Modeling and optimizing a parabolic trajectory for the largest horizontal distance

I. Introduction

Time and again, while studying different phenomena in my physics HL class, I have been intrigued by all the different mathematical tools which can be used to model, explain, and predict reality. From highly complex integrals used to describe interactions between particles with an astounding level of precision, to small angle approximations for trigonometric identities -such as $\sin\theta \approx \theta$- mathematics seems to hold the necessary tools for every occasion and every level of complexity . They can be used to simplify, perfect and expand our understanding of physical reality; hence I became very interested in the tools which math offers to do so. Although I can definitely appreciate the beauty of pure, abstract mathematics, I am more on the side of applied mathematics. Yes, they should be elegant and simple, but there is nothing more elegant and simple than providing practical and useful tools to model reality.

I am particularly interested in describing and predicting the behaviour of moving physical objects, because I feel it gives an additional sense of tangibility to the calculations. Therefore, I decided to explore one of my favourite topics in physics, which is also very present in my everyday life, but in a deeper mathematical sense: parabolic motion.

During this pandemic I often take online classes in my living room. Let´s imagine I have just completed a two page long mathematical proof and realized I made a sign mistake in the second step, which makes the whole thing be wrong (resemblance with reality is mere coincidence). Angrily, I take out a new blank sheet to start all over again, not before squeezing my initial attempt into a tight paper ball (as tight as possible so that air resistance affects the trajectory as little as possible). Right before throwing it to my paper bin across the room, I stop and wonder: without simply throwing the ball harder, what is the best angle at which I should launch the ball if I want to reach a paper bin that is as far as possible?

II. Aim:

My aim is to calculate the launch angle which will result in the largest horizontal distance covered by the object undergoing parabolic motion, taking into account the surface the object will impact against. After arriving at a general expression for any surface, I want to apply it to a personal situation by modeling the surface of my living room and calculating the most efficient angle for a parabolic motion I employ very often: trying to score a basket, but to a paper bin across the room.

III. The motion equations for the different components

In parabolic motion the vertical and horizontal components can be -and usually are- analyzed completely separately. Hence, by decomposing the initial velocity vector into its components we can arrive at equations for vertical and horizontal displacement as functions of time y(t) and x(t), respectively. Since the motion equations are not included in the math curriculum and their derivation using calculus will be useful later, I will explain how to arrive at them.

Firstly, let us establish the initial situation such that the starting point of the trajectory is at the origin (image1):

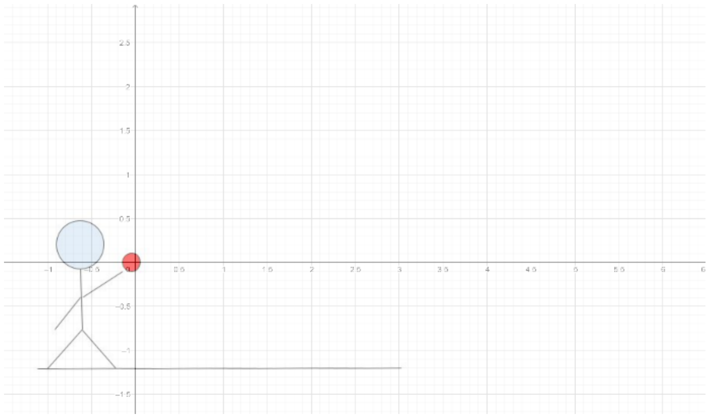

Image 1: Cartesian plane used to model the situation. The projectile (shown as a red circle) starts its trajectory at the origin.

This means that y(0) = 0 and x(0) = 0. Then, knowing that the horizontal component of a vector $\overrightarrow{v}$ with an angle α to the horizontal is $\vec{v}\cos(\alpha)$ and that the vertical component is $\vec{v}\sin(\alpha)$, we can use the fact that velocity is the rate of change of displacement to arrive at:

$$y'(t) = \vec{v}\sin(\alpha) \qquad (1)$$

$$x'(t) = \vec{v}\cos(\alpha) \qquad (2)$$

However, equation 2 is incomplete: the velocity is not constant in the vertical direction because the object is under the influence of the gravitational force g which provides an acceleration of 9.81ms^{-2}. In other words, the rate of change of vertical velocity is:

$$y''(t) = g \qquad (3)$$

Whereas the rate of change of horizontal velocity is:

$$x''(t) = 0 \qquad (4)$$

Now, to arrive at $y(t)$ and $x(t)$ we need to derive on both sides with respect to t and consider the terms we had arrived to in equations 1 and 2. Also, let us define the upwards direction as positive so that the next calculations are more intuitive; therefore, g must turn negative:

$$y'(t) = -gt + \vec{v}\sin(\alpha) \qquad (5)$$

$$y(t) = -\frac{1}{2}gt^2 + \vec{v}t\sin(\alpha) \qquad (6)$$

And

$$x'(t) = \vec{v}\cos(\alpha) \qquad (7)$$

$$x(t) = \vec{v}t\cos(\alpha) \qquad (8)$$

IV. Modelling the whole parabolic motion:

To optimize the trajectory, a function relating the angle α and both horizontal and vertical positions needs to be created. For that, let us use the substitution method to get rid of t and combine equations 8 and 6.

$$x(t) = \vec{v}t\cos(\alpha) => t = \frac{x(t)}{\vec{v}\cos(\alpha)}$$

$$\therefore y(t) = -\frac{1}{2}g\left(\frac{x(t)}{\vec{v}\cos(\alpha)}\right)^2 + \vec{v}\left(\frac{x(t)}{\vec{v}\cos(\alpha)}\right)\sin(\alpha)$$

Which simplifies to:

$$y(t) = -\frac{gx(t)^2}{2\vec{v}^2\cos^2(\alpha)} + x(t)\tan(\alpha) \qquad (9)$$

Although both displacements are functions of time, equation 9 describes the vertical position of the object in relation the horizontal position, taking the initial angle and velocity into account. Therefore, a more accurate notation would be:

$$y(x) = -\frac{gx^2}{2\vec{v}^2 cos^2(\alpha)} + x\tan(\alpha) \qquad (9.1)$$

Since the starting point is (0|0), we need to define a function $g(x)$ for the surface against which the projectile will impact. For example, if I were to toss a coin into a wish-granting well, the impact function (the bottom of the well) would be located in the fourth quadrant of the cartesian plane because the coin would fall further downwards than the initial x-position. Mathematically speaking, the projectile will hit the ground at $y(x) = g(x)$ and in this exploration the aim is to make the impact x-coordinate x_i as big as possible. Therefore, we must take into account the factors influencing x_i, namely velocity, initial angle and gravitational acceleration (from equation 9.1).

V. <u>Why is this investigation focused on the optimization of the initial angle?</u>

From experience we can tell that the harder someone throws a projectile, the longer the distance it will travel (given a fixed initial angle); therefore, there exists a positive relationship between x_i and $\vec{v}$. This is explored much more in detail and explicitly shown in section VI. d)"Finding the maximum horizontal impact distance in terms of the initial angle" and it turns out that there is no "most efficient $\vec{v}$"; x_i just keeps increasing as $\vec{v}$ does, following the relationship $x_i(\alpha) \propto \vec{v}^2$. Similarly, there is also no "most efficient g"and x_i simply decreases as g increases, following the relationship $x_i(\alpha) \propto \frac{1}{g}$. Furthermore, it would be impossible to significantly alter g without leaving the planet, so it would not make any practical sense to explore the effect of different values of it. Therefore, optimization of the trajectory will be focused on the initial angle which results in the longest x displacement, given a constant initial velocity and gravitational acceleration.

VI. <u>Optimizing angle α</u>

Let $x_i(\alpha)$ be the function describing the impact x-coordinate with respect to the initial angle. The maximum horizontal distance will be achieved at the maximum of $x_i(\alpha)$. However, we must consider that there are many values of α in which the projectile would either head straight towards the floor without any parabolic motion, or land backwards (in negative x-coordinates). Therefore, let us restrict the domain of the impact function ($\frac{\pi}{2}$ is not included because at that angle the projectile would head upwards and downwards in a straight line, without making a parabola):

$$dom(x_i) = \left[0, \frac{\pi}{2}\right)$$

a. <u>The classical parabola case</u>

If the trajectory were a classical parabola (i.e. if $g(x) = 0$, which could occur, for example if the motion were to start on ground level and to end on ground level), the problem would be relatively simple, as long as air resistance and every other force except gravity is negligible (Figure 2) .

Using equation 5 we can solve for t to get the time the projectile will travel upwards. Then, that time multiplied by 2 will be the total time the projectile will be in the air because, given the symmetry a parabola, it can be known that the time the projectile travels upwards is *exactly* the same time it travels downwards. Notice that this feature is only true when the trajectory is a complete, or "classical", parabola.

Figure 2: plot of a theoretical parabolic trajectory where the motion begins and ends at the intersects of the function with the horizontal axis.

$$y'(t) = -gt + \vec{v}\sin(\alpha) => t = \frac{\vec{v}\sin(\alpha)}{g} \therefore t_{total} = \frac{2\vec{v}\sin(\alpha)}{g}$$

If we substitute the total time into the equation for horizontal displacement (8) we get an equation for x_i:

$$x_i(\alpha) = \vec{v}\cos(\alpha)\left(\frac{2\vec{v}\sin(\alpha)}{g}\right) = \frac{2\vec{v}}{g}\cos(\alpha)\sin(\alpha)$$

The maximum of $x_i(\alpha)$ is achieved when $x_i'(\alpha) = 0$, so let us take the derivative of both sides of the equation and set it equal to zero:

$$x_i'(\alpha) = \frac{2\vec{v}}{g}(-\sin(\alpha)\sin(\alpha) + \cos(\alpha)\cos(\alpha)) = \frac{2\vec{v}}{g}(-\sin^2(\alpha) + \cos^2(\alpha))$$

$$\frac{2\vec{v}}{g}(-\sin^2(\alpha) + \cos^2(\alpha)) = 0$$

$$\cos^2(\alpha) = \sin^2(\alpha)$$

$$\arctan(\alpha) = 1$$

$$\alpha = \frac{\pi}{4}$$

Hence, the angle at which the trajectory will achieve the longest horizontal distance is $\frac{\pi}{4}$, or 45°. However, many of the everyday situations of parabolic motion (at least the one I personally encounter) do not form complete "classical" parabolas. Basketball throws, soccer ball clearance, and even throwing a ball of wrinkled paper into a paper bin across my room, are all examples of incomplete parabolas, so let us model a more complex -and interesting- situation:

b. <u>The incomplete parabola case</u>

For these cases we could try to arrive at the ideal angle by a similar method to the one used for the complete parabola and calculate the total time spent in the air, but it is shorter -and more useful in case we wish to change the surface $g(x)$ against which the projectile will impact (for $g(x) \neq 0$) - if we do it the following way:

Let us imagine all the possible parabolic trajectories given a constant velocity and changing only the initial angle: every possible value of $x_i(\alpha)$ with $\vec{v}$ and g being constant. Now, let us create a function which is tangent to each of those parabolas, so that it contains every possible trajectory that the projectile could make; this is called the envelope function. More generally, an envelope function of a family of curves is a curve which is -at every point- tangent to one of the curves of the family, where a family of curves can be defined as a set of functions written in the same form, but with some constraints and some variable parameters (in our case there is only one variable parameter). A family can be written as $f(x, y, P)$, where the set of points (x, y) satisfy a specific equation form, and P is the variable parameter in that equation. An example is shown in figure 3[1].

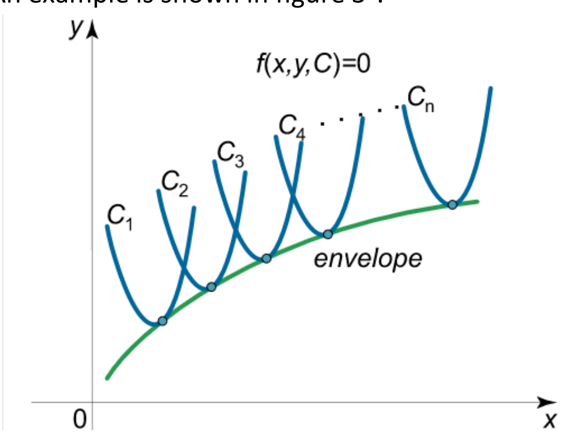

Figure 3: example of the envelope function (green) of the family $f(x, y, C)$, where the set of points (x, y) satisfy a specific equation (not given) which equals zero, for different values of C.

In our case, the family of possible trajectories T can be written as:

$$T(x, y, \alpha)$$

Where the set of points (x, y) satisfy the equation $y(x) = -\frac{gx^2}{2\vec{v}^2 cos^2(\alpha)} + x\tan(\alpha)$ for every value of α in the previously defined domain, and given that g and $\vec{v}$ remain constant. This envelope function is important because it will intersect the parabola that will provide the longest horizontal distance at exactly one point. Which point? The point at which the parabola is no longer contained within all the other parabolas: the maximum horizontal impact distance. Therefore, the point at which the enveloping function intersects $g(x)$ will be $x_{i_{max}}(\alpha)$.

[1] "Envelope of a Family of Curves", *Math24,* accessed March 22, 2021, https://www.math24.net/envelope-family-curves

Also notice that since there are infinitely many real numbers between 0 and $\frac{\pi}{2}$, and every value of α will produce a different parabola in equation 9.1, there is an infinite number of impact points between 0 and the maximum distance; hence, $x_i(\alpha)\epsilon[0, x_{i_{max}}(\alpha)]$. Since $x_i(\alpha)$ is a continuous function, it is safe to assume that it is differentiable, which will be very useful later on.

c. Calculating the envelope function

My first step for arriving at the envelope function was graphing equation 9.1 for several values of α, using the graphing software GeoGebra, while keeping an arbitrary initial velocity of $8.0ms^{-1}$ and considering the gravitational acceleration as $9.81ms^{-2}$ (Figure 4).

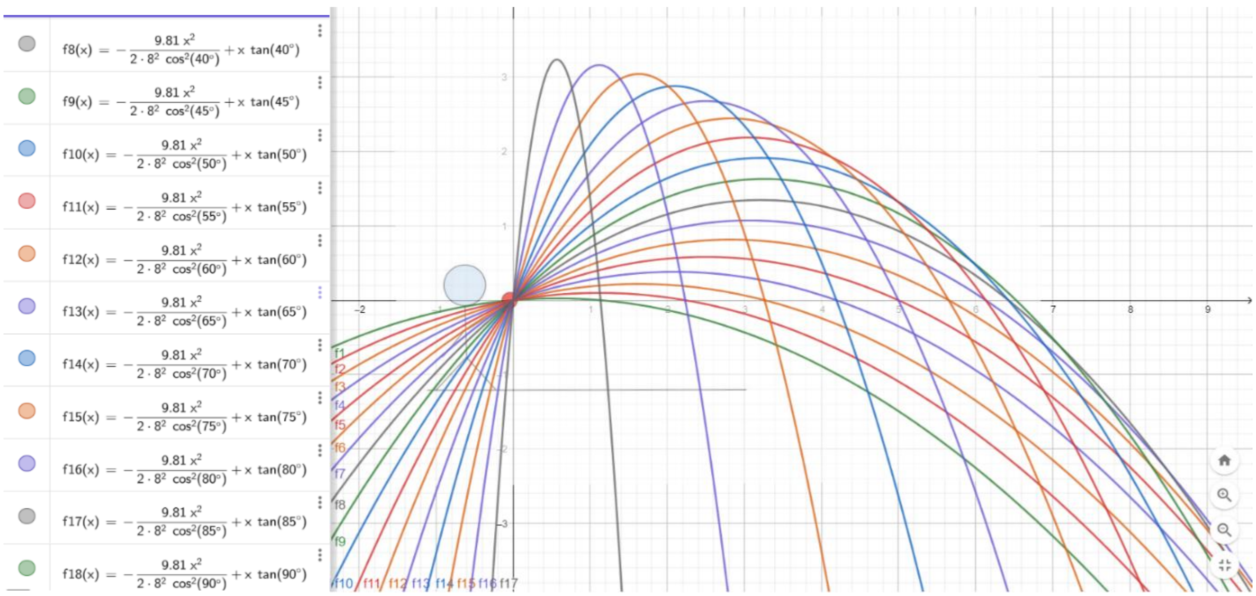

Image 3: plot of 17 parabolas of the form $y(x) = -\frac{gx^2}{2\vec{v}^2cos^2(\alpha)} + x\tan(\alpha)$ where $g = 9.81, \vec{v} = 8$, and α was changed in intervals of 5 from 0 to 85.

It is immediately noticeable that the enveloping function resembles a parabola; therefore, let us begin by defining the enveloping function as a parabola $e(x) = ax^2 + bx + c$. This will, of course, be more rigorously proved later in the procedure. By definition, $e'(x)$ will equal $y'(x)$ at *one* point (because $e(x)$ is *tangent* to every trajectory). However, by integrating both sides of the equation we can know that $e(x)$ equals $y(x)$ also at the *one* point:

$$y'(x) = e'(x)$$

$$y(x) + c = e(x) + c$$

$$y(x) = e(x) \qquad (10)$$

Now we can solve the system to arrive at the equation for the enveloping parabola:

$$-\frac{gx^2}{2\vec{v}^2\cos^2(\alpha)} + x\tan(\alpha) = ax^2 + bx + c$$

$$0=ax^2+bx+c\ +\frac{gx^2}{2\vec{v}^2\cos^2(\alpha)}-x\tan(\alpha)+c$$

$$0=x^2\left(a+\frac{g}{2\vec{v}^2\cos^2(\alpha)}\right)+x(b-\tan(\alpha))+c$$

This is basically a quadratic equation, and since it is known that there is only one solution the discriminant must be zero:

$$(b-\tan(\alpha))^2+4c\left(a+\frac{g}{2\vec{v}^2\cos^2(\alpha)}\right)=0$$

Multiplying both sides by $cos^2(\alpha)$ gives:

$$(bcos(\alpha)-\sin(\alpha))^2-4c\left(a\cos^2(\alpha)\ +\frac{g}{2\vec{v}^2}\right)=0$$

Which is true for every $\alpha\epsilon\mathbb{R}$, so by manipulating the value of α we can find the coefficients a, b and c of the enveloping parabola:

- Let $\alpha=\frac{\pi}{2}$, then

  $$(0-1)^2-4c\left(0+\frac{g}{2\vec{v}^2}\right)=0$$
  $$1=4c\left(\frac{g}{2\vec{v}^2}\right)$$
  $$1=\frac{2cg}{\vec{v}^2}$$
  $$\boldsymbol{c=\frac{\vec{v}^2}{2g}} \qquad (11)$$

- Let $\alpha=0$, then

  $$b^2=4c\left(a+\frac{g}{2\vec{v}^2}\right)$$

  Substituting c for the expression previously found we have:

  $$b^2=\left(\frac{4\vec{v}^2}{2g}\right)a+\left(\frac{4\vec{v}^2}{2g}\right)\left(\frac{g}{2\vec{v}^2}\right)$$
  $$\boldsymbol{b^2=\frac{2a\vec{v}^2}{g}+1} \qquad (12)$$

- Let $\alpha=\frac{\pi}{4}$

  $$\left(b\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)^2=4c\left(a\frac{2}{4}+\frac{g}{2\vec{v}^2}\right)$$
  $$\frac{1}{2}(b^2-2b+1)=2c\left(a+\frac{g}{\vec{v}^2}\right)$$
  $$(b^2-2b+1)=4c\left(a+\frac{g}{\vec{v}^2}\right)$$

  Substituting c for $\frac{\vec{v}^2}{2g}$ and b² for $\frac{2a\vec{v}^2}{g}+1$:

  $$(\frac{2a\vec{v}^2}{g}+1-2b+1)=\frac{4\vec{v}^2}{2g}(a)+(\frac{4\vec{v}^2}{2g})(\frac{g}{\vec{v}^2})$$

$$\frac{2a\vec{v}^2}{g} - 2b + 2) = \frac{2a\vec{v}^2}{g} + 2$$

$$-2b = 0 \therefore \boldsymbol{b} = \mathbf{0} \qquad (13)$$

Form equations 12 and 13 we get that:

$$\frac{2a\vec{v}^2}{g} + 1 = 0$$

$$a = -\frac{g}{2\vec{v}^2}$$

$$\therefore e(x) = -\frac{gx^2}{2\vec{v}^2} + \frac{\vec{v}^2}{2g} \qquad (14)$$

Graphing $e(x)$ in the coordinate plane with the previous test functions (also considering an initial velocity of 8ms^{-1} and gravitational acceleration of 9.81ms^{-2}) shows that it is indeed tangent to each of them (Figure 5):

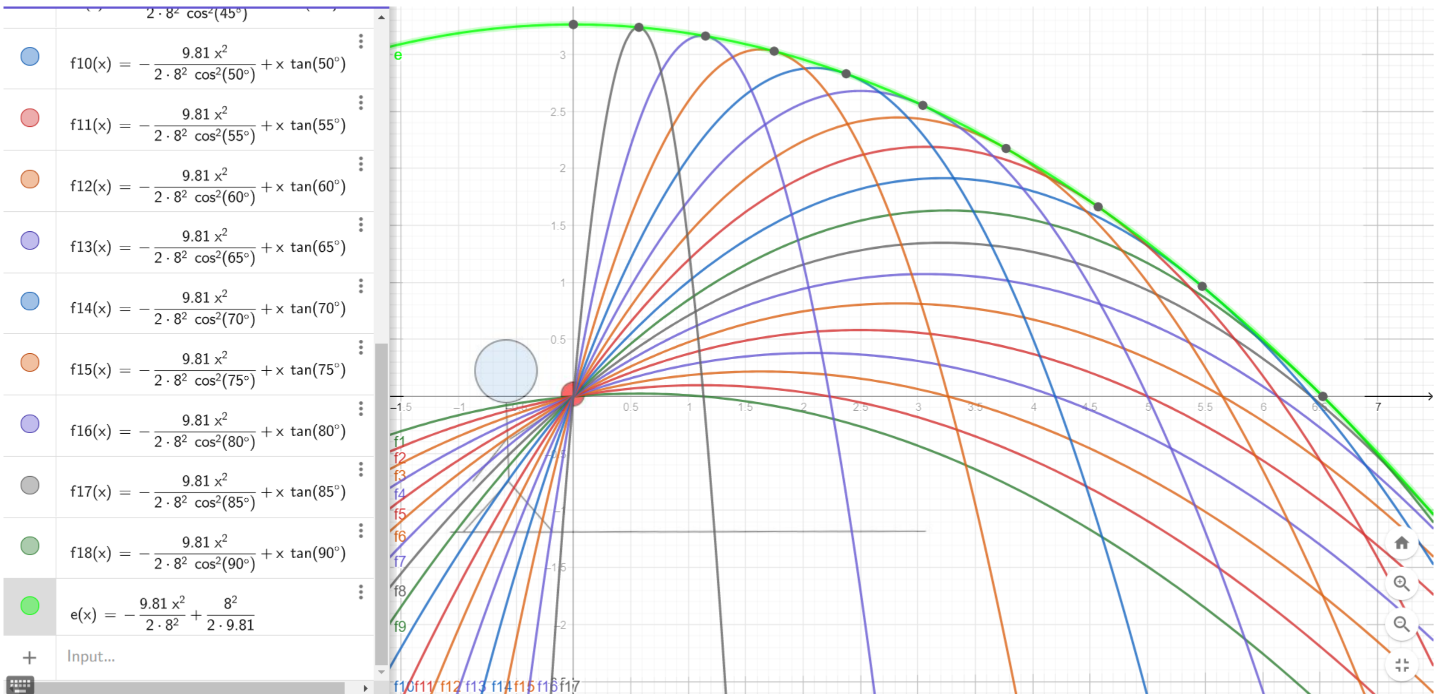

Figure 5: graph of the same functions as shown in Figure 4, but with the envelope function at which we arrived (green)

Furthermore, to prove in a more rigorous way that the initial assumption of the enveloping function being a parabola is correct, let us substitute the values found for a, b and c into our original system and see if there is indeed only one solution:

$$-\frac{gx^2}{2\vec{v}^2\cos^2(\alpha)} + x\tan(\alpha) = ax^2 + bx + c$$

$$-\frac{gx^2}{2\vec{v}^2\cos^2(\alpha)} + x\tan(\alpha) = -\frac{gx^2}{2\vec{v}^2} + \frac{\vec{v}^2}{2g}$$

$$0 = x^2\left(-\frac{g\cos^2(\alpha) + g}{2\vec{v}^2\cos^2(\alpha)}\right) - x\tan(\alpha) + \frac{\vec{v}^2}{2g}$$

If this system has only one solution for every α in our domain, then the two functions intersect only once, given any possible angle (in our domain). Therefore, let us check whether the discriminant is indeed equal to zero:

$$b^2 - 4ac = tan^2(\alpha) - 4\left(-\frac{gcos^2(\alpha)+g}{2\vec{v}^2\cos^2(\alpha)}\right)\left(\frac{\vec{v}^2}{2g}\right)$$

$$= tan^2(\alpha) - \frac{-4\vec{v}^2 gcos^2(\alpha)+4\vec{v}^2 g}{4\vec{v}^2\cos^2(\alpha)\,g}$$

$$= tan^2(\alpha) - \frac{-cos^2(\alpha)+1}{\cos^2(\alpha)}$$

From the Pythagorean identity we know that:

$$sin^2(\alpha)+cos^2(\alpha)=1 => -cos^2(\alpha)+1 = sin^2(\alpha)\ \forall\alpha$$

Therefore, we can substitute into the expression for the discriminant:

$$tan^2(\alpha) - \frac{sin^2(\alpha)}{\cos^2(\alpha)} = 0\ \ \forall\alpha$$

QED

Hence, the envelope function is indeed a parabola, as suggested by the experiments made in GeoGebra.

As said before, the point at which the enveloping parabola intercepts the impact function will be $x_{i_{max}}$; however, to find which angle will yield that distance we need to find an expression for $x_{i_{max}}(\alpha)$. Then, we can substitute the x value $g(x)=e(x)$ into $x_{i_{max}}(\alpha)$ and get the angle.

d. <u>Finding the maximum horizontal impact distance in terms of the initial angle</u>

To maximize $x_i(\alpha)$ for more general cases we need to substitute for it in function 9.1 so that we can have a trajectory in terms of the angle. Just as in the complete parabola case, when $x_i'(\alpha)=0$ the distance will be the longest.

So, $y(x) = -\frac{gx^2}{2\vec{v}^2cos^2(\alpha)} + x\tan(\alpha)$ evaluated at the coordinate of impact $x_i(\alpha)$ becomes:

$$y(x_i(\alpha)) = -\frac{gx_i(\alpha)^2}{2\vec{v}^2\cos^2(\alpha)} + x_i(\alpha)\tan(\alpha)$$

Now we derive with respect to α and set the equation equal to zero. Notice that since the derivative of a constant multiple of a function is simply the constant multiplied by the derivative of the function (as long as the multiple is an element of the real numbers), $\frac{g}{\vec{v}^2}$ can be factored out:

$$y'\big(x_i(\alpha)\big)(x_i'(\alpha)) = -\frac{g}{2\vec{v}^2}\frac{\cos(\alpha)2x_i(\alpha)x_i'(\alpha)-x_i(\alpha)^2 2\sin(\alpha)\cos(\alpha)}{cos^4(\alpha)} + x_i'(\alpha)\tan(\alpha) + x_i(\alpha)sec^2(\alpha)$$

$$0 = \frac{g}{\vec{v}^2}\left(\frac{-x_i(\alpha)^2}{cos^2(\alpha)}\right)\tan(\alpha) + x_i(\alpha)sec^2(\alpha)$$

$0 = \frac{g}{\vec{v}^2}(-x_i(\alpha)^2)\tan(\alpha) + x_i(\alpha)$

$0 = x_i(\alpha)(-gx_i(\alpha)\tan(\alpha) + \vec{v}^2$

$gx_i(\alpha)\tan(\alpha) = \vec{v}^2$

$x_i(\alpha) = \frac{\vec{v}^2}{g}\cot(\alpha)$ (15)

This means that the longest distance possible for a parabolic trajectory using the most efficient angle, will be as well dependent upon initial velocity and gravitational acceleration (as mentioned in section V). Also notice that it is impossible to find the most efficient value of either $\vec{v}$ or g using derivatives. If we derive equation 15 with respect to g and set the equation equal to zero, we get:

$$0 = -\frac{\vec{v}^2}{g^2}\cot(\alpha) => g^2 = -\vec{v}^2\cot(\alpha)$$

And since no real number squared can equal a negative number, the above equation has no solution. Therefore g cannot be maximized. Likewise, if we derive equation 15 with respect to $\vec{v}$ and set the equation equal to zero we get:

$$0 = \frac{2\vec{v}}{g}\cot(\alpha)$$

And the only value of $\vec{v}$ that satisfies the equation is zero, which makes no physical sense (a projectile thrown at zero speed will certainly not make it very far).

Now let us create the impact function so we can find its intersection with the envelope function:

e. Creating the impact function

A mathematical model of the surface against which the projectile might impact needed to be made in order to get the impact function. To do this, I took a picture of my living room with the camera positioned perpendicular to the floor and at a low height so that the distortions of dimensions due to the camera angle were minimized. Also, a one-meter ruler was included in the photo for the scaling to be easier. Next, the photo was pasted into a cartesian plane in the graphing software GeoGebra so I could "trace out" the impact function on top of the picture.

I decided to model the surface using polynomials, namely through Lagrange interpolating polynomials[2]. This is a polynomial of degree $n-1$ which passes through n points. Since it is a proven mathematical theorem that given *n* distinct x-values $x_1, x_2, x_3 \dots, x_n \forall x \in \mathbb{R}$ and *n* -not necessarily distinct- y-values $y_1, y_2, y_3, \dots y_n \forall x \in \mathbb{R}$, there is a *unique*

[2]Archer, Branden and Weisstein, „Lagrange Interpolating Polynomia", Wolfram MathWorld, consulted February 12, 2021, https://mathworld.wolfram.com/LagrangeInterpolatingPolynomial.html

polynomial $P(x)$ which passes through every point $(x_i, y_i) \forall i \in [1, 2, 3, \dots, n]$,[3] the Lagrange formula is simply one way of arriving at that unique polynomial. The Lagrange form of $P(x)$ is given by:

$$P(x) = \sum_{i=1}^{n} L_i(x) P(x_i)$$

Where $L_i(x)$ is a discrete product of a sequence, which can be expressed as follows:

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Or, written explicitly:

$$P(x) = \frac{(x - x_2)(x - x_3) \dots (x - x_n)}{(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n)} P(x_1) + \frac{(x - x_1)(x - x_3) \dots (x - x_n)}{(x_2 - x_1)(x_2 - x_3) \dots (x_2 - x_n)} P(x_2) + \cdots$$
$$+ \frac{(x - x_1)(x - x_2) \dots (x - x_{n-1})}{(x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1})} P(x_n)$$

Bear in mind that $P(x_i) = y_i$

However, before starting to select random points along the surface the coordinate (0|0) needed to be determined. To do that I recorded myself throwing a paper ball 10 times and, using the tracking software Tracker, I measured for every test the coordinates -using the floor as my x-axis and the right margin of the video as my y-axis- at which the paper ball left my hand (Figure 6). Additionally, Tracker also calculates the vertical and horizontal components of the velocity, to determine the initial velocity I simply took the square root of the sum of the squares of both components (the magnitude of the resulting vector).

Table 1: The coordinates of the starting point of the parabola for different throws. The software gave the coordinates with ten significant figures, however, I decided to use only two because the 1-meter ruler used to calibrate the software had an uncertainty of $\pm$ 0.5 cm, since the smallest unit it could measure was 1 cm. The uncertainty of the average was calculated using the standard deviation of a sample formula:

Test number	X-coordinate (m)	Y-coordinate (m)	Initial velocity (ms-1)
1	0.36	1.2	3.663

[3] Trevor Arashiro, Lurker Zer, Patrick Corn, „Lagrange Interpolation", Brilliant, consulted February 12, 2021, https://brilliant.org/wiki/lagrange-interpolation/#:~:text=The%20Lagrange%20interpolation%20formula%20is,proof%20of%20the%20theorem%20below.

2	0.34	1.2	5.801
3	0.37	1.3	3.667
4	0.25	1.2	4.273
5	0.24	1.2	4.332
6	0.20	1.2	4.409
7	0.32	1.3	5.262
8	0.32	1.3	3.83
9	0.27	1.3	3.60
10	0.23	1.2	4.37
Average:	0.29±0.059	1.2±0.052	4.32±0.72

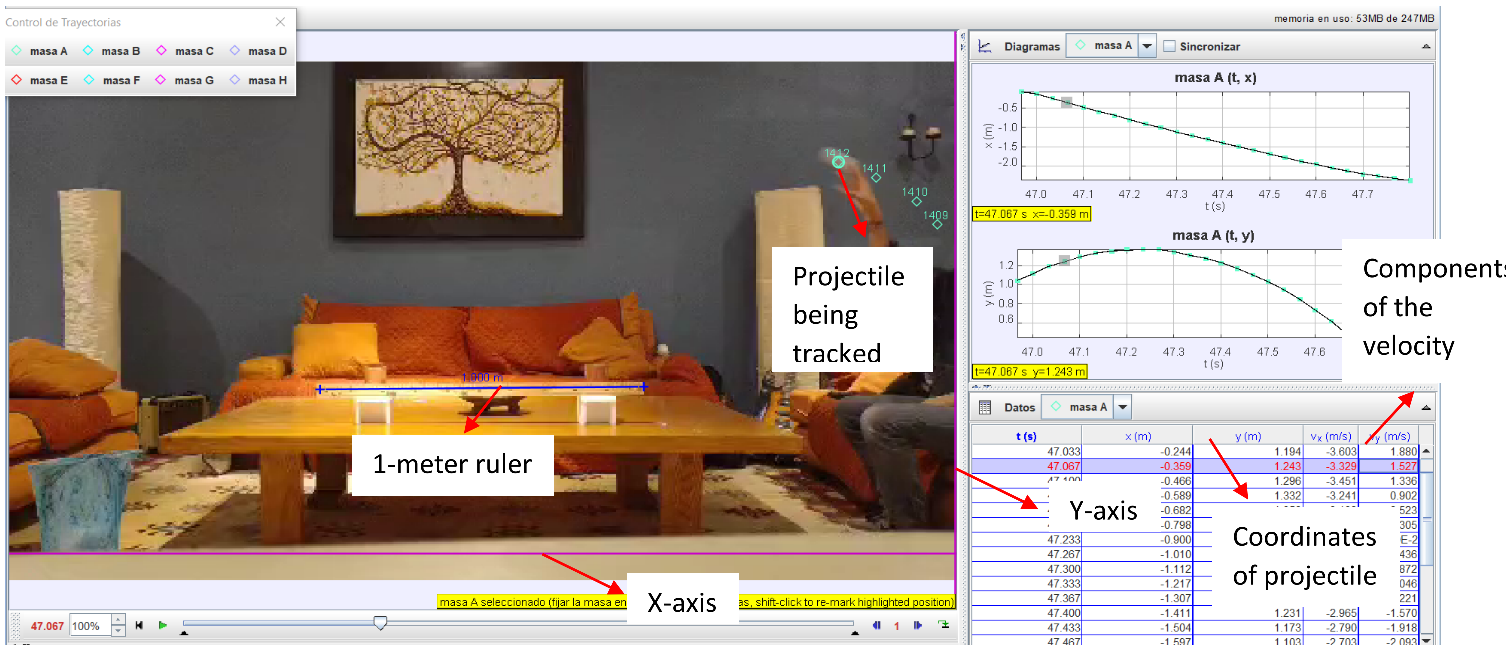

Figure 6: Analysis, using Tracker, of one of the videos of me throwing the ball

Therefore, from the average coordinates of launch, it can be known that the origin of the cartesian plane in the video must be located on coordinates (-0.29|-1.2) in GeoGebra).

Now, given that it would be extremely long and not very reliable to calculate a high-degree Lagrange polynomial “by hand” I found an online site where it is calculated automatically by providing the points with which one wishes to interpolate[4]. Afterwards, several points along the surface of the living room were arbitrarily chosen, assuming of course movement only in two dimensions. However, simply finding the Lagrange interpolation of as many points as possible did not result in an accurate approximation of the impact surface. Far from it, actually. From several “experiments” I observed that the higher-degree the polynomial, the more oscillation between the points and the less smooth the interpolation is. An example of one of these “experiments is shown in Figure 7.

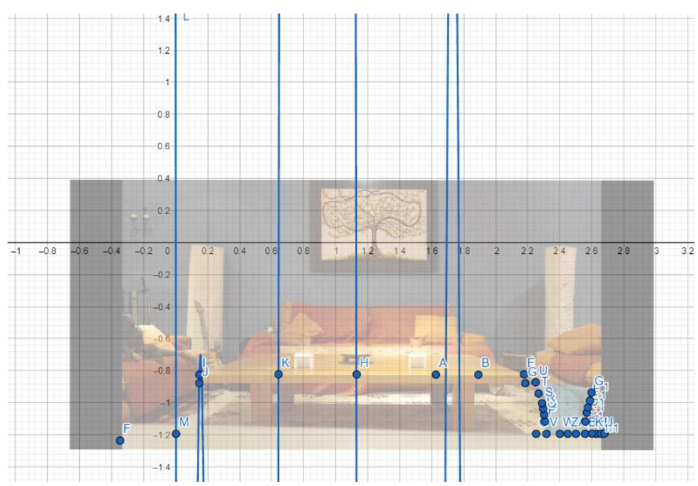

Figure 7: Lagrange interpolating polynomial for ten points along the surface I was trying to model. It can be seen that the function does not accuratelv model the surface.

[4]“Lagrange Interplation Polynomia“, DCODE, consulted February 12, 2021 https://www.dcode.fr/lagrange-interpolating-polynomial

Therefore, I decided to create several polynomials for smaller sets of points to model different parts of the surface, and restrict their domains so that I could arrive at a much more accurate model (Figure 8):

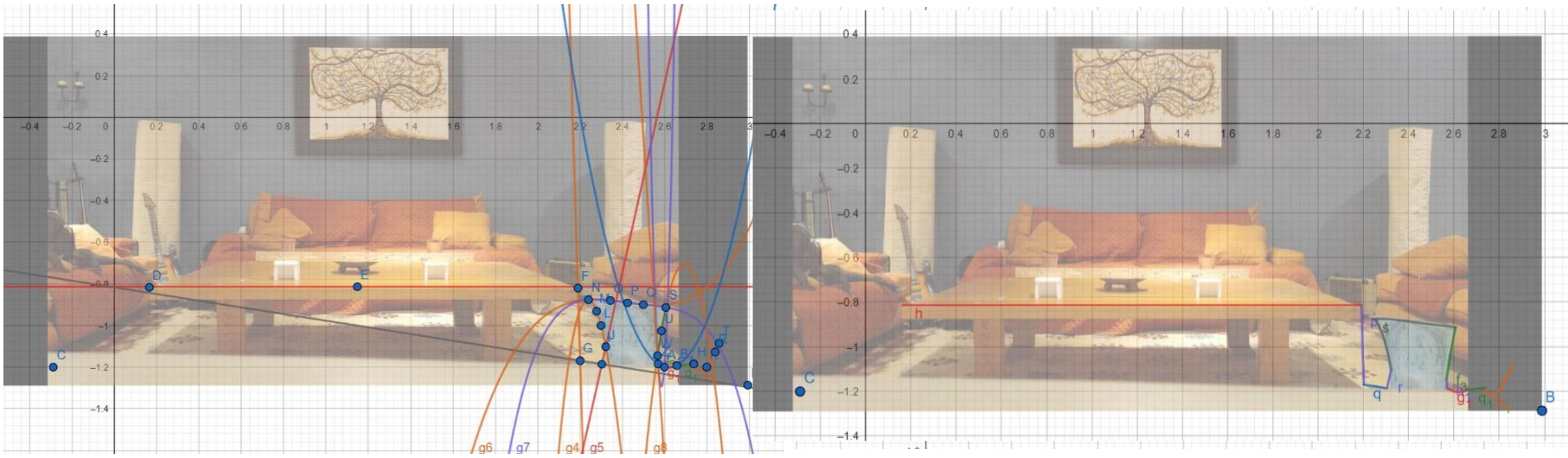

Figure 8: Model of the suface using different polynomials with restricted domain (right) and without restricted domain (left)

Final model of the surface:

$$g(x) = \begin{cases} -0.8136202562725 & 0.1643773503175 \leq x \leq 2.19073272 \\ 71.6535 - 33.082\,x & 2.19073272 \leq x \leq 2.201284907926 \\ -0.16\,x - 0.817277 & 2.20128491 \leq x \leq 2.30400121 \\ 4.55345\,x - 11.6771 & 2.30400121 \leq x \leq 2.3224524617186 \\ 28.8031\,x^3 - 227.654\,x^2 + 586.2x - 495.417 & 2.2407379740302 \leq x \leq 2.3224524617186 \\ -6.9979x^4 + 68.7197x^3 - 252.909x^2 + 413.308x - 2 & 2.24073779740302 \leq x \leq 2.6065544792214 \\ -27.7407\,x^2 + 149.43\,x - 201.936 & 2.6065544792214 \leq x \leq 2.60655448 \\ 39.2897 - 15.75\,x & 2.567304468 \leq x \leq 2.56983693 \\ -71.0854\,x^3 + 612.929\,x^2 - 1759.19x + 1679. & 2.60655448 \leq x \leq 2.85817119 \\ 0.105943\,x - 1.47478 & 2.6583839070649 \leq x \leq 2.7382799193642 \end{cases}$$

Notice that, although the domain of the model is constant, at several points the domains of different polynomials overlap briefly; therefore, this model is considered a relation rather than a function.

Lastly, let us find the x-coordinate where $g(x) = e(x)$, so we can find the best angle in equation 15. This point can be obtained by GeoGebra if we plot $e(x)$(Figure 9). The value obtained is 2.8273575577269m; this is also the longest possible horizontal distance that the projectile can travel with the average initial velocity at with I throw. Now we can find the angle which will yield that distance, substituting into equation the value for distance, initial velocity, and gravitational acceleration:

$$x_i(\alpha) = \frac{\vec{v}^2}{g}\cot(\alpha) => \frac{g\left(x_i(\alpha)\right)}{\vec{v}^2} = \cot(\alpha)$$

$$\frac{9.81ms^{-2}(2.8273575577269\text{m})}{(\overrightarrow{4.32}ms^{-1})^2} = \cot(\alpha)$$

$$\alpha = 38.5°$$

It was expressed to three significant figures because the velocity and gravitational acceleration had that also three significant figures.

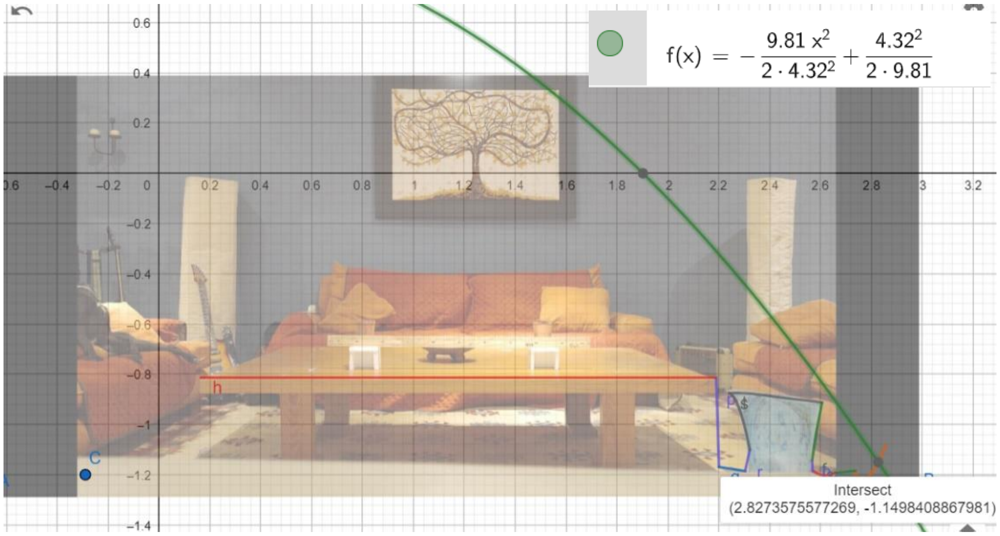

Figure 9: envelope function of the parabolic paths (green) and model of the impact surface. Intersection is marked as a point. The value for initial velocity was taken from table 1

VI. <u>Conclusion</u>

So, as said, the purpose of this work was to find the angle at which the horizontal distance traveled by an object undergoing parabolic motion was maximized, given a specific impact surface described by a function. In this case, the object undergoing parabolic motion was a ball of paper, and the function was a Lagrange interpolating polynomial which modeled the surface of my living room. It was concluded that the best angle was 38.5°, taking earth´s gravitational acceleration and my average launching initial velocity as constant values. It is also very important to note that launching the ball of paper at that angle, it would actually land *after* my rubbish bin. This means that the angles at which I usually launch are far from the most efficient one, because I have never thrown a ball that laned after the bin.

The polynomial model for the impact surface was found through an admittedly arbitrary method and more efficient approaches need to be explored. Let us keep in mind that on many cases "rough approximation" of the impact function can be made with much more simple functions. However, an expression for the most efficient angle was successfully determined, given *almost* any impact function. This knowledge can have many applications other than the explored in this work, such as in military or in many different sports where parabolic motion is involved.

Some possible further explorations could be considering air resistance and other forces acting on the projectile, such as the one generated if it happens to be spinning (magnus effect)[5]. Also, the case in which the impact surface actually touches the envelope function at more than one point. This could happen if there were tall objects, such as buildings or mountains between the launching point and the maximum distance.

It was very interesting to investigate, use, and combine different areas of mathematics like calculus, series, and functions to model a seemingly simple action of my everyday life. I managed to combine one of my favourite topic of physics with my favourite area -epistemologically speaking (i.e. with focus on the nature and purpose of mathematical knowledge rather than the knowledge itself) – of mathematics: applied mathematics.

[5] David Newman"What is the Magnus effect and how to calculate it", David Newman´s page, consulted February 12, 2021, http://ffden-2.phys.uaf.edu/211_fall2010.web.dir/Patrick_Brandon/what_is_the_magnus_effect.html

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