Exercise Sheet 2

January 2023

1 Continuity

Recall that we say that given two topological spaces $f : X \to Y$, a function is *continuous* if for each open set $U \subseteq Y$, $f^{-1}[U]$ is open. We say that it is *open* if for each open set $U \subseteq X$, $f[U]$ is open. We furthermore say that f is *closed* if for each closed set V, $f[V]$ is closed.

Exercise 1.1. Let $f : X \to Y$ be two partially ordered sets equipped with the Alexandroff topology. Recall that in the lecture it was shown that f is continuous if and only if it is monotone; and it is open if and only if it satisfies the back condition. Give conditions, like those presented in class, for f to be a closed map.

Exercise 1.2. Show the following are equivalent for a map $f : X \to Y$:

1. f is continuous;
2. Inverse images of closed sets are closed;
3. For each set B, $f^{-1}[int(B)] \subseteq int(f^{-1}[B])$.

Definition 1.1. Given a space (X, τ), an equivalence relation $\sim$ on X and the quotient map $q : x \to [x] := \{y \in X \mid x \sim y\}$, we define the quotient topology $\tau/\sim$ on $X/\sim$ as $U \in \tau/\sim$ if $q^{-1}[U] \in \tau$.

Exercise 1.3. Let $(X, \tau), (Y, \tau')$ be topological spaces and $\sim$ an equivalence relation on X. Let $f : (X, \tau) \to (Y, \tau')$ be a continuous map such that for any $x, y \in X$ such that $x \sim y$, we have that $f(x) = f(y)$. Show that there exists a unique continuous map $g : (X/\sim, \tau/\sim) \to (Y, \tau')$ such that $gq = f$.

$$\begin{tikzcd}
(X,\tau) \arrow[d, "q"] \arrow[dr, "f"] & \\
(X/\sim, \tau/\sim) \arrow[r, "g"] & (Y,\tau')
\end{tikzcd}$$

Exercise 1.4. Consider the subspaces: the interval $I = [0, 1]$ and the unit circle $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$. Consider also the equivalence relation $\sim$ on I defined as $x \sim y$ if $x = y$ or $x, y \in \{0, 1\}$.

Show that $(I/\sim, \tau/sim) \cong S^1$. **Hint:** Show that the map $x \mapsto (\sin 2\pi x, \cos 2\pi x)$ is well defined and induces a homeomorphism.

2 Metric Spaces

Given $\mathbb{R}$ with the standard topology, there is a different definition of continuity which is familiar if you have taken a calculus or analysis course:

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a map. We say that f is *analytically continuous at a point* c if and only if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in \mathbb{R}$, if $|c - x| < \delta$ then $|f(c) - f(x)| < \varepsilon$. A function is analytically continuous if and only if it is analytically continuous at every point c.

Denote: $B_r^c = \{x \in \mathbb{R} \mid |x - c| < r\}$. This is often called the *open ball* around c of radius r. Notice that analytic continuity amounts to for all B_δ^c there exists $B_\varepsilon^{f(c)}$ such that if $x \in B_\delta^c$ then $f(x) \in B_\varepsilon^{f(c)}$.

Exercise 2.1.

1. Given an open set U in the standard topology on the reals, show that for any $x \in U$ there exists ε such that $B_\varepsilon^x \subseteq U$.
2. Show that for a map $f : \mathbb{R} \to \mathbb{R}$ it is analytically continuous if and only if it is continuous with regards to the standard topology on $\mathbb{R}$.

One of the first reasons people have explored topology is to axiomatise the meaning of a metric space

Definition 2.2. A metric space is a pair (X, d) with a set X and a map $d : X \times X \to [0, \infty)$, such that for any $x, y, z \in X$, the following conditions hold:

1. $d(x, x) = 0$.
2. If $x \neq y$, then $d(x, y) > 0$ (separation).
3. $d(x, y) = d(y, x)$ (symmetry).
4. $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality).

Exercise 2.2. Verify that $\mathbb{R}$ with the distance function $|y - x|$ forms a metric space. **Hint:** Show that $|x - y|^2 \leqslant (|x - z| + |z - y|)^2$ and use the fact that for any $a, b > 0$ we have that $a^2 \leqslant b^2$ if and only if $a < b$.

In fact, we also have that $\mathbb{R}^2$ with the Euclidean distance function $d((x_1, y_1), (x_2, y_2)) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$ is a metric space.

Exercise 2.3. We define a ball in a metric space (X, d) as $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Show that the balls are closed under binary intersections and that it covers X; hence show that every metric space induces a topology, called the *metric topology*. Also show that on the real line, the metric topology is the standard topology.

Definition 2.3. Given two metric spaces (X, d_1) and (Y, d_2) we define an isometry to be a bijective map $f : (X, d_1) \to (Y, d_2)$ such that for any $x, y \in X$ we have that $d(f(x), f(y)) = d(x, y)$

Exercise 2.4. Show that every isometry is a homeomorphism.

Exercise 2.5. Given a metric space (X, d), a topological space (Y, τ) and a homeomorphism $f : (Y, \tau) \to (X, d)$, show that f induces a metric space on (Y, τ).

3 Filters and Filter Convergence

In class we saw the definition of a filter base and of filter convergence. We briefly recall these definitions here:

Definition 3.1. Let X be a set. We say that a collection of subsets $F \subseteq \mathcal{P}(X) - \{\varnothing\}$ is a *filter base* if it satisfies the following:

- $X \in F$;
- If $A, B \in F$ then $A \cap B \in F$.

We say that a given filter base is a *filter* if it is upwards closed: whenever $A \in F$ and $A \subseteq B$ then $B \in F$.

Solve the exercise from the previous exercise sheet, now with a newfound glee from understanding the definition of a filter:

Exercise 3.1. Show that neighbourhoods are a filter such that:

1. For every $U \in N(x)$, we have that $x \in U$.
2. $N(x)$ is closed under finite intersections - given a finite sequence $\{V_i\}_{i \in I} \subseteq N(x)$, we have that $\bigcap_{i \in I} V_i \in N(x)$.
3. $N(x)$ is an up set - if $U \in N(x)$ and $U \subseteq V$, then $V \in N(x)$.
4. For every $U \in N(x)$, there exists some $V \in N(x)$, such that $V \subseteq U$ and for every $y \in V$, $U \in N(y)$.

A collection of subsets satisfying conditions (1)-(4) is called a filter.

Given a set X with a map $N : X \to \mathcal{P}\mathcal{P}X$ such that $N(x)$ is a filter for every $x \in X$, show that N induces a topology on X. (**Hint:** set U to be open if and only if for every $x \in U$ we have that $U \in N(x)$).

Exercise 3.2. Let $\{a, b, c\}$ be a set with three elements. Describe all filters and filter bases on this set.

Exercise 3.3. Let (X, τ) be a topological space. Show the following:

1. If F is a filter base, then
$$\uparrow F = \{A \subseteq X : \exists B \in F, B \subseteq A\},$$
is a filter.
2. Let $F \subseteq \mathcal{P}(X)$. We say that F has the *finite intersection property* if for all $A_0, ..., A_n \in F$, $A_0 \cap ... \cap A_n \neq \varnothing$. Show that if F has the FIP then
$$F^{\cap} = \{A_0 \cap ... \cap A_n : A_0, ..., A_n \in F\}$$
is a filter base.

Definition 3.2. Given a topological space (X, τ) and a filter F on X, we say that F is a convergent filter to some point $x \in X$ if for any $U \in N(x)$ we have that $U \in F$.

Exercise 3.4. Consider the subspace $[0, 1]$ of $\mathbb{R}$. We call a sequence $\{a_n : n \in \omega\}$ a *converging sequence* to a limit x if for any $U \in \tau$, $x \in U$ implies that there exists a natural number N such that for any $n > N$ we have that $a_n \in U$.

- Show that in $[0, 1]$, the limit is unique. That is, if a sequence converges, it converges to a unique point x. **Hint:** assume the opposite and construct two open sets that contradict this assumption.
- Show that each converging sequence defines a converging filter, and each converging filter contains a converging sequence.

Exercise 3.5. Describe converging filters over the Cantor space.