Probability and Statistics: Lecture-22

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on September 30, 2020

» Online Quiz

1. Please login to gradescope
2. Attempt the online quiz ~~4~~ 5
3. You may use calculator if necessary
4. Time for the quiz is mentioned in the quiz

» Checklist

1. Turn off your microphone
2. Turn on microphone only when you have question
3. Attend Tutorials to Practice Problems or to discuss solutions or doubts

» Table of contents

» Problem-1

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Consider the PDF of the random variable X

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* Find $E[X]$ and $\mathrm{Var}[X]$

Problem-1

Consider the PDF of the random variable X

$$f_X(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$-1 \leq x \leq 1$

* Find the constant c
* Find $E[X]$ and $\mathrm{Var}[X]$
* Find $P(X \geq \frac{1}{2})$

» Solution to Problem-1

1. To find c, we have

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_X(u)\,du \\
&= \int_{-1}^{1} cu^2\,du \\
&= \frac{2}{3}c \implies c = \frac{3}{2}.
\end{aligned}
$$

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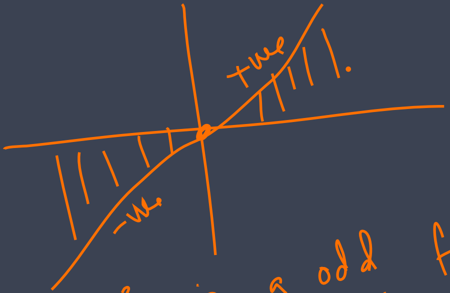

u^3 is a odd fn.

$g(-u) = -g(u)$

2. To find $E[X]$, we have

$$
\begin{aligned}
E[X] &= \int_{-1}^{1} u f_X(u)\, du \\
&= \frac{3}{2}\int_{-1}^{1} u^3\, du \\
&= 0
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$$

1. We have

0

$$
\begin{aligned}
\text{Var} &= E[X^2] - E[X]^2 \\
&= \int_{-1}^{1} u^2 f_X(u)\, du \\
&= \frac{3}{2} \int_{-1}^{1} u^4\, du = \frac{3}{5}
\end{aligned}
$$

» Solution to Problem-1

1. To find c, we have

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2. To find $P(X \geq \frac{1}{2})$, we have

$$\int_{1/2}^{\infty} P_X(x)$$

$$= \quad P(X \geq \frac{1}{2}) = \frac{3}{2}\int_{1/2}^{1} x^2\,dx = \frac{7}{16}.$$

Problem 2

Consider the PDF of continuous random variable X

$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad \text{for all } x \in \mathbb{R}$$

If $Y = X^2$, find the CDF of Y.

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

$R_X: (-\infty, \infty)$

» Solution to Problem-2

1. We have $R_Y = [0, \infty)$.

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» Solution to Problem-2

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2. For $y \in [0, \infty)$,

$g(-u) = g(u)$

Recall Calculus

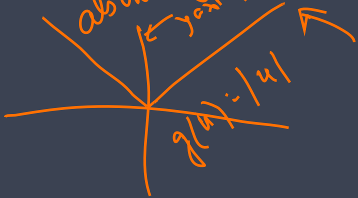
even or odd?

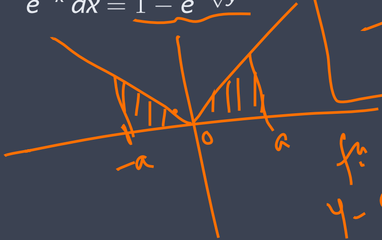
For odd: symmetric about origin

For even: symmetric about y-axis

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \underbrace{\frac{1}{2} e^{-|x|}}_{\text{even}} \, dx \\ &= \int_0^{\sqrt{y}} e^{-x} \, dx = 1 - e^{-\sqrt{y}} \end{aligned}$$

$$\boxed{\int_{-a}^{a} g(u)\,du = 2\int_0^a g(u)\,du \quad g \text{ even}}$$





f: symm. about y-axis ⇒ area under curve is also symm abt y-axis

» Solution to Problem-2

1. We have $R_Y = [0, \infty)$.
2. For $y \in [0, \infty)$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|}\, dx \\ &= \int_0^{\sqrt{y}} e^{-x}\, dx = 1 - e^{-\sqrt{y}} \end{aligned}$$

Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3

Consider the PDF of the continuous random variable

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$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \leq \frac{2}{3} \mid X > \frac{1}{3})$.

conditional

» Solution to Problem 3

We have

$$
\begin{aligned}
P(X \le \frac{2}{3} \mid X > \frac{1}{3}) &= \frac{\overbrace{P(1/3 < X \le 2/3)}^{\cap \text{ interval}}}{P(X > \frac{1}{3})} \\
&= \frac{\int_{1/3}^{2/3} 4x^3 \, dx}{\int_{1/3}^{1} 4x^3 \, dx} \\
&= 3/16
\end{aligned}
$$

Problem 4

Consider the PDF of random variable X

» Problem 4

Problem 4

Consider the PDF of random variable X

$$f_X(x) = \begin{cases} x^2(2x + \frac{3}{2}) & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

check for validity

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

» Problem 4

$V(aX+b) = a^2 \, Var(X)$

Problem 4

Consider the PDF of random variable X

$$f_X(x) = \begin{cases} x^2(2x + \frac{3}{2}) & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 3$, find $\text{Var}(Y)$.

» Solution to Problem 4

* We have

$$\mathrm{Var}(Y) = \mathrm{Var}(\frac{2}{X} + 3) = 4\,\mathrm{Var}(\frac{1}{X})$$

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- We now find $\mathrm{Var}(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[X])^2$
- We have

$$E[\frac{1}{X}] = \int_0^1 x(2x + \frac{3}{2})\,dx = \frac{17}{12}$$

$$E[\frac{1}{X^2}] = \int_0^1 (2x + \frac{3}{2})\,dx = \frac{5}{2}$$

» Solution to Problem 4

- We have

$$\mathsf{Var}(Y) = \mathsf{Var}(\frac{2}{X} + 3) = 4\,\mathsf{Var}(\frac{1}{X})$$

- We now find $\mathsf{Var}(\frac{1}{X}) = \underbrace{E[\frac{1}{X^2}] - (E[X])^2}$ $= \frac{5}{2} - \left(\frac{17}{12}\right)^2$
- We have

$$E[\frac{1}{X}] = \int_0^1 x(2x + \frac{3}{2})\, dx = \frac{17}{12}$$

$$E[\frac{1}{X^2}] = \int_0^1 (2x + \frac{3}{2})\, dx = \frac{5}{2}$$

- Hence,

$$\underbrace{\mathsf{Var}(\frac{1}{X})} = \underline{\underline{\frac{71}{144}}}$$

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- Hence,

$$\mathrm{Var}(\frac{1}{X}) = \frac{71}{144}$$

- $\mathrm{Var}(Y) = 4\,\mathrm{Var}(\frac{1}{X}) = \frac{71}{36}$

» Problem 5

Problem 5

Let $X \sim \text{Uniform}(-\frac{\pi}{2}, \pi)$ and $Y = \sin(X)$. Find $f_Y(y)$.

method of Transform

diff.

① Cont. & diff.

② g is incr. or decr.

g is piecewise incr. or decr.

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* Here $Y = g(X)$, where g is a differentiable function
* g is not monotone, but it can be divided to a finite number of regions in which it is monotone
* We have $R_X = [-\pi/2, \pi]$, $\quad R_Y = [-1, 1]$

Partition of R_X : $[-\pi/2, 0] \cup [0, \pi/2] \cup [\pi/2, \pi]$

$= [-\pi/2, \pi/2] \cup [\pi/2, \pi]$

$[-1, 0] \cup [0, 1]$

$[0, \pi/2] \cup [\pi/2, \pi]$

» Answer to previous problem 5...

$\cancel{y \in (0,1)}$ $y(-1,0)$ $\underline{y = \sin(x_1) = g(x_1)}$

$Y = g(X)$, $R_X = [-\pi/2, \pi]$ $R_Y = [-1, 1]$

$f_X(\cancel{\mathrm{Uniform}(-1,0)})$

$$\underline{f_Y(y)} = \frac{f_X(x_1)}{|g'(x_1)|} = \frac{f_X(\sin^{-1} y)}{|\cos(\sin^{-1} y)|}$$

$$f_X(\sin^{-1}(-1,0))$$

$$= \frac{\cancel{\pi/2}\ 2/3\pi}{\sqrt{1-\sin^2(x_1)}} = \frac{\cancel{\pi/2}\ 2/3\pi}{\sqrt{1-y^2}}$$

$$= \frac{f_X(\sin^{-1} y)}{|\cos(\sin^{-1} y)|} + \frac{f_X(\pi - \sin^{-1} y)}{\cos(\pi - \sin^{-1} y)}$$

$\underline{y \in (0,1)}$

$$f_Y(y) = \underbrace{\frac{f_X(x_1)}{|g'(x_1)|}} + \underbrace{\frac{f_X(x_2)}{|g'(x_2)|}} = \frac{\cancel{\pi/2}\ 2/3\pi}{\sqrt{1-y^2}} + \frac{\cancel{\pi/2}\ 2/3\pi}{\sqrt{1-y^2}}$$

» Answer to previous problem 5...

$$f_Y(y) = \begin{cases} \dfrac{2}{3\pi\sqrt{1-y^2}} & -1 < y < 0 \\ \dfrac{4}{3\pi\sqrt{1-y^2}}, & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

check that this is valid PDF

$$\boxed{\int_{-\infty}^{\infty} f_Y(y) = 1}$$

Note: In video there was a mistake, refer to this corrected slide.

» Special Continuous Distributions: Uniform Distribution...

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Recall: Uniform Distribution

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The PDF of $X \sim \text{Uniform}(a, b)$ is given by

$$f_X = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

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$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^b x^2 \left(\frac{1}{b-a}\right)\, dx = \frac{a^2+ab+b^2}{3}$$

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* Hence the variance is: $\text{Var}(X) = E[X^2] - (E[X])^2 = \dfrac{(b-a)^2}{12}$

» Special Distribution: Exponential Distributions...

Definition of Exponential Distribution

Let X be a continuous random variable. Here X is said to have exponential distribution with parameter $\lambda > 0$ shown as $X \sim \text{Exponential}(\lambda)$, if its PDF is given as follows

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

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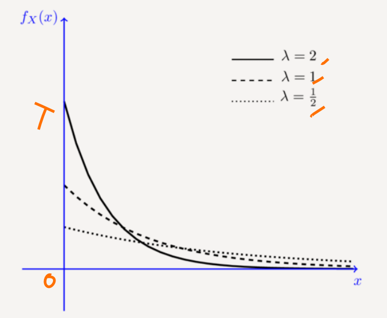

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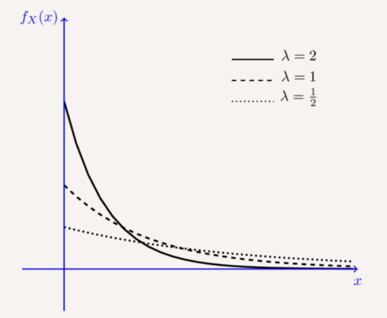

The CDF is given by

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda t}$$

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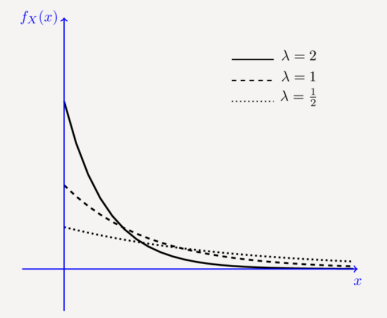

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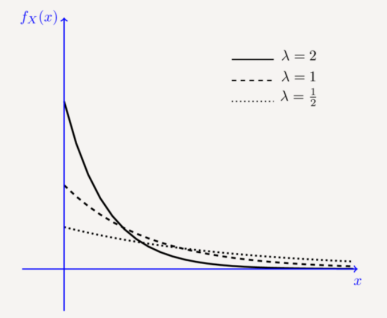

The CDF is given by

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda t}$$

$\lambda x = y$

The expectation is

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x}\, dx = \frac{1}{\lambda}\int_0^\infty ye^{-y}$$

$$= \frac{1}{\lambda}[-e^{-y} - ye^{-y}]_0^\infty = \frac{1}{\lambda}$$

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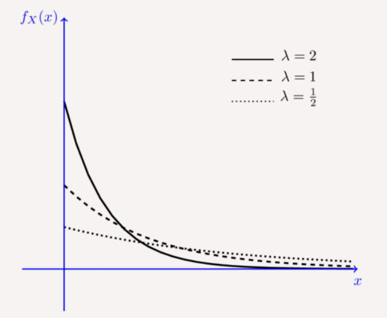

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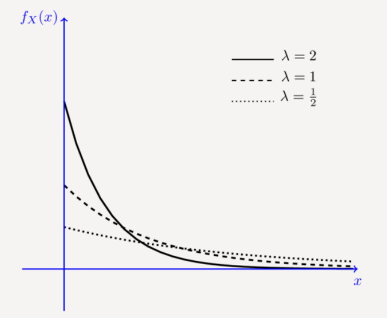

Var(X) is given by:

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y}\, dy$$
$$= \frac{1}{\lambda^2} \left[-2e^{-y} - 2ye^{-y} - y^2 e^{-y}\right]_0^\infty = \frac{2}{\lambda^2}$$

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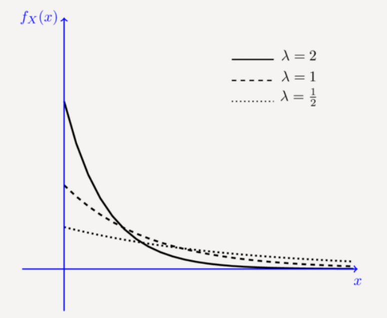

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$$= \frac{1}{\lambda^2} \left[-2e^{-y} - 2ye^{-y} - y^2 e^{-y} \right]_0^{\infty} = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$