# Probability and Statistics: Lecture-30

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad) on October 21, 2020



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\* For an example, let X=aY+b. Then  $E[X\mid Y=y]=E[aY+b\mid Y=y]=ay+b$   $g(y)=ay+b,\quad E[X\mid Y]=aY+b$ 

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- \* Since  $E[X \mid Y]$  is a RV, we can find its PMF, CDF, Variance, etc

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Let 
$$Z = E[X \mid Y]$$
.

1. Find the Marginal PMFs of X and Y.

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- 1. Find the Marginal PMFs of X and Y.
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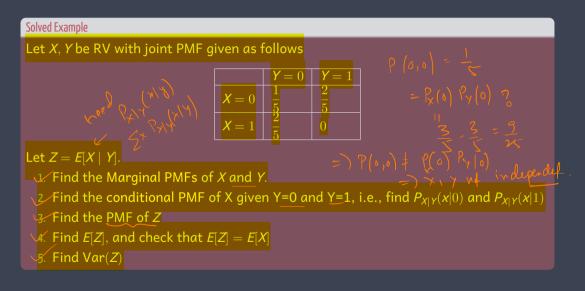
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- 3. Find the PMF of Z
- 4. Find E[Z], and check that E[Z] = E[X]



» Answer to previous problem... 2 3/5, y=0 } Boroulli X1 y independent? And No

We need 
$$P_2(z)$$
, that what is the probability of  $Z=Z$ 

(i.e.  $P(Z=2/3)$  of  $P(Z=0)$ 

We need  $P_{2}(2)$ , that what is the probable of Z = Z15 the probable of Z = ZThe P(Z = 2/3) + P(Z = 0)? P(Z = 0) = P(Z = 1)

\* Answer to previous problem...

$$P_{2}(7) = \begin{cases} 3 \mid c \text{ if } 2 = 0 \\ 2 \mid c \text{ if } 2 = 0 \end{cases}$$

$$0, \text{ otherwise}$$

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$$0 \in [2] = \begin{cases} 2 = 0 \\ 3 = 1 \end{cases}$$

$$0 \in [2] = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

» Fact...

Fact

Let X, Y be two RVs and g, h be two functions of X and Y respectively. Show that

$$E[g(X)h(Y) \mid X] = g(X)E[h(Y) \mid X]$$

Solution

$$= \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left( \frac{1}{2} \right) \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} \right) \left$$



where the definitions...

Law of Iterated Expectations

Let 
$$X, Y$$
 be two RVs, then we have

$$E[X] = E[E[X | Y]]$$

Proof

Let  $g(Y) = E[X | Y]$ 

Applying Law & Total Probable of the definition of the probable of the proba

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Number of customers N visiting a fast food restaurant follows Poisson distribution  $N \sim \text{Poisson}(\lambda)$ .



#### Solved Example 2

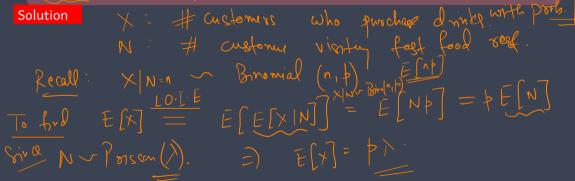
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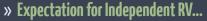
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Number of customers N visiting a fast food restaurant follows Poisson distribution  $N \sim \text{Poisson}(\lambda)$ . Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers. What is the average number of customers who purchase drinks?





# » Expectation for Independent RV...



#### **Expectation for Independent RVs**

Let X, Y be two independent RVs. Then we have the following

$$1 \cdot E[X \mid Y] = E[X]$$

$$2. E[g(X) \mid Y] = E[g(X)]$$

3. 
$$E[XY] = E[X]E[Y]$$

$$\mathscr{L}E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

$$D = \{x \mid x = x\} = \sum_{x \in R_{x}} x \in \mathbb{R}_{x}$$

$$E[g(x) \mid x] = E[g(x) \mid x] = \sum_{x \in R_{x}} x \in \mathbb{R}_{x}$$

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» Answer to previous problem...

 $= \sum_{x} n R_{x}(x) Z$  = E[x] E[7]

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- (1) Early !

**Definition of Conditional Variance** 

Let X, Y be two RVs.

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#### **Definition of Conditional Variance**

Let X, Y be two RVs. By  $Var(X \mid Y = y)$  the conditional variance of X given Y = y. Let  $\mu_{X|Y}(y) = E[X \mid Y = y]$ . Then

$$Var(X | Y = y) = E[X^2 | Y = y] - \mu_{X|Y}(y)^2$$

Proof