(10 points) Let $A = \{a, b, c, d\}$, $B=\{0,1\}$.

(a) How many relations there are from A to B?

(b) Let $B^A=\{f \mid f: A \to B, f \text{ is everywhere defined}\}$. Compute $|B^A|$.

(c) Is there a bijection $k: B^A \to 2^A$, where 2^A is the power set of A? If the answer is yes, please define such a function. If the answer is no, explain why.

Answers:

(a). $A \times B = \{(a,0),(a,1),(b,0),(b,1),(c,0),(c,1),(d,0),(d,1)\}$

A到B有多少种关系等价于 A×B有多少个子集.

$\because |A\times B| = 8$. $\therefore$ 子集个数 $2^8 = 256$

$\therefore$ 从A到B有256种关系.

(b). $|B^A| = C_4^0 + C_4^1 + C_4^2 + C_4^3 + C_4^4 = 16$.

(c). Yes. $|2^A| = 16$.

$k(B) =$ B的特征函数

4. (10 points) Suppose that following assumptions:

(1) Logic is not difficult, or not many students like logic;

(2) If mathematics is easy, then logic is not difficult.

By translating these assumptions into statements involving propositional variables and connectives, deciding whether each of the following is a valid conclusion of these assumptions:

(a) That mathematics is not easy, if many students like logic;

(b) That not many students like logic, if mathematics is not easy;

(c) That logic is not difficult or mathematics is not easy.

Answer: "p" stand for "Logic is difficult"; "q" stand for "Many students like Logic"

"r" stand for "Mathematics is easy".

(1). $\neg p \vee \neg q$. (a). invalid.

(2). $r \to \neg p$. (b). valid —2

(c). valid.

5. (10 points) Define the following propositions and answer the following questions by drawing their truth tables.

A. $p \Rightarrow \sim p$ B. $(p \Rightarrow q) \vee (q \Rightarrow p)$

C. $((p \Rightarrow q) \wedge \sim q) \Rightarrow p$ D. $(p \Rightarrow q) \wedge (r \Rightarrow q) \wedge (p \vee r) \Rightarrow q$

(a) Which of the propositions above are contingencies?

(b) Which of the propositions above are tautologies?

(c) Which of the propositions above are absurdities?

Answers:

$\neg p$	p	q	r	$p\to q$	$q\to p$	$r\to q$	$p\vee r$	A	B	C	D
F	T	T	T	T	T	T	T	F	T	T	T
F	T	T	F	/	/	T	T	/	/	/	T
F	T	F	T	F	T	F	/	/	T	T	T
F	T	F	F	/	/	T	/	/	/	/	T
T	F	T	T	T	F	/	T	T	T	T	T
T	F	T	F	/	/	/	F	/	/	/	T
T	F	F	T	T	T	/	/	/	T	F	T
T	F	F	F	/	/	/	/	/	/	/	T

~~故~~ So:

(a). A, C are contingencies

(b). B, D is tautologies.

~~(c) B~~

(c). No absurdities.

Answer: (a). If $p \leftrightarrow q$ is a tautology, then p, q are both true or both false.

①. ~~$p \leftrightarrow q$~~ $p \leftrightarrow p$ is a tautology, so PRP, then R is reflexive.

②. If PRq, then qRP, that is to say $q \leftrightarrow P$. then R is symmetric

③. If $p \leftrightarrow q$, $q \leftrightarrow r$, we can say p, q have the same truth value. and q, r have the same truth value, so p, r have the same truth value, so $p \leftrightarrow r$, then R is transitive.

so, we prove R is an equivalence relation on W.

(b).

$$|W/R| = 2^8 = 256.$$

7. (10 points) Let A={1, 2, 3, 4}, and R={(1, 4), (3, 1), (3, 2), (3, 3), (4, 2)}

(a) Show the corresponding matrix M_R and draw the digraph of R;

(b) Compute the matrix of R^2;

(c) Compute the matrix of the transitive closure of R.

Answer: (a).

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(b).

$$M_{R^2} = M_R \odot M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c). R的传递闭包为{(1,4),(4,2),(1,2),(3,1),(3,4),(3,2),(3,3)}

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

. (10 points) Let S = {1, 2, 3, 4, 5} and A=S×S. Define the following relation R on A: (a, b) R (a', b') if and only if b=b'.

(a) Show that R is an equivalence relation

(b) Compute A/R and |A/R|.

Answer: (a). ①. For (a,b),(a,b). since b=b. so (a,b) R (a,b).
then R is reflexive.
②. For (a,b)R(c,d). since b=d. so (c,d) R (a,b)
then R is symmetric.
③. If (a,b) R (c,b). and (c,b) R (d,b). so (a,b) R (d,b).
then R is transitive.
So, we prove R is an equivalence relation.

(b). A/R = {{(1,1),(2,1),(3,1),(4,1),(5,1)},{(1,2),(2,2),(3,2),(4,2),(5,2)},{(1,3),(2,3),(3,3),(4,3),(5,3)},
{(1,4),(2,4),(3,4),(4,4),(5,4)},{(1,5),(2,5),(3,5),(4,5),(5,5)}}
|A/R| = 5

9. (10 points) Let A={1, 2, 3, 4, 5, 6} and $p=\begin{pmatrix}1&2&3&4&5&6\\3&1&2&5&4&6\end{pmatrix}$ be a permutation of A, i.e. a bijection from A to A: p(1) = 3, p(2) = 1, p(3) = 2, ..., p(6) = 6

(a) Compute p^{-1} and $p^2 = p\circ p$, and write them in the form above.

(b) Is it possible that p^n is the identity function for some n? If yes, why and what is such a possible integer? If not, why?

Answer: (a). $p^{-1}=\begin{pmatrix}1&2&3&4&5&6\\2&3&1&5&4&6\end{pmatrix}$. $p^2=\begin{pmatrix}1&2&3&4&5&6\\2&3&1&4&5&6\end{pmatrix}$

(b). p is injective and surjective. and $p^5 = p^{-1}$.
so when n=6. $p^6 = p^5 \circ p = p^{-1} \circ p$ = identity function.
So, when n = 6, 12, 18, 24, ..., p^n is the identity function.

10. (10 points) Let f: A->B and g: B ->C be two functions that are everywhere defined. Prove

when $n = 6, 12, 18, 24, \cdots$, P^n is the identity function.

10. (10 points) Let f: A->B and g: B->C be two functions that are everywhere defined. Prove

(a) If f and g are surjections, then g·f is a surjection from A to C.

(b) If g·f is surjective(onto), then g is surjective.

(c) Disprove that the following proposition: If g·f is surjective, then f is surjective.

Answer:

(a). If f is surjection, then every element in B correspond ~~an~~ element in A.
If g is surjection. then every element in C correspond element in B.
So, every element in C correspond element in A.
then $g \circ f$ is a surjection from A to C.

(b). Because $g \circ f$ is surjective, then ~~ele~~ every element in C correspond element in A.
since $g: B \to C$.
so, g is surjective.

(c). For example. ~~$A=\{1,2,3,4\}$ $B=C=\{1,2,3\}$~~ ~~$A=\{1,2,3\}$, $B=C=\{1,2,3,4\}$~~

$A=B=\{1,2,3,4\}$
$C=\{1,2,3\}$

f		g	
$1 \to$	1		2
$2 \to$	2		1
$3 \to$	3	$\to$	3
4	4	$\nearrow$	

In this example, $g \circ f$ is surjection, but f is not surjection.