Pitts problem and Higher-Order Uniform Interpolation

Rodrigo Nicolau Almeida – ILLC-UvA – Seminario LAC
Ongoing joint work with: Marco Abbadini, Igor Arrieta, Daniël Otten, Lingyuan Ye

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Structure of this talk

1. Pitts problem on Heyting algebras and Elementary toposes.

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2. A survey of what is known about this problem.
3. An approach to the problem inspired by the work of Dito Pataraia.

Pitts Problem

Definition

An algebra $(H, \wedge, \vee, \to, 0, 1)$ is called a *Heyting algebra* if:

1. $(H, \wedge, \vee, 0, 1)$ is a (distributive) lattice.
2. The following law holds for all $a, b, c \in H$:

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As a special example, if $(P, \leq)$ is a poset, $\mathrm{Up}(P)$ the upwards closed sets of P, are a Heyting algebra.

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3. The effective topos Eff.

Toposes and Heyting algebras

Given a topos $\mathcal{E}$ and $A \in \mathcal{E}$, the lattice of subobjects $\mathrm{Sub}_{\mathcal{E}}(A)$ has the structure of a Heyting algebra. The Heyting algebra $\mathrm{Sub}_{\mathcal{E}}(1)$ – the lattice of subterminal objects – is often called the lattice of truth values.

Example

1. In **Set**, $\mathrm{Sub}(1)$ is 2, the two element Boolean algebra;
2. If L is a complete Heyting algebra then in $\mathrm{Sh}(L)$ we have $\mathrm{Sub}(1) \cong L$.

Pitts Problem

Figure 1: Andrew Pitts (1956-)

For which Heyting algebras H is there a topos $\mathcal{E}$ such that $\mathrm{Sub}_{\mathcal{E}}(1) \cong H$?

Known cases

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1. Chain Heyting algebras?
2. Countable Heyting algebras?
3. Free and finitely generated Heyting algebras?

Reducing the problem (but not by much)

The filter-quotient construction of toposes ensures that if H is a Heyting algebra arising as $\mathrm{Sub}_{\mathcal{E}}(1)$, and $f : H \to H'$ is a surjective homomorphism of HA, then there is a topos $\mathcal{E}'$ such that $\mathrm{Sub}_{\mathcal{E}'}(1) \cong H'$.

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Hence if there is any algebra which fails to appear in this way, $\mathsf{F}_{HA}(X)$ does not appear so.

Pitts' intuition: $\mathsf{F}_{HA}(\omega)$ must fail to appear in this way.

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Pitts implication and Uniform Interpolation

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Proposition (Pitts, 1981 (unpublished))

Assume that every Heyting algebra H is isomorphic to $\mathsf{Sub}_{\mathcal{E}}(1)$. Then the inclusion:

$$\mathsf{F}_{HA}(n) \hookrightarrow \mathsf{F}_{HA}(n+1)$$

has a left and a right adjoint.

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Now consider $\mathsf{F}_{HA}(\omega)[x]$, the Heyting algebra of polynomials. There is a map $p : \mathsf{F}_{HA}(\omega)[x] \to \mathsf{F}_{HA}(\omega)(x)$, fixing $\mathsf{F}_{HA}(\omega)$, and sending x to id_{Ω}.

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Then $\forall p$ is right adjoint to $i : \mathsf{F}_{HA}(\omega) \to \mathsf{F}_{HA}(\omega)[x]$, and similar with $\exists p$. This then restricts to adjoint maps on finitely many generators. $\square$

Uniform Interpolation

The logical implications of this are that for each formula $\phi(\overline{p}, q)$ of intuitionistic logic, there are formulas $\tilde{\forall} q\phi(\overline{p}, q)$ and $\tilde{\exists} q\phi(\overline{p}, q)$ such that for each formula $\psi(\overline{p})$:

$$\vdash \tilde{\exists} q\phi(\overline{p}, q) \to \psi(\overline{p}) \iff \vdash \phi(\overline{p}, q) \to \psi(\overline{p}) \text{ and}$$
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Pitts believed (for some years) that there should be some formula of intuitionistic logic without this property. Then:

Theorem (Pitts, 1992)
The above property holds for each formula $\phi(\overline{p}, q)$ of intuitionistic logic.

Pataraia doctrines

Pitts problem continued

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Dito Pataraia had the opposite intuition:

Figure 2: Dito Pataraia (1963-2011)

Pataraia doctrines

Pataraia worked with so called Higher-Order cylindric Heyting algebras. We work with an equivalent, but more categorical-logically inspired formalism, which we called Pataraia doctrines. These are intimately related to *triposes* and more generally, first-order hyperdoctrines with comprehension.

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Definition

The *initial type structure* is composed by pure types and indecomposable types, and is defined inductively as follows:

- 1 is an indecomposable type;
- If T is a pure type, then we have an indecomposable type PT.
- For every $n \in \mathbb{N}$, and every n-tuple $T_1, \ldots, T_n$ of indecomposable types such that $T_i \neq 1$, we have a pure type, denoted $\langle T_1, \ldots, T_n \rangle$ (or $T_1 \times \cdots \times T_n$);

$I :=$ set of indecomposable type; $V := I \times \mathbb{N}$ set of variables.

Definition

A *Pataraia doctrine* consists of

- A family $(\mathcal{H}(X))_{X \subseteq_\omega V}$ of Heyting algebras.
- For all $X, Y \subseteq_\omega V$ and every function $\sigma\colon X \to Y$ that preserves types (i.e. for each $(T, n) \in X$, the first coordinate of the pair $\sigma(T, n)$ should be T), a Heyting homomorphism
$$\mathcal{H}_\sigma \colon \mathcal{H}(X) \to \mathcal{H}(Y)$$
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- For every $X \subseteq_\omega V$ and every $x \in V \setminus X$, functions
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$$(\forall x)_X,\ (\exists x)_X\colon \mathcal{H}(X \cup \{x\}) \to \mathcal{H}(X).$$
- For every pure type T, every $x : PT$ and $t : T$, an element (t, x) in $\mathcal{H}(|t| \cup \{x\})$.
- For every indecomposable type T, and every $x, x' : T$, an element $\mathrm{eq}(x, x')$ in $\mathcal{H}(\{x, x'\})$.

Subject to a few axioms:

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3. For every indecomposable type T, for any $x : T$,
$$\mathrm{eq}(x, x) = \top_{\mathcal{H}(\{x\})}.$$
4. For any indecomposable type T, for any $x, x' : T$, every $X \subseteq_\omega V$ and every surjective type-preserving function $\sigma \colon \{x, x'\} \to X$, we have
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$$\mathcal{H}_\sigma(\mathrm{eq}(x, x')) = \mathrm{eq}(\sigma(x), \sigma(x')).$$
5. For every pure type T, all $s = \langle s_1, \ldots, s_n \rangle, t = \langle t_1, \ldots, t_n \rangle, t' = \langle t'_1 \ldots, t'_n \rangle : T$ with $s_1, \ldots, s_n$ all distinct and $t_1, \ldots, t_n, t'_1, \ldots, t'_n$ all distinct, denoting with $\sigma, \sigma' \colon |s| \to |t| \cup |t'|$ the function mapping s_i to t_i and t'_i, respectively, we have
$$\mathrm{eq}_{|t| \cup |t'|}(t, t') \wedge \mathcal{H}_\sigma(\alpha) \leq \mathcal{H}_{\sigma'}(\alpha).$$

Axioms of Pataraia doctrines

6. For every pure type T, every $x, x' : PT$ and every $t : T$ with no repeated variables,

$$\mathrm{eq}_{\{x,x'\}}(x, x') = (\forall t)_{\{x,x'\}}(\epsilon(t, x) \leftrightarrow \epsilon(t, x'))$$

7. (Comprehension) For all pure types T and S, every $t = \langle t_1, \ldots, t_n \rangle : T$ without repeated variables, every $x : P(T)$, every $s = \langle s_1, \ldots, s_m \rangle : S$ without repeated variables such that $|s|$ is disjoint from $|t| \cup \{x\}$, and every $\alpha \in \mathcal{H}(|t| \cup |s|)$,

$$(\exists x)_{|s|}(\forall |t|)_{\{x\} \sqcup |s|}\big(\mathcal{H}_{|t| \sqcup \{x\} \hookrightarrow |t| \sqcup \{x\} \sqcup |s|}(\epsilon(t, x)) \leftrightarrow \mathcal{H}_{|t| \sqcup |s| \hookrightarrow |t| \sqcup \{x\} \sqcup |s|}(\alpha)\big) = \top_{\mathcal{H}(|s|)}.$$

The following essentially follows from the work of Pitts, Pataraia and Johnstone:

Theorem
If F is a functor defining a Pataraia doctrine, there is an elementary topos $\mathcal{E}$ such that $\mathrm{Sub}_{\mathcal{E}}(1) \cong F(\emptyset)$. Moreover, for any elementary topos $\mathcal{E}$, the functor $Hom(-, \Omega_{\mathcal{E}})$ on the initial type system is a Pataraia doctrine.

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To solve Pitts problem we can work with Pataraia doctrines instead.

Pataraia doctrines and Higher-Order Uniform Interpolation

By restricting the type system, we can obtain a more modular notion:

Definition
We say that a type $x \in I$ has power depth 0 if $x : 1$; and that it has power depth at most $n+1$ if $x : PT$ where $T = \langle t_1, \ldots, t_n \rangle$ and t have power depth at most n.

We denote by V_n the set of variables of types of power depth at most n.

Then $V_0 = \emptyset$, and $V_1 \cong \mathbb{N}$.

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Then $V_0 = \emptyset$, and $V_1 \cong \mathbb{N}$.

Definition
An n-Pataraia doctrine is a hyperdoctrine on V_n sastisfying the axioms of Pataraia doctrine for types of power depth at most n.

A 0-Pataraia doctrine is a Heyting algebra.

1-Pataraia doctrines

Note that (n-) Pataraia doctrines are many-sorted algebraic theories. Hence they admit free objects.

One theorem derived by Pitts from his Uniform Interpolation, is the following:

Theorem
Every 0*-Pataraia doctrine can be extended to a* 1*-Pataraia doctrine.*

Proof.
For H a Heyting algebra, one defines $\mathcal{H}(\emptyset) = H$ and for each set of variables $\{x_1, \ldots, x_n\}$, $\mathcal{H}(\{x_1, \ldots, x_n\}) = H[x_1, \ldots, x_n]$. $\square$

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One possible formulation of this: free quantifier-free 1-Pataria doctrines over a Heyting algebra H are 1-Pataraia doctrines.

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There, starting from a Boolean algebra B we need to generate a (Boolean) algebra corresponding to:

$$\mathcal{H}(\{x : PP1\})$$

this must contain expressions of the form:

$$\epsilon(p, x)$$

for $p \in B$. As it turns out this can be reduced to:

$$\epsilon(p, x) \cong \epsilon(p, 0) \vee \epsilon(p, 1)$$

Using this we can construct, for a given Boolean algebra B, a Pataraia doctrine over B.

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Part of our current work concerns the following kinds of algebras:

$$H[[x]]$$

which is the Heyting algebra of formulas of a calculus containing:

1. $\forall$-and $\exists$-quantifiers defined for variables of type $P1$;
2. Equalities $eq(y, y')$ for $y, y' : PP1$;
3. Elements $\epsilon(x, y)$ for $x : P1$ and $y : PP1$;
4. It satisfies extensionality, and comprehension for formulas of type $P1$, and invariance under equality.

We denote by $F_{HA^2}(n)$ the free algebra on ω many generators for this theory. Then Pitts implication says:

Theorem
Assume that every Heyting algebra is isomorphic to $\mathrm{Sub}_{\mathcal{E}}(1)$. *Then the inclusion:*

$$F_{HA^2}(n) \hookrightarrow F_{HA^2}(n+1)$$

has a left and right adjoint.

This entails a form of higher-order uniform interpolation. So one way of furthering the study of this problem lies in studying these higher order properties and checking whether they still hold.

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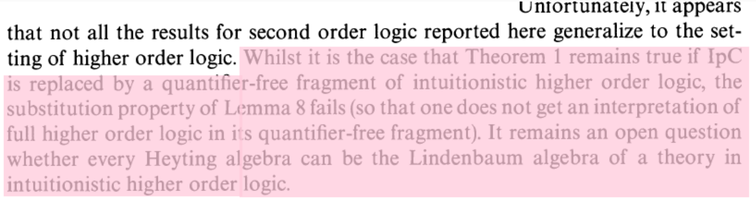
For example, what should be equivalent to:

$$\exists x\, \epsilon(\phi, x)$$

in intuitionistic logic, where ϕ is any formula?

Some considerations on this were already made by Pitts:



Unfortunately, it appears that not all the results for second order logic reported here generalize to the setting of higher order logic. Whilst it is the case that Theorem 1 remains true if IpC is replaced by a quantifier-free fragment of intuitionistic higher order logic, the substitution property of Lemma 8 fails (so that one does not get an interpretation of full higher order logic in its quantifier-free fragment). It remains an open question whether every Heyting algebra can be the Lindenbaum algebra of a theory in intuitionistic higher order logic.

Figure 3:

Higher-Order Uniform Interpolation

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However, this particular failure seems to only happen if one does not impose axioms on equality or the $\epsilon(y, x)$ elements. This is why we work with quantifiers of smaller depth.

There is still much work to be done to understand how these quantifiers can be added freely, given they must often be Pitts-like quantifiers.

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Summary and Future Work

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Regardless of the importance of the question itself, it has stimulated several directions of research in algebraic logic, namely uniform interpolation.

Our short to medium term goals will be to find answers to some reasonable sounding problems:

1. Whether every chain arises as the truth values of a topos;
2. How to prove the higher-order uniform interpolation property above;
3. Several challenges lie in understanding the connection between quantifiers, Heyting algebras, and higher order logics.

Thank you!

Questions?