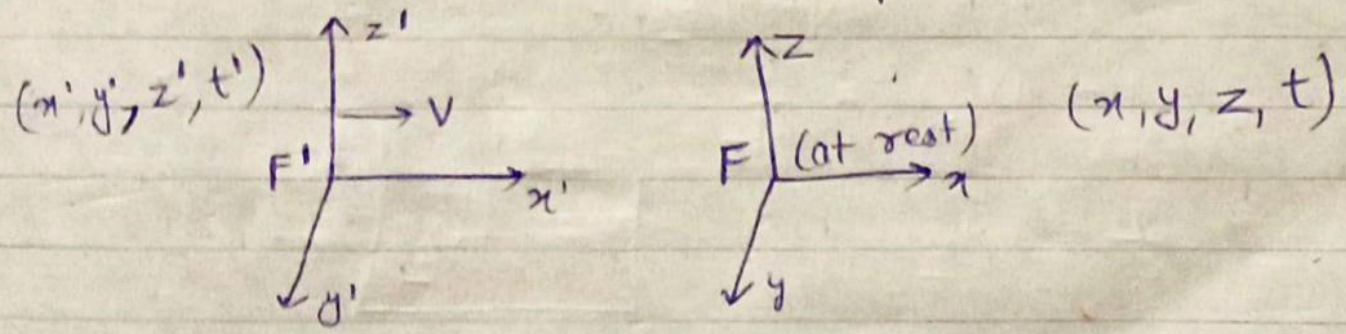
Science-1 (Special theory of Relativity)

Ans 1 Let there be 2 ~~frames~~ inertial frames F and F'. F' is moving with velocity V. (but V is comparable to c) in x-direction.



So Einstein formulated that if $(V \sim c)$ then

$$x' = \gamma(V)\,[x - Vt] \qquad (F \to F')$$
$$x = \gamma(-V)\,[x' + Vt'] \qquad (F' \to F)$$

where $\gamma(V) = \dfrac{1}{\sqrt{1-(V/c)^2}}$

Continued

So Lorentz transformation are:-

Forward Transformaⁿ	Inverse Transformaⁿ
$\because x' = \frac{1}{\sqrt{1-(v/c)^2}}(x - vt)$	$x = \frac{1}{\sqrt{1-(v/c)^2}}(x' + vt')$
$y' = y \quad , \quad z' = z$	$y = y' \quad ; \quad z = z'$
$\therefore \boxed{t' = \gamma\left(t - \frac{xv}{c^2}\right)}$	$\therefore \boxed{t = \gamma\left(t' + \frac{x'v}{c^2}\right)}$

Ans. 2 Let observer be represented by O and object as B (for body). M means moving and F means stationary (F for fixed). The arrow from the moving entity, if points to the other entity means moving towards, else it means moving away from.

The 4 cases can now be written as:-

a) OF BM →

b) OF ← BM

c) BF OM →

d) BF ← OM

Notice that since motion is relative, (a) and (c) are equivalent and (b) and (d) are equivalent. Also, since γ is a defining factor in ~~relativity~~

relativity given by $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$

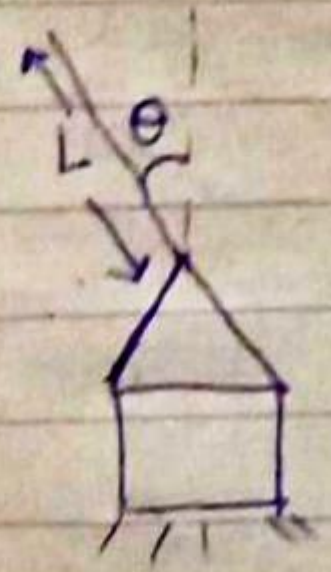
We get that γ is independent of the direction of motion, as the velocity is squared. This tells us that (a) and (b) are equivalent and (c) and (d) are equivalent.

From these 2 observations, we see that all 4 scenarios are equivalent. So, the table should look like :-

Scenario	Rod (L)	Clock (τ)
OF BM →	$L\sqrt{1-\beta^2}$	$\tau/\sqrt{1-\beta^2}$
OF ← BM	$L\sqrt{1-\beta^2}$	$\tau/\sqrt{1-\beta^2}$
BF OM →	$L\sqrt{1-\beta^2}$	$\tau/\sqrt{1-\beta^2}$
BF ← OM	$L\sqrt{1-\beta^2}$	$\tau/\sqrt{1-\beta^2}$

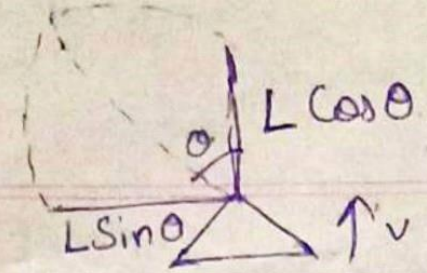
where $\beta = v/c$. v rel. velocity b/w observer and object.

Ans. 3



$L_{observed}$ from Earth if rocket moves up by v velocity.

We can take components of length in horizontal and vertical direction.



Since rocket is moving up at velocity v so horizontal length observed will remain same.

Vertical length can be calculated by applying Length Contraction principle.

$$L_{vertical\ (observed)} = L\cos\theta \cdot \sqrt{1-(v/c)^2}.$$

θ' : Angle observed from earth (with vertical line)

$$\therefore \cancel{\tan\theta' = \frac{L_{vertical\ (observed)}}{L_{Horizontal\ (observed)}}}$$

$$\cancel{\tan\theta'}$$

$$\therefore \tan(90-\theta') = \frac{L_{vertical\ (observed)}}{L_{Horizontal\ (observed)}}$$

$$\tan(90-\theta') = \frac{L\cos\theta\sqrt{1-(v/c)^2}}{L\sin\theta}$$

$$\therefore \tan\theta' = \frac{\sin\theta}{\cos\theta\sqrt{1-(v/c)^2}} = \frac{\tan\theta}{\sqrt{1-(v/c)^2}}$$

$$\therefore \boxed{\theta' = \tan^{-1}\left(\frac{\tan\theta}{\sqrt{1-(v/c)^2}}\right)}$$

Ans 4 ~~Time~~ We just have to apply formula of time dialation.

$$\tau = \frac{\tau_0}{\sqrt{1-(v/c)^2}}$$

$$\therefore \quad 2 = \frac{1}{\sqrt{1-(v/c)^2}}$$

$$1-(v/c)^2 = \frac{1}{4} \qquad \left(\frac{v}{c}\right)^2 = \frac{3}{4}$$

$$\frac{v}{c} = \frac{\sqrt{3}}{2} \qquad \boxed{v = \frac{\sqrt{3}c}{2}}$$

Ans.5 We will use the time dialation concept. Problem is similar to twin paradox problem.

$$\therefore \quad \tau = \frac{\tau_0}{\sqrt{1-(v/c)^2}}$$

Where

τ : 2 weeks = 14 days

$\tau_0 = ?$ $\quad v = 0.8c$.

$$\therefore \quad 14 = \frac{\tau_0}{\sqrt{1-\left(\frac{0.8c}{c}\right)^2}}$$

$$\therefore \quad \tau_0 = 14 \times 0.6 = 8.4 \text{ days.}$$

$\therefore$ Height of plant $= 8.4 \times 1$ (inch/day)

$= \boxed{8.4 \text{ inches}}$

Ans 6

$$\lambda = \lambda_0 \sqrt{\frac{1+v/c}{1-v/c}}$$

$v = 15{,}000 \text{ km/s}$

$= 15 \times 10^3 \times 10^4 \text{ m/s} = 1.5 \times 10^8 \text{ m/s} = c/2$

$\therefore$ ~~$\lambda = \lambda_0 \sqrt{}$~~

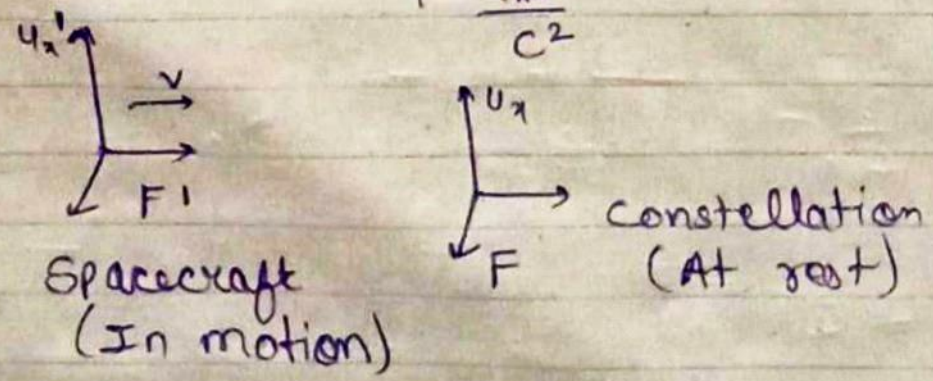
$$\lambda = \lambda_0 \sqrt{\frac{1+(c/2)/c}{1-(c/2)/c}} = \lambda_0 \sqrt{\frac{1.5}{0.5}} = \sqrt{3}\,\lambda_0$$

$\therefore \lambda = \sqrt{3} \times 550 = 952.628 \text{ nm}$

~~Ans~~ ($\lambda > \lambda_0 \Rightarrow$ Red Shift).

Ans 7 (a) By Velocity-Addition Rule

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$



v: Velocity of frame F' wrt F = 0.2c

u_x: Velocity of constella^n wrt F (= 0)

u_x': Velocity of object wrt F' = ?

$$\therefore \quad u_x' = \frac{0 - v}{1 - \frac{0 \times v}{c^2}} = -v = -0.2c$$

$$\boxed{u_x' = -0.2c}$$

$\therefore$ Speed of constellation from the spacecraft $= 0.2c$ towards it.

(b) For an observer on Earth

$$t_0 = \frac{2L}{v} = \frac{2 \times 330\,000 \times c}{(0.2)\,c} = 3.3 \times 10^6 \text{ year}$$

For round-trip

(c) Time elapsed for a person is spaceship will be

$$t = \frac{t_0}{\gamma} = \frac{3.3 \times 10^6}{\sqrt{1 - \left(\frac{0.2c}{c}\right)^2}} = \frac{3.3 \times 10^6}{\sqrt{0.96}} \text{ years.}$$

$$\text{Size required} = \frac{3.3 \times 10^6}{\sqrt{0.96}} \times 1 \text{ micrometercube}$$

$$\approx 3.23 \times 10^6 \text{ micrometer cube.}$$

Ans.8

$\uparrow v(t),\ M(t),\ \gamma(t)$

$\downarrow v_g\ (\Delta M') \rightarrow \gamma_g$: 0 gravity

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \qquad \gamma_g = \frac{1}{\sqrt{1-(v_g/c)^2}}$$

$v(t)$: Velocity of Rocket w.r.t. ground

v_g: velocity of gases w.r.t. rocket

Let $v_{g\gamma}$: velocity of gases w.r.t. ground

Applying Velocity-Addition Rule:-

$$\boxed{v_{g\gamma} = \frac{v - v_g}{1 - \frac{v v_g}{c^2}}}$$

Applying Momentum Conservation w.r.t. ground:-

~~Mass is losing so ΔM is -ve~~

$$\gamma M v = (\gamma + \Delta\gamma)(M + \Delta M)(v + \Delta v) + \gamma_g \Delta M' v_{g\gamma}$$

$$\gamma M v = (\gamma + \Delta\gamma)(Mv + M\Delta v + v\Delta M) + \gamma_g \Delta M' v_{g\gamma}$$

$$\cancel{\gamma MV} = \cancel{\gamma MV} + \gamma M\Delta V + \gamma V\Delta M + \Delta\gamma MV + \gamma_g \Delta M' V_{g\gamma}$$

(Neglecting smaller terms)

$$\gamma = (1-\beta^2)^{-1/2} \quad \text{where } \beta = V/c.$$

$$d\gamma = -\frac{1}{2} \cdot (1-\beta^2)^{-3/2} \cdot (-2\beta)\, d\beta$$

$$d\gamma = \beta \cdot \left[\frac{1}{(1-\beta^2)^{1/2}}\right]^3 d\beta = \gamma^3 \beta\, d\beta.$$

$$\boxed{0 = \gamma M\Delta V + \gamma V\Delta M + \gamma^3 \beta\, d\beta\, MV + \gamma_g \Delta M' V_{g\gamma}} \rightarrow (1)$$

Applying Energy conservation

$$\cancel{\therefore \gamma m c^2 = \gamma M c^2 = (\gamma + d\gamma)}$$

$$\gamma M \cancel{c^2} = (\gamma + \Delta\gamma)(M + \Delta M)\cancel{c^2} + \gamma_g \Delta M' \cancel{c^2}$$

$$\cancel{\gamma M} = \cancel{\gamma M} + \gamma\Delta M + \Delta\gamma M + \gamma_g \Delta M'$$

$$\boxed{\gamma_g \Delta M' = -(\gamma \Delta M + \Delta\gamma M)} \rightarrow (2)$$

Substitute this value in (1)

$$\therefore\ 0 = \gamma M \Delta V + \gamma V \Delta M + \gamma^3 \beta d\beta MV + V_{g\gamma}(-\gamma \Delta M + \Delta\gamma M)$$

$$\beta = V/c \qquad \therefore\ d\beta = \frac{dV}{c}$$

Put value of $d\beta$ & $d\gamma$ in eq^n

$$\cancel{0 = M\Delta V + V\Delta M + \gamma^3 \beta \frac{MVdV}{c}}$$

$$\cancel{0 = \gamma M \Delta V + \gamma V \Delta M + \frac{\gamma^3 \beta MVdV}{c} + V_{g\gamma}(-\gamma \Delta M + M\gamma^3 \beta d\beta)}$$

$$\cancel{0 = M\Delta V + V\Delta M + \frac{\gamma^2 \beta MV\Delta V}{c} + V_{g\gamma}(-\Delta M + M\gamma^2 \beta d\beta)}$$

$$\cancel{0 = \Delta V\left(M + \frac{\gamma^3 \beta M}{c}\right) + \Delta M(V}$$

$$\cancel{0 = \Delta V\left(M + \frac{\gamma^2 \beta MV}{c}\right) + \Delta M(V - V_{g\gamma})}$$

$$\cancel{0 = \gamma M \Delta V + \gamma V \Delta M + \frac{\gamma^3 \beta MV\Delta V}{c} + V_{g\gamma}\left(-\gamma \Delta M + \frac{\gamma^3 \beta M\Delta V}{c}\right)}$$

$$0 = \gamma M \Delta V + \gamma V \Delta M + \frac{\gamma^3 \beta MV\Delta V}{c} + V_{g\gamma}\left(-\gamma \Delta M + \frac{\gamma^3 \beta M\Delta V}{c}\right)$$

$$0 = M\Delta V + V\Delta M + \frac{\gamma^2 \beta MV\Delta V}{c} + V_{g\gamma}\left(-\Delta M + \frac{\gamma^2 \beta M\Delta V}{c}\right)$$

$$0 = \Delta V\left(M + \frac{\gamma^2 \beta MV}{c} + \frac{\gamma^2 \beta MV_{g\gamma}}{c}\right) + \Delta M(V - V_{g\gamma})$$

$$\Rightarrow \gamma^2 m\,dV\left(1 - \frac{\beta V_{g\gamma}}{c}\right) + dm(V - V_{g\gamma}) = 0.$$

$$0 = \gamma^2 m\,dv\left[1 - \beta/c\left(\frac{v - v_g}{1 - \frac{v v_g}{c^2}}\right)\right] + dm\left(v - \frac{v - v_g}{1 - \frac{v v_g}{c^2}}\right)$$

$$0 = \gamma^2 m\,dv\left(\frac{1 - \frac{v v_g}{c^2} - \frac{v}{c^2}(v - v_g)}{1 - \frac{v v_g}{c^2}}\right) + dm\left(\frac{v - \frac{v^2 v_g - v + v_g}{c^2}}{1 - \frac{v v_g}{c^2}}\right)$$

$$\gamma^2 m\,dv\left(1 - \left(\frac{v}{c}\right)^2\right) + dm\left(v_g - \frac{v_g v^2}{c^2}\right) = 0.$$

$$\frac{\cancel{\gamma^2} m\,dv}{\cancel{\gamma^2}} + dm\,(v_g)(1 - \beta^2) = 0$$

$$\therefore \boxed{\frac{m\,dv}{dt} + v_g \frac{dm}{dt}(1 - \beta^2) = 0}$$

we have $m\frac{dv}{dt} + v_g\frac{dm}{dt}\left(1-\left(\frac{v}{c}\right)^2\right) = 0.$

$$\therefore \int \frac{dv}{1-(v/c)^2} = -v_g\int dm$$

$$\int \frac{(1+v/c)+(1-v/c)}{(1+v/c)\cdot(1-v/c)}\cdot dv = -v_g \ln M + c'$$

$$\int \frac{dv}{1-v/c} + \int \frac{dv}{1+v/c} = -v_g \ln M + c'$$

$$\frac{c}{2}\left[\ln\left(1+\frac{v}{c}\right) - \ln\left(1-\frac{v}{c}\right)\right] = -v_g \ln M + c'$$

$$\boxed{\ln\left(\frac{c+v}{c-v}\right) = -\frac{2v_g \ln M}{c} + c'} \rightarrow (4)$$

Roughly solving it by plugging some values

Let $M = e$ $\quad v_g = c/2$ $\quad c' = 3$

$$\therefore \ln\left(\frac{c+v}{c-v}\right) = -\frac{2\cdot c/2\cdot 1}{c} + c'$$

$$\frac{c+v}{c-v} = e^2 = \frac{e^2}{1}$$

Apply componendo & Dividendo

$$\therefore \frac{(c+v)+(c-v)}{(c+v)-(c-v)} = \frac{e^2+1}{e^2-1}$$

$$\therefore \frac{c}{v} = \frac{e^2+1}{e^2-1} \qquad \therefore \boxed{v = c\left[\frac{e^2-1}{e^2+1}\right]}$$

Solving eqⁿ ④ further:-

$$\therefore \frac{c+v}{c-v} = \lambda e^{-\frac{2v_g \ln M}{c}}$$

Apply C & D

$$\therefore \frac{c}{v} = \frac{\lambda e^{-\frac{2v_g \ln M}{c}} + 1}{\lambda e^{-\frac{2v_g \ln M}{c}} - 1}$$

$$v = c\left[\frac{\lambda e^{-\frac{2v_g \ln M}{c}} - 1}{\lambda e^{-\frac{2v_g \ln M}{c}} + 1}\right]$$

<u>Assumption</u>:- Assuming Mass to be decreasing exponentially with time

i.e.

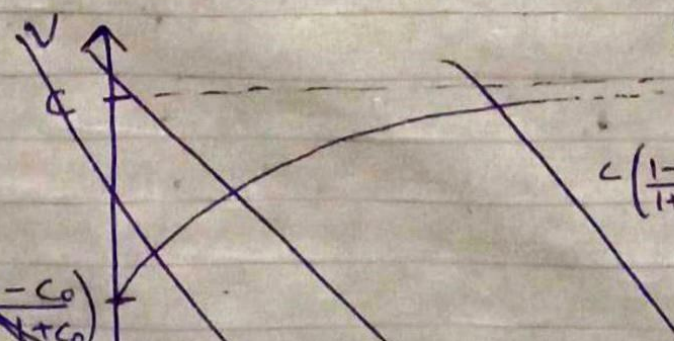
$$M = M_0 e^{-\mu t} \qquad \text{let } M_0 = e^{\gamma}$$

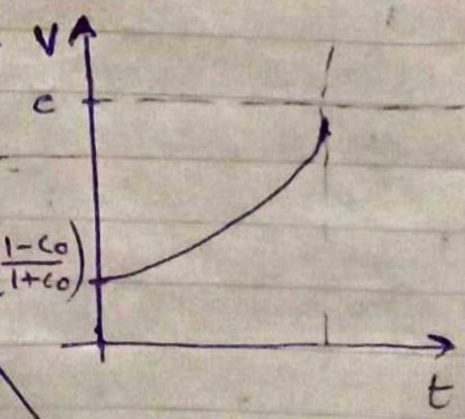
$$\therefore M = e^{-\mu t + \gamma}.$$

$$\therefore v = c\left(\frac{\lambda e^{\alpha t + \beta} - 1}{\lambda e^{\alpha t + \beta} + 1}\right)$$

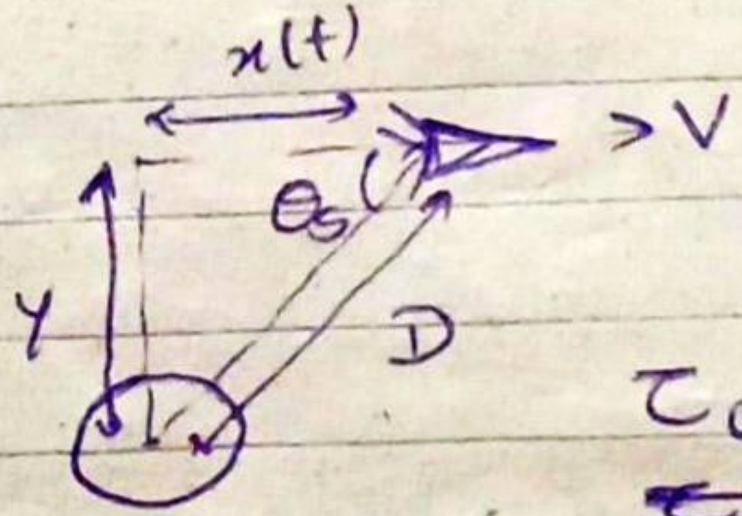
λe^{β} : const.

$$\therefore \boxed{v = c\left[\frac{e^{\alpha t} - C_0}{e^{\alpha t} + C_0}\right]}$$





Ans 9



$$D^2 = x^2 + y^2$$

τ_0 = rest time interval

$$\tau = \gamma\tau_0 + \frac{dD(t)}{c} = \gamma\tau_0\left(1 + \frac{x(t)}{D(t)} \cdot \frac{v}{c}\right)$$

$$= \gamma\tau_0\left(1 + \cos(\theta_s) \cdot \frac{v}{c}\right)$$

$$\nu = \frac{1}{\tau} = \frac{1}{\gamma\tau_0} \cdot \frac{1}{(1 + (\cos\theta_s)\, v/c)}$$

$$\therefore \nu = \frac{\nu_0}{\gamma} \cdot \left(\frac{1}{1 + \cos\theta_s (v/c)} \right).$$

given $v = 0.2c$

$$\therefore \gamma = \frac{1}{\sqrt{1 - \left(\frac{0.2c}{c}\right)^2}} = \frac{1}{\sqrt{0.96}} = \frac{10}{4\sqrt{6}}.$$

$$\therefore \nu = \frac{\nu_0 \cdot 4\sqrt{6}}{10} \cdot \left(\frac{1}{1 + (0.2c/c) \cos\theta_s} \right)$$

$$\therefore \boxed{\nu = \frac{4\sqrt{6}\,\nu_0}{10 + 2\cos\theta_s} = \frac{2\sqrt{6}\,\nu_0}{5 + \cos\theta_s}}$$

Ans. D

$e^{\ominus} + e^{\ominus} \longrightarrow e^{\ominus} + e^{\ominus} + (e^{\ominus} + e^{\oplus})$

↳ positron

Let mass of $e^{\ominus}$ be m.

COM: Center of Momentum

Total KE = KE (COM) + KE (of particles in COM frame)

Given all particles have same velocity after the reaction

⇒ Velocity in COM frame = 0

⇒ KE (in COM frame) = 0.

$\therefore$ Total KE = KE(COM)

$\downarrow$

The KE is minimum and is required KE

In COM frame :-

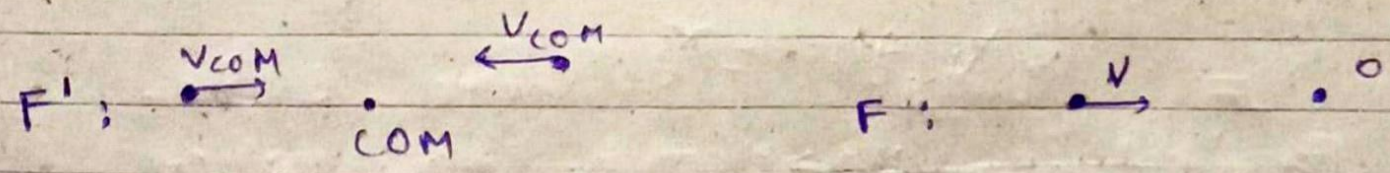
By Energy Conservation $E_i = E_f$

$E_i = 2\gamma mc^2$ $\qquad E_f = 4mc^2$

$\Rightarrow \boxed{\gamma = 2}$

$\cancel{\gamma = \sqrt{1-\left(\frac{V_{COM}}{c}\right)^2} = 2} \quad \therefore \cancel{\boxed{V_{COM} = \frac{\sqrt{3}c}{2}}}$

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{V_{COM}}{c}\right)^2}} = 2 \qquad \therefore \boxed{V_{COM} = \frac{\sqrt{3}c}{2}}$$



By velocity addition rule

$$u_x = \frac{u_x' + V}{1 + \frac{u_x' V}{c^2}}$$

$u_x' = V_{COM}$

$u_x = ?$

$V = V_{COM}$

where

u_x : Velocity of $e^{\ominus}$ in frame F

u_x' : Velocity of $e^{\ominus}$ in frame F'

V : Relative velocity of F' wrt F.

$$\therefore \quad u_x = \frac{V_{COM} + V_{COM}}{1 + \frac{V_{COM} \cdot V_{COM}}{c^2}} = \frac{2V_{COM}}{1 + \frac{V_{COM}^2}{c^2}}$$

$$\boxed{u_x = \frac{4\sqrt{3}}{7}c}$$

Total $KE = KE_{req.} = (\gamma - 1) mc^2$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{4x}{c}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{4\sqrt{3}c}{7c}\right)^2}} = 7.$$

$$\boxed{KE_{req.} = (7-1) mc^2 = 6mc^2}$$

(a)

Ans:- If possible let only 1 photon be released.

Let λ be the wavelength of photon released.

Mass of $e^{\ominus}$ = Mass of $e^{\oplus} = m$.

Let the velocities of $e^{\ominus}$ and $e^{\oplus}$ before collision be $\vec{v}_e$ and $\vec{v}_p$ resp..

Energy of photon $= \frac{hc}{\lambda} = k$.

Momentum of photon $= \frac{h}{\lambda}\hat{v} = \vec{p}$

$\hat{v}$: unit vector along direction of motion of photon.

Total Relativistic Energy of $e^{\ominus}$ and $e^{\oplus}$ is transferred to the photon. (Energy Conservation).

By Conservation of Momentum:-

$$\gamma(u_e)\, mc^2 + \gamma(u_p)\, mc^2 = \frac{hc}{\lambda}$$

$$\therefore \frac{h}{\lambda} = mc\left(\gamma(u_e) + \gamma(u_p)\right).$$

$$|\vec{p}| = h/\lambda.$$

$$\therefore \left|\gamma(u_e)\, m\, \vec{u}_e + \gamma(u_p)\, m\, \vec{u}_p\right| = mc\left(\gamma(u_e) + \gamma(u_p)\right)$$

$$\therefore \left|\gamma(u_e)\frac{\vec{u}_e}{c} + \gamma(u_p)\frac{\vec{u}_p}{c}\right| = \gamma(u_e) + \gamma(u_p)$$

Let $\vec{\beta}_e = \frac{\vec{u}_e}{c}$, $\vec{\beta}_p = \frac{\vec{u}_p}{c}$.

then $\cancel{\beta_e = \frac{u_e}{c}}$ $|\vec{\beta}_e| = \frac{u_e}{c} < 1$

and $|\vec{\beta}_p| = \frac{u_p}{c} < 1$.

$\therefore \; |\gamma(u_e)\,\vec{\beta}_e + \gamma(u_p)\,\vec{\beta}_p| = \gamma(u_e) + \gamma(u_p)$

But $|\vec{\beta}_e| < 1$ and, $|\vec{\beta}_p| < 1$ when $u_e > 0$, $u_p > 0$.

Hence even if $\vec{u}_e$ and $\vec{u}_p$ are in the same direction.

$$|\gamma(u_e)\,\vec{\beta}_e + \gamma(u_p)\,\vec{\beta}_p| < \gamma(u_e) + \gamma(u_p)$$

if $u_e = u_p = 0$. $h/\lambda > 0$. This violates conservation of linear Momentum. Hence ~~Contra~~ contradiction.

Now let 2 photons be emitted. Let us assume the simplest case where both the photon travels in the same direction. Then we eventually get the condition.

$$2\,|\gamma(u_e)\,\vec{\beta}_e + \gamma(u_p)\,\vec{\beta}_p| = \gamma(u_e) + \gamma(u_p)$$

There is no contradiction to this statement in a general condition. Also if net ~~mom~~ relativistic momentum of the ~~$e^{\oplus}$ and $e^{\ominus}$~~ $e^{\ominus}$ and $e^{\oplus}$ before collision was 0, the 2 photons can have equal wavelength

and travel in opposite directions to satisfy conservation of momentum and energy.

★ Hence, a ~~mim~~ minimum of 2 photons must be released per $e^{\ominus} - e^{\oplus}$ pair annihilation.

(b) Mass of spacecraft + reactants $= m = 10{,}000$ tonnes

~~Mass of the reactant (annihilation) = m'~~

Mass after reaction (annihilation) $= m'$,

$m' = m -$ (Mass of reactants destroyed)

Now $E_{init} = mc^2$ $\quad V \rightarrow$ volume

$E_{after} = \gamma_{v_t} m'c^2$ $\quad v_t$: terminal velocity

$\gamma_{v_t} m'c^2 - mc^2 =$ Energy by reaction

$\gamma_{v_t} m'c^2 - mc^2 = V_{(in\,cm^3)} \times M$ Mev

(since $1\,cm^3 \rightarrow M$ Mev)

$= V \times M \times 1.6 \times 10^{-13}$ Joules $\rightarrow$ (1)

Now Mass of reactant's destroyed $\rightarrow V \times 10^{-9}$ kg

(since $1\,cm^3 \rightarrow 1$ mg)

$\therefore \quad m' = m - 10^{-9} \times V \rightarrow$ (2)

Plugging m' in (1)

$$\gamma_{v_t}(m - 10^{-9} \times V)c^2 - mc^2 = M \times V \times 1.6 \times 10^{-13}$$

$$V = \frac{\gamma(m - 10^{-9}V)c^2 - mc^2}{M \times 1.6 \times 10^{-13}}$$

$$V \times M \times 1.6 \times 10^{-13} = \gamma m c^2 - \gamma \times 10^{-9} v c^2 - m c^2$$

$$V \left(M \times 1.6 \times 10^{-13} + \gamma c^2 \times 10^{-9} \right) = m c^2 (\gamma - 1)$$

$$\therefore \boxed{V = \frac{(\gamma - 1) m c^2}{M \times 1.6 \times 10^{-13} + \gamma c^2 \times 10^{-9}}}$$

where $\gamma_{v_t} = \dfrac{1}{\sqrt{1 - (v_t/c)^2}}$ $\quad v_t$: terminal velocity.

$c = 3 \times 10^8$ m/s $\quad m = 10{,}000$ tonnes $= 10^7$ kg.