УМФ. Лекция (13.09.24)

$$\frac{\partial U}{\partial x} = U_x,\ \frac{\partial^2 U}{\partial x \partial y} = u_{xy}, \dots$$

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + h u = f \tag{1}$$

$$b^2 - ac \begin{cases} > 0, & \text{гиперболический тип} \\ < 0, & \text{эллиптический тип} \\ = 0, & \text{параболический тип} \end{cases}$$

Замена: $\begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases}$ $\qquad D(\xi,\eta) = \det \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \neq 0$

$$2\tilde{b}\,\frac{\tilde{U}(\xi,\eta)}{\partial\xi\partial\eta} + \tilde{d}\tilde{u}_\xi + \tilde{e}\tilde{u}_\eta + h\tilde{u} = \tilde{f} \quad \text{(гиперболический тип КВ)} \tag{2}$$

$$\tilde{u}(\xi,\eta) = u\left(\xi(x,y), \eta(x,y)\right) = u(x,y) \tag{3}$$

$$\begin{aligned} \tilde{u}_x &= \frac{\partial}{\partial x}\tilde{u}(\xi,\eta) = \tilde{u}_\xi \xi_x + \tilde{u}_\eta \eta_x \\ \tilde{u}_{xx} &= \frac{\partial}{\partial x} u_x = \frac{\partial}{\partial x}(\tilde{u}_\xi \xi_x + \tilde{u}_\eta \eta_x) = (\frac{\partial \tilde{u}_\xi}{\partial \xi}\xi_x + \frac{\partial \tilde{u}_\xi}{\partial \eta}\eta_x)\xi_x + \tilde{u}_\xi \xi_{xx} + \dots \end{aligned}$$

$$\tilde{a}\tilde{u}_{\xi\xi} + 2\tilde{b}\tilde{u}_{\xi\eta} + \tilde{c}\tilde{u}_{\eta\eta} + \tilde{d}\tilde{u}_\xi + \tilde{e}\tilde{u}_\eta + \tilde{h}\tilde{u} = \tilde{f}$$

$$\begin{aligned} \tilde{a} &= a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 \\ \tilde{c} &= a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 \\ \tilde{b} &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_x\eta_y \\ \tilde{d} &= a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y \\ \tilde{e} &= a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y \\ \tilde{h} &= h \\ \tilde{f} &= f \end{aligned} \tag{4}$$

Гиперболический тип

гиперболическое ур-е: $\begin{cases} \tilde{a} & = 0 \\ \tilde{c} & = 0 \end{cases}$

$$a(\phi_x)^2 + 2b(\phi_x\phi_y) + c(\phi_y)^2 = 0 \quad \text{- характеристическое уравнение} \tag{5}$$

$$a \neq 0 \quad (\phi_x)_{1,2} = \frac{-b \pm \sqrt{b^2 - ac}}{a} \cdot \phi_y$$

$$\left[\begin{aligned} a\phi_x + (b - \sqrt{b^2 - ac})\phi_y & = 0 \\ a\phi_x + (b + \sqrt{b^2 - ac})\phi_y & = 0 \end{aligned} \right. \tag{6}$$

$$\alpha(x,y)\phi_x(x,y) + \beta(x,y)\phi_y(x,y) = 0$$

$$\tilde{b}^2 - \tilde{a}\tilde{b} = (b^2 - ac)\left(\det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\right)^2$$

$$\left[\begin{aligned} \tfrac{dx}{a} & = \tfrac{dy}{b - \sqrt{b^2 - ac}} \to \Phi \equiv \xi \\ \tfrac{dx}{a} & = \tfrac{dy}{b + \sqrt{b^2 - ac}} \to \Phi \equiv \eta \end{aligned} \right. \tag{7}$$

Параболический тип

$$\begin{cases} \tilde{b} &= 0 \\ \tilde{a} &= 0 \end{cases} \quad \text{или} \quad \begin{cases} \tilde{b} &= 0 \\ \tilde{c} &= 0 \end{cases}$$

$$a\phi_x + b\phi_y = 0| \cdot b$$
$$ab\phi_x + b^2\phi_y = 0$$
$$ab\phi_x + ac\phi_y = 0$$
$$b\phi_x + c\phi_y = 0$$

Замена: $\begin{cases} \phi \to \xi &= \xi(x,y) \\ \eta &= \eta(x,y) \end{cases}$

$\tilde{b} = \eta_x(a\xi_x + b\xi_y) + \eta_y(b\xi_x + c\xi_y) = 0$

Эллиптический тип

$$b^2 - ac < 0 \quad \tilde{a} = \tilde{c} \quad \tilde{b} = 0$$

$$\left[\begin{array}{ll} a\phi_x + (b - i\sqrt{ac - b^2})\phi_y &= 0 \\ a\phi_x + (b + i\sqrt{ac - b^2})\phi_y &= 0 \end{array} \right. \tag{8}$$

$\Phi(x,y)$

$$\frac{dx}{a} = \frac{dy}{b - i\sqrt{ac - b^2}}$$

$$\Phi(x,y) = Re\Phi(x,y) + iIm\Phi(x,y) \quad \Phi = \xi + i\eta$$

$\begin{cases} \xi &= Re\Phi \\ \eta &= Im\Phi \end{cases}$

$$a(\xi_x^2 - \eta_x^2) + 2b(\xi_x\xi_y - \eta_x\eta_y) + c(\xi_y^2 - \eta_y^2) + 2i(a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y) = 0$$

В получившимся уравении действительная часть равна $(\tilde{a} - \tilde{c})$ и действительная часть равна $\tilde{b}$. И притом обе равны 0.