Session 2: Methods in modelling (Part 1)

Primer for Mathematical Modelling for Biologists
EASTBIO Foundation Masterclasses

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23th May 2023

Overview

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- ordinary differential equation (ODE) models - how to formulate, solve and analyse them

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- chemical reaction ODE models
- ODE models in ecology (animal/human/cell/microbe systems)
- stochastic models

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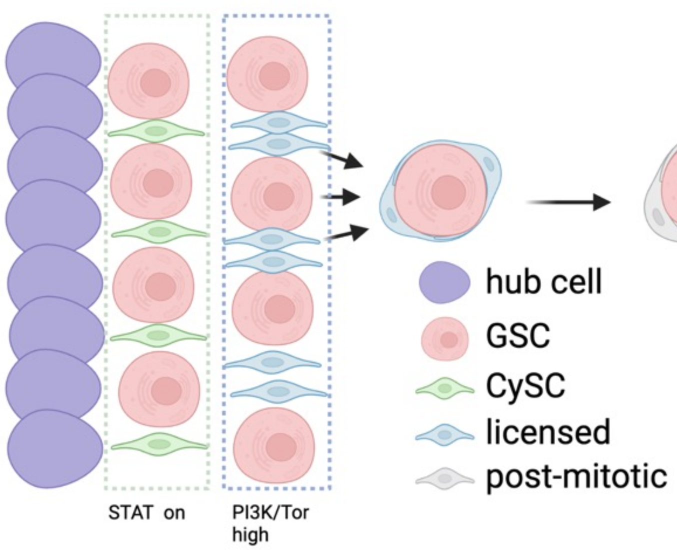
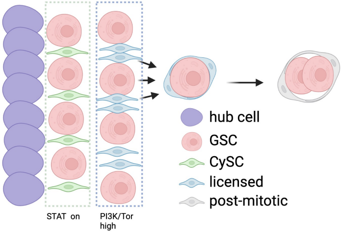
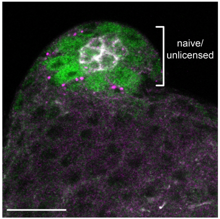
- f can depend on y and/or x or be a constant.
- ODEs can be solved and analysed (1) analytically, (2) numerically, (3) using visual graphs

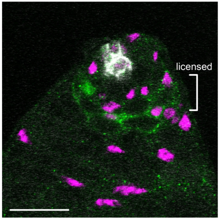
Worked example







STAT activity Hh activity Hub



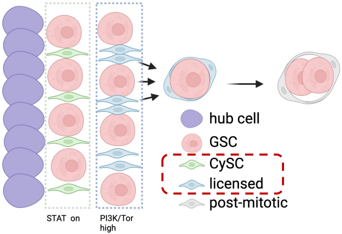
Tor activity cyst cells Hub

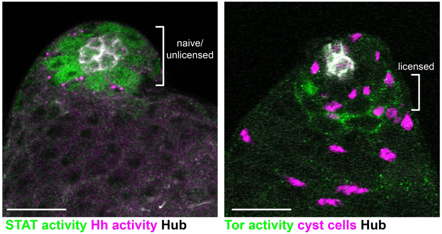
Marc Amoyel

UCL

Amoyel et al. (2014)

Worked example





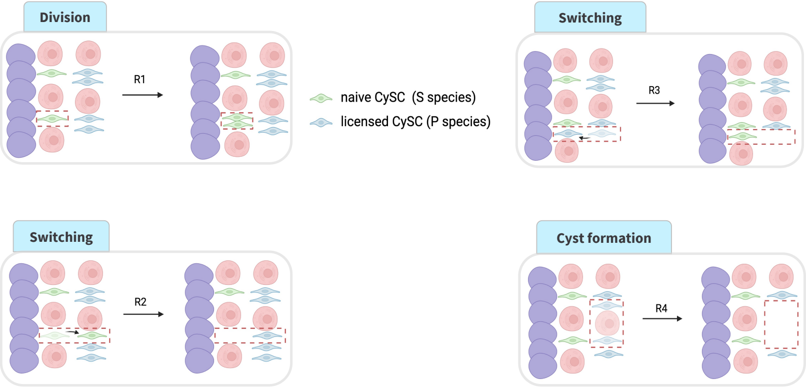
Marc Amoyel

What's the role of licensed CySC states in the stability of the homeostatic state, recovery after injury and regulation of fluctuations?

Can we detect the presence of licensed states from snapshots of the population at different times?

Amoyel et al. (2014)

Worked example



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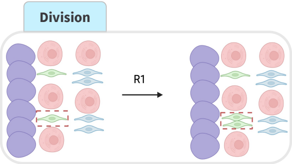
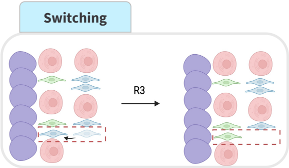


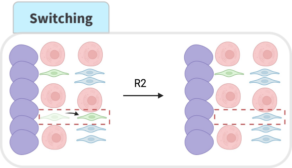
naive CySC (S species)

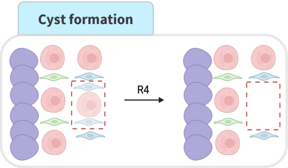
licensed CySC (P species)



SP model

$$\begin{cases} S \to 2S & (R1) \\ S \to P & (R2) \\ P \to S & (R3) \\ 2P \to \emptyset & (R4) \end{cases}$$





Worked example

Reaction network: $S \xrightarrow{k_1} 2S$; $S \underset{k_3}{\overset{k_2}{\rightleftharpoons}} P$; $2P \xrightarrow{k_4} \emptyset$

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- We solve the system **numerically** (analytical methods are available as well)

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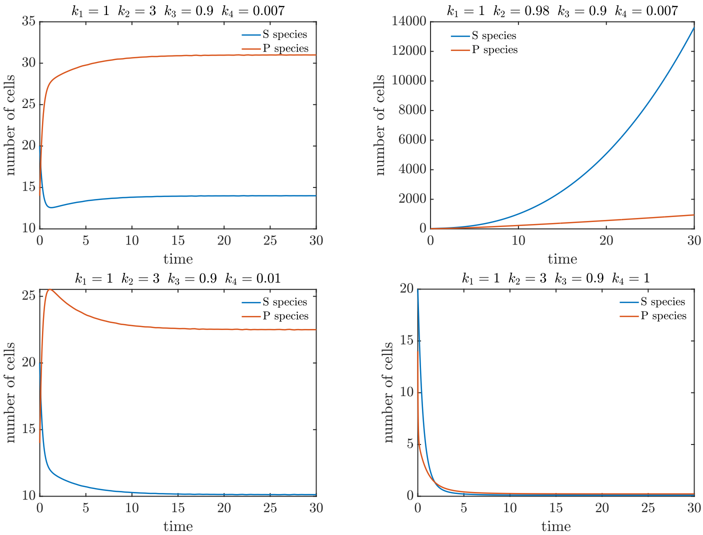

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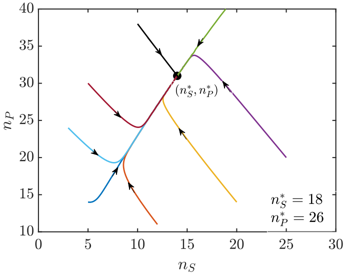

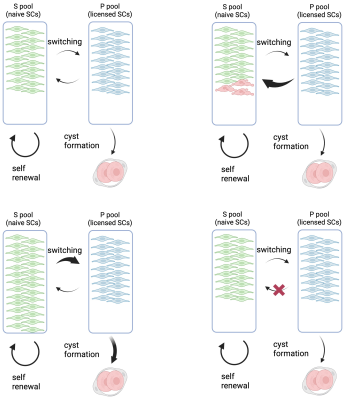

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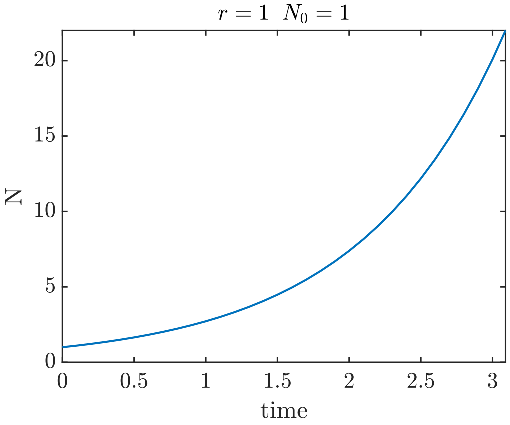

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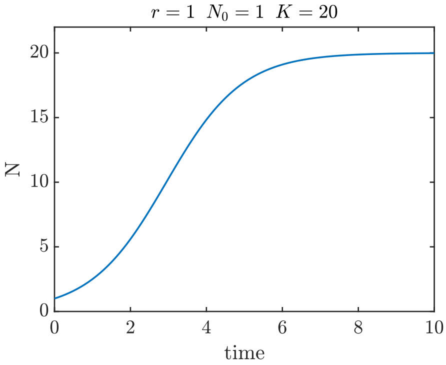

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- \+ there are a lot of well-established ODEs for biological systems
- – do not assume individual agents in the system
- – one variable - assumes that the system is well-mixed and do not capture changes in both space and time simultaneously

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- In comparison, a stochastic process occurs when a random process is evolving in time
- Compared to the definition of a deterministic process, this implies that irrelevant of what we know about the current process we cannot be sure about it's future state

Stochastic modelling

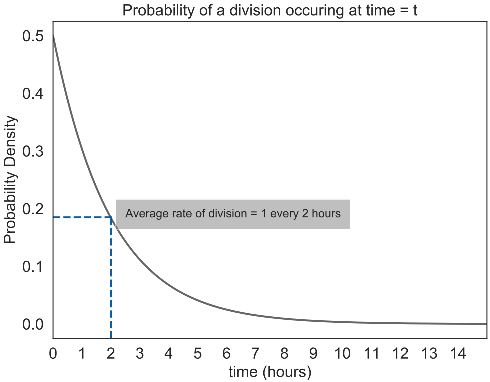
Exponential population growth of bacteria from a stochastic view!
First let's revisit the deterministic model:

- We assume that with the deterministic model: $N(t) = N(0)e^{rt}$, we can calculate the exact population at any time t
- We can choose some starting amount, for N(0), lets say 5 bacteria
- Our new deterministic model is now: $N(t) = 5e^{rt}$,where 5 is our initial number of bacteria
- Lets also assume that our bacteria divide on average every 2 hours, so we can set r $= \frac{1}{2}$
- Now with our final equation of $N(t) = 5e^{\frac{t}{2}}$ We can now plot this over time and see that our population grows exponentially from 5 bacteria at a rate of one division every 2 hours

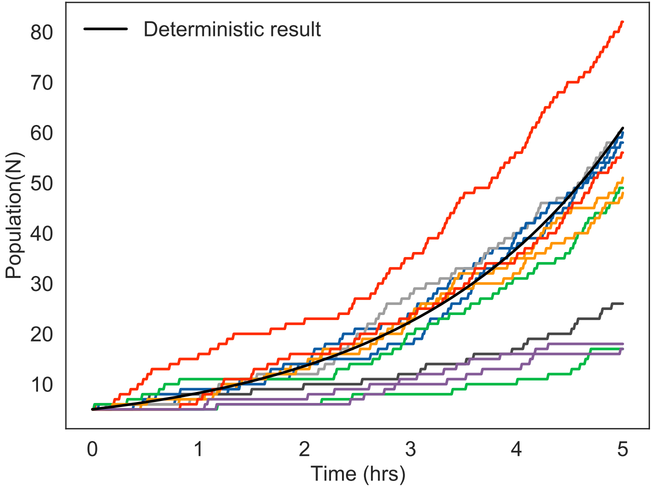
Stochastic modelling

Lets now create the stochastic model:

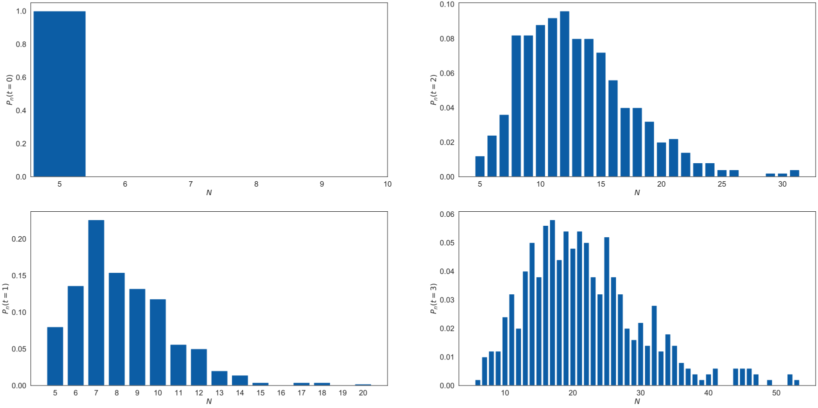
- We will again start with an initial population of 5 bacteria
- Instead of using the solution to the above ODE, we will instead assume that the bacteria divides after a random interval of time with an average rate of 2 hours



Lets compare the deterministic model against the stochastic model!



We can now see how the population changes over time:



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– Caution has to be taken when choosing suitable probability distributions to describe data

Coffee break!