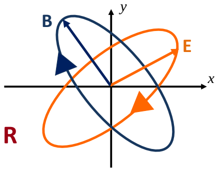
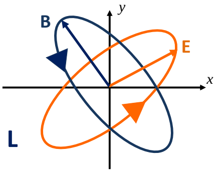
Elliptical polarization

$$\vec{E} = \hat{\imath} E_{0x} e^{i(kz-\omega t)} + \hat{\jmath} E_{0y} e^{i(kz-\omega t+\delta)}$$

Right-elliptically polarized (**R**): $E_{0x} \neq E_{0y}$ $\delta = -\dfrac{\pi}{2}$

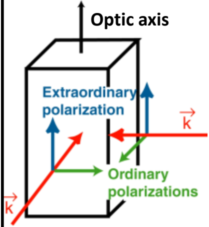
Left-elliptically polarized (**L**): $E_{0x} \neq E_{0y}$ $\delta = +\dfrac{\pi}{2}$





Clockwise (**counter-clockwise**) rotation if looking from the receiver

Propagation in crystals



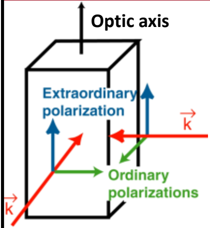
For any given direction of the propagation vector ($\vec{k}$), 2 possible values of the refractive index (n) of the crystal exist, corresponding to orthogonal polarization states.

$$\mathrm{v} = \frac{c}{n}$$

Birefringence

Optic axis: direction of the propagation vector ($\vec{k}$) for which the refractive indexes of orthogonal polarization states are the same.

Propagation in crystals



Ordinary wave:
$\boldsymbol{E} \perp$ **optic axis,** $\perp \vec{\boldsymbol{k}}$

Extraordinary wave:
$\boldsymbol{E} \parallel$ **optic axis,** $\perp \vec{\boldsymbol{k}}$

ISOTROPIC

cubic $\quad n = \sqrt{1+\chi_1}$

UNIAXIAL

trigonal
tetragonal
hexagonal

$n_o = \sqrt{1+\chi_1}$ **ordinary**

$n_e = \sqrt{1+\chi_3}$ **extraordinary**

BIAXIAL

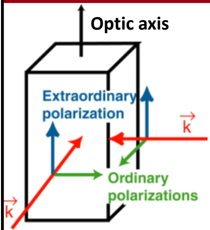
triclinic
monoclinic
orthorhombic

$n_1 = \sqrt{1+\chi_1}$

$n_2 = \sqrt{1+\chi_2}$

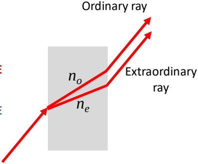
$n_3 = \sqrt{1+\chi_3}$

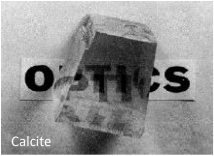
Birefringence



$n_e > n_o$ uniaxial **POSITIVE**

$n_e < n_o$ uniaxial **NEGATIVE**





Calcite

Refractive indexes of birefringent uniaxial crystals (λ_0=589 nm)		
Crystal	n_o	n_e
Tormaline	1.669	1.638
Calcite	1.6584	1.4864
Quartz	1.5443	1.5534
Sodium nitrate	1.5854	1.3369
Ice	1.309	1.313
Rutile (TiO_2)	2.616	2.903

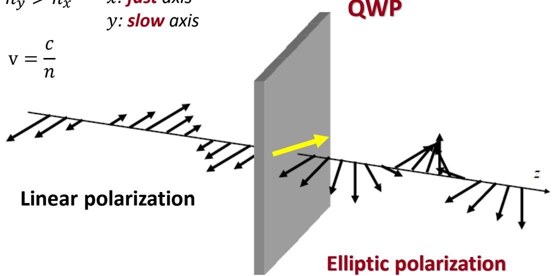
Quarter waveplate: QWP

$$\Delta\phi = k\Delta L = \frac{2\pi}{\lambda} d\left(n_y - n_x\right) = \frac{\pi}{2} \quad \Longrightarrow \quad d\left(n_y - n_x\right) = \frac{\lambda}{4}$$

$n_y > n_x$

x: ***fast*** *axis*
y: ***slow*** *axis*

$$\mathrm{v} = \frac{c}{n}$$

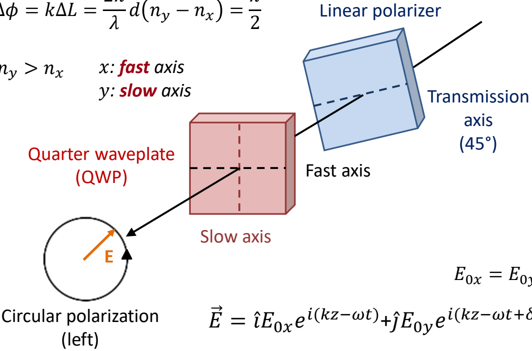


Circular polarizer

$$\Delta\phi = k\Delta L = \frac{2\pi}{\lambda} d\left(n_y - n_x\right) = \frac{\pi}{2}$$

$n_y > n_x$

x: ***fast*** *axis*
y: ***slow*** *axis*



$$E_{0x} = E_{0y}$$

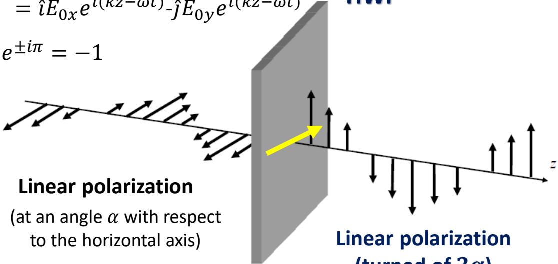
$$\vec{E} = \hat{\imath}E_{0x}e^{i(kz-\omega t)} + \hat{\jmath}E_{0y}e^{i(kz-\omega t+\delta)}$$

Half waveplate: HWP

$$\Delta\phi = k\Delta L = \frac{2\pi}{\lambda} d\left(n_y - n_x\right) = \pi \quad \Rightarrow \quad d\left(n_y - n_x\right) = \frac{\lambda}{2}$$

$$\begin{aligned}\vec{E} &= \hat{\imath}E_{0x}e^{i(kz-\omega t)} + \hat{\jmath}E_{0y}e^{i(kz-\omega t+\pi)} \\ &= \hat{\imath}E_{0x}e^{i(kz-\omega t)} - \hat{\jmath}E_{0y}e^{i(kz-\omega t)}\end{aligned}$$

$$e^{\pm i\pi} = -1$$



Linear polarization

(at an angle α with respect to the horizontal axis)

Linear polarization (turned of 2α)

Jones' notation

Robert Clark Jones, 1941

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)} = \hat{\imath} E_{0x} e^{i(kz-\omega t)} + \hat{\jmath} E_{0y} e^{i(kz-\omega t+\delta)}$$

$$\vec{E}_0 = \hat{\imath} E_{ox} + \hat{\jmath} E_{oy}$$

$E_{ox}\,,\,E_{oy} \in \mathbb{R}$ ➡ **Linear** polarization

$E_{ox}\,,\,E_{oy} \in \mathbb{C}$ ➡ **Elliptical** polarization

$|E_{ox}| = |E_{oy}|$ and $\delta = \pm\frac{\pi}{2}$ ➡ **Circular** polarization, right (-) or left (+)

$$\bar{E}_0 = \begin{bmatrix} E_{ox} \\ E_{oy} \end{bmatrix}$$ **Jones' vector** $$\bar{E}_0' = \frac{1}{\sqrt{|E_{ox}|^2 + |E_{oy}|^2}} \begin{bmatrix} E_{ox} \\ E_{oy} \end{bmatrix}$$

normalized form

Jones' notation

$$\begin{bmatrix} A \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$ **linearly polarized** wave along the **x-axis**

$$\begin{bmatrix} 0 \\ B \end{bmatrix} = B \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$$ **linearly polarized** wave along the **y-axis**

$$\begin{bmatrix} A \\ A \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$$ **linearly polarized** wave at 45°

$$\begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow$$ **Left-circularly polarized** wave

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \Rightarrow$$ **Right-circularly polarized** wave

$$\bar{E}_0 = \begin{bmatrix} E_{ox} \\ E_{oy} \end{bmatrix}$$ **Jones' vector**

$$\bar{E}'_0 = \frac{1}{\sqrt{|E_{ox}|^2 + |E_{oy}|^2}} \begin{bmatrix} E_{ox} \\ E_{oy} \end{bmatrix}$$

normalized form

Jones' notation

$$\begin{bmatrix} A \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$ **linearly polarized** wave along the **x-axis**

$$\begin{bmatrix} 0 \\ B \end{bmatrix} = B \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$$ **linearly polarized** wave along the **y-axis**

$$\begin{bmatrix} A \\ A \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$$ **linearly polarized** wave at 45°

$$\begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow$$ **Left-circularly polarized** wave

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \Rightarrow$$ **Right-circularly polarized** wave

$$\begin{bmatrix} 1 \\ i \end{bmatrix} + \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1+1 \\ i-i \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Jones' notation

Any linear optical element acting on the polarization states can be described by a 2x2 matrix:

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Jones' matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}}_{\text{Last optical element}} \cdots \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \underbrace{\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}}_{\text{First optical element}}$$

Optical system ↔ Matrix 2 x 2

$$\begin{bmatrix} A \\ B \end{bmatrix} \longrightarrow \boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \longrightarrow \begin{bmatrix} A' \\ B' \end{bmatrix}$$

Jones' matrixes

Linear polarizer

Horizontal transmission axis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Vertical transmission axis $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Transmission axis at ±45° $\frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$

Quarter waveplate (QWP)

Vertical fast axis $\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$

Horizontal fast axis $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

Fast axis at ±45° $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mp i \\ \mp i & 1 \end{bmatrix}$

Circular polarizer

Right $\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

Left $\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$

Half waveplate (HWP)

Vertical and horizontal fast axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

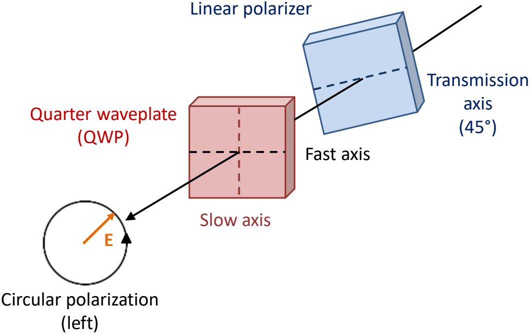
Phase retarder $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$

Jones' notation

What is the polarization state of the outcoming wave when a linearly polarized wave at 45° is impinging on a QWP with horizontal fast axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow$$ **Left-circularly polarized** wave

Circular polarizer

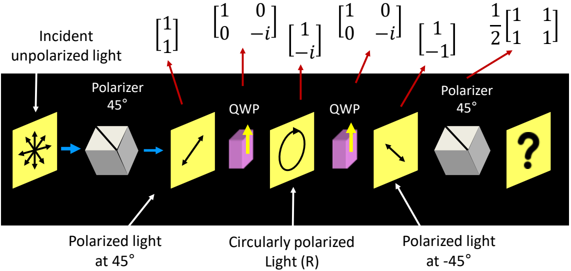


Jones' notation

What is the polarization state of the outcoming wave when a linearly polarized wave at 45° is impinging on a QWP with horizontal fast axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \Rightarrow \quad \textbf{Left-circularly polarized} \text{ wave}$$

NB: There is no Jones' vector for unpolarized light!



$$\begin{bmatrix}0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

Eigenvectors of Jones' matrices

The **eigenvectors** of a Jones' matrix are those specific **polarization states that pass unaffected** through the optical element represented by such Jones' matrix:

eigenvalue

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \lambda \begin{bmatrix} A \\ B \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = 0$$

eigenvector

$$\begin{cases} (a-\lambda)A + bB = 0 \\ cA + (d-\lambda)B = 0 \end{cases} \quad \Rightarrow \quad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0 \qquad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \longleftarrow \lambda_{1,2} \longrightarrow \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$$

Eigenvectors of Jones' matrices

Determine the **eigenvectors** of a QWP with horizontal fast axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \lambda \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & i-\lambda \end{vmatrix} = 0$$

$$\begin{cases} (1-\lambda)A = 0 \\ (i-\lambda)B = 0 \end{cases} \Rightarrow (1-\lambda)(i-\lambda) = 0$$

$\lambda_1 = 1 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ linearly polarized wave along the x-axis (fast axis)

$\lambda_2 = i \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ linearly polarized wave along the y-axis (slow axis)

$$\frac{\lambda_1}{\lambda_2} = i = e^{i\frac{\pi}{2}}$$