Probability and Statistics: Lecture-17

Monsoon-2020

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» Higher Order Moments...

Define nth moment

The nth moment about the mean or nth central moment of a real valued random variable X is defined as follows

$$\mu_n = E[(X - E[X])^n],$$

where E is the expectation operator.

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where E is the expectation operator.

$\mu_1 = E[X - E[X]] = 0$

* The zeroth central moment μ_0 is 1
* The first central moment μ_1 is 0 (not the same as expected value!)

» Higher Order Moments...

Define *n*th moment

The *n*th moment about the mean or *n*th central moment of a real valued random variable X is defined as follows

$$\mu_n = E[(X - \underbrace{\overset{0}{E[X]}}_{\text{about mean}})^n],$$

where E is the expectation operator.

* The zeroth central moment μ_0 is 1
* The first central moment μ_1 is 0 (not the same as expected value!)
    * First central moment about origin is indeed expectation!

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Generating Moments...

Is there a quick way to generate moments?

» Moment Generating Function...

» Moment Generating Function...

$g(x) = e^{tx}$

Moment Generating Function

The moment generating function $M_X(t)$ is the expectation value

fn. t

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} p_X(x)$$

Lemma

* $M_X(0) = 1$
* $E[X] = M'_X(0)$, where ' is the derivative w.r.t. t

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» Moment Generating Function for Binomial Distribution...

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$$M_X(t) = \sum_{x=0}^{n} \underbrace{\binom{n}{x} p^x (1-p)^{n-x}}_{\text{PMF of } X} e^{tx}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$$

$$= (e^t p + 1 - p).$$

» Moment Generating Function for Binomial Distribution...

- Let X be a discrete random variable whose PMF is given by Binomial distribution with parameters n and p
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$$\begin{aligned} M_X(t) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} e^{tx} \\ &= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} \\ &= (e^t p + 1 - p). \end{aligned}$$

Differentiating w.r.t. t, we have

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&= (e^t p + 1 - p).
\end{aligned}
$$

Differentiating w.r.t. t, we have

$$M'_X(t) = n(e^t p + 1 - p)^{n-1} p e^t.$$

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Differentiating w.r.t. t, we have

$$M'_X(t) = n(e^t p + 1 - p)^{n-1} p e^t.$$

* Setting $t = 0$, $M'_X(0) = np = E[X]$

» Moment Generating Function for Poisson Distribution...

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* Let X be the random variable whose PMF is *a* Poisson distribution with parameter λ.

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$$M_X(t) = \sum_{x=0}^{\infty} \underbrace{e^{-\lambda} \frac{\lambda^x}{x!}}_{\text{PMF}} e^{tx} \quad \text{of Poisson}(\lambda)$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{\lambda(e^t - 1)} \leftarrow$$

» Moment Generating Function for Poisson Distribution...

* Let X be the random variable whose PMF is a Poisson distribution with parameter λ.
* The moment generating function is

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\begin{aligned}
M_X(t) &= \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} e^{tx} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!} \\
&= e^{\lambda(e^t - 1)}
\end{aligned}
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* Differentiating w.r.t. to t,

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$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!}$$
$$= e^{\lambda(e^t-1)}$$

- Differentiating w.r.t. to t,

$$M_X'(t) = e^{\lambda(e^t-1)} \lambda e^t$$

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- Let X be the random variable whose PMF is a Poisson distribution with parameter λ.
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M_X(t) &= \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} e^{tx} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!} \\
&= e^{\lambda(e^t - 1)}
\end{aligned}
$$

- Differentiating w.r.t. to t,

$$M'_X(t) = e^{\lambda(e^t-1)}\, \lambda e^t$$

- Setting $t = 0$, $\quad M'_X(0) = \lambda$

» Variance Using Moment Generating Function...

$E[x^2]$ $E[x]^2$

Variance Using Moment Generating Function

$$\text{Var}(X) = M_X''(0) - M_X'(0)^2$$

$Var(X) = E[X^2] - (E[X])^2$

» Computing Variance Using Moment Generating Function...

Computing Variance using moment generating function

Let X be a discrete random variable whose PMF is a binomial distribution with parameters n and p. It has mean $\mu = np$ and the moment generating function is

$$M_X(t) = (e^t p + 1 - p)^n$$

$$M_X'(t) = \underbrace{n(e^t p + 1 - p)^{n-1}}_{2nd} \underbrace{pe^t}_{1st}$$

$$M_X''(t) = pe^t \cdot n(n-1)(e^t p + 1 - p)^{n-2} + n(e^t p + 1 - p)^{n-1} pe^t$$

$$M_X''(0) = n(n-1)p^2 + np$$

$$Var(X) = M_X''(0) - \underbrace{(M_X'(0))^2}_{(np)^2} = n(n-1)p^2 + np - (np)^2$$

$$= np(1-p)$$

» Computing Variance Using Moment Generating Function...

Computing Variance using moment generating function

Let X be a discrete random variable whose PMF is a binomial distribution with parameters n and p. It has mean $\mu = np$ and the moment generating function is

$$M_X(t) = (e^t p + 1 - p)^n$$

Find the Variance using $M_X(t)$.

» Computing Variance Using Moment Generating Function...

Computing Variance using moment generating function

Let X be a discrete random variable whose PMF is a Poisson distribution with mean λ. The moment generating function is

$$M_X(t) = e^{\lambda(e^t-1)}$$

$f(t) = e^{u(t)}$

$f'(t) = e^{u(t)} \cdot u'(t)$

$M_X'(t) = e^{\lambda(e^t-1)} \cdot e^t \cdot \lambda$

$M_X'(0) = \lambda \Leftarrow$ ✓

$M_X''(t) = e^{\lambda(e^t-1)} \lambda e^t + e^{\lambda(e^t-1)} (\lambda e^t)^2$

At $t=0$

$M_X''(0) = \lambda + \lambda^2$

$Var(X) = M_X''(0) - \{M_X'(0)\}^2$

$= \lambda + \lambda^2 - (\lambda)^2$

$= \lambda$

» Computing Variance Using Moment Generating Function...

Computing Variance using moment generating function

Let X be a discrete random variable whose PMF is a Poisson distribution with mean λ. The moment generating function is

$$M_X(t) = e^{\lambda(e^t - 1)}$$

Find the Variance using $M_X(t)$.

» Problem 1

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Problem

Let X be a discrete R.V. with PMF

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Let X be a discrete R.V. with PMF

$$P_X(x) = \begin{cases} 0.1 & \text{for } x = 0.2 \\ 0.2 & \text{for } x = 0.4 \\ 0.2 & \text{for } x = 0.5 \\ 0.3 & \text{for } x = 0.8 \\ 0.2 & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

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Answer the following:

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Answer the following:

1. Find R_X

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2. Find $P(X \leq 0.5)$

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Answer the following:

1. Find R_X
2. Find $P(X \leq 0.5)$
3. Find $P(0.25 < X < 0.75)$

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$X < 0.6$

$X \in \{0.2, 0.4, 0.5\}$

$X = 0.2$

$X \in \{0.2\}$

$x > 0.5$

Answer the following:

1. Find R_X
2. Find $P(X \leq 0.5)$
3. Find $P(0.25 < X < 0.75)$
4. Find $P(X = 0.2 \mid X < 0.6)$

» Answer to previous problem...

a) Find $R_x = \{0.2, 0.4, 0.5, 0.8, 1\}$

b) $P(X \leq 0.5) = P(X \in \{0.2, 0.4, 0.5\})$

$= P(0.2) + P(0.4) + P(0.5)$

$= P(X=0.2) + P(X=0.4) + P(X=0.5)$

$= 0.1 + 0.2 + 0.2 = \underline{\underline{0.5}}$

c) $P(0.25 < X < 0.75)$

$= P(X \in \{\underline{0.4}, \underline{0.5}\})$

$= P(X=0.4) + P(X=0.5)$

$= 0.2 + 0.2$

$= \underline{\underline{0.4}}$

d) $P(X=0.2 \mid X<0.6)$

$= \dfrac{P((\underline{\underline{X=0.2}}) \text{ and } (X<0.6))}{P(X<0.6)}$

$= \dfrac{P(X=0.2)}{P(X<0.6)} = \dfrac{0.1}{0.5} = \underline{\underline{0.2}}$

» Answer to previous problem...

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1. Find R_X, R_Y and the PMF of X and Y

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1. Find R_X, R_Y and the PMF of X and Y
2. Find $P(X = 2, Y = 6)$
3. Find $P(X > 3 \mid Y = 2)$

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1. Find R_X, R_Y and the PMF of X and Y
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3. Find $P(X > 3 \mid Y = 2)$
4. Let $Z = X + Y$. Find the range and PMF of Z

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1. Find R_X, R_Y and the PMF of X and Y
2. Find $P(X = 2, Y = 6)$
3. Find $P(X > 3 \mid Y = 2)$
4. Let $Z = X + Y$. Find the range and PMF of Z
5. Find $P(X = 4 \mid Z = 8)$

» Answer to previous problem...

» Answer to previous problem...

» Problem 3

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Problem

Consider an exam that contains 20 MCQs.

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Consider an exam that contains 20 MCQs. Each question has 4 choices. Assume that you can answer 10 questions correctly, but you are not sure about other 10 questions, so you answer them randomly. Let the random variable X denote the score, which is equal to the total number of correct answers. Answer the following:

1. What is PMF of X?

» Problem 3

$X = 10 + 10$ Y $10 + \geq 5$

Problem

Consider an exam that contains 20 MCQs. Each question has 4 choices. Assume that you can answer 10 questions correctly, but you are not sure about other 10 questions, so you answer them randomly. Let the random variable X denote the score, which is equal to the total number of correct answers. Answer the following:

1. What is PMF of X?
2. What is $P(X > 15)$?

Sol: Let Y = # Correct answer to the remaining 10 MCQs that you answer randomly.

$\Rightarrow X = Y + 10$

Goal: Find PMF of Y.

» Answer to previous problem...

For each question, succ. prob. is $\frac{1}{4}$

We perform 10 independent Bernoulli$(1/4)$ trials and Y is the no. of success.

$\Rightarrow Y \sim \text{Binomial}(10, 1/4)$.

So, $$P_Y(y) = \begin{cases} \binom{10}{y}\left(\frac{1}{4}\right)^y\left(\frac{3}{4}\right)^{10-y} & \text{for } y = 0, 1, 2, \ldots, 9, 10 \\ 0, & \text{otherwise} \end{cases}$$

Need to find PMF of X.

$X = Y + 10$

$R_X = \{10, 11, \ldots, 20\}$

$P_X(10) = P(X=10) = P(Y+10=10)$

$= P(Y=0) = \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10-0}$

$= \left(\frac{3}{4}\right)^{10}$

$P_X(11) = P(Y+10=11) = P(Y=1)$

$= \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{10-1} = 10\,\frac{1}{4}\left(\frac{3}{4}\right)^9$

» Answer to previous problem...

$$P_X(k) = P(X=k) = P(Y+10=k) = P(Y=k-10)$$

$$= \binom{10}{k-10}\left(\frac{1}{4}\right)^{k-10}\left(\frac{3}{4}\right)^{20-k}$$

To summarize

$$P_X(k) = \begin{cases} \binom{10}{k-10}\left(\frac{1}{4}\right)^{k-10}\left(\frac{3}{4}\right)^{20-k} & k = 10, 11, \dots, 20 \\ 0 & \text{otherwise} \end{cases}$$

$P(X > 15)$

$10 + \{6, 7, 8, 9, 10\}$

> 15

$\Rightarrow Y = 6, 7, 8, 9, 10$

$$P_Y(y) = \begin{cases} \binom{10}{y}\left(\frac{1}{4}\right)^{y}\left(\frac{3}{4}\right)^{10-y} & y = 6, 7 \dots 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P_X(k) = \begin{cases} \binom{10}{k-10}\left(\frac{1}{4}\right)^{k-10}\left(\frac{3}{4}\right)^{20-k} \\ k = 16, \dots 20 \end{cases}$$

» Problem 4

$$P(X > 15) = P_X(16) + P_X(17) + \cdots + P_X(20)$$

$$= \binom{10}{6}\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^4 + \cdots \binom{10}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^0$$

» Problem 4

Problem

Average number of customers arriving at a grocery store per hour is 10.

» Problem 4

Problem

Average number of customers arriving at a grocery store per hour is 10. Let X denote the number of customers arriving from $10AM$ to $11:30AM$.

» Problem 4

average .

time

Problem

Average number of customers arriving at a grocery store per hour is 10. Let X denote the number of customers arriving from $10AM$ to $11:30AM$. What is $P(10 < X \leq 15)$?

1.5 hours.

Sol^n^: We have interval of length 1.5 hours

$\Rightarrow$ no. of customers in this interval is

$X \sim \text{Poisson}\left(\lambda = 1.5 \times 10 = 15\right)$

$$\Rightarrow P(10 < X \leq 15) = \sum_{k=11}^{15} P_X(k) = \sum_{k=11}^{15} \frac{e^{-15} \cdot 15^k}{k!} = \ldots$$

$=$