Probability and Statistics: Lecture-41

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
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» Asymptotic Properties of MLEs...

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converges in distribution to $N(0,1)$.

» Solved Example 1 ...

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$$\hat{\Theta}_2 = \hat{\Theta}_1 + W$$

is also an unbiased estimator for θ

$E[\hat{\Theta}_2] = E[\hat{\Theta}_1] + E[W]$

$= E[\hat{\Theta}_1]$

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2. Let $\hat{\Theta}_1$ be an estimator for θ such that $E[\hat{\Theta}_1] = a\theta + b$, where $a \neq 0$.

» Solved Example 1 ...

$$E[\hat{\Theta}_2] = \frac{E[\hat{\Theta}_1] - b}{a} = \frac{\cancel{a}\theta + \cancel{b} - \cancel{b}}{\cancel{a}} = \underline{\underline{\theta}}$$

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» Solved Example...

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Order statistics

$X_{(1)}$

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1. Find the bias of $\hat{\Theta}_n$, $B(\hat{\Theta}_n)$
2. Find the MSE of $\hat{\Theta}_n$, $\text{MSE}(\hat{\Theta}_n)$

$B[\hat{\Theta}_n] = E[\hat{\Theta}_n] - \theta.$

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1. Find the bias of $\hat{\Theta}_n$, $B(\hat{\Theta}_n)$
2. Find the MSE of $\hat{\Theta}_n$, MSE$(\hat{\Theta}_n)$
3. Is $\hat{\Theta}_n$ a consistent estimator of θ?

» Answer to previous problem...

$X \sim$ Uniform $(0, \theta)$, then PDF and CDF

$$f_X(x) = \begin{cases} 1/\theta & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/\theta & 0 \le x \le \theta \\ 1 & x > \theta \end{cases}$$

Since $\hat{\theta}_n = X_{(1)}$, PDF of $\hat{\theta}_n$ is

$$f_{\hat{\theta}_n}(y) = n f_X(y) \left[F_X(y)\right]^{n-1}$$

[Recall order stat.]

$$= \begin{cases} n \cdot \frac{1}{\theta} \cdot \left(\frac{y}{\theta}\right)^{n-1} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{n}{\theta^n} y^{n-1} & 0 \le y \le \theta \\ 0 & \text{otherwise.} \end{cases}$$

To find the Bias of $\hat{\theta}_n$

$$E[\hat{\theta}_n] = \int_0^\theta y \cdot \frac{n\, y^{n-1}}{\theta^n} dy$$

$$= \frac{n}{\theta^n} \int_0^\theta y^n dy = \frac{n}{\theta^n} \left[\frac{y^{n+1}}{n+1}\right]_0^\theta$$

$$= \frac{n}{n+1} \theta$$

Thus Bias

$$B(\hat{\theta}_n) = E[\hat{\theta}_n] - \theta$$

$$= \frac{n}{n+1}\theta - \theta = \frac{-\theta}{n+1}$$

(b) $$MSE[\hat{\theta}_n] = Var(\hat{\theta}_n) + B(\hat{\theta}_n)^2$$

$$= \underline{Var(\hat{\theta}_n)} + \frac{\theta^2}{(n+1)^2}$$

↑ need this

$$E(\hat{\theta}_n^2) = \int_0^\theta y^2 \frac{n y^{n-1}}{\theta^n} dy = \frac{n}{n+2}\theta^2$$

$$Var(\hat{\theta}_n) = E[\hat{\theta}_n^2] - (E[\hat{\theta}_n])^2$$

$$= \frac{n}{(n+2)(n+1)^2}\theta^2 \quad \text{(check)}$$

$$\Rightarrow MSE[\hat{\theta}_n] = \frac{n}{(n+2)(n+1)^2}\theta^2 + \frac{\theta^2}{(n+1)^2} = \frac{2\theta^2}{(n+2)(n+1)}$$

» Answer to previous problem...

(c) $\lim_{n \to \infty} MSE(\hat{\theta}_n)$

$$= \lim_{n \to \infty} \frac{2\theta^2}{(n+2)(n+1)} = 0$$

$\Rightarrow \hat{\theta}_n$ is a consistent estimator of θ.

» Solved Example...

Example

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a Geometric(θ) distribution, where θ is unknown.

$$L(x_1, x_2, \ldots x_n; \theta) = P(x_1, x_2, \ldots x_n; \theta)$$
$$= P_{X_1}(\theta) P_{X_2}(\theta) \ldots P_{X_n}(\theta)$$

Example

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a Geometric(θ) distribution, where θ is unknown. Find the maximum likelihood estimator (MLE) of θ based on this random sample.

» Answer to previous problem...

Solution: $X_i \sim \text{Geometric}(\theta)$, then

$$P_{X_i}(x_i;\theta) = \underline{(1-\theta)^{x_i-1}}\,\theta$$

Likelihood fn

$$\underline{\underline{L(x_1,x_2,\dots,x_n;\theta)}} = P_{X_1X_2\dots X_n}(x_1,x_2,\dots x_n;\theta)$$

$$= P_{X_1}(x_1;\theta)\,P_{X_2}(x_2;\theta)\cdots P_{X_n}(x_n;\theta)$$

$$= (1-\theta)^{\sum_{i=1}^{n} x_i - n}\,\theta^n \longleftarrow$$

Better to use log likelihood to maximize

$$\ln L(x_1,x_2,\dots,x_n;\theta)$$

$$= \left(\sum x_i - n\right)\ln\underline{\underline{(1-\theta)}} + n\ln\theta$$

Maximize:

$$\frac{d}{d\theta}\ln L(x_1,\dots x_n;\theta)$$

$$= \left(\sum x_i - n\right)\frac{-1}{1-\theta} + \frac{n}{\theta} = 0$$

Solve for θ

$$\Rightarrow \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i}$$

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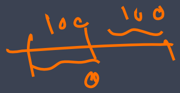

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6. The smaller the interval, the higher the precision with which we can estimate θ, and higher the confidence level

» Interval Estimation with Confidence Level...

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- Equivalently, we say that $[\hat{\Theta}_l, \hat{\Theta}_h]$ is a $(1-\alpha)100\%$ confidence interval for θ
- The randomness in these terms is due to $\hat{\Theta}_l$ and $\hat{\Theta}_h$, not θ
- Here $\hat{\Theta}_l$ and $\hat{\Theta}_h$ are random variables because they are functions of $X_1, \ldots, X_n$

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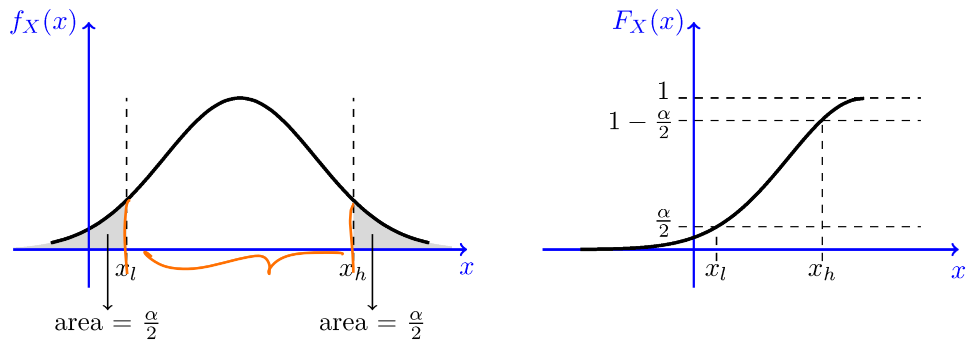
5. Rewriting these equations by using inverse, we have

$$x_l = F_X^{-1}\left(\frac{\alpha}{2}\right) \quad \text{and} \quad x_h = F_X^{-1}\left(1 - \frac{\alpha}{2}\right)$$

» Plot of confidence Interval...



* $[x_l, x_h]$ is a $(1-\alpha)$ interval for X, that is, $P(x_l \leq X \leq x_h) = 1 - \alpha$

» Example of Interval Estimation...

Example

Let $Z \sim N(0,1)$, find x_l and x_h such that

$$P(x_l \leq Z \leq x_h) = 0.95 = 1-\alpha$$

$$\Rightarrow \alpha = 0.05$$

CDF of $Z = \Phi$

$x_l = \Phi^{-1}\left(\frac{\alpha}{2}\right) = \Phi^{-1}(0.025) = -1.96$

$x_h = \Phi^{-1}\left(1-\frac{\alpha}{2}\right) = \Phi^{-1}(1-0.025) = 0.95$

» Answer to previous problem...

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» Recall: Statistical Inference...

Statistical Inference: Compare frequentist and Bayesian

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Frequentist Approach

In that approach, the unknown quantity θ is assumed to be a **fixed (non-random)** quantity that is to be estimated by the observed data.

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Frequentist Approach

In that approach, the unknown quantity θ is assumed to be a fixed (non-random) quantity that is to be estimated by the observed data.

Bayesian Approach

In the Bayesian framework, we treat the unknown quantity, Θ, as a random variable. More specifically, we assume that we have some initial guess about the distribution of Θ. This distribution is called the prior distribution. After observing some data, we update the distribution of Θ (based on the observed data).

» Motivating Example...

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Example (Motivating Example)

Suppose that you would like to estimate the portion of voters in your town that plan to vote for Party A in an upcoming election. To do so, you take a random sample of size n from the likely voters in the town. Since you have a limited amount of time and resources, your sample is relatively small. Specifically, suppose that $n = 20$. After doing your sampling, you find out that 6 people in your sample say they will vote for Party A.

* Let θ be the true portion of voters in your town who plan to vote for Party A. You might want to estimate θ as

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* In fact, in absence of any other data, that seems to be a reasonable estimate. However, you might feel that $n = 20$ is too small.

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* you can then proceed to find an updated distribution for Θ, called the posterior distribution, using Bayes' rule:

prior

$$f_{\Theta|D}(\theta|D) = \frac{P(D|\theta) f_\Theta(\theta)}{P(D)}. \tag{1}$$

* We can now use the posterior density, $f_{\Theta|D}(\theta|D)$ to further draw inferences about Θ

» Answer to previous problem...

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5. Note that in the above setting, X or Y (or possibly both) could be random vectors

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4. Here $f_{X|Y}(x|y)$ is called posterior distribution

» Solved Example...

Example

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» Solved Example...

① Baye's rule $f_{X|Y}(x|2) = \dfrac{P_{Y|X}(2|x)\, f_X(x)}{P_Y(2)}$

Since $Y|X \sim \text{Geometric}(x)$

$P_{Y|X}(y|x) = x(1-x)^{y-1}$ for $y = 1, 2, \ldots \Rightarrow P_{Y|X}(2|x) = x(1-x)$

Example

Let $X \sim$ Uniform$(0, 1)$. Suppose that we know

$$Y \mid X = x \sim \text{Geometric}(x).$$

Find the posterior density of X given $Y = 2$, $f_{X|Y}(x|2)$

$P_Y(2) = \int_{-\infty}^{\infty} P_{Y|X}(2|x)\, f_X(x)\, dx = \int_0^1 x(1-x)\cdot 1\, dx = \frac{1}{6}$

$\Rightarrow f_{X|Y}(x|2) = \dfrac{x(1-x)\cdot 1}{1/6} = 6x(1-x) \quad 0 \le x \le 1$

» Answer to previous problem...

» Maximum Apriori Estimation (MAP)...

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$$\frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$$

does not depend on X

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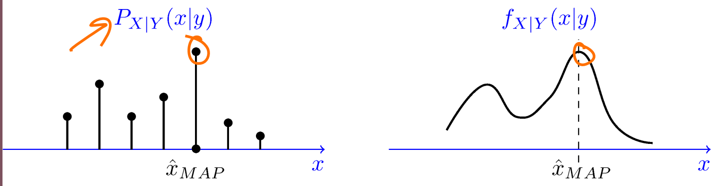

Here $\hat{x}_{MAP}$ is the value of X for which the posterior $f_{X|Y}(x|y)$ is maximized

» Finding the MAP Estimate...

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$$f_{Y|X}(y|x)f_X(x).$$

(forget $f_Y(y)$ in denominator)

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Whenever, X or Y is discrete, we replace PDF by its PMF.

» Example of MAP Estimate...

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$P_{Y|X}(y|x) = x(1-x)^{y-1}$ $y = 1, 2, \ldots$

$\Rightarrow P_{Y|X}(3|x) = \underline{x(1-x)^2}.$

Need to find value $x \in [0,1]$ that maximize. $\underline{P_{Y|X}(y|x)}\, \underline{f_X(x)}$

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Find the MAP estimate of X given $Y = 3$.

$= x(1-x)^2 \cdot 2x = 2x^2(1-x)^2$

$\underline{\text{Maximize}}$: $\frac{d}{dx}\left(x^2(1-x)^2\right] = 2x(1-x)^2 - 2(1-x)x^2 = 0$ ($\underline{\text{Check}}$.

$\Rightarrow \hat{x}_{MAP} = \frac{1}{2}.$

» Answer to previous problem...

» Comparison of MAP to ML Estimator...

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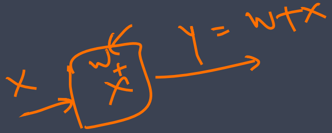
4. If X is uniformly distributed over a finite interval, then ML and MAP estimate is same

» Solved Example...

Example

Suppose that the signal $X \sim N(0, \sigma_X^2)$ is transmitted over a communication channel.



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1. Find the ML estimate of X, given $Y = y$ is observed

» Solved Example...

Sol^n: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{x^2}{2\sigma_X^2}}$

$\underbrace{Y|X = x \sim N(x, \sigma_W^2)}$ (Check) $\Rightarrow f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_W} e^{-\frac{(y-x)^2}{2\sigma_W^2}}$

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(1) $f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_W} e^{-(y-x)^2/2\sigma_W^2}$

Maximize: $\min (y-x)^2 \Rightarrow \hat{x}_{ML} = y$

2) $f_{Y|X}(y|x) f_X(x) = C e^{-\left[\frac{(y-x)^2}{2\sigma_W^2} + \frac{x^2}{2\sigma_X^2}\right]}$

Maximize $\Rightarrow$ minimize $-\frac{(y-x)^2}{2\sigma_W^2} + \frac{x^2}{2\sigma_X^2}$

$\Rightarrow \hat{x}_{MAP} = \sigma_X^2 / (\sigma_X^2 + \sigma_W^2)$

» Minimum Mean Squared Error (MMSE) Estimation...

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» Example of MMSE Computation...

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Exercise

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» Answer to previous problem...

» Mean Squared Error (MSE) ...

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has the lowest MSE among all possible estimators.