Rodrigo Becerril Ferreyra
Dr. Alireza Mehrnia
Homework 2
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The code used to generate all plots is given at the end of this document.

P2.2

Below are the plots generated according to the instructions in P2.2 of the class textbook.

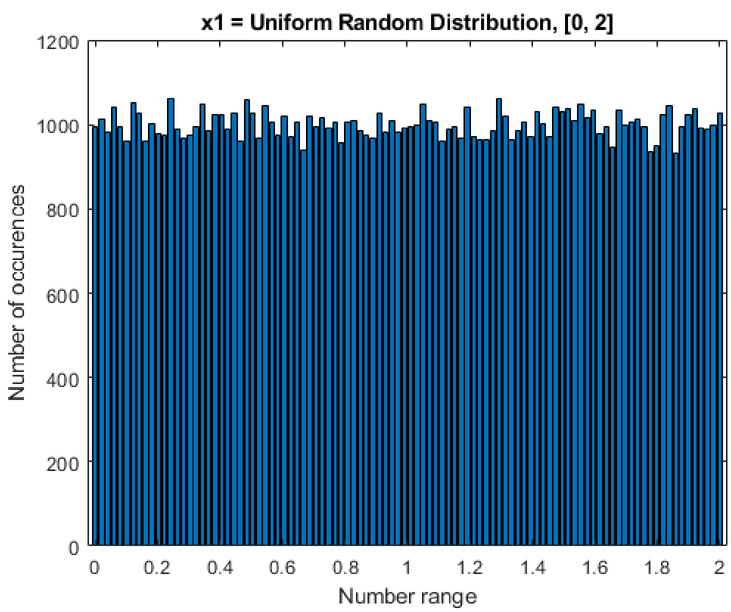

Figure 1: P2.2.1

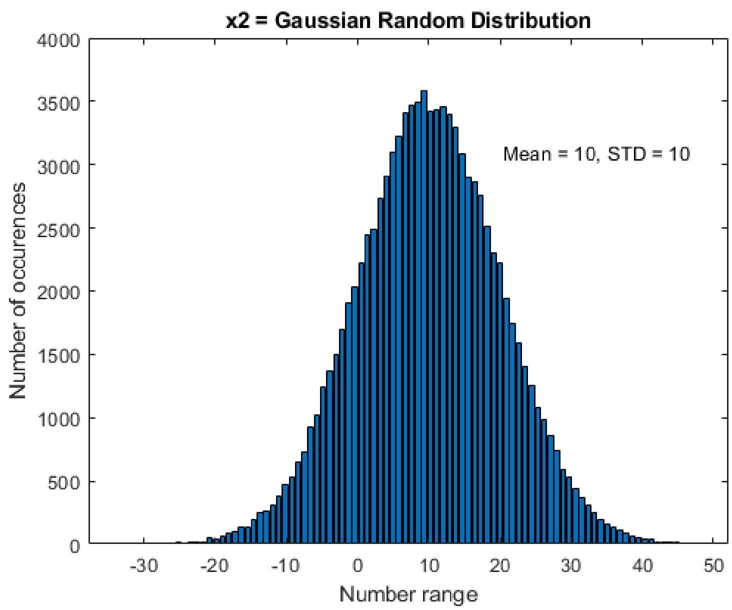

Figure 2: P2.2.2

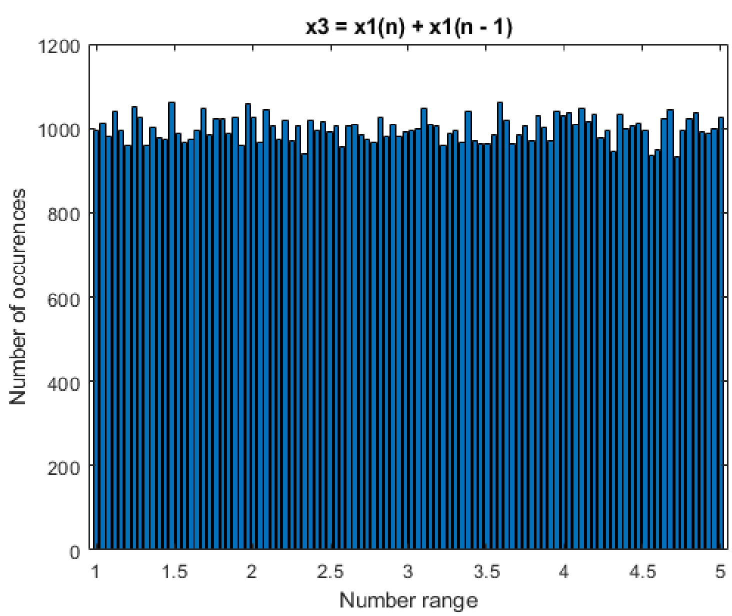

Figure 3: P2.2.3

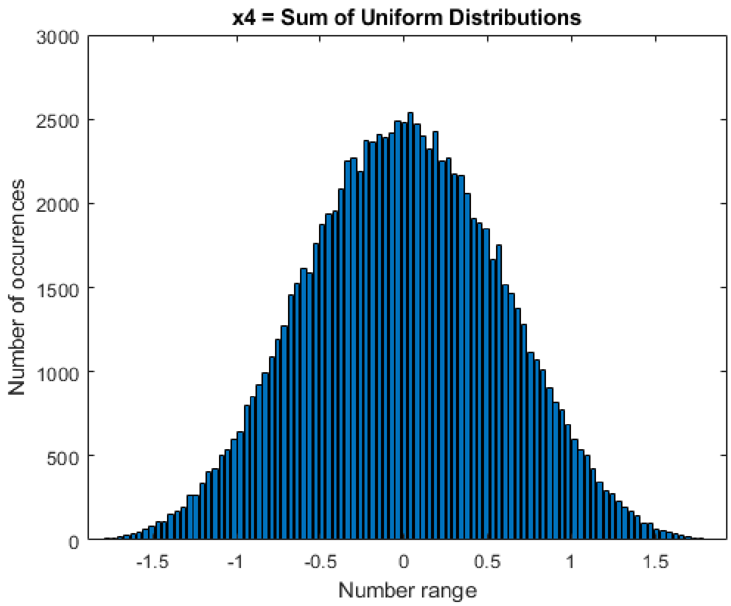

Figure 4: P2.2.4

Figures 3 and 4 were generated similarly: They are sums of multiple random variables uniformly distributed. Figure 3 is the sum of the random variable shown in Figure 1 added with an index-shifted version of itself, while Figure 4 is the sum of four individual random variables uniformly distributed over the range $[-1/2, 1/2]$. The difference between the two plots is self-evident: the former resembles a uniform distribution, while the latter resembles a normal distribution. This is due to the central limit theorem: informally, as one sums independent random variables with the same distribution, the sum of the random variables is itself normally distributed.

P2.3

The following are plots generated for sections 2 and 3 of Problem 2.3.

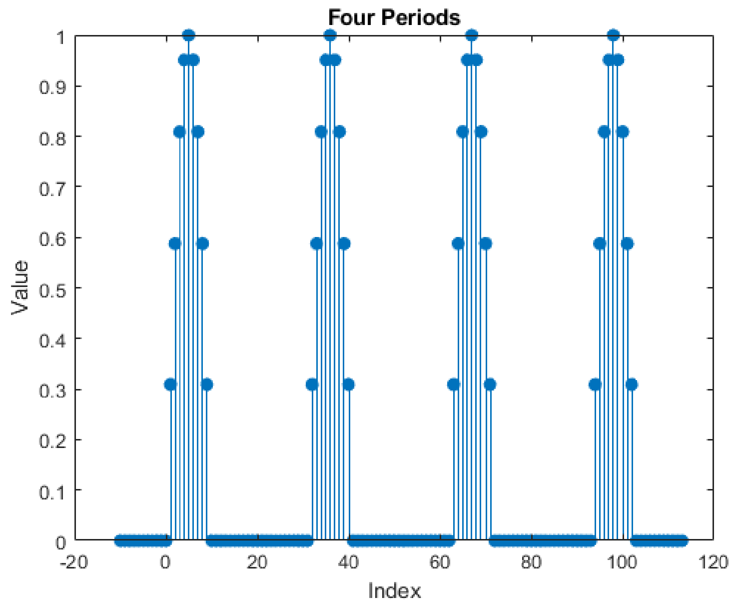

Figure 5: P2.3.3

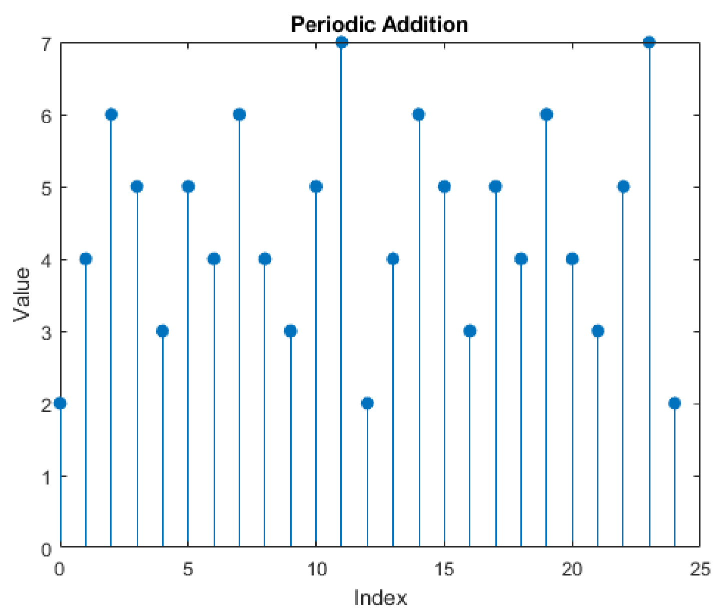

Figure 6: P2.3.4

The period of Figure 6 is the number of indexes between the lowest two points (in this case, when the value is 2); this means that the period is 12. The value 2 occurs at index 0, 12, and 24. This is because the periods of the two periodic signals used to create this plot are 3 and 4 (and $3 \times 4 = 12$).

P2.5

The following theorem was given: "The complex exponential sequence $\exp(j\omega_0 n)$ or the sinusoidal sequence $\cos(\omega_0 n)$ are periodic if the *normalized* frequency $f_0 \equiv \omega_0/2\pi$ is a rational number; that is, $f_0 = K/N$ such that $K, N \in \mathbb{Z}$."

Note that the following proof is for $\cos(\omega_0 n)$, but applies to both the real and imaginary parts of $\exp(j\omega_0 n)$ as well, due to Euler's formula that states that $\mathrm{Re}\left(e^{jx}\right) = \cos(x)$ and $\mathrm{Im}\left(e^{jx}\right) = \sin(x)$. In addition, $\sin(\pi/2 - x) = \cos(x)$. Therefore, this proof applies to both cases.

In the continuous case, it is always true that a sinusoid such as the cosine function is periodic. However, for the discrete case, the normalized frequency $f_0 \equiv \omega_0/2\pi$ must be a rational number. Remember that a sequence $x(n)$ is periodic if

$$x[n] = x[n + P] \text{ for all } n, \tag{1}$$

where P is the fundamental period (textbook, p. 25). In the case of $f_0 \equiv K/N$, N will be the fundamental period (i.e., $x(n) = x(n+N)$ for all n). For example, if $f_0 = 3/8$, then 8 will be the fundamental period ($x(n) = x(n+8)$ for all n).[1] The number 3 is the amount of continuous periods the sinusoid must go through before a period starts on an integer index. For example, for the signal $x(n) = \cos(0.75\pi n)$, $x(0) = 1$. $x(8/3)$, $x(2 \times 8/3)$, and $x(3 \times 8/3)$ all also equal 1, but $3 \times 8/3 = 8$ is the only index that is an integer and thus valid for a discrete signal; in the discrete case, $x[0] = 1$, and the first sample that will also equal 1 is $x[8]$; thus, 8 will be the fundamental period. If f_0 is not a rational number, N is not defined/does not exist. Therefore, for a discrete signal to be periodic, f_0 must be rational.

The following are plots generated for Problem 2.5. Figure 7 was generated using $z = \exp(j0.1\pi n)$, and Figure 8 was generated using $x = \cos(0.1n)$.

[1] This was discovered numerically by using the Desmos graphing calculator to graph sinusoids, available at https://www.desmos.com/calculator.

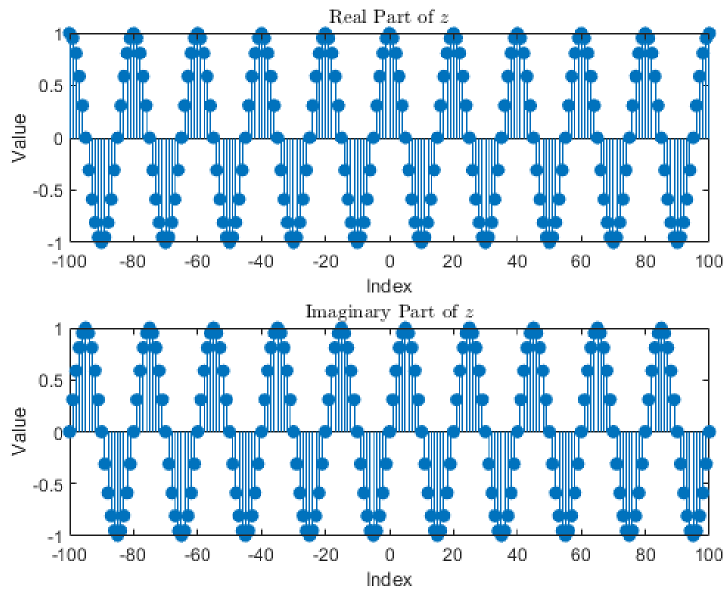

Figure 7: P2.5.2

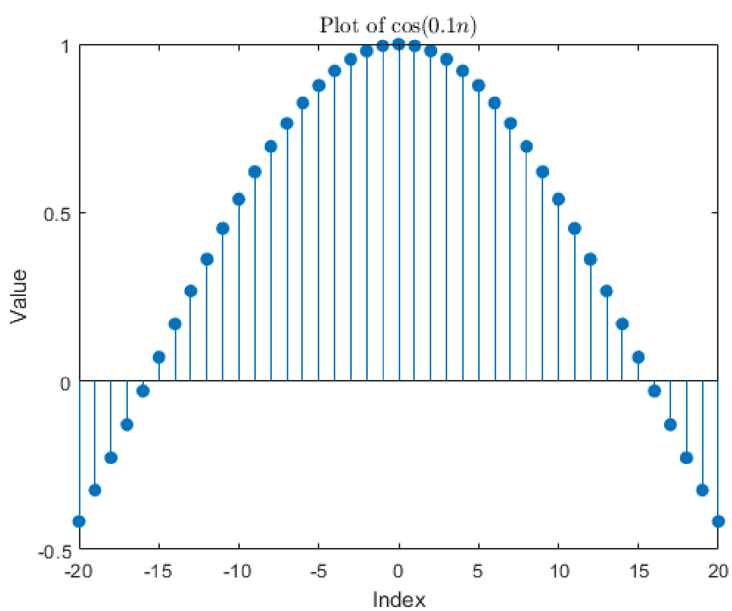

Figure 8: P2.5.3

Both the real and imaginary parts of Figure 7 are indeed periodic. The normalized frequency is

$$f_0 = \frac{0.1\pi}{2\pi} = \frac{1}{20}.$$

Thus, $K = 1$ and $N = 20$, and its fundamental period is $N = 20$. This can be seen graphically by noticing that the value of zero is achieved on the indexes that are integer multiples of 20 ($\{20k \mid k \in \mathbb{Z}\}$).

The sinusoid presented in Figure 8 is not periodic. It certainly looks like it is, but numerically, it does not satisfy the definition of periodicity (Equation (1)). Its normalized frequency is

$$f_0 = \frac{0.1}{2\pi} = \frac{1}{20\pi}.$$

$K = 1$ and $N = 20\pi$, so its fundamental frequency is 20π. However, since this is not an integer, it cannot be used in the practice of discrete signals, where all indexes are integers. Thus, it is not periodic.

P2.6

The following are plots generated for sections 3 and 4 of Problem 2.6. These graphs were obtained by using the `evenodd` function provided in the textbook (p. 34).

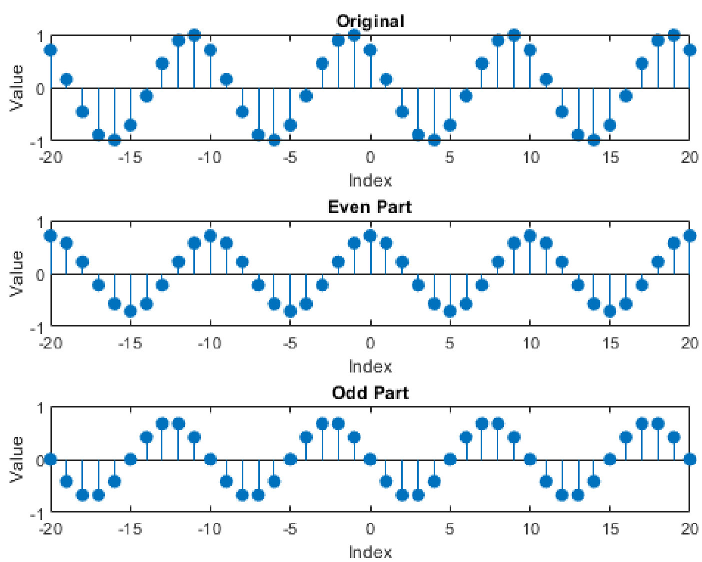

Figure 9: P2.6.3

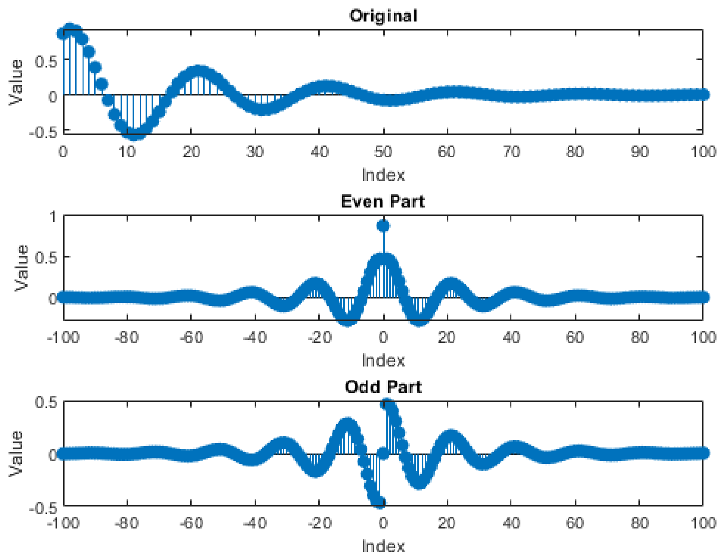

Figure 10: P2.6.4

P2.10

Let $y(n) = x(n) + \alpha x(n-k)$, where $x(n)$ is a signal, and $\alpha x(n-k)$ is its echo (noise). The cross-correlation between y and x is given by

$$r_{yx}(\ell) = \sum_{n=-\infty}^{\infty} y(n)x(n-\ell)$$

by definition (the complex conjugate is not needed because both functions are real-valued). Substituting $y(n) = x(n) + \alpha x(n-k)$, we have

$$\begin{aligned} r_{yx}(\ell) &= \sum_{n=-\infty}^{\infty} (x(n) + \alpha x(n-k))x(n-\ell) \\ &= \sum_{n=-\infty}^{\infty} [x(n)x(n-\ell) + \alpha x(n-k)x(n-\ell)] \\ &= \sum_{n=-\infty}^{\infty} x(n)x(n-\ell) + \sum_{n=-\infty}^{\infty} \alpha x(n-k)x(n-\ell) \\ &= (x(n) \star x(n))(\ell) + \alpha(x(n-k) \star x(n))(\ell) \end{aligned}$$

where $(x(n) \star x(n))(\ell) = r_{xx}(\ell)$ is the autocorrelation of $x(n)$.

To generate Figure 11, I used the following definitions:

$$x(n) = \cos(0.2\pi n) + 0.5\cos(0.6\pi n)$$
$$\alpha = 0.1, k = 50.$$

Note that the plot of $x(n)$ ends at index 200, and the plot of $\alpha x(n-k)$ starts at index 50. Theoretically, you can find the value of k by observing where $r_{yx}(\ell)$ reaches its maximum. It is not possible to find the value of α, however.

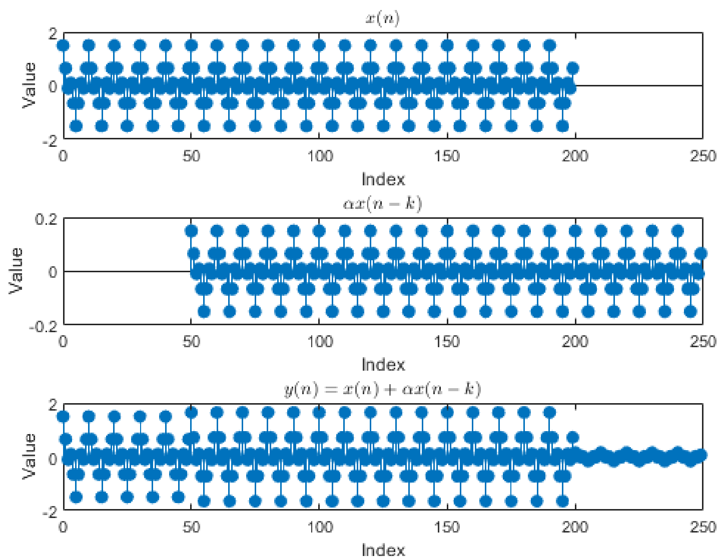

Figure 11: P2.10.2

P2.11

System 1

$$T[x(n)] = x(n)u(n)$$
$$aT[x(n)] + bT[y(n)] = ax(n)u(n) + by(n)u(n)$$
$$T[ax(n) + by(n)] = (ax(n) + by(n))u(n) = ax(n)u(n) + by(n)u(n)$$
$$ax(n)u(n) + by(n)u(n) \stackrel{?}{=} ax(n)u(n) + by(n)u(n)$$

The above equality is satisfied; thus, System 1 is indeed linear.

System 2

$$
\begin{aligned}
T[x(n)] &= x(n) + nx(n+1) \\
aT[x(n)] + bT[y(n)] &= ax(n) + anx(n+1) + by(n) + bny(n+1) \\
T[ax(n) + by(n)] &= ax(n) + by(n) + n(ax(n+1) + by(n+1)) \\
&= ax(n) + by(n) + anx(n+1) + bny(n+1) \\
&ax(n) + by(n) + anx(n+1) + bny(n+1) \\
&\stackrel{?}{=} ax(n) + anx(n+1) + by(n) + bny(n+1)
\end{aligned}
$$

The above equality is satisfied; thus, System 2 is indeed linear.

System 3

$$
\begin{gathered}
T[x(n)] = x(n) + \frac{1}{2}x(n-2) - \frac{1}{3}x(n-3)x(2n) \\
aT[x(n)] + bT[y(n)] = ax(n) + \frac{a}{2}x(n-2) - \frac{a}{3}x(n-3)x(2n) \\
+by(n) + \frac{b}{2}y(n-2) - \frac{b}{3}y(n-3)y(2n) \\
T[ax(n) + by(n)] \\
= ax(n) + by(n) + \frac{1}{2}(ax(n-2) + by(n-2)) \\
-frac13(ax(n-3) + by(n-3))(ax(2n) + by(2n))
\end{gathered}
$$

I can stop here, because as you can see on the last line, I am going to get a^2, which is not present in $aT[x(n)] + bT[y(n)]$. Therefore, System 3 is not linear.

System 4

$$
\begin{gathered}
T[x(n)] = \sum_{k=-\infty}^{n+5} 2x(k) \\
aT[x(n)] + bT[y(n)] = 2a \sum_{k=-\infty}^{n+5} x(k) + 2b \sum_{k=-\infty}^{n+5} y(k) \\
T[ax(n) + by(n)] = 2 \sum_{k=-\infty}^{n+5} (ax(k) + by(k)) \\
= 2a \sum_{k=-\infty}^{n+5} x(k) + 2b \sum_{k=-\infty}^{n+5} y(k) \\
aT[x(n)] + bT[y(n)] \stackrel{?}{=} T[ax(n) + by(n)]
\end{gathered}
$$

The above equality is satisfied; thus, System 4 is indeed linear.

System 5

$$
\begin{gathered}
T[x(n)] = x(2n) \\
aT[x(n)] + bT[y(n)] = ax(2n) + by(2n) \\
T[ax(n) + by(n)] = ax(2n) + by(2n) \\
aT[x(n)] + bT[y(n)] \stackrel{?}{=} T[ax(n) + by(n)]
\end{gathered}
$$

The above equality is satisfied; thus, System 5 is indeed linear.

System 6

$$
\begin{gathered}
T[x(n)] = \mathrm{round}(x(n)) \\
aT[x(n)] + bT[y(n)] = a\,\mathrm{round}(x(n)) + b\,\mathrm{round}(y(n)) \\
T[ax(n) + by(n)] = \mathrm{round}(ax(n) + by(n)) \\
aT[x(n)] + bT[y(n)] \stackrel{?}{=} T[ax(n) + by(n)]
\end{gathered}
$$

The above equality is not satisfied; thus, System 6 is not linear.

System 7

$$
\begin{gathered}
T[x(n)] = x(-n) \\
aT[x(n)] + bT[y(n)] = ax(-n) + by(-n) \\
T[ax(n) + by(n)] = ax(-n) + by(-n) \\
aT[x(n)] + bT[y(n)] \stackrel{?}{=} T[ax(n) + by(n)]
\end{gathered}
$$

Yes this is linear.

$$
\begin{gathered}
T[x(n-k)] = x(-n-k) \\
y(n-k) = x(-(n-k)) = x(-n+k)
\end{gathered}
$$

Time variant.