Probability and Statistics: Lecture-39

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on November 13, 2020

» Convex Functions and Jensen's Inequality...

Definition of convex function

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- Here $\alpha x + (1-\alpha)y$ is the weighted average of x and y

$X \in \{x, y\}$

$g(X) \in \{g(x), g(y)\}$

$g[E[X]] \leq E[g(X)]$

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$z(\alpha) = \alpha x + (1-\alpha) y \qquad \alpha \in [0,1]$

if $\alpha = 0 \Rightarrow z(0) = y$

if $\alpha = 1 \Rightarrow z(1) = x$

if $\alpha = 1/2 \Rightarrow z(1/2) = \frac{x+y}{2} \in$ mid-pt. of x & y.

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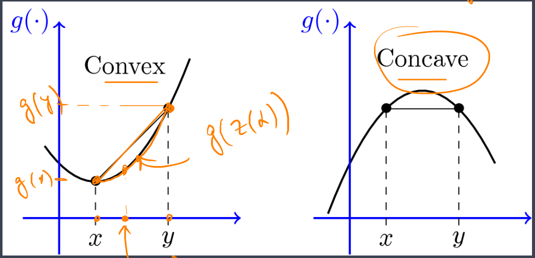

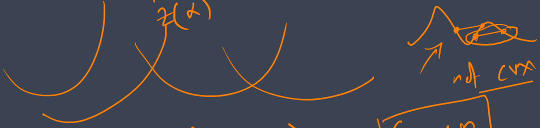

$-(\text{Concave}) = \boxed{\text{Convex}}$

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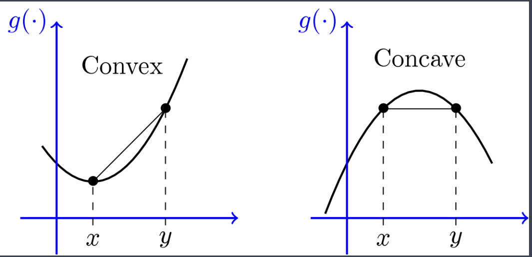

* From the definition of convexity on left, we conclude

$$E[g(X)] \geq g(E[X])$$

» Jensen's Inequality...

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* For example, $g(x) = x^2$ is convex in $\mathbb{R}$

$g'(x) = 2x$, $\rightarrow g''(x) = 2 > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow g$ is cvx

» Application of Jensen's Inequality...

Example (Jensen's Inequality)

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2. $E[e^{\frac{1}{X+1}}]$
3. $E[\ln \sqrt{X}]$

$g(x) = \frac{1}{x+1}$

» Answer to previous problem...

a) $g(x) = \frac{1}{x+1}$, $g'(x) = \frac{-1}{(x+1)^2}$

$g''(x) = \frac{2}{(1+x)^3} > 0$ for $x > 0$

$\Rightarrow g$ cvx on $(0, \infty)$.

Using Jensen's ineq

$E\left[\frac{1}{X+1}\right] \geq g(E[X]) = \frac{1}{E[X]+1}$

$= \frac{1}{10+1} = \frac{1}{11}$

$E[g(X)] \geq g(E[X])$

(b) $E\left[e^{\frac{1}{X+1}}\right]$, $g(x) = e^{\frac{1}{x+1}}$

$g'(x) = e^{\frac{1}{x+1}} \cdot \frac{-1}{(x+1)^2}$. tedious

try something else

Let, $h(x) = e^{x}$
$g(x) = \frac{1}{x+1}$

Observation: (1) $h(x)$ is cvx, non-decreasing

(2) $g(x)$ is convex

$f(x) = h(g(x)) = e^{\frac{1}{x+1}}$

» Answer to previous problem...

$f'(x) = h' \cdot g'$ (Using chain rule of diff)

$= \frac{dh}{dg} \cdot \frac{dg}{dx}$

$f''(x) = \frac{dh}{dg} \cdot \frac{d^2g}{dx^2} + \frac{d^2h}{dg^2} \cdot \left[\frac{dg}{dx}\right]^2$

$\frac{dh}{dg} \geq 0$ (Since h non decr.), $\frac{d^2g}{dx^2} \geq 0$ (Since g is cvx), $\frac{d^2h}{dg^2} \geq 0$ (Since h is cvx), $\left[\frac{dg}{dx}\right]^2 \geq 0$ (obv.)

≥ 0

$E\left[e^{\frac{1}{x+1}}\right] \geq e^{\frac{1}{E[x]+1}}$

$= \frac{1}{e^{11}}$

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$E[\bar{X}] = \frac{1}{n}(E[X_1] + \cdots + E[X_n])$

$= \frac{\cancel{n} E[X]}{\cancel{n}} = E[X]$

It is easy to establish the following:

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$Var(\underline{a}X) = a^2 \, Var(X)$

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1. $E[\overline{X}] = E[X]$
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» Weak Law of Large Numbers...

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$$\lim_{n\to\infty} P(|\bar{X} - \mu| \geq \epsilon) = 0$$

Proof. Assume $Var(X) = \sigma^2$ is finite $(< \infty)$

$$P(|\bar{x} - \mu| \geq \epsilon) \overset{\text{Chebyshev}}{\leq} \frac{Var(\bar{X})}{\epsilon^2} = \frac{Var(X)}{n\epsilon^2}$$

$\left(\text{Since } Var(\bar{X}) = \frac{Var(X)}{n}\right)$

$$\Rightarrow \lim_{n\to\infty} P(|\bar{x} - \mu| \geq \epsilon) = 0$$

» Central Limit Theorem...

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with expected value $E[X_i] = \mu < \infty$ and variance $0 < \text{Var}(X_i) = \sigma^2 < \infty$. Then, the random variable

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$$\lim_{n\to\infty} P(X_n \le x) = \Phi(x), \quad \text{for all } x \in \mathbb{R},$$

CDF of Std. Normal

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- It does not matter what the distribution of X_i is
- The X_i can be discrete, continuous, or mixed random variables

1. Let X_i be Bernoulli(p)
2. Then $E[X_i] = p, \text{Var}(X_i) = p(1-p)$
3. $Y_n = X_1 + X_2 + \cdots + X_n$ has Binomial((n, p))
4. Hence,

$$Z_n = \frac{Y_n - np}{\sqrt{np(1-p)}}$$

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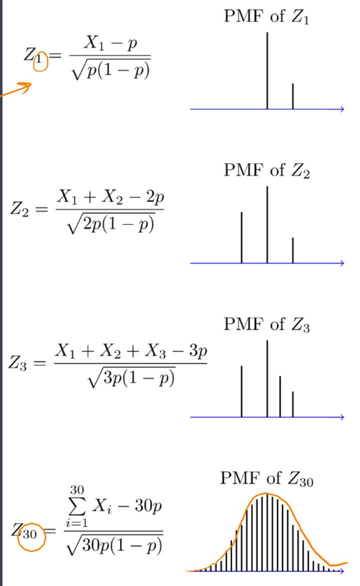

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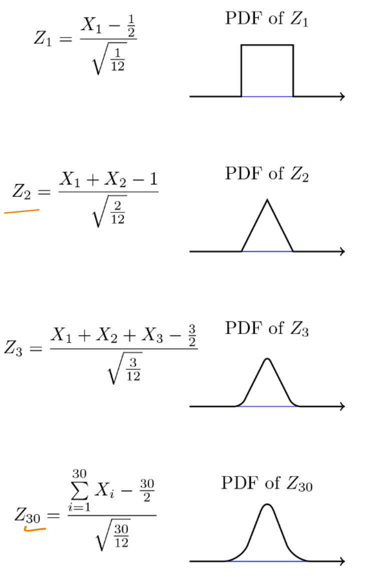

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$$P(y_1 \le Y \le y_2) = P\left(\frac{y_1 - n\mu}{\sqrt{n}\sigma} \le \underbrace{\frac{Y - n\mu}{\sqrt{n}\sigma}}_{\text{Std. Normal}} \le \frac{y_2 - n\mu}{\sqrt{n}\sigma}\right) \approx \Phi\left(\frac{y_2 - n\mu}{\sqrt{n}\sigma}\right) - \Phi\left(\frac{y_1 - n\mu}{\sqrt{n}\sigma}\right)$$

(approx)

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» Answer to previous problem...

We have

$$Y = X_1 + X_2 + \cdots + X_n,$$

where $n = 50$, $E[X_i] = \mu = 2$ (given)

and $Var(X_i) = \sigma^2 = 1$ (given)

$$P(90 < Y < 110)$$

$$= P\left(\frac{90 - 50\cdot 2}{\sqrt{50}\cdot 1} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \frac{110 - 50\cdot 2}{\sqrt{50}\cdot 1}\right)$$

$$= P\left(\frac{-10}{\sqrt{50}} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \frac{10}{\sqrt{50}}\right)$$

$$= \underline{\Phi}\left(\frac{10}{\sqrt{50}}\right) - \underline{\Phi}\left(\frac{-10}{\sqrt{50}}\right)$$

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X_i : R.V. for the ith bit

$X_i = 1$ if the ith bit is received

$X_i = 0$ otherwise.

Example (Applications of CLT)

In a communication system each data packet consists of 1000 bits. Due to the noise, each bit may be received in error with probability 0.1. It is assumed bit errors occur independently. Find the probability that there are more than 120 errors in a certain data packet.

X_i's are i.i.d $\quad X_i \sim$ Bernoulli $(p = 0.1)$

$E[X_i] = \mu = p = 0.1$, $\quad Var(X_i) = \sigma^2 = p(1-p) = 0.09$

$Y = X_1 + X_2 + \dots + X_n$,

Using CLT: $\rightarrow P(Y > 120) = P\left(\frac{Y - n\mu}{\sqrt{n}\,\sigma} > \frac{120 - n\mu}{\sqrt{n}\,\sigma}\right)$

$= 1 - P\left(\underbrace{\frac{Y - n\mu}{\sqrt{n}\,\sigma}}_{\text{std norm}} \le \frac{120 - n\mu}{\sqrt{n}\,\sigma}\right) = 1 - \left[\Phi\left(\frac{120 - 1000 \cdot 0.1}{\sqrt{1000} \cdot \sqrt{0.09}}\right)\right]$