Introduction to Topology in and via Logic

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Chapter 1

Introduction

Topology is the study of space as an abstract concept. Its origins lie in the 19th century, where it served an attempt to unify work from analysis, and provide a solid foundation for geometry, unifying a number of hitherto separate subjects. Thus, the modern notion of topological space appears as the culmination of a process of abstraction which started with an axiomatisation of "usual space" – what is now called Euclidean space – and now includes an unending range of applications.

Some particularly relevant connections have always existed betwen topology and its geometric intuition, and logic. In a sense, logic and geometry have a very flirtatious relationship. For instance, one sees easily a connection between logical connectives and the basic set theory operations:

Logic	**Sets**
$\Rightarrow$	$\subseteq$
$\wedge$	$\cap$
$\vee$	$\cup$
$\neg$	$(\cdot)^C$
$\bot$	$\varnothing$

This relationship is made further clear when one studies set theory, as there it becomes clear that some form of the axiom of comprehension is used to specify the basic set theoretic definitions on the basis of their logical counterparts.

However, all of this theory is developed on the back of first-order logic. Often times logicians are concerned with spicy types of logics. In this sense, topology can be thought to stand with respect to *epistemic logic* as set theory does to first-order logic.

Logic	**Geometry**
First-order logic	Set theory
Epistemic logic	Topology

In the following lecture notes we will introduce topology, keeping this connection in mind. These notes are part of a coordinated project at the ILLC, taking place in January 2023. They contain proposed exercises, as well as several propositions which are left as an exercise for the reader.[1]

[1] These lecture notes are inspired by and partly based on three sources: Steven Vickers (1989), *Topology via Logic*; James Munkres (2014), *Topology*; Ryszard Engelking (1968), *General Topology*. The former one for the epistemic intuition, and the latter two for the actual mathematical content.

In the next sections we fix some notation that will be needed throughout these notes. We assume the reader is familiar with elementary set theory, and has a passing familiarity with the real numbers.

1.1 Set-Theoretic Notation

We write ω to mean the natural numbers. We refer to sets of sets as *families* or *collections* of sets. We write $\cap$ for intersection and $\cup$ for union of sets. When necessary we also have indexed versions of these operations: if $(U_i)_{i\in I}$ is an indexed collection of sets, we write:

$$\bigcup_{i\in I} U_i \text{ and } \bigcap_{i\in I} U_i,$$

for the sets $\{x : \exists i \in I, x \in U_i\}$ and $\{x : \forall i \in I, x \in U_i\}$. We write $\bigsqcup_{i\in I} U_i$ for the *disjoint union*:

$$\bigcup \{(x,i) : x \in U_i\}$$

Given a collection of sets indexed on the natural numbers $(U_n)_{n\in\omega}$ we say that this is a *non-decreasing* (resp. non-increasing) collection if $U_n \subseteq U_{n+1}$ for each n (resp. $U_n \supseteq U_{n+1}$). We say that it is *increasing* (resp. decreasing) if the inclusion is strict.

Given a set X, we write $X \times X$ for the set of ordered pairs of elements of X, and denote its elements by (x,y) (or sometimes $\langle x,y\rangle$) where $x,y \in X$, and call this the *cartesian product.* We call a subset $R \subseteq X \times X$ a *relation.* We say that R is an *equivalence relation* if it is:

- (Reflexive): for every $x \in X$, xRx;
- (Symmetric): for every $x,y \in X$, xRy implies yRx;
- (Transitive): for every $x,y,z \in X$, xRy and yRz implies xRz.

Given an equivalence relation we write $[x]_R$ for the equivalence class of x, often dropping the subscript when it is understood. We write X/R for the quotient set, i.e., the set $\{[x]_R : x \in X\}$.

Throughout we write $\mathcal{P}(X)$ for the power set of X. We denote by $\mathcal{P}(X)^{fin}$ the set of finite subsets of the set X. We also denote by $X^{<\omega}$ the set of finite sequences of elements from X.

Given a function $f : X \to Y$ between two sets, we associate with it two natural operators:

$$\begin{aligned} \overline{f} : \mathcal{P}(X) &\to \mathcal{P}(Y) \\ A &\mapsto f[A] := \{f(w) : w \in A\} \end{aligned}$$

and

$$\begin{aligned} f^{-1} : \mathcal{P}(Y) &\to \mathcal{P}(X) \\ B &\mapsto f^{-1}[B] := \{z : f(z) \in B\}. \end{aligned}$$

We call the former the *direct image* of f and the latter the *inverse image* or *preimage* of f. We recall that the preimage interacts naturally with both unions and intersections, i.e., for each family $(U_i)_{i\in I}$ of sets:

$$f^{-1}\left[\bigcup_{i\in I} U_i\right] = \bigcup_{i\in I} f^{-1}[U_i] \text{ and } f^{-1}\left[\bigcap_{i\in I} U_i\right] = \bigcap_{i\in I} f^{-1}[U_i].$$

1.2 Partial Orders

If R is a relation on X we say that R is a *partial order* if it is reflexive, transitive and antisymmetric:

- (Antisymmetry): for every $x, y \in X$, if xRy and yRx then $x = y$.

We say that a partial order is furthermore *total* or *linear* if for all $x, y \in X$ either xRy or yRx. We normally use the symbol $\leqslant$ to mean a partial order and refer to a pair $(X, \leqslant)$ as a *partially ordered set* or poset. We often use $<$ to denote the irreflexive variant of this structure:

$$x < y \iff x \leqslant y \text{ and } x \neq y$$

Given a subset $S \subseteq X$, we say that S *has an upper bound* if there is some $z \in X$ such that for each $a \in S$, $a \leqslant z$; we say that S has a *least upper bound*, if there is some a, an upper bound, such that whenever b is an upper bound, then $a \leqslant b$. We say that X has the *least upper bound property* if every $S \subseteq X$ has a least upper bound.

Given a poset $(X, \leqslant)$, we say that a collection of elements $(x_i)_{i \in I}$ is *totally ordered* or a *chain* if for each $i, j \in I$ either $x_i \leqslant x_j$ or $x_j \leqslant x_i$. We say that an element $x \in X$ is *maximal* if for each $y \in X$ if $x \leqslant y$ then $y = x$.

Given a totally ordered set $(X, <)$, we say that this is *dense* if whenever $x < y$ there is some z such that $x < z < y$. We assume the reader is familiar with a few basic ordered sets:

- The natural numbers $(\mathbb{N}, <)$; this is countable, and has the property that every subset has a least element;
- The integers $(\mathbb{Z}, <)$;
- The rationals $(\mathbb{Q}, <)$; this is countable and dense;
- The reals $(\mathbb{R}, <)$; this is dense, and has the least upper bound property.

An important kind of ordered set we will sometimes need is the following:

Definition 1.2.1. Let $(P, <)$ be a linearly ordered set. We say that P is a *well-order* if each subset $S \subseteq P$ has a least element. $\dashv$

One can look at the isomorphism types of these well-orders and pick specific representatives; these are what we call *ordinals*. We usually denote them by Greek letters, α, β, etc. The main fact we will need about ordinals is that there exist uncountable ordinals; we will denote the least such by ω_1.

1.3 Products and the Axiom of Choice

Given a collection $(X_i)_{i \in I}$ we write $\prod_{i \in I} X_i$ for the following collection of functions:

$$\left\{ f : I \to \bigcup_{i \in I} X_i \mid \forall i \in I, f(i) \in X_i \right\}.$$

We also call this the *Cartesian product* of these sets.

We will in some specific points require forms of the *Axiom of Choice*.[2] This says the following:

Axiom 1.3.1. For every collection of sets $(X_i)_{i\in I}$, there is a function $f : I \to \bigsqcup_{i\in I} X_i$ such that for each $i \in I$, there is an $x \in X_i$ such that $f(i) = (x, i)$.

We will need in particular the following equivalent formulation of this:

Lemma 1.3.2. (Zorn's Lemma) Let $(X, \leqslant)$ be a partially ordered set. If for each chain $(x_i)_{i\in I}$, there exists some $x_0 \in X$ such that $x_i \leqslant x_0$ for each $i \in I$, then X has a maximal element.

This will prove important especially when we meet the concept of an ultrafilter, and look more broadly at compact topological spaces.

[2]Indeed, the knowing reader will note that this is needed to ensure that the above definition of cartesian product yields a non-empty collection of elements whenever the underlying sets are non-empty.

Chapter 2

Basic Definitions and Examples

The first and fundamental concept of topology is that of a topological space. It took mathematicians a good chunk of the 20th century to agree on our current standard definition: on one hand, they sought to have a definition general enough to include an array of disparate "space-like" mathematical concepts as instances of topological spaces; on the other, they wanted a definition that was restrictive enough to still capture our intuitions about what "space" is. Having the actual definition – which was only settled upon after decades of inquiry – simply presented to you, it might appear awfully abstract and arbitrary. However, as we will argue throughout, such a notion can be given a rich epistemic meaning, which can also help us understand how it relates to these various notions of space.

Definition 2.0.1. Let X be a set. We say that a collection of subsets $\tau \subseteq \mathcal{P}(X)$ is a *topology on* X if it satisfies the following conditions:

(O1) $\varnothing$ and X are in τ; i.e., $\varnothing \in \tau$ and $X \in \tau$.

(O2) τ is closed under *arbitrary* unions; i.e., if $(U_i)_{i \in I}$ is a collection of sets in τ, then $\bigcup_{i \in I} U_i \in \tau$.

(O3) τ is closed under *finite* intersections; i.e., if $\{U_1, ..., U_n\} \subseteq \tau$, then $(U_1 \cap \cdots \cap U_n) \in \tau$.

Given a set and a topology τ on X, we say that the pair (X, τ) is a *topological space*. When no confusion can arise, we will simply say that X is a topological space. We call the elements $x \in X$ *points* and say that a set $U \in \mathcal{P}(X)$ is *open* if $U \in \tau$. $\dashv$

2.1 Topology and Logic: an intuition

As mentioned, to aid our intuitions, we will develop an informal epistemic interpretation of topological spaces (X, τ). To do so, we must interpret such sets X and topologies τ on X so that the three conditions, (O1)-(O3), are fulfilled:

(X) We think of X as a set of 'epistemic worlds'.

(τ) We think of $\tau \subseteq \mathcal{P}(X)$ as corresponding to the set of 'verifiable propositions', under a given epistemological framework (i.e., a way to determine what can be verified/falsified).

Before checking that the three conditions are satisfied under this interpretation (i.e., that $\varnothing, X$ are verifiable propositions, and that verifiable propositions are closed under arbitrary unions and finite intersections), let us be a bit more precise on what we mean by a 'verifiable proposition'.

First, for a proposition P, we identify it with the set of epistemic worlds of X in which P is true; that is,

$$P \mapsto [\![P]\!] = \{x \in X \mid P \textit{ is true in the world } x\} \subseteq X.$$

This explains why τ (being a set of subsets of X) can be thought of as a set of propositions. As an interlude, note that under this identification disjunction corresponds to union; conjunction to intersection; and negation to set-theoretical complement (recall the table in the introduction):

$$\begin{aligned}
(P \vee Q) \quad &\mapsto \quad [\![P \vee Q]\!] &&= \{x \in X \mid P \vee Q \textit{ is true in the world } x\} \\
& &&= \{x \in X \mid P \textit{ is true in the world } x\} \cup \{x \in X \mid Q \textit{ is true in the world } x\} \\
& &&= [\![P]\!] \cup [\![Q]\!]; \\
(P \wedge Q) \quad &\mapsto \quad [\![P \wedge Q]\!] &&= \{x \in X \mid P \wedge Q \textit{ is true in the world } x\} = [\![P]\!] \cap [\![Q]\!]; \\
(\neg P) \quad &\mapsto \quad [\![\neg P]\!] &&= \{x \in X \mid \neg P \textit{ is true in the world } x\} = X - [\![P]\!].
\end{aligned}$$

In the infinitary case – with disjunctions of the form $\bigvee_{i \in I} P_i$ – the exact same reasoning applies. Thus, under our epistemic interpretation, condition (O2) becomes closure of verifiable propositions under arbitrary *disjunctions*, and (O3) becomes closure of verifiable propositions under finite *conjunctions*.

Second, to explain what we mean by 'verifiable', consider the proposition

$(\exists\neg WS)$ There is a non-white swan.

and its negation

$(\forall WS)$ All swans are white.

Now, if we were to observe a black swan, we would *verify* $(\exists\neg WS)$ (and *falsify* $(\forall WS)$). In contrast, it does not seem a priori possible to verify the proposition $(\forall WS)$, or, equivalently, falsify the proposition $(\exists\neg WS)$. No matter how many white swans we come across, we cannot be sure what colour the next one will be – this is the (in)famous problem of induction. In our words, $(\exists\neg WS)$ is verifiable, while $(\forall WS)$ is not. Spelling it out, we say:

(ver) A proposition P is *verifiable* if and only if: whenever P is true at a world x (i.e., $x \in [\![P]\!]$), it is possible to verify P at x (i.e., verify $x \in [\![P]\!]$).

And dually:

(fal) A proposition P is *falsifiable* if and only if: whenever P is false at a world x (i.e., $x \notin [\![P]\!]$), it is possible to falsify P at x (i.e., falsify $x \in [\![P]\!]$).[1]

With this terminology defined, let us check that (O1)-(O3) are satisfied under our epistemic interpretation:

(O1) We first check that $X \in \tau$ is sensible given our interpretatio, that is, that X corresponds to a verifiable proposition. Since

$$\top \mapsto [\![\top]\!] = \{x \in X \mid \top \textit{ is true in the world } x\} = X,$$

we find that the full set X corresponds to the proposition of logical truth. And since logical truth can always be verified (our domain being X we know by default that $x \in X$), X corresponds to a verifiable proposition.

Next, we check that $\varnothing \in \tau$, that is, that $\varnothing$ corresponds to a verifiable proposition. Since

$$\perp \mapsto [\![\perp]\!] = \{x \in X \mid \perp \textit{ is true in the world } x\} = \varnothing,$$

we find that the empty set $\varnothing$ corresponds to falsity. Since falsity is never true, it is vacuously verifiable (note the clause 'whenever P is true at a world x [...]' in the definition of (ver)).

(O2) We have to check that verifiable propositions are closed under arbitrary disjunctions. Accordingly, suppose that $(U_i)_{i \in I} \subseteq \tau$, where each U_i corresponds to a verifiable proposition P_i. We will show that $\bigvee_{i \in I} P_i$ is verifiable. So suppose $\bigvee_{i \in I} P_i$ is true at a world x. Then there must be some $i \in I$ such that P_i is true at x. Now since P_i is verifiable, it must be possible to verify P_i at x. But then it is possible to verify $\bigvee_{i \in I} P_i$ at x, namely by verifiying P_i – showing that $\bigvee_{i \in I} P_i$ is verifiable, as required.

(O3) We have to check that verifiable propositions are closed under finite conjunctions. Accordingly, suppose that $\{U_1, \ldots, U_n\} \subseteq \tau$. Then these correspond to verifiable propositions $P_1, \ldots, P_n$. We have to show that $(P_1 \wedge \cdots \wedge P_n)$ is verifiable. So suppose $(P_1 \wedge \cdots \wedge P_n)$ is true at a world x. Then each of the P_is are true, and since they all are verifiable, they are all possible to verify at x. But then it is possible to verify $(P_1 \wedge \cdots \wedge P_n)$ at x, namely by verifying $P_1, \ldots, P_n$. Thus, $(P_1 \wedge \cdots \wedge P_n)$ is verifiable, as desired.

Example 2.1.1. Let us make the former more concrete by formalising the example of swans. If our intuitions are to hold water, we should end up with a topological space.

We take the setting to consist of two possible worlds, namely

1. A world a in which all swans are white; and
2. a world b in which there is at least one non-white swan.

[1]Notice the similarity, almost to the extent of paraphrasing, to Karl Popper's criterion of falsifiability.

Then $X = \{a, b\}$. We have four subsets of X:

- $\varnothing$, which corresponds to the verifiable proposition $\perp$, hence it is open.
- X, which corresponds to the verifiable proposition $\top$, hence it is open.
- $\{a\}$, which corresponds to the proposition $(\forall WS)$ which is not verifiable, hence it is not open.
- $\{b\}$, which corresponds to the verifiable proposition $(\exists\neg WS)$, hence it is open.

Thus, we get the collection $\tau = \{\varnothing, X, \{b\}\}$, which is indeed a topology on X:

(O1) We have that $\varnothing \in \tau \ni X$.

(O2) Clearly, any union of elements of τ equals either $\varnothing, X$ or $\{b\}$, which all are in τ.

(O3) Clearly, any finite (or infinite) intersection of elements of τ equals either $\varnothing, X$ or $\{b\}$, which all are in τ. $\dashv$

While we have argued how an epistemic interpretation can make sense of the requirements (O1)-(O3), one can wonder whether the asymmetries in the definition of topological spaces can be made more symmetric.

- **Closure under arbitrary intersections:** It might seem arbitrary that we only require closure under *finite* intersections and not under *arbitrary* intersections (unlike how we require closure under arbitrary unions). So suppose there are countably infinitely many swans, one for each natural number $n \in \omega$, and let P_n be the proposition that swan number n is white. Whilst it seems reasonable to say that each P_n is verifiable (we can simply go check swan number n), it appears less reasonable to say their conjunction

$$\bigwedge_{n\in\omega} P_n$$

  is: we would have to check all *infinitely* many of the swans. In a nutshell, the salient distinction is that while infinite disjunctions can be verified by finite information (one disjunct is enough), infinite conjunctions cannot (all conjuncts are required). Notice that it is not that a topology may not be closed under arbitrary intersections, but simply that we do not require it to be so. If it is, we call it an *Alexandroff topology.*

- **Closure under complements:** Whilst in some cases both a proposition and its negation might be verifiable, it seems equally reasonable that this does not always hold, as exemplified by the case of swans above: $(\exists\neg W)$ is reasonably said to be verifiable, while its negation $(\forall WS)$ is not.

To summarize, (O1)-(O3) seem to model closure conditions of verifiable propositions fairly well, and throughout the project we will be using this epistemic interpretation to gain intuition for various topological definitions, notions and concepts.

Logic	**Topology**
Epistemic worlds/situations/models/objects satisfying a property	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$

2.2 (Sub)basis and Examples

We have seen how the definition of a topological space formalises our intuition of verifiable propositions. However, once we have the abstract definition, we can use it in a number of different settings. To get a feel for this, let us look at a few more examples of topological spaces.

Example 2.2.1. Let $X = \{x, y, z\}$, and consider the following subsets of $\mathcal{P}(X)$:

$$\tau_1 = \{\varnothing, X, \{x\}\},\ \tau_2 = \{\varnothing, X, \{x\}, \{x, y\}\},\ \tau_3 = \{\varnothing, X, \{x\}, \{y\}\},\ \text{and } \tau_4 = \{\varnothing, \{x\}\}.$$

Then τ_1 and τ_2 are both topologies on X (check this); however, τ_3 is not, nor is τ_4. τ_3 – although satisfying (O1) and (O3) – is not closed under unions:

$$\{x\} \in \tau_3 \ni \{y\},\ \text{but } \{x\} \cup \{y\} = \{x, y\} \notin \tau_3.$$

And τ_4 fails to satisfy (O1) because $X \notin \tau_4$. ⊣

Example 2.2.2. (Discrete and Indiscrete Topology)
Fix a set X. Then the collection $\{\varnothing, X\}$ satisfies the axioms of a topology; it is called the *indiscrete topology*. Similarly, the collection $\mathcal{P}(X)$ also satisfies the axioms of a topology, and it is called the *discrete topology*. By definition, all topologies contain (in terms of inclusion) the indiscrete topology, and are contained in the discrete topology; that is, if τ is a topology on X, then $\{\varnothing, X\} \subseteq \tau \subseteq \mathcal{P}(X)$. ⊣

Example 2.2.3. (Kripke frames)
Let $\mathfrak{F} = (W, R)$ be a reflexive and transitive Kripke frame. Then W can be given a topology, called the *Alexandroff topology* on the preorder $\mathfrak{F}$, in the following way: $U \subseteq W$ is open if and only if

$$\forall x, y \in W\, [(x \in U \text{ and } xRy) \implies y \in U]\,.$$

In other words, the open sets are precisely the *R-upsets* (check that this forms a topology). Notice that this topology has the property that arbitrary intersections of opens are again open, hence the name "Alexandroff topology". ⊣

In each of the previous examples, we defined the topological space (X, τ) by explicitly specifying all open sets (i.e., all members of τ). Sometimes this is unwieldy, and it is more practical to only specify a subcollection of τ from which we can *generate* the entire topology:

Definition 2.2.4. (Basis and Subbasis) Let (X, τ) be a topological space. We say that $\mathcal{B} \subseteq \tau$ is a *basis for the topology* τ if for each $U \in \tau$ there is a collection $(V_i)_{i \in I} \subseteq \mathcal{B}$ such that

$$U = \bigcup_{i \in I} V_i.$$

We say that $\mathcal{S} \subseteq \tau$ is a *subbasis for the topology* if the set of finite intersections of elements from $\mathcal{S}$

$$\left\{ \bigcap_{V \in M} V \mid M \subseteq \mathcal{S}, M \textit{ is finite} \right\} [2]$$

forms a basis for the topology.

Moreover, given a (sub)basis $\mathcal{B} \subseteq \tau$, we call members $U \in \mathcal{B}$ *(sub)basic opens*. ⊣

[2] We adopt the convention that the nullary intersection is the full set; i.e., $\bigcap_{V \in \varnothing} V = X$. And similarly that $\bigcup_{V \in \varnothing} V = \varnothing$, and correspondingly for propositions in logic: $\bigwedge_{P \in \varnothing} P = \top$ and $\bigvee_{P \in \varnothing} P = \bot$.

We can express the relationships between these concepts in various way: a topology is a basis closed under arbitrary unions, and closing a subbasis under finite intersections one obtains a basis.[3] We also note the following easy but important facts:

Proposition 2.2.5. Let X be a set and $C \subseteq \mathcal{P}(X)$ a collection of sets. Then there is a (unique) topology on X for which C is a subbasis.

Moreover, if (1) C covers X (i.e., $\bigcup_{U \in C} U = X$) and (2) C is closed under binary intersections, then there is a (unique) topology on X for which C is a basis.[4]

Proof. For the former claim, first close C under finite intersections, and then close the resulting collection of sets under arbitrary unions. This will yield a topology (check this). Uniqueness is left as an exercise.

For the latter, close C under arbitrary unions. This will yield a topology (check this). Uniqueness is left as an exercise. ■

Remark 2.2.1. (*Epistemic intuition: what is a (sub)basis?*) Given a topological space (X, τ), we have seen that the topology τ can be thought of as the set of verifiable propositions on the set of epistemic worlds X.

What about $\mathcal{B}$ then, for $\mathcal{B}$ a basis for τ? Since $\mathcal{B} \subseteq \tau$, it too can be thought of as a set of verifiable propositions. And since τ is generated by closing $\mathcal{B}$ under arbitrary unions, we can think of $\mathcal{B}$ as a basis of verifiable propositions from which all other verifiable propositions can be inferred by a form of weakening, namely, by forming disjunctions of these basic propositions.

Similarly, for $\mathcal{S}$ a subbasis for τ, we can think of $\mathcal{S}$ as a subbasis of verifiable propositions from which all other verifiable propositions can be generated by inference, through both strengthening pieces of knowledge (conjunctions) and weakening (disjunctions).

Example 2.2.6. Let $X = \{a, b, c, d\}$ where a, b, c, d are worlds described as follows:

	All ravens are black	Some raven is non-black
All swans are white	a	b
Some swan is non-white	c	d

I.e., the world c, for instance, is a world in which all ravens are black and with some non-white swan.

If we want to specify the verifiable propositions in this setting, instead of listing all of them, we could simply say that our subbasic verifiable propositions are $(\exists \neg WS)$ and the analogous $(\exists \neg BR)$. That is, all verifiable propositions are obtained by first taking (finite) conjunctions of these two propositions, and then combining them in disjunctions. Doing so corresponds to having the subbasis

$$\mathcal{S} = \{\{c, d\}, \{b, d\}\},$$

which generates the topology

$$\tau = \{\varnothing, X, \{c, d\}, \{b, d\}, \{d\}, \{b, c, d\}\}$$

on X – exactly corresponding to the set of verifiable propositions (check this). ⊣

[3] However, it should be noted that a basis need not be closed under intersection.

[4] In fact, '(2)' can be weakened to: for all $x \in X$, if $x \in (U_1 \cap U_2)$ for some $\{U_1, U_2\} \subseteq C$, then there is some $U_3 \in C$ such that $x \in U_3 \subseteq (U_1 \cap U_2)$.

Example 2.2.7. (Real Line Topology)
When first learning mathematics in school, one is often introduced to the real numbers $\mathbb{R}$ and basic properties of these: they are ordered, and one can then look at the so-called "open intervals", typically written as

$$(x, y) = \{z \in \mathbb{R} : x < z < y\} \text{ where } x, y \in \mathbb{R}.$$

The choice of terminology is not a coincidence: the open intervals form a basis for the *Euclidean topology* on $\mathbb{R}$ (check this), in which a set $U \subseteq \mathbb{R}$ is open if and only if,

$$\forall z \in U \exists x, y \in U (x < z < y). \qquad \dashv$$

Example 2.2.8. (Cantor and Baire Spaces)
Consider the set 2^ω of infinite binary sequences. We call this the *Cantor set*. Given a finite sequence $s \in 2^{<\omega}$, and a finite or infinite sequence $t \in 2^{<\omega} \cup 2^\omega$ we write $s \lhd t$ to mean that s is an initial subsequence of t. Given any $s \in 2^{<\omega}$ we can consider the following set:

$$C(s) = \{x \in 2^\omega : s \lhd x\}.$$

Now consider the collection of the sets of the form $\{C(s) : s \in 2^{<\omega}\}$. You can check that this (1) covers 2^ω, and (2) is closed under binary intersections, and hence, defines a basis for a topology (cf. Proposition 2.2.5) which we call the *Cantor space*.

Similarly, we topologise ω^ω with a basis of sets $C(s) = \{x \in \omega^\omega : s \lhd x\}$. We call this latter space the *Baire space*. $\dashv$

The concepts of basis and subbasis are thus instrumental when working with topological spaces. In fact, when proving a topological statement, it is often enough to only consider *basic opens*. An instance of this occurs when *comparing topologies*, which can be used to show that two topologies are the same:

Definition 2.2.9. Let X be a set, and τ and τ' two topologies on this set. We say that τ is a *coarser topology* than τ' if $\tau \subseteq \tau'$. Conversely, we say that τ' is *finer* than τ. $\dashv$

Lemma 2.2.10. Suppose X is a set with two topologies τ and τ', and $\mathcal{B}_\tau$ and $\mathcal{B}_{\tau'}$ are bases for these topologies, respectively. Then $\tau \subseteq \tau'$ holds if and only if for all points $x \in X$ and all basic τ-open $U \in \mathcal{B}_\tau$ containing x, there is some basic τ'-open $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$.

Proof. ($\Rightarrow$) Let $x \in X$ and $x \in U \in \mathcal{B}_\tau$ be arbitrary. We are then to find a $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$. Since $U \in \mathcal{B}_\tau$, we have that $U \in \tau$, so since $\tau \subseteq \tau'$ by assumption, we also have that $U \in \tau'$. Now since τ' is generated by $\mathcal{B}_{\tau'}$, U must be the union of some elements from $\mathcal{B}_{\tau'}$. And since $x \in U$, one of these must contain x, hence we have our $U' \in \mathcal{B}_{\tau'}$ such that $x \in U' \subseteq U$, as required.

($\Leftarrow$) Let $U \in \tau$ be arbitrary. We are then to show that $U \in \tau'$. For each $x \in U$, fix some $U_x \in \mathcal{B}_\tau$ such that $x \in U_x \subseteq U$ (such U_x must exist, since otherwise U would not be the union of elements from $\mathcal{B}_\tau$). By assumption, for each of these, there must be some $U'_x \in \mathcal{B}_{\tau'}$ such that $x \in U'_x \subseteq U_x$. But then

$$U \subseteq \bigcup_{x \in U} U'_x \subseteq \bigcup_{x \in U} U_x \subseteq \bigcup_{x \in U} U = U,$$

which shows that

$$U = \bigcup_{x \in U} U'_x.$$

But $(\bigcup_{x \in U} U'_x) \in \tau'$ because each $U'_x \in \tau'$, hence $U \in \tau'$, as desired. ■

Example 2.2.11. Consider the sets of the form

$$(l, \infty) = \{z \in \mathbb{R} \mid x < z\}.$$

These (1) cover $\mathbb{R}$, and (2) are closed under binary intersections, hence form a basis for a topology τ_F on $\mathbb{R}$. Using the above proposition, we show that $\tau_F \subsetneq \tau_E$, where τ_E is the Euclidean topology on $\mathbb{R}$.

($\subseteq$) Let $x, l \in \mathbb{R}$ be arbitrary such that $x \in (l, \infty)$. By the above proposition, it then suffices to find an open interval $(a, b) \subseteq \mathbb{R}$ such that $x \in (a, b) \subseteq (l, \infty)$. However, this is easy: setting, e.g., $a = l$ and $b = x + 1$ does the job.

($\not\supseteq$) By the above proposition, it suffices to find some $x \in \mathbb{R}$ and open interval $x \in (a, b) \subseteq \mathbb{R}$ such that there is no $l \in \mathbb{R}$ for which $x \in (l, \infty) \subseteq (a, b)$. Again, this is easy: set, e.g., $x = 0$, $a = -1$ and $b = 1$. Then $x \in (a, b)$, and for no $l \in \mathbb{R}$, do we have $(l, \infty) \subseteq (a, b)$. $\dashv$

2.3 Generating New Topologies

Often we are interested in getting new topologies from existing ones. In this course we will encounter several such procedures. The most elementary is to generate a topological space from a subset of an already existing topological space:

Definition 2.3.1. Let (X, τ) be a topological space and $S \subseteq X$ a subset. We denote by τ_S the *subspace topology on* S defined as

$$\tau_S := \{U \cap S \mid U \in \tau\}.$$

I.e., the τ_S-open sets are our original τ-open sets restricted to the subset S by means of intersection. We then say that (S, τ_S) is a *subspace* of (X, τ) (you should check that (S, τ_S), indeed, is a topological space). $\dashv$

Subspaces can in fact be given by only looking at a basis for the original space:

Lemma 2.3.2. Let (X, τ) be a topological space with a basis $\mathcal{B}$, and let $S \subseteq X$. Then the set

$$\mathcal{B}_S = \{U \cap S : U \in \mathcal{B}\}$$

is a basis for τ_S.

Proof. To see this, let $T \subseteq S$ be open in the subspace topology. Then by definition, $T = V \cap S$ for some V open in X. Hence, because $\mathcal{B}$ is a basis, we have that $V = \bigcup_{i \in I} U_i$, where the $U_i \in \mathcal{B}$. Thus, putting this together

$$T = V \cap S = \left(\bigcup_{i \in I} U_i \right) \cap S = \bigcup_{i \in I} (U_i \cap S),$$

which was to show. ■

Example 2.3.3. Consider $\mathbb{R}$ the reals, and look at $\mathbb{Z}$ with the subspace topology.[5] We claim that the latter coincides with the discrete topology. Indeed, for any $n \in \mathbb{Z}$, we can consider the interval

$$(n - \frac{1}{2}, n + \frac{1}{2}).$$

This is open in $\mathbb{R}$, hence its intersection with $\mathbb{Z}$ – which is the singleton $\{n\}$ – will be open in the subspace topology on $\mathbb{Z}$. Thus, every singleton is open in the subspace, which implies that every subset is open (because topologies, in particular, are closed under arbitrary unions).

Question: is $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) also discrete? ⊣

Also interesting is the following very frequent construction:

Definition 2.3.4. Let X and Y be topological spaces. We define a topology on the product $X \times Y$, called the *product topology*, as follows: a set $U_0 \times U_1 \subseteq X \times Y$ is basic open if and only if U_0 is open in X and U_1 is open in Y (you should check that this, indeed, defines a basis for a topology on $X \times Y$, cf. Proposition 2.2.5). ⊣

With subspace and product topologies introduced, we prove the following proposition as a sanity check:

Proposition 2.3.5. Suppose X and Y are topological spaces and that $S_X \subseteq X$ and $S_Y \subseteq Y$. Then first constructing the product topology $X \times Y$ and then constructing the subspace topology $S_X \times S_Y \subseteq X \times Y$ is the same as first constructing the subspace topologies $S_X \subseteq X$ and $S_Y \subseteq Y$ and then taking their product $S_X \times S_Y$ (we say the constructions *commute*).

Proof. For the topology obtained by the former sequence of constructions, a basic open is of the form $(U_X \times U_Y) \cap (S_X \times S_Y)$ for U_X open in X and U_Y open in Y. And for the topology obtained by the latter sequence of constructions, a basic open is of the form $(U_X \cap S_X) \times (U_Y \cap S_Y)$ for U_X open in X and U_Y open in Y. So since

$$(U_X \times U_Y) \cap (S_X \times S_Y) = (U_X \cap S_X) \times (U_Y \cap S_Y),$$

the bases are the same, hence the topologies are the same. ■

Naturally, to define the product topology, one can also just take the bases to generate a new basis:

Lemma 2.3.6. Let X and Y be topological spaces with bases $\mathcal{B}_X$ and $\mathcal{B}_Y$, respectively. Then

$$\{U_X \times U_Y \mid U_X \in \mathcal{B}_X, U_Y \in \mathcal{B}_Y\}$$

forms a basis for the product topology on $X \times Y$.

Proof. Exercise. ■

The product topology construction easily generalises to any *finite* product, and leads to many usual spaces:

[5] When nothing else is mentioned, we take the topology on the reals to be the Euclidean topology. This topology is also known as the *standard topology* on $\mathbb{R}$.

Example 2.3.7. Let $\mathbb{R}^n$ be the set of n-dimensional tuples of reals. The n-dimensional topology on this set is given by the basis of n-dimensional balls; i.e., the basis consists of all sets of the form

$$B_\varepsilon(\overline{x}) := \left\{\langle y_1, \dots, y_n\rangle \in \mathbb{R}^n \mid \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2} < \varepsilon\right\}$$

where $\overline{x} = \langle x_1, \dots, x_n\rangle \in \mathbb{R}^n$ and $\varepsilon > 0$.

We can show that this topology is the same as the product topology of $\mathbb{R}$ with the usual Euclidean topology, n many times. For simplicity we do this for $n = 2$, though the argument is analogous for higher dimensions.

We will show equality of the topologies going inclusion by inclusion using Lemma 2.2.10. First, let $B_{\varepsilon'}(y_1, y_2)$ and $\langle x_1, x_2\rangle \in B_{\varepsilon'}(y_1, y_2)$ be arbitrary, and set $\varepsilon := \varepsilon' - \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$. Then $\varepsilon > 0$ and $\langle x_1, x_2\rangle \in B_\varepsilon(x_1, x_2) \subseteq B_{\varepsilon'}(y_1, y_2)$, hence – for one inclusion – it suffices to find some open intervals (a, b) and (c, d) such that $\langle x_1, x_2\rangle \in (a, b) \times (c, d) \subseteq B_\varepsilon(x_1, x_2)$ (because $(a, b) \times (c, d)$ is a basic open in the product topology, cf. the preceding lemma). So consider the following choice of intervals:

$$(x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}) \times (x_2 - \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2})$$

Clearly, $\langle x_1, x_2\rangle \in (x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}) \times (x_2 - \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2})$. Now let

$$(z_1, z_2) \in (x_1 - \frac{\varepsilon}{2}, x_1 + \frac{\varepsilon}{2}) \times (x_2 - \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2})$$

be arbitrary; for this inclusion of topologies, it then suffices to show that $\langle z_1, z_2\rangle \in B_\varepsilon(x_1, x_2)$. Observe that $|z_i - x_i| \leqslant \frac{\varepsilon}{2}$. Further, we have that the following inequality holds for all $\langle z_1, z_2\rangle \in \mathbb{R}^2$ (and all $\langle x_1, x_2\rangle \in \mathbb{R}^2$):

$$\sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2} \leqslant |z_1 - x_1| + |z_2 - x_2|.$$

Thus,

$$\sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that $\langle z_1, z_2\rangle \in B_\varepsilon(x_1, x_2)$, as required.

Second, for the other inclusion of topologies, let $(x_1, x_2) \times (y_1, y_2)$ and $\langle x, y\rangle \in (x_1, x_2) \times (y_1, y_2)$ be arbitrary. We can think of $(x_1, x_2) \times (y_1, y_2)$ as an (open) rectangle. Since the point $\langle x, y\rangle$ is in this rectangle, we can consider an open ball around $\langle x, y\rangle$, that is small enough that it is entirely contained in the rectangle. For concreteness, we can set $\varepsilon := \min\{|x - x_1|, |x - x_2|, |y - y_1|, |y - y_2|\}$, the minimal distance from the point to the edges of the rectangle. Then

$$\langle x, y\rangle \in B_\varepsilon(x, y) \subseteq (x_1, x_2) \times (y_1, y_2),$$

which proves the claim. $\dashv$

So far we have only discussed the *finite* product topology, but there is, of course, also a product topology simpliciter which generalises to cover the infinite case as well. The reason we discussed the finite product topology separately, is that the product topology in general is quite complex, as we will now see:

Definition 2.3.8. Let $(X_i)_{i \in I}$ be a collection of topological spaces. We define a topology on their product $\prod_{i \in I} X_i$ by saying that a set

$$U = \prod_{i \in I} U_i \subseteq \prod_{i \in I} X_i$$

is *basic open* if (1) all the $U_i \subseteq X_i$ are open (in X_i) and (2) for all but finitely many coordinates, do we have $U_i = X_i$ (you should check that this, indeed, defines a basis for a topology on $\prod_{i \in I} X_i$, cf. Proposition 2.2.5). Equivalently, we say that a set U is a subbasic open if it is basic open and *on all but one* position i it is equal to X_i. $\dashv$

Notice that this definition does not reduce down to a set being open if it is the product of infinitely many open sets. This condition is crucial to ensure that many properties are preserved under products, as we will see later.

Example 2.3.9. Recall the Cantor space 2^ω. We can prove that the topology on this set is the product topology of ω many copies of $\mathbf{2} = \{0, 1\}$, the two element set with the discrete topology. To do so, we, as usual, go inclusion by inclusion using Lemma 2.2.10.

Accordingly, first, suppose $(x_n)_{n \in \omega} \in U$ for some basic open $U = \prod_{n \in \omega} U_n$ in the product topology. By definition, we then know that for all but finitely many coordinates, $U_n = \{0, 1\}$. Hence there must be some greatest natural number m such that $U_m \neq \{0, 1\}$ (or in case $U_n = \{0, 1\}$ for all $n \in \omega$, we set $m := 0$). Then $(x_i)_{i \leqslant m} \lhd (x_i)_{i \in \omega}$ and $(x_i)_{i \in \omega} \in C((x_i)_{i \leqslant m}) \subseteq U$, which shows the first inclusion.

Second, for the other inclusion, it suffices to show that any $C(s)$ is a basic open in the product topology. Since s is a finite sequence it is of the form $(x_i)_{i \leqslant m}$ for some $m \in \omega$ and $x_0, \dots, x_m$, hence we get that

$$C(s) = C((x_i)_{i \leqslant m}) = \{x_0\} \times \cdots \times \{x_m\} \times \prod_{n > m} \{0, 1\},$$

which, indeed, is a basic open in the product topology. This shows the other inclusion, and thus, concludes the proof of the two topologies coinciding. $\dashv$

We now take a look at the *topological sum*:

Definition 2.3.10. Let $(X_i)_{i \in I}$ be a collection of topological spaces. We define the *topological sum* of this collection to be their disjoint union $\bigsqcup_{i \in I} X_i$ endowed with the following topology: $U \subseteq \bigsqcup_{i \in I} X_i$ is open if and only if for each $i \in I$, the following set is open in X_i:

$$U(i) := \{a \in X_i \mid (a, i) \in U\}.$$

(As always, you should check that this actually does define a topological space). $\dashv$

One can prove that a collection of bases for each X_i induces a basis for their topological sum:

Lemma 2.3.11. Let $(X_i)_{i \in I}$ be a collection of topological spaces, and let $(\mathcal{B}_i)_{i \in I}$ be a collection of bases for each of these spaces. Then the set

$$\mathcal{B} = \left\{ Z \subseteq \bigsqcup_{i \in I} X_i \mid Z(i) \in \mathcal{B}_i \right\}$$

is a basis for the topological sum of the spaces.

Proof. Exercise. ■

Example 2.3.12. Suppose $(\mathfrak{F}_i)_{i \in I}$ is a collection of transitive and reflexive Kripke frames. Then their disjoint union is also a transitive and reflexive Kripke frame, hence it admits an Alexandroff topology as per Example 2.2.3. We can show that the topological space obtained in this way is precisely the topological sum of the Alexandroff spaces $(\mathfrak{F}_i)_{i \in I}$. ⊣

2.4 Closed sets, Neighbourhoods, Closure and Interior Operators

Having seen some examples of topological spaces and ways of constructing them, we close off the chapter by covering some important notions related to topological spaces, beginning with the notion of a *closed set*:

Definition 2.4.1. Let (X, τ) be a topological space. We say that a set $U \in \mathcal{P}(X)$ is *closed* if its complement is open; i.e., if $(X - U) \in \tau$. ⊣

An immediate consequence of this definition and our definition of a topological space is the following:

Proposition 2.4.2. Let (X, τ) be a topological space. Then:

(C1) X and $\varnothing$ are closed sets.

(C2) Arbitrary intersections of closed sets are closed; i.e., if $(U_i)_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} U_i$ is a closed set.

(C3) Finite unions of closed sets are closed; i.e., if $U_1, \ldots, U_n$ are closed, then so is $(U_1 \cup \cdots \cup U_n)$.

Proof. Follows from the set-theoretical complement operator taking unions to intersections (and vice versa). ■

In fact, conditions (C1)-(C3) work as an equivalent definition of a topological space: specifying a collection of subsets satisfying (C1)-(C3) (which we call closed), determines a collection of open sets – namely the complements of the closed ones – which satisfies (O1)-(O3).

Remark 2.4.1. (Epistemic intuition: what are the closed sets?) We know that, given a topological space (X, τ), members of τ are called open and that these can be thought of as verifiable propositions. We also now know that the complement $(X - U)$ of an open set $U \in \tau$ is called closed and corresponds to the negation of a verifiable proposition. But, as previously argued, the negation of a verifiable proposition is not always verifiable (recall $(\exists \neg WS)$ and its negation $(\forall WS)$). However, the negation of a verifiable proposition is always falsifiable; e.g., to falsify $(\forall WS)$, it suffices to show the existence of a, say, black swan. That is, we can think of closed sets as falsifiable propositions.

Remark 2.4.2. The reader familiar with contemporary service work will be aware of the notion of a "clopen". Just as in this case, in topology the concept of being closed and open are not mutually exclusive nor exhaustive. One can have sets, just like the whole space and the empty set, which are both open and closed – these are often shortened to *clopen sets*. And one can have sets which are neither (see Exercise 2.2).

Our intuition of open sets as verifiable propositions and closed sets as falsifiable propositions can help us make sense of this: The proposition

It is raining outside.

can reasonably be said to be both verifiable and falsifiable (I can simply go out and check); while the proposition

John F. Kennedy's last thought was "What is the One True Logic?"

neither seems verifiable nor falsifiable. ⊣

	Verifiable (open)	Falsifiable (closed)
All swans are white		x
Some swan is non-white	x	
It is raining outside	x	x
JFK's last thought was "What is the OTL?"		

Recall that all opens of a subspace S of X are of the form $(U \cap S)$ for U some open in X. An analogous result holds for closed sets in subspace topologies:

Lemma 2.4.3. Suppose (S, τ_S) is a subspace of (X, τ). Then a set $U \in \mathcal{P}(S)$ is closed in S if and only if there is some closed set V in X (i.e., $(X - V) \in \tau$) such that $U = V \cap S$.

Proof. Exercise. ■

Definition 2.4.4. Let (X, τ) be a topological space and $S \subseteq X$ arbitrary. We denote by $cl(S)$ or $\overline{S}$ the *closure of* S, the smallest closed set K such that $S \subseteq K$; that is, $cl(S)$ is the intersection of all closed sets containing S. We denote by $int(S)$ the *interior of* S, the largest open set K such that $K \subseteq S$; that is, $int(S)$ is the union of all open sets contained in S. ⊣

Remark 2.4.3. Using this definition, we have that a set S is closed if and only if $S = \overline{S}$, and open if and only if $S = int(S)$. We call the operators

$$int : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto int(S)$$

and

$$cl : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto cl(S)$$

the *topological interior* and *topological closure*, respectively. As the reader will find in the exercises, interior and closure operators provide an alternative, but equivalent, form of describing topologies. ⊣

The last notion we will introduce in this chapter is that of a *neighbourhood*:

Definition 2.4.5. Given a topological space (X, τ) and a point $x \in X$, we say that $V \subseteq X$ is a *neighbourhood* of x if and only if there is an open set U such that $x \in U \subseteq V$.

Moreover, observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x if and only if $x \in V$ and V is open.[6] ⊣

[6] In the literature, you will sometimes find that a neighbourhood simpliciter already is required to be open. We do not adopt that convention, but simply speak of 'open neighbourhoods' when needed.

Remark 2.4.4. (*Epistemic intuition: what is an (open) neighbourhood?*) The open neighbourhoods of a point x have a neat epistemic interpretation: they are precisely the verifiable propositions true at world x (i.e., the propositions that in fact can be verified at x − assuming that only true propositions can be verified). One can also come up with an epistemic interpretation of a neighbourhood simpliciter, but it seems a rather artificial concept; all intuitions, including our epistemic one, have their shortcomings. $\dashv$

Using the definition of an open neighbourhood, we can give another equivalent definition of the closure of a set, which is particularly useful when proving a set is open or closed:

Proposition 2.4.6. Suppose X is a topological space and $S \subseteq X$. Then the following are equivalent for a point $x \in X$:

- x is in the closure of S; i.e., $x \in cl(S)$.
- All open neighbourhoods U of x have non-empty intersection with S; i.e., $U \cap S \neq \varnothing$.

Proof. ($\Rightarrow$) Suppose for contraposition that there is some open neighbourhood U of x such that $U \cap S = \varnothing$. Then $(X - U)$ is closed, $S \subseteq (X - U)$ and $x \notin (X - U)$. So since $cl(S)$ equals the intersection of all closed sets containing S – which includes the set $(X - U)$ – we find that $x \notin cl(S)$, as required.

($\Leftarrow$) Suppose for contraposition that $x \notin cl(S)$. Then since $cl(S)$ is closed, its complement $(X - cl(S))$ is an open set containing x, i.e., an open neigbourhood of x. And clearly, $(X - cl(S)) \cap S = \varnothing$, which proves the claim. ■

Logic	**Topology**
Epistemic worlds/situations/models/objects satisfying a property	Points, $x \in X$
Verifiable propositions	Open sets, $U \in \tau$
Falsifiable propositions	Closed sets, $U^C \in \tau$
Verifiable propositions true at x	Open neighbourhoods U of x
(Sub)basic verifiable propositions	(Sub)basic opens

2.5 Exercises

Exercise 2.1. Let $(\tau_i)_{i \in I}$ be a collection of topologies on a set X.

(a) Is their intersection $\bigcap_{i \in I} \tau_i$ (necessarily) a topology on X?

(b) Is their union $\bigcup_{i \in I} \tau_i$ (necessarily) a topology on X?

(c) Show that there is a greatest topology τ on X such that $\tau \subseteq \tau_i$ for all $i \in I$. (with “greatest” we mean that if τ' is some other topology such that $\tau' \subseteq \tau_i$ for all $i \in I$, then $\tau' \subseteq \tau$)

(d) Show that there is a least topology τ on X such that $\tau_i \subseteq \tau$ for all $i \in I$.

Now let $X = \{x, y, z\}$, $\tau_0 = \{\varnothing, X, \{x\}, \{x, y\}\}$ and $\tau_1 = \{\varnothing, X, \{x\}, \{y, z\}\}$.

(e) Find the greatest topology τ on X such that $\tau \subseteq \tau_0$ and $\tau \subseteq \tau_1$.

(f) Find the least topology τ on X such that $\tau_0 \subseteq \tau$ and $\tau_1 \subseteq \tau$.

Exercise 2.2. Consider the space $(\mathbb{R}, \tau_{Euc})$, with its Euclidean topology.

- Give an example of a set which is neither open nor closed.
- Show that the intervals of the form (x, y) where $x, y \in \mathbb{Q}$ form a basis for this topology.
- Show that $\mathbb{Q}$ is a countable union of closed sets.

Exercise 2.3. Let (S, τ_S) be a subspace of (X, τ), and $T \subseteq S$. Show that the topology the set T inherits as a subspace of (S, τ_S) is the same as the topology it inherits as a subspace of (X, τ).

Exercise 2.4. Prove Lemma 2.3.6.

Exercise 2.5. Prove Lemma 2.3.11.

Exercise 2.6. Prove Lemma 2.4.3.

Definition 2.5.1. Let $T \subseteq \omega^{<\omega}$ be a collection of finite sequences of natural numbers. We say that T is a *tree* if whenever $s \in T$ and $t \lhd s$, then $t \in T$.

Given a tree T we write $[T]$ for the set of all branches, i.e. the infinite sequences all of whose finite approximations belong to T:

$$[T] = \{s \in \omega^\omega : \forall n, s\restriction n \in T\}$$

Exercise 2.7. Let $A \subseteq \omega^\omega$ be a set of natural number sequences. We write $T(A)$ for the *tree of initial sequences*, that is:

$$T(A) = \{s \in \omega^{<\omega} : s \lhd\}$$

1. Show that whenever T is a tree, $[T]$ is a closed subset.
2. Show that the assignment $A \mapsto [T(A)]$ is the topological closure in the Baire space.

Chapter 3

Continuous Maps

Having acquainted ourselves with the basic objects, notions and constructions of topology, we proceed to a central, but elusive, mathematical notion: continuity. Back when the main mathematical concerns about continuity were geometrically rich settings, such as the real line, this could be captured by the following intuition:

A continuous function is one which has 'no gaps'; it can be drawn using a pencil without lifting it.

Or putting it in a slightly different way:

Points close to each other in the domain of f, are mapped to points close to each other in the codomain of f.

However, rigorously formalising this notion was one of the greatest challenges of 19th century mathematics. Under Richard Dedekind's formalisation of Euclidean space, continuity is defined by the "ε-δ"-definition (see exercises). Historically, this lead to the first real generalisation of Euclidean space: so-called metric spaces. For the reader familiar with metric spaces, the topological definition of continuity comes as no surprise: on metric spaces, it is easily seen to be equivalent to the usual metric definition but with the advantage of not referring to any metric. However, for the reader unfamiliar with metric spaces, it probably appears shockingly different from our basic intuition of continuity:

Definition 3.0.1. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and $f : X \to Y$ a map between them. We say that f is *continuous* if for every open U in Y, its preimage $f^{-1}[U] = \{x \in X \mid f(x) \in U\}$ is open in X; that is,

$$\forall U \subseteq Y (U \in \tau_Y \implies f^{-1}[U] \in \tau_X).$$

Strictly speaking, we should say that "f is continuous *relative to the topologies* τ_X and τ_Y": had we equipped X and Y with other topologies than τ_X and τ_Y, the condition that preimages of opens are open might not have been met any longer. Nonetheless, in almost all cases, we go for brevity and omit explicit mention of the topologies. $\dashv$

3.1 Understanding Topological Continuity

As you have seen in lecture, through the following proposition, we can, fortunately, reconcile this seemingly bizarre definition with our basic intuition of continuity:

Proposition 3.1.1. Let $f : X \to Y$ be a map between topological spaces. Then f is continuous if and only if

($*$) for every $S \subseteq X$: $f\left[\overline{S}\right] \subseteq \overline{f[S]}$, i.e., if $x \in cl_X(S)$ then $f(x) \in cl_Y(f[S])$.

Remark 3.1.1. Before proving the proposition, let us recall from lecture why this equivalent ($*$) definition of continuity agrees with our intuition. We make the following interpretation:

> Given a topological space (X, τ), $S \subseteq X$ and $x \in X$, we say that x is *close to* S if x is in the closure of S, i.e., $x \in cl(S)$.

Under this interpretation,[1] the proposition amounts to the following: f is continuous if and only if

> for every $S \subseteq X$, f maps points close to S to points close to $f[S]$.

Thus, this otherwise seemingly bizarre definition – upon further scrutiny – concurs rather well with our informal idea of continuity. $\dashv$

Proof (of Proposition 3.1.1). ($\Rightarrow$) Suppose f is continuous, and let $S \subseteq X$ and $x \in cl_X(S)$ be arbitrary. We are then to show that $f(x) \in cl_Y(f[S])$. To show so, let $V \subseteq Y$ be an arbitrary open neighbourhood of $f(x)$. By Proposition 2.4.6, it then suffices to show that $V \cap f[S] \neq \varnothing$. But since (i) $f(x) \in V$, (ii) V is open in Y, and (iii) f, by assumption, is continuous, we get that $f^{-1}[V]$ is an open neighbourhood of x. So because $x \in cl_X(S)$, we get by Proposition 2.4.6 that there is some $y \in (f^{-1}[V] \cap S)$. Thus, $f(y) \in (V \cap f[S])$, as required.

($\Leftarrow$) Suppose $V \subseteq Y$ is open in Y. We are then to show that $f^{-1}[V]$ is open in X. We have that the complement $(Y - V)$ is closed in Y, and since obviously $f^{-1}[Y - V] \subseteq X$, we can use ($*$) for $S = f^{-1}[Y - V]$ to get that

$$f\left[\overline{f^{-1}[Y - V]}\right] \subseteq \overline{f[f^{-1}[Y - V]]} \subseteq \overline{Y - V} = Y - V.$$

Thus by taking preimages, we get that

$$\overline{f^{-1}[Y - V]} \subseteq f^{-1}\left[f\left[\overline{f^{-1}[Y - V]}\right]\right] \subseteq f^{-1}[Y - V],$$

which implies that

$$\overline{f^{-1}[Y - V]} = f^{-1}[Y - V],$$

hence $f^{-1}[Y - V]$ is closed in X, and thus the complement

$$f^{-1}[V] = (X - f^{-1}[Y - V])$$

is open in X, as desired. ■

One might ask whether we also have an epistemic understanding of continuity.

[1] And this interpretation does indeed make a lot of sense. For instance, in $\mathbb{R}$ with the standard topology, we get that x is close to $(0, 1)$ if and only if $x \in [0, 1] = \{r \in \mathbb{R} \mid 0 \leqslant r \leqslant 1\}$.

Remark 3.1.2. (*Epistemic intuition: what is continuity?*)

Say that X and Y are two topological spaces, understood as epistemic domains. As we specified them, epistemic domains consist of worlds, an epistemic framework, and propositions. But it might be that we wish to *compare* two epistemic domains: for instance, we want to compare the knowledge about topology we have in English and in Polish.[2] In order to do that, we need a form of *translation* between the propositions. Since we identified propositions with sets of worlds, we can just as well think of such a translation as a map $f : X \to Y$.

Now it is an often remarked fact that translations can sometimes be lossy – some facts that were known can be lost, since there is no way to express them unambiguously in the new language. On the other hand, if this translation is to be sound, we would expect that *knowledge should be stable under translation*: if we could known that $y \in U$ for $U \subseteq Y$ and $y \in Y$, then whatever U corresponds to under the translation (i.e., $f^{-1}[U]$) we should have that we could already decide whether $x \in f^{-1}[U]$ – in short, translations do not increase knowledge. This is precisely the criterion of continuity we have. $\dashv$

Besides from the definitions of continuity just considered, there are many other equivalent ones. The following gathers some of the most common and useful of these:

Proposition 3.1.2. Let $f : X \to Y$ be a map between topological spaces and $\mathcal{B}_Y$ a (sub)basis for the topology on Y. Then the following are equivalent:

1. f is continuous.

2. For every (sub)basic open $U \in \mathcal{B}_Y$, its preimage $f^{-1}[U]$ is open in X.

3. For every closed set U in Y, its preimage $f^{-1}[U]$ is closed in X.

4. For every $x \in X$, whenever $V \subseteq Y$ is a (basic) open neighbourhood of $f(x)$, there is an open neighbourhood $U \subseteq X$ of x such that $f[U] \subseteq V$.

Proof. We show that 1. ⇔ 2., and the leave the rest as an exercise.

Clearly, 1. implies 2., so suppose 2. holds for $\mathcal{B}_Y$ a basis, and let $V \subseteq Y$ be an arbitrary open in Y. We are then to show that $f^{-1}[V]$ is open in X. By assumption, (a) there are $(V_i)_{i \in I} \subseteq \mathcal{B}_Y$ such that

$$V = \bigcup_{i \in I} V_i,$$

and (b) each

$$f^{-1}[V_i]$$

is open in X. So using that preimages commute with unions, we get that

$$f^{-1}[V] = f^{-1}\left[\bigcup_{i \in I} V_i\right] = \bigcup_{i \in I} f^{-1}[V_i],$$

which suffices because topologies are closed under unions.

[2]This is a reference to the fact that the Polish school of topology was extremely influential, and continues to this day to have a hold on conjectures and directions for some areas of general topology.

Now suppose that 2. holds for $\mathcal{B}_Y$ a subbasis. Then, by definition, the set of finite intersections of $\mathcal{B}_Y$ is a basis. Thus, using what was just proven, it suffices to show that

$$f^{-1}[V_1 \cap \cdots \cap V_n]$$

is open for any $n \in \omega$, $\{V_1, \ldots, V_n\} \subseteq \mathcal{B}_Y$. Now, using that preimages also commute with intersections, we get that

$$f^{-1}[V_1 \cap \cdots \cap V_n] = f^{-1}[V_1] \cap \cdots \cap f^{-1}[V_n],$$

which suffices because topologies are closed under finite intersections. ■

Remark 3.1.3. f is said to be *continuous at a point* $x \in X$ if condition 4. holds for x.

You should show that under the "close to"-interpretation of Remark 3.1.1, we have that f is continuous at a point $x \in X$ if and only if

$(*)_{local}$ for every $S \subseteq X$, if x is close to S then $f(x)$ is close to $f[S]$. ⊣

To get some hands-on experience with the topological notion of continuity, let us look at some specific examples – starting with checking that the quadratic function $x \mapsto x^2$, indeed, is continuous:

Example 3.1.3. Consider the function

$$f : \mathbb{R} \to \mathbb{R}$$

given by setting

$$f(x) = x^2$$

for all $x \in \mathbb{R}$. We check that this is a continuous function (relative to the Euclidean topology on both copies of $\mathbb{R}$). By 2. of the preceding proposition, it suffices to show that

$$f^{-1}[(a, b)]$$

is open for all open intervals $(a, b) \subseteq \mathbb{R}$. However, this is clearly the case, since

$$f^{-1}[(a,b)] = \{x \in \mathbb{R} \mid a < x^2 < b\} = \begin{cases} (\sqrt{|a|}, \sqrt{|b|}) \cup (-\sqrt{|b|}, -\sqrt{|a|}) & \textit{if } 0 \leqslant a < b \\ (0, \sqrt{|b|}) \cup (-\sqrt{|b|}, 0) & \textit{if } a < 0 < b \\ \varnothing & \textit{otherwise} \end{cases}$$

which, as a union of open intervals, is open in the Euclidean topology on $\mathbb{R}$. ⊣

Even if the reader (hopefully) finds that we have given sufficient reason for why and how the topological definition of continuity formalises our basic intuitions about continuity, it is instructive to consider another kind of maps whose definition might—but only at first sight—appear like a more obvious candidate for a topological definition of continuity.

Definition 3.1.4. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and $f : X \to Y$ a map between them. We say that f is *open* if for every open U in X, its image $f[U] = \{f(x) \in Y \mid x \in U\}$ is open in Y; that is,

$$\forall U \subseteq X (U \in \tau_X \implies f[U] \in \tau_Y).$$

As with the definition of continuity, strictly speaking, we should say that "f is open *relative to the topologies* τ_X and τ_Y", but, for brevity, we will typically omit such mention. ⊣

Although the concept of an open map is useful, it is not – as the following example shows – what captures our intuitions about continuity. Making use of our epistemic intuition, this actually becomes quite clear:

Remark 3.1.4. (*Epistemic intuition: What are open maps?*) We have said that maps $f : X \to Y$ between epistemic domains are translations, and that continuous maps are translations where "no knowledge is gained". Open maps are, on the other hand, translations where "no knowledge is lost". Indeed, if U is verifiable in X, and f is such a translation, then since no knowledge is lost, we should be able to verify $f[U]$; and conversely, if we have an open map, it is clear that no knowledge will be lost.

Intuitively then, it seems plausible that amongst translations, some will have lost knowledge – some things will be lost in translation – though the class of such translations is certainly interesting. ⊣

If the former intuition is to hold, we should be able to find some examples of continuous maps which are not open, and vice-versa. Let us see an instance of the former:

Example 3.1.5. Consider the function

$$f : \mathbb{R} \to \mathbb{R}$$

given by setting

$$f(x) = \begin{cases} x & \textit{if } x \leqslant 0 \\ 0 & \textit{otherwise} \end{cases}$$

for all $x \in X$. Then f is continuous but not open. Indeed, we find that, e.g.,

$$f[(-1,1)] = \{x \in \mathbb{R} \mid -1 < x \leqslant 0\} = (-1,0]$$

which is not open (check this), hence f is not open. To see that it is continuous, observe that for any open interval (a,b), we have that

$$f^{-1}[(a,b)] = \begin{cases} (a,b) & \textit{if } a < b \leqslant 0 \\ (a,\infty) & \textit{if } a \leqslant 0 < b \\ \varnothing & \textit{otherwise} \end{cases},$$

which, as a union of open intervals, is open. ⊣

We have seen how the topological definition of continuity, as a special case, agrees with our geometric intuition. But since the definition of topological spaces go far beyond the standard topology on $\mathbb{R}$, so must the topological definition of continuity. To see an instance of this, let us look at the reflexive and transitive Kripke frames with their Alexandroff topologies. What are the continuous maps between such spaces?

Definition 3.1.6. Let $\mathfrak{F} = (W,R)$ and $\mathfrak{F}' = (W',R')$ be two Kripke frames. We say that a map $f : W \to W'$ satisfies

- *the forth condition* if whenever xRy, we also have that $f(x)R'f(y)$; and
- *the back condition* if whenever $f(x)R'y'$, there is some $y \in W$ such that xRy and $f(y) = y'$.⊣

Proposition 3.1.7. Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be two reflecive and transitive Kripke frames, equipped with the Alexandroff topology, and $f : W \to W'$ a map between them. Then:

1. f satisfies the forth condition if and only if f is continuous.

2. f satisfies the back condition if and only if f is open.

Proof. (1) First, suppose that f satisfies the forth condition, and assume that $U' \subseteq W'$ is an R'-upset of W'. We will show that $f^{-1}[U']$ is an R-upset. Indeed, assume that $x \in f^{-1}[U']$ and xRy; we are then to show that $y \in f^{-1}[U']$. By the forth condition, we get $f(x)R'f(y)$, so since $f(x) \in U'$, we have that $f(y) \in U'$, hence $y \in f^{-1}[U']$ – as required.

Conversely, assume that f is continuous and xRy. Consider the R'-upset:

$$\uparrow f(x) = \{z' \in W' : f(x)R'z'\}.$$

Then look at $f^{-1}[\uparrow f(x)]$. Because the map is continuous, this is an R-upset. So since $x \in f^{-1}[\uparrow f(x)]$ (by reflexivity of (W', R')) and xRy, we have that $y \in f^{-1}[\uparrow f(x)]$. Hence $f(y) \in \uparrow f(x)$, and so $f(x)Rf(y)$.

(2) Now assume that f satisfies the back condition. Let $U \subseteq W$ be an R-upset; we will show that $f[U]$ is an R'-upset. Indeed, if $f(x) \in f[U]$, and $f(x)R'y'$, we get by the back condition that there is some $y \in W$ such that xRy and $f(y) = y'$; since U is an upset, we have $y \in U$, so $y' = f(y) \in f[U]$ – as required.

Conversely, if f is an open map, assume that $f(x)R'y'$. Because $\uparrow x$ is an R-upset, its image $f[\uparrow x]$ must be an R'-upset, so since $f(x) \in f[\uparrow x]$ by reflexivity of (W, R), we have that $y' \in f[\uparrow x]$. But that means that there is some $y \in W$ such that xRy and $f(y) = y'$. This shows the back condition. ∎

Having Proposition 3.1.7 in mind, it is easy to give examples of continuous maps which are not open, and vice-versa. Consider for instance Figure 3.1.

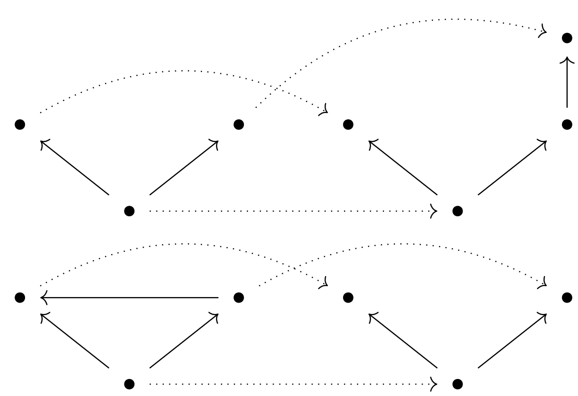

Figure 3.1: Continuous map, but not open; open map but not continuous

3.2 Homeomorphisms, Embeddings and Quotient Maps

Combining the notion of an open and a continuous map, we can define the correct notion of an "isomorphism" for topological spaces, denoted a *homeomorphism*. This principle will be seen very often in these notes: whenever we have two spaces which are *homeomorphic*, and one of them has a specific property, we will be able to transfer that property along the homeomorphism.

Definition 3.2.1. Let $f : X \to Y$ be a map between topological spaces. We say that f is

- a *quotient map* if (i) it is surjective and (ii) for all $U \subseteq Y$, U is open in Y if and only if $f^{-1}(U)$ is open in X;
- a *homeomorphism* if it is bijective, continuous and open; and
- a *(topological) embedding* if the map

$$f' : X \to f[X]$$

obtained by restricting the codomain of f to its image is a homeomorphism (where $f[X] \subseteq Y$ is endowed with the subspace topology). $\dashv$

As you will show in your first assignment, there are other elegant equivalent definitions of homeomorphisms. Alas, this is not the case when it comes to quotient maps and embeddings, but – as the lemma following the next definition shows – the situation is not all bad.

Definition 3.2.2. Let $f : X \to Y$ be a map between topological spaces. We say that f is *closed* if for every closed U in X, its image $f[U] = \{f(x) \in Y \mid x \in U\}$ is closed in Y. $\dashv$

Lemma 3.2.3. Let $f : X \to Y$ be a map between topological spaces. Then the following all hold:

(0.q) f is a quotient map if and only if (i) f is surjective and (ii') for all $U \subseteq Y$, U is closed in Y if and only if $f^{-1}(U)$ is closed in X.

(1.q) If f is a quotient map, then f is surjective and continuous.

(2.q) If f is surjective, continuous and open, then f is a quotient map.

(3.q) If f is surjective, continuous and closed, then f is a quotient map.

(1.e) If f is an embedding, then f is injective and continuous.

(2.e) If f is injective, continuous and open, then f is an embedding.

(3.e) If f is injective, continuous and closed, then f is an embedding.

Proof. (0.q) to (3.q) all follow almost directly by definition (you should still check these) and (3.e) matches one of the exercises of your first assignemnt, so we only cover (1.e) and (2.e) as an aid in unpacking the definition of an embedding.

(1.e) Suppose f is an embedding. Injectivity follows by the restriction

$$f' : X \to f[X]$$

being a homeomorphism, hence in particular injective. For continuity, suppose $V \subseteq Y$ is open in Y. Then $V \cap f[X]$ is open in the subspace topology on $f[X]$, hence, since homeomorphisms are, in particular, continuous, we find that

$$f^{-1}[V \cap f[X]]$$

is open in X. But

$$f^{-1}[V \cap f[X]] = f^{-1}[V],$$

so the claim has been proven.

(2.e) Suppose f is injective, continuous and open. Then

$$f' : X \to f[X]$$

is also surjective, hence bijective. Moreover, f' is continuous since for any open V in Y, we have that

$$f'^{-1}[V \cap f[V]] = f^{-1}[V],$$

which is open in X. Lastly, f' is open because for any open U in X,

$$f'[U] = f[U] = f[U] \cap f[X],$$

which is open in the subspace topology on $f[X]$ because $f[U]$ is open in X by f being an open map. Thus, $f' : X \to f[X]$ is a homeomorphism, hence f is an embedding.

■

Remark 3.2.1. Statements (2.q) (and (3.q)) of the preceding lemma are only partial converses to (1.q). This is for good reason: there are quotient maps which are not open (and quotient maps which are not closed). And similarly, as regards embeddings and the statements (1.e), (2.e), (3.e). ⊣

Example 3.2.4. Recall the Cantor space 2^ω. Then we can construct a topological embedding from this space to $[0, 1]$ (as a subspace of $\mathbb{R}$) through the map taking an infinite binary sequence $(a_i)_{i \in \omega}$ to:

$$\sum_{n=0}^{\infty} \frac{2a_n}{3^{n+1}}.$$

You can check that this is indeed a topological embedding. Additionally, you can see that the corresponding subset of $\mathbb{R}$ is *not open* in $\mathbb{R}$. ⊣

Quotient maps are of further interest because their definition also works as a recipe for constructing a topology:

Definition 3.2.5. Let X be a topological space, $\sim$ an equivalence class on X, and

$$q : X \to X/{\sim}$$

the canonical map projecting each element to its equivalence class; i.e.,

$$q : x \mapsto [x]_{\sim}.$$

We then define the *quotient topology* on $X/{\sim}$ as follows: $U \subseteq X/{\sim}$ is open if $q^{-1}[U]$ is open in X (you should check that (a) this, indeed, defines a topology on $X/{\sim}$, and (b) q does become a quotient map when $X/{\sim}$ is equipped with this topology). $\dashv$

An often repeated adage is that quotienting "is like gluing". Let us see a few examples to get an understanding of this:

Example 3.2.6. If the previous example made you reconsider your choices for project season, there are much simpler examples of quotient topologies you can think of. Consider the unit interval with the subspace topology: $I = [0, 1] \subseteq \mathbb{R}$. Define a relation on this space by saying that

$$x \sim y \qquad \text{if and only if} \qquad x = y \textit{ or } x = 0, y = 1 \textit{ or } x = 1, y = 0.$$

It is not too difficult to see that this is an equivalence relation. Consider the quotient $[0, 1]/{\sim}$. If quotienting is like gluing, this space – having glued the interval endpoints 0 and 1 together – should give you a circle in $\mathbb{R}^2$. And topologically that is precisely what happens: $[0, 1]/{\sim}$ is homeomorphic to, e.g., the circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$$

equipped with the subspace topology. $\dashv$

Example 3.2.7. Consider $I^2 = [0, 1] \times [0, 1]$. This can be visualised as a unit square. Consider the following equivalence relation:

$$(x, y) \sim (x', y') \qquad \textbf{:iff} \qquad (x, y) = (x', y') \textit{ or } (x = 1 - x' \in \{0, 1\}, y = 1 - y').$$

This can be visualised as in Figure 3.2.

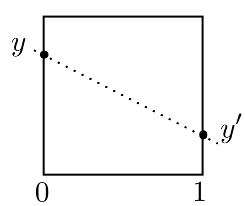

Figure 3.2: Gluing of Möbius Strip

Now consider the quotient of I^2 through $\sim$. In our ordinary 3D space this corresponds to taking a rectangle, folding half of it and gluing the edges – obtaining a so-called *Möbius Strip*. If we look at the topology on I^2 obtained canonically (i.e., by taking the subspace topology on $[0, 1] \subseteq \mathbb{R}$ and the product topology on these), then the result of the quotient topology is precisely the topological structure one would desire from such a geometric object.[3] $\dashv$

[3]The Möbius Strip and the *torus* (i.e., the "surface of a doughnut") are plausibly the most commonly used examples of topological spaces within popular math education. In this (wonderful) 3Blue1Brown video, you can see an instance of how these topological spaces actually are used in mathematics.

3.3 Exercises

Exercise 3.1. Prove the rest of Proposition 3.1.2.

Exercise 3.2. Consider the Baire space defined in Example 2.2.8. Show that a function $f : \omega^\omega \to \omega^\omega$ is continuous if and only if whenever $s \lhd f(x)$, there is $t \lhd x$ such that for all y, $t \lhd y$ implies $s \lhd f(y)$.

Exercise 3.3. (Continuous maps preserve the structure of topological spaces) Given continuous maps $f : A \to B, g : B \to C$ show that

- Composition: $g \circ f$ is continuous; and
- Identity: $Id : A \to A, a \mapsto a$ is continuous.

Exercise 3.4. (Topological spaces are stretchy) Prove that $(-1, 1)$ and $\mathbb{R}$ are homeomorphic. Use your homeomorphism to show that for any real numbers a and b, the interval (a, b) is homeomorphic to the real line.

Remark 3.3.1. You now have sufficient topological knowledge to understand the jokes made about topologists, doughnuts and coffee mugs. Topology is the study of spaces *up to homeomorphism*, which means that spaces which can be obtained by this sort of "stretching" behaviour are homeomorphic. But what out! There can be very wild homeomorphisms between spaces.

Exercise 3.5. ("Being homeomorphic to" forms an equivalence class)

(Re.) Show that for any topological space X, there is a homeomorphism $f : X \to X$.

(Sy.) Suppose $f : X \to Y$ is a homeomorphism. Show that the inverse $f^{-1} : Y \to X$ is a homeomorphism as well.

(Tr.) Suppose $f : X \to Y, g : Y \to Z$ are homeomorphisms. Show that their composition $g \circ f : X \to Z$ is a homeomorphism as well.

Exercise 3.6. Show that the following are equivalent for a continuous function $f : X \to Y$ between topological spaces:

1. f is a homeomorphism;
2. There is a continuous function $g : Y \to X$ such that $f \circ g = id_Y$, and $g \circ f = id_X$.

Exercise 3.7. (Subspaces) Let X be a topological space. Suppose that $U \subseteq X$ is an open subset. Show that:

1. The inclusion of the topological subspace U into X is an interior map.
2. Is the same true if U is arbitrary, i.e., not assumed to be open? What properties does the inclusion have?

Exercise 3.8. Let $\mathbb{R}$ be the set of reals with the Euclidean topology. Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point a if and only if it is ε-δ continuous at that point: for all real numbers $\varepsilon > 0$, there exists some real $\delta > 0$ such that for all $x \in \mathbb{R}$,

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Hint: Use the so-called triangle inequality, which holds of the absolute value: $|x - z| \leqslant |x - y| + |y - z|$.

Exercise 3.9. (Quotient maps) Let X be a topological space and $\sim$ an equivalence relation.

1. Show that the map $q : X \to X/\sim$ is a quotient map.
2. Give an example of a continuous surjection which is not a quotient map.
3. Given another topological space Y, show that for every continuous map $f : X \to Y$, there exists a unique map $\tilde{f} : X/\sim \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "f"] & \\ X/\sim \arrow[r, dashed, "\tilde{f}"] & Y \end{tikzcd}$$

Exercise 3.10. (Universality of product) Consider a family $(X_i)_{i\in I}$ of topological spaces. Show that:

1. The projection maps $p_j : \prod_{i\in I} X_i \to X_i$ given by sending $(x_i)_{i\in I} \mapsto x_j$ are continuous;
2. The topology on the set theoretic product is the coarsest making the projections continuous: if τ' is some topology on $\prod_{i\in I} X_i$ such that all the maps p_i are continuous, then τ' contains all open sets from the product topology.

Exercise 3.11. (Very Tricky!) Show that there exists a surjective continuous function $f : [0,1] \to [0,1] \times [0,1]$. *Hint: Define this function in successive stages, such that in the limit it covers the whole space.*

Chapter 4

Separation Axioms

We have now encountered all the main concepts we need to discuss some properly topological properties. One of the most relevant such properties is "separation". This is in some sense a complementary discussion to the one we have had so far. So far we have only talked about recognising that certain points $x \in X$ belong to certain subsets $U \subseteq X$ – open subsets, understood as intervals, verifiable propositions, or some other such structure. However, one can ask whether not only our epistemic structure allows us to talk about knowable facts, but whether we can actually *use* this epistemic structure to determine things about the possible worlds.

To motivate the technical developments, we will begin by introducing an informal collection of ideas, motivated epistemically, which will eventually lead us to the notion of a *filter*.

4.1 Epistemic Theories

Throughout this section, let X be some topological space. Our topology $\tau \subseteq \mathcal{P}(X)$ is our epistemic structure. Now consider a subset $T \subseteq \mathcal{P}(X)$. We can think of T as an *epistemic theory*. Not all theories will be good: some, like $\{X\}$ would support every state. Hence if we want to use these theories to provide us with knowledge of our epistemic position, we need to impose some conditions. Here is one intuitive idea: the facts about our theory should consist of facts which support that we are in a given world. Indeed, we can ask that if U is open and a neighbourhood of x – i.e., we can verify $x \in U$ – there is some piece of evidence V such that $V \subseteq U$. In words: if we can know something about the world x, then this must be implied by some piece of evidence of our theory. Hence we will say that:

$$T \vdash x \iff \text{Whenever } x \in U \text{ is open}, \exists V \subseteq U, V \in T.$$

On the other hand, we can ask that a theory T is in some sense *saturated*: we cannot add any more information to the theory T, that is, we have included in this theory every proposition (whether verifiable or not) we can, without obtaining a contradiction. Namely, for a theory T, we write $sat(T)$ to mean T is a saturated theory, defined as follows:

$$sat(T) := \text{For each } U \subseteq X,\ U \in T \text{ or } X - U \in T$$

Now, if a theory is saturated, then it seems that we should be able to collect all of our information, and have it correspond to a world – i.e., saturated theories should be definitive. And likewise, if we have a definitive theory, it seems plausible that this should be saturated. Hence we want:

$$sat(T) \iff \exists x, T \vdash x$$

However, two problems arise from this:

1. There might *not be enough worlds*, hence, we might have saturated theories which correspond to no definitive theory;
2. There might be *too many worlds*, hence we might have definitive theories which are not saturated (as the worlds will then disagree on some proposition).

The purpose of the present and subsequent chapter will be to show under which circumstances we can make the identification that "definitive=saturated". Note that this has a strong epistemic meaning: if definitive implies saturated, this means that from our theory built on verifying properties we can obtain complete knowledge of our epistemic position; and if saturated implies definitive, then essentially there are "no holes" epistemically – all complete epistemic positions correspond to some epistemically possible world.

In the next section we will formalise this, and introduce a concurrent geometric intuition for this. We will then, in this chapter, show how we can show the first inclusion. In the next chapter we will take the other inclusion in turn.

4.2 Filter Bases and Extension of Filters

The first notion that we will need moving forward is some concept which allows us to capture the idea laid above of being a "definitive theory". We can see by the way we defined it that this also has a in some sense something to do with "approaching a point", or in more technical-sounding terms "convergence". One – of multiple possible – ways that this can be formalised is using the idea of a "filter base" and associatedly, a "filter":

Definition 4.2.1. Let X be a set. We say that a collection of subsets $F \subseteq \mathcal{P}(X) - \{\varnothing\}$ is a *filter base* if it satisfies the following:

- $X \in F$;
- If $A, B \in F$ then $A \cap B \in F$.

We say that a given filter base is a *filter* if it is upwards closed: whenever $A \in F$ and $A \subseteq B$ then $B \in F$.

Example 4.2.2. Consider the set $\{a, b, c, d\}$. Then the following collection defines a filter base (check this):

$$F = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c, d\}\}$$

We see that in fact this filter base seems to be structured "around" a. To obtain a filter we could simply take the upwards closure of F:

$$F^{\uparrow} := \{C \subseteq \{a, b, c, d\} : \exists G \in F, G \subseteq C\};$$

an easy calculation then yields that $F^{\uparrow}$ is the set of all subsets containing a. ■

The construction used in the previous example is extremely standard, and allows us to pass from a filter base to a filter:

Definition 4.2.3. Let X be a topological space, and F be a filter base. Then we define the *upwards closure of* F as

$$F^{\uparrow} := \{C \subseteq X : \exists G \in F, G \subseteq C\},$$

and $F^{\uparrow}$ is a filter.

An important example of a filter is the collection of all neighbourhoods of a point, which we denote by $\mathcal{N}(x)$.

Proposition 4.2.4. Let (X, τ) be a topological space, and $x \in X$. Then $\mathcal{N}(x)$ is a filter.

Proof. First, note that since $x \in X$, and X is a neighbourhood, we have that $X \in \mathcal{N}(x)$. Additionally, if $U, V \in \mathcal{N}(x)$, this means that there are open sets $U' \subseteq U$ and $V' \subseteq V$ containing x. Hence $x \in U' \cap V' \subseteq U \cap V$, so $U \cap V$ is again a neighbourhood of x. So $\mathcal{N}(x)$ is closed under binary intersections. Finally, if $V \in \mathcal{N}(x)$ and $V \subseteq K$, then certainly K is also a neighbourhood of x, so $K \in \mathcal{N}(x)$, as desired. ∎

The former proposition tells us that given a point x, there is a filter corresponding to it. Motivated by our epistemic questions, we can ask whether to each filter we can associate a given point, and whether this association can be made uniquely. In general, we certainly cannot expect that this will hold for arbitrary filters: the singleton $\{X\}$ of any topological space is always a filter, but it is not clear to what point this should correspond. However, it is possible to make this correspondence, so long as the filter can be related to the topological structure in an appropriate way.

To understand this, let us look at a very important example:

Example 4.2.5. We will say that an arbitrary collection S of intervals *converges to* a if whenever (x, y) is an arbitrary interval of real numbers such that $a \in (x, y)$, then there exists $(x', y') \in S$ such that $x < x' < y' < y$ and $a \in (x', y')$.

Clearly the collection of all intervals containing a point a – denote it by Q_a – converges to a. Now, if "converging" ought to mean anything geometrically sensible, we would not want such a collection to also converge to some other point b. Let us check whether this can happen: assume that Q_a also approaches b, and $a \neq b$. Since the set of real numbers is totally ordered, we can assume without loss of generality that $a < b$. By density and unboundedness of the rational numbers in the reals, we can pick points $c, d, e, f \in \mathbb{Q}$ such that $a \in (c, d)$ and $b \in (e, f)$ and $(c, d) \cap (e, f) = \varnothing$. By hypothesis, since Q_a converges to b, we have that there is some rational interval $(c', d') \in Q_a$ such that $e < c' < d' < f$. But this is a contradiction. Hence, we conclude that Q_a cannot converge to b unless $a = b$.

In this example, we are able to tell the two points a and b apart using the structure of intervals, which as we know is a basis for the topology. What this should tell us is that reals are separated, because we have enough topological structure to tell our points apart. We can now see how to generalise this:

Definition 4.2.6. Let (X, τ) be a topological space and $F \subseteq \tau$ a filter (base). We say that the filter (base) F *converges to a point* x, and that x is the *limit of the filter (base)*, if and only if for every $U \in \mathcal{N}(x)$, there is some $V \in F$ such that $V \subseteq U$.

Example 4.2.7. Fix a given $x \in 2^\omega$, the Cantor space (recall Example 2.2.8). Then we can look at the filter base of opens:

$$\{C(x \restriction n) : n \in \omega\}.$$

I.e., the collection of opens determined by a finite part of the sequence x. Then we have that this converges to x: indeed if U is a neighbourhood of x, we have that $U \supseteq V$ for V an open neighbourhood, and by the structure of the Cantor space:

$$V = \bigcup_{i \in I} C(s_i)$$

Now since $x \in V$, then $x \in C(s_i)$, which means that $s_i \lhd x$. Hence for some n, $s_i = x \restriction n$. So $C(x \restriction n) \subseteq V$ as intended.

Both here and in the example above, one can think of taking intersections of sets as getting to a finer and finer set, which eventually becomes a unique point. But we should stress that the notion of convergence *does not say anything about uniqueness*. Indeed, in the Exercises you will find examples of spaces – even reasonable looking ones – where filters may converge to multiple points at once. This is a consequence of working with a notion as broad as that of a topological space, and it is towards fixing this situation that we make use of Separation Axioms [1].

4.3 Hausdorff Spaces

The most frequently mentioned separation axiom is the following:

Definition 4.3.1. Let (X, τ) be a topological space. We say that X is *Hausdorff*, or T_2, if whenever $x, y \in X$ and $x \neq y$, there there exist two open neighbourhoods $x \in U_x$ and $y \in V_y$ and

$$U_x \cap V_y = \varnothing.$$

Remark 4.3.1. A favourite pun of a topologist is that in a Hausdorff space, every two points are 'housed off' each other by two open sets.

Example 4.3.2. Any set with the discrete topology is Hausdorff. No set with more than one element, when equipped with the indiscrete topology, is Hausdorff.

Example 4.3.3. It can be illustrative to see what in the topological structure of a space makes it Hausdorff. A quick glance at Example 4.2.5 shows that the argument there in fact shows that $\mathbb{R}$ is a Hausdorff space.

The Cantor space is Hausdorff. An easy way to see this goes through the following: given n a natural number, note that there are finitely many sequences of length n. Denote by $Len(n)$ the set of all such sequences. Now if $t \in Len(n)$ then:

$$2^\omega - C(t) = \bigcup_{s \in Len(n) - \{t\}} C(s).$$

[1] Historic note: this narrative, whilst tempting, reverses the historical trend. The history of topology was one of progressive rarification of the concepts; at times what we refer to as "Normal spaces" were considered to be the full extent of topological spaces; later, axioms such as Hausdorff or T_1 were taken to be fundamental.

This implies that the complement of a basic open set is again open. So assume that $x, y \in 2^\omega$ and $x \neq y$. Then this means that there is a finite sequence t such that $t \lhd x$ and $\neg(t \lhd y)$, so $x \in C(t)$ whilst $y \in 2^\omega - C(t)$. By what we just argued, both of these sets are open, and their intersection is disjoint.

Having gained some intuition from these examples, we can now see that Hausdorff spaces are, in a precise sense, exactly those spaces where filters have unique limit points whenever they converge:

Theorem 4.3.4. *Let (X, τ) be a topological space. Then the following are equivalent:*

1. *X is Hausdorff;*
2. *For each filter F on X, F converges to at most one point;*

Proof. (1) implies (2): Assume that X is Hausdorff. Let F be a filter on X, and suppose that F converges to x and y. By assumption, there are some U, V, open sets, such that $x \in U$ and $y \in V$ and $U \cap V = \varnothing$. By assumption, there are $U' \subseteq U$ and $V' \subseteq V$, and $U', V' \in F$. Because F is a filter, $U' \cap V' \in F$; but then $\varnothing \in F$, a contradiction. So by reductio, we conclude that F can converge to at most one point.

(2) implies (1): We go by contraposition, assuming that X is not Hausdorff. Assume that $x \neq y$ but for each pair of open sets U, V such that $x \in U$ and $y \in V$, then $U \cap V \neq \varnothing$. Hence consider the collection:

$$F := \{U \cap V : x \in U, y \in V, U, V \text{ are open }\}.$$

Note that F is a filter base: surely $x \in X$ and $y \in X$; if $U \cap V \in F$ and $U' \cap V' \in F$, then $U \cap U'$ is an open set containing x and $V \cap V'$ is an open set containing y, so $(U \cap U') \cap (V \cap V') \in F$; and $\varnothing \neq F$. Taking the upwards closure of F, $F^{\uparrow}$ we obtain a filter. Now we claim that $F^{\uparrow}$ converges to both x and y.

Indeed if U is a neighbourhood of x, then since $y \in X$, $U \cap X \in F$, so $U \in F$. Similar arguments hold for y. Hence $F^{\uparrow}$ converges to two distinct points. ∎

We conclude by noting that the Hausdorff property is nicely preserved under some topological constructions, but not all of them:

Proposition 4.3.5. Let X and $(X_i)_{i \in I}$ be Hausdorff spaces. Then:

1. If $Y \subseteq X$ is a subspace, then Y is Hausdorff;
2. $\prod_{i \in I} X_i$ is Hausdorff.
3. $\coprod_{i \in I} X_i$ is Hausdorff.

Proof. Exercise. ∎

Crucially, quotients need not preserve this property (or separation properties in general), as the next example shows:

Example 4.3.6. Consider $\mathbb{R} + \mathbb{R}$, the sum of two disjoint copies of the reals, where elements are denoted as $\langle x, 0\rangle$ or $\langle x, 1\rangle$ depending on the summand they belong to. Now consider the equivalence relation which identifies $\langle x, 0\rangle$ and $\langle x, 1\rangle$ if and only if $x \neq 0$. The quotient under this equivalence relation can be visualised in Figure 4.1:

$$
\begin{tikzcd}
\ldots & -2 \arrow[l] & -1 \arrow[l] & (0,0) \arrow[l] \arrow[r, bend left] & (0,1) \arrow[l, bend left] \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \ldots
\end{tikzcd}
$$

Figure 4.1: Reals with a Double Point

Then note that this double point cannot be separated by any open set: take any open neighbourhood U_0 containing $[\langle 0,0\rangle]$ and a neighbourhood U_1 containing $[\langle 0,1\rangle]$. Since they are open, $q^{-1}[U_0]$ must be open as well by the quotient topology. By the definition of the topological sum, and the usual Euclidean topology we then have that:

$$q^{-1}[U_0] = \bigcup_{i\in I}(\langle a_i, 0\rangle, \langle b_i, 0\rangle).$$

Now since $\langle 0,0\rangle$ belongs in $q^{-1}[U_0]$, it belongs to some interval, say $(\langle a,0\rangle,\langle b,0\rangle)$ where a,b are real numbers; similar for an interval $(\langle c,1\rangle,\langle d,1\rangle)$, included in $q^{-1}[U_1]$. Now define

$$\varepsilon := \min(|a-b|,|c-d|)$$

and look at $\langle \varepsilon,0\rangle$ and $\langle \varepsilon,1\rangle$. It is not hard to see that this will be in $(\langle a,0\rangle,\langle b,0\rangle)$ and also $(\langle c,1\rangle,\langle d,1\rangle)$. Hence $[\langle \varepsilon,0\rangle] = [\langle \varepsilon,1\rangle] \in U_0 \cap U_1$. Thus, we cannot separate $\langle 0,0\rangle$ and $\langle 0,1\rangle$ using disjoint open neighbourhoods. ■

4.4 Weaker Separation Axioms

Whilst Hausdorff topologies are extremely convenient, there are some cases where one might need weaker conditions. An example from geometry:

Example 4.4.1. Suppose that you take the collection $\mathbb{R}$. We denote by $Pol(R)$ the collection of all polynomials over $\mathbb{R}$ of the form:

$$f(x) = a_n x^n + \ldots + a_1 x + a_0$$

Now define a topology on $\mathbb{R}$, called the *Zariski topology* τ_Z, as follows: U is *closed* if and only if $U = \{x \in \mathbb{R} : \exists S \subseteq Pol(R), \forall f \in S, f(x) = 0\}$: that is, U is the set of zeros of the polynomials $f \in S$. Then note that given Z and Z', two open sets, $Z \cap Z'$ will be non-empty, since their complements are finite (any polynomial has only finitely many zeros). Hence this space cannot be Hausdorff.

The former is a naturally occuring example which motivates the following definition:

Definition 4.4.2. Let (X,τ) be a topological space. We say that X is *Frèchet* or T_1 if for all $x \neq y \in X$ there exists an open neighbourhood U_x such that $x \in U_x$ and $y \notin U_x$.

We can generalise the former example, to something which almost always yields a T_1 space which is not Hausdorff:

Example 4.4.3. Let X be an infinite set. Define a topology on X as follows: $U \subseteq X$ is open iff either U is empty, or U contains all but finitely many points. Then (X, τ_{cof}) is a T_1 topology: given $x \neq y$, pick the set containing all but y, and this provides an open neighbourhood of x excluding y.

However, this space is not Hausdorff: given any two open sets U,V if $U \cap V = \varnothing$, then since all non-empty subsets contain all but finitely many points, this can only happen if either U or V is empty.

Sometimes, though, the inherent symmetry of the T_1 space can cause us problems. Especially when dealing with very weak topologies, like Alexandroff topologies, it can be useful to have an even weaker property:

Definition 4.4.4. Let (X, τ) be a topological space. We say that two points x, y are *topologically distinguishable* if there exists an open neighbourhood $U_{x,y}$ such that either $x \in U_{x,y}$ and $y \notin U_{x,y}$ or $y \in U_{x,y}$ and $x \notin U_{x,y}$. We say that the space X is T_0 if all pairs of points are topologically distinguishable.

Example 4.4.5. Any set with the indiscrete topology with more than one element is not T_0.

For an example of a space which is T_0 and not T_1 consider the *Sierpinski space* (which we have briefly encountered in Chapter 2): this is the topology on the set $\{a, b\}$, where we say that $\{b\}$ is open and not closed. It is easy to see that the points are topologically distinguishable, however, the distinction can only be made in one direction.

Example 4.4.6. Consider $\mathfrak{F} = (W, R)$ a Kripke frame. Then one can prove the following two facts, related to these weak separation properties:

1. The topological space induced by $\mathfrak{F}$ is T_1 if and only if R corresponds to the identity.
2. The topological space induced by $\mathfrak{F}$ is T_0 if and only if R is a partial order (i.e., it is antisymmetric).

You can check that any T_0 Alexandroff space induces a Kripke frame when one considers the following partial order: $x \leqslant y$ if and only if whenever $x \in U$ then $y \in U$ for U an open set. In this sense, the order denotes "having more knowledge". In a sense, this means that *only T_0 spaces are interesting for epistemic settings with a dynamic component.*

A mathematically very natural example of a space satisfying the axiom T_0 but not necessarily any stronger separation, which is related to the former, is that presented by so-called "spectral spaces". These are abundant in mathematics, occurring in algebraic geometry, and in functional analysis; in logic they appear through duality theory and the study of algebraic logic and topos theory. Additionally, the notion of topological distinguishability is quite interesting in and of itself, and provides a gateway to the area of *point-free topology*; see the Exercises for more on this.

4.5 Stronger Separation Axioms

On the flip side, we might sometimes need stronger properties than Hausdorff. We mention one very common strengthening:

Definition 4.5.1. Let (X, τ) be a Hausdorff topological space. We say that X is *normal* or T_4 if whenever E, F are disjoint closed sets, then there exist open sets U, V, $E \subseteq U$ and $F \subseteq V$, such that $U \cap V = \varnothing$.

Normality is a very strong, and very desirable property when working with geometry. This is because it allows us to relate arbitrary topological spaces to the real line:

Definition 4.5.2. Let X be a topological space. We say that two disjoint closed subsets E, F are *separated by a continuous function* if there is a map $f : X \to [0, 1]$ such that $E \subseteq f^{-1}[\{0\}]$ and $F \subseteq f^{-1}[\{1\}]$.

We have the following fact, known as Urysohn's Lemma, which we mention without proof:

Lemma 4.5.3. Let X be a T_1 space. Then X is normal if and only if every pair of disjoint closed subsets can be separated by a continuous function.

A weakening of this, sometimes referred to as a *Tychonoff space*, yields an important class in its own right:

Definition 4.5.4. Let X be a space. We say that X is *Tychonoff* if it is T_1 and whenever A is closed and $x \notin A$, then A and $\{x\}$ are separated by a continuous function.

The following proposition follows immediately from Urysohn's Lemma:

Proposition 4.5.5. If X is a normal topological space, then X is Tychonoff.

Example 4.5.6. The space $\mathbb{R}$ is normal. To see this, given A an arbitrary subset, denote by $f_A : \mathbb{R} \to [0, \infty) \subseteq \mathbb{R}$ the following function:

$$x \mapsto d(x, A) := \inf\{|x - a| : a \in A\}.$$

We have that f_A is a continuous function. Indeed if (c, d) is any interval, we want to show that $f_A^{-1}[(c, d)]$ is open. Let x be a point there. Hence $0 < d(x, A) < d$. Let $\varepsilon < |d - d(x, A)|$, and look at the interval $(x - \varepsilon, x + \varepsilon)$. Then if z belongs to that interval, we have that $d(z, A) = \inf\{|z - a| : a \in A\} \leqslant \inf\{|z - x| + |x - a| : a \in A\} < |z - x| + d(x, A)$. But the latter is less than or equal to $|d - d(x, A)| + d(x, A)$, which since $d(x, A)$ and d are positive, means that $d(z, A) < d$. This shows continuity. We think of $f_A(x)$ as outputting the distance of x from the set A.

Now then, assume that E and F are disjoint closed subsets. Then look at the function:

$$h(x) := \frac{f_E(x)}{f_E(x) + f_F(x)}$$

Note that this function is well-defined, given that $A \cap B = \varnothing$, i.e., every point is at a positive distance from either E or F. It is not hard to prove that h is then a continuous function. Moreover, if $x \in E$, $h(x) = 0$, and if $x \in F$, $h(x) = 1$. Hence $E \subseteq h^{-1}[\{0\}]$ and $F \subseteq h^{-1}[\{1\}]$. Hence by Urysohn's Lemma, we have that $\mathbb{R}$ is normal. ■

On the other hand, the existence of Hausdorff spaces which are not normal is a rich area of study in the field of *set-theoretic topology*. We discuss one example here.

Example 4.5.7. (Sorgenfrey Plane) Consider the following topology τ_{sor} on $\mathbb{R}$: we say that a set U is basic open if and only if it is of the form $[a, b)$, i.e., an half-open interval. We call this the *Sorgenfrey line*. Now let $\mathbb{S} = \mathbb{R} \times \mathbb{R}$ be the plane obtained by taking the product topology on two copies of the Sorgenfrey line. The basic opens are then the squares which are open on the right and closed on the left, as detailed in Figure 4.2. Note that this space is Hausdorff: if $\langle x, y\rangle, \langle a, b\rangle$ are two distinct pairs of real numbers, then we have a few possibilities:

- If $x = a$ then $y \neq b$, so without loss of generality, assume that $y < b$, and pick c an intermediate point. Then for some $x < d$, we have that $\langle x, y\rangle \in [a, d) \times [y, c)$ and $\langle a, b\rangle \in [a, d) \times [c, b + \varepsilon)$, which are disjoint sets.

- If $x \neq a$, a similar argument to the previous point shows that they belong to disjoint open sets.

Now consider:

$$\Delta = \{\langle x, -x\rangle : x \in \mathbb{R}\}$$

We call this set the *antidiagonal*. We can show that it is *closed*: if a point $\langle a, b\rangle \notin \Delta$, then we can pick a sufficiently small neighbourhood to avoid the antidiagonal entirely. This can be done by taking the minimum of the distances from a to Δ and b to Δ, and picking a neighbourhood around $\langle a, b\rangle$ of that size, as in Example 2.3.7. Additionally, the induced subspace is *discrete*: given $\langle x, -x\rangle$, consider the rectangle $[-x, -x+\varepsilon) \times [x, x+\varepsilon)$; then this intersects the antidiagonal only at the point $\langle x, -x\rangle$, showing that all singletons are open in the subspace Δ.

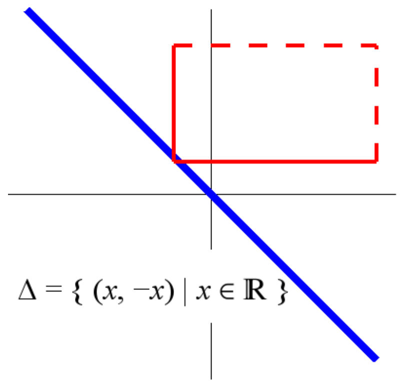

Figure 4.2: Antidiagonal on the Sorgenfrey plane

To show that the space cannot be normal, we now simply need the following Lemma due to Jones:

Lemma 4.5.8. (Jones Lemma) Let X be a normal topological space. If D is a closed and discrete subspace, then $2^{|D|} \leqslant 2^{\aleph_0}$.

Since $|\Delta| = 2^{\aleph_0}$, by Cantor's theorem, $2^{|\Delta|} > 2^{\aleph_0}$. Using the above lemma, this shows that the space cannot be normal. ■

The former example also provides a key difference with Hausdorff spaces: one can show that the Sorgenfrey *line* is a normal space, hence, the product of normal spaces need not be normal. The full extent of how products and normality relate is not yet full understood, and is related to various interesting areas of set-theoretic topology, including the existence of *Dowker spaces* (shown by Mary Ellen Rudin and Saharon Shelah) and the famous "Normal Moore space conjecture", eventually proven to be independent.

As we will see in the next chapter, normal spaces occur quite naturally in the presence of more topological ingredients, like compactness. We also note that all of the properties we mentioned (Hausdorff, Frèchet and Normality) are topological properties in the sense we specified above:

Proposition 4.5.9. If $X \cong Y$ then X is Hausdorff (resp. Frèchet, Normal) if and only if Y is Hausdorff (resp. Frèchet, Normal).

Nevertheless these properties are not in general preserved by continuous functions, as the above Example 4.3.6 shows, and you are asked to look at in the Exercies.

We conclude with a small table summarising the separation classes we have seen so far, and some important examples:

Separation Class	**Kind of Separation**	**Non-Examples**
T_0	$x \neq y$ then x, y are top. distinguishable	Ind. Top
T_1	$x \neq y$ then $\exists U \in \tau$, $x \in U \not\ni y$	Sierpinski
T_2	$x \neq y$ then $\exists U, V \in \tau$, $x \in U, y \in V$, $U \cap V = \varnothing$	Cof. Top.
T_{Π}	$T_2 + A$, $x \notin A$, $\exists f : X \to [0,1]$, $f(x) = 1$, $A \subseteq f^{-1}[\{0\}]$	
T_4	$T_2 + E \cap F = \varnothing$, closed, $\exists U, V \in \tau$ $U \cap V = \varnothing, E \subseteq U, F \subseteq V$	Sorgenfrey

4.6 Exercises

Exercise 4.1. Show the following:

1. Prove the statements in Example 4.4.6 regarding T_1 and T_0.

2. Give an example of a T_1-Kripke frame $\mathfrak{F}$, and a continuous function f to another Kripke frame $\mathfrak{F}'$, such that $\mathfrak{F}'$ is not T_1.

3. Give an example of a frame $\mathfrak{F} = (W, R)$, with its canonical Alexandroff topology, and a filter F on this frame which converges to two distinct points. Can you find an example which is T_0? What about T_1?

Exercise 4.2. Let (X, τ) be a topological space. Let $A \subseteq X$ be a subset. We say that A is *dense in* X if $\overline{A} = X$.

1. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

2. Show that is A is dense in X, then A intersects all open subsets of X.

3. Let X be a topological spaces and Y a Hausdorff space, and let $A \subseteq X$ be a dense subset. Let $f, g : X \to Y$ be two continuous functions such that $f\restriction_A = g\restriction_A$. Show that $f = g$.

4. Show that if A is dense in X, then every filter over A (i.e., $F \subseteq \mathcal{P}(A)$) converges to some point in X.

Exercise 4.3. Show that the following property is equivalent to a space X being Hausdorff: the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Exercise 4.4. Show the following for a T_1-space X:

1. If X is finite, then the topology on it is discrete.

2. For each $x \in X$, $\{x\}$ is closed.

3. For each $x \in X$, the filter
$$F(x) := \{S \subseteq X : x \in S\}$$
converges uniquely to x.

Show that the last property is an alternative definition for T_1-spaces.

Exercise 4.5. Let $L = (L, <)$ be a linearly ordered set. We can induce a topology called the *order topology*, by specifying that the following sets are subbasic opens for each $a \in L$:

$$(a, \to) := \{b \in L : a < b\} \text{ and } (\leftarrow, a) := \{b \in L : b < a\}$$

1. Show that the order topology is always a Normal topological space.

2. (* In case you know ordinals): Let α be an ordinal with the order topology. Show that the limit points in this space are exactly the limit ordinals. Show that a set is closed and unbounded in the set-theoretic sense if and only if it is unbounded and topologically closed.

Exercise 4.6. Let X be a topological space. Define an equivalence relation on X as follows: $x \sim y$ if and only if for all U open subsets, $x \in U$ if and only if $y \in U$. Show that $X/\sim$ is always a T_0 space.

Definition 4.6.1. Let X be a topological space. We say that a subset $S \subseteq X$ is *irreducible* if there do not exist two distinct closed subsets U_0, U_1 such that $U_0 \cup U_1 = S$. We say that X is *sober* if whenever S is an irreducible subset, there exists some $x \in X$ such that:

$$S = \overline{\{x\}}$$

Exercise 4.7. Show that every Hausdorff space is sober. Also show that sobriety is not comparable to the T_1 condition, i.e.:

1. There is a T_1 space which is not sober;

2. There is a sober space which is not T_1.

Definition 4.6.2. Let X be an arbitrary space. We say that a filter F over X is *completely prime* if for all families $(U_i)_{i \in I}$ of open sets:

$$\bigcup_{i \in I} U_i \in F \iff \exists i \in I, U_i \in F$$

Exercise 4.8. The purpose of the following exercise is to show that sobriety as defined above can be characterised by completely prime filters.

1. Show that if X is any topological space, for each $x \in X$, $\mathcal{N}(x)_{op}$ – the family of *open neighbourhoods* of x – is in fact a completely prime filter.

2. (Tricky!) Show that a space X is sober if and only if whenever F is a completely prime filter, there is some $x \in X$ such that $F = \mathcal{N}(x)_{op}$.

Chapter 5

Compactness

5.1 Extending Filters and Existence of Points

In the previous chapter we began by talking about two kinds of theories: definitive ones, and saturated ones. We have seen that filters implement naturally the notion of a theory, and converging filters provide us with a notion of a definitive theory. We now need something that corresponds to saturation. As one can expect, these will also be special kinds of filters:

Definition 5.1.1. Let X be a set and F a filter. We say that F is a *prime filter* (or sometimes a fluffy filter[1]) if it satisfies the following:

- For each $S \subseteq X$, either $S \in F$ or $X - S \in F$.

Before proceeding, we collect some essential properties of fluffy filters in the next lemma. In particular, part (4) will show that fluffy filters implement the notion of being a saturated theory. The proof is left as an exercise, which the reader not already familiar with filters is strongly encouraged to attempt:

Lemma 5.1.2. Let X be a set. Then:

1. If I is a totally ordered set, and $(G_i)_{i \in I}$ is a chain of filters, then $\bigcup_{i \in I} G_i$ is a filter.
2. If F is a filter and $A \subseteq X$ is a subset such that for each $B \in F$, $A \cap B \neq \emptyset$, then:
$$F \oplus A := \{C : A \cap B \subseteq C\}$$
is a filter, and is the smallest filter extending F which contains A.
3. The collection $\{U \subseteq X : x \in U\}$ is a prime filter.
4. The condition of being a prime filter is equivalent to the following: for each U, either U or $X - U$ belongs to F.

Proof. Exercise. ■

One key result by Tarski we will need is that filters can always be extended to prime filters. This involves a use of the Axiom of Choice, namely through Zorn's Lemma:

[1] See the end notes of Chapter 7 for an explanation of this nomenclature.

Theorem 5.1.3. *(Tarski, Prime Filter Theorem) Let X be a set and F a filter base on X. Then there exists a prime filter $G \supseteq F$.*[2]

Proof. We note that since each filter base can be extended to a filter by taking its upwards closure, it suffices to show that each filter can be extended to a prime filter. Let F be such a filter. Consider the following set:

$$\{G : G \text{ is a filter }, G \supseteq F\}$$

We have that ordering this set by inclusion we have a partially ordered set, such that each of its chains has an upper bound: indeed if $(G_i)_{i \in I}$ is a collection of filters extending F, then consider:

$$\bigcup_{i \in I} G_i.$$

By Lemma 5.1.2, we have that this is a filter, and it extends F. So by Zorn's Lemma, we have that there exists some maximal element, say G. We claim that G is prime. Assume that $U \notin G$, and $X - U \notin G$; then by upwards closure, there can be no $C \subseteq X - U$ such that $C \in G$. Hence for each C, $C \cap U \neq \varnothing$, since:

$$C \cap U = \varnothing \implies C \subseteq X - U.$$

So by Lemma 5.1.2, we have that there is a filter $G \oplus U$, which properly extends G; but this is a contradiction, since G was maximal. Thus either $U \in G$ or $X - U \in G$, which shows that G is a prime filter. ∎

With this in mind, let us proceed to see the topological meaning of these filters.

5.2 Compact Topological Spaces

Returning to our epistemic intuitions, we now have all the tools for a more formal discussion. We have seen in the previous chapter that definitive theories are saturated in Hausdorff spaces: they can be identified with the filters $\mathcal{N}(x)$, which as stated above, is prime, and hence, implements a saturated theory. So what about the converse?

To understand this, suppose that our epistemic landscape consists of countably many points x_n, where x_n stands for a world where an alarm clock will ring after n many seconds. Hence we have the (falsifiable) proposition:

$$P_n \coloneqq \text{“The alarm clock will ring after more than } n \text{ many seconds”}.$$

Certainly in our epistemic landscape each of the propositions P_n is consistent – there is some world where the alarm clock rings after more than n many seconds. But there is an obvious option that seems to be missing – *what happens if the alarm clock is simply broken*? Our model seems to have the finite traces of this possibility – for each n, we have a world where the alarm clock takes longer than n many seconds to play – but not a world where it simply does not play at all.

This leads us to another crucial notion of topology: *compactness*. As our example hints, compactness can be thought of as a *completeness property* of our spaces, since it forces the existence of points. However, the topological notion of compactness we often encounter appears to be very different. To properly introduce it we will need some more basic concepts.

[2]The reader who is familiar with *lattices* will find that the former carries out for any distributive lattice. This will be discussed in MSL.

Definition 5.2.1. Let X be a topological space, and $A \subseteq X$. Given a collection $(U_i)_{i\in I}$ of open sets, we say that this is an *open cover* of A if:

$$A = \bigcup_{i\in I} U_i.$$

Given such a cover, we say that a subcollection $(U_j)_{j\in J}$ for $J \subseteq I$ is a *subcover* if it is a cover of A. We say that a cover is *finite* if I is finite.

Definition 5.2.2. Let X be a topological space. We say that X is *compact* if whenever $(U_i)_{i\in I}$ is an open cover, there exists a finite $I_0 \subseteq I$ such that $(U_j)_{j\in I_0}$ is a subcover of X.

Example 5.2.3. The set $\mathbb{R}$ with the usual topology is *not* compact. For instance, we can consider the following open cover:

$$\{(k, k+2) : k \in \mathbb{Z}\}$$

which, if we remove all but finitely many of the intervals, will leave gaps. [3]

Similarly, the Baire space ω^ω is also not compact: one can consider the cover:

$$\{C((n)) : n \in \omega\}$$

Which, if we take away a single point, will leave infinitely many points uncovered. ■

As in other situations, it suffices to check compactness on a basis to obtain compactness of the whole space:

Proposition 5.2.4. Let (X, τ) be a topological space with a basis $\mathcal{B}$. Then a set is compact if and only if every cover of X by basic open sets has a finite subcover.

Proof. If the set is compact surely every such cover has a finite subcover. Conversely, assume that every cover of a set by basic opens has a finite subcover. Suppose that:

$$X = \bigcup_{i\in I} U_i$$

where each U_i is open. By the properties of the basis, we have that:

$$X = \bigcup_{i\in I} \bigcup_{j\in J_i} V_j$$

which is a cover of the set by basic opens. We can then extract a finite subcover:

$$X = V_{j_0} \cup ... \cup V_{j_k}$$

which entails the result. ■

[3]However, it is well-known that the interval $[0, 1]$, which is homeomorphic to adding a point above and below the reals, is compact. This is the so-called *Heine-Borel Theorem*. The reader is asked to show this, using the ideas of the corresponding proof for the Cantor space, in the Exercises.

The corresponding result for *subbases* is indeed true, but more difficult. This is often referred to as *Alexander's Subbase theorem*, and it is a very valuable tool when dealing with spaces defined using subbases[4]. On a first pass, the reader may want to skip the proof, and instead focus on the applications of it (e.g., below Tychonoff's theorem, or, in MSL, the proof of compactness of Priestley spaces).

Theorem 5.2.5. *(Alexander Subbase Theorem) Let X be a topological space with a subbasis $\mathcal{S}$. Then X is compact if and only if every cover of X by subbasic opens has a finite subcover.*

Proof. One direction is easy. For the other, assume that $\mathcal{B}$ is a cover of X which does not contain any finite subcover. Using a Zorn's Lemma argument, we can assume that $\mathcal{B}$ is maximal: indeed, let P be the set of open covers of X which contain no finite subcover. If $\mathcal{B}_i$ is a chain by inclusion of such objects, then we claim that:

$$\bigcup_{i \in I} \mathcal{B}_i$$

is again such an object. Indeed, it will clearly be a cover of X. Additionally, if it contained a finite subcover, say $U_0, ..., U_n$, note that since this is a chain, we can place $U_0, ..., U_n$ together in some $\mathcal{B}_k$, which contradicts our hypothesis. This shows that the poset P is in the conditions of Zorn's Lemma, and hence, there is some maximal $\mathcal{B}'$.

Now note that $\mathcal{B}' \cap \mathcal{S}$, where the latter is the subbasis, cannot cover X, since otherwise, by our assumption, we could extract a finite subcover from $\mathcal{B}'$. So let $x \in X - \bigcup(\mathcal{B}' \cap \mathcal{S})$. Since $\mathcal{B}'$ is a cover, there is some U such that $x \in U$; since $\mathcal{S}$ is a subbasis, we have that:

$$U = \bigcup_{k \in K} V_{k_0} \cap ... \cap V_{k_l}$$

so $x \in V_{k_0} \cap ... \cap V_{k_l}$. Now by choice of x, then $V_{k_l} \notin \mathcal{B}'$. Since $\mathcal{B}'$ was chosen to be maximal, we have that since $\mathcal{B}' \cup \{V_{k_l}\}$ is a cover of X (since $\mathcal{B}'$ is one already), it must contain a finite subcover. Hence, for each l we can obtain a finite subcover:

$$X = C_0^l \cup, ..., \cup C_n^l \cup V_{k_l}$$

where $C_0^i \in \mathcal{B}'$. It follows that the following is a cover of X:

$$X = \bigcap_{l=0}^{l} C_0^l \cup ... \cup C_n^l \cup V_{k_l} \subseteq (\bigcup_{l=0} (C_0^l \cup ... \cup C_n^l)) \cup (V_{k_0} \cap ... \cap V_{k_n})$$

But note that the latter is a finite cover of X by elements all in $\mathcal{B}'$, which is a contradiction to our hypothesis. So by reductio, there must be a finite subcover of $\mathcal{B}$. ■

Let us now move to some positive examples:

Example 5.2.6. The Cantor space *is compact*. To see this, we take a cover of 2^ω by basic open sets; that is, take a collection $T \subseteq 2^{<\omega}$ and assume that

$$2^\omega = \bigcup_{s \in T} C(s).$$

[4]It is also interesting to note that this result makes necessary use of a weak form of the Axiom of Choice – indeed, what is necessary is exactly the Prime Filter Theorem we have just stated.

Assume towards a contradiction that there is no finite subcover of this cover. Notice that this means that the set $C(\varnothing)$ admits no finite subcover. But this must mean that either $C(\langle 0\rangle)$ or $C(\langle 1\rangle)$ admits no finite subcover by these sets. Let s_0 be one such sequence which does not admit any finite subcover. Proceeding inductively, we assume that we have established that $C(s_n)$ admits no finite subcover; then using the same argument, we have that either $C(s_n^\frown 0)$ or $C(s_n^\frown 1)$ admits no finite subcover.

Hence, we obtain an infinite binary sequence:

$$x = \lim_{n\in\omega} s_n$$

composed of each finite part of the previous sequences, i.e., such that for each n, $C(x\restriction_n)$ admit no finite subcover from the above cover. By assumption $x \in 2^\omega = \bigcup_{s\in T} C(s)$, so $x \in C(s)$ for some s. Hence $s = s_n$ for some n. But by hypothesis, $C(s_n)$ admitted no finite subcover, a contradiction. ∎

Let us now prove some properties of compact spaces. An important such property connects back to our intuition of compactness as a completeness property, and the example provided in the beginning of the section. We begin with a reformulation of compactness using closed sets:

Definition 5.2.7. Let X be a set, and $S \subseteq \mathcal{P}(X)$ a family of subsets. We say that S has the *finite intersection property* if whenever $A_0, ..., A_n \in S$ then $A_0 \cap ... \cap A_n \neq \varnothing$.

Lemma 5.2.8. Let X be a topological space. Then X is compact if and only if, whenever $\mathcal{F}$ is a family of closed subsets with the finite intersection property, then $\bigcap \mathcal{F} \neq \varnothing$.

Proof. Assume that X is compact, and let $\mathcal{F}$ be a family of closed subsets with the f.i.p. Assuming that $\bigcap \mathcal{F} = \varnothing$ we get that $\bigcup_{U\in\mathcal{F}} X - U = X$; by compactness, we can extract a finite subcover, say:

$$X = X - U_0 \cup ... \cup X - U_n.$$

Hence the finite intersection $U_0 \cap ... \cap U_n$ is empty, a contradiction. The converse is similar and left as an exercise. ∎

Using this criterion, we can provide the desired equivalent notion to compactness, which relates it to the existence of points.

Proposition 5.2.9. Let (X, τ) be a topological space. Then X is compact if and only if whenever $\mathcal{U}$ is a fluffy filter, then $\mathcal{U}$ converges in X.

Proof. Let $\mathcal{U}$ be the prime filter and look at:

$$\bigcap \mathcal{U}_{cl} := \{C : C \in \mathcal{U}, C \text{ is closed }\}.$$

We know that $\mathcal{U}_{cl}$ is a family of closed sets with the f.i.p., so since the space is compact, by Lemma 5.2.8, the intersection is non-empty. Now assume that $x \in \bigcap \mathcal{U}_{cl}$; suppose that U is an open neighbourhood of x; we claim that $U \in \mathcal{U}$. Indeed, if not, then $X - U \in \mathcal{U}$; since U is open, this in turn implies that $X - U \in \mathcal{U}_{cl}$. Hence by definition $x \in X - U$, a contradiction. Hence $U \in \mathcal{U}$, so trivially $U \subseteq U$ and $x \in U$; this shows that $\mathcal{U}$ converges to x, as intended.

Conversely, assume that each prime filter converges to at least one point. Assume that $X = \bigcup_{i \in I} U_i$ is a cover of X by open sets which has no finite subcover. Hence consider:

$$\{X - U_{i_0} \cap ... \cap X - U_{i_n} : \{i_0, ..., i_n\} \subseteq I\}$$

Note that by hypothesis, this is a collection of non-empty sets, since if $X - U_{i_0} \cap ... \cap X - U_{i_n} = \varnothing$, then $U_{i_0} \cup ... \cup U_{i_n} = X$. Moreover, X belongs above, since we can take the empty family, and it is not difficult to see that elements of this set are closed under intersection. Hence the above forms a filter base. By the Prime filter Theorem, we have that this can be extended to a prime filter, $\mathcal{U}$; by hypothesis, we have that $\mathcal{U}$ converges to a point x. But then we have that $x \in \bigcap_{i \in I} X - U_i$, i.e., $x \notin \bigcup_{i \in I} U_i$, a contradiction to the fact that this is a cover. ■

To cap us off, we can also consider compactness relatively:

Definition 5.2.10. Let X be a topological space. We say that $A \subseteq X$ is a *compact subset* if A is compact as a topological space with the subspace topology.

Lemma 5.2.11. Let X be a compact topological space. If A is closed, then A is a compact subset.

Proof. Assume that A is closed, and suppose that:

$$A = \bigcup_{i \in I} U_i$$

where U_i are open in A. Hence for some $V_i \subseteq X$, open subsets, we have that $U_i = V_i \cap A$. Now since A is closed, we then have that:

$$X = \bigcup_{i \in I} V_i \cup X - A.$$

Indeed if $x \in X$, then either $x \notin A$, or, $x \in A$, and then for some i, $x \in V_i$ as well. So since X is compact, there is a finite subcover, i.e.:

$$X = V_0 \cup ... \cup V_n \cup X - A.$$

Then note that $A = U_0 \cup ... \cup U_n$: if $x \in A$, then for some $j \leqslant n$, $x \in V_j$; so $x \in U_j$. The converse inclusion is clear. ■

The former thus explains, at least in part, how compactness is preserved under subspaces. It is not very hard to see that topological sums will not preserve compactness in general, and that quotients will. The most interesting case with respect to the constructions we studied arises from products. We have the following theorem, which puts Alexanders' subbase theorem to good use:

Theorem 5.2.12. *(Tychonoff's Theorem) Let $(X_i)_{i \in I}$ be a collection of compact spaces. Then $\prod_{i \in I} X_i$ is compact.*

Proof. By Alexander's subbase theorem, note that it suffices to show that every cover of the product by a cover from subbasic opens contains a finite subcover. Recall by definition of the product space that the subbasic opens are sets of the form

$$p_i^{-1}(U_i),$$

where $i \in I$ is any coordinate and U_i is an open in X_i. Now indeed assume that:

$$\prod_{i \in I} X_i = \bigcup_{j \in J} p_{i_j}^{-1}(U_{i_j}).$$

By definition of the projection function, if we let $p_{i_0}^{-1}(U_{i_0})$ be arbitrary, then only finitely many coordinates $i_j \in I$ are left uncovered, since $p_{i_0}^{-1}(U_{i_0})$ is equal to X_{i_j} in all but one position. Now assume that i_0 is that position. Then by assumption we can cover X_{i_j} using some of the sets U_k ocurring in the cover above. By compactness of these spaces we can extract a finite subcover on each of these spaces, which yields a finite subcover of $\prod_{i \in I} X_i$, as intended. ∎

5.3 Compact Hausdorff spaces and Compactifications

In practice many important spaces one works with are compact Hausdorff, and these are very amenable, both for geometry and logic.

Theorem 5.3.1. *Let X be a topological space. If X is compact and Hausdorff, then:*

- *X is Normal;*
- *The compact subsets are precisely the closed ones.*
- *If $f : X \to Y$ is a continuous bijection between compact Hausdorff spaces, then f is a homeomorphism.*

Proof. (1) Assume that E and F are disjoint closed subsets. Note that by Lemma 5.2.11, we know these are compact. Fix $x \in E$; then since the space is Hausdorff, for each $y \in F$ we have that there is some $x \in U_{x,y}$ and $y \in V_{x,y}$ such that $U_{x,y} \cap V_{x,y} = \varnothing$; hence we have that:

$$F \subseteq \bigcup_{y \in F} V_{x,y}$$

so since F is compact, $F \subseteq V_{x,y_0} \cup ... \cup V_{x,y_n} = V_x$, where the latter does not contain x; hence look at $U_x = U_{x,y_0} \cap ... \cap U_{x,y_n}$, and note that this is open, and disjoint from V_x; hence we obtain a cover:

$$E \subseteq \bigcup_{x \in E} U_x$$

and because of compactness, then $E \subseteq U_{x_0} \cup ... \cup U_{x_n}$. So look at $V_{x_0} \cap ... \cap V_{x_n}$. Note the latter is an open set containing F, and is disjoint from $U_{x_0} \cup ... \cup U_{x_n}$, as intended.

(2) We already know that closed subsets are compact in any compact space. Now assume that A is compact. Suppose that $x \notin A$. Then for each $y \in A$, there is some U such that $x \in U_y$ and $y \in V_y$, where these are disjoint open sets. Hence $A \subseteq \bigcup_{y \in A} V_y$, and by compactness we can extract a finite subcover, say $A \subseteq V_{y_0} \cup ... \cup V_{y_n}$. Hence $x \in U_{y_0} \cap ... \cap U_{y_n}$, which is an open subset which is disjoint from A. Hence $X - A$ is open, which shows that A is closed.

(3) Assume that $f : X \to Y$ is a continuous bijection from a compact to a Hausdorff space. Let U be a closed set; then because U is compact, so $f[U]$ is compact by Exercise 5.3; hence $f[U]$ is a compact and since Y is compact Hausdorff, $f[U]$ is closed. Thus by an Assignment Exercise (See Assignment 1, Exercise 3), we have that f is a homeomorphism. ∎

Given these (and other) properties, we might want to take our spaces and turn them into compact Hausdorff spaces. Taking a space and making it Hausdorff is somewhat hopeless: we cannot separate a space in a way that relates naturally to the original topology. The situation is much better for compactness: finding such a space is not difficult, but it can be difficult to find one preserving all properties we are interested in. We thus need a general idea of what it is to turn a space into a compact one.

Definition 5.3.2. Let X, Y be topological spaces such that $f : X \to Y$ is a continuous function. We say that the pair (Y, f) is a *topological extension* of X if $f[X]$ is dense in Y (i.e., $\overline{f[X]} = Y$).[5]. We say that an extension is

- A *compactification*: if Y is compact;
- A *proper extension* if f is a homeomorphism and X is non-compact.
- A *strong compactification* if it is a proper extension, a compactification, and $f[X]$ is open in Y.

Let us look at some examples: first consider the set ω, and give it the discrete topology. It is easy to see that such a topology is Hausdorff, and is not compact (since the collection of all singletons is an open cover with no finite subcover). Now consider the space in Figure 5.1:

$$0 \qquad 1 \qquad 2 \qquad \dots \qquad \omega$$

Figure 5.1: One-point compactification

We give this space the following topology: a subset is open if and only if either it is a subset of the naturals, or a cofinite subset containing ω. This space is often denoted $\alpha(\omega)$, and is called the *Alexandroff one-point compactification*; you can verify that indeed it is compact, and also Hausdorff.

There are some general facts one can say about compactifications, which relate to some separation properties we have mentioned before, and which we mention without proof:

Proposition 5.3.3. Let X be a topological space. Then X can be topologically embedded in a compact space if and only if X is Tychonoff.

As such, we cannot expect a procedure that yields proper extensions in general. But for compactifications, it turns out that there is a general procedure following the same recipe as the construction of $\alpha(\omega)$, that works for all spaces:

Definition 5.3.4. Let X be a topological space. Let $X^* := X \sqcup \{\infty\}$, and topologise this as follows: a subset $U \subseteq X^*$ is open either if it is open in X, or if $U = X - C \cup \{\infty\}$ where C is a compact and closed subset of X.

Proposition 5.3.5. Let X be a non-compact topological space. Then (X^*, i) is a strong compactification of X.

Proof. Note that the inclusion is by definition a topological embedding. It is also dense: consider the element ∞, and let U be an open neighbourhood of this point. Since X is not compact, then

[5]Note: None of this terminology is standard, since the existing terminology seems to differ a lot between authors.

$U \neq \{\infty\}$, so $U \cap X \neq \varnothing$. Thus $\infty \in \overline{X}$. Finally, it is easy to see that the resulting space is compact: if $X = \bigcup_{i \in I} U_i$ is a cover by opens, then there must be some $U_i = X - C \sqcup \{\infty\}$; hence this space covers everything but a compact subset of X, for which we can extract a finite subcover from the cover $\bigcup_{i \in I} U_i$. ■

This compactifition is useful, as in most cases it adds the minimum number of points to make the space compact. However, it has the drawback that even when starting from a Normal space it might result in a non-Hausdorff space (see Exercise 5.7)!

A way to avoid this happening is to add some compactness to begin with. The following is one of many "local" properties which play a big role in all areas of topology:

Definition 5.3.6. Let X be a Hausdorff space. We say that X is *locally compact* if for each $x \in X$ there is a compact neighbourhood of x.

With this we can show the following:

Proposition 5.3.7. Let X be a non-compact Hausdorff space. Then $\alpha(X)$ is Hausdorff if and only if X is locally compact.

Proof. See Exercise 5.8. ■

However, this solution might not be particularly pleasant. One may want to *always* obtain a compact Hausdorff extension of our space[6]. This way we are lead to the notion of a *Stone-Cech compactification*.

Definition 5.3.8. Let X be a topological space. We say that a pair (Y, i) where $i : X \to Y$ is a *Stone-Cech compactification* if it satisfies the following property: if Z is a compact and Hausdorff space, and $f : X \to Z$ is a continuous function, there is a unique continuous function $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ i$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Z \arrow[d, dashed, "\overline{f}"] \\ & Y \end{tikzcd}$$

Figure 5.2: Stone-Cech Compactification

Whilst it might not be immediate from the definition, this construction *is unique*. Indeed, if (Y, i) and (Y', i') were two such compactifications, note that by definition then there are:

- $\overline{i} : Y \to Y'$ such that $i = \overline{i} \circ i'$
- $\overline{i'} : Y' \to Y$ such that $i' = \overline{i'} \circ i$.

[6] If you are familiar with the concept of an *adjunction* another way of saying this is as follows: is there a left adjoint to the inclusion of the category of compact Hausdorff spaces in the category of all topological spaces? By abstract nonsense (e.g., Freyd's adjunction theorem) it is possible to see that such a left adjoint must exist, and what we give here is a concrete description.

It follows that $i = \bar{i} \circ \overline{i'} \circ i$. But then by the same property, we have that 1_Y is the unique function such that $i = 1_Y \circ i$. So $\bar{i} \circ \overline{i'} = 1_Y$. Similarly $\overline{i'} \circ \bar{i} = 1_{Y'}$. This shows that the two spaces are homeomorphic, showing the uniqueness.

In light of this, the Stone-Cech compactification of a space X is usually denoted by βX. The construction of this is quite technical, so we delay the full definition. Instead, let us look at an example.

Example 5.3.9. Consider again the space ω with the discrete topology. Now let $\beta\omega$ denote the set of all prime filters over ω: all subsets of $\mathcal{P}(x)$ which are prime filters in the usual sense. We give this space the following topology: a basic open looks like

$$\varphi(U) := \{f \in \beta\omega : U \in f\}$$

where $U \subseteq X$. Then we can show that this space is compact and Hausdorff, and indeed, is a compactification in the sense outlined above.

We now proceed to give a general construction of the Stone-Cech compactification. As noted, the reader may wish to skip this in a first reading, and we provide it here only for completeness. To give it we will need some properties of the space $[0,1]$.

Lemma 5.3.10. Let X, Y be two compact Hausdorff spaces, and assume that $f, g : X \to Y$ are two distinct continuous maps. Then there is $h : Y \to [0,1]$ such that $hf \neq hg$.

Definition 5.3.11. Let X be an arbitrary topological space. Let $C(X)$ denote the set of continuous functions from X to $[0,1]$. Let $i : X \to [0,1]^{C(X)}$ (where the latter has the product topology) be given by: x maps to the function $g : C(X) \to [0,1]$ which, on function f, returns $f(x)$.

Lemma 5.3.12. The map $i : X \to [0,1]^{C(X)}$ is continuous.

Proof. We first show that the preimage of a subbasic open is open. Let $U = p_f^{-1}(a,b)$ where $0 \leqslant a < b \leqslant 1$ are real numbers. Hence these are functions from $C(X)$ to $[0,1]$ which map f to this interval. Hence we claim that:

$$i^{-1}[U] = \bigcup_{f \in C(X)} f^{-1}(a,b)$$

Indeed, if $i(x) \in U$, then $i(x)(f) = f(x) \in (a,b)$, so $x \in f^{-1}(a,b)$. Conversely, if $x \in f^{-1}(a,b)$, then certainly $i(x)(f) \in (a,b)$. Since the functions f are all continuous, and (a,b) is open, then $i^{-1}[U]$ is a union of open sets, and hence, open as well. ∎

By Tychonoff's Theorem 5.2.12, we have that $[0,1]^{C(X)}$ is compact, and by Proposition 4.3.5, it is Hausdorff. Hence let $K = \overline{i[X]}$, i.e., take the closure of the image of i in this space. It is clear that then $(K(X), i)$ is a Hausdorff extension of X, since it is a closed subset of a compact space, and compact as well.

Lemma 5.3.13. If X is Compact and Hausdorff, then $X \cong i[X]$.

Proof. We will show that the map i is injective. Assume that $x \neq y$. Hence because the space is compact and Hausdorff, by Proposition 4.5.3 there is a function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. Thus $f \in C(X)$, and so $i(x)(f) \neq i(y)(f)$. This shows that i is injective. Since X is compact, and $i[X]$ is Hausdorff, then we have by Theorem 5.3.1 that i is a homeomorphism. ∎

The next lemma, which we mention without proof, is useful below: [7]

Lemma 5.3.14. Assume that $m : X \to Y$ is a continuous map. Let $\hat{m} : [0,1]^{C(X)} \to [0,1]^{C(Y)}$ be given as follows: for each $g \in [0,1]^{C(X)}$, consider the function $g_m : C(Y) \to [0,1]$, such that if $f : Y \to [0,1]$ is arbitrary, it is mapped to $g(f \circ m)$. Then this is a continuous function on the product space.

Proposition 5.3.15. Given a topological space X, the pair $(K(X), i)$ is the Stone-Cech compactification of X.

Proof. Let Z be an arbitrary compact and Hausdorff space, and $m : X \to Z$ be a continuous function. By the previous lemma, consider $\hat{m} : [0,1]^{C(X)} \to [0,1]^{C(Z)}$, and consider the restriction of $\hat{m}$ to $K(X)$. This is a continuous function from $K(X)$ to $[0,1]^{C(Z)}$. Now let $g \in i[X]$ be arbitrary. Any such function, by construction, is determined by an element $x \in X$, hence $g = k_x$ for some $x \in X$. Now note that then $\hat{m}(k_x) = k_{m(x)}$: indeed, given $f : Z \to [0,1]$, $\hat{m}(k_x)(f) = k_x(f \circ m) = f(m(x))$. This then shows that:

$$\hat{m}(i[X]) \subseteq i[Z]$$

Since $\hat{m}$ is continuous, we have that $\hat{m}[\overline{i[X]}] \subseteq \overline{\hat{m}[i[X]]} \subseteq i[Z]$, where the last inclusion holds since the latter set is closed by Proposition 5.3.13. It follows that $\hat{m}[K(X)] \subseteq Z$, which means that $\hat{m}$ defines a continuous function from $K(X)$ to Z. It is also easy to see that $\hat{m} = i \circ m$.

We now note that the map is unique: if $l : K(X) \to Z$ was another map such that $l \circ i = m$, then this immediately says that $l \upharpoonright i[X] = \overline{m} \upharpoonright i[X]$, so because the space is Hausdorff (see Exercise 1.3 of Assignment 2), $l = m$. ■

Just like with the Alexandroff compactification, some of the properties of this construction work much better for special classes of spaces. We mention the following, without proof:

Proposition 5.3.16. Let X be a topological space. Then:

1. $\beta X = X$ if and only if the space is compact Hausdorff;
2. X is Tychonoff if and only if $(\beta X, i)$ is a proper extension.
3. X is normal, non-compact and locally compact, if and only if $(\beta X, i)$ is a strong compactification.

In the case where X is Tychonoff, there is a space of particular interest: the space $\beta X - X$, called the *remainder* is in that case closed, and hence has a rich topological and set-theoretical structure. From the algebraic and set-theoretic point of view, the space $\beta\omega - \omega$ in particular is a well-known structure called the *set of non-principal prime filters on* ω, and sometimes, the Parovicenko space. In the latter, it has special connections with combinatorial set theory and forcing; in the former it appears in the context of the duals of products of algebras. But this would take us much too far.

[7] Unlike other mentions without proof, the proof of this result is not necessarily very difficult, but it would be long in an already very long construction. We encourage the brave reader to try it as an exercise.

5.4 Exercises

Exercise 5.1. Prove the following form of the Heine-Borel theorem: the interval $[0, 1]$, with the subspace topology, is compact. *Hint: Use a similar idea as in Example 5.2.6*

Exercise 5.2. (*) Discuss the compactness of the following spaces (i.e., specify whether they are compact, and if not, whether there are natural subclasses of compact spaces):

1. Finite spaces;
2. Spaces with the Cofinite topology (see Example 4.4.3);
3. Spaces with the Discrete topology;
4. Transitive and reflexive Kripke frames with the Alexandroff topology;
5. Linear orders with the order topology.

Exercise 5.3. Let $f : X \to Y$ be a continuous map where X is a compact space. Show that $f[X]$ with the subspace topology is a compact space.

Exercise 5.4. Give an example of a compact space that is not Hausdorff *Hint: Consider the Alexandroff topology on linear orders.*

Exercise 5.5. Given an arbitrary topological space X show the following: X is compact if and only if for every topological space Y, $\pi_Y : X \times Y \to Y$ is a closed map.

Exercise 5.6. Show that if X is a topological space, and $U \subseteq X$ is an open subspace, then the inclusion $i : U \to X$ is a topological embedding.

Let $X = \mathbb{N}^2$ be the set of pairs of natural numbers. We construct a topology on this space as follows. A given $U \subseteq X$ is open iff:

- U does not contain $(0, 0)$ or,
- U contains $(0, 0)$ and for all but finitely many m, U contains all but finitely many elements from $\{(m, n) : n \in \omega\}$.

Then we call this the *Arens-Forts space*. A typical open of the second kind is shown in Figure 5.3.

Exercise 5.7. Consider the Arens-Forts Space, $\mathcal{U}$. Show that:

1. $\mathcal{U}$ is a normal space;
2. The space $\alpha(\mathcal{U})$ is *not* Hausdorff.

Hint: Show that the compact closed subsets can only contain finitely many points distinct from $(0, 0)$.

Exercise 5.8. Let X and X_i throughout be Hausdorff spaces.

1. Show that every compact space is locally compact.
2. Show that the converse inclusion does not hold *Hint: Think of* $\mathbb{R}$.

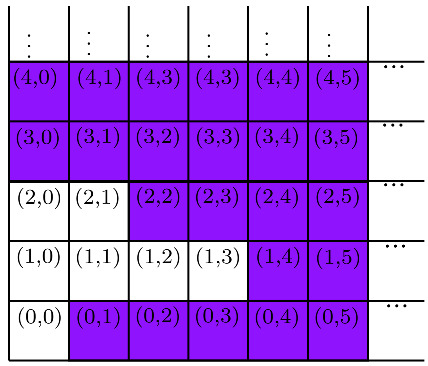

Figure 5.3: A typical Open in the Arens-Forts Space

3. Show that if X_i is a family of locally compact spaces, then $\bigsqcup_{i\in I} X_i$ is locally compact.

4. Show that if X is any Hausdorff and non-compact space, then $\alpha(X)$ is compact Hausdorff if and only if X is locally compact.

Definition 5.4.1. Let X be a topological space. We say that X has is *a Baire space*[8] if whenever $(U_n)_{n\in\omega}$ is a countable collection of dense open sets, then $\bigcap_{n\in\omega} U_n$ is a dense open set.

Exercise 5.9. (Tricky!) Show that:

1. The Baire space is a Baire space.

2. (*For the set-theoretically inclined*) Show the previous result as a consequence of the generic filter existence lemma/Rasiowa-Sikorski Lemma: if $(P, \leqslant)$ is a poset with a countable family of dense subsets, there is a generic filter which intersects all of them.

3. Show that a locally compact Hausdorff space is a Baire space.

4. Use the previous results to show the following: $\mathbb{Q}$ cannot be a countable intersection of open sets.

Exercise 5.10. (*For the categorically inclined*) The purpose of this exercise is to show that **Top** is *not cartesian closed*.

1. Show that if **Top** was cartesian closed, then for every space X, the functor $(-) \times X : \mathbf{Top} \to \mathbf{Top}$ must preserve coequalizers.

2. Provide an explicit description of a coequalizer of two topological spaces. I.e., construct coequalizers on the basis of the topological structures we have discussed. *Hint: Think about quotients.*

3. Consider the sets $\mathbb{Q}, \mathbb{Z}, \mathbb{R}$ and the two maps $i : \mathbb{Z} \to \mathbb{R}$ be the inclusion and $j : \mathbb{Z} \to \mathbb{R}$ given by $j(n) = i(n+1)$. Show that the coequalizer of these maps, $coeq(i, j)$ (in **Set**) is the quotient of $\mathbb{R}$ by the equivalence relation generated by the following: $x \sim y$ if and only if both x and y are integers.

[8] If you are wondering why anyone decided this was a nice name, in the presence of a space called "the Baire space", you are not alone.

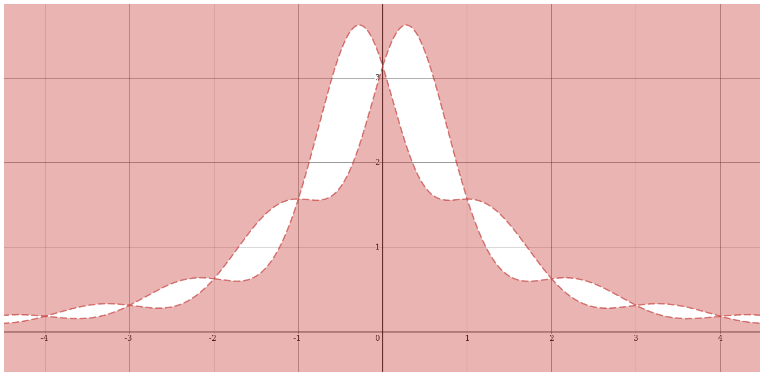

Figure 5.4: Functions from Exercise on cartesian closedness

4. Now consider the maps $i \times Q, j \times Q : \mathbb{Z} \times \mathbb{Q} \to \mathbb{R}$ defined in the obvious way. Show that the set-theoretic coequalizer of these maps has the same underlying set as $coeq(i, j) \times \mathbb{Q}$.

5. Show that there is a canonical map taking $coeq(i \times \mathbb{Q}, j \times \mathbb{Q}) \to coeq(i, j) \times \mathbb{Q}$, which is a bijection.

6. Assume that there are two functions $f, g : \mathbb{R} \to \mathbb{R}$ with the following properties: both $f(x)$ and $g(x)$ are strictly positive for all x but tend to 0 as x tends to plus and minus infinity; we have $f(x) = g(x)$ if x is an integer, and in this case their shared value is irrational. A (stolen) example of two such functions is described in Figure 5.4. Let U be the subset of $\mathbb{R} \times \mathbb{Q}$ containing those points (r, q) where either $q < f(r)$ and $q < g(r)$ or $f(r) < q$ and $g(r) < q$. Show that U is open.

7. Show that $q[U]$ is open in $coeq(i, j) \times \mathbb{Q}$ but not open in $coeq(i \times \mathbb{Q}, j \times \mathbb{Q})$. Conclude that **Top** is not cartesian closed.

Chapter 6

Connectedness

As a final basic topological properties, we will briefly take a look at the idea of connectedness. In the geometric sense, if we think of a plane, this is the property that one can, informally speaking, "go from one point to another without falling". In our epistemic analogy, this can likewise be captured: an epistemic space is connected if there are no facts where both that fact and its negation is knowable. The epistemological results of such a model would be very strange: whenever one a fact would be knowable, its negation could not be knowable. For instance, this would imply that propositions like "The sky is purple", in so far as they were verifiable, could not be falsifiable. Thus, connectedness once again nicely illustrates the different modelling requirements we might bring to topology: a geometer will want connected spaces, whilst a philosopher might want them to be quite heavily disconnected. We will take a look at both concepts.

6.1 Connected and Path-Connected Spaces

As with other concepts we have encountered so far, formalising the idea that the space should be "whole" is a task that can be done in many ways. Following the intuition of the real line, one can consider what happens if one takes out a single point x; this creates two sets (x, ∞) and $(-\infty, x)$, which are both open and closed, something which in terms of intervals we know could not happen before. So we propose the following as a first-definition:

Definition 6.1.1. Let (X, τ) be a topological space. We say that X is *connected* if the only clopen subsets of X are X and $\varnothing$.

We can give a trivial, but useful, reformulation of this definition:

Proposition 6.1.2. Let (X, τ) be a topological space. Then X is connected if and only if the only continuous functions $f : X \to \{0, 1\}$ are constant.

Proof. Exercise. ■

Example 6.1.3. The real line $\mathbb{R}$ is connected [1]. Similarly, the subspace $[0, 1]$ is connected as well.

No space with the discrete topology with more than one element is connected, whilst every space with the indiscrete topology is connected.

[1] This follows from the Intermediate Value Theorem

In fact we have that connectedness is a rather rare property for "small" spaces, as witnessed in the following criterion, which we state without proof:

Proposition 6.1.4. Let X be a normal connected space with at least two points. Then X is at least of size $2^{\aleph_0}$.

Additionally, connectedness can be decided on a dense subset:

Proposition 6.1.5. If X is a topological space, and $A \subseteq X$ is a dense set which is connected in the subspace topology, then X is connected.

Proof. Assume that $U \subseteq X$ is a clopen subset. Then look at $A \cap U$ and $A \cap (X - U)$; since these are both open subsets, we have that A has a non-empty intersection with both of them. Hence $A \cap U$ and $A \cap X - U$ are two clopen subsets in the subspace topology distinct from A and $\varnothing$. Thus $A \neq \varnothing$. ■

Corollary 6.1.6. *If a space is connected, then so is any compactification of that space.*

However, another concept is perhaps more intuitively deserving of the name "connectedness": if I have two points in a space x, y, then I can connect them by a continuous path. Formally, given a space X, we can think of a path from x to y as a continuous function:

$$f : [0, 1] \to X$$

such that $f(0) = x$ and $f(1) = y$. This follows by identifying the function with its image in X.

Definition 6.1.7. Let X be a topological space. We say that X is *path-connected* if whenever $x, y \in X$, there is some path p from x to y.

Indeed, we have that path-connectedness implies connectedness:

Proposition 6.1.8. Let X be a path-connected space. Then X is connected.

Proof. Assume that A and B are two non-empty clopen subsets of X, such that $A \cup B = X$ and $A \cap B = \varnothing$. Let x and y be respectively a point in A and B, and assume that p is a path from x to y. Then consider $p^{-1}[A]$; since A is clopen, and p is continuous, $p^{-1}[A]$ is clopen as well. But then $[0, 1]$ would contain a non-empty clopen different from the whole set, which is a contradiction to $[0, 1]$ being connected. ■

However, in general the two notions come apart (See the Exercises). For subsets of the real line, the two notions do coincide, and they do as well for finite topological spaces.

Connectedness can be seen as the first of a series of "invariants" of the space, which are somewhat difficult to capture. The intuition is the following: if we want to describe something which is *not there* – a hole in the space – how would we do so? The answer that we have given in this section is the following: try to overlay a line on top of any two points. If this is not possible you have found a whole. However, this cannot detect all kinds of holes as the next example shows:

Example 6.1.9. Consider $\mathbb{R}^2 - \{(0, 0)\}$, the so called *punctured plane.* We claim that this space is path connected. Indeed, if (x, y) and (z, w) are two points, then either the unique line defined by them does not pass through the origin, or it does; in the latter case, pick some third point (a, b) such that the line from (x, y) to (a, b) does not cross the origin, and neither does the line from (z, w) to (a, b); otherwise simply pick the unique line segment from (x, y) to (a, b) By composing the two lines we obtain the desired path.

This situation is intuitively correct: whilst removing one point from the real line breaks the line into two parts, removing a point from the real plane only creates a hole, as it is interwoven as one continuous piece of fabric. However, it also seems intuitive that $\mathbb{R}^2$ and the punctured plane are not homeomorphic, and they are geometrically different – after all, one has a hole and the other does not! The difficulty of saying exactly wherein the difference lies is the beginning of *algebraic topology*, and opens way for the concepts of homotopy and homology, which are incredibly rich fields of study. They have also in recent years become intimately related to logic, as recent work on type theory casts these concepts as ways to formulate a new potential foundations for mathematics.

6.2 Disconnectedness

For logical purposes, as alluded in the preamble, it is often *disconnectedness* which tend to be more interested in. Indeed, we often want our spaces to be *as disconnected as possible*:

Definition 6.2.1. Let X be a topological space. We say that X is *totally disconnected* if whenever $x, y \in X$ are distinct points then there exists some clopen set U such that $x \in U$ and $y \notin U$.

Example 6.2.2. The Cantor space is totally disconnected; to see this, note that if x and y are distinct infinite binary sequences, then they must disagree on some initial segment, say s; then $x \in C(s)$ whilst $y \notin C(s)$; but as argued in Example 5.2.6, we have that the complement of $C(s)$ is a finite union of basic opens, and hence, is open as well, so $C(s)$ is clopen.

A related notion is so-called *zero-dimensionality*:

Definition 6.2.3. Let X be a topological space. We say that X is *zero-dimensional* if it has a basis of clopen subsets.

There is an intimate connection between these two notions:

Definition 6.2.4. Let X be a topological space. We say that X is a *Stone space* if it is compact, Hausdorff and zero-dimensional.

Theorem 6.2.5. *Let X be a topological space. Then:*

1. *If X is zero-dimensional and Hausdorff, then it is totally disconnected.*
2. *X is compact and totally disconnected if and only if X is a Stone space.*

Proof. (1) Assume that X is zero-dimensional; suppose that $x \neq y$; then there exists an open set which separates them, by the Hausdorff property, say U and Z; since $U = \bigcup_{i \in I} V_i$, where these later sets are clopen, we obtain the result.

(2) If X is a Stone space, then it is compact, Hausdorff and zero-dimensional, so by (1), it is compact and totally disconnected. Now assume the latter. Let U be an arbitrary open set. Fix $x \in U$ and pick $y \notin U$; then there is a clopen $U_{x,y}$ which contains y and not x. Then $X - U \subseteq \bigcup_{y \notin U} U_{x,y}$, so by compactness we can pick a clopen set $X - U \subseteq V_x = U_{x,y_9} \cup ... \cup U_{x,y_n}$. Hence $X - V_x \subseteq U$ is a clopen set, and it contains x. Thus:

$$U = \bigcup_{x \in U} X - V_x$$

and hence X has a basis of clopens. ■

These forms of disconnectedness, just like connectedness, are also preserved under compactifications, when working with nice enough spaces:

Proposition 6.2.6. If X is a normal totally disconnected space, then so is βX.

However, one might not be content with totally disconnected spaces. Here is one property which one can argue makes some sense: let us say that a proposition is knowable if it is open (as we have been doing so far), and closed if it can be believed (we cannot refute its negation). Then the closure operator can be thought of as taking a given proposition P and returning the proposition "It is reasonable to know P".

Now consider P a knowable proposition. Then how would we get to know that $cl(P)$? In those worlds where P holds, we would need to assert that; and in those where P does not hold, we would assert the reasonable evidence we have that the converse of P is false. So a case could be made that $cl(P)$ should be knowable as well. This leads to the following extreme form of disconnectedness:

Definition 6.2.7. Let X be a topological space. We say that X is *extremally disconnected* [2] if whenever U is an open subset, then $cl(U)$ is open.

Our allusion to a sort of escalation is justified in the following:

Proposition 6.2.8. Let X be a Hausdorff extremally disconnected space. Then X is totally disconnected.

Proof. Assume that $x \neq y$; let $x \in U$ and $y \in V$ be disjoint neighbourhoods. Consider $cl(U)$. We have that $x \in cl(U)$, and if $y \in cl(U)$, then the neighbourhood V of y must intersect U, which is false. Hence by extremal disconnectedness, $cl(U)$ is open, and it is also closed, hence, it is clopen. This shows the result. ■

The spaces we have just discussed, in particular Stone spaces, are remarkably abundant in logic. Through the famous Stone duality, they correspond to Boolean algebras, which are the algebraic models of classical logic. Through that correspondence many connections can be made. In the last section we tease some such connections out, and take the opportunity to introduce one last crucial topological concept.

6.3 Isolated Points and an End

To understand it, fix $\mathbb{R}$, the reals with their topology. In general, the subsets of the reals are easy to draw in big clusters. But sometimes we might run into a picture such as the following:

Indeed, it seems intuitive that the leftmost point is "isolated" from the remaining ones, and this should make a difference when considering, for instance, how far apart two points are, as perhaps we do not want to consider such an "outlier" to be part of the set for those purposes. Hence we formalise this concept of isolation:

Definition 6.3.1. Let (X, τ) be a topological space, and let $A \subseteq X$. We say that x is an *accumulation point* of A if $x \in cl(A - \{x\})$. We denote the set of accumulation points of A by $d(A)$. We say that a point $x \in A$ is *isolated* if $x \in A - d(A)$. We denote by $iso(A)$ the set of isolated points of A.

[2]This is not a typo: the root word is extremal and not extreme. Though this is indeed a pretty extreme form of disconnectedness.

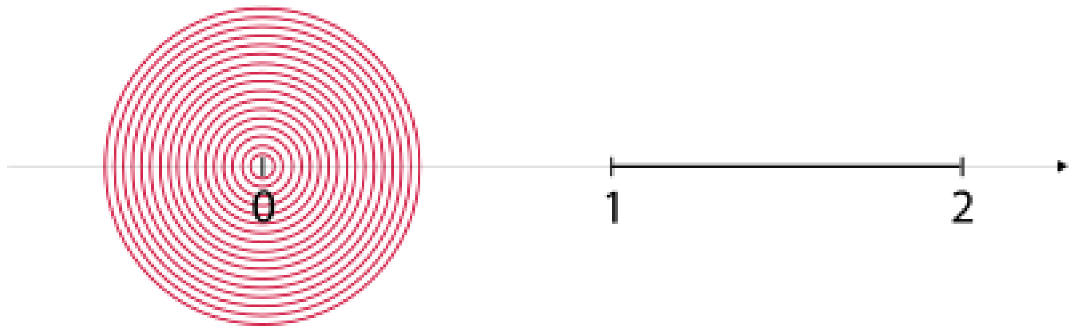

Figure 6.1: Isolated point (credit: Wikipedia)

Proposition 6.3.2. Let (X, τ) be a topological space with a basis $\mathcal{B}$. Then:

1. $x \in X$ is isolated in X if and only if $\{x\}$ is open.
2. Given $A \subseteq X$, $x \in d(A)$ if and only if whenever U is a basic neighbourhood of x, then $U - \{x\} \cap A \neq \varnothing$.

Proof. (1) If x is isolated in X, then by definition $x \notin cl(X - \{x\})$; hence clearly $cl(X - \{x\}) = X - \{x\}$, which shows that $\{x\}$ is open. The converse is similar.

(2) Assume that $x \in d(A)$, and let U be a basic neighbourhood of x. Since $x \in cl(A - \{x\})$, we have that $U \cap A - \{x\}$ have non-empty intersection. The converse follows the same way. ■

Definition 6.3.3. Let X be a topological space. We say that X is *scattered* if each non-empty subset of X contains an isolated point. We say that X is *crowded* if X has no isolated point.

Scattered spaces have found many uses in topology and logic:

- They provide models for provability logics, such as GL and GLP;
- They are intimately connected to the order-theoretic properties of structures like the ordinal numbers.
- They provide basic structures, and form the backbone of many complex order-theoretic and topological structures.

As an example of how these concepts interact fruitfully with what we have seen so far, we mention the following two results:

Proposition 6.3.4. Let X be a T_1 scattered space. Then X is totally disconnected.

Theorem 6.3.5. *Let X be a crowded topological space. Then the modal logic of X is precisely* $\mathsf{S4}$.

All that we have mentioned hints that there are many connections one can be ready to explore once we have the concepts we have developed so far. We conclude by teasing one more such connection. Consider the following theorem, due originally to Brouwer:

Proposition 6.3.6. The Cantor space 2^ω is the unique topological space which is:

1. A Stone space;

2. Crowded;

3. The basis of clopen sets has the property that any two clopen sets are homeomorphic.

Looking at such a result prima facie, it might seem like a very difficult result; but as it turns out, one can give multiple, relatively easy model-theoretic proofs of this fact; and one can give a very elementary universal algebraic proof of the same fact, when the right tools have been developed.

6.4 Exercises

Exercise 6.1. Let X and Y be two topological spaces. Assume that $f : X \to Y$ is a homeomorphism. Show that X is connected if and only if Y is connected.

Definition 6.4.1. Let L and T be two linearly ordered sets. We define the *lexicographic order on* $L \times T$ as follows: $(a, b) \prec_{lex} (x, y)$ if and only if $a <_L x$ or $a = x$ and $b <_T y$.

Given two linear orders L and T, we denote by $L + T$ the *sum of linear orders* on $L \sqcup T$ as follows: $(a, i) \prec (b, j)$ for $i, j \in \{0, 1\}$ if and only if either $i = 0$ and $j = 1$ or $i = j$ and $a < b$ in the respective order.

Given a linear order L, we denote by L^{op} the converse linear order: $x <_{op} y$ if and only if $y <_{op} x$.

Exercise 6.2. The purpose of this exercise is to show that connectedness does not imply path connectedness.

1. Show that $\mathbb{R}$ is order-isomorphic to the following order $((0, 1) \times \omega)^{op} + \circ + ((0, 1) \times \omega)$. Deduce that $\mathbb{R}$ as a topological space is homeomorphic to the order topology on the same set.

2. Consider the set $\mathbb{L}$ given as follows:

$$\circ + ((0, 1) \times \omega_1)^{op} + \circ + ((0, 1) \times \omega_1) + \circ$$

   Show that the order topology on $\mathbb{L}$ is connected.

3. Show that $\mathbb{L}$ is not path connected. *Hint: Consider the points at extremities.*

Exercise 6.3. Let X and Y be topological spaces. Show that if both X and Y are connected then so is $X \times Y$.

Exercise 6.4. Let $L = (L, <)$ be a linearly ordered set. We say that L is *dense* if whenever $a < b$ then there exists some c such that $a < c < b$. We say that L is a *linear continuum* if whenever $S \subseteq L$ is a subset, and S is bounded above, then there exists a least upper bound.

- Show that the order topology on $\mathbb{R}$ coincides with the Euclidean topology.

- Show that an arbitrary linearly ordered set is a linear continuum if and only if it is connected.

Exercise 6.5. Let X be a topological space. Given a subset $A \subseteq X$ we consider the following (transfinite) sequence over all ordinals:

$$\begin{aligned} d_0(A) &= A \\ d_{\alpha+1}(A) &= d(d_n(A)) \\ d_\lambda(A) &= \bigcap_{\beta<\lambda} d_\beta(A) \end{aligned}$$

where λ is a limit ordinal and $d(A)$ is the collection of accumulation points.

- Give an example of a space X and a subset A such that $d_2(A) \neq d(A)$. Give another such example such that $d_{\omega+1}(A) \neq d_\omega(A)$.
- Show that if a space X is scattered, then there exists some β such that $d_\beta(X) = \varnothing$.

Exercise 6.6. Let L be a linear order. We say that L is *scattered* if there is no subset $S \subseteq L$ such that S is densely ordered (i.e. whenever $a < b \in S$ then there is some $c \in S$ such that $a < c < b$). Show that then the order topology on L is scattered. Moreover, show that the converse implication does not hold.

Definition 6.4.2. Let X be a topological space. We say that X satisfies the T_D axiom if and only if for each $x \in X$, we have that $\{x\}$ is an open subset in $\overline{\{x\}}$.

Exercise 6.7. Show that the following are equivalent for a topological space X:

1. X is T_D;
2. For each $x \in X$ there is an open U such that $U - \{x\}$ is open as well;
3. For each subset S in X, $d(d(S)) \subseteq d(S)$.

Exercise 6.8. Recall the Cantor space 2^ω and the Baire space ω^ω.

1. Show that there are bijections between the sets 2^ω, $\mathbb{R}$ and ω^ω *Hint: Think about encoding subsets of the natural numbers, and decimal expansions.*
2. Show that there can be no homeomorphism between any of these three sets. *Hint: Find topological properties which are distinct between these sets.*

Exercise 6.9. (Gluing lemma) Let X be a topological space. Suppose that $(U_i)_{i\in I}$ is a collection of open subsets of X, and $f_i : U_i \to Y$ are continuous functions, such that:

- For each i, j, $f_i{\restriction}U_i \cap U_j = f_j{\restriction}U_i \cap U_j$.

Show that then there exists a function $f : X \to Y$ such that for each i, $f{\restriction}U_i = f_i$.

Definition 6.4.3. Let X, Y be topological spaces. We denote by $Fun(X, Y)$ the collection of *partial functions* from X to Y, i.e., functions which domain is a subset of X. We say that a set $\mathcal{S} \subseteq Fun(X, Y)$ is a *presheaf* if it satisfies the following:

1. For each $f \in \mathcal{S}$, $dom(f)$ is open in X;

2. If $U \subseteq V$ are open sets and $f \in \mathcal{S}$ is such that $dom(f) = V$ then $dom(f)\restriction V \in \mathcal{S}$;

We say that a presheaf $\mathcal{S}$ is a *sheaf* if it further satisfies:

1. (Locality) If U is an open set, and $U = \bigcup_{i \in I} U_i$, then: if $f, g \in \mathcal{S}$ are such that $dom(f) = dom(g) = U$ and $f\restriction U_i = g\restriction U_i$ for each i, then $f = g$.

2. (Gluing) if $U = \bigcup_{i \in I} U_i$ and $f_i \in \mathcal{S}$ are such that $dom(f_i)$ and for each i, j:

$$f_i\restriction_{U_i \cap U_j} = f_j\restriction_{U_i \cap U_j}$$

   Then there exists some $f \in \mathcal{S}$ such that $f\restriction_{U_i} = f_i$.

Exercise 6.10. Let X and Y be two topological spaces.

1. Show that the set $Cont(X, Y)$ of continuous functions from $U \subseteq X$ open to Y, is a sheaf.

2. Let $f : Y \to X$ be a continuous function. Consider $Sec(f)$ the following set:

$$Sec(f) = \{s : U \to Y : U \subseteq X \text{ is open }, f \circ s = id_U\}$$

   Show that $Sec(f)$ is a sheaf.

3. Let X be a topological space. Consider the set $Const(X)$ of constant partial functions on X, i.e., functions defined on $U \subseteq X$ such that U is open. Show that $Const(X)$ is a presheaf but not in general a sheaf.