Heyting algebras one step at a time

Rodrigo Nicolau Almeida – ILLC-UvA – University of Birmimgham TCS Seminar
Friday 4, 2025

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2. A particular technical problem I have worked on related to free Heyting algebras.
3. Some hopes for these, and related, techniques in furthering the theory concerning those open problems.

Intuitionistic Logic and its Extensions

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Its algebraic models are also mathematically quite natural:

Definition
An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is called a *Heyting algebra* if:

1. $(H, \wedge, \vee, 0, 1)$ is a (distributive) lattice.
2. The following law holds for all $a, b, c \in H$:

$$a \wedge c \leq b \iff c \leq a \rightarrow b.$$

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For example, given a topological space (X, τ), the topology defines a Heyting algebra with Heyting implication given by $int(X - U \cup V)$ for U, V open sets. Such algebras are called topological Heyting algebras. As a special example, if $(P, \leq)$ is a poset, $\mathrm{Up}(P)$ the upwards closed sets of P, are a Heyting algebra. Also every locale is a (complete) Heyting algebra.

Often we are interested not just in a single logic, but in the class of its extensions:

Definition
Let L be a set of formulas in the language $\mathcal{L}_{IPC}$ of intuitionistic logic. We say that L is a superintuitionistic logic if IPC $\subseteq L$ and L is closed under Modus Ponens and Uniform Substitution.

We denote by $\Lambda(\text{IPC})$ the lattice of extensions of IPC.

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An example:

Theorem
There are exactly seven superintuitionistic logics with the interpolation property; equivalently seven varieties of Heyting algebras with the amalgamation property.

Duality

Definition
An ordered topological space $(X, \leq, \tau)$ is called a Priestley space if it is compact and whenever $x \nleq y$ there is a clopen upwards closed set (upset) $x \in U \not\ni y$. It is called Esakia if whenever U is clopen, then $\downarrow U = \{x \in X : \exists y \in U, x \leq y\}$ is clopen.

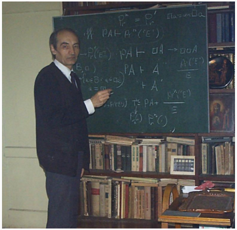

Figure 1: Hilary Priestley (info missing); Leo Esakia (1934-2010)

Definition
A given order-preserving map $p : X \to Y$ between partial orders is called a *p-morphism* if whenever $p(x) \leq y$ there is some $x' \geq x$ such that $p(x') = y$.

Duality (cont.d)

Theorem
There is a dual equivalence between DL *distributive lattices and* Pries *the category of Priestley spaces with order-preserving continuous functions. This restricts to a dual equivalence between* HA, *Heyting algebras and their homomorphisms, and* Esa, *Esakia spaces with continuous p-morphisms.*

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All finite posets are Esakia spaces. Moreover, given a class $\mathcal{K}$ of Esakia spaces we can look at their logic: it is the logic of the dual Heyting algebras.

What we know that we don't know

The extensions of IPC have been extensively studied for more than 60 years. Nevertheless some longstanding open problems have remained open.

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The extensions of IPC have been extensively studied for more than 60 years. Nevertheless some longstanding open problems have remained open.

I will briefly mention three such problems which are old and have revealed themselves quite sticky.

Consider the family of finite posets of topless Boolean algebras:

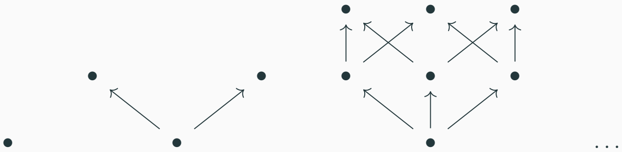

Figure 2: Medvedev frames

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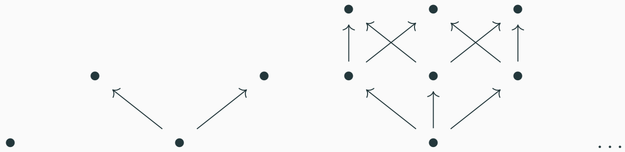

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1. Medvedev's Logic

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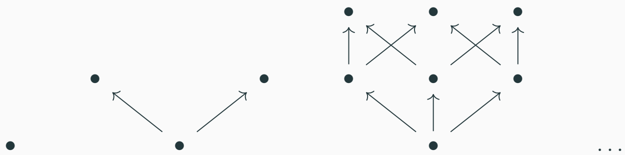

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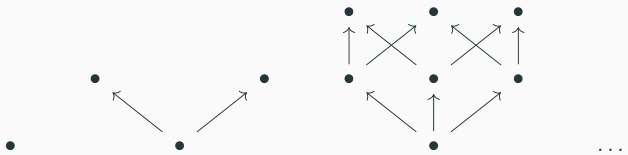

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Question: can you recursively axiomatize this logic?

Other question: is this the only intermediate logic which is structurally complete and has the disjunction property?

This logic appears very naturally elsewhere: it can be formulated as a logic of finite problems; it is the natural model of inquisitive logic; etc.

Given an intermediate logic L, sometimes we do not need all (or most) algebras of L:

1. L is tabular if $L = Log(A)$ for a single finite algebra;
2. L has the FMP if L is the logic of a class of finite algebras;
3. L is *Kripke complete* if its class of algebras $\mathsf{Alg}(L)$ is generated by the algebras of the form $\mathsf{Up}(L)$;
4. L is *topologically complete* if it is generated by its topological Heyting algebras;
5. L is *locallically complete* if it is generated by its locales.

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This establishes a link between propositional and First-order intuitionistic logic.

3. Pitts-Pataraia Problem

Given an elementary topos $\mathcal{E}$, the subobjects of the terminal object $\mathrm{Sub}(1)$ form a Heyting algebra.

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This establishes a link with Propositional Higher-order logic, i.e., with particular models of the lambda calculus.

Free Heyting Algebras

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Every logic L is generated by its free algebras. Hence understanding the free algebras gives us crucial information about the structure of L.

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Example

1. Free Boolean algebras are very well-understood: for a set X they can be described as the duals of
$$2^X$$
understood as the X-fold product of the discrete space with two elements.
2. Similarly, for Distributive Lattices, the description of the free algebras is similarly straightforward: for a set X,
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Having a good grasp of free algebras is intimately related to having a good grasp of colimits of these algebras. Logically, this has several natural desirable applications: interpolation, conservativity, etc.

A beautiful disaster

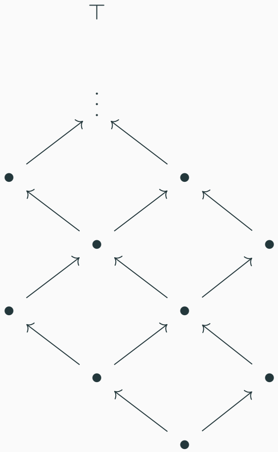

Figure 3: Rieger-Nishimura Lattice

Intuitionistic Monsters (Cont.d)

Free n-generated Heyting algebras for $n > 1$ are very complex, in all the ways this can happen.

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Free n-generated Heyting algebras for $n > 1$ are very complex, in all the ways this can happen.

The consequence of this complexity is that a number of natural related constructions which would otherwise be at our disposal have not been considered within the reasonable toolkit:

1. Coproducts of Heyting algebras (except for some locally finite subvarieties);
2. Pushouts of Heyting algebras;
3. Heyting subalgebras generated by a given subset;
4. Etc.

Describing the freely generated algebras amounts to studying the adjunction:

$$\mathsf{HA} \underset{\textit{Forget}}{\overset{\textit{Free}}{\rightleftarrows}} \mathsf{Set}$$

Figure 4: Free-Forgetful Adjunction

Heyting Algebras from Distributive Lattices

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$$
\begin{tikzcd}
\mathsf{HA} \arrow[r, shift left, "\mathit{Free}"] & \mathsf{Set} \arrow[l, shift left, "\mathit{Forget}"]
\end{tikzcd}
$$

Figure 4: Free-Forgetful Adjunction

Instead of studying it directly, we "split" this adjunction:

$$
\begin{tikzcd}
\mathsf{HA} \arrow[rr, shift right, "\mathit{Forget}"'] \arrow[dr, dashed, shift left] & & \mathsf{Set} \arrow[ll, shift right, "\mathit{Free}"'] \arrow[dl, dotted, shift left, "\mathit{Free}_{DL}"] \\
& \mathsf{DL} \arrow[ul, dotted, shift left, "\mathit{Free}_{HA}"] \arrow[ur, dashed, shift left] &
\end{tikzcd}
$$

Figure 5: Split Free Forgetful Adjunction

Key Intuition: We think of generating Heyting algebras as adding infinitely many layers of implications to a distributive lattice step-by-step.

Heyting Algebras from Distributive Lattices, algebraically

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$$\mathcal{D}_1 = F_{DL}(\{a \to b : a, b \in D\})/\Theta$$

where Θ contains:

1. Axioms of Heyting algebras;
2. Axioms enforcing elements of the form $1 \to a$, for $a \in D$, to behave like the elements from D.

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We set the map $i_1 : \mathcal{D}_0 \to \mathcal{D}_1$ sending a to $[1 \to a]$, which is a homomorphism by force. We then iterate; but for $\mathcal{D}_2$ we need to also add one more rule to Θ:

3. Axioms ensuring that if $a, b \in D$, then:

$$1 \to_{D_2} (a \to_{D_1} b) \equiv (1 \to_{D_1} a) \to_{D_2} (1 \to_{D_2} b).$$

This is how we define $\mathcal{D}_2$, and i_2 is defined similarly.

We then iterate infinitely:

1. $\mathcal{D}_{n+1} = F_{DL}(\{a \to b : a, b \in D_n\})/\Theta$, where Θ is defined as above;
2. $i_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$ sends a to $[1 \to a]$.

Problem: We would like to say that the free Heyting algebra generated by $\mathcal{D}$ is the union of all of these. But we do not know whether this is a chain of embeddings, or really anything about this construction.

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Problem: We would like to say that the free Heyting algebra generated by $\mathcal{D}$ is the union of all of these. But we do not know whether this is a chain of embeddings, or really anything about this construction.

For this we turn to duality.

One-step dual constructions

The idea of dual one-step constructions is not-new, and well-studied in Modal Logic. In the case where we generate from finite Heyting algebras, this was studied by Ghilardi (1992), generalizing previous work of Urquhart (1973); Bezhanishvili and Gehrke (2009) gave a detailed outline of this method for various classes.

Vietoris Spaces

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$V(X)$ is the set of closed subsets of X; it has a basis consisting of sets:

$$[U] = \{C \in V(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in V(X) : C \cap V \neq \emptyset\}$$

where $U, V \in \mathrm{Clop}(X)$. It is always a Priestley space.

Definition
Let X, Y, Z be Priestley spaces, and $g : X \to Y$ and $f : Z \to X$ be Priestley morphisms. We say that f is open relative to g (g-open for short) if it satisfies the following:

$$\forall a \in Z, \forall b \in X, (f(a) \leq b \implies \exists a' \in Z, (a \leq a' \;\&\; g(f(a')) = g(b)). \qquad (*)$$

Given $S \subseteq X$ a closed subset, we say that S is g-open (understood as a poset with the restricted partial order relation) if the inclusion is itself g-open.

Definition
Let X, Y be a Priestley spaces, $g : X \to Y$ Priestley morphism. We denote by $V_r(X)$ the set of closed and rooted subsets of X. We denote by $V_g(X)$ the set of closed, rooted and g-open subsets of X.

This is a Priestley space as well.

We can now proceed with the main construction:

Definition
Let $g : X \to Y$ be a Priestley morphism. The g-Vietoris complex over X $(V^g_\bullet(X), \leq_\bullet)$, is a sequence

$$(V_0(X), V_1(X), ..., V_n(X))$$

connected by morphisms $r_i : V_{i+1}(X) \to V_i(X)$ such that:

1. $V_0(X) = X$;
2. $r_0 = g$
3. For $i \geq 0$, $V_{i+1}(X) := V_{r_i}(V_i(X))$;
4. $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ is the root map.

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We denote the projective limit of this family by $V^g_G(X)$, and omit g when this is the terminal map to $\mathbf{1}$.

Then one can show that $V^g_G(X)$ is always an Esakia space.

By considering maps appropriately, one can use this to prove:

Corollary
The assignment V_G is a functor mapping **Pries** *of Priestley spaces and Priestley morphisms, to the category* **Esa** *of Esakia spaces and p-morphisms. It is the right adjoint of the inclusion.*

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In the short term we could capitalize on this construction and associated ideas for several results:

1. In (A., 2025, preprint available), I presented this theory as well as a similar adjunction for image-finite posets and profinite Heyting algebras;
2. In (A., Bezhanishvili, 2024 and A.,Dukic, 2025), we applied this to coalgebraic representations of intuitionistic modal logic, resolving an open problem in the field;
3. In (A. 2025, forthcoming), we obtained a proof of uniform interpolation using a dual representation of coproducts of Heyting algebras.

Step by Step as a technique

Consider the functor:

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One considers a specific Vietoris complex:

Let $M_\infty(Y) = V_G^r(V_{\max}(Y))$. Explicitly, this is the inverse limit of the sequence:

$$V(Y) \xleftarrow{r_1} V_{\max}(Y) \xleftarrow{r_2} V_2(Y) \xleftarrow{r_3} \dots$$

where $V_{n+1}(Y) = V_{r_n}(V_n(Y))$.

Proposition
The functor FreeM : Stone $\to$ Esa *assigning each Stone space* X *to* $M_\infty(X)$ *is right adjoint to* max : Esa $\to$ Stone.

A Heyting algebra is said to be regular if it is generated by its regularization.

Theorem
Given a Stone space X, $M_\infty(X)$ is always a regular Esakia space, and moreover, regular Esakia spaces are the coalgebras for the comonad induced by this functor.

Theorem
The logic ML *is precisely the logic of all the spaces $V(X)$ for X a Stone space.*

Hope: Understanding the role of Medevedev spaces in relation to general Esakia spaces might allow us to eventually find a way to mechanically axiomatize Medvedev logic, as well as understand how the disjunction property and admissibility of it work.

Definition
Let $(H, \Box)$ be a Heyting algebra with an operator. We say that this is a KM-algebra if $\Box$ satisfies for every $a, b \in H$:

1. (K axiom) $\Box(a \to b) \leq \Box a \to \Box b$;
2. (Lax) $a \leq \Box a$;
3. (Strictness) $\Box a \leq b \vee b \to a$;
4. (WF) $\Box a \to a \leq a$.

Heyting algebras admit a KM-algebra structure uniquely. But not every HA admits such a structure (e.g., the $\mathbf{2}$-generated free Heyting algebra).

Definition
Let $(H, \Box)$ be a Heyting algebra with an operator. We say that this is a $\mathbf{KM}$-algebra if $\Box$ satisfies for every $a, b \in H$:

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3. (Strictness) $\Box a \leq b \vee b \to a$;
4. (WF) $\Box a \to a \leq a$.

Heyting algebras admit a $\mathbf{KM}$-algebra structure uniquely. But not every HA admits such a structure (e.g., the $\mathbf{2}$-generated free Heyting algebra).

Question: Which logics are generated by their $\mathbf{KM}$-algebras?

'Little' Kuznetsov

Kuznetsov, in an inaccessible paper, proved that:

Theorem
Every superintuitionistic logic L is generated by its KM*-algebras.*

A recent (2025) algebraic proof was obtained by Jibladze and Kuznetsov *(no relation)*.

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We have been able to dualize this proof; a scary picture:

$$
\begin{tikzcd}
\vdots & \vdots & & \vdots \\
M_1 \arrow[d] & \overline{Rep_2} \arrow[l] \arrow[d] \arrow[rr, bend left] & \vdots & V_r(X_H \times 2) \arrow[d] \\
M_0 & \overline{Rep_1} \arrow[l] \arrow[d] \arrow[rr, bend left] & Rep_2 \arrow[ul] \arrow[ur] \arrow[d] & V_{\pi_H}(X_H \times 2) \arrow[d] \\
 & \overline{Rep_0} \arrow[rr, bend left] & Rep_1 \arrow[ul] \arrow[ur] \arrow[d] & X_H \times 2 \\
 & & Rep_0 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Figure 6:

The interest of this is that to solve Kuznetsov’s localic problem in the positive the minimal requirements go through generalizing the ideas of Jibladze and Kuznetsov:

1. Turning a Heyting algebra into a **KM**-algebra in the same variety, one adds for each $a \in H$ a given “fronton”, which is represented by the filter of dense elements over a;
2. To “complete” a Heyting algebra in the same variety, one should at least be able to add for each $S \subseteq H$ a given join to this subset, which is represented by the filter of upper bounds of S.

Look upon the board.

Thank you!

Questions?