场论与无穷级数习题解答

1. $\mathrm{f(x)} = \mathrm{x} + \frac{1}{2}e^{-2x} - \frac{1}{2}$

2. $\mathrm{f(x)} = (4\pi t^2 + 1)e^{4\pi t^2}$

3. （1）$\mathrm{s'(x) - xs(x)} = \frac{x^3}{2}$

（2）$\mathrm{s(x)} = e^{\frac{x^2}{2}} - \frac{x^2}{2} - 1$

4. $\mathrm{y} = -\mathrm{x}e^x + x + 2$

5. $\mathrm{f(u)} = C_1e^u + C_2e^{-u}$

6. $\mathrm{y'' - y' - 2y} = (1 - 2\mathrm{x})e^x$

7. $\because \left|\frac{\sin(na)}{n^2}\right| < \frac{1}{n^2}$ $\therefore \sum_{n=1}^{\infty}\frac{\sin(na)}{n^2}$绝对收敛

$\because \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$发散 $\therefore \sum_{n=1}^{\infty}(\frac{\sin(na)}{n^2} - \frac{1}{\sqrt{n}})$发散

8. A 反例：$a_n = \frac{(-1)^n}{\sqrt{n}},\ b_n = \frac{(-1)^n}{\sqrt{n}}$;

B 反例：$a_n = \frac{1}{n},\ b_n = \frac{1}{n}$;

D 反例：$a_n = \frac{1}{n},\ b_n = \frac{1}{n}$;

C 证明：$\exists\mathrm{N}$, 当$\mathrm{n} > \mathrm{N}$时, $|a_n| < 1, |b_n| < 1$, 即$b_n^2 < |b_n|$

$$\therefore \sum_{n=1}^{\infty} a_n^2b_n^2 < \sum_{n=1}^{N} a_n^2b_n^2 + \sum_{n=N+1}^{\infty}|b_n|, \text{故收敛}$$

9. 考察其部分和：$S_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{u_k} + \sum_{k=1}^{n}\frac{(-1)^{k+1}}{u_{k+1}}$

$$= \sum_{k=1}^{n}\frac{(-1)^{k+1}}{u_k} + \sum_{t=2}^{n+1}\frac{(-1)^t}{u_t}$$

$$= \frac{1}{u_1} + \frac{(-1)^{n+1}}{u_{n+1}}$$

$\therefore \lim_{n\to+\infty} S_n = \frac{1}{u_1}$, 故原级数收敛

$\because \lim\limits_{n\to+\infty}\left(\frac{n}{u_n}+\frac{n}{u_{n+1}}\right)=\lim\limits_{n\to+\infty}\frac{n}{u_n}+\lim\limits_{n\to+\infty}\frac{n+1}{u_{n+1}}\cdot\frac{n}{n+1}=2$

$\therefore \sum_{n=1}^{\infty}(\frac{1}{u_n}+\frac{1}{u_{n+1}})$发散

∴原级数条件收敛

10. (1) $a_n+a_{n+2}=\int_0^{\frac{\pi}{4}}sec^2xtanxdx=\frac{1}{n+1}$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n}(a_n+a_{n+2})=\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{n+1}=\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})=1$$

(2) $\because a_n<\frac{1}{n+1}\quad \therefore \frac{a_n}{n^\lambda}<\frac{1}{(n+1)n^\lambda}<\frac{1}{n^{\lambda+1}}$

$\because \sum_{n=1}^{\infty}\frac{1}{n^{\lambda+1}}$ 对 $\forall$常数 $\lambda>0$ 都收敛，故原级数收敛

11. (1) $a_n-a_{n+1}=\frac{a_n^2-1}{2a_n}$

$$\because a_n=\frac{1}{2}(a_{n-1}+\frac{1}{a_{n-1}})\geq 1$$

$\therefore a_n-a_{n+1}\geq 0$, 即$\{a_n\}$单调递减且有下界

(2) 设$u_n=\frac{a_n}{a_{n+1}}-1$, 则$\lim\limits_{n\to+\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to+\infty}\frac{a_{n+1}^2-a_{n+1}a_{n+2}}{a_na_{n+2}-a_{n+1}a_{n+2}}$

易证$a_{n+1}^2<a_na_{n+2}$

$\therefore \lim\limits_{n\to+\infty}\frac{u_{n+1}}{u_n}<1$, ∴原级数收敛

12. $(-2,4)$

13. $\mathrm{S(x)}=(x^2+3\mathrm{x}+1)e^x$

14. $\mathrm{S(x)}=\begin{cases}\frac{1}{x}ln\sqrt{\frac{1+x}{1-x}}-\frac{x^2}{1-x^2}-1,\ \mathrm{x}\neq 0\\ 0,\ \mathrm{x}=0\end{cases}$

15. $\mathrm{S(x)}=\sum_{n=1}^{\infty}(n^2-n+1)x^n=\frac{x^3+x}{(1-x)^3}, \mathrm{S}\left(-\frac{1}{2}\right)=-\frac{5}{27}$

16. $\mathrm{f(x)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^n-1}{n}x^n$

17. $\mathrm{f(x)}=\arctan\frac{1-2x}{1+2x}=arctan1-arctan2x$

$$= \frac{\pi}{4} - \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{2n+1}$$

$$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} - \mathrm{f}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

18. $\mathrm{f(x)} = 1 - \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}[\frac{(-1)^{n-1}}{n^2} cosnx] \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

19. D

20. D

21. 6

22. π

23. $\frac{\pi}{2}$

24. ln2

25. $\mathrm{S} = \int_{-\infty}^{1} e^x dx - \frac{e}{2} = \frac{e}{2}$