Probability and Statistics: Lecture-34

Monsoon-2020

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on November 2, 2020

» Computing CDF of Function of Two RVs...

Computing PDF of Function of Two RVs

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F_Z(z) &= P(Z \leq z) \\
&= P(g(X, Y) \leq z) \\
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$0 < y < 1$, $xy < z$, $x < z/y$

where $D = \{(x, y) \mid g(x, y) < z\}$. To compute the PDF, we need to $\boxed{\text{differentiate } F_Z(z)}$.

Example (Example)

Let X, Y be two independent Uniform$(0, 1)$ RVs, and $Z = XY$. Find the CDF and PDF of Z.

» Answer to previous problem...

$R_Z = [0,1]$. Then

$F_Z(z) = 0$ for $z \le 0$

$F_Z(z) = 1$ for $z \geq 1$

For $0 < z < 1$, we have

$$F_Z(z) = P(Z \le z) = P(XY \le z)$$

$$= P\left(X \le \frac{z}{y}\right)$$

$$P\left(X \le \frac{z}{y}\right) = \int_0^1 \int_0^{z/y} f_{XY}(x,y)\, dx\, dy$$

limit of y — range of x

Since $X \sim \text{Uniform}(0,1)$

$$= \int_0^1 \int_0^{\min(1, z/y)} 1.\ dx\, dy$$

$$= \int_0^1 \min(1, z/y)\, dy$$

Let $g(y) = \min(1, z/y)$

$$g(y) = \begin{cases} 1 & \text{for } 0 < y < z \\ z/y & \text{for } z \le y \le 1 \end{cases}$$

$$P\left(X \le \frac{z}{y}\right) = \int_0^1 g(y)\, dy = \int_0^z 1\, dy + \int_z^1 \frac{z}{y}\, dy$$

$P(X \le \frac{z}{Y}) = z - z\ln z$.

Another way (using total prob.)

$$P(X \le \frac{z}{Y}) = \int_0^1 P(X \le \frac{z}{y} \mid Y = y) f_Y(y)\, dy$$

$$= \int_0^1 P(X \le \frac{z}{y}) f_Y(y)\, dy \quad [\text{Since } X, Y \text{ ind}]$$

$$P(X \le \frac{z}{y}) = \begin{cases} 1 & \text{for } 0 < y < z \\ z/y & \text{for } z \le y \le 1 \end{cases}$$

$$P(X \le z/y) = \int_0^z dy + \int_z^1 \frac{z}{y}\, dy$$

$$= z - z\ln z.$$

To summarize.

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ z - z\ln z & 0 < z < 1 \\ 1 & z \ge 1 \end{cases}$$

Diff. $F_Z(z)$:

$$f_Z(z) = \begin{cases} -\ln z & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

» Method of Transformation...

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Method of Transformation

Let X and Y be two jointly continuous RVs.

» Method of Transformation...

$f(x,y) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$

$f_1 = x+y, \quad f_2 = x-y.$

Method of Transformation

Let X and Y be two jointly continuous RVs. Let $(Z, W) = g(X, Y) = (g_1(X, Y), g_2(X, Y))$, where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous one-to-one function with continuous partial derivatives.

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$$f_{ZW}(z, w) = f_{XY}(h_1(z, w), h_2(z, w)) \, |J|,$$

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where J is the Jacobian of h defined by

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \frac{\partial h_1}{\partial z} \cdot \frac{\partial h_2}{\partial w} - \frac{\partial h_2}{\partial z} \cdot \frac{\partial h_1}{\partial w}$$

» Example of Method of Tranformation...

Example (Example)

Let X and Y be two independent standard normal RVs.

Q3

$$\begin{bmatrix} z \\ w \end{bmatrix} = g(x,y) = A \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} z \\ w \end{bmatrix}$$

Q① Is this a Linear Transformation

Example (Example)

Let X and Y be two independent standard normal RVs. Let

$$Z = 2X - Y$$

$$W = -X + Y$$

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x - y \\ -x + y \end{bmatrix}$$

Q2 When is a Lin. Transf invertible.

A. T is 1-1 & onto

$\iff$ $[T] = A$ is invertible matrix.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ is a Lin transfm

iff $\exists\, A \in \mathbb{R}^{2\times 2}$ s.t

$$\begin{bmatrix} z \\ w \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

~~det~~ $\boxed{\det = 1}$

Example (Example)

Let X and Y be two independent standard normal RVs. Let

$$Z = 2X - Y$$
$$W = -X + Y$$

Find $f_{ZW}(z, w)$.

» Answer to previous problem...

Given that x & Y are jointly continuous & their joint PDF is

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) \longrightarrow (*)$$

Since $X \sim$ Std. Normal $\equiv N(0,1)$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$(*) \Rightarrow \underbrace{f_{XY}(x,y)} = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

The fn g is defined by

$$(z,w) = g(x,y)$$
$$= (g_1(x,y),\, g_2(x,y))$$
$$= (2x-y,\, -x+y)$$

Solving for x, y

$$2x-\cancel{y} = z$$
$$(+)\quad -x+\cancel{y} = w$$
$$x = z+w$$

& $y = z+2w$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z+w \\ z+2w \end{bmatrix} = \begin{bmatrix} h_1(z,w) \\ h_2(z,w) \end{bmatrix}$$

$h_1 = z + w$

$h_2 = z + 2w$

$$f_{zw}(z,w) = f_{xy}\big(h_1(z,w), h_2(z,w)\big)\,|J|$$

$$J = \det \begin{bmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1$$

Recall

$$f_{xy}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

$$\boxed{f_{zw}(z,w) = \frac{1}{2\pi} e^{-\frac{\left((z+w)^2 + (z+2w)^2\right)}{2}} \times J}$$

$J = 1$

=

Example (Example of Method of Transform)

Let X, Y be two RVs with joint PDF $f_{XY}(x, y)$. Let $Z = X + Y$. Find $f_Z(z)$.

Exercise

» Answer to previous problem...

» Convolution and PDF...

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Let X, Y be two jointly continuous RVs and $Z = X + Y$, then

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Let X, Y be two jointly continuous RVs and $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(w, z - w)\, dw = \int_{-\infty}^{\infty} f_{XY}(z - w, w)\, dw.$$

Convolution and PDF

Let X, Y be two jointly continuous RVs and $Z = X + Y$, then

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If X, Y are also independent, then we have

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Let X, Y be two jointly continuous RVs and $Z = X + Y$, then

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If X, Y are also independent, then we have

convolution operator.

$$f_Z(z) = f_X(x) * f_Y(y)$$

$$= \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\, dw = \int_{-\infty}^{\infty} f_Y(w) f_X(z-w)\, dw$$

» Solved Example...

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Example (Solved example)

Let X and Y be two independent standard normal RVs, and let $Z = X + Y$.

$x - y \quad x + v \quad v = -y.$

Example (Solved example)

Let X and Y be two independent standard normal RVs, and let $Z = X + Y$. Find the PDF of Z.

We have $f_Z(z) = f_X(z) * f_Y(z) = \int_{-\infty}^{\infty} f_X(w)\, f_Y(z-w)\, dw$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-w^2/2} \cdot e^{-\frac{(z-w)^2}{2}} \, dw = \frac{1}{\sqrt{4\pi}} e^{-z^2/4} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(w - z/2)^2} dw}_{1}$$

PDF of Normal RV with mean $z/2$ & var $= \frac{1}{2}$.

$$= \frac{1}{\sqrt{4\pi}} e^{-z^2/4}.$$

» Solved Problem 1

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Example (Solved Problem)

Let X, Y be jointly continuous RVs with joint PDF

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Example (Solved Problem)

Let X, Y be jointly continuous RVs with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx + 1 & x, y \geq 0,\ x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

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Let X, Y be jointly continuous RVs with joint PDF

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1. Find the range of (X, Y) and plot it

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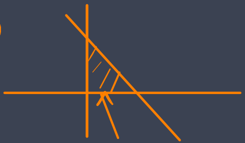
1. Find the range of (X, Y) and plot it
2. Find the constant c
3. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$
4. Find $P(Y < 2X^2)$

» Answer to previous problem...

Sol^n: (1)



R_{XY}

(2) $$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} cx+1\,dx\,dy$$

easy

$$= \int_0^1 ((cx+1)(1-x)$$

$=$

$$\Rightarrow \int_0^1 (cx - (cx^2 + 1 - x = 1$$

$$\Rightarrow \left[\frac{cx^2}{2} - \frac{cx^3}{3} + x - \frac{x^2}{2}\right]_0^1$$

$$\Rightarrow \frac{c}{2} - \frac{c}{3} + 1 - \frac{1}{2} = 1$$

$$\Rightarrow \frac{c}{6} + \frac{1}{2} = 1$$

$$\Rightarrow c = 3.$$

» Answer to previous problem...

③ $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$= \int_0^{1-x} (3x+1)\, dy = (3x+1)(1-x)$

for $x \in [0,1]$

$f_X(x) = \begin{cases} (3x+1)(1-x) & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$

Similarly

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

$= \int_0^{1-y} (3x+1)\, dx = \frac{1}{2}(1-y)(5-3y)$

$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2}(1-y)(5-3y) \\ 0 & \text{otherwise} \end{cases}$

(4) $$P(Y < 2X^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\min\{2x^2,\, 1-x\}} f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^{\min\{2x^2,\, 1-x\}} (3x+1)\, dx\, dy = \int_0^1 (3x+1) \min(2x^2, 1-x)\, dx$$

$$= \int_0^{1/2} 2x^2(3x+1)\, dx + \int_{1/2}^{1} (3x+1)(1-x)\, dx$$

$$=$$