Π_2-Rule Systems and Inductive Classes of Gödel Algebras

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November 10, 2023

Abstract

In this paper we present a general theory of Π_2-rules for systems of intuitionistic and modal logic. We introduce the notions of Π_2-rule system and of an Inductive Class, and provide model-theoretic and algebraic completeness theorems, which serve as our basic tools. As an illustration of the general theory, we analyse the structure of inductive classes of Gödel algebras, from a structure theoretic and logical point of view. We show that unlike other well-studied settings, there are continuum many Π_2-rule systems extending LC, and show how our methods allow easy proofs of the admissibility of the Takeuti-Titani rule. Our final results show that, nevertheless, the theory of inductive classes of $\mathsf{LC} = \mathsf{IPC} + (p \to q) \vee (q \to p)$ is still quite rich: (1) we present a full classification of those inductive classes which are inductively complete, i.e., where all Π_2-rules which are admissible are derivable, and (2) show that the problem of admissibility of Π_2-rules over LC is decidable.

1 Introduction

When analysing modal and intuitionistic logical systems, most research has focused on sets of theorems (sometimes called simply "logics") and single-conclusion rule systems. Given a fixed language $\mathscr{L}$ and $\Gamma = \{\varphi_i : i \leqslant n\}$ and ψ, formulas in this language, we write

$$\Gamma \vdash \psi$$

for a single-conclusion rule. In this setting, with such a rule we can associate a first-order formula

$$\chi(\Gamma, \psi) := \forall \overline{x}(\bigwedge_{i \leqslant n} \varphi_i(\overline{x}) \approx 1 \to \psi(\overline{x}) \approx 1)$$

such that an algebraic model $\mathcal{H}$ validates the rule $\Gamma \vdash \psi$ if and only if $\mathcal{H} \vDash \chi(\Gamma, \psi)$. These systems have been studied in a variety of settings, and a rich theory, connecting universal algebra and algebraic logic, has emerged out of their analysis (see e.g. [17]), which covers several classical concepts, like admissibility and structural completeness [20, 14, 11].

Nevertheless, more complex rules have also been considered for a long time: from the pioneering work of Gabbay [18] and Burgess [9], to the work of Takeuti and Titani [24], which has lead to axiomatisations of first-order Gödel logic [4, 23], to the work on "Anti-Axioms" [25], up to recent work axiomatising certain logics of compact Hausdorff spaces and studying structural properties of these calculi [7, 8], Π_2-rules have often been seen as an expedient way of obtaining axiomatisations for logical systems. With some exceptions, mentioned above, such objects have not been understood as deserving independent study.

In this paper, we argue that the theory of Π_2-rules presents an interesting picture, both theoretically and in establishing concrete connections between algebraic logic, model theory, duality theory and computer science. In particular, through its systematic study, several observations made in the literature can be brought together under a unified umbrella, and several tools can be developed to facilitate the use of Π_2-rules in applications – for example, the admissibility proof developed in [23] of the Takeuti-Titani density rule, which occupies a substantial part of that paper through syntactic methods, is derived here with great ease from the structure of models of LC, the linear calculus of Gödel and Dummett[1]. This provides a natural test-bed, lying at the intersection of multi-valued and fuzzy logics, on one hand, and intuitionistic logic, on the other, and having a very simple structure in the basic equational setting – for instance, the algebraic models of such a calculus, Gödel algebras, are locally finite, and the lattice of extensions of LC is isomorphic to $(\omega+1)^*$.

The structure of this paper is as follows: in Section 2 we present the notion of Π_2-rule systems in the setting of modal and intuitionistic logics [2]. In Section 3 we introduce the notion of a $\forall$-subalgebra and $\forall$-embedding, proving a Mal'tsev style definability theorem for classes closed under these special kinds of subalgebras, products and ultraproducts. In Section 4 we conclude our general theoretical contributions by providing a Birkhoff-style completeness theorem for Π_2-rules systems yielding a dual isomorphism between Π_2-rule systems and inductive classes.

In Section 5 we define a general notion of admissibility of Π_2-rule systems, recovering many results in the algebraic theory of admissibility to this setting. This provides a number of universal algebraic tools which can be used to show that certain rules are admissible over specific calculus of interest. As an exemplary use-case, we narrow in on the analysis of LC. In Section 6 we mobilise the previous theory for an analysis of LC, proving that unlike in other settings of interest, there are uncountably many Π_2-rule systems of Gödel logic. Whilst this reveals the structure of Π_2-admissibility in LC to be deeper than in the case of single-conclusion rule systems, our key contributions to this study, presented in Section 7, show two structural results, showing that this setting can still be fruitfully analysed:

1. We show that the problem of admissibility of Π_2-rules is decidable, through a connection with the theory of model completions.

2. We give a complete classification of all inductively complete extensions of LC, which are the Π_2-analogue of structurally complete logical systems.

We conclude in Section 8 with some final remarks and future directions for this line of research.

2 Π_2-rules and Π_2-Rule Systems

In this section we introduce the notion of Π_2-rules. Throughout we fix a given language $\mathscr{L}$, which may be assumed to be a language of intuitionistic or modal logic; as noted before such an assumption is not necessary, but makes the notation simpler, and will be sufficient for the examples at hand.

[1]Throughout the paper we refer to this often as Gödel Logic, as is common practice in the literature.

[2]Our choice to restrict to this setting is made to simplify notation, and with an eye towards the examples at hand. Nevertheless, we expect that the reader with an interest in algebraizability and other classes of algebraic models will be able to adapt our methods towards a Blok-Pigozzi style theory of algebraizability which encompasses Π_2-rules; we leave this entirely for future work.

Definition 2.1. Let $\Gamma = \{\varphi_i(\overline{p}, \overline{q}) : i \leqslant n\}$ for some $n \in \omega$ and $\psi(\overline{p})$ be formulas in the language $\mathscr{L}$, where ψ does not contain any proposition letter occurring in $\overline{q}$. We define the rule associated with this sequence of formulas, denoted $\forall \overline{q}\Gamma(\overline{q})/^2\psi$ for short (sometimes without the universal quantifier, when the variables are clear from context or do not matter)[3]:

$$\frac{\forall \overline{q}(\varphi_0(\overline{p}, \overline{q}) \wedge \ldots \wedge \varphi_n(\overline{p}, \overline{q}))}{\psi(\overline{p})}$$

Given such a collection of formulas φ_n, we refer to $\overline{q}$ as the *bound context of* φ_n, or generally, the *bound context associated to the formula*, and sometimes denote it as $B_c(\varphi_n)$ or $B_c(\Gamma)$; we refer to propositional variables not ocurring in B_c as the *free context*.

We say that a given rule $\Gamma/^2\psi$ is a Π_2-rule if it is of the above shape, for a collection of formulas φ_n, some collection of propositional variables F, and a formula ψ.

Example 2.2. Let $\mathscr{L} = (\wedge, \vee, \neg, 0, 1, \Diamond)$ be the language of modal algebras. Recall from [18], Gabbay's, "Irreflexivity rule":

$$\frac{\forall p(\neg(p \to \Diamond p) \vee \varphi)}{\varphi}$$

Where p does not occur in φ. This was used to axiomatise irreflexive frames over temporal logics.

Example 2.3. More generally, in [8], rules of the following form were considered:

$$\frac{\forall \overline{p}(F(\overline{p}, \overline{q}) \to \chi)}{G(\overline{q}) \to \chi}$$

where $\overline{p}$ does not occur in χ or G.

The following example has also been extensively studied, and was noted in [8] as not fitting in the existing frameworks:

Example 2.4. Let $\mathscr{L} = (\wedge, \vee, \to, 0, 1)$ be the language of intuitionistic logic. Recall the following rule by Takeuti and Titani, used for axiomatisation purposes in [23] and [4]:

$$\frac{\forall r((p \to r) \vee (r \to q) \vee c)}{(p \to q) \vee c}$$

Such a rule is often called the "Density rule", since it is valid on a linear Heyting algebra if and only if such an algebra is dense.

More than purely syntactic objects, we can provide a semantics for such rules. We refer to an algebra $\mathcal{A}$ as an $\mathscr{L}$-algebra if it is of type $\mathscr{L}$.

Definition 2.5. Let $\mathcal{A}$ be an $\mathscr{L}$-algebra and $\Gamma/^2\psi$ a Π_2-rule. Let $v : \mathsf{Prop} \to \mathcal{A}$ be a valuation over $\mathcal{A}$, and $(\mathcal{A}, v)$ be an algebraic model.

We say that $(\mathcal{A}, v)$ makes this rule true, and write $(\mathcal{A}, v) \vDash \Gamma/^2\psi$ if and only if: if for each $\forall \overline{q}\varphi_n(\overline{p}, \overline{q}) \in \Gamma$, and each valuation v' which is equal to v up to variables in the free context, $v'(\varphi_n(\overline{p}, \overline{q})) = 1$; then $v(\psi(\overline{q})) = 1$. We write $\mathcal{A} \vDash \Gamma/^2\psi$ if for each model $(\mathcal{A}, v)$ over $\mathcal{A}$, $(\mathcal{A}, v) \vDash \Gamma/^2\psi$. Given a collection of $\mathscr{L}$-algebras $\mathbf{C}$, we write $\mathbf{C} \vDash \Gamma/^2\psi$ if $\mathcal{A} \vDash \Gamma/^2\psi$ for each $\mathcal{A} \in \mathbf{C}$.

[3]This notation serves to emphasise the fact both that these are distinct from usual rules, and the second-order nature of these rules, but we emphasise that it is purely formal.

Note that in light of this definition, given any Π_2-rule there is a canonical way to assign to it a first-order formula. Namely, given $\Gamma(\overline{p}, \overline{q})$ and $\psi(\overline{q})$, and $F = \{\overline{p}\}$ we let

$$\chi(\Gamma, \psi) = \forall \overline{y}(\forall \overline{x}(\bigwedge \Gamma(\overline{x}, \overline{y}) \approx 1) \rightarrow \psi(\overline{y})).$$

And it is clear to see then that:

$$\mathcal{H} \vDash \Gamma/^2\psi \iff \mathcal{H} \vDash \chi(\Gamma, \psi),$$

where the right hand side consequence is the usual first-order logic modelling relation. We will use this fact, often tacitly, throughout.

Having these rules, we can now come to the crucial definition of this section: a Π_2-Rule system.

Definition 2.6. Let $\vdash$ be a set of Π_2-rules in the language $\mathscr{L}$. We say that $\vdash$ is a Π_2-*rule system* if it satisfies the following:

- (Strong Reflexivity) If Γ is some set of formulas, $F = \{\overline{p}\}$ a set of propositional variables, $\chi \in \Gamma$, and $\overline{\varphi}$ a set of formulas not containing any variable in F, we have $\forall \overline{p}\Gamma \vdash \chi[\overline{\varphi}/\overline{p}]$.
- (Monotonicity) If $\forall \overline{p}\Gamma \vdash \psi$, then for any finite set of formulas Γ', we have $\forall \overline{p}(\Gamma, \Gamma') \vdash \psi$.
- (α-Renaming) if $\forall \overline{p}\Gamma \vdash \psi$, then $\forall \overline{q}\Gamma' \vdash \psi$, where $\Gamma' = \Gamma[\overline{q}/\overline{p}]$, so long as $\overline{q}$ do not occur free in either Γ or ψ.
- (Rule cut) If $\Gamma(\overline{p}, \overline{r})$ and $\Delta(\overline{q}, \overline{r}) = \{\mu_i(\overline{q}, \overline{r}) : i \leqslant n\}$, are two finite sets of formulas, ψ does not contain any variables in $\overline{r}$ or $\overline{p}$, no variable from $\overline{q}$ occurs in Γ, and

  $$\forall \overline{p}(\Gamma(\overline{p}, \overline{r})) \vdash \mu_i(\overline{q}, \overline{r}) \text{ for each } i \leqslant n \text{ and } \forall \overline{q}(\Delta(\overline{q}, \overline{r})) \vdash \psi,$$

  Then we must have $\forall \overline{p}(\Gamma(\overline{p}, \overline{r})) \vdash \psi$.
- (Bound Structurality) if $\forall \overline{p}\Gamma \vdash \psi$ and σ is a substitution such that all variables in $F = \{\overline{p}\}$ are fixed, and such that $q \in F$ does not occur in $\sigma(p)$ for p occuring free in Γ, $\sigma[\Gamma]$ or $\sigma(\varphi)$, then $\sigma_*[\Gamma] \vdash \sigma(\psi)$.

We can motivate some of these definitions on the intuitive appeal of our notation: strong reflexivity can be seen as an instance of universal instantiation; α-renaming amounts to requiring that the names of bound variables be arbitrary; rule cut amounts to an introduction rule of universal quantifiers. The case of bound structurality amounts to the fact that the substitutions we are interested in should commute appropriately with the universal quantifiers. Indeed, the following proposition establishes that the satisfaction relation for Π_2-rules forms a Π_2-rule system as above, and levies these ideas.

Proposition 2.7. Let $\mathcal{A}$ be an $\mathscr{L}$-algebra. Then the set:

$$\Pi_2(\mathcal{A}) = \{\Gamma/^2\psi : \mathcal{H} \vDash \Gamma/^2\psi\}$$

forms a Π_2-rule system. More generally, given a class $\mathbf{K}$ of algebras, $\Pi_2(\mathbf{K})$ also forms a Π_2-rule system.

Proof. As noted above, we have that for each algebra $\mathcal{A}$, and each Π_2-rule $\Gamma/^2\psi$:

$$\mathcal{A} \vDash \Gamma/^2\psi \iff \mathcal{A} \vDash \chi(\Gamma, \psi).$$

With this in mind, one can check that the conditions of Definition 2.6 follow from the rules for universal quantifiers of first-order logic. The result for classes of logics follows similarly. ∎

With the former proposition in mind, we also note the following:

Lemma 2.8. Let $\mathscr{L}$ be a fixed language. If S is a set of Π_2-rules, then there is a smallest Π_2-rule system containing S.

Proof. Note that the collection of Π_2-relations is closed under arbitrary intersections, since each of the conditions for being a Π_2-rule system is a closure condition; moreover, there is at least one such relation extending S (namely, the relation containing all pairs of the form $\Gamma/^2\psi$). ∎

The above implies that we can construct the smallest Π_2-rule system containing any given set of axioms. This is, as usual, obtained by closing a given collection of sequents under the above operations, though this appears to be far from a "nice" closure construction.

Having this notion, and keeping in mind the classical universal algebraic and logico-algebraic results connecting quasivarieties and rule systems, we might ask whether the class of formulas above, like the Universal Horn formulas, amounts to any natural closure conditions. This will be the subject of the next section.

3 Inductive Rule Classes

Recall that given a first-order formula φ, we say that φ is a $\forall\exists$-*Special Horn formula* if it is of the form:

$$\forall \overline{x}(\forall \overline{y}(\bigwedge_{i=1}^{n} \lambda_i(\overline{x}, \overline{y})) \to \gamma(\overline{x})),$$

where λ_i and γ are atomic formulas. Such formulas, and some generalizations therein, were first studied by Lyndon [22], in the context of seeking a characterisation of those formulas preserved under Horn formulas. They have moreover appeared several times in algebraic contexts as well as in the algebra of logic.

Example 3.1. (Subordination Algebras) Consider the language of Boolean algebras together with a binary symbol $\rightsquigarrow$. This is the language of so-called *subordination algebras*. Then consider the following formula:

$$\forall a, b, d(\forall c(a \rightsquigarrow c \wedge c \rightsquigarrow b \leqslant d) \to a \rightsquigarrow b \leqslant d)$$

This formula was introduced in [7]; it defines an important class of *transitive subordination algebras*. ∎

Example 3.2. (Orthoimplicative systems) Consider the language of orthoimplicative systems; this consists of the language of ortholattices (and Boolean algebras), together with infinitely many implications $(\multimap_n)_{n\in\omega}$. Consider the following formula:

$$\forall c_0, ..., c_n, d_0, ..., d_k, e((c_i = \bigvee c_i \wedge d_j \;\&\; \forall f(f \wedge (\bigvee c_i \wedge d_j) \leqslant \bigvee(f \wedge c_i \wedge d_j))) \\ \implies e \multimap_n (c_0, ..., c_n) \leqslant e \multimap_n (d_0, ..., d_k))$$

This was shown in [2] to be an interesting property of ortholattices, allowing a nicer relationship between ortholattices and modal algebras. Intuitively, it says that when the sequences of elements $c_0, .., c_n$ and $d_0, ..., d_k$ form what is called an *admissible join*, and the joins of the first sequence is below the second sequence, then the implication should respect this structure. ∎

From the discussion of Section 2, we can see that in model-theoretic terms, Π_2-rules amount to ∀∃-Special Horn formulas, and they are essentially the fragment corresponding to such rules. As such, in this section we will provide a number of results, analogous to Birkhoff's theorem on equational classes and Mal'tsev's theorem on quasi-equational classes. For that purpose, below, given a class of algebras $\mathbf{K}$, denote by $Th^H_{\forall\exists}(\mathbf{K})$ the following:

$$Th^H_{\forall\exists}(\mathbf{K}) = \{\varphi : \varphi \text{ is } \forall\exists \text{ Special Horn}, \mathbf{K} \vDash \varphi\}. \tag{1}$$

In other words, the inductive Special Horn consequences of $\mathbf{K}$. The following construction will play a key role in the whole theory:

Definition 3.3. Let $\mathscr{L}$ be a language, and $\mathcal{A}, \mathcal{B}$ be two $\mathscr{L}$-algebras. We say that A is a ∀-*subalgebra* of B, and write $\mathcal{A} \leqslant_\forall \mathcal{B}$ if it is a subalgebra, and for each atomic formula $\varphi(\overline{x}, \overline{y})$, and sequence of elements $\overline{a} \in A$:

$$\mathcal{A} \vDash \forall\overline{x}\varphi(\overline{x}, \overline{a}) \implies \mathcal{B} \vDash \forall\overline{x}\varphi(\overline{x}, \overline{a}).$$

We define the notion of ∀-embedding in the obvious way. Given a class of algebras $\mathbf{K}$, we write $\mathbb{S}_\forall(\mathbf{K})$ for the collection of ∀-subalgebras of elements of $\mathbf{K}$.

Note that the latter operator[4] is monotone, extensive and idempotent. We now check that this construction indeed preserves the kinds of formulas we are interested in:

Lemma 3.4. If φ is a ∀∃-Special Horn sentence, then φ is preserved under ∀-subalgebras and direct products.

Proof. Assume that $\mathcal{A} \leqslant_\forall \mathcal{B}$, through an inclusion f, and we have a ∀∃-Special Horn formula $\varphi = \forall\overline{x}(\forall\overline{y}(\lambda_0(\overline{x}, \overline{y}) \wedge ... \wedge \lambda_n(\overline{x}, \overline{y})) \to \gamma(\overline{x}))$ which holds in $\mathcal{B}$. Now let $\overline{a}$ be any sequence in A. Assume that $\mathcal{A} \vDash \forall\overline{y}(\overline{\lambda}(\overline{a}, \overline{y})$. Then by the assumption of this being a ∀-subalgebra, the same formula holds in $\mathcal{B}$. Since we assumed that B satisfied the formula φ, then $\mathcal{B} \vDash \gamma(\overline{a})$; but since $\mathcal{A}$ is a subalgebra, then $\mathcal{A} \vDash \mu(\overline{a})$, as desired.

The fact that this sentence is preserved under direct products follows from general results on preservation of Horn sentences by direct products. ∎

An important example which witnesses the usefulness of the concept of ∀-subalgebras is the case of subdirect products:

Proposition 3.5. Let $\mathbf{K}$ be an arbitrary class of $\mathscr{L}$-algebras. Then:

$$\mathbb{P}_S(\mathbf{K}) \subseteq \mathbb{S}_2\mathbb{P}(\mathbf{K})$$

Proof. Assume that $f : \mathcal{A} \to \prod_{i \in I} \mathcal{B}_i$ is a map witnessing the fact that $\mathcal{A}$ is a subdirect product of the $\mathcal{B}_i$. Then we claim that A is a ∀-subalgebra of the product. Indeed, assume that $\mathcal{A} \vDash \forall\overline{x}\varphi(\overline{x}, \overline{a})$, where φ is an atomic formula. Suppose that $\prod_{i \in I} \mathcal{B}_i \nvDash \forall\overline{x}\varphi(x, f(\overline{a}))$; hence for some tuple $\overline{b}$ of

[4] Technically, it is the operator $\mathbb{IS}_2$ which is idempotent, but this will not affect anything.

elements in $\prod_{i\in I}\mathcal{B}_i$, $\prod_{i\in I}\mathcal{B}_i \not\models \varphi(\overline{b}, f(\overline{a}))$. Thus for some $\mathcal{B}_i$, $B_i \not\models \varphi(\overline{b}(i), f(\overline{a})(i))$; since the product is subdirect, $\mathcal{B}_i$ is a homomorphic image of $\mathcal{A}$, so:

$$\mathcal{B}_i \not\models \varphi(f(\overline{c})(i), f(\overline{a})(i))$$

for some $c_k \in \mathcal{A}$ such that $f(c_k)(i) = b_k(i)$, for each element of the tuple $\overline{b}$. But this in turn implies that $\prod_{i\in I}\mathcal{B}_i \not\models \varphi(f(\overline{c}), f(\overline{a}))$. Since this is an atomic formula, and f is an embedding, we then have that $\mathcal{A} \not\models \varphi(\overline{c}, \overline{a})$, which contradicts the hypothesis on $\mathcal{A}$. So by reductio, we have that $\mathcal{B} \models \forall\overline{x}\varphi(x, f(a_0), ..., f(a_n))$. ■

Definition 3.6. Let $\mathbf{K}$ be a class of algebras. We say that $\mathbf{K}$ is an *inductive rule class* if it is closed under $\forall$-subalgebras, products and ultraproducts.

Note that as usual, the arbitrary intersection of inductive rule classes is again an inductive rule class. Hence we can consider an operator $\mathbb{IR}$, which constructs the smallest inductive rule class containing the given collection of algebras. As in the case of Birkhoff and Matlsev's results – and owing to the proofs of the latter – we also have a characterization of this operator:

Theorem 3.7. *Let $\mathbf{K}$ be an arbitrary collection of algebras. Then we have that:*

$$\mathbb{IR}(\mathbf{K}) = \mathbb{IS}_2\mathbb{PP}_U(\mathbf{K})$$

Proof. Since inductive rule classes are closed under $\forall$-subalgebras, products and ultraproducts, we must have that $\mathbb{IS}_2\mathbb{PP}_U(\mathbf{K}) \subseteq \mathbb{IR}(\mathbf{K})$. For the other inclusion, note first that:

$$\mathbb{PS}_2 \leqslant \mathbb{S}_2\mathbb{P}$$

Indeed if $\mathcal{A} = \mathcal{B}_i$, where each of the factors is a $\forall$-subalgebra of $\mathcal{C}_i$, then $\mathcal{A}$ will be a $\forall$-subalgebra of $\prod_{i\in I}\mathcal{C}_i$; it is surely a subalgebra, and if $\forall\overline{x}\varphi(\overline{x}, \overline{a})$ is true in $\mathcal{A}$, then $\forall\overline{x}\varphi(\overline{x}, \overline{a}(i))$ is true in $\mathcal{B}_i$ for each i, hence $\forall\overline{x}\varphi(\overline{x}, \overline{a}(i))$ is true in $\mathcal{C}_i$, so $\forall\overline{x}\varphi(\overline{x}, \overline{a})$ is true in $\prod_{i\in I}\mathcal{C}_i$.

Similarly, we have that

$$\mathbb{P}_U\mathbb{S}_2 \leqslant \mathbb{S}_2\mathbb{P}_U$$

Indeed, if $\mathcal{A}$ is an ultraproduct of $\mathcal{B}_i$ which are $\forall$-subalgebras of $\mathcal{C}_i$ via g_i, then consider $\prod_{i\in I}\mathcal{C}_i$, and take the ultraproduct by the same ultrafilter. Let $g : \prod_{i\in I}\mathcal{B}_i \to \prod_{i\in I}\mathcal{C}_i$ be the composite function. Define the map $f : \mathcal{A} \to \prod_{i\in I} C_i/U$ as $f(a/U) = g(a)/U$. Then this is well-defined and injective: if $a/U \neq b/U$, then $I - [\![a = b]\!] \in U$, so $I - [\![g(a) = g(b)]\!] \in U$ (because g is injective, and U is upwards closed), so $g(a)/U \neq g(b)/U$; the injectivity condition follows by dual arguments. It is a homomorphism by usual arguments. Moreover, if $\varphi(\overline{x}, \overline{a})$ is any formula such that $\mathcal{A} \models \forall\overline{x}\varphi(\overline{x}, \overline{a})$, then $[\![\forall\overline{x}\varphi(\overline{x}, \overline{a})]\!] \in U$; for each $i \in [\![\forall\overline{x}\varphi(\overline{x}, \overline{a})]\!]$, then $\mathcal{B}_i \models \forall\overline{x}\varphi(\overline{x}, \overline{a}(i))$, so by assumption, $\mathcal{C}_i \models \forall\overline{x}\varphi(\overline{x}, g_i(\overline{a}(i)))$. Hence:

$$[\![\forall\overline{x}\varphi(\overline{x}, g(\overline{a}))]\!] \in U$$

Which shows the result, by Los' theorem.

Now as noted above, we have that $\mathbb{IS}_2(\mathbb{IS}_2\mathbb{PP}_U(\mathbf{K})) = \mathbb{IS}_2\mathbb{PP}_U(\mathbf{K})$. Additionally, note that:

$$\mathbb{P}(\mathbb{IS}_2\mathbb{PP}_U(\mathbf{K})) \subseteq \mathbb{IS}_2(\mathbb{PPP}_U(\mathbf{K})) = \mathbb{IS}_2\mathbb{PP}_U(\mathbf{K}))$$

and also:
$$\mathbb{P}_U(\mathbb{I}\mathbb{S}_2\mathbb{P}\mathbb{P}_U(\mathbf{K})) \subseteq \mathbb{I}\mathbb{S}_2\mathbb{P}_U\mathbb{P}\mathbb{P}_U(\mathbf{K}))$$
But now notice that $\mathbb{P}_U\mathbb{P} \leqslant \mathbb{P}_R\mathbb{P}_R \leqslant \mathbb{P}_R$. Additionally, by [19, Theorem 1.2.12], we have that $\mathbb{P}_R \leqslant \mathbb{P}_S\mathbb{P}_U$. Also by our Proposition 3.5, we have that $\mathbb{P}_S \leqslant \mathbb{S}_2\mathbb{P}$, and hence:
$$\mathbb{P}_U\mathbb{P} \leqslant \mathbb{I}\mathbb{S}_2\mathbb{P}\mathbb{P}_U$$
This thus entails above that:
$$\mathbb{P}_U(\mathbb{I}\mathbb{S}_2\mathbb{P}\mathbb{P}_U(\mathbf{K})) \subseteq \mathbb{I}\mathbb{S}_2\mathbb{P}\mathbb{P}_U(\mathbf{K})$$
As desired. Hence $\mathbb{I}\mathbb{S}_2\mathbb{P}\mathbb{P}_U(\mathbf{K})$ is closed under all operations, and is contained in the smallest inductive rule class, implying they must be equal. ■

As usual when working with model theory, the method of diagrams turns out to be quite useful.

Definition 3.8. Let $\mathcal{A}$ be an $\mathscr{L}$-algebra. We denote the $\forall$-*open diagram* of $\mathcal{A}$ as follows:
$$T_0(A) = \{\forall\overline{x}\varphi(\overline{x},\overline{a}) : \varphi(\overline{x},\overline{a}) \text{ is atomic }\} \cup \{\chi(\overline{a}) : \chi(\overline{a}) \text{ is negated atomic}\},$$
where these formulas are considered in the language with constants from A.

The following lemma will be very useful, and runs along the same lines as usual proofs using the method of diagrams:

Lemma 3.9. For each $\mathscr{L}$-algebras $\mathcal{A}, \mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \leqslant_\forall \mathcal{B}$;
2. $\mathcal{B}$ is a model of $T_0(\mathcal{A})$.

Putting all of this together gives us the main result of this section:

Theorem 3.10. *Let $\mathbf{K}$ be a collection of algebras. Then the following are equivalent:*

1. *$\mathbf{K}$ is an inductive rule class;*
2. *$\mathbf{K}$ is axiomatised using $\forall\exists$ Special Horn formulas;*
3. *$\mathbf{K} = \mathbb{I}\mathbb{S}_2\mathbb{P}\mathbb{P}_U(\mathbf{K}^*)$ for some class $\mathbf{K}^*$.*

Proof. (3) is equivalent to (1) by Theorem 3.7. (2) implies (1) by Proposition 3.4. So we show that (1) implies (2). For that purpose, we will consider $Th^H_{\forall\exists}(\mathbf{K})$, and show that this axiomatises $\mathbf{K}$. So assume that $\mathcal{A}$ is a $\mathscr{L}$-algebra satisfying this set of formulas.

Now consider $\{\forall\overline{x}\varphi_1(\overline{a},\overline{x}), ..., \forall\overline{x}\varphi_n(\overline{a},\overline{x}), \neg\chi_0(\overline{a}), ..., \neg\chi_k(\overline{a})\} \subseteq T_0(A)$, where all formulas are atomic; then
$$\mathbf{A} \vDash \exists\overline{a}(\forall\overline{x}(\varphi_1(\overline{x},\overline{a}) \wedge ... \wedge \varphi_n(\overline{x},\overline{a})) \wedge \neg\chi_0(\overline{a}) \wedge ... \wedge \neg\chi_k(\overline{a})).$$

Note that this follows from the fact that universal quantifiers commute with conjunction, and hence we can pull them outside of the conjunction of the φ_i. We will show that some member of $\mathbb{P}(\mathbf{K})$ satisfies this sentence as well. For this purpose it suffices to show that:
$$\mathbb{P}(\mathbf{K}) \nvDash \forall\overline{y}(\forall\overline{x}(\varphi_1(\overline{y},\overline{x}) \wedge ... \wedge \varphi_n(\overline{y},\overline{x})) \to \chi_0(\overline{y}) \vee ... \vee \chi_k(\overline{y}))$$

If there were at most one χ_i, then this would be logically equivalent to a $\forall\exists$ Special Horn formula which is not true of A, and since the latter satisfies all $\forall\exists$ Special Horn consequences of the theory of $\mathbf{K}$, not of $\mathbf{K}$. So let us suppose that there are at least two χ_i. Then for $1 \leqslant i \leqslant k$ one can take the formula:

$$\mu_i := \forall \overline{y}(\forall \overline{x}(\varphi_1(\overline{y}, \overline{x}) \wedge ... \wedge \varphi_n(\overline{y}, \overline{x})) \to \chi_i(\overline{y}).$$

And we will obtain that $\mathbf{K} \not\models \mu_i$, since otherwise, we would have a contradiction to the fact that $\mathcal{A}$ satisfies all $\forall\exists$-Special Horn consequences of $\mathbf{K}$. Hence for some $\mathcal{A}_i \in \mathbf{K}$, $\mathcal{A}_i \models \exists \overline{y} \forall \overline{x}(\varphi_1(\overline{x}, \overline{y}) \wedge \varphi_n(\overline{x}, \overline{y})) \wedge \neg\chi_i(\overline{y}))$. Now choose elements $a_1(i), ..., a_n(i) \in \mathcal{A}_i$, such that:

$$\mathcal{A}_i \models \forall \overline{x}(\varphi_1(\overline{x}, a_1(i), ..., a_n(i)) \wedge ... \wedge \varphi_n(\overline{x}, a_1(i), ..., a_n(i))) \wedge \neg\chi_i(a_1(i), ..., a_n(i))$$

Then look at $\mathcal{A}_0 \times ... \times \mathcal{A}_t$, and consider the elements $a_1, ..., a_n$ defined to coincide with the witnesses above on each i. Then we have:

$$\mathcal{A}_0 \times ... \times \mathcal{A}_t \models \bigwedge_{i=1}^{n} \forall \overline{x}(\varphi_i(\overline{x}, a_1, ..., a_n) \wedge \bigwedge_{j=1}^{k} \neg\chi_i(a_1, ..., a_n)$$

Which follows from the facts we had about $\forall\exists$-Special Horn formulas. Hence, we have shown the desired fact about $\mathbb{P}(\mathbf{K})$.

Now let I be the set of finite subsets of $T_0(\mathcal{A})$. By the argument we just showed, for each $i \in I$, there is some $\mathcal{A}_i \in \mathbb{P}(\mathbf{K})$, and elements $\overline{a}(i) \in \mathcal{A}_i$ such that the formulas in i become true of $\mathcal{A}_i$ when $\overline{a}$ is interpreted as $\overline{a}(i)$.

Hence consider $J_i = \{j \in I : i \subseteq j\}$. Let $F = \uparrow\{J_i : i \in I\}$; then note that F is a filter: it is clearly uwpards closed, and if $J_{i_0} \subseteq K$ and $J_{i_1} \subseteq K'$, note that $J_{i_0} \cap J_{i_1} = \{j \in I : i_0 \subseteq j \text{ and } i_1 \subseteq j\} = J_{i_0 \cup i_1}$ so $J_{i_0 \cup i_1} \subseteq K \cap K'$. Hence by the ultrafilter principle, let U be an ultrafilter extending F.

Now let $\hat{a}$ be the tuple of elements in $\prod_{i \in I} A_i$ whose ith coordinate is $\overline{a}(i)$. Then note that for $\psi(a_0, ..., a_n) \in T_0(\mathbf{A})$, we have:

$$[\![\psi(\hat{a}_0, ..., \hat{a}_n)]\!] \supseteq J_i \in U$$

where $i = \{\psi(a_0, ..., a_n)\}$; hence $[\![\psi(\hat{a}_0, ..., \hat{a}_n)]\!] \in U$, so $\prod_{i \in I} \mathbf{A}_i/U \models \psi(\hat{a}_0/U, ..., \hat{a}_n/U)$.

By hypothesis on the factors of the product, $\prod_{i \in I} \mathbf{A}_i/U \in \mathbb{P}_U\mathbb{P}(\mathbf{K})$. By what we just showed, this satisfies $T_0(\mathbf{A})$, so by Lemma 3.9, we have that $\mathbf{A} \in \mathbb{IS}_2\mathbb{P}_U\mathbb{P}(\mathbf{K})$. By the closure conditions, we then obtain that:

$$\mathbf{A} \in \mathbf{K},$$

which shows the axiomatisation, as intended. ■

The sharp reader may have noticed that in the proofs above, the fact that the formulas in the antecedent of the special Horn formulas were universal did not play any special role − one can force the variables to be disjoint, ensuring all the commutations of quantifiers ensure. One might then wonder whether a similar result could be proved for subdirect product classes; the key obstacle of this seems to be the use of the method of diagrams: whilst we know that satisfying the diagram $T_0(A)$ implies being a $\forall$-subalgebra, no similar characterisation seems available for subdirect products. Nevertheless, the study of Π_n-classes seems entirely plausible, and the work of this chapter should adapt easily to this setting. One criterion which does not seem to easily generalize, and which turns out to be the most useful notion in practice, is the following form of verifying whether a given subalgebra is a $\forall$-subalgebra.

Proposition 3.11. Assume that $f : \mathcal{A} \to \mathcal{B}$ is an embedding. Then f is a $\forall$-embedding if and only if for every finite sequence $\overline{a}$, $(\mathcal{B}, f\overline{a}) \in \mathbb{V}(\mathcal{A}, \overline{a})$.

Proof. First assume that f is not a $\forall$-embedding. Hence there is some $\overline{a} \in A$, and some equation $\varphi(\overline{a}, \overline{x})$ such that $A \models \forall\overline{x}\varphi(\overline{a}, \overline{x})$ and $B \not\models \forall\overline{x}\varphi(f\overline{a}, \overline{x})$. But then clearly $(\mathcal{B}, f\overline{a})$ cannot belong to the variety generated by $(\mathcal{A}, \overline{a})$: the equation

$$\varphi(\overline{a}, \overline{x})$$

is valid in $(\mathcal{A}, \overline{a})$, but the same equation, interpreted over $(\mathcal{B}, f\overline{a})$ fails for some $\overline{x}$; so by Birkhoff's theorem, we have that $(\mathcal{B}, f\overline{a}) \notin \mathbb{V}(\mathcal{A}, \overline{a})$.

Conversely, assume that $\mathcal{A} \leqslant_{\forall} \mathcal{B}$. Then whenever $(\mathcal{A}, \overline{a})$ validates some equation, so does $(\mathcal{B}, f\overline{a})$ by assumption; hence by Birkhoff's theorem, $(\mathcal{B}, f\overline{a})$ belongs to the variety generated by $(\mathcal{A}, \overline{a})$. ∎

As a simple corollary of this we can obtain the following fact about Boolean algebras:

Corollary 3.12. *There exists a unique inductive rule class of Boolean algebras.*

Proof. Let $\mathbf{B}$ be an arbitrary Boolean algebra. It is clear that $\mathbf{2}$ is a subalgebra of $\mathbf{B}$; we claim that moreover it is a universal subalgebra. Indeed, note that $\mathbf{2}$ admits no proper extension with constants (since all of its elements are already constants in the language). Additionally, $\mathbf{B}$ will certainly belong to the variety generated by $\mathbf{2}$. So by Proposition 3.11, $\mathbf{2}$ is indeed a universal subalgebra.

So notice that whenever $\mathbf{K}$ is a non-empty inductive rule class, $\mathbf{2} \in \mathbf{K}$; since the latter is closed by subdirect products by our results above, then $\mathbb{P}_S(\mathbf{2}) \subseteq \mathbf{K}$; but by Stone's theorem, the former already contains all Boolean algebras. ∎

4 Algebraic Completeness for Π_2-rule Systems

The results of the previous section can in some sense be called a form of model-theoretic completeness. However, this is not the same as completeness with respect to the notion of Π_2-rule system we have previously developed. In this section, we show, using a Lindenbaum-Tarski style argument, that this can furthermore be made into a dual isomorphism between the lattice of Π_2-rule Systems and the lattice of Inductive rule classes. Our results in this section follow the usual recipe for establishing such results, and the notion of proofs using Π_2-rules is generalised from [7] and [8].

Throughout this section we assume that we have a logical calculus L of modal or intuitionistic logic, which is algebraized by some class of (Heyting or modal) algebras $\mathbf{K}$.

Definition 4.1. Let $\mathscr{L}$ be an algebraic language, and let Σ be a set of Π_2-rules. Let L be a logical calculus. For a set of formulas Γ with bound context F and a formula φ we say that φ is *derivable* from Γ in L using the Π_2-rules in Σ, and write $\forall\overline{p}\Gamma \vdash_{\mathsf{L}\oplus\Sigma} \varphi$, provided there is a sequence $\psi_0, ..., \psi_n$ of formulas such that:

- $\psi_n = \varphi$;
- For each ψ_i we have that either:

1. ψ_i is an instance of Γ where possibly some free variables are replaced by some substitution instance;
2. ψ_i is an instance of an axiom of L or,
3. ψ_i is obtained using Modus Ponens from some previous ψ_j or,
4. $\psi_i = \chi(\overline{\xi})$ and $\psi_{j_k} = \mu_k(\overline{r}, \overline{\xi})$ for $j_k < i$, where
   - $\Delta = \{\mu_k(\overline{q}, \overline{p}) : k \in \{0, ..., m\}\}$;
   - $\Delta/^2\chi \in \Sigma$;
   - $\chi = \chi(\overline{p})$;
   - $\overline{r}$ does not appear free in Γ.[5]

Lemma 4.2. Let Σ be a Π_2-rule system. Then:

$$\Gamma \vdash_{\mathsf{L}\oplus\Sigma} \varphi \iff \Gamma/^2\varphi \in \Sigma$$

Proof. Right to left is immediate. Left to right follows by an induction on the structure of the derivation: Strong Reflexivity ensures condition (1); bound structurality ensures that the substitutions work; from rule cut we get both the use of Modus Ponens, and allows us to pull through the case for other inductive rules being used in the derivation. ■

Theorem 4.3 (Completeness Theorem for Inductive rules)**.** *Let Σ be a Π_2-rule system in the language $\mathscr{L}$. Assume that $\Gamma/^2\varphi \notin \Sigma$. Then there is some $\mathscr{L}$-algebra $\mathcal{A}$, such that $\mathcal{A} \vDash \Sigma$, and $\mathcal{A} \nvDash \Gamma/^2\varphi$.*

Proof. Since $\Gamma/^2\varphi \notin \Sigma$, by the previous Lemma, $\forall\overline{p}\Gamma \nvdash_{\mathsf{IPC}\oplus\Sigma} \varphi$. Hence, let Prop be the set of proposition letters. Let $\mathcal{F}_{\mathsf{P}}$ be the Lindenbaum-Tarski algebra defined over the term algebra, in the following way

$$[\psi] \leqslant [\mu] \iff \forall\overline{p}\Gamma \vdash_{\mathsf{IPC}\oplus\Sigma} \psi \to \mu$$

The standard argument then shows that $\mathcal{F}_{\mathsf{P}}$ yields a $\mathscr{L}$-algebra. We additionally claim that $\mathcal{F}_{\mathsf{P}}$ validates the axioms of Σ. Indeed take $\Delta/^2\psi$ a certain rule, and $v : \mathsf{Prop} \to \mathcal{F}_{\mathsf{P}}$ a valuation, with the property that for each valuation v' differing from it at most in $B_c(\Delta)$, the bound context of Δ, $v'(\Delta) = [\top]$. Note that then, in particular, we can take this for some propositional variables not ocurring free in Γ, since only finitely many do so; hence set $v'(q) = q_i$ for q_i not occuring free in Γ, and not ocurring in $v(p)$ for any p which is not in $B_c(\Delta)$, and $v'(p) = v(p)$ otherwise, and consider v' as a substitution, i.e. for each $p \notin B_c(\Delta)$, $v'(p) = \chi_p$ is some formula. Then we note that:

$$[\psi_i(\chi_p, \overline{q})] = [\top]$$

It follows that for each i:

$$\Gamma \vdash_{\mathsf{IPC}\oplus\Sigma} \psi_i(\overline{\chi_p}, \overline{q})$$

Hence note that since $\Delta/^2\psi$ is in the calculus (and there are no variable clashes), we can then derive:

$$\Gamma \vdash_{\mathsf{IPC}\oplus\Sigma} \psi(\overline{\chi_p}).$$

[5] Note that this definition works implicitly up to α-renaming, which is done to avoid complicated tracking of substitutions renaming variables. All of this could be developed using such artifacts, though we will not pursue it here.

Which means that $v(\psi(\overline{p})) = 1$. This shows that $\mathcal{F}_{\mathsf{P}} \models \Delta/^2\mu$. We conclude that $\mathcal{F}_{\mathsf{P}} \models \Sigma$.

Now consider the valuation over $\mathcal{F}_{\mathsf{P}}$ defined by $v(p) = p$. Note that since $\Gamma \vdash \mu$ for each $\mu \in \Gamma$, we have that $[\bigwedge \Gamma] = 1$. Additionally, for each valuation v' differing at most in the value of the free context of Γ, we can consider v' as a substitution, and obtain that $\Gamma \vdash_{\mathsf{IPC} \oplus \Sigma} v'(\mu)$ again, by our assumption on the derivations. On the other hand, by hypothesis:

$$\Gamma \nvdash_{\mathsf{IPC} \oplus \Sigma} \varphi$$

and so $v(\varphi) \neq [\top]$. Hence, $\mathcal{F}_{\mathsf{P}}, v \nvDash \Gamma/^2\varphi$, as desired. ■

Having this completeness theorem, we now have all the tools necessary to prove the dual isomorphism as desired. The following definition was mentioned above:

Definition 4.4. Let $\mathbf{K}$ be a class of $\mathscr{L}$-algebras. Define the collection of Π_2-rules associated with $\mathbf{K}$ as follows:

$$\Pi_2(\mathbf{K}) := \{\Gamma/^2\psi : \mathcal{A} \models \Gamma/^2\psi, \forall \mathcal{A} \in \mathbf{K}\}$$

The following is a straightforward consequence of Proposition 2.7:

Proposition 4.5. For each class $\mathbf{K}$ of $\mathscr{L}$-algebras, $\Pi_2(\mathbf{K})$ is a Π_2-rule system.

As before we have that that the operations defining an inductive rule class preserve satisfaction of these rules, essentially given that such rules correspond to $\forall\exists$-Special Horn formulas:

Proposition 4.6. Let $\mathbf{K}$ be a class of Heyting algebras. Then:

$$\Pi_2(\mathbf{K}) = \Pi_2(\mathbb{S}_2(\mathbf{K})) = \Pi_2(\mathbb{P}(\mathbf{K})) = \Pi_2(\mathbb{P}_U(\mathbf{K}))$$

Consequently:

$$\Pi_2(\mathbf{K}) = \Pi_2(\mathbb{IR}(\mathbf{K}))$$

Likewise, we have the following operation, moving in the dual direction:

Definition 4.7. Given a collection Φ of Π_2-rules, define:

$$Ind(\Phi) := \{\mathcal{A} : \mathcal{A} \models \Gamma/^2\psi, \forall \Gamma/^2\psi \in \Phi\}$$

We call this the *inductive rule class* generated by Φ.

We can now prove the following:

Proposition 4.8. For each inductive rule class of $\mathscr{L}$-algebras $\mathbf{K}$, we have that:

$$Ind(\Pi_2(\mathbf{K})) = \mathbf{K}$$

Proof. One inclusion is clear: if $\mathcal{A} \in \mathbf{K}$, then it certainly validates all rules validated by all algebras in that class. We focus on the other. By Theorem 3.10, $\mathbf{K}$ is axiomatised by $\forall\exists$-formulas. So if $\mathcal{A} \in Ind(\Pi_2(\mathbf{K}))$, then $\mathcal{A} \models Th^H_{\forall\exists}(\mathbf{K})$, and so $\mathcal{A} \in \mathbf{K}$. ■

The other direction follows from our algebraic completeness proof:

Corollary 4.9. *Let Σ be a Π_2-rule system. Then:*

$$\Pi_2(Ind(\Sigma)) = \Sigma.$$

Proof. The right to left inclusion surely holds. Now assume that $\Gamma/^2\varphi \notin \Sigma$. By Theorem 4.3, we have that there is an algebra $\mathcal{A}$ such that $\mathcal{A} \models \Sigma$ and $\mathcal{A} \not\models \Gamma/^2\varphi$. The former implies that $\mathcal{A} \in Ind(\Sigma)$, which together with the latter fact implies that $\Gamma/^2\varphi \notin \Pi_2(Ind(\Sigma))$. ∎

We then have the following theorem:

Theorem 4.10. *There is a dual isomorphism, Ind, between the lattice of Π_2-rule systems, and the lattice of inductive rule classes of $\mathscr{L}$-algebras.*

Proof. By Proposition 4.8 and Corollary 4.9, we have that Ind is a bijection and that Π_2 is its inverse. Also note that Ind and Π_2 are order-reversing maps. Given $\vdash$ and $\vdash^*$, two Π_2-rule systems, note that:

$$Ind(\vdash \cap \vdash^*) = Ind(\vdash) \vee Ind(\vdash^*).$$

Indeed, note that if $\mathcal{A} \in Ind(\vdash)$, or $\mathcal{A} \in Ind(\vdash^*)$, then $\mathcal{A} \in Ind(\vdash \cap \vdash^*)$, so the right to left inclusion holds. Now to show the other inclusion, we show that:

$$\Pi_2(Ind(\vdash) \vee Ind(\vdash^*)) \subseteq \vdash \cap \vdash^*.$$

Indeed if $\Gamma/^2\varphi \notin \vdash \cap \vdash^*$, then it does not belong to one of them, say $\vdash$. Let $\mathcal{A}$ be an algebra validating $\vdash$ such that $\mathcal{A}$ refutes this rule. Then since $\mathcal{A} \in Ind(\vdash)$, surely $\mathcal{A} \in Ind(\vdash) \vee Ind(\vdash^*)$, so $\Gamma/^2\varphi$ is not valid in such an inductive class, i.e., $\Gamma/^2\varphi \notin \Pi_2(Ind(\vdash) \vee Ind(\vdash^*))$.

Additionally we have that:

$$Ind(\vdash \vee \vdash^*) = Ind(\vdash) \cap Ind(\vdash^*),$$

Indeed, if $\mathcal{A} \in Ind(\vdash)$ and $\mathcal{A} \in Ind(\vdash^*)$, note that mimicking the derivation relation using the consequence relation in $\mathcal{H}$, we have that $\mathcal{A} \in Ind(\vdash \vee \vdash^*)$. Now assume that $\mathcal{A} \in Ind(\vdash \vee \vdash^*)$. Then surely it will validate all rules in $\vdash$ and all rules in $\vdash^*$, so $\mathcal{H} \in Ind(\vdash) \cap Ind(\vdash^*)$. This shows the result. ∎

In the next section, making use of the results of this section[6], we will outline some basic notions of admissibility which make sense in the setting of Π_2-rule systems.

5 Admissibility of Inductive Rules

The question of admissibility often comes up in the context of axiomatising logics of interest, wherein one first assumes some rule (for example, Gabbay's irreflexivity rule, or Takeuti-Titani's rule), uses it to prove completeness, and then shows the rule to be admissible. Due to the interest of using such rules, one can formulate several general questions concerning this:

[6] We briefly remark that the former arguments and techniques can likewise obtain for us a dual isomorphism between the lattice of Π_2-*multi-conclusion* rule systems and general inductive rule classes algebras (i.e., classes closed under ultraproducts and $\forall$-subalgebras). Whilst we do not pursue this here, we believe such a concept would be fruitful in furthering some connections with the model theory of Boolean and Heyting algebras.

- When is an inductive rule admissible?
- When is an admissible inductive rule derivable?
- How does this relate to admissibility and structural completeness in the classical case?

The above asks whether there are natural analogues of all of these concepts for the case of Π_2-rules. Keeping in mind the usual notion of admissibility from algebraic logic (see e.g. [6]), and again fixing a given logical calculus of interest, we can give the following definition:

Definition 5.1. Let $\Gamma/^2\psi$ be a Π_2-rule, and $\vdash_{\mathsf{L}\oplus\Sigma}$ some calculus, possibly extended by some additional Π_2-rules Σ. We say that the rule $\Gamma/^2\psi$ is *admissible* in $\vdash_{\mathsf{L}\oplus\Sigma}$ if for all χ:

$$\vdash_{\mathsf{L}\oplus\Sigma\oplus\Gamma/^2\psi} \chi \implies \vdash_{\mathsf{L}\oplus\Sigma} \chi$$

We say that the rule is *derivable* if $\Gamma \vdash_{\mathsf{L}\oplus\Sigma} \psi$.

Note that every derivable rule is admissible. The next Lemma establishes a first semantic criterion to recognise admissible rules, even if a very general and crude one:

Lemma 5.2. Let $\mathbf{K}$ be an inductive rule class of $\mathscr{L}$-algebras, and let $\vdash_S$ be its dual Π_2-rule system. For each $\Gamma/^2\psi$, we have that $\Gamma/^2\psi$ is admissible over $\vdash_S$ if and only if for each substitution σ leaving the free context fixed, $\mathbf{K} \vDash \sigma(\Gamma)$, then $\mathbf{K} \vDash \sigma(\psi)$.

Proof. Assume that $\Gamma/^2\psi$ is admissible. Assume that σ is some substitution such that $\mathbf{K} \vDash \sigma(\Gamma)$. In other words, if $\lambda_1, ..., \lambda_n$ are the terms occuring in Γ, $\mathbf{K} \vDash \sigma(\lambda_1 \wedge ... \wedge \lambda_n)$, where the latter is understood as a formula. Hence by completeness, $\vdash_S \sigma(\lambda_1 \wedge ... \wedge \lambda_n)$, and so a fortriori, $\vdash_{S\oplus\Gamma/^2\psi}$ $\sigma(\lambda_1 \wedge ... \wedge \lambda_n)$, where the free context is left fixed by σ; using the rule, we have that $\vdash_{S\oplus\Gamma/^2\varphi} \sigma(\psi)$; so by admissibility, $\vdash_S \sigma(\psi)$, which means that $\mathbf{K} \vDash \sigma(\psi)$.

For the other direction, asssume that $\vdash_{S\oplus\Gamma/^2\varphi} \psi$. We wish to show that then $\vdash_S \psi$ as well. Using completeness, we have that $\mathbf{K} \vDash S$ certainly, and we can preserve the relation that $\mathbf{K}$ validates a set of formulas through the derivation. The fact that whenever $\mathbf{K} \vDash \sigma(\Gamma)$ then $\mathbf{K} \vDash \sigma(\psi)$ thus works to mimick the applications of $\Gamma/^2\psi$, which allows us to conclude, by induction, that $\mathbf{K} \vDash \psi$, i.e., $\vdash_S \psi$. ■

The former definition might appear slightly odd at first glance to those who are familiar with the algebraic definition of admissibility of the usual rules. Indeed, in general we require that a rule Γ/φ be admissible if for *each substitution* σ, if $\mathbf{K} \vDash \sigma(\Gamma)$ then $\mathbf{K} \vDash \sigma(\varphi)$. It thus appears that being admissible as a Π_2-rule is easier than being admissible as a usual rule, which might appear counterintuitive. As we will see when we relate admissibility to generation by algebras, however, this simply reflects the fact that Π_2-rules are more expressive, and so can capture finer classes than usual rules.

Because Π_2-rules have a higher complexity, there are alternative notions of admissibility that also make themselves available in this context. Namely, the following makes sense here:

Definition 5.3. Let $\Gamma/^2\varphi$ be a Π_2-rule and $\vdash_{\mathsf{IPC}\oplus\Sigma}$ some extended calculus. We say that the rule $\Gamma/^2\varphi$ is *hereditarily admissible* if for all Δ/ψ a Π_1-rule, we have that

$$\Delta \vdash_{\mathsf{IPC}\oplus\Sigma\oplus\Gamma/^2\varphi} \psi \implies \Delta \vdash_{\mathsf{IPC}\oplus\Sigma} \psi.$$

Note that a derivable rule is hereditarily admissible: whenever Δ/ψ can be derived by using the derived rule, we can carry out the derivation of $\Gamma/^2\varphi$ to obtain the conclusion without relying on the rule. Hereditary admissibility is a priori a stronger property than admissibility: every Π_1-rule which is hereditarily admissible is derivable by definition.

Lemma 5.4. Let $\mathbf{K}$ be an inductive rule class, and $\vdash_S$ the dual Π_2-intuitionistic rule system. For each $\Gamma/^2\varphi$ we have that $\Gamma/^2\varphi$ is hereditarily admissible if and only if for each Δ a set of formulas, and σ a substitution leaving the free context fixed, $\mathbf{K} \vDash \sigma[\Delta] \to \sigma(\psi_i)$ for each $\psi_i \in \Gamma$, then $\mathbf{K} \vDash \sigma[\Delta] \to \sigma(\varphi)$.

Proof. Assume that $\Gamma/^2\varphi$ is hereditarily admissible. Assume that Δ is arbitrary and σ is some substitution such that $\mathbf{K} \vDash \sigma[\Delta] \to \sigma(\psi_i)$ for each ψ_i. Hence we have that $\sigma[\Delta] \vdash_S \sigma(\psi_i)$, through a substitution which preserves the free context of Γ. Hence surely $\sigma[\Delta] \vdash_{S\oplus\Gamma/^2\varphi} \sigma(\psi_i)$; by using the rule, we then have that $\sigma[\Delta] \vdash_{S\oplus\Gamma/^2\varphi} \sigma(\varphi)$, and by hereditary admissibility, then $\sigma[\Delta] \vdash_S \sigma(\varphi)$, hence $\mathbf{K} \vDash \sigma[\Delta] \to \sigma(\varphi)$.

Now assume the condition. Suppose that $\Delta \vdash_{S\oplus\Gamma/^2\varphi} \psi$. Assume that $\psi_1, ..., \psi_n$ is a derivation. Then find $\mathbf{A} \in \mathbf{K}$ which validates the axioms of Δ; then $\mathbf{A}$ can reproduce the derivation given; admissibility is used to witness the use of the rule $\Gamma/^2\varphi$. Hence we conclude with $\mathbf{K} \vDash \Delta/\psi$, so $\Delta \vdash_S \psi$. ■

As promised we can now relate the concepts of admissibility and hereditary admissibility to generation by varieties and quasivarieties:

Lemma 5.5. Let $\mathbf{K}$ be an inductive rule class. Suppose that $\mathbb{V}(\mathbf{K}) = \mathbb{V}(\mathbf{S})$ where $\mathbf{S}$ is a class of $\mathscr{L}$-algebras validating a rule $\Gamma/^2\varphi$. Then $\Gamma/^2\varphi$ is admissible in $\mathbf{K}$. Similarly if $\mathbf{K}$ is a quasivariety and $\mathsf{QVar}(\mathbf{K}) = \mathsf{QVar}(\mathbf{S})$, then $\Gamma/^2\varphi$ is hereditarily admissible.

Proof. Assume that $\mathbf{K} \vDash \sigma(\Gamma)$ through a substitution not affecting the bound context of Γ. Hence $\mathbf{S} \vDash \sigma(\Gamma)$ as well. Now given the hypothesis, this means that for each $\mathcal{H} \in \mathbf{S}$, and each model v, $\mathcal{H}, v \vDash \forall\overline{p}\Gamma$: indeed, one can simply change the valuation only for propositions occuring in $\forall\overline{p}$, and note that the assumption will then be that $\mathcal{H}, v' \vDash \sigma(\Gamma)$ as well; so by hypothesis, $\mathcal{H}, v \vDash \sigma(\varphi)$. So $\mathbf{S} \vDash \sigma(\varphi)$, which immediately implies that $\mathbf{K} \vDash \sigma(\varphi)$. ■

We can also provide a form of converse of the previous:

Lemma 5.6. Let $\Gamma/^2\psi$ be a rule which is admissible in an inductive rule class $\mathbf{K}$. Then $\mathbb{V}(\mathbf{K})$ is generated as a variety by:

$$\{\mathcal{H} \in \mathbf{K} : \mathcal{H} \vDash \Gamma/^2\psi\}$$

Similarly if $\Gamma/^2\psi$ is hereditarily admissible in an inductive rule class $\mathbf{K}$ then $\mathsf{QVar}(\mathbf{K})$ is generated as a quasivariety by the same set.

Proof. Consider:

$$\mathbb{V}(\{\mathcal{H} \in \mathbf{K} : \mathcal{H} \vDash \Gamma/^2\psi\})$$

We wish to show that this is the same as $\mathbb{V}(\mathbf{K})$. One inclusion is clear. For the other, assume that $\{\mathcal{A} \in \mathbf{K} : \mathcal{A} \vDash \Gamma/^2\psi\} \vDash \chi$ for some formula χ. Then note that:

$$\vdash_{\Pi_2(\mathbf{K})\oplus\Gamma/^2\psi} \chi,$$

So by admissibility, we have that $\vdash_{\Pi_2(\mathbf{K})} \chi$, i.e., $\mathbf{K} \vDash \chi$. By Birkhoff's theorem this shows that the two classes are equal. The result for hereditary admissibility and quasivarieties is similar. ■

Some logical systems have the desirable property that all admissible rules are derivable – such systems are often called *structurally complete*. We can also formulate an analogous concept to this one for the setting of Π_2-rule systems:

Definition 5.7. Let $\mathbf{K}$ be an inductive rule class. We say that $\mathbf{K}$ is *inductively complete* if whenever $\Gamma/^2\varphi$ is an admissible rule in the corresponding calculus, then it is derivable.

Lemma 5.8. Let $\mathbf{K}$ be an inductive rule class, and let $\mathbf{F}_\omega(\mathbf{K})$ be the free algebra on ω-many generators of this class. For each rule $\Gamma/^2\psi$, $\Gamma/^2\psi$ is admissible over $\mathbf{K}$ if and only if it is derivable in $\mathbf{F}_\omega(\mathbf{K})$.

Proof. First assume that $\Gamma/^2\psi$ is admissible over $\mathbf{K}$. Let v be a valuation over $\mathbf{F}_\omega(\mathbf{K})$ such that each variant of it in the bound context of Γ evaluates to 1. Then note that we can turn v into a substitution σ which leaves the bound context of Γ fixed, and does not induce any clash of variables, and such that $\mathbf{F}_\omega(\mathbf{K}) \vDash \sigma(\Gamma)$. Hence $\mathbf{K} \vDash \sigma(\Gamma)$, so because the rule is admissible, by Lemma 5.2, $\mathbf{K} \vDash \sigma(\psi)$, i.e., $(\mathbf{F}_\omega(\mathbf{K}), v) \vDash \psi$. This shows the rule is derivable.

For the converse, note that if $\mathbf{F}_\omega(\mathbf{K})$ makes the rule valid, then since $\mathbb{V}(\mathbf{K}) = \mathbb{V}(\mathbf{F}_\omega(\mathbf{K}))$, then any derivable rule in $\mathbf{F}_\omega(\mathbf{K})$ will be admissible in $\mathbf{K}$ by Lemma 5.5. ∎

Using this we easily obtain a characterization of inductive completeness: a class is inductively complete if and only if it is generated as an inductive rule class by the free algebra. Using this, we can obtain a more useful characterization of inductive completeness, which applies when we assume the variety generated by the class to be itself structurally complete:

Lemma 5.9. Let $\mathbf{K}$ be an inductive rule class, and assume that $\mathbb{V}(\mathbf{K})$ is structurally complete, i.e., $\mathbb{V}(\mathbf{K}) = \mathsf{QVar}(\mathbf{F}_\omega(\mathbf{K}))$. Then $\mathbf{K}$ is inductively complete if and only if every proper sub-inductive class generates a proper subquasivariety of the quasivariety generated by $\mathbf{K}$.

Proof. First assume that the inductive rule class is inductively complete, and hence by Lemma 5.8, it is generated by the free algebra. Let $\mathbf{K}'$ be a subinductive class of $\mathbf{K}$, and assume that $\mathbb{S}(\mathbf{K}') = \mathbb{S}(\mathbf{K})$. Then $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbb{S}(\mathbf{K}')$, so clearly $\mathbf{K} \subseteq \mathbb{V}(\mathbf{K}')$ so by the Corollary 11.10 of [10], we have that $\mathbf{F}_K(\omega) = \mathbf{F}_{K'}(\omega)$. But since $\mathbf{F}_{K'}(\omega) \in K'$, we know that $\mathbf{F}_K(\omega) \in \mathbf{K}'$, which means that $\mathbf{K} \subseteq \mathbb{IR}(\mathbf{F}_K(\omega)) \subseteq \mathbf{K}'$; in other words, $\mathbf{K}' = \mathbf{K}$.

For the other direction, assume that $\mathbf{K}$ is not generated as a subinductive class by $\mathbf{F}_K(\omega)$; then $\mathbb{IR}(\mathbf{F}_K(\omega))$ is a proper inductive subclass. Since $\mathbb{S}(\mathbf{K})$ is structurally complete, $\mathbb{S}(\mathbf{K}) = \mathsf{QVar}(\mathbf{F}_K(\omega))$, and surely this will be the same as $\mathbb{S}(\mathbb{IR}(\mathbf{F}_K(\omega))$. So this is a proper subinductive class which does not generate a proper subquasivariety. ∎

We will have opportunity to see that these notions can be fruitfully analysed in specific cases. However, in this section, we wish to draw one more relevant connection, namely to the theory of model companions and model completions. The next theorem is analogous to [8, Theorem 5.4], and gives us a very useful criterion for hereditary admissibility.

Theorem 5.10. *Let $\mathbf{K}$ be an inductive rule class, and $\Gamma/^2\psi$ a Π_2-rule. Then $\Gamma/^2\psi$ is hereditarily admissible in $\mathbf{K}$ if and only if every $\mathcal{A} \in \mathbf{K}$ embeds into $\mathcal{B} \in \mathbf{K}$ such that $\mathcal{B} \vDash \Gamma/^2\psi$.*

Proof. First assume that $\Gamma/^2\psi$ is hereditarily admissible in $\mathbf{K}$. Let $\mathcal{A}$ be some algebra in $\mathbf{K}$. Look at $T(\mathcal{A})$, the *open diagram* of $\mathcal{A}$, i.e.

$$T(\mathcal{A}) = \{\varphi : \varphi \text{ is atomic or negated atomic, and } \mathcal{A} \vDash \varphi\}$$

We will show that $T(\mathcal{A})$ is consistent with the theory of $\mathbf{K}$ together with $\Gamma/^2\psi$. For this purpose, let $S = \{\varphi_0, ..., \varphi_n, \neg\psi_0, ..., \neg\psi_m\}$ be finitely many formulas from $T(\mathcal{A})$, such that all φ_i and ψ_j are atomic. Then note that for each i, it cannot be that

$$\Pi_2(\mathbf{K}) \cup \{\Gamma/^2\psi\} \vDash \forall\overline{x}(\varphi_0(\overline{x}) \wedge ... \wedge \varphi_n(\overline{x}) \rightarrow \chi_i(\overline{x}))$$

To see why, note that if this was the case, then we could consider $\Delta = \{\varphi_0, ..., \varphi_n\}$ and the rule Δ/χ_i. The above would imply that $\Pi_2(\mathbf{K}) \cup \{\Gamma/^2\psi\} \vDash \Delta/\chi_i$. Then by completeness, this would imply that $\Delta \vdash_{\Pi_2(\mathbf{K})\oplus\Gamma/^2\psi} \chi_i$; but this latter fact would imply, by hereditary admissibility, that $\Delta \vdash_{\Pi_2(\mathbf{K})} \chi_i$. This contradicts the fact that $\mathcal{A} \nvDash \Delta/\psi_i$ and $\mathcal{A} \vDash \Pi_2(\mathbf{K})$. Hence for each i, we can find $\mathcal{A}_i \in \mathbf{K}$ such that $\mathcal{A}_i \vDash \exists\overline{x}(\varphi_0(\overline{x}) \wedge ... \wedge \varphi_n(\overline{x}) \wedge \neg\psi_i(\overline{x}))$. Like in Theorem 3.10, we can then consider $\mathcal{A}_0 \times ... \times \mathcal{A}_n$, which yields the desired witness to the consistency of S.

Hence by compactness we have that we can find $\mathcal{B} \in \mathbf{K}$ such that $\mathcal{B} \vDash \Gamma/^2\psi$, and $\mathcal{B} \vDash T(\mathcal{A})$; but note that satisfying $T(\mathcal{A})$ implies that there is an embedding from $\mathcal{A}$ to $\mathcal{B}$.

Now assume that the condition holds. Assume that $\mathbf{K} \vDash \sigma[\Delta] \rightarrow \sigma(\psi_i)$ for each $\psi_i \in \Gamma$. Assume that $\mathcal{A} \in \mathbf{K}$, and $\mathcal{A} \nvDash \sigma[\Delta] \rightarrow \sigma(\varphi)$. Then for some model v, $(\mathcal{A}, v) \vDash \sigma[\Delta]$, but $(\mathcal{A}, v) \nvDash \sigma(\varphi)$. By assumption, there is some $\mathcal{B}$ which validates the rule in $\mathbf{K}$ into which $\mathcal{A}$ embeds. This means that $\mathcal{B} \nvDash \sigma[\Delta] \rightarrow \sigma(\varphi)$ as well. By assumption on $\mathbf{K}$, we then have that $(\mathcal{B}, v) \vDash \sigma(\psi_i)$ for each ψ_i. Since the substitution leaves the free context invariant, and $\mathcal{C}$ validates the rule, then $\mathcal{C}, v \vDash \sigma(\varphi)$, a contradiction. ■

The previous theorem implies that the Π_2-rules holding in a system are exactly those which are valid in the model completion, whenever this exists:

Theorem 5.11. *Let $\mathbf{K}$ be an inductive rule class, and suppose that it has a model companion T^*. Then a Π_2-rule $\Gamma/^2\psi$ is hereditarily admissible in $\mathbf{K}$ if and only if $T^* \vDash \Gamma/^2\psi$.*

Proof. First assume that $T^* \vDash \Gamma/^2\psi$. By general model theoretic facts, every algebra in $\mathbf{K}$ embeds into an existentially closed extension, which will be a model of T^*. Hence by Theorem 5.10, the rule $\Gamma/^2\psi$ is hereditarily admissible.

Conversely, suppose that $\Gamma/^2\psi$ is hereditarily admissible in $\mathbf{K}$. By the above theorem, we have that every algebra in $\mathbf{K}$ embeds into some algebra satisfying $\Gamma/^2\psi$. Let $\mathcal{A} \vDash T^*$. Let $\mathcal{B}$ be the extension satisfying $\Gamma/^2\psi$. Let $\overline{a} \in A$ be arbitrary; then $\mathcal{A} \vDash \exists\overline{y}\varphi(\overline{a}, \overline{y})$, certainly; since $\mathcal{A}$ is existentially closed, being a model of the model companion, also satisfies $\exists\overline{y}\varphi(\overline{a}, \overline{y})$. Thus $\mathcal{A} \vDash \Gamma/^2\psi$. ■

We also note the following relevant idea:

Proposition 5.12. Let $\mathbf{K}$ be an inductive rule class. Then $\Pi_2(\mathbf{K})$ has a model companion if and only if $\mathsf{Log}(\mathsf{QVar}(\mathbf{K}))$ has a model companion.

Proof. Assume that $\mathsf{Log}(\mathsf{QVar}(\mathbf{K}))$ has a model companion, say a theory T^*. Certainly every model of $\mathbf{K}$ is a model of $\mathsf{QVar}(\mathbf{K})$, and so can be embedded in a model of T^*; conversely, every model of T^* can be embedded in a model of the generated quasivariety, which in turn embeds into a model of $\mathbf{K}$. Hence T^* is a model companion of $\Pi_2(\mathbf{K})$.

Conversely, assume that $\Pi_2(\mathbf{K})$ has a model companion T^*. Then every model of T^* can be embedded in a model of $\mathsf{QVar}(\mathbf{K})$; additionally, every model of the quasivariety embeds into a model of $\mathbf{K}$, and through that, into a model of T^*. So T^* is a model companion of the quasivariety. ■

We also remark that if the quasivariety has a *model completion*, then that theory will be a model completion also of the inductive rule class; we do not know if the converse holds. Now we know that a quasivariety generated by finitely many finite elements has a model companion, which is additionally $\aleph_0$-categorical. Hence to study the hereditary admissibility of these theories it suffices to study these unique countable models, as well as the structure of the model companions of these theories. We leave such a study for future elaboration.

In the next section we will give some examples of how to use these results in a concrete setting.

6 Inductive Rule Classes of Gödel algebras

In these last two sections we provide some examples of how the general theory from before can be fruitfully analysed in specific cases. Throughout fix IPC as our base calculus. We recall that the *Gödel-Dummett logic* is axiomatised over this system as

$$\mathsf{LC} := \mathsf{IPC} \oplus (p \to q) \vee (q \to p).$$

Such a system was considered by Gödel and Dummett, and extensively studied from thereon (see e.g. [1] for a recent survey containing several open problems in the area). They are algebraized by *Gödel algebras*, with the same axiom defining them. An especially important case of such algebras are the *linear algebras*, which are Gödel algebras which lattice reduct is a linear order; throughout we also refer to these as "linear Heyting algebras". These algebras enjoy several desirable properties:

1. The variety of Gödel algebras is locally finite;
2. It is primitive, meaning that all extensions of LC are structurally complete [15];
3. Its subdirect irreducible elements are exactly the chains containing a second largest element.

In the following subsections we will analyse some natural questions related to LC.

6.1 Inductive Completeness and Cardinality of Inductive Rule Classes of Gödel Algebras

As discussed in the Introduction, one natural setting which was studied as an elaboration of Gödel logic was first-order Gödel logic [5]. Using the results from Section 5 we can give a particularly simple proof of the admissibility of some such rules:

Example 6.1. Recall from Example 2.4 the Takeuti-Titani "density rule". As mentioned before, such a rule is valid in a linear Heyting algebra if and only if it is dense. So by our above remarks, we obtain:

Corollary 6.2. *The density rule is admissible over* LC.

Proof. One simply notes that the variety of all Gödel algebras is generated by the dense linear Heyting algebras, since every finite linear Heyting algebra embeds into a dense one. So by Lemma 5.5 we have the result. ■

This has the following consequence:

Corollary 6.3. *The class of Gödel algebras is not inductively complete.*

Proof. It suffices to consider a linear Heyting which does not validate the Takeuti-Titani; simply consider for instance [3]. ■

We can contrast the former with the proof provided in [23], which involved a delicate syntactic argument. This indicates that the tools developed in this chapter might be useful in carrying out proofs of admissibility in other more general settings where, like in LC, there is a good grasp of the subdirect irreducibles.

Having a theory to discuss Π_2-rule systems, it is natural to ask what is the structure of such systems in the setting of extensions of LC, which as noted above are quite tame. Indeed, as just mentioned, it is well-known that there are only countably many Gödel logics, all of which are structurally complete, and all Gödel quasivarieties are already varieties. Additionally, there are also only countably many first-order Gödel logics [5]. As it turns out, though we do not focus on this setting in the present paper, there are also only countably many multiple conclusion consequence relations[7].

In this section we will show that the situation for inductive classes changes dramatically, on precise account of the reflection enforced by $\forall$-subalgebras.

For the sequel, given $n \in \omega$, denote by $[n]$ the unique finite linear Gödel algebra with n many elements; denote by λ_n the formula which defines the variety generated by this algebra. The crucial fact we will need about λ_n is that for each $m, k \in \omega$, we have $m \leqslant k$, if and only if $[m] \Vdash \lambda_k$. Given a subset $X \subseteq \omega$, write $\mathbb{IR}(X)$ for the inductive rule class generated by all $[k]$ for $k \in X$.

Lemma 6.4. Let $X, Y \subseteq \omega$ be infinite subsets, and assume that $X \neq Y$. Then $\mathbb{IR}(X) \neq \mathbb{IR}(Y)$.

Proof. Assume that the inductive class generated by X is the same as the inductive class generated by Y. Since they are different, assume without loss of generality that $[n] \in \mathbb{IR}(X)$ for $n \in Y - X$. By our Theorem 3.10, giving the model theoretic completeness, then $[n] \leqslant_\forall \mathcal{P}$ where $\mathcal{P} = \prod_{i \in I} \mathcal{M}_i$ where $\mathcal{M}_i$ are ultraproducts of algebras in X. Now, by assumption, $[n]$ satisfies λ_n. Hence because this is a $\forall$-subalgebra, $\mathcal{P} \Vdash \lambda_n$ as well, and so will each of its factors, by preservation of equations in products. Hence by definition of satisfaction of a formula in an ultraproduct, given any $\mathcal{M}_i$ there must be ultrafilter many coordinates where the defining equation is valid; hence, for ultrafilter many coordinates, the algebra at that coordinate is smaller than or equal to $[n]$.

Now, since $n \notin X$, for each M_i, there must be a greatest $k < n$ such that $k \in X$ (and this must be non-empty, by the assumption on the ultrafilter), and for ultrafilter many coordinates m, $\mathcal{M}_i(m) = [k]$. Hence by Los' theorem, we have that $\mathcal{M}_i \vDash \lambda_k$.

Since this argument holds regardless of $\mathcal{M}_i$, we have that $\mathcal{P} \vDash \lambda_k$ as well. But then we have that since $[n]$ is a subalgebra of P, then $[n] \Vdash \lambda_k$, from which we obtain that $n \leqslant k$ – a contradiction. Hence by reductio, we have that $[n] \notin \mathbb{IR}(X)$. ■

Corollary 6.5. *There are at exactly $2^{\aleph_0}$ inductive rule classes of Gödel algebras.*

Proof. The former gives us a lower bound, which is an upper bound since inductive rule classes are bounded by the size of the language. ■

[7]This result, which is not currently available in the literature, follows from some work communicated to the author by Ian Hodkinson. The key idea lies in showing that the order on co-trees given by surjective p-morphism (i.e., given trees T, Q, $T \leqslant Q$ if there is a surjective p-morphism from Q to T) is a well-partial order, which can be done by adapting the classical proof of Kruskal's theorem.

The former result should sound quite strange indeed if one is used to working with varieties and quasivarieties of Heyting algebras: it implies that with respect to inductive rule classes, the order which makes classical logic the unique maximal extension disappears: the inductive rule class generated by **2** is incomparable with the one generated by **3**. Let us look at this situation further. Throughout we write a/b for $(b \to a) \to b$. This connective is frequently used in the literature on Gödel algebras, for example by [3]. We note the following about a/b, which is easily shown.

Lemma 6.6. For each $a, b \in \mathcal{H}$ where $\mathcal{H}$ is a linear Heyting algebra, we have that:

- If $a < b$, then $a/b = 1$;
- If $b \leqslant a$, then $a/b = b$.

To give some logical substance to the phenomenon just outlined of the classes being separate, we will give some examples of inductive rules which separate these classes. For instance, consider the following:

$$\forall p(\forall q(\neg\neg q \leqslant q \vee p) \to p = 1)$$

Notice that in **2** this rule fails: we have that $0 \neq 1$, but if q is arbitrary, then $\neg\neg q = q \leqslant q \vee p$. So the rule fails in this algebra. On the other hand, it is valid in **3**: if $p \neq 1$, then let a be the intermediate element, and let $q = a$. Then we have that $\neg\neg q = 1$, whilst $q \vee p \leqslant a$, so the rule holds.

Also note that $\neg\neg q \approx \neg q \to q$ for linear Heyting algebras, which can be written in our notation as $0/q$. More generally, consider the following, which are patterned on rules from [3].

$$\rho_n := \forall p(\forall p_1, ..., p_n(0/p_1 \wedge p_1/p_2 \wedge ... \wedge p_{n-1}/p_n \leqslant p_1 \vee ... \vee p_n \vee p) \to p = 1)$$

Then we claim that such a rule is valid in $[n+1]$ but not $[n]$. To see why it is not valid in $[n]$, note that given $p \neq 1$, in order for the inequality not to hold, then $p_{i+1} \to p_i$ can never be 1; hence $p_i < p_{i+1}$, which is impossible since this would draw a chain of $n+1$ many elements in $[n]$.

On the other hand, $[n+1]$ validates this: given $p \neq 1$, simply pick precisely the chain of n many elements from 0 to the element covered by 1. Then $p_{i+1} \to p_i$ will always be p_i, so $(p_{i+1} \to p_i) \to p_{i+1}$ will be 1, this holding for all terms on the left; whilst on the right no element is 1. More generally, any chain of size greater than or equal to n will validate this rule.

6.2 $\forall$-subalgebras of Gödel algebras and Implicit Connectives

Whilst in general the notion of a $\forall$-subalgebra appears to be difficult to outline, in the context of *linear* Heyting algebras we can provide a more concrete description of such embeddings.

Definition 6.7. Let $f : \mathcal{H} \to \mathcal{H}'$ be an embedding of linear Heyting algebras. We say that f is *cover-preserving* if whenever $a \prec b$ then $f(a) \prec f(b)$.

Theorem 6.8. *Let $f : \mathcal{H} \to \mathcal{H}'$ be an embedding of linear Heyting algebras. Then f is a $\forall$-embedding if and only if it is cover-preserving.*

Proof. First we note that every $\forall$-embedding is cover-preserving. To see this, consider the formula:

$$\forall c(a/c \wedge c/b \leqslant c \vee a)$$

Assume that $a \prec b$ but there is some c such that $f(a) < c$ and $c < f(b)$. Hence note that $f(a)/c = 1$ and $c/f(b) = 1$, which is not below $f(a) \vee c$; so f could not be a $\forall$-embedding. On the other hand, if c is arbitrary in $\mathcal{H}$; then $c \leqslant a$ or $b \leqslant c$, and one can verify that the above equation holds. Hence since f is a $\forall$-embedding, we obtain that $f(a) \prec f(b)$.

Now assume that f is a cover-preserving embedding. Let $\overline{a}$ be some collection of elements in $\mathcal{H}$; we will show that $(\mathcal{H}', f\overline{a}) \in \mathbb{SP}_U(\mathcal{H}, \overline{a})$. This amounts to showing, by a classical result, that every finite subgraph can be embedded into $(\mathcal{H}, \overline{a})$. So let $\mathbb{X}$ be such a local subgraph of $(\mathcal{H}, f\overline{a})$, and assume without loss of generality that it contains all constants.

Now define an embedding from $\mathbb{X}$ to $(\mathcal{H}, \overline{a})$ as follows: first, send $f(a)$ to a; now for each other $x \in X$, notice that we can identify this element, in $\mathcal{H}'$, by its position relative to all elements previously identified. We order the constants as $0, a_0, ..., a_n, 1$, and proceed to define the mapping within the blocks.

So first, assume that only the constants have been defined. We work between 0 and a_0, but the same argument will work wherever. Enumerate the elements ocurring in $0 < x < f(a_0)$, in order, as $x_0, ..., x_n$. Notice that if the interval $[0, a_0]$ is finite with cardinality n, then so is the interval $[0, f(a_0)]$: formally one can prove this by induction, by noting that each element in $[0, a_0]$ will be a cover of the previous element, and using the fact that the map is cover-preserving. Hence if the interval is finite, we have an onto isomorphism, and can map $x_0, ..., x_n$ in the same order. If the interval is infinite, then we can always find new images for the elements.

Proceeding in this way, we have that we can define the map g from $\mathbb{X}$ to $\mathcal{H}'$. Such a map is a local subgraph embedding, since the order is preserved (and hence the meet and join properties are respected, and so is the implication). Hence $(\mathcal{H}', f\overline{a}) \in \mathbb{SP}_U(\mathcal{H}, \overline{a})$ and so:

$$(\mathcal{H}', f\overline{a}) \in \mathbb{V}(\mathcal{H}, \overline{a})$$

Hence by Proposition 3.11, the map f is a $\forall$-embedding, as desired. ■

We will have occasion to make use of the above ideas further in the next section. Before that, we will briefly make a few notes about $\forall$-embeddings.

The first concerns the dual meaning of such maps. Whilst in this paper we have focused solely on the algebraic meaning of maps, readers familiar with Esakia duality will have wondered what the dual of a $\forall$-embedding ought to be. Whilst we are far from a definitive answer, we have a few hints in this direction: as the reader may verify, when attention is restricted to linear Heyting algebras, the dual of a $\forall$-embedding will be a map which acts as an isomorphism except on limit points of the space, where it is allowed to collapse entire connected components of the linear order into a single point. Moreover, such a condition appears to be related to the notion of a local homeomorphism as discussed in [21].

The second aspect which is worthy to note has to do with the notion of "implicit connectives". The reader familiar with Kuznetsov's Δ-pseudo-Boolean algebras – memorably called by Esakia "frontons" [16] – will have noted that such embeddings are precisely the ones preserving the frontal structure whenever it exists. The general behaviour of such "intuitionistic connectives" was studied by Caicedo and Cignoli [12].

Definition 6.9. Let $f : H \to H'$ be a map of Heyting algebras. We say that f is a *frontal map* if for all $a, b \in H$:

$$b = \bigwedge\{q \vee q \to a : q \in H\} \text{ and } b \to a \leqslant b$$

Then:

$$f(b) = \bigwedge\{q \vee q \to f(a) : q \in H'\} \text{ and } f(b) \to f(a) \leqslant f(b)$$

We can see this easily through the following lemma:

Lemma 6.10. Let $\mathcal{H}$ be a Heyting algebra, and $p \in H$. Then we have that:

$$S_0 := \{q \in H : q \to p \leqslant q\} = S_1 := \{q \vee q \to p : q \in H\}$$

Proof. Assume that $q \to p \leqslant q$. Then note that $q \to p \vee q = q$, so $S_0 \subseteq S_1$. On the other hand, note that:

$$\begin{aligned}(q \vee q \to p) \to p &= q \to p \wedge (q \to p) \to p \\ &\leqslant p \\ &\leqslant q \to p \\ &\leqslant q \vee q \to p\end{aligned}$$

Hence $S_1 \subseteq S_0$. ■

Then we have the following lemma:

Lemma 6.11. Let $\mathcal{H}$ and $\mathcal{H}'$ be Heyting algebras and $f : H \to H'$ a $\forall$-embedding. Then f is a frontal map.

Proof. Consider the following:

$$\forall c(b \leqslant c \vee c \to a) \text{ and } b \to a \leqslant b$$

By assumption, if b is the fronton of a, then we must have that:

$$\forall c(f(b) \leqslant c \vee c \to f(a)) \text{ and } f(b) \to f(a) \leqslant f(b)$$

But by an elementary calculation, supported in the above lemma, this shows that f is frontal. ■

In general, being a frontal map does not seem sufficient to be a $\forall$-embedding. Hence there is a natural question to be asked whether the $\forall$-embeddings are precisely those which preserve the implicit connectives in the sense of Cignoli and Caicedo. We leave a full analysis of this situation for future work.

7 Inductively Complete Inductive Classes and Admissibility of Π_2-rules

In this final section we provide two theorems which provide a silverlining to the complex picture painted by the previous section: despite the fact that there are continuum many inductive classes of Gödel algebras, there are only countably many inductively complete ones, which are in 1-1 correspondence with the varieties and quasivarieties. These are precisely the minimal inductive classes. We also prove that the admissibility problem of Π_2-rules for LC is decidable, following from our work from Section 5, given that LC has a model completion.

Both of our results are consequences of the following technical structure theorem; we note that $\mathbb{Q}$ below denotes the chain of the rationals with a minimal and maximal point added to it.

Theorem 7.1. *Let $\mathcal{H}$ be a Gödel algebra. Then either $[n] \leqslant_{\forall} \mathcal{H}$ or $\mathbb{Q}$ is a $\forall$-subalgebra of an ultrapower of $\mathcal{H}$.*

Proof. Because LC is locally finite, given $\mathcal{H}$ we have two options:

1. There is a uniform bound n on the size of chain-subalgebras of $\mathcal{H}$. Then first we claim that $[n] \leqslant_{\forall} \mathcal{H}$. It is clear that it is a subalgebra, say through an embedding f. Now pick all constants $\overline{n}$, and we need to show that:

$$(\mathcal{H}, f\overline{n}) \in \mathbb{V}([n], \overline{n}).$$

   To see this in turn, pick some finite subalgebra of $\mathcal{H}$ containing $f\overline{n}$, say $(\mathcal{L}, f\overline{n})$. By universal algebra, we have that

$$(\mathcal{L}, f\overline{n}) \leqslant \prod_{i=1}^{n} \mathcal{L}_i$$

   through a subdirect inclusion, in the variety $[n]$, with no added constants, where $\mathcal{L}_i$ gets its constants interpreted by forcing their interpretation. Now first notice that $(\mathcal{L}_i, f\overline{n}(i))$ can only identify constants with 1: indeed if $p(f(n)) \neq 1$, and $f(n) \neq f(m)$ then without loss of generality, since they are linearly ordered, $f(n) < f(m)$, so $f(m) \rightarrow f(n) = f(n)$. If $p : \mathcal{L} \rightarrow \mathcal{L}_i$ is the homomorphism, then $p(f(m) \rightarrow f(n)) = p(f(n))$; but if $p(f(m)) = p(f(n))$, then $p(f(m) \rightarrow f(n)) = 1$, a contradiction.

   Hence the constants will appear in $\mathcal{L}_i$ in the correct order, possibly with some of them identified with 1. Hence we can take a homomorphic image of $[n]$ which obtains an isomorphic copy to $(\mathcal{L}_i, pf\overline{n})$. Hence we obtain that $(\mathcal{L}, \overline{f}\overline{n})$ will be a subdirect product of such homomorphic images, and so it will belong to $\mathbb{V}([n], \overline{n})$. Hence we have that each finite subgraph of $(\mathcal{H}, f\overline{n})$ belongs to $\mathbb{V}([n], \overline{n})$, which shows, because the latter is then a universal class, that $(\mathcal{H}, f\overline{n})$ belongs there as well.

2. There is no bound on the chains. Then note that $\mathbb{Q}$ will embed in an ultrapower of $\mathcal{H}$ by usual model theoretic reasoning, using compactness. Hence $\mathbb{Q} \leqslant \mathcal{D}$ for some such ultrapower. Then we show that for each collection of constants $f\overline{a} \in \mathbb{Q}$,

$$(\mathcal{D}, f\overline{a}) \in \mathbb{V}(\mathbb{Q}, \overline{a}).$$

   The argument proceeds as above, by taking a finite subgraph, $(\mathcal{L}, f\overline{a})$, which again can be decomposed into linear factors. The additional argument needed is thaat $\overline{a}$ forms a linear subalgebra of $\mathbb{Q}$, from which we then proceed to construct a homomorphism; the argument concludes then in the same way.

■

7.1 Inductively Complete LC-Inductive Rule Classes

We have shown that LC is not inductively complete, so a natural question arises whether anything can be said about those extensions of it which are inductively complete. For general logical systems this appears to be a difficult question, but making use of the structural completeness of LC and the previous structure theorem, we can provide a complete characterisation. This goes through the following lemma:

Lemma 7.2. The inductive rule class $\mathbb{IR}(\mathbb{Q})$, and the inductive rule classes $\mathbb{IR}([n])$ are all minimal; they are precisely the minimal inductive classes.

Proof. Assume that $\mathbf{K} \subseteq \mathbb{IR}(\mathbb{Q})$ and this is non-empty. Assume that they are different; then we may assume that $\mathbb{Q} \notin \mathbf{K}$. Let $\mathcal{H}$ be some element in $\mathbf{K}$, which exists by non-emptiness. By the previous theorem, if $\mathbb{Q}$ did not embed in an ultrapower of $\mathcal{H}$, then $[n] \leqslant_{\forall} \mathcal{H}$ for some n; but this is a contradiction, since then the equation characterising $[n]$ would be valid on a product of dense linear algebras, a contradiction. So $\mathbb{Q}$ embeds in an ultrapower of $\mathcal{H}$; but such an ultrapower will also be $\mathbf{K}$, and hence, so will $\mathbb{Q}$.

Similarly, assume that $\mathbf{K} \subseteq \mathbb{IR}([n])$. Again suppose that $[n] \notin \mathbf{K}$. Let $\mathcal{H}$ be some algebra in $\mathbf{K}$. Note that then $\mathcal{H}$ must have a uniform bound on its chains (otherwise $\mathbb{Q}$ would be in $[n]$, which is absurd). Moreover this uniform bound must be n, otherwise by the arguments above, $[m] \leqslant_{\forall} \mathcal{H}$ which would belong to $\mathbb{IR}([m])$, in violation of Lemma 6.4. Hence by the above result $[n] \leqslant_{\forall} \mathcal{H}$, showing that $[n] \in \mathbf{K}$.

Now assume that $\mathbf{K}$ is a minimal class (hence non-empty). Let $\mathcal{H}$ be some algebra in $\mathbf{K}$. Then by the above theorem, either $[n] \in \mathbf{K}$ or $\mathbb{Q} \in \mathbf{K}$. Since $\mathbf{K}$ is minimal, then it will coincide with one of the former, as desired. ■

From this we get a characterization of the inductively complete inductive rule classes:

Theorem 7.3. *The inductively complete inductive rule classes in* LC *are precisely the minimal ones.*

Proof. Certainly all minimal inductive rule classes are inductively complete, since they do not have any proper subinductive classes. Now let $\mathbf{K}$ be some inductive rule class which is not minimal. First assume that the algebras in $\mathbf{K}$ all have a uniform bound of n. Then note that $[n]$ is a proper subinductive class (because $\mathbf{K}$ is not minimal), but the quasivariety generated by it generates the quasivariety generated by $\mathbf{K}$. Now assume that there is no such bound. Then $\mathbb{Q} \in \mathbf{K}$, so again we get the same conclusion by looking at the minimal inductive class $\mathbb{IR}(\mathbb{Q})$. Hence $\mathbf{K}$ cannot be inductively complete. ■

One may be interested for an explicit axiomatisation of these classes. To construct an axiomatisation of $\mathbb{IR}([n])$, one may extract such an axiomatisation from an axiomatisation of the model companion of $\mathbb{V}([n])$. In the special case of $\mathbb{V}(\mathbb{Q})$, this can be effectively studied using the results from [13], characterising the model completion of LC, and implies in particular that the Π_2-rule system associated to $\mathbb{IR}(\mathbb{Q})$ is recursively axiomatisable, and decidable. In both cases, this problem appears to be difficult and intimately related to the study of the model companions and completions of such varieties, and therefore it is left outside of the scope of the present article.

7.2 Decidability of Admissibility of Π_2-Rules of LC

In [8] the authors studied several procedures to show the decidability of the problem of admissibility of Π_2-rules over several logical systems. These concerned rules as outlined in Example 2.3, which crucially leaves out settings such as the present one. As it turns out, some of the results presented therein can be generalized. Key to our work will be the connection between hereditary admissibility and admissibility spelled out in Theorem 5.11. We start with a decidability result that is somewhat weaker:

Theorem 7.4. *Hereditary Admissibility of Π_2-rules over LC is effectively recognizable.*

Proof. By Theorem 5.11, this amounts to showing that in the model completion LC^* we can decide whether $\mathsf{LC}^* \vDash \Gamma/^2\varphi$. We know that the theory LC^* will be complete[8], which implies that it is decidable. But additionally, quantifier elimination is effectively recognizable by the same arguments as used in [8, Corollary 5.7, Lemma 5.8]. Hence, since LC is decidable (and therefore we can decide the validity of quantifier-free formulas as computed by our theory), this implies the whole problem is decidable. ∎

One immediately obtains that in fact hereditary admissibility of Π_2-rules of a system with a model completion will be effectively recognizable. But one might naturally wonder about admissible rules which are not hereditarily admissible. As we will show, in the case of LC, the former already gives us everything we need, since all admissible rules are hereditarily admissible. To see this, we begin by stating an easy fact.

Proposition 7.5. A rule $\Gamma/^2\varphi$ is admissible/hereditarily admissible over LC if and only if it is admissible/hereditarily admissible over the class of linear Heyting algebras.

Proof. Obvious, since LC is generated as an inductive rule class by the linear algebras. ∎

Proposition 7.6. Let $\Gamma/^2\varphi$ be a rule which is not hereditarily admissible over LC. Then $\Gamma/^2\varphi$ is derivable in only finitely many chains and no infinite chain.

Proof. If $\Gamma/^2\varphi$ is not hereditarily admissible, by Theorem 5.10, there is an algebra $\mathcal{H}$ which is not embeddable in an algebra satisfying this rule. Now if each finite chain-factor of $\mathcal{H}$ was embeddable in such an algebra, then any product would be embeddable in a product of the targets, and so $\mathcal{H}$ (which is a subdirect product of its finite chain subalgebras) would also embed there. So not all chains of $\mathcal{H}$ can embed into such an algebra, say, $[n]$ does not embed into any such algebra. But if $n \leqslant m$ and $[m]$ can embed there, then so can $[n]$; moreover no infinite chain can embed there, otherwise all finite chains would embed as well. So there must be only finitely many chains which embed into such algebras which validate the rule, and a fortriori, only finitely many chains where the rule is derivable. ∎

Now we combine Lemma 5.6, with the facts proven at the end of the last section:

Proposition 7.7. Let $\Gamma/^2\varphi$ be a rule which is admissible over LC. Then LC is generated as a variety by the chains validating $\Gamma/^2\varphi$.

Proof. By Lemma 5.6 we know that LC is generated by

$$\{\mathcal{H} : \mathcal{H} \vDash \Gamma/^2\varphi\}.$$

Now for each such $\mathcal{H}$ either it has a bound on chains or it does not. If it does, then if n is such a bound, $[n] \leqslant_\forall \mathcal{H}$, and so $[n] \vDash \Gamma/^2\varphi$, and certainly $\mathcal{H}$ can be recovered from $[n]$ through operations generating a variety; if not $\mathbb{Q} \leqslant_\forall \mathcal{H}$ and $\mathbb{Q}$ certainly also generates $\mathcal{H}$ in this manner. So this proves the result. ∎

[8] Here is a short argument: By exercise 2 of Section 8.3 of Hodges, a first order theory which is model complete and has the joint embedding property is complete. Now LC certainly has the joint embedding property, and this implies that its model companion does as well.

This has the following strong consequence:

Proposition 7.8. Let $\Gamma/^2\varphi$ be a rule which is admissible over LC. Then the class:

$$\mathsf{Lin} \cap \{\mathcal{H} : \mathcal{H} \models \Gamma/^2\varphi\}$$

must either include $\mathbb{Q}$ or cofinally many finite chains.

Proof. Indeed, if not, then there would only be finitely many finite chains in such a set. But then the variety generated by this finite set of chains would not be LC. ∎

Corollary 7.9. *Over* LC*, a rule is admissible if and only if it is hereditarily admissible.*

Proof. Assume that $\Gamma/^2\varphi$ is admissible but not hereditarily admissible. By Proposition 7.6 then the class of finite chains validating this rule must be finite, and $\mathbb{Q}$ cannot validate it. But then this contradicts Proposition 7.8. ∎

Corollary 7.10. *Admissibility of* Π_2*-rules over* LC *is effectively recognizable.*

Proof. This follows from Corollary 7.9 together with Theorem 7.4. ∎

Note that the previous proof involved a great deal of universal algebraic concepts. Namely:

1. Interpolation and Uniform interpolation (in the form of model completions);
2. Local finiteness of algebras;
3. A Theorem, like Theorem 7.1 which entails that generating classes can always be taken from the 'tame ones' (which in these case are the linear algebras).

It is thus natural to wonder whether these concepts are enough to ground the arguments at hand. Several questions could be asked in this direction, but we restrict ourselves to one here. We note that in the above proof the structural completeness of LC is implicitly used. Additionally, it makes conceptual sense that in the presence of structural completeness the problem of hereditary admissibility should resemble admissibility – after all, only the Π_1-fragment should remain. However, it is not obvious how one would prove such a general result. Nevertheless, the importance of such a concept merits the following conjectures, in increasing degree of generality:

Conjecture 7.11. Let L be an intermediate logic which is hereditarily structurally complete. Then hereditary admissibility is equivalent to admissibility over L.

Conjecture 7.12. Let L be an intermediate logic which is structurally complete. Then hereditary admissibility is equivalent to admissibility over L.

8 Conclusion

In this article I have proposed an analytic framework to study Π_2-rules using tools stemming from model theory and universal algebra. In it I introduce the notions of a Π_2-rule system and an inductive rule class, and establish an algebraic completeness connecting the two concepts. As an example use-case, I study Gödel algebras, showing that the present tools allow for easier admissibility proofs for interesting rules like the Takeuti-Titani rule. Additionally, I present a preliminary study of inductive rule classes over LC, emphasising two aspects: the structure of the lattice of inductive rule classes; and admissibility of rules over such classes. Our key contribution in this respect lies in providing a full characterisation of the inductively complete inductive rule classes, as well as showing the problem of admissibility of inductive rule classes to be decidable over LC.

In this respect there are several follow-up problems that stem immediately from the current work. With respect to LC, these concern, for instance, the axiomatisation of the inductively complete rule classes. Given the results presented in this paper, this can be carried out in parallel and in support of a study of the model companions of the finitely generated varieties of Gödel algebras. Similarly, it would be interesting to understand how the structure of such inductive rule classes through a duality-theoretic perspective.

From a theoretical point of view, it would be interesting to understand the connections between the concepts of hereditary admissibility, admissibility, and the theory of model completions, and the role of structural completeness in establishing these connections. Moreover, it would be interesting to understand how such concepts in turn relate to the theory of unification, as pointed out in [8], and whether Π_2-rules can provide logical meaning to natural computational problems of this sort. Finally, as a more long term ambition, it would be interesting to provide a systematic definability theory which explains which classes of frames can be captured using Π_2-rules, and providing conditions for such definitions to hold.

9 Acknowledgements

I would like to thank Silvio Ghilardi for several suggestions and discussions about the results of this paper. I also thankful Nick Bezhanishvili for several discussions and suggestions for improvement, as well as for his continued support and help in carrying out this project.

Bibliography

[1] Juan P. Aguilera and Matthias Baaz. "Ten problems in Gödel logic". In: *Soft Computing* 21.1 (Sept. 2016), pp. 149–152. DOI: 10.1007/s00500-016-2366-9. URL: https://doi.org/10.1007/s00500-016-2366-9.

[2] Rodrigo N. Almeida. "Polyatomic Logics and Generalised Blok-Esakia Theory with Applications to Orthologic and KTB". MA thesis. University of Amsterdam, 2022.

[3] M. Baaz and R. Zach. "Compact propositional Godel logics". In: *Proceedings. 1998 28th IEEE International Symposium on Multiple- Valued Logic (Cat. No.98CB36138)*. IEEE Comput. Soc. DOI: 10.1109/ismvl.1998.679315. URL: https://doi.org/10.1109/ismvl.1998.679315.

[4] Matthias Baaz and Helmut Veith. "An Axiomatization of Quantified Propositional Gödel Logic Using the Takeuti-Titani Rule". In: *Logic Colloquium '98*. Cambridge University Press, Mar. 2017, pp. 91–104. DOI: 10.1017/9781316756140.008. URL: https://doi.org/10.1017/9781316756140.008.

[5] Arnold Beckmann, Martin Goldstern, and Norbert Preining. "Continuous Fraïssé Conjecture". In: *Order* 25.4 (Sept. 2008), pp. 281–298. DOI: 10.1007/s11083-008-9094-4. URL: https://doi.org/10.1007/s11083-008-9094-4.

[6] Clifford Bergman. “Structural Completeness in Algebra and Logic”. In: *Colloquia Mathematica Societatis János Bolyai* 54 (1988).

[7] Guram Bezhanishvili, Nick Bezhanishvili, Thomas Santoli, and Yde Venema. “A strict implication calculus for compact Hausdorff spaces”. In: *Annals of Pure and Applied Logic* 170.11 (2019), p. 102714. ISSN: 0168-0072. DOI: https://doi.org/10.1016/j.apal.2019.06.003.

[8] Nick Bezhanishvili, Luca Carai, Silvio Ghilardi, and Lucia Landi. “Admissibility of 2-Inference Rules: interpolation, model completion, and contact algebras”. In: *Annals of Pure and Applied Logic* 174.1 (2023), p. 103169. ISSN: 0168-0072. DOI: https://doi.org/10.1016/j.apal.2022.103169. URL: https://www.sciencedirect.com/science/article/pii/S0168007222000847.

[9] John P. Burgess. “Decidability for branching time”. In: *Studia Logica* 39.2-3 (1980), pp. 203–218. DOI: 10.1007/bf00370320. URL: https://doi.org/10.1007/bf00370320.

[10] Stanley Burris and H.P. Sankappanavar. *A Course in Universal Algebra*. New York: Springer, 1981.

[11] Leonardo Cabrer and George Metcalfe. “Admissibility via natural dualities”. In: *Journal of Pure and Applied Algebra* 219.9 (Sept. 2015), pp. 4229–4253. DOI: 10.1016/j.jpaa.2015.02.015. URL: https://doi.org/10.1016/j.jpaa.2015.02.015.

[12] Xavier Caicedo and Roberto Cignoli. “An Algebraic Approach to Intuitionistic Connectives”. In: *The Journal of Symbolic Logic* 66.4 (2001), pp. 1620–1636. ISSN: 00224812. URL: http://www.jstor.org/stable/2694965 (visited on 10/09/2023).

[13] Luck Darnière and Markus Junker. “Model-completion of varieties of co-Heyting algebras”. In: *Houston Journal of Mathematics* 44 (Jan. 2010).

[14] Wojciech Dzik and Michał M. Stronkowski. “Almost structural completeness; an algebraic approach”. In: *Annals of Pure and Applied Logic* 167.7 (2016), pp. 525–556. ISSN: 0168-0072. DOI: https://doi.org/10.1016/j.apal.2016.03.002. URL: https://www.sciencedirect.com/science/article/pii/S0168007216300197.

[15] Wojciech Dzik and Andrzej Wroński. “Structural Completeness of Gödel’s and Dummett’s Propositional Calculi”. In: *Studia Logica: An International Journal for Symbolic Logic* 32 (1973), pp. 69–75. ISSN: 00393215, 15728730. URL: http://www.jstor.org/stable/20014670 (visited on 10/10/2023).

[16] Leo Esakia. “The modalized Heyting calculus: a conservative modal extension of the Intuitionistic Logic”. In: *Journal of Applied Non-Classical Logics* 16.3-4 (Jan. 2006), pp. 349–366. DOI: 10.3166/jancl.16.349-366. URL: https://doi.org/10.3166/jancl.16.349-366.

[17] Josep Maria Font. *Abstract algebraic logic – An introductory textbook*. London: College Publications, 2016.

[18] Dov M. Gabbay. “An Irreflexivity Lemma with Applications to Axiomatizations of Conditions on Tense Frames”. In: *Aspects of Philosophical Logic*. Springer Netherlands, 1981, pp. 67–89. DOI: 10.1007/978-94-009-8384-7_3. URL: https://doi.org/10.1007/978-94-009-8384-7_3.

[19] Viktor A Gorbunov. *Algebraic theory of quasivarieties*. en. 1998th ed. Siberian School of Algebra and Logic. New York, NY: Kluwer Academic/Plenum, Sept. 1998.

[20] Rosalie Iemhoff. “Consequence Relations and Admissible Rules”. In: *Journal of Philosophical Logic* 45.3 (2016), pp. 327–348. ISSN: 00223611, 15730433. URL: http://www.jstor.org/stable/43895434 (visited on 03/03/2023).

[21] Evgeny Kuznetsov. “Properties of Local Homeomorphisms of Stone spaces and Priestley spaces”. In: 2023.

[22] Roger C Lyndon. “Properties preserved in subdirect products”. In: *Pacific J. Math* 9.1 (1959), pp. 155–164.

[23] George Metcalfe and Franco Montagna. “Substructural Fuzzy Logics”. In: *The Journal of Symbolic Logic* 72.3 (2007), pp. 834–864. ISSN: 00224812. URL: http://www.jstor.org/stable/27588573 (visited on 07/17/2023).

[24] Gaisi Takeuti and Satoko Titani. “Intuitionistic Fuzzy Logic and Intuitionistic Fuzzy Set Theory”. In: *Journal of Symbolic Logic* 49.3 (1984), pp. 851–866. DOI: 10.2307/2274139.

[25] Yde Venema. “Derivation Rules as Anti-Axioms in Modal Logic”. In: *The Journal of Symbolic Logic* 58.3 (1993), pp. 1003–1034.