# Probability and Statistics: Lecture-3

## Monsoon-2020

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on August 14, 2020

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## » Combinations with or without repetitions...

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- Let us try to find out...

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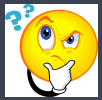
**Question**

So, what could be your answer?



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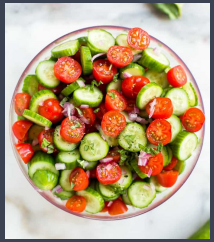

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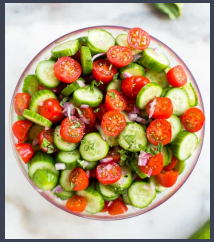

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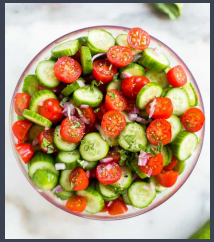

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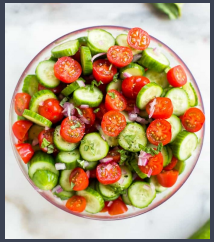

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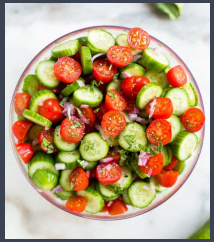

- We pick 4 items out of 3 options with repetitions
- Order does not matter; Still do not know how to count
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- But we want to do it wisely!

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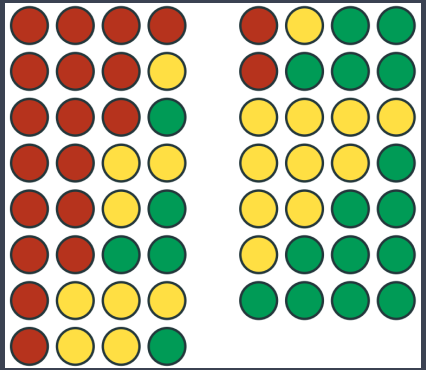

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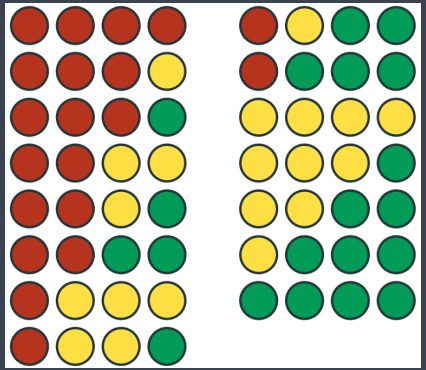

* There are 15 possible combinations. Do we see any structure?

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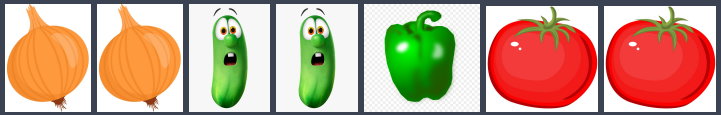

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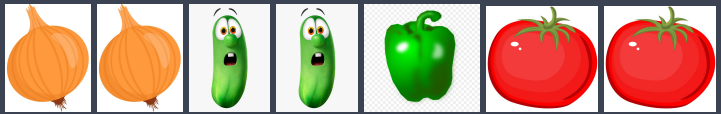

* Do you already see a way to find all possible combinations?

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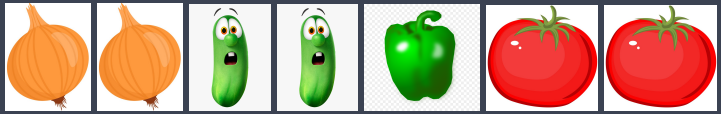

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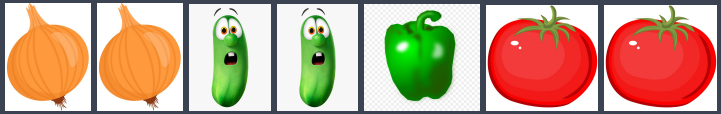

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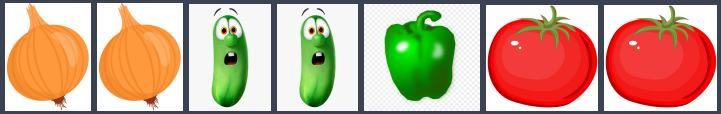

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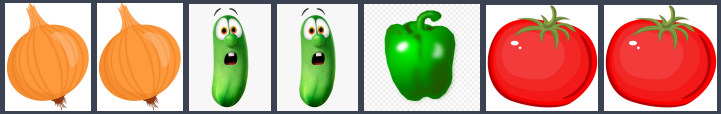

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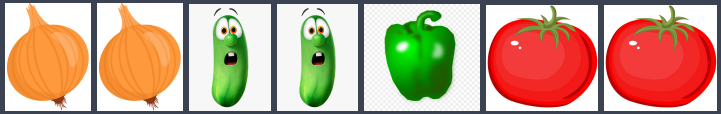

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* The number of ways we can put the delimiters determine the number of combinations

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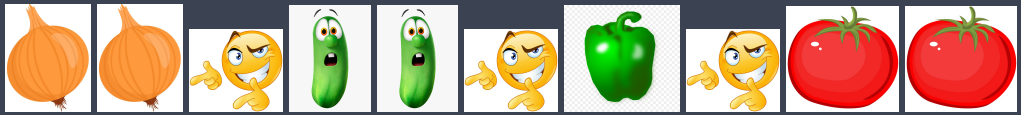

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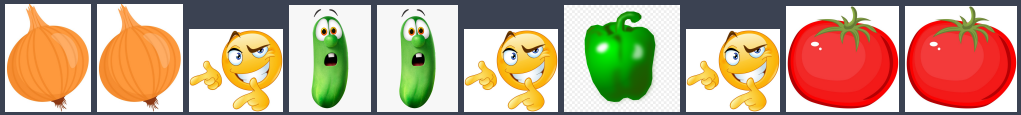

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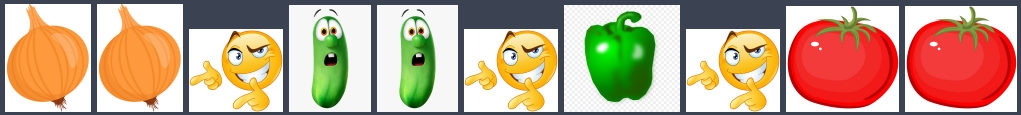

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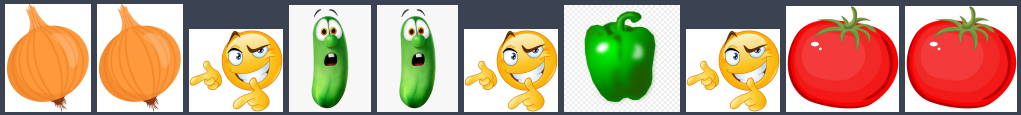

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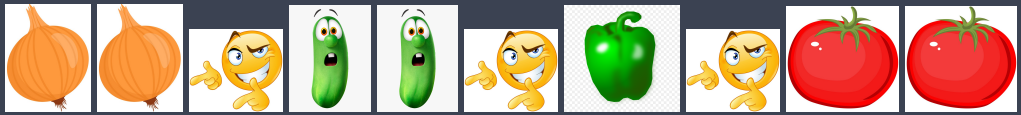

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- The problem now reduces to arranging 3 delimiters among 10 items! Voila!

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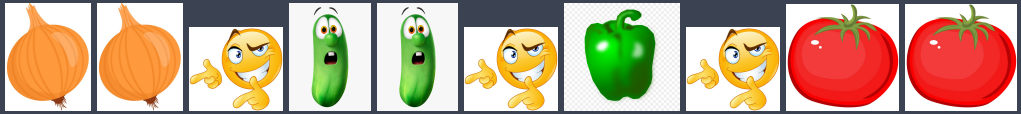

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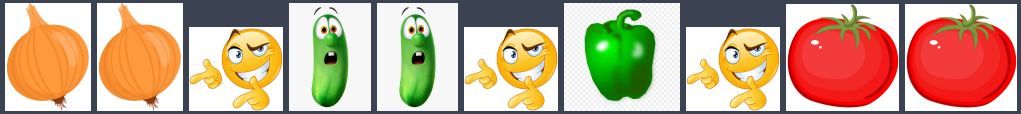

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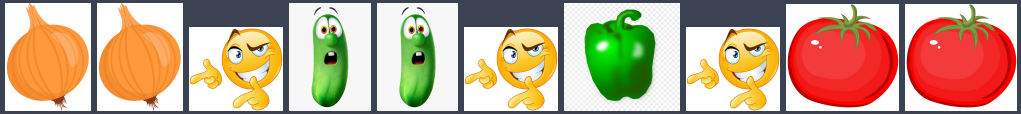

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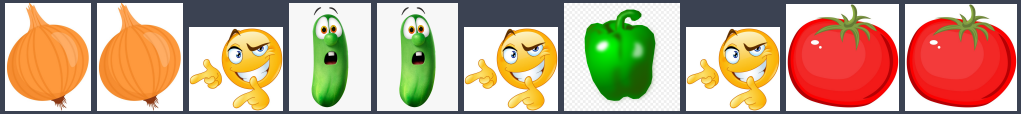

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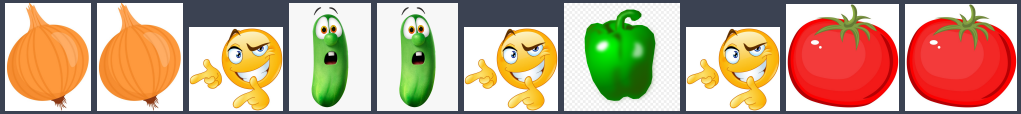

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**Quiz**

Where does this problem fit in our combination and permutation table?

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- Here assignments and people are different: so a ordered case
- No repetitions: same assignment is not given to two persons
- It is a case of ordered without repetitions, i.e., $k-$permutations

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* The answer is $\binom{9!}{(9-4)!} = 9 \times 8 \times 7 \times 6 = 3024$

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Which category this problem belongs to? 

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- First person can be assigned arbitrary number of assignments
  - It is a case of counting all possible subsets of assignments
- Same assignment can't be given to two persons, the number of subsets of assignments for second person depends on what we chose for first person! How to attack this problem?

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| Assignments | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Options | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

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* There are $4^9 = 262144$ choices! This was a case of Tuples!

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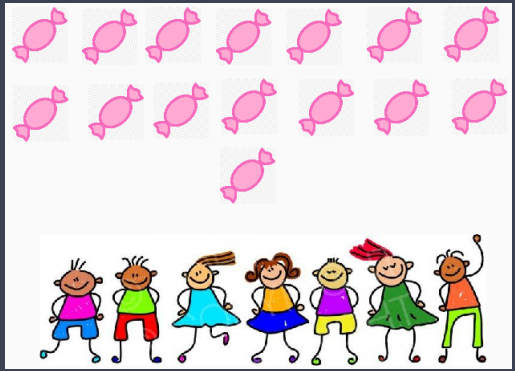

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- Since the candies to be distributed are identical, ordering doesn't matter
- This is a case of combinations with repetitions

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* Number of combinations = size of combination
* Number of options = Number of kids
* The answer is $\binom{15+(7-1)}{(7-1)} = \binom{21}{3} = 54264$