# TOPOLOGY IN AND VIA LOGIC HOMEWORK ASSIGNMENT 1

• Deadline: February 10 at 23:59.

• All exercises are worth the same points.

• Good luck!

## TOPOLOGICAL SPACES

**Exercise 1.** Consider the space  $(\mathbb{R}, \tau_{Euc})$ , with its Euclidean topology.

(1) Give an example of a set which is neither open nor closed.

(2) Show that the open intervals of the form (x, y) where  $x, y \in \mathbb{Q}$  form a basis for this topology.

(3) Show that  $\mathbb{Q}$  is a countable union of closed sets.

**Exercise 2.** Let X be a set. We say that an operation  $\square : \mathcal{P}(X) \to \mathcal{P}(X)$  is called an *interior operator* if it satisfies for each  $U, V \in \mathcal{P}(X)$ ,

• (All set):  $\Box X = X$ ;

• (Normality):  $\Box(U \cap V) = \Box U \cap \Box V$ ;

• (Inflationarity):  $\Box U \subseteq U$ ;

• (Idempotence):  $\Box U \subseteq \Box \Box U$ .

(1) Show that if  $(X, \tau)$  is a topological space, the topological interior *int* is an interior operator in this sense.

(2) Given a set  $(X, \square)$  equipped with an interior operator, define a topology for which  $\square$  is the topological interior operator.

(3) We say that an interior operator  $\square$  is *completely multiplicative* if for each  $(U_i)_{i\in I}$  we have that:

$$\Box(\bigcap_{i\in I}U_i)=\bigcap_{i\in I}\Box U_i$$

Show that Alexandroff topology are in 1-1 correspondence with completely multiplicative interior operators.

(4) Let  $(X, \square)$  be a set, equipped with a completely multiplicative interior operator, with the following property: if  $x \neq y$ , then there is some  $U \subseteq X$  such that either  $x \in \square U$  and  $y \notin \square U$  or  $y \in \square U$  and  $y \notin \square U$ . Show that then there is a poset  $(P, \leq)$  such that the Alexandroff topology on P is the same as the topology induced on X by the interior operator.

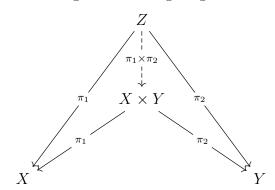
#### CONTINUITY AND CONTINUOUS FUNCTIONS

### Exercise 3. Show the following:

- (1) Given an example of a bijective continuous map which is not a homeomorphism.
- (2) Show that all functions from a discrete space to another space are continuous. If  $(X, \tau)$  is a space with the *indiscrete* topology, which functions from this space to some other space are continuous?
- (3) Show that if  $f: X \to Y$  is a bijective continuous map between topological spaces, then the following are equivalent:
  - $f^{-1}$  is continuous;
  - f is closed;
  - $\bullet$  f is a homeomorphism.

## **Exercise 4.** (*Product maps*) Let X, Y be topological spaces.

(1) Show that for any other topological space Z, if there exists continuous functions  $\pi_1: Z \to X$  and  $\pi_2: Z \to Y$ , then there exists a unique continuous function  $\pi_1 \times \pi_2: Z \to X \times Y$  making the following diagram commute



(2) Show that this defines the product topology up to homeomorphism: whenever a topological space A satisfies the condition in (1), then there exists a homeomorphism between A and  $X \times Y$ .