Notions of Structural Completeness in an algebraic setting*

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July 2021

Abstract

A logical system is said to be structurally complete if every admissible rule is derivable. This is an attractive property, since it means the system contains all rules it could need to make derivations of theorems. In this report we present various notions adjacent to structural completeness in the context of algebraizable logics, showing the equivalence of these with certain algebraic and model theoretic properties. We illustrate this with some examples coming from logic and algebra.

Keywords— Structural completeness, Quasivarieties, Passive Structural Completeness, Joint Embedding Property

1 Introduction

Given a logical language L, we call a tuple of formulas $(\alpha_1, ..., \alpha_n, \beta)$ (occasionally written as $\alpha_1, ..., \alpha_n/\beta$) a *rule*. If we fix a given logic $\vdash$, we say that the rule $(\alpha_1, ..., \alpha_n, \beta)$ is *derivable* if $\alpha_1, ..., \alpha_n \vdash \beta$; we say that the rule is *admissible* if for every substitution σ whenever $\vdash \sigma(\alpha_i)$ for each i, then $\vdash \sigma(\beta)$. We say that a logic $\vdash$ is *structurally complete* if every admissible rule is derivable.

In general, structural completeness yields as strong a deductive system as one can wish, without increasing the set of theorems, which makes it generally a desirable feature. This motivated specific questions such as *Friedman's problem* (5) which asks whether admissibility in intuitionistic logic is decidable, and was eventually solved by Rybakov in the positive (cf. (8)). However, the situation is complicated on multiple fronts. On one hand, many widely used logical systems fail to be structurally complete: for example, **IPC** and **S5** are both not structurally complete. On the other hand, one may want to study extensions of a logical system, and wish that all these extensions are structurally complete. This calls for more nuanced notions of "completeness" with regards to admissible rules.

In this report, we study various analogues of structural completeness, from an

*This report is the result of a coordinated project, supervised by Tommaso Moraschini. All work presented was done in collaboration by the two authors.

algebraic and model theoretic point of view. We begin by characterising suitably weaker notions, which are connected with "reasonable" properties of a deductive system, such as the so-called *Relevance Principle*, and connecting them with the joint embedding property. We then present the notion of *passive* structural completeness, giving a result of Wronski that equates this with the positive existential theory of all nontrivial members of a quasi-variety being the same. We then present some equivalent characterisations of structural completeness. We conclude by showing that passive structural completeness implies the joint embedding property, establishing a hierarchy of structural properties of a logical system with regards to admissibility of rules.

2 Joint Embedding Property and Relevance Principles

We begin by proving Maltsev's theorem, which says that a quasi-variety is generated by a single algebra if and only if it has the joint embedding property.

Definition 1. *Let K be a class of similar algebras. We say that K has the* ***joint embedding property*** *(JEP) if given nontrivial $A, B \in K$, there exists some $C \in K$ such that A and B both embed into C.*

As some easy examples, one can think of groups or Heyting algebras.

Theorem 2. *(Maltsev) Let K be a quasi-variety. Then the following are equivalent:*

- *(1) K is generated by a single algebra A, i.e, $K = \mathbb{Q}(A)$;*
- *(2) K has the joint embedding property.*

Proof
(1) to (2) follows from the equivalent characterisation of the least quasi-variety generated by an algebra A as $ISP_R(A)$, where P_R is the class operator of reduced products. Indeed, if we have $A, B \in ISP_R(A)$, then A, B are subalgebras of two reduced powers of the same algebra A^I/F_1 and A^J/F_2. Take the algebra $A^{I\times J}/F$, where F is the filter generated by the union of the projection preimages of F_1 and F_2. Then, letting for $f : I \to A$ f^* be the map $f^*(i, j) = f(i)$, we can check that the map:

$$\begin{aligned} g : A^I/F_1 &\to A^{I\times J}/F \\ f/F_1 &\mapsto f^*/F \end{aligned}$$

is a well-defined embedding.
Finally, we look at (2) implies (1). First, let $\{\phi_\alpha : \alpha < \kappa\}$ be a sequence of quasi-equations which fail in K; for each one, pick $A_\alpha \in K$ a witness of the failure of ϕ_α; note that such algebras must all be non-trivial. We will construct by transfinite recursion an algebra B in K such that each A_α embeds into B.

This will be enough, for then $\mathbb{Q}(B) \subseteq K$, since $B \in K$, and $K \subseteq \mathbb{Q}(B)$, since every quasi-equation that fails in K also fails in B.
So let $B_0 = A_0$. For each α, let $B_{\alpha+1}$ be an algebra that embeds B_α and $A_{\alpha+1}$. Finally, if α is a limit ordinal, let B_α be the direct union of B_β for $\beta < \alpha$. Since K is a quasi-variety, it is axiomatised by inductive sentences, and so by Chang-Łoś-Suszko theorem[1], it is closed under directed unions. So $B_\alpha \in K$, and every B_β for $\beta < \alpha$ embeds into it.$\square$

The most logically relevant concept that we wish to prove equivalent to these is the following, which has its origins in the work of Łoś and Suszko (6):

Definition 3. *Let K be a quasi-variety. Suppose that for all $\Gamma \cup \Delta \cup \{\phi\} \subseteq Eq(X)$, where:*

- *(LS 1) Γ and Δ are finite*
- *(LS 2) $Var(\Gamma) \cap Var(\Delta \cup \{\phi\}) = \emptyset$*
- *(LS 3) Γ is consistent over K: there are terms α and β such that $K \not\models \bigwedge \Gamma \to \alpha \approx \beta$.*

Then whenever:

$$\bigwedge \Gamma \wedge \bigwedge \Delta \models_K \phi$$

it is the case that:

$$\bigwedge \Delta \models_K \phi$$

*Then we say that K has the **Relevance Principle** (RP). If K is the equivalent algebraic semantics of a logic $\vdash$, we likewise say that $\vdash$ has the RP.*

This property can also be formulated in terms of a logic $\vdash$: if we are given two finite sets of formulas $\Gamma, \Delta, \{\phi\}$, such that $Var(\Gamma) \cap Var(\Delta \cup \{\phi\}) = \emptyset$, and Γ is consistent, then whenever $\Gamma, \Delta \vdash \phi$ we have that $\Delta \vdash \phi$. For algebraizable logics, it can be seen that the two concepts will coincide.

Theorem 4. *Let K be a quasi-variety. Then K has the Relevance Principle iff K has the Joint Embedding Property.*

Proof
First suppose that K has the JEP, and let Γ, Δ, ϕ satisfy (LS1-3). By Theorem 2, K is generated as a quasi-variety by some $A \in K$. We proceed by contraposition: suppose that $K \not\models \bigwedge \Delta \to \phi$. Then because $K = \mathbb{Q}(A)$, we obtain that $A \not\models \bigwedge \Delta \to \phi$, witnessed by valuation v. Since Γ is consistent over K, we know there is a valuation w such that $A, w \models \bigwedge \Gamma$. Since the variables in Γ are all distinct from the ones in $\Delta \cup \{\phi\}$, we can combine the valuations v and w to obtain a valuation k such that $A, k \models \bigwedge \Gamma \wedge \bigwedge \Delta$ but $A, k \not\models \phi$.

Conversely, suppose that K has the RP, and let $A, B \in K$ be two algebras. Let $\pi_A : T(A) \to A$ be the unique homomorphism from the algebra

[1]You can see this in Tutorial 2 of the Model Theory course 2021.

of terms with variables from the set A to the algebra A, extending the identity function on A. Analogously, define $\pi_B : T(B) \to B$. Furthermore, define $\theta \coloneqq \mathbf{Cg}_K^{T(A\cup B)}(\mathrm{Ker}(\pi_A) \cup \mathrm{Ker}(\pi_B))$ and $F \coloneqq T(A \cup B)/\theta$. We claim that the map $f : A \to F$ where $a \mapsto [a]_\theta$ is an embedding (and similarly for B). The interesting part is proving injectivity so assume toward contradiction that $a_1, a_2 \in A, a_1 \neq a_2$ but $f(a_1) = f(a_2)$. We have the following chain of implications:

$$\begin{aligned}
f(a_1) = f(a_2) &\implies (a_1, a_2) \in \theta \\
&\implies (a_1, a_2) \in \mathbf{Cg}_K^{T(A\cup B)}(\mathrm{Ker}(\pi_a) \cup \mathrm{Ker}(\pi_b)) \\
&\implies a_1 \approx a_2 \in \mathbf{Cn}_K(\mathrm{Ker}(\pi_A) \cup \mathrm{Ker}(\pi_B)) \\
&\implies \mathrm{Ker}(\pi_A) \cup \mathrm{Ker}(\pi_B) \vDash_K a_1 \approx a_2 \\
&\implies \Delta \cup \Gamma \vDash_K a_1 \approx a_2 \\
&\implies \Delta \vDash_K a_1 \approx a_2
\end{aligned}$$

where Δ is a finite subset of $\mathrm{Ker}(\pi_a)$ and Γ is a finite subset of $\mathrm{Ker}(\pi_b)$. Here the third implication is due to the correspondence between theories and congruences, the fifth implication uses the fact that $\vDash_K$ is finitary and the last implication is an application of RP. But now we have a contradiction, because A with the natural assignment validates Δ but refutes $a_1 \approx a_2$.$\square$

The relevance principle seems perhaps at first sight too intuitive to fail. The previous theorems however allow us to easily find logics failing JEP and the relevance principle.

Example 5. Consider the class of all commutative rings with identity K, which forms a quasi-variety (indeed, a variety). As discussed during the course, there is a logic $\vdash$ which is algebraized by this quasi-variety.
On the other hand, consider $\mathbb{Z}$ and $\mathbb{Z}/2$, and let A be some algebra which embeds both of them. Let $f : \mathbb{Z}/2 \to A$ be such an embedding. Then $f(1) = 1^A$ and $f(0) = 0^A$ because it is a homomorphism, and so:

$$1^A + 1^A = f(1) + f(1) = f(1 + 1) = f(0) = 0^A$$

Thus, this tells us that $g : \mathbb{Z} \to A$ cannot be injective since $g(2) = g(1 + 1) = g(1) + g(1) = 1^A + 1^A = 0^A = g(0)$, despite $0 \neq 2$. Thus K does not have the JEP, and so by the previous theorem, the logic $\vdash$ does not satisfy the relevance principle.

By contrast, the variety of abelian groups has the joint embedding property: given A, B the product $A \times B$ is the coproduct of the two groups, and it is clear to see that the canonical coprojections are injective. So the logic of abelian groups satisfies the relevance principle.

3 Structural Completeness

We will now make the notion of structural completeness a bit more precise.

Definition 6. *Let K be a quasi-variety. We say that a quasi-equation $\phi = \bigwedge a_1, \ldots, a_n \to b$ is **admissible** if for each substitution σ, whenever $\vDash_K \sigma(a_i)$ for each i then $\vDash_K \sigma(b)$. We say that ϕ is **derivable** if $a_1, \ldots, a_n \vDash_K b$.*

*Similarly, given a logic $\vdash$, and a rule $(\phi_1, ..., \phi_n, \psi)$, we say that it is **admissible** if for all substitutions σ whenever $\vdash \sigma(\phi_i)$ for each i then also $\vdash \sigma(\psi)$. We say that the rule is **derivable** if $\phi_1, \ldots, \phi_n \vdash \psi$.*

We will also need a special kind of quasi-equations and rules:

Definition 7. *Given a quasi-variety K, we say that a quasi-equation $\bigwedge a_1, ..., a_n \to b$ is **passive** if there is no substitution σ such that $\vDash_K \sigma(a_1), \ldots, \vDash_K \sigma(a_n)$.*
*Given a logic $\vdash$, we say that a rule $(\phi_1, ..., \phi_n, \psi)$ is **passive** if there is no substitution σ such that $\vdash \sigma(\phi_1), \ldots, \vdash \sigma(\phi_n)$.*

Remark: Note that every passive rule or quasi-equation is vacuously admissible.

Definition 8. *We say that a quasi-variety K is **(passively) structurally complete** if every (passive) admissible quasi-equation is derivable.*
*Similarly, we say that a logic $\vdash$ is **(passively) structurally complete** if every (passive) admissible rule is derivable.*

Also note that for algebraizable logics, the semantic and syntactic consequence relations will be structurally complete as long as one of them is - this follows straightforwardly from the conditions of algebraizability.

Whilst passive rules may not seem the most intuitive, we will see that restricting to them in discussions of structural completeness gives us a useful intermediate property. Indeed, the following was what motivated Wronski to introduce the concept (9).

Definition 9. *Let ϕ be a formula. We say that ϕ is **positive existential** if ϕ is of the form:*

$$\phi = \exists x_1, ..., x_n \psi$$

where ψ is quantifier free and a positive formula (i.e contains no negations or implications). Let $Th_P(K)$ be the positive existential theory of a class of algebras K.

Theorem 10. *Let $\vdash$ be an algebraizable logic, witnessed by quasi-variety K. Then $\vdash$ is PSC iff whenever $A, B \in K$ are nontrivial, $Th_P(A) = Th_P(B)$.*

Proof

By our previous remark, it suffices to show that K is passively structurally complete as a quasi-variety, if and only if all of its algebras satisfy the same positive existential theory.

First suppose that all elements in K satisfy the same positive existential theory. Suppose toward contradiction that $\bigwedge \phi_1, ..., \phi_n \to \phi_{n+1}$ is a passive quasi-equation which is not derivable in K. Thus, there is a witness for the failure of this quasi-equation: let $A \in K$ be an algebra and v a valuation such that $A, v \vDash \phi_1, ..., \phi_n$ but $A, v \nvDash \phi_{n+1}$. Then consider the following sentence, which is true in A:

$$\psi := \exists \overline{x} \bigwedge_i \phi_i(\overline{x})$$

Now, because this sentence is positive existential, we know that all algebras in K satisfy it. Thus in particular, we know this holds for the free algebra $F_K(\aleph_0) := T(\aleph_0)/\theta$ where θ is the least K-congruence in $T(\aleph_0)$. Let $p_1, ..., p_n$ be terms such that $[p_1]_\theta, \ldots, [p_n]_\theta$ witness $F_K(\aleph_0) \vDash \psi$. Let σ be a substitution that sends x_i to p_i. Now, this means that $\sigma(x_i) \in [p_i]_\theta = x([p_i]_\theta)$ for each i; by construction of the terms, then, $\sigma(\varepsilon_i) \in \varepsilon_i([p_1]_\theta, \ldots, [p_n]_\theta)$ for each term ε.

Suppose $\phi_i = \epsilon_i \approx \eta_i$. Then since $\epsilon_i([p_1]_\theta, \ldots, [p_n]_\theta) = \eta_i([p_1]_\theta, \ldots, [p_n]_\theta)$, by what we argued:

$$(\sigma(\epsilon_i), \sigma(\eta_i)) \in \theta$$

By the correspondence of K-congruences on the term algebra and theories of K:

$$\begin{aligned}(\sigma(\epsilon_i), \sigma(\eta_i)) \in \theta &\iff \sigma(\epsilon_i) \approx \sigma(\eta_i) \in Cg_K^{T(X)}(\emptyset)\\ &\iff \sigma(\epsilon_i) \approx \sigma(\eta_i) \in Th_K(\emptyset)\\ &\iff K \vDash \sigma(\epsilon_i) \approx \sigma(\eta_i)\end{aligned}$$

So this means that the rule is not passive – contradiction.

Now suppose that $\exists \overline{x} \Phi(\overline{x})$ is some positive existential sentence, and assume that it is true in some nontrivial $A \in K$. Then write $\Phi(x)$ in disjunctive normal form, i.e, $\Phi(\overline{x}) = \phi_1(\overline{x}) \vee ... \vee \phi_n(\overline{x})$. So $\exists \overline{x} \phi_i(\overline{x})$ is true in A for some $1 \leq i \leq n$. In order to complete the proof, it suffices to show that $\exists \overline{x} \phi_i(\overline{x})$ is true in each element of K.

Notice that $\phi_i(\overline{x})$ is of the form:

$$\bigwedge_{j=1}^{m} p_j \approx q_j$$

Where all p_j, q_j are terms. Assume toward contradiction that

$$K \nvDash \exists \overline{x} (\bigwedge_{j=1}^{m} p_j \approx q_j)$$

.

Consequently, let $B \in K$ such that $B \vDash \bigvee_{j=1}^{m} p_j \not\approx q_j$ for every valuation v. Let y_1, y_2 be two variables not occurring in any of the equations in $\phi_i(\overline{x})$. We will

show that $\phi_i(\overline{x}) \to y_1 \approx y_2$ is a passive quasi-equation.
Let σ be an arbitrary substitution; then by the Substitution Lemma, $B, v \vDash \bigvee_{j=1}^{m} \sigma(p_j) \not\approx \sigma(q_j)$ for every valuation v. In particular, this means there exists a valuation v such that $B, v \nvDash \bigwedge_{j=1}^{m} \sigma(p_j) \approx \sigma(q_j)$, i.e. $\sigma(\phi_i)$ is not valid in K. So the quasi-equation $\phi_i \to y_1 \approx y_2$ is passive. By PSC, the quasi-equation is also derivable. But now observe that $\{p_j \approx q_j\}_{j=1}^{m}$ is true in A for some valuation, so $y_1 \approx y_2$ is also true in A, i.e. A is trivial – contradiction.□

Example 11. Heyting algebras are passively structurally complete: given A, B, we use the fact that the HA 2 is a homomorphic image of, and embeds into, every Heyting algebra. By preservation of truth under homomorphisms and embeddings, this shows that A and B satisfy the same positive existential theory. So by the previous result, they are passively structurally complete. (7)

The following uses this characterisation, and will be needed later on:

Corollary 12. Suppose that a quasi-variety K is PSC. Then for each $A, B \in K$, each of them can be mapped homomorphically to an ultrapower of the other.

Proof
We want to show that:

$$S = AtDiag(A) \cup ElDiag(B)$$

is consistent where AtDiag(A) is the set of atomic sentences true in A in the language expanded with constants for all elements of A; ElDiag(B) is the set of all first-order sentences in a similarly expanded language. So take a finite subset Σ, and let ϕ be the conjunction of these sentences. Since the constants of ϕ do not occur in $ElDiag(B)$ we have ϕ is consistent with $ElDiag(B)$ iff $\exists\overline{x}\phi$ is consistent with $ElDiag(B)$. Since this is a positive primitive formula, if B does not satisfy it, neither does A - a contradiction.
Thus by compactness, let C be a model of S. Since C is an elementary extension of B, then we know that it can be embedded in an ultrapower of B, B^I/U. So since $C \vDash AtDiag(A)$ means that A maps homomorphically to C, and so by composition, A maps homomorphically to B^I/U, as intended.□

The notion of structural completeness likewise has some algebraic analogues, first noted by Bergman (1), which we provide for completeness:

Theorem 13. *Let K be a quasi-variety. Then the following are equivalent:*

- *(i) $K = \mathbb{Q}(\boldsymbol{F}_K(\aleph_0))$, i.e, it is generated by the free K-algebra on countably many generators;*
- *(ii) Every proper subquasi-variety of K generates a proper subvariety of the variety generated by K (the latter is equal to $H(K)$).*
- *(iii) K is structurally complete.*

Proof

For (i) to (ii) we proceed by contraposition. Recall that for any quasi-variety J, $H(J)$ is the variety generated by J. Let K' be a subquasi-variety of K, and suppose that $H(K) = H(K')$. Then we have that $K' \subseteq K \subseteq H(K')$, so it is well know (Corollary 11.10, in (3)) that $\mathbf{F}_{H(K')}(\aleph_0) = \mathbf{F}_K(\aleph_0)) = \mathbf{F}_{K'}(\aleph_0)$. But since $\mathbf{F}_{K'}(\aleph_0) \in K'$, we know that $\mathbf{F}_K(\aleph_0) \in K'$, which means the subvariety cannot be proper.

For (ii) to (i) we again go by contraposition: if we suppose that K is not generated as a subquasi-variety by $\mathbf{F}_K(\aleph_0)$, then $\mathbf{Q}(\mathbf{F}_K(\aleph_0))$ is a proper subquasi-variety, but every variety is generated by its free algebra on countably many generators. So $V(\mathbb{Q}(\aleph_0)) = V(K)$, as intended.

From (i) to (iii), suppose that ϕ is a quasi-equation that fails in K. Then w.l.o.g., it fails in $\mathbf{F}_K(\aleph_0)$. So if we take a valuation v on this algebra that witnesses the failure, we note that such a valuation corresponds to a substitution h such that $\mathbf{F}_K(\aleph_0) \vDash h(\phi_i)$ for all i, but $\mathbf{F}_K(\aleph_0) \nvDash h(\psi)$. Since K is generated by this algebra, the same holds for K.

Finally, (iii) to (i): take $\phi = \bigwedge \phi_i \to \psi$ a quasi-equation and suppose that $K \nvDash \phi$. We want to show this fails in $\mathbf{F}_K(\aleph_0)$ as well. Since K is structurally complete, that means that whenever $K \nvDash \phi$, there is a substitutition h such that $K \vDash h(\phi_i)$ for each i but $K \nvDash h(\psi)$. Then in particular $\mathbf{F}_K(\aleph_0) \vDash h(\phi_1), ..., h(\phi_n)$, where these are all equations. Let A be an algebra in K such that $A \nvDash h(\psi)$, with valuation v. Then v is a map from $T(Var)$ to A. Take the mapping of the variables, and extend that, using freeness, to a homomorphism from $\mathbf{F}_K(\aleph_0)$ to A. Then $\mathbf{F}_K(\aleph_0) \nvDash h(\psi)$, under the same valuation. So we obtain that $\mathbf{F}_K(\aleph_0) \nvDash \phi$. This shows that $K = \mathbb{Q}(A)$, as desired.$\square$

The flexibility of this characterisation allows us an abundant source of examples:

Example 14.
- CPC is structurally complete, since the quasi-variety of Boolean algebras is generated by 2, and this is clearly a subalgebra of $\mathbf{F}_{BA}(\aleph_0)$;
- IPC is not structurally complete, as witnessed by Harrop's rule:

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

  So by the previous characterisation, HA, the equivalent algebraic semantics of IPC, is not generated by the free algebra on countably many generators.

The following notion is also of interest, if one wants to preserve structural completeness:

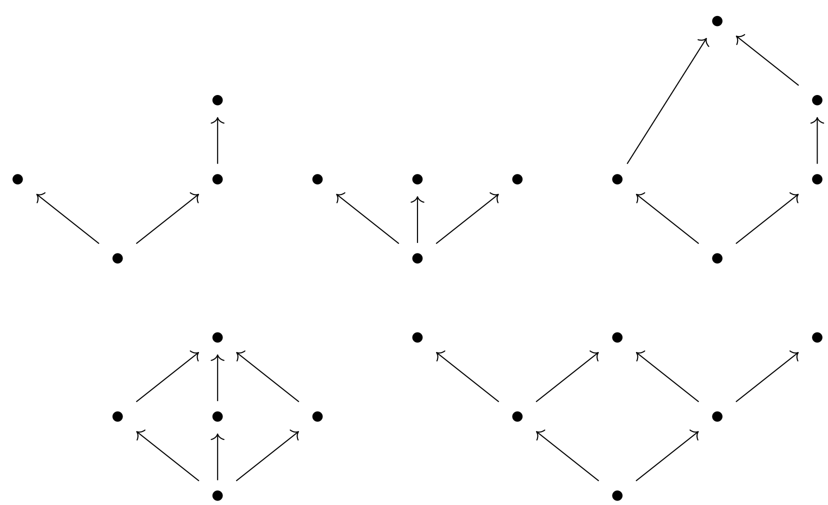
Definition 15. *A quasi-variety K is hereditarily structurally complete (HSC) if all of its subquasi-varieties are structurally complete.*

Similarly, a logic $\vdash$ is hereditarily structurally complete if all of its extensions are structurally complete.

We mention the following without proof (cf.(7), where this is discussed further):

Theorem 16. *Let K be a quasi-variety. Then K is hereditarily structurally complete if and only if whenever K' is a sub-quasivariety of K, then K' is a relative subvariety.*

Remark: Note that if K is a variety with HSC, this statement says that if K' is a subquasivariety, then it is already a variety. For the special cases of Heyting and Modal algebras, the varieties of Hereditary Structurally Complete algebras are particularly well understood. Namely, a variety of Heyting Algebras is HSC if and only if it omits the following five posets: This result was originally proved



by Citkin (4), and a recent proof was given using duality-theoretic methods in (2). A similar result holds for varieties of modal algebras ommitting a larger but finite class of frames (8).

To complete the connection with the previous section, we have the following:

Theorem 17. *Let K be a quasi-variety that is PSC. Then it and all of its subquasivarieties have JEP.*

Proof

Let A and B be two nontrivial members of K. By the previous corollary, let $f : A \to B^I/U$ and $g : B \to A^J/V$ be homomorphisms into ultrapowers. Let e_A and e_B be the diagonal elementary embeddings from A and B to the ultrapowers. So define:

$$(e_A, f) : A \to A^J/V \times B^I/U$$
$$(g, e_B) : B \to A^J/V \times B^I/U$$

defined pointwise in the expected way. Then these are embeddings: if $a_1 \neq a_2$, then since $e_A(a_1) \neq e_A(a_2)$ they must be distinct. It is clear they are homomorphisms. So $A, B \in IS(A^J/V \times B^I/U)$ and $A^J/V \times B^I/U \in \mathbf{Q}(A, B) \subseteq K$, as intended.
Now notice that the PSC condition is hereditary, which justifies the second statement.□

All of our work thus proves the following:

Theorem 18. *The following implications hold between classes of quasi-varieties, and by extent, between algebraizable logics:*

$$HSC \implies SC \implies PSC \implies RP$$

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