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Mathematical Structures in Logic

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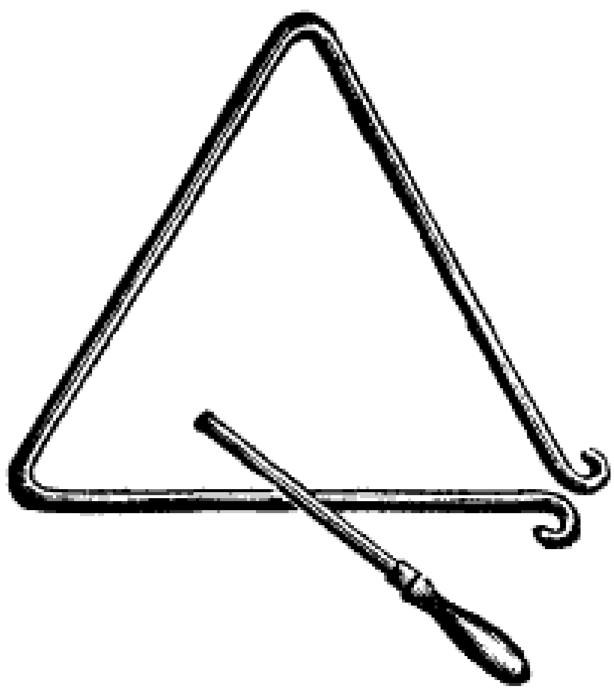

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Chapter 1

Introduction

Since the early 20th century it has been possible to undertake a study of propositional logic in several ways: as a set of syntactic derivations, following some specified set of rules – broadly the domain of *Proof Theory*; as the set of validities in a collection of structures which are taken to capture truth – broadly the domain of *Model Theory*. We assume you to be familiar with both: the proof theory of classical logic is given either by Natural Deduction, Hilbert-style derivation systems, or sequent calculus; the model theory is typically given by *truth-tables*. Using such tools it was possible from early on to not only find interesting applications for logic, but also to find *properties of logic*: completeness, decidability, finite axiomatizability, the finite model property, interpolation, etc, are all properties that classical propositional logic enjoys, and which were discovered through using one or both of the above methods.

Quite early it was observed that a possible further way of analysing logic could be obtained by mimicking elementary algebra, and making the necessary modifications: instead of expressions like

$$(x + y)^2 = zy + 2z^3,$$

where our variables are assumed to range over numbers, and the symbols denote arithmetic operations, one could form expressions of the form

$$(x \wedge y) \vee (\neg z \wedge y) = (y \vee z) \wedge 1,$$

where the variables are now taken to be *truth-values*, and the operations are *logical operations*: conjunction, disjunction, negation, *verum* (1). The result of this was the invention of a new form of analysis of logic – *algebraic logic* – which held a certain hybrid status with respect to Proof Theory and Model Theory: it is both syntactic, since it manipulates expressions through equivalences, and semantic, since it concerns specific mathematical structures which are taken to be models for the laws of logic.

As it turns out, the algebraic method was particularly amenable to generalization to *non-classical logics*: modal logics such as $\mathsf{S4}$ or $\mathsf{S5}$, intuitionistic logic IPC, amongst others, were from early on studied using algebraic tools, since they originally lacked an obvious model theory paralleling the elegance and simplicity of truth-tables. In more specificity, after setting up appropriate *algebraic semantics*, one uses specific subcollections of algebraic models called *varieties* to study specific logics, and due to the nature of the set-up, the validities of the variety always coincide with the truths of the logic.

The 20th century brought another innovation: the development of Kripke semantics, and several related systems of *relational semantics* provided a simple model theory for non-classical logics of various kinds, and began an industry of analysing all sorts of non-classical logics. However, as you may have studied[1], this lead to the realization that not all good properties of classical logic were shared by every non-classical logic – for instance, not every such logic was *complete* with respect to its relational semantics. As such, a slightly more general semantics – *general frame semantics* – needed to be introduced, where a collection of admissible subsets are identified. Such a semantics carried implicitly with it several notions from *topology*; as such, we refer to these kinds of relational semantics possibly together with a class of admissible subsets, as *frame semantics*.

For quite deep mathematical reasons, it was expected that there should be a relationship between these two forms of analysing classical logic. Namely, algebras of classical logic – often called *Boolean algebras* – correspond to specific topological spaces – now called *Stone spaces* – and vice-versa, in a correspondence which is unique up to isomorphism[2]. Such a situation is called a *duality* between algebras and spaces.

We thus arrive at the following picture, which aptly summarizes all of this:

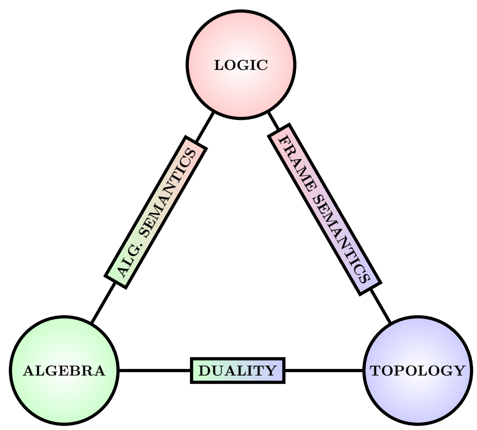

Figure 1.1: Triangle of Logic, Algebra and Topology

In this text, we will introduce these topics – algebraic logic, topological semantics and duality. We will focus exclusively on *propositional, (non-)classical logics* (to the detriment of first and higher-orders, and other variations), in a two-fold way:

1. Introducing the algebraic and topological semantics of several logical systems, and especially the duality theory connecting the algebra and the topology;

2. Studying some properties of non-classical systems using these tools.

As examples of the former, we will study **algebraic completeness**, **Stone duality**, **Priestley duality** and **Esakia duality**; as examples of the latter we will prove some basic facts about the *size* of **lattice of extensions of** IPC as well as some techniques giving **axiomatizability** using *refutation patterns*. We will also include several more advanced topics which can be chosen as final

[1] Namely in a course like *Introduction to Modal Logic*.

[2] This course does not require any knowledge of category theory; nevertheless, having such knowledge allows one to appreciate the theoretical results in a broader structural light. Having such readers in mind, we will include specific results and exercises that deal with category theory in end-notes, footnotes, and special sections.

topics to address in a course on the subject, which illustrate the full power of all the machinery developed.

We assume the reader is familiar with basic set-theoretic notation, as can be expected from any formal course. We will also assume throughout that the reader is familiar with basic concepts from point-set topology[3]; some concepts from universal algebra will be used, and will be assumed throughout; we have included all necessarily preliminaries in chapter 3, in an optional section, though we refer to reader to "A Course in Universal Algebra" for any facts left unproven, or additional facts which are necessary[4].

These notes were originally designed for the course *Mathematical Structures in Logic*, taught in 2025 at the ILLC; they contain proposed exercises as well as some propositions which are left as exercises for the reader[5]. In order to make the reading to students more accessible, we have marked those Sections, Propositions and Theorems which do not constitute examinable material for the course using coloured boxes, with a disclaimer "Extra Content", as well as a $\Diamond$ marker around sections which are extra, so they can easily be identified from the index; they are included here to give a more complete view of the subject, and for reference to the interested student.

We also hope in a future version of these notes to provide a detailed reference to all results, though for reasons of time, this has not been possible to include here.

1.1 Partial orders

Definition 1.1.1. *A* partially ordered set *or* poset *is a structure* $(P, \leqslant)$*, where* P *is a set and* $\leqslant$ *is a partial order, i.e., a reflexive* ($a \leqslant a$)*, transitive* ($a \leqslant b$ *and* $b \leqslant c$ *implies* $a \leqslant c$) *and antisymmetric* ($a \leqslant b$ *and* $b \leqslant a$ *implies* $a = b$) *relation.*

We say that a poset is *totally ordered* if for each $a, b \in P$, $a \leqslant b$ or $b \leqslant a$.

Whenever there is no risk of confusion, we will identify the carrier set of $(P, \leqslant)$ with the poset, i.e., we will refer to P as the poset. We will also use the abbreviation $x < y$ to mean that $x \leqslant y$ and $x \neq y$, and similarly for $x > y$.

Definition 1.1.2. Let $(P, \leqslant)$ be a poset.

1. For $A \subseteq P$, $c \in P$ is the *supremum* of A (denoted as $\sup A$) if it is an upper bound ($a \leqslant c$ for all $a \in A$) and it is the least such ($c \leqslant b$ for each upper bound b of A).

   $c \in P$ is the *infimum* of A (denoted $\inf A$) if it is a lower bound ($c \leqslant a$ for all $a \in A$) and it is the greatest such.

2. Given $a, b \in P$ we write $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ and call them the *join* and the *meet* of a and b, respectively.

Definition 1.1.3. Let $(P, \leqslant)$ be a poset, and $A \subseteq P$. We say that the pair $(A, \leqslant_A)$ is a *subposet* of P (or that it has the *induced order* on A) if we for all $a, b \in A$, $a \leqslant_A b$ if and only if $a \leqslant b$.

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[3] For necessary preliminaries on topology, consult Ryskard Engelking's "General Topology"; for a dedicated set of lecture notes, see "Introduction to Topology in and via Logic", available here.

[4] For preliminaries on Universal Algebra, consult Stanley Burris and H.P. Sankappanavar's "A Course in Universal Algebra", Chapter 2.

[5] These notes are based on earlier notes by Nick Bezhanishvili; special thanks to Saul Fernandez Gonzalez for his class notes which were used as a basis for the present document, and to Ruiting Jiang for spotting numerous mistakes and typos in previous versions of these notes.

We will occasionally need to make use of Zorn's Lemma, which refers to *chains on posets*:

Definition 1.1.4. A subset $C \subseteq P$ of a poset P is called a *chain* if for all $a, b \in C$ either $a \leqslant b$ or $b \leqslant a$.

Equivalently, $C \subseteq P$ is a chain if the subposet $(C, \leqslant_C)$ is totally ordered.

Definition 1.1.5. Let $(P, \leqslant)$ be a poset. We say that $a \in P$ is *maximal* if for each $b \in P$ if $a \leqslant b$ then $a = b$.

Zorn's Lemma is the the following statement, known to be equivalent to the Axiom of Choice:

Theorem 1.1.1 (Zorn's Lemma). *Let $(P, \leqslant)$ be a non-empty poset such that for each $C \subseteq P$ a non-empty chain, there is an upper bound a of C. Then P has a maximal element.*

Definition 1.1.6. Let $p : P \to Q$ be a map betweeen posets. We say that p is *monotone* if for all $x, y \in P$, whenever $x \leqslant y$ then $p(x) \leqslant p(y)$. We say that p is a *p-morphism* if p is monotone and for all $x \in P$ and $y \in Q$ whenever $p(x) \leqslant y$ then there is some $x' \geqslant x$ such that $p(x') = y$.

1.2 Logical languages and semantics

We assume the reader is familiar with modal logic as presented, e.g. in [4]. We recall here the basic concepts which will be useful to have in mind throughout the text:

Definition 1.2.1. The *basic modal language* over a set of propositions Prop, denoted $\mathcal{ML}(\mathsf{Prop})$ consists of the formulas φ obtained by the following rules:

$$\varphi ::= \ p \mid \bot \mid \top \mid \neg\varphi \mid \psi \wedge \varphi \mid \psi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \mathsf{Prop}$.

Abbreviations such as $\varphi \to \psi$ are given as usual in classical logic. From this signature we can exclude a few connectives to obtain two other logics which are somewhat familiar:

Definition 1.2.2. The *classical logic language* over a set of propositions Prop, denoted $\mathcal{CL}(\mathsf{Prop})$ consists of the formulas φ obtained by the following rules:

$$\varphi ::= \ p \mid \bot \mid \top \mid \neg\varphi \mid \psi \wedge \varphi \mid \psi \vee \varphi,$$

where $p \in \mathsf{Prop}$.

Definition 1.2.3. The *positive logic language* over a set of propositions Prop, denoted $\mathcal{PL}(\mathsf{Prop})$ consists of the formulas φ obtained by the following rules:

$$\varphi ::= \ p \mid \bot \mid \top \mid \psi \wedge \varphi \mid \psi \vee \varphi,$$

where $p \in \mathsf{Prop}$.

The most immediate semantics of classical (and positive) logic is given by truth-tables, allowing one to compute truth-values quickly and efficiently. For modal logic, one needs other kinds of semantics.

Definition 1.2.4. A pair $\mathfrak{F} = (F, R)$ where R is a relation is called a *Kripke frame.*

As usual we say that the Kripke frame $\mathfrak{F}$ is *reflexive, transitive, directed, etc* if R satisfies those conditions.

Definition 1.2.5. Let $\mathfrak{F} = (F, R)$ be a Kripke frame. A pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $V : \mathsf{Prop} \to \mathcal{P}(F)$ is called a *Kripke model* over $\mathfrak{F}$. Given $x \in F$, we define the relation $\mathfrak{M}, x \Vdash \varphi$ for $\varphi \in \mathcal{CL}$ as follows:

1. $\mathfrak{M}, x \Vdash p$ if and only if $x \in V(p)$ for $p \in \mathsf{Prop}$;
2. $\mathfrak{M}, x \Vdash \top$ always;
3. $\mathfrak{M}, x \Vdash \bot$ never;
4. $\mathfrak{M}, x \Vdash \varphi \wedge \psi$ if and only if $\mathfrak{M}, x \Vdash \varphi$ and $\mathfrak{M}, x \Vdash \psi$;
5. $\mathfrak{M}, x \Vdash \varphi \vee \psi$ if and only if $\mathfrak{M}, x \Vdash \varphi$ or $\mathfrak{M}, x \Vdash \psi$.
6. $\mathfrak{M}, x \Vdash \Diamond\varphi$ if and only if there is $y \in F$ such that xRy and $\mathfrak{M}, y \Vdash \varphi$;
7. $\mathfrak{M}, x \Vdash \Box\varphi$ if and only if whenever xRy then $\mathfrak{M}, y \Vdash \varphi$.

We write $\mathfrak{F} \Vdash \varphi$ to mean that for any Kripke model over $\mathfrak{F}$, $\mathfrak{M}$, and any $x \in F$, $\mathfrak{M}, x \Vdash \varphi$. Given a collection $\mathbf{C}$ of frames, we write $\mathbf{C} \Vdash \varphi$ to mean that for every $\mathfrak{F} \in \mathbf{C}$, $\mathfrak{F} \Vdash \varphi$.

Given a Kripke frame $\mathfrak{F} = (F, R)$ and a set $U \subseteq F$, we write

$$R^{-1}[U] = \{x \in F : \exists y, xRy \text{ and } y \in U\}$$

, and

$$\Box_R U = \{x \in F : \forall y(xRy \text{ implies } y \in U)\}.$$

Given a Kripke model $\mathfrak{M} = (\mathfrak{F}, V)$ we define

$$V(\varphi) = \{x \in F : \mathfrak{M}, x \Vdash \varphi\};$$

Using the above clauses we can obtain a purely set-theoretic calculation of $V(\varphi)$; namely:

$$\begin{aligned}
V(\top) &= F \\
V(\bot) &= \varnothing, \\
V(\neg\varphi) &= F - V(\varphi), \\
V(\varphi \wedge \psi) &= V(\varphi) \cap V(\psi), \\
V(\varphi \vee \psi) &= V(\varphi) \cup V(\psi), V(\Diamond\varphi) && = R^{-1}[V(\varphi)] \\
V(\Box\varphi) &= \Box_R V(\varphi).
\end{aligned}$$

Note that, since some of the signature symbols are redundant, so are some of these clauses. This perspective segues naturally into a more general semantics:

Definition 1.2.6. Let $\mathfrak{F} = (F, R)$ be a Kripke frame. We say that $\varnothing \neq \mathcal{A} \subseteq \mathcal{P}(X)$ is a *collection of admissible subsets* for (F, R) if:

1. We have $F \in \mathcal{A}$;

2. Whenever $U \in \mathcal{A}$ then $F - U \in \mathcal{A}$;

3. Whenever $U, V \in \mathcal{A}$ then $U \cap V \in \mathcal{A}$;

4. Whenever $U \in \mathcal{A}$ then $R^{-1}[U] = \{x \in F : \exists y, xRy, y \in U\} \in \mathcal{A}$.

Definition 1.2.7. A *general frame* is a triple $\mathfrak{F} = (F, R, \mathcal{A})$ where (F, R) is a Kripke frame and $\mathcal{A}$ is a collection of subsets admissible for (F, R). A *Kripke model over* $\mathfrak{F}$ is a Kripke model $(\mathfrak{F}, V)$ where for each $p \in \mathsf{Prop}$, $V(p) \in \mathcal{A}$. We write $\mathfrak{F} \Vdash \varphi$ if any Kripke model over $\mathfrak{F}$ satisfies the formula, and similarly for collections of general frames.

1.3 Completeness of modal logics

Definition 1.3.1. A *normal modal logic* is a set L of formulas in the language $\mathcal{ML}$ that contains all classical tautologies and the following axioms:

1. (Dual) $\Diamond p \leftrightarrow \neg\Box\neg p$;

2. $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$.

3. $\Box\top$,

or equivalently,

1. (Dual) $\Diamond p \leftrightarrow \neg\Box\neg p$;

2. $\Box(p \to q) \to (\Box p \to \Box q)$

and is closed under Modus Ponens, Uniform Substitution and Generalization.

As usual we denote the least normal modal logic by K, and use the usual names for logics $\mathsf{S4}, \mathsf{S5}$, etc. Given a collection $\mathbf{F}$ frames (which may be a collection consisting only of Kripke frames, or of general frames), we write $\mathsf{Log}(\mathbf{F}) = \{\varphi : \mathbf{F} \Vdash \varphi\}$, which forms a normal modal logic. Conversely, given a collection of formulas S, we write $\mathsf{KFr}(S) = \{\mathfrak{F} : \mathfrak{F} \text{ is a Kripke frame}, \mathfrak{F} \Vdash S\}$, and write $\mathsf{GFr}(S) = \{\mathfrak{F} : \mathfrak{F} \text{ is a general frame}, \mathfrak{F} \Vdash S\}$. The issue of completeness has precisely to do with the interaction of these two operations:

Definition 1.3.2. Let L be a normal modal logic. We say that L is *Kripke complete* if $\mathsf{L} = \mathsf{Log}(\mathsf{KFr}(\mathsf{L}))$; we say that L is *general frame complete* if $\mathsf{L} = \mathsf{Log}(\mathsf{GFr}(\mathsf{L}))$.

Recall that to prove completeness (whether Kripke or general frame), it suffices to find some subcollection $\mathbf{C} \subseteq \mathsf{Fr}(L)$ for which we can prove soundness and completeness, and this will follow for the whole class. The way that completeness results are usually proved is through the *canonical model*. Recall that given L a normal modal logic, and $\Gamma \cup \{\varphi\}$ a set of formulas, we write $\Gamma \vdash_{\mathsf{L}} \varphi$ to mean that there are finitely many formulas $\psi_0, .., \psi_n \in \Gamma$ such that $\psi_0 \wedge ... \wedge \psi_n \to \varphi \in \mathsf{L}$.

Definition 1.3.3. We say that a set of $\mathcal{ML}$ formulas Γ is L-consistent if $\Gamma \nvdash_{\mathsf{L}} \bot$. we say that Γ is *maximally* L*-consistent* (or that Γ is an L-MCS) if it is consistent and no extension of it is cosnsistent.

Definition 1.3.4. Given a normal modal logic L, the *canonical model of* L is the triple (W^L, R^L, V^L) where:

1. W^L is the set of L-MCS;
2. For $\Gamma, \Delta \in W^L$, $\Gamma R^L \Delta$ if and only if one of the two following equivalent conditions holds:
   - For each φ, if $\Box\varphi \in \Gamma$ then $\varphi \in \Delta$;
   - For each φ if $\varphi \in \Delta$ then $\Diamond\varphi \in \Gamma$.
3. $V^L(p) = \{\Gamma \in W^L : p \in \Gamma\}$.

We can also consider a particular general frame based on the canonical model:

Definition 1.3.5. Given a normal modal logic L, the *canonical general frame* of L is the quadruple $\mathfrak{F}^L = (W^L, R^L, V^L, \mathsf{Form}(L))$ such that (W^L, R^L, V^L) is the canonical model of L and $\mathsf{Form}(\mathsf{L}) = \{V^L(\varphi) : \varphi \in \mathcal{ML}\}$.

Proposition 1.3.6. Given a normal modal logic L, the canonical general frame of L is a general frame.

Proof. Simply note that applying the operations to admissible subsets will again obtain formulas, e.g., $R^{-1}[V^L(\varphi)] = V^L(\Diamond\varphi)$. ■

A simple way of rephrasing the well-known *Truth Lemma*, which is instrumental to proving completeness, is the following:

Lemma 1.3.7. Given a normal modal logic L, and each formula φ, we have:

$$V^L(\varphi) = \{\Gamma \in W^L : \varphi \in \Gamma\}.$$

Having the truth lemma at hand, the one issue for proving Kripke completeness lies in showing that the logic is sound with respect to the Kripke frame (W^L, R^L); indeed, it is known that there are Kripke incomplete modal logics (see e.g. [4, Section 4.4]). In the case of the general frame completeness this never happens:

Theorem 1.3.1. *For every normal modal logic* L*,* L *is general frame complete.*

Proof. First we note that $\mathfrak{F}^L$ is in fact a frame for L: indeed, let $\varphi(p_1, ..., p_n) \in \mathsf{L}$; let $T : \mathsf{Prop} \to \mathsf{Form}(\mathsf{L})$ be an admissible valuation. Let $\psi_i = T(p_i)$ be formulas such that $T(p_i) = V^L(\psi_i)$. Consider the formula

$$\chi := \varphi(\psi_1, ..., \psi_n),$$

i.e., the result of applying uniform substitution of p_i by ψ_i in φ. Then by closure of L under uniform substitution, $\chi \in \mathsf{L}$. Note that then by definition:

$$T(\varphi) = V^L(\chi).$$

By assumption, since the elements of W^L are L-consistent sets, then $\mathsf{L} \subseteq \Gamma$ whenever $\Gamma \in W^L$. Hence $\chi \in \Gamma$ for each such element. By the truth lemma, then $V^L(\chi) = W^L$, and so $T(\varphi) = W^L$, i.e., φ is true globally over the model (W^L, R^L, T). Since this holds for any admissible valuation, $\mathfrak{F} \Vdash \mathsf{L}$.

Now assume that $\varphi \notin \mathsf{L}$. Using Lindenbaum's Lemma for L (see [4, Lemma 4.17]) we can extend $\{\neg\varphi\}$, an L-consistent set, to an L-MCS Γ; then by the truth lemma, $(\mathfrak{F}^L, \Gamma) \Vdash \neg\varphi$, i.e., $\mathfrak{F}^L \nVdash \varphi$.

We have shown that $\mathsf{L} = \mathsf{Log}(\mathfrak{F}^L)$; by our remark after Definition 1.3.2, we have that L is general frame complete. ■

We will return to this result when we talk about varieties and algebraic completeness; as we will see, such a result actually stems from the combination of two tools we will learn about: free algebras and the completeness results they ensure; and duality theory for modal algebras.

1.4 Collections of logics

Definition 1.4.1. Given two normal moda logics L, T, we say that L is a *sublogic* of T, or that T is an *extension* of L if $\mathsf{L} \subseteq \mathsf{T}$. Given a normal modal logic L we denote by $\Lambda(\mathsf{L})$ the set of its sublogics.

Given any normal modal logic L we may consider $\Lambda(\mathsf{L})$ as a poset, with the partial order being given by inclusion. Moreover, we have:

Proposition 1.4.2. For each normal modal logic L, and $(L_i)_{i\in I} \subseteq \Lambda(\mathsf{L})$, we have that $\bigcap_{i\in I} L_i$ is a normal modal logic, and it is the infimum of $(L_i)_{i\in I}$. Moreover, there is a least normal modal logic containing $\bigcup_{i\in I} L_i$, denoted by $\bigoplus_{i\in I} L_i$, which is the supremum of $(L_i)_{i\in I}$.

Proof. Exercise. ■

As we will immediately see, the above property means that $\Lambda(\mathsf{L})$ always has the structure of a *complete lattice*. Often understanding the structure of this lattice can give us important information about the logical system, on one hand, and about how certain logical properties are distributed. Here is a small example:

Definition 1.4.3. Let $(P, \leqslant, \top)$ be a poset with maximal element $\top$. We say that an element $x \in P$ is a *co-atom* if $x < \top$ and whenever $x \leqslant y$ and $y < \top$ then $x = y$.

The following was originally shown in [7]:

Theorem 1.4.1 (Makinson, 1971). *The lattice* $\Lambda(\mathsf{K})$ *has exactly two co-atoms.*

As we will see later, there is an easy proof of this fact using the duality theory and some basic facts about K.

Chapter 2

Algebra and Duality Theory

The purpose of this chapter is to introduce the main algebraic structures we will study in the context of logic: lattices, especially distributive lattices, Boolean algebras, Heyting algebras and modal algebras. In parallel, and stemming from the original work of Marshall Stone, we will introduce special classes of topological spaces, often equipped with additional relations, which will be used to provide *duals* of the algebraic structures.

2.1 Lattices

Our first main concept will be that of a lattice; this admits two different presentations, which we will look at in turn. The first starts from some basic concepts we have encountered before.

Definition 2.1.1. A *lattice* is a poset $(L, \leqslant)$ such that for each $a, b \in L$, $a \vee b$ and $a \wedge b$ both exist. We say that a lattice $(L, \leqslant)$ is *complete* if additionally for each $S \subseteq L$, $\inf S$ and $\sup S$ both exist.

Remark 2.1.1. For P a poset, it is equivalent that it contains all infima and that it contains all suprema and that it is a complete lattice (see Exercises).

Example 2.1.2. Typically one represents lattices, like posets in general, using *Hasse diagrams* – where given $x, y \in P$, we draw a line between x and y if y is an immediate successor of x (i.e., $x \leqslant y$ and whenever $x \leqslant z \leqslant y$ then $x = z$ or $z = y$). We can see several Hasse diagrams of posets on Figure 2.1.

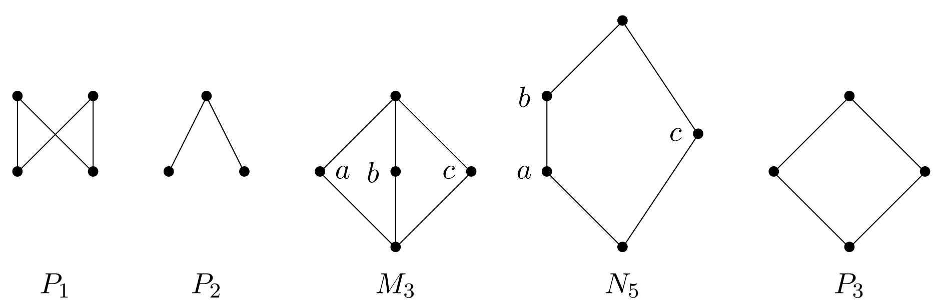


Figure 2.1: Posets and Lattices

We can see that M_3, N_5 and P_3 are all lattices; M_3 is sometimes referred to as the *diamond lattice* and N_5 as the *pentagon lattice*. On the other hand, neither P_1 nor P_2 are lattices: P_1 is lacking both infima and suprema for some subsets, whilst P_2 contains all suprema but is missing one infimum.

Example 2.1.3. Let $(P, \leqslant)$ be a linear order. Given two elements $a, b \in P$ we can consider the *maximum* between a, b to be the greastest element c such that $a \leqslant c$ and $b \leqslant c$; dually we define the *minimum*. Then $(P, \leqslant)$ forms a lattice. Lattices of this sort are called *linear lattices*.

Example 2.1.4. Let L be a normal modal logic. Then $\Lambda(\mathsf{L})$ is a complete lattice: by Proposition 1.4.2, we have that the infimum of a set of logics is given by arbitrary intersection, and the supremum of a set of logics is the least normal modal logic containing the elements.

The following is easy to verify.

Proposition 2.1.5. Let L be a lattice. Then the following identities are satisfied:

c1. $a \vee b = b \vee a$

c2. $a \wedge b = b \wedge a$

a1. $a \vee (b \vee c) = (a \vee b) \vee c$

a2. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

i1. $a \vee a = a$

i2. $a \wedge a = a$

abs1. $a = a \vee (a \wedge b)$

abs2. $a = a \wedge (a \vee b)$

In fact, one can give an alternative definition of a lattice by taking the operations $\wedge$ and $\vee$ as primitives: then a lattice *in the algebraic sense* (as opposed to the *order-theoretic sense*) will be a triple $(L, \wedge, \vee)$ where $\wedge, \vee : L^2 \to L$ are operations satisfying the equations of Proposition 2.1.5. In this context, c_1, c_2 are called the *commutative laws*, a_1, a_2 are called the *associative laws*, i_1, i_2 are called the *idempotence laws* and abs_1, abs_2 are called the *absorption laws*. It is easy to prove that these two definitions are equivalent: given a lattice $(L, \wedge, \vee)$, we define a partial order on L by $a \leqslant b$ if and only if $a = a \wedge b$, or equivalently (by the absorption laws), if and only if $b = a \vee b$ (see the Exercises).

To finish our definition of these structures, we will describe what are the *structure-preserving maps* which we will be interested in when dealing with them:

Definition 2.1.6. If L_1 and L_2 are lattices, a *lattice homomorphism* is a map

$$h : L_1 \to L_2$$

satisfying for every $a, b \in L_1$:

$$\begin{aligned} h(a \vee b) &= h(a) \vee h(b), \\ h(a \wedge b) &= h(a) \wedge h(b). \end{aligned}$$

An isomorphism of lattices is a bijective homomorphism.

2.1.1 Distributive and modular lattices; forbidden configurations

Lattices are very general structures, and arise in many areas of mathematics. In logic, a lot of focus is placed on *distributive* lattices:

Definition 2.1.7. A lattice $(L, \wedge, \vee,)$ is *distributive* if for all $a, b, c \in L$:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Proposition 2.1.8.

1. A lattice satisfies one of the above two expressions iff it satisfies the other one.
2. The inequality $(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c)$ holds in every lattice.
3. A lattice is distributive iff $a \wedge (b \vee c) \leqslant (a \wedge b) \vee (a \wedge c)$.

Proof. (1) Suppose L satisfies the first of equation. Then,

$$\begin{aligned}
a \vee (b \wedge c) &\overset{\text{abs1}}{=} (a \vee (a \wedge c)) \vee (b \wedge c) \\
&\overset{\text{a1}}{=} a \vee ((a \wedge c) \vee (b \wedge c)) \\
&\overset{\text{c1}}{=} a \vee ((c \wedge a) \vee (c \wedge b)) \\
&\overset{\text{hyp.}}{=} a \vee (c \wedge (a \vee b)) \\
&\overset{\text{abs2}}{=} (a \wedge (a \vee b)) \vee (c \wedge (a \vee b)) \\
&\overset{\text{c2}}{=} ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) \\
&\overset{\text{hyp.}}{=} (a \vee b) \wedge (a \vee c).
\end{aligned}$$

The other direction is similar.

(2) $a \wedge b \leqslant a$ and $a \wedge b \leqslant b \leqslant b \vee c$; $a \wedge c \leqslant a$ and $a \wedge c \leqslant c \leqslant b \vee c$. (3) follows directly from (1) and (2). ■

In lattice theory there is a weakening of distributivity which is often useful:

Definition 2.1.9 (Modular lattices)**.** A lattice $(L, \wedge, \vee)$ is *modular*[1] if it satisfies the *modular law* for each $a, b, c \in L$:

$$a \leqslant b \text{ implies } a \vee (b \wedge c) = b \wedge (a \vee c).$$

The modular law ia equivalent to the following equation

$$(a \wedge b) \vee (b \wedge c) = b \wedge ((a \wedge b) \vee c).$$

It is easy to show that distributivity implies modularity (see the Exercises).

[1]The terminology comes from the structures where such a law was first observed: Dedekind noted that the lattice of *submodules* of a module (generalizations of vector spaces) always satisfied this law.

Example 2.1.10. Looking again at Figure 2.1, we see that P_3 is the only distributive lattice. You should check that N_5 and M_3 are non-distributive: to give the example of N_5, we will verify it is not modular. Indeed, $a \leqslant b$ but:

$$a \vee (b \wedge c) = a \text{ and } b \wedge (a \vee c) = b;$$

hence N_5 is neither modular nor distributive. With some work one can verify that M_3 is modular.

Definition 2.1.11. If L is a lattice, $\varnothing \neq L' \subseteq L$ is a *sublattice* of L if $\forall a, b \in L'$, $a \vee b, a \wedge b \in L'$. A lattice L_1 can be *embedded* into a lattice L_2 if L_1 is isomorphic to some sublattice of L_2. Notation: $L_1 \hookrightarrow L_2$.

Lattice embeddings can be used to characterize modularity and distributivity:

Theorem 2.1.1. *Let L be a lattice:*

1. *(Dedekind, 1900)[6] L is modular if and only if N_5 cannot be embedded into L;*
2. *(Birkhoff, 1934)[3] L is distributive if and only if N_5 and M_3 cannot be embedded into L.*

Extra Content

You should know this result; the proof is included for completeness.

Proof. It is clear that if L is modular – respectively, distributive – then M_3 (resp. M_3 and N_5) cannot be a sublattice. We focus on the converse.

(1) First assume that L is a non-modular lattice. Hence there exist elements a, b, c such that $a \leqslant b$ but $x = a \vee (b \wedge c) \neq b \wedge (a \vee c) = y$. First note that $x \leqslant y$: indeed $a \leqslant b \wedge (a \vee c)$ since $a \leqslant b$, and $b \wedge c \leqslant b \leqslant y$, so y is an upper bound of both a and $b \wedge c$, i.e., $x \leqslant y$. Moreover, note that:

$$\begin{aligned} c \vee x &= c \vee (a \vee (b \wedge c)) \\ &= a \vee (c \vee (b \wedge c)) \\ &= a \vee c, \end{aligned}$$

using commutativity, associativity and absorption, and by similar calculations, $c \wedge y = c \wedge b$. Hence we can consider the sublattice as depicted in Figure 2.2:

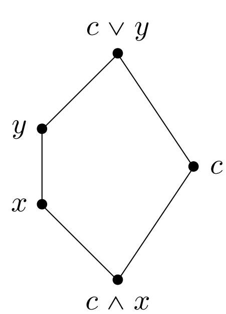


Figure 2.2: Pentagon sublattice

(2) Now assume that L is non-distributive; assume further that L does not contain a copy of N_5 as a sublattice; hence by (1) L is modular. Since the distributive law fails, there must be elements a, b, c such that $(a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c)$. We define the following elements:

$$\begin{aligned}
d &= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \\
e &= (a \vee b) \wedge (a \vee c) \wedge (b \vee c) \\
x_a &= (a \wedge e) \vee d \\
x_b &= (b \wedge e) \vee d \\
x_c &= (c \wedge e) \vee d.
\end{aligned}$$

Certainly $d \leqslant x_a, x_b, x_c$. Moreover, since $(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c)$ always, then also $(a \wedge b) \vee (a \wedge c) \leqslant (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$; and hence $d \leqslant e$. From this it is easy to conclude that $x_a, x_b, x_c \leqslant e$.

Using absorption, note that $a \wedge e = a \wedge (b \vee c)$. Moreover, since $a \leqslant (a \wedge b) \vee (a \wedge c)$, by the modular law we have that:

$$\begin{aligned}
a \wedge (((a \wedge b) \vee (a \wedge c)) \vee (b \vee c)) &= (a \wedge b) \vee (a \wedge c) \vee (a \wedge (b \wedge c)) \\
&= (a \wedge b) \vee (a \wedge c).
\end{aligned}$$

Now if $d = e$, then $a \wedge d = a \wedge e$. But by the above, $a \wedge e = a \wedge (b \vee c) > (a \wedge b) \vee (a \wedge c) = a \wedge d$. Hence $d \neq e$.

Thus consider the diagram in Figure 2.3:

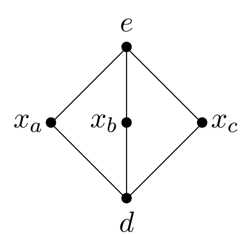


Figure 2.3: Diamond sublattice

To show that this diagram indeed corresponds, inside of L, to a copy of M_3, we need to verify that x_a, x_b, x_c are all distinct, they are distinct from d, e and that their pairwise meets are d and pairwise joins are e. For this, note that it suffices to show that $x_a \vee x_b = x_a \vee x_c = x_b \vee x_c = e$ and $x_a \wedge x_b = x_a \wedge x_c = x_b \wedge x_c = d$. To see why, assume that this holds, and towards a contradiction suppose that $x_a = d$; then since $x_a \vee x_b = e$, and $x_b \leqslant e$, then $x_b = e$. Since $x_b \wedge x_c = d$, for similar reasons, $x_c = d$. But then $x_a \vee x_c = e$, which is a contradiction. Similar arguments rule out any of the middle elements coinciding with either e or d. Moreover if $x_a = x_b$, then $x_a \vee x_b = x_a = e$, which is again a contradiction.

So we show the equalities. We will show that $x_a \wedge x_b = d$; similar arguments prove the remaining cases. For that, note that certainly $d \leqslant d \vee (b \wedge e)$, so using modularity we have

$$x_a \wedge x_b = ((a \wedge e) \vee d) \wedge ((b \wedge e) \vee d)) = ((a \wedge e) \wedge ((b \wedge e) \vee d)) \vee d.$$

Since $d \leqslant e$, by modularity again:

$$((a \wedge e) \wedge ((b \vee d) \wedge e)) \vee d;$$

Using commutativity and idempotence this simplifies to:

$$((a \wedge e) \wedge (b \vee d)) \vee d.$$

Since $a \wedge e = a \wedge (b \vee c)$, and by similar arguments, $b \vee d = b \vee (a \wedge c)$, this further simplifies to:

$$(a \wedge (b \vee c) \wedge (b \vee (a \wedge c))) \vee d;$$

Since $b \leqslant b \vee c$, once more by modularity, this is equal to:

$$(a \wedge (b \vee ((b \vee c) \wedge (a \wedge c)))) \vee d;$$

by absorption this further simplifies to:

$$(a \wedge (b \vee (a \wedge c))) \vee d;$$

One final application of modularity to $a \wedge c \leqslant a$ gets us:

$$(a \wedge c) \vee (a \wedge b) \vee d = d.$$

Hence we get $x_a \wedge x_b = d$, as desired. This obtains us that the diagram of Figure 2.3 is indeed a copy of M_3 proving the result. ■ ■

End of extra content.

2.1.2 Bounded lattices; filters and ideals

Definition 2.1.12 (Bounded lattices)**.** A lattice L is *bounded* if there are $0, 1 \in L$ such that $0 \leqslant a \leqslant 1$ for all $a \in L$.

Definition 2.1.13. Let L, L' be two bounded lattices, and $f : L \to L'$ a lattice homomorphism. We say that f is a *bounded lattice homomorphism* if $f(0_L) = 0_{L'}$ and $f(1_L) = 1_{L'}$.

Some particular subsets of posets and bounded lattices will turn out to be very useful.

Definition 2.1.14. Let P be a poset. A subset $U \subseteq P$ is called an *upset* if for all $a, b \in P$ whenever $a \in U$ and $a \leqslant b$ then $b \in U$. A subset $D \subseteq P$ is called a *downset* if for all $a, b \in P$ whenever $a \in D$ and $b \leqslant a$ then $b \in D$.

Given a subset $S \subseteq P$, we denote the *upwards closure* of S by $\uparrow S = \{x \in P : \exists y \in S, y \leqslant x\}$. Similarly, we denote the *downwards closure* of S $\downarrow S = \{x \in P : \exists y \in S, x \leqslant y\}$.

Definition 2.1.15. Let L be a lattice. $\varnothing \neq F \subseteq L$ is a *filter* if for all $a, b \in L$,

i. F is an upset;

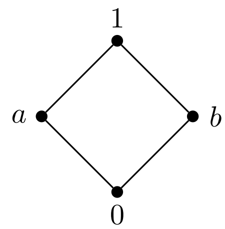
ii. $a, b \in F$ implies $a \wedge b \in F$.

An *ideal* is $\varnothing \neq I \subseteq L$ such that for all $a, b \in L$,

i. I is a downset;

ii. $a, b \in I$ implies $a \vee b \in I$.

Example 2.1.16. Consider



$\{1\}$ is a filter, $\{0\}$ and ideal. Both $\uparrow a = \{a, 1\}$ and $\uparrow b = \{b, 1\}$ are filters and both $\downarrow a = \{0, a\}$ and $\downarrow b = \{0, b\}$ are ideals.

Not all upsets are filters: $\{a, 1, b\}$ is not a filter.

Lemma 2.1.17. Point generated upsets (resp. downsets) are filters (resp. ideals): for all $a \in A$, $\uparrow a$ is a filter and $\downarrow a$ is an ideal.

Proof. Exercise. ■

For $a \in L$, $\uparrow a$ and $\downarrow a$ are called the *principal filter* and *principal ideal generated by a*. Such filters and ideals are called *principal filters* and *principal ideals*, respectively. Moreover, if our lattice is finite, these are the only kinds of filters which exist (see Exercises). The idea of filters is thus to act as "pseudo-elements" – which as we will see is one of the key ingredients of duality.

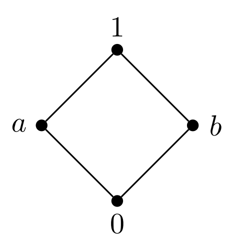
Definition 2.1.18. Let L be a lattice.

1. A filter $F \subseteq L$ is *proper* if $F \neq L$. An ideal I is *proper* if $I \neq L$.

2. A proper filter F is *prime* if for all $a, b \in L$, $a \vee b \in F$ implies $a \in F$ or $b \in F$.

3. A proper ideal I is *prime* if for all $a, b \in L$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Example 2.1.19. In



Take $F = \{1\}$. It is obviously not prime. But $F = \{a, 1\}$ is. Note that $\{a, 1\}$ is a maximal proper filter. Note as well that the complement of $\{a, 1\}$ is $\{0, b\}$ which is a prime ideal.

Definition 2.1.20. Let L be a bounded lattice. We say that an element $a \in L$ where $a \neq 0$ is *join-irreducible* if for all $b, c \in L$, whenever $a = b \vee c$ then $a = b$ or $a = c$. We say that an element $a \in L$ is *join-prime* if for all $b, c \in L$ whenever $a \leqslant b \vee c$ then $a \leqslant b$ or $a \leqslant c$.

Lemma 2.1.21. Let L be a lattice.

1. If $a \in L$ is join-prime then a is join-irreducible.

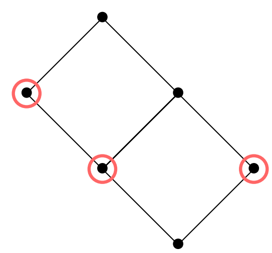
2. If L is a distributive lattice, then $a \in L$ is join-prime if and only if it is join-irreducible.

3. If L if a finite distributive lattice, then a filter F is prime if and only if $F = \uparrow a$ for $a \in L$ where a is join-irreducible.

Proof. Exercise. ∎

Lemma 2.1.21 allows us to calculate the prime filters of a finite distributive lattice with relative ease: one simply needs to identify the elements which are not joins of two strictly smaller elements. We will have many occasions to practice this, but can see a simple example now:

Example 2.1.22. Consider the following lattice:



It is not very difficult to see that this lattice is distributive using Theorem 2.1.1: it does not embed any pentagon or diamond. Hence to know its prime filters, we need only look at the join-irreducible elements. The marked red circles are exactly these elements: they are not joins of smaller elements, and every other element (except for the bottom, where its principal filter is not proper) is the join of two smaller elements.

We now turn to some more theoretical results on prime filters and prime ideals:

Theorem 2.1.2. *Let L be a lattice. Then $F \subseteq L$ is a prime filter iff $L \backslash F$ is a prime ideal.*

Proof. Suppose F is a prime filter. and take $a \in L\backslash F$, and $b \leqslant a$. But if $b \in F$, then $a \in F$, hence it has to be the case that $b \in L\backslash F$. Now take $a, b \in L\backslash F$ and consider $a \vee b$. If $a \vee b \notin L\backslash F$, this means $a \vee b \in F$, hence either $a \in F$ or $b \in F$, which is a contradiction. So $L\backslash F$ is an ideal. To check that it is a *prime* ideal, let $a \wedge b \in L\backslash F$. If $a \notin L\backslash F, b \notin L\backslash F$, then both $a \in F$ and $b \in F$, and hence $a \wedge b \in F$, which is a contradiction.
The other direction is similar. ∎

Lemma 2.1.23. Let L be a bounded lattice, and $S \subseteq L$ be a subset. Then there is a smallest filter containing S, denoted by $\mathsf{Fil}(S)$. More concretely:

$$\mathsf{Fil}(S) = \uparrow\{a_0 \wedge ... \wedge a_n : a_i \in S\}$$

Proof. Exercise. ∎

Theorem 2.1.3 (Prime filter theorem). *Let L be a distributive lattice, let F be a filter and I an ideal in L such that $F \cap I = \varnothing$. Then there is a prime filter $F' \supseteq F$ such that $F' \cap I = \varnothing$.*

Proof. Let

$$\mathrm{P} := \{G : G \text{ is a filter}, F \subseteq G, G \cap I = \varnothing\}.$$

Then ordering P by inclusion, this forms a nonempty poset ($F \in \mathrm{P}$) and if C is a chain of elements of P, then $\bigcup C$ is a filter (easy, verify!). So by Zorn's Lemma (see Theorem 1.1.1), P has a maximal element, which we denote by F'.

It remains to show that F' is in fact a prime filter. It is proper, for $F' \cap I = \varnothing$. Now let $a \vee b \in F'$ and suppose $a \notin F'$, $b \notin F'$. Now consider $F'_a = \mathsf{Fil}(F' \cup \{a\})$. By Lemma 2.1.23, we know that

$$F'_a = \uparrow\{a \wedge c : c \in F'\}$$

is a filter properly extending F'. This necessarily means that $F'_a \cap I \neq \varnothing$. Let $x \in F'_a \cap I$. We have $x \geqslant a \wedge c$ for some $c \in F'$. This means that $a \wedge c \in I$. Analogously, $b \wedge d \in I$ for some $d \in F'$. But then $a \wedge c \geqslant (a \wedge c) \wedge d \in I$ and $b \wedge d \geqslant (b \wedge d) \wedge c \in I$, and hence, using distributivity:

$$\begin{aligned} F' \ni (a \vee b) \wedge (c \wedge d) &= [a \wedge (c \wedge d)] \vee [b \wedge (c \wedge d)] \\ &= [(a \wedge c) \wedge d] \vee [(b \wedge d) \wedge c] \in I, \end{aligned}$$

contradiction. ■

2.1.3 Complements in lattices

Definition 2.1.24 (Complements)**.** If L is a bounded lattice, and $a \in L$, we say that b is *a complement of* a if $a \vee b = 1$ and $a \wedge b = 0$. We say that L is *uniquely complemented* if for each a there is a unique b which is its complement; in such a case we write $b = \neg a$.

Proposition 2.1.25. If L is a distributive lattice, then for each element $a \in L$, if there exists a complement, it is unique.

Proof. Suppose there are two complements a' and a'' of an element a. Then

$$a' = a' \wedge 1 = a' \wedge (a \vee a'') = (a' \wedge a) \vee (a' \wedge a'') = 0 \vee (a' \wedge a'') = a' \wedge a''.$$

So $a' \leqslant a''$. Analogously we show that $a'' \leqslant a'$. Therefore, $a' = a''$. ■

If the lattice is not distributive then an element may have several complements: simply consider M_3, where each of the elements in the center is the complement of the other ones.

2.2 Boolean algebras

We now turn to the notion of a Boolean algebra. Here it will be convenient to use the definition of a bounded lattice as a tuple $(L, \wedge, \vee, 0, 1)$ with binary operations and constants.

Definition 2.2.1 (Boolean algebra)**.** A *Boolean algebra* is a structure $(B, \vee, \wedge, \neg, 0, 1)$, where $\neg : B \to B$ is a unary operation, such that

i. $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice.

ii. each element a has a complement.

Proposition 2.2.2. Let B be a Boolean algebra. Then for each $a, b \in B$ we have:

i. $\neg 0 = 1$, $\neg 1 = 0$,

ii. $\neg\neg a = a$,

iii. $\neg(a \vee b) = \neg a \wedge \neg b$,

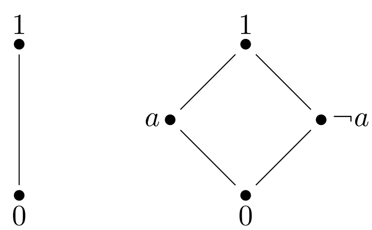
iv. $\neg(a \wedge b) = \neg a \vee \neg b$,

v. $a \wedge \neg b = 0$ iff $a \leqslant b$.

Proof. Exercise. ■

Example 2.2.3.

i. The following are Boolean algebras:



ii. $(\mathcal{P}(X), \cup, \cap, \setminus, \varnothing, X)$ is a Boolean algebra. Every finite Boolean algebra is isomorphic to one of this form for a finite X.

iii. $(\mathrm{FinCofin}(\mathbb{N}), \cup, \cap, \setminus, \varnothing, \mathbb{N})$ is an infinite Boolean algebra. This one is not complete.

iv. Let P be a set of propositional letters. Let $\mathsf{Form}(P)$ be the set of classical logic formulas over the propositions in P. Consider the relation $\approx$ on this set which says that $\varphi \approx \psi$ if and only if $\vdash \varphi \leftrightarrow \psi$. Then this is an equivalence relation. The set $B := \mathsf{Form}(P)/\approx$ is a Boolean algebra, where the operations are defined on equivalence classes: $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$. In this algebra, we have that for two formulas, φ, ψ then $[\varphi] \leqslant [\psi]$ if and only if $\vdash \varphi \to \psi$. (This example will be studied in detail in section 3.2.3).

Definition 2.2.4. Given B, B' two Boolean algebras, a map $f : B \to B'$ is called a *Boolean algebra homomorphism* if it is a lattice homomorphism and $f(\neg a) = \neg f(a)$ for each $a \in B$.

Proposition 2.2.5. Let B and B' be BAs and $h : B \to B'$ a lattice homomorphism. Then h is a bounded lattice homomorphism iff h is a BA homomorphism.

Proof. We need to show that the following two conditions are equivalent.

1. $h(0) = 0$ and $h(1) = 1$.

2. $h(\neg a) = \neg h(a)$ for each $a \in A$.

1. $\Rightarrow$ 2. Consider

$$0 = h(0) = h(a \wedge \neg a) = h(a) \wedge h(\neg a) \text{ and } 1 = h(1) = h(a \vee \neg a) = h(a) \vee h(\neg a).$$

Since in Boolean algebras the negation is unique (by Proposition 2.1.25) we obtain that $h(\neg a) = \neg h(a)$.

2. $\Rightarrow$ 1. We have

$$h(0) = h(a \wedge \neg a) = h(a) \wedge h(\neg a) = h(a) \wedge \neg h(a) = 0$$
$$\text{and}$$
$$h(1) = h(a \vee \neg a) = h(a) \vee h(\neg a) = h(a) \vee \neg h(a) = 1.$$

■

An injective and surjective homomorphism between BAs is called an isomorphism. Two BAs are called is isomorphic if there is an isomorphism between them.

We will now consider some basic constructions of Boolean algebras:

Definition 2.2.6. Let B be a Boolean algebra, and $B' \subseteq B$ a homomorphism. We say that B' is a *subalgebra* of B if B' is a sublattice, $0, 1 \in B'$ and whenever $a \in B'$ then $\neg a \in B'$.

Lemma 2.2.7. Let B be a Boolean algebra and $S \subseteq B$ a subset. There is a smallest subalgebra of B containing S, denoted by $\langle S \rangle$.

Proof. Note that an arbitrary intersection of subalgebras is again a subalgebra. Hence since there is at least one subalgebra of B containing S (namely B itself), there will be a least one (the intersection of all the ones which contain S). ■

Definition 2.2.8. Let $f : B \to B'$ be a homomorphism. We say that B' is a *homomorphic image* of B if f is surjective.

There is an intimate relationship between homomorphic images and filters:

Definition 2.2.9. If $F \subseteq B$ is a filter, then define the equivalence relation:

$$a \approx_F b \iff a \leftrightarrow b \in F,$$

where $a \leftrightarrow b := (\neg a \vee b) \wedge (\neg b \vee a)$. We let $B/F := B/ \approx_F$.

Lemma 2.2.10. The set B/F has a Boolean algebra structure if we define it as:

- $0_{B/F} := [0_B]_F$ and $1_{B/F} := [1_B]_F$;
- $[a]_F \wedge_{B/F} [b]_F := [a \wedge b]_F$ and $[a]_F \vee_{B/F} [b]_F := [a \vee b]_F$;
- $\neg_{B/F}[a]_F := [\neg a]_F$.

Proof. One needs to verify that these operations are well-defined, and that they satisfy the Boolean algebra equations. Certainly the constants are well-defined; to show the binary and unary operations are well-defined, assume that $b \in [a]_F$ and $d \in [c]_F$. To show that the meet is well-defined we need to show that then $b \wedge d \in [a \wedge c]_F$, hence that:

$$(b \wedge d) \leftrightarrow (a \wedge c) \in F.$$

To see this first note that:

$$a \leftrightarrow b \wedge c \leftrightarrow d \leqslant (b \wedge d) \leftrightarrow (a \wedge c);$$

Indeed, note that $(\neg a \vee b) \wedge (\neg c \vee d) \leqslant \neg a \vee \neg c \vee d$, and also $(\neg a \vee b) \wedge (\neg c \vee d) \leqslant \neg a \vee \neg c \vee b$, so using distributivity, we obtain $(\neg a \vee b) \wedge (\neg c \vee d) \leqslant (\neg a \vee \neg c) \vee (b \wedge d)$; hence by Proposition 2.2.2, $(\neg a \vee b) \wedge (\neg c \vee d) \leqslant \neg(a \wedge c) \vee (b \wedge d)$; similarly, $(\neg b \vee a) \wedge (\neg d \vee c) \leqslant \neg(b \wedge d) \vee (a \wedge c)$. This proves the desired inequality.

Since F is a filter, we then have that $(b \wedge d) \leftrightarrow (a \wedge c) \in F$. Similar arguments show that the join is well-defined. Finally for the negation, note that if $b \in [a]_F$, then $a \leftrightarrow b = \neg a \leftrightarrow \neg b$, by definition.

The verification that B/F is a bounded lattice and that it has unique complements is left as an exercise. ∎

Proposition 2.2.11 (First Homomorphism Theorem for Boolean algebras)**.**

1. If B is a Boolean algebra, and F is a filter, the map $q_F : B \to B/F$ given by $q_F(a) = [a]_F$ is a surjective homomorphism.

2. If $f : B \to B'$ is a surjective homomorphism, then $F = f^{-1}[1_{B'}]$ is a filter. Moreover, $B/F \cong B'$.

Proof. We leave (1) as an exercise. To see (2), We define the isomorphism $g : B' \to B/F$ as follows: for each $a \in B'$, by surjectivity, there is some $b \in B$ such that $f(b) = a$. Then set $g(a) = [b]_F$. Note that this is independent of the choice of b: if $f(b) = f(c) = a$, then $f(b \leftrightarrow c) = 1$ in B', so $b \leftrightarrow c \in F$, hence $g(a) = [b]_F = [c]_F$. Also note that this is injective: if $g(a) = g(c)$, then there is some b and some d both in B such that $f(b) = a$ and $f(d) = c$, and $[b]_F = [d]_F$. Then $b \leftrightarrow d \in F$, so by definition, $f(b \leftrightarrow d) = 1$; this means that $a \leftrightarrow c = 1$ in B', which means precisely that $a = c$. Moreover, it is surjective: for each $[a]_F \in B/F$, consider $f(a) \in B'$, then certainly $g(f(a)) = [a]_F$ by our remarks. Finally, g is a homomorphism, because f is one (check each of the clauses). ∎

Definition 2.2.12. Let B, C be two Boolean algebras. We define the *product* of B and C, denoted by $B \times C$ as follows: for each operation we define it pointwise, e.g.:

$$(a_1, b_1) \wedge_{B \times C} (a_2, b_2) := (a_1 \wedge a_2, b_1 \wedge b_2),$$

and likewise for $\vee, \neg, 0, 1$.

2.2.1 Representation of finite BAs

Definition 2.2.13. Let B be a Boolean algebra.

1. An element $a \neq 0$ is called and *atom* if for each $b \in B$ we have that $b < a$ implies $b = 0$.

2. An element $a \neq 1$ is called and *co-atom* if for each $b \in B$ we have that $b > a$ implies $b = 1$.

For a Boolean algebra B let $At(B) = \{x \in B : x \text{ is an atom}\}$.

Proposition 2.2.14. Let B be a finite BA. Then for each $a \in B$ we have

$$a = \bigvee \{x \in At(B) : x \leqslant a\}.$$

Proof. Let $S = \{x \in At(B) : x \leqslant a\}$. Then a is an upper bound of S. We will show that it is the least upper bound of S. Let b an upper bound of S. We show that $a \leqslant b$. Suppose otherwise, then $a \nleqslant b$. By Proposition 2.2.2(v) $a \wedge \neg b > 0$. Let x be an atom such that $x \leqslant a \wedge \neg b$. Then $x \leqslant a$ and $x \leqslant \neg b$. The former implies that $x \in S$, and so $x \leqslant b$. But then $x \leqslant b$ and $x \leqslant \neg b$. Therefore, $x \leqslant b \wedge \neg b = 0$, which is a contradiction. Thus, $a \leqslant b$ and $a = \bigvee S$. ∎

Definition 2.2.15. A Boolean algebra B is called *atomic* if each $a \in B$ is a join of atoms.

Thus, Proposition 2.2.14 states that every finite BA is atomic.

Proposition 2.2.16. Every finite BA B is isomorphic to the powerset BA $\mathcal{P}(X)$ for some finite set X.

Proof. Let B be a finite BA and let $X = At(B)$. Define $\eta : B \to \mathcal{P}(X)$ by

$$\eta(a) = \{x \in At(B) : x \leqslant a\}.$$

We show that η is an isomorphism. First assume $a, b \in B$ are such that $a \neq b$. If $\eta(a) = \eta(b)$. Then by Proposition 2.2.14, $a = \bigvee \eta(a) = \bigvee \eta(b) = b$, which is a contradiction. So η is injective. For surjectivity, assume $S \subseteq X$. Let $a = \bigvee S$. We show that $\eta(a) = S$. Since a is an upper bound of S we have $S \subseteq \eta(a)$. Now suppose $x \in \eta(a)$ and let $S = \{a_1, \ldots, a_n\}$. Consider two cases:

1. $x \wedge a_i = 0$, for each $i = 1, \ldots, n$.
2. $x \wedge a_i > 0$, for some $i = 1, \ldots, n$.

Then in Case 1, using distributivity:

$$x = x \wedge a = x \wedge (a_1 \vee \cdots \vee a_n) = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n) = 0.$$

A contradiction. So $x \wedge a_i > 0$, for some $i = 1, \ldots, n$. However, as x and a_i are atoms this is only possible if $x = a_i$. Therefore, $x \in S$.

The verifications that $\eta(0) = \varnothing$, $\eta(1) = X$, $\eta(a \wedge b) = \eta(a) \cap \eta(b)$ and $\eta(a \vee b) = \eta(a) \cup \eta(b)$ are all routine and left as an exercise. ∎

However, there exist infinite BAs without any atoms (e.g., $\mathcal{RO}(\mathbb{R})$, see the tutorial sheet).

2.2.2 Special filters and ideals in Boolean algebras

In section 2.1.2 we have studied filters and ideals for arbitrary lattices and distributive lattices. It turns out that for Boolean algebras the situation is quite a bit more simple.

Lemma 2.2.17. $F \subseteq B$ is a filter if and only if $\neg F := \{\neg a \mid a \in F\}$ is an ideal.

Proof. Assume that $b \in \neg F$, and $c \leqslant b$. Hence $b = \neg a$ for $a \in F$. By Proposition 2.2.2, $c \wedge \neg\neg a = 0$, so using $\neg\neg a = a$, $a \leqslant \neg c$. Then $\neg c \in F$ because F is an upset, so $c \in \neg F$. This shows that $\neg F$ is downwards closed.

Similarly, if $b, d \in \neg F$, then $b = \neg a$ and $d = \neg c$, and $a, c \in F$. Since F is a filter, $a \wedge c \in F$, so $\neg(a \wedge c) = \neg a \vee \neg c \in \neg F$.

The other direction is similar. ∎

Definition 2.2.18 (Maximal Filters in Lattices)**.** Let L be a lattice. We say that a filter $F \subseteq L$ is *maximal* if F is proper and for each $G \supseteq F$ a proper filter, $F = G$.

Definition 2.2.19 (Ultrafilter)**.** Let B be a Boolean algebra. We say that $U \subseteq B$ a proper filter is an *ultrafilter* if for all $a \in B$ either $a \in U$ or $\neg a \in U$.

Theorem 2.2.1. *Let B be a Boolean algebra. Let $F \subseteq B$ be a proper filter. Then the following are equivalent:*

1. *F is maximal.*
2. *F is prime.*
3. *F is an ultrafilter.*

Proof. 1.⇒2. Suppose $a \vee b \in F$ and $a \notin F$. Let us prove that $b \in F$. Consider the set

$$F_a := \uparrow\{a \wedge c : c \in F\} = \{x \in B : x \geqslant a \wedge c \text{ for some } c \in F\}.$$

F_a is a filter: it is upward closed by definition and let $e, d \in F_a$, i.e. $e \geqslant a \wedge c'$, $d \geqslant a \wedge c''$. Then

$$e \wedge d \geqslant (a \wedge c') \wedge (a \wedge c'') = a \wedge (c' \wedge c''),$$

hence $e \wedge d \in F_a$. Also, $F \subseteq F_a$, for, if $c \in F$, then $c \geqslant a \wedge c$, hence $c \in F_a$. Since F is maximal, and $F \neq F_a$ (because $a \notin F$ but $a = a \wedge 1 \in F_a$), we have $F_a = B$. In particular, $0 \in F_a$. But then $0 = a \wedge c$ for some $c \in F$. Since $a \vee b \in F$, we have

$$F \ni (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = 0 \vee (b \wedge c) = b \wedge c.$$

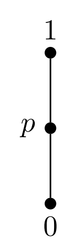
As $b \wedge c \leqslant b$, we obtain that $b \in F$.

2.⇒3. For every filter, $1 = a \vee \neg a \in F$. As F is prime, $a \in F$ or $\neg a \in F$.

3.⇒1. Let F' be a filter such that $F' \supsetneq F$. There is an element $a \in F' \setminus F$. But then $\neg a \in F$ and consequently, $\neg a \in F'$. But then $0 = a \wedge \neg a \in F'$, which means that $F' = B$. ■

This is particular to Boolean algebras:

Example 2.2.20. Consider a three element chain



and notice that in this lattice $\{1\}$ is a prime filter, for if $a \vee b = 1$, then $a = 1$ or $b = 1$. But it is not maximal, for $\{p, 1\}$ is a proper filter.

Putting all of this together, we can obtain a version of the Prime Filter Theorem (Theorem 2.1.3) that is more common for Boolean algebras:

Theorem 2.2.2 (Ultrafilter Theorem)**.** *Let B be a Boolean algebra, and $F \subseteq B$ a proper filter. Then there exists an ultrafilter $U \supseteq F$.*

Proof. By Lemma 2.2.17, since F is a proper filter, $\neg F$ is an ideal, and certainly $F \cap \neg F = \varnothing$: if $a \in F$ and $a \in \neg F$ then $\neg a \in F$, so $a \wedge \neg a = 0 \in F$, meaning by upwards closure that F is not proper. Hence by the Prime Filter Theorem, there is a prime filter $U \supseteq F$; by Theorem 2.2.1, U is an ultrafilter, as desired. ■

Example 2.2.21. Consider again Example 2.2.3 iv. Given a subset S in this algebra, let S^+ be the subset of the set of formulas containing all formulas in all equivalence classes $[\varphi] \in S$. Then note that S is an upset if and only if S^+ is deductively closed. Moreover, S is a maximal filter if and only if S^+ is a maximally consistent set. In this sense, Theorem 2.2.2 is a general version of *Lindenbaum's Lemma.*

2.2.3 Stone duality

In the previous section we have been introduced to the basic theory of Boolean algebras, and to representations of these in the finite case. Now we will push the situation to account for all Boolean algebras, and in addition, introduce a class of spaces which correspond precisely to Boolean algebras.

Definition 2.2.22. A *Stone space* is a totally disconnected, compact Hausdorff space, or, equivalently [2], a compact, Hausdorff space with a basis of clopens.

Let B be a BA. We call $X_B := \{\text{all maximal filters of } B\}$, the *spectrum* of B. It is sometimes denoted by $\mathrm{Spec}(B)$; we will often call it the *dual* of B.

Consider the map $\varphi : B \to \mathcal{P}(X_B)$, defined as

$$\varphi(a) = \{F \in X_B : a \in F\}.$$

Proposition 2.2.23. For each Boolean algebra B, φ is an injective homomorphism (an embedding of B into $\mathcal{P}(X_B)$).

Proof. That φ is a homomorphism follows from the following:

$$\begin{aligned}
\varphi(1_B) &= \{F \in X_B : 1 \in F\} = X_B = 1_{\mathcal{P}(X_B)} \\
\varphi(0_A) &= \{F \in X_B : 0 \in F\} = \varnothing = 0_{\mathcal{P}(X_B)} \\
\varphi(\neg a) &= \{F \in X_B : \neg a \in F\} = \{F \in X_B : a \notin F\} = X_B \backslash \varphi(a) = \neg_{\mathcal{P}(X_B)} \varphi(a) \\
\varphi(a \wedge b) &= \{F \in X_B : a \wedge b \in F\} = \{F \in X_B : a \in F \;\&\; b \in F\} \\
&= \{F \in X_B : a \in F\} \cap \{F \in X_B : b \in F\} = \varphi(a) \cap \varphi(b) = \varphi(a) \wedge_{\mathcal{P}(X_B)} \varphi(b) \\
\varphi(a \vee b) &= \{F \in X_B : a \vee b \in F\} = \{F \in X_B : a \in F \text{ or } b \in F\} \\
&= \{F \in X_B : a \in F\} \cup \{F \in X_B : b \in F\} = \varphi(a) \cup \varphi(b) = \varphi(a) \vee_{\mathcal{P}(X_B)} \varphi(b).
\end{aligned}$$

It is injective: if $a \neq b$, then either $a \nleq b$ or $b \nleq a$. Assume without loss of generality the former. Then $\uparrow a \cap \downarrow b = \varnothing$. Since $\uparrow a$ is a filter and $\downarrow b$ is an ideal, by the Prime Filter Theorem (Theorem 2.1.3) there is a maximal filter F' extending $\uparrow a$ that has an empty intersection with $\downarrow b$. In particular $b \notin F \ni a$, hence $\varphi(a) \ni F \notin \varphi(b)$. ■

[2] See [1, Proposition 6.2.5] for a proof of this equivalence. The property of being Hausdorff with a basis of clopens is equivalent to the following: whenever $x \neq y$ there is a clopen U such that $x \in U$ and $y \notin U$. This property is what is commonly referred to in the literature as being *totally separated.*

The set $\{\varphi(a) : a \in B\}$ contains X_B and is closed under finite intersections, hence it forms the basis of a topology on X_B. Moreover, it is Hausdorff, for if $F_1 \neq F_2$, take, without loss of generality, $a \in F_1 \backslash F_2$, and we have $F_1 \in \varphi(a)$, $F_2 \in X \backslash \varphi(a) = \varphi(\neg a)$.

This space is called the *Stone space of B* or the *dual of B*, because of the following:

Proposition 2.2.24. Let B be a BA. Then X_B is a Stone space.

Proof. We already showed above that X_B is Hausdorff with a basis of clopens. It remains to prove that it is compact. Let $X_B = \bigcup_{i \in I} U_i$, where each U_i is clopen. Then $X_B = \bigcup_{a \in \Lambda} \varphi(a)$, and let

$$F := \{b \in B : b \geqslant \neg a_1 \wedge \ldots \wedge \neg a_n \text{ for some } a_1, \ldots, a_n \in \Lambda\}.$$

(By Lemma 2.1.23, this is the filter generated by $\{\neg a : a \in \Lambda\}$.)

If $F \subsetneq B$, i.e, F is a proper filter, then by Ultrafilter Theorem (Theorem 2.2.2) we can find U an ultrafilter extending F. By assumption, $U \in X_B$, so $U \in \varphi(a)$ for some $a \in \Lambda$. But then $a \in U$, and since by construction $\neg a \in U$, this would mean that $0 \in U$, contradicting the fact that U is proper. So we can assume that $F = B$, i.e. $0 \in F$; this means that $0 = \neg a_1 \wedge \ldots \wedge \neg a_n$ for some $a_1, \ldots, a_n \in \Lambda$. Hence

$$1 = \neg(\neg a_1 \wedge \ldots \wedge \neg a_n) = a_1 \vee \ldots \vee a_n,$$

and therefore $X_A = \varphi(1) = \varphi(a_1 \vee \ldots \vee a_n) = \varphi(a_1) \cup \ldots \cup \varphi(a_n)$. Thus, we found a finite sub-covering and X_B is compact. ∎

Knowing that X_B is a Stone space allows us to know the exact structure of the clopens of this space, and to know that they are exactly of the shape we wanted:

Proposition 2.2.25.

1. Given a Stone space X, the structure $(\mathsf{Clop}(X), \cap, \cup, \backslash, \varnothing, X)$ is a Boolean algebra.

2. Every clopen set U in X_B is of the form $\varphi(a)$ for some $a \in A$. So $\mathsf{Clop}(X_B) = \{\varphi(a) : a \in B\}$, and $\varphi : B \to \mathsf{Clop}(X_B)$ is an isomorphism.

Proof. (1) Exercise.

(2) Proposition 2.2.23 shows that φ is an embedding; we only need to show that it is surjective. Let U be an arbitrary clopen. Since U is open, $U = \bigcup_{a \in \Lambda} \varphi(a)$. But since it is also a closed subset of a compact space, it is compact, hence, for some $a_1, \ldots, a_n \in \Lambda$,

$$U = \varphi(a_1) \cup \ldots \cup \varphi(a_n) = \varphi(a_1 \vee \ldots \vee a_n).$$

∎

We thus obtain the following:

Corollary 2.2.26 (Stone representation theorem). *Every Boolean algebra B is isomorphic to $\mathsf{Clop}(X_B)$ for some Stone space X_B.*

Remark 2.2.1. If X is a finite Stone space, then $\{x\}$ is closed for all $x \in X$, hence every set is closed, thus every set is clopen, i.e. $\mathsf{Clop}(X) = \mathcal{P}(X)$.

As a result we obtain that if B is a finite Boolean algebra, as B has only finitely many prime filters, $B \simeq \mathsf{Clop}(X) = \mathcal{P}(X)$. Moreover, note that if F is a prime filter in B, it is maximal, and principal, generated by a; hence we must have that a is an atom. Thus, we recover the representation from section 2.2.1 as a special case.

One can ask whether every Stone space arises this way. This is indeed true:

Theorem 2.2.3. *Let X be a Stone space. Then $X \cong X_{\mathsf{Clop}(X)}$.*

Proof. See Exercise 2.16 for an extended outline of the proof. ∎

More than simply allowing us to swap between the algebraic and spatial perspective, the connection between Boolean algebras and Stone spaces also allows us to transfer *morphisms*.

Proposition 2.2.27. Let X and X' be Stone spaces and $f : X \to X'$ be a continuous map. Then $f^* : \mathsf{Clop}(X') \to \mathsf{Clop}(X)$, defined by

$$f^*(U) := f^{-1}(U) = \{x \in X : f(x) \in U\},$$

is a BA-homomorphism.

Proof. Exercise. ∎

Proposition 2.2.28. Let $h : A \to B$ be a Boolean algebra homomorphism. Then $h_* : X_B \to X_A$, defined by

$$h_*(x) := h^{-1}(x) = \{a \in A : ha \in x\},$$

is a well-defined continuous map.

Proof. Let us see that it is well defined, i.e., that $h^{-1}x$ is a prime filter for $x \in X_A$.

- If $a \in h^{-1}x$ and $a \leqslant b$, then $a \wedge b = a$, hence $ha = h(a \wedge b) = ha \wedge hb$, hence $ha \leqslant hb$. So if $ha \in x$, $ha \leqslant hb$, then $hb \in x$ and hence $b \in h^{-1}x$.
- If $a, b \in h^{-1}x$, then $ha, hb \in x$, and since x is a filter, $h(a \wedge b) = ha \wedge hb \in x$, hence $a \wedge b \in h^{-1}x$.
- If $a \vee b \in h^{-1}x$, then $ha \vee hb = h(a \vee b) \in x$, and, as x is prime, $ha \in x$ or $hb \in x$, i.e. $a \in h^{-1}x$ or $b \in h^{-1}x$.

Let us see that it is continuous. It suffices to see that $(h_*)^{-1}(U) \in \mathsf{Clop}(X_B)$ for $U \in \mathsf{Clop}(X_A)$. But we showed that $\varphi : A \simeq \mathsf{Clop}(X_A)$, so for $U \in \mathsf{Clop}(X_A)$ there is $a \in A$ such that $\varphi(a) = U$. But then (recall that $x \in \varphi(a)$ iff $a \in x$):

$$\begin{aligned}(h_*)^{-1}(\varphi(a)) &= \{x \in X_B : h^{-1}x \in \varphi(a)\} = \{x \in X_B : a \in h^{-1}x\} \\ &= \{x \in X_B : ha \in x\} = \{x \in X_B : x \in \varphi(ha)\} = \varphi(ha) \in \mathsf{Clop}(X_B).\end{aligned}$$

∎

Let us consider some interesting examples that will pop up several times below:

Example 2.2.29.

1. *The Alexandroff compactification $\alpha\mathbb{N}$ of $\mathbb{N}$:*
   Consider the BA $A = \mathrm{FinCofin}(\mathbb{N})$. We denote by $\alpha\mathbb{N}$ the dual Stone space X_A. Then $\alpha\mathbb{N} = \mathbb{N} \cup \{\omega\}$, where ω is the unique non-principal ultrafilter of A. It is easy to see (exercise) that clopen sets of $\alpha\mathbb{N}$ are finite subsets of $\mathbb{N}$ and cofinite subsets of $\mathbb{N}$ together with the point ω.

2. *The Stone-Cech compactification $\beta\mathbb{N}$ of $\mathbb{N}$:*
   Consider the BA $B = \mathcal{P}(\mathbb{N})$. We denote by $\beta\mathbb{N}$ the dual Stone space X_B. Then $\beta\mathbb{N}$ is an extremally disconnected Stone space (see the end of the next Section).

2.2.4 Duals of constructions

In light of the previous sections, we have found that any concept which involves Boolean algebras, and maps between them, ought to correspond to some concept between Stone spaces. In this section we will study some basic examples of these sorts of correspondences.

Our first place of consideration will be *filters*. As we saw in Theorem 2.2.11, filters are in 1-1 correspondence with homomorphic images of Boolean algebras. Our dual representation will allow us a faster route between these two. Denote by $\mathsf{Fil}(B)$ the set of filters of a Boolean algebra B, and by $\mathsf{Closed}(X)$ the set of closed sets of a Stone space X. Also note that a closed subspace of a Stone space is again a Stone space (check this!):

Theorem 2.2.4. *Let B be a Boolean algebra. There is a dual isomorphism, $\xi : (\mathsf{Fil}(B), \subseteq) \to (\mathsf{Closed}(X_B), \subseteq)$ with the following property: for each $F \in \mathsf{Fil}(B)$ if $C = B/F$ then $X_C \cong \xi(F)$.*

Proof. For a filter F, define $\xi(F) = \bigcap\{\varphi(a) : a \in F\}$. Note that ξ is order-reversing, and injective by the prime filter theorem. Moreover, it is surjective: given C a closed set, write $C = \bigcap_{a \in G} \varphi(a)$ where $C \subseteq \varphi(a)$ for any $a \in G$. Then G is a filter in B, and $\xi(G) = C$ by definition. It is easy to see that ξ is order-reversing, and that it reflects the order appropriately.

Now assume that $C = B/F$, and let $q_F : B \to B/F$ be the canonical surjection. By Proposition 2.2.28 we have a homomorphism $q_F^{-1} : X_C \to X_B$. Moreover, because q_F is surjective, this map is injective: given $x \neq y$, let $\varphi([a]_F)$ separate them. Then $[a]_F \in x$ and $[a]_F \notin y$. Thus $a \in q_F^{-1}[x]$ and $a \notin q_F^{-1}[y]$ by construction of the map; hence $q_F^{-1}[x] \in \varphi(a)$ and $q_F^{-1}[y] \notin \varphi(a)$, showing that the images are distinct. Being a continuous map between Stone spaces, it will be an embedding. Hence $X_C \cong q_F^{-1}[X_C]$, which is a closed subspace of X_B. It suffices to show that $q_F^{-1}[X_C] = \xi(F)$.

Indeed, if $x \in q_F^{-1}[X_C]$, then for some $y \in X_C$, $q_F^{-1}[y] = x$. Let $a \in F$. Then $[a]_F \in y$, given the definition of the quotient, and so $a \in x$ by construction; hence $x \in \xi(F)$. Conversely if $x \in \xi(F)$, consider:

$$y = \{[a]_F : a \in x\}.$$

We can check that this is a prime filter over B/F: $[1]_F \in y$; if $[a]_F, [b]_F \in y$, then for some $c \in [a]_F$ and $d \in [b]_F$, $c, d \in x$. Then $c \wedge d \in x$, and using the equivalences, $a \wedge b \in x$ (since $[a \wedge b]_F = [c \wedge d]_F$, and so the equivalence of the two belongs to the filter, using that $x \in \xi(F)$); hence $[a \wedge b]_F \in y$; if $[a]_F \in y$ and $[a]_F \leqslant [b]_F$, then by definition, $(\neg a \vee b) \in F$, so $\neg a \vee b \in x$; also $a \in x$ so $b \in x$, meaning that $[b]_F \in x$. Finally it is clear that y will be an ultrafilter: if $[a]_F$ is arbitrary, and $a \notin x$, then $\neg a \in x$, so $\neg [a]_F = [\neg a]_F \in y$. It is clear to see that $q_F^{-1}[y] = x$, showing that $x \in q_F^{-1}[X_C]$. We conclude that $q_F^{-1}[X_C] = \xi(F)$, proving the result. ∎

Denote by $\mathsf{Id}(B)$ the set of ideals of a Boolean algebra, by $\mathsf{Open}(X)$ the set of open subsets of a Stone space X. From the former we get:

Corollary 2.2.30. *Let B be a Boolean algebra. There is an isomorphism, $\xi : (\mathsf{Id}(B), \subseteq) \to (\mathsf{Open}(X_B), \subseteq)$.*

Proof. By Lemma 2.2.17 there is an isomorphism between $\mathsf{Fil}(B)$ and $\mathsf{Id}(B)$, and by usual set-theoretic considerations, there is a dual isomorphism between $\mathsf{Closed}(X_B)$ and $\mathsf{Open}(X_B)$. Together with the dual isomorphism from Theorem 2.2.4 this gives us the result. ∎

The explicit form of the isomorphism is given by:

$$\xi(I) = \bigcup\{\varphi(a) : a \in I\}$$

for I an ideal. As a consequence of this we get that for each $x \in X_B$, the same way that x corresponds to a maximal filter, $X_B - \{x\}$ corresponds to a maximal ideal, indeed, to the maximal ideal $B - x$.

A similar dual presentation is possible for subalgebras:

Theorem 2.2.5. *Given $f : B \to C$ a Boolean homomorphism, f is an embedding if and only if $f^{-1} : X_C \to X_B$ is a surjective continuous map.*

Proof. Assume first that $f : B \to C$ is an embedding. Let $x \in X_B$ be a prime filter. Consider:

$$y = \{f(a) : a \in x\}.$$

It is not hard to see that y is a filter in C. We check that it is proper: if $f(a) = 0$ then $f(a) = f(0)$, so $a = 0$ because f is an embedding; but then $a \notin x$. The fact that y is prime is then equally straightforward.

Then note that $f^{-1}[y] = x$, again because f is an embedding. This shows that f^{-1} is surjective, as desired.

Conversely, assume that f^{-1} is surjective. Suppose that $a \neq b$. We want to show that $f(a) \neq f(b)$. Indeed, let $x \in \varphi(a) - \varphi(b)$; let y be such that $f^{-1}[y] = x$. Then $f(a) \in y$, since $a \in f^{-1}[y]$, but $f(b) \notin y$, otherwise $b \in x$. This shows that $f(a) \neq f(b)$, and hence that f is an embedding. ∎

We now turn to the dual of products of Boolean algebras. This admits a particularly simple representation:

Theorem 2.2.6. *Let B_1, B_2 be two Boolean algebras. Then $X_{B_1 \times B_2} \cong X_{B_1} \sqcup X_{B_2}$.*

Proof. Given an ultrafilter x of $B_1 \times B_2$ we claim that either $\pi_{B_1}[x]$ or $\pi_{B_2}[x]$ is a proper filter; and whichever is a proper filter will then be an ultrafilter. Indeed, the fact that they are filters is easy to check; assume without loss of generality that $0 \in \pi_{B_1}[x]$, i.e., there is some $a \in C$ such that $(0, a) \in x$; if also $0 \in \pi_{B_2}[x]$, so there is some $b \in B$ such that $(b, 0) \in x$; then by closure under meets, $(0, 0) \in x$, a contradiction. So $0 \notin \pi_{B_2}[x]$. Moreover if $a \notin \pi_{B_2}[x]$, then $(0, a) \notin x$, so by assumption on x being an ultrafilter, $(1, \neg a) \in x$, hence $\neg a \in \pi_{B_2}[x]$. It is likewise impossible that both $\pi_{B_1}[x]$ and $\pi_{B_2}[x]$ are both proper filters, since if $\pi_{B_1}[x]$ is an ultrafilter, then $(1, 0) \in \pi_{B_1}[x]$, and so $0 \in \pi_{B_2}[x]$.

Hence, for each x, there is a unique factor $i \in \{1, 2\}$ such that $\pi_{B_i}[x]$ is an ultrafilter. Let $p : \mathsf{Spec}(B_1 \times B_2) \to \{1, 2\}$ be a function making that selection. Then define an isomorphism:

$$\begin{aligned} i : i : X_{B \times C} &\to X_B \sqcup X_C \\ x &\mapsto \pi_{p(x)}[x]. \end{aligned}$$

The above argument shows that this function is well-defined. It is clearly surjective, since for example given $x \in X_{B_i}$ we can consider $F = \{(a, 0) \in B_1 \times B_2 : a \in x\}$, and use the Ultrafilter theorem (Theorem 2.2.2) to extend it in $B \times C$ to an ultrafilter U such that $\pi_1[U] = x$. It is injective, since if $i(x) = i(y)$, say without loss of generality, $i(x) = \pi_1[x] = \pi_1[y]$, then given any $b \in B_2$, if $(a, b) \in x$, then either $(0, \neg b)$ or $(1, b) \in y$; since $0 \notin \pi_1[y]$, then $(1, b) \in y$, and since $a \in \pi_1[x]$, then $(a, 1) \in y$ as well, showing that $(a, b) \in y$. Hence $x \subseteq y$, which since these are maximal filters, means that $x = y$. Finally, note that i is continuous: a clopen of $B_1 \sqcup B_2$ corresponds to a disjoint union of a clopen from B_1 and a clopen from B_2, of the form $\varphi(a) \sqcup \varphi(b)$; it is not hard to see that $i^{-1}[\varphi(a) \sqcup \varphi(b)] = \varphi(a \times b)$. This shows that i is continuous, and since the two spaces are compact Hausdorff, i is a homeomorphism as desired. ∎

We finish this section by mentioning one more use of duality: to connect distinct concepts with relevance both in the field of algebra and in the field of topology. This can be seen in the case of *atoms* and in the case of *complete Boolean algebras*, as well as related concepts.

Definition 2.2.31. Let B be a Boolean algebra. We say that B is *superatomic* if whenever C is a homomorphic image of B, then C is atomic.

Theorem 2.2.7. *Let B be a Boolean algebra. Then:*

1. *For each $a \in B$, a is an atom if and only if $\varphi(a) = \{x\}$ where x is an isolated point.*
2. *B is complete if and only if X_B is extremally disconnected.*
3. *B is atomic if and only if the set of isolated points of X_B is dense.*
4. *B is atomless if and only if X_B is dense-in-itself.*
5. *B is superatomic if and only if X_B is scattered.*
6. *B is complete and atomic if and only if $X_B \cong \beta(Y)$ for Y a discrete topological space.*

Proof. Exercise. ■

In light of Theorem 2.2.7(5), we can see the connection between Stone duality and the Tarski duality between **CABA** and **Set** (see Exercise 2.17): for a Boolean algebra of the form $\mathcal{P}(X)$, the Tarski dual of it will be precisely X, whilst the Stone dual will be $\beta(X)$ [3].

Extra Content

The following section is intended for the categorically minded reader with an interest in seeing duality presented categorically.

2.2.5 ◊ A categorical view of Stone duality

The results of the previous sections give us a vague notion of what is meant by a *duality*. The reader familiar with category theory may wonder whether the subject can give some crispness to these intuitions. The purpose of this section is to discuss things from this point of view.

We denote by **BA** be the category of Boolean algebras and Boolean algebra homomorphisms and by **Stone** the category of Stone spaces and continuous functions. Given two categories $\mathbf{C}, \mathbf{D}$ we write $\mathbf{C} \simeq \mathbf{D}$ to mean that the two categories are *equivalent*. The goal of this section will be to show the following:

Theorem 2.2.8. *We have that:*

$$\mathbf{BA}^{op} \simeq \mathbf{Stone}.$$

[3] The reasons for this are in fact fairly deep and have to do with the way in which **CABA** and **BA** arise as completions of the category $\mathbf{BA}_{fin}$, on one hand, and some adjunctions holding between these categories.

For this purpose, define a contravariant functor:

$$\begin{aligned} \mathsf{Spec} : \mathbf{BA} &\to \mathbf{Stone} \\ B &\mapsto X_B \\ f : B \to C &\mapsto f^{-1} : X_C \to X_B. \end{aligned}$$

We will show that this functor is an equivalence of categories; for that purpose we will show:

Proposition 2.2.32. The functor Spec is full, faithful and essentially surjective.

Proof. Essential surjectivity follows from Theorem 2.2.3. For fullness, assume that $f : X_C \to X_B$ is a continuous map. Then $f^{-1} : \mathsf{Clop}(X_B) \to \mathsf{Clop}(X_C)$ is a Boolean algebra homomorphism. Consider the homomorphism $h : B \to C$ defined as follows:

$$h(a) := \varphi_C^{-1}[f^{-1}[\varphi_B(a)]];$$

Note that then $h^{-1} : X_C \to X_B$ maps $x \in X_C$ to $h^{-1}[x] = \{a \in B : h(a) \in x\}$; we want to show that $h^{-1}[x] = f(x)$. Assume that $a \in f(x)$, then $f(x) \in \varphi_B(a)$, so $x \in f^{-1}[\varphi_B(a)]$, and so $\varphi_C^{-1}[f^{-1}[\varphi_B(a)]] \in x$ (using the fact that φ is an isomorphism); the converse direction is similar. Hence $\mathsf{Spec}(h) = f$.

Now if $h \neq g$, then for some $a \in B$, $h(a) \neq g(a)$; we can construct some $x \in h(a) - g(a)$, and then $a \in h^{-1}[x]$ and $a \notin g^{-1}[x]$, i.e., $\mathsf{Spec}(h) \neq \mathsf{Spec}(g)$. ■

The pseudo-inverse of this functor is of course the contravariant functor:

$$\begin{aligned} \mathsf{Clop} : \mathbf{Stone} &\to \mathbf{BA} \\ X &\mapsto \mathsf{Clop}(X) \\ f : X \to Y &\mapsto f^{-1} : \mathsf{Clop}(Y) \to \mathsf{Clop}(X). \end{aligned}$$

The fact that Spec is a dual equivalence means that Clop is also a dual equivalence. The natural transformations which witness this are the maps φ, defined in the previous sections, and ε, discussed in Exercise 2.16.

Having a categorical point of view, the results from section 2.2.4 are to be expected, and have a principled explanation:

1. Surjective homomorphisms between Boolean algebras correspond to epimorphisms; hence by duality, they will correspond via Spec to monomorphisms. Monomorphisms of Stone spaces are exactly the embeddings. Hence surjective homomorphisms will correspond to closed subspaces of a Stone space.

2. Injective homomorphisms of Boolean algebras are monomorphisms in **BA**. They will correspond via Spec to epimorphisms of Stone spaces. These are exactly the surjective homomorphisms (since the spaces are compact Hausdorff).

3. The product of Boolean algebras is the categorical product in **BA**. Hence its dual will be the coproduct of **Stone**.In the binary case, this turns out to be the topological sum of Stone spaces.

A lot of facts concerning duality can be obtained using categorical intuition. This perspective also allows us to be careful about operating the categories at hand. For example, note that whilst the I-fold product of Boolean algebras makes sense – one often denotes it, for a family $(B_i)_{i\in I}$ as $\prod_{i\in I} B_i$ – the dual of it will *not* be the topological sum of Stone spaces, $\bigsqcup_{i\in I} X_{B_i}$. This is because it is not difficult to see that the topological sum of infinitely many Stone spaces can never be compact. Nevertheless, **Stone** is a cocomplete category – the dual is the *Stone-Cech compactification* of the topological sum, $\beta(\bigsqcup_{i\in I} X_{B_i})$ [4].

We conclude this section by looking at a construction which is dramatically simplified by the presence of duality theory and a categorical mindset: the *free product of Boolean algebras*.

Definition 2.2.33. Let B_1 and B_2 be two Boolean algebras. We define the *free product of* B_1, B_2 as follows: let $\mathsf{Form}(B_1 + B_2)$ be the set of classical logic formulas with variables c_a for each $a \in B_1$ or $a \in B_2$. Let Γ denote the *theory* of B_1, B_2, i.e., it contains all the following formulas:

$$\begin{aligned}
&\{c_{1_{B_1}} \leftrightarrow \top, c_{1_{B_2}} \leftrightarrow \top, c_{0_{B_1}} \leftrightarrow \bot, c_{0_{B_2}} \leftrightarrow \bot\} \\
&\{c_{\neg a} \leftrightarrow \neg c_a : a \in B_1 \sqcup B_2\} \\
&\{c_{a\wedge b} \leftrightarrow c_a \wedge c_b, c_{a\vee b} \leftrightarrow c_a \vee c_b : a, b \in B_1 \sqcup B_2\}.
\end{aligned}$$

Now define an equivalence on $\mathsf{Form}(B_1 + B_2)$, $\approx$ as follows: for φ, ψ formulas there:

$$\varphi \approx \psi \iff \Gamma \vdash \varphi \leftrightarrow \psi.$$

Let $B_1 \oplus B_2$ be the quotient Boolean algebra $\mathsf{Form}(B_1 + B_2)/\approx$.

The following is not very difficult to prove – it follows from algebraic consideration of the definitions:

Proposition 2.2.34. Given Boolean algebras B_1, B_2, the algebra $B_1 \oplus B_2$ is the categorical coproduct of B_1, B_2 in the category of Boolean algebras with Boolean algebra homomorphisms.

Proof. Exercise. ∎

Nevertheless, one can find a much easier and more concrete description:

Theorem 2.2.9. *Let B_1, B_2 be two Boolean algebras. Then $\mathsf{Clop}(X_{B_1} \times X_{B_2})$ is the categorical coproduct of B_1, B_2.*

Proof. By duality, it suffices to show that $X_{B_1} \times X_{B_2}$ is the product in the category of Stone spaces; first note that it is indeed a Stone space. The proof that it is the product of X_{B_1} and X_{B_2} is simply the usual proof showing that cartesian products are categorical products. ∎

End of extra content.

[4] See [1, pp.52-55].

2.3 Distributive lattices and Priestley duality

Just like we were able to represent Boolean algebras, the case of distributive lattices turns out to be only slightly more complicated. We follow the structure of the previous chapter, first outlining some basic constructions of distributive lattices, then proceeding with the representation of finite distributive lattices, before finally turning to *Priestley duality*.

Recall that given a distributive lattice D, a *bounded sublattice* D' is a subset containing $0, 1$ which is closed under $\wedge$ and $\vee$. The following is similar to Lemma 2.2.7:

Lemma 2.3.1. Let D be a bounded distributive lattice. Then:

1. Any sublattice of D is distributive.
2. If $S \subseteq D$ is a subset, there is a smallest bounded sublattice containing S, denoted by $\langle S \rangle$.

Proof. Exercise. ∎

Definition 2.3.2. Let $f : D \to D'$ be a homomorphism. We say that D' is a *homomorphic image* of B if f is surjective.

Unlike the case of Boolean algebras, *not every distributive lattice homomorphic image corresponds to a filter*:

Example 2.3.3. Consider the following homomorphism: This is a lattice homomorphism, and if

$$\begin{tikzcd}
1 \arrow[r, dotted] & 1 \\
a \arrow[u] \arrow[r, dotted] & 0 \arrow[u] \\
0 \arrow[u] \arrow[ur, dotted] &
\end{tikzcd}$$

one considers $F = f^{-1}[1] \subseteq D$, this is a filter containing only $\{1\}$. On the other hand, the identity map $i : D \to D$ also forms the same filter. Hence the same filter can correspond to two distinct homomorphic images.

Later we will see that this situation arises because of the lack of the $\to$ connective. For now we turn to one final basic construction of distributive lattices:

Definition 2.3.4. Let D, D' be two distributive lattices. We define the *product* of D and D', denoted by $D \times D'$ as follows: for each operation we define it pointwise, e.g.:

$$(a_1, b_1) \wedge_{D \times D'} (a_2, b_2) := (a_1 \wedge a_2, b_1 \wedge b_2),$$

and likewise for $\vee, 0, 1$.

2.3.1 Representation of finite distributive lattices

Recall from Lemma 2.1.21 that given a distributive lattice D an element $a \in D$ is called *join-irreducible* if whenever $a = b \vee c$ then $a = b$ or $a = c$, and that this coincides for distributive lattices with being *join-prime*. It is clear from examples like Example 2.1.22 that atoms will not be sufficient to represent finite distributive lattices. But as it turns out, adding a bit more structure – an order – we can represent all distributive lattices. This will also give us occasion to study some more specific kinds of elements in distributive lattices:

Definition 2.3.5. Let P be a poset. We denote by $(\mathsf{Up}(P), \subseteq)$ the set of upsets of P ordered by inclusion.

In many ways, the set of upsets of a poset acts like the powerset of a regular set – indeed, if X is a set, and we equip it with the discrete order (i.e., $x \leqslant y$ if and only if $x = y$), the two will coincide. This also obtains us the following:

Definition 2.3.6. Let D be a distributive lattice. An element $a \in D$ is called *completely join-prime* if for each subset $S \subseteq D$, if $a \leqslant \bigvee S$, then for some $s \in S$, $a \leqslant s$.

It is clear that if D is a finite distributive lattice, then an element $a \in D$ is completely join-prime if and only if a is join-prime if and only if a is join-irreducible. Given any distributive lattice D, denote by $\mathcal{J}(D)$ the set of join-irreducible elements of D, and by $\mathcal{J}^{\infty}(D)$ the set of completely join-irreducible elements of D.

Definition 2.3.7. A distributive lattice D is called *completely join-prime generated* if each $a \in D$ is of the form

$$a = \bigvee \{s \in \mathcal{J}^{\infty}(D) : s \leqslant a\}.$$

Indeed, we have the following:

Proposition 2.3.8. Every finite distributive lattice D is isomorphic to $\mathsf{Up}(P)$ for some finite poset P. Consequently, all of them are completely join-prime generated.

Proof. Let D be a finite distributive lattice and let $P = \mathcal{J}(D)$, with the order given by the opposite of the order in D, i.e., $x \leqslant_P y$ if and only if $y \leqslant x$ in D. Define $\eta : D \to \mathsf{Up}(\mathcal{J}(D))$ by:

$$\eta(a) = \{x \in \mathcal{J}(D) : x \leqslant a\}.$$

Indeed, let $S = \{x \in \mathcal{J}(D) : x \leqslant a\}$. Then a is an upper bound of S; we will show that it is the least upper bound. Let b be an upper bound. Suppose that $a \nleqslant b$. Consider $\uparrow a$ and $\downarrow b$; then these are a pair of a filter and an ideal which are disjoint. By the Prime Filter Theorem, there is some prime filter $F \supseteq \uparrow a$; because the lattice is finite, this is principal, generated by d, and by Lemma 2.1.21, d will be join-irreducible. Hence $d \leqslant a$ and $d \nleqslant b$ by assumption, a contradiction. So $a \leqslant b$, showing that $a = \bigvee S$. Similar arguments show that η is injective. Now assume that $S \in \mathsf{Up}(D)$. Consider $a = \bigvee S$. Then we show that $\eta(a) = S$. For this, note that certainly $S \subseteq \eta(a)$; conversely if $b \leqslant \eta(a)$ is join-irreducible, then by Lemma 2.1.21 it is join-prime, so $b \leqslant s$ for some $s \in \eta(a)$; since S is an upset, $b \leqslant s$ implies $s \leqslant_P b$, so $b \in S$ as well. This shows equality, whilst order-preservation is clear. ■

Just like in the case of Boolean algebras, there exist infinite DLs which are not completely join-prime generated, and in fact contain no completely join-prime elements; however, an analysis of completely join-prime elements for arbitrary distributive lattices seems to be rather tricky, and hence we will reserve it for later, when we can access the simplifying assumption that the distributive lattice is in fact a Heyting algebra.

2.3.2 Priestley duality

Moving from the finite case, we consider the case of arbitrary DL; for this we will need the concept of a *Priestley space*:

Definition 2.3.9. A *Priestley space* is a pair $(X, \leqslant)$ where X is a compact space and $\leqslant$ is a partial order that satisfies the *Priestley separation axiom*:

$$x \nleqslant y \Rightarrow \exists U \in \mathrm{Clop}(X) : U \text{ is an upset}, x \in U \text{ and } y \notin U.$$

Definition 2.3.10. Let X, Y be two Priestley spaces. We say that X and Y are *isomorphic* and write $X \cong Y$ if there exists a map $f : X \to Y$ which is both a homeomorphism and an order-isomorphism.

Remark 2.3.1. Note that the Priestley separation axiom implies total separation, hence the underlying topological space of every Priestley space is a Stone space.

Example 2.3.11. Take $X = \alpha\mathbb{N} = \mathbb{N} \cup \{\omega\}$, where $\leqslant$ the standard order on natural numbers, and $x \leqslant \omega$ for each $x \in X$. Let τ be the topology of $\alpha\mathbb{N}$, i.e., generated by the following basis:

$$\{U : U \text{ is finite in } \mathbb{N}\} \cup \{U \cup \{\omega\} : U \text{ is cofinite in } \mathbb{N}\}.$$

This is a Priestley space: $\alpha\mathbb{N}$ is a Stone space (see Example 2.2.29) and if $n \nleqslant m$, then $m < n$, then take $U = \{x \in \mathbb{N} : x \geqslant n\} \cup \{\omega\}$. Similar arguments show that by considering the *reverse* of the standard order, $\alpha\mathbb{N}$ is likewise a Priestley space.

We can also consider the following order on $\alpha\mathbb{N}$: $\omega \leqslant x$ for $x \in \alpha\mathbb{N}$ and n and m are unrelated. In this case, if $x \nleqslant y$, take $U = \{x\}$. Alternatively, keep n and m unrelated and let $x \leqslant \omega$ for $x \in \alpha\mathbb{N}$. This produces the spaces depicted in Figure 2.4:

Priestley spaces give us duality for distributive lattices, by considering, for an arbitrary Priestley space $(X, \leqslant)$, $\mathsf{ClopUp}(X) = \{U \subseteq X : U \text{ is a clopen upset}\}$:

Proposition 2.3.12. If $(X, \leqslant)$ is a Priestley space, the structure $(\mathsf{ClopUp(X)}, \cap, \cup, X, \varnothing)$ is a bounded distributive lattice.

Proof. Just note that clopen upsets are preserved under binary union and intersection. ∎

The difficult bit is constructing a Priestley space from a distributive lattice D. This will be called the *Priestley dual* of D. Let X_D be the set of all prime filters of D. Let us define an order and topology on X_D:

- We order prime filters under inclusion: $x \leqslant x'$ iff $x \subseteq x'$.

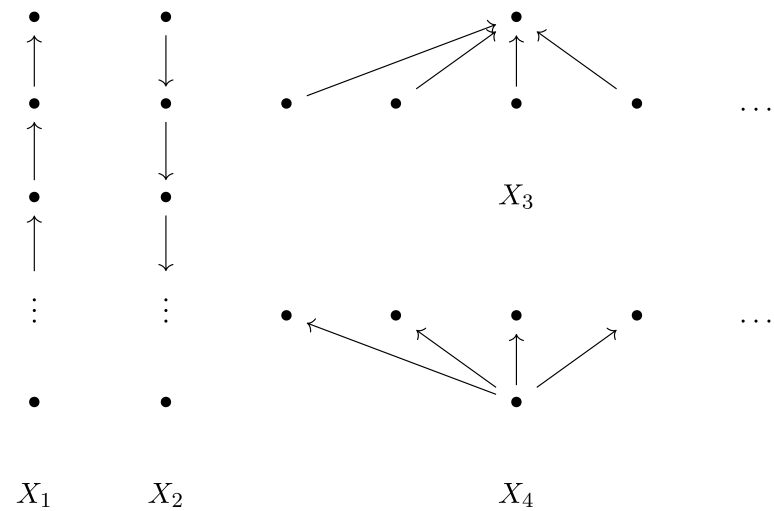


Figure 2.4: Four Examples of Priestley Spaces

- For every $a \in D$, let $\varphi(a) = \{x \in X_D : a \in x\}$. Let $\mathcal{S} = \{\varphi(a) : a \in D\} \cup \{X \backslash \varphi(a) : a \in D\}$, and let τ be the topology that has $\mathcal{S}$ as a subbasis (i.e. the opens are arbitrary unions of finite intersections of elements of $\mathcal{S}$, or equivalently, $\mathcal{B} = \{U_1 \cap ... \cap U_n : n \in \mathbb{N}, U_1, ..., U_n \in \mathcal{S}\}$ is a basis of τ).

Proposition 2.3.13. Given D a distributive lattice, $(X_D, \tau, \subseteq)$ is a Priestley space.

Proof. It satisfies the Priestley Separation Axiom: for $x, y \in X_D$, suppose $x \nleqslant y$, i.e., $x \nsubseteq y$. Then there is $a \in x \backslash y$, i.e., $y \notin \varphi(a) \ni x$. But note that $\varphi(a)$ is clopen and moreover if $x \in \varphi(a)$ and $x \subseteq y$, then $y \in \varphi(a)$. So it is also an upset.

The space is compact; for this we make use of Alexander's Subbase Theorem: suppose that $X_D = \bigcup_{i \in I} \varphi(a_i) \cup \bigcup_{j \in J} X \backslash \varphi(b_j)$. Let $F = \mathsf{Fil}(\{b_j : j \in J\})$ and $I = \mathsf{Id}(\{a_i : i \in I\})$. If F and I are *not disjoint*, then by the definition of the smallest filter and smallest ideal generated by sets, we have that there is an element $a \in F \cap I$, and hence there are $a_1, ..., a_m, b_1, ..., b_k$ such that

$$b_1 \wedge ... \wedge b_k \leqslant a \leqslant a_1 \vee ... \vee a_m.$$

In particular, this means that $\varphi(b_1) \cap ... \cap \varphi(b_k) \subseteq \varphi(a_1) \cup ... \cup \varphi(a_m)$. (Note that $\varphi(a \wedge b) = \varphi(a) \cap \varphi(b)$ and $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$ can be proven here without any problems). But then

$$\begin{aligned} X_D &= (X \backslash (\varphi(b_1) \cap ... \cap \varphi(b_k))) \cup \varphi(a_1) \cup ... \cup \varphi(a_m) \\ &= X \backslash \varphi(b_1) \cup ... \cup X \backslash \varphi(b_n) \cup \varphi(a_1) \cup ... \cup \varphi(a_m), \end{aligned}$$

so X_D has a finite cover.

If F and I are disjoint, by the Prime Filter Theorem, let F' be the prime filter extending F and disjoint with I. Then

$$F' \in X_D = \bigcup_{i \in I} \varphi(a_i) \cup \bigcup_{j \in J} X \backslash \varphi(b_j),$$

which means that either $F' \in \varphi(a_i)$ (i.e. $a_i \in F'$) for some $i \in I$ or $F' \in X \backslash \varphi(b_j)$ (i.e. $b_j \notin F'$) for some $j \in J$. But we have built F' so that $\{b_j\} \subseteq F'$ and $\{a_i\} \cap F' = \varnothing$ for all i, j. Contradiction. ∎

We have a map $\varphi : D \to \mathsf{ClopUp}(X_D)$; as before we get:

Proposition 2.3.14. Given a distributive lattice D, $\varphi : D \to \mathsf{ClopUp}(X_D)$is an isomorphism.

Proof. This is a bounded lattice homomorphism: $\varphi(0) = \varnothing$, as no prime filter contains 0; $\varphi(1) = X$, and it preserves meets and joins. We check that it is injective and surjective:

For injectivity, take $a \neq b$. Without loss of generality, $a \nleqslant b$. But then $\uparrow a \cap \downarrow b = \varnothing$. Since $\uparrow a$ is a filter and $\downarrow b$ is an ideal, we can use the prime filter theorem and take the prime filter F extending $\uparrow a$ and disjoint with $\downarrow b$. In particular, $b \notin F \ni a$. So $\varphi(a) \ni F \notin \varphi(b)$.

For surjectivity, let $U \in \mathsf{ClopUp}(X_D)$. We want to see that it can be written in the form $U = \bigcup_{i \in I} \varphi(a_i)$.
For each $x \in U$, we have to find $\varphi(a) \subseteq U$ such that $x \in \varphi(a)$. Take $y \notin U$. Then $x \nleqslant y$, because U is an upset. We have proved above that there is a_y such that $x \in \varphi(a_y)$ and $y \notin \varphi(a_y)$. Then $X \backslash U \subseteq \bigcup_{y \notin U} X \backslash \varphi(a_y)$, and by compactness

$$X \backslash U \subseteq (X \backslash \varphi(a_1)) \cup ... \cup (X \backslash \varphi(a_n)) = X \backslash (\varphi(a_1) \cap ... \cap \varphi(a_n)),$$

hence $\varphi(a_1 \wedge ... \wedge a_n) = \varphi(a_1) \cap ... \cap \varphi(a_n) \subseteq U$ and $x \in \varphi(a)$ for $a = a_1 \wedge \cdots \wedge a_n$.
This proves that U can be written in the form $U = \bigcup_{i \in I} \varphi(a_i)$ and, since it is a closed subset of a compact space, $U = \varphi(a_1) \cup ... \cup \varphi(a_k)$ for some $a_1, ..., a_n$, hence $U = \varphi(a_1 \vee .... \vee a_k)$, which proves surjectivity. ■

Similarly, we can prove:

Theorem 2.3.1. *Let X be a Priestley space. Then $X \cong X_{\mathsf{ClopUp}(X)}$.*

Proof. (Sketch) $(X, \leqslant) \mapsto \mathsf{ClopUp}(X) \mapsto X_{\mathsf{ClopUp}(X)}$, this last one is topologically homeomorphic to X via the map

$$\varepsilon(x) := \{U \in \mathsf{ClopUp}(X) | x \in U\} \in X_{\mathsf{ClopUp}(X)}.$$

Moreover, note that: $x \leqslant y$ iff $\varepsilon(x) \subseteq \varepsilon(y)$. The latter can be proved using the Priestley separation axiom. ■

Moving on to homomorphisms, we will need to introduce the morphisms between Priestley spaces; given the order-structure, these are inherently more complicated than simply continuous functions:

Definition 2.3.15. Let $(X, \leqslant)$ and $(Y, \leqslant')$ be Priestley spaces. A map $f : X \to Y$ is called a *Priestley morphism* if it is continuous and order-preserving, i.e., for every $x, y \in X$:

$$x \leqslant y \text{ implies } f(x) \leqslant' f(y).$$

Proposition 2.3.16. Let $(X, \leqslant)$ and $(Y, \leqslant')$ be Priestley spaces and $f : X \to Y$ a Priestley morphism. Then $f^* : \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$ defined by $f^*(U) = f^{-1}(U)$, for $U \in \mathsf{Clop}(Y)$ is a bounded lattice morphism.

Proof. Exercise. ■

Proposition 2.3.17. Let A and B be distributive lattices and $h : A \to B$ a bounded lattice morphism. Then $h_* : X_B \to X_A$ defined by $h_*(x) = h^{-1}(x)$ is a Priestley morphism.

Proof. Exercise. ■

2.3.3 Duals of constructions

The duals of constructions for distributive lattices are in many ways similar to those for Boolean algebras. We list them here without proof; proofs can be obtained by adapting the ideas given in Section 2.2.4, doing the appropriate modifications – though the reader should be warned that some of these modifications are far from trivial!

Note that given two Priestley spaces X, Y, we denote by $X \sqcup Y$ the Priestley space obtained by taking the disjoint union of the orders and the topological sum.

Theorem 2.3.2. *Given $f : D \to D'$ a bounded lattice homomorphism between distributive lattices we have:*

1. *f is an embedding if and only if f^{-1} is a surjective continuous map.*
2. *f is surjective if and only if f^{-1} is an order-embedding and a topological embedding.*
3. *Given D_1, D_2 two Distributive lattices, $X_{D_1 \times D_2} \cong X_{D_1} \sqcup X_{D_2}$.*

Proof. Exercise (see Exercise 2.25). ■

Despite filters not corresponding to homomorphic images, it is nevertheless the case that filters correspond to specific subsets of the dual Priestley space:

Theorem 2.3.3. *Let D be a distributive lattice. There is a dual isomorphism $\xi : (\mathsf{Fil}(D), \subseteq) \to (\mathsf{ClosedUp}(X_D), \subseteq)$.*

Proof. One uses the same idea as in Theorem 2.2.4. ■

Similarly, one can characterize ideals, which come out slightly less expected:

Theorem 2.3.4. *Let D be a distributive lattice. There is an isomorphism $\xi : (\mathsf{Id}(D), \subseteq) \to (\mathsf{OpenUp}(X_D), \subseteq)$.*

Proof. Exercise. ■

Finally we note that just like atoms can be given dual meaning, join-irreducible elements admit a simple characterization:

Theorem 2.3.5. *Let D be a distributive lattice. An element $a \in D$ is join-irreducible if and only if $\varphi(a)$ is rooted, i.e., there is an element $x \in X_D$ such that $\varphi(a) = \uparrow x$.*

Proof. If $\varphi(a)$ is rooted at x, assume that $a \leqslant b \vee c$; then $\varphi(a) \subseteq \varphi(b) \cup \varphi(c)$. Then either $x \in \varphi(b)$ or $x \in \varphi(c)$, and by upwards closure, either $\varphi(a) \subseteq \varphi(b)$ or $\varphi(a) \subseteq \varphi(c)$, showing that a is join-irreducible.

Conversely, assume that $\varphi(a)$ is join-irreducible but not rooted. Since $\varphi(a)$ is a Priestley space, by Exercise 2.23 we have that its minimum is non-empty. Since it is not rooted, there must be two elements x and y both minimal in $\varphi(a)$. Now, let $x \in U \not\ni y$ be a clopen upset; for each $z \in \mathsf{Min}(\varphi(a))$, $x \neq z$, let U_z be a clopen upset such that $x \notin U_z$ and $z \in U_z$. Then we have:

$$\varphi(a) \subseteq U_x \cup \bigcup_{z \in \mathsf{Min}(\varphi(a)), z \neq x} U_z.$$

Because $\varphi(a)$ is clopen, we can take a finite subcover to obtain:

$$\varphi(a) \subseteq U_x \cup U_{z_0} \cup \ldots \cup U_{z_n}.$$

By join-irreducibility, we have that either $\varphi(a) \subseteq U_x$ – which is impossible since $y \notin U_x$ – or $\varphi(a) \subseteq U_{z_0} \cup \ldots \cup U_{z_n}$ – which is impossible since $x \notin U_{z_0} \cup \ldots \cup U_{z_n}$. Hence by reductio we have that $\varphi(a)$ must be rooted. ■

Extra Content

The following sections are intended for the categorically minded readers and readers interested in the furhter relationships between Boolean algebras and Distributive lattices.

2.3.4 ◊ A categorical view of Priestley duality

Just like when considering Stone duality, one can consider **DL** the category of distributive lattices and distributive lattice homomorphisms, and **Pries** the category of Priestley spaces with Priestley morphisms. One defines now the *patch-spectrum* functor[5] functor:

$$\begin{aligned} \mathsf{PSpec} : \mathbf{DL} &\to \mathbf{Pries} \\ D &\mapsto X_D \\ f : D \to D' &\mapsto f^{-1} : X_D \to X_{D'}. \end{aligned}$$

Then we have:

Proposition 2.3.18. The functor PSpec is full, faithful and essentially surjective.

This yields:

Theorem 2.3.6. *We have that* $\mathbf{DL}^{op} \simeq \mathbf{Pries}$.

This allows us to recover all duality results from the previous section. Moreover, as before, we have:

Definition 2.3.19. Let D_1 and D_2 be two distributive lattices. We define the *free product of* D_1, D_2 as follows: let $\mathsf{Form}(D_1 + D_2)$ be the set of positive logic formulas with variables c_a for each $a \in D_1$ or $a \in D_2$. Let Γ denote the *theory* of D_1, D_2, i.e., it contains all the following formulas:

$$\{c_{1_{D_1}} \leftrightarrow \top, c_{1_{D_2}} \leftrightarrow \top, c_{0_{D_1}} \leftrightarrow \bot, c_{0_{D_2}} \leftrightarrow \bot\}$$
$$\{c_{a \wedge b} \leftrightarrow c_a \wedge c_b, c_{a \vee b} \leftrightarrow c_a \vee c_b : a, b \in D_1 \sqcup D_2\}.$$

Now define an equivalence on $\mathsf{Form}(D_1 + D_2)$, $\approx$ as follows: for φ, ψ formulas there:

$$\varphi \approx \psi \iff \Gamma \vdash \varphi \approx \psi.$$

I.e., in every model M which validates the whole of Γ, $v(\varphi) = v(\psi)$. Let $D_1 \oplus D_2$ be the quotient distributive lattice $\mathsf{Form}(B_1 + B_2)/\approx$.

[5] For more on this nomenclature, see the next subsection.

Proposition 2.3.20. For each D_1, D_2:

1. The free product $D_1 \oplus D_2$ is the coproduct in the category of distributive lattices of these two algebras.
2. $X_{D_1 \oplus D_2} \cong X_{D_1} \times X_{D_2}$, i.e., the Priestley space with the product order and product topology.

More interestingly, we can see that Boolean algebras, being distributive lattices, represent, in light of Proposition 2.2.5 a full subcategory of Distributive lattices. From a categorical point of view a natural question is the following: does this inclusion functor have any adjoints?

Verifying whether a functor has adjoints for these kinds of categories is in general quite easy from a very abstract point of view – one can typically check whether the conditions of some Adjoint Functor Theorem hold. Using such conditions, and knowing a bit about the structure of colimits in each of the categories, it is easy to verify that in fact the inclusion has *both adjoints*. To simplify the discussion, we will focus here on the case of the *left adjoint*.

Indeed, by a version of the adjoint functor theorem applying to so-called Locally Presentable categories[6] – which both **BA** and **DL** are – the functor $I : \mathbf{BA} \to \mathbf{DL}$ admits a left adjoint if and only if it preserves all small limits. For this it is sufficient that it preserves all products and all equalizers. Equalizers in categories of algebras are exactly the regular monomorphisms, which are exactly the injective homomorphisms. Hence it suffices to verify that the inclusion preserves injective homomorphisms and products – which in light of our definitions it certainly does. Hence I has a left adjoint.

Naturally simply knowing the existence of the adjoint is not very useful. Hence we may wish to construct it. Here duality can once again be of use, together with some heuristics: the functor $I : \mathbf{BA} \to \mathbf{DL}$ is "forgetful", hence its left adjoint will be "free" – and in principle difficult to describe. This functor induces a functor $I^* : \mathbf{Stone} \to \mathbf{Pries}$; this is the functor which assigns to each Stone space X the Priestley space $(X, =)$ (see the Exercises). Moreover, I^* will have a right adjoint. Such a right adjoint must morally be a "forgetful" type functor. Hence we are left wondering: given a Priestley space $(X, \leqslant)$ how do we obtain a Stone space? Certainly we can simply consider X, and given a Priestley morphism $f : X \to Y$ simply treat it as a continuous function. Denote this by the functor $\mathsf{For} : \mathbf{Pries} \to \mathbf{Stone}$:

Theorem 2.3.7. *The functor* $\mathsf{For} : \mathbf{Pries} \to \mathbf{Stone}$ *is left adjoint to* I^**.*

Verifying this adjunction is substantially easier, and it immediately provides us with an algebraic description of the functor $\mathsf{Free} : \mathbf{DL} \to \mathbf{BA}$:

Definition 2.3.21. Given a distributive lattice D, let $\mathsf{Free}(D) = \mathsf{Clop}(X_D)$; we call this the *free Boolean extension* of D.

One can then prove several interesting properties of the free Boolean extension; for now we leave these considerations, leaving the interested reader to ponder on the right adjoint functor as well as related issues.

One final issue which the astute reader may ask is the following: is there any analogue of **CABA** for distributive lattices? Following the intuitions from the finite duality, one can make the following definition:

[6] See e.g. this reference in the section on locally presentable categories.

Definition 2.3.22. Let D be a complete distributive lattice. We say that D is a *Birkhoff lattice* if D is complete and completely join-prime generated.

Denote by **Birk** the category of Birkhoff lattices with complete distributive lattice homomorphisms.

Theorem 2.3.8. *There is a duality between* **Birk** *and the category* **Pos** *of posets with monotone maps.*

Indeed, in the course of this duality, one shows that any Birkhoff lattice is of the form $\mathsf{Up}(P)$ for P a poset. However, to know what this corresponds to in Priestley duality one needs the concept of *order-compactifications* (namely the so-called *Nachbin order-compactification*, $\nu(P)$) – this would take us too far even for extra content.

2.3.5 ♢ Spectral duality for distributive lattices

The duality given in the previous sections is not the only duality for distributive lattices; in fact, it is not the first presented either. Originally, Marshall Stone provided a description of the dual of a distributive lattice by a duality that more closely resembles the one for Stone spaces, by involving only topological spaces. However, the price to pay is that such spaces are very badly behaved: they are in general T_0 but are only T_1 in the case of Boolean algebras, whereby they are automatically Stone spaces.

Nevertheless, rather surprisingly, such spaces appear both in the context of distributive lattices and in the duality for commutative rings with unit – where a similar notion of "spectrum" is used. This connection with algebraic geometry makes it worthwhile to consider this duality on its own merits. Given X a topological space, let $K^\circ(X)$ be the set of compact and open sets:

Definition 2.3.23. Let X be a topological space. We say that X is a *spectral space*:

1. X is compact and sober;
2. $K^\circ(X)$ forms a basis for the topology of X;
3. $K^\circ(X)$ is closed under finite intersections.

Given X a spectral space, the set $K^\circ(X)$ is easily seen to form a distributive lattice. Moreover, given a distributive lattice D, we can equip $\mathsf{Spec}(D) = \{x \in D : x \text{ is a prime filter}\}$ with the basis given by:

$$\{\varphi(a) : a \in D\}.$$

Then we have:

Theorem 2.3.9. *Let D be a distributive lattice. Then $\mathsf{Spec}(D)$ is a spectral space.*

Definition 2.3.24. A continuous map $f : X \to Y$ between topological spaces is said to be *spectral* if whenever $U \in K^\circ(Y)$ then $f^{-1}[U] \in K^\circ(X)$.

Proposition 2.3.25. A map $f : D \to D'$ between distributive lattices is a bounded lattice homomorphism if and only if $f^{-1} : \mathsf{Spec}(D') \to \mathsf{Spec}(D)$ is a spectral map.

Let **Spectral** be the category of spectral spaces with spectral maps. Then we have:

Theorem 2.3.10. *We have that* $\mathbf{DL}^{op} \simeq \mathbf{Spectral}$.

Putting together this theorem and Priestley duality, one obtains that **Spectral** and **Pries** are equivalent categories. But this can be seen in a much more direct way:

1. Given a Priestley space $(X, \leqslant)$, one can see by the above definition that losing the order and taking the topology given by sets of the form $\varphi(a)$ yields a spectral space;

2. Given a spectral space X, one can consider the *specialization partial order* (since the spaces are T_0), i.e., $x \leqslant y$ if and only if for each $U \in K^{\circ}(X)$, $x \in U$ implies that $y \in U$. Moreover, one defines the *patch topology* by considering the subbase of X as a Priestley space be given by
$$\{U : U \in K^{\circ}(X)\} \cup \{X - U : U \in K^{\circ}(X)\}.$$

3. One verifies that Priestley morphisms induce spectral maps and vice-versa.

Using this one obtains a rather strong connection between these categories:

Theorem 2.3.11. *The categories* **Spectral** *and* **Pries** *are isomorphic under the above correspondence.*

End of extra content.

2.4 Heyting Algebras and Esakia Duality

So far we have remained eminently in the realm of lattices. However, it is often useful to consider signatures enriched with further operations. The primary example of this throughout the course will be our next objects of analysis: Heyting algebras. We will begin by outlining their algebraic theory and basic properties, before turning to their duality theory. We will then touch on some of the many peculiarities of Heyting algebras *viz.* Boolean algebras and Distributive lattices.

2.4.1 Basic theory of Heyting algebras

Definition 2.4.1. A lattice H is called a *Heyting algebra* if for all $a, b \in A$ there is an element $a \to b \in A$ (called the *relative pseudocomplement*) with the property that, for all $c \in A$,

$$c \leqslant a \to b \text{ iff } c \wedge a \leqslant b.$$

We write $\neg a$ for $a \to 0$; this is called the *pseudocomplement* of a. Just like with lattices and Boolean algebras, is in our interest to provide an equational definition of a Heyting algebra:

Theorem 2.4.1. *A bounded lattice* $(H, \vee, \wedge, 0, 1)$ *is a Heyting algebra iff there is a binary operation* $\to: H^2 \to H$, *satisfying, for all* $a, b, c \in H$:

h1. $a \to a = 1$,

h2. $a \wedge (a \to b) = a \wedge b$,

h3. $b \wedge (a \to b) = b$,

h4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Proof. From left to right, is easy (Exercise!). From right to left, assume (h1)-(h4) are satisfied and let $c \leqslant a \to b$. Then $c \wedge a \leqslant (a \to b) \wedge a \overset{\text{h2.}}{=} a \wedge b \leqslant b$. Conversely, if $c \wedge a \leqslant b$,

$$\begin{aligned} c &\overset{\text{h3.}}{=} c \wedge (a \to c) \\ &\leqslant 1 \wedge (a \to c) \\ &\overset{\text{h1.}}{=} (a \to a) \wedge (a \to c) \\ &\overset{\text{h4.}}{=} a \to (a \wedge c) \\ &\underset{(*)}{\leqslant} a \to b. \end{aligned}$$

Here, $*$ is a result of the fact that $a \to \cdot$ is monotone, for if $b_1 \leqslant b_2$, then $a \to b_1 = a \to (b_1 \wedge b_2) = (a \to b_1) \wedge (a \to b_2)$. ■

It will be useful to have some algebraic identities of Heyting algebras to work with:

Lemma 2.4.2. Let H be a Heyting algebra, and $a, b, c \in H$. Then:

1. $(a \vee b) \to c = a \to c \wedge b \to c$.

2. $\neg(a \vee b) = \neg a \wedge \neg b$.

3. $\neg\neg\neg a = \neg a$;

4. $a \to (b \to c) = (a \wedge b) \to c$;

5. $\neg(a \wedge b) = a \to \neg b$.

Being a Heyting algebra turns out to be intimately related to satisfying some strong distributivity laws, so long as we work with complete lattices:

Theorem 2.4.2. *Let H be a complete lattice. Then H is a Heyting algebra iff H satisfies the infinite distributivity law:*

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i).$$

Proof. Let H be a complete Heyting algebra. We will show it satisfies the infinite distributivity law. For all $i_0 \in I$, $a \wedge b_{i_0} \leqslant a \wedge \bigvee_i b_i$, so $\bigvee_{i \in I}(a \wedge b_i) \leqslant a \wedge \bigvee_i b_i$. Now, if $\bigvee_i (a \wedge b_i) \leqslant c$, then for all $i_0 \in I$, $a \wedge b_{i_0} \leqslant c$, hence $b_{i_0} \leqslant a \to c$, therefore $\bigvee_{i \in I} b_i \leqslant a \to c$, thus $a \wedge \bigvee_i b_i \leqslant c$. In particular, $a \wedge \bigvee_i b_i \leqslant \bigvee_i (a \wedge b_i)$.

Conversely, let $a \to b := \bigvee\{c \in A : c \wedge a \leqslant b\}$. Then it is easy to show that $c \leqslant a \to b$ iff $c \wedge a \leqslant b$ (Exercise). ■

Corollary 2.4.3.

1. *Every finite distributive lattice is a Heyting algebra.*

2. *Every Heyting algebra is distributive.*

Proof. Both (1) and (2) follow from Theorem 2.4.2 by taking I to be finite. ∎

The definition of the relative pseudocomplement given in Theorem 2.4.2 is in fact fully general:

Lemma 2.4.4. If H is a Heyting algebra and $a, b \in H$, then

$$a \to b = \bigvee \{c \in H : c \wedge a \leqslant b\}.$$

Proof. Exercise. ∎

Example 2.4.5.

1. Every Boolean algebra is a Heyting algebra with $a \to b := \neg a \vee b$.

2. Every linear lattice (see Example 2.1.3) is a Heyting algebra: note that

$$\{c \in H : c \wedge a \leqslant b\} = \begin{cases} H & \text{if } a \leqslant b \\ \{c \in H : c \leqslant b\} & \text{if } b < a, \end{cases}$$

so

$$a \to b := \bigvee \{c \in H : c \wedge a \leqslant b\} = \begin{cases} 1, & \text{if } a \leqslant b, \\ b, & \text{if } b < a. \end{cases}$$

3. Take a topological space (X, τ). Then

$$(\tau, \cap, \cup, \varnothing, X) \text{ is a distributive lattice.}$$

It is complete for $S \subseteq \tau \Rightarrow \sup S = \bigcup S \in \tau$. This means that there is $\bigwedge S$, but it does not equal to $\bigcap S$; indeed,

$$\bigwedge S = \mathrm{Int}\left(\bigcap S\right).$$

This structure is a Heyting algebra, for $U \wedge \bigvee_i V_i = U \cap \bigcup_i V_i = \bigcup_i (U \cap V_i)$[7].

In this Heyting algebra,

$$U \to V = \bigvee \{C \in \tau : C \cap U \subseteq V\} = \bigcup \{C \in \tau : C \subseteq (X \backslash U) \cup V\} = \mathrm{Int}[(X \backslash U) \cup V].$$

4. Given $(P, \leqslant)$ a poset, consider $\mathsf{Up}(P)$, the set of upsets; this forms an Alexandroff topology on P. For $A \subseteq P$, we have that $\mathrm{Int}(A) = \{x \in P : \uparrow x \subseteq A\}$ and $\mathrm{Cl}(A) = \downarrow A := \{y \in P : \exists x \in A (y \leqslant x)\}$.

In this case,

$$\begin{aligned} U \to V &= \mathrm{Int}(U^c \cup V) = \mathrm{Cl}((U^c \cup V)^c)^c \\ &= (\downarrow (U \cap V^c))^c = (\downarrow (U \backslash V))^c \\ &= \{x \in P : \uparrow x \subseteq U^c \cup V\}. \end{aligned}$$

[7] Notice, however that $a \vee \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \wedge b_i)$ does not hold in general; the algebraic structures where such a law holds are called *co-Heyting algebras* or *dual Heyting algebras*. Structures where both laws hold are called *bi-Heyting algebras*. These provide very interesting algebraic structures with a rich internal structure, but we will not address them in the forecoming pages.

Using the above examples we can find counterexamples for $a \vee \neg a = 1$ holding uniformly: take $\tau_{\mathbb{R}}$, $U = (0, \infty)$, for instance.

Just like Boolean algebras and distributive lattices, we can consider Heyting *subalgebras* in the obvious way:

Definition 2.4.6. Given a Heyting algebra H, a subset $S \subseteq H$ is called a *Heyting subalgebra* if S is a bounded sublattice and it is closed under $\to$.

Unlike Boolean algebras, though, not every bounded sublattice of a Heyting algebra H will be a Heyting subalgebra (see Exercise 2.27). Even more striking, there can be bounded sublattices which are not Heyting algebras, as we will see soon.

Similarly to what we have done before, we can define *homomorphic images* of Heyting algebras as the images of surjective homomorphisms. Like for Boolean algebras, these correspond to filters:

Definition 2.4.7. If $F \subseteq H$ is a filter, then define the equivalence relation:

$$a \approx_F b \iff a \leftrightarrow b \in H,$$

where $a \leftrightarrow b := a \to b \wedge b \to a$. We let $H/F := H/\approx_F$.

Lemma 2.4.8. The set H/F has a Heyting algebra structure if we define it as:

- $0_{H/F} := [0_H]_F$ and $1_{H/F} := [1_H]_F$;
- $[a]_F \wedge_{H/F} [b]_F := [a \wedge b]_F$ and $[a]_F \vee_{H/F} [b]_F := [a \vee b]_F$ and $[a]_F \to_{H/F} [b]_F := [a \to b]_F$.

Proof. Exercise. ∎

Proposition 2.4.9 (First Homomorphism Theorem for Heyting algebras)**.**

1. If H is a Heyting algebra, and F is a filter, the map $q_F : H \to H/F$ given by $q_F(a) = [a]_F$ is a surjective homomorphism.
2. If $f : H \to H'$ is a surjective homomorphism, then $F = f^{-1}[1_{H'}]$ is a filter. Moreover, $H/F \cong H'$.

Proof. Exercise. ∎

Finally, we can also consider the product of Heyting algebras in the obvious way:

Definition 2.4.10. Let H, H' be two Heyting algebras. We define the *product* of H and H', denoted by $H \times H'$ as follows: this has the same structure as the distributive lattice product, together with the following definition for the implication:

$$(a_1, b_1) \to_{H \times H'} (a_2, b_2) := (a_1 \to a_2, b_1 \to b_2).$$

Heyting algebras and Boolean algebras are very intimately linked: as noted above every Boolean algebra is a Heyting algebra. Indeed, one can capture Boolean algebras in an easy way:

Proposition 2.4.11. Let H be a Heyting algebra. Then H is a Boolean algebra if and only if for each $a \in H$, $\neg a \vee a = 1$.

Proof. Simply note that this is the only axiom that the negation $\neg$ does not necessarily satisfy from those of Boolean algebras. ■

Moreover, from every Heyting algebra one can construct a Boolean algebra in a canonical way.

Definition 2.4.12. Let H be a Heyting algebra. An element $a \in H$ is said to be *regular* if $a = \neg\neg a$. Let $H_{\neg\neg} = \{a \in H : a \text{ is regular}\}$. We equip this structure with the following operations for every $a, b \in H_{\neg\neg}$:

$$\begin{aligned} 0_{\neg\neg} &:= 0 \\ 1_{\neg\neg} &:= 1 \\ a \wedge_{\neg\neg} b &:= a \wedge b \\ a \rightarrow_{\neg\neg} b &:= a \rightarrow b \\ a \vee_{\neg\neg} b &:= \neg\neg(a \vee b) \end{aligned}$$

Proposition 2.4.13. Let H be a Heyting algebra.

1. The structure $(H_{\neg\neg}, \wedge_{\neg\neg}, \vee_{\neg\neg}, \rightarrow_{\neg\neg}, 0_{\neg\neg}, 1_{\neg\neg})$ is well-defined (i.e., regular elements are closed under these operations).

2. $H_{\neg\neg}$ is a Boolean algebra.

Proof. (1) Certainly 0 and 1 are regular since they are each others complement. Assume that $a, b \in H_{\neg\neg}$ to show that $a \wedge b$ is regular note that:

$$\begin{aligned} \neg\neg(a \wedge b) &= \neg\neg(\neg\neg a \wedge \neg\neg b) \\ &= \neg\neg\neg(\neg a \vee \neg b) \\ &= \neg(\neg a \vee \neg b) \\ &= \neg\neg a \wedge \neg\neg b \\ &= a \wedge b. \end{aligned}$$

The first and last equality follow from the assumption of regularity, and the remaining ones follow from Lemma 2.4.2.

Certainly $\neg\neg(a \vee b)$ will be regular, since by the above stated Lemma, triple negation is equivalent to single negation. To see why the structure is closed under implication note that:

$$\begin{aligned} \neg\neg(a \rightarrow b) &= \neg\neg(\neg\neg a \rightarrow \neg\neg b) \\ &= \neg\neg(\neg\neg a \wedge \neg b \rightarrow 0) \\ &= \neg\neg\neg(\neg\neg a \wedge \neg b) \\ &= \neg(\neg\neg a \wedge \neg b) \\ &= \neg\neg a \rightarrow \neg\neg b \\ &= a \rightarrow b. \end{aligned}$$

(2) Note that given $a, b \in H_{\neg\neg}$, $a \wedge b$ is certainly their infimum. Define $a \leqslant_{\neg\neg} b := a \leqslant b$. On the other hand, if $a \leqslant c$ and $b \leqslant c$ and c is regular, then we have that $a \vee b \leqslant c$ in H, so $\neg\neg(a \vee b) \leqslant \neg\neg c = c$, so $\neg\neg(a \vee b)$ is the supremum of a, b. This means that $H_{\neg\neg}$ forms a lattice (though not a sublattice). From this fact, and the fact that it is closed under $\rightarrow$, we conclude that $H_{\neg\neg}$ is a Heyting algebra. Now for any $a \in H_{\neg\neg}$, note that $\neg a \vee_{\neg\neg} a = \neg\neg(\neg a \vee a) = \neg(\neg\neg a \wedge \neg a) = \neg 0 = 1$, hence by Proposition 2.4.11, $H_{\neg\neg}$ is a Boolean algebra. ■

2.4.2 Esakia duality

We now move on to providing the duality theory for Heyting algebras. This will build on Priestley duality, enriching it with additional requirements. First let us see some motivation for our definitions.

We know that for Alexandroff spaces, $\mathrm{Int} = -\downarrow -$ and $\mathrm{Cl} = \downarrow$. So:

$$U \to V = -\downarrow -(-U \cup V) = -\downarrow(U \cap -V) = -\downarrow(U \setminus V).$$

We can abbreviate for each U, $\Box_{\leqslant} U := -\downarrow - U$, or when the order is understood, $\Box U$. Note that this operation is normal in the following sense: for each U, V, $\Box(U \cap V) = \Box U \cap \Box V$.

If $(X, \leqslant)$ is a Priestley space, and U, V are clopen upsets, we can check that if any subset of X is to be the relative pseudocomplement it will be $\Box(-U \cup V)$. Hence we need to ensure that clopen upsets are closed under it; the set will always be an upset, but it may fail to be clopen. However we can make the following simplification:

Lemma 2.4.14. Let $(X, \leqslant)$ be a Priestley space. Then the following are equivalent:

1. For each U, V clopen upsets, $\Box(-U \cup V)$ is a clopen (upset);
2. For each clopen U, $\downarrow U$ is clopen.

Proof. In light of the above definitions, (2) implies (1) straightforwardly. To see that (1) implies (2), note that for each U a clopen subset of a Priestley space, we can write:

$$-U = \bigcap_{i=1}^{n} -A_i \cup B_i.$$

And so by our remarks on normality:

$$\Box - U = \bigcap_{i=1}^{n} \Box(-A_i \cup B_i).$$

Since finite intersections of clopens are clopen, then $\Box - U$ is clopen, and hence $\downarrow U$ is clopen. ■

We can now make the key definition of this section:

Definition 2.4.15. An *Esakia space* is a Priestley space $(X, \leqslant)$ such that for every $U \in \mathsf{Clop}(X)$, we have $\downarrow U \in \mathsf{Clop}(X)$.

Example 2.4.16. Recall the Priestley spaces from fig. 2.4. It is easy to see that both X_1 and X_2 are Esakia spaces. Moreover, X_3 is also an Esakia space. But X_4 is not: $\{1\}$ is clopen but $\downarrow\{1\} = \{1, \omega\}$ is not.

Again, starting with Esakia spaces we obtain Heyting algebras by taking clopen upsets, together with the operation:

$$U \to V := \Box(-U \cup V)$$

for U, V clopen upsets.

Proposition 2.4.17. Let $(X, \leqslant)$ be an Esakia space. Then $(\mathsf{ClopUp}(X), \cap, \cup, \varnothing, X, \to)$ is a Heyting algebra.

Proof. Exercise. ■

Proposition 2.4.18. Let H be a Heyting algebra. Let X_H be the Priestley dual of H. Then for each $a, b \in H$ we have

$$\varphi(a \to b) = \varphi(a) \to \varphi(b).$$

Proof. Let $x \in \varphi(a \to b)$, $x \subseteq y$ and $y \in \varphi(a)$. Then $a \in y$, $a \to b \in x$ and hence $a \to b \in y$. Then $(a \to b) \wedge a = a \wedge b$ belongs to y and as $a \wedge b \leqslant b$, we have that $b \in y$. So $y \in \varphi(b)$ and $x \in \varphi(a) \to \varphi(b)$.

Conversely, let $x \notin \varphi(a \to b)$. Then $a \to b \notin x$. Let $F = \mathsf{Fil}(x \cup \{a\})$, and let $I = \mathsf{Id}(\{b\})$. If $F \cap I \neq \varnothing$, we have $a \wedge c \leqslant b$ for some $c \in x$. This yields $c \leqslant a \to b$. So $a \to b \in x$, which is a contradiction. Thus, $F \cap I = \varnothing$ and by the Prime Filter Theorem, there is a prime filter y such that $F \subseteq y$ and $y \cap I = \varnothing$. Therefore, $x \subseteq y$, $a \in y$ and $b \notin y$. So $y \in \varphi(a)$ and $y \notin \varphi(b)$, and $x \notin \varphi(a) \to \varphi(b)$. ■

Proposition 2.4.19. Let H be a Heyting algebra. Let X_H be the Priestley dual of H. Then $(X_H, \leqslant)$ is an Esakia space.

Proof. By Lemma 2.4.14, in order for X_H to be Esakia it suffices to require that given U, V clopen upsets, $U \to V$ is a clopen. But this is the content of Proposition 2.4.18. ■

We are yet to define the morphisms in the category of Esakia spaces. We want these morphisms to satsfy certain things. The condition we want to hold for an Esakia space morphism $f : X \to Y$ is, for $U, V \subseteq Y$, to have:

$$f^{-1}(U \to V) = f^{-1}(U) \to f^{-1}(V).$$

Definition 2.4.20. Let $(X, \leqslant)$ and $(Y, \leqslant')$ be Esakia spaces. A map $f : X \to Y$ is an *Esakia morphism* (or *continuous p-morphism*) if it is a Priestley morphism satisfying in addition the following condition for each $x \in X$ and $y \in Y$:

$$f(x) \leqslant y \text{ implies } \exists z \in X\ (z \geqslant x\ \&\ f(z) = y).$$

We sometimes refer to p-morphisms between posets, which simply remove the continuity assumption and the topology. One of the attractive features of working with p-morphisms is the following:

Proposition 2.4.21. Let $f : X \to Y$ be a p-morphism between posets. Then for each upset $U \subseteq X$, $f[U]$ is an upset.

Proof. Let U be an upset and $y \in f[U]$. Then $f(x) = y$ for some $x \in U$. If $y \leqslant y'$, there is a $z \geqslant x$ (hence $z \in U$) such that $f(z) = y'$. Therefore, $y' \in f[U]$. ■

Remark 2.4.1. The above condition is no more than the bounded morphism condition known from modal logic.

Proposition 2.4.22. Let $(X, \leqslant)$ and $Y, \leqslant')$ be Esakia spaces and $f : X \to Y$ an Esakia morphism. Then for every $U, V \in \mathsf{ClopUp}(Y)$ we have:

$$f^{-1}(U \to V) = f^{-1}(U) \to f^{-1}(V).$$

Proof. Assume that $x \in f^{-1}[U \to V]$. Suppose that $x \leqslant y$, and $y \in f^{-1}[U]$. Then $f(x) \leqslant f(y)$ together with the former implies that $f(y) \in U$ so $f(y) \in V$. Thus $x \in f^{-1}[U] \to f^{-1}[V]$. Conversely, assume that $x \in f^{-1}[U] \to f^{-1}[V]$. We want to show that $f(x) \in U \to V$. So suppose that $f(x) \leqslant y$, and $y \in U$. By the p-morphism condition, there is some $x' \geqslant x$ such that $f(x') = y$. Then $x' \in f^{-1}[U]$, so by assumption, $x' \in f^{-1}[V]$. Thus $y \in V$, as was to show. ∎

Proposition 2.4.23. Let $(X, \leqslant)$ and $(Y, \leqslant')$ be Esakia spaces and $f : X \to Y$ an Esakia morphism. Then $f^* : \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$ is a Heyting algebra homomorphism.

Proof. This follows from Proposition 2.4.22. ∎

Proposition 2.4.24. Let A and B be Heyting algebras and $h : A \to B$ a Heyting algebra homomorphism. Then $h_* : X_B \to X_A$ defined by $h_*(x) = h^{-1}(x)$ is an Esakia morphism.

Proof. Let $x \in X_B$, $y \in X_A$ and $h^{-1}(x) \subseteq y$. Consider the filter $F = [x, h[y])$ generated by x and $h[y]$. Let also $I = (h[A \backslash y]]$ be the ideal generated by $h[A \backslash y]$. Suppose $F \cap I \neq \varnothing$. Then there is $b \in x$, $a \in y$ and $a' \in A \backslash y$ such that (verify!) $b \wedge h(a) \leqslant h(a')$. Then $b \leqslant h(a) \to h(a') = h(a \to a')$. So $h(a \to a') \in x$ and $a \to a' \in h^{-1}(x) \subseteq y$. We have $a \in y$ and $a \to a' \in y$ implying $a' \in y$ (verify!), which is a contradiction. So $F \cap I = \varnothing$. Then by the Prime Filter Theorem, there is $z \in X_B$ such that $F \subseteq z$ and $z \cap I = \varnothing$. So $x \subseteq z$, and on the one hand $y \subseteq h^{-1}(z)$, and on the other $h^{-1}(z)$ has empty intersection with $A \backslash y$. Therefore, $h^{-1}(z) = y$. ∎

2.4.3 Duals of constructions

We now study the duals of specific constructions for Heyting algebras. The first is to be expected:

Proposition 2.4.25. Given $f : H \to H'$ a bounded lattice homomorphism between distributive lattices we have that f is injective if and only if $f^{-1} : X_{H'} \to X_H$ is a surjective p-morphism.

Proof. Exercise. ∎

If H and H' are two *finite* Heyting algebras there is a very convenient way of decomposing p-morphisms which allows us to easily analyze all subalgebras of a given finite Heyting algebra:

Given a poset $(P, \leqslant)$ it will sometimes be useful for theoretical reasons to consider quotients of P.

Definition 2.4.26. Let P be a poset, and E an equivalence relation on P. We say that E is a *bisimulation equivalence* if:

1. Whenever $w \leqslant w'$ and xEw then there is some $y \geqslant x$ such that yEw'.
2. If $\neg(xEw)$ then there is some U, an upset of P such that whenever $x \in U$ and $[x] = [y]$ then $y \in U$, and we have: either $x \in U$ and $w \notin U$ or $w \in U$ and $x \notin U$.

Lemma 2.4.27. Let P be a poset and E a bisimulation equivalence. Then the order defined on P/E given by:

$$[x] \leqslant_E [y] \iff \exists x' \in [x] \text{ and } y' \in [y], x' \leqslant y',$$

is a partial order, and the map $p : P \to P/E$ is a p-morphism of posets.

Definition 2.4.28. Let $f : P \to Q$ be a p-morphism between finite posets. We say that f is:

1. An α-*reduction* if there are two points x, y such that $x \leqslant y$ and $\uparrow x - \{x\} = \uparrow y$, $f(x) = f(y)$ and for every $w, z \in P$, if $f(w) = f(z)$ then $w = x$ and $z = y$ or vice-versa.

2. A β-*reduction* if there are two points x, y such that $\uparrow x - \{x\} = \uparrow y - \{y\}$, $f(x) = f(y)$ and for every $w, z \in P$, if $f(w) = f(z)$ then $w = x$ and $z = y$ or vice-versa.

Equivalently, a map $f : P \to Q$ is an α or β-reduction, if Q is isomorphic to the quotient poset $P/E(x, y)$, where $E(x, y)$ is the bisimulation equivalence relation which identifies x and y.

Theorem 2.4.3 (Finite P-morphism Decomposition Theorem). *Let $f : P \to Q$ be an onto p-morphism between two finite posets. There exists a sequence $f_n, f_{n-1}, ..., f_0$ such that for $0 \leqslant i \leqslant n-1$, with $P_0 = P$, $f_i : P_i \to P_{i+1}$ is an α or β-reduction, and $f_n : P_n \to Q$ is an isomorphism.*

Proof. Let $f : P \to Q$ be the p-morphism. Let w be a maximal point of Q such that there are two points $x, y \in P$ such that $f(x) = f(y) = w$. Then note that $\uparrow x - \{x\} = \uparrow y - \{y\}$: for each a such that $x \leqslant a$, we have that $f(y) \leqslant f(a)$, so there is some k such that $y \leqslant k$ and $f(k) = f(a)$; but since $w \leqslant f(a)$, then $a = k$, so $y \leqslant a$, and similar vice-versa. This means that there are two options:

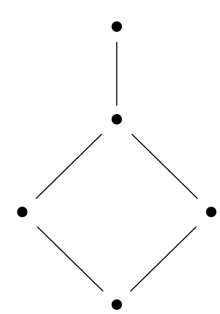
1. Case 1: x and y are incompatible. Then consider the equivalence relation on P, $E(x, y)$, which only identifies x, y. It is easy to see that this is a bisimulation equivalence, and letting $P_1 = P/E(x, y)$ we get a surjective p-morphism $f_0 : P \to P_1$ which is a β-reduction. Moreover, we get $f_r : P_1 \to Q$ by sending for each $[x]$, $f([x]) = f(x)$; this is certainly well-defined, and will produce a surjective p-morphism, and $f_r \circ f_1 = f$.

2. Case 2: Similar to Case 1, except we get an α-reduction.

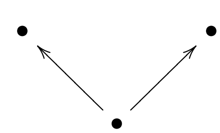
Hence we have that $f = f_r \circ f_1$. By now applying the same reasoning to f_r, since the posets are finite, we will eventually obtain the desired decomposition. ∎

Let us look at an example applying this construction:

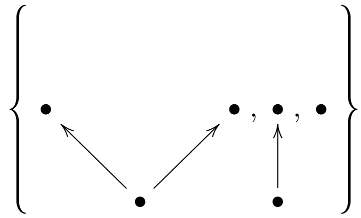
Example 2.4.29. Let H be the following Heyting algebra:



The Esakia space of H has the shape



whose p-morphic images are



Now if we go back to Heyting algebras, by taking the upsets of these spaces, we obtain an exhaustive list of all Heyting subalgebras of H, namely $\{H, \mathbf{3}, \mathbf{2}\}$, respectively, where $\mathbf{2}$ and $\mathbf{3}$ are the 2 and 3 element linear orders respectively.

As concerns homomorphic images, the filter-homomorphic image connection once again allows us to have a correspondence as in the case of Boolean algebras, and the Theorems 2.3.3 and 2.3.4 still hold true for Heyting algebras. The case for products is likewise the same, and all concepts which depend only on the bounded lattice reduct remain similar.

One thing which substantially improves is our understanding of infinitary concepts:

Theorem 2.4.4. *Let D be a distributive lattice. Assume that $S \subseteq D$ is a subset such that $\bigvee S$ exists. Then:*

1. *We have that $\varphi(\bigvee S) = {\uparrow}\overline{\bigcup_{s\in S} \varphi(s)}$*

2. *If additionally D is a Heyting algebra, then $\varphi(\bigvee S) = \overline{\bigcup_{s\in S} \varphi(s)}$.*

Proof. (1) Certainly $\varphi(s) \subseteq \varphi(\bigvee S)$, so $\bigcup_{s\in S} \varphi(s) \subseteq \varphi(\bigvee S)$, and since the latter is closed and an upset, ${\uparrow}\overline{\bigcup_{s\in S} \varphi(s)} \subseteq \varphi(\bigvee S)$. Now assume that $x \notin {\uparrow}\overline{\bigcup_{s\in S} \varphi(s)}$. Hence for each $y \in \overline{\bigcup_{s\in S} \varphi(s)}$, $y \nleq x$. Hence we can find a clopen upset U_y containing y and not x. Then $\overline{\bigcup_{s\in S} \varphi(s)} \subseteq U$, some U obtained by taking unions. Since $\varphi(s) \subseteq U$ for each $s \in S$, $\varphi(\bigvee S) \subseteq U$; since $x \notin U$, then $x \notin \varphi(\bigvee S)$.

(2) By Exercise 2.32, and by (1), we have that:

$$\varphi(\bigvee S) = {\uparrow}\overline{\bigcup_{s\in S} \varphi(s)} = \overline{{\uparrow}\bigcup_{s\in S} \varphi(s)} = \overline{\bigcup_{s\in S} \varphi(s)},$$

where the last equality follows because $\bigcup_{s\in S} \varphi(s)$ is an upset. ■

Theorem 2.4.5. *Let H be a Heyting algebra, and X_H its dual Esakia space.*

1. *$a \in H$ is a completely join-prime element if and only if $\varphi(a)$ is rooted at $x \in X_H$ and x is isolated.*

2. *$a \in H$ is an atom if and only if $\varphi(a)$ is a singleton containing a maximal element of X_H which is isolated.*

Proof. (1) If $a \in H$ is completely join-prime, consider $\varphi(a)$, with root x. Look at $U = \varphi(a) - \{x\}$; this is an open upset, so we can write it as:

$$U = \bigcup\{\varphi(c) : \varphi(c) \subseteq U\}.$$

Let $S = \{c \in H : \varphi(c) \subseteq U\}$. Now suppose that x is not isolated. Then U is not closed, so $\overline{U} = \varphi(a)$. Thus, by Theorem 2.4.4 $a = \bigvee S$. Then because a is completely join-prime, $a \leqslant s$ for some s, which means that $\varphi(a) \subseteq U$ – which is absurd. Hence x must be isolated.

Conversely, if x is isolated, and it is the root of $\varphi(a)$, assume that $a \leqslant \bigvee S$; then $\varphi(a) \subseteq \varphi(\bigvee S) = \overline{\bigcup_{s \in S} \varphi(s)}$. Then $x \in \overline{\bigcup_{s \in S} \varphi(S)}$, so consider the neighbourhood $\{x\}$. By definition of closure, $\{x\} \cap \varphi(s) \neq \varnothing$ for some s, and so $\varphi(a) \subseteq \varphi(s)$ for some $s \in S$.

(2) Note that an atom is a completely join-prime element: if $a \in H$ is an atom, and $a \leqslant \bigvee S$, then using infinite distributivity, $a = \bigvee_{s \in S} a \wedge s$, and since a is an atom, $a \wedge s = 0$ or $a \wedge s = a$, and for some s, $a \wedge s = a$ must hold, i.e., $a \leqslant s$.

Using this fact, if a is an atom, then $\varphi(a) = \{x\}$ must be isolated; it must also be maximal, since if $x \leqslant y$, there would need to be some $b \in y$, such that $a \wedge b \in y$, so $a \leqslant b$ must hold and so $b \in x$. The converse is obvious to see. ∎

Moreover, note that the maximum filters sit well within the Esakia space:

Proposition 2.4.30. Let X be an Esakia space. Then $\mathsf{Max}(X)$ is a closed subset.

Proof. Exercise. ∎

Extra Content

The following section is intended for the categorically minded reader and readers interested in further subtleties of related dualities.

2.4.4 ◊ A categorical view of Esakia duality

As before, we can consider the categories **HA** of Heyting algebras and Heyting algebra homomorphisms and **Esa** of Esakia spaces and Esakia space homomorphisms. These are respectively a subcategory of **DL** and **Pries** – but not that they are not *full* subcategories, as the class of morphisms considered is smaller. Then the previous sections show:

Theorem 2.4.6. *The functor* PSpec *restricts to a dual equivalence between* **HA** *and* **Esa***.*

Just like in previous cases, we can also define the *free product* of two Heyting algebras H_1, H_2[8]:

Definition 2.4.31. Let H_1 and H_2 be two Heyting algebras. We define the *free product of* H_1, H_2 as follows: let $\mathsf{Form}(H_1 + H_2)$ be the set of intuitionistic logic formulas with variables c_a for each $a \in H_1$ or $a \in H_2$. Let Γ denote the *theory* of H_1, H_2, i.e., it contains all the following formulas:

$$\{c_{1_{H_1}} \leftrightarrow \top, c_{1_{H_2}} \leftrightarrow \top, c_{0_{H_1}} \leftrightarrow \bot, c_{0_{H_2}} \leftrightarrow \bot\}$$
$$\{c_{a \wedge b} \leftrightarrow c_a \wedge c_b, c_{a \vee b} \leftrightarrow c_a \vee c_b, c_{a \to b} \leftrightarrow c_a \to c_b : a, b \in H_1 \sqcup H_2\}.$$

Now define an equivalence on $\mathsf{Form}(H_1 + H_2)$, $\approx$ as follows: for φ, ψ formulas there:

$$\varphi \approx \psi \iff \Gamma \vdash \varphi \leftrightarrow \psi.$$

Let $H_1 \oplus H_2$ be the quotient distributive lattice $\mathsf{Form}(B_1 + B_2)/\approx$.

[8] For this definition make sense you should be familiar with intuitionistic logic. For a very terse introduction, see chapter 3.

And indeed we have that $H_1 \oplus H_2$ is the coproduct of H_1 and H_2. One would want a more concrete description of this. So following the previous categorical sections, we consider $X_{H_1} \times X_{H_2}$. Indeed we have:

Proposition 2.4.32. If X and Y are Esakia spaces, then $X \times Y$ are Esakia spaces, and the projection maps π_X and π_Y are p-morphisms.

Nevertheless, $X \times Y$ is *not* the categorical product, as shown by the following example:

Example 2.4.33. Let $P_1 \cong P_2$ be a poset of the form $0 < 1$. Then $P_1 \times P_2$ is isomorphic to the 2-element Boolean algebra. Now consider the poset Q from Figure 2.5:

$$\begin{tikzcd}
 & (1,1) & \\
(1,0) \arrow[ur] & (0,0)^* \arrow[u] & (0,1) \arrow[ul] \\
(0,0) \arrow[u] \arrow[ur] & & (0,0) \arrow[u] \arrow[ul] \\
 & (0,0) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 2.5

The annotation on the figure denotes two maps: $p_1 : Q \to P_1$ sends the elements in the first-coordinate to their corresponding value in P_1, and similarly $p_2 : Q \to P_2$ sends the elements in the second-coordinate to P_2. Certainly these are p-morphisms. Hence if $P_1 \times P_2$ was the categorical product, the induced product map would be a p-morphism. However, notice that $p(0,0)^* = (0,0) \leqslant (1,0)$, but there is no element above $(0,0)^*$ which maps to $(1,0)$.

This problem necessitates using much more complicated tools – and turns out to be related to the structure of free Heyting algebras.

As noted before, there is a duality between **Birk** the category of Birkhoff algebras, and the category **Pos** of posets with monotone maps. We note the following:

Proposition 2.4.34. Let H be a Birkhoff algebra. Then H is a Heyting algebra.

Proof. By our results above, $H \cong \mathsf{Up}(P)$ for some poset P. Then this is the algebra of open sets on the Alexandroff topology on P, and hence is a Heyting algebra. ∎

We can thus naturally ask whether this duality restricts to $\mathbf{Birk}_{HA}$, the category of Birkhoff algebras with complete Heyting homomorphisms, and $\mathbf{Pos}_p$, the category of posets with p-morphisms. The result, due to De Jongh, is positive. Indeed, the key fats left to observe concern only morphisms:

Proposition 2.4.35.

1. If $p : P \to Q$ is a p-morphism, then $p^{-1} : \mathsf{Up}(Q) \to \mathsf{Up}(P)$ is a complete Heyting algebra homomorphism.

2. If $f : H \to H'$ is a complete Heyting algebra homomorphism, then $f^{-1} : \mathcal{J}^{\infty}(H') \to \mathcal{J}^{\infty}(H)$ is a p-morphism.

Proof. The proof of (1) is standard. We focus on (2), whereby we prove that the map indeed satisfies the p-morphism condition. To see the latter fact, assume that $x \in \mathcal{J}^{\infty}(H')$ and $y \in \mathcal{J}^{\infty}(H)$ and

$$f^{-1}[x] \subseteq y.$$

Let x' be the principal generator of the filter x. Consider the following sets:

$$Im(y) = \{f(a) : a \in y\} \text{ and } CoIm(y) = \{f(b) : b \notin y\}.$$

Then first we claim that

$$x' \wedge \bigwedge Im(y) \nleqslant \bigvee CoIm(y)$$

For suppose it was. Then

$$x' \leqslant \bigwedge\{f(a) : a \in y\} \to \bigvee\{f(b) : b \notin y\};$$

since f is a complete Heyting homomorphism, we have that

$$x' \leqslant f(\bigwedge\{a : a \in y\} \to \bigvee\{b : b \notin y\}).$$

So $f(\bigwedge\{a : a \in y\} \to \bigvee\{b : b \notin y\}) \in x$, so $\bigwedge\{a : a \in y\} \to \bigvee\{b : b \notin y\} \in f^{-1}[x]$, and by assumption, it is also in y. Since $\bigwedge\{a : a \in y\} \in y$ as well, then this means that $\bigvee\{b : b \notin y\} \in y$; but since y is completely prime, this is a contradiction.

Hence, by complete join-prime generation, we have that there must be a completely join-prime element c' such that $c' \leqslant x' \wedge \bigwedge\{f(a) : a \in y\}$ and $c \nleqslant \bigvee\{f(b) : b \notin y\}$. Let $c := \uparrow c'$. Then clearly $x \leqslant c$; moreover, we have that

$$f^{-1}[c] = y;$$

indeed if $a \in y$, then $f(a) \in c$; and if $b \notin y$, then $f(b) \notin c$, by construction. This shows that f^{-1} is a p-morphism as desired. ■

Some related questions appear here, for which we need the notion of a *profinite object*:

Definition 2.4.36. Let **C** be a concrete category (i.e., containing set-based objects) with quotient objects. An object $A \in \mathbf{C}$ is said to be *profinite* if A is the projective limit of its finite quotients.

Duals of projective limits are directed limits. Because of this, profinite Boolean algebras are exactly **CABA**s: by Tarski duality, this amounts to saying that each set X is the directed union of its finite subsets, which is trivially true. Similarly, Profinite Distributive lattices are exactly Birkhoff lattices: by Birkhoff duality, this amounts to saying that each poset P is the directed union of its finite subposets, which again is obviously true. But note that this tells us nothing about profinite *Heyting algebras*, since the morphisms connecting the algebras must be Heyting homomorphisms.

Definition 2.4.37. Let P be a poset. We say that P is *image-finite* if for each $x \in P$, $\uparrow x$ is finite.

We will now prove Bezhanishvili & Bezhanishvili's characterization of profinite Heyting algebras:

Lemma 2.4.38. Let P be a poset. Then P is a directed colimit of finite posets through a diagram consisting only of p-morphisms if and only if P is an image-finite poset.

Proof. Recall that directed colimits in the category of posets consist of disjoint unions with identification along the images. Hence, if P is an image-finite poset, then we can certainly represent it as the directed colimit of its finite upsets, all of which will be finite posets as well.

Now assume that P can be represented in this way. Suppose that $x \in P$; then by definition, x must have been introduced at some stage, say P', such that $g : P' \to P$ is a p-morphism. Now since P' is finite, this then implies that $\uparrow x$ must be finite as well, otherwise the back condition could not be satisfied on g. ■

We will also help ourselves to the following lemma (see [2, Lemma 2.6]:

Lemma 2.4.39. Let H be a profinite Heyting algebra. Then H is complete and completely join-prime generated.

We will briefly need to talk about the dual Esakia space of H when H is a Birkhoff lattice. The key fact we will need is the following:

Proposition 2.4.40. Let $H \cong \mathsf{Up}(P)$. Then P is a generated subframe of X_H through a map $p : P \to X_H$, and for each $x \in P$, $\uparrow p(x)$ is clopen.

With this we can prove the following:

Lemma 2.4.41. Let H be a complete and completely join-prime generated Heyting algebra. Then for each F a finite Heyting algebra, it is a quotient of H if and only if it is a complete quotient of H.

Proof. It is a standard fact that quotients correspond to filters, and that complete quotients correspond to complete filters, i.e., filters F where if $S \subseteq F$ then $\bigwedge S \in F$ whenever the latter exists. Complete filters in complete algebras must always be principal; so in the dual Esakia space, they are represented by closed upsets of the form $\varphi(a)$. Hence complete quotients into finite algebras correspond, dually, to finite upsets of the form $\varphi(a)$.

On the other hand, a finite quotient of such an algebra corresponds to a filter, and via duality, to a finite closed upset. Let U be such a subset in the Esakia space. Since U is finite note that $U = \bigcup_{y \in U} \uparrow y$ is a finite union of clopen subsets (by Proposition 2.4.40), so U is a clopen upset. Hence, we have that finite quotients correspond to complete finite quotients, as desired. ■

Let **ImFinPos** denote the category of image-finite posets with p-morphisms, and let **ProHA** denote the category of Profinite Heyting algebras. We obtain the following:

Theorem 2.4.7. *The equivalence between* $\mathbf{Pos}_p$ *and* $\mathbf{Birkh}_{HA}$ *restricts to an equivalence between* **ImFinPos** *and* **ProHA***.*

Proof. By definition, a Heyting algebra is profinite if and only if it is an inverse limit of the finite Heyting algebras which are its finite quotients, through Heyting surjections. Note that since these algebras are all complete, by Lemmas 2.4.39 and 2.4.41, the inverse limit is taken in the category $\mathbf{Birk}_{HA}$. Hence using the above duality, a poset P is dual to a profinite Heyting algebra if and only if it is a directed colimit of its finite upsets, through maps which are p-morphisms. But by Lemma 2.4.38, we have that these are exactly the image-finite posets. ■

To conclude this section, we note that similarly to the relationship between **DL** and **BA**, there are several adjunctions between **HA** and **BA**. In fact consider the functor $\mathsf{Reg} : \mathbf{HA} \to \mathbf{BA}$ which to H assigns $H_{\neg\neg}$, and takes the restriction of Heyting algebra homomorphisms. Then we invite the dedicated reader to prove the following:

Theorem 2.4.8. *Let $I : \mathbf{BA} \to \mathbf{HA}$ be the inclusion of Boolean algebras into Heyting algebras. Then* Reg *is the left adjoint of I.*

End of extra content.

2.5 Modal Algebras and Jónnson-Tarski Duality

The last stop in our algebraic tour will be *modal algebras* – which will provide us with a way of re-framing some facts we already know from modal logic in the setting of algebra and duality. As for the previous classes, we begin with the basic theory of these algebras, and then present the duality for them.

2.5.1 Basic theory of Modal algebras

Definition 2.5.1. A *modal algebra* is a pair $(B, \Box)$ where B is a Boolean algebra and $\Box : B \to B$ is a map satisfying:

i. $\Box 1 = 1$.

ii. $\Box(a \wedge b) = \Box a \wedge \Box b$.

Note that we can always define a map $\Diamond : B \to B$ as $\Diamond a = \neg\Box\neg a$. The analogous conditions are: i. $\Diamond 0 = 0$ and ii. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. We will denote modal algebras by $(B, \Box)$ or by $(B, \Diamond)$, depending on whichever is more convenient. A modal algebra $(B, \Box)$ is a $\mathbf{K}4-$algebra if it satisfies

iii. $\Box a \leqslant \Box\Box a$;

and an $\mathbf{S4}-$algebra if, in addition,

iv. $\Box a \leqslant a$.

The $\Diamond$ analogues of these are: iii. $\Diamond\Diamond a \leqslant \Diamond a$, and iv. $a \leqslant \Diamond a$.

Example 2.5.2. For any topological space (X, τ), we have that $(\mathcal{P}(X), \mathrm{Int})$ is an interior algebra. Conversely, if $\Box : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies (i.)-(iv.), then $\tau = \{U \in \mathcal{P}(X) : \Box(U) = U\}$ becomes a topology in which $\mathrm{Int}_\tau = \Box$.

Example 2.5.3. Let B be a Boolean algebra. Define $\Box a = a$. Then $(B, \Box)$ is a modal algebra, called a *discrete modal algebra*.

Similarly, define $\Box a = 0$ for every $a \neq 1$. Then this forms a modal algebra, called the *indiscrete modal algebra*.

Example 2.5.4. Let (X, R) be a Kripke frame. Consider $(\mathcal{P}(X), \Box_R)$ (see definition 1.2.5). Then this is a complete and atomic modal algebra. Note that this algebra furthermore has the property that for each collection $(S_i)_{i \in I}$ where $S_i \subseteq X$:

$$\Box(\bigcap_{i \in I} S_i) = \bigcap_{i \in I} \Box S_i,$$

i.e., it satisfies the infinitary analogue of normality.

Let us collect some basic algebraic facts about modal algebras:

Lemma 2.5.5. Let M be a modal algebra. Then:

1. $\Box(a \to b) \leqslant \Box a \to \Box b$;
2. $\Box a \vee \Box b \leqslant \Box(a \vee b)$;
3. $\Diamond(a \wedge b) \leqslant \Diamond a \wedge \Diamond b$.
4. If $a \leqslant b$ then $\Box a \leqslant \Box b$ and $\Diamond a \leqslant \Diamond b$.

Proof. Exercise. ■

Modal algebra homomorphisms are defined as usual. Moreover, we have the typical definition of a subalgebra:

Definition 2.5.6. Given M a modal algebra, a subset $S \subseteq M$ is a *modal subalgebra* if S is a Boolean subalgebra and it is closed under $\Box$.

Homomorphic images are again images of surjective homomorphisms. Like in the previous cases we can make them correspond to filters, but with additional requirements:

Definition 2.5.7. A subset $F \subseteq M$ of a modal algebra M is said to be a $\Box$-*filter* (or an *open filter* if M is an **S4**-algebra) if whenever $a \in F$, then $\Box a \in F$.

Definition 2.5.8. If $F \subseteq M$ is a filter, then define the equivalence relation:

$$a \approx_F b \iff a \leftrightarrow b \in M,$$

where $a \leftrightarrow b := a \to b \wedge b \to a$. We let $M/F := M/\approx_F$.

Lemma 2.5.9. The set M/F has a Heyting algebra structure if we define it as:

- $0_{M/F} := [0_M]_F$ and $1_{M/F} := [1_M]_F$;
- $[a]_F \wedge_{M/F} [b]_F := [a \wedge b]_F$ and $[a]_F \vee_{M/F} [b]_F := [a \vee b]_F$;
- $\neg_{M/F}[a]_F := [\neg a]_F$ and $\Box_{M/F}[a]_F := [\Box a]_F$.

Proof. Exercise. ■

Proposition 2.5.10 (First Homomorphism Theorem for modal algebras)**.**

1. If M is a modal algebra, and F is a $\Box$-filter, the map $q_F : M \to M/F$ given by $q_F(a) = [a]_F$ is a surjective homomorphism.

2. If $f : M \to M'$ is a surjective homomorphism, then $F = f^{-1}[1_{M'}]$ is a $\Box$-filter. Moreover, $M/F \cong M'$.

Proof. Exercise. ■

Products of modal algebras are defined in the way you would expect: the $\Box$ is given pointwise.

Zooming in briefly to the case of **S4**-algebras, we can consider the following interesting construction:

Definition 2.5.11. Let M be an **S4**-algebra. We say that an element $a \in M$ is *open* if $\Box a = a$. We write $M_\Box = \{a \in M : a \text{ is open }\}$. We equip this structure with the following operations for every $a, b \in M_\Box$:

$$
\begin{aligned}
0_\Box &:= 0 \\
1_\Box &:= 1 \\
a \wedge_\Box b &:= a \wedge b \\
a \vee_\Box b &:= a \vee b \\
a \to_\Box b &:= \Box(a \to b).
\end{aligned}
$$

Proposition 2.5.12. Let M be an **S4**-algebra.

1. The structure $(M_\Box, \wedge_\Box, \vee_\Box, \to_\Box, 0_\Box, 1_\Box)$ is well-defined (i.e., open elements are closed under these operations).

2. $M_\Box$ is a Heyting algebra.

Proof. (1) 1 is an open element by definition of being a modal algebra, and since $\Box a \leqslant a$, $\Box 0 \leqslant 0$, so 0 is open. Now if a, b are open, note that $\Box(a \wedge b) = \Box a \wedge \Box b = a \wedge b$, so meets of open elements are open. Moreover, note that certainly $\Box(a \vee b) \leqslant a \vee b$ always, and also by Lemma 2.5.5:

$$a \vee b = \Box a \vee \Box b \leqslant \Box(a \vee b).$$

Finally, the combination of axioms iii and iv means that $\Box\Box a = \Box a$, so closure under implication is clear.

(2) It is obvious that $M_\Box$ is a lattice, seeing that it is a bounded sublattice of a Boolean algebra. To see that it is a Heyting algebra, note that for $a, b, c \in M_\Box$:

$$
\begin{aligned}
a \wedge c \leqslant b &\iff c \leqslant a \to b \\
&\iff c \leqslant \Box(a \to b);
\end{aligned}
$$

for the second equivalence, note that $c \leqslant a \to b$ implies by the above Lemma that $\Box c \leqslant \Box(a \to b)$, and $\Box c = c$; conversely, if $c \leqslant \Box(a \to b)$, since $\Box(a \to b) \leqslant a \to b$, we get the equivalence. ■

It is natural to wonder whether Heyting algebras also can be turned into **S4**-algebras. We will return later to this question.

2.5.2 Jónnson-Tarski duality

For this section, for convenience, we work with modal algebras in the signatura with $\Diamond$.

Definition 2.5.13. A *modal space* is a pair (X, R) where X is a Stone space and $R \subseteq X^2$ such that

1. $R[x]$ is closed for every $x \in X$ (point-closedness).
2. If $U \in \mathsf{Clop}(X)$, then $R^{-1}[U] \in \mathsf{Clop}(X)$.

Example 2.5.14. Let $(X, \leqslant)$ be an Esakia space where $R = \leqslant$, and let $U \subseteq X$ be a clopen set. Then $U = \bigcup_{i=1}^{n} U_i - V_i$. If $R^{-1}[U_i - V_i]$ is clopen for each i, then so will $R^{-1}[U]$. The fact that $R^{-1}[U_i - V_i]$ is clopen follows directly from $U_i \to V_i$ being clopen. Moreover, $\uparrow x$ is closed for each $x \in X$. Hence Esakia spaces are modal spaces.

Lemma 2.5.15. If (X, R) is a modal space, then $(\mathsf{Clop}(X), R^{-1})$ is a modal algebra.

Proof. Exercise. ■

Just like Esakia duality is built on Priestley duality, Jónnson-Tarski duality will be built on top of Stone duality. Given $(M, \Box)$, a modal algebra, we take $X = X_M$ to be the Stone dual of M and on it we define

$$\begin{aligned} xRy &\text{ iff } \Box a \in x \Rightarrow a \in y \\ &\text{ iff } b \in y \Rightarrow \Diamond b \in x. \end{aligned}$$

These two definitions are equivalent. Indeed, suppose ($\Box a \in x \Rightarrow a \in y$) holds and let $b \in y$. Now, if $\Diamond b \notin x$, then, since it is an ultrafilter, $x \ni \neg\Diamond b = \Box\neg b$, which implies $\neg b \in y$: contradiction. The other direction is analogous. Note that with such a definition, for each $x \in X$:

$$R[x] = \bigcap_{\Box a \in x} \varphi(a),$$

hence we immediately get point-closedness.

The following might appear somewhat reminiscent of the Existence Lemma:

Theorem 2.5.1. *The Stone map φ defined by $\varphi(a) = \{x \in X : a \in x\}$ is a modal algebra isomorphism, i.e., $\varphi(\Diamond a) = R^{-1}[\varphi(a)]$.*

Proof. Suppose $x \in R^{-1}[\varphi(a)]$. Then there is a $y \in X$ such that xRy and $y \in \varphi(a)$, i.e., xRy and $a \in y$. But this implies that $\Diamond a \in x$ by the definition of R. So $x \in \varphi(\Diamond a)$.

Conversely, suppose $x \in \varphi(\Diamond a)$. Then $\Diamond a \in x$. We need to find an ultrafilter y with xRy and $y \in \varphi(a)$. Consider $y_0 := \mathsf{Fil}(\{b \in M : \Box b \in x\} \cup \{a\})$. We will show that this is a proper filter: indeed, if not, then for some $\Box b_0, ..., \Box b_n \in x$, we would have:

$$b_0 \wedge ... \wedge b_n \wedge a = 0.$$

By usual facts of Boolean algebras, this means:

$$b_0 \wedge ... \wedge b_n \leqslant \neg a,$$

so $\Box(b_0 \wedge ... \wedge b_n) \leqslant \Box\neg a$. But by normality the left-hand side belongs to x, so $\Box\neg a \in x$, which contradicts the fact that $\Diamond a \in x$. Thus by reductio, y_0 is proper. So by the Ultrafilter Theorem (see Theorem 2.2.2), we have that there is some $y \supseteq y_0$ an ultrafilter. By construction, xRy, and $a \in y$, which is what we needed to show. ■

Definition 2.5.16. Let $p : X \to Y$ be a continuous map between modal spaces. We say that p is a *continuous p-morphism* if it satisfies the following:

1. If $x, y \in X$, xRy then $p(x)Rp(y)$.

2. If $x \in X$ and $z \in Y$ and $p(x)Rz$, then there is some $y \in X$ such that xRy and $p(y) = z$.

A continuous map between modal spaces is an isomorphism if it is a homeomorphism and xRy if and only if $p(x)Rp(y)$.

Theorem 2.5.2. *For each modal space (X, R), $X \cong X_{\mathsf{Clop}(X)}$.*

Proof. Exercise. ∎

The representation can be extended to a full duality between the category of modal algebras and modal algebra homomorphisms and the category of modal spaces and continuous p-morphisms.

Proposition 2.5.17. Let $p : X \to Y$ be a continuous p-morphism. Then $p^{-1} : \mathsf{Clop}(Y) \to \mathsf{Clop}(X)$ is a modal homomorphism.

Proof. Exercise. ∎

Proposition 2.5.18. Let $f : M \to M'$ be a modal homomorphism. Then $f^{-1} : X_{M'} \to X_M$ is a continuous p-morphism.

Proof. Exercise, using the Ultrafilter theorem. ∎

In light of this duality, it is natural to ask: what are the modal spaces that correspond to **K4** algebras? And the ones that correspond to **S4** algebras? This turns out to be a question of *correspondence*, and can be elegantly answered:

Proposition 2.5.19. Let $(M, \Diamond)$ be a modal algebra and (X, R) its dual modal space. Then:

1. $(M, \Diamond)$ is a **K4**-algebra if and only if R is transitive.

2. $(M, \Diamond)$ is a **T**-algebra if and only if R is reflexive.

3. $(M, \Diamond)$ is an **S4**-algebra if and only if R is reflexive and transitive.

Proof. By duality, we show that $(\mathsf{Clop}(X_B), R^{-1})$ is an **L**-algebra if and only if R satisfies the condition. We show (1) and leave the remaining ones as exercises.

Indeed, if R is transitive, we will show that for each clopen U:

$$R^{-1}[R^{-1}[U]] \subseteq R^{-1}[U];$$

indeed if x belongs to the first set, xRy and yRz where $z \in U$. By transitivity, xRz, so $x \in R^{-1}[U]$.

Conversely, assume that xRy and yRz but $\neg(xRz)$. By definition, this means that there is some clopen $\varphi(a)$ such that $z \in \varphi(a)$ but $x \notin R^{-1}[\varphi(a)]$. By the assumptions it is clear to see that $x \in R^{-1}[R^{-1}[\varphi(a)]]$, which shows the axiom fails. ∎

2.5.3 General frames and modal spaces

It is worthy to pause quickly and consider the relationship between modal spaces as outlined in the previous section and general frames. Indeed, one fact is obvious:

Lemma 2.5.20. Let (X, R) be a modal space. Then the triple $(X, R, \mathsf{Clop}(X))$ is a general frame.

Proof. Immediate from the definition of general frame and the facts proven in section 2.5.2. ■

Not every general frame arises in this way. To see why, we recall some definitions pertaining to general frames (see e.g. [4]):

Definition 2.5.21. Let $\mathfrak{F} = (F, R, \mathcal{A})$ be a general frame. We say that $\mathfrak{F}$ is:

- *differentiated* if whenever $x, y \in F$ and $x \neq y$ then there is some $U \in \mathcal{A}$ such that $x \in U$ and $y \notin U$;
- *tight* if whenever $x, y \in F$ and $\neg(xRy)$ the there is some $U \in \mathcal{A}$ such that $y \in U$ and $x \notin R^{-1}[U]$;
- *compact* if $\bigcap A_0 \neq \varnothing$ for each $A_0 \subseteq \mathcal{A}$ a subset with the finite intersection property;
- *refined* if it is differentiated and tight;
- *descriptive* if it is differentiated, tight and compact.

A cursive look at the above reveals that a differentiated and compact general frame enforces that $\mathcal{A}$ be the basis for a Stone topology, whilst tightness enforces that the relation R be continuous (i.e., $R[x]$ being closed and $R^{-1}[U]$ being clopen whenever U is clopen). Hence descriptive general frames are definitionally equivalent to modal spaces.

Some examples of general frames where this fails give rise to interesting constructions:

Definition 2.5.22. Let $\mathfrak{F} = (F, R, \mathcal{A})$ be a refined general frame. We say that $\mathfrak{C}(\mathfrak{F}) = (F_{\mathcal{A}}, R_{\mathcal{A}}, \mathsf{Clop}(F_{\mathcal{A}}))$, where $\mathcal{A} \cong \mathsf{CLop}(F_{\mathcal{A}})$, and $(F_{\mathcal{A}}, R_{\mathcal{A}})$ is the modal space of $\mathcal{A}$, is the *ultrafilter extension* of $\mathfrak{F}$.

Lemma 2.5.23. For each $\mathfrak{F}$ a refined general frame, there is an embedding $i : \mathfrak{F} \to \mathfrak{C}(\mathfrak{F})$.

Proof. Each point $x \in \mathfrak{F}$ is sent to:

$$\varepsilon(x) = \{U \in \mathcal{A} : x \in U\},$$

which is always an ultrafilter. Now if xRy, it is easy to see that $\varepsilon(x)R\varepsilon(y)$. Conversely if $\neg(xRy)$, by refinedness, there is some U such that $y \in U$ and $x \notin R^{-1}[U]$. Then $U \in \varepsilon(y)$ and $R^{-1}[U] \notin \varepsilon(x)$, showing that $\neg(\varepsilon(x)R_{\mathcal{A}}\varepsilon(y))$. ■

Example 2.5.24. Consider $\mathfrak{F} = (\omega, <, \mathsf{Fin}(\omega) + \mathsf{CoFin}(\omega))$. Then the ultrafilter extension will coincide with the Alexandroff compactification, adding a single point and obtaining $(\omega + 1, <, \mathsf{Fin}(\omega) + \mathsf{CoFin}(\omega + 1)^*)$.

In the case of Kripke frames (F, R), we also speak of the *ultrafilter extension* of F, by considering the refined general frame $(F, R, \mathcal{P}(F))$, and taking the ultrafilter extension there.

2.5.4 Duals of constructions

As regards the dual picture, the situation is analogous to what we have come to expect:

Proposition 2.5.25. Let $f : M \to M'$ be a homomorphism between modal algebras.

1. f is injective if and only if f^{-1} is a surjective p-morphism.
2. f is surjective if and only if f^{-1} is an isomorphism onto a generated subframe Y of X_M.
3. $X_{M\times M'} \cong X_M \sqcup X_{M'}$.

Proof. Exercise. ∎

The correspondence between $\Box$-filters and homomorphic images likewise gives us some meaning to $\Box$-filters; given a modal algebra let $\Box\mathsf{Fil}(M)$ be the set of its boxed filters, and given a modal space (X, R) let $\mathsf{ClosedG}(X)$ be the set of its closed generated subframes:

Theorem 2.5.3. *There is a dual isomorphism between* $(\Box\mathsf{Fil}(M), \subseteq)$ *and* $(\mathsf{ClosedG}(X))$.

Let us quickly turn to the case of **S4**-algebras. As noted in section 2.5.1, we have that given an **S4**-algebra M, $M_\Box$ is a Heyting algebra. This can also be seen dually:

Definition 2.5.26. Let (X, R) be a transitive and reflexive space. Define an equivalence relation on this space by: $x \approx y$ if and only if xRy and yRx. Let $\rho(X) = X/\approx$, and let $[x]R_\rho[y]$ if and only if xRy.

Proposition 2.5.27. The space $(\rho(X), R_\rho)$ with the quotient topology from (X, R) is an Esakia space.

Proof. Exercise. ∎

Extra Content

The following section is intended for the categorically minded reader.

2.5.5 ♢ A categorical view of Jónnson-Tarski duality

Let **MA** be the category of modal algebras and modal algebra homomorphisms, and **MS** be the category of modal spaces and continuous p-morphisms. Our results show:

Theorem 2.5.4. *We have that* $\mathbf{MA}^{op} \simeq \mathbf{MS}$.

End of extra content.

2.6 Duality dictionaries

For ease of access, we summarise the results concerning duality in some *duality dictionaries*, allowing us to move between algebraic and (order, relational) topological concepts:

[9] There is such a description available, but that would take us very, very far.

BA	**Stone**
Boolean homomorphism	Continuous function
Filter	Closed set
Homomorphic image	Closed set
Ideal	Open set
Maximal Filter	Point
Maximal Ideal	$X - \{x\}$
Atom	Isolated Point
Subalgebra	Image under a continuous map
Product	Topological sum
Free product	Cartesian Product
Complete Boolean algebra	Extremally disconnected Stone space
Superatomic Boolean algebra	Scattered Stone space
CABA ($\mathcal{P}(X)$)	$\beta(X)$ for set X

Table 2.1: Stone duality dictionary

2.7 Exercises

2.7.1 Lattices

Exercise 2.1. Prove (some cases of) Proposition 2.1.5.

Exercise 2.2. Suppose $(L, \vee, \wedge)$ is a lattice in the algebraic sense. Show that:

1. If one defines for $a, b \in L$ $a \leqslant b$ if and only if $a \wedge b = a$ then $\leqslant$ is a partial order. Similarly, if one defines $a \leqslant' b$ if and only if $a \vee b = b$ then $\leqslant'$ is a partial order.

2. We have that $\leqslant \, = \, \leqslant'$.

3. Given $a, b \in L$, the supremum of a, b with respect to $\leqslant$ is $a \vee b$, and the infimum of them is $a \wedge b$.

Exercise 2.3. Let $(P, \leqslant)$ be a poset. Show that the following are equivalent:

1. $(P, \leqslant)$ is a complete lattice;

2. For each $S \subseteq P$, $\inf S$ exists.

3. For each $S \subseteq P$, $\sup S$ exists.

Exercise 2.4 (Funny Examples)**.**

a. Give an example of a lattice $(L, \leqslant)$ such that no infinite subset $X \subseteq L$ has a least upper bound.

b. Consider the poset $(\mathbb{N}, \leqslant)$. Is this a lattice? Is it complete?

c. Find an example of a lattice $(L, \leqslant)$ that contains a subset $A \subseteq L$ such that $\inf A$ and $\sup A$ exist but $\sup A \neq \inf A$ and $\sup A \leqslant \inf A$.

DL	Pries
DL homomorphism	Priestley morphism
Filter	Closed upset
Homomorphic image	Closed subset
Ideal	Open upset
Prime Filter	Point
Maximal Filter	?
Join-irreducible	$\uparrow x$ for $x \in X$
Completely Join-Irreducible	?
Atom	?
Subalgebra	Image under a continuous map
Product	Priestley sum
Free product	Product of Posets
Complete Distributive Lattice	Extremally order-disconnected Priestley space
Birkhoff Lattice ($\mathsf{Up}(P)$)	$\nu(P)$ for P a poset

Table 2.2: Priestley duality dictionary

d. Find an example of a poset where $\inf \varnothing$ does not exist.

e. Give an example of a lattice $(A, \leqslant)$ and a subset B of A such that $(B, \leqslant|_{B\times B})$ is a lattice, but B is not a sublattice of A.

Exercise 2.5. Give an example of a poset $(P, \leqslant)$ in which there are three elements x, y, z satisfying simultaneously the following three conditions:

1. $\{x, y, z\}$ is an antichain (a set $A \subseteq P$ is an *antichain* if $a \nleqslant b$ for distinct $a, b \in A$),

2. $x \vee y$, $y \vee z$ and $z \vee x$ fail to exist,

3. $\bigvee\{x, y, z\}$ exists.

Exercise 2.6. Consider a finite poset $(P, \leqslant)$.

a. Suppose there exist two elements $a, b \in P$ such that $\{a, b\}$ has *upper bounds*, but not a *l.u.b.* Show that there exist two elements $c, d \in P$ such that $\{c, d\}$ has *lower bounds*, but not a *g.l.b.*

b. Does the property hold for infinite posets? Justify or give a counterexample.

Given a lattice L, and two elements $a, b \in L$ such that $a \leqslant b$ we can consider the interval $[a, b]$ defined as:

$$[a, b] = \{c \in L : a \leqslant c \leqslant b\}.$$

It is immediate to see that $[a, b]$ is a sublattice of L.

Exercise 2.7. Let L be a lattice.

1. Show that if L is distributive then L is modular.

HA	**Esa**
Heyting homomorphism	Continuous p-morphism
Filter	Closed upset
Homomorphic image	Closed upset
Ideal	Open upset
Prime Filter	Point
Maximal Filter	$x \in \mathsf{Max}(X)$
Join-irreducible	$\uparrow x$ for $x \in X$
Completely Join-Irreducible	$\uparrow x$ for $x \in X$, x isolated
Atom	$x \in \mathsf{Max}(X)$, x isolated
Subalgebra	P-morphic image
Product	Esakia sum=Priestley sum
Free product	.[9]
Complete Heyting Algebra	Extremally order-disconnected Esakia space
Birkhoff Lattice ($\mathsf{Up}(P)$)	$\nu(P)$ for P a poset
Profinite Heyting Algebra	$\mathsf{Up}(P)$ for image-finite P

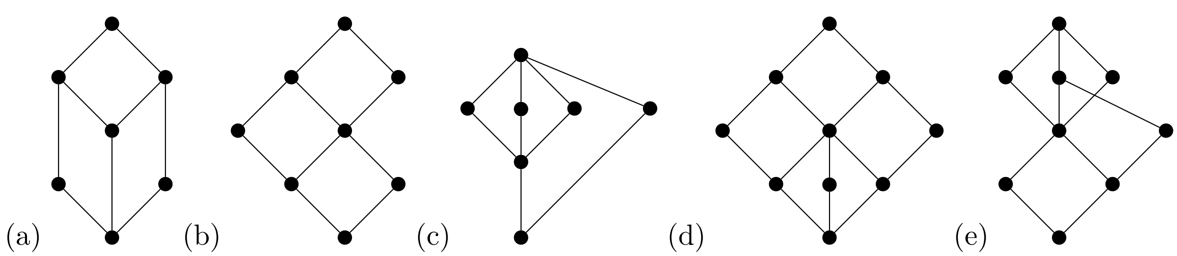
Table 2.3: Esakia duality dictionary

MA	**MS**
modal homomorphism	Continuous p-morphism
$\Box$-Filter	Closed generated subframe
Homomorphic image	Closed generated subframe
Filter	Closed subset
Subalgebra	P-morphic image
Product	Disjoint union of Modal Spaces
Complete modal algebra	Extremally disconnected modal space

Table 2.4: Jónnson-Tarski duality dictionary

2. (*) Show that a lattice L is modular if and only if for each $a, b \in L$, the maps $j_a : [a \wedge b, b] \to [a, a \vee b]$ given by $j_a(x) = x \vee a$ and $m_b : [a, a \vee b] \to [a \wedge b, b]$ given by $m_b(x) = x \wedge b$ are lattice isomorphisms.

Exercise 2.8. Which of the following lattices are modular and which of them are distributive?



Exercise 2.9. Prove Lemma 2.1.17.

Exercise 2.10. Let L be a bounded lattice.

1. Show that if L is finite, then every filter is principal.

2. Does the converse to (1) hold?

Exercise 2.11. Prove Lemma 2.1.21.

2.7.2 Boolean algebras

Exercise 2.12. Prove Proposition 2.2.2.

Definition 2.7.1. Let $(B, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra. Let $a \in B$ be arbitrary. Let $B{\restriction}a$ be the algebra

$$([0, a], \vee, \wedge, ^*, 0, a)$$

where $[0, a] = \{x \in \mathbf{B} : 0 \leqslant x \leqslant a\}$, $\vee$ and $\wedge$ are the same except restricted to $[0, a]$, and $b^* = a \wedge \neg b$.

Definition 2.7.2. Let B, C be two Boolean algebras. We define the *product* of B and C, denoted by $B \times C$ as follows: for each operation we define it pointwise, e.g.:

$$(a_1, b_1) \wedge_{B \times C} (a_2, b_2) := (a_1 \wedge a_2, b_1 \wedge b_2),$$

and likewise for $\vee, \neg, 0, 1$.

Exercise 2.13. Let B be a Boolean algebra.

1. Let $a \in B$. Show that $B{\restriction}a$ is a Boolean algebra.

2. Define a map $f_a : B \to B{\restriction}a$ by $f_a(b) = a \wedge b$. Show that f_a is a surjective homomorphism.

3. Show that the map $f_a \times f_{\neg a} : B \to B{\restriction}a \times B{\restriction}\neg a$, given by

$$f_a \times f_{\neg a}(b) = (f_a(b), f_{\neg a}(b))$$

   is an isomorphism.

4. We say that a Boolean algebra B is *directly indecomposable* if for each Boolean algebras C, D such that $B \cong C \times D$, then either $B \cong C$ or $B \cong D$. Use the previous parts to show that the algebra $\mathbf{2}$ is the unique directly indecomposable Boolean algebra.

Exercise 2.14. (**Atoms and co-Atoms**) Recall that, if $(L, \leqslant)$ is a bounded lattice, $a \in L$ is called an *atom* if $b < a$ implies $b = 0$ and a *co-atom* if $a < b$ implies $b = 1$. Define $\mathsf{At}(a) := \{x \in \mathsf{At}(L) \mid x \leqslant a\}$.

1. Describe atoms and co-atoms on a Boolean algebra of the form $\mathcal{P}(X)$.

2. Show that in every Boolean algebra, if a is an atom, then $\neg a$ is a co-atom.

3. Show that in every Boolean algebra, $\mathsf{At}(\neg b) = \mathsf{At}(X) \backslash \mathsf{At}(b)$.

Exercise 2.15. Let A be a finite Boolean algebra. We order the set of all filters of A by inclusion. Show that A has a least non-unital filter iff A is isomorphic to a two-element Boolean algebra. Note that a least non-unital filter is a filter $F \subseteq A$ such that (i) $F \neq \{1\}$ and (ii) for each filter $F' \neq \{1\}$ we have $F \subseteq F'$.

Exercise 2.16. Let X be a Stone space.

1. Define the map $\varepsilon : X \to X_{\mathsf{Clop}(X)}$ as follows: for each $x \in X$:

$$\varepsilon(x) = \{U \in \mathsf{Clop}(X) : x \in U\}.$$

   Show that ε is a well-defined continuous map.

2. Show that ε is a bijective continuous map.

3. Conclude that $X \cong X_{\mathsf{Clop}(X)}$.

Exercise 2.17.

The aim of this exercise is to understand *Tarski duality* between complete and atomic Boolean algebras and sets. This duality is closely related to Stone duality, but still differs from it.
Let **CABA** be the class of complete and atomic Boolean algebras. Let also **Set** be the class of all sets. To each set X we associate the powerset Boolean algebra $\mathcal{P}(X)$. To each complete atomic Boolean algebra B we associate the set $\mathsf{At}(B)$ of its atoms. Show that

1. Every complete and atomic Boolean algebra B is isomorphic to $\mathcal{P}(\mathsf{At}(B))$.

2. Every set X is bijective to $\mathsf{At}(\mathcal{P}(X))$.

3. Show that $\mathbf{Set}^{\mathrm{op}} \cong \mathbf{CABA}$ (this is for people who know (want to learn a bit more) category theory).

Exercise 2.18. Let α be an ordinal. Consider the order topology on $\alpha + 1$, i.e., the following is a subbasis:

$$(\infty, a) = \{x \in \alpha + 1 : x < a\} \text{ and } (a, \infty) = \{x \in \alpha + 1 : a < x\},$$

for $a \in \alpha + 1$. Then show:

1. $\alpha + 1$ is a Stone space.

2. $\alpha + 1$ is scattered and extremally disconnected.

Exercise 2.19. Let A be an infinite Boolean algebra, and let B be a finite Boolean algebra. Show that B is isomorphic to a subalgebra of A. (*Hint*: use duality.)

2.7.3 Distributive lattices and Priestley duality

Exercise 2.20. Complete the proofs of Priestley duality in Section 2.3.2.

Exercise 2.21. (Priestley duality generalizes Stone duality)

1. Let $(X, \tau, \leqslant)$ be a Priestley space. Show that if the partial order '$\leqslant$' is the identity '$=$', then the Priestley-dual distributive lattice of $(X, \tau, \leqslant)$ is the Stone-dual Boolean algebra of (X, τ).

2. Let D be a distributive lattice. Show that if D is a Boolean algebra, then its dual Priestley space is its dual Stone space with the identity relation.

Exercise 2.22. Let X be a Priestley space.

1. Show that $\downarrow\{y\}$ and $\uparrow\{y\}$ are closed for each $y \in X$.

2. Show that if $Y \subseteq X$ is closed in X, then $\uparrow Y$ and $\downarrow Y$ are closed in X.

3. Show that if $Y, Z \subseteq X$ are closed and $\uparrow Y \cap \downarrow Z = \varnothing$, then there is a clopen upset C s.t. $Y \subseteq C$ and $Z \cap C = \varnothing$.

4. Show that for each U a closed upset, $U = \bigcap\{V \in \mathsf{ClopUp}(X) : U \subseteq V\}$ and if U is a closed downset, $U = \{V \in \mathsf{ClopDown}(X) : U \subseteq V\}$.

(Observe that (a) follows from (b). But (a) is easier, and it is good practice to first do (a) before generalizing to (b).)

Exercise 2.23. For a poset P, recall that $x \in P$ is *minimal* if for every $y \in P$, we have $y \leqslant x$ implies $x = y$; and *maximal* if $x \leqslant y$ implies $x = y$. Let $\mathsf{Min}(P)$ and $\mathsf{Max}(P)$ denote the set of minimal elements and maximal elements, respectively, of the poset P. Prove that if X is a non-empty Priestley space then both $\mathsf{Min}(X)$ and $\mathsf{Max}(X)$ are non-empty.

Exercise 2.24. A Priestley space X is called extremally order-disconnected if for every open upset U in X, the smallest closed upset containing U is open. Prove that a bounded distributive lattice L is complete if and only if the space $D(L)$ is extremally order-disconnected. (Hint: use the fact that Y is a closed upset in X iff Y is an intersection of clopen upsets).

Exercise 2.25. Let $f : D \to D'$ be a bounded lattice homomorphism between distributive lattices:

1. f is an embedding if and only if f^{-1} is a surjective continuous map.

2. (*) f is surjective if and only if f^{-1} is an order-embedding and a topological embedding.

   Hint: Given $\varphi(b) \subseteq X_{D'}$ consider $f^{-1}[\varphi(b)]$ and $f^{-1}[X_{D'} - \varphi(b)]$; then use Exercise 2.22(3).

3. Show that f^{-1} can be injective without f being surjective.

4. Given D_1, D_2 two Distributive lattices, $X_{D_1 \times D_2} \cong X_{D_1} \sqcup X_{D_2}$.

2.7.4 Heyting algebras and Esakia duality

Exercise 2.26. Check that axioms (h1)-(h4) hold of the operation $\to$ in a Heyting algebra.

Exercise 2.27. Let A_1, A_2 and A_3 be the posets in Figure 2.6.

1. Convince yourself that A_1, A_2 and A_3 are all Heyting algebras.

2. Identify the pseudo-complements of elements, and the join-irreducible elements in A_1, A_2 and A_3.

3. Is A_1 isomorphic to a bounded sublattice of A_2 or A_3? Is it isomorphic to a Heyting subalgebra of A_2 and A_3?

4. Show that A_1, A_2 and A_3 contain non-maximal prime filters.

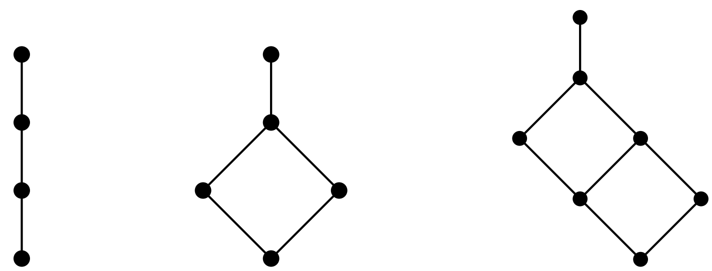


Figure 2.6: The lattices A_1, A_2, A_3.

Exercise 2.28. Let H be a complete lattice satisfying the infinite distributivity law. Show that for each $a, b \in H$, defining:

$$a \to b := \bigvee \{c \in H : c \wedge a \leqslant b\},$$

gives us a Heyting algebra.

Exercise 2.29. Show that in every Heyting algebra

1. $a \wedge \neg a = 0$;
2. $a \leqslant b$ iff $a \to b = 1$;
3. $a \leqslant \neg\neg a$;
4. $\neg\neg\neg a \leqslant \neg a$;
5. $\neg a \vee \neg b \leqslant \neg(a \wedge b)$;
6. $b \leqslant c$ implies $a \to b \leqslant a \to c$;
7. $b \leqslant c$ implies $c \to a \leqslant b \to a$.
8. $a \to (b \to c) = (a \wedge b) \to c$.

Exercise 2.30.

1. Prove that the definition of H/F given in Lemma 2.4.8 is correct.
2. Prove the First homomorphism theorem for Heyting algebras.

Exercise 2.31. Let H be a Heyting algebra and X_H be its Esakia space.

1. Show that $p : H \to H_{\neg\neg}$ given by $p(a) = \neg\neg a$ is a surjective homomorphism.
2. Show that dually, $X_{H_{\neg\neg}}$ is a closed upset of X_H, and that in fact $X_{H_{\neg\neg}}$ is isomorphic to $\mathsf{Max}(X)$.

Exercise 2.32. Let X be a Priestley space. Show that the following are equivalent:

1. X is an Esakia space;
2. For each subset $S \subseteq X$, $cl(\uparrow S) = \uparrow cl(S)$

Exercise 2.33. Let X be a copy of the Priestley space X_1 from Figure 2.4, and let $Y = \{y\}$ be a single element poset. Let Z be the space obtained by taking the topological sum of X and Y (i.e., its carrier is $X \sqcup Y$), and the order on Z is the disjoint union of the orders, except $\omega \leqslant y$.

1. Show that $X \sqcup Y$ is an Esakia space, and Z is a Priestley space which is not an Esakia space.

2. Show that Z is a Priestley morphic image of $X \sqcup Y$.

3. Conclude by duality and the last two parts that there exist Heyting algebras H which have bounded sublattices which are not Heyting algebras.

2.7.5 Modal algebras and Jonnson-Tarski duality

Exercise 2.34. Prove Lemma 2.5.5.

Exercise 2.35. Let (X, R) be a modal space. Prove that $R[U]$ is closed if U is closed.

Exercise 2.36. Let $(\mathcal{P}(W), \Box)$ be a modal algebra where $\Box$ distributes over infinite meets; that is, for all $\mathcal{A} \subseteq \mathcal{P}(W)$:

$$\Box\left(\bigcap \mathcal{A}\right) = \bigcap\{\, \Box A \mid A \in \mathcal{A} \,\}.$$

Show that there exists a binary relation $R \subseteq W \times W$ such that $\Box_R = \Box$.

Exercise 2.37. Let $(M, \Box)$ be a modal algebra and (X_M, R) its dual modal space.

1. Show that M satisfies for all a:
$$a \leqslant \Box\Diamond a,$$
if and only if R is symmetric.

2. Show that M satisfies for all a:
$$\Diamond a \leqslant \Box\Diamond a,$$
if and only if R is Euclidean.

3. Show that M satisfies for all a:
$$\Diamond\Box a \leqslant \Box\Diamond a,$$
if and only if R is directed.

4. Assume that (X, R) is transitive and reflexive. Show that M satisfies for all a:
$$\Box\Diamond a \leqslant \Diamond\Box a,$$
if and only if for each $x \in X$, there exists some y such that xRy and $R[y] = \{y\}$.

Exercise 2.38. Show that if (X, R) is an **S4**-space, then $(\rho(X), R_\rho)$ (see definition 2.5.26) with the quotient topology is an Esakia space.

Chapter 3

Varieties and Algebraic Completeness

Thus far we have considered algebras as single structures, and spaces as single structures, and the dualities that can hold there. We now wish to bring these tools back to logic. For that purpose, we need to: 1) find a way of making algebras into models for our logical systems; 2) collect together all models of a given logical system in an algebraically meaningful way. As it turns out, *universal algebra* contains all the tools we might want for achieving this. As such, in this chapter we start by recalling all the important concepts from universal algebra which we will need to study logical systems as well as lattices of logics; we then move on to showing how these concepts turn out in the cases we have seen so far: classical logic (**BA**), positive logic (**DL**), intuitionistic logic (**HA**) and modal logic (**MA**).

> Extra Content
>
> The following section is included for completeness, for the sake of the reader not familiar with universal algebra. Readers which know this material may safely skip it.

3.1 ◊ Universal Algebra and Varieties

Definition 3.1.1. A *language of algebras* $\mathcal{F}$ is a set of function symbols such that for each $f \in \mathcal{F}$ there is a unique non-negative integer n called its *arity*. A *term* of language $\mathcal{F}$, over variables X, is defined inductively as follows:

1. For each $x \in X$, x is a term;
2. If $f \in \mathcal{F}$ is of arity n, and $t_1, ..., t_n$ are terms of language $\mathcal{F}$, then $f(t_1, ..., t_n)$ is a term of language $\mathcal{F}$.

We denote the set of terms of language $\mathcal{F}$ over X by $Term_{\mathcal{F}}(X)$; we suppress the subscript when this is understood.

An *algebra of type* $\mathcal{F}$ is a structure $\mathcal{A} = (A, \mathcal{F})$ where for each $f \in \mathcal{F}$ of arity n, there is a function $f^{\mathcal{A}} : A^n \to A$ (by convention, $A^0 = \{\varnothing\}$. The set A is called *carrier* or the *universe*. Often when the context is clear we simply write f for $f^{\mathcal{A}}$. Given elements $a_1, ..., a_n \in A$, we write $f(a_1, ..., a_n)$ for the element of A obtained by applying the operation to those elements. We extend this to terms inductively, writing, for a term on variables $x_1, ..., x_n$, $t(x_1, ..., x_n)$, and the same elements, $t(a_1, ..., a_n)$ to mean the element of A obtained by applying the operations in the term successively.

An algebra $\mathcal{A}$ is called *trivial* if $|A| = 1$.

Note that $Term_{\mathcal{F}}(X)$ carries a natural algebraic structure: for each operation symbol $f \in \mathcal{F}$, of arity n, we define:

$$f^{Term_{\mathcal{F}}(X)} : Term_{\mathcal{F}}(X)^n \to Term_{\mathcal{F}}(X)$$

by setting for each $t_1, ..., t_n$ terms, $f^{Term_{\mathcal{F}}(X)}(t_1, ..., t_n) = f(t_1, ..., t_n)$. We call this the *term algebra*.

Definition 3.1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras in the language $\mathcal{F}$. We say that $\mathcal{A}$ is *isomorphic* to $\mathcal{B}$ (and write $\mathcal{A} \cong \mathcal{B}$) if there is a bijective function $g : A \to B$ such that for each $f \in \mathcal{F}$ of arity n, and each $a_1, ..., a_n \in A$:

$$g(f(a_1, ..., a_n)) = f(g(a_1), ..., g(a_n)).$$

Definition 3.1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras of type $\mathcal{F}$. We say that a map $g : A \to B$ is a *homomorphism* if for each $f \in \mathcal{F}$ of arity n, and each $a_1, ..., a_n \in A$:

$$g(f(a_1, ..., a_n)) = f(g(a_1), ..., g(a_n)).$$

We say that g is an:

- *Embedding* if g is injective;
- *Isomorphism* if g is injective and surjective.

Moreover, if g is surjective, we say that $\mathcal{B}$ is a *homomorphic image* of $\mathcal{A}$.

Note that given homomorphisms $g : A \to B$ and $h : B \to C$ the composition $h \circ g : A \to C$ is again a homomorphism. This likewise holds for embeddings, surjective homomorphisms, and isomorphisms. A related definition is that of a subalgebra:

Definition 3.1.4. Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. We say that $S \subseteq \mathcal{A}$ is a *subalgebra* if S is closed under all the operations from $\mathcal{F}$. We denote this by $\mathcal{S} \leqslant \mathcal{A}$.

Note that if $g : \mathcal{A} \to \mathcal{B}$ is an embedding, then $A \cong g[A]$ is a subalgebra of B. In this case, $g[A]$ is called the *image* of A under g. As before, given a subset $S \subseteq A$ there is a smallest subalgebra $\langle S \rangle \leqslant A$ which contains A. This can be given an explicit description:

Lemma 3.1.5. Given $\mathcal{A}$ an algebra of type $\mathcal{F}$, and $S \subseteq A$ a subset, and X an infinite set, then:

$$\langle S \rangle = \{t(a_1, ..., a_n) : a_1, ..., a_n \in S, t \in Term_{\mathcal{F}}(X)\}$$

Proof. Exercise. ∎

In this case we say that the algebra $\langle S \rangle$ is *generated* by S. Algebras generated by subsets have some interesting properties as far as homomorphisms go: they are determined by the mapping of generators.

Theorem 3.1.1. *Let $g, h : \mathcal{A} \to \mathcal{B}$ be two homomorphisms between algebras of type $\mathcal{F}$. Suppose that $\mathcal{A}$ is generated by $S \subseteq A$. Then $g = h$ if and only if for each $s \in S$, $g(s) = h(s)$.*

Proof. By induction on construction of terms. ■

Example 3.1.6. Given X a set of variables, the algebra $Term_{\mathcal{F}}(X)$ is generated by X. Any homomorphism $\sigma : Term_{\mathcal{F}}(X) \to Term_{\mathcal{F}}(X)$ is determined by the image of X, and lifted in the obvious way. Such homomorphisms are called *substitutions*.

We now turn to the general construction of *products*. For notation, given an element $a \in \prod_{i\in I} S_i$, we denote by $a(i)$ the i-th component of the function $a : I \to \bigsqcup_{i\in I} S_i$.

Definition 3.1.7. Let $(\mathcal{A}_i)_{i\in I}$ be an indexed family of algebras of type $\mathcal{F}$. The *direct product* $\prod_{i\in I} \mathcal{A}_i$ is an algebra with univere $\prod_{i\in I} A_i$ and such that for every $f \in \mathcal{F}$ of arity n and every $a_1, ..., a_n \in \prod_{i\in I} A_i$:

$$f^{\prod_{i\in I} A_i}(a_1, ..., a_n)(i) = f^{A_i}(a_1(i), ..., a_n(i));$$

we say that in this case the operations are defined *pointwise*. The *projection maps* $\pi_i : \prod_{i\in I} A_i \to A_i$ are then (surjective) homomorphisms.

Given an algebra $\mathcal{B}$, and a family of homomorphisms $g_i : B \to A_i$, there is a unique way of forming a homomorphism $g : B \to \prod_{i\in I} A_i$ such that $\pi_i \circ g = g_i$ for each i: let $g(b)(i) = g_i(b)$. You can verify that this yields a homomorphism as desired.

3.1.1 ♢ Congruences

One of the most fundamental concepts of universal algebra has to do with a generalization of the concept of quotient algebra which we saw before. For comparison see Example 2.2.3, and Definition 2.2.9 and related examples in the previous chapter.

Definition 3.1.8. Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. Let $\Theta \subseteq A$ be an equivalence relation. We say that Θ is a *congruence* if for each $f \in \mathcal{F}$ of arity n, and $(a_1, b_1), .., (a_n, b_n) \in \Theta$

$$(f(a_1, ..., a_n), f(b_1, ..., b_n)) \in \Theta.$$

Congruences are useful in that they allow us an internal way of describing quotients:

Definition 3.1.9. Given $\mathcal{A}$ an algebra of type $\mathcal{F}$, and Θ a congruence, we define an $\mathcal{F}$-algebra structure on A/Θ as follows: for each $f \in \mathcal{F}$ of arity n, and each $[a_1], ..., [a_n]A/\Theta$:

$$f^{A/\Theta}([a_1], ..., [a_n]) = [f(a_1, ..., a_n)].$$

The definition of congruence ensures that the above is well-defined. Congruences correspond to homomorphic images in the way which we have seen in Theorems 2.2.11, 2.4.9 and 2.5.10:

Theorem 3.1.2. *Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$:*

1. *Given a congruence Θ on A, the quotient map $p_\Theta : A \to A/\Theta$ is a surjective homomorphism.*
2. *Given a homomorphism $g : A \to B$, the set*

$$\mathsf{Ker}(g) = \{(a, b) : g(a) = g(b)\},$$

called the kernel of g, is a congruence on A.

3. If $g : A \to B$ is a surjective homomorphism, then:

$$A/Ker(g) \cong B.$$

Proof. Exercise. ■

Note that one can organize congruences in a lattice: indeed, given θ_1, θ_2 two congruences, $\theta_1 \cap \theta_2$ is again a congruence, and indeed, this works for arbitrary intersections. Hence, given an algebra $\mathcal{A}$, the set $\mathsf{Cong}(A)$ is a complete lattice. Given the same two congruences, the congruence join $\theta_1 \vee \theta_2$ is the smallest congruence containing θ_1, θ_2, sometimes called the *congruence generated by* θ_1, θ_2, denoted also by $\Theta(\theta_1 \cup \theta_2)$. This admits an explicit description:

Proposition 3.1.10. For each congruence θ_1, θ_2, define $\Theta_n(\theta_1 \cup \theta_2)$ recursively as follows: if $n = 0$, then $\Theta_n = \theta_1$. If $n = 2k$, then $\Theta_{n+1} = \Theta_n \circ \theta_2$ and if $n = 2k + 1$, $\Theta_{n+1} = \Theta_n \circ \theta_1$. Then we have that:

$$\theta_1 \vee \theta_2 = \bigcup_{n \in \omega} \Theta_n.$$

The structure of this lattice of congruences plays a remarkable role in universal algebra, as we will quickly see.

Definition 3.1.11. Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. We denote by $\Delta \in \mathsf{Cong}(A)$ the smallest congruence of A, defined by $\Delta = \{(a, a) : a \in A\}$. We denote by ∇ the largest congruence of A defined by $\nabla = \{(a, b) : a, b \in A\}$. For each congruence $\theta \in \mathsf{Cong}(A)$, we denote by $[\theta, \nabla]$ the sublattice of $\mathsf{Cong}(A)$ which has as its least element θ.

The first fact we will need concerns the relationship between congruences of an algebra and that of its quotient algebras. This is important since it allows us to validate the intuition that going to a quotient algebra “hones in” on a part of the congruence lattice; you can find the proof of this for example in [5, Theorem 6.20]:

Theorem 3.1.3. *Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$ and let $\theta \in \mathsf{Cong}(A)$. Then there is a bounded lattice isomorphism $\alpha : [\theta, \nabla] \to \mathsf{Cong}(A/\theta)$.*

Definition 3.1.12. Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. We say that $\mathcal{A}$ is *congruence-distributive* if $\mathsf{Cong}(A)$ is a distributive lattice.

Also of great importance is the following property concerning both congruences and subalgebras:

Definition 3.1.13. An algebra $\mathcal{A}$ of type $\mathcal{F}$ has the *congruence-extension property* (CEP) if for every $\mathcal{B} \leqslant \mathcal{A}$ and every $\theta \in \mathsf{Cong}(B)$ there is $\varphi \in \mathsf{Cong}(A)$ such that $\theta = \varphi \cap B^2$. A class $\mathbf{K}$ of algebras has the CEP if every algebra in the class has the CEP.

We finish by mentioning one construction which will occasionally be of use: *ultraproducts.*

Definition 3.1.14. Let $(\mathcal{A}_i)_{i \in I}$ be a family of algebras of type $\mathcal{F}$, and let U be an ultrafilter of the algebra $\mathcal{P}(I)$. Consider the relation on $\prod_{i \in I} A_i$ defined by:

$$a \sim_U b \iff \{i \in I : a(i) = b(i)\} \in U.$$

This defines an equivalence relation (check this!) and indeed a congruence. We denote $\prod_{i \in I} A_i / \sim_U$ by $\prod_{i \in I} A_i / U$ and call this an *ultraproduct* of A_i.

The most important facts we will need about ultraproducts are the following, which are left as an exercise:

Proposition 3.1.15. Let K be a class of algebras.

1. If I is finite, then $\prod_{i\in I} A_i/U$ is isomorphic to A_i for some $i \in I$.
2. If K is a finite class of finite algebras, and $\prod_{i\in I} A_i/U$ is an ultraproduct of algebras from K, then $\prod_{i\in I} A_i/U$ is isomorphic to A_i for some $i \in I$.

3.1.2 ♢ Subdirect Irreducibility

It was a substantial discovery by Birkhoff that one single construction, integrating direct products, homomorphic images and subalgebras, is enough to generate all algebras from a much smaller class of algebras. For that we need the concept of a *subdirectly irreducible algebra*:

Definition 3.1.16. Let $\mathcal{A}, (B_i)_{i\in I}$ be algebras of type $\mathcal{F}$. An embedding $g : A \to \prod_{i\in I} B_i$ is called *subdirect* if for each $i \in I$, $\pi_i \circ g : A \to B_i$ is surjective. In this case we say that A is a *subdirect product* of B_i.

We say that an algebra A is *subdirectly irreducible* if whenever $g : A \to \prod_{i\in I} B_i$ is a subdirect embedding, then there is some $i \in I$ such that $\pi_i \circ g : A \cong B_i$.

This notion turns out to be intimately connected to congruences, in the following way:

Theorem 3.1.4. *Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. Then $\mathcal{A}$ is subdirectly irreducible if and only if* $\mathsf{Cong}(A)$ *has a least element $\theta \neq \Delta$.*

Proof. Assume that $\theta \neq \Delta$ is the least element, and suppose that $g : A \to \prod_{i\in I} B_i$ is a subdirect embedding. For each i, we have that $\pi_i \circ g : A \to B_i$ is surjective, so by Theorem 3.1.2, it corresponds to some congruence $\mu_i \in \mathsf{Cong}(A)$. If $A \ncong B_i$, then $\Delta \neq \mu$, so $\theta \subseteq \mu$. Hence let $\mu' = \bigcap_{i\in I} \mu_i$. If $(a, b) \in \mu'$, then $\pi_i \circ g(a) = \pi_i \circ g(a)$, and so by definition, $g(a) = g(b)$. But then since g is injective, $a = b$. Hence $\mu' = \Delta$, which is absurd, since $\theta \subseteq \mu'$.

Conversely, assume that A is subdirectly irreducible. For each $\theta \in \mathsf{Cong}(A)$, $\theta \neq \Delta$, let $A_\theta = A/\theta$. Consider the induced map:

$$g : A \to \prod_{\theta\in\mathsf{Cong}(A)} A/\theta.$$

Suppose towards a contradiction that there is no least congruence distinct from Δ. Then for each $a, b \in A$ there is at least one $\theta \in \mathsf{Cong}(A)$ where $\theta \neq \Delta$ such that $(a, b) \notin \theta$. Hence $g(a) \neq g(b)$. This means that g is injective, and hence a subdirect embedding; but then by subdirect irreducibility, $\pi_\theta \circ g : A \cong A/\theta$, which is absurd since these were proper congruences. Hence by reductio there must be a least congruence distinct from Δ. ∎

The following is Birkhoff's subdirect decomposition theorem:

Theorem 3.1.5. *Every algebra $\mathcal{A}$ of type $\mathcal{F}$ is isomorphic to a subdirect product of subdirectly irreducible algebras $\mathcal{A}_i$, which are homomorphic images of $\mathcal{A}$.*

Proof. Note tha trivial algebras are subdirectly irreducible, so we assume that A is non-trivial. For each $a, b \in A$, $a \neq b$, we can use Zorn's Lemma on the poset $\mathsf{Cong}(A)$ to show that there is a maximal congruence $\theta_{a,b}$ with the property that $(a, b) \notin \theta_{a,b}$. Then in the lattice $[\theta_{a,b}, \nabla]$, the congruence $\Theta(a, b) \vee \theta_{a,b}$ is smallest. Moreover by Theorem 3.1.3, the latter is the congruence lattice of $A/\theta_{a,b}$. Hence $A/\theta_{a,b}$ is subdirectly irreducible by Theorem 3.1.4.

Consider the homomorphism:

$$g : A \to \prod_{a \neq b \in A} A/\theta_{a,b};$$

by similar arguments to used in Teorem 3.1.4, this is a subdirect embedding. ■

3.1.3 ◊ Classes of Algebras and Operators

Having collected all relevant operations on algebras we will need, we now turn to the issue of how to axiomatize classes of algebras. For that we will need some basic definitions.

Definition 3.1.17. Given an algebraic type $\mathcal{F}$, and two terms $\lambda, \gamma \in Terms_{\mathcal{F}}(X)$, an expression of the form $\lambda \approx \gamma$ is called an *equation*. Given an algebra $\mathcal{A}$ of this type, we write $\mathcal{A} \models \lambda \approx \gamma$ if for every $a_1, ..., a_n \in A$,

$$\lambda(a_1, ..., a_n) = \gamma(a_1, ..., a_n).$$

Given a class of algebras $\mathbf{K}$ we write $\mathbf{K} \models \lambda \approx \gamma$ to mean that this holds for each algebra $\mathcal{A} \in \mathbf{K}$.

Example 3.1.18. All of the definitions from chapter 2 involved only equations.

Definition 3.1.19. A class $\mathbf{K}$ of algebras of type $\mathcal{F}$ is called an *equational class* if there is a collection Γ of equations, such that for each algebra $\mathcal{A}$ of this type, $\mathcal{A} \in \mathbf{K}$ if and only if $\mathcal{A} \models \Gamma$.

Given a class $\mathbf{K}$ of algebras, we write $\mathsf{Eq}(\mathbf{K}) = \{\lambda \approx \gamma : \forall \mathcal{A} \in \mathbf{K}, \mathcal{A} \models \lambda \approx \gamma\}$.

The original goal of universal algebra was to give equational classes a mathematical meaning, i.e., show what mathematical operations on algebras correspond to being an equational class. This turn out to be ones which we have already seen:

Definition 3.1.20. Let $\mathbf{K}$ be a class of algebras. We write:

- $\mathbb{I}(\mathbf{K})$ for the class of isomorphic copies of algebras from $\mathbf{K}$.
- $\mathbb{S}(\mathbf{K})$ for the class of subalgebras of algebras from $\mathbf{K}$;
- $\mathbb{H}(\mathbf{K})$ for the class of homomorphic images of algebras from $\mathbf{K}$;
- $\mathbb{P}(\mathbf{K})$ for the class of products of algebras from $\mathbf{K}$;
- $\mathbb{P}_S(\mathbf{K})$ for the class of subdirect products of algebras from $\mathbf{K}$;
- $\mathbb{P}_U(\mathbf{K})$ for the class of ultraproducts of algebras from $\mathbf{K}$.

Given two operators O_1, O_2, we write $O_1 \leqslant O_2$ to mean that for any class $\mathbf{K}$, $O_1(\mathbf{K}) \subseteq O_2(\mathbf{K})$ (with slight abuse of notation since these classes may be large). We write O_1O_2 to mean $O_1(O_2(\mathbf{K}))$ for any given class. The following facts are well-known and are a good exercise:

Lemma 3.1.21.

- The following inequalities hold $\mathbb{SH} \leqslant \mathbb{HS}$, $\mathbb{PS} \leqslant \mathbb{SP}$ and $\mathbb{PH} \leqslant \mathbb{HP}$.
- The operators $\mathbb{H}, \mathbb{S}, \mathbb{IP}$ are idempotent.
- For any class $\mathbf{K}$:
$$\mathsf{Eq}(\mathbf{K}) = \mathsf{Eq}(\mathbb{H}(\mathbf{K})) = \mathsf{Eq}(\mathbb{S}(\mathbf{K})) = \mathsf{Eq}(\mathbb{P}(\mathbf{K})).$$

Definition 3.1.22. A non-empty class $\mathbf{K}$ is called a *variety* if $\mathbf{K}$ is closed under $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$.
Given $\mathbf{K}$, there is a smallest variety containing $\mathbf{K}$, denoted by $\mathbb{V}(\mathbf{K})$.

Indeed, arbitrary intersections of varieties form a complete lattice. Given a variety $\mathbf{K}$ we denote by $\Lambda(\mathbf{K})$ the lattice of subvarieties of $\mathbf{K}$.

The following gives us a way of constructing varieties directly:

Theorem 3.1.6 (Tarski's theorem)**.** *For each class* $\mathbf{K}$*,* $\mathbb{V}(\mathbf{K}) = \mathbb{HSP}(\mathbf{K})$*.*

Proof. It is clear that $\mathbb{HSP}(\mathbf{K}) \subseteq \mathbb{V}(\mathbf{K})$. For the other inclusion, we show that $\mathbb{HSP}(\mathbf{K})$ is closed under all three operations:

- For $\mathbb{H}$, by idempotence of $\mathbb{H}$ (by Lemma 3.1.21, we have that $\mathbb{HHSP}(\mathbf{K}) = \mathbb{HSP}(\mathbf{K})$.
- For $\mathbb{S}$, by the same lemma note that $\mathbb{SHSP}(\mathbf{K}) \subseteq \mathbb{HSSP}(\mathbf{K}) = \mathbb{HSP}(\mathbf{K})$.
- Finally, for $\mathbb{P}$, by the same inequations, note that $\mathbb{PHSP}(\mathbf{K}) \subseteq \mathbb{HPSP}(\mathbf{K}) \subseteq \mathbb{HSPP}(\mathbf{K}) = \mathbb{HSP}(\mathbf{K})$.

This shows that $\mathbb{HSP}(\mathbf{K})$ is a variety. ■

Given a class $\mathbf{K}$, let $\mathbf{K}_{si}$ be the subclass of subdirectly irreducible elements. By Birkhoff's subdirect decomposition theorem (Theorem 3.1.5) we have:

Proposition 3.1.23. For each variety $\mathbf{K}$, $\mathbf{K} = \mathbb{P}_S(\mathbf{K}_{si})$.

Proof. One inclusion is obvious: $\mathbf{K}_{si} \subseteq \mathbf{K}$, and $\mathbb{P}_S \leqslant \mathbb{SP}$, so $\mathbb{P}_S(\mathbf{K}_{si}) \subseteq \mathbf{K}$ because the latter is a variety. Conversely, if $\mathcal{A} \in \mathbf{K}$, then by closure under homomorphic images, each of its subdirectly irreducible factors is still in $\mathbf{K}$, and hence in $\mathbf{K}_{si}$, and $\mathcal{A}$ is a subdirect product of them. ■

Similarly to how we can generate a set of equations from a class of algebras, we can move in the opposite direction:

Definition 3.1.24. Let Γ be a set of equations. We denote by $\mathsf{Alg}(\Gamma) = \{\mathcal{A} : \mathcal{A} \vDash \Gamma\}$.

Then it follows from Lemma 3.1.21 that $\mathsf{Alg}(\Gamma)$ is always a variety. Just like varieties form the "good" classes of algebras, certain sets of equations correspond to the "good" sets of equations:

Definition 3.1.25. Let Γ be a set of equations in type $\mathcal{F}$ over variables X. We say that Γ is an *equational theory* if:

- (Reflexivity) For each $\lambda \in Term_{\mathcal{F}}(X)$ we have $\lambda \approx \lambda \in \Gamma$;
- (Symmetry) For each $\lambda, \gamma \in Term_{\mathcal{F}}(X)$ if $\lambda \approx \gamma \in \Gamma$ then $\gamma \approx \lambda \in \Gamma$;
- (Transitivity) For each $\lambda, \gamma, \theta \in Term_{\mathcal{F}}(X)$ if $\lambda \approx \gamma, \gamma \approx \theta \in \Gamma$ then $\lambda \approx \theta \in \Gamma$;

- (Congruence) For each $f \in \mathcal{F}$ of arity n, if $\lambda_1 \approx \gamma_1, ..., \lambda_n \approx \gamma_n \in \Gamma$ then $f(\lambda_1, ..., \lambda_n) \approx f(\gamma_1, ..., \gamma_n) \in \Gamma$;
- (Substitution invariance) If $\lambda, \gamma \in Term_{\mathcal{F}}(X)$ and $\lambda \approx \gamma \in \Gamma$ then $\sigma(\lambda) \approx \sigma(\gamma) \in \Gamma$.

It is not difficult to see that given a class $\mathbf{K}$, $\mathsf{Eq}(\mathbf{K})$ is always an equational theory. Moreover, this presentation makes it clear that equational theories also form a complete lattice. For a given equational class $\mathbf{K}$ of algebras, we denote by $Eq(\mathbf{K})$ the lattice of equational theories of algebras in $\mathbf{K}$.

In the next section we will show that if $\mathbf{K}$ is a variety, and Γ is the set of equations true in $\mathbf{K}$, then

$$\Lambda(\mathbf{K})^{op} \cong Eq(\Gamma).$$

This will make crucial use of a key technical ingredient: the *free algebras*.

3.1.4 $\Diamond$ Free Algebras and Birkhoff's Isomorphism

Definition 3.1.26. Let $\mathbf{K}$ be a class of algebras of type $\mathcal{F}$, and X be a set of variables. We say that an algebra $\mathcal{F}(X)$ is *free over X in* $\mathbf{K}$ if there is a map $i : X \to \mathcal{F}(X)$, and for each $\mathcal{A} \in \mathbf{K}$, for each function $f : X \to \mathcal{A}$ there is a unique homomorphism $\tilde{f} : \mathcal{F}(X) \to \mathcal{A}$ such that $\tilde{f} \circ i = f$.

The above definition is quite categorical in nature, and consequently, it substantially tightens the scope of what free algebras can be:

Lemma 3.1.27. Let $\mathbf{K}$ be a class of algebras and let X, Y be two sets such that $|X| = |Y|$. Then $\mathcal{F}(X) \cong \mathcal{F}(Y)$.

Example 3.1.28. Let $\mathbf{K}$ be the class of all algebras of type $\mathcal{F}$, and let X be any set. Then the term algebra $Term_{\mathcal{F}}(X)$ is free over X in $\mathbf{K}$: given any map $f : X \to A$, we extend this to a homomorphism inductively, by setting, for each term $t(x_1, ..., x_n)$:

$$\tilde{f}(t) = t(f(x_1), ..., f(x_n)).$$

The term algebra can thus be seen as the "mother of all algebras" of a given type. If we restrict attention to specific subclasses then free algebras must be constructed by quotienting. This can be achieved as follows:

Definition 3.1.29. Let Γ be a set of equations. The Γ-free algebra is the algebra $\mathcal{F}_{\Gamma}(X) = Term_{\mathcal{F}}(X)/\Theta(\Gamma)$ where $\Theta(\Gamma) = \{(\lambda, \gamma) : \lambda \approx \gamma \in \Gamma\}$ (note that this forms a congruence because Γ is an equational theory).

Note the following:

Lemma 3.1.30. For each equational theory Γ the Γ-free algebra $\mathcal{F}_{\Gamma}(X)$ is free over $\mathsf{Alg}(\Gamma)$ and it belongs to $\mathsf{Alg}(\Gamma)$.

Proof. Given any map $f : X \to \mathcal{A}$ where $\mathcal{A} \in \mathsf{Alg}(\Gamma)$, we have by the above a map $\tilde{f} : Term_{\mathcal{F}}(X) \to \mathcal{A}$. Use this to define a map on the quotient as follows: for each $[\lambda]_{\Gamma} \in \mathcal{F}_{\Gamma}(X)$, $g([x]_{\Gamma}) = \tilde{f}(x)$. Note that this is well-defined: if $(\lambda, \gamma) \in \Theta(\Gamma)$, then by definition, $\lambda \approx \gamma \in \Gamma$, and $\lambda(f(x_1), ..., f(x_n)) = \tilde{f}(\lambda(x_1, ..., x_n))$ and $\gamma(f(x_1), ..., f(x_n)) = \tilde{f}(\gamma(x_1, ..., x_n))$. Since $\mathcal{A} \vDash \lambda \approx \gamma$, then these two images coincide. It is clear to see that g is a homomorphism. Moreover, because Γ is closed under substitutions, $\mathcal{F}_{\Gamma}(X)$ will validate all equations in Γ. ∎

Corollary 3.1.31. *For each equational theory* Γ*,* $\Gamma = \mathsf{Eq}(\mathsf{Alg}(\Gamma))$*.*

Proof. The fact that $\Gamma \subseteq \mathsf{Eq}(\mathsf{Alg}(\Gamma))$ follows from the definitions. Now assume that $\lambda \approx \gamma \notin \Gamma$. Then by definition, there are elements $x_1, ..., x_n \in \mathcal{F}_\Gamma(X)$ such that $\lambda(x_1, ..., x_n) \neq \gamma(x_1, ..., x_n)$. Since this algebra belongs to $\mathsf{Alg}(\Gamma)$, this shows that $\lambda \approx \gamma \notin \mathsf{Eq}(\mathsf{Alg}(\Gamma))$. ∎

The key lemma connecting varieties and free algebras is the following:

Proposition 3.1.32. Let $\mathbf{K}$ be a variety. Then for each X, $\mathcal{F}_{\mathsf{Eq}(\mathbf{K})}(X) \in \mathbf{K}$.

Proof. To show this we will make a different construction of the same algebra. Indeed, consider

$$\Theta_{\mathbf{K}} = \{\theta \in \mathsf{Cong}(Term_{\mathcal{F}}(X)) : Term_{\mathcal{F}}(X)/\theta \in \mathbf{K}\},$$

For each $\theta \in \mathsf{Cong}(Term_{\mathcal{F}}(X))$, let $F_\theta = Term_{\mathcal{F}}(X)/\theta$; let $\theta_0 = \bigcap \Theta_{\mathbf{K}}$. Then first observe that

$$F_{\theta_0} \leqslant \prod_{\theta \in \Theta_{\mathbf{K}}} F_\theta.$$

Indeed, we define the map $g_\theta : F_{\theta_0} \to F_\theta$ by setting $g([x]_{\theta_0}) = [x]_\theta$ – which is well-defined, since if $(a, b) \in \theta_0$ then $(a, b) \in \theta$, and surjective for obvious reasons, and certainly a homomorphism. This lifts to a map $g : F_{\theta_0} \to \prod_{\theta \in \Theta} F_\theta$ in the obvious way, and it will be injective, since if $(a, b) \notin \theta_0$, then $(a, b) \notin \theta$ for some θ. Hence since $\mathbf{K}$ is a variety, $F_{\theta_0} \in \mathbf{K}$.

Next we show that $F_{\theta_0}(X)$ is free over X in $\mathbf{K}$: if $\mathcal{A} \in \mathbf{K}$ is arbitrary, and $f : X \to \mathcal{A}$ is a homomorphism, then $\tilde{f} : Term_{\mathcal{F}}(X) \to \mathcal{A}$ will be a homomorphism. Let $\tilde{f} : Term_{\mathcal{F}}(X) \to Im(\tilde{f})$ be the surjective homomorphism; then note that $Im(\tilde{f}) \leqslant \mathcal{A}$, so the former will be in $\mathbf{K}$. By construction, then $\mathsf{Ker}(\tilde{f}) \in \Theta_{\mathbf{K}}$; thus define $\tilde{\tilde{f}} : F_{\theta_0}(X) \to \mathcal{A}$ by sending $[x]_{\theta_0}$ to $\tilde{f}(x)$. This extends f, and it is well-defined by our arguments. This shows that $F_{\theta_0}(X)$ is free as desired.

Now for any equation $\lambda \approx \gamma$, suppose that $(\lambda, \gamma) \in \theta_0$; then we claim that $\mathbf{K} \vDash \lambda \approx \gamma$. Indeed if $\mathcal{A} \in \mathbf{K}$, and $a_1, ..., a_n \in A$, choose a map $p : X \to A$ such that $p(x_i) = a_i$. This lifts to a homomorphism $\tilde{p} : Term_{\mathcal{F}}(X) \to A$, and since $\mathsf{Ker}(\tilde{p}) \in \theta_0$ (by construction), $\theta_0 \subseteq \mathsf{Ker}(\tilde{p})$. Hence we can define a map $q : F_{\theta_0} \to A$ on the quotient, by setting $q([x]) = \tilde{p}(x)$, and know that this will be well-defined. Since $(\lambda, \gamma) \in \theta_0$, we obtain that $\lambda(a_1, ..., a_n) = \gamma(a_1, ..., a_n)$, i.e., $A \vDash \lambda \approx \gamma$.

Using this fact, note that:

$$\theta_0 = \Theta(\mathsf{Eq}(\mathbf{K})).$$

Indeed, if $\lambda \approx \gamma \in \mathsf{Eq}(\mathbf{K})$, then $F_{\theta_0} \vDash \lambda \approx \gamma$, so in particular, $(\lambda, \gamma) \in \theta_0$; we already showed the converse. Hence we conclude that $F_{\theta_0} = F_{\mathsf{Eq}(\mathbf{K})}$, obtaining the result. ∎

Remark 3.1.1. Note that in Proposition 3.1.32 we only used the fact that $\mathbf{K}$ is closed under *subdirect products.*

We are now able to conclude with Birkhoff's theorem:

Theorem 3.1.7. *For each variety* $\mathbf{K}$*,* $\mathbf{K} = \mathsf{Alg}(\mathsf{Eq}(\mathbf{K}))$*. Moreover, given a variety* $\mathbf{K}$ *and* Γ *its equational class,* $\mathsf{Eq} : \Lambda(\mathbf{K}) \to Eq(\Gamma)$ *is a dual isomorphism.*

Proof. The fact that $\mathbf{K} \subseteq \mathsf{Alg}(\mathsf{Eq}(\mathbf{K}))$ is obvious from the definitions. For the converse, assume that $\mathcal{A} \vDash \mathsf{Eq}(\mathbf{K})$. Let X be a set of the same cardinality as A, and $f : X \to A$ a bijection. Then we have that $\mathcal{F}_{\mathsf{Eq}(\mathbf{K})}(X)$ has A as a homomorphic image by Lemma 3.1.30. By Proposition 3.1.32, this algebra belongs to $\mathbf{K}$, so since the latter is a variety, $\mathcal{A} \in \mathbf{K}$ as well.

The fact that this map respects lattice operations is left as an exercise to the reader – it uses the same tools as given above. ∎

Corollary 3.1.33. *Given a class of similar algebras* $\mathbf{K}$*,* $\mathbf{K}$ *is a variety if and only if it is an equational class.*

3.1.5 ◊ Jónnson's Lemma and Malcev Conditions

Having established our major result, we now zoom in on one of the most powerful general results available in universal algebra: Jónnson's Lemma. This concerns the relationship between subdirectly irreducible algebras and a class $\mathbf{K}$ of similar algebras.

We will need one additional piece of machinery concerning congruences:

Proposition 3.1.34. Let B be an algebra, $\theta \subseteq \mu$ congruences. Then there is a congruence $\mu/\theta \in \mathsf{Cong}(B/\theta)$ such that:

$$B/\mu \cong (B/\theta)/(\mu/\theta).$$

Here is the full statement of the result:

Theorem 3.1.8 (Jónnsons Lemma)**.** *Let* $\mathbf{K}$ *be a class of algebras of type* $\mathcal{F}$*, and let* $\mathbb{V}(\mathbf{K})$ *be a congruence-distributive variety. If* $\mathcal{A} \in \mathbb{V}(\mathbf{K})$ *is a subdirectly irreducible algebra, then*

$$\mathcal{A} \in \mathbb{HSP}_U(\mathbf{K}).$$

Proof. Let A be a non-trivial subdirectly irreducible (if it is trivial the result easily follows); by Tarski's theorem (Theorem 3.1.6) we have that for some $A_i \in \mathbf{K}$, there is some B, $B \leqslant \prod_{i\in I} A_i$ through a map k, and a surjective homomorphism $g : B \to A$. Let $\theta = \mathsf{Ker}(g)$.

For each $J \subseteq I$, let $\theta_J = \{(a,b) \in (\prod_{i\in I} A_i)^2 : \forall j \in J, a(j) = b(j)\}$. It is easy to verify that θ_J is a congruence on the product. Then $\mu_J = \theta_J \cap B^2$ is a congruence on B. Then define

$$U = \{J \subseteq I : \mu_J \subseteq \theta\}.$$

Certainly $\mu_I = \Delta_B$, so $I \in U$, and $\varnothing \notin U$, since $\mu_\varnothing = \nabla_B$, and then A would be trivial. Moreover, if $J \subseteq K$, and $J \in U$, then $\theta_K \subseteq \theta_J$, so $\mu_K \subseteq \mu_J$, meaning that $K \in U$. If $J, K \in U$, then similar arguments show closure under intersections.

Now assume that $J \cup K \in U$. Then note that $\theta_{J\cup K} = \theta_J \cap \theta_K$, and hence we obtain $\mu_{J\cup K} = \mu_J \cap \mu_K$, and that $\mu_J \cap \mu_K \subseteq \theta$.

Now since $\theta = \theta \vee (\mu_J \cap \mu_K)$, by distributivity we have $\theta = (\theta \vee \mu_J) \cap (\theta \cap \mu_K)$. Now, by definition, $B/\theta \cong A$ is subdirectly irreducible, and $\mathsf{Cong}(B/\theta) = [\theta, \nabla] \subseteq \mathsf{Cong}(B)$; if $\theta \subsetneq \theta \vee \mu_J$ and $\theta \subsetneq \theta \vee \mu_K$, then their intersection would be the second least congruence in B/θ, which is distinct from θ. Hence $\theta = \theta \vee \mu_J$ or $\theta = \theta \vee \mu_K$ must hold for J or K, meaning that $J \in U$ or $K \in U$.

Now note that we have shown that U is an ultrafilter on I. Hence consider the natural homomorphism $g_U : \prod_{i\in I} A_i \to \prod_{i\in I} A_i/U$; let $C = g_U \circ k(B)$, and let $m : B \to C$ be the restriction. Note that by construction, $\mathsf{Ker}(m) \subseteq \theta$: if $m(a) = m(b)$, then $g_U k(a) = g_U k(b)$, so $J = \{i \in I : k(a)(i) = k(b)(i)\} \in U$; then by definition $\mu_J \subseteq \theta$, so by construction, $(a,b) \in \theta$. Hence by Proposition 3.1.34:

$$A \cong B/\theta \cong (B/\mathsf{Ker}(m))/(\theta/\mathsf{Ker}(m)).$$

Since $B/\mathsf{Ker}(m) \leqslant \prod_{i\in I} A_I/U$, $B/\mathsf{Ker}(m) \in \mathbb{SP}_U(\mathbf{K})$, and os $A \in \mathbb{HSP}_U(\mathbf{K})$, as desired. ∎

The following corollary is one of the key applications of this result:

Corollary 3.1.35. *Let* $\mathbf{K}$ *be a finite set of finite algebras. Then if* $\mathcal{A} \in \mathbb{V}(\mathbf{K})$ *is subdirectly irreducible, then*
$$A \in \mathbb{HS}(\mathbf{K}).$$

Proof. By Proposition 3.1.15, we have that any ultraproduct of the finitely many algebras will be isomorphic to one of them; hence the ultraproduct operator disappears. ■

In light of this result, it is quite useful to know when our varieties are congruence-distributive. However, in all of the cases in which we are interested this will hold, by an old result:

Theorem 3.1.9. *Let* $\mathbf{K}$ *be a variety of algebra of type* $\mathcal{F}$. *Suppose that there is a term* $M(x, y, z)$ *in this language such that:*

$$\mathbf{K} \models M(x, x, y) \approx M(x, y, x) \approx M(y, x, x) \approx x,$$

called a "majority term". Then $\mathbf{K}$ *is congruence-distributive.*

Proof. Let $\theta, \varphi, \psi \in \mathsf{Cong}(A)$. We will show that:

$$\theta \cap (\varphi \vee \psi) \subseteq (\theta \vee \varphi) \cap (\theta \vee \psi).$$

Assume that (a, b) belongs to the first set. Then by Proposition 3.1.10, there are $c_1, ..., c_n$ such that, without loss of generality:
$$a\varphi c_1 \psi c_2 ... c_{n-1} \varphi c_n \psi b;$$
Note that since $M(a, c_i, b)\theta M(a, c_i, a)$, then applying the term above:

$$a = M(a, a, b)(\theta \cap \varphi)M(a, c_1, b)(\theta \cap \psi)M(a, c_2, b)(\theta \cap \varphi)...M(a, c_n, b)(\theta \cap \psi)M(a, b, b) = b$$

so $(a, b) \in (\theta \cap \varphi) \vee (\theta \cap \psi)$, as desired. ■

Corollary 3.1.36. *Let* $\mathbf{K}$ *be a variety of algebras of type* $\mathcal{F}$, *such that* $\{\wedge, \vee\} \subseteq \mathcal{F}$, *and for each algebra* $\mathcal{A} \in \mathbf{K}$, $\mathcal{A}$ *validates the lattice equations. Then* $\mathbf{K}$ *is congruence-distributive.*

Proof. In light of Theorem 3.1.9 we need only show a majority term. Consider the following term[1]:

$$M(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

It can easily be checked that by the lattice equations this is a majority term. ■

End of extra content.

3.2 Basic algebraic logic of non-classical logics

The previous section provides us with all the necessary background in universal algebra to tackle many issues of interest in logic. In this section we will begin by refreshing our understanding of what logics are, and showing how algebras can be used to model these logics. We will then show how to apply many of the general universal algebraic facts to the special cases we are interested in.

[1]Compare this term to the term used in the proof that distributivity is equivalent to forbidding the diamond and pentagon lattices in Theorem 2.1.1.

3.2.1 Intermediate Logics, Modal Logics and Algebraic Completeness

Recall from theorem 3.1.9 the definition of a normal modal logic, and the lattice of sublogics of a logic. We will also at last need the definition of *intuitionistic logic*:

Definition 3.2.1. The logic **IPC** in the language $\{\wedge, \vee, \rightarrow, \top, \bot\}$ is by considering the axioms

$$\begin{aligned}
&\vdash q \rightarrow (p \rightarrow q)\\
&\vdash (r \rightarrow (p \rightarrow q)) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q))\\
&\vdash p \wedge q \rightarrow p\\
&\vdash p \wedge q \rightarrow q\\
&\vdash p \rightarrow (q \rightarrow (p \wedge q))\\
&\vdash p \rightarrow p \vee q\\
&\vdash q \rightarrow p \vee q\\
&\vdash (p \rightarrow q) \rightarrow ((r \rightarrow q) \rightarrow ((p \vee r) \rightarrow q))\\
&\vdash \bot \rightarrow p. \qquad\qquad \vdash p \rightarrow \top,
\end{aligned}$$

and letting **IPC** be the least set of formulas which contains all of these axioms, and is closed under uniform substitution, i.e., $\vdash \varphi(p_1, ..., p_n)$ then for each set of formulas $\psi_1, ..., \psi_n$, $\vdash \varphi(\psi_1, ..., \psi_n)$, and Modus Ponens, i.e., $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ then $\vdash \psi$.

The following is routine, although remarkably annoying to prove:

Lemma 3.2.2. The following are theorems of **IPC**:

1. $\vdash p \wedge q \leftrightarrow q \wedge p$;
2. $\vdash p \vee q \leftrightarrow q \vee p$;
3. $\vdash p \wedge (p \vee q) \leftrightarrow p$;
4. $\vdash p \vee (p \wedge q) \leftrightarrow p$;
5. $\vdash p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$;
6. $\vdash p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$;
7. $\vdash p \rightarrow p$;
8. $\vdash p \wedge (p \rightarrow q) \leftrightarrow p \wedge q$;
9. $\vdash p \wedge (p \rightarrow q) \leftrightarrow q$
10. $\vdash p \rightarrow (b \wedge c) \leftrightarrow p \rightarrow b \wedge p \rightarrow c$.

Indeed, the smallest set of formulas of the language of **IPC** containing these theorems also derives the axioms of **IPC**, i.e., this is an equivalent axiomatization.

Proof. Try doing this if you cannot sleep. ∎

Definition 3.2.3. Let L be a set of formulas in the language of **IPC**. We say that L is a *superintuitionistic logic* if $\mathbf{IPC} \subseteq L$ and L is closed under uniform substitution and Modus Ponens. Such a logic L is said to be trivial if L contains all formulas. Given a superintuitionistic logic L, let $\Lambda(L)$ be the set of its sublogics.

Example 3.2.4. The logic **CPC** extends **IPC** and is closed under uniform substitution and Modus Ponens, hence it is a superintuitionistic logic. In fact, as we will soon see, it is the greatest non-trivial such logic.

Example 3.2.5. The following are superintuitionistic logics:

$$\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p) \qquad \text{the logic of the law of excluded middle.}$$
$$\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p) \qquad \text{Gödel-Dummet logic.}$$

There is a strict connection between logics in this sense and equational theories as defined in Definition 3.1.25. For that purpose we will first define what it means for an algebra to satisfy a formula:

Definition 3.2.6. An *model* (A, v) over a (Heyting or modal) algebra A is a pair (A, v) where $v : \mathsf{Prop} \to A$ is a map. Denote $Term(\mathsf{Prop})$ as $\mathsf{Form}(\mathsf{Prop})$. We extend the map to a homomorphism $v : \mathsf{Form}(\mathsf{Prop}) \to A$ by the unique lifting, i.e. by setting $v(\varphi(p_1, ..., p_n)) = \varphi(v(p_1), ..., v(p_n))$.

We write $(A, v) \vDash \varphi$ for a formula φ if $v(\varphi) = 1$. We write $A \vDash \varphi$, i.e., the formula is valid in A, if for all models (A, v) over A, $(A, v) \vDash \varphi$. Given a class $\mathbf{K}$ of algebras, we write $\mathbf{K} \vDash \varphi$ if each algebra in $\mathbf{K}$ validates the formula.

In the foregoing definition, in the case of intuitionistic logic or modal logic one of course should set the language and algebras to be the appropriate ones.

Definition 3.2.7. Given L a superintuitionistic logic, let $Eq(L) = \{\lambda \approx \gamma : \lambda \leftrightarrow \gamma \in L\}$. Conversely, if Γ is an equational theory of Heyting algebras, let $Log(\Gamma) = \{\varphi : \varphi \approx 1 \in \Gamma\}$.

Lemma 3.2.8. For each L a superintuitionistic logic, and Γ an equational theory of Heyting algebras we have:

1. $Eq(L)$ is an equational theory and $Log(\Gamma)$ is a superintuitionistic logic.
2. $Eq(Log(L)) = L$ and $Log(Eq(\Gamma)) = \Gamma$.
3. $Log : Eq(\mathbf{HA}) \to \Lambda(\mathbf{IPC})$ is an isomorphism.

Proof. If L is a superintuitionistic logic, the conditions of being an equational theory amount to basic properties of the biconditional, namely:

$$\vdash p \leftrightarrow p$$
$$\vdash p \leftrightarrow q \implies \vdash q \leftrightarrow p$$
$$\vdash p \leftrightarrow q \text{ and } \vdash q \leftrightarrow r \implies \vdash p \leftrightarrow r$$
$$\text{For any connective } \circ \in \{\wedge, \vee, \to\} \vdash p_1 \leftrightarrow q_1 \text{ and } \vdash p_2 \leftrightarrow q_2 \implies \vdash p_1 \circ p_2 \leftrightarrow q_1 \circ q_2,$$

together with the requirement that if $\vdash \varphi \leftrightarrow \psi$, then any substitution instance must also be there, which follows by uniform substitution. Conversely, if Γ is an equational theory of Heyting algebras, then by Lemma 3.2.2, we have that $\mathbf{IPC} \subseteq Log(\Gamma)$; the latter is closed because Γ is substitution invariant; and if $\varphi \in Log(\Gamma)$ and $\varphi \to \psi \in Log(\Gamma)$, then $\varphi \approx 1 \in \Gamma$ and $\varphi \to \psi \approx 1$, hence $\varphi \wedge \varphi \to \psi \approx 1 \in \Gamma$ by the congruence properties; also $\varphi \wedge \psi \approx \varphi \wedge \varphi \to \psi$ is always there, so by symmetry and transitivity, $\varphi \wedge \psi \approx 1 \in \Gamma$, and hence $\psi \approx 1 \in \Gamma$, showing that it is closed under Modus Ponens.

Now if L is a superintuitionistic logic, note that $\varphi \in L$ if and only if $\varphi \leftrightarrow 1 \in L$ if and only if $\varphi \approx 1 \in Eq(L)$ if and only if $\varphi \in Log(Eq(L))$. Similar arguments show the inclusion of equational theories. The lattice homomorphism facts are left as an exercise. ∎

Theorem 3.2.1. *There is a dual isomorphism $Log : \Lambda(\mathbf{HA}) \to \Lambda(\mathbf{IPC})$ with inverse $Var(L) = \{A \in \mathbf{HA} : A \models L\}$. Similarly, there is a dual isomorphism $Log : \Lambda(\mathbf{MA}) \to \Lambda(\mathbf{K})$.*

This dual isomorphism is the celebrated *algebraic completeness*: for each superintuitionistic or normal modal logic L, the class $Var(L)$ will necessarily be complete for L, since $Log(Var(L)) = L$. Moreover, in the case of normal modal logics, it allows us to derive the completeness with respect to general frames mentioned in Theorem 1.3.1:

Theorem 3.2.2. *For each $L \in \Lambda(\mathbf{K})$, L is complete with respect to is class of general frames.*

Proof. Certainly L is sound with respect to its general frames. If $\varphi \notin L$, then let $\mathcal{A} \in Var(L)$ be an algebra and (A, v) a model such that $(A, v) \not\models \varphi$. then consider the general frame $\mathfrak{F} = (X_A, R, \mathsf{Clop}(X_A))$. Note that for each formula ψ, $\mathfrak{F} \models \psi$ if and only if $A \models \psi$; hence $\mathfrak{F}$ is an L-general frame. Moreover, considering $v' : \mathsf{Prop} \to \mathsf{Clop}(X_A)$ induced by v (i.e., $v'(p) = \varphi(a)$ whenever $v(p) = a$), we obtain a model such that $v'(\varphi) \neq X_A$, i.e., there is some $x \in X_A$ such that $(\mathfrak{F}, v'), x \not\Vdash \varphi$. This shows completeness. ∎

From this perspective we also observe the following:

Proposition 3.2.9. Let L be a normal modal logic. Let F_L be the free $Var(L)$-algebra. Then (X_{F_L}, R) is the canonical L-model.

Proof. Simply note that L-MCS will be ultrafilters on F_L, and the rest follows by duality. ∎

The algebra F_L is called in this context the *Lindenbaum-Tarski algebra* of L. Constructions of F_L will often go by considering the algebra of formulas $\mathsf{Form}(\mathsf{Prop})$, and quotienting this by derivability of L, which in light of Lemma 3.2.8 is of course exactly the same as considering the free algebra over the equations in L.

Using our algebraic results we can already show something about the structure of $\Lambda(\mathbf{IPC})$:

Proposition 3.2.10. The variety $\mathbf{BA}$ is generated by the algebra $\mathbf{2}$. Consequently, the logic $\mathbf{CPC}$ is the largest superintuitionistic logic.

Proof. For each Boolean algebra B, and each ultrafilter U in B, note that $B/U \cong \mathbf{2}$. Now consider:

$$i : B \to \prod_{U \in \mathsf{Spec}(B)} B/U;$$

the ultrafilter theorem ensures that i is injective (for if $a \neq b$, there is an ultrafilter $a \in U \not\ni b$, hence $i(a)(U) \neq i(b)(U)$). Hence i presents B as a subdirect product of copies of $\mathbf{2}$. So $B \in \mathbb{HSP}(\mathbf{2})$, and hence $\mathbf{BA} = \mathbb{HSP}(\mathbf{2})$.

Now to show that $\mathbf{CPC}$ is the largest non-trivial superintuitionistic logic, we show that $\mathbf{BA}$ is the least subvariety of $\mathbf{HA}$. Given any Heyting algebra H, note that $\mathbf{2} \leqslant H$. Hence given any non-trivial variety K of Heyting algebras, $2 \in K$. ∎

3.2.2 Subdirectly Irreducible Algebras in the classes $\mathbf{BA}, \mathbf{DL}, \mathbf{HA}, \mathbf{MA}$

We now begin to study in more detail some universal-algebraically interesting classes of algebras. First up we will look at subdirectly irreducible algebras in each of the equational classes we have been looking at. We begin with **DL** where the situation is not very interesting:

Proposition 3.2.11. A distributive lattice D is subdirectly irreducible if and only if $D \cong \mathbf{2}$.

Proof. Exercise. ■

For the case of **HA** – from which the case for **BA** will follow – first note that by Proposition 2.4.9 we can deduce the following:

Proposition 3.2.12. Let H be a Heyting algebra. There is an isomorphism between $(\mathsf{Fil}(H), \subseteq)$ and $\mathsf{Cong}(H)$. Hence $(\mathsf{Fil}(H), \subseteq)$ is a distributive lattice.

Proof. If F is a filter, let $q_F : H \to H/F$ be the homomorphic image described in Proposition 2.4.9. Then $\mathsf{Ker}(q_F)$ is a congruence. This map is clearly injective. If $F \subseteq F'$, then note that this holds if and only if $\mathsf{Ker}(q_F) \subseteq \mathsf{Ker}(q_{F'})$. Moreover, given any congruence θ, we can look at H/θ, and then the filter $F_\theta = q_\theta^{-1}[1]$ will have kernel equal to θ. So we have an isomorphism.

The additional fact can be proven directly, but it also follows from Corollary 3.1.36. ■

Hence we can derive the following:

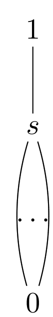
Theorem 3.2.3. *Let H be a Heyting algebra. Then H is subdirectly irreducible if and only if $\mathsf{Fil}(H)$ has a second least element (i.e., a least non-unital filter).*

Proof. By Theorem 3.1.4, we have that H is subdirectly irreducible if and only if $\mathsf{Cong}(H)$ has a second least element; by Proposition 3.2.12 we obtain the stated result. ■

Note that this characterization of subdirect irreducible algebras is independent of the variety we are considering. Since by Proposition 3.1.23 each variety is generated by its subdirect irreducible algebras, this allows us to study varieties of Heyting algebras by looking at what subdirect irreducible algebras they contain. Using duality, this can be made even more palatable:

Proposition 3.2.13. Let H be a Heyting algebra. Then:

1. H is subdirect irreducible iff H has a second greatest element, i.e., if it is isomorphic to



2. H is subdirectly irreducible if and only if X_H is rooted and its root is isolated.

Proof. Exercise (see Exercise 3.8). ∎

We thus conclude the following:

Proposition 3.2.14. For B a Boolean algebra, B is subdirectly irreducible if and only if $B \cong \mathbf{2}$.

Corollary 3.2.15. *Both* **BA** *and* **DL** *have no non-trivial proper subvariety.*

In fact, using the above we can make computations of lattices of subvarieties. Indeed, for this purpose, we put together the corollary to Jónnson's Lemma, Corollary 3.1.35, and our duality results for Heyting algebras. First we mention one interesting technical fact:

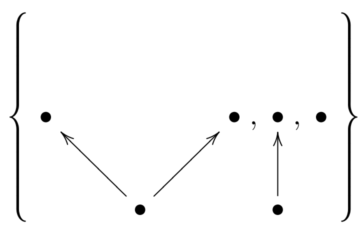
Proposition 3.2.16. The variety **HA** has the Congruence Extension Property. Consequently, for each class **K** of Heyting algebras, $\mathbb{HS}(\mathbf{K}) = \mathbb{SH}(\mathbf{K})$.

Proof. Exercise. ∎

Proposition 3.2.17. Let H be a finite Heyting algebra. Then for each $K \subseteq Var(H)$ a subvariety, there is a finite set $\{H_0, ..., H_n\}$ of finite Heyting algebras, such that $K = Var(\{H_0, ..., H_n\})$ and for each i, X_{H_i} is a generated subframe of a p-morphic image of X_H, or equivalently, a p-morphic image of a generated subframe.

Let us make use of this Proposition to calculate lattices of subvarieties:

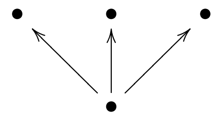
Example 3.2.18. Recall the algebras from Example 2.4.29. There we obtained the following p-morphic images: whose p-morphic images are



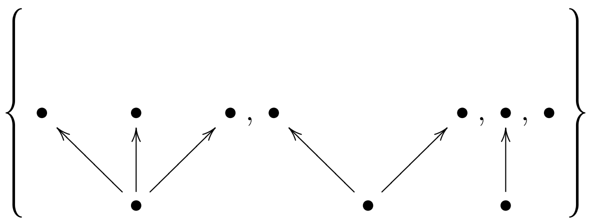
If we let D_2 be the the dual Heyting algebra of the leftmost poset, we obtain that the above is an exhaustive list of all duals of subdirectly irreducible algebras which are homomorphic images of subalgebras of H. Since $\mathbf{2} \hookrightarrow \mathbf{3} \hookrightarrow D_2$, this means that $\mathbb{V}(\mathbf{2}) \subseteq \mathbb{V}(\mathbf{3}) \subseteq \mathbb{V}(D_2)$, and it is also the case that $\mathbb{V}(\{\mathbf{2}, \mathbf{3}\}) = \mathbb{V}(\mathbf{3})$, so the lattice of subvarieties of $\mathbf{V} = \mathbb{V}(D_2)$ looks like this (check that all these varieties are distinct!)

$$\begin{array}{c} \mathbf{V} \\ | \\ \mathbb{V}(3) \\ | \\ \mathbb{V}(2) = \mathbf{BA} \\ | \\ \text{Trivial variety} \end{array}$$

Example 3.2.19. Now let D_3 be the Heyting algebra that we obtain when we take the 2^3 Boolean algebra and attach a top element to it, which dually corresponds with the Esakia space



And let $\mathbf{V} = \mathbb{V}(D_3)$. The $p-$morphic images of this are

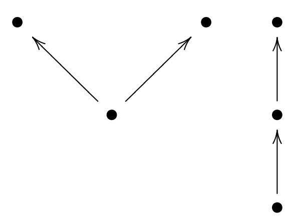


which correspond with the set $\{D_3, D_2, 3, 2\}$, all the subalgebras of D_3. Again, they are all subdirectly irreducible, and similarly we get a linearly ordered lattice of subvarieties of $\mathbf{V}$:

$$\text{Trivial} \to \mathbf{BA} \to \mathbb{V}(A_1) \to \mathbb{V}(D_2) \to \mathbb{V}(D_3) = \mathbf{V}.$$

More generally, we get this exact linear order if we call D_n the Heyting algebra which is dual to the poset P_n, which consists of n distinct points with a single point below them, and $\mathbf{V}_n = \mathbb{V}(L_n)$).

Example 3.2.20. For an example of one of the lattices of varieties that is not a linear order, we can take the variety $\mathbf{V} = \mathbb{V}(D_2 \times \mathbf{4})$, whose Esakia space is the disjoint union



We can easily compute the $p-$morphisms of this that happen to be rooted (these will correspond to the s.i. subalgebras of $D_2 \times \mathbf{4}$) and from this we compute $\mathbf{V}_{SI}$ to be $\{D_2, \mathbf{4}, \mathbf{3}, \mathbf{2}\}$, hence getting incomparable subvarieties, because $\mathbb{V}(D_2) \nsubseteq \mathbb{V}(\mathbf{4}) \nsubseteq \mathbb{V}(D_2)$.

The case of subdirectly irreducible *modal algebras* is quite a lot trickier. Indeed, we also have:

Proposition 3.2.21. Let M be a modal algebra. There is an isomorphism between $(\square\mathsf{Fil}(M), \subseteq)$ and $\mathsf{Cong}(M)$. Hence $(\square\mathsf{Fil}(M), \subseteq)$ is a distributive lattice.

Proof. Similar to the above. ■

Putting this together we obtain:

Theorem 3.2.4. *Let M be a modal algebra. Then M is subdirectly irreducible if and only if $\square\mathsf{Fil}(M)$ has a second least element.*

In order to use our duality, we need to be able to explain what subdirect irreducibility means in terms of modal spaces.

Definition 3.2.22. Let (X, R) be a modal space. We say that an element $x \in X$ is a *root* if $R^*[x] = X$. We say that X is *topo-rooted* if the set of its roots is open.

Proposition 3.2.23. Let M be a modal algebra. Then M is subdirectly irreducible if and only if X_M is topo-rooted.

Proof. Exercise. ∎

Extra Content

The following section dives deeper into the structure of free algebras of the classes of interest. The reader with an interest in some of the deeper algebraic theory may consult this; it can be safely skipped for the sake of the course.

3.2.3 $\Diamond$ Free Algebras in $\mathbf{BA}, \mathbf{DL}, \mathbf{HA}, \mathbf{MA}$

We have seen in section 3.1 that free algebras always exist in varieties. The construction given was in many ways syntactic: we constructed it out of congruences on the term algebra. Hence it is natural to ask whether there are other constructions of free algebras, and whether anything can be said about these constructions. We especially look at the case of a finite set X of generators.

We begin by the simple case of **BA**. The free Boolean algebras admit a very simple and very transparent description:

Theorem 3.2.5. *Let X be a set. Then $\mathsf{Clop}(\{0,1\}^X)$ is the free Boolean algebra on X many generators.*

Proof. Note that $\{0,1\}^X$, the product space, is a Stone space. For each $a \in X$, let π_a be the projection. Let $i : X \to \mathsf{Clop}(\{0,1\}^X)$ be the map which assigns to $a \in X$, the subset $P_a = \pi_a^{-1}[1]$.

We now show that this algebra satisfies the universal mapping property. Indeed, let B be a Boolean algebra, and $p : X \to B$ be a map. Note that for each $x \in X_B$, and each $a \in X$, either $x \in \varphi(p(a))$ or $x \notin \varphi(p(a))$. Let $h_x : X \to \{0,1\}$ be a function such that $h_x(a) = 0$ if and only if $x \notin \varphi(p(a))$. Then let $\tilde{p}(x) = h_x$.

First note that $\tilde{p}$ is continuous: for this, note that $\tilde{p}^{-1}[P_a] = \{x \in X_B : h_x \in P_a\} = \varphi(p(a))$ and $\tilde{p}^{-1}[\pi_a^{-1}[\{0\}]] = X_B - \varphi(p(a))$, by construction. since the projections form a subbase of $\{0,1\}^X$, this shows continuity. Then finally note that:

$$\tilde{p}^{-1} \circ i = p,$$

showing the universal mapping property. ∎

Corollary 3.2.24.

1. *If X is a finite set, then $F_{BA}(X)$ is finite.*

2. *Finitely generated Boolean algebras are finite.*

Proof. (1) By Theorem 3.2.5, we have that $F_{BA}(X) \cong \mathsf{Clop}(\{0,1\}^X)$, which is finite if X is finite. (2) follows since every finitely generated Boolean algebra is a homomorphic image of a free one. ■

In light of this theorem, note that not every finite Boolean algebra is free – the ones which are, are exactly the ones of the form $\mathbf{2}^{\mathbf{2}^X}$ for X a finite set. A similar, but slightly more involved argument, shows the following:

Proposition 3.2.25. Let X be a set. Then $\mathsf{ClopUp}(\mathbf{2}^X)$ is the free distributive lattice on X many generators, where $\mathbf{2} = \{0 < 1\}$ is the two element chain.

Proof. Exercise. ■

Corollary 3.2.26.

1. *If X is a finite set, then $F_{DL}(X)$ is finite.*
2. *Finitely generated distributive lattices are finite.*

We will see a different proof of the aforementioned corollary shortly. The natural next question is what is the nice description of free Heyting algebras. Unfortunately this turns out to not be the case: as you are asked to show in Exercise 3.11, the one-generated free Heyting algebra is infinite; the n-generated free Heyting algebras for $n > 1$ have an extremely subtle structure.

Because of these difficulties, several techniques exist to analyze the structure of such algebras. We briefly discuss, without going into details, the two most important ones:

1. n-**Universal Models**: The free n-generated Heyting algebras are completely join-prime generated (though they are not complete). Hence it is possible to describe them by analyzing the completely join-prime elements. A recursive construction can be given of such elements called the *n-universal model*. The source of the name comes from the fact that every finite n-generated Heyting algebra will at some point occur as a generated subframe of the model.

   This technique is especially powerful when combined with frame-based characterizations of logics, as discussed in section 4.3. Using this, it has been used to give strong structural results concerning subvarieties of Heyting algebras.

2. **Step-by-Step Decomposition**: The free n-generated Heyting algebras can also be constructed by approximating them by distributive lattices. This allows us to decompose them into *join-irreducible* elements, and construct the Heyting algebras recursively. This procedure is purely algebraic, although duality ensures that the algebraic constructions go through as desired. Such finite approximations are similar, albeit not exactly the same as, the n-universal model layers. This method has been used to prove that such algebras have additional adjoints (they are so-called *bi-Heyting algebras*), and to provide normal forms for intuitionistic formulas.

Overall the study of free Heyting algebras – both finitely generated and not – and especially of free algebras in *subvarieties* of Heyting algebras is still an ongoing area of research, with several open questions in the vicinity.

Finally we consider free modal algebras. For these, the situation is even worse than for Heyting algebras: there seems to be no universal model-like construction, with the only constructions of such algebras being through step-by-step. However, the situation improves if one goes to subvarieties, since free algebras can become dramatically simpler by adding some equations.

Definition 3.2.27. Let M be a modal algebra. We say that M is an **S5**-algebra if M is an **S4**-algebra and additionally it satisfies the axiom:

$$\Diamond a \leqslant \Box\Diamond a.$$

Note the following:

Proposition 3.2.28. Let (X, R) be a modal space. Then $\mathsf{Clop}(X)$ is an **S5**-algebra if and only if R is an equivalence relation.

Proof. Exercise. ■

We can thus show the following, which we leave as a challenging exercise for the interested reader:

Theorem 3.2.6. *Let X be a finite set. Let $S_X = \bigsqcup\{S : S \in \mathcal{P}(2^X)\}$. Let (S_X, R) be a Kripke frame where for each $(x, S), (y, T) \in S_X$, $(x, S)R(y, T)$ if and only if $S = T$. Then $\mathsf{Clop}(S_X, R)$ is the free* **S5***-algebra.*

End of extra content.

3.2.4 Hierarchy of Varieties

In this final section we look at how we can stratify logical systems using tools from universal algebra. Namely we will consider a *hierarchy of varieties* from the simplest to the most complex.

We begin by laying out the key definitions. This will be done in the case of $\Lambda(\mathbf{IPC})$ although similar definitions obviously work for $\Lambda(\mathbf{K})$:

Definition 3.2.29. Let $\mathbf{V}$ be a variety of Heyting algebras, and L its corresponding superintuitionistic logic. We say that $\mathbf{V}$ is:

- *finitely generated* if $\mathbf{V} = \mathbb{V}(H))$ for H a finite Heyting algebra. In this case we say that L is *tabular*;
- *locally finite* if every n-generated Heyting algebra is finite. In this case we say that L is *locally tabular*;
- *finitely approximable* (or occasionally, that it has the FMP) if $\mathbf{V} = \mathbb{V}(\mathbf{V}_{fin})$ where $\mathbf{V}_{fin}$ is the set of finite algebras of $\mathbf{V}$. In this case we say that L has the FMP.
- complex if $\mathbf{V} = \mathbb{V}(\mathbf{V}_{birk})$ where $\mathbf{V}_{birk}$ are the algebras H which are Birkhoff algebras, i.e., $H = \mathsf{Up}(P)$ for a poset P. In this case we say that L is *Kripke complete.*
- *topological* if $\mathbf{V} = \mathbb{V}(\mathbf{V}_{top})$ where $\mathbf{V}_{top}$ are the algebras H which are of the form $H \cong Op(X)$ for X a topological space. In this case we say that L is *topologically complete.*
- *localic* if $\mathbf{V} = \mathbb{V}(\mathbf{V}_{loc})$ where $\mathbf{V}_{loc}$ is the class of complete algebras of $\mathbf{V}$.

The definitions do not make this obvious, so we note the following:

Proposition 3.2.30. The above classes are contained in each other.

Proof. See Exercise 3.12. ∎

Finitely Generated Varieties: As shown before, **BA** is finitely generated, which means that **CPC** is tabular; this is after all the same as saying that one can use *truth-tables* as semantics, which is not surprising at all. The examples from section 3.2.2 are also finitely generated varieties of Heyting algebras. Also note that by the arguments using Jonnson's Lemma, we have that finite generation is a hereditary property: if $\mathbf{V}$ then $\Lambda(\mathbf{V})$ contains only finitely generated varieties. One fact which we can easily prove is the following:

Proposition 3.2.31. **IPC** is not tabular.

Proof. If it was, assume that $\mathbf{HA} = \mathbb{V}(H)$ for H a finite algebra. Then note that each subdirectly irreducible will be a homomorphic image of a subalgebra of H; hence they will all be of size at most $|H|$. But by Proposition 3.2.13, any rooted finite poset – of any size – will be a subdirectly irreducible Heyting algebra. ∎

The structure of tabularity in Heyting algebras is very well-understood, and their position in $\Lambda(\mathbf{IPC})$ is quite precise. We mention the following result, although a proof lies outside the scope of our presentation:

Proposition 3.2.32. The set $\Lambda_{tab}(\mathbf{IPC})$ is an upset in $\Lambda(\mathbf{IPC})$. Moreover, every tabular logic has finitely many immediate predecessors in $\Lambda(\mathbf{IPC})$, all of which are tabular.

Locally finite varieties are the next step in the hierarchy. Not every locally finite variety is finitely generated: the variety **GA** of Gödel algebras, which corresponds to the logic **LC**, is an example of a locally finite variety which is not finitely generated (see Exercise 3.12(4)). However, it is clear that **IPC** is not amongst these ones:

Theorem 3.2.7. *The logic* **IPC** *is not locally tabular.*

Proof. See Exercise 3.11. ∎

Local finiteness is much harder to understand, and it is the subject of ongoing research to characterize it amongst extensions of **IPC**. Until recently there was an open conjecture by Bezhanishvili and Grigolia that said

A variety is locally finite if and only if all of its 2-generated algebras are finite.

This conjecture was disproven in 2023, by two independent teams consisting of, amongst others, *two former MSL alumni* (to whom Nick had spoken about this problem).

Varieties with the FMP: First let us check that indeed locally finite varieties have the FMP:

Theorem 3.2.8. *Let* V *be a locally finite variety. Then* V *has the FMP.*

Proof. Let $\mathbf{V}_{Fin}$ be the collection of finite algebras in $\mathbf{V}$. We will see that $\mathbf{V} = \mathbb{HSP}(\mathbf{V}_{Fin})$. This amounts to saying that, if $\mathbf{V} \not\models \varphi \approx \psi$, there is some $H \in \mathbf{V}_{Fin}$ such that $H \not\models \varphi \approx \psi$.

So, suppose that $\varphi(x_1, ..., x_n)$, $\psi(y_1, ..., y_m)$, and $H \not\models \varphi \approx \psi$ for some $H \in \mathbf{V}$. Then there are $a_1, ..., a_n, b_1, ..., b_m \in H$ such that $\varphi_H(a_1, ..., a_n) \neq \psi_H(b_1, ..., b_n)$. Let $S = \{a_1, ..., a_n, b_1, ..., b_m\}$. Generate a subalgebra $\langle M \rangle$ of H by S. M will be finite because $\mathbf{V}$ is locally finite and it is obvious that $M \not\models \varphi \approx \psi$. Hence $M \in \mathbf{V}_{fin}$, because it is a subalgebra of H, and it is finite, as desired. ∎

We also note that when dealing with varieties of Heyting algebras – indeed of arbitrary algebras – one may replace finite by finite and subdirectly irreducible:

Lemma 3.2.33. Let $\mathbf{V}$ be a variety of Heyting algebras. Then $\mathbf{V} = \mathbb{V}(\mathbf{V}_{fin})$ if and only if $\mathbf{V} = \mathbb{V}(\mathbf{V}_{fin.si})$.

Proof. One direction is obvious. For the other, we show that $\mathbf{V}_{fin} \subseteq \mathbb{V}(\mathbf{V}_{fin.si})$. Indeed if H is a finite algebra, note that by Birkhoff's subdirect decomposition theorem (Theorem 3.1.5), H is a subdirect product of subdirectly irreducible algebras which are its homomorphic images; since H is finite, the factors will be finite as well. ∎

To show that **IPC** does have the FMP we will make use of the technique of *algebraic filtration*. This has close parallels to filtration in modal logic. To use it we need to work with reducts of Heyting algebras. One of them we have not met before:

Definition 3.2.34. An algebra $(I, \wedge, \to, \top, \bot)$ is called a bounded *implicative semilattice* if:

1. $\bot$ is the least element and $\top$ is the greatest element;
2. $\wedge$ is associative, commutative and idempotent, and so defines an order: $a \leqslant b$ if and only $a \wedge b = a$;
3. For each $a, b, c \in I$:
$$c \wedge a \leqslant b \iff c \leqslant a \to b.$$

Implicative semilattices are the $\{\wedge, \to, \bot, \top\}$ are the reducts of Heyting algebras to that signature. For our purposes we will use the following purely logical lemma:

Lemma 3.2.35. Let φ be a formula in classical logic. If φ is positive (i.e., contains only positive occurrences of propositional variables), then φ is equivalent to a disjunction of conjunctions of propositional formulas.

Proof. Note a formula is positive if and only if its negation normal form contains no negation symbols. Hence we can assume that φ is constructed using only conjunctions, disjunctions, propositional variables, $\top$ and $\bot$. We will do induction on such a construction.

Indeed, for propositional variables, $\top$ and $\bot$, this follows easily. If φ, ψ satisfy the requirement, then $\varphi \vee \psi$ certainly does. Finally if $\varphi \equiv \bigvee_{i=1}^{n} \chi_i$ and $\psi \equiv \bigvee_{j=1}^{n} \chi'_j$ then

$$\varphi \wedge \psi \equiv \bigvee_{i=1}^{n} \chi_i \wedge \bigvee_{j=1}^{n} \chi'_j \equiv \bigvee_{1 \leqslant i < j \leqslant n} \chi_i \wedge \chi_j,$$

where we use distributivity. This gets us the result. ∎

Now we have:

Theorem 3.2.9.

1. *If $I = \langle S \rangle$ is an implicative semilattice generated by a finite set S, then I is finite.*
2. *If $D = \langle S \rangle$ is a distributive lattice generated by a finite set S, then D is finite.*

Proof. (1) is known as Diego's theorem; its proof is long, and requires techniques beyond the scope of our presentation.

We show (2). For this purpose, recall that

$$\langle S\rangle = \{t(s_1, ..., s_n) : s_i \in S, t \in Term_{DL}(\omega)\}.$$

Now let t be a term in the language of distributive lattices. Then t is a formula $\varphi(x_1, ..., x_n)$ in the language of classical logic, which is moreover positive. By Lemma 3.2.35, we have that $\varphi \equiv \psi$ for ψ a disjunction of conjunctions of propositional variables. But by idempotence, there are up to equivalence only finitely many such formulas. Hence $\langle S\rangle$ is finite. ■

As a corollary we get that **DL** is locally finite[2]; **IS**, the class of implicative semilattices, is also a variety, and the above also shows that it is locally finite.

We put this to work in the following:

Proposition 3.2.36. **HA** is finitely approximable, so **IPC** has the FMP.

Proof idea. Although Heyting algebras are not locally finite, by Theorem 3.2.9 we can use locally finite reducts of the signature $\vee, \to, \wedge, 0, 1$:

- The signature of distributivelattices: $\vee, \wedge, 0, 1$ (we get $\to$-free reducts)
- The signature of implicative semilattices: $\wedge, \to, 0, 1$ ($\vee$-free reducts)

Either of these two results will give us that **HA** has the FMP. We will illustrate the idea with the latter reduct.

To show that **HA** is finitely approximable, we will show that $\mathbb{V}(\mathbf{HA}_{fin}) = \mathbf{HA}$. By the logic-variety correspondence, this means that they have the same logic, i.e., if a formula fails for a Heyting algebra, it fails for some finite Heyting algebra.

Suppose that H is some Heyting algebra such that $H \not\models \varphi \approx 1$;, then there is a valuation v on H such that $v(\varphi) \neq 1$. Let $\mathsf{Sub}(\varphi)$ be the set of all subformulas of φ and let $\Sigma = \{v(\psi) : \psi \in Sub(\varphi)\} \subseteq H$. We generate an implicative subsemilattice $(I, \wedge, \to, 0, 1)$ of H by Σ, which will be finite by Theorem 3.2.9.

Now, we turn this into a Heyting algebra by defining a "fake" $\dot\vee$ on it: $a \,\dot\vee\, b = \bigwedge\{s \in A | s \geqslant a, b\}$. This new $(A, \wedge, \dot\vee, \to, 0)$ is a finite Heyting algebra with the property that for each $a, b \in A$ we have $a \vee b \leqslant a \,\dot\vee\, b$, and $a \vee b = a \,\dot\vee\, b$ whenever $a \vee b \in \Sigma$. It now follows (exercise) that this finite HA $(A, \wedge, \dot\vee, \to, 0)$ also refutes φ. ■

Remark 3.2.1. The above proof in fact entails the following result known as *McKay's theorem*: if $L = \mathbf{IPC} \oplus \Sigma$ such that every formula in Σ does not contain $\vee$, then $\mathbf{V}_L$ has the FMP (verify this claim).

Complex varieties: The next step up in the hierarchy are complex varieties, which are generated by duals of Kripke frames. There are many such logics, and they certainly include every logic with the FMP. Nevertheless, there are logics which are Kripke complete and do not have the FMP. A particularly easy example is the logic of the frame in Figure 3.1. Such a logic is Kripke complete, being the logic of a given Kripke frame. A proof that this logic does not have the FMP is beyond the scope of our methods, but see [2, Chapter 3] for a reference.

[2] Of course an abstract argument for this could be obtained by noting that $\mathbf{DL} = \mathbb{V}(\mathbf{2})$, by a similar argument to that given for Boolean algebras. But that argument is much less informative as to what is obtained when forming finitely generated distributive lattices.

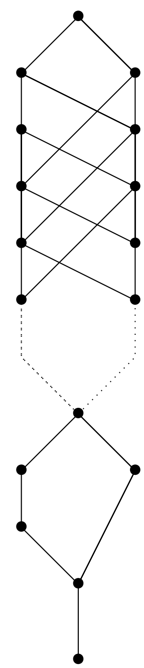


Figure 3.1: The frame $\mathcal{E}$

Topological varieties: Every Kripke frame has a dual Heyting algebra which also doubles as a topological space on the frame. Hence Kripke completeness implies topological completeness. Nevertheless the converse does not hold: Shehtman gave an example of a topologically complete logic which is Kripke incomplete. Such an example is much too complicated to present here, but is intimately related to one of longest-standing open problems in the field. *Kuznetsov's conjecture* says that every variety of **HA** is topologically complete. Despite over 50 years of research, this has neither been proven nor disproven. It is not known either whether every **localic variety** if topologically complete, or whether every variety is localic. Nevertheless, there has been progress in related problems: for varieties of bi-Heyting algebras, Bezhanishvili, Gabelaia and Jibladze showed in 2021 that Kuznetsov's conjecture is false.

To conclude we note the following result which also concerns topological completeness, and shows that **HA**, more than being the logic of arbitrary topological spaces, is also the logic of some very well-behaved topological spaces:

Theorem 3.2.10 (McKinsey-Tarski)**.** *Let A be the Heyting algebra of open sets on $\mathbb{R}$ with the standard topology. Then* $\mathbf{HA} = \mathbf{HSP}(A)$.

Proof strategy. As A is a HA we obviously have that $\mathbf{HSP}(A) \subseteq HA$. For the converse inclusion, we know that HA is generated by finite s.i. HAs by Lemma 3.2.33 and Proposition 3.2.36. So if $\mathbf{HA} \not\models \varphi \approx \top$, then there is a finite s.i. B such that $B \not\models \varphi \approx \top$. We show that B is embeddable in A, which will imply that $A \not\models \varphi \approx \top$. To show that B is embeddable in A it is sufficient to construct a continuous and open map from $\mathbb{R}$ onto X_B (seen as the Alexandroff space), which is the main technical content of the proof of the theorem. ■

3.3 Exercises

3.3.1 Universal Algebra and Varieties

Exercise 3.1. 1. Let $\varphi = \varphi(\bar{x})$ be a formula, where $\bar{x}$ denotes a tuple of variables. Prove the following:

(a) If $h : A \to B$ is a homomorphism, then $h(\varphi^A(\bar{a})) = \varphi^B(h\bar{a})$.

(b) If A is a subalgebra of B and $\bar{a} \in A$, then $\varphi^A(\bar{a}) = \varphi^B(\bar{a})$.

(c) If $A = \prod_{i \in I} A_i$ and $\bar{a} = (\langle a_i^1 \rangle_{i \in I}, \dots, \langle a_i^n \rangle_{i \in I}) \in A$, then

$$\varphi^A(\bar{a}) = \langle \varphi^{A_i}(a_i^1, \dots, a_i^n) \rangle_{i \in I}.$$

2.

Let $\mathbf{K}$ be a class of algebras. Use the previous exercise to show that, if an equation $\varphi \approx \psi$ is valid on $\mathbf{K}$, then it is also valid on $\mathbb{H}(\mathbf{K})$, $\mathbb{S}(\mathbf{K})$ and $\mathbb{P}(\mathbf{K})$.

(b) Deduce the following corollary of Birkhoff's theorem: For any class of algebras $\mathbf{K}$ and any variety $\mathbf{V}$,

$$Var(\mathbf{K}) = \mathbf{V} \qquad \text{if and only if} \qquad \forall \varphi \approx \psi : \mathbf{K} \models \varphi \approx \psi \Leftrightarrow \mathbf{V} \models \varphi \approx \psi$$

Exercise 3.2. Show that if $\mathbf{K}$ is a variety with the CEP then for each subclass of algebras $\mathbf{K}' \subseteq \mathbf{K}$, $\mathbb{HS}(\mathbf{K}') = \mathbb{SH}(\mathbf{K}')$.

Exercise 3.3. Let $\mathbf{V}$ be a variety of algebras of type $\mathcal{F}$, generated by a finite set K of finite algebras. Show that $\mathbf{K}$ is a finitely generated variety, i.e., it is generated by a single finite algebra.

Definition 3.3.1. Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$. We say that $\mathcal{A}$ is *simple* if $\mathsf{Cong}(A) = \{\Delta, \nabla\}$.

Exercise 3.4.

1. Show that for each algebra A in $\mathbf{DL}, \mathbf{HA}$ and $\mathbf{BA}$, A is simple if and only if $A \cong \mathbf{2}$.

2. Show that there exist infinitely many non-isomorphic simple modal algebras.

Exercise 3.5. Show that a variety $\mathbf{V}$ of algebras of type $\mathcal{F}$ is locally finite if and only if each free n-generated algebra of $\mathbf{V}$ is finite.

3.3.2 Applications to Logic

Exercise 3.6. Show that a distributive lattice D is subdirectly irreducible if and only if $D \cong \mathbf{2}$. *Hint: Use duality.*

Exercise 3.7. Prove Makinson's theorem: for each $L \in \Lambda(\mathbf{K})$ either $L \subseteq Log(B_1)$ or $L \subseteq Log(B_2)$ where B_1 is the dual of a single reflexive point and B_2 is the dual of a single irreflexive point.

Definition 3.3.2. Let H be a Heyting algebra. We say that H is *finitely subdirectly irreducible* if the unit filter $\{1\} \in \mathsf{Fil}(H)$ is a join-irreducible element.

Exercise 3.8. Show the following for H a Heyting algebra:

1. H is finitely subdirectly irreducible if and only if X_H is rooted.

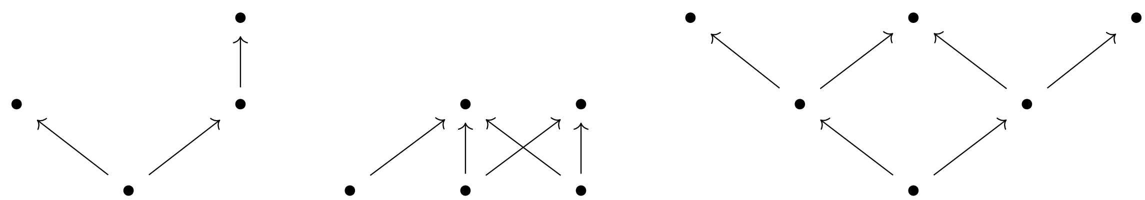
2. H is subdirectly irreducible if and only if $\{1\} \in \mathsf{Fil}(H)$ is a completely join-prime element.

3. H is subdirectly irreducible if and only X_H is rooted and the root is isolated.

Hint: Recall the dual characterizations of these concepts.

Exercise 3.9. Show that **HA** has the Congruence Extension Property (see Definition 3.1.13).

Exercise 3.10. Describe the lattice of subvarieties of the varieties generated by the dual Heyting algebras of the following posets:



S K P

Exercise 3.11. Define the formulas f_n, g_n by the following recursive clauses:

$$\begin{cases} g_0 & := p \\ g_1 & := \neg p \\ g_2 & := \neg\neg p \\ g_3 & := \neg\neg p \to p \\ g_{n+4} & := g_{n+3} \to g_n \vee g_{n+1} \end{cases} \qquad \begin{cases} f_0 & := g_0 = p \\ f_1 & := g_1 = \neg p \\ f_{n+2} & := g_n \vee g_{n+1} \end{cases}$$

1. What is the evaluation of the formulas f_n, g_n on the Rieger-Nishimura lattice (depicted in Figure 3.2)? Label the nodes with the corresponding formulas.

2. Deduce that the Rieger-Nishimura lattice is an example of finitely generated, non-finite Heyting algebra.

3. Deduce that the variety of all Heyting algebras **HA** is *not* locally finite.

4. *Extra:* Show that the join-irreducible elements are exactly those labeled with a formula g_n.

Exercise 3.12. In this exercise we will show that the hierarchy of varieties holds for **IPC**, and is strict: show that

$$\text{fin. gen.} \implies \text{loc. finite} \implies \text{FMP} \implies \text{complex} \implies \text{Topological} \implies \text{Localic}\ ,$$

and show that some of these arrows cannot be reversed,

1. Show each of these implications.

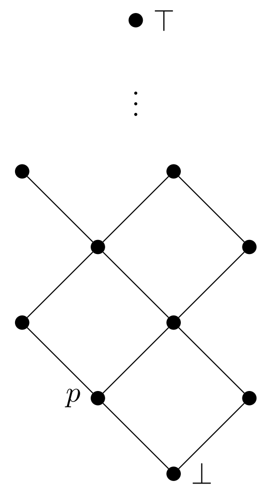


Figure 3.2: Rieger-Nishimura lattice.

2. Give an example of a finitely generated variety with a non-linear lattice of logics.

3. Show that there are varieties with the FMP which are not locally finite.

4. (*) Consider the system $\mathbf{LC} = \mathbf{IPC} \oplus (p \to q) \vee (q \to p)$. Its algebras are called *Gödel algebras*, and denoted by $\mathbf{GA}$.

   (a) Show that an algebra H is a Gödel algebra if and only if for each $x \in X_H$, $\uparrow x$ is a chain.

   (b) Given a finite set X, let 2^X be the poset with the pointwise order. Let $C(2^X) = \{S \subseteq 2^X : S \text{ is a chain}\}$, and order this poset as follows: $S \leqslant T$ if and only if T is a final interval of S. Show that $\mathsf{ClopUp}(C(2^X), \leqslant)$ is the free Gödel algebra on X many generators[3].

   (c) Use the previous result, together with Exercise 3.5, to conclude that $\mathbf{GA}$ is locally finite.

   (d) Show that $\mathbf{GA}$ is not finitely generated.

Exercise 3.13. Let $(\mathbb{Z}^-, \leqslant)$ be the set of non-positive integers with the usual order. Consider the $\mathbf{S4}$-algebra $(\mathcal{P}(\mathbb{Z}^-), \Box_\leqslant)$. Show that the subalgebra generated by $E = \{0, -2, -4, -6, ...\}$ is infinite. Conclude that the variety of $\mathbf{S4}$-algebras is not locally finite.

[3] This description of free Gödel algebras is, as far as we are aware, due to Aguzzolli et.al., and was extended to the infinite case by Carai.

Chapter 4

Topics in Algebraic Logic

We have now collected all the necessary tools and concepts to begin proving some strong results about the structure of non-classical logic systems. The approach taken here is one which emphasizes several steps:

- First one identifies a property of interest in the logical system – such as completeness. Here the emphasis remains on analysing *the system*.
- One then develops general enough methods to try to prove that a large class of subsystems have or do not have the property. Here the emphasis moves to subsystems. For instance, a general completeness result showing that all varieties axiomatized by a class of formulas is Kripke complete.
- Eventually, one tries to characterize exactly those subsystems which have the property. Here the emphasis is more on *the property*: such results allow us to understand what is *the structure of the property*, such as knowing that completeness happens in specific subvarieties.

The example is not felicitous, since completeness is a very wild property. But for other tamer properties, we do have complete solutions.

The topics mentioned here are all of relevance to modal logic and intuitionistic logic. The latter topics are included for the interested reader, as well as optional topics for future year of teaching MSL.

4.1 Sahlqvist Canonicity

Let $(B, \Box)$ be a modal algebra. Then $(B, \Box) \simeq (\mathsf{Clop}(X), \Box_R)$ for its dual modal space (X, R). The algebra $(\mathcal{P}(X), \Box_R)$ is called the *canonical extension* of $(B, \Box)$, which is an extension of $(B, \Box)$ as an algebra. We denote it by $(B^\sigma, \Box^\sigma)$.

Definition 4.1.1. A variety of modal algebras $\mathbf{V}$ is called *canonical* if $(B, \Box) \in \mathbf{V}$ entails $(B^\sigma, \Box^\sigma) \in \mathbf{V}$. In such a situation we say that the corresponding normal modal logic is *strongly canonical* or *d-persistent* [1].

[1] The reason for this terminology is as follows: a normal modal logic L is called *canonical* if its canonical frame is a frame for the logic. This is the canonical extension of the canonical general frame, and hence a strongly canonical logic will be canonical. The converse is an open problem: it is not known whether every canonical logic is strongly canonical.

Theorem 4.1.1. *Every canonical variety is complex, i.e.,* $\mathbf{V}$ *is generated by the class*

$$\{(\mathcal{P}(X), \Box_R) \in \mathsf{V} : \text{ for some modal space } (X, R)\}.$$

Proof. Suppose $\mathbf{V} \not\models \varphi \approx \top$. Then there is $(B, \Box) \not\models \varphi \approx \top$. But then $(B^\sigma, \Box^\sigma) \not\models \varphi \approx \top$. By our assumption $(B^\sigma, \Box^\sigma) \in \mathbf{V}$, which finishes the proof. ■

Next we show that every variety axiomatized by Sahlqvist formulas is canonical and hence Kripke complete. Before that we will need the following technical result due to Esakia:

Lemma 4.1.2 (Esakia's lemma)**.** Let (X, R) be a modal space. If I is a directed set of closed sets[2], then

$$R^{-1}[\bigcap I] = \bigcap\{R^{-1}[U] : U \in I\}.$$

Proof. Obviously, $\bigcap I \subseteq U$ for each $U \in I$. So $R^{-1}(\bigcap I) \subseteq R^{-1}[U]$ for each $U \in I$ and therefore $R^{-1}[\bigcap I] \subseteq \bigcap\{R^{-1}[U] : U \in I\}$. Conversely, assume $x \in \bigcap\{R^{-1}[U] : U \in I\}$. Then $R[x] \cap U \neq \varnothing$ for each $U \in I$. By directedness of I this gives us that $\{R[x] \cap U : U \in I\}$ has the finite intersection property. By compactness, $\bigcap\{R[x] \cap U : U \in I\} = R[x] \cap \bigcap I \neq \varnothing$, which means that $x \in R^{-1}[\bigcap I]$. ■

We now recall the definition of Sahlqvist formulas.

Definition 4.1.3. *Positive modal formulas* are defined via the following schema:

$$p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi$$

Negative formulas are negations of positive formulas.

Theorem 4.1.2 (The intersection lemma)**.** *Let (X, R) be a modal space and consider the modal algebra $(\mathcal{P}(X), \Box_R)$. Then for each positive formula $\psi(p_0, \ldots, p_n)$ and a closed sets $F, F_1, \ldots, F_n \subseteq X$ we have*

$$\psi(F, F_1, \ldots, F_n) = \bigcap\{\psi(C, F_1, \ldots, F_n) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}.$$

Proof. For simplicity we prove the lemma only for fomrulas in one variable. The cases $\psi = \top$ and $\psi = \bot$ are trivial. The case $\psi = p$ follows from the fact that in compact Hausdorff spaces every closed set is an intersection of clopen sets containing it. Now assume $\psi = \chi \wedge \xi$. Then $\psi(F) = \chi(F) \cap \xi(F) = \bigcap\{\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} \cap \bigcap\{\xi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{\chi(C) \cap \xi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{\chi \wedge \xi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{\psi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}$.

Now let $\psi = \chi \vee \xi$. First note that if $U \subseteq V$, then for every positive ψ we have $\psi(U) \subseteq \psi(V)$ (Exercise). So $F \subseteq C$ implies $\psi(F) \subseteq \bigcap\{\psi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}$. Now suppose $x \notin \psi(F)$. Then $x \notin \chi(F)$ and $x \notin \xi(F)$. By induction hypothesis, this implies that there are $C \in \mathsf{Clop}(\mathsf{X})$ and $D \in \mathsf{Clop}(\mathsf{X})$ such that $x \notin \chi(C)$ and $x \notin \xi(D)$. Let $E = C \cap D$. Then $x \notin \chi(E)$ and $x \notin \xi(E)$. So $x \notin \psi(E)$ and $x \notin \bigcap\{\psi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}$.

Next suppose $\psi = \Diamond\chi$. Then $\Diamond_R\chi(F) = \Diamond_R \bigcap\{\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}$. Note that $\{\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}$ is directed. So by Esakia's lemma we have $\Diamond_R \bigcap\{\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{\Diamond_R\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{\psi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}$.

Finally, let $\psi = \Box\chi$. Then $\Box_R\chi(F) = -\Diamond_R - \bigcap\{\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = -\Diamond_R \bigcup\{-\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = -\bigcup\{\Diamond_R\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{-\Diamond_R - \chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{\Box_r\chi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\} = \bigcap\{\psi(C) : C \in \mathsf{Clop}(\mathsf{X}), F \subseteq C\}$.

■

[2]A subset $I \subseteq \mathcal{P}(X)$ is called *directed* if for every $F, G \in I$, there is an $H \subseteq F \cap G$.

Definition 4.1.4.

1. A *boxed atom* is a modal formula of the form $\Box^n p$, for some $n \in \mathbb{N}$, where p is a propositional variable, and $\Box^n p$ is defined by the rule: $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$, $n \in \mathbb{N}$.

2. A *Sahlqvist antecedent* is built from $\bot, \top$, boxed atoms and negative formulas by applying $\Diamond$ and $\wedge$.

3. A *simple Sahlqvist formula* is a modal formula of the form $\varphi \to \psi$, where φ is a Sahlqvist antecedent and ψ is a positive formula.

4. A *Sahlqvist formula* is built from simple Sahlqvist formulas by applying $\Box$ and $\vee$.

An equation $\varphi \approx \top$ is a *Sahlqvist equation* if φ is a Sahlqvist formula. A variety of modal algebra V is called *Sahlqvist* if it is axiomatized by some set Σ of Sahlqvist equations.

Theorem 4.1.3. *Every Sahlqvist variety of modal algebras is canonical.*

Proof. Let $\mathbf{V}$ be a Sahlqvist variety axiomatized by a set Σ of Sahlqvist equations. Let $(B, \Box)$ be a modal algebra such that $(B, \Box) \in \mathbf{V}$. We will show that $(B^\sigma, \Box^\sigma) \in \mathsf{V}$. Let (X, R) be a modal space dual to $(B, \Box)$. Then $(B, \Box) \simeq (\mathsf{Clop}(X), \Box_R)$ and $(B^\sigma, \Box^\sigma) = (\mathcal{P}(X), \Box_R)$. Suppose $(B^\sigma, \Box^\sigma) \notin \mathbf{V}$. This means that there is a Sahlqvist equation $\varphi \approx \top \in \Sigma$ such that $(B^\sigma, \Box^\sigma) \not\models \varphi \approx \top$. Therefore, $(\mathcal{P}(X), \Box_R) \not\models \varphi \approx \top$. We will now show that then $(\mathsf{Clop}(X), \Box_R) \not\models \varphi \approx \top$, which is a contradiction.

We will illustrate this by going through a concrete example. Let

$$\varphi = \Diamond^2 \Box p \wedge \Diamond \Box^2 p \to \psi(p)$$

where ψ is a positive formula.

Since $(\mathcal{P}(X), \Box_R) \not\models \varphi \approx \top$, we have that there is $U \in \mathcal{P}(X)$ such that $\Diamond_R^2 \Box_R(U) \cap \Diamond_R \Box_R^2(U) \nsubseteq \psi(U)$. So there is $x \in \Diamond_R^2 \Box_R(U) \cap \Diamond_R \Box_R^2(U)$ such that $x \notin \psi(U)$. The former implies that there are $x_1, x_2 \in X$ such that $x R x_1 R x_2$ and $x_2 \in \Box_R(U)$, and there is $x_3 \in X$ such that $x R x_3$ and $x_3 \in \Box_R^2(U)$. So $R[x_2] \subseteq U$ and $R[R[x_3]] \subseteq U$. We let $F = R[x_2] \cup R[R[x_3]]$. Then F is a closed set (why?). Moreover, as ψ is a positive formula $\psi(F) \subseteq \psi(U)$. So $x \notin \psi(F)$. Therefore, it follows from the definition of F that $x \in \Diamond_R^2 \Box_R(F) \cap \Diamond_R \Box_R^2(F)$. By the intersection lemma (Lemma 4.1.2), $\psi(F) = \bigcap\{\psi(C) : C \in \mathsf{Clop}(X), F \subseteq C\}$. So there is $C \in \mathsf{Clop}(X)$ such that $F \subseteq C$ and $x \notin \psi(C)$. But as $F \subseteq C$ we also have that $x \in \Diamond_R^2 \Box_R(C) \cap \Diamond_R \Box_R^2(C)$. Therefore, $(\mathsf{Clop}(X), \Box_R) \not\models \varphi \approx \top$. ■

One might wonder why Sahlqvist formulas would have this privileged role, and whether they belong to a larger class of formulas which are canonical. After decades, still no one seems to have an answer to either of these questions.

4.2 Translations and Modal Companions

One of the interesting features connecting the various logical systems we have been working with is that not only do they often extend each other – classical logic can be seen as an extension of both intuitionistic and modal logic – but there are translations connecting these systems. In this section we focus on two: the *Glivenko negative translation* and the *Godel-McKinsey-Tarski translation.*

The first one traces its origins back to Kolmogorov, and connects intuitionistic logic very tightly to classical logic:

Definition 4.2.1. The *Kolmogorov-Glivenko Double Negation Translation* $(\neg\neg)$, denoted by K maps the language of classical logic $\mathcal{L}_{CPC}$ to the language of intuitionistic logic $\mathcal{L}_{IPC}$, through the following assignment:

1. For each $p \in \mathsf{Prop}$, $K(p) = \neg\neg p$;
2. $K(\top) = \top$ and $K(\bot) = \bot$;
3. $K(\varphi \wedge \psi) = K(\varphi) \wedge K(\psi)$, and $K(\neg\varphi) = \neg K(\varphi)$;
4. $K(\varphi \vee \psi) = \neg\neg(K(\varphi) \vee K(\psi))$.

Remark 4.2.1. It can be shown that for the above translation, we have that for each formula $\varphi \in \mathcal{L}_{CPC}$:

$$\neg\neg\varphi \equiv K(\varphi).$$

Theorem 4.2.1 (Glivenko's theorem)**.** *For each formula $\varphi \in \mathcal{L}_{CPC}$:*

$$\varphi \in \mathbf{CPC} \iff K(\varphi) \in \mathbf{IPC}$$

Proof. If $K(\varphi) \in \mathbf{IPC} \subseteq \mathbf{CPC}$, $K(\varphi) \in \mathbf{CPC}$, and by our remark this is equivalent to $\neg\neg\varphi$; hence in classical logic it is equivalent to φ.

For the converse, assume that $K(\varphi) \notin \mathbf{IPC}$. Let H be a Heyting algebra such that $H \nvDash K(\varphi)$, and let $v : \mathsf{Prop} \to H$ be a valuation witnessing this. Let $v' : \mathsf{Prop} \to H$ be a valuation such that $v'(p) = \neg\neg v(p)$, and induce the clauses upwards. Then $v' : \mathsf{Prop} \to H_{\neg\neg}$, as given in Definition 2.4.12. We then have that $H_{\neg\neg}, v' \nvDash \varphi$ by induction, which shows that $\varphi \notin \mathbf{CPC}$. ■

The situation is similar with the Gödel-McKinsey-Tarski translation; however such a translation is much stronger, in the sense that it connects a whole lattice of logics.

Definition 4.2.2. The *Gödel-McKinsey-Tarski* (G) translation maps the language of intuitionistic logic $\mathcal{L}_{IPC}$ to the language of modal logic $\mathcal{L}_K$ through the following assignment:

1. For each $p \in \mathsf{Prop}$, $G(p) = \Box p$;
2. $G(\top) = \top$ and $G(\bot) = \bot$;
3. $G(\varphi * \psi) = G(\varphi) * G(\psi)$, $* \in \{\wedge, \vee\}$;
4. $G(\varphi \to \psi) = \Box(\neg G(\varphi) \vee G(\psi))$.

Then we have:

Theorem 4.2.2. *For each formula $\varphi \in \mathcal{L}_{IPC}$:*

$$\varphi \in \mathbf{IPC} \iff G(\varphi) \in \mathbf{S4}.$$

Proof. Assume that $\varphi \notin \mathbf{IPC}$. Let H be a Heyting algebra refuting this, and consider (X_H, v) its Esakia space with a valuation such that $v(\varphi) \neq X_H$. Recall by Remark 2.5.14 that Esakia spaces are modal spaces, and given the relation is transitive and reflexive, **S4**-spaces. Hence by induction we can show that:

$$(X_H, v) \nVdash \varphi \iff (X_H, v) \nVdash G(\varphi),$$

and the latter implies that $G(\varphi) \notin \mathbf{S4}$ since **S4** is complete with respect to **S4** modal spaces.

Conversely assume that $G(\varphi) \notin \mathbf{S4}$. Let M be an S4 algebra such that $M \nvDash G(\varphi)$. Then consider $M_{\Box}$, the algebra of open elements (see Definition 2.5.11). Then we can show by induction that

$$M_{\Box} \nvDash \varphi \iff M \nvDash G(\varphi)$$

which since $M_{\Box}$ is a Heyting algebra, means that $\varphi \notin \mathbf{IPC}$. ∎

Once one arrives here, there are several observations to make: first, that this translation is not restricted to **IPC**, and hence we could consider it on extensions; second that its not clear whether **S4** is the strongest logic which does not derive any more translations of intuitionistic formulas. The former question leads to the definition of a *modal companion*:

Definition 4.2.3. Let $L \supseteq \mathbf{IPC}$ and $M \supseteq \mathbf{S4}$. If for each intuitionistic formula φ we have

$$\varphi \in L \text{ if and only if } G(\varphi) \in M$$

then M is called a *model companion* of L and L is called an *intuitionistic fragment* of M.

Let us show that another well-known logic is a modal companion of **IPC**: recall the logic

$$\mathbf{Grz} := \mathbf{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p.$$

Recall that (you were asked to prove this in Introduction to Modal Logic, Homework 4):

Theorem 4.2.3. *The logic* **Grz** *is sound and complete with respect to finite posets.*

Then we have:

Proposition 4.2.4. The logic **Grz** is a modal companion of **IPC**.

Proof. If $\varphi \in \mathbf{IPC}$, then $G(\varphi) \in \mathbf{S4} \subseteq \mathbf{Grz}$ by the above result. Conversely, if $\varphi \notin \mathbf{IPC}$, let $H \nvDash \varphi$ be a finite algebra. Then X_H is a finite poset. Then $X_H \nvDash \varphi$ means that $G(\varphi) \notin \mathbf{Grz}$. ∎

It is thus natural to ask: how far does this extend?

Extra Content

The following section dives deeper into the properties of modal companions, giving a brief idea of Blok-Esakia theory.

4.2.1 $\Diamond$ The Blok-Esakia isomorphism

The theory concerning modal companions is extensive, and comprises several technical tools connecting the lattice $\Lambda(\mathbf{IPC})$ and $\Lambda(\mathbf{S4})$, and their algebraic analogues. The key result in the area is the following:

Theorem 4.2.4 (Blok-Esakia isomorphism)**.** *Let L be a superintuitionistic logic, and let $\sigma(L) = \mathbf{Grz} \oplus \{G(\varphi) : \varphi \in L\}$. Then $\sigma(L)$ is the greatest modal companion of L, and $\sigma : \Lambda(\mathbf{IPC}) \to \Lambda(\mathbf{Grz})$ is an isomorphism of lattices of logics.*

This result was obtained indepenently by Wim Blok and Leo Esakia, building on previous work by a variety of different authors. More than simply giving an isomorphism, this method allows the transfer of many properties backwards and forwards (in some directions) – FMP, tabularity, local tabularity, decidability, axiomatizability by special classes of formulas, amongst other properties.

We will give some routes towards the proof – which is too extensive to give here. Using algebraic tools, we first find a corresponding map between $\Lambda(\mathbf{HA})$ and $\Lambda(\mathbf{Grz})$, the lattice of extensions of $\mathbf{Grz}$-algebras. Indeed, given a Heyting algebra H, let $B(H) \cong (\mathsf{Clop}(X_H), \Box_{\leqslant})$ be the $\mathbf{S4}$-algebra corresponding to the Jonnson-Tarski dual of the Esakia space X_H. The key observation to make is:

Proposition 4.2.5. A reflexive and transitive modal space (X, R) is dual to a $\mathbf{Grz}$-algebra if and only if whenever $U \subseteq X$ is clopen, and $x \in \mathsf{qmax}(U)$, then there does not exist some $y \notin U$ such that xRy and some $z \in U$ such that yRz.

Proof. Assume first that the property holds. Consider the converse version of the axiom:

$$p \to \Diamond(p \wedge \neg\Diamond(\Diamond p \wedge \neg p)).$$

Suppose that $x \in U$ for U a clopen. Let xRy be some element such that $y \in \mathsf{qmax}(U) = \{w \in U : wRk \wedge k \in U \implies kRw\}$ (one exists by a Zorn's Lemma argument). Then we claim that $y \in X - R^{-1}[R^{-1}[U] \cap X - U]$. Otherwise, yRz and zRw where $w \in U$ and $z \notin U$, contradicting the assumption.

Conversely, assume that the axiom holds. then if U is clopen, assume that $x \in \mathsf{qmax}(U)$. If there exist $y \notin U$ and $z \in U$ in the described conditions then because points seen by y which are in U will always see y, $U \nsubseteq R^{-1}[U \cap X - R^{-1}[R^{-1}[U] \cap X - U]]$. ∎

Corollary 4.2.6. *If X_H is an Esakia space, then seen as a modal space, X_H is a* **Grz***-space.*

Proof. Since the order in X_H is partial, quasi-maximal points are maximal, so the situation described above could never happen. ∎

With this we can describe the following map:

$$\begin{aligned} \sigma : \Lambda(\mathbf{HA}) &\to \Lambda(\mathbf{Grz}) \\ \mathbf{V} &\mapsto \mathbb{V}(\{B(H) : H \in \mathbf{V}\}). \end{aligned}$$

Since $B(H)$ are always $\mathbf{Grz}$-algebras, this is well-defined. It can be shown that this map is precisely the algebraic dual of the σ-map on logics, hence the ambiguous terminology.

It is not very difficult to show that σ is order-preserving and reflecting, and that it is injective – though it requires some more careful analysis of the way in which the $B(-)$ construction relates to other algebraic constructions, like products, injective homomorphisms and surjective homomorphisms. Part of this goes by showing that its inverse is well-defined:

$$\rho : \Lambda(\mathbf{S4}) \to \Lambda(\mathbf{HA})$$
$$\mathbf{V} \mapsto \{M_{\Box} : M \in \mathbf{V}\},$$

and it can be shown that $\rho(\mathbf{V})$ is a variety without too much effort.

The most difficult step lies in showing that σ is surjective. This goes fundamentally by showing *Blok's lemma*, which in one of its equivalent formulations, says the following:

Lemma 4.2.7 (Blok's Lemma)**.** For each M a **Grz**-algebra, M embeds into an ultrapower of $B(M_{\Box})$, which is its subalgebra.

This implies surjectivity: for each variety $\mathbf{V}$ of **Grz**-algebras, certainly $\mathbb{V}(\{B(M_{\Box}) : M \in \mathbf{V}\}) \subseteq \mathbf{V}$ since $B(M_{\Box}) \leqslant M$. Conversely, if $M \in \mathbf{V}$, then $M \in \mathbb{SP}_U(\{B(M_{\Box})\}) \subseteq \mathbb{V}(B(M_{\Box}) : M \in \mathbf{V}\})$, showing the equality.

The proof of the lemma is highly technical and requires many steps, including some highly non-trivial algebraic manipulations. Zacharyaschev obtained an alternative proof using canonical formulas, which is itself quite involved due to the complexity of working with canonical formulas. Recently Bezhanishvili and Cleani, in a MoL thesis, provided an easier and geometrically intuitive proof of this result. Recent work has focused on studying further transfer properties of this isomorphism, as well as attempting to establish other isomorphisms in related systems, some of which extend intuitionistic logic and **S4**.

End of extra content.

4.3 The Method of Jankov Formulas

In what follows we will prove two important results:

1. There are precisely continuum many intermediate logics; and
2. Every logic corresponding to a locally finite variety is axiomatizable by Jankov formulas.

Both of these will follow from the use of Jankov formulas, so we begin by introducing these first.

4.3.1 Jankov (de Jongh, Fine) formulas

Let H and H' be two Heyting algebras. We define the following order:

$$H \leqslant H' \Leftrightarrow H \in \mathbf{HS}(H')$$

Notice that because Heyting algebras have the CEP, and hence $\mathbb{HS} = \mathbb{SH}$ (see Exercise 3.9), this is a partial order.

Definition 4.3.1 (Jankov formula)**.** Let $H \in \mathbf{HA}_{fin.si}$ and for each $a \in H$, let p_a be a variable. Then the *Jankov formula of H* is defined as:

$$\chi(H) = \alpha \to p_s$$

where:

- s is the second largest element of H (which exists by Proposition 3.2.13);
- $\alpha = (p_0 \leftrightarrow \bot) \wedge \bigwedge\{(p_a \wedge p_b) \leftrightarrow p_{a\wedge b} \mid a, b \in A\} \wedge$
  $\bigwedge\{(p_a \vee p_b) \leftrightarrow p_{a\vee b} \mid a, b \in A\} \wedge$
  $\bigwedge\{(p_a \to p_b) \leftrightarrow p_{a\to b} \mid a, b \in A\}$.

Theorem 4.3.1 (Jankov)**.** *Let $H \in \mathbf{HA}_{fin.si}$. Then for any Heyting algebra H' we have*

$$H' \nvDash \chi(H) \text{ iff } H \leqslant H'.$$

Proof. Exercise. ■

It follows from Theorem 4.3.1 that for H a finite s.i., $H \nvDash \chi(H)$. Jankov introduced these formulas in an attempt to analyze independent axiomatizations of varieties of Heyting algebras. The idea that they seek to generalize, and which went in the direction proposed by McKenzie, is one which we have seen before in Theorem 2.1.1 about lattices: trying to characterize a variety by *forbidding some configurations*. As it turns out this also has its own lattice theoretic import, in terms of the lattice of subvarieties:

Definition 4.3.2 (Splitting Pair)**.** Let Λ be a lattice of varieties. For $\mathbf{V}_1, \mathbf{V}_2 \in L$, we say that $(\mathbf{V}_1, \mathbf{V}_2)$ *splits* Λ iff $\mathbf{V}_1 \nleqslant \mathbf{V}_2$ and for all $\mathbf{V} \in \Lambda$, $\mathbf{V}_1 \nsubseteq \mathbf{V} \Rightarrow \mathbf{V} \subseteq \mathbf{V}_2$. We call $(\mathbf{V}_1, \mathbf{V}_2)$ a *splitting pair.*

Notice that a splitting pair *completely* splits a lattice of varieties, in the sense that it partitions the lattice in two complementary classes (check this!).

Example 4.3.3. Recall $\Lambda(\mathbf{K})$ and Makinson's theorem (Exercise 3.7). Then the logics $\mathbb{V}(B_1)$ and $\mathbb{V}(B_2)$ are splittings of this lattice.

We then have the following proposition:

Proposition 4.3.4. Let $A \in \mathbf{HA}_{fin.si}$. Then $(\mathbb{V}(H), \mathbf{HA} + \chi(H))$ is a splitting pair in $\Lambda(\mathbf{HA})$.

Proof. Notice first that $\mathbb{V}(H) \nsubseteq \mathbf{HA} + \chi(H)$; this is so because $H \in \mathbb{V}(H)$ but, by Theorem 4.3.1, $H \nvDash \chi(H)$ and thus $H \notin \mathbf{HA} + \chi(H)$.

Let then $\mathbf{V}$ be a variety of Heyting algebras such that $\mathbb{V}(H) \nsubseteq \mathbf{V}$; we show that $\mathbf{H} \vDash \chi(H)$ (and thus $\mathbf{V} \subseteq \mathbf{HA} + \chi(H)$). Suppose towards contradiction that $H' \nvDash \chi(H)$, for some $H' \in \mathbf{V}$. Then, by Jankov's Theorem 4.3.1, $H \in \mathbb{HS}(H')$ and therefore $H \in \mathbf{V}$; but then $\mathbb{V}(A) \subseteq \mathbf{V}$, contradicting our assumption. Therefore for all $H' \in \mathbf{V}$, we have $H' \vDash \chi(H)$ and hence we obtain that $\mathbf{V} \vDash \chi(H)$. ■

We may wonder whether there are any other splittings; as follows by results of McKenzie, these are in fact the only ones.

4.3.2 Size of the lattice of extensions of HA

It is not difficult to show that **HA** has infinitely many subvarieties. Indeed, above (see Example 3.2.19) we have alluded to the fact that there are infinitely many finitely generated varieties. Indeed, there can obviously be at most countably infinitely many finitely generated varieties of Heyting algebras. But the question of how many varieties there are in general is still left in the open – it could potentially be as many as continuum. We will now use Jankov's theorem and duality to show that this is true: there are exactly continuum many varieties of Heyting algebras, and consequently, continuum many intermediate logics.

We prove the desired result in two steps.

Lemma 4.3.5. Let $\mathbf{V}$ be a variety of Heyting algebras. Suppose that Δ is an infinite anti-chain in $\mathbf{V}_{FSI}$. Then there are (precisely) continuum many subvarieties of $\mathbf{V}$.

Proof. Since our language is countable, we know that there are at most continuum many elements of $\Lambda(\mathbf{V})$. Notice that as $\mathbf{V}_{fin.si}$ is a collection of finite algebras, then it is at most countable and therefore Δ is countable. Therefore $|\Lambda(V)| \leqslant \mathfrak{c} = |\mathcal{P}(\Delta)|$. Let Γ_1 and Γ_2 be subsets of Δ; it suffices to show that if $\Gamma_1 \neq \Gamma_2$, then $\mathbb{V}(\Gamma_1) \neq \mathbb{V}(\Gamma_2)$.

So suppose that $\Gamma_1 \neq \Gamma_2$. Without loss of generality, assume that $\Gamma_1 \nsubseteq \Gamma_2$. Then there is $H \in \Gamma_1$ such that $H \notin \Gamma_2$. Consider $\chi(H)$; by Theorem 4.3.1, $H \not\models \chi(H)$ and thus $\mathbb{V}(\Gamma_1) \not\models \chi(A)$. Suppose for contradiction that also $\mathbb{V}(\Gamma_2) \not\models \chi(H)$; hence by an easy corollary to the correspondence between varieties and logics (see Exercise 3.1), there is $H' \in \Gamma_2$ such that $H' \not\models \chi(H)$. Then, by Theorem 4.3.1, $H \leqslant H'$; as $H, H' \in \Delta$, this contradicts the assumption that Δ is an anti-chain. Therefore $\mathbb{V}(\Gamma_2) \models \chi(H)$ and thus $H \notin \mathbb{V}(\Gamma_2)$. Hence $\mathbb{V}(\Gamma_1) \neq \mathbb{V}(\Gamma_2)$. ■

But do infinite anti-chains *exist* in $\mathbf{HA}_{fin.si}$? We use *duality* for answering this question in the affirmative.

Lemma 4.3.6. There exist infinite anti-chains in $\mathbf{HA}_{fin.si}$.

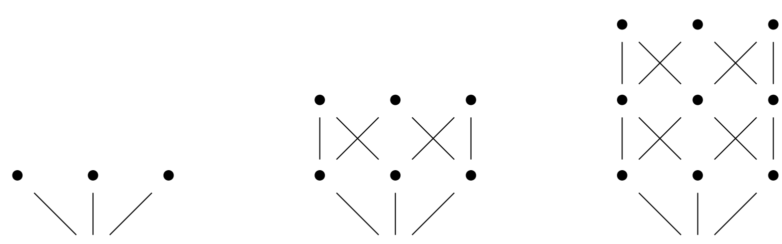
Proof. Using *Esakia duality*, we will present two infinite anti-chains in $\mathbf{HA}_{fin.si}$. Recall that if $H \in \mathbf{HA}_{fin.si}$, then its dual space X is a *finite rooted poset*. For F, G finite rooted posets, the following order corresponds to the one defined on $\mathbf{HA}_{fin.si}$:

$$F \leqslant G \text{ iff } F \text{ is an upset of a p-morphic image of } G$$

We now present two collections of finite rooted posets which are not related by the previous order, namely whose elements are:

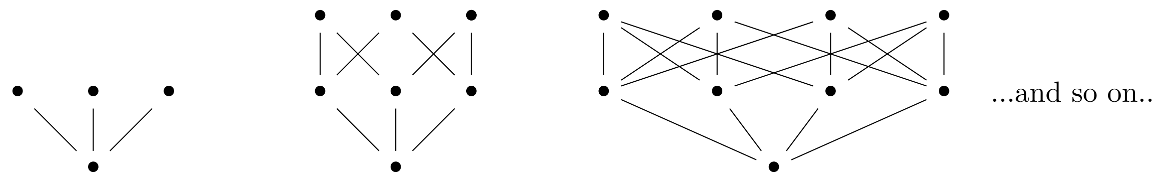
1. of arbitrary height and finite width;
2. of finite height and arbitrary width.

For 1, consider the following infinite collection of finite rooted posets:



...and so on...

As for 2, the following is a infinte collection of posets:



Just as before one can establish that no two distinct elements of this collection are comparable (Exercise).

Now, for each of the previous two collections of finite rooted posets, the collections of Heyting algebras dual to their elements constitute infinite anti-chains in $\mathbf{HA}_{fin.si}$. ■

It is then easy to establish our first main result:

Theorem 4.3.2. *There are (precisely) continuum many subvarieties of* $\mathbf{HA}$*. Consequently, there are (precisely) continuum many intermediate logics.*

Proof. By Lemmas 4.3.5 and 4.3.6, there are continuum many subvarieties of $\mathbf{HA}$. Thus this follows by the isomorphism of logics and varieties. ■

Remark 4.3.1. Let bd_n and bw_n be the following formulas for $n \geqslant 1$:

$$\mathbf{bw}_n \coloneqq \bigvee_{i=0}^{n}(p_i \to \bigvee_{j \neq i} p_j)$$

$$\mathbf{bd}_1 \coloneqq p_1 \vee \neg p_1$$

$$\mathbf{bd}_{n+1} \coloneqq p_{n+1} \vee (p_{n+1} \to \mathbf{bd}_n).$$

One can show (Exercise!) that for a finite poset, $(X, \leqslant)$, it validates bw_n if and only if it contains an antichain of n points and there is no antichain of greater cardinality; and it validates bd_n if it contains a chain of n points and no chain of greater cardinality.

Then by the shape of anti-chains, something more has been shown: there are continuum many intermediate logics of depth 3 and also continuum many such logics of width 3.

4.3.3 Axiomatizability over locally finite varieties using Jankov formulas

We now turn to the second important application of Jankov formulas: axiomatizating certain classes of algebras. In particular, we will show that this happens if the corresponding variety is locally finite. This gives us both a method for proving that a class of logics is finitely axiomatizable and a characterization of classes of logics in terms of Jankov formulas.

Let $\mathbf{V}' \in \Lambda(\mathbf{V})$, be varieties. Our question is then how should we axiomatize $\mathbf{V}'$ with respect to $\mathbf{V}$? For locally finite varieties, *Jankov's method* provides such a means in terms of *minimal counterexamples.*

Definition 4.3.7 (Minimal counterexample)**.** Let $\mathbf{V}' \in \Lambda(\mathbf{V})$. $H \in \mathbf{V}_{fin.si}$ is a minimal counterexample to $\mathbf{V}'$ iff $H \notin \mathbf{V}'$ and for all $H' \in \mathbf{V}$, if $H' \leqslant H$ and $H' \not\cong H$, then $H' \in \mathbf{V}'$.

Jankov's method is based on the following theorem.

Theorem 4.3.3. *If* $\mathbf{V}$ *is a locally finite variety, then for all* $\mathbf{V}' \in \Lambda(\mathbf{V})$*,* $\mathbf{V}'$ *is axiomatizable by Jankov formulas relative to* $\mathbf{V}$*.*

Proof. Let $\mathbf{V}$ be a locally finite variety, $\mathbf{V}' \in \Lambda(\mathbf{V})$ and define $\mathbf{V}''$ such that

$$\mathbf{V}'' = \mathbf{V} + \{\chi(H) : H \in min(\mathbf{V}_{fin.si} \backslash \mathbf{V}')\}$$

We show that $\mathbf{V}' = \mathbf{V}''$; since $\mathbf{V}$ is locally finite then both $\mathbf{V}'$ and $\mathbf{V}''$ are locally finite and thus both have FMP; therefore it suffices to show that $\mathbf{V}'_{fin.si} = \mathbf{V}''_{fin.si}$. For the $\subseteq$ direction, suppose that $H' \in \mathbf{V}'_{fin.si}$ and assume for contradiction that $H' \notin \mathbf{V}''_{fin.si}$. By the latter, there is $H \in min(\mathbf{V}_{fin.si} \backslash \mathbf{V}')$ such that $H' \not\models \chi(H)$. Therefore, by Theorem 4.3.1, $H \leqslant H'$ and thus $H \in \mathbf{HS}(H')$. Since $H' \in \mathbf{V}'$, it follows that $H \in \mathbf{V}'$. But $H \in min(\mathbf{V}_{FSI} \backslash \mathbf{V}')$ and thus $H \notin \mathbf{V}'$: contradiction.

For the $\supseteq$ direction, suppose that $H' \notin \mathbf{V}'_{FSI}$, for some $H' \in \mathbf{V}_{FSI}$. There is then $H \in min(\mathbf{V}_{FSI} \backslash \mathbf{V}')$ such that $H \leqslant H'$; this is because H' is finite, so either H' is minimal or a minimal element is reachable from H' in finitely many steps. Therefore, by Theorem 4.3.1, $H' \not\models \chi(H)$; hence $H' \notin \mathbf{V}''$, proving the counterpositive. Hence $V' = V''$. ■

Corollary 4.3.8. *Let* $\mathbf{V}$ *be a locally finite variety. If* $\mathbf{V}_{fin.si}$ *doesn't contain any infinite anti-chains, then for all* $\mathbf{V}' \in \Lambda(\mathbf{V})$*,* $\mathbf{V}'$ *is finitely axiomatizable over* $\mathbf{V}$*.*

Proof. If there is no infinite anti-chain in $\mathbf{V}_{FSI}$, then $min(\mathbf{V}_{FSI} \backslash \mathbf{V}')$ is finite (because $min(\mathbf{V}_{FSI} \backslash \mathbf{V}')$ is always an anti-chain!) and thus $\{\chi(H) : H \in min(\mathbf{V}_{FSI} \backslash \mathbf{V}')\}$ is also finite, thus Theorem 4.3.3 yields a finite axiomatization of $\mathbf{V}'$. ■

The following theorem give us an easy method for characterizing $\Lambda(V)$ in terms of Jankov formulas, for locally finite varieties V.

Theorem 4.3.4. *Let* $\mathbf{V}$ *be a locally finite variety. Then* $\Lambda(\mathbf{V}) \cong Downset(\mathbf{V}_{fin.si})$*.*

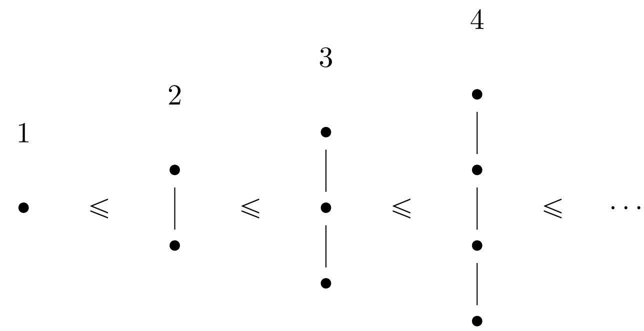
Proof. Let $\mathbf{V}' \in \Lambda(V)$ and consider the map

$$\mathbf{V}' \mapsto \mathbf{V}'_{fin.si}$$

Notice that $\mathbf{V}'_{fin.si} \in Downset(\mathbf{V}_{fin.si})$, because varieties are closed under $\mathbb{HS}$ and therefore $\mathbf{V}'_{fin.si}$ is also closed under $\mathbb{HS}$. We show that this is a bijection. For *injectivity*, suppose $\mathbf{V}' \neq \mathbf{V}''$, for $\mathbf{V}'$, $\mathbf{V}'' \in \Lambda(V)$.

Since $\mathbf{V}$ is locally finite, then both $\mathbf{V}'$ and $\mathbf{V}''$ are locally finite and thus both $\mathbf{V}'$ and $\mathbf{V}''$ have the FMP; therefore $\mathbf{V}' = \mathbb{V}(\mathbf{V}'_{fin.si})$ and $\mathbf{V}'' = \mathbb{V}(\mathbf{V}''_{fin.si})$. Therefore $\mathbf{V}'_{fin.si} \neq \mathbf{V}''_{fin.si}$. For *surjectivity*, let $D \in Downset(\mathbf{V}_{fin.si})$ and $Var(D) = \mathbf{V}'$; then $\mathbf{V}' \in \Lambda(\mathbf{V})$ and we show that $\mathbf{V}'_{fin.si} = D$. For the $\supseteq$ direction, it suffices to notice that $D \subseteq \mathbb{V}(D)_{fin.si} = \mathbf{V}'_{fin.si}$. As for the $\subseteq$ direction, let $H \in \mathbf{V}'_{fin.si}$ (we show that then $H \in D$). Consider $\chi(H)$. By Jankovs theorem, $H \not\models \chi(H)$. Since $H \in \mathbf{V}'$, then $\mathbf{V}' = \mathbb{V}(D) \not\models \chi(H)$. Therefore, there is $H' \in D$ such that $H' \not\models \chi(H)$ and hence, $H \leqslant H'$. As D is a downset, therefore $H \in D$, as required. ■

For illustration, let's characterize $\Lambda(\mathbf{LC})$. By Exercise 3.12 we have that the variety $\mathbf{GA}$ of Gödel algebras is locally finite. From this, one can deduce that $\mathbf{LC}_{fin.si}$ is the set of all finite chains. Notice that the order on $\mathbf{GA}_{fin.si}$ is



Observe that there is no anti-chain in $\mathbf{GA}_{fin.si}$ and thus by Corollary 4.3.8, together with the finite axiomatization of $\mathbf{LC}$, every subvariety of $\mathbf{GA}$ is finitely axiomatizable.

For the purposes at hand, call each chain by its cardinality. By Theorem 4.3.4, it follows that $\Lambda(\mathbf{GA}) \cong Downset(\mathbf{GA}_{fin.si})$. If $\mathbf{V} \in \Lambda(\mathbf{GA})$, then $\mathbf{V}_{fin.si}$ is its image in $Downset(\mathbf{GA}_{fin.si})$; notice that $\mathbf{V}_{fin.si}$ is the downset of some element n of $\mathbf{GA}_{FSI}$. But notice that there is then only one minimal counterexample to $\mathbf{V} = Var(\mathbf{V}_{FSI})$, namely $n+1$; therefore by Theorem 4.3.3, $\mathbf{V} = \mathbf{GA} + \chi(n+1)$. Therefore, we can characterize $\Lambda(\mathbf{GA})$ as $\{\mathbf{GA} + \chi(n+1) : n \in \omega\}$ (ordered, of course, by inclusion), which is, $\{Trivial, \mathbf{BA}, \cdots, \mathbf{GA}\}$, or, in terms of the corresponding logics, $\{\mathbf{LC}, \cdots, \mathbf{CPC}, Inconsistent\}$.

It should be noted that the above theorem 4.3.3 can in fact be strengthened:

Theorem 4.3.5. *If $\mathbf{V}$ is a locally finite variety of $\mathbf{HA}$, then $\mathbf{V}$ can be axiomatized by Jankov formulas over $\mathbf{HA}$.*

However, the tools used to prove this theorem are much more refined than we have access to. We also note the following fact:

Proposition 4.3.9. Not every subvariety of $\mathbf{HA}$ is axiomatizable by Jankov formulas.

The study of these questions leads to several interesting open fields of research – the most challenging being the question of how to characterize those varieties which are axiomatizable by Jankov formulas.

Extra Content

The following two sections are included as extra content covering two classical topics of algebraic logic with deep dual connections.

4.4 ◊ MacNeille completions

Definition 4.4.1. Let P and Q be posets. A map $\eta : P \to Q$ is called an *order-embedding* if it is injective and

$$x \leqslant y \Leftrightarrow \eta x \leqslant \eta y.$$

If $\eta : P \hookrightarrow L$ is an order-embedding and L is a complete lattice, then L is called a *completion of P via η*.

Let us define $\eta : x \in (P, \leqslant) \mapsto \downarrow x \in (\mathcal{D}own(P), \subseteq)$. This is a completion.

Definition 4.4.2. Let $A \subseteq P$ be a subset of a poset. Then we define

$$A^u = \{x \in P | (\forall a \in A) a \leqslant x\} = \{\text{upper bounds of } A\}$$

$$A^l = \{x \in P | (\forall a \in A) x \leqslant a\} = \{\text{lower bounds of } A\}.$$

The following lemma provides us with some basic properties about the operations u and l.

Lemma 4.4.3. Let P be a poset and $A, B \subseteq P$. Then:

1. $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$.
2. If $A \subseteq B$, then $A^u \supseteq B^u$ and $A^l \supseteq B^l$.
3. If $A \subseteq B$ then $A^{ul} \subseteq B^{ul}$ and $A^{lu} \subseteq B^{lu}$.
4. $A^u = A^{ulu}$ and $A^l = A^{lul}$.
5. A^u is an upset and A^l is a downset.
6. If P is a lattice, $A \subseteq P$, then A^u is a filter and A^l is an ideal.

Notice that $(_)^u$can be seen as a map $Down(P) \to Up(P)$. And $(\_)^l : Up(P) \to Down(P)$. Notice also that

$$\begin{aligned} A^u \supseteq B &\Leftrightarrow (\forall x \in A)(\forall y \in B) x \leqslant y \\ &\Leftrightarrow A \subseteq B^l \end{aligned}$$

These two operators form what is called a *Galois connection*[3].

Definition 4.4.4. Let P be a poset. Define

$$DM(P) := \{A \subseteq P : A^{ul} = A\}.$$

Then $(DM(P), \subseteq)$ is called the *Dedekind-MacNeille completion of P*.

Lemma 4.4.5. Let P be a poset.

1. For all $x \in P$, $(\downarrow x)^{ul} = \downarrow x$, hence $\downarrow x \in DM(P)$.
2. If $A \subseteq P$ and $\bigvee A$ exists in P, then $A^{ul} = \downarrow (\bigvee A)$.

Proof. (1). $z \in (\downarrow x)^u$ iff $x \leqslant z$. And hence, $y \in (\downarrow x)^{ul}$ iff $y \leqslant z$ for all $z \geqslant x$ iff $y \leqslant x$.

(2). If $x \in A^{ul}$, this means x is smaller or equal than all upper bounds of A, in particular $x \leqslant \bigvee A$. Conversely, if $x \leqslant \bigvee A$, then for every upper bound y of A, $x \leqslant \bigvee A \leqslant y$, so $x \in \{z \in P | z \leqslant y (\forall y \in A^u)\} = A^{ul}$. ■

Theorem 4.4.1. *Let P be a poset. Define $\eta : P \to DM(P)$ by $\eta(x) = \downarrow x$. Then*

[3] The reader familiar with category theory can recognize a Galois connection as nothing more than an adjunction between the posets, understood as categories.

1. *$DM(P)$ is a completion of P via η.*

2. *η preserves joins and meets that exist in P, i.e. $\eta(\bigvee A) = \bigvee_{DM(P)} \eta[A]$ and $\eta(\bigwedge A) = \bigwedge_{DM(P)} \eta[A]$ whenever they exists.*

Proof. (1). *$DM(P)$ is a complete lattice.* Let $\{A_i\}_{i\in I} \subseteq DM(P)$. Define

$$\bigvee_{i\in I} A_i := \left(\bigcup_{i\in I} A_i\right)^{ul} \text{ and } \bigwedge_{i\in I} A_i := \bigcap_{i\in I} A_i.$$

Now, $A_i \subseteq \bigcup_{i\in I} A_i \subseteq (\bigcup_{i\in I} A_i)^{ul}$. Moreover, if $A_i \subseteq B$ for all $i \in I$, with $B \in DM(P)$ (i.e. $B^{ul} = B$) we have that $\bigcup_{i\in I} A_i \subseteq B$, hence $(\bigcup_{i\in I} A_i)^{ul} \subseteq B^{ul} = B$. So $(\bigcup_{i\in I} A_i)^{ul}$ is the least upper bound. This is enough, but we can also easily show that the meet is the intersection.

(2). Suppose $\bigvee A$ exists in P. Let us see that $\bigvee \eta[A] = \bigvee\{\downarrow a | a \in A\} = (\bigcup_{a\in A} \downarrow a)^{ul}$ is equal to $\downarrow \bigvee A$.
If $a' \leqslant \bigvee A$, take b to be an upper bound of $\bigcup_{a\in A} \downarrow a$. In particular, $b \geqslant a$ for all $a \in A$, hence $b \geqslant \bigvee A \geqslant a'$. Hence a' is a lower bound of the set of upper bounds of $\bigcup_{a\in A} \downarrow a$, or in other words $a' \in (\bigcup_{a\in A} \downarrow a)^{ul}$.
Conversely, if $a' \in (\bigcup_{a\in A} \downarrow a)^{ul}$, then, since $\bigvee A$ is an upper bound of A and hence an upper bound of the set $\bigcup_{a\in A} \downarrow a$, we have that, necessarily, $a' \leqslant \bigvee A$.
Similar for $\bigwedge A$. ∎

4.4.1 ◊ MacNeille completion of a Boolean algebra

MacNeille completions provide a very convenient way to complete a poset, but it is not at all clear that the resulting complete lattice will preserve any more of the structure. For Boolean algebras, their MacNeille completion turns out to again be a Boolean algebra. To see this, note that if B is a Boolean algebra, and $A \subseteq B$, then A^u is in fact a filter, and A^l is in fact an ideal. So the fixed points are fixed points of the pair of maps

$$(-)^u : \mathsf{Fil}(B) \to \mathsf{Id}(B) : (-)^l.$$

Now we note the following, recalling the isomorphism $\xi : \mathsf{Id}(B) \to \mathsf{Open}(X_B)$:

Lemma 4.4.6. For each $A \subseteq B$ a Boolean algebra, $A^{ul} = A$ if and only if $\xi(A)$ is regular open (i.e., $\xi(A) = int(cl(\xi(A))))$.

Proof. Assume that $A^{ul} = A$. Unfolding what this means, we have that $\xi(A^{ul}) = \bigcup\{\varphi(c) : \varphi(c) \subseteq \bigcap\{\varphi(a) : \xi(A) \subseteq \varphi(a)\}\}$. Note that certainly $\xi(A) \subseteq int(cl(\xi(A)))$. So suppose that $x \in int(cl(\xi(A)))$; this means that there is some $\varphi(c) \subseteq cl(\xi(A))$ such that $x \in \varphi(c)$. Now note that if $\xi(A) \subseteq \varphi(a)$, then $cl(\xi(A)) \subseteq \varphi(a)$ since the latter is clopen. Hence $\varphi(c) \subseteq \varphi(a)$. Thus $\varphi(c) \subseteq \xi(A)$ as desired. The converse is similar. ∎

Corollary 4.4.7. *Given a Boolean algebra B, the complete lattice $\mathcal{RO}(X_B)$ is the Dedekind-MacNeille completion of B.*

The harder fact to establish is that in fact such a lattice is a Boolean algebra: we equip it with the structure $(\mathcal{RO}(X), \cap, \vee, \neg, \varnothing, X)$ where $U \vee V := int(cl(U \cup V))$ and $\neg U = int(X - U)$.

Proposition 4.4.8. Given X a Stone space, $(\mathcal{RO}(X), \cap, \vee, \neg, \varnothing, X)$ is a complete Boolean algebra.

Proof. We know that it is a bounded complete lattice. Moreover, note that $\neg$ works as a complement: certainly $U \cap \neg U = \varnothing$, and $U \vee \neg U = int(cl(U \cup int(X - U)) = int(cl(U) \cup cl(int(X - U))) \supseteq int(cl(U) \cup X - U) \supseteq X$. So the key argument lies in distributivity.

Assume that U, V, W are regular open. We need to show that:

$$U \cap (V \vee W) \subseteq (U \cap V) \vee (U \cap W).$$

For that we will first show that for any open sets U, V (not necessarily regular open):

$$int(cl(U)) \cap int(cl(V)) = int(cl(U \cap V))$$

One inclusion is clear, since $int(cl(-))$ is a monotone operator. For the other, first note that $U \cap cl(V) \subseteq cl(U \cap V)$ by usual topological reasoning. Thus applying the interior to both sides, and distributing over the intersection $U \cap int(cl(V)) \subseteq int(cl(U \cap V))$. Similarly we obtain $V \cap int(cl(U)) \subseteq int(cl(U \cap V))$, from which the equality easily follows.

Now to obtain the final result notice that:

$$\begin{aligned} U \cap (V \vee W) &= int(cl(U)) \cap int(cl(V \cup W)) \\ &= int(cl(U \cap (V \cup W)) \\ &= int(cl((U \cap V) \cup (U \cap W)) \\ &= (U \cap V) \vee (U \cap W). \end{aligned}$$

∎

Corollary 4.4.9. *Given a Boolean algebra B, the inclusion of* $\mathsf{Clop}(X_B)$ *into* $\mathcal{RO}(X_B)$ *is a Boolean homomorphism which preserves all existing joins and meets.*

4.4.2 ◊ Completeness of First-Order Logic

Completions have several purely theoretical uses in algebraic logic – they serve for instance to characterize the injective Boolean algebras. But they also have logical uses:

Definition 4.4.10. Let B be a complete Boolean algebra. Let Σ be a first-order language with no function symbols and no equality[4], and $\mathsf{FOForm}(\Sigma)$ the set of first-order logic formulas over Σ, and let X be a set. A map $p : \mathsf{Var} \to X$ is called a *variable* assignment.

For each n-ary predicate symbol $R \in \Sigma$, let $p_R : X^n \to B$ be a map. We call $\mathfrak{X} = (X, \{p_R : R \in \Sigma\})$ a Σ-B-*structure.* is called a B-model.

Given a pair A pair $(\mathfrak{X}, v)$ of a Σ-B-structure and a map $v : \mathsf{Var} \to X$, we interpret formulas $\varphi \in \mathsf{FOForm}(\Sigma)$ in B as follows:

$$\begin{aligned} [\![R(x_1, ..., x_n)]\!](v) &= p_R(v(x_1), ..., v(x_n)); \\ [\![\varphi \wedge \psi]\!](v) &= [\![\varphi]\!] \wedge [\![\psi]\!] \\ [\![\varphi \vee \psi]\!](v) &= [\![\varphi]\!] \vee [\![\psi]\!] \\ [\![\neg\varphi]\!](v) &= \neg[\![\varphi]\!] \\ [\![\exists x \varphi(x)]\!] &= \bigvee_{m:\mathsf{Var} \to X} [\![\varphi(x)]\!](m). \end{aligned}$$

[4] This can be adapted to include equality, but we have chosen this presentation for ease of understanding.

We call this the Boolean semantics of first-order logic. Given a complete Boolean algebra B, and φ a closed formula of first-order logic, we write $B \vDash \varphi$ if in every B-structure $(\mathfrak{X}, v)$ we have that $[\![\varphi]\!](v) = 1$. Let $\mathsf{FOL}(\Sigma)$ be the set of first-order formulas in this language given by the usual Hilbert-style axiomatization. Then we can show the following:

Theorem 4.4.2. *For each relational vocabulary Σ, first-order logic without equality, and every $\varphi \in \mathsf{FOForm}$:*

$$\mathsf{FOL}(\Sigma) \vdash \varphi \iff \forall B \text{ a complete Boolean algebra }, B \vDash \varphi.$$

Proof. The soundness follows easily by usual arguments. For completeness, assume that $\mathsf{FOL}(\Sigma) \nvdash \varphi$. Consider the Lindenbaum-Tarski algebra of first-order logic: we consider FOForm, and define the relation $\approx$ saying that two formulas $\varphi, \approx$ are equivalent if and only if $\mathsf{FOL}(\Sigma) \vdash \varphi \leftrightarrow \psi$. Then $\mathsf{FOForm}/\approx$ is a Boolean algebra $B(FOL)$. This Boolean algebra need not be complete, but note that:

$$[\exists x \varphi(x)] = \bigvee_{t \in \mathsf{Var}} [\varphi(t)] \text{ and } [\forall x \varphi(x)] = \bigwedge_{t \in \mathsf{Var}} [\varphi(t)].$$

Hence consider $DM(B(FOL))$. By Corollary 4.4.9 this is a complete Boolean algebra, and the inclusion $i : B(FOL) \to DM(B(FOL))$ preserves all existing meets and joins.

Let $X = \mathsf{Var}$ be an infinite countable set of the same size as the set of variables. Then pick for each predicate R, let:

$$p_R(x_1, ..., x_n) = i([R(x_1, ..., x_n)]).$$

Then considering the identity valuation id, we obtain a model $\mathfrak{X}$. In this model over $DM(B(FOL))$ we have in particular that:

$$DM(B(FOL)) \nvDash \varphi.$$

This shows completeness as desired. ∎

Stronger results may be obtained in several directions, but we leave that for further developments.

4.4.3 ◊ MacNeille closure for classes of algebras

The results of the previous section allow us to obtain first-order completeness for classes of algebras which are closed under MacNeille completions. Hence this presents an interesting question. However, we should not expect every natural class of algebras to be closed under MacNeille completions: the following example, due to Funayama, shows that MacNeille completions of distributive lattices are not necessarily distributive. The proof given will not be self-contained, making use of some heavy machinery from the theory of completions; nevertheless, we believe the dual argument makes this worth it.

Example 4.4.11. Let X be a Priestley space. Within this space, namely, between its lattice of open and closed sets, we can consider operators $\mathbf{D}(-) = \uparrow cl(-)$ and $\mathbf{J}(-) = \Box int(-)$; a subset $S \subseteq X$ is called a *regular open fixed point* if and only if $S = \mathbf{JD}(S)$. Note that such a set will always be open. We denote by $RO^*(X)$ the set of regular open fixed points, which forms a complete lattice.

Given a distributive lattice D, a result by Bezhanishvili and Harding shows that $DM(D) \cong RO^*(X)$. We now consider the following Priestley space outlined in Figure 4.1, which we call the *Funayama space*:

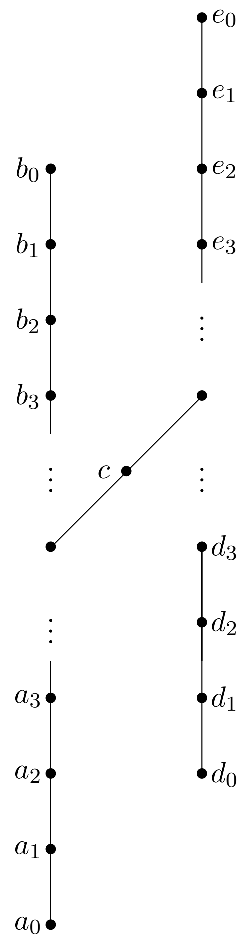


Figure 4.1: Funayama space

One can see this space in the following way: consider the Priestley space $\omega + 1 + \omega$ with the order topology, and the one element Priestley space $\mathbf{1}$; and form the disjoint union

$$X = (\omega + 1 + \omega) \sqcup \mathbf{1} \sqcup (\omega + 1 + \omega).$$

Then X^* is the Priestley surjective image corresponding to adding the two arrows, whilst keeping the topology intact. It is not hard to see that this space cannot be an Esakia space: $\{c\}$ will be clopen, but $\downarrow c$ is not clopen (since it does not contain any of the b_i elements).

For ease of notation, we make the following abbreviations:

$$A = \{a_i : i \in \omega\}$$
$$B = \{b_i : i \in \omega\}$$
$$D = \{d_i : i \in \omega\}$$
$$E = \{e_i : i \in \omega\}.$$

With this in place we can prove that the Dedekind-MacNeille completion of $\mathsf{ClopUp}(X^*)$ is not distributive (in fact, not even modular). For that, by Birkhoff/Dedekind's theorem, it suffices to exhibit a pentagon sublattice. So consider the following subsets:

1. $S_0 = B \cup E$;
2. $S_1 = \uparrow d_0 \cup \uparrow c \cup \{b_0\}$;
3. $S_2 = \uparrow d_0 \cup \{b_0\}$.

Note that S_i are all open upsets of X^*. Moreover, a simple calculation shows that they are indeed regular open fixed points. Note that $S_1 \subseteq S_2$, and that $S_1 \cap S_0 = S_2 \cap S_0$. Similarly, $\mathbf{JD}(S_1 \cup S_0) = \mathbf{JD}(S_2 \cup S_0)$. Hence we consider $S_0, S_1, S_2, S_0 \cap S_1, \mathbf{JD}(S_0 \cup S_1)$, which forms a pentagon sublattice of $\mathcal{RO}(X^*)$. We have thus shown the following theorem of Funayama:

Theorem 4.4.3. *The sublattice $\{S_0, S_1, S_2, S_3, S_4\}$ is a pentagon sublattice of $\mathcal{RO}(X^*)$. As such, the latter is not a distributive (or modular) lattice.*

What can we expect in the setting of Heyting algebras? Somehow, we have the following, which we mention without proof:

Theorem 4.4.4. *If H is a Heyting algebra, then $DM(H)$ is a Heyting algebra.*

The perplexity comes from a result by Bezhanishvili and Harding:

Theorem 4.4.5. *The only varieties of Heyting algebras closed under MacNeille completions are* **HA** *and* **BA**.

In the setting of modal algebras quite a few more varieties are closed under MacNeille, but again very natural examples fail: whilst **S4** and **S5** are closed, **Grz** for instance is not. These results by Bezhanishvili and Harding give us limiting cases. Some general results by Venema, independently as well as in collaboration with Gehrke, give conditions on when varieties of tense algebras can be closed under MacNeille, which happens substantially more often. Nevertheless, as far as we are aware the following is open: what are the varieties of *bi-Heyting algebras* which are closed under MacNeille completions?

4.5 $\lozenge$ Interpolation in intuitionistic and modal logics

Interpolation is a property often learned about in an introductory course in logic, though its relevance is often brushed aside. Its role in computation is also sometimes mentioned, though it is not clear what this property means. Using our tools we will see that interpolation has a clear mathematical meaning when one works with algebraic models. We will prove the interpolation of IPC, and some closely related systems, and then move on to discuss some more general issues regarding interpolation.

Definition 4.5.1. A logic $L \in \Lambda(\mathsf{IPC})$ is said to have the *Craig interpolation property* if whenever $\varphi(\overline{x}, \overline{y}), \psi(\overline{y}, \overline{z})$ are two formulas and $\varphi \to \psi \in L$ then there is a formula $\chi(\overline{y})$ such that $\varphi \to \chi \in L$ and $\chi \to \psi \in L$.

Example 4.5.2. Classical logic **CPC** is well-known to have the Craig Interpolation property.

4.5.1 ◊ Amalgamation and Interpolation

Definition 4.5.3. Given an ordered tuple $(H_0, H_1, H_2, i_1, i_2)$ of three Heyting algebras, where $i_1 : H_0 \to H_1$ and $i_2 : H_0 \to H_2$ are embeddings, we say that (H_3, j_1, j_2) where $j_1 : H_1 \to H_3$ and $j_2 : H_2 \to H_3$ is called an *amalgam* of $(H_0, H_1, H_2, i_1, i_2)$ if:

1. j_1, j_2 are embeddings;

2. $j_1 i_1 = j_2 i_2$.

See Figure 4.2. We say that an amalgam is a *superamalgam* if additionally for each $a \in H_1$ and $b \in H_2$ if $j_1(a) \leqslant j_2(b)$ then there is some $c \in H_0$ such that $a \leqslant i_1(c)$ and $i_2(c) \leqslant b$.

$$\begin{tikzcd}
& A_3 & \\
H_1 \arrow[ur, dashed, hook] & & H_2 \arrow[ul, dashed, hook'] \\
& H_0 \arrow[ul, hook', "i_1"] \arrow[ur, hook, "i_2"'] &
\end{tikzcd}$$

Figure 4.2

Definition 4.5.4. Given a class $\mathbf{K}$ of Heyting algebras we say that $\mathbf{K}$ has the *amalgamation property* if each tuple $(H_0, H_1, H_2, i_1, i_2)$ of $\mathbf{K}$-algebras has an amalgam $H_3 \in \mathbf{K}$. We say that it has the *superamalgamation property* if each tuple has a superamalgam in the class of algebras.

The relevance of these notions is captured by Maksimova's theorem, connecting them. We will prove this theorem, for which we will need the following lemma:

Lemma 4.5.5. Let $H_0 \leqslant H_1, H_2$ be a subalgebra. Suppose that $a \in H_1$ and $b \in H_2$ but there exists no $c \in H_0$ such that both $a \leqslant c$ and $c \leqslant b$. Then there exist prime filters $F_1 \subseteq H_1$ and $F_2 \subseteq H_2$ such that $a \in F_1$, $b \notin F_2$ and $F_1 \cap H_0 = F_2 \cap H_0$.

Proof. Consider the sets $A_1 = \{c \in H_0 : a \leqslant_1 c\}$ and $A_2 = \{c \in H_0 : c \leqslant_2 b\}$. Then $\mathsf{Fil}(A_1) \cap \mathsf{Id}(A_2 \cup \{b\}) = \varnothing$, by our assumption. By the Prime filter theorem, extend the former to a prime filter F_2 disjoint from the ideal. Then let $F_0 = F_2 \cap H_0$. Let $A_0 = (A_2 - F_2) \cap H_0$.

Now in turn consider $\mathsf{Fil}(F_0 \cup \{a\})$ and $\mathsf{Id}(A_0)$ in H_1. We claim that they are disjoint. Indeed if not, then for some $c \in F_0$ and $d \in A_0$, $a \wedge c \leqslant d$. Then $a \leqslant_1 c \to d$, so by construction, $c \to d \in A_1 \subseteq F_0$. Since $c \in F_0$, then $d \in F_0$, which is a contradiction. So by the Prime filter theorem, we find $F_1 \supseteq F_0 \cup \{a\}$ a prime filter disjoint from the ideal. Note that:

$$F_1 \cap H_0 = F_0;$$

indeed one inclusion is clear. For the other if $x \notin F_0$, then $x \in A_0$, so $x \notin F_1$. We thus have that $F_1 \cap H_0 = F_2 \cap H_0$ as desired. ■

Theorem 4.5.1 (Maksimovas characterization of interpolation theorem)**.** *The following are equivalent for $L \in \Lambda(\mathbf{IPC})$:*

1. *L has the interpolation property;*

2. *$\mathbf{V}_L$ has the superamalgamation property;*

3. *$\mathbf{V}_L$ has the amalgamation property.*

4. *For each triple $(H_0, H_1, H_2, i_1, i_2)$ of finitely subdirectly irreducible algebras, they have an amalgam.*

Proof. Obviously (2) implies (3), and (3) implies (1). We first show that (4) implies (1): suppose that $\overline{x}, \overline{y}, \overline{z}$ are pairwise disjoint sets of variables, and $L \vdash \varphi(\overline{x}, \overline{y}) \to \psi(\overline{y}, \overline{z})$, but there is no interpolant. Consider the three free algebras $F_0 = F_{HA}(\overline{x})$, $F_1 = F_{HA}(\overline{x}, \overline{y})$ $F_2 = F_{HA}(\overline{y}, \overline{z})$. Dually, consider

$$X_0, X_1, X_2$$

the dual spaces of these algebras; let $p_1 : X_1 \to X_0$ be the dual map to the inclusion, and similarly, $p_2 : X_2 \to X_0$. Since $\varphi \to \psi$ has no interpolant, by Lemma 4.5.5 we have that there are prime filters $x_1 \in X_1$ and $x_2 \in X_2$ such that their intersection with $F(\overline{x})$ is the same, i.e., $p_1(x_1) = p_2(x_2) = x_0$. Consider the algebras $U_i = \mathsf{ClopUp}(\uparrow x_i)$; these are quotients of F_i, and they are finitely subdirectly irreducible algebras, since they have a root. Elements of U_i are thus denoted as $[\varphi]_{x_i}$. Moreover (U_0, U_1, U_2), together with the inclusions i_1, i_2 forms an amalgam, and hence there is by hypothesis an algebra D, and inclusions $j_1 : U_1 \to D$ and $j_2 : U_2 \to D$ witnessing amalgamation.

Now let $v(x) = j_1([x]_{x_1})$, $v(y) = j_1([y]_{x_1})$ and $v(z) = j_2([z]_{x_2})$. Note that $v(x)$ is also equal to $j_1 i_1([x]_{x_0}) = j_2 i_2([x]_{x_0}) = j_2([x]_{x_2})$. With this valuation, $v(\varphi) = j_1([\varphi]_{x_1}) = 1$ since $\varphi(\overline{x}, \overline{y}) \in x_1$, and $v(\psi) = j_2([\psi]_{x_2} \neq 1$ since j_2 is injective and $\psi(\overline{y}, \overline{z}) \notin x_2$. But then $D \nvDash \varphi \to \psi$, which is absurd since we assumed this was valid on this class of algebras. We thus conclude the interpolation theorem.

Finally we show that (1) implies (2). Assume that $(H_0, H_1, H_2, p_1, p_2)$ is an amalgam, and without loss of generality, assume that $H_0 = H_1 \cap H_2$. Let $|X_i| = |H_i|$ and let $F_i = F_{HA}(X_i)$. Using the above bijections, we can lift them to surjective homomorphisms $\beta_i : F_i \to H_i$ obtaining the situation in the diagram of Figure 4.3:

$$\begin{tikzcd}
 & H_2 \arrow[rr, dashed] & & F_3/G \\
F_2 \arrow[ur, "\beta_2"] \arrow[rr, "i_{2,3}"] & & F_3 \arrow[ur] & \\
 & H_0 \arrow[uu, "p_1"] \arrow[rr, "p_2"] & & H_1 \arrow[uu, dashed] \\
F_0 \arrow[uu, "i_{0,2}"] \arrow[ur, "\beta_0"] \arrow[rr, "i_{0,1}"] & & F_1 \arrow[uu, "i_{1,3}"] \arrow[ur, "\beta_1"] &
\end{tikzcd}$$

Figure 4.3

The algebra $F_3 = F_{HA}(H_1 \cup H_2)$, and there are obvious inclusions from F_1, F_2 into it. Now let $G_i = \mathsf{Ker}(\beta_i)$ be the filter associated to this quotient. Note that if $\chi \in F_0$, and $\chi \in G_1$, then $\chi \in G_2$ by the definitions. Then we can generate a filter G on F_3 with $i_{1,3}[G_1] \cup i_{2,3}[G_2]$. Then note that the map $q_1 : H_1 \to F_3/G$, given by $q_i(x) = [x]_G : H_i \to F/\Theta$ will be a well-defined homomorphism.

Now we will prove that this enjoys the following property: for each each formula $\varphi \in F_1$ and $\psi \in F_2$, $\varphi \to \psi \in G$ if and only if there is $\chi \in F_0$ such that $\varphi \to \chi \in G_1$ and $\chi \to \psi \in F_2$. The right to left direction is easy. For the left to right direction, note that if $\varphi \to \psi \in G$, then by construction, there are $\mu_1 \in F_1$ and $\mu_2 \in F_2$ such that $\mu_1 \wedge \mu_2 \leqslant \varphi \to \psi$. This means that $\chi_1 = \mu_1 \wedge \varphi \leqslant \mu_2 \to \psi = \chi_2$. Applying interpolation, we have that for some formula $\chi \in F_0$, $\chi_1 \to \chi$, and $\chi \to \chi_2$. Hence $\mu_1 \leqslant \varphi \to \chi$, which means that $\varphi \to \chi \in G_1$; and $\chi \leqslant \mu_2 \to \psi$ implies that $\mu_2 \leqslant \chi \to \psi$, meaning the latter is in G_2. Note that this completely symmetric between F_1 and F_2.

Now to see that this indeed enjoys the right properties, first note that $q_1(x) = q_2(y)$ if and only if $x \leftrightarrow y \in G$; that means that the formula $\top \to (x \leftrightarrow y) \in G$, so there is some formula $\chi \in F_0$ such that $\top \to \chi \in G_2$, and $\chi \to (x \leftrightarrow y) \in G_1$. Since $\chi \in G_2$, and it belongs to the common language, $\chi \in G_1$, so $x \leftrightarrow y \in G_1$. This shows injectivity of q_i. Now assume that $q_1(\varphi) \leqslant q_2(\psi)$. The above then implies the existence of $\chi \in F_0$ such that $q_1(\varphi) \leqslant q_1 p_1(\chi)$ and $q_2 \varphi_2(\chi) \leqslant q_2(\psi)$. Hence this a superamalgam. ■

Having Maksimova's characterization of interpolation at hand, we can now easily prove interpolation for **IPC** using duality.

Theorem 4.5.2. *The variety* **HA** *has the amalgamation property. Hence* **IPC** *has the interpolation property.*

Proof. Let $(H_0, H_1, H_2, i_1, i_2)$ be a tuple. Dualizing, this gives us a tuple $(X_0, X_1, X_2, p_1, p_2)$ where p_1 and p_2 are p-morphisms. We need to find an Esakia space Z with two surjective p-morphisms $q_1 : Z \to X_1$ and $q_2 : Z \to X_2$ such that $p_1 q_1 = p_2 q_2$.

For that purpose, let $Z = \{(x, y) \in X_1 \times X_2 : p_1(x) = p_2(y)\}$. As a poset, Z satisfies the desired conditions: to see surjectivity, note that for each $x \in X_1$, by surjectivity of p_2, there is some $y \in X_2$ such that $p_1(x) = p_2(y)$, so $(x, y) \in Z$ will project onto x. Now consider $\mathsf{Up}(Z)$. Then this algebra will be an amalgamation of the algebras as desired. ■

Having this proof in mind, we might ask which other logical systems we have seen have interpolation. One fact which follows essentially by the same proof, and the specific structure of duals of **KC** algebras, is the following:

Proposition 4.5.6. The logic **KC** has the interpolation property.

Proof. Exercise. ■

For the system **LC** a bit more care is needed but a key trick takes care of the problem: given X_1, X_2 two duals of Gödel algebras, take $Z \subseteq X_1 \times X_2$ as above. Now let $C(Z)$ be the poset of chains with a smallest element over Z, and let $r : C(Z) \to Z$ be the map which takes the root of the chain. Then this will be surjective, $C(Z)$ will be a prelinearly ordered poset (i.e., $x \leqslant y, z$ then $y \leqslant z$ or $z \leqslant y$) and taking $C(Z)_*$ again gives us the result

Proposition 4.5.7. The logic **LC** has the interpolation property.

Of course **CPC** also has interpolation. The reader might find it an interesting challenge to show that

$$\mathbf{BW}_2 := \mathbf{IPC} \oplus \mathbf{bw}_2,$$

has interpolation as well.

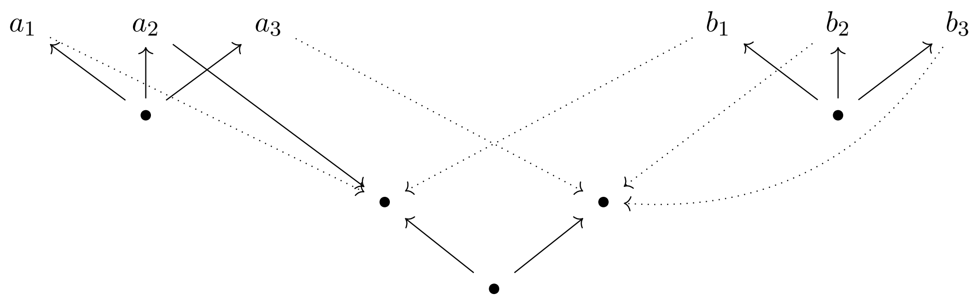


Figure 4.4

Then, consider the logics $\mathbb{V}(\mathbf{3})$, and $\mathbb{V}(D_2)$ (see Example 3.2.18). Both of these also have interpolation, a fact which can likewise be shown by some additional arguments (see Exercise 4.3).

What other intuitionistic systems will there be?

4.5.2 $\lozenge$ Maksimova's Characterization

A famous result by Larissa Maksimova, which made her name in the 70s, says the following:

Theorem 4.5.3. *The logics* $\mathbf{IPC}, \mathbf{KC}, \mathbf{LC}, \mathbf{BW}_2, Log(\mathbf{3}), Log(D_2), \mathbf{CPC}$ *are exactly the superintuitionistic logics with the interpolation property.*

The most common reaction to this theorem is bewilderment. In its even more bewildering form it is stated as "there are exactly 7 superintuitionistic propositional logics with the interpolation property". We will not even begin to hint at the proof, but will say a few words on it.

The logics under consideration split into two big groups: the logics **IPC** and **KC** on one side are logics with a large class of models – any poset, respectively, any poset with a final element, will be a model; the other logics have very strong restrictions on the shape of their models. Hence this tells us that interpolation as far as intuitionistic logic goes is extremal: either a logic L is so small, that it does not have almost any sequents $\varphi \to \psi$ to interpolate; or if it has some sequents, it needs to have a lot of them, which forces it to be an extremelly large logic.

Now let us narrow in on the large logics. We have $\mathbf{BA}, \mathbb{V}(\mathbf{3})$ and $\mathbb{V}(D_2)$ on one side, and **LC** and $\mathbf{BW}_2$ on the other. Classical logic can be put aside: the other two form pairs, the logic of the 3-chain being contained in **LC** and the logic of the 2-fork being contained in $\mathbf{BW}_2$. Here again we have a similar thing happening: the lattice of logics of $\mathbf{BW}_2$ is a chain, with the logic of the 2-fork on top, so interpolation appears extremally.

The key insight of Maksimova is that we can indeed observe this extremality in terms of amalgams one tries to put together. Indeed, consider for instance the following amalgam 4.4:

A preliminary result of Maksimova shows that if one can amalgamate subdirect irreducibles, amalgamation follows. Hence in $\mathbb{V}(D_3)$, to try to amalgamate, the poset on top would need to be the dual of D_3. But any attempt to do so will lead to trouble. This means that in $\mathbf{BW}_2$ one needs to proceed to $\mathbb{V}(D_4)$, where the argument can be repeated inductively. Proceeding in this way it turns out the only logic with sufficiently many models to make the interpolation work is $\mathbf{BW}_2$. Similar reasoning goes for **LC**.

The idea of Maksimova's proof then extends this core idea: if one has both width greater than 2, and depth greater than 2, then one can inductively generate almost all models, with the only caveat being whether there can be forking or not. This is how one obtains the final logics.

Maksimova was not content with this characterization, and she has countless results exhaustively classifying all the logics of many other systems with interpolation. The situation for **S4** is further complicated because there amalgamation and superamalgamation are no longer equivalent. Properties such as Lyndon interpolation, and uniform interpolation, are also worthy of consideration, and lead to many interesting directions of study.

4.6 $\Diamond$ Admissibility of Rules and Unification

In proof theory it is common and usual to talk about *derivation rules*. These are often denoted by $\Gamma \vdash \varphi$ for Γ a set of formulas and φ a given formula. In this setting one often has in mind a proof system, and talks about *derivable rules* as well as *admissible rules*. These are important concepts in logic, which happen to make a lot of sense from an algebraic point of view. For that we will need some basic definitions and results which extend Birkhoff's theorem.

Definition 4.6.1. Let $\mathbf{K}$ be a class of similar algebras. We say that $\mathbf{K}$ is a *quasivariety* if it is closed under isomorphisms, subalgebras, products and ultraproducts. Let $\mathbb{Q}(\mathbf{K})$ be the smallest quasivariety containing $\mathbf{K}$.

Definition 4.6.2. Let $\mathcal{L}$ be a language of algebras. A sequence $(\lambda_1 \approx \gamma_1, ..., \lambda_n \approx \gamma_n, \lambda \approx \gamma)$ is called a *quasi-equation*. These are often written:

$$\lambda_1 \approx \gamma_1 \wedge ... \wedge \lambda_n \approx \gamma_n \to \lambda \approx \gamma.$$

For simplicity, this is often written as $\Gamma \vdash \lambda \approx \gamma$ where Γ is a finite set of equations. An algebra $\mathcal{A}$ validates a quasi-equation if it models

$$\forall \overline{x}(\bigwedge \Gamma \to \lambda \approx \gamma).$$

We say that a class $\mathbf{K}$ is a *quasiequational class* if it is the class of algebras satisfying a quasiequation.

Theorem 4.6.1. *Let* $\mathbf{K}$ *be a class of algebras. Then:*

1. $\mathbb{Q}(\mathbf{K}) = \mathbb{ISPP}_U(\mathbf{K})$.
2. *If* $\mathbf{K}$ *is a quasivariety, there is a dual isomorphism between the lattice of subquasivarieties of* $\mathbf{K}$ *and the lattice of quasiequational theories of algebras of* $\mathbf{K}$.

Having the above in place, given a logical language (such as that of **IPC** or **K**), a rule $\Gamma \vdash \varphi$, where $\Gamma = \{\psi_1, ..., \psi_n\}$, will simply be a quasi-equation

$$\psi_1 \approx 1 \wedge ... \wedge \psi_n \approx 1 \to \varphi \approx 1.$$

Definition 4.6.3. Let $\mathbf{K}$ be a variety of algebras. We say that a rule $\Gamma \vdash \varphi$ is *derivable* if every algebra $\mathcal{A} \in \mathbf{K}$ validates the rule. We say that the rule $\Gamma \vdash \varphi$ is *admissible* if for each substitution σ, if $\mathbf{K} \models \sigma(\Gamma)$, then $\mathbf{K} \models \sigma(\varphi)$.

Example 4.6.4. Consider the rule $\varphi \vdash \Box\varphi$. It is clear that this rule is admissible in the class of all modal algebras. However, it is certainly not derivable, since the only algebras which validate it are the duals of discrete modal spaces.

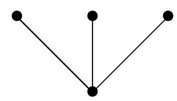


Figure 4.5

Example 4.6.5. Consider the following rule, written in deductive rule form, called *Harrop's rule*:

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C).}$$

This rule is admissible over **IPC**. To see why, assume that it is valid that $\neg A \to (B \vee C)$. Suppose that $\nVdash (\neg A \to B) \vee (\neg A \to C)$. Then by Kripke completeness, let $\mathfrak{M}_1, x \nVdash \neg A \to B$ and $\mathfrak{M}_2, y \nVdash \neg A \to C$, both rooted models. Let T be the model obtained by taking the disjoint union of $M_1 \sqcup M_2 \sqcup \{\bullet\}$ where $\bullet$ is below x and y, and copying the same valuations. Then $\bullet \Vdash \neg A$, because this is a negative formula, so by assumption, either $\bullet \Vdash B$ or $\bullet \Vdash C$ – which is a contradiction.

On the other hand, this formula is not derivable: for instance, the 3-fork in Figure 4.5, is not a model for the formula.

Admissible rules are intimately related to quasivarieties:

Theorem 4.6.2. *The following are equivalent for a variety* $\mathbf{K}$ *of Heyting or modal algebras:*

1. *The rule* $\Gamma \vdash \varphi$ *is admissible in* $\mathbf{K}$*;*
2. $\mathbf{K}$ *is generated by algebras where* $\Gamma \vdash \varphi$ *is derivable;*
3. *The rule* $\Gamma \vdash \varphi$ *is valid in* $\mathbf{F}_K(\omega)$*.*

Proof. Exercise. ■

Issues of admissibility are a rich field of analysis in algebraic logic, establishing connections to many interesting other areas of logic and computer science. We will mention two: issues of *unification* and decidability, on one hand, and issues of *structural completeness*, on te other.

4.6.1 ♢ Unification and Decidability of Admissibility

A basic question – sometimes called *Friedman's problem* for the special case of **IPC** – concerns the decidability question of admissibility: given a rule $\Gamma \vdash \varphi$, is this rule admissible over the given logical system?

The analysis of such problems tends to go through a related area of study:

Definition 4.6.6. Let $\mathbf{K}$ be a class of similar algebras, and $\lambda \approx \gamma$ a formula. A *unifier* for this equation is a substitution σ such that

$$\mathbf{K} \vDash \sigma(\lambda) \approx \sigma(\gamma).$$

As usual, for logical languages one considers only formulas φ, since unifying equations is the same as unifying the equivalence they represent. Now, given two unifiers σ, δ – which are homomorphisms of the free algebra – one can compare them: $\sigma \leqslant \delta$ if there is some homomorphism θ such that $\sigma = \theta \circ \delta$. In some circumstances, like classical logic, this situation reduces substantially, since there is always a *most general unifier*. In general though, this may not be the case.

Definition 4.6.7. Given a class $\mathbf{K}$ of algebras, we say that $\mathbf{K}$ has *finitary unification type* if for each formula φ, there are finitely many unifiers $\theta_0, ..., \theta_n$ such that each unifier σ of φ is such that $\sigma \leqslant \theta_i$ for some i; this is called a *basis of unifiers.* We say that $\mathbf{K}$ has *computable bases of unifiers* if for each formula φ, one can calculate a finite basis of unifiers.

Lemma 4.6.8. Suppose that L is a logic which is decidable, and such that $\mathbb{V}(L)$ has finitary unification type, and computable basis of unifiers. Then the admissibility problem for L is decidable as well.

Proof. One shows that, taking $\Gamma = \bigwedge \Gamma$ to be a single formula, and $\theta_1, ..., \theta_n$ a basis of unifiers for this formula:

$$\forall \sigma \text{ substitution}(\sigma[\Gamma] \in L \implies \sigma(\varphi) \in L) \iff \forall 1 \leqslant i \leqslant n(\theta_i[\Gamma] \in L \implies \theta_i(\varphi) \in L).$$

One direction is obvious. For the other, suppose that $\sigma(\Gamma) \in L$ but $\sigma(\varphi) \notin L$. Then by assumption, $\sigma \leqslant \theta_i$ for some i. Hence $\theta_i(\Gamma) \in L$ by assumption. But since $\sigma \leqslant \theta_i$, if $\theta_i(\varphi) \in L$, by substitution, we would have $\sigma(\varphi) \in L$ – a contradiction. ■

This problem reduces the decidability of admissibility to solving unification problems for super-intuitionistic as well as normal modal logics. But this is far from trivial in many cases. In the case of **IPC** this leads to a careful analysis of *projective algebras*, as well as several other sophisticated algebraic tools, and indeed we have:

Theorem 4.6.3. *The logic* **IPC** *has finitary unification type; hence its admissibility problem is decidable.*

Alternatively, from a proof theoretic point of view, it opens the way for calculi of admissible rules, a subject of contemporary research, in search of the answer to a very old open problem: the decidability of rules over the normal modal logic **K**.

4.6.2 ◊ Structural Completeness

Let us begin with a basic definition:

Definition 4.6.9. Let L be a logic. We say that L is *structurally complete* if whenever $\Gamma \vdash \varphi$ is admissible in $\mathbb{V}(L)$, then it is derivable. We say that L is *hereditarily structurally complete* if it and all its extensions are structurally complete.

Lemma 4.6.10. For L a logic the following are equivalent:

1. L is structurally complete;
2. $\mathbb{V}(L) = \mathbb{Q}(\mathbf{F}_L(\omega))$.

And the following are equivalent:

1. L is hereditarily structurally complete.
2. Whenever $\mathbf{K} \leqslant \mathbb{V}(L)$ is a subquasivariety, then $\mathbf{K}$ is already a variety.

Proof. Exercise. ■

We can see that **IPC** is not structurally complete. In fact many interesting logics fail to be structurally complete, but not all – **LC** is structurally complete (a fact which you may attempt to prove!). Structural completeness is in general too hard to characterize due to the possibility that an extension of a structurally complete logic may fail to be structurally complete. But hereditary structural completeness is somewhat easier. In fact, the following striking result exists:

Theorem 4.6.4. *A logic $L \in \Lambda(\mathbf{IPC})$is hereditarily structurally complete if and only if L omits the posets in Figure 4.6.*

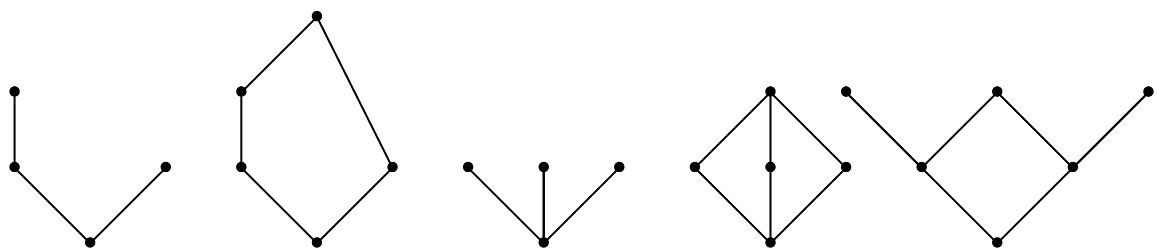


Figure 4.6: Citkin's Five Forbidden Posets

Similar characterizations exist for **S4** and **K4**, and have recently been extended to some logics such as **wK4** in MoL theses (see the work of James Carr and Simon Lemal). However, adjacent problems are quite abundant: can one obtain a similar characterization saying what the *degree* of hereditary structural completeness is (i.e., *how many* logics share the same underlying quasivariety)? What happens in settings such as tense logics, modal logics, or bi-intuitionistic logics?

End of extra content.

4.7 Exercises

4.7.1 The Method of Jankov Formulas

Exercise 4.1. Let $A \in \mathsf{HA}_{FSI}$. Show that for any algebra B we have

$$B \not\models \chi(A) \qquad \text{iff} \qquad A \leqslant B$$

(*Hint:* Use (and prove) Wronski's Lemma: Given a finite algebra B and an element $b \neq 1_B$, there exists a subdirectly irreducible algebra C and a surjective morphism $f : B \twoheadrightarrow C$ such that $f(b)$ is the second-greatest element of C.)

Exercise 4.2. In this exercise we will show McKenzie's theorem that every splitting of **HA** is of the form $\mathbf{HA} + \chi(H)$.

1. Show that $\mathbf{HA} = \bigvee_{H \in \mathbf{HA}_{fin.si}} \mathbb{V}(H)$.

2. Show that if $(\mathbf{V}_1, \mathbf{V}_2)$ is a splitting of $\Lambda(\mathbf{HA})$, then $\mathbf{V}_1$ is completely join-prime in this lattice.

3. Deduce that in a splitting $(\mathbf{V}_1, \mathbf{V}_2)$, $\mathbf{V}_1 \subseteq \mathbf{V}$ for $\mathbf{V}$ a finitely generated variety. Deduce that then $\mathbf{V}_1$ must also be finitely generated. *Hint: Jónnsons Lemma.*

4. Conclude that then $(\mathbf{V}_1, \mathbf{V}_2)$ is of the form $(\mathbb{V}(H), \mathbf{HA} + \chi(H))$.

Exercise 4.3. Let $\mathbf{V}$ be a variety of Heyting algebras.

1. Show that if $\mathbf{V}$ is locally finite, then $\mathbf{V}$ has amalgamation if and only if every tuple $(H_1, H_2, H_3, i_1, i_2)$ where i_1, i_2 are finite has an amalgam. *Hint: Use compactness*

2. Use the above to show that $\mathbb{V}(\mathbf{3})$ and $\mathbb{V}(D_2)$ have amalgamation.

Chapter 5

Open Problems

We have now given a comprehensive introduction to very basic theory of the field. Nevertheless, the reader who has made it this far will already have a sufficient appreciation of the tools at hand to understand some of the key open problems guiding research in the algebraic logic of non-classical logics done via duality theory, and with a structural and often categorical emphasis. We list these problems here, with brief discussion.

5.1 Long-standing open problems

1. **Representation of Priestley posets**: Posed by Gratzer we have the following problem: *which posets P admit a Priestley topology?* Some facts are known, but a full description seems far, and may be out of our reach.

2. **Medvedev's Logic**: Let $B_n - \{\top\}$ be the finite poset of a Boolean algebra with n atoms and no top element. Let $Med = \bigcap_{n\in\omega} Log(B_n)$. This is called *Medvedev's logic*. For decades it has not been known whether this logic is axiomatizable – it is known that it is not finitely axiomatizable. Indeed, if the logic was recursively axiomatizable, it would be decidable, which is how this problem is often presented. Moreover, this logic has several other peculiar properties which seem unique to it, even though proofs of that uniqueness have been elusive.

3. **Degrees of Completeness of IPC**: Given a superintuitionistic or modal logic L, let $scop(L)$ be the set of all logics which have the same set of Kripke frames as L. The cardinality of this set is called the *degree of completeness* of L. Blok showed that for logics $L \in \Lambda(\mathbf{K})$, this degree is always 1 or $2^{\aleph_0}$. But no such result is known for superintuitionistic logics.

4. **Kuznetsovs Conjecture**: Is every variety topological, i.e., complete with respect to topological spaces? The overall direction of study has been to disprove this, using the so-called "Fine general frame". It is apparently not known whether this logic is topologically complete or not, and whether it can be used to provide a counterexample. Relatedly there is the issue of **completions**: the MacNeille completion introduced above is not very useful for Heyting algebras. Canonical extensions are also completions, and they are more prevalent in varieties of Heyting algebras. Nevertheless, there is no general theory outlining how one completes Heyting algebras within varieties, or why this should fail.

5. **Criterion for local finiteness**: as noted, local finiteness is quite complicated in **HA**. This contrasts with **S4**, where Maksimova proved a simple cirterion for local tabularity. Developing a criterion for **HA**, or more generally, studying the category of locally finite Heyting algebras, is as far as we know open.

6. **The Pitts-Pataraia Problem**: given a Heyting algebra H, does H arise as the lattice of truth-values of an elementary topos? The story of this problem is rather circuitous and complicated, but it is an active driving force of research.

7. **Morleys theorem of modal logic**: Is every canonical logic strongly canonical?

8. **Admissibility of rules in K**: Say that a rule in some logical system $\varphi \vdash \psi$ is *admissible* in L if whenever $\sigma(\varphi) \in L$ then $\sigma(\psi) \in L$. Say that it is *derivable* if $\varphi \to \psi \in L$. IPC has admissible rules which are not derivable, and this has been studied extensively. Namely, the problem of saying whether a rule is admissible or not is decidable. Describe this problem for **K** – say whether a rule being admissible in basic modal logic **K** is decidable or not.

5.2 Other Open Problems

1. **Characterizing the FMP**: Similar to local finiteness and Kripke completeness, there has been interest in understanding what makes logics have the FMP. This lead recently to the notion of *degree of the FMP* (analogous to degree of Kripke completeness for FMP), which was solved by Bezhanishvili, Bezhanishvili and Moraschini, showing that intuitionistic logics and extensions of **S4** can have any degree of the FMP amongst $\omega \cup \{\omega, 2^{\aleph_0}\}$[1]. But it is still not known how this degree function operates.

2. **Dualizing the center**: Let H be a Heyting algebra. Let $\mathsf{Cen}(H) = \{a : a \vee \neg a = 1\}$. This is a Boolean algebra. Can this operation be described dually? In the finite case this corresponds to taking the connected components of a poset. In the infinite case something more needs to be done.

3. **Dualizing ultraproducts**: Let $(H_i)_{i \in I}$ be a family of Heyting algebras. Can you give a dual description of ultraproducts of this family? Describe what happens when the family consists of finite algebras.

4. **Axiomatizing the logic of free algebra**: The Rieger-Nishimura lattice $\mathbf{F}_{HA}(1)$ has a logic which enjoys a lot of nice properties, and which crucially can be axiomatized by Jankov formulas. Is the logic $\mathbf{F}_{HA}(2)$ recursively axiomatizable? Can it be axiomatized by Jankov formulas? Answering this may lead to clues in answering the questions surrounding Medvedev's logic.

5. **Decidability of Sahlvqist**: Are finitely axiomatizable Sahqlvist logics extending **K4** decidable?

6. **Modal Reduction Principles**: Are the logics $\mathbf{K} \oplus \Box^n p \to \Box^m p$ decidable? Do they have the FMP?

[1] The dichotomy between ω and $2^{\aleph_0}$ follows, as far as we know, by unpublished work of Aguillera.

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