线性代数

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线性代数理论

- 大学数学课的基础课程

- 理工科、经济类专业课的数学基础

- 自动化、人工智能、计算机、电子信息、光学、通信、金融、经济等学科数学基础

- 工作和科研必备的数学储备

考核方式

- 比例：平时成绩20％，期末测试80％。
- 考核形式：闭卷，不准使用计算器，不准自带草稿纸，不准用手机，带着准考证/学生证等能证明自己身份的有照片的证件。
- 考试时间：100分钟。
- 试卷结构：

　　1．总分：100分

　　2．题型：

　　单选题；　　　　　　判断题；

　　计算题；　　　　　　证明题。

　　3．容易、中等难度、难题的分值比例约为3:6:1。

考试范围

- 第1－7章所讲内容
- 下面内容特殊处理

　　厄米特矩阵和酉矩阵不考

　　线性变换的核不考

　　(*Jordan*)标准形不考

　　补充的分块矩阵应用建议使用

如何学好线性代数

- 理解最重要

- 没有理解的时候先记住

- 大量的练习既可以帮助记忆，又可以帮助理解

- 建议读其他教材，看其他视频

- 科研实例中体会

第一章 行列式

本章主要内容

➤ *n*阶行列式的定义

➤ 行列式的主要性质

➤ 行列式按行（列）展开

第一节 n阶行列式的定义

§ 1.1.1 二、三阶行列式的定义

一、二阶行列式的引入

用消元法解二元线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, & (1) \\ a_{21}x_1 + a_{22}x_2 = b_2. & (2) \end{cases}$$

$(1) \times a_{22}: \quad a_{11}a_{22}x_1 + \boxed{a_{12}a_{22}}x_2 = b_1a_{22},$

$(2) \times a_{12}: \quad a_{12}a_{21}x_1 + \boxed{a_{12}a_{22}}x_2 = b_2a_{12},$

两式相减消去 x_2，得

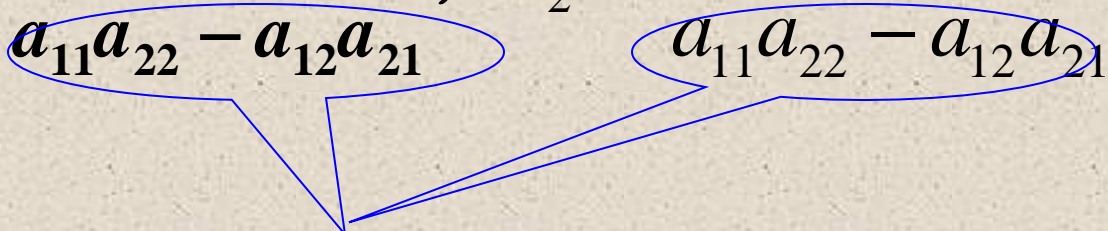$$\left(a_{11}a_{22} - a_{12}a_{21}\right)x_1 = b_1a_{22} - a_{12}b_2;$$

$$（a_{11}a_{22} - a_{12}a_{21}）x_1 = b_1a_{22} - a_{12}b_2;$$

类似地，消去 x_1，得
$$（a_{11}a_{22} - a_{12}a_{21}）x_2 = a_{11}b_2 - b_1a_{21},$$

当 $a_{11}a_{22} - a_{12}a_{21} \neq 0$ 时，方程组的解为

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}.$$

由方程组的四个系数确定.

为便于记忆和表达

用记号 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 表示 $a_{11}a_{22} - a_{12}a_{21}$ 并称之为一个
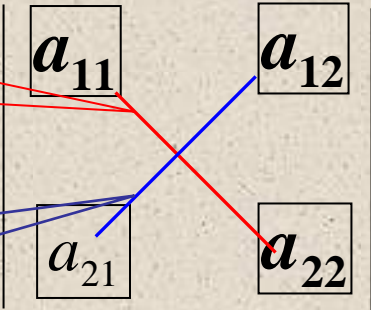
二阶(级)行列式, 即

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$
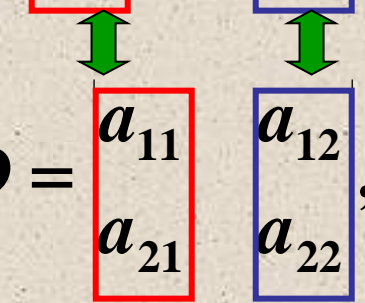
二阶行列式的计算——对角线法则

主对角线

副对角线

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

对于二元线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

若记

系数行列式

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2, \end{cases}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1,} \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2,} \end{cases} \qquad \boxed{x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}},}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1a_{22} - a_{12}b_2,$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1,} \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2,} \end{cases}$$

$$D = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & \boxed{a_{22}} \end{vmatrix},$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2}, \end{cases}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - a_{12} b_2,$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2}, \end{cases} \qquad x_2 = \frac{a_{11}b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}.$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = a_{11}b_2 - b_1 a_{21}.$$

则二元线性方程组的解为

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

注意　分母都为原方程组的系数行列式.

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}, \qquad x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}.$$

二、三阶行列式

定义 设有9个数排成3行3列的数表

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \qquad (3)$$

记

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \qquad (4)$$
$$ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31,}$$

（4）式称为数表（3）所确定的**三阶行列式**.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

· 列标

行标

三阶行列式的计算

沙路法 $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

$-$ $-$ $-$ $+$ $+$ $+$

$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$

三阶行列式包括3!项, 每一项都是位于不同行, 不同列的三个元素的乘积, 其中三项为正, 三项为负.

利用三阶行列式求解三元线性方程组

如果三元线性方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3; \end{cases}$

的系数行列式 $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3; \end{cases}$$

若记
$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

或
$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3; \end{cases}$$

若记

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

或

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

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$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3; \end{cases}$$
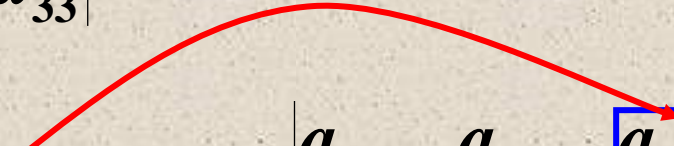
$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \boxed{b_2}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \boxed{b_3}; \end{cases}$$

得 $\qquad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \boxed{b_2}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \boxed{b_3}; \end{cases} \qquad D = \begin{vmatrix} a_{11} & a_{12} & \boxed{a_{13}} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}$$

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$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \boxed{b_2}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \boxed{b_3}; \end{cases}$$

得 $\qquad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \boxed{b_2}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \boxed{b_3}; \end{cases} \Rightarrow D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

则三元线性方程组的解为:

$$x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}, \qquad x_3 = \frac{D_3}{D}.$$

三、小结

二阶和三阶行列式是由解二元和三元线性方程组引入的.

二阶和三阶行列式的计算 ——对角线法则

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31},$$

思考

- 1. n阶行列式如何定义？
- 2. n元线性方程组的解是什么形式？