CSCI 2824 – Fall 2021

Homework 8

<u>Name:</u> XXXXXXX XXXXXXX <u>Student ID:</u> ZZZZZZZZ

This assignment is due on Friday, November 5th to Gradescope by 6PM. There are 5 questions on this homework. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for extra credit on the assignment. Make sure that your images/scans are clear or you will lose points/possibly be given a 0. Additionally, please be sure to match the problems from the Gradescope outline to your uploaded images.

Do not just give the answer, show the steps or give the reasoning behind all your answers.

1. Using Binomial Theorem, give the closed form expression for: $\sum_{k=0}^{n} \binom{n}{k} 3^n \cdot 2^k$

   Solution:

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} 3^n \cdot 2^k &= 3^n \cdot \sum_{k=0}^{n} \binom{n}{k} 2^k \\ &= 3^n \cdot \sum_{k=0}^{n} \binom{n}{k} 2^k \cdot 1^{n-k}\end{aligned}$$

   using binomial theorem, we know that:

$$(2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k \cdot 1^{n-k}$$

   So,

$$\begin{aligned}3^n \cdot \sum_{k=0}^{n} \binom{n}{k} 2^k \cdot 1^{n-k} &= 3^n \cdot (2+1)^n \\ &= 3^n \cdot (3)^n = (3 \cdot 3)^n = 9^n\end{aligned}$$

2. Answer the following:

   (a) You have a standard 52 card deck. If you draw two cards at random, what is the probability that they are both hearts?

   (b) You reshuffle your cards into the deck, but unfortunately now a mischievous dog (a golden retriever named Bubbles), decides to eat one of the cards! Assuming that the missing card is a diamond, now when you draw two cards at random what is the probability that they are both hearts?

   Solution:

   (a) Since both the probabilities are independent, we just multiply the probability of drawing a hearts card first with the probability of drawing a hearts card second.

$$P(\text{Hearts First} \cap \text{Hearts Second}) = P(\text{Hearts First}) \times P(\text{Hearts Second}|\text{Hearts first})$$

$$P(\text{Hearts First} \cap \text{Hearts Second}) = \frac{13}{52} \times \frac{12}{51} = 0.059$$

   (b) P(both hearts | one diamond missing)
   $=P(\text{Hearts First}|\text{missing diamond}) \times P(\text{Hearts Second}|\text{Hearts first and missing diamond})$

$$= \frac{13}{51} \times \frac{12}{50} = 0.0612$$

3. How many solutions in non-negative integers are there to the equation $x_1 + x_2 + x_3 + x_4 = 19$?

Solution:

To count the number of solutions, we note that a solution corresponds to a way of selecting 19 items from a set with three elements so that x_1 items of type one, x_2 items of type two and so on. Hence, the number of solutions is equal to the number of 19-combinations with repetition allowed from a set with four elements.

This now becomes a stars and bars problem, where r is the 19 and n is 4.

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Here:

$$\binom{n+r-1}{n-1} = \binom{4+19-1}{4-1} = \binom{22}{3} = 1540$$

4. Consider all strings of length 12, consisting of all uppercase letters. Letters may be repeated. Please do *not* simplify your answers.

   (a) How many such strings are there?

   (b) How many such strings contain the word "SCOOBY"?

   (c) How many such strings contain neither the word "SCOOBY" nor the word "DAPHNE"?

Solution:

(a) Since there are 12 spots, all of which can be occupied by any of the 26 alphabets. Therefore, 26^{12}.

(b) By removing the letters used for SCOOBY, there are 6 remaining location. $(12 - 6 = 6)$ The number of length 6 substrings is given by: 26^6
The number of ways in which the substring SCOOBY can be inserted into each length 6 substring is $C(7, 1) = 7$ since there are 7 possible locations for the 6 characters and the substring SCOOBY and you want to pick one location for the substring.
We must check to see if we double counted any strings which we did for the string SCOOBYSCOOBY. This is because we count as 'SCOOBY S C O O B Y' and as 'S C O O B Y SCOOBY'. However, this is the same string and should only be counted one so we need to subtract 1 to account for this.

Thus, the total number of strings with the substring SCOOBY is $7 * 26^6 - 1$

(c) The total length 12 strings was calculated in part A as 26^{12}. To calculate the number of strings containing SCOOBY or DAPHNE we can use the following rule about the cardinality of sets where S represents the substrings with SCOOBY and D represents the substrings with DAPHNE.

$$|S \cup D| = |S| + |D| - |S \cap D|$$

The number of strings containing the substring SCOOBY was calculated in part B. The number of strings containing the substring DAPHNE is the same as that of SCOOBY since it has the same number of letters. The number of strings containing both SCOOBY and DAPHNE is 2: SCOOBYDAPHNE and DAPHNESCOOBY. Thus the number of strings with either word is:

$$|S \cup D| = |S| + |D| - |S \cap D| = \left(7 * 26^6 - 1\right) + \left(7 * 26^6 - 1\right) - 2 = 14 * 26^6 - 4$$

So the total number of strings with neither word is:

$$|T| - |S \cup D| = 26^{12} - \left(14 * 26^6 - 4\right) = 26^{12} - 14 * 26^6 + 4$$

5. State and prove Pascal's identity using the formula for $\binom{n}{k}$.

Solution: Pascal's identity is: $\left(\begin{array}{c} n+1 \\ k \end{array} \right) = \left(\begin{array}{c} n \\ k-1 \end{array} \right) + \left(\begin{array}{c} n \\ k \end{array} \right)$

Proof: Starting with the R.H.S

$$\begin{aligned}
\left(\begin{array}{c} n \\ k-1 \end{array} \right) + \left(\begin{array}{c} n \\ k \end{array} \right) &= \frac{n!}{(n-(k-1))! \cdot (k-1)!} + \frac{n!}{(n-k)! \cdot k!} \\
&= \frac{n! \times k}{(n-k+1)! \cdot (k-1)! \times k} + \frac{n! \times (n-k+1)}{(n-k)! \cdot k! \times (n-k+1)} \\
&= \frac{n! \times k}{(n-k+1)! \cdot k!} + \frac{n! \times (n-k+1)}{(n-k+1)! \cdot k!} \\
&= \frac{n! \cdot k + n! \cdot (n-k+1)}{(n-k+1)! \cdot k!} \\
&= \frac{n! \cdot (k + (n-k+1))}{(n-k+1)! \cdot k!} \\
&= \frac{n! \cdot (k+n-k+1)}{(n-k+1)! \cdot k!} \\
&= \frac{n! \cdot (n+1)}{(n+1-k)! \cdot k!} \\
&= \frac{(n+1)!}{(n+1-k)! \cdot k!} = \left(\begin{array}{c} n+1 \\ k \end{array} \right)
\end{aligned}$$

Since R.H.S = L.H.S, pascal's identity is proved.

6. The English alphabet contains 21 consonants and 5 vowels. How many strings of **five** lowercase letters can be formed using the following constraints? Give two answers for each of the following - one where repetition is allowed in the string and one where repetition is not allowed.

   (a) Only one vowel (placed anywhere)

   (b) Maximum two consonants (placed anywhere)

   (c) Starts with x, y or z

   Solution: This can be thought of as 5 empty spaces being filled by either a vowel (v) or a consonant (c).

   (a) Only one vowel:

   $\boxed{v}\boxed{c}\boxed{c}\boxed{c}\boxed{c} + \boxed{c}\boxed{v}\boxed{c}\boxed{c}\boxed{c} + \boxed{c}\boxed{c}\boxed{v}\boxed{c}\boxed{c} + \boxed{c}\boxed{c}\boxed{c}\boxed{v}\boxed{c} + \boxed{c}\boxed{c}\boxed{c}\boxed{c}\boxed{v}$

   So, we can calculate the number of possible strings with a single vowel and 4 consonants and multiply that by 5 to indicate the permutations shown above.

   $\boxed{v} \times \boxed{c} \times \boxed{c} \times \boxed{c} \times \boxed{c} \times 5$

   With Repetition this would be:
   $= \boxed{5} \times \boxed{21} \times \boxed{21} \times \boxed{21} \times \boxed{21} \times 5$

   Without Repetition this would be:
   $= \boxed{5} \times \boxed{21} \times \boxed{20} \times \boxed{19} \times \boxed{18} \times 5$

   (b) Maximum two consonants (placed anywhere):

   Note that this also includes: no consonants and one consonant. So finally our answer should have -
   No consonants + One consonant (placed anywhere) + Two consonants (placed anywhere)

   With repetition:
   No consonant would mean all 5 spots taken by vowels $= \boxed{v} \times \boxed{v} \times \boxed{v} \times \boxed{v} \times \boxed{v} = \boxed{5} \times \boxed{5} \times \boxed{5} \times \boxed{5} \times \boxed{5}$

Using the logic from part (a), having single consonant would be $= \boxed{\text{c}} \times \boxed{\text{v}} \times \boxed{\text{v}} \times \boxed{\text{v}} \times \boxed{\text{v}} \times 5 = \boxed{21} \times \boxed{5} \times \boxed{5} \times \boxed{5} \times \boxed{5} \times 5$

As of two consonants being placed anywhere, the permutations can be more than just 5. We can figure out the numbers of permutations by calculating how many distinct strings can be formed using two "c"s and three "v"s. Finally the permutation part is: $\frac{5!}{2!3!} = 2 \times 5 = 10$. Therefore, the total number of 5 length strings having two consonants $= \boxed{21} \times \boxed{21} \times \boxed{5} \times \boxed{5} \times \boxed{5} \times 10$

Finally, strings having maximum two consonants:
$= (5 \times 5 \times 5 \times 5 \times 5) + (21 \times 5 \times 5 \times 5 \times 5 \times 5) + (21 \times 21 \times 5 \times 5 \times 5 \times 10)$

Now, without repetition:
Using same logic as above, we get
$= \boxed{5} \times \boxed{4} \times \boxed{3} \times \boxed{2} \times \boxed{1} + \boxed{21} \times \boxed{5} \times \boxed{4} \times \boxed{3} \times \boxed{2} \times 5 + \boxed{21} \times \boxed{20} \times \boxed{5} \times \boxed{4} \times \boxed{3} \times 10$

(c) Starts with x, y or z:

Here the first position can only be taken by 3 possibilities - x, y or z. There are no other restrictions so the rest of the positions can be taken up by any of the 26 alphabets (when repetition is allowed) and 25 (minus the *one* of x/y/z already used in the first position). The permutation is set so we don't need to multiply this by 5.

Now, with repetition:
$= \boxed{3} \times \boxed{26} \times \boxed{26} \times \boxed{26} \times \boxed{26}$

Without repetition:
$= \boxed{3} \times \boxed{25} \times \boxed{24} \times \boxed{23} \times \boxed{22}$