Probability and Statistics: Lecture-21

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on September 28, 2020

» Online Quiz

1. Please login to gradescope
2. Attempt the online quiz 4
3. You may use calculator if necessary
4. Time for the quiz is mentioned in the quiz

» Checklist

1. Turn off your microphone
2. Turn on microphone only when you have question
3. Attend Tutorials to Practice Problems or to discuss solutions or doubts

» Table of contents

Method of Transformation

We are interested in finding the PDF of the function of random variable

$$Y = g(X).$$

» Method of Transformation

Method of Transformation

We are interested in finding the PDF of the function of random variable

$$Y = g(X).$$

To this end, method of transform can be used if g satisfies the following

» Method of Transformation

Method of Transformation

We are interested in finding the PDF of the function of random variable

$$Y = g(X).$$

To this end, method of transform can be used if g satisfies the following

1. $g(x)$ is differentiable

» Method of Transformation

Method of Transformation

We are interested in finding the PDF of the function of random variable

$$Y = g(X).$$

To this end, method of transform can be used if g satisfies the following

1. $g(x)$ is differentiable
2. $g(x)$ is a strictly increasing function

» Method of Transformation

Method of Transformation

We are interested in finding the PDF of the function of random variable

$$Y = g(X).$$

To this end, method of transform can be used if g satisfies the following

1. $g(x)$ is differentiable
2. $g(x)$ is a strictly increasing function
   * That is, if $x_1 < x_2$, then $g(x_1) < g(x_2)$

Method of Transformation

We are interested in finding the PDF of the function of random variable

$$Y = g(X).$$

To this end, method of transform can be used if g satisfies the following

1. $g(x)$ is differentiable
2. $g(x)$ is a strictly increasing function
   * That is, if $x_1 < x_2$, then $g(x_1) < g(x_2)$

$$\frac{dy}{dx_1} = g'(x_1)$$

We can directly find the PDF of Y using the following formula

$$f_Y(x) = \begin{cases} \dfrac{f_X(x_1)}{g'(x_1)} = f_X(x_1) \cdot \dfrac{dx_1}{dy} & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ does not have a solution} \end{cases}$$

PDF of Y

$$\frac{dx_1}{dy} = \frac{1}{g'(x_1)}$$

Since g is strictly increasing
$\Rightarrow$ g^{-1} is well defined, that is
g is 1-1 and onto $\Rightarrow$ g^{-1} exists.

For each $y \in R_X$, $\exists$ a unique x_1
s.t. $g(x_1) = y \Rightarrow \frac{dy}{dx_1} = g'(x_1)$

i.e., $x_1 = g^{-1}(y) \longrightarrow (*)$

PDF of Y

$F_Y(y) = P(Y \le y) = P(g(X) \le y)$
$= P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$

To find PDF of Y, we diff.

$f_Y(y) = \frac{d}{dy} F_Y(y)$

chain rule
$y = g(f(x))$
$\frac{dy}{dx} = \frac{dg}{df} \cdot \frac{df}{dx}$

$= \frac{d}{dy} F_X(g^{-1}(y))$

$= \frac{d}{dy} F_X(x_1)$ (from $*$)

$= F_X'(x_1) \cdot \frac{dx_1}{dy} = f_X(x_1) \cdot \frac{dx_1}{dy}$

$$\boxed{f_Y(y) = f_X(x_1) \cdot \frac{1}{g'(x_1)}, \quad g(x_1) = y}$$

$F_Y(y) = P(Y \le y) = P(g(X) \le y)$

Note: Now g is strictly decreasing

$= P(X \ge g^{-1}(y)) = 1 - P(X < g^{-1}(y))$

$= 1 - F_X(g^{-1}(y))$

To find PDF, we diff.

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right)$

$= -\frac{d}{dy} F_X(g^{-1}(y))$

absorb − sign inside mod

Since $g(x_1) = y \Rightarrow x_1 = g^{-1}(y)$

g^{-1} exists because g is monotonically decr. i.e g is 1-1 & onto.

$= -\frac{d}{dy} F_X(x_1)$

$= -F'_X(x_1) \cdot \frac{dx_1}{dy} = -f_X(x_1) \cdot \frac{1}{\frac{dy}{dx_1}}$

$= \ominus \frac{f_X(x_1)}{g'(x_1)}$

$\boxed{\frac{f_X(x_1)}{|g'(x_1)|}}$

» Method of Transformation for Monotonic Function for Monotonic Functions...

Method of Transformation for Monotonic Function

Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonic differentiable function.

Method of Transformation for Monotonic Function

Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonic differentiable function. Let $Y = g(X)$.

» Method of Transformation for Monotonic Function for Monotonic Functions...

Method of Transformation for Monotonic Function

Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonic differentiable function. Let $Y = g(X)$. Then the PDF of Y is given by

Method of Transformation for Monotonic Function

Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonic differentiable function. Let $Y = g(X)$. Then the PDF of Y is given by

$$f_Y(y) = \begin{cases} \dfrac{f_X(x_1)}{|g'(x_1)|} = f_X(x_1) \cdot |\dfrac{dx_1}{dy}| & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ does not have a solution} \end{cases}$$

» Example: Using Method of Transformation to Find PDF of Function of Random Variable

Example: Method of Transformation

Consider the PDF of the continuous random variable X

» Example: Using Method of Transformation to Find PDF of Function of Random Variable

Example: Method of Transformation

Consider the PDF of the continuous random variable X

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

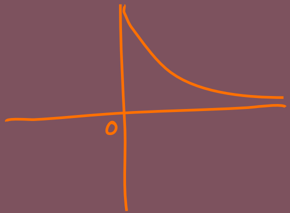
» Example: Using Method of Transformation to Find PDF of Function of Random Variable

Example: Method of Transformation

Consider the PDF of the continuous random variable X

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $Y = \dfrac{1}{X}$. Find $f_Y(y)$.



$R_X = [0,1]$. $R_Y = [1, \infty)$. Step: $g(x) = \frac{1}{x}$ is strictly decreasing and diff. in $(0,1]$. We have $g'(x) = -\frac{1}{x^2} \le 0$. For $y \in [1, \infty)$

$x_1 = g^{-1}(y) = \frac{1}{y}$. ~~For $y \in [1,\infty)$~~

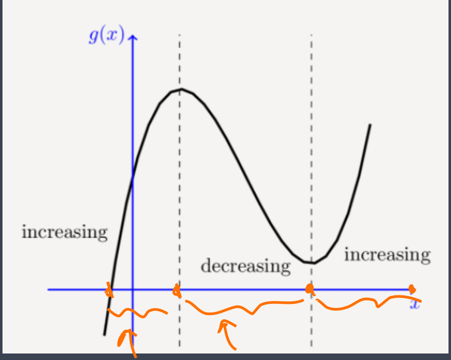
$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{4x_1^3}{\left|-\frac{1}{x_1^2}\right|} = 4x_1^5 = \frac{4}{y^5}$$

Summary

$$f_Y(y) = \begin{cases} \frac{4}{y^5}, & y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

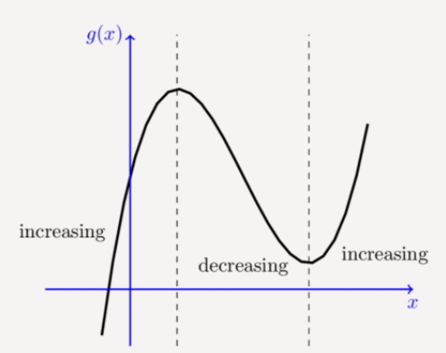
» Method of Transformation for Piecewise Continuous Functions...

» Method of Transformation for Piecewise Continuous Functions...



Partiton a function to monotone parts

» Method of Transformation for Piecewise Continuous Functions...

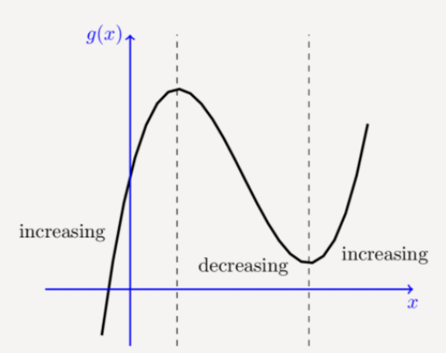


Partition a function to monotone parts

Method of Transform

Let X be a continuous random variable with domain R_X.

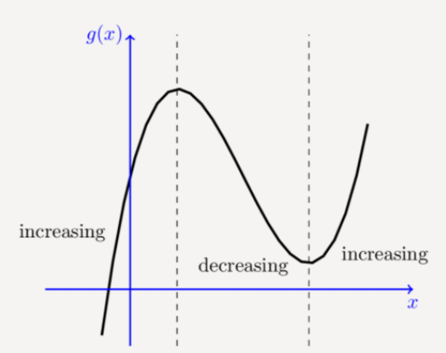
» Method of Transformation for Piecewise Continuous Functions...



Partition a function to monotone parts

Method of Transform

Let X be a continuous random variable with domain R_X. Let $Y = g(X)$.



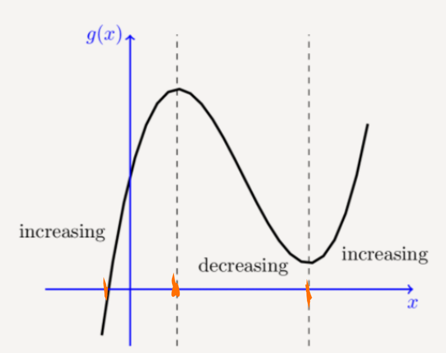
Partition a function to monotone parts

Method of Transform

Let X be a continuous random variable with domain R_X. Let $Y = g(X)$. Assuming that we can partition R_X into finite number of intervals such that $g(x)$ is strictly monotone and differentiable on each partition.

» Method of Transformation for Piecewise ~~Continuous~~ Functions...

Monotone



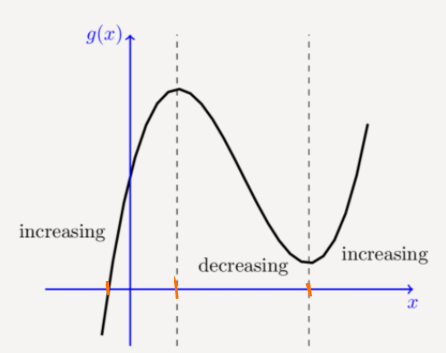
Partition a function to monotone parts

Method of Transform

Let X be a continuous random variable with domain R_X. Let $Y = g(X)$. Assuming that we can partition R_X into finite number of intervals such that $g(x)$ is strictly monotone and differentiable on each partition. Then the PDF of Y is

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|} = \sum_{i=1}^{n} f_X(x_i) \cdot \left|\frac{dx_i}{dy}\right|$$

» Method of Transformation for Piecewise Continuous Functions...



Partition a function to monotone parts

Method of Transform

Let X be a continuous random variable with domain R_X. Let $Y = g(X)$. Assuming that we can partition R_X into finite number of intervals such that $g(x)$ is strictly monotone and differentiable on each partition. Then the PDF of Y is

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|} = \sum_{i=1}^{n} f_X(x_i) \cdot \left|\frac{dx_i}{dy}\right|$$

where $x_1, x_2, \ldots, x_n$ are real solutions to $g(x) = y$.

» Example: Method of Transformation...

» Example: Method of Transformation...

Example

Consider the PDF of the random variable X

» Example: Method of Transformation...

Example

Consider the PDF of the random variable X

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \in \mathbb{R}$$

and let $Y = X^2$.

» Example: Method of Transformation...

Example

Consider the PDF of the random variable X

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \in \mathbb{R}$$

and let $Y = X^2$. Find the PDF $f_Y(y)$.

» Example: Method of Transformation...

Example

Consider the PDF of the random variable X

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \in \mathbb{R}$$

cont · dif

and let $Y = X^2$. Find the PDF $f_Y(y)$.

* Does this satisfy the criteria for applying method of transformation?

» Example: Method of Transformation...

Example

Consider the PDF of the random variable X

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \in \mathbb{R}$$

and let $Y = X^2$. Find the PDF $f_Y(y)$.

* Does this satisfy the criteria for applying method of transformation?
* Can we partition R_X into intervals such that $g(x)$ is monotone?

» Example: Method of Transformation...

Example

Consider the PDF of the random variable X

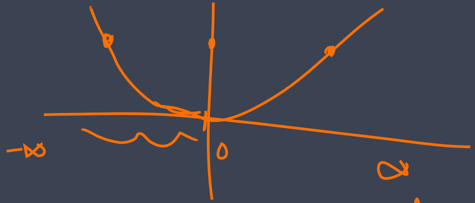
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \in \mathbb{R}$$

$R_X = \mathbb{R}$

and let $Y = X^2$. Find the PDF $f_Y(y)$.

* Does this satisfy the criteria for applying method of transformation?
* Can we partition R_X into intervals such that $g(x)$ is monotone?
* On which intervals $g(x)$ is monotone?

$R_X = \mathbb{R}, \quad R_Y = (0, \infty)$



$g(x) = x^2$ is strictly decr. on $(0, -\infty)$ and strictly incr. on $(0, \infty)$

$g'(x) = 2x$.

Recall

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|} = \boxed{\sum_{i=1}^{n} f_X(x_i) \cdot \left|\frac{dx_i}{dy}\right|}$$

For any $y \in (0, \infty)$, we have two soln for $y = g(x_*)$

i.e., $y = x^2$

$x_1 = \sqrt{y}, \quad x_2 = -\sqrt{y}$

$$f_Y(y) = \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{g'(x_2)}$$

$$= \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|-2\sqrt{y}|}$$

$$= \frac{1}{2\sqrt{2\pi y}} e^{-y/2} + \frac{1}{2\sqrt{2\pi y}} e^{-y/2}$$

$= \frac{1}{2}$

» Solution to Previous Question...

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \qquad y \in (0, \infty)}$$