Linear Algebra – MAT 2610

Section 1.7 (Linear Independence)

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Definition: A set of vectors $\{u_1, u_2, \dots, u_k\}$ is **linearly dependent** if there exist scalars $c_1, c_2, \dots, c_k$, not all zero, such that $c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = 0$

Example: The set $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-3\end{bmatrix}, \begin{bmatrix}3\\5\\-2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ is linearly dependent since

$$2\begin{bmatrix}1\\2\\1\end{bmatrix} + 1\begin{bmatrix}2\\1\\-3\end{bmatrix} - 1\begin{bmatrix}3\\5\\-2\end{bmatrix} - 1\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Definition: A set of vectors $\{u_1, u_2, \dots, u_k\}$ is **linearly independent** if the only scalars $c_1, c_2, \dots, c_k$ that satisfy $c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$ are $c_1 = c_2 = \cdots = c_k = 0$

Example: The set $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-3\end{bmatrix}, \begin{bmatrix}3\\5\\-2\end{bmatrix}\right\}$ is linearly independent since the only solution to

$$a_1\begin{bmatrix}1\\2\\1\end{bmatrix} + a_2\begin{bmatrix}2\\1\\-3\end{bmatrix} + a_3\begin{bmatrix}3\\5\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

is $a_1 = a_2 = a_3 = 0$

Example: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Example: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Linear Independence of Matrix Columns

The columns of a matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.

Facts:

- The columns of a matrix A are linearly **independent** if the matrix equation $Ax = 0$ has $x = 0$ as its unique solution.
- The columns of a matrix A are linearly **dependent** if the matrix equations $Ax = 0$ has an infinite number of solutions.
- The columns of a matrix A are linearly **independent** if the matrix equation $Ax = b$ has at most one solution.
- The columns of a matrix A are linearly **dependent** if the matrix equation $Ax = b$ has an infinite number of solutions.

Example: For what values of r is $\begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -3 & r \end{bmatrix}$ linearly dependent?

Theorem 7

An indexed set $S = \{v_1, \dots, v_n\}$ of two or more vectors is linearly dependent if and nly if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors $v_1, \dots, v_{j-1}$.

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \dots, v_p\}$ in R^n is linearly dependent if $p > n$

Theorem 9

If a set $\{v_1, \ldots, v_p\}$ in R^n contains the zero vector, then the set is linearly dependent.

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