Introduction to Topology in and via Logic

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Contents

Chapter 1

Introduction

Topology is the study of space as an abstract concept. Its origins lie in the 19th century, where it served an attempt to unify work from analysis, and provide a solid foundation for geometry, unifying a number of hitherto separate subjects. Thus, the modern notion of topological space appears as the culmination of a process of abstraction which started with an axiomatisation of "usual space" – what is now called Euclidean space – and now includes an unending range of applications.

Some particularly relevant connections have always existed betwen topology and its geometric intuition, and logic. In a sense, logic and geometry have a very flirtatious relationship. For instance, one sees easily a connection between logical connectives and the basic set theory operations:

Logic	**Sets**
$\Rightarrow$	$\subseteq$
$\wedge$	$\cap$
$\vee$	$\cup$
$\neg$	$(\cdot)^C$

This relationship is made further clear when one studies set theory, as there it becomes clear that some form of the axiom of comprehension is used to specify the basic set theoretic definitions on the basis of their logical counterparts.

However, all of this theory is developed on the back of first order logic. Often times logicians are concerned with spicy types of logics. In this sense, topology can be thought to stand with respect to *epistemic logic* as set theory does to first-order logic.

Logic	**Geometry**
First order logic	Set theory
Epistemic logic	Topology

In the following lecture notes we will introduce topology, keeping this connection in mind. These notes are part of a coordinated project at the ILLC, taking place in January 2022. They contain exercises. Exercises marked with an asterisk are part of the homework for the project.

In the next sections we fix some notation that will be needed throughout these notes. We assume the reader is familiar with elementary set theory, and has a passing familiarity with the real numbers.

1.1 Set-Theoretic Notation

We write ω to mean the natural numbers. We refer to sets of sets as *families* or *collections* of sets. We write $\cap$ for intersection and $\cup$ for union of sets. When necessary we also have indexed versions of these operations: if $(U_i)_{i\in I}$ is an indexed collection of sets, we write:

$$\bigcup_{i\in I} U_i \text{ and } \bigcap_{i\in I} U_i,$$

for the sets $\{x : \exists i \in I, x \in U_i\}$ and $\{x : \forall i \in I, x \in U_i\}$. We write $\bigsqcup_{i\in I} U_i$ for the *disjoint union*:

$$\bigcup \{(x,i) : x \in U_i\}$$

Given a collection of sets indexed on the natural numbers $(U_n)_{n\in\omega}$ we say that this is a *non-decreasing* (resp. non-increasing) collection if $U_n \subseteq U_{n+1}$ for each n (resp. $U_n \supseteq U_{n+1}$). We say that it is *increasing* (resp. decreasing) if the inclusion is strict.

Given a set X, we write $X \times X$ for the set of ordered pairs of elements of X, and denote its elements by (x, y) where $x, y \in X$, and call this the *cartesian product*. We call a subset $R \subseteq X \times X$ a *relation*. We say that R is an *equivalence relation* if it is:

- (Reflexive): for every $x \in X$, xRx;
- (Symmetric) for every $x, y \in X$, xRy implies yRx;
- (Transitive) for every $x, y, z \in X$, xRy and yRz implies xRz.

Given an equivalence relation we write $[x]_R$ for the equivalence class of x, often dropping the subscript when it is understood. We write X/R for the quotient set, i.e., the set $\{[x]_R : x \in X\}$.

Throughout we write $\mathcal{P}(X)$ for the power set of X. We denote by $\mathcal{P}(X)^{fin}$ the set of finite subsets of the set X. We also denote by $X^{<\omega}$ the set of finite sequences of elements from X.

Given a function $f : X \to Y$ between two sets, we associate with it two natural operators:

$$\begin{aligned} \overline{f} : \mathcal{P}(X) &\to \mathcal{P}(Y) \\ A &\mapsto f[A] := \{f(w) : w \in A\} \end{aligned}$$

and

$$\begin{aligned} f^{-1} : \mathcal{P}(Y) &\to \mathcal{P}(X) \\ B &\mapsto f^{-1}[B] := \{z : f(z) \in B\} \end{aligned}$$

We call the former the *direct image* of f and the latter the *inverse image* or *preimage* of f. We recall that the preimage interacts naturally with both unions and interactions, i.e., for each family $(U_i)_{i\in I}$ of sets:

$$f^{-1}(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} f^{-1}[U_i] \text{ and } f^{-1}(\bigcap_{i\in I} U_i) = \bigcap_{i\in I} f^{-1}[U_i]$$

1.2 Partial Orders

If R is a relation on X we say that R is a *partial order* if it is reflexive, transitive and antisymmetric:

- (Antisymmetry) for every $x, y \in X$, if xRy and yRx then $x = y$;

We say that a partial order is furthermore *total* if for all $x, y \in X$ either xRy or yRx. We normally use the symbol $\leqslant$ to mean a partial order and refer to a pair $(X, \leqslant)$ as a *partially ordered set* or poset. We often use $<$ to denote the irreflexive variant of this structure:

$$x < y \iff x \leqslant y \text{ and } x \neq y$$

Given a subset $S \subseteq X$, we say that S *has an upper bound* if there is some $z \in X$ such that for each $a \in S$, $a \leqslant z$; we say that S has a *least upper bound*, if there is some a, an upper bound, such that whenever b is an upper bound, then $a \leqslant b$. We say that X has the *least upper bound property* if every $S \subseteq X$ has a least upper bound.

Given a poset $(X, \leqslant)$, we say that a collection of elements $(x_i)_{i \in I}$ is *totally ordered* or a *chain* if for each $i \neq j \in I$ either $x_i \leqslant x_j$ or $x_j \leqslant x_i$. We say that an element $x \in X$ is *maximal* if for each $y \in X$ if $x \leqslant y$ then $y = x$.

Given a totally ordered set $(X, <)$, we say that this is *dense* if whenever $x < y$ there is some z such that $x < z < y$. We assume the reader is familiar with a few basic ordered sets:

- The natural numbers $(\mathbb{N}, <)$; this is countable, and has the property that every subset has a least element;
- The integers $(\mathbb{Z}, <)$;
- The rationals $(\mathbb{Q}, <)$; this is countable and dense;
- The reals $(\mathbb{R}, <)$; this is dense, and has the least upper bound property.

An important kind of ordered set we will sometimes need is the following:

Definition 1.2.1. Let $(P, <)$ be a linearly ordered set. We say that P is a *well-order* if each subset $S \subseteq P$ has a least element.

One can look at the isomorphism types of these well-orders and pick specific representatives; these are what we call *ordinals*. We usually denote them by Greek letters, α, β, etc. The main fact we will need about ordinals is that there exist uncountable ordinals; we will denote the least such by ω_1.

1.3 Products and the Axiom of Choice

Given a collection $(X_i)_{i \in I}$ we write $\prod_{i \in I} X_i$ for the following collection of functions:

$$\{f : I \to \bigsqcup_{i \in I} X_i : \forall i \in I, \exists x \in X_i, f(i) = (x, i)\}.$$

We also call this the *Cartesian product* of these sets.

We will in some specific points require forms of the *Axiom of Choice* [1]. This says the following:

[1] Indeed, the knowing reader will note that this is needed to ensure that the above definition of cartesian product yields a non-empty collection of elements whenever the underlying sets are non-empty

Axiom 1.3.1. For every collection of sets $(X_i)_{i \in I}$, there is a function $f : I \to \bigsqcup_{i \in I} X_i$ such that for each $i \in I$, there is an $x \in X_i$ such that $f(i) = (x, i)$.

We will need in particular the following equivalent formulation of this:

Lemma 1.3.2. (Zorn's Lemma) Let $(X, \leqslant)$ be a partially ordered set. If for each chain $(x_i)_{i \in I}$, there exists some $x_0 \in X$ such that $x_i \leqslant x_0$ for each $i \in I$, then X has a maximal element.

This will prove important especially when we meet the concept of an ultrafilter, and look more broadly at compact topological spaces.