1. Let f_n be the n^{th} Fibonacci number. Prove that, for $n > 0$: $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$

   Solution: By the recursive definition of the Fibonacci sequence: $f_{n+2} = f_n + f_{n+1}$ or $f_n = f_{n-1} + f_{n-2}$ or $f_{n+1} = f_n + f_{n-1}$

   Proof Using Mathematical Induction

   $P(n) : f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for $n > 0$

   Basis Step: $P(1) : f_1^2 = 1^2 = 1$, $f_n f_{n+1} = (1)(1) = 1$, 1=1 True

   Inductive Step: Show $P(k) \implies P(k+1)$

   $P(k) : f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}$

   $P(k+1) : f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 = f_{k+1} f_{k+2}$

   We can replace the part of P(k+1) that is the same as P(k) with the equivalent to P(k)

   Now, $f_k f_{k+1} + f_{k+1}^2 = f_{k+1} f_{k+2}$

   $f_k f_{k+1} + f_{k+1}^2 = f_k f_{k+1} + f_{k+1} f_{k+1} = f_{k+1}(f_k + f_{k+1})$

   By using the definition of the Fib. Seq. we can rewrite $f_k + f_{k+1}$ as f_{k+2}. This gives us $f_{k+1} f_{k+2}$ which equals P(k+1), so the inductive step is proved Q.E.D

2. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = -1$, $f(1) = 2$, and for $n = 1, 2, 3, ...$

(a) $f(n+1) = f(n) + 3f(n-1)$

Solution:
$f(2) = f(1+1) : f(1) + 3f(0) = 2 + 3(-1) = -1$ (n = 1)
$f(3) : f(2) + 3f(1) = -1 + 3(2) = 5$ (n = 2)
$f(4) : f(3) + 3f(2) = 5 + 3(-1) = 2$ (n = 3)
$f(5) : f(4) + 3f(3) = 2 + 3(5) = 17$ (n = 4)

(b) $f(n+1) = (f(n))^2 \cdot f(n-1)$

Solution:
$f(2)$: $(f(1))^2 \cdot f(0) = 4 * -1 = -4$
$f(3)$: $(f(2))^2 \cdot f(1) = 16 * 2 = 32$
$f(4)$: $(f(3))^2 \cdot f(2) = 32^2 * -4 = -4,096$
$f(5)$: $(f(4))^2 \cdot f(3) = 4096^2 * 32 = 536,870,912$

(c) $f(n+1) = 3(f(n))^2 - 4f(n-1)$

Solution:
$f(2)$: 16
$f(3)$: 760
$f(4)$: 1732736
$f(5)$: 9007122134048

(d) $f(n+1) = \dfrac{f(n-1)}{f(n)}$

Solution:
$f(2) : f(0)/f(1) = -1/2 = -0.5$
$f(3) : f(1)/f(2) = 2/-0.5 = -4$
$f(4) : f(2)/f(3) = -0.5/-4 = 1/8 = 0.125$
$f(5) : f(3)/f(4) = -4/0.125 = -32$

3. You have a locked box which has a five-digit pin. Since this is 2021, it is fair to assume that you will open the box on the very last attempt. How many attempts would it take?

   Solution:
   There are 5 spaces that contain a digit 0-9 which is 10 digits. We can repeat numbers and the order does not matter, so the last attempt will occur at the 10^5 th attempt.

4. Suppose you are a minimalist and have only 4 shirts of colors - `blue, red, black, orange` and pants of colors - `blue, black and tan`.

   (a) How many combinations of shirts and pants are possible?
   Solution: $4 \times 3 = 12$

   (b) How many combinations of shirts and pants are possible if you are cannot wear a pant and a shirt of the same color?
   Solution: Total possibilities- $4 \times 3 - 2 = 10$. Because you cannot wear a combination of blue or black clothes.

5. How many bitstrings of length 5 contain the sub-string 010 or 000?

   Solution: Let's suppose we have a string x which can take values of length 3 - 010 and 000. We now have 3 positions to work with. Therefore total number of combinations are $3 \times 2 \times 2^2 = 24$, because x can be at 3 positions and can take on 2 values. But there are some overlap values. For example, 01000 will be counted twice. The possible overlapping cases are $01000, 00010, 00000(thrice), 01010, 10000, 00001$. Total possible cases $24 - 7 = 17$.

6. There are 28 different time periods during which classes at a university can be scheduled per room. If there are 539 different classes, what is the minimum number of different rooms that will be needed?

   Solution: There exists a time period that will have at least $\lceil 539/28 \rceil = 20$ classes during it. So 20 different rooms will be needed.

7. Suppose that valid passwords must be strings of length 10 containing only numbers and letters.

(a) How many passwords exist if uppercase and lowercase letters are indistinguishable from one another?

Solution: Each character can be either a number - $0-9$ or a letter $a-z$. Since the case doesn't matter, each character can be chosen from a possible 36 character set. Number of valid passwords - 36^{10}.

(b) How many passwords exist if uppercase and lowercase letters are distinguishable?

Solution: Each character can be either a number - $0-9$ or a letter $a-z$. Since the case does matter, each character can be chosen from a possible 62 character set (Because each letter and its capital are distinct). Number of valid passwords - 62^{10}.

(c) How many passwords exist if passwords must contain at least one uppercase and at least one lowercase letter?

Hint: Let U = { Passwords with at-least 1 uppercase letter}, let L = {Passwords with at-least 1 lowercase letter}, let T = { All possible passwords }.

Solution:

$$|U \cup L| = |U| + |L| - |U \cap L| \text{ (Principle of Inclusion-Exclusion / Subtraction Rule)}$$

$$|T| - |U^C| + |T| - |L^C| - |T| + |(U \cup L)^C| \text{ (Universal set - complement of a set = original set)}$$

$$= |T| - |U^C| - |L^C| + |(U \cup L)^C| \text{ (Cancel)}$$

$$= 62^{10} - 36^{10} - 36^{10} + 10^{10}$$