CSCI 2824 – Fall 2021

Homework 5

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This assignment is due on Friday, October 8th to Gradescope by 6PM. There are 7 questions on this homework. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for extra credit on the assignment. Make sure that your images/scans are clear or you will lose points/possibly be given a 0. Additionally, please be sure to match the problems from the Gradescope outline to your uploaded images.

1. Solve for x if $(g \circ f)(x) = 1$. Here, $f(x) = (x^{log(x)} \cdot x^2)$ and $g(x) = log(x) + 1$.

   Solution:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= log(f(x)) + 1 \\ &= log((x^{log(x)} \cdot x^2)) + 1 \\ &= (log(x^{log(x)}) + log(x^2)) + 1 \\ &= log(x) \cdot log(x) + 2 \cdot log(x) + 1 \\ &= log(x)^2 + 2 \cdot log(x) + 1\end{aligned}$$

   Since $(g \circ f)(x) = 1$

$$log(x)^2 + 2 \cdot log(x) + 1 = 1$$

   Let $y = log(x)$, then:

$$\begin{aligned}y^2 + 2y + 1 &= 1 \\ y^2 + 2y &= 0 \\ y(y+2) &= 0 \\ \therefore y &= 0, -2\end{aligned}$$

   For $log(x) = 0$, $x = 1$.
   For $log(x) = -2$, $x = e^{-2}$.

2. Compute the following:

   (a)
$$\lim_{t\to\infty} \frac{\log(2t)}{t^2}$$

   (b)
$$\lim_{n\to\infty} \frac{2^n}{\frac{n(n+1)}{2}}$$

   Solution:

   (a) Since $\lim_{t\to\infty} \frac{\log(2t)}{t^2} = \frac{\infty}{\infty}$

   We apply L'hopital rule:

$$\lim_{t\to\infty} \frac{\log(2t)}{t^2} = \lim_{t\to\infty} \frac{d(\log(2t))/dt}{d(t^2)/dt}$$

$$\left[\because d(\log(2t))/dt = \frac{1}{2t} \cdot d(2t)/dt = \frac{1}{2t} \cdot 2 = \frac{1}{t}\right]$$

$$\left[\because d(t^2)/dt = 2t\right]$$

$$= \lim_{t\to\infty} \frac{1/t}{2t}$$

$$= \lim_{t\to\infty} \frac{1}{t \times 2t} = \lim_{t\to\infty} \frac{1}{2t^2} = 0$$

(b) Since $\lim_{n\to\infty} \frac{2^n}{\frac{n(n+1)}{2}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n^2+n)} = \frac{\infty}{\infty}$

We apply L'Hopital rule:

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{f'(x)}{g'(x)}$$
$$\implies \lim_{n\to\infty} \frac{2^{n+1}}{(n^2+n)} = \lim_{n\to\infty} \frac{\log(2)\cdot 2^{n+1}\cdot(1+0)}{2n+1} = \frac{\infty}{\infty}$$

We apply L'Hopital rule again!

$$\implies \lim_{n\to\infty} \frac{\log(2)\cdot 2^{n+1}}{2n+1} = \lim_{n\to\infty} \frac{\log(2)^2\cdot 2^{n+1}}{2+0}$$
$$= \lim_{n\to\infty} \frac{\log(2)^2\cdot 2^{n+1}}{2} = \log(2)^2 \lim_{n\to\infty} 2^n = \infty$$

3. Find the $big-O$, $big-\Omega$ estimate for $x^7y^3 + x^5y^5 + x^3y^7$. [Hint: Big- O, big- Ω, and big-Θ notation can be extended to functions in more than one variable. For example, the statement $f(x,y)$ is $O(g(x,y))$ means that there exist constants C, k_1, and k_2 such that $|f(x,y)| \le C|g(x,y)|$ whenever $x > k_1$ and $y > k_2$.]

Solution:

$x^7y^3 \le x^7y^7$ where $x > 1$ and $y > 1$
$x^5y^5 \le x^7y^7$ where $x > 1$ and $y > 1$
$x^3y^7 \le x^7y^7$ where $x > 1$ and $y > 1$
to find the big-O bound for $f(x,y)$ we must add up the big-O bounds of its terms, which we found above. [Here x and y are considered to be > 1 and not > 0 since the the inequalities above would not hold for x,y values as fractions.]
$f(n) \le \mathrm{x}^7y^7 + x^7y^7 + x^7y^7$
f(n) $\le 3x^7y^7$
$f(n)$ is $O\left(x^7y^7\right)$ where C $= 3$ and $k_1 = 1$ and $k_2 = 1$

$x^7y^3 \ge x^3y^3$ where $x > 1$ and $y > 1$
$x^5y^5 \ge x^3y^3$ where $x > 1$ and $y > 1$
$x^3y^7 \ge x^3y^3$ where $x > 1$ and $y > 1$
to find the big-Omega bound for $f(x,y)$ we must add up the big-Omega bounds of its terms, which we found above.
$f(n) \ge \mathrm{x}^3y^3 + x^3y^3 + x^3y^3$
f(n) $\ge 3x^3y^3$
$f(n)$ is $\Omega\left(x^3y^3\right)$ where C $= 3$ and $k_1 = 1$ and $k_2 = 1$

4. Consider the function $f(n) = 35n^3 + 2n^3\log(n) - 2n^2\log\left(n^2\right)$ which represents the complexity of some algorithm.

   (a) Find a tight big-O bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-O definition?

   (b) Find a tight big- $\boldsymbol{\Omega}$ bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big- Ω definition?

   (c) Can we conclude that f is big$-\Theta\left(n^p\right)$ for some natural number p?

Solution:

(a) First, we simplify using our rules for logarithms: $f(n) = 35n^3 + 2n^3\log(n) - 4n^2\log(n)$. Next, the natural first guess should be the leading order term, which is n^3. But two terms have n^3 in them. since the $\log n$ is only going to make the second term larger than n^3 for $\log(n) > 1 (n > e)$ this means the second term is the dominant one (upper bound). The lowest power of n that can be an upper bound for that second term is n^4. We get upper bounds for each term in terms of n^4 :

$$35n^3 \le 35n^4, \text{ for } n \ge 1$$
$$2n^3\log(n) \le 2n^3 \cdot n = 2n^4, \quad \text{for } n \ge 1$$
$$-4n^2\log(n) \le 0, \text{ for } n \ge 1$$

So we have:

$$f(n) \leq 35n^4 + 2n^4 + 0 = 37n^4, \text{ for } n \geq 1$$

Thus with $C = 37$ and $k = 1$, f is $\mathrm{O}\left(n^4\right)$

(b) Again, the natural first guess should be the leading order term, which is n^3. But two terms have n^3 in them. since the $\log n$ is only going to make the second term larger than n^3 for $\log(n) > 1 (n > e)$ this means the first term will be smaller, and so this is our first guess for the big- Ω bound.

The lowest power of n that can serve as a lower bound is n^3. We get the lower bounds for each term in terms of n^3 :

$$35n^3 \geq 35n^3, \text{ for } n \geq 1$$
$$2n^3 \log(n) \geq 2n^3, \text{ for } n \geq e$$
$$4n^2 \log(n) \leq 4n^3 \longrightarrow -4n^2 \log(n) \geq -4n^3, \text{ for } n \geq 1$$

where the second line comes from the fact that $\log(n) > 1$ exactly when $n > e$ (by exponentiating both sides and using the fact that $e^{\log n} = n$). So we have:

$$f(n) \geq 35n^3 + 2n^3 - 4n^3 = 33n^3, \text{ for } n \geq e$$

Thus, with $C = 33$ and $k = e$, f is $\Omega\left(\mathbf{n}^3\right)$

(c) No, because it is not both big-O and big- Ωn^p for some function n^p.

5. Find a div b *and* a mod b when:

   (a) $a = 30303, b = 333$

   (b) $a = -765432, b = 38271$

   Solution:

   (a) a div b = 30303 div 333 = 91

   a mod b = 30303 mod 333 = 0 (no remainder)

   (b) $-765432 \operatorname{div} 38271 = -21$

   -765432 mod 38271
   $= -21 \cdot 38271 = -803691$
   $= -765432 + 803691$
   $= 38259$

6. Show that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b(\bmod m)$, where a and b are integers, then $a \equiv b(\bmod n)$.

   Solution:
   (1) If $a \equiv b(\bmod m)$, then $m \mid (a-b)$ by definition of congruence
   (2) There exists an integer k such that $(a-b) = mk$ by definition of division
   (3) Also, since $n \mid m$, there exists an integer b such that $m = nb$ by definition of division
   (4) $(a-b) = mk = (nb)k = n(bk)$ from combining steps 2 and 3
   (5) $\therefore n \mid (a-b)$ from step 4 by definition of division (and since b and k are integers $b \times k$ should be an integer too)
   (6) $\therefore a \equiv b(\bmod n)$ by definition of congruence

7. What is the big- O estimate of the function given in the pseudocode below if the size of the input is n? (a function that takes in a list of numbers as input and returns the biggest number) Justify your answer.

```
define function(input_list):

    for i from range 0 to length(input_list):

        min_idx = i

        for j from range i+1 to length(input_list):

            if input_list[min_idx] > input_list[j]:
                min_idx = j

        input_list[i], input_list[min_idx] = input_list[min_idx], input_list[i]

    return input_list[length(input_list)]
```

Solution:
If the input_list if of length n; then the outer for-loop (i) will loop over each element, and the nested for-loop will loop over (n-i) elements. This amounts to

$$n + (n-1) + (n-2) + ... + 1 = \frac{n(n+1)}{2} = \frac{n^2+n}{2} = O(n^2)$$