Electrons vs photons

Propagation and confinement

Maxwell Equations

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r},t) = -\dfrac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \\ \nabla \times \mathbf{H}(\mathbf{r},t) = \dfrac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} + \mathbf{j}(\mathbf{r},t) \\ \nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t), \\ \nabla \cdot \mathbf{B}(\mathbf{r},t) = 0. \end{cases}$$

$$\nabla \cdot \mathbf{j}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t),$$

$$\mathbf{H}(\mathbf{r},t) = \mu_0^{-1} \mathbf{B}(\mathbf{r},t) - \mathbf{M}(\mathbf{r},t)$$

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E} \qquad (\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}),$$

$$\mathbf{B} = \mu_0 \mu \mathbf{H} \qquad (\mathbf{M} = \chi_m \mathbf{H}),$$

$$\mathbf{j}_c = \sigma \mathbf{E}.$$

$$\varepsilon_1 = n^2 - \kappa^2$$

$$\varepsilon_2 = 2n\kappa$$

$$n^2 = \frac{\varepsilon_1}{2} + \frac{1}{2}\sqrt{\varepsilon_1^2 + \varepsilon_2^2}$$

$$\kappa = \frac{\varepsilon_2}{2n}.$$

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{-i\omega t}$$

Plane waves solutions

Maxwell Equations

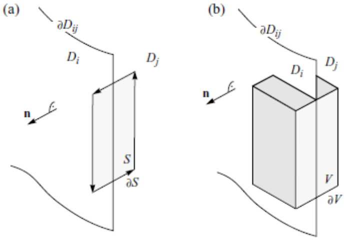
$$
\begin{cases}
\nabla \times \mathbf{E}(\mathbf{r}) = i\omega \mathbf{B}(\mathbf{r}) \,, \\
\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega \mathbf{D}(\mathbf{r}) + \mathbf{j}(\mathbf{r}) \\
\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}) \,, \\
\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \,,
\end{cases}
$$

$$\nabla \times \nabla \times = -\nabla^2 + \nabla\nabla$$

$$(\nabla^2 + k_i^2)\,\mathbf{E}_i = -i\omega\mu_0\mu_i\,\mathbf{j}_i + \frac{\nabla\rho_i}{\varepsilon_0\varepsilon_i}$$

$$(\nabla^2 + k_i^2)\,\mathbf{H}_i = -\nabla \times \mathbf{j}_i \,,$$

Boundary conditions



$$\partial D_{ij}$$

$$
\begin{cases}
\boldsymbol{n} \times (\boldsymbol{E}_i - \boldsymbol{E}_j) = 0 \\
\boldsymbol{n} \times (\boldsymbol{H}_i - \boldsymbol{H}_j) = \boldsymbol{J}_{surf} & \text{(surf. curr. density)} \\
\boldsymbol{n} \cdot (\boldsymbol{D}_i - \boldsymbol{D}_j) = \sigma & \text{(surf. charge density)} \\
\boldsymbol{n} \cdot (\boldsymbol{B}_i - \boldsymbol{B}_j) = 0
\end{cases}
$$

Equazioni di Maxwell

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}) = i\omega \mathbf{B}(\mathbf{r}) \,, \\ \nabla \times \mathbf{H}(\mathbf{r}) = -i\omega \mathbf{D}(\mathbf{r}) + \mathbf{j}(\mathbf{r}) \\ \nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}) \,, \\ \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \,, \end{cases}$$

$$\nabla \times \nabla \times = -\nabla^2 + \nabla\nabla$$

$$(\nabla^2 + k_i^2)\, \mathbf{E}_i = -i\omega\mu_0\mu_i\, \mathbf{j}_i + \frac{\nabla \rho_i}{\varepsilon_0 \varepsilon_i}$$

$$(\nabla^2 + k_i^2)\, \mathbf{H}_i = -\nabla \times \mathbf{j}_i \,,$$

$$\mathbf{K}(\mathbf{K} \cdot \mathbf{E}) - K^2 \mathbf{E} = -\varepsilon(\mathbf{K}, \omega) \frac{\omega^2}{c^2} \mathbf{E},$$

For Transverse waves: $\mathbf{K} \cdot \mathbf{E} = 0$, $K^2 = \varepsilon(\mathbf{K}, \omega) \dfrac{\omega^2}{c^2}$ Dispersion Relation

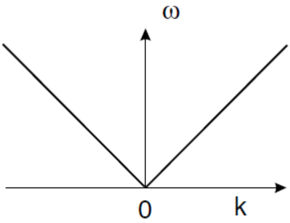
For Longitudinal waves: $\mathbf{K} \parallel \mathbf{E}$ $\varepsilon(\mathbf{K}, \omega) = 0$

Photons vs Electrons

	photons	electrons
mass	0	m_0
charge	0	e
statistics	Bosons	Fermions
wavelength	$\lambda = \frac{h}{p} = \frac{c}{f}$	$\lambda = \frac{h}{p} = \frac{h}{mv}$
Eigenvalue Equation	$-\nabla^2 \boldsymbol{E} = k^2 \boldsymbol{E}$ $\boldsymbol{E} = \boldsymbol{p_1} E$ $-\nabla^2 E = k^2 E$	$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi = \varepsilon\,\Psi$ $-\nabla^2\Psi = \left[\frac{2m}{\hbar^2}(\varepsilon - V)\right]\Psi = k^2\Psi$
Free Space solution	$\boldsymbol{E} = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{-i\omega t}$	$\Psi = \Psi_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{-i\omega t}$
Interaction	$\varepsilon(\omega, \boldsymbol{r})$	$V(\boldsymbol{r})$
Classically forbidden zone	Photon tunneling (evanescent wave) $\boldsymbol{k} \in \mathbb{C}$	Electron tunneling (exp. decay probability) $\boldsymbol{k} \in \mathbb{C}$
Band Structure	Photonic crystals (periodic modulation of $\varepsilon(\omega, \boldsymbol{r})$)	Crystals (periodic modulation of V($\boldsymbol{r}$))
Cooperative effects	Nonlinear optics	Many-body correlations Cooper pairs

Free-space dispersion laws

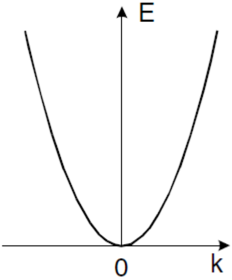
photons



$$\omega = c\,k$$
$$\varepsilon = \hbar c\,k$$

Non-dispersive

electrons



$$\varepsilon = \frac{\hbar^2}{2m}k^2$$

Dispersive

Phase velocity

$$v_p = \frac{\omega}{k} \qquad v_p = c \qquad v_p = \frac{\hbar}{2m}k$$

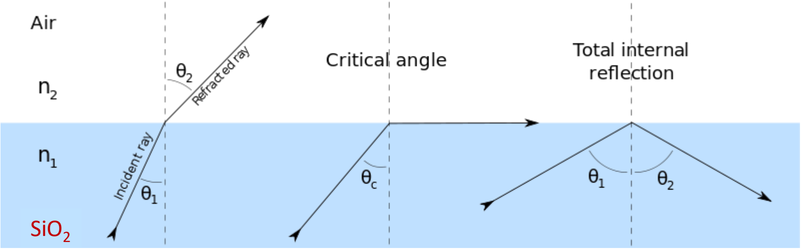
Group velocity

$$|\boldsymbol{v}_g| = |\bar{\nabla}_{\boldsymbol{k}}\omega| = \frac{1}{\hbar}|\bar{\nabla}_{\boldsymbol{k}}\varepsilon| \qquad v_g = c = v_p \qquad v_g = \frac{\hbar}{m}k = 2v_p$$

Snell Law and Total Internal Reflection

$n_1 > n_2$

Snell $\quad n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad \sin\theta_2 = \dfrac{n_1}{n_2}\sin\theta_1$



$$\theta_2 = 90°$$

$$\sin\theta_1 = \sin\theta_c = \frac{n_2}{n_1}\sin\theta_2 = \frac{n_2}{n_1} \qquad \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

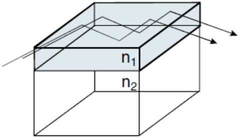
$$\theta_c(SiO_2 - air) = \sin^{-1}\left(\frac{1}{1.46}\right) = 43.23°$$

Confinement

Photon confinement

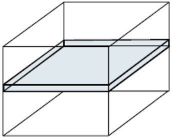
Electron confinement

$n_1 > n_2$

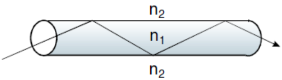


Optical planar waveguide

1D

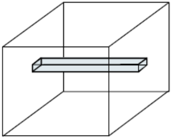


Quantum well

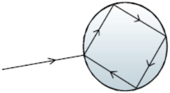


Optical fiber

2D

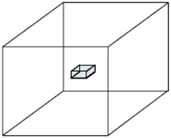


Quantum wire



Microsphere optical cavity

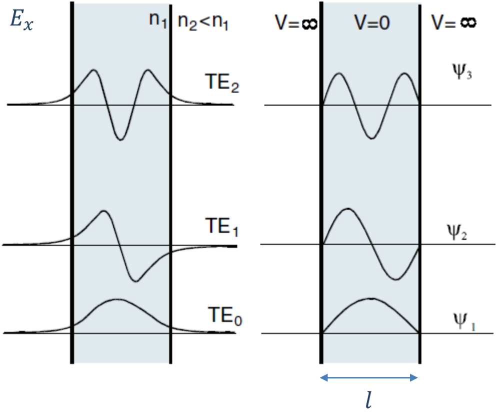
3D



Quantum dot

1D Confinement

photons electrons



Particle-in-a-box

$$\Psi_n(x) = \left(\frac{2}{l}\right)^{1/2} \sin\left(\frac{n\pi x}{l}\right)$$

$$k = n\pi/l \qquad n = 1, 2, 3, \ldots$$

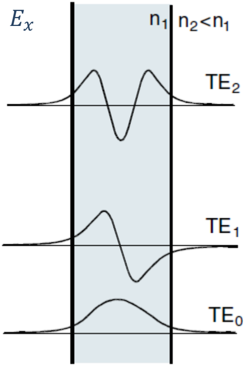
$$l = n\frac{\lambda}{2}$$

$$E_n = \frac{n^2h^2}{8ml^2}$$

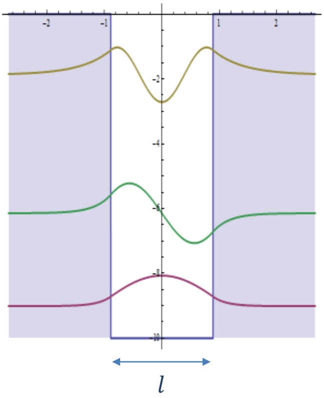
$$\Delta E = (2n+1)\frac{h^2}{8ml^2}$$

photons

electrons



Finite V_0



Particle-in-a-box

$$\Psi_n(x) = \left(\frac{2}{l}\right)^{1/2} \sin\left(\frac{n\pi x}{l}\right)$$

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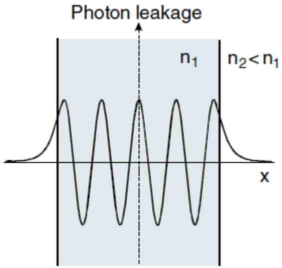
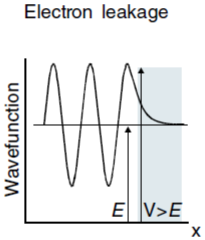
$$l = n\frac{\lambda}{2}$$

$$E_n = \frac{n^2h^2}{8ml^2}$$

$$\Delta E = (2n+1)\frac{h^2}{8ml^2}$$

Leakage and tunneling

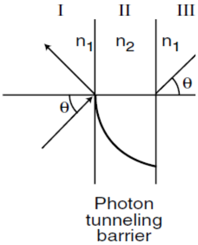


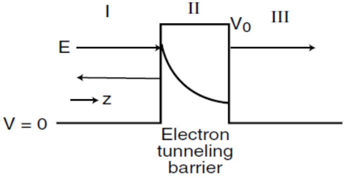


$$E = E_0 e^{-\frac{x}{d}}$$

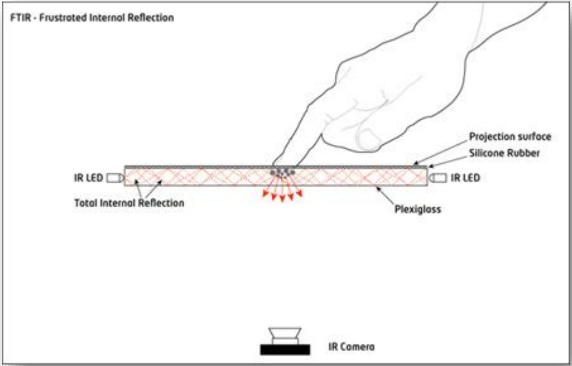
$$\Psi = \Psi_0 e^{-\frac{x}{d}}$$

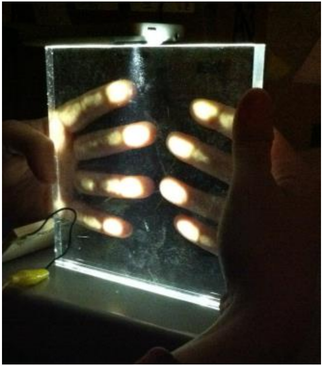
Frustrated Total Internal Reflection





FTIR and Multitouch Screen





(source: Cornell Univ.,US)

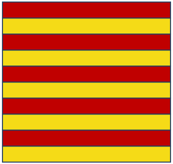
Photonic Crystals

Photonic Crystals

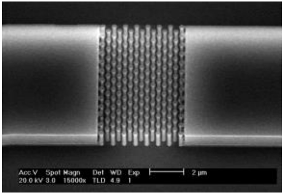
Definition:

A photonic crystal (PC) is a periodic structure in which the dielectric function $\varepsilon(\omega, \boldsymbol{r})$ varies periodically.

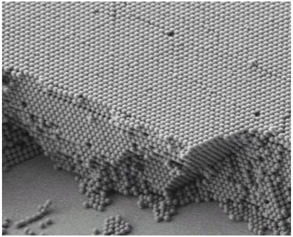
$$\varepsilon(\omega, \boldsymbol{r}) = \varepsilon(\omega, \boldsymbol{r} + \boldsymbol{R})$$



1D: Bragg Reflector



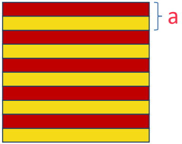
2D: Si pillar crystal



3D: colloidal crystal

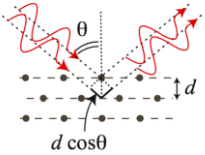
1D Photonic Crystals

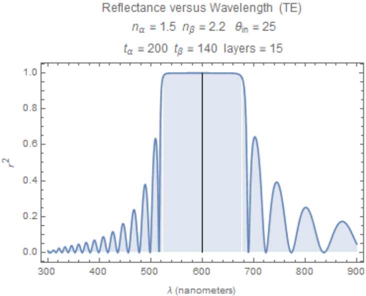
- Distributed Bragg Reflectors
- Antireflection coatings



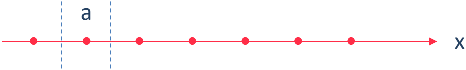
Bragg's law:

$$2n_{average}\, d\cos\theta = m\lambda$$

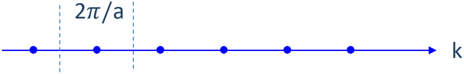




$$\varepsilon(\omega, \boldsymbol{r}) = \varepsilon(\omega, \boldsymbol{r} + \boldsymbol{R})$$



Direct space (Bravais lattice)



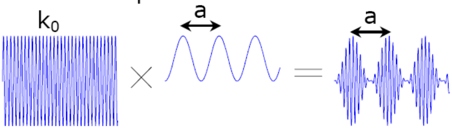
Reciprocal space (Bravais lattice)

Photonic Crystals: optical semiconductors

A Bloch wave is a plane wave with a modulation:

$$\psi_{n\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n\mathbf{k}}(\mathbf{r})$$

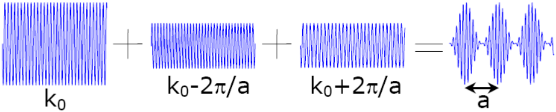
A



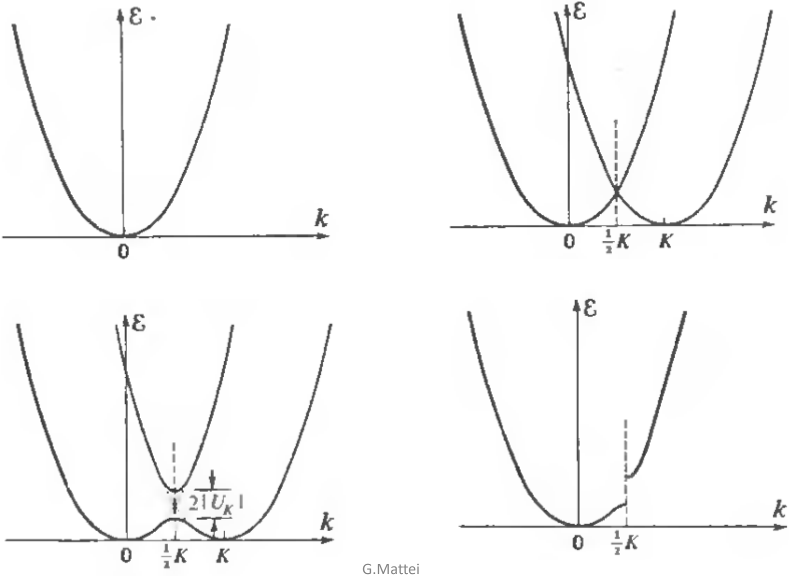
A Bloch wave consists of multiple wavevectors:

$$\psi(\mathbf{r}) = \Sigma_m c_m e^{i(k+m2\pi/a)r}$$

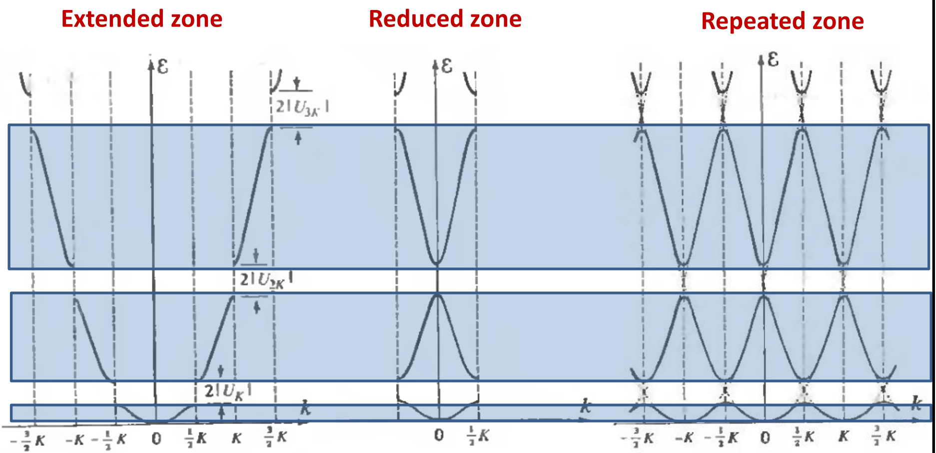
B



Electrons in a weak Periodic Potential



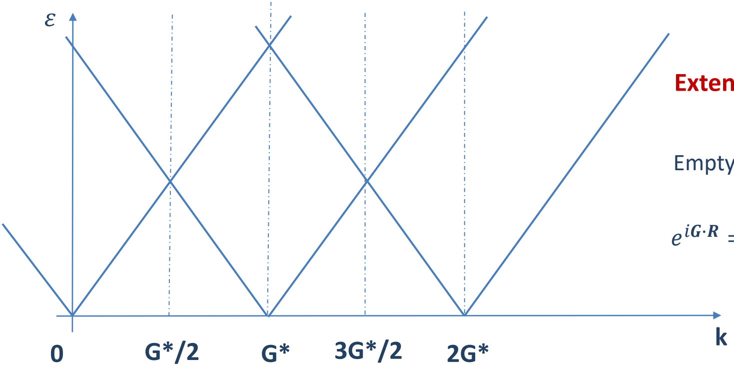
Electrons in a weak Periodic Potential



$$\mathcal{E} = \tfrac{1}{2}(\mathcal{E}^0_{\mathbf{q}} + \mathcal{E}^0_{\mathbf{q}-\mathbf{K}}) \pm \left[\left(\frac{\mathcal{E}^0_{\mathbf{q}} - \mathcal{E}^0_{\mathbf{q}-\mathbf{K}}}{2}\right)^2 + |U_{\mathbf{K}}|^2\right]^{1/2}$$

$$U(\boldsymbol{r}) = U(\boldsymbol{r}+\boldsymbol{R}) = \textstyle\sum_K U_K e^{iK\cdot r}$$

$$|\psi(\mathbf{r})|^2 \propto (\cos \tfrac{1}{2}\mathbf{K}\cdot\mathbf{r})^2, \quad \mathcal{E} = \mathcal{E}^0_{\mathbf{q}} + |U_{\mathbf{K}}|,$$
$$|\psi(\mathbf{r})|^2 \propto (\sin \tfrac{1}{2}\mathbf{K}\cdot\mathbf{r})^2, \quad \mathcal{E} = \mathcal{E}^0_{\mathbf{q}} - |U_{\mathbf{K}}|,$$



Extended zone

Empty Lattice Model

$$e^{i\boldsymbol{G}\cdot\boldsymbol{R}} = 1$$

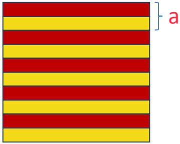
$$\varepsilon(\omega, \boldsymbol{r}) = \varepsilon(\omega, \boldsymbol{r} + \boldsymbol{R})$$

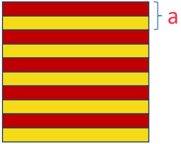
Remove degeneracy at **G*/2** by modulating the dielectric function

$$\varepsilon(\omega, \boldsymbol{r}) = \sum_{\boldsymbol{G}} \varepsilon_{\boldsymbol{G}} e^{i\boldsymbol{G}\cdot\boldsymbol{r}}$$

Photonic band structure

$$\varepsilon_{\boldsymbol{G}} \equiv \frac{1}{V_c} \int_{V_c} d\boldsymbol{r}\ \varepsilon(\omega, \boldsymbol{r}) e^{-i\boldsymbol{G}\cdot\boldsymbol{r}}$$





$$\nabla^2 \boldsymbol{E} + \left(\frac{\omega}{c}\right)^2 \varepsilon\ \boldsymbol{E} = \boldsymbol{0}$$

$$e^{i\boldsymbol{G}\cdot\boldsymbol{R}} = 1$$

$$\begin{cases} \varepsilon(\omega,\boldsymbol{r}) = \varepsilon(\omega,\boldsymbol{r}+\boldsymbol{R}) = \sum_{\boldsymbol{G}} \varepsilon_{\boldsymbol{G}} e^{i\boldsymbol{G}\cdot\boldsymbol{r}} \qquad \varepsilon_{\boldsymbol{G}} \equiv \dfrac{1}{V_c}\displaystyle\int_{V_c} d\boldsymbol{r}\ \varepsilon(\omega,\boldsymbol{r}) e^{-i\boldsymbol{G}\cdot\boldsymbol{r}} \\ \boldsymbol{E}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \boldsymbol{A}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \qquad \boldsymbol{E}(\boldsymbol{r}) \sim \boldsymbol{A}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{G}^*) e^{i(\boldsymbol{k}-\boldsymbol{G}^*)\cdot\boldsymbol{r}} \end{cases}$$

$$\left.\begin{cases} k^2 \boldsymbol{A}(\boldsymbol{k}) - \left(\dfrac{\omega}{c}\right)^2 \displaystyle\sum_{\boldsymbol{G}'} \varepsilon_{\boldsymbol{G}'}\ \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{G}') = \boldsymbol{0} \\ (\boldsymbol{k}-\boldsymbol{G}^*)^2 \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{G}^*) - \left(\dfrac{\omega}{c}\right)^2 \displaystyle\sum_{\boldsymbol{G}'} \varepsilon_{\boldsymbol{G}'-\boldsymbol{G}^*}\ \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{G}') = \boldsymbol{0} \end{cases}\right\} \quad \boldsymbol{G}' = \boldsymbol{0}, \boldsymbol{G}^*$$

$$\begin{cases} k^2 \boldsymbol{A}(\boldsymbol{k}) - \left(\dfrac{\omega}{c}\right)^2 \varepsilon_0\ \boldsymbol{A}(\boldsymbol{k}) - \left(\dfrac{\omega}{c}\right)^2 \varepsilon_{\boldsymbol{G}}\ \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{G}^*) = \boldsymbol{0} \\ (\boldsymbol{k}-\boldsymbol{G}^*)^2 \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{G}^*) - \left(\dfrac{\omega}{c}\right)^2 \varepsilon_{-\boldsymbol{G}^*}\ \boldsymbol{A}(\boldsymbol{k}) - \left(\dfrac{\omega}{c}\right)^2 \varepsilon_0\ \boldsymbol{A}(\boldsymbol{k}-\boldsymbol{G}^*) = \boldsymbol{0} \end{cases}$$

1D Photonic Crystals

$$\begin{cases} \left(k^2 - \left(\frac{\omega}{c}\right)^2 \varepsilon_0\right) A(\boldsymbol{k}) - \left(\frac{\omega}{c}\right)^2 \varepsilon_{\boldsymbol{G}}\; \boldsymbol{A}(\boldsymbol{k} - \boldsymbol{G}^*) = \boldsymbol{0} \\ -\left(\frac{\omega}{c}\right)^2 \varepsilon_{-\boldsymbol{G}^*}\, \boldsymbol{A}(\boldsymbol{k}) + \left((\boldsymbol{k} - \boldsymbol{G}^*)^2 - \left(\frac{\omega}{c}\right)^2 \varepsilon_0\right) \boldsymbol{A}(\boldsymbol{k} - \boldsymbol{G}^*) = \boldsymbol{0} \end{cases}$$

$$\left(\frac{\omega}{c}\right)^4 \left(\varepsilon_0{}^2 - \varepsilon_{\boldsymbol{G}^*}\varepsilon_{-\boldsymbol{G}^*}\right) - \left(\frac{\omega}{c}\right)^2 \varepsilon_0\left(k^2 + (\boldsymbol{k} - \boldsymbol{G}^*)^2\right) + k^2(\boldsymbol{k} - \boldsymbol{G}^*)^2 = 0$$

$$\left(\frac{\omega}{c}\right)^2 = \frac{\varepsilon_0\left(k^2 + (\boldsymbol{k} - \boldsymbol{G}^*)^2\right)}{2\left(\varepsilon_0{}^2 - \varepsilon_{\boldsymbol{G}^*}\varepsilon_{-\boldsymbol{G}^*}\right)} \pm \frac{\sqrt{\left(\varepsilon_0\left(k^2 - (\boldsymbol{k} - \boldsymbol{G}^*)^2\right)\right)^2 + 4\left(\varepsilon_{\boldsymbol{G}^*}\varepsilon_{-\boldsymbol{G}^*}\right)k^2(\boldsymbol{k} - \boldsymbol{G}^*)^2}}{2\left(\varepsilon_0{}^2 - \varepsilon_{\boldsymbol{G}^*}\varepsilon_{-\boldsymbol{G}^*}\right)}$$

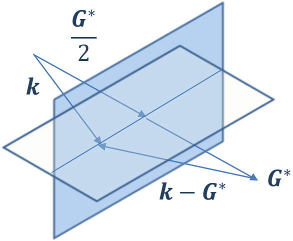
1D Photonic Crystals

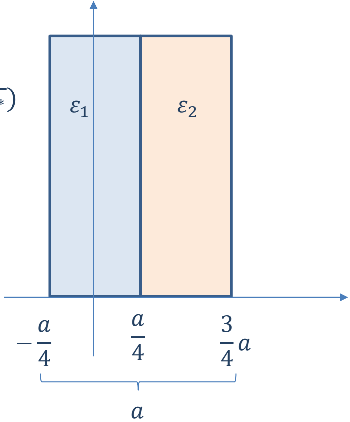
$$\varepsilon(\omega,\boldsymbol{r}) = \varepsilon(\omega,-\boldsymbol{r}) \qquad \varepsilon_{\boldsymbol{G}} \in \mathbb{R}$$

$$\boldsymbol{k} = \pm\frac{\boldsymbol{G}^*}{2}$$

$$\left(\frac{\omega}{c}\right)^2 = \frac{(\pi/a)^2}{\left({\varepsilon_{\boldsymbol{0}}}^2 - \varepsilon_{\boldsymbol{G}^*}\varepsilon_{-\boldsymbol{G}^*}\right)}\left(\varepsilon_{\boldsymbol{0}} \pm \sqrt{\varepsilon_{\boldsymbol{G}^*}\varepsilon_{-\boldsymbol{G}^*}}\right)$$

$$\omega_{\pm}^2 = \frac{\pi^2 c^2}{a^2\left(\varepsilon_{\boldsymbol{0}} \mp |\varepsilon_{\boldsymbol{G}^*}|\right)}$$





$$\begin{cases} \varepsilon_{\boldsymbol{0}} = \dfrac{\varepsilon_1 + \varepsilon_2}{2} \\[2ex] \varepsilon_{\boldsymbol{G}} = \dfrac{\varepsilon_1 - \varepsilon_2}{\pi} \end{cases} \qquad G = \frac{2\pi}{a}$$

1D Photonic Crystals

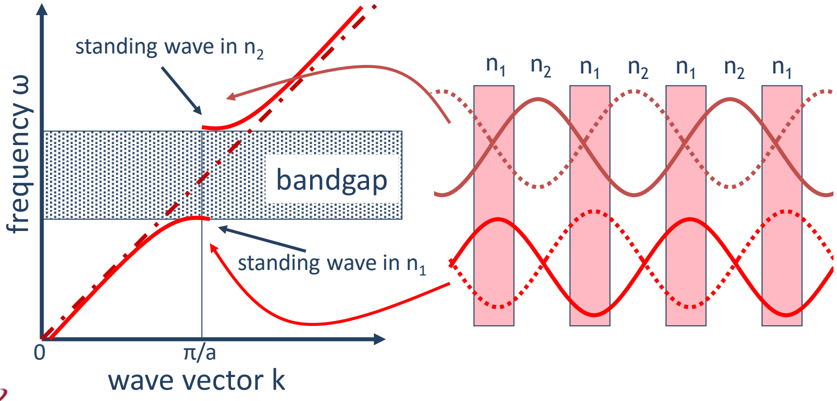
Dispersion relation

$$\omega = \omega(k)$$

n_1: high index material

n_2: low index material

$$\omega = k\frac{c}{n}$$



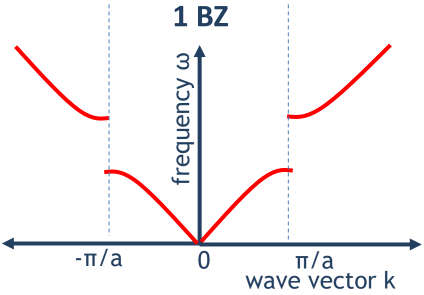
1D Photonic Crystals

Dispersion relation

$$\omega = \omega(k)$$

n_1: high index material

n_2: low index material



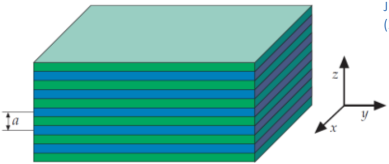
Bloch wave with wave vector $\boldsymbol{k}$ is equal to Bloch wave with wave vector $\boldsymbol{k}+m2\pi/a$:

A light pulse with a center frequency ω propagates with *group velocity* v_g, which is altered near the band gap

$$v_g = \frac{d\omega}{dk}$$

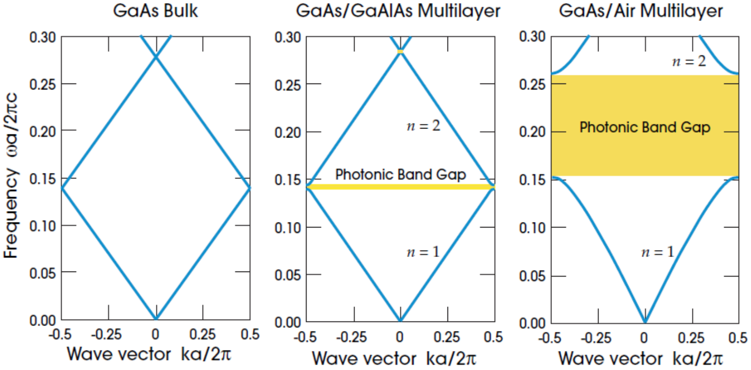
1D Photonic Crystals

J D Joannopoulos et al. (2008), *Photonic Crystals: Molding the Flow of Light* (2nd ed.), Princeton NJ: Princeton University Press



$$\varepsilon(\omega, z) = \varepsilon(\omega, z + na)$$

$$d_1 = d_2 = 0.5\ a$$



$\varepsilon_1 = 13$
$\varepsilon_2 = 13$

$\varepsilon_1 = 13$
$\varepsilon_2 = 12$

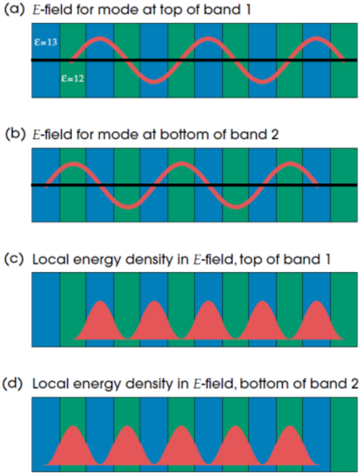
$\varepsilon_1 = 13$
$\varepsilon_2 = 1$

1D Photonic Crystals

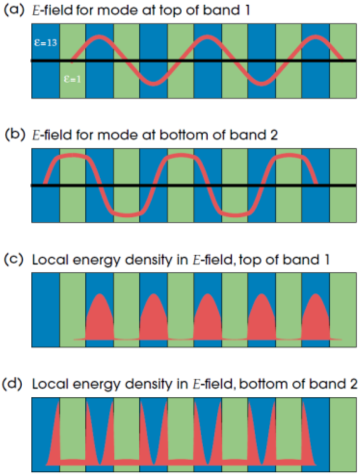
$$d_1 = d_2 = 0.5\, a$$

J D Joannopoulos et al. (2008), *Photonic Crystals: Molding the Flow of Light* (2nd ed.), Princeton NJ: Princeton University Press

$$\varepsilon_1 = 13$$
$$\varepsilon_2 = 12$$

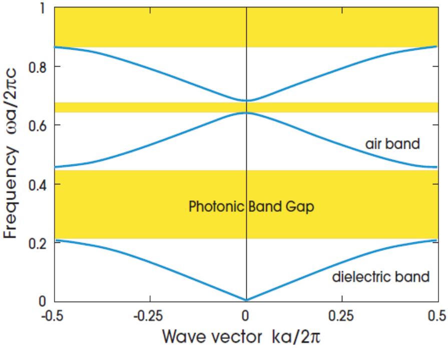


$$\varepsilon_1 = 13$$
$$\varepsilon_2 = 1$$



1D Photonic Crystals

$$\varepsilon_1 = 13 \qquad d_1 = d = 0.2\,a$$
$$\varepsilon_2 = 1 \qquad d_2 = a - d = 0.8\,a$$



J D Joannopoulos et al. (2008), *Photonic Crystals: Molding the Flow of Light* (2nd ed.), Princeton NJ: Princeton University Press

Perturbation Theory:

$$\Delta\omega = -\frac{\omega}{2}\frac{\int d^3\mathbf{r}\,\Delta\varepsilon(\mathbf{r})|\mathbf{E}(\mathbf{r})|^2}{\int d^3\mathbf{r}\,\varepsilon(\mathbf{r})|\mathbf{E}(\mathbf{r})|^2} + O(\Delta\varepsilon^2).$$

$$\varepsilon_1 = \varepsilon \qquad \varepsilon_2 = \varepsilon + \Delta\varepsilon \qquad \left[\begin{array}{l} \dfrac{\Delta\varepsilon}{\varepsilon} \ll 1 \\[2ex] \dfrac{d}{a} \ll 1 \end{array}\right.$$

Gap-midgap ratio

$$\frac{\Delta\omega}{\omega_m} \approx \frac{\Delta\varepsilon}{\varepsilon}\cdot\frac{\sin(\pi d/a)}{\pi}$$

More refined theory:

$\Delta\omega$ max if $\quad d_1 n_1 = d_2 n_2$

$$\left[\begin{array}{l} \omega_m = \dfrac{n_1+n_2}{4n_1n_2}\cdot\dfrac{2\pi c}{a}. \\[3ex] \dfrac{\Delta\omega}{\omega_m} = \dfrac{4}{\pi}\sin^{-1}\left(\dfrac{|n_1-n_2|}{n_1+n_2}\right) \end{array}\right.$$