$$v \in T_e(G)$$

$$\begin{cases} \frac{d\theta}{dt} = dL_{\theta(t)}(v) \\ \theta(0) = e \end{cases} \tag{1}$$

$\exists!\ \theta(t)$ - решение (1) при $t \in (-\varepsilon, \varepsilon)$

$$\psi(t) = \theta\left(\frac{t}{N}\right)^N,\ t \in \mathbb{R}$$

$$N \in \mathbb{N}\ :\ \left|\frac{t}{N}\right| < \frac{\varepsilon}{2}$$

$$t \in (t_0 - \delta, t_0 + \delta)$$

$$\left|\frac{t}{N}\right| < \frac{\varepsilon}{2} \implies \frac{t}{N} \text{ гладко зависит от } t \in (t_0 - \delta, t_0 + \delta)$$

$\theta\left(\frac{t}{N}\right)$ - гладкая
$\theta\left(\frac{t}{N}\right)^N$ - гладко зависит от t
$\implies \psi(t)$ - гладко зависит от $t \in \mathbb{R}$

Рассмотрим:

$$\psi(s+t) = \theta\left(\frac{s+t}{N}\right)^N = \left(\theta\left(\frac{s}{N}\right)\theta\left(\frac{t}{N}\right)\right)^N = \theta\left(\frac{s}{N}\right)^N \cdot \theta\left(\frac{t}{N}\right)^N = \psi(s)\cdot\psi(t)$$

$$\theta\left(\frac{s}{N}\right)\theta\left(\frac{t}{N}\right) = \theta\left(\frac{s+t}{N}\right) = \theta\left(\frac{t+s}{N}\right) = \theta\left(\frac{t}{N}\right)\theta\left(\frac{s}{N}\right)$$

$$\left.\frac{d\psi}{dt}\right|_{t=0} = \left.\frac{d\theta}{dt}\right|_{t=0} = v$$

Экпоненциональное отображение

$\theta_v(t)$ - однопараметрическая подгруппа такая, что $\left.\frac{d\theta_v}{dt}\right|_{v=0} = v$
Определим:

$$\exp:\ T_e(G) \to G,\ \exp(v) = \theta_v(1) \in G$$

$G \sim (\mathbb{R}^\star, \cdot)$.

$$T_1(G) = \mathbb{R},\ \exp v = e^v$$

$$\frac{d\theta}{dt} = \theta(t)\cdot v$$

$G = GL(n, \mathbb{R}), \quad T_e(G) = M_n(\mathbb{R})$

$$\theta_v(t) = A(t)$$

$$\frac{dA(t)}{dt} = A(t)\cdot v \quad v \in M_n(\mathbb{R})$$

$$\exp(B) = E + B + \frac{B^2}{2!} + \dots + \frac{B^m}{m!} + \dots$$

$$\left|\left|\frac{B^m}{m!}\right|\right| \le \frac{||B||^m}{m!}$$

$$\exp(||B||) = 1 + ||B|| + \dots + \frac{||B||^m}{m!} + \dots$$

$$\exp(At) = 1 + A\cdot t + \frac{A^2t^2}{2} + \dots$$

$$\frac{d\exp(At)}{dt} = A + A^2t + \frac{A^3t^2}{2!} + \dots = \exp(At)\cdot A$$

$$\left.\exp(At)\right|_{t=0} = 1 = E$$

$$ab = ba \quad \exp(a+b) = \exp a \cdot \exp b$$

$$\theta_A(t) = \exp(At)$$

Теорема. G, H *- группы Ли.* $\phi: G \to H$ *- гомоморфизм групп Ли*

$$\begin{tikzcd} T_e(G) \arrow[r, "(d\phi)_e"] \arrow[d, "\exp"'] & T_e(H) \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

коммутативна

Доказательство.

$$\phi(\exp(v)) = \exp(d\phi_e(v))$$

$\theta_v(t)$ - однопар. подгруппа в G

$$\left.\frac{d\phi(\theta_v(t))}{dt}\right|_{t=0} = (d\phi)_e \cdot \left.\frac{d\theta_v(t)}{dt}\right|_{t=0} = d\phi_e(v)$$

$$w = \left.\frac{d\tilde{\theta}_w(t)}{dt}\right|_{t=0} = d\phi_e(v)$$

$$(\theta_v(t)) = \widetilde{\theta}_{(d\phi)_e(v)} \mid t = 0 \implies$$

■

$$O(n) \hookrightarrow GL(n, \mathbb{R})$$

$$v \in O(n)$$

$$v \in T_e(G) \subset M_n(\mathbb{R}) = T_e(H)$$