Probability and Statistics: Lecture-19

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on September 23, 2020

» Online Quiz

1. Please login to gradescope
2. Attempt the online quiz-2
3. You may use calculator if necessary
4. Time for the quiz is mentioned in the quiz

» Checklist

1. Turn off your microphone
2. Turn on microphone only when you have question
3. The grades will be uploaded by Today

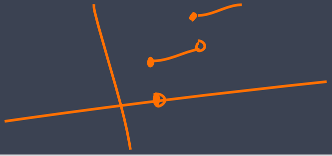
» Table of contents

» Continuous Random Variable

Definition: Continuous Random Variable

A random variable X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.

» Continuous Random Variable



Definition: Continuous Random Variable

A random variable X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.

* CDF is *always* a continuous function whereas PMF may not be continuous

» Continuous Random Variable

Definition: Continuous Random Variable

A random variable X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.

* CDF is always a continuous function whereas PMF may not be continuous
* We will usually assume that the CDF of a continuous R.V. is differentiable

» Continuous Random Variable

PMF → $P(X = x) = 0$

Definition: Continuous Random Variable

A random variable X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.

* CDF is always a continuous function whereas PMF may not be continuous
* We will usually assume that the CDF of a continuous R.V. is differentiable
* Although PMF does not makes sense for continuous random variable, we define probability density function

» Probability Density Function...

» Probability Density Function...

* The probability density function (PDF) would be defined as probability per unit length

» Probability Density Function...

- The probability density function (PDF) would be defined as probability per unit length
- Consider a continuous random variable X and define $f_X(x)$ as follows

PDF

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta}$$

$X \in (x, x+\Delta)$

$\Delta \to 0$

» Probability Density Function...

- The probability density function (PDF) would be defined as probability per unit length
- Consider a continuous random variable X and define $f_X(x)$ as follows

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta}$$

- Here, $f_X(x)$ gives the probability density at point x

» Probability Density Function...

- The probability density function (PDF) would be defined as probability per unit length
- Consider a continuous random variable X and define $f_X(x)$ as follows

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta}$$

- Here, $f_X(x)$ gives the probability density at point x
- We recall that $P(x < X \leq x + \Delta) = F_X(x + \Delta) - F_X(x)$.

$P(a < X \leq b) = F_X(b) - F_X(a)$

» Probability Density Function...

- The probability density function (PDF) would be defined as probability per unit length
- Consider a continuous random variable X and define $f_X(x)$ as follows

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta}$$

Recall Calculus

$$\lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} = f'(x) = \frac{df}{dx}$$

- Here, $f_X(x)$ gives the probability density at point x
- We recall that $P(x < X \leq x + \Delta) = F_X(x + \Delta) - F_X(x)$.
- We then have

CDF

$$f_X(x) = \lim_{\Delta \to 0} \frac{F_X(x + \Delta) - F_X(x)}{\Delta} = \frac{dF_X(x)}{dx} = F'_X(x)$$

» Probability Density Function...

* The probability density function (PDF) would be defined as probability per unit length
* Consider a continuous random variable X and define $f_X(x)$ as follows

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \le x + \Delta)}{\Delta}$$

* Here, $f_X(x)$ gives the probability density at point x
* We recall that $P(x < X \le x + \Delta) = F_X(x + \Delta) - F_X(x)$.
* We then have

$$f_X(x) = \lim_{\Delta \to 0} \frac{F_X(x + \Delta) - F_X(x)}{\Delta} = \frac{dF_X(x)}{dx} = F'_X(x)$$

Definition of Probability Density Function

Let X be a continuous R.V. with continuous CDF $F_X(x)$. The function $f_X(x)$ defined by

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

is called the probability density function of X. We assume that $F_X(x)$ is differentiable.

» Example of CDF and PDF for Continuous Random Variable...

» Example of CDF and PDF for Continuous Random Variable...

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable.

» Example of CDF and PDF for Continuous Random Variable...

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random.

» Example of CDF and PDF for Continuous Random Variable...

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability.

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. The CDF is given by:

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$

$\boxed{X \sim \text{Uniform}(a, b)}$

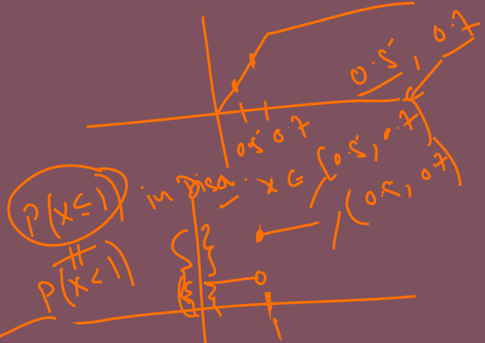
$X \sim \text{Uniform}(0, 1)$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

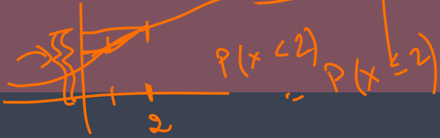
Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. The CDF is given by:

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$



- Does not matter if we use $<$ or $\leq$.
  That is, $P(X < 2) = P(X \leq 2)$



» Example of CDF and PDF for Continuous Random Variable...

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. The CDF is given by:

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

* Does not matter if we use $<$ or $\le$. That is, $P(X < 2) = P(X \le 2)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

PDF

a=0, b=1

1 ≠ 0

$\lim_{\epsilon \to 0}$

does not exist

0 ≠ Derivative = 0

at x = 0, x = 1

Why?

$\frac{f(x+\epsilon)-f(x)}{\epsilon}$

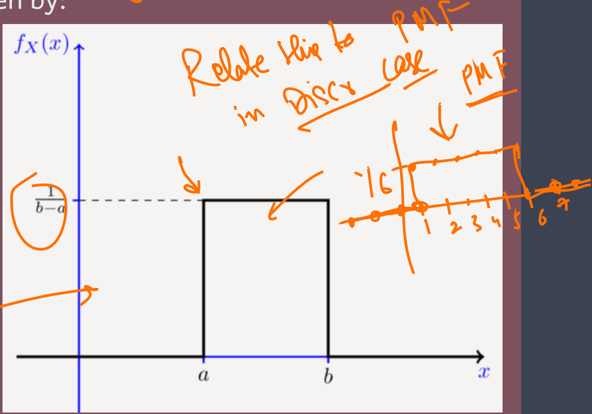
$\lim_{\epsilon \to 0} \frac{f(x+\epsilon)-f(x)}{\epsilon}$

» Example of CDF and PDF for Continuous Random Variable...

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. The CDF is given by:

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$



- Does not matter if we use $<$ or $\le$. That is, $P(X < 2) = P(X \le 2)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

» Properties of PDF...

» Properties of PDF...

- Since the PDF is the derivative of CDF, we have

» Properties of PDF...

derivative ⟷ anti derivative

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$\frac{d}{dx} F_X(x) = f_X(x)$$

$f_X \quad \int^x$

» Properties of PDF...

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\,du$

$$\int_{-\infty}^{b} f_X(u)\,du - \int_{-\infty}^{a} f_X(u)\,du$$

$a < b$

$$\int_a^b f_X(u)\,du.$$

» Properties of PDF...

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$\boxed{\sum_x p_X(x) = 1}$

* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
* $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

$P(-\infty < X \leq -\infty) =$ Sample space

$P(S) = 1$ (By axiom)

» Properties of PDF...

- Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$\lim_{\epsilon \to 0}$ $\frac{f(x+\epsilon) - f(x)}{\epsilon} \geq 0$ ≥ 0

$\epsilon \geq 0$

$\frac{df}{dx} =$

Since f is $\uparrow$

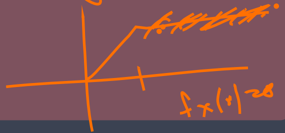
$f(x+\epsilon) - f(x) \geq 0$

- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
- $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

Properties of PDF

- $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

$f_X(x)$ is the derivative of a non-decreasing fn, CDF



» Properties of PDF...

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
* $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

Properties of PDF

* $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
* $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

» Properties of PDF...

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\,du$
* $\int_{-\infty}^{\infty} f_X(u)\,du = 1$

Properties of PDF

* $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
* $\int_{-\infty}^{\infty} f_X(u)\,du = 1$
* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\,du$

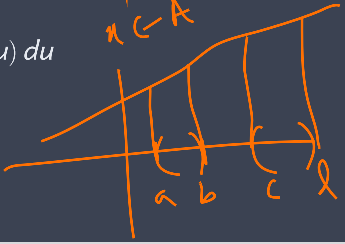
» Properties of PDF...

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$A = (a,b) \cup (c,d)$

$x \in A$



* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
* $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

Properties of PDF

* $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
* $\int_{-\infty}^{\infty} f_X(u)\, du = 1$
* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
* More generally, for a set A, $P(x \in A) = \int_A f_X(u)\, du$

» Properties of PDF...

- Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
- $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

Properties of PDF

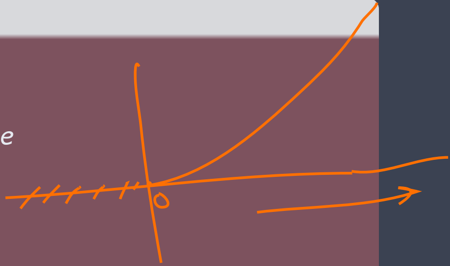
- $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(u)\, du = 1$
- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
- More generally, for a set A, $P(x \in A) = \int_A f_X(u)\, du$
- If $A = [0, 1] \cup [3, 4]$: $P(X \in A) = \int_0^1 f_X(u)\, du + \int_3^4 f_X(u)\, du$

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

PDF

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$



where c is a positive constant.

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

where c is a positive constant.

1. Find c

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$

where c is a positive constant.

1. Find c
2. Find the CDf of X, $F_X(x)$

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$P(1 < x < 3) = \int_1^3 f_x(x)\,dx =$

where c is a positive constant.

1. Find c
2. Find the CDf of X, $F_X(x)$
3. Find $P(1 < X < 3)$

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

where c is a positive constant.

1. Find c
2. Find the CDf of X, $F_X(x)$
3. Find $P(1 < X < 3)$

» Answer to previous problem...

a) Find c

We know that

$$\underbrace{\int_{-\infty}^{\infty} f_X(u)\,du}_{} = 1 \qquad (\text{Prop. } 2)$$

$$= \int_0^{\infty} c e^{-u}\,du = c\left[-e^{-u}\right]_0^{\infty} = c$$

$$\Rightarrow \boxed{c = 1}$$

b) CDF of X

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

For $x < 0 \Rightarrow F_X(x) = 0$

For $x \geq 0$

$$F_X(x) = \int_0^{x} e^{-u}\,du$$

$$= 1 - e^{-x}$$

» Answer to previous problem...

$$\Rightarrow F_X(x) = \begin{cases} 1-e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

c) $P(1 < X < 3)$

We have both CDF & PDF.

Using CDF

$$P(1 < X < 3) = F_X(3) - F_X(1)$$
$$= (1-e^{-3}) - (1-e^{-1})$$

Equivalently using PDF

$$P(1 < X < 3) = \int_1^3 f_X(x)\,dx$$

$$= \int_1^3 c \cdot e^{-x}\,dx \qquad c = 1$$

$$= \int_1^3 e^{-x}\,dx = \left[-e^{-x}\right]_1^3$$

$$= e^{-1} - e^{-3}$$

» Range, Expectation, Variance of Continuous Random Variable...

» Range, Expectation, Variance of Continuous Random Variable...

Definition: Range of Continuous Random Variable

The range of a random variable X is the set of possible values of the random variable. If X is a continuous random variable, we can define the range of X as the set of real numbers x for which the PDF is larger than zero, i.e,

$$R_X = \{x \mid f_X(x) > 0\}$$

» Range, Expectation, Variance of Continuous Random Variable...

Definition: Range of Continuous Random Variable

The range of a random variable X is the set of possible values of the random variable. If X is a continuous random variable, we can define the range of X as the set of real numbers x for which the PDF is larger than zero, i.e,

$$R_X = \{x \mid f_X(x) > 0\}$$

Definition: Expected Value of Continuous R.V

Recall that the expected value of discrete R.V. is

» Range, Expectation, Variance of Continuous Random Variable...

Definition: Range of Continuous Random Variable

The range of a random variable X is the set of possible values of the random variable. If X is a continuous random variable, we can define the range of X as the set of real numbers x for which the PDF is larger than zero, i.e,

$$R_X = \{x \mid f_X(x) > 0\}$$

Definition: Expected Value of Continuous R.V

Recall that the expected value of discrete R.V. is

$$EX = \sum_{x_k \in R_x} x_k P_X(x_k)$$

x

» Range, Expectation, Variance of Continuous Random Variable...

Definition: Range of Continuous Random Variable

The range of a random variable X is the set of possible values of the random variable. If X is a continuous random variable, we can define the range of X as the set of real numbers x for which the PDF is larger than zero, i.e,

$$R_X = \{x \mid f_X(x) > 0\}$$

Definition: Expected Value of Continuous R.V

Recall that the expected value of discrete R.V. is

$$EX = \sum_{x_k \in R_x} x_k P_X(x_k)$$

Replacing sum by integral, and PMF by PDF we have

$$EX = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

» Example of Expected Value of Continuous Random Variable...

» Example of Expected Value of Continuous Random Variable...

Example

Let $X \sim \text{Uniform}(a, b)$. Find EX.

PDF of X

$$f_X(x) : \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$(-\infty, a] = 0$ $\quad$ $[b, \infty) = 0$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{a+b}{2}$$