VARIOUS QUESTERS

THE HITCHIKER'S GUIDE TO C.ESPINDOLA'S PROOF OF

SHELAH'S EVENTUAL CATEGORICITY CONJECTURE

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1
Preliminary Notions

This chapter serves to set up the main concepts needed to begin reading the paper. (...)

1.1 Morley's Theorem

This section is heavily based on Chapter 7.1 of [CK12], the reader is encouraged to consult this for more information.

We start off by restating the target theorem, Morley's theorem.

Theorem 1.1.1 (Morley)**.** *Let $\mathcal{L}$ be a countable language, and let T be a first-order $\mathcal{L}$-theory that has infinite models, and is categorical in some uncountable power. Then T is categorical in every uncountable power.*

Here is a refresher on saturated models, which will be used in the proof:

Definition 1.1.2 (Saturated model)**.** Let $\mathfrak{M}$ be a structure, and $X \subseteq M$. We write $\mathfrak{M}_X$ to denote the model $\mathfrak{M}$ expanded to the language containing a new constant symbol, c_x, for every element $x \in X$, with each c_x interpreted as the element x.

Let $\mathfrak{M}$ be a structure, and κ an infinite cardinal. We say that $\mathfrak{M}$ is *κ-saturated* iff, for any $X \subseteq M$ with $|X| \ \kappa$, every type that is finitely realised in $\mathfrak{M}_X$ is realised in $\mathfrak{M}_X$.

We say that $\mathfrak{M}$ is *saturated* iff $\mathfrak{M}$ is κ-saturated for $\kappa = |M|$.

Theorem 1.1.3. *Let $\mathfrak{M}$ be an infinite model, and let κ be an infinite cardinal. Then $\mathfrak{M}$ has a κ-saturated elementary extension $\mathfrak{N}$. (Note that $\mathfrak{N}$ might not be same cardinality of $\mathfrak{M}$! It might be much bigger)*

Proof. (Adapted from van den Berg (2018), 'Syllabus Model Theory 2018/2019')

We will find a model which is κ^+-saturated; we need a regular cardinal to make the proof work, and it is immediate from the definition of κ-saturation that such a model must also be κ-saturated.

We build the new model $\mathfrak{N}$ in κ^+ many stages. For successor ordinals $\alpha + 1$, let $\mathfrak{M}_\alpha$ be the model constructed so far. Let A denote the universe of $\mathfrak{M}_\alpha$. Let $(p_i(x))_{i \in I}$ be an enumeration of all types over $Th((\mathfrak{M}_\alpha)_A)$. Let $\{b_i \mid i \in I\}$ be a collection of fresh constant symbols. Consider the theory $T_{\alpha+1} := Th((\mathfrak{M}_\alpha)_A) \cup \{p_i(b_i) : i \in I\}$. This theory is finitely satisfiable by assumption, so by the Compactness theorem, it's satisfiable. Set $\mathfrak{M}_{\alpha+1}$ to be some model of $T_{\alpha+1}$- it will then be an elementary extension of $\mathfrak{M}_\alpha$, since $T_{\alpha+1}$ includes the elementary diagram of $\mathfrak{M}_\alpha$.

For limit ordinals γ, define $\mathfrak{M}_\gamma$ to be the union of the chain $(\mathfrak{M}_\alpha)_{\alpha\gamma}$. Then $\mathfrak{M}_\gamma$ is an elementary extension of each of the $\mathfrak{M}_\alpha$.

Here we use the closure property of elementary classes under unions of chains. Recall that AECs also have this property!

Keep going until we reach $\mathfrak{M}_{\kappa^+}$. Now consider any $X \subseteq \mathfrak{M}_{\kappa^+}$ with $|X| \leq \kappa^+$, and any finitely realisable type $p(x)$ over $(\mathfrak{M}_{\kappa^+})_X$. By the regularity of κ^+, there must be some $\alpha \ \kappa^+$ such that $X \subseteq \mathfrak{M}_\alpha$. But then $p(x)$ is realised in $\mathfrak{M}_{\alpha+1}$ by construction, and the same element will realise p in $\mathfrak{M}_{\kappa^+}$. □

To prove Morley's theorem, we will need many, many lemmas. Here's the first one:

Lemma 1.1.4. Let $\mathcal{L}$ be a countable language, and let T be a complete $\mathcal{L}$- theory such that every model of T of cardinality ω_1 is saturated. Then every uncountable model of T is saturated.

Proof. We proceed by contraposition: we suppose that T has an uncountable model which is not saturated, and find a model of T of size ω_1 which is not saturated.

Let $\mathfrak{A}$ be a model of T which is not saturated. Then let X be a subset of A, with $|X| \ |A|$, and let $p(x)$ be a type over $\mathfrak{A}_X$, such that p is finitely satisfiable in $Th(\mathfrak{A}_X)$ but not realised in $\mathfrak{A}_X$. We now let $U \subseteq A$ be any subset of A such that $|U| = |p|$. (Observe that $|U| = |p| \ |A|$, because $\mathcal{L}$ is countable and $|X| \ |A|$.)

□

1.2 Abstract Elementary Classes

This section is modelled after [Gro02]; the reader is encouraged to consult this for more information.

Definition 1.2.1. Let $\langle K, \preceq_K \rangle$ be a pair consisting of a collection of structures K for some language $L(K)$, and a relation $\preceq_K$ holding between these structures, such that:

1. $\preceq_K$ is a partial order.
2. If $M \preceq_K N$ then M is a substructure of N.
3. (Isomorphism closure): K is closed under isomorphism, and if $M, N, M', N' \in K$, $f : M \cong M'$ and $g : N \cong N'$, $f \subseteq g$ and $M \preceq_K N$ then $M' \preceq_K N'$.
4. (Coherence): If $M \preceq_K N$ and $P \preceq_K N$, and $M \subseteq P$, then $M \preceq_K P$.
5. (Tarski-Vaught Axioms): If γ is an ordinal and $\{M_\alpha : \alpha \in \gamma\} \subseteq K$ is a chain under $\preceq_K$, then
   - $\bigcup_{\alpha\in\gamma} M_\alpha \in K$;
   - If $M_\alpha \preceq_K N$ for all $\alpha \in \gamma$ then $\bigcup_{\alpha\in\gamma} M_\alpha \preceq_K N$.

6. (Lowenheim-Skolem Axiom) : There exists a cardinal $\mu \geq |L(K)| + \aleph_0$, such that if A is a subset of $M \in K$, then there is N such that $A \subseteq N$, $|N| \leq |A| + \mu$ and $N \preceq_K M$.

Given an AEC $\langle K, \preceq_K \rangle$ we denote by $\mathsf{LS}(K)$ the least μ in the conditions of the LS-axiom, and call it the *Lowenheim-Skolem number*.

Given such an AEC, a map $f : M \to N$ where $M, N \in K$ is a K-*embedding* if $f[M] \preceq_K N$, and f is an isomorphism from M onto $f[M]$.

Let us consider some examples:

Example 1.2.2. If K is an elementary class, i.e., $K = \mathsf{Mod}(T)$ for some theory T, then it is an abstract elementary class with the relation $\preceq_K$ being given by elementary *substructure*. The two first axioms are trivial, and isomorphism closure, coherence, the Tarski-Vaught axioms and the Lowenheim-Skolem axiom are all properties known in classical model theory. The Lowenheim-Skolem number is $|T| + \aleph_0$.

Originally this paragraph claims when $\preceq_K$ is given elementary *embedding*, the resulting class will be an AEC, which is *not* true. Again, axiom 2 means $M \preceq_K N$ implies M is a substructure of N. Of course, none of these nitpicking things survive once we consider accessible categories, which generalise AECs (cf. next Section).

Of course we would not be interested in abstract elementary classes if this were the only example on hand. The key motivation of the theory lies in the fact that some classes are, from a certain point of view, very natural, and do not look too wild to be analysed through model-theoretic methods. For example we have

- Finitely generated groups;
- Archimedean fields;
- Connected graphs;
- Noetherian rings;
- The class of algebraically closed fields with infinite transcendence degree.

It seems like there should be some setting in which one could study these models that offered tools to classify them. But it is not immediately obvious what that would be. For the majority of the 20th century, it was judged that the way forward would be to construct languages which could “tame" these classes. Let us turn to some examples of this kind; for general references on infinitary logic, the reader can consult [Kar64; Mar02; Dic85]:

Definition 1.2.3. Let κ and λ be regular cardinals and $\lambda \leq \kappa$. Let τ be a first order vocabulary. We denote by $\mathcal{L}_{\kappa,\lambda}(\tau)$ the language constructed using $\bigvee_\kappa$, $\bigwedge_\kappa$ and $\exists_\lambda$ and $\forall_\lambda$:

Regularity of the cardinals is assumed to ensure that any formula ϕ has at most κ many subformulas; also $\lambda \leq \kappa$ is a sanity condition given we maintain our atomic formulas finite (which is not necessary, but very convenient).

(1) Terms and atomic formulas are as in first order logic;

(2) If ϕ is a formula then so is $\neg\phi$;

(3) If (ϕ_α) is a collection of less than κ formulas, then so is $\bigwedge_\kappa \phi_\alpha$ and $\bigvee_\kappa \phi_\alpha$;

(4) If $\phi(x_\alpha)$ for $\alpha \in \lambda$ is a formula with less than λ free variables and all its free variables amongst x_α, then so is $\exists_\lambda \phi(x_\alpha)$ and $\forall_\lambda \phi(x_\alpha)$.

We let:

$$\mathcal{L}_{\infty,\lambda} = \bigcup_{\kappa \in \mathsf{Ord}} \mathcal{L}_{\kappa,\lambda}$$

We briefly mention some notions which are relevant here. Namely, given two first-order theories M, N, we write:

$$M \simeq_{\infty,\omega} N,$$

If and only if the two structures satisfy the same formulas from $\mathcal{L}_{\infty,\omega}$. An equivalent, and more useful, characterization due to Karp uses the notion of a *partial isomorphism system*.

Definition 1.2.4. Let $\mathbf{M}, \mathbf{N}$ be two first order structures in a language τ. We say that a collection P of partial τ-embeddings $f : A \to \mathbf{N}$ where $A \subseteq M$ is a *partial isomorphism system* if:

- For all $f \in P$, and $a \in \mathbf{M}$, there is some $g \in P$ such that $f \subseteq g$ and $a \in \mathsf{dom}(g)$;
- For all $f \in P$ and $b \in \mathbf{N}$, there is $g \in P$ such that $f \subseteq g$ and $b \in \mathsf{img}(g)$

We write $\mathbf{M} \cong_p \mathbf{N}$ if there is a partial isomorphism system between these two structures.

The following relates this concept to Ehrenfeucht-Fraisse games, to the above notion of infinitary equivalence, as well as to some set-theoretic notions.

Theorem 1.2.5. *The following are equivalent for* $\mathbf{M}, \mathbf{N}$*:*

1. $\mathbf{M} \simeq_{\infty,\omega} \mathbf{N}$*;*
2. $\mathbf{M} \cong_p \mathbf{N}$*;*
3. *Player II has a winning strategy in the Ehrenfeucht-Fraisse game* $G(\mathbf{M}, \mathbf{N})$*;*
4. *There is a forcing extension* $V[G]$ *such that* $V[G] \vDash \mathbf{M} \cong \mathbf{N}$ *(i.e., the structures are isomorphic in a forcing extension).*

Proof. The equivalence of 1-3 is known as *Karp's theorem*; for a good proof see [Mar02, Theorem 2.1.4]. The equivalence with (4) is a known set-theoretic fact, mentioned in [Mar02, Exercise 2.1.8] □

This uses essentially the fact that the language $\mathcal{L}_{\infty,\omega}$ is absolute, and constructs the extension using a forcing poset consisting of a partial isomorphism system.

For the most part we will look at infinitary logics with finite quantifiers. We will also need the notion of a *fragment*:

Definition 1.2.6. Let $\mathbb{A} \subseteq \mathcal{L}_{\infty,\omega}$ be a set of formulas in the language τ such that there is an infinite set of variables V, such that if $\phi \in \mathbb{A}$ then all of its variables occur in V. We say that $\mathbb{A}$ is a *fragment* of τ if $\mathbb{A}$ satisfies the following closure properties:

1. All atomic formulas using only the constant symbols in the vocabulary τ and the variables in V are in $\mathbb{A}$;
2. $\mathbb{A}$ is closed under subformulas;
3. $\mathbb{A}$ is closed under substitution of terms assembled from V: if $\phi \in \mathbb{A}$ and v is free in ϕ and t is a term with all of its variables in V, then the formula obtained by replacing all instances of v in ϕ by t is in $\mathbb{A}$;

4. $\mathbb{A}$ is closed under formal/single negations;

5. $\mathbb{A}$ is closed under $\neg, \wedge, \vee, \exists v, \forall v$ for $v \in V$.

If $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$, and $|\mathbb{A}| \leq \aleph_0$, we say that it is a *countable fragment*.

The following definition is the crucial one:

Definition 1.2.7. Let M and N be structures in a language L, and $\mathbb{A}$ is an L-fragment. We write $M \subseteq_{TV,\mathbb{A}} N$ if and only if:

1. $M \subseteq N$ and,

2. For every $\bar{a} \in M$ and every formula $\phi(y, \bar{x}) \in \mathbb{A}$, if $N \vDash \exists y \phi(y, \bar{a})$, then there exists some $b \in M$, such that $N \vDash \phi(b, \bar{a})$.

Example 1.2.8 (Models of a countable infinitary theory). Let T be a countable theory in a language $\mathcal{L}$, and let $\mathbb{A}$ be a fragment containing T. Let $K = \mathsf{Mod}(T)$. Let $M \preceq_K N$ if and only if $M \subseteq_{TV,\mathbb{A}} N$. Then $\langle K, \preceq_K \rangle$ is an abstract elementary class. The trickier parts to verify are the Lowenheim-Skolem and the union axiom; but both of these follow by the same proofs as their first-order correspondents.

The restriction to countable theories is sharp: it is not hard to find a theory T, in a countable language, of $\mathcal{L}_{\omega_1,\omega}$ which models are at least of size $2^{\aleph_0}$.

However AEC's are not at all limited to examples coming from logic. Let us see some preliminary examples, and then conclude with a wild, unexpected, example, which breathed new life to the field.

Example 1.2.9 (Noetherian Rings). Let K be the class of noetherian rings. We define $R \preceq_K S$ if and only if R is a subring of S, and $R \simeq_{\infty,\omega} S$. Note that then R is noetherian if and only if S is noetherian. To see this, note that if we assume that R is noetherian and S is not, then (by an equivalent characterization), there is $f_1, f_2, ...$, a sequence of elements such that for every integer n there is some f_i, such that f_i cannot be written in terms of the smaller elements. Then we claim that Player I has winning strategy in an unbounded Ehrenfeucht-Fraisse game: successively pick elements from that sequence. Once the game is played out, whatever Player II has chosen, say a sequence $g_1, g_2, ...$, there must be an integer n such that each g_i is a linear combination of g_k for $k \leq n$. But then this sequence cannot be isomorphic to the former.

It is clear that if $R \preceq_K S$ then R is a substructure of S, and isomorphism closure and coherence are obvious. The Tarski-Vaught axiom follows from the fact that chains of models respect the $\simeq_{\infty,\omega}$ relation. I could not prove the Lowenheim-Skolem axiom, though Grossberg's notes claim it (Shrug).

Perhaps the most striking example – and one which in part revived the interest in this topic from the point of view of mainstream mathematics – is in the work of Boris Zil'ber's "Schanuel's Structures".

Definition 1.2.10. Let $\mathcal{K}_e$ be defined as:

$$\mathcal{K}_e := \{\langle F, +, \cdot, exp \rangle : F \text{ is an alg. closed field of characteristic zero, } \forall x \forall y (exp(x+y) = exp(x) \cdot exp(y))\}$$

also let

$$\mathcal{K}_{pexp} := \{\langle F, +, \cdot, exp\rangle \in \mathcal{K}_e : \mathsf{ker}(exp) = \pi\mathbb{Z}\}.$$

We consider the class of *Schanuel structures* to be the class $\mathcal{K}_{exp} \subseteq \mathcal{K}_{pexp}$ which satisfies some conditions, amongst them the *Schanuel condition*.

We are willfully vauge on the extra conditions, as they are not important, except for the Schanuel condition. For more information on this, check [Mar02, Chapter 8], and see also Will Boney's notes on this topic.

This essentially imposes that the so-called "Schanuel conjecture" be true:

Conjecture 1.2.11 (Schanuel,1960). Assume that $x_0, ..., x_n \in F$ are linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(x_0, ..., x_n, exp(x_0), ..., exp(x_n))$ has transcendence degree at least n over $\mathbb{Q}$.

To understand the significance of this, it should be noted that the study of transcendental numbers is one of the most unexplored and difficult areas of number theory.

Schanuel's conjecture is a piece of machinery that would clarify many difficult conjectures in transcendental number theory. As a toy example, recall that it is widely assumed that $e + \pi$ is transcendental, though no proof of it is in sight; this would immediately fall off from the above result: if we set $x_0 = 1$ and $x_1 = \pi * i$, then $\mathbb{Q}(\pi, e)$ (the result of the field extension) would have transcendence degree at least 2, showing that there is no polynomial $f(x, y)$ such that $f(\pi, e) = 0$; this implies that $e + \pi$ is transcendental.

Now what Zil'ber did was note that $\mathcal{K}_{exp}$ can be given a relation $\preceq$, forming an abstract elementary class. Additionally, using some heavy model-theoretic and number-theoretic weaponry, he managed to prove that:

Theorem 1.2.12. *The theory $\mathcal{K}_{exp}$ has a unique model of cardinality $2^{\aleph_0}$.*

Thus, the only problem lies in proving that this model is indeed the model of the complex field $\mathbb{C}$, i.e., prove that the latter has the model-theoretically desirable properties. This is an active research area today.

Such examples motivate the idea that abstract elementary classes are indeed ubiquitous, and serve as a strong foundation for exploring non-trivial solutions to mathematical problems. However, as discussed in the logical dream, just like for first-order logic, this appears as a matter of finding the right "dividing lines". Hence we can encounter our appropriate generalization of Los' conjecture:

Conjecture 1.2.13 (Shelah). Let $\mathcal{K}$ be an AEC. If there is a $\lambda \geq \beth_{2^{LS(\mathcal{K})}})^+$ such that $\mathcal{K}$ is categorical in λ, then $\mathcal{K}$ is categorical in all μ for $\mu \geq \beth_{2^{LS(\mathcal{K})}})^+$.

1.3 Accessible Categories

In the category of sets, we have the special property that the elements of a set X (its internal structure) correspond exactly to maps from the terminal object to X (its external structure), i.e. $\mathrm{Hom}_{\mathbf{Set}}(1, X) \cong X$. This is not the case for all categories in general. For a wider class of categories, the internal structure of objects can still be reflected externally, but we require more than just one object to "probe" its internal structure.

A warning: we do not mention *sketches* in our presentation. While they play a central role in the books that introduced accessible categories, they are not as necessary for modern treatments, which is why we have chosen not to include them here. Similarly goes for *synthetic categories* as theories, but for different reasons.

Example 1.3.1. In the category of undirected graphs, the terminal object is the single node graph with a reflexive edge. However, graph homomorphisms from this terminal graph will never reveal anything about irreflexive nodes.

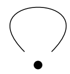

Figure 1.1: The terminal graph.

What we can do instead is to use two graphs: one graph with a single irreflexive node (targetting the nodes) and one graph with two nodes and an edge between them(targetting the edges). Any graph can be constructed by taking multiple copies of these diagrams (with some gluing homomorphisms f and g that designate a node to be the endpoint of some edge). Another way of viewing this is that any graph can be reconstructed as a colimit of a diagram inside the subcategory consisting of these two graphs and f, g.

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "f"] \arrow[r, bend left=20, "g"'] & (\bullet \arrow[r, dash] & \bullet) \end{tikzcd}$$

Figure 1.2: The probing subcategory.

In general the objects that we consider will have to be simple building blocks: a first candidate is for the object to be "finite".

Locally Finitely Presentable Categories

Definition 1.3.2. A diagram $D : \mathcal{I} \to \mathcal{C}$ is *directed* if $\mathcal{I}$ is a directed poset (considered as a category): every finite subset has an upper bound.

Definition 1.3.3 (Finitely Presentable Object [AR94])**.** An object C of a category $\mathcal{C}$ is *finitely presentable* if for each directed diagram $D : (I, \leq) \to \mathcal{C}$, and for each colimit cone $(D_i \xrightarrow{c_i} C)_{i \in I}$ with C as vertex, any morphism $f : K \to C$ uniquely factorizes through some c_i, i.e. there is a unique c_i and $g : K \to D_i$ s.t. $f = c_i \circ g$.

More succintly, C is finitely presentable if the hom-functor $\mathrm{Hom}_{\mathcal{C}}(C, -)$ preserves directed colimits.

Example 1.3.4. The finitely presentable objects in **Set** are exactly the finite sets. We sketch why this is the case.

One can see this example as the motivating example for the above category-theoretic definition - after all, finiteness for an object is only well defined in categories of sets.

For any set X, the diagram D_X consisting of its finite subsets with inclusion functions is directed with X being the colimit of X. If X is finitely presentable, then the identity map id_X factors as $id_X = (D_X(i) \hookrightarrow X) \circ g$, but this means g has to be the identity map with $D_X(i) = X$, where $D_X(i)$ is finite.

On the other hand, suppose X is finite, and take any $f : X \to C$ and any directed diagram D with C as colimit. For each $x \in X$, $f(x)$ must be an element of $D(i_x)$ for some i_x. However, since D is a directed diagram and X is finite we can find a $D(i)$ which contains all the $D(i_x)$, and therefore all the $f(x)$. We can then factor f through $D(i)$.

$$\begin{tikzcd} C & & & \\ & & f[X] \subseteq D(i) \arrow[ull] & \\ f(x_1) \in D(i_{x_1}) \arrow[urr] & f(x_2) \in D(i_{x_2}) \arrow[ur] & \dots & f(x_k) \in D(i_{x_k}) \arrow[ul] \\ X \arrow[uuu, bend left, "f"] \arrow[uurr] & & & \end{tikzcd}$$

Since any set X is a colimit of a directed diagram of finitely presentable sets D_X, we say that **Set** is locally finitely presentable.

Definition 1.3.5. A category $\mathcal{C}$ is *locally finitely presentable* (LFP) if it is cocomplete and has a set of finitely presentable objects $\mathcal{A}$ s.t. every object is a directed colimit of objects from $\mathcal{A}$.

Intuition: $\mathcal{C}$ is locally finitely presentable if the finitely presentable objects essentially determine the rest of the category.

Many categories are LFP - in particular, the category of models of an equational theory (i.e. finitary varieties of algebras) is LFP. We illustrate this by way of the category of groups.

Example 1.3.6 ([AR94]). A *finitely presentable* group is a group which has a presentation $\langle A|R\rangle$ with A a finite set of generators and R a finite set of relations of said generators. The finitely presentable objects in **Grp** are exactly the finitely presentable groups[1].

[1] In fact the general terminology is derived from this.

Suppose G is a finitely presentable group. Then analogously to **Set**, we can consider the diagram D_G of subgroups generated by finitely many generators (with inclusion maps between subgroups), which has G as its colimit. Then the identity map id_G factors through some $D_G(i)$ and therefore $A = D_G(i)$.

For the relations, we again use the same technique of constructing a diagram and factoring the identity map. We have shown G can be finitely generated (let's call this finite set of generators X), so we can consider the canonical map $k : F(X) \to G$ from the free group generated by X to G. The kernel set

$$\ker k = \{(t,t') \in F(X) \mid k(t) = k(t')\}$$

contains all possible relations that hold between the terms when interpreted as elements of A. Hence, X generates G using the equivalence relation $\ker k$. Now, we can once again construct a directed diagram of the groups $D_G(i)$ generated by X and using finite subsets E_i of $\ker k$[2] - this diagram has G as colimit, and we can once again factor the identity map id_G through this diagram (see [AR94, p. 144] for details, as well as for the proof of the other direction).

We can think of k as the mapping from syntax (i.e. elements of the free group) to semantics (elements of G).

$$\begin{aligned}
&k : F(X) \to G\\
&k(x \in X) := x\\
&k(e) := e_G\\
&k(t_1 \cdot t_2) := k(t_1) \cdot_G k(t_2)\\
&k(t^{-1}) := k(t_1)^{-1}
\end{aligned}$$

Categorically, the kernel set can be constructed as the pullback

$$\begin{tikzcd}
\ker k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "|k|"] \\
|F(X)| \arrow[r, "|k|"'] & |G|
\end{tikzcd}$$

[2] i.e. take the congruence closure of E_i.

Any group is a filtered colimit of its finitely generated subgroups. Hence, the category **Grp** is locally finitely presentable. ■

In fact, even some non-equational theories have LFP categories of models. Consider the theory of partially ordered sets.

Example 1.3.7. In the same way as *Set* is a *LFP*, *Pos*, the category of posets and monotone maps, is an *LFP*. Since every poset is the union of all its finite subsets under the (restricted) ordering, the *FP*-objects in *Pos* are exactly the finite posets.

It can be enlightening to see what a locally finitely presentable poset.

Example 1.3.8. A poset, seen as a category, is *LFP* if and only if it is a complete algebraic lattice. Since algebraic lattices are those which are generated by joins of finite (compact) elements, it is exactly the finite elements that correspond to the finitely presentable objects in the category.

Example 1.3.9. A *non-example* is the category **FinSet** of finite sets and functions. We will come back to this example later.

Limit Theories

In general, we can characterise the locally finitely presentable categories as the categories of models of a finitary limit theory.

Definition 1.3.10. Let Σ be a first-order signature of function and relation symbols. We define the category $\mathbf{Str}\Sigma$ of structures interpreting Σ, with the morphisms being structure homomorphisms.

Definition 1.3.11 (Limit Theory)**.** A set of $\mathcal{L}_{\omega,\omega}$ sentences T is a *limit theory* if every sentence in T is of the form

$$\forall \overrightarrow{\mathbf{x}}(\varphi(\overrightarrow{\mathbf{x}}) \rightarrow \exists! \overrightarrow{\mathbf{y}} \psi(\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}}))$$

where φ and ψ are conjunctions of atoms.

Definition 1.3.12 (Model Category)**.** Given a theory T in signature Σ, the category $\mathbf{Mod}T$ of models of T is the full subcategory of $\mathbf{Str}\Sigma$ containing structures that satisfy T.

Theorem 1.3.13 ([AR94, p. 207])**.** *$\mathcal{C}$ is LFP iff it is equivalent to* $\mathbf{Mod}T$ *for some limit theory T in $\mathcal{L}_{\omega,\omega}$.*

Proof. (sketch) ($\Leftarrow$) The category $\mathbf{Str}\Sigma$ is complete, cocomplete and LFP [AR94, p. 201][3]. We show that $\mathbf{Mod}T$ is closed under limits and directed colimits when T is a limit theory. This is enough to show that $\mathbf{Mod}T$ is LFP. TODO

($\Rightarrow$) TODO □

[3] $\mathbf{Str}\Sigma$ inherits a lot of its structure from considering the algebraic structures interpreting its function symbols (forgetting the relations).

We use the theories of groups and posets as concrete examples of limit theories.

Example 1.3.14. We axiomatise the theory of groups with a constant symbol e, unary function symbol $-^{-1}$ and $- \cdot -$ in the following way:

$$\forall x(\top \Rightarrow x \cdot e = x \wedge e \cdot x = x)$$
$$\forall x(\top \Rightarrow \exists! y(x \cdot y = e \wedge y \cdot x = e))$$
$$\forall x_1, x_2, x_3(\top \Rightarrow ((x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)))$$

Since each sentence is a limit sentence, the theory of groups is a limit theory. The category of models is exactly $\mathbf{Grp}$.

Example 1.3.15. We similarly axiomatise posets by limit sentences:

$$\forall x(\top \Rightarrow x \leq x)$$
$$\forall x_1, x_2(x_1 \leq x_2 \wedge x_2 \leq x_1 \Rightarrow x_1 = x_2)$$
$$\forall x_1, x_2, x_3(x_1 \leq x_2 \wedge x_2 \leq x_3 \Rightarrow x_1 \leq x_3)$$

Hence, the theory of posets is also a limit theory. The category of models is exactly $\mathbf{Pos}$.

Locally Presentable Categories

The correspondence between locally finitely presentable[4] categories and models of finite limit theories suggests that if we move to infinitary logic, we must also loosen the finite presentability[5].

[4] A more suggestive name in this context would be "compact categories"

[5] i.e. infinitary logic is no longer compact Is this true? idk I'm tired

Definition 1.3.16. Let λ be a regular cardinal. The diagram $D : \mathcal{I} \rightarrow \mathcal{C}$ is *λ-directed* if $\mathcal{I}$ is a poset where every subset of cardinality λ has an upper bound.

We consider only regular cardinals because otherwise if λ is singular, we may get an upper bound whose "size" is greater than or equal to λ. In particular, consider the example of the $\aleph_\omega$-directed diagram of subsets of $\aleph_\omega$.

Definition 1.3.17. Let λ be a regular cardinal. An object C of a category $\mathcal{C}$ is *λ-presentable* if $\mathrm{Hom}_{\mathcal{C}}(C, -)$ preserves λ-directed colimits.

$\mathcal{C}$ is *locally λ-presentable* if it is cocomplete and has a set of λ-presentable objects $\mathcal{A}$ s.t. every object is a directed colimit of objects from $\mathcal{A}$.

$\mathcal{C}$ is *locally presentable* if it is *locally λ-presentable* for some λ.

The less succint definition is analogously obtained by replacing "directed diagram" with "λ-directed diagram".

Going back to our example of a poset, seen as a category, we can see the difference between an *LFP* category and an *LP* category.

Example 1.3.18. A poset, seen as a category, is locally presentable if and only if it is a complete lattice. Notice here the lack of requirement of algebraicity, that the lattice is generated by finite (compact) elements.

There is an interesting theorem, which we will note for posterity's sake but not use in the sequel.

Theorem 1.3.19 (Gabriel-Ulmer)**.** *CITE HERE If $\mathbb{C}$ is a locally presentable category (locally finitely presentable), $\mathbb{C}^{op}$ is never locally presentable (locally finitely presentable), unless it is a poset.*

We end with going back to our earlier non-example.

Example 1.3.20. **FinSet** is *not* locally presentable since it is not cocomplete. To prove this, take all singletons $\{x\}$ with $x \in \mathbb{N}$. Their directed colimit, seen as union, is all of $\mathbb{N}$, which clearly is not finite.

This shows us that while local presentability generalises local *finite* presentablility, it does not include some quite natural categories.

On the logical side, the corresponding language for FPs is $\mathcal{L}\infty$. Still allowing the same limit theories, we obtain an analogous representation theorem as for LFPs.

Theorem 1.3.21. *$\mathcal{C}$ is locally presentable iff it is equivalent to the category of models of some limit theory in $\mathcal{L}\infty$.*

In other words, the generalisation to LPs changes the logical language to an infinitary one, but the theories are the same. As noted above, there are many familiar theories that are not limit theories, such as the theory of finite sets and the theory of linear orders.

Accessible Categories

Accessible categories are locally presentable categories that aren't necessarily cocomplete.

Definition 1.3.22. A category $\mathbb{C}$ is *λ-accessible* for a regular cardinal λ if $\mathbb{C}$ has λ-directed colimits and $\mathbb{C}$ has a set of λ-presentable objects such that every object in $\mathbb{C}$ is a λ-directed colimit of objects from this set. A category $\mathbb{C}$ is *accessible* if it is *λ-accessible* for some regular cardinal λ.

Compare this with the definition of locally presentable categories. Note that the condition for cocompleteness has been removed.

Example 1.3.23. The category of linear orders, **LinOrd** is accessible. The finitely presentable objects are exactly the finite linear orders and every linear order is a directed colimit of these.

Note that the reason **LinOrd** is not LP is due to the fact that **LinOrd** is not cocomplete.

Example 1.3.24. **FinSet** is accessible. Every finite set is a presentable object and thus every object in **FinSet** is finitely presentable.

Since we no longer require **FinSet** to be cocomplete, it now fits in our definition.

Example 1.3.25. All the previous examples of LFPs and LPs are trivially accessible categories.

Basic Theories

Just as LP categories correspond to limit theories, accessible categories correspond as well to basic theories.

Definition 1.3.26 ([AR94, p. 227]). A formula in $\mathcal{L}_{\infty,\infty}$ is called

1. *positive-primitive* if it has the form $\exists Y \psi(X,Y)$[6] where $\psi(X,Y)$ is a conjunction of atomic formulas.
2. *positive-existential* if it is a disjunction of positive-primitive formulas.
3. *basic* if it has the form $\forall X(\phi(X) \to \psi(X))$ where ϕ and ψ are positive-existential formulas.

A set of $\mathcal{L}_{\infty,\infty}$ sentences T is a *basic theory* if every sentence in T is basic.

[6] We use capital letters to denote a set of variables being quantified over.

Note that basic theories indeed generalises limit theories because limit sentences of the form

$$\forall X(\phi(X) \to \exists! Y \psi(X,Y))$$

can be replaced by two basic sentences:

$$\forall X(\phi(X) \to \exists Y \psi(X,Y))$$

$$\forall X,Y,Z(\phi(X) \wedge \psi(X,Y) \wedge \psi(X,Z) \to Y = Z)$$

Example 1.3.27. **FinSet** is basic since we can axiomatise finite sets via the sentence

$$\forall x_0, x_1, \ldots (\bigvee_{i \neq j \in \omega} x_i = x_j)$$

.

Example 1.3.28. **LinOrd** is basic - in particular the sentence

$$\forall x,y(x \leq y \vee y \leq x)$$

is basic.

In some sense[7], working with basic theories only causes no loss in generality as compared to working with all theories.

[7] I'm not really sure in what "sense" this causes no loss in generality, but this seems like an important thing to know.

Definition 1.3.29. $\mathcal{F} \subseteq \mathcal{L}_{\infty,\infty}$ is a *fragment* if

1. all atomic formulas are in $\mathcal{F}$.
2. $\mathcal{F}$ is closed under substitution of terms for free variables.
3. $\mathcal{F}$ is subformula closed.
4. $\forall X \phi(X) \in \mathcal{F}$ implies $\neg \exists X \neg \phi(X) \in \mathcal{F}$ and if $\phi \to \psi \in \mathcal{F}$ then $\neg \phi \vee \psi \in \mathcal{F}$.

The $\mathcal{F}$-basic formulas are defined just like basic formulas, except atomic formulas are replaced by formulas of $\mathcal{F}$

Definition 1.3.30. Given a fragment $\mathcal{F}$, a structure map $h: M \to N$ is an $\mathcal{F}$*-elementary map* if h preserves the meaning of all formulas in $\mathcal{F}$, i.e. $M \vDash \varphi[A \subseteq M]$ iff $N \vDash \varphi[h[A]]$.

Define the category $\mathbf{Mod}^{\mathcal{F}} T$ as models of T but where the morphisms are the $\mathcal{F}$-elementary maps.

Theorem 1.3.31 ([MP89, p. 52]). *Given any small fragment $\mathcal{F}$ and a $\mathcal{F}$-basic theory T, the category $\mathbf{Mod}^{\mathcal{F}} T$ is equivalent to $\mathbf{Mod} T'$ for some other basic theory T' in some other language L'.*

AECs are Accessible Categories

This whole subsection follows from CITE HERE, Lieberman

We can see an AEC as a category by letting its models be the objects and morphisms the strong embeddings between these. The following theorem connects the story of accessible categories to that of AECs.

Theorem 1.3.32. CITE HERE *Let K be an AEC and μ its cardinal defined in the Löwenheim-Skolem Axiom. Then, K is μ^+-accessible. More generally, K is λ-accessible for all regular cardinals $\lambda \;\; \mu$.*

It follows that every AEC is accessible. Moreover, we have a characterisation theorem which needs just a little preamble.

Given a signature L over an AEC K, denoted $L(K)$, we can define the category of L-structures as the category with objects being the L-structures and morphisms are injective L-morphisms that preserve and reflect relations in L. We will denote this category by L-**Struct**.

We now need two definitions to obtain our characterisation. That of *repleteness* and being *almost full*.

Definition 1.3.33. A subcategory $\mathbb{D}$ of some category $\mathbb{C}$ is *replete* if there are no isomorphic objects $X \cong Y$ such that $X \in \mathbb{D}_0$ while $Y \notin \mathbb{D}_0$.

In other words, replete subcategories respect isomorphisms. Examples of these abound: take topological spaces and continuous maps. Any combination of topological properties forms a replete subcategory, with *Haus* for Hausdorff spaces, *Stone* for Stone spaces, and Sob_ω for Sober spaces of cardinality ω.

Definition 1.3.34. A subcategory $\mathbb{D}$ of some category $\mathbb{C}$ is *almost full* if for any two objects X, Y, Z in $\mathbb{D}$, with $f : X \to Z$ and $g : Y \to Z$, if there is a $h : X \to Y$ in $\mathbb{C}$ such that $g \circ h = f$, then h is in $\mathbb{D}$ as well.

So, $\mathbb{D}$ is almost full if it has the base of every commutative triangle of which it has the other sides. Diagrammatically:

$$\begin{tikzcd} X \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

This property corresponds to the coherency axiom for AECs.

Now, we can characterise AECs more precisely as a special form of subcategory of L-**Struct**.

Theorem 1.3.35. *Let K be an AEC. Then K is a nearly full, replete subcategory of L-**Struct** which is λ-accessible for every $\lambda \gneq \mu$ (where μ is the cardinal from the Löwenheim-Skolem axiom) and which has all directed colimits. (The directed colimits are computed as in L-**Struct**.)*

We have a converse result as well.

Theorem 1.3.36. *Any nearly full, replete subcategory of L-**Struct** which is λ-accessible for every $\lambda \gneq \mu$, for some cardinal μ, and which has all directed colimits which are computed as in L-**Struct** can be seen as an AEC.*

Give the specific construction here.

Bibliography

[AR94] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. Lecture note series / London mathematical society. Cambridge University Press, 1994. ISBN: 9780521422611. URL: https://books.google.nl/books?id=iXh6rOd7of0C.

[CK12] C. Chang and H. Keisler. *Model Theory*. Mineola, New York: Dover, 2012. ISBN: 9780486488219.

[Dic85] M. Dickman. “Large Infinitary Languages”. In: *Model Theoretic Logics*. Ed. by J. Barwise and S. Feferman. New York, Berlin, Heidelberg, Tokyo: Springer-Verlag, 1985. Chap. IX, pp. 317–348.

[Gro02] R. Grossberg. *Classification theory for abstract elementary classes*. 2002. DOI: 10.1090/conm/302/05080. URL: https://doi.org/10.1090/conm/302/05080.

[Kar64] C. R. Karp. *Languages with Expressions of Infinite Length*. Amsterdam: North-Holland Publishing Company, 1964.

[Mar02] D. Marker. *Model Theory*. Springer-Verlag, 2002. DOI: 10.1007/b98860. URL: https://doi.org/10.1007/b98860.

[MP89] M. Makkai and R. Paré. *Accessible Categories: The Foundations of Categorical Model Theory: The Foundations of Categorical Model Theory*. Contemporary mathematics - American Mathematical Society. American Mathematical Society, 1989. ISBN: 9780821851111. URL: https://books.google.nl/books?id=ExIcCAAAQBAJ.