Ans) $\therefore$ $mgf_x(t) = \sum_x e^{tx} p(x)$

$$\therefore \sum_x e^{tx} p(x) = \frac{1}{10} e^{-20t} + \frac{1}{5} e^{-3t} + \frac{3}{10} e^{4t} + \frac{2}{5} e^{5t}$$

$\therefore e^{tx} p(x) = \frac{1}{10} e^{-20t}$ $\quad\therefore p(-20) = 1/10$

$e^{tx} p(x) = \frac{1}{5} e^{-3t}$ $\quad\therefore P(-3) = 1/5$

$e^{tx} p(x) = \frac{3}{10} e^{4t}$ $\quad\therefore p(4) = 3/10.$

$e^{tx} p(x) = \frac{2}{5} e^{4t}$ $\quad\therefore p(5) = 2/5$

$\therefore \boxed{P(|x| \le 2) =}$ ~~$P(-20) + P(-3) = \frac{1}{10} + \frac{1}{5} = \frac{3}{10}$.~~

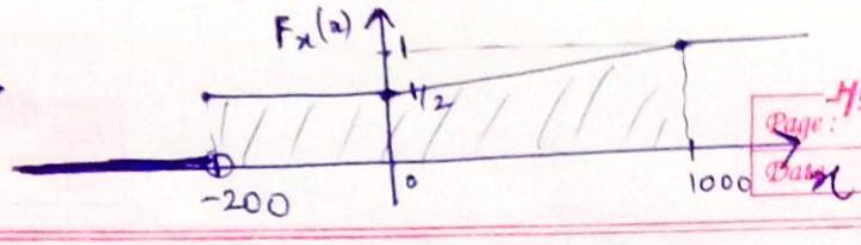
$\boxed{P(-2 \le x \le 2) = 0}$

An. 2 Min amount we can get is -200 and max amount $= 1000$.

Also note that at the end of game we can only obtain ~~denomination~~ profit $= -200, 0, 1, 2, \ldots 1000$.

We can't get any ~~dem~~ denomination b/w -200 and 0. (Exclusive).

$P(X = -200) = \frac{1}{2}$ (guessed incorrectly)



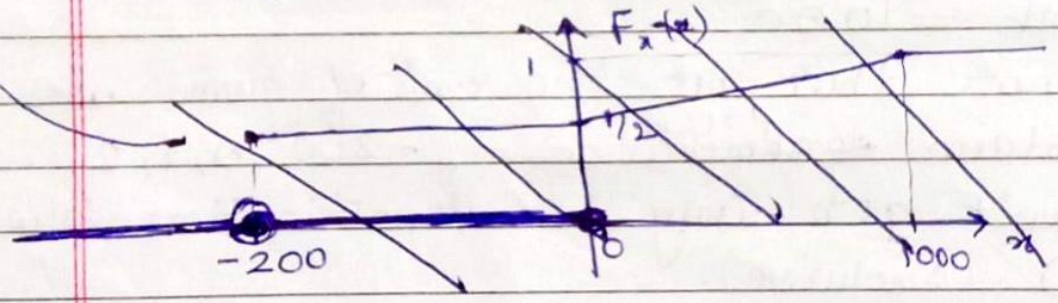
for $X \geq 0$ and $X \leq 100$

$$P(x \leq X) = \frac{1}{2} \times \frac{x-0}{1000-0} = \frac{x}{2000}$$

guessed correctly

$$\therefore F_x(x) = \begin{cases} 0 & x < -200 \\ 1/2 & -200 \leq x < 0 \\ \cancel{x = 200} & \\ \frac{1}{2} + x/2000 & 0 \leq x \leq 1000 \\ \cancel{0} \; 1 & x \geq 1000 \\ \cancel{\text{otherwise}} & \end{cases}$$

CDF of x



Probability of not winning atleast 500 rupees

$$= 1 - P(X \geq 500) = P(X < 500) = F(500)$$

$$\therefore F(500) = \frac{1}{2} + \frac{1}{2}\left(\frac{500-0}{1000-0}\right) = \frac{1}{2} + \frac{500}{2000}$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Ans 3 Given $P \sim \text{Uniform}(0, 3)$

$$\therefore \quad F(x) \begin{cases} \frac{x-0}{3-0} = x/3 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) \begin{cases} 1/3 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$Q = g(x) = \log\left(\frac{x}{3-x}\right) \qquad E[Q] = \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx$$

$(\because Q = g(x))$

$$E[Q] = \cancel{\int_{-\infty}^{0} g(x) \cdot f(x) \cdot dx}^{f(x)=0 \to 0} + \int_{0}^{3} g(x) \cdot f(x) \cdot dx + \cancel{\int_{3}^{\infty} g(x) f(x) \cdot dx}^{f(x)=0 \to 0}$$

$$E[Q] = \int_{0}^{3} \log\left(\frac{x}{3-x}\right) \cdot \frac{1}{3} \cdot dx$$

$$= \frac{1}{3}\left[\int_{0}^{3} [\log x - \log(3-x)]\, dx\right]$$

$$\cancel{E\left[\frac{1}{3}\right]} \int \log x \, dx = \int \underbrace{1}_{II} \cdot \underbrace{\log x}_{I} \, dx$$

By parts

$$= x \log x - x.$$

$$\int \log(3-x) \cdot dx = \int \underbrace{1}_{II} \cdot \underbrace{\log(3-x)}_{I} \cdot dx$$

By parts

$$\cancel{-(x-3)\log(3-x) + x}$$

$$= (3-x) - (3-x)\log(3-x)$$

$$\therefore E[Q] = \frac{1}{3}\left[x \log x - \cancel{x} + \cancel{(x-3)}(3-x)\log(3-x) + \cancel{x-3}\right]_0^3$$

$$= \frac{1}{3}\left[x \log x + (3-x)\log(3-x) \cancel{-3x}\right]_0^3$$

$$= \frac{1}{3}\left[3\log 3 \cancel{-3} - (0 + 3\log 3 - 3)\right]$$

$$= \cancel{\frac{1}{3}[-6] = -2} = \frac{1}{3}[0] = 0.$$

Ans. 4 Let X : waiting time of Ashish

$\therefore P(X=0) = 2/3$ (Ashish get's a ~~[illegible]~~ taxi waiting)

It is to note that bus arrives in every 5 min so Ashish do not have to wait for more than 5 min.

~~i.e. P(~~

$E[X] = ?$ $\quad \because f_x(x) = 1/30$ for $0 < x < 5$

$$\therefore E[X] = \underbrace{\int_0^5 x \cdot \frac{1}{30} \cdot dx}_{\text{continuous case}} + \underbrace{\frac{1}{2} \times \frac{2}{3} \times 0}_{\text{Taxi at } 0^{th} \text{ min}} + \underbrace{\frac{1}{2} \times \frac{1}{3} \times 5}_{\text{Bus}} = \frac{5}{12} + \frac{5}{6} = \frac{15}{12} = 1.25 \text{ min.}$$

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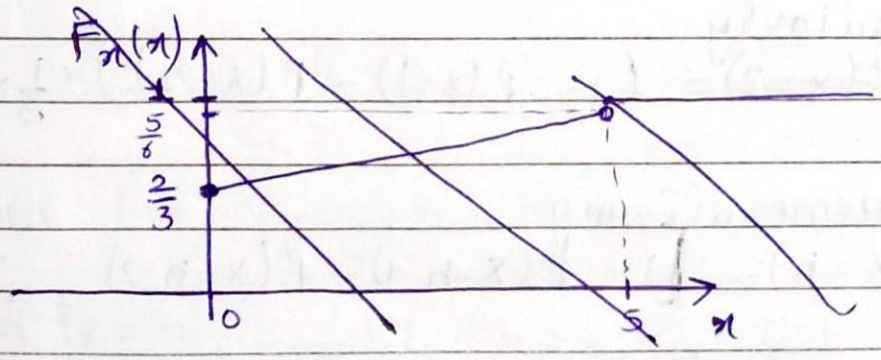
B/w 0 and 5 min there is only possibility that taxi can come.

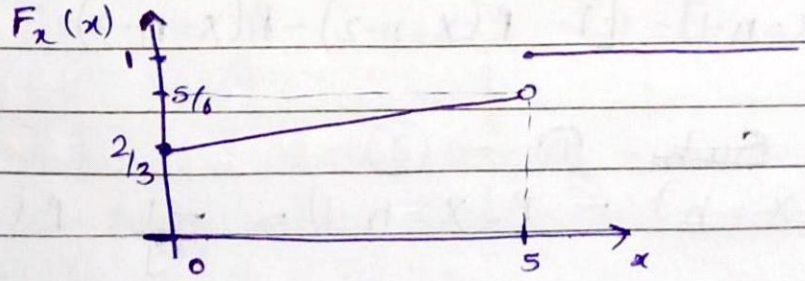
$$\therefore P(0 < X < 5) = \underbrace{\frac{1}{3}}_{\text{taxi is not waiting.}} \times \left(\frac{x-0}{10-0}\right) = \frac{x}{30}.$$

$\therefore$ Writing CDF of x

$$\therefore F_x(x) \begin{cases} 2/3 & x = 0 \\ 2/3 + x/30 & 0 < x < 5 \\ 1 & x \geq 5 \end{cases}$$

Because after waiting for more than 5 min we will definetely get bus (doesn't matter whether taxi comes or not).





$$E[X] = \underbrace{\frac{2}{3} \times 0}_{0^{th} \text{ min}} + \underbrace{\frac{1}{2} \times \frac{1}{3} \times 5}_{\text{Taxi do not comes first (Bus comes)}} + \underbrace{\frac{1}{2} \times \frac{1}{3} \times (2.5)}_{\text{Taxi comes first (Taken avg. of uniform(0,10) for 5 min interval)}} = 1.25 \text{ min.}$$

Ans 5 (a) X : No of rounds

P(X): Probability that the first time where a loss by Mahesh immediately follows a loss by Vinay in the "x^{th} round".

$\therefore$ ~~P(X=1)~~ Prob(loss) = 1/3 .

$\therefore$ $P(X=1) = 1/3 \times 1/3$ (Both Mahesh & Vinay loses)

$$P(X=2) = \underbrace{\left(1 - P(X=1)\right)}_{\text{M and V should not lose in 1st round}} \times \underbrace{\frac{1}{3} \times \frac{1}{3}}_{\text{Both M and V loses in 2nd round}}$$

Similarly

$$P(X=3) = \left(1 - P(X=1) - P(X=2)\right) \times \frac{1}{3} \times \frac{1}{3}$$

Generalizing

① $P(X=n) = \left[1 - P(X=n-1) - P(X=n-2) \cdots - P(X=1)\right] \times \frac{1}{3} \times \frac{1}{3}$

Similarly ② $P(X=n-1) = \left[1 - P(X=n-2) - P(X=n-3) \cdots - P(X=1)\right] \times \frac{1}{3} \times \frac{1}{3}$

~~Sub~~ ① − ②

$\therefore$ $P(X=n) - P(X=n-1) = -\frac{1}{9} \cdot P(X=n-1)$

$$\boxed{P(X=n) = \frac{8}{9} P(X=n-1)}$$

$$P(X=2) = \frac{8}{9} P(X=1)$$

$$\therefore \boxed{P(X=n) = \left(\frac{8}{9}\right)^{n-1} \cdot P(X=1) = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}}$$

$\therefore$ we obtain a general formula for $P(X)$.

$\therefore$ we can easily write PMF as

$$P(X) \begin{cases} 0 & X=0 \\ \frac{1}{9} \cdot \left(\frac{8}{9}\right)^0 & X=1 \\ \frac{1}{9} \cdot \left(\frac{8}{9}\right)^1 & X=2 \\ \vdots & \vdots \\ \frac{1}{9} \left(\frac{8}{9}\right)^{n-1} & X=n \end{cases}$$

(b) Z: Number of trials

P(Z): Probability that Mahesh has his third loss.

$P(Z=3) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$ (Mahesh loses in first 3 rounds) ~~loss, loss~~

$\hookrightarrow$ we have not written any probability term for Vinay as it doesn't actually matter.

$$P(Z=4) = \frac{1}{3} \times \left({}^3C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \right)$$

for the 4th round

Choosen 2 rounds out of 3 rounds in which Mahesh will lose

In one of the remaining round Mahesh wins

$$P(Z=5) = \frac{1}{3} \times \left({}^4C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \right).$$

Generalizing

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$$\cancel{P(X} \; P(Z=n) = \frac{1}{3} \times \left({}^{n-1}C_2 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^{n-3} \right)$$

Now we can write PMF of Z as

$$P(Z) \begin{cases} 0 & 0 \le Z < 3 \quad \cancel{Z=3} \quad \cancel{Z=0,1,2} \\ (1/3)^3 & Z=3 \quad \cancel{Z=3} \\ {}^3C_2 \cdot 2/3 \cdot (1/3)^3 & Z=4 \\ {}^4C_2 \, (2/3)^2 (1/3)^3 & Z=5 \\ \vdots & \\ {}^{n-1}C_2 \, (2/3)^{n-3} (1/3)^3 & Z=n \end{cases}$$

Ans. 5(c) $N \to$ Number of rounds until each of them won atleast once.

$E[N] = ?$

Now define 4 events: $E_1 \to$ Both win 1st round

$E_2 \to$ Vimal wins, Mahesh loses the 1st round

$E_3 \to$ Mahesh wins, Vinay loses 1st round

$E_4 \to$ Both lose 1st round.

$E(N|E_1) = 1$ { If they both win 1st round }

$E(N|E_2)$: If vimal wins 1st round, now we are only looking for Mahesh to win and stop.

$$E(N|E_2) = \underbrace{2 + \frac{2}{3}}_{\text{wins } 2^{nd} \text{ round}} + \underbrace{3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}_{\text{wins } 3^{rd} \text{ round}} + \underbrace{4\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)}_{\text{wins } 4^{th}} + \cdots$$

$$= \frac{2}{3} \times \sum_{i=2}^{\infty} i \times \left(\frac{1}{3}\right)^{i-2}$$

let $r = 1/3$

$$S = 2 + 3r + 4r^2 + 5r^3 + \cdots$$
$$-\quad rS = 2r + 3r^2 + 4r^3 + \cdots$$

$$S(1-r) = 2 + r + r^2 + r^3 + \cdots$$
$$S(1-r) = 2 + \frac{r}{1-r} = \frac{2-r}{1-r}$$

$$S = \frac{2-r}{(1-r)^2} \qquad r = 1/3 \qquad \therefore S = 15/4$$

$$\therefore E(N|E_2) = \frac{2}{3} \times \frac{15}{4} = \frac{5}{2}.$$

Similarly $E(N|E_3) = 5/2$

Now $E(N|E_4) = 1 + E[N]$

$$\therefore E[N] = E[N|E_1] \cdot P(E_1) + E(N|E_2) \cdot P(E_2) + E[N|E_3] \cdot P(E_3) + E(N|E_4) \cdot P(E_4)$$

$$\cancel{E[N] = 1 \cdot \left(\frac{2}{3} \times \frac{2}{3}\right) + \frac{5}{2}\left(\frac{1}{3} \times \frac{2}{3}\right) + (1 + E[N]) \times \frac{1}{3} \times}$$

$$E[N] = 1 \cdot \left(\frac{2}{3} \times \frac{2}{3}\right) + \frac{5}{2}\left(\frac{1}{3} \times \frac{2}{3}\right) + \frac{5}{2}\left(\frac{1}{3} \times \frac{2}{3}\right) + (1 + E[N])\left(\frac{1}{3} \times \frac{1}{3}\right)$$

$$E[N] = \frac{4}{9} + \frac{10}{9} + \frac{(1 + E[N])}{9}.$$

$$\therefore 9E[N] = 15 + E[N].$$

$$\therefore \boxed{E[N] = \frac{15}{8}}$$

Ans.6 500 pages contains 500 misprints on average

$\therefore$ 1 page contains 1 misprint on average.

$\therefore$ For one page $\lambda = 1$ (Considering Poisson process)

Asked $= P(k \geq 3)$

$= 1 - P(k < 3)$ ~~$(k \geq 0$ and~~

$= 1 - [P(k=0) + P(k=1) + P(k=2)]$

$\because P(k) = \dfrac{e^{-\lambda} \cdot \lambda^{k}}{k!}$ where $\lambda = 1$ and $k \geq 0$

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$\therefore \; P(k=0) = \dfrac{e^{-1} \cdot 1^{0}}{0!}$, $P(k=1) = \dfrac{e^{-1} \cdot 1^{1}}{1!}$. $P(k=2) = \dfrac{e^{-1} \cdot 1^{2}}{2!}$

$\therefore$ Asked $= 1 - \left(e^{-1} + e^{-1} + \dfrac{e^{-1}}{2} \right)$

$= 1 - \dfrac{5}{2e}$ ~~$= \dfrac{2e-5}{2e}$~~ .

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Ans 7 $M(t) = E[e^{tx}]$

a) for RV (kx)

$$mgf_{kx}(t) = E[e^{tkx}] = M(kt)$$

b) for RV $(k+x)$

$$mgf_{(k+x)}(t) = E[e^{t(k+x)}]$$
$$= e^{tk} E[e^{tx}]$$
$$= e^{tk} \cdot M(t).$$

c) $\sum_{i=0}^{N} x_i$

$$mgf_{(\sum x_i)}(t) = E[e^{t(\sum x_i)}] = M(t) \text{ only.}$$

d) $PDF(y) = PDF(x+k)$

$\Rightarrow CDF(y) = CDF(x+k)$

$\Rightarrow M(y) = M(x+k)$

$= E[e^{t(x+k)}]$

$= e^{tk} M(t)$

using uniqueness theorem

e) $PDF(y)$ $\quad M(y) = M(2x)$

$= E[e^{2xt}] = M(2t)$

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Ans. 8 Let's say for the ith person we define the random variable X_i.

$X_i = 1$: Person get's his own hat

$X_i = 0$: Person do not get's his own hat

$\therefore P(X_i = 1) = 1/n$ $\qquad P(X_i = 0) = 1 - 1/n$

$$E[X_i] = 1 \times \frac{1}{n} + 0 \times \left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

$\because$ There are n persons

$\therefore X = X_1 + X_2 + \cdots X_n$

$\therefore E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times \frac{1}{n} = 1.$

Ans 9 $f(y) = \dfrac{f(x_1)}{g'(x_1)}$ where $y = g(x) = 6x - 3$

$\therefore \quad x = \dfrac{y+3}{6}$

$6X - 3 = Y = \Phi(x)$: differential at all points

$$f(y) = \frac{x_1/2}{6} = \frac{x_1}{12} = \frac{(y+3)/6}{12} = \frac{(y+3)}{72}$$

$\because \quad x \in [0, 2] \quad$ ~~p(x~~ $f_x(x) = x/2$.

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$x = \frac{y+3}{6}$ $\quad \therefore \quad 0 \leq \frac{y+3}{6} \leq 2$

$\therefore \quad -3 \leq y \leq 9$

$$\therefore \quad f_y(y) \quad \begin{cases} (y+3)/72 & -3 \leq y \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

Ans: 10 Whenever time b/w arrival of 2 cars $> \tau$ he will make a U-turn.

Now if time b/w 2 cars $> t = P(0 \text{ cars } t \text{ unit})$

$$= \frac{e^{-\lambda t} \cdot (\lambda t)^0}{\lfloor 0}$$

$$= e^{-\lambda t}.$$

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time b/w 2 cars $< t$ = P(at least 1 car in t interval)

$$= 1 - e^{-\lambda t}$$

Now for a car before U turn.

Time b/w (i) and $(i-1)$ $< \tau$ $(i \le n)$
Time b/w $(n+1)$ and n^{th} car $> \tau$.

$$P(x = N) = \underbrace{(1-e^{-\lambda\tau})^n}_{\text{time diff less than } \tau} \times \underbrace{e^{-\lambda\tau}}_{\text{time diff} > \tau}$$

$$E[N] = \sum_{0}^{\infty} i \times (1-e^{-\lambda\tau})^i \cdot e^{-\lambda\tau}$$

$$= e^{-\lambda\tau} \cdot \underbrace{\sum_{0}^{\infty} i \cdot (1-e^{-\lambda\tau})^i}_{\text{simply an AGP}}$$

$$S = r + 2r^2 + 3r^3 + \ldots$$
$$- \quad rS = \quad r^2 + 2r^3 + \ldots$$
$$(1-r)S = r + r^2 + r^3 = \frac{r}{1-r} \qquad S = \frac{r}{(1-r)^2}$$

$$\therefore S = \frac{r}{(1-(1-e^{-\lambda\tau}))^2} = \frac{1-e^{-\lambda\tau}}{(1-(1-e^{-\lambda\tau}))^2}$$

$$S = \frac{1-e^{-\lambda\tau}}{e^{-2\lambda\tau}}$$

$$S = e^{2\lambda\tau} - e^{\lambda\tau}$$

$$E[N] = e^{-\lambda\tau}(e^{2\lambda\tau} - e^{\lambda\tau})$$

$$\therefore \boxed{E[N] = e^{\lambda\tau} - 1}$$

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Ans 11 (a) Let 2 ballons of same color are shot at the xth turn.

$\therefore$ $(x-1)^{th}$ shots should be distinct color.

$x-1$ $\quad$ $x-1$ $\quad$ n

choosing $(x-1)$ distinct colors and permutating $\rightarrow$ ${}^{n}P_{x-1}$.

Probability of choosing each $= 1/n$.

for last slot $(x-1)$ possible colors are there

$\therefore$ $P(\text{last slot}) = (x-1)/n$.

$$PMF_x = \frac{{}^{n}P_{x-1} \quad (x-1)}{(n)^{x-1} \cdot (n)}$$

$$PMF(X) \begin{cases} \dfrac{{}^{n}P_{x-1} \cdot (x-1)}{n^x} & x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) i colors have been shot already.

Let at time y_i unique color is shot

$$\Rightarrow (i/n)^{y_i - 1} \times \left(\frac{n-i}{n}\right)$$

one of i is choosen $\hookrightarrow$ Different color is choosen

$$PMF\ y_i \begin{cases} \left(\dfrac{i}{n}\right)^{y_i - 1} \left(\dfrac{n-i}{n}\right) & y_i \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

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Ans 12 Let T_A be the time after 1 pm (in hrs) when you leave pt. A
and T_B be the time when our friend ~~leaves~~ leaves pt. B.

$\therefore 0 \le T_A, T_B \le 1$

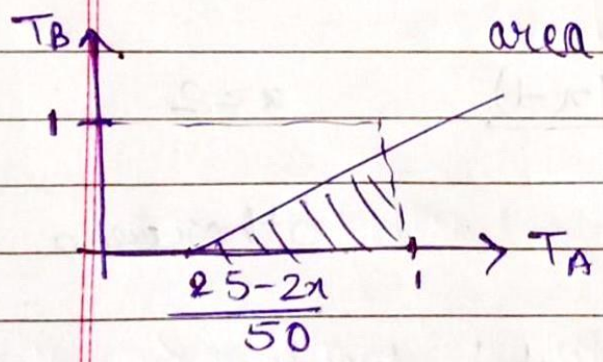
$\therefore$ ~~$0 \le x \le 12.5$ where~~

Case (1) $T_A \ge T_B$

$\therefore 0 \le x \le 12.5$ where x is dist. in kms. from pt. A at which both of you meet

$$\therefore T_A - T_B = \frac{25-2x}{50} \qquad T_B = T_A + \frac{2x-25}{50}$$

$\therefore P(X \le x) = F_x(x)$ is represented by the area under curve.



$$\therefore F_x(x) = \frac{1}{2}\left[1 - \left(\frac{25-2x}{50}\right)\right]^2$$

$$= \frac{1}{2}\left(\frac{25+2x}{50}\right)^2$$

Case (2) $T_B > T_A$

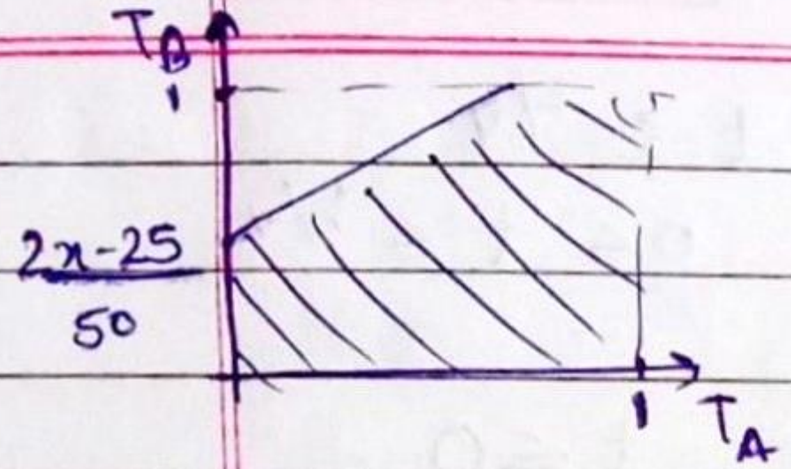
$\therefore 25 \ge x > 12.5$

Again $T_B = T_A + \frac{2x-25}{50}$

$\therefore P(X \le x) = F_x(x)$ is represented by the shaded area.

$$\therefore F_x(x) = 1 - \frac{1}{2}\left[1 - \left(\frac{2x-25}{50}\right)\right]^2$$

$$= 1 - \frac{1}{2}\left(\frac{25-2x}{50}\right)^2$$



Hence

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}\left(\frac{25+2x}{50}\right)^2, & 0 \le x \le 12.5 \\ 1 - \frac{1}{2}\left(\frac{25-2x}{50}\right)^2 & 12.5 \le x \le 25 \end{cases}$$

Ans 13 $\lambda_1 = 0.25$ years $\lambda_2 = 0.5$ years

(1) Exactly 3 bulbs replacements.

$\Rightarrow$ Acc. to poisson the time t for which A and B bulb work, the prob. of t is :-

$$\frac{(\lambda_1)^t e^{-\lambda_1}}{t!}, \frac{(\lambda_2)^t e^{-\lambda_2}}{t!} \text{ respectively.}$$

Now 3 replacements

$$\text{Tot time} = \underbrace{t_{A_1} + t_{B_1}}_{1^{st}\text{ replacement } A \to B} + \overbrace{t_{A_2}}^{2^{nd}\text{ replacement } B \to A} + \underbrace{t_{B_2}}_{3^{rd}\text{ replacement } A \to B}$$

$$= \frac{(\lambda_1)^{t_{A_1}} \cdot e^{-\lambda_1}}{(t_{A_1})!} + \frac{(\lambda_2)^{t_{B_1}} e^{-\lambda_2}}{(t_{B_1})!} + \frac{(\lambda_1)^{t_{A_2}} e^{-\lambda_1}}{(t_{A_2})!} + \frac{(\lambda_2)^{t_{B_2}} \cdot e^{-\lambda_2}}{(t_{B_2})!}$$

All these 4 terms are mutually independent, as an A bulb will not affect B and vice versa and two A bulbs are also distinct.

$\therefore$ E(total time) $= E_{A_1} + E_{B_1} + E_{A_2} + E_{B_2}$

For a poisson distribu^n ~~variation~~ expectation $= \lambda$. (average)

$$\therefore \text{Expectation} = \lambda_A + \lambda_B + \lambda_A + \lambda_B$$
$$= 2(\lambda_A + \lambda_B)$$
$$= 2(0.25 + 0.5) = 1.5 \text{ years.}$$

(ii) Let us define $T_{n,A}$ as total time such that I can use n bulbs starting from bulb of type A and then I can use both A and B type with a prob. of p and (1-p), similarly we can define $T_{n,B}$ and let

$$F_1 = \frac{(\lambda_1)^{t_1} e^{-\lambda_1}}{t_1!} \quad \text{and} \quad F_2 = \frac{(\lambda_2)^{t} \cdot e^{-\lambda_2}}{t_2!}$$

$$\therefore T_{n,A} = F_1 + \underbrace{p(T_{n-1,A})}_{} + (1-p)\cdot(T_{n-1,B}) \rightarrow (1)$$

Rem. (n-1) bulbs and since I choose p $\therefore$ A must be the starting point

Similarly

$$T_{n,B} = F_2 + p(T_{n-1,A}) + (1-p)(T_{n-1,B}) \rightarrow (2)$$

(1) - (2)

$$\therefore \left.\begin{aligned} T_{n,A} - T_{n,B} &= F_1 - F_2 \\ T_{n-1,A} - T_{n-1,B} &= F_1 - F_2 \end{aligned}\right\} \text{substituting } n \text{ with } (n-1).$$

Replacing $T_{n-1,B}$ in (1)

$$T_{n,A} = F_1 + p(T_{n-1,A}) + (1-p)(T_{n-1,A} + F_2 - F_1);$$

$$T_{n,A} = F_1 + p(T_{n-1,A}) + T_{n-1,A} + F_2 - F_1 - p\cdot T_{n-1,A} - pF_2 + p\cdot F_1$$

$$T_{n,A} - T_{n-1,A} = F_2 + p(F_1 - F_2)$$

where RHS is a const. $\therefore$ $T_{n,A}$'s are in AP

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$T_{1,A} = F_1$ { straight forward, just a single bulb }
n replacement means (n+1) bulbs, ∴ $(n+1)^{th}$ term of AP is

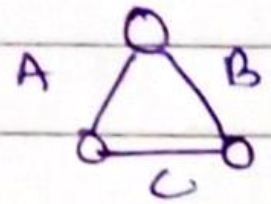
$$T_{n+1,A} = F_1 + n(F_2 + p(F_1 - F_2))$$

$$\therefore \text{Expectation} = \lambda_1 + n(\lambda_2 + p(\lambda_1 - \lambda_2))$$

Because all F_1, F_2, p, n are independent from each other where p, n are constants.

Unit 4 Probability of an edge being non-monochromatic = 2/3

a) Counter-Example:



Given A & B be monochromatic, then C has to be monochromatic by default.

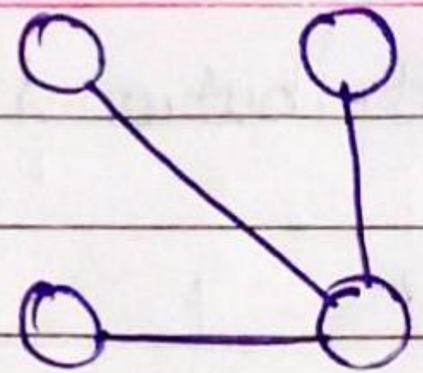
b) $f(e) \begin{cases} 0 & e \text{ is non-chromatic} \\ 1 & \text{chromatic} \end{cases}$

$E[\text{No of non-chromatic edges}] = E\left[\sum_{e \in E} f(e)\right]$

From linearity of expectation

$= \sum_{e \in E} E[f(e)]$ → probability of e being non monochromatic

$= |E| \cdot \frac{2}{3}$

c.) 

Let's color the graph to get more than $2/3^{rd}$ edges are non-monochromatic

- If 3 neighbours have at most 2 different colors then make this node the other color.

- Else choose one with the least occurence in the neighbourhood. This will make it $2/3^{rd}$ at minimum different from others.

Ans 15 Let X denote the posn of ambulance and Y denote the position of the accident.

$X, Y \sim \text{uniform}(0, L)$

Let $Z = |X - Y| = vt$

Now let's calculate CDF.

$$F_Z(z) = P(|X-Y| < z) = P(-z < X - Y < z)$$

$$= \int_0^L P(y - z < X < y + z \mid Y = y)\, f_Y(y) \cdot dy$$

$f_Y(y) = 1/L$ as uniformly distribn over $[0, L]$

$$= \int_0^z \left(\frac{y+z}{L}\right) \cdot \frac{1}{L} \cdot dy + \int_z^{L-z} \frac{2z}{L} \cdot \frac{1}{L} \cdot dy + \int_{L-z}^{L} \left(\frac{L+z-y}{L}\right) \cdot \frac{dy}{L}$$

$$= \frac{3z^2}{2L^2} + \frac{(L-2z)(2z)}{L^2} + \frac{3z^2}{2L^2}$$

$$= \frac{2Lz - z^2}{L^2} = \frac{2vt}{L} = \frac{v^2 t^2}{L^2}$$

$$\therefore F_Z(z) \begin{cases} 1 & z \geq L \\ \dfrac{2Lz - z^2}{L^2} & 0 < z < L \\ 0 & z \leq 0 \end{cases}$$

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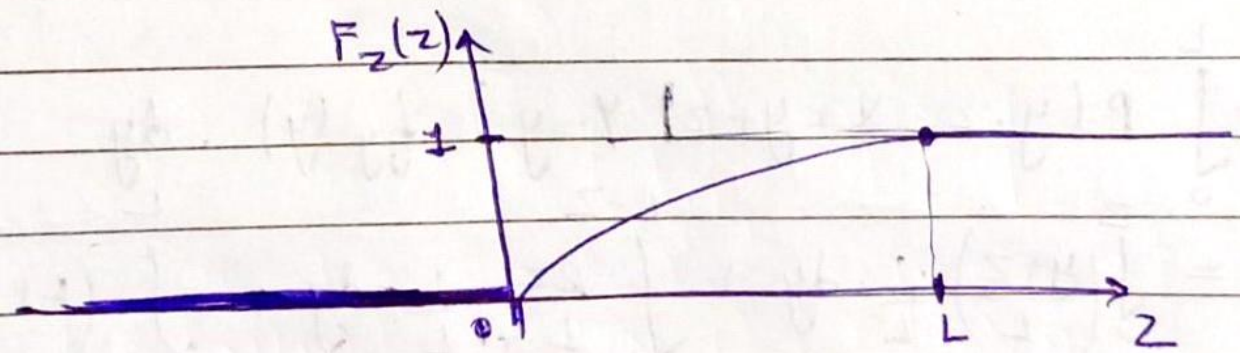
$$F_T(t) = \begin{cases} 1 & t \geq L/v \\ \dfrac{2Lvt - v^2t^2}{L^2} & 0 < t < L/v \\ 0 & t \leq 0 \end{cases}$$

$f_T(t)$ can be calculated by differentiating the CDF.

$$\frac{d(F_T(t))}{dt} = \frac{2v}{L} - \frac{2v^2t}{L^2} \qquad t \in \left[0, \frac{L}{v}\right]$$

$$f_T(t) = \begin{cases} \dfrac{2(vL - v^2t)}{L^2} & 0 \leq t \leq L/v \\ 0 & \text{otherwise} \end{cases}$$



Ans: 16 a) Getting 2 tails is independent of getting 2 tails again.

$\therefore$ Required prob = 2 trials of 2 tails

$$= (1/2)^2 \cdot (1/2)^2 = \frac{1}{2^{2 \cdot 2}}$$

b) After firing the 2nd success we have k ways to place 1st success.

Success $\rightarrow$ 2 heads or 2 tails

$$\Rightarrow 2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{2^{2-1}}$$

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$$\text{Failure} = \left(1 - \frac{1}{2^{z-1}}\right).$$

$$PMF_k = {}^{k}C_1 \times \left(1 - \left(\frac{1}{2}\right)^{z-1}\right)^{k-1} \cdot \left(\frac{1}{2}\right)^{z-1} \times \left(\frac{1}{2}\right)^{z-1}$$

Ans (ii) $X_i \rightarrow$ No of tails in 1 unsuccessfull trial

Now $S \rightarrow$ 8 - {HHH, TTT}

$$E[X_i] = 1 \times 3/6 + 2 \times 3/6 = 9/6 = 3/2.$$

Permutaⁿ of THH Permutaⁿ of HHT

$$Var(x_i) = E[X_i^2] - (E[X_i])^2$$
$$= \left(1^2 \times \frac{3}{6} + 2^2 \times \frac{3}{6}\right) - \left(\frac{3}{2}\right)^2 = \frac{1}{4}.$$

Now for N

$$P(N=n) = (3/4)^n \cdot (1/4)$$

This is a geometric random variable $p = 1/4$.

$$\therefore E[N] = 1/p = 4. \qquad Var[N] = \frac{1-p}{p^2} = 12.$$

Now $M = x_1 + x_2 + \cdots x_n$

Now $E(M/N=n) = E(x_1 + x_2 + \cdots x_n)$

$$= E[X_1] + E[X_2] + \cdots E[X_n]$$

Now $E(M/N=0) = n \times E[X_1] = 3n/2.$

$\therefore$ From law of total expectation

$$E(M) = \sum_0^\infty P(N=i) \times E(M/N=i)$$

$$= \sum_0^\infty \left(\frac{3}{4}\right)^i \cdot \frac{1}{4} \cdot \frac{3i}{2} = \frac{3}{8} \underbrace{\sum_0^\infty i \cdot (3/4)^i}_{AGP}.$$

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$= \frac{3}{8} \times 12$ This A.G.P. sum $S = \frac{r}{(1-r)^2}$ where $r = 3/4$

$\therefore E(M) = \frac{3}{8} \times 12 = 4.5$

Now,

$Var(M \mid N=i) = Var(x_1 + x_2 \ldots x_n)$

$= i \times Var(x_i)$

$= i \times 1/4 = i/4.$

$Var(M) = E(Var(M/N)) + Var(E(M/N))$

Law of total expectation

$E(Var(M/N)) = \sum_0^\infty P(N=i) \cdot Var(M \mid N=i)$

$= \sum_0^\infty \left(\frac{3}{4}\right)^i \times \frac{1}{4} \times \frac{i}{4} = \frac{1}{16} \times 12 = \frac{3}{4}.$

$Var(E(M/N)) = \sum_0^\infty P(N=i) \left(E(M/N=i) - E(M/N)\right)^2$

$= \sum_0^\infty P(N=i) \left(E(M/N=i) - E(M)\right)^2$

$= \sum_0^\infty P(N=i) \left(E(M/N=i)^2 + E(M)^2 - 2E(M/N=i)E(M)\right)$

$= \sum_0^\infty P(N=i) \left[\frac{9i^2}{4} + \frac{81}{4} - \frac{27i}{2}\right]$

$= \frac{1}{16} \sum_0^\infty \left(\frac{3}{4}\right)^i (9i^2 + 81 - 54i)$

$Var(M) = E(Var(M/N)) + Var(E(M/N))$

$= \frac{3}{4} + \frac{432}{16} = 27.75.$

c) Let X_M be trial with m coins

$E[X_m] = \frac{1}{p}$ and here $p = \left(\frac{1}{2}\right)^{m-1}$

$E[X_m] = 2^{m-1}$

Now $X = X_m + X_{m-1} + \cdots X_2$

$E[X_1] = E[X_m] + E[X_{m-1}] + \cdots E[X_2]$

$= \sum_{i=2}^{n} 2^{i-1} = 2^M - 2.$

Ans 17 a) 225 different colors = m

Let X_i be a indicator variable which is 0 when color is chosen else 1.

~~i ∈~~ $i \in [1, 225]$.

So now, no of colors not chosen $= X_1 + X_2 + \cdots X_m$

$$E[X_i] = 1 \times \left(1 - \frac{1}{m}\right)^k.$$

$$E\left(\sum_{j=1}^{n} X_j\right) = \sum_{i=1}^{n} E(X_i) = m\left(1 - \frac{1}{m}\right)^k$$

E[No of colors assigned to k balls]

$$= m - m\left(1 - \frac{1}{m}\right)^k = m\left[1 - \left(1 - \frac{1}{m}\right)^k\right].$$

b) E[No of colors assigned to ~~each~~ more than 1 ball]

$$= m - E[\text{No assigned to 0 ball}] - E[\text{No assigned to 1 ball}]$$

$$E[\text{No of colors assigned to 0 ball}] = m\left(1 - \frac{1}{m}\right)^n$$

Calculated in part (a)

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Now calculate E[No. of colors assigned to 1 ball]

$$P(i^{th} \text{ color assigned to 1 ball}) = \frac{(m-1)^{n-1} \times n}{m^n}$$

$$= \frac{1}{m}\left(1-\frac{1}{m}\right)^{n-1} \times n$$

$\therefore$ E[No. of colors assigned to 1 ball]

$$= m \times \frac{n}{m}\left(1-\frac{1}{m}\right)^{n-1} = \left(1-\frac{1}{m}\right)^{n-1} \times n$$

$\therefore$ E[Required] $= m - m\left(1-\frac{1}{m}\right)^{n} - \left(1-\frac{1}{m}\right)^{n-1} \times n$

c) $m - m\left(1-\frac{1}{m}\right)^{n} - \left(1-\frac{1}{m}\right)^{n-1} \times n > 1$

$$m - \left(1-\frac{1}{m}\right)^{n-1}\left[m-1+n\right] > 1$$

$$\therefore \quad m-1 > \left(1-\frac{1}{m}\right)^{n-1}(m+n-1)$$

$$\left(\frac{224}{225}\right)^{n-1}(224+n) < 224$$

$$\left(\frac{224}{225}\right)^{n-1} < \frac{224}{224+n}$$

Using C++ code we get the value of n as 23

$(n \geq 23)$.

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Obs. 18 $P(\text{accepted}) = p$ $\quad P(\text{Not accepted}) = 1-p = q.$

Mean $= E[X]$ $\quad X$: No of trials needed

$X \in [1, \infty)$

$\therefore E[X] = \sum_{x=1}^{\infty} x \cdot P(x) = (\text{a/c in } 1^{st} \text{ attempt}) + (\text{a/c in } 2^{nd} \text{ attempt}) + \cdots$

$E[X] = 1 \times p + 2 \times q \times p + 3 \times q \times q \times p + 4 \times q \times q \times q \times p + \cdots \infty$

$= p[1 + 2q + 3q^2 + 4q^3 + \cdots \infty]$

$S = 1 + 2q + 3q^2 + 4q^3 + \cdots$ ← A.G.P.

Subtract $qS = q + 2q^2 + 3q^3 + \cdots$

$(1-q)S = 1 + q + q^2 + q^3 + \cdots$

$(1-q)S = \frac{1}{1-q}$ $\quad 0 < q < 1$

$\therefore S = \frac{1}{(1-q)^2}$

$\therefore \cancel{\boxed{E[X] = \frac{p}{(1-q)^2}}}$ $\quad E[X] = p \cdot \frac{1}{(1-q)^2} = \frac{p}{(1-q)^2}$

$\because 1 - p = q \quad \therefore 1 - q = p$

$\therefore \boxed{E[X] = \frac{p}{p^2} = \frac{1}{p}}$

$V[X]$: variance of X

$\therefore V[X] = \sum_{x=1}^{\infty} x^2 \cdot p(x) - \sum_{x=1}^{\infty} x\, p(x) = E[X^2] - (E[X])^2.$

$E[X^2] = 1^2 \cdot p + 2^2 \cdot q \cdot p + 3^2 \cdot q^2 p + \cdots$

$E[X^2] = p[1^2 + 2^2 q + 3^2 q^2 + 4^2 q^3 + \cdots \infty]$

Let $S = 1^2 + 2^2 q + 3^2 q^2 + 4^2 q^3 + \cdots \infty$

$qS = 1^2 q + 2^2 q^2 + 3^2 q^3 + \cdots \infty$

$(1-q)S = 1 + 3q + 5q^2 + 7q^3 + \cdots \infty$

Subtract $q(1-q)S = q + 3q^2 + 5q^3 + \cdots$

$(1-q)^2 S = 1 + 2q + 2q^2 + 2q^3 + \cdots \infty$

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$$(1-q)^2 S = 1 + 2q(1 + q + q^2 + \cdots \infty)$$

$$(1-q)^2 S = 1 + \frac{2q \cdot 1}{1-q}$$

$$(1-q)^2 S = \frac{1-q+2q}{1-q} = \frac{1+q}{1-q}.$$

$$\therefore S = \frac{1+q}{(1-q)^3}$$

$$E[X^2] = p \cdot \frac{(1+q)}{(1-q)^3} = \frac{p \cdot (2-p)}{p^3} = \frac{2-p}{p^2}$$

$$V[X] = E[X^2] - (E[X])^2$$

$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \quad \therefore \quad \boxed{V[X] = \frac{1-p}{p^2}}$$

Ans 19 Student arrive with poisson distribution λ.
in t time

$$\text{avg. No of students} = \lambda t - \frac{t}{\text{Avg. time for sanitization}}$$

$$\text{Avg time for sanitization} = \sum_{i=1}^{n} i p_i$$

$$\text{Avg. no of students in the queue} = \lambda t - \frac{t}{\sum i p_i} = \lambda_{new}$$

$$PMF(N_t) = \left\{ \frac{e^{-\lambda_{new}} (\lambda_{New})^N}{\lfloor N} \right.$$

where N: no of students.

Ans.20 given $g(x) = y = x^{1/2}$

$$\therefore f_y(y) = \frac{f_x(x_1)}{g'(x_1)} = \frac{e^{-x_1}}{\frac{1}{2} \cdot x_1^{-1/2}}$$

$$f_y(y) = 2x_1^{1/2} \cdot e^{-x_1}$$

$y = x^{1/2}$ ~~$x = y^2$~~ → differentiable at points where $x > 0$ (for $x < 0$ it is not defiene

$$\therefore f_y(y) = 2(y^2)^{1/2} \cdot e^{-y^2}$$

$$\boxed{f_y(y) = 2ye^{-y^2}}$$

~~As~~ $x \in [0, \infty]$ ~~$\because x = y^2 \therefore y \in R$~~

$\because y = x^{1/2} \quad \therefore y \in [0, \infty]$

$$\therefore f_y(y) \to \begin{cases} 2ye^{-y^2}, & y \in R^+ \\ 0 & \text{otherwise} \end{cases}$$

~~only 1~~

But at $x = 0$ $y = x^{1/2}$ is not differentiable $\therefore$ we have to exclude $x = 0$