Probability and Statistics: Lecture-29

Monsoon-2020

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by Dr. Pawan Kumar (IIIT, Hyderabad) on October 19, 2020
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Definition of Conditional Expectation

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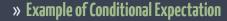
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$$E[X \mid A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i)$$

$$E[X \mid Y = y_j] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i \mid y_j)$$



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- 2. Find $E[X \mid -1 < Y < 2]$

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If we pick a point (X, Y) from this grid at random, then the probability of choosing a point is 1/13.

- 1. Find E[X | Y = 1]
- 2. Find $E[X \mid -1 < Y < 2]$
- 3. Find $E[|X| \mid -1 < Y < 2]$

**Answer to previous problem...

Recall from last class that

Given
$$y=1$$
, x is uniformly distributed

Over the set $\xi=1,0.11$?

P(-1 < $y < 2$) = $\frac{1}{12}$ for $x=0$ distributed

P(x|y) = $\frac{1}{12}$ for $x=0$ distributed

Note $x = -1,0.11$ because $y=1$

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 $|x|+|x| \leq 2$
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» Answer to previous problem...

So we can write
$$P(x=-2,A) = \frac{13}{8}P(x=2,0)$$

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$$R_{X|A}(-1) = \frac{13}{8} P(x=-1)A) = \frac{13}{8} \left[\frac{1}{13} = \frac{1}{6} \right]$$

$$R_{X|A}(-1) = \frac{13}{8} P(x=-1)A) = \frac{13}{8} \left[\frac{1}{13} \cdot \frac{1}{13} + \frac{1}{13} \cdot \frac{1}{13} - \frac{1}{13} \right]$$

$$= \frac{13}{8} \cdot \frac{1}{13} = \frac{1}{6}$$



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Law of Total Probability

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P(A) = \$ P(A(Bi))P(B)Shu | Parktim

Law of Total Probability

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$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A \mid Y = y_j) P_Y(y_j), \quad \text{for any set } A$$

Law of Total Probability

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$$extit{P}(extit{X} \in extit{A}) = \sum_{ extit{y}_j \in extit{R}_Y} extit{P}(extit{X} \in extit{A} \mid extit{Y} = extit{y}_j) extit{P}_Y(extit{y}_j), \quad ext{for any set } extit{A}$$

* Law of Total Expectation:

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 - 1. If B_1, B_2, \ldots is a partition of sample space S

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- * Law of Total Expectation:
 - 1. If B_1, B_2, \ldots is a partition of sample space S

Proof
$$E[x|B_i] = \sum_i E[x|B_i]P(B_i)$$

Multi by $P(B_i) + \epsilon_{im}$ over

$$E[x|B_i] = \sum_i E[x|B_i]P(B_i)$$

$$E[x|B_i] + \epsilon_{im}$$
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Law of Total Probability

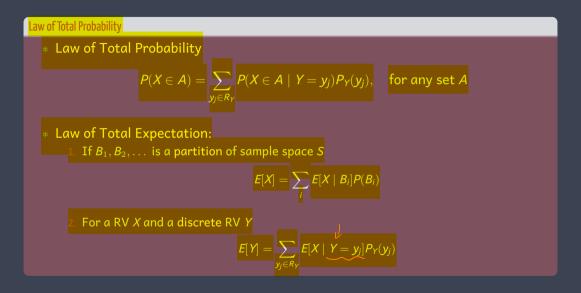
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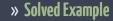
$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A \mid Y = y_j) P_Y(y_j), \quad ext{for any set } A$$

- * Law of Total Expectation:
 - 1. If B_1, B_2, \ldots is a partition of sample space S

$$E[X] = \sum_{i} E[X \mid B_{i}] P(B_{i})$$

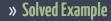
2. For a RV X and a discrete RV Y





Solved Example 1

Let $X \sim \text{Geometric}(p)$. Find E[X]. [Hint: condition on first coin toss.]



in day

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$.

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Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers. What is the average number of customers who purchase drinks?

» Answer to previous problem... (x:) no of curformy purchase

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» PMF and Expectation of Two Random Variables...

PMF and Expectations of Two Random Variables

* Let X, Y be two RVs and suppose $Z = g(X, Y), g : \mathbb{R}^2 \to \mathbb{R}$.

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The expectation is given as follows

$$E[g(X,Y)] = \sum_{(x_i,y_j) \in R_{XY}} g(x_i,y_j) \underbrace{P_{XY}(x_i,y_j)}_{P_{XY}(x_i,y_j)}$$

» Linearity of Expectation for Two Random Variable...

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Linearity of Expectation for Two RV

Let X, Y be two discrete RVs. Then E[X + Y] = E[X] + E[Y]



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» PMF of Difference of Two Geometric Distributions...

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PMF of Difference

Let $X, Y \sim Geometric(p)$ be two random variables. Let Z = X - Y. Find the PMF of Z.

** Answer to previous problem...
$$2 = x - y$$

** Rx = Ry = $\{1, 2, 3, \dots, 3 = 1 \text{ N}$

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** P(x=j+k) P(y=j)

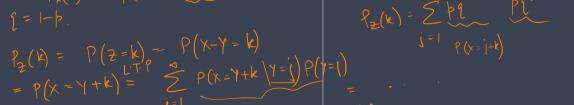
** = Z

** P(x=j+k) P(y=j)

** = Z

** Sino X, Y \(\text{ Geometric Quometric (P)} \)

** Rx(k) = Ry(k) = \(\text{ P}(k) = \(\text{ P}(k) = \text{ P}(k) =



» Answer to previous problem...