Novel Heuristic and Metaheuristic Approaches to Cutting and Packing

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ABSTRACT

This thesis develops and investigates new approaches for two-dimensional stock cutting problems. These problems occur in several important manufacturing industries where the application of automatic packing algorithms can yield considerable cost savings through the redeployment of human 'solvers' and better utilisation of raw material (which also has ecological benefits).

Three rectangle packing approaches and two irregular packing approaches are developed and presented within this thesis which have produced the best-known solutions for all of the readily available benchmark data drawn from over twenty years of cutting and packing literature (34 rectangular instances and 26 irregular instances), some of which had not been improved upon for over 10 years. Further to this, initial benchmark solutions are given for new problem instances that are presented in this thesis for both variants of the problem. In particular, the new irregular benchmark problems contain shapes which consist of arcs and holes. Shapes with these features are infrequently represented within the literature. Another major contribution of this work is the development of a no-fit polygon generation algorithm that does not suffer from degenerate cases that are found in other approaches from the literature. As far as the author is aware, this is the first time that no-fit polygons can be generated both accurately and robustly. Furthermore, the approach is then extended to handle shapes which contain arcs. This is the first no-fit polygon algorithm that is able to deal with shapes consisting of arcs as well as lines. The underlying consideration for this research programme has been to produce algorithms that are suitable for use in the real world industrial setting. The developed approaches are able to quickly and accurately pack irregular shapes (consisting of lines, arcs and holes) and have produced the best-known solutions on all of the readily available datasets. This makes these algorithms very strong candidates for use in real world industrial settings.

PUBLICATIONS PRODUCED

During the course of this Ph.D. research programme, the following publications have been produced from the work presented within this thesis. The chapter of the thesis on which each publication is based is also shown.

Burke, E. K., Kendall, G., Whitwell, G., 2004, "*A New Placement Heuristic for the Orthogonal Stock Cutting Problem*", **Operations Research**, Vol. 52, No. 4, July-August 2004, pp. 655-671. [**1**]
(This work was drawn from **chapter three** of this thesis)

Burke, E. K., Hellier, R. S. R., Kendall, G., Whitwell, G., 2005, "*A New Bottom-Left-Fill Algorithm for the Two-Dimensional Irregular Packing Problem*", Accepted for **Operations Research**. [**2**]
(This work was drawn from **chapter six** and **chapter seven** of this thesis)

Burke, E. K., Kendall, G., Whitwell, G., 2005, "*Metaheuristic Enhancements of the Best-Fit Heuristic for the Orthogonal Stock Cutting Problem*", **INFORMS Journal on Computing** (in review). [**3**]
(This work was drawn from **chapter four** of this thesis)

Burke, E. K., Hellier, R. S. R., Kendall, G., Whitwell, G., 2005, "*Complete and Robust No-Fit Polygon Generation for the Irregular Stock Cutting Problem*" **Operations Research** (in review). [**4**]
(This work was drawn from **chapter eight** of this thesis)

Burke, E. K., Kendall, G., Whitwell, G., 2005, "*The Two-Dimensional Orthogonal Stock Cutting Problem: A User-Guided Hybrid Approach*", **Journal of the ACM** (in review). [**5**]
(This work was drawn from **chapter five** of this thesis)

Burke, E. K., Kendall, G., Whitwell, G., 2005, "*Irregular Packing using the Line and Arc No-Fit Polygon*", In Preparation for **Operations Research**. [**6**]
(This work was drawn from **chapter eight** and **chapter nine** of this thesis)

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PART A – CUTTING AND PACKING

Cutting and packing is an increasingly active area within the academic community and can yield considerable cost benefits within industrial and manufacturing sectors as well as providing environmental benefits through a reduction of raw materials required. This thesis specifically addresses the two-dimensional packing problem which is commonplace within a wide variety of industrial settings. This work has been developed under two Engineering and Physical Sciences Research Council research grants. The first involved a CASE award (Cooperative Awards in Science and Engineering) in partnership with a local manufacturing company, Esprit Automation Limited, who manufacture oxy-fuel plasma cutting machines and a software suite that offers both computer aided design and automated packing. The second of these grants had been obtained to further develop algorithms for the robust generation of a geometric tool known as the no-fit polygon (discussed in chapters six, eight and nine).

Part A of this thesis provides an introduction to this work and presents a review of the related literature. Chapter one discusses the motivation for the research and examines the involvement of our industrial partner, Esprit Automation Limited. This thesis also presents an overview of the research programme and a discussion of the contributions and impact that this work brings to the research area. The second chapter provides a review of the relevant areas within the field of cutting and packing. Within this chapter, a discussion of the published literature is presented, with emphasis on the two-dimensional rectangular and irregular variants of cutting and packing. Also, some of the unique problems and requirements that exist within the industrial setting are identified and discussed.

CHAPTER ONE

Introduction

This thesis is the culmination of four years research investigating state-of-the-art automated packing algorithms. This chapter introduces the background and requirements for the research programme and then details its aims and objectives. Following this, an overview of each chapter of the thesis is provided and the key contributions of this work are discussed.

1.1 Background and Motivation

This research programme was conceived to provide a continuation of the work into cutting and packing conducted by Dr. Graham Kendall during his PhD research programme which was completed in December 2000 [**7,8,9,10,11,12,13**]. During his research, Dr. Kendall (supervised by Professor Edmund Burke) had worked closely with a local company, Esprit Automation Limited, who manufacture cutting machines for the metal cutting industry. Esprit Automation also contains a small software development team who produce CAD (Computer Aided Design) software and CNC (Computer Numerical Control) code to interface with their cutting machines through their "Procut" software suite. Automated packing is an important part of their software. Approximately 10 years previous to the start of this research programme, Esprit's automated packing algorithm was regarded as one of the best in the world both in terms of solution quality and speed. However, in the intervening years, Esprit's competitors have conducted research and development into automated packing whereas Esprit has been pursuing other software developments. Further to this, the extra processing power of hardware has allowed for more varied and intelligent approaches to automated packing and this has resulted in a significant increase in the competition within the marketplace. This has led to the Procut suite failing to keep up with state-of-the-art automated packing technology and losing its status as one of the industry leaders. The cutting machine / software partnership is important to Esprit's business model because the provision of superior software and packing algorithms provides greater marketing power for selling cutting machines and vice versa. In order to rectify the situation with their packing algorithms, Esprit Automation contacted Professor Edmund Burke and Dr. Graham Kendall at the University of Nottingham and committed £53,600 for the part funding of an Engineering and Physical Sciences Research Council (EPSRC) programme and a Teaching Company Scheme (TCS) which were to operate in conjunction for a period of three years:

(i) EPSRC CASE for New Academics (Ref: CNA 00802329)

"Applying Metaheuristics and Hyperheuristic to the Stock Cutting Problem"

The author of this thesis was the investigator on the CASE award and primarily focussed on the development of new algorithms for automated packing. The award required the author to work at the company for a period of three months of each year of the programme.

(ii) TCS (Teaching Company Scheme) Award (No. 3074)

"New Approaches to Produce Efficient Nesting Patterns"

The TCS award (now known as Knowledge Transfer Partnerships) programme involved pre-defined objectives that had been set out by the company and involved the transfer of technology into their Procut software suite. The investigator on this award, Robert Hellier, was required to work at the company on a full-time basis, although he was employed by the University of Nottingham.

The industrial involvement in this project not only enabled the comparison of results against a commercial algorithm but also allowed, where applicable, for the reuse and modification of Esprit Automation's existing geometry library. A further benefit to using the company's software framework was that the CAD interface could be utilised for the input, representation and displaying of shapes.

The final year of this research has been conducted under a fully funded EPSRC research grant and focuses on the robust generation of no-fit polygons and their applications to packing problems:

iii) EPSRC Research Grant (Ref: GR/S52414/01)

"An Investigation of Cutting, Packing and Planning using Automated Algorithm Selection"

The work conducted under this research grant has already led to a significant commercial opportunity which is discussed in chapter eleven of the thesis.

1.2 Aims and Scope

The fundamental aim of this project was to investigate two-dimensional packing problems and to develop state-of-the-art algorithms that could be used in an industrial setting. For this to be achieved it was important that any packing algorithm that was developed could produce very good solutions within reasonable time where "reasonable time" is considered to be around twenty minutes. This is the approximate time taken by a human operator to produce a good quality layout within Esprit's client base and was found from consultation with the industrial partner and attendance at industrially focussed workshops. Due to these observations, heuristic and metaheuristic searches were identified as the most suitable algorithms and part of this

project was to hybridise packing algorithms with such optimisation techniques. The development and evaluation of novel metaheuristic hybridised packing algorithms forms a major contribution of this thesis.
Obviously, any industrial applicable packing algorithm must be flexible enough to allow for solutions to be produced more quickly if the user so requires. The user may also wish to allow longer durations if the solution will be used as a template to cut multiple identical layouts from several sheets or if material unit costs are high. In this instance, any additional improvements in solution quality, material and cost savings are multiplied several times and, therefore, justification can be made for the additional time taken to produce a solution.

As mentioned above, part of the funding for the research programme was provided by Esprit Automation who would like to include successfully developed algorithms into their own software. Therefore, a further necessity was that any developed packing algorithms must be able to robustly handle the geometry involved in the automated packing process. This is an important factor in the industrial setting as users expect software to work without errors. Any software malfunctions are unacceptable as they would severely affect our industrial partner's reputation for developing software.

A further objective was to produce packing algorithms that could correctly handle floating point based line *and* arc geometry. The inclusion of such geometry can offer considerable advantages in terms of accuracy and speed gains. Academically, this is a challenging undertaking and offers improvements over previous approaches from the literature as any shapes that involve arc were previously represented by their line approximations which inevitably affects accuracy and is undesirable within the industrial environment. The inclusion of arcs has been a major consideration for the irregular packing approaches that have been developed during this work (and described within Part C of this thesis).

1.3 Contributions

This thesis makes several contributions to the field of rectangular and irregular two-dimensional packing. The contributions are briefly summarised in the list below (each contribution is discussed in greater detail within section 10.1):

- Development of a New Fast and Effective Rectangle Packing Heuristic (see section 10.1.1)
- Hybridisation of Two Packing Algorithms to Combine Relative Strengths (see section 10.1.2)
- Introduction of User Interaction (see section 10.1.3)
- Presentation of a New Placement Technique for Irregular Packing (see section 10.1.4)
- Development of a Complete and Robust Algorithm for Generation of No-Fit Polygons involving Non-Convex Polygons (see section 10.1.5)
- Modification of the Developed No-Fit Polygon Generation Algorithm to Allow Shapes Containing Arcs (see section 10.1.6)
- Identification of No-Fit Polygon Properties and Applicability of Caching within the Generation Process for Packing Problems (see section 10.1.7)
- Generation of the Best-Known Solutions for 60 Benchmark Datasets (Both Rectangular and Irregular) Gathered from over Twenty Years in Cutting and Packing Research (see section 10.1.8)
- Introduction of New Benchmark Problem Instances to the Literature (see section 10.1.9)

1.4 Overview of the Thesis

This thesis is divided into four logical sections:

Part A - Chapters One and Two

This part is concerned with the introduction of this thesis and a discussion of the related literature. Chapter one has placed the thesis in the context in which this research has been undertaken. Chapter two provides an overview to the problem that has been attempted and the related literature.

Part B - Chapters Three to Five

The initial research focussed on the rectangle packing problem. This is the subject of Part B which is arranged such that three successive chapters introduce three newly developed packing strategies, with each improving over the previous algorithm. Chapter three introduces a fast new placement heuristic that uses the principles of best-fit. Chapter four modifies this strategy to work with floating-point data and hybridises it with another metaheuristic based packing strategy from the literature. Chapter five presents an extension to the work of chapter four that allows the user to interact with the automation process in order to obtain further improvements in solution quality.

Part C - Chapters Six to Nine

Part C specifically focuses on the irregular packing problem and automation approaches and consists of four chapters. Firstly, chapter six introduces some of the geometric considerations that have been utilised within this work and presents a review of the no-fit polygon. Next, chapter seven develops a new approach for the packing of irregular shapes through trigonometric intersection approaches and is shown to outperform other methods from the literature in terms of solution quality and time taken. Chapter eight, for the first time, presents a complete and robust no-fit polygon generation algorithm (the no-fit polygon is a geometric construct that defines the intersection relationship between two shapes, see section 6.3) that does not suffer from the degenerate cases that are found within the other methods from the literature. In chapter nine, the no-fit polygon is modified to work with circular arcs and the packing strategy of chapter seven is reimplemented using the no-fit polygon. Once more, experiments are conducted on the benchmark problems and comparisons are made between using trigonometric and no-fit polygon based intersection testing.

Part D – Chapters Ten and Eleven

The final part summarises the work produced in this thesis. Chapter ten presents conclusions from the approaches of the previous chapters and indicates possible extensions to this research. Chapter eleven provides an insight to how this research has been disseminated into the academic community and discusses the benefits for our industrial partner. The chapter finally discusses the new commercialisation opportunities that have arisen as a direct result of the work in this thesis and describes the launch of a new spin-out company to exploit the packing algorithms that have been developed.

Chapter Two

The Stock Cutting Problem

Packing problems occur in many different situations within everyday life. An individual is subjected to several circumstances where packing 'skills' are required such as the placing of clothes into a suitcase or placing foodstuffs in a freezer. Humans seem to be able to solve packing problems relatively well through the use of intuition and spatial awareness. However in an industrial setting, where there are a multitude of similar instances of packing problems, it is usually not feasible or cost effective to solve such problems through manual means alone and more than likely it is not cost effective if several human 'solvers' have to be employed. The computer aided automation of packing problems offers a solution to this problem and has been the focus of over fifty years of research by the academic and industrial communities. Unlike their human counterparts, computers have no intuition or spatial awareness and thus algorithmic strategies must be developed for the generation of layouts. Industrially, the problem is often more complicated and requires further representation and the modelling of additional constraints and objectives. Due to the involvement of an industrial partner within this research programme, the work presented in this thesis brings together both academic and industrial aspects of the problem. This chapter firstly provides a broad overview of stock cutting problems and methods of categorising them. In particular, the chapter provides a special focus on the related literature to the two-dimensional rectangular and irregular stock cutting problems. Related literature to computational geometry and optimisation search methods are also reviewed. Finally, the chapter presents a discussion of considerations that are specific to automated packing within the industrial environment.

2.1 Cutting and Packing: An Overview

Since their formulation in the 1950's [**14**], stock cutting problems have received interest from manufacturing industries as a means of increasing profitability by reducing costs and have also become an increasingly active research area within the academic community. There are many facets of research that exist under the 'umbrella' term of "cutting and packing". However, the basic form of such problems remains consistent: (i) a number of available *resources* and (ii) a number of *items* that must be *assigned to* the *resources*. In the general case, the main objective is to *assign* all of the *items* whilst minimising the usage of the *resources* and satisfying certain problem specific constraints. In the industrial setting, this has particular financial benefits when *resources* have high unit cost. Relaxations of stock cutting problems also lend themselves to research within the academic community as they are simple, well-formed but still NP-Complete and thus it is believed that they cannot be solved with a polynomial time algorithm (Garey and Johnson, 1979) **[15]**.

2.1.1 Types of Problem

The term "cutting and packing" has become synonymous with a wide range of subtly different problems. These are identified below:

Bin Packing

This problem is concerned with the determination of the minimum number of bins that are needed to pack a set of items. Several different versions of the problem exist and can have single dimension or multidimensional items and bins. The problem appears in a broad range of applications including: industrial manufacturing, vehicle loading, scheduling, vehicle routing and integrated circuit manufacturing. For a detailed survey on bin packing, see Coffman, Garey and Johnson (1997) [**16**].

Knapsack Problem

The problem consists of an object (the knapsack) of fixed capacity and a set of items (camping equipment) which each have a size and evaluation (or measure of usefulness). The objective is to pack the subset of items with the maximal total evaluation whilst observing the capacity constraint of the knapsack. The knapsack problem is closely related to bin packing [**17**].

Space Allocation / Capacity Allocation

Space (or capacity) allocation problems are closely related to bin packing and knapsack problems and involve the distribution of space to a set of items (i.e. assigning people to office space) whilst observing additional constraints (more details can be found in [**18,19**]).

Orthogonal Packing / Strip Packing

This involves the packing of rectangles where their sides are always parallel to the *x* and *y* axis (i.e. only 90° rotations are allowed). The problem can exist with sheets of limited or unlimited height. With unlimited height (also known as "Strip Packing"), a roll of material is assumed.

Trim-Loss Problem

This involves the minimisation of the "trim-loss" or sheet wastage that is incurred as a result of laying out irregular shapes (i.e. non-rectangular). Industrially, the problem can become more complicated when partially used sheets can be restocked for use in future packing operations but requires the correct identification and definition of regions that are "trim".

Nesting

This is a term used to represent two-dimensional irregular shape packing and is usually found within the metal cutting industry. The problem is often made more difficult by packing onto multiple or non-rectangular sheets. A packed arrangement of shapes is referred to as a "nest" within the metal cutting industry.

Loading Problem

The loading problem is explicitly used for regular three-dimensional packing whereby boxes must be placed within a container. There can often be further constraints and objectives involved within industry. An example of this is the loading of a delivery lorry. An additional constraint might be "larger boxes may not be placed on smaller fragile boxes" whereas additional objectives could be to place boxes that will be delivered first at the back of the lorry, or to spread weight evenly over the axles.

Marker Layout Problem (Textiles)

This problem is concerned with the two-dimensional irregular packing problem. The term 'marker' is used to refer to an item that must be cut within the textile industry. As clothing is generally made up of several pieces, the problem can involve highly irregular pieces. A further complication to this style of problem is that defects may be present within the sheet or roll of material. An example of this is that differing qualities and strengths exist within certain areas of cow hides that are used for production of shoes within the footwear industry. It is important that certain parts of the shoe, such as the toe, are produced from the strongest leather but less structural pieces can be produced from lesser quality regions. The defective regions that cannot be used in the manufacture of the shoe can be utilised in other ways (e.g. one manufacturer uses the offcut pieces to produce "classy" labels for the shoe). This information was relayed to the author at an industrial focussed workshop into the leather and footwear industry.

Assortment Problem

This problem is a concern to industry and involves stock sheet holdings. The idea is to determine which stock sheet sizes should be maintained within the warehouse such that wastage is reduced (or so that fewer cuts are needed to simplify the cutting operations).

2.1.2 Typological Categorisation

Due to the multitude of problem names that exist within the cutting and packing industry (and academic literature) and the fact that several of these refer to the same types of problem, it was important to classify the problems in a more sensible way to investigate the underlying structure within cutting and packing problems and to facilitate the cross-fertilisation of research within the academic community. In 1990, Dyckhoff proposed such a typology that could describe problem types based on four characteristics [**20**].

The first characteristic involves identification of the dimensionality of the problem: one, two, three and N-dimensional. These are all self explanatory except for the N-dimensional packing. As an example of an N-dimensional problem consider lorry loading. This intrinsically involves three spatial dimensions but a further non-spatial dimension can be added if weight is also an important factor. Now each item has four dimensions: length, height, width and weight. An example of the loading problem including weight considerations is presented in Davis and Bischoff (1999) [**21**]. Others examples that can be modelled as an N-dimensional stock cutting are the capital budgeting problem (Lorie and Savage, 1955) [**22**] and the dynamic allocation of computer memory for data storage (Garey and Johnson, 1981) [**23**].

The second characteristic is drawn from the type of assignment that is involved. Two options are provided: i) resources accommodate all of the items, ii) resources are limited and cannot accommodate all of the items. In the first case, the emphasis is placed on finding a good arrangement of the items, whereas, in the second case, the aim is to assign as many items as possible to the limited resources to minimise some objective function. An example of the first case is the packing of a few markers onto a long roll of material. The roll is sufficient to produce all of the markers. An example of where resources are limited might be where multiple letter shapes are required to be cut out of a metal sheet for stock piling. As there are no specific quantities required, the sheet should be used in order to maximise utilisation.

The third characteristic involves the assortment of the available resources (sheets in two-dimensional packing). The options discussed are: i) a singular resource, ii) multiple identical resources, and iii) multiple different resources. The practical difference in the first case is that there is no interaction between resources (this can be seen when a roll of material is used). With multiple identical resources, the order in which resources are used has no impact on the solution quality but a new resource must be started if the current resource becomes full. In the final case, the order in which resources are used has a direct impact on the quality of solutions produced and, once again, we must identify when to start a new resource if another becomes full.

The final characteristic is concerned with the assortment of the items (shapes in two-dimensional packing). The types take the following forms: i) identical items, ii) few and different items, iii) many copies but few different items, and iv) many copies and many different items. Examples of case i) include the cutting of metal blanks where a sheet is cut into multiple identical smaller sheets by a stamping procedure and the pallet loading problem. The other cases are dependent on specific instances of problem rather than problem types.

These four characteristics can be combined to form a classification for stock cutting problems. The following examples show the categorisations for the pallet loading and the nesting problem:

Pallet Loading Problem:

2-dimensional | resources cannot accommodate all items | singular resource | identical items

Nesting Problem:

2-dimensional | resources accommodate all items | many identical resources | many different items

However, the typology proposed by Dyckhoff [**20**] has not been as widely used because of several drawbacks that have been discussed by Wäscher, Haußner and Schumann (2004) [**24**]. Firstly, not every cutting and packing problem can be uniquely defined by just one of the classifications. For example, the vehicle loading problem can be categorised into the following two classifications:

i) 1-dimensional | resources accommodate all items | many identical resources | <u>many different items</u>

ii) 1-dimensional | resources accommodate all items | many identical resources | <u>few different items</u>

Further to this, the representation of the typology is partially inconsistent because quite different problems can be classified under the same classification (such as the strip packing and bin packing problems). As such, the method does not result in homogeneous problem categories (Gradisar et. al, 2002) [**25**]. It is partly due to these reasons that there has been a concerted effort by the research community to define a more descriptive typology to eliminate the problems that have been highlighted in [**24**].

2.1.3 Applicability, Surveys and Bibliographies

Aspects of cutting and packing can be applied to many diverse problems within many different disciplines. Research has been conducted within the following areas, as identified by Dyckhoff: computer science, operations research, logistics, management science, industrial engineering, engineering science, combinatorial optimisation, production research, mathematics and, finally, manufacturing. There have been several survey papers and research bibliographies into stock cutting problems from the many disciplines listed above. Many of these are shown in table 2.1 which have been derived from [**8,20,26**] but include several papers identified by the author:

Author(s)	**Year**	**Author(s)**	**Year**
Brown [**27**]	1971	Haessler and Sweeney [**46**]	1991
Salkin and de Kluyver [**28**]	1975	Dowsland and Dowsland [**47**]	1992
Golden [**29**]	1976	Dyckhoff and Finke [**48**]	1992
Hinxman [**30**]	1980	Sweeney and Paternoster [**49**]	1992
Garey and Johnson [**23**]	1981	Haessler [**50**]	1992
Israni and Sanders [**31**]	1982	Lirov [**51**]	1992
Sarin [**32**]	1983	Ram [**52**]	1992
Rayward-Smith and Shing [**33**]	1983	Gritzmann and Wills [**53**]	1993
Coffman, Garey and Johnson [**34**]	1984	Gerasch and Wang [**54**]	1994
Berkey and Wang [**35**]	1985	Cheng, Feiring and Cheng [**55**]	1994
Dowsland [**36**]	1985	Dowsland and Dowsland [**56**]	1995
Dyckhoff, Kruse, Abel and Gal [**37**]	1985	Sixt [**57**]	1996
Israni and Sanders [**38**]	1985	Hopper and Turton [**58**]	1997
Dudzinski and Walukiewicz [**39**]	1987	Dyckhoff, Scheithauer and Terno [**59**]	1997
Martello and Toth [**40**]	1987	Mavridou and Pardalos [**60**]	1997
Rode and Rosenberg [**41**]	1987	Hopper [**26**]	2000
Dyckhoff, Finke and Kruse [**42**]	1988	Hopper and Turton [**61**]	2001
Coffman and Shor [**43**]	1990	Lodi, Martello and Vigo [**62**]	2002
Martello and Toth [**17**]	1990	Cagan, Shimada and Yin [**63**]	2002
Dyckhoff and Wäscher [**44**]	1990	Lodi, Martello and Monaci [**64**]	2003
Dowsland [**45**]	1991	Oliveira [**65**]	2003

Table 2.1. Selected surveys and bibliographies for cutting and packing problems

2.2 Two Dimensional Orthogonal Packing

Early cutting and packing research focussed on the two-dimensional orthogonal stock cutting problem. The problem is also of interest because it has applications to other areas such as dynamic memory allocation, multi-processor scheduling problems and general layout problems (Coffman, Garey and Johnson, 1978 [**66**]; Garey and Johnson, 1981 [**23**]; Coffman and Leighton, 1989 [**67**]; Dyckhoff, 1990 [**20**]). These problems have a similar logical structure and can be modelled by a set of rectangular pieces that must be arranged on a pre-defined stock sheet (or sheets) so that each rectangular piece does not overlap with another and also remains within the confines of the stock sheet. The problem occurs, with different constraints, within manufacturing industries including paper, wood, glass, and metal cutting. For example, paper cutting is generally concerned with the guillotine packing (where only straight vertical or horizontal cuts across the entire sheet region are allowed) of rectangular items from a stock roll of fixed width whereas applications in metal and shipbuilding are often concerned with the cutting of irregular shapes from a stock sheet. However, in most industrial applications the goals are similar: to produce good quality arrangements of items on the stock sheet in order to maximise material utilisation and, therefore, minimise wastage. The time allowed for any specific problem is usually dependent on the material cost and the urgency of a solution. For example, when packing onto a metal sheet with a two inch thickness (and thus, probably expensive) you would probably allow the packing algorithm more time as a small improvement in the solution quality can result in large cost savings. With a metal sheet of 1mm thickness, there may be more emphasis in finding solutions quickly as small reductions in packing quality may only yield negligible savings. These processes are most heavily associated with mass production operations and it is usually very important to produce better quality solutions, in quicker time, with less wastage in order to maximise profits. In some industrial situations, this optimisation task is still undertaken by skilled experts. However, due to the large costs, performance fall-off and the liability inherent with employed labour, automated packing approaches have been more widely used in recent years. The solution quality of automated packing approaches can often be equal or better than that of their human counterparts (Roberts, 1984 [**68**]; Li and Milenkovic, 1995 [**69**]) and are usually performed more quickly (Hower, Rosendahl and Köstner, 1996) [**70**].

2.2.1 Related Literature for Rectangular Problems

This area has been the subject of several decades of research from its formulation in the 1950s where Paull addressed the newsprint layout problem (Paull, 1956) [**14**]. There have been many approaches to producing automated rectangular packing algorithms. The approaches can be broadly categorised into three methods: exact, problem specific heuristics and, more recently, metaheuristic algorithms. The key papers within these groupings are discussed below:

Exact methods were originally investigated by Gilmore and Gomory in the 1960s and form the first era of cutting and packing research [**71**]. They used linear programming techniques to solve instances to optimality for one-dimensional problems. However, only small problem instances could be solved due to the computational time required. In 1963, the authors extended their algorithms to solve instances from the paper roll cutting industry in, what is considered to be, the first real industry applicable research into stock cutting. They state that the problem instances can involve several million possible cutting layouts but also show how this complexity can be reduced by using column generation techniques [**72**]. In 1965, Gilmore and Gomory extended their linear programming algorithms for solving the two-dimensional problem [**73**]. Due to an infeasible number of columns being required, they imposed a restriction to the guillotine variant of the problem. They show that this significantly reduces the number of required columns and is justified because many of these restrictions exist within the paper industry anyway. A year later, Gilmore and Gomory used dynamic programming techniques to solve the knapsack problem (Gilmore and Gomory, 1966) [**74**]. The research conducted by Gilmore and Gomory is considered to be the seminal work for the domain of cutting and packing with their papers being cited most frequently within the literature. Further to this, many approaches have been based on modified versions of the Gilmore and Gomory linear programming model (Dyson, 1974 [**75**]; Haessler, 1980 [**76**]; Dyckhoff, 1981 [**77**]; Viswanathan and Bagchi, 1993 [**78**]; Hifi and Roucairol, 2001 [**79**]).

In 1967, Barnett and Kynch produced an algorithm that was simpler than previous exact approaches [**80**]. A requirement for this was that the problem had to be relaxed to allow non-guillotine arrangements. This is the first instance of research into the non-guillotine rectangle packing that has been found.

Haessler (1971) [**81**] and, subsequently, Haessler (1975) [**82**] approached the stock cutting problem from a dual evaluation perspective based on industrial observations. Within industry, there is usually a cost involved in the setting up of cutting machines and cutting operations. Haessler suggests that this must also be considered during layout generation even if it produces less optimal solutions. Haessler indicates the need for a balance between packing quality and the cost of cutting. This was achieved through mathematical modelling and aspiration levels.

Dyson (1974) also attempted a problem that is typically found within the industrial setting [**75**]. This involved the situation where the pieces within many cutting jobs/orders pass through a production line in a fragmented manner. That is, a particular job might not be completed on a contiguous number of sheets which requires temporary storage of pieces until all parts of the order can be shipped. A Gilmore and Gomory style linear program was used as the main solver with an extension that penalised greater distances between the individual pieces of a job as it is more desirable to keep pieces of a job together.

An iterative recursive approach was presented in Herz (1972) [**83**] for the multistage cutting problem in the guillotine variant. Further improvements in performance were reported over previous exact approaches for small instances. Adamowicz and Albano (1976) [**84**] used dynamic programming methods for the placing of rectangles that had been sorted into groups consisting of dimensional commonality. These groups were then packed into columns or strips. The authors report that good quality solutions are produced in "reasonable time" although wastage occurs when rectangles do not have dimensions that can form groups. In the same year, the authors also utilised rectangle packing techniques for the packing of irregular shapes [**85**]. This was achieved by firstly placing the shapes into rectangular modules that were then, in turn, packed.

Christofides and Whitlock (1977) used a tree search method to solve the two-dimensional guillotine stock cutting problem to optimality [**86**], as does Beasley (1985a) with the non-guillotine variant [**87**]. These methods can solve problems involving up to 20 and 10 rectangles respectively. In the same year Beasley also compares both optimal and heuristic algorithms using dynamic programming on the guillotine variant [**88**]. However, on medium to large problem instances exact methods become time infeasible even on restricted instances of the rectangular problem (Albano and Orsini, 1980) [**89**]. In 1983, Wang presented two exact algorithms for the constrained cutting problem. These methods joined rectangular pieces together to form good arrangements for guillotine cutting. The exactness is found by enumerating all of the possible combinations in which pieces can be placed [**90**].

The following ten years of rectangular stock cutting research (1985-1995) seemed to be mainly dominated by heuristic methods. However, there was resurgence in the use of exact approaches in 1995 when Christofides and Hadjiconstantinou (1995) [**91**] produced a new tree-search procedure that is guaranteed to solve medium sized problems for the guillotine variant. Their approach achieved the optimal solution for 8 of 18 problems but report computation times for the procedure of between 0.8 and 167 minutes depending on the size and complexity of the problem instances. In the same year, Fayard and Zissimopoulos approached the same problem by producing optimal subsets of strips by solving a sequence of one-dimensional knapsack problems. They reported a good average worst-case approximation of around 0.98 and optimal solutions are produced for 91% of the attempted solutions. The largest problem size used involved 60 rectangles [**92**]. A further linear-programming approach was presented by Valerio de Carvalho and Guimaraes Rodrigues in the same year. The approach is worthy of note because it solved the two-stage cutting problem within a made-to-order steel roll cutting facility. The authors indicate that, once again, reduction in the setup cost of cutting layouts must also be factored alongside layout quality [**93**].

More recently, Hifi and Zissimopolous (1997) presented an exact algorithm that improves on Christofides and Whitlocks' approach [**94**]. In 2002, this was improved upon by GG and Kang who describe an approach to calculate a new upper bound [**95**]. This is achieved by solving two knapsack problems at the beginning of the algorithm. In comparisons against [**94**], the authors reported that a 95% reduction is observed when the new upper bound is applied to Hifi and Zissimopolous' exact algorithm. Hifi (1997) modifies Viswanathan and Bagchi's tree search procedure presented in (Viswanathan and Bagchi, 1993) [**78**] by introducing one-dimensional bounded knapsacks and dynamic programming techniques to allow for considerable branch cuts to be made to reduce the complexity of search trees. The author reports an average computation time saving of, on average, 34.89% although it is also suggested that further gains are achieved as the size of the problem increases [**96**].

Parada et al. examines the behaviour of five algorithms for generating guillotine layouts [**97**]. The algorithms are: i) exact method proposed by Wang in [**90**], ii) improvement over [**90**] presented by Oliveria and Ferreira in [**98**], iii) and/or graph approach, iv) heuristic with simulated annealing and v) heuristic with evolutionary algorithm. The authors determine that on medium to large instances the exact method (i) requires high running times, the improved exact method (ii) requires medium running times but the other approaches are all capable of producing solutions using little computation time. The authors report that only the simulated annealing and evolutionary approaches can produce solutions for all instances with the evolutionary method producing the best solutions but requiring greater time than the simulated annealing approach.

Cung, Hifi and Le Cun (2000) developed a new version of the algorithm proposed in [**94**] that uses a best-first branch-and-bound approach to solve exactly some variants of two-dimensional cutting stock problems [**99**]. Although branch-and-bound eliminates some of the computational time required with the exact approaches, the major drawback of these methods still persists, they cannot provide good results for large instances of the problem.

In 1998, Faggioli and Bentivoglio investigated both heuristic and exact methods in a three phase approach [**100**]. This involved firstly producing a good starting solution with a greedy algorithm, improving with tabu search (or other local searches) and then using exact implicit enumeration procedures with the previously best found solution as the upper bound. The authors report that their approach improved upon the heuristic approaches in [**101,102**] but involves greater computational effort. The authors also state that their algorithm can solve instances up to 40 patterns but requires a whole day's computation in the worst case which, in some situations, may not be acceptable. The research conducted by Faggioli and Bentivoglio typifies the need for fast heuristic approaches that can provide good, although not necessarily optimal, solutions on medium to large problem instances (50 rectangles or more).

There have been many heuristic approaches that have been formulated to produce good packings in an acceptable timeframe, even with large stock cutting instances. Examples of approaches that have handled large problem instances includes Albano and Orsini (1980) [**89**] and Benati (1997) [**103**]. Albano and Orsini used a heuristic method that packs similar rectangles into strips for large problem instances consisting of between 400 to 4000 rectangles (Albano and Orsini, 1980) [**89**]. Benati approached two-stage cutting guillotine packing using a heuristic approach based on the partial enumeration of all feasible layouts [**103**]. Benati demonstrated that the approach could produce layouts with, on average, 2% trim loss on real world problem instances of 1000 rectangles although, as problem size decreases, the trim loss percentage increases.

Bengtsson presented a heuristic solution that achieved trim loss of between 2% and 5% by sorting rectangles and then arranging them into piles (Bengtsson, 1982) [**104**]. One of the features of Bengtsson's approach was the dismissal of partial solutions for which it was impossible to produce a better solution than the best already seen. This required an assumption that all remaining rectangles would be packed with no wastage.

The most documented heuristic approaches are the bottom-Left (BL) and bottom-left-fill (BLF) methods. These are described in greater detail because they are used extensively within the experimentation of Part B of this thesis. The bottom-left heuristic was presented in Baker, Coffman and Rivest in 1980. It involved placing rectangles sequentially in a bottom-left most position. They examined different input sequences and determined that a sorted order of decreasing width results in a layout with no more than three times the optimal height [**105**].

Jakobs (1996) used a bottom-left method that takes as input a list of rectangles and places each one in turn onto the stock sheet [**106**]. The bottom-left placement strategy firstly places the rectangle in the top right location and makes successive moves of sliding it as far down and left as possible (figure 2.1). Jakobs then applied a genetic algorithm to manipulate permutations of rectangle sequences which are then 'decoded' into a layout by the bottom-left placement heuristic and evaluated based on their height. In 1980, Brown examined the lower bound of the bottom-left placement strategy and showed that there are groups of rectangles for which it is unable to produce the optimal solution [**107**]. This is the main disadvantage of the bottom-left placement heuristic.

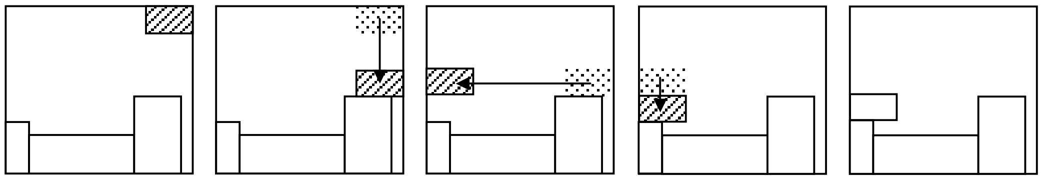


Figure 2.1. A Bottom-Left Method (Jakobs, 1996) [**106**]

Lui and Teng developed an improved bottom-left heuristic giving downward movement priority so that shapes only slide leftwards if no downwards movement is possible (figure 2.2) [**108**]. It was shown that, unlike the Jakobs method, there was always at least one rectangle sequence that could be decoded into the optimal solution.

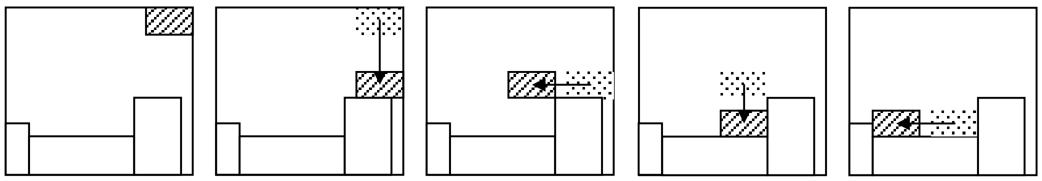


Figure 2.2. An Improved Bottom-Left Method (Liu and Teng, 1999) [**108**]

The second method, bottom-left-fill, is a modified version of the bottom-left placement heuristic. One implementation (used in [**109**]) maintains a list of location points in bottom left ordering to indicate where shapes may be placed. When placing a shape, the algorithm starts with the lowest and leftmost point, places the shape and left justifies it, then checks whether the shape would overlap with any other shape and that it stays within the confines of the sheet. If it does not overlap, the shape is placed and the point list is updated to indicate new placement positions. If the shape would overlap, the next point in the point list is selected until the shape can be placed without overlap occurring. Figure 2.3 shows an example of available points for placement.

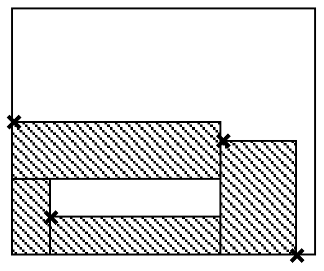


Figure 2.3. Storing placement locations for one implementation of bottom-left-fill

In figure 2.3, when placing a fifth shape the algorithm tries to place at the bottommost point first (with points at the same height being resolved by leftmost first). Therefore bottom-left-fill can overcome the problem of holes by the storage of possible location points. Specifically, this location-point procedure for the bottom-left-fill approach has been used for comparisons with the newly developed algorithms of this thesis in Part B. The difference between the two placement heuristics is shown when adding a fifth shape to the above packing (figure 2.4).

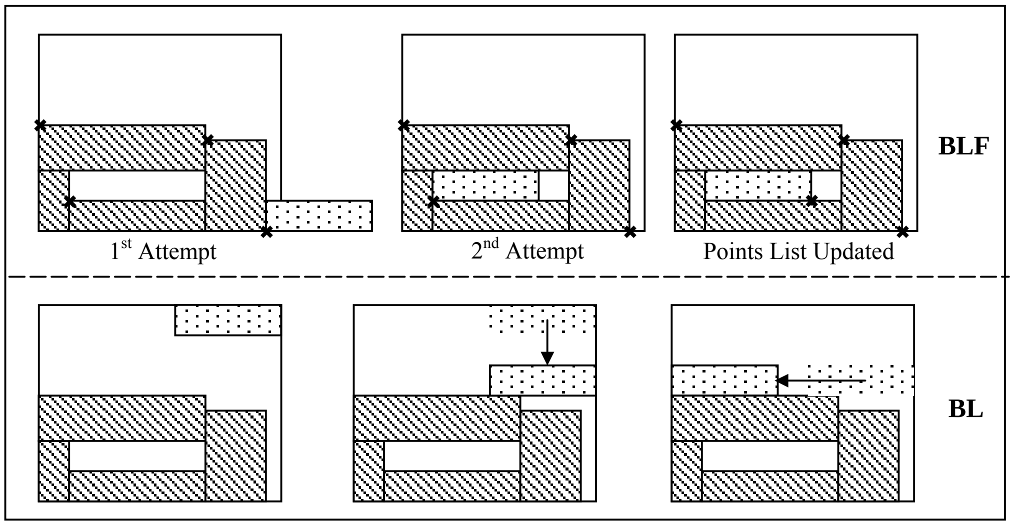


Figure 2.4. A comparison of the BL and BLF placement heuristics when adding a rectangle

Figure 2.4 shows that bottom-left-fill is able to fill holes by using rectangles later in the packing, whereas using bottom-left results in a hole in the final packing. This hole will never be filled (even if a later rectangle would fit) and therefore the space can be considered to be wasted. An important aspect of these algorithms is that the sequence of rectangles supplied can greatly influence the quality of the solution.

In experiments between the algorithms, Hopper and Turton found that bottom-left-fill outperformed bottom-left by up to 25% and that pre-ordering the shapes by decreasing widths or decreasing heights for both algorithms increased the packing quality by up to 10% compared to random sequences. The main advantage for using bottom-left is its time complexity of $O(N^2)$ (Hopper and Turton, 2001) [**110**]. The bottom-left-fill algorithm is disadvantaged by its worse time complexity of $O(N^3)$ (Chazelle, 1983) [**111**]. Consequently, execution times can become large for bottom-left-fill with increasing problem sizes.

Healy, Creavin and Kuusik presented a new generalised placement heuristic for the bottom-left placement of rectangles [**112**]. They reduce the problem of finding a feasible location for any given rectangle by performing a series of one-dimensional sweeps over the partial layout. Although the authors report that the approach performs with O(n log n) time, no experimental results have been given. An overriding feature of these placement heuristics is that they are deterministic and therefore the generated solution will be the same if supplied with an identical rectangle input sequence. They attempt to produce good quality packings in one attempt, reducing the time required to achieve solutions.

Nagao et al. approach the rectangle packing problem from the perspective of VLSI (very large scale integration) chip design [**113**]. This type of problem has the property that the size of the sheet is not prespecified but its rectangular area should be minimised (an important factor within microelectronics). Their approach involves calculating the boundary of the shapes that have been placed. This boundary has the property that it always travels across and up the boundary of the shapes within the partial layout and allows for definition of placement locations in the positions whereby the boundary element moves upwards. They show two examples produced by the approach, with and without rotations, and generate solutions of 2.91% over optimal without rotation and 1.65% with rotations within a few minutes. Further work has been conducted by Wu et al. who developed a “quasi-human” heuristic for approaching a similar non-bounded rectangle packing problem [**114**]. Their “less-flexibility-first” heuristic has been developed from observations of the techniques employed by skilled human ‘solvers’ and involves placement of larger pieces into the corners (where they can fit well) and packing the smaller objects in the spaces. The solutions have an average trim-loss (called the ‘unpack ratio’) of 2.68%.

In 2003, Lins, Lins and Morabito draw on their experience with the pallet loading problem to apply a recursive approach for congruent non-guillotine rectangle packing (multiple copies of just one rectangle) [**115**]. The algorithm groups rectangles into larger rectangles or L-shaped pieces which, in turn, are also grouped into rectangles or L-shapes. The authors claim that the approach has produced optimal solutions for all congruent problem instances (including 18 ‘hard’ instances which other heuristics cannot solve). The main drawback is that the approach can require significant memory overheads.

A recent trend has been to utilise metaheuristic approaches in conjunction with placement heuristics in order to search for better packings. These are usually hybridised algorithms involving the generation of input sequences that are then interpreted by placement heuristics such as bottom-left or bottom-left-fill (Jakobs, 1996 [**106**]; Ramesh Babu and Ramesh Babu, 1999 [**109**]; Healy, Creavin and Kuusik, 1999 [**112**]; Hopper and Turton, 2001 [**110**]).

However, alternative methods have been developed to allow temporarily infeasible solutions by placing all of the shapes onto the stock sheet and then applying small movements to the shapes in order to maximise the packing density whilst minimising an overlap penalty function (Dagli and Hajakbari, 1990 [**116**]; Dowsland and Dowsland, 1992 [**47**]).

Dagli and Hajakbari utilised metaheuristics and applied them to the rectangle packing problem. This was one of the first applications of metaheuristics to packing problems. They used a simulated annealing algorithm to pack irregular shapes using their rectangular bounding boxes onto a sheet of finite dimensions (Dagli and Hajakbari, 1990) [**116**]. As such, the generated layouts contain significant wastage when shapes are non-rectangular.

Other work has since appeared on the applications of simulated annealing to the stock cutting problem. Lai and Chan also use simulated annealing and test their difference process algorithm with real data from a printing company and report that their algorithm performs well with problems where less than 15 rectangles need to be packed (Lai and Chan, 1997) [**117**]. Further to this, the authors show how their packing approach can be generalised to the three-dimensional packing problem. In 2001, this work was extended by Leung, Yung, and Troutt who compared the sequence-pair difference process algorithm, which recursively defines rectangular regions of free space and holes within the sheet when placing shapes, to the bottom-left algorithm (presented in [**105**]) when both algorithms are hybridised with simulated annealing and genetic algorithms [**118**]. The authors report that the genetic algorithm approach generated layouts with less trim-loss than when hybridising with simulated annealing and, in general, the difference process placement algorithm outperformed the bottom-left approach. The bottom-left-fill approach, which is known to perform better than bottom-left, was not compared. Further examination of the sequence-pair approach was presented in the same year by Imahori, Yagiura and Ibaraki [**119**].

Faina applies simulated annealing to both the guillotine and non-guillotine variants of the problem and show that, for larger problems, non-guillotine significantly outperforms guillotine with only a small computational overhead (Faina, 1999) [**120**]. In the non-guillotine algorithm, a randomised bottom-left placement heuristic (although it was not called this in the paper) is employed that places the first shape in the bottom leftmost corner and then places subsequent rectangles randomly at the bottom right or top left corner of the first item. This placement algorithm is unlike the previously described bottom-left methods due to the randomness within placements.

Genetic algorithms have also been extensively studied for this problem over the last decade. Kröger used genetic algorithms for guillotine bin packing and introduced the idea of 'metarectangles', which are groupings of rectangles that can pack well together and can be treated as one single rectangle in future search iterations (Kröger, 1995) [**121**]. Kröger examined the genetic approach against random search and simulated annealing with the conclusion that the genetic method is superior. Ramesh Babu and Ramesh Babu detail improvements over the genetic method proposed by Jakobs in [**106**]. Their approach is able to pack onto many finite sheets by also including an ordering of sheets at the start of the chromosome of each population member (Ramesh Babu and Ramesh Babu, 1999) [**109**]. Hopper and Turton provide a review of genetic approaches that have been applied to the rectangle packing problem. They found that a genetic algorithm with bottom-left-fill placement outperformed a genetic algorithm with bottom-left placement (Hopper and Turton, 1999) [**122**]. Genetic methods have also been applied to bin packing problems (Falkenauer, 1996) [**123**] and guillotine variants of the orthogonal stock cutting problem (Valenzuela and Wang, 2001) [**124**]. Valenzuela and Wang compare three approaches: i) a simple level oriented approaches (whereby rectangles are placed in successive rows), ii) split algorithm (which involves defining two new sheets each time a shape is placed), and iii) the authors genetic bottom-up constructive approach (combining rectangles into pairs and then recombining recursively). The authors report that the genetic approach produced better results than the other methods for instances of less than 100 rectangles but was outperformed by a simple first-fit-decreasing-height level oriented heuristic on larger problems.

In 2003, Onwubolu and Mutingi proposed a genetic approach for the guillotine rectangle packing problem [**125**]. The approach firstly places the initial rectangle and then searches for a rectangle that can be attached to the first whilst maintaining the guillotine cut constraint. The chromosome within the genetic algorithm represents the piece pairs that are to be attached and the orientation in which they should be placed. The authors reported an improvement on over 60% of the problems used in the work of Benati in [**103**]. In 2004, Mukhacheva and Mukhacheva approached the rectangle packing problem using block structures (defining the horizontal and vertical divisions between placed rectangles) in conjunction with local search and genetic algorithms [**126**]. The genetic approach produces the best solutions although only small to medium problem instances consisting of between 20 and 100 rectangles are used. The authors state that the block structure representation offers an alternative to the more common placement heuristic and can be used in conjunction with exact, heuristic and parallel approaches.

Dagli and Poshyanonda compare the nesting of rectangular patterns using two methods involving artificial neural networks [**127**]. This is a continuation from their previous work presented in [**128**]. The first neural network method was trained through back propagation. The second method involved a neural network / genetic algorithm combination. Their methods produced an average waste of 7.88%.

Hopper and Turton compare several metaheuristics including genetic algorithms, simulated annealing, naïve evolution, hill climbing, and random searches using both bottom-left and bottom-left-fill placement heuristics. Genetic algorithms, simulated annealing, and naïve evolution (with the bottom-left-fill placement heuristic) all gave similar results and the authors show that they are better than using the bottom-left-fill heuristic with height or width sorted input sequence although extra computation time is required (Hopper and Turton, 2001) [**110**]. Metaheuristic algorithms generate a number of different solutions i.e. the metaheuristic algorithms are guided through the problem's search space by previous attempts. However, there are usually search variables that must be tuned to achieve best performance. Metaheuristic techniques are used for rectangle packing in Part B of this thesis.

In 2003, Lesh et al. examined the effectiveness of the bottom-left-fill approach and hybridised the algorithm with a localised random search algorithm that generates random orderings chosen from a distribution determined by the Kendall-tau distance (also known as the 'bubblesort distance') from the presorted orderings (decreasing height, width, area and perimeter) [**129**]. In this way, the authors generate orderings that are slight variations on the presorted sequences that are known to perform well. Lesh et al. showed that the bottom-left-fill heuristic alone is outperformed by the random search version. This is not surprising as the random search tries several orderings that have been drawn from variations of the presorted inputs whereas the bottom-left-fill heuristic alone only tries the presorted input. The authors also develop a user interface and allow the user to interact with the system to move individual rectangles, 'freeze' rectangles, or apply the randomised search to the 'unfrozen' rectangles. Through this user interaction, they report improved solutions on problem instances involving up to 97 rectangles.

In chapter five, an interactive user-guided approach is demonstrated and investigated using the new algorithms that are developed within chapters three and four on the set of literature benchmark problems (see section 2.3).

2.3 Rectangular Benchmark Problems from the Literature

In order to compare the relative performance of the rectangular packing approaches that have been developed in this thesis to other heuristic and metaheuristic approaches, thirty-four rectangular test problems from the literature have been gathered and are summarised in table 2.2. Perhaps the most extensive data given for this problem is found in Hopper and Turton (2001) [**110**] where 21 problem instances of rectangle data are presented in seven different sized categories (each category has three problems of similar size and object dimension). Valenzuela and Wang provide floating-point data sets of both similarly dimensioned rectangles (named 'nice' data) and vastly differing dimensions (named 'pathological' data). Each category has data ranging from 25 to 1000 rectangles (Valenzuela and Wang, 2001) [**124**]. Ramesh Babu and Ramesh Babu [**109**] use a test problem to compare their GA method against the GA method proposed in [**106**]. The optimal solution heights are known for all of these instances as they are created by the deconstruction of a larger rectangular sheet with random dissections.

Data Source	Problem Category	Test Problems Given	Number of Rectangles	Optimal Height	Object Dimensions
Hopper and Turton, 2001	C1	P1, P2, P3	16 or 17	20	20 x 20
	C2	P1, P2, P3	25	15	40 x 15
	C3	P1, P2, P3	28 or 29	30	60 x 30
	C4	P1, P2, P3	49	60	60 x 60
	C5	P1, P2, P3	72 or 73	90	60 x 90
	C6	P1, P2, P3	97	120	80 x 120
	C7	P1, P2, P3	196 or 197	240	160 x 240
Valenzuela and Wang, 2001	Nice	P1, P2, P3, P4, P5, P6	25,50,100,200, 500,1000	100	100 x 100
	Path	P1, P2, P3, P4, P5, P6	25,50,100,200, 500,1000	100	100 x 100
Ramesh Babu and Ramesh Babu, 1999		P1	50	375	1000 x 375

Table 2.2. Rectangular benchmark instances from the literature

2.4 Two Dimensional Irregular Packing

Whereas rectangle packing problems have been the focus of considerable academic research since the formulation of cutting and packing in the 1950s, the irregular variant of the problem had not received as much interest until the last 20 years. This is inevitably due to the irregular geometry that must be modelled and computationally expensive geometric operations that are needed to perform manipulations upon the shapes. However, as the irregular problem occurs within several important industries, it has seen considerable academic and industrial interest. Along with improvements in computing technology, this has rapidly driven forward the research area with an increasing number of novel strategies being developed and disseminated within the academic community and industry. The related literature for irregular variants of two-dimensional packing is the subject of the following section (2.4.1). Once again, the literature is discussed chronologically.

2.4.1 Related Literature for Irregular Problems

The majority of the work conducted on irregular packing before 1980 was concerned with firstly clustering into rectangles and then using exact approaches to pack the rectangular modules. One of the first instances is found in Haims (1966) who firstly clustered irregular shapes into their minimum bounding rectangles and then packed them using an approach based on dynamic programming [**130**]. The author conducted this work within a Ph.D. thesis but then extended the work in Haims and Freeman (1970) [**131**].

Adamowicz and Albano (1976) also cluster shapes into their minimum bounding rectangle by the use of the no-fit polygon which defines the positions with which two polygons touch but do not intersect (originally proposed by Art in 1966 [**132**] and discussed further in chapters six and eight of this thesis). These modules are then packed onto the stock sheet using dynamic programming approaches. This work was extended in [**85**] where the authors investigated methods of determining whether two polygons can be packed together efficiently to produce a sub-cluster. Albano appreciated that unless the rectangular modules could be completely filled by the shapes, there would be intra-rectangle trim-loss that could not be recovered during the packing phase [**133**]. In order to rectify this, the solution produced by their dynamic programming methods was displayed and the user was allowed to manually adjust the layouts in order to create tighter packings.

Some of the most significant of the earlier papers on irregular packing was conducted by Albano and Sapuppo (1980) who approached the two dimensional problem using a nesting algorithm which also utilised the no-fit polygon (NFP) in addition to a profile simplification method to reduce the geometric complexity of the nesting process (the profile represents the leading edge of the shapes placed on the nest). Available space behind the leading edge is ignored causing this method to perform as a bottom-left (BL) packing algorithm (i.e. with no hole filling capabilities) [**134**]. Each shape was placed in turn by examining the right profile of the partial layout and using the no-fit polygon to place the shape in the left and lowest feasible position. This was one the first times that an irregular packing approach was not required to pack into rectangular modules first (this was also the first time that the specific term “no-fit polygon” had been used). Other non-rectangular based approaches have been presented that operate directly on the irregular shapes in [**135**] and [**136**] although they did not include an improvement phase unlike the approach of Albano and Sapuppo which used a simple A* heuristic search.

In 1984, Dori and Ben-Bassat considered the template layout problem involving the tiling of a shape onto a stock sheet [**137**]. The first stage of their proposed algorithm was to calculate the convex hull of the shape. The convex hull is the minimum convex shape that encloses the shape (this can be pictured by imagining the path which an elastic band would take when stretched around a shape). Finally, the convex hull was placed inside one of a set of tessellatable tiles which was then repeated over the sheet.

Qu and Sanders (1987) proposed an approach that used a bottom-left style placement algorithm and two possible strategies of shape representation [**138**]. The first of these involved representing the irregular shape as a set of rectangular building blocks and the second involved the rectangular enclosure. The building block representation inherently introduces a certain degree of inaccuracy into the problem but the authors explain that this can be avoided by using smaller blocks. Therefore, irregular shape data is converted to a set of rectangles which allows for the faster rectangle overlap testing to be employed. A vector based representation was discarded because the geometric calculation was too great (with the improvement of computer technology, this is no longer such a significant factor).

Prasad and Somasundaram (1991) [**139**], and subsequently Prasad et al. (1995) [**140**], addressed the metal blank nesting problem. This problem usually involves only one shape (although the authors report up to three being nested). They used an algorithm based on sliding techniques to find all possible arrangements of two shapes. After clustering two items, their boundaries are combined to create a 'supershape'. This supershape can then be clustered with another shape and so forth. In 1990, Jain et al. also approached the blank nesting problem but used a simulated annealing heuristic approach [**141**]. When the authors compared their proposed method to a multi-start hill climbing approach, they report solutions with significantly improved trim-loss.

Marques, Bispo and Sentieiro (1991) approached the problem by using a grid approximation to reduce the complexity of the nesting process [**142**]. This, in combination with a simulated annealing search (to control compaction via Markov chains), produces a result for their data set in just over 24 hours. Whilst it is difficult to determine the exact quality of the produced solution because numerical results are not given, the density of the solution has been estimated as 69% from the figure within the paper.

Blazewicz, Hawryluk & Walkowiak (1993) presented an extension to the work performed by Albano & Sapuppo (1980) [**134**]. Their method produces a bottom-left-fill style algorithm which performs well against their chosen data sets. Their approach attempts to backfill holes in the existing layout before attempting to place a shape on the leading edge of the nest. The technique utilised a tabu search method to produce moves from one nest to another. The authors noted that the main problem is in the definition of an algorithm for checking the feasibility of the move produced by the process [**143**].

Fujita, Akagi and Hirokawa (1993a) presented a two part approach consisting of a GA combinatorial search and a local minimisation function for nesting [**144**]. The GA manipulates shape pairs which hold information about their positional relationship with one another. A hybrid crossover operator produces child chromosomes which are used to create a nest via the local minimisation function. The method then attempts to place these shape pairs in a bottom left fashion. Overlapping solutions are not allowed but overhang from the required sheet width is acceptable, although penalised in the evaluation function. They only use convex shapes to reduce the time complexity of identifying shape intersections.

Oliveira & Ferreira (1993) discussed two approaches to the problem. Both are driven by a simulated annealing algorithm [**145**]. The first approach is based on a bitmap raster representation of the shapes to be nested. This approximation allows for the fast generation of solutions but suffers from inaccuracy, caused by the approximation inherent in the raster representation. The second approach uses a polygon-based representation where D-Functions (Konopasek, 1981) [**146**] are used to identify and resolve overlap in the solutions. Both methods allow overlap in the solutions, the extent of which is penalised by the evaluation function. The algorithm aims to reduce the overlap to zero, but allows worse moves based on the cooling schedule applied. The second implementation always produces feasible results but performs five times slower than the first method where overlap is allowed.

Dowsland & Dowsland (1993) discussed simple geometric techniques for the identification of overlaps in a nested solution and they present several new algorithms which use these techniques [**147**]. They also presented and discussed computational results for the various methods that they develop. One method in particular, the jostle algorithm, produces excellent results when used to improve an existing nest. The jostle procedure iteratively shifts all shapes to the right and then the leftmost boundaries of the plate. Hole filling is attempted on each move. The method gradually fills holes in the packing and improves the solution. A significant benchmark of 63 units length was set for one particular benchmark instance, SHAPES0 (see table 2.3), using this algorithm. This represents an improvement of 20.6% on the previous best solution from Oliveira & Ferreira (1993) [**145**]. Other algorithms presented in the work also surpass the previous best solution for SHAPES0 but not to the degree of the jostle algorithm, or with the speed of the jostle algorithm which is able to improve solutions within a "few minutes".

Li and Milenkovic investigated compaction and separation methods that can be used to generate solutions of less trim-loss [**69**]. The authors stated that compaction can be thought of as simulating 'forces' upon the shapes and develop a velocity based optimisation model based on the laws of physics. Also presented is a positional optimisation model which uses linear programming techniques and runs in half the time of the velocity based method. The authors also consider the problem of separating overlapping polygons using the least amount of motion. Using the compaction and separation methods, solutions of a quality that approach human performance are reported. A key feature of their compaction and separation algorithms is the no-fit polygon which can be used to calculate the distance required to move one polygon such that it touches another or, conversely, to remove overlap using the smallest distance. Milenkovic has conducted a considerable amount of further work using similar approaches for rotational polygon overlap minimisation, compaction, containment and for calculating minimum enclosures [**148,149**]. Milenkovic summarised that compaction and separation approaches can further improve layout quality in these problems. An example is given for a textile manufacturer who outlays $100 million in cloth costs. It is approximated that $250,000 is saved each year through compaction processes.

Dighe & Jakiela (1996) extended the work of Smith (1985) [**150**] by adding geometric extensions to cope with irregular polygons [**151**]. Their use of a complex genetic algorithm chromosome which is able to avoid the generation of solutions with overlap significantly improves the performance of previous work. Their two stage approach generates clusters of shapes, in valid and tightly packed positions, which are then manipulated through a tree structure by a GA searching for the best arrangement of those clusters. The generated solutions are evaluated by the area of the rectangular enclosure around the clusters. Dighe and Jakiela's problems are *jigsaw* in nature in that they were generated from a single rectangle through a process of subdivision and therefore have known optima.

Jakobs (1996) tackled problems found within the sheet steel stamping/punching industry [**106**]. The work developed upon the rectangle packing approaches from Baker, Coffman and Rivest (1980) [**105**], using a GA approach to improve on the orthogonal results of earlier work. The paper then extends this to irregular polygons. Jakobs's approach used polygons in their minimum bounding rectangle orientation and places them using the orthogonal method followed by a compaction phase. The compaction is a step wise bottom left algorithm that continues until no shape can be moved any further towards the bottom or left of the sheet respectively. Jakobs also discussed the idea of clustering several shapes, finding the minimum bounding rectangle of these clusters and packing them orthogonally with the use of the same compaction phase to improve the layout but the results from this approach are not reported.

Bounsaythip & Maouche (1997) utilised an evolutionary algorithm approach which improves on the results from Fujita, Akaji & Hirokawa (1993) [**144**]. The solution uses 'comb codes' to represent the rectangular deficiency (the free regions within the bounding rectangle) that exists when shapes are placed together [**152**]. They approach the problem from a textile industry perspective, where the practicalities of cutting mean that strips of a user defined length across the sheet of fixed width are preferable. This means that direct comparison of results is problematic. The nesting of shapes is again performed on two levels: the lower level uses the comb approach to find the minimum bounding rectangle for two shapes and the upper level employs an evolutionary algorithm to search through a tree of possible combinations.

Ratanapan & Dagli (1997) described an improved approximation method, similar to the computer graphics technique of anti-aliasing (Foley et. al, 2003) [**153**], with the aim of reducing the computational time required to generate solutions. The authors reported high densities achieved during runs with a higher resolution approximation [**154**].

In 1998, Dowsland, Dowsland and Bennell developed a 'jostle' approach that is inspired by the settling of granular products within a container when repeatedly shaken up and down [**155**]. The algorithm starts by placing all shapes onto the sheet using a bottom-left placement heuristic (given an initial ordering). After all pieces have been placed, they are reordered in decreasing order of their rightmost *x* coordinates and packed using a rightmost placement policy. The pieces are then reordered in an increasing order of their leftmost points and the process repeats for a fixed number of iterations. The authors used both traditional trigonometry intersection calculations and the no-fit polygon. The authors report promising results for data with different characteristics with the exception of problems in which pieces have the same or similar lengths.

Grinde and Daniels (1999) presented an approach that tackles a problem from the textile industry and, in particular, with layouts for the cutting of trouser parts [**156**]. This style of problem involves distinct large and small pieces (such as leg panels and pockets respectively). The authors reported that in an ideal situation the larger pieces are placed onto the material and the smaller pieces can fill the holes that occur. They presented a heuristic for the assigning of smaller items to the set of container objects (holes) that are defined by the larger pieces and, therefore, both the sheets (holes) and the items are highly irregular in nature. High quality solutions are reported (including some best-known solutions) but it is reported that the required time varies considerably. Anand, McCord and Sharma (1999) also tackled problems for the apparel industries and propose a genetic algorithm approach which the authors stated can also be adapted to fit the special requirements of other problems [**157**]. The sheets and parts can involve irregular shapes although arcs are represented by line approximations (polygonisation) which, as the authors conceded, is introduced at the cost of inaccuracy and error. In Part C of this thesis approaches are developed for the fast and accurate nesting of irregular shapes that can include full arc primitives *without line approximations*.

Cheng and Rao (2000) tackled the clustering and nesting of multiple congruent pieces. They proposed an algorithm that evaluates the possible orientations of the piece and spatial relationships that exist between copies [**158**]. Then they utilise a genetic algorithm to modify both the nesting direction and the orientation in which pieces are to be placed (the orientation is maintained for all copies due to machine setups). This work was an extension from the authors' previous work on blank nesting and clustering of pieces that was presented in [**159**] and [**160**]. The authors also proposed a method for the nesting of multiple pieces but did not investigate this extensively and conclusions cannot be drawn [**160**]. Other work on the packing of congruent polygons has been conducted by Joshi and Sudit (1994) [**161**], Watson and Tobias (1999) [**162**] and Stoyan and Pankratov (1999) [**163**].

The packing of circles still offers a considerable academic challenge but benefits from more simple intersection detection. If the distance between two circle centres is smaller than the sum of the radii then the circles must be overlapping. The problem has been the subject of two recent papers. In 2001, WenQi, Yu and RuChu explained that there has been no significant research into the packing of different sized circles into a circular container and propose the modelling of the physical system of potential energy and elasticity (found within particle systems) to solve the problem [**164**]. The aim is to minimise the radius of the containing circle of an arrangement of smaller circular pieces. The authors show how this approach can be represented as a steepest decent gradient search (i.e. hill-climbing) and report results for problems of up to 19 pieces. Although the authors present very good results, execution times can become large and can take several hours on the largest of the instances. Hifi and M'Hallah approached the problem of packing different sized circles onto rectangular sheets in 2004 [**165**]. They use two heuristic methods to produce solutions: i) a constructive heuristic and a genetic algorithm based heuristic. In comparisons against other exact approaches the simple constructive heuristic produces solutions with a worse material utilisation whereas the genetic method improves upon the previous literature methods (obtaining material utilisations of around 80% in each instance).

In 1999, Bennell and Dowsland utilised a tabu-thresholding approach for irregular packing. The main feature of the approach is that overlaps are allowed but are penalised within a cost function and a method for approximating the horizontal overlap of pieces using shape bounding boxes is presented [**166**]. The authors extensively investigate the parameters for tabu search and report that their presented tabu method is competitive to many other generic approaches within the literature. In 2001, the authors extended this work by hybridising the tabu approach with a further optimisation phase to find the best solution within an infinite neighbourhood [**167**]. Also presented was a second hybrid that allowed neighbourhood moves that were not in the scope of the original algorithm. The authors reported significant improvements over their previous work. One important aspect of their work is that rotations were disallowed.

Hopper (2000) was the first practitioner in this area to collate numerous examples of problems from different papers and to attempt to produce results for all the gathered data [**26**]. A genetic algorithm in combination with bottom-left and bottom-left-fill approaches are reported in this work for both orthogonal and irregular nesting problems. Within the orthogonal field both guillotine and non-guillotine problems are attempted. The irregular nesting approach also includes results gathered from commercial nesting software, which sometimes outperformed the presented methods. The work contains information on many authors' approaches. Further to the collection of data from other authors, usually through a process of scanning shapes from illustrations in publications, Hopper added nine new polygon based problems to her results. These problems are randomly generated convex and concave polygons ranging from fifteen to seventy five in quantity.

Oliveira, Gomes & Ferreira (2000) approached the irregular nesting problem with a no-fit polygon solution that produces good results [**168**]. They tackled several problems; most of which have been generated by the authors in previous papers (from the fabric cutting industry). They also tackle one problem from the paper presented by Blazewicz, Hawryluk & Walkowiak (1993) [**143**]. They produced new best-known benchmarks for the majority of the problems. Their approach involved generating an outline polygon for all shapes already nested onto a sheet. This polygon is then used in the generation of a no-fit polygon for the next shape to be nested. The algorithm is controlled with either the aim of producing a nest with the minimum bounding rectangle or minimum length. The shape will be nested in a non-overlapping position touching at least one other shape already on the plate, allowing its addition to the profile of the shapes on the sheet before beginning the process again. Oliveira et al. compared their results against an implementation of the Albano & Sapuppo (1980) algorithm [**134**] and against results achieved in Blazewicz, Hawryluk & Walkowiak (1993) [**143**] and Dowsland & Dowsland (1993) [**147**]. They report improvements on three of the five attempted problems, the best result being 6.2% better than any known solution and the worst being 4% worse than the best known solution. The work also provides the data for these problems, in vertex form, as an appendix.

Dickinson and Knopf (2000) presented a method for evaluating the quality of a produced layout. The motivation for the new metric is that traditional material utilisation evaluation functions are unable to distinguish between layouts that exist in the same amount of space [**169**]. The authors stated that the optimal clustering of shapes exists in when they are placed within a circle and describe a point moment based approach. Several examples of it use are provided and it is shown how the approach can be used for three-dimensional and N-dimensional problems with which the optimal arrangements exist in a sphere and hyper-sphere respectively.

In 2001, Ramesh Babu and Ramesh Babu approached the problem of packing pieces onto irregular shaped sheet through the use of a novel indexed bitmap representation that stores horizontal scan information of the distance that each pixel of the shape is from its rightmost non-shape pixel [**170**]. This allows overlaps to be detected more efficiently when coupled with a layout generation process (which is hybridised with genetic and heuristic algorithms). The authors reported improved material utilisation compared to previous approaches from the literature including their own work presented [**109**] and the work of Jakobs in [**106**].

The problem of nesting onto irregular sheets with genetic algorithms was also tackled by Tay, Chong and Lee (2002) [**171**]. The fundamental principal of the procedure is that any shape can be placed such that at least one of its vertices touches the stock sheet boundary. The optimal position along the boundary is found using a genetic algorithm. The chromosome indicates both the distance along the boundary from an origin point (also on the boundary) and the angle with which the shape is to be placed. Once a shape is placed, it is negated from the sheet boundary which creates a new boundary for future shape placements. The process continues until all shapes have been placed. The authors state that as the boundary space becomes smaller the genetic algorithm's parameters need to be further tuned to perform a more extensive search. In particular, they recommend that the population size and the number of generations should be increased in such circumstances.

Gomes and Oliveira (2002) [**172**] developed shape ordering heuristics for an extended nesting algorithm similar to that used in the authors' previous work of [**168**] and [**173**]. The nesting algorithm is improved by the introduction of the inner-fit rectangle which is the interior no-fit polygon for a shape about to be nested and the rectangle of the sheet on which the nest is generated. The vertices and intersection points of the inner-fit rectangle with the separate no-fit polygons of the shapes already nested, and the shape to be nested, produces all the non-overlapping feasible positions for the shape. In addition to the extension of the geometric techniques, the paper introduces a 2-exchange heuristic for manipulating the order of shapes which are then nested onto a sheet. The paper presented results for numerous experiments which use various initial ordering criteria, e.g. random, longest length and greatest area, using the 2-exchange heuristic to develop better solutions over a number of iterations. The results for the improved geometry and new heuristic set new benchmarks for the SHAPES1, SHAPES2, SHIRTS and TROUSERS data sets. Only the SHAPES0 data set is not improved upon, the best solution of 63 units from Dowsland & Dowsland (1993) [**147**]. However, Gomes and Oliveira improve on their own best solution for the problem [**168**].

Dowsland, Vaid and Dowsland (2002) presented an irregular packing approach using the bottom-left methodology [**174**]. In particular the authors used the no-fit polygon (although no implementation details are given for its creation) in order to resolve overlaps and pack shapes in a bottom-left strategy. One simplification employed by the authors is that the shapes must remain in their original orientations (i.e. they may not be rotated). They suggested methods to improve the algorithm efficiency that includes maintaining the front and back edges of the polygons and storing the location of previously placed shapes (as future shape copies may start from the position that the last copy was placed). The algorithm was only investigated with a one-pass solution based on sorted orderings, that is to say, there was no iterative improvement searches. Nesting specific algorithmic efficiencies that have been employed in the developed approaches during this research programme for irregular packing are described in chapter seven of this thesis.

In 2003, Wu et al. addressed the problem of nesting onto multiple sheets [**175**]. They propose an algorithm that revisits the packing of irregular shapes into rectangular modules and suggest a greedy heuristic rule that can determine the number of embedding rectangles to use in order to maximise material utilisation. An additional constraint is that each piece has an associated 'minimum demand' which must be observed. Both simulated annealing and genetic algorithms are proposed for an iterative improvement phase (extended from [**109**]) and a compaction phase is included to move pieces together at the end of the search. The authors report computational results on data from the literature and examples from the footwear industry with which an average material utilisation of around 70% is reported.

2.5 Irregular Benchmark Data from the Literature

In order to compare the irregular packing algorithms that have been developed within Part C of this thesis, all the relevant test problems from the literature have been gathered (at least, all that could be found). Most of these problems were first collected by Hopper (2000) [**26**] and now also feature on the EURO Special Interest Group on Cutting and Packing (ESICUP) website[1].

Hopper introduced 10 new problems, 9 of which were randomly generated and consist of varying quantities of similar polygonal shapes [**26**]. Hopper used the minimum bounding rectangle orientations for these problems and, therefore, it is necessary to rotate the shapes into their minimum bounding rectangle orientation before applying each problem's rotation criteria. The remaining problem is an existing literature instance with new rotational criteria applied.

Oliveria, Gomes & Ferreira (2002) presented 5 new problems, 3 of which are drawn from the textile industry [**172**]. Blazewicz, Hawryluk & Walkowiak (1993) [**176**], Jakobs (1996) [**106**] and Dighe & Jakiela (1996) [**151**] introduced two problems each to the collection. The remaining problems have each been contributed by different practitioners.

Table 2.3 shows the 21 problems and provides the best known results using a length based evaluation. Table 2.4 contains 5 problems where the best known solution is evaluated by density measures.

Further to these problems, the author has provided ten more irregular problem instances involving shapes that are common to the metal cutting industry (presented in section 7.3).

[1] http://www.apdio.pt/sicup/

Original Author	**Problem Name**	**Shapes**	**Rotational Constraints**	**Sheet Width**	**Best Length**	**Best Result Reference**
Blazewicz, Hawryluk & Walkowiak(1993) [**143**]	Blasz1	28	0, 180 Absolute	15	27.3	Gomes & Oliveira (2002) [called SHAPES2] [**172**]
Ratanapan & Dagli (1997) [**154**]	Dagli	30	90 Incremental	60	65.6	Hopper (2000) [using NestLib] [**26**]
Fujita, Akagi & Kirokawa (1993) [**144**]	Fu	12	90 Incremental	38	34	Fujita & Akagi (1993) [**144**]
Jakobs (1996) [**106**]	Jakobs1	25	90 Incremental	40	13.2	Hopper (2000) [using SigmaNest] [**26**]
Jakobs (1996) [**106**]	Jakobs2	25	90 Incremental	70	28.2	Hopper (2000) [Simulated Annealing] [**26**]
Marques, Bispo & Sentieiro (1991) [**142**]	Marques	24	90 Incremental	104	83.6	Hopper (2000) [Naïve Evolution] [**26**]
Hopper (2000) [**26**]	Poly1A	15	90 Incremental	40	14.7	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	Poly2A	30	90 Incremental	40	30.1	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	Poly3A	45	90 Incremental	40	40.4	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	Poly4A	60	90 Incremental	40	56.9	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	Poly5A	75	90 Incremental	40	71.6	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	Poly2B	30	90 Incremental	40	33.1	Hopper (2000) [using SigmaNest] [**26**]
Hopper (2000) [**26**]	Poly3B	45	90 Incremental	40	41.8	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	Poly4B	60	90 Incremental	40	52.9	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	Poly5B	75	90 Incremental	40	63.4	Hopper (2000) [using NestLib] [**26**]
Hopper (2000) [**26**]	SHAPES	43	90 Incremental	40	63	Hopper (2000) [Simple Heuristic] [**26**]
Oliveira & Ferreira (1993) [**145**]	SHAPES0	43	0 Absolute	40	63	Dowsland & Dowsland (1993) [**147**]
Oliveira & Ferreira (1993) [**145**]	SHAPES1	43	0, 180 Absolute	40	59	Gomes & Oliveira (2002) [**172**]
Oliveira & Ferreira (1993) [**145**]	SHIRTS	99	0, 180 Absolute	40	63.13	Gomes & Oliveira (2002) [**172**]
Oliveira, Gomes & Ferreira (2000) [**168**]	SWIM	48	0, 180 Absolute	5752	6568	Hopper (2000) [using NestLib] [**26**]
Oliveira, Gomes & Ferreira (2000) [**168**]	TROUSERS	64	0, 180 Absolute	79	245.75	Gomes & Oliveira (2002) [**172**]

Table 2.3. Length evaluated irregular benchmark problems from the literature

Original Author	**Problem Name**	**Shapes**	**Rotational Constraints**	**Sheet Width**	**Best Density %**	**Best Result Reference**
Albano & Sappupo (1980) [**134**]	Albano	24	90 Incremental	4900	86	Hopper (2000) [Simulated Annealing] [**26**]
Blazewicz, Hawryluk & Walkowiak (1993) [**143**]	Blasz2	20	90 Incremental	15	68.6	Blaszewicz, Hawryluk & Walkowiak(1993) [**143**]
Dighe & Jakiela (1996) [**151**]	Dighe1	16	90 Incremental	100	72.4	Hopper (2000) [using NestLib] [**26**]
Dighe & Jakiela (1996) [**151**]	Dighe2	10	90 Incremental	100	74.6	Hopper (2000) [Genetic Algorithm] [**26**]
Bounsaythip & Maouche (1997) [**152**]	Mao	20	90 Incremental	2550	71.6	Hopper (2000) [Genetic Algorithm] [**26**]

Table 2.4. Density evaluated irregular benchmark problems from the literature

2.6 Optimisation and Search Methods

This section details some of the search methods that have been used for automated packing approaches. The seminal literature for each search method is provided where known and some examples of where they have been applied to stock cutting problems are given.

2.6.1 Iterative Improvement Searches

The first class of searches involve so-called "iterative improvement searches" whereby the *next solution* to evaluate belongs to the *neighbourhood* of the *current solution*. Such searches require the definition of a suitable neighbourhood function to obtain a neighbouring solution. In terms of rectangle packing, it is common to use a placement heuristic that places rectangles onto the sheet in a given sequence. The sequence of rectangle identification numbers can be used to generate a neighbouring solution, for example, by swapping two of the elements of the sequence such as:

$$1,2,3,4,5,6,7,8 \rightarrow 1,6,3,4,5,2,7,8$$

In this example, rectangles 2 and 6 have been swapped to produce a neighbouring solution which can then be evaluated. After evaluating the sequence (by packing the rectangles using a placement heuristic and examining the height and/or density), an action must be taken based on the knowledge that is drawn from the evaluation. The nature of this action depends on the specific search that is undertaken.

A further point of note is that all searches require an initial solution to be generated. Generally, this is achieved by applying a constructive (normally quick) heuristic. In the case of the work within this thesis, the initial solution is always based on the sequence produced by sorting the shapes by decreasing height, width or area. This offers a reasonably good solution because the larger shapes are placed first and the smaller shapes can be placed in the holes between the larger shapes or at the end of the layout. If the sequence was generated in the reverse, then the larger shapes would all be placed at the end of the layout which would adversely affect the solution quality.

2.6.1.1 Hill Climbing

The hill climbing algorithm (also known as the "gradient descent algorithm" when an evaluation function is to be minimised) probably offers one of the simplest strategies for optimisation problems. It involves maintaining one current solution and generating one neighbouring solution, which is then evaluated. If the neighbour evaluates to a better quality solution than the current solution, the current solution is replaced by the neighbour and the search continues by generating a neighbour solution of the new current solution. If the neighbour is of worse quality than the current solution, it is discarded and the process continues with the original solution. At the end of the search, the best solution seen is returned (this will always be the current solution in the case of hill climbing).

The main drawback of hill climbing is that it can pursue areas of the search space that result in only a local optimum being found and because there is no means of backtracking or acceptance of worse solutions. One proposed solution to avoid this problem is to randomly restart the search if there has not been an improvement in solution quality for a specified number of iterations.

Multi-neighbourhood hill climbing techniques have been utilised within chapter seven and chapter nine of this thesis.

2.6.1.2 Tabu Search

The tabu search algorithm extends the idea behind hill climbing and was first proposed in Glover (1977) [**177**] but was not widely known and adopted until after Glover (1989) [**178,179**]. Unlike hill climbing, tabu search evaluates a subset of neighbouring solutions at each iteration of the search and moves to the solution with the best evaluation value. The current solution is always replaced with the best neighbour even if the new solution has a worse quality evaluation. In addition to this, the approach also maintains a bounded list that contains a history of previously seen solutions. This "tabu" list is used to avoid revisiting areas that have already been examined in the recent history of the search. If this was not employed with a full neighbourhood search, the search could potentially get stuck by alternating between the two solutions (or a cycle of more than two moves) with the locally best quality evaluations. The tabu list is not so important when only evaluating a subset of the neighbourhood as long as the neighbouring solutions are chosen randomly.

Tabu search is used within chapters three, four, seven and nine of this thesis. Other examples can be drawn from the following cutting and packing literature: [**143,166,167,176,180**].

2.6.1.3 Simulated Annealing

Simulated annealing originates from an algorithm presented by Metropolis et al. for the simulation of the annealing of solids (Metropolis et al., 1953) [**181**] although the first application of simulated annealing to searching within combinatorial optimisation problems was proposed by (Kirkpatrick, Gelatt and Vecchi, 1983) [**182**]. The algorithm behaves the same as hill climbing in accepting the neighbour solution when it is better than the current solution (i.e. the neighbour replaces the current solution). However, one of the main features behind simulated annealing is that it is able to accept worse solutions based on a probability function consisting of two parameters: i) the difference in the evaluation quality of the current and neighbour solutions (known as the *change in energy*, ΔE), and ii) the duration of the search (known as the *temperature*, T). The standard formula to calculate the probability of a worse solution being accepted is $e^{(-\Delta E/T)}$. The temperature parameter is directly controlled by the *cooling schedule* which is usually a linear or geometric scale that is applied to reduce the temperature after each or a certain number of iterations.

It can be seen that as the temperature decreases or the neighbour solution becomes increasingly worse than the current solution, there is less probability of accepting the neighbour solution. Therefore, at the outset of the search there is a less stringent acceptance criterion and the probability of accepting worse solutions is high whereas, as the search progresses and the temperature is cooled, there is less probability of accepting a solution. When the temperature becomes zero, the algorithm becomes the equivalent of hill climbing.

Simulated annealing is used in chapters three, four and five of this thesis but has also been applied to packing problems in [**116,117,118,120,142,183,184,185,186,187,188,189,190**].

2.6.2 Genetic and Evolutionary Algorithms

Genetic algorithms are population based search methods inspired by the principal of evolution found within the natural world. Although genetic algorithms were first conceived by Fraser in 1957 [**191**] and Bremermann in 1958 [**192**], Holland's book of 1975 [**193**] is recognised as an important work on the subject. A genetic algorithm maintains several solutions which are known as *individuals* (or chromosomes). Each individual is composed of several separate parts which are called *genes*. The actual values that are assigned to genes are called *alleles*. The individuals of a population represent different solutions and, thus, it is important to evaluate the quality of each individual. The numerical evaluation of an individual is known as its *fitness*.

Throughout the course of the genetic algorithm, the population is constantly being replaced by new individuals (i.e. new populations). This is achieved by applying two types of operator that mimic the evolution of natural systems. The first operator is known as *crossover* and can be thought of as the process of breeding. In the genetic algorithm, this is achieved by taking two individuals, known as the *parents*, and recombining genetic material from within their genes in order to produce new individuals or *children*. An important aspect of the algorithm is that parents are selected in proportion of their fitness. This means that the fitter members of the population are selected to breed more often than less fit individuals. Hopefully, the good properties from a chromosome are propagated into new individuals and the population should steadily improve as individuals become increasingly better. The second operator that is commonly used is *mutation*. This aims to introduce diversity and allows for new genetic material to be injected into the population. Usually, this takes the form of random value changes to genes with a low probability. Genetic algorithms require specification of the following operators and parameters:

Population Size:	the number of individuals within the population
Measure of Fitness:	the method used to determine the quality of individuals
Crossover Operators:	how two chromosomes will be combined to produce children
Crossover Probability:	individuals breed with some probability
Mutation Operators:	how to mutate values within chromosomes
Mutation Probability:	mutation is carried out with some (usually small) probability
Elitism:	the best *m* individuals are carried forward into the next generation

A detailed analysis of genetic algorithms is provided by Goldberg [**194**]. A bottom-left-fill hybridised genetic approach is examined as part of chapter three of this thesis but other examples of genetic algorithms used for packing problems are found in the following: [**106,108,109,118,121,123,125,128,144,151,157,170,171,195,196,197,198,199,200,201,202,203 204,205,206**]

2.6.3 Other Search Methods

Two other optimisation techniques that have been applied to packing problems are: i) artificial neural networks and ii) ant algorithms. Although these are not investigated within this work, they are included here for completeness.

i) Artificial Neural Networks

Artificial neural networks are based on the modelling of information processing that occurs in the brain. The system is made up of many simple nodes or *neurons*. Each *neuron* receives one or more weighted inputs and then outputs some value depending on whether the total has exceeded an *activation level threshold* value. A neuron is said to have ‘fired’ when it outputs a value. Inputs can be *excitatory* or *inhibitory*. Excitatory inputs contribute positively to a neuron’s total and inhibitory inputs contribute negatively. The outputs from neurons are inputs to other neurons and, hence, complex networks can be created. Some neurons receive external inputs which propogate through the neural network, inducing some to fire and others not to fire. The network as a whole exhibits some behaviour through output nodes. The seminal study of neural networks was conducted in 1943 by McCulloch and Pitts [**207**]. Within the domain of cutting and packing, neural networks are generally combined with other search techniques and can be used to produce input sequences to dictate the order in which shapes are to be packed. They have also been used to create placement strategies for use in automated packing [**26**].

Examples of neural networks can be found in the following literature within the cutting and packing field: [**128,186,208,209**].

ii) Ant Algorithms

The *ant algorithm* is a relatively new search methodology based on the observation that ants can find the shortest route between their habitat and a food source, even though they are almost completely blind. Ant algorithms (also known as *ant systems*) are also population based strategies with each ant representing one potential solution. The ants 'communicate' by depositing of *pheromone* trails. An example of the ant algorithm in operation can be drawn from a simple problem, discussed in the seminal work by Dorigo in 1996, involving a problem whereby ants must travel between two locations (i.e. a nest and a food source) [**210**]. In the simplest case, the ants only need to travel in a straight line. However, if an object is blocking the path of the ants then the ants must decide whether to move left or right around the object. Assume that the path to the right is shorter than the left path. Also assume that ants deposit pheromone trails as they move and that ants prefer to follow paths which have been taken by other ants. When the first ant reaches the object, there will be no pheromone trail to guide the ant's decision. Therefore a random choice will be made. When subsequent ants reach the object they will either go left or right around the object (laying pheromone trails as they move). Now the first ant returns from the food source and reaches the object once more. This time there will be higher levels of pheromone on the shorter path as more ants have reached the end of this path (because the ants have travelled less distance than ants taking the other path) and, therefore, the ants returning from the food source are under greater influence to take this path as it has a stronger pheromone trail. The system works with positive feedback whereby the greater the number of ants that follow a particular path, the larger the pheromone trail becomes on that path and the more desirable it becomes for future ants.

Ant algorithms have been used in the following packing approaches within the following literature: [**7,9,211,212**].

2.7

Packing within Industry

Packing problems occur in many important mass-production industries. Within this section, these industries are identified and the related academic literature that has specifically attempted problems arising from the industrial setting are discussed.

2.7.1 Identification of Applicable Industries

2.7.1.1 Paper

The first work into cutting and packing was conducted in the 1950-1960s and addressed problems from within the paper industry. This initial research was conducted to find the optimal way that a roll of paper may be cut into desired pieces and used linear programming techniques on a restricted guillotine variant of the orthogonal packing problem [**213,214**]. More recent examples of research that has been based on the paper industry are presented by Johnson, Rennick and Zak (1997) who extend the one-dimensional stock cutting problem to include the process of "skiving" (joining of lengths of paper) [**215**]. The authors stated that, through the skiving process, solutions can be found for instances that could not be solved previously. A heuristic and exact method is described that can produce a solution to a 30 roll problem within around one minute of computation time. Harjunkoski, Westerlund, Porn and Skrifvars (1998) presented two integer programming algorithms for cutting larger stock rolls of paper into product paper rolls and compare them using a real-world instance from a Finnish paper converting mill [**216**]. Aboudi and Barcia (1998) used linear programming methods for the paper cutting industry but also deal with defective areas that can appear through manufacturing glitches [**217**]. A defective area cannot be used when cutting out product rolls. Defective sheets were first investigated by Hahn in 1968. Hahn produced an optimal algorithm based on Gilmore and Gomory (1966) [**74**] for cutting out multiple identical sheets containing rectangularly defined defective areas [**218**]. An important feature of the algorithm was that larger rectangles were placed first because the sheets had more restricted potential positions because of the defects. McDiamid (1999) presented heuristic methods for a very restricted case of the stock cutting problem [**219**]. Once more, jumbo reels (or “raws”) must be cut into smaller product reels (or “finals”) but it is known that it is only ever possible to cut two product reels out of each jumbo (never three or more). McDiamid stated that even this problem is strongly NP-hard. Menon and Schrage (2002) provided an integer programming approach to solve instances of the problem involving the order allocation problem on multiple parallel cutting machines [**220**]. Their approach produced solutions in 10.12, 37.04 and 118.39 seconds, respectively, for two, three and four parallel machine problems. Multiple parallel machines are the focus of Giannelos and Georgiadis (2001) who also used exact techniques [**221**].

2.7.1.2 Wood

The wood cutting industry has also been the subject of stock cutting research. Morabito and Garcia (1998) tackled a similar problem to the paper industry except that sheets of hardboard are dissected into smaller rectangular by circular saws (and thus the problem is guillotine as the blades may not turn mid-cut) [**222**]. They compared two techniques to solve problem instances that are taken from the Brazilian hardboard industry. The first is a dynamic programming based integer program and the second is a simple extension of the algorithm presented in Gilmore and Gomory (1963) [**72**]. The flat-pack furniture producing industry is the subject of Morabito and Arenales (2000) [**223**]. The authors draw particular attention to the balance between producing less wastage and producing more simple layouts that can be cut with less complication. The unit cost of material is factored into the algorithms such that, for cheap material, the layout becomes secondary to the simplicity of the cutting process. However as unit cost rises, there becomes more emphasis on the reduction of waste (at the expense of cutting difficulty).

2.7.1.3 Glass

Glass cutting almost always involves the rectangular guillotine variant of the stock cutting problem. This is because the glass must firstly be 'scored' and then snapped over a flat edge. Bisotto, Corno, Prinetto, Rebaudengo and Reorda (1997) [**224**] hybridise a greedy placement heuristic with a genetic algorithm in order to solve the trim loss glass production problem. The purpose of the genetic algorithm is to generate orderings of the rectangles which are then decoded into different layouts by the placement heuristic. The approach compares favourably against other approaches and has now been adopted by one manufacturer of glass cutters. Puchinger, Raidl and Koller (2004) [**225**] developed evolutionary algorithms to solve glass cutting problems from the real world. They compared a greedy first-fit heuristic, two branch-and-bound tree search procedures and different variants of an evolutionary approach. They suggested that the evolutionary algorithms perform well in practice although they are occasionally outperformed by the other methods.

2.7.1.4 Textiles

The textiles industry also provides a test bed in which to apply automated packing research. The cutting of clothes from material rolls offers many difficulties and nuances that must be handled by any automated approach. Such problems may include patterned rolls of materials and, hence, rotational constraints on the orientations in which clothing can be cut (when pieces are stitched together, patterns must be aligned). Further to this, highly irregular shapes are usually involved, requiring complex geometric handling, and defective areas may be present within the material rolls (which, in general, must be identified by human operators). Although the general irregular problem has its foundations with Adamowicz and Albano (1976) [**85**], there has not been substantial research interest in the textile problem until the last ten years. This could potentially be explained by complex modelling that is required by the high irregularities within shapes *and* sheets. However, there are now many instances of literature on this problem due to additional cooperation and funding from within the textile industry. Chryssolouris, Papakostas and Mourtzis (2000) tackled the scheduling of nesting operations that takes into account both nesting efficiency and further production objectives such as meeting due dates, minimising cost, maximising machine throughput and maximising material utilisation [**226**]. They produce a scheduling approach and then use rule based transformations to also produce nesting solutions for the rectangular version of the problem. The work was conducted with a carpet weaving firm. In 2002, Degraeve, Gochet and Jans extended the work presented by Degraeve and Bandebreok (1998) [**227**] using mixed integer programming methods for solving marker layout problems in the fashion industry. They factor 'bill of material' requirements (the demand and requirement of individual pieces to fulfil an order) into the objective and evaluation criteria. The authors then further extended the model to include different colours of materials [**228**]. Yeung and Tang (2003) approach the strip packing problem for the cutting of larger rolls of material into smaller rectangular pieces within Hong Kong's clothing industry (note the similarity to the paper industry above) [**201**]. They utilised a genetic algorithm based approach to generate permutations for the so-called lowest-fit-left-aligned decoder heuristic (although this is similar to the bottom-left class of heuristics, they use a similar representation of the height 'skyline' of a partial layout as is presented for the best-fit heuristic in chapter three).

2.7.1.5 Leather and Footwear

The leather cutting industry has also been the focus of some of the latest work into cutting and packing. Crispin, Clay, Taylor, Bayes and Reedman, in 2003, addressed the problem of leather-lay planning for the footwear industry [**200**]. Leather 'hides' have considerable variability in: i) the strength of the leather, ii) direction of this strength and iii) regions of differing quality. These factors often severely restrict potential placements of shapes and the variability in hides generally requires human operator input to either oversee the entire process or to define the areas of strength (unless X-ray techniques are used). The authors use an imaging technique to generate no-fit polygons of the sheet and the shapes and then use genetic algorithms to generate shape angles and nesting direction. This is complicated by almost free rotation being allowed (shape angles are mutated in 5° increments) and many no-fit polygon generations must be made on-line. The algorithm cannot perform within the same timeframes as skilled experts. Other approaches that focus on leather lay-planning can be found in Bounsaythip, Maouche and Neus (1995) [**229**] where the authors approached the layout problem by using evolutionary search techniques and in Heistermann and Lengauer (1995) [**230**] who highlighted some of the problems that are found within the industry.

2.7.1.6 Plastics and Foam

Whilst the author has been unable to establish academic literature that specifically addresses the plastic cutting industry, the problem does not differ from those found in the other industries and, therefore, the majority of techniques from other methods can be applied. The problem usually includes the multisheet two-dimensional irregular variant but shapes are generally convex. This information was gathered during this research programme through correspondence and visits to Ultraframe Limited, a UK conservatory manufacturer who are required to cut roofing panels from polycarbonate sheets.

2.7.1.7 Metals

Metal cutting is of interest because of the wide range of operational considerations that may occur. For example, the material costs can vary considerably, from thin aluminium to processed metals that can have thicknesses of several centimetres (or even metres in the case of power station reactor linings). This generally has an impact on the amount of computation time that one would be willing to allow an automated strategy. In the aluminium case the importance of generating the optimal layout is probably secondary to producing a quick solution as minimal cost savings can be achieved and it is important to keep the production line operating. However, thick processed metals have high unit cost and every improvement in material utilisation can yield significant cost savings. Nonås and Thorstenson (2000) [**231**] attempted a stock cutting problem from a Norwegian off-road truck production company whereby the majority of the parts required for production of the trucks are made in-house. The authors presented three local searches and column generation approaches but restrict the problem such that only sheets with common dimensions are used. In 2002, Kos and Duhovnik used the steel structure production industry where cutting beams into smaller pieces is of importance [**232**]. They showed that a genetic algorithm with domain specific information and a local search can achieve improvements of around 5% over initial solutions. Further examples can be drawn from the literature that was discussed in section 2.3.

2.7.2 Cutting Technologies and their Implications

The purpose of this section is to highlight some of the real world difficulties that occur from the use of different types of cutting technology. This information is reported to show the complex nature of industrial problems and was gathered from discussions with our industrial partner.

2.7.2.1 Stamping

Stamping is mainly used to cut multiple identical shapes out of a sheet of material (mainly metals) as different shapes require costly production stoppages to change the cutting plate. The process involves applying a large downward force onto the sheet by a heavy block of metal with a cut plate attached to its face.

2.7.2.2 Knife Cutting

Knife cutting offers its own challenges because the blade has limited freedom once a cut is started (i.e. it cannot make lateral movements). Therefore, when cutting shape corners, the blade must be removed, rotated and then pierced back into the material. Another complication is that due to the knife's triangular shape, the blade must travel past the boundary of shape edges by the length of the blade (see figure 2.5). Knife cutting is usually used for paper, polycarbonates and other plastics.

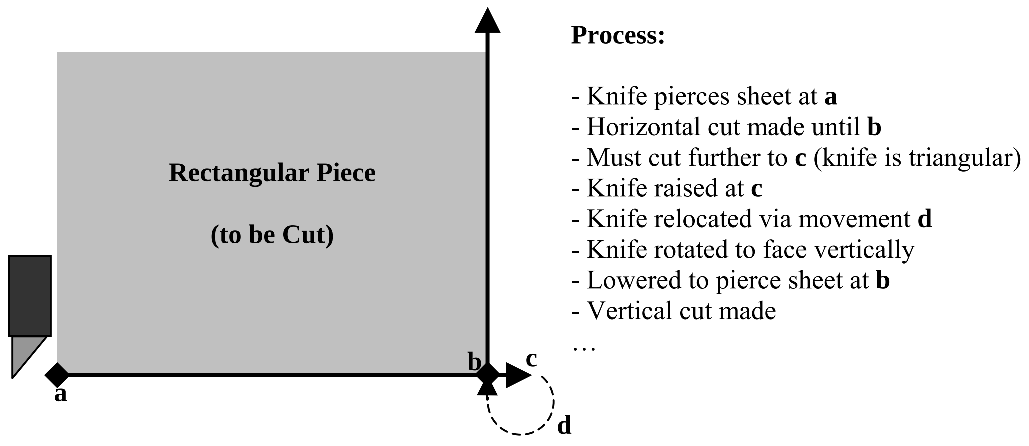

Rectangular Piece
(to be Cut)
a
b
c
d
Process:
- Knife pierces sheet at a
- Horizontal cut made until b
- Must cut further to c (knife is triangular)
- Knife raised at c
- Knife relocated via movement d
- Knife rotated to face vertically
- Lowered to pierce sheet at b
- Vertical cut made
…

Figure 2.5. Nuances of the knife cutting procedure

2.7.2.3 Oxy-Fuel Plasma Cutting

Oxy-fuel plasma cutting operates through intense heat and is uniquely used for metal cutting. This brings about difficulties within the cutting processes because the metal sheets become very hot and can warp during cutting. This can be significant because the cutting head is usually close to the sheet and can collide with the sheet if such warping occurs. There are two potential ways for the possibility of this to be reduced. The first involves generating a cut path (the order in which shapes are cut) that maximises heat distribution and can be thought of as a travelling salesman style optimisation problem where the objective is to maximise the path length taken. The second method involves avoidance of previously cut areas. For example, the cutting head needs to travel over a region that has previously been cut and, thus, the cutting head must either: i) go around the area, or more likely ii) be raised a certain distance before moving over the area. A further complication that is important to automatic nesting is that the cutting flame has a certain degree of kerf (width). This means that, during automatic nesting, a distance equivalent to kerf must be left between each shape. This can be achieved by expanding each shape by a ½ kerf before automatic nesting commences. This expansion operation generally introduces arcs into shapes where none existed before (see figure 2.6).

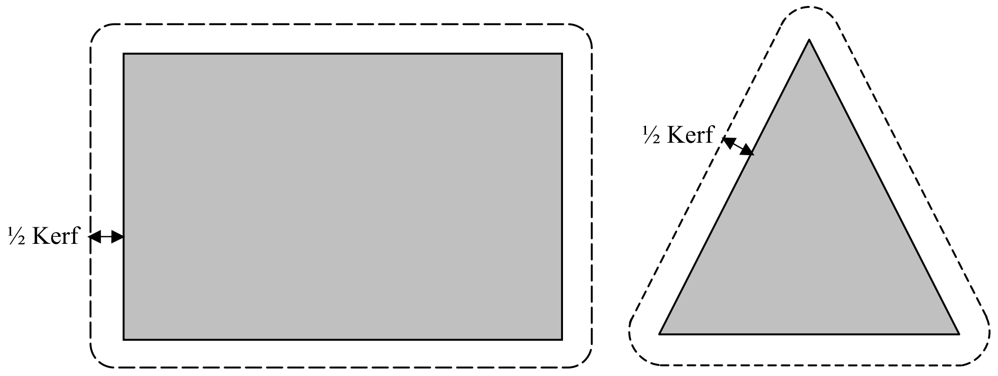


Figure 2.6. Expanding shapes by ½ kerf width

Further problems can occur from 'burring' when travelling around corners. This is mainly due to the cutting flame being in contact with a singular point for longer duration and causing untidy edges. Referring to figure 2.6, when the cutting flame cuts around the triangle corners, it is in contact with the triangle vertices the whole time and the continuous high temperatures can result in less defined corners due to heat damage. A solution to this is to perform what are known as "butterfly cuts" which perform a similar cutting path to the knife cutting example of figure 2.5. This ensures that crisp edges are cut but it requires the use of additional parts of the sheet.

2.7.2.4 Water-jet Cutting

Water-jet cutting involves the high-powered streaming of water reinforced with sand grits. This cutting technology is the most flexible and can, generally, be used on any material. It benefits from being a cool cutting method, and so the problems of oxy-fuel plasma cutting are avoided, but it still has a certain amount of kerf. Also as the cutter moves, the angle of the jet will alter in proportion to its speed. These are all practical factors that must be dealt with during cutting but it is only really kerf width (and hence, arcs) that needs to be handled appropriately for the purposes of automatic nesting within the industrial setting.

2.7.3 Additional Objectives and Considerations

Automatic nesting software within the industrial setting may be required to cope with other objectives (apart from utilisation). Whilst these considerations do not feature in this work, they do provide an interesting insight into problems that appear within the industrial domain and, therefore, some of these are briefly described below and have been discovered from time spent with the industrial partner.

2.7.3.1 Multi-head Cutting

Some cutting technologies can have multiple cutting heads to reduce the throughput time of cutting operations. Generally, the cutting heads move using the same CNC cutting machine controller code and therefore make identical cuts (at an offset). In order to utilise such facilities, nesting algorithms should place multiple shapes copies within the correct offset bounds.

2.7.3.2 Multiple Sheets

The inclusion of multiple different-dimensioned sheets also add extra complication as algorithms must 'decide' which sheet sizes to employ for particular jobs. The correct selection of sheets can drastically reduce trim-loss and restocking costs.

2.7.3.3 Partial Sheet Usage

One important aspect of increased profitability is the reutilisation of partly used sheets. Any areas of the sheet which have not been used can be restocked and presented in future jobs. However, if the task of reintroducing the sheet to nesting is performed through manual input of the sheet through CAD definitions, then the process can be slow. Previously, industry has taken straight line offcuts of the sheet to avoid this problem (by creating another rectangular sheet). If nesting operations can effectively store offcuts (as a post process) into a database of available sheets, the entire offcut can be reused (even with considerable irregularities and holes).

2.8 Cutting and Packing: Summary

In this chapter, an overview of the field of cutting and packing was provided and some of the interrelated problems that exist within the research area were described. The typological categorisation of problems was discussed, indicating the deficiencies in the current system (presented by Dyckhoff in [**20**]), and reported on the concerted efforts to produce a better typological categorisation model within the academic community. Further to this, 42 survey papers were identified that provide good overviews and bibliographies for the field of cutting and packing.

After this, two literature reviews were provided for the packing problems that are the focus of the remainder of this thesis, namely non-guillotine two-dimensional rectangular packing and irregular packing. These reviews included the publications that the author of this thesis has found the most useful for the work presented in this thesis. Both reviews concluded with a summary of the readily available benchmark datasets and their sources from within the literature.

The chapter concluded by identifying the different industries for which cutting and packing processes are an important consideration. Further to this, a summary of published work that has addressed problems from each of the identified industries was given. The implications that cutting technology has on the process of automatic packing for real world applications of the problem have also been discussed. Finally, additional considerations that are related to production management within the industrial setting were highlighted. A large proportion of this information was obtained through discussions with the industrial partner and is, therefore, biased towards the metal cutting industry. However, many of the issues found in metal cutting processes can be directly related to problems within other industries.

PART B – RECTANGULAR PACKING

The two-dimensional orthogonal stock cutting problem consists of the assignment or placement of rectangular pieces onto a larger rectangular sheet of material such that pieces do not overlap and the total height required is minimised. This problem has been shown to be NP-hard and, therefore, it is strongly believed that it cannot be solved within polynomial time (Garey and Johnson, 1978) [**15**]. The problem exists in two variants, guillotine and non-guillotine, which mainly depends on the cutting technology being employed. The key feature of the guillotine variant is that pieces must be assigned to the sheet so that they may be cut out from the sheet by successive guillotine cuts that span the entire width or length of the sheet. The non-guillotine variant does not have this constraint but does require a cutting technology that can either change direction whilst cutting or one that can pierce the middle of a sheet. These are practical issues that are important when a particular layout will be physically cut from a sheet of material.

The work presented in this section of the thesis specifically addresses the non-guillotine variant of the problem in which we are not restricted to only performing full horizontal or vertical cuts from one sheet edge to another (unlike guillotine packing). All items in the set of rectangles must be packed onto the object sheet and pieces can undergo rotations of 90° to allow rectangles to be placed in either their portrait or landscape orientations. The author has evaluated the presented approaches on a wide range of benchmark data involving 34 problem instances from various sources within the literature and, further to this, has generated and presented 28 new rectangular benchmark problem instances to the academic community to aid comparison between previous approaches and the new methods presented in this thesis. The author also hopes that this will help to facilitate further research within the field of cutting and packing.

The first chapter in this section, chapter three, introduces an extremely fast heuristic placement approach that is based on the principals of best-fit. The heuristic yields significant improvements over the previous state-of-the-art approaches from within the literature when applied to readily available benchmark problems (of which the author of this thesis is aware from previously published datasets) that have been provided by other researchers within the field. Specifically, the quality of solutions generated with the best-fit heuristic can be within 2% of the optimal solutions and, furthermore, are produced within a few seconds. Two implementations are provided to cope with integer and floating point data respectively. This was the first significant contribution that the author has brought to the field of cutting and packing during the course of the research programme and was published in the following publication:

> Burke, E. K., Kendall, G., **Whitwell, G.**, 2004, "*A New Placement Heuristic for the Orthogonal Stock Cutting Problem*", **Operations Research**, Vol. 52, No. 4, July-August 2004, pp. 655-671. [**1**]

Chapter four presents a new hybrid approach that combines the respective strengths of the best-fit heuristic (from chapter three) and state-of-the-art metaheuristic bottom-left-fill approaches that had previously produced the best quality solutions to benchmark datasets. The main principle of the approach is that the best-fit heuristic produces the initial layout and then the metaheuristic approach is applied to the last *m* rectangles to evaluate different rectangle arrangements. Once again, experimentation is performed on the benchmark problem instances to i) identify the optimal value for *m,* ii) compare against the best-fit heuristic and the metaheuristic bottom-left-fill methods alone, and iii) investigate the impact of longer run times. Further improvements in solution quality are achieved over best-fit and other previous approaches by using the presented hybrid approach. The hybrid approach appears in the following contribution:

> Burke, E. K., Kendall, G., **Whitwell, G.**, 2005, "*Metaheuristic Enhancements of the Best-Fit Heuristic for the Orthogonal Stock Cutting Problem*", **INFORMS Journal on Computing** (in review). [**3**]

The last chapter of this section, chapter five, investigates the advantages that can be achieved, in terms of solution quality, through the utilisation of user interaction in order to 'guide' the automation. This is achieved by revisiting the hybrid approach of chapter four except that it is modified to allow user interaction through the selection of unsatisfactory regions within the generated layouts (that is the rectangles are now selected by the user). A metaheuristic search can then be invoked to specifically concentrate on the user-defined regions. This process can be repeated until no more improvements in quality are achieved or until the user is satisfied with the solution. One of the main principles of the approach is that the user retains control over the automation process but is abstracted away from directly moving individual rectangles. Once again, large improvements in solution quality are made over the previous techniques from the literature (and methods from chapters four and five) against the benchmark problems. The interactive approach presented within chapter five is the focus of the following paper:

> Burke, E. K., Kendall, G., **Whitwell, G.**, 2005, "*The Two-Dimensional Orthogonal Stock Cutting Problem: A New User-Guided Hybrid Approach*", **Journal of the ACM** (in review). [**5**]

The data for the newly generated rectangular problem instances that are used within this thesis are provided in appendix C and the solutions produced for the entire set of rectangle benchmark problem instances are displayed in appendix A of the thesis.

CHAPTER THREE

The Best-Fit Placement Heuristic

3.1 Overview

Unlike the bottom-left and bottom-left-fill methods (described in section 2.2.1) that make placements which are based on the sequence of rectangles supplied to them, the proposed heuristic method dynamically selects the next rectangle for placement during the packing stage. This enables the algorithm to make informed decisions about which rectangle should be packed next and where it should be placed. A *best-fit* type strategy is adopted for this purpose. This is, essentially, a greedy algorithm that attempts to produce good quality packings by examining the available space within the stock sheet and then placing the rectangle that best fits the lowest space available. There are some inherent problems with respect to the time complexity of the algorithm and the quality of the solution. In order to find the lowest available *gap*, the stock sheet and current assignment of shapes must be examined. Once the lowest gap is found, the list of rectangles must be searched in order to find the best fitting shape. Both of these operations must be conducted at each step of the algorithm and therefore they contribute towards increasing the time complexity. In section 3.2.1, methods are addressed that can be employed to reduce this complexity bottleneck.

At the beginning of the packing process there will not be any shapes assigned to locations on the stock sheet. Therefore, the lowest available gap or 'niche' for placement of a shape will be the entire width of the stock sheet. As shapes are placed, the *lowest gap* will change with respect to both location and width. The best fitting shape is always selected and placed within this gap.

There are three possibilities: i) there is a shape with a dimension (either width or height) that exactly fits the gap, ii) there is a shape with a dimension smaller than the gap, or iii) there are no shapes that will fit the gap. In the first case, the rectangle placement is easy because there is a perfect fit (where there are several shapes that would fit exactly, the rectangle with the largest area is chosen). However, in the second case the shape that consumes the largest portion of the gap is placed. As this shape does not completely fill the gap, it needs to be defined how a shape should be placed within the gap. This has been termed the *niche placement policy*. The third case gives rise to an important property of using the best-fit methodology. If none of the available rectangles can fit within the lowest gap then the relevant space can be considered wastage. This is clear because if none of the shapes fit the space now then none of the remaining shapes will be able to fit in the space in future iterations because new rectangles are never added to the list of shapes to be placed. Therefore, unlike the bottom-left placement heuristic, this method only ever creates holes that cannot be filled and unlike bottom-left-fill, all holes that are created can be ‘forgotten’. This leads to another important feature of the best-fit heuristic. Many algorithms, such as bottom-left and bottom-left-fill, require a costly ‘overlap’ function. This performs an overlap test between the current shape and each of the shapes that have previously been placed onto the sheet. Obviously the more rectangles that have been packed, the more overlap tests have to be performed thus resulting in the process becoming increasingly slower as each rectangle is placed. However, because of the best-fit approach and the implementation (presented in section 3.2), this operation is not required as it is always the case that the shapes being placed do not overlap with other rectangles.

3.1.1 Niche Placement Policies

The best-fit heuristic is a combination of three niche placement policies with each policy indicating how a shape could be placed when it does not exactly fit the lowest niche. The heuristic never deliberately creates a niche but, if no rectangle can fit the gap completely, there is no option but to create one. In the following niche placement policies, it is assumed that the stock sheet sides are 'infinitely tall' to simulate a roll of material.

Place at Leftmost

The leftmost niche placement policy places a non-exact fitting rectangle at the left side of the niche. Note: if a rightmost niche placement policy were used, the resultant packing would be a mirror image of the leftmost policy.

Place Next to Tallest Neighbour

In this placement policy the two rectangles on the stock sheet that define the lowest gap are examined. The best fitting rectangle is then placed within the gap next to the tallest gap-defining rectangle (see figure 3.1). If the lowest gap is defined by a rectangle and the sheet side, then the rectangle is placed next to the sheet side.

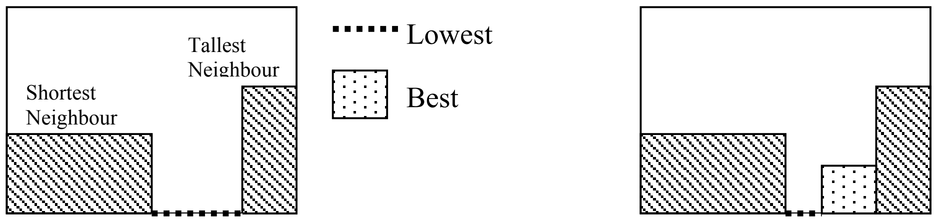


Figure 3.1. Placement next to tallest neighbour

Place Next to Shortest Neighbour

This placement policy is the opposite of the tallest neighbour policy described above. This time, rectangles are placed next to the shortest neighbour. An example is given in figure 3.2.

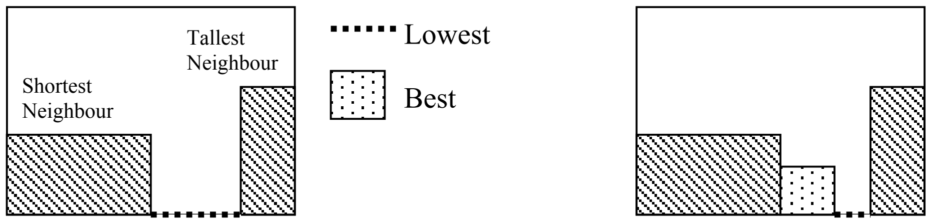


Figure 3.2. Placement next to shortest neighbour

3.1.2 Improving the Packing

A drawback of using the proposed method is that it can create poorer quality packings due to 'towers'. Towers are produced when long thin rectangles have not been placed until the latter stages of the layout. The algorithm may place these rectangles in portrait orientation near to the top of the layout where they can negatively affect the solution quality. Due to this problem, a further step is conducted after all rectangles have been placed. This searches for the shape that is currently adding the greatest height to the packing and determines if it is a tower by examining if the rectangle's height is greater than its width. If the rectangle is identified as a tower, the rectangle is removed from the packing, is rotated by 90 degrees and then repacked in the new orientation on top of the layout. If the solution quality is improved by this step then it is repeated on a new 'tower'. This process continues until there is no improvement in solution quality. The procedure is described in more detail in implementation section 3.2.3.

3.2 Implementation

The description of the algorithm above requires a search for both the best fitting rectangle and the lowest space within the stock sheet at every time step of the algorithm. These are computationally expensive operations and the algorithm would benefit if these could be simplified or, better still, removed. The inputs to the algorithm are a list of rectangles in random order and the width of the stock sheet.

3.2.1 Pre-Processing Stage

Firstly, the problem of representation is addressed. Some algorithms store this information using location points as in the bottom-left-fill algorithms (Chazelle, 1983) [**111**] and/or collision detection by the sliding of shapes such as the bottom-left algorithms (Jakobs, 1996 [**106**]; Liu and Teng, 1999 [**108**]). Others avoid the problem by packing in rows with a new row starting from the highest point of the previous one (Bengtsson, 1982) [**104**]. However, these methods require the maintenance of potential location lists that must be traversed and/or collision detections on each rectangle placement. Collision detection is computationally expensive due to the need to examine all of the already placed rectangles when adding a shape. Therefore when adding the last piece in a problem of *n* shapes, *n – 1* collision tests are required to ensure that there are no overlaps with other placed shapes. In the case of Bengtsson's row packing approach, there is no overlap detection but there can be a significant increase in sheet wastage between rows. The proposed approach eliminates the need for intersection testing by representing the stock sheet as a linear array in which the number of elements is equal to the width of the stock sheet. Whilst this is sufficient for integer data, floating point data must be scaled and rounded to the desired accuracy for the representation (see section 3.2.4). Each element of the array holds the total height of the packing at that *x* coordinate of the stock sheet. Thus this array can be thought of as maintaining the height profile or "skyline" of the layout. Two examples are given in figure 3.3 to show a sheet of 9 units width when empty and the same sheet after packing seven rectangles.

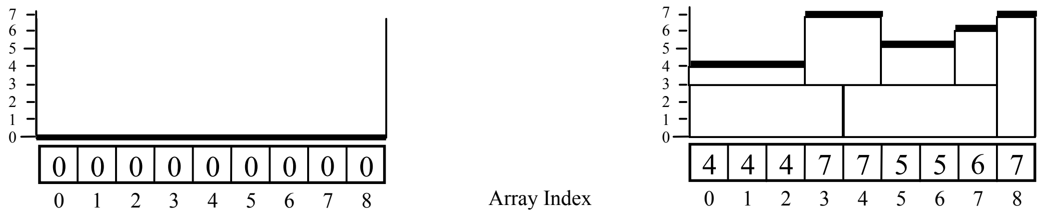


Figure 3.3. Storing the skyline heights of the layout on a sheet of width 9 units when empty and after packing seven rectangles

Therefore, the coordinate of the lowest space of the stock sheet can be found by locating the smallest valued entry of the array. The width of the gap can be found by examining how many consecutive array items of equal value exist. In figure 3.3, the lowest available space is located at $x = 0$ and has a width of 3.

The other problem encountered is caused by the best-fit methodology. The rectangle data is defined as a list of rectangles each denoted by a (width, height) pair. In an unsorted list all *n* rectangles must be examined to be sure that there is not a 'better' fitting rectangle available for placement. However, the list of rectangles can be sorted once before packing commences so that the number of rectangles that need to be examined can be reduced to ½ *n* (on average per rectangle placement). The first stage of this restructuring is to rotate any rectangle for which the height is greater than the width. For example:

{ (3,5), (5,2), (1,1), (7,3), (1,2) } becomes { (5,3), (5,2), (1,1), (7,3), (2,1) }

Next, the rectangles are sorted into decreasing width order (resolving equal widths by decreasing height order):

{ (5,3), (5,2), (1,1), (7,3), (2,1) } becomes { (7,3), (5,3), (5,2), (2,1), (1,1) }

This list of rectangles can now be examined for the best fitting rectangle without the need to search the entire list. For example, suppose we require a shape to fill a gap of 6 units. The first rectangle in the list is examined, (7,3). Note that this rectangle could fill 3 units of the gap if rotated. The second rectangle in the list, (5,3), can occupy a gap of 5 units. At this point the search can be terminated, as all remaining rectangles have dimensions of equal or lower value than 5. Assume that this time there is a gap of 4 units. The first rectangle in the list is examined, (7,3). It can fill 3 units if rotated. The second rectangle is examined, (5,3). Although it has an equal capacity as the first rectangle to fill 3 units of the gap, it is preferable to pack larger rectangles first and so the search continues. The third rectangle, (5,2), is not better than rectangle 1. The search must continue because there may be a rectangle with a width of 4 units. The fourth rectangle is examined, (2,1). Now the width of this rectangle is worse than the current best so the search can terminate. The first rectangle would be returned as the best fitting rectangle.

Also note that as soon as a rectangle that fits exactly is found, it is selected. This reduces the search time of the process and, due to the list structure, rectangles will have smaller dimensions as the list is traversed. In general, it is better to place shapes with larger dimensions earlier in the packing than towards the end of the packing where they may be allowed to protrude at the top of the layout and affect solution quality.

3.2.2 Packing Stage

Firstly, the stock sheet is examined to find the lowest available gap (initially at $x = 0$, $y = 0$, and lasting the entire sheet width). The rectangle list is examined and the best fitting rectangle returned. This is placed within the gap depending on the current placement policy (as described in section 2). The rectangle is assigned coordinates and removed from the rectangle list. Finally, the relevant stock sheet array elements are incremented by the rectangle height. The process continues: find the position of the lowest gap, find the width of the lowest gap, find the best fitting rectangle, assign coordinates, remove the rectangle from the rectangle list and update the stock sheet array. If the best fitting rectangle does not completely fill the gap, then there is no need to locate the lowest gap for the next rectangle because it is a portion of the recent gap and can be found by the following (figure 3.4):

If last shape placed left in gap then:	New Gap Location = (Gap Location) + (Placed Rectangle Width)
If last shape placed right in gap then:	New Gap Location = Gap Location
The gap's width is found by:	New Gap Width = (Gap Width) – (Placed Rectangle Width)

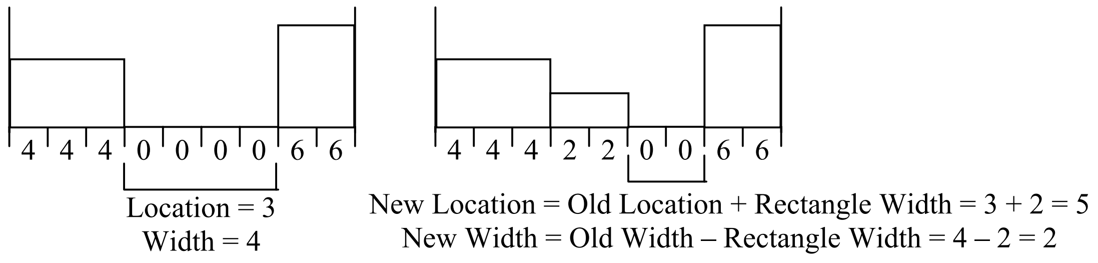

4 4 4 0 0 0 0 6 6
Location = 3
Width = 4
4 4 4 2 2 0 0 6 6
New Location = Old Location + Rectangle Width = 3 + 2 = 5
New Width = Old Width – Rectangle Width = 4 – 2 = 2

Figure 3.4. Finding a new gap when the old gap has not been completely filled

If a gap is found for which no remaining rectangle can fit, then this is wasted space and the stock sheet array elements that reference the gap are raised up to the lowest neighbour. Assume there is a gap of 2 units but that there are no rectangles with a dimension of 2 units or less within the rectangle list (figure 3.5).

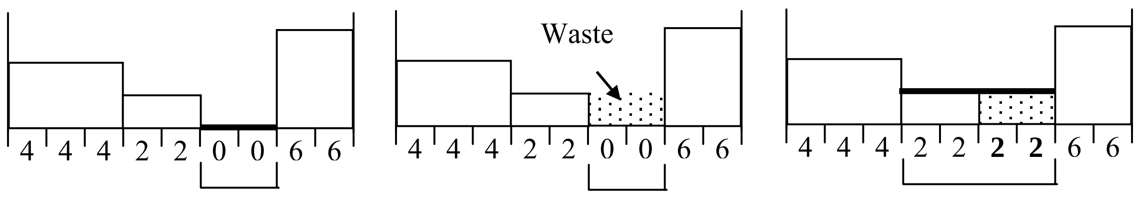


Figure 3.5. Procedure when no rectangle will fit gap

In figure 3.5 there is a gap for which no rectangle is small enough to fit. Each neighbour of the gap is examined, a height of 2 to the left and 6 to the right. The array elements that define the gap are incremented to the height of the lowest neighbour element (in this case to a height of 2). The array is now re-checked for the lowest gap and packing continues. In raising some of the array elements it cannot be assumed that the new lowest gap is at a height of 2 because there may be more gaps at a height of 0 or 1, so the array must be rechecked.

3.2.3 Post-Processing Stage

Once every rectangle is packed, all of the rectangles are examined to find if any are protruding from the top of the packing and negatively affecting solution quality. When the highest positioned rectangle is found, if the rectangle is orientated in such a way that its height is greater than its width then it is removed from the packing and the stock sheet array is reduced by the relevant rectangle height. Note that if the rectangle is found orientated with its width greater than its height, then the height of the packing cannot be improved (reduced) as this rectangle is in the lowest position possible. Then the rectangle is rotated so that its width has the larger dimension and try to pack the rectangle as before in 'normal' packing but with the constraint that it must be packed in the *width* > *height* orientation. If the rectangle were allowed to rotate again then it would be placed back in exactly the same position. This process continues until the solution cannot be improved (figure 3.6). This occurs when the highest shape of the layout is removed, rotated and, then by placing it in this new orientation, a worse solution is achieved.

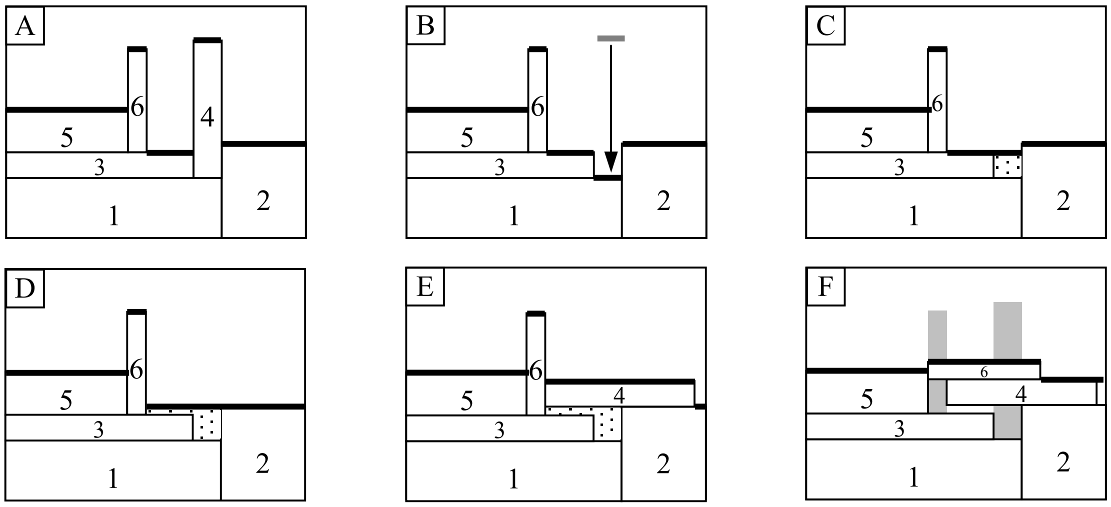


Figure 3.6. Processing of the 'tower' rectangles

Figure 3.6A shows the solution after packing with the best-fit algorithm. To apply post-processing to give better solutions the tallest shape is removed (shape 4) and the skyline is decreased appropriately as in figure 3.6B. The removed shape is rotated and an attempt is made to re-insert it in the lowest part of the nest. As this shape will not fit, the lowest gap is raised to its lowest neighbour to make a more sizable gap as in figure 3.6C. As it still will not fit, the gap is raised once more (see figure 3.6D). Now this gap is large enough to accommodate the shape so it is placed as shown in figure 3.6E. If this new arrangement enhances the solution, it is accepted (as in this case). The same operation is performed with the next highest shape (shape 6). Figure 3.6F shows shape 6 placed in its new position. If it enhances the quality of solution it is accepted (as in this case). As all previous attempts have produced better quality packings, the highest shape is selected once more. The highest shape is shape 6 once again and its width is greater than its height so the procedure terminates and the layout is returned as the final solution.

3.2.4 Floating Point Data

As the implementation is based on the faster integer data type, floating-point data must be converted to integer format by multiplying each rectangle by a scaling factor depending on the degree of accuracy required. Some of the literature test problems which have been used for the experimentation of this chapter are based on readily available floating point data (Valenzuela and Wang, 2001) [**124**]. This data has been scaled to maintain accuracy for the experiments of this chapter.

Note: in chapter four of this thesis, a fully accurate alternative for representation of the skyline is presented (i.e. without requiring floating point to integer data conversions). However, it is also shown that this comes at a slight expense of speed.

3.2.5 Summary of the Process

The final algorithm is based on the best solution returned after trying three packings with each utilising a different placement policy. The whole process can be summarised by the following pseudocode:

```
Obtain Stock Sheet Dimensions
Obtain List of n Rectangles
Rotate each Rectangle so that Width >= Height
Sort Rectangle List by Decreasing Width (resolving equal widths by decreasing heights)

Initialize Skyline Array of n Elements

for Each Placement Policy (Leftmost, Tallest Neighbour, Smallest Neighbour) do

    while Rectangles Not Packed do
        Find Lowest Gap
        if (Find Best-Fitting Rectangle == true) then
            Place Best-Fitting Rectangle Using Placement Policy
            Raise Array to Appropriately Reflect Skyline
        else
            Raise Gap to Lowest Neighbour
        end if
    end while

    while Optimisation Not Finished do
        Find Highest Shape
        if (Shape Width >= Shape Height) then
            Optimisation Finished
        end if
        Remove Highest Shape
        Reduce Array to Reflect Skyline
        Rotate Shape by 90 Degrees
        if (Shape Fits) then
            Place Best-Fitting Rectangle Using Placement Policy
            Raise Array to Appropriately Reflect Skyline
        else
            Raise Gap to Lowest Neighbour
        end if
        if (Packing Better == False) then
            Optimisation Finished
        end if
    end while

end for

Return Best Solution
```

3.3 Benchmark Problems

In order to compare the relative performance of the presented best-fit heuristic to other heuristic and metaheuristic approaches, the 34 test problems from the literature (presented in table 2.2) have been used. Further to this, 12 new test problems were generated at random in order to extensively test the new approach, N1–12. The method chosen to do this involved supplying the dimensions of a large rectangle, the number of smaller rectangles to be cut from this larger rectangle, and finally the smallest dimension allowed for any rectangle. A list of rectangles was maintained – initially with only the specified large rectangle. The algorithm selects any rectangle from the list and makes a random vertical or horizontal guillotine cut through it at a random point to create two new rectangles. This process continues, making sure that the minimum dimension is observed, until the larger rectangle has been divided into the desired number of rectangles. This allows data sets to be produced for which the optimal solution is known. The final problem, N13, was obtained by a 4 by 4 multiplication of problem C7P2 from (Hopper and Turton, 2001) [**110**]. Table 3.1 displays a summary of these new problems.

New Test Problems	**Number of Rectangles**	**Optimal Height**	**Object Dimensions**
N1	10	40	40 x 40
N2	20	50	30 x 50
N3	30	50	30 x 50
N4	40	80	80 x 80
N5	50	100	100 x 100
N6	60	100	50 x 100
N7	70	100	80 x 100
N8	80	80	100 x 80
N9	100	150	50 x 150
N10	200	150	70 x 200
N11	300	150	70 x 200
N12	500	300	100 x 300
N13	3152	960	640 x 960

Table 3.1. Generated Benchmark Problems

3.4 Experimentation and Results

For these experiments the proposed heuristic method is compared to conventional methods such as the bottom-left, bottom-left-fill and published metaheuristic methods. First of all, however, a series of experiments are presented that examine the performance of the different niche placement policies that were identified in section 3.1.1.

3.4.1 A Comparison of the Placement Policies

The proposed best-fit heuristic was applied to each of the problem instances from Hopper and Turton's datasets and the performance of the three different placement policies were compared (LM – Left Most, TN – Tallest Neighbour, SN – Smallest Neighbour). Table 3.2 shows that each policy appears to have equal ability in finding good solutions (ticks indicate the best placement policy for each dataset).

Category	Problem	Placement Policy Solution Height				Best Policy		
		LM	TN	SN	Optimal	LM	TN	SN
C1	P1	21	22	21	20	✓		✓
	P2	22	22	22	20	✓	✓	✓
	P3	24	24	24	20	✓	✓	✓
C2	P1	17	16	17	15		✓	
	P2	16	17	16	15	✓		✓
	P3	18	16	17	15		✓	
C3	P1	32	32	32	30	✓	✓	✓
	P2	34	34	34	30	✓	✓	✓
	P3	33	35	35	30	✓		
C4	P1	63	64	65	60	✓		
	P2	64	66	62	60			✓
	P3	62	63	63	60	✓		
C5	P1	94	93	94	90		✓	
	P2	93	92	96	90		✓	
	P3	94	93	93	90		✓	✓
C6	P1	124	123	124	120		✓	
	P2	124	124	122	120			✓
	P3	124	128	125	120	✓		
C7	P1	246	247	250	240	✓		
	P2	246	244	246	240		✓	
	P3	245	246	248	240	✓		
	TOTALS	1796	1801	1806	1725	12	11	9

Table 3.2. Comparison of Placement Policies and their Cumulative Performances

After 21 problems, all three policies perform within a cumulative height of only 10 units difference and each policy creates an outright best solution in several of the problems. This result validates the decision to allow all of three policies to be tried within the heuristic. As an illustrative example, Figure 3.7 shows the solutions obtained by each placement policy with problem C2P3 from (Hopper and Turton, 2001). The problem has 25 shapes and an optimal solution of 15. The leftmost placement policy obtains a solution with a height of 18. The tallest neighbour policy obtains a height of 16. The smallest neighbour policy achieves a height of 17.

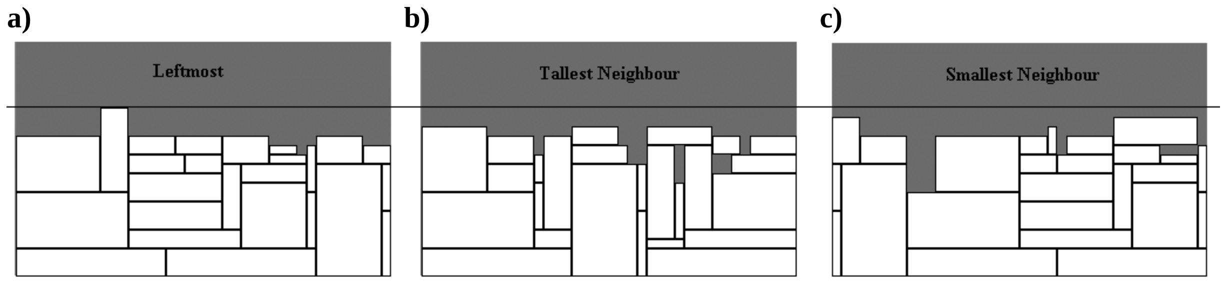


Figure 3.7. The solutions achieved by each of the placement policies: a) Leftmost, b) Tallest Neighbour, and c) Smallest Neighbour on problem C2P3 from Hopper and Turton (2001) [**110**]

3.4.2 Comparison against Bottom-Left and Bottom-Left-Fill Algorithms

In Hopper and Turton's comparison of bottom-left and bottom-left-fill algorithms, bottom-left-fill produced the better packings although it took additional computation time. Also, the use of pre-ordering shapes by decreasing width or decreasing height gave better quality solutions. As a different quality measure is being used to that of Hopper and Turton, involving the total height of the packing and not the density of the packing, the bottom-left and bottom-left-fill algorithms have been implemented and tested using Hopper's data. Table 3.3 shows that the bottom-left-fill heuristic performs better than bottom-left and that the proposed best-fit heuristic outperforms bottom-left-fill in all but two of the instances even when pre-ordering is allowed (DW = decreasing width, DH = decreasing height, for unsorted random sequences are applied and the average is taken). The best solutions are shown in bold type.

	C1			C2			C3			C4			C5			C6			C7		
Problem	**P1**	**P2**	**P3**	**P1**	**P2**	**P3**	**P1**	**P2**	**P3**	**P1**	**P2**	**P3**	**P1**	**P2**	**P3**	**P1**	**P2**	**P3**	**P1**	**P2**	**P3**
No. of Rectangles	16	17	16	25	25	25	28	29	28	49	49	49	72	73	72	97	97	97	196	197	296
BL	45	40	35	53	80	67	40	43	40	32	37	30	27	32	30	33	39	34	22	41	31
BL-DW	30	20	20	13	27	27	10	20	17	17	22	22	16	18	13	22	25	18	16	19	17
BL-DH	15	**10**	**5**	13	73	13	10	10	13	12	13	6.7	4.4	10	7.8	8.3	8.3	9.2	5	10	7.1
BLF	30	35	25	47	73	47	37	50	33	25	25	27	20	23	21	20	18	21	15	20	17
BLF-DW	10	15	15	13	20	20	10	13	13	10	5	10	5.6	6.7	5.6	5	4.2	4.2	4.6	3.3	2.9
BLF-DH	10	**10**	**5**	13	73	13	10	**6.7**	13	10	5	5	4.4	5.6	4.4	5	2.5	6.7	3.8	2.9	3.8
NEWBF	**5**	**10**	20	**6.7**	**6.7**	**6.7**	**6.7**	13	**10**	**5**	**3.3**	**3.3**	**3.3**	**2.2**	**3.3**	**2.5**	**1.7**	**3.3**	**2.9**	**1.7**	**2.1**

Table 3.3. Comparison of the best-fit heuristic to bottom left and bottom-left-fill heuristics (percentage over optimal)

3.4.3 Comparison against Metaheuristic Approaches

Hopper and Turton also show that the best packings are achieved when using the bottom-left-fill algorithm along with a metaheuristic/evolutionary algorithm (simulated annealing or genetic algorithm). Both of these algorithms were implemented to obtain a comparison with the new best-fit heuristic. The metaheuristics both operate on the rectangle input sequence which is then decoded into a layout using the bottom-left-fill placement heuristic. The fitness is based on two distinct evaluators, a primary evaluator of the height of the layout and a secondary evaluator based on the sheet area used. This is necessary in order to distinguish packings with identical heights. The metaheuristics was allowed 5 runs on each problem.

Genetic Algorithm / Bottom-Left-Fill Decoder (GA+BLF)

A genetic algorithm approach was implemented with a population size of 50 and generation size of 1000. Initially, each member of the population is a random ordering and orientation of the input rectangles. At each generation, the sequences are decoded using bottom-left-fill to obtain layouts. To improve the population after each generation, two parents are selected using linearly normalised roulette wheel selection and then partially matched crossover is applied with a probability of 60%. Two mutations are used with each occurring with a probability of 3%. The first mutation randomly swaps two rectangles within an input sequence and the second for the flipping of a random rectangle's orientation. Elitism is also used to preserve the best two solutions of the population and the others are replaced by the children. The best solution seen during the search is returned. The search parameters were set through initial experimentation.

Simulated Annealing / Bottom-Left-Fill Decoder (SA+BLF)

Simulated annealing was implemented and allowed 50000 iterations (same as the genetic algorithm method). The initial solution is a randomly generated input sequence. To move to a new candidate solution one of three neighbourhood operators are randomly chosen: i) swaps the orientation of a randomly chosen rectangle, ii) moves a randomly chosen rectangle to a random position, iii) swaps two randomly chosen rectangles. The starting temperature was calculated based on the total area of the input rectangles for each problem. A geometric cooling schedule of 0.9999 is applied after each iteration. If there is no improvement for 10 consecutive iterations, the temperature is heated by applying a 1.01 times increase to the temperature. This scheme has the effect of cooling to about a 10% acceptance rate where the search remains until completion. After 50000 iterations, the best solution found during the process is returned. The parameters for the simulated annealing metaheuristic were chosen through initial experimentation.

3.4.4 Experiments on Published Data

Table 3.4 shows the resultant solutions from using the metaheuristic methods and the new best-fit heuristic with test problems from the literature. For the metaheuristics, all times shown indicate the time taken to achieve the solution and not the time taken to complete the run. Therefore, the total time taken to finish the search will be of a longer duration. All experiments were performed on a PC with an 850MHz CPU and 128MB RAM. For all instances the best solution is shown in bold.

Data Set	Cat.	Problem	No. of Shapes	Opt. Height	GA+BLF			SA+BLF			Best-Fit		GA_{best} - BF		SA_{best} - BF	
					Best	Worst	Time(s)	Best	Worst	Time(s)	Sol.	Time(s)	abs.	% impv.	abs.	% impv.
Hopper and Turton (2001)	C1	P1	16	20	**20**	21	3.4	**20**	21	1.1	21	<0.01	-1	-5.0%	-1	-5.0%
		P2	17	20	**21**	21	0.5	**21**	21	0.8	22	<0.01	-1	-4.8%	-1	-4.8%
		P3	16	20	**20**	21	7.1	**20**	21	0.8	24	<0.01	-4	-20.0%	-4	-20.0%
	C2	P1	25	15	16	16	1.3	16	16	6.5	**16**	<0.01	**0**	**0%**	**0**	**0%**
		P2	25	15	16	16	2.2	16	16	13.9	**16**	<0.01	**0**	**0%**	**0**	**0%**
		P3	25	15	16	16	1.0	16	16	13.6	**16**	<0.01	**0**	**0%**	**0**	**0%**
	C3	P1	28	30	32	32	7.4	32	33	20.3	**32**	<0.01	**0**	**0%**	**0**	**0%**
		P2	29	30	**32**	32	12.4	**32**	32	22.5	34	<0.01	-2	-6.3%	-2	-6.3%
		P3	28	30	**32**	32	11.6	**32**	33	18.3	33	<0.01	-1	-3.1%	-1	-3.1%
	C4	P1	49	60	64	64	35	64	64	65	**63**	<0.01	**1**	**1.6%**	**1**	**1.6%**
		P2	49	60	63	64	48	64	64	46	**62**	<0.01	**1**	**1.6%**	**2**	**3.1%**
		P3	49	60	62	63	61	63	64	70	**62**	<0.01	**0**	**0%**	**1**	**1.6%**
	C5	P1	72	90	95	95	236	94	95	501	**93**	0.01	**2**	**2.1%**	**1**	**1.1%**
		P2	73	90	95	96	440	95	96	285	**92**	0.01	**3**	**3.2%**	**3**	**3.2%**
		P3	72	90	95	96	150	95	95	425	**93**	0.01	**2**	**2.1%**	**2**	**2.1%**
	C6	P1	97	120	127	127	453	127	128	854	**123**	0.01	**4**	**3.1%**	**4**	**3.1%**
		P2	97	120	126	127	866	126	128	680	**122**	0.01	**4**	**3.2%**	**4**	**3.2%**
		P3	97	120	126	127	946	126	126	912	**124**	0.01	**2**	**1.6%**	**2**	**1.6%**
	C7	P1	196	240	255	255	4330	255	256	4840	**247**	0.01	**8**	**3.1%**	**8**	**3.1%**
		P2	197	240	251	253	5870	253	253	5100	**244**	0.01	**7**	**2.8%**	**9**	**3.6%**
		P3	196	240	254	255	5050	255	255	6520	**245**	0.01	**9**	**3.5%**	**10**	**3.9%**
Valenzuela and Wang (2001)	Nice	P1	25	100	108.2	109.5	185	109.3	110.9	192	**107.4**	0.02	**0.8**	**0.7%**	**1.9**	**1.7%**
		P2	50	100	112.0	113.5	956	112.6	113.3	1358	**108.5**	0.01	**3.5**	**3.1%**	**4.1**	**3.6%**
		P3	100	100	113.0	114.0	2580	113.3	113.8	4130	**107.0**	0.01	**6.0**	**5.3%**	**6.3**	**5.6%**
		P4	200	100	113.2	114.0	$1.6x10^4$	113.4	113.8	$2.3x10^4$	**105.3**	0.03	**7.9**	**7.0%**	**8.1**	**7.1%**
		P5	500	100	111.9	112.3	$5.1x10^4$	111.9	112.2	$7.3x10^4$	**103.5**	0.08	**8.4**	**7.5%**	**8.4**	**7.5%**
		P6	1000	100	-	-	-	-	-	-	**103.7**	0.26				
	Path	P1	25	100	**106.7**	108.4	148	109.6	110.8	122	110.1	0.04	-3.4	-3.2%	-0.5	-0.5%
		P2	50	100	**107.0**	108.0	2417	108.7	109.5	1010	113.8	0.24	-6.8	-6.4%	-5.1	-4.7%
		P3	100	100	109.0	110.1	4190	109.0	111.2	6865	**107.3**	0.15	**1.7**	**1.6%**	**1.7**	**1.6%**
		P4	200	100	108.8	109.0	$1.6\ x10^4$	110.3	111.3	$2.7x10^4$	**104.1**	0.33	**4.7**	**4.3%**	**6.2**	**5.6%**
		P5	500	100	111.1	111.5	$8.8x10^4$	112.4	113.0	$1.4x10^5$	**103.7**	0.19	**7.4**	**6.7%**	**8.7**	**7.7%**
		P6	1000	100	-	-	-	-	-	-	**102.8**	0.23				
Ramesh Babu (1999)		P1	50	400	400	400	5.7	400	400	6.4	**400**	0.01	**0**	**0%**	**0**	**0%**

Table 3.4. Comparison of the Best-Fit heuristic against the metaheuristic methods (GA+BLF, SA+BLF)

The final columns of table 3.4 show that the best-fit heuristic equals or outperforms both the GA+BLF and the SA+BLF metaheuristic methods on most of the data. On closer inspection of the cases where best-fit cannot better the metaheuristic methods, we see that this occurs on problems involving fewer shapes. It seems that with up to around 50 shapes, which are small problems, in general the metaheuristic methods produce the better solutions. As there are fewer possible sequences of shapes, they are able to evaluate a larger area within the search space and, are therefore, able to produce better solutions. On large problems, where there are more than 50 shapes to be packed, the best-fit heuristic provides the better solutions, in general. Figure 3.8 shows the best-fit solution to the problem given in (Ramesh Babu and Ramesh Babu, 1999).

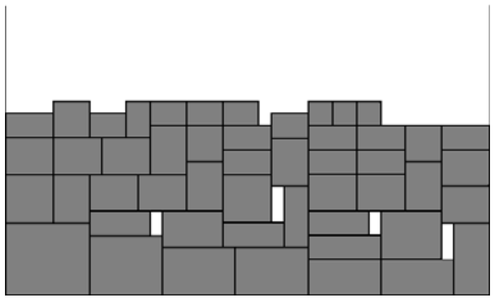


Figure 3.8. The resultant packing of the best-fit heuristic with Ramesh Babu's data.

With Ramesh Babu's data, the Best-Fit heuristic obtains a solution of 40mm which equals the height returned from the implemented metaheuristic methods and Ramesh Babu's genetic method. It gives a better result than the solution of Jakobs' genetic method, which only achieved a solution of height of 45mm (Ramesh Babu and Ramesh Babu, 1999). The execution time of Ramesh Babu's genetic algorithm is given as 521s on a 100MHz processor. However, the solution obtained using best-fit was achieved in 0.01 seconds. Although the best-fit solution was achieved on a significantly faster processor, the reduction in the time required is significant.

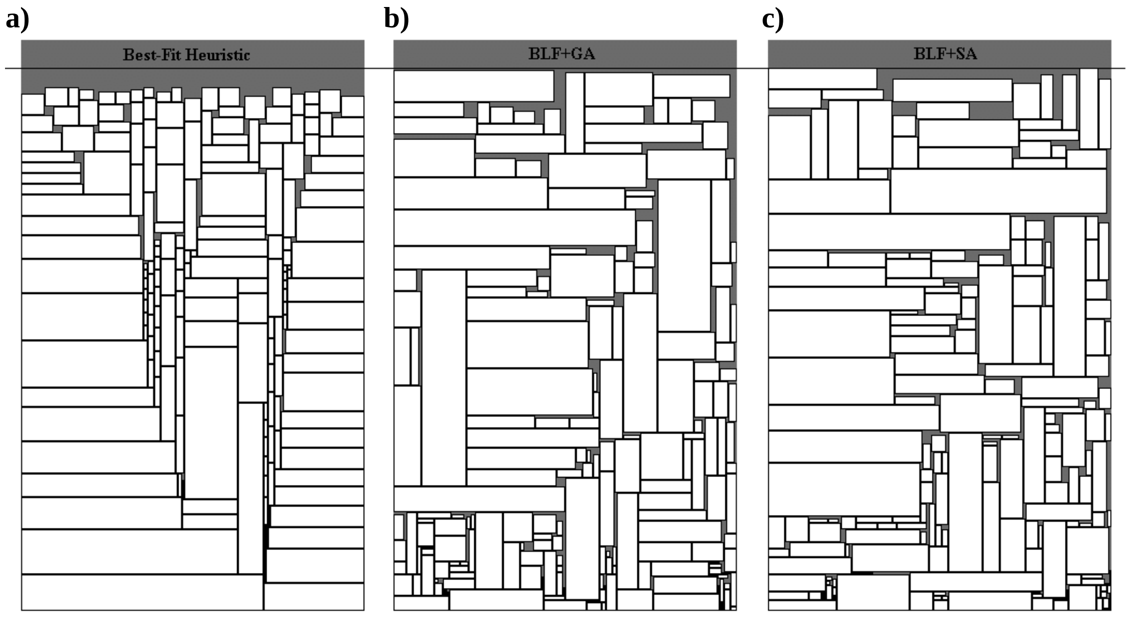


Figure 3.9. A comparison of the a) best-fit heuristic, b) bottom-left-fill + GA and c) bottom-left-fill + SA using problem C7P2 from Hopper and Turton (2001) [**110**]

As an illustrative example a comparison of the solutions obtained by each of the examined methods is given in figure 3.9 using problem C7P2 of 197 shapes and where the optimal solution is known to be 240 from (Hopper and Turton, 2001). Best-fit achieves a solution with height of 244, GA+BLF finds a solution with height 251 using a population of 50 and several runs of 1000 generation each, SA+BLF finds a solution with height 253 after several 50000 iteration runs.

With Valenzuala's floating point based data, the best-fit heuristic performed consistently better than the other methods. Due to the floating point nature of the data and the integer based implementations, all methods took longer to execute than similarly sized integer problems due to scaling. However, because the best-fit heuristic is a one-pass method, the overhead is not exceptional.

3.4.5 Experiments on Randomly Generated Data

In many real world industrial applications there are time constraints and economic reasons for reaching good solutions quickly, so to compare the various methods we should consider the time used in finding the solution. Table 3.5 uses the newly presented data sets to show both solution quality and difference in time taken to find the solution. Once again the best solutions are presented in bold.

The execution times from tables 3.4 and 3.5 indicate that the best-fit heuristic can reach good solutions in significantly quicker time than the other methods. In fact, on the newly generated problems in table 3.5, best-fit completes all solutions in a combined time of less than 2 seconds. Due to the use of populations with the genetic algorithm and neighbourhood functions with simulated annealing, there is an increase in execution time (because of their associated operations).

Problem	No. of Shapes	Optimal Height	GA+BLF		SA+BLF		BF Heuristic		GA_{best} - BF		SA_{best} - BF	
			Result	Time(s)	Result	Time(s)	Result	Time(s)	Abs.	% imp.	Abs.	% imp.
N1	10	40	**40**	1.02	**40**	0.24	45	<0.01	-5	-12.5%	-5	-12.5%
N2	20	50	**51**	9.2	52	8.14	53	<0.01	-2	-3.9%	-1	-1.9%
N3	30	50	52	2.6	52	39.5	**52**	**<0.01**	**0**	**0%**	**0**	**0%**
N4	40	80	83	12.6	83	84	**83**	**<0.01**	**0**	**0%**	**0**	**0%**
N5	50	100	106	52.3	106	228	**105**	**0.01**	**1**	**0.9%**	**1**	**0.9%**
N6	60	100	103	261	103	310	**103**	**0.01**	**0**	**0%**	**0**	**0%**
N7	70	100	**106**	671	**106**	554	107	0.01	-1	-0.9%	-1	-0.9%
N8	80	80	85	1142	85	810	**84**	**0.01**	**1**	**1.2%**	**1**	**1.2%**
N9	100	150	155	4431	155	1715	**152**	**0.01**	**3**	**1.9%**	**3**	**1.9%**
N10	200	150	154	2×10^4	154	6066	**152**	**0.02**	**2**	**1.3%**	**2**	**1.3%**
N11	300	150	155	8×10^4	155	3×10^4	**152**	**0.03**	**3**	**1.9%**	**3**	**1.9%**
N12	500	300	313	4×10^5	312	6×10^4	**306**	**0.06**	**7**	**2.2%**	**6**	**1.9%**
N13	3152	960	-	-	-	-	**964**	**1.37**				

Table 3.5. A comparison of execution time

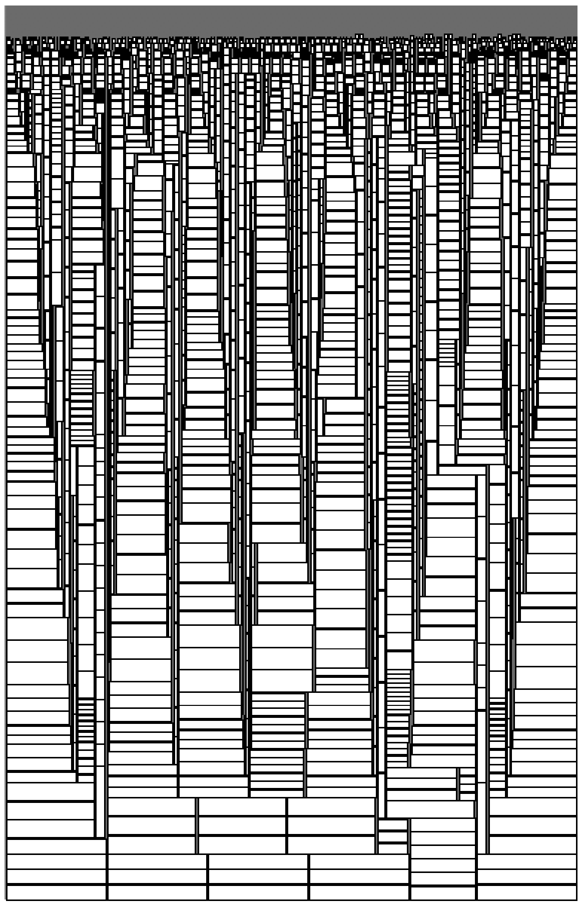


Figure 3.10. The resultant packing from data N13 involving 3152 rectangles. A packing of height 964 is achieved using best-fit, which is only 4 over optimum

The power of the approach that has been presented is particularly illustrated by considering the largest of the new problems N13. Note that the GA+BLF and SA+BLF methods were unable to complete N13, due to its excessive time requirements (extrapolated to approximately 26-40 weeks of execution time). However, the best-fit heuristic was able to pack the 3152 shapes in less than two seconds and gave a solution of 964, which is only 4 units above optimal. The solution is presented in figure 3.10.

3.5 Summary

In this chapter, a new heuristic based on a best fit methodology has been described and an efficient implementation of the heuristic has been produced. Using data provided by other researchers in the field of cutting and packing, the new best-fit heuristic has produced better results than the bottom-left and bottom-left-fill heuristics for most of the problems. The new heuristic has also produced better quality packings than the metaheuristic hybrids when given problems of 50 shapes or more. Also, solutions were obtained in a significantly shorter period of time by several orders of magnitude. The method also performs very well with datasets containing similar sized rectangles and ones containing differing sized rectangles (such as the problems represented by Valenzuela's data). In Hopper and Turton (2001) [**110**], the authors concluded that the methods that yielded the best results were GA+BLF and SA+BLF. It has been demonstrated that, even with extremely large problems, the proposed best-fit heuristic is able to outperform currently published and established heuristic and metaheuristic methods to produce solutions that are very close to optimal using very small execution times.

In the next chapter, the best-fit placement heuristic is modified to perform accurately with floating point data and the heuristic is hybridised with bottom-left-fill metaheuristic methods to enable the automatic generation of even better quality layouts.

CHAPTER FOUR

A Metaheuristic Hybrid Approach

In this chapter, a hybridised best-fit / bottom-left-fill strategy is proposed. The chapter begins with a broad overview of the approach and then a description of the modifications that are made to the representation of the skyline from the previous chapter. It then describes how the best-fit and bottom-left-fill *elements* can be combined to produce a strategy that outperforms other approaches from the literature. Finally, comparisons are drawn to the previous literature and the best-fit heuristic presented in chapter three on the set of literature benchmark problems as detailed in table 2.2, the new data from the previous chapter (N1 – N13) and ten more randomly generated benchmark datasets (MT1 – MT10).

4.1 Overview of the Hybrid Metaheuristic Approach

The method that is described in this section utilises two different packing algorithms to produce a solution for a given orthogonal packing problem. These algorithms are: the best-fit heuristic as presented in chapter three (modified to operate on floating point data without discretisation) and the bottom-left-fill heuristic as developed by Hopper and Turton (2001) [**110**]. A solution is generated by a two-step process firstly involving the application of the best-fit heuristic to place an initial number of rectangles onto the stock sheet. The remaining unassigned rectangles are placed using the bottom-left-fill heuristic with different input orderings as guided by tabu, simulated annealing and genetic algorithm based metaheuristic search procedures. The positions of the rectangles that are assigned to the sheet in the initial best-fit stage of the process remain fixed throughout and form the first part of the solution whereas the rectangles placed during the second stage will be tried in many different arrangements according to the orderings produced by the metaheuristic. Figure 4.1 displays a pictorial overview of the proposed hybrid algorithm where *n* is the total number of rectangles in the problem and *m* is the number of rectangles that are passed to the metaheuristic bottom-left-fill procedure of phase two.

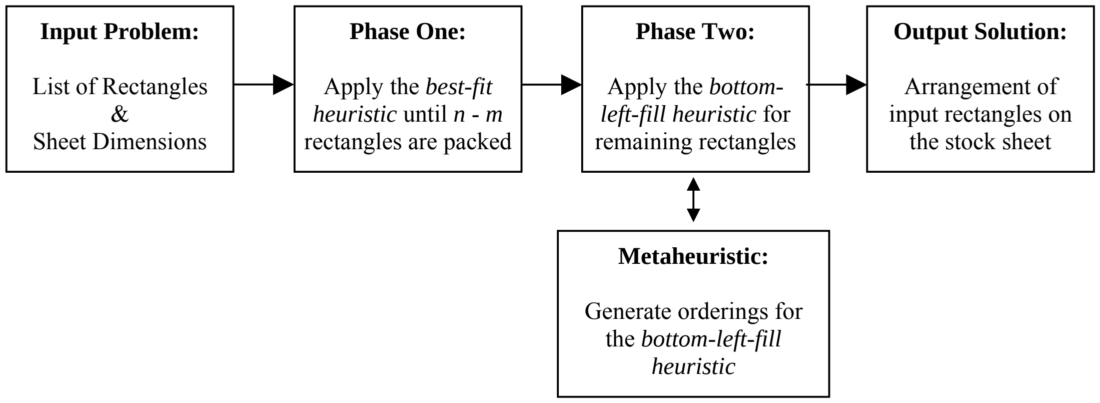

Input Problem:
List of Rectangles & Sheet Dimensions
Phase One:
Apply the *best-fit heuristic* until *n* - *m* rectangles are packed
Phase Two:
Apply the *bottom-left-fill heuristic* for remaining rectangles
Output Solution:
Arrangement of input rectangles on the stock sheet
Metaheuristic:
Generate orderings for the *bottom-left-fill heuristic*

Figure 4.1. General overview of the proposed hybrid approach

4.2 Motivation of the Hybrid Strategy

In this section, a more detailed description of the approach is presented and its motivation is discussed. Chapter three compared the best-fit heuristic to the previous best performing approaches to the problem. These were the bottom-left based metaheuristic methods (Jakobs, 1996 [**106**]; Hopper and Turton, 1999 [**122**]; Hopper and Turton, 2001 [**110**]; Valenzuela and Wang, 2001 [**124**]). In the comparison, conclusions were drawn that the best-fit heuristic is able to consistently outperform the other approaches when used on problems of medium to very large sizes and is able to achieve solutions that can be within 2% of optimal. However, it was also shown that, in general, the best-fit heuristic obtains worse solutions for smaller problems involving less than 50 shapes. Motivated by these observations, a packing strategy has been developed that can further improve upon the best-fit heuristic by utilising the respective strengths of both the best-fit heuristic and the metaheuristic bottom-left methods. The advantages and disadvantages of the two strategies are summarised in table 4.1.

Best-Fit Heuristic (Phase 1)		***Metaheuristic Bottom-Left Methods (Phase 2)***	
Advantages	**Disadvantages**	**Advantages**	**Disadvantages**
Produces best published results on a range of medium/large standard benchmark problems	Worse solutions found when problem size < 50	Optimal solutions found on small problems	Does not represent the best approaches on medium/large problems
Fast solutions obtained ≈ 1000 rectangles/second	Arrangements at the end of the packing often create holes/waste	Can execute for as long as user allows	Similar input sequences may produce vastly differing solutions
No parameters			Definition of search parameters is required

Table 4.1. Advantages and disadvantages of the best-fit heuristic and metaheuristic bottom-left methods

Given an orthogonal stock cutting problem involving n rectangles, the best-fit heuristic is invoked to pack the first $n - m$ rectangles where m is the number of rectangles that are to be carried over to phase 2. Phase 2 involves the packing of the remaining rectangles using the metaheuristic bottom-left-fill combinatorial method for a suitable duration of time. At the commencement of phase 2 it should be noted that we already have the foundations of a solution because the best-fit heuristic (phase 1) has already placed a number of the rectangles on the sheet. These rectangles will remain unchanged during the guided search procedure of phase 2.

The reasoning for this approach can be seen by further examination of table 4.1. Phase 1 aims to utilise the best-fit heuristic's ability to produce high quality solutions in quick time. The earlier termination of the best-fit process, before the entirety of the rectangles are packed, aims to reduce the untidy arrangements that usually occur during the latter stages of best-fit solutions due to the smaller number of rectangles that are available for selection. If a relatively small sub-problem is passed to the metaheuristic method of phase 2, we can limit the problem of excessively large search spaces therefore allowing good phase 2 partial solutions to be found relatively quickly which can be concatenated to the partial solution of phase 1 in order to create the overall layout. Although other factors may also be important such as sheet width or relative size of the problem shapes, it can be seen how the proposed model aims to promote the advantages of both methods whilst inhibiting their disadvantages.

The transition from one phase to the other is guided by the m variable. If $m = 0$ then all rectangles are packed using the best-fit heuristic. However, if $m >= n$ then only phase 2 is used and the approach only uses the metaheuristic bottom-left-fill strategy. From the observations of the previous chapter, the following prediction can be made: if $m > 50$ then the metaheuristic of phase two will probably obtain a worse solution than if we used the standalone best-fit heuristic; therefore an effective value for m may be somewhere in the range $0 < m < 50$. In section 4.5.1 experiments are conducted to evaluate the given strategy and to examine the prediction for the most productive value for m.

In the next two sections, 4.3 and 4.4, a new floating point representation for the best-fit skyline is described and, because the bottom-left-fill placement heuristic uses a representation utilising placement positions rather than the skyline, a method to convert the skyline to location points is described. Only this representation conversion is needed to enable the new hybrid approach to function properly. All other parameter settings remain the same as in chapter three.

4.3 Development of a Floating Point Skyline Representation

In order to achieve full accuracy within the best-fit heuristic, this chapter uses a new floating point capable representation. This is achieved through the use of a data structure that involves a linked list containing the location and length of skyline sub-elements. Figure 4.2 displays the new linked list representation of the skyline constructed from four skyline elements after packing five rectangles and raising the gap once (due to no rectangle fitting the lowest space).

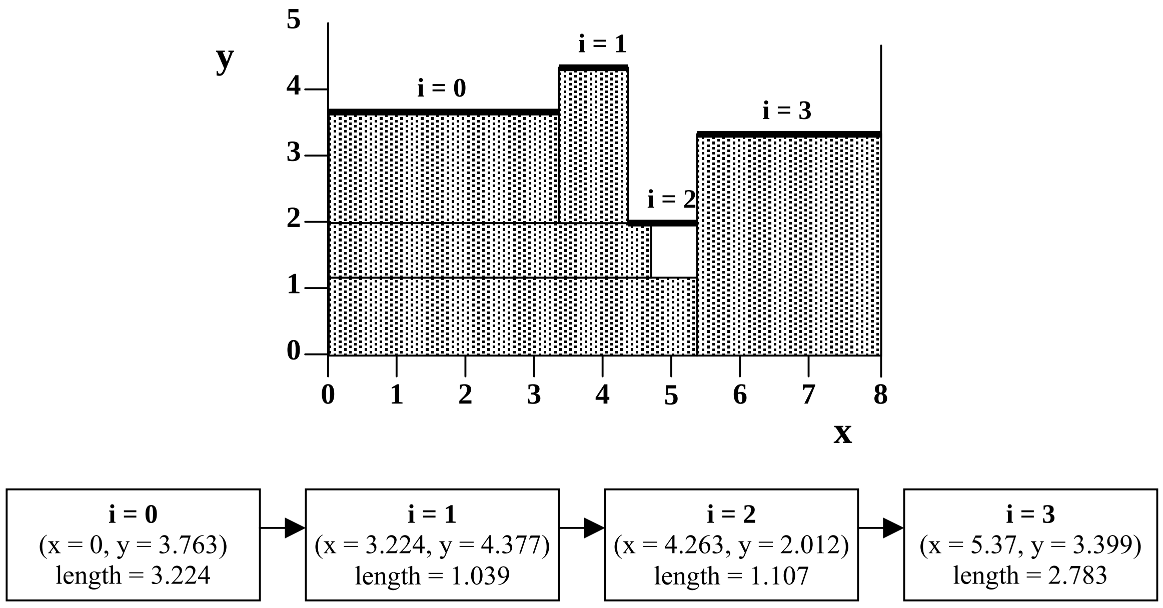

y
x
i = 0
i = 1
i = 2
i = 3
0 1 2 3 4 5 6 7 8
0 1 2 3 4 5
i = 0 (x = 0, y = 3.763) length = 3.224
i = 1 (x = 3.224, y = 4.377) length = 1.039
i = 2 (x = 4.263, y = 2.012) length = 1.107
i = 3 (x = 5.37, y = 3.399) length = 2.783

Figure 4.2. Floating-point representation of the skyline

Using this new floating-point data structure the lowest location and the size of this location on the stock sheet can be found by traversing the linked-list for the skyline element with the smallest *y* component. Due to the floating-point operations and the importance of handling mathematical processor rounding errors, this method can take up to 5 times longer than the array based method. However, this enhancement can be justified because of the extra accuracy on floating-point datasets (as given in Valenzuela and Wang (2001) [**124**]) and also because the execution times are still less than 1 second for all but the largest of the benchmark problems.

4.4 Best-Fit Conversion to Bottom-Left-Fill Representation

One important consideration is how the solution of the best-fit heuristic can be converted into the considerably different representation that is used by the bottom-left-fill approach. The best-fit heuristic produces solutions by using a data structure to maintain the ‘skyline’ or the height profile of a layout whereas the bottom-left-fill approach requires the maintenance of height sorted (and left to right sorted) potential location points (see figure 2.3). After phase one of this approach, the skyline data structure can be converted to a set of potential rectangle locations for use with bottom-left-fill by extending each subelement of the skyline leftwards until it intersects another part of the skyline or the sheet side (see figure 4.3). The placement locations must then be sorted in increasing *y* coordinate order (and sorted by left to right where *y* coordinates are equal). This overall state is stored so that it can be easily revisited because it is always the starting point for the bottom-left-fill heuristic to evaluate the rectangle sequences that are generated by the metaheuristic search techniques.

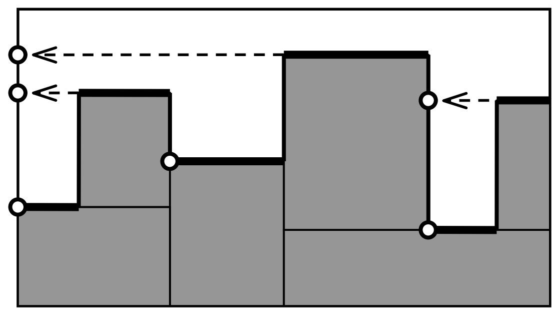


Figure 4.3. Conversion of best-fit’s skyline into bottom-left-fill’s placement positions

Given a specific ordering of the selected rectangles, the bottom-left-fill heuristic can now produce a layout by taking the first rectangle of the sequence and placing it at each location point in turn until it does not intersect with an already placed rectangle. Once a valid position is found, new potential location points, given by the top left and bottom right corners of the placed rectangle, are added to the list of locations and the heuristic can place the second rectangle and so forth until all rectangles have been placed.

4.5 Hybrid Approach Summary

The whole process is summarised pictorially in figure 4.4.

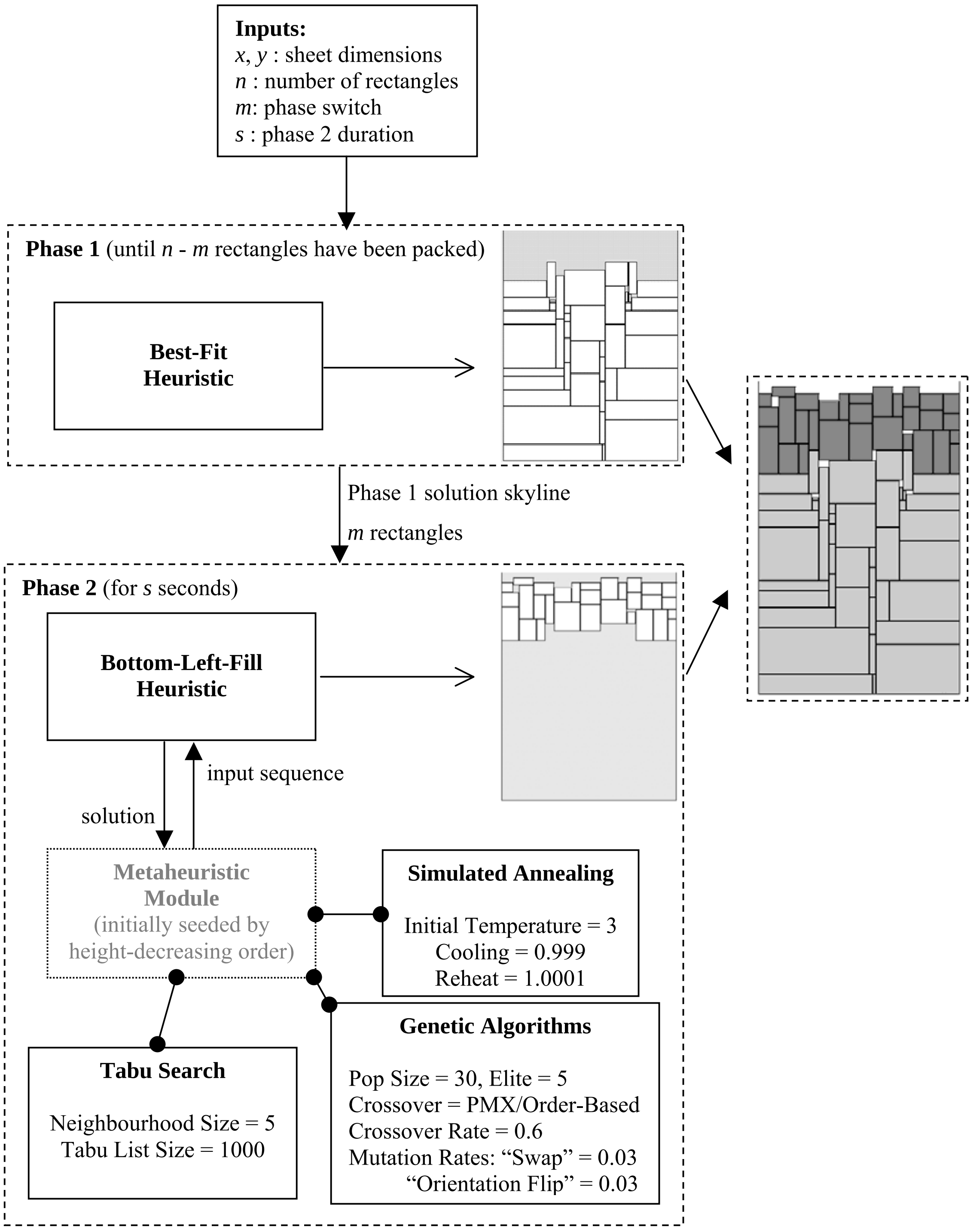


Figure 4.4. A summary of the proposed strategy

4.6 Benchmark Test Data

In order to rigorously evaluate the proposed approach, the entire set of literature benchmark problems (see table 2.2) and the problems introduced in the last chapter, N1-13 (see table 3.1), are used once more. Further to these, ten new problem instances have been introduced for this chapter which are used to examine how solution quality is affected by different phase switch values for *m* and longer search durations in section 4.5.4.

The first five of these problems, MT1-MT5, offer more realistic industrial problems because the optimal solutions are not known (lower bounds of feasibility are calculated by dividing the total area of the rectangles by the sheet width). The last five problems, MT5-MT10, have been created through the dissection of a larger rectangular object and so for these problems the optimal height is known. Table 4.2 summarises the first five datasets, MT1-MT5, for which the optimal packing is unknown. The new datasets, for which optimal solutions are known, MT6-MT10, are given in table 4.3. The data for all of these problems can be found in appendix C.

New Test Problem	Number of Rectangles	Lower Bound Height	Object Dimensions
MT1	100	492.63	400 x 600
MT2	150	760.90	300 x 900
MT3	200	767.31	400 x 900
MT4	300	1174.94	600 x 1300
MT5	500	1264.86	800 x 1400

Table 4.2. New test data with unknown optimal heights

New Test Problem	Number of Rectangles	Optimal Height	Object Dimensions
MT6	100	500	400 x 500
MT7	150	500	400 x 500
MT8	200	800	500 x 800
MT9	300	800	600 x 800
MT10	500	800	700 x 800

Table 4.3. New test data with known optimal heights

4.7 Experimentation and Results

In this section, the results of the proposed method are compared with the results from the approach of the previous chapter. Previously, a singular evaluation method of the total layout height was used to compare the approaches. In order to make comparisons in this chapter, a dual evaluation approach is introduced. This involves the height of the solution obtained (as before) but a secondary measure of the total area underneath the skyline. The main objective is to minimise the height of the packing but, as several different arrangements may have identical heights, the area under the skyline can also be important because it quantifies the amount of wasted space. If a smaller area is required, it means that the rectangles have been packed more densely and therefore the solution has less wasted material and, hence, more of the sheet may be reused in future cutting operations.

As an example, two arrangements that produce solutions of identical height but differing skyline areas are shown in figure 4.5. The solution on the right in figure 4.5 is intuitively (and visually) more desirable as it requires less sheet area so there is less trim-loss and more of the sheet could normally be reused in future cutting processes.

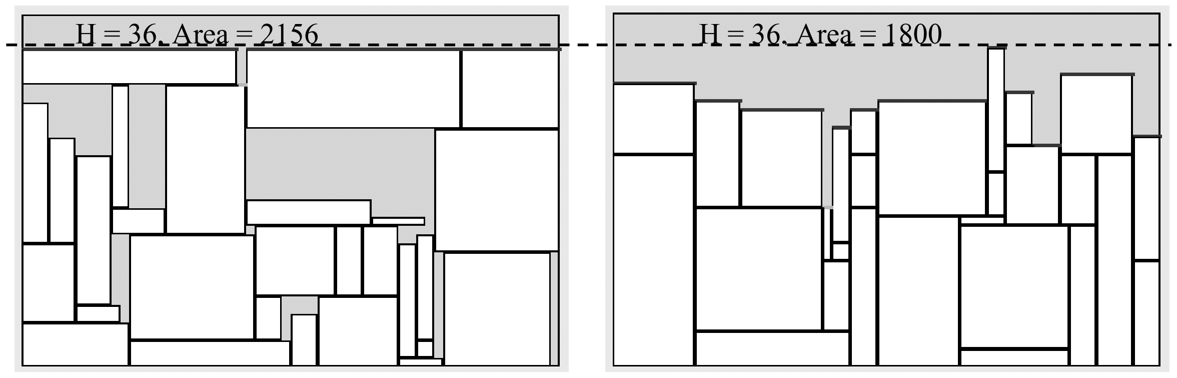


Figure 4.5. Two possible solutions of identical total height but vastly different skyline area evaluations

We use an evaluation measure that is a function of both evaluation methods. The first term is simply the total height required and is therefore the highest location of the skyline. The second term is the average height across the width of the sheet and is found by dividing the area under the skyline by the width of the sheet. This is shown below:

Solution Evaluation = Total Height + (Area Under Skyline / Sheet Width)

Although both solutions of figure 4.3 require the same total height, the left solution has a large amount of wasted space and therefore achieves an evaluation value of 36+(2156/60) = 71.9 and the right solution does not have any wasted space and thus obtains a far better evaluation of 36+(1800/60) = 66. For problems that have been created from the dissection of a larger rectangle (where the optimal solution is known), the optimal evaluation value is twice that of the optimal height. This evaluation measure is used both in this chapter and the next. The aim is to generate layouts that minimise the evaluation measure (indicating more dense solutions). All of the experiments in this chapter have been conducted on a 2GHz Pentium 4 computer with 256Mb RAM.

4.5.1 Value of the *m* Variable

The first experiments varied the value of the phase switch variable, *m*, in order to test the assumption drawn from section 4.2.2 that a value in the range $0 < m < 50$ should produce better results. The following values for *m* were used: 0, 5, 10, 20, 30, 40, 50, 60, and 70 on a subset of the test problems using tabu search, simulated annealing, and genetic algorithms. We have chosen four different data sets to ensure a range of problems sizes, sheet widths, and rectangle sizes. For each test, we allow each metaheuristic method 60 second durations and report the average solution evaluation of 10 runs. Table 4.4 displays the results for each method whereby the metaheuristic that achieves the best average for each value of *m* is shown in bold type.

Problem	Optimal Evaluation	*m* Value	Average Evaluation (10 runs)					
			Tabu		SA		GA	
			Eval	% Over Opt	Eval	% Over Opt	Eval	% Over Opt
Hopper C6P1 (97 shapes)	240	0	244.01	**1.67**	244.01	**1.67**	244.01	**1.67**
		5	243.98	**1.66**	243.98	**1.66**	243.98	**1.66**
		10	243.88	**1.62**	243.88	**1.62**	243.88	**1.62**
		20	242.70	1.13	242.69	**1.12**	242.73	1.14
		30	242.66	**1.11**	242.71	1.13	243.15	1.31
		40	243.03	**1.26**	243.47	1.45	244.08	1.70
		50	243.57	**1.49**	244.05	1.69	244.95	2.06
		60	243.25	**1.35**	244.54	1.89	245.95	2.48
		70	243.64	**1.52**	245.10	2.13	246.91	2.88
Hopper C7P1 (196 shapes)	480	0	489.12	**1.90**	489.12	**1.90**	489.12	**1.90**
		5	489.12	**1.90**	489.12	**1.90**	489.12	**1.90**
		10	489.71	**2.02**	489.71	**2.02**	489.71	**2.02**
		20	487.72	1.61	487.65	**1.59**	487.71	1.61
		30	487.70	**1.60**	487.74	1.61	488.11	1.69
		40	487.59	**1.58**	487.69	1.60	488.50	1.77
		50	488.10	**1.69**	488.58	1.79	488.26	1.72
		60	487.63	**1.59**	488.89	1.85	489.29	1.93
		70	488.08	**1.68**	490.10	2.10	491.30	2.35
Valenzuela Nice P4 (200 shapes)	200	0	209.63	**4.82**	209.63	**4.82**	209.63	**4.82**
		5	209.54	**4.77**	209.54	**4.77**	209.54	**4.77**
		10	209.11	**4.55**	209.11	**4.55**	209.11	**4.55**
		20	207.50	3.75	207.14	**3.57**	207.64	3.82
		30	207.48	3.74	207.38	**3.69**	208.07	4.03
		40	207.81	**3.91**	208.21	4.10	208.39	4.19
		50	208.01	**4.00**	208.51	4.26	208.94	4.47
		60	208.52	**4.26**	209.80	4.90	209.41	4.71
		70	208.79	**4.39**	209.78	4.89	210.19	5.10
New Data MT2 (150 shapes)	Optimal Not Known (Lower Bound Eval =1521.8)	0	1557.27	**2.33**	1557.27	**2.33**	1557.27	**2.33**
		5	1552.70	**2.03**	1552.70	**2.03**	1552.70	**2.03**
		10	1552.77	**2.03**	1552.77	**2.03**	1552.77	**2.03**
		20	1553.47	2.08	1549.90	**1.85**	1553.30	2.07
		30	1544.01	**1.46**	1544.06	**1.46**	1545.14	1.53
		40	1546.37	1.61	1544.71	**1.51**	1548.27	1.74
		50	1548.63	1.76	1547.78	**1.71**	1556.38	2.27
		60	1548.67	**1.77**	1549.93	1.85	1557.56	2.35
		70	1549.80	**1.84**	1552.92	2.04	1561.85	2.63

Table 4.4. Varying the phase switch parameter, *m*

From these results, the best setting for *m* is consistently between 20 and 40 for these sets of data and a 60 second run. The average result returned over the range of *m* values is shown in the graphs of figure 4.6. At lower values, the metaheuristics are not given enough shapes to be able to make significant improvements on the best-fit result. Larger values of *m* seem to impair the metaheuristics because of the larger search spaces but may produce better solutions if a greater execution time were allowed (see sections 4.5.3 and 4.5.4). Other problem specific factors may also be important such as the sheet width, rectangle sizes, and more importantly the quantity of wasted space that is inherited from the result given by the best-fit heuristic. Due to these results, the value of *m* is fixed at 30 for the experiments on the literature benchmark problems in section 4.5.2.

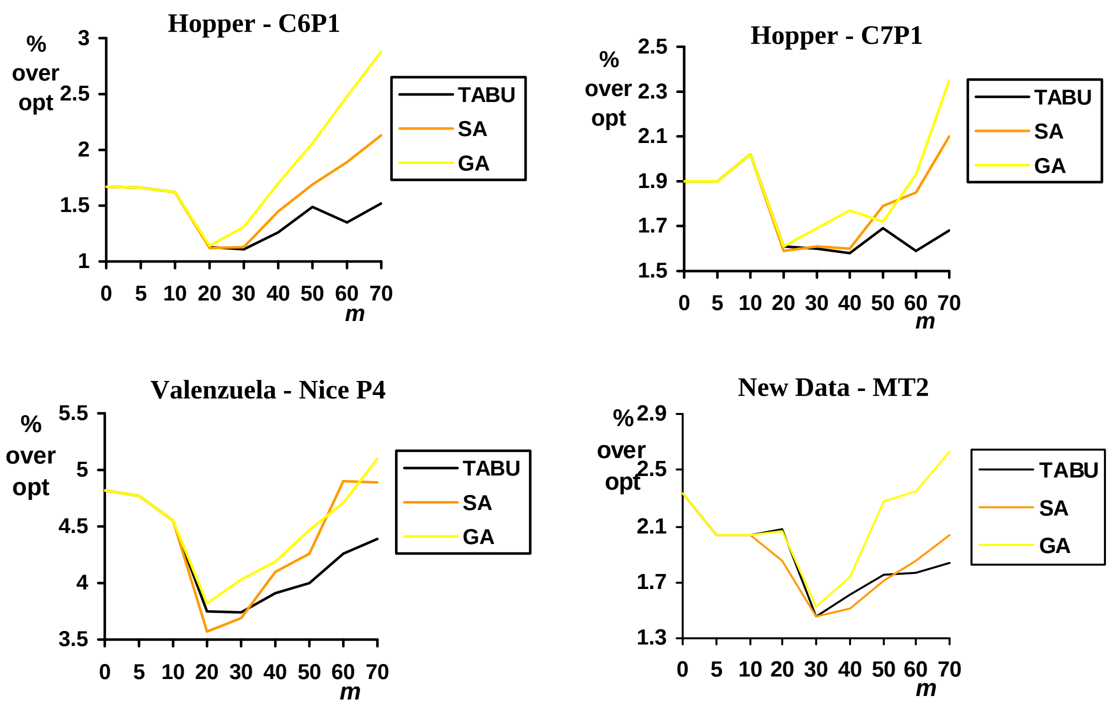


Figure 4.6. Comparison graphs of the percentage above optimal when varying phase switch parameter, *m*

4.5.2 Evaluation of the Hybrid Metaheuristic Approach with the Benchmark Problems of the Literature

The second set of experiments aimed to compare the proposed hybrid metaheuristic method to other approaches from the literature including the standalone best-fit heuristic as proposed in chapter three and Burke, Kendall, and Whitwell (2004) [**1**] and the metaheuristic bottom-left-fill methods proposed in Hopper and Turton (2001) [**110**]. For all the experiments in this section *m* remains fixed at 30 rectangles (given the results of section 4.5.1) and 60 seconds are allowed for the metaheuristic search of phase 2. The best and average solution of ten runs is given for all the problems from the literature in table 4.6. The best solution for each method is given in the final columns as a percentage over optimal and the best method is highlighted in bold type.

Data Set	Cat.	Prob	No. Shapes	Opt Height	Opt Eval	Best-Fit			Best-Fit + Metaheuristic Bottom-Left-Fill												Best Sol. % Over Optimal			
									Tabu Search				Simulated Annealing				Genetic Algorithms							
						Solution Obtained			Best Solution			Avr. Eval	Best Solution			Avr. Eval	Best Solution			Avr. Eval	BF	BF + Tabu	BF + SA	BF + GA
						Height	Area	Eval.	Height	Area	Eval.		Height	Area	Eval.		Height	Area	Eval.					
Hopper and Turton (2001)	C1	P1	16	20	40	21	410	41.50	20	400	40.00	40.71	20	400	40.00	**40.00**	20	400	40.00	40.60	3.75	**0.00**	**0.00**	**0.00**
		P2	17	20	40	22	400	42.00	21	400	41.00	41.02	20	400	40.00	**40.50**	21	400	41.00	41.00	5.00	2.50	**0.00**	2.50
		P3	16	20	40	24	400	44.00	20	400	40.00	40.61	20	400	40.00	**40.00**	20	400	40.00	40.70	10.00	**0.00**	**0.00**	**0.00**
	C2	P1	25	15	30	16	609	31.23	16	600	31.00	31.02	16	600	31.00	**31.00**	16	600	31.00	31.02	4.08	**3.33**	**3.33**	**3.33**
		P2	25	15	30	16	600	31.00	16	600	31.00	31.01	16	600	31.00	**31.00**	16	600	31.00	**31.00**	**3.33**	3.33	3.33	3.33
		P3	25	15	30	16	604	31.10	16	600	31.00	**31.00**	16	600	31.00	**31.00**	16	600	31.00	31.01	3.67	**3.33**	**3.33**	**3.33**
	C3	P1	28	30	60	32	1814	62.23	31	1810	61.17	61.95	31	1818	61.30	**61.59**	31	1823	61.38	61.98	3.72	**1.94**	2.17	2.31
		P2	29	30	60	34	1801	64.02	32	1800	62.00	62.06	31	1814	61.23	**61.72**	32	1809	62.15	62.49	6.69	3.33	**2.06**	3.58
		P3	28	30	60	33	1806	63.10	31	1826	61.43	61.98	31	1808	61.13	**61.46**	31	1826	61.43	62.15	5.17	2.39	**1.89**	2.39
	C4	P1	49	60	120	63	3623	123.38	62	3622	122.37	122.51	61	3621	121.35	**122.08**	62	3623	122.38	122.75	2.82	1.97	**1.13**	1.99
		P2	49	60	120	62	3621	122.35	62	3617	122.28	122.46	61	3618	121.30	**122.20**	62	3624	122.40	122.90	1.96	1.90	**1.08**	2.00
		P3	49	60	120	62	3621	122.35	61	3616	121.27	122.01	61	3608	121.13	**121.53**	62	3610	122.17	122.34	1.96	1.06	**0.94**	1.81
	C5	P1	72	90	180	93	5444	183.73	92	5415	182.25	182.40	91	5419	181.32	**182.33**	92	5432	182.53	182.74	2.07	1.25	**0.73**	1.41
		P2	73	90	180	92	5413	182.22	92	5414	182.23	182.52	91	5410	181.17	**182.18**	92	5422	182.37	182.94	1.23	1.24	**0.65**	1.31
		P3	72	90	180	93	5469	184.15	92	5437	182.62	**182.72**	92	5438	182.63	182.79	92	5446	182.77	182.91	2.31	**1.45**	1.46	1.54
	C6	P1	97	120	240	123	9681	244.01	122	9648	242.60	**242.75**	122	9652	242.65	242.80	122	9678	242.98	243.33	1.67	**1.08**	1.10	1.24
		P2	97	120	240	122	9632	242.40	121	9632	241.40	241.93	121	9632	241.40	**241.42**	121	9634	241.43	241.84	1.00	**0.58**	**0.58**	0.59
		P3	97	120	240	124	9664	244.80	122	9645	242.56	**242.60**	122	9647	242.59	242.63	122	9649	242.61	242.68	2.00	**1.07**	1.08	1.09
	C7	P1	196	240	480	246	38899	489.12	245	38818	487.61	**487.70**	244	38839	486.74	487.75	245	38855	487.84	488.47	1.90	1.59	**1.40**	1.63
		P2	197	240	480	244	38715	485.97	244	38663	485.64	**485.73**	244	38667	485.67	487.02	244	38699	485.87	486.70	1.24	**1.18**	**1.18**	1.22
		P3	196	240	480	245	38864	487.90	245	38853	487.83	488.74	245	38850	487.81	**488.17**	245	38858	487.86	488.75	1.65	**1.63**	**1.63**	1.64
Valenzuela and Wang (2001)	Nice	P1	25	100	200	107.99	10267.7	210.67	106.93	10242.0	209.35	210.70	104.024	10049.5	204.52	207.25	108.42	10190.1	210.33	211.30	5.33	4.67	**2.26**	5.16
		P2	50	100	200	109.69	10423.8	213.93	106.64	10388.5	210.53	211.10	104.413	10221.0	206.62	208.24	106.87	10409.4	210.97	212.59	6.97	5.26	**3.31**	5.48
		P3	100	100	200	107.92	10273.2	210.65	105.23	10276.1	207.99	208.45	105.04	10335.0	208.39	208.69	105.94	10336.3	209.31	211.11	5.33	**4.00**	4.19	4.65
		P4	200	100	200	106.89	10274.1	209.63	104.21	10288.1	207.09	207.39	104.743	10266.7	207.41	207.59	104.89	10307.8	207.97	208.90	4.82	**3.55**	3.71	3.98
		P5	500	100	200	103.39	10195.4	205.35	103.18	10193.4	205.11	205.89	103.507	10199.3	205.50	206.01	103.48	10197.7	205.46	206.25	2.67	**2.56**	2.75	2.73
		P6	1000	100	200	103.84	10267.6	206.52	103.80	10268.2	206.48	206.73	103.756	10269.8	206.45	206.72	103.80	10271.7	206.52	207.02	3.26	3.24	**3.23**	3.26
	Path	P1	25	100	200	110.19	10253.7	212.73	105.82	10345.9	209.28	209.87	103.09	10038.3	203.47	207.21	108.17	10250.6	210.68	212.88	6.36	4.64	**1.74**	5.34
		P2	50	100	200	113.65	10285.1	216.50	103.66	10080.2	204.46	204.79	103.4	10060.9	204.01	204.59	103.66	10085.0	204.51	204.92	8.25	2.23	**2.00**	2.26
		P3	100	100	200	106.75	10180.6	208.56	104.73	10084.9	205.58	206.47	103.037	10125.7	204.29	205.92	104.37	10217.6	206.55	207.52	4.28	2.79	**2.15**	3.27
		P4	200	100	200	104.12	10157.4	205.70	103.53	10058.3	204.11	204.51	103.419	10116.1	204.58	204.63	103.85	10068.3	204.53	204.73	2.85	**2.06**	2.29	2.26
		P5	500	100	200	103.81	10189.0	205.70	103.15	10210.4	205.25	205.38	103.479	10200.1	205.48	205.60	103.41	10209.0	205.50	206.22	2.85	**2.63**	2.74	2.75
		P6	1000	100	200	103.09	10173.0	204.82	102.80	10162.1	204.42	204.69	102.867	10165.0	204.52	204.68	102.90	10169.1	204.59	205.11	2.41	**2.21**	2.26	2.30
Ramesh Babu		P1	50	375	750	400	378125	778.13	400	367875	767.88	767.88	400	367875	767.88	767.88	400	367875	767.88	767.96	3.75	**2.38**	**2.38**	**2.38**
Burke, Kendall, Whitwell (2004)		N1	10	40	80	45	1800	90.00	40	1600	80.00	80.00	40	1600	80.00	80.00	40	1600	80.00	80.00	12.50	**0.00**	**0.00**	**0.00**
		N2	20	50	100	53	1542	104.40	50	1500	100.00	101.68	50	1500	100.00	101.68	50	1500	100.00	102.01	4.40	**0.00**	**0.00**	**0.00**
		N3	30	50	100	52	1525	102.83	51	1503	101.10	101.51	51	1500	101.00	101.32	52	1508	102.27	102.13	2.83	1.10	**1.00**	2.27
		N4	40	80	160	86	6547	167.84	83	6437	163.46	163.70	82	6455	162.69	163.50	83	6443	163.54	163.82	4.90	2.16	**1.68**	2.21
		N5	50	100	200	105	10154	206.54	103	10124	204.24	204.34	103	10122	204.22	204.29	104	10112	205.12	205.48	3.27	2.12	**2.11**	2.56
		N6	60	100	200	102	5061	203.22	102	5047	202.94	202.95	102	5047	202.94	202.94	102	5047	202.94	202.97	1.61	**1.47**	**1.47**	**1.47**
		N7	70	100	200	108	8230	210.88	105	8094	206.18	206.26	104	8117	205.46	205.97	104	8113	205.41	206.30	5.44	3.09	2.73	**2.71**
		N8	80	80	160	83	8116	164.16	82	8097	162.97	162.97	82	8097	162.97	162.97	82	8097	162.97	162.97	2.60	**1.86**	**1.86**	**1.86**
		N9	100	150	300	152	7556	303.12	152	7544	302.88	317.88	152	7544	302.88	317.88	152	7544	302.88	317.92	1.04	0.96	0.96	0.96
		N10	200	150	300	152	10567	302.96	152	10570	303.00	303.00	152	10570	303.00	303.00	152	10570	303.00	303.00	**0.99**	1.00	1.00	1.00
		N11	300	150	300	153	10586	304.23	153	10586	304.23	304.23	153	10586	304.23	304.23	153	10586	304.23	304.23	**1.41**	1.41	1.41	1.41
		N12	500	300	600	306	30431	610.31	306	30431	610.31	610.31	306	30431	610.31	610.31	306	30431	610.31	610.31	**1.72**	1.72	1.72	1.72
		N13	3152	960	1920	964	615637	1925.9	964	615637	1925.9	1925.9	964	615629	1925.9	1925.9	964	615629	1925.9	1925.9	**0.31**	0.31	0.31	0.31

Table 4.5. Comparison of new hybrid approach to the best-fit heuristic using test problems from the literature

Data Set	Cat	Prob	No. Shapes	Opt Height	Opt Eval	Metaheuristic Bottom-Left-Fill								Best-Fit + Metaheuristic Bottom-Left-Fill							
						Tabu Search				Simulated Annealing				Tabu Search				Simulated Annealing			
						Best Solution			Avr. Eval	Best Solution			Avr. Eval	Best Solution			Avr. Eval	Best Solution			Avr. Eval
						Height	Area	Eval.		Height	Area	Eval.		Height	Area	Eval.		Height	Area	Eval.	
Hopper and Turton (2001)	C1	P1	16	20	40	20	400	40.00	40.82	20	400	40.00	**40.00**	20	400	40.00	40.71	20	400	40.00	**40.00**
		P2	17	20	40	21	400	41.00	41.00	20	400	40.00	40.73	21	400	41.00	41.02	20	400	40.00	**40.50**
		P3	16	20	40	20	400	40.00	40.50	20	400	40.00	40.10	20	400	40.00	40.61	20	400	40.00	**40.00**
	C2	P1	25	15	30	16	600	31.00	31.00	15	600	30.00	**30.90**	16	600	31.00	31.02	16	600	31.00	31.00
		P2	25	15	30	16	600	31.00	**31.00**	16	600	30.00	**31.00**	16	600	31.00	31.01	16	600	31.00	**31.00**
		P3	25	15	30	16	600	31.00	**31.00**	16	600	30.00	**31.00**	16	600	31.00	**31.00**	16	600	31.00	**31.00**
	C3	P1	28	30	60	31	1810	61.17	61.85	31	1800	61.00	**61.36**	31	1810	61.17	61.95	31	1818	61.30	61.59
		P2	29	30	60	32	1800	62.00	62.20	31	1801	61.02	61.63	32	1800	62.00	62.06	31	1801	61.02	**61.59**
		P3	28	30	60	31	1822	61.37	61.96	31	1810	61.17	61.52	31	1826	61.43	61.98	31	1808	61.13	**61.46**
	C4	P1	49	60	120	62	3617	122.28	122.91	62	3643	122.72	123.92	62	3622	122.37	122.51	61	3621	121.35	**122.08**
		P2	49	60	120	62	3617	122.28	122.33	61	3620	121.33	122.66	62	3617	122.28	122.46	61	3618	121.30	**122.20**
		P3	49	60	120	62	3612	122.20	122.32	62	3619	122.32	122.43	61	3616	121.27	122.01	61	3608	121.13	**121.53**
	C5	P1	72	90	180	92	5415	182.25	183.13	92	5462	183.03	183.84	92	5415	182.25	182.40	91	5419	181.32	**182.33**
		P2	73	90	180	92	5407	182.12	182.48	91	5422	181.37	183.24	92	5414	182.23	182.52	91	5410	181.17	**182.18**
		P3	72	90	180	92	5431	182.52	183.02	93	5460	184.00	184.12	92	5437	182.62	**182.72**	92	5438	182.63	182.79
	C6	P1	97	120	240	122	9695	243.19	243.75	122	9694	243.18	244.32	122	9648	242.60	**242.75**	122	9652	242.65	242.80
		P2	97	120	240	122	9635	242.44	242.99	122	9668	242.85	244.20	121	9632	241.40	241.93	121	9632	241.40	**241.42**
		P3	97	120	240	122	9726	243.58	244.14	123	9703	244.30	244.52	122	9645	242.56	**242.60**	122	9647	242.59	242.63
	C7	P1	196	240	480	246	38741	488.13	488.47	245	38915	488.22	489.39	245	38818	487.61	**487.70**	244	38839	486.74	487.75
		P2	197	240	480	244	38767	486.29	487.12	246	38873	487.72	488.13	244	38663	485.64	**485.73**	244	38667	485.67	487.02
		P3	196	240	480	245	38864	487.90	488.40	245	38899	488.12	488.52	245	38853	487.83	488.74	245	38850	487.81	**488.17**
Valenz-uela and Wang (2001)	Nice	P1	25	100	200	106.63	10230.3	208.93	209.83	104.11	10049.5	204.61	207.31	106.93	10242.0	209.35	210.70	104.024	10049.5	204.52	**207.25**
		P2	50	100	200	109.13	10400.8	213.14	213.77	109.411	10392.2	213.33	214.50	106.64	10388.5	210.53	211.10	104.413	10221.0	206.62	**208.24**
		P3	100	100	200	108.83	10589.4	214.72	215.71	109.21	10614.8	215.36	216.79	105.23	10276.1	207.99	**208.45**	105.04	10335.0	208.39	208.69
		P4	200	100	200	110.93	10836.1	219.29	220.06	111.45	10787.1	219.32	219.67	104.21	10288.1	207.09	**207.39**	104.743	10266.7	207.41	207.59
		P5	500	100	200	110.45	10817.9	218.63	219.45	110.68	10736.5	218.04	218.45	103.18	10193.4	205.11	**205.89**	103.507	10199.3	205.50	206.01
		P6	1000	100	200	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	103.80	10268.2	206.48	206.73	103.756	10269.8	206.45	**206.72**
	Path	P1	25	100	200	106.22	10302.2	209.24	209.99	101.82	10030.6	202.13	**205.61**	105.82	10345.9	209.28	209.87	103.09	10038.3	203.47	207.21
		P2	50	100	200	105.01	10046.9	205.48	206.06	105.01	10072.8	205.74	206.32	103.66	10080.2	204.46	204.79	103.4	10060.9	204.01	**204.59**
		P3	100	100	200	104.03	10162.0	205.65	205.89	104.417	10216.5	206.58	208.03	104.73	10084.9	205.58	206.47	103.037	10125.7	204.29	**205.92**
		P4	200	100	200	104.37	10210.2	206.47	206.84	104.821	10236.0	207.18	207.67	103.53	10058.3	204.11	**204.51**	103.419	10116.1	204.58	204.63
		P5	500	100	200	106.71	10438.3	211.10	211.25	105.914	10452.5	210.44	211.49	103.15	10210.4	205.25	**205.38**	103.479	10200.1	205.48	205.60
		P6	1000	100	200	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	102.80	10162.1	204.42	204.69	102.867	10165.0	204.52	**204.68**
Ramesh Babu		P1	50	375	750	400	375000	775.00	**775.00**	400	375000	775.00	**775.00**	400	376875	776.88	776.88	400	376875	776.88	776.88
Burke, Kendall and Whitwell (2004)		N1	10	40	80	40	1600	80.00	**80.00**	40	1600	80.00	**80.00**	40	1600	80.00	**80.00**	40	1600	80.00	**80.00**
		N2	20	50	100	50	1500	100.00	101.49	50	1500	100.00	**101.42**	50	1500	100.00	101.68	50	1500	100.00	101.68
		N3	30	50	100	51	1501	101.03	101.36	51	1500	101.00	101.34	51	1503	101.10	101.51	51	1500	101.00	**101.32**
		N4	40	80	160	82	6400	162.00	163.12	81	6403	161.04	**162.98**	83	6437	163.46	163.70	82	6455	162.69	163.50
		N5	50	100	200	104	10147	205.47	206.21	104	10071	204.71	205.56	103	10124	204.24	204.34	103	10122	204.22	**204.29**
		N6	60	100	200	102	5039	202.78	203.10	102	5016	202.24	203.36	102	5047	202.94	202.95	102	5047	202.94	**202.94**
		N7	70	100	200	101	8009	201.11	203.34	102	8022	202.18	**203.12**	105	8094	206.18	206.26	104	8117	205.46	205.97
		N8	80	80	160	82	8061	162.61	163.02	82	8057	162.57	163.41	82	8097	162.97	**162.97**	82	8097	162.97	**162.97**
		N9	100	150	300	152	7556	303.12	**304.66**	153	7557	304.14	304.80	152	7544	302.88	317.88	152	7544	302.88	317.88
		N10	200	150	300	152	10582	303.14	303.46	153	10595	304.36	304.62	152	10570	303.00	**303.00**	152	10570	303.00	**303.00**
		N11	300	150	300	152	10583	303.19	**304.18**	153	10620	304.71	305.81	153	10586	304.23	304.23	153	10586	304.23	304.23
		N12	500	300	600	307	30524	612.24	612.51	306	30508	611.08	612.56	306	30431	610.31	**610.31**	306	30431	610.31	**610.31**
		N13	3152	960	1920	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	964	615637	1925.9	**1925.9**	964	615629	1925.9	**1925.9**

Table 4.6. Comparison of the new hybrid approach with the metaheuristic bottom-left-fill algorithm as used in [**109,110**]

Table 4.5 shows that of the 57 literature problems, the proposed hybrid metaheuristic method can significantly improve upon 52 of these (91%) using just one minute of extra computation time for the metaheuristic search. Of the five problems that are not improved by phase 2 of the approach, one problem achieves a very good solution by employing best-fit alone and can only be improved by achieving the optimal solution (C2P2). The other four problems have very small rectangles at the top of the layout which consume all of the 30 rectangles and therefore there is not enough scope for improvement (N10, N11, N12, N13). For these problems the phase switch variable may need to be increased but this will also require a larger increase in the execution time to achieve suitable exploration and exploitation of the search space. However, with a phase switch setting of 30 rectangles, the proposed approach still manages to produce the equivalent quality solution (to using best-fit alone) for these five problems. Experimentation is extended for these five problems in section 4.5.3.

Table 4.6 compares the average solutions achieved by the new hybrid algorithm to one of the best-performing approaches of the literature (and the phase 2 method), the metaheuristic bottom-left-fill algorithm (the best average result is shown in bold). In this table, shading is used to indicate all of the problem instances involving fewer than 30 rectangles because for these problems both algorithms will perform equally as the hybrid approach will not use the best-fit heuristic (phase 1) as the phase switch variable, *m*, has been fixed at 30 rectangles. Indeed this is evident from the table because both algorithms achieve similar average solutions. The most interesting problems occur when there are greater than 30 rectangles, thus allowing the approach to utilise the respective advantages of both phases. In most of the datasets the approach obtains much better average solutions than the metaheuristic bottom-left-fill algorithm on problems with more than 30 rectangles. In the new datasets from the previous chapter, N1-N13, the algorithm achieves improved results on over half of the problems and is competitive on the remaining problems. The explanation of why the metaheuristic bottom-left-fill algorithm performs better on this dataset is due to each of these problems involving relatively few large rectangles and many small rectangles that can be used to fill holes. Therefore it can place the larger rectangles in any orientation and in potentially poor intermediary arrangements but, because of the vast number of smaller rectangles, the holes can easily be filled to produce good layouts. The other literature datasets do not share this property and thus it cannot perform so well on these whereas the proposed approach can utilise best-fit to lay the foundations of good solutions.

It has been shown that the hybrid method can achieve better solutions than best-fit alone and metaheuristic bottom-left-fill alone on most problems with only 60 seconds of additional execution time. In terms of how the individual metaheuristics compared during the phase two process, simulated annealing finds the best solution on 30 problems, and tabu search on 22 problems. The genetic algorithm suffers due to the loss of genetic structure when decoded in a solution by the bottom-left-fill heuristic. It can only find the best solution on 10 of the problems and outright best on only 1. This can be explained because, when using a metaheuristic with a 'decoder heuristic', such as bottom-left-fill, two input sequences that are very similar can potentially produce very different solutions in terms of both visual appearance and solution quality. This is because the bottom-left-fill heuristic is a 'building block' heuristic where each successive rectangle depends on the placement of all the rectangles before it in the sequence. As an example, the two input strings: i) 1, 2, 3, 4, 5, 6 and ii) 2, 1, 3, 4, 5, 6 are similar but for the swap of rectangles at the start of the sequence. However, this change may be enough to dramatically affect the placements of rectangles 3, 4, 5, and 6. If we take the two sequences: i) 1, 2, 3, 4, 5, 6 and ii) 1, 2, 3, 4, 6, 5 then these will be visually similar when decoded because rectangles 1, 2, 3, and 4 will appear in the same locations in both solutions because they are not affected by the later swap. This can create potential problems for genetic algorithms where the best individuals are bred with the assumption that the children inherit the good (and possibly bad) properties of the parents. However, it should be noted that all methods consistently outperformed best-fit alone which had previously produced the best results in the literature for problem sizes larger than 50. Simulated annealing produced the best overall results on the test problems although, as it required tuning of its search parameters to be most effective for the testing criteria of 60 second runs with *m* = 30 rectangles, the tabu search may provide a better alternative in the general case.

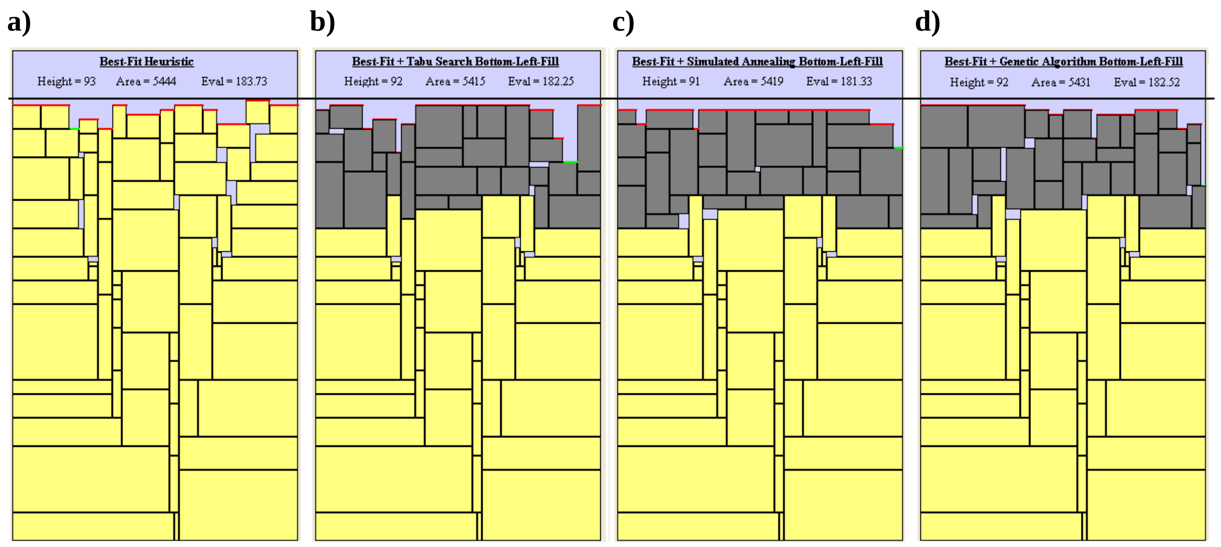

a)
Best-Fit Heuristic
Height = 93 Area = 5444 Eval = 183.73
b)
Best-Fit + Tabu Search Bottom-Left-Fill
Height = 92 Area = 5415 Eval = 182.25
c)
Best-Fit + Simulated Annealing Bottom-Left-Fill
Height = 91 Area = 5419 Eval = 181.33
d)
Best-Fit + Genetic Algorithm Bottom-Left-Fill
Height = 92 Area = 5431 Eval = 182.52

Figure 4.7. Solutions obtained by a) solitary best-fit heuristic, b) best-fit with tabu bottom-left-fill, c) best-fit with simulated annealing bottom-left-fill and d) best-fit with genetic algorithm bottom-left-fill on problem C5P1.

As an example, figure 4.7 shows the best solutions achieved by each method on the dataset C5P1 from Hopper and Turton. From figure 4.7a, it can be seen that by applying the best-fit heuristic alone, a very good overall solution can be generated (3 units above optimal). Also it is apparent that the heuristic produces very dense packings at the bottom of the sheet. However, at the top of the packing there are fewer rectangles for the heuristic to select from and therefore the arrangement begins to suffer from holes and thus wastage. Figures 4.7b, 4.7c and 4.7d show the results of applying the three metaheuristic bottom-left-fill methods to the last 30 rectangles of the initial packing given by the best-fit heuristic. In less than 60 seconds, all of the methods are able to improve upon the best-fit heuristics attempt at packing the last 30 rectangles both in terms of total packing height and sheet area required. For this dataset, the best solution was obtained by applying the simulated annealing bottom-left-fill method yielding a solution height of 91 which is only 1 unit above the optimal solution. The best solutions for two other problems, C7P1 and PathP2, are shown in figure 4.8.

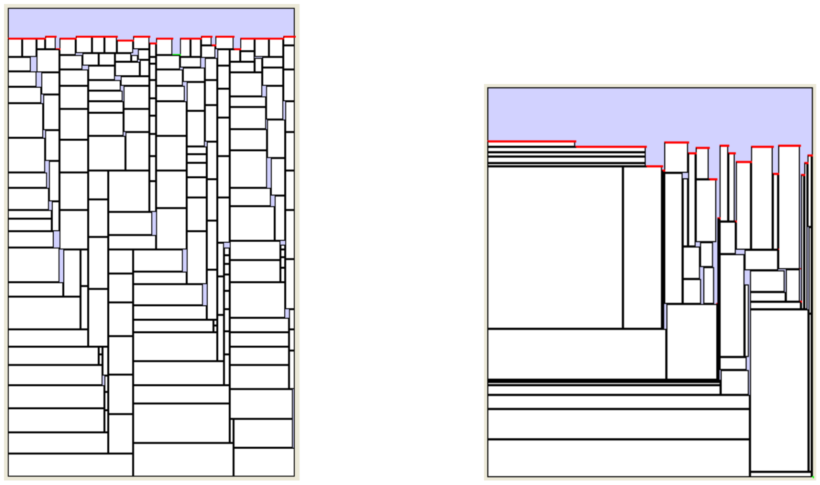


Figure 4.8. Best solutions for C7P1 (Height = 244 units) and PathP2 (Height = 103.4 units)

4.5.3 Extended Experiments on Benchmark Problems from the Literature

In section 4.5.2 new best solutions were produced for 52 of the 57 literature benchmark problems. In this section the experiments are extended for the 5 problems for which the metaheuristic bottom-left-fill methods were unable to improve upon the best-fit heuristic's results. For these 5 problems further experiments are conducted using a phase switch value of 150 rectangles and the run times are extended to 5 minutes. In allowing more rectangles into the second phase, each iteration takes longer due to the extra collision detections required within the bottom-left-fill algorithm. For this reason, the starting temperature has been reduced to 2 units for the simulated annealing algorithm so that cooling could progress thoroughly. All other search parameters remain the same for all methods. The best and average solutions of 10 runs for each of these extended experiments are shown in table 4.7 (best results in bold).

Test Data Set	Size	Best-Fit Heuristic			Best-Fit + Metaheuristic Bottom-Left-Fill (*m* = 150, 5 min run)											
					Tabu Search				Simulated Annealing				Genetic Algorithm			
		Best Solution			Best Solution			Avr. Eval	Best Solution			Avr. Eval	Best Solution			Avr. Eval
		Height	Area	Eval	Height	Area	Eval		Height	Area	Eval		Height	Area	Eval	
C2P2	25	16	600	31.00	**15**	**600**	**30.00**	30.90	**15**	**600**	**30.00**	30.80	16	600	31.00	31.00
N10	200	152	10567	302.96	**151**	**10530**	**301.43**	302.07	152	10551	302.73	302.77	152	10590	303.29	303.36
N11	300	153	10586	304.23	152	10585	303.21	303.23	**152**	**10581**	**303.16**	303.20	152	10589	303.27	303.30
N12	500	306	30431	610.31	**304**	**30281**	**606.81**	606.86	304	30288	606.88	607.00	304	30291	606.91	607.20
N13	3152	**964**	**615637**	**1925.9**	**964**	**615637**	**1925.9**	1925.9	**964**	**615637**	**1925.9**	1925.9	**964**	**615637**	**1925.9**	1925.9

Table 4.7. Results of extended experiments using *m* = 150 and run times of 5 minutes

From table 4.7, the best performing methods were tabu search and simulated annealing with each improving over best-fit alone on four of the five problems. Once again, the genetic algorithm was the least productive method during phase two with no overall best results (although it produced better results than best-fit alone on two problems). In section 4.5.2 it was suggested that the solution given for problem C2P2 could not be improved by the proposed method because the best-fit heuristic alone produces an excellent solution. However, the optimal solution for this problem was found by allowing more time to explore the search space. It should be noted that only phase two is active in this problem anyway because the problem size is smaller than *m*. This solution is shown in figure 4.9.

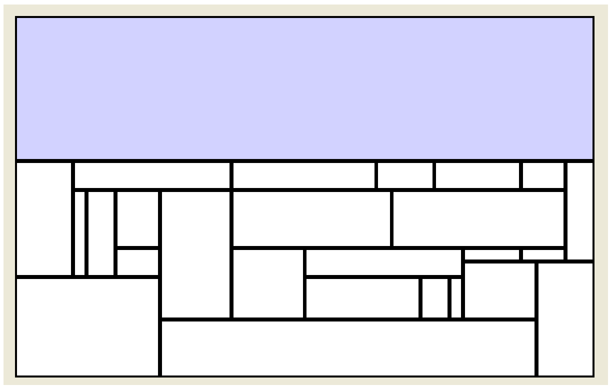


Figure 4.9. The optimal solution found for C2P2 using tabu (height = 15 units, area = 600)

The final four problems, N10-N13, had the property that small rectangles consumed all of the rectangles that are passed to phase two of the proposed method and thus the approach did not have enough scope for improvement. Increasing the value of the phase switch variable has allowed enough scope for improvement on three of these problems when using the proposed hybrid approach. For example, the height of the solution obtained for N10 is only 1 unit above optimal. The approach was unable to improve upon the largest of the problems, N13, due to the slowdown inherent with the collision detection tests for the bottom-left-fill heuristic but an equal solution of 4 units over optimal was achieved. The best solutions for the other three extended problems for which the algorithm produced the best known results, N10-N12, are shown in figure 4.10. Further improvements may be found by extending the problems further but this is ultimately a time versus quality decision that would depend on the user’s requirements.

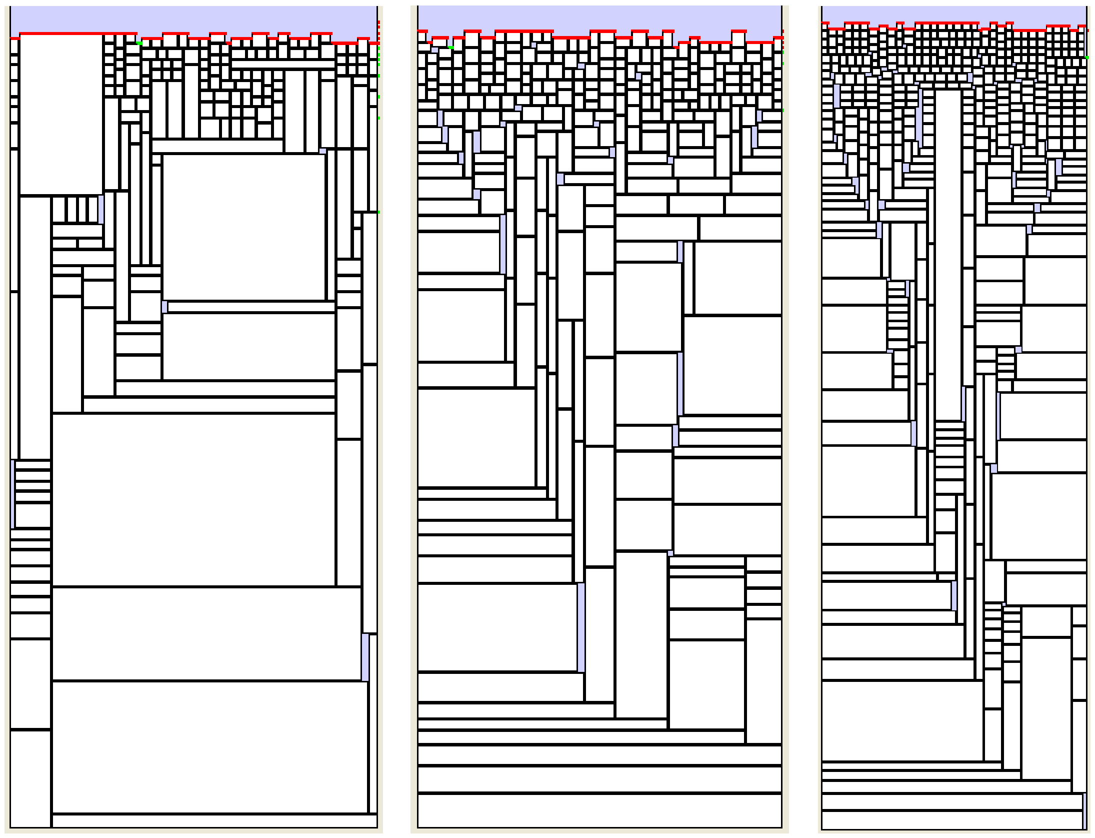


Figure 4.10. Best solutions obtained for the extended problems (from left to right): N10 (height = 151, area = 10530), N11 (height = 152, area = 10581) and N12 (height = 304, area = 30281)

4.5.4 Experiments on New Test Problems

In the final set of experiments, further comparisons are drawn into the improvements that the new hybrid method can achieve with larger periods of time and larger values of *m*. Solutions are generated for the ten new test problems given in table 4.2 and table 4.3 using the proposed approach with values of 30, 50, and 100 rectangles for the phase switch variable, *m*. In order to compare how time affects the results obtained, 60, 300, and 600 second phase 2 durations are used (shown in tables 4.8, 4.9 and 4.10 respectively). In each case the best and average result of ten runs is given. The best result for each row of the table is highlighted (black text on grey shading). Also highlighted is the best evaluation average for each dataset within each table (white text on black). This identifies the phase switch value that performed the best under each phase two time duration. The tables show that allowing longer time durations, in general, produces better average solutions given equal *m* values. This is not surprising as the metaheuristic search has more time to explore the search space in order to find good solutions. However, it is more interesting to observe how the best value of the phase switch variable *m* changes with different time durations. The results show that low values of *m* yield the best results given only small time durations for phase two. This is shown in table 4.8 where *m* = 30 gives the best average evaluation for eight of the ten problems. For the other two problems, the best-fit heuristic creates solutions with several holes during the latter stages of packing and therefore a larger value for *m* can allow these holes to be eliminated within the metaheuristic packing phase. In order to obtain further improvements more rectangles may be passed to phase two, although inevitably this requires longer execution times. This can be demonstrated by problems MT1 and MT4. With 60 second durations, the best *m* value for these problems is 30 rectangles (see table 4.8). However, if a longer duration of 600 seconds is allowed both problems exhibit better average evaluations using an *m* value of 50 rectangles (see table 4.10). Increasing the number of rectangles passed to phase two *can* yield improvements, although ultimately this depends somewhat on the quality of solution given by the best-fit heuristic. If the best-fit heuristic produces very good solutions with very few holes towards the latter stages of the packing then by increasing the number of rectangles passed to phase two, more of the best-fit solution's *desirable* layout is destroyed and must be *rediscovered* by the metaheuristic search. This inevitably requires much longer search durations and could potentially not find an equivalent layout given reasonable time. The results indicate this is especially the case with problems created from the dissection of a large rectangle, MT6-MT10, because the best average solutions are consistently obtained using an *m* value of 30 rectangles. Another factor to consider is that the bottom-left-fill heuristic becomes slower with a greater number of rectangles due to the extra collision detection tests that are required. Therefore this also favours smaller *m* phase switch values.

Data Set	**Prob. Size**	***m***	**Best-Fit + Metaheuristic Bottom-Left-Fill**											
			Tabu Search				**Simulated Annealing**				**Genetic Algorithms**			
			Best Solution			**Avr. Eval**	**Best Solution**			**Avr. Eval**	**Best Solution**			**Avr. Eval**
			Height	Area	Eval.		Height	Area	Eval.		Height	Area	Eval.	
MT1	100	30	505.05	198672.2	1001.73	1002.52	503.08	198729.6	999.90	1001.28	506.10	199240.2	1004.20	1004.80
		50	502.50	198926.4	999.82	1001.97	501.38	199053.4	999.01	1001.31	508.30	200137.3	1008.64	1009.30
		100	504.30	199886.0	1004.02	1007.24	504.50	199437.8	1003.09	1006.92	516.40	200379.8	1017.35	1017.35
MT2	150	30	773.70	231121.1	1544.10	1545.99	775.20	231146.9	1545.69	1549.14	776.50	231554.6	1548.35	1549.88
		50	775.00	231083.9	1545.28	1548.12	774.00	230946.8	1543.82	1548.40	783.20	231162.7	1553.74	1555.70
		100	783.40	232513.5	1558.45	1562.37	781.50	232762.1	1557.37	1560.56	798.50	235898.6	1584.83	1592.19
MT3	200	30	795.70	314251.7	1581.33	1583.14	793.20	313991.5	1578.18	1582.21	798.90	314288.2	1584.62	1586.73
		50	795.80	314391.0	1581.78	1583.07	794.00	313337.4	1577.34	1581.23	803.60	314885.5	1590.81	1594.01
		100	791.00	313942.5	1575.86	1578.53	793.00	313505.3	1576.76	1580.67	801.70	316915.2	1593.99	1594.43
MT4	300	30	1201.00	714758.0	2392.26	2393.45	1199.00	714773.0	2390.29	2390.97	1201.00	715017.0	2392.70	2393.43
		50	1199.00	714822.0	2390.37	2392.27	1200.00	715022.0	2391.70	2392.79	1202.00	715905.0	2395.18	2396.07
		100	1208.00	717337.0	2403.56	2411.83	1211.00	720719.0	2412.20	2414.83	1220.00	722800.0	2424.67	2424.67
MT5	500	30	1286.60	1020366.1	2562.06	2564.07	1286.60	1020656.6	2562.42	2563.63	1290.70	1021301.9	2567.33	2568.57
		50	1283.40	1020694.3	2559.27	2561.12	1283.10	1021290.7	2559.71	2563.03	1288.60	1022191.4	2566.34	2568.91
		100	1286.20	1021081.4	2562.55	2564.61	1285.00	1022023.3	2562.53	2564.48	1289.90	1023482.0	2569.25	2569.25
MT6	100	30	513.83	203341.3	1022.18	1023.74	512.25	203351.0	1020.63	1022.84	516.03	203903.0	1025.79	1027.58
		50	516.81	203893.8	1026.54	1028.87	518.06	204174.6	1028.49	1029.95	518.68	204745.5	1030.55	1032.03
		100	526.54	206505.4	1042.80	1048.83	530.39	206802.9	1047.39	1050.49	542.82	209445.3	1066.43	1066.68
MT7	150	30	508.34	201766.2	1012.75	1012.87	508.16	201621.9	1012.22	1012.67	510.05	201876.6	1014.74	1015.14
		50	509.37	201756.8	1013.76	1014.59	509.29	202175.7	1014.73	1015.69	512.98	202053.9	1018.11	1018.76
		100	511.99	202735.2	1018.83	1019.77	514.16	202695.8	1020.90	1022.02	516.50	203308.2	1024.77	1024.77
MT8	200	30	819.02	406547.5	1632.12	1632.64	818.10	406701.3	1631.50	1632.51	820.96	407041.2	1635.04	1635.34
		50	821.99	406758.8	1635.51	1637.07	821.05	406762.7	1634.57	1635.83	825.00	407762.5	1640.53	1641.29
		100	825.45	407336.5	1640.12	1642.73	823.86	406911.6	1637.69	1642.60	830.75	409210.7	1649.17	1650.55
MT9	300	30	816.18	485304.1	1625.02	1625.52	815.44	485287.6	1624.26	1624.85	817.60	485334.1	1626.49	1627.07
		50	816.41	485611.2	1625.77	1628.11	816.41	485544.2	1625.65	1627.76	820.16	485996.8	1630.16	1631.09
		100	896.13	485051.5	1704.55	1705.21	896.13	487239.4	1708.19	1709.21	823.74	489353.5	1639.33	1643.10
MT10	500	30	1515.78	906963.9	3027.38	3028.89	1516.00	906931.1	3027.55	3028.31	1517.75	907106.3	3029.59	3030.54
		50	1517.34	906915.1	3028.87	3030.13	1520.56	907169.7	3032.51	3032.99	1521.41	907982.3	3034.71	3035.82
		100	1524.82	908585.9	3039.13	3040.60	1526.27	908840.3	3041.00	3042.56	1530.77	910407.6	3048.12	3048.77

Table 4.8. Best and average results using phase two durations of 60 seconds

Data Set	Prob. Size	m	Best-Fit + Metaheuristic Bottom-Left-Fill											
			Tabu Search				Simulated Annealing				Genetic Algorithms			
			Best Solution			Avr. Eval	Best Solution			Avr. Eval	Best Solution			Avr. Eval
			Height	Area	Eval.		Height	Area	Eval.		Height	Area	Eval.	
MT1	100	30	502.78	198569.3	999.20	1000.36	502.78	198626.9	999.34	1001.36	502.78	198830.6	999.85	1002.09
		50	501.38	198663.1	998.04	1000.49	502.50	199153.5	1000.38	1001.49	505.68	199635.2	1004.77	1006.24
		100	505.50	199195.6	1003.49	1006.93	504.50	199024.8	1002.06	1004.70	513.00	200670.3	1014.68	1016.82
MT2	150	30	773.70	231121.1	1544.10	1544.60	773.70	231126.6	1544.12	1544.80	774.00	231287.3	1544.96	1545.82
		50	774.00	230771.8	1543.24	1545.41	773.20	230840.4	1542.67	1545.88	782.30	231121.1	1552.70	1553.47
		100	782.40	232019.1	1555.80	1560.46	780.90	231832.8	1553.68	1558.23	796.70	235685.0	1582.32	1587.04
MT3	200	30	795.10	313719.0	1579.40	1580.43	793.00	313508.1	1576.77	1579.05	798.90	313646.3	1583.02	1585.10
		50	796.50	313715.2	1580.79	1582.78	796.00	313531.9	1579.83	1582.36	799.60	315657.6	1588.74	1590.43
		100	790.70	313718.7	1575.00	1576.29	790.60	312996.5	1573.09	1578.56	802.80	314891.6	1590.03	1593.11
MT4	300	30	1199.00	714827.0	2390.38	2391.88	1198.00	715312.0	2390.19	2390.45	1199.00	714839.0	2390.40	2390.78
		50	1198.00	714600.0	2389.00	2390.13	1197.00	714443.0	2387.74	2390.25	1200.00	715764.0	2392.94	2394.92
		100	1199.00	715191.0	2390.99	2392.19	1198.00	715015.0	2389.69	2392.50	1205.00	718542.0	2402.57	2403.67
MT5	500	30	1286.60	1020264.6	2561.93	2562.72	1285.60	1020681.0	2561.45	2561.83	1289.50	1020601.9	2565.25	2566.71
		50	1282.50	1020156.8	2557.70	2558.95	1283.20	1020722.6	2559.10	2559.75	1288.80	1021177.9	2565.27	2566.14
		100	1284.10	1021141.7	2560.53	2561.91	1283.40	1021242.7	2559.95	2562.39	1289.90	1023482.0	2569.25	2569.25
MT6	100	30	515.03	203354.7	1023.41	1023.76	512.93	203303.9	1021.19	1022.29	514.46	203798.8	1023.96	1025.59
		50	517.43	204089.1	1027.66	1028.71	518.07	204032.7	1028.15	1029.32	515.85	204121.6	1026.15	1029.95
		100	524.08	206168.9	1039.50	1046.21	527.48	206714.3	1044.27	1047.22	533.90	209937.2	1058.74	1063.47
MT7	150	30	508.05	201628.7	1012.12	1012.61	507.19	201631.8	1011.27	1011.75	508.84	201698.2	1013.09	1013.78
		50	508.77	201837.4	1013.36	1014.23	508.56	201864.1	1013.22	1014.55	511.86	202116.2	1017.15	1017.56
		100	511.39	202443.2	1017.50	1018.40	512.02	203152.6	1019.91	1021.52	516.59	202982.6	1024.05	1024.58
MT8	200	30	818.73	406615.6	1631.96	1632.11	817.93	406622.6	1631.18	1631.53	819.74	406706.7	1633.16	1633.80
		50	819.76	406629.7	1633.02	1634.14	818.11	406667.9	1631.45	1633.72	823.59	407370.1	1638.33	1639.70
		100	820.62	407242.2	1635.10	1637.84	823.91	407206.7	1638.32	1640.44	829.48	408452.6	1646.39	1647.62
MT9	300	30	815.00	485178.9	1623.63	1624.08	814.67	485194.4	1623.32	1623.69	816.00	485144.7	1624.58	1625.77
		50	815.81	485471.6	1624.93	1626.19	814.21	485310.4	1623.07	1626.03	817.52	486130.0	1627.74	1629.06
		100	896.13	484926.6	1704.34	1704.98	820.84	485492.0	1629.99	1676.41	824.47	488191.5	1638.13	1640.62
MT10	500	30	1515.76	906799.0	3027.09	3027.48	1514.95	906697.6	3026.11	3026.62	1517.71	906998.5	3029.37	3029.85
		50	1516.68	906829.1	3028.07	3029.48	1517.93	907245.6	3030.01	3030.71	1520.99	907669.6	3033.78	3034.81
		100	1521.59	908957.4	3036.52	3037.49	1523.89	909188.6	3039.21	3040.59	1527.53	910011.8	3044.21	3045.48

Table 4.9. Best and average results using phase two durations of 300 seconds

Data Set	Prob. Size	m	Best-Fit + Metaheuristic Bottom-Left-Fill											
			Tabu Search				Simulated Annealing				Genetic Algorithms			
			Best Solution			Avr. Eval	Best Solution			Avr. Eval	Best Solution			Avr. Eval
			Height	Area	Eval.		Height	Area	Eval.		Height	Area	Eval.	
MT1	100	30	503.08	198616.3	999.62	1000.61	503.08	198574.8	999.51	1000.79	505.80	198630.2	1002.38	1002.93
		50	502.60	198995.1	1000.09	1001.58	502.50	198559.3	998.90	999.85	504.08	199835.3	1003.67	1005.70
		100	504.78	199031.6	1002.36	1007.23	502.88	199352.5	1001.26	1002.63	516.40	200379.8	1017.35	1017.35
MT2	150	30	774.00	231117.6	1544.39	1544.46	774.00	231059.3	1544.20	1544.50	774.00	231126.6	1544.42	1546.39
		50	774.30	230803.5	1543.65	1545.86	772.80	230876.5	1542.39	1545.30	779.10	231039.2	1549.23	1551.19
		100	781.00	232070.6	1554.57	1558.70	779.70	232071.8	1553.27	1557.91	796.10	236024.1	1582.85	1588.56
MT3	200	30	792.60	313922.3	1577.41	1579.66	794.10	313836.8	1578.69	1579.80	797.40	313929.2	1582.22	1584.12
		50	797.30	313759.2	1581.70	1582.84	794.00	313368.5	1577.42	1581.48	799.40	314650.1	1586.03	1588.58
		100	791.80	312864.3	1573.96	1576.73	790.30	312690.7	1572.03	1575.14	800.70	315384.6	1589.16	1590.92
MT4	300	30	1199.00	714748.0	2390.25	2390.75	1198.00	715012.0	2389.69	2390.21	1198.00	715133.0	2389.89	2390.46
		50	1198.00	714250.0	2388.42	2389.62	1197.00	715158.0	2388.93	2390.13	1200.00	715988.0	2393.31	2393.97
		100	1198.00	715013.0	2389.69	2393.16	1199.00	714488.0	2389.81	2391.65	1206.00	717656.0	2402.09	2403.15
MT5	500	30	1286.60	1020295.3	2561.97	2562.05	1286.60	1020261.4	2561.93	2563.60	1286.60	1020758.7	2562.55	2564.69
		50	1283.20	1020484.0	2558.81	2559.26	1282.50	1020353.7	2557.94	2559.29	1288.08	1021134.3	2564.50	2565.79
		100	1283.10	1021257.1	2559.67	2561.09	1282.80	1020929.0	2558.96	2561.89	1289.90	1023482.0	2569.25	2569.25
MT6	100	30	513.00	203521.4	1021.81	1023.18	512.77	203451.1	1021.40	1022.17	514.15	203616.0	1023.19	1024.34
		50	516.97	204263.1	1027.62	1029.26	517.17	203872.5	1026.85	1028.06	518.83	204077.2	1029.02	1029.78
		100	526.24	206319.8	1042.03	1046.47	525.25	206668.7	1041.93	1048.11	533.59	209332.3	1056.92	1062.14
MT7	150	30	507.71	201642.4	1011.82	1012.36	507.08	201600.3	1011.08	1011.45	508.54	201697.6	1012.79	1013.61
		50	509.67	201798.9	1014.17	1014.74	508.49	202030.2	1013.57	1014.27	510.61	202105.7	1015.87	1016.49
		100	511.54	202572.1	1017.97	1019.55	512.87	203155.2	1020.76	1022.40	515.42	203108.3	1023.19	1024.29
MT8	200	30	817.83	406818.4	1631.47	1631.72	817.82	406520.0	1630.86	1631.09	819.94	406634.7	1633.21	1633.39
		50	819.41	406432.1	1632.27	1633.95	817.85	406691.2	1631.24	1632.99	821.80	407189.2	1636.18	1638.06
		100	823.98	406826.9	1637.63	1639.38	823.91	407351.6	1638.61	1639.88	829.40	407566.9	1644.54	1646.12
MT9	300	30	814.91	485146.4	1623.49	1623.92	814.20	485107.0	1622.71	1623.25	816.78	485207.4	1625.46	1625.81
		50	814.82	485298.6	1623.65	1624.61	815.46	485521.1	1624.66	1626.56	818.12	486059.5	1628.22	1628.94
		100	890.89	486519.3	1701.76	1704.95	820.64	487090.6	1632.45	1662.91	824.55	488241.2	1638.28	1639.96
MT10	500	30	1516.20	906891.7	3027.69	3028.09	1514.91	906824.1	3026.28	3026.53	1517.30	907073.6	3029.08	3029.95
		50	1517.48	906813.8	3028.84	3029.62	1517.44	906775.1	3028.73	3029.87	1519.49	907608.6	3032.18	3033.99
		100	1521.20	908621.9	3035.57	3037.12	1522.51	908946.6	3037.42	3037.98	1528.09	910557.0	3045.69	3046.27

Table 4.10. Best and average results using phase two durations of 600 seconds

4.8 Summary

In chapter three, it was concluded that the best-fit heuristic outperformed other approaches from the literature on problems which had more than 50 rectangles. Metaheuristic bottom-left-fill methods produced better results on these smaller problems. In this chapter, a new hybrid has been proposed which combines the relative strengths of both the best-fit heuristic and the metaheuristic based bottom-left-fill methods for the two-dimensional orthogonal stock cutting problem. A new floating point skyline representation was implemented which is fully accurate, although best-fit execution times were marginally slower. Initial experiments involved using a small subset of the test problems to discover the best value for *m* and it was shown that a value of around 30 rectangles produced the best solutions with sixty-second execution times. Using 57 test problems (most of which are provided by other researchers in the field of cutting and packing) the proposed method can make significant improvements upon the solutions given by using the best-fit heuristic alone on 52 of the literature problems using only small amounts of additional time. For the remaining 5 literature problems, which could not initially be improved upon, longer execution times were allowed for the metaheuristic search and the number of rectangles being passed to phase 2 was increased. This resulted in a further 4 new overall best solutions being produced. The problem for which the algorithm could not obtain improved results was the largest of the instances, N13 (3152 rectangles) but, as the best-fit heuristic is involved, a solution which had equal quality with the best result from chapter three was found. Overall, the approach described within this chapter has obtained the best published results for 56 of the 57 problems that have previously appeared in the literature and an equivalent result for the 57th problem (N13). No method in the literature produces better results than the hybrid approach presented in this chapter on such a wide range of standard benchmark problems. The hybrid method can quickly and significantly improve upon layouts produced by other techniques of the literature and it can achieve good solutions that are less than 1% over optimum in some cases.

CHAPTER FIVE

An Interactive Approach

In the last two chapters, the best-fit placement heuristic was described and developed to allow hybridisation with a metaheuristic bottom-left-fill method. The motivation for the hybrid approach was that the best-fit heuristic produces excellent layouts when it can choose from many rectangles but suffers in the latter stages of layout generation whereby holes can be created due to the heuristic being too "greedy". The last chapter has improved upon the best results from within the literature but seem to be dependent on the relationship of the phase switch parameter, *m*, the type of problem and the quality of the phase one layout produced by the best-fit heuristic.

The aim of this chapter is to eliminate the phase switch parameter, instead allowing a human user to select the rectangles which should be presented to the metaheuristic bottom-left-fill second phase through the definition of poorer regions within the layout. This approach enables the specific targeting of unacceptable areas which, as shown in this chapter, yields significant improvements in solution quality.

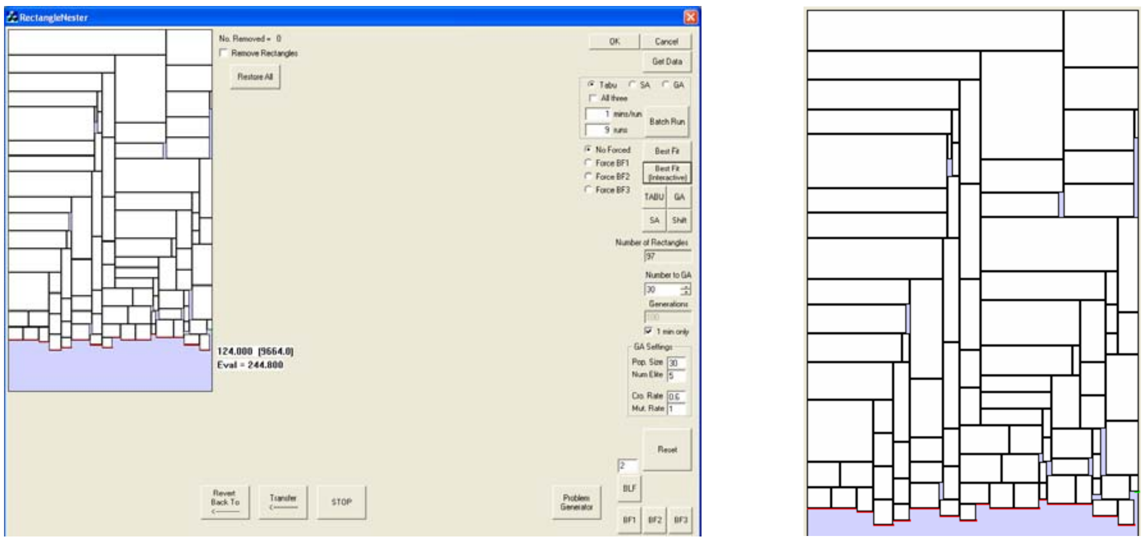
5.1 An Overview of the New User-Guided Approach

In this section, a description and the implementation details for the user-guided procedure are given. The aim of the presented approach is to bring together the respective advantages of the current state-of-the-art rectangle packing heuristics, metaheuristic algorithms and a mechanism that allows the user to interact with the automation process. An important aspect of the presented approach is that the user does not need to "micromanage" the packing process. This means that the user is abstracted away from directly altering the individual placement or rotation of the rectangles and, instead, can direct the packing algorithms to concentrate on poorer regions of the solution. An analogy can be drawn to a manufacturing business involving several skilled engineers and a manager. The engineers are very proficient within their field of expertise (i.e. producing products). However, the manager takes a broader view of the entire business to ensure that orders are satisfied and, ultimately, that the business is profitable. The manager must provide a link between the production of products by the engineers and the orders that must be satisfied for the customers. The manager does not want to become involved in the process of welding piece A to piece B because this distracts from their key role of managing the business. In the same way the user within the automated packing model is better placed to direct the automation algorithms rather than moving and adjusting individual rectangles. Automation approaches are proficient at quickly examining a large number of possible layouts and are good at placing rectangles together so that they touch whereas the user must use drag-and-drop style interfaces which take longer and can result in rectangles that do not touch because of the pixelisation of displayed solutions. However, humans excel in the quick analysis of layout quality and identification of poor regions through the visualisation of holes (large amounts of space within the layouts). Of course, such spatial awareness is an integral part of human life and it is this that we seek to exploit.

The algorithm remains unchanged from the previous chapter except that the phase switch parameter is removed in favour of user definition and, due to its success within the previous experimentation, only the simulated annealing metaheuristic has been used within the experiments of this chapter (the method and parameters for simulated annealing remain unchanged). The method of user interaction and an illustrated example of the process is presented in the next section.

5.2 User-Guided Procedure

As it is imperative to provide a user-friendly graphical interface for interactive approaches, a simple graphical interface testbed was developed. This provided the user with a display of the last accepted solution and a display of the best solution currently produced throughout the current metaheuristic search. For the following explanation of the procedure, it is assumed that the best-fit placement heuristic has already provided an initial solution to the user. This is shown on the left side of the screen within the software (see figure 5.1). The solution is also displayed in larger scale on figure 5.1 for clarity. Note that the solutions are displayed with height increasing downscreen.



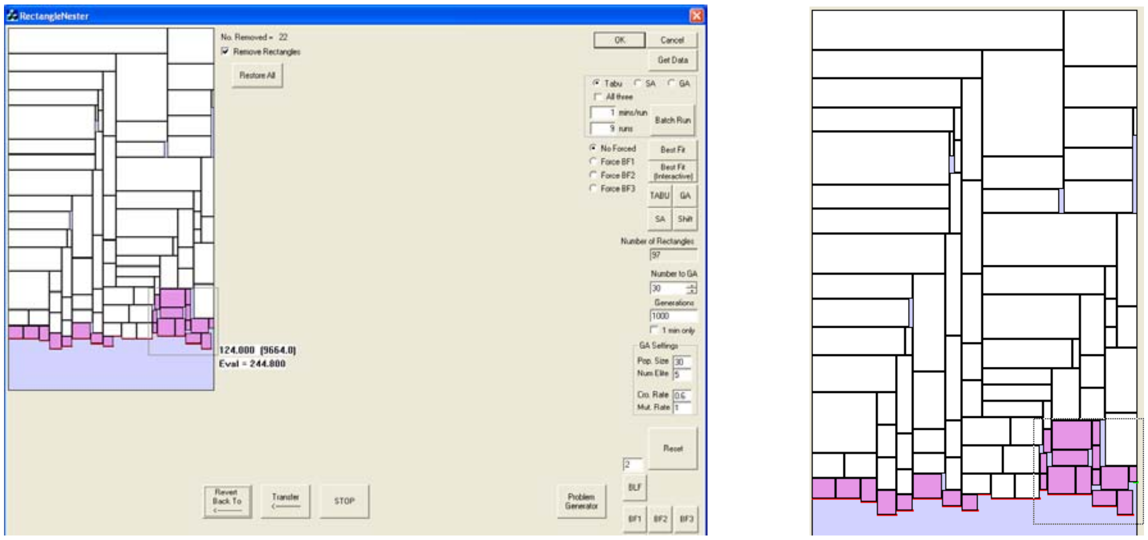
No. Removed = 0
Remove Rectangles
124.000 [9664.0]
Eval = 244.800
[✓ Tabu] SA GA
All three
1 mins/run
9 runs
[✓ No Forced] Force BF1 Force BF2 Force BF3
Number of Rectangles 97
Number to GA 30
Generations 100
[✓ 1 min only]
GA Settings
Pop. Size 30
Num Elite 5
Cro. Rate 0.6
Mut. Rate 1
2

Figure 5.1. Initial solution produced by the best-fit placement heuristic (height = 124, area = 9664, eval = 244.8)

5.2.1 Selection of Regions

The first procedure involves the identification of regions within which the arrangement of rectangles is not acceptable. As with all interactive approaches, there is an element of subjectivity involved in this process because different users will have differing ideas about acceptable wastage. Of course, this may also be a factor of the job being undertaken. For example, if the solution will be used as a "template" for the cutting of multiple identical layouts from an expensive material then the user will probably have a lower wastage acceptance threshold as small improvements in solution quality are magnified several times in terms of saved material and profitability. In the case of a one-off layout with cheaper material, the user may have a higher acceptable wastage threshold as improvements may not yield significant cost savings.

Within the proposed system the user defines a region, which they think could be improved, by dragging a rectangle around the area that they wish to improve. All rectangles that are contained within this region are now 'selected' (identified with shading) and will be available to the metaheuristic search when invoked. This is shown in figure 5.2 where the user has selected some of the regions involving holes and rectangles that increase the height of the layout.



No. Removed = 22
[✓] Remove Rectangles
124.000 (9664.0)
Eval = 244.800

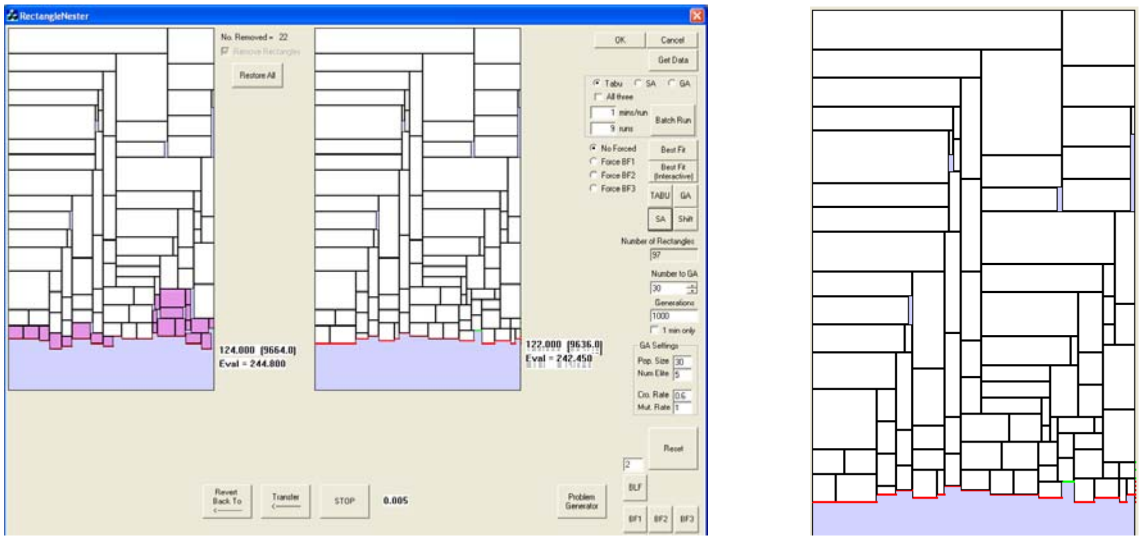
[✓ Tabu] SA GA
All three
1 mins/run
9 runs
[✓ No Forced] Force BF1 Force BF2 Force BF3
Number of Rectangles 97
Number to GA 30
Generations 1000
1 min only
GA Settings
Pop. Size 30
Num Elite 5
Cro. Rate 0.6
Mut. Rate 1
2

Figure 5.2. Selection of unacceptable regions involving holes

Conversely, the rectangles that remain unselected (white rectangles within figure 5.2) are locked in position and may not be moved by the automated approach. The user may identify any number of regions, select individual rectangles or deselect rectangles as they wish. However, as discussed in chapter four the metaheuristic search works best with fewer rectangles as the search space can be investigated more extensively. The user is advised to respect this observation although is free to select as many rectangles as they wish (the user may even select all of them).

5.2.2 Invoking the Metaheuristic Search

The user must be able to specify the termination criterion for the metaheuristic bottom-left-fill algorithm. In the proposed implementation this is achieved by either defining the time or the number of iterations allowed. In reality, the user in industry would be more interested in using the time allowed as the termination measure but the iteration based measure has been included to facilitate easier comparisons whilst conducting this research. If the user is content with the best solution produced so far from the metaheuristic bottom-left-fill algorithm or wants to define new regions, the current search can be stopped before the termination criterion has been met.



No. Removed = 22
[✓ Remove Rectangles]
124.000 (9664.0)
Eval = 244.800
122.000 (9636.0)
Eval = 242.450
[✓ Tabu] SA GA
All three
1 mins/run
9 runs
[✓ No Forced] Force BF1 Force BF2 Force BF3
Number of Rectangles: 97
Number to GA: 30
Generations: 1000
1 min only
GA Settings
Pop. Size: 30
Num Elite: 5
Cro. Rate: 0.6
Mut. Rate: 1
2
0.005

Figure 5.3. Solution before (left) and after (right) metaheuristic is invoked (height = 122, area = 9636, eval = 242.45)

Figure 5.3 shows a screenshot after the metaheuristic bottom-left-fill search runs for a few seconds (the best solution is shown on the right side of the screenshot and is shown in larger scale on the right of figure 5.3). The holes that existed previously are no longer present and the overall height has been reduced by two units because of the new arrangement of the rectangles within the selected regions.

After the metaheuristic search has stopped because of the termination criterion being met or from user intervention then the new solution can be accepted, rejected or further time can be allowed for continued searching. In the case that the solution is accepted, the new layout becomes the editable solution and the user is once again free to define unacceptable regions on the layout. If rejected the solution is deleted and the original solution, that existed before the metaheuristic was invoked, is maintained and can be used for redefinition of regions once more. It is possible that the solution is still improving when the search is terminated. In this case the user may decide to allow the metaheuristic further search time to examine more layouts.

The procedure of defining unacceptable regions and then invoking the metaheuristic bottom-left-fill algorithm in order to search for better quality layouts can continue indefinitely until the user is satisfied with the solution.

5.3 Benchmark Data

In order to rigorously test the new approach against the other approaches of the literature all of the literature benchmarks presented in table 2.2 and all of the newly generated benchmarks from the previous chapters, N1-N13 and MT1-MT10, are used. Also presented are 5 new floating point benchmark instances, U1-U5, for which the optimal solutions are not known (the lower bound height is found by summing the rectangle area and dividing by the sheet width).

The author of this thesis feels it is important to introduce more of these benchmarks to the literature as they are typical of the style of problem that exists within an industrial setting (in that the optimal solution is not known) and offer particular challenges for generating good quality solutions. The problem sizes range from 50 to 200 rectangles and involve different combinations of small and large pieces to represent the wide range of problem characteristics that could occur (see table 5.1).

Problem	**Number of Rectangles**	**Lower Bound Height**	**Sheet Dimensions**
U1	50	77.42	50 x 120
U2	100	72.79	80 x 150
U3	100	250.00	150 x 300
U4	200	163.57	80 x 250
U5	200	207.13	80 x 250

Table 5.1. New benchmark data

5.4 Experimentation and Results

This section compares the solutions generated using the new user-guided approach to that of the approaches of chapter three, chapter four and previously published methods within the literature. Once again, the secondary measure based on the containment area (the area beneath the solution skyline) is also used. This enables layouts with identical height to be distinguished in terms of quality because the secondary measure will quantify the used and, therefore, reusable area of the sheet. An example of this was shown in figure 4.5.

Once again, all experiments have been conducted on a 2GHz Pentium 4 processor with 256MB RAM. The author took on the role of the end user for the interactive experiments and set an upper time limit of 30 minutes as this reflects the conditions that would be acceptable within many industrial settings (within the author's experience).

5.4.1 Comparisons against Previous Approaches of the Literature

The first set of experiments involved producing layouts for the 57 existing benchmark problems in order to compare against the previous approaches that had produced the best results on these problems. The interactive approach is compared against: i) the best-fit heuristic, ii) metaheuristic bottom-left-fill and iii) a hybrid best-fit heuristic combined with metaheuristic bottom-left-fill. Table 5.2 shows the best result obtained using each method and the time taken for it to be achieved. The best results given for methods involving metaheuristics (ii and iii) are the best found during 30 runs of the algorithm (10 with each of tabu search, simulated annealing and genetic algorithms) and are drawn from chapter four.

The table shows that the new approach produces solutions that are generally significantly better or equal to the previous state-of-the-art approaches from the literature but for three problem instances. However, this is at the expense of longer time durations because of the addition of the user feedback. In particular, the user interaction seems to be able to produce almost optimal solutions within reasonable time frames for many of the larger problems and in the case of C5P1, involving 72 rectangles, the optimal solution is found using just 11 minutes of operation (see figure 5.4).

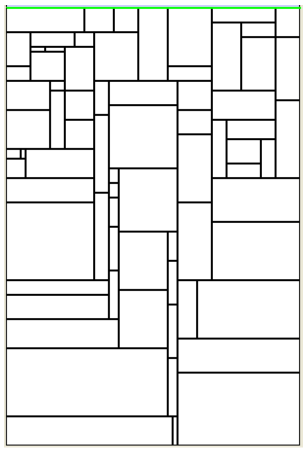
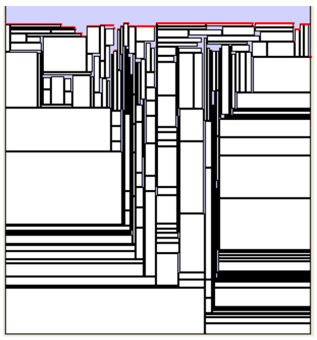


Figure 5.4. Solution generated for dataset C5P1 with user interactivity (size = 72, height = 90 units, time = 11 mins)

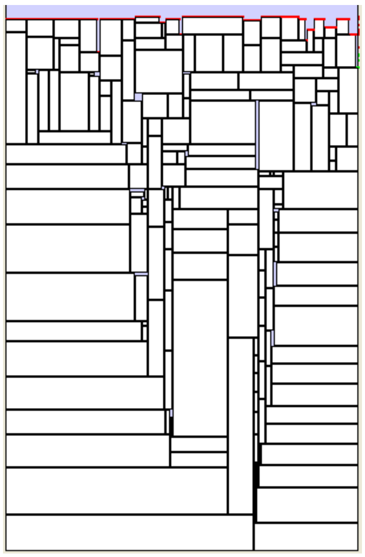
Data Set	Cat.	Prob	No. Shapes	Opt Height	Opt Eval	Best-Fit (BF)				Metaheuristic Bottom-Left-Fill (MBLF)				Best-Fit + Metaheuristic Bottom-Left-Fill (BF+MBLF)				Proposed Interactive Approach (IA)				Best Sol. % Over Optimal			
						Best Solution				Best Solution				Best Solution				Solution Obtained				BF	MBLF	BF + MBLF	IA
						Height	Area	Eval.	Time	Height	Area	Eval.	Time	Height	Area	Eval.	Time	Height	Area	Eval.	Time				
Hopper and Turton (2001)	C1	P1	16	20	40	21	410	41.50	<0.1 s	20	400	40.00	60 s	20	400	40.00	60 s	20	400	40.00	<1 min	3.75	**0.00**	**0.00**	**0.00**
		P2	17	20	40	22	400	42.00	<0.1 s	20	400	40.00	60 s	20	400	40.00	60 s	20	400	40.00	<1 min	5.00	**0.00**	**0.00**	**0.00**
		P3	16	20	40	24	400	44.00	<0.1 s	20	400	40.00	60 s	20	400	40.00	60 s	20	400	40.00	<1 min	10.00	**0.00**	**0.00**	**0.00**
	C2	P1	25	15	30	16	609	31.23	<0.1 s	15	600	30.00	60 s	16	600	31.00	60 s	15	600	30.00	<1 min	4.08	**0.00**	3.33	**0.00**
		P2	25	15	30	16	600	31.00	<0.1 s	16	600	31.00	60 s	15	600	30.00	5 min	15	600	30.00	4 min	3.33	3.33	**0.00**	**0.00**
		P3	25	15	30	16	604	31.10	<0.1 s	16	600	31.00	60 s	16	600	31.00	60 s	15	600	30.00	3 min	3.67	3.33	3.33	**0.00**
	C3	P1	28	30	60	32	1814	62.23	<0.1 s	31	1800	61.00	60 s	31	1810	61.17	60 s	31	1800	61.00	3 min	3.72	**1.67**	1.94	**1.67**
		P2	29	30	60	34	1801	64.02	<0.1 s	31	1801	61.02	60 s	31	1801	61.02	60 s	31	1800	61.00	4 min	6.69	1.69	1.69	**1.67**
		P3	28	30	60	33	1806	63.10	<0.1 s	31	1810	61.17	60 s	31	1808	61.13	60 s	31	1800	61.00	2 min	5.17	1.94	1.89	**1.67**
	C4	P1	49	60	120	63	3623	123.38	<0.1 s	62	3617	122.28	60 s	61	3621	121.35	60 s	61	3600	121.00	6 min	2.82	1.90	1.13	**0.83**
		P2	49	60	120	62	3621	122.35	<0.1 s	61	3620	121.33	60 s	61	3618	121.30	60 s	61	3600	121.00	8 min	1.96	1.11	1.08	**0.83**
		P3	49	60	120	62	3621	122.35	<0.1 s	62	3612	122.20	60 s	61	3608	121.13	60 s	61	3600	121.00	8 min	1.96	1.83	0.94	**0.83**
	C5	P1	72	90	180	93	5444	183.73	<0.1 s	92	5415	182.25	60 s	91	5419	181.32	60 s	90	5400	180.00	11 min	2.07	1.25	0.73	**0.00**
		P2	73	90	180	92	5413	182.22	<0.1 s	91	5422	181.37	60 s	91	5410	181.17	60 s	91	5400	181.00	7 min	1.23	0.76	0.65	**0.56**
		P3	72	90	180	93	5469	184.15	<0.1 s	92	5431	182.52	60 s	92	5437	182.62	60 s	91	5400	181.00	11 min	2.31	1.40	1.45	**0.56**
	C6	P1	97	120	240	123	9681	244.01	<0.1 s	122	9694	243.18	60 s	122	9648	242.60	60 s	121	9616	241.20	19 min	1.67	1.32	1.08	**0.50**
		P2	97	120	240	122	9632	242.40	<0.1 s	122	9635	242.44	60 s	121	9632	241.40	60 s	121	9615	241.19	12 min	1.00	1.02	0.58	**0.49**
		P3	97	120	240	124	9664	244.80	<0.1 s	122	9726	243.58	60 s	122	9645	242.56	60 s	121	9617	241.21	17 min	2.00	1.49	1.07	**0.51**
	C7	P1	196	240	480	246	38899	489.12	0.1 s	245	38915	488.22	60 s	244	38839	486.74	60 s	243	38607	484.29	21 min	1.90	1.71	1.40	**0.89**
		P2	197	240	480	244	38715	485.97	0.1 s	244	38767	486.29	60 s	244	38663	485.64	60 s	242	38492	482.58	11 min	1.24	1.31	1.18	**0.54**
		P3	196	240	480	245	38864	487.90	0.1 s	245	38864	487.90	60 s	245	38850	487.81	60 s	243	38657	484.61	16 min	1.65	1.65	1.63	**0.96**
Valenzuela and Wang (2001)	Nice	P1	25	100	200	107.99	10267.7	210.67	<0.1 s	104.11	10049.5	204.61	60 s	104.02	10049.5	204.52	60 s	102.62	10147.2	204.09	8 min	5.33	2.30	2.26	**2.05**
		P2	50	100	200	109.69	10423.8	213.93	<0.1 s	109.13	10400.8	213.14	60 s	104.41	10221.0	206.62	60 s	102.33	10147.6	203.81	21 min	6.97	6.57	3.31	**1.90**
		P3	100	100	200	107.92	10273.2	210.65	<0.1 s	107.92	10273.2	210.65	60 s	105.23	10276.1	207.99	60 s	104.72	10272.0	207.44	22 min	5.33	5.33	4.00	**3.72**
		P4	200	100	200	106.89	10274.1	209.63	0.2 s	108.83	10589.4	214.72	60 s	104.21	10288.1	207.09	60 s	103.10	10188.5	204.98	13 min	4.82	7.36	3.55	**2.49**
		P5	500	100	200	103.39	10195.4	205.35	1.3 s	110.93	10836.1	219.29	60 s	103.18	10193.4	205.11	61 s	102.31	10155.2	203.87	26 min	2.67	9.65	2.56	**1.93**
		P6	1000	100	200	103.84	10267.6	206.52	6.9 s	-	-	-	-	103.76	10269.8	206.45	67 s	103.26	10244.0	205.70	21 min	3.26	-	3.23	**2.85**
	Path	P1	25	100	200	110.19	10253.7	212.73	<0.1 s	101.82	10030.6	202.13	60 s	103.09	10038.3	203.47	60 s	101.59	10062.7	202.22	10 min	6.36	**1.06**	1.74	1.11
		P2	50	100	200	113.65	10285.1	216.50	<0.1 s	105.01	10046.9	205.48	60 s	103.40	10060.9	204.01	60 s	102.66	10095.8	203.61	11 min	8.25	2.74	2.00	**1.81**
		P3	100	100	200	106.75	10180.6	208.56	0.1 s	104.03	10162.0	205.65	60 s	103.04	10125.7	204.29	60 s	102.63	10098.6	203.61	15 min	4.28	2.83	2.15	**1.81**
		P4	200	100	200	104.12	10157.4	205.70	0.1 s	104.37	10210.2	206.47	60 s	103.53	10058.3	204.11	60 s	102.24	10151.6	203.76	12 min	2.85	3.24	2.06	**1.88**
		P5	500	100	200	103.81	10189.0	205.70	1.1 s	105.91	10452.5	210.44	60 s	103.15	10210.4	205.25	61 s	102.94	10194.1	204.88	18 min	2.85	5.22	2.63	**2.44**
		P6	1000	100	200	103.09	10173.0	204.82	6.0 s	-	-	-	-	102.80	10162.1	204.42	66 s	102.45	10176.0	204.21	28 min	2.41	-	2.21	**2.11**
Ramesh Babu		P1	50	375	750	400	378125	778.13	<0.1 s	400	375000	775.00	60 s	400	376875	776.88	60 s	375	375000	750.00	4 min	3.75	3.33	3.58	**0.00**
Burke, Kendall, Whitwell (Chapter Three)		N1	10	40	80	45	1800	90.00	0.1 s	40	1600	80.00	60 s	40	1600	80.00	60 s	40	1600	80.00	<1 min	12.50	0.00	0.00	**0.00**
		N2	20	50	100	53	1542	104.40	<0.1 s	50	1500	100.00	60 s	50	1500	100.00	60 s	50	1500	100.00	<1 min	4.40	0.00	0.00	**0.00**
		N3	30	50	100	52	1525	102.83	<0.1 s	51	1500	101.00	60 s	51	1500	101.00	60 s	50	1500	100.00	2 min	2.83	1.00	1.00	**0.00**
		N4	40	80	160	86	6547	167.84	<0.1 s	81	6403	161.04	60 s	82	6455	162.69	60 s	81	6400	161.00	6 min	4.90	0.65	1.68	**0.63**
		N5	50	100	200	105	10154	206.54	<0.1 s	104	10071	204.71	60 s	103	10122	204.22	60 s	101	10000	201.00	14 min	3.27	2.35	2.11	**0.50**
		N6	60	100	200	102	5061	203.22	<0.1 s	102	5016	202.32	60 s	102	5047	202.94	60 s	101	5000	201.00	13 min	1.61	1.16	1.47	**0.50**
		N7	70	100	200	108	8230	210.88	<0.1 s	101	8009	201.11	60 s	104	8117	205.46	60 s	101	8000	201.00	22 min	5.44	0.56	2.73	**0.50**
		N8	80	80	160	83	8116	164.16	<0.1 s	82	8057	162.57	60 s	82	8097	162.97	60 s	81	8005	161.05	25 min	2.60	1.61	1.86	**0.66**
		N9	100	150	300	152	7556	303.12	0.1 s	152	7556	303.12	60 s	152	7544	302.88	60 s	151	7500	301.00	22 min	1.04	1.04	0.96	**0.33**
		N10	200	150	300	152	10567	302.96	0.2 s	152	10582	303.17	60 s	151	10530	301.43	5 min	151	10515	301.21	16 min	0.99	1.06	0.48	**0.40**
		N11	300	150	300	153	10586	304.23	0.2 s	152	10583	303.19	60 s	152	10581	303.16	5 min	151	10545	301.64	14 min	1.41	1.06	1.05	**0.55**
		N12	500	300	600	306	30431	610.31	0.4 s	306	30508	611.08	60 s	304	30281	606.81	5 min	303	30223	605.23	22 min	1.72	1.85	1.13	**0.87**
		N13	3152	960	1920	964	615637	1925.9	5.7 s	-	-	-	-	964	615637	1925.9	5 min	964	615605	1925.9	30 min	**0.31**	-	**0.31**	0.31
Burke, Kendall, Whitwell (Chapter Four)		MT1	100	492.63	985.26	506.80	198632.1	1003.38	<0.1 s	-	-	-	-	501.38	198663.1	998.04	5 min	500.90	198418.2	996.95	22 min	1.84	-	1.30	**1.19**
		MT2	150	760.90	1521.80	789.00	231582.9	1560.90	<0.1 s	-	-	-	-	772.80	230876.5	1542.39	10 min	771.70	230077.8	1538.63	17 min	2.57	-	1.35	**1.11**
		MT3	200	767.31	1534.62	803.30	314210.0	1588.83	0.1 s	-	-	-	-	790.30	312690.7	1572.03	10 min	779.70	309977.3	1554.64	19 min	3.53	-	2.44	**1.30**
		MT4	300	1174.94	2349.88	1204.00	715045.0	2395.74	0.2 s	-	-	-	-	1197.00	714443.0	2387.74	5 min	1197.00	712945.0	2385.24	21 min	1.95	-	1.61	**1.50**
		MT5	500	1264.86	2529.72	1299.50	1021018.5	2575.77	0.6 s	-	-	-	-	1282.50	1020156.8	2557.70	5 min	1280.60	1019857.8	2555.42	18 min	1.82	-	1.11	**1.02**
		MT6	100	500	1000	523.24	204072.4	1033.42	<0.1 s	-	-	-	-	512.25	203351.0	1020.63	1 min	513.53	203798.1	1023.03	15 min	3.34	-	**2.06**	2.30
		MT7	150	500	1000	516.93	201707.3	1021.20	<0.1 s	-	-	-	-	507.08	201600.3	1011.08	10 min	509.15	201501.4	1012.90	19 min	2.12	-	**1.11**	1.29
		MT8	200	800	1600	822.35	406800.7	1635.95	0.1 s	-	-	-	-	817.82	406520.0	1630.86	10 min	816.01	404173.7	1624.35	20 min	2.25	-	1.93	**1.52**
		MT9	300	800	1600	818.12	485271.2	1626.90	0.2 s	-	-	-	-	814.20	485107.0	1622.71	10 min	812.45	484752.7	1620.37	25 min	1.68	-	1.42	**1.27**
		MT10	500	1500	3000	1521.88	907096.3	3033.71	0.8 s	-	-	-	-	1514.95	906697.6	3026.11	10 min	1514.69	905969.9	3024.64	28 min	1.12	-	0.87	**0.82**

Table 5.2. Comparison of the new interactive approach against previous methods from the literature (no result available indicated by '-')



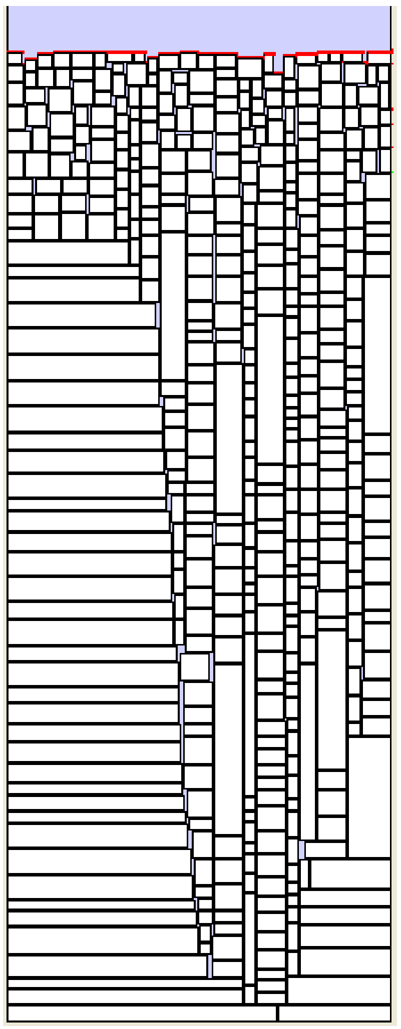
Path200 (200 rectangles, optimal height = 100)

Height = 102.24, Time taken = 10 minutes



C7P2 (197 rectangles, optimal height = 240)

Height = 242, Time taken = 8 minutes



MT10 (500 rectangles, optimal height = 1500)

Height = 1514.69, Time taken = 21 minutes

Figure 5.5. Example solutions generated for some of the larger problem instances using the new interactive approach

Figure 5.5 shows the solutions obtained for some larger problem instances. All of the solutions produced with the proposed interactive approach for the literature benchmark problems are presented in appendix A.

The best-fit heuristic (i) produces solutions using the quickest execution times out of all of the approaches but suffers due to holes being present during the latter stages of packing. The metaheuristic bottom-left-fill algorithm can produce better solutions than the best-fit heuristic for smaller problems (involving less than 50 rectangles) but cannot achieve this on larger problem instances in reasonable time. The best solutions created through automation alone are produced by the hybrid best-fit + metaheuristic bottom-left-fill algorithm (iii). This is because the approach bases its foundations on the best-fit heuristic but also evaluates many arrangements of the rectangles within the latter regions of the layouts (where holes are more likely to appear). Further to this, it can produce some of the best known results using quick execution times of a few minutes. However, the disadvantage of the approach is that it requires the predetermination of the number of rectangles that are passed to the metaheuristic search bottom-left-fill phase of the algorithm. In an industrial environment, where the characteristics of each problem may be significantly different, it is not possible to generalise the optimal number of rectangles without user specification or considerable computational pre-investigation of the problem and, hence, user interaction can be of considerable benefit in such circumstances.

5.4.2 Evaluation of the Performance of Metaheuristic Bottom-Left-Fill

In the first set of experiments it was demonstrated that the interactive approach can produce almost optimal solutions for the majority of the benchmark problems. Of course, this is dependent upon the interaction of the user and, in particular, the user's skill in picking out areas of the layout that contain wasted spaced. Of course, this also leads to the solutions being generated using longer durations than the other approaches. However, in the author's experience, this combination of the user and automated approach is a good balance between the amount of time that a user *wants* to spend and the *quality* of the solution obtained. In practice the user will only interact for as long as is necessary to produce the quality of solution that is required. In the next set of experiments, ten of the benchmark problems are used to investigate the performance of the metaheuristic bottom-left-fill algorithm when it is allowed the same time durations taken to produce the solutions with the interactive procedure. These instances have been selected to reflect a wide range of different problem sizes (72 to 500 rectangles) and durations (11 to 28 minutes). The metaheuristic bottom-left-fill algorithm was run three times with both tabu search and simulated annealing using the same times taken to produce the solutions with the interactive approach (the tabu and simulated annealing algorithms and parameters remain unchanged from the previous chapters). Table 5.3 summarises the best results achieved (and also average evaluation for the three metaheuristic bottom-left-fill runs). As usual the best evaluations are shown in bold.

Problem	Number of Shapes	Duration (mins)	Proposed Interactive Approach (IA)			Metaheuristic Bottom-Left-Fill (MBLF)							
						Tabu Search				Simulated Annealing			
			Height	Area	Eval	Height	Area	Eval	Average Eval	Height	Area	Eval	Average Eval
C5P3	72	11	91	5400.0	**181.00**	92	5436.0	182.60	183.15	92	5444.0	182.73	182.87
N8	80	25	81	8005.0	**161.05**	82	8017.0	162.17	162.53	82	8019.0	162.19	162.84
C6P1	97	19	121	9616.0	**241.20**	122	9632.0	242.40	243.78	123	9698.0	244.23	245.01
Nice P3	100	22	104.72	10272.0	**207.44**	109.52	10452.9	214.05	215.03	109.47	10615.5	215.63	216.03
MT2	150	17	771.70	230077.8	**1538.63**	790.50	233263.8	1568.05	1568.49	785.90	232253.5	1560.08	1561.31
C7P2	197	11	242	38492.0	**482.58**	243	38666.0	484.66	485.78	245	38805.0	487.53	488.03
Path P4	200	12	102.24	10151.6	**203.76**	104.46	10211.3	206.57	206.75	105.07	10211.2	207.18	207.66
MT8	200	20	816.01	404173.7	**1624.35**	842.47	415406.6	1673.28	1676.69	841.90	414640.4	1671.18	1674.06
N11	300	14	151	10545.0	**301.64**	152	10547.0	302.67	303.09	153	10610.0	304.57	305.22
MT10	500	28	1514.69	905969.9	**3024.64**	1580.43	942142.2	3150.67	3160.10	1591.40	948014.2	3171.43	3178.17

Table 5.3. Comparison of the proposed interactive approach and metaheuristic bottom-left-fill algorithm when employed with identical time

From table 5.3 it can be seen that the metaheuristic bottom-left-fill approaches are unable to find equivalent or better solutions than the interactive approach within the same timeframes. In fact, both the average solutions and the best solutions found using the metaheuristic bottom-left-fill were considerably worse than the interactive approach. This can be seen more clearly in table 5.4 that shows the percentage over optimal for the best solutions produced by each approach.

Problem	Number of Shapes	Duration (mins)	Proposed Interactive Approach (IA)			Metaheuristic Bottom-Left-Fill Approach (MBLF)			% Over Optimal	
			Height	Area	Eval	Height	Area	Eval	IA	MBLF
C5P3	72	11	91	5400.0	181.00	92	5436.0	182.60	**0.56**	1.44
N8	80	25	81	8005.0	161.05	82	8017.0	162.17	**0.66**	1.36
C6P1	97	19	121	9616.0	241.20	122	9632.0	242.40	**0.50**	1.00
Nice P3	100	22	104.72	10272.0	207.44	109.52	10452.9	214.05	**3.72**	7.03
MT2	150	17	771.70	230077.8	1538.63	785.90	232253.5	1560.08	**1.11**	2.52
C7P2	197	11	242	38492.0	482.58	243	38666.0	484.66	**0.54**	0.97
Path P4	200	12	102.24	10151.6	203.76	104.46	10211.3	206.57	**13.20**	14.76
MT8	200	20	816.01	404173.7	1624.35	841.90	414640.4	1671.18	**1.52**	4.45
N11	300	14	151	10545.0	301.64	152	10547.0	302.67	**0.55**	0.89
MT10	500	28	1514.69	905969.9	3024.64	1580.43	942142.2	3150.67	**0.82**	5.02

Table 5.4. Percentage over optimal for the best solution generated by the user interactivity approach and the metaheuristic bottom-left-fill methods using equivalent execution times

5.4.3 Experiments on New Benchmark Data

Finally, 5 new problems have been generated to aid further comparison and promote further research within the academic community in this area. The results achieved by the interactive approach are given in table 5.5.

Problem	Number of Shapes	Optimal Evaluation (Lower Bound)	Height	Area	Eval	Time Taken (mins)
U1	50	77.42	79.37	3946.2	158.29	15
U2	100	72.79	75.13	5906.6	148.96	15
U3	100	250.00	256.57	38059.2	510.30	22
U4	200	163.57	166.43	13201.4	331.45	9
U5	200	207.13	211.87	16775.7	421.57	17

Table 5.5. Solutions on the newly presented benchmark problems using the user interaction approach

5.5 Summary

In this chapter, a modified version of the hybrid approach of chapter four was proposed to allow user-interaction through the definition of poor regions which are then rearranged by the metaheuristic bottom-left-fill algorithm. Using the 57 benchmark problems from several sources within the literature and chapter three and four of this thesis, it was demonstrated that, through user interaction, the approach can produce optimal solutions up to 72 rectangles and almost optimal solutions on larger problem instances. Furthermore, the generated solutions are significantly better than previously published methods of the literature and approaches in the earlier chapters of this thesis on 54 of the 57 benchmark problems. Due to the interaction within the proposed approach, longer durations are required than the automated approaches but the amount of interaction is the choice of the user. In the second set of experiments, it was shown that the metaheuristic bottom-left-fill approach was unable to achieve such good solutions using identical durations to the interactivity method on a subset of the problem instances (even with multiple runs). Finally, initial solutions have been produced for 5 new industry-typical benchmark problems (i.e. where the optimal is not known) to promote further research and comparison between this and future approaches.

It has been shown that automation approaches can achieve significant performance benefits when guided by their human counterparts and that very good quality solutions can be achieved within reasonable time.

PART C – IRREGULAR PACKING

In Part B, the packing of rectangular items was discussed and three distinct packing strategies were developed that produced the best known results for all of the gathered benchmark problems from the literature. Irregular packing, the focus of this part of the thesis, follows a similar problem definition to the rectangle variant where shapes must be placed onto a stock sheet such that the total length of material required is minimised. The main difference between the two problems is that intersection testing between irregular shapes is considerably more complicated than is the case with their rectangular counterparts. With the rectangular intersection calculations, determination whether two rectangles, A and B, overlap was achieved by examining if rectangle B's bottom left corner was above or right of rectangle A's top right corner. If this is satisfied then the two rectangles do not overlap. The rectangles also do not overlap if rectangle B's top right corner is lower or left of rectangle A's bottom left corner. Therefore the intersection testing between rectangles is straightforward as if these two tests are false, then the rectangles intersect. It should be noted that rectangle intersection testing becomes more complicated if the pieces are allowed to rotate by any angle other than 90° increments because of the transition away from the orthogonal *x* and *y* axis representation. Whereas rectangles have one standard form and can be examined using these specific properties, irregular shapes have an infinite number of possible forms and require the development of more general intersection routines.

The development of fast geometry routines and the presentation of reliable automated packing strategies is the topic of the next four chapters.

As the geometry library is important for any technique that manipulates irregular shapes, this is the purpose of chapter six. A significant contribution of this thesis is the packing of irregular shapes that can be defined using lines *and* circular arcs. There have been few attempts to use arcs within the literature because of the additional handling that must be contained within the geometry library. Instead, practitioners have preferred to represent such shapes by approximating arcs using several small lines. This is unsatisfactory because it can result in shapes that are created from a larger number of primitives and therefore any geometrical operation can require more computation time. Of course, speed may be retained by using fewer lines to approximate the arc but this highlights a further drawback to arc approximation, there exists an accuracy to speed trade-off. Using fewer lines can reduce this overhead but will inevitably affect accuracy which may be acceptable for academic work but is definitely not acceptable for the industrial setting. Further to the inclusion of arc primitives, the geometry library that has been developed from this thesis can also handle shapes containing holes (defined by lines and arcs once more). This means that smaller shapes can be packed in the 'hole' and further savings can be made in material wastage. Currently, the literature does not provide the facility of methods that can cope with shapes involving holes as it further complicates the geometry routines.

Chapter seven presents a new approach for packing of irregular shapes using trigonometric based intersection routines and a method of resolving intersection in the *y* axis. The method has been applied to 26 literature benchmark problems drawn from over 20 years of cutting and packing research and has achieved better solutions than the previous state-of-the-art published methods on 25 of these. The datasets that have been gathered from the literature do not contain shapes involving arcs or holes and so 10 new benchmark problems have been created to examine the capabilities of the new approach and to promote the development of more algorithms that can accurately and speedily cope with real shapes that are present within the industrial environment. The work within chapter seven is presented in the following publication:

> Burke, E. K., Hellier, R. S. R., Kendall, G., **Whitwell, G.**, 2005, "*A New Bottom-Left-Fill Algorithm for the Two-Dimensional Irregular Packing Problem*", Accepted for **Operations Research**. [**2**]

The no-fit polygon is a geometric concept that can significantly simplify intersection testing. The no-fit polygon defines the path which one shape takes when orbiting around another fixed shape such that the two shapes always touch. The polygon that is defined from this operation (by tracing the path of one reference point on the orbiting polygon) reflects the intersection state between the two polygons. If the two shapes are placed such that the reference point is inside the no-fit polygon, then the shapes overlap. If the reference point exists on the boundary of the no-fit polygon then the two shapes touch. Finally, if the reference point is outside of the no-fit polygon then the two shapes do not intersect or touch. However, there has not been a robust implementation that can generate no-fit polygons for all possible polygons involving lines. Chapter eight presents a new approach for the generation of no-fit polygons that is robust and can cope with the degenerate cases that the other approaches of the literature cannot handle. This is the first no-fit polygon algorithm to not suffer from such degeneracies. Furthermore, the approach can deal with polygons that contain holes. This algorithm is then extended to produce no-fit polygons for shapes involving arcs. Previously, this has not been attempted within the literature (possibly because of the degeneracy that existed within the line only case). The robust no-fit polygon algorithm is presented in the following paper:

> Burke, E. K., Hellier, R. S. R. Kendall, G., **Whitwell, G.**, 2005, "*Complete and Robust No-Fit Polygon Generation for the Irregular Stock Cutting Problem*", **Operations Research** (in review). [**4**]

The final chapter within this part of the thesis, chapter nine, combines the irregular packing algorithm presented in chapter seven and the no-fit polygon of chapter eight to produce a packing algorithm that achieves good quality layouts in quick time. Experiments are conducted on the 36 benchmark problems once more (the 26 problems from the literature and the 10 new benchmarks introduced in this thesis) and an investigation is made with respect to the speed benefits that can be achieved from using faster no-fit polygon intersection testing. It is also shown how the overlap resolution technique is simplified from using the no-fit polygon construct. Although the general concept of the packing algorithm is identical to that of chapter seven, the extra speed obtained from the use of the no-fit polygon results in significant improvement over the majority of the best-known solutions. Furthermore, a 5% improvement is found in just a few minutes over the best solution for the only dataset for which the best-known solution was not found in chapter seven. The following work has been drawn from the investigations of chapter nine:

> Burke, E. K., Hellier, R. S. R., Kendall, G., **Whitwell, G.**, 2005, "*Irregular Packing using the Line and Arc No-Fit Polygon*", In Preparation for **Operations Research**. [**6**]

The data for the 10 newly generated irregular problem instances that are used within this thesis are provided in appendix D and the solutions produced for the entire set of irregular benchmark instances are displayed in appendix B of the thesis.

CHAPTER SIX

Geometry, Trigonometry and the No-Fit Polygon

One of the most important aspects of any automated packing routine is the correct handling of geometry. This is not only important from a functional perspective but can also significantly affect solution quality. For example, the misreporting of results from intersections tests can result in shapes not being positioned where they can be physically placed (if intersections are reported where there are none) or, in the worst case, shapes may be placed overlapping with other shapes (when reporting that no intersections are present when some really exist). Added complications are found due to rounding errors inherent with computational floating point arithmetic and geometry libraries should be able to cope with such problems.

6.1 Geometry Libraries

There are two main libraries that have been produced within the academic community and are available for the representation and manipulation of geometric structures. The first of these is LEDA, or Library of Efficient Datatypes and Algorithms, and was initiated from research conducted by Mehlhorn and Naher in 1988 and produced a set of tools and algorithms for combinatorial optimisation and computational geometry (amongst others) [**233**]. The second library, CGAL (Computational Geometry Algorithm Library), is a joint project between several universities and commercial companies to produce a general set of geometry representation and operation tools (Overmars, 1996) [**234**].

During this study, a robust geometry library has been of paramount importance. Early in this research programme, it was decided that the LEDA and CGAL libraries would not be used because they required licensing for use in commercial software. This was not only unacceptable to the industrial partner because of cost issues but also because there was already a significant amount of geometry that could be reused from the existing Procut software suite. Furthermore, these computational geometry libraries have been provided to only cover general geometric disciplines and most of such functionality would remain redundant within nesting approaches. It is not possible to describe all of the algorithms and approaches that have been developed for the new vector geometry library. However, most of these algorithms can be found in computational geometry texts such as Preparata and Shamos [**235**] and O'Rourke [**236**]. Details are given for the geometric algorithms that may be of significance to the academic community that have been developed during the course of this research (with particular emphasis on no-fit polygon generation and its usage).

6.2 Shape Representation

Irregular shapes have been represented in many different ways within the literature. The approaches used range from bitmap and other discretised methods to vector based representations. Discretised methods are usually faster than the vector representations when their resolutions are reduced but can be more computationally expensive if the resolutions of the bitmapped shapes are high. Vector approaches do not suffer from inaccuracy other than the floating point rounding errors inherent with mathematical operations on processors. These observations have been drawn from experience with the industrial partner whose previous implementation was based on bitmapped shapes. One of the reasons this research programme was conceived is because the industrial partner recognised that, although very fast, their algorithms suffered due to lack of accuracy. Therefore, the irregular packing algorithms developed within this part of the thesis utilise full vector representation of shapes. The accuracy issues are also a concern when representing arcs with their line approximations. Although this is a common approach within the previous approaches of the literature, the author has strived to represent arcs in their true vector form for the purposes of full and accurate geometry whereby arcs are represented by the centre, radius, start angle and offset of the arc (with positive offsets depicting convex arcs and negative offsets representing concave arcs). Lines can be represented easily using just the start and end points.

In the vector form, shapes are represented using either clockwise or anticlockwise 'paths' (a collection of consecutive lines and arcs that form a loop which begins and ends at the same location). It is important that this is maintained throughout as this directionality can be used in several geometric operations (such as the point-in-polygon test). Traditionally, the representation of holes can be achieved by including further paths that have opposite directionality. In the geometry representation of this research programme, an anticlockwise path direction was chosen for both shape boundaries *and* holes. With this representation the holes require a separate flag to indicate that they are holes. This allows the same functions to be used for both types of path. An example can be seen with the point-in-polygon test. When testing if a point is inside a polygon with holes, we firstly test if the point is inside the shape boundary path. If it is not contained by the path then the point is not inside the shape whereas if the point is inside the path we need to test if the point is inside any of the holes (internal paths). If the point is inside any of the internal paths then the point is not inside the polygon. The benefit of this representation can be seen because only one point-in-polygon function is required for both external and internal paths.

6.3 The No-Fit Polygon

In this section the functionality of the no-fit polygon is described and a comparison to the more traditional trigonometry based intersection testing is drawn. Also given are brief overviews of the many techniques that have been used to generate no-fit polygons within the previous literature.

The first application of no-fit polygon techniques within the field of cutting and packing was given in Art (1966) although the term "shape envelop" was used instead [**132**]. It was ten years later that the term "no-fit polygon" was introduced by Adamowicz and Albano who approached the irregular stock cutting problem by using the no-fit polygons to pack shapes together using their minimum enclosing rectangles [**85**]. The term "configuration space obstacle" is often used to denote the NFP within the field of engineering and robot motion planning but the term has also been used with respect to cutting and packing in Cuninghame-Green (1989) [**237**]. The term "hodograph" is often used to describe the no-fit polygon by the mathematics community (Stoyan and Ponomarenko, 1976 [**238**]; Bennell, Dowsland and Dowsland, 2001 [**239**]; Scheithauer and Terno, 1993 [**240**]). The main function of the no-fit polygon is to describe the region in which two polygons intersect. The following example gives an overview as to the construction of the no-fit polygon.

Given two polygons, A and B, the no-fit polygon can be found by tracing one shape around the boundary of another. One of the polygons remains fixed in location and other polygon traverses around the fixed polygon's edges whilst ensuring that the two polygons always touch but never intersect. Throughout this thesis, the convention of the first polygon being fixed and the second being the traversing/orbiting polygon is adopted. Therefore when polygon B traces around the fixed polygon A, the resulting no-fit polygon is denoted by NFP_{AB}. In order to create the NFP_{AB}

object we must choose a reference point from B which will be traced as B moves around A. In the NFP implementations within this thesis the first point of a shape is always used as its reference point (see figure 6.1).

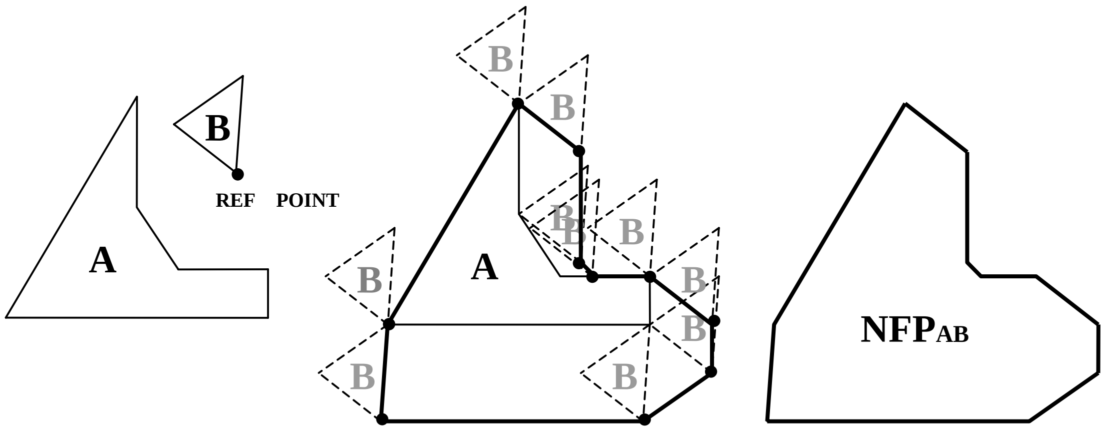


Figure 6.1. The no-fit polygon of two shapes A and B

It should be noted that the reference point can be any arbitrary point providing it always undergoes identical movements to the orbiting polygon. It is also important to maintain the relative position of reference point with respect to polygon B as this is required when using the NFP to test for overlap.

In order to test whether polygon B overlaps polygon A we use NFP_{AB} and B's reference point. If polygon B is positioned such that its reference point is inside the polygon NFP_{AB} then it overlaps with polygon A. If the reference point is on the boundary of NFP_{AB} then polygon B touches polygon A. Finally, if the reference point is outside of NFP_{AB} then polygons A and B do not overlap or touch. The three possibilities that have been described above are displayed in figure 6.2.

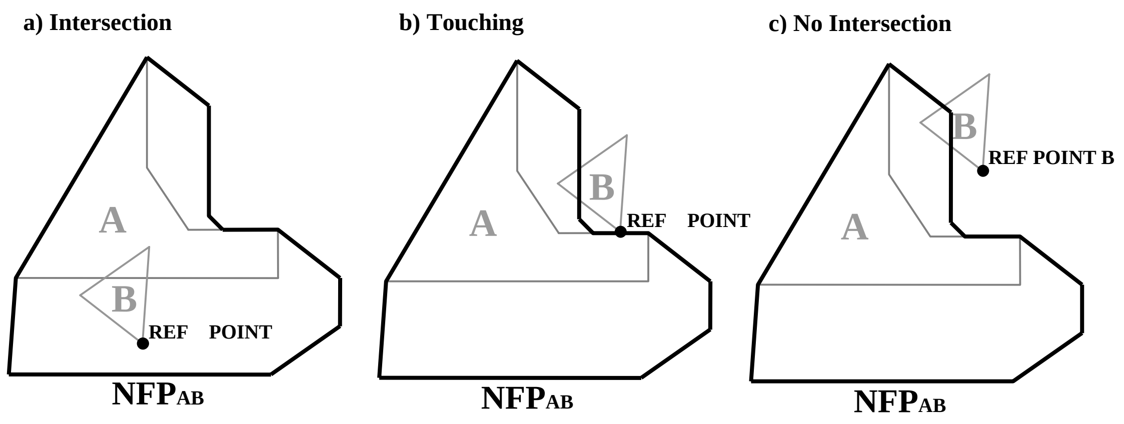


Figure 6.2. Using the no-fit polygon to test for intersection between polygons A and B

The no-fit polygon is used within the following selection of papers from the literature (Grinde and Calvalier, 1995 [**241**]; Ramkumar, 1996 [**242**]; Cheng and Rao, 1997 [**159**]; Dowsland, Dowsland and Bennell, 1998 [**155**]; Milenkovic, 1999 [**243**]; Gomes and Oliveira, 2002 [**172**]; Dowsland, Vaid and Dowsland, 2002 [**174**]).

6.4 No-Fit Polygon Versus Standard Trigonometry Overlap Detection

Although the no-fit polygon is an excellent tool for conducting intersection tests between pairs of polygons, it has not been widely applied for two-dimensional packing problems in both the literature and in real world manufacturing industries. Undoubtedly this is due to the no-fit polygon's complex implementation and the lack of complete and robust algorithms. Instead, many intersection implementations use standard trigonometry approaches which especially occur in the case of packing software for real-world applications wherein it is important that distributed software is able to handle all possible polygons without errors. However, whilst both approaches have the same overall effect, use of the no-fit polygon can be several times quicker than even the most optimised trigonometry routines. For instance where we wish to attempt numerous iterations of the same layout problem the pre-generation of no-fit polygons can significantly improve the total computation time. Over numerous iterations, trigonometric approaches are likely to repeatedly detect and resolve the same overlapping shapes in repeated orientations and positions. Where a nesting overlap resolution approach requires that you calculate all intersection points between two intersecting shapes the benefits of a no-fit polygon approach are multiplied further as numerous intersection calculations are required for the full detection of collision when the no-fit polygon method is not used. Therefore, by utilising no-fit polygons, we can reduce the overlap detection problem, a major factor in the computational overhead of nesting process, to a significantly less expensive point inside polygon test (Dowsland, Vaid & Dowsland, 2002) [**174**].

In addition to this when utilising the trigonometric overlap resolution technique, used in chapter seven and published in [**1**], whereby intersecting shapes are resolved through the repeated resolution of intersecting edges in the *y*-axis direction, by utilising the no-fit polygon the resolution technique is more efficient, resolving the *y*-axis overlap in one complete movement (overlap resolution with the no-fit polygon is investigated in chapter nine). Further to this, the no-fit polygon would also allow for the overlap to be resolved in *any direction* by casting out a ray from the relevant reference point and finding the nearest intersection with the no-fit polygon boundary.

The following section compares the computation necessary to detect and resolve the overlap between two polygons where we have pre-calculated the NFP and where we have not. Given a nesting method where we wish to detect all points of intersection for polygons A and B shown in figure 6.3, every edge is tested against every other edge leading to 42 intersection tests to determine the intersection status (given that polygon A has 7 edges and polygon B has 6 edges). Furthermore if no intersection is found, in order to eliminate the possibility of the polygons intersecting one another through their vertices or one polygon completely containing the other, a point inside test must be performed for all points on both shapes, requiring 55 tests in total, in the worse case. The number of tests can be generalised as: mn + m + n where m and n are the number of edges within the two polygons, A and B, respectively (e.g. in the example provided, 7 x 6 = 42 line intersection tests, 7 + 6 = 13 point inside tests). These can be regarded as a large, but in the main unavoidable, overhead for the detection of intersection between any two polygons in a layout, which can only increase as we add an increasing number of polygons (or edges to the polygons) to the solution. Whilst many optimisations can be attempted and fast intersection libraries can be developed this remains a considerable proportion of the computational overhead in the generation of packing solutions. Additionally, where you wish to develop numerous solutions using a combinatorial optimisation approach these, often repeated, calculations will have considerable impact on the overall computation time.

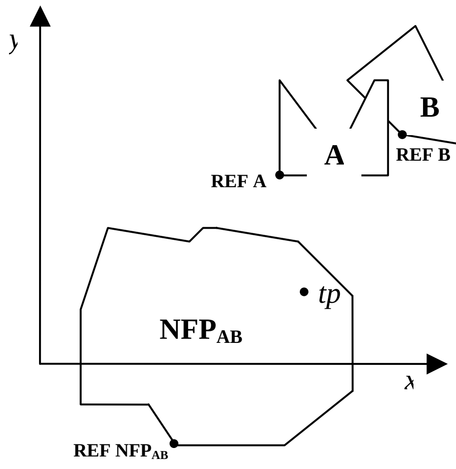


Figure 6.3. Intersection testing with the no-fit polygon

Figure 6.3 shows a possible arrangement of polygons A and B. The No Fit Polygon of these polygons NFP$_{AB}$ is also shown at some arbitrary position in the problem space. In this case the

NFP_{AB} has been generated using the proposed edge sliding technique where polygon B traversed the edges on the polygon A and the NFP was formed by tracking the reference point on the traversing polygon, REF B. In the context of the overlap detection process the position of the NFP is rendered inconsequential by generating the test point (*tp*) using a simple translation, namely (REF NFP_{AB} + REF B – REF A – *Offset*), where *Offset* is the positional offset from the reference point of polygon A and the reference point of the no-fit polygon during generation (REF NFP_{AB} – REF A).

The status of *tp* with respect to NFP_{AB} can be calculated using a ray-crossing algorithm described in O'Rourke (1998) [**236**]. If the point *tp* is found to be within NFP_{AB} then the polygons A and B are colliding with one another, colliding includes both intersection and containment of one polygon by another. If *tp* falls on the NFP_{AB} then the polygons are known to be touching and if *tp* falls outside of NFP_{AB} then the shapes are neither touching nor colliding. Both touching and not colliding are viable states for polygon placement in numerous nesting approaches. By undertaking the calculation of all no-fit polygons for all possible pairs of polygon rotations, considerable computation time can be saved for multiple iteration nesting.

6.5 Generation of the No-Fit Polygon: Previous Literature Approaches and their Degeneracies

In this section, the main techniques that have previously been used for construction of no-fit polygons within the literature are reviewed. For each method a brief overview of the approach is given and any benefits and drawbacks that may occur from their usage are discussed.

1.1.1 Convex Shapes

The basic form of no-fit polygon generation occurs when both polygons are convex. Given two convex shapes, A and B, the no-fit polygon is created by the following steps:

- Orientate shape A anticlockwise and shape B clockwise (see figure 6.4a)
- Translate all edges from A and B to a single point (see figure 6.4b)
- Concatenating these edges in anticlockwise order yields the no-fit polygon (see figure 6.4c)

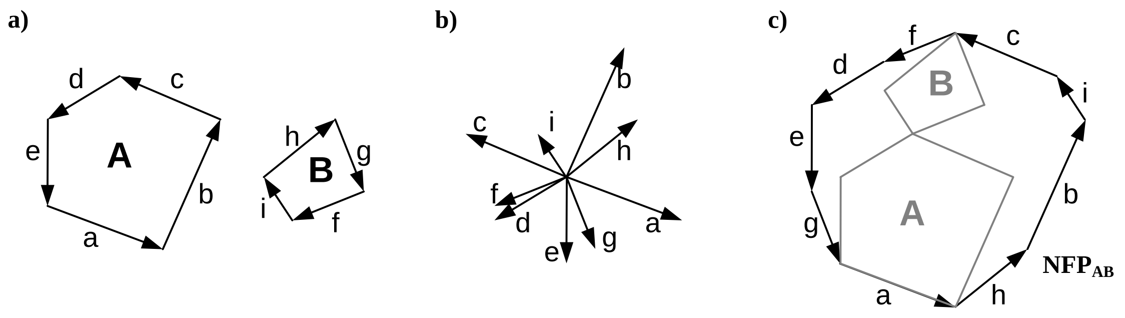


Figure 6.4. No-fit polygon generation with convex shapes

Cuninghame-Green used this approach to produce "configuration space obstacles" between pairs of convex polygons which are then used for intersection tests during the packing of shapes (Cuninghame-Green, 1989) [**237**]. For instances involving non-convex pieces, Cuninghame-Green firstly calculates the convex hull of each non-convex shape (the minimal containing convex polygon) and then produces no-fit polygons using the respective convex hulls. The benefit of convex no-fit polygon generation is that it is simple and is extremely quick using a standard sorting algorithm in combination with edge reordering through translation. The obvious disadvantage is that no-fit polygons cannot be generated for non-convex shapes and the reduction to convex hulls results in concavities being unavailable in packing and non-traversable in robot motion planning. As this can adversely affect solution quality, other approaches are required.

1.1.2 Non-Convex Shapes

There have been many different approaches to producing no-fit polygons from non-convex shapes. These can be placed into four general categories: digitisation, decomposition, minkowski sums and orbital approaches.

1) Digitisation

One of the simplest approaches to producing no-fit polygons from non-convex shapes is to discretise the shapes through digitisation. Firstly, the shapes are represented as 0-1 bitmap objects whereby "ones" represent the solid body of a shape and "zeros" indicate empty spaces (see figure 6.5a). As intersection detection can be implemented easily with bitmaps through boolean operations, the no-fit polygon can be created through multiple scans of shape B across shape A or through orbital approaches. The problem with such an approach is that there is a speed/accuracy tradeoff. For example, with a low resolution grid the process can be very fast but higher resolution grids require more and longer scanlines which will compromise speed.

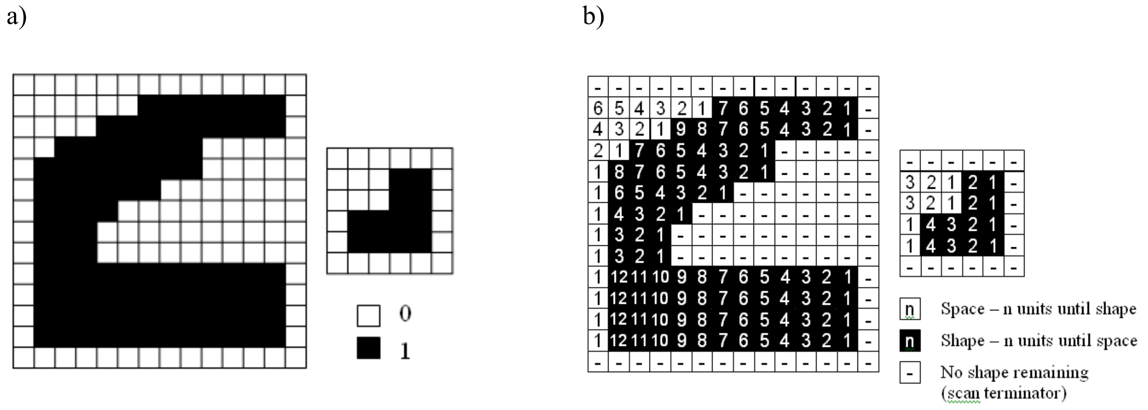

a)
b)
0
1
Space – n units until shape
Shape – n units until space
No shape remaining (scan terminator)

Figure 6.5. Bitmap representations of non-convex pieces with a 0-1 representation or index optimisations

The discrete representation of the bitmaps may also allow for certain optimisations. Ramesh Babu and Ramesh Babu utilise indexing within their bitmap representations whereby "zeros" and “ones” are replaced with index entries that indicate how many ‘pixels’ must be traversed horizontally until an empty space is reached [**170**]. Although they do not calculate no-fit polygons within their layout generation implementation, it is apparent that the fundamentals of their indexed scanning technique are also applicable to digitised no-fit polygon generation (see figure 6.5b). Another imaging technique has been used by Crispin et al. for the generation of leather lay plan layouts within the footwear industry. The no-fit polygon is found using image processing techniques by filtering an image of the cow hide boundary with an image of the shape to be placed. The convolution between the two shape images yields the no-fit polygon (Crispin et al., 2003) [**200**]. This is another form of the Minkoski sum that is described later. Ultimately, all digitisation or imaging approaches will result in inaccurate no-fit polygons because of the loss of information resulting from the discretisation of a continuous space. While it is true that the resolution could be set to an arbitrarily high value, this may severely affect computation times. One benefit of the digitisation approach is that it is able to handle most of the traditional problem cases, such as holes and interlocking concavities. However, one disadvantage may be that that jigsaw shapes may not fit together after digitisation whereas they would in their pure vector form.

2) Decomposition

An alternative to the digitisation techniques described above is to decompose any non-convex shapes into subpieces that can be ‘managed’ more easily. However, using decomposition usually results in several subpieces and therefore several no-fit polygons must be created. In order to conduct intersection testing, the no-fit polygons can remain decomposed or they can be recombined through union operations. Where possible, a single no-fit polygon offers the fastest computation time for intersection but will require additional calculations to recombine constituent parts. This can be computationally expensive and is a difficult undertaking if several subparts or holes are present. For example, Agarwal, Flato and Halperin conduct experiments on different decomposition and recombination operations with respect to constructing Minkowski sums of non-convex polygons without holes [**244**]. They conclude that it is counterproductive to use optimal decompositions because the computation times that are required to calculate them outweigh the benefits achieved during recombination. The authors report that the recombination operations are the most costly and give example execution times that range from a few seconds to produce the no-fit polygon for shapes involving a small amount of concavities up to twenty minutes for highly irregular shapes.

a) Convex Pieces

It was already discussed that no-fit polygon generation is trivial when convex shapes are involved. If shapes with concavities can be divided into convex pieces, the fast convex no-fit polygon generation techniques can be employed. The simplification of geometrical intersection through the decomposition of non-convex shapes to convex pieces was suggested and discussed in Avnaim and Boissonnat (1987) [**245**] and Cuninghame-Green and Davis (1992) [**246**]. The main difficulties with such an approach are the decomposition and recombination algorithms.

There are many well-known approaches that can be used for decomposition including triangulation and convex partitioning. Seidal suggests a fast polygon triangulation algorithm which only has O(n log n) complexity [**247**]. Further implementation details for triangulation can be found in Amenta (1997) [**248**]. However, the triangulation approaches produce more subpieces than is necessary and ultimately impacts upon computation time within the generation process. Unlike triangulation, which can be seen as a specialised case of convex partitioning, generalised convex partitioning algorithms aim to represent polygons with as few convex pieces as possible. The two key approaches are suboptimal partitioning, which has an O(n) complexity (Hertel and Mehlhorn, 1983) [**249**] and optimal partitioning which has an $O(n^3)$ complexity (Chazelle and Dobkin, 1985) [**250**]. Agarwal, Flato and Halperin (2002) state that it is generally more efficient to use suboptimal partition algorithms because of the computational overhead inherent with optimal partitioning but they also suggest that an alternative and possibly more efficient approach is to perform convex covering instead [**244**]. Some possible decompositions of an irregular polygon into convex pieces are shown in figure 6.6.

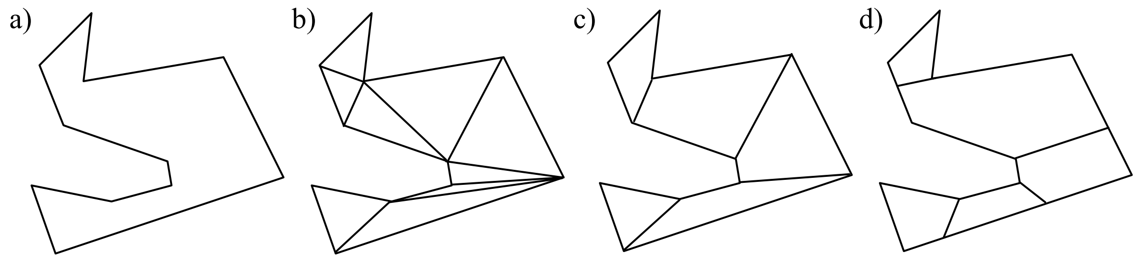


Figure 6.6. Convex Decompositions: a) irregular polygon, b) triangulation, c) convex division (vertex to vertex), d) convex division (vertex to edge)

Once irregular polygons have been decomposed into convex pieces, the no-fit polygon may be generated by recombining the no-fit polygons produced by passing each convex piece of shape B around each convex piece of shape A. Providing these no-fit regions have been generated in relation to a reference point, they may then be recombined. The disadvantage with this approach is that recombination can be difficult because the no-fit polygons of the convex partitions may cross and, therefore, care must be taken when constructing the no-fit polygon entity. Particular difficulty occurs if the original shapes contain holes as it is unclear whether intersecting no-fit polygon subsections define holes or whether they define regions that may be discarded.

b) Star-Shaped Polygons

Li and Milenkovic decompose shapes into convex and star-shaped polygons. A star-shaped polygon has the property that there exists at least one internal point, or 'kernel point', that can see the entirety of the polygon (Preparata and Shamos, 1985 [**235**]; O'Rourke, 1998 [**236**]). Figure 6.7 shows one non-star-shaped polygon and one star-shaped polygon. By extending the concavity edges and elimination of invisible regions it is evident that the star-shaped polygon has a region, R_{kernel}, that is defined within which a kernel point can be placed to see the entire polygon boundary. Conversely, this is not the case with the non-star-shaped polygon because no region can be defined in which to place a kernel. Thus, star-shaped polygons are situated somewhere between convex and non-convex shapes in terms of generality.

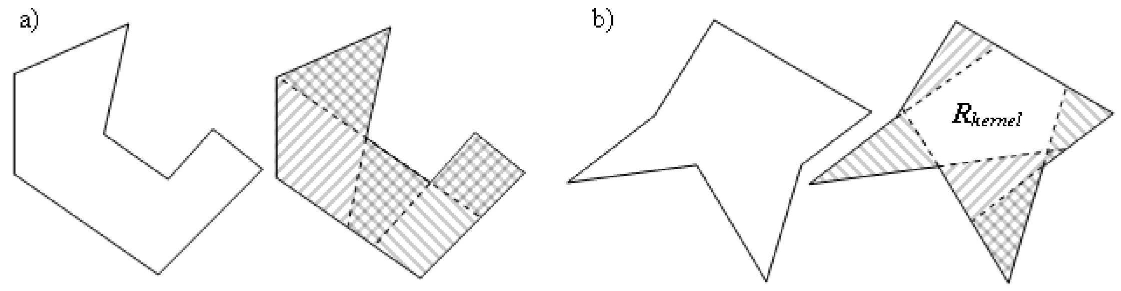


Figure 6.7. a) Non-star-shaped polygon, b) Star-shaped polygon

The authors state that star-shaped polygons are 'closed' under Minkowski sum operations and provide a proof that the Minkowski sum of two star-shaped polygons also yields a star-shaped polygon (Li and Milenkovic, 1995) [**69**]. The no-fit polygon is then created through use of an angular sweep line algorithm. The authors do not state whether they recombine the no-fit polygon regions into one no-fit polygon entity or whether they perform multiple no-fit polygon intersection tests during the layout generation stage.

c) Phi-Functions

An alternative approach is presented by Stoyan and is based on the use of "phi-functions" (Stoyan, Novozhilova and Kartashov, 1996) [**251**]. Phi-functions define mathematical intersection relationships between pairs of standard or "primary" objects such as rectangles, circles and convex polygons (Stoyan et al., 2001) [**252**]. The authors further develop their work to enable the definition of mathematical intersection relationships for non-convex polygons through the union, intersection and complement of primary objects (Stoyan et al., 2002) [**253**]. The resultant intersection test between two shapes during layout generation is performed through comparisons of phi-functions between all pairs of the primary objects that define shape A and shape B.

3) Minkowski Sums

The no-fit polygon construct can be unified through the use of Minkoski sums (a form of vector addition) and was first suggested by (Stoyan and Ponomarenko, 1977) [**238**]. The concept is as follows: given two arbitrary point sets, A and B, the Minkowski sum of A and B is defined by the following:

$$A \oplus B = \{a + b: a \in A, b \in B\}$$

In order to produce no-fit polygons the Minkowski difference can be used, $A \oplus -B$,. This is equivalent to the two input polygons in opposing orientations and is easily shown through simple vector algebra (Bennell, Dowsland and Dowsland, 2001) [**239**]. We can verify that this is the case using the simple convex only case (section 2.3.1) whereby we place shape A in its anticlockwise orientation and shape B in its clockwise orientation. We can state that the method of Cuninghame-Green [**237**], involving convex shapes only, is *also* using the Minkowski difference in its most basic form.

Whilst non-mathematical implementation details of such approaches are scarce, Ghosh and Bennell provide excellent explanations and implementation details for no-fit region calculation. As such, their contributions are briefly summarised. Ghosh details the production of no-fit polygons through the notion of Minkowski difference and demonstrates a simple extension to the convex only case that allows for the generation of no-fit polygons for non-convex shapes. The drawback is that this approach will only work providing that the concavities of the two shapes do not interfere or interlock (Ghosh, 1991 [**254**]; Ghosh, 1993 [**255**]). Where this is not the case, a modification is proposed but is only outlined briefly. Bennell, Dowsland and Dowsland (2001) state that Ghosh's modification would cause a "cumbersome tangle of intersecting edges" which would be difficult to recombine to form the no-fit polygon [**239**]. They introduce a further implementation that reduces the amount of tangled edges and give thorough implementation details and pseudocode of the entire process. They also report fast generation times of around 0.3 seconds for each of five separate datasets from the literature. However, they state that their approach cannot deal with internal holes as it is difficult to detect which of the internal no-fit polygon edges can be discarded and which form the internal no-fit regions. In chapter eight the robustness and speed of a new orbital approach is compared through the reporting of generation times for 32 datasets of the literature some of which include holes.

4) Orbital Sliding Approach

The orbital sliding approach is the main focus of chapter eight whereby a new approach is presented based on the physical sliding of one polygon around another using standard trigonometry. The no-fit polygon is defined by tracing the motion of a locus point of the sliding polygon when orbiting. The first discussion and implementation of an orbiting approach for the generation of 'envelopes' is detailed in Mahadevan (1984) [**256**] and is further investigated by Kendall (2000) [**8**]. The key elements of Mahadevan's approach are: calculation of touching vertices and edges, determination of the translation vector and calculation of the translation length. The calculation of intersecting edges is performed using the notion of D-functions which were introduced by Konopasek in 1981 [**146**]. Mahadevan modifies the D-function test to also calculate touching points which is a necessity for both Mahadevan's algorithm and the new approach presented in chapter eight. This information is then used to select a translation vector based on the touching edge. The translation vector is then projected through each vertex of the orbiting shape and then intersecting edge testing is used to calculate the translation distance. It is also important to project the translation vector in the reverse direction of the stationary polygon. The orbiting shape is then translated along the translation vector by the smallest distance (from projection and intersection tests). This ensures the two polygons never intersect but always touch. The process continues until the orbiting polygon returns to its original starting position. A more detailed description of Mahadevan's approach is deferred because many of its features are also used within the newly proposed orbital approach which is described in chapter eight (where applicable, Mahadevan's approach is compared and improvements of the new approach are stated). The major disadvantage with Mahadevan's algorithm is that it cannot generate the full no-fit polygon for shapes involving holes or some concavities. The problem occurs when the orbiting polygon can be placed inside a concavity of the stationary polygon but the concavity has a narrow entrance. In this case the orbiting polygon is too wide and will simply slide over the concavity. Oliveria, Gomes and Ferreira also use Mahadevan's implementation within their work on irregular packing and include steps to detect such problems (Oliveria, Gomes and Ferreira, 1996) [**257**].

This chapter provided a brief review of geometric considerations for automated packing and a summary of the different approaches that have been used for the generation of no-fit polygons. In chapter eight, a new orbital no-fit polygon procedure is described and is shown to be able to handle all of the known classes of degenerate shapes.

CHAPTER SEVEN

Automated Packing using Trigonometric Approaches

This chapter presents a method for irregular packing using an improved bottom-left fill strategy through the use of new shape primitive overlap resolution techniques. These techniques efficiently deal with the irregular nature of shapes in order to produce high quality solutions in quick time. Furthermore, the proposed approach allows problems containing shapes consisting of circular arcs and holes to be nested. In the vast majority of the irregular packing literature (discussed in section 2.3), there have only been approaches that can handle circular arcs by using line approximations. By combining the new techniques with both hill climbing and tabu local search methods the proposed irregular packing approach has been tested using the 26 existing problems from the literature (as discussed in section 2.3.2). Also introduced are 10 new irregular benchmark problems that have been produced from industrial style shapes, five of which contain circular arcs and three which include holes.

7.1 Motivation for the Approach

There have been many different approaches for producing solutions for the irregular two-dimensional stock cutting problem. It should be noted that, in general, the approaches that have achieved the best known results have used a no-fit polygon based technique to generate potential placement positions and/or test for overlaps [**172**]. While the no-fit polygon is a powerful geometric technique, there are several issues that make it limited in scalability for industrial applications. No-fit polygon techniques are notorious for the large number of degenerate cases that must be handled in order to be completely robust (as discussed in section 6.6). A further problem with no-fit polygon generation techniques is that, as far as the author is aware, there has previously been no algorithm that successfully handles true arc geometry.

As the previous no-fit polygon based packing techniques are unable to handle arcs, most methods and test problems approximate arcs using a sequence of lines. The problem with approximations is that there is an accuracy to time trade-off. That is to say, that if we model the arc with fewer lines then the accuracy of the approximation is reduced but the shape is less complex and if we increase the quantity of lines then accuracy is improved but the resultant shape is made more complex. When performing translation or rotation operations, obviously shapes with a larger number of lines will take longer to manipulate than a shape with fewer lines. Ultimately, there will always be inaccuracy when modelling arcs as a series of lines and in adding more lines to approximations, operating on that approximation required greater time. The geometry implementation that has been developed is able to model shapes composed of both lines and non-approximated arcs and is able to conduct fast shape intersection operations to an accuracy of 10^{-9} metres. The development of such a geometry system is a complex and time-consuming undertaking and the author would like to suggest that other practitioners may like to consider general geometry libraries such as LEDA and CGAL (as discussed in section 6.1).

Note: a new no-fit polygon algorithm that is able to cope with such degeneracies is developed in chapter eight but this had not been attempted until the latter stages of this research and, therefore, was not available during the development for the approach in this chapter.

In real world industrial settings, and especially for profiling (sheet metal cutting) within the steel cutting industry, it is imperative that automated packing approaches are able to handle arcs, concavities and holes. These requirements mean that the no-fit polygon techniques that have previously been discussed are not currently suitable. Traditionally, industry has used a bottom-left-fill approach based on an iterative grid approximation in which arcs are represented by approximated lines. The grid approach aims to reduce the infinite number of potential positions (due to the continuous nature of space) to a fixed set of potential locations. The algorithm works as follows: when placing a shape on the sheet the algorithm places the shape in the first grid location (bottom-left point) and then checks for intersection with shapes already assigned to the sheet. If there are no intersecting shapes then the shape is assigned to the sheet in its current position and a new shape is packed starting from the first grid location. However, if the current shape does intersect with another shape on the sheet then it is moved to the next grid position and is tested for intersection again, continuing the process until a valid position is found. This means that using a lower resolution grid will, in the general case, adversely affect the quality of the solution because shapes are placed in later positions than they could otherwise be placed if a higher resolution grid was used. This causes, as is often found in cutting and packing, an accuracy to time trade-off.

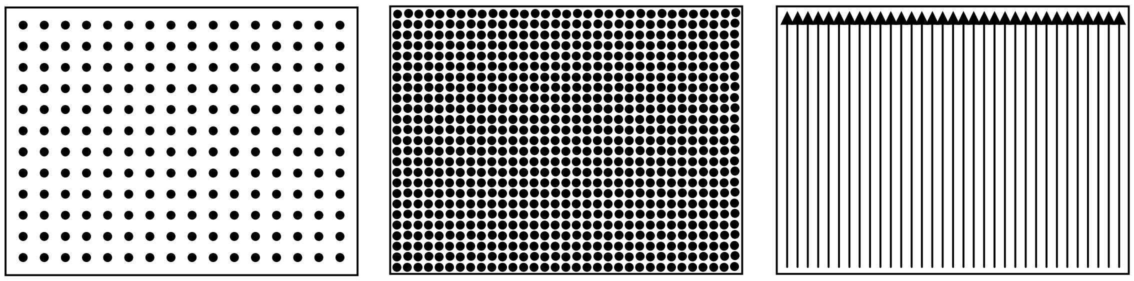


Figure 7.1. a) low resolution grid approach b) high resolution grid approach c) variable shift approach

Figure 7.1 shows the potential locations for two iterative grid approaches with differing resolutions and also the proposed overlap resolution method. The packing method involves placing shapes leftmost (and then bottommost) within the sheet. Therefore, in the two iterative grid approaches, the first location tried is the bottom-left grid location and then the grid location above it and so on until the shape is placed or moves off the top of the sheet. After moving off the top of the sheet, the shape is relocated at the bottom of the next column of grid points and the process continues. Therefore, with the grid approaches, there are a finite number of locations for which shapes may be placed. The iterative grid approach of figure 7.1b has a higher resolution than that in figure 7.1a and therefore has more potential placement locations and should result in more compact packings although this will require longer computation times as each shape must be tried in many more potential positions than in figure 7.1a. The proposed variable shift approach is shown in figure 7.1c and has an infinite number of potential locations due to its continuous *y* axis property and therefore can produce more compact packings.

The approach that is described in the next section, unlike the iterative grid approach, is not restricted to moving shapes by a fixed translation when intersecting with another shape. This is achieved by using the underlying geometric primitives of intersecting shapes to resolve the overlap. This has two main advantages in that intersecting shapes can be resolved so that they touch exactly and, secondly, that this accuracy is achieved in a smaller number of steps than the iterative grid mechanism. Although the grid method and the proposed approach follow a similar conceptual procedure when placing shapes, the proposed approach has both a faster and more accurate method of producing solutions. For irregular packing, it is convention to minimise horizontal length rather than height (as in rectangle packing), but as the two are conceptually identical, the recognised convention will be used.

7.2 The New Bottom-Left-Fill Approach

In this section, the techniques used to resolve overlap are explained. The overall algorithm pseudocode can be found in section 7.2.4.

7.2.1 Geometrical Definitions

In order to illustrate the new packing method the terms "primitive", "loop" and "shape" are defined. For the purposes of this discussion, a "primitive" is defined as either an arc or a line. A line is represented by its start and end points whereas an arc is circular and is represented through a centre point, radius, start angle and offset angle. A "loop" is defined as an anti-clockwise closed list of primitives where each primitive's end point is the start point of the next primitive. Non-simple polygons are not allowed (essentially these are polygons where the edges are allowed to intersect one another) as we are only interested in shapes that can be physically cut from a material (O'Rourke, 1998) [**236**]. A "shape" is defined as one outer loop and 0….*n* internal loops which can be thought of as holes in the shape. Figure 7.2 shows an example shape where the solid line is used to represent the outer loop and dashed lines the holes (internal loops).

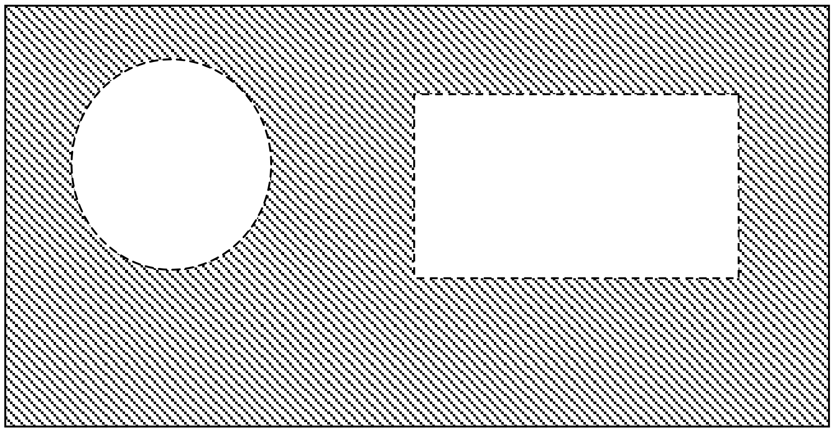


Figure 7.2. Shape containing holes

Most of the problems from the current literature do not include shapes with arcs or holes and numerous examples only contain convex shapes, where it is easier to detect overlaps. Furthermore, it is necessary for these algorithms to cope with floating-point data in order to establish high accuracy and realism on real world problems. The geometry library developed during this research programme can cope with these extra complications.

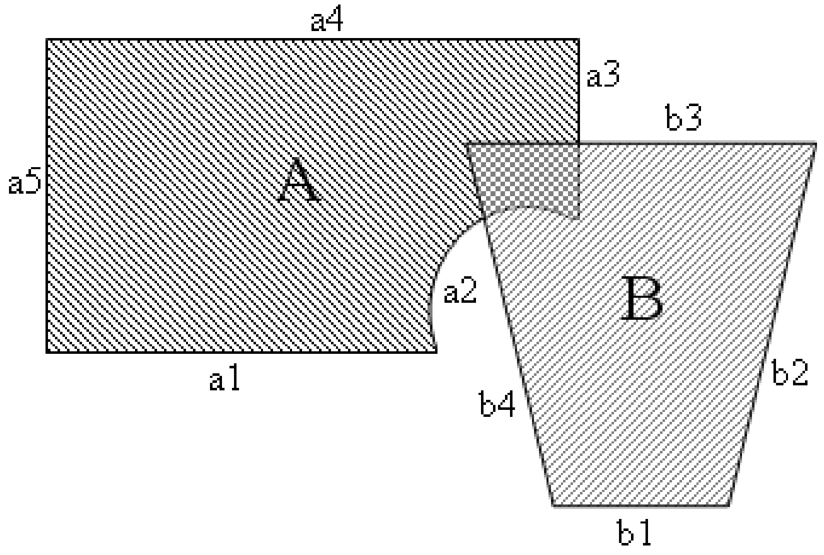


Figure 7.3. Overlapping Shapes

Figure 7.3 demonstrates an instance of overlap between two shapes, A and B. It can be seen that the arc primitive a2 intersects with line primitive b4 and that the line primitives, a3 and b3, also intersect. The following section shows how the information about intersecting primitives can be used to accurately resolve overlap between shapes.

7.2.2 Resolving Overlapping Primitives

There are four possible cases that must be handled in order to resolve intersecting primitives. One of these primitives is always part of the shape that is being placed, termed the "free shape", and the other is always part of a shape that has already been placed on the sheet, named the "locked shape". The techniques that are described in the following sections involve calculating the positive vertical distance required to translate the free shape such that the two primitives no longer intersect. Whilst this resolves the overlap between the two intersecting primitives, the two shapes may still not be fully resolved in that other primitives belonging to the shapes may also intersect or the free shape may be entirely contained in the locked shape (discussed in section 7.2.3). However, repeated application of these techniques will always resolve overlapping shapes with the smallest positive vertical distance required. Table 7.1 shows the cases that must be handled and the subsections in which the resolution method is discussed.

Free Shape's Intersecting Primitive	**Locked Shape's Intersecting Primitive**	**Section**
Line	Line	7.2.2.1
Line	Arc	7.2.2.2
Arc	Line	7.2.2.3
Arc	Arc	7.2.2.4

Table 7.1. Possible intersection types

It should be noted that throughout all of the cases, the locked shape's intersecting primitive, which is called the "locked primitive", is a primitive belonging to a shape that has already been assigned to the sheet and its position may not change whereas the intersecting primitive, belonging to free shape, is termed the "free primitive". An explanation of the steps required in resolving the intersections of these cases is provided but firstly some terminology is introduced.

The "x span" of a primitive can be thought of as the horizontal span of its bounding rectangle. Figure 7.4 highlights the x spans for both primitive types. Another concept used is that of an "infinite vertical line". This is a vertical line that spans from negative infinity to positive infinity congruent to the y axis. The notations that are used in the diagrams and descriptions of the following subsections are presented in table 7.2.

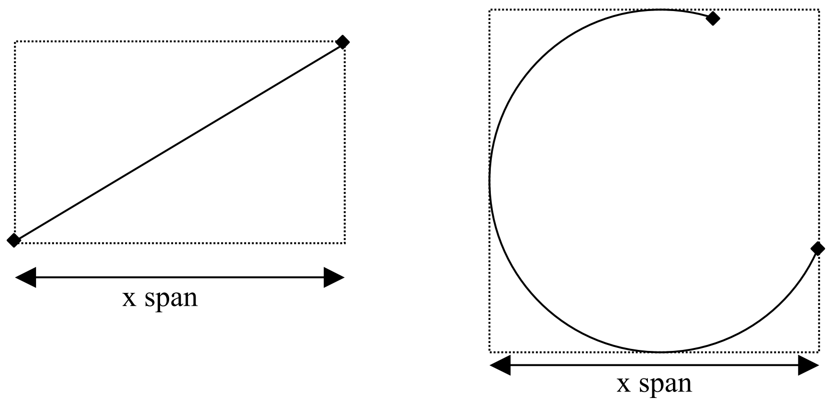


Figure 7.4. Line and arc x spans

Symbol	Description
A1→A2 A1, A2	Primitive A (the free primitive). Start point (A1) and end point (A2) of primitive A
B1→B2 B1, B2	Primitive B (the locked primitive). Start point (B1) and end point (B2) of primitive B
CP	Centre point of an arc primitive
c1…cn	Intersection points
t1, t2	Tangent points of a line on an arc
¦	Infinite vertical line (short-dashed vertical line)
– – – – – –	Perpendicular to a line primitive (long-dashed line)
↑	Translation used to resolve overlap (bolded vertical arrow)

Table 7.2. Notation for diagrams and descriptions

Having established the required terminology and notation, each case is explained as outlined in table 7.1.

7.2.2.1 Line & Line (Free Line moving through Locked Line)

In order to resolve any two intersecting lines we find the end points of each line, A and B, that are within the x span of the other, these points are known as the points in range (pir). An infinite vertical line is passed through each of the pir originating from line A and the intersection points of these lines with line B are found. The distance is calculated for each pir from line A and its corresponding intersection point on line B by using formula **(1)**.

distancePirA = intersectionPointB.y - pirA.y **(1)**

Infinite vertical lines are also passed through each of the pir originating from line B to find their intersection points with line A. A different formula, formula **(2)**, is used to calculate the distances when the pir originates from the locked primitive (line B).

distancePirB = pirB.y - intersectionPointA.y **(2)**

For the resolution method, the overlap must always be resolved by translating line A in the positive vertical direction. The distance formulae, given in formulas **(1)** & **(2)**, may yield negative results and, therefore, such results are not valid positive vertical movements and can be eliminated. These distance formulae also form part of the strategy for resolving the other intersection cases. For two lines, there always exists one valid positive result which can be used to vertically translate line A in order to fully resolve the intersection between the lines. An example of this approach is shown in figure 7.5.

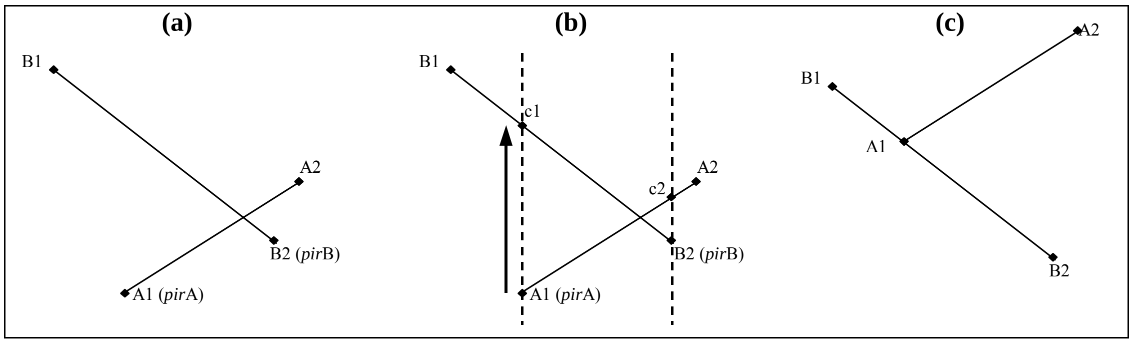


Figure 7.5. Resolution of intersecting line primitives

Here, point A1 is within the x span of B1 → B2 and B2 is within the x span of A1 → A2. Let these points be labelled pirA and pirB respectively (see figure 7.5a). An infinite vertical line is passed through pirA to create intersection point c1 and another through pirB to create intersection point c2 (shown in figure 7.5b). The distance between pirA and c1 is calculated using formula **(1)** and the distance between pirB and c2 is calculated using formula **(2)**. In this example, the first distance yields a positive result whilst the second is negative. Formula **(3)** shows how the distance formulae **(1)** & **(2)** may be combined into a "max" function call:

$$\text{yTranslation} = \max(\text{c1.y} - \text{pirA.y}, \text{pirB.y} - \text{c2.y}) \qquad \textbf{(3)}$$

The result of formula **(3)** is shown pictorially as the bolded arrow in figure 7.5b. Figure 7.5c shows how the overlap has been resolved by vertically translating line A by this positive translation. In practice, all of the primitives of shape A are translated vertically by this positive distance, not just the line involved in the overlap.

7.2.2.2 Line & Arc (Free Line moving through Locked Arc)

In this case, where a line is intersecting with an arc, the positive vertical translation must be found with which the line should be translated in order to completely resolve its intersection with the arc. As with the Line & Line case (section 7.2.2.1), the points in range of each primitive can be utilised. However, because an arc is involved, the use tangent points between the arc and line primitives may be required. Figure 7.6 shows an example where applying the points in range method alone is not sufficient to resolve the overlap.

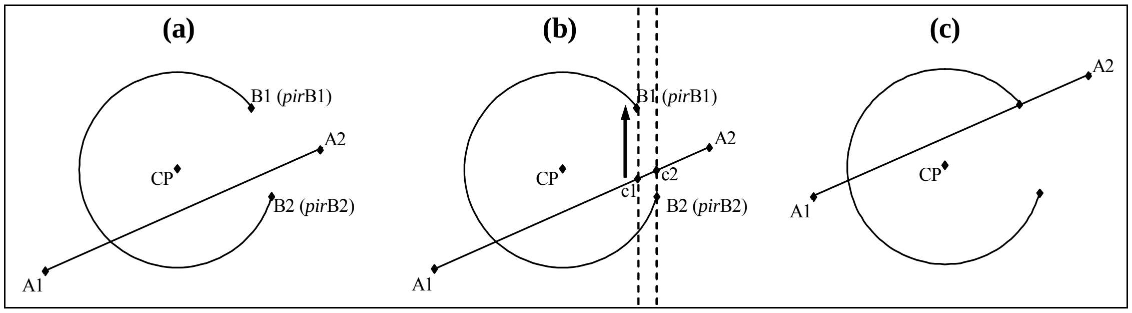


Figure 7.6. A line arc example where points in range are insufficient to resolve overlap

Figure 7.6a shows that the only points within range are B1 (pirB1) and B2 (pirB2) from the arc (both end points of the line, A1 and A2, are outside the x span of the arc and, therefore, are not in range). Once again, infinite vertical lines are passed through each pir and are intersected with the line A1→A2. This creates intersection points c1 from pirB1 and c2 from pirB2 (see figure 7.6b). The distance between each pir and its respective intersection point on the other primitive is calculated using the distance formulae, **(1)** and **(2)**. In this example, both pir originate from the locked primitive (arc B) and therefore both distances are calculated using formula **(2)**. This yields one positive result that is shown by the bolded arrow in figure 7.6b. Figure 7.6c shows that this vertical translation is not sufficient to resolve the overlap. Figure 7.7 shows how tangent points can be used to fully resolve the overlap.

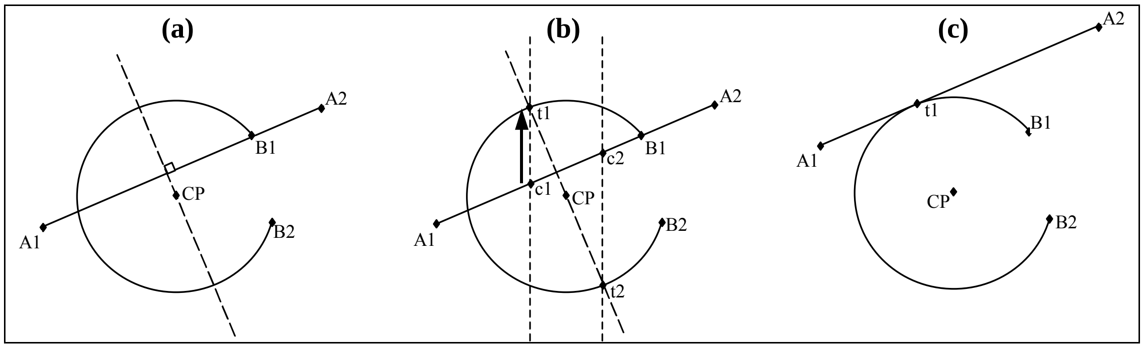


Figure 7.7. Using tangent points to resolve overlap with the line arc case

The tangent points can be found by translating the perpendicular (or “normal”) of the line such that it passes through the centre point of the arc, CP, as shown in figure 7.7a. The intersection of the perpendicular with the arc gives the tangent points, t1 and t2 (see figure 7.7b). These tangent points are then used in a similar manner to the point in range technique whereby infinite vertical lines are passed through each tangent point and intersected with line A1→A2 to give points c1 and c2. The translation distances can then be calculated by substituting each tangent point, t1 and t2, for pirB into formula **(2)**. This gives formula **(4)**.

distanceTangentB = tangentB.y - intersectionPointA.y **(4)**

In the example, it can be seen that t1 would yield a positive translation whereas t2 would give a negative translation distance. Therefore, translating the line A1→A2 by the distance given by the calculation: t1 - c1 will resolve the overlap (see figure 7.7c). It should be noted that if the intersection of the perpendicular line with the arc yields no tangent points or the tangent points result in negative translation distances using formula **(4)** then the point in range technique must be able to resolve the overlapping primitives.

7.2.2.3 Arc & Line (Free Arc moving through Locked Line)

This case, where there is a free arc through a locked line, involves a similar approach to the free line & locked arc case. Once again, the same technique for points in range applies and, therefore, will not be repeated here. However, because the arc is now the free primitive (arc A1→A2) and the line is now the locked primitive (line B1→B2), we must substitute calculated tangent points and their intersections into formula **(1)** instead of formula **(2)** (as was the previous case in section 7.2.2.2). An example of this is shown in figure 7.8.

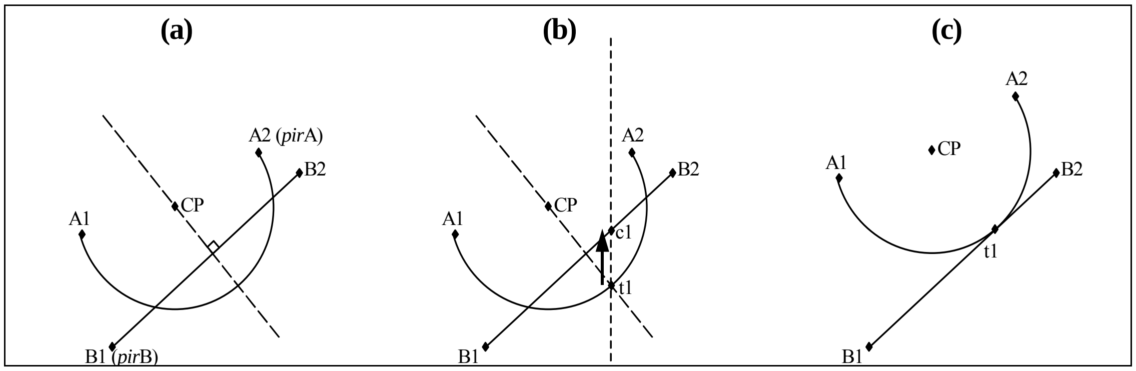


Figure 7.8. Resolving overlap using the tangent point method with the arc line case

Figure 7.8a shows that points A2 and B1 are the points in range, pirA and pirB. However, both of the pir produce negative translations (using formulae **(1)** and **(2)**) thus they cannot be used to resolve the intersection). Therefore, the tangent points method must be used again. In the example, only one tangent point is found because the perpendicular line intersects the arc in only one place. Figure 7.8b shows that an infinite vertical line is passed through the tangent point, t1, and is intersected with line B1→B2 to produce point c1. The translation distances can then be calculated by substituting tangent points for pirA in formula **(1)**. This gives formula **(5)**.

distanceTangentA = intersectionPointB.y - tangentA.y **(5)**

Using formula **(5)** with the example yields a positive translation distance as shown by the bolded arrow in figure 7.8b. The intersection is resolved by translating the arc, A1→A2, by this vertical distance as shown in figure 7.8c. If the tangent point method does not find any tangent points or no positive translation is found then the points in range method can fully resolve the intersection.

7.2.2.4 Arc & Arc (Free Arc moving through Locked Arc)

The arc through arc case initially uses the point in range technique. This will not be repeated in detail here as it is used throughout the previous cases. However, a brief example of the method is shown in figure 7.9. Figure 7.9a shows the points in range, figure 7.9b shows the infinite line intersections passing through the pir to create the intersection points, c1, c2 and c3, and figure 7.9c shows the resolved overlap after translating arc A by the result of the distance formulae. An example of arcs of opposite orientations is shown in figure 7.10.

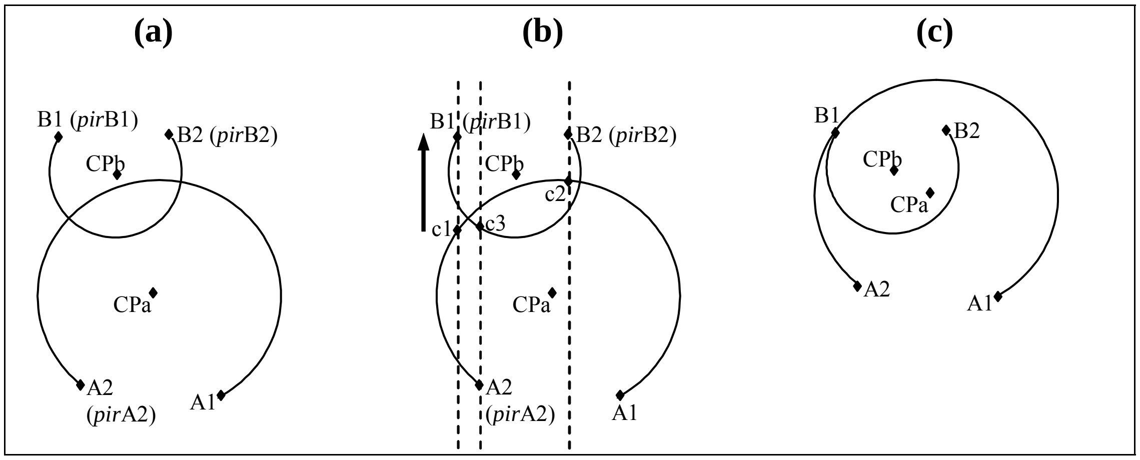


Figure 7.9. Resolving arc intersections using the points in range

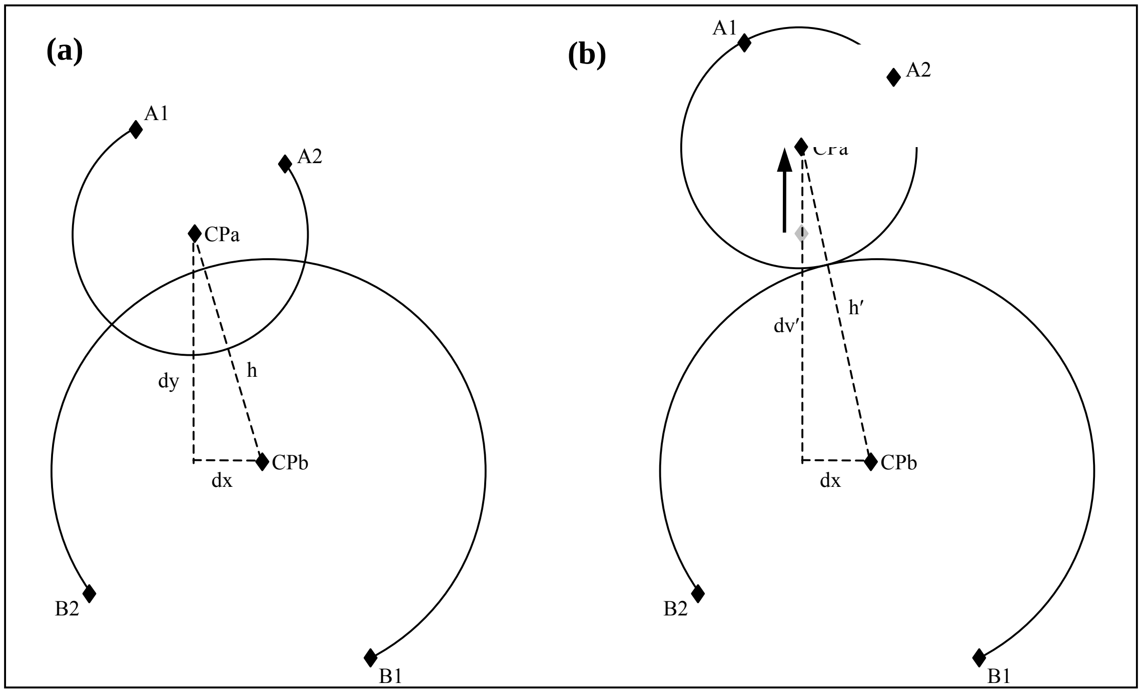


Figure 7.10. Using the Pythagorean theorem to resolve arc intersections (method 1)

For the situation of figure 7.10, where the point in range method is unable to resolve the intersection between the two arc primitives, we use two circle tangent methods that utilise the radii of the arcs and the Pythagorean theorem.

Given that rA is the radius of the free arc A1→A2 and rB is the radius of the locked arc B1→B2 we can make the following observations:

From figure 7.10a, when the arcs are intersecting: $(rB - rA) < h < (rA + rB)$

From figure 7.10b, when the intersection has been resolved: $h' = (rA + rB)$

Therefore:

yTranslation = (dy′ - dy) : where dy′ = sqrt((h′ * h′) – (dx * dx)) **(6)**

This intersection can then be resolved by translating arc A by the result of formula **(6)**.

A further arc through arc case that can occur involves two arcs of similar orientation as shown in figure 7.11.

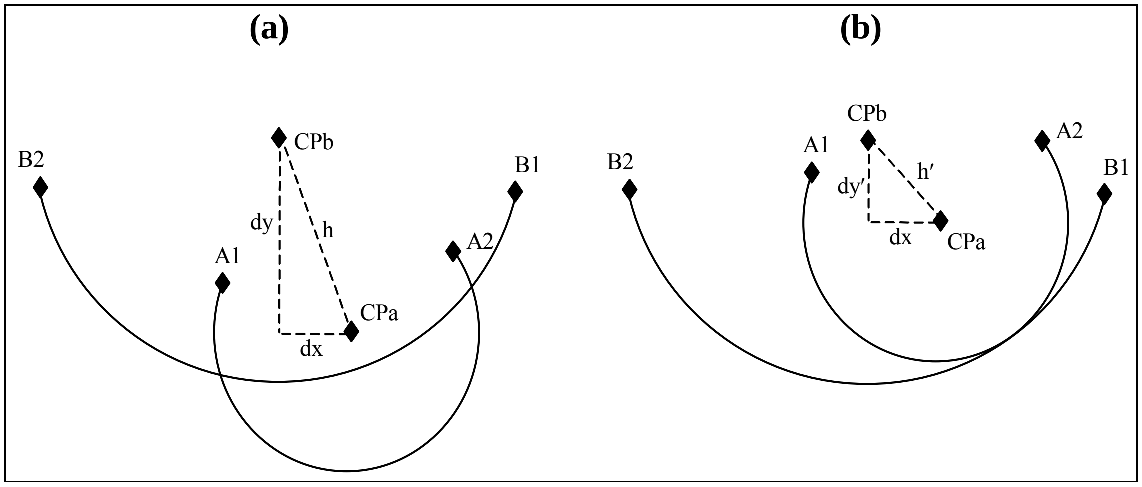


Figure 7.11. Using the Pythagorean theorem to resolve arc intersections (method 2)

Given that rA is the radius of the free arc A1→A2 and rB is the radius of the locked arc B1→B2, the following observations can be made:

From figure 7.11a, when the arcs are intersecting: $(rB - rA) < h < (rA + rB)$

From figure 7.11b, when the intersection has been resolved: $h' = (rB - rA)$

Therefore, the translation distance can be calculated by formula **(7)**.

$$yTranslation = (dy - dy') \quad : \text{where } dy' = \text{sqrt}((h' * h') - (dx * dx)) \qquad \textbf{(7)}$$

If the result of formula **(7)** is positive, applying this vertical translation to arc A will resolve the overlap. It can be seen that, whereas the first circle tangent resolution method translates the free arc to the exterior of the locked arc circle, the second method translates the free arc to be inside of the arc circle. This is important for the correct manipulation of both convex and concave arcs.

7.2.2.5 Intersection Resolution Summary

The four possible intersection cases that may occur within intersecting shapes have been described. It has been shown that each case may be resolved by using the *points-in-range* method by using formulae **(1)** and **(2)**. This will always resolve the intersection where two lines are involved (see formula **(3)**). If arcs are involved, the point in range method may not be sufficient to resolve intersections fully and other supplementary tangent based techniques may be required. When an arc and line are intersecting, the perpendicular of the line primitive is displaced through the arc's centre point and is intersected with the arc to find tangent points. Where the arc is the "locked primitive" and the line is the "free primitive" formula **(4)** is used to calculate the vertical translation required. Formula **(5)** is utilised when the arc is the "free primitive" and the line is locked. The final case, where two arcs are intersecting, introduced two circle tangent methods that can resolve intersections. The first method is calculated using formula **(6)** and results in the respective arc's parent circles being separate. The second method results in the respective free arc's parent circle being contained by the locked arc's parent circle (and is given in formula **(7)**).

It is of note that the most simple of the cases is where two lines are involved, as no extra tangent calculations are required. This presents optimisation possibilities (which are included in the implementation) whereby it is preferred to resolve line only cases if many intersecting pairs of primitives between two shapes exist. Although repeated applications of these techniques may be necessary to fully resolve the overlap between two shapes, this is more efficient and accurate than the iterative grid method outlined in section 3.2. One further circumstance is the overlapping case in which one shape is completely contained within another. This requires another resolution strategy as there are no intersecting primitives.

7.2.3 Dealing with Contained Shapes

During the nesting process it is possible that a shape may become entirely contained within other already assigned shapes, as shown in figure 7.12.

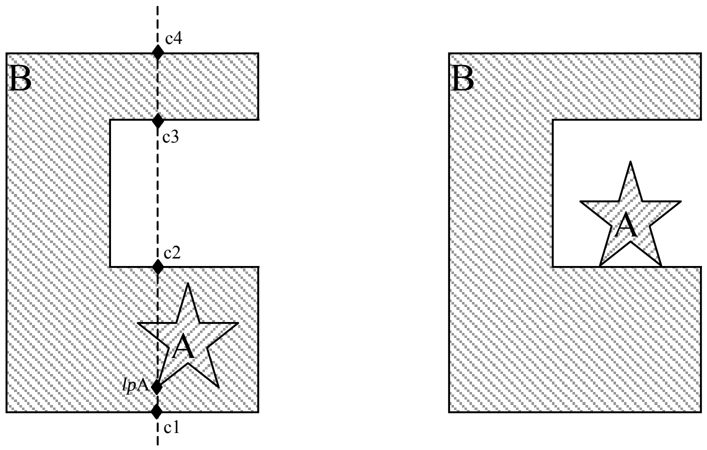


Figure 7.12. Resolving a contained shape

Here there are no intersecting primitives to resolve so another approach is required. As the free shape A is contained by locked shape B the lowest point on shape A is used, lpA, through which an infinite vertical line is passed. The resultant intersection of the infinite vertical line and shape B gives the points c1 … cn. The translation that is performed is defined by formula **(8)**:

$$yTranslation = \min(c1.y - lpA.y, \dots cn.y - lpA.y) \quad : \text{where } (ci.y - lpA.y) > 0, i = 1 \dots n \quad \textbf{(8)}$$

In the example, the value for vertical translation would be c2.y – lpA.y.

Figure 7.13 shows an instance where employing this technique does not fully resolve overlap between the shapes. However, shape A is no longer contained by shape B and there are intersecting primitives once more. This allows the techniques that have been described for resolving primitive intersection to be employed. This process continues until the shape intersection is completely resolved.

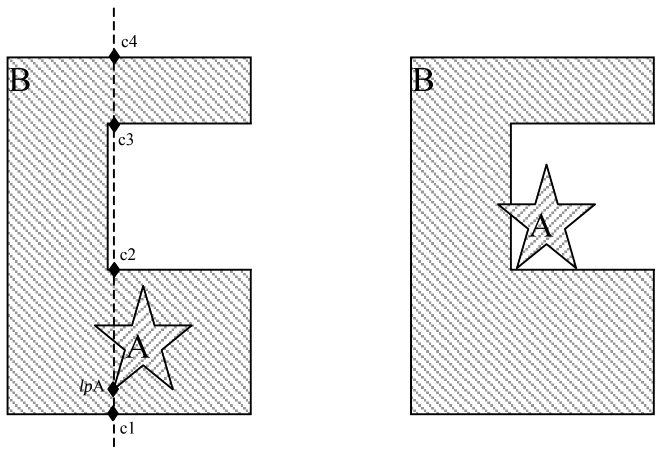


Figure 7.13. Contained shape and overlap is not fully resolved

In resolving shape intersections, the vertical translations may cause the free shape to intersect with other already assigned shapes. These intersections must also be resolved until the shape does not intersect and can be assigned to the sheet or until the shape has moved off the top of the sheet. In the latter case, the shape must be translated back to the bottom of the sheet and the shape's x location is incremented and the process resumes until all shapes are placed.

7.2.4 Summary – The Bottom-Left-Fill Algorithm

Given the functionality outlined in the previous sections the bottom-left-fill process can be describes using the following pseudo code:

```
Input Sequence : shapes[1..m][1..n] where    m = number of different shapes,
       n = number of rotations for given shape[m]
sheetshape[1..q] where array holds placed shapes on the sheet
positions[1..n] first available position for rotation of shape shapes[][1..n]
x = current x position
resolution = x increment step value


Begin

Place shapes[1][1] at bottom left of sheet (0,0);
Add shapes[1][1] to sheetshape[1..q];

for (i = 2; i <= m; i++)
{
       for (j = 1; j <= n; j++)
       {
              place shapes[i][j] at bottom left of sheet;

              while (Overlap(shapes[i][j], sheetshape[1..q]))
              {
                     Get Intersecting Primitives of shape[i][j] and sheetshape[1..q];

                     Resolve Overlapping primitives;

                     if (shapes[i][j] off top of sheet)
                     {
                            x = x + resolution;
                            place shape[i][j] at (x,0);
                     }

              }

              Add shapes[i][j]'s coordinate to positions[1..n];
       }

       Add shapes[i][1..j] with leftmost position[1..n] to sheetshape[1..q]
}

Return Evaluation (total length of packing);

End
```

This bottom-left-fill placement algorithm takes a sheet size and an input sequence of shapes and their allowable rotations. The algorithm starts by placing the first shape in the lower left corner of the sheet in its most efficient orientation. The most efficient orientation is the rotation which yields the smallest bounding width for the shape, with equal widths being resolved by the rotation that faces concavities to the right. This enables later shapes to easily be placed within the concavity regions. With subsequent shapes, if a copy of this shape has not been placed on the sheet, the shape starts at the lower left corner of the sheet. If a copy of the shape has previously been assigned to the sheet, then the new copy starts from where the previous copy of the shape was placed. A valid location for placement is found by testing for intersections and containment. If the shape is not intersecting or contained by (or containing) other already placed shapes and the shape is within the remits of the sheet, then the location of the shape is valid and therefore can be assigned to the sheet. When a shape is in a position that intersects with already assigned shapes, the resolution techniques (described earlier in this section) are used to resolve the overlap in a positive vertical direction (upwards congruent to the *y* axis of the sheet). If resolving overlap results in the shape moving off the top of the sheet, then it is returned to the bottom of the sheet and incremented along the positive x axis by a certain value (known as the resolution).

The process continues as before with overlap/intersection testing and resolution of the intersections until the shape can be placed. Packing is completed when all shapes have been assigned to the sheet and the solution can be returned to the user. Shapes are always packed in the order they appear in the input sequence that is provided by the local search techniques.

7.2.5 Local Search

It is usual to apply some sorting criteria to the shapes of a given problem before packing, often by decreasing length or area. Although these often yield solutions of reasonable quality, further improvements can be found if a local search mechanism is applied to generate new input orderings. This approach has been used during the experiments of section 7.4 by applying hill climbing and tabu search for both area and length pre-sorted arrangements.

A standard hill climbing algorithm is used during the experiments which simply applies operators to the current solution in order to find a neighbour of increased quality. If an improved neighbour is found it is adopted as the current solution and the search continues. If the neighbour is not an improvement on the current solution it is discarded and the search continues with other neighbours. The best solution is returned at the end of the search.

The tabu search mechanism implemented for the experiments is similar in nature to that described in Glover (1993) [**258**]. The process generates a given number of neighbours and moves to the best solution in this subset of the neighbourhood. This solution is then used to generate the next set of neighbours and the cycle continues. The use of a tabu list means that recently seen solutions, within a given list length, will not be revisited. Throughout this chapter, a neighbourhood size of 5 solutions and a tabu list length of 200 solutions are used. These values were chosen from a set of possible values during initial experimentation.

The operators used throughout both search techniques are 1 Opt, 2 Opt, 3 Opt, 4 Opt and *N* Opt. 1 Opt removes a randomly chosen shape and inserts it at a random location in the sequence. 2 Opt swaps two randomly chosen shapes in the order, although not two of the same type as this would produce the same result, this is extended up to 4 Opt where four randomly selected shapes are swapped. *N* Opt selects a random number of shapes to swap and is likely to produce a radically different solution, and thus diversify the search. The solution operator is chosen by means of a random number that is then compared to a weighted scale, which gives the particular operator to be used. Each operator has a different chance of selection, from 1 Opt which has the largest chance of being selected down to *N* Opt which has a much lower chance of being selected. This is because the less radical operators allow for concentration of the search and the highly radical operators, e.g. *N* Opt, enable the search to escape local optima. The following pseudocode shows how both of the local searches interface with bottom-left-fill:

```
INPUT:     Problem Shapes, Quantities and Allowable Rotations, Sheet Size
           current.ordering = Sort Ordering(decreasing area, decreasing length)

Begin

current.evaluation = Bottom-Left-Fill(current.ordering);
best = current;

while (!Stopping Criteria)    // either max number of iterations or time based
{
   opt = Select Operator (1,2,3,4,n Opt);

   if (TABU)
   {
       for (i = 0; i < neighbourhood size; i++)
       {
          neighbour[i].ordering = Generate NotTabu Neighbour(current ordering, opt);
          neighbour[i].evaluation = Bottom-Left-Fill(neighbour[i].ordering);
       }
       current = GetBestNeighbour(neighbour[]);
   }
   else if (HILL CLIMBING)
   {
       neighbour.ordering = Generate Neighbour(current.ordering, opt);
       neighbour.evaluation = Bottom-Left-Fill(neighbour.ordering);
       if (neighbour.evaluation <= current.evaluation) { current = neighbour; }
   }

   if (current.evaluation < best.evaluation) { best = current; }

}

return best;

End
```

7.3 Benchmark Problems

In order to compare the proposed algorithm with the current state of the art, the literature benchmarks are used (see table 2.3). Furthermore, ten new benchmark problems have been created for the irregular stock cutting problem which include some shapes that are typical of the metal cutting industry. Table 7.3 gives the details of these new problems including the sheet width and rotational constraints. The raw data for these instances can be found in appendix D.

Problem Name	Number of Shapes	Rotational Constraints	Sheet Width	Optimal Known
Profiles1	32	90 Incremental	2000	No
Profiles2	50	90 Incremental	2500	No
Profiles3	46	45 Incremental	2500	No
Profiles4	54	90 Incremental	500	No
Profiles5	50	15 Incremental	4000	No
Profiles6	69	90 Incremental	5000	No
Profiles7	9	90 Incremental	500	Yes
Profiles8	18	90 Incremental	1000	Yes
Profiles9	57	90 Incremental	1500	No
Profiles10	91	0 Absolute	3000	No

Table 7.3. New benchmark problems

Line & Arc - Profiles1 to Profiles5

These new problems introduce arcs and holes to the set of benchmark problems. Some of these shapes, in particular Profiles1 and Profiles2, have been chosen from a library of standard shapes within the sheet metal profiling industry. All of these problems contain at least one shape consisting of one or more arcs. For these problems the optimal solutions are not known.

Line Only - Profiles6 to Profiles10

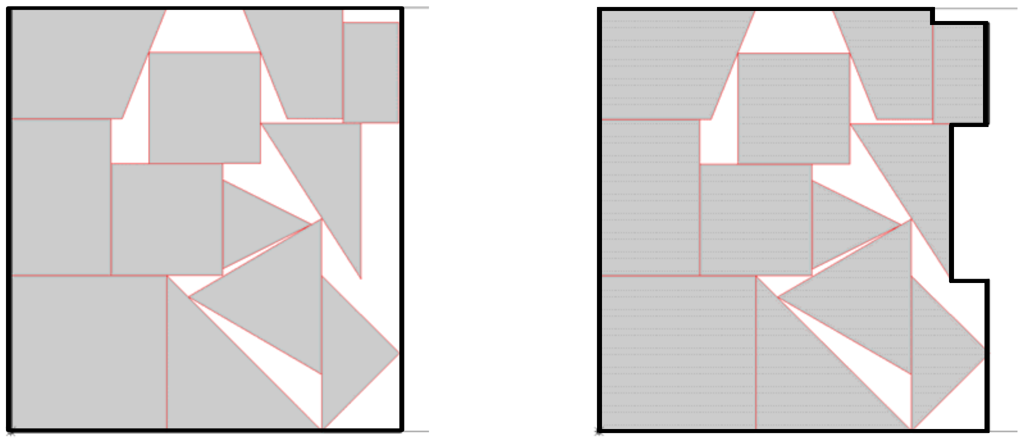
A further five problems have been created which can be tackled by non-arc implementations (but also include shapes with holes). Once again the optimal solutions are not known with the exception of Profiles7 and Profiles8 which are 'jigsaw' problems where the optimal solutions are known (1000 units for both problems). Profiles9 is a novelty dataset involving a subset of letters from the English alphabet. Profiles10 combines polygons from numerous literature benchmarks.

7.4 Experiments

For the purposes of these experiments, the proposed irregular bottom-left-fill placement algorithm is used to pack input shape orderings as generated by the local search techniques. During the search process, the generated solutions are evaluated by the total length of the layout. Two different density measures are recorded, the first is a simple straight line density (Density1) whilst the second density measure, used by Hopper (2000) [**26**], is based on the union of all individual shape bounding rectangles; this gives a non-rectangular final density measurement which provides an indication of the sheet area that is used (Density2). Both methods are pictured in figure 7.14, where the thicker line represents the containing area.

Density can then calculated by applying the following formula:

Density = total shape area / containing area



Straight Line Density Measure (Density1) Hopper (2000) Density Measure (Density2)

Figure 7.14. Density Measures

All of the experiments conducted have, once again, been conducted on a PC with a 2GHz Intel Pentium 4 processor and 256MB RAM.

7.4.1 Experiments on Literature Benchmark Problems

The literature problems have been divided into two groups, those for which the best known result is measured using overall nest length and those which have been evaluated using density measures (see table 2.3 and 2.4). Both the length and density measures are used with all of the problem instances to facilitate easier comparison with future research. Table 7.4 presents the results for the length evaluated problems using the tabu and hill climbing search methods outlined previously (best results shown in bold). Each problem was run 10 times using 100 iteration runs, for both length and area initial orderings (resulting in a total of 40 runs for each problem). It seems that tabu search outperforms hill climbing for most cases. Table 7.5 shows the results for the density evaluated problems. In table 7.7, the best results from table 7.4 & 7.5 are shown and include the average times per nest, time taken to find the best solution and the percentage improvement on the best known result from the literature. For problems where there has been an improvement upon the best known solution, the percentage is shown in bold typeface. From these initial experiments, the algorithm has improved upon 20 of the 26 available literature problems. The average overall improvement over the 26 problems is slightly over 4% and is nearer 6% for the 20 problems where new best solutions have been found. On average the best solutions have been obtained within a few minutes execution time, although for many of the problems only a few seconds are necessary due to the high performance of the presented algorithm. The average time per nest compares favourably with other literature approaches utilising the no-fit polygon. For example, Gomes & Oliveira (2002) state that a layout for the problem SHAPES1 can be generated within 6.18 seconds, on a 450MHz CPU, in comparison with 0.82 seconds for a solution generated by the proposed automated nesting approach [**172**]. The best solution for dataset SWIM was obtained in 607 seconds and is shown in figure 7.15.

		Hill Climbing								Tabu Search							
		Length Ordered				Area Ordered				Length Ordered				Area Ordered			
Problem	Best	Average	Best Result			Average	Best Result			Average	Best Result			Average	Best Result		
	Literature	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2
Blasz1	27.30	28.72	28.60	75.5%	78.0%	28.57	28.40	76.1%	**81.0%**	28.42	28.00	77.1%	78.5%	28.64	**27.80**	**77.7%**	79.9%
Dagli	65.60	62.57	61.82	82.0%	84.1%	62.28	61.67	82.2%	**89.0%**	62.15	61.10	83.0%	84.2%	61.98	**60.57**	**83.7%**	84.3%
Fu	34.00	33.57	33.00	86.4%	90.1%	33.90	33.00	86.4%	90.3%	33.58	**32.80**	**86.9%**	90.0%	33.67	33.00	86.4%	**90.8%**
Jakobs1	13.20	12.40	12.00	81.7%	89.1%	12.02	12.00	81.7%	87.8%	12.08	12.00	81.7%	**92.6%**	11.95	**11.86**	**82.6%**	89.5%
Jakobs2	28.20	26.40	26.00	74.2%	**83.3%**	26.62	26.00	74.2%	77.8%	26.53	26.00	74.2%	80.8%	26.37	**25.80**	**74.8%**	83.1%
Marques	83.60	82.76	81.00	85.4%	88.6%	81.97	80.84	85.6%	88.2%	84.20	83.00	83.3%	86.0%	81.10	**80.00**	**86.5%**	**89.3%**
Poly1A	14.70	14.73	14.27	71.8%	76.0%	14.40	14.03	73.1%	77.5%	14.80	14.56	70.4%	76.0%	14.46	**14.00**	**73.2%**	**78.2%**
Poly2A	30.10	28.82	28.41	72.1%	**77.5%**	28.68	28.43	72.1%	75.3%	28.76	28.26	72.5%	76.3%	28.42	**28.17**	**72.8%**	75.9%
Poly3A	40.40	43.14	42.65	72.1%	74.3%	42.49	41.77	73.6%	**75.5%**	43.24	42.48	72.4%	74.0%	42.61	**41.65**	**73.8%**	74.8%
Poly4A	56.90	56.89	56.30	72.8%	74.6%	56.93	55.32	74.1%	75.8%	56.14	**54.93**	**74.6%**	**75.9%**	56.07	55.51	73.9%	75.4%
Poly5A	71.60	70.64	**69.37**	**73.9%**	**75.7%**	71.25	70.72	72.5%	74.0%	70.96	70.69	72.5%	73.4%	70.98	70.55	72.6%	73.8%
Poly2B	33.10	31.14	30.98	73.0%	77.1%	31.15	30.70	73.7%	77.0%	31.26	31.09	72.7%	76.2%	31.05	**30.00**	**75.4%**	**77.5%**
Poly3B	41.80	41.33	40.91	74.5%	**77.1%**	41.13	40.77	74.8%	76.4%	41.16	**40.74**	**74.9%**	76.5%	41.31	40.88	74.6%	76.4%
Poly4B	52.90	52.21	51.91	74.5%	76.5%	52.56	52.12	74.2%	75.5%	52.11	**51.73**	**74.8%**	**77.4%**	52.05	51.78	74.7%	76.5%
Poly5B	63.40	61.97	60.83	75.5%	**77.2%**	61.19	**60.54**	**75.8%**	76.9%	61.87	61.04	75.2%	76.6%	61.43	61.43	74.7%	75.3%
SHAPES	63.00	60.00	59.20	67.4%	68.4%	60.22	60.00	66.5%	**69.1%**	59.72	**59.00**	**67.6%**	69.0%	60.00	60.00	66.5%	**69.1%**
SHAPES0	63.00	66.50	**66.00**	**60.5%**	**62.6%**	67.26	66.70	59.8%	62.0%	66.60	**66.00**	**60.5%**	62.3%	67.22	67.00	59.6%	61.6%
SHAPES1	59.00	63.38	62.10	64.3%	65.5%	61.32	**60.00**	**66.5%**	**68.9%**	62.36	61.00	65.4%	68.1%	62.18	62.00	64.4%	66.9%
SHIRTS	63.13	64.18	64.00	84.4%	85.7%	65.06	64.30	84.0%	86.4%	64.24	**63.80**	**84.6%**	86.0%	64.99	64.04	84.3%	**87.3%**
SWIM	6568.00	6610.98	**6462.40**	**68.4%**	**71.6%**	6713.08	6601.80	67.0%	69.2%	6551.88	6489.80	68.2%	69.6%	6804.80	6723.70	65.8%	70.2%
TROUSERS	245.75	251.39	248.67	87.7%	89.0%	248.88	246.98	88.3%	**90.1%**	251.54	248.38	87.8%	88.6%	249.56	**246.57**	**88.5%**	89.7%

Table 7.4. Experiments on length evaluated literature benchmark problems

		Hill Climbing								Tabu Search							
		Length Ordered				Area Ordered				Length Ordered				Area Ordered			
Problem	Best	Average	Best Result			Average	Best Result			Average	Best Result			Average	Best Result		
	Literature	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2
Albano	86.0%(D2)	10465.16	10354.80	84.1%	85.9%	10505.64	**10292.90**	**84.6%**	86.2%	10426.76	10315.10	84.4%	**86.5%**	10406.74	10298.60	84.5%	85.8%
Blasz2	68.6%(D1)	25.85	25.60	73.6%	79.3%	25.87	**25.28**	**74.5%**	78.4%	25.86	25.38	74.2%	**79.9%**	25.70	25.38	74.2%	78.2%
Dighe1	72.4%(D2)	1351.10	1320.00	75.8%	76.9%	1342.74	1324.60	75.5%	77.8%	1327.72	1307.20	76.5%	78.2%	1332.98	**1292.30**	**77.4%**	**78.9%**
Dighe2	74.6%(D2)	1302.20	**1260.00**	**79.4%**	83.3%	1303.36	1270.70	78.7%	81.5%	1279.38	1270.70	78.7%	81.5%	1295.14	**1260.00**	**79.4%**	**84.3%**
Mao	71.6%(D2)	1875.52	**1854.30**	**79.5%**	82.0%	1889.32	1863.40	79.1%	82.5%	1875.46	**1854.30**	**79.5%**	82.0%	1890.28	1867.00	78.9%	**82.9%**

Table 7.5. Experiments on density evaluated literature benchmark problems

Problem	Best Literature	Best Result			Time/Nest (s)	Time to best (s)	Percentage Improvement
		Length	Density1	Density2			
Blasz1	27.3	27.80	77.7%	81.0%	0.32	21	-1.83%
Dagli	65.6	60.57	83.7%	89.0%	2.04	188.8	**7.66%**
Fu	34	32.80	86.9%	90.8%	0.24	20.78	**3.53%**
Jakobs1	13.2	11.86	82.6%	92.6%	0.74	43.49	**10.17%**
Jakobs2	28.2	25.80	74.8%	83.3%	2.13	81.41	**8.51%**
Marques	83.6	80.00	86.5%	89.3%	0.25	4.87	**4.31%**
Poly1A	14.7	14.00	73.2%	78.2%	0.36	12.48	**4.76%**
Poly2A	30.1	28.17	72.8%	77.5%	1.24	120.56	**6.42%**
Poly3A	40.4	41.65	73.8%	75.5%	2.01	210.07	-3.10%
Poly4A	56.9	54.93	74.6%	75.9%	2.43	203.17	**3.47%**
Poly5A	71.6	69.37	73.9%	75.7%	5.04	475.63	**3.12%**
Poly2B	33.1	30.00	75.4%	77.5%	2.50	179.86	**9.36%**
Poly3B	41.8	40.74	74.9%	77.1%	4.26	417.67	**2.54%**
Poly4B	52.9	51.73	74.8%	77.4%	8.24	95.66	**2.20%**
Poly5B	63.4	60.54	75.8%	77.2%	14.70	676.61	**4.51%**
SHAPES	63	59.00	67.6%	69.1%	0.60	31.36	**6.35%**
SHAPES0	63	66.00	60.5%	62.6%	0.93	21.33	-4.76%
SHAPES1	59	60.00	66.5%	68.9%	0.82	2.19	-1.69%
SHIRTS	63.13	63.80	84.6%	87.3%	4.99	58.36	-1.06%
SWIM	6568	6462.40	68.4%	71.6%	12.39	607.37	**1.61%**
TROUSERS	245.75	246.57	88.5%	90.1%	7.89	756.15	-0.33%
Albano	86.0%	10292.90	84.6%	86.5%	1.18	93.39	**0.5%**
Blasz2	68.6%	25.28	74.5%	79.9%	0.16	10.94	**11.3%**
Dighe1	72.4%	1292.30	77.4%	78.9%	0.22	8.87	**6.5%**
Dighe2	74.6%	1260.00	79.4%	84.3%	0.10	7.12	**9.7%**
Mao	71.6%	1854.30	79.5%	82.9%	0.38	29.74	**11.3%**

Table 7.6. Summary of best results using 100 iteration runs

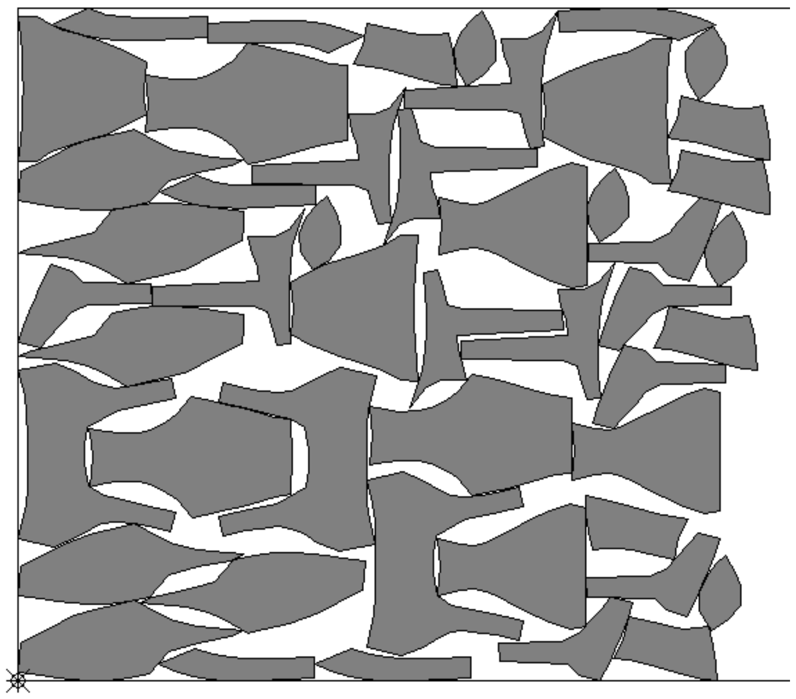


Figure 7.15. SWIM 100 iteration solution (6462.4 units, 607 seconds)

Further extended experiments were then conducted for the problems in which the approach has not set new benchmarks namely, Blasz1, SHAPES1, SHIRTS and TROUSERS. For these extended runs the durations 551.73s, 2019.77s, 6367.57s and 13613.67s respectively are used as upper limits in each run, as noted by the authors who have presented the best solutions (previous to this thesis) for these problems (Gomes & Oliveira, 2002) [**172**]. For the problems SHAPES and Poly3A the experiments were extended to 1000 iterations per run, as guidance on computation times is not available in the literature. The algorithm was allowed a further 10 runs on each of these benchmark problems for the extended durations specified. Table 7.7 shows the best results of these runs including the time taken to find the best solution, the method that generated that solution and percentage improvement on the best known solution. Of the 6 extended experiments, 5 new best benchmark solutions have been set for Blasz1, Poly3A, SHAPES1, SHIRTS and TROUSERS. For the remaining problem, SHAPES0, the solution matches the work of [**172**] (length of 65 units) but has not reached the best known solution of 63 units found in Dowsland & Dowsland (1993) [**147**]. Note: the approach of chapter nine consistently improves over the literature best for the SHAPES0 benchmark within three minutes.

Problem	Best Literature	Average Length	Best Result			Method	Time to best (s)	Percentage Improvement
			Length	Density1	Density2			
Blasz1	27.3	27.47	27.20	79.4%	80.7%	Hill Climb/Length	501.91	**0.37%**
Poly3A	40.4	41.45	40.33	76.2%	77.3%	Tabu/Area	1515.49	**0.18%**
SHAPES0	63	65.68	65.00	61.4%	63.6%	Hill Climb/Area	332.39	-3.17%
SHAPES1	59	60.53	58.40	68.3%	71.5%	Tabu/Area	1810.14	**1.02%**
SHIRTS	63.13	63.66	63.00	85.7%	88.1%	Tabu/Length	806.5	**0.21%**
TROUSERS	245.75	246.40	243.40	89.6%	91.1%	Tabu/Area	3611.99	**0.96%**

Table 7.7. Summary of the results from the extended experiments

Of the 26 problems from the literature, using the proposed nesting algorithm, 25 new best solutions have been produced. The author believes that this is the first time that an irregular packing approach has attempted such a broad range of problems with such success. The solution obtained for TROUSERS is shown in figure 7.16. All best solutions obtained for the literature problems are shown in appendix B.

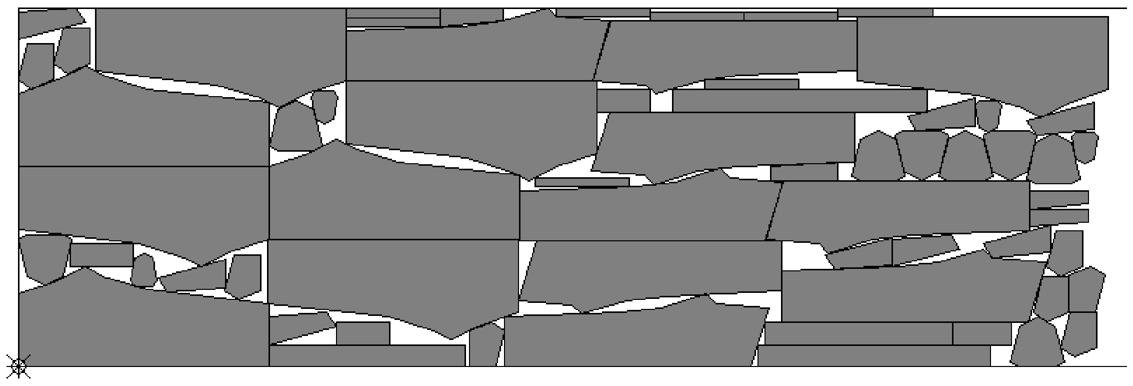


Figure 7.16. TROUSERS extended run solution (243.4 units, 3612 seconds)

7.4.2 Experiments on New Benchmark Problems

The next set of experiments involved setting initial benchmarks for the new problem instances (Profiles1 – Profiles10). Each problem was run 10 times for 30 minute durations using hill climbing and tabu search (with both length and area initial orderings) resulting in a total of 40 runs. Table 7.8 shows the results of these experiments, the best solutions are indicated in bold typeface, and table 7.9 is summary of the best results. Figure 7.17 shows the best solutions for datasets Profiles1 and Profiles9. All best solutions obtained for these new problems are also shown in appendix B.

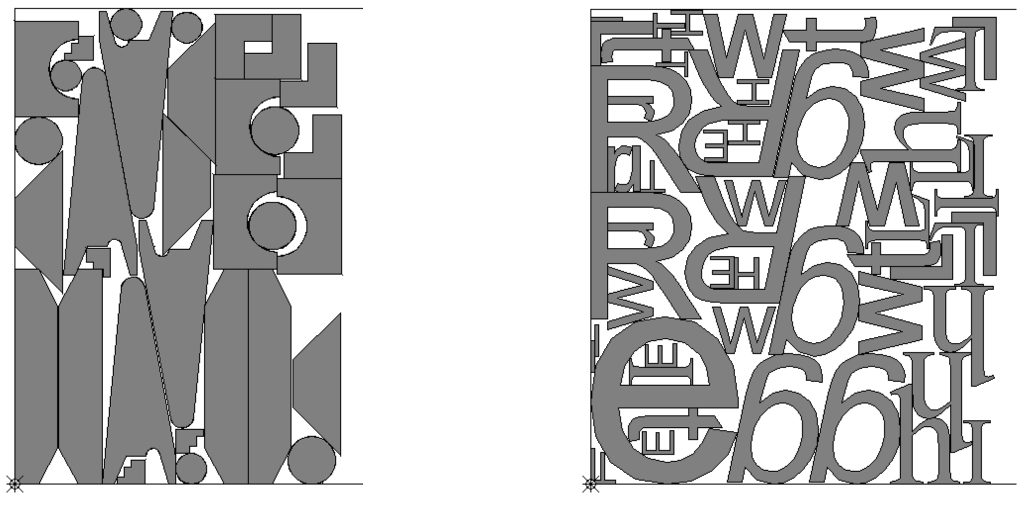


Figure 7.17. Best solutions for Profiles1 (1377.74 units) & Profiles9 (1290.67 units)

Problem	Hill Climbing								Tabu Search							
	Length Ordered				Area Ordered				Length Ordered				Area Ordered			
	Average	Best Result			Average	Best Result			Average	Best Result			Average	Best Result		
	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2
Profiles1	1417.50	**1377.74**	**82.0%**	**85.2%**	1443.21	1436.20	78.4%	80.4%	1431.45	1405.64	80.3%	84.1%	1453.12	1427.64	79.1%	82.3%
Profiles2	3285.67	3261.00	49.3%	50.5%	3291.01	3225.54	49.8%	**51.3%**	3291.86	**3216.10**	**50.0%**	50.8%	3344.81	3280.00	49.0%	50.0%
Profiles3	8311.23	8228.55	50.7%	51.1%	8252.02	**8193.89**	**50.9%**	**52.6%**	8323.90	8231.14	50.7%	51.5%	8392.79	8290.49	50.3%	51.7%
Profiles4	2489.54	2480.96	74.3%	75.1%	2492.88	2479.30	74.3%	74.8%	2488.86	2460.60	74.9%	**75.7%**	2486.56	**2453.12**	**75.1%**	75.7%
Profiles5	3427.47	3360.57	69.6%	73.1%	3381.38	**3332.70**	**70.2%**	71.1%	3405.22	3352.90	69.7%	**73.6%**	3383.31	3360.94	69.6%	71.3%
Profiles6	3180.16	3170.25	73.9%	76.7%	3177.91	3146.82	74.4%	77.5%	3147.11	**3097.86**	**75.6%**	**77.8%**	3162.74	3145.76	74.5%	77.0%
Profiles7	1326.71	**1296.30**	**77.1%**	78.3%	1326.57	1309.10	76.4%	**80.2%**	1324.38	1309.10	76.4%	79.6%	1339.99	1325.40	75.4%	78.3%
Profiles8	1330.88	**1318.70**	**75.8%**	77.2%	1361.02	1341.92	74.5%	77.7%	1337.57	1322.55	75.6%	**78.0%**	1355.04	1338.47	74.7%	76.4%
Profiles9	1341.70	1332.98	51.4%	53.9%	1323.44	1314.97	52.1%	54.3%	1350.47	1318.68	51.9%	54.2%	1305.70	**1290.67**	**53.1%**	**54.9%**
Profiles10	11496.12	11410.60	64.7%	65.5%	11758.50	11555.60	63.9%	64.6%	11401.28	**11160.10**	**66.2%**	**66.8%**	11657.30	11552.30	63.9%	64.7%

Table 7.8. Experiments on new benchmark problems

Problem	Best Result		
	Length	Density1	Density2
Profiles1	1377.74	82.0%	85.2%
Profiles2	3216.10	50.0%	51.3%
Profiles3	8193.89	50.9%	52.6%
Profiles4	2453.12	75.1%	75.7%
Profiles5	3332.70	70.2%	73.6%
Profiles6	3097.86	75.6%	77.8%
Profiles7	1296.30	77.1%	80.2%
Profiles8	1318.70	75.8%	77.2%
Profiles9	1290.67	53.1%	54.9%
Profiles10	11160.10	66.2%	66.8%

Table 7.9. Summary of best results for new benchmark problems

7.5 Summary

In this chapter, a new technique of primitive overlap resolution was introduced and it was demonstrated how an efficient bottom-left-fill nesting algorithm could be generated that is able to handle profiles with both circular arcs and holes. This algorithm produces layouts for realistic problems both quickly and to a level of accuracy that makes it a strong candidate for industrial application. The proposed algorithm has been applied to 26 problems drawn from over 20 years of cutting and packing research and, using only simple local search mechanisms, was able to produce the best-known results for 25 of the 26 problems. The majority of these best solutions were generated within 100 search iterations and yielded an average improvement of 5% over the existing best solutions from the literature. Finally, the chapter introduced 10 new problems, some of which involve profiles with both arcs and holes that have not previously been represented within the literature, and set initial benchmark solutions to facilitate further comparisons.

To the author's knowledge this is the first time that an automated approach has attempted such a broad range of problems from the literature and with such success.

CHAPTER EIGHT

The No-Fit Polygon: A Robust Implementation

Chapter six focussed on the geometric difficulties that this research has faced with specific attention given to the no-fit polygon. The previous generation approaches were also discussed with the main point that they all have disadvantages or degenerate shapes that cannot be handled. In this chapter, a new no-fit polygon generation approach is presented that modifies and extends the orbital trigonometry approach (as used by Mahadevan in 1984 [**256**]). Furthermore, examples are provided to show how the developed approach is able to robustly generate no-fit polygons for several known degenerate shape cases and shapes involving holes. Finally, reasonable generation times are reported for the set of benchmark data that include shapes consisting of lines (all of the literature datasets of tables 2.3 & 2.4 and new benchmark problems, Profiles6 – Profiles10).

8.1 The New No-Fit Polygon Construction Algorithm

For the intersection calculations within this presented implementation, it is important to use a robust geometry library such as the one developed during this work. It is also beneficial to implement routines that are as fast and optimised as possible in order to promote fast no-fit polygon generation and packing algorithms. Once again, the author recommends that practitioners who do not wish to develop such a library use the geometry modules of the LEDA and CGAL libraries.

8.1.1 Overview

The new approach can be divided into two logical stages. Section 8.2 describes the procedure for the first of these where one polygon slides around another to create the outer path of no-fit polygon of the two shapes. This follows a similar approach to that of Mahadevan's algorithm although a modified implementation is proposed. The second section introduces the notion and identification of starting positions that allow the algorithm to find the remaining paths of the no-fit polygon (see section 8.3). These paths will be internal holes of the no-fit polygon (i.e. contained by the outer path) and are not found using Mahadevan's algorithm alone. For this section we will assume that we have two anticlockwise oriented polygons, A and B, that exist somewhere within two-dimensional space and we are required to produce the no-fit polygon NFP_{AB} (orbiting polygon B around polygon A). The first operation that must be performed is to translate polygon B such that it is touching, but not intersecting, polygon A. This can be achieved by translating polygon B such that its largest y-coordinate is placed at the lowest y-coordinate of polygon A (see figure 8.1).

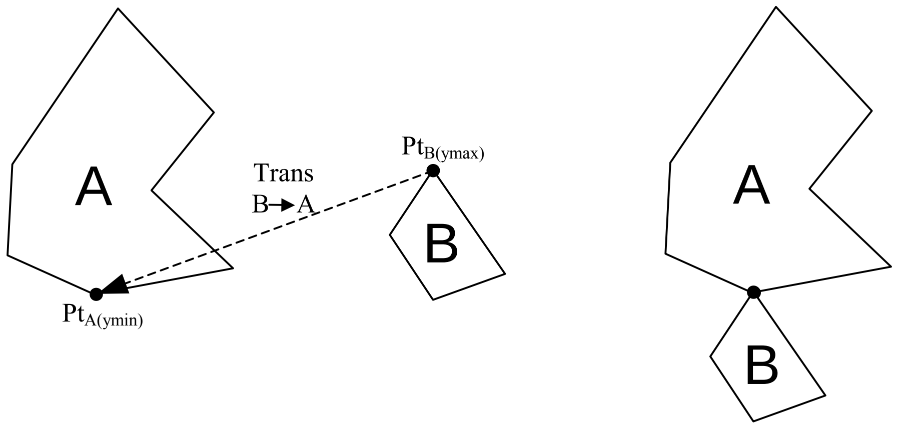


Figure 8.1. Initial translation of the orbiting polygon to touch the stationary polygon

Using these two vertices for alignment guarantees that polygons A and B will be non-intersecting and touching. The translation that results in polygon B touching (but not intersecting) polygon A is given by formula **(1)**:

$$\text{Trans } B \rightarrow A = Pt_{A(ymin)} - Pt_{B(ymax)} \qquad \textbf{(1)}$$

In reality, any starting position may be used providing that it results in polygons A and B touching and not intersecting. The orbital approach can now commence to generate the outer path of the no-fit polygon by orbiting polygon B in an anticlockwise direction.

8.2 Orbiting / Sliding

The main aim of the orbiting (or sliding) section of the algorithm is to detect the correct movements that B must move to traverse around A in order to return to its original position. This is an iterative procedure with each translation step creating an edge of the no-fit polygon. This can be broken down into the following subparts which will be discussed in turn:

Section 8.2.1 - Detection of Touching Edges

Section 8.2.2 - Creation of Potential Translation Vectors

Section 8.2.3 - Finding a Feasible Translation

Section 8.2.4 - Trimming the Feasible Translation

Section 8.2.5 - Applying the Feasible Translation

8.2.1 Detection of Touching Edges

The ability to correctly detect touching and intersecting edges is paramount to the success of this approach. This is achieved by testing each edge of polygon A against each edge of polygon B.

Each pair of edges that touch (one from polygon A and one from polygon B) is stored along with the position of the touching vertex.

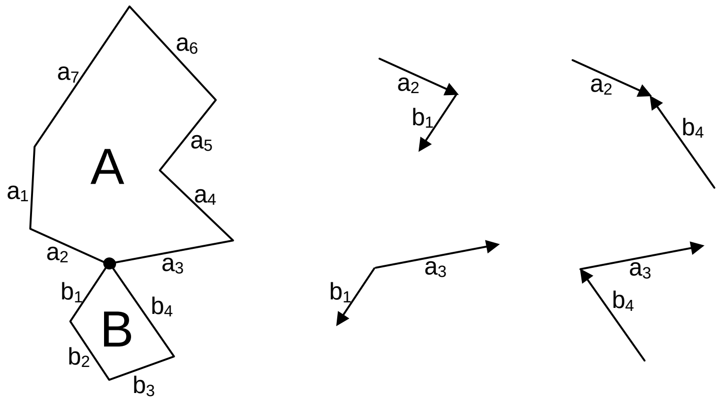


Figure 8.2. Identification of touching edge pairs

Figure 8.2 shows the resulting set of touching edge pairs. This is in contrast to the approach of Mahadevan who performs calculations using all edges at a touching vertex. For example on figure 8.2, Mahadevan would present edges a2, a3, b1 and b4 on the same diagram. In the authors experience with such algorithms, this makes the required calculations more complicated and difficult to explain. These edge-pair diagrams will be used in the next stages to create the potential translation vectors for polygon B and to eliminate the non-viable translation vectors.

8.2.2 Creation of Potential Translation Vectors

The vector with which polygon B must be translated to orbit polygon A must either be derived from an edge of polygon A or from polygon B depending on the situation. Figure 8.3 shows an example of each case. Note that in the second case (figure 8.3b), because polygon B slides along one of its own edges, the translation vector is found by reversing the edge. This is further examined through the relative movement of the polygon A with respect to B. Polygon A is relatively moved along the edge b_1. In reality, polygon A must remain fixed and B must be translated and therefore, the translation vector is reversed in this situation.

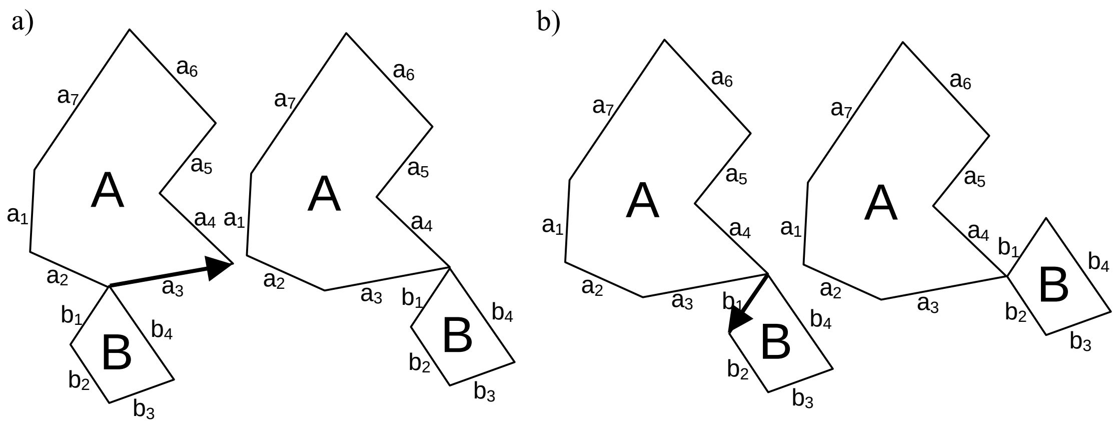


Figure 8.3. Translation vector: a) derived from edge a3, b) derived from edge b1

The set of potential translation vectors can be derived by using the touching edge pairs. There are three possibilities: i) both edges touch at a vertex, ii) a vertex of orbiting edge touches the middle of the stationary edge, or iii) a vertex of the stationary edge touches the middle of the orbiting edge. These cases are depicted in figure 8.4.

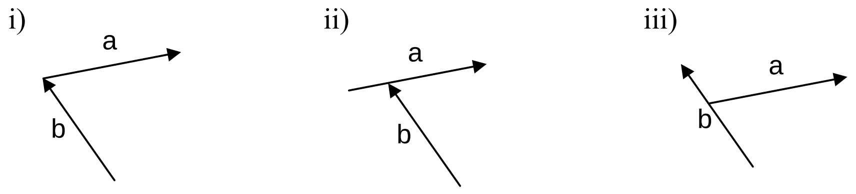


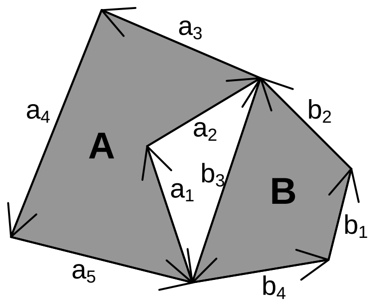
Figure 8.4. Touching edge-pair types

Each pair of touching edges yields one potential translation vector. In case (ii), the translation vector is simply defined using the point at which the two edges touch and the stationary edge's end vertex. In case (iii), we use a similar process except we use the end vertex of the orbiting edge and we also reverse the vector direction. Case (i) requires the correct identification of whether the potential translation vector is derive from the stationary or orbiting edge. This can be identified by the following set of rules based on the touching vertices and a test for whether the orbiting edge, b, is left or right of the stationary edge, a. Table 8.1 shows the different possibilities and the edge from which a potential translation vector is derived under each circumstance.

Case	**Touching Edge Vertex**		**Relative Position of the Orbiting Edge to the Stationary Edge**	**Translation Derived From**
	Stationary	**Orbiting**		
1	Start	Start	Left	Orbiting Edge
2	Start	Start	Right	Stationary Edge
3	Start	End	Left	-
4	Start	End	Right	Stationary Edge
5	End	Start	Left	-
6	End	Start	Right	Orbiting Edge
7	End	End		-
8			Parallel	Either Edge

Table 8.1. Deriving the potential translation when both edges touch at vertices

Clarification is now given as to why some potential translations can be eliminated in certain cases of the table (an '-' entry in the table indicates that no translation is derived). Remember that a translation that is derived from an edge is the resultant vector defined by touching point → end vertex (and then the vector is reversed for an orbiting edge). This allows the elimination of case 7 of the table because both the stationary and orbiting edges touch on the end vertices and this would yield a null translation vector. In cases 3 and 4, the orbiting edge touches at its end vertex thus a translation cannot be derived from the orbiting edge. In cases 5 and 6, the stationary edge cannot be used as it would produce a null vector. Figure 8.5 shows an example for each of the cases using two polygons that touch at two separate places.



Case	Edges	Derived From
1	(a_3,b_3)	b_3
2	(a_1,b_4)	a_1
3	(a_3,b_2)	-
4	(a_1,b_3)	a_1
5	(a_5,b_4)	-
6	(a_2,b_3)	b_3
7	(a_2,b_2) , (a_5,b_3)	-

Figure 8.5. Two polygons touching at two separate positions and vertex-vertex touch cases

In cases 3 and 5, no translation can be derived because the edge of the orbiting polygon is left of the edge from the stationary polygon. For example, edges a_3 and b_2 touch on their start and end vertices respectively. As a null translation is produced using edge b_2 (due to touching with the end vertex), the only possibility is to derive a translation vector from the stationary edge, a_3. However, edge b_2 is left of edge a_3 and, if this translation vector were used, edge b_2 would slide along the inside of edge a_3. Therefore, the positional "left test" eliminates the translations whereby polygons would slide along the inside of an edge rather than the outside of the edge. Where edges are parallel, either the stationary or the orbiting edge may be used.

8.2.3 Finding a Feasible Translation Vector

Once the potential translation vectors have been produced, the next stage is to select a translation vector that does not result in an immediate intersection. For example, returning to figure 8.5, two potential translation vectors, a_1 and $-b_3$, have been generated. It can easily be seen that the orbiting polygon B must be translated using vector $-b_3$ in order to move anticlockwise around polygon A. If polygon B was to be translated along vector a_1 instead, this would result in an immediate intersection between edges a_2 and b_3 (and also a_3 and b_3). Once again the set of touching edge-pairs, generated from section 8.2.1, contains all of the information needed to determine a feasible translation vector.

The process here is simplified due to the proposed representation of touching edges and involves taking each of the potential translation vectors in turn (identified in section 8.2.2) and placing them on the touching position at each touching edge-pair. Positional relationships between the stationary and orbiting edge can be defined based on the union of left/right regions that indicate whether a particular potential translation will be suitable for those edges. For a potential translation vector to be identified as feasible, it must be suitable for every pair of touching edges. Figure 8.6 shows some examples of touching edge-pairs and the angular range of feasible translations.

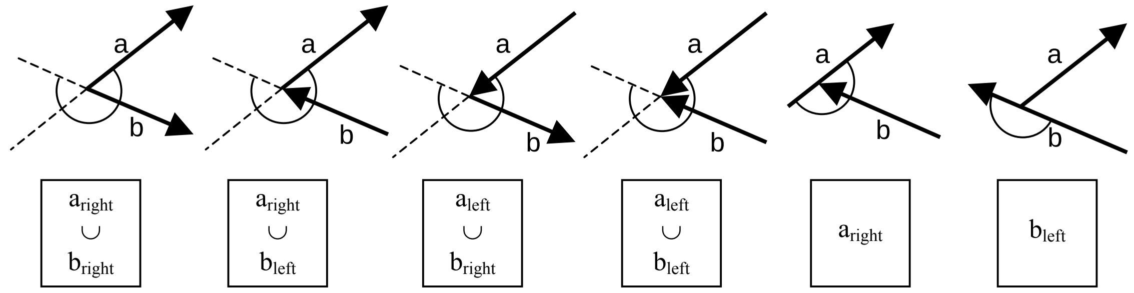

a
b
$a_{right} \cup b_{right}$
a
b
$a_{right} \cup b_{left}$
a
b
$a_{left} \cup b_{right}$
a
b
$a_{left} \cup b_{left}$
a
b
a_{right}
a
b
b_{left}

Figure 8.6. Identifying the feasible angular range of translations (indicated by the arc)

There are other possibilities that can occur which are not shown. These have been omitted for brevity and because they are reasonably straightforward (the examples shown in figure 8.6 provide enough information to derive the omitted relationships). In figure 8.5 two potential translation vectors, $–b_3$ and a_1, were calculated for two touching polygons. Figure 8.7 uses the same example to demonstrate how the approach eliminates translation vector a_1 (the edge-pairs involving edge a_1 have been omitted for brevity but these will be feasible translations because the translation vector is also a_1). In reality, once a potential translation fails on one of these tests, it is infeasible and can be eliminated without further testing.

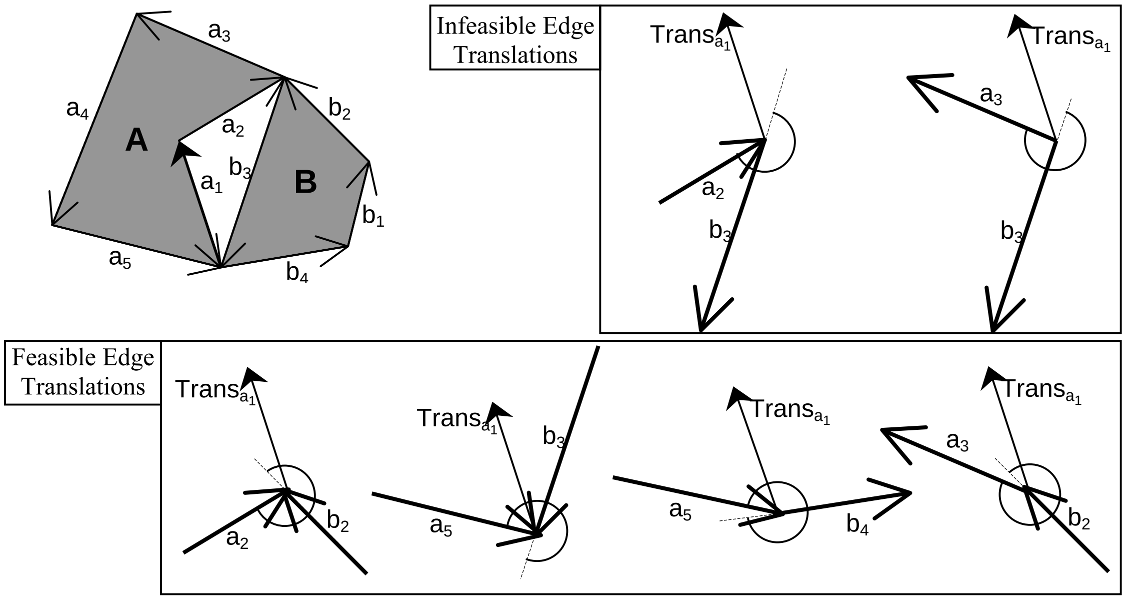


Figure 8.7. Elimination of potential translation vector, a1

At this stage, one (or more) feasible translation vectors have been found. Usually there will only be one translation vector but several may be present under the special circumstances as shown in figure 8.8 (this was an actual screenshot taken from the implementation). The reference point of the orbiting square has been set to the centre to aid clarity.

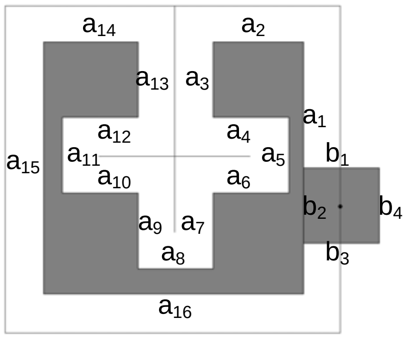


Figure 8.8. Two polygons involving exactly fitting 'passageways'

This introduces another important aspect of the algorithm; we must maintain the edge that was used to generate the previous translation vector. When several feasible translation vectors are present, the edge that is nearest (in edge order) to the previous move is chosen. Therefore, returning to figure 8.8, the square enters the passageway using edge a_3 instead of sliding straight over the opening using edge a_{14}. Then, after sliding into the centre of the stationary polygon, the square has four feasible translation vectors derived from edges a_4, a_7, a_{10} and a_{13}. However, the translation vector derived from edge a_4 will be used because the edge that was used previously was a_3. This is another modification that has been made to the approach of Mahadevan.

8.2.4 Trimming the Feasible Translation Vector

The last step before polygon B can be translated is to trim the translation vector. This important because there may be other edges that can interfere with the translation of the orbiting polygon. Figure 8.9a provides an example whereby applying the entire translation vector results in the two shapes intersecting. In order to prevent the orbiting polygon entering the body of the stationary polygon, the feasible translation, a_7, must be trimmed to edge a_1 (see figure 8.9b).

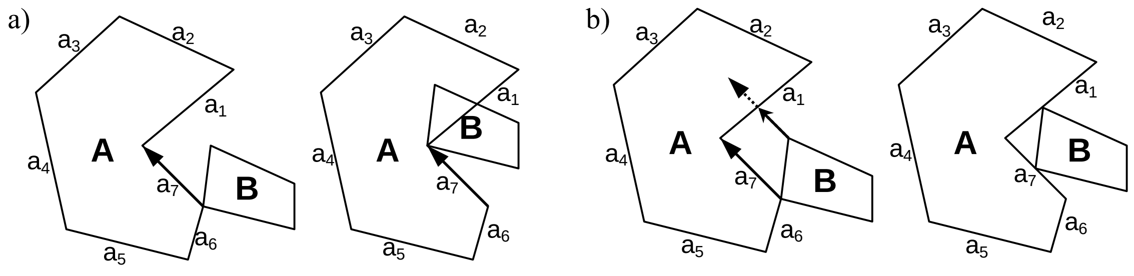


Figure 8.9. A translation vector that requires 'trimming' to avoid intersection

In order to find the correct non-intersecting translation the translation vector is projected at each of the vertices of polygon B and is tested for intersection with all edges of polygon A. This ensures that we correctly identify the vertices of polygon B that will cross into polygon A and reduce the translation distance accordingly using formula **(2)**.

$$\text{New Translation} = \text{IntersectionPt} - \text{Translation}_{\text{StartPt}} \qquad \textbf{(2)}$$

The translation vector must also be projected back from all vertices of polygon A (by translating the end vertex of the translation vector onto each vertex) and then test for intersection with all edges of polygon B to examine if any will cross into polygon B. This is depicted in figure 8.10 using the same example but a different orbiting polygon.

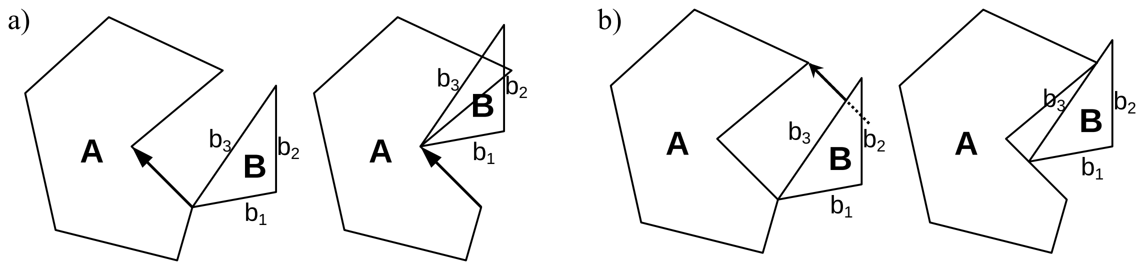


Figure 8.10. Trimming with projections from polygon A

Once again, if there is an intersection, the translation distance should be reduced to reflect this using formula **(3)**.

$$\text{New Translation} = \text{Translation}_{\text{EndPt}} - \text{IntersectionPt} \qquad \textbf{(3)}$$

The new approach is fundamentally the same as used by Mahadevan except that the translation vector is trimmed at the time of intersection whereas Mahadevan keeps the translation vector the same but stores the minimal intersection distance and trims once after all projections have been completed. As the proposed algorithm trims at every intersection, the translation vector becomes shorter throughout the testing process which can reduce the number of intersections that occur due to fast elimination tests through the use of bounding boxes which are considerably faster than performing full line intersection routines. The approach of Mahadevan may need to calculate several more full intersection tests because the entire translation vector is used at each projection and thus could potentially require more computation.

8.2.5 Applying the Feasible Translation Vector

The final step is to append the trimmed translation vector to the end of the partial no-fit polygon created so far and polygon B is translated by the trimmed translation vector. This will move the polygon to the next 'decision' point and the process can restart from the detection of touching edges (section 8.2.1). The only nuance to this process is that a test must be performed to detect if the reference point of polygon B has returned to its initial starting position.

8.3 Start Points

Section 8.2 identified an approach for the orbiting of one polygon around another using edge comparisons and edge sliding. The overall effect of the approach is similar to the algorithm proposed by Mahadevan [**256**]. However, some improvements have been proposed that result in the procedure requiring less computation and allow the approach to be explained more easily. Therefore, the proposed approach currently has the same limitations as Mahadevan's approach (the inability to generate complete no-fit polygons for shapes involving interlocking concavities, jigsaw pieces or holes). Figure 8.11a shows two polygons for which the sliding algorithm alone does not produce the complete no-fit polygon. The polygons have concavities that can interlock with each other to create a further no-fit polygon region but this is never found because of the narrow entrance to the stationary polygon's concavity (see figure 8.11b).

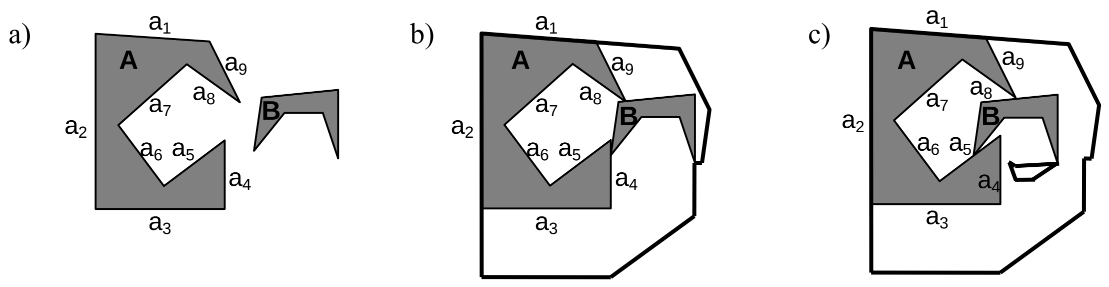


Figure 8.11. Interlocking concavities: a) polygons, b) no-fit polygon using sliding alone, c) the complete no-fit polygon

In this section, an approach is given based on the identification of feasible touching but non-intersecting start positions with which the previously described sliding technique can be employed to generate the remaining inner loops of the no-fit polygon construct. For example, if polygon B can be placed in the location shown in figure 8.11c, then the sliding algorithm can be employed (without change) to generate the internal no-fit polygon region (note that this loop will be a clockwise loop indicating that it is an inner-fit polygon region). Feasible starting positions are found through a modification of the approach detailed in section 8.2. Once again, this involves sliding one polygon around the edges of the other. However, we are not interested in sliding to trace the no-fit polygon but to resolve edge intersections whilst translating along a particular edge vector.

In order to explain the process, an example of finding the start positions along one edge of polygon A is given. This process can then easily be reproduced for each edge of polygon A and polygon B to create all feasible starting positions to allow polygon B to orbit around polygon A in order to create the entire no-fit polygon.

Given an edge, e, of polygon A, the potential starting positions of an orbiting polygon B along e can be found. The process involves translating polygon B such that each of its vertices, in turn, are aligned to the start vertex of e. The following steps are performed for each position. If the polygons do not intersect in this position, then this is noted as a feasible start position for the two polygons. If polygon B intersects with polygon A then further testing and edge sliding must be performed to traverse along edge e until a non-intersecting position is found or the end of the edge is reached. In such an instance, the first test involves examining whether the two connected edges of polygon B (joined at the touching vertex) are both right or parallel to the edge e. If either of polygon B's connected edges are left of e then they will translate on the inside of polygon A and can be eliminated immediately because sliding along the vector derived from edge e will never yield a feasible starting position.

Figure 8.12a shows an example of where the connected edge pair, b_3 and b_4, are eliminated from sliding along a_5 because b_4 is left of a_5 (note that this will not be eliminated when using edge b_4 of the polygon B and connected edges a_4 and a_5 of polygon A). Figure 8.12b shows an example of where both connected edges, b_1 and b_2, are right of edge a_5.

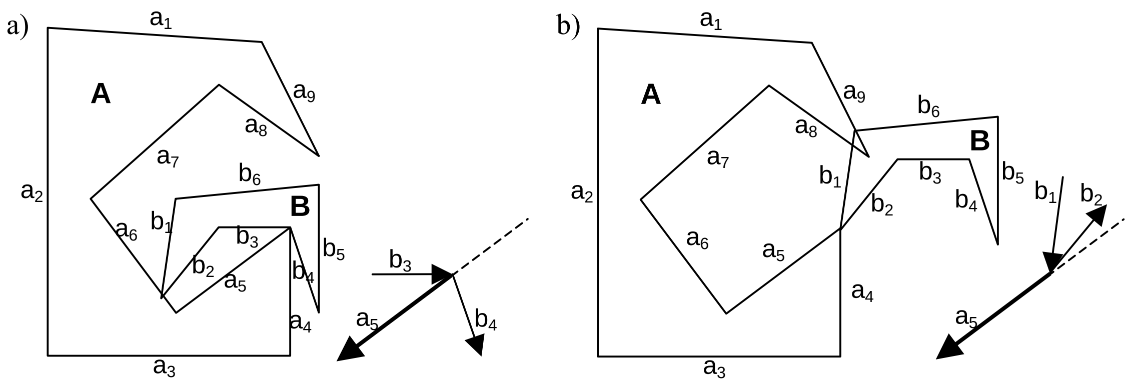


Figure 8.12. Vertex alignment of polygon B to polygon A using edge a_5: a) invalid alignment, b) valid alignment

Now assuming both connected edges are right of edge e and polygons A and B intersect we can attempt to resolve the overlap by translating the orbiting polygon along the translation vector defined by e. As with the previous section, trimming can now be employed to this vector (the procedure is identical). The resultant translation vector can then be applied to slide polygon B along edge e and then another intersection test is performed. If the two polygons still intersect then the process repeats with the translation vector derived from the touching point to the end vertex of edge e. If the intersection has been resolved then the reference point of polygon B is a potential start position. If the entirety of edge e is traversed and there is still an intersection then no feasible starting position can be found along edge e and the aligned vertex / connected edges of polygon B. The next vertex of polygon B is then considered and so forth until all edge vertex possibilities have been examined. Note it is also important to examine the edges of polygon B with the vertices from polygon A but as this is an identical procedure and will not be discussed. Figure 8.13 shows this process on the example presented in figure 8.12b. When aligned using a_5 and the vertex connecting edges b_1 and b_2, the two polygons are initially intersecting so we need to resolve the intersection along the vector derived from edge a_5.

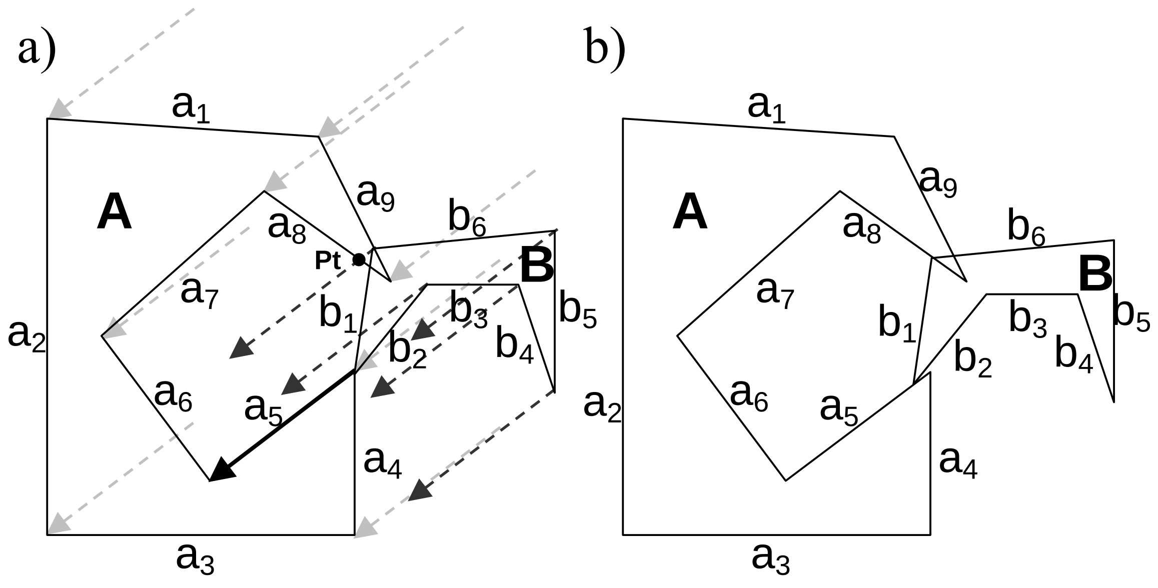
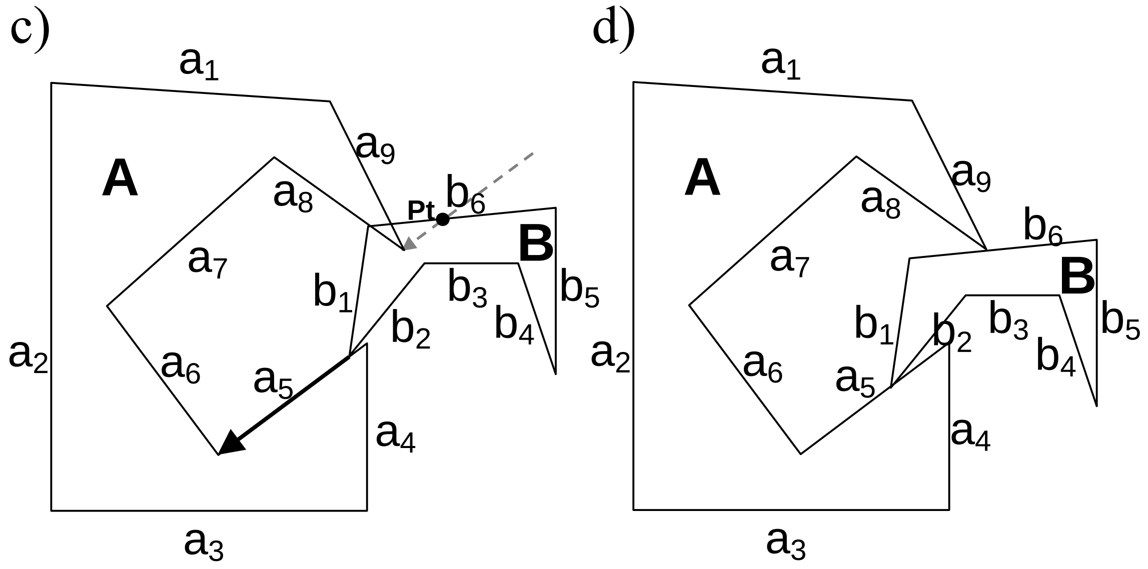



Figure 8.13. Start point generation process

Figure 8.13a shows the trimming process and identification of the closest intersection point, Pt. Polygon B is then translated by the trimmed translation vector. The resulting position is shown in figure 8.13b. The polygons are still intersecting so the procedure must be repeated. The translation vector is calculated from the touching point to the end vertex of edge a_5 and is then trimmed as shown in figure 8.13c (only the important trim is shown). After applying this translation to polygon B, the polygons are no longer intersecting and therefore a feasible start point has been found and the standard orbital approach can be employed once more to generate the no-fit polygon loop.

An optimisation that has been included in the implementation is that the outer no-fit polygon loop is calculated using the approach in section 8.2 and *then* the starting position procedure is applied on the untraversed / unflagged edges from both polygon A and polygon B in turn. During the process, as soon as a feasible start position is found, the inner loop of the no-fit polygon can be calculated using the orbital algorithm (as before, the edges are flagged as they are traversed). This ensures that the algorithm does not attempt to find start points which already exist within one of the no-fit polygon loops that have already been calculated. Therefore, this avoids unnecessary calculations because more edges become flagged as new no-fit polygon loops are produced, thus reducing the computational requirements for generating new feasible start positions. The procedure repeats until all edges have been seen or no more feasible start positions are available and the complete no-fit polygon is returned.

8.4 Pseudocode

The no-fit polygon algorithm can be described using the following pseudocode given the functionality outlined in the previous sections:

```
Input : Polygons A and B

Pt$_{A(ymin)}$ = point of minimum x value of polygon A
Pt$_{B(ymax)}$ = point of maximum x value of polygon B
IFP = initial feasible position
Bool bStartPointAvailable = true;
Point NFPLoopStartRefPoint;
Point PolygonB_RefPoint;
Array[Line[]] nfpEdges; // an array of arrays of lines to store NFP edges
Int loopCount = 0; // counter for number of loops in NFP

Begin

Place polygons in IFP using translation Pt$_{A(ymin)}$ – Pt$_{B(ymax)}$
NFPLoopStartRefPoint = Pt$_{B(ymax)}$;
PolygonB_RefPoint = Pt$_{B(ymax)}$;

While(bStartPointAvailable)
{
    bStartPointAvailable = false;
    // find touching segments & touching points on those, generate touching structures

    Touchers[] toucherStructures = FindTouchers(A, B);

    // Eliminate non feasible touchers, ones that cause immediate intersection
    Touchers[] feasibleTouchers = CanMove(A, B, toucherStructures);

    // Trim feasible translations against polygon A and B
    Touchers[] trimmedTouchers = Trim(feasibleTouchers, A, B);

    // Sort trimmed translations by length
    Touchers[] lengthSortedTouchers = Sort(trimmedTouchers);

    //Translate polygon B along longest feasible translation vector
    B.Translate(lengthSortedTouchers[0].Translation);

    // Add translation to nfpEdges & mark traversed edge on static
    nfpEdges[loopCount].Add(lengthSortedTranslations[0].Translation);
    A.MarkEdge(lengthSortedTranslations[0].StaticEdgeID);

    if(NFPLoopStartRefPoint == PolygonB_RefPoint) // completed an NFP loop
    {
        Point nextStartPoint;
        // find next feasible start point – reset PolygonB_RefPoint to relevant point
        bStartPointExists = FindNextStartPoint(A,B,&nextStartPoint, &PolygonB_RefPoint);

        if(bStartPointExist)
        {
             //Translate polygon B to nextStartPoint
             B.Translate(PolygonB_RefPoint - nextStartPoint);

             NFPLoopStartRefPoint = nextStartPoint;
             loopCount++;
        }
    }
    else
    {
        bStartPointAvailable = true; // allow edge traversal to continue
    }
}

NFP$_{AB}$  = Complete(nfpEdges); //Reconstitute NFP from nfpEdges

End
```

Pseudocode for the **`FindNextStartPoint`** function:

```
Input : PolygonA, PolygonB, reference to Point nextStartPoint, reference to Point
PolygonB_RefPoint

Int A_EdgeCount = PolygonA.EdgeCount;
Int B_EdgeCount = PolygonB.EdgeCount;
Edge StaticEdge;
Edge movingEdge;


Begin

for(int i = 0; i < A_EdgeCount; i++)
{
    if(PolygonA.IsEdgeMarked(i))
    {
        continue;
    }
    else
    {
        staticEdge = PolygonA.GetEdge(i);
    }

    for(int j = 0; j < B_EdgeCount; j++)
    {
        movingEdge = PolygonB.GetEdge(j);

        // translate PolygonB so that movingEdge start is on start of the static edge
        PolygonB.Translate(movingEdge.Start – staticEdge.Start);

        Bool bFinishedEdge = false;
        Bool bIntersects = PolygonB.IntersectsWith(PolygonA);

        while(bIntersects AND !FinishedEdge)
        {
            //Edge slide until not intersecting or end of staticEdge reached
            Toucher currentToucher = MakeToucher(staticEdge.Start);
            Toucher trimmedToucher = Trim(currentToucher, PolygonA, PolygonB);
            PolygonB.Translate(trimmedToucher.Translation);

            bIntersects = PolygonB.IntersectsWith(PolygonA);

            if(bIntersects)
            {
                if(movingEdge.Start == staticEdge.End)
                bFinishedEdge = true;
            }
        }

        // mark the traversed edge as seen (whether edge start point found or not)
        staticEdge.Mark(true);

        if(!bIntersects)
        {
            // set the references to the points passed in to be the nextStartPoint
            // and return true;
            nextStartPoint = movingEdge.Start;
            PolygonB_RefPoint = movingEdge.Start;

            return true;
        }
    }
}

// all edges on moving tried on all unmarked edges on static, nothing found – no more starts
return false;

End
```

8.5 Problem Cases

In this section each of the problem cases that cause difficulties with other methods are discussed and it is shown how the proposed algorithm is able to handle them without specific case-by-case implementations. In each figure of this section, the stationary and orbiting polygons are represented by dark and light grey shading respectively and the reference point of the orbiting polygon is shown as a black dot. This is used to trace the no-fit polygon loops (numbered).

8.5.1 Interlocking Concavities

The interlocking of concavities is a typical problem case for the previous orbital sliding approaches of the literature such as (Mahadevan, 1984). Figure 8.14 displays a screenshot after the algorithm has generated the complete no-fit polygon of identical shapes with one rotated through 180°. The shapes in these orientations involve many different interactions of the concavities and, therefore, multiple feasible start points. The no-fit polygon of these two shapes results in *six* loops within the no-fit polygon (one outer loop, 1, and five internal loops, 2-6).

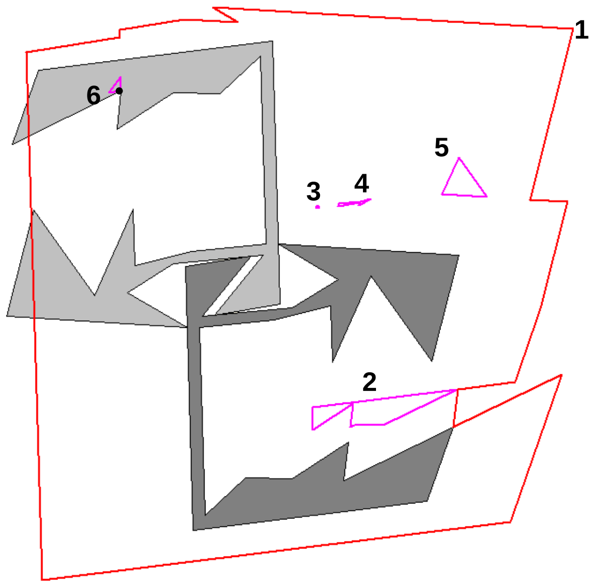


Figure 8.14. Multiple interlocking positions

8.5.2 Exact Fit: Sliding

The next case involves the sliding through exactly fitting "passageways" which cannot be handled by either the Minkowski sum approach of (Bennell, Dowsland and Dowsland, 2001) or the orbital approach given in (Mahadevan, 1984). An extension was proposed to Mahadevan's algorithm (in section 8.2.3) that can deal with such problems through the maintenance of the previously traversed edge which enables consecutive edges of the stationary polygon to be used in the next translation iteration. An example is shown in figure 8.15 (also see figure 8.8).

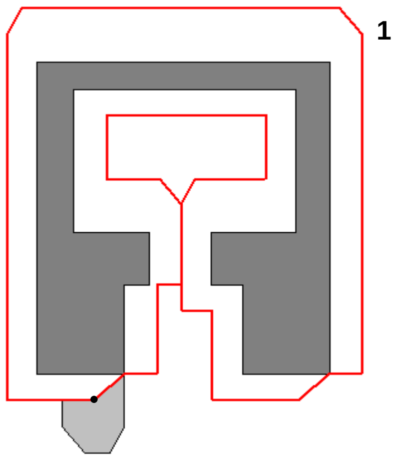


Figure 8.15. Exact fit sliding through a "passageway"

8.5.3 Exact Fit: Jigsaw Pieces

The problem case involving jigsaw pieces that link together exactly (also called "lock-and-key") is another form of the interlocking concavity case. However, the distinguishing feature of jigsaw pieces is that they fit together with no movement, thus creating a singular feasible point within the no-fit polygon (rather than an internal loop in the interlocking concavity case). In the proposed approach, this position is found by the generation of feasible starting positions. Of course, the algorithm will still try to slide the orbiting polygon along edges but, in this case, there is no feasible translation vector which distinguishes that this position is just a singular feasible point location (see figure 8.16). Most of the previous approaches have suffered from such degeneracy including: the Minkowski sum approach where no boundary exists to identify lock-and-key positions within the no-fit polygon, digitisation which suffers from loss of accuracy, convex decomposition whereby it is not possible to find such positions through recombination alone and the sliding algorithm of Mahadevan (which suffers identical difficulties to that of interlocking concavities).

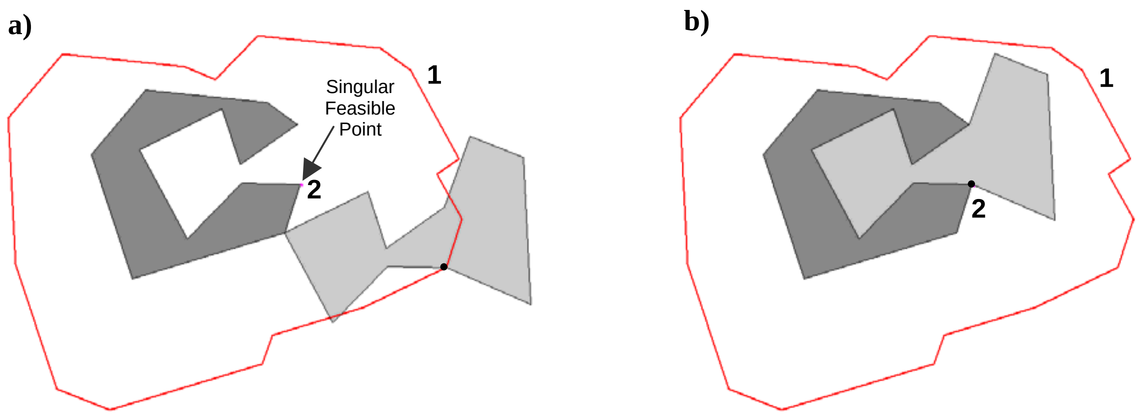


Figure 8.16. Jigsaw pieces: a) outer NFP loop, b) singular feasible internal position

8.5.4 Holes

There have been few approaches within the literature that operate on polygonal shapes containing holes. This could be due to the difficulty of generation process for no-fit polygons involving holes. However, through the detection of feasible starting positions, the no-fit polygon can be generated easily and, more importantly, completely. Figure 8.17 shows an example involving a shape with two holes and a shape that can be placed within several distinct regions of the holes. The figure shows a mixture of cases involving holes and interlocking concavities: loop 1 is the outer no-fit polygon loop, loops 2,3,5,6,7 are all hole cases, and loops 4 and 8 are cases whereby the concavity of the smaller orbiting shape interacts with the narrow gaps within the larger hole of the stationary shape.

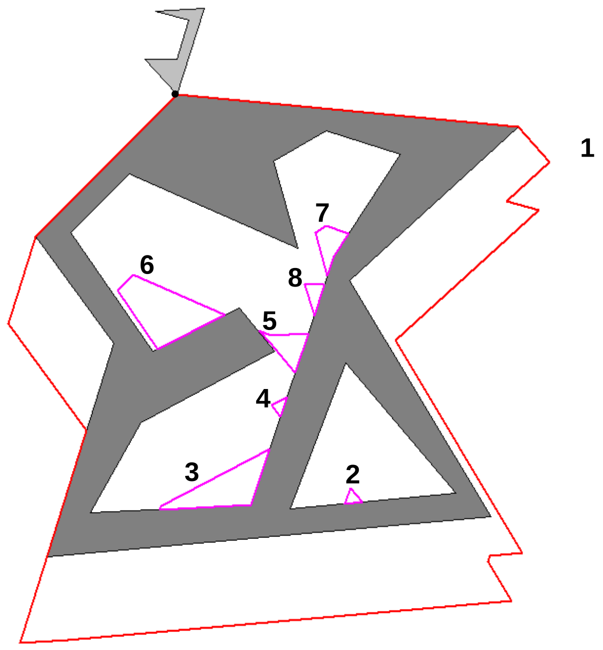


Figure 8.17. No-fit polygon of two polygons, one of which involves multiple holes

Convex decompositions can result in a larger number of pieces and become more difficult with the introduction of holes. Therefore union or recombination is also compromised due to the larger number of intersecting sub-no-fit polygon elements that need to be untangled. Digitisation can generate no-fit polygons with holes but this is achieved with inaccuracy due to the discretisation of a continuous space. The phi-function approach of Stoyan could deal with many of the degeneracies through the resolution of primary objects. However, this approach will ultimately impact on the efficiency of intersection testing as many separate tests must be performed (especially with very complex shapes). Previous orbital approaches have only produced the outer no-fit polygon loop and Minkowski sum approaches can become difficult due to the untangling of edges. The author is unaware of any previous no-fit polygon generation methods that have specifically presented no-fit polygon constructs for shapes involving such degeneracies.

8.6 Generation Times on the Benchmark Data

In order to demonstrate the speed and capabilities of the proposed no-fit polygon procedure, the generation times for all of the literature benchmarks of table 2.3 & 2.4 and the new benchmark problems containing shapes consisting of lines only, Profiles5 – Profiles10 are reported. For each problem the total generation time and number of no-fit polygons generated per second are given (see table 8.2). The "Logical Total Number of Shapes" (column E) is found by E = B * D and the "Total Number of NFPs" (column F) is calculated by $F = E^2$. All experiments have been conducted on a Pentium 4 2GHz processor with 256MB RAM.

A	B	C	D	E	F	G	H
Dataset	**Number of Different Shapes**	**Rotational Constraints**	**Number of Rotations per Shape**	**Logical Total Number of Shapes**	**Total Number of NFPs**	**Total Generation Time (s)**	**NFPs Per Second**
Albano180	8	180	2	16	256	0.32	800
Albano90	8	90	4	32	1024	0.71	1442
Blasz1	7	180	2	14	196	0.21	933
Blasz2	4	90	4	16	256	0.19	1347
Dagli	10	90	4	40	1600	0.93	1720
Dighe1	16	90	4	64	4096	1.28	3200
Dighe2	10	90	4	40	1600	0.62	2581
Fu	12	90	4	48	2304	0.52	4431
Jakobs1	25	90	4	100	10000	5.57	1795
Jakobs2	25	90	4	100	10000	5.07	1972
Mao	9	90	4	36	1296	1.41	919
Marques	8	90	4	32	1024	0.79	1296
Poly1a	15	90	4	60	3600	1.37	2628
Poly2a	15	90	4	60	3600	1.37	2628
Poly3a	15	90	4	60	3600	1.37	2628
Poly4a	15	90	4	60	3600	1.37	2628
Poly5a	15	90	4	60	3600	1.37	2628
Poly2b	30	90	4	120	14400	7.54	1910
Poly3b	45	90	4	180	32400	27.14	1194
Poly4b	60	90	4	240	57600	68.45	841
Poly5b	75	90	4	300	90000	141.90	634
Shapes	4	90	4	16	256	0.38	674
Shapes0	4	0	1	4	16	0.11	145
Shapes1	4	180	2	8	64	0.19	337
Shirts	8	180	2	16	256	0.33	776
Swim	10	180	2	20	400	6.08	66
Trousers	17	180	2	34	1156	0.73	1584
Profiles6	9	90	4	36	1296	0.86	1507
Profiles7	9	90	4	36	1296	0.58	2234
Profiles8	9	90	4	36	1296	0.56	2314
Profiles9	16	90	4	64	4096	44.30	92
Profiles10	13	0	1	13	169	0.55	307

Table 8.2. No-fit polygon generation times for 32 datasets of the literature

Table 8.2 shows that the no-fit polygon constructs can be generated within reasonable time frames on most of the benchmark data. Typically the algorithm can generate about 1000 no-fit polygons per second although this can vary drastically depending on the number of line segments in each problem. For example, the two problems with the lowest no-fit polygons generated per second, "Swim" and "Profiles9", involve pieces with many line segments (up to 67) due to approximation of arcs and, further to this, "Profiles9" contains several shapes with holes (see figure 8.18).

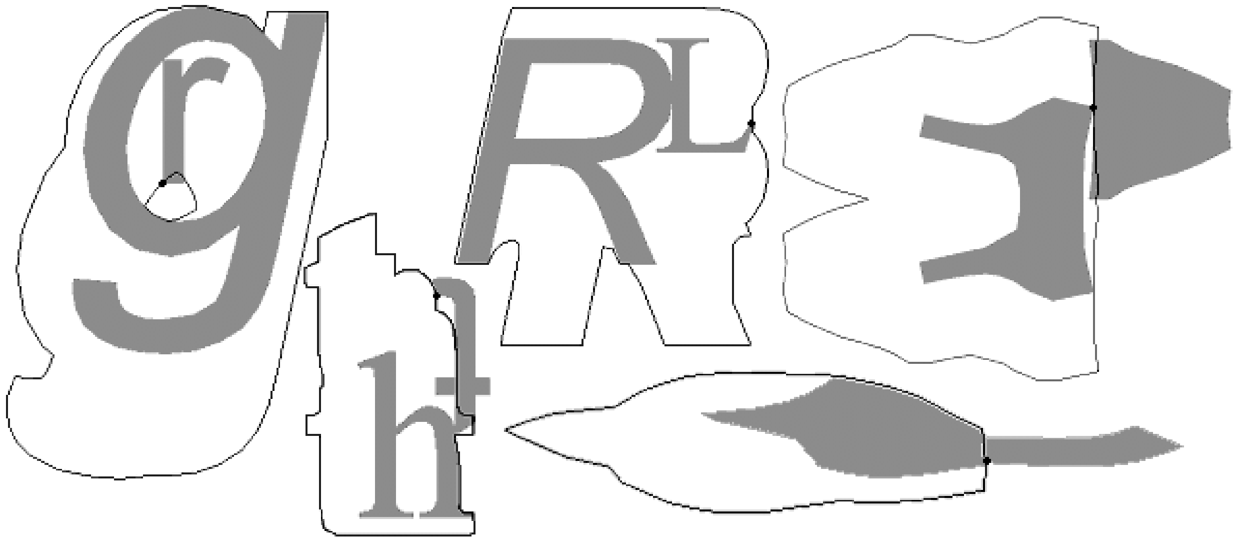


Figure 8.18. A selection of pieces and no-fit polygons from the "Profiles 9" (letters) and "Swim" datasets

The slowest overall generation times occur within the Poly4b and Poly5b problems and are simply due to the large number of no-fit polygons that must be generated. As mentioned in section 8.3.2, other reported generation times for available benchmark problems of the literature can be found in Bennell, Dowsland and Dowsland (2001) [**239**]. Table 8.3 shows a comparison of the generation times of the new orbital approach to the Minkowski based approach of Bennell, Dowsland and Dowsland using a DEC Alpha 3000/400. Each problem is set to use 0°

rotations for comparative purposes (polygons in the orientation they are drawn) as this was the rotational constraint that was used in Bennell, Dowsland and Dowsland (2001).

Dataset (0° rotation only)	Total Generation Times (seconds)	
	Minkowski Approach (Bennell et al, 2002)	Orbital Approach (Presented)
DS1	0.32	0.23
DS2	0.22	0.04
DS3 (Shirts)	0.23	0.17
DS4 (Shapes0)	0.38	0.11
DS5 (Blasz1)	0.36	0.13

Table 8.3. Comparison of generation times of the Minkowski approach of Bennell, Dowsland and Dowsland (2001) with the presented orbital approach on 5 literature datasets.

The table shows that the new orbital approach generates the no-fit polygons quicker than the Minkowski sum approach for all of the five datasets. Although there is an obvious difference in the computing power available to the two approaches, the fact that they both perform with very quick execution times shows that the overhead incurred in generating no-fit polygons is almost negligible. The proposed algorithm is able to quickly and robustly generate all of the no-fit polygons for the literature benchmark problems (including several datasets from the textiles and metal cutting industries).

Whilst very large problems may require a few minutes to generate all of no-fit polygons, such as the Poly4b and Poly5b test problems (requiring 68 and 142 seconds respectively), for repeated automatic nesting it is likely that this overhead will more than be recovered by being able to use the faster no-fit polygon based intersection testing.

8.7 Summary

In this chapter, a complete and robust implementation of the orbital no-fit polygon has been presented. The capabilities of the approach have been demonstrated using several examples of degenerate cases that the previous methods cannot resolve easily. It was shown that the execution times required to produce the no-fit polygons on the available benchmark problems from the literature are reasonable and that they are consistent with other approaches such as minkowski sums (Bennell, Dowsland and Dowsland, 2001) [**239**].

As far as the author is aware, this is the first time that a no-fit polygon has been so successful and rigorously investigated for robustness on the known degenerate cases. The approach is also able to robustly generate all of the no-fit polygons for the entire set of the literature benchmark problems and the five line-only new benchmarks that have been introduced in this thesis. Hopefully, this will help in the further dissemination of the benefits of using the no-fit polygon (as opposed to traditional trigonometry based approaches) within both industry and the academic community.

In the next chapter, the presented no-fit polygon algorithm is modified to correctly handle shapes that consist of circular arcs. Also shown is how the trigonometry overlap resolution of the packing approach of chapter six can be simplified and significantly improved in terms of speed using the no-fit polygon. The additional speed also enables more of the search space to be evaluated and, thus, better quality solutions can be produced.

CHAPTER NINE

Automated Packing using the No-Fit Polygon

This chapter unifies the work from the previous two chapters to produce a faster packing strategy that could be used in an industrial setting. It has been stated throughout this thesis that the correct handling of arcs and holes was an important feature of this research. The new no-fit polygon generation algorithm of chapter eight is not suitable due to it being incapable of dealing with shapes that contain arcs. Therefore, the no-fit polygon generation algorithm is modified to give extra functionality in order to deal with shapes that contain arcs. The author is unaware of any previous no-fit polygon algorithm that can generate no-fit polygons for such shapes. The packing algorithm presented in chapter seven is then reimplemented to utilise the newly proposed arc capable no-fit polygon for intersection testing and overlap resolution (instead of the previous trigonometric techniques). The concept of the packing strategy remains the same whereby local searches are employed to generate input orderings which are then evaluated by the new bottom-left-fill approach. In fact, both the trigonometric and no-fit polygon implementations of the bottom-left-fill approach will produce identical layouts when presented with the same input orderings although, as will be shown, the packing algorithm of this chapter can obtain the layout in much quicker time. Experimentation is repeated for all of the benchmark datasets to highlight this improvement in speed. This yields improved average and best-known solutions for the majority of the literature benchmarks using just 5 minutes computation time.

9.1 Modifying the No-Fit Polygon Generation Algorithm to Handle Arcs

A complete and robust algorithm for the generation of no-fit polygons was presented in chapter eight which offered a considerable improvement over previous approaches. However, although the algorithm created accurate no-fit polygons and could deal with all of the known degenerate cases (for the first time), it cannot be employed for shapes that contain arcs in the current form. Indeed, the use of arcs within the academic literature is very scarce and, furthermore, no-fit polygon generation of shapes containing arcs has not previously been attempted.

In order to explain how the algorithm has been modified to generate no-fit polygons for shapes involving arcs, two simplified cases are used: i) two circles and ii) a circle and a line (see section 9.1.1). After this, each stage of the no-fit polygon construction algorithm of chapter eight is revisited and details are given for the additional handling of arcs (see section 9.1.2). Finally, section 9.1.3 provides some examples of no-fit polygons that have been generated using the modified algorithm on shapes containing arcs.

9.1.1 Simplified Cases using Circles

In order to describe the arc modifications needed for no-fit polygon generation, it is beneficial to firstly examine the cases involving circles. These concepts are then generalised to arcs and are used throughout the no-fit polygon modifications of this chapter. The first case that will be examined involves the no-fit polygon of two circles. The no-fit polygon of two circles, A and B (the convention of shape B moving around shape A is maintained), is defined as the path followed by the reference point of B (the circle's centre point) when circle B orbits circle A. The no-fit polygon of two circles is a circle of radius equal to the sum of A and B's radii (see figure 9.1). From figure 9.1, it can also be seen that NFP_{AB} shares circle A's centre point.

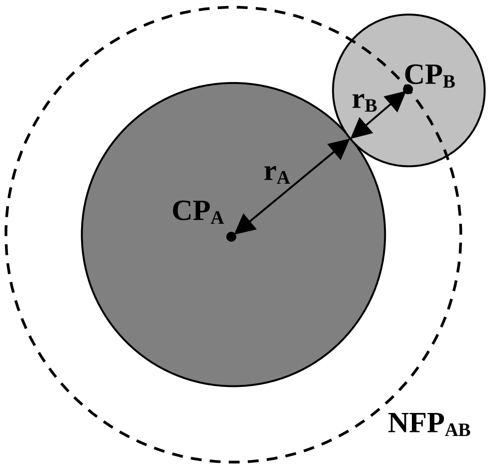


Figure 9.1. No-Fit Polygon of Two Circles

This draws an interesting observation; the edges within a line-only no-fit polygon remain the same as the shape edges from which they were derived (albeit trimmed). However, this is no longer the case with arcs and can be seen from figure 9.1 whereby the no-fit polygon does not take the form of the shape edges from which it was derived. This is because the touching position of each arc moves position on *both* arcs as B moves around A (see figure 9.2).

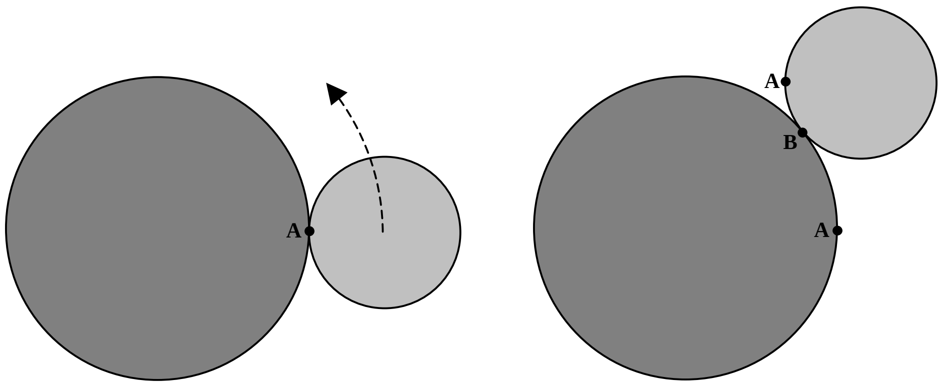


Figure 9.2. Movement of the Touching Positions of the Circles

The circular no-fit polygon example presented in figure 9.1 can be generalised to convex arcs by eliminating the segment of the no-fit polygon for which the arcs will not touch on their radial extents. This can be created by examining the start and end angles of the two arcs (figure 9.3a).

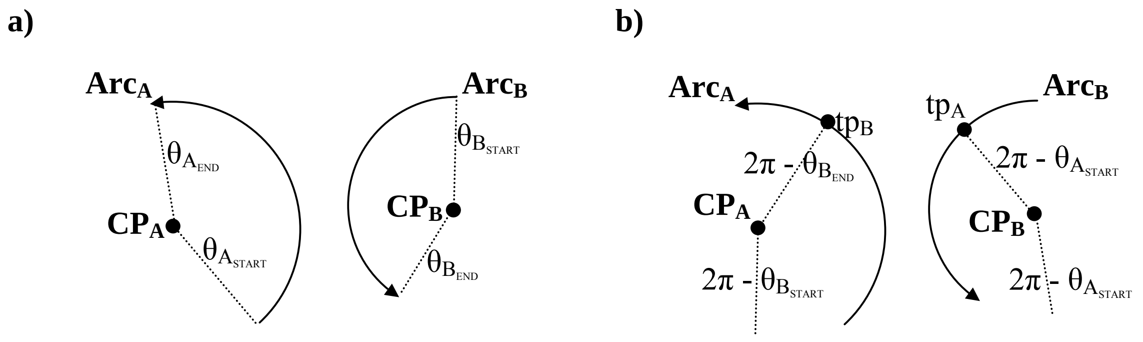


Figure 9.3. Finding circle tangents (A and B are convex)

From figure 9.3b, the angular position of tangent point tp_A on arc B's parent circle is identified as the angle represented by $2\pi - \theta_{AStart}$. If this angle exists on arc B then it is a valid tangent. Likewise, tangent point tp_B's angular position on parent circle A is given by $2\pi - \theta_{BEnd}$. The partial no-fit polygon is then defined as the arc of radius $r_A + r_B$ through the angular range of tp_A through to tp_B. Figure 9.4 shows the start position whereby arc B touches arc A at tp_A and, after completing the partial no-fit polygon, the arcs touch at tp_B.

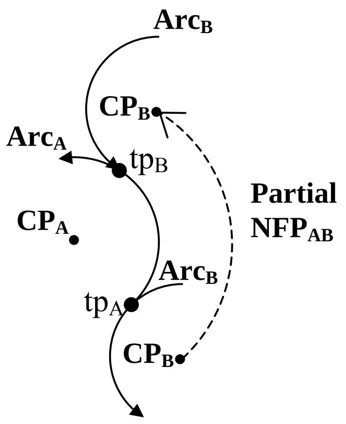


Figure 9.4. Creating the partial no-fit polygon circle from the arc tangents

If there are no tangent points, then the two arcs are unable to create any of the circular no-fit polygon (see figure 9.5).

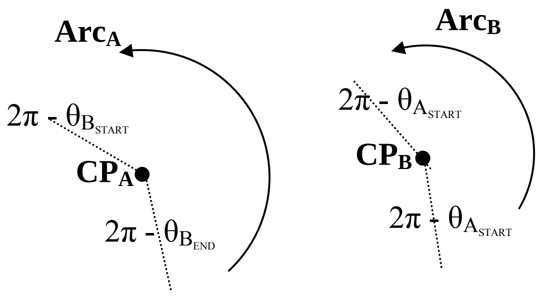

Arc_A
$2\pi - \theta_{B_{START}}$
CP_A
$2\pi - \theta_{B_{END}}$
Arc_B
$2\pi - \theta_{A_{START}}$
CP_B
$2\pi - \theta_{A_{START}}$

Figure 9.5. No arc tangent points available

A further example must be examined whereby one of the circles exists in its concave form (i.e. a hole) as this will allow for generalisations to be made for concave arcs. Figure 9.3 shows an example of the no-fit polygon produced when a circle travels around the inside edge of another circle. In figure 9.6, the no-fit polygon is a circle once again, but this time it has a radius equal to r_A - r_B. Its centre point is the same as A's centre point.

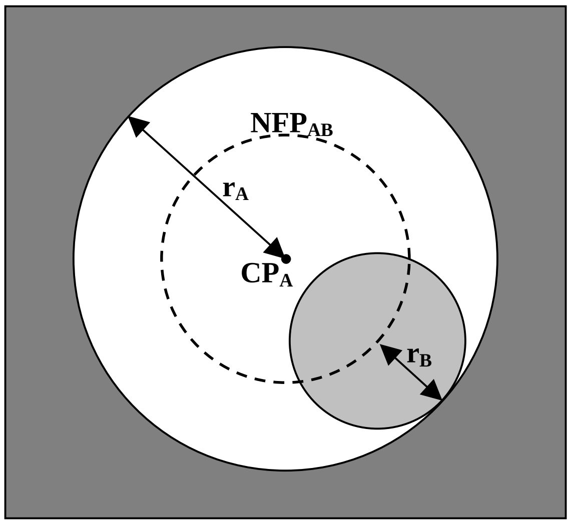


Figure 9.6. No-fit polygon of two circles in the concave case

To create the partial circular no-fit polygon in this instance, the same tangent point procedure ensues except that the start/end angles are not subtracted from 2π. An example is shown in figure 9.7.

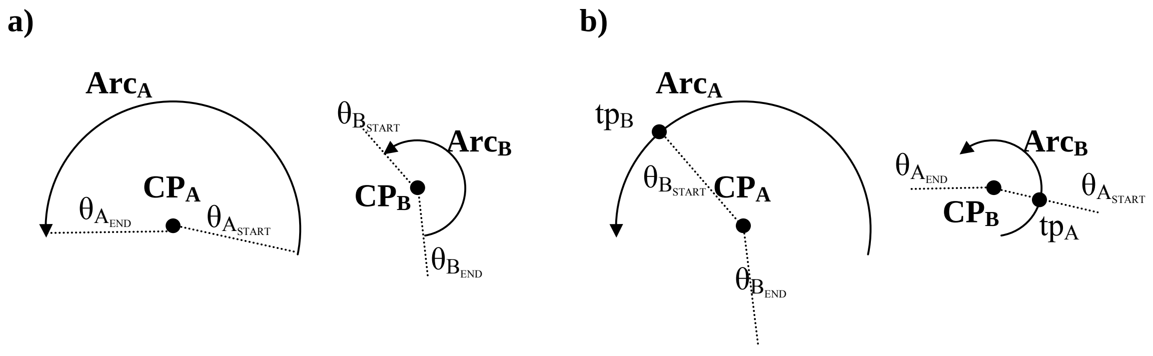


Figure 9.7. Finding circle tangents (A is concave, B is convex)

The partial circular no-fit polygon is defined by the arc of radius rA - rB and start and end angles defined by the tangent point angles, tp_B and tp_A respectively.

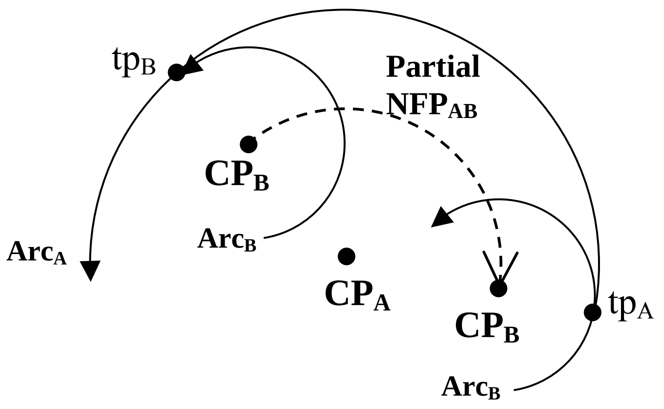


Figure 9.8. Creating the partial no-fit polygon (A is concave, B is convex)

In the case where arc A is convex and B is concave, the partial circular no-fit polygon can be obtained by the same procedure as above. For this, arc A and B must be switched such that arc B is now the convex arc and A is concave. Now by rotating the resultant partial circular no-fit polygon by 180°, the correct no-fit polygon arc for the original situation is produced. As discussed in chapter eight, this is because as B moves in one direction, A can also be thought of as moving relatively in the opposite direction. If one concave and one convex arc are involved then a partial circular no-fit polygon is only possible if the concave arc has a larger radius than the convex arc.

Note: it is not physically possible for two concave arcs to orbit each other via the circular no-fit polygon as the shapes with which they belong would have to be intersecting.

9.1.2 Arc Modifications for the No-Fit Polygon Generation Algorithm

Now that the concept of the circular no-fit polygon has been described, the modifications that are required to extend the previously described no-fit polygon generation algorithm are presented. The modifications are described using the same stages as the previous chapter:

Section 9.1.2.1 - Detection of Touching Edges (Arc Modifications)

Section 9.1.2.2 - Creation of Potential Translation Vectors (Arc Modifications)

Section 9.1.2.3 - Finding a Feasible Translation (Arc Modifications)

Section 9.1.2.4 - Trimming the Feasible Translation (Arc Modifications)

Section 9.1.2.5 - Applying the Feasible Translation (Arc Modifications)

9.1.2.1 Detection of Touching Edges (Arc Modifications)

In itself, the detection of touching edges does not require modification of the previous no-fit polygon generation algorithm except that the geometry must be able to detect that lines are touching arcs or that arcs are touching other arcs. Once again the touching edge pairs are created (as in section 8.2.1) and therefore, it is also important to identify the touching point of the touching edges. With concave arcs this opens the possibility of two edges touching at two separate points. However, this additional complication can be eliminated by storing two sets of touching edge pairs as shown in figure 9.9.

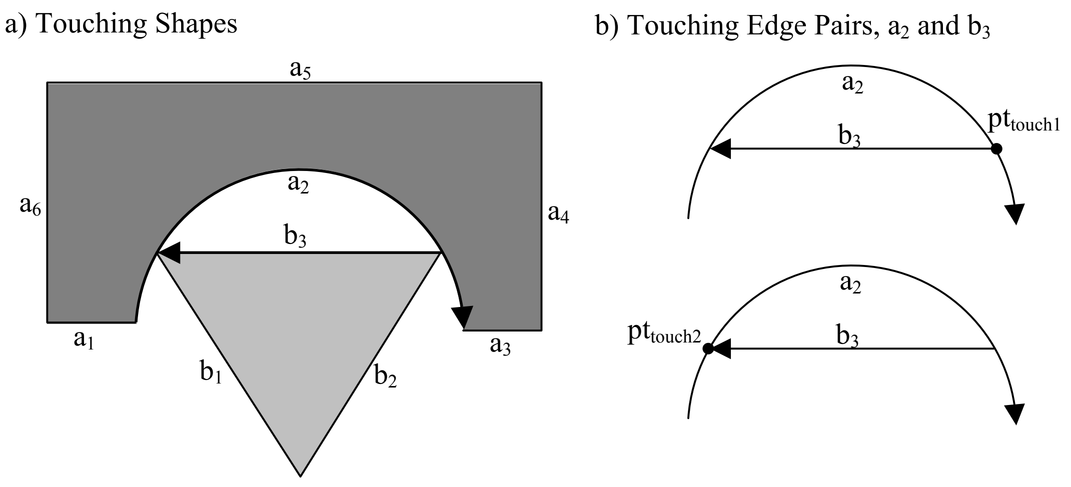


Figure 9.9. Touching edges with two distinct touch positions (edge a_2 and b_3)

In figure 9.9a, both the start and end points of line, b_3, touches the concave arc edge, a_2. Therefore, two touching entries are stored in the touching edge pair list to reflect this and are shown by figure 9.9b (of course, edges b_1 and b_2 touch a_2 and would also be included in the touching edge pair list). There are no further modifications for this stage of the algorithm.

9.1.2.2 Creation of Potential Translation Vectors (Arc Modifications)

The next stage is to create potential translation vectors using the touching edge pairs. In this stage the inclusion of arc edges requires modifications based on the notion of the partial circular no-fit polygon when two arcs are involved (described in section 9.1.1). Further to this, we must also create potential translation vectors for touching edge pairs consisting of a line and an arc.

Potential Translation Vectors for Touching Pairs Involving Two Arcs

Firstly, the situation for producing potential translation vectors for touching arc pairs will be addressed. As already discussed, the circular no-fit polygon can easily be created when convex arcs are touching at their tangent points. This is when the distance between the two arc centres is equal to the sum of the two radii (the maximal distance where the arcs can touch). In the case of touching concave and convex arcs, the concave radius must be greater than the convex radius and the distance between the two arc centres must equal to the convex radius subtracted from the concave radius. In these situations, the potential translation is simply the partial circular no-fit polygon (as defined in section 9.1.1).

Of course, arcs can also touch such that the distance between the centre points does not allow the partial circular no-fit polygon to be used. Examples of this situation are shown in figure 9.10a with touching convex arcs and figure 9.10b with touching concave and convex arcs.

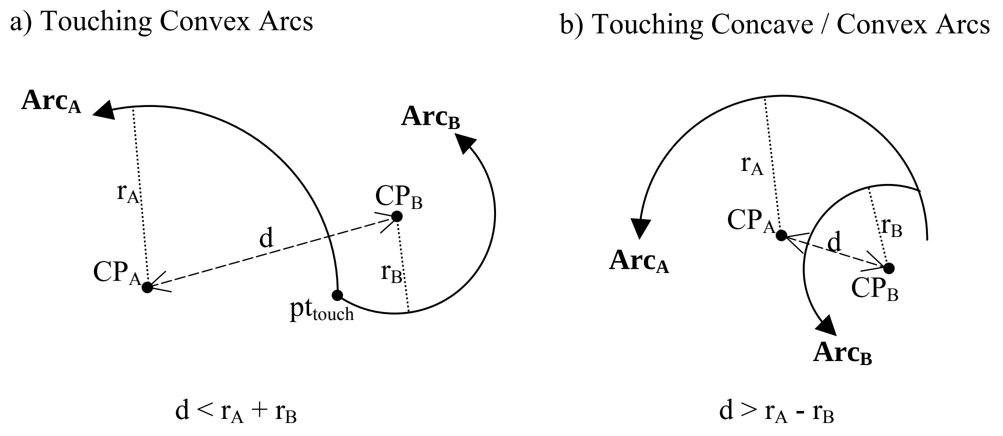


Figure 9.10. Non-tangential touching arcs

In such situations two potential translations can be derived from firstly the stationary arc, A, and then the orbiting arc, B. There are three possible cases (note the similarity to figure 8.4): i) the two arcs both touch on start/end points, ii) arc B's start/end point touches the middle of arc A, or iii) arc A's start/end point touches the middle of arc B.

Cases (ii) and (iii) will be examined first as these were the most simple within the line case. In section 8.2.2, given two touching lines, A and B, the potential translation vector for case (ii) was the line defined by the touching point to the end point of line A (stationary line). The same principle is used for the arc case (ii), the potential translation vector is the arc defined by the touch point to the end point of arc A (stationary arc) which also has the same centre point and radius of arc A. Figure 9.11 shows an example of a potential translation vector for case (i).

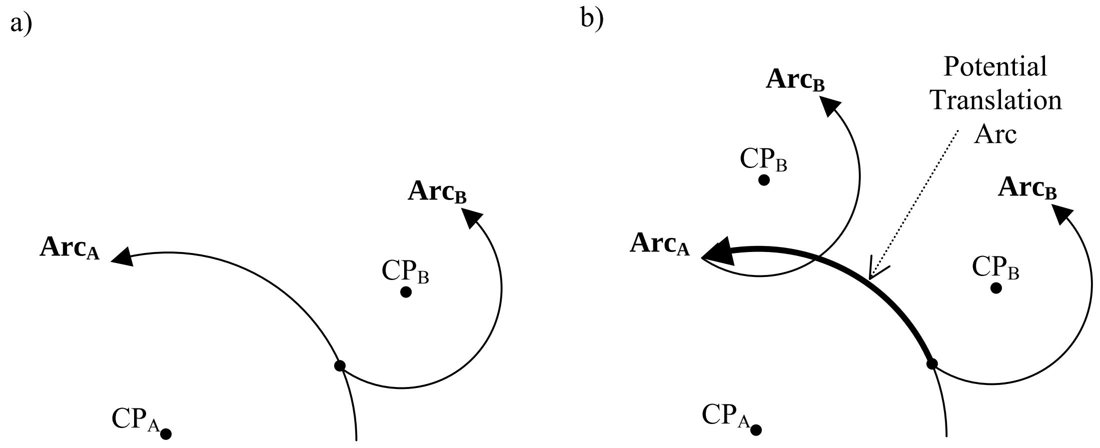


Figure 9.11. Potential translation vector derived in case (ii): arc B's start/end point touches the middle of arc A

However, as shown in figure 9.11b, this potential translation could result in arc B intersecting with arc A. On further examination, this can be explained because a tangent point exists before the end point of arc A (see figure 9.12a). Therefore, the potential translation should be trimmed to any tangent point that exists along the potential translation (figure 9.12b).

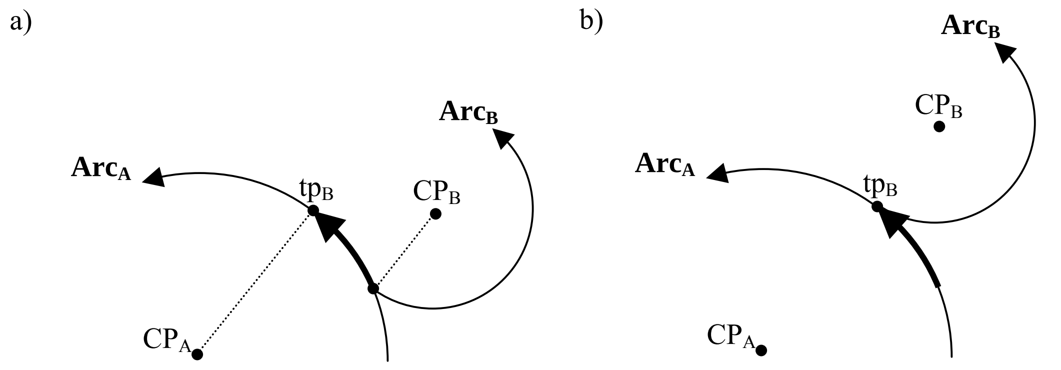


Figure 9.12. Potential translation vector derived in case (ii) when a tangent point is present: arc B's start/end point touches the middle of arc A

Assuming that the translation of figure 9.12 is performed (and was not trimmed by later stages), the next potential translation can now be derived from the circular no-fit polygon of the two arcs (i.e. the distance between the arc centre points is equal to $r_A + r_B$ in the convex case and equal to $r_A - r_B$ in the concave case).

The same principles found in case (ii) also apply for case (iii) whereby the start/end point of arc A touches the middle of arc B (see figure 9.13a). Once again, we must firstly examine whether the two arcs are tangential about their touch point. If this is the case then the partial circular no-fit polygon can be derived as in section 9.1.1. If not then an arc is created from the touch point to the end point of arc B (with the same centre point and radius of B). However, this defines the potential translation vector with which arc A moves (relatively) so the translation vector must be rotated by 180° to obtain the correct translation vector for arc B (see figure 9.13b).

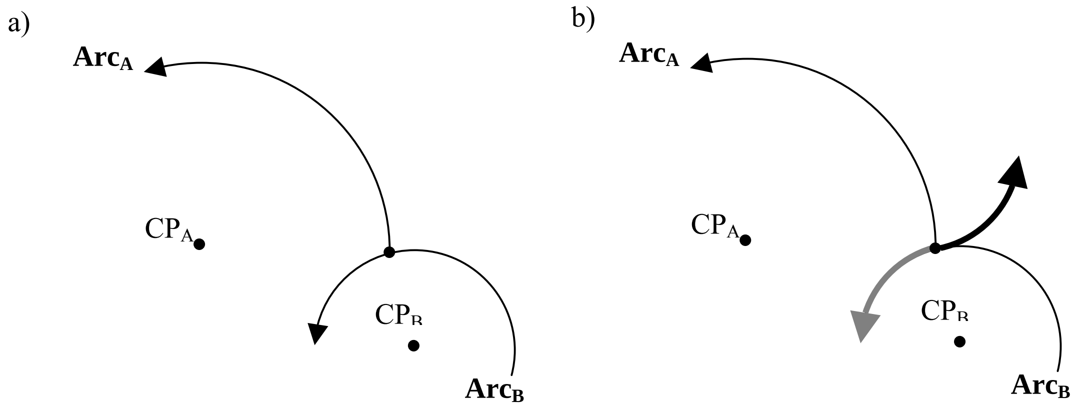


Figure 9.13. Potential translation vector derived in case (iii): arc A's start/end point touches the middle of arc B

From figure 9.13b, it can be seen that when arc B is translated by the potential vector, the two arcs remain in contact, without intersecting, throughout arc B's translation. Once again, this only holds true if there are no tangential positions along the translation. If one exists, the translation should be trimmed (as with the previous case). Although only convex cases have been presented within the figures, the concave cases follow the same procedure except for the calculation of the circular no-fit polygon (as detailed in section 9.1.1).

The final case to explain is case (i) whereby both arcs touch on their start or end points. Firstly we must examine whether the two arcs are tangential using the centre point distance calculation. If they are tangential then the partial circular no-fit polygon can be used as before. If not, then potential translation vectors are derived from both of the arcs based on cases (ii) and (iii) described previously. If the arc's touching point is the end point, then no potential translation can be created (i.e. a zero length arc translation would be created as the end point is also the touch point). Also, if both arcs touch on their start points, then two potential translation vectors will be produced.

Potential Translation Vectors for Touching Pairs Involving a Line and an Arc

The second type of touching edge pairs is touching lines and arcs. These are less complicated than the touching arcs case but tangential points must still be identified (see figure 9.14).

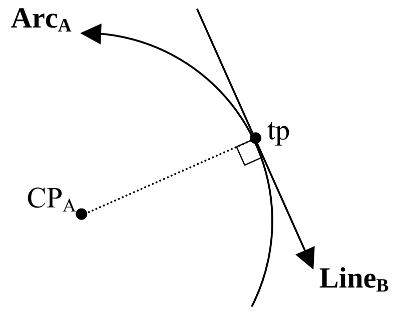


Figure 9.14. Tangents with lines and arcs

The same three cases will be briefly summarised for touching lines and arcs (as the principles are the same as the above). A potential translation arc is created from the touch point to the end point of arc A in case (ii) where no tangent point exists and from the touch point to the tangent point where one does exist. It is important to detect whether there is a tangent point to avoid translations that result in intersections. In case (iii) the potential translation vector is defined by the end point of line B to the touch point (tangent points are not required here). Finally, case (i) is handled by the techniques from cases (ii) and (iii). Once again, two potential translation vectors are produced (one derived from the arc and one from the line). As before, no potential translation can be derived from a primitive whereby its end point is also the touching point.

9.1.2.3 Finding a Feasible Translation (Arc Modifications)

The next stage is to find a feasible translation from the set of potential translations. In the line no-fit polygon approach, a potential translation is only feasible if it is feasible for each of the touching edge pairs. In section 8.2.3, a set of rules was developed to indicate translation feasibility through the use of left/right line tests. Now that arcs are involved, it initially seems that the problem is more complicated. However, the same method can be used providing that both the arcs within the touching pairs and potential arc translations are reduced to a tangential line at the touch point. Figure 9.15 shows how the tangential line is created depending on the position of the touching point: i) at a touching start point, the tangential line must start at the touch point, ii) a mid-arc touch results in a tangential line with its mid point on the touch point and iii) a touching end point requires that the tangential line also ends at the arc's end point.

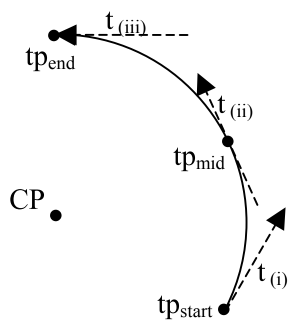


Figure 9.15. Conversion to tangential lines at the touch point of an arc

It is important that the directionality of the arc is maintained within the tangential line for the method of section 8.2.3 to be applicable (the length of the tangential line does not matter as it is only used for left/right tests). Figures 9.16, 9.17 and 9.18 demonstrate the procedure.

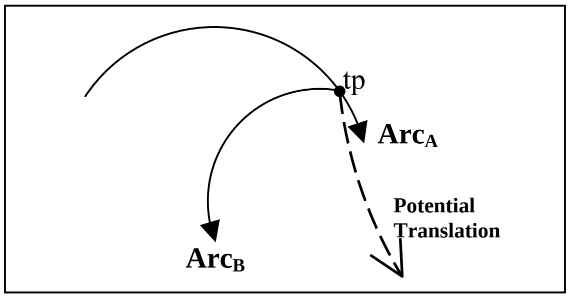


Figure 9.16. Two touching arcs and a potential arc translation

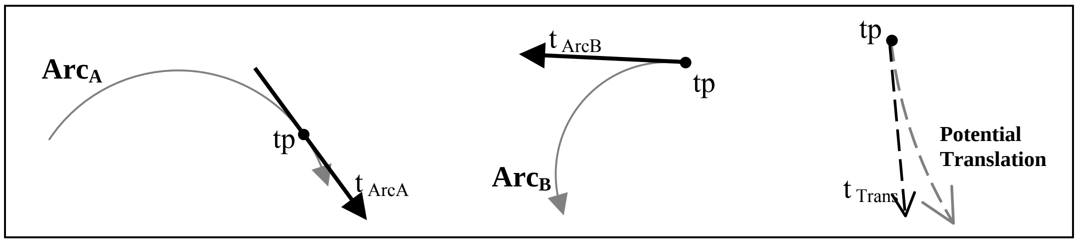


Figure 9.17. Creating the tangential lines

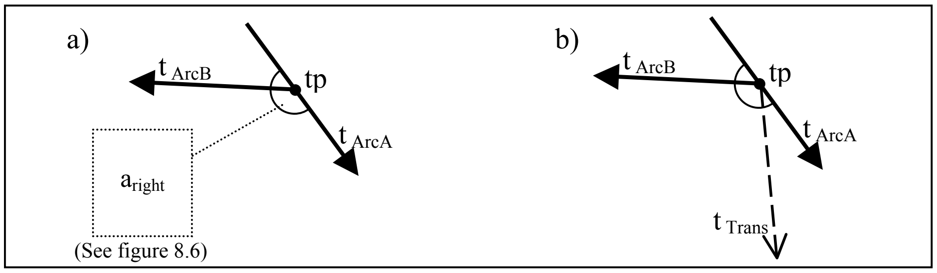


Figure 9.18. Revisiting the line method of section 8.2.3 using tangential lines

Figure 9.16 shows an example of two touching arcs and a potential arc translation (arc_A is concave, arc_B is convex and the touch point is tp). In figure 9.17, tangential lines are created for the two touching arcs and the potential translation arc. In figure 9.18a, the method developed for touching line edges (section 8.2.3) is used to define the feasible region and figure 9.18b shows that the potential translation arc is feasible for this pair of touching edges.

On further examination why the tangential lines can be used, we must examine what is being calculated. In order to show that a potential translation is feasible for a particular edge pair, it only needs to shown that the translation will not *immediately* result in an intersection. As the tangential lines of arc primitives define the vector path of the arc at a particular position, the same test routines can be used from the line-only case. The feasibility of any potential translation and combination of touching edge pairs can be identified provided that the tangential lines of the arcs are used.

9.1.2.4 Trimming the Feasible Translation (Arc Modifications)

Referring back to section 8.2.4, once a feasible translation has been identified, it must be trimmed to avoid intersections. In the line case, this was achieved by projecting the translation vector through each vertex of shape B and testing for its intersection with lines of shape A. The translation vector was also projected back from the vertices of shape A in order to test for intersection with shape B's primitives. In order to correctly handle arcs, the trimming procedure only requires a simple modification in order to deal with: i) arc to arc trims, ii) line to arc trims and iii) arc to line trims. For each of these, the same approach is taken as with the line case (projecting the translation through each start/end vertex etc.). However, an extra projection must also be conducted from the tangent point of any arcs. Given an arc and a line, the tangent point of the line and arc must be detected. This tangent point is then treated as if it were another vertex of the arc (i.e. the translation is also projected from the point). Similarly, when trimming is performed on two arcs, we not only need to project from each start/end point but also from the centre point of either of the arcs to test for any intersection with the partial circular no-fit polygon of the two arcs. If there is an intersection, then the translation is trimmed as normal. The line/arc trim procedure is demonstrated in figure 9.19 and the process for an arc/arc trim is demonstrated in figure 9.20. In the arc/arc case, if no partial circular no-fit polygon exists for two arcs then the start/end point projections will result in the correct trim.

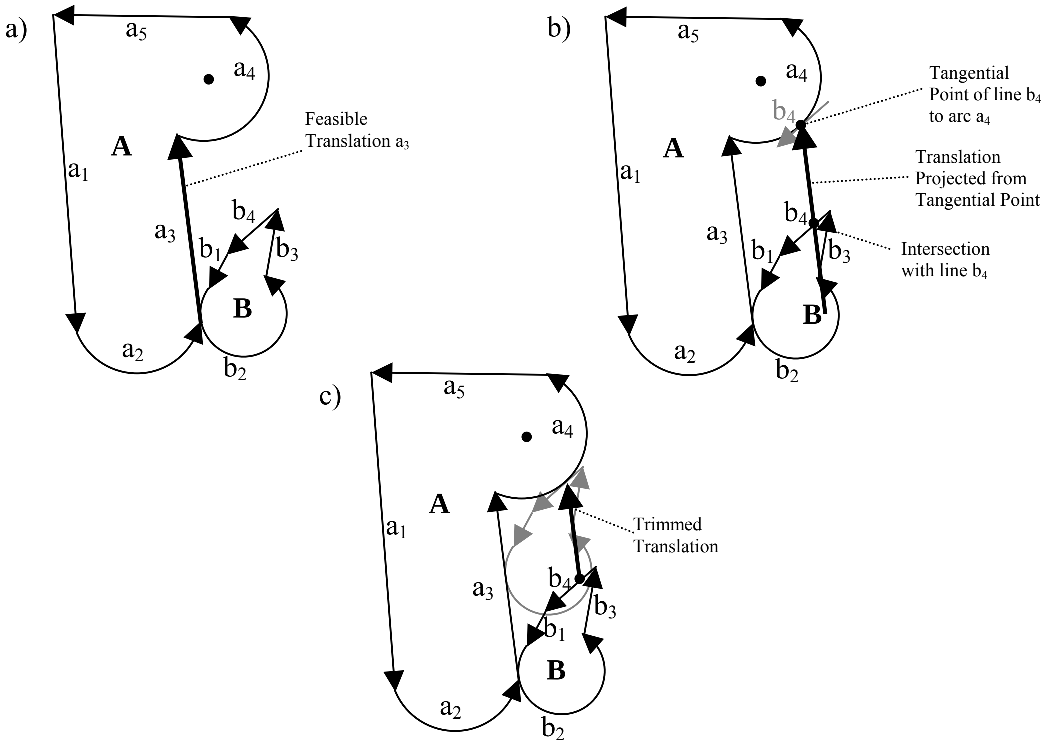

a)
b)
c)
A
B
a_1
a_2
a_3
a_4
a_5
b_1
b_2
b_3
b_4
Feasible Translation a_3
Tangential Point of line b_4 to arc a_4
Translation Projected from Tangential Point
Intersection with line b_4
Trimmed Translation

Figure 9.19. Tangential trimming of the feasible translation from an arc and a line, a_4 and b_4

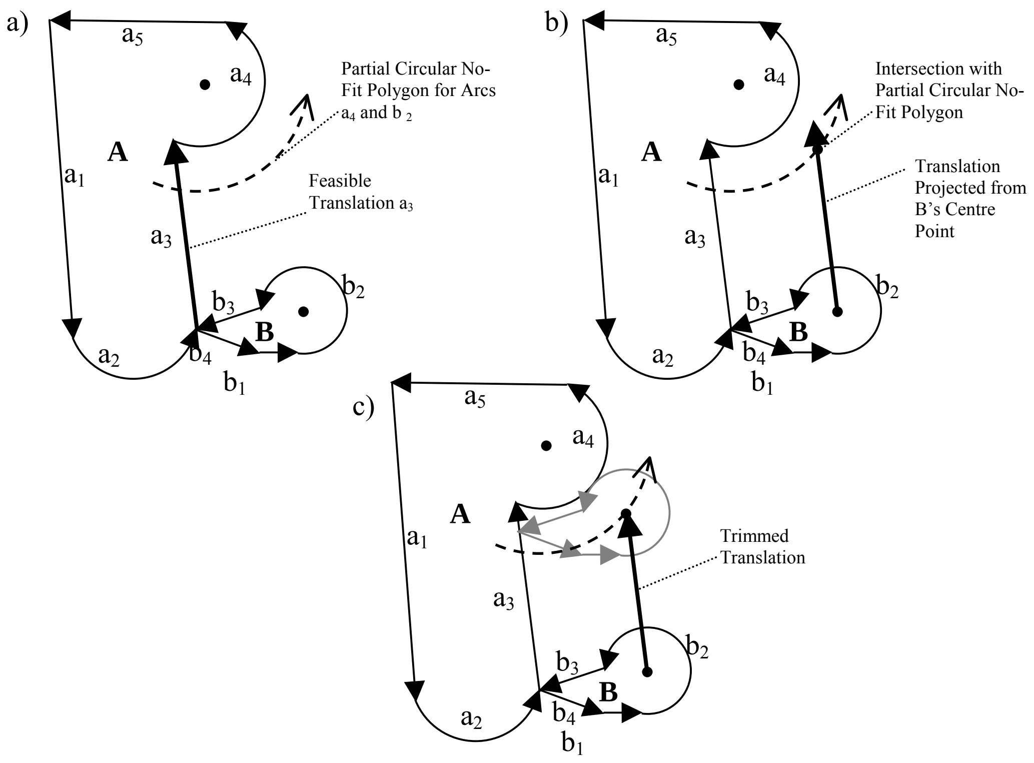

a)
b)
c)
A
B
a_1
a_2
a_3
a_4
a_5
b_1
b_2
b_3
b_4
Partial Circular No-Fit Polygon for Arcs a_4 and b_2
Feasible Translation a_3
Intersection with Partial Circular No-Fit Polygon
Translation Projected from B's Centre Point
Trimmed Translation

Figure 9.20. Tangential trimming of the feasible translation from two arcs

9.1.2.5 Applying the Feasible Translation (Arc Modifications)

There are no arc modifications in this stage but it has been included for completeness. Now that a feasible trimmed translation has been produced, shape B can be translated and then the procedure continues until the shape returns to its initial location (indicating the no-fit polygon loop has been created). For further implementation details refer back to chapter eight.

9.1.3 Examples of Generated No-Fit Polygons with Arcs

Examples of no-fit polygons that have been generated using the proposed no-fit polygon algorithm are shown in figures 9.21 and 9.22. All of these shapes feature within the new datasets of this thesis (specifically from the arc benchmark problems, Profiles1 to Profiles5).

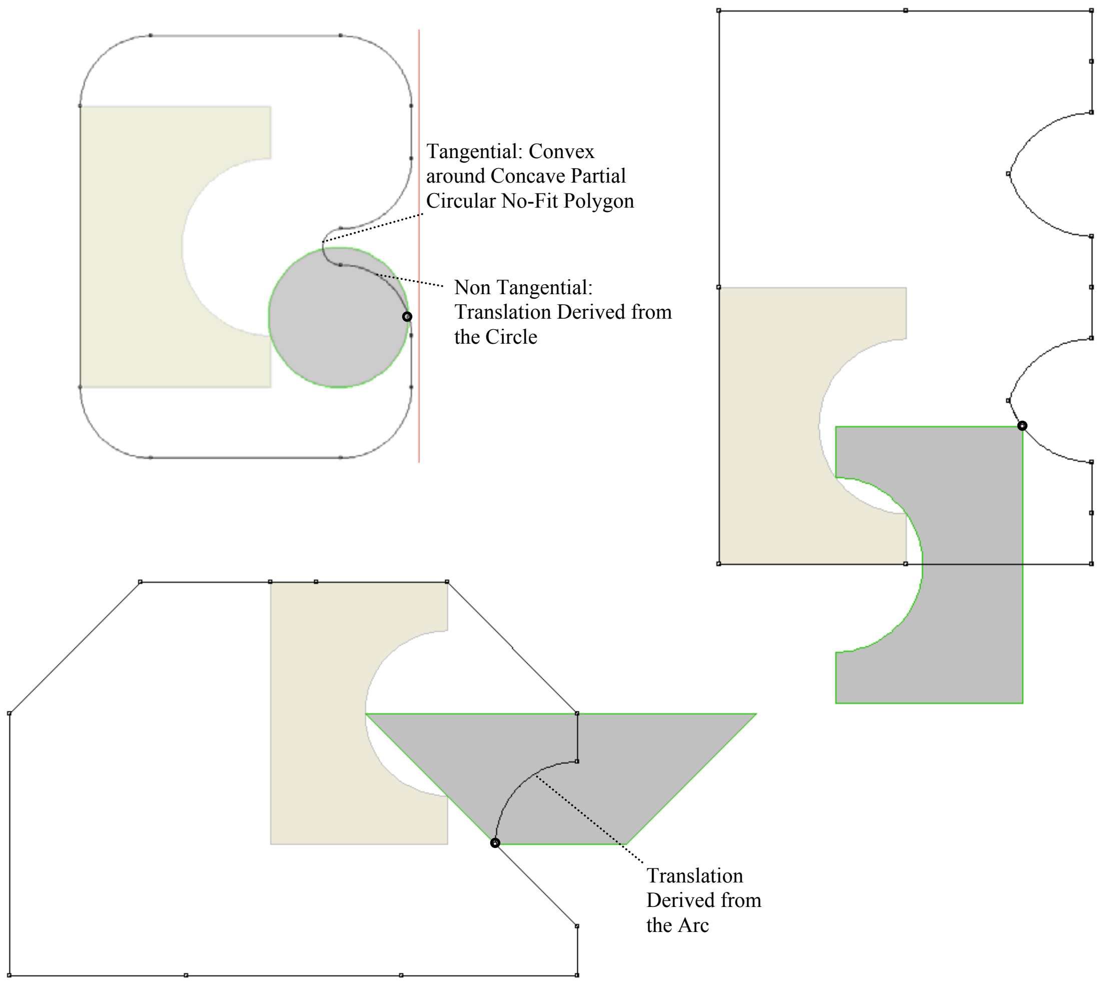

Tangential: Convex around Concave Partial Circular No-Fit Polygon
Non Tangential: Translation Derived from the Circle
Translation Derived from the Arc

Figure 9.21. Annotated examples of no-fit polygons including arcs

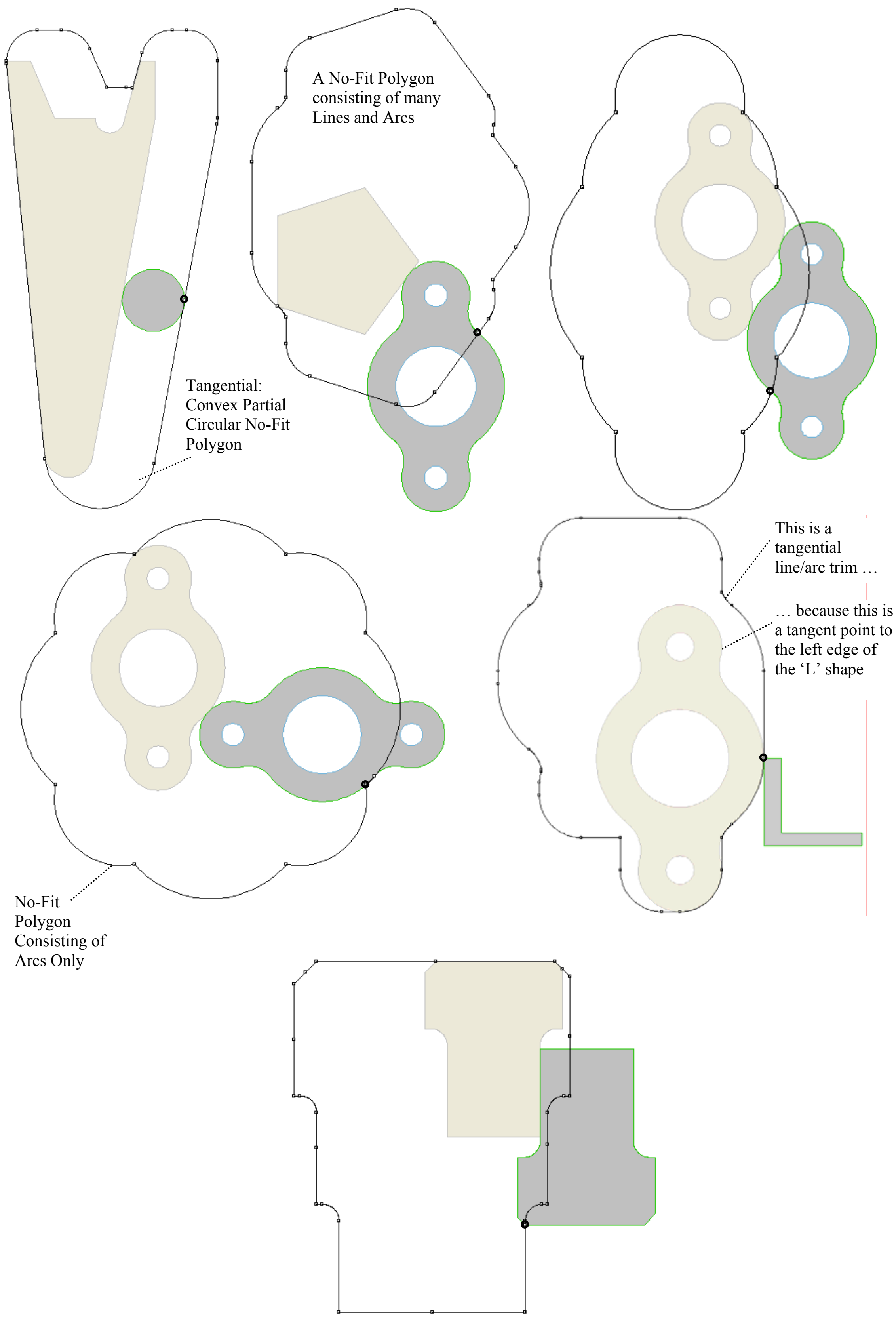

A No-Fit Polygon consisting of many Lines and Arcs
Tangential: Convex Partial Circular No-Fit Polygon
This is a tangential line/arc trim …
… because this is a tangent point to the left edge of the 'L' shape
No-Fit Polygon Consisting of Arcs Only

Figure 9.22. More annotated examples of no-fit polygons including arcs

9.2 An Amended Packing Algorithm using the No-Fit Polygon

The packing algorithm that was developed in chapter seven used trigonometry based intersection and resolution. When a shape was placed in a position that intersected another shape that had already been placed on the sheet, the overlap was resolved in the vertical axis by iteratively resolving individual edge intersections until the shapes no longer intersected. Now, through the development of the no-fit polygon, intersection testing between pairs of shapes can be performed using just one point-in-polygon test and furthermore the overlap can be resolved by casting an infinite vertical line from the test point and detecting its first intersection within the no-fit polygon.

Whilst the use of no-fit polygons for intersection detection and overlap resolution offers considerable advantages for nesting, the creation of no-fit polygons for all pairs of shapes and each rotation would be costly both in terms of generation speed and memory requirements and, therefore, two optimisations are proposed.

Faster Generation Times

The no-fit polygon of two shapes in their 180° orientations is identical to that of the same two shapes in their 0° orientations albeit rotated by 180°. As long as the shapes remain in the same relative orientations, the no-fit polygon is identical albeit rotated by the difference in angles between the two pairs of orientations. Therefore, if a no-fit polygon has been generated for shape B in its 180° rotation and shape A in its 45° rotation, when shape B is in its 190° rotation and shape A is in its 55° rotation the previous no-fit polygon can simply be copied and rotated by the difference (i.e. 10°). This can drastically reduce the number of no-fit polygons that *actually* need to be calculated which reduces the overall generation times.

Reduced Memory Overheads

Although the required no-fit polygons can be generated faster using the previous method, the problem of the memory overhead still remains. When explaining the no-fit polygon generation algorithm it was discussed that the movement of shape B to shape A is the same relative movement of shape A to shape B albeit in opposite directions. This leads to another property that is exploited by the presented algorithms to reduce the total number of no-fit polygons that must be generated. The no-fit polygon of shape B around shape A is the same as the no-fit polygon of shape A around shape B. This property can be used so that only the no-fit polygon of shape B orbiting A is required (and not A around B). This yields a ½ reduction in the number of generated no-fit polygons. However, this requires a modification to the intersection resolution method that has been described as we do not have the required no-fit polygon for shape A around shape B. The correct resolution can be achieved by using NFP_{AB} and casting an infinite line downwards (instead of upwards as before) and finding the distance, *d*, to the first intersection with the no-fit polygon. Then by moving shape A by this distance, d, in the positive vertical direction, the overlap is correctly resolved.

Other implementation details within the packing approach remain the same as with chapter seven.

9.3 Experimental Results

In this section, solutions are generated for the entire set of literature benchmark problems and the newly presented problem instances using the new bottom-left-fill implementation based on the no-fit polygon. For comparative purposes, all of the experiments are conducted with the same local search settings as used in chapter seven (unchanged hill climbing and tabu search with a neighbourhood of 5 and history of 25). Once again, input orderings are produced by the local search and 1, 2, 3, 4 and *N* opt operators are used to find neighbour solutions (see chapter seven). The no-fit polygon implementation of the bottom-left-fill algorithm is used to 'decode' the orderings into actual layouts. All of the experiments within this chapter have been conducted on a 2GHz Intel Pentium 4 processor with 256MB RAM.

9.3.1 Experiments on the Literature Benchmarks

In the previous experimentation on the literature benchmark problems (section 7.4.1), 100 iterations were allowed. However, as both approaches produce identical layouts given the same orderings, only allowing 100 iterations in these experiments would not yield useful comparisons in the quality of solutions or average solutions produced. In light of this, each problem is only allowed 5 minutes computation in these experiments as this was the timeframe in which most of the solutions were produced using the trigonometric implementation of the algorithm. This will allow comparisons to be made both in terms of the speed benefits (average time taken per nest) and the effectiveness of finding better quality solutions (on average) using the no-fit polygon approach as opposed to the trigonometric approach of chapter seven. Table 9.1 shows the average solution and best solution found using 10 runs (each of 5 minutes) for hill climbing and tabu search (once again with both length and area pre-sorted initial orderings) on the length evaluated literature problems. Table 9.2 shows the results obtained on the literature problems for which density is the evaluation measure. In each table, the best results are shown in bold.

		Hill Climbing								Tabu Search							
		Length Ordered				Area Ordered				Length Ordered				Area Ordered			
Problem	Best Literature	Average Length	Best Result			Average Length	Best Result			Average Length	Best Result			Average Length	Best Result		
			Length	Density1	Density2		Length	Density1	Density2		Length	Density1	Density2		Length	Density1	Density2
Blasz1	27.30	27.54	**26.80**	**80.6%**	**82.8%**	27.73	27.20	79.4%	81.2%	27.33	27.10	79.7%	81.0%	27.51	27.00	80.0%	81.3%
Dagli	65.60	60.81	**59.94**	**84.6%**	**85.8%**	60.87	60.36	84.0%	85.1%	61.03	60.24	84.2%	84.8%	61.15	60.38	84.0%	84.8%
Fu	34.00	32.98	32.89	86.7%	89.1%	32.33	**31.60**	**90.2%**	**92.1%**	32.94	32.70	87.2%	87.7%	32.92	**31.60**	**90.2%**	**92.1%**
Jakobs1	13.20	11.90	**11.50**	**85.2%**	88.5%	12.00	12.00	81.7%	86.0%	11.97	11.86	82.6%	88.9%	11.79	**11.50**	**85.2%**	**90.3%**
Jakobs2	28.20	26.00	26.00	74.2%	77.5%	26.00	26.00	74.2%	78.4%	26.00	26.00	74.2%	80.8%	25.41	**24.70**	**78.1%**	**82.5%**
Marques	83.60	78.80	**78.00**	**88.7%**	89.6%	79.59	79.00	87.6%	89.1%	79.71	78.93	87.6%	**89.9%**	79.60	79.00	87.6%	88.0%
Poly1A	14.70	13.67	**13.30**	**77.1%**	**82.8%**	13.81	13.70	74.8%	78.5%	13.78	13.69	74.8%	79.3%	13.97	13.69	74.9%	77.1%
Poly2A	30.10	27.53	**27.09**	**75.7%**	77.7%	27.79	27.58	74.3%	75.2%	27.19	**27.09**	**75.7%**	**78.8%**	27.55	27.13	75.6%	77.3%
Poly3A	40.40	41.35	**41.07**	**74.9%**	76.5%	41.75	41.53	74.0%	75.7%	41.55	41.25	74.5%	**77.7%**	41.77	41.67	73.8%	75.4%
Poly4A	56.90	55.78	55.14	74.4%	76.0%	55.73	55.53	73.8%	75.7%	55.75	**54.60**	**75.1%**	**76.7%**	55.73	55.10	74.4%	75.7%
Poly5A	71.60	69.98	69.84	73.4%	74.6%	70.20	69.56	73.7%	74.2%	70.06	69.13	74.1%	75.4%	69.79	**68.84**	**74.4%**	**75.5%**
Poly2B	33.10	30.37	30.11	75.1%	77.4%	30.44	**29.63**	**76.3%**	**77.5%**	30.54	30.40	74.4%	76.6%	30.52	30.28	74.7%	76.1%
Poly3B	41.80	41.00	40.66	75.0%	**76.5%**	40.97	40.63	75.1%	76.0%	41.20	41.06	74.3%	75.8%	41.02	**40.50**	**75.3%**	76.4%
Poly4B	52.90	52.66	52.33	73.9%	74.7%	52.32	51.84	74.6%	75.3%	52.10	51.72	74.8%	75.7%	51.67	**51.18**	**75.6%**	**76.9%**
Poly5B	63.40	61.68	61.14	75.1%	76.1%	61.62	61.31	74.9%	**76.4%**	61.61	**60.86**	**75.4%**	75.9%	61.81	61.71	74.4%	75.8%
SHAPES	63.00	57.40	**56.00**	**71.2%**	**73.7%**	57.80	57.00	70.0%	71.1%	57.90	57.50	69.4%	70.7%	58.20	**56.00**	**71.2%**	72.5%
SHAPES0	63.00	62.00	**60.00**	**66.5%**	**70.2%**	62.40	62.00	64.4%	66.5%	62.30	61.00	65.4%	67.3%	64.22	62.50	63.8%	65.5%
SHAPES1	59.00	56.20	**55.00**	**72.5%**	**74.3%**	57.00	57.00	70.0%	71.6%	58.46	58.00	68.8%	71.1%	58.20	58.00	68.8%	70.4%
SHIRTS	63.13	63.71	**63.40**	**85.2%**	**86.9%**	64.10	63.98	84.4%	86.3%	64.04	63.93	84.5%	86.5%	64.09	63.85	84.6%	86.2%
SWIM	6568.00	6464.73	6311.28	70.1%	71.3%	6531.24	6305.94	70.1%	**71.5%**	6416.59	**6270.88**	**70.5%**	71.4%	6566.58	6499.91	68.1%	70.0%
TROUSERS	245.75	248.56	245.95	88.7%	89.5%	247.75	246.80	88.4%	89.4%	246.60	**245.28**	**88.9%**	**89.8%**	247.98	247.17	88.2%	89.1%

Table 9.1. Experiments on length evaluated literature benchmark problems

		Hill Climbing								Tabu Search							
		Length Ordered				Area Ordered				Length Ordered				Area Ordered			
Problem	Best Literature	Average Length	Best Result			Average Length	Best Result			Average Length	Best Result			Average Length	Best Result		
			Length	Density1	Density2		Length	Density1	Density2		Length	Density1	Density2		Length	Density1	Density2
Albano	86.0%(D2)	10203.84	**9980.86**	**87.2%**	**88.3%**	10196.87	10169.47	85.6%	86.4%	10130.41	10010.73	87.0%	87.4%	10174.15	10037.38	86.7%	87.5%
Blasz2	68.6%(D1)	24.94	24.80	75.9%	79.9%	24.92	24.90	75.6%	79.9%	24.96	**24.80**	**75.9%**	**80.4%**	25.06	24.90	75.6%	79.0%
Dighe1	72.4%(D2)	1257.96	1239.60	80.7%	81.1%	1260.38	1250.00	80.0%	81.9%	1289.62	1270.00	78.7%	80.1%	1257.72	**1210.00**	**82.6%**	**83.8%**
Dighe2	74.6%(D2)	1224.66	1215.70	82.3%	83.0%	1205.66	**1180.00**	**84.7%**	**86.5%**	1221.12	1190.00	81.1%	84.7%	1229.32	1226.60	81.5%	83.6%
Mao	71.6%(D2)	1857.70	1847.20	79.8%	83.1%	1841.52	1821.70	80.9%	**83.9%**	1853.90	1821.20	80.9%	83.1%	1838.78	**1811.50**	**81.4%**	83.6%

Table 9.2. Experiments on density evaluated literature benchmark problems

The best results from tables 9.1 and 9.2 are summarised in table 9.3 where they are compared to the best results from the literature and the results obtained by the trigonometric packing approach of chapter seven using 100 iteration runs. Also shown is the percentage improvement that is achieved on the benchmark problems by using the approach presented in this chapter (the best results are shown in bold).

Problem	Prev. Best Lit.	Trigonometry Packing Best (Chapter 7)					No-Fit Polygon Packing Best (Chapter 9)					% Improvement over	
		Length	Density1	Density2	Time / Nest (s)	Time To Best (s)	Length	Density1	Density2	Time / Nest (s)	Time To Best (s)	Best Literature	Chapter 7
Blasz1	27.3	27.80	77.7%	81.0%	0.32	21	26.80	80.6%	82.8%	0.09	281	**1.83%**	**3.60%**
Dagli	65.6	60.57	83.7%	89.0%	2.04	189	59.94	84.6%	85.8%	0.42	252	**8.63%**	**1.05%**
Fu	34	32.80	86.9%	90.8%	0.24	21	31.60	90.2%	92.1%	0.17	139	**7.06%**	**3.66%**
Jakobs1	13.2	11.86	82.6%	92.6%	0.74	43	11.50	85.2%	90.3%	0.19	29	**12.88%**	**3.02%**
Jakobs2	28.2	25.80	74.8%	83.3%	2.13	81	24.70	78.1%	82.5%	0.64	51	**12.41%**	**4.26%**
Marques	83.6	80.00	86.5%	89.3%	0.25	5	78.00	88.7%	89.6%	0.07	21	**6.70%**	**2.50%**
Poly1A	14.7	14.00	73.2%	78.2%	0.36	12	13.30	77.1%	82.8%	0.11	254	**9.55%**	**5.03%**
Poly2A	30.1	28.17	72.8%	77.5%	1.24	121	27.09	75.7%	78.8%	0.50	239	**10.01%**	**3.84%**
Poly3A	40.4	41.65	73.8%	75.5%	2.01	210	41.07	74.9%	76.5%	1.00	159	-1.66%	**1.40%**
Poly4A	56.9	54.93	74.6%	75.9%	2.43	203	54.60	75.1%	76.7%	1.71	224	**4.04%**	**0.59%**
Poly5A	71.6	69.37	73.9%	75.7%	5.04	476	68.84	74.4%	75.5%	1.76	300	**3.85%**	**0.76%**
Poly2B	33.1	30.00	75.4%	77.5%	2.50	180	29.63	76.3%	77.5%	0.90	189	**10.48%**	**1.24%**
Poly3B	41.8	40.74	74.9%	77.1%	4.26	418	40.50	75.3%	76.4%	2.22	114	**3.11%**	**0.58%**
Poly4B	52.9	51.73	74.8%	77.4%	8.24	96	51.18	75.6%	76.9%	6.00	176	**3.25%**	**1.07%**
Poly5B	63.4	60.54	75.8%	77.2%	14.70	677	60.86	75.4%	75.9%	12.62	299	**4.01%**	-0.52%
SHAPES	63	59.00	67.6%	69.1%	0.60	31	56.00	71.2%	73.7%	0.27	226	**11.11%**	**5.09%**
SHAPES0	63	66.00	60.5%	62.6%	0.93	21	60.00	66.5%	70.2%	0.14	274	**4.76%**	**9.09%**
SHAPES1	59	60.00	66.5%	68.9%	0.82	2	55.00	72.5%	74.3%	0.22	239	**6.78%**	**8.33%**
SHIRTS	63.13	63.80	84.6%	87.3%	4.99	58	63.40	85.2%	86.9%	0.77	194	-0.43%	**0.62%**
SWIM	6568	6462.40	68.4%	71.6%	12.39	607	6270.88	70.5%	71.4%	1.24	141	**4.52%**	**2.96%**
TROUSER	245.75	246.57	88.5%	90.1%	7.89	756	245.28	88.9%	89.8%	1.02	253	**0.19%**	**0.52%**
Albano	86.0%(D2)	10292.90	84.6%	86.5%	1.18	93	9980.86	87.23%	88.25%	0.25	299	**2.55%**	**2.00%**
Blasz2	68.6%(D1)	25.28	74.5%	79.9%	0.16	11	24.80	75.94%	80.41%	0.07	14	**9.67%**	**1.88%**
Dighe1	72.4%(D2)	1292.30	77.4%	78.9%	0.22	9	1210.00	82.65%	83.84%	0.15	3	**11.02%**	**5.94%**
Dighe2	74.6%(D2)	1260.00	79.4%	84.3%	0.10	7	1180.00	84.75%	86.50%	0.70	148	**13.75%**	**2.51%**
Mao	71.6%(D2)	1854.30	79.5%	82.9%	0.38	30	1821.70	80.91%	83.86%	0.13	152	**14.62%**	**1.14%**

Table 9.3. Summary of the best results from the no-fit polygon approach of this chapter to that of the previous best results from the literature and the trigonometry approach of chapter seven

On examination of table 9.3, we can see that, using just 5 minutes computation time, the no-fit polygon packing implementation has further improved upon the best known results for 23 of the 26 literature problems. Furthermore, the new approach yields an average improvement of 7%

over the previous best solutions from the literature and 3% over the approach of chapter seven (which had set 25 of 26 new best-known solutions for the benchmarks). Such reductions in sheet wastage are not to be underestimated as they could yield significant cost savings within the industrial setting. These savings could then either be reinvested into the company or passed on to customers in order to be more competitive within the marketplace. The reduction in the use of raw material also has ecological benefits in a world where natural resources are being depleted steadily.

Of the three problems for which further improvements were not found, two of them were previously extended (see section 7.4.1) to reflect the durations used by the authors of the approaches which had, prior to the work of this thesis, produced the best known solutions, Poly3A and SHIRTS. These extended durations are significantly longer than the five minutes that was allowed in the initial experiments (30 minutes / 1000 iterations and 1 hour 45 minutes respectively). With the third problem, Poly5B, the approach of this chapter improves upon the previous literature best but is unable to beat the solution produced by the 100 iteration run of chapter seven. On closer inspection of the average generation time per layout, it can be seen that both methods produce layouts within similar time timeframes (14.70 seconds and 12.62 seconds). With these packing times, only around 20 layouts can be examined within the 5 minute duration (i.e. the proposed method of this chapter only has a fifth of the time of the previous approach). This also highlights an interesting observation about the performance of the two implementations. When datasets consist of simple shapes (mainly convex with relatively few edges), the no-fit polygon approach provides less dramatic speed increases (there may also be penalties in finding the correct no-fit polygon when the number of no-fit polygons within the cache increases). This can also be seen within the other problems that involve ‘simple’ shapes (i.e. Fu, Dighe1, Dighe2).

However, the benefit of the no-fit polygon is shown in the benchmark problems that have been taken from the textile industry. For example, for the problems Marques and Mao, layouts are generated three times as quickly as the trigonometry approach. With Albano and Dagli there is a fourfold increase in nesting speed, SHIRTS and TROUSERS are both seven times quicker and, finally, layouts for the SWIM dataset (consisting of very complex pieces) can be produced in a tenth of the time that is required by the trigonometry based approach. Therefore it can be seen that the no-fit polygon provides the greatest speed improvements for the problems that contain highly irregular shapes or shapes that are created from many primitives. However, on the problems involving simple shapes (some of which have been identified) the no-fit polygon still yields speed improvements, albeit less dramatic. These speed increases can be used to examine more of the search space (as successfully demonstrated with the presented method) or it may provide the necessary additional speed to allow more complicated packing strategies to be developed.

Referring back to the experiments of chapter seven (using the trigonometric implementation), the literature benchmark problems that were not improved upon in the first set of experiments were extended to allow the same time as reported by the authors who produced the best-known solutions (section 7.4.1). These problems were Blasz1, SHAPES1, SHIRTS and TROUSERS which were extended to allow durations of 551.73s, 2019.77s, 6367.57s and 13613.67s respectively. Two other problems, SHAPES0 and Poly3A, were extended to allow 10,000 iterations as run times were not provided. This yielded new best solutions for five of these six problems (all except SHAPES0). Table 9.4 compares the previous best-known solutions for these problems and the solutions generated from the new approaches presented in this thesis. Once again, the best solutions are shown in bold typeface.

Problem	Previous Best Literature		Trigonometry Best Result (Chapter 7 - Extended Run)				No-Fit Polygon Best Result (Chapter 9 - 5 Minute Run)			
	Length	Time to best (s)	Length	Density1	Density2	Time to best (s)	Length	Density1	Density2	Time to best (s)
Blasz1	27.3	551	27.20	79.4%	80.7%	502	**26.80**	**80.6%**	**82.8%**	**281**
Poly3A	40.4	-	**40.33**	**76.2%**	**77.3%**	**1515**	41.07	74.9%	76.5%	159
SHAPES0	63	-	65.00	61.4%	63.6%	332	**60.00**	**66.5%**	**70.2%**	**274**
SHAPES1	59	2019	58.40	68.3%	71.5%	1810	**55.00**	**72.5%**	**74.3%**	**239**
SHIRTS	63.13	6367	**63.00**	**85.7%**	**88.1%**	**807**	63.40	85.2%	86.9%	194
TROUSERS	245.75	13613	**243.40**	**89.6%**	**91.1%**	**3612**	245.28	88.9%	89.8%	253

Table 9.4. Comparison of the best solutions achieved by the proposed approach (using 5 minute runs) to the literature and the approach of chapter seven on the previously extended problems

It can be seen from table 9.4 that the new implementation has improved upon three of these problems (Blasz1, SHAPES0, and SHAPES1) and, furthermore, the solutions have been found within just five minutes computation. Before the work of this thesis, the previous best-known solutions for Blasz1 and SHAPES1 had layout lengths of 27.3 and 59.0 respectively. In chapter seven of this thesis, both of these results were improved upon to 27.2 and 58.4. The experiments in this chapter have significantly improved upon these two problems to 26.8 and 56.5 respectively.

However, it is the problem instance of SHAPES0 that deserves special mention because the previous literature best of 63 units length was found by Dowsland and Dowsland in 1993 [**147**] and has remained unbeaten until now. Also, this is the only problem that the trigonometry packing approach of chapter seven could not better. Using the approach of this chapter, a new best solution with a length of 60 units has been produced and was found in under 5 minutes. This yields a considerable improvement over the previous best of 5%. Furthermore, the new algorithm consistently equalled or improved upon the previous best in the majority of the runs. Figure 9.23 shows the new best-known layouts that have been produced within 5 minutes for the three benchmark problems discussed above.

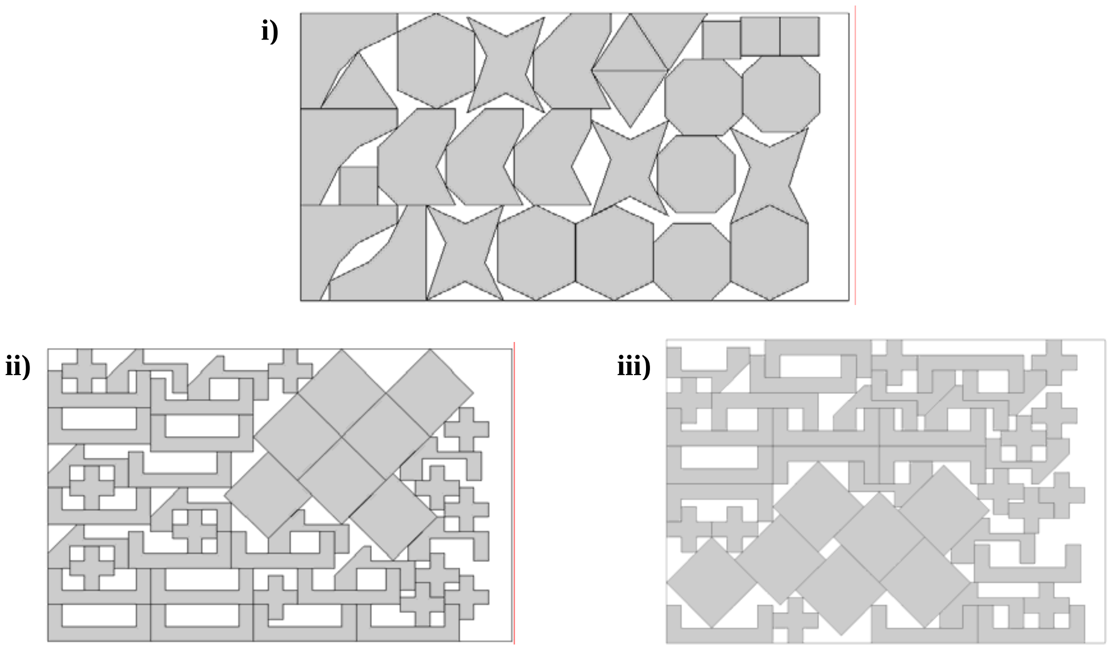


Figure 9.23. New best layouts for previously extended literature benchmark problems: i) Blasz1 (Length = 26.8), ii) SHAPES0 (Length = 60), iii) SHAPES1 (Length = 55)

9.3.2 Extended Experiments on the Literature Benchmarks

The next set of experiments involved extending the same problems that were extended in chapter seven. The presented approach has already improved upon three of these during the 5 minute runs and, therefore, only Poly3A, SHIRTS and TROUSERS have been extended. Once again, 10 runs were allowed for each problem using both hill climbing and tabu search with decreasing length and area initial orderings. The best and average solutions are presented in table 9.5.

Whilst the proposed approach has only produced one new best-known solution for the TROUSERS dataset (see figure 9.24), the solutions for the other two problems have similar length measures to the best solutions of chapter seven (both Poly3A and SHIRTS are within 0.003%). However, the average solutions produced by the proposed approach of this chapter are better than the averages of chapter seven for all three of the problems.

Problem	Previous Literature Best		Trigonometry Best Result (Chapter 7)					No-Fit Polygon Best Result (Chapter 9)				
			Avr Length	Best Solution				Avr Length	Best Solution			
	Length	Time to best (s)		Length	Density1	Density2	Time to best (s)		Length	Density1	Density2	Time to best (s)
Poly3A	40.40	-	41.45	**40.33**	76.2%	77.3%	1515	**40.95**	40.45	76.0%	78.1%	429
SHIRTS	63.13	6368	63.66	**63.00**	85.7%	88.1%	807	**63.61**	63.16	85.5%	87.9%	806
TROUSERS	245.75	13614	246.40	243.40	89.6%	91.1%	3612	**244.10**	**243.00**	89.8%	90.8%	12484

Table 9.5. Extended experiments with the no-fit polygon packing approach

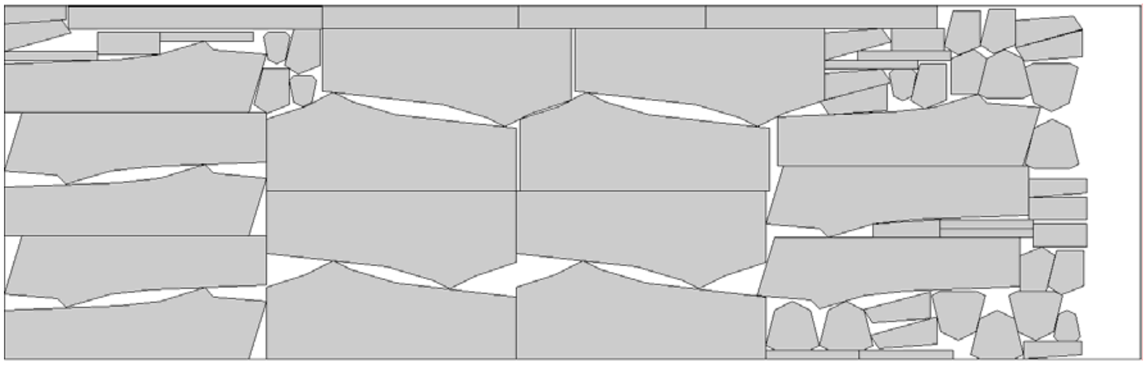


Figure 9.24. New best layout for TROUSERS data (Length = 243.0, Density1 = 89.8%)

9.3.3 Experiments on the New Benchmark Problems (Profiles1 – Profiles10)

Finally, experiments were conducted on the new benchmark problems that were introduced in chapter seven of this thesis (see section 7.3). For these problems, 10 runs were conducted with both hill climbing and tabu search (with both length and area sorted orderings) where each run was allowed 30 minutes computation time. These experiments have been repeated using the packing approach presented within this chapter and the results are presented in table 9.6. The best results are summarised in table 9.7 (best results are shown in bold).

These experiments have produced new best solutions for 7 of the 10 datasets. For the datasets in which the new approach did not achieve improvements over chapter seven, solutions were only slightly worse (within 1%).

Problem	Hill Climbing								Tabu Search							
	Length Ordered				Area Ordered				Length Ordered				Area Ordered			
	Average	Best Result			Average	Best Result			Average	Best Result			Average	Best Result		
	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2	Length	Length	Density1	Density2
Profiles1	1389.14	1380.10	81.6%	83.0%	1391.40	1386.00	81.3%	82.5%	1412.88	1404.80	80.2%	81.4%	1394.54	**1359.90**	**82.8%**	**84.5%**
Profiles2	3262.10	3216.06	50.0%	50.8%	3230.98	**3194.19**	**50.3%**	**51.5%**	3264.40	3252.70	49.4%	50.3%	3267.78	3223.30	49.8%	50.8%
Profiles3	8073.88	**7881.13**	**52.9%**	**54.0%**	8230.33	8189.66	50.9%	51.7%	8074.34	7999.74	52.1%	53.0%	8177.60	8045.84	51.8%	53.0%
Profiles4	2476.46	2452.42	75.2%	75.7%	2475.58	2464.35	74.8%	75.4%	2466.55	**2425.26**	**76.0%**	**76.5%**	2486.49	2482.72	74.2%	74.8%
Profiles5	3394.32	3367.88	69.4%	70.6%	3401.17	**3351.94**	**69.8%**	**70.8%**	3385.32	3364.35	69.5%	70.5%	3399.48	3384.90	69.1%	70.4%
Profiles6	3134.01	**3121.36**	**75.0%**	**78.6%**	3172.97	3156.02	74.2%	75.6%	3161.51	3146.56	74.4%	76.8%	3176.52	3161.22	74.1%	76.3%
Profiles7	1305.78	**1292.30**	**77.4%**	**79.9%**	1307.44	**1292.30**	**77.4%**	**79.9%**	1316.32	1296.30	77.1%	77.7%	1313.44	1309.10	76.4%	79.6%
Profiles8	1303.17	1268.98	78.8%	79.8%	1283.19	1268.98	78.8%	79.8%	1308.61	1293.88	77.3%	77.5%	1285.44	**1263.11**	**79.1%**	**82.1%**
Profiles9	1314.73	**1278.21**	**52.7%**	**54.2%**	1298.42	1290.00	52.2%	53.2%	1308.93	1298.62	51.9%	52.9%	1303.43	1292.63	52.1%	53.3%
Profiles10	11373.40	**11219.60**	**65.8%**	**66.6%**	11515.03	11403.99	64.8%	65.4%	11392.36	11302.50	65.3%	66.0%	11653.40	11585.81	63.7%	64.7%

Table 9.6. Experiments on new benchmark problems

Problem	Trigonometry Approach Best (Chapter 7)			No-Fit Polygon Approach Best (Chapter 9)			Percentage Improvement over Chapter 7
	Best Solution			Best Solution			
	Length	Density1	Density2	Length	Density1	Density2	
Profiles1	1377.74	82.0%	85.2%	1359.90	82.8%	85.4%	**1.29%**
Profiles2	3216.10	50.0%	51.3%	3194.19	50.3%	51.5%	**0.68%**
Profiles3	8193.89	50.9%	52.6%	7881.13	52.9%	54.0%	**3.82%**
Profiles4	2453.12	75.1%	75.7%	2425.26	76.0%	76.5%	**1.14%**
Profiles5	3332.70	70.2%	73.6%	3351.94	69.8%	70.8%	-0.58%
Profiles6	3097.86	75.6%	77.8%	3121.36	75.0%	78.6%	-0.76%
Profiles7	1296.30	77.1%	80.2%	1292.30	77.4%	79.9%	**0.31%**
Profiles8	1318.70	75.8%	77.2%	1263.11	79.1%	82.1%	**4.22%**
Profiles9	1290.67	53.1%	54.9%	1278.21	52.7%	54.2%	**0.97%**
Profiles10	11160.10	66.2%	66.8%	11219.60	65.8%	66.6%	-0.53%

Table 9.7. Summary of best results for new benchmark problems

9.4 Summary

The aim of this chapter was to develop a no-fit polygon packing algorithm that could be used within industrial settings. However, as has previously discussed, the fast and efficient handling of shapes containing both holes *and* arcs has been an underlying feature of this work. Therefore, in the first part of the chapter, an algorithm was presented that can robustly generate no-fit polygons for shapes, which may also consist of arcs, through extension of the previous no-fit polygon generation algorithm of chapter eight. This is a significant milestone for the academic community as there have not been any published approaches which address no-fit polygon generation with shapes involving arcs. In the second part of the chapter, the packing approach of chapter seven was reimplemented to utilise the arc/line no-fit polygon for the detection and resolution of overlaps. The performance of the no-fit polygon based implementation of the packing algorithm was investigated through repeated experiments on the literature benchmark problems and the new datasets. The packing approach yielded an average improvement in solution quality of 7% over the previous best solutions from the literature and an average improvement of 3% was achieved over the packing method presented within chapter seven (which had, itself, produced the best known results for 25 of the 26 literature benchmark problems). The presented packing approach of this chapter was able to achieve new best-known solutions for 30 of the 36 benchmark problems (including a 5% improvement for the SHAPES0 problem which could not be improved upon by the packing approach of chapter seven). Therefore, the two irregular packing approaches of this thesis have produced the <u>best-known solutions for all 26 of the literature benchmark problems</u>. The approach also improved upon 7 of the 10 new benchmark datasets which contain shapes with lines, arcs and holes. The additional speed gain from use of the no-fit polygon enables more layouts to be examined in less computational time which, on average, allows for higher quality solutions to be obtained. These features make the presented packing approach a strong candidate for use in the industrial setting.

PART D – DISCUSSION

In Part B and Part C, new strategies for automating layout generation for the rectangular and irregular variants of the two-dimensional packing problem were presented. The purpose of this part of the thesis is to draw conclusions and to analyse the presented work, to discuss the future research direction that could be undertaken and to show how the work has been disseminated.

Chapter ten provides a summary of the research conducted within this thesis and places it in context to the field of cutting and packing, conclusions are drawn for the developed approaches presented within this thesis and the possible future research direction that could be followed in order to improve or extend this work.

Chapter eleven provides an insight into why this work has been of significant benefit to both the academic community and the industrial partner. Finally, the chapter presents an overview of the commercialisation opportunities that have arisen as a direct result of this research.

CHAPTER TEN

Conclusions

10.1 Discussion

This thesis has provided several milestones for two-dimensional cutting and packing problems. The main contributions of this thesis are summarised below giving particular attention to the context of the work related to the previous literature.

10.1.1 Development of a New Fast and Effective Rectangle Packing Heuristic

The rectangular variant of the packing problem benefits from simpler implementation than the irregular variant because the form of each shape is known and, therefore, specialised intersection detection routines can be employed. However, the problem has still been shown to be NP-Complete [**15**]. The best performing placement heuristics, prior to this thesis, have been bottom-left-fill style algorithms. These operate by firstly sorting the rectangles into some order (usually by decreasing height or decreasing width) and then placing each rectangle as close to the bottom left corner of the sheet as possible. In order to fill holes, a list of potential placement locations must be maintained and intersection testing must be performed between the rectangles already assigned to the sheet and the current rectangle being placed. Both of these operations can be expensive when the problem size becomes large. The bottom-left-fill algorithms generally produce better results when hybridised with metaheuristics to generate different input orderings that can then be evaluated. One of the first contributions of this work was the development of a fast rectangle placement heuristic that is based on the principles of best-fit. This is a completely different methodology to the previous state-of-the-art algorithms because the list of available rectangles is examined at each placement step and the best-fitting rectangle is chosen dynamically. A novel method of representing the available sheet space is presented using the height profile of the layout. This representation, along with the best-fit method, enables the heuristic to produce a layout very quickly because holes are never created unless absolutely necessary and intersection testing is redundant (unlike the previous algorithms). The algorithm produces solutions that are much denser (lower overall height required) than the other approaches from the literature for medium and large problem instances and, furthermore, are produced within just a few seconds. Although subsequent algorithms proposed in this thesis have produced better solutions than the best-fit heuristic, the speed of the approach would allow it to be employed where real-time packing is required such as dynamic problems (whereby new orders arrive periodically and layouts should be repacked to include any new data) or online quoting systems (whereby an indication of the material required to fulfil an order is necessary to provide a competitive quotation).

10.1.2 Hybridisation of Two Packing Algorithms to Combine Relative Strengths

A further contribution of this thesis is the hybridisation of the best-fit heuristic with the metaheuristic bottom-left-fill placement heuristic method. The author believes this to be one of the first approaches to combine the relative strengths of two separate packing algorithms to generate layouts of a better quality than either of the constituent methods. It was emphasised that the best-fit heuristic can produce extremely dense solutions at the start of packing when there is a large selection of rectangles from which to choose. However, after placing the majority of the rectangles, there is inevitably a smaller selection and therefore more wastage occurs at the top of layouts. Conversely, the metaheuristic bottom-left-fill algorithms produce excellent layouts on small problems as they can search many potential arrangements. It is intuitive that the metaheuristic bottom-left-fill approach can be applied to the latter rectangles of the solution produced by the best-fit heuristic. It has been shown that this approach makes significant improvements over the best-known solutions on all of the benchmark problems of the literature using, in general, just one minute of computation time.

10.1.3 Introduction of User Interaction

A method is also presented to allow the user to interact with the algorithms by identifying poor regions within the layout. The metaheuristic bottom-left-fill algorithm can then be invoked to target the rectangles within the identified regions. The user interaction approach further improves on the newly presented rectangle packing algorithms within this thesis (which, as stated above, have already improved on all benchmarks from the literature). However, the method requires more computational time due to the interactivity and obviously requires a human operator to oversee the procedure. To the author's knowledge, this is the first time that user-interactive approaches have been evaluated using such a wide range of benchmark data from the literature.

10.1.4 Presentation of a New Placement Technique for Irregular Packing

The irregular variant of the packing problem involves similar objectives to rectangle packing, that is to minimise the total length of sheet required and, as mentioned previously, the irregular variant of the problem involves significantly more complex geometric modelling, which yields its own difficulties. This thesis contributes a new heuristic algorithm which, through the use of overlap resolution in the vertical axis, generates significantly better results than the previous state-of-the-art on a wide range of established benchmark problems. The developed algorithm is not only able to pack shapes involving the traditional line polygonal representation but can also pack shapes that incorporate circular arcs and holes. This in itself represents a significant improvement over the previous state-of-the-art. In general, an average improvement of 5% is achieved over previous methods and solutions are produced within quick timeframes (5 minutes for the majority of problems).

10.1.5 Development of a Complete and Robust Algorithm for Generation of No-Fit Polygons involving Non-Convex Polygons

The no-fit polygon defines the intersection relationship between pairs of shapes and is considerably faster than its trigonometric counterpart. For this reason, there has been considerable interest in using the no-fit polygon for automated packing. However, the generation of the no-fit polygon becomes complicated in all but the simplest convex shape case and previous generation algorithms for non-convex shapes have reported degenerate cases for which the algorithms cannot derive the complete no-fit polygon. The author feels that one of the major milestones that has been provided to the academic community through this research programme has been the development of a complete and robust algorithm for calculating the no-fit polygon between pairs of shapes. As this algorithm can generate no-fit polygons (for all of the degenerate cases including shapes with holes) robustly and quickly, it offers significant commercial advantages for faster automated packing within an industrial setting.

10.1.6 Modification of the No-Fit Polygon Generation Algorithm to Allow Shapes Containing Arcs

In addition to presenting the first no-fit polygon not to suffer from degeneracies, large steps have been taken into producing an arc-capable no-fit polygon algorithm through the modification of the line based algorithm. Using the property that the no-fit polygon of two circles is also a circle, it was shown how arcs create a section of the circular no-fit polygon (defined by tangent points).

There have not been any previous attempts to generate no-fit polygons for shapes containing arcs and the author considers this to be an important development for both industrial and academic study.

10.1.7 Identification of No-Fit Polygon Properties and Applicability of Caching within the Generation Process for Packing Problems

In order to use the faster no-fit polygon based intersection during the packing phase, no-fit polygons must be generated for all pairs of shapes. In normal circumstances, $(nm)^2$ no-fit polygons must be generated (where n is the number of unique shapes and m is the number of rotations). This generation procedure can be expensive and results in significant memory overheads for larger problems. For example, with sixteen different shapes and 90° rotations (four rotations: 0°, 90°, 180°, 270°), $(4*16)^2 = 4096$ no-fit polygons must be produced. This thesis identifies the rotational properties of no-fit polygons and shows that a reduction in the number required can be achieved. The proposed method of reduction would result in only 408 no-fit polygons being generated in the example given above (as opposed to 4096 without reductions). The reduced number of no-fit polygons that must be generated is generalised in equation **(1)**.

$$\text{Number of NFPs} = \left(\frac{n(n+1)}{2}\right)\left(\left\lceil\frac{m+1}{2}\right\rceil\right) \quad \text{where } n > 0, m > 0 \qquad \textbf{(1)}$$

The possibility of reductions has not been reported previously in the literature and, thus, is included in this work. Further to this, a modified version of the irregular packing technique (discussed in 10.1.3 above) is presented that utilises the faster no-fit polygon intersection testing during the nesting process. Although the packing algorithm is fundamentally the same, the introduction of the no-fit polygon produces faster layouts and yields considerable time savings which can be used to evaluate more of the problem search space. This is reflected in the quality of solutions that are produced which are better than any other reported method within the literature and the other trigonometric irregular packing implementation presented in chapter seven of this thesis (and discussed in 10.1.3 above).

10.1.8 Generation of the Best-Known Solutions for 60 Benchmark Datasets (Both Rectangular and Irregular) Gathered from over Twenty Years in Cutting and Packing Research

The techniques presented within this thesis have *produced the best-known results for all of the sixty benchmark problems* that are readily available within the literature for both rectangular and irregular packing. These datasets have been taken from over twenty years of cutting and packing literature involving some invented problems and some problems that are drawn from the textile industry. As a particular example of the advances that the new algorithms provide, the solution to one irregular dataset, SHAPES0, has not been improved upon for over ten years but the no-fit polygon packing approach of chapter nine yields an improvement of around 5% over the previous best and consistently obtains better layouts than the previous literature best within three minutes of execution time.

10.1.9 Introduction of New Benchmark Problem Instances to the Literature

Twenty-eight new rectangle benchmark problems have been produced and provided to the academic community. In addition to this, ten new benchmark problems for irregular packing have been presented. The irregular benchmarks are of particular interest as they present shapes that have not been represented previously within the literature. Namely, the new benchmarks include irregularities that are commonplace within the industrial setting and consist of shapes that not only contain lines but arcs and holes as well. The author hopes that the introduction of these new benchmark problems will facilitate greater research into the areas of rectangular and irregular packing and, in the latter case, will promote the use of more typical shapes, containing arcs and holes, which occur frequently in the industrial setting.

10.2 Future Work

This research programme has been very successful in producing state-of-the-art packing algorithms and has achieved the best-known solutions on all of the readily available benchmark problems from the literature. However, the author believes several modifications to this work may yield further improvements in performance (both in the quality of solutions produced and the time required to achieve them). The author is currently pursuing the following:

10.2.1 Algorithmic Analysis

Although the algorithms presented within this thesis have produced excellent layouts in very quick time on a wide range of existing literature benchmark problems, it was beyond the scope of this PhD programme to provide full algorithmic complexity analyses. However, it is recognised that this represents an interesting future research direction to further investigate these algorithms and to define what is and is not possible computationally in the best and worst cases.

10.2.2 Problem Categorisation and Automatic Algorithm Selection

One area that has not been explored within this work is the categorisation of problems. Fundamentally, it could be said that the problems used within this thesis have been manually categorised into rectangular and irregular data with different algorithms being applied to these types. However, further benefits might be achieved through automatic categorisation and statistical analysis of input data before an approach is selected from a *suite* of different packing algorithms. As an example, assume we have some data which we analyse and find to contain only rectangles. In such instances, we know that there are good packing algorithms to tackle this data (i.e. the best-fit heuristic or metaheuristic bottom-left-fill approaches). An indication of problem size should also be considered (e.g. the best-fit heuristic was found to perform the best, in general, for problems involving 50 or more rectangles). Other data characteristics may also influence the choice of one algorithm over another such as the distribution of rectangle sizes (maybe there are only a few distinct sizes of rectangle but with multiple instances of each). As a further example, assume we have some irregular data (i.e. can contain any shape). Analysis of the data may show that there are a number of rectangles, circles and unidentified shapes. In such instances, several algorithms that are tailored to specific data may be used to produce the overall layout (perhaps the best-fit heuristic algorithm to pack the rectangles then a circle packing approach for the circular objects and finally the packing approach of chapter nine). In addition to the automatic selection of packing algorithms, it may also be the case that different search algorithms (and search parameters) perform better on certain types of data and this would also be an interesting avenue for further investigation. The informed automatic selection of algorithms and search parameters should allow for better layouts to be produced.

10.2.3 Hyperheuristics and Packing Problems

Hyperheuristics are a new search paradigm for solving optimisation problems and can be thought of as heuristics that *choose* low-level heuristics [**259,260**]. One of the main features of a hyperheuristic is that there is a layer of abstraction between the hyperheuristic and the problem domain (and related low-level heuristics). This layer of abstraction means that the hyperheuristic cannot utilise any domain specific information (unlike other specific solution approaches) and, instead, may only use general domain-independent feedback from the search to guide low-level heuristic selection (e.g. improvement in evaluation, time taken, recently employed heuristics, good sequences of heuristics). This maintains the generality of the hyperheuristic which enables it to be transferred and applied across a wide range of optimisation problems without change. Therefore, hyperheuristic methods will be appreciably cheaper in terms of implementation costs and easier to use than knowledge-intensive methods and may deliver better average and worst-case performance than other heuristic methods. Hyperheuristic approaches have already been successfully applied to bin packing in Ross et al. (2002) [**261**] and subsequently in Ross et al. (2003) [**262**]. Therefore, it would be an interesting extension to apply and evaluate hyperheuristic approaches on the two-dimensional packing problems that have been addressed within this thesis.

10.2.4 Automatic Identification of Poor Regions within a Layout

In chapter four, the best-fit heuristic was hybridised with a metaheuristic bottom-left-fill method. This involved packing the first *n* rectangles with the best-fit heuristic and then allowing the metaheuristic bottom-left-fill method to produce good layouts with the remaining rectangles. It was shown that this outperforms both of the constituent algorithms but requires the pre-specification of the number of rectangles to pass to the metaheuristic bottom-left-fill phase. However, because different problems can have vastly different characteristics, an interactive approach was developed to allow users to target the metaheuristic bottom-left-fill at the poorer regions of a layout (thus removing the need for pre-specification of the number of rectangles to be passed to the metaheuristic bottom-left-fill phase). Although the solutions produced through the user interaction method achieved significant benefits in solution quality it has two drawbacks: i) the approach can require longer computational time due to requiring feedback from the user and, more importantly, ii) the approach requires a human operator. If we could develop algorithms to detect poorer regions within layouts then both pre-specification *and* the user can be removed from the system. This has benefits because the algorithms can then operate out of working hours (i.e. overnight). Thus, the algorithm can use longer durations and this alone should facilitate better quality solutions.

10.2.5 Applying Similar User Interaction Models to Irregular Packing

One improvement that could be advantageous for irregular packing is the provision of a similar user interaction procedure as was developed for the rectangle packing approach of chapter five. As both the rectangular and irregular packing approaches operate by decoding piece orderings, the presented user interaction method can be directly applied to irregular packing without change (the user selects poor quality regions or clusters of shapes which are then removed and repacked in different combinations whilst unselected shapes remain fixed). The author expects similar improvements in solution quality to be achieved for the irregular packing as was achieved with rectangle packing.

10.2.6 Extension of Arc No-Fit Polygon Generation to Include Start Points

In chapter nine, the complete and robust line no-fit polygon generation algorithm from chapter eight was extended to allow shapes containing arcs. Although this is the first time any no-fit polygon approach has attempted shapes containing arcs, the idea of start points (presented in section 8.3) must be extended to ensure no-fit polygon completeness. The extension of start points to arcs requires similar concepts to that provided in chapter nine and the author is currently pursuing this development.

10.2.7 Preprocessing into Clusters using the No-Fit Polygon

In this thesis, the no-fit polygon was identified as a geometric tool for faster detection and resolution of overlapping shapes. However, analysis of the no-fit polygon may also be beneficial for the pre-clustering of shapes. For example, figure 10.1 revisits the no-fit polygon of two shapes from the SWIM dataset, it can be seen that the most efficient clustering of these two shapes occurs when one shape is placed inside the other shape’s concavity (it is beneficial to fill concavities as, in general, the space is more difficult to utilise). We can see that this is also identified by a large fluctuation in the path of the no-fit polygon (i.e. a ‘sharp’ concavity).

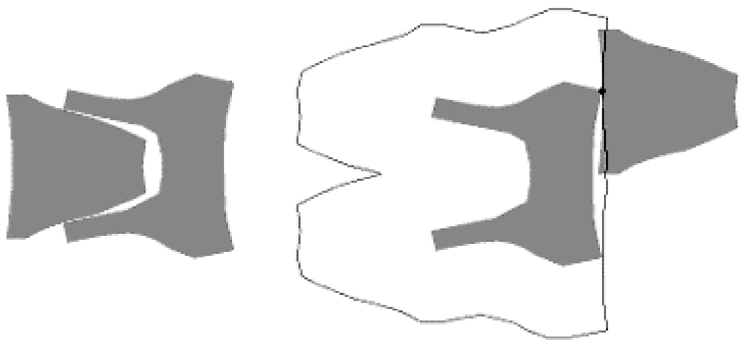


Figure 10.1. Clustering potential of no-fit polygons

If this fluctuation can be detected algorithmically, we can cluster a large proportion of the shapes before packing commences. A nesting algorithm would then treat these clusters as a single shape which would reduce the search space for shape permutation approaches (such as the ones presented in this thesis).

10.2.8 Development of More Intelligent Packing Algorithms

The irregular packing approaches developed within this thesis follow the well-known bottom-left packing methodology. This involves a greedy placement heuristic that places a shape in its leftmost position on the sheet. However, this may not always be the best policy. For example, take the shape of figure 10.1 that contains the concavity and assume that it may be placed in either its 0° or 180° orientation (with concavity facing left or right respectively). Generally, it is better to face concavities to the right (i.e. facing the open sheet) as a left facing concavity becomes 'closed' (i.e. a hole) which limits the number of shapes that can utilise the space within the concavity. Figure 10.2 shows the leftmost position of the example shape in each of its orientations in a typical packing scenario. In figure 10.2 we can see that the two orientations can be placed at almost the same position along the *x* axis. However, the shape is placed in its less desirable orientation (left-facing concavity) because it is marginally the leftmost position for the shape. In this instance, the other position may allow for better future placements as the concavity's space can be exploited by more shapes (i.e. the other shape from figure 10.1).

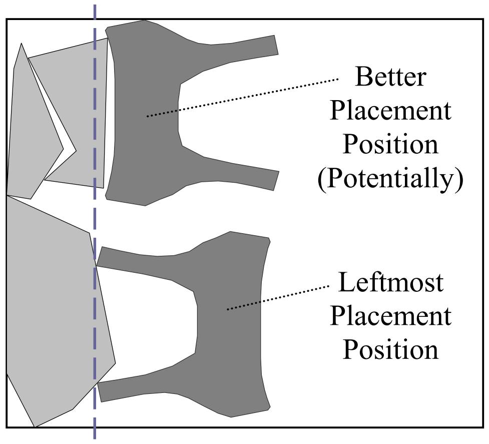

Better Placement Position (Potentially)
Leftmost Placement Position

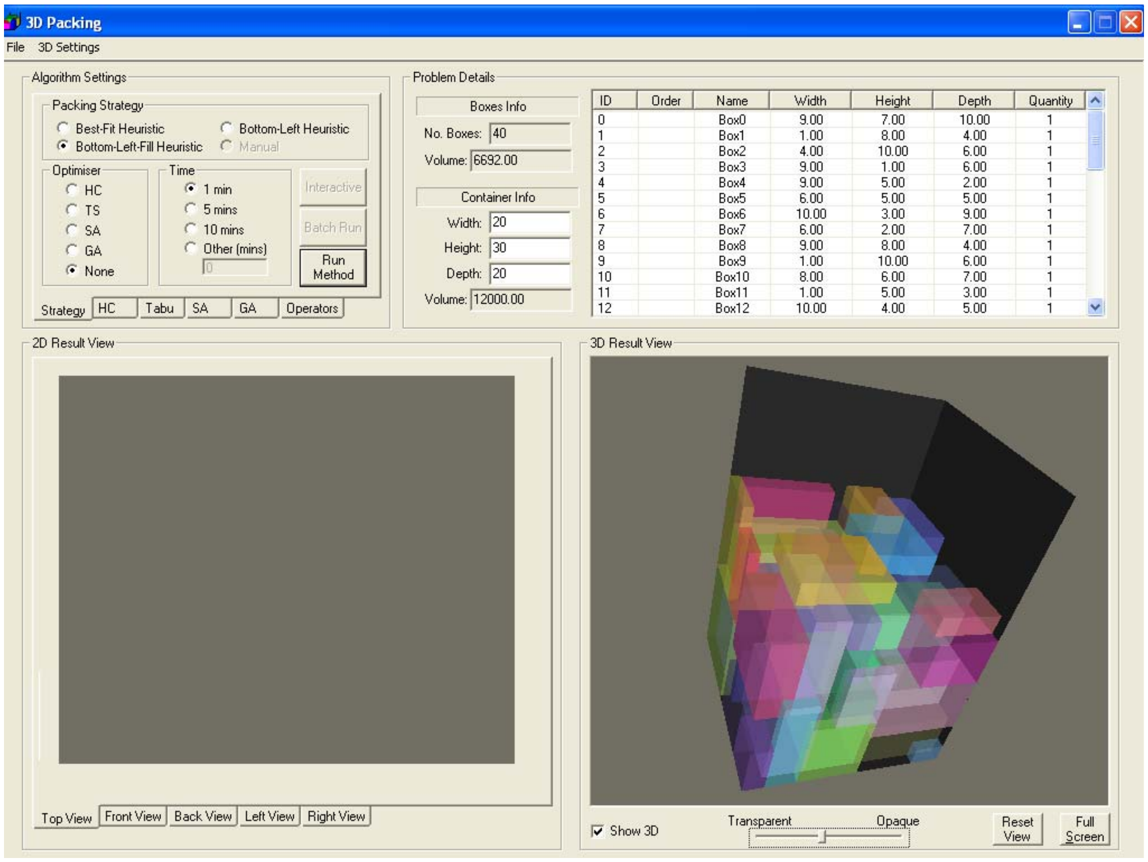
Figure 10.2. Greedy leftmost placement and a better placement position

10.2.9 Dynamic Layout Analysis and Online Clustering

During the packing stages, a considerable amount of information is created about the positional relationships of groups of shapes. However, in the current packing model, this information is not exploited. One possible extension for the developed packing approaches is the algorithmic analysis of layouts to identify high (and low) density sub-regions of the sheet. These regions could be frozen or could be dynamically clustered for use in later iterations.

10.2.10 Three-Dimensional Packing: Lorry and Pallet Loading

As the rectangular packing algorithms presented in Part B of this thesis have been so successful for the two-dimensional variant of the packing problem, it should be possible to extend these algorithms for packing in three dimensions. Figure 10.3 shows a screenshot of the test bed application that has been produced by the author to facilitate future research into three-dimensional packing. The initial experimentation into the packing of boxes looks promising and it is hoped that the modified two-dimensional packing approaches can be applied to three-dimensional benchmark problems from the literature in the near future.



Packing Strategy: Best-Fit Heuristic, Bottom-Left Heuristic, [✓ Bottom-Left-Fill Heuristic], Manual

Optimiser: HC, TS, SA, GA, [✓ None]

Time: [✓ 1 min], 5 mins, 10 mins, Other (mins) 0

Problem Details

Boxes Info

No. Boxes: 40

Volume: 6692.00

Container Info

Width: 20

Height: 30

Depth: 20

Volume: 12000.00

ID	Order	Name	Width	Height	Depth	Quantity
0		Box0	9.00	7.00	10.00	1
1		Box1	1.00	8.00	4.00	1
2		Box2	4.00	10.00	6.00	1
3		Box3	9.00	1.00	6.00	1
4		Box4	9.00	5.00	2.00	1
5		Box5	6.00	5.00	5.00	1
6		Box6	10.00	3.00	9.00	1
7		Box7	6.00	2.00	7.00	1
8		Box8	9.00	8.00	4.00	1
9		Box9	1.00	10.00	6.00	1
10		Box10	8.00	6.00	7.00	1
11		Box11	1.00	5.00	3.00	1
12		Box12	10.00	4.00	5.00	1

[✓ Show 3D]

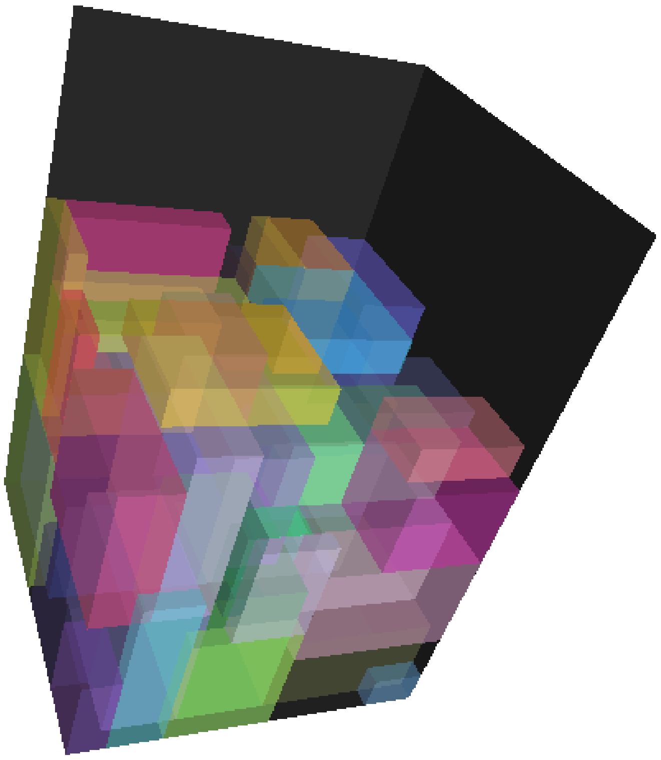


Figure 10.3. Developed Test Bed Application for Three-Dimensional Packing

CHAPTER ELEVEN

Dissemination

The dissemination of knowledge is an important part of any research programme. The focus of this chapter is to provide an overview of the different ways in which this work has been of benefit to the academic community and to outline avenues of commercial exploitation. Details are presented about new commercial opportunities that have arisen as a direct result of the work conducted within this thesis.

11.1 Academic Community

During the course of this research, the author has added to the state-of-the-art in two-dimensional packing both in the regular and irregular variants. As far as the author is aware, the research has produced the best published results for the sixty readily available benchmark problems from over twenty years of cutting and packing literature.

The developed algorithms and strategies have been the focus of six papers that have been targeted at internationally leading journals. One of these has already been published (and another accepted) by ***Operations Research*** which is one of the flagship journals for the OR community. The other journals that have been targeted for this work are ***INFORMS Journal on Computing*** which publishes a broad range of papers on the intersection of operations research to computing and the ***Journal of the ACM*** which publishes significant work to the general field of computer science.

Also, several new benchmark datasets have been introduced to the academic community for both the regular and irregular problems. In particular, the ten irregular problem instances that have been produced contain shapes involving both arcs and holes. These types of shapes have not been widely represented and certainly are not readily available for use by other practitioners. The author hopes that this will facilitate further research and comparison into automated packing algorithms of a more industrial standard (shapes consisting of arcs and holes) within the academic community.

Furthermore, this research has been presented in several seminars and conferences during the course of the research programme. These are listed below:

"*Automated Packing*" – Invited Seminar, 9th June 2004, The University of York, United Kingdom.

"*A New Placement Heuristic for the Orthogonal Stock Cutting Problem*" – CORS / INFORMS Joint International Meeting, May 16-19, 2004, The Banff Centre, Banff Alberta, Canada.

"*Automated Packing*" – Invited Seminar, 21st April 2004, Institute of Computing Science, Poznań University of Technology, Poland

"A *New Placement Heuristic for the Orthogonal Stock Cutting Problem*"– 1st ESICUP Meeting, 18-20 March, 2004, Lutherstadt Wittenberg, Germany.

"*The Irregular Two-Dimensional Packing Problem*" – Seminar, 12th Novemeber 2003, The University of Nottingham, United Kingdom.

"*A New Placement Heuristic for the Orthogonal Stock Cutting Problem*" – Seminar, 16th January 2002, The University of Nottingham, United Kingdom.

11.2 Industrial Partner

As described in chapter 1 of this thesis, an important part of the research programme was the close collaboration with an industrial partner, Esprit Automation Ltd. Before the commencement of this work, Esprit's automated packing algorithms had fallen behind their competitors' packing algorithms. In particular, the irregular algorithm was limited in scope due to being optimised for speed and not for producing good quality solutions. The software consisted of a bitmap geometry representation and single-pass nesting algorithm which resulted in the production of quick layouts but this was at the expense of quality. The advancement of computer technology has lessened the need for such a fast (but inaccurate) algorithm.

During the course of the three year period of industrial funding, this research programme developed the best-fit rectangle packing heuristic that quickly produces the best-known results for medium to large problem instances. In addition to the rectangle packing heuristic, the trigonometric packing algorithm was also produced and shown to perform better than the other approaches of the literature (which include some comparisons of commercial software such as "NestLib" and "Sigmanest"). This algorithm improves upon the company's old packing algorithm in several aspects: i) a new accurate vector based geometry environment, ii) a new nesting algorithm in which shapes touch (this in itself results in significant improvements over Esprit's previous software) and iii) inclusion of combinatorial optimisation techniques that are used within nesting and have already been applied to other problems such as cut path minimisation. These facilities have been developed within the company's software environment (through both the CASE and the TCS research grants) and will significantly improve Esprit's position within the marketplace. A further benefit to Esprit is that they can exploit their relationship with the university to be perceived to be a company that is proactive in the conducting of research. This may be desirable to Esprit's current clients and is a possible marketing point when trying to sell to potential clients.

11.3 Commercial Exploitation

The work developed within the last year of this research programme has led to significant improvements over the state-of-the-art and the layouts produced by the algorithms presented within this thesis have achieved considerable material savings over existing algorithms. Specifically, the robust generation of no-fit polygons has led to faster geometrical testing and its application within the industrial domain offers considerable benefits. An example is to utilise some of the saved computation time for the development of more intelligence within the packing algorithms.

In June 2004 a spin-out company, Aptia Solutions Limited, was formed (with the author of this thesis as its Managing Director) to further develop and commercially exploit these algorithms within areas that are of no commercial interest to our industrial partner (Esprit Automation Limited) such as the textile, polycarbonate and carbon fibre industries. Aptia Solutions Limited has received full backing and support from The University of Nottingham (a major shareholder) and has already successfully won two business grants to conduct market research, draw up a business plan and to continue its commercial software development.

Aptia Solutions will not only commercialise existing algorithms but will draw upon the knowledge gained from this research programme to develop new strategies for all aspects of cutting and packing problems (eventually including one-dimensional cutting and three-dimensional loading automation strategies). Aptia Solutions will release a library of their automatic nesting algorithms that can be "plugged" into existing software produced by development houses through the use of a "black-box" style DLL (dynamic link library). This will commercially protect the algorithms whilst still providing access to the performance benefits that are obtained from their usage. Of course, many companies do not have in-house development teams and Aptia will produce fully integrated software packages for distribution in these situations. A further opportunity that will be exploited is integration into AutoCad, which is the world's leading CAD/CAM software package, through the use of AutoCad's developers environment. This will allow further accessibility for those without tailored software suites.

Aptia Solution's will exploit these algorithms, developed software packages and plug-ins (software that integrates with existing products) by targeting industries on a global scale. This will ensure that the work conducted under this research programme is widely disseminated but will also show that research conducted within United Kingdom is at the forefront of a domain which had typically been dominated by companies from the United States of America. Furthermore, the commitment of the author, the supervisors, Automated Scheduling Optimisation and Planning research group (within the School of Computer Science and Information Technology at the University of Nottingham) and Aptia Solutions Limited to conducting innovative cutting and packing research will mean that this work is further improved and should remain the state-of-the-art for the foreseeable future.

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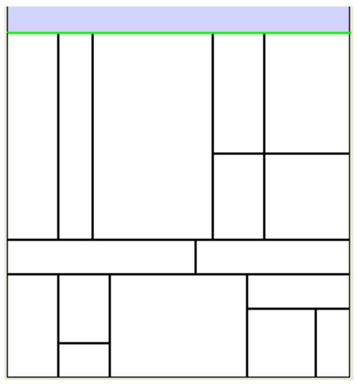
261. Ross, P., Schulenburg, S., Marín-Blázquez, J.G., Hart, E., 2002, *Hyper-heuristics: Learning To Combine Simple Heuristics In Bin-packing Problems*, Proceedings of the 2002 Genetic and Evolutionary Computation Conference (GECCO 2002), New York, NY, USA, 942-948

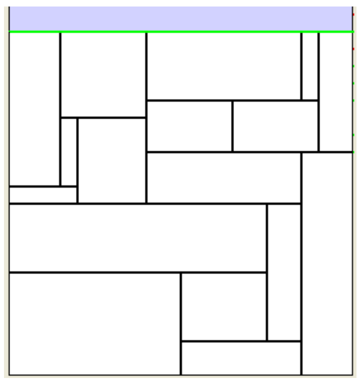
262. Ross, P., Marín-Blázquez, J.G., Schulenburg, S., Hart, E., 2003, *Learning a Procedure That Can Solve Hard Bin-Packing Problems: A New GA-Based Approach to Hyper-heuristics*, Proceedings of the 2003 Genetic and Evolutionary Computation Conference (GECCO 2003), Chicago, IL, USA, July 12-16, 1295-1306

APPENDICES

The following appendices are included to encourage further research and greater comparison between the approaches presented in this thesis and future approaches in the area of two-dimensional cutting and packing:

APPENDIX A – Layouts for Rectangular Benchmarks


C1P1 (h = 20)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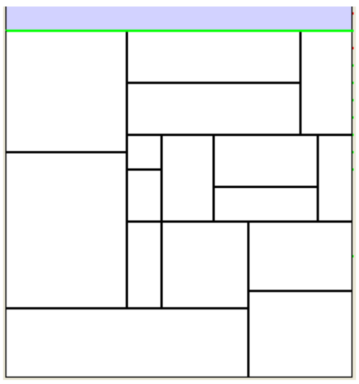
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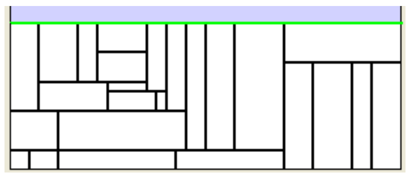
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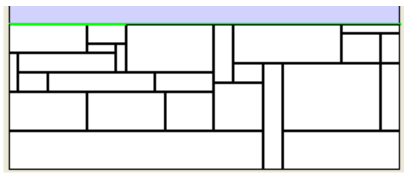
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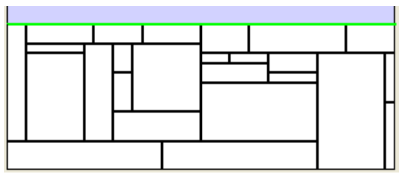
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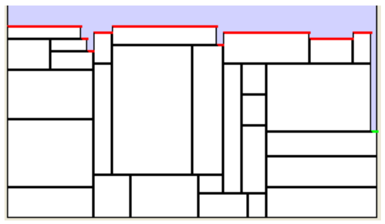
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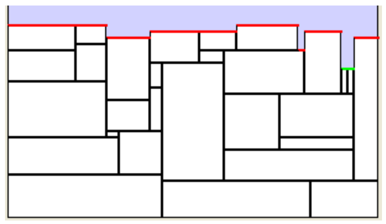
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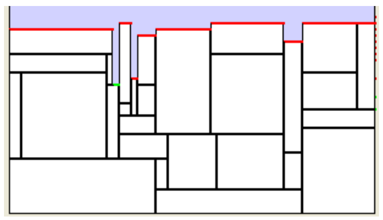
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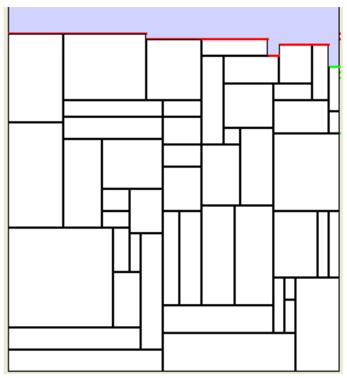
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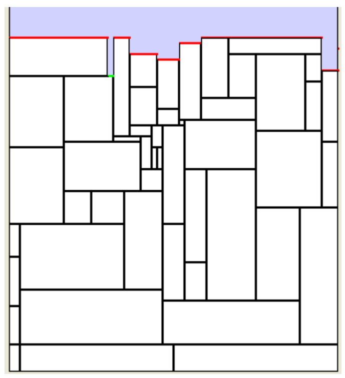
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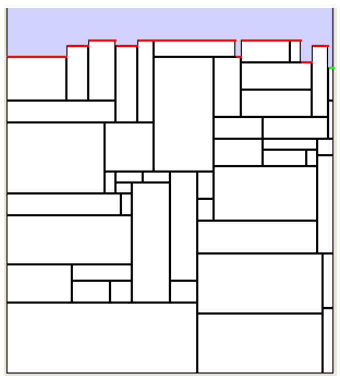
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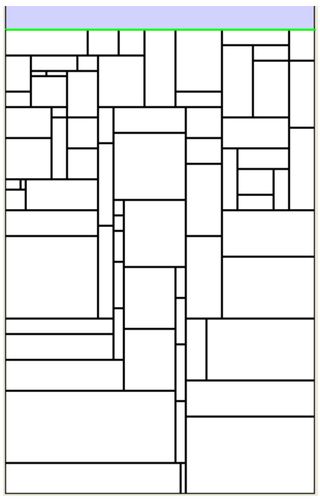
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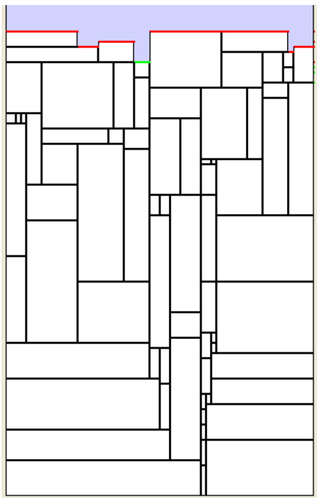
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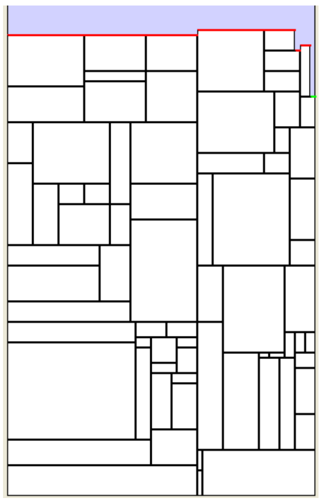
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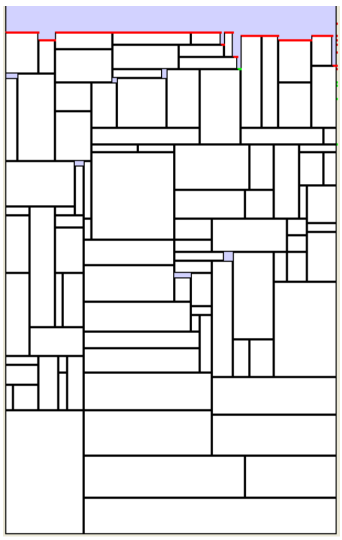
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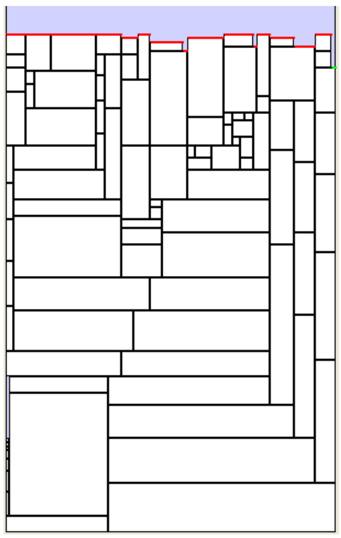
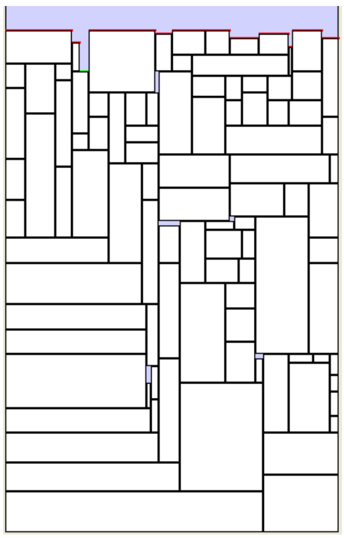

C6P2 (h = 121)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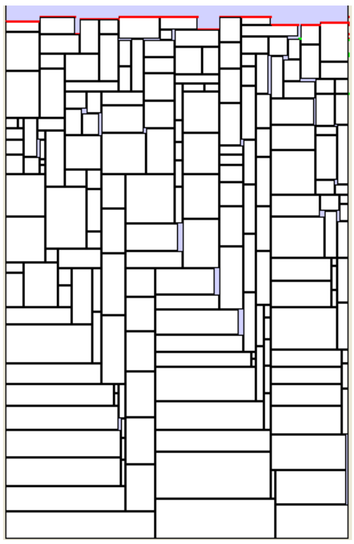
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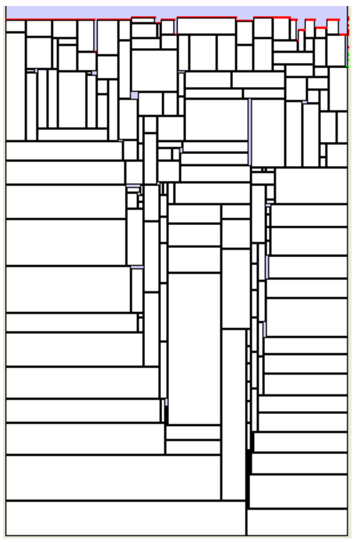
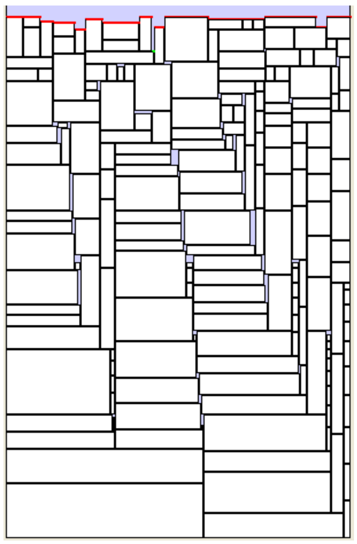
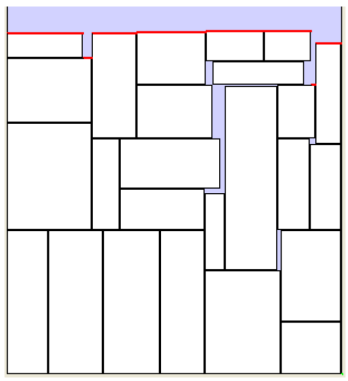

C7P2 (h = 242)


C7P3 (h = 243)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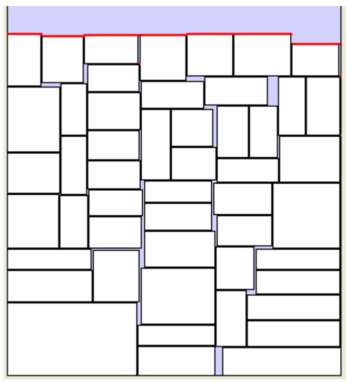
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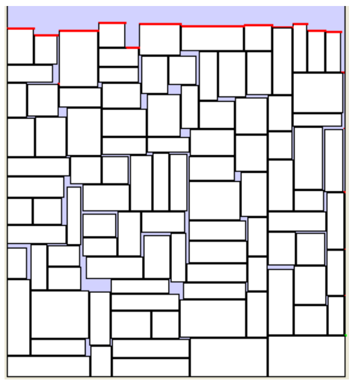
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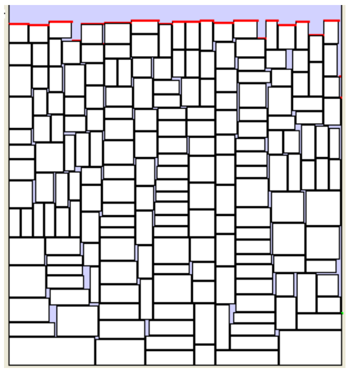
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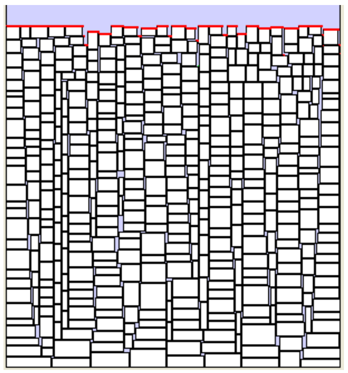
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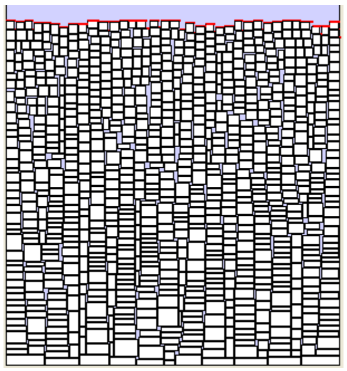
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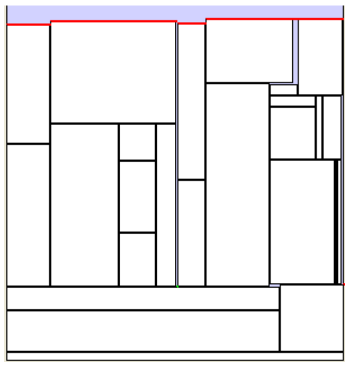
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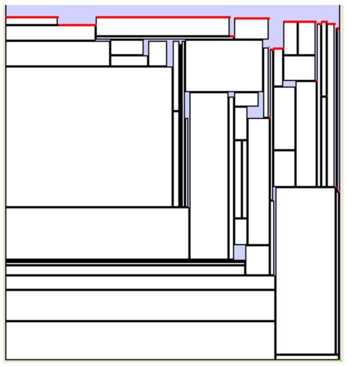
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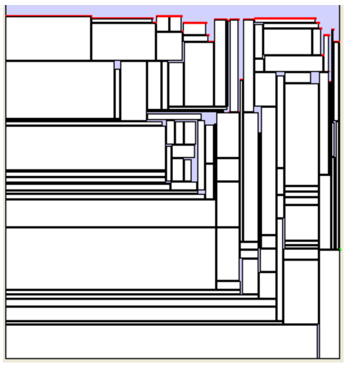
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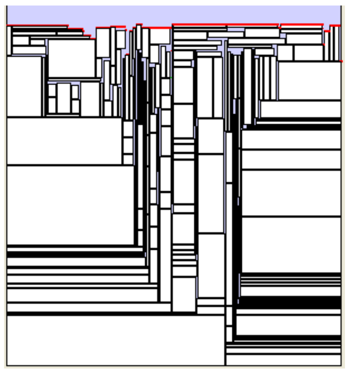
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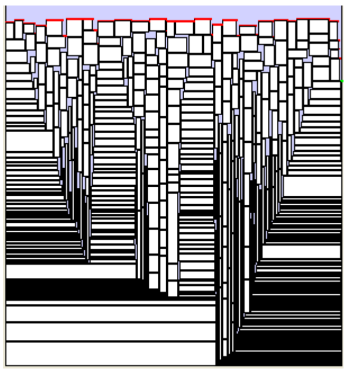
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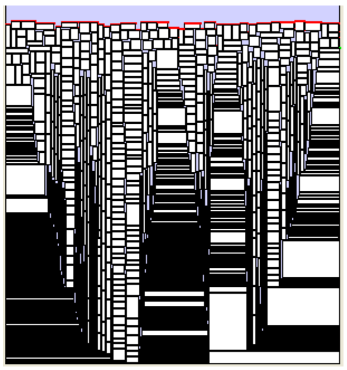
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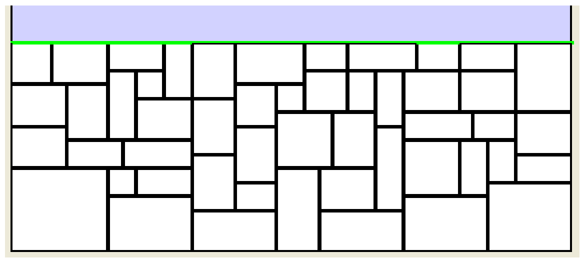
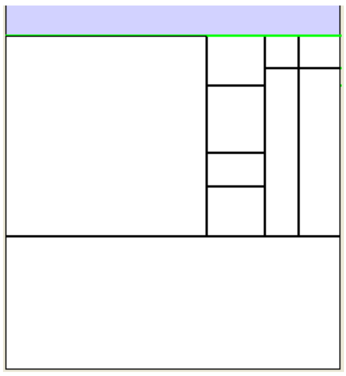


RAMESH BABU (h = 375)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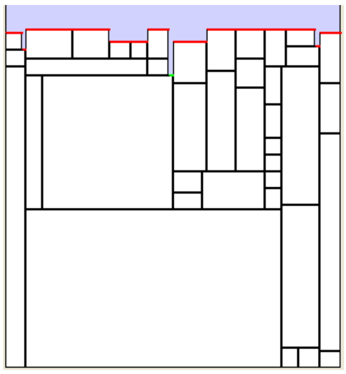
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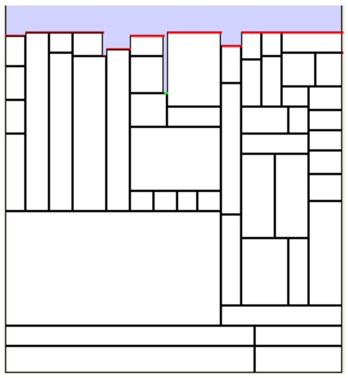
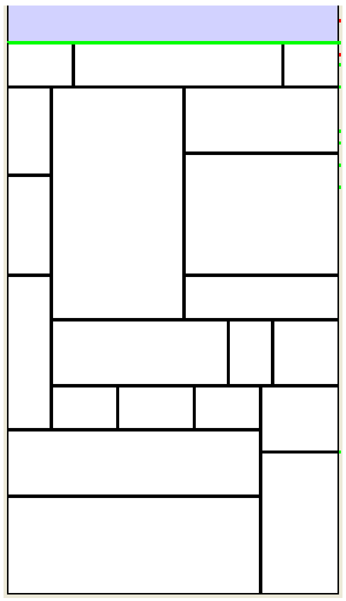

N5 (h = 101)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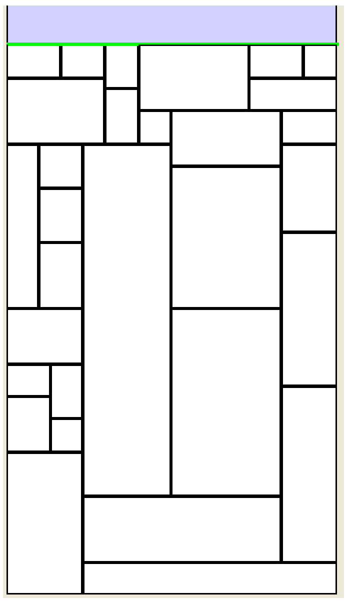
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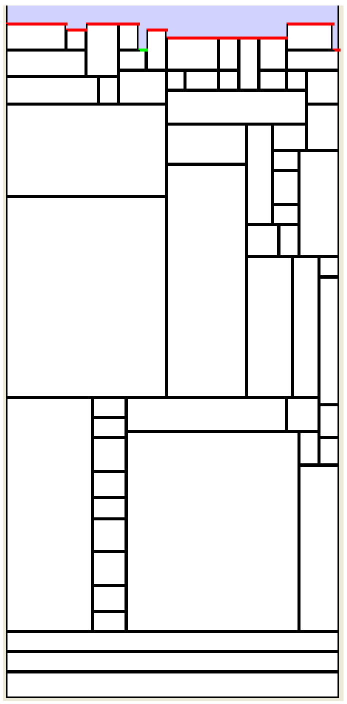
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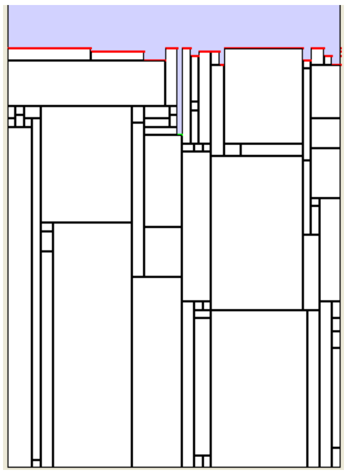
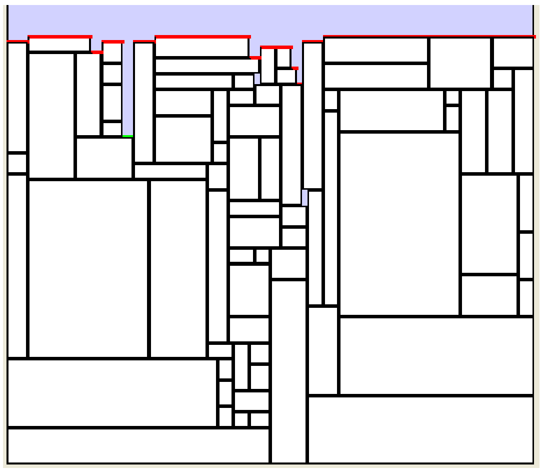
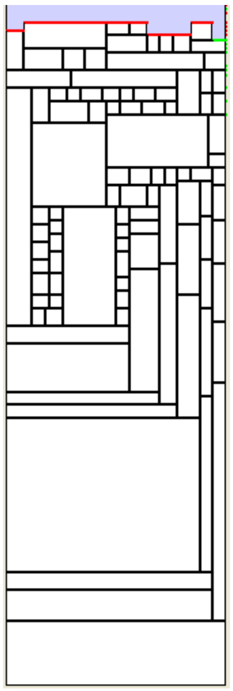
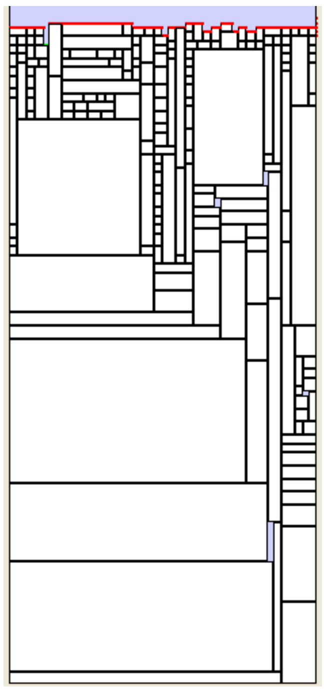

N7 (h = 101)


N8 (h = 81)


N9 (h = 151)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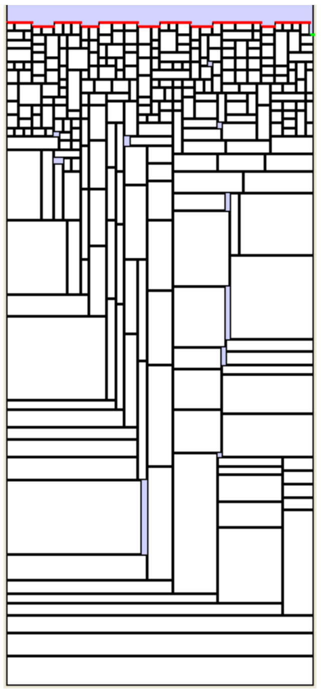
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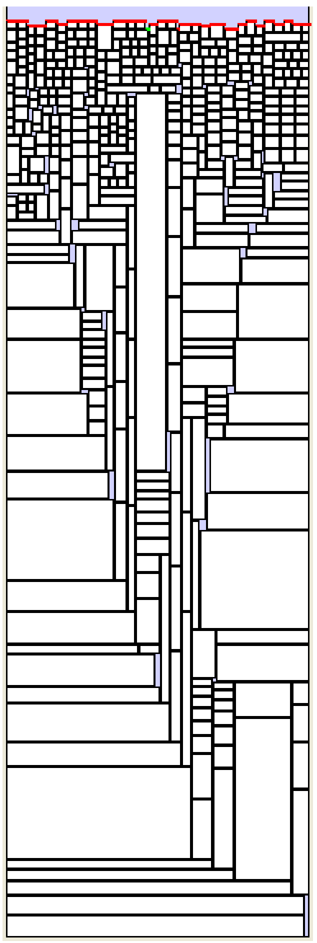
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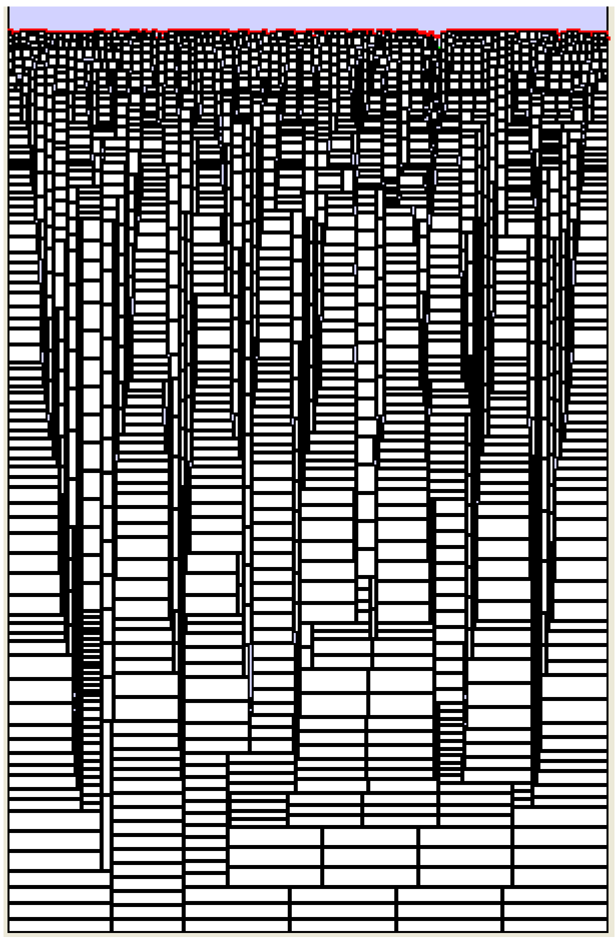
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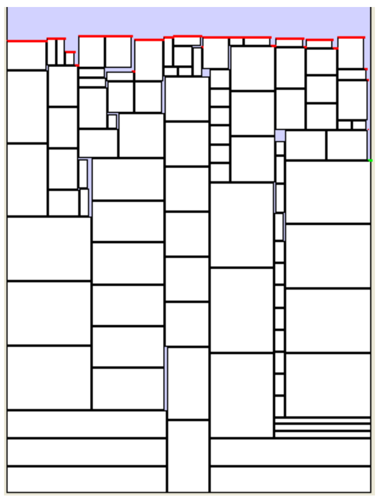
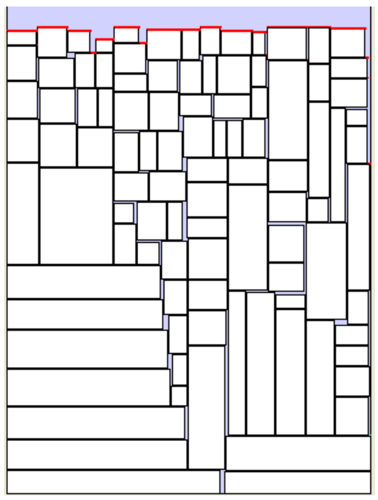
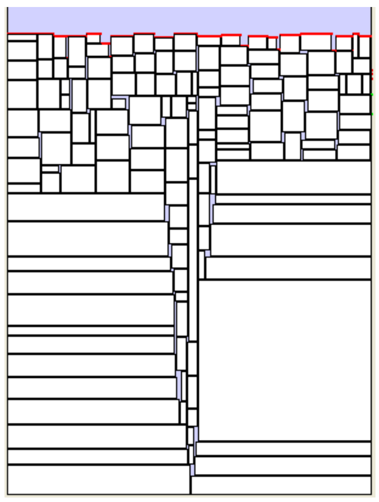
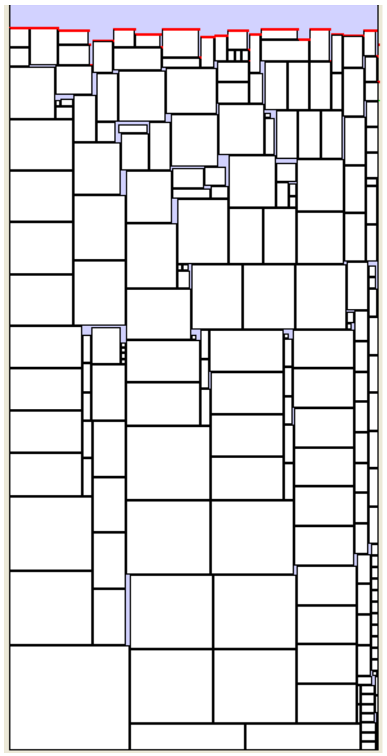
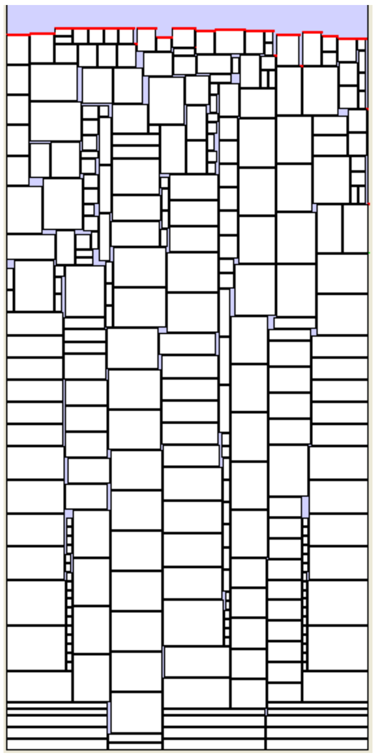
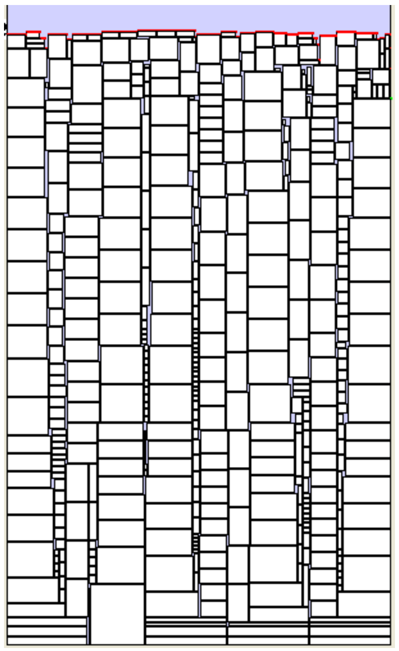

MT1 (h = 500.90)


MT6 (h = 513.53)

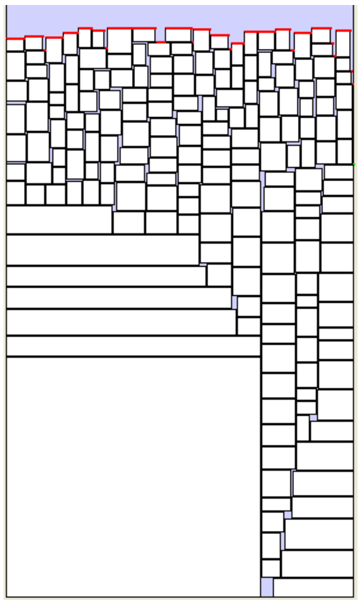

MT7 (h = 509.15)

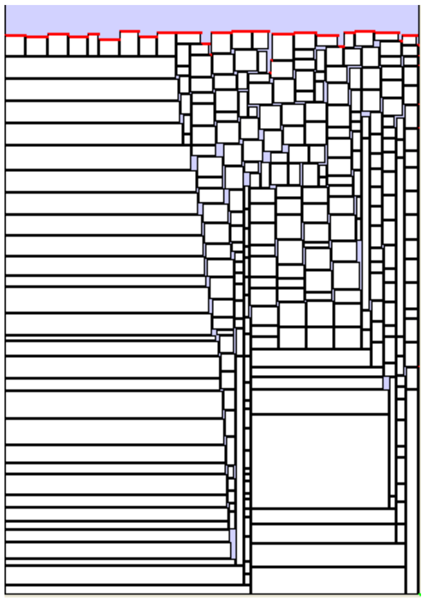


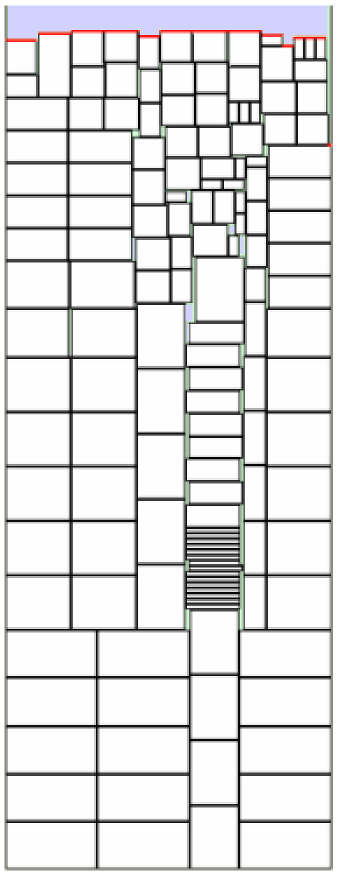


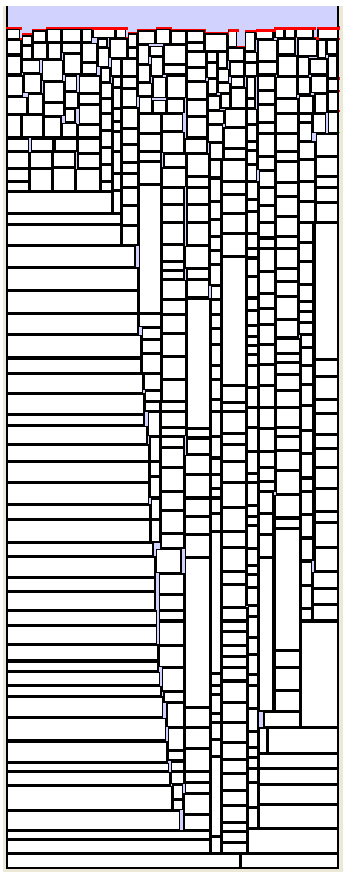


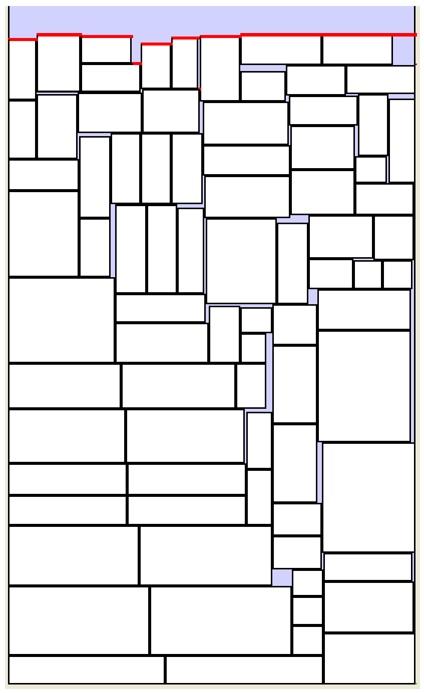
MT3 (h = 779.70) **MT4** (h = 1197.00) **MT5** (h = 1280.60)

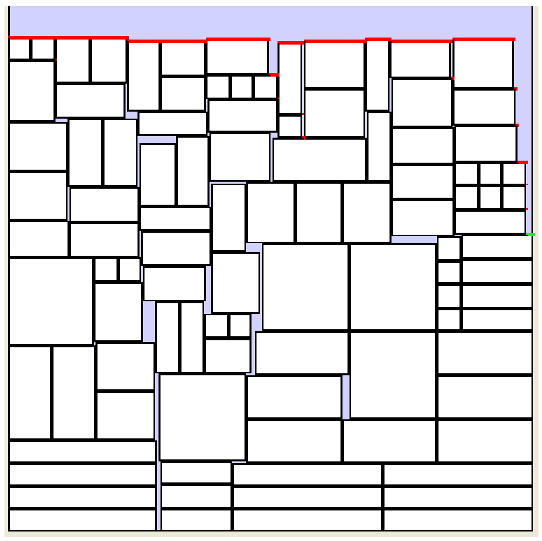

MT8 (h = 816.01)

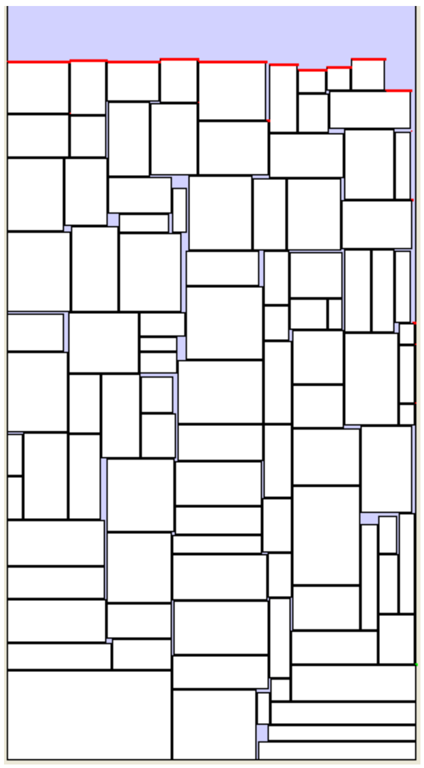

MT9 (h = 812.45)

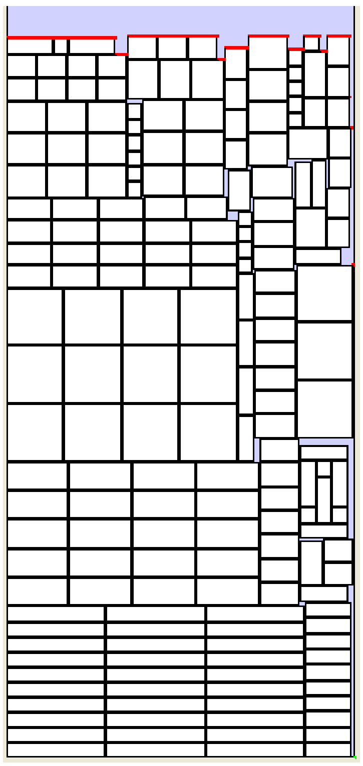

MT2 (h = 771.70)

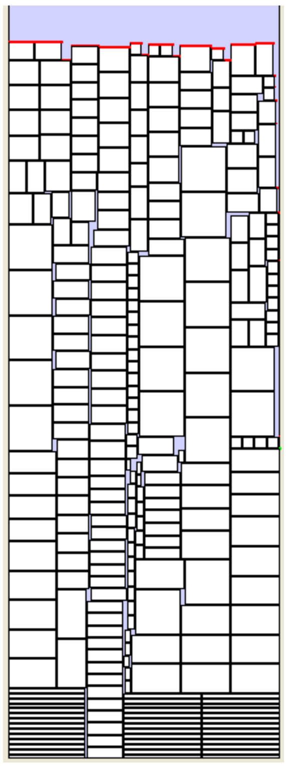

MT10 (h = 1514.69)


U1 (h = 79.37)


U2 (h = 75.13)

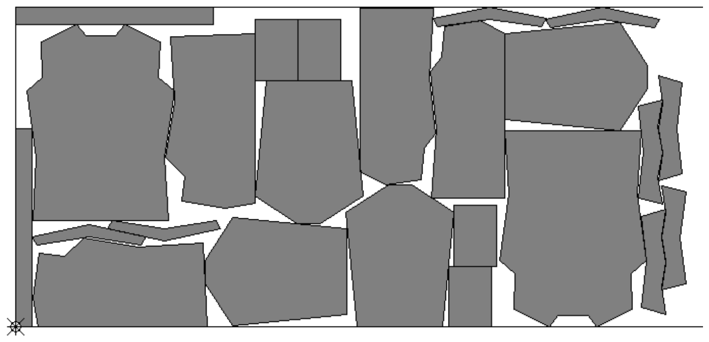

U3 (h = 256.57)


U4 (h = 166.43)

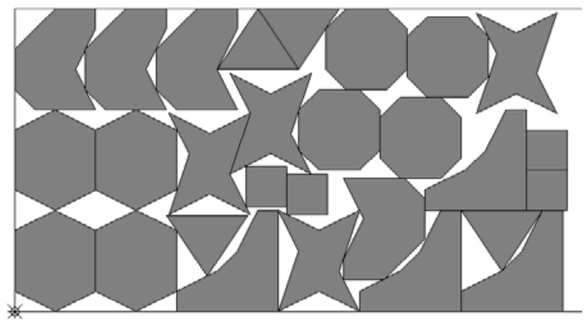

U5 (h = 211.87)

APPENDIX B – Layouts for Irregular Benchmarks

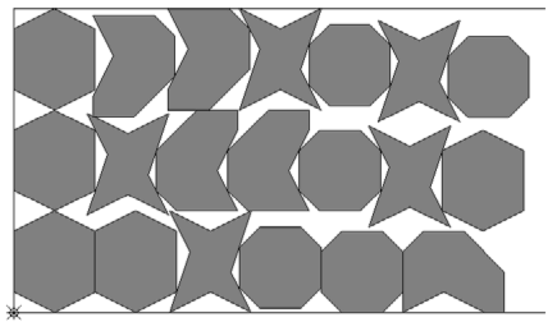
B.1 – Solutions from Chapter Seven (Trigonometry Packing)



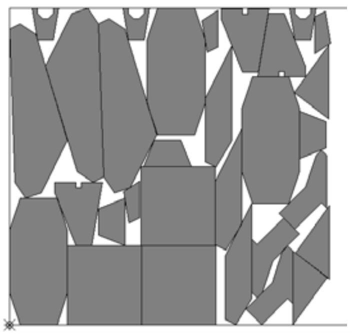
Albano: 10292.90 units (93.3s)

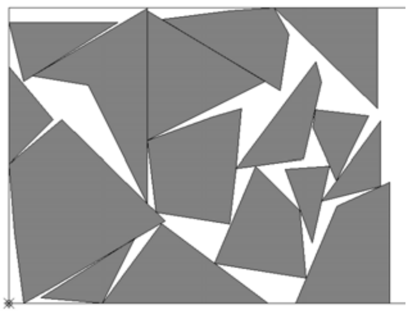


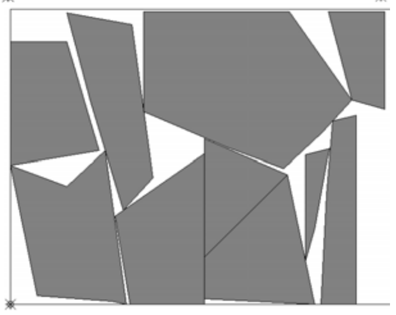
Blasz1: 27.20 units (501.9s)



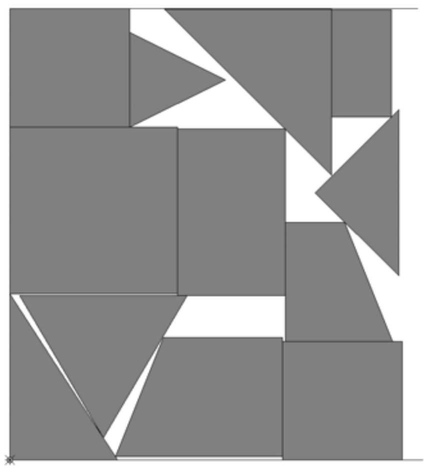
Blasz2: 25.28 units (10.9s)

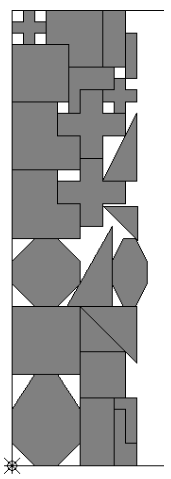


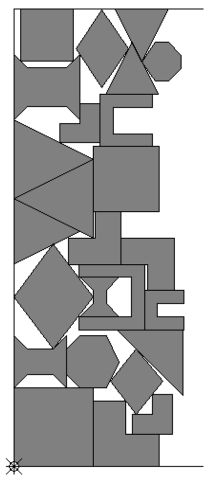




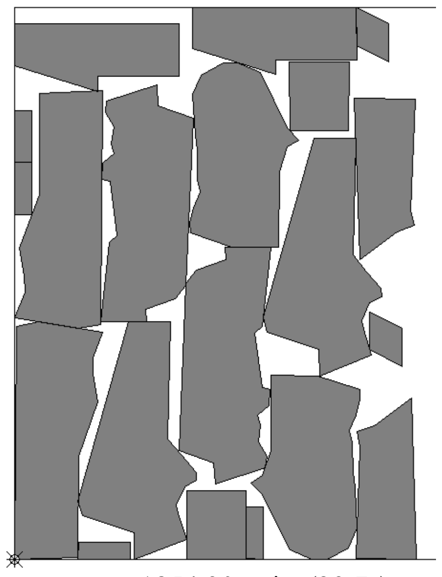
Dagli: 60.57 units (188.8s) **Dighe1**: 1292.30 units (8.8s) **Dighe2**: 1260.00 units (7.1s)

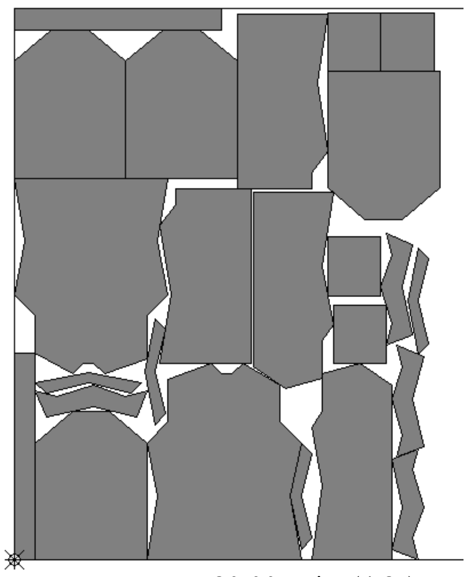


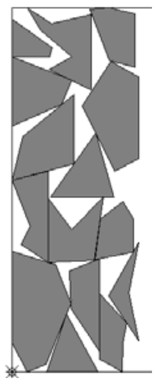


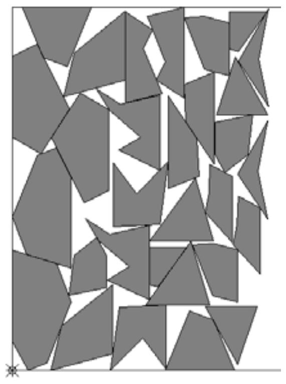


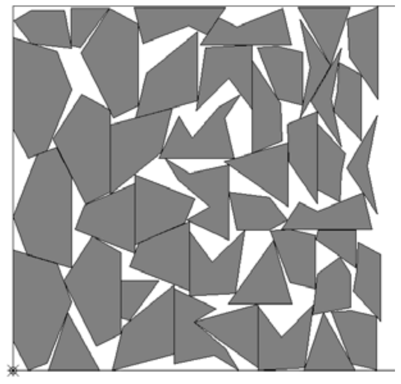
Fu: 32.80 units (20.7s) **Jakobs1**: 11.86 units (43.4s) **Jakobs2**: 25.80 units (81.4s)

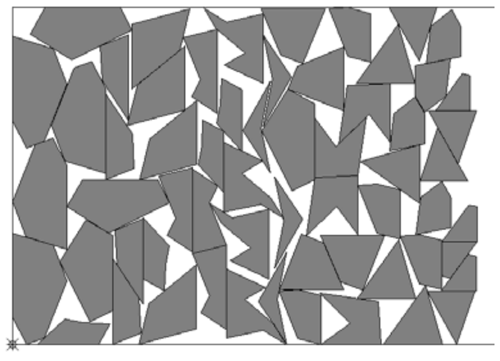

Mao: 1854.30 units (29.7s)

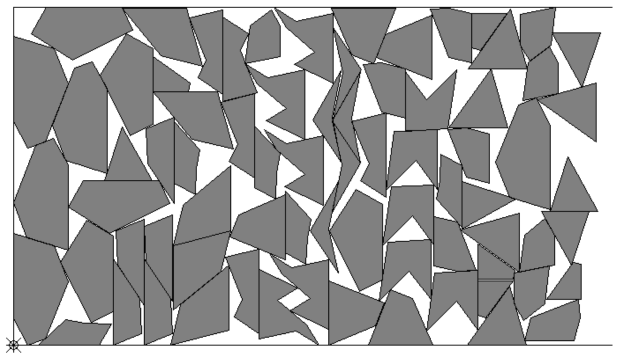

Marques: 80.00 units (4.8s)

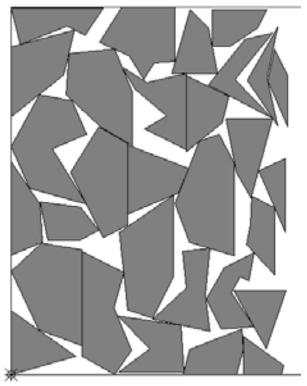

Poly1A: 14.00 units (12.5s)

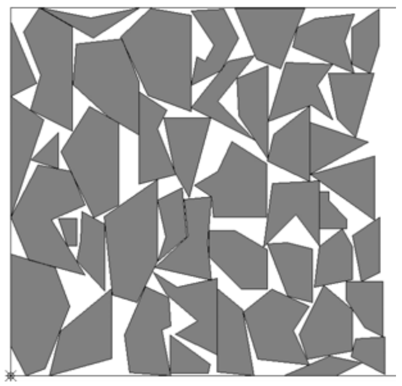

Poly2A: 28.17 units (120.6s)

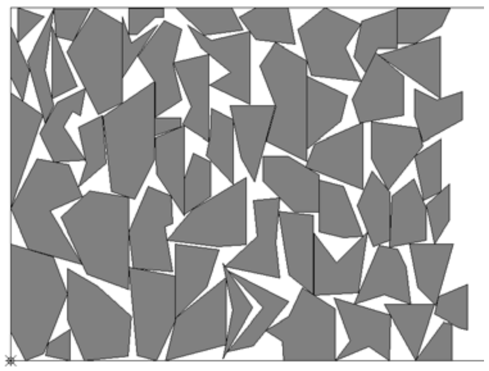

Poly3A: 40.33 units (1515.5s)

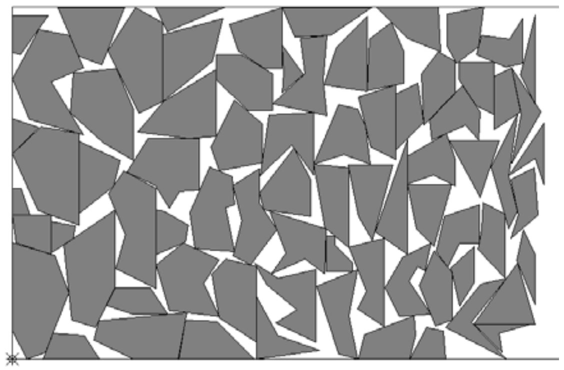

Poly4A: 54.93 units (203.2s)

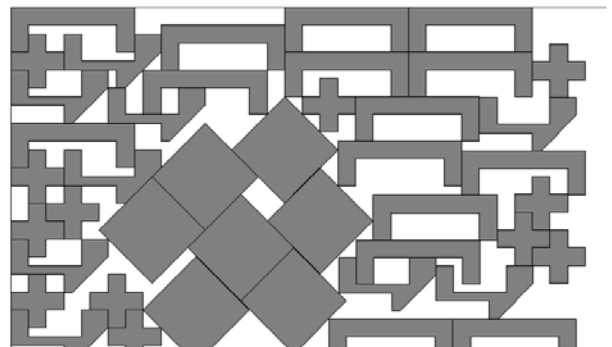

Poly5A: 69.37 units (475.6s)

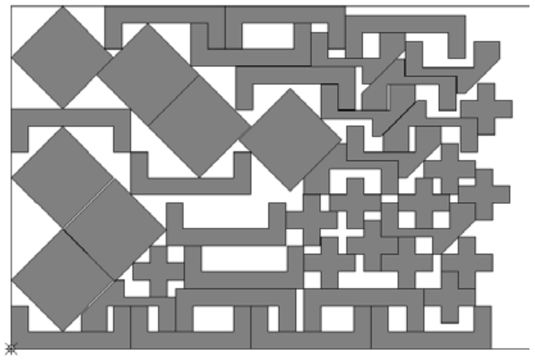

Poly2B: 30.0 units (179.9s)

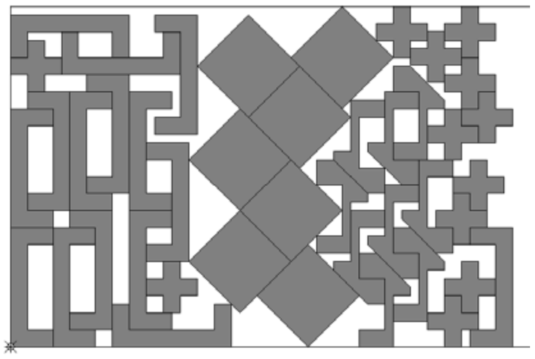

Poly3B: 40.74 units (417.7s)

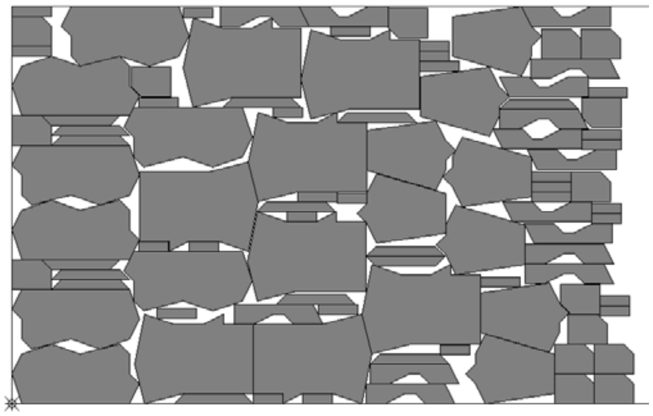

Poly4B: 51.73 units (95.7s)

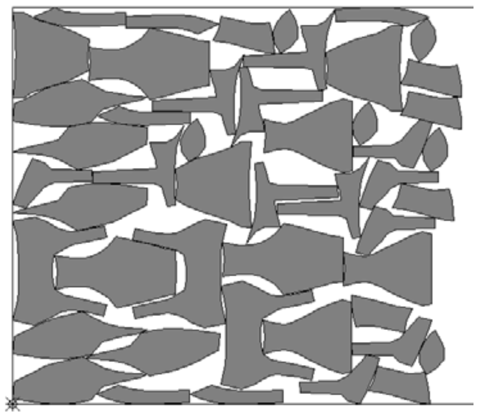

Poly5B: 60.54 units (676.6s)

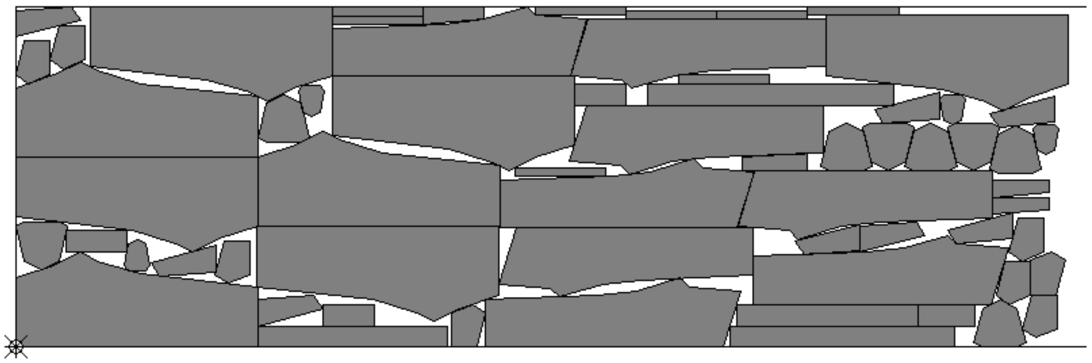

SHAPES0: 65.00 units (332.4s)

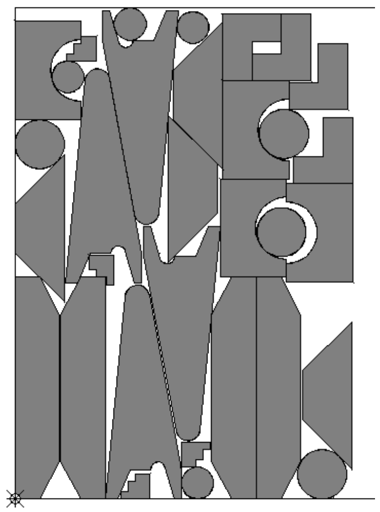

SHAPES1: 58.40 units (1810.1s)

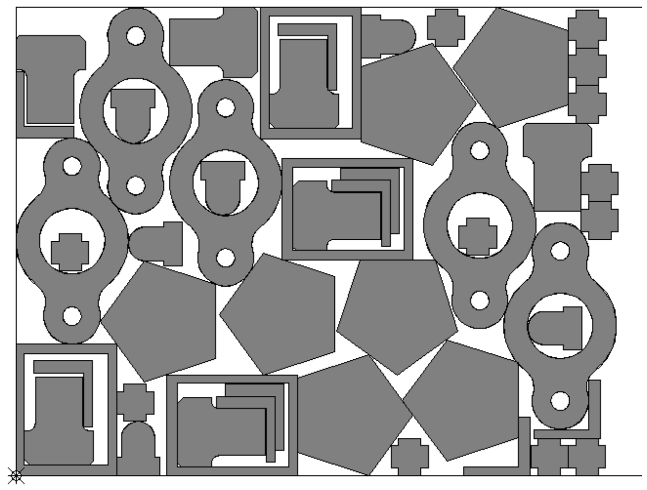

SHAPES: 59.00 units (31.4s)

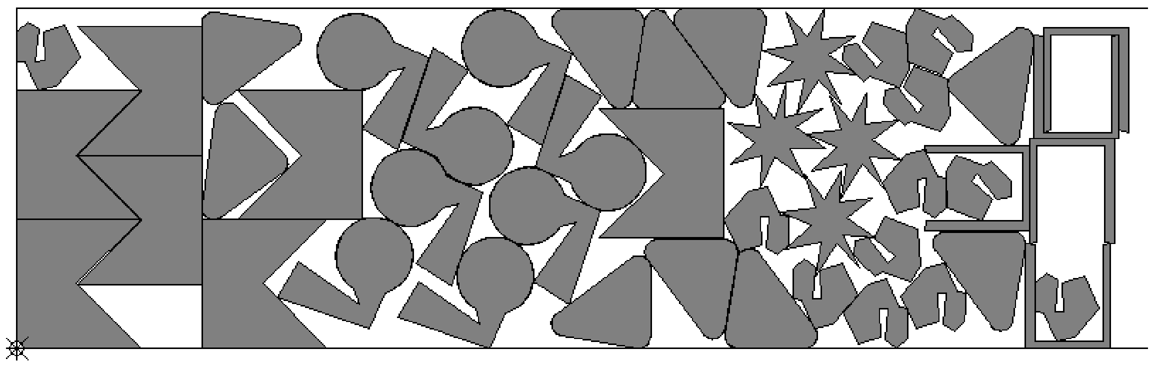

SHIRTS: 63.00 units (806.5s)

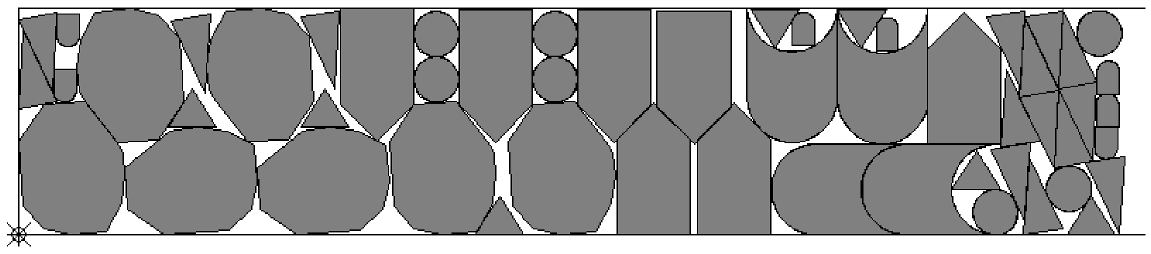

SWIM: 6462.40 units (607.4s)

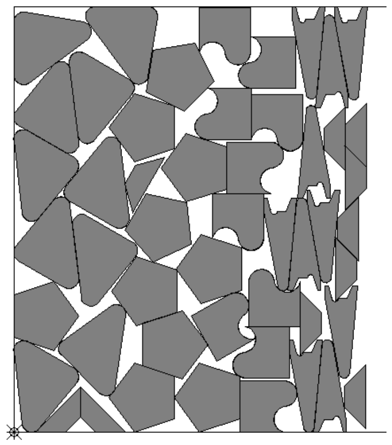

TROUSERS: 243.40 units (3612.0s)

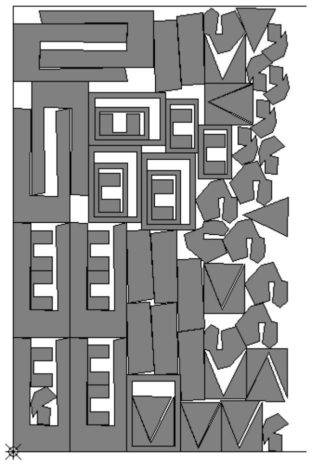

Profiles1: 1377.74 units

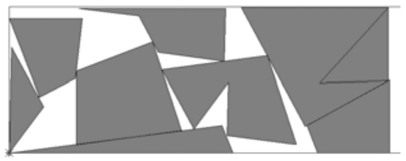

Profiles2: 3216.10 units

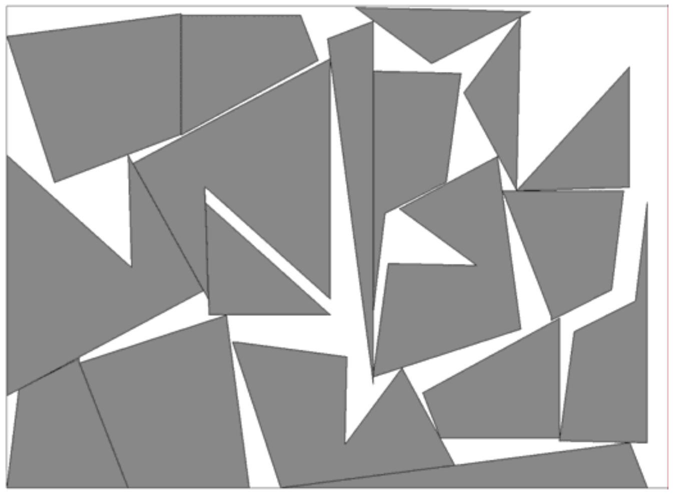

Profiles3: 8193.89 units

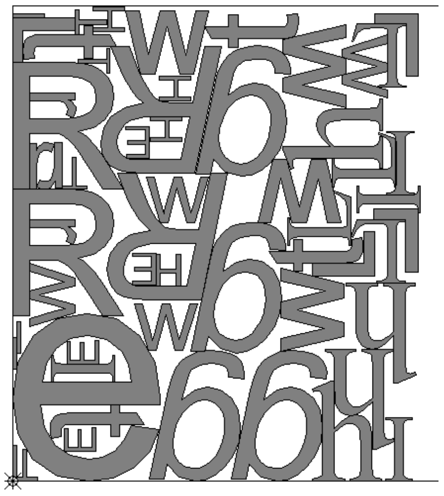

Profiles4: 2453.12 units

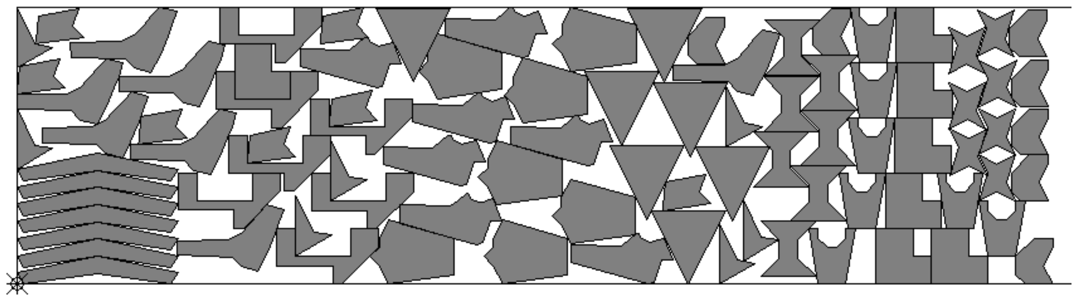

Profiles5: 3332.70 units


Profiles6: 3097.86 units

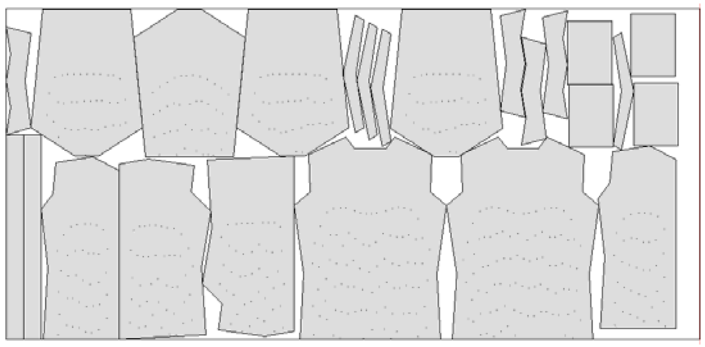

Profiles7: 1296.30 units


Profiles8: 1318.70 units

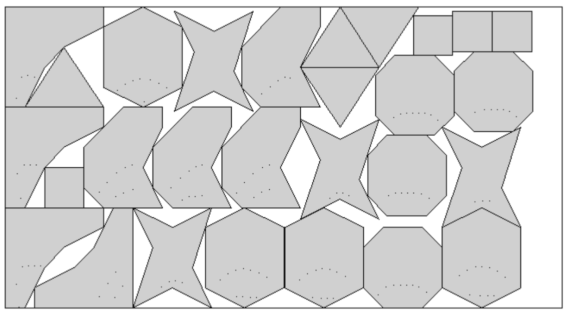

Profiles9: 1290.67 units


Profiles10: 11160.10 units

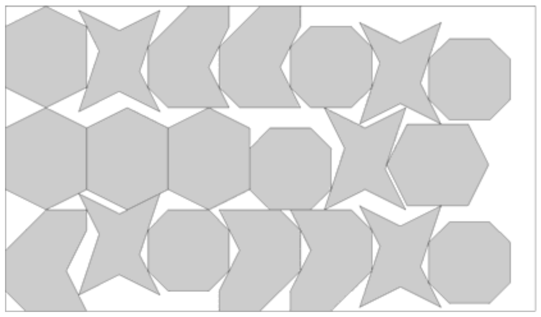
B.2 – Solutions from Chapter Nine (No-Fit Polygon Packing)



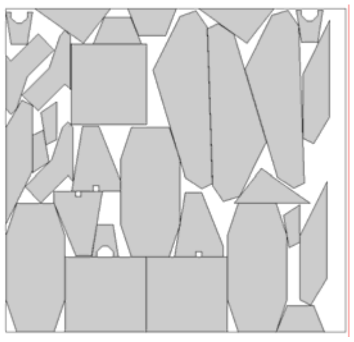
Albano: 9980.86 units (299s)



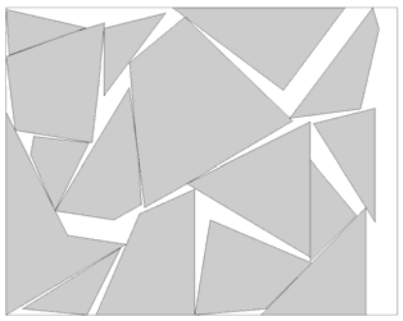
Blasz1: 26.80 units (281s)



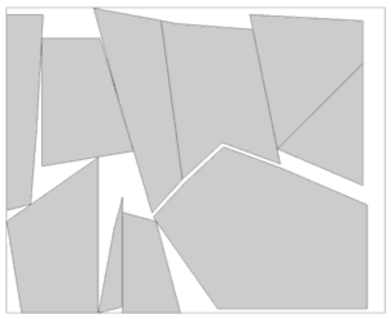
Blasz2: 24.80 units (14s)



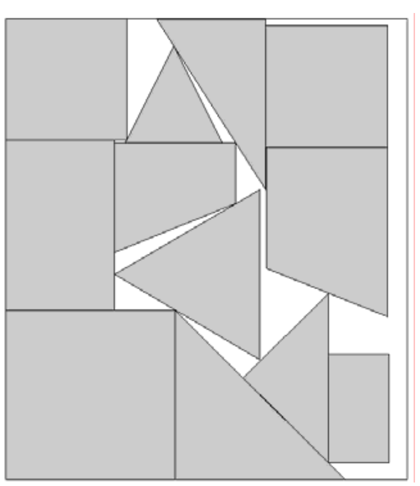
Dagli: 59.94 units (252s)

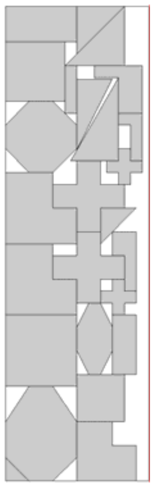


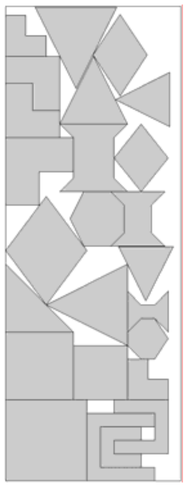
Dighe1: 1210.00 units (3s)

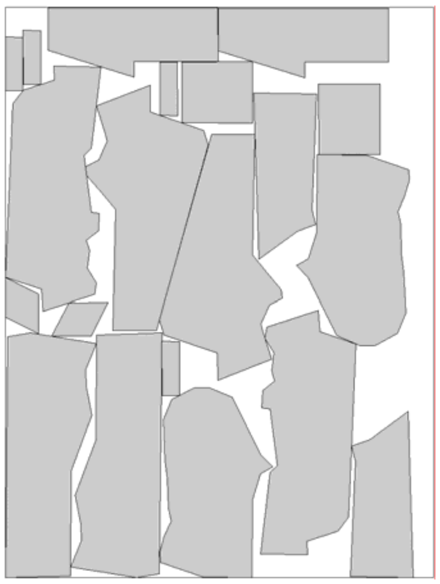


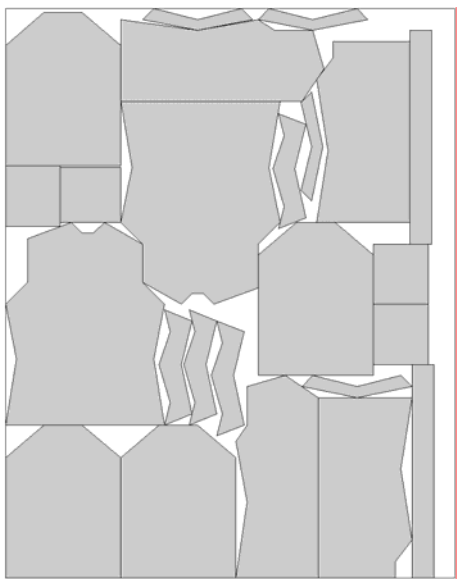
Dighe2: 1180.00 units (148s)

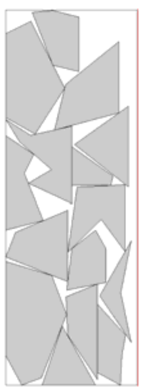

Fu: 31.60 units (139s)

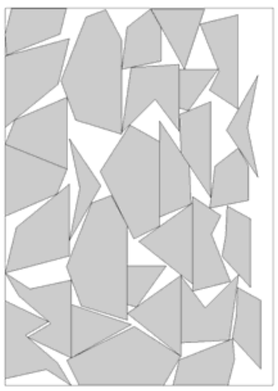

Jakobs1: 11.50 units (29s)

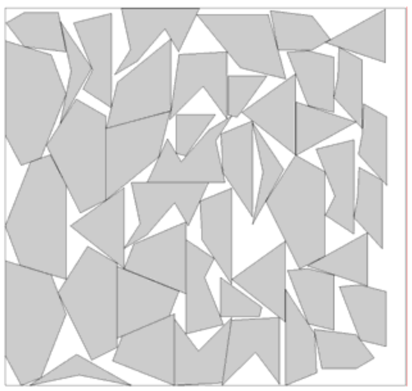

Jakobs2: 24.70 units (51s)

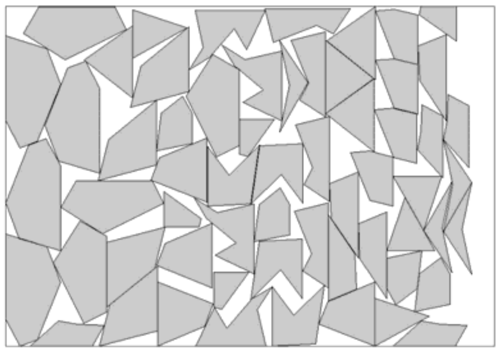

Mao: 1821.70 units (152s)

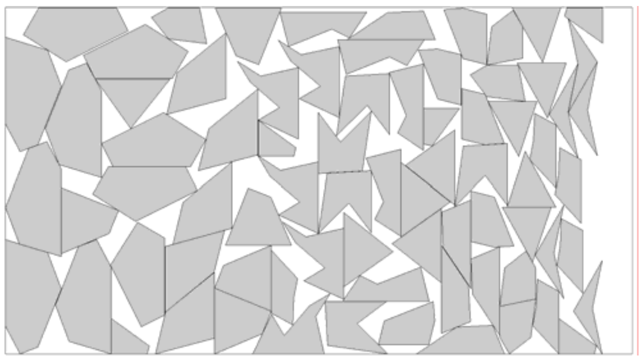

Marques: 78.00 units (21s)

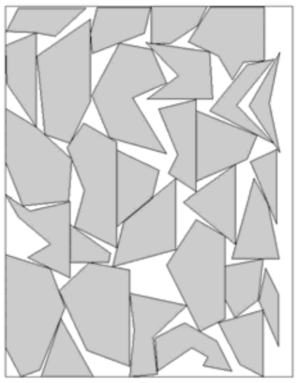

Poly1A: 13.30 units (254s)

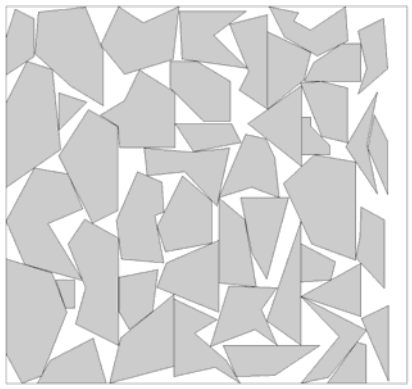

Poly2A: 27.09 units (239s)

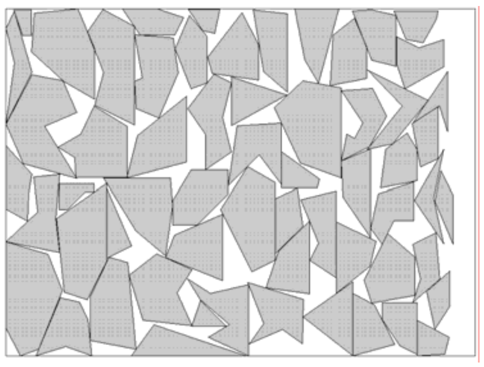

Poly3A: 40.45 units (429s)

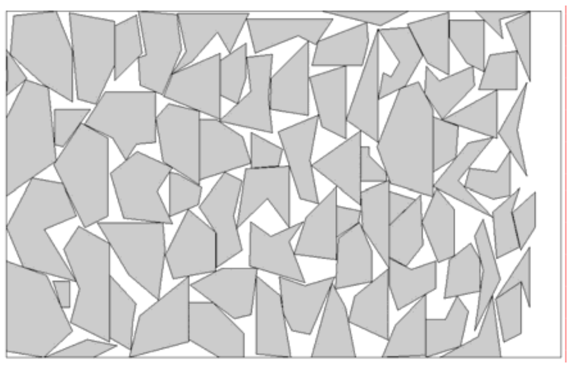

Poly4A: 54.60 units (224s)

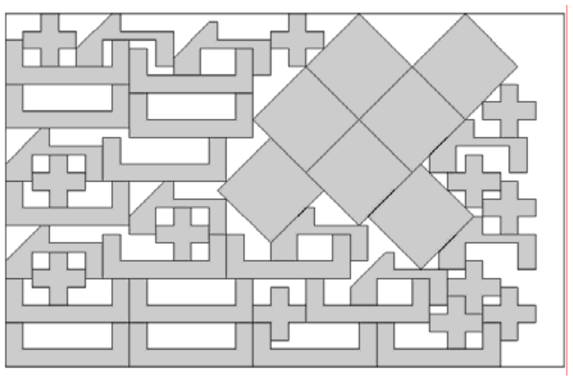

Poly5A: 68.84 units (300s)

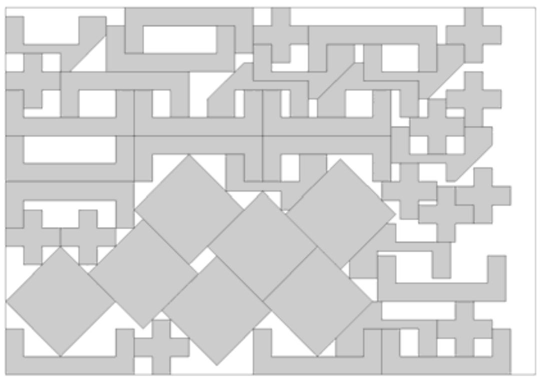

Poly2B: 29.63 units (189s)

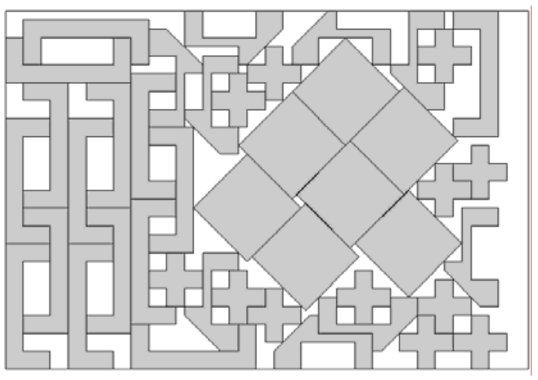

Poly3B: 40.50 units (114s)

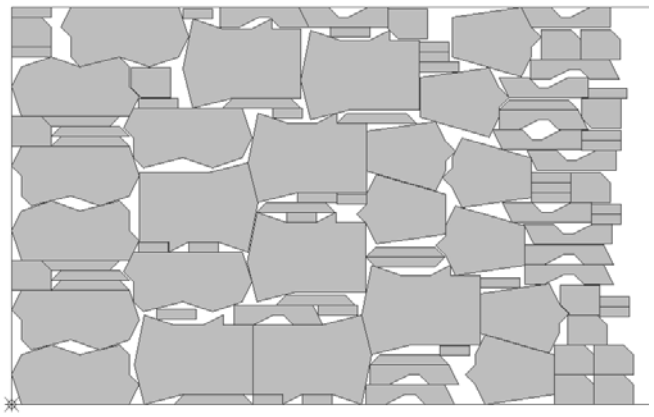

Poly4B: 51.18 units (176s)

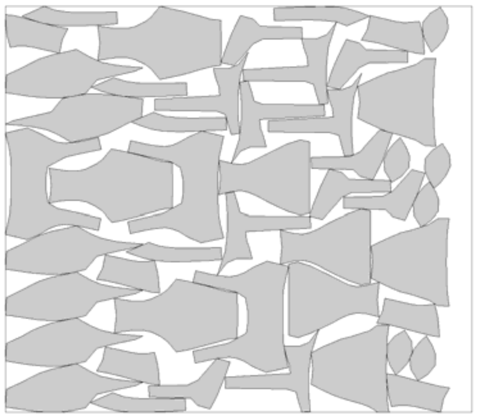

Poly5B: 60.86 units (299s)

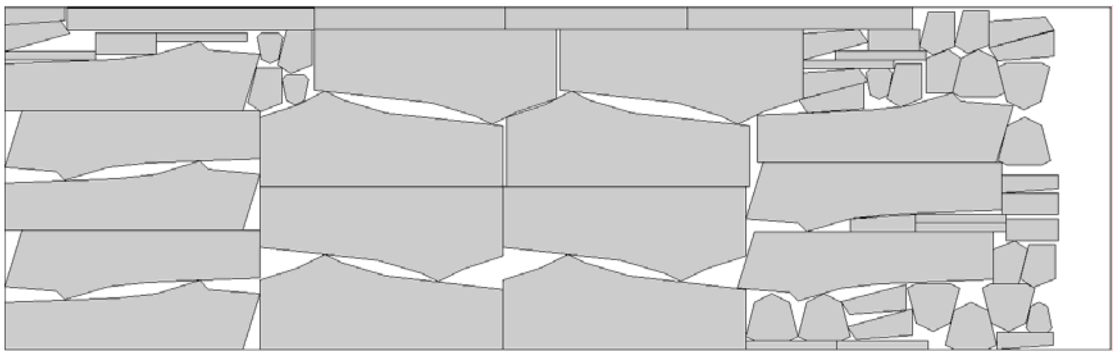

SHAPES0: 60.00 units (274s)

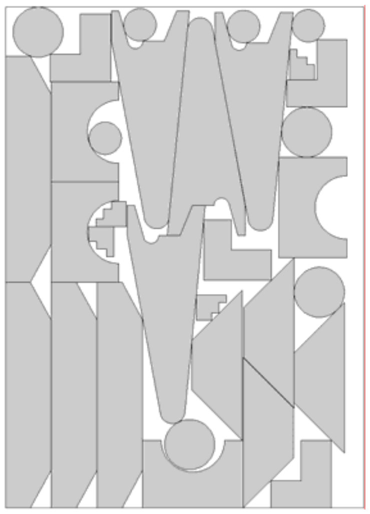
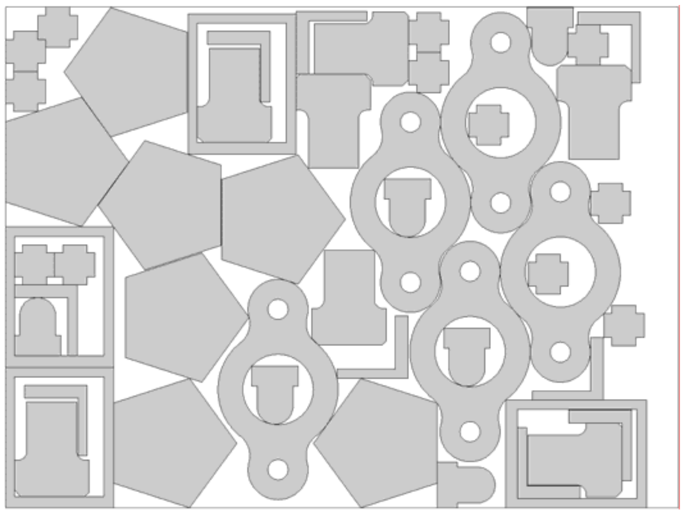

SHAPES1: 55.00 units (166s)


SHAPES: 56.00 units (226s)

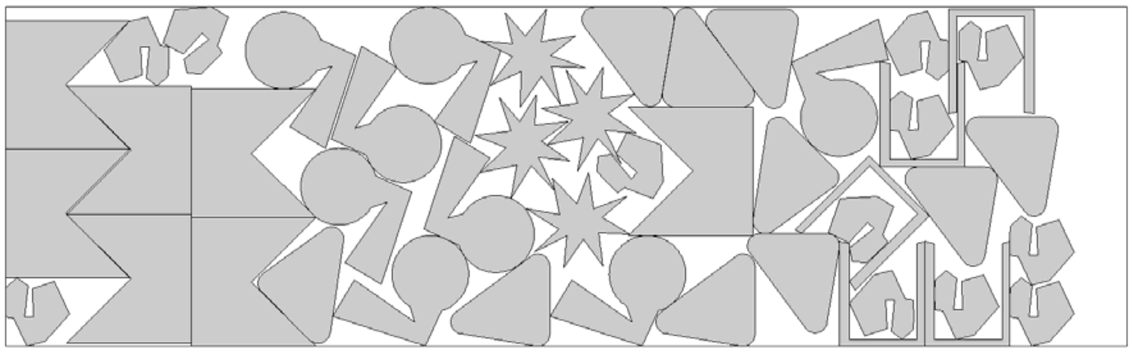

SHIRTS: 63.16 units (806s)


SWIM: 6270.88 units (141s)

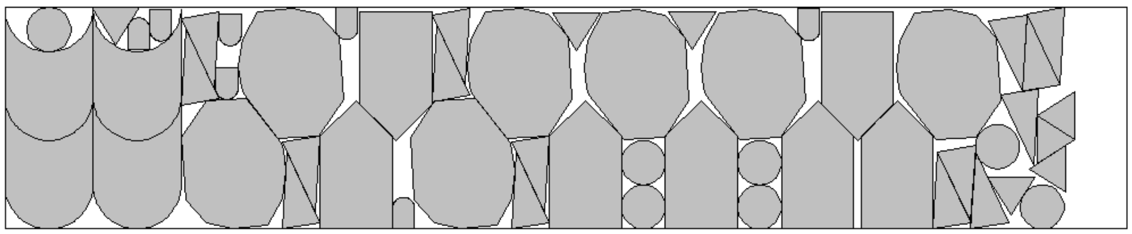

TROUSERS: 243.00 units (12484s)

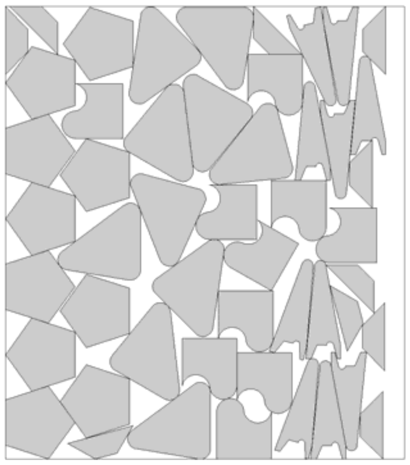
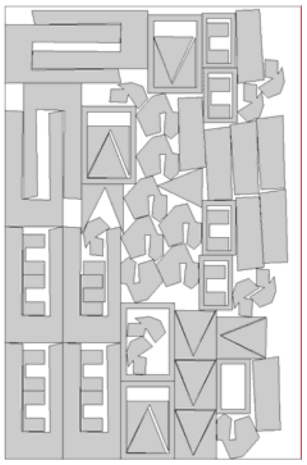
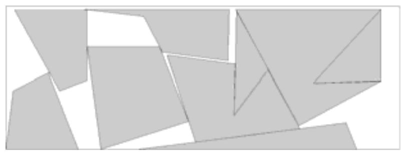
Profiles1: 1359.90 units **Profiles2**: 3194.19 units



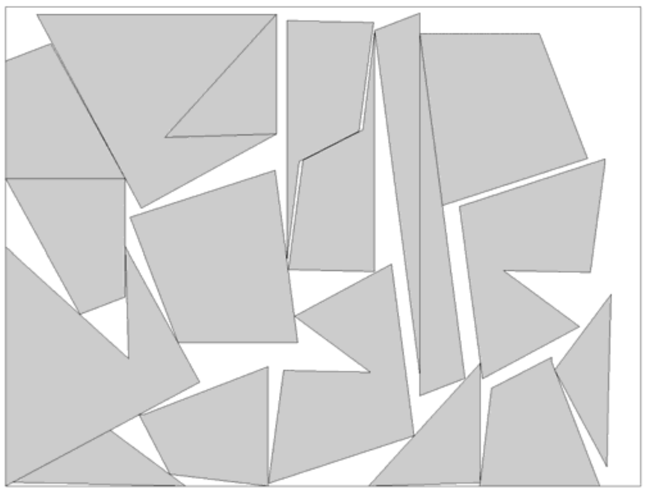
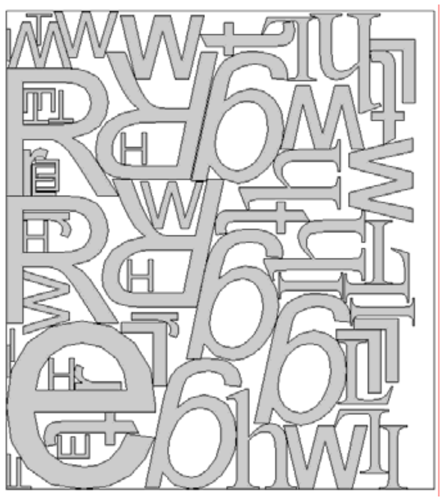
Profiles3: 7881.13 units



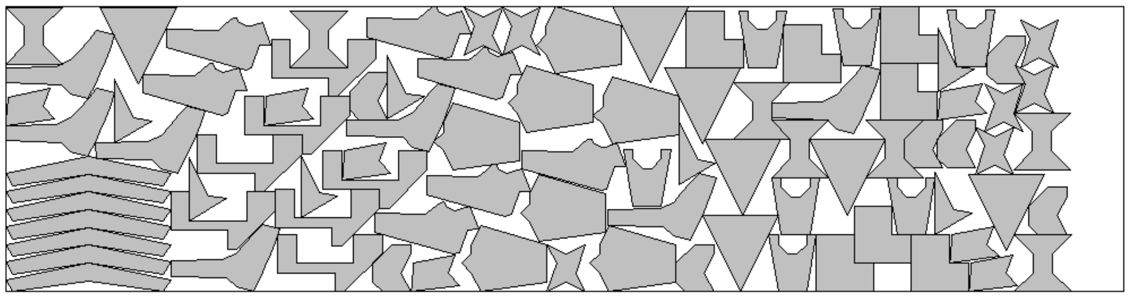
Profiles4: 2425.26 units

Profiles5: 3351.94 units **Profiles6**: 3121.36 units **Profiles7**: 1292.30 units

Profiles8: 1263.11 units **Profiles9**: 1278.21 units



Profiles10: 11219.60 units

APPENDIX C – New Rectangular Benchmark Instances

C1 – Problems N1 to N13

N1: 10 shapes, 40 x 40

Width	Height
7	6
40	16
5	20
24	24
7	4
4	4
7	8
4	20
5	4
7	6

N2: 20 shapes, 30 x 50

Width	Height	Width	Height
23	9	14	6
19	4	6	6
12	21	5	4
6	4	4	6
7	13	6	4
9	4	7	6
4	6	14	11
23	6	4	7
16	6	8	4
4	14	14	4

N3: 30 shapes, 30 x 50

Width	Height	Width	Height	Width	Height
10	6	6	4	5	4
4	4	3	15	3	5
4	5	9	6	3	3
3	5	3	3	5	8
10	5	3	4	5	16
3	5	6	18	10	13
7	13	7	5	3	3
3	4	3	4	10	17
8	32	5	3	5	14
3	23	3	8	5	3

N4: 40 shapes, 80 x 80

Width	Height	Width	Height	Width	Height	Width	Height
61	38	4	4	5	4	5	72
7	4	5	4	10	7	5	52
9	5	8	10	4	7	5	4
5	4	4	4	20	7	9	7
5	7	5	5	5	7	5	12
7	7	32	4	5	12	9	33
9	15	7	7	5	4	4	8
4	4	5	4	11	7	9	34
4	4	5	8	8	21	4	4
32	31	7	24	9	4	29	4

N5: 50 shapes, 100 x 100

Width	Height	Width	Height	Width	Height
25	10	53	7	20	6
74	8	20	6	16	22
27	19	46	10	20	14
64	34	18	6	10	6
74	6	10	6	14	8
39	6	10	10	8	6
48	7	9	6	7	6
11	10	7	6	16	6
8	10	11	6	8	6
14	6	7	6	15	6
6	10	6	8		
26	6	9	7		
27	6	47	7		
6	10	10	7		
8	10	7	6		
31	10	25	10		
23	6	26	8		
7	10	36	6		
6	10	6	6		
11	10	6	7		

N6: 60 shapes, 50 x 100

Width	Height	Width	Height	Width	Height
3	3	50	4	3	5
9	4	6	25	5	8
26	30	5	3	50	3
5	5	24	30	3	19
4	21	4	12	8	3
14	24	5	3	3	8
4	5	5	4	4	3
3	4	5	3	4	3
3	4	5	4	4	3
3	5	12	6	35	13
5	5	5	5	4	3
5	5	16	6	4	3
3	3	3	3	5	5
7	5	7	5	3	3
3	5	4	14	5	21
3	5	5	5	4	7
5	3	4	5	35	12
3	6	7	21	4	15
3	3	5	24	4	5
3	5	5	8	50	3

N7: 70 shapes, 80 x 100

Width	Height	Width	Height	Width	Height
6	82	3	6	4	2
3	2	2	2	12	7
3	38	37	22	2	4
19	23	24	2	3	2
7	36	13	2	3	52
46	12	11	2	3	14
3	4	4	33	3	2
3	2	9	12	18	4
2	82	5	2	19	59
3	15	25	5	23	2
3	2	2	2		
3	22	2	2		
2	2	3	2		
3	37	2	2		
2	2	3	4		
2	2	3	22		
3	2	9	22		
3	40	3	5		
6	2	3	40		
7	2	3	4		
7	7	8	2		
2	2	3	20		
18	2	2	2		
13	7	2	2		
2	4	3	4		
2	2	2	2		
8	2	3	2		
3	2	23	38		
28	22	2	2		
22	2	11	38		

N8: 80 shapes, 100 x 80

Width	Height	Width	Height	Width	Height
17	6	3	15	12	10
3	5	12	6	18	4
16	5	37	15	3	5
4	4	3	4	5	16
4	3	3	3	20	5
4	5	34	11	4	5
4	4	4	5	4	5
12	4	23	4	8	6
3	11	23	4	10	3
3	4	37	3	3	4
40	13	7	50	3	7
8	20	20	3	11	19
3	9	3	5	13	43
4	28	4	7	5	20
9	11	10	6	5	11
3	22	7	6	4	4
4	3	5	8	9	3
4	3	3	3	3	4
20	4	4	4	3	14
4	35	9	24	10	6
4	4	11	8		
23	34	4	3		
4	29	3	5		
3	12	8	10		
11	8	7	3		
4	21	3	3		
7	35	3	5		
23	35	16	5		
3	4	7	4		
4	5	3	10		

N9: 100 shapes, 50 x 150

W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H
5	3	3	5	4	3	3	4	3	3	3	6	4	28	5	3	4	32	3	4	3	10	3	14
3	5	4	3	5	3	12	23	3	5	4	5	3	5	4	3	3	5	3	39	4	9		
5	3	47	7	4	4	28	7	3	3	6	5	3	3	28	11	8	3	19	17	15	50		
5	5	4	4	4	3	4	3	9	5	7	3	4	4	3	3	3	5	10	5	12	27		
4	5	11	3	8	7	3	40	3	3	3	8	4	4	4	3	3	4	6	54	22	4		
4	15	3	4	3	3	4	5	5	3	4	5	3	35	10	3	4	3	9	4	8	4		
5	5	4	5	4	4	24	4	3	20	3	5	3	13	10	5	4	9	7	3	3	4		
3	3	19	6	28	5	4	3	3	3	47	4	4	3	3	8	5	3	3	54	16	5		
44	35	3	5	9	5	3	3	3	4	3	3	3	3	3	4	4	5	3	5	18	4		

N10: 200 shapes, 70 x 150

W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H
2	3	2	2	2	2	6	2	2	2	2	2	4	6	2	2	3	12	3	2	5	2	6	2
3	3	2	2	3	2	3	2	2	2	2	2	51	3	2	3	2	2	14	3	3	3	8	3
3	21	2	3	3	2	2	2	2	8	13	2	3	2	6	3	16	31	2	2	4	4	2	2
54	33	3	3	25	3	25	3	20	2	2	2	9	2	34	2	2	2	3	12	42	3	2	2
8	5	3	2	2	2	2	8	2	3	2	3	5	9	2	2	3	2	2	2	2	2	3	3
2	2	2	2	2	2	2	2	4	2	2	2	2	2	13	33	13	3	4	2	3	11	2	3
3	2	3	62	2	2	2	2	2	3	2	3	25	2	2	2	2	2	3	3	2	3	2	3
2	2	3	5	2	4	2	3	3	8	2	2	2	2	16	3	2	6	2	3	2	2	2	3
3	2	2	2	60	25	3	3	2	11	2	3	3	2	4	2	16	4	27	2	2	2	2	2
5	3	2	3	2	2	5	13	2	3	19	8	3	3	2	7	8	3	3	2	31	28	19	2
2	2	2	2	59	18	4	3	5	12	9	4	2	3	2	5	2	5	2	4	2	2	2	2
29	3	6	6	5	28	4	2	32	2	17	8	32	2	2	2	3	2	7	5	2	3	2	2
3	3	3	3	3	2	2	2	2	2	5	2	11	3	2	2	3	3	5	2	2	2	20	2
2	3	2	2	12	3	2	2	3	2	2	2	17	6	2	2	2	2	3	10	2	2		
6	50	2	2	29	2	48	3	5	2	2	3	3	2	7	2	2	2	5	3	2	7		
3	2	2	2	2	2	2	2	2	3	5	3	2	2	8	3	2	2	14	3	2	2		
22	6	7	2	3	6	3	3	3	3	23	2	2	2	2	2	2	3	18	3	5	6		

N11: 300 shapes, 70 x 150

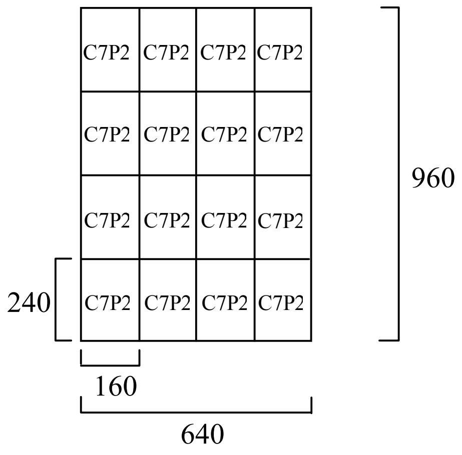
W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H
3	3	3	4	3	20	3	2	3	17	70	7	2	2	3	3	2	3	2	2	2	6	32	10
17	3	3	2	2	5	19	23	3	2	2	3	2	2	6	9	2	3	3	3	3	3	3	3
21	9	2	3	3	3	3	2	2	2	32	6	9	11	3	2	3	3	8	4	11	4	2	2
2	3	2	5	2	2	3	2	63	3	4	2	4	16	2	13	2	2	9	3	5	5	2	3
2	11	2	10	3	2	2	2	2	2	4	2	3	3	3	2	6	17	9	2	9	5	2	2
2	3	2	3	7	3	2	3	2	2	2	7	3	3	2	2	3	5	8	2	4	6	3	12
3	3	3	2	2	3	19	19	4	3	2	3	2	3	2	3	3	2	6	2	13	17	3	9
31	17	2	3	10	11	3	2	3	2	2	2	2	2	15	2	2	2	2	4	2	2	2	5
2	2	3	2	5	4	2	5	8	2	15	17	7	2	2	2	4	2	2	3	2	14	14	17
2	3	3	3	3	2	2	3	4	3	3	2	2	6	2	3	3	2	8	3	14	17	12	2
20	3	2	5	3	2	3	3	3	10	2	3	23	6	2	2	2	7	10	2	3	3	2	2
2	3	2	3	38	3	2	3	3	2	2	2	18	2	2	2	10	4	3	6	3	8	11	4
2	3	2	2	48	2	3	2	3	2	3	3	3	2	3	4	3	2	30	5	27	2	3	4
70	4	12	14	2	5	3	3	2	2	3	2	4	8	3	3	2	2	3	2	10	21	3	3
5	4	2	3	6	2	4	11	2	2	30	4	3	2	24	2	2	3	10	3	2	4	7	3
23	2	3	30	3	3	2	3	2	2	21	3	10	2	4	13	4	3	3	4	6	2	2	3
5	3	3	3	8	16	2	3	3	3	3	3	8	3	3	16	2	3	2	2	21	2	3	2
2	12	2	3	2	3	3	3	2	23	2	2	10	2	26	6	3	2	3	2	4	2	4	3
2	3	2	3	2	3	2	3	2	3	2	3	2	8	2	2	3	3	5	2	2	4	3	2
2	3	8	3	3	2	3	5	2	3	3	3	5	2	3	2	3	3	7	3	7	3	16	2
7	2	4	2	15	2	2	2	18	2	3	5	3	5	2	3	6	16	3	4	2	3	5	5
2	3	2	3	15	6	2	3	2	3	3	3	3	2	5	11	2	2	2	2	3	2	4	11
5	16	2	2	2	3	2	2	3	2	3	3	2	2	70	5	2	2	3	3	3	2	5	17
2	2	24	7	2	3	4	6	2	3	27	4	5	4	3	5	25	2	12	4	19	5	3	4
15	6	3	3	3	3	5	3	3	3	3	3	5	16	2	2	2	2	2	2	10	4	2	4

N12: 500 shapes, 100 x 300

W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H	W	H
5	4	4	3	5	3	3	5	3	4	9	34	4	28	3	7	5	3	4	75	3	7	5	3
3	13	12	33	5	20	5	5	3	12	3	3	3	4	4	3	3	4	8	6	10	125	4	4
49	3	3	5	3	26	3	3	3	3	3	3	3	4	3	4	3	5	22	5	3	5	6	98
3	5	3	3	5	3	4	4	3	3	4	3	5	4	3	29	3	10	4	6	4	5	3	3
5	5	4	4	17	3	3	3	3	3	5	16	5	5	3	3	3	4	9	23	5	6	9	5
3	5	3	5	5	7	3	3	3	3	3	36	3	4	3	5	3	4	3	7	5	94	3	3
5	5	5	5	5	9	3	3	5	10	4	5	3	4	4	7	3	4	12	31	6	8	3	3
3	3	5	4	3	7	3	16	3	3	5	4	3	3	12	3	3	3	31	3	4	3	3	4
4	10	7	3	54	13	3	5	3	3	3	21	3	3	3	3	8	4	15	8	4	4	10	27
3	4	5	3	10	10	12	3	3	3	3	3	4	3	8	58	3	4	4	3	3	3	3	21
4	11	5	4	14	3	3	5	3	5	4	3	3	3	14	5	3	68	3	9	5	18	10	17
3	7	5	3	3	4	3	3	3	13	3	4	3	3	5	4	7	15	3	12	4	5	3	3
9	8	3	4	3	3	3	4	3	3	5	3	3	3	4	5	5	4	3	3	5	5	3	9
7	33	3	4	3	5	5	3	3	3	3	4	3	3	3	4	3	3	36	27	3	3	3	51
5	5	3	5	3	3	4	4	4	5	3	3	5	3	5	4	5	20	3	8	3	10	8	98
5	3	3	3	3	3	5	3	4	5	3	5	3	6	3	4	6	5	5	51	4	3	6	7
4	5	3	4	3	3	5	4	3	3	3	4	3	3	4	7	3	3	3	35	3	3	3	33
5	4	7	3	3	3	5	5	54	19	4	10	3	12	3	4	3	4	3	4	3	25	7	20
5	3	5	8	3	3	3	18	5	3	21	3	3	3	3	5	4	3	5	8	4	3	3	8
3	5	34	12	5	6	3	3	12	3	3	3	22	10	6	16	3	5	3	3	4	9	4	3
12	5	5	5	5	3	3	6	5	10	4	5	4	6	3	5	4	10	3	10	3	3	4	13
3	3	3	6	5	5	3	3	3	3	3	4	20	4	3	5	9	3	5	3	3	6	3	4
3	4	3	3	3	4	3	4	15	23	3	7	4	5	5	5	5	11	4	16	3	3	6	35
3	3	3	3	5	3	3	3	3	3	7	4	3	3	3	5	4	3	3	4	18	24	3	5
3	5	5	34	4	3	3	5	5	3	11	3	3	3	4	5	5	4	3	3	9	3	3	4
5	3	3	5	3	3	4	3	3	3	3	3	4	4	3	5	3	3	4	5	4	5	3	6
3	5	3	3	3	4	3	3	3	4	3	3	3	7	3	4	5	3	3	3	5	7	3	4
3	7	5	3	14	27	7	6	19	12	4	3	3	3	3	5	3	4	3	3	4	11	4	3
5	5	3	3	3	3	4	3	4	3	3	5	3	3	3	5	4	14	18	3	9	3	24	4
5	3	3	4	3	4	3	3	5	18	8	7	3	4	4	3	18	25	5	4	3	12	3	6
5	3	3	5	11	3	12	19	3	3	3	3	12	4	3	4	4	3	3	21	36	33	4	3
3	7	5	3	3	3	3	17	3	4	4	24	5	4	3	5	3	4	3	23	3	4	5	6
5	3	3	21	4	4	3	3	33	18	5	3	3	8	3	12	10	25	3	3	16	8	30	4
3	3	4	4	3	3	3	3	3	4	3	3	5	5	3	4	11	49	5	13	3	6	3	3
3	4	3	28	3	4	3	5	3	7	4	4	3	44	5	5	3	3	5	23	3	4	17	3
5	4	3	3	3	3	3	5	3	10	3	5	18	5	3	3	5	4	43	11	3	5	4	5
4	4	3	14	4	5	3	3	4	3	4	3	3	5	3	3	5	3	3	3	4	26	5	5
3	4	31	5	4	5	4	3	4	5	18	25	3	3	5	3	15	4	31	61	4	3	3	6
3	3	3	3	3	5	4	3	5	11	3	3	4	3	28	5	3	3	3	3	4	10		
4	8	4	4	18	9	3	3	4	4	3	3	4	6	4	5	8	3	3	3	3	4		
5	4	3	5	4	16	4	3	3	3	3	3	9	5	3	3	4	5	4	5	40	10		
5	3	5	3	3	4	3	4	3	10	3	4	3	5	3	5	6	12	3	11	9	18		

N13: 3152 shapes, 640 x 960

Obtained by a 4 x 4 extension of problem c7p2 (197 shapes) from (Hopper and Turton, 2001) [**110**].



C2 – Problems MT1 to MT10

MT1 – Rectangles = 100, Sheet = 400 x 600

Width	Height	Quantity
49.5	49.5	10
12.5	24.1	8
45.9	80.1	10
32.8	45.2	12
15.7	10.2	8
35.2	29.777	10
30.5	176.8	5
7.5	105.8	3
20.5	22.8	7
31.4	10.8	9
70.4	93.4	10
10	30	8

MT2 – Rectangles = 150, Sheet = 300 x 900

Width	Height	Quantity
44.7	60	15
31	20.3	15
50.6	5	15
10	20	15
30	58.8	15
60.5	50	15
44.2	85	15
32.3	30	15
21	50.2	15
30.5	29	15

MT3 – Rectangles = 200, Sheet = 400 x 900

Width	Height	Quantity
42.5	65.5	10
90.3	80.3	9
52.5	24.1	16
45.9	80.1	10
32.8	15.2	15
15.7	10.2	8
31.2	9.777	10
60.5	36.8	7
70.5	55.8	12
20.5	22.8	7
11.4	40.8	15
50.4	56.4	10
14	28	8
30.3	125.1	2
8.3	12.4	14
14.1	20.1	4
5.9	6.1	12
130.4	114.8	1
13.2	8.1	7
3.4	20.3	6
40.3	10.2	5
40.2	30.6	5
34.8	22.5	7

MT4 – Rectangles = 300, Sheet = 600 x 1300

Width	Height	Quantity
20	70	5
43	71	10
80	20	8
25	26	20
101	76	5
30	13	19
170	20	9
10	15	20
37	13	15
99	55	12
111	52	4
40	50	10
50	60	10
10	15	20
95	37	17
67	87	24
43	82	10
34	57	20
10	169	5
15	90	2
18	18	5
20	15	10
32	39	20
62	80	20

MT5 – Rectangles = 500, Sheet = 800 x 1400

Width	Height	Quantity	Width	Height	Quantity
20	70	5	42.5	65.5	10
43	71	10	90.3	80.3	9
80	20	8	52.5	24.1	16
25	26	20	45.9	80.1	10
101	76	5	32.8	15.2	15
30	13	19	15.7	10.2	8
170	20	9	31.2	9.777	10
10	15	20	60.5	36.8	7
37	13	15	70.5	55.8	12
99	55	12	20.5	22.8	7
111	52	4	11.4	40.8	15
40	50	10	50.4	56.4	10
50	60	10	14	28	8
10	15	20	30.3	125.1	2
95	37	17	8.3	12.4	14
67	87	24	14.1	20.1	4
43	82	10	5.9	6.1	12
34	57	20	130.4	114.8	1
10	169	5	13.2	8.1	7
15	90	2	3.4	20.3	6
18	18	5	40.3	10.2	5
20	15	10	40.2	30.6	5
32	39	20	34.8	22.5	7
62	80	20			

MT6 – Rectangles = 100, Sheet = 400 x 500

Width	Height	Width	Height
27.512	234.999	39.306	31.992
199.36	33.779	31.982	39.161
196.456	35.04	39.306	31.492
180.646	41.972	44.37	26.295
177.68	42.868	28.545	41.972
170.417	37.106	44.37	25.494
33.478	172.228	35.51	33.779
39.306	159.304	26.956	41.972
31.982	157.449	44.531	23.544
106.471	81.204	42.603	25.451
25.378	160.305	33.328	33.779
158.302	19.435	33.153	33.779
41.07	135.208	39.306	27.559
139.246	33.795	42.496	23.544
137.903	25.451	22.971	42.868
115.684	44.195	42.209	23.544
32.775	126.51	21.769	42.868
44.37	109.042	19.971	44.195
106.471	45.306	41.07	22.685
111.667	36.305	38.817	23.544
17.658	125.495	36.809	25.451
104.507	23.544	38.595	23.544
47.236	41.972	24.477	37.106
44.44	44.195	24.354	37.106
40.17	44.195	26.753	33.779
47.525	36.305	23.07	37.106
44.37	38.539	16.851	42.868
39.421	42.868	23.09	36.305
38.791	42.868	16.251	42.868
41.07	40.182	17.063	41.972
44.37	36.489	16.514	41.972
42.835	37.106	15.596	41.972

MT7 – Rectangles = 150, Sheet = 400 x 500

Width	Height	Width	Height	Width	Height	Width	Height	Width	Height	Width	Height
202.076	33.656	37.913	12.586	27.544	35.951	16.711	33.656	7.302	25.987	14.725	21.891
17.307	199.243	37.913	24.104	17.425	35.951	19.55	33.656	25.208	25.987	8.172	21.891
26.547	197.95	37.913	30.366	13.814	35.951	16.388	33.656	25.922	21.339	10.01	21.891
197.837	21.891	37.913	35.627	30.359	35.951	33.566	21.891	25.754	10.646	16.5	21.891
193.599	21.756	37.913	12.889	31.091	35.951	32.804	25.578	24.034	25.578	17.517	21.891
16.023	193.293	37.913	17.879	33.608	35.951	32.592	19.854	14.79	25.578	11.937	21.756
190.63	28.737	37.913	12.162	14.279	35.951	32.508	25.987	19.477	25.578	20.915	21.756
190.63	177.112	37.913	18.428	35.444	35.951	8.393	31.436	6.61	25.578	14.659	21.756
23.793	187.358	37.913	27.532	35.851	31.436	18.788	31.436	18.738	25.578	6.919	21.339
186.312	25.177	37.908	10.646	35.427	10.266	6.096	31.436	19.083	25.177	17.307	21.275
185.174	19.854	37.473	19.854	34.974	21.339	15.889	31.436	23.094	25.177	20.212	10.646
184.718	10.614	37.154	11.027	34.839	25.578	31.162	25.987	21.316	25.177	8.307	19.854
34.457	184.483	37.154	28.582	23.793	34.602	30.954	21.339	20.426	25.177	12.866	19.854
183.673	25.987	37.154	11.165	34.457	24.599	30.29	25.578	21.269	25.177	18.025	19.854
181.449	25.578	37.154	22.976	34.457	17.647	17.307	27.967	24.503	10.646	8.683	19.854
16.176	180.713	37.154	6.785	16.176	34.04	27.336	25.987	24.228	19.854	11.245	19.854
178.305	10.646	37.154	6.75	33.905	25.177	26.547	26.855	23.793	22.539	16.176	18.205
37.913	177.419	37.154	11.989	22.309	33.656	26.547	6.365	23.793	14.862	16.588	10.646
176.29	35.951	37.154	10.495	6.897	33.656	26.547	17.964	23.793	10.598	16.176	9.681
174.217	21.339	36.969	25.578	21.813	33.656	26.547	14.298	23.793	17.671	16.176	10.348
173.4	31.436	16.176	36.758	14.269	33.656	26.547	18.662	23.793	13.63	16.023	12.556
171.226	10.266	36.555	25.177	8.266	33.656	26.547	23.889	23.494	19.854	13.898	10.646
37.154	170.152	16.176	36.494	31.774	33.656	26.547	17.168	16.176	22.899	13.719	10.614
38.572	10.646	36.316	10.646	18.591	33.656	15.029	25.987	22.585	21.339	7.944	10.646
38.04	25.177	20.146	35.951	21.356	33.656	23.693	25.987	21.037	21.891	8.216	10.614

MT8 – Rectangles = 200, Sheet = 500 x 800

Width	Height	Width	Height	Width	Height	Width	Height
345.889	367.32	40.963	32.484	43.229	19.342	21.938	32.151
29.314	367.32	40.344	33.069	21.033	41.48	23.624	30.393
38.752	332.136	35.817	36.728	25.437	36.728	20.869	33.069
325.393	31.617	40.787	30.393	26.039	35.984	20.786	33.069
45.082	278.932	40.963	29.819	31.417	30.393	20.614	33.069
288.682	30.713	26.548	43.956	29.523	32.151	20.55	33.069
40.963	154.009	40.963	29.31	34.781	26.83	23.088	30.393
103.746	41.48	41.476	28.511	24.862	36.728	21.11	32.151
115.92	28.511	50.594	19.342	19.53	41.906	24.613	28.511
98.582	43.956	45.082	24.833	34.489	26.83	29.589	23.522
82.801	41.906	36.479	33.069	39.076	22.217	26.218	26.83
87.041	35.984	40.647	28.511	34.268	26.83	19.955	33.069
88.961	32.151	40.963	28.037	28.451	32.151	33.554	19.342
51.745	43.956	32.112	36.728	24.524	35.984	24.382	28.511
45.082	47.894	36.4	32.151	23.6	36.728	22.156	30.393
62.51	30.393	36.29	32.151	31.706	28.511	19.32	33.069
50.979	41.906	26.627	41.48	40.963	19.159	19.286	33.069
45.082	47.278	37.457	30.393	29.421	30.393	19.159	33.069
47.76	43.956	30.935	36.728	27.005	32.151	20.051	32.151
50.141	41.48	30.817	36.728	38.752	20.333	32.842	19.342
49.463	41.906	23.537	43.956	22.321	36.728	21.79	30.393
40.963	48.607	38.752	28.737	26.833	32.151	19.029	33.069
46.989	41.48	25.273	41.906	25.853	33.069	21.557	30.393
46.907	41.48	36.775	30.393	22.895	35.984	19.53	32.151
45.518	41.48	25.32	41.48	39.076	19.689	23.048	28.511
49.747	36.728	25.237	41.48	38.752	19.981	24.68	26.83
50.219	35.984	24.704	41.906	21.862	36.728	31.603	19.505
49.466	36.728	24.703	41.906	28.126	30.393	19.693	31.155
44.015	41.906	35.297	30.713	39.173	19.342	31.372	19.156
40.963	44.058	22.007	43.956	22.268	35.984	19.059	31.155
40.963	43.085	29.946	35.984	27.735	30.393	29.589	20.434
40.963	42.93	38.905	26.83	30.726	26.83	23.183	26.83
39.195	43.956	21.521	43.956	21.486	35.984	20.564	28.511
41.135	41.906	28.727	36.728	38.011	19.342	19.628	28.511
49.859	33.069	23.007	41.906	30.218	26.83	25.296	22.832
46.517	35.984	37.916	26.83	24.738	32.151	20.232	26.83
49.81	31.617	32.484	32.151	26.427	30.393	25.854	20.976
44.709	35.984	38.752	25.819	24.578	32.151	19.843	26.83
44.454	35.984	33.988	30.393	37.267	19.342	27.318	19.342
46.633	33.069	31.213	33.069	19.617	36.728	19.723	26.83
48.971	30.713	22.101	41.906	19.359	36.728	25.296	21.124
46.303	33.069	22.468	41.48	27.571	28.511	25.854	20.504
37.767	41.48	22.393	41.48	31.603	24.451	19.313	26.83
50.73	28.511	21.841	41.906	19.863	35.984	26.439	19.342
47.335	30.713	40.963	22.647	19.027	36.728	24.285	19.342
33.585	43.956	33.142	30.393	28.835	26.83	24.113	19.342
43.748	33.069	19.403	43.956	36.317	19.342	23.059	19.342
39.969	36.728	31.065	32.151	19.076	35.984	23.031	19.342
49.41	26.83	31.043	32.151	31.372	22.75	21.668	19.342
29.629	43.956	26.325	36.728	27.26	26.83	19.1	19.342

MT9 – Rectangles = 300, Sheet = 600 x 800

Width	Height	Width	Height	Width	Height	Width	Height	Width	Height	Width	Height
19.532	36.64	32.646	40.085	21.805	40.085	37.77	15.477	17.811	19.724	15.883	21.714
31.32	31.678	15.769	8.611	37.77	14.441	17.627	32.348	35.505	28.834	31.531	15.098
37.77	318.604	297.215	38.57	31.32	28.832	13.251	39.678	35.437	17.443	134.869	199.671
19.532	36.704	31.851	18.53	39.982	38.57	38.608	40.085	21.786	33.574	28.049	31.194
31.32	323.323	296.857	19.201	27.682	32.156	31.851	20.963	33.358	15.115	28.049	12.482
20.043	32.725	31.851	14.541	19.54	40.085	15.238	10.084	33.679	32.725	17.811	29.487
320.724	25.985	27.682	12.495	31.531	27.984	31.851	36.29	25.455	328.907	17.317	32.185
312.159	33.574	38.574	31.4	20.894	38.57	31.851	20.665	27.682	347.555	18.743	17.443
37.77	24.023	25.455	8.721	28.049	9.243	37.77	22.987	27.659	17.443	37.185	28.834
17.627	26.323	31.32	38.204	19.532	324.538	17.627	32.353	252.745	31.668	20.42	15.115
334.163	9.373	15.238	27.424	37.77	16.174	307.25	8.498	28.638	40.085	16.29	19.201
17.627	323.76	33.86	15.115	31.749	27.329	11.869	40.085	25.455	19.276	26.695	8.498
37.77	15.832	31.851	35.765	37.77	22.867	30.159	25.985	27.682	22.034	28.049	28.852
320.796	15.115	31.32	21.439	31.531	40.333	13.16	38.57	31.531	12.824	25.455	9.332
19.532	32.309	31.531	17.958	248.426	31.4	17.627	10.008	30.042	15.115	31.531	33.621
22.331	15.115	19.532	15.567	25.455	33.022	25.455	25.97	31.531	36.641	28.049	30.586
8.636	25.985	31.531	35.371	31.531	17.44	8.371	38.57	31.851	11.295	25.455	8.998
35.074	40.085	15.883	39.931	37.77	20.129	17.811	33.048	31.851	11.087	31.851	37.69
28.049	35.227	31.851	40.596	27.682	12.29	8.091	35.113	9.741	12.281	15.883	18.071
28.049	22.245	31.851	34.789	20.381	37.01	28.049	14.467	25.455	8.115	13.251	17.534
30.788	38.57	19.532	39.174	37.77	35.893	11.916	9.755	28.049	14.309	15.784	15.115
9.536	11.55	31.851	12.732	33.694	26.512	25.455	22.867	28.049	14.264	31.32	13.91
31.531	40.021	17.811	12.621	13.151	25.985	15.091	34.899	22.9	210.408	15.238	16.725
31.846	33.885	27.682	8.408	25.455	31.342	37.77	23.087	13.251	301.511	28.049	18.538
13.251	25.887	34.255	31.668	18.215	38.57	19.532	33.523	15.883	25.441	26.027	40.085
31.846	281.883	10.69	19.201	37.77	37.147	23.926	25.985	27.682	13.505	31.32	9.3
31.851	279.375	27.682	37.948	19.532	36.856	37.77	13.089	17.627	33.091	15.238	14.37
15.883	293.938	37.77	27.606	37.77	18.569	31.531	13.948	37.77	14.363	19.532	18.287
25.455	21.132	300.699	32.185	31.32	24.966	316.172	40.085	35.434	27.329	31.32	21.119
259.938	32.725	31.851	38.22	34.108	25.985	37.77	11.585	23.134	15.115	16.87	32.185
288.937	28.834	24.627	32.185	31.851	27.296	11.916	37.37	17.627	22.288	15.883	22.529
17.627	27.392	31.851	16.82	19.173	9.528	13.86	15.317	28.049	32.77	11.415	17.081
8.506	25.985	37.77	8.75	36.98	199.671	31.531	29.923	31.274	8.745	17.627	10.804
258.097	12.294	11.916	325.991	271.909	34.899	17.627	40.701	31.274	8.698	28.049	9.866
29.809	25.985	15.238	357.933	31.851	23.689	27.682	21.091	37.77	16.051	17.627	10.388
248.356	12.281	31.32	19.811	15.238	40.184	15.238	13.646	13.484	15.115	18.525	9.373
222.857	34.471	31.32	30.86	12.606	15.115	17.627	28.436	31.32	8.41	31.32	22.526
28.108	210.408	30.188	30.369	13.251	13.51	31.531	287.962	15.238	24.277	37.77	13.87
37.77	29.076	15.919	11.55	19.958	35.113	31.32	16.564	16.987	24.575	17.811	326.744
39.164	27.329	25.455	36.184	15.238	39.407	21.877	25.985	24.075	32.725	11.595	15.317
27.867	25.985	19.532	25.441	31.851	40.276	8.939	9.373	28.049	11.084	31.531	14.533
192.669	12.36	174.264	27.329	19.532	27.97	15.238	32.54	23.324	15.115	28.049	8.629
31.851	26.83	33.767	27.329	28.049	12.846	37.77	9.952	19.532	19.454	34.099	32.725
20.381	311.531	31.851	29.041	31.851	8.395	31.32	16.298	31.531	39.145	13.251	9.136
13.251	40.599	14.04	17.081	22.658	32.725	31.32	27.788	19.05	19.201	13.091	15.115
13.251	23.228	31.32	33.718	28.049	25.785	15.238	21.302	28.049	12.362	31.32	30.069
171.849	10.737	254.703	32.773	18.858	38.57	37.77	16.387	18.74	19.201	222.857	40.52
36.567	38.57	25.455	35.267	312.91	17.443	19.567	27.329	28.049	290.855	8.876	9.528
33.694	22.947	27.682	33.002	15.009	38.57	192.669	18.009	25.455	16.212	25.455	22.341
34.979	32.185	37.77	26.947	19.532	33.782	37.77	27.094	11.616	25.985	9.686	8.611

MT10 – Rectangles = 500, Sheet = 700 x 800

Width	Height	Width	Height	Width	Height	Width	Height	Width	Height	Width	Height	Width	Height	Width	Height
19.499	19.803	45.522	43.102	44.592	19.211	41.627	28.916	17.156	48.523	19.813	35.089	22.675	35.089	29.44	17.364
24.349	369.233	40.017	42.207	32.06	28.129	40.017	30.497	20.341	45.337	32.784	37.653	25.267	36.77	27.398	19.346
40.193	38.178	46.281	42.141	43.241	34.171	24.367	28.129	40.017	35.95	68.631	191.739	31.125	23.263	27.007	19.552
34.878	314.222	40.017	16.742	29.44	47.905	31.125	271.041	21.483	44.033	47.729	25.521	34.878	18.066	26.163	20.351
40.017	16.016	17.267	42.141	40.017	37.304	40.017	38.606	26.697	38.729	25.267	26.923	35.815	18.389	24.501	21.711
144.76	44.427	41.694	25.318	34.878	42.189	144.76	27.943	23.044	42.141	40.55	18.389	31.125	21.373	25.85	20.351
29.292	176.546	40.017	47.512	29.973	47.059	39.097	36.981	17.935	47.245	22.372	37.426	34.878	21.623	47.134	22.943
298.58	25.521	44.592	42.746	40.017	21.77	28.021	42.141	21.98	43.102	29.255	40.856	25.878	27.652	42.413	20.298
285.122	37.426	43.986	43.327	32.88	43.327	25.878	40.967	31.125	33.88	32.836	26.803	34.878	19.977	250.273	20.351
47.134	269.997	29.44	25.1	45.373	43.102	126.825	47.245	273.429	34.171	44.592	18.385	34.878	18.311	20.161	25.583
270.36	39.989	48.034	38.729	22.308	38.306	20.309	33.029	22.723	34.171	40.017	19.32	34.878	18.102	22.146	19.346
25.878	38.901	29.44	45.649	44.592	31.399	42.488	19.979	25.878	27.897	26.181	33.029	244.818	43.996	276.139	28.916
42.488	22.494	29.44	46.162	20.341	45.321	42.488	27.202	34.878	24.905	25.878	33.323	42.4	34.171	42.488	19.232
44.592	28.418	36.62	38.548	29.44	46.424	40.017	44.475	42.413	30.264	19.187	39.989	19.828	33.029	40.017	30.275
38.913	38.729	30.169	18.664	42.488	33.346	20.341	32.264	20.448	25.521	25.878	33.228	16.118	20.351	23.388	43.102
19.203	46.777	44.592	19.219	45.482	41.13	31.125	37.911	68.631	28.516	19.147	25.521	40.017	44.542	20.341	24.803
22.873	27.258	42.488	42.806	42.488	27.456	20.341	41.708	144.76	27.084	41.694	38.734	29.44	20.122	25.641	19.42
20.341	282.291	144.76	18.681	26.316	41.13	37.956	28.916	24.309	42.141	48.278	25.189	18.767	38.729	20.341	24.701
38.477	20.351	37.166	19.346	42.488	30.716	25.878	43.002	20.341	47.991	31.125	27.743	19.248	43.102	24.656	20.351
272.463	28.129	46.238	38.729	31.748	44.033	42.488	25.954	44.081	19.346	29.973	44.369	31.125	21.287	19.978	22.655
23.781	20.351	40.017	44.788	25.878	46.776	26.065	28.916	41.694	21.723	44.592	43.125	19.303	33.029	36.043	44.465
23.249	34.171	42.413	42.381	20.653	34.171	44.592	23.772	34.878	47.284	16.194	19.346	31.125	42.709	42.488	30.522
25.878	270.435	28.143	20.351	43.277	43.102	20.341	25.479	47.22	38.178	37.826	41.13	29.44	22.669	21.716	22.655
275.81	18.389	26.973	25.583	42.488	44.159	31.125	47.517	20.341	42.549	41.694	16.852	25.878	25.803	260.276	38.548
268.605	25.189	22.937	33.029	29.44	45.556	25.878	42.442	25.878	36.933	20.18	38.306	23.436	24.831	29.44	38.125
259.574	30.463	41.694	40.813	16.846	43.102	29.44	38.808	32.888	25.189	44.592	35.285	29.44	21.891	41.358	34.171
20.341	25.114	40.017	44.171	44.592	24.918	38.258	28.916	40.954	44.033	230.319	40.856	32.919	18.389	20.341	22.522
41.694	29.302	41.694	42.464	31.125	43.629	23.653	44.465	20.341	42.217	25.878	28.738	25.878	17.898	22.989	19.803
29.973	26.921	44.731	34.171	31.993	38.178	42.488	46.492	17.744	21.901	40.017	17.637	20.341	30.602	40.017	35.661
261.119	26.803	41.694	42.325	44.592	46.079	246.811	36.189	42.488	33.285	34.878	22.743	34.878	47.461	42.249	38.627
249.873	36.981	34.878	37.716	40.017	34.519	29.44	38.405	25.878	36.511	29.44	20.885	28.763	33.029	20.341	21.629
40.017	21.673	39.458	43.327	41.37	33.029	31.125	36.448	42.689	41.13	18.139	369.233	30.169	20.065	16.608	25.189
40.017	34.527	44.592	38.143	42.488	37.944	163.562	44.037	19.127	43.102	27.007	23.55	34.878	33.299	25.878	19.843
235.971	44.465	44.592	39.887	31.125	43.183	29.44	48.055	29.039	42.141	25.878	31.568	16.851	21.977	22.056	19.359
40.017	239.954	34.878	47.481	25.267	29.866	20.341	19.844	42.413	288.09	259.574	38.552	43.267	27.683	16.581	24.831
240.328	38.627	41.694	19.648	40.017	34.058	29.973	275.225	29.44	46.254	31.822	25.521	34.878	23.713	20.341	21.055
231.695	44.033	41.694	261.502	25.878	19.732	48.65	20.351	34.878	34.152	25.878	29.601	20.341	29.62	20.526	20.351
255.094	19.346	40.017	45.681	17.679	44.465	29.44	32.34	45.921	41.13	20.341	25.323	30.56	19.169	42.488	33.56
245.939	27.683	42.413	39.698	41.694	31.723	18.759	34.171	31.125	30.771	20.341	30.928	20.168	19.346	47.134	21.19
293.955	16.299	17.158	43.102	16.708	19.346	47.134	40.777	47.742	18.389	25.878	40.338	30.232	19.346	20.341	19.553
28.9	28.916	20.341	18.624	23.285	46.777	31.125	33.135	31.125	26.486	29.973	26.856	29.44	24.412	25.267	36.846
16.463	44.465	34.878	29.042	22.895	37.653	39.056	27.683	34.104	27.683	47.134	41.195	30.081	19.169	17.987	21.711
46.837	220.255	233.868	37.653	30.973	42.141	20.341	25.036	34.878	39.891	44.592	37.079	20.341	28.523	20.48	19.169
21.243	28.916	41.797	17.207	41.694	25.346	40.017	29.146	20.341	46.653	40.017	30.461	29.973	36.975	239.714	42.141
244.774	43.327	39.655	19.346	20.341	24.302	40.017	17.576	21.926	38.627	27.529	28.916	29.973	38.074	33.476	28.129
191.937	38.729	46.132	20.351	34.878	31.121	29.44	33.104	32.826	28.916	29.44	26.708	17.722	48.523	29.44	17.774
208.479	19.169	34.878	45.117	42.488	30.476	23.03	43.327	42.488	41.619	44.592	301.283	20.341	28.25	29.44	423.136
41.694	23.196	42.413	21.551	29.44	43.473	29.44	19.207	29.44	19.314	31.125	30.685	32.081	38.729	34.878	26.048
27.86	44.465	42.488	37.355	34.345	38.548	29.44	36.799	21.722	33.083	17.255	38.627	20.341	28.078	19.638	19.359
40.017	45.274	40.017	44.726	31.125	33.016	18.295	33.083	29.973	44.214	36.661	19.169	21.791	20.351	44.592	18.033
144.76	36.05	40.608	38.627	33.096	25.521	40.017	33.215	23.747	37.426	29.44	26.159	22.872	25.521	21.696	25.521
17.134	21.901	40.017	17.028	29.292	43.709	208.724	41.13	44.592	16.574	29.973	35.9	25.878	22.459	40.017	34.905
25.267	34.91	16.131	19.042	28.332	44.037	22.83	43.327	20.341	27.491	29.44	26.033	46.883	19.346	21.061	16.725
25.878	36.34	20.341	22.881	291.007	38.178	34.878	31.278	20.341	17.152	25.878	29.458	27.796	20.206	256.348	33.029
29.44	17.916	42.488	36.165	25.878	46.392	25.267	22.482	40.017	20.841	36.105	19.169	29.44	18.536	34.878	28.525
44.592	37.02	18.027	21.977	40.017	31.49	32.984	33.029	16.8	44.033	17.023	33.029	19.686	44.033	25.878	18.904
33.37	44.465	21.563	18.152	29.066	42.141	44.592	45.892	35.153	25.521	41.694	26.672	20.341	17.26	29.44	18.973
44.592	46.074	20.341	29.779	27.119	43.102	34.878	39.831	16.28	38.102	31.125	23.798	34.878	23.916	42.736	36.189
34.878	17.56	34.878	22.323	20.341	17.202	20.341	19.173	29.44	48.001	25.878	34.326	20.341	26.999	18.454	18.152
42.488	47.632	28.312	38.102	34.878	36.084	42.413	23.44	40.017	21.758	20.341	17.31	25.267	277.925		
41.694	26.368	44.592	33.166	45.497	33.029	40.579	25.189	18.747	19.042	42.488	33	31.125	18.5		
44.729	43.996	18.924	27.258	42.271	36.981	29.44	36.293	42.488	23.749	42.488	39.502	27.751	19.346		
18.956	16.725	25.878	34.195	16.053	19.42	16.796	20.206	22.423	38.178	40.017	20.105	16.524	19.346		

APPENDIX D – New Irregular Benchmark Instances

The data for the new benchmark problems introduced in this thesis is detailed below. For each dataset, the problem name is stated followed by the sheet size (width, height), number of different shapes and finally the rotational constraints. After the problem definition, each shape of that dataset is defined. The shape number, number of loops and quantity that must be packed onto the sheet are given. This is followed by a list of the closed loops that define the shape. Each loop of the shape is defined by its number, whether it is external or internal and the number of line/arc primitives that define the loop. Then a list of line/arc primitives is given for that closed loop. A line is defined by its start and end coordinates. An arc is described by its start and end coordinates, a centre point coordinate, radius, start angle and offset angle (-ve offset is convex, +ve offset is concave). Although arcs can be defined using less information than is given in these datasets, they are declared in this manner for improved compatibility between different implementations.

Profiles1: (4000,2000), Shapes: 8, Rotations: 90 incremental

```
   Shape 1 (Loops: 1, Quantity: 4)
      Loop 1 (external): 6 Primitives
         Line: (99.952271, 330.356628),(209.952271, 330.356628)
         Line: (209.952271, 330.356628),(209.952271, 210.356644)
         Line: (209.952271, 210.356644),(369.952271, 210.356644)
         Line: (369.952271, 210.356644),(369.952271, 90.356644)
         Line: (369.952271, 90.356644),(99.952271, 90.356644)
         Line: (99.952271, 90.356644),(99.952271, 330.356628)

   Shape 2 (Loops: 1, Quantity: 4)
      Loop 1 (external): 6 Primitives
         Line: (128.702362, 153.247681),(128.702362, 423.247681)
         Line: (128.702362, 423.247681),(202.820251, 423.247681)
          Arc: (202.820251, 423.247681),(454.584534, 423.247681),
               Cen: (328.702393, 423.247681), Rad: 125.882141,
               StAng: -3.141593, Offset 3.141593
         Line: (454.584534, 423.247681),(528.702393, 423.247681)
         Line: (528.702393, 423.247681),(528.702393, 153.247681)
         Line: (528.702393, 153.247681),(128.702362, 153.247681)

   Shape 3 (Loops: 1, Quantity: 4)
      Loop 1 (external): 8 Primitives
         Line: (158.517273, 238.100677),(158.517273, 338.100677)
         Line: (158.517273, 338.100677),(278.517273, 338.100677)
         Line: (278.517273, 338.100677),(278.517273, 308.100677)
         Line: (278.517273, 308.100677),(248.517273, 308.100677)
         Line: (248.517273, 308.100677),(248.517273, 268.100677)
         Line: (248.517273, 268.100677),(218.517273, 268.100677)
         Line: (218.517273, 268.100677),(218.517273, 238.100677)
         Line: (218.517273, 238.100677),(158.517273, 238.100677)

   Shape 4 (Loops: 1, Quantity: 4)
      Loop 1 (external): 1 Primitives
          Arc: (315.091614, 272.041870),(315.091614, 272.041870),
               Cen: (250.091621, 272.041870), Rad: 64.999992,
               StAng: 0.000000, Offset -6.283185

   Shape 5 (Loops: 1, Quantity: 4)
      Loop 1 (external): 1 Primitives
          Arc: (301.109985, 305.983063),(301.109985, 305.983063),
               Cen: (201.109993, 305.983063), Rad: 99.999992,
               StAng: 0.000000, Offset -6.283185

   Shape 6 (Loops: 1, Quantity: 4)
      Loop 1 (external): 4 Primitives
         Line: (381.943237, 381.199280),(581.943237, 381.199280)
         Line: (581.943237, 381.199280),(781.943237, 181.199265)
         Line: (781.943237, 181.199265),(181.943268, 181.199265)
         Line: (181.943268, 181.199265),(381.943237, 381.199280)

   Shape 7 (Loops: 1, Quantity: 4)
      Loop 1 (external): 10 Primitives
         Line: (117.980972, 190.649583),(246.121307, 226.775071)
          Arc: (246.121307, 226.775071),(237.980957, 285.649583),
               Cen: (237.980934, 255.649556), Rad: 30.000026,
               StAng: -1.296006, Offset: 2.866801
         Line: (237.980957, 285.649583),(237.980957, 370.649567)
         Line: (237.980957, 370.649567),(117.980972, 420.649567)
         Line: (117.980972, 420.649567),(117.980972, 470.649567)
         Line: (117.980972, 470.649567),(942.821777, 390.414704)
          Arc: (942.821777, 390.414704),(947.053282, 291.479534),
               Cen: (937.980947, 340.649583), Rad: 50.000009,
               StAng: 1.473828, Offset -2.862167
         Line: (947.053282, 291.479534),(237.980957, 160.649583)
         Line: (237.980957, 160.649583),(117.980972, 160.649583)
         Line: (117.980972, 160.649583),(117.980972, 190.649583)

   Shape 8 (Loops: 1, Quantity: 4)
      Loop 1 (external): 6 Primitives
         Line: (110.497086, 255.303055),(110.497086, 355.303040)
         Line: (110.497086, 355.303040),(261.497070, 437.303040)
         Line: (261.497070, 437.303040),(861.497070, 437.303040)
         Line: (861.497070, 437.303040),(1012.497070, 355.303040)
         Line: (1012.497070, 355.303040),(1012.497070, 255.303055)
         Line: (1012.497070, 255.303055),(110.497086, 255.303055)
```

Profiles2: (5000,2500), Shapes: 7, Rotations: 90 incremental

```
   Shape 1 (Loops: 4, Quantity: 5)
      Loop 1 (internal): 1 Primitives
          Arc: (202.238806, 283.582090),(202.238806, 283.582090),
               Cen: (152.238806, 283.582090), Rad: 50.000000,
               StAng: 0.000000, Offset -6.283185
      Loop 2 (internal): 1 Primitives
          Arc: (1002.238806, 283.582090),(1002.238806, 283.582090),
               Cen: (952.238806, 283.582090), Rad: 50.000000,
               StAng: 0.000000, Offset -6.283185
      Loop 3 (internal): 1 Primitives
          Arc: (727.238806, 283.582090),(727.238806, 283.582090),
               Cen: (552.238806, 283.582090), Rad: 175.000000,
               StAng: 0.000000, Offset -6.283185
      Loop 4 (external): 8 Primitives
          Arc: (787.506663, 469.724603),(913.002695, 428.359600),
               Cen: (881.613806, 544.181609), Rad: 120.000000,
               StAng: -2.472244, Offset 1.166101
          Arc: (913.002695, 428.359600),(913.002695, 138.804579),
               Cen: (952.238806, 283.582090), Rad: 150.000000,
               StAng: 1.835449, Offset -3.670898
          Arc: (913.002695, 138.804579),(787.506663, 97.439576),
               Cen: (881.613806, 22.982570), Rad: 120.000000,
               StAng: 1.306144, Offset 1.166101
          Arc: (787.506663, 97.439576),(316.970949, 97.439576),
               Cen: (552.238806, 283.582090), Rad: 300.000000,
               StAng: -0.669348, Offset -1.802896
          Arc: (316.970949, 97.439576),(191.474917, 138.804579),
               Cen: (222.863806, 22.982570), Rad: 120.000000,
               StAng: 0.669348, Offset 1.166101
          Arc: (191.474917, 138.804579),(191.474917, 428.359600),
               Cen: (152.238806, 283.582090), Rad: 150.000000,
               StAng: -1.306144, Offset -3.670898
          Arc: (191.474917, 428.359600),(316.970949, 469.724603),
               Cen: (222.863806, 544.181609), Rad: 120.000000,
               StAng: -1.835449, Offset 1.166101
          Arc: (316.970949, 469.724603),(787.506663, 469.724603),
               Cen: (552.238806, 283.582090), Rad: 300.000000,
               StAng: 2.472244, Offset -1.802896

   Shape 2 (Loops: 1, Quantity: 7)
      Loop 1 (external): 12 Primitives
         Line: (3.735266, 36.876257),(3.735266, 356.876257)
         Line: (3.735266, 356.876257),(33.735266, 386.876257)
         Line: (33.735266, 386.876257),(183.735266, 386.876257)
         Line: (183.735266, 386.876257),(183.735266, 371.876257)
          Arc: (183.735266, 371.876257),(228.735266, 326.876257),
               Cen: (228.735266, 371.876257), Rad: 45.000000,
               StAng: 3.141593, Offset 1.570796
         Line: (228.735266, 326.876257),(473.735266, 326.876257)
         Line: (473.735266, 326.876257),(473.735266, 76.876257)
         Line: (473.735266, 76.876257),(228.735266, 76.876257)
          Arc: (228.735266, 76.876257),(183.735266, 31.876257),
               Cen: (228.735266, 31.876257), Rad: 45.000000,
               StAng: 1.570796, Offset 1.570796
         Line: (183.735266, 31.876257),(183.735266, 16.876257)
         Line: (183.735266, 16.876257),(23.735266, 16.876257)
         Line: (23.735266, 16.876257),(3.735266, 36.876257)

   Shape 3 (Loops: 2, Quantity: 4)
      Loop 1 (internal): 4 Primitives
         Line: (244.630072, 188.544153),(244.630072, 633.651551)
         Line: (244.630072, 633.651551),(840.095465, 633.651551)
         Line: (840.095465, 633.651551),(840.095465, 188.544153)
         Line: (840.095465, 188.544153),(244.630072, 188.544153)
      Loop 2 (external): 4 Primitives
         Line: (196.897375, 143.198091),(196.897375, 681.384248)
         Line: (196.897375, 681.384248),(898.568019, 681.384248)
         Line: (898.568019, 681.384248),(898.568019, 143.198091)
         Line: (898.568019, 143.198091),(196.897375, 143.198091)

   Shape 4 (Loops: 1, Quantity: 6)
      Loop 1 (external): 8 Primitives
         Line: (193.648896, 229.989415),(193.648896, 129.989415)
         Line: (193.648896, 129.989415),(218.648896, 129.989415)
         Line: (218.648896, 129.989415),(218.648896, 29.989415)
         Line: (218.648896, 29.989415),(-11.351104, 29.989415)
         Line: (-11.351104, 29.989415),(-11.351104, 129.989415)
         Line: (-11.351104, 129.989415),(13.648896, 129.989415)
         Line: (13.648896, 129.989415),(13.648896, 229.989415)
          Arc: (13.648896, 229.989415),(193.648896, 229.989415),
               Cen: (103.648896, 229.989415), Rad: 90.000000,
               StAng: 3.141593, Offset -3.141593
   Shape 5 (Loops: 1, Quantity: 7)
      Loop 1 (external): 5 Primitives
         Line: (398.524900, 658.183912),(633.639000, 334.577114)
         Line: (633.639000, 334.577114),(398.524900, 10.970317)
         Line: (398.524900, 10.970317),(18.102293, 134.577114)
         Line: (18.102293, 134.577114),(18.102293, 534.577114)
         Line: (18.102293, 534.577114),(398.524900, 658.183912)

   Shape 6 (Loops: 1, Quantity: 9)
      Loop 1 (external): 6 Primitives
         Line: (236.459853, 24.359731),(-75.862367, 24.359526)
         Line: (-75.862367, 24.359526),(-75.862598, 377.685979)
         Line: (-75.862598, 377.685979),(-29.782271, 377.686009)
         Line: (-29.782271, 377.686009),(-29.782082, 88.652454)
         Line: (-29.782082, 88.652454),(236.459811, 88.652628)
         Line: (236.459811, 88.652628),(236.459853, 24.359731)

   Shape 7 (Loops: 1, Quantity: 12)
      Loop 1 (external): 12 Primitives
         Line: (4.588083, 72.575156),(4.588083, 192.575156)
         Line: (4.588083, 192.575156),(44.588083, 192.575156)
         Line: (44.588083, 192.575156),(44.588083, 232.575156)
         Line: (44.588083, 232.575156),(164.588083, 232.575156)
         Line: (164.588083, 232.575156),(164.588083, 192.575156)
         Line: (164.588083, 192.575156),(204.588083, 192.575156)
         Line: (204.588083, 192.575156),(204.588083, 72.575156)
         Line: (204.588083, 72.575156),(164.588083, 72.575156)
         Line: (164.588083, 72.575156),(164.588083, 32.575156)
         Line: (164.588083, 32.575156),(44.588083, 32.575156)
         Line: (44.588083, 32.575156),(44.588083, 72.575156)
         Line: (44.588083, 72.575156),(4.588083, 72.575156)
```

Profiles3: (10000,2500), Shapes: 6, Rotations: 45 incremental

```
   Shape 1 (Loops: 1, Quantity: 4)
      Loop 1 (external): 16 Primitives
         Line: (404.534606, 794.749403),(440.334129, 531.026253)
         Line: (440.334129, 531.026253),(686.157518, 616.945107)
         Line: (686.157518, 616.945107),(505.966587, 463.007160)
         Line: (505.966587, 463.007160),(706.443914, 340.095465)
         Line: (706.443914, 340.095465),(479.713604, 353.221957)
         Line: (479.713604, 353.221957),(508.353222, 107.398568)
         Line: (508.353222, 107.398568),(366.348449, 303.102625)
         Line: (366.348449, 303.102625),(219.570406, 95.465394)
         Line: (219.570406, 95.465394),(244.630072, 328.162291)
         Line: (244.630072, 328.162291),(-0.000000, 206.443914)
         Line: (-0.000000, 206.443914),(183.770883, 377.088305)
         Line: (183.770883, 377.088305),(-57.279236, 531.026253)
         Line: (-57.279236, 531.026253),(187.350835, 485.680191)
         Line: (187.350835, 485.680191),(134.844869, 751.789976)
         Line: (134.844869, 751.789976),(317.422434, 565.632458)
         Line: (317.422434, 565.632458),(404.534606, 794.749403)

   Shape 2 (Loops: 1, Quantity: 12)
      Loop 1 (external): 13 Primitives
         Line: (338.902148, 745.823389),(348.448687, 807.875895)
         Line: (348.448687, 807.875895),(474.940334, 809.069212)
         Line: (474.940334, 809.069212),(564.439141, 806.682578)
         Line: (564.439141, 806.682578),(637.231504, 725.536993)
         Line: (637.231504, 725.536993),(579.952267, 643.198091)
         Line: (579.952267, 643.198091),(348.448687, 667.064439)
         Line: (348.448687, 667.064439),(350.835322, 601.431981)
         Line: (350.835322, 601.431981),(558.472554, 610.978520)
         Line: (558.472554, 610.978520),(609.785203, 448.687351)
         Line: (609.785203, 448.687351),(371.121718, 335.322196)
         Line: (371.121718, 335.322196),(164.677804, 515.513126)
         Line: (164.677804, 515.513126),(132.458234, 652.744630)
         Line: (132.458234, 652.744630),(338.902148, 745.823389)

   Shape 3 (Loops: 1, Quantity: 7)
      Loop 1 (external): 5 Primitives
         Line: (58.303887, -30.035336),(58.303887, 888.692580)
         Line: (58.303887, 888.692580),(531.802120, 415.194346)
         Line: (531.802120, 415.194346),(1005.300353, 888.692580)
         Line: (1005.300353, 888.692580),(1005.300353, -30.035336)
         Line: (1005.300353, -30.035336),(58.303887, -30.035336)

   Shape 4 (Loops: 1, Quantity: 5)
      Loop 1 (external): 8 Primitives
         Line: (84.725537, 180.190931),(84.725537, 807.875895)
         Line: (84.725537, 807.875895),(855.608592, 807.875895)
         Line: (855.608592, 807.875895),(843.675418, 735.083532)
         Line: (843.675418, 735.083532),(136.038186, 735.083532)
         Line: (136.038186, 735.083532),(136.038186, 236.276850)
         Line: (136.038186, 236.276850),(843.675418, 236.276850)
         Line: (843.675418, 236.276850),(855.608592, 180.190931)
         Line: (855.608592, 180.190931),(84.725537, 180.190931)

    Shape 5 (Loops: 1, Quantity: 10)
      Loop 1 (external): 6 Primitives
         Line: (38.394054, 159.795716),(128.394054, 709.795716)
          Arc: (128.394054, 709.795716),(289.369894, 749.051604),
               Cen: (217.212767, 695.261745), Rad: 90.000000,
               StAng: 2.979394, Offset -2.338808
         Line: (289.369894, 749.051604),(699.369894, 199.051604)
          Arc: (699.369894, 199.051604),(627.212767, 55.261745),
               Cen: (627.212767, 145.261745), Rad: 90.000000,
               StAng: 0.640586, Offset -2.211382
         Line: (627.212767, 55.261745),(127.212767, 55.261745)
          Arc: (127.212767, 55.261745),(38.394054, 159.795716),
               Cen: (127.212767, 145.261745), Rad: 90.000000,
               StAng: -1.570796, Offset -1.732995

   Shape 6 (Loops: 1, Quantity: 8)
      Loop 1 (external): 6 Primitives
         Line: (-1036.992840, 529.832936),(-920.047733, 789.976134)
         Line: (-920.047733, 789.976134),(-237.470167, 565.632458)
         Line: (-237.470167, 565.632458),(-399.761337, 300.715990)
         Line: (-399.761337, 300.715990),(-843.675418, 613.365155)
         Line: (-843.675418, 613.365155),(-808.113597, 497.949032)
          Arc: (-808.113597, 497.949032),(-1036.992840, 529.832936),
               Cen: (-958.233890, 257.756563), Rad: 283.246403,
               StAng: 1.012197, Offset -5.442814
```

Profiles4: (5000,500), Shapes: 7, Rotations: 90 incremental

```
   Shape 1 (Loops: 1, Quantity: 4)
      Loop 1 (external): 4 Primitives
         Line: (38.121547, 49.490946),(38.121547, 249.490946)
          Arc: (38.121547, 249.490946),(238.121547, 249.490946),
               Cen: (138.121547, 244.490946), Rad: 100.124922,
               StAng: 3.091634, Offset -3.041676
         Line: (238.121547, 249.490946),(238.121547, 49.490946)
          Arc: (238.121547, 49.490946),(38.121547, 49.490946),
               Cen: (138.121547, 44.490946), Rad: 100.124922,
               StAng: 0.049958, Offset 3.041676

   Shape 2 (Loops: 1, Quantity: 7)
      Loop 1 (external): 11 Primitives
         Line: (239.856802, 730.310263),(255.369928, 735.083532)
         Line: (255.369928, 735.083532),(303.102625, 741.050119)
         Line: (303.102625, 741.050119),(365.155131, 732.696897)
         Line: (365.155131, 732.696897),(426.014320, 699.284010)
         Line: (426.014320, 699.284010),(434.367542, 619.331742)
         Line: (434.367542, 619.331742),(420.047733, 559.665871)
         Line: (420.047733, 559.665871),(369.928401, 513.126492)
         Line: (369.928401, 513.126492),(227.923628, 503.579952)
         Line: (227.923628, 503.579952),(145.584726, 559.665871)
         Line: (145.584726, 559.665871),(139.618138, 651.551313)
         Line: (139.618138, 651.551313),(239.856802, 730.310263)

   Shape 3 (Loops: 1, Quantity: 7)
      Loop 1 (external): 5 Primitives
         Line: (5.966587, 5.966587),(5.966587, 214.797136)
         Line: (5.966587, 214.797136),(87.708831, 296.539379)
         Line: (87.708831, 296.539379),(169.451074, 214.797136)
         Line: (169.451074, 214.797136),(169.451074, 5.966587)
         Line: (169.451074, 5.966587),(5.966587, 5.966587)

   Shape 4 (Loops: 1, Quantity: 15)
      Loop 1 (external): 3 Primitives
         Line: (137.231504, 836.515513),(223.150358, 850.835322)
         Line: (223.150358, 850.835322),(212.410501, 677.804296)
         Line: (212.410501, 677.804296),(137.231504, 836.515513)

   Shape 5 (Loops: 1, Quantity: 7)
      Loop 1 (external): 3 Primitives
         Line: (-0.000000, 0.000000),(53.776627, 84.309397)
         Line: (53.776627, 84.309397),(107.466293, -0.055405)
         Line: (107.466293, -0.055405),(-0.000000, 0.000000)

   Shape 6 (Loops: 1, Quantity: 7)
      Loop 1 (external): 1 Primitives
          Arc: (50.000000, 0.000000),(50.000000, 0.000000),
               Cen: (0.000000, 0.000000), Rad: 50.000000,
               StAng: 0.000000, Offset -6.283185

   Shape 7 (Loops: 1, Quantity: 7)
      Loop 1 (external): 4 Primitives
         Line: (0.000000, 0.000000),(0.000000, 50.000000)
          Arc: (0.000000, 50.000000),(50.000000, 50.000000),
               Cen: (25.000000, 46.743828), Rad: 25.211161,
               StAng: 3.012075, Offset -2.882557
         Line: (50.000000, 50.000000),(50.000000, 0.000000)
         Line: (50.000000, 0.000000),(0.000000, 0.000000)
```

Profiles5: (8000,4000), Shapes: 5, Rotations: 15 incremental

```
   Shape 1 (Loops: 1, Quantity: 10)
      Loop 1 (external): 10 Primitives
         Line: (117.980972, 190.649583),(246.121307, 226.775071)
          Arc: (246.121307, 226.775071),(237.980957, 285.649583),
               Cen: (237.980934, 255.649556), Rad: 30.000026,
               StAng: -1.296006, Offset 2.866801
         Line: (237.980957, 285.649583),(237.980957, 370.649567)
         Line: (237.980957, 370.649567),(117.980972, 420.649567)
         Line: (117.980972, 420.649567),(117.980972, 470.649567)
         Line: (117.980972, 470.649567),(942.821777, 390.414704)
          Arc: (942.821777, 390.414704),(947.053282, 291.479534),
               Cen: (937.980947, 340.649583), Rad: 50.000009,
               StAng: 1.473828, Offset -2.862167
         Line: (947.053282, 291.479534),(237.980957, 160.649583)
         Line: (237.980957, 160.649583),(117.980972, 160.649583)
         Line: (117.980972, 160.649583),(117.980972, 190.649583)

   Shape 2 (Loops: 1, Quantity: 10)
      Loop 1 (external): 4 Primitives
         Line: (381.943237, 381.199280),(581.943237, 381.199280)
         Line: (581.943237, 381.199280),(781.943237, 181.199265)
         Line: (781.943237, 181.199265),(181.943268, 181.199265)
         Line: (181.943268, 181.199265),(381.943237, 381.199280)

   Shape 3 (Loops: 1, Quantity: 10)
      Loop 1 (external): 5 Primitives
         Line: (398.524900, 658.183912),(633.639000, 334.577114)
         Line: (633.639000, 334.577114),(398.524900, 10.970317)
         Line: (398.524900, 10.970317),(18.102293, 134.577114)
         Line: (18.102293, 134.577114),(18.102293, 534.577114)
         Line: (18.102293, 534.577114),(398.524900, 658.183912)

   Shape 4 (Loops: 1, Quantity: 10)
      Loop 1 (external): 5 Primitives
         Line: (280.429594, 193.317422),(280.429594, 613.365155)
          Arc: (280.429594, 613.365155),(522.076372, 613.365155),
               Cen: (401.252983, 609.425990), Rad: 120.887586,
               StAng: 3.109002, Offset -3.076410
          Arc: (522.076372, 613.365155),(763.723150, 613.365155),
               Cen: (642.899761, 637.483107), Rad: 123.207008,
               StAng: -2.944569, Offset 2.747545
         Line: (763.723150, 613.365155),(763.723150, 193.317422)
         Line: (763.723150, 193.317422),(280.429594, 193.317422)

   Shape 5 (Loops: 1, Quantity: 10)
      Loop 1 (external): 6 Primitives
         Line: (38.394054, 159.795716),(128.394054, 709.795716)
```

```
     Arc: (128.394054, 709.795716),(289.369894, 749.051604),
          Cen: (217.212767, 695.261745), Rad: 90.000000,
          StAng: 2.979394, Offset -2.338808)
    Line: (289.369894, 749.051604),(699.369894, 199.051604)
     Arc: (699.369894, 199.051604),(627.212767, 55.261745),
          Cen: (627.212767, 145.261745), Rad: 90.000000,
          StAng: 0.640586, Offset -2.211382)
    Line: (627.212767, 55.261745),(127.212767, 55.261745)
     Arc: (127.212767, 55.261745),(38.394054, 159.795716),
          Cen: (127.212767, 145.261745), Rad: 90.000000,
          StAng: -1.570796, Offset -1.732995)
```

Profiles6: (10000,5000), Shapes: 9, Rotations: 90 incremental

```
Shape 1 (Loops: 1, Quantity: 8)
   Loop 1 (external): 8 Primitives
      Line: (-569.212411, 1789.976134),(-569.212411, 2429.594272)
      Line: (-569.212411, 2429.594272),(714.797136, 2429.594272)
      Line: (714.797136, 2429.594272),(705.250597, 2229.116945)
      Line: (705.250597, 2229.116945),(-263.723150, 2238.663484)
      Line: (-263.723150, 2238.663484),(-273.269690, 1928.400955)
      Line: (-273.269690, 1928.400955),(681.384248, 1942.720764)
      Line: (681.384248, 1942.720764),(714.797136, 1789.976134)
      Line: (714.797136, 1789.976134),(-569.212411, 1789.976134)

Shape 2 (Loops: 1, Quantity: 8)
   Loop 1 (external): 4 Primitives
      Line: (1125.298329, 2214.797136),(1898.568019, 2262.529833)
      Line: (1898.568019, 2262.529833),(1941.527446, 1971.360382)
      Line: (1941.527446, 1971.360382),(1134.844869, 1952.267303)
      Line: (1134.844869, 1952.267303),(1125.298329, 2214.797136)

Shape 3 (Loops: 2, Quantity: 4)
   Loop 1 (internal): 4 Primitives
      Line: (203.980100, 167.910448),(203.980100, 644.278607)
      Line: (203.980100, 644.278607),(935.323383, 644.278607)
      Line: (935.323383, 644.278607),(935.323383, 167.910448)
      Line: (935.323383, 167.910448),(203.980100, 167.910448)
   Loop 2 (external): 4 Primitives
      Line: (138.059701, 109.452736),(138.059701, 706.467662)
      Line: (138.059701, 706.467662),(1002.487562, 706.467662)
      Line: (1002.487562, 706.467662),(1002.487562, 109.452736)
      Line: (1002.487562, 109.452736),(138.059701, 109.452736)

Shape 4 (Loops: 1, Quantity: 8)
   Loop 1 (external): 13 Primitives
      Line: (338.902148, 745.823389),(348.448687, 807.875895)
      Line: (348.448687, 807.875895),(474.940334, 809.069212)
      Line: (474.940334, 809.069212),(564.439141, 806.682578)
      Line: (564.439141, 806.682578),(637.231504, 725.536993)
      Line: (637.231504, 725.536993),(579.952267, 643.198091)
      Line: (579.952267, 643.198091),(348.448687, 667.064439)
      Line: (348.448687, 667.064439),(350.835322, 601.431981)
      Line: (350.835322, 601.431981),(558.472554, 610.978520)
      Line: (558.472554, 610.978520),(609.785203, 448.687351)
      Line: (609.785203, 448.687351),(371.121718, 335.322196)
      Line: (371.121718, 335.322196),(164.677804, 515.513126)
      Line: (164.677804, 515.513126),(132.458234, 652.744630)
      Line: (132.458234, 652.744630),(338.902148, 745.823389)

Shape 5 (Loops: 1, Quantity: 8)
   Loop 1 (external): 5 Primitives
      Line: (207.637232, 108.591885),(207.637232, 657.517900)
      Line: (207.637232, 657.517900),(412.887828, 147.971360)
      Line: (412.887828, 147.971360),(669.451074, 657.517900)
      Line: (669.451074, 657.517900),(669.451074, 108.591885)
      Line: (669.451074, 108.591885),(207.637232, 108.591885)

Shape 6 (Loops: 1, Quantity: 8)
   Loop 1 (external): 3 Primitives
      Line: (214.797136, 686.157518),(653.937947, 680.190931)
      Line: (653.937947, 680.190931),(414.081146, 182.577566)
      Line: (414.081146, 182.577566),(214.797136, 686.157518)

Shape 7 (Loops: 2, Quantity: 5)
   Loop 1 (internal): 4 Primitives
      Line: (1453.056148, 201.492537),(1453.056148, 467.661692)
      Line: (1453.056148, 467.661692),(1951.812367, 467.661692)
      Line: (1951.812367, 467.661692),(1951.812367, 201.492537)
      Line: (1951.812367, 201.492537),(1453.056148, 201.492537)
   Loop 2 (external): 4 Primitives
      Line: (1404.548685, 150.497512),(1404.548685, 521.144279)
      Line: (1404.548685, 521.144279),(2010.270078, 521.144279)
      Line: (2010.270078, 521.144279),(2010.270078, 150.497512)
      Line: (2010.270078, 150.497512),(1404.548685, 150.497512)

Shape 8 (Loops: 1, Quantity: 4)
   Loop 1 (external): 10 Primitives
      Line: (825.775656, 343.675418),(713.603819, 231.503580)
      Line: (713.603819, 231.503580),(607.398568, 394.988067)
      Line: (607.398568, 394.988067),(588.305489, 305.489260)
      Line: (588.305489, 305.489260),(398.568019, 307.875895)
      Line: (398.568019, 307.875895),(414.081146, 449.880668)
      Line: (414.081146, 449.880668),(559.665871, 449.880668)
      Line: (559.665871, 449.880668),(488.066826, 501.193317)
      Line: (488.066826, 501.193317),(647.971360, 523.866348)
      Line: (647.971360, 523.866348),(769.689737, 479.713604)
      Line: (769.689737, 479.713604),(825.775656, 343.675418)

Shape 9 (Loops: 1, Quantity: 12)
   Loop 1 (external): 8 Primitives
      Line: (468.435543, 34.696133),(21.435543, 34.696133)
      Line: (21.435543, 34.696133),(21.435543, 274.696133)
      Line: (21.435543, 274.696133),(170.435543, 274.696133)
      Line: (170.435543, 274.696133),(170.435543, 64.696133)
      Line: (170.435543, 64.696133),(319.435543, 64.696133)
      Line: (319.435543, 64.696133),(319.435543, 274.696133)
      Line: (319.435543, 274.696133),(468.435543, 274.696133)
      Line: (468.435543, 274.696133),(468.435543, 34.696133)
```

Profiles7: (2500,500), Shapes: 9, Rotations: 90 incremental

```
Shape 1 (Loops: 1, Quantity: 1)
   Loop 1 (external): 4 Primitives
      Line: (48.342460, 448.434459),(48.342460, 698.434459)
      Line: (48.342460, 698.434459),(293.727356, 698.193599)
      Line: (293.727356, 698.193599),(330.361696, 603.837311)
      Line: (330.361696, 603.837311),(48.342460, 448.434459)

Shape 2 (Loops: 1, Quantity: 1)
   Loop 1 (external): 6 Primitives
      Line: (48.342623, 198.434459),(281.342081, 455.735420)
      Line: (281.342081, 455.735420),(48.342460, 448.434459)
      Line: (48.342460, 448.434459),(330.361696, 603.837311)
      Line: (330.361696, 603.837311),(439.352160, 401.135967)
      Line: (439.352160, 401.135967),(548.342623, 198.434622)
      Line: (548.342623, 198.434622),(48.342623, 198.434459)

Shape 3 (Loops: 1, Quantity: 1)
   Loop 1 (external): 3 Primitives
      Line: (48.342623, 198.434459),(48.342460, 448.434459)
      Line: (48.342460, 448.434459),(281.342081, 455.735420)
      Line: (281.342081, 455.735420),(48.342623, 198.434459)

Shape 4 (Loops: 1, Quantity: 1)
   Loop 1 (external): 4 Primitives
      Line: (330.361696, 603.837311),(293.727356, 698.193599)
      Line: (293.727356, 698.193599),(1048.342623, 698.434786)
      Line: (1048.342623, 698.434786),(689.352160, 651.136048)
      Line: (689.352160, 651.136048),(330.361696, 603.837311)

Shape 5 (Loops: 1, Quantity: 1)
   Loop 1 (external): 7 Primitives
      Line: (552.514464, 378.712089),(556.686306, 558.989555)
      Line: (556.686306, 558.989555),(439.352160, 401.135967)
      Line: (439.352160, 401.135967),(330.361696, 603.837311)
      Line: (330.361696, 603.837311),(689.352160, 651.136048)
      Line: (689.352160, 651.136048),(788.225768, 348.309401)
      Line: (788.225768, 348.309401),(778.839033, 348.309401)
      Line: (778.839033, 348.309401),(552.514464, 378.712089)

Shape 6 (Loops: 1, Quantity: 1)
   Loop 1 (external): 4 Primitives
      Line: (439.352160, 401.135967),(556.686306, 558.989555)
      Line: (556.686306, 558.989555),(552.514464, 378.712089)
      Line: (552.514464, 378.712089),(548.342623, 198.434622)
      Line: (548.342623, 198.434622),(439.352160, 401.135967)

Shape 7 (Loops: 1, Quantity: 1)
   Loop 1 (external): 4 Primitives
      Line: (788.225768, 348.309401),(689.352160, 651.136048)
      Line: (689.352160, 651.136048),(1048.342623, 698.434786)
      Line: (1048.342623, 698.434786),(1048.342623, 448.434786)
      Line: (1048.342623, 448.434786),(788.225768, 348.309401)
Shape 8 (Loops: 1, Quantity: 1)
   Loop 1 (external): 5 Primitives
      Line: (552.514464, 378.712089),(778.839033, 348.309401)
      Line: (778.839033, 348.309401),(843.503207, 223.152873)
      Line: (843.503207, 223.152873),(1048.342623, 198.434786)
      Line: (1048.342623, 198.434786),(548.342623, 198.434622)
      Line: (548.342623, 198.434622),(552.514464, 378.712089)

Shape 9 (Loops: 1, Quantity: 1)
   Loop 1 (external): 5 Primitives
      Line: (788.225768, 348.309401),(1048.342623, 448.434786)
      Line: (1048.342623, 448.434786),(1048.342623, 198.434786)
      Line: (1048.342623, 198.434786),(843.503207, 223.152873)
      Line: (843.503207, 223.152873),(778.839033, 348.309401)
      Line: (778.839033, 348.309401),(788.225768, 348.309401)
```

Profiles8: (2500,1000), Shapes: 9, Rotations: 90 incremental

```
Shape 1 (Loops: 1, Quantity: 2)
   Loop 1 (external): 4 Primitives
      Line: (48.342460, 448.434459),(48.342460, 698.434459)
      Line: (48.342460, 698.434459),(293.727356, 698.193599)
      Line: (293.727356, 698.193599),(330.361696, 603.837311)
      Line: (330.361696, 603.837311),(48.342460, 448.434459)

Shape 2 (Loops: 1, Quantity: 2)
   Loop 1 (external): 6 Primitives
      Line: (48.342623, 198.434459),(281.342081, 455.735420)
      Line: (281.342081, 455.735420),(48.342460, 448.434459)
      Line: (48.342460, 448.434459),(330.361696, 603.837311)
      Line: (330.361696, 603.837311),(439.352160, 401.135967)
      Line: (439.352160, 401.135967),(548.342623, 198.434622)
      Line: (548.342623, 198.434622),(48.342623, 198.434459)

Shape 3 (Loops: 1, Quantity: 2)
   Loop 1 (external): 3 Primitives
      Line: (48.342623, 198.434459),(48.342460, 448.434459)
      Line: (48.342460, 448.434459),(281.342081, 455.735420)
      Line: (281.342081, 455.735420),(48.342623, 198.434459)

Shape 4 (Loops: 1, Quantity: 2)
   Loop 1 (external): 4 Primitives
      Line: (330.361696, 603.837311),(293.727356, 698.193599)
      Line: (293.727356, 698.193599),(1048.342623, 698.434786)
      Line: (1048.342623, 698.434786),(689.352160, 651.136048)
      Line: (689.352160, 651.136048),(330.361696, 603.837311)

Shape 5 (Loops: 1, Quantity: 2)
   Loop 1 (external): 7 Primitives
      Line: (552.514464, 378.712089),(556.686306, 558.989555)
      Line: (556.686306, 558.989555),(439.352160, 401.135967)
      Line: (439.352160, 401.135967),(330.361696, 603.837311)
      Line: (330.361696, 603.837311),(689.352160, 651.136048)
      Line: (689.352160, 651.136048),(788.225768, 348.309401)
      Line: (788.225768, 348.309401),(778.839033, 348.309401)
      Line: (778.839033, 348.309401),(552.514464, 378.712089)

Shape 6 (Loops: 1, Quantity: 2)
   Loop 1 (external): 4 Primitives
```

```
      Line: (439.352160, 401.135967),(556.686306, 558.989555)
      Line: (556.686306, 558.989555),(552.514464, 378.712089)
      Line: (552.514464, 378.712089),(548.342623, 198.434622)
      Line: (548.342623, 198.434622),(439.352160, 401.135967)

  Shape 7 (Loops: 1, Quantity: 2)
    Loop 1 (external): 4 Primitives
      Line: (788.225768, 348.309401),(689.352160, 651.136048)
      Line: (689.352160, 651.136048),(1048.342623, 698.434786)
      Line: (1048.342623, 698.434786),(1048.342623, 448.434786)
      Line: (1048.342623, 448.434786),(788.225768, 348.309401)

  Shape 8 (Loops: 1, Quantity: 2)
    Loop 1 (external): 5 Primitives
      Line: (552.514464, 378.712089),(778.839033, 348.309401)
      Line: (778.839033, 348.309401),(843.503207, 223.152873)
      Line: (843.503207, 223.152873),(1048.342623, 198.434786)
      Line: (1048.342623, 198.434786),(548.342623, 198.434622)
      Line: (548.342623, 198.434622),(552.514464, 378.712089)

  Shape 9 (Loops: 1, Quantity: 2)
    Loop 1 (external): 5 Primitives
      Line: (788.225768, 348.309401),(1048.342623, 448.434786)
      Line: (1048.342623, 448.434786),(1048.342623, 198.434786)
      Line: (1048.342623, 198.434786),(843.503207, 223.152873)
      Line: (843.503207, 223.152873),(778.839033, 348.309401)
      Line: (778.839033, 348.309401),(788.225768, 348.309401)
```

Profiles9: (3000,1500), Shapes: 16, Rotations: 90 incremental

```
  Shape 1 (Loops: 2, Quantity: 4)
    Loop 1 (internal): 24 Primitives
      Line: (-677.378261, 221.415713),(-675.544340, 239.043940)
      Line: (-675.544340, 239.043940),(-670.042576, 263.659067)
      Line: (-670.042576, 263.659067),(-660.738634, 288.980848)
      Line: (-660.738634, 288.980848),(-647.899507, 310.632831)
      Line: (-647.899507, 310.632831),(-632.399816, 327.938869)
      Line: (-632.399816, 327.938869),(-616.159420, 340.222816)
      Line: (-616.159420, 340.222816),(-598.947519, 347.552493)
      Line: (-598.947519, 347.552493),(-581.177225, 349.995719)
      Line: (-581.177225, 349.995719),(-554.312876, 344.661407)
      Line: (-554.312876, 344.661407),(-532.146704, 328.658473)
      Line: (-532.146704, 328.658473),(-517.283892, 303.951212)
      Line: (-517.283892, 303.951212),(-512.329621, 271.566001)
      Line: (-512.329621, 271.566001),(-515.925405, 238.345949)
      Line: (-515.925405, 238.345949),(-526.712759, 203.777611)
      Line: (-526.712759, 203.777611),(-544.254302, 172.258543)
      Line: (-544.254302, 172.258543),(-566.101545, 150.412045)
      Line: (-566.101545, 150.412045),(-590.270768, 137.654385)
      Line: (-590.270768, 137.654385),(-614.376922, 133.401832)
      Line: (-614.376922, 133.401832),(-632.023501, 136.189838)
      Line: (-632.023501, 136.189838),(-648.176747, 144.553856)
      Line: (-648.176747, 144.553856),(-661.808235, 158.134199)
      Line: (-661.808235, 158.134199),(-671.810168, 176.571183)
      Line: (-671.810168, 176.571183),(-675.986238, 195.399154)
      Line: (-675.986238, 195.399154),(-677.378261, 221.415713)

    Loop 2 (external): 43 Primitives
      Line: (-739.095874, 67.026286),(-689.247606, 62.368353)
      Line: (-689.247606, 62.368353),(-688.405312, 47.281215)
      Line: (-688.405312, 47.281215),(-685.162972, 36.505273)
      Line: (-685.162972, 36.505273),(-679.124707, 29.350968)
      Line: (-679.124707, 29.350968),(-670.295967, 23.845011)
      Line: (-670.295967, 23.845011),(-654.979333, 19.549745)
      Line: (-654.979333, 19.549745),(-635.764846, 18.117990)
      Line: (-635.764846, 18.117990),(-597.276299, 23.726281)
      Line: (-597.276299, 23.726281),(-571.320852, 40.551154)
      Line: (-571.320852, 40.551154),(-559.023723, 65.491684)
      Line: (-559.023723, 65.491684),(-547.750966, 108.586042)
      Line: (-547.750966, 108.586042),(-542.725429, 131.994950)
      Line: (-542.725429, 131.994950),(-582.414232, 102.919107)
      Line: (-582.414232, 102.919107),(-624.612823, 93.227160)
      Line: (-624.612823, 93.227160),(-664.874692, 101.284406)
      Line: (-664.874692, 101.284406),(-698.037871, 125.456144)
      Line: (-698.037871, 125.456144),(-720.097749, 164.681204)
      Line: (-720.097749, 164.681204),(-727.451042, 216.960493)
      Line: (-727.451042, 216.960493),(-721.681772, 263.833180)
      Line: (-721.681772, 263.833180),(-704.373964, 306.706193)
      Line: (-704.373964, 306.706193),(-678.764276, 342.865029)
      Line: (-678.764276, 342.865029),(-648.490692, 369.193856)
      Line: (-648.490692, 369.193856),(-616.185737, 384.489576)
      Line: (-616.185737, 384.489576),(-582.792782, 389.588150)
      Line: (-582.792782, 389.588150),(-532.389149, 376.259195)
      Line: (-532.389149, 376.259195),(-495.059123, 336.272330)
      Line: (-495.059123, 336.272330),(-485.047734, 383.183492)
      Line: (-485.047734, 383.183492),(-439.543455, 383.183492)
      Line: (-439.543455, 383.183492),(-497.934558, 103.220662)
      Line: (-497.934558, 103.220662),(-509.033900, 61.746333)
      Line: (-509.033900, 61.746333),(-523.000759, 30.581500)
      Line: (-523.000759, 30.581500),(-541.593737, 7.973198)
      Line: (-541.593737, 7.973198),(-566.170110, -8.793312)
      Line: (-566.170110, -8.793312),(-596.048235, -19.177521)
      Line: (-596.048235, -19.177521),(-630.145143, -22.638925)
      Line: (-630.145143, -22.638925),(-662.667950, -20.497999)
      Line: (-662.667950, -20.497999),(-690.415071, -14.075222)
      Line: (-690.415071, -14.075222),(-712.369865, -3.259548)
      Line: (-712.369865, -3.259548),(-727.917021, 12.060068)
      Line: (-727.917021, 12.060068),(-737.174523, 30.734281)
      Line: (-737.174523, 30.734281),(-740.260357, 51.613745)
      Line: (-740.260357, 51.613745),(-739.969237, 59.016971)
      Line: (-739.969237, 59.016971),(-739.095874, 67.026286)

  Shape 2 (Loops: 1, Quantity: 4)
    Loop 1 (external): 6 Primitives
      Line: (-727.937351, 911.991882),(-727.937351, 1110.536278)
      Line: (-727.937351, 1110.536278),(-687.471557, 1110.536278)
      Line: (-687.471557, 1110.536278),(-687.471557, 945.761897)
      Line: (-687.471557, 945.761897),(-587.325997, 945.761897)
      Line: (-587.325997, 945.761897),(-587.325997, 911.991882)
      Line: (-587.325997, 911.991882),(-727.937351, 911.991882)

  Shape 3 (Loops: 1, Quantity: 4)
    Loop 1 (external): 38 Primitives
      Line: (-320.620115, 971.301784),(-315.903134, 970.195269)
      Line: (-315.903134, 970.195269),(-332.884266, 916.270338)
      Line: (-332.884266, 916.270338),(-487.601247, 916.270338)
      Line: (-487.601247, 916.270338),(-487.601247, 921.616250)
      Line: (-487.601247, 921.616250),(-480.204316, 921.616250)
      Line: (-480.204316, 921.616250),(-469.371360, 923.678791)
      Line: (-469.371360, 923.678791),(-462.087687, 929.866413)
      Line: (-462.087687, 929.866413),(-459.760593, 937.714363)
      Line: (-459.760593, 937.714363),(-458.984895, 951.804929)
      Line: (-458.984895, 951.804929),(-458.984895, 1080.735747)
      Line: (-458.984895, 1080.735747),(-460.037222, 1096.139998)
      Line: (-460.037222, 1096.139998),(-463.194203, 1104.359294)
      Line: (-463.194203, 1104.359294),(-470.297851, 1109.283143)
      Line: (-470.297851, 1109.283143),(-480.204316, 1110.924426)
      Line: (-480.204316, 1110.924426),(-487.601247, 1110.924426)
      Line: (-487.601247, 1110.924426),(-487.601247, 1116.270338)
      Line: (-487.601247, 1116.270338),(-397.349675, 1116.270338)
      Line: (-397.349675, 1116.270338),(-397.349675, 1110.924426)
      Line: (-397.349675, 1110.924426),(-410.879461, 1110.223193)
      Line: (-410.879461, 1110.223193),(-419.625197, 1107.926288)
      Line: (-419.625197, 1107.926288),(-425.046228, 1104.499772)
      Line: (-425.046228, 1104.499772),(-428.385146, 1100.409709)
      Line: (-428.385146, 1100.409709),(-430.108542, 1092.359168)
      Line: (-430.108542, 1092.359168),(-430.683008, 1077.905558)
      Line: (-430.683008, 1077.905558),(-430.683008, 951.686891)
      Line: (-430.683008, 951.686891),(-430.106932, 941.347898)
      Line: (-430.106932, 941.347898),(-428.378706, 934.884979)
      Line: (-428.378706, 934.884979),(-425.986530, 932.167621)
      Line: (-425.986530, 932.167621),(-422.957121, 930.253332)
      Line: (-422.957121, 930.253332),(-415.293373, 929.200049)
      Line: (-415.293373, 929.200049),(-399.736593, 928.848954)
      Line: (-399.736593, 928.848954),(-385.296945, 928.848954)
      Line: (-385.296945, 928.848954),(-365.715033, 929.673357)
      Line: (-365.715033, 929.673357),(-352.958530, 932.146564)
      Line: (-352.958530, 932.146564),(-344.097549, 936.840602)
      Line: (-344.097549, 936.840602),(-335.985448, 944.110740)
      Line: (-335.985448, 944.110740),(-328.329473, 955.372200)
      Line: (-328.329473, 955.372200),(-320.620115, 971.301784)

  Shape 4 (Loops: 1, Quantity: 4)
    Loop 1 (external): 12 Primitives
      Line: (-219.013076, 1404.317897),(-219.013076, 1504.317897)
      Line: (-219.013076, 1504.317897),(-205.767079, 1504.317897)
      Line: (-205.767079, 1504.317897),(-205.767079, 1463.269862)
      Line: (-205.767079, 1463.269862),(-153.802013, 1463.269862)
      Line: (-153.802013, 1463.269862),(-153.802013, 1504.317897)
      Line: (-153.802013, 1504.317897),(-140.556016, 1504.317897)
      Line: (-140.556016, 1504.317897),(-140.556016, 1404.317897)
      Line: (-140.556016, 1404.317897),(-153.802013, 1404.317897)
      Line: (-153.802013, 1404.317897),(-153.802013, 1451.479469)
      Line: (-153.802013, 1451.479469),(-205.767079, 1451.479469)
      Line: (-205.767079, 1451.479469),(-205.767079, 1404.317897)
      Line: (-205.767079, 1404.317897),(-219.013076, 1404.317897)

  Shape 5 (Loops: 1, Quantity: 4)
    Loop 1 (external): 67 Primitives
      Line: (-45.880464, 1644.169894),(-45.880464, 1495.652611)
      Line: (-45.880464, 1495.652611),(-23.885520, 1517.801102)
      Line: (-23.885520, 1517.801102),(-7.053898, 1530.260460)
      Line: (-7.053898, 1530.260460),(7.280781, 1536.093479)
      Line: (7.280781, 1536.093479),(21.697951, 1538.037818)
      Line: (21.697951, 1538.037818),(37.685989, 1535.626611)
      Line: (37.685989, 1535.626611),(51.349725, 1528.392988)
      Line: (51.349725, 1528.392988),(62.175649, 1516.200875)
      Line: (62.175649, 1516.200875),(69.975385, 1498.589064)
      Line: (69.975385, 1498.589064),(72.941989, 1478.566045)
      Line: (72.941989, 1478.566045),(73.930857, 1446.798875)
      Line: (73.930857, 1446.798875),(73.930857, 1375.301969)
      Line: (73.930857, 1375.301969),(74.684291, 1359.099630)
      Line: (74.684291, 1359.099630),(76.944593, 1348.988309)
      Line: (76.944593, 1348.988309),(79.964140, 1344.382422)
      Line: (79.964140, 1344.382422),(84.489348, 1340.565366)
      Line: (84.489348, 1340.565366),(92.035914, 1338.315932)
      Line: (92.035914, 1338.315932),(104.119536, 1337.566120)
      Line: (104.119536, 1337.566120),(104.119536, 1329.547252)
      Line: (104.119536, 1329.547252),(4.591234, 1329.547252)
      Line: (4.591234, 1329.547252),(4.591234, 1337.566120)
      Line: (4.591234, 1337.566120),(9.082857, 1337.566120)
      Line: (9.082857, 1337.566120),(21.044744, 1338.724045)
      Line: (21.044744, 1338.724045),(28.734781, 1342.197818)
      Line: (28.734781, 1342.197818),(33.341498, 1347.265328)
      Line: (33.341498, 1347.265328),(36.378555, 1354.595554)
      Line: (36.378555, 1354.595554),(36.948442, 1361.478649)
      Line: (36.948442, 1361.478649),(37.138404, 1375.301969)
      Line: (37.138404, 1375.301969),(37.138404, 1446.755403)
      Line: (37.138404, 1446.755403),(36.252366, 1474.168837)
      Line: (36.252366, 1474.168837),(33.594253, 1490.226498)
      Line: (33.594253, 1490.226498),(29.177914, 1499.309102)
      Line: (29.177914, 1499.309102),(22.692065, 1505.797366)
      Line: (22.692065, 1505.797366),(14.421046, 1509.812611)
      Line: (14.421046, 1509.812611),(4.649197, 1511.151026)
      Line: (4.649197, 1511.151026),(-6.406124, 1509.681403)
      Line: (-6.406124, 1509.681403),(-17.912615, 1505.272535)
      Line: (-17.912615, 1505.272535),(-30.770615, 1496.622385)
      Line: (-30.770615, 1496.622385),(-45.880464, 1482.103781)
      Line: (-45.880464, 1482.103781),(-45.880464, 1375.301969)
      Line: (-45.880464, 1375.301969),(-45.339332, 1358.482837)
      Line: (-45.339332, 1358.482837),(-43.715935, 1349.438271)
      Line: (-43.715935, 1349.438271),(-40.320879, 1344.758120)
      Line: (-40.320879, 1344.758120),(-34.789898, 1340.945290)
      Line: (-34.789898, 1340.945290),(-26.424917, 1338.410913)
      Line: (-26.424917, 1338.410913),(-13.333294, 1337.566120)
      Line: (-13.333294, 1337.566120),(-13.333294, 1329.547252)
      Line: (-13.333294, 1329.547252),(-113.804992, 1329.547252)
      Line: (-113.804992, 1329.547252),(-113.804992, 1337.566120)
      Line: (-113.804992, 1337.566120),(-101.759369, 1338.578347)
      Line: (-101.759369, 1338.578347),(-92.501294, 1341.615026)
      Line: (-92.501294, 1341.615026),(-88.534917, 1344.848686)
      Line: (-88.534917, 1344.848686),(-85.396841, 1350.035554)
      Line: (-85.396841, 1350.035554),(-83.353898, 1359.433894)
      Line: (-83.353898, 1359.433894),(-82.672917, 1375.301969)
      Line: (-82.672917, 1375.301969),(-82.672917, 1559.264234)
      Line: (-82.672917, 1559.264234),(-83.106577, 1586.932649)
      Line: (-83.106577, 1586.932649),(-84.407558, 1601.443403)
      Line: (-84.407558, 1601.443403),(-86.552728, 1608.105328)
      Line: (-86.552728, 1608.105328),(-89.518954, 1612.227252)
      Line: (-89.518954, 1612.227252),(-93.478690, 1614.366196)
      Line: (-93.478690, 1614.366196),(-98.604388, 1615.079177)
      Line: (-98.604388, 1615.079177),(-104.694200, 1614.215064)
      Line: (-104.694200, 1614.215064),(-113.635935, 1611.622724)
      Line: (-113.635935, 1611.622724),(-116.635181, 1619.641592)
      Line: (-116.635181, 1619.641592),(-56.122577, 1644.169894)
      Line: (-56.122577, 1644.169894),(-45.880464, 1644.169894)

Shape 6 (Loops: 1, Quantity: 4)
   Loop 1 (external): 13 Primitives
      Line: (-303.830940, 523.279099),(-351.129642, 723.279099)
      Line: (-351.129642, 723.279099),(-309.623546, 723.279099)
      Line: (-309.623546, 723.279099),(-279.709647, 586.048939)
      Line: (-279.709647, 586.048939),(-243.325508, 723.279099)
      Line: (-243.325508, 723.279099),(-194.957717, 723.279099)
      Line: (-194.957717, 723.279099),(-160.058992, 583.779594)
      Line: (-160.058992, 583.779594),(-129.524099, 723.279099)
      Line: (-129.524099, 723.279099),(-88.829787, 723.279099)
      Line: (-88.829787, 723.279099),(-137.051505, 523.279099)
      Line: (-137.051505, 523.279099),(-180.642119, 523.279099)
      Line: (-180.642119, 523.279099),(-220.029880, 672.740339)
      Line: (-220.029880, 672.740339),(-259.622311, 523.279099)
      Line: (-259.622311, 523.279099),(-303.830940, 523.279099)

Shape 7 (Loops: 1, Quantity: 4)
   Loop 1 (external): 19 Primitives
      Line: (-1.353104, 520.752469),(-45.600487, 665.730635)
      Line: (-45.600487, 665.730635),(-20.023590, 665.730635)
      Line: (-20.023590, 665.730635),(3.260113, 582.033913)
      Line: (3.260113, 582.033913),(11.494966, 550.857156)
      Line: (11.494966, 550.857156),(13.651542, 559.497878)
      Line: (13.651542, 559.497878),(19.028334, 580.774035)
      Line: (19.028334, 580.774035),(42.011973, 665.730635)
      Line: (42.011973, 665.730635),(67.125871, 665.730635)
      Line: (67.125871, 665.730635),(89.093056, 581.715962)
      Line: (89.093056, 581.715962),(96.175350, 553.830967)
      Line: (96.175350, 553.830967),(104.407222, 581.917651)
      Line: (104.407222, 581.917651),(129.357161, 665.730635)
      Line: (129.357161, 665.730635),(153.235030, 665.730635)
      Line: (153.235030, 665.730635),(107.882785, 520.752469)
      Line: (107.882785, 520.752469),(82.528445, 520.752469)
      Line: (82.528445, 520.752469),(59.461336, 607.774769)
      Line: (59.461336, 607.774769),(53.671711, 632.460417)
      Line: (53.671711, 632.460417),(24.316206, 520.752469)
      Line: (24.316206, 520.752469),(-1.353104, 520.752469)

Shape 8 (Loops: 1, Quantity: 4)
   Loop 1 (external): 28 Primitives
      Line: (344.874051, 934.129998),(349.677545, 901.352077)
      Line: (349.677545, 901.352077),(334.898575, 898.945474)
      Line: (334.898575, 898.945474),(321.797021, 898.143274)
      Line: (321.797021, 898.143274),(303.951221, 899.710225)
      Line: (303.951221, 899.710225),(290.561667, 904.411081)
      Line: (290.561667, 904.411081),(281.189405, 911.885527)
      Line: (281.189405, 911.885527),(275.094488, 921.069806)
      Line: (275.094488, 921.069806),(271.752304, 937.158819)
      Line: (271.752304, 937.158819),(270.638243, 964.644025)
      Line: (270.638243, 964.644025),(270.638243, 1089.846330)
      Line: (270.638243, 1089.846330),(243.564007, 1089.846330)
      Line: (243.564007, 1089.846330),(243.564007, 1118.667291)
      Line: (243.564007, 1118.667291),(270.638243, 1118.667291)
      Line: (270.638243, 1118.667291),(270.638243, 1172.459570)
      Line: (270.638243, 1172.459570),(307.319466, 1194.472496)
      Line: (307.319466, 1194.472496),(307.319466, 1118.667291)
      Line: (307.319466, 1118.667291),(344.874051, 1118.667291)
      Line: (344.874051, 1118.667291),(344.874051, 1089.846330)
      Line: (344.874051, 1089.846330),(307.319466, 1089.846330)
      Line: (307.319466, 1089.846330),(307.319466, 962.594767)
      Line: (307.319466, 962.594767),(307.793038, 949.670191)
      Line: (307.793038, 949.670191),(309.213754, 942.391169)
      Line: (309.213754, 942.391169),(311.894383, 938.447623)
      Line: (311.894383, 938.447623),(315.685719, 935.529474)
      Line: (315.685719, 935.529474),(321.081457, 933.363186)
      Line: (321.081457, 933.363186),(328.414313, 932.641090)
      Line: (328.414313, 932.641090),(335.621789, 933.013317)
      Line: (335.621789, 933.013317),(344.874051, 934.129998)

Shape 9 (Loops: 1, Quantity: 4)
   Loop 1 (external): 8 Primitives
      Line: (405.382646, 981.289111),(405.382646, 1069.498718)
      Line: (405.382646, 1069.498718),(372.485994, 1069.498718)
      Line: (372.485994, 1069.498718),(372.485994, 1081.289111)
      Line: (372.485994, 1081.289111),(451.525295, 1081.289111)
      Line: (451.525295, 1081.289111),(451.525295, 1069.498718)
      Line: (451.525295, 1069.498718),(418.628643, 1069.498718)
      Line: (418.628643, 1069.498718),(418.628643, 981.289111)
      Line: (418.628643, 981.289111),(405.382646, 981.289111)

Shape 10 (Loops: 1, Quantity: 4)
   Loop 1 (external): 4 Primitives
      Line: (541.980923, 1481.914894),(541.980923, 1581.914894)
      Line: (541.980923, 1581.914894),(555.226920, 1581.914894)
      Line: (555.226920, 1581.914894),(555.226920, 1481.914894)
      Line: (555.226920, 1481.914894),(541.980923, 1481.914894)

Shape 11 (Loops: 1, Quantity: 4)
   Loop 1 (external): 32 Primitives
      Line: (687.322990, 1424.745161),(687.322990, 1419.399249)
      Line: (687.322990, 1419.399249),(601.788399, 1419.399249)
      Line: (601.788399, 1419.399249),(601.788399, 1424.745161)
      Line: (601.788399, 1424.745161),(608.812902, 1424.745161)
      Line: (608.812902, 1424.745161),(619.480047, 1426.562192)
      Line: (619.480047, 1426.562192),(626.785732, 1432.013287)
      Line: (626.785732, 1432.013287),(629.499997, 1440.116406)
      Line: (629.499997, 1440.116406),(630.404751, 1454.933840)
      Line: (630.404751, 1454.933840),(630.404751, 1583.864658)
      Line: (630.404751, 1583.864658),(629.901129, 1596.531828)
      Line: (629.901129, 1596.531828),(628.390261, 1604.054847)
      Line: (628.390261, 1604.054847),(626.184701, 1607.644381)
      Line: (626.184701, 1607.644381),(622.367116, 1610.414997)
      Line: (622.367116, 1610.414997),(615.796877, 1613.143752)
      Line: (615.796877, 1613.143752),(608.812902, 1614.053337)
      Line: (608.812902, 1614.053337),(601.788399, 1614.053337)
      Line: (601.788399, 1614.053337),(601.788399, 1619.399249)
      Line: (601.788399, 1619.399249),(687.322990, 1619.399249)
      Line: (687.322990, 1619.399249),(687.322990, 1614.053337)
      Line: (687.322990, 1614.053337),(680.380097, 1614.053337)
      Line: (680.380097, 1614.053337),(669.626009, 1612.209211)
      Line: (669.626009, 1612.209211),(662.304223, 1606.676834)
      Line: (662.304223, 1606.676834),(659.606034, 1598.600809)
      Line: (659.606034, 1598.600809),(658.706638, 1583.864658)
      Line: (658.706638, 1583.864658),(658.706638, 1454.933840)
      Line: (658.706638, 1454.933840),(659.210537, 1442.185387)
      Line: (659.210537, 1442.185387),(660.722235, 1434.635274)
      Line: (660.722235, 1434.635274),(662.858563, 1431.176155)
      Line: (662.858563, 1431.176155),(666.706739, 1428.275123)
      Line: (666.706739, 1428.275123),(673.276173, 1425.627652)
      Line: (673.276173, 1425.627652),(680.380097, 1424.745161)
      Line: (680.380097, 1424.745161),(687.322990, 1424.745161)

Shape 12 (Loops: 2, Quantity: 2)
   Loop 1 (internal): 13 Primitives
      Line: (763.797818, 619.325467),(763.797818, 751.494317)
      Line: (763.797818, 751.494317),(890.436397, 751.494317)
      Line: (890.436397, 751.494317),(929.032496, 746.897496)
      Line: (929.032496, 746.897496),(955.584857, 733.107033)
      Line: (955.584857, 733.107033),(971.133224, 712.365909)
      Line: (971.133224, 712.365909),(976.316013, 686.515778)
      Line: (976.316013, 686.515778),(973.618930, 667.986381)
      Line: (973.618930, 667.986381),(965.527681, 651.005001)
      Line: (965.527681, 651.005001),(952.477410, 636.496658)
      Line: (952.477410, 636.496658),(934.501934, 626.646247)
      Line: (934.501934, 626.646247),(910.272857, 621.155662)
      Line: (910.272857, 621.155662),(877.603233, 619.325467)
      Line: (877.603233, 619.325467),(763.797818, 619.325467)
   Loop 2 (external): 30 Primitives
      Line: (710.813830, 395.744681),(710.813830, 795.744681)
      Line: (710.813830, 795.744681),(888.457148, 795.744681)
      Line: (888.457148, 795.744681),(935.511914, 793.027476)
      Line: (935.511914, 793.027476),(969.662994, 784.875860)
      Line: (969.662994, 784.875860),(994.773399, 770.078864)
      Line: (994.773399, 770.078864),(1014.304811, 746.916873)
      Line: (1014.304811, 746.916873),(1026.861247, 718.186090)
      Line: (1026.861247, 718.186090),(1031.046726, 686.575400)
      Line: (1031.046726, 686.575400),(1024.117212, 647.406981)
      Line: (1024.117212, 647.406981),(1003.328671, 614.920925)
      Line: (1003.328671, 614.920925),(968.277166, 591.114835)
      Line: (968.277166, 591.114835),(918.157434, 577.986311)
      Line: (918.157434, 577.986311),(936.660001, 567.839330)
      Line: (936.660001, 567.839330),(950.174761, 557.882113)
      Line: (950.174761, 557.882113),(972.599029, 534.121484)
      Line: (972.599029, 534.121484),(993.920391, 504.713298)
      Line: (993.920391, 504.713298),(1063.236956, 395.744681)
      Line: (1063.236956, 395.744681),(996.998540, 395.744681)
      Line: (996.998540, 395.744681),(943.996665, 479.482206)
      Line: (943.996665, 479.482206),(922.884444, 511.063086)
      Line: (922.884444, 511.063086),(906.043640, 534.084733)
      Line: (906.043640, 534.084733),(891.621981, 550.340547)
      Line: (891.621981, 550.340547),(878.812432, 560.793321)
      Line: (878.812432, 560.793321),(866.891617, 567.193508)
      Line: (866.891617, 567.193508),(854.734831, 571.291557)
      Line: (854.734831, 571.291557),(842.781504, 572.819173)
      Line: (842.781504, 572.819173),(825.159751, 573.328378)
      Line: (825.159751, 573.328378),(763.797818, 573.328378)
      Line: (763.797818, 573.328378),(763.797818, 395.744681)
      Line: (763.797818, 395.744681),(710.813830, 395.744681)

Shape 13 (Loops: 2, Quantity: 2)
   Loop 1 (internal): 17 Primitives
      Line: (1232.136879, 612.351320),(1260.523628, 747.431378)
      Line: (1260.523628, 747.431378),(1390.411208, 747.431378)
      Line: (1390.411208, 747.431378),(1416.705986, 746.289253)
      Line: (1416.705986, 746.289253),(1433.753462, 742.862877)
      Line: (1433.753462, 742.862877),(1445.336437, 736.115442)
      Line: (1445.336437, 736.115442),(1454.514428, 725.010138)
      Line: (1454.514428, 725.010138),(1460.496053, 710.946906)
      Line: (1460.496053, 710.946906),(1462.489927, 694.602406)
      Line: (1462.489927, 694.602406),(1459.792844, 674.465649)
      Line: (1459.792844, 674.465649),(1451.701596, 655.726365)
      Line: (1451.701596, 655.726365),(1438.931267, 639.546056)
      Line: (1438.931267, 639.546056),(1421.795616, 627.086225)
      Line: (1421.795616, 627.086225),(1399.585753, 618.393823)
      Line: (1399.585753, 618.393823),(1371.994114, 613.515803)
      Line: (1371.994114, 613.515803),(1347.716315, 612.642441)
      Line: (1347.716315, 612.642441),(1305.576833, 612.351320)
      Line: (1305.576833, 612.351320),(1232.136879, 612.351320)
   Loop 2 (external): 28 Primitives
      Line: (1132.358923, 391.099500),(1216.201718, 791.099500)
      Line: (1216.201718, 791.099500),(1383.603173, 791.099500)
      Line: (1383.603173, 791.099500),(1427.173433, 788.983914)
      Line: (1427.173433, 788.983914),(1458.771965, 782.637154)
      Line: (1458.771965, 782.637154),(1481.624204, 770.798318)
      Line: (1481.624204, 770.798318),(1498.955581, 751.805177)
      Line: (1498.955581, 751.805177),(1510.034288, 725.595361)
      Line: (1510.034288, 725.595361),(1513.727191, 693.581014)
      Line: (1513.727191, 693.581014),(1505.803348, 648.912601)
      Line: (1505.803348, 648.912601),(1482.031820, 612.604711)
      Line: (1482.031820, 612.604711),(1439.959878, 586.142761)
      Line: (1439.959878, 586.142761),(1378.315302, 571.012164)
      Line: (1378.315302, 571.012164),(1397.789607, 554.938429)
      Line: (1397.789607, 554.938429),(1411.389794, 539.143890)
      Line: (1411.389794, 539.143890),(1433.899488, 503.312694)
      Line: (1433.899488, 503.312694),(1451.510807, 466.506778)
      Line: (1451.510807, 466.506778),(1482.345764, 391.099500)
      Line: (1482.345764, 391.099500),(1422.500082, 391.099500)
      Line: (1422.500082, 391.099500),(1393.785415, 465.336333)
      Line: (1393.785415, 465.336333),(1377.240251, 503.012117)
      Line: (1377.240251, 503.012117),(1357.972617, 535.109934)
      Line: (1357.972617, 535.109934),(1344.617066, 552.129835)
      Line: (1344.617066, 552.129835),(1330.622166, 562.074895)
      Line: (1330.622166, 562.074895),(1312.222679, 567.031122)
      Line: (1312.222679, 567.031122),(1285.144716, 568.683198)
      Line: (1285.144716, 568.683198),(1222.928332, 568.683198)
      Line: (1222.928332, 568.683198),(1185.694679, 391.099500)
      Line: (1185.694679, 391.099500),(1132.358923, 391.099500)
   Shape 14 (Loops: 1, Quantity: 4)
      Loop 1 (external): 20 Primitives
         Line: (1608.863518, 441.554308),(1608.863518, 586.532474)
         Line: (1608.863518, 586.532474),(1630.988699, 586.532474)
         Line: (1630.988699, 586.532474),(1630.988699, 564.496725)
         Line: (1630.988699, 564.496725),(1639.224554, 577.278093)
         Line: (1639.224554, 577.278093),(1646.570138, 584.817051)
         Line: (1646.570138, 584.817051),(1654.180012, 588.505365)
         Line: (1654.180012, 588.505365),(1662.578801, 589.734803)
         Line: (1662.578801, 589.734803),(1674.991424, 587.792096)
         Line: (1674.991424, 587.792096),(1687.757258, 581.963973)
         Line: (1687.757258, 581.963973),(1678.884548, 559.208853)
         Line: (1678.884548, 559.208853),(1669.954196, 563.107682)
         Line: (1669.954196, 563.107682),(1661.153008, 564.407292)
         Line: (1661.153008, 564.407292),(1653.379430, 563.186797)
         Line: (1653.379430, 563.186797),(1646.522440, 559.525313)
         Line: (1646.522440, 559.525313),(1641.107896, 553.727118)
         Line: (1641.107896, 553.727118),(1637.661654, 546.096492)
         Line: (1637.661654, 546.096492),(1634.403663, 532.367118)
         Line: (1634.403663, 532.367118),(1633.317666, 517.379869)
         Line: (1633.317666, 517.379869),(1633.317666, 441.554308)
         Line: (1633.317666, 441.554308),(1608.863518, 441.554308)

   Shape 15 (Loops: 2, Quantity: 1)
      Loop 1 (internal): 9 Primitives
         Line: (1260.398733, 1357.025795),(1273.602588, 1416.977096)
         Line: (1273.602588, 1416.977096),(1304.770623, 1463.536327)
         Line: (1304.770623, 1463.536327),(1349.941837, 1493.697096)
         Line: (1349.941837, 1493.697096),(1404.452905, 1503.750685)
         Line: (1404.452905, 1503.750685),(1463.990361, 1491.162831)
         Line: (1463.990361, 1491.162831),(1511.888457, 1453.399268)
         Line: (1511.888457, 1453.399268),(1532.785016, 1413.303920)
         Line: (1532.785016, 1413.303920),(1543.659287, 1357.025795)
         Line: (1543.659287, 1357.025795),(1260.398733, 1357.025795)
      Loop 2 (external): 28 Primitives
         Line: (1543.057715, 1219.471209),(1631.411857, 1208.263058)
         Line: (1631.411857, 1208.263058),(1601.320620, 1138.923728)
         Line: (1601.320620, 1138.923728),(1553.340847, 1087.172962)
         Line: (1553.340847, 1087.172962),(1488.618378, 1054.955600)
         Line: (1488.618378, 1054.955600),(1408.299054, 1044.216479)
         Line: (1408.299054, 1044.216479),(1308.581965, 1061.321259)
         Line: (1308.581965, 1061.321259),(1231.868894, 1112.635600)
         Line: (1231.868894, 1112.635600),(1182.960597, 1194.827134)
         Line: (1182.960597, 1194.827134),(1166.657831, 1304.563495)
         Line: (1166.657831, 1304.563495),(1183.077977, 1418.257271)
         Line: (1183.077977, 1418.257271),(1232.338413, 1503.175198)
         Line: (1232.338413, 1503.175198),(1308.122961, 1556.336071)
         Line: (1308.122961, 1556.336071),(1403.413115, 1574.056362)
         Line: (1403.413115, 1574.056362),(1495.859496, 1556.640362)
         Line: (1495.859496, 1556.640362),(1569.601878, 1504.392362)
         Line: (1569.601878, 1504.392362),(1618.158035, 1421.008036)
         Line: (1618.158035, 1421.008036),(1634.343421, 1309.480735)
         Line: (1634.343421, 1309.480735),(1634.177377, 1300.104362)
         Line: (1634.177377, 1300.104362),(1633.679246, 1286.720118)
         Line: (1633.679246, 1286.720118),(1255.304119, 1286.720118)
         Line: (1255.304119, 1286.720118),(1269.366070, 1213.199046)
         Line: (1269.366070, 1213.199046),(1302.482367, 1158.695803)
         Line: (1302.482367, 1158.695803),(1350.328701, 1125.565568)
         Line: (1350.328701, 1125.565568),(1408.580766, 1114.522156)
         Line: (1408.580766, 1114.522156),(1452.846116, 1120.730901)
         Line: (1452.846116, 1120.730901),(1489.279129, 1139.357137)
         Line: (1489.279129, 1139.357137),(1519.486442, 1172.401294)
         Line: (1519.486442, 1172.401294),(1543.057715, 1219.471209)

   Shape 16 (Loops: 1, Quantity: 4)
      Loop 1 (external): 12 Primitives
         Line: (1719.395136, 1384.292866),(1719.395136, 1484.292866)
         Line: (1719.395136, 1484.292866),(1791.593098, 1484.292866)
         Line: (1791.593098, 1484.292866),(1791.593098, 1472.502473)
         Line: (1791.593098, 1472.502473),(1732.641133, 1472.502473)
         Line: (1732.641133, 1472.502473),(1732.641133, 1441.934787)
         Line: (1732.641133, 1441.934787),(1787.808528, 1441.934787)
         Line: (1787.808528, 1441.934787),(1787.808528, 1430.144394)
         Line: (1787.808528, 1430.144394),(1732.641133, 1430.144394)
         Line: (1732.641133, 1430.144394),(1732.641133, 1396.083259)
         Line: (1732.641133, 1396.083259),(1793.922065, 1396.083259)
         Line: (1793.922065, 1396.083259),(1793.922065, 1384.292866)
         Line: (1793.922065, 1384.292866),(1719.395136, 1384.292866)
```

Profiles10: (15000,3000), Shapes: 13, Rotations: 0 absolute

```
   Shape 1 (Loops: 1, Quantity: 7)
      Loop 1 (external): 8 Primitives
         Line: (200.000000, -0.000000),(100.000000, 200.000000)
         Line: (100.000000, 200.000000),(0.000000, 300.000000)
         Line: (0.000000, 300.000000),(100.000000, 400.000000)
         Line: (100.000000, 400.000000),(100.000000, 500.000000)
         Line: (100.000000, 500.000000),(200.000000, 700.000000)
         Line: (200.000000, 700.000000),(900.000000, 500.000000)
         Line: (900.000000, 500.000000),(900.000000, 100.000000)
         Line: (900.000000, 100.000000),(200.000000, -0.000000)

   Shape 2 (Loops: 1, Quantity: 7)
      Loop 1 (external): 6 Primitives
         Line: (74.000000, -0.000000),(0.000000, 125.000000)
         Line: (0.000000, 125.000000),(870.000000, 305.000000)
         Line: (870.000000, 305.000000),(1740.000000, 125.000000)
         Line: (1740.000000, 125.000000),(1666.000000, -0.000000)
         Line: (1666.000000, -0.000000),(870.000000, 119.000000)
         Line: (870.000000, 119.000000),(74.000000, -0.000000)

   Shape 3 (Loops: 1, Quantity: 7)
      Loop 1 (external): 11 Primitives
         Line: (-0.000000, 200.000000),(-0.000000, 600.000000)
         Line: (-0.000000, 600.000000),(200.000000, 600.000000)
         Line: (200.000000, 600.000000),(200.000000, 300.000000)
         Line: (200.000000, 300.000000),(800.000000, 300.000000)
         Line: (800.000000, 300.000000),(800.000000, 600.000000)
         Line: (800.000000, 600.000000),(1100.000000, 600.000000)
         Line: (1100.000000, 600.000000),(1100.000000, 400.000000)
         Line: (1100.000000, 400.000000),(700.000000, -0.000000)
         Line: (700.000000, -0.000000),(600.000000, -0.000000)
         Line: (600.000000, -0.000000),(600.000000, 200.000000)
         Line: (600.000000, 200.000000),(-0.000000, 200.000000)

   Shape 4 (Loops: 1, Quantity: 7)
      Loop 1 (external): 8 Primitives
         Line: (-0.000000, 0.000000),(100.000000, 300.000000)
         Line: (100.000000, 300.000000),(0.000000, 500.000000)
         Line: (0.000000, 500.000000),(200.000000, 400.000000)
         Line: (200.000000, 400.000000),(400.000000, 500.000000)
         Line: (400.000000, 500.000000),(300.000000, 200.000000)
         Line: (300.000000, 200.000000),(400.000000, 0.000000)
         Line: (400.000000, 0.000000),(200.000000, 100.000000)
         Line: (200.000000, 100.000000),(-0.000000, 0.000000)

   Shape 5 (Loops: 1, Quantity: 7)
      Loop 1 (external): 8 Primitives
         Line: (0.000000, 0.000000),(200.000000, 200.000000)
         Line: (200.000000, 200.000000),(200.000000, 400.000000)
         Line: (200.000000, 400.000000),(0.000000, 600.000000)
         Line: (0.000000, 600.000000),(600.000000, 600.000000)
         Line: (600.000000, 600.000000),(400.000000, 400.000000)
         Line: (400.000000, 400.000000),(400.000000, 200.000000)
         Line: (400.000000, 200.000000),(600.000000, 0.000000)
         Line: (600.000000, 0.000000),(0.000000, 0.000000)

   Shape 6 (Loops: 1, Quantity: 7)
      Loop 1 (external): 4 Primitives
         Line: (0.000000, 0.000000),(0.000000, 660.000000)
         Line: (0.000000, 660.000000),(200.000000, 260.000000)
         Line: (200.000000, 260.000000),(390.000000, 230.000000)
         Line: (390.000000, 230.000000),(0.000000, 0.000000)

   Shape 7 (Loops: 1, Quantity: 7)
      Loop 1 (external): 14 Primitives
         Line: (0.000000, 232.000000),(0.000000, 426.000000)
         Line: (0.000000, 426.000000),(539.000000, 414.000000)
         Line: (539.000000, 414.000000),(616.000000, 477.000000)
         Line: (616.000000, 477.000000),(694.000000, 544.000000)
         Line: (694.000000, 544.000000),(732.000000, 547.000000)
         Line: (732.000000, 547.000000),(761.000000, 490.000000)
         Line: (761.000000, 490.000000),(845.000000, 453.000000)
         Line: (845.000000, 453.000000),(884.000000, 463.000000)
         Line: (884.000000, 463.000000),(1003.000000, 500.000000)
         Line: (1003.000000, 500.000000),(1097.000000, 260.000000)
         Line: (1097.000000, 260.000000),(975.000000, 258.000000)
         Line: (975.000000, 258.000000),(894.000000, 20.000000)
         Line: (894.000000, 20.000000),(827.000000, -0.000000)
         Line: (827.000000, -0.000000),(0.000000, 232.000000)

   Shape 8 (Loops: 1, Quantity: 7)
      Loop 1 (external): 15 Primitives
         Line: (882.000000, 623.000000),(963.000000, 713.000000)
         Line: (963.000000, 713.000000),(1147.000000, 666.000000)
         Line: (1147.000000, 666.000000),(1098.000000, 518.000000)
         Line: (1098.000000, 518.000000),(1052.000000, 396.000000)
         Line: (1052.000000, 396.000000),(998.000000, 265.000000)
         Line: (998.000000, 265.000000),(954.000000, 169.000000)
         Line: (954.000000, 169.000000),(868.000000, 0.000000)
         Line: (868.000000, 0.000000),(689.000000, 49.000000)
         Line: (689.000000, 49.000000),(592.000000, 148.000000)
         Line: (592.000000, 148.000000),(4.000000, 165.000000)
         Line: (4.000000, 165.000000),(0.000000, 325.000000)
         Line: (0.000000, 325.000000),(546.000000, 334.000000)
         Line: (546.000000, 334.000000),(688.000000, 402.000000)
         Line: (688.000000, 402.000000),(780.000000, 508.000000)
         Line: (780.000000, 508.000000),(882.000000, 623.000000)

   Shape 9 (Loops: 1, Quantity: 7)
      Loop 1 (external): 6 Primitives
         Line: (0.000000, 0.000000),(10.000000, 50.000000)
         Line: (10.000000, 50.000000),(30.000000, 300.000000)
         Line: (30.000000, 300.000000),(470.000000, 390.000000)
         Line: (470.000000, 390.000000),(400.000000, 200.000000)
         Line: (400.000000, 200.000000),(520.000000, 70.000000)
         Line: (520.000000, 70.000000),(0.000000, 0.000000)

   Shape 10 (Loops: 1, Quantity: 7)
      Loop 1 (external): 10 Primitives
         Line: (100.000000, 0.000000),(-0.000000, 600.000000)
         Line: (-0.000000, 600.000000),(100.000000, 600.000000)
         Line: (100.000000, 600.000000),(100.000000, 500.000000)
         Line: (100.000000, 500.000000),(200.000000, 400.000000)
         Line: (200.000000, 400.000000),(300.000000, 400.000000)
         Line: (300.000000, 400.000000),(400.000000, 500.000000)
         Line: (400.000000, 500.000000),(400.000000, 600.000000)
         Line: (400.000000, 600.000000),(500.000000, 600.000000)
         Line: (500.000000, 600.000000),(400.000000, 0.000000)
         Line: (400.000000, 0.000000),(100.000000, 0.000000)

   Shape 11 (Loops: 1, Quantity: 7)
      Loop 1 (external): 3 Primitives
         Line: (0.000000, 800.000000),(800.000000, 800.000000)
         Line: (800.000000, 800.000000),(400.000000, 0.000000)
         Line: (400.000000, 0.000000),(0.000000, 800.000000)

   Shape 12 (Loops: 1, Quantity: 7)
      Loop 1 (external): 6 Primitives
         Line: (0.000000, 0.000000),(-0.000000, 600.000000)
         Line: (-0.000000, 600.000000),(400.000000, 600.000000)
         Line: (400.000000, 600.000000),(400.000000, 300.000000)
         Line: (400.000000, 300.000000),(600.000000, 300.000000)
         Line: (600.000000, 300.000000),(600.000000, -0.000000)
         Line: (600.000000, -0.000000),(0.000000, 0.000000)

   Shape 13 (Loops: 1, Quantity: 7)
      Loop 1 (external): 8 Primitives
         Line: (100.000000, 0.000000),(0.000000, 100.000000)
         Line: (0.000000, 100.000000),(-0.000000, 300.000000)
         Line: (-0.000000, 300.000000),(200.000000, 500.000000)
         Line: (200.000000, 500.000000),(400.000000, 500.000000)
         Line: (400.000000, 500.000000),(400.000000, 400.000000)
         Line: (400.000000, 400.000000),(300.000000, 200.000000)
         Line: (300.000000, 200.000000),(400.000000, -0.000000)
         Line: (400.000000, -0.000000),(100.000000, 0.000000
```