CSE251
Basics of Computer Graphics
Module: Graphics Pipeline

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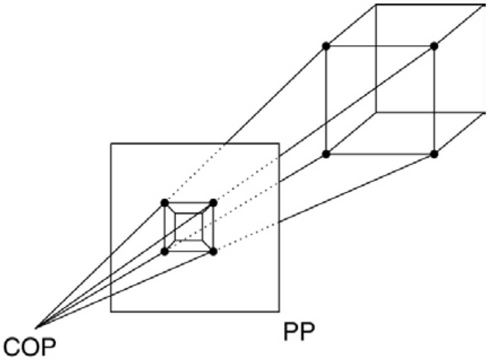
Projections

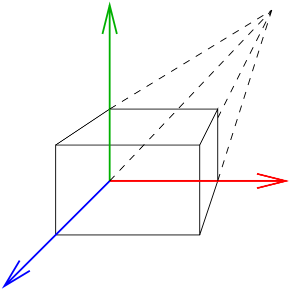
- Projection involves *projectors* starting from 3D points and hitting the 2D *projection plane*, forming the *image* of the point.
- Two types of projections.
- **Parallel projection**: Projectors are parallel to each other, all have the same *direction of projection* (**DOP**).
- **Perspective projection**: All projects pass through a point in space called the *centre of projection* (**COP**).

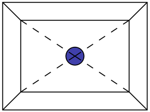
Projections (cont.)



Perspective Projections

- Can be characterized by the number of **vanishing points**. (projections of points at infinity).
- Depends on the number of axes the projection plane intersects.
- 1-point, 2-point, and 3-point perspective projections.

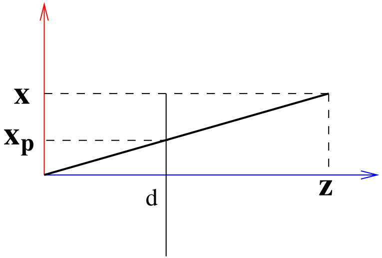




Geometry of Perspective Projection

- What is x_p, y_p, z_p?
- We know x, z, and d.
- Remember similar triangles?

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ w \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

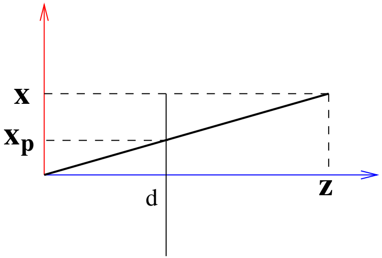


Geometry of Perspective Projection

- $\frac{x_p}{d} = \frac{x}{z}, \quad \frac{y_p}{d} = \frac{y}{z}, \quad z = d.$
- In matrix form,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

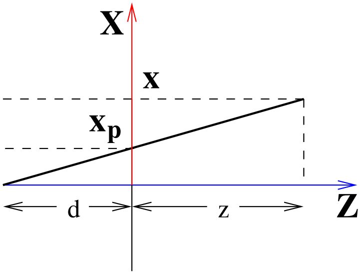
- Coordinates scaled down proportional to the depth or z values.



Another View

- Shift origin to lie on the projection plane, CoP at $(0, 0, -d)$, we get the matrix:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$



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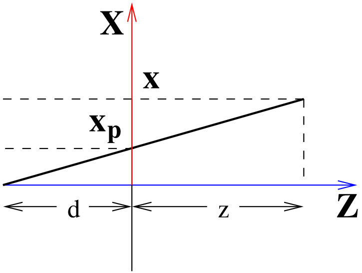
- Orthographic Projection matrix is a special case when $d \to \infty$.



$$x_p = \frac{xd}{z+d} = \frac{x}{1+\frac{z}{d}}, \quad z_p = 0$$

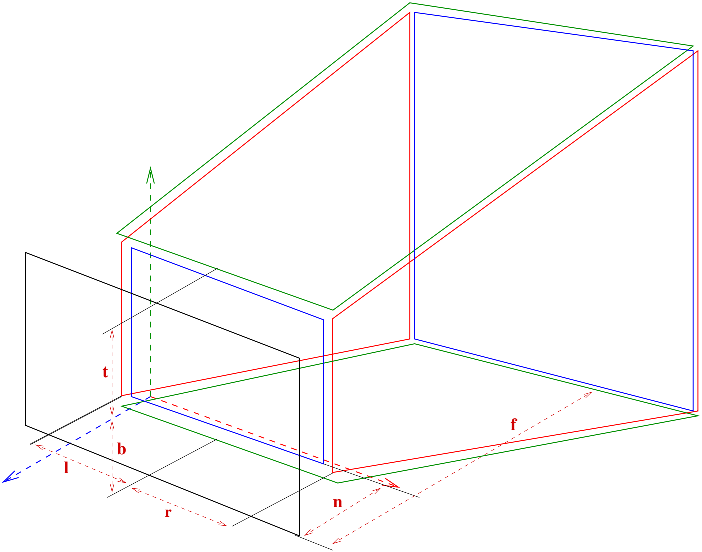
Volume of Visibility

- Cameras have finite fields of view in horizontal and vertical directions.
- What is the shape of its visible space?
- A cylinder for orthographic projections and a cone starting from the CoP for perspective may seem natural.
- Mathematics is difficult for cones; rectangular structures are easier!
- **View Volume:** The volume of potentially visible space.

View Volume

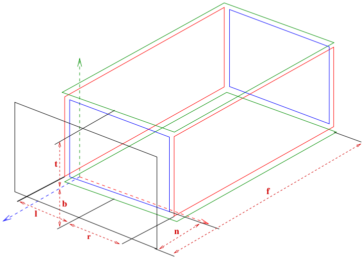
- View volume is a cube for orthographic cameras and a (truncated) pyramid for perspective projections.
- 4 planes (left, right, top, bottom) define the view volume.
- Graphics cameras use 2 additional planes to limit visibility: **near & far!**
- Planes are specfied in VRC; they move with the camera

Perspective View Volume



Orthographic & Perspective

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{d} & 1 \end{bmatrix}$$



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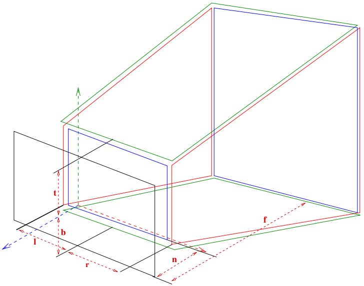


View Volume Specification

- View volume is specified by 6 planes:
  left, right, top, bottom, near, far. All values in VRC

  `glm::frustum()` or `glm::ortho()`

- left, right, top, bottom: signed distances.
  near, far: positive distances to planes.

- Needn't be symmetric!

Alternate Specification

- Symmetric view volumes: horizontal and vertical fields of view $\theta_{\mathbf{h}}, \quad \theta_{\mathbf{v}}$
- For symmetric perspective view volumes:

  $\tan \frac{\theta_h}{2} = \ldots ?$

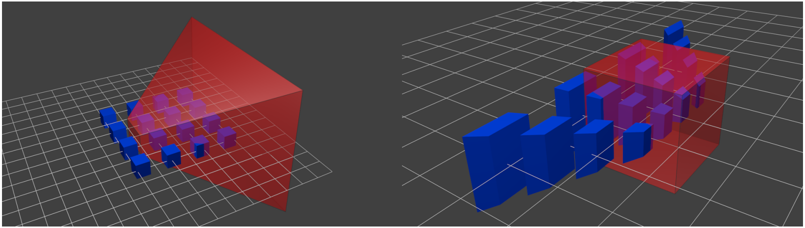
  $\tan \frac{\theta_v}{2} = \ldots ?$
- Aspect ratio: top / right.

  Using `glm::perspective()`

Canonical View Volume

- Projection is not performed right away; instead, map the view volume to a cube of fixed dimensions, called the **canonical view volume** or a standard view volume
- A **normalizing matrix** performs this transformation.
- Why?
  - Easier to eliminate objects outside the view volume.
  - Orthographic & perspective aren't different.
  - The z-coordinates not thrown away. (Used later!)

Canonical View Volume: Visualization

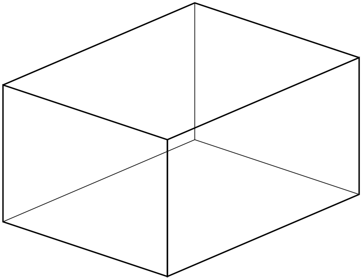


http://www.opengl-tutorial.org/beginners-tutorials/tutorial-3-matrices/

Canonical View Volume: Dimensions

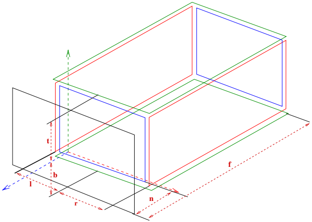
- OpenGL:

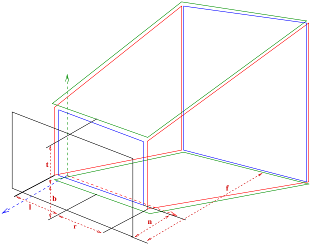
$$\mathbf{-1 \leq x \leq 1}$$
$$\mathbf{-1 \leq y \leq 1}$$
$$\mathbf{-1 \leq z \leq 1}$$



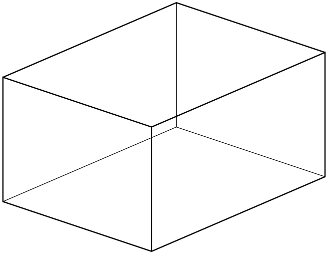
- Canonical view volume is the Orthographic View Volume, with appropriate scaling.

Orthographic & Perspective





To target view volume:



Perspective Normalizing Matrix

- General case: Match the vertices and solve!
  $(left, bottom, -near) \to (-1, -1, 1)$,
  $(l, t, -n) \to (-1, 1, 1)$, $(r, t, -n) \to (1, 1, 1)$,
  $(rf/n, bf/n, -f) \to (1, -1, -1)$,
  $(lf/n, tf/n, -f) \to (-1, 1, -1)$

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  $(lf/n, tf/n, -f) \to (-1, 1, -1)$

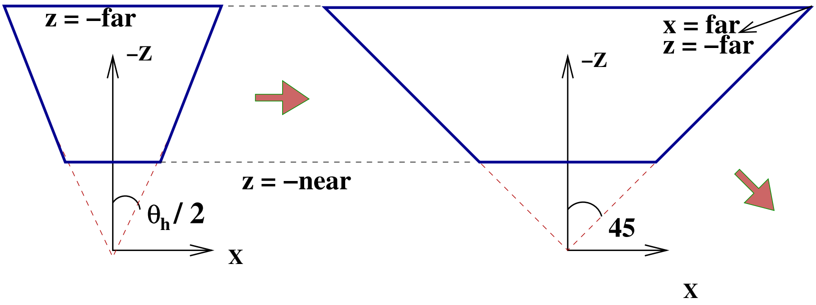
- Each match gives 3 equations in the 16 unknowns m_{ij}.
  (Only 15 unknowns upto scale!)

- Can solve for them given 5 point matchings.
  We have 8, hence easy!

Symmetric Perspective Proj Matrix

- More complicated than orthographic case, as a frustum has to be mapped to a cube. Do it in steps.
- First, scale the horizontal and vertical extents so that the vertical and horizontal fields of view are 90 degres.
- A scaling transformation with $s_x = ??, s_y = ??, s_z = 1$
- View volume is almost right except for a uniform scale.
- Next, scale uniformly so that the far plane is at -1. We will also have $-1 \leq x, y \leq 1$ at the far plane after this.
- $s_x = s_y = s_z = ??$

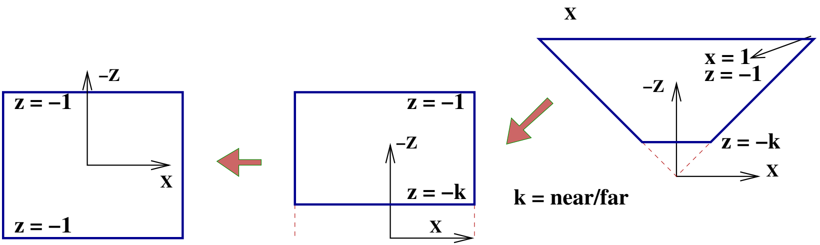
Symmetric Perspective Proj Matrix (cont.)



Symmetric Perspective Proj Matrix (cont.)

- M_1 with $s_x = \cot\frac{\theta_h}{2},\quad s_y = \cot\frac{\theta_v}{2},\quad s_z = 1$
- M_2 with $s_x = s_y = s_z = \frac{1}{\textbf{far}}$
- $M_2 M_1 = \begin{bmatrix} \frac{\cot\theta_h/2}{\textbf{far}} & 0 & 0 & 0 \\ 0 & \frac{\cot\theta_v/2}{\textbf{far}} & 0 & 0 \\ 0 & 0 & \frac{1}{\textbf{far}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- The near plane is now at $\mathbf{k} = \frac{\textbf{near}}{\textbf{far}}$.
- View volume fits into the canonical view volume, but is still a frustum!

Symmetric Perspective Proj Matrix (cont.)



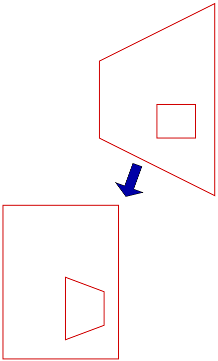
- Scale by to $-z$ to convert to a cube using a matrix with last row $[0 \quad 0 \quad -1 \quad 0]$.
- Simultaneously send third component to range $[-1, 1]$. $z = -k$ maps to 1 and $z = -1$ maps to -1.
- Scale z by $\frac{1+k}{1-k}$ and translate by $\frac{2k}{1-k}$.

Perspective Normalizing Txform

- Matrix M_3 for this step:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+k}{1-k} & \frac{2k}{1-k} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- $(x, y, -k)$ & $(x, y, -1)$ go to?
- Final matrix: $M = M_3 M_2 M_1$
- Frustum becomes a cube



Final 2D Coordinates

$$(u, v, d) \equiv \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = M_3 M_2 M_1 \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{\text{VRC}}$$

- **Perspective division:** Divide x', y' coordinates by the w to get the normalized coordinates (u, v). (z' maintains ordering and can be used without division.)
- The normalized d component has non-linear precision. Higher around the *near* plane and lower around the *far* plane due to the division by z.

OpenGL Normalizing/Perspective Matrix

- The projection matrix in OpenGL is given by

$$P = \begin{bmatrix} \frac{2n}{r-l} & 0 & A & 0 \\ 0 & \frac{2n}{t-b} & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

where,

$$A = \frac{r+l}{r-l}, B = \frac{t+b}{t-b}, C = \frac{f+n}{f-n}, D = \frac{2nf}{f-n}$$

*Please refer to the support material for derivation.

Where is the Film?

- Turns out: **It does not matter**.
- A final scaling is in the viewport transformations.
- As long as the film is in front of the camera, we will see an upright image.
- Can consider the near plane as the film plane.

Viewport Txformation: To Window

- Image of size -1 to +1 in X and Y is ready. The **viewport** transformation maps it to the actual window on screen.
- From $[1, 1]$, map x and y to $[0, W]$ and $[0, H]$.
- First step: set sizes by scaling: $\mathbf{S}(\frac{\mathbf{W}}{\mathbf{2}}, \frac{\mathbf{H}}{\mathbf{2}})$.
  Next: Translate origin to South-West corner: $\mathbf{T}(\frac{\mathbf{W}}{\mathbf{2}}, \frac{\mathbf{H}}{\mathbf{2}})$
- Overall: $\mathbf{M} = \mathbf{T}(\frac{\mathbf{W}}{\mathbf{2}}, \frac{\mathbf{H}}{\mathbf{2}})\ \mathbf{S}(\frac{\mathbf{W}}{\mathbf{2}}, \frac{\mathbf{H}}{\mathbf{2}}) = \begin{bmatrix} \frac{W}{2} & 0 & \frac{W}{2} \\ 0 & \frac{H}{2} & \frac{H}{2} \\ 0 & 0 & 1 \end{bmatrix}$

General Viewport Txform

- General command: `glViewport(l, b, r, t)`.
- Translate so the range of x, y is $0 \cdots 2$.
- Scale so x varies from 0 to $(r - l)$ and y varies from 0 to $(t - b)$.
- Translate so x range is l to r and y range is b to t.
- Matrix for this? $\mathbf{T}(l, b)\ \mathbf{S}(\frac{r-l}{2}, \frac{t-b}{2})\ \mathbf{T}(1, 1) = \begin{bmatrix} \frac{r-l}{2} & 0 & \frac{r+l}{2} \\ 0 & \frac{t-b}{2} & \frac{t+b}{2} \\ 0 & 0 & 1 \end{bmatrix}$
- $[-1 \quad -1 \quad 1]^{\mathsf{T}}$ maps to $[l \quad b \quad 1]^{\mathsf{T}}$.
- $[1 \quad 1 \quad 1]^{\mathsf{T}}$ maps to $[r \quad t \quad 1]^{\mathsf{T}}$.

(Point) Pipeline in Action

- Points are transformed from Object to World to Canonical to Window coordinates.
- Each 3D point maps to a pixel (i, j) in the window space.
- Lines are made out of two points. Triangles and polygons are made out of 3 or more points.