CSCI 2824 – Fall 2021

Homework 2

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This assignment is due on Friday, Sept. 10 to Gradescope by 6PM. There are 6 questions on this homework. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for extra credit on the assignment. Make sure that your images/scans are clear or you will lose points/possibly be given a 0. Additionally, please be sure to match the problems from the Gradescope outline to your uploaded images.

1. The two parts of this question are about the Island of Knights and Knaves. Every inhabitant of the island is either a Knight or a Knave. Knights always tell the truth, and Knaves always lie. Consider the propositional function $K(x) =$ "x is a knight", where the domain for x is all of the inhabitants of the Island.

   (a) While performing an academic survey on the Island of Knights and Knaves you manage to speak to **every** inhabitant on the island and each one tells you "Some of us are Knights and some of us are Knaves".

      i. Translate this statement into a predicate statement using quantifiers, connectives, and $K(x)$. You may not define any other propositional functions.

   (b) While performing a much lazier academic survey on another Island of Knights and Knaves you speak to **only one** inhabitant on the island and they tell you "All of us are Knaves".

      i. Translate this statement into a predicate statement using quantifiers, connectives, and $K(x)$. You may not define any other propositional functions.

   Solution:

   (a)

      i. In predicate logic, the statement "Some of us are Knights and some of us are Knaves" can be written as:
      $$\exists x[K(x)] \wedge \exists x[\neg K(x)]$$

   (b)

      i. In predicate logic, the statement "All of us are Knaves" can be written as either:
      $$\forall x[\neg K(x)] \quad \text{or} \quad \neg \exists x[K(x)]$$

2. You arrive on yet another Island of Knights and Knaves. Knights always tell the truth and Knaves always lie. You meet three inhabitants: A, B, and C. A claims "I am a knight or B is a knave." B tells you, "A is a knight and C is a knave." C says, "Myself and B are different." Use a truth table to determine who is a knight and who is a knave, if possible. Justify and explain your answer.

   Solution: First we assign propositions in the following way:
   p: A is a knight.
   q : B is a knight.
   r : C is a knight.

Next we translate each person's statement.
A: $p \vee \neg q$
B: $p \wedge \neg r$
C: $q \oplus r$

The solution of the problem will be found by identifying the truth values of $p, q,$ and r that make the compound proposition $(p \leftrightarrow (p \vee \neg q)) \wedge (q \leftrightarrow (p \wedge \neg r)) \wedge (r \leftrightarrow (q \oplus r))$ true.

p	q	r	$\neg r$	$\neg q$	$p \vee \neg q$	$p \wedge \neg r$	$q \oplus r$	$(p \leftrightarrow (p \vee \neg q))$	$(q \leftrightarrow (p \wedge \neg r))$	$(r \leftrightarrow (q \oplus r))$	$(**)$
T	T	T	F	F	T	F	F	T	F	F	F
T	T	F	T	F	T	T	T	T	T	F	F
T	**F**	**T**	**F**	**T**	**T**	**F**	**T**	**T**	**T**	**T**	**T**
T	F	F	T	T	T	T	F	T	F	T	F
F	T	T	F	F	F	F	F	T	F	F	F
F	T	F	T	F	F	F	T	T	F	F	F
F	F	T	F	T	T	F	T	F	T	T	F
F	F	F	T	T	T	F	F	F	T	T	F

From the truth table, we see that A and C are Knights and B is a Knave.

3. (a) Show that $(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow q)$ is unsatisfiable using **both** (i) a truth table and (ii) a logical argument (not a chain of logical equivalences, but rather a written-out argument in English).

   (b) Show that $(p \to r) \vee (q \to r)$ is logically equivalent to $(p \wedge q) \to r$ using **both** (i) a truth table and (ii) a chain of logical equivalences. Note that you may only use logical equivalences from Table 6 (p. 27 of Rosen textbook) and the other four named equivalences given in lecture. At each step you should cite the name of the equivalence rule you are using, and please only use one rule per step. Is this compound proposition satisfiable? Why or why not?

Solution:

(a) $(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow q)$

i. Truth Table

p	q	$\neg p$	$p \leftrightarrow q$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	T	F	F

ii. Logical Argument

Given a truth value for p, The only way for the statement to be true is if both sides of the bi-conditional are of the same truth value. In order for both sides to have the same truth value, both $(p \leftrightarrow q)$ and $(\neg p \leftrightarrow q)$ must have the same truth value, for a given truth value of p. This would mean that q must be the same as p and ¬p at the same time, or that q is the opposite of p and the opposite of ¬p at the same time, both of which would lead to a contradiction of p ∧ ¬p.

(b) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

i. Truth Table.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$	$((p \to r) \vee (q \to r)) \leftrightarrow ((p \wedge q) \to r)$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	F	T	T
T	F	F	F	T	T	F	T	T
F	T	T	T	T	T	F	T	T
F	T	F	T	F	T	F	T	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

ii. Logical Equivalences

$$
\begin{aligned}
(p \to r) \vee (q \to r) &\equiv (\neg p \vee r) \vee (\neg q \vee r) && \text{RBI} && (1)\\
&\equiv \neg p \vee r \vee \neg q \vee r && \text{Associativity} && (2)\\
&\equiv \neg p \vee \neg q \vee r \vee r && \text{Commutativity} && (3)\\
&\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{Commutativity} && (4)\\
&\equiv (\neg p \vee \neg q) \vee r && \text{Idempotent} && (5)\\
&\equiv \neg (p \wedge q) \vee r && \text{DeMorgans Law} && (6)\\
&\equiv (p \wedge q) \to r && \text{RBI} && (7)
\end{aligned}
$$

The truth table tells us that for all values of p, q, r, the compound proposition is True. Therefore, the given compound proposition is satisfiable.

4. This semester there are 498 students in Discrete Structures. The newest CU club, called DST (Discrete Structures Travelers), has decided that all its members (the 498 students in this years class) are going to map various Colorado hiking trails. DST put forth the following criteria: Each club member will walk and map a set of trails, and no two students in the club will walk/map the same set of trails. This means that although some trails will be walked/mapped by more than one club member, we must ensure that for any two club members, their list of trails walked/mapped must differ by at least one trail. It is required that all members walk at least one trail.
Answer the following questions and fully explain your answer (points are given for the quality of your explanation):

(a) If there are 498 club members that will be mapping, what is the smallest number of trails that will be mapped?

(b) In general, with n different trails to map, what is the maximum number of club members that can walk/map so that the criteria are still met?

Solution: (a) Let us represent each club members set of trails walked as a sequence of bits. Each of the bits represents a single trail walked and mapped. If the bit is 1, then the club member walked the trail. If the bit is 0, then the club member did not walk the trail.

For example, consider club members: Aiden, Paige, Quinn, and Majed

Name	trail 1	trail 2	trail 3
Aiden	1	1	0
Paige	1	1	1
Quinn	0	1	1
Majed	1	0	1

From the data, it follows that Aiden is walking trail 1 and trail 2; Paige is walking all 3 trails; Quinn is walking trail 2 and trail 3; and Majed is walking trail 1 and trail 3.
Since each trail mapped is represented by a bit, we need enough bits to represent the number of club members in DST.
Note that every member must walk at least one trail.

With 8 bits (or 8 trails), we can *uniquely* represent $2^8 - 1 = 256 - 1 = 255$ sets of trails to walk. We had to subtract 1 because a club member cannot be given the number 0.

However, $255 < 498$. So we need to find more trails. Adding a 9-th trail, we have $2^9 - 1 = 512 - 1 = 511$. Therefore, the smallest number of trails we need is 9. Thus, the club will need to explore at least 9 trails.

(b) We just saw that with 8 trails, we can have at most $2^8 - 1$ sets of trails to explore, and with 9 trails, $2^9 - 1$ sets of trails.
We can generalize to n trails. With n trails to explore, we can have at most $2^n - 1$ club members enrolled.

5. Consider the following satisfiability problem: Donatello, Rafael, Michelangelo, Leonardo, and Splinter are going to order a pizza. First they need to agree on some toppings. Splinter is happy to eat any toppings. The other members of the group, however, are very particular about their pizza topping preferences.

   They will order a pizza that can have 1, 2, or 3 toppings, and the entire pizza must have the same topping(s) on all portions of it. (e.g. it can't be part pepperoni and part cheese). The group's preferences are:

   i. Rafael wants licorice and not peanut butter.
   ii. Michelangelo does not want salami.
   iii. If the pizza has peanut butter on it, then Donatello does not want licorice.
   iv. Leonardo wants licorice if and only if there is salami or granola.

   Let $Z(x)$ represent the propositional function "the pizza must have topping x", where the domain for x is the set of possible pizza toppings: granola (G), licorice (L), peanut butter (P), and salami (S). Note that statements like "Rafael wants a pizza with licorice" does not imply that Rafael wants no other toppings. For example, Rafael would be perfectly happy with a licorice and salami pizza.

   (a) Translate each of the group's pizza topping requirements $i - iv$ from English into a proposition using the given propositional function notation.
   (b) Are the group's pizza topping requirements satisfiable? If they are, provide a set of pizza toppings that satisfies the requirements. If they are not, provide a **concise** written argument explaining why not. Do **not** use a truth table.

   Solution:

   (a)

   i. $Z(L) \wedge \neg Z(P)$
   ii. $\neg Z(S)$
   iii. $Z(P) \rightarrow \neg Z(L)$
   iv. $Z(L) \leftrightarrow (Z(S) \vee Z(G))$

   (b) From (ii), we cannot have salami on the pizza. Condition (i) tells us that the pizza must have licorice but not peanut butter.

   Condition (iv) stipulates that if licorice is on the pizza, then there also must be either salami or granola. We've already established that there needs to be licorice and no salami, so we can conclude that granola must be included on the pizza.

   Lastly, since the pizza will not have peanut butter (ruled out by (ii)), then Donatello will be ok with having licorice.

   Thus, the pizza that will satisfy everyone is a pizza with **licorice and granola**. This problem **is satisfiable**. Delicious.

6. Lets say Q is a quadrilateral. If you were given the statements: If Q is a rhombus, then Q is a parallelogram. Q is not a parallelogram. Then what statement follows by *modus tollens*?

   Solution:
   Modus Tollens, or 'denial mode', has the structure: $\neg q \wedge (p \rightarrow q)$ results in $\neg p$. So, 'Q is not a parallelogram', and 'If Q is a rhombus, then Q is a parallelogram' results in **'Q is not a rhombus'**