Probability and Statistics: Lecture-12

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
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$$P(E \cap F) = P(E)P(F).$$

Otherwise, E and F are called dependent events.
If E and F are independent, then

$$P(E \mid F) = P(E).$$

Solution:

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)\cancel{P(F)}}{\cancel{P(F)}} = P(E)$$

» Examples of independent events...

$E \cap G = \{(1,4)\}$

$P(E \cap G) = \frac{1}{36}$ ✗

$P(G) = \frac{4}{36}$

Example

Roll two 6-sided dice, yielding values D_1 and D_2. Let us consider the following events:

* $E : D_1 = 1$
* $F : D_2 = 6$
* $G : D_1 + D_2 = 5$
* ~~Taht~~ That is, $G = \{(1,4),(2,3),(3,2),(4,1)\}$

1. Are E and F independent?

$E = \{(1,1),(1,2),\cdots,(1,6)\}$

$F = \{(1,6),(2,6),\cdots(6,6)\}$

$E \cap F = \{(1,6)\}$

$P(E \cap F) = \frac{1}{36}$, $P(E) = \frac{6}{36} = \frac{1}{6}$

$P(F) = \frac{1}{6}$

$\therefore P(E \cap F) = P(E) \cdot P(F)$

» Examples of independent events...

Example

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1. Are E and F independent? Ye
2. Are E and G independent? No

» General Definition of Independence...

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Definition of independence for 3 events

Three events E, F, and G are independent if

* $P(E \cap F \cap G) = P(E)P(F)P(G)$

» General Definition of Independence...

Definition of independence for 3 events

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General Definition of Independence...

Definition of independence for 3 events

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General Definition for many events

The n events $E_1, E_2, \ldots, E_n$ are independent if

for $r = 1, \ldots, n$:

for every subset $E_1, E_2, \ldots, E_r$:

$P(E_1 \cap E_2 \cap \cdots \cap E_r) = P(E_1)P(E_2)\cdots P(E_r)$

$1 \leq r \leq n$

» Example of general independence...

» Example of general independence...

$\boxed{F \cap G} = \{(1,6)\}$ $P(F \cap G) = \frac{1}{36}$
$= P(F)\,P(G)$

Question

Each roll of 6-sided die is an independent trial. Two rolls with output D_1 and D_2. Consider the following events:

* $E : D_1 = 1$
* $F : D_2 = 6$
* $G : D_1 + D_2 = 7$
* $G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$\boxed{E \cap G} = \{(1,6)\}$

$P(G) = \frac{6}{36} = \frac{1}{6}$

$P(E) = \frac{6}{36} = \frac{1}{6}$

Answer the following:

1. Are E and F independent? Yes
2. Are E and G independent? Yes
3. Are F and G independent? Yes
4. Are E, F, G independent? No

$P(E \cap G) = \frac{1}{36} = P(G)\,P(E)$

$\Rightarrow E$ & G are ind.

$E \cap F \cap G = \{(1,6)\}$

$P(E \cap F \cap G) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$

» Solution to problem on previous slide...

We have $E : D_1 = 1 \quad F : D_2 = 6 \quad G : D_1 + D_2 = 7$

» Independent Trials...

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Definition of Independent Trials

A set of n trials are called independent trials if

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Definition of Independent Trials

A set of n trials are called independent trials if

1. Each of the n trials have same set of possible outcomes

Event : Toss a coin : → {H, T}

T_1 : Toss a coin : ← {H, T}

T_2 : Toss a coin ← {H, T}

T_3 :

T_4 :

Definition of Independent Trials

A set of n trials are called independent trials if

1. Each of the n trials have same set of possible outcomes
2. The trials are independent if an event in one subset of trials is independent of events in other subsets of trials

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Examples of Independent Trials...

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Examples of Independent Trials...

* Flip a coin n times

» Independent Trials...

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Examples of Independent Trials...

* Flip a coin n times
* Roll a die n times

» Independent Trials...

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Examples of Independent Trials...

* Flip a coin n times
* Roll a die n times
* Send a multiple choice survey to n people

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Examples of Independent Trials...

* Flip a coin n times
* Roll a die n times
* Send a multiple choice survey to n people
* Send n web requests to k different servers

» Examples involving independent trials...

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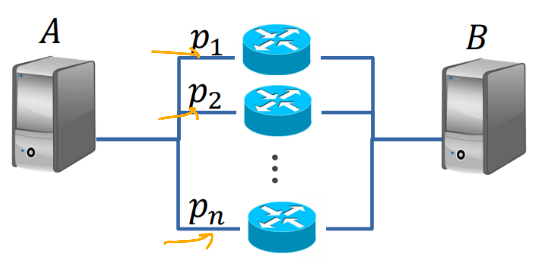
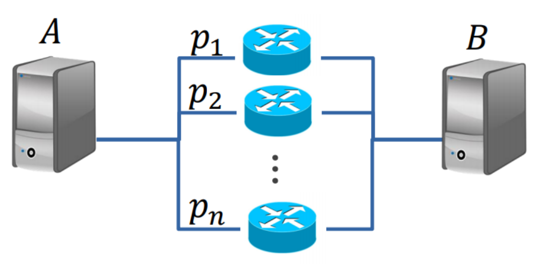


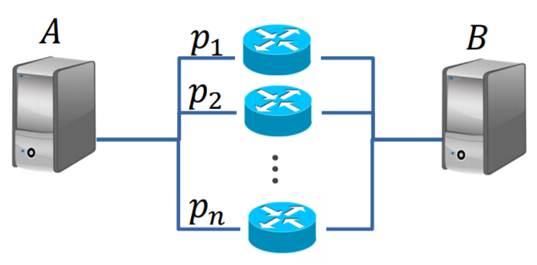
» Examples involving independent trials...



Problem

Consider the parallel network above:

» Examples involving independent trials...

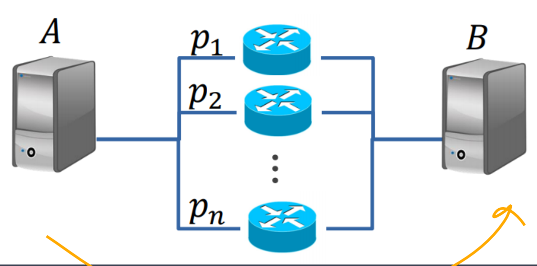


Problem

Consider the parallel network above:

* n independent routers, each with probability p_i of functioning, where $1 \leq i \leq n$

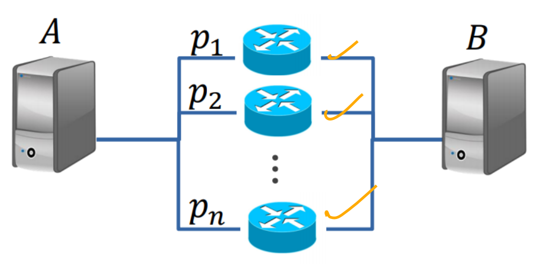
» Examples involving independent trials...



Problem

Consider the parallel network above:

* n independent routers, each with probability p_i of functioning, where $1 \leq i \leq n$
* $E =$ functional path from A to B exists.



$P(E) = P(\text{at least } 1 \text{ router works})$

$= 1 - P(\text{no router works})$

$= 1 - P(R_1 \text{ Fail} \wedge R_2 \text{ Fail} \wedge \ldots \wedge R_n \text{ Fail})$

$= 1 - P(R_1 \text{ Fail}) P(R_2 \text{ Fail}) \cdots P(R_n \text{ Fail})$

$= 1 - \underline{(1-p_1)(1-p_2)\cdots(1-p_n)}$

Problem

Consider the parallel network above:

* n independent routers, each with probability p_i of functioning, where $1 \leq i \leq n$
* $E =$ functional path from A to B exists.

What is $P(E)$?

» Examples involving independent trials...

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

» Examples involving independent trials...

H_i : Head on the ith trial

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

* P(n heads on n coin flips) $= P(H_1 \cap H_2 \cdots \cap H_n)$

$= P(H_1)\, P(H_2) \cdots P(H_n)$

$= p \cdot p \cdots p$

$= p^n$

» Examples involving independent trials...

$F_i := H_i^c$

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

* P(n heads on n coin flips)
* P(n tails on n coin flips)

$$= P(H_1)\,P(H_2)\cdots P(H_n)$$

$$= (1-P(H_1))\,(1-P(H_2))\cdots(1-P(H_n))$$

$$= \underbrace{(1-p)\cdots(1-p)}_{}$$

$$= (1-p)^n.$$

» Examples involving independent trials...

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

* P(n heads on n coin flips)
* P(n tails on n coin flips)
* P(first k heads, then $n - k$ tails)

$$\underbrace{P(H_1)\,P(H_2)\cdots P(H_k)}_{\text{1st k heads}}\;\underbrace{P(T_1)\,P(T_2)\cdots P(T_{n-k})}$$

$$p^k(1-p)^{n-k}$$

» Examples involving independent trials...

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

- P(n heads on n coin flips)
- P(n tails on n coin flips)
- P(first k heads, then $n-k$ tails)
- P(exactly k heads on n coin flips)

T H T H . . H

n coin flips

$\binom{n}{k} p^k (1-p)^{n-k}$

k - H

n - k T

» Solution to parallel network problem...

» Biased Coin, Independence, Infinite Sample Space, Total Probability, and Bayes Theorem...

Problem

A biased coin (with probability of obtaining a Head equal to $p > 0$ is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even numbered toss.

$k \rightarrow 2m$

$\rightarrow 2m+1$

T T T . . . . T H T T . . H

even numbers

Problem

A biased coin (with probability of obtaining a Head equal to $p > 0$ is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even numbered toss.

Solution to the problem...

What are sample space and events in this problem?

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A **biased** coin (with probability of obtaining a Head equal to $p > 0$ is tossed **repeatedly and independently** until the first head is observed. Compute the probability that the first head appears at an even numbered toss.

Solution to the problem...

What are sample space and events in this problem?

1. Sample space, $S =$ all possible infinite binary sequences of coin toss.

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What are sample space and events in this problem?

1. Sample space, $S =$ all possible infinite binary sequences of coin toss.
2. Consider event H_1 : head on first toss.

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What are sample space and events in this problem?

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2. Consider event H_1 : head on first toss.
3. Consider event E : first head on even numbered toss.

We want to compute $P(E)$.

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We want to compute $P(E)$.
How do we solve problems like this?

» Solution to problem on previous slide...Part-1

$E_k = \underbrace{\text{T . . . . . T}}_{2k-1}\ H$

Consider the partition of E into $E_1, E_2, \ldots$

where E_k = event that 1^{st} head occurs on the $(2k)^{th}$ toss

$$E = \bigcup_{k=1}^{\infty} E_k \qquad E_k\text{'s are Mutually exclusive}$$

$$\Rightarrow P(E) = P\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} P(E_k)$$

$$P(E_k) = (1-p)^{2k-1}\, p \qquad P(E) = \sum_{k=1}^{\infty} (1-p)^{2k-1}\, p$$

$$= \frac{1-p}{2-p}$$

» Solution to problem on previous slide...Part-2

» Coin Toss Example...

» Coin Toss Example...

$- - - - - \longrightarrow \infty$

Problem

A coin for which P(Heads) = p is tossed until two successive Tails are obtained. Find the probability that the experiment is completed on the nth toss.

Ω = all possible infinite seq^n of tosses

TT

E_1 : first toss is H

E_2 : first two tosses are TH

E_3 : " " " " TT

F_n : experiment completes on the n^{th} toss.

» Coin Toss Example...

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Solution

What are sample space and events?

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1. Sample space, S: all possible infinite sequences of tosses
2. event E_1 : first toss is H
3. event E_2 : first two tosses are TH
4. event E_3 : first two tosses are TT

» Coin Toss Example...

E_2: T H _ _ _ _ _ _ _

E_3: T T _ _ _ _ . E_1: H _ _ _ _ _ _ _ _

Problem

A coin for which P(Heads) = p is tossed until two successive Tails are obtained. Find the probability that the experiment is completed on the nth toss.

Solution

What are sample space and events?

1. Sample space, S: all possible infinite sequences of tosses
2. event E_1 : first toss is H
3. event E_2 : first two tosses are TH
4. event E_3 : first two tosses are TT
5. event F_n : experiment completed on the nth toss.

» Solution to problem in previous slide...part-1

$\{E_1, E_2, E_3\}$ is a partition of Ω

We want $P(F_n)$, $n = 2, 3, \ldots$

For $n = 2$: $P(F_2) = P(E_3) = (1-p)(1-p) = (1-p)^2$

and for $n > 2$:

$$P(F_n \mid E_1) = P(F_{n-1})$$

$$P(F_n \mid E_2) = P(F_{n-2})$$

$$P(F_n \mid E_3) = 0$$

$\underbrace{T\,H}_{E_1} \mid \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{(n-1)^{th}}$ n^{th}

$P(F_n)$

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{(n-1)^{th}}$

$T\,T\ \underbrace{\qquad\qquad}_{n-2}$ n^{th}

$\underset{1}{T}\ \underset{2}{H} \mid \_\ \_\ \_\ \_\ \_\ \_$ n^{th}

» Solution to problem in previous slide...part-2

$p_n = P(\bar{F}_n)$, $p_2 = P(\bar{F}_2) = (1-p)^2$

p_n satisfies