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Homework 3
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The code used to generate the following results can be found at the end of this document.

# P3.1

Below are the plots of magnitude and argument for the DTFT of the given sequences.

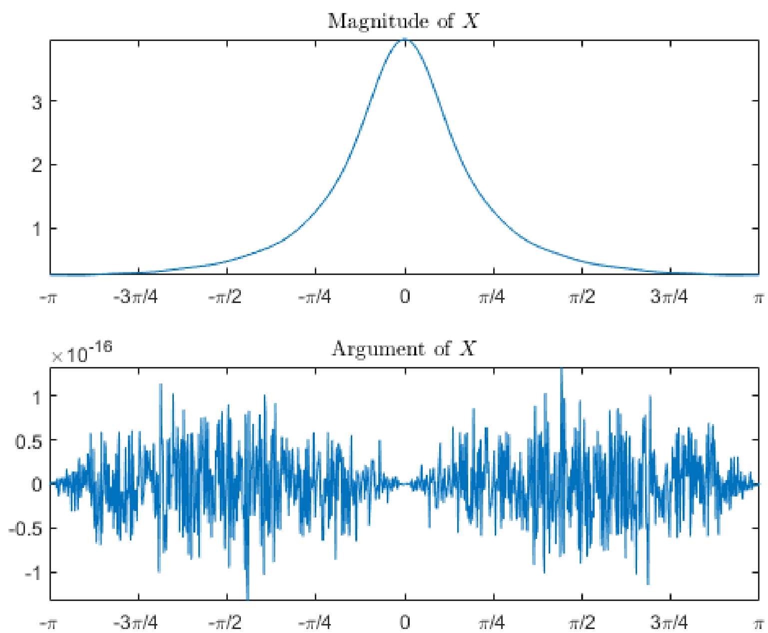


Figure 1: Problem 3.1 Part 1

Note how this DTFT has very little (negligible) argument; for all purposes, the DTFT is purely real.

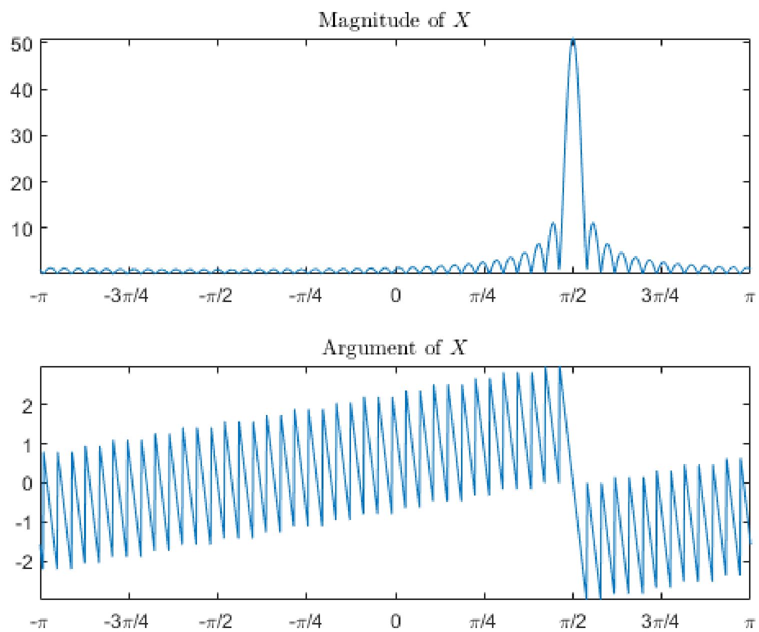


Figure 2: Problem 3.1 Part 3

Note how both the magnitude and argument seem to be centered around $\pi/2$.

# P3.3

The following DTFTs were found analytically (using the definition of DTFT), then plotted using MATLAB. The definition is as follows:

$$\mathcal{F}(x[n]) = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{1}$$

## Part 2

In this example, $x[n] = 0.6^{|n|}(u(n+10) - u(n-11))$, which has the following graph. Note that the value is zero at all indexes $n < -10$ and $n > 10$, where $n$ is the index.

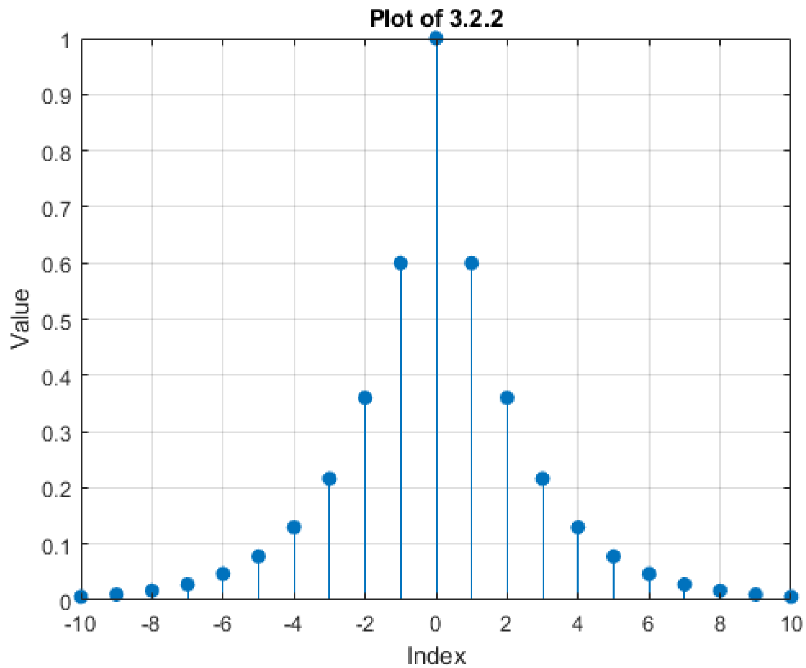


Figure 3: Graph of $x[n]$.

Below is the calculation for the DTFT.

$$
\begin{gathered}
\mathcal{F}[0.6^{|n|}(u(n+10)-u(n-11))] \\
= X\left(e^{j\omega}\right) = \sum_{n=-10}^{10} 0.6^{|n|} e^{-j\omega n} \\
= 0.6^{10}e^{10j\omega} + 0.6^{9}e^{9j\omega} + 0.6^{8}e^{8j\omega} + 0.6^{7}e^{7j\omega} + 0.6^{6}e^{6j\omega} \\
+0.6^{5}e^{5j\omega} + 0.6^{4}e^{4j\omega} + 0.6^{3}e^{3j\omega} + 0.6^{2}e^{2j\omega} + 0.6e^{j\omega} + 1 \\
+0.6e^{-j\omega} + 0.6^{2}e^{-2j\omega} + 0.6^{3}e^{-3j\omega} + 0.6^{4}e^{-4j\omega} + 0.6^{5}e^{-5j\omega} \\
+0.6^{6}e^{-6j\omega} + 0.6^{7}e^{-7j\omega} + 0.6^{8}e^{-8j\omega} + 0.6^{9}e^{-9j\omega} + 0.6^{10}e^{-10j\omega}
\end{gathered}
$$

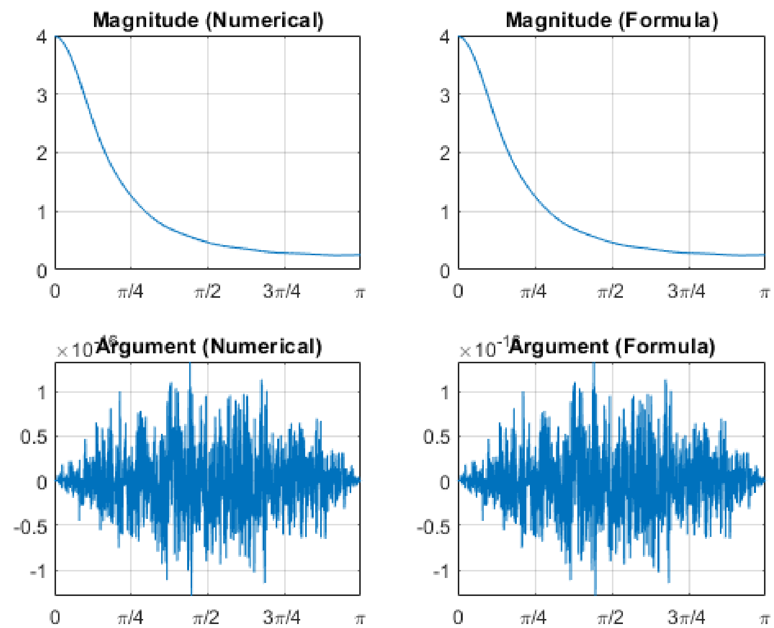


Figure 4: Problem 3.3 Part 2

## Part 5

In this example, $x[n] = 4(-0.7)^n \cos(0.25\pi n)u(n)$. The graph of this is below.

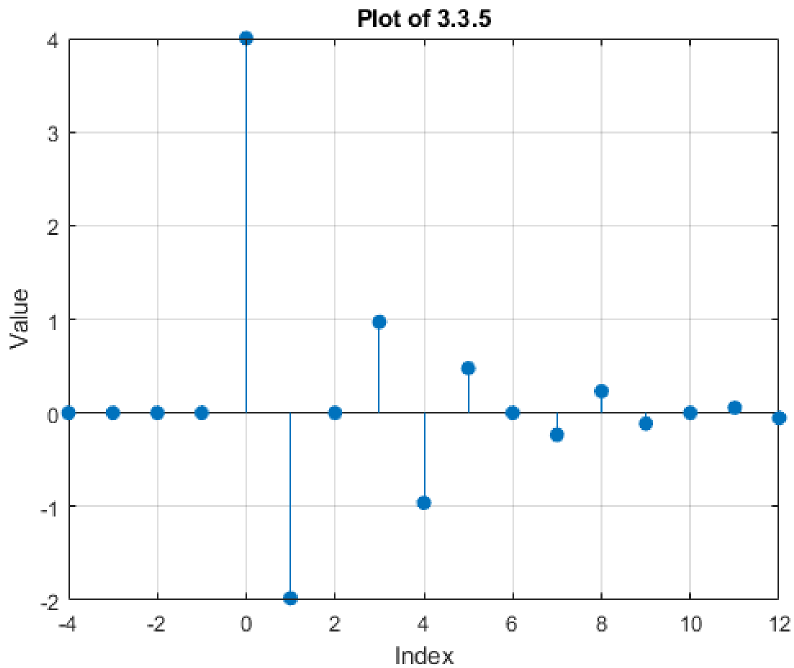


Figure 5: Graph of $x[n]$.

Below is the calculation for the DTFT.

$$
\begin{aligned}
X\left(e^{j\omega}\right) &= \sum_{n=0}^{\infty} 4(-0.7)^n \cos(0.25\pi n) e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} 2(-0.7)^n \left(e^{j0.25\pi n} + e^{-j0.25\pi n}\right) e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} 2(-0.7)^n \left(e^{j\pi n/4} e^{-j\omega n} + e^{-j\pi n/4} e^{-j\omega n}\right) \\
&= \sum_{n=0}^{\infty} \left(2(-0.7)^n e^{j\pi n/4} e^{-j\omega n} + 2(-0.7)^n e^{-j\pi n/4} e^{-j\omega n}\right) \\
&= \sum_{n=0}^{\infty} 2(-0.7)^n e^{j\pi n/4} e^{-j\omega n} + \sum_{n=0}^{\infty} 2(-0.7)^n e^{-j\pi n/4} e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} 2\left[(-0.7) e^{j\pi/4} e^{-j\omega}\right]^n + \sum_{n=0}^{\infty} 2\left[(-0.7) e^{-j\pi/4} e^{-j\omega}\right]^n \\
&= \frac{2}{1 + 0.7 e^{j\pi/4} e^{-j\omega}} + \frac{2}{1 + 0.7 e^{-j\pi/4} e^{-j\omega}}
\end{aligned}
$$

The summation reduction formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ only works if $|r| < 1$; in this case, since $r$ is the complex number $(-0.7)e^{\pm j\pi/4 - j\omega}$, its magnitude $|r| = 0.7$, making the summation reduction valid.

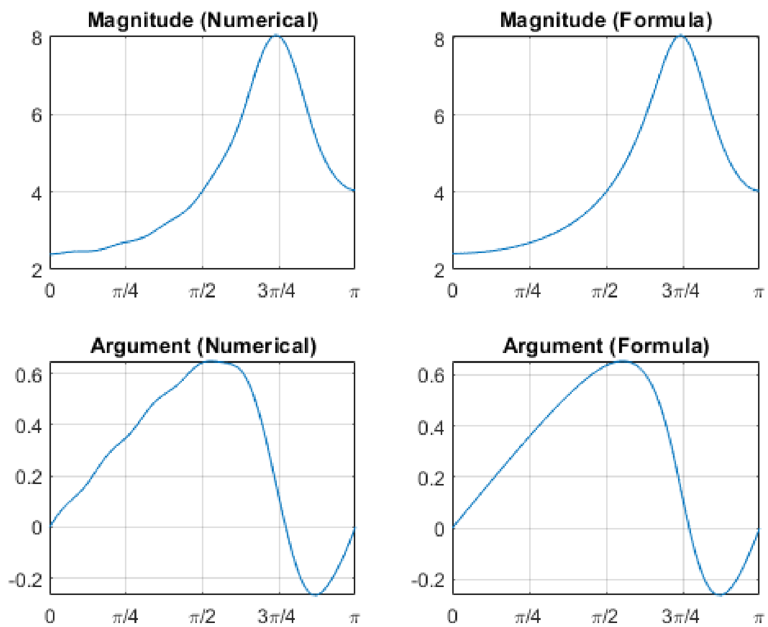


Figure 6: Problem 3.3 Part 5

# P3.4

Below is the plot for the rectangular window function for $M = 50$. Other values of $M$ can be obtained via the code.

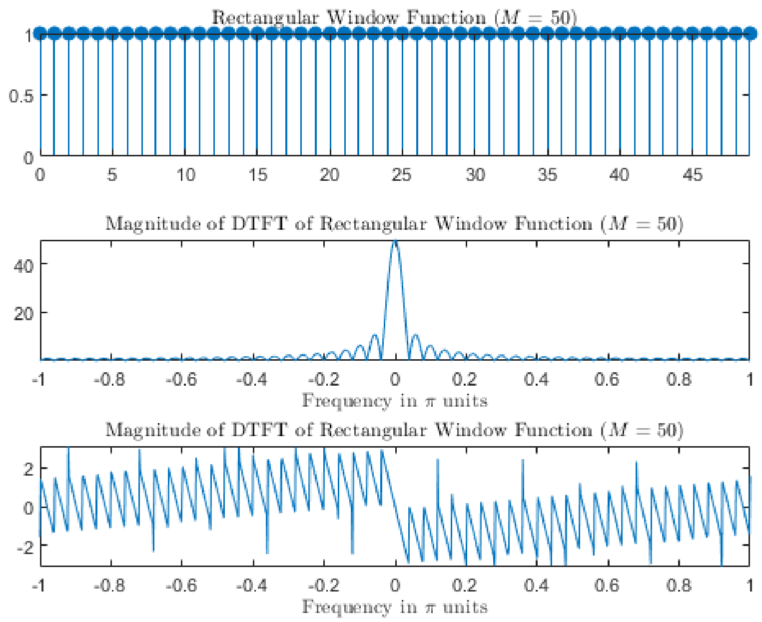


Figure 7: Problem 3.4 Part 1

Similarly, below is the plot for the rectangular window function for $M = 50$. Other values of $M$ can be obtained via the code.

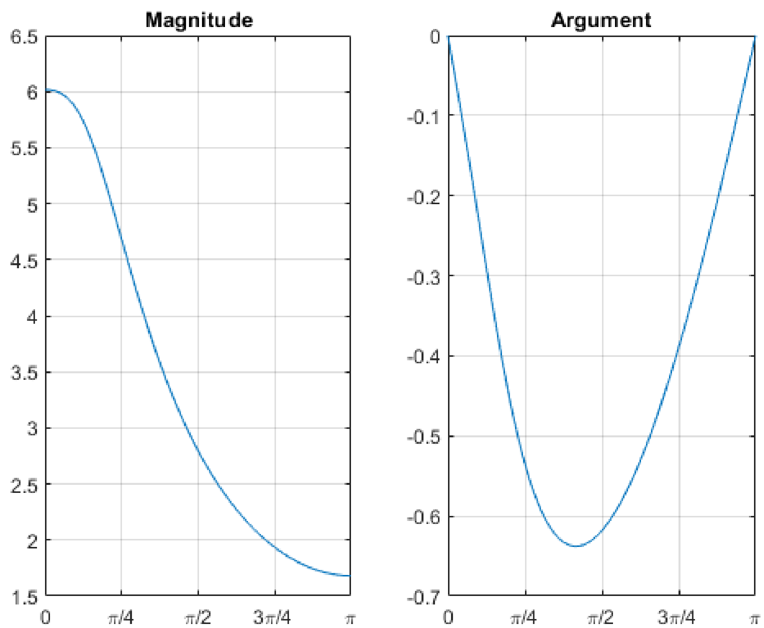


Figure 8: Problem 3.4 Part 4

# P3.5

Since the DTFT is an invertible transformation, one can find the original signal $x[n]$ given the transform $\mathcal{F}(x[n])$.

## Part 1

$$
\begin{gathered}
X\left(e^{j\omega}\right) = 3 + 2\cos(\omega) + 4\cos(2\omega) \\
= 3 + 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 4\left(\frac{e^{2j\omega} + e^{-2j\omega}}{2}\right) \\
= 3 + e^{j\omega} + e^{-j\omega} + 2e^{2j\omega} + 2e^{-2j\omega} \\
= 2e^{2j\omega} + e^{j\omega} + 3 + e^{-j\omega} + 2e^{-2j\omega} \\
x[n] = \{2, 1, \underset{\uparrow}{3}, 1, 2\}
\end{gathered}
$$

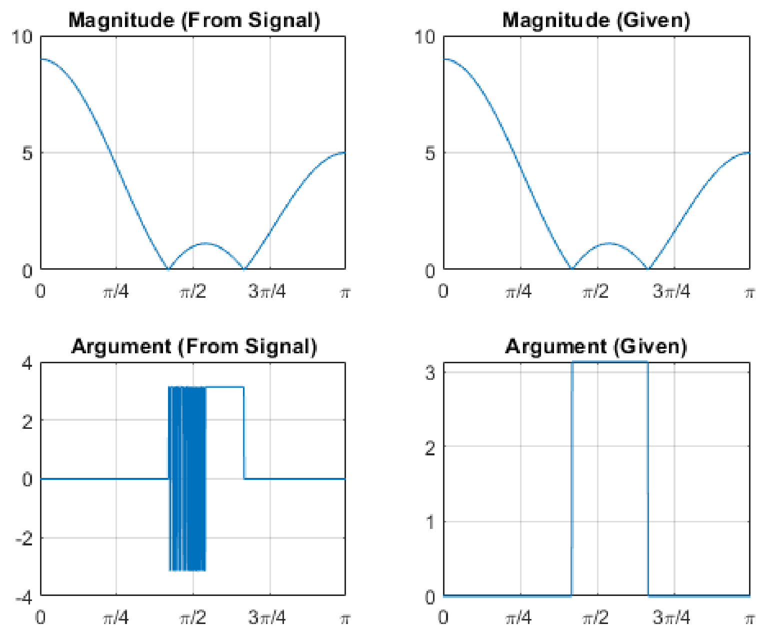


Figure 9: Problem 3.5 Part 1

## Part 2

$$
\begin{aligned}
X\left(e^{j\omega}\right) &= \left[1 - 6\cos(3\omega) + 8\cos(5\omega)\right] e^{-3j\omega} \\
&= \left[1 - 6\left(\frac{e^{3j\omega} + e^{-3j\omega}}{2}\right) + 8\left(\frac{e^{5j\omega} + e^{-5j\omega}}{2}\right)\right] e^{-3j\omega} \\
&= \left[1 - 3e^{3j\omega} - 3e^{-3j\omega} + 4e^{5j\omega} + 4e^{-5j\omega}\right] e^{-3j\omega} \\
&= e^{-3j\omega} - 3e^{3j\omega-3j\omega} - 3e^{-3j\omega-3j\omega} + 4e^{5j\omega-3j\omega} + 4e^{-5j\omega-3j\omega} \\
&= e^{-3j\omega} - 1 - 3e^{-6j\omega} + 4e^{2j\omega} + 4e^{-8j\omega} \\
&= 4e^2 j\omega - 1 + e^{-3j\omega} - 3e^{-6j\omega} + 4e^{-8j\omega} \\
x[n] &= \{4, 0, \underset{\uparrow}{-1}, 0, 0, 1, 0, 0, -3, 0, 4\}
\end{aligned}
$$

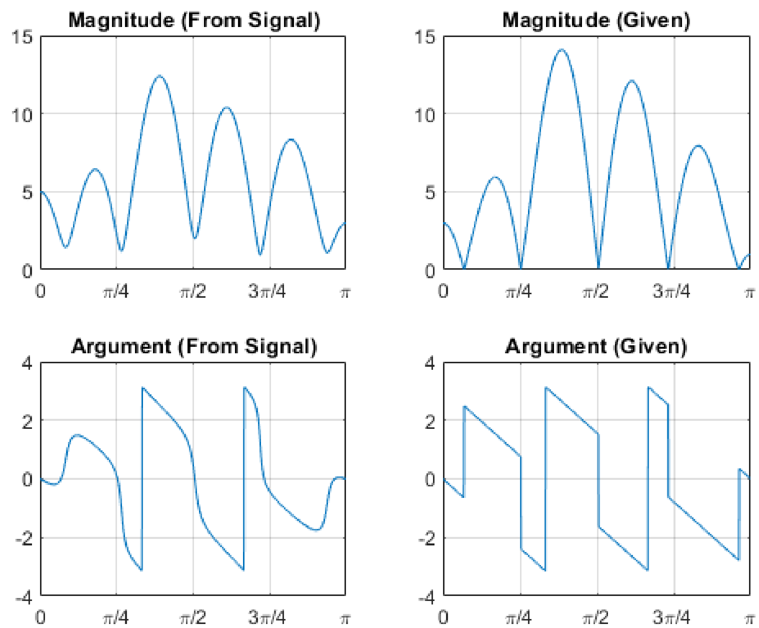


Figure 10: Problem 3.5 Part 2

## Part 4

$$
\begin{gathered}
X\left(e^{j\omega}\right) = \left[1 + 2\cos(\omega) + 3\cos(2\omega)\right]\cos(\omega/2)e^{-5j\omega/2} \\
= \cos(\omega/2)e^{-5j\omega/2} + 2\cos(\omega)\cos(\omega/2)e^{-5j\omega/2} + 3\cos(2\omega)\cos(\omega/2)e^{-5j\omega/2} \\
= \cos(\omega/2)e^{-5j\omega/2} + \left[\cos(\omega/2) + \cos(3\omega/2)\right]e^{-5j\omega/2} \\
+\frac{3}{2}\left[\cos(3\omega/2) + \cos(5\omega/2)\right]e^{-5j\omega/2} \\
= 2\cos(\omega/2)e^{-5j\omega/2} + \cos(3\omega/2)e^{-5j\omega/2} \\
+\frac{3}{2}\cos(3\omega/2)e^{-5j\omega/2} + \frac{3}{2}\cos(5\omega/2)e^{-5j\omega/2} \\
= \left(e^{j\omega/2} + e^{-j\omega/2}\right)e^{-5j\omega/2} + \frac{1}{2}\left(e^{3j\omega/2} + e^{-3j\omega/2}\right)e^{-5j\omega/2} \\
+\frac{3}{4}\left(e^{3j\omega/2} + e^{-3j\omega/2}\right)e^{-5j\omega/2} + \frac{3}{4}\left(e^{5j\omega/2} + e^{-5j\omega/2}\right)e^{-5j\omega/2} \\
= e^{-2j\omega} + e^{-3j\omega} + \frac{1}{2}e^{-j\omega} + \frac{1}{2}e^{-4j\omega} + \frac{3}{4}e^{-j\omega} + \frac{3}{4}e^{-4j\omega} + \frac{3}{4} + \frac{3}{4}e^{-5j\omega} \\
= \frac{3}{4} + \frac{5}{4}e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + \frac{5}{4}e^{-4j\omega} + \frac{3}{4}e^{-5j\omega} \\
x[n] = \left\{\underset{\uparrow}{\frac{3}{4}}, \frac{5}{4}, 1, 1, \frac{5}{4}, \frac{3}{4}\right\}
\end{gathered}
$$

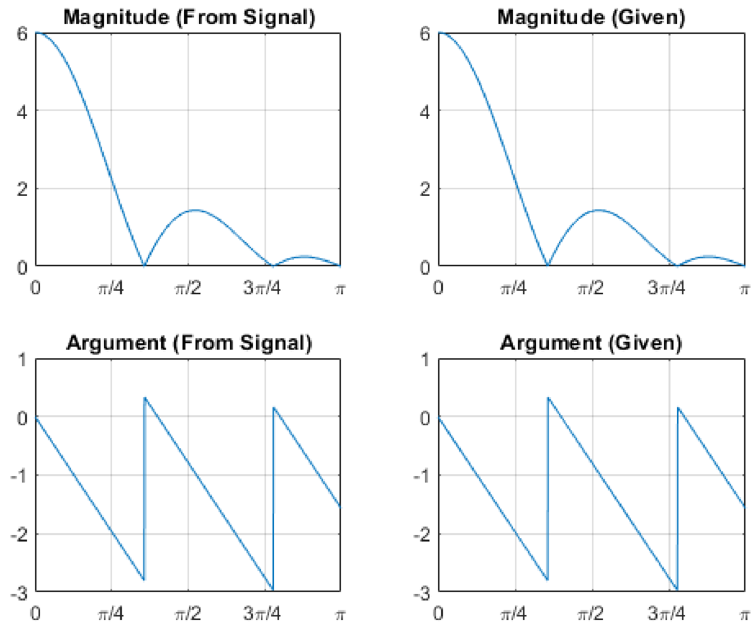


Figure 11: Problem 3.5 Part 4

# P3.6

The definition of the inverse DTFT is as follows:

$$\mathcal{F}^{-1}[\mathcal{F}[x[n]]] = \mathcal{F}^{-1}\left[X\left[e^{j\omega}\right]\right] \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \tag{2}$$

## Part 1

$$X\left(e^{j\omega}\right) = \begin{cases} 1 & \text{if } 0 \le |\omega| \le \pi/3 \\ 0 & \pi/3 \le |\omega| \le \pi \end{cases}$$

$$x[n] = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/3} 0\, d\omega + \int_{-\pi/3}^{\pi/3} e^{j\omega n}\, d\omega + \int_{\pi/3}^{\pi} 0\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{0 + jn}\right]_{\omega=-\pi/3}^{\omega=\pi/3}$$

$$= \frac{1}{2\pi}\left(\frac{e^{jn\pi/3}}{ne^{j\pi/2}} - \frac{e^{-jn\pi/3}}{ne^{j\pi/2}}\right)$$

$$= \frac{e^{j(n\pi/3-\pi/2)} - e^{j(-n\pi/3-\pi/2)}}{2\pi n}$$

## Part 5

$$X\left(e^{j\omega}\right) = \omega e^{j(\pi/2-10\omega)}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega e^{j(\pi/2-10\omega)} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \omega e^{j(\pi/2-10\omega-\omega n)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\left.\frac{\omega e^{j(\pi/2-10\omega-\omega n)}}{-j\left(10+n\right)}\right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{j(\pi/2-10\omega-\omega n)}}{-j\left(10+n\right)}\, d\omega\right]$$

$$\frac{1}{2\pi}\left[\frac{\pi e^{j(-19\pi/2-\pi n)}}{-j\left(10+n\right)} + \frac{\pi e^{j(21\pi/2+\pi n)}}{-j\left(10+n\right)}\right.$$

$$\left. - \left.\frac{e^{j(\pi/2-10\omega-\omega n)}}{\left(-j\left(10+n\right)\right)^2}\right|_{-\pi}^{\pi}\right]$$

$$= \frac{e^{j(-19\pi/2-\pi n)}}{-2j\left(10+n\right)} + \frac{e^{j(21\pi/2+\pi n)}}{-2j\left(10+n\right)}$$

$$+\frac{e^{j(-19\pi/2-\pi n)}}{(100+20n+n^2)} - \frac{e^{j(21\pi/2-\pi n)}}{(100+20n+n^2)}$$

# P3.11

## Part 1

$$h(n) = 0.9^{|n|}$$

$$\mathcal{F}[h(n)] = H\left(e^{j\omega}\right) = \frac{1-0.9^2}{1-2(0.9)\cos(\omega)+0.9^2}$$

$$= \frac{0.19}{1.81-1.8\cos(\omega)}$$

This result was found using the given DTFT pairs.

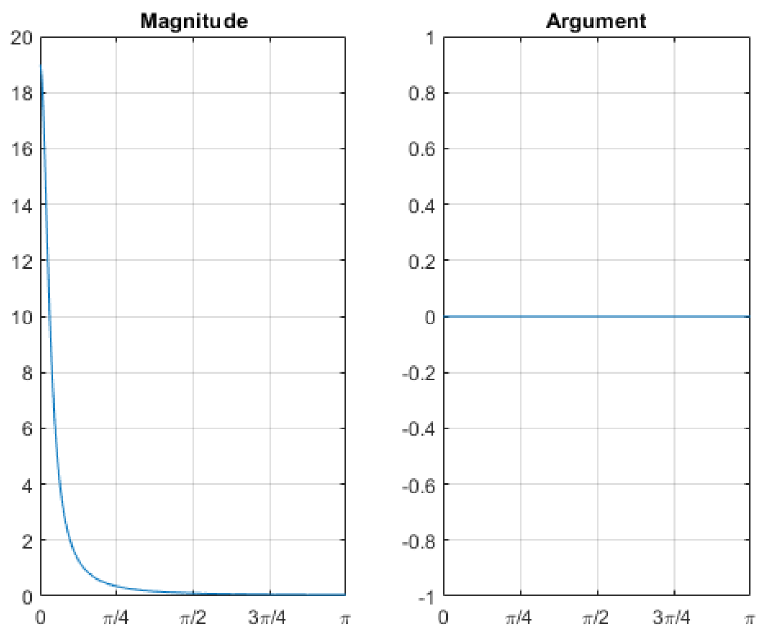


Figure 12: Problem 11 Part 1

**Part 5**

$$h(n) = 0.5^{|n|}\cos(0.1\pi n)$$
$$\mathcal{F}[h(n)] = \sum_{n=-\infty}^{\infty} 0.5^{|n|}\cos(0.1\pi n)e^{-j\omega n}$$
$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} 0.5^{|n|}\left(e^{0.1\pi nj} + e^{-0.1\pi nj}\right)e^{-j\omega n}$$
$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} 0.5^{|n|}\left[\left(e^{0.1\pi j - j\omega}\right)^n + \left(e^{-0.1\pi j - j\omega}\right)^n\right]$$
$$= \frac{1}{2}\sum_{n=0}^{-\infty}\left[\left(0.5e^{j(0.1\pi-\omega)}\right)^n + \left(0.5e^{j(-0.1\pi-\omega)}\right)^n\right]$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\left[\left(0.5e^{j(0.1\pi-\omega)}\right)^n + \left(0.5e^{j(-0.1\pi-\omega)}\right)^n\right]$$
$$-1 \text{ (this is because we're adding n=0 twice)}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(0.5e^{j(0.1\pi-\omega)}\right)^{-n} + \left(0.5e^{j(-0.1\pi-\omega)}\right)^{-n}\right]$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\left[\left(0.5e^{j(0.1\pi-\omega)}\right)^n + \left(0.5e^{j(-0.1\pi-\omega)}\right)^n\right] - 1$$
$$= \frac{1}{1-0.5e^{j(0.1\pi-\omega)}} + \frac{1}{1-0.5e^{j(-0.1\pi-\omega)}}$$
$$+\frac{1}{1-0.5e^{j(0.1\pi-\omega)}} + \frac{1}{1-0.5e^{j(-0.1\pi-\omega)}} - 1$$
$$= \frac{2}{1-0.5e^{j(0.1\pi-\omega)}} + \frac{2}{1-0.5e^{j(-0.1\pi-\omega)}} - 1$$

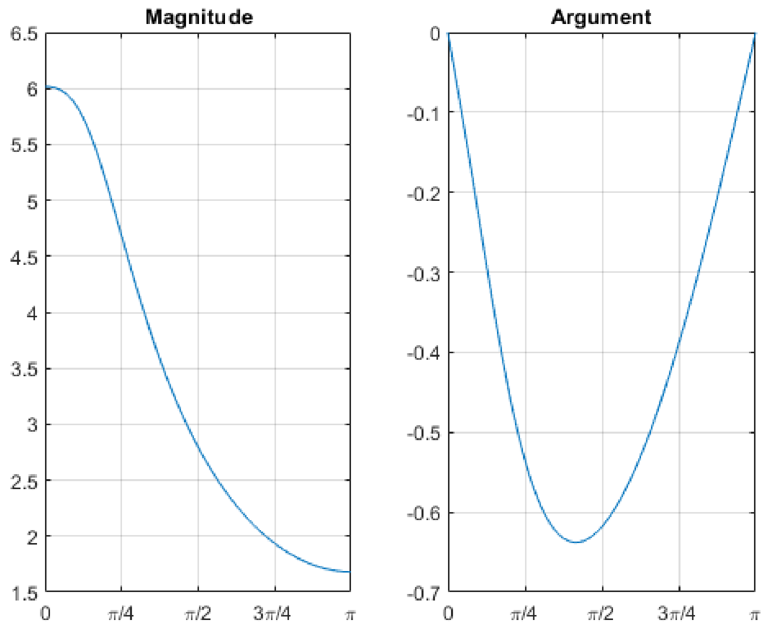


Figure 13: Problem 11 Part 5