Universal mapping class groups

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Mapping class group

S: orientable surface, of finite or infinite topological type.

- $\mathrm{Mod}(S)$: the group of isotopy classes of homeos $S \to S$.
- $\mathrm{PMod}(S)$: the *pure* subgroup (elements pointwise fix boundary and punctures).

Example

$\mathrm{Mod}(\text{torus}) = \mathrm{PMod}(\text{torus}) = \mathrm{GL}(2, \mathbb{Z})$.

Finite presentation

Theorem (Dehn-Lickorish, Hatcher-Thurston, etc)

If S has finite type, then $\mathrm{Mod}(S)$ *is finitely presented. In fact,* $\mathrm{PMod}(S)$ *is generated by a finite collection of Dehn twists.*

Proof.

Construct a nice action of $\mathrm{Mod}(S)$ on a simply-connected complex (built from simple closed curves on S). $\square$

Universal mapping class groups

Groups that contain *every* mapping class group (of genus g).

Motivation

- Homological stability
- Dynamics

Homological stability I

$S_{g,n}$: surface of genus g with n boundary components. Observe that $S_{g,n} \hookrightarrow S_{g,n+1}$ for all n (glue a pair of pants). Can form

$$\lim_{\to_n} \mathrm{PMod}(S_{g,n})$$

Remark

- $\lim_{\to_n} \mathrm{PMod}(S_{g,n})$ *contains* $\mathrm{PMod}(S_{g,n})$ *for all* $n \geq 1$.
- *It is infinitely generated.*

Homological stability II

Theorem (Harer's stability theorem, 1985)

If $g >> i$, then

$$H_i(\mathrm{PMod}(S_{g,n}), \mathbb{Z}) \cong H_i(\lim_{\to_k} \mathrm{PMod}(S_{g,k}), \mathbb{Z})$$

(i.e. H_i does not depend on number of boundary components)

Big mapping class groups

Notation: Σ_g is $S_{g,0}$ minus a Cantor set. Consider $\mathrm{Mod}(\Sigma_g)$.

Big mapping class groups II

Remark

- *Appear naturally in dynamics (D. Calegari)*
- $\mathrm{Mod}(\Sigma_g)$ *contains* $\mathrm{PMod}(S_{g,n})$ *for all* $n \geq 1$.
- $\mathrm{Mod}(\Sigma_g)$ *is uncountable!*
- $\lim_{\to_k} \mathrm{PMod}(S_{g,k}) = \mathrm{PMod}_c(\Sigma_g)$

Asymptotically rigid homeos

Defined in terms of a *rigid structure* on Σ_g: a pants decomposition P, plus some other things. (Point: if f fixes rigid structure then $f = id$.)

Definition

A homeomorphism $f : \Sigma_g \to \Sigma_g$ is *asymptotically rigid* if there exists a P-suited compact subsurface Z with $f(Z)$ P-suited, such that f preserves the rigid structure outside these.

Asymptotic mapping class group

Definition

B_g: the group of isotopy classes of asymptotically rigid homeomorphisms.

Theorem (A-Funar)

One has

$$1 \to P\operatorname{Mod}_c(\Sigma_g) \to B_g \to V \to 1$$

where V is Thompson's group V.

(V is a simple, infinite, finitely-presented group with lots of other cool properties. It's a subgroup of $\mathrm{Homeo}(\mathrm{Cantor})$)

Finite presentation

Theorem (A-Funar)

For all g, the group B_g is finitely presented.

(Known by Funar-Kapoudjian (2004) for $g = 0$.)

Proof.

Construct a nice action on a simply-connected complex (built from simple closed curves on Σ_g). $\square$

Homology

Theorem (A-Funar)

For $g >> i$, $H_i(B_g, \mathbb{Z})$ is the stable homology group of the m.c.g of genus g. (Corollary: B_g is perfect for $g \geq 3$.)

Proof.

Promote Harer's stability to B_g using the short exact sequence above. □

Injections

Theorem (A-Funar)

If $h > g$ then no (weakly) injective homomorphisms $B_h \to B_g$. (Corollary: Class. of the B_g's up to isomorphism.)

Proof.

Use properties of V, plus the analogous statement for $\mathrm{PMod}(S_{g,n})$ (Castel, A-Souto). $\square$

Algebraic rigidity

Theorem (Ivanov)

If S has finite-type then

$$\mathrm{Aut}(\mathrm{Mod}(S)) = \mathrm{Aut}(\mathrm{PMod}(S)) = \mathrm{Mod}(S).$$

Theorem (A-Funar)

For every g, $\mathrm{Aut}(B_g) = N_{\mathrm{Mod}(\Sigma_g)}(B_g)$.

Proof.

Prove that Dehn twists go to Dehn twists. □

Homomorphisms from lattices

Theorem (Farb-Masur)

Let Γ be an higher-rank lattice, S a surface of finite type. Then every homomorphism $\Gamma \to \mathrm{Mod}(S)$ has finite image.

Theorem (A-Funar)

Let Γ be an higher-rank lattice. Then every homomorphism $\Gamma \to B_g$ has finite image.

Proof.

Use properties of V plus the analogous statement for $\mathrm{PMod}(S_{g,n})$ (Farb-Masur...) $\square$

Kazhdan's Property (T)

Question

Do finite-type mapping class groups have Property (T)?

(The expected answer is "no" (Andersen).)

Theorem (A-Funar)

B_g does not have Property (T).

Proof.

B_g surjects onto V, which is infinite and has the *Haagerup property*. □

Linearity

Question

Are mapping class groups linear?

Known only for the closed surface of genus 2 (Bigelow-Budney).

Theorem (A-Funar)

B_g is not linear.

Proof.

B_g contains a copy of Thompson's group F, which is not linear. $\square$

A related group

H_g (contains B_g) consists of asymptotically rigid homeos, in a weaker sense.

Theorem (A-Funar)

H_g is finitely presented, and is dense in $\mathrm{Mod}(\Sigma_g)$.

Proof.

Given a mapping class, construct *by hand* a sequence in B_g that converges to it. □