9.4 Closures of Relations

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Closures of Relations

- Definition
  - The *closure*(闭包) of a relation R with respect to property P is the relation obtained by adding the *minimum number of ordered pairs* to R to obtain property P.
- 3 elements:
  - R_1 contains R
  - R_1 possesses the property P
  - If R_2 contains R and possesses the property P, then R_2 contains R_1

Closures of Relations

- In terms of the digraph representation of *R*
  - To find the *reflexive closure*
    - add loops.
  - To find the *symmetric closure*
    - add arcs in the opposite direction.
  - To find the *transitive closure*
    - if there is a path from *a* to *b*, add an arc from *a* to *b*.

Reflexive Closure

- **Theorem:**
  - Let R be a relation on A.
  - The *reflexive closure* of R, denoted r(R), is $R \cup \Delta$.
- Method:
  - Add loops to all vertices on the digraph representation of R.
  - Put 1's on the diagonal of the connection matrix of R.

Symmetric closure

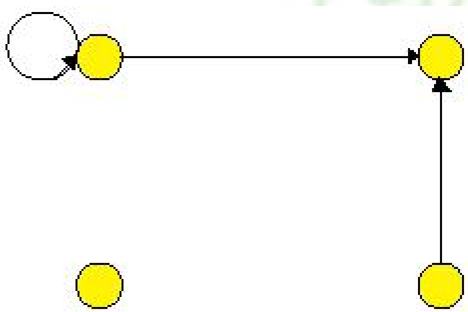
- **Theorem**
  - Let R be a relation on A.
  - The *symmetric closure* of R, denoted $s(R)$, is the relation $R \cup R^{-1}$.
  - $R^{-1}=\{(b,a) \mid (a,b) \in R\}$

Theorem

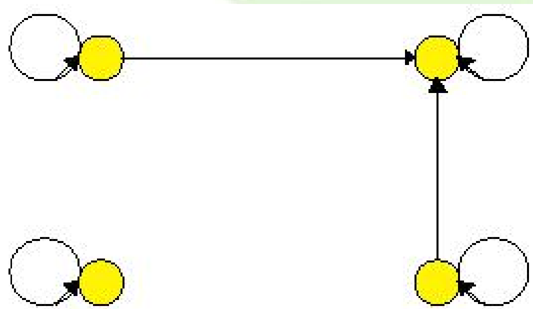
- R is symmetric
  - If and only if
- $R = R^{-1}$

  - Note: in digraph of a symmetric relation, use *undirected edges* instead of *arcs*

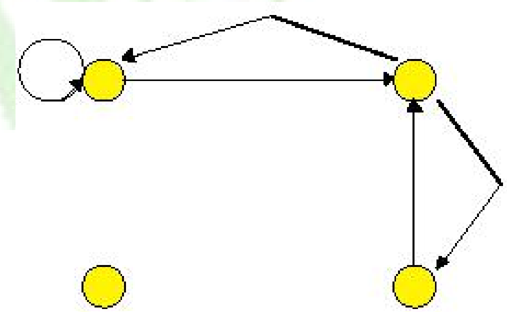
Example



R



$r(R)$

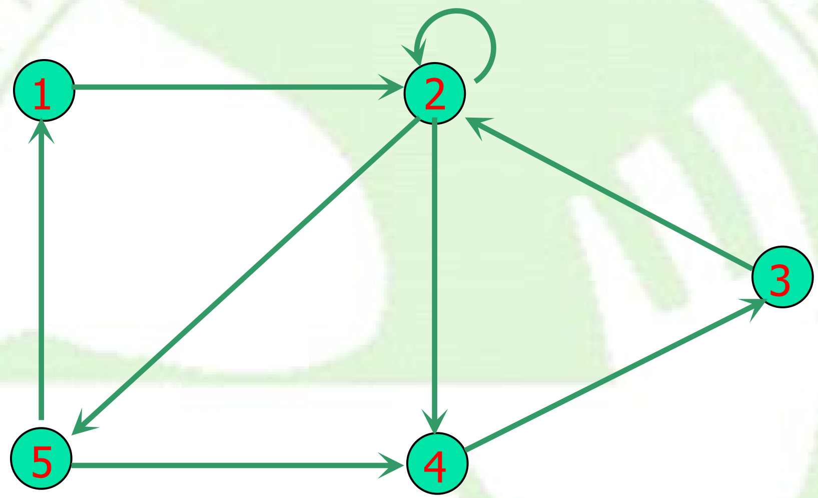


$s(R)$

Paths

- Suppose that R is a relation on a set A. A *path of length n* in R from a to b is a finite sequence $\pi : a, x_1, x_2, \ldots, x_{n-1}, b$, beginning with a and ending with b, such that
  - $a\ R\ x_1, x_1\ R\ x_2, \ldots, x_{n-1}\ R\ b$

Example



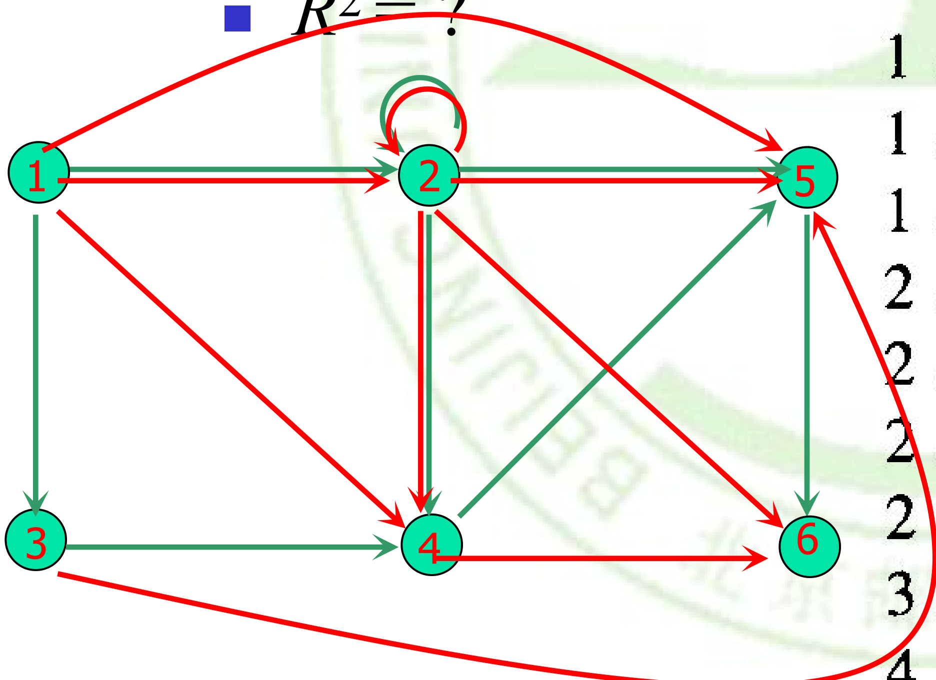
- π_1 : 1, 2, 5, 4, 3 is a path of length 4 from vertex 1 to vertex 3
- π_2 : 1, 2, 5, 1 is a path of length 3 from vertex 1 to itself
- π_3 : 2, 2 is a path of length 1 from vertex 2 to itself

Some definitions

- A path that begins and ends at the same vertex is called a *cycle*.
- R^n : $x\ R^n\ y$ means that there is a path of length n from x to y in R.
  - $R^n(x)$
- R^∞ : $x\ R^\infty\ y$ means that there is some path in R from x to y.
  - $R^\infty(x)$
- The relation R^∞ is sometimes called the *connectivity relation* for R .

Example

- Let $A = \{1, 2, 3, 4, 5, 6\}$
- R is shown as in figure
- $R^2 = ?$



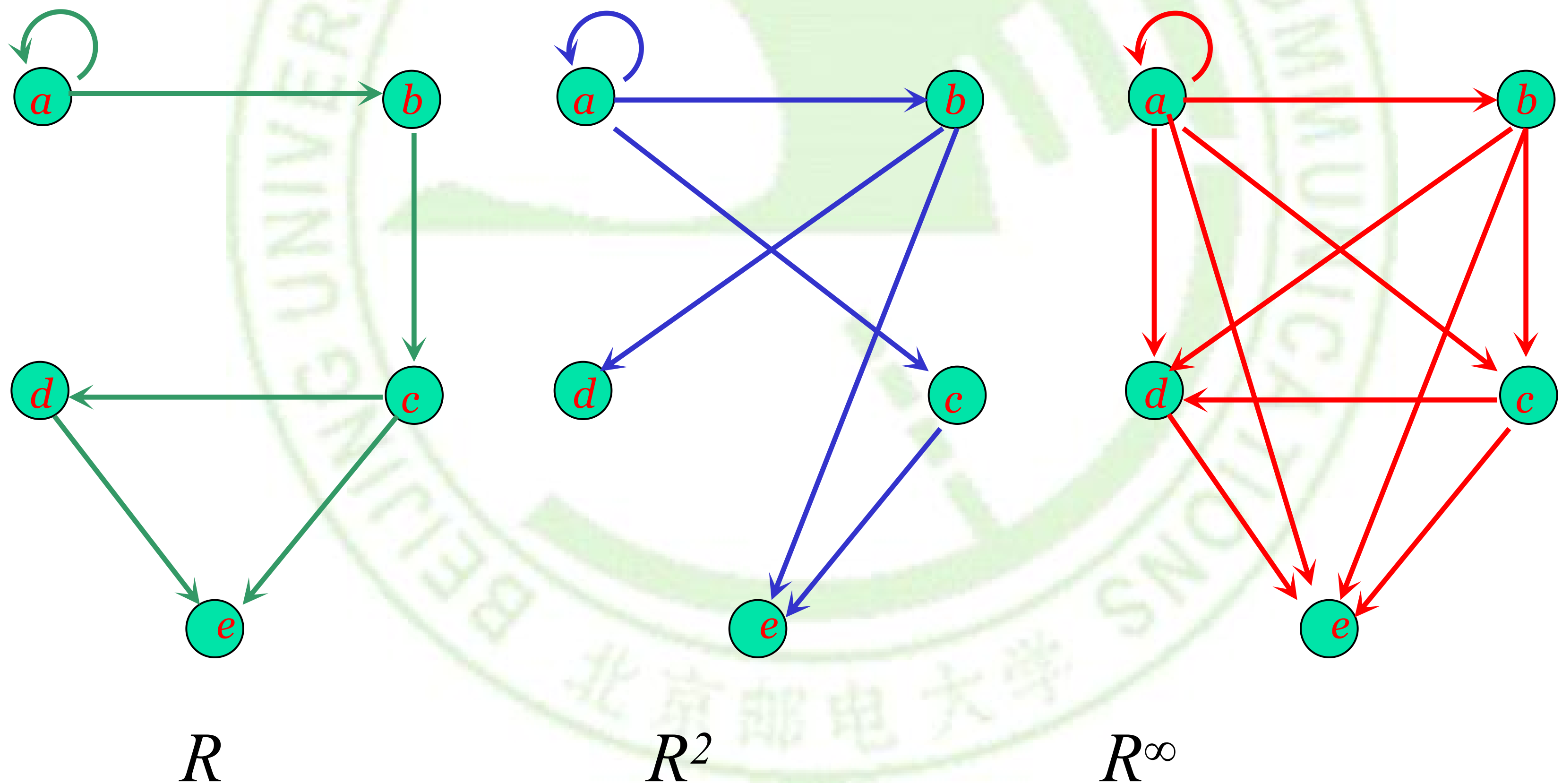
$$
\begin{array}{llll}
1\ R^2\ 2 & \text{since} & 1\ R\ 2 & \text{and} \quad 2\ R\ 2 \\
1\ R^2\ 4 & \text{since} & 1\ R\ 2 & \text{and} \quad 2\ R\ 4 \\
1\ R^2\ 5 & \text{since} & 1\ R\ 2 & \text{and} \quad 2\ R\ 5 \\
2\ R^2\ 2 & \text{since} & 2\ R\ 2 & \text{and} \quad 2\ R\ 2 \\
2\ R^2\ 4 & \text{since} & 2\ R\ 2 & \text{and} \quad 2\ R\ 4 \\
2\ R^2\ 5 & \text{since} & 2\ R\ 2 & \text{and} \quad 2\ R\ 5 \\
2\ R^2\ 6 & \text{since} & 2\ R\ 5 & \text{and} \quad 5\ R\ 6 \\
3\ R^2\ 5 & \text{since} & 3\ R\ 4 & \text{and} \quad 4\ R\ 5 \\
4\ R^2\ 6 & \text{since} & 4\ R\ 5 & \text{and} \quad 5\ R\ 6.
\end{array}
$$

Example

- Let $A = \{a, b, c, d, e\}$
  - $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$.
- Compute (a) R^2 ; (b) R^∞

Solution

- $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$.



Theorem

- If R is a relation on $A = \{a_1, a_2, \ldots, a_n\}$, then

$$M_{R^2} = M_R \odot M_R$$

$$M_{R^2} = M_R \odot M_R \triangleq (M_R)^2_{\odot}$$

Proof

- Let $M_R = [m_{ij}]$ and $M_{R^2} = [n_{ij}]$.
  - the i, jth element of $M_R \otimes M_R$ is equal to 1
  - $m_{ik} = 1$ and $m_{kj} = 1$ for some k, $1 \le k \le n$.
- By definition of the matrix M_R
  - $a_i\ R\ a_k$ and $a_k\ R\ a_j$
  - $a_i\ R^2\ a_j$, and so $n_{ij} = 1$.
- Therefore
  - position i, j of $M_R \otimes M_R$ is equal to 1
  - $n_{ij} = 1$.
- So $M_R \otimes M_R = M_{R^2}$

  - QED

Example

- Let $A = \{a, b, c, d, e\}$
  - $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$.
- Compute R^2

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example cont.

$$M_{R^2} = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem

- For $n \geq 2$ and R a relation on a finite set A, we have

$$M_{R^n} = M_R \odot M_R \odot \cdots \odot M_R \quad (n \text{ factors})$$

$$\triangleq (M_R)_{\odot}^{n}$$

Proof by induction

- Let $P(n)$ be the assertion that the statement holds for an integer $n \geq 2$.
- *Basis Step*: $P(2)$ is true by Theorem 1.

Induction Step

- Consider the matrix $M_{R^{k+1}}$. Let $M_{R^{k+1}} = [x_{ij}]$, $M_{R^k} = [y_{ij}]$, and $M_R = [m_{ij}]$
- If $x_{ij} = 1$, we must have a path of length $k + 1$ from a_i to a_j.
- If we let a_s be the vertex that this path reaches just before the last vertex a_j, then there is a path of length k from a_i to a_s and a path of length 1 from a_s to a_j.
- Thus $y_{is} = 1$ and $m_{sj} = 1$, so $M_{R^k} \odot M_R$ has a 1 in position i, j.
- similarly, if $M_{R^k} \odot M_R$ has a 1 in position i, j, then $x_{ij} = 1$.
- So

$$M_{R^{k+1}} = M_{R^k} \odot M_R$$

Induction Step

$\because$ P(k): $M_{R^k} = M_R \odot \cdots \odot M_R$ (k factors)

$\therefore M_{R^{k+1}} = M_{R^k} \odot M_R = (M_R \odot M_R \odot \cdots \odot M_R) \odot M_R$

hence

P(k+1): $M_{R^{k+1}} = M_R \odot \cdots \odot M_R \odot M_R$ ($k+1$ factors)

- is true.
- Thus by the principle of mathematical induction, P(n) is true for all n

- QED

The reachability relation

- Let R be a relation on a set A that has n elements, The *reachability relation* R^* consists of the pairs (a,b) such that there is a path of length at least one from a to b in R .
  - $x\ R^*\ y$ if and only if $x\ R^\infty\ y$

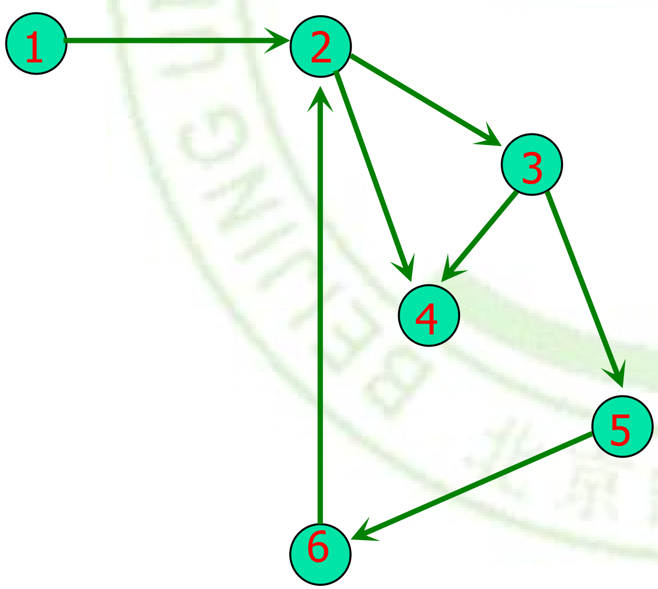
$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

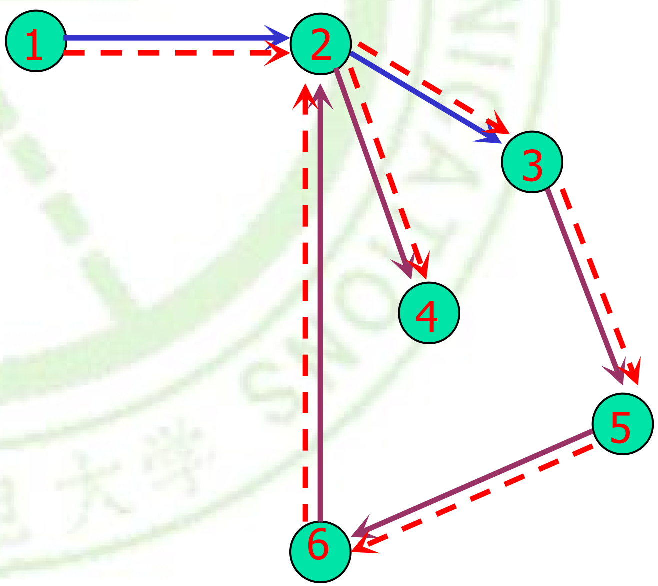
Composition of paths

- Let
  - $\pi_1: a, x_1, x_2, \ldots, x_{n-1}, b$
  - $\pi_2: b, y_1, y_2, \ldots, y_{m-1}, c$
- The composition of π_1 and π_2 is the path
  - $\pi_2 \circ \pi_1: a, x_1, x_2, \ldots, x_{n-1}, b, y_1, y_2, \ldots, y_{m-1}, c$
  - *Note the order of composition!*

Example

- Consider the relation whose digraph is given in Figure and the paths
  - π_1: 1, 2, 3 π_2: 3, 5, 6, 2, 4 $\pi_2 \circ \pi_1$





Transitive closure

- The transitive closure of a relation R is the smallest transitive relation containing R.

Transitive Closure

- Also recall R is transitive iff R^n is contained in R for all n
- Hence, if there is a path from x to y then there must be an arc from x to y, or (x, y) is in R.

Useful Results for Transitive Closure

- **Theorem:**
  - If $A \subseteq B$ and $C \subseteq B$, then $A \cup C \subseteq B$.
- **Theorem:**
  - If $R \subseteq S$ and $T \subseteq U$ then $R \circ T \subseteq S \circ U$.
- **Corollary:**
  - If $R \subseteq S$ then $R^n \subseteq S^n$

Useful Results for Transitive Closure

- **Theorem:**
  - If R is transitive then so is R^n
  - Trick proof: Show $(R^n)^2 = (R^2)^n \subset R^n$
- **Theorem:**
  - If $R^k = R^j$ for some $j > k$, then $R^{j+m} = R^n$ for some $n \geq j$.
  - We don't get any new relations beyond R^j.

Theorem

- Let R be a relation on a set A. then R^∞ is the transitive closure of R.

$$t(R) = R^\infty = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{i=1}^{\infty} R^i$$

- Proof: we must show that R^∞
  - 1) is a transitive relation
  - 2) contains R
  - 3) is the smallest transitive relation which contains R

Proof of Part 1)

- Suppose (x, y) and (y, z) are in R^∞. Show (x, z) is in R^∞.
  - By definition of R^∞, (x, y) is in R^m for some m and (y, z) is in R^n for some n.
  - Then (x, z) is in $R^n \circ R^m = R^{m+n}$ which is contained in R^∞.
  - Hence, R^∞ must be transitive.

Proof of Part 2)

- Easy from the definition of R^∞

Proof of Part 3)

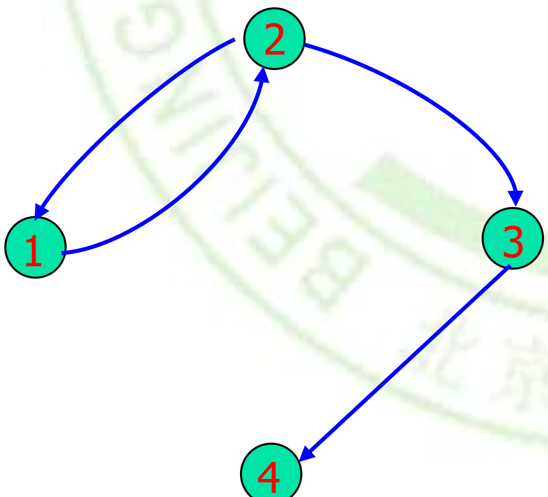
- Now suppose S is any transitive relation that contains R, show S contains R^∞ (that is R^∞ is the smallest such relation).
- $R \subseteq S$ so $R^2 \subseteq S^2 \subseteq S$ since S is transitive
- Therefore $R^n \subseteq S^n \subseteq S$ for all n. (why?)
- Hence S must contain R^∞ since it must also contain the union of all the powers of R.

  - Q. E. D.

  - In fact, we need only consider paths of length n or less.

Example

- Let
  - $A=\{1, 2, 3, 4\}$
  - $R=\{(1, 2), (2, 3), (3, 4), (2, 1)\}$
- Find the transitive closure of R.



$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$(M_R)^2_{\odot}=\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}=(M_R)^4_{\odot}=(M_R)^6_{\odot}=\ldots$$

$$(M_R)^3_{\odot}=\begin{bmatrix}0&1&0&1\\1&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=(M_R)^5_{\odot}=(M_R)^7_{\odot}=\ldots$$

$$M_{R^\infty}=M_R\vee(M_R)^2_{\odot}\vee(M_R)^3_{\odot}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Theorem

- Let A be a set with $|A|=n$, and let R be a relation on A. Then

$$R^{\infty} = \bigcup_{i=1}^{n} R^{i} = R \cup R^{2} \cup \cdots \cup R^{n}$$

Proof

$$R^{\infty} = \bigcup_{i=1}^{\infty} R^{i} = R \cup R^{2} \cup R^{3} \cup \cdots$$

$$? = \bigcup_{i=1}^{n} R^{i} = R \cup R^{2} \cup \cdots \cup R^{n}$$

- The equality will hold, if, for $k \le n < m$, we have
  - $R^m \subseteq R^k$
  - $(a, b) \in R^m \rightarrow (a, b) \in R^k$

Proof

- Let a and b be A and suppose that $a, x_1, x_2, \ldots, x_{m\text{-}1}, b$ is a path of length m from a to b in R
  - $(a, x_1) \in R$
  - $(x_1, x_2) \in R$
  - …
  - $(x_{m\text{-}1}, b) \in R$

Proof

- There are m+1 elements in the path, but we have only n distinct elements in A.
  - So, there must be some same vertex in the path, say $x_i = x_j = c,\ i<j$
    - $(a, x_1) \in R$
    - $(x_1, x_2) \in R$
    - …
    - $(x_{i\text{-}1}, x_i) \in R$
    - $(x_i, x_{i+!}) \in R$
    - …
    - $(x_{j\text{-}1}, x_j) \in R$
    - $(x_j, x_{j+1}) \in R$
    - …
    - $(x_{m\text{-}1}, b) \in R$
- The red edges form a cycle in the path, we get a new path by deleting the cycle

Proof

- A new path from a to b by deleting the cycle
  - $(a, x_1) \in R$
  - $(x_1, x_2) \in R$
  - …
  - $(x_{i-1}, x_i) \in R$
  - $(x_j, x_{j+1}) \in R$
  - …
  - $(x_{m-1}, b) \in R$

Proof

- A path from a to b ($x_i = x_j = c$)
  - $a, x_1, x_2, \ldots, x_{i\text{-}1}, c, x_{j+1}, \ldots, x_{m\text{-}1}, b$
- The length is $k = m - j + i$.
- The process can continue until $k \le n$, so we have
  - $R^m \subseteq R^k$
  - $\forall m\ (m > n \wedge (a, b) \in R^m \rightarrow \exists k\ (k \le n \wedge (a, b) \in R^k))$
- Therefore

$$R^{\infty} = \bigcup_{i=1}^{n} R^{i} = R \cup R^{2} \cup \cdots \cup R^{n}$$

- QED

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return $\mathbf{B}${$\mathbf{B}$ is the zero–one matrix for R^*}

Some definitions

- Let
  - $A = \{a_1, a_2, \ldots, a_n\}$
  - R be a relation on A
- *Interior vertices*
  - $a, x_1, x_2, \ldots, x_i, b$

Some definitions

- W_k: a Boolean matrix, for $1 \le k \le n$
  - W_k has a 1 in position i, j
    - If and only if
  - there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_k\}$
- What about W_0 W_n ?
  - Let $W_0 = W_R$
  - $W_n = W_R^{\infty}$
  - $W_0, W_1, W_2, \ldots, W_n$

Warshall's Algorithm

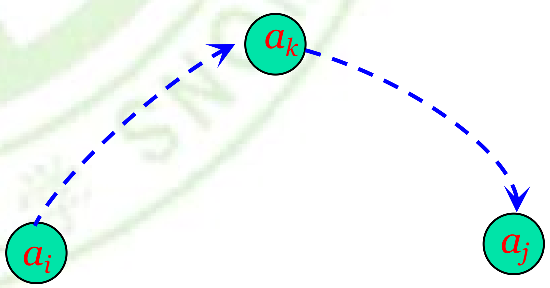
- Procedure
  - begin with the matrix of R, and
  - compute each matrix W_k from the previous matrix W_{k-1}, and,
  - reach W_{R^∞} in n steps,

Warshall's Algorithm

- Suppose
  - $W_k = [t_{ij}]$
  - $W_{k\text{-}1} = [s_{ij}]$
- If $t_{ij} = 1$, then there must be a path from a_i to a_j whose interior vertices come from the set $\{a_1, a_2, \ldots, a_k\}$.
  - Whether a_k is an interior vertex ?
  - Two cases

Warshall's Algorithm

- a_k is not an interior vertex
  - then all interior vertices must actually come from the set $\{a_1, a_2, \ldots, a_{k-1}\}$
  - so $s_{ij} = 1$.
- a_k is an interior vertex
  - Assume a_k appears only once (why?)
  - Two subpaths
    - a_i to a_k and a_k to a_j
  - $s_{ik} = 1$ and $s_{kj} = 1$



Warshall's Algorithm

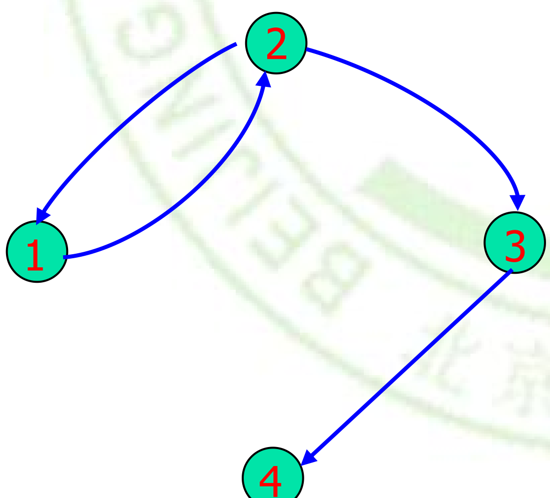
- The basis for Warshall's Algorithm
  - $t_{ij} = 1$
    - If and only if
  - either
    - $s_{ij} = 1$
    - $s_{ik} = 1$ and $s_{kj} = 1$

Warshall's Algorithm

- *Step1*:
  - First transfer to W_k all 1's in $W_{k\text{-}1}$.
- *Step2*:
  - List the locations p_1, p_2, …, in column k of $W_{k\text{-}1}$, where the entry is 1
  - List the locations q_1, q_2, …, in row k of $W_{k\text{-}1}$, where the entry is 1
- *Step3*:
  - Put 1's in all the positions p_i, q_j of W_k (if they are not already there)

Example (1)

- Let
  - $A=\{1, 2, 3, 4\}$
  - $R=\{(1, 2), (2, 3), (3, 4), (2, 1)\}$
- Find the transitive closure of R.



$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$W_0 = \begin{bmatrix} \underline{0} & \underline{\underline{1}} & \underline{0} & \underline{0} \\ \underline{\underline{1}} & 0 & 1 & 0 \\ \underline{0} & 0 & 0 & 1 \\ \underline{0} & 0 & 0 & 0 \end{bmatrix} = M_R, \qquad W_1 = \begin{bmatrix} 0 & \underline{\underline{1}} & 0 & 0 \\ \underline{\underline{1}} & \underline{\underline{1}} & \underline{\underline{1}} & \underline{0} \\ 0 & \underline{0} & 0 & 1 \\ 0 & \underline{0} & 0 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 1 & \underline{\underline{1}} & 0 \\ 1 & 1 & \underline{\underline{1}} & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{\underline{1}} \\ 0 & 0 & \underline{0} & 0 \end{bmatrix}, \qquad W_3 = \begin{bmatrix} 1 & 1 & 1 & \underline{\underline{1}} \\ 1 & 1 & 1 & \underline{\underline{1}} \\ 0 & 0 & 0 & \underline{\underline{1}} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix} = W_4 = W^{\infty}$$

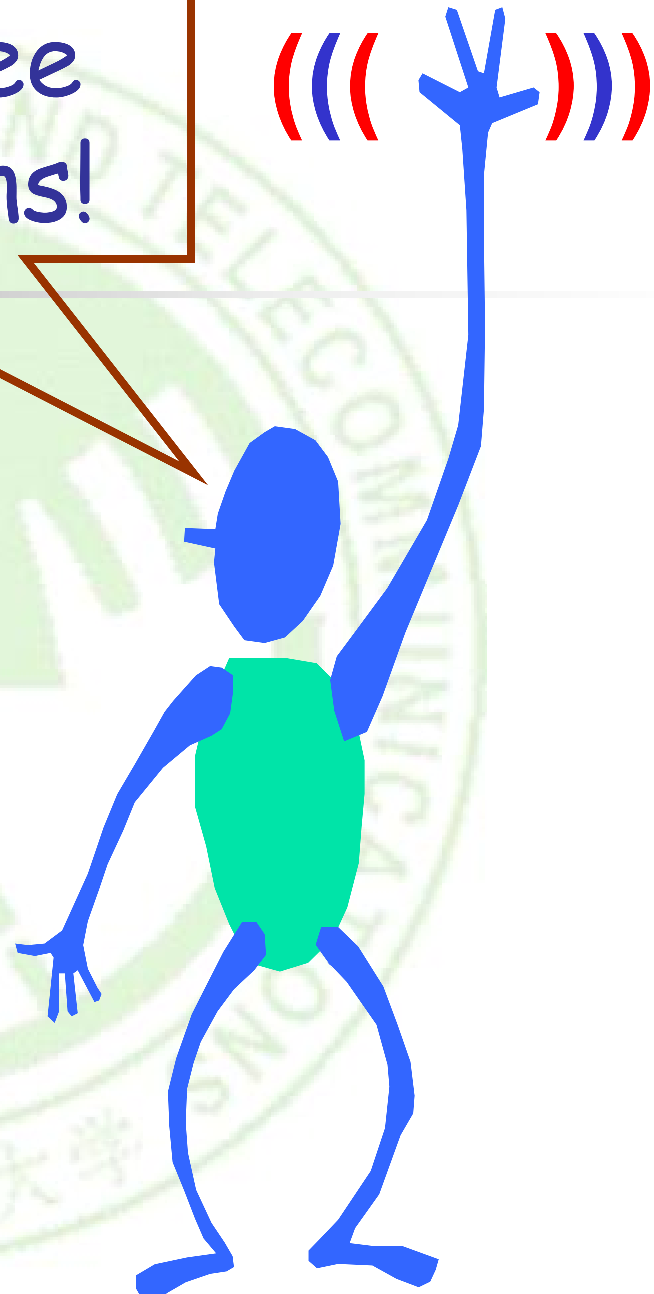
Warshall's Algorithm

```
Algorithm Warshall
CLOSURE←MAT
   For k = 1 thru N
      For i = 1 thru N
         For j = 1 thru N
            CLOSURE[i,j]←CLOSURE[i,j]
                         ∨(CLOSURE[i,k]∧CLOSURE[k,j])
End of Algorithm Warshall
```

Analysis

- Complexity of Algorithm
  - Warshall
    - n^3
  - $M_R^{\infty} = M_R \vee (M_R)_{\odot}^{2} \vee \ldots \vee (M_R)_{\odot}^{n}$
    - n^4

Please feel free to ask questions!



homework

- § 9.4
  - 20, 22, 28