Relations

Zhang Yanmei

ymzhang@bupt.edu.cn

QQ: 11102556

College of Computer Science & Technology

Beijing University of Posts & Telecommunications

9 Relations

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Relations and their properties

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Ordered pair(序偶)

- An ordered pair (a, b) is a listing of the objects a and b in a prescribed order, with a appearing first and b appearing second.
- The ordered pairs $(a_1, b_1) = (a_2, b_2)$
  - if and only if
- $a_1 = a_2$ and $b_1 = b_2$

Cartesian product (笛卡尔积)

- If A and B are two nonempty sets, we define the *product set* or *Cartesian product* $A \times B$ as the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$
  - $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

$A_1 \times A_2 \times \cdots \times A_m$

- The *Cartesian product* $A_1 \times A_2 \times \cdots \times A_m$ of the nonempty sets $A_1, A_2, \cdots, A_m$ is the set of all ordered m-tuples (m元组) $(a_1, a_2, \ldots, a_m)$, where $a_i \in A_i$, $i = 1, 2, \ldots, m$
- Thus
  - $A_1 \times A_2 \times \cdots \times A_\mathrm{m} = \{(a_1, a_2, \ldots, a_m) \mid a_i \in A_i, i = 1, 2, \ldots, m\}$.

Partitions

- A *partition*(划分) or *quotient set*(商集) of a nonempty set A is a collection $\mathcal{P}$ of nonempty subsets of A such that
  - Each element of A belongs to one of the sets in $\mathcal{P}$
  - If A_1 and A_2 are distinct elements of $\mathcal{P}$, then $A_1 \cap A_2 = \varnothing$
- The sets in $\mathcal{P}$ are called the *blocks* or *cells* of the partition.

Example

- Let $A = \{a, b, c, d, e, f, g, h\}$.
- Consider the following subsets of A
  - $A_1 = \{a, b, c, d\}$
  - $A_2 = \{a, c, e, f, g, h\}$
  - $A_3 = \{a, c, e, g\}$
  - $A_4 = \{b, d\}$
  - $A_5 = \{f, h\}$

Relations

- The notion of a relation between two sets of objects is quite common and intuitively clear
- Examples
  - x is the father of y
  - $x < y$
- A relation is often described verbally and may be denoted by a familiar name or symbol.

Relations

- Discuss any possible relation from one abstract set to another.
- The *only thing that really matters* about a relation *is* that we know precisely *which elements in A are related to which elements in B*

Definition

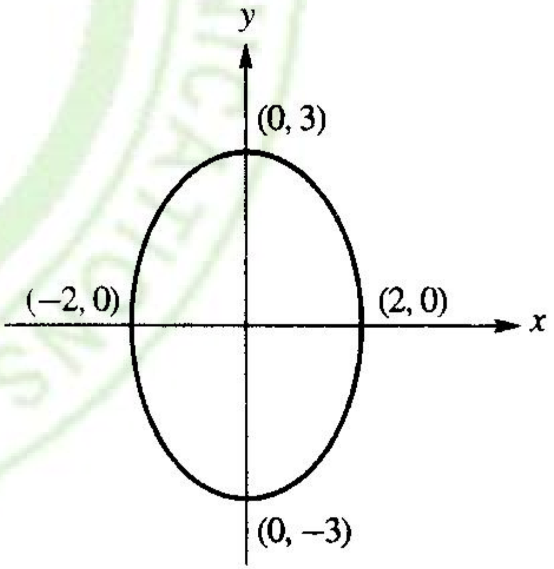
- Let A and B be nonempty sets. A *relation R from A to B* is a subset of $A \times B$.
  - If $R \subseteq A \times B$ and $(a, b) \in R$, we say that a is related to b by R, and we also write $a\ R\ b$.
  - If a is not related to b by R, we write $a \not R\ b$.
- Frequently, A and B are equal. In this case, we often say that $R \subseteq A \times A$ is a relation on A.

Examples

- Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$. Then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B.
- Let A and B be sets of real numbers. We define the following relation R (*equals*) from A to B:
  - $a\ R\ b$ if and only if $a = b$
- Let $A = \{1, 2, 3, 4, 5\}$. Define the following relation R (*less than*) on A:
  - $a\ R\ b$ if and only if $a < b$
  - $R = \{(1,2), (1, 3), (1,4), (1,5), (2, 3), (2. 4), (2, 5), (3,4), (3, 5), (4,5)\}$

Example

- Let $A = \mathbb{R}$, the set of real numbers. Define the following relation R on A:
  - $x\ R\ y$
- if and only if
  - $x^2/4 + y^2/9 = 1$
- The set R consists of all points on the ellipse shown in Figure



Sets Arising from Relations

- Let $R \subseteq A \times B$ be a relation from A to B.
- Dom(R), the *domain* of R is a subset of A, is the set of all first elements in the pairs that make up R.
- Ran(R), the *range* of R is the set of elements in B that are second elements of pairs in R.

Sets Arising from Relations

- If R is a relation from A to B and $x \in A$.
- Define $R(x)$, the *R-relative set of x*, to be the set of all y in B with the property that x is R-related to y.
  - $R(x) = \{y \in B \mid x\, R\, y\}$.
- Similarly, if $A_1 \subseteq A$, then $R(A_1)$, the *R-relative set of A_1*, is the set of all y in B with the property that x is R-related to y for some x in A_1.
  - $R(A_1) = \{y \in B \mid x\, R\, y \text{ for some } x \text{ in } A_1\}$

Theorem

- Let R be a relation from A to B, and let A_1 and A_2 be subsets of A. Then
  - (a) If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$
  - (b) $\mathrm{R}(A_1 \cup A_2) = R(A_1) \cup R(A_2)$
  - (c) $\mathrm{R}(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Proof of If $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$

- If $y \in R(A_1)$, then $x\ R\ y$ for some $x \in A_1$.
- Since $A_1 \subseteq A_2$, $x \in A_2$.
- Thus
  - $y \in R(A_2)$
- which proves part (a).

Proof of $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

- If $y \in R(A_1 \cup A_2)$, then by definition x R y for some x in $A_1 \cup A_2$.
  - If x is in A_1, then, since $x\ R\ y$, we must have $y \in R(A_1)$.
  - By the same argument, if x is in A_2, then $y \in R(A_2)$.
  - In either case, $y \in R(A_1) \cup R(A_2)$.
  - Thus $R(A_1 \cup A_2) \subseteq R(A_1) \cup R(A_2)$.
- Conversely,
  - since $A_1 \subseteq (A_1 \cup A_2)$, part (a) tells us that $R(A_1) \subseteq R(A_1 \cup A_2)$.
  - Similarly, $R(A_2) \subseteq R(A_1 \cup A_2)$.
  - Thus $R(A_1) \cup R(A_2) \subseteq R(A_1 \cup A_2)$, and therefore part (b) is true.

Proof of $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

- If $y \in R(A_1 \cap A_2)$, then, for some x in $A_1 \cap A_2$, $x\ R\ y$.
- Since x is in both A_1 and A_2, it follows that y is in both $R(A_1)$ and $R(A_2)$;
  - $y \in R(A_1) \cap R(A_2)$.
- Thus part (c) holds.

  - QED

- Why "$\subseteq$" instead of "="?

Remark

- The strategy of this proof is one we have seen many times in earlier sections:
  - Apply a relevant definition to a generic object.

Example

- Let $A = \mathbb{Z}$, R be "$\leq$", $A_1 = \{0, 1, 2\}$, and $A_2 = \{9, 13\}$. Then
  - $R(A_1)$ consists of all integers n such that $0 \leq n$, or $1 \leq n$, or $2 \leq n$. Thus $R(A_1) = \{0, 1, 2, ...\}$
  - Similarly, $R(A_2) = \{9, 10, 11, ...\}$
  - So $R(A_1) \cap R(A_2) = \{9, 10, 11, ...\}$
  - On the other hand, $A_1 \cap A_2 = \varnothing$; thus $R(A_1 \cap A_2) = \varnothing$
- This shows that the containment in theorem 1(c) is not always an equality.

Theorem

- Let R and S be relations from A to B.
  - If $R(a) = S(a)$ for all a in A,
  - then $R = S$
- Proof
  - If $a\ R\ b$, then $b \in R(a)$. Therefore, $b \in S(a)$ and $a\ S\ b$.
  - A completely similar argument shows that, if $a\ S\ b$, then $a\ R\ b$.
  - Thus $R = S$.

- QED

Special Properties of Binary Relations

- Given:
  - A Universe *U*
  - A binary relation *R* on a subset *A* of *U*
- Special Properties:
  - *Reflexive* and *Irreflexive*（自反、反自反）
  - *Symmetric*, *Asymmetric*, and *Antisymmetric*（对称、非对称、反对称）
  - *Transitive*（传递）

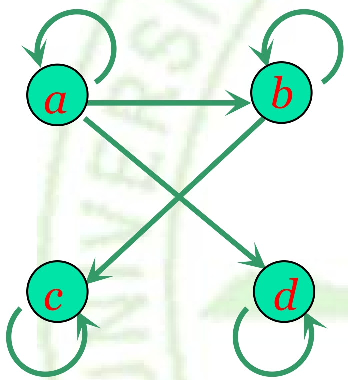
Definition: [re]

- R is *reflexive* iff
  - $\forall x[x \in A \rightarrow (x, x) \in R]$
- Note:
  - if $A = \varnothing$ then the implication is true vacuously
  - The void relation on a void Universe is reflexive!
  - If U is not void then all vertices in a reflexive relation must have loops!

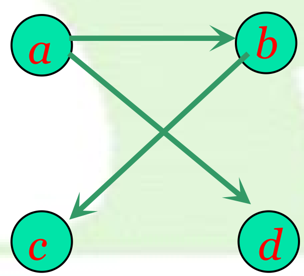
Definition : [ir]

- R is *irreflexive* iff
  - $\forall x[x \in A \rightarrow (x, x) \notin R]$
- Note:
  - if $A = \varnothing$ then the implication is true vacuously
  - Any void relation is irreflexive!

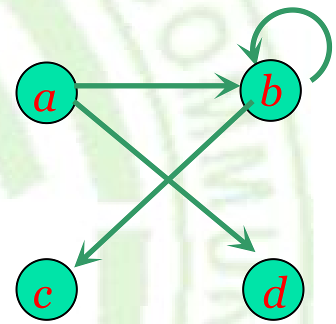
Examples



- **[*Re*]**
- Not [*Ir*]



- Not [*Re*]
- **[*Ir*]**



- Not [*Re*]
- Not [*Ir*]

Definition: [Sy]

- R is *symmetric* iff
  - $\forall x \forall y[(x, y) \in R \rightarrow (y, x) \in R]$

- Note:
  - If there is an arc (x, y) there must be an arc (y, x)

Definition: [As]

- R is *Asymmetric* iff
  - $\forall x \forall y[(x, y) \in R \rightarrow (y, x) \notin R]$

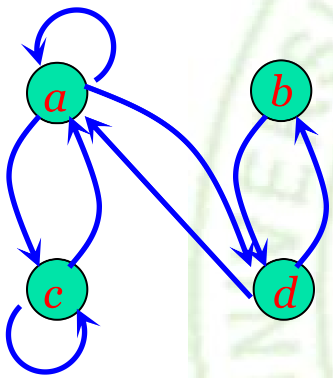
- Note:
  - If there is an arc (x, y) there must not be an arc (y, x)

Definition: [Ats]

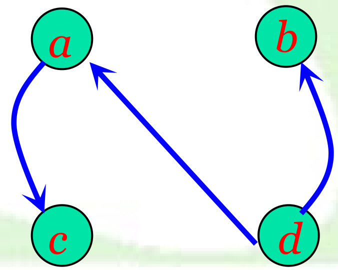
- R is *antisymmetric* iff
  - $\forall x \forall y[(x, y) \in R \wedge (y, x) \in R \rightarrow x = y]$

- Note:
  - If there is an arc from x to y there cannot be one from y to x if $x \neq y$.
  - You should be able to show that logically: if (x, y) is in R and $x \neq y$ then (y, x) is not in R.

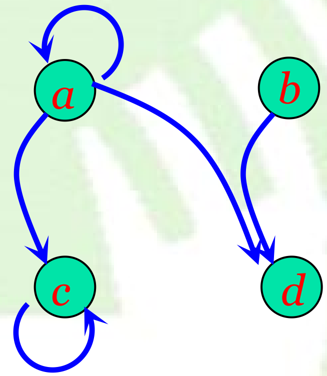
Examples



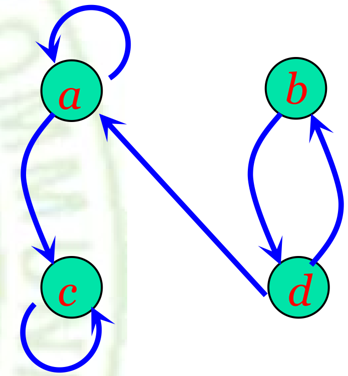
- **[*Sy*]**
- Not [*As*]
- Not [*Ats*]



- Not [*Sy*]
- **[*As*]**
- [*Ats*]



- Not [*Sy*]
- Not [*As*]
- **[*Ats*]**



- Not [*Sy*]
- Not [*As*]
- Not [*Ats*]

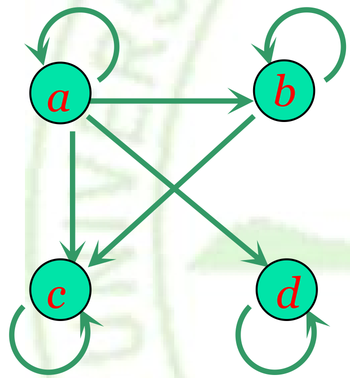
What about other 4 cases?

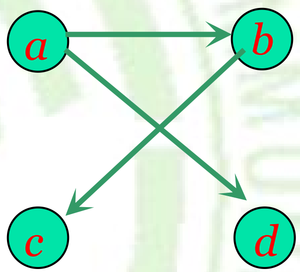
Definition: [tr]

- R is *transitive* iff
  - $\forall x \forall y \forall z[(x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R]$

- Note:
  - if there is an arc from x to y and one from y to z then there must be one from x to z.

Examples





- **[*Tr*]**
- Not [*Tr*]

Examples in Mathematics

- $=$ (equality)
- $\neq$ (inequality)
- $\varnothing$ (empty relation)

- Example 15
  - Is the “divides” relation on the set of positive integers transitive?
- Example 16
  - How many reflexive relations are there on a set with n elements?

Combing relations

- $R_2 \cup R_1$
- $R_2 \cap R_1$
- $R_2 - R_1$
- $R_2 \oplus R_1$
- R_1
- Example 17-19

Composition

- Now suppose
  - A, B, and C are sets,
  - R is a relation from A to B, and
  - S is a relation from B to C.
- The *composition of R and S*, written as $S \circ R$, is a relation from A to C and defined as: if a is in A and c is in C, then
  - $a \; S \circ R \; c$
    - if and only if
  - for some b in B, $a \, R \, b$ and $b \, S \, c$.

Example

- Let
  - $A=\{1, 2, 3, 4\}$
  - $R=\{(1, 1), (1, 2), (1, 3), (2, 4), (3, 2)\}$
  - $S=\{(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)\}$
- Then
  - $S \circ R=\{(1, 4), (1, 3), (1, 1), (2, 1), (3, 3)\}$

Theorem

- Let R be a relation from A to B and let S be a relation from B to C. Then, if A_1 is a subset of A, we have

$$(S \circ R)(A_1) = S(R(A_1))$$

Proof of $(S \circ R)(A_1) = S(R(A_1))$

$$\forall c \in S \circ R(A_1)$$

$$\Leftrightarrow \exists a \in A_1, (a, c) \in S \circ R$$

$$\Leftrightarrow \exists a \in A_1, \exists b \in B, (a, b) \in R \wedge (b, c) \in S$$

$$\Leftrightarrow \exists b \in B, (\exists a \in A_1, (a, b) \in R) \wedge (b, c) \in S$$

$$\Leftrightarrow \exists b \in B, b \in R(A_1) \wedge (b, c) \in S$$

$$\Leftrightarrow c \in S(R(A_1))$$

$$(S \circ R)(A_1) = S(R(A_1))$$

- Let R be a relation on the set A. The power $R^n, n=1,2,3\ldots$ are defined recursively by $R^1=R$ and $R^{n+1}=R^n \circ R$
- Example 22

Theorem 1

- The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n=1,2,3\ldots$

Homework

- § 9.1
  - 40, 47, 48

§ 9.2: *n*-ary Relations

- An n-ary relation R on sets $A_1,\ldots,A_n$, written (with signature) $R{:}A_1 \times \ldots \times A_n$ or $R{:}A_1,\ldots,A_n$, is simply a subset
$$R \subseteq A_1 \times \ \ldots \ \times \ A_n.$$
- The sets A_i are called the *domains* of R.
- The *degree* of R is n.
- R is *functional in the domain* A_i if it contains at most one n-tuple $(\ldots, a_i, \ldots)$ for any value a_i within domain A_i.

Example 1~2 P584

- Let R be the relation on N×N×N consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$.
- Let R be the relation on Z×Z×Z consisting of all triples of integers (a, b, c) in which a, b, and c form an arithmetic progression. That is, $(a, b, c) \in R$ if and only if there is an integer k such that $b = a + k$ and $c = a + 2k$, or equivalently, such that $b - a = k$ and $c - b = k$.

Example 3~4 P584

- Let R be the relation on $Z \times Z \times Z^+$ consisting of triples (a, b, m), where $a, b,$ *and* m are integers with $m \geq 1$ *and* $a \equiv b \pmod{m}$.
- Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time.

Relational Databases

- A *relational database* is essentially just an n-ary relation R.
- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.
- A *composite key* for the database is a set of domains $\{A_i, A_j, \ldots\}$ such that R contains at most 1 n-tuple $(\ldots,a_i,\ldots,a_j,\ldots)$ for each composite value $(a_i, a_j,\ldots)\in A_i \times A_j \times \ldots$
- Example 5
- Example 6

Selection Operators

- Let A be any n-ary domain $A=A_1 \times \ldots \times A_n$, and let C:$A\rightarrow\{\mathbf{T},\mathbf{F}\}$ be any *condition* (predicate) on elements (n-tuples) of A.
- Then, the *selection operator* s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.
  - *I.e.*, $\forall R\subseteq A,\ s_C(R) = \{a\in R \mid s_C(a) = \mathbf{T}\}$

Selection Operator Example

- Suppose we have a domain $A = \text{StudentName} \times \text{Standing} \times \text{SocSecNos}$
- Suppose we define a certain condition on A, $UpperLevel(name,standing,ssn) :\equiv [(standing = \text{junior}) \vee (standing = \text{senior})]$
- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of *just* the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \ldots \times A_n$ be any n-ary domain, and let $\{i_k\}=(i_1,\ldots,i_m)$ be a sequence of indices all falling in the range 1 to n,
  - That is, where $1 \leq i_k \leq n$ for all $1 \leq k \leq m$.
- Then the *projection operator* on n-tuples

is defined by:

$$P_{\{i_k\}} : A \rightarrow A_{i_1} \times \ldots \times A_{i_m}$$

$$P_{\{i_k\}}(a_1,\ldots,a_n) = (a_{i_1},\ldots,a_{i_m})$$

Projection Example

- Suppose we have a ternary (3-ary) domain *Cars=Model* × *Year* × *Color*. (note n=3).
- Consider the index sequence $\{i_k\}$= 1,3. (m=2)
- Then the projection $P_{\{i_k\}}$ simply maps each tuple (a_1,a_2,a_3) = (*model*,*year*,*color*) to its image:

$$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

- This oper whole relation $R \subseteq Cars$ (a database of cars) to obtain a list of the model/color combinations available.
- Example 8

Join Operator

- Puts two relations together to form a sort of combined relation.
- If the tuple (A,B) appears in R_1, and the tuple (B,C) appears in R_2, then the tuple (A,B,C) appears in the join $J(R_1,R_2)$.
  - A, B, and C here can also be sequences of elements (across multiple fields), not just single elements.

Join Example

- Suppose R_1 is a teaching assignment table, relating *Professors* to *Courses*.
- Suppose R_2 is a room assignment table relating *Courses* to *Rooms*,*Times*.
- Then $J(R_1,R_2)$ is like your class schedule, listing *(professor,course,room,time)*.
- Example 11

Sql

- Example 12,13

Homework

- § 9.2
  - 8, 20

§ 9.3: Representing Relations

- Some ways to represent *n*-ary relations:
  - With an explicit list or table of its tuples.
  - With a function from the domain to **{T,F}**.
    - Or with an algorithm for computing this function.
- Some special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

The Matrix of a Relation

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, R is a relation from A to B.
- The $m \times n$ matrix $M_R = [\ m_{ij}\ ]$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- M_R is called the *connection matrix of* R.
- M_R provides an easy way to check whether R has a given property.

Example

- Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- Since M is 3 × 4, let
  - $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$.
- Then $(a_i, b_j) \in R$ if and only if $m_{ij} = 1$.
  - R = $\{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$

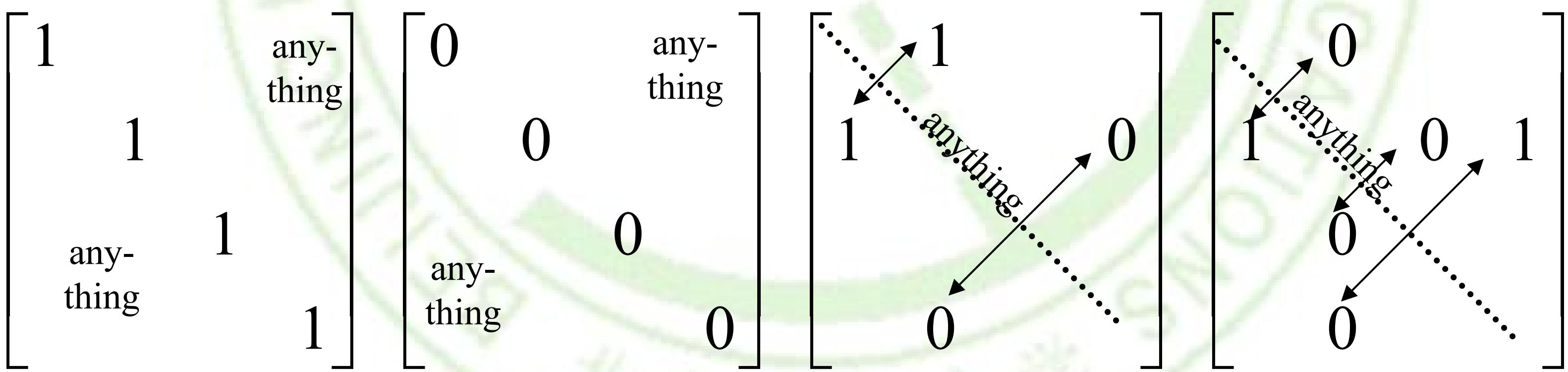
Using Zero-One Matrices

- To represent a binary relation $R:A\times B$ by an $|A|\times|B|$ 0-1 matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ iff $(a_i,b_j)\in R$.
- *E.g.*, Suppose Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally.
- Then the 0-1 matrix representation of the relation Likes:Boys×Girls relation is:

$$\begin{array}{r|ccc|} & \text{Susan} & \text{Mary} & \text{Sally} \\ \text{Joe} & 1 & 1 & 0 \\ \text{Fred} & 0 & 1 & 0 \\ \text{Mark} & 0 & 0 & 1 \end{array}$$

Zero-One Reflexive, Symmetric

- Terms: *Reflexive*, *non-reflexive*, *irreflexive*, *symmetric*, *asymmetric*, *and antisymmetric*.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal	*Irreflexive*: all 0's on diagonal	*Symmetric*: all identical across diagonal	*Antisymmetric*: all 1's are across from 0's

Obvious questions

- Given the connection matrix for two relations, how does one find the connection matrix for
  - The complement?
  - The relative complement?
  - The symmetric difference?

Combining Connection Matrices

- **Definition:** the *join* of two matrices M_1, M_2, denoted $M_1 \vee M_2$, is the component wise *boolean 'or'* of the two matrices.
- **Fact:** If M_1 is the connection matrix for R_1 and M_2 is the connection matrix for R_2 then the join of M_1 and M_2, $M_1 \vee M_2$ is the connection matrix for $R_1 \cup R_2$.

Combining Connection Matrices

- **Definition:** the *meet* of two matrices M_1, M_2, denoted $M_1 \wedge M_2$ is the component wise *boolean 'and'* of the two matrices.
- **Fact:** If M_1 is the connection matrix for R_1 and M_2 is the connection matrix for R_2 then the meet of M_1 and M_2, $M_1 \wedge M_2$ is the connection matrix for $R_1 \cap R_2$.

Composite of relations

- Exmaple 5 P593
  - Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } Ms = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Example 6

- Find the matrix representing the relation R^2, where the matrices representing R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The Digraph of a Relation

- If A is a finite set and R is a relation on A.
  - Draw a small circle, called a *vertex*, for each element of A and label the circle with the corresponding element of A.
  - Draw an arrow, called an *arc* or *edge*, from vertex a_i to vertex a_j if and only if $a_i \ R \ a_j$.
- The resulting pictorial representation of R is called a *directed graph* or *digraph* of R.

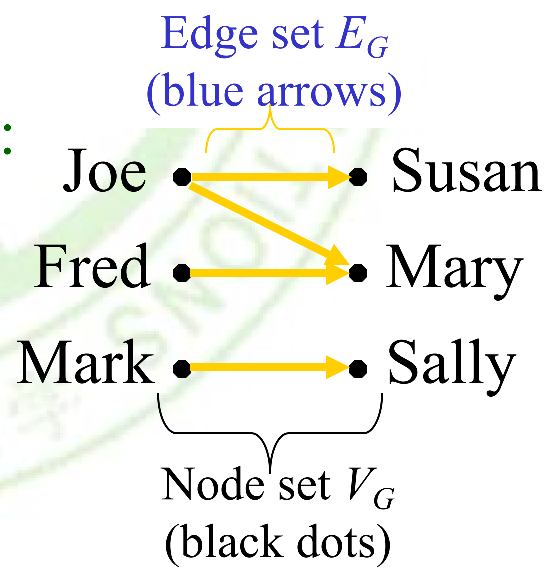
Using Directed Graphs

- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs,links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R:A \times B$ can be represented as a graph $G_R=(V_G=A\cup B,\ E_G=R)$.

Matrix representation $\mathbf{M}_R$:

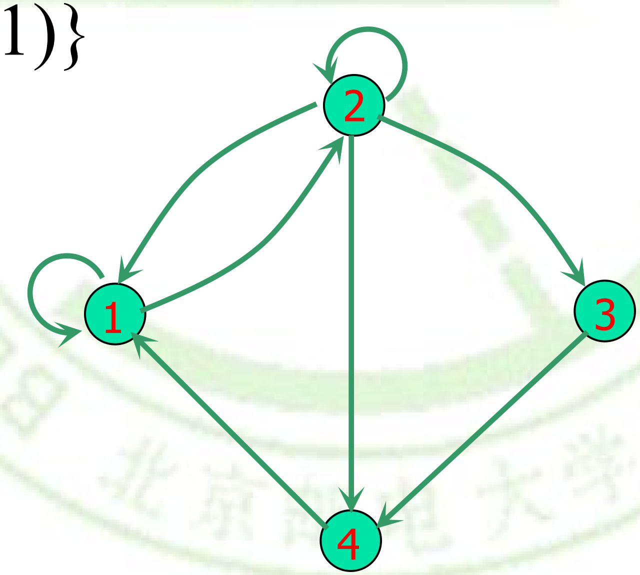
$$\begin{array}{r} \\ \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Graph rep. G_R:



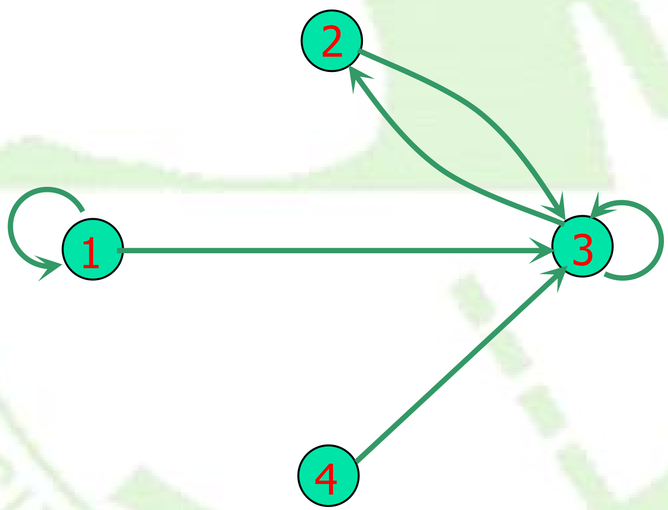
Example

- Let
  - A = {1, 2, 3, 4}
  - R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)}
- Then



Example

- Find the relation determined by



- $R = \{(1, 1), (1, 3), (2, 3), (3, 2), (3, 3), (4, 3)\}$

Definitions

- If R is a relation on a set A and $a \in A$, then
  - the *in-degree* of a (relative to the relation R) is the number of $b \in A$ such that $(b, a) \in R$.
  - the *out-degree* of a is the number of $b \in A$ such that $(a, b) \in R$.

Example

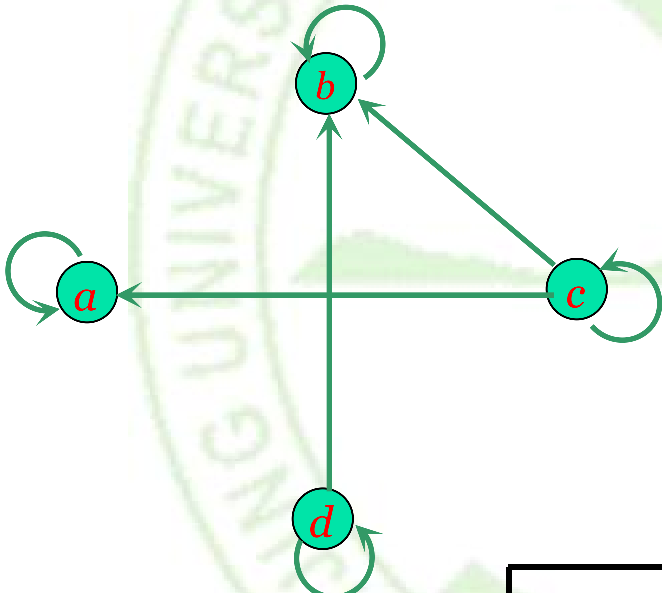
- Let $A = \{a, b, c, d\}$, and R be the relation on A that has the matrix

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

  - Construct the digraph of R, and list in-degrees and out-degrees of all vertices

Solution

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$



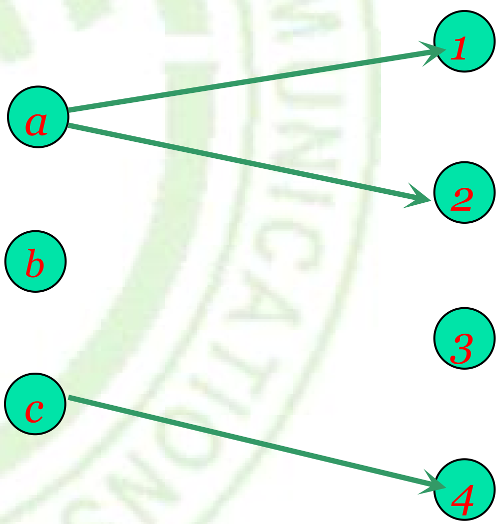
	a	*b*	*c*	*d*
in-degree	2	3	1	1
out-degree	1	1	3	2

Digraph of combined relations

- Given the digraphs for R_1 and R_2, find the digraphs for
  - $R_2 \cup R_1$
  - $R_2 \cap R_1$
  - $R_2 - R_1$
  - $R_2 \oplus R_1$
  - R_1

Example: R from A to B

- Let
  - $A = \{ a, b, c\}$
  - $B = \{1, 2, 3, 4\}$
  - $R = \{(a, 1), (a, 2), (c, 4)\}$
- Then R can be represented by the digraph



Definition

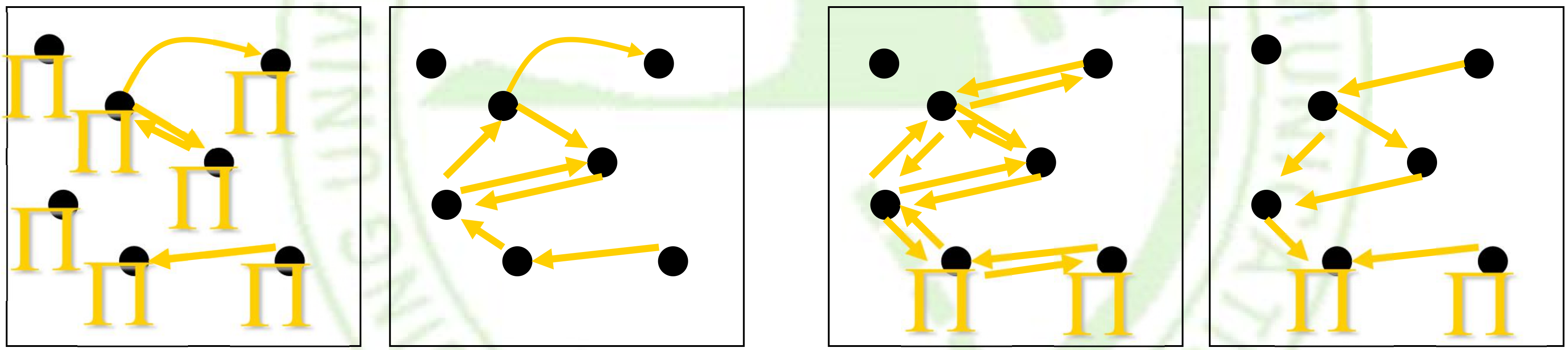
- If R is a relation on a set A, and B is a subset of A, the *restriction of R to B* is
  - $R \cap (B \times B)$

Example

- Let
  - $A = \{a, b, c, d, e, f\}$
  - $R = \{(a, a), (a, c), (b, c), (a, e). (b, e), (c, e)\}$
  - $B = \{a, b, c\}$
- Then
  - $B \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$
  - The restriction of R to B is $\{(a, a), (a, c), (b, c)\}$

Digraph Reflexive, Symmetric

It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop

Irreflexive: No node links to itself

Symmetric: Every link is bidirectional

Antisymmetric: No link is bidirectional

These are non-asymmetric & non-antisymmetric

These are non-reflexive & non-irreflexive

Ways to …

- Identify which properties does a relation R has:
  - R
  - Digraph of R
  - Matrix of R

Reflexive or Irreflexive

- R is a relation on a set A, Let M_R be the matrix of a relation R and Δ be the equality relation, then R is
  - *Reflexive* $\Leftrightarrow \Delta \subseteq R \Leftrightarrow$ all 1's on its main diagonal
  - *Irreflexive* $\Leftrightarrow \Delta \cap R = \varnothing \Leftrightarrow$ all 0's on its main diagonal

Symmetric

- The matrix $M_R = [m_{ij}]$ of a symmetric relation satisfies the property that
  - if $m_{ij} = 1$, then $m_{ji} = 1$
  - moreover, if $m_{ij} = 0$, then $m_{ji} = 0$
  - so that M_R is a symmetric matrix

Asymmetric

- The matrix $M_R = [m_{ij}]$ of a asymmetric relation satisfies the property that
  - if $m_{ij} = 1$, then $m_{ji} = 0$
  - note: $m_{ii} = 0$

Antisymmetric

- The matrix $M_R = [m_{ij}]$ of a antisymmetric relation satisfies the property that
  - if $i \neq j$, then $m_{ij} = 0$ or $m_{ji} = 0$

Transitive

- A relation R is transitive if and only if its matrix $M_R = [m_{ij}]$ has the property
  - if $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$

- $R^2 \subseteq R$
  - if a and c are connected by a path of length 2 in R, then they must be connected by a path of length 1.
  - if $a\ R^2\ c$, then $a\ R\ c$

Example

$$M_{R_1}=\begin{bmatrix}1&0&1\\0&0&1\\1&1&1\end{bmatrix}\qquad M_{R_2}=\begin{bmatrix}0&1&1&0\\1&1&0&0\\1&0&1&1\\0&0&1&1\end{bmatrix}\qquad M_{R_2}^2=\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&1&1\\1&0&1&1\end{bmatrix}$$

$$M_{R_3}=\begin{bmatrix}1&1&1\\0&1&0\\0&0&0\end{bmatrix}\qquad M_{R_4}=\begin{bmatrix}0&0&1&1\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\qquad M_{R_4}^2=\begin{bmatrix}1&0&0&1\\0&0&0&1\\1&0&0&0\\0&0&1&1\end{bmatrix}$$

$$M_{R_5}=\begin{bmatrix}1&0&0&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{bmatrix}\qquad M_{R_6}=\begin{bmatrix}0&1&1&1\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\qquad M_{R_6}^2=\begin{bmatrix}0&0&1&1\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Theorem

- R is transitive
  - If and only if
- $R^n \subseteq R$ for $n > 0$.

Proof: R transitive $\rightarrow R^n \subseteq R$

- Use a direct proof and a proof by induction:
  - Assume R is transitive.
  - Now show $R^n \subseteq R$ by induction.
- *Basis*: Obviously true for $n = 1$.
- *Induction:*
  - The induction hypothesis:
    - 'assume theorem is true for n'.
  - Show it must be true for $n + 1$

Proof: R transitive $\rightarrow R^n \subseteq R$

- $R^{n+1} = R^n \circ R$ so if (x, y) is in R^{n+1} then there is a z such that (x, z) is in R and (z, y) is in R^n.
- But since $R^n \subseteq R$, (z, y) is in R.
- R is transitive so (x, y) is in R.
- Since (x, y) was an arbitrary edge the result follows.

Proof: $R^n \subseteq R \rightarrow R$ transitive

- Use the fact that $R^2 \subseteq R$ and the definition of transitivity. Proof left to the ......
  - Q. E. D.

Combining Relations

- A very large set of potential questions -
- Let R_1 and R_2 be binary relations on a set A:
- If
  - R_1 has property 1
  - R_2 has property 2,
- does
  - $R_1 * R_2$ have property 3
- where * represents an arbitrary binary set operation?

Example

- If
  - R_1 is symmetric,
  - and
  - R_2 is antisymmetric,
- does it follow that
  - $R_1 \cup R_2$ is transitive?
- If so, prove it. Otherwise find a counterexample.

Example

- Let R_1 and R_2 be transitive on A. Does it follow that
  - $R_1 \cup R_2$ is transitive?
- Consider
  - $A = \{1, 2\}$
  - $R_1 = \{(1, 2)\}$
  - $R_2 = \{(2, 1)\}$
- Then $R_1 \cup R_2 = \{(1, 2), (2,1)\}$ which is not transitive! (Why?)

Homework

- § 9.3
  - 14, 32