Regular Heyting algebras and free Heyting extensions of Boolean algebras

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2. How to study this adjunction using duality.
3. Regular Heyting algebras and Inquisitive logic.
4. Some connections with Medvedev's logic.

Heyting Algebras and Boolean algebras

Definition
An algebra $(H, \wedge, \vee, \to, 0, 1)$ is called a *Heyting algebra* if:

1. $(H, \wedge, \vee, 0, 1)$ is a (distributive) lattice.
2. The following law holds for all $a, b, c \in H$:

$$a \wedge c \leq b \iff c \leq a \to b.$$

We write $\neg a := a \to 0$. It is called a *Boolean algebra* if it satisfies:

$$\forall a \in H(a \vee \neg a = 1) \text{ or } \forall a \in H(\neg\neg a = a).$$

- **HA** – category of Heyting algebras with Heyting algebra homomorphisms.
- **BA** – (full sub)category of Boolean algebras with Boolean algebra homomorphisms.

Double Negation Translation

The double negation translation of classical logic into intuitionistic logic:

Definition
Given $\phi \in \mathcal{L}_{CPC}$ we define the double negation translation into $\mathcal{L}_{IPC}$, as follows:

1. $K(p) = \neg\neg p$ and $K(\bot) = \bot$;
2. $K(\phi \wedge \psi) = K(\phi) \wedge K(\psi)$;
3. $K(\neg\phi) = \neg K(\phi)$.

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Theorem (Glivenko,1929)
For every formula ϕ, $\phi \in \mathrm{CPC}$ if and only if $K(\phi) \in \mathrm{IPC}$.

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But who is F?

Free Heyting Extension

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No shocking results: mostly categorical housekeeping, with some logical consequences.

Heyting Extensions and Esakiafication of Stone Spaces

Esakia Duality

Our main tool will be the duality between Heyting algebras and Esakia spaces:

Definition
An ordered topological space $(X, \leq, \tau)$ is said to be a Priestley space if:

1. (X, τ) is compact;
2. Whenever $x \not\leq y$ there is a clopen upwards-closed set U such that $x \in U$ and $y \notin U$;

A Priestley space is called an Esakia space if:

3. Whenever U is a clopen set, ${\downarrow}U = \{x \in X : \exists y \in U, x \leq y\}$ is clopen.

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A continuous map $p : X \to Y$ between Esakia spaces is said to be a *p-morphism* if it is order-preserving, and whenever $p(x) \leq y$, there is some $x' \geq x$ and $p(x') = y$.

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Theorem
There is a categorical equivalence between HA^{op} *and the category* Esa *of Esakia spaces and p-morphisms, which restricts to the Stone duality of* BA^{op} *and* Stone.

Dual Constructions

To describe $F : \mathbf{BA} \to \mathbf{HA}$, we can instead describe a dual functor $M : \mathbf{Stone} \to \mathbf{Esa}$ which is adjoint to $\mathrm{Max} : \mathbf{Esa} \to \mathbf{Stone}$ (the dual functor to Reg).

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This amounts to the following:

$$\max X \xrightarrow{f} Y$$

$$X \xrightarrow{\tilde{f}} M(Y)$$

Figure 1: Adjunction Property

Example

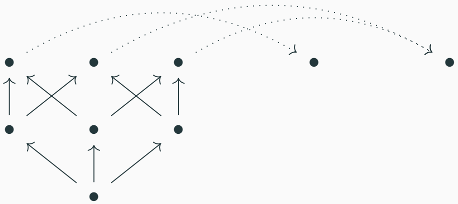

Figure 2: Example of the Problem

Inquisitive Extensions

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Then:

Proposition
If $(X, \leq)$ is a Priestley space then:

1. $(V(X), \supseteq)$ *is an Esakia space;*
2. *If X is an Esakia space and Y is a Stone space, and $f : \max(X) \to Y$ is a continuous map, there is a unique order-preserving map $\tilde{f} : X \to V(Y)$, a p-morphism on maximal elements, which agrees on f.*

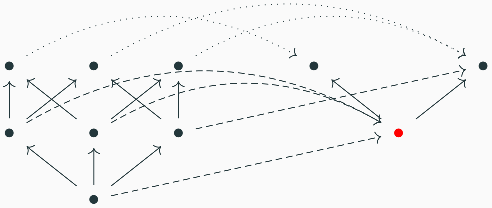

Figure 3: Back to the Example

This does not give us an adjunction because $\tilde{f}$ need not be a p-morphism. But this situation can be fixed, at a certain cost.

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Definition
Given two Priestley spaces X, Y and a continuous and order-preserving map $g : X \to Y$ between them, we say that a subset $S \subseteq X$ is *g-open* if it satisfies:

$$\forall x \in S, y \in X(x \leq y \to \exists z \in S(x \leq z \land g(z) = g(y))).$$

We denote by $V_g(X)$ the set of closed, rooted and g-open subsets of X.

Obtaining Freeness

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Obtaining Freeness (Cont.d)

Note that if $Y = \{\bullet\}$, and g is the terminal map, $V_g(X)$ is the set of all closed and rooted subsets, which we denote by $V_r(X)$. Recall that there is a map called the *root map* $r : V_g(X) \to X$ which is a surjective order preserving map.

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Proposition
Let Y be a Stone space and let $V_{\max}(Y) = \{C \in V_r(V(Y)) : \forall D \in C, \forall x \in D, \{x\} \in C\}$. Then $V_{\max}(Y)$ is a Priestley space, and the restriction $r : V_{\max}(Y) \to V(Y)$ is such that for any map $f : \max X \to Y$, and its unique lifting $\tilde{f} : X \to V(Y)$, there is a unique r-open $g_f : X \to V_{\max}(Y)$ making the diagram commute.

Obtaining Freeness (Cont.d)

Let $M_\infty(Y) = V^r_G(V_{\max}(Y))$. The latter is constructed as follows: we consider the following sequence:

$$V(Y) \xleftarrow{r_1} V_{\max}(Y) \xleftarrow{r_2} V_2(Y) \xleftarrow{r_3} \dots$$

where $V_{n+1}(Y) = V_{r_n}(V_n(Y))$, and $r_{n+1} : V_{n+1}(Y) \to V_n(Y)$ is the root map. Then $V^r_G(V_{\max}(Y))$ is the inverse limit of this sequence.

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$M_\infty(Y)$ is then an Esakia space, with the property that $\max(M_\infty(Y)) \cong Y$ through a natural isomorphism; moreover this assignment is functorial by using the functoriality of $V(-)$, $V_{\max}(-)$ and $V_G^r(-)$.

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Proposition
The functor $\mathrm{FreeM} : \mathbf{Stone} \to \mathbf{Esa}$ *assigning each Stone space* X *to* $M_\infty(X)$ *is right adjoint to* $\max : \mathbf{Esa} \to \mathbf{Stone}$.

Inquisitive Logic and Regular Heyting Algebras

Inquisitive Logic

Inquisitive Logic was introduced to study questions. In the work of Ciardelli, this has been reveleated to have intimate ties to intuitionistic logic; in the view of Bezhanishvili, Grilletti and Quadrellaro (2019), inquisitive logic can be seen as a non-standard logic extending intuitionistic logic.

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In the work above, algebraic semantics are given for inquisitive logic in the form of regular Heyting algebras:

Definition
Let H be a Heyting algebra. We say that H is *regular* if $H = \langle \mathrm{Reg}(H) \rangle$. We say that an Esakia space X is regular, if its dual Heyting algebra is regular.

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In Grilletti and Quadrellaro (2023) a study of regular Esakia spaces was carried out. One of the questions left there is whether one can describe this class in some categorically natural way. Our main result, following from the above analysis, gives an answer:

Theorem
Given a Stone space X, $M_{\infty}(X)$ is always a regular Esakia space, and moreover, regular Esakia spaces are those spaces for which the unit of the adjunction is injective. The algebras for this monad are exactly the co-freely generated regular Esakia spaces.

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WARNING: Do not get confused: these are algebras on the dual side, and coalgebras on the algebraic side.

The above categorical machinery makes it easy to adapt known tools to the study of inquisitive logic:

Definition
Given $n \in \omega$ the *n-universal regular model* if the (unique) poset $(\mathcal{R}_n, \leq)$ satisfying the following:

1. $\max(P)$ contains 2^n points.
2. For each antichain $S \subseteq R_n$ where $|S| \geq 1$, there is a unique point $x \in P$ which covers S.

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Theorem
Inquisitive logic InqL *is sound and complete with respect to the class* $\{\mathcal{R}_n : n \in \omega\}$.

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The functors in that case are:

1. Con : **ImFinPos** $\to$ **Set** the connected components functor;
2. I : **Set** $\to$ **ImFinPos** the discretization functor;
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The universal regular model as given provides the discrete analogue of the right adjoint to Max.

Definition
Let $n \in \omega$. We define $\mathrm{R}_n := \mathrm{Log}(\{M_n(B) : B \text{ is a (finite) Boolean algebra}\})$,

where $M_n(B)$ is the n-th step of the above step by step construction. Finiteness may be necessary: it is not obvious whether $M_n(B)$ is always a Heyting algebra.

We finally bring the discussion to Medevedev's Logic.

Definition
Medvedev's logic Med is the logic of the frames:

$$\{V(X) : |X| = n, n \in \omega\}.$$

With some topological arguments, it is not difficult to show:

Theorem
The logic ML *is precisely the logic of all the spaces* $V(X)$ *for* X *a Stone space; hence* $\mathsf{ML} = \mathsf{R}_0$.

It was shown by Grilletti and Quadrellaro (2023) that the logic of all n-regular algebras for any n is simply IPC. From this it easily follows that:

$$\mathrm{IPC} = \bigcap_{n \in \omega} \mathrm{R}_n.$$

One basic observation:

Proposition
The logic $\mathrm{R}_1 \neq \mathrm{R}_0$.

It would be interesting to know what the logics R_n yield, and how they can be axiomatized.

Thank you!

Questions?