# An investigation on the physics behind the electric guitar technique of bending strings

**Research question:** How does the angle at which a guitar string is bent affect the frequency of its fundamental harmonic?

## Introduction:

I have played the electric guitar for many years and have always been fascinated by the wide variety of sounds it is capable of producing. For me, the guitar has been not just a mean for relaxing and clearing my head, but also an instrument to experiment with, to see which dream-like sounds were different techniques and distortions capable of producing. This sparked my curiosity for the physical mechanisms behind such sounds, and when we saw in physics class standing waves and harmonics applied to musical instruments I became instantly very interested.

From all the different techniques that I have learned through the years, the one I find most interesting -and perhaps one of the most characteristic techniques of electric guitar- is bending. This consists on pressing down on a string while exerting a force parallel to the bridge so that the string bends, thereby producing a higher pitch. This technique is hard because it demands intuition and a trained ear to obtain the desired increase in pitch. Therefore, I decided to use the bending angle as my independent variable and the change in frequency as my dependent variable. Since the note produced is basically the fundamental harmonic (although other harmonics are still present and are called overtones[1]) the aim of this investigation is analyze particularly the changes to the fundamental frequency, or harmonic, of a string.

## Theoretical background

### Guitar anatomy

Throughout this work I will mention several parts of the guitar; therefore, a prior overview of its main structure shall make several of the descriptions and calculations easier to follow and understand (Figure1):

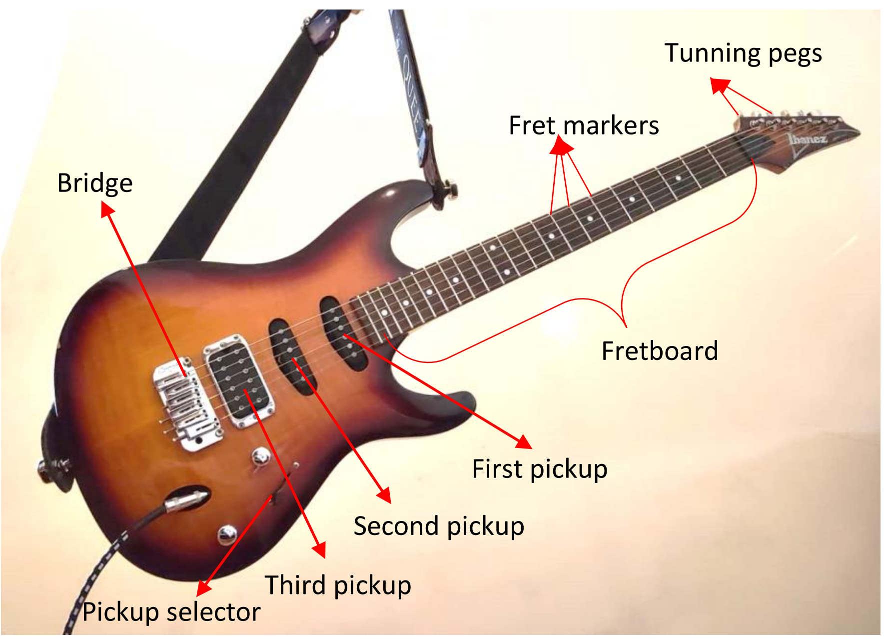

Figure 1: Labeled image of my guitar, which was the one used to perform all the experiments

### Music, strings and waves

Guitars, like every string instrument, produce a sound by making a string vibrate at different frequencies. The vibrations of the strings produce longitudinal waves which travel through the air and into our ears. In acoustic guitars, these vibrations are amplified thanks to different resonance patterns produced in the guitar´s hollow body[2], while in electric guitars the sound is amplified through a fascinating (although outside the scope of this work) electrical system involving mainly electromagnetic induction[3]. Either way, since the strings are fixed at both ends they produce a

[1]Jeremy Burns, Mathew Scott Philips, “The Overtone Series”, *Music Student 101,* accessed March 14, 2021, https://musicstudent101.com/46-the-overtone-series.html

[2]Neville H. Fletcher, Thomas D. Rossing, *The Physics of Musical Instruments*(Australia/USA: Springer, 1998), pages 239-263)

[3] Chris Woodford, “The Physics of Electric Guitars”, *Explainthatstuff,* November 5, 2020, https://www.explainthatstuff.com/electricguitars.html

stationary wave whose frequencies of vibration are given by:

$$f_n = \frac{nc}{2L} \quad (1)$$

Where $n$ is the number of the harmonic, $L$ is the length of the string and $c$ is the speed of the wave. To produce different sounds, the player presses down on a string so that it touches one of the fret markers, making the length of the string effectively shorter. Since $f_n \propto \frac{1}{L}$, manipulating the length of the string gives rise to different frequencies: different musical notes. Now, assuming that the tension on the string and string density are constant, that it is vibrating at small angles, and that there are no further external forces, the speed of a wave in one dimension is described by the second-order differential equation[4]:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$$

Where $u(x,t)$ is the vertical displacement of the string at position $u$ and time $t$. It can be solved for c to yield[5]:

$$c = \sqrt{\frac{T}{\mu}} \quad (2)$$[6]

Where $T$ is the tension and $\mu$ is the linear density, or mass per unit length. Combining equations 1 and 2, we get that the frequency of the fundamental harmonic of an unbent string is given by:

$$f_1 = \frac{\sqrt{\frac{T}{\mu}}}{2L} \quad (3)$$

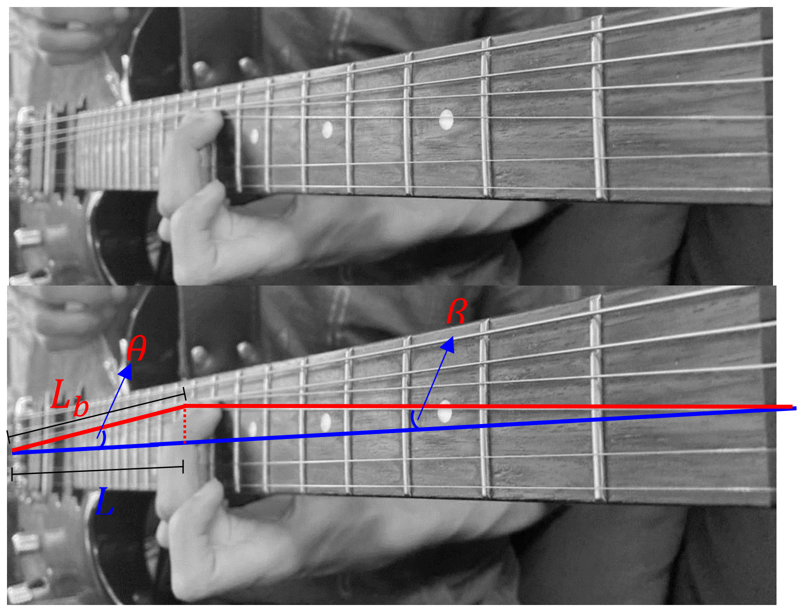


Figure 2: labeled and non-labeled photo of a bending technique

**<u>Physical changes caused by the bending technique</u>**

Let us consider a string of length $L$ bent by an angle $\theta$ (Figure 2). Notice that the only part of the string vibrating is the one going from the bridge to where the finger is pressing down; therefore $L$ can be identified as that length. Angle $\beta$, together with the effects on the rest of the string beyond $L$, have no effect on the note produced.

When bending occurs, there is an additional force $\vec{F}_b$ applied to the string, which causes it to elongate. So, its length changes to $L_b = L\,sec\theta$ . Since $L\,sec\theta > L \;\forall\; \theta \in ]0, \frac{\pi}{2}[$, if all the other variables remained constant, this elongation would cause a decrease in frequency (the bending angle is never greater than $\frac{\pi}{2}$; strings snap before that point). However, an increase in length comes with a decrease in mass per unit length because mass stays constant. Furthermore, $\vec{F}_b$ also causes an increase in tension, so the net effect of bending is actually an increase in frequency.

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[4] Joel Feldman, "Derivation of the Wave Equation", *University of British Columbia,* accessed February 28, 2021, http://www.math.ubc.ca/~feldman/m256/wave.pdf

[5] The explicit solving process lies beyond the scope of this work:

[6] David Robert Grimes, "String Theory – The Physics of String Bending and Other Electric Guitar Techniques", *National Center for Biotechnology Information*, July 23, 2014, https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4108333/#pone.0102088-Engineering

To calculate the mass per unit length of the bent string $\mu_b$, let us consider the definition of $\mu$:

$$\mu = \frac{m}{L} \therefore \mu_b = \frac{m}{L_b} = \frac{m}{L\,sec\theta} = \mu\cos\theta \text{ (4)}$$

To find out the increase in tension let us draw a free-body diagram showing a string under the effects of $F_b$:

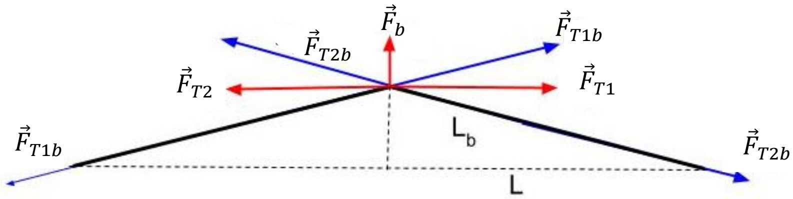


Diagram 1: Forces acting on a bent string

The total force of tension on the segment L of the bent string, $\vec{F}_{T1b}$, can be expressed as the vector sum $\vec{F}_{T1b} = \vec{F}_{T1} + \overrightarrow{F_b}$, where $\vec{F}_{T1}$ is the original tension force and $F_b$ is the bending force which causes the elongation of the string (both $\vec{F}_{T1}$ and $\vec{F}_b$ are represented with red arrows).

Notice that the extra force $\vec{F}_b$, which adds stress and induces deformation along the length of the string is equal to the restoring force described by Hooke's law[7]. This is a safe assumption to make, because the elongation is small compared to the length of the string (i.e. the strain is very small).

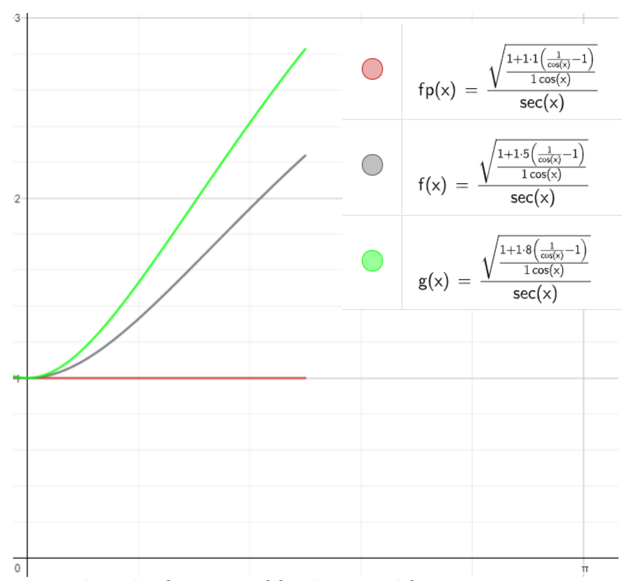
$$\vec{F}_b = k\Delta L = k(L_b - L) = k(Lsec\theta - L) = kL\left(\frac{1}{cos\theta} - 1\right) \text{ (5)}$$

Where $k$ is the spring constant. Therefore, substituting the new parameters into equation 3, we have that the fundamental frequency of a bent string $f_{0b}$ is given by:

$$f_{0b} = \frac{\sqrt{\frac{\vec{F}_{T1}+\vec{F}_b}{\mu_b}}}{2L_b} = \frac{\sqrt{\frac{\vec{F}_{T1}+kL\left(\frac{1}{cos\theta}-1\right)}{\mu\cos\theta}}}{2L\,sec\theta} \text{ (6)}$$

**<u>Hypothesis</u>:**

The relationship between the bending angle and the change in fundamental frequency will be non-linear. Furthermore, as the angle increases the rate of change in frequency will also increase; in other words, small angles will cause a smaller change than big angles. This prediction was arrived to by setting every constant value in equation (6) equal to one and graphing it. Then, testing the behaviour of the function for different values of $\vec{F}_{T1}, \vec{F}_b, \mu_b$ and $L_b$, considering that $\vec{F}_{T1} >> \vec{F}_{Cb}, \mu_b, L_b$ because the initial tension is much greater than the extra pressure exerted by the player, the length of the fretboard is less than 1 meter, and since guitar strings are very thin their mass is small compared to their length. Furthermore, the function intersect with the vertical axis will be the frequency reported in literature of the note played (329.63 Hz[8]) because a bending angle of zero will not change the frequency.



Graph 1: the function of fundamental frequency against bending angle for different, arbitrarily chosen, values of $\vec{F}_{T1}, \vec{F}_b, \mu_b$ and $L_b$

[7] "What is Hooke's Law?", *Khan Academy,* accessed, Februery 28, 2021, https://www.khanacademy.org/science/physics/work-and-energy/hookes-law/a/what-is-hookes-law

[8] Bryan H. Suits, "Frequencies for equal-tempered scale, A4=440 Hz", *Michigan Technological University,* accessed February 28, 2021, https://pages.mtu.edu/~suits/notefreqs.html

## Experimental method

| Materials: |
|---|
| ❖ Vernier Go direct Sound Sensor (order code: GDX-SND) |
| ❖ Electric guitar Ibanez SA260FM |
| ❖ Electric guitar strings with nickel-plated steel wrap brand Elixir. The G-string has a reported diameter of 0.04064 cm, which was not reported with uncertainty[9] |
| ❖ Guitar amplifier brand Novation PG-15E without distortions (gain was set to 0; treble, middle and bass were set to the highest values) |
| ❖ Computer with software Vernier Graphical Analysis and Logger Pro 3.9 |
| ❖ Measuring tape $\pm 0.05$ cm |
| ❖ Guitar capo |
| ❖ Scale Mettler AE 200 $\pm 0.0001$g |

**Variables:**

- **Independent:** The displacement $d$ of the string from its equilibrium position. The bending angle is a consequence of the displacement; therefore it is a more direct -and easier to manipulate since the angles used will be very small and a highly precise instrument would be needed- measurement. The displacements used were 0, 0.3, 0.6, 0.9 and 1.2 $\pm 0.1$ cm. The bending angle is given by $\arctan\left(\frac{d}{L}\right)$ where $L$ is the original length of the pressed string. Therefore, in a way, there will be two independent variables: one directly measured and one calculated from those measurements.

- **Dependent:**
  - **Fundamental frequency of the string:** The waveform was recorded using the Vernier sound sensor. Then, the different frequencies which composed the sound were isolated by transferring the data to Logger Pro and applying a Fast Fourier Transformation (FFT). This is an efficient computational algorithm[10] that applies a discrete Fourier transformation[11] to a set of data. It is used for decomposing a complex waveform into the "pure" frequencies (sine and cosine waves) that compose it[12].

- **Controlled**
  - **First note played:** The initial note from which bending was made was an E4 located in the 9$^{th}$ fret of the G-string. This note sounds at a fundamental frequency of 329.63 Hz (reported without uncertainty)[13], which should -in theory- be the frequency measured at a bending angle of 0 rad.

  - **Initial length of the pressed string:** It is important to note that the when the player presses down on a string so that it makes contact with the fret marker there is also an increase in tension and a small elongation taking place (diagram 2). Therefore, the length $L$ of the string will be ever so slightly bigger that the measurement from the fret marker

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[9] "Guitar String Tension Charts", *Elixir Strings,* accessed February 28, 2021, https://www.elixirstrings.com/support/string-tension-for-tuning-guitar

[10] Steve Brunton, *The Fast Fourier Transform Algorithm,* University of Washington, April 4, 2020, YouTube video, 10:17, https://www.youtube.com/watch?v=toj_IoCQE-4

[11] Eric W. Weisstein, „Discrete Fourier Transformation" *Mathworld- a Wolfram Web Resource* , accessed February 28, 2021, https://mathworld.wolfram.com/DiscreteFourierTransform.html

[12] Grant Sanderson, *But what is a Fourier Transformation?*, January 26, 2018, YouTube video, 19:42, https://www.youtube.com/watch?v=spUNpyF58BY

[13] Bryan H. Suits, "Frequencies for equal-tempered scale, A4=440 Hz", *Michigan Technological University,* accessed February 28, 2021, https://pages.mtu.edu/~suits/notefreqs.html

with which the string makes contact to the bridge, noted as $L_0$. Specifically, $L = \frac{D_f}{sin\theta} = \frac{D_f}{\sin\left(\arctan\frac{D_f}{L_0}\right)}$ .The distance $D_f$ from the strings to the fretboard changes between different guitars, and since the same guitar was used for all the experiments it stayed constant. For my guitar, $D_f$ was measured to be $5 \pm 0.5$ mm and $L_0$ to be $391 \pm 0.5$ mm; therefore $L$ was $391.031968 \pm 43.996766$. Nevertheless, $L$ is within the error bars of $L_0$, and making calculations with $L$ would only add unnecessary uncertainties to the data processing. Hence, the extension of the string in this dimension was consider to be negligible: $L \approx L_0$.

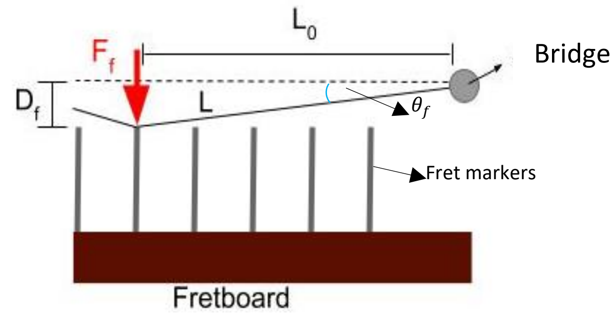



Diagram 2: Forces acting on a pressed string (without bending)

- **Initial Linear mass per unit length of the string:** All tests were performed using the same string which has a constant mass; the initial length is also known as explained in the previous point. Taking that into account we can calculate that $\mu = \frac{m}{L} = \frac{m}{\frac{D_f}{\sin\left(\arctan\frac{D_f}{L_0}\right)}} = \frac{0.785g}{75.3\ cm} = 1.04 \times 10^{-3} \pm 2.1 \times 10^{-6} kg\ m^{-1}$. Where the experimental values were obtained by cutting an arbitrarily long segment of the string used (after the experiments had been made) and weighting it. Keep in mind that equation (4 )tells us that $\mu_b = \mu cos\theta$ .

- **Initial tension of the string:** The string used was tuned to the same frequency (in this case, the open G-string must vibrate at 196 Hz[14]) for every test. Bearing equation (3) in mind, since the initial length, initial mass per unit length and initial frequency stay constant, tension must also stay constant. Solving for it in equation (3) gives $T = 4L^2 f_0^2 \mu$.

All the previous controlled variables are components of equation (6) an therefore affect change in frequency. Conversely, the following controlled variables either affect the change in frequency in an indirect way.

- **Guitar:** For all the experiments I used my guitar Ibanez SA260FM[15]. The material and shape of the guitar influences mainly the overtones of the note, which should be isolated either way by the Fast Fourier Transformation done in the data analysis but it is best to take them into account either way.

- **Pickups used:** As previously mentioned, string vibrations are amplified by different "microphones" called pickups which measure the current induced by the metal strings moving in an electromagnetic field. Since tension is not truly constant throughout the whole string (that is only true on a massless string)[16], the place at which the vibrations are picked up will slightly change the frequency detected and amplified. There is a switch in the guitar which allows the player to choose several combinations of the 3 pickups available, so for all the experiments I used the first two pickups.

**<u>Experimental setup:</u>**

To measure the displacement from the equilibrium position as accurately as possible, the measuring tape was wrapped around the neck of the guitar, and secured with a clamp. The mark at 15 centimeters was taken as the "zero" since it

[14] Neville H. Fletcher, Thomas D. Rossing, *The Physics of Musical Instruments*(Australia/USA: Springer, 1998), pages 239-263)
[15] "SA260FM", *Ibanez,* accessed February 28, 2021, https://www.ibanez.com/na/products/detail/sa260fm_4l_02.html

[16] Roland Newburgh, "Why is the tension constant in a massless string?", *Physics Education,* 1997, doi:10.1088/0031-9120/32/3/020

was put exactly at the equilibrium position. The capo is a tool widely used by guitar players, which presses down on the strings so it was used to hold the G string in place for the different displacements (Figures 3 and 4).

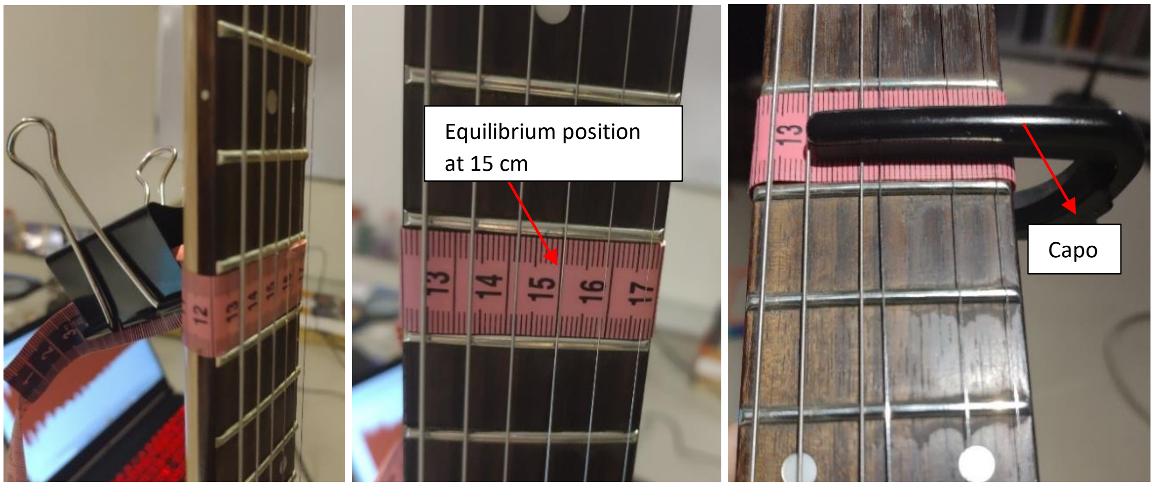



Figure 3: setup for fixing and measuring the displacement of the string from the equilibrium position

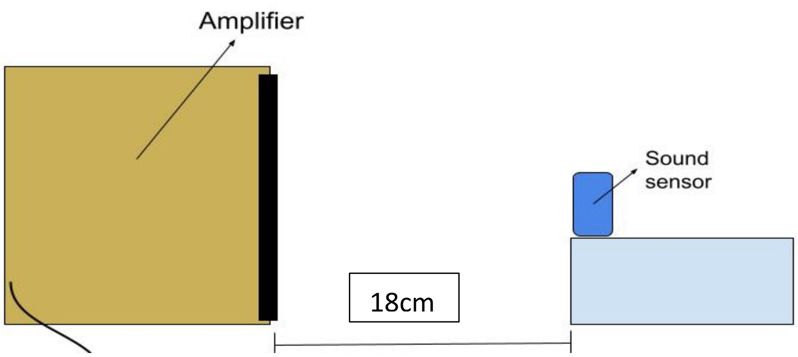



Figure 4: Setup for measuring the frequency of the string

**Procedure**:

1. The string was tuned to its respective standard frequency (in this case, the open G-string must vibrate at 196 Hz[17]) using a Digitech element pedal ELMT.
2. The guitar was connected to an amplifier without distortion and pickups 1 and 2 were selected. The base, middle and tremble values of the amplifier were set to maximum value, and gain was set to zero.[18]
3. The sound sensor was placed 18 cm away (an arbitrary distance) from the amplifier and parallel to it.
4. The displacement of the G string from the mean position was fixed with the capo.
5. The string was plucked and immediately after, measurements with the sensor started to be taken at 1665 samples per second during three seconds because the sensor can only store up to 5000 data points[19] (since sampling time was very short in order to maximize the number of samples per second, it was decided to pluck the string first and then start taking measurements).
6. The process was repeated 5 times, after which the string was tuned again.
7. Steps 1 to 5 were repeated another 5 times.
8. Steps 1 to 6 were repeated for each displacement from the mean position.

[17] Nevile H. Fletcher, "The physics of musical instruments".

[18] These are all parameters that control the way the sound signal is amplified.

[19] "Go Direct Sound", *Vernier,* accessed March 14, 2021, https://www.vernier.com/files/manuals/gdx-snd/gdx-snd.pdf

## Safety, ethical and environmental considerations:

The greatest risk when performing this experiment is that the tension makes the string snap and that it hits someone (probably myself). This is highly unlikely but not too uncommon among guitar players. If it were to hit someone, considering that the tension of the string was later calculated to be at least $69.6\pm$ $\pm0.58$ N and that the surface impact area is very small because the string is very thin, it could cause a nasty cut. Therefore, I used a jacket and protective glasses at all times while performing the experiments. Other than that case, this is a very safe experiment: no toxic waste products are produced and no ethical or environmental factors need to be taken into account.

## Qualitative Observations:

There was an audible increase in pitch as the bending angle increased (which I already knew from previous experiences). Also, for 9 and 12 mm of displacement, the string had gone slightly out of tune by the time it was tuned again after 5 tests, which suggests that it does not undergo fully elastic deformation. When taking raw data, a damping effect was clearly visible in the waveform (Figure 5); however that should not affect the frequency at all, rather only the amplitude. A further interesting observation was that the number of frequencies additional to the fundamental harmonic showing up in the FFT graph increased as bending angle increased.

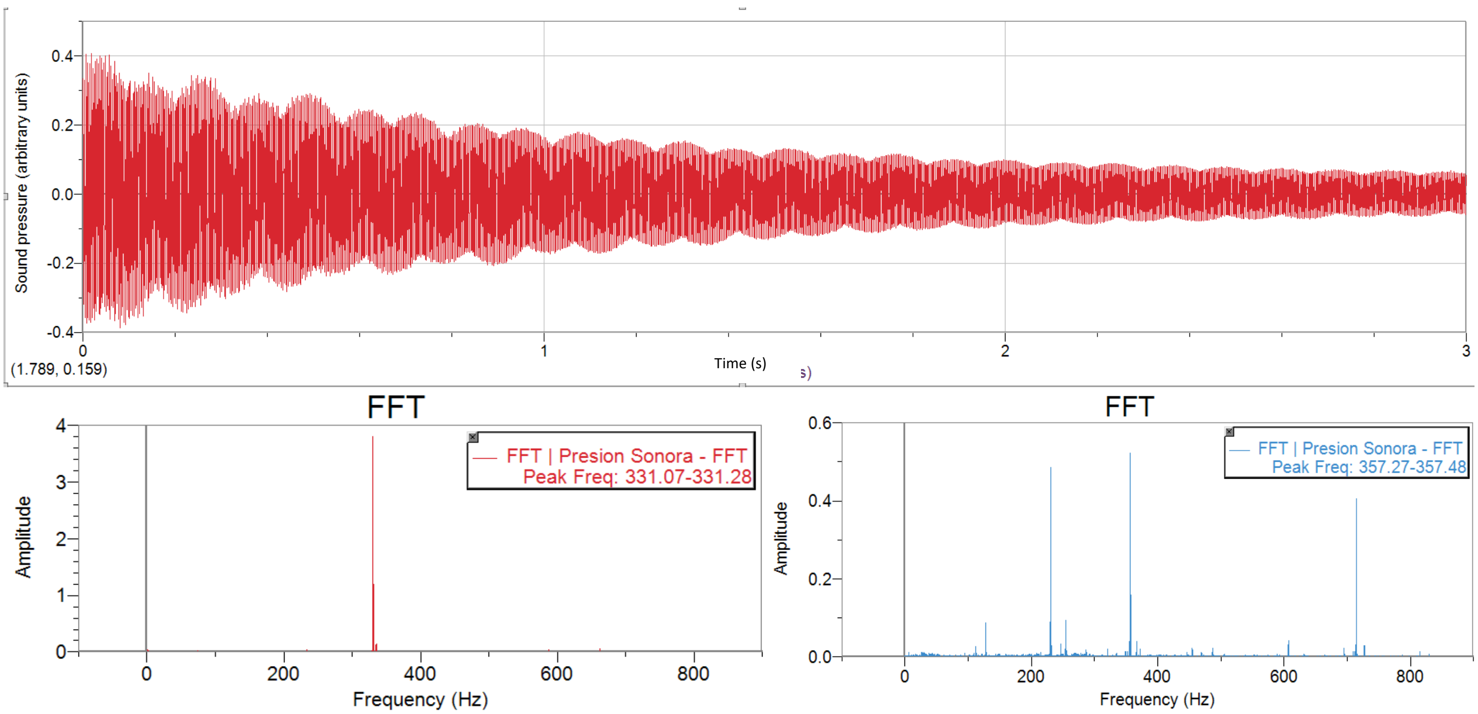



Figure 5: Waveform of the sound produced by 3 mm of displacement and its corresponding FFT graph (red). Next to it is the FFT graph of the waveform produced by a 12 mm displacement (blue)

## Quantitative data and data analysis:

As shown in Figure 5, from the FFT graph Logger Pro gave a range for the peak frequency measured (in this case, the fundamental harmonic). In table 1, the average of the range given is shown for every trial; the associated uncertainty reported is the largest deviation that the range had from the average, rounded to two significant figures. Normally, since the values for every trial are averages, the uncertainty of the average of the ten trials should be given by the standard error of the mean formula. However, to avoid an underestimation of the uncertainty due to small sample size in each trial (the number of samples is two, because Logger Pro only gave a minimum and a maximum possible value) the standard deviation of sample was used instead:

$$\sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})}{n-1}}$$

Where $\bar{x}$ is the average fundamental frequency recorded for each displacement and $n$ is the number of samples per displacement (in this case, ten). The uncertainty of the displacement is 1mm rather than 0.5 mm because it was not measured from zero (as explained in the "experimental setup" section).

| | Peak Frequency (Hz) | | | | |
|---|---|---|---|---|---|
| **Trial** | 0 mm$\pm$1 displacement | 3$\pm$1 mm displacement | 6 $\pm$1 mm displacement | 9$\pm$1 mm displacement | 12$\pm$1 mm displacement |
| **1** | 330.565$\pm$0.11 | 333.61$\pm$0.10 | 339.11$\pm$0.10 | 347.03$\pm$0.10 | 356.78$\pm$0.10 |
| **2** | 330.77$\pm$0.10 | 333.61$\pm$0.10 | 338.63$\pm$0.12 | 346.63$\pm$0.30 | 357.375$\pm$0.11 |
| **3** | 330.16$\pm$0.10 | 332.385$\pm$0.11 | 338.955$\pm$0.15 | 346.83$\pm$0.30 | 356.54$\pm$0.11 |
| **4** | 330.77$\pm$0.10 | 332.19$\pm$0.10 | 338.96$\pm$0.05 | 347.03$\pm$0.10 | 356.78$\pm$0.12 |
| **5** | 330.16$\pm$0.10 | 333.14$\pm$0.05 | 338.8$\pm$0.17 | 347.53$\pm$0.40 | 356.355$\pm$0.14 |
| **6** | 331.58$\pm$0.10 | 334.175$\pm$0.27 | 338.5$\pm$0.10 | 347.03$\pm$0.10 | 357.84$\pm$0.08 |
| **7** | 330.58$\pm$0.10 | 331.965$\pm$0.02 | 340.11$\pm$0.10 | 347.07$\pm$0.09 | 357.37$\pm$0.11 |
| **8** | 330.97$\pm$0.10 | 333.305$\pm$0.2 | 340.135$\pm$0.11 | 347.36$\pm$0.20 | 357.65$\pm$0.39 |
| **9** | 330.18$\pm$0.20 | 332.55$\pm$0.39 | 340.11$\pm$0.10 | 347.79$\pm$0.14 | 357.425$\pm$0.11 |
| **10** | 331.38$\pm$0.10 | 332.815$\pm$0.11 | 340.06$\pm$0.05 | 346.715$\pm$0.59 | 357.32$\pm$0.1 |
| **Average** | 330.7115$\pm$0.49 | 332.9745$\pm$0.71 | 339.337$\pm$0.68 | 347.1015$\pm$0.36 | 357.1435$\pm$0.50 |

Table 1: average frequencies of the fundamental harmonic for different displacements of the string from its mean position

Recalling the parameters for equation $f_{0b} = \frac{\sqrt{\frac{F_{T1}+kL(\frac{1}{cos\theta}-1)}{\mu\cos\theta}}}{2L\ sec\theta}$: we now have measured that $L$ =391$\pm$0.5 mm; using the average peak frequency at 0 mm displacement and equation 3 we get that $F_{T1}$= 69.6$\pm$ $\pm$0.58 N; and $\mu$ of the G-string was calculated to be $1.04 \times 10^{-3}$ kg m$^{-1}$. The percentual uncertainty of $F_{T1}$ was calculated as follows:

$$F_{T1} = 4L^2 f_0^2 \mu \therefore \left(\frac{\Delta F_{T1}}{F_{T1}}\right)100\% = \left(\frac{2\Delta L}{L} + \frac{2\Delta f_0}{f_0} + \frac{\Delta\mu}{\mu}\right)100\%$$

Therefore, the only parameter missing is the spring constant k. Now, to calculate it is necessary to use a material property called Young´s Modulus (E), which is the gradient of a stress ($\sigma$) versus strain ($\epsilon$) graph of a material. Young´s modulus is related to Hooke's spring constant in a similar way in which resistivity is to resistance: k is E adapted to the dimensions of a sample. This means that if we know the Young´s Modulus of the material of which the string is made, we can obtain k.

As said, $E = \frac{\sigma}{\epsilon}$, where $\sigma = \frac{F}{A}$ and $\epsilon = \frac{\Delta L}{L}$. $A$ would be the cross-sectional area of the string and $\Delta L$ would be the elongation of the string. Recalling that $F = k\Delta L => k = \frac{F}{\Delta L}$, the previous equations can be combined to obtain:

$$E = \frac{F}{A}\frac{L}{\Delta L} = \frac{L}{A}k => k = \frac{EA}{L}$$

Substituting k for E in equation (5) gives:

$$F_b = kL\left(\frac{1}{cos\theta} - 1\right) = EA\left(\frac{1}{cos\theta} - 1\right) = E\pi r^2\left(\frac{1}{cos\theta} - 1\right)$$

The radius of the G-string of my guitar is -based on the diameter reported by the manufacturer- 0.2032 mm, and although I could not find the Young´s Modulus of Nickel plated steel reported in literature, I did find the Young´s Modulus of stainless steel, which is 180 GPa[20].

So, now we have all the values needed for the theoretical prediction. The new expression for $F_b$ can be plugged into equation 6 to obtain a complete function where the theoretical frequency of the fundamental harmonic $f_{0b}$ is given by:

$$f_{ob} = \frac{\sqrt{\frac{F_{T1}+E\pi r^2\left(\frac{1}{cos\theta}-1\right)}{\mu\cos\theta}}}{2L\ sec\theta} = \frac{\sqrt{\frac{69.6\ \mathrm{N}+180\times10^{9}Pa\ \pi\left(2.032\times10^{-4}m\right)^2\left(\frac{1}{cos\theta}-1\right)}{1.04\times10^{-3}\pm1.3\times10^{-4}kg\ m^{-1}\cos\theta}}}{2(3.91\times10^{-1}\ \mathrm{m}\ )sec\theta} \quad (7)$$

Where the percentual uncertainty is: $\left(\frac{\Delta f_{ob\ theoretical}}{f_{ob\ theoretical}}\right)100\% = \left(\frac{\Delta F_{T1}}{F_{T1}}+\frac{2\Delta r}{r}+\frac{\Delta\mu}{\mu}+\frac{\Delta L}{L}+\frac{3\Delta\theta}{\theta}\right)100\%$

**Note:** Example of the process followed to calculate the uncertainty of operations involving trigonometric identities:

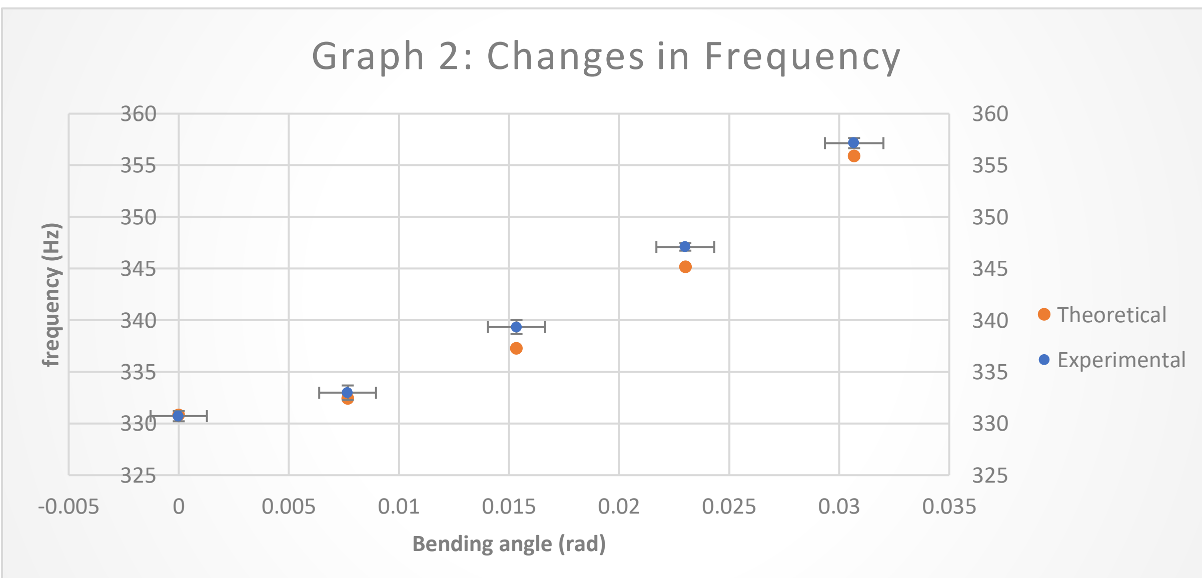
$$\theta = \arctan\left(\frac{d\pm\Delta d}{L\pm\Delta L}\right) => \ \theta_{max} = \left(\frac{d+\Delta d}{L-\Delta L}\right), \theta_{min} = \left(\frac{d-\Delta d}{L+\Delta L}\right),$$

The uncertainty $\Delta\theta$ is the largest deviation of $\theta_{max}$ or $\theta_{min}$ from $\theta$:

$$\theta = \arctan\left(\frac{3\pm0.5mm}{391\pm0.5mm}\right) = 7.67\times10^{-3}rad\ ; \theta_{max} = 8.96\times10^{-3}rad;\ \theta_{min} = 6.39\times10^{-3}rad$$

$$(\theta-\theta_{min}) < (\theta_{max}-\theta) \therefore \Delta\theta = \theta_{max}-\theta = 1.29\times10^{-3}rad$$

Graph 2 shows the experimentally measured frequencies against the bending angle (blue) and the theoretically predicted ones. This last ones graphed without uncertainties




Both data sets show the same behaviour, namely that the rate of change in frequency increases with the angle; however, experimental values were significantly higher than theoretical predictions for the last 3 bending angles. This strongly suggests a systematic error in data sampling, because the theoretical predictions do not lie within the uncertainty bars of the experimental measurements (notice that the vertical uncertainty bars of experimental measurements are so small that they barely show in graph 2). Nevertheless, a look at the uncertainties of the theoretical data (Table 2)reveals that that might not necessarily be the case at all:

[20] "Young's Modulus - Tensile and Yield Strength for common Materials ", *The Engineering ToolBox,* accessed February 28, 2021, https://www.engineeringtoolbox.com/young-modulus-d_417.html

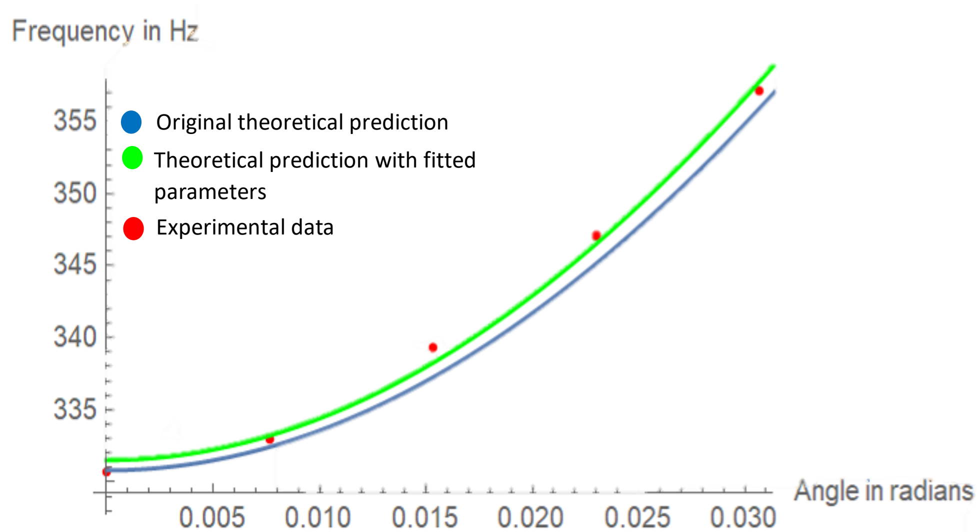
| Displacement (mm $\pm 1$) | Angle (rad) | Experimental frequency (Hz) | Theoretical frequency (Hz) |
|---|---|---|---|
| 0 | $0 \pm 1.28 \times 10^{-3}$ | $330.7115 \pm 0.16$ | $330.6269524 \pm 12$ |
| 3 | $7.67 \times 10^{-3} \pm 1.29 \times 10^{-3}$ | $332.9745 \pm 0.23$ | $332.109333 \pm 170$ |
| 6 | $1.53 \times 10^{-2} \pm 1.30 \times 10^{-3}$ | $339.337 \pm 0.22$ | $336.516524 \pm 88$ |
| 9 | $2.30 \times 10^{-2} \pm 1.32 \times 10^{-3}$ | $347.1015 \pm 0.11$ | $343.7337736 \pm 61$ |
| 12 | $3.07 \times 10^{-2} \pm 1.33 \times 10^{-3}$ | $357.1435 \pm 0.16$ | $353.5853783 \pm 48$ |

Table 2: theoretically predicted and experimentally measured frequencies for different bending angles

The uncertainty of the predicted frequencies is huge; therefore the experimental values always fell within the uncertainty bars. The reason for these big uncertainties is mainly the uncertainty of the bending angle because it was summed three times due to it being used three times in equation 7. Furthermore, since the absolute uncertainty was calculated by multiplying the percentual uncertainty times the resulting value, the error bars of the smallest values are much bigger. The value at bending angle zero was an exception because $\frac{3\Delta\theta}{\theta}$ is not defined (it would be dividing by zero) and was therefore not taken into account. Despite the fact that the experimental values always fell within the error bars of the prediction, a new approach taken in an attempt to refine and improve the prediction formula.

**Development of a computational model to fit the values of the variables to experimental data**

Computer programs and computational models are, without a doubt, one of the most powerful tools that science has at its disposition nowadays. Therefore, I decided to try to obtain the values of the different variables of equation 7 that best fit experimental observations, using a computer model.  This was made because, either due to the uncertainties of measurement equipment or  because some values were not measured directly but rather taken from outside sources (this will be further discussed in the "evaluation section"), the values used for $F_{T1}, r, E, L$ or $\mu$ may not be the ones closer to the real values nor the ones that give the best prediction. I made a program in a Software called *Matematica* to generate a non-linear best-fit line with the equation $f(\theta) = \left(\frac{1}{2L\ sec\theta}\right)\sqrt{\frac{F_{T1}+E\pi r^2\left(\frac{1}{cos\theta}-1\right)}{\mu\cos\theta}}$. First, where $\mu$ and $L$ were given as the experimentally measured values, and the rest of the variables were given as fit parameters (because the uncertainties of $\mu$ and $L$ are very small), and second -a more radical approach- where *all* the variables were given as fit parameters. In each case, the program found the values of the variables given as fit parameters which were most in accordance to my data points, following some constraints I previously defined (all values had to be greater than zero and the algorithm used had to start iterating using the previously obtained values of those variables). The results (graphs 3 and 4, and table 3) showed that both of the computationally fitted models generated essentially the exact same graph -therefore, only one is showed here, although both can be found in the appendix-, yet slightly




Graph 3: comparison of experimental data, the original theoretical prediction and the best-fit model generated in *Matematica*.  Error bars were omitted for the sake of clarity, but the uncertainties of experimental data are reported on table 1, uncertainties of the original theoretical prediction on table 2, and the uncertainty of the fitted function was not computed.



| Variable | Consulted/calculated value | Fitted value (5 parameters) | Fitted value (3 parameters) |
|---|---|---|---|
| $F_{T1}$ (N) | $69.9 \pm 0.58$ | 69.6 | 69.89 |

different values for $F_{T1}, r, E, L$ and $\mu$.

Interestingly, the values which showed the greatest difference between the two fitted models and the original calculated/consulted data, were $F_{T1}, r, E,$ and $L$: the ones not measured directly.

| Variable | Consulted/calculated value | Fitted value (5 parameters) | Fitted value (3 parameters) |
|---|---|---|---|
| $F_{T1}$ (N) | 69.9$\pm$0.58 | 69.6 | 69.89 |
| $r$ (m) | 2.032$\times 10^{-4}$ | 2.085$\times 10^{-4}$ | 2.090$\times 10^{-4}$ |
| $E$(Pa) | 180$\times 10^{9}$ | 180$\times 10^{9}$ | 179.7$\times 10^{9}$ |
| $L$(m) | 0.391$\pm$5$\times 10^{-4}$ | 0.391 | 0.391$\pm$5$\times 10^{-4}$ |
| $\mu$ (Kg m$^{-1}$) | 1.04$\times 10^{-3}$ | 1.04$\times 10^{-3}$ | 1.04$\times 10^{-3}$ |

Table 3: comparison between the values of different variables of equation 7 calculated and consulted, and using two different models.

## Conclusions

As mentioned before, the measured changes in frequency as a function of the bending angle showed a similar behaviour than those predicted by equation 7. This means that the results were partially in accordance with the hypothesis because they showed indeed a non-linear tendency. However, the intersection with the y-axis was not at the frequency that the note played (E4) should have, but rather approximately 1 Hz more, even taking the uncertainty of the measurement into account. A difference of 1 Hz might not seem much, however it is still perceivable by the human ear[21] (especially a trained musical ear). Potential causes, implications and improvements of this, and other experimental weaknesses will be discussed in the evaluation. The theoretical prediction was also slightly lower than the real value, although the real value did lie within its error bars. Computational non-linear models showed that small changes on the parameters which were not experimentally measured could yield a significantly better prediction.

How does the angle at which a guitar string is bent affect the frequency of its fundamental harmonic?

It depends on the string´s initial tension, linear density, cross sectional area and the note from which bending is made (that determines the length of the string, as explained in the "music, strings and waves" section). However, frequency increases, and the rate of increase is not constant. The formula $f_{1b}(\theta) = \left(\frac{1}{2L\,sec\theta}\right)\sqrt{\frac{F_{T1}+E\pi r^2\left(\frac{1}{cos\theta}-1\right)}{\mu \cos\theta}}$ describes the frequency obtained by a bending angle $\theta$, although that way it is formulated greatly increases the effect of uncertainties in angle measurement. This result is significant because it provides insight into why this technique is so hard to learn and brings musical intuition into the realm of physics.

## Evaluation:

### Strengths:

The experimental results show great precision (as the low uncertainty of the average frequency per angle shows). This means that there was a very consistent experimental method which gave my results a high degree of repeatability and reproducibility.

The experimental setup was very simple which minimized the possibility of random errors and facilitated controlling all the relevant variables. Furthermore, all the materials used -except for the sound sensor- were very common (for electric guitar players) and no permanent modifications to the guitar needed to be made.

### Weaknesses, improvements and extensions:

[21] "Sensitivity of Human Ear", HyperPhysics, *Georgia State University,* accessed February 28, 2021, http://hyperphysics.phy-astr.gsu.edu/hbase/Sound/earsens.html

The high uncertainty in the angle measurements is attributed both to the fact that it was an indirect measurement and to the uncertainty of the measuring device. The way angle uncertainties were calculated resulted in a very high uncertainty because, since the displacements were small, the percentual uncertainty was very high. To minimize this uncertainty a more precise measuring device, like a Vernier caliper, should be used.

The aforementioned (in the conclusion) difference between the measured fundamental frequency of the unbent string and the frequency that the note E4 should correspond to, can be due to my guitar´s intonation being slightly off. The intonation is how in tune a guitar is along the entire fretboard. In other words, how much do the discrete changes in frequency caused by the discrete changes in length of the string correspond to the changes in frequency between the notes of the western musical scale.[22] Normally, the intonation of a guitar needs to be corrected around every six months, but due to the Covid-19 pandemic, I have not been able to take mine to the guitar shop for maintenance.

Two prior experimental assumptions were made which only approximated the reality: The first was that the effect of Nikel plating in the strig´s Young's Modulus was negligible. This need not be the case and might be another factor influencing the discrepancy between predictions and measurements. A possible improvement, and extension to the investigation, would be to experimentally determine the Young´s Modulus of the string used and replace that value into equation 7. If it turns out to be significantly higher than that of plain stainless steel, it could correct the deviation showed in graphs 2 and 3. Although the computational models suggest that the difference is very small, they also show that a small difference can have significant changes in the theoretical predictions.

The second one was that guitar strings exhibit perfect elastic deformation for all bending angles used i.e. when the force stops being applied, the length of the string -and therefore the tension- returns to the original value. This is certainly not 100% true because, from personal experience and qualitative observations during the experiments, strings go out of tune after being bent repeatedly. Also after being played normally, or even without being played at all, strings go out of tune after some hours[23]. Recalling equation 3: $f_0 = \frac{\sqrt{\frac{T}{\mu}}}{2L}$ we can deduce that this means that tension and length do not remain constant (because mass must remain so). Therefore, equation 7 will never provide absolutely accurate predictions: for the frequency to be calculated the length of the string and its fundamental frequency need to be previously measured, but measuring the frequency changes the length and tension itself. Of course, these changes are negligible given an imperfect measuring device, short periods of time and small bending angles. However, it was shown that it does not take very big angles for this non constant tension and length to play a role in measurements.

Another probable cause of the systematic errors could have been the fact that most of the values of the variables used in equation (7) were not directly measured: $F_{T1}$ was calculated from experimental data and could have been affected by the aforementioned sources of errors, and both $r$ and $E$ were taken from outside sources. Hence, potential imprecisions in the fabrication process of my guitar strings, resulting in a slightly different radius than the reported by the manufacturer, and the fact that Young´s Modulus for the string was "elegantly guessed" to be essentially equal to Young´s Modulus of stainless steel also remain possible errors that could not be accounted for experimentally. However, the computer models were a very efficient way of trying to indirectly account for these potential errors. The models suggested most clearly that the radius of the string was bigger that the reported by the manufacturer; therefore, a further possible extension could be to check that prediction by measuring the radius with a high-precision tool, again like a Vernier caliper.

[22] Gabriele U. Varieschi, Christina M. Gower, "Intonation and Compensation of Fretted String Instruments", *Department of Physics, Loyola Marymount University, American Journal of Physics*, ResearchGate, 2009, DOI: 10.1119/1.3226563

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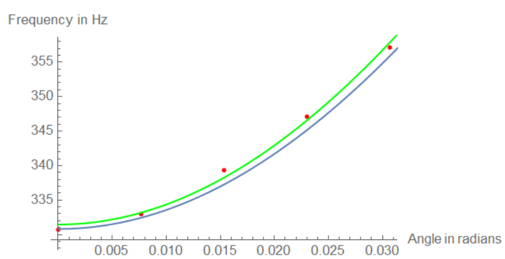
**Appendix 1**: Code in Matematica to generate the models from experimental data

```
In[1]:= ExperimentalData = {{0, 330.7115}, {0.007671856, 332.9745}, {0.01534281, 339.337}, {0.023011958, 347.1015}, {0.0306784, 357.1435}};


ExperimentalErrors2 = {0.1565, 0.2254, 0.2156, 0.1148, 0.1566} (*frequency uncertainties*);
WeightedErrors2 = 1 / ExperimentalErrors2^2;

ErrorAngle = {0.001280, 0.001291, 0.001303, 0.001316, 0.001332};

InitTension = 69.6;
Young = 180 * 10^9;
radious = 0.02032 * 10^-2;
LinDensity = 1.04 * 10^-3;
length = 0.391;

ExpFit3Variables = Sqrt[(Ft1 + YoungsM * Pi * R^2 * (1 / Cos[r] - 1)) / (LinDensity * Cos[r])] / (2 * length * Sec[r])
fit3Variables = NonlinearModelFit[ExperimentalData, {ExpFit3Variables, Ft1 > 0, YoungsM > 0, R > 0},
  {{Ft1, 69.6}, {YoungsM, 180 * 10^9}, {R, 0.02032 * 10^-2}}, r, MaxIterations → 10000, Weights → WeightedErrors2]
Print[fit3Variables["BestFitParameters"]]
Show[ListPlot[ExperimentalData, PlotStyle → Red, AxesLabel → {(Angle in radians), ("Frequency in Hz")}],
 Plot[((Sqrt[InitTension + Young * Pi * (radious)^2 * (1 / Cos[d] - 1)]) / (Sqrt[LinDensity * Cos[d]])) / (2 * length * Sec[d]), {d, 0, 0.04}],
 Plot[fit3Variables[r], {r, 0, 0.04}, PlotStyle → Green]]



ExpFitTest = Sqrt[(Ft1 + YoungsM * Pi * R^2 * (1 / Cos[r] - 1)) / (mu * Cos[r])] / (2 * L * Sec[r])
fitTest = NonlinearModelFit[ExperimentalData, {ExpFitTest, Ft1 > 0, Ft1 < 100},
   {{Ft1, 69.6}, {YoungsM, 180 * 10^9}, {R, 0.02032 * 10^-2}, {mu, 1.04 * 10^-3}, {L, 0.391}}, r, MaxIterations → 10000, Weights → WeightedErrors2]
 Print[fitTest["BestFitParameters"]]

 Show[ListPlot[ExperimentalData, PlotStyle → Red, AxesLabel → {(Angle in radians), ("Frequency in Hz")}],
  Plot[((Sqrt[InitTension + Young * Pi * (radious)^2 * (1 / Cos[d] - 1)]) / (Sqrt[LinDensity * Cos[d]])) / (2 * length * Sec[d]), {d, 0, 0.04}],
  Plot[fitTest[r], {r, 0, 0.04}, PlotStyle → Magenta]]
```

Out[10]= $39.653\,\mathrm{Cos}[r]\sqrt{(\mathrm{Ft1}+\pi R^2\,\mathrm{YoungsM}\,(-1+\mathrm{Sec}[r]))\,\mathrm{Sec}[r]}$

Out[11]= FittedModel[ $39.653\,\mathrm{Cos}[r]\sqrt{(69.8721+24675.2\,(-1+\mathrm{Sec}[r]))\,\mathrm{Sec}[r]}$ ]

{Ft1 → 69.8721, YoungsM → $1.79745\times10^{11}$, R → 0.000209039}

Out[13]=



Out[14]= $\dfrac{\mathrm{Cos}[r]\sqrt{\dfrac{(\mathrm{Ft1}+\pi R^2\,\mathrm{YoungsM}\,(-1+\mathrm{Sec}[r]))\,\mathrm{Sec}[r]}{\mathrm{mu}}}}{2\,L}$

Out[15]= FittedModel[ $39.7305\,\mathrm{Cos}[r]\sqrt{(69.6+24579.2\,(-1+\mathrm{Sec}[r]))\,\mathrm{Sec}[r]}$ ]

{Ft1 → 69.6, YoungsM → $1.8\times10^{11}$, R → 0.000208484, mu → 0.00103584, L → 0.39102}

Out[17]=