Linear Algebra – MAT 2610

Section 1.8 (Introduction to Linear Transformations)

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Definitions

Definition: A **function (or transformation)** $f: A \rightarrow B$ is a rule that assigns every element in a set A to an element in set B. (Note that A and B are not matrices here, but sets).

Set A is called the **domain** of f and B is called the **co-domain** of f

Definition: The **range** of a function $f: A \rightarrow B$ is the set of all possible values $f(a)$ where $a \epsilon A$

Example

Suppose $A = \{1,2,3,4,5\}$ and $B = \{red, green, blue\}$ and $f: A \rightarrow B$ is defined by:

$$f(n) = \begin{cases} red\ if\ n\ is\ odd \\ blue\ if\ n\ is\ even \end{cases}$$

Note that we can define $f(n)$ for any n in A (such as $f(2)$ or $f(5)$). But not every element of B is obtained.

Domain of f: $\{1,2,3,4,5\}$
Co-Domain of f: $\{red, green, blue\}$
Range of f: $\{red, blue\}$

We can define functions $f: S_1 \to S_2$ where S_1 and S_2 are sets of vectors.

Example: Suppose $f: R_2 \to R_2$ is defined as $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ x - 2y \end{bmatrix}$

Find $f\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$

Example: Suppose $f: R_5 \to R_3$ is defined as $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ x_3 + x_4 \\ x_2 - x_3 + 2x_5 \end{bmatrix}$

Find $f\left(\begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \\ -5 \end{bmatrix}\right)$

Matrix Transformation

If A is and $m \times n$ matrix, then if x is any vector in R^n, let

$$T(x) = Ax$$

Observe that this takes any vector in R^n and “transforms” it into a vector in R^m. This is a **matrix transformation** denoted by $x \longmapsto Ax$

Matrix Transformation Example

$T: R^3 \to R^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 & -3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Matrix Transformation Example

Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the range of $T: R^2 \rightarrow R^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Linear Transformation

Definition: A **linear transformation** $T: R^n \to R^m$ is a function where the following holds for any vectors $u, v \in R^n$ and c any scalar:

1. $T(u+v) = T(u) + T(v)$
2. $T(cu) = cT(u)$

Note that every matrix transformation is a linear transformation.

Example: Is $T: R^3 \to R^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 & -3 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ a linear transformation?

For any linear transformation $T: R^n \to R^m$ the following are true:

1. $T(0) = 0$
2. $T(-u) = -T(u)$ for all $u \in R^n$
3. $T(u - v) = T(u) - T(v)$ for all $u, v \in R^n$
4. $T(au + bv) = aT(u) + bT(v)$ for all $u, v \in R^n$ and scalars a, b

Note: this means that

$$T(a_1u_1 + \cdots + a_nu_n) = a_1T(u_1) + \cdots a_nT(u_n)$$

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