# Probability and Statistics: Lecture-2

## Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on August 12, 2020

# » Table of contents

# » Permutations

# » Permutations

**Definition**

Tuples of length $k$ without repetitions are called $k-$permutations

# » Permutations

**Definition**

Tuples of length $k$ without repetitions are called $k-$permutations

**Question**

## » Permutations

**Definition**

Tuples of length $k$ without repetitions are called $k-$permutations

**Question**

Suppose we have a set of $n$ symbols. How many different sequences of length $k$ we can form out of these symbols if we are not allowed to use the same symbol twice?

# » Permutations

**Definition**

Tuples of length $k$ without repetitions are called $k-$permutations

**Question**

Suppose we have a set of $n$ symbols. How many different sequences of length $k$ we can form out of these symbols if we are not allowed to use the same symbol twice?

* Observe that if $n < k$ then there are no $k-$permutations: there are simply not enough different letters

## » Permutations

**Definition**

Tuples of length $k$ without repetitions are called $k$–permutations

**Question**

Suppose we have a set of $n$ symbols. How many different sequences of length $k$ we can form out of these symbols if we are not allowed to use the same symbol twice?

* Observe that if $n < k$ then there are no $k$–permutations: there are simply not enough different letters
* So it is enough to solve the problem for the case $k \leq n$

# » Permutations

**Definition**

Tuples of length $k$ without repetitions are called $k-$permutations

**Question**

Suppose we have a set of $n$ symbols. How many different sequences of length $k$ we can form out of these symbols if we are not allowed to use the same symbol twice?

* Observe that if $n < k$ then there are no $k-$permutations: there are simply not enough different letters
* So it is enough to solve the problem for the case $k \leq n$

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & k \\ * & * & * & \cdots & * \\ n & n-1 & n-2 & & \end{array}$$

# » Permutations

**Definition**

Tuples of length $k$ without repetitions are called $k-$permutations

**Question**

Suppose we have a set of $n$ symbols. How many different sequences of length $k$ we can form out of these symbols if we are not allowed to use the same symbol twice?

* Observe that if $n < k$ then there are no $k-$permutations: there are simply not enough different letters
* So it is enough to solve the problem for the case $k \leq n$

| 1 | 2 | 3 | ... | $k$ |
|---|---|---|---|---|
| * | * | * | ... | * |
| $n$ | $n-1$ | $n-2$ | | |

* Hence there are

$$n \times (n-k) \times \cdots (n-k+1)$$

$k-$permutations, which is $n!/(n-k)!$

# » Permutation Examples

# » Permutation Examples

**Question**

In how many ways we can *arrange* $n$ different books in $n$ different bins on shelf?

# » Permutation Examples

**Question**

In how many ways we can *arrange* $n$ different books in $n$ different bins on shelf?

**Answer**

Hint: Use previous result with $k = n$.

## » Combinations

# » Combinations

**Question**

You are organizing a car journey.

# » Combinations

**Question**

You are organizing a car journey. You have five friends,

# » Combinations

**Question**

You are organizing a car journey. You have five friends, but there are only three vacant places in you car.

# » Combinations

**Question**

You are organizing a car journey. You have five friends, but there are only three vacant places in you car. What is the number of ways of taking three of your five friends to the journey?

# » Combinations

### Question

You are organizing a car journey. You have five friends, but there are only three vacant places in you car. What is the number of ways of taking three of your five friends to the journey?

### Above Question Reformulated

We are essentially asking: What is the number of ways of choosing 3 elements out of a set containing 5 elements?

## » Answer to Previous Question...

# » Answer to Previous Question...

**Answer**

* There are five choices of the first friend,

# » Answer to Previous Question...

**Answer**

* There are five choices of the first friend, four choices of the second friend, and three choices of the third friend

# » Answer to Previous Question...

**Answer**

* There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
* How many choices are there for choosing 3 friends, assuming ordering?

# » Answer to Previous Question...

## Answer

* There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
* How many choices are there for choosing 3 friends, assuming ordering?
* In total, there are $5 \times 4 \times 3 = 60$ choices assuming ordering

# » Answer to Previous Question...

**Answer**

* There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
* How many choices are there for choosing 3 friends, assuming ordering?
* In total, there are $5 \times 4 \times 3 = 60$ choices assuming ordering
* But each group of three friends is counted $3! = 6$ times,

# » Answer to Previous Question...

**Answer**

* There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
* How many choices are there for choosing 3 friends, assuming ordering?
* In total, there are $5 \times 4 \times 3 = 60$ choices assuming ordering
* But each group of three friends is counted $3! = 6$ times, that is, a group $\{a, b, c\}$ is counted as $abc$, $acb$, $bac$, $bca$, $cab$, $cba$

## » Answer to Previous Question...

### Answer

* There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
* How many choices are there for choosing 3 friends, assuming ordering?
* In total, there are $5 \times 4 \times 3 = 60$ choices assuming ordering
* But each group of three friends is counted $3! = 6$ times, that is, a group $\{a, b, c\}$ is counted as $abc, acb, bac, bca, cab, cba$
* Thus, we need to divide by 3!, the answer is $(5 \times 4 \times 3)/3! = 10$

## » Answer to Previous Question...

### Answer

- There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
- How many choices are there for choosing 3 friends, assuming ordering?
- In total, there are $5 \times 4 \times 3 = 60$ choices assuming ordering
- But each group of three friends is counted $3! = 6$ times, that is, a group $\{a, b, c\}$ is counted as $abc, acb, bac, bca, cab, cba$
- Thus, we need to divide by 3!, the answer is $(5 \times 4 \times 3)/3! = 10$
- Since ordering does not matter, we call them combinations!

## » Answer to Previous Question...

**Answer**

- There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
- How many choices are there for choosing 3 friends, assuming ordering?
- In total, there are $5 \times 4 \times 3 = 60$ choices assuming ordering
- But each group of three friends is counted $3! = 6$ times, that is, a group $\{a, b, c\}$ is counted as $abc, acb, bac, bca, cab, cba$
- Thus, we need to divide by 3!, the answer is $(5 \times 4 \times 3)/3! = 10$
- Since ordering does not matter, we call them combinations!
- We define combinations in next slide...

## » Combinations: $k-$combination

**Definition of $k-$combination**

For a set $S$, a $k-$combination is a subset of $S$ of size $k$

## » Combinations: $k-$combination

**Definition of $k-$combination**

For a set $S$, a $k-$combination is a subset of $S$ of size $k$

**Definition of $k-$combinations**

The number of $k-$combinations of an $n$ element set is denoted by $\binom{n}{k}$.

Pronounced: "$n$ choose $k$". Proof by example!

# » Combinations: $k$—combination

### Definition of $k$—combination

For a set $S$, a $k$—combination is a subset of $S$ of size $k$

### Definition of $k$—combinations

The number of $k$—combinations of an $n$ element set is denoted by $\binom{n}{k}$.

Pronounced: "$n$ choose $k$". Proof by example!

### Number of $k$—combinations

The number of $k$—combinations of an $n$ element set is given by

$$\frac{n!}{(n-k)!}$$

# » Derive formula for n choose k...

$\binom{5}{3}$

$3!$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| abc | abd | abe | acd | ace | ade | bcd | bce | bde | cde |
| acb | adb | aeb | adc | aec | aed | bdc | bec | bed | ced |
| bac | bad | bae | cad | cae | dae | cbd | cbe | dbe | dce |
| bca | bda | bea | cda | cea | dea | cdb | ceb | deb | dec |
| cba | dba | eba | dca | eca | eda | dcb | ecb | edb | edc |
| cab | dab | eab | dac | eac | ead | dbc | ebc | ebd | ecd |

$$3!\binom{5}{3} = \frac{5!}{(5-3)!}$$

## » Pascal triangle...

# » Pascal triangle...

**Question**

There are $n$ students. What is the number of ways of forming a team of $k$ students out of them?

## » Pascal triangle...

**Question**

There are $n$ students. What is the number of ways of forming a team of $k$ students out of them?

**Answer**

$$\binom{n}{k}$$

## » Pascal triangle...

**Question**

There are $n$ students. What is the number of ways of forming a team of $k$ students out of them?

**Answer**

$$\binom{n}{k}$$

**A result...**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

## » Proof of previous identity...

# » Proof of previous identity...

**Theorem**

*Prove the following* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

# » Proof of previous identity...

**Theorem**

*Prove the following* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

* Fix one of the students, call him Ramesh

# » Proof of previous identity...

**Theorem**

*Prove the following* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

* Fix one of the students, call him Ramesh
* Then there are two types of teams:

## » Proof of previous identity...

**Theorem**

*Prove the following* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

* Fix one of the students, call him Ramesh
* Then there are two types of teams:
  * Teams without Ramesh: $\binom{n-1}{k-1}$

## » Proof of previous identity...

**Theorem**

*Prove the following* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

- Fix one of the students, call him Ramesh
- Then there are two types of teams:
  - Teams without Ramesh: $\binom{n-1}{k-1}$
  - Teams with Ramesh: $\binom{n-1}{k}$

## » Proof of previous identity...

**Theorem**

*Prove the following* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

- Fix one of the students, call him Ramesh
- Then there are two types of teams:
  - Teams without Ramesh: $\binom{n-1}{k-1}$
  - Teams with Ramesh: $\binom{n-1}{k}$
  - Apply sum rule to conclude the proof

# » Proof of previous identity...

**Theorem**

*Prove the following* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

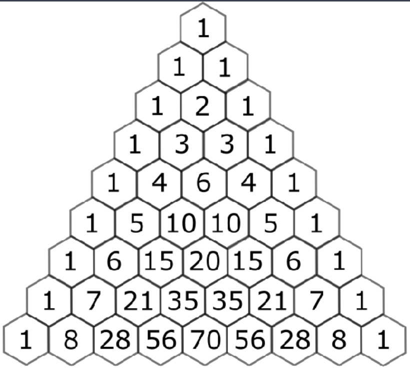
* Fix one of the students, call him Ramesh
* Then there are two types of teams:
  * Teams without Ramesh: $\binom{n-1}{k-1}$
  * Teams with Ramesh: $\binom{n-1}{k}$
  * Apply sum rule to conclude the proof

**Hence recursion for $n$ choose $k$ is...**

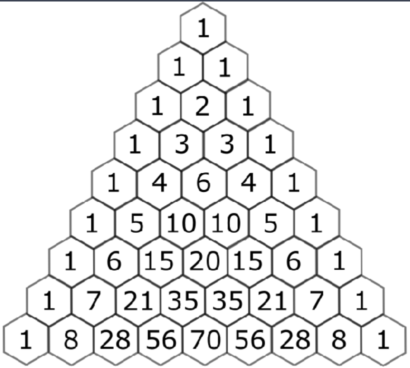
$$\binom{n}{k} = \binom{n-2}{k-2} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k} = \cdots$$

## » Pascal’s Triangle

# » Pascal's Triangle



## » Pascal's Triangle

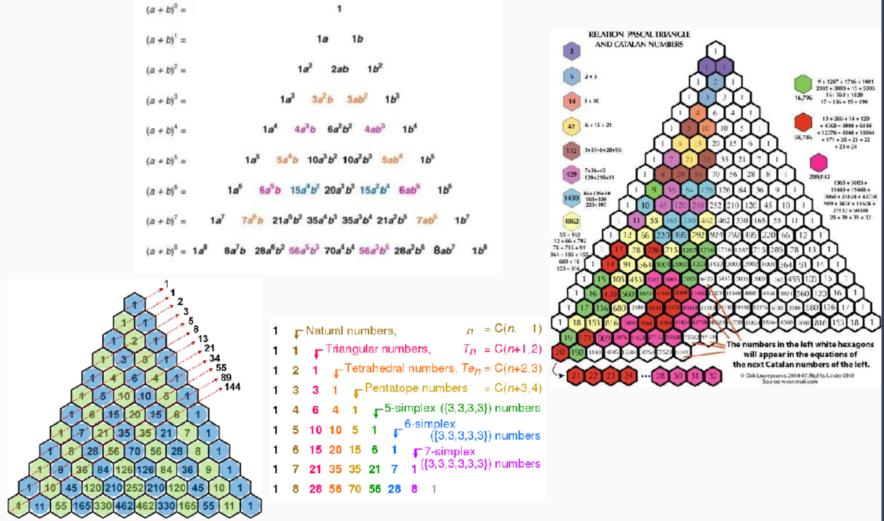



**Quiz**

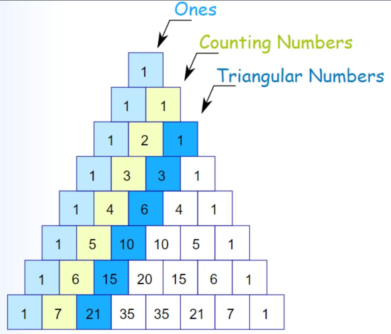
Do you know how to grow Pascal's triangle? What is the rule?

# » Pascal's Triangle and Many Relations...

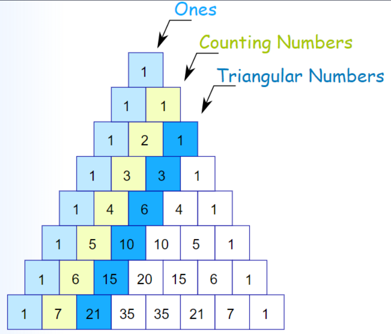
# » Pascal's Triangle and Many Relations...

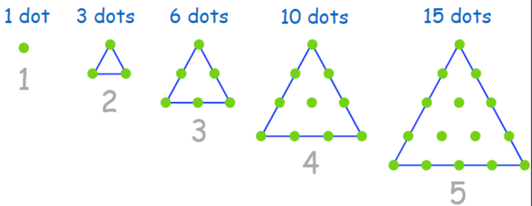


# » Pascal's Triangle and Triangular Numbers



# » Pascal's Triangle and Triangular Numbers

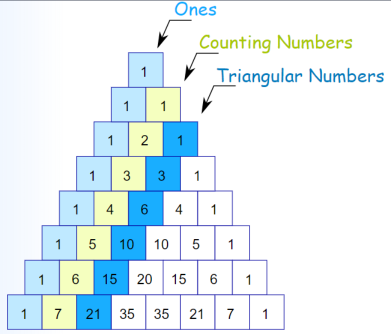


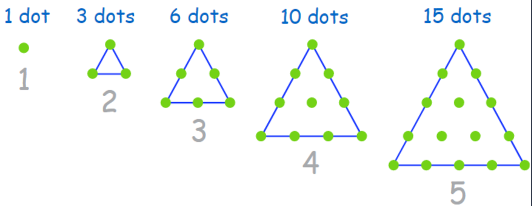




* Triangular numbers are the number of dots

* If we look at the indicated colors, we obtain counting numbers, triangular numbers, etc

# » Pascal's Triangle and Triangular Numbers



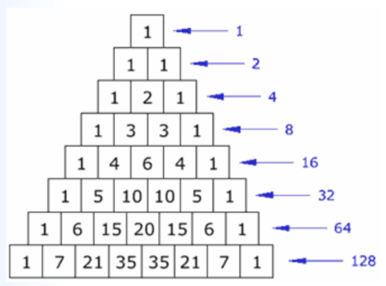




- If we look at the indicated colors, we obtain counting numbers, triangular numbers, etc

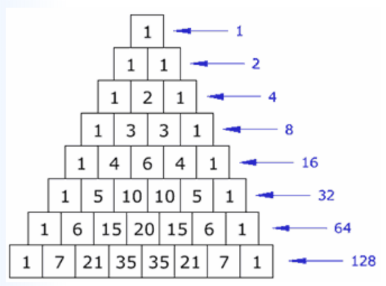
- Triangular numbers are the number of dots
- Add one more row and dots to get next triangular number

# » Pascal’s Triangle: Horizontal Sums and Exponents of 11

# » Pascal's Triangle: Horizontal Sums and Exponents of 11

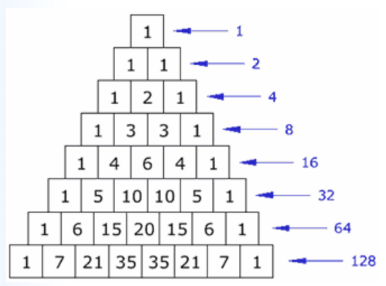


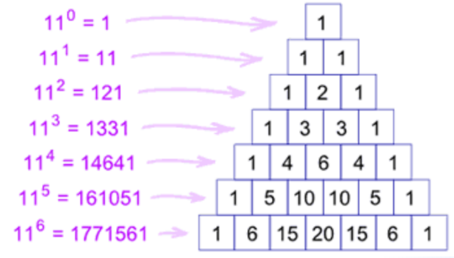
# » Pascal's Triangle: Horizontal Sums and Exponents of 11




* The horizontal sums are $2^i$, $i$ is the $i$th row

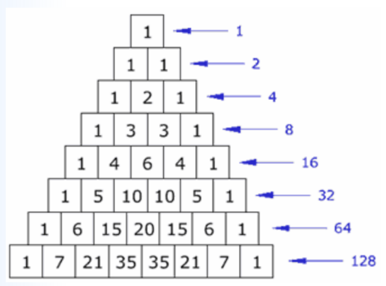
# » Pascal's Triangle: Horizontal Sums and Exponents of 11

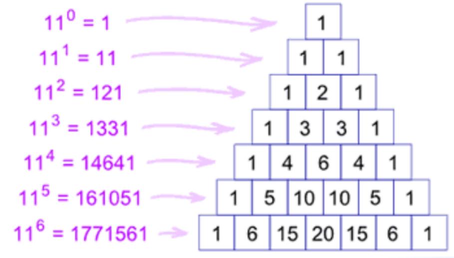



* The horizontal sums are $2^i$, $i$ is the $i$th row

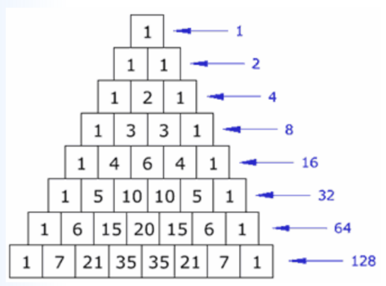
# » Pascal's Triangle: Horizontal Sums and Exponents of 11



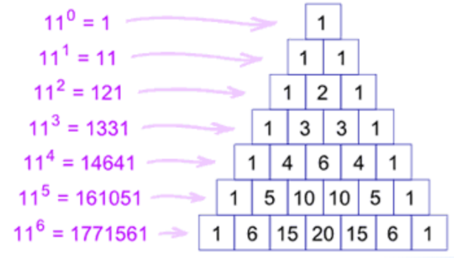




- The horizontal sums are $2^i$, $i$ is the $i$th row
- The row entries are digits of powers of 11

# » Pascal's Triangle: Horizontal Sums and Exponents of 11



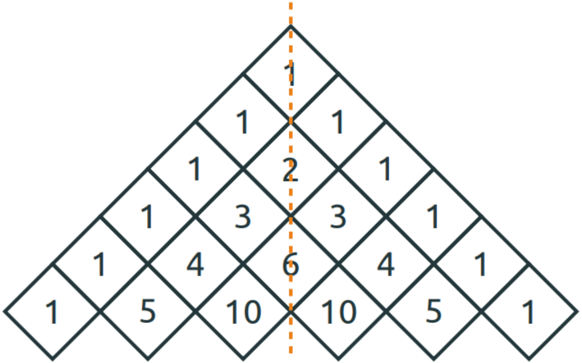

* The horizontal sums are $2^i$, $i$ is the $i$th row




* The row entries are digits of powers of 11
* The entries of the $i$th row are digits of $11^i$

## » Pascal’s Triangle and Symmetry

# » Pascal's Triangle and Symmetry




$$\binom{n}{k} = \binom{n}{n-k}$$

# » Proof of symmetry...

**Theorem**

*Prove that*

$$\binom{n}{k} = \binom{n}{n-k}$$

# » Proof of symmetry...

**Theorem**

*Prove that*

$$\binom{n}{k} = \binom{n}{n-k}$$

* $\binom{n}{k}$ is the number of ways of selecting a team of size $k$ out of $n$ students

# » Proof of symmetry...

**Theorem**

*Prove that*

$$\binom{n}{k} = \binom{n}{n-k}$$

* $\binom{n}{k}$ is the number of ways of selecting a team of size $k$ out of $n$ students
* $\binom{n}{n-k}$ is the number of ways of selecting a team of size $n-k$ out of $n$ students

## » Proof of symmetry...

**Theorem**

*Prove that*

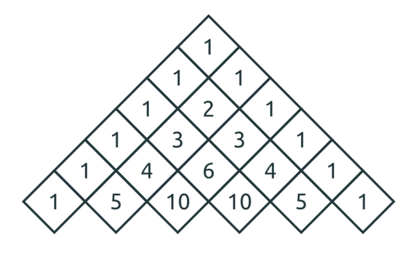
$$\binom{n}{k} = \binom{n}{n-k}$$

- $\binom{n}{k}$ is the number of ways of selecting a team of size $k$ out of $n$ students
- $\binom{n}{n-k}$ is the number of ways of selecting a team of size $n-k$ out of $n$ students

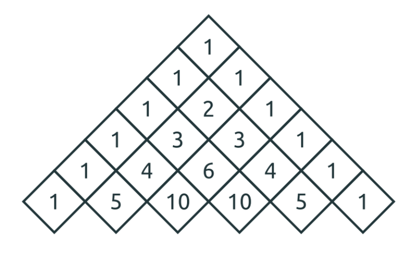
**Answer**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}$$

# » Row Sums of Pascal's Triangle...

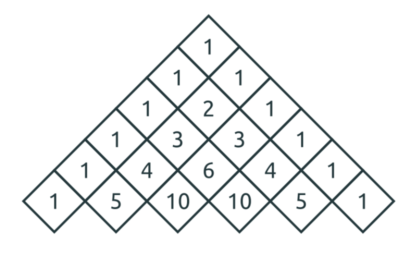


# » Row Sums of Pascal's Triangle...




$$
\begin{array}{c}
1 \\
1 + 1 \\
1 + 2 + 1 \\
1 + 3 + 3 + 1 \\
1 + 4 + 6 + 4 + 1 \\
1 + 5 + 10 + 10 + 5 + 1
\end{array}
$$

# » Row Sums of Pascal's Triangle...




$$1$$
$$1 + 1$$
$$1 + 2 + 1$$
$$1 + 3 + 3 + 1$$
$$1 + 4 + 6 + 4 + 1$$
$$1 + 5 + 10 + 10 + 5 + 1$$

**Theorem**

*The sum of all the numbers in the n-th row of Pascal's triangle is equal to* $2^n$ :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

**Theorem (Prove this...)**

*The sum of all the numbers in the n-th row of Pascal's triangle is equal to* $2^n$ :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

**Theorem (Prove this...)**

*The sum of all the numbers in the n-th row of Pascal's triangle is equal to* $2^n$ :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

* The base case (0-th row) holds

**Theorem (Prove this...)**

*The sum of all the numbers in the n-th row of Pascal's triangle is equal to* $2^n$ :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

* The base case (0-th row) holds
* We'll show that the sum of each row is twice the sum of the previous row

**Theorem (Prove this...)**

*The sum of all the numbers in the n-th row of Pascal's triangle is equal to* $2^n$ :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

- The base case (0-th row) holds
- We'll show that the sum of each row is twice the sum of the previous row
- $\binom{n}{k}$ is the number of $k$-subsets of a set of size $n$

**Theorem (Prove this...)**

*The sum of all the numbers in the n-th row of Pascal's triangle is equal to* $2^n$ :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

* The base case (0-th row) holds
* We'll show that the sum of each row is twice the sum of the previous row
* $\binom{n}{k}$ is the number of *k*-subsets of a set of size $n$
* The sum $\binom{n}{k}$ for all $k$ (from $0$ to $n$) is the number of all subsets of an $n$ element set; this is $2^n$ by the product rule (how?)

# » Alternating Row Sum in Pascal

# » Alternating Row Sum in Pascal

$$
\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & \\
 & & & & 1 & - & 1 & & & & & & = 0 \\
 & & & 1 & - & 2 & + & 1 & & & & & = 0 \\
 & & 1 & - & 3 & + & 3 & - & 1 & & & & = 0 \\
 & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & & = 0 \\
1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
$$

# » Alternating Row Sum in Pascal

$$
\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & \\
 & & & & 1 & - & 1 & & & & & & = 0 \\
 & & & 1 & - & 2 & + & 1 & & & & & = 0 \\
 & & 1 & - & 3 & + & 3 & - & 1 & & & & = 0 \\
 & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & & = 0 \\
1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
$$

**Theorem**

*For* $n > 0, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

# » Alternating Row Sum in Pascal

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & - & 1 & & & & & = 0 \\
 & & & & 1 & - & 2 & + & 1 & & & & = 0 \\
 & & & 1 & - & 3 & + & 3 & - & 1 & & & = 0 \\
 & & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & = 0 \\
1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
$$

**Theorem**

*For* $n > 0, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

- Hint: Number of odd size subsets = Number of even size subsets

# » Counting Problems...

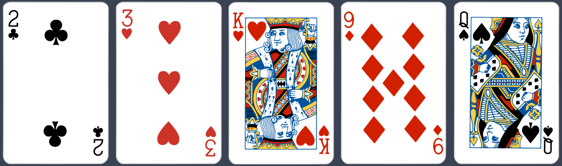
# » Counting Problems...

**Question**

What is the number of 5-card hands dealt off of a standard 52-card deck?

**Question**

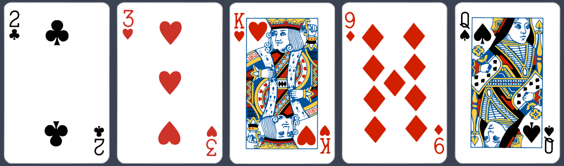
What is the number of 5-card hands dealt off of a standard 52-card deck?



# » Counting Problems...

**Question**

What is the number of 5-card hands dealt off of a standard 52-card deck?



**Answer**

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \times 52 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2598960$$

# » Counting Problems ...

# » Counting Problems ...

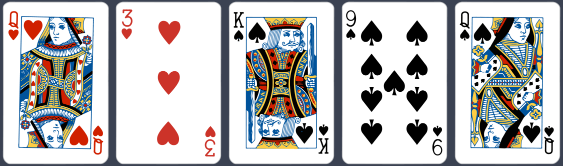
**Question**

What is the number of 5-card hands with two hearts and three spades?

# » Counting Problems ...

**Question**

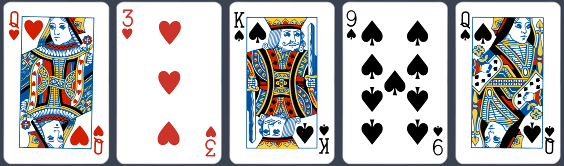
What is the number of 5-card hands with two hearts and three spades?



## » Counting Problems ...

**Question**

What is the number of 5-card hands with two hearts and three spades?



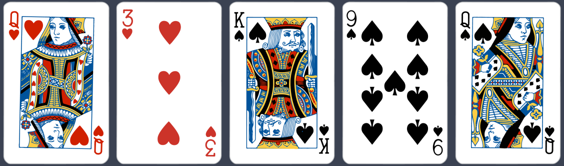
**Answer**

* Number of ways of picking 2 hearts from 13 hearts

# » Counting Problems ...

**Question**

What is the number of 5-card hands with two hearts and three spades?



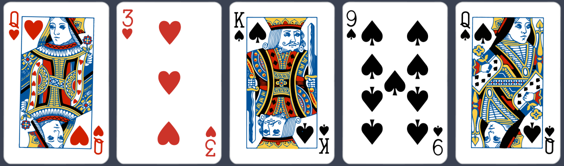
**Answer**

* Number of ways of picking 2 hearts from 13 hearts
* Number of ways of picking 3 spades from 13 spades

# » Counting Problems ...

**Question**

What is the number of 5-card hands with two hearts and three spades?

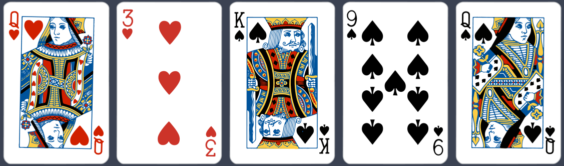


**Answer**

* Number of ways of picking 2 hearts from 13 hearts
* Number of ways of picking 3 spades from 13 spades
* Now apply product rule!

**Question**

What is the number of 5-card hands with two hearts and three spades?



**Answer**

* Number of ways of picking 2 hearts from 13 hearts
* Number of ways of picking 3 spades from 13 spades
* Now apply product rule!
* The answer is: $\binom{13}{2}\binom{13}{2} = 22308$

**Question**

What is the number of non-negative integers with *at* most four digits *at* least one of which is equal to 7?

# » Counting Problems...

**Question**

What is the number of non-negative integers with at most four digits at least one of which is equal to 7?

* Total number of 4 digit numbers = $10^4$

**Question**

What is the number of non-negative integers with *at most* four digits *at least* one of which is equal to 7?

- Total number of 4 digit numbers = $10^4$
- Total number of 4 digit number that does not contain 7 = $9^4$

# » Counting Problems...

**Question**

What is the number of non-negative integers with at most four digits at least one of which is equal to 7?

* Total number of 4 digit numbers = $10^4$
* Total number of 4 digit number that does not contain 7 = $9^4$
* Hence, the answer is $10^4 - 9^4 = 3439$

# » Counting Problems...

**Question**

What is the number of non-negative integers with *at most* four digits whose digits are increasing?

**Question**

What is the number of non-negative integers with at most four digits whose digits are increasing?

* We can choose 4 different digits and we can rearrange them in increasing order

**Question**

What is the number of non-negative integers with *at most* four digits whose digits are increasing?

* We can choose 4 different digits and we can rearrange them in increasing order
* Hence, it is nothing but picking 4 different digits out of 10 digits!

**Question**

What is the number of non-negative integers with at most four digits whose digits are increasing?

* We can choose 4 different digits and we can rearrange them in increasing order
* Hence, it is nothing but picking 4 different digits out of 10 digits!
* Hence, the answer is $\binom{10}{4} = 210$

# » Counting Problems...

# » Counting Problems...

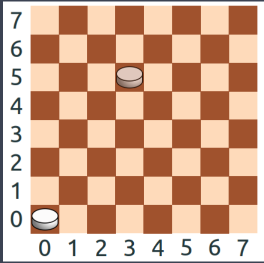
**Question**

A piece can move one step up or one step to the right. What is the number of ways of getting from the cell [0, 0] (bottom left corner) to the cell [5, 3]?

# » Counting Problems...

**Question**

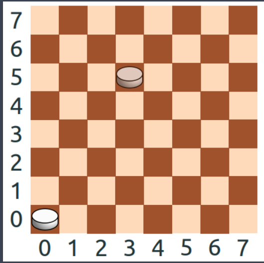
A piece can move one step up or one step to the right. What is the number of ways of getting from the cell [0, 0] (bottom left corner) to the cell [5, 3]?



# » Counting Problems...

**Question**

A piece can move one step up or one step to the right. What is the number of ways of getting from the cell [0, 0] (bottom left corner) to the cell [5, 3]?

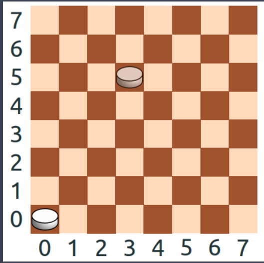



- We want to go to the cell [5,3]. How many ways we can go?

# » Counting Problems...

**Question**

A piece can move one step up or one step to the right. What is the number of ways of getting from the cell [0, 0] (bottom left corner) to the cell [5, 3]?

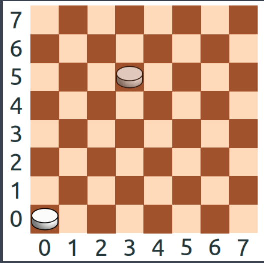



- We want to go to the cell [5,3]. How many ways we can go?
- Any path to [5,3] must involves 3 moves right and 5 moves up!

# » Counting Problems...

**Question**

A piece can move one step up or one step to the right. What is the number of ways of getting from the cell [0, 0] (bottom left corner) to the cell [5, 3]?




- We want to go to the cell [5,3]. How many ways we can go?
- Any path to [5,3] must involves 3 moves right and 5 moves up!
- Hence, answer is $\binom{8}{3} = 56$

## » Answer using Pascal's triangle...

# » Answer using Pascal's triangle...

The answer to the previous problems can be found using Pascal's triangle:

## » Answer using Pascal’s triangle...

The answer to the previous problems can be found using Pascal’s triangle:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 1 | | | | | | | |
| 1 | | | | | | | |
| 1 | | | | | | | |
| 1 | 4 | | | | | | |
| 1 | 3 | 6 | | | | | |
| 1 | 2 | 3 | 4 | | | | |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

## » Answer using Pascal's triangle...

The answer to the previous problems can be found using Pascal's triangle:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 1 | | | | | | | |
| 1 | | | | | | | |
| 1 | | | | | | | |
| 1 | 4 | | | | | | |
| 1 | 3 | 6 | | | | | |
| 1 | 2 | 3 | 4 | | | | |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

It is now only *a* matter of filling the (5,3) cell...

## » Combinations with or without repetitions...

# » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. Consider $n = 2$ and $n = 3$ options: $a, b, c$

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | (a,a), (a,b), (a,c)<br>(b,a), (b,b), (b,c)<br>(c,a), (c,b), (c,c) | |
| Unordered | | |

# » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. Consider $n = 2$ and $n = 3$ options: $a, b, c$

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | (a,a), (a,b), (a,c)<br>(b,a), (b,b), (b,c)<br>(c,a), (c,b), (c,c) | (a,b), (a,c)<br>(b,a), (b,c)<br>(c,a), (c,b) |
| Unordered | | |

# » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. Consider $n = 2$ and $n = 3$ options: $a, b, c$

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | (a,a), (a,b), (a,c)<br>(b,a), (b,b), (b,c)<br>(c,a), (c,b), (c,c) | (a,b), (a,c)<br>(b,a), (b,c)<br>(c,a), (c,b) |
| Unordered | | {a,b}, {a,c}, {b,c} |

## » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. Consider $k = 2$ and $n = 3$ options: $a, b, c$

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | (a,a), (a,b), (a,c)<br>(b,a), (b,b), (b,c)<br>(c,a), (c,b), (c,c) | (a,b), (a,c)<br>(b,a), (b,c)<br>(c,a), (c,b) |
| Unordered | {a,b}, {a,c}, {b,c}<br>{a,a}, {b,b}, {c,c} | {a,b}, {a,c}, {b,c} |

# » Combinations with or without repetitions...

## » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. The formulas we have derived are the following:

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | Tuples $n^k$ | $k-$permutations $\frac{n!}{(n-k)!}$ |
| Unordered | | Combinations $\binom{n}{k}$ |

## » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. The formulas we have derived are the following:

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | Tuples $n^k$ | $k-$permutations $\frac{n!}{(n-k)!}$ |
| Unordered | | Combinations $\binom{n}{k}$ |

* What about the case ”unordered” and ”with repetitions”?

# » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. The formulas we have derived are the following:

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | Tuples $n^k$ | $k$–permutations $\dfrac{n!}{(n-k)!}$ |
| Unordered | | Combinations $\dbinom{n}{k}$ |

* What about the case "unordered" and "with repetitions"?
* Is it worth knowing? Is there any formula?

# » Combinations with or without repetitions...

So far we have considered selections of $k$ items out of $n$ possible options. The formulas we have derived are the following:

| | With repetitions | Without repetitions |
|---|---|---|
| Ordered | Tuples $n^k$ | $k$–permutations $\dfrac{n!}{(n-k)!}$ |
| Unordered | | Combinations $\dbinom{n}{k}$ |

- What about the case "unordered" and "with repetitions"?
- Is it worth knowing? Is there any formula?
- Let us try to find out...