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Prob & Stats 201910056 Aryan Jain

Ans 1 (1) $Y = aX + b$

Let CDF of $x = F_x(x)$

$$F_y(y) = P(X \le y) = P(aX + b \le y)$$

$$* \; a > 0 \qquad F_y(y) = P\left(X \le \frac{y-b}{a}\right) = F_x\left(\frac{y-b}{a}\right)$$

$$* \; a < 0 \qquad F_y(y) = P\left(X \ge \frac{y-b}{a}\right) = 1 - F_x\left(\frac{y-b}{a}\right)$$

$$\frac{d F_y(y)}{dy} = \frac{d F_x\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} \quad \text{\{chain Rule\}}$$

$$= \frac{1}{a} f_x\left(\frac{y-b}{a}\right) \quad \forall \; a > 0$$

and similarly

$$= -\frac{1}{a} f_x\left(\frac{y-b}{a}\right) \quad \forall \; a < 0$$

$$\Rightarrow f_y(y) = \begin{cases} \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right) & , \left\{\frac{y-b}{a}\right\} \in D_x \rightarrow \text{Domain of } x. \\ 0 & \text{otherwise} \end{cases}$$

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Ans 1(2) $\therefore f_Y(y) = \dfrac{f_X(x)}{a}$ where $x = \dfrac{y-b}{a}$.

given $f_X(x) = \lambda e^{-\lambda x}$ where $\lambda > 0$.

$$\therefore f_Y(y) = \frac{\lambda e^{-\lambda x}}{a} = \frac{\lambda}{a} e^{-\lambda\left(\frac{y-b}{a}\right)}$$

$$f_Y(y) = \frac{\lambda}{a} \cdot e^{\lambda b/a} \cdot e^{-\lambda y/a} = \lambda' e^{-\lambda' y} \qquad \lambda' > 0$$

$\rightarrow$ Y is exponential random variable

$\cancel{\frac{\lambda}{a} \cdot e^{\lambda b/a} = \lambda'}$ and $\cancel{\frac{-\lambda y}{a} = -\lambda' y}$

$\cancel{\Rightarrow \boxed{\lambda = \lambda'}}$

$\cancel{\therefore \frac{\lambda}{a} e^{\lambda b/a} = \lambda}$ $\cancel{\therefore e^{\lambda b/a} = a}$

$\cancel{\therefore \ln\left(\frac{\lambda b}{a}\right)}$

$$-\frac{\lambda y}{a} = -\lambda' y \Rightarrow \boxed{\lambda' = \lambda/a} \quad \because \lambda' > 0, \ \lambda > 0 \Rightarrow \boxed{a > 0}$$

and $\dfrac{\lambda}{a} \cdot e^{\lambda b/a} = \lambda'$

$$\therefore \frac{\lambda}{a} e^{\lambda b/a} = \lambda/a$$

$$\therefore e^{\lambda b/a} = 1 \Rightarrow \frac{\lambda b}{a} = 0$$

$$\because a, \lambda > 0 \Rightarrow \boxed{b = 0} \ \times$$

But given in ques a and b are non-zero real constants $\rightarrow b \neq 0$

$\Rightarrow$ NO solution.

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Ans. (3) $\because f_Y(y) = \dfrac{f_X(x)}{|a|}$ where $x = \dfrac{y-b}{a}$

$\because$ X is a normal random variable

$\therefore f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\therefore \boxed{f_Y(y) = \frac{1}{|a|\sigma.\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\frac{y-b}{a}-\mu}{\sigma}\right)^2}} \rightarrow (1)$$

$\because Y = aX + b$ $\quad \therefore E[Y] = a\mu + b$

$V[Y] = a^2\sigma^2$.

$\therefore \mu', \sigma'$: Mean, Variance of Y

$\therefore \mu' = a\mu + b \quad \sigma' = |a|\sigma \quad (\because \sigma > 0)$

$\therefore f_Y(y)$ can be written from this assuming Y to be a normal random variable

$\therefore$ $$f_Y(y) = \frac{1}{\sigma'.\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu'}{\sigma'}\right)^2}$$

substitute values

$$\therefore f_Y(y) = \frac{1}{|a|\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-(a\mu+b)}{|a|\sigma}\right)^2}$$

$$\boxed{f_Y(y) = \frac{1}{|a|\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{\frac{y-b}{a}-\mu}{\sigma}\right)^2}} \rightarrow (2)$$

(1) = (2)

Both (1) and (2) are equal $\Rightarrow$ Y will always be normal random variable if X is a normal random variable.
(No conditions on a & b).

Ans.2 (1.) $\because \quad F_x(x) = P\left(z \le \frac{x-\mu}{\sigma}\right) = \phi\left(\frac{x-\mu}{\sigma}\right) = P(X \le x)$

CDF of y: $F_y(y) = P(Y \le y) = P(e^x \le y)$ $\quad \because x = \sigma z + \mu$

$$= P(e^{\sigma z + \mu} \le y) = P(\sigma z + \mu \le \ln y)$$

$$\Rightarrow P\left(z \le \frac{\ln y - \mu}{\sigma}\right)$$

$$= \phi\left(\frac{\ln y - \mu}{\sigma}\right)$$

$$\Rightarrow \boxed{F_y(y) = \phi\left(\frac{\ln y - \mu}{\sigma}\right)}$$

$\phi \rightarrow$ CDF of z (standard normal).

Ans.2 (2.) ~~pdf~~ without using Method of Transformation

$$f_y(y) = \frac{d}{dy} F_y(y) = F_y'(y)$$

where $F_y(y) = \phi\left(\frac{\ln y - \mu}{\sigma}\right)$

$$\therefore f_Y(y) = \frac{1}{\sigma y} \cdot \phi'\left(\frac{\ln y - \mu}{\sigma}\right)$$

$$f_Y(y) = \frac{1}{\sigma y} \cdot f_Z\left(\frac{\ln y - \mu}{\sigma}\right)$$

$\sim N(0,1)$

Z : Standard Normal Random Variable

$$\cancel{f_Y(y) = \frac{1}{\sigma y}} \quad \because f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Put $z = \frac{\ln y - \mu}{\sigma}$

$$\therefore f_Y(y) = \frac{1}{\sigma \cdot y} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}$$

$$\therefore \boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma \cdot y} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}} \rightarrow (1)$$

By applying Method of transformation

$$f_Y(y) = \frac{f_X(x)}{g'(x)} \quad \text{where} \quad y = g(x) = e^x$$

$$g'(x) = e^x$$

$$f_Y(y) = \frac{f_X(x)}{e^x} = \frac{1}{e^x} \cdot \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}_{f_X(x)}$$

$\because y = e^x$

$\therefore x = \ln y$

$$\therefore \boxed{f_Y(y) = \frac{1}{y \cdot \sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}} \rightarrow (2)$$

~~By bo~~ (1) = (2) By both methods we got the same PDF.

Ans 3 Let X: Prob. of obtaining 2 $\therefore P(X) = 1/6 \quad P(X') = 5/6$

$\therefore$ Asked prob. $= \sum_{i=1901}^{2149} {}^{12000}C_i \, (1/6)^i \, (5/6)^{12000-i}$

By Binomial distribu^n

But we can use Normal Approximation of Binomial Distribution.

$\rightarrow$ n should be sufficiently large (condition)

i.e. $np \geq 5$ and $n(1-p) \geq 5$

$12000 \times \frac{1}{6} = 2000 \geq 5$ ✓ $\quad 12000 \times \frac{5}{6} = 10000 \geq 5$ ✓

Both cond^{n}s satisfied

$\Rightarrow$ We can apply Normal Approximo^n to Binomial distribu^n.

X: Bernoulli random variable because we just have only 2 possibilities.

$\hookrightarrow$ either $X = 2$ or $X \neq 2$.

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Applying the theorm we can say that

$$\text{Asked} = P(1900 < X < 2150)$$

$$\mu = np = 12000 \times 1/6 = 2000$$

$$\sigma^2 = np(1-p) = 12000 \times \frac{1}{6} \times \frac{5}{6} = \frac{10^4}{6} \qquad \sigma = \frac{100}{\sqrt{6}}$$

$$\therefore P(1900 < X < 2150) = P\left(\frac{1900-2000}{\frac{100}{\sqrt{6}}} < X < \frac{2150-2000}{\frac{100}{\sqrt{6}}}\right)$$

$$= P\left(\frac{-100}{\frac{100}{\sqrt{6}}} < X < \frac{150}{\frac{100}{\sqrt{6}}}\right)$$

$$= P\left(-\sqrt{6} < X < \frac{3\sqrt{3}}{\sqrt{2}}\right)$$

$$= \phi\left(\frac{3\sqrt{3}}{\sqrt{2}}\right) - \phi(-\sqrt{6})$$

$$\because \phi(x) + \phi(-x) = 1 \qquad \phi(\sqrt{6}) + \phi(-\sqrt{6}) = 1$$

$$= \phi\left(\frac{3\sqrt{3}}{\sqrt{2}}\right) - (1 - \phi(\sqrt{6})$$

$$= \phi\left(\frac{3\sqrt{3}}{\sqrt{2}}\right) + \phi(\sqrt{6}) - 1.$$

$$\cancel{= \phi(2.5980) + \phi(2.4495) - 1}$$

$$\cancel{= 0.993}$$

$$= \phi(3.67423) + \phi(2.44948) - 1$$

$$\cancel{= 0.993}$$

Getting values of ϕ from table.

$$= 1.000 + 0.993 - 1$$

$$= 0.993$$

Ans 4 a) Sample space (MM, MF, FM, FF)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{3} \cdot \frac{\{MF, FM\}}{\{MM, MF, FM\}}$$

Ans 4 b) Now atsi is being fixed as a boy
Sample space [{Atsi, Male}, {Atsi, Female}].

$$P(A|B) = \frac{1}{2} \quad \frac{\{Atsi, Female\}}{\{sample\ space\}}$$

In (c) by telling atsi ~~as a boy~~ is a boy, sample space is reduced ⇒ we gained some information. ~~from at~~

Ans 4 c) Now, since we fixed Atsi being the boy. The sample space for one of them being a male is reduced, as initially ordering of MF and FM was different.
But now as we know atsi is a boy, other child can be one of two boy or girl ∴ 1/2.

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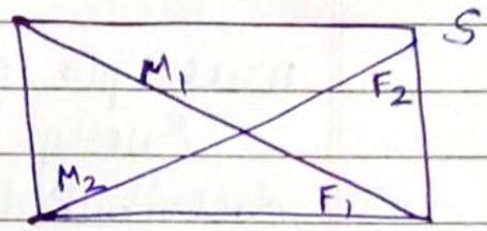
Ans 1 (d) Let E be the event that one of the children is female.

$\therefore P(E \mid M_1 \cap M_2) = \frac{2}{3}$

First child is Male

Second child is male

Let Atsi be the first child.



$\therefore$ We need to find

$$P(E \mid M_1) = \frac{P(E \cap M_1)}{P(M_1)}$$

$$\frac{P(E \cap (M_1 \cup M_2))}{P(M_1 \cup M_2)} = \frac{2}{3}, \quad P(E \cap (M_1 \cup M_2)) = \frac{2}{4}, \quad P(M_1 \cap M_2) = \frac{3}{4}$$

$$P(E \cap M_1) = P(E \cap M_1 \cap (M_1 \cup M_2)) \quad (\text{as } M_1 \subset M_1 \cup M_2)$$
$$= P(M_1 \mid E \cap (M_1 \cup M_2)) \times P(E \mid M_1 \cup M_2) \times P(M_1 \cup M_2)$$
$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

$$P(M_1) = P(M_1 \cap (M_1 \cup M_2)) \quad (\text{as } M_1 \subset M_1 \cup M_2)$$
$$= P(M_1 \mid M_1 \cup M_2) \cdot P(M_1 \cup M_2)$$
$$= \frac{2}{3} \times \frac{3}{4} = \frac{2}{4}$$

$$\therefore P(E \mid M_1) = \frac{P(E \cap M_1)}{P(M_1)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

$\boxed{\text{Hence, the probability is } \frac{1}{2}}$

Ans.5 (1.) $f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

and so if $z > 0$,

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y) \cdot f_Y(y) \cdot dy$$

$$= \int_0^z \lambda e^{-\lambda(z-y)} \cdot \lambda e^{-\lambda y} \cdot dy$$

$$= \int_0^z \lambda^2 e^{-\lambda z} \cdot dy = \lambda^2 z e^{-\lambda z}.$$

while if $z < 0$, $f_Z(z) = 0$ Hence,

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & \text{if } z \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(2.) $f_X(x) = f_Y(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

and the density function for the sum is given by

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y) \cdot f_Y(y)\, dy$$

Since $f_Y(y) = 1$ if $0 \leq y \leq 1$ and 0 otherwise, this becomes

$$f_Z(z) = \int_0^1 f_X(z-y)\, dy$$

Now the integrand is 0 unless $0 \leq z-y \leq 1$ (i.e. unless $z-1 \leq y \leq z$) and then it is 1. So if $0 \leq z \leq 1$, we have

$$f_Z(z) = \int_0^z dy = z.$$

while if $1 < z \leq 2$, we have

$$f_Z(z) = \int_{z-1}^{1} dy = 2 - z.$$

and if $z < 0$ or $z > 2$ we have $f_Z(z) = 0$. Hence,

$$f_Z(z) = \begin{cases} z & \text{if } 0 \leq z \leq 1 \\ 2-z & \text{if } 1 < z \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Ans. 6 (b) $F_T(t) = P(T \le t) = P(T_1 - T_2 \le t)$

$= P(T_1 + t \le T_2)$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{t_1+t_2} f(t_1, t_2)\, dt_1 \right] dt_2$$

as T_1 and T_2 are independent and as $t_1 \ge 0$ (exponential RV)

$$\Rightarrow \int_{-\infty}^{\infty} \left[\int_{0}^{t_1+t_2} \lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda t_2} \cdot dt_1 \right] dt_2$$

$$= \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda t_2} \left[\int_0^{t_0+t_2} e^{-\lambda t_1} \, dt_1 \right] dt_2$$

→ Similarly changing limits for t_2

$$\int_0^{\infty} \lambda e^{-\lambda t_2} \left(1 - e^{-\lambda(t_0+t_2)}\right] dt_2$$

$$\int_0^{\infty} \lambda e^{-\lambda t_2} \cdot dt_2 - \int_0^{\infty} \lambda e^{-\lambda t} \cdot e^{-2\lambda t_2} \cdot dt_2$$

$$= 1 - \int_0^{\infty} \lambda e^{-\lambda t} e^{-2\lambda t_2} \, dt_2$$

$$= 1 - \frac{\lambda e^{-\lambda t}}{(-2\lambda)} [-1] = 1 - \frac{e^{-\lambda t}}{2}.$$

(2) for $T < 0$

$\cancel{F_T(t) + F_T(t) = 1}$ $\quad F_T(t) + F_T(-t) = 1.$

$$F_T(t) = 1 - \left(1 - \frac{e^{\lambda t}}{2}\right)$$

$$F_T(t) = \frac{e^{\lambda t}}{2} \quad \forall \; t > 0$$

$$F_T(t) = \begin{cases} 1 - \frac{e^{-\lambda t}}{2} & t \geq 0 \\ e^{\lambda t}/2 & t < 0 \end{cases}$$

(3) $f_T(t) = \frac{d}{dt} F_T(t)$

$$f_T(t) = \begin{cases} \frac{\lambda e^{-\lambda t}}{2} & t \geq 0 \\ \frac{\lambda e^{\lambda t}}{2} & t < 0 \end{cases}$$

(4) Laplace Distribution

Ans 1) (1.) Given time b/w 2 successive trains is exactly 10 min.
So waiting time for atsi will always be b/w 0 to 10 min.
and 24 hours = 24×60 min.

multiple of 10
∴ Every instance b/w 0 to 10 is uniformly distributed.
hence waiting time (t) ~ Uniform (0, 10).

$$F_T(t) = \begin{cases} 0 & t < 0 \\ t/10 & 0 \le t \le 10 \\ 1 & t > 10 \end{cases}$$

$$E[t] = \int_0^{10} \frac{1}{10} \cdot x \, dx = \frac{10^2}{2 \times 10} = 5.$$

Ans 1 (2.) Given: Already waiting time for 6 min. then waits for atleast 3 more mins.

$$\frac{P(3 \text{ more mins} \cap \text{already } 6 \text{ min})}{P(\text{already } 6 \text{ min})}$$

$$= \frac{P(t \ge 9)}{P(t \ge 6)} = \frac{1 - 9/10}{1 - 6/10} = \frac{1}{4}$$

Ans 1 (3) Let's say atsi arrives x seconds after the previous ~~turn~~ train.
Now, arrival of train is exponential.

By the memory less property

$$P(T > t+x \mid T > x) = \frac{e^{-\lambda(t+x)}}{e^{-\lambda x}} = e^{-\lambda t}$$

$$(P(T > t))$$

The distribution of time after x seconds has passed is still the same as if it would have started from 0.

$\therefore$ t seconds after train left has same distribution as t seconds after x sec. passed.

$\boxed{\text{Avg time or Mean} = 10 \text{ mins.}}$

Distribution is exponential with $\lambda = 1/10$

Ans. 8 1) Let T be the time of arrival until first arrival (from any start point)

$P(T > t) \rightarrow$ No arrival in first t

New Avg $\rightarrow \lambda t$

$\therefore$ For no arrival in t units

$$P(N=0) = \frac{e^{-\lambda t}}{\lfloor 0} (\lambda t)^0 = e^{-\lambda t}$$

$$P(T \leq t) = 1 - e^{-\lambda t} \qquad F_T(t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \forall\ t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ Exponential distribution

Since arrival is independent, this can extend to time b/w arrivals being exponential (for memoryless property).

Ans 8 (2) Let one time unit corresponds to 1 trial.

1st success at $t \rightarrow (t-1)$ failures before 1st success

$\therefore$

$P(T=t) = (1-p)^{t-1} p \rightarrow$ geometric

And it doesn't depend upon where we start.

In bernoulli, trials are independent of each other.

$\Rightarrow P(T=t+x \mid T>x) = (1-p)^{t-1} p$

if we start after x, still geometric

Ans 8 (3) Let X_n be geometric RV with parameter λ/n

Then;

$$P(X_n \leq nx) = \sum_{k=1}^{\lfloor nx \rfloor} \left(1-\frac{\lambda}{n}\right)^{k-1} \cdot \lambda/n$$

$$= \frac{1}{n} \sum_{k=0}^{\lfloor nx \rfloor - 1} \left(1-\frac{\lambda}{n}\right)^{n \cdot k/n} \cdot \lambda$$

and we know that

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} = e^{-\lambda x}$$

$$\therefore \approx \frac{1}{n} \sum_{k=0}^{\lfloor nx \rfloor - 1} \lambda e^{-\lambda k/n}$$

Now $y = k/n$ and integrating wrt y

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$$\simeq \int_0^x \lambda e^{-\lambda y} \cdot dy = 1 - e^{-\lambda x} \qquad \frac{d(1-e^{-\lambda x})}{dx} = \lambda e^{-\lambda x}$$

Exponential Distribution

This implies that if X_n is geometric with parameter λ/n then distribution of RV $T_n = \frac{1}{n} X_n$ is approximately exponential with parameter λ when n is large.

Ans 9 (1.) (a.) Rayleigh Distribu^n is defeined for non-negative x

$$E[X] = \int_0^\infty \underbrace{x}_{I} \cdot \underbrace{\frac{x}{\sigma^2} e^{-x^2/2\sigma^2}}_{II} dx$$

By parts

$$= \int \frac{x}{\sigma^2} \cdot e^{-x^2/2\sigma^2} \cdot dx = ?$$

$$t = \frac{-x^2}{2\sigma^2}$$

$$\therefore dt = -\frac{x}{\sigma^2} dx$$

$$\int -e^t \cdot dt = -e^t = -e^{-x^2/2\sigma^2}$$

$$\therefore E[X] = \left[-x e^{-x^2/2\sigma^2}\right]_0^\infty + \int_0^\infty e^{-x^2/2\sigma^2} \cdot dx$$

$$= 0 + \int_0^\infty e^{-x^2/2\sigma^2} \cdot dx$$

From PDF of standard normal random variable we know that

$$\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}_{\text{even fun.}} \cdot dz = 1$$

$$\therefore 2 \times \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \cdot dz = 1$$

$$\therefore \int_0^\infty e^{-z^2/2} \cdot dz = \sqrt{\frac{\pi}{2}}$$

Put $x = \sigma z$

$\therefore dx = \sigma\, dz$

$$\therefore \int_0^\infty e^{-x^2/2\sigma^2} \cdot dx = \int_0^\infty e^{-z^2/2} \cdot \sigma\, dz = \sigma \int_0^\infty e^{-z^2/2} \cdot dz$$

$$= \boxed{\sigma \sqrt{\frac{\pi}{2}}}$$

Ans 9 (1.) (b.) $V[X] = E[X^2] - (E[X])^2$

$$E[X^2] = \int_0^\infty x^2 \cdot \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} \cdot dx$$

$$t = -\frac{x^2}{2\sigma^2} \qquad dt = -\frac{x\, dx}{\sigma^2}$$

$$E[X^2] = \int e^t \cdot 2\sigma^2\, t\, dt$$

$$= 2\sigma^2 \int \underbrace{t}_{I} \underbrace{e^t}_{II} \, dt$$

$$= 2\sigma^2 \left[t\, e^t - \int 1 \cdot e^t dt \right]$$

$$= 2\sigma^2 \left[t e^t - e^t \right]$$

$$= 2\sigma^2 \left[-\frac{x^2}{2\sigma^2} e^{-x^2/2\sigma^2} - e^{-x^2/2\sigma^2} \right]_0^\infty$$

$$= 2\sigma^2 \left[0 - 0 - (0 - 1) \right] = 2\sigma^2.$$

$$\therefore V[X] = 2\sigma^2 - (E[X])^2$$
$$= 2\sigma^2 - \left(\sigma\sqrt{\frac{\pi}{2}}\right)^2$$

$$V[X] = 2\sigma^2 - \frac{\sigma^2\pi^2}{2} = \frac{\sigma^2}{2}\left(4 - \pi^2\right)$$

Ans 9 (i) (c.) Mode is that Value of x whose frequency is max.

$\Rightarrow$ We want to find that value of x at which maxima of Probability Density fun. (PDF) occurs.

$$\therefore \cancel{f_x(x)} = f_x(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}$$

$$f_x'(x) = \frac{e^{-x^2/2\sigma^2}}{\sigma^2} + \frac{x}{\sigma^2}\left(-\frac{x}{\sigma^2}\right) \cdot e^{-x^2/2\sigma^2}$$

always + ve

$$f_x'(x) = \left(1 - \frac{x^2}{\sigma^2}\right)\frac{e^{-x^2/2\sigma^2}}{\sigma^2} = 0 \;?$$

when $1 - \frac{x^2}{\sigma^2} = 0$

$$\Rightarrow \boxed{x = \sigma}$$

Ans. 9 (2) (a) $F_X(x) = \dfrac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]} = \dfrac{\gamma}{\pi}\left[\dfrac{1}{\gamma^2+(x-x_0)^2}\right]$

$$E[X] = \frac{\gamma}{\pi}\int_{-\infty}^{\infty} \frac{x}{\gamma^2+(x-x_0)^2}\cdot dx$$

$$= \frac{\gamma}{\pi}\left[\frac{1}{2}\ln\left(\gamma^2+(x-x_0)^2\right)\right]_{-\infty}^{\infty} + \frac{x_0}{\gamma}\tan^{-1}\left(\frac{x-x_0}{\gamma}\right)\Bigg|_{-\infty}^{\infty}$$

$$= \frac{\gamma}{2\pi}\left(\ln\left(\gamma^2+(x-x_0)^2\right)_{-\infty}^{\infty}\right) + x_0$$

We can consider $\ln\left(\gamma^2+(x-x_0)^2\right)\Big|_{-\infty}^{\infty}$

$$= \underset{a\to\infty}{Lt}\ \ln\left(\gamma^2+(x-x_0)^2\right)\Big|_{-a}^{a}$$

$$= \underset{a\to\infty}{Lt}\ \ln\left(\frac{\gamma^2+(a-x_0)^2}{\gamma^2+(a+x_0)^2}\right) = \underset{a\to\infty}{Lt}\ \ln\left(1-\frac{4ax_0}{\gamma^2+(a+x_0)^2}\right)$$

$$= \ln(1) = 0.$$

But if we consider $\ln\left(\gamma^2+(x-x_0)^2\right)\Big|_{-\infty}^{\infty}$

$$= \underset{a\to\infty}{Lt}\ \ln\left(\gamma^2+(x-x_0)^2\right)\Big|_{-2a}^{a}$$

$$= \underset{a\to\infty}{Lt} \ln\left(\frac{\gamma^2+(a-x_0)^2}{\gamma^2+(2a+x_0)^2}\right) = \underset{a\to\infty}{lt} \ln\left(1-\frac{3a^2+6ax_0}{\gamma^2+(2a+x_0)^2}\right)$$

$$= \underset{a\to\infty}{Lt} \ln\left(1-\frac{3+\frac{x_0}{a}}{\frac{x^2}{a^2}+\left(2+\frac{x_0}{a}\right)^2}\right)$$

$$\ln\left(1-\frac{3}{4}\right) = -\ln(4) \neq 0.$$

$\therefore$ The limits doesn't match

$\boxed{\text{Hence, the mean of the distribu}^n \text{ does not exist.}}$

Ans 9 (2.) (b.) The characteristic function of Cauchy distribu^n is

$$E[e^{ixt}] = e^{ix_0t-\gamma|t|} = \varphi_x(t)$$

$$\therefore \varphi_x'(t) = \left.\frac{d(\varphi_x)}{dt}\right|_t = \begin{cases}(ix_0-\gamma)e^{ix_0t-\gamma t}, & t>0\\ (ix_0+\gamma)e^{ix_0t+\gamma t}, & t<0\end{cases}$$

$\varphi_x'(t)$ is not defiened at $t=0$.
γ cannot be 0, hence $t=0$ is not a removable discontinuity.
$\therefore$ None of the derivatives can be defeined at $t=0$.

$\boxed{\text{Hence, the values of the derivatives at 0 cannot be defiened.}}$

Ans 9 (2.) (c.) n^{th} moment of X around 0 is $E[X^n]$

$$E[X^n] = \int_{-\infty}^{\infty} \frac{\gamma}{\pi}\left(\frac{x^n}{\gamma^2+(x-x_0)^2}\right)dx$$

If n is even and $n \geq 2$, $\frac{x^n}{\gamma^2 + (x-x_0)^2}$ diverges as $x \to \infty$ and $x \to -\infty$

$\therefore$ The integral does not take a finite value.
$\therefore$ The nth higher moment does not exist when n is even.

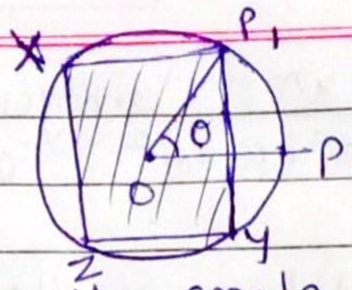
If n is odd and $n \geq 3$, $E[X^n] = \underbrace{\int_{-\infty}^{0} \frac{\gamma}{\pi}\left(\frac{x^n}{\gamma^2 + (x-x_0)^2}\right) dx}_{I_1}$

$$+ \underbrace{\int_{0}^{\infty} \frac{\gamma}{\pi}\left(\frac{x^n}{\gamma^2 + (x-x_0)^2}\right) dx}_{I_2}$$

Here $I_1 \to \infty$ and $I_2 \to -\infty$, and we get $E[X^n]$ as $\infty - \infty$ form.
$\therefore$ The nth higher moment does not exist when n is odd.
Hence, Higher moments do not exist for Cauchy distribution.

This also means that $var(X)$ is not defiened. The consequence of undefiened variance is that the Central Limit Theorm will not apply on this distribution, and certain parameter estimation techniques will break down.

Ans 10



Let P be such that ~~x coord.~~ x-coordinate of $P > 0$, y-co-ordinate of $P = 0$. Let θ be the angle OP_1 makes with OP.
(θ measured in anticlockwise sense)

$\therefore \theta \in [0, 2\pi)$ and a θ denotes a unique position of P_1 on C.
Let A be the RV denoting the value of θ.
$\therefore A \sim \text{Uniform}(0, 2\pi)$

$$\therefore f_A(\theta) = \begin{cases} 1/2\pi & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Also for each position of P_1, the shaded area denotes the acceptable positions of P_2 such that no point of the rectangle lie outside circle.
The shaded area is rectangle with corners $P_1(\cos\theta, \sin\theta)$, $X(-\cos\theta, \sin\theta)$, $Y(\cos\theta, -\sin\theta)$, $Z(-\cos\theta, -\sin\theta)$.
Let E be the event where no point of rectangle with diagonal $\overline{P_1P_2}$ lies outside circle.

$$\therefore P(E \mid A = \theta) = \begin{cases} \dfrac{4\sin\theta\cos\theta}{\pi} & 0 \le \theta \le \frac{\pi}{2} \text{ or } \pi \le \theta \le \frac{3\pi}{2} \\ -\dfrac{4\sin\theta\cos\theta}{\pi} & \frac{\pi}{2} < \theta < \pi \text{ or } \frac{3\pi}{2} < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore P(E) = \int_0^{2\pi} P(E \mid A = \theta) \cdot P(A = \theta) \, d\theta$$

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$$= \int_0^{\pi/2} \frac{4\sin\theta\cos\theta}{\pi} \times \frac{1}{2\pi}\, d\theta + \int_{\pi/2}^{\pi} -\frac{4\sin\theta\cos\theta}{\pi} \cdot \frac{1}{2\pi}\, d\theta$$

$$+ \int_{\pi}^{3\pi/2} \frac{4\sin\theta\cos\theta}{\pi} \cdot \frac{1}{2\pi}\, d\theta + \int_{3\pi/2}^{2\pi} -\frac{4\sin\theta\cos\theta}{\pi} \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{\pi^2}\left[\left(\int_0^{\pi/2} \sin 2\theta\, d\theta + \int_{\pi}^{3\pi/2} \sin 2\theta\, d\theta\right) - \left(\int_{\pi/2}^{\pi} \sin 2\theta\, d\theta + \int_{3\pi/2}^{2\pi} \sin 2\theta\, d\theta\right)\right]$$

$$= \frac{1}{2\pi^2}\left[\left(-\cos 2\theta \Big|_0^{\pi/2}\right) + \left(-\cos 2\theta \Big|_{\pi}^{3\pi/2}\right) - \left(\cos 2\theta \Big|_{\pi/2}^{\pi}\right) - \left(-\cos 2\theta \Big|_{3\pi/2}^{2\pi}\right)\right]$$

$$= \frac{1}{2\pi^2}\left(2 + 2 - (-2) - (-2)\right) = \frac{4}{\pi^2}.$$

Hence, the probability that no point of the rectangle lies outside circle is $4/\pi^2$.