Topics in Algebraic Logic and Duality Theory
Lecture 3

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Plan for the Day

- Announcements.
- Local Uniform interpolation and Adjoints.
- Deductive Uniform interpolation and Model Completions.

Announcements

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2. Topics should be decided by the middle of next week, to ensure you have enough time.

Uniform interpolation

Interpolation seen uniformly

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As we will see, this situation is not exclusive of classical logic.

Local Uniform Interpolation

Adjoints

Let $\mathcal{K}$ be a class of ordered algebras, possessing free algebras; we will say that $\mathcal{K}$ has the *uniform definability property* if whenever $[k] = \{p_1, ..., p_k\}$ then the inclusion:

$$i : \mathcal{F}_{\mathcal{K}}([k]) \hookrightarrow \mathcal{F}_{\mathcal{K}}([k+1]),$$

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has a left and a right adjoint;

Explicitly, this means that there are two order preserving maps $\forall_i : \mathcal{F}_{\mathcal{K}}([k+1]) \to \mathcal{F}_{\mathcal{K}}([k])$ and $\exists_i : \mathcal{F}_{\mathcal{K}}([k+1]) \to \mathcal{F}_{\mathcal{K}}([k])$, such that for each $a \in \mathcal{F}_{\mathcal{K}}([k])$ and $b \in \mathcal{F}_{\mathcal{K}}([k+1])$ we have:

$$\exists_i(b) \leqslant a \iff b \leqslant i(a) \text{ and } i(a) \leqslant b \iff a \leqslant \forall_i(b).$$

Uniform Craig interpolation

Definition

We say that a logic L has the *uniform Craig interpolation property* if and only if whenever $\vdash \phi(\overline{p}, \overline{r}) \rightarrow \psi(\overline{q}, \overline{r})$, there are two formulas χ_0 and χ_1, in the common language, such that:

1. χ_i are interpolants for $\vdash \phi \rightarrow \psi$;
2. Whenever μ is an interpolant, then $\vdash \chi_0 \rightarrow \mu$ and $\vdash \mu \rightarrow \chi_1$.

Uniform Craig Interpolation (cont.d)

Proposition

Let L be a logic with the Craig interpolation property and the uniform definability property. Then L has the uniform Craig interpolation property.

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Proposition

Let L be a locally tabular logic. Then L has the uniform definability property.

What to do for non-locally tabular logics?

Definition

Let $\phi \in \mathcal{L}$. We define the *modal depth* of ϕ by recursion as follows:

1. $md(p) = 0$;
2. $md(\phi \wedge \psi) = \max(md(\phi), md(\psi))$ and $md(\neg\phi) = md(\phi)$;
3. $md(\Box\phi) = md(\phi) + 1$.

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Definition

Let $(\mathfrak{M}, x) = (W, R, V)$ and $(\mathfrak{N}, y) = (W', R', V')$ be two models over $\overline{p}$. We say that a relation $S_n \subseteq W \times W'$ is an *n-bisimulation* based on (x, y) if there are relations $S_n \subseteq ... \subseteq S_k \subseteq ... \subseteq S_0$ for each $0 \leqslant k \leqslant n$, such that xS_ny and:

1. Whenever wS_0w' then $w \in V(p) \iff w' \in V'(p)$ for each $p \in \overline{p}$;
2. For each $k < n$, whenever $wS_{k+1}w'$ and wRv there is some $w'Rv'$ such that vS_kv'.
3. For each $k < n$, whenever $wS_{k+1}w'$ and $w'Rv'$ there is some wRv such that vS_kv'.

N-bisimulation and modal equivalence

One of the reasons we care about this is the following:

Proposition

Given two models $\mathfrak{M}, x$, and $\mathfrak{N}, y$ we have that $\mathfrak{M}, x \leftrightarroweq_n \mathfrak{N}, y$ if and only if $\mathfrak{M}, x$ and $\mathfrak{N}, y$ satisfy the same formulas of modal depth n (resp. implication rank n).

Proof.

(See board) $\square$

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Given two models $\mathfrak{M}, x$, and $\mathfrak{N}, y$ we have that $\mathfrak{M}, x \leftrightarrows_n \mathfrak{N}, y$ if and only if $\mathfrak{M}, x$ and $\mathfrak{N}, y$ satisfy the same formulas of modal depth n (resp. implication rank n).

Proof.

(See board) $\square$

From this we derive:

Proposition

Let L be a logic with the finite model property. Let $\mathcal{K}$ be a class of (finite or infinite) models of L, over $\overline{p}$. Then the following are equivalent:

1. *$\mathcal{K}$ is closed under n-bisimulation;*
2. *There is a formula ϕ of modal depth at most n such that for each $\mathfrak{M}, x$ a finite model, $\mathfrak{M}, x \Vdash \phi$ if and only if $(\mathfrak{M}, x) \in \mathcal{K}$.*

Bisimulation quantifiers

One way to look at uniform definability is through *bisimulation quantifiers*:

$$\mathfrak{M}, x \Vdash \tilde{\exists} p_i \, \phi \iff \text{there is a model } (\mathfrak{N}, y), \mathfrak{M}^{p_i}, x \leftrightarroweq \mathfrak{N}, y, \text{ and } \mathfrak{N}, y \Vdash \phi.$$

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$$\mathfrak{M}, x \Vdash \tilde{\exists} p_i\, \phi \iff \text{there is a model } (\mathfrak{N}, y), \mathfrak{M}^{p_i}, x \leftrightarrows \mathfrak{N}, y, \text{ and } \mathfrak{N}, y \Vdash \phi.$$

Proposition

For each model $(\mathfrak{M}, x)$ over $(\overline{p}, q)$, and each formula $\phi(\overline{p}, q)$ we have:

1. $\mathfrak{M}, x \Vdash \phi(\overline{p}, q) \rightarrow \tilde{\exists} q\, \phi(\overline{p}, q)$*;*
2. *Whenever* $\mathfrak{M}, x \Vdash \chi(\overline{p}) \rightarrow \phi(\overline{p}, q)$*, then* $\mathfrak{M}, x \Vdash \chi(\overline{p}) \rightarrow \tilde{\exists} q\, \phi(\overline{p}, q)$*.*

Moreover, if for each finite model, $\mathfrak{M}, x \Vdash \phi \rightarrow \psi$*, then for each finite model* $\mathfrak{M}, x \Vdash \tilde{\exists} q\, \phi \rightarrow \tilde{\exists} q\, \psi$*.*

Corollary

Assume that L is a logic which has the FMP. Suppose that for each $\phi(\overline{p}, q)$ in the language $\mathcal{L}(M)(\overline{p}, q)$, there is a formula $\psi(\overline{q})$ in the same language such that for each finite model $\mathfrak{M}, x$:

$$\mathfrak{M}, x \Vdash \psi(\overline{p}) \iff \mathfrak{M}, x \Vdash \tilde{\exists} q \; \phi(\overline{p}, q).$$

Then L has the uniform definability property.

Corollary

Assume that L is a logic which has the FMP. Suppose that for each $\phi(\overline{p}, q)$ in the language $\mathcal{L}(M)(\overline{p}, q)$, there is a formula $\psi(\overline{q})$ in the same language such that for each finite model $\mathfrak{M}, x$:

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Then L has the uniform definability property.

We note that under very reasonable assumptions, the above is an equivalence.

Theorem

Let L be a logic with the FMP. Then L has uniform definability whenever the following holds:

1. *Whenever $(\mathfrak{M}, x) \in \pi_q[\mathcal{K}]$, so that $(\mathfrak{M}, x) \leftrightarrows (\mathfrak{M}', x')^q$ and $(\mathfrak{M}, x) \leftrightarrows_k (\mathfrak{N}, y)$, then there is some model $(\mathfrak{N}', y')$ over $(\overline{p}, q)$ such that $(\mathfrak{N}', y')^q \leftrightarrows (\mathfrak{N}, y)$ and $(\mathfrak{N}', y') \leftrightarrows_n (\mathfrak{M}', x')$.*

In a picture

$$\begin{tikzcd}
(\mathfrak{M}, x) \arrow[r, dash, "\leftrightarrow_k"] & (\mathfrak{N}, y) \\
(\mathfrak{M}', x') \arrow[u, "\pi_q"] & (\mathfrak{N}', y') \arrow[l, "\leftrightarrow_n"'] \arrow[u, "\pi_q"']
\end{tikzcd}$$

Figure 1: Combinatorial condition for uniform definability

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\end{tikzcd}$$

Figure 1: Combinatorial condition for uniform definability

It kind of looks like amalgamation. The details of how precisely this is connected are a mystery to everyone.

We show the uniform definability for modal logic $\mathbf{K}$. As we will see, this exploits the rigid structure of $\mathbf{K}$-bisimulations:

Lemma (Combinatorial Lemma)

Let $(\mathfrak{M}, x)$ *be a finite model over* $\overline{p}$ *such that* $(\mathfrak{M}, x) \leftrightarroweq (\mathfrak{M}', x')^q$, *and* $(\mathfrak{M}, x) \leftrightarroweq_n (\mathfrak{N}, y)$. *Then there is some* $(\mathfrak{N}', y')$ *over* $(\overline{p}, q)$ *such that* $(\mathfrak{N}', y')^q \leftrightarroweq (\mathfrak{N}, y)$ *and* $(\mathfrak{N}', y') \leftrightarroweq_n (\mathfrak{M}', x')$.

(See the board)

Suppose that the three models in case are the ones from Figure 2.

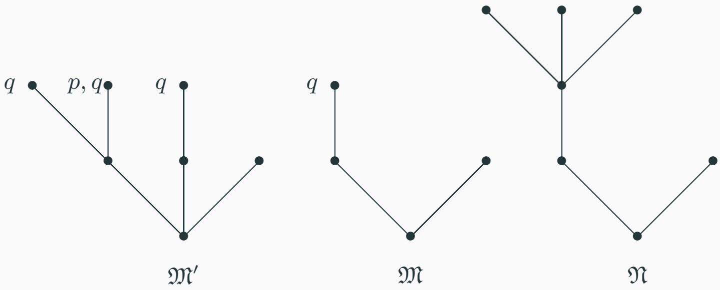

Figure 2: The models $\mathfrak{M}$, $\mathfrak{M}'$ and $\mathfrak{N}$

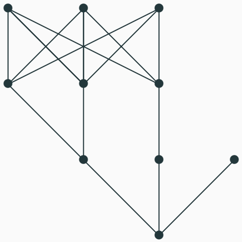

Figure 3: The witness to the combinatorial lemma

The above proof can be adapted for some simple modal systems – $\mathbf{KB}$ and $\mathbf{KD}$ for example – but fails for $\mathbf{S4}$. This is where the equivalences become very useful.

Other modal logics

The above proof can be adapted for some simple modal systems – $\mathbf{KB}$ and $\mathbf{KD}$ for example – but fails for $\mathbf{S4}$. This is where the equivalences become very useful.

The case of other logics – $\mathbf{Grz}$, $\mathbf{GL}$, etc – also goes through with some non-trivial modifications. Similar for $\mathbf{IPC}$.

Uniform deductive interpolation

Uniform deductive interpolation

Definition

A logic L has *uniform deductive interpolation* if and only if whenever $\phi(\overline{p}, \overline{r}) \vdash_L \psi(\overline{q}, \overline{r})$, there two formulas χ_0 and χ_1 in the common language such that:

1. χ_i are deductive interpolants for $\phi \vdash_L \psi$;
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1. χ_i are deductive interpolants for $\phi \vdash_L \psi$;
2. Whenever μ is a deductive interpolant, then $\chi_0 \vdash \mu$ and $\mu \vdash \chi_1$.

As it can be expected, these notions are interrelated:

Definition

Let L be a logic. We say that L has a *deduction theorem* if there is a term $t(x)$ such that for each formulas ϕ, ψ, we have

$$\phi \vdash_L \psi \iff \vdash_L t(\phi) \to \psi.$$

Proposition

Let L be a logic with a deduction theorem. Then if L has uniform Craig interpolation, then L has uniform deductive interpolation.

Model completions

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Definition

Let T be a universal first-order theory. We write $T_\forall$ for the set of universal consequences of T. We say that a theory U is a *cotheory of* T if $U_\forall = T_\forall$. Equivalently, every model of T can be extended to a model of U, and every model of U can be extended to a model of T.
We say that a theory T^* is a *model companion* of T if T and T^* are cotheories and T^* is model-complete.

Definition

Let T be a theory and let T^* be its model companion. We say that T^* is a *model completion* if for every model $\mathfrak{M} \models T$, $T^* \cup Diag(\mathfrak{M})$ is complete.

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Proposition

Let T^* be a model companion of T, a theory axiomatised by $\forall\exists$ axioms. The following are equivalent:

1. *T^* is a model completion of T;*
2. *T has the amalgamation property.*

Examples

Let T be the theory of Boolean algebras. Then one can show that T has a model completion.

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The theory of T is the theory of atomless Boolean algebras.

Proposition

Let L be a logic. If L has an algebraic model completion, then L has uniform deductive interpolation.

Notably, the converse of this is also true:

Proposition

Let L be a logic which has uniform deductive interpolation. Then L has an algebraic model completion.

Definition

Let $\mathcal{K}$ be a variety of algebras. We say that $\mathcal{K}$ is *coherent* if whenever $\mathcal{A}$ is a finitely presented algebra, and $\mathcal{B} \leqslant \mathcal{A}$ is a finitely generated subalgebra, then $\mathcal{B}$ is finitely presented as well.

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Theorem

Let L be a modal logic such that $\mathsf{Alg}(L)$ is coherent. Then L has a deduction theorem. Moreover, if L has uniform deductive interpolation, then $\mathsf{Alg}(L)$ is coherent.

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Theorem

Let L be a modal logic such that $\mathsf{Alg}(L)$ *is coherent. Then L has a deduction theorem. Moreover, if L has uniform deductive interpolation, then* $\mathsf{Alg}(L)$ *is coherent.*

Consequently $\mathbf{K}$ does not have a model completion.

Next time

- Maksimova's characterization of the seven superintuitionistic logics with Craig interpolation.

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- The end (?)

Thank you!
Questions?