Topics in Algebraic Logic and Duality Theory
Lecture 4

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Plan for the Day

- Maksimova’s characterisation of superintuitionistic logics with the interpolation property

A superintuitionistic logic has the interpolation property iff it is one of

$$
\begin{aligned}
&\mathsf{CPC},\\
\mathsf{LC}_2 &= \mathsf{IPC} + \mathsf{bd}_2 + \mathsf{bw}_1 &&= \text{logic of the } 2\text{-chain},\\
\mathsf{LC} &= \mathsf{IPC} + p \to q \vee q \to p &&= \text{logic of chains},\\
\mathsf{KC} &= \mathsf{IPC} + \neg p \vee \neg\neg p &&= \text{logic of directed frames},\\
\mathsf{BD}_2 &= \mathsf{IPC} + \mathsf{bd}_2 &&= \text{logic of frames of depth at most } 2,\\
\mathsf{BDW}_2 &= \mathsf{BD}_2 + \mathsf{bw}_2 &&= \text{logic of the } 2\text{-fork},\\
&\mathsf{IPC}.
\end{aligned}
$$

Understanding the statement

Logics of bounded depth

Definition

We define a sequence of formulas bd_n inductively:

$$\begin{aligned} \mathsf{bd}_0 &= \bot, \\ \mathsf{bd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathsf{bd}_n), \end{aligned}$$

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Proof.

Exercise. $\square$

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Preliminary tools

Theorem (Jankov)

Let F be a finite rooted frame (i.e. the dual of a finite SI algebra). Then there is a formula $\chi(F)$ such that for any frame G, we have $G \not\models \chi(F)$ iff F is a p-morphic image of a subframe of G.

Jankov formulas

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Corollary

Let F be a finite rooted frame and L be a logic. We have $\chi(F) \notin L$ iff $F \models L$.

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Theorem

A locally finite logic L can be axiomatised by the Jankov formulas

$$\{\chi(P) \mid P \not\models L\}.$$

Corollary

The logic of a finite rooted frame is axiomatisable by Jankov formulas.

Definition

We turn the class of finite rooted frames into a poset by setting $F \leqslant G$ iff G is a p-morphic image of F.

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Proposition

Let $\mathcal{F}' \subseteq \mathcal{F}$ be classes of finite rooted frames such that every frame in $\mathcal{F}$ is below a frame in $\mathcal{F}'$. Then

$$\mathsf{IPC} + \{\chi(F) \mid F \in \mathcal{F}\} = \mathsf{IPC} + \{\chi(F) \mid F \in \mathcal{F}'\}.$$

Lemma

Let F be a frame and $x, y \in F$ be distinct.

1. *Suppose that y is the only immediate successor of x, i.e. $\uparrow y = \uparrow x \backslash \{x\}$. Then the equivalence relation identifying x and y is a bisimulation equivalence, and the quotient map an α-reduction.*

Reductions

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2. *Suppose that x and y have the same immediate successors, i.e. $\uparrow x \setminus \{x\} = \uparrow y \setminus \{y\}$. Then the equivalence relation identifying x and y is a bisimulation equivalence, and the quotient map an β-reduction.*

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Proof.

Routine check. $\square$

Proposition

Let $f\colon F \to G$ be an onto p-morphism between finite frames. Then f factors as a sequence of α- and β-reductions.

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Proposition

Let $f \colon F \to G$ be an onto p-morphism between finite frames. Then f factors as a sequence of α- and β-reductions.

Proof.

Exercise. Hint: induction on the number of points that are identified. $\square$

See board.

Maksimova’s characterisation

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Proof of Maksimova's characterisation

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$\Rightarrow$: Assume that L is a superintuitionistic with interpolation. If $L = \mathsf{CPC}$, we are done. Otherwise, observe that CPC is the logic of the 1-frame, whose only cover is the 2-chain.

(a) The 1-frame

(b) The 2-chain

Proof of Maksimova's characterisation

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Therefore CPC is axiomatised by the Jankov formula χ_{2c} of the 2-chain.

Proof of Maksimova's characterisation

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$\Rightarrow$: Assume that L is a superintuitionistic with interpolation. If $L = \mathsf{CPC}$, we are done. Otherwise, observe that CPC is the logic of the 1-frame, whose only cover is the 2-chain.

(a) The 1-frame

(b) The 2-chain

Therefore CPC is axiomatised by the Jankov formula χ_{2c} of the 2-chain. As $L \subsetneq \mathsf{CPC}$, we have $\chi_{2c} \notin L$, thus the 2-chain is an L-frame.

If L is the logic of the 2-chain, then $L = \mathsf{LC}_2$, and we are done.

Otherwise, observe that the only covers of the 2-chain are the 3-chain and the 2-fork.

(a) 3-chain

(b) 2-fork

Therefore, LC_2 is axiomatized by the Jankov formulas χ_{3c} of the 3-chain and χ_{2f} of the 2-fork.

Otherwise, observe that the only covers of the 2-chain are the 3-chain and the 2-fork.

(a) 3-chain

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Therefore, LC_2 is axiomatized by the Jankov formulas χ_{3c} of the 3-chain and χ_{2f} of the 2-fork. As $L \subsetneq \mathsf{L}_2$, we have $\chi_{3c} \notin L$ or $\chi_{2f} \notin L$, thus either the 3-chain or the 2-fork are L-frames.

Appendix

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Proof.

Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$.

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Let $L' = \mathsf{IPC} + \{\chi(P) \mid P \not\models L\}$. We claim that $L = L'$.

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$\supseteq$: Exercise.

$\subseteq$: $\square$

Thank you!
Questions?