CSCI 2824 – Fall 2021

Homework 4

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This assignment is due on Friday, Oct. 1st to Gradescope by 6PM. There are 7 questions on this homework. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for extra credit on the assignment. Make sure that your images/scans are clear or you will lose points/possibly be given a 0. Additionally, please be sure to match the problems from the Gradescope outline to your uploaded images.

1. Give an example of two uncountable sets A and B with a nonempty intersection, such that $A - B$ is
   (a) Finite
   (b) Countably infinite
   (c) Uncountably infinite

   Solution:

   (a) $A = (-\infty, 1], B = (-\infty, 1)$ or some variation where only one number remains in the set.
   (b) $A = \{x \mid x \in \mathbb{R}\}, B = \{x \mid x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$ or some variation where a set containing all irrational numbers is subtracted from a larger set.
   (c) $A = [0, 2], B = [0, 1]$ or any variations where irrational numbers remain in the set $A - B$

2. Let A and B be two sets, then prove that: $A \oplus B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

   Solution: Let us consider the membership table for the sets given above. Here, 1 means that the item belong to the set and 0 means that the item does not belong to that set. The sets are given as columns:

A	B	**A ⊕ B**	A ∪ B	A ∩ B	**(A ∪ B)-(A ∩ B)**	A-B	B-A	**(A-B) ∪ (B-A)**
1	1	**0**	1	1	**0**	0	0	**0**
1	0	**1**	1	0	**1**	1	0	**1**
0	1	**1**	1	0	**1**	0	1	**1**
0	0	**0**	0	0	**0**	0	0	**0**

   From the above we can conclude that *all the members* $\in A \oplus B$ must also $\in (A \cup B) - (A \cap B)$ and $\in (A - B) \cup (B - A)$. and Therefore $A \oplus B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

3. Given two sets$A$ and B, for *each* of the following statements, what can you conclude about the sets?

   For example: consider the statement $A - B = \emptyset$, this could be possible if -
   Senario - 1: if $A = B$ then $A - B = \emptyset$
   Senario - 2: Since $A - B = A - (A \cap B)$, if $(A \cap B) = A$ then $A - B = \emptyset$
   Senario - 3: $A = \emptyset$ in which case, no matter what B is $A - B = \emptyset$.
   Therefore we can conclude that if $A - B = \emptyset$, then one of the above scenarios must be true. You do not need to draw exactly 3 conclusions. Try to answer with an exhaustive list of conclusions you can draw from each of the following statements:

   (a) $A \cup B = A$
   (b) $A \cap B = A$
   (c) $A - B = A$
   (d) $A \cap B = B \cap A$
   (e) $A - B = B - A$

   Solution:

   (a) $A \cup B = A$ - For all $b \in B$, we can know $b \in A$ also (B is subset of A).
   (b) $A \cap B = A$ - For all $a \in A$, we can know $a \in B$ also (A is subset of B).

(c) $A - B = A$ - For all $a \in A$, we can know $a \notin B$ (A and B disjoint).

(d) $A \cap B = B \cap A$ - We can not conclude anything from this.

(e) $A - B = B - A$ - We can conclude $A = B$.

4. Consider the following functions and determine if they are bijective. [A function is said to be bijective or bijection, if a function $f : A \to B$ is both one-to-one and onto.]

   (a) $f : Z \times Z \to Z, f(n,m) = n^2 + m^2$

   (b) $f : R \to R, f(x) = x^3 - 3$

   (c) $f : R \times R \to R, f(n,m) = 2m - n$

   Solution

   (a) $f(n,m)$ is not bijective because it is not one to one. For example, $f(1,m) = f(-1,m)$, for any m.

   (b) $g(x)$ is bijective because its inverse can be found. $g(x)^{-1} = (x+3)^{\frac{1}{3}}$

   (c) $f(n,m)$ is not bijective because it is not one to one. For example, $f(0,0) = f(-2,1) = 0$.

5. (a) Find the solution to $a_n = a_{n-1} + 2n + 3$ with the initial conditions $a_0 = 4$.

   (b) Consider the recurrence $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$, show that this recurrence is solved by:

      i. $a_n = 2 - n$

      ii. $a_n = 2 - n + b \cdot 2^n$ for any real b

   Solution:

   (a) $a_0 = 4 = 4n + 4$
   $a_1 = 4 + 2 + 3 = 5n + 4$
   $a_2 = 9 + 4 + 3 = 6n + 4$
   $a_n = (n+4)n + 4 = n^2 + 4n + 4$
   $\therefore a_n = (n+2)^2$

   (b) i. Assuming $a_n = 2 - n$

   then, $a_{n-1} = 2 - (n-1) = 2 + 1 - n = 3 - n$ and $a_{n-2} = 2 - (n-2) = 2 + 2 - n = 4 - n$

$$
\begin{aligned}
a_n &= a_{n-1} + 2a_{n-2} + 2n - 9 \\
2 - n &= (3-n) + 2(4-n) + 2n - 9 \\
&= 3 - n + 2 \cdot 4 - 2n + 2n - 9 \\
&= 3 - n + 8 - 2n + 2n - 9 \\
&= 3 - 9 + 8 - 2n + 2n - n \\
&= 3 - 1 - n = 2 - n
\end{aligned}
$$

   $\because$ LHS = RHS, $a_n = 2 - n$

   ii. Assuming $a_n = 2 - n + b \cdot 2^n$
   then, $a_{n-1} = 2 - (n-1) + +b \cdot 2^n = 2 + 1 - n + b \cdot 2^{n-1} = 3 - n + b \cdot 2^{n-1}$
   and $a_{n-2} = 2 - (n-2) + b \cdot 2^{n-2} = 2 + 2 - n + b \cdot 2^{n-2} = 4 - n + b \cdot 2^{n-2}$
   $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$
   $2 - n + b \cdot 2^n = (3 - n + b \cdot 2^{n-1}) + 2(4 - n + b \cdot 2^{n-2}) + 2n - 9$
   $= 3 - n + b \cdot 2^{n-1} + 2 \cdot 4 - 2n + 2 \cdot b \cdot 2^{n-2} + 2n - 9$
   $= 3 - n + b \cdot 2^{n-1} + 8 - 2n + b \cdot 2^{n-1} + 2n - 9$
   $= 3 + 8 - 9 - n - 2n + 2n + b \cdot 2^{n-1} + b \cdot 2^{n-1}$
   $= 2 - n + b \cdot 2 \cdot 2^{n-1}$
   $= 2 - n + b \cdot 2^n$
   LHS = RHS, So $a_n = 2 - n + b \cdot 2^n$.

6. (a) Compute the double summation:

$$\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$$

(b) Solve the following in terms of x [Here $|x| < 1$]:

$$\sum_{i=3}^{\infty} x^i$$

Solution:

(a) For a double summation, we must open the inner summation first:

$$\begin{aligned}
\sum_{i=0}^{2}\sum_{j=0}^{3} i^2 j^3 &= 0^3 \sum_{i=0}^{2} + 1^3 \sum_{i=0}^{2} + 2^3 \sum_{i=0}^{2} + 3^3 \sum_{i=0}^{2} \\
&= 0 \sum_{i=0}^{2} + 1 \sum_{i=0}^{2} + 8 \sum_{i=0}^{2} + 27 \sum_{i=0}^{2} \\
&= 0[0^2 + 1^2 + 2^2] + 1[0^2 + 1^2 + 2^2] + 8[0^2 + 1^2 + 2^2] + 27[0^2 + 1^2 + 2^2] \\
&= 0 \times 5 + 1 \times 5 + 8 \times 5 + 27 \times 5 \\
&= 0 + 5 + 40 + 135 = 180
\end{aligned}$$

(b) Since $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ for $|x| < 1$, we add and subtract $x^0 + x^1 + x^2$ to the given summation.

$$\sum_{i=3}^{\infty} x^i = \sum_{i=3}^{\infty} x^i + x^0 + x^1 + x^2 - x^0 - x^1 - x^2$$

By definition of summation, adding those terms will change the starting point

$$\begin{aligned}
&= \sum_{i=0}^{\infty} x^i - x^0 - x^1 - x^2 \\
&= \frac{1}{1-x} - x^0 - x^1 - x^2 = \frac{1}{1-x} - 1 - x - x^2
\end{aligned}$$

7. Find out if the following sets are Countable, Uncountable, Finite or if it cannot be determined. give the reasoning behind your answer for each.

(a) Subset of a countable set

(b) integers divisible by 5 but not by 7

(c) $(3, 5)$

(d) A - B (A is an Uncountable set and B is a Countable set)

(e) $\mathcal{P}(C)$ where C is a finite set

Solution:

(a) *Subset of a countable set*: Can be both - finite or countable. Since it is a subset,it's cardinality can range from 0 to cardinality of the superset.

(b) *integers divisible by 5 but not by 7*: Countable since we can still map the items in this set to Natural Numbers. If we map the factor of the number in the set (product of all factors except 5) we can map these numbers to N.

$$\begin{aligned}
5 \times 1 &\rightarrow 1 \\
5 \times 2 &\rightarrow 2 \\
&\ldots \\
5 \times 6 &\rightarrow 6 \\
5 \times 8 &\rightarrow 7 \\
5 \times 9 &\rightarrow 8
\end{aligned}$$

and so on. This way the countably infinite set has only items divisible by 5 and not 7.

(c) $(3, 5)$: Depends on what this range of numbers belongs to. By default we will assume $(3, 5) \in R$, so this set is uncountably infinite.

(d) *A - B (A is an Uncountable set and B is a Countable set)*: Uncountable. Removing the countably infinite would still not remove the elements that were uncountable.

(e) $\mathcal{P}(C)$ *where C is a finite set*: Finite, since $|C|$ is finite so $2^{|C|}$ must also be finite.