Probability and Statistics: Lecture-23

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on October 5, 2020

» Checklist for online class

1. Turn off your microphone, when you are listening
2. Turn on microphone only when you have question
3. Attend tutorials to practice problems or to discuss solutions or doubts
4. Chat is not always reliable, I may not look at chat

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» Special Distribution: Exponential Distributions...

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Definition of Exponential Distribution

Let X be a continuous random variable. Here X is said to have exponential distribution with parameter $\lambda > 0$ shown as $X \sim \text{Exponential}(\lambda)$, if its PDF is given as follows

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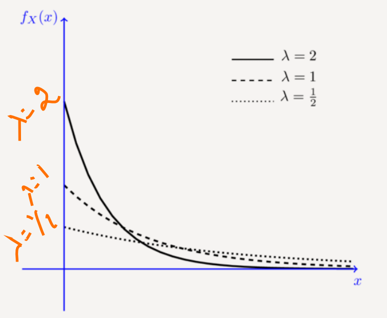

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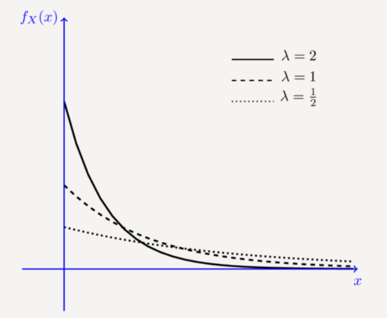

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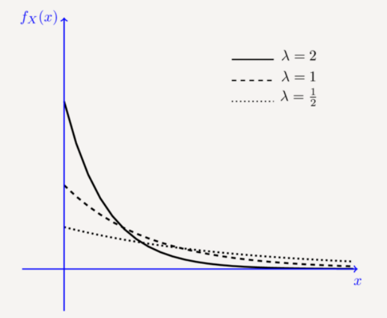

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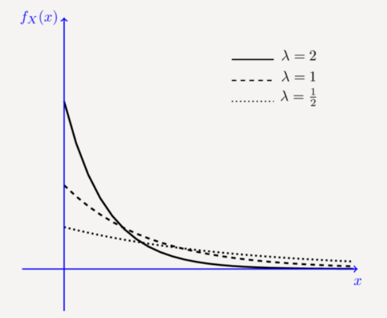

The CDF is given by

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda t}$$

The expectation is

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda}\int_0^\infty y e^{-y}$$

$$= \frac{1}{\lambda}[-e^{-y} - y e^{-y}]_0^\infty = \frac{1}{\lambda}$$

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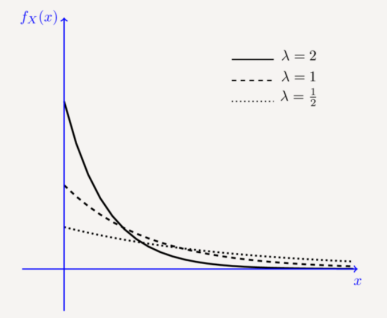

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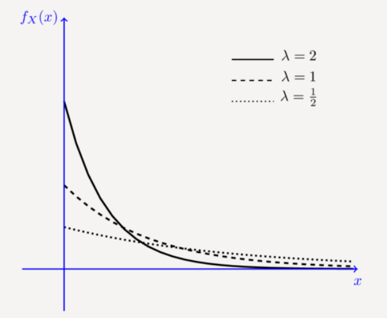

Var(X) is given by: $E[X^2] - (E[X])^2$ $\lambda x = y$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y}\, dy$$

int. by part.

$$= \frac{1}{\lambda^2} \left[-2e^{-y} - 2ye^{-y} - y^2 e^{-y}\right]_0^\infty = \frac{2}{\lambda^2}$$

verify.

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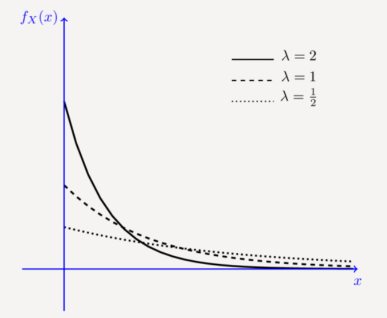

Var(X) is given by:

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y}\, dy$$
$$= \frac{1}{\lambda^2} \left[-2e^{-y} - 2ye^{-y} - y^2 e^{-y}\right]_0^\infty = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

» Exponential Distribution is Memoryless...

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Exponential distribution is memoryless

If X is exponential random variable with parameter $\lambda > 0$, then X is a memoryless variable:

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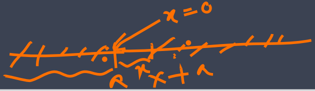

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If X is exponential random variable with parameter $\lambda > 0$, then X is a memoryless variable:

$$P(X > x + a \mid X > a) = P(X > x), \quad \text{for } a, x \geq 0.$$

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$$
\begin{aligned}
P(X > x + a \mid X > a) &= \frac{P(X > x + a, X > a)}{P(X > a)} \\
&= \frac{P(X > x + a)}{P(X > a)} = \frac{1 - F_X(x + a)}{1 - F_X(a)} \\
&= \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} = e^{-\lambda x} \\
&= P(X > x)
\end{aligned}
$$

$F_X(x) = \int_{-\infty}^{x}$

$= P(X \leq x)$

$\Rightarrow P(X > x + a)$

$= 1 - P(X \leq x + a)$

$= 1 - F_X(x + a)$

$= 1 - e^{-\lambda x}$

$F_X(x) = 1 - e^{-\lambda x}$

$e^{-\lambda x} = 1 - F_X(x)$

$P(X > x)$

» Normal (Gaussian) Distribution...

Definition of Standard Normal Random Variable

A continuous random variable Z is said to be a standard normal (standard Gaussian) random variable,

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Normal

$Z \sim N(0,1)$

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$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad \text{for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_Z(z) = 1$$

check

$$\boxed{\int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \sqrt{2\pi}}$$

Indefinite integral

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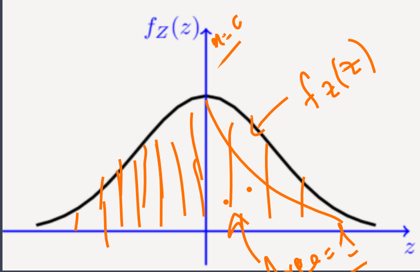
even or odd

$f(-z) = f(z)$



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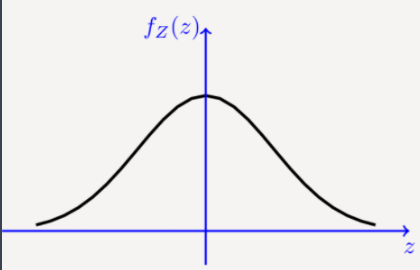

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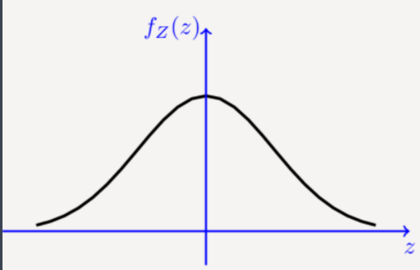

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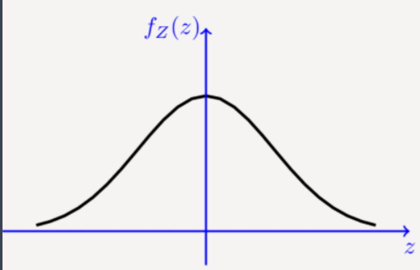

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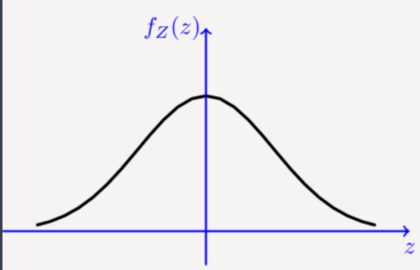

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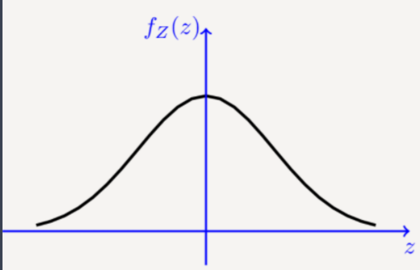

* Most important Probability Distribution!
* **Central Limit Theorem** (TODO):
  * If we add **large** number of random variable, then the distribution of the **sum is normal** (proof later)
* Here $1/\sqrt{2\pi}$ is there to make area under curve 1

» Mean and Variance of Standard Normal Distribution...

Mean and Variance of Standard Normal Distribution

Let Z be a normal distribution, i.e., $Z \sim N(0, 1)$, then $E[Z] = 0$ and $\text{Var}(Z) = 1$.

Recall

If $g(u) : \mathbb{R} \to \mathbb{R}$. If $g(u)$ is an odd function, i.e., $g(-u) = -g(u)$, and

$$\left| \int_0^\infty g(u)\, du \right| < \infty,$$

then

$$\int_{-\infty}^{\infty} g(u)\, du = 0.$$

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» Answer to previous problem...

$Z \sim N(0,1)$

$$E[Z] = \underbrace{\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{\text{odd fn}} \, dx$$

$$= 0$$

$$E[Z^2] = \underbrace{\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}\, dx$$

Int. by parts

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \underbrace{\int_0^{\infty} x^2 e^{-x^2/2} dx}$$

$$I = \int_0^{\infty} \underbrace{x^2}_{1st} \underbrace{e^{-x^2/2}}_{2nd} dx$$

$$= \left[x^2 \int_0^{\infty} e^{-x^2/2} dx\right]_0^{\infty} - \underbrace{\int_0^{\infty} 2x \int_0^{x} e^{-x^2/2} dx}$$

» Answer to previous problem...

$$\cancel{= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2}\, dx} =$$

$$= 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2}\, dx$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{2\pi}}} \frac{1}{\cancel{2}} \underbrace{\int_{-\infty}^\infty e^{-x^2/2}\, dx}_{\cancel{\sqrt{2\pi}}} = 1$$

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* The integral does not have a closed form solution!

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Definition of CDF of Standard Normal Distribution

The CDF of the standard normal distribution is denoted by Φ $Z \sim N(0,1)$

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du$$

$= F_Z(x)$

* The integral does not have a closed form solution!
* However, values of $F(Z)$ have been tabulated

» CDF of Standard Normal Distribution...

$F_Z(x) := P(Z \le x)$

Definition of CDF of Standard Normal Distribution

The CDF of the standard normal distribution is denoted by Φ

$$\Phi(x) = P(Z \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du$$

- The integral does not have a closed form solution!
- However, values of $F(Z)$ have been tabulated
- The CDF of any normal distribution can be written in terms of Φ function

» CDF of Standard Normal Distribution...

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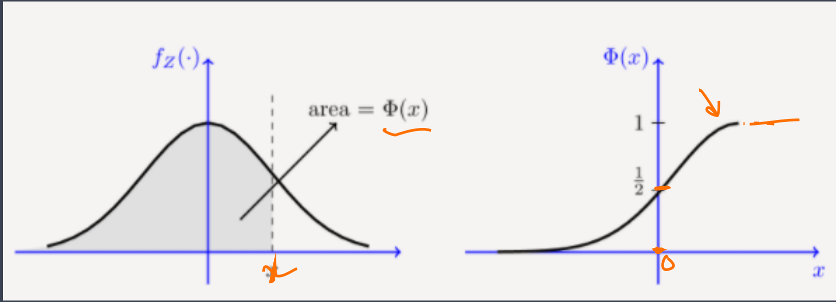

The Φ function satisfies the following properties:

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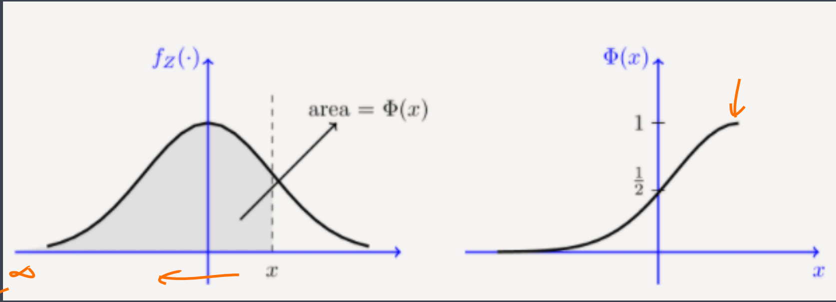

The Φ function satisfies the following properties:

* $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$

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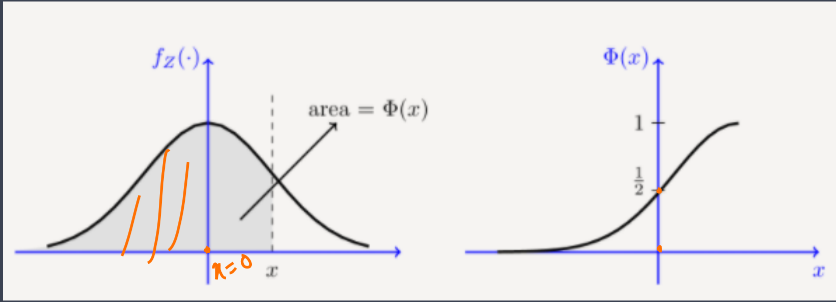

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* $\Phi(0) = \dfrac{1}{2}$

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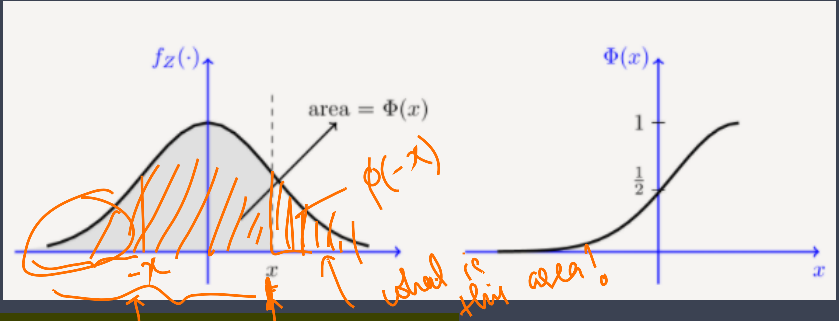

The Φ function satisfies the following properties:

* $\lim_{x \to \infty} = 1, \quad \lim_{x \to -\infty} = 0$
* $\Phi(0) = \frac{1}{2}$
* $\Phi(-x) = 1 - \Phi(x)$, for all $x \in \mathbb{R}$

» Bound for Φ Function...

Bound for Φ Function

Let $Z \sim N(0,1)$. We recall that

$$\Phi(x) = P(Z \leq x).$$

$1 - \phi(x) = 1 - P(Z \leq x)$

$= P(Z > x)$

For all $x \geq 0$, the $\Phi-$function satisfies the following bound

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

$P(Z > x)$

» Answer to previous problem...

To show upper bound:

$$P(Z>x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \underbrace{e^{-u^2/2}}_{f(u)} \, du$$

$$\le \frac{1}{\sqrt{2\pi}} \int_x^\infty \underbrace{\frac{u}{x} e^{-u^2/2}}_{g(u)} \, du$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \int_x^\infty u \, e^{-u^2/2} \, du$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{x} \left[-e^{-u^2/2} \right]_x^\infty = \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

Note: obviously

$u \ge x \ge 0$

Recall

$f(x) \le g(x)$ $\quad x \in [a,b]$

$$\int_a^b f(x) \, dx \le \int_a^b g(x) \, dx$$