Introduction to Topology in and via Logic

Amity Aharoni Rodrigo Nicolau Almeida Søren Brinck Knudstorp

January 9, 2023

Contents

Chapter 1

Introduction

Topology is the study of space as an abstract concept. Its origins lie in the 19th century, where it served an attempt to unify work from analysis, and provide a solid foundation for geometry, unifying a number of hitherto separate subjects. Thus, the modern notion of topological space appears as the culmination of a process of abstraction which started with an axiomatisation of “usual space” – what is now called Euclidean space – and now includes an unending range of applications.

Some particularly relevant connections have always existed betwen topology and its geometric intuition, and logic. In a sense, logic and geometry have a very flirtatious relationship. For instance, one sees easily a connection between logical connectives and the basic set theory operations:

Logic	**Sets**
$\Rightarrow$	$\subseteq$
$\wedge$	$\cap$
$\vee$	$\cup$
$\neg$	$(\cdot)^C$
$\bot$	$\varnothing$

This relationship is made further clear when one studies set theory, as there it becomes clear that some form of the axiom of comprehension is used to specify the basic set theoretic definitions on the basis of their logical counterparts.

However, all of this theory is developed on the back of first-order logic. Often times logicians are concerned with spicy types of logics. In this sense, topology can be thought to stand with respect to *epistemic logic* as set theory does to first-order logic.

Logic	**Geometry**
First-order logic	Set theory
Epistemic logic	Topology

In the following lecture notes we will introduce topology, keeping this connection in mind. These notes are part of a coordinated project at the ILLC, taking place in January 2023. They contain proposed exercises, as well as several propositions which are left as an exercise for the reader.[1]

[1] These lecture notes are inspired by and partly based on three sources: Steven Vickers (1989), *Topology via Logic*; James Munkres (2014), *Topology*; Ryszard Engelking (1968), *General Topology*. The former one for the epistemic intuition, and the latter two for the actual mathematical content.

In the next sections we fix some notation that will be needed throughout these notes. We assume the reader is familiar with elementary set theory, and has a passing familiarity with the real numbers.

1.1 Set-Theoretic Notation

We write ω to mean the natural numbers. We refer to sets of sets as *families* or *collections* of sets. We write $\cap$ for intersection and $\cup$ for union of sets. When necessary we also have indexed versions of these operations: if $(U_i)_{i\in I}$ is an indexed collection of sets, we write:

$$\bigcup_{i\in I} U_i \text{ and } \bigcap_{i\in I} U_i,$$

for the sets $\{x : \exists i \in I, x \in U_i\}$ and $\{x : \forall i \in I, x \in U_i\}$. We write $\bigsqcup_{i\in I} U_i$ for the *disjoint union*:

$$\bigcup \{(x,i) : x \in U_i\}$$

Given a collection of sets indexed on the natural numbers $(U_n)_{n\in\omega}$ we say that this is a *non-decreasing* (resp. non-increasing) collection if $U_n \subseteq U_{n+1}$ for each n (resp. $U_n \supseteq U_{n+1}$). We say that it is *increasing* (resp. decreasing) if the inclusion is strict.

Given a set X, we write $X \times X$ for the set of ordered pairs of elements of X, and denote its elements by (x, y) (or sometimes $\langle x, y\rangle$) where $x, y \in X$, and call this the *cartesian product.* We call a subset $R \subseteq X \times X$ a *relation.* We say that R is an *equivalence relation* if it is:

- (Reflexive): for every $x \in X$, xRx;
- (Symmetric): for every $x, y \in X$, xRy implies yRx;
- (Transitive): for every $x, y, z \in X$, xRy and yRz implies xRz.

Given an equivalence relation we write $[x]_R$ for the equivalence class of x, often dropping the subscript when it is understood. We write X/R for the quotient set, i.e., the set $\{[x]_R : x \in X\}$.

Throughout we write $\mathcal{P}(X)$ for the power set of X. We denote by $\mathcal{P}(X)^{fin}$ the set of finite subsets of the set X. We also denote by $X^{<\omega}$ the set of finite sequences of elements from X.

Given a function $f : X \to Y$ between two sets, we associate with it two natural operators:

$$\begin{aligned} \overline{f} : \mathcal{P}(X) &\to \mathcal{P}(Y) \\ A &\mapsto f[A] := \{f(w) : w \in A\} \end{aligned}$$

and

$$\begin{aligned} f^{-1} : \mathcal{P}(Y) &\to \mathcal{P}(X) \\ B &\mapsto f^{-1}[B] := \{z : f(z) \in B\}. \end{aligned}$$

We call the former the *direct image* of f and the latter the *inverse image* or *preimage* of f. We recall that the preimage interacts naturally with both unions and intersections, i.e., for each family $(U_i)_{i\in I}$ of sets:

$$f^{-1}\left[\bigcup_{i\in I} U_i\right] = \bigcup_{i\in I} f^{-1}[U_i] \text{ and } f^{-1}\left[\bigcap_{i\in I} U_i\right] = \bigcap_{i\in I} f^{-1}[U_i].$$

1.2 Partial Orders

If R is a relation on X we say that R is a *partial order* if it is reflexive, transitive and antisymmetric:

- (Antisymmetry): for every $x, y \in X$, if xRy and yRx then $x = y$.

We say that a partial order is furthermore *total* or *linear* if for all $x, y \in X$ either xRy or yRx. We normally use the symbol $\leqslant$ to mean a partial order and refer to a pair $(X, \leqslant)$ as a *partially ordered set* or poset. We often use $<$ to denote the irreflexive variant of this structure:

$$x < y \iff x \leqslant y \text{ and } x \neq y$$

Given a subset $S \subseteq X$, we say that S *has an upper bound* if there is some $z \in X$ such that for each $a \in S$, $a \leqslant z$; we say that S has a *least upper bound*, if there is some a, an upper bound, such that whenever b is an upper bound, then $a \leqslant b$. We say that X has the *least upper bound property* if every $S \subseteq X$ has a least upper bound.

Given a poset $(X, \leqslant)$, we say that a collection of elements $(x_i)_{i \in I}$ is *totally ordered* or a *chain* if for each $i, j \in I$ either $x_i \leqslant x_j$ or $x_j \leqslant x_i$. We say that an element $x \in X$ is *maximal* if for each $y \in X$ if $x \leqslant y$ then $y = x$.

Given a totally ordered set $(X, <)$, we say that this is *dense* if whenever $x < y$ there is some z such that $x < z < y$. We assume the reader is familiar with a few basic ordered sets:

- The natural numbers $(\mathbb{N}, <)$; this is countable, and has the property that every subset has a least element;
- The integers $(\mathbb{Z}, <)$;
- The rationals $(\mathbb{Q}, <)$; this is countable and dense;
- The reals $(\mathbb{R}, <)$; this is dense, and has the least upper bound property.

An important kind of ordered set we will sometimes need is the following:

Definition 1.2.1. Let $(P, <)$ be a linearly ordered set. We say that P is a *well-order* if each subset $S \subseteq P$ has a least element. $\dashv$

One can look at the isomorphism types of these well-orders and pick specific representatives; these are what we call *ordinals*. We usually denote them by Greek letters, α, β, etc. The main fact we will need about ordinals is that there exist uncountable ordinals; we will denote the least such by ω_1.

1.3 Products and the Axiom of Choice

Given a collection $(X_i)_{i \in I}$ we write $\prod_{i \in I} X_i$ for the following collection of functions:

$$\left\{ f : I \to \bigcup_{i \in I} X_i \mid \forall i \in I, f(i) \in X_i \right\}.$$

We also call this the *Cartesian product* of these sets.

We will in some specific points require forms of the *Axiom of Choice*.[2] This says the following:

Axiom 1.3.1. For every collection of sets $(X_i)_{i \in I}$, there is a function $f : I \to \bigsqcup_{i \in I} X_i$ such that for each $i \in I$, there is an $x \in X_i$ such that $f(i) = (x, i)$.

We will need in particular the following equivalent formulation of this:

Lemma 1.3.2. (Zorn's Lemma) Let $(X, \leqslant)$ be a partially ordered set. If for each chain $(x_i)_{i \in I}$, there exists some $x_0 \in X$ such that $x_i \leqslant x_0$ for each $i \in I$, then X has a maximal element.

This will prove important especially when we meet the concept of an ultrafilter, and look more broadly at compact topological spaces.

[2]Indeed, the knowing reader will note that this is needed to ensure that the above definition of cartesian product yields a non-empty collection of elements whenever the underlying sets are non-empty.

Chapter 2

Basic Definitions and Examples

The first and fundamental concept of topology is that of a topological space. It took mathematicians a good chunk of the 20th century to agree on our current standard definition: on one hand, they sought to have a definition general enough to include an array of disparate "space-like" mathematical concepts as instances of topological spaces; on the other, they wanted a definition that was restrictive enough to still capture our intuitions about what "space" is. Having the actual definition – which was only settled upon after decades of inquiry – simply presented to you, it might appear awfully abstract and arbitrary. However, as we will argue throughout, such a notion can be given a rich epistemic meaning, which can also help us understand how it relates to these various notions of space.

Definition 2.0.1. Let X be a set. We say that a collection of subsets $\tau \subseteq \mathcal{P}(X)$ is a *topology on* X if it satisfies the following conditions:

(O1) $\varnothing$ and X are in τ; i.e., $\varnothing \in \tau$ and $X \in \tau$.

(O2) τ is closed under *arbitrary* unions; i.e., if $(U_i)_{i \in I}$ is a collection of sets in τ, then $\bigcup_{i \in I} U_i \in \tau$.

(O3) τ is closed under *finite* intersections; i.e., if $\{U_1, ..., U_n\} \subseteq \tau$, then $(U_1 \cap \cdots \cap U_n) \in \tau$.

Given a set and a topology τ on X, we say that the pair (X, τ) is a *topological space*. When no confusion can arise, we will simply say that X is a topological space. We call the elements $x \in X$ *points* and say that a set $U \in \mathcal{P}(X)$ is *open* if $U \in \tau$. $\dashv$

2.1 Topology and Logic: an intuition

As mentioned, to aid our intuitions, we will develop an informal epistemic interpretation of topological spaces (X, τ). To do so, we must interpret such sets X and topologies τ on X so that the three conditions, (O1)-(O3), are fulfilled:

(X) We think of X as a set of 'epistemic worlds'.

(τ) We think of $\tau \subseteq \mathcal{P}(X)$ as corresponding to the set of 'verifiable propositions', under a given epistemological framework (i.e., a way to determine what can be verified/falsified).

Before checking that the three conditions are satisfied under this interpretation (i.e., that $\varnothing, X$ are verifiable propositions, and that verifiable propositions are closed under arbitrary unions and finite intersections), let us be a bit more precise on what we mean by a 'verifiable proposition'.

First, for a proposition P, we identify it with the set of epistemic worlds of X in which P is true; that is,

$$P \mapsto \llbracket P \rrbracket = \{x \in X \mid P \textit{ is true in the world } x\} \subseteq X.$$

This explains why τ (being a set of subsets of X) can be thought of as a set of propositions. As an interlude, note that under this identification disjunction corresponds to union; conjunction to intersection; and negation to set-theoretical complement (recall the table in the introduction):

$$\begin{array}{llll}
(P \vee Q) & \mapsto & \llbracket P \vee Q \rrbracket & = \{x \in X \mid P \vee Q \textit{ is true in the world } x\} \\
 & & & = \{x \in X \mid P \textit{ is true in the world } x\} \cup \{x \in X \mid Q \textit{ is true in the world } x\} \\
 & & & = \llbracket P \rrbracket \cup \llbracket Q \rrbracket; \\
(P \wedge Q) & \mapsto & \llbracket P \wedge Q \rrbracket & = \{x \in X \mid P \wedge Q \textit{ is true in the world } x\} = \llbracket P \rrbracket \cap \llbracket Q \rrbracket; \\
(\neg P) & \mapsto & \llbracket \neg P \wedge Q \rrbracket & = \{x \in X \mid \neg P \textit{ is true in the world } x\} = X - \llbracket P \rrbracket.
\end{array}$$

In the infinitary case – with disjunctions of the form $\bigvee_{i \in I} P_i$ – the exact same reasoning applies. Thus, under our epistemic interpretation, condition (O2) becomes closure of verifiable propositions under arbitrary *disjunctions*, and (O3) becomes closure of verifiable propositions under finite *conjunctions*.

Second, to explain what we mean by 'verifiable', consider the proposition

($\exists \neg WS$) There is a non-white swan.

and its negation

($\forall WS$) All swans are white.

Now, if we were to observe a black swan, we would *verify* ($\exists \neg WS$) (and *falsify* ($\forall WS$)). In contrast, it does not seem a priori possible to verify the proposition ($\forall WS$), or, equivalently, falsify the proposition ($\exists \neg WS$). No matter how many white swans we come across, we cannot be sure what colour the next one will be – this is the (in)famous problem of induction. In our words, ($\exists \neg WS$) is verifiable, while ($\forall WS$) is not. Spelling it out, we say:

(ver) A proposition P is *verifiable* if and only if: whenever P is true at a world x (i.e., $x \in [\![P]\!]$), it is possible to verify P at x (i.e., verify $x \in [\![P]\!]$).

And dually:

(fal) A proposition P is *falsifiable* if and only if: whenever P is false at a world x (i.e., $x \notin [\![P]\!]$), it is possible to falsify P at x (i.e., falsify $x \in [\![P]\!]$).[1]

With this terminology defined, let us check that (O1)-(O3) are satisfied under our epistemic interpretation:

(O1) We first check that $X \in \tau$ is sensible given our interpretatio, that is, that X corresponds to a verifiable proposition. Since

$$\top \mapsto [\![\top]\!] = \{x \in X \mid \top \textit{ is true in the world } x\} = X,$$

we find that the full set X corresponds to the proposition of logical truth. And since logical truth can always be verified (our domain being X we know by default that $x \in X$), X corresponds to a verifiable proposition.

Next, we check that $\varnothing \in \tau$, that is, that $\varnothing$ corresponds to a verifiable proposition. Since

$$\bot \mapsto [\![\bot]\!] = \{x \in X \mid \bot \textit{ is true in the world } x\} = \varnothing,$$

we find that the empty set $\varnothing$ corresponds to falsity. Since falsity is never true, it is vacuously verifiable (note the clause 'whenever P is true at a world x [...]' in the definition of (ver)).

(O2) We have to check that verifiable propositions are closed under arbitrary disjunctions. Accordingly, suppose that $(U_i)_{i \in I} \subseteq \tau$, where each U_i corresponds to a verifiable proposition P_i. We will show that $\bigvee_{i \in I} P_i$ is verifiable. So suppose $\bigvee_{i \in I} P_i$ is true at a world x. Then there must be some $i \in I$ such that P_i is true at x. Now since P_i is verifiable, it must be possible to verify P_i at x. But then it is possible to verify $\bigvee_{i \in I} P_i$ at x, namely by verifiying P_i – showing that $\bigvee_{i \in I} P_i$ is verifiable, as required.

(O3) We have to check that verifiable propositions are closed under finite conjunctions. Accordingly, suppose that $\{U_1, \ldots, U_n\} \subseteq \tau$. Then these correspond to verifiable propositions $P_1, \ldots, P_n$. We have to show that $(P_1 \wedge \cdots \wedge P_n)$ is verifiable. So suppose $(P_1 \wedge \cdots \wedge P_n)$ is true at a world x. Then each of the P_is are true, and since they all are verifiable, they are all possible to verify at x. But then it is possible to verify $(P_1 \wedge \cdots \wedge P_n)$ at x, namely by verifying $P_1, \ldots, P_n$. Thus, $(P_1 \wedge \cdots \wedge P_n)$ is verifiable, as desired.

Example 2.1.1. Let us make the former more concrete by formalising the example of swans. If our intuitions are to hold water, we should end up with a topological space.

We take the setting to consist of two possible worlds, namely

1. A world a in which all swans are white; and
2. a world b in which there is at least one non-white swan.

[1] Notice the similarity, almost to the extent of paraphrasing, to Karl Popper's criterion of falsifiability.

Then $X = \{a, b\}$. We have four subsets of X:

- $\varnothing$, which corresponds to the verifiable proposition $\perp$, hence it is open.
- X, which corresponds to the verifiable proposition $\top$, hence it is open.
- $\{a\}$, which corresponds to the proposition $(\forall WS)$ which is not verifiable, hence it is not open.
- $\{b\}$, which corresponds to the verifiable proposition $(\exists \neg WS)$, hence it is open.

Thus, we get the collection $\tau = \{\varnothing, X, \{b\}\}$, which is indeed a topology on X:

(O1) We have that $\varnothing \in \tau \ni X$.

(O2) Clearly, any union of elements of τ equals either $\varnothing, X$ or $\{b\}$, which all are in τ.

(O3) Clearly, any finite (or infinite) intersection of elements of τ equals either $\varnothing, X$ or $\{b\}$, which all are in τ. $\dashv$

While we have argued how an epistemic interpretation can make sense of the requirements (O1)-(O3), one can wonder whether the asymmetries in the definition of topological spaces can be made more symmetric.

- **Closure under arbitrary intersections:** It might seem arbitrary that we only require closure under *finite* intersections and not under *arbitrary* intersections (unlike how we require closure under arbitrary unions). So suppose there are countably infinitely many swans, one for each natural number $n \in \omega$, and let P_n be the proposition that swan number n is white. Whilst it seems reasonable to say that each P_n is verifiable (we can simply go check swan number n), it appears less reasonable to say their conjunction

  $$\bigwedge_{n\in\omega} P_n$$

  is: we would have to check all *infinitely* many of the swans. In a nutshell, the salient distinction is that while infinite disjunctions can be verified by finite information (one disjunct is enough), infinite conjunctions cannot (all conjuncts are required). Notice that it is not that a topology may not be closed under arbitrary intersections, but simply that we do not require it to be so. If it is, we call it an *Alexandroff topology.*

- **Closure under complements:** Whilst in some cases both a proposition and its negation might be verifiable, it seems equally reasonable that this does not always hold, as exemplified by the case of swans above: $(\exists\neg W)$ is reasonably said to be verifiable, while its negation $(\forall WS)$ is not.

To summarize, (O1)-(O3) seem to model closure conditions of verifiable propositions fairly well, and throughout the project we will be using this epistemic interpretation to gain intuition for various topological definitions, notions and concepts.

Logic	**Topology**
Epistemic worlds/situations/models/objects satisfying a property	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$

2.2 (Sub)basis and Examples

We have seen how the definition of a topological space formalises our intuition of verifiable propositions. However, once we have the abstract definition, we can use it in a number of different settings. To get a feel for this, let us look at a few more examples of topological spaces.

Example 2.2.1. Let $X = \{x, y, z\}$, and consider the following subsets of $\mathcal{P}(X)$:

$$\tau_1 = \{\varnothing, X, \{x\}\},\ \tau_2 = \{\varnothing, X, \{x\}, \{x, y\}\},\ \tau_3 = \{\varnothing, X, \{x\}, \{y\}\},\ \text{and } \tau_4 = \{\varnothing, \{x\}\}.$$

Then τ_1 and τ_2 are both topologies on X (check this); however, τ_3 is not, nor is τ_4. τ_3 – although satisfying (O1) and (O3) – is not closed under unions:

$$\{x\} \in \tau_3 \ni \{y\},\ \text{but } \{x\} \cup \{y\} = \{x, y\} \notin \tau_3.$$

And τ_4 fails to satisfy (O1) because $X \notin \tau_4$. ⊣

Example 2.2.2. (Discrete and Indiscrete Topology)
Fix a set X. Then the collection $\{\varnothing, X\}$ satisfies the axioms of a topology; it is called the *indiscrete topology*. Similarly, the collection $\mathcal{P}(X)$ also satisfies the axioms of a topology, and it is called the *discrete topology*. By definition, all topologies contain (in terms of inclusion) the indiscrete topology, and are contained in the discrete topology; that is, if τ is a topology on X, then $\{\varnothing, X\} \subseteq \tau \subseteq \mathcal{P}(X)$. ⊣

Example 2.2.3. (Kripke frames)
Let $\mathfrak{F} = (W, R)$ be a reflexive and transitive Kripke frame. Then W can be given a topology, called the *Alexandroff topology* on the preorder $\mathfrak{F}$, in the following way: $U \subseteq W$ is open if and only if

$$\forall x, y \in W\, [(x \in U \text{ and } xRy) \implies y \in U]\,.$$

In other words, the open sets are precisely the *R-upsets* (check that this forms a topology). Notice that this topology has the property that arbitrary intersections of opens are again open, hence the name "Alexandroff topology". ⊣

In each of the previous examples, we defined the topological space (X, τ) by explicitly specifying all open sets (i.e., all members of τ). Sometimes this is unwieldy, and it is more practical to only specify a subcollection of τ from which we can *generate* the entire topology:

Definition 2.2.4. (Basis and Subbasis) Let (X, τ) be a topological space. We say that $\mathcal{B} \subseteq \tau$ is a *basis for the topology* τ if for each $U \in \tau$ there is a collection $(V_i)_{i \in I} \subseteq \mathcal{B}$ such that

$$U = \bigcup_{i \in I} V_i.$$

We say that $\mathcal{S} \subseteq \tau$ is a *subbasis for the topology* if the set of finite intersections of elements from $\mathcal{S}$

$$\left\{ \bigcap_{V \in M} V \mid M \subseteq \mathcal{S}, M \textit{ is finite} \right\}$$ [2]

forms a basis for the topology.

Moreover, given a (sub)basis $\mathcal{B} \subseteq \tau$, we call members $U \in \mathcal{B}$ *(sub)basic opens*. ⊣

[2] We adopt the convention that the nullary intersection is the full set; i.e., $\bigcap_{V \in \varnothing} V = X$. And similarly that $\bigcup_{V \in \varnothing} V = \varnothing$, and correspondingly for propositions in logic: $\bigwedge_{P \in \varnothing} P = \top$ and $\bigvee_{P \in \varnothing} P = \bot$.

We can express the relationships between these concepts in various way: a topology is a basis closed under arbitrary unions, and closing a subbasis under finite intersections one obtains a basis.[3] We also note the following easy but important facts:

Proposition 2.2.5. Let X be a set and $C \subseteq \mathcal{P}(X)$ a collection of sets. Then there is a (unique) topology on X for which C is a subbasis.

Moreover, if (1) C covers X (i.e., $\bigcup_{U \in C} U = X$) and (2) C is closed under finite non-empty intersections, then there is a (unique) topology on X for which C is a basis.

Proof. For the former claim, first close C under finite intersections, and then close the resulting collection of sets under arbitrary unions. This will yield a topology (check this). Uniqueness is left as an exercise.

For the latter, close C under arbitrary unions. This will yield a topology (check this). Uniqueness is left as an exercise. ■

Remark 2.2.1. (*Epistemic intuition: what is a (sub)basis?*) Given a topological space (X, τ), we have seen that the topology τ can be thought of as the set of verifiable propositions on the set of epistemic worlds X.

What about $\mathcal{B}$ then, for $\mathcal{B}$ a basis for τ? Since $\mathcal{B} \subseteq \tau$, it too can be thought of as a set of verifiable propositions. And since τ is generated by closing $\mathcal{B}$ under arbitrary unions, we can think of $\mathcal{B}$ as a basis of verifiable propositions from which all other verifiable propositions can be inferred by a form of weakening, namely, by forming disjunctions of these basic propositions.

Similarly, for $\mathcal{S}$ a subbasis for τ, we can think of $\mathcal{S}$ as a subbasis of verifiable propositions from which all other verifiable propositions can be generated by inference, through both strengthening pieces of knowledge (conjunctions) and weakening (disjunctions).

Example 2.2.6. Let $X = \{a, b, c, d\}$ where a, b, c, d are worlds described as follows:

	All ravens are black	Some raven is non-black
All swans are white	a	b
Some swan is non-white	c	d

I.e., the world c, for instance, is a world in which all ravens are black and with some non-white swan.

If we want to specify the verifiable propositions in this setting, instead of listing all of them, we could simply say that our subbasic verifiable propositions are $(\exists \neg WS)$ and the analogous $(\exists \neg BR)$. That is, all verifiable propositions are obtained by first taking (finite) conjunctions of these two propositions, and then combining them in disjunctions. Doing so corresponds to having the subbasis

$$\mathcal{S} = \{\{c, d\}, \{b, d\}\},$$

which generates the topology

$$\tau = \{\varnothing, X, \{c, d\}, \{b, d\}, \{d\}, \{b, c, d\}\}$$

on X – exactly corresponding to the set of verifiable propositions (check this). ⊣

[3] However, it should be noted that a basis need not be closed under intersection.

Example 2.2.7. (Real Line Topology)
When first learning mathematics in school, one is often introduced to the real numbers $\mathbb{R}$ and basic properties of these: they are ordered, and one can then look at the so-called "open intervals", typically written as

$$(x, y) = \{z \in \mathbb{R} : x < z < y\} \text{ where } x, y \in \mathbb{R}.$$

The choice of terminology is not a coincidence: the open intervals form a basis for the *Euclidean topology* on $\mathbb{R}$ (check this), in which a set $U \subseteq \mathbb{R}$ is open if and only if,

$$\forall z \in U \exists x, y \in U (x < z < y). \qquad \dashv$$

Example 2.2.8. (Cantor and Baire Spaces)
Consider the set 2^ω of infinite binary sequences. We call this the *Cantor set*. Given a finite sequence $s \in 2^{<\omega}$, and a finite or infinite sequence $t \in 2^{<\omega} \cup 2^\omega$ we write $s \lhd t$ to mean that s is an initial subsequence of t. Given any $s \in 2^{<\omega}$ we can consider the following set:

$$C(s) = \{x \in 2^\omega : s \lhd x\}.$$

Now consider the collection of the sets of the form $\{C(s) : s \in 2^{<\omega}\}$. You can check that this (1) covers 2^ω, and (2) is closed under finite non-empty intersections, and hence, defines a basis for a topology (cf. Proposition 2.2.5) which we call the *Cantor space*.

Similarly, we topologise ω^ω with a basis of sets $C(s) = \{x \in \omega^\omega : s \lhd x\}$. We call this latter space the *Baire space*. $\dashv$

The concepts of basis and subbasis are thus instrumental when working with topological spaces. In fact, when proving a topological statement, it is often enough to only consider *basic opens*. An instance of this occurs when *comparing topologies*, which can be used to show that two topologies are the same:

Definition 2.2.9. Let X be a set, and τ and τ' two topologies on this set. We say that τ is a *coarser topology* than τ' if $\tau \subseteq \tau'$. Conversely, we say that τ' is *finer* than τ. $\dashv$

Lemma 2.2.10. Suppose X is a set with two topologies τ and τ', and $\mathcal{B}_\tau$ and $\mathcal{B}_{\tau'}$ are bases for these topologies, respectively. Then $\tau \subseteq \tau'$ holds if and only if for all points $x \in X$ and all basic τ-open $U \in \mathcal{B}_\tau$ containing x, there is some basic τ'-open $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$.

Proof. ($\Rightarrow$) Let $x \in X$ and $x \in U \in \mathcal{B}_\tau$ be arbitrary. We are then to find a $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$. Since $U \in \mathcal{B}_\tau$, we have that $U \in \tau$, so since $\tau \subseteq \tau'$ by assumption, we also have that $U \in \tau'$. Now since τ' is generated by $\mathcal{B}_{\tau'}$, U must be the union of some elements from $\mathcal{B}_{\tau'}$. And since $x \in U$, one of these must contain x, hence we have our $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$, as required.

($\Leftarrow$) Let $U \in \tau$ be arbitrary. We are then to show that $U \in \tau'$. For each $x \in U$, fix some $U_x \in \mathcal{B}_\tau$ such that $x \in U_x \subseteq U$ (such U_x must exist, since otherwise U would not be the union of elements from $\mathcal{B}_\tau$). By assumption, for each of these, there must be some $U'_x \in \mathcal{B}_{\tau'}$ such that $x \in U'_x \subseteq U_x$. But then

$$U \subseteq \bigcup_{x \in U} U'_x \subseteq \bigcup_{x \in U} U_x \subseteq \bigcup_{x \in U} U = U,$$

which shows that

$$U = \bigcup_{x \in U} U'_x.$$

But $(\bigcup_{x \in U} U'_x) \in \tau'$ because each $U'_x \in \tau'$, hence $U \in \tau'$, as desired. ■

Example 2.2.11. Consider the sets of the form

$$(x, \infty) = \{z \in \mathbb{R} \mid x < z\}.$$

These (1) cover $\mathbb{R}$, and (2) are closed under finite non-empty intersections, hence form a basis for a topology τ_F on $\mathbb{R}$. Using the above proposition, we show that $\tau_F \subsetneq \tau_E$, where τ_E is the Euclidean topology on $\mathbb{R}$.

($\subseteq$) Let $x, l \in \mathbb{R}$ be arbitrary such that $x \in (l, \infty)$. By the above proposition, it then suffices to find an open interval $(a, b) \subseteq \mathbb{R}$ such that $x \in (a, b) \subseteq (l, \infty)$. However, this is easy: setting, e.g., $a = l$ and $b = x + 1$ does the job.

($\not\supseteq$) By the above proposition, it suffices to find some $x \in \mathbb{R}$ and open interval $x \in (a, b) \subseteq \mathbb{R}$ such that there is no $l \in \mathbb{R}$ for which $x \in (l, \infty) \subseteq (a, b)$. Again, this is easy: set, e.g., $x = 0$, $a = -1$ and $b = 1$. Then $x \in (a, b)$, and for no $l \in \mathbb{R}$, do we have $(l, \infty) \subseteq (a, b)$. $\dashv$

2.3 Generating New Topologies

Often we are interested in getting new topologies from existing ones. In this course we will encounter several such procedures. The most elementary is to generate a topological space from a subset of an already existing topological space:

Definition 2.3.1. Let (X, τ) be a topological space and $S \subseteq X$ a subset. We denote by τ_S the *subspace topology on* S defined as

$$\tau_S := \{U \cap S \mid U \in \tau\}.$$

I.e., the τ_S-open sets are our original τ-open sets restricted to the subset S by means of intersection. We then say that (S, τ_S) is a *subspace* of (X, τ) (you should check that (S, τ_S), indeed, is a topological space). $\dashv$

Subspaces can in fact be given by only looking at a basis for the original space:

Lemma 2.3.2. Let (X, τ) be a topological space with a basis $\mathcal{B}$, and let $S \subseteq X$. Then the set

$$\mathcal{B}_S = \{U \cap S : U \in \mathcal{B}\}$$

is a basis for τ_S.

Proof. To see this, let $T \subseteq S$ be open in the subspace topology. Then by definition, $T = V \cap S$ for some V open in X. Hence, because $\mathcal{B}$ is a basis, we have that $V = \bigcup_{i \in I} U_i$, where the $U_i \in \mathcal{B}$. Thus, putting this together

$$T = V \cap S = \left(\bigcup_{i \in I} U_i \right) \cap S = \bigcup_{i \in I} (U_i \cap S),$$

which was to show. ■

Example 2.3.3. Consider $\mathbb{R}$ the reals, and look at $\mathbb{Z}$ with the subspace topology.[4] We claim that the latter coincides with the discrete topology. Indeed, for any $n \in \mathbb{Z}$, we can consider the interval

$$(n - \frac{1}{2}, n + \frac{1}{2}).$$

This is open in $\mathbb{R}$, hence its intersection with $\mathbb{Z}$ – which is the singleton $\{n\}$ – will be open in the subspace topology on $\mathbb{Z}$. Thus, every singleton is open in the subspace, which implies that every subset is open (because topologies, in particular, are closed under arbitrary unions).

Question: is $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) also discrete? ⊣

Also interesting is the following very frequent construction:

Definition 2.3.4. Let X and Y be topological spaces. We define a topology on the product $X \times Y$, called the *product topology*, as follows: a set $U_0 \times U_1 \subseteq X \times Y$ is basic open if and only if U_0 is open in X and U_1 is open in Y (you should check that this, indeed, defines a basis for a topology on $X \times Y$, cf. Proposition 2.2.5). ⊣

With subspace and product topologies introduced, we prove the following proposition as a sanity check:

Proposition 2.3.5. Suppose X and Y are topological spaces and that $S_X \subseteq X$ and $S_Y \subseteq Y$. Then first constructing the product topology $X \times Y$ and then constructing the subspace topology $S_X \times S_Y \subseteq X \times Y$ is the same as first constructing the subspace topologies $S_X \subseteq X$ and $S_Y \subseteq Y$ and then taking their product $S_X \times S_Y$ (we say the constructions *commute*).

Proof. For the topology obtained by the former sequence of constructions, a basic open is of the form $(U_X \times U_Y) \cap (S_X \times S_Y)$ for U_X open in X and U_Y open in Y. And for the topology obtained by the latter sequence of constructions, a basic open is of the form $(U_X \cap S_X) \times (U_Y \cap S_Y)$ for U_X open in X and U_Y open in Y. So since

$$(U_X \times U_Y) \cap (S_X \times S_Y) = (U_X \cap S_X) \times (U_Y \cap S_Y),$$

the bases are the same, hence the topologies are the same. ∎

Naturally, to define the product topology, one can also just take the bases to generate a new basis:

Lemma 2.3.6. Let X and Y be topological spaces with bases $\mathcal{B}_X$ and $\mathcal{B}_Y$, respectively. Then

$$\{U_X \times U_Y \mid U_X \in \mathcal{B}_X, U_Y \in \mathcal{B}_Y\}$$

forms a basis for the product topology on $X \times Y$.

Proof. Exercise. ∎

The product topology construction easily generalises to any *finite* product, and leads to many usual spaces:

[4] When nothing else is mentioned, we take the topology on the reals to be the Euclidean topology. This topology is also known as the *standard topology* on $\mathbb{R}$.

Example 2.3.7. Let $\mathbb{R}^n$ be the set of n-dimensional tuples of reals. The n-dimensional topology on this set is given by the basis of n-dimensional balls; i.e., the basis consists of all sets of the form

$$B_\varepsilon(\overline{x}) := \left\{\langle y_1, \ldots, y_n\rangle \in \mathbb{R}^n \mid \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2} < \varepsilon\right\}$$

where $\overline{x} = \langle x_1, \ldots, x_n\rangle \in \mathbb{R}^n$ and $\varepsilon > 0$.

We can show that this topology is the same as the product topology of $\mathbb{R}$ with the usual Euclidean topology, n many times. For simplicity we do this for $n = 2$, though the argument is analogous for higher dimensions.

We will show equality of the topologies going inclusion by inclusion using Lemma 2.2.10. First, let $B_{\varepsilon'}(y_1, y_2)$ and $\langle x_1, x_2\rangle \in B_{\varepsilon'}(y_1, y_2)$ be arbitrary, and set $\varepsilon := \varepsilon' - \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$. Then $\varepsilon > 0$ and $\langle x_1, x_2\rangle \in B_\varepsilon(x_1, x_2) \subseteq B_{\varepsilon'}(y_1, y_2)$, hence – for one inclusion – it suffices to find some open intervals (a, b) and (c, d) such that $\langle x_1, x_2\rangle \in (a, b) \times (c, d) \subseteq B_\varepsilon(x_1, x_2)$ (because $(a, b) \times (c, d)$ is a basic open in the product topology, cf. the preceding lemma). So consider the following choice of intervals:

$$(x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}) \times (x_2 - \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2})$$

Clearly, $\langle x_1, x_2\rangle \in (x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}) \times (x_2 - \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2})$. Now let

$$(z_1, z_2) \in (x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}) \times (x_2 - \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2})$$

be arbitrary; for this inclusion of topologies, it then suffices to show that $\langle z_1, z_2\rangle \in B_\varepsilon(x_1, x_2)$. Observe that $|z_i - x_i| \leqslant \frac{\varepsilon}{2}$. Further, we have that the following inequality holds for all $\langle z_1, z_2\rangle \in \mathbb{R}^2$ (and all $\langle x_1, x_2\rangle \in \mathbb{R}^2$):

$$\sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2} \leqslant |z_1 - x_1| + |z_2 - x_2|.$$

Thus,

$$\sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that $\langle z_1, z_2\rangle \in B_\varepsilon(x_1, x_2)$, as required.

Second, for the other inclusion of topologies, let $(x_1, x_2) \times (y_1, y_2)$ and $\langle x, y\rangle \in (x_1, x_2) \times (y_1, y_2)$ be arbitrary. We can think of $(x_1, x_2) \times (y_1, y_2)$ as an (open) rectangle. Since the point $\langle x, y\rangle$ is in this rectangle, we can consider an open ball around $\langle x, y\rangle$, that is small enough that it is entirely contained in the rectangle. For concreteness, we can set $\varepsilon := \min\{|x - x_1|, |x - x_2|, |y - y_1|, |y - y_2|\}$, the minimal distance from the point to the edges of the rectangle. Then

$$\langle x, y\rangle \in B_\varepsilon(x, y) \subseteq (x_1, x_2) \times (y_1, y_2),$$

which proves the claim. ⊣

So far we have only discussed the *finite* product topology, but there is, of course, also a product topology simpliciter which generalises to cover the infinite case as well. The reason we discussed the finite product topology separately, is that the product topology in general is quite complex, as we will now see:

Definition 2.3.8. Let $(X_i)_{i\in I}$ be a collection of topological spaces. We define a topology on their product $\prod_{i\in I} X_i$ by saying that a set

$$U = \prod_{i\in I} U_i \subseteq \prod_{i\in I} X_i$$

is *basic open* if (1) all the $U_i \subseteq X_i$ are open (in X_i) and (2) for all but finitely many coordinates, do we have $U_i = X_i$ (you should check that this, indeed, defines a basis for a topology on $\prod_{i\in I} X_i$, cf. Proposition 2.2.5). ⊣

Notice that this definition does not reduce down to a set being open if it is the product of infinitely many open sets. This condition is crucial to ensure that many properties are preserved under products, as we will see later.

Example 2.3.9. Recall the Cantor space 2^ω. We can prove that the topology on this set is the product topology of ω many copies of $\mathbf{2} = \{0,1\}$, the two element set with the discrete topology. To do so, we, as usual, go inclusion by inclusion using Lemma 2.2.10.

Accordingly, first, suppose $(x_n)_{n\in\omega} \in U$ for some basic open $U = \prod_{n\in\omega} U_n$ in the product topology. By definition, we then know that for all but finitely many coordinates, $U_n = \{0,1\}$. Hence there must be some greatest natural number m such that $U_m \neq \{0,1\}$ (or in case $U_n = \{0,1\}$ for all $n \in \omega$, we set $m := 0$). Then $(x_i)_{i\leqslant m} \lhd (x_i)_{i\in\omega}$ and $(x_i)_{i\in\omega} \in C((x_i)_{i\leqslant m}) \subseteq U$, which shows the first inclusion.

Second, for the other inclusion, it suffices to show that any $C(s)$ is a basic open in the product topology. Since s is a finite sequence it is of the form $(x_i)_{i\leqslant m}$ for some $m \in \omega$ and $x_0, \dots, x_m$, hence we get that

$$C(s) = C((x_i)_{i\leqslant m}) = \{x_0\} \times \cdots \times \{x_m\} \times \prod_{n>m} \{0,1\},$$

which, indeed, is a basic open in the product topology. This shows the other inclusion, and thus, concludes the proof of the two topologies coinciding. ⊣

We now take a look at the *topological sum*:

Definition 2.3.10. Let $(X_i)_{i\in I}$ be a collection of topological spaces. We define the *topological sum* of this collection to be their disjoint union $\bigsqcup_{i\in I} X_i$ endowed with the following topology: $U \subseteq \bigsqcup_{i\in I} X_i$ is open if and only if for each $i \in I$, the following set is open in X_i:

$$U(i) := \{a \in X_i \mid (a,i) \in U\}.$$

(As always, you should check that this actually does define a topological space). ⊣

One can prove that a collection of bases for each X_i induces a basis for their topological sum:

Lemma 2.3.11. Let $(X_i)_{i\in I}$ be a collection of topological spaces, and let $(\mathcal{B}_i)_{i\in I}$ be a collection of bases for each of these spaces. Then the set

$$\mathcal{B} = \left\{ Z \subseteq \bigsqcup_{i\in I} X_i \mid Z(i) \in \mathcal{B}_i \right\}$$

is a basis for the topological sum of the spaces.

Proof. Exercise. ■

Example 2.3.12. Suppose $(\mathfrak{F}_i)_{i \in I}$ is a collection of transitive and reflexive Kripke frames. Then their disjoint union is also a transitive and reflexive Kripke frame, hence it admits an Alexandroff topology as per Example 2.2.3. We can show that the topological space obtained in this way is precisely the topological sum of the Alexandroff spaces $(\mathfrak{F}_i)_{i \in I}$. ⊣

2.4 Closed sets, Neighbourhoods, Closure and Interior Operators

Having seen some examples of topological spaces and ways of constructing them, we close off the chapter by covering some important notions related to topological spaces, beginning with the notion of a *closed set*:

Definition 2.4.1. Let (X, τ) be a topological space. We say that a set $U \in \mathcal{P}(X)$ is *closed* if its complement is open; i.e., if $(X - U) \in \tau$. ⊣

An immediate consequence of this definition and our definition of a topological space is the following:

Proposition 2.4.2. Let (X, τ) be a topological space. Then:

(C1) X and $\varnothing$ are closed sets.

(C2) Arbitrary intersections of closed sets are closed; i.e., if $(U_i)_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} U_i$ is a closed set.

(C3) Finite unions of closed sets are closed; i.e., if $U_1, \ldots, U_n$ are closed, then so is $(U_1 \cup \cdots \cup U_n)$.

Proof. Follows from the set-theoretical complement operator taking unions to intersections (and vice versa). ■

In fact, conditions (C1)-(C3) work as an equivalent definition of a topological space: specifying a collection of subsets satisfying (C1)-(C3) (which we call closed), determines a collection of open sets – namely the complements of the closed ones – which satisfies (O1)-(O3).

Remark 2.4.1. (Epistemic intuition: what are the closed sets?) We know that, given a topological space (X, τ), members of τ are called open and that these can be thought of as verifiable propositions. We also now know that the complement $(X - U)$ of an open set $U \in \tau$ is called closed and corresponds to the negation of a verifiable proposition. But, as previously argued, the negation of a verifiable proposition is not always verifiable (recall $(\exists \neg WS)$ and its negation $(\forall WS)$). However, the negation of a verifiable proposition is always falsifiable; e.g., to falsify $(\forall WS)$, it suffices to show the existence of a, say, black swan. That is, we can think of closed sets as falsifiable propositions.

Remark 2.4.2. The reader familiar with contemporary service work will be aware of the notion of a "clopen". Just as in this case, in topology the concept of being closed and open are not mutually exclusive nor exhaustive. One can have sets, just like the whole space and the empty set, which are both open and closed – these are often shortened to *clopen sets*. And one can have sets which are neither (see Exercise 2.2).

Our intuition of open sets as verifiable propositions and closed sets as falsifiable propositions can help us make sense of this: The proposition

It is raining outside.

can reasonably be said to be both verifiable and falsifiable (I can simply go out and check); while the proposition

John F. Kennedy's last thought was "What is the One True Logic?"

neither seems verifiable nor falsifiable. ⊣

	Verifiable (open)	Falsifiable (closed)
All swans are white		x
Some swan is non-white	x	
It is raining outside	x	x
JFK's last thought was "What is the OTL?"		

Recall that all opens of a subspace S of X are of the form $(U \cap S)$ for U some open in X. An analogous result holds for closed sets in subspace topologies:

Lemma 2.4.3. Suppose (S, τ_S) is a subspace of (X, τ). Then a set $U \in \mathcal{P}(S)$ is closed in S if and only if there is some closed set V in X (i.e., $(X - V) \in \tau$) such that $U = V \cap S$.

Proof. Exercise. ■

Definition 2.4.4. Let (X, τ) be a topological space and $S \subseteq X$ arbitrary. We denote by $cl(S)$ or $\overline{S}$ the *closure of* S, the smallest closed set K such that $S \subseteq K$; that is, $cl(S)$ is the intersection of all closed sets containing S. We denote by $int(S)$ the *interior of* S, the largest open set K such that $K \subseteq S$; that is, $int(S)$ is the union of all open sets contained in S. ⊣

Remark 2.4.3. Using this definition, we have that a set S is closed if and only if $S = \overline{S}$, and open if and only if $S = int(S)$. We call the operators

$$int : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto int(S)$$

and

$$cl : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto cl(S)$$

the *topological interior* and *topological closure*, respectively. As the reader will find in the exercises, interior and closure operators provide an alternative, but equivalent, form of describing topologies. ⊣

The last notion we will introduce in this chapter is that of a *neighbourhood*:

Definition 2.4.5. Given a topological space (X, τ) and a point $x \in X$, we say that $V \subseteq X$ is a *neighbourhood* of x if and only if there is an open set U such that $x \in U \subseteq V$.

Moreover, observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x if and only if $x \in V$ and V is open.[5] ⊣

[5] In the literature, you will sometimes find that a neighbourhood simpliciter already is required to be open. We do not adopt that convention, but simply speak of 'open neighbourhoods' when needed.

Remark 2.4.4. (*Epistemic intuition: what is an (open) neighbourhood?*) The open neighbourhoods of a point x have a neat epistemic interpretation: they are precisely the verifiable propositions true at world x (i.e., the propositions that in fact can be verified at x – assuming that only true propositions can be verified). One can also come up with an epistemic interpretation of a neighbourhood simpliciter, but it seems a rather artificial concept; all intuitions, including our epistemic one, have their shortcomings. $\dashv$

Using the definition of an open neighbourhood, we can give another equivalent definition of the closure of a set, which is particularly useful when proving a set is open or closed:

Proposition 2.4.6. Suppose X is a topological space and $S \subseteq X$. Then the following are equivalent for a point $x \in X$:

- x is in the closure of S; i.e., $x \in cl(S)$.
- All open neighbourhoods U of x have non-empty intersection with S; i.e., $U \cap S \neq \varnothing$.

Proof. ($\Rightarrow$) Suppose for contraposition that there is some open neighbourhood U of x such that $U \cap S = \varnothing$. Then $(X - U)$ is closed, $S \subseteq (X - U)$ and $x \notin (X - U)$. So since $cl(S)$ equals the intersection of all closed sets containing S – which includes the set $(X - U)$ – we find that $x \notin cl(S)$, as required.

($\Leftarrow$) Suppose for contraposition that $x \notin cl(S)$. Then since $cl(S)$ is closed, its complement $(X - cl(S))$ is an open set containing x, i.e., an open neigbourhood of x. And clearly, $(X - cl(S)) \cap S = \varnothing$, which proves the claim. ■

Logic	**Topology**
Epistemic worlds/situations/models/objects satisfying a property	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$
Falsifiable propositions	Closed sets, $U^C \in \tau$
Verifiable propositions true at x	Open neighbourhoods U of x
(Sub)basic verifiable propositions	(Sub)basic opens

2.5 Exercises

Exercise 2.1. Let $(\tau_i)_{i \in I}$ be a collection of topologies on a set X.

(a) Is their intersection $\bigcap_{i \in I} \tau_i$ (necessarily) a topology on X?

(b) Is their union $\bigcup_{i \in I} \tau_i$ (necessarily) a topology on X?

(c) Show that there is a greatest topology τ on X such that $\tau \subseteq \tau_i$ for all $i \in I$. (with "greatest" we mean that if τ' is some other topology such that $\tau' \subseteq \tau_i$ for all $i \in I$, then $\tau' \subseteq \tau$)

(d) Show that there is a least topology τ on X such that $\tau_i \subseteq \tau$ for all $i \in I$.

Now let $X = \{x, y, z\}$, $\tau_0 = \{\varnothing, X, \{x\}, \{x, y\}\}$ and $\tau_1 = \{\varnothing, X, \{x\}, \{y, z\}\}$.

(e) Find the greatest topology τ on X such that $\tau \subseteq \tau_0$ and $\tau \subseteq \tau_1$.

(f) Find the least topology τ on X such that $\tau_0 \subseteq \tau$ and $\tau_1 \subseteq \tau$.

Exercise 2.2. Consider the space $(\mathbb{R}, \tau_{Euc})$, with its Euclidean topology.

- Give an example of a set which is neither open nor closed.
- Show that the intervals of the form (x, y) where $x, y \in \mathbb{Q}$ form a basis for this topology.
- Show that $\mathbb{Q}$ is a countable union of closed sets.

Exercise 2.3. Let (S, τ_S) be a subspace of (X, τ), and $T \subseteq S$. Show that the topology the set T inherits as a subspace of (S, τ_S) is the same as the topology it inherits as a subspace of (X, τ).

Exercise 2.4. Prove Lemma 2.3.6.

Exercise 2.5. Prove Lemma 2.3.11.

Exercise 2.6. Prove Lemma 2.4.3.

Definition 2.5.1. Let $T \subseteq \omega^{<\omega}$ be a collection of finite sequences of natural numbers. We say that T is a *tree* if whenever $s \in T$ and $t \lhd s$, then $t \in T$.

Given a tree T we write $[T]$ for the set of all branches, i.e. the infinite sequences all of whose finite approximations belong to T:

$$[T] = \{s \in \omega^\omega : \forall n, s \upharpoonright n \in T\}$$

Exercise 2.7. Let $A \subseteq \omega^\omega$ be a set of natural number sequences. We write $T(A)$ for the *tree of initial sequences*, that is:

$$T(A) = \{s \in \omega^{<\omega} : s \lhd\}$$

1. Show that whenever T is a tree, $[T]$ is a closed subset.
2. Show that the assignment $A \mapsto [T(A)]$ is the topological closure in the Baire space.

Exercise 2.8. Let X be a set. We say that an operation $\Box : \mathcal{P}(X) \to \mathcal{P}(X)$ is called an *interior operator* if it satisfies for each $U, V \in \mathcal{P}(X)$,

- (All set): $\Box X = X$;
- (Normality): $\Box(U \cap V) = \Box U \cap \Box V$;
- (Inflationarity): $\Box U \subseteq U$;
- (Idempotence): $\Box U \subseteq \Box\Box U$.

1. Show that if (X, τ) is a topological space, the topological interior *int* is an interior operator in this sense.
2. Given a set $(X, \Box)$ equipped with an interior operator, define a topology for which $\Box$ is the topological interior operator.