Примеры алгебр Ли

$$G = GL(n, \mathbb{R}) \subset M_n(\mathbb{R}) \text{ - открытое подмножество}$$

$$M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

$$A \in T_e(G), \quad e = E$$

x_{ij} - координаты на G

$$A = \sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_l, \quad A = (a_{ij})$$

$$\forall\, g \in GL(n, \mathbb{R}) \quad T_g(G) = \mathbb{R}^{n^2} = M_n(\mathbb{R})$$

$$dL_g(A)$$

$$g = (g_{ij}),\ x = (x_{ij})$$

$$L_g(x) = gx = ((gx)_{ij})$$

$$(gx)_{ij} = \sum_{k=1}^{n} g_{ik} x_{kj}$$

$$x \to$$

$$\begin{aligned} &x'_1 = b_{11}x_1 + \dots + b_{1n}x_n = f_1(x_1, \dots, x_n) \\ &x'_2 = b_{21}x_1 + \dots + b_{2n}x_n = f_2(x_1, \dots, x_n) \\ &\dots\dots\dots \\ &x'_n = b_{n1}x_1 + \dots + b_{nn}x_n = f_n(x_1, \dots, x_n) \end{aligned}$$

$$J = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ b_{21} & \dots & b_{2n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

$$df_x(v) = (\partial_1|_{x'} \dots \partial_n|_{x'})B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 l_1 + \dots + \alpha_n l_n$$

$$(x') = B(x)$$

$$dL_g(A) = g \cdot A$$

$$X_A \text{ - левое инвариантное поле на } G$$

$$X_A(g) = g \cdot A$$

$$x \in G, \quad X_A(x) = x \cdot A = \sum_{i,j} \left(\sum_k x_{ik} a_{kj} \right) \frac{\partial}{\partial x_{ij}}$$

$$Z = [X_A, X_B] = \sum h_{ij} \cdot \frac{\partial}{\partial x_{ij}}$$

$$h_{ij} = Z(x_{ij})$$

$$Z = X_A X_B - X_B X_A$$

$$Z(x_{ij}) = X_A(X_B(x_{ij}) - X_B(X_A(x_{ij}) = X_A \left(\sum_k x_{ik} b_{kj} \right) - X_B \left(\sum_k x_{ik} a_{kj} \right)$$

$$= \sum_{i,j} \left(\sum_s (x_{is} a_{sj}) \frac{\partial}{\partial x_{ij}} \left(\sum_k x_{ij} b_{kj} \right) - \dots \right)$$

$$= \sum_k \left(\sum_s x_{is} a_{sk} \right) b_{kj} - \sum_k \left(\sum_s x_{is} b_{kj} \right) a_{sk} = \sum_s x_{is} \left(\sum_k a_{sk} b_{kj} \right) - \sum_s x_{is} \left(\sum_k b_{kj} a_{sk} \right)$$

$$= \sum_s x_{is} (AB - BA)_{sj} = \sum_s x_{is} ([A, B])_{sj}$$

$$Z = [X_A, X_B] = \sum h_{ij} \cdot \frac{\partial}{\partial x_{ij}} = \sum_{i,j} \left(\sum_s x_{is} C_{sj} \right) \frac{\partial}{\partial x_{ij}} = X_C, \quad C = [A, B]$$

$$X_A = \sum_{i,j} \sum_k x_{ik} a_{kj} \frac{\partial}{\partial x_{ij}}$$

$$Z(e) = \left(\sum_{i,j,s} \delta_{is} e_{si} \right) \frac{\partial}{\partial x_{ij}} \bigg|_e = \sum_{i,j} c_{ij} \frac{\partial}{\partial x_{ij}}$$

$$g = gl(n, \mathbb{R})$$

$$X_A, X_B, \ (X_A, X_B)(e) = X_C$$

$$(A, B) \mapsto C = [A, B]$$

L - алгебра Ли

$$\dim L = 1, \quad L = \langle e_1 \rangle$$

$$[a, b] = 0, \ \forall\, a, b \in L$$

$$a = \alpha e_1, \ b = \beta e_1$$

$$[a, b] = [\alpha e_1, \beta e_1] = \alpha\beta[e_1, e_1] = 0$$

$\left.\frac{d}{dx}\right|_0$ - базисный вектор в 0

$\left.\frac{d}{dx}\right|_x$ - базисный вектор а x

$$L_a(x) = a + x$$

$$J = (1)$$

$$dL_a = \left(\frac{d}{dx}\bigg|_0 \right) = \frac{d}{dx}\bigg|_a (1)$$

$$(1) = \frac{d}{dx}\bigg|_a$$

$$X_{\frac{d}{dx}\big|_0} = \frac{d}{dx}$$

$$L(G) = \langle \frac{d}{dx} \rangle_{\mathbb{R}}$$

$$\mathbb{R}^\star = (\mathbb{R} \setminus \{0\}, \cdot)$$

$$\frac{d}{dx}\bigg|_1$$

$$L_a x = ax$$

$$J = (a)$$

$$dL_a \left(\frac{d}{dx}\bigg|_1 \right) = \frac{d}{dx}\bigg|_a J(1) = \frac{d}{dx}\bigg|_a \cdot a = a \frac{d}{dx}\bigg|_a$$

$$X_{\frac{d}{dx}\big|_1}(x) = x \cdot \frac{d}{dx}$$

$$G = (\mathbb{R}, +)$$

$$L(G) = \langle x \frac{d}{dx} \rangle$$

$e^x: \ (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ - изоморфизм (не алгебраический)