Probability and Statistics: Lecture-29

Monsoon-2020

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» Definition of Conditional Expectation...

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$$E[X \mid A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i)$$

$P_{x|y}(x_i|\,)$

$$E[X \mid Y = y_j] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i \mid y_j)$$

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1. Find $E[X \mid Y = 1]$ $P_{X|Y}(x|1)$
2. Find $E[X \mid -1 < Y < 2]$

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If we pick a point (X, Y) from this grid at random, then the probability of choosing a point is $1/13$.

1. Find $E[X \mid Y = 1]$
2. Find $E[X \mid -1 < Y < 2]$
3. Find $E[|X| \mid -1 < Y < 2]$

» Answer to previous problem...

Recall from last class that

Given $Y=1$, X is uniformly distributed over the set $\{-1,0,1\}$

$$\underline{P_{X|Y}} = \begin{cases} \frac{1}{3} & \text{for } x=-1 \\ \frac{1}{3} & \text{for } x=0 \\ \frac{1}{3} & \text{for } x=1 \end{cases}$$

Uniform distr.

Note $X = -1, 0, 1$ because $Y=1$

$(|x|+|y| \leq 2)$

$$\Rightarrow E[X|Y=1] = -1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3}$$
$$= 0$$

(b) $E[X \mid -1 < Y < 2]$

$-1 < Y < 2 \Rightarrow Y \in \{0,1\} \equiv A$

$$P(\underbrace{-1<Y<2}_{A}) = P_Y(0) + P_Y(1)$$
$$= \frac{5}{13} + \frac{3}{13} = \frac{8}{13}.$$

$$P_{X|A}(k) = \frac{P(X=k, A)}{P(A)}$$
$$= \frac{P(X=k, A)}{8/13} = \frac{13}{8} P(X=k, A)$$

» Answer to previous problem...

So we can write

$$P_{X|A}(-2) = \frac{13}{8} P(X=-2, A) = \frac{13}{8} P(\cancel{\;}2, 0)$$

$$= \frac{13}{8} \cdot \frac{1}{13} = \frac{1}{8}.$$

$$P_{X|A}(-1) = \frac{13}{8} P(X=-1, A) = \frac{13}{8} \left[P_{XY}(-1, 0) + P_{XY}(-1, 1) \right]$$

$$= \frac{13}{8} \cdot \frac{2}{13} = \frac{1}{4}.$$

Similarly,

$$P_{X|A}(0) = \frac{1}{4}, \quad P_{X|A}(2) = \frac{1}{8}$$

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i)$$

$$= -2 \cdot \frac{1}{8} + (-1) \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} = 0$$

(c) Try!

» Law of Total Probability...

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$$\overline{P(A)} = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$ when B_i's form a partition

Law of Total Probability

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$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A \mid Y = y_j) P_Y(y_j), \quad \text{for any set } A$$

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  1. If $B_1, B_2, \ldots$ is a partition of sample space S

$$E[X] = \sum_i E[X \mid B_i] P(B_i)$$

Proof $E[X|B_i] \overset{\text{def}}{=} \sum_j x_j P(x_j|B_i)$

Mult by $P(B_i)$ & sum over i

$$\sum_i E[X|B_i] P(B_i) = \sum_i \sum_j x_j P(x_j|B_i) P(B_i)$$

$$= \sum_j x_j \sum_i \frac{P(x_j|B_i) P(B_i)}{P(x_j)} = E[X]$$

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  2. For a RV X and a discrete RV Y

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$$E[Y] = \sum_{y_j \in R_Y} E[X \mid Y = y_j] P_Y(y_j)$$

» Solved Example

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Solved Example 1

Let $X \sim$ Geometric(p). Find $E[X]$. [Hint: condition on first coin toss.]

Geometric(p) : toss the coin repeatedly until 1st head.

Here $P(H) = p$, X = total no. of coin tosses.

Two possible choices : H or T.

Using Law of total Expectation

$$E[X] = E[X|H]\,P(H) + E[X|T]\,P(T)$$

$$= p\,E[X|H] + (1-p)\,E[X|T]$$

$$= p \cdot 1 + (1-p)\left(1 + E[X]\right)$$

Solving for $E[X] = \frac{1}{p}$.

$E[X|H] = 1$

$E[X|T] = 1 + E[X]$

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in day

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$.

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Uniform, Bernoulli, Geometric, Binomial

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers. What is the average number of customers who purchase drinks?

» Answer to previous problem...

$\boxed{X:}$ no of customers purchasing drinks

Sol^n. Given $N=n$, then X is a sum of n ind. Bernoulli(p), which means it is

$$X \mid N=n \sim \text{Binomial}(n, p)$$

$$P_{X|N}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X|N=n] = np$$

$E[X]$ = expected no of customers purchasing drink.

$$= \sum_{n=0}^{\infty} E[X|N=n]\, P_N(n)$$

$$= \sum_{n=0}^{\infty} n p\, P_N(n)$$

$$= p \sum_{n=0}^{\infty} n\, P_N(n)$$

$$= p\, E[N] = p\lambda$$

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PMF and Expectations of Two Random Variables

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$$P_Z(z) = P(\underbrace{g(X, Y)}_{Z} = z) = \sum_{(x_i, y_j) \in A_z} P_{XY}(x_i, y_j),$$

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* Let X, Y be two RVs and suppose $Z = g(X, Y), g : \mathbb{R}^2 \to \mathbb{R}$. Then the PMF of Z is

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* The expectation is given as follows

$$E[g(X, Y)] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j)\, P_{XY}(x_i, y_j)$$

» Linearity of Expectation for Two Random Variable...

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Linearity of Expectation for Two RV

Let X, Y be two discrete RVs. Then $E[X+Y] = E[X] + E[Y]$.

Exercise

» PMF of Difference of Two Geometric Distributions...

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PMF of Difference

Let $X, Y \sim Geometric(p)$ be two random variables. Let $Z = X - Y$. Find the PMF of Z.

» Answer to previous problem...

$Z = X - Y$

Ans. $R_X = R_Y = \{1, 2, 3, \dots\} = \mathbb{N}$

$\Rightarrow R_Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} = \mathbb{Z}.$

Since $X, Y \sim \text{Geometric}(p)$

$P_X(k) = P_Y(k) = \underbrace{p q^{k-1}}, \quad k = 1, 2, 3, \dots$

$q = 1 - p.$

$P_Z(k) = P(Z = k) = P(X - Y = k)$

$= P(X = Y + k) \overset{L.T.P}{=} \sum_{j=1}^{\infty} \underbrace{P(X = Y + k \mid \underline{Y = j}) P(Y = j)}$

$= \sum_{j=1}^{\infty} P(X = j + k \mid Y = j) P(Y = j)$

$= \sum_{j=1}^{\infty} \underbrace{P(X = j + k) P(Y = j)}_{(X, Y \text{ ind})}$

$\underline{k \geq 0}$

$P_Z(k) = \sum_{j=1}^{\infty} \underbrace{p q^{j+k-1}}_{P(X = j+k)} \; \underbrace{p q^{j-1}}$

$= \dots$

$\underline{k < 0}$

» Answer to previous problem...

For $k < 0$

$$P_Z(k) = \sum_{j=1}^{\infty} \underbrace{P_X(j+k)}\, P_Y(j)$$

$$= \sum_{j=-k+1}^{\infty} \underbrace{p\,q^{j+k-1}} \cdot p\,q^{j-1}$$

$=$