Universal and Inductive Classes of Goedel Algebras

Rodrigo N. Almeida

Institute ofr Logic, Language and Computation – University of Amsterdam

Classical analysis of intuitionistic and modal logical systems has put the most emphasis on sets of theorems (sometimes called simply "logics") and single-conclusion rule systems. In recent years, variations on this picture have arisen in the analysis of multi-conclusion rule systems (see e.g., [9]). Given a fixed language $\mathcal{L}$ and $\Gamma = \{\phi_i : i \leq n\}$ and $\Delta = \{\psi_j : j \leq k\}$, sets of formulas in this language, we write

$$\Gamma \vdash \Delta$$

for a multiple-conclusion rule. With such a rule we associate a first-order formula

$$\chi(\Gamma, \Delta) := \forall \overline{x}(\bigwedge_{i \leq n} \phi_i(\overline{x}) \rightarrow \bigvee_{j \leq k} \psi_j(\overline{x}))$$

such that an algebraic model $\mathbf{H}$ validates the rule $\Gamma \vdash \Delta$ if and only if $\mathbf{H} \vDash \chi(\Gamma, \Delta)$. These systems have been studied in a variety of concepts, and classical concepts of admissibility and structural completeness of rules have been discussed in [9, 11, 6]. These have been used extensively in modal and intuitionistic settings, owing to the possibility of coding refutation patterns as specific rules, with applications to the study of properties such at the FMP [3], the study of translations [7] or admissibility [8].

More complex rules recently begun to be considered: for instance in [4, 5] so-called Π_2-rule systems were introduced with the goal of axiomatising compact Hausdorff spaces. Similar use was made in [1] to axiomatise classes of models not necessarily closed under subalgebras. To define them, given sets $\Gamma \cup \Delta$ of formulas (of a modal of intermediate logic[1]), we specify an extra datum, called the *free context of* Γ, consisting of a subset of the propositional variables occuring in Γ, denoted $Fr(\Gamma)$, and write

$$\forall \overline{p}_{p \in F} \Gamma(\overline{p}) \vdash_{\forall} \Delta$$

for the Π_2-rule specified by Γ, Δ and the free context $F = Fr(\Gamma)$. Often we will omit the free context, and write simply $\forall \overline{p} \Gamma(\overline{p}) \vdash \Delta$, with the meaning that the free context is precisely the variables in $\overline{p}$. With such a rule we associate a first-order formula

$$\chi_{\forall}(\Gamma, \Delta, F) := \forall \overline{x}(\forall \overline{y}(\bigwedge_{i \leq n} \phi_i(\overline{x}, \overline{y}) \approx 1) \rightarrow \bigvee_{j \leq k} \psi_j(\overline{x}) \approx 1)$$

such that an algebraic model $\mathbf{H}$ validates the rule $\forall \overline{p}_{p \in F} \Gamma(\overline{p}) \vdash \Delta$ if and only if $\mathbf{H} \vDash \chi_{\forall}(\Gamma, \Delta, F)$. Single-conclusion versions of these rules have been recently discussed in a number of articles and preprints [4, 5], though the overall idea of using $\forall\exists$-formulas (which the former is, modulo some rearrangement of quantifiers) has a much longer history (see e.g., [10]).

In this paper we report on ongoing work, focused on providing a systematic analysis of multiple-conclusion rule systems and Π_2-multiple conclusion rule systems of Heyting and modal algebras. Our basic definitions are as follows:

Definition 1. Let $\mathcal{L}$ be a modal or intermediate language, and let $\vdash$ be a collection of Π_2-rules. We say that this is a Π_2-rule system if it is closed under the following:

- (Reflexivity) $\phi \vdash \phi$;
- (Monotonicity) $\Gamma \vdash \Delta$ then for any finite Γ', Δ', we have $\Gamma, \Gamma' \vdash \Delta, \Delta'$;
- (Cut) If $\forall \overline{p} \Gamma(\overline{p}) \vdash \Delta, \phi$, and $\phi, \forall \overline{p} \Gamma(\overline{p}) \vdash \Delta$, then $\forall \overline{p} \Gamma(\overline{p}) \vdash \Delta$.

[1] Though we will not pursue it here, it should nevertheless be clear how to adapt these results to a more general universal-algebraic context, by using finite sets of equations in the place of Γ and Δ.

- (Bound Structurality) if $\Gamma \vdash \psi$ and σ is a substitution leaving all variables in $Fr(\Gamma)$ fixed, then $\sigma_*[\Gamma] \vdash \sigma(\psi)$

In this work, we provide some basic results establishing model-theoretic and universal algebraic characterizations of classes axiomatised by Π_2-rules. The key tool here lies in a special class of subalgebras.

Definition 2. Let $\mathbf{A}, \mathbf{B}$ be algebras of the same type, and assume that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. We say that $\mathbf{A}$ is a $\forall$-*subalgebra*, if for each quantifier free formula $\phi(\overline{x}, \overline{y})$, and each $\overline{a} \in \mathbf{A}$ we have that:

$$\mathbf{A} \vDash \forall \overline{x} \phi(\overline{x}, \overline{a}) \implies \mathbf{B} \vDash \forall \overline{x} \phi(\overline{x}, \overline{a})$$

Given a class of algebras $\mathbf{K}$ we write $\mathbb{S}_\forall(\mathbf{K})$ for the class of $\forall$-subalgebras of algebras in $\mathbf{K}$.

Definition 3. Given a class $\mathbf{K}$ of algebras, we say that it is an *inductive rule class* if it is closed under $\forall$-subalgebras, products and ultraproducts; we say that $\mathbf{K}$ is an *inductive-universal rule class* if it is closed under $\forall$-subalgebras and ultraproducts.

Proposition 4. *Given $\mathbf{K}$ a class of algebras we have that:*

1. *$\mathbf{K}$ is an inductive rule class if and only if $\mathbf{K} = \mathbb{I}\mathbb{S}_\forall\mathbb{P}\mathbb{P}_U(\mathbf{K})$;*
2. *$\mathbf{K}$ is an inductive-universal rule class if and only if $\mathbf{K} = \mathbb{I}\mathbb{S}_\forall\mathbb{P}_U(\mathbf{K})$.*

We also have the following analogue of Mal'tsev and Tarski's results:

Theorem 5. *For a class $\mathbf{K}$ of similar algebras then:*

1. *$\mathbf{K}$ is an inductive rule class if and only if $\mathbf{K}$ is axiomatised by formulas of the shape $\forall \overline{x}(\forall \overline{y}(\bigwedge_{i \leq n} \phi_i(\overline{x}, \overline{y})) \rightarrow \chi(\overline{x}))$.*
2. *$\mathbf{K}$ is an inductive-universal rule class if and only if $\mathbf{K}$ is axiomatised by formulas of the shape $\forall \overline{x}(\forall \overline{y}(\bigwedge_{i \leq n} \phi_i(\overline{x}, \overline{y})) \rightarrow \bigvee_{j \leq k} \psi_j(\overline{x}))$.*

Analogously to the results in [5], given an algebra $\mathbf{A}$ and a rule $\forall \overline{p} \Gamma(\overline{p}) \vdash_\forall \Delta$, we write $\mathbf{A} \vDash \forall \overline{p} \Gamma(\overline{p}) \vdash_\forall \Delta$ to mean that $\mathbf{A}$ validates the first-order translation of the formula. Given a Π_2-rule system $\vdash$ we consider:

$$Ind(\vdash) = \{\mathbf{A} : \forall \overline{p} \Gamma(\overline{p}) \vdash_\forall \Delta \in \vdash, \mathbf{A} \vDash \forall \overline{p} \Gamma(\overline{p}) \vdash_\forall \Delta\}$$

which yields an inductive-universal rule class. Then we can show that:

Proposition 6. *Given a modal or intuitionistic language $\mathcal{L}$, the mapping Ind is a dual isomorphism between the lattice of Π_2-rule systems, and the lattice of inductive-universal rule classes of (modal or Heyting) algebras. This restricts to a dual isomorphism between the lattices of single-conclusion Π_2-rule systems and inductive rule classes.*

This allows us to move between logic and algebra in formulating our results; though they were obtained as results about classes of algebras, as a matter of consistency we formulate all of these results in terms of rule systems.

As a testing bed for our approach, we focus specifically on the case of Goedel algebras. In the case of multiple-conclusion rule systems, we show, making use of a combinatorial lemma proved by Ian Hodkinson (private communication), that there are only countably many multiple-conclusion rule systems of Goedel algebras, all of which are finitely axiomatised and have the finite model-property. The lattice of extensions is more complex than in the case of varieties and quasivarieties. Nevertheless we can show that:

Proposition 7. *The lattice of extensions of the rule system generated by $\vdash p \rightarrow q, q \rightarrow p$ is isomorphic to $(\omega + 1)^*$.*

In the case of inductive rule systems the picture changes dramatically. In contrast to both multiple-conclusion rule systems, and also first-order Goedel logics [2],we show that

Theorem 8. *There are $2^{\aleph_0}$ single conclusion Π_2-rule systems of Goedel logic, all of which are generated by their finite members.*

This hints that the structure of Π_2-rule systems even of Goedel logic becomes too difficult to analyse in depth. In the specific case of *linear Goedel algebras* more can be achieved by using a more concrete criterion of $\forall$-subalgebras:

Definition 9. Let $\mathbf{H} \leq \mathbf{H}'$ be subalgebras. We say that $\mathbf{H}$ is a *cover-preserving subalgebra* if whenever $a \in \mathbf{H}$, and there is $b \in \mathbf{H}'$ which is the immediate successor of a, then $b \in \mathbf{H}$.

Theorem 10. *Given $\mathbf{H} \leq \mathbf{H}'$ linear Heyting algebras, then $\mathbf{H}$ is a $\forall$-subalgebra of $\mathbf{H}'$ if and only if it is a cover-preserving subalgebra..*

We call an Π_2-rule system *infinitely modelled* if it contains only infinite algebras as models, and none of the algebras which are its models contain any atoms or co-atoms. Using the previous criterion, and some classical model-theory of countable linear orders, we are able to show the following classification result:

Theorem 11. *The class of infinitely modelled multiple conclusion Π_2-rule systems of Goedel logics is countable, and is isomorphic to $(\omega + 1)^*$.*

We also highlight some directions for further exploration, namely connections with admissibility, the theory of model-completions and the study of implicit connectives in intuitionistic and modal logic.

Bibliography

[1] Rodrigo N. Almeida. "Polyatomic Logics and Generalised Blok-Esakia Theory with Applications to Orthologic and KTB". MA thesis. University of Amsterdam, 2022.

[2] Arnold Beckmann, Martin Goldstern, and Norbert Preining. "Continuous Fraïssé Conjecture". In: *Order* 25.4 (Sept. 2008), pp. 281–298. DOI: 10.1007/s11083-008-9094-4. URL: https://doi.org/10.1007/s11083-008-9094-4.

[3] Guram Bezhanishvili, Nick Bezhanishvili, and Julia Ilin. "Cofinal Stable Logics". In: *Studia Logica* 104.6 (June 2016), pp. 1287–1317.

[4] Guram Bezhanishvili, Nick Bezhanishvili, Thomas Santoli, and Yde Venema. "A strict implication calculus for compact Hausdorff spaces". In: *Annals of Pure and Applied Logic* 170.11 (2019). ISSN: 0168-0072.

[5] Nick Bezhanishvili, Luca Carai, Silvio Ghilardi, and Lucia Landi. "Admissibility of 2-Inference Rules: interpolation, model completion, and contact algebras". In: *Annals of Pure and Applied Logic* 174.1 (2023), p. 103169. ISSN: 0168-0072. DOI: https://doi.org/10.1016/j.apal.2022.103169. URL: https://www.sciencedirect.com/science/article/pii/S0168007222000847.

[6] Leonardo Cabrer and George Metcalfe. "Admissibility via natural dualities". In: *Journal of Pure and Applied Algebra* 219.9 (Sept. 2015), pp. 4229–4253. DOI: 10.1016/j.jpaa.2015.02.015. URL: https://doi.org/10.1016/j.jpaa.2015.02.015.

[7] Antonio Maria Cleani. "Translational Embeddings via Stable Canonical Rules". MA thesis. Universiteit van Amsterdam, 2021.

[8] Nick Bezhanishvili Guram Bezhanishvili and Rosalie Iemhoff. "Stable Canonical Rules". In: *The Journal of Symbolic Logic* 81.1 (2016), pp. 284–315. (Visited on 03/03/2023).

[9] Rosalie Iemhoff. "Consequence Relations and Admissible Rules". In: *Journal of Philosophical Logic* 45.3 (2016), pp. 327–348. ISSN: 00223611, 15730433. URL: http://www.jstor.org/stable/43895434 (visited on 03/03/2023).

[10] Sara Negri. "Proof analysis beyond geometric theories: from rule systems to systems of rules". In: *Journal of Logic and Computation* 26.2 (2016), pp. 513–537.

[11] Michał M. Stronkowski. *On the Blok-Esakia theorem for universal classes.* 2018. URL: https://arxiv.org/abs/1810.09286.