}_\infty = [0 \quad 0 \quad 1]^\mathrm{T}$$



Visualizing Projective Geometry of a Plane



- $\mathbf{x} = [x_1\ \ x_2\ \ x_3]^{\mathrm{T}}$ represents rays from the origin in 3-space.
- The plane can be any cross section $\perp^{r}$ to x_3.
- Ideal points are rays on the $x_3 = 0$ plane.
- Lines are planes passing through the origin.

- Line at infinity, $\mathbf{l}_\infty$, corresponds to $x_3 = 1,\ x_1, x_2 = 0$.

Line joining 2 points

- Let p and q be points. We have: $\mathbf{l}^{\mathrm{T}}\mathbf{p} = \mathbf{l}^{\mathrm{T}}\mathbf{q} = 0$.
- Equation of $\mathbf{l}$: $y = y_1 + \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

  or: $(y_2 - y_1)x - (x_2 - x_1)y + (x_2 y_1 - x_1 y_2) = 0$

  or: $\mathbf{l} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathrm{T}}$
- Considering them as vectors in 3-space, we want to find a vector **l** orthogonal to both $\mathbf{p}$ and $\mathbf{q}$.
- The cross-product $\mathbf{x} \times \mathbf{y}$ is a solution. Thus, $\mathbf{l} = \mathbf{p} \times \mathbf{q}$.
- $\mathbf{p} \times \mathbf{q} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathrm{T}}$

Example: Line connecting 2 points

- Line through (5,2) and (3,2): $\begin{bmatrix} i & j & k \\ 5 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$
- i.e., $y = 2$.

- Ideal point of the line $[0 \;\; 1 \;\; -2]^T$ is $[1 \;\; 0 \;\; 0]^T$.
  This is same as $[0 \;\; 1 \;\; k]^T$ for any k.

- Line joining $[3 \;\; 4 \;\; 0]^T$ and $[2 \;\; 3 \;\; 0]^T$ is $[0 \;\; 0 \;\; 1]^T$ or $\mathbf{l}_\infty$.

Point of Intersection of 2 lines

- Lines $\mathbf{l}$, $\mathbf{m}$ intersect at a point $\mathbf{x}$ with $\mathbf{l}^{\mathrm{T}}\mathbf{x} = \mathbf{m}^{\mathrm{T}}\mathbf{x} = 0$.
- $\mathbf{x} = \mathbf{l} \times \mathbf{m}$.
- $\mathbf{l}$: $a_1\, x + b_1\, y + c_1 = 0$; and $\mathbf{m}$: $a_2\, x + b_2\, y + c_2 = 0$.
- $x = (b_2c_1 - b_1c_2)/(a_2b_1 - a_1b_2)$.
- $y = (a_1c_2 - a_2c_1)/(a_2b_1 - a_1b_2)$.
- $\mathbf{x} = [(b_2c_1 - b_1c_2)\ \ (a_1c_2 - a_2c_1)\ \ (a_2b_1 - a_1b_2)]^{\mathrm{T}} = \mathbf{l} \times \mathbf{m}$.
- Duality at work: points and lines are interchangeable.

Example: Intersection of Lines

- Intersection of x=1 and y=2: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- Same as: $(1,2)$.
- Intersection of x=1 and x=2: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- Ideal point of the line $\mathbf{l} = [a \;\; b \;\; c]^{\mathrm{T}}$ is $[\mathrm{b} \;\; -a \;\; 0]^{\mathrm{T}}$
- This is $\mathbf{l} \times \mathbf{l}_{\infty}$, the intersection of l with line at infinity!

Conics: 2nd order Entities

- General quadratic entity:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

- Rewrite using homogeneous coordinates as:

$$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0.$$

- Rewrite as:

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

- A symmetric C represents a conic: $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$.
  Covers circle, ellipse, parabola, hyperbola, etc.
- Degenerate conics include a line $(a = b = c = 0)$ and two lines when $\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$.

General Camera Equation

- General projection equation in world coordinates:

  $$\mathbf{x} = \mathbf{K}\,[\mathbf{R} \mid -\mathbf{RC}]\,\mathbf{X_w} = [\mathbf{KR} \mid -\mathbf{KRC}]\,\mathbf{X_w} = [\mathbf{M} \mid \mathbf{p_4}]\,\mathbf{X_w}$$

- 3x4 matrix $\mathbf{P}$ maps/projects World-C to Image-C
  - Left 3 × 3 submatrix is non-singluar for finite cameras
  - Orthographic projection: left submatrix is singular
- Any 3 × 4 matrix P with non-singular left submatrix represents a camera! It can be decomposed as:
  - A non-singular upper diagonal matrix K
  - An orthonormal matrix R and a vector C with the usual meanings!!

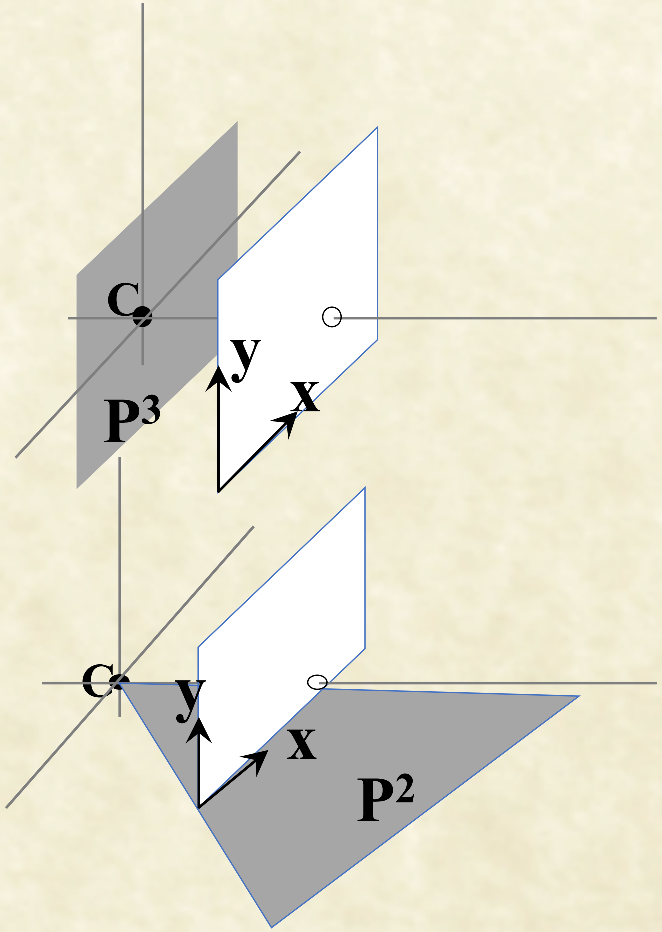
Camera Matrix Anatomy

$$\mathbf{P} = [\, \mathbf{p}_1 \;\; \mathbf{p}_2 \;\; \mathbf{p}_3 \;\; \mathbf{p}_4 \,] = [\, \mathbf{P}^1 \;\; \mathbf{P}^2 \;\; \mathbf{P}^3 \,]^T$$

- 4-vector $\mathbf{C}$ with $\mathbf{PC} = \mathbf{0}$ is the camera center.
  - Camera center is the only point with no projection or projects to the vector **0**, which is undefined in $\mathbf{P}^2$.
- Columns $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are the images of vanishing points of the world X, Y and Z directions.
  - $\mathbf{p}_1 = \mathbf{P}\,[1\ 0\ 0\ 0]^T$, the image of the ideal point in X direction and similarly the rest.
- $\mathbf{p}_4$ is the image of world origin: $\mathbf{p}_4 = \mathbf{P}\,[0\ 0\ 0\ 1]^T$.

Prove the Following

- Row vector $\mathbf{P}^3$ is the principal plane.
- Row vectors $\mathbf{P}^1$ and $\mathbf{P}^2$ are axis planes for image Y and X axes respectively.
- The principal point (or image center) is given by $x_0 = M\, m_3$, with m_3, the third row vector of matrix M.



Thank You