Definition (Homomorphism)

Let

- $(S, *)$ and $(T, *')$ be two groupoids, and
- f be a function from S to T

If for all a and b in S

$$f(a * b) = f(a) *' f(b) \tag{1}$$

Then

- f is called a homomorphism from $(S, *)$ to $(T, *')$
- S is homomorphic to T, denoted by $S \sim T$
- $f(S)$ is the homomorphic image of S

Definition (Homo- and Isomorphism)

Let f be a homomorphism from $(S, *)$ to $(T, *')$

- If f is an onto from S to T, that is $f(S) = T$
  - f is called an onto homomorphism from $(S, *)$ to $(T, *')$
- If f is a bijection from S to T
  - f is called an isomorphism from $(S, *)$ to $(T, *')$
  - S is isomorphic to T, or S and T are isomorphic, denoted by $S \cong T$

Notice for both an isomorphism and a homomorphism

- The image of a product is the product of the images

Example (Isomorphic?)

Let

- $S = \{a, b, c\}$ and $T = \{x, y, z\}$ with following operations

It is easy to verify that S and T are group.

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$*$	x	y	z
x	z	x	y
y	x	y	z
z	y	z	x

Example (Free Semigroup)

Let

- $A = \{0, 1\}$
- $(A^*, \cdot)$, $\cdot$ is the catenation operation
- $(A, +)$, $+$ is defined by the table on the right

$+$	0	1
0	0	1
1	1	0

Define the function $f : A^* \to A$ by

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ has an odd number of } 1\text{'s} \\ 0 & \text{if } \alpha \text{ has an even number of } 1\text{'s} \end{cases}$$

If $\alpha, \beta \in A^*$, then it is easy to verify

- $f(\alpha \cdot \beta) = f(\alpha) + f(\beta)$
- Therefore, f is a homomorphism.

Steps To Determine Whether ...

Homomorphism?

- Step1: Define a function $f : S \to T$
- Step2: Show that $f(a * b) = f(a) *' f(b)$

Isomorphism? - two more steps

- Step3: Show that f is one-to-one.
- Step4: Show that f is onto.

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Example

Let

- $\mathbb{Z}$ be the set of all integers
- $\mathbb{E}$ be the set of all even integers

Show that the groups $(\mathbb{Z}, +)$ and $(\mathbb{E}, +)$ are isomorphic.

Proof.

1. Define $f : \mathbb{Z} \to \mathbb{E}$ by $f(a) = 2a$.
2. $f(a+b) = 2(a+b) = 2a + 2b = f(a) + f(b)$
3. f is one-to-one.
4. f is onto.

Hence $(\mathbb{Z}, +)$ and $(\mathbb{E}, +)$ are isomorphic groups.

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   - Suppose that $f(a_1) = f(a_2)$.
   - Then $2a_1 = 2a_2$, so $a_1 = a_2$.
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2. $f(a+b) = 2(a+b) = 2a + 2b = f(a) + f(b)$
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4. f is onto.
   - Suppose that b is any even integer.
   - Then $a = b/2 \in \mathbb{Z}$
   - $f(a) = f(b/2) = 2(b/2) = b$
   - So f is onto.

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Group of Order 4: Different or Isomorphic?

Klein 4 group

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Cyclic Group

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

$*$	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

Theorem

Let

- $(G, *)$ *and* $(G', *')$ *be two groupoids, and*
- $f : G \to G'$ *be an onto homomorphism*

Then

- *If* e *is the identity in* G *and* e' *is the identity in* G', *then* $f(e) = e'$.
- *If* $a \in G$, *then* $f(a^{-1}) = (f(a))^{-1}$.
- *If* H *is a subgroup of* G, *then* $f(H) = \{f(h) | h \in H\}$ *is a subgroup of* G'.
- *If* G *is groupoid, semigroup, monoid, group, or abelian, so is* G'.

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Remember

Isomorphism preserves all properties defined in terms of the group operations.

- The groups S_3 and $\mathbb{Z}_6$ are both of order 6. However, S_3 is not Abelian and $\mathbb{Z}_6$ is Abelian. Hence they are not isomorphic.

Example (Non-Isomorphic)

Let

- $\mathbb{Z}$ be the set of all integers
- $\mathbb{E}$ be the set of all even integers
- $\times$ be ordinary multiplication

Then

- The semigroups $(\mathbb{Z}, \times)$ and $(\mathbb{E}, \times)$ are not isomorphic
  - Since $\mathbb{Z}$ has an identity and $\mathbb{E}$ does not.