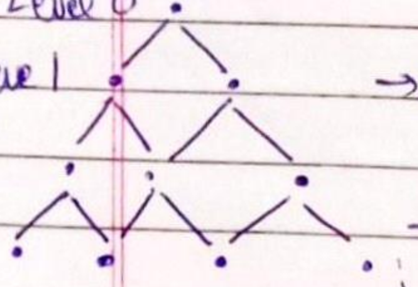
Ans) All the 4 parts of the problem can be modeled by building ~~choice~~ probability choice tree by following the principles of Galton Board Experiment and Pascal triangle.

Level 0

Level 1



→ Assume that at each level the beams are splitted evenly.

→ Probability that ball hits current beam is sum of probability of this ball hitting left and right beam above.

→ If x is the top left probability & y is the top right probability, then probability of hitting the current bin is $(x+y)/2$.

$n=0$		1		
$n=1$		1/2	1/2	
$n=2$	1/4	2/4	1/4	
$n=3$	1/8	3/8	3/8	1/8
	$r=0$	$r=1$	$r=2$	$r=3$

write numerators of each term →

1

1 1

1 2 1

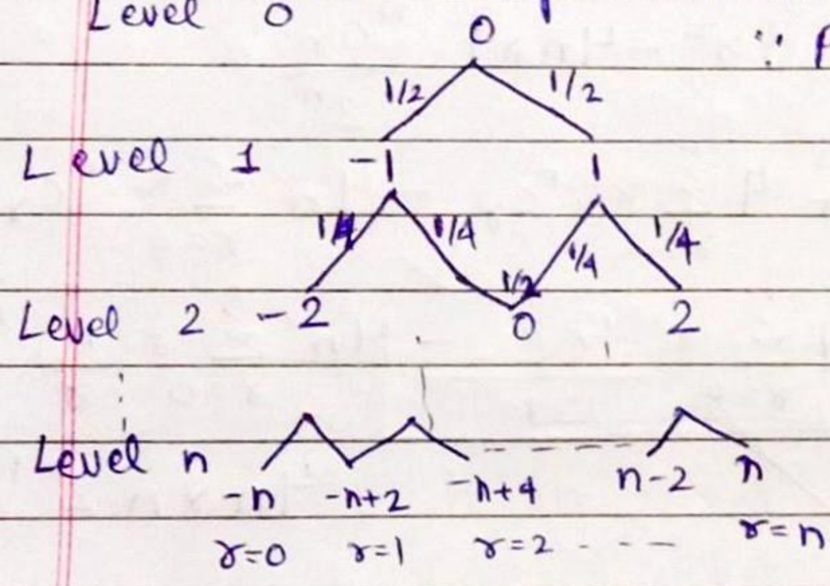
1 3 3 1

(Pascal's triangle)

$\therefore$ we can conclude that probability of current bin $= \dfrac{{}^nC_r}{2^n}$

$\boxed{\text{We will be using this result directly now.}}$

(a) Mean Displacement ($\bar{x}$)



∵ Probability of drunken at level n, for general position r is given by $\dfrac{{}^nC_r}{2^n}$

$r \in [0, n]$.

position as a fun of r ↑ ; probability as a fun of r ↑

∴ ~~Mean Displac~~ $\bar{x} = \sum_{r=0}^{n} \text{position}(r) \times \text{probability}(r)$

$$\therefore \quad \bar{x} = \sum_{r=0}^{n} (-n + 2r) \cdot \frac{{}^nC_r}{2^n}$$

$$\therefore \quad \bar{x} = \frac{1}{2^n}\left[-n \sum_{r=0}^{n} {}^nC_r + 2 \sum_{r=0}^{n} r \cdot {}^nC_r\right]$$

$$= \frac{1}{2^n}\left[-n \cdot 2^n + 2 \cdot \sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= \frac{1}{2^n}\left[-n \cdot 2^n + 2n \cdot 2^{n-1}\right] = \frac{1}{2^n}\left[-n \cdot 2^n + n \cdot 2^n\right]$$

$$\bar{x} = \boxed{0}.$$

OR

at each step we can step either +1 or −1 with equal 1/2 probability.

~~For one step~~.

For 1 step $\quad \bar{X} = \frac{1}{2}\times(+1) + \frac{1}{2}(-1) = 0$

$\therefore$ For n steps $\quad \bar{X} = n \times 0 = 0.$

(b) Mean Squared Displacement $(\overline{X^2})$

From (a) part $\quad \bar{X}^2 = \sum_{r=0}^{n} [\text{position}(r)]^2 * \text{prob}(r)$

$$\bar{X}^2 = \sum_{r=0}^{n} (-n+2r)^2 \cdot \frac{{}^nC_r}{2^n}$$

$$= \sum_{r=0}^{n} (n^2 + 4r^2 - 4nr) \cdot \frac{{}^nC_r}{2^n}$$

$$= \frac{1}{2^n}\left[n^2 \cdot \sum_{r=0}^{n} {}^nC_r + 4\sum_{r=0}^{n} r^2 \cdot {}^nC_r - 4n \sum_{r=0}^{n} r \cdot {}^nC_r \right]$$

$$= \frac{1}{2^n}\left[n^2 \cdot 2^n + 4\underbrace{\sum_{r=0}^{n} r^2 \cdot {}^nC_r} - 4n \underbrace{\sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1}} \right]$$

$$4n \times n \cdot 2^{n-1} = n^2 \cdot 2^{n+1}$$

$$\sum_{r=0}^{n} r^2 \cdot {}^nC_r = \sum_{r=0}^{n} r \cdot \cancel{r} \cdot \frac{n}{\cancel{r}} \cdot {}^{n-1}C_{r-1} = n\sum_{r=0}^{n} (r-1+1) \cdot {}^{n-1}C_{r-1}$$

$$= n\left[\sum_{r=0}^{n} (r-1) \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^{n-1}C_{r-1} \right]$$

$$= n\left[\sum_{r=0}^{n} \cancel{(r-1)} \cdot \frac{(n-1)}{\cancel{(r-1)}} \cdot {}^{n-2}C_{r-2} + 2^{n-1} \right]$$

$$= n\left[(n-1) \cdot 2^{n-2} + 2^{n-1} \right] = n \cdot 2^{n-2} - 2^{n-2} + 2 \cdot 2^{n-2}$$

$$= n\left[n \cdot 2^{n-2} + 2^{n-2} \right] = n^2 \cdot 2^{n-2} + n \cdot 2^{n-2}$$

$$= \frac{1}{2^n}\left[n^2 \cdot 2^n + 4(n^2 \cdot 2^{n-2} + n \cdot 2^{n-2}) - n^2 \cdot 2^{n+1} \right]$$

$$= \frac{1}{2^n}\left[n^2 \cdot 2^n + n^2 \cdot 2^n + n \cdot 2^n - n^2 \cdot 2^{n+1}\right]$$

$$= \frac{1}{2^n}\left[n^2 \cdot \cancel{2^{n+1}} + n \cdot 2^n - n^2 \cdot \cancel{2^{n+1}}\right] = \frac{n \cdot 2^n}{2^n} = n.$$

OR

At each step we can step either to +1 or -1 with equal 1/2 probability

For 1 step $\overline{x^2} = \frac{1}{2} \times (-1)^2 + \frac{1}{2} \times (1)^2 = 1.$

~~For 2 step~~ $\cancel{\overline{x}^2 = \frac{1}{2} \times (1)^2 + \frac{1}{2} \times (2)^2 =}$

$\therefore$ For n steps $\overline{x}^2 = n \times 1 = n.$

(c) Probability that 2 drunken meet at ~~any~~ same pos^n.
From the Galton Board we constructed in part (a), we can say that probability of drunken 1 to be at position r

$$= \frac{^nC_r}{2^n}.$$

If they both meet then they must be at same position [i.e. same r]

$$\therefore \text{Probability} = \underbrace{\sum_{r=0}^{n}}_{\text{For all possible } r} \underbrace{\left(\frac{^nC_r}{2^n}\right)}_{\text{drunken1}} \cdot \underbrace{\left(\frac{^nC_r}{2^n}\right)}_{\text{drunke 2}} = \frac{\sum_{r=0}^{n} (^nC_r)^2}{2^{2n}}$$

$$\sum_{r=0}^{n} (^nC_r)^2 = (^nC_0)^2 + (^nC_1)^2 + (^nC_2)^2 + \cdots (^nC_n)^2 = {}^{2n}C_n$$

(Alredy known)

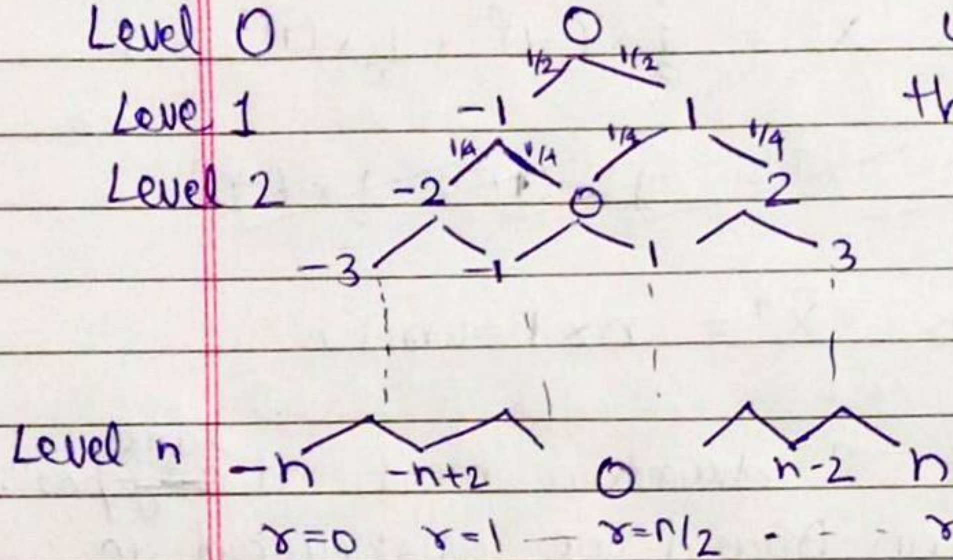
$$\therefore \boxed{\text{Probability} = \frac{^{2n}C_n}{2^{2n}}}$$

(d) Probability that drunken lands up at origin after n steps.

Clearly if n is odd then drunken can never return to origin.

$\therefore$ For n = odd probability $= 0$

For n = even



we know that probability that drunken is at position r at level n is $\dfrac{{}^nC_r}{2^n}$

For origin $r = n/2$

$$\therefore \boxed{\text{Probability} = \frac{{}^nC_{n/2}}{2^n}}$$