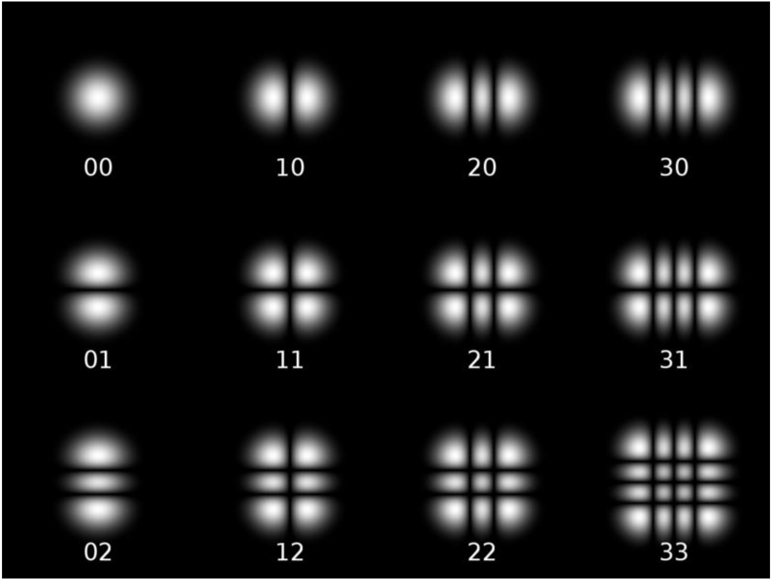
Transverse modes (TEM_{mn})

Hermite-Gaussian

Brewster's windows in cavity



Gaussian modes (TEM_{mn})

Hermite-Gaussian

$$E_{mn}(x,y,z) = E_0 \frac{w_0}{w(z)} H_m\left(\frac{\sqrt{2}x}{w}\right) H_n\left(\frac{\sqrt{2}y}{w}\right) \times$$
$$\times \exp\left[-\frac{(x^2+y^2)}{w^2(z)} - ik\frac{(x^2+y^2)}{2R(z)} - ikz + i(m+n+1)\phi(z)\right]$$

$$H_0(u) = 1$$
$$H_1(u) = 2u$$
$$H_2(u) = 2(2u^2-1)$$
$$\dots$$
$$H_m(u) = (-1)^m e^{u^2} \frac{d^m\left(e^{-u^2}\right)}{du^m}$$

Hermite's polynomials

Gaussian modes (TEM_{mn})

$$w^2(z) = w_0^2\left[1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2\right] = w_0^2\left[1+\left(\frac{z}{z_0}\right)^2\right]$$

Beam spot radius

$$w_0 = w(z=0)$$

Beam waist

$$z_0 = \frac{\pi w_0^2}{\lambda}$$

Rayleigh range

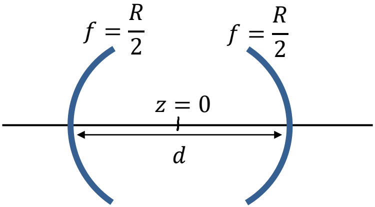
$$R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right]$$

Wavefront curvature radius

$$\phi(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$$

Phase factor

Let's consider a symmetrical cavity formed by two concave mirrors with radius of curvature R, at a distance d:



curvature radius of the mirrors (at $z = \frac{d}{2}$)

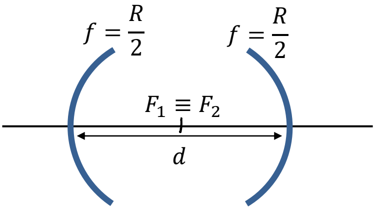
Wavefront curvature radius

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right] \implies R\left(z = \frac{d}{2}\right) = \frac{d}{2} + \frac{2z_0^2}{d} = R$$

$$R = \frac{d^2 + 4z_0^2}{2d} \implies z_0^2 = \frac{d(2R-d)}{4} \implies z_0 = \frac{\sqrt{d(2R-d)}}{2} = \frac{\pi w_0^2}{\lambda}$$

$$\implies w_0^2 = \frac{\lambda}{2\pi}\sqrt{d(2R-d)}$$

For a **confocal resonator**: $d = R$



$$z_0 = \frac{\sqrt{d(2R-d)}}{2} = \frac{d}{2} \quad \Longrightarrow \quad w_0^2 = \frac{\lambda}{2\pi}\sqrt{d(2R-d)} = \frac{\lambda d}{2\pi} \qquad w_0 = \sqrt{\frac{\lambda d}{2\pi}}$$

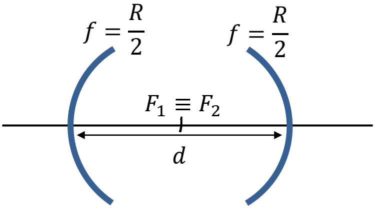
$$w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right] \quad \Longrightarrow \quad w^2\left(z = \frac{d}{2}\right) = w_0^2\left[1 + \left(\frac{d/2}{d/2}\right)^2\right]$$

$$\Longrightarrow \quad w\left(\frac{d}{2}\right) = \sqrt{2}w_0 = \sqrt{\frac{\lambda d}{\pi}}$$

For a **confocal resonator**: $d = R$

$$d = R = 1\ m$$

$$\lambda = 532\ nm$$

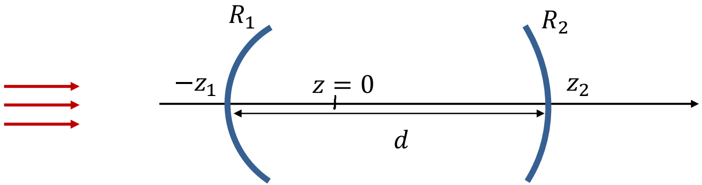


$$z_0 = \frac{d}{2} = 0.5\ m \qquad w_0 = \sqrt{\frac{\lambda z_0}{\pi}} = \sqrt{\frac{\lambda d}{2\pi}} = 0.29\ mm$$

$$w\left(\frac{d}{2}\right) = w(z_0) = \sqrt{2}w_0 = 0.41\ mm$$

$$\vartheta = \frac{2\lambda}{\pi w_0} = 1.2\ 10^{-3}\ rad = 0.07°$$ Far-field beam divergence

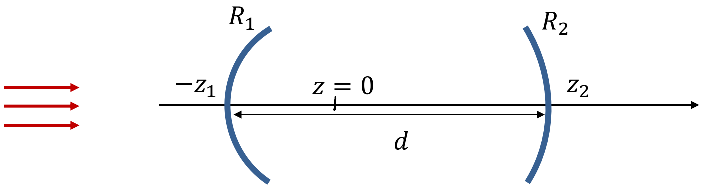
Let's consider a two-mirror **asymmetrical** cavity:



$$
\begin{cases}
d = z_2 + z_1 \\
R(z_2) = z_2\left[1 + \left(\dfrac{z_0}{z_2}\right)^2\right] = R_2 \\
R(-z_1) = -z_1\left[1 + \left(\dfrac{z_0}{-z_1}\right)^2\right] = -R_1
\end{cases}
$$

The "$-$" sign has to be considered here since the system of reference is unique (from left to right) and thus mirror 1 is seen having an opposite radius with respect to how it is seen from inside the cavity (e.g., convex instead of concave in the example in the figure) for a light beam coming from left.

Let's consider a two-mirror **asymmetrical** cavity:



$$\begin{cases} z_2^2 + z_0^2 = R_2 z_2 \\ z_1^2 + z_0^2 = R_1 z_1 \end{cases} \Longrightarrow \begin{cases} z_0^2 = R_2 z_2 - z_2^2 & (*) \\ z_1^2 + R_2 z_2 - z_2^2 - R_1 z_1 = 0 & (**) \end{cases}$$

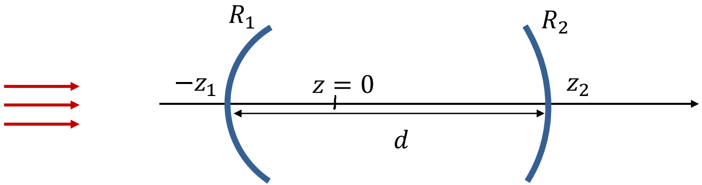
(*) $\quad z_0^2 = R_2 z_2 - z_2^2 = z_2(R_2 - z_2) \quad \Longrightarrow \quad z_0 = \sqrt{z_2(R_2 - z_2)}$

$z_1 = d - z_2 \quad \Longrightarrow \quad$ substituting in (**) we get

$$(d - z_2)^2 + R_2 z_2 - z_2^2 - R_1(d - z_2) = 0 \quad \Longrightarrow \quad z_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d}$$

Optical resonators

Let's consider a two-mirror **asymmetrical** cavity:



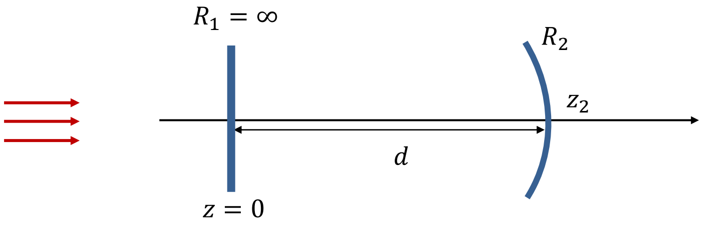
$$z_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d} \qquad z_0 = \sqrt{z_2(R_2 - z_2)} \qquad z_1 = d - z_2$$

$0 < z_2 < d \quad \Longrightarrow \quad$ the beam waist is inside the cavity

If z_0 is complex number ($z_0 \in \mathbb{C}$) the cavity becomes **unstable**

Optical resonators

Let's consider a two-mirror **asymmetrical** cavity:



$$z_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d} \qquad z_0 = \sqrt{z_2(R_2 - z_2)} \qquad z_1 = d - z_2$$

If one of the two mirrors is a plane mirror ($R = \infty$), then the beam waist will be at the plane mirror and its position will be at $z = 0$

$$w(z=0) = w_0 \qquad R(z=0) = \infty \qquad \Longrightarrow \begin{cases} z_2 = d \\ z_0 = \sqrt{d(R_2 - d)} \end{cases}$$

Optical resonators

Q: A CO_2 laser ($\lambda = 10.6\ \mu m$) has an asymmetric two-mirror optical cavity formed by two concave mirrors with curvature radii of $R_1 = 20\ m$ and $R_2 = 10\ m$, at a distance $d = 1.5\ m$.

Determine:

- size and position of the beam waist
- beam radius at each mirror
- Rayleigh range of the beam extracted from the cavity

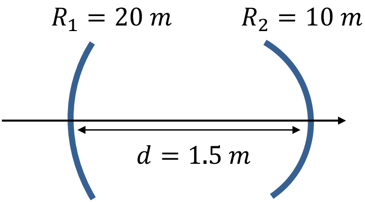
Is the resonator stable or unstable?

A: Stability condition $0 \leq g_1 g_2 \leq 1$

$$g_1 = 1 - \frac{d}{R_1} = 1 - \frac{1.5}{20} = 0.925$$

$$g_2 = 1 - \frac{d}{R_2} = 1 - \frac{1.5}{10} = 0.85$$

$$0 \leq g_1 g_2 = 0.786 \leq 1 \quad \Longrightarrow \quad \text{stable}$$



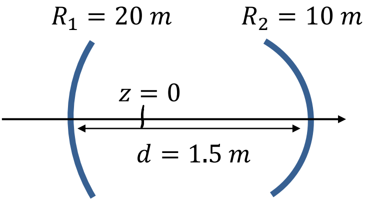
Q: A CO_2 laser ($\lambda = 10.6\ \mu m$) has an asymmetric two-mirror optical cavity formed by two concave mirrors with curvature radii of $R_1 = 20\ m$ and $R_2 = 10\ m$, at a distance $d = 1.5\ m$.
Determine:

- size and position of the beam waist
- beam radius at each mirror
- Rayleigh range of the beam extracted from the cavity

Is the resonator stable or unstable?

A:

$$z_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d} = \frac{1.5(20 - 1.5)}{20 + 10 - 3} = 1.03\ m$$



$$0 < z_2 < d = 1.5\ m$$

$\Longrightarrow$ the beam waist is inside the cavity

Optical resonators

Q: A CO_2 laser ($\lambda = 10.6\ \mu m$) has an asymmetric two-mirror optical cavity formed by two concave mirrors with curvature radii of $R_1 = 20\ m$ and $R_2 = 10\ m$, at a distance $d = 1.5\ m$.
Determine:

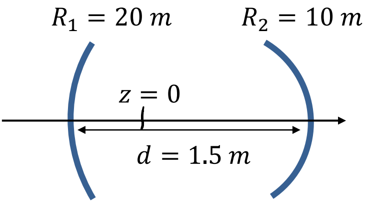
- size and position of the beam waist
- beam radius at each mirror
- Rayleigh range of the beam extracted from the cavity

Is the resonator stable or unstable?

A:

$$z_0 = \sqrt{z_2(R_2 - z_2)} = \sqrt{1.03(10 - 1.03)} = 3.04\ m = \frac{\pi w_0^2}{\lambda}$$

$$\Longrightarrow \quad w_0 = \sqrt{\frac{\lambda z_0}{\pi}} = 3.2\ mm$$



Q: A CO_2 laser ($\lambda = 10.6\ \mu m$) has an asymmetric two-mirror optical cavity formed by two concave mirrors with curvature radii of $R_1 = 20\ m$ and $R_2 = 10\ m$, at a distance $d = 1.5\ m$.

Determine:

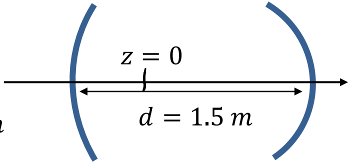
- size and position of the beam waist
- beam radius at each mirror
- Rayleigh range of the beam extracted from the cavity

Is the resonator stable or unstable?

A:

$$w_2 = w(z_2) = w_0\sqrt{1 + \left(\frac{z_2}{z_0}\right)^2} = 3.38\ mm$$

$$w_1 = w(z_1) = w_0\sqrt{1 + \left(\frac{z_2 - d}{z_0}\right)^2} = 3.24\ mm$$



Gaussian beam propagation

The properties of a **Gaussian beam** are totally determined **in every point** of the space if it is known the **beam waist** and the **curvature** radius **in one specific point**.

The ABCD matrix method allows us to calculate the propagation of a Gaussian beam through any optical element.

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\,\frac{\lambda}{n\pi w^2(z)}$$

Complex parameter of the Gaussian beam

$q(z)$ contains information on λ, the radius of curvature $R(z)$ and the waist $w(z)$ of the beam in the point z

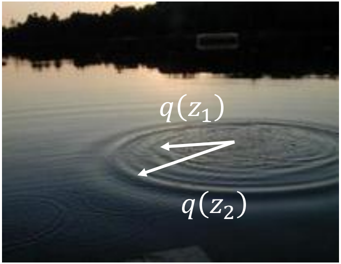
n = refractive index of the medium

$$w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right] \qquad z_0 = \frac{n\pi w_0^2}{\lambda} \qquad R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

Gaussian beam propagation

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi w^2(z)} = \frac{1}{q_0 + z}$$

$$q_0 = q(z=0) = i\frac{n\pi w_0^2}{\lambda} = iz_0$$



$\Longrightarrow$ $q(z)$ is the **complex curvature radius** of the Gaussian beam

it transforms itself as the curvature radius of a spherical wave:

$$q(z_2) = q(z_1) + (z_2 - z_1)$$

n = refractive index of the medium

$$w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right] \qquad z_0 = \frac{n\pi w_0^2}{\lambda} \qquad R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

Gaussian beam propagation

Given the parameter $q_1 = q(z_1)$ in a point z_1, the parameter $q_2 = q(z_2)$ in the point z_2 after the interaction of the beam with an optical element with $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ matrix can be obtained applying the **propagation relation**:

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \quad \Longrightarrow \quad \frac{1}{q_2} = \frac{C + D\left(\frac{1}{q_1}\right)}{A + B\left(\frac{1}{q_1}\right)} = \frac{1}{R_2(z)} - i\frac{\lambda}{n\pi w_2^2(z)}$$

Free-space propagation (for a distance z)

Let's assume that at the coordinate $z_1 = 0$ the beam has plane wavefront $R(z_1) = \infty$ and beam waist $w(z_1) = w_0$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \qquad \frac{1}{q_1} = -i\frac{\lambda}{n\pi w_0^2} = -\frac{i}{z_0} = \frac{1}{iz_0} \qquad z_0 = \frac{n\pi w_0^2}{\lambda}$$

Gaussian beam propagation

$$\frac{1}{q_2} = \frac{C + D\left(\frac{1}{q_1}\right)}{A + B\left(\frac{1}{q_1}\right)} = \frac{0 + \frac{1}{q_1}}{1 + z\frac{1}{q_1}} = \frac{\frac{1}{iz_0}}{1 + \frac{z}{iz_0}} = \frac{1}{z + iz_0} = \frac{z - iz_0}{(z + iz_0)(z - iz_0)}$$

$$= \frac{z - iz_0}{z^2 + z_0^2} = \frac{z}{z^2 + z_0^2} - i\frac{z_0}{z^2 + z_0^2} = \frac{1}{z\left[1 + \left(\frac{z_0}{z}\right)^2\right]} - i\frac{1}{z_0\left[1 + \left(\frac{z}{z_0}\right)^2\right]}$$

$$= \frac{1}{R(z)} - i\frac{\lambda}{n\pi w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right]} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi w^2(z)}$$

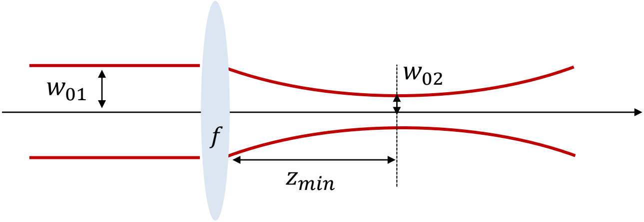
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \qquad \frac{1}{q_1} = -i\frac{\lambda}{n\pi w_0^2} = -\frac{i}{z_0} = \frac{1}{iz_0} \qquad z_0 = \frac{n\pi w_0^2}{\lambda}$$

Gaussian beam with plane wavefront impinging on a thin lens (with focal length f)

$$R(z_1) = \infty$$

$$w(z_1) = w_{01}$$

$$z_{01} = \frac{n\pi w_{01}^2}{\lambda}$$



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{z}{f} & z \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

$$\frac{1}{q_1} = -i\,\frac{\lambda}{n\pi w_{01}^2} = -i\,\frac{1}{z_{01}} \quad \Longrightarrow \quad \frac{1}{q_2} = \frac{C + D\left(\dfrac{1}{q_1}\right)}{A + B\left(\dfrac{1}{q_1}\right)} = \frac{-\dfrac{1}{f} + \dfrac{1}{q_1}}{1 - \dfrac{z}{f} + \dfrac{z}{q_1}}$$

$$\frac{1}{q_2} = \frac{-\frac{1}{f} + \frac{1}{q_1}}{1 - \frac{z}{f} + \frac{z}{q_1}} = \frac{-\frac{1}{f} - \frac{i}{z_{01}}}{1 - \frac{z}{f} - \frac{iz}{z_{01}}} = \frac{\left(-\frac{1}{f} - \frac{i}{z_{01}}\right)\left(1 - \frac{z}{f} + \frac{iz}{z_{01}}\right)}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}$$

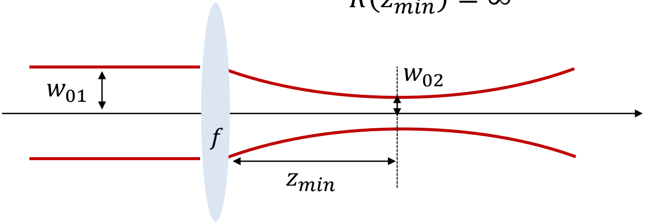
$$= \frac{-\frac{1}{f} + \frac{z}{f^2} - \cancel{\frac{iz}{z_{01}f}} - \frac{i}{z_{01}} + \cancel{\frac{iz}{z_{01}f}} + \frac{z}{z_{01}^2}}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}$$

$$= \frac{-\frac{1}{f} + z\left(\frac{1}{f^2} + \frac{1}{z_{01}^2}\right)}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2} - i\frac{\frac{1}{z_{01}}}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}$$

plane wavefront

$$= \frac{1}{R(z)} - i\frac{\lambda}{n\pi w^2(z)} \quad \Longrightarrow \quad z = z_{min} \qquad R(z_{min}) = \infty$$

$$\frac{1}{R(z_{min})} = 0 \qquad R(z_{min}) = \infty$$



$$\frac{1}{q_2} = \boxed{\frac{-\frac{1}{f} + z\left(\frac{1}{f^2} + \frac{1}{z_{01}^2}\right)}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}} - i\,\frac{\frac{1}{z_{01}}}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2} = \boxed{\frac{1}{R(z)}} - i\,\frac{\lambda}{n\pi w^2(z)}$$

$$\Longrightarrow \; -\frac{1}{f} + z_{min}\left(\frac{1}{f^2} + \frac{1}{z_{01}^2}\right) = 0 \quad \Longrightarrow \quad z_{min} = \frac{\frac{1}{f}}{\frac{1}{f^2} + \frac{1}{z_{01}^2}} = \frac{f}{1 + \frac{f^2}{z_{01}^2}} < f \quad !!$$

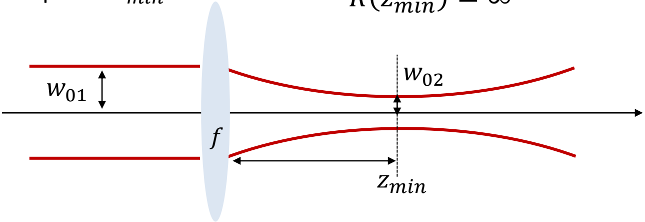
$$z_{01} \gg f \quad \Longrightarrow \quad z_{min} \cong f$$

To determine the beam spot at z_{min}:

$$R(z_{min}) = \infty$$

$$w_{02} = w(z_{min})$$

$$z_{01} = \frac{n\pi w_{01}^2}{\lambda}$$



$$\frac{1}{q_2} = \frac{-\frac{1}{f} + z\left(\frac{1}{f^2} + \frac{1}{z_{01}^2}\right)}{\left(1-\frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2} - i\boxed{\frac{\frac{1}{z_{01}}}{\left(1-\frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}} = \frac{1}{R(z)} - i\boxed{\frac{\lambda}{n\pi w^2(z)}}$$

$$\Longrightarrow \quad \frac{1}{z_{01}\left[\left(1-\frac{z_{min}}{f}\right)^2 + \left(\frac{z_{min}}{z_{01}}\right)^2\right]} = \frac{\lambda}{n\pi w_{01}^2\left[\left(1-\frac{z_{min}}{f}\right)^2 + \left(\frac{z_{min}}{z_{01}}\right)^2\right]} = \frac{\lambda}{n\pi w^2(z_{min})}$$

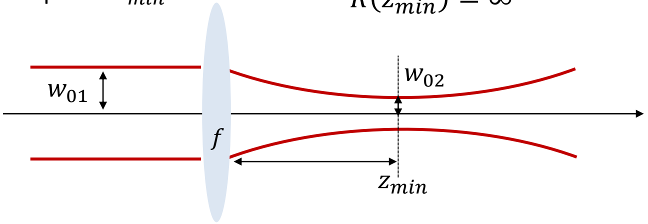
Gaussian beam propagation

To determine the beam spot at z_{min}:

$$R(z_{min}) = \infty$$

$$w_{02} = w(z_{min})$$

$$z_{01} = \frac{n\pi w_{01}^2}{\lambda}$$



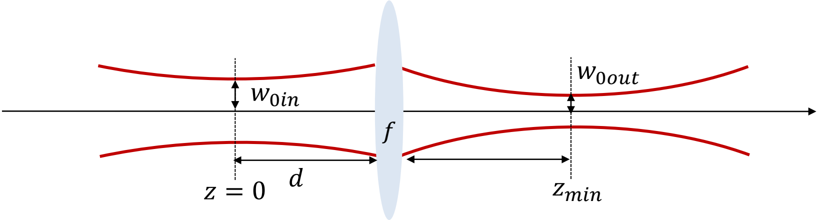
$$w_{02}^2 = w^2(z_{min}) = w_{01}^2\left[\left(1-\frac{z_{min}}{f}\right)^2 + \left(\frac{z_{min}}{z_{01}}\right)^2\right] \qquad z_{min} = \frac{f z_{01}^2}{f^2+z_{01}^2}$$

$$= \frac{w_{01}^2 f^2}{f^2+z_{01}^2} \quad \Longrightarrow \quad w_{02} = \frac{w_{01} f}{z_{01}\sqrt{1+\left(\frac{f}{z_{01}}\right)^2}} \cong \frac{w_{01} f}{z_{01}} \qquad z_{01} \gg f$$

$$= \frac{w_{01}^2 f^2}{z_{01}^2\left[1+\left(\frac{f}{z_{01}}\right)^2\right]}$$

$$w_{02} \cong \frac{\lambda f}{n\pi w_{01}}$$

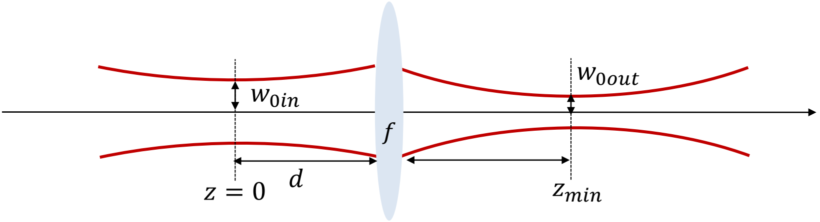
Determine the output waist of a Gaussian beam impinging on a thin lens with focal length f, whose input waist is at a distance d before the lens.



$$z_{0in} = \frac{n\pi w_{0in}^2}{\lambda} \qquad R(z=0) = \infty \qquad \frac{1}{q_{in}} = -i\frac{\lambda}{n\pi w_{0in}^2} = -i\frac{1}{z_{0in}}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{z}{f} & d + z\left(1 - \frac{d}{f}\right) \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix}$$

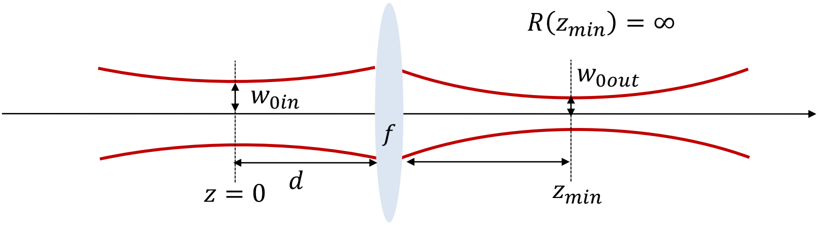
Determine the output waist of a Gaussian beam impinging on a thin lens with focal length f, whose input waist is at a distance d before the lens.



$$z_{0in} = \frac{n\pi w_{0in}^2}{\lambda} \qquad R(z=0) = \infty \qquad \frac{1}{q_{in}} = -i\frac{\lambda}{n\pi w_{0in}^2} = -i\frac{1}{z_{0in}}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 - \frac{z}{f} & d + z\left(1 - \frac{d}{f}\right) \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix} \qquad \frac{1}{q_{out}} = \frac{C + D\left(\frac{1}{q_{in}}\right)}{A + B\left(\frac{1}{q_{in}}\right)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi w^2(z)}$$

Determine the output waist of a Gaussian beam impinging on a thin lens with focal length f, whose input waist is at a distance d before the lens.



$$\frac{1}{R(z_{min})} = 0 \quad \Longrightarrow \quad z\left[(f-d)^2 + z_{0in}^2\right] + fd(f-d) - z_{0in}^2 f = 0$$

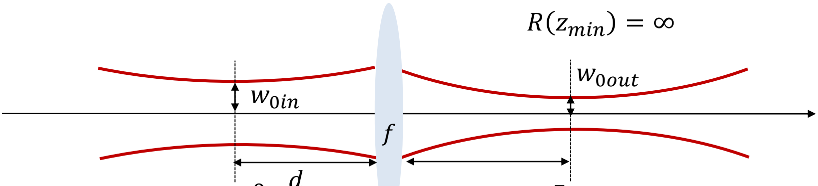
$$z_{min} = \frac{z_{0in}^2 f - fd(f-d)}{(f-d)^2 + z_{0in}^2}$$

Let's assume the lens is converging ($f > 0$) and $d = f$

$$\Longrightarrow \quad z_{min} = \frac{z_{0in}^2 f - f^2(f-f)}{(f-f)^2 + z_{0in}^2} = \frac{z_{0in}^2 f}{z_{0in}^2} = f \quad !!$$

A Gaussian beam with input waist at a distance $d = f$ from a converging lens will be focalized at $z_{min} = f$ independently of the Rayleigh range!

Determine the output waist of a Gaussian beam impinging on a thin lens with focal length f, whose input waist is at a distance d before the lens.



$z = 0$ $\qquad\qquad\qquad\qquad z_{min}$

$$\frac{\lambda}{n\pi w^2(z_{min})} = \frac{1}{z_{0out}} = z_{0in}\frac{fd + f(f-d)}{[fd + z_{min}(f-d)]^2 + z_{0in}^2(f - z_{min})^2}$$

$$\boxed{d = f} \implies \boxed{z_{min} = f} \qquad = z_{0in}\frac{f^2}{f^4} = \frac{z_{0in}}{f^2} \implies z_{0out} = \frac{f^2}{z_{0in}}$$

$$\implies \frac{n\pi w_{0out}^2}{\lambda}\frac{n\pi w_{0in}^2}{\lambda} = f^2 \implies \boxed{w_{0out} = \frac{\lambda f}{n\pi w_{0in}}}$$

The larger the input waist at $d = f$, the smaller the output waist!