УМФ. Лекция

20 декабря 2024 г.

$$u_{tt}(x,y,t) - a^2(u_{xx}(x,y,t) + u_{yy}(x,y,t)) = 0 \tag{1}$$

$$u(x,y,t)|_{t=0} = \phi(x,y) \tag{2}$$

$$u_t(x,y,t)|_{t=0} = \psi(x,y) \tag{3}$$

$$u|_{x=0} = u|_{x=p} = u|_{y=0} = u|_{y=q} = 0 \tag{4}$$

$$\Pi = [0,p] \times [0,q]$$

$$(x,y) \in \Pi,\ t \geq 0$$

$$u(x,,y,t) = T(t)v(x,y) \tag{5}$$

$$T''(t)v(x,y) - a^2 T(t)\Delta v(x,y) = 0 \mid\ :\ a^2 Tv$$

$$\frac{T''(t)}{a^2 T(t)} = \frac{\Delta v(x,y)}{v(x,y)} = -k$$

$$T''(t) + a^2 kT(t) = 0 \tag{6}$$

$$\Delta v(x,y) + kv(x,y) = 0 \tag{7}$$

$$v(0,y) = v(p,y) = v(x,0) = v(x,p) = 0 \tag{8}$$

Мы доказали, что все собственные значения вещественные и положительные

$$k = \lambda^2,\ \lambda > 0$$

$$T''(t) + a^2\lambda^2 T(t) = 0$$

$$\Delta v(x,y) + \lambda^2 v(x,y) = 0$$

$$v(0,y) = v(p,y) = v(x,0) = v(x,p) = 0$$

$$v(x,y) = X(x)Y(y) \tag{9}$$

$$X''(x)\cdot Y(y) + X(x)\cdot Y''(y) + \lambda^2 X(x)Y(y) = 0 \mid X(x)Y(y)$$

$$\frac{X''(x)}{X(x)} + \frac{Y(y)}{Y(y)} + \lambda^2 = 0$$

$$\frac{X''(x)}{X(x)} = -\lambda^2 - \frac{Y''(y)}{Y(y)} = \widetilde{k}$$

$$X''(x) - \widetilde{k}X(x) = 0$$

$$X(0) = X(p) = 0$$

$$Y''(y) + (\widetilde{k} + \lambda^2)Y(y) = 0$$

$$Y(0) = Y(q) = 0$$

$$\begin{cases} \widetilde{-k} = \left(\frac{\pi n}{p}\right)^2,\ n \in \mathbb{N} \\ X_n(x) = \sin\frac{\pi n x}{p} \end{cases}$$

$$\begin{cases} \widetilde{-\lambda^2} + \lambda^2 = \left(\frac{\pi m}{q}\right)^2, m \in \mathbb{N} \\ Y_m(x) = \sin\frac{\pi n y}{q} \end{cases}$$

$$v_{m,n}(x,y) = X_n(x)Y_m(y) = \sin\frac{\pi n x}{p}\cdot\sin\frac{\pi m y}{q} \tag{10}$$

$$\lambda^2_{n,m} = \left(\frac{\pi n}{p}\right)^2 + \left(\frac{\pi m}{q}\right)^2 \tag{11}$$

$$T''_{m,n}(t) + (a\lambda_{m,n})^2 T_{m,n}(t) = 0$$

$$T_{m,n}(t) = A_{m,n}\cos(a\lambda_{m,n}t) + B_{m,n}\sin(a\lambda_{m,n}t) \tag{12}$$

$$u_{m,n}(x,y,t) = T_{m,n}(t)v_{m,n}(x,y) = (A_{m,n}\cos(a\lambda_{m,n}t) + B_{m,n}\sin(a\lambda_{m,n}t))\sin\frac{\pi n x}{p}\sin\frac{\pi m y}{q}$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n}(x,y,t) = \sum_{m,n=1}^{\infty} T_{m,n}(t)v_{m,n}(x,y) =$$

$$\sum_{m,n=1}^{\infty}(A_{m,n}\cos(a\lambda_{m,n}t) + B_{m,n}\sin(a\lambda_{m,n}t))\sin\frac{\pi n x}{p}\sin\frac{\pi m y}{q} \tag{13}$$

$$u(x,y,t)|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\sin\frac{\pi n x}{p}\sin\frac{\pi m y}{q} = \phi(x,y)$$

$$\sum_{m=1}^{\infty} A_{m,n}\sin\frac{\pi m y}{q} = \frac{2}{p}\int_0^p \phi(x,y)\sin\frac{\pi n x}{p}dx$$

$$A_{m,n} = \frac{4}{pq}\int_0^p\int_0^q \phi(x,y)\sin\frac{\pi n x}{p}\sin\frac{\pi m y}{q}dxdy \tag{14}$$

$$u(x,y,t)|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a\lambda_{m,n}B_{m,n}\sin\frac{\pi n x}{p}\sin\frac{\pi m y}{q} = \psi(x,y)$$

$$B_{m,n} = \frac{4}{pqa\lambda_{m,n}}\int_0^p\int_0^q \psi(x,y)\sin\frac{\pi n x}{p}\sin\frac{\pi m y}{q}dxdy \tag{15}$$

Отметим, что выражение вида:

$$a\lambda_{m,n} = a\sqrt{\left((\frac{\pi n}{p}\right)^2 + \left(\frac{\pi m}{q}\right)} = \omega_{m,n}$$