Linear Algebra – MAT 2610

Section 1.2 (Row Reductions and Echelon Forms)

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Definitions

A matrix is in **row echelon form** if:

1. Each nonzero row lies above every zero row (i.e.: if there is a row that is all zeros, it is at the bottom of the matrix).
2. The leading entry (i.e. the first non-zero entry in a row of the matrix) of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row (i.e.: the leading entries keep getting further to the right in the matrix as you go down the rows).
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0 (some entries above the leading entry **may** be non-zero).

A matrix is said to be in **reduced row echelon form** if it also satisfies the following conditions:

4. If a column contains the leading entry of some row, then all the other entries of that column are 0 (both above and below the leading entry).

The leading entry of each nonzero row is 1.

Definitions

A **non-zero** row or column in a matrix means a row or column that contains at least one non-zero entry.

A **leading entry** of a row refers to the leftmost nonzero entry.

$$\begin{bmatrix} 1 & 3 & -4 & 5 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Echelon Form

A matrix is in **row echelon form** (or echelon form) if:

1. Each nonzero row lies above every zero row (i.e.: if there is a row that is all zeros, it is at the bottom of the matrix).
2. The leading entry (i.e. the first non-zero entry in a row of the matrix) of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row (i.e.: the leading entries keep getting further to the right in the matrix as you go down the rows).
3. If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0 (some entries above the leading entry **may** be non-zero).

Row Echelon Form

Are the following in Row Echelon Form?

$$\begin{bmatrix} 1 & 3 & -4 & 5 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 2 & 1 & 7 \end{bmatrix}$$

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** (or reduced echelon form) if:

1. It is in row echelon form.
2. The leading entry of each nonzero row is 1
3. If a column contains the leading entry of some row, then all the other entries of that column are 0 (both above and below the leading entry).

Row Echelon Form

Are the following in Reduced Row Echelon Form?

$$\begin{bmatrix} 1 & 0 & -4 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Row Reduction

Any matrix can be row reduced (transformed by elementary row operations into more than one matrix in echelon form. The matrix obtained depends on the elementary row operations selected.

However, the reduced echelon matrix obtained for any matrix by row reductions is unique. It does not matter what elementary row operations you choose, the reduced echelon matrix will always be the same

$$\begin{bmatrix} -2 & 3 & 4 \\ -2 & 5 & 8 \end{bmatrix}$$

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Theorem 1

Each matrix is equivalent to one and only one reduced echelon matrix

Pivot position: the positions in a matrix that contain the leading entries of non-zero rows

Pivot column: A column that contains a pivot position

The **Gaussian Elimination** method is a methodical process that converts any matrix into a row reduced echelon matrix. It identifies pivot positions, then converts that pivot position value to 1 and all other entries in the pivot column to 0

Row Reduction Algorithm (Gaussian Elimination Method):

1. Locate the left-most non-zero column. This is your pivot column. Locate the top-most entry in the pivot column. This is your pivot position.
2. Use the operation $r_i \leftrightarrow r_j$ if necessary to move a non-zero entry into the pivot position.
3. For each row r_i below the pivot row, transform the entry in the pivot column to 0 using elementary row operations.
4. If your matrix is in Row Echelon form, identify the bottom-most non-zero row and go to step 5. Otherwise, repeat steps 1-4 but choose your pivot column in step 1 from the submatrix obtained by eliminating all previous pivot rows.
5. Find the leading entry in this row. This is a pivot position. Use a row operation to transform the leading entry into a 1. Then use row operations to transform all entries above this leading 1 into a 0.
6. If the matrix is in Reduced Row Echelon form, stop. Otherwise repeat step 5 for the next row up.

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

Solutions of Linear Systems

We have seen how a system of linear equations can be written as an augmented matrix. The row reduction method gives us a technique to solve any system of linear equations.

Example:

$$3x + 2y = 5$$
$$6x + y = 8$$

Solve

$$\begin{bmatrix} 3 & 2 & 5 \\ 6 & 1 & 8 \end{bmatrix}$$

Basic and Free Variables

If a column of R contains a leading entry then the variable corresponding to that column is called a **basic variable**.

If a variable is not basic, then it is called a **non-basic variable** or **free variable.**

Example:

$$4x_1 - 2x_2 + x_3 = 0$$
$$x_1 + 3x_2 - 4x_3 = 2$$
$$2x_1 - 8x_2 + 9x_3 = -4$$

$$\begin{bmatrix} 4 & -2 & 1 & 0 \\ 1 & 3 & -4 & 2 \\ 2 & -8 & 9 & -4 \end{bmatrix}$$

Can be transformed into:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} 1 & 0 & -5/14 & 2/7 \\ 0 & 1 & 17/14 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Basic Variables: x_1, x_2

Free Variables: x_3

Parametric Description of Solution Set

When a system of linear equations is consistent and it has at least one free variable, it has infinitely many solutions. In this case, the value of the basic variables can be expressed in terms of the value of the free variables.

Example

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Write the general solution:

$$x_1 + 4x_3 + 2x_5 = 0$$
$$x_2 - 5x_3 - 3x_5 = 2$$
$$x_4 - 4x_5 = 4$$

Basic: x_1, x_2, x_4

Free: x_3, x_5

Solve for the basic variables:

$$x_1 = -4x_3 - 2x_5$$
$$x_2 = 2 + 5x_3 + 3x_5$$
$$x_3 = x_3 \text{ (free)}$$
$$x_4 = 4 + 4x_5$$
$$x_5 = x_5 \text{ (free)}$$

Theorem 2

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column.

If a linear system is consistent then the solution set contains either

1. A unique solution (when there are no free variables), or
2. Infinitely many solutions (when there is at least one free variable)

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