

1 Lecture 3

Slide 1

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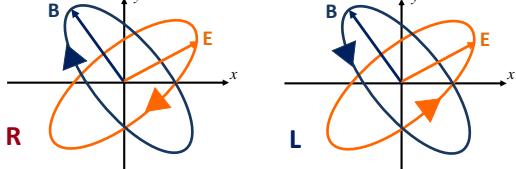
Elliptical polarization

Optics and
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T. Cesca

$$\vec{E} = \hat{i}E_{0x}e^{i(kz-\omega t)} + \hat{j}E_{0y}e^{i(kz-\omega t+\delta)}$$

Right-elliptically polarized (R): $E_{0x} \neq E_{0y}$ $\delta = -\frac{\pi}{2}$

Left-elliptically polarized (L): $E_{0x} \neq E_{0y}$ $\delta = +\frac{\pi}{2}$



Clockwise (counter-clockwise) rotation if looking from the receiver

In the last lecture we have seen how to obtain elliptical polarization. If we have a left-elliptically polarized wave we have a counter-clock-wise rotation, while for right-elliptically we have a clockwise rotation.

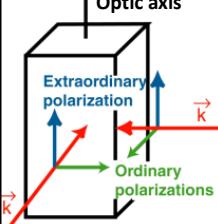
How can we physically get the overlap of two components with a given phase shift in order to get an elliptical polarization?

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Propagation in crystals

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For any given direction of the propagation vector (\vec{k}), 2 possible values of the refractive index (n) of the crystal exist, corresponding to orthogonal polarization states.

$$v = \frac{c}{n}$$

Birefringence

Optic axis: direction of the propagation vector (\vec{k}) for which the refractive indexes of orthogonal polarization states are the same.

To do this, we have to make a step back and say few things about the propagation on crystals. In general, we think that is always possible to cut such crystal in such a way that you have for any given direction of propagation vector two possible refractive index (which means two possible velocities) corresponding to orthogonal polarization state. In particular you get **birefringence**. So the idea is that you can always cut a crystal such that you have two possible refractive indexes.

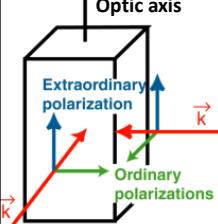
The **optic axis** is the direction for which you have the same refractive indexes.

Slide 3

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**Propagation
in crystals**

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ISOTROPIC
cubic $n = \sqrt{1 + \chi_1}$

UNIAXIAL
trigonal $n_o = \sqrt{1 + \chi_1}$ ordinary
tetragonal
hexagonal $n_e = \sqrt{1 + \chi_3}$ extraordinary

BIAXIAL
triclinic $n_1 = \sqrt{1 + \chi_1}$
monoclinic $n_2 = \sqrt{1 + \chi_2}$
orthorhombic $n_3 = \sqrt{1 + \chi_3}$

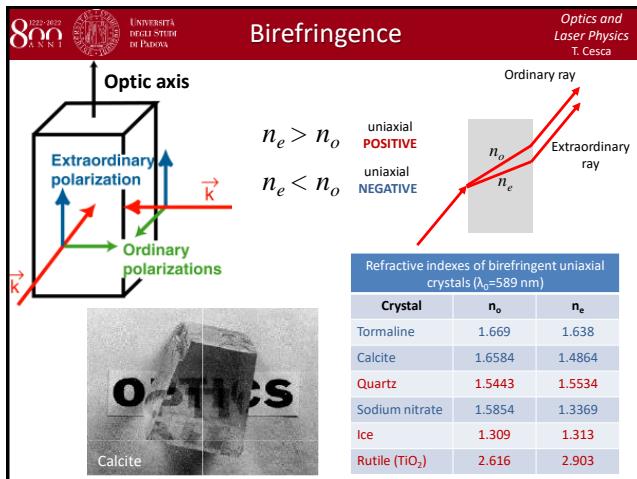
Ordinary wave: $E \perp \text{optic axis}, \perp \vec{k}$

Extraordinary wave: $E \parallel \text{optic axis}, \perp \vec{k}$

You can distinguish three classes of crystals.

- **isotropic:** there is no birefringence for any propagation direction. We have only one refractive index.
- **uniaxial:** this family of crystals have one optic axis. We have two different refractive indexes associated to orthogonal polarizations. The **ordinary waves** are electromagnetic waves linearly polarized perpendicularly to the optic axis and to the propagation direction. The **extraordinary waves** for those field parallel to the optic axis and perpendicular to the propagation direction.
- **biaxial:** we have two different optics axis, such that we can have 3 different refractive indexes for the 3 different orientation.

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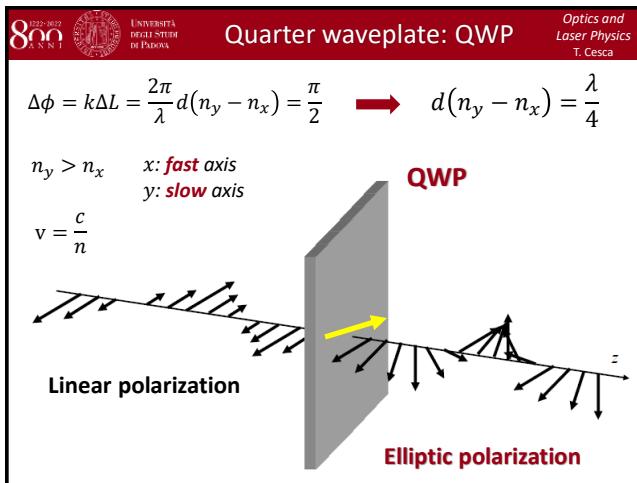
In order to answer to the question: how can we physically get the overlap of two components with a given phase shift in order to get an elliptical polarization? We will consider **uniaxial crystals**. We can distinguish two families:

- **positive**: the extraordinary refractive index is larger than the ordinary.
- **negative**: the contrary of positive.

Here, is reported a table of different uniaxial crystals. The most used one is *calcite*.

How can we use birefringence to control the polarization state of our wave?

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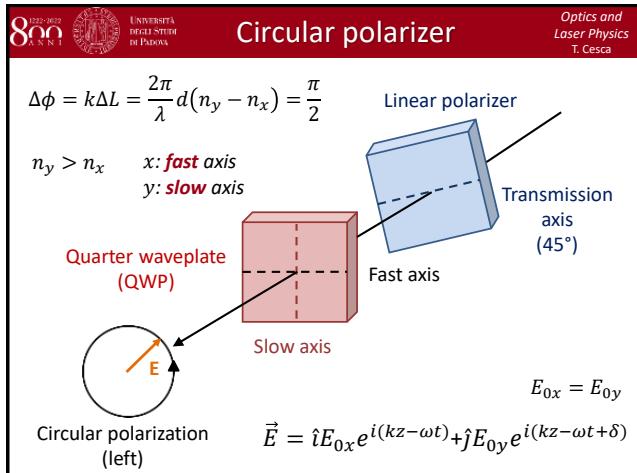


Let us suppose that now we have a material with two different refractive indexes related to orthogonal polarization space. We can imagine to align our system of reference in order to have an x axis related to one refractive index and a y axis related to the other. When wave propagates through a crystal aligned in this way, we can introduce a **phase shift** due to the difference of optical path. If you cut the material with the thickness in such a way that we have $\Delta\Phi = \pi/2$, we have that:

$$d(n_y - n_x) = \lambda/4$$

so we are realizing a **quarter wave plate**. In this way, we will get an elliptical polarization. The x axis is the **fast axis** (the wave velocity is faster). If the **fast axis** is vertical (instead of being horizontal) we obtain $\Delta\Phi = -\pi/2$. So the sign depends on where is oriented the fast axis.

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This can be used to obtain also a circular polarization. If you start with a linear polarization state and if you orient the transmission axis to 45 with respect to the fast axis, you will obtain $E_{0x} = E_{0y}$.

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Half waveplate: HWP

$\Delta\phi = k\Delta L = \frac{2\pi}{\lambda} d(n_y - n_x) = \pi \rightarrow d(n_y - n_x) = \frac{\lambda}{2}$

$\vec{E} = iE_{0x}e^{i(kz-\omega t)} + jE_{0y}e^{i(kz-\omega t+\pi)}$
 $= iE_{0x}e^{i(kz-\omega t)} - jE_{0y}e^{i(kz-\omega t)}$

$e^{\pm i\pi} = -1$

HWP

The diagram shows a grey rectangular block labeled "HWP". To its left, a horizontal line with arrows at both ends represents "Linear polarization (at an angle α with respect to the horizontal axis)". An arrow from this line points to the HWP. From the right side of the HWP, two parallel lines with arrows pointing upwards represent "Linear polarization (turned of 2α)".

Linear polarization (at an angle α with respect to the horizontal axis)

Linear polarization (turned of 2α)

With the same idea, you can also realize **Half waveplate**. You select the thickness of your material in such a way to obtain $\Delta\Phi = \pi$.

This is the expression of the electric field after passing through this material, you have to sum up two components with a phase of π . If we remember that $e^{\pm i\pi} = -1$, we are just flipping the component and we obtain a rotation. If we incide with a linear polarized wave with angle α , we will get still a linearly polarized wave but rotating by 2α . This is the same both if you introduce a phase shift of π or $-\pi$. So in this case does not matter what you are orienting along the horizontal axis. In any case you will have a rotation. This is useful if you want to obtain a rotation of 90° which you cannot obtain with a linear polarizer as we have already seen in the last lectures!).

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Jones' notation

Robert Clark Jones, 1941

$\vec{E}(z, t) = \vec{E}_0 e^{i(kz-\omega t)} = iE_{0x}e^{i(kz-\omega t)} + jE_{0y}e^{i(kz-\omega t+\delta)}$

$\vec{E}_0 = iE_{0x} + jE_{0y}$

$E_{0x}, E_{0y} \in \mathbb{R} \rightarrow$ Linear polarization

$E_{0x}, E_{0y} \in \mathbb{C} \rightarrow$ Elliptical polarization

$|E_{0x}| = |E_{0y}|$ and $\delta = \pm \frac{\pi}{2} \rightarrow$ Circular polarization, right (-) or left (+)

$\vec{E}_0 = \begin{bmatrix} E_{0x} \\ E_{0y} \end{bmatrix}$ **Jones' vector** $\vec{E}'_0 = \frac{1}{\sqrt{|E_{0x}|^2 + |E_{0y}|^2}} \begin{bmatrix} E_{0x} \\ E_{0y} \end{bmatrix}$
 normalized form

Jones' notation is a very useful notation for polarized waves. Let us consider a wave and decompose it into two directions with a phase shift contained in the y direction (δ).

We can write this in a more compact wave if we introduce \vec{E}_0 . If the two components are both real we are talking about **linear polarization**, while if they are complex we are dealing with an **elliptic polarization** state. We have a **circular polarization** if the modulus are the same.

With this idea in mind, we introduce a **Jones' vector**, which is simply a vector which contains the two components. We can have also its normalized form.

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Jones' notation

$\begin{bmatrix} A \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$ linearly polarized wave along the x -axis

$\begin{bmatrix} 0 \\ B \end{bmatrix} = B \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$ linearly polarized wave along the y -axis

$\begin{bmatrix} A \\ A \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow$ linearly polarized wave at 45°

$\begin{bmatrix} 1 \\ i \end{bmatrix} \rightarrow$ Left-circularly polarized wave $\begin{bmatrix} 1 \\ -i \end{bmatrix} \rightarrow$ Right-circularly polarized wave

$\vec{E}_0 = \begin{bmatrix} E_{0x} \\ E_{0y} \end{bmatrix}$ **Jones' vector** $\vec{E}'_0 = \frac{1}{\sqrt{|E_{0x}|^2 + |E_{0y}|^2}} \begin{bmatrix} E_{0x} \\ E_{0y} \end{bmatrix}$
 normalized form

In this way, we can immediately write a Jones' vector for a linearly polarized wave and so on.

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Jones' notation

$\begin{bmatrix} A \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → linearly polarized wave along the x-axis

$\begin{bmatrix} 0 \\ B \end{bmatrix} = B \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → linearly polarized wave along the y-axis

$\begin{bmatrix} A \\ A \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → linearly polarized wave at 45°

$\begin{bmatrix} 1 \\ i \end{bmatrix}$ → Left-circularly polarized wave $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ → Right-circularly polarized wave

$\begin{bmatrix} 1 \\ i \end{bmatrix} + \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1+1 \\ i-i \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

If we combine the left and right circularly polarized way we apply the Jones'notation and we sum the two components. We obtain a Jones'vector for a linearly polarized wave.

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Jones' notation

Any linear optical element acting on the polarization states can be described by a 2x2 matrix:

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \quad \text{Jones' matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \dots \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

Last optical element First optical element

Optical system ↔ Matrix 2 x 2

$$\begin{bmatrix} A \\ B \end{bmatrix} \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} A' \\ B' \end{bmatrix}$$

Jones'notation is useful particularly because you can also associate to any optical element which produce an effect on the polarization state you can introduce a **Jones'matrix**.

By simply applying in sequence all the matrices of the different optical elements to specific order to the incidence polarization state, you will have the final polarization state of your wave. Pay attention to the order of applications of the operators!

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Jones' matrixes

Linear polarizer	Quarter waveplate (QWP)
Horizontal transmission axis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Vertical fast axis $\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$
Vertical transmission axis $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	Horizontal fast axis $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
Transmission axis at ±45° $\frac{1}{2} \begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$	Fast axis at ±45° $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mp i \\ \mp i & 1 \end{bmatrix}$

Circular polarizer	Half waveplate (HWP)
Right $\frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$	Vertical and horizontal fast axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Left $\frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$	Phase retarder $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$

In this slide there is a summary of the main Jones'matrices for the different optical elements.

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Jones' notation

What is the polarization state of the outcoming wave when a linearly polarized wave at 45° is impinging on a QWP with horizontal fast axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \rightarrow \text{Left-circularly polarized wave}$$

Let us try to apply the Jones' notation. Let us see this exercise. We have just seen the result in this lecture. We have a linearly polarized wave at 45 degrees. We apply QWP Jones' matrix.

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Circular polarizer

This is the graphical answer to the exercise as we have already seen.

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Jones' notation

What is the polarization state of the outcoming wave when a linearly polarized wave at 45° is impinging on a QWP with horizontal fast axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \rightarrow \text{Left-circularly polarized wave}$$

NB: There is no Jones' vector for unpolarized light!

An important point is that **there is no Jones' vector for describing unpolarized light!** The vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ does not mean unpolarized light! Instead this means that there is no light coming out, does not mean an unpolarized state.

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$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Let us try to solve this exercise with this notation. Let us consider an incident unpolarized light, which impinges on a linear polarizer of 45° . We have then a QWP with vertical fast axis, another QWP with vertical fast axis and finally a linear polarizer of 45° degrees. What is coming out at the end?

With an unpolarized light which passed through a linear polarizer at 45° , we obtain a linear polarized wave at 45° . If then we enter on a QWP, we obtain a circularly polarized wave right (the fast axis is vertical). Then, if we enter again on a QWP we obtain polarized light at -45° . **The two QWP in sequence work as one HWP.** If we enter with a light with a -45° degree on a polarizer at 45° the result is zero. There is no light coming out.

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800 ANNI UNIVERSITÀ DELL'ISTITUTO DI PADOVA Eigenvectors of Jones' matrices Optics and Laser Physics T. Cesca

The **eigenvectors** of a Jones' matrix are those specific **polarization states** that pass unaffected through the optical element represented by such Jones' matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \lambda \begin{bmatrix} A \\ B \end{bmatrix}$$

eigenvalue
eigenvector

$$\begin{bmatrix} (a - \lambda)A + bB = 0 \\ cA + (d - \lambda)B = 0 \end{bmatrix} \rightarrow \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0 \quad \lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \xleftarrow{\lambda_{1,2}} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$$

Another interesting exercise is to answer to the question: what are the polarization states which pass unaffected through the optical elements represented by such Jones' matrix? We are solving an eigenvectors and eigenvalue problem for a given optical element or for a sequence of them. We obtain two eigenvalues. The eigenvectors are the polarization states which pass unaffected through the sequence of optical elements.

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800 ANNI UNIVERSITÀ DELL'ISTITUTO DI PADOVA Eigenvectors of Jones' matrices Optics and Laser Physics T. Cesca

Determine the **eigenvectors** of a QWP with horizontal fast axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \lambda \begin{bmatrix} A \\ B \end{bmatrix} \rightarrow \begin{vmatrix} 1 - \lambda & 0 \\ 0 & i - \lambda \end{vmatrix} = 0$$

$$\begin{cases} (1 - \lambda)A = 0 \\ (i - \lambda)B = 0 \end{cases} \rightarrow (1 - \lambda)(i - \lambda) = 0$$

$$\lambda_1 = 1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ linearly polarized wave along the x-axis (fast axis)}$$

$$\lambda_2 = i \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ linearly polarized wave along the y-axis (slow axis)}$$

$$\frac{\lambda_1}{\lambda_2} = i = e^{i\frac{\pi}{2}}$$

For instance, let us determine the eigenvectors of a QWP with horizontal fast axis. The ratio of the two eigenvalues is i , so the two eigenvectors have a phase shift of $\pi/2$ which is what we get for QWP.

The important idea of Jones' notation is that you can determine the final polarization state (or an intermediate one) by computing matrix-product.