Science-1 (classical Mechanics Assignment)

Ans.) Let m_0 be the initial mass of Rocket (including mass of fuel)

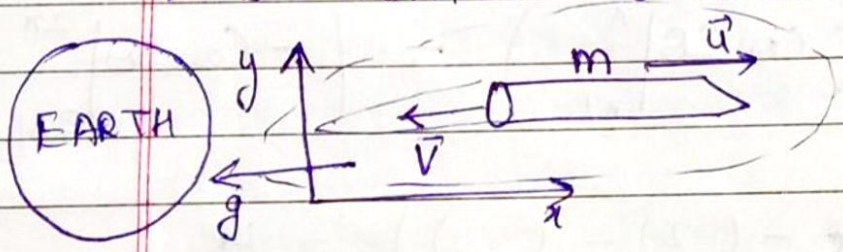
m : Instantaneous mass of Rocket (including wt of fuel)

$\vec{u}$: Instantaneous velocity of Rocket.

$\vec{v}$: Rel. velocity of exhaust gases.
(velocity of gas w.r.t. Rocket)

The problem has the mass and velocity of the rocket changing, also, the total mass of ejected gases is changing. If we define our system to be the (rocket + fuel), then this is a closed system ~~(in deep space, no external force on rocket)~~ (only gravitational force is experienced)

~~→ Momentum is conserved~~



System

Assuming g to be const. throughout

$\vec{p}_i$: Initial Momentum $= m u \hat{i}$.

$\because$ $\vec{v}$: velocity of gas w.r.t. Rocket

$\vec{v}_g$: Velocity of gas wrt Earth $= (u-v)\hat{i}$.

As a consequence of ejection of the feul gas, the rocket's mass decreases by dm_g, and it's velocity increases by $du\hat{i}$. ~~Therefore, including both~~

An external force $\vec{F} = -mg$ is acting on the system. This force applies an impulse

$$dJ = F\,dt = -mg\,dt$$
$$= \Delta \text{ Momentum}$$

i.e. $dp = dJ$ $\qquad p_f - p_i = -mg\,dt$

$$(m - dm_g)(u + du) + dm_g(u - v) - mu = -mg\,dt$$

and so $\qquad m\,du - dm_g\,v = -mg\,dt$

~~Neglecting~~ $dm_g\,dv \approx 0$ and $dm_g = -dm$

$$\therefore \quad m\,du + dm \cdot V = -mg\,dt$$

$$m\,du = -dm\,V - mg\,dt$$

~~$dv = -u\,dt$~~

$$du = -v \cdot \frac{dm}{m} - g\,dt$$

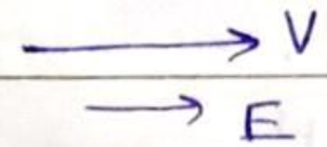
~~$\Delta u = u$~~ $\boxed{\Delta u = v \ln\left(\frac{m_0}{m}\right) - g\Delta t}$

Burn time of fuel

Ans 2 Let us fix the velocity V in the +ve x direction. Doing so, we won't lose the generality as we can always move our axes.

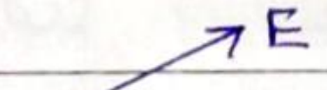
(a) If $\vec{B} \to 0$

(i) $\bar{E} \parallel \bar{v}$



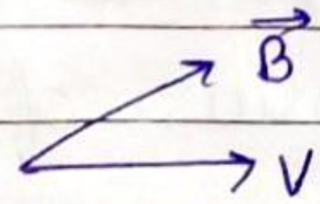
Straight line motion

(ii).



$F = qE$ 2D accelerated motion.

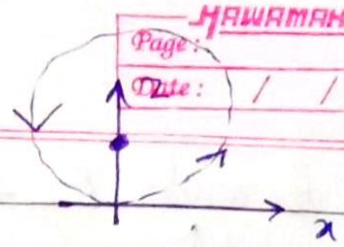
(b) If $\vec{E} = 0$



(i) $\vec{V} \parallel \vec{B}$

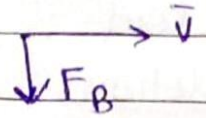
$F = q(\bar{v} \times \bar{B}) = 0$ No force is acting

$\Rightarrow$ Uniform Straight ~~lim~~ Line Motion



(ii) $\bar{v} \perp \bar{B}$ $\quad v\hat{i}, \; B\hat{j}$

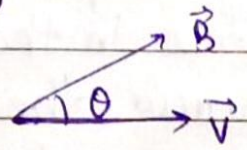
At each point

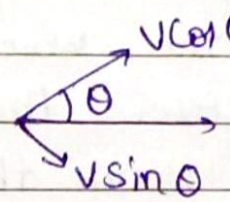


F_B will act as centripetal force.

$qvB = \frac{mv^2}{r} \Rightarrow r = \frac{mv}{qB}$ → Circular Motion with radius $= \frac{mv}{qB}$.

(iii) general case

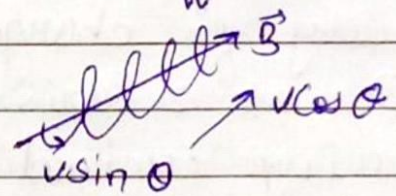




For V component $\perp$ to $\vec{B}$. only it will experience a force.

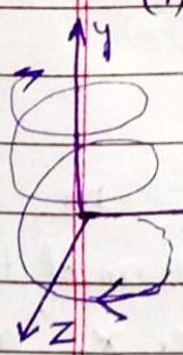
Other component will be unaffected as no force.

∴ Helical Motion



(c) ~~$\bar{E} \neq$~~ $\bar{E} \neq 0$

(i) $\bar{B}$ and $\bar{E}$ in same direction $\perp$ to $\bar{v}$



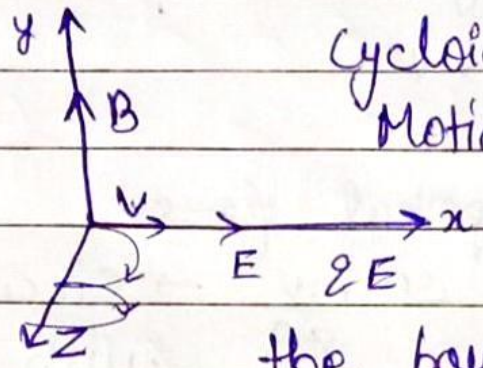
Helical Motion

The $\bar{B}$ will cause it to have circular motion around y axis. And $\bar{E}$ will produce an acceleration in y-direction.

∴ Helix will get more and more speed.

Time period $= \frac{2\pi \frac{mv}{qB}}{v} = \frac{2\pi m}{qB}$.

(ii) $\bar{B}$ and $\bar{E}$ $\perp$ to each other

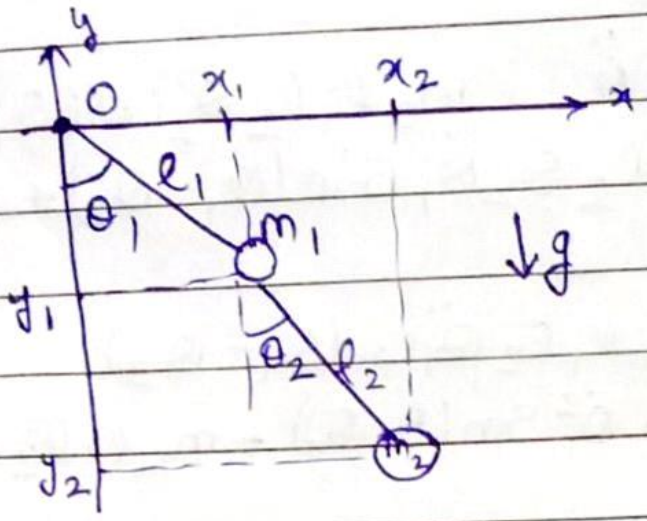


Cycloid Motion

As the particle acquires velocity in z direction, the magnetic force comes to play and tries to ~~protect~~ rotate the particle about center on z-axis.

However, x component of velocity keeps increasing with time due to electric force in that direction. The magnetic force though draws the charged particle away from the x-axis along a curved path. This action of Magnetic force is countered by electric force in x direction. The velocity of charged particle ultimately reduces to 0 at z-axis. This cycle repeats itself forming cycloid Motion.

Ans3



$x_1 = l_1 \sin\theta_1$

$y_1 = -l_1 \cos\theta_1$

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$

$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$

Differentiating above 4 quantities for velocities.

$\dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1$ $\qquad$ $\dot{y}_1 = l_1 \dot{\theta}_1 \sin\theta_1$

$\dot{x}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$ $\qquad$ $\dot{y}_2 = l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2$

The Lagrangian for the double pendulum is given by

$L = T - U.$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]$$

~~Similarly~~ & $V = m_1 g y_1 + m_2 g y_2$

$$= -m_1 g l_1 \cos\theta_1 - m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2)$$
$$= -(m_1+m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

$$\therefore L = \frac{1}{2}(m_1+m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1-\theta_2)$$
$$+ (m_1+m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

The equations of motion of the System are the Euler- Lagrange eq^{n}s :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) = \frac{\partial L}{\partial \theta_i} \Rightarrow \frac{dp_{\theta_i}}{dt} = \frac{\partial L}{\partial \theta_i} \quad \text{for } i=1,2$$

Writing E-L eq^{n}s :-

$$\frac{dp_{\theta_1}}{dt} = (m_1+m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2)$$

$$\frac{dp_{\theta_2}}{dt} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - (m_1+m_2) g l_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2$$

On Equating for $i=1$ & for $i=2$ we get

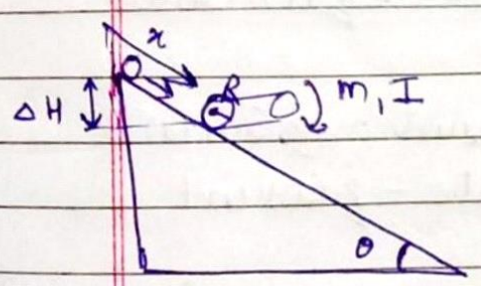
$$(m_1+m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) \cancel{+m_2}$$
$$+ m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g \sin\theta_1 = 0.$$

$$l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + g \sin\theta_2 = 0.$$

Ans. 4 Let ϕ denote the angle of rotation about cylinder's symmetry axis. Let x denote position of cylinder's COM measured from starting point. Since the plane is inclined at an angle θ, if cylinder travels distance x it's ht. will change by $x\sin\theta$. Assuming that the cylinder rolls without slipping, we have $dx = R\,d\phi$. K.E., P.E. and Lagrangian of the system are given by:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 \qquad \dot{x} = R\dot{\phi}$$

$$U = mg(-x\sin\theta)$$



$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{R^2} + mgx\sin\theta$$

Using Euler-Lagrange eqn, we find:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\left(m + \frac{I}{R^2}\right)\ddot{x} = mg\sin\theta$$

$$\therefore \boxed{\ddot{x} = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}}$$

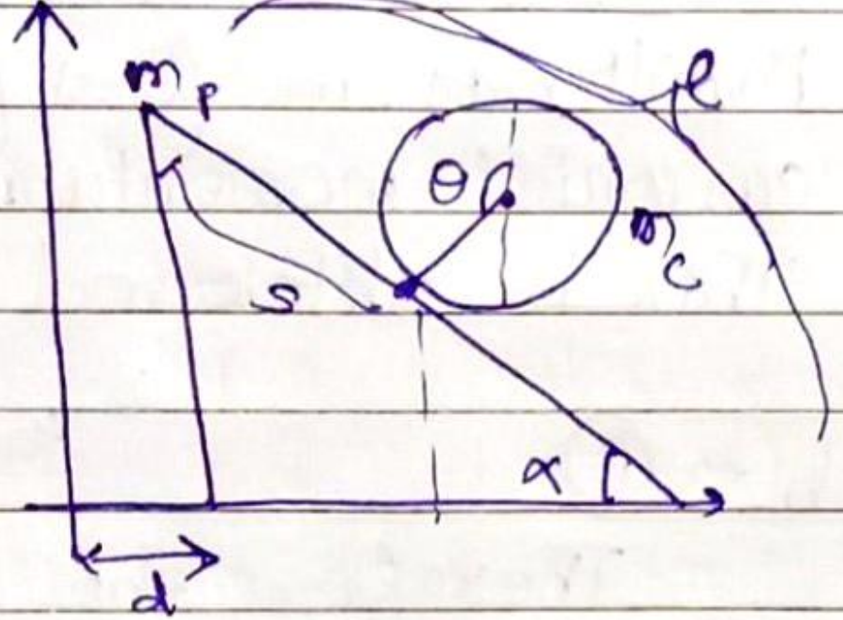
By integrating we will get $\dot{x}$, x as follows.

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$

Similarly

$$\therefore \boxed{\dot{x} = \frac{g\sin\theta\, t}{1 + \frac{I}{mR^2}}} \quad \boxed{x = \frac{t^2}{2}\cdot\frac{g\sin\theta}{1 + \frac{I}{mR^2}}}$$

Ans 5



m_p: mass of plane
m_c: Mass of cylinder
s: distance rolled by cylinder
l: length of inclined surface
θ: Angular displacement of the cylinder
d: distance moved by the plane.
α: Angle of Incline.

a) Constraint eqns: We can take d and x as our generalised coordinates. We can represent the cartesian coordinates of the cylinder as

$$x = d + s\cos\alpha + r\sin\alpha$$

$$y = r\cos\alpha + (l - s)\sin\alpha$$

b.) Generalized velocities:

$\dot{x} = \dot{d} + \dot{s}\cos\alpha \qquad \dot{y} = -\dot{s}\sin\alpha$

K.E. $\quad T = T_{plane} + T_{cylinder}$

$T_{cylinder} = \frac{1}{2} m_c (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2$

$= \frac{1}{2} m_c \left[(\dot{d} + \dot{s}\cos\alpha)^2 + (-\dot{s}\sin\alpha)^2 \right] + \frac{1}{2}\left(\frac{1}{2} m_c r^2\right)\dot{\theta}^2$

$= \frac{1}{2} m_c \left[\dot{d}^2 + \dot{s}^2\cos^2\alpha + 2\dot{d}\dot{s}\cos\alpha + \dot{s}^2\sin\alpha \right] + \frac{1}{4} m_c \cancel{r^2} \frac{\dot{s}^2}{\cancel{r^2}}$

$= \frac{1}{2} m_c (\dot{d}^2 + \dot{s}^2 + 2\dot{d}\dot{s}\cos\alpha) + \frac{1}{4} m_c \dot{s}^2$

$T_{cylinder} = \frac{1}{2} m_c \left(\frac{3\dot{s}^2}{2} + \dot{d}^2 + 2\dot{d}\dot{s}\cos\alpha \right) \qquad T_{plane} = \frac{1}{2} m_p \dot{d}^2$

$$\therefore \boxed{T = \frac{3 m_c \dot{s}^2}{4} + \left(\frac{m_c + m_p}{2}\right) \times \dot{d}^2 + m_c \dot{d}\dot{s}\cos\alpha}$$

P.E.

$$\boxed{V = m_c g y = m_c g \left[r\cos\alpha + (l - s)\sin\alpha \right]}$$

Now the Lagrangein $L = T - V$

$$\boxed{L = \frac{3 m_c \dot{s}^2}{4} + \frac{(m_c + m_p)}{2} \cdot \dot{d}^2 + m_c \dot{d}\dot{s}\cos\alpha - m_c g \left[r\cos\alpha + (l - s)\sin\alpha \right]}$$

c.) Euler-Lagrange Eq^n:- $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$.

Eq^n 1 : $d, \dot{d}$

L.H.S. $\quad \frac{\partial L}{\partial \dot{d}} = (m_c + m_p)\dot{d} + m_c \dot{s}\cos\alpha$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}}\right) = (m_c + m_p)\ddot{d} + m_c \ddot{s}\cos\alpha$

R.H.S. $\partial L/\partial \alpha = 0$ L.H.S. = R.H.S.

$$\therefore \boxed{(m_c + m_p)\ddot{d} + m_c \ddot{s}\cos\alpha = 0}$$

Eqn 2 : $s, \dot{s}$

L.H.S. $\dfrac{\partial L}{\partial \dot{s}} = \dfrac{3}{2} m_c \dot{s} + m_c \dot{d}\cos\alpha$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{3}{2} m_c \ddot{s} + m_c \ddot{d}\cos\alpha$$

R.H.S. $\dfrac{\partial L}{\partial s} = -m_c g(-\sin\alpha) = m_c g \sin\alpha$

L.H.S. = R.H.S. $\therefore \dfrac{3}{2} m_c \ddot{s} + m_c \ddot{d}\cos\alpha = m_c g \sin\alpha$

$$\therefore \boxed{3m_c \ddot{s} + 2m_c \ddot{d}\cos\alpha - 2m_c g \sin\alpha = 0}$$

iv.) The Hamiltonian :-

$$H = \sum q_i p_i - L$$

Now, $p_i = \dfrac{\partial L}{\partial \dot{q}_i} \Rightarrow H = \sum \dfrac{\partial L}{\partial \dot{q}_i} q_i - L$

We have 2 generalized coordinates : d & s

$$H = \frac{\partial L}{\partial \dot{d}} \cdot \dot{d} + \frac{\partial L}{\partial \dot{s}} \cdot \dot{s} - L$$

$$H = 2T - (T - V) = T + V.$$

$$\Rightarrow \boxed{H = \frac{3m_c \dot{s}^2}{4} + \frac{(m_c + m_p)\dot{d}^2}{2} + m_c \dot{d}\dot{s}\cos\alpha + \cancel{m_c g[r\cos\alpha} + m_c g[r\cos\alpha - (l - s)\sin\alpha]}$$

e) Hamiltonian's Canonical Equation :-

$$\dot{q} = \frac{\partial H}{\partial p} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial q}$$

For generalized coordinates d :-

$$\frac{\partial H}{\partial d} = -\dot{p}_d = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}}\right)$$

$$0 = -\frac{d}{dt}\left((m_c + m_p)\dot{d} + m_c \dot{s}\cos\alpha\right)$$

$$\Rightarrow \boxed{(m_c + m_p)\ddot{d} + m_c \ddot{s}\cos\alpha = 0}$$

For generalized coordinates s:

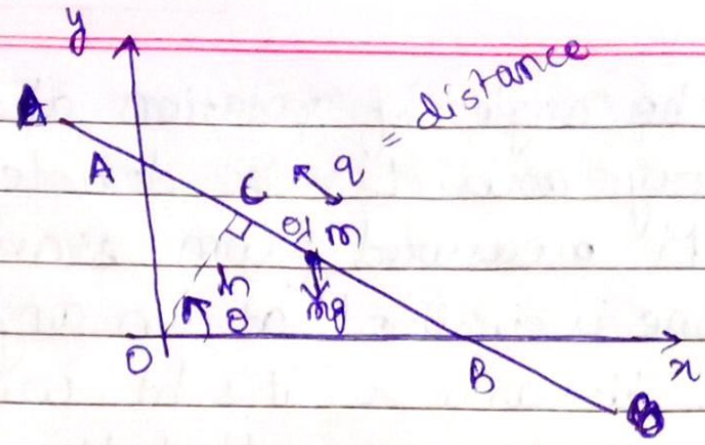
$$\frac{\partial H}{\partial s} = -\dot{p}_s = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right)$$

$$\Rightarrow mg\sin\alpha = \frac{d}{dt}\left(\frac{3}{2}m_c\dot{s} + m_c\dot{d}\cos\alpha\right)$$

$$\Rightarrow \cancel{\frac{3}{2}\ddot{s} + \ddot{d}}\ mg\sin\alpha = \frac{3}{2}m_c\ddot{s} + m_c\ddot{d}\cos\alpha$$

$$\Rightarrow \boxed{\frac{3}{2}\ddot{s} + \ddot{d}\cos\alpha - g\sin\alpha = 0}$$

Ans 6



$\dot{\theta} = \omega$.

Assuming the initial conditions as

$\theta(0) = 0 \quad q(0) = 0$

$\dot{q}(0) = 0$.

Using q and θ ($= \omega t$ and since $\theta(0) = 0$), the x, y coordinates of the particle are expressed as

$\cancel{x = h\cos\theta + q\sin\theta}$

$x = h\cos\theta + q\sin\theta = h\cos\omega t + q(t)\sin\omega t$

$y = h\sin\theta - q\cos\theta = h\sin\omega t - q(t)\cos\omega t$

$\dot{x} = -h\omega\sin\omega t + q\omega\cos\omega t + \dot{q}\sin\omega t$

$\dot{y} = h\omega\cos\omega t + q\omega\sin\omega t - \dot{q}\cos\omega t$

Therefore, K.E. of particle is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$T = \frac{1}{2}m(h^2\omega^2 + q^2\omega^2 + \dot{q}^2) - mh\omega\dot{q}$$

The P.E. $U = mgy = mg(h\sin\omega t - q\cos\omega t)$

$\therefore$ Langrangein $L = T - U$.

$$\therefore L = \frac{1}{2}mh^2\omega^2 + \frac{1}{2}mq^2\omega^2 + \frac{1}{2}m\dot{q}^2 - mgh\sin\omega t + mgq\cos\omega t - mh\omega\dot{q}$$

Langrangein eqn for the co-ordinate is

$$\ddot{q} - \omega^2 q = g\cos\omega t \rightarrow (A)$$

The complementary & particular soln for (A) will be

The Hamiltonian is therefore different from total energy, $T+U$. The energy is not conserved in this problem since the Hamiltonian contains time explicitly (The particle gains energy from gravitational field).

$$q_c(t) = A\cos(i\omega t + \delta) \qquad q_p(t) = \frac{-g}{2\omega^2}\cos\omega t.$$

so that the general solution is

$$q(t) = A\cos(i\omega t + \delta) - \frac{g}{2\omega^2}\cos\omega t$$

Using initial conditions we have

$$q(0) = A\cos\delta - \frac{g}{2\omega^2} = 0$$

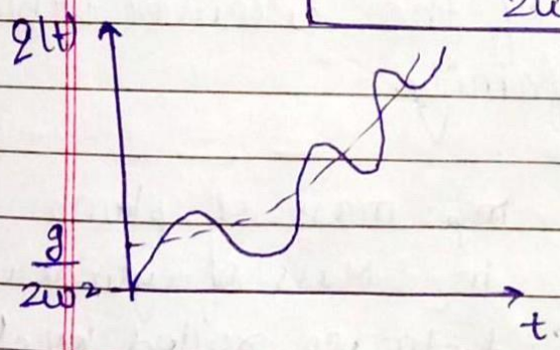
$$\dot{q}(0) = -i\omega A\sin\delta = 0.$$

Therefore,

$$\delta = 0, \quad A = g/2\omega^2$$

and $q(t) = g/2\omega^2 \times (\cos i\omega t - \cos\omega t)$

or $$\boxed{q(t) = \frac{g}{2\omega^2}(\cosh\omega t - \cos\omega t)}$$



In order to compute the Hamiltonian, we first find the canonical momentum of q. This is obtained by

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} - m\omega h \rightarrow (B)$$

Therefore, the Hamiltonian H becomes

$$H = p\dot{q} - L = m\dot{q}^2 - m\omega h\dot{q} - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 - \frac{1}{2}m\dot{q}^2$$

$$+ mgh\sin\omega t - mgq\cos\omega t + m\omega\dot{q}h$$

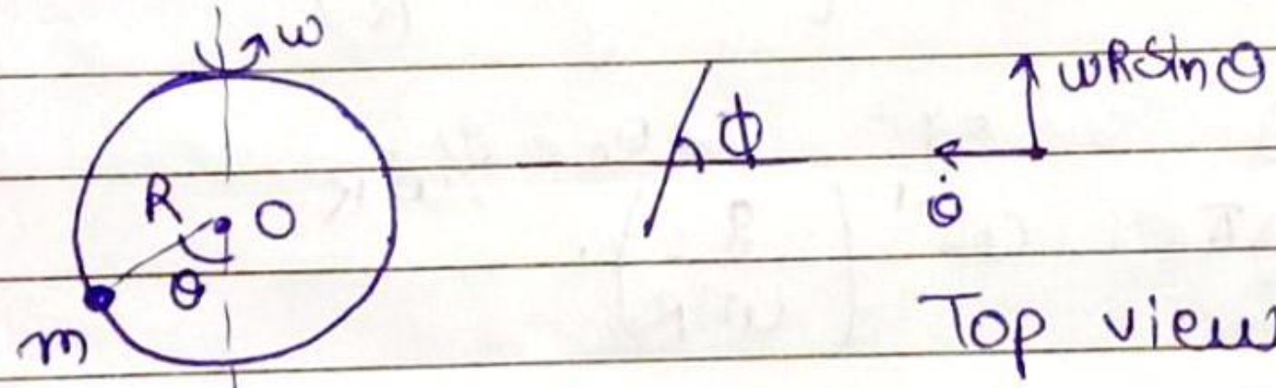
so that

$$H = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t$$

From (B) $$\boxed{H = \frac{p^2}{2m} + \omega hp - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t}$$

continued

Ans. 7



Let ~~φ be~~ the angle b/w initial and final angle of the circle in the top view be ϕ.

$\therefore \phi = \omega t$ and $\omega = \dot{\phi}$

$\therefore$ Each state of the system can be determined by unique values of (θ, ϕ).

$\therefore$ The generalized coordinates are $\{\theta, \phi\}$.

$\therefore$ ~~The~~ $KE_{bead} = \frac{1}{2} m (\omega^2 R^2 \sin^2\theta + \dot{\theta}^2 R^2)$.

$$\therefore K = \frac{m\omega^2 R^2 \sin^2\theta}{2} + \frac{m\dot{\theta}^2 R^2}{2}.$$

P.E. bead $= U = mgR(1-\cos\theta)$

$\therefore$ Lagrangein : $L = K - U$

$$L = \frac{m\omega^2 R^2 \sin^2\theta}{2} + \frac{mR^2\dot{\theta}^2}{2} - mgR(1-\cos\theta)$$

Since we are only concerned ~~with~~ about motion of bead on circle, and there is no term with ϕ in the Lagrangein, we apply Euler Lagrange eq^n to θ only

$$\therefore \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow \frac{m\omega^2 R^2 \sin^2\theta}{2} - mgR\sin\theta = \frac{d}{dt}(mR^2\dot{\theta})$$

$$\Rightarrow \ddot{\theta} = \omega^2 \sin\theta\cos\theta - \frac{g\sin\theta}{R}$$

For the system to achieve equilibrium at $\theta = \theta_0$

$$\ddot{\theta}\big|_{\theta=\theta_0} = 0.$$

$$\therefore \cancel{\sin\theta_0(\omega^2} \quad \sin\theta_0\left(\omega^2\cos\theta_0 - \frac{g}{R}\right) = 0.$$

$\therefore \sin\theta_0 = 0$ or $\cos\theta_0 = g/\omega^2 R$

$\therefore \theta_0 = 0, \pi$ or $\cos^{-1}\left(\frac{g}{\omega^2 R}\right)$.

Note that the eqlibribrium at $\theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$ exists only when $\omega \geq \sqrt{\frac{g}{R}}$

We consider $\alpha = \theta - \theta_0$ and let α be a small angle.

$\therefore$ ~~$\theta = \alpha + \alpha_0$~~ $\theta = \alpha + \theta_0$, $\sin\alpha \approx \alpha$ $\cos\alpha \approx 1$, $\ddot{\alpha} = \ddot{\theta}$

$$\therefore \ddot{\alpha} = \omega^2 \sin(\theta_0+\alpha)\cos(\theta_0+\alpha) - \frac{g}{R}\sin(\theta_0+\alpha)$$

$$= (\sin\theta_0\cos\alpha + \cos\theta_0\sin\alpha)\left(\omega^2(\cos\theta_0\cos\alpha - \sin\theta_0\sin\alpha) - \frac{g}{R}\right)$$

$$\therefore \ddot{\alpha} = \omega^2\left(\cos\theta_0 - \sin\theta_0\,\alpha - \frac{g}{\omega^2 R}\right)(\sin\theta_0 + \alpha\cos\theta_0)$$

$$\therefore \ddot{\alpha}\Big|_{\theta_0=0} = \omega^2\left(1-\frac{g}{\omega^2 R}\right)\alpha$$

$$\therefore \ddot{\alpha}\Big|_{\theta_0=\pi} = \omega^2\left(1+\frac{g}{\omega^2 R}\right)\alpha$$

$$\therefore \ddot{\alpha}\Big|_{\theta_0=\cos^{-1}\left(\frac{g}{\omega^2 R}\right)} = \omega^2\left(\cancel{\frac{g}{\omega^2 R}} - \alpha\left(\sqrt{1-\frac{g^2}{\omega^2 R^2}}\right) - \cancel{\frac{g}{\omega^2 R}}\right) \times \left(\sqrt{1-\frac{g^2}{\omega^2 R^2}} + \frac{\alpha g}{\omega^2 R}\right) \quad (\alpha^2 \simeq 0)$$

$$= -\omega^2 \sin^2\theta_0\, \alpha$$

A SHM differential eq^n is of the form $\ddot{x} = -kx$

$\therefore$ At $\theta_0 = \pi$ is not a SHM, $\theta_0 = 0$ is a SHM when $\omega < \sqrt{g/R}$ and $\theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$ is a SHM when $\omega \geq \sqrt{g/R}$.

The Hamiltonian $H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$

$$\therefore H = \dot{\theta}(mR^2\dot{\theta}) - \frac{mR^2\dot{\theta}^2}{2} - m\omega^2 R^2 \sin^2\theta + mgR(1-\cos\theta)$$

$$= \frac{mR^2\dot{\theta}^2}{2} - \frac{m\omega^2 R^2 \sin^2\theta}{2} + mgR(1-\cos\theta)$$

Let $\omega_c = \sqrt{g/R}$.

$$\therefore \text{Let } T = \frac{H}{m\omega_c^2 R^2} = \frac{1}{2}\left[\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\right)^2 \sin^2\theta\right] + 1 - \cos\theta$$

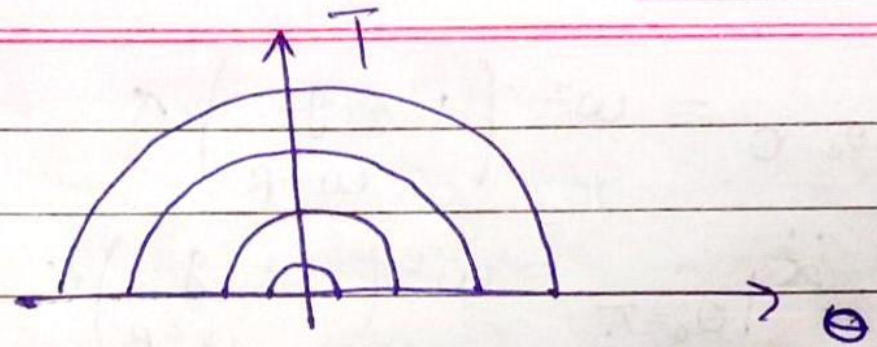
$\therefore$ We now plot $\left(\frac{\dot{\theta}}{\omega_c}\right)^2 = 2(T + \cos\theta - 1) + \left(\frac{\omega}{\omega_c}\right)^2 \sin^2\theta$

for various values of T and one value of ω in 2 cases.

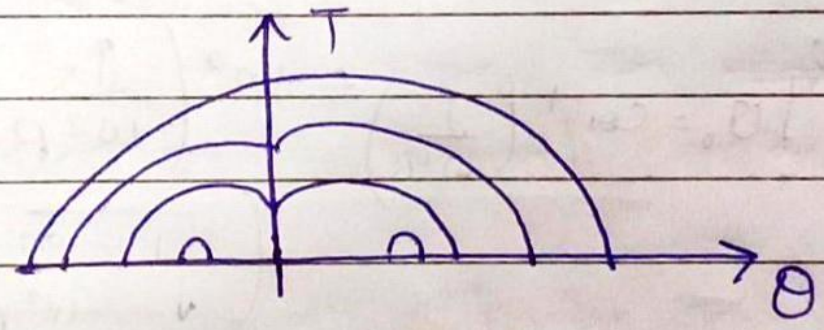
(i) $\omega < \omega_c$

(ii) $\omega > \omega_c$.

(i) $\omega < \omega_c$



(ii) $\omega > \omega_c$

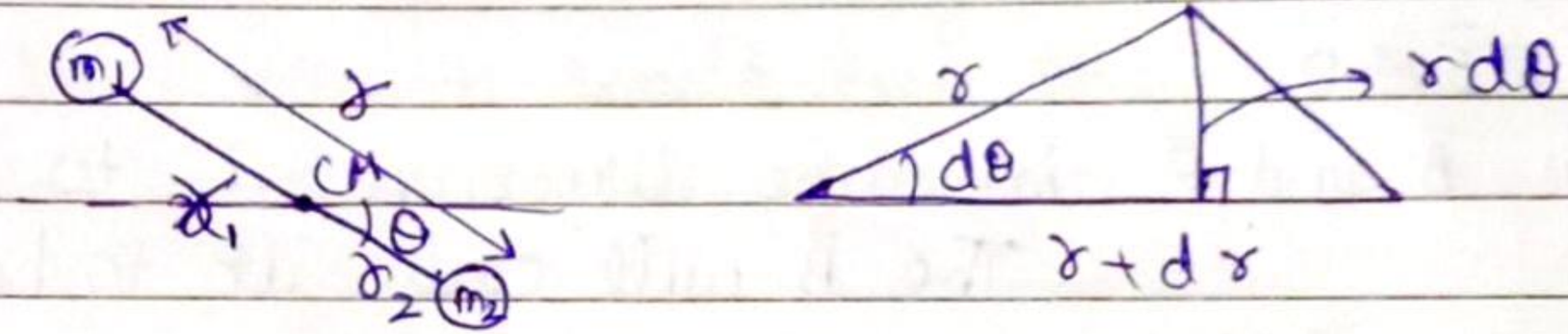


Hence, the two different regimes are :-

(i) $\omega < \sqrt{\frac{g}{R}}$: The equilibrium is at $\theta = 0$ and the frequency of oscillation is $\sqrt{\frac{g}{R} - \omega^2}$.

(ii) $\omega > \sqrt{\frac{g}{R}}$: The equilibrium is at $\theta = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$ and the frequency of oscillation is $\omega \sin\theta$.

Ans.8



Let the two bodies have masses m_1, m_2 and their motion be considered in the frame of the CM.
Let the distance from the CM be r_1, r_2 and distance b/w them be r.
Let the angle made by the line joining the two masses with the horizontal axis be θ.

$$r_1 = \frac{m_2 r}{m_1+m_2} \qquad r_2 = \frac{m_1 r}{m_1+m_2}$$

also $r_1 + r_2 = r \Rightarrow \dot{r} = \dot{r}_1 + \dot{r}_2$

$\therefore$ Each state of the system can be uniquely determined by value of (r, θ).

$\therefore$ The generalized coordinates of the system is $\{r, \theta\}$.

~~$\therefore KE_{system} = \frac{1}{2}m$~~

$$KE_{system} = \frac{1}{2}m_1(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2) + \frac{1}{2}m_2(\dot{r}_2^2 + r_2^2\dot{\theta}_2^2)$$

$$= \frac{1}{2}(m_1\dot{r}_1^2 + m_2\dot{r}_2^2) + \frac{\dot{\theta}^2}{2}(m_1 r_1^2 + m_2 r_2^2)$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 \quad \left(\mu = \frac{m_1 m_2}{m_1 + m_2} \text{ is reduced mass}\right)$$

$$= \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

The Potential Energy of the system $U = \frac{-Gm_1m_2}{r}$ ~~$L = K - U$~~

$$\therefore L = \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\theta}^2}{2} + \frac{Gm_1m_2}{r}$$

Lagrangein: $L = (K - U)$.

Applying Euler-Lagrangein eqn :-

For θ,

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow 0 = \frac{d}{dt}(\mu r^2\dot{\theta})$$

$\rightarrow$ Angular Momentum

$\Rightarrow \mu r^2\dot{\theta} = l$ (l is a constant)

The area covered by their motion in time interval dt is

$$dA = \frac{1}{2}r_1 d\theta(r_1 + dr_1) + \frac{1}{2}r_2 d\theta(r_2 + dr_2)$$

$$= \frac{r_1}{2}(r_1 d\theta + \cancelto{0}{dr_1 d\theta}) + \frac{r_2}{2}(r_2 d\theta + \cancelto{0}{dr_2 d\theta})$$

$$\therefore dA = \frac{r_1^2 d\theta}{2} + \frac{r_2^2 d\theta}{2}$$

$$dA = \frac{r^2 d\theta}{2(m_1+m_2)^2} \times (m_1^2 + m_2^2)$$

$$\therefore \frac{dA}{dt} = \frac{(m_1^2+m_2^2)}{2(m_1+m_2)^2} r^2 \dot{\theta} = \frac{(m_1^2+m_2^2)\, l}{2(m_1+m_2)^2 \mu}$$

$\therefore \frac{dA}{dt}$ is constant

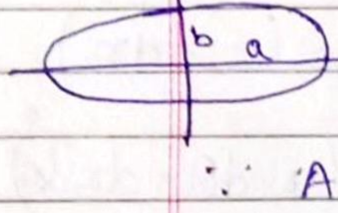
Kepler's 2nd law states that :-
" A line segment joining the planet and the sun sweeps equal areas during equal intervals of time".

$\because U = -\frac{G m_1 m_2}{r} \Rightarrow U \propto 1/r$

And Newton founded out that if $U \propto 1/r$ then Motion will be ellipse
→ Kepler 1st law :- "Each planet's orbit around Sun is an ellipse".

~~$\because l = \mu r^2 \dot{\theta} \Rightarrow \boxed{\dot{\theta} \propto 1/r^2}$ → Kepler's 3rd Law.~~

Kepler's 3rd law :- $T^2 \propto a^3$
The square of the orbital period varies like the cube of the major axis.



From Kepler's 2nd law

$$\frac{dA}{dt} = \frac{l}{2\mu} \Rightarrow A = \int_0^T \frac{dA}{dt} \cdot dt = \frac{l}{2\mu} \cdot T$$

$\because A = \pi ab$ and $a^2 e^2 = a^2 + b^2$

$$\therefore b = a\sqrt{1-e^2} = a\sqrt{\frac{-2El^2}{\mu k^2}} = a\sqrt{\frac{l^2}{\mu k a}} = \frac{l}{\sqrt{\mu k}} a^{1/2}$$

$$T = \frac{2\mu}{l} A = \frac{2\pi\mu}{l} ab = \frac{2\pi\mu}{l} \cdot \frac{l}{\sqrt{\mu k}} a^{3/2} = 2\pi\sqrt{\frac{\mu}{k}}\, a^{3/2}$$

$\boxed{T^2 \propto a^3}$ H.P.

Ans 9 If we denote the total energy and the potential of the family of orbits by E and $U(r)$, we have the relation

$$\frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) = E$$

from which

$$l^2 = 2\mu r^2 \left(E - U(r) - \frac{1}{2}\mu \dot{r}^2\right)$$

Here E and $U(r)$ are same for all orbits, and the different values of l result from different values of $(1/2)\mu r^2$. For stable circular motion, $\dot{r} = 0$, but for all other motions, $\dot{r} \neq 0$. Therefore, for non-circular motions, $\dot{r}^2 > 0$ and l is smaller than for the circular case. That is, the angular momentum of the circular orbit is largest among the family.

Ans 10) By the virial theorem $\langle T \rangle = -\frac{1}{2}\langle U \rangle$ for U [illegible]

The firing of the rocket doesn't change U, so

$$\boxed{U_f = U_i}$$

But $T_f = \frac{1}{2} m (v^2 + v^2) = 2 T_i$

So $E_f = 2T_i + U_i = -U_i + U_i = 0.$

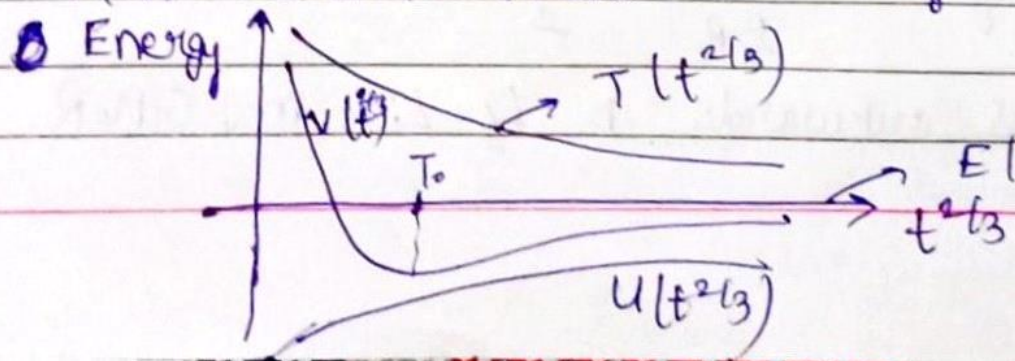
$$\boxed{\frac{E_f}{E_i} = 0}$$

The firing of the rocket doesn't change the angular momentum since it fires in radial direction

$$\therefore \boxed{\frac{l_f}{l_i} = 1}$$

(b) Since $T^2 \propto r^3$ $\therefore T^{2/3} \propto R$

(c) $E = 0$ means the orbit is parabolic. The satellite will be lost.

$E(r) = 0$ $\quad U(r) = \frac{-GMem}{r}$ $\quad T(r) = E - U = \frac{GMem}{r}$

$V(r) = U(r) + l^2/2\mu r^2 = \frac{GMem}{r} + \frac{l^2}{2\mu r^2}$