**CSCI 2824 – Fall 2021**

**Homework 7**

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This assignment is due on Friday, October 29th to Gradescope by 6PM. There are 5 questions on this homework. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

**Important**: Make sure to clearly write your full name and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for extra credit on the assignment. Make sure that your images/scans are clear or you will lose points/possibly be given a 0. Additionally, please be sure to match the problems from the Gradescope outline to your uploaded images.

**For any question, If you use proof by Mathematical Induction: be sure to clearly state your induction hypothesis, and state whether you're using weak induction or strong induction.**

1. Using mathematical induction, prove that for every positive integer $n$,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

**Solution:**

Basis step: $1(1+1) = \frac{1(1+1)(1+2)}{3}$
$2 = 2$
Therefore, $P(1)$ is true.

Inductive hypothesis: Assuming $P(k)$ to be true.

Inductive step: $P(k) \to P(k+1)$

$$P(k) : 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$
$$P(k+1) : 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

By substituting the right side of the $P(k)$ expression into the $P(k+1)$ expression we get:

$$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$
$$(k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3} - \frac{k(k+1)(k+2)}{3}$$
$$(k+1)(k+2) = \frac{(k+1)(k+2)((k+3)-k)}{3}$$

$$(k+1)(k+2) = \frac{(k+1)(k+2)(3)}{3}$$

$$(k+1)(k+2) = (k+1)(k+2)$$

since $P(1)$ is true, and $P(k) \to P(k+1)$ is true, by mathematical induction, $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ is true for any positive integer $n$

2. Let $f_n$ be the $n^{th}$ Fibonacci number. Prove that $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ [**Hint:** use strong induction]

**Solution:**

Base case $= (LHS) F_1 = 1$ and

$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}}$

$= \frac{\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right)}{\sqrt{5}}$

$= \frac{\left(\frac{\sqrt{5}\times 2}{2}\right)}{\sqrt{5}}$

$= \frac{\sqrt{5}}{\sqrt{5}} = 1 (RHS)$

Inductive hypothesis: Assume the following to be true for all $F_n$ from $F_1$ upto $F_k$:

$F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$

To prove:

$F_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$

Since $F_{k+1} = F_k + F_{k-1}$,

$F_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$

$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k \left(1 + \frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^k \left(1 + \frac{2}{1-\sqrt{5}}\right)\right]$

Since:

$\left(1 + \frac{2}{1+\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right) = \frac{1+\sqrt{5}}{2}$ and $\left(1 + \frac{2}{1-\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{1+\sqrt{5}}\right) = \frac{1-\sqrt{5}}{2}$

So, $F_{k+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{1-\sqrt{5}}{2}\right)\right]$

Therefore, $F_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$

3. There is a long line of eager children outside of your house for trick-or-treating, and with good reason! Word has gotten around that you will give out $3^k$ pieces of candy to the $k^{th}$ trick-or-treater to arrive. Children love you, dentists despise you.

   (a) Expressed in summation notation (using a $\Sigma$), what is $c_n$, the total amount of candy that you should buy to accommodate $n$ children total?

   (b) Use induction to prove that the total amount of candy that you need is given by the closed-form solution:

$$c_n = \frac{3^{n+1} - 3}{2}$$

**Solution:**

(a) The first person receives $3^1 = 3$ pieces of candy.

The second person receives $3^2 = 9$ pieces of candy.

The third person receives $3^3 = 27$ pieces of candy.

And so on... So we have: $c_n = 3 + 3^2 + 3^3 + \ldots = \sum_{k=1}^{n} 3^k$

(b) Base case: For $n = 1$, the formula gives $c_1 = \frac{3^{1+1} - 3}{2} = \frac{6}{2} = 3$, which matches $\sum_{k=1}^{1} 3^1 = 3$.

Inductive step: Suppose the $c_m = \frac{3^{m+1} - 3}{2}$ for some $m > 0$ (inductive hypothesis). Now we need to show that this same formula works for $n = m + 1$.

$$
\begin{aligned}
c_{m+1} &= \sum_{k=1}^{m+1} 3^k && \text{from the definition in terms of total amounts of candy from (a)} \\
&= 3^{m+1} + \sum_{k=1}^{m} 3^k && \text{separating off the } m+1^{st} \text{ term} \\
&= 3^{m+1} + \frac{3^{m+1}-3}{2} && \text{by inductive hypothesis} \\
&= \frac{2 \cdot 3^{m+1} + 3^{m+1} - 3}{2} \\
&= \frac{3 \cdot 3^{m+1} - 3}{2} \\
&= \frac{3^{m+2} - 3}{2}
\end{aligned}
$$

which is exactly what we want when we plug $n = m+1$ into the closed form $c_{m+1}$. Thus, if the formula is true for $n = m$, then it must be true for $n = m+1$, and **by weak induction**, it must be true for any integer $n \geq 1$.

4. Let the sequence $T_n$ be defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Use induction to prove that

$$T_n < 2^n \quad \text{for} \quad n \geq 4$$

**Solution:** We will use strong induction. Let $P(n) : T_n < 2^n$ for $n \geq 4$.
Basis step:
The following are all true:

$$
\begin{aligned}
P(4) &: T_3 + T_2 + T_1 = 1+1+1 = 3 < 2^4 = 16, \text{ or } 3 < 16 \\
P(5) &: T_4 + T_3 + T_2 = 3+1+1 = 5 < 2^5 = 32, \text{ or } 5 < 32 \\
P(6) &: T_5 + T_4 + T_3 = 5+3+1 = 9 < 2^6 = 64, \text{ or } 9 < 64
\end{aligned}
$$

Inductive step: We will temporarily assume (the inductive hypothesis:) $P(j)$ is true for $4 \leq j \leq k$ where $k \geq 6$ in order to show $P(k+1)$

$$\text{Consider } P(k+1) : T_k + T_{k-1} + T_{k-2} < 2^{k+1}$$

Knowing $4 \leq k, k-1, k-2 \leq k$, we assume $P(k), P(k-1), P(k-2)$ to be true.
Respectively, $T_k < 2^k, T_{k-1} < 2^{k-1}$, and $\bar{T}_{k-2} < 2^{k-2}$ are true.

By the inductive hypothesis, it follows that the sum of these inequalities is also true:
$T_k + T_{k-1} + T_{k-2} < 2^k + 2^{k-1} + 2^{k-2}$

If we factor out $2^k$ from the right side of the inequality, we find $T_k + T_{k-1} + T_{k-2} < 2^k \left(1 + 2^{-1} + 2^{-2}\right)$

Looking at the right side again, $2^k \left(1 + 2^{-1} + 2^{-2}\right) < 2^{k+1}$ because $2^k \left(1 + 2^{-1} + 2^{-2}\right) < 2^k \cdot 2$
This is because when comparing the terms being multiplied by $2^k$ on either side of this inequality, $1 + 2^{-1} + 2^{-2} < 2$.

Thus, $T_k + T_{k-1} + T_{k-2} < 2^k \left(1 + 2^{-1} + 2^{-2}\right) < 2^{k+1}$
and then $T_k + T_{k-1} + T_{k-2} < 2^{k+1}$, which is $P(k+1)$. Therefore $P(k+1)$ is true. This finishes the inductive step.

By strong induction, and showing the basis step and the inductive step, $P(n)$, and in turn $T_n < 2^n$ for $n \geq 4$, are true.

5. For the following recursive functions, Prove that $a_{m,n} = m \cdot n$. Here $a_{m,n}$ is defined recursively for $(m,n) \in \mathbf{N} \times \mathbf{N}$.

$$a_{m,n} = \begin{cases} 0 & \text{if } m = 0 \\ n + a_{m-1,n} & \text{if } m \neq 0 \end{cases}$$

**Solution:** Base step: base case here would be $m = 0$ and any value of n: $a_{0,n} = 0$(Since $0 \times n = 0$) So the base case is true.

Inductive hypothesis: According to the recursive definition, for $1 \leq m \leq k$: $a_{m,n} = n + a_{m-1,n}$. Now, for inductive hypothesis, we assume that for $1 \leq m \leq k$: $a_{m,n} = m \times n$ using Strong Induction.

Inductive step: now, we prove $a_{k+1,n} = (k+1) \times n$
According to the recursive definition, $a_{k+1,n} = n + a_{(k+1)-1,n} = n + a_{k,n}$.
Using the inductive hypothesis we know that $a_{k,n} = k \times n$.
Finally, we have:
$a_{k+1,n} = n + ak, n = n + k \times n = n \times (k+1)$
Therefore, since the claim $a_{m,n} = m \cdot n$ is true for $ak + 1, n$, we proved that it must be true for all $(m, n) \in \mathbf{N} \times \mathbf{N}$ using Strong Induction.