VARIOUS QUESTERS

THE HITCHIKER'S GUIDE TO C.ESPINDOLA'S PROOF OF

SHELAH'S EVENTUAL CATEGORICITY CONJECTURE

Contents

1
Preliminary Notions

This chapter serves to set up the main concepts needed to begin reading the paper. (...)

1.1 Morley's Theorem

(?)

1.2 Abstract Elementary Classes

This section is modelled after [Gro02]; the reader is encouraged to consult this for more information.

Definition 1.2.1. Let $\langle K, \preceq_K \rangle$ be a pair consisting of a collection of structures K for some language $L(K)$, and a relation $\preceq_K$ holding between these structures, such that:

1. $\preceq_K$ is a partial order.

2. If $M \preceq_K N$ then M is a substructure of N.

3. (Isomorphism closure): K is closed under isomorphism, and if $M, N, M', N' \in K$, $f : M \cong M'$ and $g : N \cong N'$, $f \subseteq g$ and $M \preceq_K N$ then $M' \preceq_K N'$.

4. (Coherence): If $M \preceq_K N$ and $P \preceq_K N$, and $M \subseteq P$, then $M \preceq_K P$.

5. (Tarski-Vaught Axioms): If γ is an ordinal and $\{M_\alpha : \alpha \in \gamma\} \subseteq K$ is a chain under $\preceq_K$, then

   - $\bigcup_{\alpha\in\gamma} M_\alpha \in K$;
   - If $M_\alpha \preceq_K N$ for all $\alpha \in \gamma$ then $\bigcup_{\alpha\in\gamma} M_\alpha \preceq_K N$.

6. (Lowenheim-Skolem Axiom) : There exists a cardinal $\mu \geq |L(K)| + \aleph_0$, such that if A is a subset of $M \in K$, then there is N such that $A \subseteq N$, $N \in K$, $|N| \leq |A| + \mu +$ and $N \preceq_K M$.

Given an AEC $\langle K, \preceq_K \rangle$ we denote by $\mathsf{LS}(K)$ the least μ in the conditions of the LS-axiom, and call it the *Lowenheim-Skolem number*.

Given such an AEC, a map $f : M \to N$ where $M, N \in K$ is a K-*embedding* if $f[M] \preceq_K N$, and f is an isomorphism from M onto $f[M]$.

Let us consider some examples:

Example 1.2.2. If K is an elementary class, i.e., $K = \mathsf{Mod}(T)$ for some theory T, then it is an abstract elementary class with the relation $\preceq_K$ being given by elementary *substructure*. The two first axioms are trivial, and isomorphism closure, coherence, the Tarski-Vaught axioms and the Lowenheim-Skolem axiom are all properties known in classical model theory. The Lowenheim-Skolem number is $|T| + \aleph_0$.

Originally this paragraph claims when $\preceq_K$ is given elementary *embedding*, the resulting class will be an AEC, which is *not* true. Again, axiom 2 means $M \preceq_K N$ implies M is a substructure of N. Of course, none of these nitpicking things survive ones we consider accessible categories, which generalise AECs (cf. next Section).

Of course we would not be interested in abstract elementary classes if this were the only example on hand. The key motivation of the theory lies in the fact that some classes are, from a certain point of view, very natural, and do not look too wild to be analysed through model-theoretic methods. For example we have

- Finitely generated groups;
- Archimedean fields;
- Connected graphs;
- Noetherian rings;
- The class of algebraically closed fields with infinite transcendence degree.

It seems like there should be some setting in which one could study these models that offered tools to classify them. But it is not immediately obvious what that would be. For the majority of the 20th century, it was judged that the way forward would be to construct languages which could "tame" these classes. Let us turn to some examples of this kind; for general references on infinitary logic, the reader can consult [Kar64; Mar02; Dic85]:

Definition 1.2.3. Let κ and λ be regular cardinals and $\lambda \leq \kappa$. Let τ be a first order vocabulary. We denote by $\mathcal{L}_{\kappa,\lambda}(\tau)$ the language constructed using $\bigvee_\kappa$, $\bigwedge_\kappa$ and $\exists_\lambda$ and $\forall_\lambda$:

Regularity of the cardinals is assumed to ensure that any formula ϕ has at most κ many subformulas; also $\lambda \leq \kappa$ is a sanity condition given we maintain our atomic formulas finite (which is not necessary, but very convenient).

(1) Terms and atomic formulas are as in first order logic;

(2) If ϕ is a formula then so is $\neg\phi$;

(3) If (ϕ_α) is a collection of less than κ formulas, then so is $\bigwedge_\kappa \phi_\alpha$ and $\bigvee_\kappa \phi_\alpha$;

(4) If $\phi(x_\alpha)$ for $\alpha \in \lambda$ is a formula with less than λ free variables and all its free variables amongst x_α, then so is $\exists_\lambda \phi(x_\alpha)$ and $\forall_\lambda \phi(x_\alpha)$.

We let:

$$\mathcal{L}_{\infty,\lambda} = \bigcup_{\kappa \in \mathsf{Ord}} \mathcal{L}_{\kappa,\lambda}$$

We briefly mention some notions which are relevant here. Namely, given two first-order theories M, N, we write:

$$M \equiv_{\infty,\omega} N,$$

If and only if the two structures satisfy the same formulas from $\mathcal{L}_{\infty,\omega}$. An equivalent, and more useful, characterization due to Karp uses the notion of a *partial isomorphism system*.

Definition 1.2.4. Let $\mathbf{M}, \mathbf{N}$ be two first order structures in a language τ. We say that a collection P of partial τ-embeddings $f : A \to \mathbf{N}$ where $A \subseteq M$ is a *partial isomorphism system* if:

- For all $f \in P$, and $a \in \mathbf{M}$, there is some $g \in P$ such that $f \subseteq g$ and $a \in \mathsf{dom}(g)$;
- For all $f \in P$ and $b \in \mathbf{N}$, there is $g \in P$ such that $f \subseteq g$ and $b \in \mathsf{img}(g)$

We write $\mathbf{M} \cong_p \mathbf{N}$ if there is a partial isomorphism system between these two structures.

The following relates this concept to Ehrenfeucht-Fraisse games, to the above notion of infinitary equivalence, as well as to some set-theoretic notions.

Theorem 1.2.5. *The following are equivalent for* $\mathbf{M}, \mathbf{N}$*:*

1. $\mathbf{M} \equiv_{\infty,\omega} \mathbf{N}$*;*
2. $\mathbf{M} \cong_p \mathbf{N}$*;*
3. *Player II has a winning strategy in the Ehrenfeucht-Fraisse game* $G(\mathbf{M}, \mathbf{N})$*;*
4. *There is a forcing extension* $V[G]$ *such that* $V[G] \vDash \mathbf{M} \cong \mathbf{N}$ *(i.e., the structures are isomorphic in a forcing extension).*

Proof. The equivalence of 1-3 is known as *Karp's theorem*; for a good proof see [Mar02, Theorem 2.1.4]. The equivalence with (4) is a known set-theoretic fact, mentioned in [Mar02, Exercise 2.1.8] □

This uses essentially the fact that the language $\mathcal{L}_{\infty,\omega}$ is absolute, and constructs the extension using a forcing poset consisting of a partial isomorphism system.

For the most part we will look at infinitary logics with finite quantifiers. We will also need the notion of a *fragment*:

Definition 1.2.6. Let $\mathbb{A} \subseteq \mathcal{L}_{\infty,\omega}$ be a set of formulas in the language τ such that there is an infinite set of variables V, such that if $\phi \in \mathbb{A}$ then all of its variables occur in V. We say that $\mathbb{A}$ is a *fragment* of τ if $\mathbb{A}$ satisfies the following closure properties:

1. All atomic formulas using only the constant symbols in the vocabulary τ and the variables in V are in $\mathbb{A}$;
2. $\mathbb{A}$ is closed under subformulas;
3. $\mathbb{A}$ is closed substitution of terms assembled from V: if $\phi \in \mathbb{A}$ and v is free in ϕ and t is a term with all of its variables in V, then the formula obtained by replacing all instances of v in ϕ by t is in $\mathbb{A}$;
4. $\mathbb{A}$ is closed under formal/single negations;
5. $\mathbb{A}$ is closed under $\neg, \wedge, \vee, \exists v, \forall v$ for $v \in V$.

If $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$, and $|\mathbb{A}| \leq \aleph_0$, we say that it is a *countable fragment*.

The following definition is the crucial one:

Definition 1.2.7. Let M and N be structures in a language L, and $\mathbb{A}$ is an L-fragment. We write $M \subseteq_{TV,\mathbb{A}} N$ if and only if:

1. $M \subseteq N$ and,

2. For every $\bar{a} \in M$ and every formula $\phi(y,\bar{x}) \in \mathbb{A}$, if $N \models \exists y \phi(y,\bar{a})$, then there exists some $b \in M$, such that $N \models \phi(b,\bar{a})$.

Example 1.2.8 (Models of a countable infinitary theory). Let T be a countable theory in a language $\mathcal{L}$, and let $\mathbb{A}$ be a fragment containing T. Let $K = \mathsf{Mod}(T)$. Let $M \preceq_K N$ if and only if $M \subseteq_{TV,\mathbb{A}} N$. Then $\langle K, \preceq_K \rangle$ is an abstract elementary class. The trickier parts to verify are the Lowenheim-Skolem and the union axiom; but both of these follow by the same proofs as their first-order correspondents.

The restriction to countable theories is sharp: it is not hard to find a theory T, in a countable language, of $\mathcal{L}_{\omega_1,\omega}$ which models are at least of size $2^{\aleph_0}$.

However AEC's are not at all limited to examples coming from logic. Let us see some preliminary examples, and then conclude with a wild, unexpected, example, which breathed new life to the field.

Example 1.2.9 (Noetherian Rings). Let K be the class of noetherian rings. We define $R \preceq_K S$ if and only if R is a subring of S, and $R \equiv_{\infty,\omega} S$. Note that then R is noetherian if and only if S is noetherian. To see this, note that if we assume that R is noetherian and S is not, then (by an equivalent characterization), there is $f_1, f_2, ...$, a sequence of elements such that for every integer n there is some f_i, such that f_i cannot be written in terms of the smaller elements. Then we claim that Player I has winning strategy in an unbounded Ehrenfeucht-Fraisse game: successively pick elements from that sequence. Once the game is played out, whatever Player II has chosen, say a sequence $g_1, g_2, ...$, there must be an integer n such that each g_i is a linear combination of g_k for $k \leq n$. But then this sequence cannot be isomorphic to the former.

It is clear that if $R \preceq_K S$ then R is a substructure of S, and isomorphism closure and coherence are obvious. The Tarski-Vaught axiom follows from the fact that chains of models respect the $\equiv_{\infty,\omega}$ relation. I could not prove the Lowenheim-Skolem axiom, though Grossberg's notes claim it (Shrug).

Perhaps the most striking example – and one which in part revived the interest in this topic from the point of view of mainstream mathematics – is in the work of Boris Zil'ber's "Schanuel's Structures".

Definition 1.2.10. Let $\mathcal{K}_e$ be defined as:

$$\mathcal{K}_e := \{\langle F, +, \cdot, exp \rangle : F \text{ is an alg. closed field of characteristic zero,} \\ \forall x \forall y (exp(x+y) = exp(x) \cdot exp(y))\}$$

also let

$$\mathcal{K}_{pexp} := \{\langle F, +, \cdot, exp \rangle \in \mathcal{K}_e : \ker(exp) = \pi\mathbb{Z}\}.$$

We consider the class of *Schanuel structures* to be the class $\mathcal{K}_{exp} \subseteq \mathcal{K}_{pexp}$ which satisfies some conditions, amongst them the *Schanuel condition*.

We are willfully vauge on the extra conditions, as they are not important, except for the Schanuel condition. For more information on this, check [Mar02, Chapter 8], and see also Will Boney's notes on this topic.

This essentially imposes that the so-called "Schanuel conjecture" be true:

Conjecture 1.2.11 (Schanuel,1960). Assume that $x_0, ..., x_n \in F$ are linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(x_0, ..., x_n, exp(x_0), ..., exp(x_n))$ has transcendence degree at least n over $\mathbb{Q}$.

To understand the significance of this, it should be noted that the study of transcendental numbers is one of the most unexplored and difficult areas of number theory.

Schanuel's conjecture is a piece of machinery that would clarify many difficult conjectures in transcendental number theory. As a toy example, recall that it is widely assumed that $e + \pi$ is transcendental, though no proof of it is in sight; this would immediately fall off from the above result: if we set $x_0 = 1$ and $x_1 = \pi * i$, then $\mathbb{Q}(\pi, e)$ (the result of the field extension) would have transcendence degree at least 2, showing that there is no polynomial $f(x, y)$ such that $f(\pi, e) = 0$; this implies that $e + \pi$ is transcendental.

Now what Zil'ber did was note that $\mathcal{K}_{exp}$ can be given a relation $\preceq$, forming an abstract elementary class. Additionally, using some heavy model-theoretic and number-theoretic weaponry, he managed to prove that:

Theorem 1.2.12. *The theory $\mathcal{K}_{exp}$ has a unique model of cardinality $2^{\aleph_0}$.*

Thus, the only problem lies in proving that this model is indeed the model of the complex field $\mathbb{C}$, i.e., prove that the latter has the model-theoretically desirable properties. This is an active research area today.

Such examples motivate the idea that abstract elementary classes are indeed ubiquitous, and serve as a strong foundation for exploring non-trivial solutions to mathematical problems. However, as discussed in the logical dream, just like for first-order logic, this appears as a matter of finding the right "dividing lines". Hence we can encounter our appropriate generalization of Los' conjecture:

Conjecture 1.2.13 (Shelah). Let $\mathcal{K}$ be an AEC. If there is a $\lambda \geq \beth_{2^{LS(\mathcal{K})})^+}}$ such that $\mathcal{K}$ is categorical in λ, then $\mathcal{K}$ is categorical in all μ for $\mu \geq \beth_{2^{LS(\mathcal{K})})^+}}$.

1.3 Accessible Categories

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