4.2 The Dirac delta-function (impulse function)

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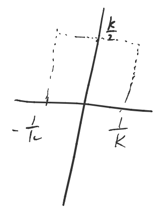
Recall the mean-value theorem for integrals: If $f:[a,b]\to\mathbb{R}$ is continuous, then $\int_a^b f(x)\,dx = (b-a)f(x_0)$ for at least one x_0 with $a \le x_0 \le b$.

Geometrically, this means the area under the continuous curve is equivalent to that of a rectangle with length equal to the interval of integration.

Consider the "top hat" function

$$f_k(x) = \begin{cases} \frac{k}{2}, & |x| < \frac{1}{k} \\ 0, & |x| > \frac{1}{k}. \end{cases}$$



Clearly, we can see an important property of this function is that

$$\int_{-\infty}^{+\infty} f_k(x)\,dx = 1.$$

As k increases, $f_k(x)$ gets taller and thinner. The Dirac delta function is defined as

$$\delta(x) = \lim_{k\to\infty} f_k(x).$$

Note that this limit does not exist in the usual mathematical sense. Effectively, $\delta(x)$ is infinite at $x=0$ and zero at all other values of x.

The key property however, is

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1.$$

Remark: There are alternatives to define the Dirac delta. For example,

$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}};$$

or

$$\delta(x) = \lim_{y\to 0} \frac{y/\pi}{x^2+y^2}.$$

Mathematically, the Dirac delta in such limits is a distribution.

The Dirac delta is most useful in how it interacts with other (test) functions.

Consider in 1D
$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx,$$

where f is continuous on $\mathbb{R}$. Using the definition of the Dirac delta, we have
$$\lim_{k\to\infty}\int_{-\frac{1}{k}}^{\frac{1}{k}} \frac{k}{2} f(x)dx = \lim_{k\to\infty} f(x_0)\int_{-\frac{1}{k}}^{\frac{1}{k}} \frac{k}{2}dx \quad (£££)$$

for some $x_0 \in [-\frac{1}{k}, \frac{1}{k}]$ by the mean-value theorem for integrals. As $k\to\infty$, we must have $x_0 \to 0$. Then

$$(£££) \Rightarrow f(0)\frac{k}{2}\cdot\frac{2}{k} = f(0).$$

We have therefore established that for any continuous function f,
$$\delta[f] \underbrace{:=}_{\text{definition}} \int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0).$$

This result can easily be generalised to
$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a).$$

4.2.1 Properties of Dirac delta function

(i). If f is infinitely differentiable, then we can define the derivative of the δ-function by
$$\delta'[f] := -\delta[f'] = -f'(0).$$

and
$$\delta^{(n)}[f] := (-1)^n\delta[f^{(n)}] = (-1)^n f^{(n)}(0).$$

(ii). Since $\delta(x)$ vanishes whenever $x \neq 0$, we can write
$$\delta[f] = \int_a^b \delta(x)f(x)dx$$

where $[a,b]$ is any interval containing point $x=0$. Otherwise where $[a,b]$ is any interval containing point $x=0$. Otherwise, if $0 \notin [a,b]$, then the integral is zero.

(iii). If $g: \Omega \to \mathbb{R}$ is continuous in a neighbourhood of the origin $0 \in \Omega$, then

$$(g\delta)[f] = \delta[gf] = g(0)f(0) = g(0)\delta[f],$$

or $f(x)\delta(x) = f(0)\delta(x)$.

(iv). For any $c \in \mathbb{R}\setminus\{0\}$,

$$\int_{\mathbb{R}} \delta(cx) f(x)dx = \frac{1}{|c|}\int_{\mathbb{R}} \delta(y) f(\tfrac{y}{c})\, dy = \frac{1}{|c|} f(0),$$

or $\delta(cx) = \dfrac{\delta(x)}{|c|}$.

(v). For any continuously differentiable $g: \Omega \to \mathbb{R}$, $\delta(g(x))$ is zero everywhere except at zeros of g. In particular, if g has only simple zeros at points $\{x_1, x_2, \cdots, x_n\} \in \Omega$, then

$$\int_{\mathbb{R}} \delta(g(x)) f(x)\, dx = \sum_{i=1}^{n} \int_{x_i^-}^{x_i^+} \delta(g(x)) f(x) dx$$

$$\approx \sum_{i=1}^{n} \left[\frac{1}{|g'(x_i)|} \int_{x_i^-}^{x_i^+} \delta(x - x_i) f(x)\, dx\right]$$

$$= \sum_{i:\, g(x_i)=0} \frac{f(x_i)}{|g'(x_i)|},$$

where to obtain the 2nd line, we used the fact that

$$g(x) \approx (x - x_i)\, g'(x_i)$$

when x is near the root x_i.

Remark: In physics, the δ-function models point sources in a continuum. For example, suppose we have a unit point charge at $x=0$ in 1D. The charge density should satisfy $\rho(x)=0$ for $x\neq 0$ and total charge $Q = \int \rho(x) = 1$, which are exactly the properties of the δ-function, so we set $\rho(x) = \delta(x)$ and the physical total charge $Q = \int \rho(x) = 1$, which are exactly the properties of the δ-function, so we set $\rho(x) = \delta(x)$ and the physical intuition is well modelled by the "top hat" function above.

In mechanics, δ-functions model impulses. Suppose a particle travelling with momentum $p = mv$ in 1D. Newton's law gives $F = dP/dt$, so

$$P(t_2) - P(t_1) = \int_{t_1}^{t_2} F\, dt.$$

If the particle is suddenly struck by a hammer at $t=0$, then we might imagine that the force acts only over a vanishing small time $|\delta t| < 2\varepsilon$ for some ε and results in a finite momentum change, say C. Then $\int_{-\varepsilon}^{\varepsilon} F\, dt = \Delta P$ while F is non-zero only very near $t=0$. In the limit of vanishing time interval δt, $F(t) = C\delta(t)$ models the impulsive force.

4.2.2. Eigenfunction expansion of the Dirac-δ

The excellent differentiability of distributions also allows us to make sense of divergent Fourier series.

Consider $\delta(x)$ defind for $x \in [-L, L)$ and write

$$\delta(x) = \sum_{n \in \mathbb{Z}} \hat{\delta}_n e^{\frac{in\pi x}{L}} dx$$

with
$$\hat{\delta}_n = \frac{1}{2L} \int_{-L}^{L} e^{-\frac{in\pi x}{L}} \delta(x)$$

$$= \frac{1}{2L} \quad \forall n \in \mathbb{Z}$$

$$\Rightarrow \quad \delta(x) = \frac{1}{2L} \sum_{n \in \mathbb{Z}} e^{\frac{in\pi x}{L}}.$$

The LHS is an object that does not exist as a function whereas on the RHS we have a series that diverges.

Check: Let
$$S_N \delta(x) \equiv \frac{1}{2L} \sum_{n=-N}^{N} e^{\frac{in\pi x}{L}}$$

denote the partial Fourier sum. $\forall N < \infty$, $S_N \delta(x)$ is well-behaved (even, analytic). For any smooth function $f: [-L, L] \to \mathbb{R}$, we have

$$\begin{aligned}(S_N\delta)[f] &= \int_{-L}^{L} S_N \delta(x) f(x)\,dx \\ &= \frac{1}{2L}\int_{-L}^{L}\left[\sum_{n=-N}^{N} e^{\frac{in\pi x}{L}}\right] f(x)\,dx \\ &= \sum_{n=-N}^{N}\left[\frac{1}{2L}\int_{-L}^{L} e^{\frac{in\pi x}{L}} f(x)\,dx\right] \\ &= \sum_{n=-N}^{N} \hat{f}_{-n},\end{aligned}$$

which is a partial Fourier series of $f(x)$, evaluated at $x=0$.

Since $f(x)$ is everywhere smooth, its Fourier series converges everywhere. In particular, we have

$$\lim_{N\to\infty}\left(\sum_{n=-N}^{N}\hat{f}_{-n}\right) = \lim_{N\to\infty}\left(\sum_{n=-N}^{N}\hat{f}_{-n}\, e^{-\frac{in\pi x}{L}}\right)\Bigg|_{x=0} = f(0).$$

Therefore, the limiting value of δ is

$$\lim_{N\to\infty}(S_N\delta)[f] = f(0) = \delta[f].$$

In this sense, the partial sum $\frac{1}{2L}\sum_{n=-N}^{N} e^{\frac{i\pi x}{L}}$ converges to $\delta(x)$.

Note that it would not converge as a function because there is no such function "$\delta(x)$". Rather, it converges as a distribution.

Example: Recall the sawtooth function given by the 2π-periodic function $f(x) = x$ for $x \in [-\pi, \pi)$ has Fourier series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx),$$

which converges $\forall x \in \mathbb{R}$. For $x \neq n\pi$, it converges to the value of the sawtooth function while converges to zero when $x = n\pi$. Differentiating the series term by term, we have

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\cos(nx) = 2[\cos x - \cos 2x + \cos 3x - \dots]$$

which diverges as a function at $x = n\pi$. We use the step function which diverges as a function at $x=n\pi$. We use the step function $\Theta(x) \equiv \begin{cases} 1, & x>0 \\ 0, & x\leq 0 \end{cases}$, to write $f(x)$ as

$$f(x) = x - 2\pi - 2\pi \sum_{n=1}^{\infty} \Theta(x-\pi-2n\pi) + 2\pi \sum_{n=0}^{+\infty} \Theta(-x+\pi-2n\pi), \; x \in \mathbb{R}$$

where the step functions provide jumps in the sawtooth. Using the fact that the derivative of the step function is the Dirac δ-function (check it!), we have

$$f'(x) = 1 - 2\pi \sum_{n\in\mathbb{Z}} \delta(x-\pi-2n\pi)$$

The sawtooth function has a constant gradient $(=1)$ everywhere except at $x=(2n+1)\pi$, $n\in\mathbb{Z}$, where it has a δ-function spike.

Following the same step as above, one can check that

$$2\lim_{N\to\infty}\left[\sum_{n=1}^{N}(-1)^{n+1}\cos(nx)\right] = 1 - 2\pi\sum_{n\in\mathbb{Z}}\delta(x-\pi-2n\pi)$$

so that the sequence of partial series $\sum_{n=1}^{\infty}(-1)^{n+1}\cos(nx)$ does converge (as a distribution).

For $n\in\mathbb{Z}$, let $\{Y_n(x)\}$ be a complete set of orthonormal eigenfunctions of a Sturm–Liouville operator on the domain $[a,b]$, with the weight function $w(x)$. For $\xi\in(a,b)$, the Dirac delta-function $\delta(x-\xi)$ is zero for the boundary $x=a$ or $x=b$, so the Sturm-Liouville operator will be self-adjoint. We expect to expand

$$\delta(x-\xi) = \sum_{n\in\mathbb{Z}} c_n Y_n(x),$$

with
$$c_n = \int_a^b Y_n^*(x)\,\delta(x-\xi)\,w(x)\,dx = Y_n^*(\xi)\,w(\xi).$$

Using the fact $\delta(x-\xi) = \dfrac{w(x)\,\delta(x-\xi)}{w(\xi)}$, we can write

$$\delta(x-\xi) = w(\xi)\sum Y_n^*(\xi)\,Y_n(x)$$

$$\delta(x-\xi) = w(\xi) \sum_{n\in\mathbb{Z}} Y_n^*(\xi) Y_n(x)$$
$$= w(x) \sum_{n\in\mathbb{Z}} Y_n^*(\xi) Y_n(x).$$

This expansion is consistent with the sampling property. If $g(x) = \sum_{m\in\mathbb{Z}} d_m Y_m(x)$, then

$$\int_a^b g^*(x)\,\delta(x-\xi)\,dx = \sum_{m,n\in\mathbb{Z}} Y_n^*(\xi)\, d_m^* \int_a^b w(x)\, Y_m^*(x)\, Y_n(x)\,dx$$
$$= \sum_{m\in\mathbb{Z}} d_m^* Y_m^*(\xi) = g^*(\xi).$$

Remark: The eigenfunction expansion of the δ-function is intimately related to the eigenfunction expansion of the Green's function introduced in the last chapter. We can develop a theory of Green's functions for solving inhomogeneous ODEs.

4.2.3. Fourier transform of the Dirac δ.

We have
$$\mathcal{F}[\delta(x)] = \int_{-\infty}^{+\infty} e^{-ikx}\,\delta(x)\,dx = e^{-ik0} = 1.$$

From this and using the inversion formula, we can have an alternative representation of the Dirac δ-function as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm ikx}\,dk,$$
$$\Leftrightarrow \quad \delta(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm ikx}\,dx.$$

If we are prepared to work in terms of Dirac δ functions, we can now take the Fourier transforms of functions that do not decay as $x \to \pm\infty$.

Example: $$\mathcal{F}[\cos(lx)] = \int_{-\infty}^{+\infty} \frac{1}{2}(e^{ilx} + e^{-ilx})\, e^{-ikx}\,dx$$
$$= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-i(k-l)x}\,dx + \frac{1}{2}\int_{-\infty}^{+\infty} e^{-i(l+k)x}\,dx$$
$$= \pi\,\delta(k-l) + \pi\,\delta(k+l).$$