Ans:1 $x+y=10$ where x: No of blue marbles
y: No of red marbles

$$R_{XY} = \{(i,j) \mid i+j=10;\ i,j \in W\}$$
$$= \{(10,0), (9,1), (8,2), (7,3), (6,4), (5,5), (4,6), (3,7), (2,8), (1,9), (10,0)\}.$$

So we can write PMF of X and Y in general form as:-

$$P_{XY}(i,j) = \begin{cases} \dfrac{{}^{40}C_i \cdot {}^{60}C_j}{{}^{100}C_{10}} & i \geq 0,\ j \geq 0,\ i+j=10. \\ 0 & \text{otherwise} \end{cases}$$

Ans 2(a): $(i,j) \in Z^2$, $i,j \geq 0$ and $|i-j| \leq 1$

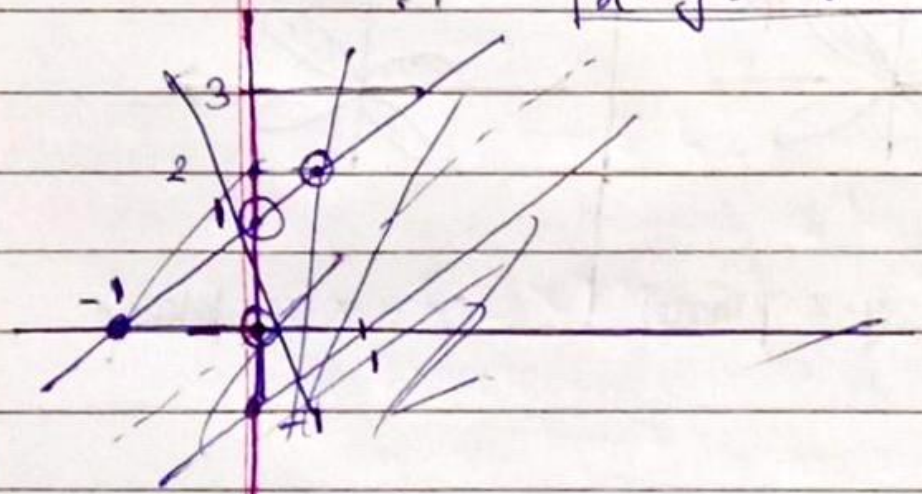
$\Rightarrow -1 \leq i-j \leq 1$

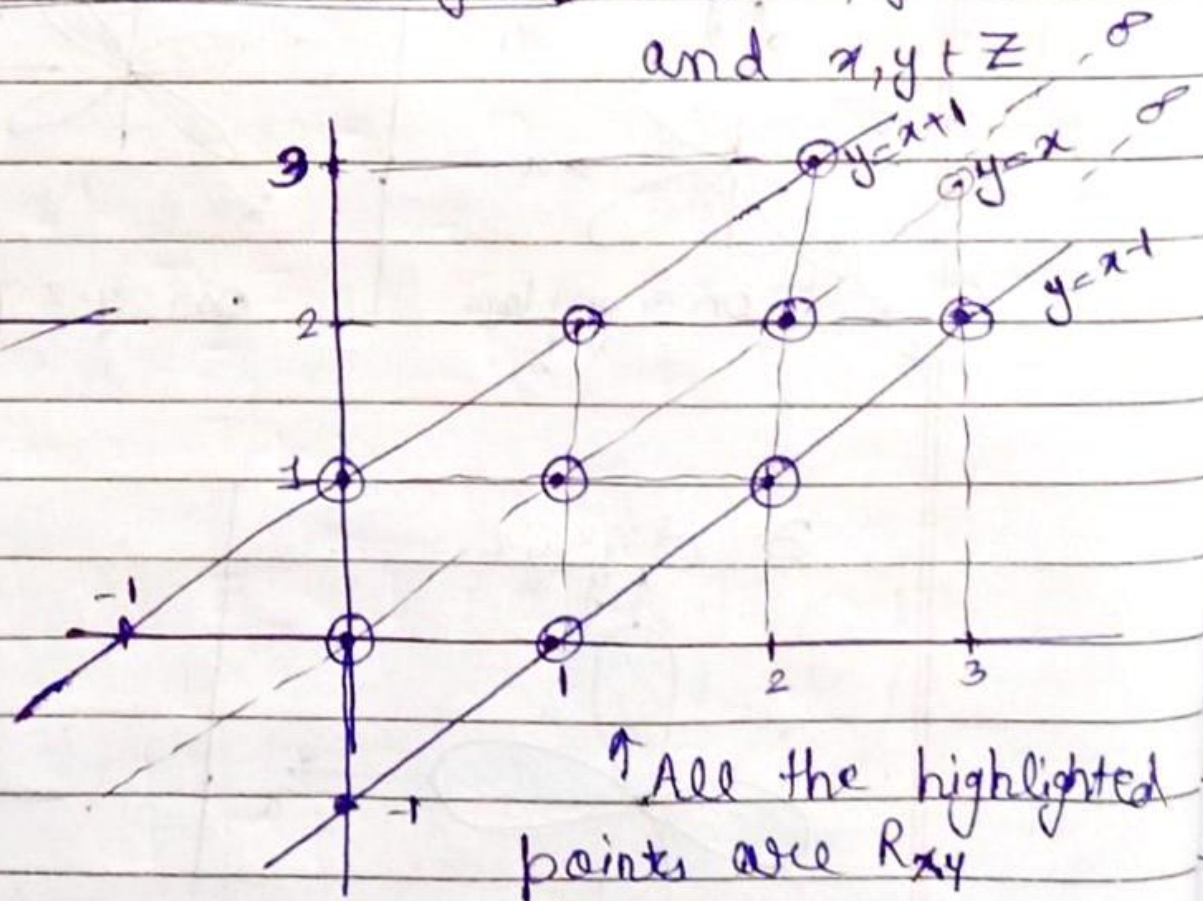
$\Rightarrow i-j \geq -1$ & $i-j \leq 1$

~~Let~~ Put i as x and j as y.

$\therefore$ $\boxed{x-y \geq -1 \quad \& \quad x-y \leq 1}$ also $x, y \geq 0$.

and $x, y \in Z$





So the points $\longrightarrow$ corresponding to R_{xy} are all the points which have $x \geq 0$, $y \geq 0$ & lie b/w ~~xy~~ $y = x-1$ and $y = x+1$ and also $x, y \in Z$.

↑ All the highlighted points are R_{xy}

↳ Those highlighted points are going till ∞.

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Ans 2(b) $P_{XY}(i,j) = \dfrac{1}{6 \cdot 2^{\min(i,j)}}$

$\rightarrow \boxed{P_X(i)}$

$P_X(0) = P_{XY}(0,0) + P_{XY}(0,1) = 1/6 + 1/6 = 1/3$

$P_X(1) = P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2) = 1/6\,(1 + 1/2 + 1/2) = 1/3$

$P_X(2) = P_{XY}(2,1) + P_{XY}(2,2) + P_{XY}(2,3) = 1/6\,(1/2 + 1/4 + 1/4) = 1/6.$

$\vdots$

$P_X(k) = P_{XY}(k,k-1) + P_{XY}(k,k) + P_{XY}(k,k+1)$

$= \dfrac{1}{6}\left(\dfrac{1}{2^{k-1}} + \dfrac{1}{2^k} + \dfrac{1}{2^k}\right) = \dfrac{1}{6}\left(\dfrac{1}{2^{k-2}}\right) = \dfrac{1}{3}\left(\dfrac{1}{2^{k-1}}\right)$

So $P_X(k) = \begin{cases} 1/3 & k = 0 \\ 1/3\,(1/2^{k-1}) & k = 1, 2, \ldots \in N \\ 0 & \text{otherwise} \end{cases}$

Due to symmetry $P_X(k) = P_Y(k)$

$\Rightarrow$ Both X and Y will have same marginal PDF.

Ans. $P_X(k) = P_Y(k) = \begin{cases} 1/3 & k = 0 \\ 1/3\,(1/2^{k-1}) & k = 1, 2, \ldots \in N \\ 0 & \text{otherwise} \end{cases}$

Ans 2(c) $P(X=Y \mid X<2) = \dfrac{P(X=Y,\ X<2)}{P(X<2)} = \dfrac{P_{XY}(0,0) + P_{XY}(1,1)}{P_X(0) + P_X(1)}$

$= \dfrac{1/6 + 1/12}{1/3 + 1/3} = 3/8.$

Ans 2(d) $P(1 \le X^2 + Y^2 \le 5) = P_{XY}(0,1) + P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2) + P_{XY}(2,1)$

$= \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{12} + \dfrac{1}{12} + \dfrac{1}{12}$

$= \dfrac{7}{12}.$

Ans.2 (e) $P(X=Y) = \sum_{i=0}^{\infty} P_{XY}(i,i) \cancel{= P_{XY}(0,0)} = \sum_{i=0}^{\infty} \frac{1}{6 \cdot 2^{\min(i,i)}}$

$$= \sum_{i=0}^{\infty} \frac{1}{6 \cdot 2^i} = \frac{1}{6} \left(\sum_{i=0}^{\infty} \frac{1}{2^i} \right) \infty \text{ G.P.}$$

$$\cancel{S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.}$$

$$\therefore P(X=Y) = \frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \infty\right)$$

$$= \frac{1}{6} \cdot \frac{1}{1-1/2} = \frac{1}{6} \times 2 = \frac{1}{3}.$$

Ans.2 (f) $E[X \mid Y=2]$

$\cancel{P_{X|Y}(k \mid 2)}$

$$P_{X|Y}(x=k \mid Y=2) = \frac{P_{XY}(k,2)}{P_Y(2)} = \frac{P_{XY}(k,2)}{1/6}$$

$$= 6 \cdot P_{XY}(k,2).$$

So $P_{XY}(k,2) = \begin{cases} 1/2 & k=1 \\ 1/4 & k=2,3 \\ 0 & \text{otherwise} \end{cases}$

So $E[X \mid Y=2] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 0$

$$= 7/4$$

Ans.2 (g) $Var(X \mid Y=2) = E[X^2 \mid Y=2] - (E[X \mid Y=2])^2$

$(7/4)^2$ (from previous part)

$$1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} + 3^2 \times \frac{1}{4} = \frac{15}{4}.$$

$$\therefore Var(X \mid Y=2) = \frac{15}{4} - \left(\frac{7}{4}\right)^2 = \frac{11}{16}.$$

Ans. 3 Given X and Y are bivariate normal ; ~~i.e.~~
$(X \sim N(\mu_x, \sigma_x^2), \quad Y \sim N(\mu_y, \sigma_y^2))$

$\therefore$ Their joint PDF will be given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\}$$

Where ρ = correlation coefficient.
Given in the question X and Y are uncorrelated
$\Rightarrow \rho = 0$
Put it in joint PDF

$$\therefore f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{\frac{-1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}. \rightarrow (1)$$

$$\because f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right].$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right].$$

$\because$ X and Y are normal RV

$$\therefore f_X(x) \cdot f_Y(y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\} \rightarrow (2)$$

(1) = (2)

$$\Rightarrow f_{XY}(x,y) = f_X(x) \cdot f_Y(y).$$

This occurs only when X and Y are independent.
$\Rightarrow$ If X and Y are bivariate normal and uncorrelated, then they are independent.

Hence Proved

Ans. 4 $\because Y = AX + b$ Take expectation on both sides

$\therefore E[Y] = E[AX+b] = AE[X]+b$

(By linearity of Expectation)

$\therefore C_y = E[(Y - E[Y])(Y - E[Y])^T]$

$= E[((AX+b) - (AE[X]+b)).(AX+b - (AE[X]+b))^T]$

$= E[A(X - E[X])(X - E[X])^T A^T]$

$= E[A(X - E[X])(X - E[X])^T A^T]$

$= A E[(X - E[X])(X - E[X])^T] A^T$

$= A C_x A^T$ Hence Proved

Ans-5 Both the formula expressions given in the question are in the form of

$$a e^{(-1/2)b}.$$

So it is sufficient to show that they have the same a and b.

We have $m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$

Also,

$$C = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

Take determinant

$$\therefore \quad |C| = \det C = \sigma_x^2 \sigma_y^2 (1-\rho^2).$$

Thus, in both formulas for PDF,

$$a = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

Let's check b. We have

$$C^{-1} = \frac{1}{\sigma_x^2 \sigma_y^2 (1-\rho^2)} \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix}$$

By Matrix-multiplication we obtain:-

$$(x-m)^T C^{-1} (x-m) =$$

$$\frac{1}{\sigma_x^2 \sigma_y^2 (1-\rho^2)} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

$$= \frac{-1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{x-\mu_y}{\sigma_y}\right) \right]$$

which agrees with formula given in question.

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Ans.6 Jensen's Inequality states that if $g(x)$ is a convex function of R_X and $E[g(x)]$ and $g(E[X])$ are finite, then

$$E[g(X)] \geq g(E[X])$$

If $g(x)$ is a concave function then.

$$g(E[X]) \geq E[g(X)].$$

Given RV X with $E[X]=10$; $X>0$

Let $g(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$.

$\therefore g'(x) = \frac{1}{2x}$ for $x>0$ $\quad g''(x) = -\frac{1}{2x^2}$.

$\because x>0 \Rightarrow g''(x)$ will always be -ve

$\Rightarrow g$ is a concave fxn. in $(0,\infty)$.

$\therefore$ By Jensen's Inequality we have

$$g(E[X]) \geq E[g(X)].$$

$\therefore E[\ln\sqrt{x}] = E[\frac{1}{2}\ln X] \leq \frac{1}{2}\ln E[X]$

$\leq \frac{1}{2}\ln 10$

$\therefore \boxed{E[\ln\sqrt{x}] \leq \frac{\ln 10}{2}}$

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Day 7 Proof for the Generalized Union bound

To prove:- $P\left(\bigcup_{i=1}^{n+1} A_i\right) \leq \sum_{i=1}^{n+1} P(A_i)$

This inequality may be proved for finite collections of events using the method of induction.

n = 1 case $\therefore P(A_1) \leq P(A_1) \rightarrow$ True

n = n case $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

$\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and because the union operation is associative, we have

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right)$$

~~Since~~ $\because P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right) \geq 0$ [First Axiom of Probability]

$\therefore$ We have

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) \leq P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1})$$

$$\therefore P\left(\bigcup_{i=1}^{n+1} A_i\right) \leq \sum_{i=1}^{n} P(A_i) + P(A_{n+1}) = \sum_{i=1}^{n+1} P(A_i)$$

i.e. $P\left(\bigcup_{i=1}^{n+1} A_i\right) \leq \sum_{i=1}^{n+1} P(A_i)$

Hence Proved

Now proving other bounds:-

By inclusion-exclusion principle:-

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$+ \cdots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Now

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

if no of terms in RHS $\rightarrow$ odd

$\Rightarrow$ RHS $\rightarrow$ $\sum P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots + \sum P(A_i \cap A_j \cdots)$

odd terms

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$$P\left(\bigcup_{i=1}^{n} A_i\right) = RHS - c \quad ; \text{ where } c \geq 0$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq RHS \text{ for odd terms}$$

if even no of terms

RHS →

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \cdots - \sum P(A_i \cap A_j \cdots)$$

even no of terms

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = RHS + c \; ; \; c \geq 0$$

$$\therefore P\left(\bigcup_{i=1}^{n} A_i\right) \geq RHS \text{ for even terms}$$

Ans 7 (2) There are n nodes in the network. Let's call them Node 1, Node 2, ..., Node n.
Let A_i be the event that the ith node is isolated. Then we have :-

$$B_n = \bigcup_{i=1}^{n} A_i$$

$\therefore$ ~~Inclusion- Exclusio~~ By using first two terms of Inclusion- Exclusion, we have

$$P(B_n) = P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

Through symmetry, we can say that

$$\sum_{i=1}^{n} P(A_i) = n P(A_1) = n P(A_2) \cdots$$

$$\therefore \sum_{i<j} P(A_i \cap A_j) = {}^nC_2 \cdot P(A_1 \cap A_2).$$

These are the total ~~possible~~ terms in this summation.

Thus, we conclude $P(B_n) \geq n P(A_1) - {}^nC_2 P(A_1 \cap A_2)$

$P(A_1) = ?$

~~Thus using the union bound we conclude that~~

$$\cancel{P(B_n) = P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)}$$

~~By symmetry $\forall i, j$ we have $P(A_i) = P(A_j)$, so~~

$$\cancel{P(B_n) \leq n P(A_1)}$$

The event A_1 occurs if Node 1 is not connected to any of the other $n-1$ nodes. Since the connections are independent, we conclude that

$$P(A_1) = (1-p)^{n-1}.$$

$P(A_1 \cap A_2) = ?$ By the same argument above we can say that

$$P(A_1 \cap A_2) = (1-p)^{2(n-2)-1} = (1-p)^{2n-3}.$$

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The reason for this is that $A_1 \cap A_2$ is the event that Nodes 1 and 2 are isolated. There are $2(n-2)$ potential edges from the rest of the graph to Nodes 1 and 2, and there also is a potential edge from Node 1 to Node 2. These edges exist independently from each other with probability p_n.

$\therefore$ By substituting the values obtained for $P(A_1)$ and $P(A_1 \cap A_2)$ we obtain:-

$$P(B_n) \geq n(1-p)^{n-1} - {}^nC_2 (1-p)^{2n-3}$$

Hence Proved

Ans. 8 $P(X \geq \alpha n) = P(X - np \geq \alpha n - np)$

$$\leq P(|X - np| \geq n\alpha - np)$$

$$\leq \frac{Var(X)}{n^2(\alpha - p)^2} \quad \text{(Chebyshev's Inequality)}$$

$$\cancel{\leq p(1-p)}$$

$\because X \sim \text{Binomial}(n, p)$ $\therefore Var(X) = np(1-p)$

$$P(X \geq \alpha n) \leq \frac{np(1-p)}{n^2(\alpha - p)^2}$$

$$\therefore P(X \geq \alpha n) \leq \frac{p(1-p)}{n(\alpha - p)^2}$$

For $p = 1/2$ and $\alpha = 3/4$, we obtain

$$P\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{n}$$

Ans 9 <u>Markov's Inequality</u>:- $P(X \geq a) \leq \frac{E[X]}{a}$

where X: Non-negative RV.

$\because X \sim \text{Binomial}(n, p)$ $\therefore E[X] = np$

$$\therefore \boxed{P(X \geq \alpha n) = \frac{np}{\alpha n} = \frac{p}{\alpha}}$$

<u>Chebyshev's Inequality</u>:- $P(|X - E[X]| \geq b) \leq \frac{Var(X)}{b^2}$

where $b > 0$.

$$\therefore P(X \geq \alpha n) = P(X - np \geq \alpha n - np)$$
$$\leq P(|X - np| \geq n(\alpha - p))$$
$$\leq \frac{Var(X)}{n^2(\alpha - p)^2}$$

$\because X \sim \text{Binomial}(n, p)$ $\therefore Var(X) = np(1-p)$

$$\therefore \boxed{P(X \geq \alpha n) \leq \frac{np(1-p)}{n^2(\alpha - p)^2} = \frac{p(1-p)}{n(\alpha - p)^2}}$$

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<u>Chernoff bound:-</u> $P(X \geq a) \leq \min_{s \geq 0} e^{-sa} M_X(s)$

$\downarrow$

Moment Generating fun. of X.

For $X \sim \text{Binomial}(n, p)$

$$M_X(s) = (pe^s + q)^n \text{ where } q = 1-p.$$

$$\therefore P(X \geq \alpha n) \leq \min_{s>0} e^{-sa} M_X(s) = \min_{s>0} \underbrace{e^{-sa} (pe^s + q)^n}_{f(s)}$$

we have to find $\min_{s>0} f(s)$

{ Let $a = \alpha n$ }

$$\therefore \frac{d}{ds}\left(e^{-sa}(pe^s+q)^n\right) = 0.$$

$$\therefore -ae^{-sa}(pe^s+q)^n + e^{-sa} \cdot n \cdot (pe^s) \cdot (pe^s+q)^{n-1} = 0$$

$$\therefore -a(pe^s + q) + npe^s = 0.$$

$$e^s p(n-a) = aq$$

$$\therefore e^s = \frac{aq}{p(n-a)} = \frac{\alpha n q}{p(n-\alpha n)} = \boxed{\frac{\alpha q}{p(1-\alpha)}}$$

$\frac{d^2}{ds^2} f(s) > 0$ (Minima) where $q = 1-p$

at s obtained

By using this value of s in $f(s)$ we get

$$\boxed{P(X \geq \alpha n) \leq \left(\frac{1-p}{1-\alpha}\right)^{(1-\alpha)n} \left(\frac{p}{\alpha}\right)^{\alpha n}}$$

Ans: 10 Let's assume each people sitting around table to be an indicator random variable. Let them be $X_1, X_2, \ldots X_n$.

$$X_i = \begin{cases} 0 & \text{If } i^{th} \text{ person do not receive present} \\ 1 & \text{If } i^{th} \text{ person receives present.} \end{cases}$$

$$\therefore X = X_1 + X_2 + \ldots X_n.$$

$P(X_i = 1) = 1/4$

↳ It is the probability that the person to the right has a different outcome times the probability that the person to the left has a different outcome. In other words, if we define H_i and T_i be the events that the ith person's outcome is heads and tails resp., we can write:-

$$E[X_i] = P(X_i = 1)$$
$$= P(H_{i-1}, T_i, H_{i+1}) + P(T_{i-1}, H_i, T_{i+1})$$
$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

Thus, we find

$$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = \boxed{N/4}$$

Next, we can write

$$Var(X) = \sum_{i=1}^{N} Var(X_i) + \sum_{i=1}^{N} \sum_{j \neq i} Cov(X_i, X_j)$$

$\because$ $X_i \sim$ Bernoulli $(1/4)$, we have

$$Var(X_i) = (1/4)(3/4) = 3/16.$$

$Cov(X_i, X_j) = ?$

$\because$ X_i & X_j are independent if there are at least 2 people b/w the ith person and the jth person. In other words, if $2 < |i-j| < N-2$, then X_i and X_j are independent, so

$$Cov(X_i, X_j) = 0 \quad \text{for} \quad 2 < |i-j| < N-2.$$

Also, By symmetry:-

$$Cov(X_1, X_2) = Cov(X_2, X_3) = Cov(X_3, X_4) = \cdots Cov(X_{n-1}, X_n) = Cov(X_n, X_1)$$

$$Cov(X_1, X_3) = Cov(X_2, X_4) = Cov(X_3, X_5) = \cdots Cov(X_{n-1}, X_1) = Cov(X_n, X_2).$$

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$\therefore \; Var(X) = N\,Var(X_1) + 2N\,Cov(X_1, X_2) + 2N\,Cov(X_1, X_3)$

$= \frac{3N}{16} + 2N\,Cov(X_1, X_2) + 2N\,Cov(X_1, X_3).$

$Cov(X_1, X_2) = ?$ $\qquad Cov(X_1, X_3) = ?$

$E[X_1 X_2] = P(X_1 = 1, X_2 = 1)$

$= P(H_N, T_1, H_2, T_3) + P(T_N, H_1, T_2, H_3)$

$= 1/16 + 1/16 = 1/8.$

Thus, $Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$

$= 1/8 - 1/16 = 1/16.$

Similarly $E[X_1 X_3] = P(X_1 = 1, X_3 = 1)$

$= P(H_N, T_1, H_2, T_3, H_4) + P(T_N, H_1, T_2, H_3, T_4)$

$= 1/32 + 1/32 = 1/16.$

Thus, $Cov(X_1, X_3) = E[X_1 X_3] - E[X_1]E[X_3]$

$= 1/16 - 1/16 = 0.$

$\therefore \; Var(X) = \frac{3N}{16} + 2N\,Cov(X_1, X_2) + 2N\,Cov(X_1, X_3)$

$= \frac{3N}{16} + 2N \times \frac{1}{16} + 2N \times 0$

$= \boxed{\frac{5N}{16}}$

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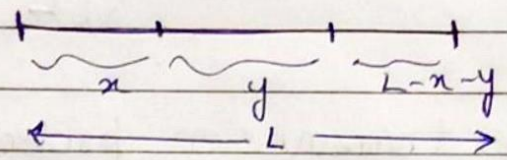
Ans'll Let us assume, the length of the 1st piece be x and the piece obtained after 2nd cut ~~be~~ has length y.

$\therefore$ Length of third piece $= l-(x+y)$.

Applying Δ inequality $A+B>C$, we have

$x+y > L-(x+y)$; $L-y > y$; $L-x > x$

$\therefore \boxed{x+y > \frac{L}{2}}$; $\boxed{y < \frac{L}{2}}$; $\boxed{x < \frac{L}{2}}$



Let's say $x = k$, then since $y \le l/2$ and $x+y > l/2$ $\quad y > \frac{l}{2} - k$

$$\therefore y \in \left(\frac{l}{2} - k, \frac{l}{2}\right).$$

$$\therefore P(Y \mid x=k) = \frac{\frac{L}{2} - \left(\frac{L}{2} - k\right)}{L-k} \quad \left\{\frac{\text{Available interval}}{\text{Total interval}}\right\}$$

$$= \frac{k}{L-k}.$$

Now $P(\text{choosing } x \,\&\, y) = \int_0^{L/2} P(Y \mid x=k)\,(\text{PDF})\,dx$

$$P(\text{choosing}) = \int_0^{L/2} \frac{k}{L-k} \cdot \underbrace{\frac{dk}{L}}$$

Since for choosing x, we have the whole rod available.

$$\therefore P(\text{choosing}) = \int_0^{L/2} \frac{L-(L-k)}{(L)(L-k)}\,dk$$

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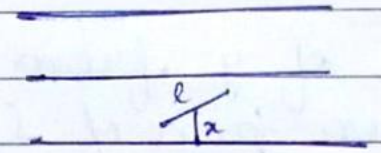
$$= \int_0^{L/2} \frac{dk}{L-k} - \int_0^{L/2} \frac{dk}{L}$$

$$= \left[-\ln(L-k)\right]_0^{L/2} - \left[\frac{k}{L}\right]_0^{L/2} = \boxed{\ln 2 - \frac{1}{2}}$$

And P(choosing) = P(triangle formation) because x and y were choosen like that.

Ans 12



Let x be the distance from the center of needle to closest ‖ line and θ be the acute angle b/w the needle and one of the ‖ lines.

$$\therefore f_X(x) = \begin{cases} 2/d & 0 \le x \le d/2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_\Theta(\theta) = \begin{cases} 2/\pi & 0 \le \theta \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

The two RV x and θ are independent

$$\therefore \quad f_{X,\Theta}(x,\theta) = f_X(x) \cdot f_\Theta(\theta)$$

$$= \begin{cases} 4/d\pi & 0 \le x \le d/2 \; ; \; 0 \le x \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

The needle will cross if $x \le \frac{l}{2} \sin\theta$

$$\therefore \text{Probability (Asked)} = \int_0^{\pi/2} \int_0^{l/2 \sin\theta} \frac{4}{d\pi} \, dx \, d\theta$$

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$$= \int_0^{\pi/2} \frac{4}{\pi d} \frac{l}{2} \sin\theta \, d\theta = \frac{2l}{\pi d} \left[-\cos\theta \right]_0^{\pi/2}$$

$$= \boxed{\frac{2l}{\pi d}}$$