Probability and Statistics: Lecture-37

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on November 9, 2020

» Multiple Random Variable: Joint PDF, Joint CDF...

Joint PDF, Joint CDF

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$$f_{XYZ}(x, y, z) = \begin{cases} c(x + 2y + 3z) & 0 \leq x, y, z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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1. Find the constant c
2. Find the marginal PDF of X

① We have

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 c\,(x+2y+3z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 c\left[\left(\frac{x^2}{2} + 2yx + 3zx\right)\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 c\left[\frac{1}{2} + 2y + 3z\right] dy\,dz$$

$$= \int_0^1 c\left[\frac{1}{2}y + \frac{2y^2}{2} + 3zy\right]_0^1 dz$$

$$= \int_0^1 c\left[\frac{3}{2} + 3z\right] dz$$

$$= c\left[\frac{3}{2}z + \frac{3z^2}{2}\right]_0^1$$

$$= c\cdot 3$$

$$\Rightarrow \quad c = \frac{1}{3}$$

② $$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\,dy\,dz$$

$$= \int_0^1\int_0^1 \frac{1}{3}(x+2y+3z)\,dy = \underline{\qquad\qquad}$$

» Independence of Multiple Random Variables...

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* If random variables $X_1, X_2, ..., X_n$ are independent, then we have

$$E[X_1, X_2, \cdots, X_n] = E[X_1]E[X_2]\cdots E[X_n]$$

» Independent and Identically Distributed Random Variables (I.I.D.)

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n^{th} flip

* If we flip the same coin N times and record the outcome, then $X_1, \ldots, X_n$ are I.I.D.
* Verify that these I.I.D. variables will have same mean and variances

» Expectation and Variance...

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$Cov(X,Y) = E(XY) - E[X]E[Y]$

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» Example...

Example (Who will receive the present?)

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Try

» Answer to previous problem...

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» PDF of the Sum of Multiple Random Variables...

PDF of the Sum of Multiple RVs

We recall that if $Y = X_1 + X_2$, and X_1 and X_2 being independent, we have

$\bar{X} = [X_1, X_2, \cdots X_n]$
$\bar{Y} = [Y_1, Y_2, \cdots Y_n]$

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Convolution

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y-x)\,dx$$

For multiple variable case, i.e., if $Y = X_1 + X_2 + \cdots + X_n$, we have

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * \cdots * f_{X_n}(y)$$

- However, it is computationally difficult!

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$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \qquad E[X] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

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We call X a random vector and $E[X]$ is the expectation of random vector.

* CDF is $F_X(x) = F_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = P(X_1 \leq x_1, \dots, X_n \leq x_n)$
* If X is jointly continuous, the PDF is denoted as

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Shorthand notation

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Variance of Multiple Random Variables

Let $X_1, X_2, \ldots, X_n$ be n random variables. Then we have the following

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» Correlation and Covariance Matrix...

Definition of Correlation and Covariance Matrix

For a random vector X, the correlation matrix R_X and covariance matrix C_X is

Outer Product

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2^2 & \cdots & x_2x_n \\ x_nx_1 & x_nx_2 & \cdots & x_nx_n \end{bmatrix}$$

Definition of Correlation and Covariance Matrix

For a random vector X, the correlation matrix R_X and covariance matrix C_X is

$$R_X = E[XX^T] = E\begin{bmatrix} X_1^2 & X_1X_2 & \dots & X_1X_n \\ X_2X_1 & X_2^2 & \dots & X_2X_n \\ \vdots & \vdots & \vdots & \vdots \\ X_nX_1 & X_nX_2 & \dots & X_n^2 \end{bmatrix} = \begin{bmatrix} E[X_1^2] & E[X_1X_2] & \dots & E[X_1X_n] \\ E[X_2X_1] & E[X_2^2] & \dots & E[X_2X_n] \\ \vdots & \vdots & \vdots & \vdots \\ E[X_nX_1] & E[X_nX_2] & \dots & E[X_n^2] \end{bmatrix}$$

$$C_X = E[(X - E[X])(X - E[X])^T] = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \dots & \mathrm{Cov}(X_1, X_n) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \dots & \mathrm{Cov}(X_2, X_n) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Cov}(X_n, X_1) & \mathrm{Cov}(X_n, X_2) & \dots & \mathrm{Var}(X_n) \end{bmatrix}$$

$U = \begin{bmatrix} X \\ Y \end{bmatrix}$

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1. $C_X = R_X - E[X]E[X]^T$

Recall: $Cov(X,Y) = Cov(Y,X)$

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$(i,j) = (j,i)$

1. $C_X = R_X - E[X]E[X]^T$
2. If $Y = AX + b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, then $C_Y = AC_XA^T$

» Answer to previous problem...

» Example of Correlation and Covariance Matrices...

Example (Example of correlation and covariance matrices)

Let X and Y be jointly continuous random variables with joint PDF

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Let $U = \begin{bmatrix} X \\ Y \end{bmatrix}$ be the random vector. Find the correlation and covariance matrices of U.

» Answer to previous problem...

Solution: First find the marginal PDFs of X & Y.

$$f_X(x) = \int_0^1 \left(\frac{3}{2}x^2 + y\right) dy = \frac{3}{2}x^2 + \frac{1}{2} \quad 0 < x < 1.$$

Similarly,

$$f_Y(y) = \int_0^1 \left(\frac{3}{2}x^2 + y\right) dx$$

$$= y + \frac{1}{2} \quad 0 < y < 1.$$

We recall

$$R = \begin{bmatrix} E[X^2] & E[XY] \\ E[YX] & E[Y^2] \end{bmatrix}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_0^1 x^2 \cdot \left(\frac{3}{2}x^2 + \frac{1}{2}\right) dx$$

||ly $E[Y^2] = \ldots$

$$E[XY] = \int_0^1 \int_0^1 (xy)\left(\frac{3}{2}x^2 + y\right) dx\, dy$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]\,E[Y]$$

Covariance Matrix

$$C_u = E\left[(U - E[U])\,(U - E[U])^T\right]$$

$$= \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}(Y) \end{bmatrix}$$

» Properties of Covariance Matrix...

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2. The covariance matrix C_X is positive semi-definite (PSD)

Recall definition of semi Positive definite (SPD), Assume A symmetric.

A matrix A is SPD if $x^T A x \geq 0 \quad \forall x \neq 0$

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2. The covariance matrix C_X is positive semi-definite (PSD)
3. The covariance matrix is positive definite if and only if all its eigenvalues are larger than zero

$x^T E[zz^T] x \geq 0$

$$C_x = E\left[\underbrace{(x - E[x])}_{z}(x - E[x])^T\right] = E[\underbrace{zz^T}_{A}]$$

① Symmetry: $A^T = (zz^T)^T = zz^T = A \Rightarrow A$ is symmetric

② S.P.D.

$x^T(zz^T)x = (z^Tx)^T(z^Tx)$

$\|z^Tx\|_2 \geq 0$

arb.

$x^T A x > 0 \quad \forall x \neq 0$

If λ is eig. val. of $A \Rightarrow \quad Au = \lambda u, \quad u$ eig. vec.

u eig. vec. $\Rightarrow u \neq 0 \Rightarrow u^T A u > 0$ (A is P.D) $\Rightarrow u^T \lambda u > 0$

$\Rightarrow \lambda \|u\|_2 > 0$

$\Rightarrow \boxed{\lambda > 0}$

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2. The covariance matrix C_X is positive semi-definite (PSD)
3. The covariance matrix is positive definite if and only if all its eigenvalues are larger than zero
4. The covariance matrix is positive definite if and only if $\det(C_X) > 0$

$\det(C_X) = \prod \lambda_i$

But $\lambda_i > 0 \Rightarrow \det(C_X) > 0$

» Answer to previous problem...

» Example of Covariance Matrix...

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Are the matrices C_U and C_V positive definite?

» Answer to previous problem...

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» Denition of Cross-Correlation and Cross-Covariance Matrix...

$$R_x = E[x x^T]$$

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$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} H_1(Y_1, Y_2, \ldots, Y_n) \\ H_2(Y_1, Y_2, \ldots, Y_n) \\ \vdots \\ H_n(Y_1, Y_2, \ldots, Y_n) \end{bmatrix}$$

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$$f_Y(y) = f_X(H(y))\,|J|, \qquad \text{where } J = \det \begin{bmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} & \cdots & \frac{\partial H_1}{\partial y_n} \\ \frac{\partial H_2}{\partial y_1} & \frac{\partial H_2}{\partial y_2} & \cdots & \frac{\partial H_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial H_n}{\partial y_1} & \frac{\partial H_n}{\partial y_2} & \cdots & \frac{\partial H_n}{\partial y_n} \end{bmatrix}$$

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$X = A^{-1}(y-b) = H(y)$

$J = \det(A^{-1}) = \frac{1}{\det(A)}$

Example (Example of Method of Transform for Function of Random Vector)

invertible

Let $Y = AX + b$, where X is a n dimensional random vector, A be a fixed (non-random) n by n matrix, and b be a fixed n-dimensional vector. Find the PDF of Y in terms of X.

$f_Y(y) = f_X(H(y))\,|J|$

$= f_X(A^{-1}(y-b))\,|J| = \frac{1}{\det(A)} f_X(A^{-1}(y-b))$

» Answer to previous problem...

» Definition of Jointly Normal or Gaussian Random Vectors...

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