

TOPOLOGY IN AND VIA LOGIC HOMEWORK ASSIGNMENT 2

- Deadline: February 10 at 23:59.
- All exercises are worth the same points.
- Good luck!

SEPARATION

Exercise 1. Let (X, τ) be a topological space. Let $A \subseteq X$ be a subset. We say that A is *dense in X* if $\overline{A} = X$.

- (1) Show that \mathbb{Q} is dense in \mathbb{R} .
- (2) Show that if A is dense in X , then A intersects all open subsets of X .
- (3) Let X be a topological space and Y a Hausdorff space, and let $A \subseteq X$ be a dense subset. Let $f, g : X \rightarrow Y$ be two continuous functions such that $f|_A = g|_A$. Show that $f = g$.
- (4) Show that if A is dense in X , then every filter over A (i.e., $F \subseteq \mathcal{P}(A)$) converges to some point in X .

Exercise 2. Show the following for a T_1 -space X :

- (1) If X is finite, then the topology on it is discrete.
- (2) For each $x \in X$, $\{x\}$ is closed.
- (3) For each $x \in X$, the filter

$$F(x) := \{S \subseteq X : x \in S\}$$

converges uniquely to x .

Show that the last property is an alternative definition for T_1 -spaces.

COMPACTNESS

Exercise 3. Let X be a compact Hausdorff space, and A a closed subspace. Define the equivalence relation $x \sim y$ if and only if either $x = y$ or x and y are both in A . Show that the quotient space X/\sim is compact Hausdorff.

Exercise 4. Let X and X_i throughout be Hausdorff spaces.

- (1) Show that every compact space is locally compact.
- (2) Show that the converse inclusion does not hold *Hint: Think of \mathbb{R} .*
- (3) Show that if X is any Hausdorff and non-compact space, then $\alpha(X)$ is compact Hausdorff if and only if X is locally compact.

CONNECTEDNESS

Exercise 5. Prove Proposition 6.1.4 of the lecture note. *Hint: The cardinality of the interval $[0, 1]$ is exactly 2^{\aleph_0} .*