CSCI 2824 – Fall 2021

Homework 9

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This assignment is due on Friday, November 19th to Gradescope by 6PM. There are 6 questions on this homework. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for extra credit on the assignment. Make sure that your images/scans are clear or you will lose points/possibly be given a 0. Additionally, please be sure to match the problems from the Gradescope outline to your uploaded images.

Show the steps or give the reasoning behind all your answers.

1. In Monopoly, your token is allowed to leave the "jail" cell if you roll doubles: you roll two 6 -sided dice and each shows the same face. Zach hates being in jail, So he invents a couple of weighted dice that are not independent. In particular, if you roll either die on its own it's a fair die: each outcome has probability 1/6. But if you roll one die and then the other, the red die will take the same outcome as the blue die exactly half the time: all other outcomes are equally likely.

   (a) Suppose you roll a 3 on the blue die. What is the probability distribution of the red die given this outcome on the blue die?

   (b) What is the probability you roll doubles?

   (c) What is the probability that you roll a 7 as the sum of the two dice?

   Solution:

   (a) Since the red die will take the same outcome as the blue die half the time. Therefore, the probability of getting a 3 on the red die is $\frac{1}{2}$. Since the other outcomes are equally likely then the other $\frac{1}{2}$ must be equally divided into 5 parts. Therefore, the probability of getting any other number on the red die is $\frac{1}{2} \times \frac{1}{5}$.

X	1	2	3	4	5	6
P (X-Red—3-Blue)	0.1	0.1	0.5	0.1	0.1	0.1

   (b) Probability of Rolling any doubles, i.e. $\mathrm{P(x - Red \cap x - Blue)} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

   Total probability of rolling doubles:
   $\mathrm{P(1 - Red \cap 1 - Blue) + P(2 - Red \cap 2 - Blue) + P(3 - Red \cap 3 - Blue) + P(4 - Red \cap 4 - Blue) +}$
   $\mathrm{P(5 - Red \cap 5 - Blue) + P(6 - Red \cap 6 - Blue)}$
   $= 1/12 + 1/12 + 1/12 + 1/12 + 1/12 + 1/12 = 6/12 = 1/2$

   (c) $P(\sum = 7)$
   $= P(1\text{– Red} \cap 6 - \text{Blue}) + P(2\text{– Red} \cap 5\text{– Blue}) + P(3\text{– Red } 4\text{– Blue}) + P(4\text{– Red} \cap 3\text{– Blue}$
   $) + P(5\text{– Red} \cap 2\text{– Blue}) + P(6\text{– Red} \cap 1 - \text{Blue})$
   $= 0.0167 \times 6$
   $= 0.1002$

2. You have two machines A and B that, each, generate binary digits. On each machine, when the "run" button is pressed, it will generate a single binary digit. Machine A generates a **0** - 52% of the times and a **1** - 48% of the times. Machine B, however, has a memory slot that stores the latest bit generated. Machine B always starts by generating a **0** and storing this in the memory slot. Each time, after this initialization, machine B generates the new bit by checking the bit in the memory slot, and generates the new bit by flipping the bit 61% of times and overwrites the memory with this new bit. When machine B is turned off, the memory slot clears itself.

   (a) On using the 'run' feature 7 times on Machine A, what is the probability that the outcome has exactly five **0**'s?

   (b) What is the probability of each machine generating '00110' and '1001'?

   Solution:

(a) For Machine A: P(0) = 0.52 and P(1) = 0.48. P(0) and P(1) are also independent event, so we can use the Bernoulli trials. Let's consider the machine generating a **0** as a 'success' and the number of times 'run button' is used as the number of independent trials. Finally, we have the probability of five **0**'s in 7 trials:

$$\begin{aligned}\binom{7}{5} \cdot P(0)^5 \cdot P(1)^{7-5} &= \frac{7!}{5! \cdot 2!} \cdot (0.52)^5 \cdot (0.48)^2 \\ &= \frac{6 \cdot 7}{2} \cdot 0.038 \cdot 0.2304 \\ &= 21 \cdot 0.0087552 = 0.1838592\end{aligned}$$

(b) Probability of Machine A generating the strings "00110":
$P(0) \times P(0) \times P(1) \times P(1) \times P(0)$
$= 0.52 \times 0.52 \times 0.48 \times 0.48 \times 0.52 = 0.0324$

Probability of Machine A generating the strings "1001":
$P(1) \times P(0) \times P(0) \times P(1)$
$= 0.48 \times 0.52 \times 0.52 \times 0.48 = 0.0623$

What we know about Machine B:
P(current = 0 | memory = 1) = 0.61
P(current = 0 | memory = 0) = 1 - 0.61 = 0.39
P(current = 1 | memory = 1) = 1 - 0.61 = 0.39
P(current = 1 | memory = 0) = 0.61
P(current = 0 | memory = null) = 1
P(current = 1 | memory = null) = 0

Now, Probability of Machine B generating the strings "00110""
P(current = 0 | memory = null) × (current = 0 | memory = 0) × (current = 1 | memory = 0) × (current = 1 | memory = 1) × (current = 0 | memory = 1)
= 1 × 0.39 × 0.61 × 0.39 × 0.61 = 0.0566

Probability of Machine B generating the strings "1001""
P(current = 1 | memory = null) × (current = 0 | memory = 1) × (current = 0 | memory = 0) × (current = 1 | memory = 0)
= 0 × 0.61 × 0.39 × 0.61 = 0

3. Let R be the relation on the set $\{1, 2, 3, 4\}$, where R = $\{(1, 1), (1, 2), (2, 3), (3, 1), (3, 4)$ (4,2)$\}$. Find $-\mathrm{R}^2, \mathrm{R}^3$ and R^4

Solution:

$R^2 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 4), (3, 1), (3, 2), (4, 3)\}$
$R^3 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 4)\}$
$R^4 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)\}$

4. You have to maintain a record of residents in a neighborhood. The field are given as: Name of the resident, address, Phone number, date of birth and social security number. Which would you choose as the primary key, composite key and why?

Primary Key: SSN, since no two entries will have the same SSN.
Composite key: (open ended question, multiple correct answers) unique identifier composed of a Cartesian product of two or more fields.

5. Let S be the relation on the set **R** (real numbers) defined by xSy, if and only if x − y is an integer. Prove that S is an equivalence relation on **R**.

Solution:

To prove that S is an equivalence relation on R, we need to show that it is reflexive, symmetric, and Transitive.

Reflexive:
If x is an element of all real numbers, then we can assume that $x - x = 0.0$ is an integer and therefore is also reflexive xSx.

Symmetric:
If both x and y are an element of all real numbers and x Sy, then $x - y$ is an integer. The equation "$\mathrm{y} - \mathrm{x}''$ is also an integer that can be proven, $\mathrm{y} - \mathrm{x} = -(\mathrm{x} - \mathrm{y})$. Therefore, the relation is also symmetric ySx.

Transitive:
If both x and y are an element of all real numbers and the previous statement is true (xSy and ySx), then both $x - y$ and $y - x$ are integers. We can now show that the relation is transitive using the equation, $(x - y) + (y - z) = x - z$. This equation would also be an integer and therefore, xSz.

6. *Prove or disprove* that if R and S are antisymmetric, then so is:

   (a) $(\mathrm{R} \cup \mathrm{S})$

   (b) $(\mathrm{R} \cap \mathrm{S})$

   Solution:

   (a) For $(R \cup S)$, we can disprove that the union must be antisymmetric with direct proof by counter example.
   Suppose $R = \{(1,2)\}$ and $S = \{(2,1)\}$. Each of those individual relations are antisymmetric, however the union is $(R \cup S) = \{(1,2),(2,1)\}$ which is therefor symmetric and not anti symmetric as the symmetry is not from $1 = 2$. This directly shows that R and S being antisymmetric does not mean the union will be anti symmetric.

   (b) The intersection $(R \cap S)$ on the other hand can be proved that if R and S are antisymmetric then the intersection will be too.
   Using a direct proof we can see that R being antisymmetric will have no elements of $a, b \in A$ where both $(a, b) \in R$ and $(b, a) \in R$ unless $a = b$. Therefor, when an intersection takes place with S, the elements in R that are not in S are removed, however removing those elements will not create any additional symmetry as elements must be added to create symmetry. Therefore the intersection of R and S will remain antisymmetric.