Linear Algebra – MAT 2610

Section 2.6 (The Leontief Input-Output Model)
Section 1.10 (Linear Models in Business, Science, and Engineering

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Leontief Input-Output Model

The goal of the Leontief Input-Output Model is to model the economy of a region. The basis of this model uses what we have learned in the course so far and won a Nobel Prize in Economics.

The general idea of the model is:

- Divide an economy into different sectors
- Determine the portion of each sector consumed by other sectors
- Using the resulting equations we can now either
  - Determine the surplus or net production, or
  - Determine production required to meet demand

Farming
Forestry
Manufacturing
Technology

Farming
Forestry
Manufacturing
Technology

2015

THE UNIVERSITY OF PRINCE EDWARD ISLAND
ECONOMIC IMPACT ASSESSMENT
PREPARED BY:
JUPIA CONSULTANTS INC.

Output and GDP impacts	UPEI direct expenditures
Direct output (expenditures)	$137,477,355
Direct GDP contribution	$109,937,450
Total GDP contribution**	$142,557,329

** Includes direct, indirect, and induced economic impacts

How do we get the difference between “Direct GDP Contribution” and “Total GDP Contribution”? How do we know the impact of “indirect and induced economic impacts”?

Example: Suppose we have a town with only two industries: farming and forestry.

To produce one unit of product from farming, we require:

- 0.2 units of farming products
- 0.3 units of forestry products

To produce one unit of product from forestry, we require:

- 0.4 units farming products
- 0.1 units of forestry products

Suppose there is an external demand for a certain amount of farming products and of forestry products, what must be produced to meet the demand?

Step 1: Define the variables:

Let x_1 = the number of units of farming product produced
Let x_2 = the number of units of forestry product produced

Let d_1 = the surplus units of farming product available
Let d_2 = the surplus units of forestry product available

Step 2: Build the consumption equations for each sector in turn

Sector 1: Farming

Let x_1 = the number of units of farming product produced

Let d_1 = the surplus units of farming product available

For farming, any product produced (x_1) can go to three possible places:

- Some is required to support farming production itself (*to produce one unit of product from farming, we require 0.2 units of farming products*)
- Some is required to support forestry production (*to produce one unit of product from forestry, we require 0.4 units of farming products*)
- Surplus availability (represented by d_1)

This leads to the following equation:

$$x_1 = 0.2x_1 + 0.4x_2 + d_1$$

Step 2: Build the consumption equations for each sector in turn

Sector 1: Forestry

Let x_2 = the number of units of forestry product produced

Let d_2 = the surplus units of forestry product available

For forestry, any product produced (x_2) can go to three possible places:

- Some is required to support farming production (*to produce one unit of product from farming, we require 0.3 units of forestry products*)
- Some is required to support forestry production itself (*to produce one unit of product from forestry, we require 0.1 units of forestry products*)
- Surplus availability (represented by d_2)

This leads to the following equation:

$$x_2 = 0.3x_1 + 0.1x_2 + d_2$$

Step 3: Write the model as a vector/matrix equation

$$x_1 = 0.2x_1 + 0.4x_2 + d_1$$
$$x_2 = 0.3x_1 + 0.1x_2 + d_2$$

Can be rewritten as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Production Vector
(denoted as p)

Technology Matrix
Or
Consumption Matrix
(denoted as A)

Demand Vector
(denoted as d)

This equation can be rewritten and simplified:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$p = Ap + d$$
$$Ip = Ap + d$$
$$Ip - Ap = d$$
$$(I - A)p = d$$

Example: Suppose we have an external demand for 10 units farming and 8 units forestry. How much of each must be produced to meet that demand?

Use the equation $(I - A)p = d$. We know $\mathrm{A} = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.1 \end{bmatrix}$ and $\mathrm{d} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 0.9 \end{bmatrix}$$

So we need to solve

$$\begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 0.9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & -0.4 & 10 \\ -0.3 & 0.9 & 8 \end{bmatrix} \xrightarrow[10r_2 \to r_2]{10r_1 \to r_1} \begin{bmatrix} 8 & -4 & 100 \\ -3 & 9 & 80 \end{bmatrix} \xrightarrow{3r_1 + 8r_2 \to r_2} \begin{bmatrix} 8 & -4 & 100 \\ 0 & 60 & 940 \end{bmatrix}$$

$$\xrightarrow{15r_1 + r_2 \to r_1} \begin{bmatrix} 120 & 0 & 2440 \\ 0 & 60 & 940 \end{bmatrix} \xrightarrow[\frac{1}{60} r_2 \to r_2]{\frac{1}{120} r_1 \to r_1} \begin{bmatrix} 1 & 0 & \frac{61}{3} \\ 0 & 1 & \frac{32}{3} \end{bmatrix}$$

$$p = \begin{bmatrix} \frac{61}{3} \\ \frac{32}{3} \end{bmatrix} \approx \begin{bmatrix} 20.3 \\ 10.7 \end{bmatrix}$$

To meet the demand of 10 units of farming products and 8 units of forestry products, we need to produce approximately 20.3 units of farming products and 10.7 units of forestry products.

The idea behind this technique is the basis of many of the economic studies that we see today. In Canada, the technique has been refined – sectors are grouped into “commodities”, “industries” and “demand categories”.

Statistics Canada maintains the consumption matrix for the country (it is normally called the “Canadian Input-Output tables”). It includes

- 627 Commodities
- 216 Industries
- 136 Demand Categories

Example: Suppose we have a region with three principal industries: agriculture, manufacturing, and services. If the region produces 2 units of agriculture, 3 units of manufacturing, and 2 units of services, what is available for export?

To produce one unit of product from agriculture, we require:

- 0.2 units of agriculture products
- 0.2 units of manufacturing products
- 0.1 units of service products

To produce one unit of product from manufacturing, we require:

- 0.3 units agriculture products
- 0.4 units of manufacturing products
- 0.1 units of service products

To produce one unit of product from services, we require:

- 0.1 units agriculture products
- 0.2 units of manufacturing products
- 0.3 units of service products

Step 1: Define the variables:

Step 2: Build the consumption equations for each sector in turn

Sector 1: For agriculture, any product produced (x_1) can go to three possible places:

- Some is required to support agriculture production (*to produce one unit of product from agriculture, we require 0.2 units of agriculture products*)
- Some is required to support manufacturing production (*to produce one unit of product from manufacturing, we require 0.3 units of agriculture products*)
- Some is required to support services production (*to produce one unit of product from services, we require 0.1 units of agriculture products*)
- Surplus availability (represented by d_1)

Step 2: Build the consumption equations for each sector in turn

Sector 1: For manufacturing, any product produced (x_2) can go to three possible places:

- Some is required to support agriculture production (*to produce one unit of product from agriculture, we require 0.2 units of manufacturing products*)
- Some is required to support manufacturing production (*to produce one unit of product from manufacturing, we require 0.4 units of manufacturing products*)
- Some is required to support services production (*to produce one unit of product from services, we require 0.2 units of manufacturing products*)
- Surplus availability (represented by d_2)

Step 2: Build the consumption equations for each sector in turn

Sector 1: For services, any product produced (x_3) can go to three possible places:

- Some is required to support agriculture production (*to produce one unit of product from agriculture, we require 0.1 units of service products*)
- Some is required to support manufacturing production (*to produce one unit of product from manufacturing, we require 0.1 units of service products*)
- Some is required to support services production (*to produce one unit of product from services, we require 0.3 units of service products*)
- Surplus availability (represented by d_2)

Step 3: Write the model as a vector/matrix equation

Recall the equation for our model:

$$(I - A)p = d$$

We can often simplify multiple calculations by determining $(I - A)^{-1}$ then using

$$p = (I - A)^{-1}d$$

Example: Suppose for an Input-Output Model we have a consumption matrix $\mathrm{A} = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0 \end{bmatrix}$. What production is needed if demand is $\begin{bmatrix} 12 \\ 8 \end{bmatrix}$? What if demand jumps to $\begin{bmatrix} 12 \\ 16 \end{bmatrix}$?

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 \\ -0.5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & -0.2 & 1 & 0 \\ -0.5 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 9 & -2 & 10 & 0 \\ -5 & 10 & 0 & 10 \end{bmatrix} \xrightarrow{5r_1+9r_2 \to r_2} \begin{bmatrix} 9 & -2 & 10 & 0 \\ 0 & 80 & 50 & 90 \end{bmatrix}$$

$$\xrightarrow{40r_1+r_2 \to r_1} \begin{bmatrix} 360 & 0 & 450 & 90 \\ 0 & 80 & 50 & 90 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5/4 & 1/4 \\ 0 & 1 & 5/8 & 9/8 \end{bmatrix}$$

So $(\mathrm{I}-\mathrm{A})^{-1}=\begin{bmatrix}\frac{5}{4} & \frac{1}{4}\\ \frac{5}{8} & \frac{9}{8}\end{bmatrix}$. If demand is $\begin{bmatrix}12\\8\end{bmatrix}$, required production is

$$p=(I-A)^{-1}d=\begin{bmatrix}\frac{5}{4} & \frac{1}{4}\\ \frac{5}{8} & \frac{9}{8}\end{bmatrix}\begin{bmatrix}12\\8\end{bmatrix}=\begin{bmatrix}17\\16.5\end{bmatrix}$$

Now, if demand increases to $\begin{bmatrix}12\\16\end{bmatrix}$, required production is

$$p=(I-A)^{-1}d=\begin{bmatrix}\frac{5}{4} & \frac{1}{4}\\ \frac{5}{8} & \frac{9}{8}\end{bmatrix}\begin{bmatrix}12\\16\end{bmatrix}=\begin{bmatrix}19\\25.5\end{bmatrix}$$

Difference Equations

Linear equations can often be used to determine future changes for certain systems. If we can divide all the entities into discrete, non-overlapping groups, and we know the percentage that transitions from one group into another group at each step, we can solve many questions using linear equations

Example: Population

In any given year in the city of Metropolis, 15% of the city dwellers move to the suburbs and 10% of surburbanites move to the city. Suppose there are no other changes in population. If in 2020, there are 500K city dwellers and 400K in the suburbs, how many will there be in each group in 2021? How many were in each group in 2019?

Step 1: Define the variables:

Let c_{year} be the number of city dwellers in $year$
Let s_{year} be the number of suburban dwellers in $year$

So, $\begin{bmatrix} c_{2020} \\ s_{2020} \end{bmatrix} = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$

Step 2: Determine the equations that describe the transitions at each step

City: If c_t is the population in a given year, how does it change as we transition to c_{t+1}, the next year:

- 15% of city dwellers move, so 85% stay
- 10% of suburban dwellers move to the city

Putting this in equation form, we get:

$$c_{t+1} = 0.85c_t + 0.1s_t$$

Similarly for suburban dwellers we get:

$$s_{t+1} = 0.15c_t + 0.9s_t$$

Step 3: Write the model as a vector/matrix equation

$$c_{t+1} = 0.85c_t + 0.1s_t$$
$$s_{t+1} = 0.15c_t + 0.9s_t$$

can be written as:

$$\begin{bmatrix} c_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix} \begin{bmatrix} c_t \\ s_t \end{bmatrix}$$

For our example $\begin{bmatrix} c_{2020} \\ s_{2020} \end{bmatrix} = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$

Then $\begin{bmatrix} c_{2021} \\ s_{2021} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix} \begin{bmatrix} 500 \\ 400 \end{bmatrix} = \begin{bmatrix} 465 \\ 435 \end{bmatrix}$

So, in 2021, there will be 465K city residents and 435K suburban residents

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