Linear Algebra – MAT 2610

Section 1.4 (The Matrix Equation $Ax = b$)

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Matrix Multiplication

Let A be an $m \times n$ matrix and v be a $n \times 1$ vector. The **product** Av is the linear combination of the columns of A using the corresponding entries in v as weights, and is given by : $Av = v_1 a_1 + \cdots + v_n a_n$

Examples:

$$\begin{bmatrix} 1 & 4 \\ -4 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Theorem 3

If A is an $m \times n$ matrix, with columns $a_1, \dots, a_n$ and if b is in R^m, the matrix equation

$$Ax = b$$

Has the same solution set as the vector equation

$$x_1a_1 + \cdots + x_na_n = b$$

Which, in turn, as the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & \dots & a_n & b \end{bmatrix}$$

Existence of Solutions

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Example: is the following consistent for all values of b_1, b_2, b_3?

$$\begin{aligned} x + y + z &= b_1 \\ 2x + 3y &= b_2 \\ 2x + 2z &= b_3 \end{aligned}$$

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$$

Theorem 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. For each b in R^m, the equation $Ax = b$ has a solution.
2. Each b in R^m is a linear combination of the columns of A.
3. The columns of A span R^m.
4. There is a pivot position in every row of A.

Note: This theorem uses the coefficient matrix A, not the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$

Computation of Ax

Row-Vector Rule for Computing Ax: If the product Ax is defined, then the ith entry in Ax is the sum of the products of corresponding entries from row i of A and from the vector x.

Example:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Theorem 5

If A is an $m \times n$ matrix, u and v are vectors in R^n, and c is a scalar, then:

1. $A(u + v) = Au + Av$
2. $A(cu) = c(Au)$

Proof of Theorems 4 and 5 are in the textbook.

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