

1 Lecture 6

Slide 1

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Multi-layers (normal incidence)

First interface $B = \frac{E}{v} = n \frac{E}{c}$

$$E_0 + E'_0 = E_1 + E'_1$$

$$B_0 - B'_0 = B_1 - B'_1$$

$$n_0 E_0 - n_0 E'_0 = n_1 E_1 - n_1 E'_1$$

Second interface

$$E_1 e^{ikl} + E'_1 e^{-ikl} = E_T$$

$$B_1 e^{ikl} - B'_1 e^{-ikl} = B_T$$

$$n_1 E_1 e^{ikl} - n_1 E'_1 e^{-ikl} = n_T E_T$$

For the second interface, we have to consider the propagation of the wave inside n_1 . This is the solution of the problem in the steady state (we have a large number of cycles, interference of multiple waves until we reach a steady state solution, results independent on time).

In the end, we end up with four equations.

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Multi-layers (normal incidence)

Transfer matrix

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} r = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} t$$

$$r = \frac{E'_0}{E_0} \quad t = \frac{E_T}{E_0}$$

$$M = \begin{bmatrix} \cos kl & -\frac{i}{n_1} \sin kl \\ -in_1 \sin kl & \cos kl \end{bmatrix}$$

$$R = |r|^2 \quad T = |t|^2 \quad R + \frac{n_T}{n_0} T = 1$$

We shine a beam on a thin film of material. We can use this configuration in order to realize coatings with high reflectivity.

Let us consider normal incidence (TE and TM are degenerate). The substrate has a semiinfinite length.

Light is coming from left with electric field E_0 (the magnetic field coming out from the screen). We have the reflected wave E'_0 . Inside the layer, we have the forward wave E_1 and the backward (reflected) which is E'_1 . We have the trasmitted wave E_T .

This is an interference problem: how to imagine the interference of the wave forawrd and backward inside the layer. Instead of inference, we can use the cointinuity equations for the tangentials components of electric and magnetic field at the interface.

It is possible to simply this problem, instead of considering the amplitudes, we can consider the reflection and trasmission coefficients.

We rewrite the system in a matrix formulation. M is the **transfer matrix**.

In this framework the reflectivity is $R = |r|^2$ and the transmittance $T = |t|^2$. With this assumption:

$$R + \frac{n_T}{n_0} T \equiv 1$$

This convention is used for multi-layers typically.

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Multi-layers (normal incidence)

N layers

$$k_j = \frac{2\pi}{\lambda_j} = \frac{2\pi}{\lambda_0} n_j$$

Transfer matrix (multi-layer)

$$M = M_1 M_2 \dots M_N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

First illuminated layer

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} r = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} t$$

$$r = \frac{E'_0}{E_0} \quad t = \frac{E_T}{E_0}$$

$$R = |r|^2 \quad T = |t|^2 \quad R + \frac{n_T}{n_0} T = 1$$

Let us see how we can use this matrix formulation. Let us imagine to have a stack of different layers with different n with different thickness. The propagation vector inside each layer is related to the refractive index.

The transfer matrix is given by the product of the transfer matrix of each layer!

The order in which you have to multiply the matrix is important, but in this case you have to multiply the matrix from left (first illuminated layer) to right. This is the **main difference** with the Jones's notation.

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Multi-layers (normal incidence)

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N layers

$$k_j = \frac{2\pi}{\lambda_j} = \frac{2\pi}{\lambda_0} n_j$$

Transfer matrix (multi-layer)

$$M = M_1 M_2 \dots M_N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

↑
First illuminated layer

$$r = \frac{An_0 + Bn_T n_0 - C - Dn_T}{An_0 + Bn_T n_0 + C + Dn_T}$$

$$t = \frac{2n_0}{An_0 + Bn_T n_0 + C + Dn_T}$$

$$R = |r|^2 \quad T = |t|^2 \quad R + \frac{n_T}{n_0} T = 1$$

If we solve the system, the results for the reflection and trasmission coefficients are shown in this slide.
In many books, n_0 (environment) is assumed to be 1.

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Anti-reflection coating

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Optical thickness of the layer: $l = \frac{\lambda}{4} \rightarrow kl = \frac{\pi}{2}$

Reflectance of a $\lambda/4$ film

$$R = |r|^2 = \left(\frac{n_T n_0 - (n_1)^2}{n_T n_0 + (n_1)^2} \right)^2$$

$$r = \frac{An_0 + Bn_T n_0 - C - Dn_T}{An_0 + Bn_T n_0 + C + Dn_T}$$

$\sqrt{n_T n_0} = n_1 \rightarrow R = 0$ **ANTI-REFLECTION!**

If $n_T = 1.5$ (glass) $\rightarrow n_1 = \sqrt{n_T} = 1.22$ to get $R = 0$

Let us see some examples of the above formulation.

Let us determine the reflectivity for a single layer embedded in an environment with n_0 and in a substrate of n_T . λ is the wavelength of the wave inside n_1 .

The results in this slide for the reflectance are valid only for $kl = \frac{\pi}{2}$.

Let us imagine that we want to obtain an **anti-reflection** coating. The condition is that $\sqrt{n_T n_0} = n_1$. In reality, you cannot get materials of any refractive index. First, the material should be transparent, easily deposited, durable... There are a lot of technological problems. For this problem, typically, for making an anti-reflection coating for glass we use MgF_2 . We can get a good value, which is not exactly zero but is very near.

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Anti-reflection coating

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Optical thickness of the layer: $l = \frac{\lambda}{4} \rightarrow kl = \frac{\pi}{2}$

Reflectance of a $\lambda/4$ film


$$R = |r|^2 = \left(\frac{n_T n_0 - (n_1)^2}{n_T n_0 + (n_1)^2} \right)^2$$

$$r = \frac{An_0 + Bn_T n_0 - C - Dn_T}{An_0 + Bn_T n_0 + C + Dn_T}$$

$\sqrt{n_T n_0} = n_1 \rightarrow R = 0$ **ANTI-REFLECTION!**

Typically anti-reflection coatings of glass lenses ($n_T = 1.5$) are of MgF_2 ($n_1 = 1.38$) $\rightarrow R \approx 0.014$ (1.4%)

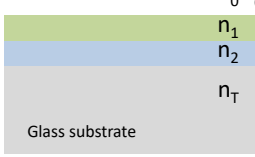
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Anti-reflection coating

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Optical thickness of the layer: $l = \frac{\lambda}{4} \rightarrow kl = \frac{\pi}{2}$

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} r = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} t$$

$$M = M_1 M_2 = \begin{bmatrix} 0 & -i \\ -in_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ -in_2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{n_2}{n_1} & 0 \\ 0 & -\frac{n_1}{n_2} \end{bmatrix}$$

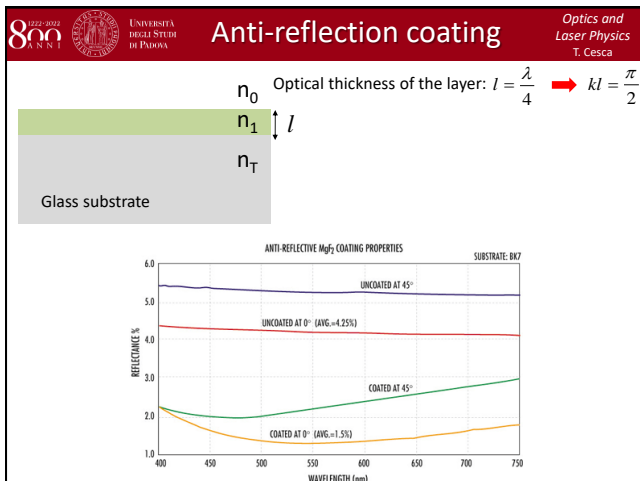
$R = |r|^2 = \left(\frac{n_0(n_2)^2 - n_T(n_1)^2}{n_0(n_2)^2 + n_T(n_1)^2} \right)^2$ Reflectance of 2 $\lambda/4$ films

$R = 0 \rightarrow \sqrt{\frac{n_T}{n_0}} = \frac{n_2}{n_1}$ **ANTI-REFLECTION!**

If $n_0 = 1, n_T = 1.5, n_1 = 1.38$ (MgF₂) $\rightarrow n_2 = 1.69 \rightarrow n_2 = 1.62$ (Al₂O₃)
 $R = 0.0018$


Let us imagine to have two layers with thickness again $l = \lambda/4$.
 This is the reflectance of two layers with this thickness.
 We get again the condition of anti-reflection.

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This is an example graph of anti-reflection coating. We have a single layer of magnesium fluoride. The red curve is the one we should consider (normal incidence).

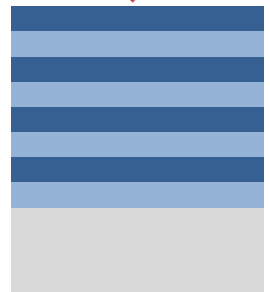
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High reflectance films

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2N layers $l = \frac{\lambda}{4}$

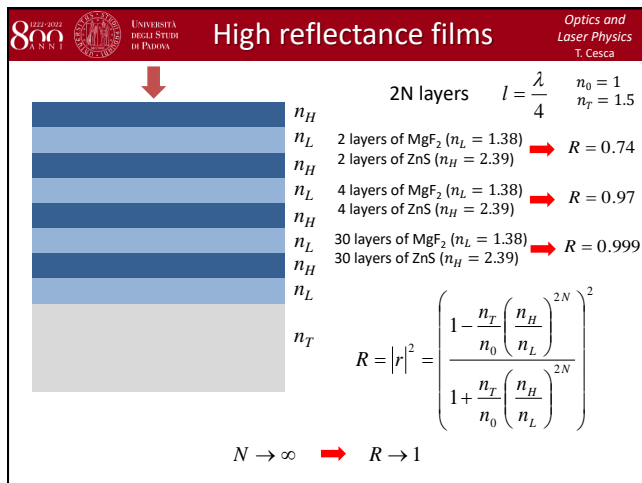
$$M = (M_H M_L)^N = \begin{bmatrix} -\frac{n_L}{n_H} & 0 \\ 0 & -\frac{n_H}{n_L} \end{bmatrix}^N$$

$$R = |r|^2 = \left(\frac{1 - \frac{n_T}{n_0} \left(\frac{n_H}{n_L} \right)^{2N}}{1 + \frac{n_T}{n_0} \left(\frac{n_H}{n_L} \right)^{2N}} \right)^2$$

$N \rightarrow \infty \rightarrow R \rightarrow 1$

This approach could be applied to determine high reflectance coatings (mirror).
 We have a substrate n_T , environment n_0 and a stack of $2N$ layers of two different materials with n_H (high) and n_L (low) and with $l = \lambda/4$.
 When $N \rightarrow \infty$ we obtain $R \rightarrow 1$!

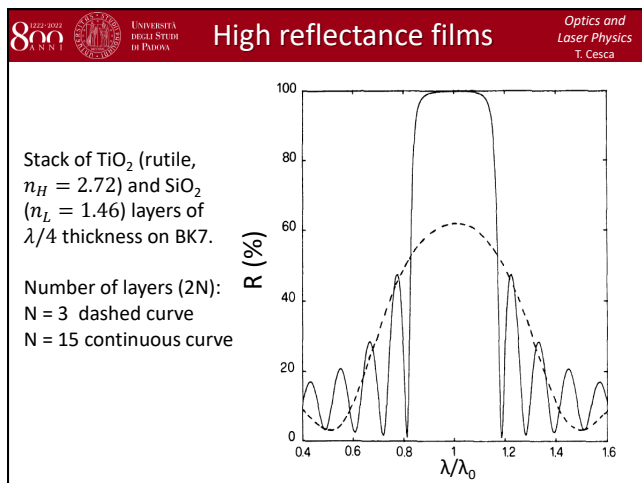
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This is a practical example.

If the number of layer is very large, the order of n_H and n_L does not play an important role. Instead, in different cases, the order is important and it makes a different.

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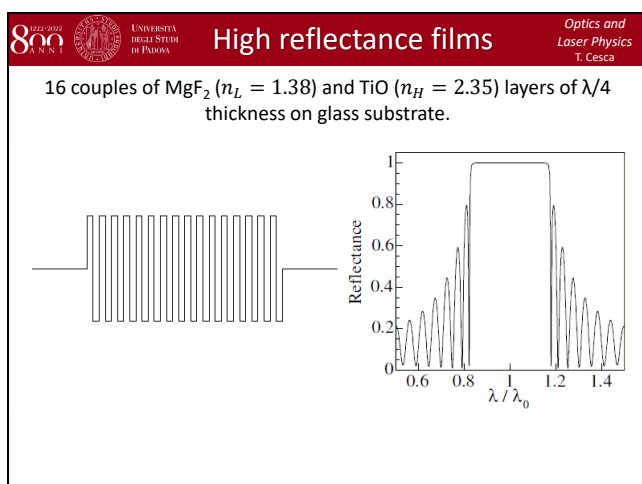


This is the reflectance for a stack of TiO_2 .

The dashed line gives the spectral trend for three couples. You obtain a reflectivity of around 60%. If you increase the number of layers, you obtain 100%.

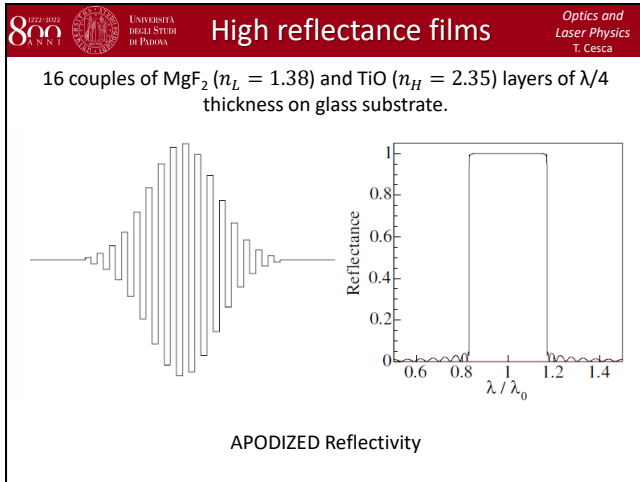
We are consider an interference problem, so we have interference effects (oscillation in the reflectivity).

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This happens every time you have a stack of a low and high refractive index

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This can be solved if you make an **apodized**, you modulate the values of the refractive index, which means modulating the phase of your wave inside the layers. You can play with this and find the configuration which reduces the oscillation in the border and give you high reflectivity.

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Multi-layers (oblique incidence)

Transverse Electric TE (s)

at normal incidence

$$r = \frac{An_0 + Bn_0n_T - C - Dn_T}{An_0 + Bn_0n_T + C + Dn_T}$$

$$t = \frac{2n_0}{An_0 + Bn_0n_T + C + Dn_T}$$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\beta & -\frac{i}{p}\sin\beta \\ -ip\sin\beta & \cos\beta \end{pmatrix}$$

$$\beta = k \cos\theta z$$

$$n_0 \rightarrow n_0 \cos\theta_0 \quad r \rightarrow r$$

$$n_1 \rightarrow n_1 \cos\theta = p \quad t \rightarrow t$$

$$n_T \rightarrow n_T \cos\theta_t \quad R + \frac{n_T \cos\theta_t}{n_0 \cos\theta_0} T = 1$$

$R = |r|^2 \quad T = |t|^2$

How to move to a situation which is no more at normal incidence. In this case, we should distinguish TE and TM.

We have the environment n_0 , the material n_1 and the substrate n_T .

This is the solution of the problem. We consider the expression for r and t at normal incidence. We consider the transfer matrix. The thickness of the layer is z . The argument of the cosine and sine is $\beta = k \cos(\theta)z$, where k is the modulus of the propagation vector within the layer, propagation angle within the layer θ .

These are the substitution we have to do. Also the mapping of the conservation of energy is changed.

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Multi-layers (oblique incidence)

Transverse Magnetic TM (p)

at normal incidence

$$r = \frac{An_0 + Bn_0n_T - C - Dn_T}{An_0 + Bn_0n_T + C + Dn_T}$$

$$t = \frac{2n_0}{An_0 + Bn_0n_T + C + Dn_T}$$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\beta & -\frac{i}{p}\sin\beta \\ -ip\sin\beta & \cos\beta \end{pmatrix}$$

$$\beta = k \cos\theta z$$

$$n_0 \rightarrow \frac{n_0}{\cos\theta_0} \quad r \rightarrow r$$

$$n_1 \rightarrow \frac{n_1}{\cos\theta} = p \quad t \rightarrow t \frac{\cos\theta_t}{\cos\theta_0}$$

$$n_T \rightarrow \frac{n_T}{\cos\theta_t} \quad R + \frac{n_T \cos\theta_t}{n_0 \cos\theta_0} T = 1$$

$R = |r|^2 \quad T = |t|^2$

This is the case for TM.

The important point is that from normal incidence to oblique incidence you have to distinguish between TE and TM.