Probability and Statistics: Lecture-37

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad) on November 9, 2020



Let X_1, X_2, \dots, X_n be n discrete RVs.

Joint PDF, Joint CDF

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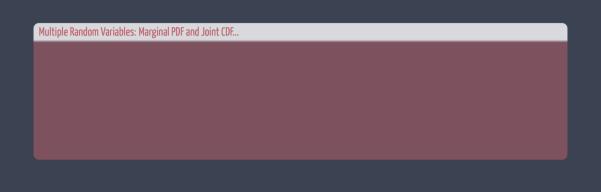
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Example (Three jointly continuous RV:

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Let X, Y and Z be three jointly continuous random variables with joint PDF

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- 1. Find the constant c
- 2. Find the marginal PDF of X

= [[c](==22+2yx+32x)]dydz $= \int_{0}^{\infty} \int_{0}^{\infty} C\left[\frac{1}{2} + 2y + 3z\right] dy dz$

= [] = (x+2y+32)dy =

= Sc[==+37] dz

@ fx(x) = [[fxyz (x17,2)dydz

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* If random variables $X_1, X_2, ..., X_n$ are independent, then we have

$$E[X_1,X_2,\cdots,X_n]=E[X_1]E[X_2]\cdots E[X_n]$$

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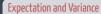
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- uIf we flip the same coin N times and record the outcome, then $(X_1,\ldots,X_n$ are I.I.D.
- * Verify that these I.I.D. variables will have same mean and variances

» Expectation and Variance...



Let $Y = X_1 + X_2 + \cdots + X_n$. The we have the following

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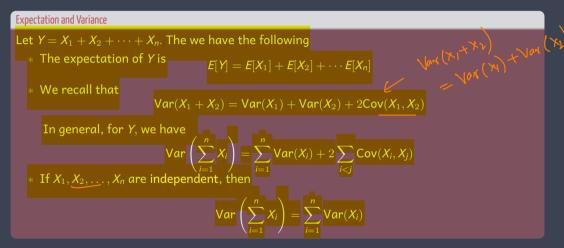
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$$f_{Y}(y) = f_{X_{1}}(y) * f_{X_{2}}(y) = \int_{-\infty}^{\infty} f_{X_{1}}(x) f_{X_{2}}(y - x) dx$$

For multiple variable case, i.e., if $Y = X_1 + X_2 + \cdots + X_n$, we have

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * \cdots * f_{X_n}(y)$$

* However, it is computationally difficult!

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We call X a random vector and E[X] is the expectation of random vector.

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* If X is jointly continuous, the PDF is denoted as

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- * If $Y = AX + b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, then E[Y] = AE[X] + b
- * Also, if $X_1, \overline{X_2, \cdots, X_k}$ are k n-dimensional RVs, then

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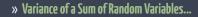
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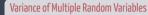
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Definition of Correlation and Covariance Matrix



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$$\underline{R}_{X} = \underline{E[XX^{T}]} = \begin{bmatrix}
X_{1}^{2} & X_{1}X_{2} & \dots & X_{1}X_{n} \\
X_{2}X_{1} & X_{2}^{2} & \dots & X_{2}X_{n} \\
\vdots & \vdots & \vdots & \vdots \\
X_{n}X_{1} & X_{n}X_{2} & \dots & X_{n}^{2}
\end{bmatrix}^{N} = \begin{bmatrix}
E[X_{1}^{2}] & E[X_{1}X_{2}] & \dots & E[X_{1}X_{n}] \\
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E[X_{n}X_{1}] & E[X_{n}X_{2}] & \dots & E[X_{n}^{2}]
\end{bmatrix}$$

$$C_{X} = E[(X - E[X])(X - E[X])^{T}] = \begin{bmatrix} Var(X_{1}) & Cov(X_{1}, X_{2}) & \dots & Cov(X_{1}, X_{n}) \\ Cov(X_{2}, X_{1}) & Var(X_{2}) & \dots & Cov(X_{2}, X_{n}) \\ \vdots & \vdots & \vdots & \vdots \\ Cov(X_{n}, X_{1}) & Cov(X_{n}, X_{2}) & \dots & Var(X_{n}) \end{bmatrix}$$

» Correlation and Covariance Matrix...



Definition of Correlation and Covariance Matrix

$$R_{X} = E[XX^{T}] = \begin{bmatrix} X_{1}^{2} & X_{1}X_{2} & \dots & X_{1}X_{n} \\ X_{2}X_{1} & X_{2}^{2} & \dots & X_{2}X_{n} \\ \vdots & \vdots & \vdots & \vdots \\ X_{n}X_{1} & X_{n}X_{2} & \dots & X_{n}^{2} \end{bmatrix} = \begin{bmatrix} E[X_{1}^{2}] & E[X_{1}X_{2}] & \dots & E[X_{1}X_{n}] \\ E[X_{2}X_{1}] & E[X_{2}^{2}] & \dots & E[X_{2}X_{n}] \\ \vdots & \vdots & \vdots & \vdots \\ E[X_{n}X_{1}] & E[X_{n}X_{2}] & \dots & E[X_{n}^{2}] \end{bmatrix}$$

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$$Cov(X_{1}, X_{2}) & \dots & Cov(X_{1}, X_{n}) \\ Var(X_{2}) & \dots & Cov(X_{1}, X_{n}) \\ Var(X_{2}) & \dots & Cov(X_{2}, X_{n}) \end{bmatrix}$$

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1.
$$C_X = R_X - E[X]E[X]^T$$

2. If
$$Y = AX + b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$
, then $C_Y = AC_XA^T$



 $\boldsymbol{\mathsf{w}}$ Example of Correlation and Covariance Matrices...

Example (Example of correlation and covariance matrices

Let $\ensuremath{\textit{X}}$ and $\ensuremath{\textit{Y}}$ be jointly continuous random variables with joint PDF

Example (Example of correlation and covariance matrices)

Let X and Y be jointly continuous random variables with joint PDF

$$f_{\mathsf{X},\mathsf{Y}} = egin{cases} rac{3}{2} \mathsf{x}^2 + \mathsf{y} & 0 < \mathsf{x},\mathsf{y} < 1 \\ 0 & \mathsf{otherwise} \end{cases}$$

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Let $U = \begin{vmatrix} A \\ Y \end{vmatrix}$ be the random vector

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Let X and Y be jointly continuous random variables with joint PDF $f_{X,Y} = \begin{cases} \frac{3}{2}x^2 + y & 0 < x,y < 1 \\ 0 & \text{otherwise} \end{cases}$ Let $U = \begin{bmatrix} X \\ Y \end{bmatrix}$ be the random vector. Find the correlation and covariance matrices of U.

» Answer to previous problem...

Solution First find the manginal
$$R = \frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \right] \right] = \frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \right] \right] = \frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \right] \right] = \frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \right] \right] = \frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \right] \right] = \frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \left[\frac{1}{2} \right] \right] = \frac{1}{2} \left[\frac{1}$$

» Answer to previous problem...

Covariona Madino
$$Cu = E[U-E[U]) (U-E[U])$$

$$Cu = (v) Cu(v)$$

Properties of Covariance

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Recall definition promise definite (PD), Assum A eymnéhi.

A matrix A is SPD if TAX > 0 Hx = 0 » Properties of Covariance Matrix... Properties of Covariance We have the following properties for covariance matrix: 1. The covariance matrix C_X is symmetric matrix 2. The covariance matrix C_X is positive semi-definite (PSD) 3. The covariance matrix is positive definite if and only if all its eigenvalues are larger than zero

$$C_{X} = E\left[(X - E[X])(X - E[X])^{T}\right] = E\left[\frac{2T}{A}\right]$$

$$OSymmetry: A^{T} = \left(\frac{2T}{A}\right)^{T} = 22^{T} = A \stackrel{?}{=} A \text{ is cosymmetry}$$

$$||2Tx||_{2} \stackrel{?}{>} O$$

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ties of Covariance Matrix...

$$x^{T}Ax > 0 \quad \forall x \neq 0$$

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- 4. The covariance matrix is positive definite if and only if $\det(C_X) > 0$





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Are the matrices C_U and C_V positive definite?





» Denition of Cross-Correlation and Cross-Covariance Matrix...

$$R_{x} = E[x \times T]$$

$$C_{x} = \left(\frac{1}{100} (x - E(x))\right) \left(x - E(x)\right)^{T}$$

 $\ensuremath{\,{\bf >\!\!\!\!>}}\,$ Denition of Cross-Correlation and Cross-Covariance Matrix...

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» Functions of Random Variables...

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$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} H_1(Y_1, Y_2, \dots, Y_n) \\ H_2(Y_1, Y_2, \dots, Y_n) \\ \vdots \\ H_n(Y_1, Y_2, \dots, Y_n) \end{bmatrix}$$

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Then the PDF of Y, denoted by $f_{Y_1,Y_2,...,Y_n}(y_1,y_2,...,y_n)$ is given as follows

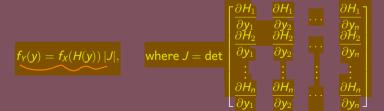
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Action of Random Vector...

$$\chi = A^{-1} (Y - b) = H(Y)$$

$$J = J(A) = J(A)$$

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Let Y = AX + b, where X is a n dimensional random vector, A be a fixed (non-random) n by n matrix, and b be a fixed n-dimensional vector. Find the PDF of Y in terms of X.

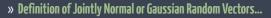
$$f_{\gamma}(\gamma) = f_{\times}(H(\gamma)) |J|$$

$$= f_{\times}(A^{-1}(\gamma-b)) |J| = \int_{A} f_{\times}(\bar{A}^{1}(\gamma^{0}))$$



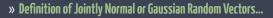
32/

» Definition of Jointly Normal or Gaussian Random Vectors...



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$$f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} \exp\left\{-\frac{1}{2}(x-m)^T C^{-1}(x-m)\right\}$$