Probability and Statistics: Lecture-38

Monsoon-2020

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» Union bound and extension...

Union Bound

Recall the inclusion exclusion principle:

Inclusion Exclusion → $|A \cup B| = |A| + |B| - |A \cap B|$

$|\cup_i A_i| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \cdots$

Union Bound

Recall the inclusion exclusion principle:

$$P((\cup_{i=1}^{n} A_i)) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P\left(\cap_{i=1}^{n} A_i\right)$$

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$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
≥ 0

Union Bound

$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

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$$P\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \quad \text{(Union Bound)}$$

» Generalized Union Bounds and Bonferroni Inequalities...

Generalized Union Bounds

Let $A_1, A_2, \ldots, A_n$ be events, then

$$P\left(\left(\cup_{i=1}^{n} A_i\right)\right) \leq \sum_{i=1}^{n} P(A_i)$$

(Union bound)

$$P\left(\left(\cup_{i=1}^{n} A_i\right)\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

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$$\cdots\cdots \quad \cdots \quad \cdots\cdots \quad \cdots\cdots$$

Generalized Union Bounds

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$$\ldots\ldots \quad \ldots \quad \ldots\ldots \quad \ldots\ldots$$

1. If we stop at the second term, we obtain a lower bound

Generalized Union Bounds

Let $A_1, A_2, \ldots, A_n$ be events, then

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$$\ldots\ldots \quad \ldots \quad \ldots\ldots \quad \ldots\ldots$$

1. If we stop at the second term, we obtain a lower bound
2. If we stop at the third term, we obtain an upper bound, etc

Generalized Union Bounds

Let $A_1, A_2, \ldots, A_n$ be events, then

(upper bound)

$$P((\cup_{i=1}^{n} A_i)) \leq \sum_{i=1}^{n} P(A_i)$$

1 term

(lower bound)

2 terms

$$P((\cup_{i=1}^{n} A_i)) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

3 terms (upper)

$$P((\cup_{i=1}^{n} A_i)) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$\ldots\ldots \quad \ldots \quad \ldots\ldots \quad \ldots\ldots$$

1. If we stop at the second term, we obtain a lower bound
2. If we stop at the third term, we obtain an upper bound, etc
3. In general, if we write an odd number of terms, we get an upper bound

Generalized Union Bounds

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$$\cdots\cdots \quad \cdots \quad \cdots\cdots \quad \cdots\cdots$$

1. If we stop at the second term, we obtain a lower bound
2. If we stop at the third term, we obtain an upper bound, etc
3. In general, if we write an odd number of terms, we get an upper bound
4. If we write an even number of terms, we get a lower bound

» Solved Example Using Union Bound...

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Example (Application of Union Bound)

Consider the random graph denoted $G(n, p)$, a graph with n nodes and p denotes the probability of an edge between pair of nodes.

Example (Application of Union Bound)

Consider the random graph denoted $G(n, p)$, a graph with n nodes and p denotes the probability of an edge between pair of nodes. Let B_n be the event that this graph has at least one node. Show that

$$P(B_n) \geq n(1-p)^{n-1} - \binom{n}{2}(1-p)^{2n-3}.$$

» Answer to previous problem...

Markov and Chebyshev Inequalities

If X is any nonnegative random variable, then

unique
|||
supremum $\equiv$ l.u.b

Markov and Chebyshev Inequalities

If X is any nonnegative random variable, then

$$P(X \geq a) \leq \frac{E[X]}{a}.$$

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$$P(\,|X - E[X]| \geq b\,) \leq \frac{\mathrm{Var}(X)}{b^2}$$

The above inequality is called Chebyshev inequality.

* Chebyshev inequality states that the difference between X and $E[X]$ is bounded by $\mathrm{Var}(X)$

Proof of Chebychev

Define $Y = (X - E[X])^2$ is non-negative

We can apply Markov. For any positive real no. b, we have

$$P(Y \geq b^2) \leq \frac{E[Y]}{b^2} \quad (\text{Markov})$$

$$E[Y] = E\left[(X - E[X])^2\right] = \mathrm{Var}(X)$$

$$\Rightarrow P(Y \geq b^2) = P\left((X - E[X])^2 \geq b^2\right)$$

$$= P\left(|X - E[X]| \geq b\right) \leq \frac{\mathrm{Var}(X)}{b^2}$$

» Example of Markov Inequality...

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Example (Markov Inequality)

Let $X \sim \text{Binomial}(n, p)$.

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Let $X \sim \text{Binomial}(n, p)$. Using Markov inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$.

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Let $X \sim \text{Binomial}(n, p)$. Using Markov inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Verify this bound for $p = 1/2$ and $\alpha = 3/4$.

X is non-negative RV. and $E[X] = np$. Applying Markov

$$P(X \geq \alpha n) \leq \frac{E[X]}{\alpha n} = \frac{\cancel{n}p}{\alpha \cancel{n}} = \frac{p}{\alpha}.$$

Verify $p = \frac{1}{2}, \alpha = \frac{3}{4} \Rightarrow P(X \geq \frac{3}{2}n) \leq \frac{1/2}{3/4} = \frac{2}{3}.$

» Answer to previous problem...

» Example of Chebychev Inequality...

Example (Chebychev Inequality)

Let $X \sim \text{Binomial}(n, p)$.

$$P(|X - E[X]| \geq b) \leq \frac{Var(X)}{b^2}$$

$$E[X] = np$$

Example (Chebychev Inequality)

Let $X \sim \text{Binomial}(n, p)$. Using Chebyshev inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$.

$$P(X \geq \alpha n) = P(X - np \geq \alpha n - np)$$

$$\leq P(|X - np| \geq n\alpha - np)$$

$$\leq \frac{Var(X)}{(n\alpha - np)^2} = \frac{np(1-p)}{n^2(\alpha - p)^2} = \frac{p(1-p)}{n(\alpha - p)^2}$$

Example (Chebychev Inequality)

Let $X \sim \text{Binomial}(n, p)$. Using Chebyshev inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Verify this bound for $p = 1/2$ and $\alpha = 3/4$.

Ex .

» Answer to previous problem...

» Chernoff Bounds...

Chernoff Bound

Let X be a random variable and $a \in \mathbb{R}$.

Moment generating fn $E[e^{sX}] = 1 + \frac{1}{sX} + \frac{1}{s^2X^2}$

Chernoff Bound

Let X be a random variable and $a \in \mathbb{R}$. Let $M_X(s) = E[e^{sX}]$ be the moment generating function.

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Let X be a random variable and $a \in \mathbb{R}$. Let $M_X(s) = E[e^{sX}]$ be the moment generating function. Then the following holds

$$P(X \geq a) \leq e^{-sa} M_X(s), \quad \text{for all } s > 0$$
$$P(X \leq a) \leq e^{-sa} M_X(s), \quad \text{for all } s < 0$$

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Since, the above holds for any s, we have the following

$X \geq a \iff sX \geq sa$

$a \geq b$ $\quad$ $\boxed{a \geq b > 0}$

$e^{sX} \geq e^{sa}$ $\quad$ $e^a \geq e^b$

Chernoff Bound

Let X be a random variable and $a \in \mathbb{R}$. Let $M_X(s) = E[e^{sX}]$ be the moment generating function. Then the following holds

$$\begin{cases} P(X \geq a) \leq e^{-sa} M_X(s), & \text{for all } s > 0 \\ P(X \leq a) \leq e^{-sa} M_X(s), & \text{for all } s < 0 \end{cases}$$

~~strict~~ strong

Since, the above holds for any s, we have the following

$s > 0$

$$\begin{cases} P(X \geq a) \leq \min\limits_{s>0} e^{-sa} M_X(s) \\ P(X \leq a) \leq \min\limits_{s<0} e^{-sa} M_X(s), \end{cases}$$

better bond

$s < 0$

» Answer to previous problem...

X, RV, $a \in \mathbb{R}$

We can write

$$P(X \geq a) = P(e^{sX} \geq e^{sa}) \text{ for } s > 0$$

Similar

$$P(X \leq a) = P(e^{sX} \geq e^{sa}) \text{ for } s < 0$$

Since e^{sX} is always pos. RV. $\forall s \in \mathbb{R}$

$$P(X \geq a) = P(e^{sX} \geq e^{sa})$$

$a < b < 0$

Markov

$$\leq \frac{E[e^{sX}]}{e^{sa}} = e^{-sa} \underbrace{E[e^{sX}]}_{M_X(s)}$$

Similarly, for $s < 0$

try!

» Example: Application of Chernoff Bound...

Example (Application of Chernoff bound)

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Let $X \sim$ Binomial(n,p). Find an upper bound for $P(X \geq \alpha n)$ using Chernoff bound.

Example (Application of Chernoff bound)

Let $X \sim$ Binomial(n,p). Find an upper bound for $P(X \geq \alpha n)$ using Chernoff bound. Assume $p < \alpha < 1$.

Sol^n: For $X \sim$ Binomial (n, p), $\underline{M_X(s)} = (pe^s + q)^n$, where $q = 1-p$

$$P(X \geq \alpha n) \leq \min_{s>0} e^{-sa} M_X(s) = \min_{s>0} \underbrace{e^{-sa}(pe^s + q)^n}_{= f(s)} \rightarrow (*)$$

Example (Application of Chernoff bound)

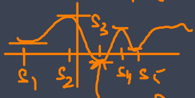
Let $X \sim$ Binomial(n,p). Find an upper bound for $P(X \geq \alpha n)$ using Chernoff bound. Assume $p < \alpha < 1$. Verify the bound for $p = 1/2$ and $\alpha = 3/4$.

We have $\min_{s>0} f(s)$

$$\frac{d}{ds} e^{-sa}(pe^s + q)^n = 0$$

$$\Rightarrow e^s = \frac{a\,q}{np(1-\alpha)}$$

Recall: (1) $f'(s) = 0$, solve for s to obtain critical points



$s \in \{s_1, s_2, s_3, s_4 \dots\}$

(2) If $f''(s)\big|_{s=s_i} > 0 \Rightarrow s = s_i$ is local minima

$f''(s)\big|_{s=s_i} < 0 \Rightarrow s = s_i$ is local maxima.

» Answer to previous problem...

From previous slide

$$e^{s} = \frac{a q}{n p(1-\alpha)}$$

(check that this is minima)

By using Chernoff bound.

$$P(X \geq \alpha n) \leq e^{-s\alpha n} \cdot (p e^{s} + q)^{n}$$

$$P(X \geq a) = e^{-sa} M_X(s)$$

» Comparison between Markov, Chebyshev, and Chernoff Bounds...

Example (Comparison between Markov, Chebyshev, and Chernoff Bound)

Let $X \sim \text{Binomial}(n, p)$.

Example (Comparison between Markov, Chebyshev, and Chernoff Bound)

Let $X \sim \text{Binomial}(n, p)$. Find the upper bounds for $P(X \geq \alpha n)$ using Markov, Chebyshev, and Chernoff bounds.

» Cauchy Schwarz Inequality...

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Linear Algebra ≡ Inner Product Space

$|\langle x,y\rangle| \le \sqrt{\langle x,x\rangle\langle y,y\rangle}$

C-S

Cauchy Schwarz Inequality

For any two random variables X and Y we have

$$E[XY] \le \sqrt{E[X^2]E[Y^2]}$$

where equality holds if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$.

$X = \alpha Y$

Left: Cauchy, Right: Schwarz

» Answer to previous problem...

Proof: Define RV $W = (X - \alpha Y)^2$

Clearly W is nonnegative RV for any α.

$\Rightarrow 0 \le E[W] = E[(X-\alpha Y)^2]$

$= E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2] = f(\alpha)$

If $\underline{f(\alpha) = 0}$, $E[W] = E[(X - \alpha Y)^2] = 0$

$\Rightarrow \underline{X} = \underline{\alpha Y}$ with Prob. 1.

To prove C-S., choose

$$\boxed{\alpha = \frac{E[XY]}{E[Y^2]}}$$

$\underline{\underline{0}} \le E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2]$

$= E[X^2] - 2\dfrac{E[XY]}{E[Y^2]} E[XY]$

$+ \dfrac{E[XY]^2}{\left(E[Y^2]\right)^{\cancel{2}}} \cancel{E[Y^2]}$

$\Rightarrow E[X^2] - \dfrac{E[XY]^2}{E[Y^2]} \ge 0$

$\Rightarrow E[XY]^2 \le E[X^2]\,E[Y^2]$

$\Rightarrow \underline{\underline{|E[XY]| \le \sqrt{E[X^2]\,E[Y^2]}}}$

» Answer to previous problem...

» Example of Cauchy Schwarz...

Example (Application of Cauchy Schwarz)

For any two random variables X and Y, show that

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$$|\rho(X, Y)| \leq 1$$

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$$|\rho(X, Y)| \leq 1$$

using Cauchy Schwarz inequality.

» Example of Cauchy Schwarz...

$$U = \frac{X - E[X]}{\sigma_x}, \quad V = \frac{Y - E[Y]}{\sigma_y}$$

$$E[U] = E[V] = 0, \quad Var(U) = Var(V) = 1$$

$$\Rightarrow |E[UV]| \leq \sqrt{E[U^2]E[V^2]} = 1$$

Example (Application of Cauchy Schwarz)

For any two random variables X and Y, show that

$$|\rho(X, Y)| \leq 1$$

using Cauchy Schwarz inequality. Furthermore, show that $|\rho(X, Y)| = 1$ if and only if $Y = aX + b$ for some constants $a, b \in \mathbb{R}$.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{E[XY] - E[X]E[Y]}{\sigma_x \sigma_y}$$

$$\rho(U,V) = \frac{E[UV] - \cancel{E[U]E[V]}^{\,0}}{1 \cdot 1} \leq 1$$