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"Square Graphs" an Algorithmic Approach

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Abstract: In this paper we have made an effort to co-op up Graph Theory with Euclidian Geometry, we adopt the notion of diameter in Graph Theory (*largest length of a path in graph)* as length and height of a graph as breadth of graph. The breadth of the graph G is defined to be the maximum of the heights taken over all the diametral paths and is denoted by $br(G)$. Therefore $br(G) = max\,\{h_P(G), P \in \mathcal{H}\}$. A graph is said to be a square graph if $diam(G) = br(G)$. An algorithm is developed to find the breadth of graph.

Keywords: height of path, breadth of graph, length of graph, diametral path, distance.

I. INTRODUCTION

The concept of square graphs was first introduced by Dr. H. B. Walikar in his lectures. Later some work had been done in the thesis [1]. Before we proceed with square graphs let us understand the basic definitions required.

A. Basic terminologies:

There are some of the most basic terms like distance, radius, diameter and eccentricity for a graph G. The distance $d(u, v)$ between two vertices u and v in G is the minimum length of a path joining them. Let G be a connected graph and let v be a vertex of G, the eccentricity $e(v)$ of v is the distance to a node farthest from v, thus $e(v) = max\,\{d(u, v): u \in V\}$. The radius is the minimum of eccentricity of nodes and diameter $diam(G)$ is the maximum eccentricity. The $radius(G) \leq diam(G)$.

B. Breadth of the graph:

By the definition of diameter we can call it as the length of the graph. Then the question arises "What is the breadth of the graph?" Definitely it's not the radius. To co-op up with Euclidian Geometry, we adopt the notion of diameter in Graph Theory (*largest length of a path in graph)* as length and height of a graph as breadth of graph. The terms breadth and length are very well defined terms in Euclidian Geometry and we adopt the same terms with new definitions in Graph Theory.

This idea led to define breadth of graph. The breadth of graph is defined as follows: [1]

Let $\mathcal{H} = \{\, P \mid P \text{ is a diameteral path} \,\}$ be the set of all diametral paths. The breadth of the graph G is defined to be the maximum of the heights taken over all the diametral paths and is denoted by $br(G)$. Therefore

$$br(G) = max\,\{h_P(G), P \in \mathcal{H}\}$$

To find the breadth of graph G we have to consider the first quadrant of $X - Y$ plane. Then perform the following steps:

a. Find the diametral path of graph G.
b. Place the diametral vertices on the auxiliary line $y = 0$.
c. Find the vertices which are adjacent to the vertices placed on auxiliary line $y = 0$, which are already not on $y = 0$. Place these vertices on the auxiliary line $y = 1$.
d. Find the vertices which are adjacent to the vertices placed on auxiliary line $y = 1$, which are already not on $y = 1$ and $y = 0$. Place these vertices on the auxiliary line $y = 2$.
e. This process is continues till all the vertices are considered.

After all the vertices are exhausted we see that the graph G is redrawn on the X-Y plane. The value of will be $y = 0,1,2,3 \ldots h$. The value of h is the *height* of the diametral path under consideration and represented $h_P(G)$ where P is the diametral path.

To find the breadth of a graph we have to find all the diametral paths and then the respective height of each diametral path. The maximum of these heights is the breadth of graph as seen in the definition. Let us consider an example to elaborate on the concept of breadth of graph.

Example: Consider the graph G given in figure 1(a). The graph has diameter equal to 2, i.e. $diam(G) = 2$. The diametral paths of the graph are:

$$P_1 = 1,2,3 \qquad P_2 = 1,2,4$$
$$P_3 = 1,2,5 \qquad P_4 = 3,5,4$$

Let us consider the path P_4 and find its height by following the method given earlier.

a. At y=0 we place the vertices 3,5,4, which are the diametral vertices.
b. Consider the vertex 3. The adjacent vertices of 3 are 2 and 5. Vertex 5 is already considered at

$y = 0$ so we will not consider it for $y = 1$. Place the vertex 2 at $y = 1$.

c. Consider the vertex 5. The adjacent vertices are 2 and 4, which have been already placed on y=1 and y=0.

d. Consider vertex 4. The adjacent vertices are 2 and 5. Both of these have been already considered. There are no more vertices at y=0. So now we consider the vertices on y=1.

e. At $y = 1$, we have only one vertex i.e. 2. Among its adjacent vertices, 3,4,5 have been alredy considered. So we take the vertex 1 for $y = 2$.

f. At this stage we see that there are no more vertices to consider for y=3. Therefore the height of path P_4 is 2.

By repeating this for all the paths we get the following heights for all paths.

$h_{P_1}(G) = 1$ $\qquad h_{P_2}(G) = 1$

$h_{P_3}(G) = 1$ $\qquad h_{P_4}(G) = 2.$

We know that, $br(G) = \max\{ h_{P_i}(G)\ for\ i > 0\}$, therefore $br(G) = \max\{1, 1, 1, 2\}$ and hence $br(G) = 2$.

C. Some of the basic properties of breadth of graph:

In this section some of properties of breadth of graph are given [1]. The proofs are not given here.

Proposition 1: For any graph G, $br(G) \le diam(G)$.

Proposition 2: For any graph G, $br(G) = 0$ if and only if G is a path.

Proposition 3: For any tree T, $br(T) = 1$ if and only if T is a caterpillar.

Proposition 4: For any cycle C_n, $br(C_n) = \left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \right\rfloor$.

Proposition 5: Let $K_{r,s}$ be a complete bipartite graph then, $br(K_{r,s}) = 1$.

Proposition 6: if G is a complete graph then $br(K_n) = 1$.

Theorem 1: For any tree T, $br(T) \le rad(T)$.

Corollary 1: For any tree T, $br(T) \le \left\lfloor \frac{k}{2} \right\rfloor$ where $diam(T) = k$.

D. Square Graphs:

A graph is said to be a **square graph** if $\boldsymbol{diam(G) = br(G)}$ [1]. Again the concept looks similar to Euclidian Geometry's square where length and breadth are same. (The same term 'square graph' is also used for the graph obtained by taking the n^2 ordered pairs of the first n positive integers as vertices and drawing an edge between all pairs having

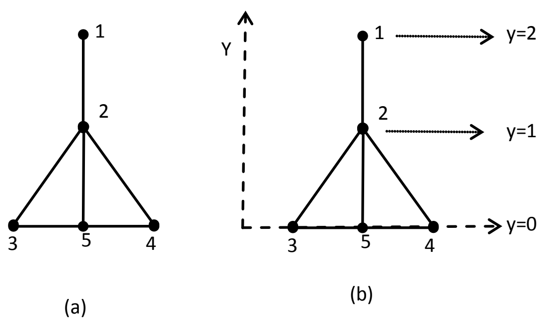

Figure 1: (a) Graph G (b) the graph G after placing the vertices on x-y plane.

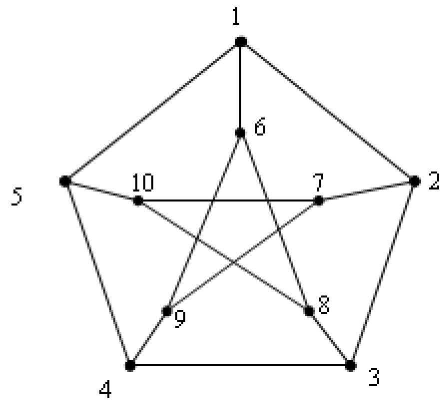

Figure 2 : Petersen Graph.

Exactly one number in common (Ball and Coxeter 1987, p. 305). But our definition of square graph is not the same; its different one). In this paper the term square graph is used as per our definition, keeping the Euclidian Geometry in view.

For graphs to be square there should be at least one diametral path such that $diam(G) = br(G)$. For example the graph of figure 1(a) is a square graph because there is a path whose height is equal to diameter of that graph, i.e. for the path P_4. Using this definition we have checked the squareness of different class of graphs. We have also introduced another category called *pure square graphs*. The details are given in next sections.

II. SQUARENESS OF SOME CLASS OF GRAPHS: (PROPOSED WORK)

Here are some observations for different class of graphs:

A. Trees with order $\boldsymbol{n \ge 2}$ are not square:

Case 1: The trivial tree with one vertex is square.

Case 2: For $n = 2$.

A tree two vertices has only one edge. Here $diam(G) = 1$ and $y = 0$. Therefore its not square.

Case 3: For $n \ge 3$

The trees with order $n \ge 3$ are not square. WKT, $(T) > rad(T)$, for $n \ge 3$ [3]. This means $diam(T) \ne rad(T)$, for $n \ge 3$. By the earlier work [1] we have $br(T) \le rad(T)$. Therefore we can write $br(T) \le rad(T) < diam(T)$. This shows $br(T) < diam(T)$ hence, $br(T) \ne diam(T)$. Therefore trees are not square. Combining cases 2 and 3 we can say that trees with order $n \ge 2$ are not square.

B. Cycles of order n>3 are not square:

Case 1: for $n = 3$, the cycle C_3 is square.

Case 2: WKT, $(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$. We also have $br(C_n) = \left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \right\rfloor$.

Comparing these two we can directly say that $diam(C_n) \ne br(C_n)$. Hence cycles are not square.

C. Complete bipartite graphs are not square:

In case of complete bipartite graphs $diam(G) = 2$ and $br(G) = 1$ [1]. Therefore complete bipartite graphs are not square.

D. *Windmill graphs are not square:*

In the mathematical field of graph theory, the windmill graph $Wd(k,n)$ is a simple undirected graph with $(k-1)n+1$ vertices and $nk(k-1)/2$ edges. It is defined for $k \geq 2$ and $n \geq 2$. The windmill graph $Wd(k,n)$ can be constructed by joining n copies of the complete graph K_k with a common vertex. It has girth 3 (if $k > 2$), radius 1 and diameter 2. Every diametral path consists of central vertex, so the breadth of these graphs is 1 for all paths. Hence windmill graphs are not square.

III. PURE SQUARE GRAPHS

If every diametral paths of graph G has the $diam(G) = br(G)$ then we call such graph as ***pure square graph***. For example: Petersen graph. In Petersen Graph, every diametral path has height equal to its diameter. The diameter of Petersen graph is 2.

IV. ALGORITHM TO FIND BREADTH AND SQUARENESS OF GRAPH:

The vertices whose value is equal to maximum eccentricity are called *diametral vertices*. The paths originating from diametral vertices, whose length is equal to diameter, are called *diametral paths*. To find the breadth of graph first we have to find all the diametral paths of the graph. Finding the diametral paths can be done using the algorithm designed to find diametral reachable index [2]. It takes $O(n^3)$ of time.

All the diametral paths can be stored in a two dimensional matrix, where each row represents the diametral path and total number of columns gives the number of diametral paths (repeated diametral paths are not handled in this matrix). The algorithm takes $O(n^4)$ time.

Steps for finding breadth and squareness of graph:

a. Find all the diametral paths.
b. Find the height of each diametral path.
c. Find the maximum of the heights of each path, this value is the breadth of graph.
d. If breadth of at least any one path is equal to $diameter(G)$ then the graph is square.
e. If the breadth of each path is equal to $diameter(G)$ then the graph is pure square.

The algorithm in pseudo code form is given as follows:

// The function *Find_Breadth ()* is used to find the breadth of the graph. This function calls *Find_height ()* function to find the height of the diametral path. The *initialize ()* function initializes the variables used. The *adjacent ()* function finds the adjacent vertices. All the diametral paths are stored in the matrix *diampath [][]*. The array d[] is used to store each of the diametral paths to be considered.

// breadth- represents the breadth of graph.

// m – is the total number of diametral paths

// L[ ] – used to store the levels of vertices. Initially all the //vertices will be initialized to -1". The diametral vertices //will be on the level $y = 0$.

// ht[ ] – used to store height of each diametral path. //Maximum of this ht[] gives diameter.

```
Find_Breadth ()
{
  for( i=1to m)
  {
    initialize();
    for( j=1 ti diam+1)
    {
      d[y]=diampath[i][j];
      y++;
      L[d[j]]=0;
    }
    Find_height ();
    ht[i]=max;
  }
  max=ht[i];
  for(i=1 to n)
    if(ht[i]>max)
      max=ht[i];
  breadth=max;
 //print the value of breadth.
  if(breadth==diam)
  Print " Graph is Square"
}
void Find_height()
{
  int u;
  for( i=1 to k)
  {
    u=adjacent(d[i]);
    while(u!=0)
    {
      L[u]=L[d[i]]+1;
      if(L[u]>max)
        max=L[u];
      d[x]=u;
      x++;
      k=x;
      u=adjacent(d[i]);
    }
  }
}
int adjacent(int u)
{
  int i,j;
  for(i=1 to n)
  {
    if(a[u][i]==1 && L[i]==-1)
      return i;
  }
  return 0;
}
void initialize()
{
  for( i=1 to n)
  {
    d[i]=0;
    L[i]=-1;
  }
  y=1;
  max=0;
  k=x=diam+2;
}
```

After running this algorithm we get an array which has the vertices with different labelling representing the y values. The array L [] is used for this purpose. Using this array we find the height of each path and in turn the maximum of these heights gives breadth of the graph.

Different algorithm and graph theory books have been referred for developing algorithms and for graph theory concepts [3, 4, 5, 6, 7].

V. CONCLUSIONS

In this paper an effort is made to see the graph theory in Euclidian geometrical view. An algorithm is also developed to find the breadth of graph. We can further try to improve the complexity of the program. We can try to develop some more properties in this regard.

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