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7 ноября 2024 г.

Тема. Формула Даламбера для неоднородного одномерного волнового уравнения

$$u_{tt} - a^2 u_{xx} = f(x,t) \quad |x| < \infty \ t > 0$$
$$u|_{t=0} = \phi(x)$$
$$u_t|_{t=0} = \psi(x)$$

Формула Даламбера решения

$$u(x,t) = \frac{1}{2}\left(\phi(x-at) + \phi(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi) d\xi + \frac{1}{2a}\int_0^t dt' \int_{x-a(t-t')}^{x+a(t-t')} f(\xi, t') d\xi$$

Note

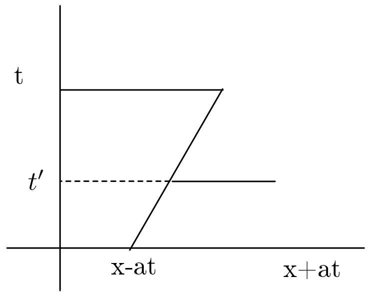

Рис. 1: pic1

Examples

a)

$$u_{tt} - a^2 u_{xx} = \sin \omega t$$
$$u|_{t=0} = 0$$
$$u_t|_{t=0} = 0$$

$$u(x,t) = \frac{1}{2a}\int_0^{t'} dt' \int_{x-a(t-t'}^{x+a(t-t'} \sin \omega t' d\xi = \frac{1}{2a}\int_0^t \sin \omega t' dt' (\xi)\Big|_{x-a(t-t')}^{x+a(t-t')} = \int_0^t (t-t') \sin \omega t' dt'$$

$$= -\frac{t-t'}{\omega}\cos\omega t'\big|_0^t - \frac{1}{\omega}\int_0^t \cos\omega t' dt' = \frac{t}{\omega} - \frac{1}{\omega^2}\sin\omega t'\big|_0^t = \frac{t}{\omega} - \frac{\sin\omega t}{\omega^2}$$

Проверка:

$$\sin\omega t = \sin\omega t$$

$$u|_{t=0} = 0$$

$$u'_t = \frac{1}{\omega} - \frac{1}{\omega}\cos\omega t|_{t=0} = 0$$

б)

$$u_{tt} - a^2 u_{xx} = \sin\omega x \quad |x| < \infty \ t > 0$$

$$u|_{t=0} = 0$$

$$u_t|_{t=0} = 0$$

$$u(x,t) = \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \sin\omega\xi d\xi = \frac{1}{2a}\int_0^t d\tau \left(\frac{-\cos\omega\xi}{\omega}\right)|_{x-a(t-\tau)}^{x+a(t-\tau)} = \frac{1}{2a\omega}$$

$$\int_0^t [\cos(\omega(x-a(t-\tau))) - \cos(\omega(x+a(t-\tau)))]d\tau = \frac{1}{2a^2\omega^2}(\sin(\omega(x-a(t-\tau))) - \sin(\omega(x+a(t-\tau)))|_0^t$$

$$= \frac{1}{2a^2\omega^2}(2\sin\omega x - \sin(\omega(x-at)) - \sin(\omega(x+at))) = \frac{1}{2a^2\omega^2}(2\sin\omega x - 2\sin\omega x\cos\omega at)$$

в)

$$u_{tt} - u_{xx} = e^x$$

$$u|_{t=0} = \sin x$$

$$u_t|_{t=0} = x + \cos x$$

$$u(x,t) = \frac{1}{2}(\phi(x+at) + \phi(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi + \frac{1}{2a}\int_0^\tau d\tau \int_{x-a(t-\tau}^{x+a(t-\tau)} f(\xi,\tau)d\xi$$

$$1)\ \phi(x-at) + \phi(x+at) = \sin(x-t) + \sin(x+t)$$

$$2)\ \frac{1}{2}\int_{x-at}^{x+at}(x+\cos x)dx = \frac{1}{2}\left(\frac{x^2}{2}\Big|_{x-t}^{x+t} + \sin x\right)\Big|_{x-t}^{x+t} = \frac{1}{2}\left(\frac{(x+at)^2-(x-t)^2}{2} + \sin(x+at) - \sin(x-t)\right)$$

$$3)\ \int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau} e^\xi d\xi = \frac{1}{2}e^\xi\Big|_{x-(t-\tau}^{x+(t-\tau)} d\tau =$$

$$= \frac{1}{2}e^{x+t}\int_0^t e^{-\tau}d\tau - e^{x-t}\int_0^t e^\tau d\tau = \frac{1}{2}(e^{x+t}(-e^{-t}+1) - x^{x-t}(e^t-1)) = \frac{1}{2}(e^x + e^{x+t} - e^x + e^{x-t} = \frac{e^{x+t}+e^{x-t}}{2} = e^x \operatorname{ch} t$$

г) Функция Хевисайда

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$u_{tt} - a^2 u_{xx} = \sin\omega t\theta(T-t)$$

$$u|_{t=0} = 0$$

$$u_t|_{t=0} = 0$$

$$u(x,t) = \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau}^{x+a(t-\tau)} \sin\omega\tau\theta(T-t)d\xi = \frac{1}{2a}\int_0^t \sin\omega\tau\theta(T-t)\cdot 2a(t-\tau)d\tau$$

1) $t \le T$

$$u(x,t) = \int_0^t \sin\omega\tau\cdot(t-\tau)d\tau \overset{\text{см. выше}}{=} \frac{t}{\omega} - \frac{\sin\omega t}{\omega^2}$$

2) $t > T$

$$u(x,t) = \int_0^T \sin\omega\tau(t-T)d\tau = \frac{T}{\omega} - \sin\frac{\omega T}{\omega^2}$$

$$u(x,t) = \begin{cases} \frac{t}{\omega} - \sin\frac{\omega t}{\omega^2} & t \le T \\ \frac{T}{\omega} - \frac{\sin\omega T}{\omega^2} & t > T \end{cases}$$