Probability and Statistics: Lecture-7

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on August 24, 2020

» Table of contents

» Shrewd Prisoner and Maximizing Chance!

Shrewd Prisoner Problem

Long ago a prisoner was to be executed.

» Shrewd Prisoner and Maximizing Chance!

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns.

» Shrewd Prisoner and Maximizing Chance!

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns,

» Shrewd Prisoner and Maximizing Chance!

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns, in any way he liked,

» Shrewd Prisoner and Maximizing Chance!

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew a white ball from one of two similar urns. The provisions were that he had to distribute 50 white and 50 black balls between the two urns, in any way he liked, after which he had to draw a ball at random from one of these urns.

» Shrewd Prisoner and Maximizing Chance!

Shrewd Prisoner Problem

Long ago a prisoner was to be executed. In response to his supplications, he was promised that he would be released if he drew *a* white ball from one of two similar urns. The provisions were that he had to distribute 50 white *and* 50 black balls between the two urns, in *any way* he liked, *after* which he had to draw *a* ball *at* random from one of these urns.

Question: How should the prisoner put the balls such that the probability of his release is maximized?

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?
* What happens if he puts equal number of white and black balls in two boxes?

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?
* What happens if he puts equal number of white and black balls in two boxes?
    * The probability of choosing a white ball (from either boxes) is $1/2$

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?
* What happens if he puts equal number of white and black balls in two boxes?
    * The probability of choosing a white ball (from either boxes) is $1/2$
    * Can he do better?

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?
* What happens if he puts equal number of white and black balls in two boxes?
    * The probability of choosing a white ball (from either boxes) is $1/2$
    * Can he do better?
* What if he puts all white in one, and all black in other urn?

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?
* What happens if he puts equal number of white and black balls in two boxes?
    * The probability of choosing a white ball (from either boxes) is $1/2$
    * Can he do better?
* What if he puts all white in one, and all black in other urn?
    * probability of choosing white from urn-1 is 1/2, but probability of choosing white from urn-2 is 0. Can he do better?

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?
* What happens if he puts equal number of white and black balls in two boxes?
  * The probability of choosing a white ball (from either boxes) is $1/2$
  * Can he do better?
* What if he puts all white in one, and all black in other urn?
  * probability of choosing white from urn-1 is 1/2, but probability of choosing white from urn-2 is 0. Can he do better?
* Can he do better than this? If yes, then how?

» Solution to Shrewd Prisoner Problem

* How did the prisoner arrange to make his chance of success as great as possible?
* What happens if he puts equal number of white and black balls in two boxes?
    * The probability of choosing a white ball (from either boxes) is $1/2$
    * Can he do better?
* What if he puts all white in one, and all black in other urn?
    * probability of choosing white from urn-1 is 1/2, but probability of choosing white from urn-2 is 0. Can he do better?
* Can he do better than this? If yes, then how?
* Put 1 white and 0 black in urn-1, and put 49 white and 100 black in urn-2

» Solution to Shrewd Prisoner Problem

- How did the prisoner arrange to make his chance of success as great as possible?
- What happens if he puts equal number of white and black balls in two boxes?
  - The probability of choosing a white ball (from either boxes) is $1/2$
  - Can he do better?
- What if he puts all white in one, and all black in other urn?
  - probability of choosing white from urn-1 is 1/2, but probability of choosing white from urn-2 is 0. Can he do better?
- Can he do better than this? If yes, then how?
- Put 1 white and 0 black in urn-1, and put 49 white and 100 black in urn-2
- The probability of success is $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{49}{99} \approx 3/4$

» Extension of the problem

Quiz-1

What is the chance of success of the prisoner if he had been allowed to distribute 100 balls among 4 urns?

» Extension of the problem

Quiz-1

What is the chance of success of the prisoner if he had been allowed to distribute 100 balls among 4 urns?

Quiz-2

What if the number of balls is increased?

» A related Problem in Game Theory: Prisoner's dilemma

» A related Problem in Game Theory: Prisoner's dilemma

* If both of them do not confess, they serve 2 years each

» A related Problem in Game Theory: Prisoner's dilemma

- If both of them do not confess, they serve 2 years each
- If they confess to crime, partner does not, then person confessing will be granted immunity. However, in this case the partner not confessing will serve 10 years

» A related Problem in Game Theory: Prisoner's dilemma

* If both of them do not confess, they serve 2 years each
* If they confess to crime, partner does not, then person confessing will be granted immunity. However, in this case the partner not confessing will serve 10 years
* If both confess, then both end up spending 5 years each in jail

» A related Problem in Game Theory: Prisoner's dilemma

* If both of them do not confess, they serve 2 years each
* If they confess to crime, partner does not, then person confessing will be granted immunity. However, in this case the partner not confessing will serve 10 years
* If both confess, then both end up spending 5 years each in jail
* Given that Wanda and Fred have no reason to trust, what is the good option?


Wanda
Fred
DON'T CONFESS
CONFESS
DON'T CONFESS
2/2
10/0
CONFESS
0/10
5/5

* Indeed, if both don't confess,

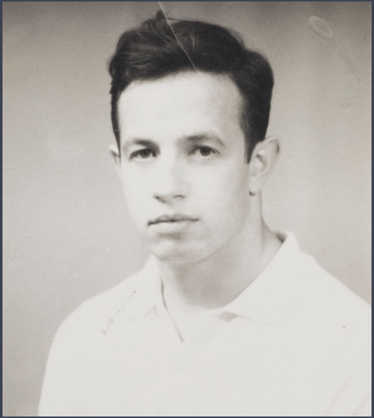
* Indeed, if both don't confess, then it is best for them;

* Indeed, if both don't confess, then it is best for them; 2 years each!

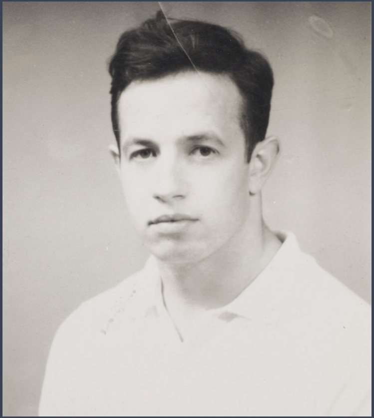
- Indeed, if both don't confess, then it is best for them; 2 years each!
- But another best option is when they both confess. Because in other cases, one of then has to spend 10 years in jail

- Indeed, if both don't confess, then it is best for them; 2 years each!
- But another best option is when they both confess. Because in other cases, one of then has to spend 10 years in jail
- This is part of co-operative games, and $5/5$ is called Nash equilibrium
- These topics are part of topic known as game theory (John Nash!)

» About John Nash

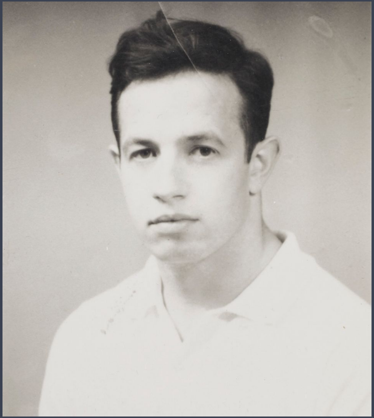


» About John Nash



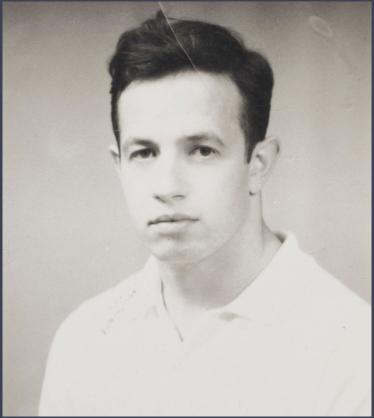
* John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations

» About John Nash



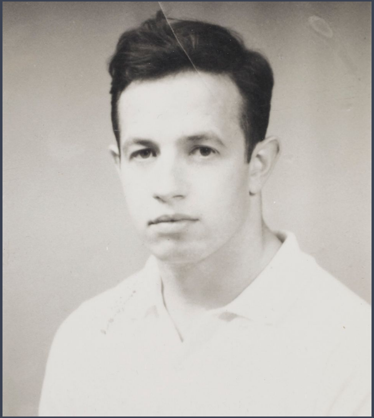
- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994

» About John Nash



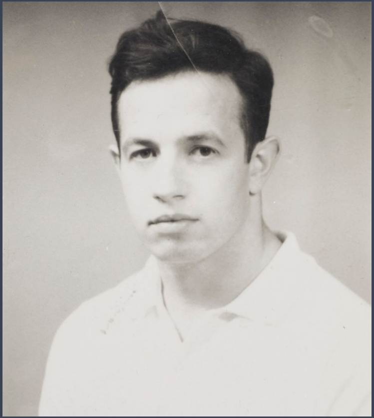
- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994
  - Abel Prize in 2015

» About John Nash



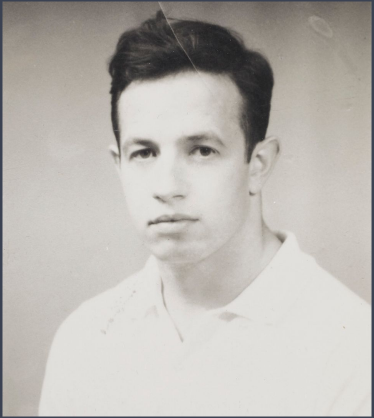
- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994
  - Abel Prize in 2015
- Movie on him: “A Beautiful Mind”

» About John Nash



* John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
* Awards
    * Nobel Prize in 1994
    * Abel Prize in 2015
* Movie on him: "A Beautiful Mind"
* Bar Scene (Game theory part...): `https://www.youtube.com/watch?v=LJS7Igvk6ZM`

» About John Nash



- John Forbes Nash Jr. (June 13, 1928 – May 23, 2015) was an American mathematician who made fundamental contributions to game theory, differential geometry, and the study of partial differential equations
- Awards
  - Nobel Prize in 1994
  - Abel Prize in 2015
- Movie on him: “A Beautiful Mind”
- Bar Scene (Game theory part...): `https://www.youtube.com/watch?v=LJS7Igvk6ZM`

More on this in Topics in Applied Optimization Elective!

» Watch Bar Scene in “A Beautiful Mind” Movie

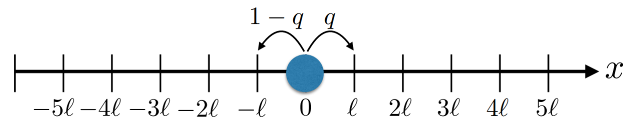
» Watch Bar Scene in “A Beautiful Mind” Movie

Movie of Bar Scene Here!

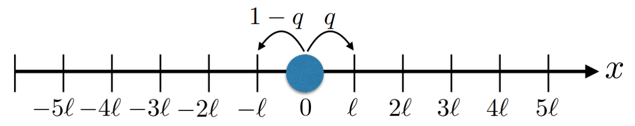
Movie of 1st Brownian Motion

Movie of 2nd Brownian Motion

» Random Walk, Probability

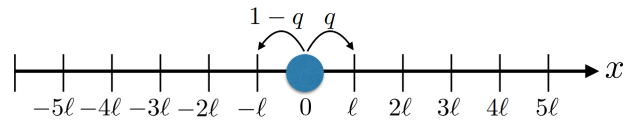


» Random Walk, Probability



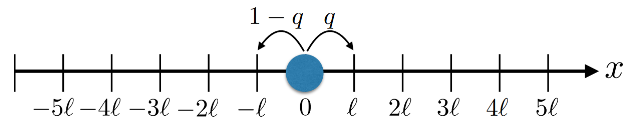
* Consider a person at $x = 0$, he can travel one step to the right or to the left

» Random Walk, Probability



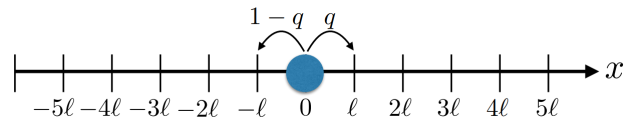
- Consider a person at $x = 0$, he can travel one step to the right or to the left
  - He can travel one step to the right with probability q

» Random Walk, Probability



* Consider a person at $x = 0$, he can travel one step to the right or to the left
  * He can travel one step to the right with probability q
  * He can travel one step to the left with probability $(1 - q)$

» Random Walk, Probability

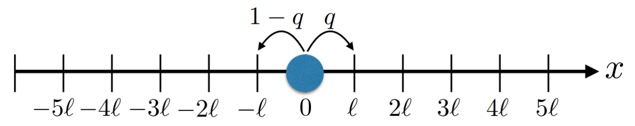


- Consider a person at $x = 0$, he can travel one step to the right or to the left
  - He can travel one step to the right with probability q
  - He can travel one step to the left with probability $(1 - q)$

Simple Random Walk in 1D

A walk is called simple random walk in 1D if there is a equal probability of either going to right or going to the left. Above, we set $p = q = 1/2$

» Random Walk, Probability



- Consider a person at $x = 0$, he can travel one step to the right or to the left
  - He can travel one step to the right with probability q
  - He can travel one step to the left with probability $(1 - q)$

Simple Random Walk in 1D

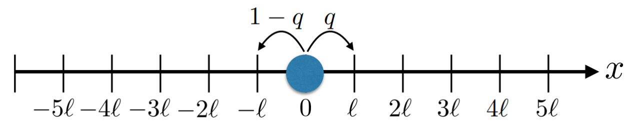
A walk is called simple random walk in 1D if there is a equal probability of either going to right or going to the left. Above, we set $p = q = 1/2$

Question

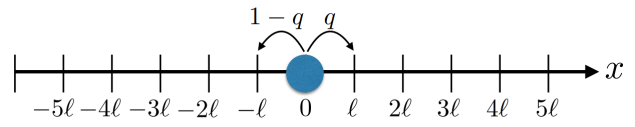
What is the probability that the person after ith step is at $x = 0$?

» Analysis of Simple Random Walk in 1D

» Analysis of Simple Random Walk in 1D



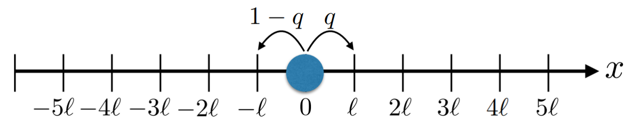
» Analysis of Simple Random Walk in 1D



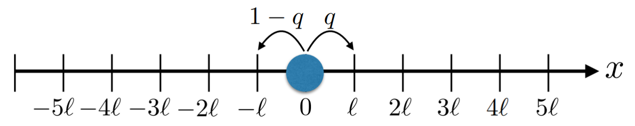
Draw the choice tree (Hint: Galton Board!):

» Analysis of Biased Random Walk in 1D, Binomial Distribution

» Analysis of Biased Random Walk in 1D, Binomial Distribution



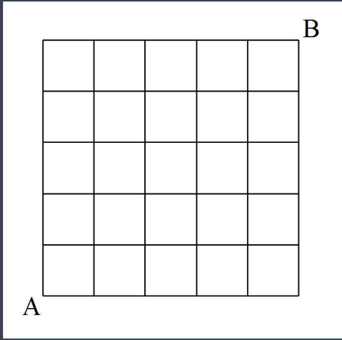
» Analysis of Biased Random Walk in 1D, Binomial Distribution



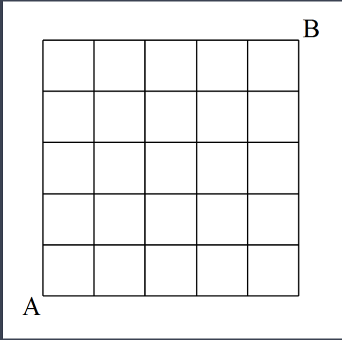
Draw the choice tree for unbiased random walk, derive binomial distribution:

» Random Walkers on 2D Grid...

» Random Walkers on 2D Grid...

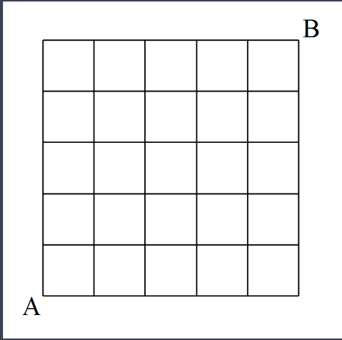


* Alice starts at point A, and each second, walks one edge right or up (if a point has two options, each direction has a 50



* Alice starts at point A, and each second, walks one edge right or up (if a point has two options, each direction has a 50
* At the same time, Bob starts at point B, and each second he walks one edge left or down (if a point has two options, each direction has a 50

» Random Walkers on 2D Grid...

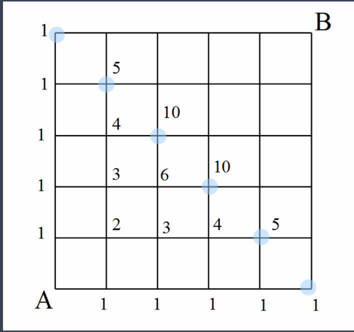


* Alice starts at point A, and each second, walks one edge right or up (if a point has two options, each direction has a 50
* At the same time, Bob starts at point B, and each second he walks one edge left or down (if a point has two options, each direction has a 50
* What is the probability Alice and Bob meet during their random walks?

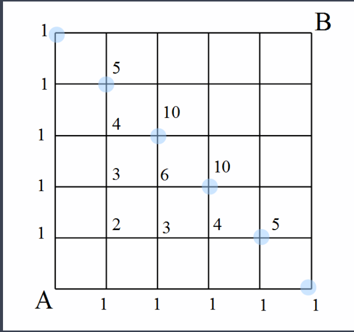
» Solution: Random Walkers on 2D Grid...

» Solution: Random Walkers on 2D Grid...

* If Alice and Bob meet, they can meet only at the blue circles, which is after both have taken 5 steps

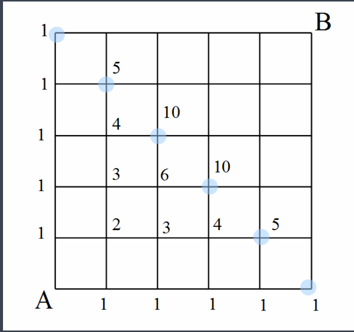


» Solution: Random Walkers on 2D Grid...



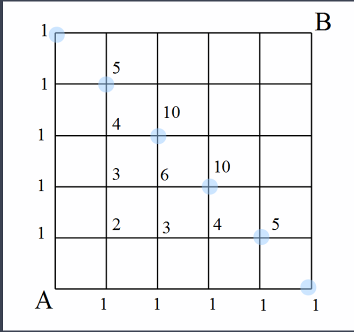
- If Alice and Bob meet, they can meet only at the blue circles, which is after both have taken 5 steps
- Number of ways they could have taken the 5 steps is 2^5 each, so combined, by product rule they can take $2^5 2^5 = 4^5 = 1024$ total paths!

» Solution: Random Walkers on 2D Grid...



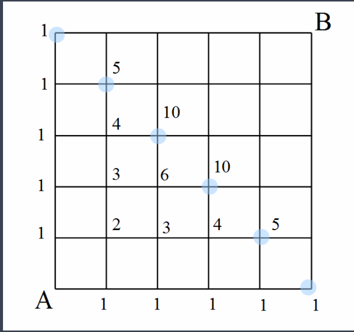
- If Alice and Bob meet, they can meet only at the blue circles, which is after both have taken 5 steps
- Number of ways they could have taken the 5 steps is 2^5 each, so combined, by product rule they can take $2^5 2^5 = 4^5 = 1024$ total paths!
- The number of ways Bob can reach the blue dots is given by binomial coefficients or Pascal's triangle! Same for Alice

» Solution: Random Walkers on 2D Grid...



- If Alice and Bob meet, they can meet only at the blue circles, which is after both have taken 5 steps
- Number of ways they could have taken the 5 steps is 2^5 each, so combined, by product rule they can take $2^5 2^5 = 4^5 = 1024$ total paths!
- The number of ways Bob can reach the blue dots is given by binomial coefficients or Pascal's triangle! Same for Alice
  - Total ways Bob and Alice could meet at blue dots is square of binomial cofficients

» Solution: Random Walkers on 2D Grid...



- If Alice and Bob meet, they can meet only at the blue circles, which is after both have taken 5 steps
- Number of ways they could have taken the 5 steps is 2^5 each, so combined, by product rule they can take $2^5 2^5 = 4^5 = 1024$ total paths!
- The number of ways Bob can reach the blue dots is given by binomial coefficients or Pascal's triangle! Same for Alice
  - Total ways Bob and Alice could meet at blue dots is square of binomial cofficients
- Hence, the total number of ways Bob and Alice could meet

$$1^2 + 5^2 + 10^2 + 5^2 + 1^2 = 252$$

- The probability that they meet is

$$252/1024 = 24.6\%$$

» Conditional Probability...

- **Experiment:** Throw two dice A and B simultaneously
- **Event:** Odd number on first die
- **Question:** What is the probability of the event E?

- **Event:** Odd number on first die A, and die B always shows even
- **Question:** What is the probability of the event E?