CENG 3420
Computer Organization & Design

Lecture 09: Floating Numbers

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Floating Point Number

Scientific notation: 6.6254×10^{-27}

- A normalized number of certain accuracy (e.g. 6.6254 is called the mantissa)
- Scale factors to determine the position of the decimal point (e.g. 10^{-27} indicates position of decimal point and is called the exponent; the **base** is implied)
- Sign bit

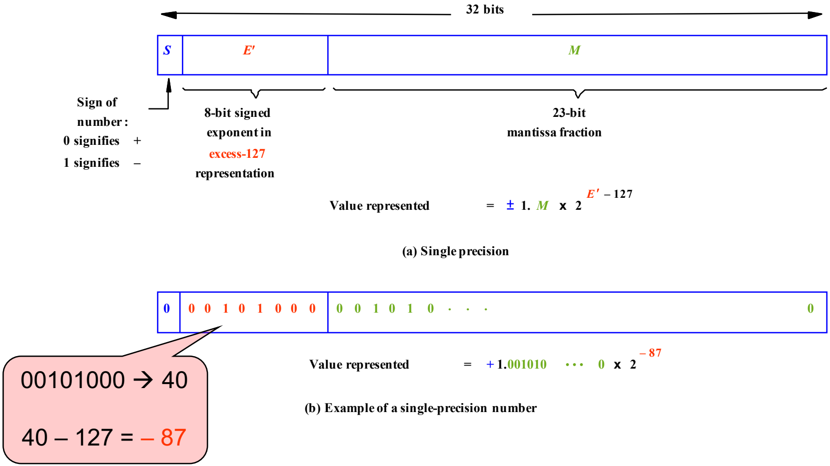
Normalized Form

- Floating Point Numbers can have multiple forms, e.g.

$$\begin{aligned} 0.232 \times 10^4 &= 2.32 \times 10^3 \\ &= 23.2 \times 10^2 \\ &= 2320. \times 10^0 \\ &= 232000. \times 10^{-2} \end{aligned}$$

- It is desirable for each number to have a unique representation => Normalized Form
- We normalize Mantissa's in the Range $[1..R)$, where R is the Base, e.g.:
  - $[1..2)$ for BINARY
  - $[1..10)$ for DECIMAL

32-bit, float in C / C++ / Java

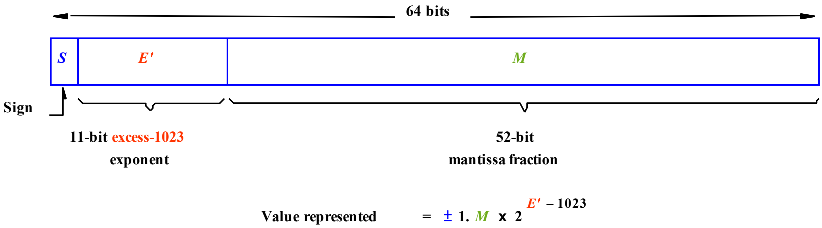


(a) Single precision

(b) Example of a single-precision number

IEEE Standard 754 Double Precision

64-bit, float in C / C++ / Java



(c) Double precision

Question:

What is the IEEE single precision number 40C0 0000_{16} in decimal?

Binary : Online cal.

Sign : +

Exponent : $129 - 127 = +2$

Mantissa : $1.1000000\ldots_2 \rightarrow 1.5_{10} \times 2^{+2}$

Decimal : $+6.0_{10}$

Question:

What is -0.5_{10} in IEEE single precision binary floating point format?

Binary : $1.0 \ldots \times 2^{-1}$

Exponent : $127 + (-1)$: 01111110

Sign : 1

Mantissa : 1. 000 0000 0000 0000 0000 0000

Bin. representation: 1 01111110 000 0000 ...

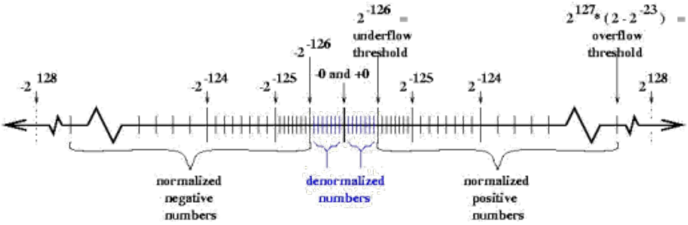
Ref: IEEE Standard 754 Numbers

- Normalized +/– 1.d…d x 2^{exp}
- ***Denormalized*** +/– 0.d…d x 2^{min_exp} → to represent *near-zero* numbers
  e.g. + 0.0000…0000001 x 2^{-126} for Single Precision

Format	# bits	# significant bits	macheps	# exponent bits	exponent range
Single	32	1+23	2^{-24} (~10^{-7})	8	2^{-126} – 2^{+127} (~$10^{\pm 38}$)
Double	64	1+52	2^{-53} (~10^{-16})	11	2^{-1022} – 2^{+1023} (~$10^{\pm 308}$)
Double Extended	>=80	>=64	<=2^{-64}(~10^{-19})	>=15	2^{-16382} – 2^{+16383} (~$10^{\pm 4932}$)

(Double Extended is *80 bits on all Intel machines*)

macheps = Machine Epsilon = ε_{mach} **=** $2^{-(\#\ \text{significand bits})}$



Special Values

Exponents of all 0's and all 1's have special meaning

- E=0, M=0 represents 0 (sign bit still used so there is ± 0)
- E=0, M$\neq$0 is a denormalized number $\pm 0.M \times 2^{-127}$ (smaller than the smallest normalized number)
- E=All 1's, M=0 represents $\pm$`Infinity`, depending on Sign
- E=All 1's, M$\neq$0 represents `NaN`

Other Features

+, -, x, /, sqrt, remainder, as well as conversion to and from integer are correctly rounded

- As if computed with infinite precision and then rounded
- Transcendental functions (that cannot be computed in a finite number of steps e.g., sine, cosine, logarithmic, , e, etc.) may not be correctly rounded

Exceptions and Status Flags

- Invalid Operation, Overflow, Division by zero, Underflow, Inexact

Floating point numbers can be treated as "integer bit-patterns" for comparisons

- If Exponent is all zeroes, it represents a denormalized, very small and near (or equal to) zero number
- If Exponent is all ones, it represents a very large number and is considered infinity (see next slide.)

Dual Zeroes: +0 (0x00000000) and -0 (0x80000000): they are treated as the same

`Infinity` is like the mathematical one

- `Finite` / `Infinity` $\rightarrow$ 0
- `Infinity` $\times$ `Infinity` $\rightarrow$ `Infinity`
- Non-zero / 0 $\rightarrow$ `Infinity`
- $\texttt{Infinity}^{\{\texttt{Finite}\ \text{or}\ \texttt{Infinity}\}} \rightarrow \texttt{Infinity}$

`NaN` (Not-a-Number) is produced whenever a limiting value cannot be determined:

- `Infinity` - `Infinity` $\rightarrow$ `NaN`
- `Infinity` / `Infinity` $\rightarrow$ `NaN`
- 0 / 0 $\rightarrow$ `NaN`
- `Infinity` $\times$ 0 $\rightarrow$ `NaN`
- If x is a `NaN`, $x \neq x$
- Many systems just store the result quietly as a `NaN` (all 1's in exponent), some

Inaccurate Floating Point Operations

- E.g. Find 1st root of a quadratic equation
  - r = (–b + sqrt(b*b – 4*a*c)) / (2*a)

Sparc processor, Solaris, gcc 3.3 (ANSI C),

```
Expected Answer   0.00023025562642476431
double            0.00023025562638524986
float             0.00024670246057212353
```

- Problem is that if c is near zero,

$$\mathrm{sqrt}(b*b - 4*a*c) \approx b$$

- Rule of thumb: use the highest precision which does not give up too much speed

- (a – b) is inaccurate when a ≈ b
- Decimal Examples
  - Using 2 significant digits to compute mean of 5.1 and 5.2 using the formula (a+b) / 2:

    a + b = 10 (with 2 sig. digits, 10.3 can only be stored as 10)
    10 / 2 = 5.0 (the computed mean is less than both numbers!!!)
  - Using 8 significant digits to compute sum of three numbers:

    $(11111113 + (-11111111)) + 7.5111111 = 9.5111111$

    $11111113 + ((-11111111) + 7.5111111) = 10.000000$
- Catastrophic cancellation occurs when

$$\left| \frac{[round(x) \text{"}\bullet\text{"} round(y)] - round(x \bullet y)}{round(x \bullet y)} \right| >> \varepsilon_{mach}$$