

# Probability and Statistics: Lecture-14

Monsoon-2020

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## » Independent Random Variables...

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We consider two random variables **independent** if

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$$P(\underline{A \cap B}) = P(\underline{A}, \underline{B})$$

### Independent Random Variables

We consider two random variables independent if

$$\underline{P(X = x, Y = y)} = \underline{P(X = x)} \underline{P(Y = y)},$$

for all x, y.

All events

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We consider two random variables **independent** if

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$$P(X = x, Y = y) = P(X = x)P(Y = y), \quad \text{for all } x, y.$$

In general, if two random variables are **independent**, then

$$P(X \in A, X \in B) = P(X \in A)P(Y \in B), \quad \text{for all sets } A \text{ and } B$$

Also, we have

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$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B), \quad \text{for all sets } A \text{ and } B$$

Also, we have

$$P(Y = y \mid X = x) = P(Y = y), \quad \text{for all } x, y$$

*Handwritten notes above the equation:  $P(A|B) = P(A)$*



## » Example: independent random variables, PMF

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### Problem

We toss a coin twice, and let  $X$  denote the number of heads we observe.

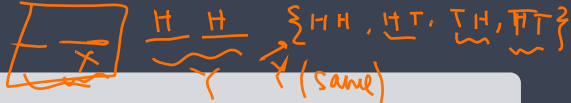
## » Example: independent random variables, PMF

### Problem

We toss a coin twice, and let  $X$  denote the number of heads we observe. After this, we then toss the coin two more times and define  $Y$  to be the number of heads we observe.



» Example: independent random variables, PMF



Problem

We toss a coin twice, and let  $X$  denote the number of heads we observe. After this, we then toss the coin two more times and define  $Y$  to be the number of heads we observe. What is  $P(X < 2)$  and  $(Y > 1)$ ?

Sol:  $P(\underline{X < 2} \text{ and } \underline{Y > 1}) = P(X < 2) P(Y > 1)$   $1 < Y \leq 2$   
 because  $X$  and  $Y$  are independent  $\Rightarrow Y = 2$

$$= (P_{\underline{X}}(0) + P_X(1)) P_Y(2)$$

$$= \left(\frac{1}{4} + \frac{1}{2}\right) \cdot \frac{1}{4} = \frac{3}{16}$$

## » Definition of Expectation..

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### Definition of Expectation

The **expectation** of a **discrete** random variable  $X$  is defined as

## » Definition of Expectation..

$$X = \{x_1, x_2, \dots, x_k\}$$

① PMF of r.v.  $X$     ②  $X = \text{value}$

### Definition of Expectation

The **expectation** of a **discrete** random variable  $X$  is defined as

$$E[X] = \sum_{\substack{x: p(x) > 0}} \underbrace{p(x)}_{\text{PMF}} \cdot x \quad E[X] = \sum_{i=1}^k x_i p(x_i)$$

$$p(x) \geq 0$$
$$\underline{p(x) = 0} \quad x$$

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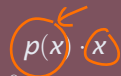
\* Note we sum over all values of  $X$  that have **non-zero** probability




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- \* Note we sum over all values of  $X$  that have **non-zero** probability
  - \* The other names of expectations are: mean, expected value, weighted average, center of mass, first moment
- 

## » Example of Expectation...

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$$X = \{1, 2, 3, 4, 5, 6\}$$

Example of Expectation...

What is the **expected** value of 6-sided die roll?

Step ① What is r.v.?

② PMF of  $X$

$$p(x) = \begin{cases} 1/6, & x \in \{1, 2, 3, \dots, 6\} \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{k=1}^6 x_k p(x_k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$
$$= \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$

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What is the **expected** value of 6-sided die roll?

Answer to the question

- \* **Define a random variable:**  $X = \text{RV for value of a roll}$

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$$P(X = x) = \begin{cases} 1/6, & x \in \{1, \dots, 6\} \\ 0, & \text{otherwise} \end{cases}$$

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- \* **Solve**

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- \* **Find PMF:**

$$P(X = x) = \begin{cases} 1/6, & x \in \{1, \dots, 6\} \\ 0, & \text{otherwise} \end{cases}$$

- \* **Solve**

$$E[X] = 1 \left( \frac{1}{6} \right) + 2 \left( \frac{1}{6} \right) + 3 \left( \frac{1}{6} \right) + 4 \left( \frac{1}{6} \right) + 5 \left( \frac{1}{6} \right) + 6 \left( \frac{1}{6} \right) = \left( \frac{7}{2} \right) = 3.5$$



## » Recall: Linear Functions...

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### Definition of linear function

Let  $A, B$  be two **vector spaces**.

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### Definition of linear function


Let  $A, B$  be two **vector spaces**. A function  $f: A \rightarrow B$  is called **linear** function if the following holds:

$$* f(u + v) = f(u) + f(v) \quad , \quad u, v \in A$$

## » Recall: Linear Functions...

### Definition of linear function

Let  $A, B$  be two **vector spaces**. A function  $f: A \rightarrow B$  is called **linear** function if the following holds:

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  - \*  $f(cu) = cf(u)$ ,  $c$  is **scalar**.
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Equivalent definition is:

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- \*  $f(u+v) = f(u) + f(v)$
- \*  $f(cu) = cf(u)$ ,  $c$  is **scalar**.

} combine

$\left. \begin{array}{l} u \in A \\ u+v \in A \end{array} \right\}$

Equivalent definition is:

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v), \quad \forall \underline{u, v \in A}, \quad \alpha, \beta \text{ scalars}$$

» Quiz on linear functions...



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Which of the following functions are linear?

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1.  $f(x) = x$  ✓ *Lin*

2.  $f(x) = ax + b$  ✗

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### Problem on linear function

Which of the following functions are linear?

1.  $f(x) = x$
2.  $f(x) = ax + b$
3.  $f(x) = 10$  ✖

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5.  $f(x) = 3 \sin x$  ✂

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6.  $f(x, y) = 3x + 4y$  ✓

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7.  $f(x, y, z) = 3 + x + y + z$  ✗



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7.  $f(x, y, z) = 3 + x + y + z$

8.  $f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y - z \\ z \end{pmatrix}$  ✓

## » Properties of Expectation...

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$$E: \underline{\text{Sp. R.V.}} \rightarrow \mathbb{R}$$

### Linearity of Expectation

$$E[\underbrace{aX + b}] = \underbrace{aE[X]} + \underbrace{b}$$

## » Properties of Expectation...

### Linearity of Expectation

$$E[aX + \underline{b}] = aE[X] + b$$

$$E[X] = \sum_{i=1}^n p(x_i) x_i = 1$$

Let  $X$  = 6 sided dice roll. Let  $Y = 2X - 1$ . If  $E[X] = 3.5$ , then what is  $E[Y]$ ?

$$\begin{aligned} E[Y] &= E[2X - 1] \\ &= 2E[X] - E[1] \\ &= 2E[X] - 1 \\ &= 2 \cdot 3.5 - 1 \\ &= 6 \end{aligned}$$

## » Properties of Expectation...

### Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

### Expectation of a sum

$$E[X + Y] = E[X] + E[Y]$$

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$$E[\underline{X + Y}] = E[X] + E[Y]$$

$$E[X] = \sum_i x_i p(x_i)$$

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## » Properties of Expectation...

### Linearity of Expectation

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Let  $X = 6$  sided dice roll. Let  $Y = 2X - 1$ . If  $E[X] = 3.5$ , then what is  $E[Y] = ?$

Consider two die rolls. Let  $X =$  roll of die 1,  $Y =$  roll of die 2, then  $E[X + Y] = ?$

$$\begin{aligned} &= E[X] + E[Y] \\ &= \underline{\underline{3.5}} + \underline{\underline{3.5}} \\ &= \underline{\underline{7}} \quad \text{Avg} \end{aligned}$$

## » Properties of Expectation...

### Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

Let  $X$  = 6 sided dice roll. Let  $Y = 2X - 1$ . If  $E[X] = 3.5$ , then what is  $E[Y]$  =?

### Expectation of a sum

$$E[X + Y] = E[X] + E[Y]$$

Consider two dice rolls. Let  $X$  = roll of dice 1,  $Y$  = roll of die 2, then  $E[X + Y]$  =?

### Expectation of a function of random variable

$$E[g(X)] = \sum_x g(x)p(x)$$



## » Properties of Expectation...

### Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

### Expectation of a sum

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Consider two dice rolls. Let  $X =$  roll of dice 1,  $Y =$  roll of die 2, then  $E[X + Y] = ?$

Let  $X = 6$  sided dice roll. Let  $g(x) = x^2$ . If  $E[X] = 3.5$ , then what is  $E[g(X)] = ?$

$$E[g(x)] = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \dots + 6^2 \cdot \frac{1}{6}$$

$$E[g(x)] = \sum_{i=1}^6 g(x_i) p(x_i) = \sum_{i=1}^6 x_i^2 p(x_i)$$

## » Proof of Linearity of Expectation...

$$E[aX + b] = aE[X] + b$$

$$\begin{aligned} E[aX + b] &= \sum_x (ax + b) p(x) = \sum_x ax p(x) + b p(x) \\ &= a \sum_x x p(x) + b \sum_x p(x) \xrightarrow{1} \\ &= a E[X] + b \end{aligned}$$

## » Proof of Expectation of Function of a Random Variable...

$$E[g(X)] = \sum_x g(x) p(x)$$

→ Assignment



## » Quiz on Expectation...

### Question on Expectation

Let  $X$  be a discrete random variable, whose PMF is given as follows

$$P_X(x_k) = \begin{cases} \frac{1}{3}, & \text{for } x \in \{-1, 0, 1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $Y = |X|$ . What is  $E[Y]$ ?

$$\begin{aligned} E[Y] &= \sum_x g(x) p(x) = g(-1)p(-1) + g(0)p(0) + g(1)p(1) \\ &= 1 \cdot \frac{1}{3} + 0 + 1 \cdot \frac{1}{3} = \frac{2}{3} \end{aligned}$$

## » Expectation and Game of Chance...

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### Expectation and Game of Chance Problem

Consider a game where a die is rolled. After the roll, the total money you gain is the output of the die. How much money would you like to pay to play this game?

$$E[X] \sim \underline{\underline{3.5}}$$





## » Saint Petersburg Paradox...

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Consider a fair coin.

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Consider a fair coin. We toss the coin, if it is heads, the game ends, and you win Rs 2.

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Consider a fair coin. We toss the coin, if it is heads, the game ends, and you win Rs 2. If it turns out to be tails, I toss it again. If it is heads on the second toss, you win Rs 4.

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Consider a fair coin. We toss the coin, if it is heads, the game ends, and you win Rs 2. If it turns out to be tails, I toss it again. If it is heads on the second toss, you win Rs 4. If not, I toss it again, and if it is heads on the third toss, you win Rs 8.

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## » Saint Petersburg Paradox...

T T T . . . T H  
 $n-1^{\text{th}}$   $2^n$

### Saint Petersburg Paradox

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There is an entrance fee to play this game. What is the maximum amount of money you would pay to play this game?

## » Analyzing Solution to Saint Petersburg Paradox Problem...



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$n$	Prize	Prob	Utility
1	Rs 2	$1/2$	Rs 2
2	Rs 4	$1/4$	Rs 4
3	Rs 8	$1/8$	Rs 8
4	Rs 16	$1/16$	Rs 16
5	Rs 32	$1/32$	Rs 32 ✓
6	Rs 64	$1/64$	Rs 64 ✓
7	Rs 128	$1/128$	Rs 128 ✓
8	Rs 252	$1/252$	Rs 252 ✓
⋮	⋮	⋮	⋮

100

Rs 5

2/15

10

1/2

## » Analyzing Solution to Saint Petersburg Paradox Problem...

$n$	Prize	Prob	Utility
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Let us make some observations.

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⋮	⋮	⋮	⋮

Let us make some observations.

- \* If we pay Rs 1, then if the game ends in 1st toss, then we get Rs 2. That is we gain Rs 1.

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Let us make some observations.

- \* If we pay Rs 1, then if the game ends in 1st toss, then we get Rs 2. That is we gain Rs 1.
- \* If we pay say Rs 10, and the game ends in 3rd toss, then we get Rs 8, and we loose Rs 2.
- \* Naturally, we wonder what is the expected value?

### Expected Value for Saint Petersburg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

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### Expected Value for Saint Petersburg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \overset{\text{Rs}}{\infty}$$

So, are you willing to pay any amount to play this game given that in theory you can win upto Rs  $\infty$ ?

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### Expected Value for Saint Petersburg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, are you willing to pay any amount to play this game given that in theory you can win upto Rs  $\infty$ ? Can you see why this seems to be a paradox?

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Expected Value for Saint Petersburg Paradox...

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Expected Value for Saint Petersburg Paradox...

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So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs  $\infty$ ?



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Expected Value for Saint Petersburg Paradox...

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Expected Value for Saint Petersburg Paradox...

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So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs  $\infty$ ? Can you see why this seems to be a paradox?

It is **paradoxical** because of the following reasons:

## » Analyzing Solution to Saint Petersburg Paradox Problem...

Expected Value for Saint Petersburg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs  $\infty$ ? Can you see why this seems to be a paradox?

It is **paradoxical** because of the following reasons:

- \* In reality, very few would pay even Rs 15 to play this game.

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- \* In reality, very few would pay even Rs 15 to play this game. **Why?**
- \* **Paradox:** you should be willing to pay any amount, but you **won't** be willing to do so!

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$n$	Prize	Prob	Utility	Exp.
1	Rs 2	$1/2$	Rs 2	1
2	Rs 4	$1/4$	Rs 4	1
3	Rs 8	$1/8$	Rs 8	1
4	Rs 16	$1/16$	Rs 16	1
5	Rs 32	$1/32$	Rs 32	1
6	Rs 64	$1/64$	Rs 32	0.5
7	Rs 128	$1/128$	Rs 32	0.25
8	Rs 252	$1/252$	Rs 32	0.125
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Handwritten calculations illustrating the expected value calculation for the Saint Petersburg Paradox:

$$2 \cdot \frac{1}{2} = 1$$

$$4 \cdot \frac{1}{4} = 1 + 1$$

$$2 \cdot \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$2 \cdot \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

The sum of the series is shown to be 10, indicating a finite expected value.



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- \* If we pay Rs 1, then if the game ends in 1st toss, then we get Rs 2. That is we gain Rs 1.
- \* If we pay say Rs 10, and the game ends in 32nd toss, then on average, we get Rs 6, and we loose Rs 2.

$$1 + 1 + 1 + 1 + 1 + 1 + \dots$$

$\approx 6$

Rs 2

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So, we should not pay more than Rs 6 or much lower than this!