Linear Algebra – MAT 2610

Section 1.1 ()

Dr. Jay Adamsson
jay@aorweb.ca
jadamsson@upei.ca

What is a Linear Equation?

$$\mathrm{a}_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers (we will only use real numbers in this course)

A System of Linear Equations

This is a collection of one or more linear equations involving the same variables.

$$
\begin{gathered}
2x_1 + 3x_2 + 7x_3 = 18 \\
-2x_1 + 2x_3 = 0 \\
x_2 = 3
\end{gathered}
$$

A Solution to a System of Linear Equations

This is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when substituted for the variables $(x_1, x_2, \dots, x_n)$ respectively.

$$
\begin{gathered}
2x_1 + 3x_2 + 7x_3 = 18 \\
-2x_1 + 2x_3 = 0 \\
x_2 = 3
\end{gathered}
\quad \Rightarrow
$$

$(1,3,1)$ substituted for (x_1, x_2, x_3) makes each equation true.

A Solution Set

The collection of all possible solutions to a linear system is called a **solution set**.

Two linear systems are **equivalent** if they have the same solution set.

Solution Set Examples

$$\begin{aligned} 2x_1 + 3x_2 &= 11 \\ -2x_1 + 2x_2 &= -6 \end{aligned}$$

Has a single solution $(x_1, x_2) = (4,1)$.

Solution Set Examples

$$x_1 + 4x_2 = 8$$
$$2x_1 - x_2 = 7$$

Also has a single solution $(x_1, x_2) = (4,1)$.

Solution Set Examples

Notice that these two systems of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 11 \\ -2x_1 + 2x_2 &= -6 \end{aligned} \qquad\qquad \begin{aligned} x_1 + 4x_2 &= 8 \\ 2x_1 - x_2 &= 7 \end{aligned}$$

are quite different but both have the same solution set $(x_1, x_2) = (4,1)$. These two systems are equivalent.

Consistent

A system of linear equations is said to be **consistent** if it has at least one solution.

If not, it is **inconsistent**.

A system of linear equations has
1. No solution, or
2. Exactly one solution, or
3. Infintely many solutions.

Matrix

A **matrix** is a rectangular array of numbers

A matrix with *m* rows and *n* columns is referred to as an **m by n matrix** (or **m x n matrix**)

We say that an $m \times n$ matrix has **size** m by n

Note that the number of rows is <u>always</u> given first

$\begin{bmatrix} 3 & 3 \\ -2 & 2 \\ 3 & 1 \end{bmatrix}$ is a 3 X 2 matrix

$\begin{bmatrix} 4 & -2 & -2 \\ 5 & -1 & 3 \end{bmatrix}$ is a 2 X 3 matrix

Matrix Notation

If we have:

$$x_1 + 3x_2 + 2x_3 = 13$$
$$2x_1 + 3x_2 - x_3 = 5$$
$$2x_1 + x_3 = 5$$

The matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ is called the **coefficient matrix.**

If we add in another column, consisting of the constants, we get the

matrix $\begin{bmatrix} 1 & 3 & 2 & 13 \\ 2 & 3 & -1 & 5 \\ 2 & 0 & 1 & 5 \end{bmatrix}$. This is the **augmented matrix.**

Solving The System

$$x_1 + 3x_2 + 2x_3 = 13$$
$$2x_1 + 3x_2 - x_3 = 5$$
$$2x_1 + x_3 = 5$$

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 2 & 3 & -1 & 5 \\ 2 & 0 & 1 & 5 \end{bmatrix}$$

Eliminate x_1 from the second equation using the first

$$x_1 + 3x_2 + 2x_3 = 13$$
$$2x_1 + 3x_2 - x_3 = 5$$

Solving The System (2)

$$x_1 + 3x_2 + 2x_3 = 13$$
$$3x_2 + 5x_3 = 21$$
$$2x_1 + x_3 = 5$$

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & 3 & 5 & 21 \\ 2 & 0 & -1 & 21 \end{bmatrix}$$

Eliminate x_1 from the third equation using the first

$$x_1 + 3x_2 + 2x_3 = 13$$
$$2x_1 + x_3 = 5$$

Solving The System (3)

$$x_1 + 3x_2 + 2x_3 = 13$$
$$3x_2 + 5x_3 = 21$$
$$6x_2 + 3x_3 = 21$$

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & 3 & 5 & 21 \\ 0 & 6 & 3 & 21 \end{bmatrix}$$

Eliminate x_2 from the third equation using the second

$$3x_2 + 5x_3 = 21$$
$$6x_2 + 3x_3 = 21$$

Solving The System (4)

$$x_1 + 3x_2 + 2x_3 = 13$$
$$3x_2 + 5x_3 = 21$$
$$7x_3 = 21$$

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & 3 & 5 & 21 \\ 0 & 0 & 7 & 21 \end{bmatrix}$$

Now notice we can easily find the value for x_3

Solving The System (5)

$$x_1 + 3x_2 + 2x_3 = 13$$
$$3x_2 + 5x_3 = 21$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & 5 & 5 & 21 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Using the value for x_3 we can solve for x_2

Solving The System (6)

$$x_1 + 3x_2 + 2x_3 = 13$$
$$x_2 = 2$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Using the value for x_2 and x_3 we can solve for x_1

Solving The System (7)

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Check your work !!

$$x_1 + 3x_2 + 2x_3 = 13$$
$$2x_1 + 3x_2 - x_3 = 5$$
$$2x_1 + x_3 = 5$$

Elementary Row Operations

1. Add a multiple of one row to another row:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 3 \\ 2 & 0 & 3 \end{bmatrix} \; (2r_1 + r_2 \rightarrow r_2)$$

2. Interchange any two rows of the matrix:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & -1 \end{bmatrix} \; (r_2 \leftrightarrow r_3)$$

3. Multiply every entry of some row by some nonzero scalar:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 6 & 3 & -3 \\ 2 & 0 & 3 \end{bmatrix} \; (3r_2 \rightarrow r_2)$$

Elementary Row Operations (2)

Any to matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 2 & 3 & -1 & 5 \\ 2 & 0 & 1 & 5 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

It is important to note that elementary row operations are reversible.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example:

$$2x + y - z = 2$$
$$x + 3y + 2z = 1$$
$$x + y - z = 2$$

Solve

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

Example:

$$3x + 2y = 5$$
$$6x + 4y = 8$$

Solve

$$\begin{bmatrix} 3 & 2 & 5 \\ 6 & 4 & 8 \end{bmatrix}$$