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Final Project

Question 1

The default filter is actually an FIR filter. This is because its poles are all located at 0. Below is an intuitive derivation of this fact (just one example; i.e. not a hard proof).

An FIR filter is a filter that only depends on the current and previous inputs, and not previous outputs.

$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + \cdots + b_Nx[n-N]$$

If we take the z-transform of both sides, we can find the transfer function and analyze the poles and zeros.

$$\begin{aligned}
Y(z) &= b_0X(z) + b_1X(z)z^{-1} + b_2X(z)z^{-2} + \cdots + b_NX(z)z^{-N} \\
Y(z) &= X(z)(b_0 + b_1z^{-1} + b_2z^{-2} + \cdots + b_Nz^{-N}) \\
H(z) = \frac{Y(z)}{X(z)} &= \frac{b_0 + b_1z^{-1} + b_2z^{-2} + \cdots + b_Nz^{-N}}{1}\left(\frac{z^N}{z^N}\right) \\
&= \frac{b_0z^N + b_1z^{N-1} + b_2z^{N-2} + \cdots + b_N}{z^N}
\end{aligned}$$

There are N poles (and N zeros) by the Fundamental Theorem of Algebra. In this case, all of the poles are all 0. N is the order of the filter, so in this case, the order is 50.

Question 2

Several things happened when the zeros were removed:

- The highest magnitude of the frequency response dropped from a little over 1 to 0.4. Before the deletion, there were 50 zeros and 50 poles. With the deletion of two zeros, there are now less zeros than poles, which decreases the total magnitude of the filter.
- The magnitude of the frequency response near 0 normalized is now lower than the magnitude at the cutoff frequency (about 0.4 normalized). This is because, by removing poles close to the angle of π, frequencies close to 1 normalized are free to come out stronger than frequencies close to 0 normalized.

- There is now positive (as opposed to 0) magnitude at the maximum 1 normalized. The reason for this is the same as before: removing the zeros gave high frequencies more freedom.
- The argument (phase) part of the frequency response is left untouched. This is because we deleted both a zero and its conjugate.

Question 3

The small effect of deleting the first two zeros has now been greatly exaggerated: the filter passes signals with frequencies close to the Nyquist rate due to the fact that there is no zero at that frequency to suppress them. The maximum magnitude is now a bit over 1, while the section from 0 to 0.4 normalized is now half of what it was previously.

Question 4

This is a high-pass filter. The zero placed at $z = 1$ suppresses frequencies at $\omega = 0$ because it was placed at a point where it has 0 argument. If more zeros were placed at this point, it would create a sharper transition between blocking and passing frequencies.

Question 5

Moving the zero from $z = 1 = e^{j0}$ to $z = e^{j\pi/4}$, we now have complex coefficients for the filter. This is because we are no longer adding zeros in conjugates, which makes sure to remove the imaginary portion of the numbers. Since symmetry is only guaranteed from 0 Hz to $f_s/2$ for real numbers, it is not applicable to this situation, and an extended $-f_s/2$ to $f_s/2$ domain is used instead.

The magnitude of the frequency response now shows a dip at $\pi/4$ normalized, or about 9.425 kHz. The maximum magnitude is 2.

Question 6

Yes, the movement of the zero displays the expected outcome, especially on normalized frequency mode. The dip in magnitude is always pointed to the argument of the zero. For arguments close to π, There appears to be two such dips on the frequency response, but this is simply due to the extended domain used, and the fact that an argument of $-\pi$ is the same as an argument of π.

Question 7

The magnitude of the frequency response theoretically goes to infinity, but this is only approximated in the filter designer due to the finite sampling rate. It is of course very high. This filter is no longer stable because the region of convergence of the transfer function (z-transform) that defines this filter does not include the circle $|z| = 1$. For any low frequencies (such as DC or $\omega = 0$; i.e. a step excitation), one can imagine what would happen: even for a low initial value, the output would be greatly amplified (around 2500 times for a filter gain of 1), and the output would quickly grow in size, constantly being added to itself due to this being an IIR filter.

Additionally, the Filter Designer tool shows that the filter is not stable.

The spike in the magnitude definitely moves according to prediction, just like the same experiment with the zero.

Question 8

This filter is not stable. This can be deduced by only looking at the frequency response; specifically, by looking at the argument part of the frequency response. Below is the phase response of the unstable filter. Note how the lowest value (corresponding to frequency $-f_s/2$ and the highest value (corresponding to frequency $f_s/2$) do not match. Due to the periodicity of the frequency response, as the graph increases in domain (above $f_s/2$), the argument will jump from about 6 to about -0.5. This means that the phase response is not continuous and therefore not stable.

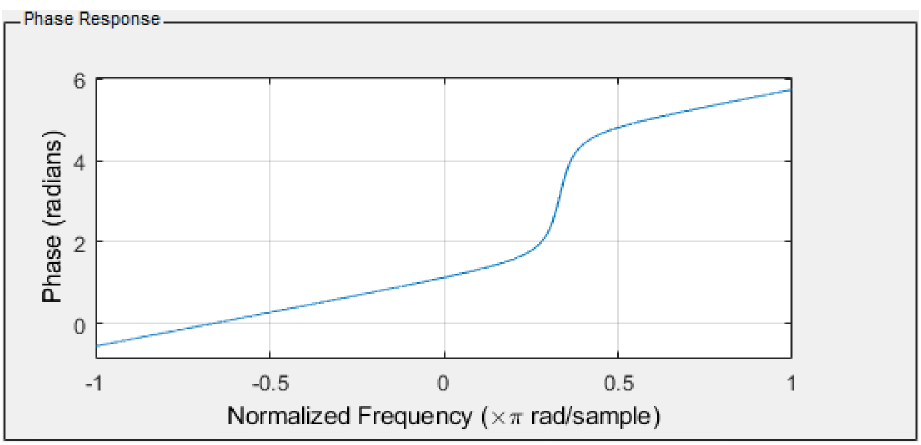

Figure 1: Unstable filter. One pole at $z = 1.1e^{j\pi/3}$.

Question 9

This filter is indeed stable. This can be deduced by looking at the phase response of the filter, as seen below. In this example, the phase starts at -0.5 at frequency $-f_s/2$ and ends at -0.5 at frequency $f_s/2$. In addition, the two derivatives at the point (quantitatively) appear the same. This leads to a nice transition as the frequency response transitions to a new period.

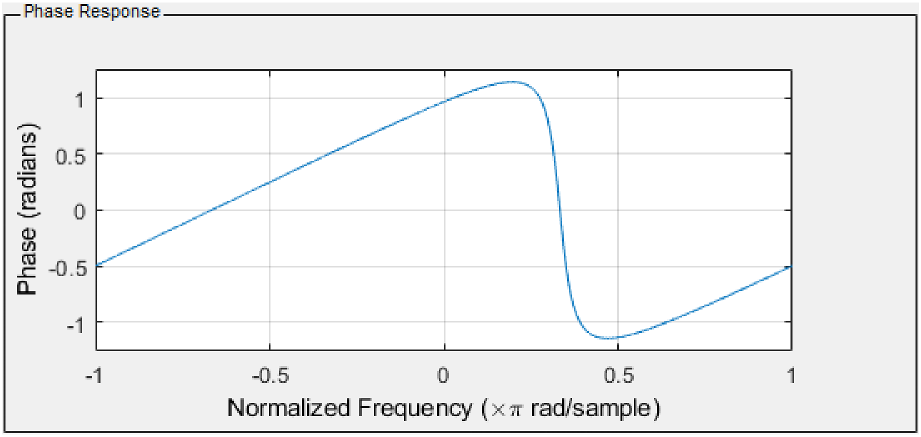

Figure 2: Stable filter. One pole at $z = \frac{1}{1.1}e^{j\pi/3}$.

Question 10

To find the coefficients of the transfer function, I will use the `poly` function available in MATLAB. The zeros are the roots of the numerator polynomial, and the poles are the roots of the denominator polynomial. Our zeros are $\{0 + 0j, 0 + 0j\}$, and our poles are $\{\left(1 - 10^{-2.275}\right) e^{\pm j5\pi/8}\}$. In addition, my gain is 1. Thus, the transfer function comes out to be equation (1).

$$H(z) = \frac{1z^2 + 0z + 0}{1z^2 + 0.7613z + 0.9894} = \frac{1}{1 + 0.7613z^{-1} + 0.9894z^{-2}} \tag{1}$$

```
>> a = 1-10^(-2.275)
a =
    0.9947
>> poly([0+0i, 0+0i])
ans =
     1     0     0
>> poly([a*exp(1i*5*pi/8), a*exp(-1i*5*pi/8)])
ans =
    1.0000    0.7613    0.9894
>>
```

Figure 3: Finding the coefficients of both polynomials.

Question 11

From the filter designer export, the actual coefficients of the filter are $[1, 0, 0]$ for the numerator and $[1, 0.761303651104029, 0.989410494944693]$ for the denominator, so my expected values are correct.