Linear Algebra – MAT 2610

Section 2.2 (The Inverse of a Matrix)

Dr. Jay Adamsson
jay@aorweb.ca
jadamsson@upei.ca

Matrix Inverse

Definition: An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. We say B is the **inverse** of A, denoted by A^{-1}

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ then $AB = BA = I_2$

A matrix that is not invertible is called a **singular** matrix. A matrix that is invertible is called a **non-singular** matrix.

Theorem 4

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The value $ad - bc$ is called the **determinant** of A and is denoted $\det(A)$

Theorem 5

If A is an invertible $n \times n$ matrix, then for each b in R^n, the equation $Ax = b$ has the unique solution $x = A^{-1}b$

Suppose A is invertible and we know A^{-1}. Then we can use this to solve $Ax = b$

$$
\begin{gathered}
Ax = b \\
A^{-1}(A\mathrm{x}) = \mathrm{A}^{-1}b \\
(A^{-1}A)x = \mathrm{A}^{-1}b \\
I_n x = A^{-1}b \\
x = A^{-1}b
\end{gathered}
$$

Example: We know that if $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ then $\mathrm{A}^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

Solve:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 5x_2 &= 3 \end{aligned}$$

Theorem 6

If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$

If A and B are $n \times n$ invertible matrices, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$

If A is an invertible matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$

Definition: An **Elementary Matrix** is a matrix obtained from performing a single Elementary Row Operation on an Identity Matrix.

Examples:

$$I_4 \xrightarrow{sr_2 \to r_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I_4 \xrightarrow{kr_1 + r_3 \to r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. The sequence of row operations that reduces A to I_n also transforms I_n to A

Example: If $A = \begin{bmatrix} 2 & -3 \\ 3 & -5 \end{bmatrix}$, find A^{-1} if it exists

Linear Algebra – MAT 2610

Section 2.2 (The Inverse of a Matrix)

Dr. Jay Adamsson
jay@aorweb.ca
jadamsson@upei.ca