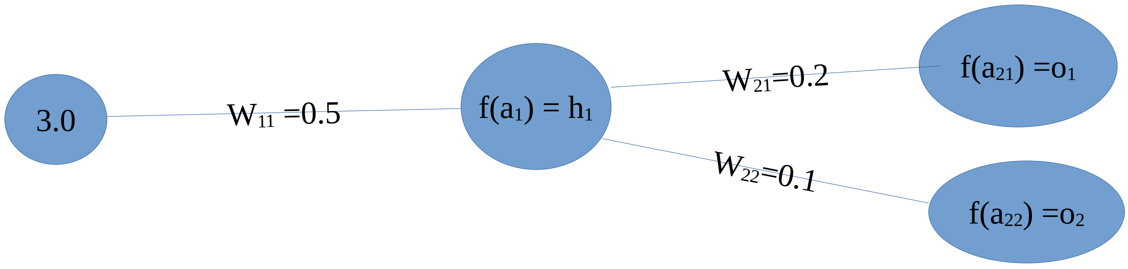
Author: Rodion Kovalenko, M.Sc. Business Information Systems



Activation function:
$f(x) = 2.0 * x$

derivative $\frac{df(x)}{dx} = 2.0$

Forward pass:

$a_1 = w_{11} * x_{11}$
$h_1 = f(a_1) = 2 * a_1$

$a_{21} = w_{21} * h_1$
$a_{22} = w_{22} * h_1$

$o_1 = f(a_{21}) = 2.0 * a_{21}$
$o_2 = f(a_{22}) = 2.0 * a_{22}$;

with real values:

$a_1 = w_{11} * x_{11} = 0.5 * 3.0 = 1.5$
$h_1 = f(a_1) = 2.0 * a_1 = 2.0 * 1.5 = 3.0$

$a_{21} = w_{21} * h_1 = 0.2 * 3 = 0.6$
$a_{22} = w_{22} * h_1 = 0.1 * 3.0 = 0.3$

$o_1 = f(a_{21}) = 2.0 * a_{21} = 1.2$
$o_2 = f(a_{22}) = 2.0 * a_{22} = 0.6$

Target values: [2.0, 1.0]
Errors = [e_1, e_2]

Loss function: mean squared error (mse)

$e_1 = (2.0 - 1.2)^2$
$e_2 = (1.0 - 0.6)^2$

Backward pass:

1. Calculate derivative of $\frac{\partial e_1}{\partial w_{21}}$

$$\frac{\partial e_1}{\partial w_{21}} = \frac{\boldsymbol{\partial e_1}}{\boldsymbol{\partial o_1}} * \frac{\boldsymbol{\partial o_1}}{\boldsymbol{\partial a_{21}}} * \frac{\partial a_{21}}{\partial w_{21}}$$

Calculation:

$$\frac{\partial e_1}{\partial o_1} = 2.0 * 0.8 * (-1)$$

$$\frac{\partial o_1}{\partial a_{21}} = 2.0$$

$$\frac{\partial a_{21}}{\partial w_{21}} = h_1 = 3.0$$

Together:

$$\frac{\partial e_1}{\partial w_{21}} = \frac{\boldsymbol{\partial e_1}}{\boldsymbol{\partial o_1}} * \frac{\boldsymbol{\partial o_1}}{\boldsymbol{\partial a_{21}}} * \frac{\partial a_{21}}{\partial w_{21}} \quad = 2.0 * 0.8 * (-1) * 2.0 * 3 = -9{,}6$$

Manual derivative check:

$a_1 = w_{11} * x_{11} = 0.5 * 3.0 = 1.5$
$h_1 = f(a_1) = 2.0 * a_1 = 2.0 * 1.5 = 3.0$

$a_{21} = (w_{21} + 0.000001) * h_1 =$ **(0.2 + 0.000001) *** $3 = 0.6$

$o_1 = f(a_{21}) = 2.0 * a_{21} = 1.200006$

$$\frac{(2.0-1.200006)^2-(2.0-1.2)^2}{0.000001} = -9{,}599964 \approx -9.6$$

2. Calculate derivative of $\frac{\partial e_2}{\partial w_{22}}$

$$\frac{\partial e_2}{\partial w_{22}} = \frac{\boldsymbol{\partial e_2}}{\boldsymbol{\partial o_2}} * \frac{\boldsymbol{\partial o_2}}{\boldsymbol{\partial a_{22}}} * \frac{\partial a_{22}}{\partial w_{22}}$$

$$\frac{\partial e_2}{\partial o_2} = 2.0 * 0.4 * (-1)$$

$$\frac{\partial o_2}{\partial a_{22}} = 2.0$$

$$\frac{\partial a_{22}}{\partial w_{22}} = h_1 = 3.0$$

Together:

$$\frac{\partial e_2}{\partial w_{22}} = \frac{\boldsymbol{\partial e_2}}{\boldsymbol{\partial o_2}} * \frac{\boldsymbol{\partial o_2}}{\boldsymbol{\partial a_{22}}} * \frac{\partial a_{22}}{\partial w_{22}}$$ =2.0 * 0.4 * (-1) * 2.0 * 3 = -4,8

Manual derivative check:

$a_1 = w_{11} * x_{11} = 0.5 * 3.0 = 1.5$
$h_1 = f(a_1) = 2.0 * a_1 = 2.0 * 1.5 = 3.0$

$a_{22} = (w_{22} + 0.00001) * h_1 = (0.1 + 0.00001) * 3.0 = 0.300003$
$o_2 = f(a_{22}) = 2.0 * a_{22} = 0.600006$

$$\frac{(1.0 - 0.600006)^2 - (1.0 - 0.6)^2}{0.000001} = -4{,}799964 \approx -4.8$$

Recollection:
Forward pass:

$a_1 = w_{11} * x_{11}$
$h_1 = f(a_1) = 2 * a_1$

$a_{21} = w_{21} * h_1$
$a_{22} = w_{22} * h_1$

$o_1 = f(a_{21}) = 2.0 * a_{21}$
$o_2 = f(a_{22}) = 2.0 * a_{22}$

$e_1 = (2.0 - 1.2)^2$
$e_2 = (1.0 - 0.6)^2$

2. Calculate derivative of $\frac{\partial e_{total}}{\partial w_{11}} = \frac{\partial e_1}{\partial w_{11}} + \frac{\partial e_2}{\partial w_{11}}$

Calculate first $\frac{\partial e_1}{\partial w_{11}}$

$$\frac{\partial e_1}{\partial w_{11}} = \frac{\boldsymbol{\partial e_1}}{\boldsymbol{\partial o_1}} * \frac{\boldsymbol{\partial o_1}}{\boldsymbol{\partial a_{21}}} * \frac{\partial a_{21}}{\partial h_1} * \frac{\partial h_1}{\partial a_1} * \frac{\partial a_1}{\partial w_{11}}$$

Derivative is already calculated in the output layer:

$$\frac{\partial \boldsymbol{e_1}}{\partial \boldsymbol{o_1}} * \frac{\partial \boldsymbol{o_1}}{\partial \boldsymbol{a_{21}}}$$

$$\frac{\partial e_1}{\partial o_1} = 2.0 * 0.8 * (-1)$$

$$\frac{\partial o_1}{\partial a_{21}} = 2.0$$

So need to further calculate the other values:

$$\frac{\partial e_1}{\partial w_{11}} = \frac{\partial \boldsymbol{e_1}}{\partial \boldsymbol{o_1}} * \frac{\partial \boldsymbol{o_1}}{\partial \boldsymbol{a_{21}}} * \frac{\partial a_{21}}{\partial h_1} * \frac{\partial h_1}{\partial a_1} * \frac{\partial a_1}{\partial w_{11}}$$

$$\frac{\partial a_{21}}{\partial h_1} = w_{21} = 0.2$$

$$\frac{\partial h_1}{\partial a_1} = 2.0$$

$$\frac{\partial a_1}{\partial w_{11}} = x11 = 3.0$$

Altogether:

$$\frac{\partial e_1}{\partial w_{11}} = \frac{\partial \boldsymbol{e_1}}{\partial \boldsymbol{o_1}} * \frac{\partial \boldsymbol{o_1}}{\partial \boldsymbol{a_{21}}} * \frac{\partial a_{21}}{\partial h_1} * \frac{\partial h_1}{\partial a_1} * \frac{\partial a_1}{\partial w_{11}} = 2.0 * 0.8 * (-1) * 2.0 * 0.2 * 2.0 * 3.0 = -3.84$$

Manual derivative check:

a_1 = **(w_{11} + 0.000001)** * x_{11} = 1.500003
$h_1 = f(a_1)$ = 2.0 * a_1 = 2.0 * 1.500003 = 3.000006
$a_{21} = w_{21} * h_1$ = 0.2 * 3.000006 = 0.6000012
$o_1 = f(a_{21})$ = 2.0 * a_{21} = 2.0 * a_{21} = $w_{21} * h_1$ = 2.0 * 0.6000012 = 1,2000024

$$\frac{(2.0 - 1.2000024)^2 - (2.0 - 1.2)^2}{0.000001} = -3{,}83999424 \approx -3.84$$

Calculate then $\frac{\partial e_2}{\partial w_{11}}$

$$\frac{\partial e_2}{\partial w_{11}} = \frac{\partial \boldsymbol{e_2}}{\partial \boldsymbol{o_2}} * \frac{\partial \boldsymbol{o_2}}{\partial \boldsymbol{a_{22}}} * \frac{\partial a_{22}}{\partial h_1} * \frac{\partial h_1}{\partial a_1} * \frac{\partial a_1}{\partial w_{11}}$$

$$\frac{\partial e_2}{\partial o_2} = 2.0 * 0.4 * (-1)$$

$$\frac{\partial o_2}{\partial a_{22}}=2.0$$

$$\frac{\partial a_{22}}{\partial h_1}=w_{22}=0.1$$

$$\frac{\partial h_1}{\partial a_1}=2.0$$

$$\frac{\partial a_1}{\partial w_{11}}=3.0$$

Altogether:

$$\frac{\partial e_2}{\partial w_{11}}=\boldsymbol{\frac{\partial e_2}{\partial o_2}}*\boldsymbol{\frac{\partial o_2}{\partial a_{22}}}*\frac{\partial a_{22}}{\partial h_1}*\frac{\partial h_1}{\partial a_1}*\frac{\partial a_1}{\partial w_{11}}=2.0*0.4*(-1)*2.0*0.1*2.0*3.0=-0.96$$

Manual derivative check:
a_1 = **(w_{11} + 0.000001)** * x_{11} = 1.500003
h_1 = f(a_1) = 2.0 * a_1 = 2.0 * 1.500003 = 3.000006

a_{22} = w_{22} * h_1 = 0.1 * 3.000006 = 0.3000006
o_2 = f(a_{22}) = 2.0 * a_{22} = 2.0 * 0.3000006 = 0.6000012

$$\frac{(1.0-0.6000012)^2-(1.0-0.6)^2}{0.000001}=-0.95999856\approx-0.96$$

$$\frac{\partial e_{total}}{\partial w_{11}}=\frac{\partial e_1}{\partial w_{11}}+\frac{\partial e_2}{\partial w_{11}}=-3.84+(-0.96)=-4.8$$

Conclusion:

As one can see the partial derivative of the output layer is propagated in direction of the input layer. Thus a general backpropagation rule can be formulated as:

Gradient o_{ji} is a partial derivative that is used to update the weight with index ji, in other words the weight from hidden neuron j to the output neuron i.

$$gradient\, o_{ji}=\boldsymbol{\frac{\partial e_j}{\partial o_j}}*\boldsymbol{\frac{\partial o_j}{\partial a_{ji}}}$$