TOPOLOGY IN AND VIA LOGIC
HOMEWORK ASSIGNMENT 1

- Deadline: February 10 at 23:59.
- All exercises are worth the same points.
- Good luck!

Topological Spaces

Exercise 1. Consider the space $(\mathbb{R}, \tau_{Euc})$, with its Euclidean topology.

(1) Give an example of a set which is neither open nor closed.

(2) Show that the open intervals of the form (x, y) where $x, y \in \mathbb{Q}$ form a basis for this topology.

(3) Show that $\mathbb{Q}$ is a countable union of closed sets.

Exercise 2. Let X be a set. We say that an operation $\Box : \mathcal{P}(X) \to \mathcal{P}(X)$ is called an *interior operator* if it satisfies for each $U, V \in \mathcal{P}(X)$,

- (All set): $\Box X = X$;
- (Normality): $\Box(U \cap V) = \Box U \cap \Box V$;
- (Inflationarity): $\Box U \subseteq U$;
- (Idempotence): $\Box U \subseteq \Box\Box U$.

(1) Show that if (X, τ) is a topological space, the topological interior *int* is an interior operator in this sense.

(2) Given a set $(X, \Box)$ equipped with an interior operator, define a topology for which $\Box$ is the topological interior operator.

(3) We say that an interior operator $\Box$ is *completely multiplicative* if for each $(U_i)_{i \in I}$ we have that:
$$\Box(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} \Box U_i$$
Show that Alexandroff topology are in 1-1 correspondence with completely multiplicative interior operators.

(4) Let $(X, \Box)$ be a set, equipped with a completely multiplicative interior operator, with the following property: if $x \neq y$, then there is some $U \subseteq X$ such that either $x \in \Box U$ and $y \notin \Box U$ or $y \in \Box U$ and $y \notin \Box U$. Show that then there is a poset $(P, \leq)$ such that the Alexandroff topology on P is the same as the topology induced on X by the interior operator.

Continuity and Continuous Functions

Exercise 3. Show the following:

(1) Given an example of a bijective continuous map which is not a homeomorphism.

(2) Show that all functions from a discrete space to another space are continuous. If (X, τ) is a space with the *indiscrete* topology, which functions from this space to some other space are continuous?

(3) Show that if $f : X \to Y$ is a bijective continuous map between topological spaces, then the following are equivalent:
- f^{-1} is continuous;
- f is closed;
- f is a homeomorphism.

Exercise 4. (*Product maps*) Let X, Y be topological spaces.

(1) Show that for any other topological space Z, if there exists continuous functions $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$, then there exists a unique continuous function $\pi_1 \times \pi_2 : Z \to X \times Y$ making the following diagram commute

$$
\begin{tikzcd}
& Z \arrow[ddl, "\pi_1"'] \arrow[d, dashed, "\pi_1 \times \pi_2"] \arrow[ddr, "\pi_2"] & \\
& X \times Y \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
X & & Y
\end{tikzcd}
$$

(2) Show that this defines the product topology up to homeomorphism: whenever a topological space A satisfies the condition in (1), then there exists a homeomorphism between A and $X \times Y$.