REGULAR HEYTING ALGEBRAS AND FREE HEYTING EXTENSIONS OF BOOLEAN ALGEBRAS

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1. Free Heyting Extensions

The[1] categories **HA** and **BA**, respectively of Heyting algebras with Heyting algebra homomorphisms and Boolean algebras with Boolean algebra homomorphisms, are related by a chain of adjunctions:

$$\mathsf{FreeM} \dashv \mathsf{Reg} \dashv I \dashv \mathsf{Center},$$

where

(1) $I : \mathbf{BA} \to \mathbf{HA}$ is the inclusion;
(2) $\mathsf{Center} : \mathbf{HA} \to \mathbf{BA}$ takes the *center* of a Heyting algebra H, namely $\mathsf{Center}(H) = \{a \in H : a \vee \neg a = 1\}$;
(3) $\mathsf{Reg} : \mathbf{HA} \to \mathbf{BA}$ takes the *regular elements* of a Heyting algebra H namely $\mathsf{Reg}(H) = \{a \in H : \neg\neg a = a\}$.

The functor $\mathsf{FreeM} : \mathbf{BA} \to \mathbf{HA}$ is guaranteed to exist by the Adjoint Functor theorem; a description using a word construction can be derived from the work of Moraschini [5] (see also [6, Example 4.18]). It was likewise studied in [7], where it was shown to be fully faithful.

In this paper we give a more concrete presentation of the functor FreeM, by studying the dual problem, employing Esakia duality (see [3] for any unexplained notions such as Priestley spaces, Esakia spaces and p-morphisms): the dual to Reg is the functor $\mathsf{Max} : \mathbf{Esa} \to \mathbf{Stone}$, mapping an Esakia space $(X, \leq)$ to its maximum. Here we present the right adjoint to this functor, also denoted by FreeM, by borrowing key ideas from [1]. Essentially, this amounts to finding some free Heyting extension of a Boolean algebra which still has the same regular elements.

Given X a Stone space, let $(V(X), \supseteq)$ be the Priestley space of its closed subsets. The following is presumably folklore:

Proposition 1. *For each Stone space X, $(V(X), \supseteq)$ is an Esakia space.*

Our goal will be, given X an Esakia space, and Y a Stone space, to extend a function $f : \max X \to Y$ to a unique p-morphism $\tilde{f} : X \to \mathsf{FreeM}(Y)$. The first step is to note that $V(X)$ enjoys a universal property related to this:

Proposition 2. *Assume that X is an Esakia space, Y is a Stone space and $f : \max X \to Y$ is a continuous map. Then there exists a unique order-preserving map $\tilde{f} : X \to V(Y)$, such that $e_Y \circ \tilde{f} \restriction_{\max} = f$ and $\tilde{f}$ is a p-morphism on maximal elements (i.e., if $\tilde{f}(x) \leq y$ and $y \in V(Y)$ is maximal, then there is some $w \geq x$ such that $\tilde{f}(w) = y$).*

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Relatedly, one can consider a variation of the rooted Vietoris construction from [1]:

Definition 3. Given two Priestley spaces X, Y and a continuous and order-preserving map $g : X \to Y$ between them, we say that a subset $S \subseteq X$ is *g-open* if it satisfies:

$$\forall x \in S, y \in X(x \leq y \to \exists z \in S(x \leq z \wedge g(z) = g(y))).$$

We denote by $V_g(X)$ the set of closed, rooted and g-open subsets of X.

Note that if $Y = \{\bullet\}$, and g is the terminal map, $V_g(X)$ is the set of all closed and rooted subsets, which we denote by $V_r(X)$. Recall that there is a map called the *root map* $r : V_g(X) \to X$ which is a surjective order preserving map.

Proposition 4. *Let Y be a Stone space and let $V_{\max}(Y) = \{C \in V_r(V(Y)) : \forall D \in C, \forall x \in D, \{x\} \in C\}$. Then $V_{\max}(Y)$ is a Priestley space, and the restriction $r : V_{\max}(Y) \to V(Y)$ is such that for any map $f : \max X \to Y$, and its unique lifting $\tilde{f} : X \to V(Y)$ from Proposition 2, there is a unique r-open $g_f : X \to V_{\max}(Y)$ making the diagram commute.*

Let $M_\infty(Y) = V_G^r(V_{\max}(Y))$. The latter is constructed as follows: we consider the following sequence:

$$V(Y) \xleftarrow{r_1} V_{\max}(Y) \xleftarrow{r_2} V_2(Y) \xleftarrow{r_3} ...$$

where $V_{n+1}(Y) = V_{r_n}(V_n(Y))$, as in [1], and $r_{n+1} : V_{n+1}(Y) \to V_n(Y)$ is the root map. Then $V_G^r(V_{\max}(Y))$ is the inverse limit of this sequence. $M_\infty(Y)$ is then an Esakia space, with the property that $\max(M_\infty(Y)) \cong Y$ through a natural isomorphism; moreover this assignment is functorial by using the functoriality of $V(-)$, $V_{\max}(-)$ and $V_G^r(-)$.

Our main result then shows the following adjunction:

Proposition 5. *The functor* $\mathsf{FreeM} : \mathbf{Stone} \to \mathbf{Esa}$ *assigning each Stone space X to $M_\infty(X)$ is right adjoint to* $\max : \mathbf{Esa} \to \mathbf{Stone}$.

2. Regular Heyting Algebras

We use the description of FreeM to study *regular Heyting algebras*, and their dual, regular Esakia spaces, in the sense of [4].

Definition 6. Let H be a Heyting algebra. We say that H is *regular* if $H = \langle \mathsf{Reg}(H) \rangle$. We say that an Esakia space X is regular, if its dual Heyting algebra is regular.

These are known to provide algebraic (respectively, order-topological) semantics for inquisitive logic [2]; indeed, given B a Boolean algebra, the Heyting algebra $\mathsf{ClopUp}(V(X_B))$ is studied there as the *inquisitive extension* of B. Using our construction, we show:

Theorem 7. *Given a Stone space X, $M_\infty(X)$ is always a regular Esakia space, and moreover, regular Esakia spaces are those spaces for which the unit of the adjunction is injective.*

This result can be seen as a dual algebraic semantics for inquisitive logic, and provides a categorical explanation for the role played by regular Heyting algebras in the study of such a logic. This furthermore answers a question of Grilleti and Quadrellaro[4], providing a categorical description of regular Esakia spaces. As an elaboration on the main ideas used to prove this theorem, we provide two theorems concerning regular Heyting algebras.

Definition 8. Given $n \in \omega$ the *n-universal regular model* if the (unique) poset $(\mathcal{R}_n, \leq)$ satisfying the following:

(1) $\max(P)$ contains 2^n points.
(2) For each antichain $S \subseteq R_n$ where $|S| \geq 1$, there is a unique point $x \in P$ which covers S.

Compared with the usual n-universal model, the regular such model can be obtained by identifying all points with the same color, and hence it is a p-morphic image of the n-universal model. Then we can show:

Theorem 9. *Inquisitive logic* InqL *is sound and complete with respect to the class* $\{\mathcal{R}_n : n \in \omega\}$.

We conclude by tracing some connections and further avenues of research connected with the intermediate logic ML, Medvedev's logic. This is well known to be the logic of the posets $V(X)$ for X a finite Stone space, i.e., a finite set, since $V(X)$ is exactly the powerset without the empty set. It likewise appears naturally in the study of inquisitive logic, as the *schematic fragment* of InqL. Here we prove:

Theorem 10. *The logic* ML *is precisely the logic of all the spaces* $V(X)$ *for* X.

We can phrase this result in more properly logical terms:

Definition 11. We define $\mathsf{R}_n := \mathsf{Log}(\{M_n(B) : B \text{ is a (finite) Boolean algebra}\})$,

where $M_n(B)$ is the n-th-step in the step-by-step construction outlined. We then have that $\mathsf{R}_0 = \mathsf{Med}$, and by [4], $\bigcap_{n\in\omega} \mathsf{R}_n = \mathsf{IPC}$. We leave it as an interesting avenue of research to obtain a further analysis of R_n for other n.

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