УМФ. Лекция (15/05/2025)

$$\Omega_\varepsilon = \Omega \setminus \left(\overline{B_\varepsilon(M_1)} \cup \overline{B_\varepsilon(M_2)}\right)$$
$$G(M_1, M)$$
$$G(M_2, M)$$

$$0 = \iint_{S_1} \underbrace{\left(G(M_1, M)\frac{\partial G(M_2, M)}{\partial \nu}\right)}_{\to 0} - \underbrace{(M_2, M)\frac{\partial}{\partial \nu}G(M_1, M)}_{\to G(M_2, M)\ \text{при}\ \ \varepsilon \to 0} d\Gamma +$$
$$\iint_{S_2} (G(M_1, M)\frac{\partial G(M_2, M)}{\partial \nu} - G(M_2, M)\frac{\partial}{\partial \nu}G(M_1, M)d\Gamma$$

Функция Грина задачи Дирихле для шара в трехмерном пространстве

$$G(M, M_0) = \frac{1}{4\pi r} + g(M, M_0) \quad r = r_{MM_0}$$

$$\Delta g(M, M_0) = 0 \tag{1}$$

$$g(M, M_0) = -\frac{1}{4\pi r_{MM_0}} \tag{2}$$

$$\Delta u = 0 \tag{3}$$

$$u|_\Gamma = f \tag{4}$$

$$u(M_0) = -\iint_\Gamma f\frac{\partial G}{\partial \nu}d\Gamma \tag{5}$$

$$G(M, M_0) = \frac{1}{4\pi r} + g(M, M_0) \quad r = r_{MM_0} \tag{6}$$

Рассмотрим задачу:

$$\Omega = B_R(0),\, S_R(0) = \Gamma$$
$$M \in B_R(0),\, M = M(x, y, z)$$
$$p \in \overline{B_R(0)},\, p = p(\xi, \eta, \zeta)$$
$$r = r_{pM} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\eta)^2}$$

Для точки M построим инверсию M_1

$$OM \cdot OM_1 = R^2$$

Рассмотрим частный случай, когда $P \in S_R$

$$OM = \rho,\, OM_1 = \rho_1$$

$$M_1(x_1, y_1, z_1)$$

$$r_1 = \sqrt{(x_1 - \xi)^2 + (y_1 - \eta)^2 + (z_1 - \eta)^2}$$

$$\triangle\, OPM \sim \triangle\, OM_1P$$

$$\frac{OP}{OM_1} = \frac{OM}{OP} = \frac{PM}{PM_1} \tag{7}$$

$$\frac{R}{\rho_1} = \frac{\rho}{R} = \frac{r}{r_1} \tag{8}$$

$$\frac{1}{4\pi}\frac{1}{R} - \frac{1R}{4\pi\rho} \cdot \frac{1}{r_1} = 0, \quad p \in S \tag{9}$$

$$g(P, M) = \frac{1}{4\pi}\frac{R}{\rho} \cdot \frac{r}{r_1} \tag{10}$$

$$\Delta_p g(P, M) = 0$$

$$G(P, M) = \frac{1}{4\pi r} + g(P, M) \tag{11}$$

Выражение (11), где g определятся (9) даёт нам функцию Грина задачи Дирихле

1.

$$\frac{\partial}{\partial \nu}\left(\frac{1}{r}\right) = \frac{\partial}{\partial \xi}\left(\frac{1}{r}\right)\cos\widehat{\nu\xi} + \frac{\partial}{\partial \eta}\left(\frac{1}{r}\right)\cos\widehat{\nu\eta} + \frac{\partial}{\partial \zeta}\left(\frac{1}{r}\right)\cos\widehat{\nu\zeta}$$

$$\frac{\partial}{\partial \xi}\frac{1}{r} = -\frac{1}{r^2}\frac{\partial r}{\partial \xi} = -\frac{1}{r^2} \cdot \frac{1}{2} \cdot \frac{2(\xi - x)}{r} = -\frac{1}{r^2} \cdot \frac{\xi - x}{r}$$

$$\begin{aligned} \frac{\partial}{\partial \nu}\left(\frac{1}{r}\right) &= -\frac{1}{r^2} \cdot \left\{\frac{\xi - x}{r} \cdot \cos\widehat{\nu\xi} + \frac{\eta - y}{r}\cos\widehat{\nu\eta} + \frac{\zeta - z}{r}\cos\widehat{\nu\zeta}\right\} = -\frac{1}{r^2} \cdot (\vec{\imath}_r \cdot \vec{\nu}) \\ &= -\frac{1}{r^2} \cdot \cos(\widehat{r\nu}) \end{aligned} \tag{12}$$

$\bar{\imath}_r$ направляющий вектор MP

$$\frac{\partial}{\partial \nu}\left(\frac{1}{r_1}\right) = -\frac{1}{r^2}\cos(\widehat{r_1\nu}) \tag{13}$$

$$\frac{\partial G(P, M)}{\partial \nu} = -\frac{1}{4\pi r^2}\cos(\widehat{r\nu}) + \frac{1}{4\pi}\frac{R}{\rho} \cdot \frac{1}{r_1^2}\cos(\widehat{r_1\nu})$$

Применим теорему косинусов, для $\triangle\, OPM$:

$$\rho^2 = R^2 + r^2 - 2Rr \cdot \cos(\widehat{\nu r}) \Rightarrow$$

$$\cos(\widehat{\nu r}) = \frac{R^2 + r^2 - \rho^2}{2Rr}$$

Для $\triangle\, OPM_1$:

$$\rho_1^2 = R^2 + r_1^2 - 2Rr_1 \cos(\widehat{\nu r_1})$$

$$\cos(\widehat{\nu r_1}) = \frac{R^2 + r_1^2 - \rho_1^2}{2Rr_1}$$

$$\frac{\partial G}{\partial \nu} = -\frac{1}{4\pi r^2} \cdot \frac{R^2 + r^2 - \rho^2}{2Rr} + \frac{1}{4\pi r_1^2} \cdot \frac{R}{\rho} \cdot \frac{R^2 + r_1^2 - \rho_1^2}{2Rr_1}$$

$$r_1 = \frac{Rr}{\rho}; \quad \rho_1 = \frac{R^2}{\rho}$$

$$\frac{\rho^2}{4\pi R^2 r^2} \cdot \frac{R}{\rho} \cdot \frac{R^2 + \frac{R^2 r^2}{\rho^2} - \frac{R^4}{\rho^2}}{2R\frac{Rr}{\rho}}$$

$$\frac{\partial G}{\partial \nu} = -\frac{1}{4\pi r^2} \cdot \frac{R^2 + r^2 - \rho^2}{2Rr} + \frac{\rho^2 + r^2 - R^2}{4\pi R r^2 \cdot 2r} = \frac{\rho^2 - R^2}{4\pi R r^3}$$

$$u(M) = \frac{R^2 - \rho^2}{4\pi R} \cdot \int_S f(P) \cdot \frac{1}{r_{MP}^2} dS$$

Получили формулу Пуассона