Probability and Statistics: Lecture-31

Monsoon-2020

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» Conditional Variance...

Definition of Conditional Variance

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» Conditional Variance...

$Var(X) = E[X^2] - (E[X])^2$

$\underline{\underline{E[X]}}$ constant

Definition of Conditional Variance

Let X, Y be two RVs. By $\mathrm{Var}(X \mid Y = y)$ the conditional variance of X given $Y = y$. Let $\underbrace{\mu_{X|Y}(y) = E[X \mid Y}$ $= y]$. Then

$$\mathrm{Var}(X \mid Y = y) = E[X^2 \mid Y = y] - \mu_{X|Y}(y)^2$$

Proof

$\underline{\text{Try}}$

» Solved Example ...

» Solved Example ...

Solved Example

Let X, Y be RV with joint PMF given as follows

	$Y = 0$	$Y = 1$
$X = 0$	$\frac{1}{5}$	$\frac{2}{5}$
$X = 1$	$\frac{2}{5}$	0

Let $Z = E[X \mid Y]$ and $V = \mathrm{Var}(X \mid Y)$. conditional variance

1. Find the PMF of V
2. Find $E[V]$
3. Verify that $\mathrm{Var}(X) = E[V] + \mathrm{Var}(Z)$

$X|Y=1 \rightarrow X = 0$

Recall:

$\boxed{X, Y \sim \text{Bernoulli}(2/5)}$

and $X|Y=0 \sim \text{Bernoulli}(2/3)$

$P(X=0 \mid Y=1) = 1$

$\mathrm{Var}(Z) = \frac{8}{75}$

$P(X=1 \mid Y=1) = 0$

(a) To find the PMF of V, we note that V is a fn of Y.

Specifically,

$$V = Var(X|Y) = \begin{cases} Var(X|Y=0) & \text{if } Y=0 \\ Var(X|Y=1) & \text{if } Y=1 \end{cases}$$

$X \sim Bernoulli(p)$

$Var(X) = p(1-p)$

$$\Rightarrow V = Var(X|Y) = \begin{cases} Var(X|Y=0) & \text{with prob. } 3/5 \\ Var(X|Y=1) & \text{with prob. } 2/5 \end{cases}$$

because $Y \sim Bernoulli\left(\frac{2}{5}\right)$

Since $X|Y=0 \sim Bernoulli(2/3) \Rightarrow \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

Given $Y=1$, $X=0$, we have

$$\mathrm{Var}(X \mid Y=1) = 0$$

$$V = \mathrm{Var}(X|Y) = \begin{cases} 2/9 & \text{with prob. } 3/5 \\ 0 & \text{with prob } 2/5 \end{cases}$$

$$P_V(v) = \begin{cases} 3/5, & \text{if } v = 2/9 \\ 2/5, & \text{if } v = 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) $E[V] = \frac{2}{9} \cdot \frac{3}{5} + 0 \cdot \frac{2}{5}$

(c) $\mathrm{Var}(X) = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}$

$$E[V] = \frac{2}{15}$$

$$\mathrm{Var}(Z) = \frac{8}{75}$$

$$\mathrm{Var}(\cancel{X}) = E[V] + \mathrm{Var}(Z)$$

$$= \frac{2}{15} + \frac{8}{75}$$

$$\frac{6}{25}$$

$$6\ \frac{\cancel{18}}{\cancel{75}}\ 25 = \frac{10+8}{75}$$

» Answer to previous problem...

» Law of Total Variance...

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Let X, Y be two RVs. The law of total variance says that

$$\mathrm{Var}(X) = E[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(E[X \mid Y])$$

Proof: Let $V = \mathrm{Var}(X|Y)$ and Let $Z = E[X|Y]$

1st

$$V = E[X^2|Y] - (E[X|Y])^2 = E[X^2|Y] - Z^2$$

$$E[V] = \underbrace{E[E[X^2|Y]]}_{L.O.T.E} - E[Z^2] = E[X^2] - E[Z^2]$$

2nd

$$\mathrm{Var}(Z) = E[Z^2] - (E[Z])^2 = E[Z^2] - (E[E[X|Y]])^2 \quad L.O.T.E$$

$$= E[Z^2] - (E[X])^2$$

1st + 2nd:

$$\underbrace{E[V] + \mathrm{Var}(Z)}_{R.H.S} = E[X^2] - \cancel{E[Z^2]} + \cancel{E[Z^2]} - (E[X])^2$$

$$= \mathrm{Var}(X) = L.H.S$$

» Solved Problem 1

Joint Discr. RVs

2 - RVs

Solved Problem 1

Let X, Y be two independent RVs with the same CDFs F_X and F_Y. Let

$$\left.\begin{aligned} Z &= \max(X, Y) \\ W &= \min(X, Y) \end{aligned}\right\} \text{fns of RVs } X \text{ \& } Y.$$

Find the CDFs of Z and W.

» Answer to previous problem...

$$F_Z(z) = P(Z \le z)$$
$$= P(\max(X,Y) \le z)$$
$$= P((X \le z) \text{ and } (Y \le z))$$
$$= P(X \le z)\, P(Y \le z)$$
$$= F_X(z)\, F_Y(z)$$

} X, Y ~~were~~ are independ

$$F_W(w) = P(W \le w)$$
$$= P(\min(X,Y) \le w)$$
$$= 1 - P(\min(X,Y) > w)$$
$$= 1 - P((X > w) \text{ and } (Y > w))$$

$\max(X,Y) \le z$

$(X \le z)$ and $(Y \le z)$

$1 = P(X \le$

$X > w$

$Y > w$

$$P(\min(X,Y) \le w)$$
① $$1 - P(\min(X,Y) > w)$$

②

$$= 1 - P(X > w)\, P(Y > w)$$
$$= 1 - (1 - P(X \le w))(1 - P(Y \le w))$$
$$= 1 - (1 - F_X(w))(1 - F_Y(w))$$
$$= F_X(w) + F_Y(w) - F_X(w) F_Y(w)$$

» Answer to previous problem...

» Solved Problem 2..

Solved Problem 2

Let X, Y be two RVs with: $R_{XY} = \{(i,j) \in \mathbb{Z}^2 \mid \quad i,j \geq 0, |i-j| \leq 1\}$.

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$$P_{XY}(i,j) = \frac{1}{6 \cdot 2^{\min(i,j)}}, \quad \text{for } (i,j) \in R_{XY}.$$

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* Find $P(X = Y \mid X < 2)$

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* Plot R_{XY} in the XY plane
* Find the marginal PMFs $P_X(i), P_Y(j)$
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* Find $P(1 \leq X^2 + Y^2 \leq 5)$

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* Find $P(X = Y)$

» Solved Problem 2..

$-1 \leq i-j \leq 1$

Solved Problem 2

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* Find $P(1 \leq X^2 + Y^2 \leq 5)$
* Find $P(X = Y)$
* Find $E[X \mid Y = 2]$

» Solved Problem 2..

$(0,0), (0,1), (1,0), (1,1)$

Solved Problem 2

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- Plot R_{XY} in the XY plane
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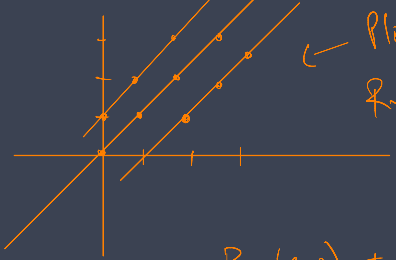
- Find $P(X = Y \mid X < 2)$
- Find $P(1 \leq X^2 + Y^2 \leq 5)$
- Find $P(X = Y)$
- Find $E[X \mid Y = 2]$
- Find $\mathrm{Var}(X \mid Y = 2)$

» Answer to previous problem...

(a)



Plot of P_{XY}

$|S| =$

(b) $P_X(0) = P_{XY}(0,0) + P_{XY}(0,1)$

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P_X(1) = P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2)$

$= \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{2+1+1}{12}$

$= \frac{4}{12} = \frac{1}{3}$

$P_X(2) = P_{XY}(2,1) + P_{XY}(2,2) + P_{XY}(2,3)$

$= \frac{1}{12} + \frac{1}{24} + \frac{1}{24}$

$= \frac{1}{6} = \frac{1}{2 \cdot 3}$

$P_X(3) = P_{XY}(3,2) + P_{XY}(3,3) + P_{XY}(3,4)$

$= \frac{1}{6 \cdot 4} + \frac{1}{6 \cdot 8} + \frac{1}{6 \cdot 8}$

$= \frac{1}{6}\left[\frac{1}{4} + \frac{1}{8} + \frac{1}{8}\right] = \frac{1}{3 \cdot 2^2}$

$= \frac{1}{12}$

» Answer to previous problem...

In general, check (by symm)

$$P_X(k) = P_Y(k) = \begin{cases} \frac{1}{3} & k=0 \\ \frac{1}{3\cdot 2^{k-1}} & k=1,2,\dots \end{cases}$$

ⓒ $$P(X=Y \mid X<2) = \frac{P(X=Y,\ X<2)}{P(X<2)}$$

$$= \frac{P(0,0)+P(1,1)}{P_X(0)+P_X(1)} = \frac{\frac{1}{6}+\frac{1}{12}}{\frac{1}{3}+\frac{1}{3}}$$

ⓓ $$P(1 \le X^2+Y^2 \le 5)$$

$$= P_{XY}(0,1) + P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2) + P_{XY}(2,1)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$$

$$= \frac{7}{12}$$

ⓔ $$P(X=Y) = \sum_{i=0}^{\infty} P_{XY}(i,i)$$

$$= \sum_{i=0}^{\infty} \frac{1}{6\cdot 2^i} =$$

» Answer to previous problem...

⑧ $E[X|Y=2]$ First we need to find PMF of $X|Y=2$.

$$P_{X|Y}(k|2) = \frac{P_{XY}(k,2)}{P(Y=2)} = 6P_{XY}(k,2)$$

Since $Y=2$, $k=1,2,3$ only.

$$P_{X|Y}(k|2) = \begin{cases} \frac{1}{2} & k=1 \\ \frac{1}{4} & k=2,3 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow E[X|Y=2] = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4}$$

$$= 7/4$$

⑨ $\text{Var}(X|Y=2)$

$= E[X^2|Y$

Exercise.