A Dual Proof of Blok's Lemma

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2. Algebraic and Dual Perspectives on Blok's Lemma – Showing the connection between old and new proofs of Blok's Lemma.
3. What we can learn from this connection: extensions and other signatures.

Intuitionistic and Modal Logic

Definition
An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is called a *Heyting algebra* if:

1. $(H, \wedge, \vee, 0, 1)$ is a (distributive) lattice.
2. The following law holds for all $a, b, c \in H$:

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An algebra $(B, \wedge, \neg, \Box, 0, 1)$ is called an *interior algebra* if $(B, \wedge, \neg, 0, 1)$ is a Boolean algebra, $\Box$ is a meet-distributive operator, and it satisfies for each $a \in B$:

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We have dual isomorphisms between:

- The lattice of extensions of IPC and the lattice of varieties of Heyting algebras.
- The lattice of extensions of S4 and the lattice of varieties of interior algebras.

Translations and Blok-Esakia

Gödel-McKinsey-Tarski Translation

Figure 1: Kurt Gödel (1906-1978); Alfred Tarski (1901-1983); J.C.C. McKinsey (1908-1953)

Definition
Given $\phi \in \mathcal{L}_{\mathsf{IPC}}$ we define the Gödel-McKinsey-Tarski (GMT) $\Box$-*translation*,into **S4**, as follows:

1. $p_\Box = \Box p$ and $\bot_\Box = \bot$;
2. $(\phi \wedge \psi)_\Box = \phi_\Box \wedge \psi_\Box$ and $(\phi \vee \psi)_\Box = \phi_\Box \vee \psi_\Box$;
3. $(\phi \rightarrow \psi)_\Box = \Box(\phi_\Box \rightarrow \psi_\Box)$.

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3. $(\phi \to \psi)_\Box = \Box(\phi_\Box \to \psi_\Box)$.

Theorem (Gödel,1933, McKinsey-Tarski, 1948)
For every formula $\phi \in \mathcal{L}_{\mathsf{IPC}}$, $\phi \in$ IPC *if and only if* $\phi_\Box \in$ **S4**.

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Let $L \in \mathrm{Ext}(\mathbf{IPC})$ and $M \in \mathrm{NExt}(\mathbf{S4})$. We say that M is a modal companion of L if:

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Definition
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Theorem (Blok, 1976, Esakia 1976)
There is an isomorphism between the lattices $\mathrm{Ext}(\mathbf{IPC})$ *and* $\mathrm{NExt}(\mathbf{S4.Grz})$*, mappings logics to their greatest modal companion.*

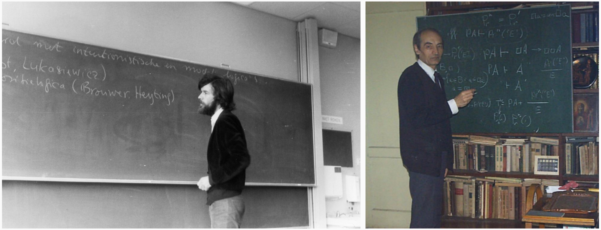

Figure 2: Wim Blok (1947-2003); Leo Esakia (1934-2010)

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The functor $\rho : \mathbf{Interior} \to \mathbf{HA}$ sends each interior algebra $(B, \Box)$ to the set:

$$\{a \in B : \Box a = a\}.$$

The adjoint functor $\sigma : \mathbf{HA} \to \mathbf{Interior}$ is the so-called Boolean envelope, which for Heyting algebras admits a (unique) interior algebra structure.

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The map $\rho : \Lambda(\mathbf{Interior}) \to \Lambda(\mathbf{HA})$ thus sends each variety V to

$$\{\rho(B) : B \in \mathsf{V}\},$$

which is in turn a variety. Conversely, we have a map $\sigma : \Lambda(\mathbf{HA}) \to \Lambda(\mathbf{Interior})$, sending V to

$$\mathbb{V}(\{\sigma(H) : H \in \mathsf{V}\}).$$

Blok-Esakia Theory (cont.d)

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The major difficulty once that is cleared is showing that σ is surjective – equivalently, that each variety of **Grz**-algebras is generated by those of the form $\sigma(H)$ for some H.

Algebraic and Dual Views of Blok-Esakia

One way to establish the aforementioned fact was done by Blok:

Lemma (Blok's Lemma)

Let $\mathcal{M}$ *be a* Grz*-algebra. Then* $\mathcal{M} \in \mathsf{ISP_U}(\sigma\rho\mathcal{M})$.

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Let $\mathcal{M}, \mathcal{N}$ be modal algebras and let $\mathcal{A}, \mathcal{B}$ be Boolean subalgebras of $\mathcal{M}, \mathcal{N}$ respectively. A mapping $h : \mathcal{A} \to \mathcal{B}$ is called a $\Box$-*homomorphism* when it is a Boolean homomorphism and $h(\Box a) = \Box h(a)$ whenever $\Box a \in A$.

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Blok's Lemma is equivalent to the following:

Lemma (Algebraic embedding lemma)

Let $\mathcal{M}$ be a Grz*-algebra and let $\mathcal{N}$ be a finite Boolean subalgebra of $\mathcal{M}$. Then there is a $\Box$-embedding $h : \mathcal{N} \to \sigma\rho\mathcal{M}$.*

One can see this lemma dually, by using stable maps:

Definition
Given modal spaces $\mathcal{X} = (X, R)$ and $\mathcal{Y} = (Y, R)$, a continuous function $f : \mathcal{X} \to \mathcal{Y}$ is said to be *stable* if $f(x)Rf(y)$ holds whenever xRy. Given $D \subseteq \mathrm{Clop}(\mathcal{Y})$ we say that f satisfies the *bounded domain condition* (BDC) with respect to D if for all $U \in D$, if $f(x)Ry$ and $y \in U$, then there is some x' such that xRx' and $f(x') \in U$.

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Bezhanishvili and Cleani proved Blok's Lemma making use of stable canonical rules. Our first observation is that this boils down to the following:

Lemma (Stable Surjection Lemma)

Given a Grz*-space* $\mathcal{X} = (X, R)$*, a finite* $\mathsf{S4}$*-space* $\mathcal{Y} = (Y, R)$*, and a surjective stable map* $p : \mathcal{X} \to \mathcal{Y}$ *satisfying the BDC for a domain* D*, there is a stable surjection* $p' : \sigma\rho\mathcal{X} \to \mathcal{Y}$ *satisfying the BDC for the domain* D*.*

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This all becomes in turn equivalent to the existence of an isomorphism of lattices of rule systems, when working in the setting of universal classes.

To see in what way one arrives at any of these proofs, it pays off to consider the specifics of the theories in consideration.

Definition
Let $(B, \Box)$ be an interior algebra. Given $a \in B$, we write:

$$\mathsf{r}(a) := a \wedge \neg\Diamond(\Diamond a \wedge \neg a).$$

We write $\mathsf{r}^0(a) = a$, $\mathsf{r}^{n+1}(a) = \mathsf{r}(a \wedge \neg \mathsf{r}^n(a))$ for $n \in \omega$. Elements such that $\mathsf{r}^k(a) = 0$ for some k are called of *finite rank*.

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Esakia (1985) showed the following:

Theorem
For an interior algebra $(B, \Box)$ we have:

1. $(B, \Box)$ *is a* Grz*-algebra if and only if* $\mathsf{r}^{k+1}(a) \neq \mathsf{r}^k(a)$ *whenever* $a \neq 0$.
2. $(B, \Box)$ *is isomorphic to* $\sigma\rho(B)$ *if and only if for each* $a \in B$*, there i* $n \in \omega$ *such that* $\mathsf{r}^n(a) = 0$.

An element $a \in B$ may have *infinite rank*; but if we consider $a \in \mathcal{M}$ a finite Boolean subalgebra, then every element has finite rank. The goal is to show that each element can be replaced locally by elements of finite rank.

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This leads naturally to the following one-step embedding lemma:

Lemma (Algebraic one-step embedding)

Let $\mathcal{M}$ be a Grz*-algebra and let $\mathcal{N}$ be a finite Boolean subalgebra, such that $h : \mathcal{N} \to \sigma\rho\mathcal{M}$ is a $\Box$-embedding fixing all open elements. If $x \in \mathcal{M}$ is arbitrary, then there are finitely many open elements C, and a $\Box$-embedding $h' : \langle \mathcal{N} \cup \{x\} \cup C \rangle \to \sigma\rho\mathcal{M}$ such that $h' \restriction_{\mathcal{N}} = h$.*

One step proofs

Enumerate $\mathcal{N} = \mathcal{N}_{op} \cup \{x_1, ..., x_n\}$ where $\mathcal{N}_{op} = \langle\{\Box a : \Box a \in \mathcal{N}\}\rangle$.

Set $\mathcal{N}_0 = \mathcal{N}_{op}$, and successively add one element, creating a sequence of algebras (not necesarily distinct):

$$\mathcal{N}_0 \to \mathcal{N}_1 \to ... \to \mathcal{N}_n$$

where $\mathcal{N} \subseteq \mathcal{N}_n$.

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given $\mathcal{N}_k$, to extend the embedding to $\langle\mathcal{N}_k \cup \{x_{k+1}\}\rangle$ one needs only define the image of x_{k+1}. This is done by picking, for each $c \in \mathcal{N}_k$, an element element w_c which is defined, by letting $u_c = \neg x_{k+1} \vee c$ as:

$$w_c := \neg(\Box(u_c \to \Box u_c) \to \Box u_c).$$

The use of the **Grz**-axiom lies in ensuring that $\Box(\Box(u_c \to \Box u_c) \to \Box u_c) \leq u_c$

Given a space $\mathcal{X}$, let $\rho : \mathcal{X} \to \rho\mathcal{X}$ be the cluster collapse continuous map. Moreover, a surjection $\varrho : \mathcal{X} \to \mathcal{Y}$ is called a *cluster-reducing map* when it is the quotient map induced by some equivalence relation on $\mathcal{Y}$ that never relates points belonging to different clusters in $\mathcal{X}$.

Lemma (Dual one-step surjection)

Let $\mathcal{X} = (X, R), \mathcal{Y} = (Y, R)$ and $\mathcal{Y}' = (Y', R)$ be S4 *spaces with the following properties.*

- $\mathcal{X} = (X, R)$ *is a* Grz *space;*
- $Y' = Y \cup \{\bullet\}$ *and there is a cluster-reducing map $\varrho : \mathcal{Y}' \to \mathcal{Y}$ which identifies $\bullet$ with some point in its cluster, but does not identify any other points;*
- *There is a stable surjection $f : \mathcal{X} \to \mathcal{Y}'$ satisfying the BDC for some $D \subseteq Y'$;*
- *There is a stable surjection $g : \sigma\rho\mathcal{X} \to \mathcal{Y}$ satisfying the BDC for $\varrho[D]$.*

Then there is a stable map $h : \sigma\rho\mathcal{X} \to \mathcal{Y}'$ satisfying the BDC for D.

One step proofs (Cont'd)

If x is the point identified with $\bullet$ by ϱ, then the preimages $f^{-1}\{x\}, f^{-1}\{\bullet\}$ in $\mathcal{X}$ do not cut clusters. So, their images $U_x := \rho[f^{-1}\{x\}]$ and $U_\bullet := \rho[f^{-1}\{\bullet\}]$ are disjoint and clopen.

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Letting $M_x := \rho[max(f^{-1}\{x\})]$ and $M_\bullet := \rho[max(f^{-1}\{\bullet\})]$, note both M_x and $M_\bullet$ are closed, so we can find a clopen cover $\{C_x, C_\bullet\}$ of $U_x \cup U_\bullet$ with $M_x \subseteq C_x$ and $M_\bullet \subseteq C_\bullet$. We then construct our map $h : \sigma\rho\mathcal{X} \to \mathcal{Y}'$ by sending elements of C_x to x and elements of $C_\bullet$ to $\bullet$.

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This can be seen as a refinement of the algebraic decomposition: Given a finite **S4**-space $\mathcal{Y} = (Y, R)$, we set $\mathcal{Y}_0 = \sigma\rho\mathcal{Y}$ and form the inverse chain

$$\mathcal{Y}_0 \leftarrow \mathcal{Y}_1 \leftarrow \ldots \leftarrow \mathcal{Y}_n$$

where $\mathcal{Y}_n = \mathcal{Y}$, and $\mathcal{Y}_{i+1}$ is obtained as some cluster-expansion of $\mathcal{Y}_i$ by one additional element. The process of choosing the elements w_c corresponds to an instance of this process.

Extensions and Other Settings

Other signatures

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In some settings such as MS4, where monadic Grz is not enough, this suggests a proof strategy: one should find sufficient conditions to make the geometric stable surjection lemma work.

One consequence of this way of thinking is that one can think of what are the base conditions necessary to get Blok's Lemma to work. This can suggest other related settings: complete Grz algebras. Given a modal algebra $\mathcal{A}$ let $\overline{A}$ be its MacNeille completion.

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Looking at the situation dually one can prove:

Theorem
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Complete Algebras

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Contrast this with Litak's result that no variety of Grz algebras is closed under MacNeille completions.

The above suggests some categorical implications:

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All complete Grz *algebras are* $\mathcal{H}$*-injective objects in the category* Grz*, where* $\mathcal{H}$ *is the class of injective homomorphisms which preserve open elements.*

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The following might be significant for applications of complete interior algebras to point-free topology.

Thank you!

Questions?