Linear Algebra – MAT 2610

Section 1.9 (The Matrix of a Linear Transformation)

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Example: Suppose $T: R^2 \to R^3$ is a linear transformation where $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\-2\end{bmatrix}$. Find $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$

Theorem 10

Let $T: R^n \to R^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(x) = Ax$$

For all $x \in R^n$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(e_j)$ where e_j is the jth column of the identity matrix in R^n

The matrix A is called the **standard matrix for the linear transformation $\boldsymbol{T}$**

Example: Suppose $\mathrm{T}: R_2 \to R_3$ is defined as $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ x - y \\ x + 2y \end{bmatrix}$

Find the standard matrix of T

This means that the range of a linear transformation equals the span of the columns of its standard matrix.

Definition: A linear transformation $\mathrm{T}: R^n \to R^m$ is **onto** if its range is all of R^m.

Definition: A linear transformation $\mathrm{T}: R^n \to R^m$ is **one-to-one** if every pair of distinct vectors in R^n has distinct images. In other words, if $u, v \in R^n$ with $u \neq v$ then $\mathrm{T}(u) \neq T(v)$.

Suppose T: $R^n \to R^m$ is a linear transformation that is one-to-one.

Since T is a linear transformation, we know that $T(0) = 0$

But since T is one-to-one, we also know that for any vector $v \in R^n$ where $v \neq 0$ we have $T(v) \neq T(0) = 0$. Therefore, the only vector whose image is 0 under a one-to-one transformation is 0.

Does this sound familiar? The only way to get to 0 is to start with 0. This is a very similar idea to *linear independent vectors.*

Theorem 11

Let $T: R^n \rightarrow R^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps R^n onto R^m if and only if the columns of A span R^m

b. T is one-to-one if and only if the columns of A are linearly independent

Example

Suppose $T: R^2 \to R^3$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \\ x_1 \end{bmatrix}$. Is T one-to-one? Is it onto?

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