Группы и Алгебры Ли. Лекция (16.05.2025)

$$\underbrace{\theta(t)}_{\in \text{ мн-во однопар. подгр}} \mapsto v = \frac{d\theta}{dt}\Big|_{t=0} \in T_e(G) \text{ - взаимно однозначное соответствие}$$

$\theta_{v(t)}$ - однопараметрическая подгруппа соответствующая вектору $v \in T_e(G)$

$$\exp : T_{e(G)} \longrightarrow G, \quad \exp(v) = \theta_v(1)$$

$$A \in gl(n, \mathbb{R}) = T_e(GL(n, \mathbb{R}))$$

$$\exp A = e^{At}|_{t=1} = e^A = E + A + \frac{A^2}{2!} + ... + \frac{A^m}{m!} + ...$$

$$e^{At} = \theta_A(t)$$

$$\left(dL_g\right)_e(A) = g \cdot A$$

$$\frac{d\theta(t)}{dt} = dL_{\theta(t)}A \overset{G=GL(n,\mathbb{R})}{=\!=\!=\!=\!=} \theta(t) \cdot A$$

$$\theta(0) = e$$

$$e^{At} = 1 + At + \frac{A^2}{2!}t^2 + ... + \frac{A^m}{m!}t^m + ...$$

$$\frac{d\left(e^{At}\right)}{dt} = A + A^2 \cdot t + ... + \frac{A^m}{(m-1)!}t^{m-1} + ... = \left(e^{At}\right)A$$

Свойства $\exp$:

1. $\exp : T_e(G) \to G$ - гладкое отображение
2. $(d\exp)_0 = \mathrm{id} : \underbrace{T_0(T_e(G))}_{=T_e(G)} \to T_e(G)$
3. $\exp$ - диффеоморфизм в окрестности $0 \in T_e(G)$ и окрестности $e \in G$

$$\theta_v(t) = dL_{\theta(t)} \cdot v$$

$$\theta_v(0) = e$$

$$\frac{d\theta_v(t)}{dt}\bigg|_{t=0} = v$$

$$\Gamma(t) = v \cdot t \in T_e(G)$$

$$\exp v = \exp\left(\frac{d\theta_v(t)}{dt}\bigg|_{t=0}\right) \quad \exp(\Gamma(t)) = \exp(t \cdot v) = \theta_{tv}(1)$$

$$\theta_v(st) = \tilde{\theta}(t)$$

$$s \text{ - фиксированный}$$

$$\frac{d\tilde{\theta}}{dt}\bigg|_{t=0} = \frac{d\theta_v(st)}{dt}\bigg|_{t=0} = s \cdot \frac{d\theta_v(st)}{d(st)}\bigg|_{t=0} = s \cdot \frac{d\theta_v(u)}{du}\bigg|_{u=0} = s \cdot v$$

$$\tilde{\theta}(t) = \theta_{sv}(t) = \theta_v(st)$$

$$\exp(t \cdot v) = \theta_{tv}(1) = \theta_v(t)$$

$$\varphi : M \longrightarrow N$$

$$\gamma(t_0) = m_0$$

$$d\varphi_{m_0} \frac{d\gamma(t)}{dt}\bigg|_{t=t_0} = \frac{d\varphi(\gamma(t))}{dt}\bigg|_{t=t_0}$$

$$(d\exp)_0 v = \frac{d\theta_v(t)}{dt}\bigg|_{t=0} = v$$