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Homework 4
17 April 2021

The code used to generate the following results can be found at the end of this document.

Problem 3.15

If $\mathcal{F}[x(n)] = e^{-j\alpha\omega}$ for $\omega_c < |\omega| \leq \pi$, then

$$
\begin{aligned}
x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\alpha\omega}e^{j\omega n}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{-j\alpha}e^{jn}\right)^{\omega}\,d\omega \\
&= \frac{1}{2\pi}\left.\frac{\left(e^{-j\alpha}e^{jn}\right)^{\omega}}{\ln\left(e^{jn-j\alpha}\right)}\right|_{-\pi}^{\pi} \\
&= \frac{\left(e^{-j\alpha}e^{jn}\right)^{\pi}-\left(e^{-j\alpha}e^{jn}\right)^{-\pi}}{2\pi(jn-j\alpha)} \\
&= \frac{e^{\pi(jn-j\alpha)}-e^{-\pi(jn-j\alpha)}}{2\pi j(n-\alpha)} \\
&= \frac{e^{j\pi(n-\alpha)}-e^{-j\pi(n-\alpha)}}{2j}\,\frac{1}{\pi(n-\alpha)} \\
&= \frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)}
\end{aligned}
$$

Problem 3.17

For the following problems, it is important to understand that if h is the impulse response, x is the excitation, and y is the result, then $Y = HX$, where Y, H, and X are the DTFTs of y, h, and x, respectively.

Part 1

$$y(n) = \frac{1}{5}\sum_{m=0}^{4} x(n-m) = \frac{1}{5}\left[x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4)\right]$$

$$Y\left(e^{j\omega}\right) = \frac{1}{5}\left[X\left(e^{j\omega}\right) + X\left(e^{j\omega}\right)e^{-j\omega}\right.$$
$$\left.+X\left(e^{j\omega}\right)e^{-2j\omega} + X\left(e^{j\omega}\right)e^{-3j\omega} + X\left(e^{j\omega}\right)e^{-4j\omega}\right]$$

$$Y\left(e^{j\omega}\right) = \frac{X\left(e^{j\omega}\right)}{5}\sum_{m=0}^{4} e^{-j\omega m}$$

$$\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = H\left(e^{j\omega}\right) = \frac{1}{5}\sum_{m=0}^{4} e^{-j\omega m}$$

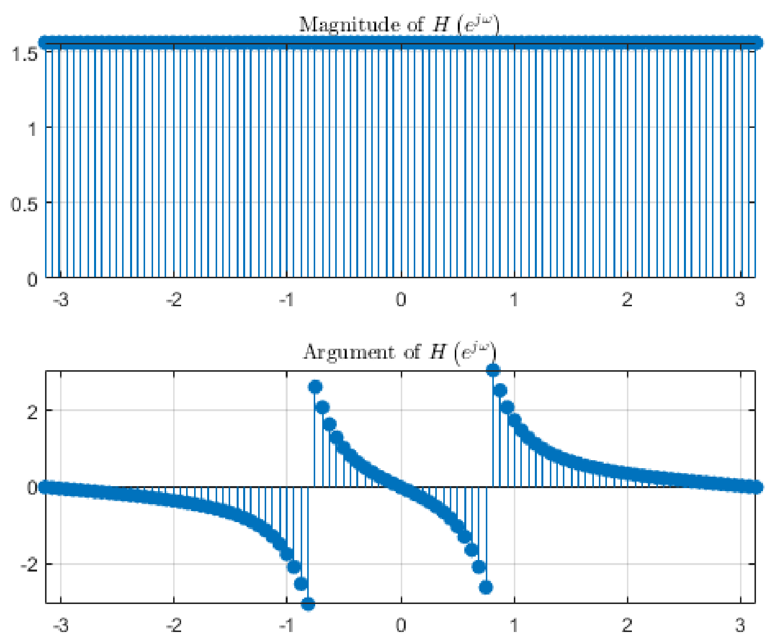

Figure 1: Magnitude and argument plot.

Part 4

$$y(n) = x(n) - 1.7678x(n-1) + 1.5625x(n-2)$$
$$+1.1314y(n-1) - 0.64y(n-2)$$
$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) - 1.7678X\left(e^{j\omega}\right)e^{-j\omega} + 1.5625X\left(e^{j\omega}\right)e^{-2j\omega}$$
$$+1.1314Y\left(e^{j\omega}\right)e^{-j\omega} - 0.64Y\left(e^{j\omega}\right)e^{-2j\omega}$$
$$Y\left(e^{j\omega}\right) - 1.1314Y\left(e^{j\omega}\right)e^{-j\omega} + 0.64Y\left(e^{j\omega}\right)e^{-2j\omega}$$
$$= X\left(e^{j\omega}\right) - 1.7678X\left(e^{j\omega}\right)e^{-j\omega} + 1.5625X\left(e^{j\omega}\right)e^{-2j\omega}$$
$$Y\left(e^{j\omega}\right)\left(1 - 1.1314e^{-j\omega} + 0.64e^{-2j\omega}\right)$$
$$= X\left(e^{j\omega}\right)\left(1 - 1.7678e^{-j\omega} + 1.5625e^{-2j\omega}\right)$$
$$\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = H\left(e^{j\omega}\right) = \frac{1 - 1.7678e^{-j\omega} + 1.5625e^{-2j\omega}}{1 - 1.1314e^{-j\omega} + 0.64e^{-2j\omega}}$$

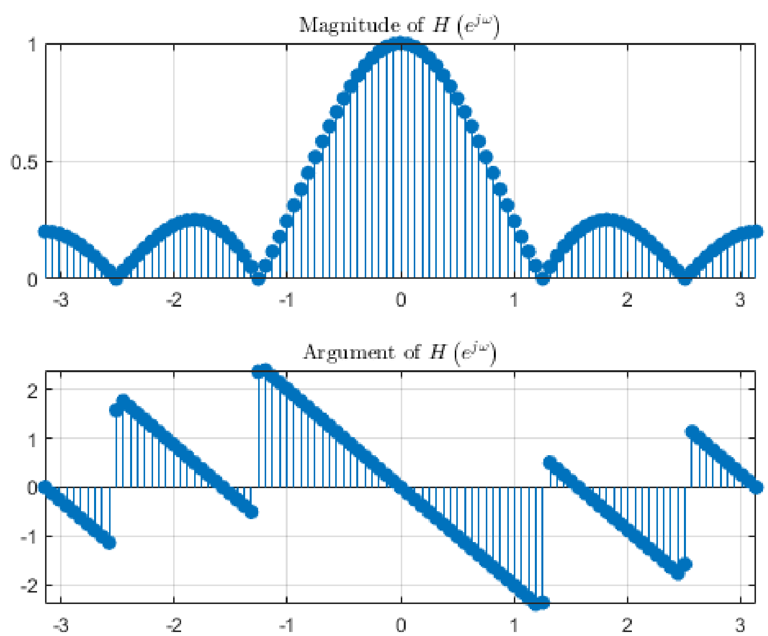

Figure 2: Magnitude and argument plot.

Problem 3.18

The frequency response can be found using the difference equation given.

$$
\begin{aligned}
y(n) &= x(n) + x(n-2) + x(n-4) + x(n-6) \\
&\quad -0.81y(n-2) - 0.81^2 y(n-4) - 0.81^3 y(n-6) \\
y(n) + 0.81y(n-2) + 0.81^2 y(n-4) + 0.81^3 y(n-6) \\
&= x(n) + x(n-2) + x(n-4) + x(n-6) \\
Y\left(e^{j\omega}\right)\left(1 + 0.81e^{-2j\omega} + 0.81^2 e^{-4j\omega} + 0.81^3 e^{-6j\omega}\right) \\
&= X\left(e^{j\omega}\right)\left(1 + e^{-2j\omega} + e^{-4j\omega} + e^{-6j\omega}\right) \\
\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = H\left(e^{j\omega}\right) &= \frac{1 + e^{-2j\omega} + e^{-4j\omega} + e^{-6j\omega}}{1 + 0.81e^{-2j\omega} + 0.81^2 e^{-4j\omega} + 0.81^3 e^{-6j\omega}}
\end{aligned}
$$

Part 1

Through Fourier analysis (numerical DTFT), it can be seen that the signal $x(n) = 5 + 10(-1)^n$ is identical (in the discrete case) to the signal $x(n) = 5\cos(0n) + 10\cos(\pi n)$.

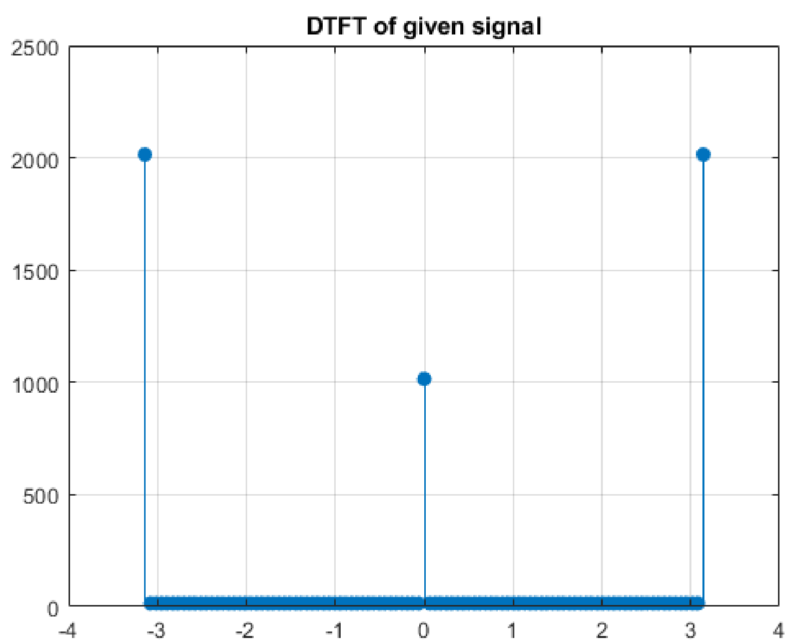

Figure 3: DTFT of signal.

Using the frequency response in MATLAB and the definition of the steady-state response, one can determine numerically the steady-state response of the system when $x(n) = 5 + 10(-1)^n$ is applied. The resulting plot can be found below.

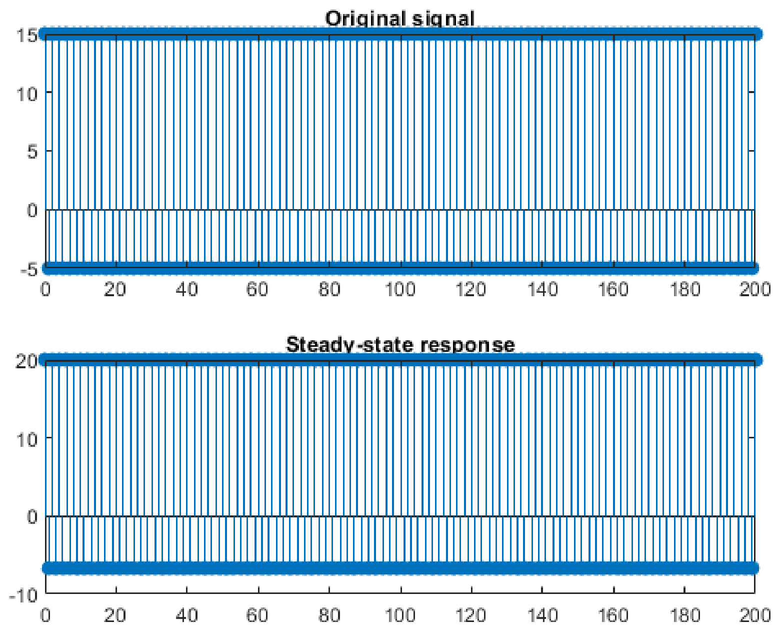

Figure 4: Magnitude and argument plot of steady-state response.

Part 2

The steady-state response for the same system when a signal of $x(n) = 1 + \cos(0.5\pi n + \pi/2)$ can be found below.

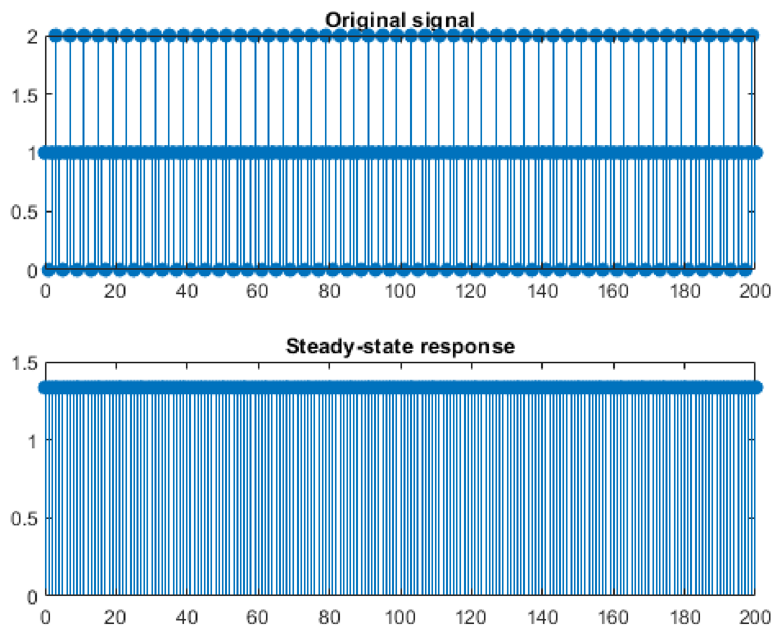

Figure 5: Magnitude and argument plot of steady-state response.

Problem 3.20

Part 1

Let $x_a(t) = 10\cos(10000\pi t)$. The angular frequency is $\omega_x = 10\,000\pi\ \mathrm{s}^{-1}$ (radian per second). If we divide by the ADC sample rate of 8000 Hz (samples per second), we arrive at the discrete angular frequency of 1.25π (unitless, or radian per sample). This translates to a discrete signal $x(n) = 10\cos(1.25\pi n)$.

Part 2

If the impulse response $h(n) = (-0.9)^n u(n)$, then the frequency response is $H\left(e^{j\omega}\right) = \frac{1}{1+0.9e^{-j\omega}}$. This comes from the DTFT pair

$$\mathcal{F}[\alpha^n u(n)] = \frac{1}{1-\alpha e^{-j\omega}}$$

for $-\pi \leq \omega \leq \pi$. The magnitude and argument of $H\left(e^{j\omega}\right)$ are calculated below.

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{1}{1+0.9e^{-j\omega}} \\
&= \frac{1}{(1+0.9\cos(\omega)) - j0.9\sin(\omega)} \cdot \frac{(1+0.9\cos(\omega)) + j0.9\sin(\omega)}{(1+0.9\cos(\omega)) + j0.9\sin(\omega)} \\
&= \frac{(1+0.9\cos(\omega)) + j0.9\sin(\omega)}{(1+0.9\cos(\omega))^2 - j^2(0.9\sin(\omega))^2} \\
&= \frac{(1+0.9\cos(\omega)) + j0.9\sin(\omega)}{1+1.8\cos(\omega)+0.81\cos^2(\omega)+0.81\sin^2(\omega)} \\
&= \frac{(1+0.9\cos(\omega)) + j0.9\sin(\omega)}{1.81+1.8\cos(\omega)} \\
&= \frac{1+0.9\cos(\omega)}{1.81+1.8\cos(\omega)} + j\frac{0.9\sin(\omega)}{1.81+1.8\cos(\omega)}
\end{aligned}
$$

$$
\begin{aligned}
\arg\left(H\left(e^{j\omega}\right)\right) &= \arctan\left(\frac{\operatorname{Im}\left(H\left(e^{j\omega}\right)\right)}{\operatorname{Re}\left(H\left(e^{j\omega}\right)\right)}\right) \\
&= \arctan\left(\frac{0.9\sin(\omega)}{1.81+1.8\cos(\omega)} \div \frac{1+0.9\cos(\omega)}{1.81+1.8\cos(\omega)}\right) = \arctan\left(\frac{0.9\sin(\omega)}{1+0.9\cos(\omega)}\right) \\
\left|H\left(e^{j\omega}\right)\right| &= \sqrt{\left(\frac{1+0.9\cos(\omega)}{1.81+1.8\cos(\omega)}\right)^2 + \left(\frac{0.9\sin(\omega)}{1.81+1.8\cos(\omega)}\right)^2} \\
&= \sqrt{\frac{1+1.8\cos(\omega)+0.81\cos^2(\omega)+0.81\sin^2(\omega)}{(1.81+1.8\cos(\omega))^2}} \\
&= \sqrt{\frac{1.81+1.8\cos(\omega)}{(1.81+1.8\cos(\omega))^2}} = \sqrt{\frac{1}{1.81+1.8\cos(\omega)}}
\end{aligned}
$$

Thus, $H\left(e^{j\omega}\right)$ can also be written as

$$
H\left(e^{j\omega}\right) = \frac{1}{1+0.9e^{-j\omega}} = \sqrt{\frac{1}{1.81+1.8\cos(\omega)}}\exp\left(j\arctan\left(\frac{0.9\sin(\omega)}{1+0.9\cos(\omega)}\right)\right)
$$

Below are four plots visually confirming the validity of this result. The left two plots graph the magnitude and argument of $1/(1+0.9e^{-j\omega})$, while the right two plots graph $\arg\left(H\left(e^{j\omega}\right)\right)$ and $\left|H\left(e^{j\omega}\right)\right|$, respectively, as derived above.

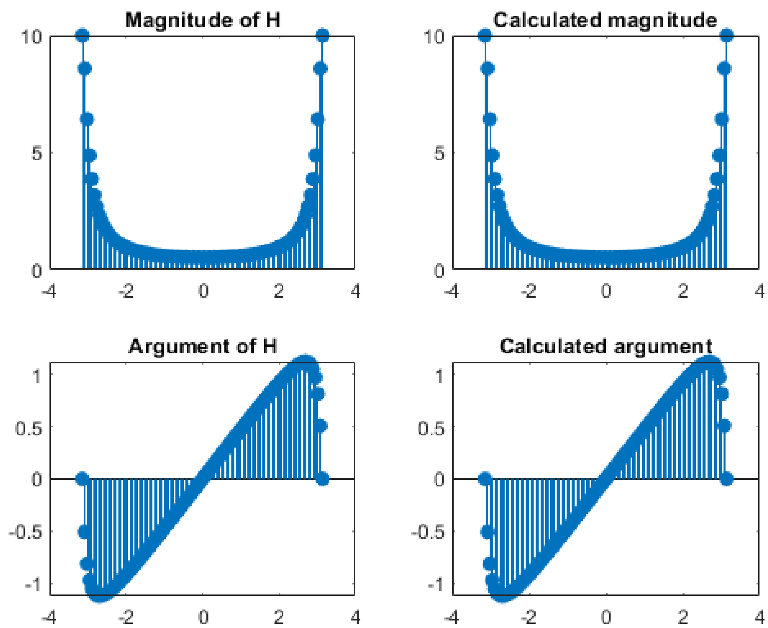

Figure 6: Magnitude and argument plots.

The steady-state response of $x(n) = 10\cos(1.25\pi n)$ is

$$\begin{aligned} y(n) &= 10\sqrt{\frac{1}{1.81 - 1.8/\sqrt{2}}}\cos\left(1.25\pi n + \arctan\left(\frac{-0.9/\sqrt{2}}{1 - 0.9/\sqrt{2}}\right)\right) \\ &\approx 13.644\cos(1.25\pi n - 1.052) \end{aligned}$$

Part 3

If the continuous signal is $x(t) = 5\sin(8000\pi t)$, then at a 8000 Hz ADC sample rate, the discrete signal will be $x(n) = 5\sin(\pi n)$. The steady-state response to this signal is

$$\begin{aligned} y(n) &= 5\sqrt{\frac{1}{1.81 - 1.8}}\sin\left(\pi n + \arctan\left(\frac{-0.9}{1 - 0.9}\right)\right) \\ &\approx 50\sin(\pi n - 1.460) \end{aligned}$$

Part 4

Both the magnitude and the argument of $H\left(e^{j\omega}\right)$ are functions of ω, and both functions are periodic with respect to 2π; this is because ω is contained solely within sine and cosine functions. Therefore, the discrete frequencies 3.25π and 5.25π. In other words, the signals $x(n) = 10\cos(3.25\pi n)$ and $x(n) = 10\cos(5.25\pi n)$ will produce the same steady-state response as the signal $x(n) = 10\cos(1.25\pi n)$.

Assuming the same ADC sample rate of 8000 Hz, this is achieved by analog radian frequencies of $26\pi\times10^3\ \mathrm{s}^{-1}$ and $42\pi\times10^3\ \mathrm{s}^{-1}$, respectively (the associated signals are $x(t) = 10\cos(26000\pi t)$ and $x(t) = 10\cos(42000\pi t)$, respectively).

Part 5

To prevent aliasing, a low-pass filter should be used. This is to remove any signals with frequencies greater than half of the desired Nyquist rate. Since the ADC is sampling at a rate of 8000 Hz, the filter would need to remove any signals with frequencies above 4000 Hz.

Problem 4.1

For the following parts, remember that the z-transform is defined as follows:

$$\mathcal{Z}[x(n)] \equiv X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Part 2

$$x(n) = 0.8^n u(n-2)$$

$$\begin{aligned}
X(z) &= \sum_{n=2}^{\infty} 0.8^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{0.8}{z}\right)^n - 0.8^0 z^0 - 0.8^1 z^{-1} \\
&= \frac{1}{1-0.8z^{-1}} - 1 - 0.8z^{-1}, \text{ROC: } |z| > 0.8 \\
&= \frac{1-(1-0.8z^{-1}) - 0.8z^{-1}(1-0.8z^{-1})}{1-0.8z^{-1}}, \text{ROC: } |z| > 0.8 \\
&= \frac{1-1+0.8z^{-1}-0.8z^{-1}+0.64z^{-2}}{1-0.8z^{-1}}, \text{ROC: } |z| > 0.8 \\
&= \frac{0.64z^{-2}}{1-0.8z^{-1}}, \text{ROC: } |z| > 0.8
\end{aligned}$$

Below is a plot of the original signal $x(n)$ and the z-transform filter when excited by a unit impulse at index $n = 2$. If the z-transform is truly correct, the two plots would be the same. In the plot, the two signals are identical save for a shift of two indexes. I could not figure out why this was the case.

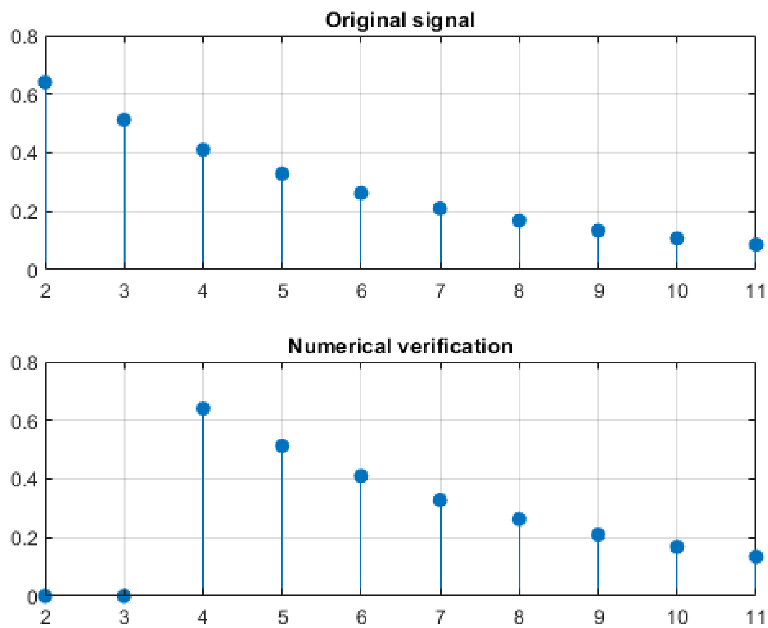

Figure 7: Verification of filter.

Part 3

$$x(n) = [0.5^n + (-0.8)^n]\, u(n)$$

$$X(z) = \sum_{n=0}^{\infty} (0.5^n + (-0.8)^n)\, z^{-n} = \sum_{n=0}^{\infty} 0.5^n z^{-n} + \sum_{n=0}^{\infty} (-0.8)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(0.5z^{-1}\right)^n + \sum_{n=0}^{\infty} (-0.8z^{-1})^n$$

$$= \frac{1}{1-0.5z^{-1}}, \text{ROC: } |z| > 0.5 + \frac{1}{1+0.8z^{-1}}, \text{ROC:} |z| > 0.8$$

$$= \frac{1}{1-0.5z^{-1}} + \frac{1}{1+0.8z^{-1}}, \text{ROC: } |z| > 0.8$$

The verification plots can be found below.

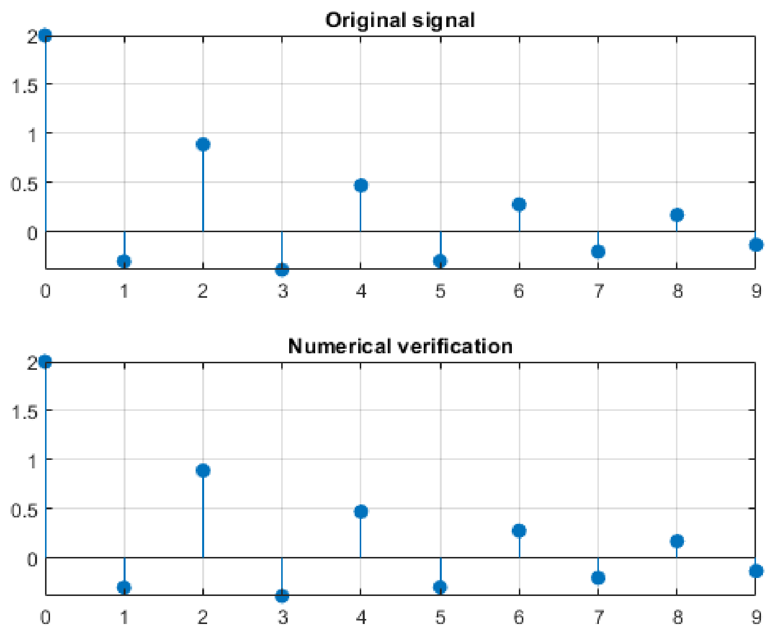

Figure 8: Verification of filter.

Problem 4.3

Part 1

$$x(n) = 2\delta(n-2) + 3u(n-3)$$

$$X(z) = 2z^{-2} + \frac{3z^{-3}}{1-z^{-1}}, \text{ROC: } |z| > 1$$

$$= \frac{2z^{-2}(1-z^{-1}) + 3z^{-}3}{1-z^{-1}}, \text{ROC: } |z| > 1$$

$$= \frac{2z^{-2} - 2z^{-3} + 3z^{-3}}{1-z^{-1}}, \text{ROC: } |z| > 1$$

$$= \frac{2z^{-2} + z^{-3}}{1-z^{-1}}, \text{ROC: } |z| > 1$$

The verification plots can be found below.

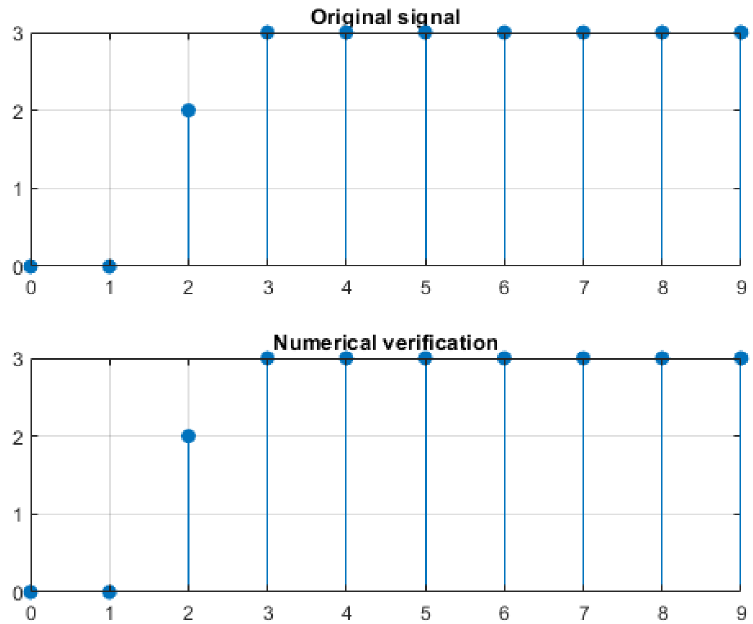

Figure 9: Verification of filter.

Part 2

$$
\begin{aligned}
x(n) &= 3(0.75)^n \cos(0.3\pi n)u(n) + 4(0.75)^n \sin(0.3\pi n)u(n)\\
X(z) &= \frac{3 - 2.25\cos(0.3\pi)z^{-1}}{1 - 1.5\cos(0.3\pi)z^{-1} + 0.5625z^{-2}}, \text{ROC: } |z| > 0.75\\
&+ \frac{3\sin(0.3\pi)z^{-1}}{1 - 1.5\cos(0.3\pi)z^{-1} + 0.5625z^{-2}}, \text{ROC: } |z| > 0.75\\
&= \frac{3 + (3\sin(0.3\pi) - 2.25\cos(0.3\pi))z^{-1}}{1 - 1.5\cos(0.3\pi)z^{-1} + 0.5625z^{-2}}, \text{ROC: } |z| > 0.75
\end{aligned}
$$

The verification plots can be found below.

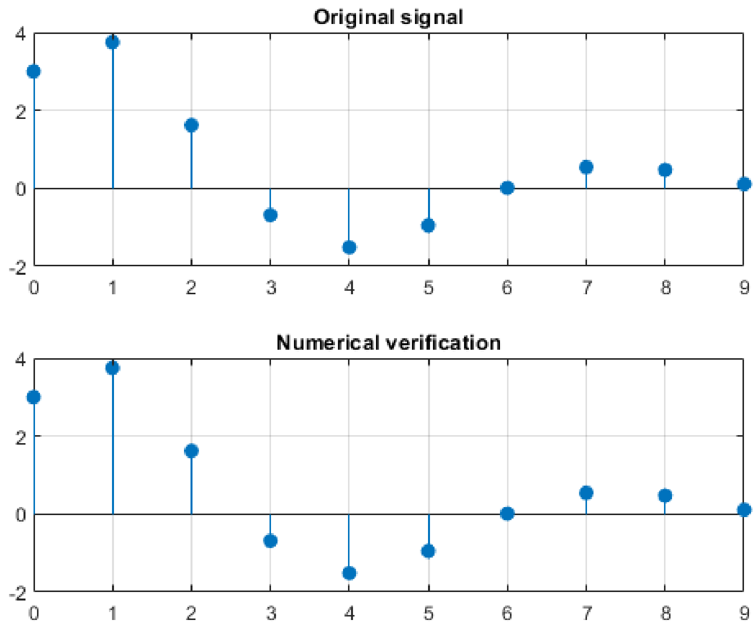

Figure 10: Verification of filter.

Problem 4.5

For the following problems, let $\mathcal{Z}[x(n)] = X(z) = \frac{1}{1+0.5z^{-1}}$, ROC:$|z| > 0.5$.

Part 1

$$\begin{aligned} x_1(n) &= x(-n+3) + x(n-3) \\ X_1(z) &= X(1/z)z^3, \text{ROC: } 0 < |z| < 2 + X(z)z^{-3}, \text{ROC:} |z| > 0.5 \\ &= \frac{z^3}{1+0.5z} + \frac{z^{-3}}{1+0.5x^{-1}}, \text{ROC: } 0.5 < |z| < 2 \end{aligned}$$

Part 2

$$x_2(n) = (1+n+n^2)x(n) = x(n) + nx(n) + n^2x(n)$$
$$X_2(z) = X(z), \text{ROC:}|z| > 0.5 + \frac{-2z}{(2z+1)^2}, \text{ROC:}|z| > 0.5$$
$$+\frac{-2z(2z-1)}{(2z+1)^3}$$
$$X_2(z) = \frac{1}{1+0.5z^{-1}} - \frac{2z}{(2z+1)^2} - \frac{2z(2z-1)}{(2z+1)^3}, \text{ROC:}|z| > 0.5$$

Part 3

$$x_3(n) = (0.5)^n\, x(n-2)$$
$$X_3(z) = X(2z)z^{-2}, \text{ROC: } |z| > 0.25$$
$$= \frac{z^{-2}}{1+0.5(2z)^{-1}}, \text{ROC: } |z| > 0.25$$
$$= \frac{z^{-2}}{1+0.25z^{-1}}, \text{ROC: } |z| > 0.25$$

Part 4

$$x_4(n) = x(n+2) * x(n-2)$$
$$X_4(z) = X(z)z^2X(z)z^{-2}, \text{ROC: } |z| > 0.5$$
$$= \left(\frac{1}{1+0.5z^{-1}}\right)^2, \text{ROC: } |z| > 0.5$$
$$= \frac{1}{1+z^{-1}+0.25z^{-2}}, \text{ROC: } |z| > 0.5$$

Part 5

$$x_5(n) = \cos(\pi n/2)x^*(n)$$
$$X_5(n) = \frac{1}{2\pi j}\oint_C \mathcal{Z}[\cos(\pi n/2)]X^*(z^*/v)v^{-1}\,dv,$$
$$\text{ROC:}|z| > 0.5$$