Distance Matrix

Introduction

Lets $\mathbf{x}_j$ be a set of points in a region $\Omega \in \mathbb{R}^s$, and $y_j \in \mathbb{R}$ such that[1]

$$y_j = f(\mathbf{x}_j) \tag{1}$$

where f is an unknown function. We want to find a function P_f such that when evaluated at the point $\mathbf{x}_j$ it matches the evaluation point y_j, that is

$$y_j = P_f(\mathbf{x}_j) \tag{2}$$

for $j = 1, \dots, N$

We can construct this function P_f as a linear combination of some *basis functions* B_k, *i.e.*,

$$P_f(\mathbf{x}) = \sum_{k=1}^{N} c_k B_k(\mathbf{x}) \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^s$. This is an interpolation problem that lead us to solve the following system of equations

$$A\mathbf{c} = \mathbf{y} \tag{4}$$

where the *interpolation matrix* A is given by $A_{j,k} = B_k(\mathbf{x}_j)$, with $j, k = 1, \dots, N$ the vector $\mathbf{c} = [c_1, c_2, ..., c_N]^T$, and the vector $\mathbf{y} = [y_1, y_2, ..., y_N]^T$.

The interpolation problem would be *well-posed* (a solution to the problem exists and it is unique) if and only if the matrix A is square.

Example

Lets say we want to construct a function P_f that approximates the following function:

$$f_s(\mathbf{x}) = 4^s \prod_{d=1}^{s} x_d(1 - x_d), \qquad \mathbf{x} = [x_1, ..., x_s] \in [0, 1]^s. \tag{5}$$

where f_s evaluates to zero at the boundaries of the unit cube $\mathbb{R}^s$ and has a maximum value of 1 at the center of the cube.

[1] The notation and exercises follows that presented in the book by Fasshauer [1].

Construction of P_f

We construct P_f as a linear combination of certain basis functions $B_k = ||\mathbf{x} - \mathbf{x}_k||_2$ such that

$$P_f(\mathbf{x}) = \sum_{k=1}^{N} c_k \underbrace{||\mathbf{x} - \mathbf{x}_k||_2}_{B_k}, \qquad \mathbf{x} \in [0,1]^s \tag{6}$$

therefore

References

[1] G. E. Fasshauer, *Meshfree approximation methods with MATLAB*, vol. 6. World Scientific, 2007.