21.09.2021

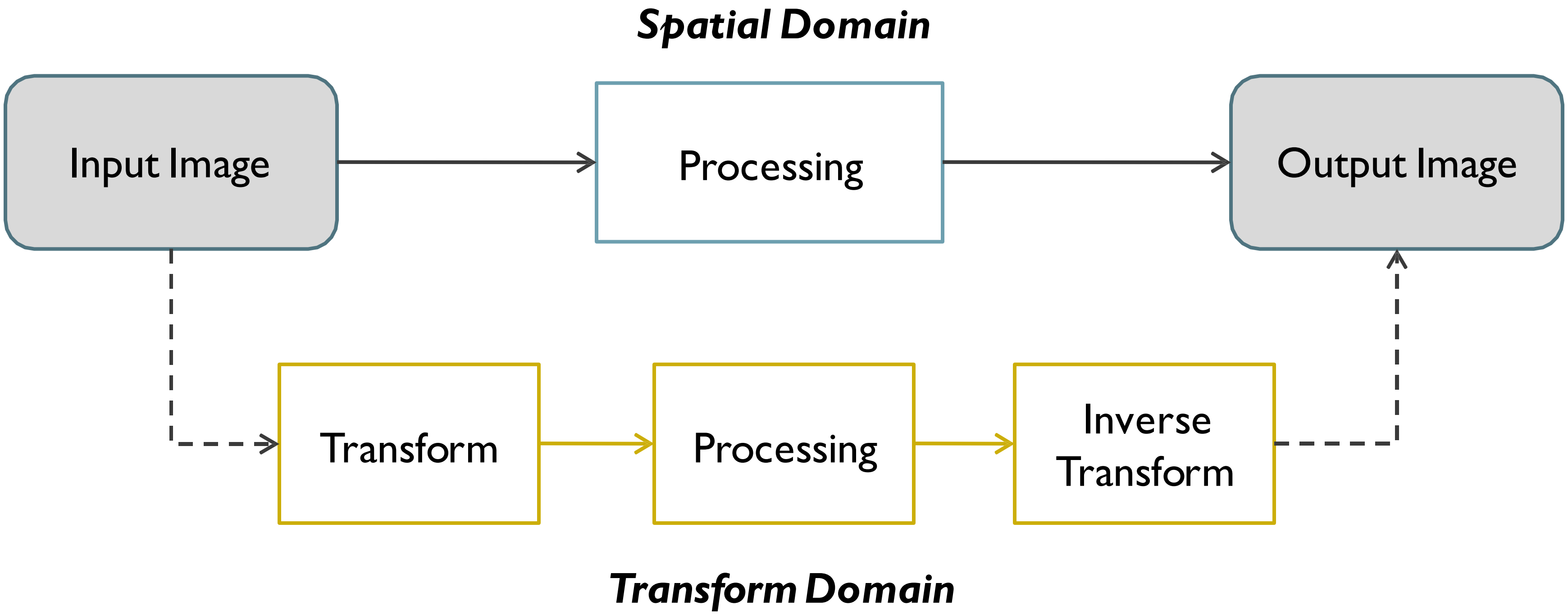
Digital Image Processing (CSE/ECE 478)

Lecture-9: Frequency Domain Processing

Ravi Kiran and Sudipta Banerjee

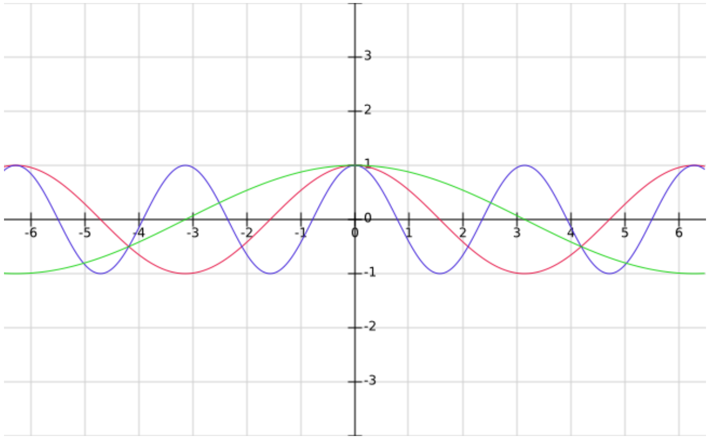
Center for Visual Information Technology (CVIT), IIIT Hyderabad

Spatial vs. Transform Domain Processing



Simple periodic signals

- $x(t) = A\cos(t)$
- $x(t) = A\cos(2t)$
- $x(t) = A\cos(t/2)$

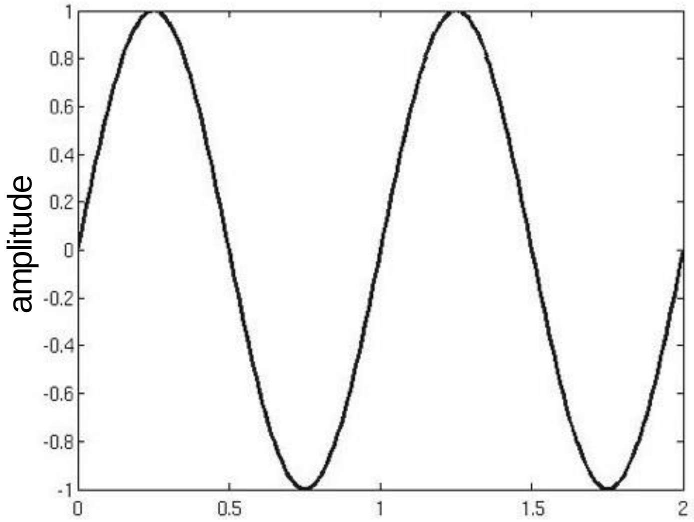


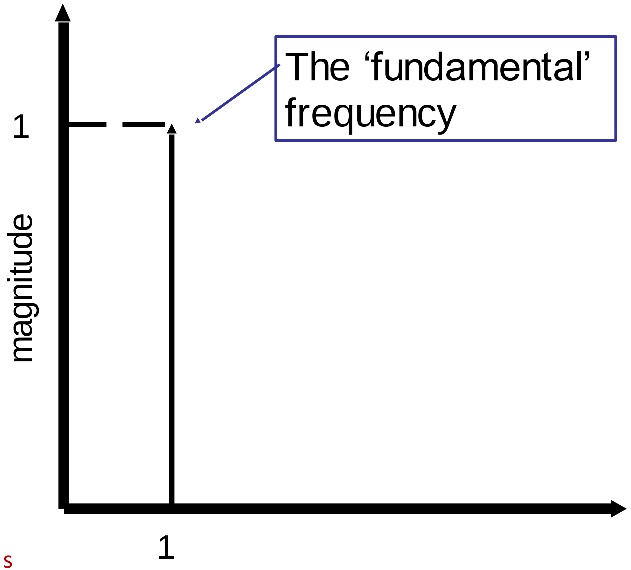
- $x(t) = A\cos(\omega t) = A\,cos(2\pi f t) = A\,cos(\frac{2\pi}{T}t)$

Angular frequency

Signal and Frequency Domains

$$\boxed{y = \sin(2\pi s)}$$





NOTE: s=0 is location from where signal is considered

spatial distance (s)

Signal domain

NOTE: s is bounded only for physical signals

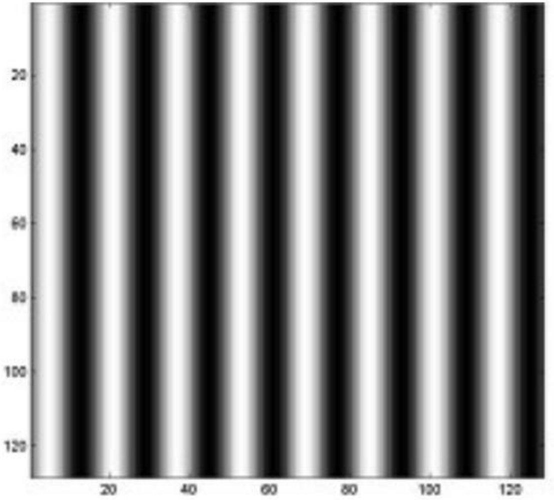
↔

Frequency (# of cycles per unit distance)

Frequency domain

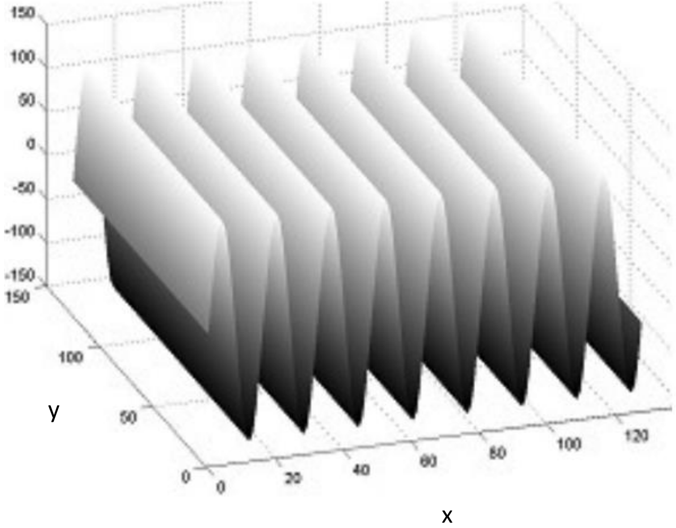
Periodic Images

128 x 128 grayscale image



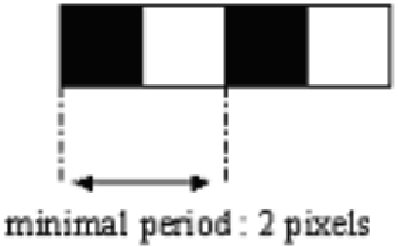
Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel

$$I(x, y) = 128\ sin(2\pi x/16)$$



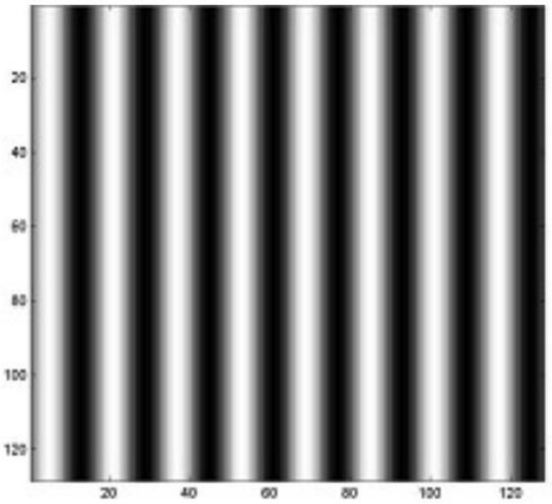
Periodic Images

Spatial period = Minimal # of pixels between two identical patterns in a "periodic" image



$$\Rightarrow \quad \nu_{max} = \frac{1}{\text{minimal period}} = \frac{1}{2}$$

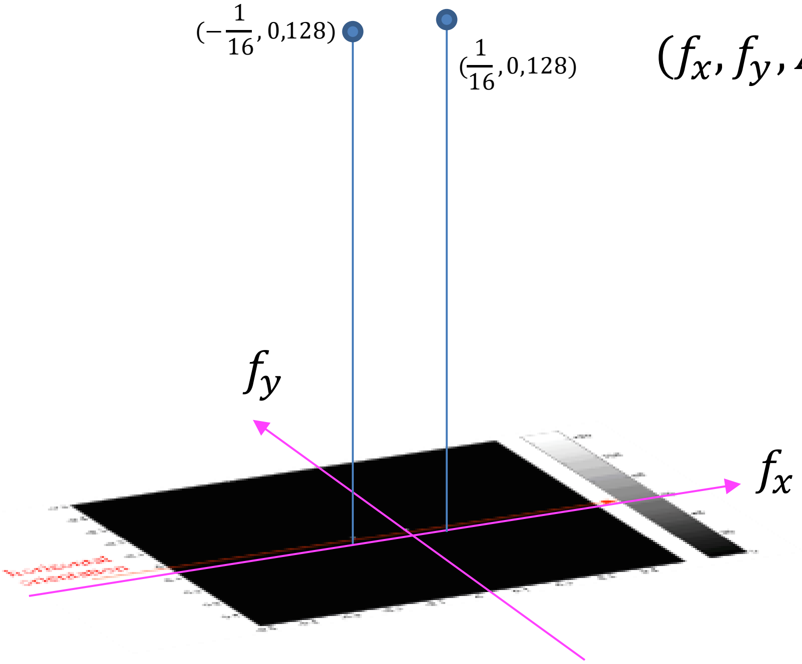
$$I(x, y) = 128\ sin(2\pi x/16)$$



Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel

Spatial domain

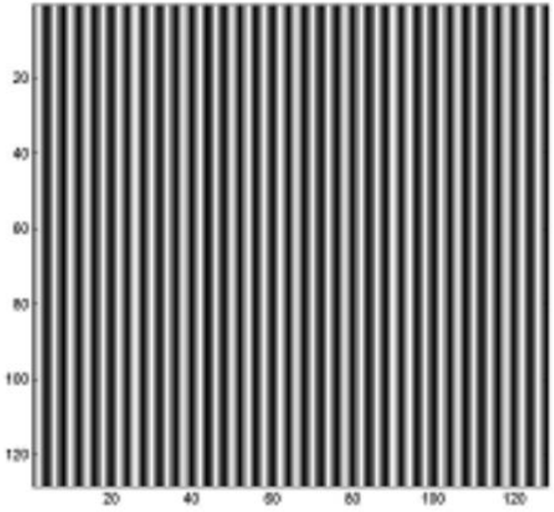
$|F|$



(f_x, f_y, A)

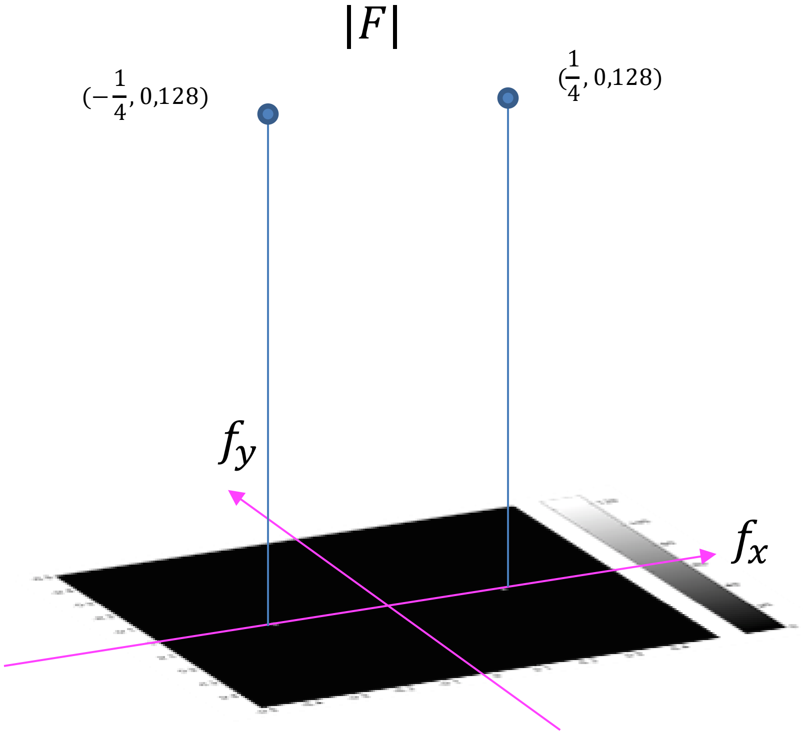
Frequency domain

$I(x, y) = 128\ sin(2\pi x/4)$



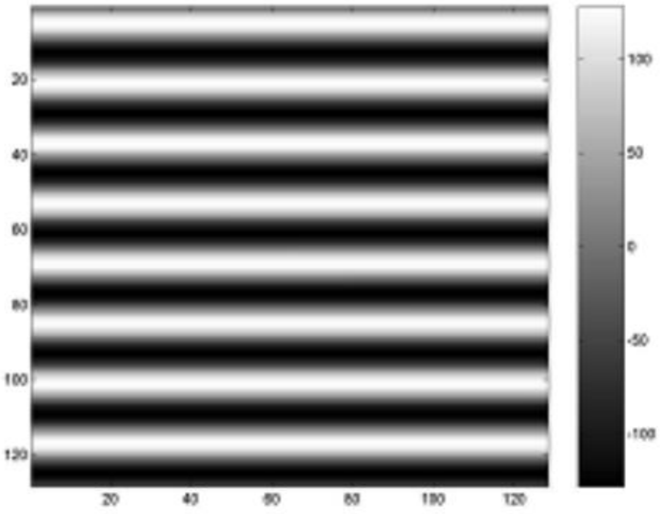
Sinusoid pattern repeats every 4 pixels
f = 1/4 cycles/pixel

Spatial domain



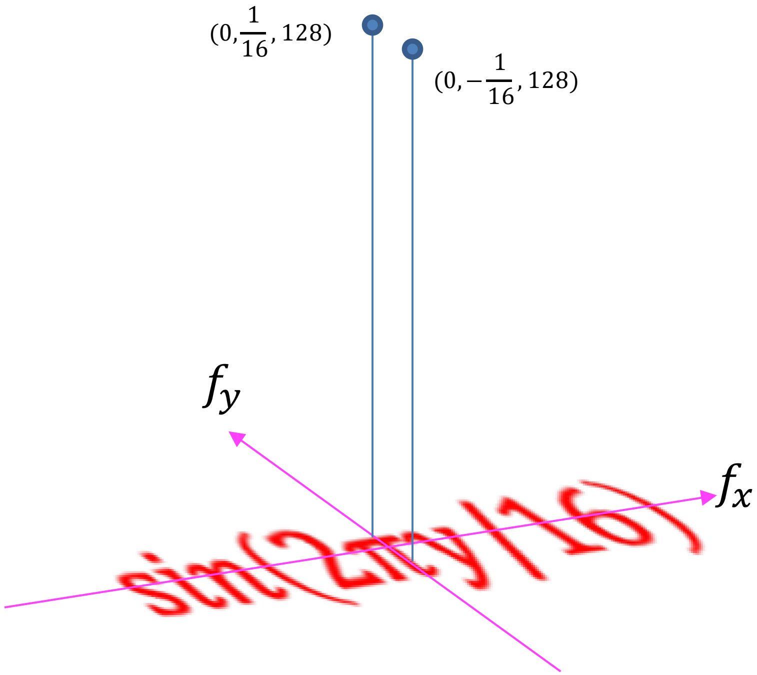
Frequency domain

$I(x, y) = 128\ sin(2\pi y/16)$



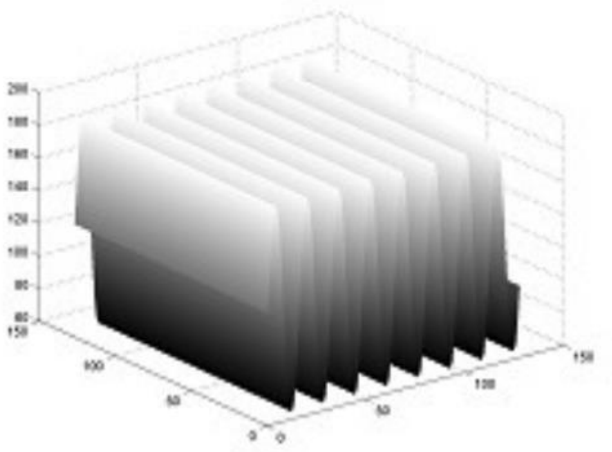
Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel

Spatial domain



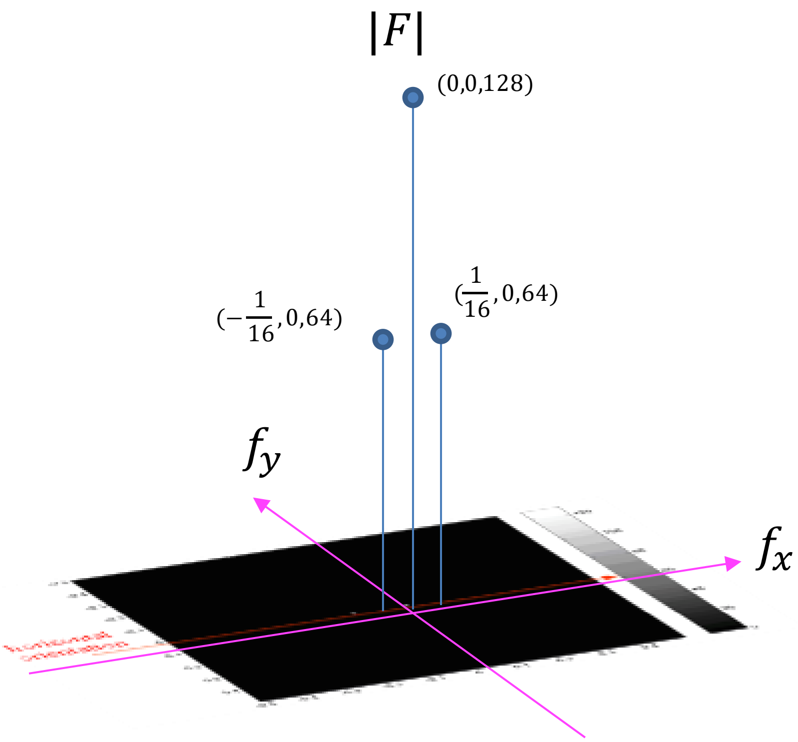
Frequency domain

$I(x, y) = 128 + 64\ sin(2\pi x/16)$



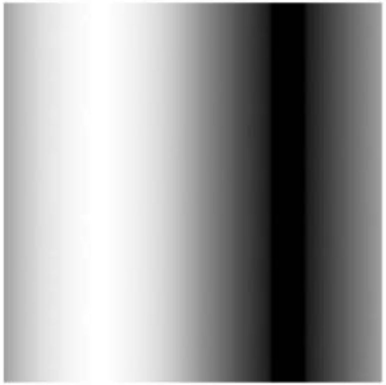
Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel

Spatial domain



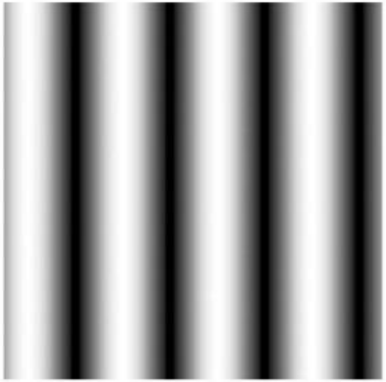
Frequency domain

- Intensity images for $s(x,y)=\sin\left[2\pi\left(u_0x+v_0y\right)\right]$

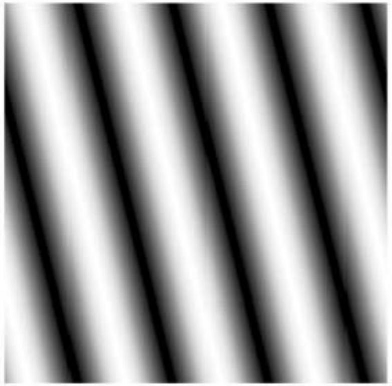


$u_0 = 1, v_0 = 0$

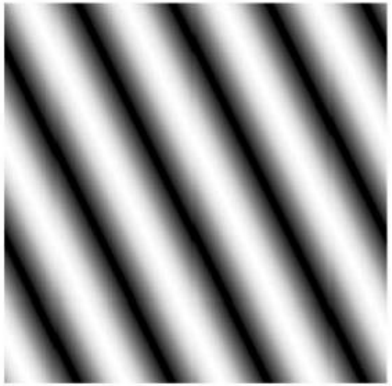
$u_0 = 2, v_0 = 0$



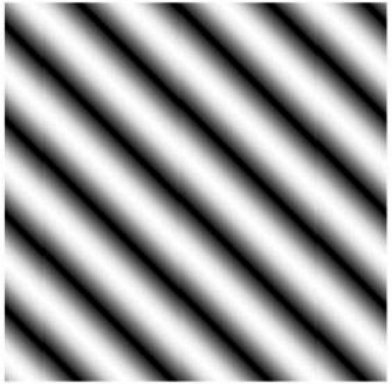
$u_0 = 4, v_0 = 0$



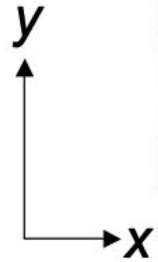
$u_0 = 4, v_0 = 1$

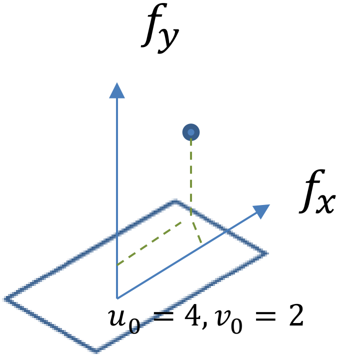


$u_0 = 4, v_0 = 2$

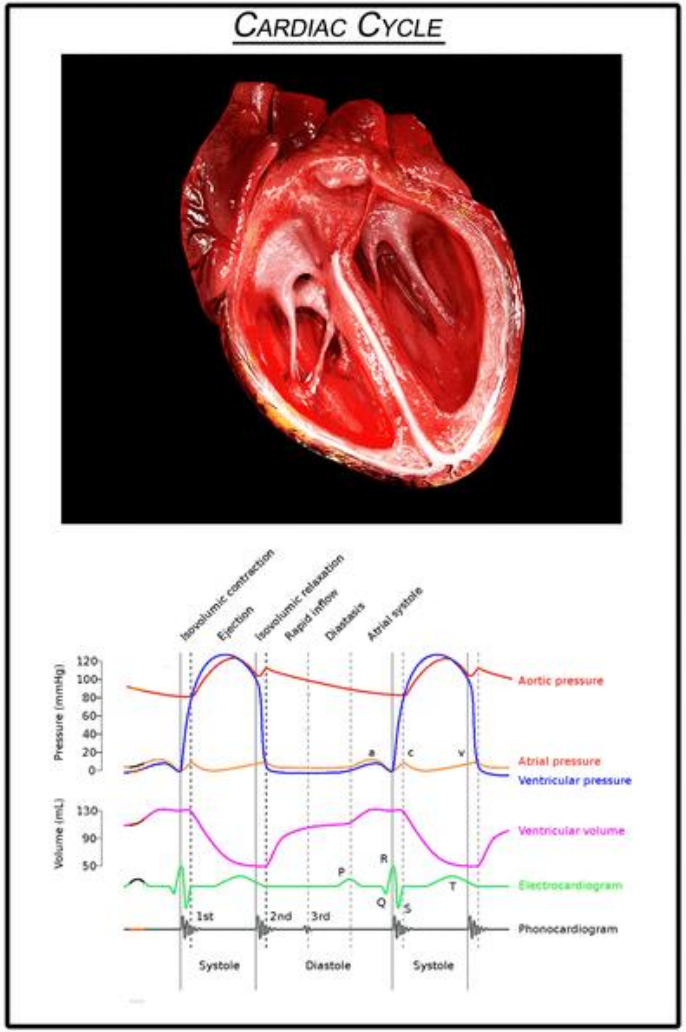


$u_0 = 4, v_0 = 4$





Many natural phenomena (signals) are periodic but not necessarily sinusoidal



https://commons.wikimedia.org/wiki/File:Cardiac-Cycle-Animated.gif

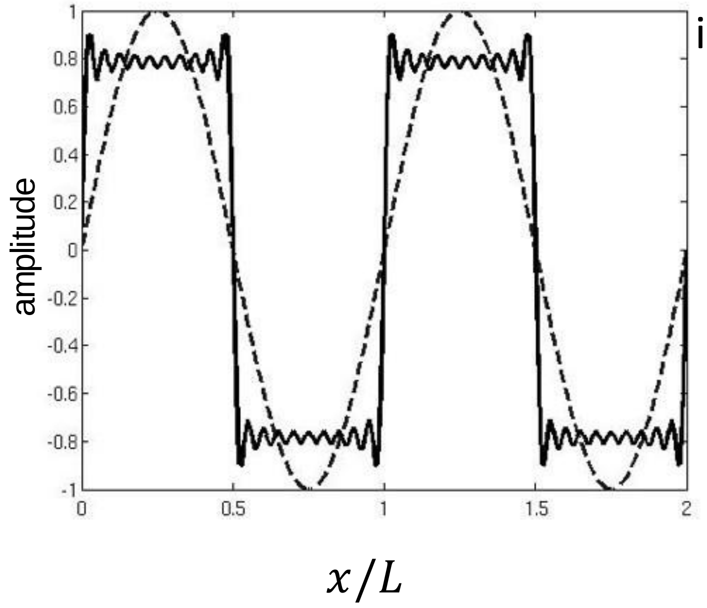
Fourier Series

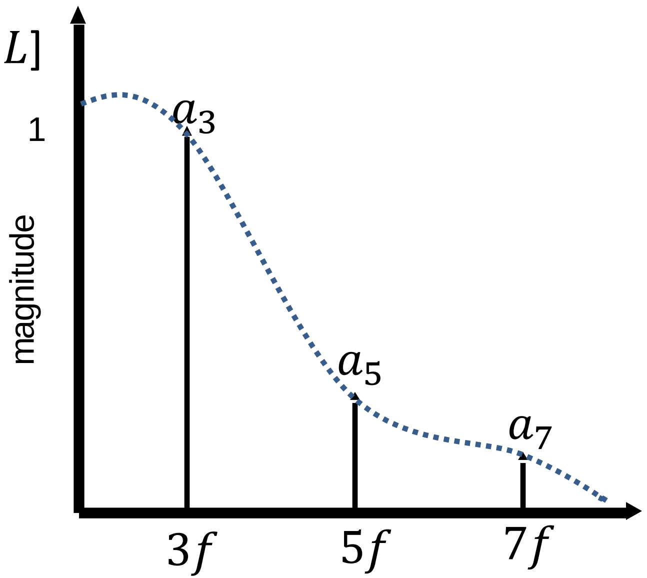
Approximate **periodic signals** with sines and cosines

Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_n cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{L}\right)$$

Periodic in interval $[-L, L]$





Signal ⟷ **Frequency Spectrum**

Fourier Series

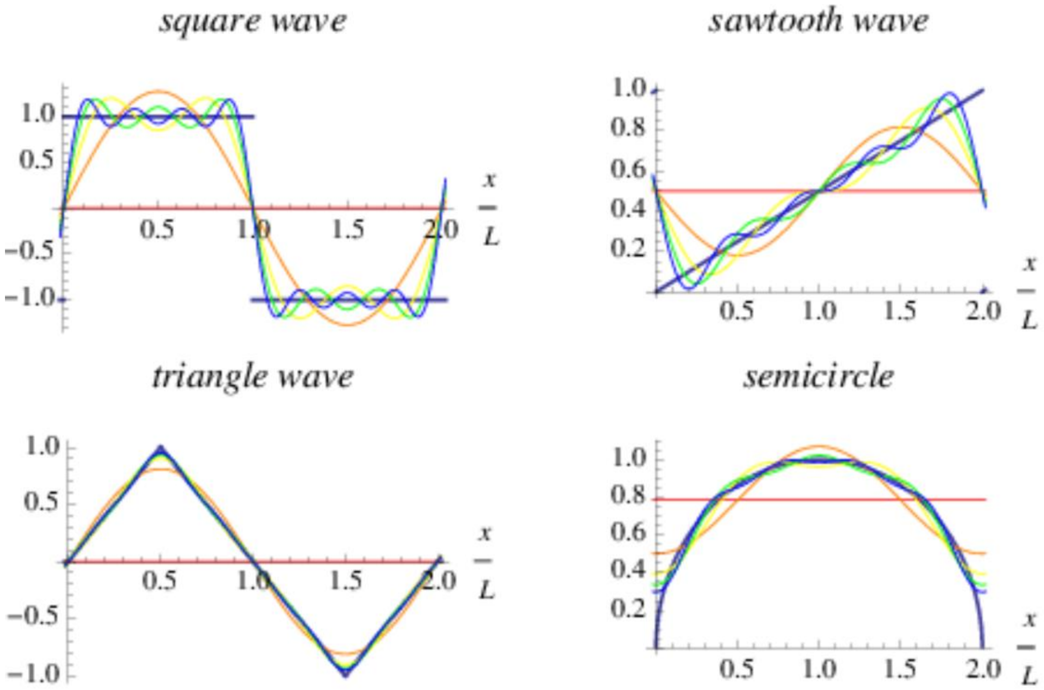
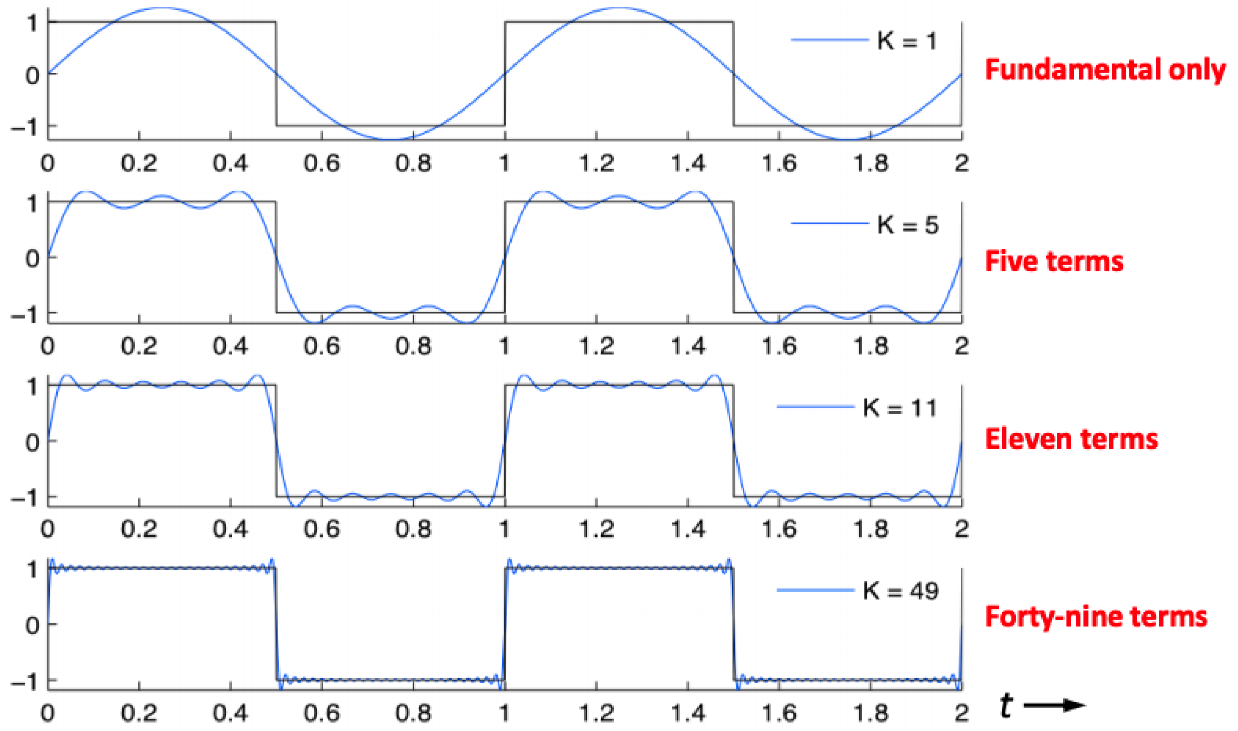
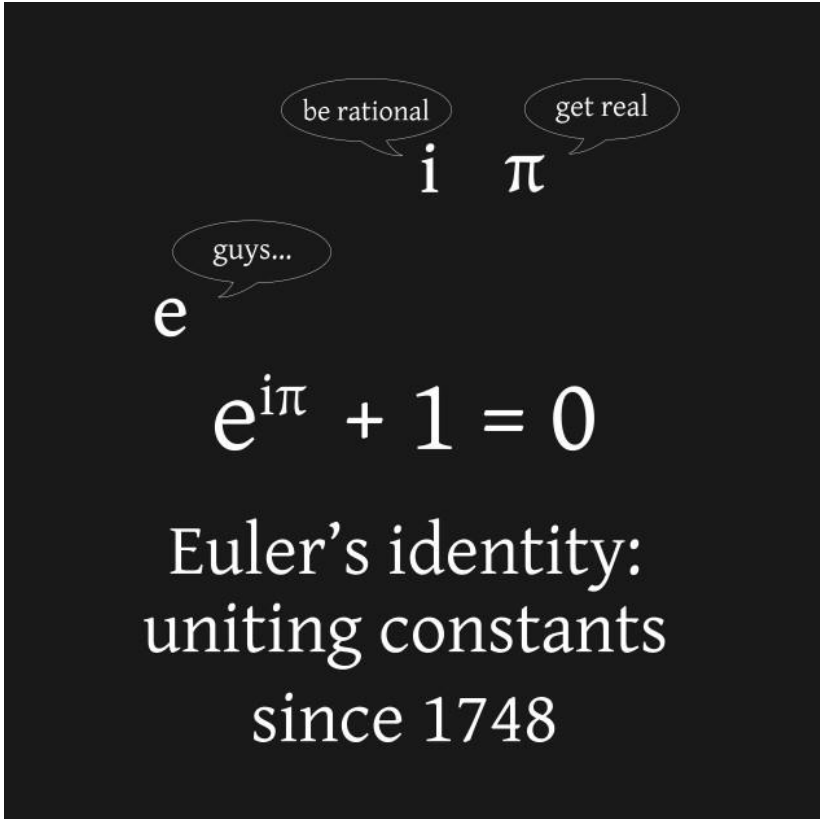


Image Courtesy: https://mathworld.wolfram.com/FourierSeries.html

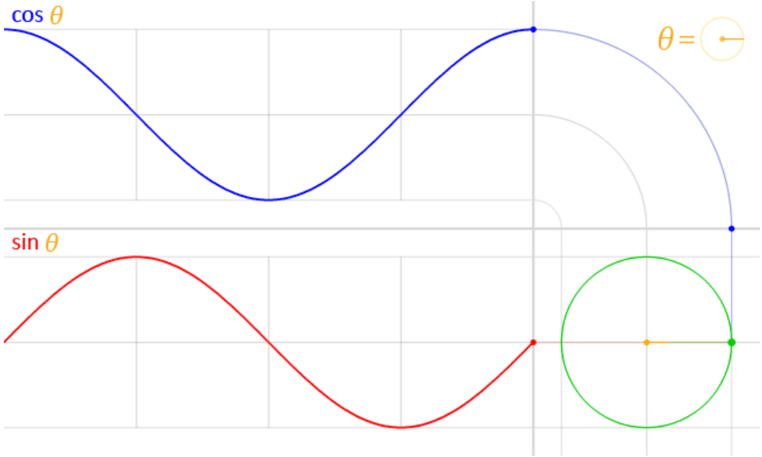
$$\hat{f}(t) = \frac{4}{\pi}\left[\sin(\pi t) + \tfrac{1}{3}\sin(3\pi t) + \tfrac{1}{5}\sin(5\pi t) + \tfrac{1}{7}\sin(7\pi t) + \cdots\right]$$



http://ceng.gazi.edu.tr/dsp/fourier_series/description.aspx



$$e^{i\theta} = \cos(\theta) + i\, sin(\theta)$$



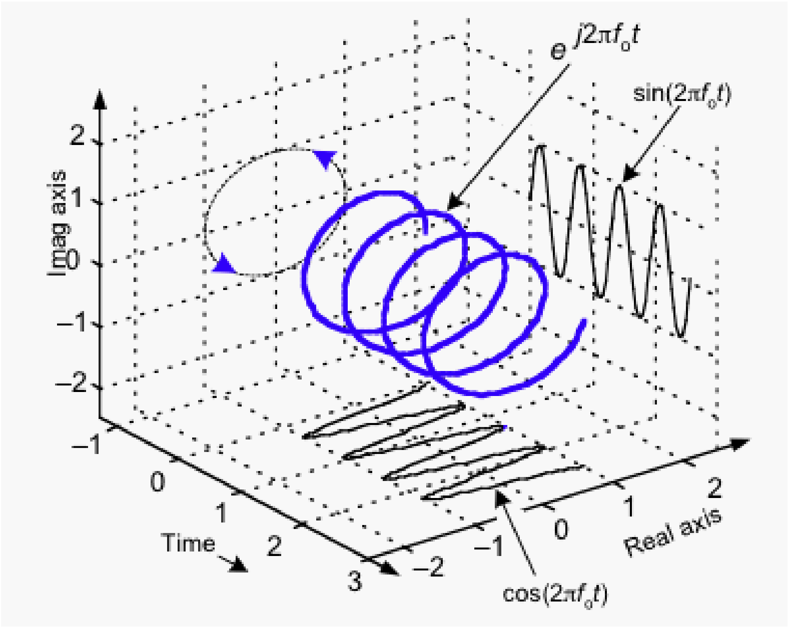
https://res.cloudinary.com/teepublic/image/private/s--s4GX0hG3--/t_Preview/b_rgb:191919,c_limit,f_jpg,h_630,q_90,w_630/v1486157785/production/designs/1177382_1.jpg

Complex sinusoid

$$e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\,\sin(2\pi f_0 t)$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$



Fourier Series in terms of complex coefficients

$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi mt}{T}\right)dt$$

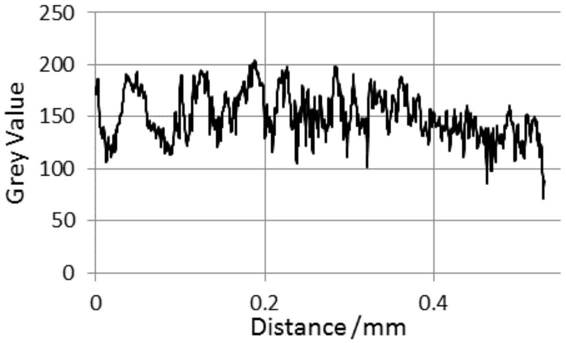
$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

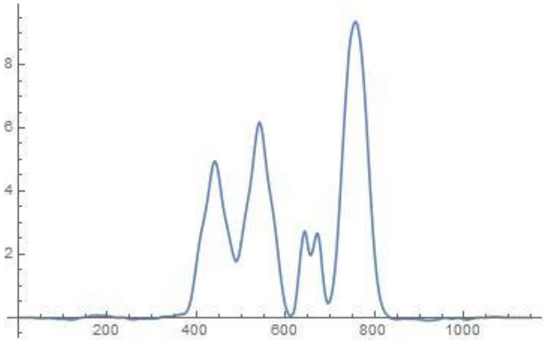
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi nt}{T}}\,dt$$

http://www.thefouriertransform.com/series/complexcoefficients.php

What if f(t) is non-periodic ?





$$\boxed{\cancel{f(t)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}}}$$

https://i.stack.imgur.com/fKznF.jpg

Fourier Transform

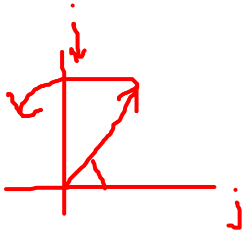
Approximate **non-periodic signals** with complex sinusoids

Definition: Fourier Transform

- the Fourier Transform of a function $f(t)$ is defined by:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

- The result is a function of ω (frequency).



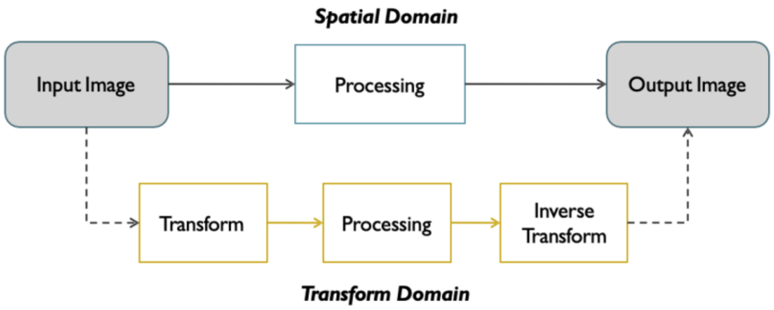
Fourier Transform and Inverse Fourier Transform

- Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad F(\omega) = \mathcal{F}[f(t)]$$

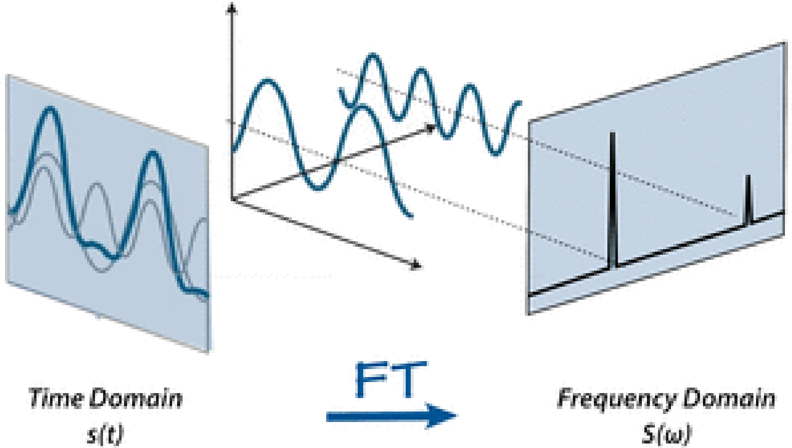
- Inverse Fourier Transform

$$f(t) = \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{2\pi} F(\omega) e^{i\omega t} d\omega \qquad f(t) = \mathcal{F}^{-1}[F(\omega)]$$



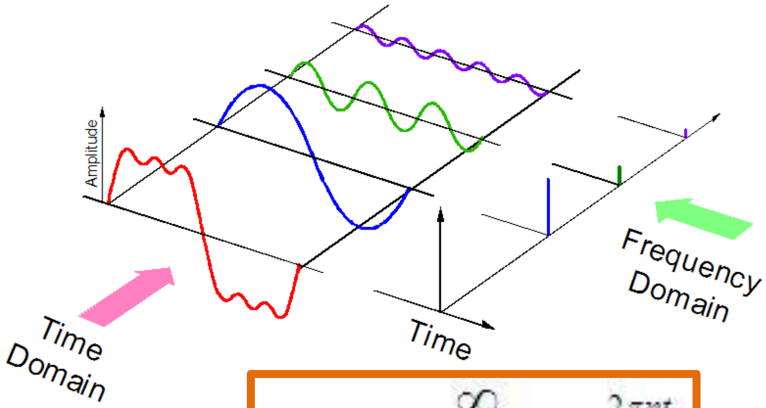
Fourier Transform vs Series

Fourier Transform



$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Fourier Series (periodic only)



$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi nt}{T}}dt$$

Unit Impulse Function

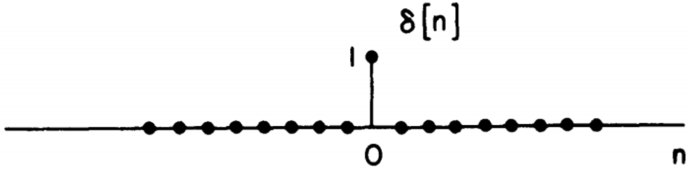
$\delta(t) = 0, \ for \ t \neq 0.$

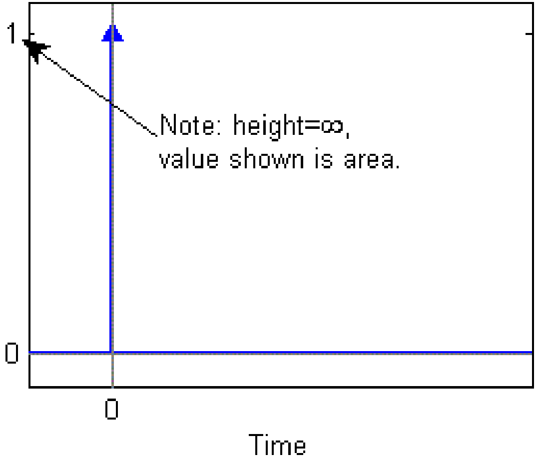
$\delta(0) = +\infty$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Discrete Impulse Function

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$





Scaled Impulse Function

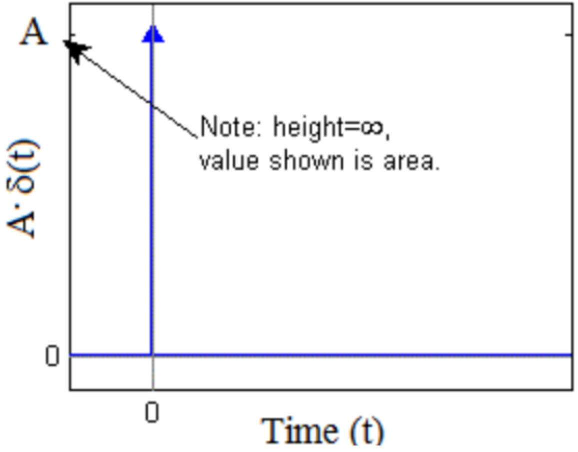


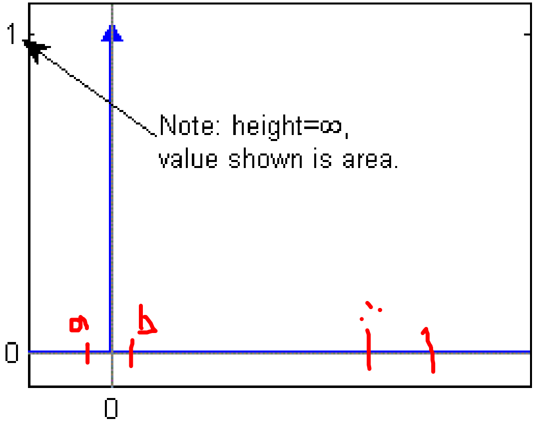
Image courtesy: https://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

Unit Impulse Function – Some properties

$$\delta(t) = 0, \ \ for \ t \neq 0.$$

$$\delta(0) = +\infty$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



$$\int_a^b \delta(t)dt = \begin{cases} 1, & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$

Integral property

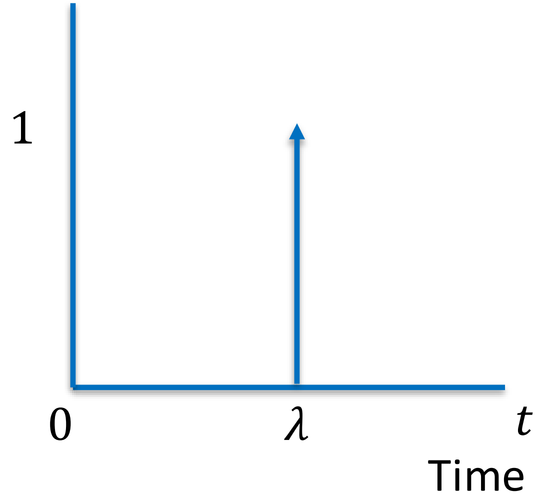
$$\begin{aligned} \int_a^b \delta(t) \cdot f(t)dt &= \int_a^b \delta(t) \cdot f(0)dt \\ &= f(0) \cdot \int_a^b \delta(t)dt \\ &= \begin{cases} f(0), & a < 0 < b \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Sifting property

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

Shifted Unit Impulse Function – Some properties

$$\delta(t-\lambda) = +\infty \; at \; t = \lambda$$
$$\delta(t-\lambda) = 0 \; at \; t \neq \lambda$$



$$\int_a^b \delta(t-\lambda)dt = \begin{cases} 1, & a < \lambda < b \\ 0, & otherwise \end{cases}$$

Integral property

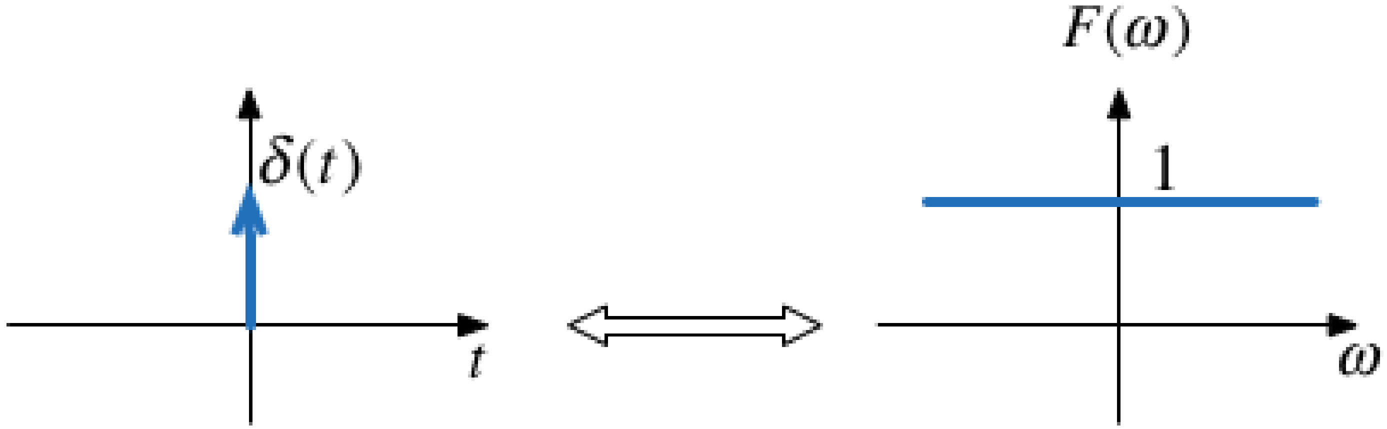
$$\int_a^b \delta(t-\lambda) \cdot f(t)dt = \begin{cases} f(\lambda), & a < \lambda < b \\ 0, & otherwise \end{cases}$$

Sifting property

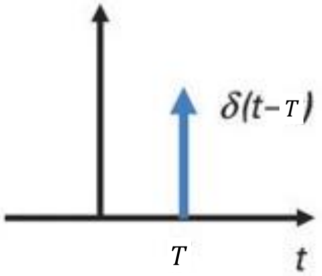
http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

FT of impulse function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$



FT of time-shifted impulse



$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$= e^{-i\omega T}$$

$$\int_{a}^{b} \delta(t-T)\, x(t)dt = x(T), \qquad a < T < b$$

$$= 0 \; otherwise$$

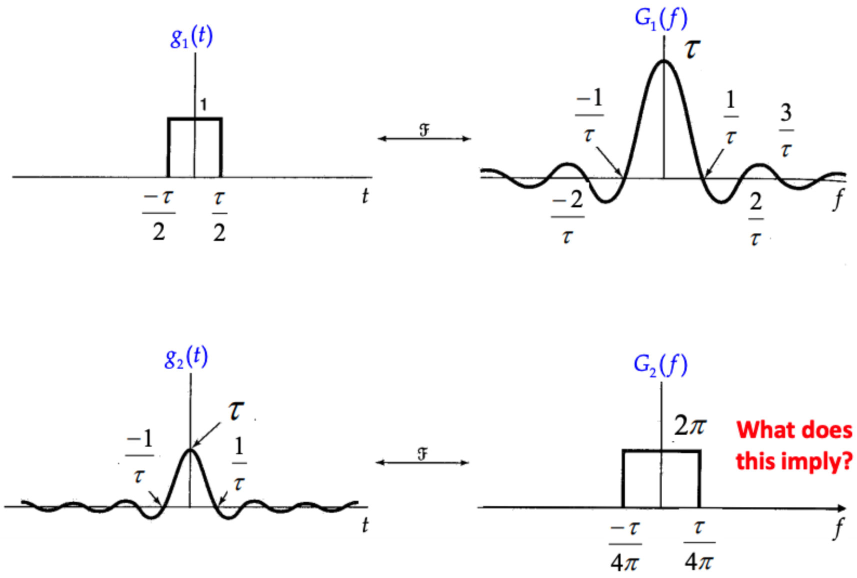
Duality property of FT

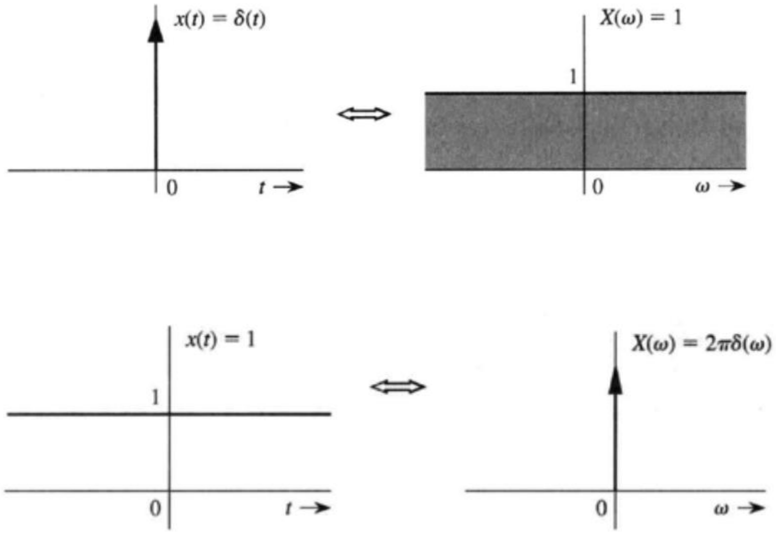
$$\mathcal{F}[f(t)] = F(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} F(\omega)e^{i\omega t}d\omega$$

$$\Longrightarrow \mathcal{F}[F(t)] = 2\pi\, f(-\omega)$$





Duality property of FT

$$\mathcal{F}[f(t)] = F(\omega)$$

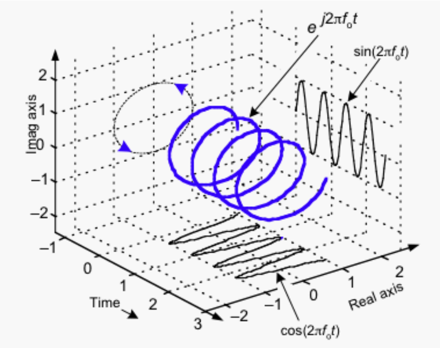
$$\Longrightarrow \mathcal{F}[F(t)] = 2\pi f(-\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} F(\omega) e^{i\omega t} d\omega$$

FT of complex exponential

$$e^{i\omega_0 t} \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$



Symmetry Property of FT

Input Domain	Frequency Domain
Real, even	Real, even
Real, odd	Imaginary, odd
Real, no symmetry	Hermitian
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd
Imaginary, no symmetry	Anti-Hermitian
Hermitian	Real, no symmetry
Anti-Hermitian	Imaginary, no symmetry
Complex, even	Complex, even
Complex, odd	Complex, odd
Complex, no symmetry	Complex, no symmetry

McCabe et al., “Theory of spatiochromatic image encoding and feature extraction,” Journal of Optical Society of America A, 2000

- The integral of product of an odd and even function over a symmetric interval is zero
- Product of functions (NOT NUMBERS):
  - Even x Even : Even
  - Odd x Even : Odd
  - Odd x Odd : Even

Even Functions

Theorem 5.3 The Fourier transform of a real even function is real.

$$
\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \\
&= \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi st) - j\sin(2\pi st)\right] dt \\
&= \int_{-\infty}^{\infty} f(t) \cos(2\pi st) dt
\end{aligned}
$$

which is real.

FT of cosine

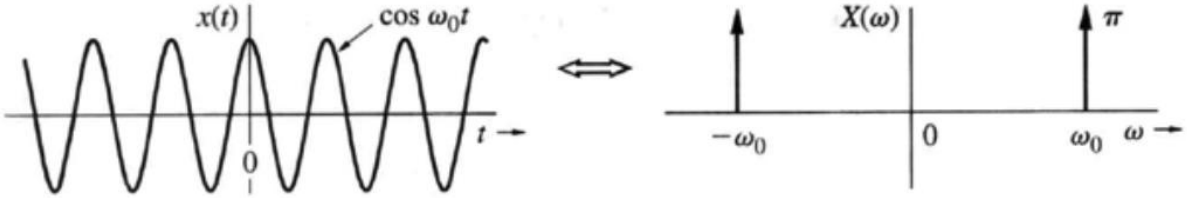
$$e^{j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega - \omega_0)$$

$$e^{-j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega + \omega_0)$$

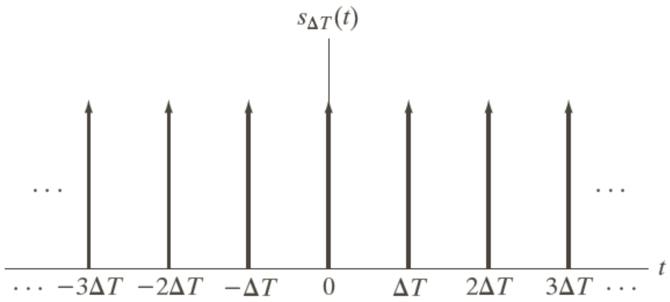
$$\cos\omega_0 t = \frac{1}{2}(e^{-j\omega_0 t} + e^{j\omega_0 t})$$

$$\mathcal{F}\big(x_1(t)\big) + \mathcal{F}\big(x_2(t)\big) = \mathcal{F}\big(x_1(t) + x_2(t)\big)$$

$$\mathcal{F}(cos\omega_0 t) = \frac{1}{2}\left[\mathcal{F}\left(e^{-j\omega_0 t}\right) + \mathcal{F}(e^{j\omega_0 t})\right]$$
$$= \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$



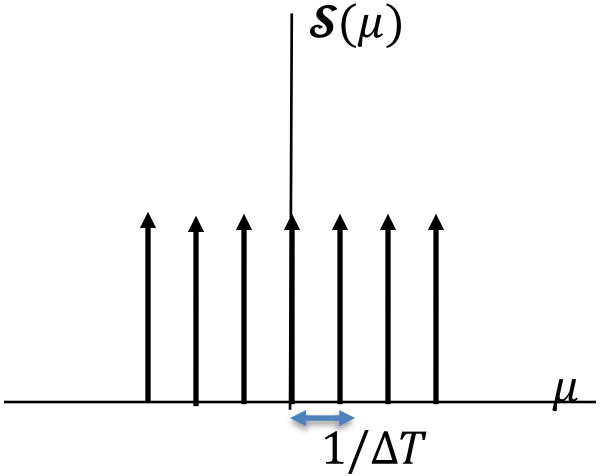
FT of impulse train(G&W, 4.2.4)



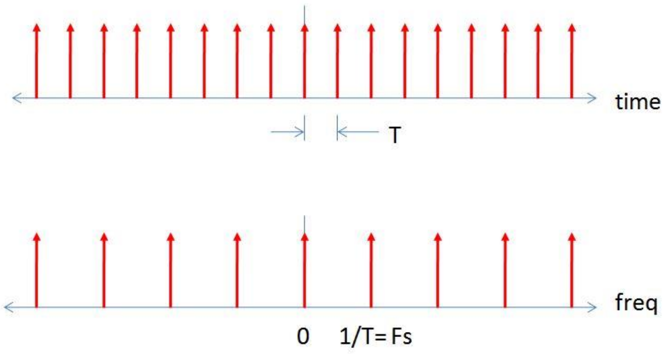
$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - n\Delta T\right)$$

$$\mathcal{F}\left(s_{\Delta T}(t)\right) = \sum_{n=-\infty}^{\infty} \delta\left(\mu - n/\Delta T\right)$$

$\mathcal{S}(\mu)$



FT of impulse train



Impulses in time $\longrightarrow \mathfrak{F}\{\} \longrightarrow$ Impulses in frequency

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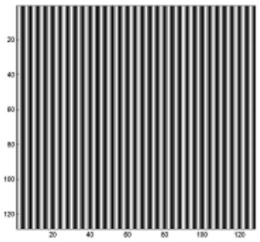
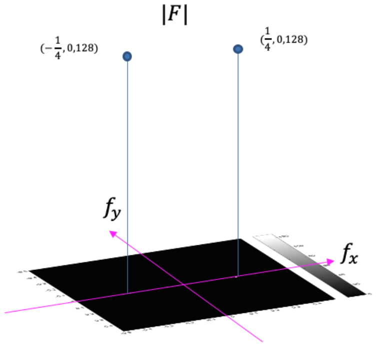
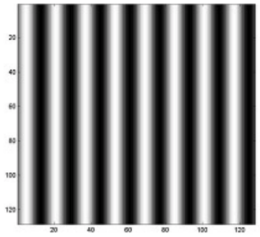
$I(x,y) = 128\,sin(2\pi x/4)$



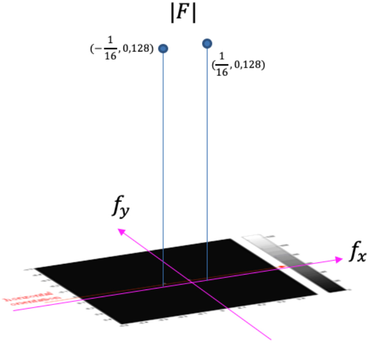
Sinusoid pattern repeats every 4 pixels
f = 1/4 cycles/pixel



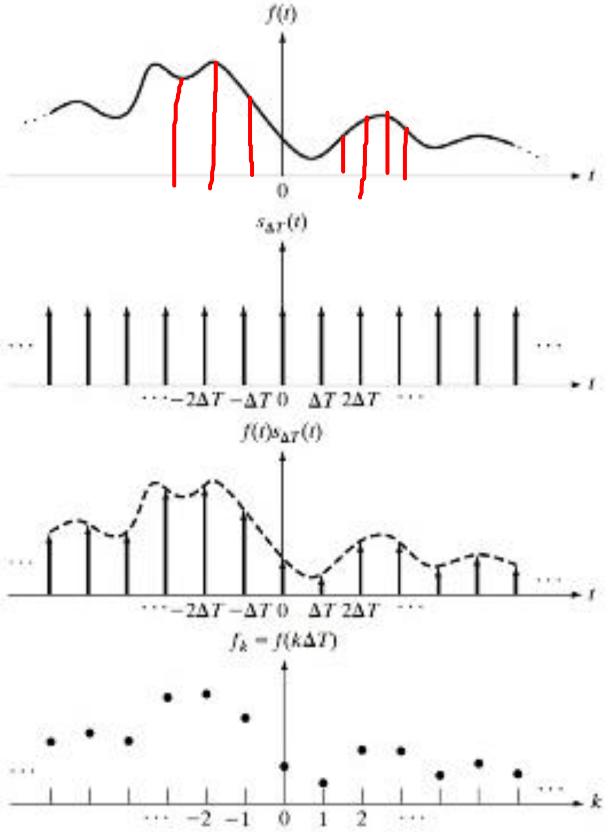
$I(x,y) = 128\,sin(2\pi x/16)$



Sinusoid pattern repeats every 16 pixels
f = 1/16 cycles/pixel



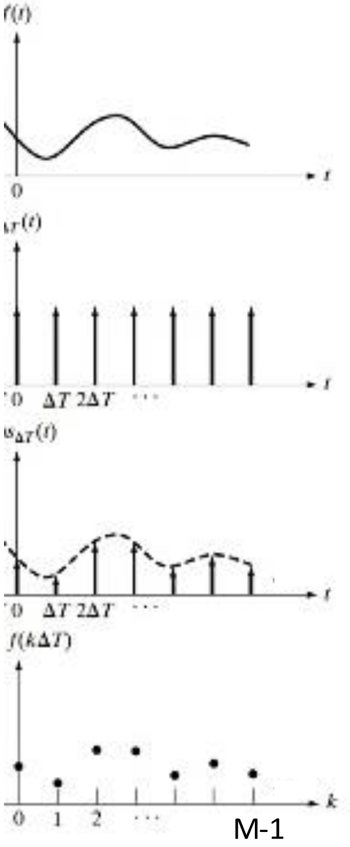
Sampling = f(t) x Impulse Train



$$s_{\Delta T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\Delta T)$$

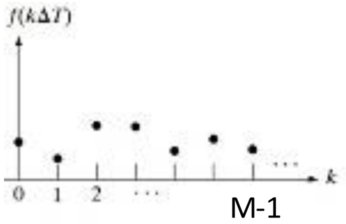
$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n \, \delta(t - n\Delta T)$$

Sampling = f(t) x Impulse Train



$$\tilde{f}(t) = \sum_{n=0}^{n=(M-1)\Delta T} f_n \, \delta(t - n\Delta T)$$

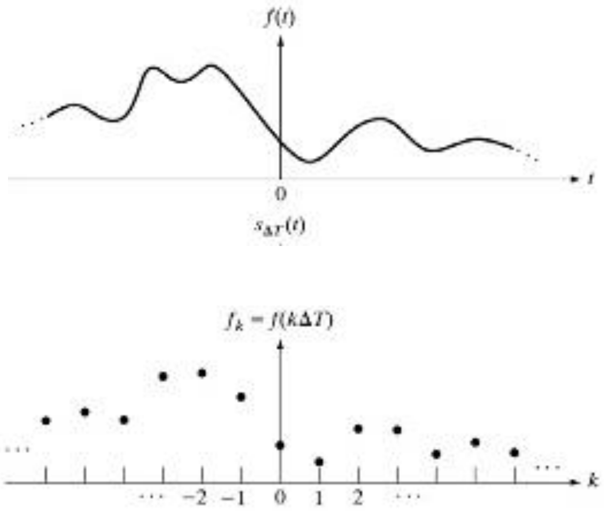
FT of sampled function

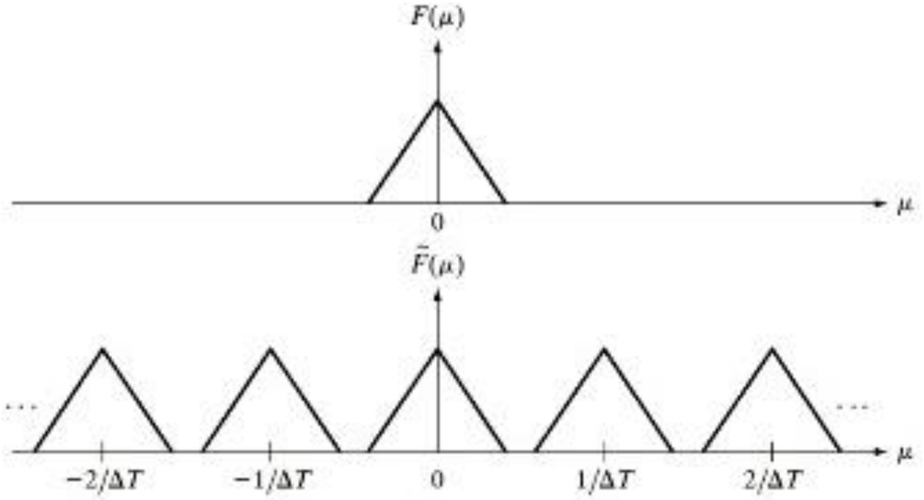


$$\tilde{f}(t) = \sum_{n=0}^{n=(M-1)\Delta T} f_n\, \delta(t - n\Delta T)$$

$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}$$

FT of sampled function (G&W 4.2.4)



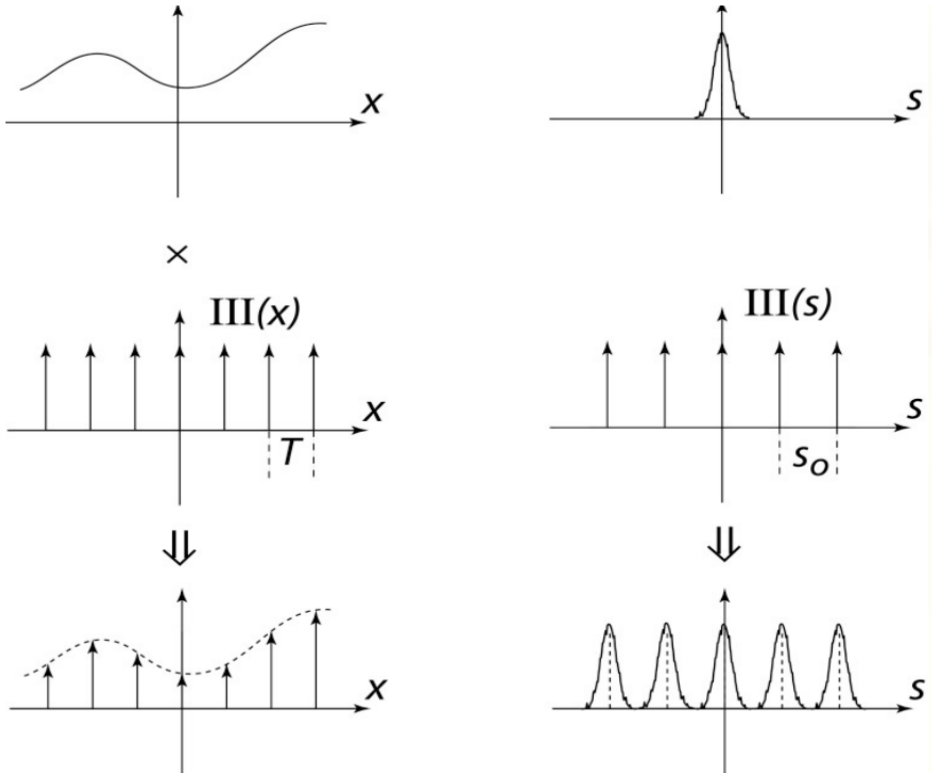


- Continuous
- Periodic (copies of f(t)'s FT)

$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n \, \delta(t - n\Delta T)$$

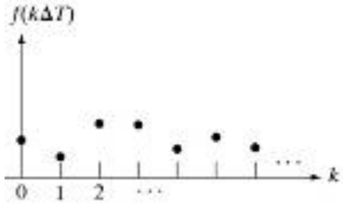
$$\tilde{F}(\mu) = \boxed{\frac{1}{\Delta T}} \sum_{n=-\infty}^{n=\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

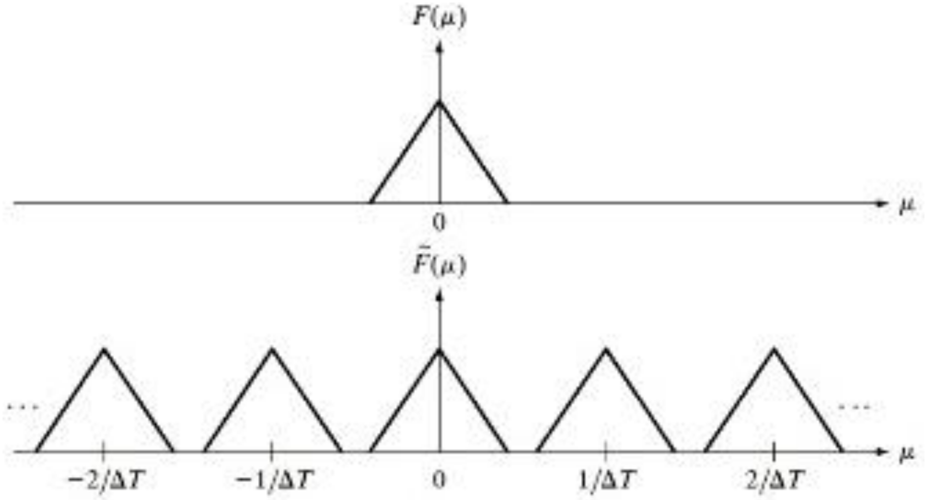
FT of sampled function



https://www.cs.utexas.edu/users/fussell/courses/cs384g-fall2011/lectures/lecture03-Sampling.pdf

Digital processing of frequencies



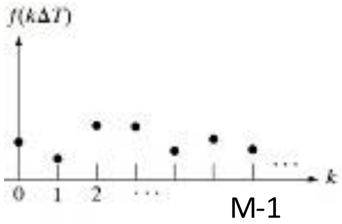


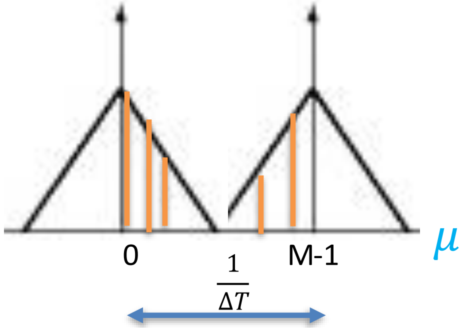
$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n\, \delta(t - n\Delta T)$$

$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}$$

- Need discrete frequency samples, but FT of sampled function is continuous
- OBSERVATION: Characterizing one period ($\frac{1}{\Delta T}$) is enough
- How do we get frequency ‘samples’ ?

FT of sampled function (G&W 4.4.1)

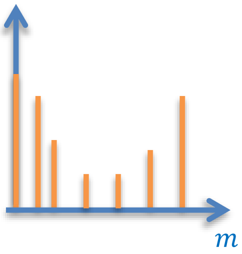




$$\tilde{f}(t) = \sum_{n=0}^{n=(M-1)\Delta T} f_n \, \delta(t - n\Delta T)$$

$$\tilde{F}(\mu) = \sum_{n=0}^{n=(M-1)} f_n e^{-j2\pi\mu n\Delta T}, \mu \in R$$

$$\mu = \frac{m}{(M-1)\Delta \mathrm{T}} \qquad m = 0, \mu = 0 \qquad m = (M-1), \mu = \frac{1}{\Delta \mathrm{T}}$$



$$F[m] = \sum_{n=0}^{n=(M-1)} f_n \, e^{\frac{-j2\pi mn}{M}}, \mathrm{m} = 0,1, \ldots (\mathrm{M}-1)$$

Discrete Fourier Transform

(Some) Properties of FT

FT Theorems and Properties

Property/Theorem	Time Domain		Frequency Domain
Notation:	$g(t)$	$\rightleftharpoons$	$G(f)$
	$g_1(t)$	$\rightleftharpoons$	$G_1(f)$
	$g_2(t)$	$\rightleftharpoons$	$G_2(f)$
Linearity:	$c_1 g_1(t) + c_2 g_2(t)$	$\rightleftharpoons$	$c_1 G_1(f) + c_2 G_2(f)$
Dilation:	$g(at)$	$\rightleftharpoons$	$\frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$
Conjugation:	$g^*(t)$	$\rightleftharpoons$	$G^*(-f)$
Duality:	$G(t)$	$\rightleftharpoons$	$g(-f)$
Time Shifting:	$g(t - t_0)$	$\rightleftharpoons$	$G(f)e^{-j2\pi f t_0}$
Frequency Shifting:	$e^{j2\pi f_c t} g(t)$	$\rightleftharpoons$	$G(f - f_c)$
Area Under $G(f)$:	$g(0)$	$=$	$\int_{-\infty}^{\infty} G(f)df$
Area Under $g(t)$:	$\int_{-\infty}^{\infty} g(t)dt$	$=$	$G(0)$
Time Differentiation:	$\frac{d}{dt} g(t)$	$\rightleftharpoons$	$j2\pi f G(f)$
Time Integration :	$\int_{-\infty}^{t} g(\tau)d\tau$	$\rightleftharpoons$	$\frac{1}{j2\pi f} G(f)$
Modulation Theorem:	$g_1(t)g_2(t)$	$\rightleftharpoons$	$\int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda)d\lambda$
Convolution Theorem:	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)d\tau$	$\rightleftharpoons$	$G_1(f)G_2(f)$
Correlation Theorem:	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t - \tau)dt$	$\rightleftharpoons$	$G_1(f)G_2^*(f)$
Rayleigh's Energy Theorem:	$\int_{\infty}^{\infty} \lvert g(t) \rvert^2 dt$	$=$	$\int_{\infty}^{\infty} \lvert G(f) \rvert^2 df$

https://www.comm.utoronto.ca/~dkundur/course_info/316/KundurFTProperties_handouts.pdf

References & Fun Reading/Viewing

- GW DIP textbook, 3rd Ed.
  - 4.1 to 4.2
  - 4.2.4, 4.2.5,4.3.1,4.3.2, 4.4.1

- https://betterexplained.com/articles/intuitive-understanding-of-sine-waves/
- A visual introduction to Fourier Transform: https://www.youtube.com/watch?v=spUNpyF58BY
- Fourier Transform, Fourier Series and Frequency Spectrum: https://www.youtube.com/watch?v=r18Gi8lSkfM
- Fourier Transform (CFT, DFT): https://blog.endaq.com/fourier-transform-basics
- LoG and DoG: http://www.cse.psu.edu/~rtc12/CSE486/lecture11.pdf

  https://medium.com/jun-devpblog/cv-3-gradient-and-laplacian-filter-difference-of-gaussians-dog-7c22e4a9d6cc
- FOURIER TRANSFORM PROPERTIES: https://ethz.ch/content/dam/ethz/special-interest/baug/ibk/structural-mechanics-dam/education/identmeth/fourier.pdf