Master theorem – General form (check the Video in Canvas)

$$T(n) = aT\left(\frac{n}{b}\right) + \theta\left(n^k log^p n\right)$$

Here, $a \geq 1, b > 1,\ k \geq 0$, $and\ p\ is\ a\ real\ number, here\ a, b, and\ k\ are\ constants.$

Cases

1. $if\ a > b^k,\ \ then\ T(n) = \Theta(n^{log_b a})$
2. $if\ a = b^k$

   a) $if\ p > -1, then\ T(n) = \Theta(n^{log_b a} log^{p+1} n)$

   b) $if\ p = -1, then\ T(n) = \Theta(n^{log_b a} loglogn)$

   c) $if\ p < -1, then\ T(n) = \Theta(n^{log_b a})$
3. $if\ a < b^k$

   a) $if\ p \geq 0, then\ T(n) = \Theta(n^k log^p n)$

   b) $if\ p < 0, then\ T(n) = O(n^k)$

Comment 1. **You may be lucky and find some cases where you can use this simplified version of the theorem. If this is the case, change in the answer O by Θ just to be more accurate. However, you will see several examples where this version is not enough to solve the problem.**

Master theorem[2] *If* $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ *for some constants* $a > 0$, $b > 1$, *and* $d \geq 0$,

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a\ . \end{cases}$$

Comment 2. **There is another version that could be used to tackle some problems solvable for the previous two version of the theorem.**

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with[1] $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$. Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

Comment 2: A common mistake is to assume that the two expressions below are the same.

$$(logn)^2 \neq log^2 n \ \ this\ are\ different\ things.$$

$(logn)^2 = logn * logn$

$log^2 n = loglogn$

Check for the video in canvas, there I solve several of them using this generalized theorem.

Let's practice.

1. $T(n) = 3T(n/2) + n^2$

2. $T(n) = 4T(n/2) + n^2$

3. $T(n) = T(n/2) + 2^n$

4. $T(n) = 2^n T(n/2) + n^n$

5. $T(n) = 16T(n/4) + n$

6. $T(n) = 2T(n/2) + n \log n$

7. $T(n) = 2T(n/2) + n/\log n$

8. $T(n) = 2T(n/4) + n^{0.51}$

9. $T(n) = 0.5T(n/2) + 1/n$

10. $T(n) = 16T(n/4) + n!$

11. $T(n) = \sqrt{2}T(n/2) + \log n$

12. $T(n) = 3T(n/2) + n$

13. $T(n) = 3T(n/3) + \sqrt{n}$

14. $T(n) = 4T(n/2) + cn$

15. $T(n) = 3T(n/4) + n \log n$

16. $T(n) = 3T(n/3) + n/2$

17. $T(n) = 6T(n/3) + n^2 \log n$

18. $T(n) = 4T(n/2) + n/\log n$

19. $T(n) = 64T(n/8) - n^2 \log n$

20. $T(n) = 7T(n/3) + n^2$

21. $T(n) = 4T(n/2) + \log n$

22. $T(n) = T(n/2) + n(2 - \cos n)$