Формула замены переменных Ито

$$X(t) = X(0) + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dw(s) \tag{1}$$

$$dX(t) = a(s,\omega)dt + b(t,\omega)dw(t) \tag{2}$$

$$f(t,x) \quad Y(t) = f(t, X(t))$$

$$dY(t) = ?$$

$$X(t) = x_0 + at + bw(t), \text{ где } a, b \text{ - случайные величины} \quad x_0 = 0$$

$$f(t, at + bW(t))$$

$$f = f(t, at + bw) = u(t, w) \equiv u$$

$$f_t(t, at + bw) + af'_w(t, w) = u'_t$$

$$bf'_w(t, at + bw) = u'_w$$

$$b^2 f'_{ww}(t, at + bw) = u''_{ww}$$

$$u(t,\omega) \quad [0,T) \ \Delta 2^{-n}$$

$$t_k - t_{k-1} = \frac{1}{2^n} = \epsilon$$

$$w(t_k) - w(t_{k-1}) = \Delta w_{t_k}$$

$$u(t, w(t)) - u(0,0) = \sum u(u\epsilon, w_k\epsilon) - u((k-1)\epsilon, w_k\epsilon) + \sum (u(k-1)\epsilon, w_k\epsilon) - u((k-1)\epsilon, w_{k-1}\epsilon) \approx$$

$$\sum u'_t(k\epsilon, w_k\epsilon)\epsilon + \sum u'_w \cdot \Delta w + \frac{1}{2}\sum u''_{ww}(\Delta w_k)^2$$

$$\int_0^t u'_s(s, w_s)ds + \int_0^t u'_w(s, w_s)dw_s + \frac{1}{2}\int_0^t u''_{ww}(s, w_s)ds$$

$$df(t, w_t) = \left[f'_t + af'_w + \frac{1}{2} f''_{ww} b^2 \right] dt + bf'_w dw_t$$

$$f(t,x) \quad \exists f'_t, f'_x, f''_{xx} \text{ - непрерывные}$$

$$z(t) = \exp\left(\int_0^t a(s)dw_s - \frac{1}{2}\int_0^t a^2(s)ds \right)$$

$$f = f(t,w) = e^y$$

$$dY(t) = -\frac{1}{2}\alpha^2(t)dt + \alpha(t)dw_t$$

$$a = \frac{1}{2}\alpha^2, \ b = \alpha(t)$$

$$f'_t = 0, \ f'_y = e^y, \ f''_{ww} = e^y$$

$$df(t, w_t) = \left[f'_t + af'_y + \frac{1}{2} f''_{yy} b^2 \right] dt + bf'_w dw_t$$

$$f'_t + af'_y + \frac{1}{2}b^2 f''_{yy} = -\frac{1}{2}\alpha^2(t)e^{Y(t)} + \frac{1}{2}\alpha(t)e^{Y(t)}$$

$$bf'_y = \alpha(t)e^{Y(t)}$$

$$\boxed{dz(t) = \alpha(t)z(t)dw(t)}$$

$$s(t) = s(0)\exp\left(\sigma w(t) + at - \frac{\sigma^2 t}{2} \right) \text{ - геометрические (экономическое) Броунское движение}$$

Рассмотрим дифференциальное уравнение

$$dN(t) = r(t)N(t)dt$$

$$\frac{dN(t)}{N(t)} = r(t)dt \tag{3}$$

$$N(t) = N(t)e^{-r(t)}$$

$$\frac{dN(t)}{N(t)} = r(t) + W(t)$$

$$dN(t) = r(t)N(t) + N(t)dw(t)$$

Простейшие СДУ

Однородные уравнения

$$dX(t) = a(t)dt + b(t)dw(t) \tag{4}$$

$$X(t) = \int_0^t a(s)ds + \int_0^t b(s)dw(s)$$

$$f(t,w) = b(t)\cdot w \quad f'_t = b'(t)w,\ f'_w = b(t),\ f''_{ww} = 0$$

$$dw(t) = dw(t) \quad a = 0,\ b = 1$$

$$d(b(t)\cdot w(t)) = b'(t)w(t)dt + b(t)dw(t)$$

$$\int_0^t f(s)w(s)ds = b(t)w(t) - \int_0^t b(s)w(s)ds$$

Линейные однородные уравнения

$$X(t) = 1 + \int_0^t \arctan(s)ds + \int_0^t s^3 dw(s)$$

$$dX(t) = \arctan(t)dt + t^3 dw(t)$$

$$dX(t) = a_1(t)X(t)dt + b_1(t)X(t)dw(t) \tag{5}$$

$$X(t) = x_0 \exp\left(\int_0^t \left(a_1(s) - \frac{1}{2}b_1^2(s)\right)ds + \int_0^t b_1(s)dw(s)\right) \tag{6}$$

$$dN(t) = r(t)N(t) + N(t)dw(t)$$

$$X(t) = N_0 \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\right)ds + w(t)\right)$$

Найти решение СДУ

$$dX(t) = 2X(t)dt + e^t X(t)dw(t)$$

$$X(t) = \exp\left(\int_0^t \left(2 - \frac{1}{2}e^s\right)ds + \int_0^t e^s dw(s)\right)$$

Линейные неоднородные уравнения

$$dX(t) = (a_1(t)X(t) + a_2(t))dt + (b_1(t)X(t) + b_2(t))dw(t)$$

$$a_1, a_2, b_1, b_2 \text{ - числа}$$

$$Y(t) = b_1(t)X(t) + b_2(t) \quad b_1 \neq 0$$

$$X(t) = \frac{Y(t) - b_2}{b_1}$$

$$dX(t) = \frac{1}{b_1}dY(t)$$

$$b_1 dX(t) = dY(t)$$

$$a_1 \cdot X(t) + a_2 = a_1\frac{Y(t) - b_2}{b_1} + a_2 = \frac{a_1 Y(t)}{b_1} - \frac{b_2}{b_1} + \frac{a_2 b_1}{b_1}$$

$$dY(t) = (a_1 Y(t) - b_2 + a_2)dt + b_1 Y(t)dw(t)$$

$$dY(t) = a_1 Y(t)dt + b_1 Y(t)dw(t)$$

$$\widetilde{Y}(t) = Y(t) + (a_2 - b_2)$$

$$f(t,x) = dx + (a_2 - b_2)t \quad f'_t = (a_2 - b_2)$$

$$dX(t) = (X^4(t) + 6X(t))dt + X(t)dw(t)$$

$$y = f(x) = -\frac{1}{3}x^{-3}$$

$$x = -\frac{1}{\sqrt[3]{3}}y^{-\frac{1}{3}} \quad c_0 = -\sqrt[3]{3}$$

$$f'_t = 0,\ f'_x = \frac{1}{x^4},\ f''_{xx} = -\frac{4}{x^5}$$

$$dY(t) = -6(1 + Y(t))dt - 3Y(t)dw(t)$$

$$d\widetilde{Y}(t) = -6\widetilde{Y}(t)dt - 3\widetilde{Y}(t)dw(t)$$

$$dz(t) = -6dt - 3dw(t)$$

$$z(t) = x_0 \exp\left(-\frac{15}{2}t - 3w(t)\right)$$