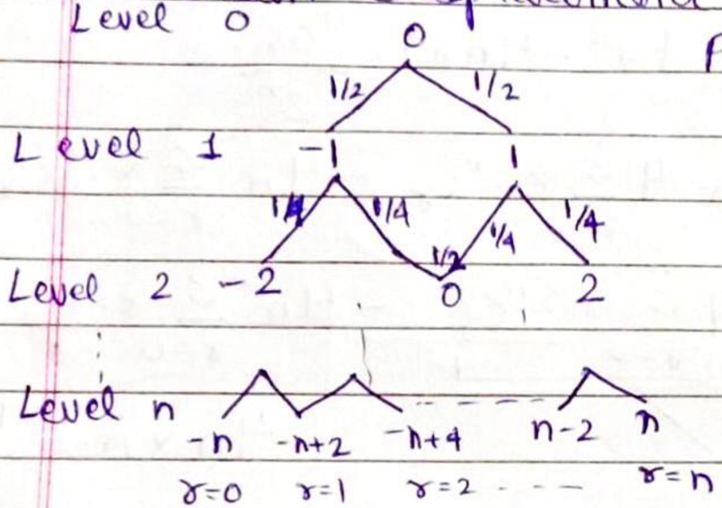
Ans. This problem can be modeled through Galton Board. We will frame Probability Tree.

(a) Mean Displacement $(\bar{x})$



Probability at level n for general position r is given by $\frac{{}^{n}C_{r}}{2^{n}}$

$r \in [0, n]$.

position as a fun of r ↑   probability as a fun of r ↑

$\therefore$ ~~Mean Displac~~ $\bar{x} = \sum_{r=0}^{n}$ position$(r) \times$ probability(r)

$$\therefore \bar{x} = \sum_{r=0}^{n} (-n+2r) \cdot \frac{{}^{n}C_{r}}{2^{n}}$$

$$\therefore \bar{x} = \frac{1}{2^{n}}\left[-n \sum_{r=0}^{n} {}^{n}C_{r} + 2 \sum_{r=0}^{n} r \cdot {}^{n}C_{r}\right]$$

$$= \frac{1}{2^{n}}\left[-n \cdot 2^{n} + 2 \cdot \sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= \frac{1}{2^{n}}\left[-n \cdot 2^{n} + 2n \cdot 2^{n-1}\right] = \frac{1}{2^{n}}\left[-n \cdot 2^{n} + n \cdot 2^{n}\right]$$

$$\bar{x} = \boxed{0}.$$

OR

at each step we can step either +1 or −1 with equal 1/2 probability.

~~For one step~~.

For 1 step $\bar{X} = \frac{1}{2}\times(+1) + \frac{1}{2}(-1) = 0$

$\therefore$ For n steps $\bar{X} = n\times 0 = 0$.

(b) Mean Squared Displacement $(\overline{X^2})$

From (a) part

$$\bar{X}^2 = \sum_{r=0}^{n} (-n+2r)^2 \cdot \frac{{}^nC_r}{2^n}$$

$$= \sum_{r=0}^{n} (n^2+4r^2-4nr) \cdot \frac{{}^nC_r}{2^n}$$

$$= \frac{1}{2^n}\left[n^2 \cdot \sum_{r=0}^{n} {}^nC_r + 4\sum_{r=0}^{n} r^2 \cdot {}^nC_r - 4n\sum_{r=0}^{n} r \cdot {}^nC_r\right]$$

$$= \frac{1}{2^n}\left[n^2 \cdot 2^n + 4\underbrace{\sum_{r=0}^{n} r^2 \cdot {}^nC_r} - \underbrace{4n\sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1}}_{4n\times n\cdot 2^{n-1} = n^2\cdot 2^{n+1}}\right]$$

$$\sum_{r=0}^{n} r^2 \cdot {}^nC_r = \sum_{r=0}^{n} r \cdot \cancel{r} \cdot \frac{n}{\cancel{r}} \cdot {}^{n-1}C_{r-1} = n\sum_{r=0}^{n} (r-1+1) \cdot {}^{n-1}C_{r-1}$$

$$= n\left[\sum_{r=0}^{n} (r-1) \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^{n-1}C_{r-1}\right]$$

$$= n\left[\sum_{r=0}^{n} \cancel{(r-1)} \cdot \frac{(n-1)}{\cancel{(r-1)}} \cdot {}^{n-2}C_{r-2} + 2^{n-1}\right]$$

$$= n\left[(n-1) \cdot 2^{n-2} + 2^{n-1}\right] = n \cdot 2^{n-2} - 2^{n-2} + 2 \cdot 2^{n-2}$$

$$= n\left[n \cdot 2^{n-2} + 2^{n-2}\right] = n^2 \cdot 2^{n-2} + n \cdot 2^{n-2}$$

$$= \frac{1}{2^n}\left[n^2 \cdot 2^n + 4(n^2 \cdot 2^{n-2} + n \cdot 2^{n-2}) - n^2 \cdot 2^{n+1}\right]$$

$$= \frac{1}{2^n}\left[n^2\cdot 2^n + n^2\cdot 2^n + n\cdot 2^n - n^2\cdot 2^{n+1}\right]$$

$$= \frac{1}{2^n}\left[\cancel{n^2\cdot 2^{n+1}} + n\cdot 2^n - \cancel{n^2\cdot 2^{n+1}}\right] = \frac{n\cdot 2^n}{2^n} = n.$$

OR

At each step we can step either to +1 or -1 with equal 1/2 probability

For 1 step $\overline{x^2} = \frac{1}{2}\times(-1)^2 + \frac{1}{2}\times(1)^2 = 1.$

~~For 2 step~~ $\cancel{\overline{x^2} = \frac{1}{2}\times(0)^2 + \frac{1}{2}\times(-2)^2 =}$

$\therefore$ For n steps $\overline{x^2} = n\times 1 = n.$

(c) Probability that 2 drunken meet at ~~any~~ same pos^n.

From the Galton Board we constructed in part (a), we can say that probability of drunken 1 to be at position r

$$= \frac{{}^nC_r}{2^n}.$$

If they both meet then they must be at same position [i.e. same r]

$$\therefore \text{Probability} = \sum_{r=0}^{n} \underbrace{\left(\frac{{}^nC_r}{2^n}\right)}_{\text{drunken 1}}\cdot\underbrace{\left(\frac{{}^nC_r}{2^n}\right)}_{\text{drunke 2}} = \frac{\sum_{r=0}^{n}({}^nC_r)^2}{2^{2n}}$$

For all possible r

$$\sum_{r=0}^{n}({}^nC_r)^2 = ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots ({}^nC_n)^2 = {}^{2n}C_n$$

(Alredy known)

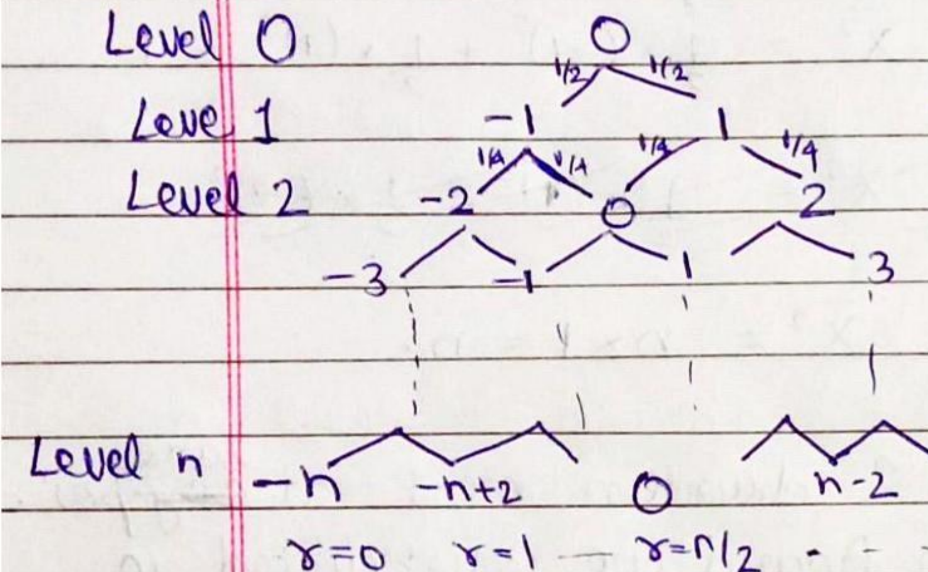
$$\therefore \boxed{\text{Probability} = \frac{{}^{2n}C_n}{2^{2n}}}$$

(d) Probability that drunken lands up at origin after n steps.

Clearly if n is odd then drunken can never return to origin.

$\therefore$ For $n = $ odd probability $= 0$

For $n = $ even ?



we know that probability that drunken is at position r at level n is $\dfrac{{}^nC_r}{2^n}$

For origin $r = n/2$

$$\therefore \boxed{\text{Probability} = \frac{{}^nC_{n/2}}{2^n}}$$