CSCI 2824 – Fall 2021

Homework 3

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This assignment is due on Friday, Sept. 17 to Gradescope by 6PM. There are 6 questions on this homework. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for extra credit on the assignment. Make sure that your images/scans are clear or you will lose points/possibly be given a 0. Additionally, please be sure to match the problems from the Gradescope outline to your uploaded images.

Note for this assignment: Remember that to prove a claim, you must prove it *in general* (i.e., for all cases), and to disprove a claim, you should present a counterexample. **If you prove a claim, be sure to indicate whether you are using a Direct Proof, a Contrapositive Proof, a Proof by Cases, a Proof by Exhaustion, a Proof by Contradiction, or a Proof by Construction (existence proof).** If you use a Proof by Cases, be sure to indicate what each case is, and in each case, which type of proof you are using.

1. For each of the following claims, prove or disprove the claim:

   (a) For the domain of all 3-digit numbers n, if the sum of the digits of n is divisible by 9, then n is divisible by 9.

   (b) For natural numbers n, $n^2 + n + 17$ is always prime.

   Solution:

   (a) Direct proof:
   Let us assume that the sum of the digits of a 3 -digit number is divisible by 9 , so if the digits are x, y and z then:
   $$x + y + z = 9 \times h$$
   The 3 digit number can be expressed as:
   $$100x + 10y + z$$
   If we substitute 'z' as $9h - x - y$($\because$ x + y + z = $9 \times h$), we get:
   $$\begin{aligned} &100x + 10y + (9h - x - y) \\ &= 100x - x + 10y - y + 9h \\ &= 99x + 9y + 9h \\ &= 9 \times (11x + y + h) \end{aligned}$$
   Therefore, if the sum of three digits of a number n is divisible by 9 then n is also divisible by 9

   (b) Proof by counter example:
   Let us consider the case where n is 17 . Then:
   $$\begin{aligned} &n^2 + n + 17 \\ &= 17^2 + 17 + 17 \\ &= 17 \times (17 + 1 + 1) \\ &= 17 \times 19 \\ &= 323 \end{aligned}$$
   So, for n = 17, $n^2 + n + 17$ has two factors: 17 and 19 . Which means it is not a prime number!

2. If n is a positive integer, then prove the following:

   (a) n is even if and only if 7n + 4 is even

   (b) n is odd if and only if 5n + 6 is odd

(a) To prove: n is even $\iff$ 7n+4 is even

($\Longrightarrow$) Forward proof -

Direct proof:

Assuming n is even then n can be expressed as $2k$, plugging in the value of n we get

$$\begin{aligned} 7n+4 &= 7 \cdot (2k) + 4 \\ &= 2 \cdot 7 \cdot k + 2 \cdot 2 \\ &= 2(7k+2) = 2 \cdot k_1 \end{aligned}$$

Therefore, 7n+4 can also be expressed as $2k_1$, meaning it is even. Finally we have n is even $\Longrightarrow$ 7n+4 is even.

($\Longleftarrow$) Backward proof -

Proof by Cases:

Case 1 - If $7n+4$ is even
$7n+4 = 2k_0$
$7n = 2k_0 - 4$
$7n = 2(k_0 - 2)$
Which makes $7n$ an even number as it can be written as $2k_1$
Since 7 is an odd number (and also prime), which means 2 is not a factor. For $7 \times n$ to be even, n has to be even.

Case 2 - If $7n+4$ is odd
$7n+4 = 2k_0 + 1$
$7n = 2k_0 + 1 - 4$
$7n = (2k_0 - 4) + 1$
$7n = 2(k_0 - 2) + 1$
Which makes $7n$ odd, as it can be written as $2k_1 + 1$
This means n cannot have 2 as a factor, odd $\times$ odd $=$ odd
Therefore, n is even $\Longleftarrow$ 7n+4 is even.

(b) Proved in a similar manner as 2(a)

3. Prove that there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 2401$.

Solution: Proof by Exhaustion:

Since x and y must be positive and $7^4 = 2401$, we need only consider the cases where $x, y < 7$.

$$
\begin{aligned}
1^4 + 1^4 &= 1 + 1 = 2 && \neq 2401 \\
1^4 + 2^4 &= 1 + 16 = 17 && \neq 2401 \\
1^4 + 3^4 &= 1 + 81 = 82 && \neq 2401 \\
1^4 + 4^4 &= 1 + 256 = 257 && \neq 2401 \\
1^4 + 5^4 &= 1 + 625 = 626 && \neq 2401 \\
1^4 + 6^4 &= 1 + 1296 = 1297 && \neq 2401 \\
\\
2^4 + 2^4 &= 16 + 16 = 32 && \neq 2401 \\
2^4 + 3^4 &= 16 + 81 = 97 && \neq 2401 \\
2^4 + 4^4 &= 16 + 256 = 272 && \neq 2401 \\
2^4 + 5^4 &= 16 + 625 = 641 && \neq 2401 \\
2^4 + 6^4 &= 16 + 1296 = 1312 && \neq 2401 \\
\\
3^4 + 3^4 &= 81 + 81 = 162 && \neq 2401 \\
3^4 + 4^4 &= 81 + 256 = 337 && \neq 2401 \\
3^4 + 5^4 &= 81 + 625 = 706 && \neq 2401 \\
3^4 + 6^4 &= 81 + 1296 = 1377 && \neq 2401 \\
\\
4^4 + 4^4 &= 256 + 256 = 512 && \neq 2401 \\
4^4 + 5^4 &= 256 + 625 = 881 && \neq 2401 \\
4^4 + 6^4 &= 256 + 1296 = 1552 && \neq 2401 \\
\\
5^4 + 5^4 &= 625 + 625 = 1250 && \neq 2401 \\
5^4 + 6^4 &= 625 + 1296 = 1921 && \neq 2401 \\
\\
6^4 + 6^4 &= 1296 + 1296 = 2592 && \neq 2401
\end{aligned}
$$

4. Prove the following:

   (a) Let a and b be integers such that c = a + b. Prove that c is even if and only if a and b have the same parity.

   (b) Suppose you have two empty jugs with volumes of 7 gallons and 11 gallons. You can assume that you have access to an unlimited water source, and that you measure out the requested amounts of water by successively filling and pouring water from one jog into the other or down the drain. Prove or disprove that an algorithm exists to put exactly 10 gallons of water in the 11 gallon jug.

Solution:

(a) ($\implies$) We will prove the forward direction **contradiction**.
Suppose $c = a + b$ is even and a, b have opposite parity. Since addition is commutative, it suffices to prove the case where a is odd and b is even.

$$
\begin{aligned}
\implies \exists k, l \in \mathbb{Z} \text{ st } a = 2k + 1, b = 2d \\
\implies c &= a + b \\
&= (2k + 1) + (2l) \\
&= 2(k + l) + 1
\end{aligned}
$$

Since $k + l \in \mathbb{Z}$, we conclude that c is odd. $\Rightarrow\Leftarrow$ This contradicts our initial assumption that c is even. Thus a and b must have the same parity.

($\Longleftarrow$) The backward direction is proven **with cases**.
Case 1 : a and b are both even

$$\begin{aligned} &\Longrightarrow \exists k, l \text{ st } a = 2k, \quad b = 2l \\ &\Longrightarrow = c = a + b \\ &= (zk+1) + (2l+1) \\ &= 2k + 2l + 2 \\ &= 2(k+l+1) \end{aligned}$$

Since $k + l + 1 \in \mathbb{Z}$, this means that c is even.
Case 2 : a and b are both odd.

$$\begin{aligned} &\Longrightarrow \exists k, l \text{ st } a = 2k+1, \quad b = 2l+1 \\ &\Longrightarrow c = a + b \\ &= (2k+1) + (2l+1) \\ &= 2k + 2l + 2 \\ &= 2(k+l+1) \end{aligned}$$

Since $k + l + 1 \in \mathbb{Z}$, then c is even.

(b) Proof By Construction:
1. Fill 7
2. Put 7 into the 11
3. Fill 7 again
4. Top off the 11 ; there are 3 gallons left in the 7
5. Pour out the 11
6. Pour the 3 gallons from the 7 into the 11
7. Fill the 7
8. Put these 7 gallons into the 11 for a total of 10 gallons

5. For the following, find if there are any errors in the methods of proof given below. List out these errors and write how you would prove/disprove the statements given below.

(a) Statement: If n is an integer and n^2 is divisible by 4, then n is divisible by 4.
Proof: Consider the number 144, which is a perfect square divisible by 4 (since $4 \times 36 = 144$). Now, considering that $\sqrt{144} = 12$ so n=12. Since 12 is also divisible by 4 ($4 \times 3 = 12$), the statement holds true. Hence, Proved!

(b) Statement: Let p and q be integers and $r = pq + p + q$, then r is even if and only if p and q are both even.
Proof: Since p and q are even we can write them as $p = 2k_1$ and $q = 2k_2$. This means -

$$r = 2k_1 \cdot 2k_2 + 2k_1 + 2k_2$$

$$r = 2(2 \cdot k_1 \cdot k_2 + k_1 + k_2)$$

$$r = 2(k_3)$$

Meaning r is an even number. Therefore, the statement above is true.

(a) Issue with the proof: We cannot prove a theorem by example. Examples, or rather counter-examples are used to disprove a theorem.

Correct Proof -

Using Proof by Counter-example:
Consider the example where n = 6. Here, $n^2 = 36$ which is divisible by 4 ($\because 4 \times 9 = 36$). However, 6 is **not** divisible by 4. Therefore the statement - "If n is an integer and n^2 is divisible by 4, then n is divisible by 4" is False. (Disproved!)

(b) Issue with the proof: In order to prove "if and only if" statements, we need to prove both sides. In this case, this means that we need to prove that p and q even $\implies$ r is even and r is even $\implies$ p and q are even. The proof above only proves p and q are even $\implies$ r is even.

Correct Proof -

p and q are even $\implies$ r is even, using Direct proof:

Since p and q are even we can write them as $p = 2k_1$ and $q = 2k_2$. This means -

$$r = 2k_1 \cdot 2k_2 + 2k_1 + 2k_2$$

$$r = 2(2 \cdot k_1 \cdot k_2 + k_1 + k_2)$$

$$r = 2(k_3)$$

Meaning r is an even number.

r is even $\implies$ p and q are even, can be proved by using - Proof by Cases:

(To prove by contrapositive, we will rewrite the statement in terms of its contrapositive form: p and q are not both even $\Rightarrow r$ is not even. Or equivalently: p or q are odd $\Rightarrow r$ is odd. (We obtain the antecedant expression by DeMorgan's Laws.)

There are three possible cases which would satisfy the antecedant expression: p and q are both odd, just p is odd, or just q is odd. We will only consider the first two cases, as the arguments for the latter two are symmetrical.

The case where p and q are both odd:
Since p and q are odd, $\exists n_1, n_2 \in \mathbb{Z}$ such that $p = 2n_1 + 1$ and $q = 2n_2 + 1$. Then we have:

$$\begin{aligned} r &= (2n_1 + 1)(2n_2 + 1) + 2n_1 + 1 + 2n_2 + 1 \\ &= 4n_1n_2 + 2n_1 + 2n_2 + 1 + 2n_1 + 1 + 2n_2 + 1 \\ &= 4n_1n_2 + 4n_1 + 4n_2 + 2 + 1 \\ &= 2(2n_1n_2 + 2n_1 + 2n_2 + 1) + 1 \end{aligned}$$

Since the integers are closed under multiplication and under addition, $(2n_1n_2 + 2n_1 + 2n_2 + 1) \in \mathbb{Z}$. Thus, r is of the form $2n + 1$ where $n \in \mathbb{Z}$ and hence is odd.

The case where only p is odd:
Since p is odd and q is even, $\exists n_1, n_2 \in \mathbb{Z}$ such that $p = 2n_1 + 1$ and $q = 2n_2$. Then we have:

$$\begin{aligned} r &= (2n_1 + 1)(2n_2) + 2n_1 + 1 + 2n_2 \\ &= 4n_1n_2 + 2n_2 + 2n_1 + 1 + 2n_2 \\ &= 4n_1n_2 + 4n_2 + 2n_1 + 1 \\ &= 2(2n_1n_2 + 2n_2 + n_1) + 1 \end{aligned}$$

So since $(2n_1n_2 + 2n_2 + n_1) \in \mathbb{Z}$, r is of the form $2n + 1$ where $n \in \mathbb{Z}$ and hence is odd.

$\therefore r$ is even $\Rightarrow p$ and q are both even.

6. (a) Prove that $(P \cap Q) \times R = (P \times R) \cap (Q \times R)$ holds by showing that each side of this equation is a subset of the other side of the equation.

   (b) Translate each of these quantifications into English and determine its truth value:

   i. $\forall x \in \mathbf{R} \left(x^2 \neq -1\right)$

   ii. $\exists x \in \mathbf{R} \left(x^3 = -1\right)$

   iii. $\forall x \in \mathbf{Z}(x - 1 \in \mathbf{Z})$

Solution:

(a) Proving both sides are subsets of each other:

$$
\begin{array}{lll}
\multicolumn{2}{l}{(\Longrightarrow) Suppose((x,y) \in (P \cap Q)) \times R} & \\
\Rightarrow & x \in (P \cap Q) \wedge (y \in R) & \text{by definition of cartesian product} \\
\Rightarrow & ((x \in P) \cap (x \in Q)) \wedge (y \in R) & \text{by definition of intersection} \\
\Rightarrow & (x \in P \wedge y \in R) \wedge (x \in Q \wedge y \in R) & \\
\Rightarrow & ((x,y) \in P \times R) \wedge ((x,y) \in Q \times R) & \text{by definition of cartesian product} \\
\Rightarrow & (x,y) \in (P \times R) \cap (Q \times R) & \text{by definition of intersection} \\
\Rightarrow & (P \cap Q) \times R \subseteq (P \times R) \cap (Q \times R) & \text{by definition of a subset}
\end{array}
$$

$$
\begin{array}{lll}
\multicolumn{2}{l}{(\Longleftarrow) Suppose((x,y) \in (P \times R) \cap (Q \times R))} & \\
\Rightarrow & (x,y) \in (P \times R) \wedge (x,y) \in (Q \times R) & \text{by definition of intersection} \\
\Rightarrow & (x \in P \wedge y \in R) \wedge (x \in Q \wedge y \in R) & \text{by definition of a cartesian product} \\
\Rightarrow & (x \in P \wedge x \in Q) \wedge y \in R & \\
\Rightarrow & (x \in P \cap Q) \wedge (y \in R) & \text{by definition of intersection} \\
\Rightarrow & (x,y) \in (P \cap Q) \times R & \text{by definition of cartesian product}
\end{array}
$$

Since $LHS \subseteq RHS$ and $RHS \subseteq LHS \subseteq LHS$, the two sides must be equal.

(b) i. $\forall x \in \mathbf{R} \left(x^2 \neq -1\right)$
In English: "There is no Real number such that the square of that number is equal to -1" OR "For all values of x such that x belongs to Real Numbers, x squared will not be equal to -1". The truth value is: True, since all squared real numbers are positive, so it will never be equal to -1.

ii. $\exists x \in \mathbf{R} \left(x^3 = -1\right)$
In English: "There exists a real number such that it's cube is -1"
Truth value: True, since $(-1)^3 = -1$

iii. $\forall x \in \mathbf{Z}(x - 1 \in \mathbf{Z})$
in English: "For all x such that x belongs to the set of integers, x-1 must also be an integer" OR "x-1 is an Integer if x is any integer"
Truth value: True, all integers are at the gap of "1" so subtracting 1 from an integer will still get you an integer.