Chapter 1 Fundamentals

Signals or waveforms are very common occurrences in everday life. Our lights and various electrical gadgets are powered by electric signals. From our basic electric circuits and networks course, we know that there are usually sinusoidal current and voltage waveforms. The computer uses, processes, and stores information that is actually made up of electrical signals. When we switch on the TV, electrical signals are received from the air and are transformed into audio and video signals.

This chapter presents fundamental concepts of signals and systems. It begins with classification of signals and its properties. The subsequent sections cover a variety of topics such as properties of systems, system interconnections and numerous examples to fortify the theoritical understanding of various concepts related to signals and systems.

1.1 Signal

A signal defines the variation of some physical quantity as function of one more independent variables, and this variation contains information that is of interest to us.

For example:

- **Electrical signals** – Voltages and currents in an electrical circuit.
- **Acoustic signals** – Audio or speech signals.
- **Video signals** – Intensity variations in an image (example, a CAT scan).
- **Biological signals** – Sequences of bases in a gene.

The independent variables can be of following type:

- **Continuous** – Trajectory of a space shuttle, mass density in the cross-section of a bone.

- **Discrete** – DNA base sequence, Digital image pixels.
- **1-dimensional, 2-dimensional,... N-dimensional and space** – Electrical field intensity as function of time.

In this book, we will focus on a single (1-dimensional) independent variable, which we call time. Summarizing, a signal is a function of set of independent variables, with time perhaps the most common single variable. A signal itself carries some kind of information available for observation.

1.2 Signal Classification

1.2.1 Continuous-time and discrete-time signals

A continuous-time (CT) signal is a funciton of time that is defined for every time instant within a given interval of time. This interval may be infinite. The signal may be a real- or complex-valued function of time. We may also define a continuous-time signal as a mapping of the set of all values of time to a set of corresponding values of the functions that are subject to certain properties. Since the continuous-time signal is well defined for all values of time in $-\infty$ to ∞, it is differentiable at all values of the independent variable t. Two examples of continuous-time signals are shown in Fig. 1.1.

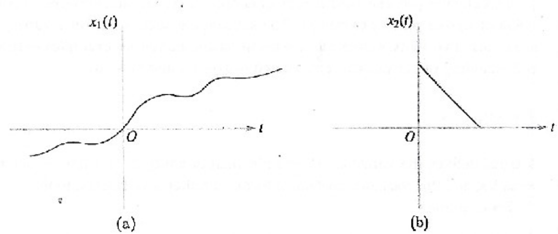


Fig. 1.1: *Two examples of continuous-time signals*

A discrete-time signal is a function that is defined only at discrete instants of time and undefined for all other values of time. A discrete-time signal is obtained from a continous-time signal by replacing t by nT_s, where T_s is known as the sampling period. If we choose to sample $x(t)$ at equal intervals of T_s seconds, we generate $x(nT_s) = x(t)|_{t=nT_s}$ as sequence of numbers. Since T_s is fixed, $x(nT_s)$ is a function of only the integer n and hence can be considered as a function of n or expressed as $x(n)$.

A continuous-time function $x(t)$ and the discrete-time function $x(n)$ are plotted in Fig. 1.2

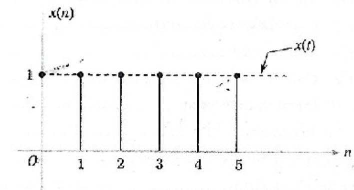


Fig. 1.2: *The continuous-time function $x(t)$ and the discrete-time function $x(n)$*

In this book, we shall denote a discrete-time (DT) function as a DT sequence, DT signal, or a DT series. Further, a DT function is a mapping of a set of all integers to a set of values of the functions that may be real-valued or complexed. For example, Fig. 1.2 may be explained as

$$n = \{0, 1, 2, 3, 4, \ldots\} \xrightarrow{\text{Mopped}} x(n) = \{1, 1, 1, 1, 1, \ldots\}$$

Most of the signals in the physical world are CT signals. For example: Voltage and current, pressure, temperature, velocity, etc. Examples of DT signals in nature are DNA base sequence, population of the n^{th} generation of certain species etc.

If a discrete-time signal $x(n)$ can only take a finite number of distinct values, then we call this signal as a digital signal.

1.2.2 Deterministic and random signals

Signals are termed *deterministic*, if their behaviour is known and can be represented, for example, by a formula. The deflection voltage of an oscilloscope is a deterministic signal, because its behaviour is known and can be represented by a sawtooth wave. By contrast, we cannot define the amplitude values of a voice signal by means of formulae or graphical elements; in addition, their continued behaviour is not known. Such signals are termed *random*. Since, it is impossible to specify their behaviour in terms of functions, such signals are described by expected values such as mean, variance and many others.

1.2.3 Even and odd signals

A signal $x(t)$ or $x(n)$ is referred to as even signal if

$$x(-t) = x(t)$$
$$x(-n) = x(n)$$

An even signal has symmetry with respect to vertical axis. That is, the signal for $t < 0$ is the mirror image of the signal for $t > 0$. The function, $x(t) = \cos 2t$ is even because $\cos(-2t) = \cos(2t)$.

A signal $x(t)$ or $x(n)$ is referred to as odd signal if

$$x(-t) = -x(t)$$
$$x(-n) = -x(n)$$

An odd signal has anti-symmetry with respect to vertical axis. The signal $x(t) = \sin 2t$ is odd because $\sin(-2t) = -\sin 2t$.

Examples of even and odd signals are shown in Fig. 1.3.

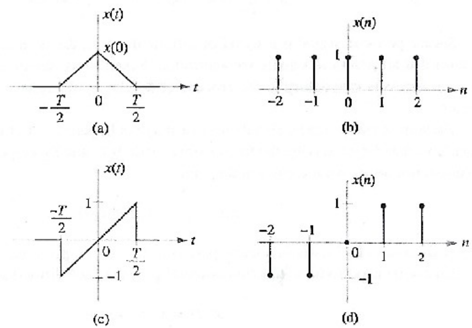


Fig. 1.3: *Examples of even signals (a and b) and odd signals (c and d).*

Even and odd signals are mutually exclusive; that is if a signal qualifies to be an even signal, it cannot be odd and vice versa. However, there could be certain class of signals that could neither be termed odd nor even.

Let us consider an arbitrary continuous-time signal, $x(t)$. The objective is to find even and odd parts of $x(t)$.

$$\text{Let} \quad x(t) = x_e(t) + x_o(t) \tag{1.1}$$

where $x_e(t)$ and $x_o(t)$ are even and odd parts of $x(t)$, respectively. Replacing t with $-t$ in Eq. (1.1) yeilds

$$x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t) \tag{1.2}$$

Adding Eqs. (1.1) and (1.2) and dividing by 2 gives

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \tag{1.3}$$

Subtracting Eq. (1.2) from Eq. (1.1) and dividing by 2 yeilds

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] \tag{1.4}$$

By analogy, for discrete-time case, we can write

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \tag{1.5}$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)] \tag{1.6}$$

Even and odd signals have the following properties:

- The sum of two even functions is even.
- The sum of two odd functions is odd.
- The sum of an even function and an odd function is neither even nor odd.
- The product of even function is even.

  This allows as to write:

  (a) $\int_{-a}^{a} x_e(t)y_e(t)dt = 2\int_{0}^{a} x_e(t)y_e(t)dt$

  (b) $\sum_{n=-a}^{a} x_e(n)y_e(n) = x_e(0)y_e(0) + 2\sum_{n=1}^{a} x_e(n)y_e(n)$

- The product of two odd functions is even.

  This allows us to write:

  (a) $\int_{-a}^{a} x_o(t)y_o(t)dt = 2\int_{0}^{a} x_o(t)y_o(t)dt$

  (b) $\sum_{n=-a}^{a} x_o(n)y_o(n) = x_o(0)y_o(0) + \sum_{n=1}^{a} x_o(n)y_o(n)$

- The product of an even function and an odd function is odd. Thus, we have

  (a) $\int_{-a}^{a} x_e(t)y_o(t)dt = 0$

  (b) $\sum_{n=-a}^{a} x_e(n)y_o(n) = 0$

The average value A_x of a signal $x(t)$ over a time interval of length $2L$ is defined as

$$A_x = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} x(t)dt \tag{1.7}$$

Example 1.1. Consider the continuous-time signal defined by

$$x(t) = \begin{cases} Ae^{-\alpha t}, & t > 0 \\ 0, & t < 0 \end{cases}$$

Find the even and odd parts of $x(t)$.

Solution: Recall:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

Hence,

$$x_e(t) = \begin{cases} \frac{1}{2}[Ae^{-\alpha t} + 0], & t > 0 \\ \frac{1}{2}[0 + Ae^{\alpha t}], & t < 0 \end{cases}$$

$$\rightarrow \quad x_e(t) = \begin{cases} \frac{1}{2}Ae^{-\alpha t}, & t > 0 \\ \frac{1}{2}Ae^{\alpha t}, & t < 0 \end{cases}$$

Recall:

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Hence,

$$x_o(t) = \begin{cases} \frac{1}{2}Ae^{-\alpha t}, & t > 0 \\ -\frac{1}{2}Ae^{\alpha t}, & t < 0 \end{cases}$$

Signals $x_e(t)$ and $x_o(t)$ are as depicted in Fig. 1.4.

Fig. 1.4: *Plots of $x_e(t)$ and $x_o(t)$*

1.2.4 Periodic and nonperiodic signals

By definition, a continuous-time signal is periodic, if

$$x(t) = x(t + mT_0); \quad m = 1, 2, 3, \ldots$$

where $T_0 > 0$ is a constant and is known as the fundamental period. A signal $x(t)$ that is not periodic is referred to as an aperiodic signal. With T_0 in seconds, the fundamental frequency in Hertz, f_0, and the fundamental frequency in rad/sec, ω_0 are given by

$$f_0 = \frac{1}{T_0} \text{ Hz}, \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} \text{ rad/sec}$$

Since a periodic signal is a signal of infinite duration, that is, defined for $-\infty \le t \le \infty$, it follows that all practical signals are aperiodic. Nevertheless, the study of system response to periodic inputs is mandatory in the process of developing the system response to all practical inputs.

The sum of two periodic signals may or may not be periodic. Let us consider two periodic signals $x_1(t)$ and $x_2(t)$ having the fundamental periods T_1 and T_2, respectively. Our objective is to investigate under what condition the sum

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$

is periodic and to find its fundamental period if the signal is periodic.

Since $x_1(t)$ is periodic with a fundamental period T_1, it follows that

$$x_1(t) = x_1(t + kT_1)$$

By the same reasoning,

$$x_2(t) = x_2(t + lT_2)$$

where k and l are integers such that

$$y(t) = a_1 x_1(t + kT_1) + a_2 x_2(t + lT_2)$$

For $y(t)$ to be periodic with a fundamental period T, it is required that

$$a_1 x_1(t + T) + a_2 x_2(t + T) = a_1 x_1(t + kT_1) + a_2 x_2(t + lT_2)$$

We hence must have

$$T = kT_1 = lT_2 \tag{1.8}$$

Equivalently,

$$\frac{T_1}{T_2} = \frac{l}{k}$$

Summarizing, the sum of two periodic signals is periodic only if the ratio of their respective periods can be expressed as a rational number. Also, if $x(t)$ is periodic, then T is the LCM of T_1 and T_2.

Periodic discrete-time signals are defined on a similar basis. A sequence or a discrete-time signal is periodic, if

$$x(n) = x(n + mN_0); \quad m = 1, 2, \ldots$$

In the above equation N_0 is known as fundamental period of $x(n)$. Any sequence which is not periodic is called an aperiodic signal.

Let $x(n)$ be the sum of two periodic signals $x_1(n)$ and $x_2(n)$ having the fundamental periods N_1 and N_2, respectively. Also, let k and l be two integers such that

$$kN_1 = lN_2 = N \tag{1.9}$$

Then $x(n)$ is periodic with a fundamental period N, since

$$\begin{aligned} x(n+N) &= x_1(n + kN_1) + x_2(n + lN_2) \\ &= x_1(n) + x_2(n) \end{aligned}$$

Since we can always find integers k and l to satisfy Eq. (1.9), it follows that sum of discrete-time periodic signals is always periodic. Also, Eq. (1.9) implies that N is the LCM of N_1 and N_2.

Fig. 1.5 depicts the examples of continuous-time and discrete-time periodic signals.

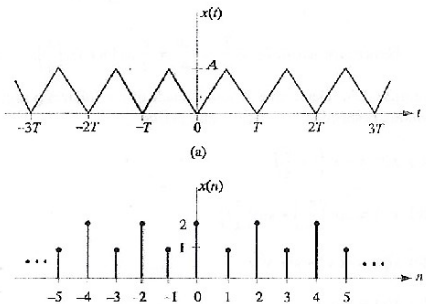


Fig. 1.5: *Examples of periodic signals.*

Example 1.2. Show that if $x_1(t)$ and $x_2(t)$ have period T, then $x_3(t) = ax_1(t) + bx_2(t)$ (a, b constant) has the same period T.

Solution: Given: $x_3(t) = ax_1(t) + bx_2(t)$.

Replacing t by $t + T$ in the above equation, we get

$$\begin{aligned} x_3(t+T) &= ax_1(t+T) + bx_2(t+T) \\ &= ax_1(t) + bx_2(t) \\ &= x_3(t) \end{aligned}$$

Hence $x_3(t)$ has the same fundamental period as that of $x_1(t)$ and $x_2(t)$.

Example 1.3. Check the following signals for periodicity. If periodic, find their fundamental periods.

(a) $x(t) = \sin\left(\frac{\pi}{3}t\right) + 2\cos\left(\frac{8\pi}{3}t\right)$

(b) $x(t) = e^{j\frac{12\pi}{7}t} + e^{j\frac{12\pi}{5}t}$

(c) $x(t) = e^{j\frac{2\pi}{5}t} + e^{j3t}$

Solution:

(a) Given: $x(t) = \sin\left(\frac{\pi}{3}t\right) + 2\cos\left(\frac{8\pi}{3}t\right)$

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\frac{\pi}{3}} = 6$$

$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{8\frac{\pi}{3}} = \frac{3}{4}$$

We have, $\frac{T_1}{T_2} = 8$, a rational number.

Hence, $x(t)$ is periodic with a fundamental period,

$$\begin{aligned} T &= \text{LCM of } T_1 \text{ and } T_2 \\ &= T_1 \text{ or } 8T_2 = 6 \end{aligned}$$

(b) Concept: Using Euler's formula, we can write

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Since $\cos\omega t$ and $\sin\omega t$ are periodic with a fundamental period, $\frac{2\pi}{\omega}$, then their linear combination is also periodic with fundamental period, $\frac{2\pi}{\omega}$.

Hence, $e^{j\frac{12\pi}{7}t}$ has a fundamental period, $T_1 = \frac{2\pi}{\frac{12\pi}{7}} = \frac{7}{6}$ and $e^{j\frac{12\pi}{5}t}$ has a fundamental period, $T_2 = \frac{2\pi}{\frac{12\pi}{5}} = \frac{5}{6}$.

We have, $\frac{T_1}{T_2} = \frac{7}{5}$, a rational number.

Therefore,

$$\begin{aligned} T &= \text{LCM of } T_1 \text{ and } T_2 \\ &= 5T_1 \text{ or } 7T_2 \\ &= \frac{35}{6} \end{aligned}$$

(c) $e^{j\frac{2\pi}{5}t}$ is periodic with a fundamental period,

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\frac{2\pi}{5}} = 5.$$

Similarly, e^{j3t} is periodic with a fundamental period, $T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{3}$.

Since the ratio, $\frac{T_1}{T_2} = \frac{5}{\frac{2\pi}{3}} = \frac{15}{2\pi}$ is not a rational number, $x(t)$ is not periodic.

Example 1.4. (a) Let $x(t)$ be a periodic signal with a fundamental period T. Show that $x(at)$, $a > 0$, is a periodic signal with a fundamental period, $\frac{T}{a}$.

(b) Show that $x\left(\frac{t}{b}\right)$, $b > 0$, is a periodic signal with period, bT.

(c) Verify the results for $a = b = 4$.

Solution:

(a) Since $x(t)$ is periodic with a fundamental period, T, we can write

$$\begin{aligned} x(t) &= x(t+T) \\ \Rightarrow \quad x(at) &= x(at+T) \\ &= x\left(a\left(t+\frac{T}{a}\right)\right) \end{aligned}$$

For $x(at)$ to be periodic with fundamental period T_1, one requires, $T_1 = \frac{T}{a}$.

(b) Similarly,

$$\begin{aligned} x\left(\frac{t}{b}\right) &= x\left(\frac{t}{b}+T\right) \\ &= x\left(\frac{1}{b}(t+bT)\right) \end{aligned}$$

For $x(\frac{t}{b})$ to be periodic with period T_2, we require that, $T_2 = bT$.

(c) For $\sin t$, the fundamental period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\pi.$$

Hence, for $\sin 4t$, $T_1 = \frac{T}{a} = \frac{2\pi}{4} = \frac{\pi}{2}$ and for $\sin\left(\frac{t}{4}\right)$, $T_2 = bT = 4 \times 2\pi = 8\pi$.

Example 1.5. Determine whether each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$

(b) $x(t) = \cos\left(\frac{\pi}{3}t\right) + \sin\left(\frac{\pi}{4}t\right)$

(c) $x(t) = \cos t + \sin\sqrt{2}t$

(d) $x(t) = \sin^2 t$

(e) $x(t) = e^{j(\frac{\pi}{2}t-1)}$

Solution:

(a)

$$x(t) = \cos\left(t + \frac{\pi}{4}\right)$$

$$\omega_0 = 1$$

$x(t)$ is periodic with a fundamental period $T_0 = \dfrac{2\pi}{\omega_0} = 2\pi$ seconds.

(b)

$$x(t) = \cos\left(\frac{\pi}{3}t\right) + \sin\left(\frac{\pi}{4}t\right)$$

$$= x_1(t) + x_2(t)$$

$$T_1 = \frac{2\pi}{\frac{\pi}{3}} = 6, \quad T_2 = \frac{2\pi}{\frac{\pi}{4}} = 8$$

Since, $\dfrac{T_1}{T_2} = \dfrac{6}{8} = \dfrac{3}{4}$ is a rational number, $x(t)$ is periodic with a fundamental period $T = 4T_1 = 3T_2 = 24$ seconds.

(c)

$$x(t) = \cos t + \sin \sqrt{2}t$$

$$= x_1(t) + x_2(t)$$

$x_1(t)$ is periodic with $T_1 = \dfrac{2\pi}{1} = 2\pi$ seconds and $x_2(t)$ is periodic with $T_2 = \dfrac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$ seconds. Since, $\dfrac{T_1}{T_2} = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

(d)

$$x(t) = \sin^2 t$$

$$\Rightarrow \quad x(t) = \frac{1}{2} - \frac{1}{2}\cos 2t$$

The first term on the RHS is a DC term, while the second term on the RHS is periodic with a period, $T_2 = \dfrac{2\pi}{\omega_2} = \dfrac{2\pi}{2} = \pi$ seconds. Therefore, $x(t)$ is periodic with a period $T = \pi$ seconds.

(e)

$$x(t) = e^{j(\frac{\pi}{2}t - 1)}$$

$$\Rightarrow \quad x(t) = e^{-j}e^{j\frac{\pi}{2}t}$$

$$= e^{-j}e^{j\omega_0 t}$$

where

$$\omega_0 = \frac{\pi}{2}$$

Therefore, $x(t)$ is periodic with a fundamental period $T = \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{\frac{\pi}{2}} = 4$ seconds.

Example 1.6. Determine, whether each of the following signals is periodic. If a signal is periodic, find its fundamental period.

(a) $x(n) = 5\cos(0.2\pi n)$

(b) $x(n) = \sin(2n)$.

Solution:

(a)

$$x(n) = 5\cos(0.2\pi n)$$

$$\Omega_0 = 0.2\pi$$

Also,

$$\frac{\Omega_0}{2\pi} = \frac{0.2\pi}{2\pi} = \frac{1}{10} = \frac{p}{q} = \text{a rational number}$$

Therefore, fundamental period of $x(n)$ is $N = q = 10$ samples.

(b)

$$x(n) = \sin(2n)$$

$$\Omega_0 = 2$$

Also,

$$\frac{\Omega_0}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi} = \text{an irrational number}$$

Therefore, $x(n)$ is not periodic.

Example 1.7. Determine which of the following signals are periodic. If periodic, find its fundamental period.

(a) $x(n) = \sin\left(\dfrac{3\pi n}{4}\right)\sin\left(\dfrac{\pi}{3}n\right)$

(b) $x(n) = e^{\frac{6\pi}{5}n}$

(c) $x(n) = \sum\limits_{n=-\infty}^{\infty} [\delta(n - 2m) + 2\delta(n - 3m)]$

Solution:

(a) Recall: $\sin A \sin B = \dfrac{1}{2}[\cos(A - B) - \cos(A + B)]$

Hence, $x(n) = \dfrac{1}{2}\cos\left(\dfrac{13\pi}{12}n\right) - \dfrac{1}{2}\cos\left(\dfrac{5\pi}{12}n\right)$

Then, we find

$$\frac{\Omega_{01}}{2\pi} = \frac{13\pi/12}{2\pi} = \frac{13}{24} = \frac{p}{q} \Rightarrow N_1 = q = 24$$

and

$$\frac{\Omega_{02}}{2\pi} = \frac{5\pi/12}{2\pi} = \frac{5}{24} = \frac{p}{q} \Rightarrow N_2 = q = 24$$

Therefore, $x(n)$ is periodic with a fundamental period, $N = 24$.

(b) Not periodic.

(c) Let

$$x(n) = x_1(n) + x_2(n)$$

where $$x_1(n) = \sum_{n=-\infty}^{\infty} \delta(n - 2m)$$

and $$x_2(n) = 2 \sum_{n=-\infty}^{\infty} \delta(n - 3m)$$

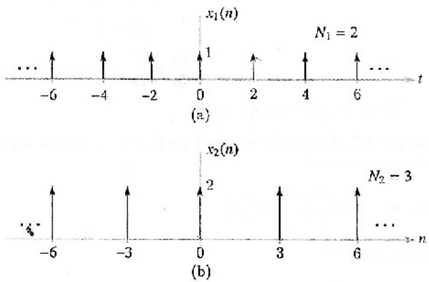


Fig. 1.6: *Sketches of $x_1(n)$ and $x_2(n)$*

From Fig. 1.6, we find that the fundamental periods of $x_1(n)$ and $x_2(n)$ are $N_1 = 2$ and $N_2 = 3$, respectively. Hence, fundamental period of $x(n)$ is

$$N = \text{LCM of } N_1 \text{ and } N_2$$
$$= 6$$

1.2.5 Energy and power signals

Let $x(t)$ be a real-valued continuous-time signal. If $x(t)$ represents the voltage across a resistance R, then the current through the resistance is $\frac{x(t)}{R}$. If we assume, $R = 1$ ohm, then the voltage across the resistance and current through the resistor are identical at $x(t)$. Hence, the instantaneous power associated with the signal $x(t)$ is $x^2(t)$. The power dissipated in the resistance, $R = 1$ ohm is called normalized power. The signal energy over a time interval of length $2t_1$ is defined as

$$E = \int_{-t_1}^{t_1} |x(t)|^2 dt \tag{1.10}$$

and the total energy in the signal $x(t)$ over the range $(-\infty, \infty)$ can be defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1.11}$$

The average power associated with the signal $x(t)$ is defined as

$$P = \lim_{t_1 \to \infty} \left[\frac{1}{2t_1} \int_{-t_1}^{t_1} |x(t)|^2 dt \right] \tag{1.12}$$

If the signal $x(t)$ is periodic, with a fundamental period, T, then the average power associated with the signal is

$$P = \frac{1}{T} \int_{0}^{T} |x(t)|^2 dt \tag{1.13}$$

Similarly, the average power and total energy associated with a discrete-time signal $x(n)$ are defined as follows:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{1.14}$$

$$P = \lim_{N_1 \to \infty} \left[\frac{1}{2N_1 + 1} \sum_{n=-N_1}^{N_1} |x(n)|^2 \right] \tag{1.15}$$

If $x(n)$ is periodic, with a fundamental period equal to N_1, then

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \tag{1.16}$$

Based on the definitions given in Eqs. (1.10) through (1.16), the following classifications are made:

- $x(t)$ or $x(n)$ is defined to be an energy signal, if and only if the total energy content of the signal is a finite quantity, that is, $0 < E < \infty$. All energy signals will have zero average power. Energy signals generally include aperiodic signals that have a finite time duration (such as rectangular function - discussed later) and signals that approach zero asymptotically so that $x(t)$ approaches zero as t approaches infinity.
- $x(t)$ or $x(n)$ is defined to be a power signal if and only if the average power associated with the signal is a finite quantity, that is, $0 < P < \infty$. As stated earlier, periodic signals are assumed to exist for all time from $-\infty$ to ∞ and therefore have infinite energy. If it happens that these periodic signals have finite average power, then they are power signals.
- Energy and power signals are mutually exclusive. However, there could be certain class of signals which do not satisfy the classifications referred above. Such signals are neither energy nor power signals.

Example 1.8. Determine whether the signals given below are power or energy signals. Justify your answers.

(a) $x_1(t) = Ae^{-2t}u(t)$

(b)

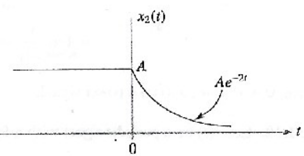


Fig. 1.7

Solution:

(a) $x_1(t)$ is aperiodic with total energy

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{0}^{\infty} A^2 e^{-4t} dt$$

$$= A^2 \left[\frac{e^{-4t}}{-4}\right]_0^{\infty} = \frac{A^2}{4}$$

Since, $0 < E < \infty$, $x_1(t)$ is an energy signal.

The average power is

$$P = \lim_{t_1 \to \infty} \left[\frac{1}{2t_1} \int_{-t_1}^{t_1} |x(t)|^2 dt\right]$$

$$= \lim_{t_1 \to \infty} \left[\frac{1}{2t_1} \int_{-t_1}^{t_1} A^2 e^{-4t} dt\right]$$

$$= \lim_{t_1 \to \infty} \frac{A^2}{8t_1} = 0$$

Average power is zero as expected.

(b) The energy of the signal $x_2(t)$ is

$$E = \int_{-\infty}^{\infty} |x_2(t)|^2 dt$$

$$= \int_{-\infty}^{0} A^2 dt + \int_{0}^{\infty} A^2 e^{-4t} dt$$

$$= \infty$$

which is clearly unbounded. Thus, $x_2(t)$ shown in Fig. 1.7 is not an energy signal.

The power associated with $x_2(t)$ is

$$P = \lim_{t_1 \to \infty} \left[\frac{1}{2t_1} \int_{-t_1}^{t_1} |x_2(t)|^2 dt\right]$$

$$= \lim_{t_1 \to \infty} \left[\frac{1}{2t_1} \int_{-t_1}^{0} A^2 dt + \frac{1}{2t_1} \int_{0}^{t_1} A^2 e^{-4t} dt\right]$$

$$= \frac{A^2}{2}$$

Since, $0 < P < \infty$, $x_2(t)$ is a power signal.

Example 1.9. Determine whether the signals given below are energy or power signals. Justify your answers.

(a) $x_1(t) = t\sin\left(\frac{\pi}{3}t\right)$

(b) $x_2(t) = e^{j\frac{5\pi}{6}t}$

(c) $x_3(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

Solution:

(a) The average power associated with the signal $x_1(t)$ is

$$P = \lim_{t_1\to\infty} \frac{1}{2t_1} \int_{-t_1}^{t_1} |x_1(t)|^2 dt$$

$$= \lim_{t_1\to\infty} \frac{1}{2t_1} \int_{-t_1}^{t_1} t^2 \sin^2\left(\frac{\pi}{3}t\right) dt$$

$$= \lim_{t_1\to\infty} \frac{1}{4t_1} \int_{-t_1}^{t_1} t^2 \left[1 - \cos\left(\frac{2\pi}{3}t\right)\right] dt$$

$$= \lim_{t_1\to\infty} \frac{1}{4t_1} \left[\frac{2t_1^3}{3} - \int_{-t_1}^{t_1} \cos\left(\frac{2\pi}{3}t\right) dt\right] \to \infty$$

Also, $$E = \int_{-\infty}^{\infty} |x_1(t)|^2 dt - \infty$$

Hence, $x_1(t)$ is neither an energy nor a power signal.

(b) Given: $x_2(t) = e^{j\frac{5\pi}{6}t}$

Hence, fundamental period, $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{5\pi/6} = \frac{12}{5}$ secs.

The average power associated with $x_2(t)$ is

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

$$= \frac{1}{T} \int_0^T \left[e^{j\frac{5\pi}{6}t}\right]^2 dt = 1$$

Since, $0 < P < \infty$, $x_2(t)$ is a power signal.

(c) Given: $x_3(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

The average power associated with $x_3(t)$ is

$$P = \lim_{t_1\to\infty} \frac{1}{2t_1} \int_{-t_1}^{t_1} |x_3(t)|^2 dt$$

$$= \lim_{t_1\to\infty} \frac{1}{2t_1} \int_0^{t_1} 1^2 dt$$

$$= \lim_{t_1\to\infty} \frac{t_1}{2t_1} = \frac{1}{2}$$

Since, $0 < P < \infty$, $x_3(t)$ is a power signal.

Example 1.10. Let $x(t)$ be a periodic signal with a fundamental period equal to T. Show that

$$\left|\int_0^T x(t)\right| \le \sqrt{PT}$$

where P is the average power associated with the signal $x(t)$.

Solution: The average power associated with signal $x(t)$ is

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

Since $\left|\int_0^T x(t)dt\right| \leq \int_0^T |x(t)|^2 dt$, the above expression can be written as

$$P \geq \frac{1}{T}\left|\int_0^T x(t)dt\right|^2$$

$$\Rightarrow \quad PT \geq \left|\int_0^T x(t)dt\right|^2$$

Hence, $$\left|\int_0^T x(t)dt\right| \leq \sqrt{PT}$$

Example 1.11. Classify the following signals as energy signals, power signals or neither.

(a) $x(n) = 2^n u(-n)$

(b) $x(n) = \cos(n\pi)$

(c) $x(n) = e^{jn\pi}$

(d) $x(n) = (j)^n + (j)^{-n}$

Solution:

(a)

$$x(n) = 2^n u(-n)$$

$$E \triangleq \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{0} 4^n$$

Putting $n = -m$, we get

$$E = \sum_{m=+\infty}^{0} 4^{-m} = \sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^m = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}J$$

Since E is finite, $x(n)$ is an energy signal.

(b)

$$x(n) = \cos(n\pi)$$

$$\frac{\Omega_0}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2} = \frac{p}{q} = \text{a rational number.}$$

Hence, $$N = q = 2$$

$$x(n) = (1, -1) \text{ for one period.}$$

So, $$P = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2$$

$$= \frac{1}{2}\sum_{n=0}^{1}|x(n)|^2$$

$$= 0.5(1+1) = 1 \text{ W}$$

Since, P is finite, $x(n)$ is a power signal.

(c)

$$x(n) = e^{jn\pi}$$

Comparing with $e^{j\Omega_0 n}$, we get

$$\Omega_0 = \pi$$

$$= \frac{2\pi}{2} = \frac{2\pi}{N}$$

Hence, $x(n)$ is periodic with a period $N = 2$.

$$P = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2$$

$$= \frac{1}{2}\sum_{n=0}^{1} 1 = \frac{1}{2}(1+1) = 1 \text{ W}$$

Since P is finite, $x(n)$ is a power signal.

(d)

$$x(n) = (j)^n + (j)^{-n} = e^{jn\frac{\pi}{2}} + e^{-jn\frac{\pi}{2}}$$

$$= 2\cos\left(\frac{n\pi}{2}\right)$$

$$\Omega_0 = \frac{\pi}{2} = \frac{2\pi}{4} = \frac{2\pi}{N}$$

Hence, $x(n)$ is periodic with $N = 4$

$$x(n) = (2, 0, -2, 0) \quad \text{for one period.}$$

So,

$$P = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2$$

$$= \frac{1}{4}\sum_{n=0}^{3}|x(n)|^2 = \frac{1}{4}(4+4) = 2\text{ W}$$

Since P is finite, $x(n)$ is a power signal.

Example 1.12. The energy in a real-valued signal, $x(n)$, is defined as the sum of the squares of the sequence values:

$$E = \sum_{n=-\infty}^{\infty} x^2(n)$$

Suppose $x(n)$ has an even part $x_e(n)$ equal to

$$x_e(n) = \left(\frac{1}{2}\right)^{|n|}$$

and the energy in $x(n)$ is $E_e = 5$ J, find the energy in the odd part, $x_o(n)$ of the signal $x(n)$.

Solution: Let

$$x(n) = x_e(n) + x_o(n)$$

$$E = \sum_{n=-\infty}^{\infty} x^2(n)$$

$$= \sum_{n=-\infty}^{\infty} [x_e(n) + x_o(n)]^2$$

$$= \sum_{n=-\infty}^{\infty} x_e^2(n) + \sum_{n=-\infty}^{\infty} x_o^2(n) + \sum_{n=-\infty}^{\infty} 2x_e(n)x_o(n)$$

Note that $x_e(n)x_o(n)$ is the product of an even and an odd sequence, the product is odd. Hence,

$$\sum_{n=-\infty}^{\infty} x_e(n)x_o(n) = 0$$

Accordingly, we get

$$E = \sum_{n=-\infty}^{\infty} x_e^2(n) + \sum_{n=-\infty}^{\infty} x_o^2(n)$$

$$= E_e + E_o$$

The next step in the analysis is to find E_e,

$$E_e = \sum_{n=-\infty}^{\infty} x_e^2(n)$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^{2|n|}$$

$$= -1 + 2\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n} = \frac{5}{3}$$

Hence,

$$E_o = E - E_e$$

$$= 5 - \frac{5}{3} = \frac{10}{3}\text{ J}$$

Example 1.13. Consider an energy signal $x(t)$, over the range $-3 \le t \le 3$, with energy $E = 12$ J. Find the range of the following signals and compute their signal energy.

(a) $x(3t)$

(b) $2x(t)$

(c) $x(t-4)$

(d) $x(-t)$

Solution:

(a) $x(3t)$ is compressed by a factor of 3 and covers $-1 \le t \le 1$. Its energy is $E = \frac{12}{3} = 4$ J.

(b) $2x(t)$ is only amplitude-scaled by a factor 2 and covers $-3 \le t \le 3$. Its energy is $E = (2)^2 \times 12 = 48$ J.

(c) $x(t-4)$ is delayed by 4 and covers $1 \le t \le 7$. Its energy is $E = 12$ J.

(d) $x(-t)$ is folded and still covers $-3 \le t \le 3$. Its energy is $E = 12$ J.

Example 1.14. Sketch each of the following signals and find its energy or power as appropriate.

(a) $x(n) = (\overset{\Downarrow}{2}, 4, 1, 1)$

(b) $x(n) = (-3, -2, \overset{\Downarrow}{-1}, 1, 0)$

(c) $x(n) = \cos\left(\frac{n\pi}{2}\right)$

(d) $x(n) = 8(0.5)^n u(n)$

where the arrow at the top or bottom of a sequence indicates the origin.

Solution:

(a) The signal is sketched as shown in Fig. 1.8 and the energy can be calculated as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = 2^2 + 4^2 + 1^2 + 1^2$$

$$= 22 \text{ J}$$

Hence, $x(n)$ is an energy signal.

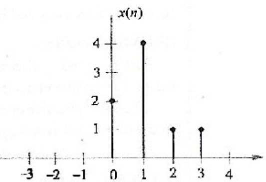


Fig. 1.8

(b) Referring the signal sketched in Fig. 1.9, we can calculate the energy as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= 3^2 + 2^2 + 1^2 + 1^2$$

$$= 9 + 4 + 1 + 1$$

$$= 15 \text{ J}$$

Hence, $x(n)$ is an energy signal.

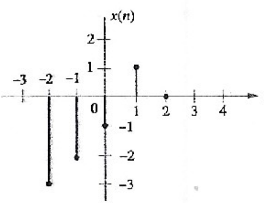


Fig. 1.9

(c) From the given expression, the signal is sketched as shown in Fig. 1.10.

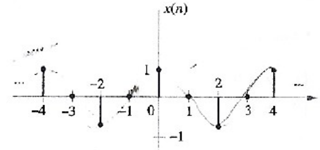


Fig. 1.10

$x(n)$ is periodic with a fundamental period $N = 4$.

Hence,

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

$$= \frac{1}{4}[1 + 1] = 0.5 \text{ W}$$

Hence, $x(n)$ is a power signal.

(d) The given signal is sketched as shown in Fig. 1.11.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} 64(0.25)^n$$

$$= \frac{64}{1 - 0.25} = 85.33 \text{ J}$$

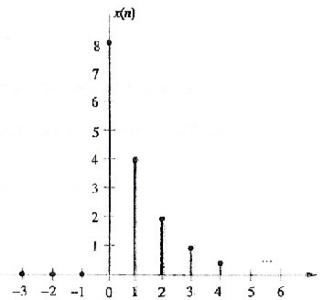


Fig. 1.11

Therefore, $x(n)$ is an energy signal.

Example 1.15. Determine whether the signals given below are energy or power signals. Justify your answers.

(a) $x(t) = A\cos(\omega_0 t + \theta)$

(b) $x(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Solution:

(a) The sinusoidal signal $x(t)$ is periodic with a fundamental period, $T = \dfrac{2\pi}{\omega_0}$.

The average power associated with $x(t)$ is

$$\begin{aligned} P &= \frac{1}{T}\int_0^T |x(t)|^2 dt \\ &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} A^2\cos^2(\omega_0 t + \theta)dt \\ &= \frac{A^2\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \frac{1}{2}[1 + \cos(2\omega_0 t + 2\theta)]dt \\ &= \frac{A^2}{2} \end{aligned}$$

Since, $0 < P < \infty$, $x(t)$ is a power signal.

(b) The average power associated with $x(n)$ is

$$\begin{aligned} P &= \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x(n)|^2 \\ &= \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=0}^{N} 1^2 \\ &= \lim_{N\to\infty} \frac{1}{2N+1}(N+1) = \frac{1}{2} < \infty \end{aligned}$$

Thus, $x(n)$ is a power signal.

1.3 Basic operations on signals

A. *Operations performed on dependent variables*

(i) Amplitude scaling

Let $x(t)$ be a continuous-time signal. The signal $y(t)$ resulting from amplitude scaling is given by

$$y(t) = C\,x(t)$$

where, C is the scaling factor.

A physical example of a device that performs amplitude scaling is an electronic amplifier.

(ii) Addition

Let $x_1(t)$ and $x_2(t)$ denote, a pair of continuous-time signals. The signal $y(t)$ obtained by addition of $x_1(t)$ and $x_2(t)$ is defined by

$$y(t) = x_1(t) + x_2(t)$$

A physical example of a device that performs addition is a frequency mixer, which combines low frequency and high frequency signals.

(iii) Multiplication

Let $x_1(t)$ and $x_2(t)$ denote, a pair of continuous-time signals. The signal $y(t)$ resulting from the multiplication of $x_1(t)$ by $x_2(t)$ is defined by $y(t) = x_1(t)x_2(t)$.

The multiplicative operation is often encountered in analog communication, where an audio frequency signal is multiplied by a high frequency sinusoid known as *carrier*. The resulting signal is known as *amplitude modulated wave*.

(iv) Differentiation

Let $x(t)$ denote, a continuous-time signal. The derivative of $x(t)$ with respect to time is defined by

$$y(t) = \frac{dx(t)}{dt}$$

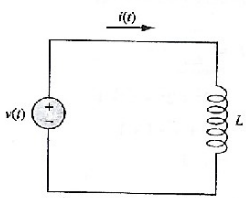


Fig. 1.12: *Inductor demonstrating the concept of differentiation.*

A physical device that performs differentiation is an inductor. The voltage across the inductor shown in Fig. 1.12 is given by,

$$v(t) = L\frac{di(t)}{dt}$$

(v) Integration

Let $x(t)$ denote a continuous-time signal. The integral of $x(t)$ with respect to t is defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

This type of operation is encountered in a capacitor. The voltage across the capacitor shown in Fig. 1.13 is given by

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

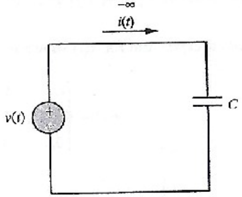


Fig. 1.13: *Capacitor demonstrating the concept of integration.*

B. *Operations performed on the independent variable*

(i) Time scaling

Let $x(t)$ be a continuous-time signal. The signal $y(t)$ obtained by scaling independent variable, t, by a factor a is given by $y(t) = x(at)$.

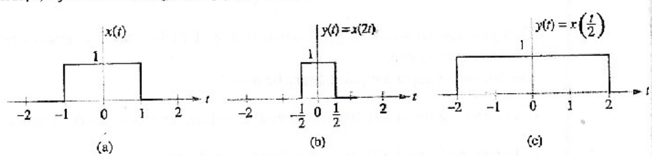


Fig. 1.14: (a) *Rectangular pulse $x(t)$,* (b) *Compression by a factor of 2* (c) *Expansion by a factor of 2.*

Fig. 1.14 demonstrate the concept of time scaling. Note that, when $a > 1$, $y(t)$ is the compressed version of $x(t)$ and if $a < 1$, $y(t)$ is the expanded version of $x(t)$. If we think $x(t)$ as the output of a videotape recorder, then $x(2t)$ is the signal obtained when the recording is played at two times the speed at which it was recorded, and $x(t/2)$ is the signal obtained when the recording is played back at half speed.

The mathematical description of $x(t)$ shown in Fig. 1.14(a) is

$$x(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$x(2t) = \begin{cases} 1, & -1 < 2t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

This justifies Fig. 1.14(b). Similarly,

$$x\left(\frac{t}{2}\right) = \begin{cases} 1, & -1 < \frac{t}{2} < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & -2 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

The above equation justifies the expansion demonstrated by Fig. 1.14(c).

In the discrete-time case, we write $y(n) = x(mn)$, $m > 0$ and m is an integer.

Fig. 1.15(a) shows an arbitrary discrete-time signal $x(n)$ and Fig. 1.15(b) shows the time-scaled version of $x(n)$.

Comparing Figs. 1.15(a) and (b), we find that in $y(n)$, samples of amplitude 1 are totally missing. Hence, time scaling of a discrete-time signal leads to inherent loss of information.

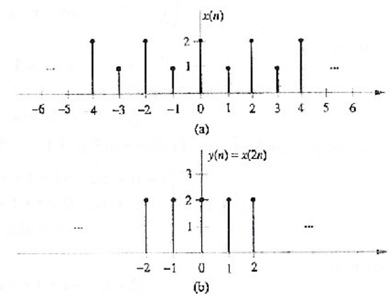


Fig. 1.15: (a) *Discrete-time signal $x(n)$ and* (b) *Time-scaled version of $x(n)$.*

(ii) Time shifting

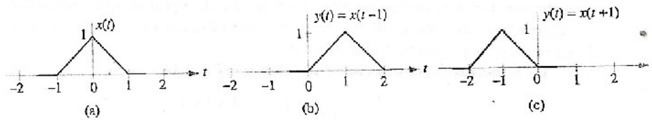


Fig. 1.16: (a) *Triangular pulse, $x(t)$,* (b) *$x(t)$ translated to right by 1 time unit,* (c) *$x(t)$ translated to left by 1 time unit.*

Let $x(t)$ be a continuous-time signal. Replacing t by $t + b$ results in a time shifted signal $y(t)$ defined below:

$$y(t) = x(t + b)$$

Figs. 1.16(a) and (b) denote the time shifted versions of $x(t)$ for $b < 0$ and $b > 0$. If $b < 0$, $x(t)$ is shifted to right by an amount, b secs and if $b > 0$, $x(t)$ is translated to left by b secs.

The mathematical description of Fig. 1.18(a) is

$$x(t) = \begin{cases} 1 + t, & -1 < t < 0 \\ 1 - t, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$x(t-1) = \begin{cases} 1 + (t-1), & -1 < t - 1 < 0 \\ 1 - (t-1), & 0 < t - 1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

or, equivalently,

$$x(t-1) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

This justifies the sketch of $x(t - 1)$ shown in Fig. 1.16(b). Similarly,

$$x(t+1) = \begin{cases} 1 + (t+1), & -1 < t + 1 < 0 \\ 1 - (t+1), & 0 < t + 1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

or, equivalently,

$$x(t+1) = \begin{cases} 2 + t, & -2 < t < -1 \\ -t, & -1 < t < 0 \\ 0, & \text{otherwise} \end{cases}$$

This justifies the sketch of $x(t + 1)$ shown in Fig. 1.16(c). Similar analysis holds good for discrete-time signals also.

(iii) Reflection

Let $x(t)$ denote a continuous-time signal. The reflected version of $x(t)$ is obtained by replacing t with $-t$. The signal, $y(t) = x(-t)$ represents a reflected version of $x(t)$ about the vertical axis.

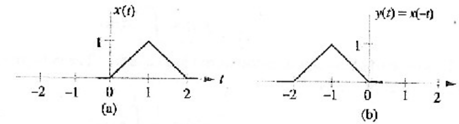


Fig. 1.17: (a) *Triangular pluse, $x(t)$ and* (b) *Reflection of $x(t)$ about vertical axis.*

The mathematical expression for $x(t)$ shown in Fig. 1.17(a) is

$$x(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$y(t) = x(-t) = \begin{cases} -t, & 0 < -t < -1 \\ 2 + t, & 1 < -t < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} t + 2, & -2 < t < -1 \\ -t, & -1 < t < 0 \\ 0, & \text{otherwise} \end{cases}$$

This justifies the sketch of $y(t)$ shown in Fig. 1.11(b). Similar analysis holds good for discrete-time signals also.

The following important points may be noted.

1. For an even signal $x(t)$, the signal and its reflected version $x(-t)$ are identical.
2. For an odd signal $x(t)$, the signal and its reflected version $x(-t)$ are negative of each other. These properties are illustrated in Fig. 1.18. A similar concept holds good for discrete-time signals also.

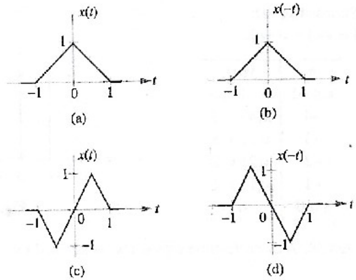


Fig. 1.18: (a) & (b) *Even signla $x(t)$ and its reflection $x(-t)$*
(c) & (d) *odd signal $x(t)$ and its reflection $x(-t)$.*

Example 1.16. A continuous-time signal $x(t)$ is shown in Fig. 1.19. Sketch and label each of the following signals.

(a) $x(t-2)$

(b) $x(2t)$

(c) $x\left(\frac{t}{2}\right)$

(d) $x(-t)$

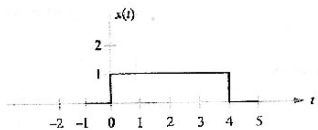


Fig. 1.19

Solution:

(a) Sketch of $x(t-2)$:

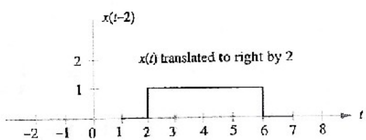


Fig. 1.20

(b) Sketch of $x(2t)$:

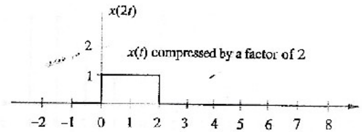


Fig. 1.21

(c) Sketch of $x\left(\frac{t}{2}\right)$:

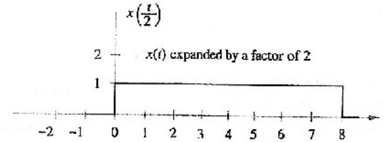


Fig. 1.22

(d) Sketch of $x(-t)$:

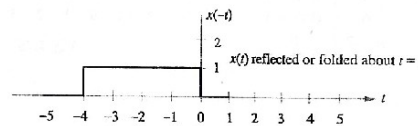


Fig. 1.23

Example 1.17. A discrete-time signal $x(n)$ is shown in Fig. 1.24. Sketch and label each of the following signals.

(a) $x(n-2)$

(b) $x(2n)$

(c) $x(-n)$

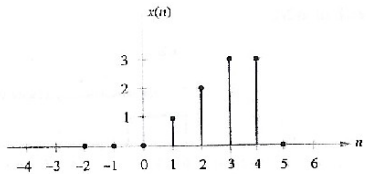


Fig. 1.24

Solution:

(a) Sketch of $x(n-2)$:
Let $y(n) = x(n-2)$. Then

n	$y(n)$
$n \le 2$	0
3	$x(1) = 1$
4	$x(2) = 2$
5	$x(3) = 3$
6	$x(4) = 3$
$n \ge 7$	0

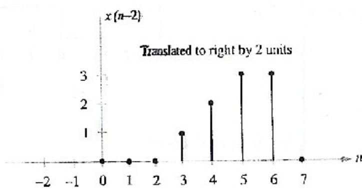


Fig. 1.25

(b) Sketch of $x(2n)$:
Let $y(n) = x(2n)$.

n	$y(n)$
$n \le 0$	0
1	$x(2) = 2$
2	$x(4) = 3$
$n \ge 3$	0

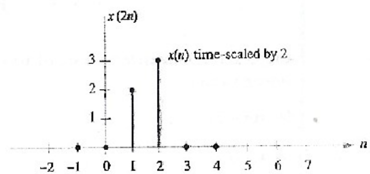


Fig. 1.26

(c) Sketch of $x(-n)$:
Let $y(n) = x(-n)$.

n	$y(n)$
$n \le -5$	0
-4	$x(4) = 3$
-3	$x(3) = 3$
-2	$x(2) = 2$
-1	$x(1) = 1$
$n \ge 0$	0

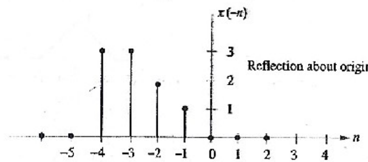


Fig. 1.27

Example 1.18. A discrete-time signal $x(n)$ is defined by

$$x(n) = \begin{cases} 1, & n = 1 \\ -1, & n = -1 \\ 0, & n = 0 \ \& \ |n| > 1 \end{cases}$$

Find the **composite signal** $y(n)$ defined in terms of $x(n)$ by

$$y(n) = x(n) + x(-n)$$

Solution:

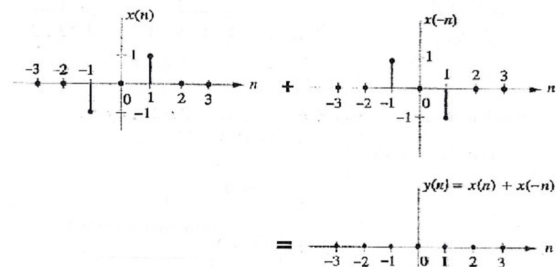


Fig. 1.28

That is, $y(n) = 0$ for all n.

C. Precedence rule for time-shifting and time scaling

Let

$$y(t) = x(at + b) \tag{1.17}$$

The above relation between $y(t)$ and $x(t)$ must satisfy the following conditions.

$$y(0) = x(b) \tag{1.18}$$

and

$$y\left(\frac{-b}{a}\right) = x(0) \tag{1.19}$$

The above relations (1.18) and (1.19) are useful checks in terms of corresponding values of $x(t)$.

As an example, let us take, $y(t) = x(2t + 3)$. First, let us perform time-shifting operation on $x(t)$, resulting in an intermediate signal $v(t)$ defined by

$$v(t) = x(t + 3)$$

Next, the time-scaling operation is performed on $v(t)$. This replaces t by $2t$, resulting in the desired output,

$$y(t) = v(2t) = x(2t + 3)$$

The above operations are depicted in Fig. 1.29.

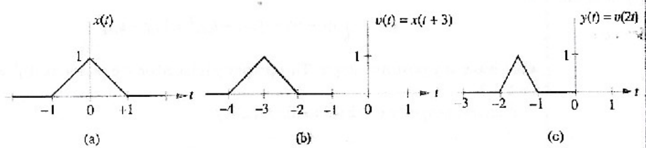


Fig. 1.29: (a) *triangular waveform, x(t),* (b) *time-shifted waveform, v(t),* (c) *time-scaled waveform, y(t).*

Verification:

$$y(0) = x(3) = 0$$

$$y\left(-\frac{3}{2}\right) = x(0) = 1$$

Hence, the precedence rule is to do the time-shifting operation first and then the time-scaling operation.

Example 1.19. A discrete-time signal $x(n)$ is described by

$$x(n) = \begin{cases} 1, & n = 1, 2, 3 \\ -1, & n = -1, -2, -3 \\ 0 & n = 0, |n| > 3 \end{cases}$$

Find $y(n) = x(2n + 2)$.

Solution:

***Step 1*:** Sketch $x(n)$.

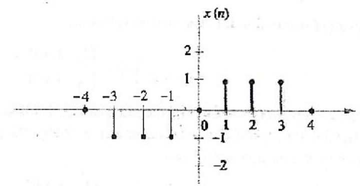


Fig. 1.30

***Step 2*:** Sketch $v(n) = x(n + 2)$.

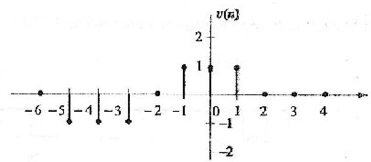


Fig. 1.31

Step 3 : Sketch $y(n) = v(2n)$.

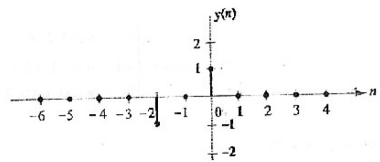


Fig. 1.32

1.4 Basic Continuous-Time Signals

1.4.1 Unit step function

The unit step function, $u(t)$ is defined as follows:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{1.20}$$

The unit step function, $u(t)$ is sketched in Fig. 1.33(a). Note that $u(t)$ is discontinuous at $t = 0$. In order to avoid the confusion at $t = 0$ regarding the amplitude of $u(t)$, it is often convenient to define $u(t)$ as follows:

$$u(t) = \begin{cases} 1, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

Similarly, the shifted unit step function is defined as follows:

$$u(t + t_0) = \begin{cases} 1, & t + t_0 > 0 \quad \text{or} \quad t > -t_0 \\ 0, & t + t_0 < 0 \quad \text{or} \quad t < -t_0 \end{cases}$$

The shifted unit step function is shown in Fig. 1.33(b).

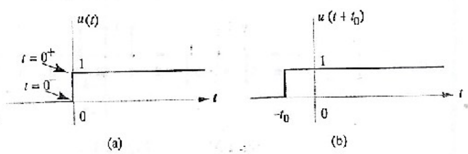


Fig. 1.33: (a) *Unit step function,* (b) *Shifted unit step function.*

The step function is used as a building block to generate several discontinuous waveforms. An example to this effect is shown in Fig. 1.34.

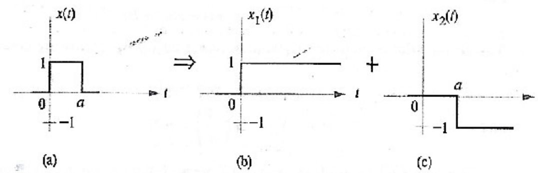


Fig. 1.34: (a) *Discontinuous waveform,* (b) *Unit step function,* (c) *Shifted and inverted unit step function.*

From Fig. 1.34,

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) \\ &= u(t) = u(t - a) \end{aligned}$$

The unit step function has the property

$$u(t - t_0) = [u(t - t_0)]^2 = [u(t - t_0)]^k$$

with k being any positive integer. This property is based on the relations $(0)^k = 0$ and $(1)^k = 1$, $k = 1, 2, 3 \ldots$.

A second property is related to time scaling:

$$u(at - t_0) = u\left(t - \frac{t_0}{a}\right); \quad a \neq 0$$

This is because, by the defining equation, we have

$$u(at - t_0) = \begin{cases} 1, & at - t_0 > 0 \\ 0, & at - t_0 < 0 \end{cases}$$

or, equivalently,

$$u(at - t_0) = \begin{cases} 1, & t > \frac{t_0}{a} \\ 0, & t < \frac{t_0}{a} \end{cases} = u\left(t - \frac{t_0}{a}\right)$$

1.4.2 Unit impulse function

Engineers have found very good use for $j = \sqrt{-1}$ inspite of the fact that it is not a real number and does not appear in nature. Electrical and electronic engineering analysis makes use of j extensively. In the same token, engineers have found tremendous use for the unit impulse function, even though this function cannot exist in nature. We introduce a unit impulse function with its definition.

The continuous-time unit impulse function is defined as follows:

$$\delta(t) = 0, \quad t \neq 0 \tag{1.21}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{1.22}$$

From Eq. 1.21, we find that, $\delta(t) = \delta(-t)$.

The inference is that $\delta(t)$ is an even function of time, t. The unit impulse function $\delta(t)$ is sketched in Fig. 1.35. Normally, at the arrow of an impulse function, we write a number. This number corresponds to the area under the pulse, also known as the weight of the impulse, and not the amplitude. The amplitude of the impulse function of $t = 0$ is unbounded.

We can visualize an impulse as a tall, narrow rectangular pulse of unit area, as shown in Fig. 1.36. The width of the rectangular pulse is a very small value ($T \to 0$). Consequently, its height is of very large value ($\frac{1}{T}$). The unit impulse therefore can be regarded as a rectangular pulse with a width that has become infinitesimally small and a height that has become infinitely large, maintaining the overall area at unity.

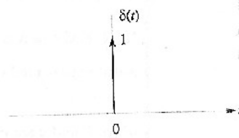


Fig. 1.35: *Unit impulse function.*

That is,

$$\delta(t) = \lim_{T\to 0} g_T(t)$$

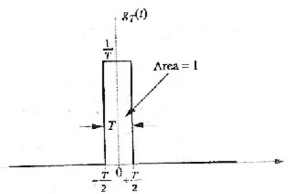


Fig. 1.36: *Approximation of unit impulse by a rectangular pulse.*

Following are some of the important properties of the unit impulse function:

(i) Product property

Let us consider, what happens when we multiply the unit impulse $\delta(t)$ by a function $\phi(t)$ that is known to be continuous at $t = 0$. Since the impulse exists only at $t = 0$, and the value $\phi(t)$ at $t = 0$ is $\phi(0)$, we obtain

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

Similarly,

$$\phi(t)\delta(t - t_0) = \phi(t_0)\delta(t - t_0)$$

(ii) Sifting property of the unit impulse function

$$\int_{t_1}^{t_2} \phi(t)\delta(t - t_0)dt = \begin{cases} \phi(t_0), & t_1 \le t_0 \le t_2 \\ 0, & \text{otherwise} \end{cases}$$

or

$$\int_{t=-\infty}^{\infty} \phi(t)\delta(t - t_0)dt = \phi(t_0)$$

That is, the function $\phi(t)$ gets evaluated at the point where the impulse is located.

(iii) Time-scaling property

$$\delta(at) = \frac{1}{a}\delta(t), \quad a > 0$$

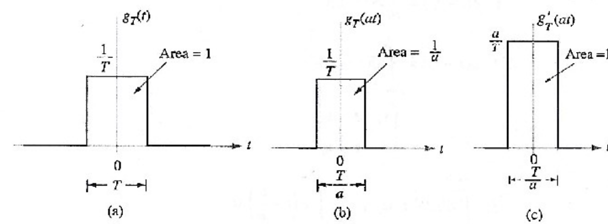


Fig. 1.37

A rectangular pulse and its scaled versions are shown in Fig. 1.37.

We know that,

$$\delta(t) = \lim_{T\to 0} g_T(t)$$

Substituting $t = at$, we get

$$\delta(at) = \lim_{T\to 0} g_T(at) \tag{1.23}$$

From Figs. 1.37(b) and 1.37(c), we can write

$$g'_T(at) = a g_T(at) \tag{1.24}$$

Substituting Eq. 1.24 in Eq. 1.23, we get

$$\delta(at) = \lim_{T\to 0} \frac{1}{a} g'_T(at)$$

Since, the area under the function $g'_T(at)$ is unity, it follows that,

$$\lim_{T\to 0} g'_T(at) = \delta(t)$$

Therefore,

$$\delta(at) = \lim_{T\to 0} \frac{1}{a}\delta(t)$$

Hence, the proof.
A more generalized scaling property for solving problems is given below:

$$\delta(at+b) = \frac{1}{|a|}\delta\left(t+\frac{b}{a}\right)$$

It may be noted that an impulse function, $\delta(t)$ is the derivative of the step function.

Additional Properties:

1. $\delta(t-t_0) = \frac{d}{dt}u(t-t_0)$
2. $u(t-t_0) = \int_{-\infty}^{t} \delta(\tau - t_0)d\tau$
   $= \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$
3. $\int_{-\infty}^{\infty} \delta(at-t_0)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta\left(t-\frac{t_0}{a}\right)dt$

1.4.3 Ramp function

The unit step function is the integral of unit impulse function with respect to time. By the sam token, the integral of the step function $u(t)$ is a ramp function of unit slope shown in Fig. 1.3 Mathematically, a ramp function is defined as follows:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Equivalently, we may write

$$r(t) = tu(t)$$

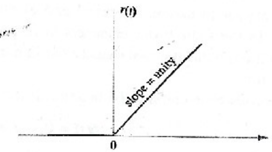


Fig. 1.38: *Unit ramp function.*

The ramp function is obtained by integrating the unit step function:

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

In contrast to unit step function, the ramp function is continuous at $t = 0$. Time scaling a unit ramp function by a factor α corresponds to a ramp function with slope α. An example of a ramp function is the linear-sweep waveform of a cathode-ray tube.

1.4.4 Exponential signals

A real exponential signal in its most general form is written as

$$x(t) = Be^{at}$$

where B and a are real parameters.

The parameter B is the amplitude of the signal $x(t)$ at $t = 0$.

If $a < 0$, $x(t)$ is said to be a decaying exponential. On the other hand, if $a > 0$, $x(t)$ is said to be a growing exponential. Figs. 1.39(a) and 1.39(b) illustrate the decaying and growing exponential functions respectively.

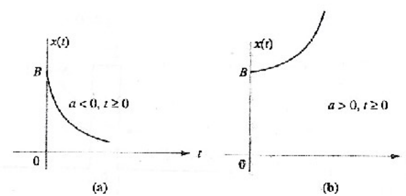


Fig. 1.39: (a) *Decaying exponential function* (b) *Growing exponential function.*

1.4.5 Sinusoidal signals

General expression for a continuous-time sinusoid is

$$x(t) = A\cos[\omega t + \phi]$$

where A is the amplitude, ω is the angular frequency in radians/sec and ϕ is the phase angle in radians.

The period of a sinusoidal signal is defined by, $T = \frac{2\pi}{\omega}$. A continuous-time sinusoid is always periodic with a period $= T$. This can be proved as follows:

Consider,

$$\begin{aligned} x(t+T) &= A\cos[\omega(t+T)+\phi] \\ &= A\cos[\omega t + \omega T + \phi] \\ &= A\cos[\omega t + 2\pi + \phi] \\ &= A\cos[\omega t + \phi] \\ &= x(t) \end{aligned}$$

which defines the condition for a periodic signal. Therefore, a continuous-time sinusoidal signal is always periodic.

1.4.6 Exponentially damped sinusoidal signals

The multiplication of a sinusoidal signal, $A\sin(\omega t + \theta)$ by a real-valued decaying exponential signal, $e^{-\alpha t}$ results in a new signal referred to as exponentially damped sinusoidal signal as shown in Fig. 1.40.

That is,

$$x(t) = Ae^{-\alpha t}\sin(\omega t + \theta), \quad \alpha > 0$$

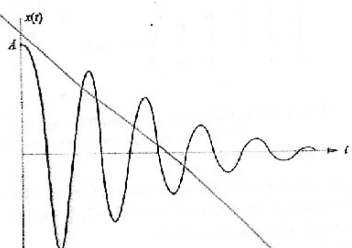
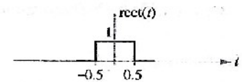


Fig. 1.40: *Exponentially damped sinusoidal signal shown for $t \geq 0$.*

1.4.7 Pulse signals

The rectangular pulse rect(t) and triangular pulse tri(t) are defined as follows:

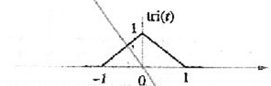
$$\text{rect}(t) = \begin{cases} 1, & |t| < 0.5 \\ 0, & \text{elsewhere} \end{cases}$$



Height = 1, Area = 1, Width = 1.

Fig. 1.41

$$\text{tri}(t) = \begin{cases} 1 - |t|, & |t| \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$



Height = 1, Area = 1, Width = 2.

Fig. 1.42

Figs. 1.41 and 1.42 depict rect(t) and tri(t) respectively. Both the pulses exhibit even symmetry. It may be noted that the signal, $x(t) = \text{rect}\left(\frac{t-b}{a}\right)$ described a rectangular pulse of width a, centered at $t = b$. Similarly, $y(t) = \text{tri}\left(\frac{t-b}{a}\right)$ describes a triangular pulse of width $2a$ centered at $t = b$.

1.5 Basic Discrete-Time Signals

1.5.1 Step function

The discrete-time version of the step function $u(n)$ is defined by

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

and sketched as shown in Fig. 1.43.

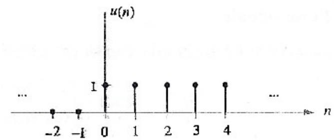


Fig. 1.43: *Discrete-time unit step function.*

1.5.2 Impulse function

The discrete-time version of the impulse function is shown in Fig. 1.44 and defined by

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

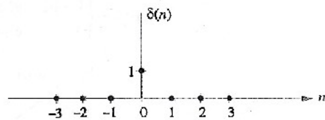


Fig. 1.44: *Discrete-time impulse function.*

Following are some of the important properties of unit impulse funciton.

(i) Product property

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

(ii) Sifting property

$$\sum_{n=N_1}^{N_2} x(n)\delta(n-k) = \begin{cases} x(k), & N_1 \le k \le N_2 \\ 0, & \text{otherwise} \end{cases}$$

or $$\sum_{n=-\infty}^{\infty} x(n)\delta(n-k) = x(k)$$

1.5.3 Ramp function

The discrete-time version of a ramp function is defined by

$$r(n) = \begin{cases} n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

or equivalently, $r(n) = nu(n)$. Fig. 1.45 shows $r(n)$.

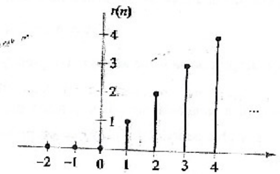


Fig. 1.45: *Discrete-time unit ramp function.*

1.5.4 Exponential function

In discrete-time, a real exponential signal is written as

$$x(n) = Br^n \quad \text{where,} \quad {}^{\ddagger}r = e^{\alpha T}$$

If $0 < r < 1$, $x(n)$ is called decaying exponential sequence and if $r > 1$, then $x(n)$ is known as the growing exponential sequence. Fig. 1.46 shows decaying and growing exponential sequences.

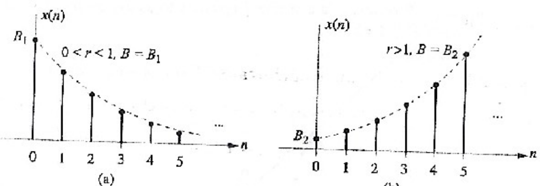


Fig. 1.46: *(a) Decaying exponential sequence shown for $n \ge 0$,*
(b) Growing exponential sequence shown for $n \ge 0$

Let us next consider the case for $r < 0$. Consider for example, $x(n) = (-2)^n$. Starting with $n = 0$, the number sequence for $x(n)$ is $1, -2, 4, -8, 16, -32, \ldots$ Hence, the number sequence is exponential with alternating sign.

‡Let us consider a continuous time signal, $x(t) = Be^{\alpha t}$. Letting $t = nT$, we get $x(nT) = x(n) = Be^{\alpha nT} = Br^n$, where $r = e^{\alpha T}$.

1.5.5 Sinusoidal signals

The discrete-time version of a sinusoidal signal is written as

$$x(n) = A\cos[\Omega n + \phi]$$

The period of a discrete-time sinusoid is measured in samples. Let the period of $x(n)$ be N. Then,

$$x(n + N) = A\cos[\Omega n + \Omega N + \phi]$$

For $x(n)$ and $x(n + N)$ to be identical,

$$\Omega N = 2\pi m \quad \text{radians}$$

$$\Rightarrow \quad \Omega = \frac{2\pi m}{N}; \quad m \text{ and } N \text{ are integers}$$

The smallest positive integer value of m gives the fundamental period.

Unlike continuous-time sinusoidal signals, not all discrete-time sinusoidal signals with ar-bitrary values of Ω are periodic.

Alternatively, a discrete-time sinusoid, $x(n)$ is periodic, if and only if $\frac{\Omega}{2\pi}$ is rational. Tha is, $\frac{\Omega}{2\pi} = \frac{p}{q}$ where p and q are integers. If $\frac{\Omega}{2\pi}$ is rational, then fundamental period, $N =$ samples. On the other hand, if $\frac{\Omega}{2\pi}$ is not rational, then $x(n)$ is not periodic.

1.5.6 Exponentially damped sinusoidal signals

The discrete-time version of the exponentially damped sinusoidal signal is described by

$$x(n) = Br^n \sin[\Omega n + \phi]$$

For the signal to decay exponentially with time, the parameter r must lie in the rang $0 < |r| < 1$.

1.5.7 Pulse signals

The discrete versions of the rectangular pulse $\text{rect}\left(\frac{n}{2N}\right)$ and triangular pulse $\text{tri}\left(\frac{n}{N}\right)$ are giv by

$$\text{rect}\left(\frac{n}{2N}\right) = \begin{cases} 1, & |n| \le N \\ 0, & \text{elsewhere} \end{cases}$$

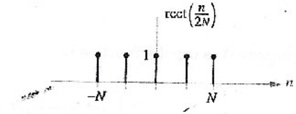


Fig. 1.47

The signal $\text{rect}\left(\frac{n}{2N}\right)$ has $(2N + 1)$ unit samples over $-N \le n \le N$ as shown in Fig. 1.47.

$$\text{tri}\left(\frac{n}{N}\right) = \begin{cases} 1 - \frac{|n|}{N}, & |n| \le N \\ 0, & \text{elsewhere} \end{cases}$$

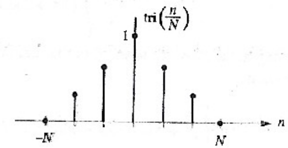


Fig. 1.48

The signal $x(n) = \text{tri}\left(\frac{n}{N}\right)$ has $(2N+1)$ samples over $-N \le n \le N$, with the terminal samples $x(N)$ and $x(-N)$ being zero as shown in Fig. 1.48.

Example 1.20. Evaluate the following integrals:

(a) $\int_{-1}^{1} (3t^2 + 1)\delta(t)dt$

(b) $\int_{1}^{2} (3t^2 + 1)\delta(t)dt$

(c) $\int_{-\infty}^{\infty} (t^2 + \cos \pi t)\delta(t - 1)dt$

(d) $\int_{-\infty}^{\infty} e^{-t}\delta(2t - 2)dt$

(e) $\int_{-\infty}^{\infty} (t - 1)\delta\left(\frac{2}{3}t - \frac{3}{2}\right)dt$

Solution:

(a) Sifting property of an impulse function is

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

Hence,

$$\int_{-1}^{+1} (3t^2+1)\delta(t)dt = 3t^2+1\Big|_{t=0} = 1$$

(b)

$$\int_{1}^{2} (3t^2+1)\delta(t)dt = 0$$

The integral vanishes due to the fact that the impulse $\delta(t)$ does not appear in the range of integration.

(c)

$$\int_{-\infty}^{\infty} [t^2+\cos\pi t]\delta(t-1)dt = t^2+\cos\pi t\Big|_{t=1} = 1+\cos\pi = 0$$

(d)

$$\int_{-\infty}^{\infty} e^{-t}\delta(2t-2)dt = \int_{-\infty}^{\infty} e^{-t}\delta[2(t-1)]dt$$

Since,

$$\delta(at) = \frac{1}{a}\delta(t), \quad a>0$$

we get,

$$\int_{-\infty}^{\infty} e^{-t}\frac{1}{2}\delta(t-1)dt = \frac{1}{2}e^{-t}\Big|_{t=1} = \frac{1}{2e}$$

(e) Recall : $\delta(at+b) = \frac{1}{|a|}\delta\left(t+\frac{b}{a}\right)$

Hence,

$$\int_{-\infty}^{\infty} (t-1)\,\delta\left(\frac{2}{3}t-\frac{3}{2}\right)dt = \int_{-\infty}^{\infty} (t-1)\frac{3}{2}\delta(t-1)dt$$

$$= \frac{3}{2}(t-1)\Big|_{t=1} = 0$$

END OF MODULE1

1.6 Systems Viewed as Interconnections of Operations

A system is a process for which cause-and-effect relations exist. The cause is the system input signal, the effect is the system output signal, and the relations are expressed as equations. An example of a physical system is the electric heater. The input signal is the ac voltage, $v(t)$, applied to the heater and the output signal is the temperature, $\theta(t)$, of a certain point in space in the vicinity of the heater. Such a system is shown by the block diagram of Fig. 1.49.

In mathematical terms, a system may be viewed as an interconnection of operations that transforms an input signal into an output signal with properties different from those of the input signal. Let the overall operator $T\{\cdot\}$ denote the action of a system. Then, $T\{\cdot\}$ operates on the input of the system to yield an output signal.

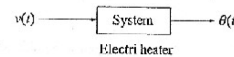


Fig. 1.49: *Electric heater.*

That is,

$$y(t) = T\{x(t)\}$$

We want to emphasize the notation $T\{\cdot\}$ indicates a transformation. This notation does not indicate a function. That is, $T\{x(t)\}$ is not a mathematical function into which we substitute $x(t)$ and directly evaluate $y(t)$. The mathematical model of a system is as shown in Fig. 1.50

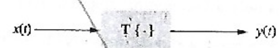


Fig. 1.50: *Continuous-time system viewed as an operator.*

Similarly, for a discrete-time case we may write,

$$y(n) = T\{x(n)\}$$

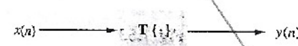


Fig. 1.51: *Discrete-time system viewed as an operator.*

In order to understand the concept of system as an operator, $T\{\cdot\}$, consider a discrete-time system characterized by an input-output relation defined by $y(n) = [x(n) + x(n-1)]\frac{1}{2}$.

Let s^k denote a system that shifts the input $x(n)$ by k times to produce an output signal equal to $x(n-k)$. Accordingly,

$$T\{\cdot\} = \frac{1}{2}[1+s]$$

Operator, $T\{\cdot\}$ could be realized in different forms as depicted in Figs. 1.52(a) and (b).