Ans.1 (a) if X follows γ distribution then

$$f(x) = \begin{cases} \dfrac{\lambda^{\alpha} e^{-\lambda x} \cdot x^{\alpha-1}}{\Gamma(\alpha)} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Compare both $f(x)$

$$\therefore \frac{\lambda^{\alpha} e^{-\lambda x} \cdot x^{\alpha-1}}{\Gamma(\alpha)} = 4x e^{-2x}$$

By comparing we get $\boxed{\alpha = 2, \lambda = 2}$

$$\therefore E[X] = \int_0^{\infty} x \cdot f(x)\, dx$$

$$= \int_0^{\infty} x \cdot \frac{\lambda^{\alpha} e^{-\lambda x} \cdot x^{\alpha-1}}{\Gamma(\alpha)} dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha} \cdot e^{-\lambda x}\, dx$$

$$\because \boxed{\int_0^{\infty} x^{\alpha-1} \cdot e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^{\alpha}}} \rightarrow \text{Property}$$

Apply

$\because \Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

$$\therefore E[X] = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\cancel{\lambda^{\alpha}}}{\cancel{\Gamma(\alpha)}} \times \frac{\alpha \cancel{\Gamma(\alpha)}}{\lambda^{\alpha+1}}$$

$$\therefore \boxed{E[X] = \frac{\alpha}{\lambda} = \frac{2}{2} = 1}$$

$$E[X^2] = \int_0^{\infty} x^2 \cdot \frac{\lambda^{\alpha} e^{-\lambda x} \cdot x^{\alpha-1}}{\Gamma(\alpha)} = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} \cdot e^{-\lambda x} dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}} = \frac{\cancel{\lambda^{\alpha}}}{\cancel{\Gamma(\alpha)}} \cdot \frac{(\alpha+1)\alpha \cancel{\Gamma(\alpha)}}{\lambda^{\alpha+2}}$$

$$\boxed{E[X^2] = \frac{\alpha(\alpha+1)}{\lambda^2} = \frac{2 \times 3}{2+2} = \frac{3}{2} = 1.5}$$

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$$E[X^3] = \int_0^\infty x^3 \cdot \frac{\lambda^\alpha \cdot e^{-\lambda x} \cdot x^{\alpha-1}}{\Gamma(\alpha)} = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty x^{\alpha+2} \cdot e^{-\lambda x} \cdot dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+3)}{\lambda^{\alpha+3}} = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{(\alpha+2)(\alpha+1)\alpha \cdot \Gamma(\alpha)}{\lambda^{\alpha+3}}$$

$$\boxed{E[X^3] = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} = \frac{2 \times 3 \times 4}{2 \times 2 \times 2} = 3}$$

Ans (b) $E[X^n] = \int_0^\infty x^n \cdot \frac{\lambda^\alpha \cdot e^{-\lambda x} \cdot x^{\alpha-1}}{\Gamma(\alpha)} = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+n-1} \cdot e^{-\lambda x} \cdot dx$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+n)}{\lambda^{\alpha+n}} = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{(\alpha+n-1)(\alpha+n-2)\ldots\alpha \cdot \Gamma(\alpha)}{\lambda^{\alpha+n}}$$

$$\therefore E[X^n] = \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)}{\lambda^n}$$

Put values of α and λ.

$$\therefore \boxed{E[X^n] = \frac{1 \cdot 2 \times 3 \times 4 \times \ldots (n+1)}{2^n} = \frac{(n+1)!}{2^n}}$$

Ans. 2 (a) $\Gamma(7/2) = ?$

$\because \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ [Property]

Proof:- $\Gamma(\alpha) = \int_0^\infty \underset{II}{x^{\alpha-1}} \cdot \underset{I}{e^{-x}}\, dx$ By parts

$$= \left[e^{-x} \cdot \frac{x^\alpha}{\alpha}\right]_0^\infty - \int_0^\infty \frac{x^\alpha}{\alpha}\left(-e^{-x}\right) \cdot dx$$

$$= 0 + \frac{1}{\alpha}\underbrace{\int_0^\infty x^\alpha \cdot e^{-x} \cdot dx}_{\Gamma(\alpha+1)}$$

$\Rightarrow \alpha\Gamma(\alpha) = \Gamma(\alpha+1)$ i.e. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$\therefore \Gamma(7/2) = 5/2\, \Gamma(5/2) = 5/2 \cdot 3/2 \cdot \Gamma(3/2) = 5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)$

$= \frac{15}{8}\Gamma(1/2)$

$\because \Gamma(1/2) = \sqrt{\pi}$ [Property]

proof:- $\Gamma(1/2) = \int_0^\infty x^{-1/2} \cdot e^{-x}\, dx$

Put $x = y^2/2$ $\therefore dx = y\,dy$

$\therefore \Gamma(1/2) = \int_0^\infty \sqrt{2} \cdot \frac{1}{\cancel{y}} \cdot e^{-y^2/2} \cdot \cancel{y}\, dy$

$\Gamma(1/2) = \sqrt{2}\int_0^\infty e^{-y^2/2} \cdot dy$

From standard normal distribution we know that

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ $\forall z \in R$.

$\therefore \int_{-\infty}^\infty \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}_{\text{even fxn.}} dz = 1$

$$\therefore \left(\int_0^\infty \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} dz\right) \times 2 = 1$$

$$\Rightarrow \int_0^\infty e^{-z^2/2} \cdot dz = \sqrt{\frac{\pi}{2}}$$

$$\therefore \Gamma(1/2) = \sqrt{2} \cdot \int_0^\infty e^{-y^2/2} \cdot dy = \sqrt{2} \cdot \sqrt{\frac{\pi}{2}} = \sqrt{\pi}.$$

$$\therefore \boxed{\Gamma(7/2) = \frac{15}{8}\Gamma\left(\frac{1}{2}\right) = \frac{15\sqrt{\pi}}{8}}$$

<u>Ans. 2(b)</u> $\because I = \int_0^\infty x^{\alpha-1} \cdot e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$ [Property]

<u>Proof</u>:- $\lambda x = y$ $\quad \therefore \lambda dx = dy$

$$\therefore I = \int_0^\infty \left(\frac{y}{\lambda}\right)^{\alpha-1} \cdot e^{-y} \cdot \frac{dy}{\lambda} = \frac{1}{\lambda^\alpha} \underbrace{\int_0^\infty y^{\alpha-1} \cdot e^{-y} dy}_{\Gamma(\alpha)}$$

$\Gamma(\alpha)$ [By definit

$$\Rightarrow I = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$ (Hence Proved)

Now put $\alpha = 7$ and $\lambda = 5$ in I

$$\therefore \int_0^\infty x^6 \cdot e^{-5x} dx = \frac{\Gamma(7)}{5^7} = \text{Asked}$$

$\because \Gamma(n) = (n-1)!$ [Property]

<u>Proof</u>:- $\because \Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha(\alpha-1)\Gamma(\alpha-1) \ldots = \alpha(\alpha-1)\ldots 1$

$= \alpha!\,\Gamma(1)$

$$\Gamma(1) = \int_0^\infty x^{1-1} \cdot e^{-x} dx = \int_0^\infty e^{-x} \cdot dx = \left[-e^{-x}\right]_0^\infty = -0 + 1 = 1$$

$\therefore \Gamma(\alpha+1) = \alpha! \times 1 = \alpha!$

$\therefore \Gamma(7) = (7-1)! = 6!$

$$\therefore \text{Asked} = \frac{\Gamma(7)}{5^7} = \frac{6!}{5^7} = \frac{144}{15625}$$

Ans. 3 (a) $\because \sum_{x \in X} \sum_{y \in Y} f_{X,Y}(x,y) = 1$

$$\therefore \sum_{x \in X} \sum_{y \in Y} c(x^2+y^2) = \sum_{x \in X} c(x^2+1) + c(x^2+9)$$

$$= \sum_{x \in X} c(2x^2+10) = 2c \sum_{x \in X} (x^2+5)$$

$$= 2c\left[(1^2+5) + (2^2+5) + (4^2+5)\right]$$

$$= 2c\left[1+5+4+5+16+5\right] = 36 \times 2c = 1$$

$$\Rightarrow \boxed{c = 1/72}$$

$$\therefore f_{X,Y}(x,y) = \begin{cases} \frac{1}{72}(x^2+y^2), & \text{if } x \in X, y \in Y \\ 0 & \text{otherwise} \end{cases}$$

Ans3(b) $P(Y<X) \rightarrow$ Possible pairs are: $\{2,1\}, \{4,1\}, \{4,3\}$

$$\therefore P(Y<X) = \frac{1}{72}(2^2+1^2) + \frac{1}{72}(4^2+1^2) + \frac{1}{72}(4^2+3^2)$$

$$= \frac{1}{72}(4+1+16+1+16+9) = \frac{47}{72}.$$

Ans3(c) $P(Y>X) \rightarrow$ Possible pairs are: $\{1,3\}, \{2,3\}$

$$\therefore P(Y>X) = \frac{1}{72}(1^2+3^2) + \frac{1}{72}(2^2+3^2)$$

$$= \frac{1}{72}(1+9+4+9) = \frac{23}{72}.$$

Ans3(d) $P(Y=X) \rightarrow$ Possible pairs are: $\{1,1\}$

$$\therefore P(Y=X) = \frac{1}{72}(1^2+1^2) = \frac{1}{36}.$$

Ans3(e) $P(Y=3) \rightarrow$ Possible pairs are: $\{1,3\}, \{2,3\}, \{4,3\}$.

$$\therefore P(Y=3) = \frac{1}{72}(1^2+3^2) + \frac{1}{72}(2^2+3^2) + \frac{1}{72}(4^2+3^2)$$

$$= \frac{1}{72}(1+9+4+9+16+9) = \frac{2}{3}.$$

ans 3 (f) $p_X(x) = \sum_{y \in y} f_{X,Y}(x,y) = \sum_{y \in y} \frac{1}{72}(x^2+y^2)$

$= \frac{1}{72}(x^2+1) + \frac{1}{72}(x^2+9) = \frac{x^2+5}{36} \Rightarrow \boxed{p_X(x) = \frac{x^2+5}{36}}$

~~$\therefore p$~~ where $x \in \{1,2,4\}$

$$\therefore p_X(x) \begin{cases} 1/6 & \text{for } x=1 \\ 1/4 & \text{for } x=2 \\ 7/12 & \text{for } x=4 \\ 0 & \text{otherwise} \end{cases}$$

Similarly $p_Y(y) = \sum_{x \in x} f_{X,Y}(x,y) = \sum_{x \in x} \frac{1}{72}(x^2+y^2)$

$p_Y(y) = \frac{1}{72}(1^2+y^2) + \frac{1}{72}(2^2+y^2) + \frac{1}{72}(4^2+y^2)$

$= \frac{1}{72}(1+y^2+4+y^2+16+y^2) = \frac{3y^2+21}{72} = \frac{y^2+7}{24}$

~~$\therefore p_Y(y)$~~ where $y \in \{1,3\}$

$$\therefore p_Y(y) \begin{cases} 1/3 & \text{for } y=1 \\ 2/3 & \text{for } y=3 \\ 0 & \text{otherwise} \end{cases}$$

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Ans 3 (g) $\because p_X(x) = \frac{x^2+5}{36}$ $\therefore E[X] = \sum_{X \in x} x \cdot p_X(x)$

$\therefore E[X] = \sum_{X \in x} x \cdot \left(\frac{x^2+5}{36}\right) = \sum_{X \in x} \left(\frac{x^3+5x}{36}\right).$

$= \frac{1}{36}[1^3+5+2^3+10+4^3+20] = \frac{108}{36} = \boxed{3}$

$\because p_Y(y) = \frac{y^2+7}{24}$ $\therefore E[Y] = \sum_{Y \in y} y \cdot p_Y(y)$

$E[Y] = \sum_{Y \in y} y \cdot \left(\frac{y^2+7}{24}\right) = \sum_{Y \in y} \left(\frac{y^3+7y}{24}\right)$

$= \frac{1}{24}[1^3+7+3^3+21] = \frac{56}{24} = \boxed{\frac{7}{3}}.$

$E[XY] = ?$ Let $g(\underset{z}{x,y}) = xy$ $\therefore E[g(z)] = \sum_z g(z) \cdot p(z)$

$\therefore E[XY] = \sum_x \sum_y xy \cdot f_{XY}(x,y) = \sum_x \sum_y xy \cdot \frac{1}{72}(x^2+y^2)$

$= \frac{1}{72} \sum_x \sum_y xy(x^2+y^2) = \frac{1}{72}[1 \cdot 1(1^2+1^2) + 1 \cdot 3 \cdot (1^2+3^2) +$

$2 \cdot 1 \cdot (2^2+1^2) + 2 \cdot 3 \cdot (2^2+3^2) + 4 \cdot 1 \cdot (4^2+1^2) + 4 \cdot 3 \cdot (4^2+3^2)]$

$= \frac{1}{72}[2+30+10+78+68+300] = \frac{61}{9}.$

(b) ~~$Var(X) = \sum_{X \in x} x^2 \cdot \frac{1}{72}$~~ $Var(X) = E[X^2] - (E[X])^2$

already known

$\therefore E[X^2] = \sum_{X \in x} x^2 \cdot \left(\frac{x^2+5}{36}\right) \rightarrow p_X(x)$

$= \sum_{X \in x} \left(\frac{x^4+5x^2}{36}\right)$

$= \frac{1}{36}\left[1^4+5+2^4+5\cdot 4+4^4+5\cdot 16\right]$

$= \frac{1}{36}\left[1+5+16+20+256+80\right] = \frac{378}{36}$

$\therefore Var(X) = \frac{378}{36} - (3)^2 = \frac{378-36\times 9}{36} = \frac{54}{36} = \frac{3}{2} = 1.5$

$Var(Y) = ?$ $E[Y^2] = \sum_{Y \in y} y^2 \cdot \left(\frac{y^2+7}{24}\right) = \sum_{Y \in y} \left(\frac{y^4+7y^2}{24}\right)$

$= \frac{1}{24}\left(1^4+7+3^4+63\right) = \frac{152}{24} = \frac{19}{3}$

~~$\therefore Var(Y) = \frac{152}{24} - \left(\frac{7}{3}\right)^2 =$~~

$\therefore Var(Y) = \frac{19}{3} - \left(\frac{7}{3}\right)^2 = \frac{8}{9}.$

$\cancel{Var(X+Y) = E[X+Y] - E}$

$$Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2.$$

$$\therefore E[X+Y] = \sum_x \sum_y (x+y) \cdot f_{XY}(x,y) \cancel{= (x+y)}$$

$$E[X+Y] = \sum_x \sum_y (x+y) \cdot \frac{1}{72}(x^2+y^2) = \frac{1}{72} \cdot \sum_x \sum_y (x+y)(x^2+y^2)$$

$$= \frac{1}{72}\Big[(1+1)(1^2+1^2) + (1+3)(1^2+3^2) + (2+1)(2^2+1^2) + (2+3)(2^2+3^2) + (4+1)(4^2+1^2) + (4+3)(4^2+3^2)\Big]$$

$$= \frac{1}{72}[4 + 40 + 15 + 65 + 85 + 175] = \frac{384}{72} = \frac{16}{3}.$$

Similarly

$$E[(X+Y)^2] = \sum_x \sum_y (x+y)^2 \cdot \frac{1}{72}(x^2+y^2) = \frac{1}{72} \sum_x \sum_y (x+y)^2 (x^2+y^2)$$

$$= \frac{1}{72}[4 \cdot 2 + 16 \cdot 10 + 9 \cdot 5 + 25 \cdot 13 + 25 \cdot 17 + 49 \cdot 25]$$

$$= \frac{2188}{72} = \frac{547}{18}.$$

$$\therefore Var((X+Y)) = \frac{547}{18} - \left(\frac{16}{3}\right)^2 = \frac{547}{18} - \frac{256}{9}$$

$$= \frac{547 - 512}{18} = \frac{35}{18}.$$

Ans 3 (i) $\underbrace{P(X \geq Y)}_{= P(A)}$: All possible pairs are $\{1,1\}, \{2,1\}, \{4,1\}, \{4,3\}$.

$\therefore E[X|A] = \sum_x x \cdot p(x|A)$ $\qquad p(x|A) = \dfrac{p(x,A)}{p(A)}$

$\therefore p(A) = \sum_{(x,y) \in A} \sum f_{XY}(x,y) = \frac{1}{72}(1^2+1^2) + \frac{1}{72}(2^2+1^2)$

$+ \frac{1}{72}(4^2+1^2) + \frac{1}{72}(4^2+3^2) = \frac{49}{72}$.

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$$\therefore E[X|A] = \sum_x x \cdot p(x|A)$$

$$= \frac{1 \cdot \frac{1}{72}(1^2+1^2)}{49/72} + \frac{2 \cdot \frac{1}{72}(2^2+1^2)}{49/72}$$

$$+ \frac{4 \cdot \frac{1}{72}(4^2+1^2)}{49/72} + \frac{4 \cdot \frac{1}{72}(4^2+3^2)}{49/72}$$

$$= \frac{1}{49}\left[1 \cdot (1+1) + 2 \cdot (4+1) + 4(16+1) + 4(16+9)\right]$$

$$= \frac{1}{49}\left[2 + 10 + 68 + 100\right] = \cancel{\frac{198}{49}} \; \frac{180}{49}$$

Similarly

$$E[X^2|A] = \cancel{\sum_x x^2} \sum_x x^2 \cdot p(x|A)$$

$$= \frac{1^2 \cdot \frac{1}{72}(1^2+1^2)}{49/72} + \frac{2^2 \cdot \frac{1}{72}(2^2+1^2)}{49/72}$$

$$+ \frac{4^2 \cdot \frac{1}{72}(4^2+1^2)}{49/72} + \frac{4^2 \cdot \frac{1}{72}(4^2+3^2)}{49/72}$$

$$= \frac{1}{49}\left[1 \cdot (1+1) + 4(4+1) + 16(16+1) + 16(16+9)\right]$$

$$= \frac{1}{49}\left[2 + 20 + 272 + 400\right] = \cancel{\frac{709}{49}} \; \frac{694}{49}$$

$$\cancel{\therefore Var(x|A) = \frac{709}{49} - \left(\frac{195}{49}\right)^2 = 34741}$$

$$\therefore Var(x|A) = \frac{694}{49} - \left(\frac{180}{49}\right)^2 = \frac{34006 - 32400}{49 \times 49}$$

$$\therefore Var(x|A) = \frac{1606}{(49)^2} = \frac{1606}{2401}$$

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Ans. 4 $P(X=1 \mid Q=q) = q \Rightarrow P(X=0 \mid Q=q) = 1-q$

By law of total probability

$$P(X=1) = \sum_{0 \le q \le 1} P(X=1 \mid Q=q) \cdot P(Q=q).$$

$$= \int_0^1 (q)\left(6q(1-q)\right) dq = \int_0^1 6q^2(1-q)\, dq$$

$$= 2(1)^3 - \frac{6}{4}(1)^4 = 1/2. \quad \therefore \boxed{P(X=0) = 1/2}$$

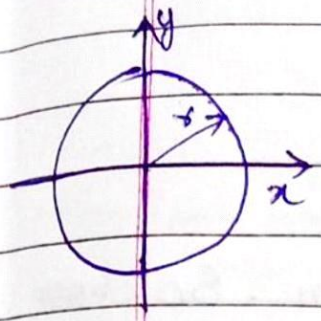
By Bayes theorem:-

$$P(Q=q \mid X=x) = \frac{P(X=x \mid Q=q) \cdot P(Q=q)}{P(X=x)}$$

$$\therefore f_{Q|x}(q) = \begin{cases} 12q^2(1-q) & , x=1 \\ 12q(1-q)^2 & , x=0 \\ 0 & \text{otherwise} \end{cases}$$

Hence, $$f_{Q|x}(q, x) = \begin{cases} 12q(1-q)^2 & 0 \le q \le 1 \ \& \ x=0 \\ 12q^2(1-q) & 0 \le q \le 1 \ \& \ x=1 \\ 0 & \text{otherwise} \end{cases}$$

Ex 5 $f_{X,Y}(x,y) = \begin{cases} 1/\text{area of the circle} & \text{if } (x,y) \text{ is in the circle} \\ 0, & 0 \text{ otherwise} \end{cases}$



$$\boxed{\underset{\text{Joint PDF}}{f_{X,Y}(x,y)} = \begin{cases} 1/\pi r^2 & x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = ? = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{x^2+y^2\le r^2} \frac{1}{\pi r^2}\cdot dx$$

$$f_Y(y) = \frac{1}{\pi r^2}\int_{x^2+y^2\le r^2} 1\cdot dx$$

$$f_Y(y) = \frac{1}{\pi r^2}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} 1\cdot dx = \frac{2\sqrt{r^2-y^2}}{\pi r^2}, \text{ if } |y|<r$$

(PDF $f_Y(y)$ is not uniform)

$$\therefore f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1/\pi r^2}{\frac{2\sqrt{r^2-y^2}}{\pi r^2}}$$

$$\therefore \boxed{f_{X|Y}(x|y) = \frac{1}{2\sqrt{r^2-y^2}} \quad \text{if } x^2+y^2\le r^2}$$

Conditional PDF

For each value of y, $f_{X|Y}(x|y)$ is uniform.

P(X)

Ans 6 Both balls and bins are labelled ∴ both ~~are~~ ~~dit~~ balls and bins are different.

$P(x) = ?$ $\quad P(x) = \dfrac{{}^4C_x \cdot 3^{(4-x)}}{4^4}$

Explanation :- we want x balls in 3rd bin. So choose x balls from 4 balls to be placed in bin 3.
∴ ${}^4C_x$.

Now we are remaining with $(4-x)$ balls which are to be distributed in remaining 3 bins ~~(1, 2, 4 bin)~~ (1st bin, 2nd bin, 4th bin).
∴ $\underbrace{3 \times 3 \times \cdots}_{(4-x) \text{ times}} = 3^{4-x}$.

Total no of ways of distributing 4 balls among 4 bins are $\underbrace{4 \times 4 \times \cdots}_{4 \text{ times}} = 4^4$.

∴ $P(x) = \dfrac{{}^4C_x \cdot 3^{4-x}}{4^4}$

~~∴ P(X) = (3/4)³ for x = 1, for x = 2, for x = 3~~

P(x)	81/256	for x = 0
	27/64	for x = 1
	27/128	for x = 2
	3/64	for x = 3
	1/256	for x = 4
	0	otherwise

~~P(N) =~~

P(N) = ? ① for $n = 2$ i.e. $P(2) = ?$ ⇒ 2 empty bins
Choose 2 bins among 4 to be keep empty
∴ ${}^4C_2$.

Now we have to distribute 4 balls to the

other 2 bins such that both are non-empty

$\Rightarrow (2^4 - 2)$

distributing 4 balls among the 2 bins

Case when either the first bin is empty or when the second bin is empty.

$$\therefore P(2) = \frac{{}^4C_2 (2^4 - 2)}{4^4} = 21/64$$

$4^4 \rightarrow$ Total cases

② for $n=3$

Explan $P(3) = ?$ Similar to the previous, but here 3 bins non-empty $\Rightarrow$ 1 empty bin.

${}^4C_1$: choosed 1 bin which we have keep empty.

$\therefore$ ~~$P(3) = {}^4C_1 (3^4$~~

Now we have to distribute the 4 balls in remaining 3 bins.

$$\therefore \underbrace{3^4}_{\text{Total possible cases}} - \underbrace{{}^3C_1 (2^4 - 2)}_{\text{when 1 bin is left empty}} - \underbrace{{}^3C_2}_{\text{when 2 bins are left empty}}$$

$\Rightarrow$ 4 balls in 2 bins

$\therefore 2^4 - 2$

Total arrangements of 4 balls in 2 bins

cases when either of the 2 bins are left empty.

$$\therefore P(3) = {}^4C_1 \times \left(\frac{3^4 - {}^3C_1 (2^4 - 2) - {}^3C_2}{4^4}\right) = \frac{4 \times 9}{64} = \frac{9}{16}.$$

~~$\therefore P(n) \begin{cases} 21/64 & \text{for } n=2 \\ 9/64 & \text{for } n=3 \end{cases}$~~

continued

③ for $n = 4$

Explanation:- $P(4) = ?$ ⇒ No bin should be empty

$\therefore 4^4 - {}^4C_1(?) - {}^4C_2(?) - {}^4C_3(1)$ → All balls in 1 bin. → 3 bins empty

Total cases			
Choose 1 bin which you will keep empty	Now we have to distribute 4 balls among 3 bins (Exactly same to the ans for part ②)	Choosing 2 bins which we have to keep empty	Now we have to distribute 4 balls among 2 bins. (Exactly same to the ans for part ①)

i.e.

$\Rightarrow 4^4 - {}^4C_1(3^4 - {}^3C_1(2^4 - 2) - {}^3C_2) - {}^4C_2(2^4 - 2) - {}^4C_3(1)$

$= 256 - 144 - 84 - 4 = 24.$ $\therefore P(4) = \frac{24}{4^4} = \frac{3}{32}.$

④ for $n = 1$

Explanation:- $P(1) = ?$ ⇒ 3 empty bins.

$\therefore {}^4C_3(1) = 4$

Choosen 3 bins out of 4 which we have to keep empty → distributing 4 balls in remaining 1 bin.

$\therefore P(1) = \frac{4}{4^4} = \frac{1}{64}.$

$$\therefore P(n) \begin{cases} 1/64 & n = 1 \\ 21/64 & n = 2 \\ 9/16 & n = 3 \\ 3/32 & n = 4 \\ 0 & \text{otherwise} \end{cases}$$

continued

(x,n)

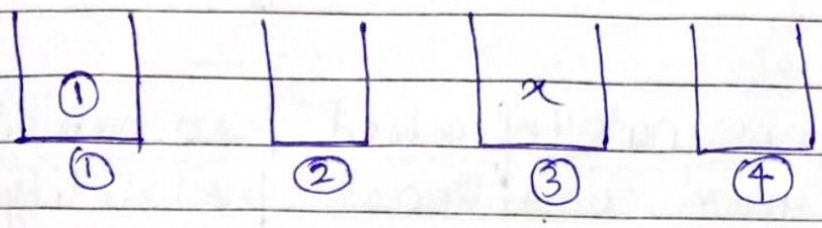
$P_{x,N|A}$

(Finding numerator first of probability):-

$P_{x,N|A}(x,n)$ = Probability of keeping x balls in 3rd bin, keeping n bins non-empty given that 1 numbered ball is in bin 1.

$$= \frac{P(x,n|A)}{P(A)}$$

divided by



for $n=2 \Rightarrow$ 2 non-empty bins.
Bin 1, Bin 3 already have some ball (2 bins) $\Rightarrow$ bin 2, bin 4 will be empty.

$\therefore$ No of ways = $^3C_x \, (1)$

choosing x balls which we have to place in bin 3

Now the remaining $(3-x)$ balls will be distributed to bin 1.

(Now balls in bin 3 are fixed i.e. we can't add balls here else no of balls will be more than x here)

for $n=3 \Rightarrow$ 3 non-empty bins $\Rightarrow$ 1 empty bin
Bin 1, Bin 3 are filled with balls. We have to choose 1 bin out of 2 or 4 which we have to keep empty.

$\therefore$ No of ways = $^3C_x \times {}^2C_1 \times (2^{3-x} - 1)$

Choosing x balls for bin 3,

Either bin 2 or bin 4 will be kept empty

Explanation

(Without the loss of generality)
Let's assume as for now that bin 4 is kept empty
$\Rightarrow$ We have to distribute remaining $(3-x)$ balls $(4 - 1 - x)$ to bins 1 and bin 2. No of balls in bin 3 is fixed (x) so we haven't included it into distribution.

in bin 1 in bin 3

$\therefore \; 2^{3-x} - 1$

All cases ↙ ↘ Only case when bin ② is kept empty and all balls are transferred to bin ①

~~Final~~

Finding Denominator (of asked probability) :-

we were given conditional probability, so denominator = ?

$P(A)$ = Probability that 1st ball is in bin 1.

$\therefore$ ~~Remain~~ 3 Remaining balls and we have to distribute them in the 4 bins in any way

$\therefore$ Total cases $= \underbrace{4 \times 4 \times 4}_{3 \text{ balls}} = 4^3$.

Now writing probability :-

~~P(~~ $P_{x,N|A}(x,2) = \dfrac{{}^3C_x}{4^3}$

and $P_{x,N|A}(x,3) = \dfrac{{}^3C_x \times {}^2C_1 \times (2^{3-x} - 1)}{4^3}$

$\therefore$ ~~P(~~ $P_{x,N|A}$

3/64	for $x=1, n=2$
3/64	for $x=2, n=2$
1/64	for $x=3, n=2$
9/32	for $x=1, n=3$
3/32	for $x=2, n=3$
0	for $x=3, n=3$

OR

~~$P_{x,N|A}(x,2)$~~	$x=1$	$x=2$	$x=3$
$P_{x,N|A}(x,2)$	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{1}{64}$
$P_{x,N|A}(x,3)$	$\frac{9}{32}$	$\frac{3}{32}$	0

Ans.] Let $S_{x,t}$ denote the event where the man moves x steps from previous position at time t. Here $x \in \{-2, 0, 1\}$ and $t \in \{1, 2, 3\}$.

Let a path be denoted by a series of such events starting with $t = 1$ and t increasing by 1 serially. Eg. $S_{1,1}$, $S_{-2,2}$, $S_{0,3}$ is a

path where the man move 1 step forward at $t=1$, 2 steps back at $t=2$ and stays for $t=3$.

a) Let G be the RV denoting the total no of steps taken by the man in any direction.

$$P(G=0) = P(S_{0,1}, T=1) + P(S_{0,1} S_{0,2}, T=2) + P(S_{0,1} S_{0,2} S_{0,3}, T=3)$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right) + \left(\frac{1}{4}\right)^3\left(\frac{1}{3}\right)$$

$$= \frac{1}{12}\left[1 + \frac{1}{4} + \frac{1}{16}\right] = \frac{7}{64}.$$

$$P(G=1) = P(S_{1,1}, T=1) + P(S_{1,1} S_{0,2}, T=2) + P(S_{0,1} S_{1,2}, T=2) + P(S_{1,1} S_{0,2} S_{0,3}, T=3) + P(S_{0,1} S_{1,2} S_{0,3}, T=3) + P(S_{0,1} S_{0,2} S_{1,3}, T=3)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right)$$

$$= \frac{1}{6}\left[1 + \frac{1}{2} + \frac{3}{16}\right] = \frac{9}{32}.$$

$$P(G=2) = P(S_{-2,1}, T=1) + P(S_{-2,1} S_{0,2}, T=2) + P(S_{0,1} S_{-2,2}, T=2) + P(S_{1,1} S_{1,2}, T=2) + P(S_{-2,1} S_{0,2} S_{0,3}, T=3) + P(S_{0,1} S_{-2,2} S_{0,3}, T=3) + P(S_{0,1} S_{0,2} S_{-2,3}, T=3) + P(S_{1,1} S_{1,2} S_{0,3}, T=3) + P(S_{1,1} S_{0,2} S_{1,3}, T=3) + P(S_{0,1} S_{1,2} S_{1,3}, T=3)$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) + 3\left(\frac{1}{4}\right)^3\left(\frac{1}{3}\right) + 3\left(\frac{1}{2}\right)^2\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$$

$$= \frac{1}{12}\left[1 + \frac{1}{2} + 1 + \frac{3}{16} + \frac{3}{4}\right] = \frac{1}{12} \times \frac{55}{16} = \frac{55}{192}.$$

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$$P(G=3) = P(S_{1,1}, S_{-2,1}, T=2) + P(S_{-2,1}, S_{1,2}, T=2)$$
$$+ P(S_{1,1}, S_{1,2} S_{1,3}, T=3) + P(S_{0,1} S_{1,2} S_{-2,3}, T=3)$$
$$+ P(S_{0,1} S_{-2,2} S_{1,3}, T=3) + P(S_{1,1} S_{0,2} S_{-2,3}, T=3) + P(S_{1,1} S_{-2,2} S_{0,3}, T=3)$$
$$+ P(S_{-2,1} S_{0,2} S_{1,3}, T=3) + P(S_{-2,1} S_{1,2} S_{0,3}, T=3)$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + 6\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right) = \frac{3}{16}.$$

$$P(G=4) = P(S_{-2,1}, S_{-2,2}, T=2) + P(S_{-2,1} S_{-2,2} S_{0,3}, T=3) + P(S_{-2,1} S_{0,2} S_{-2,3}, T=3)$$
$$+ P(S_{0,1} S_{-2,2} S_{-2,3}, T=3) + P(S_{-2,1} S_{1,2} S_{1,3}, T=3) + P(S_{1,1} S_{-2,2} S_{1,3}, T=3)$$
$$+ P(S_{1,1} S_{1,2} S_{-2,3}, T=3)$$
$$= \left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right) + 3\left(\frac{1}{4}\right)^3 \times \left(\frac{1}{3}\right) + 3\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) = \frac{19}{192}.$$

$$P(G=5) = P(S_{1,1} S_{-2,2} S_{-2,3}, T=3) + P(S_{-2,1} S_{1,2} S_{-2,3}, T=3) + P(S_{-2,1} S_{-2,2} S_{1,3}, T=3)$$
$$= 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right) = \frac{1}{32}.$$

$$P(G=6) = P(S_{-2,1}, S_{-2,2} S_{-2,3}, T=3) = \left(\frac{1}{4}\right)^3\left(\frac{1}{3}\right) = \frac{1}{192}$$

$$\therefore E[G] = \sum_{g=0}^{6} g \cdot P(G=g)$$

$$= 0\left(\frac{7}{64}\right) + 1\left(\frac{9}{32}\right) + 2\left(\frac{55}{192}\right) + 3\left(\frac{3}{16}\right) + 4\left(\frac{19}{192}\right) + 5\left(\frac{1}{32}\right)$$
$$+ 6\left(1/192\right)$$

$$= \frac{54 + 110 + 108 + 76 + 30 + 6}{192} = \frac{384}{192} = 2$$

$\boxed{\text{Hence, the expected total no° of steps taken is 2.}}$

~~Path~~ ~~Probability~~ ~~S~~

b.)

Path	Probability	S
$S_{0,1}$	1/12	0
$S_{1,1}$	1/6	1
$S_{-2,1}$	1/12	-2
$S_{0,1}$ $S_{0,2}$	1/48	0
$S_{0,1}$ $S_{1,2}$	1/24	1
$S_{0,1}$ $S_{-2,2}$	~~1/12~~ 1/48	-2
$S_{1,1}$ $S_{0,2}$	1/24	1
$S_{1,1}$ $S_{1,2}$	~~1/48~~ 1/12	2
$S_{1,1}$ $S_{-2,2}$	1/24	-1
$S_{-2,1}$ $S_{0,2}$	1/48	-2
$S_{-2,1}$ $S_{1,2}$	1/24	-1
$S_{-2,1}$ $S_{-2,2}$ ~~$S_{0,1}$ $S_{0,2}$ $S_{0,3}$~~	1/48	-4
$S_{0,1}$ $S_{0,2}$ $S_{0,3}$	1/192	0
$S_{0,1}$ $S_{0,2}$ $S_{1,3}$	1/96	1
$S_{0,1}$ $S_{0,2}$ $S_{-2,3}$	1/192	-2
$S_{0,1}$ $S_{1,2}$ $S_{0,3}$	1/96	1
$S_{0,1}$ $S_{1,2}$ $S_{1,3}$	1/48	2
$S_{0,1}$ $S_{1,2}$ $S_{-2,3}$	1/96	-1
$S_{0,1}$ $S_{-2,2}$ $S_{0,3}$	1/192	-2
$S_{0,1}$ $S_{-2,2}$ $S_{1,3}$	1/96	-1
$S_{0,1}$ $S_{-2,2}$ $S_{-2,3}$	1/192	-4
$S_{1,1}$ $S_{0,2}$ $S_{0,3}$	1/96	1
$S_{1,1}$ $S_{0,2}$ $S_{1,3}$	1/48	2
$S_{1,1}$ $S_{0,2}$ $S_{-2,3}$	1/96	-1
$S_{1,1}$ $S_{1,2}$ $S_{0,3}$	1/48	2
$S_{1,1}$ $S_{1,2}$ $S_{1,3}$	1/24	3
$S_{1,1}$ $S_{1,2}$ $S_{-2,3}$	1/48	0
$S_{1,1}$ $S_{-2,2}$ $S_{0,3}$	1/96	-1
$S_{1,1}$ $S_{-2,2}$, $S_{1,3}$	1/48	0
$S_{1,1}$ $S_{-2,2}$ $S_{-2,3}$	1/96	-3

$S_{-2,1}\ S_{0,2}\ S_{0,3}$	$1/192$	-2
$S_{-2,1}\ S_{0,2}\ S_{1,3}$	$1/96$	-1
$S_{-2,1}\ S_{0,2}\ S_{-2,3}$	$1/192$	-4
$S_{-2,1}\ S_{1,2}\ S_{0,3}$	$1/96$	-1
$S_{-2,1}\ S_{1,2}\ S_{1,3}$	$1/48$	0
$S_{-2,1}\ S_{1,2}\ S_{-2,3}$	$1/96$	-3
$S_{-2,1}\ S_{-2,2}\ S_{0,3}$	$1/192$	-4
$S_{-2,1}\ S_{-2,2}\ S_{1,3}$	$1/96$	-3
$S_{-2,1}\ S_{-2,2}\ S_{-2,3}$	$1/192$	-6

$\therefore P(S=-6) = 1/192$

$P(S=-4) = 1/48 + 1/192 + 1/192 + 1/192 = 7/192$

$P(S=-3) = 1/96 + 1/96 + 1/96 = 1/32$

$P(S=-2) = 1/12 + 1/48 + 1/48 + 1/192 + 1/192 + 1/192 = 9/64$

$P(S=-1) = 1/24 + 1/24 + 1/96 + 1/96 + 1/96 + 1/96 + 1/96 + 1/96 = \frac{7}{48}$

$P(S=0) = 1/12 + 1/48 + 1/192 + \frac{1}{48} + 1/48 = 29/192$

$P(S=1) = 1/6 + 1/24 + 1/24 + 1/96 + 1/96 + 1/96 = 9/32$

$P(S=2) = 1/12 + 1/48 + 1/48 + 1/48 = 7/48$

$P(S=3) = 1/24$

$\therefore R_S = \{-6, -4, -3, -2, -1, 0, 1, 2, 3\}.$

$$\therefore E[S^5] = \sum_{s \in R_S} s^5 P(S=s)$$

$$= (-6)^5 \times 1/192 + (-4)^5 \cdot 7/192 + (-3)^5 \cdot 1/32 + (-2)^5 \cdot 9/64 + (-1)^5 \cdot 7/48 + (0)^5 \cdot 29/192 + (1)^5 \cdot 9/132 + (2)^5 \cdot 7/48 + (3)^5 \cdot 1/24$$

$$= -75.$$

$\boxed{\text{Hence, } E[S^5] = -75}$

Ans 9 Given the die is tossed 4 times

$\therefore R_X = R_Y = \{0, 1, 2, 3, 4\}$

↳ Range of Random variable X and Y.

$\forall\ x, y \in W$ s.t. $x \geq 0,\ y \geq 0,\ x + y \leq 4.$

$$\therefore F_{X,Y}(x,y) = \frac{{}^4C_x \cdot {}^{4-x}C_y \times 4^{4-x-y}}{6^4}$$

choosing x throws out of 4, for getting chocolate of type 1

Total ways of having chocolates in 4 throws

$6 \times 6 \times 6 \times 6$

In each toss there are 6 possibilities

Choosing y throws out of rem.

4 - x throws for type 2 chocolates

choosing chocolates other than 1 & 2 for rem. tosses.

$$F_{XY}(x,y) = \frac{128}{27\, x!\, y!\, (4-x-y)!\, 4^{x+y}}$$

Hence, the joint PMF is

$P_{X,Y}(x,y)$	$Y=0$	$Y=1$	$Y=2$	$Y=3$	$Y=4$
$X=0$	$16/81$	$16/81$	$2/27$	$1/81$	$1/1296$
$X=1$	$16/81$	$4/27$	$1/27$	$1/324$	0
$X=2$	$2/27$	$1/27$	$1/216$	0	0
$X=3$	$1/81$	$1/324$	0	0	0
$X=4$	$1/1296$	0	0	0	0

Ans 10 when $N=n$ $\quad Y = X_1 + X_2 + \cdots X_n$

$\therefore E[Y|N=n] = E[X_1 + X_2 + \cdots X_n]$

$= E[X_1] + E[X_2] + \cdots E[X_n] = n\mu.$

Similarly

$Var(Y|N=n) = Var(X_1 + X_2 + \cdots X_n)$

$\therefore Var\left(\sum_{i=1}^{n} X_i\right) = E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left(E\left[\sum_{i=1}^{n} X_i\right]\right)^2$

$= \sum_{i=1}^{n} E[X_i^2] + 2\sum_{i \neq j} E[X_i \cdot X_j]$

$- \left(\sum_{i=1}^{n} (E[X_i])^2 + 2\sum_{i \neq j} E[X_i]E[X_j]\right)$

$\cancel{= \sum_{i=1}^{n} Var(X_i)}$ But for 2 independent random variable $E[AB] = E[A].E[B]$

$= \sum_{i=1}^{n} Var(X_i).$

$\therefore Var(Y|N=n) = n\sigma$

By law of total expectation:-

$E[Y] = \sum_{n \in R_N} E[Y|N=n] \cdot P(N=n)$

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$$= \sum_{n=0}^{\infty} (nk) \cdot P(N=n) = k\,E[N] = \boxed{ke}$$

Similarly by law of total variance:-

$$Var(Y) = E[Var(Y|N)] + Var(E[Y|N])$$

$$= E[mN] + Var(kN)$$

$$= m\,E[N] + k^2\,Var(N) = \boxed{me + k^2 V}$$

$$= \sum_{n=0}^{\infty} (nk) \cdot P(N=n) = k\,E[N] = \boxed{ke}$$

Similarly by law of total variance:-

$$Var(Y) = E[Var(Y|N)] + Var(E[Y|N])$$
$$= E[mN] + Var(kN)$$
$$= m\,E[N] + k^2 Var(N) = \boxed{me + k^2 V}$$

ans 11 Let T be the RV denoting the city's temperature.

$\therefore T \sim N(10, 100)$

$$\therefore f_T(t) = \frac{1}{10\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-10}{10}\right)^2\right]$$

Let X be a standard normal RV.

$$\therefore f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx$$

Note that this is not solvable as an indefinite integral.

we define $\phi : \mathbb{R} \to [0,1]$ as $\phi(x) = F_X(x)$.

$$F_{10X+10}(x) = \phi\left(\frac{x-10}{10}\right)$$

$$\therefore F_{10X+10}(x) = \left.\frac{d\,F_{10X+10}}{dx}\right|_x = \left.\frac{d\left(\phi\left(\frac{x-10}{10}\right)\right)}{dx}\right|_x$$

$$= \frac{1}{10} \times \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-10}{10}\right)^2\right] = f_T(x).$$

$\therefore \forall x \in \mathbb{R}$ $F_{10X+10}(x) = F_T(x)$ $\therefore T = 10X + 10$

$\therefore F_T(t) = \phi\left(\frac{t-10}{10}\right) = \phi\left(t/10 - 1\right)$

$\boxed{59\,F = 15°\,C}$

$\therefore \cancel{\frac{T_F - 32}{9} = \frac{T_C}{5}}$ $\Rightarrow$ $F_T(15) = \phi(0.5) = 0.6915$

$\frac{t_F - 32}{9} = \frac{t_C}{5}$

Temp in Farenheit — Temp in Celcius

$\therefore P(t \le 59\,F) = 0.6915$

t → recorded temperature

Ans. 12 X : RV denoting the amount of rice

Y : RV denoting the amount of curry

$X \sim N(370, 576)$, $Y \sim N(170, 49)$ and

X, Y are independent RV.

Let $a = 370$, $b = 24$, $c = 170$, $d = 7$.

$$f_X(x) = \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-a}{b}\right)^2\right), \quad f_Y(y) = \frac{1}{d\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-c}{d}\right)^2\right)$$

Let $Z = X + Y$, $p = a + c$, $q = \sqrt{b^2 + d^2}$.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

$$= \frac{1}{2\pi bd} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left[\left(\frac{x-a}{b}\right)^2 + \left(\frac{z-x-c}{d}\right)^2 \right] \right] dx$$

$$= \frac{1}{2\pi bd} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} \frac{\left((a^2+b^2)x^2 - 2(ad^2+(z-c)b^2)x + a^2d^2+b^2(z-c)^2 \right)}{b^2 d^2} \right] dx$$

$$= \frac{1}{2\pi bd} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \left(\frac{x^2 - 2x\left[\frac{ad^2+(z-c)b^2}{q^2}\right] + \frac{a^2d^2+b^2(z-c)^2}{q^2}}{\left(\frac{bd}{q}\right)^2} \right) \right) dx$$

~~Simplyin~~ Simplifying further, we get

$$\frac{1}{q\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-p}{q}\right)^2 \right] \Rightarrow Z \sim N(p, q^2).$$

Note that $f_{XY}(x, z-x) = f_X(x) \cdot f_Y(z-x)$ | as X and Y are independent

$\therefore Z \sim N(540, 625)$, $p = 540$, $q = 25$

As proved in Q1 $F_Z(z) = \phi\left(\frac{z-p}{q}\right)$

$\therefore F_Z(575) = \phi(1.4) = 0.9192$

Hence, the probability of plate weighing less than 575g = Probability of plate weighing less than or equal to 575g = 0.9192.

Because point probabilities are zero in the continuous distribution.

Ans 13 $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$

$\Rightarrow f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right)$ $f_Y(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)$

X & Y are independent $\Rightarrow f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

Let Z be a random variable such that $Z = X + Y$ (equals to the convolution of f_X and f_Y)

$$f_Z(z) = \int_{-\infty} f_Y(z-x) \cdot f_X(x)\, dx$$

Substituting into the convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}\right] \times \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_x\sigma_y} \exp\left[-\frac{\sigma_x^2(z-x-\mu_y)^2 + \sigma_y^2(x-\mu_x)^2}{2\sigma_x^2\sigma_y^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\sigma_x\sigma_y} \exp\left[-\frac{\sigma_x^2(z^2+x^2+\mu_y^2-2xz-2z\mu_y+2x\mu_y)+\sigma_y^2(x^2+\mu_x^2-2x\mu_x)}{2\sigma_y^2\sigma_x^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{4\pi\sigma_x\sigma_y} \exp\left[-\frac{x^2(\sigma_x^2+\sigma_y^2) - 2x\left(\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x\right)+\sigma_x^2(z^2+\mu_y^2-2z\mu_y)}{2\sigma_y^2\sigma_x^2} + \frac{\sigma_y^2+\mu_x^2}{2\sigma_y^2\sigma_x^2}\right] dx$$

Defining $\sigma_z = \sqrt{\sigma_x^2+\sigma_y^2}$ and completing the square:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_z} \frac{1}{\sqrt{2\pi}\,\frac{\sigma_x\sigma_y}{\sigma_z}} \exp\left[-\frac{x^2 - 2x\left(\frac{\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x}{\sigma_z^2}\right) + \frac{\sigma_x^2(z^2+\mu_y^2-2z\mu_y)+\sigma_y^2\mu_x^2}{\sigma_z^2}}{2\left(\frac{\sigma_x\sigma_y}{\sigma_z}\right)^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_z} \frac{1}{\sqrt{2\pi}\,\frac{\sigma_x\sigma_y}{\sigma_z}} \exp\left[-\frac{\left(x-\frac{\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x}{\sigma_z^2}\right)^2 - \left(\frac{\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x}{\sigma_z^2}\right)^2 + \frac{\sigma_x^2(z-\mu_y)^2+\sigma_y^2\mu_x^2}{\sigma_z^2}}{2\left(\frac{\sigma_x\sigma_y}{\sigma_z}\right)^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_z} \exp\left[-\frac{\sigma_z^2(\sigma_x^2(z-\mu_y)^2+\sigma_y^2\mu_x^2) - (\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x)^2}{2\sigma_z^2(\sigma_x\sigma_y)}\right] \times$$

$$\frac{1}{\sqrt{2\pi}\,\frac{\sigma_x\sigma_y}{\sigma_z}} \exp\left[-\frac{\left(x-\frac{\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_x\sigma_y}{\sigma_z}\right)^2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_z} \exp\left[-\frac{(z-(\mu_x+\mu_y))^2}{2\sigma_z^2}\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\frac{\sigma_x\sigma_y}{\sigma_z}} \times$$

$$\exp\left[-\frac{\left(x-\frac{\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_x\sigma_y}{\sigma_z}\right)^2}\right] dx$$

The expression in the integral is a normal density distribution on x, and so the integral evaluates to 1. The desired result follows:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z} \exp\left[-\frac{(z-(\mu_x+\mu_y))^2}{2\sigma_z^2}\right]$$

Let $\mu_z = \mu_x + \mu_y$

$$\therefore f_Z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z} \exp\left[-\frac{(z-\mu_z)^2}{2\sigma_z^2}\right].$$

$\therefore Z \sim N(\mu_z, \sigma_z^2)$

Hence, $X+Y \sim N(\mu_x+\mu_y, \sigma_x^2+\sigma_y^2)$ proved.

Ans. 4 Marginal PDF $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

$$= \int_y^{\infty} 4y(x-y) \cdot e^{-(x+y)}\, dx \quad (x<y \text{ is impossible as } x \geq y \text{ is given})$$

$$= \frac{4y}{e^y} \cdot \left[\int_y^{\infty} x e^{-x} - y \int_y^{\infty} e^{-x} \right]$$

$$= \frac{4y}{e^y} \left[-\left(x e^{-x} \Big|_y^{\infty} \right) + \int_y^{\infty} e^{-x} + y \left(e^{-x} \Big|_y^{\infty} \right) \right].$$

$$= \frac{4y}{e^y} \left[y e^{-y} - \left(e^{-x} \Big|_y^{\infty} \right) - y e^{-y} \right]$$

$$f_Y(y) = \frac{4y}{e^{2y}} \quad \text{for } y \geq 0.$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad \left(P_{X|Y}(x,y) = \frac{P_{XY}(x,y)}{P_Y(y)} \right)$$

$$= \cancel{4y}(x-y)\, e^{-(x+y)} \cdot \frac{e^{2y}}{\cancel{4y}}$$

$$\therefore f_{X|Y}(x,y) = \begin{cases} (x-y)\, e^{-(x-y)} & , \text{for } x \geq y \\ 0 & , \text{otherwise} \end{cases}$$

(As -ve PDF is not acceptable)

<u>Note</u>:- The constraint is also derived from intersection of domains of f_Y and f_{XY} keeping $Y=y$ fixed.

$$\therefore E[X|Y=y] = \int_{-\infty}^{\infty} (x) \cdot f_{X|Y}(x,y)\, dx$$

$$= \int_y^{\infty} (x) \times \left((x-y) \cdot e^{-(x-y)}\right) dx \quad \left(\text{Domain consideration}\right)$$

$$= e^y \left(\int_y^{\infty} x^2 \cdot e^{-x} \cdot dx - y \int_y^{\infty} x e^{-x} \cdot dx \right).$$

$$e^y \left[-\left(x^2 e^{-x} \Big|_y^{\infty}\right) + 2\int_y^{\infty} x e^{-x} dx - y\left(-\left(x e^{-x}\Big|_y^{\infty}\right) + \int_y^{\infty} e^{-x} dx\right)\right]$$

$$= e^y \left[y^2 e^{-y} - y^2 e^{-y} + y\left(e^{-x}\Big|_y^{\infty}\right) + 2\left(-\left(x e^{-x}\Big|_y^{\infty} + \int_y^{\infty} e^{-x} dx\right)\right)\right]$$

$$= e^y \left[-y e^{-y} + 2y e^{-y} - 2\left(e^{-x}\Big|_y^{\infty}\right)\right].$$

$$= e^y \left[y e^{-y} + 2e^{-y}\right] = \cancel{y+2}\, y + 2.$$

$$E[X^2|Y=y] = \int_{-\infty}^{\infty} (x^2) \cdot f_{X|Y}(x,y)\, dx$$

$$= \int_y^{\infty} x^2 (x-y) e^{-(x-y)} dx \quad \text{(Domain consideration)}$$

$$= e^y \left[\int_y^{\infty} x^3 e^{-x} dx - y \int_y^{\infty} x^2 e^{-x} dx \right].$$

In previous calculation we derived

$$(y+2) e^{-y} = \int_y^{\infty} x^2 e^{-x} dx - y \int_y^{\infty} x e^{-x} dx$$

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$\therefore$ Let $I = \int_y^\infty x^2 e^{-x}\,dx = (y+2)e^{-y} + y\int_y^\infty x e^{-x}\,dx$

$= (y+2)e^{-y} + y\left[-\left(x e^{-x}\Big|_y^\infty\right) + \int_y^\infty e^{-x}\,dx\right].$

$= (y+2)e^{-y} + y\left[y e^{-y} - \left(e^{-x}\Big|_y^\infty\right)\right].$

$= (y+2)e^{-y} + y^2 e^{-y} + y e^{-y} = (y^2+2y+2)e^{-y}.$

$\therefore E[X^2 \mid Y=y] = e^y\left[-\left(x^3 e^{-x}\Big|_y^\infty\right) + 3\int_y^\infty x^2 e^{-x}\,dx - y(I)\right]$

$= e^y\left[y^3 e^{-y} + 3I - yI\right]$

$= e^y\left[y^3 e^{-y} - (y-3)I\right] = e^y\left[y^3 e^{-y} - (y-3)(y^2+2y+2)e^{-y}\right]$

$= y^3 - (y^3 + 2y^2 + 2y - 3y^2 - 6y - 6) = y^2 + 4y + 6.$

$Var(X \mid Y=y) = E[X^2 \mid Y=y] - \left(E[X \mid Y=y]\right)^2$

$= y^2 + 4y + 6 - (y+2)^2 = 2.$

Hence, $\boxed{E[X \mid Y=y] = y+2, \quad Var(X \mid Y=y) = 2}$

Ans 15 Let X be a standard Normal RV and $Y \sim N(\mu, \sigma^2)$.

As proved in Q11 (and by definition) we can write

$Y = \sigma X + \mu$

$\forall c \in \mathbb{R}$ s.t. $c \neq 0$ let the RV $Z = cY$.

$\therefore Z = c\sigma X + c\mu$. $\quad Z = z \Rightarrow X = \dfrac{z - c\mu}{c\sigma}$.

$\therefore F_Z(z) = F_X\left(\dfrac{z - c\mu}{c\sigma}\right) \quad f_Z(z) = \left.\dfrac{dF_Z}{dz}\right|_z$

$\therefore f_Z(z) = \dfrac{1}{|c|\sigma\sqrt{2\pi}} \exp\left(\dfrac{-1}{2}\left(\dfrac{z - c\mu}{c\sigma}\right)^2\right]$ ($|c|$ is to ensure PDF is not -ve).

$\therefore Z \sim N(c\mu, (|c|\sigma)^2) \Rightarrow Z \sim N(c\mu, c^2\sigma^2)$.

$X_1 \sim N(2, 3) \quad , \quad X_2 \sim N(1, 4)$

continued

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(a) Let $Y_1 = 2X_1$, $Y_2 = 3X_2$

$\therefore Y_1 \sim N(4, 12)$, $Y_2 \sim N(3, 36)$

$Y = Y_1 + Y_2$

X_1 and X_2 are independent $\Rightarrow$ Y_1 and Y_2 are independent.

As proved in Q13.

$Y \sim N(7, 48)$

$$\boxed{\text{Hence } f_Y(y) = \frac{1}{4\sqrt{6\pi}} \exp\left(-\frac{1}{2}\frac{(y-7)^2}{48}\right)}$$

(b) Let $Y_2 = -X_2$

$\therefore Y_2 \sim N((1)(-1), (4)(-1)^2) \Rightarrow Y_2 \sim N(-1, 4)$

$Y = X_1 + Y_2$

X_1 and X_2 and independent $\Rightarrow$ X_1 and Y_2 are independent

As proved in Q.13

$Y \sim N(2 + (-1), 3 + 4) \Rightarrow Y \sim N(1, 7)$

$$\text{Hence, } \boxed{f_Y(y) = \frac{1}{\cancel{7\sqrt{2\pi}}} \frac{1}{\sqrt{14\pi}} \exp\left(\frac{-1}{2}\frac{(y-1)^2}{7}\right).}$$