Determinacy and Choice in Countable Length Games*

Rodrigo Nicolau Almeida

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Abstract

This report provides a brief state of the art of the art on Determinacy hypothesis for countable length games, with emphasis on the role of the axiom of choice. We summarise the main results proved on this field, noting the contentious role of the Axioms of Dependent Choice (DC) and Countable Choice (AC_ω) in some of these results, namely those of Martin and Woodin. In particular we dispute a claim made by Martin as to the status of a given theorem relating equivalences of long games following from ZF. We discuss some of the recent developments in this field, before showing that neither DC or AC_ω can be necessary to obtain Martin or Woodin's results. Finally, we conclude by connecting these issues to some currently open problems in the inner model theory of determinacy.
Keywords— Determinacy, Countable length games, Real Determinacy, Countable Choice, Dependent Choice

Contents

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1 Introduction

Axioms of Determinacy are deeply related to many areas of set theory, from definability results, to combinatorial properties, large cardinals and choice principles. Their flexibility has motivated research looking not only at AD, but at generalisations to larger types of games, and games with different kinds of objects. Two instances of such generalisation are given by AD^α, the principle stating that games of length α over ω are determined, and $\mathsf{AD}_\mathbb{R}$, stating that games of length ω played on the reals are determined.
A series of results proved throughout the 20th century, helped settle the way that such hypotheses relate to each other, encapsulated in the following theorem:

Theorem 1. *(Zermelo, Mycielski, Blass, Martin, Woodin) Assume* $\mathsf{ZF} + \mathsf{DC}$ *then the following holds:*

1. *($\alpha < \omega$)* AD^α *holds;*
2. *($\omega \leq \alpha < \omega \cdot 2$) All* AD^α *are equivalent, and* $\mathsf{AD}^\mathsf{n} \not\Rightarrow \mathsf{AD}$ *for all n;*
3. *($\omega \cdot 2 \leq \alpha < \omega_1$) All* AD^α *are equivalent, equivalent to* $\mathsf{AD}_\mathbb{R}$*, and* $\mathsf{AD} \not\Rightarrow \mathsf{AD}^{\omega+\omega}$*;*
4. *($\omega_1 \leq \alpha$)* AD^α *is inconsistent.*

It is important to note that some of the results involved in establishing Theorem 1, namely those related to point (3), were for a long time, and in some cases are still, unpublished. Whilst (1),(2) and (4) are known to follow from ZF, there appears to be some uncertainty as to whether (3) is currently a theorem of ZF, $\mathsf{ZF} + \mathsf{AC}_\omega$ or $\mathsf{ZF} + \mathsf{DC}$, in light of a claim made by Martin [Mar21] that would imply Theorem 1 was indeed a theorem of ZF.
In this report, we will look at the current status of Theorem 1, as it is reflected in the literature. In particular, we will discuss the role of DC and AC_ω in forming the picture given by Theorem 1. In this we dispute Martin's claim as not being grounded on the literature, as well as what can be said in light of the available proofs and methods. We then explore what can be said in their absence and to what extent such principles can be necessary.
Given the focus on trying to provide a coherent and clear description of the current state of such results, we will dispense with most proofs, providing good sources where these can be found.

2 Notation and Preliminaries

Throughout we use Greek letters to denote ordinals. As is standard, we use $\mathbb{R}$ to denote ω^ω, and refer to elements of such a set as "reals". Unless otherwise specified, we work in ZF. In this context, given two sets X and Y, we write $|X| \leq |Y|$ to mean that there is an injection from X to Y. This is relevant, since the absence of choice means precisely that not all sets need to be bijective

with a cardinal.
Given α and $X \subseteq Y^\alpha$, we define the α-game on Y with payoff set X, denoted by $G^\alpha_Y(X)$ as usual: both players take turns playing integers, and form a sequence y of length α, and player I wins if and only if $y \in X$. If Y and α are understood, we denote this simply by $G(X)$. The usual definitions of strategies and winning strategies apply, and we say a given game is determined if one of the two players has a winning strategy.
If $\beta < \alpha$, $y \in X^\beta$ and there is some z which extends y such that $z \in X^\alpha$, we say that y is the position at β. We denote by $y^\frown x$ the concatenation of the moves such that $y^\frown x = z$. If $x, y \in X^\beta$, we denote by $x * y$ the play of the game obtained by PI playing according to x, and PII playing according to y. Finally, if $\sigma : X^{<\beta} \to X$ is a strategy, then we let $\sigma * x$, respectively, $x * \sigma$, denote the play obtained by PI (resp. PII) playing σ, and PII (resp. PI) playing according to x. Given $x \in X^\beta$ denote by x_I, respectively x_{II}, the sequence of moves at positions played by PI or PII.
Given any set Y, and ordinal α, we write $\mathsf{AD}^\alpha_\mathsf{Y}$ to mean "Every α-game on Y is determined". When we omit the superscript, it is implied that $\alpha = \omega$, and when we omit the subscript it is implied that $Y = \omega$.
The following lemma is standard and allows us to relate determinacy of different sets and of different sizes of games:

Lemma 2. *Let $\alpha < \beta$ be arbitrary, and $f : X \to Y$ be an injection. Then $\mathsf{AD}^\beta_\mathsf{X}$ implies AD^α_Y.*

Given any two sets, X and Y, we denote by $\mathsf{AC_X(Y)}$ the choice principle that states that any set of nonempty subsets of Y indexed by elements of X,

$$\{X_z \subseteq Y : z \in X\}$$

admits a choice function. We write AC_ω for the principle stating that for all Y, $\mathsf{AC}_\omega\mathsf{(Y)}$ holds. Moreover, given a cardinal κ and set X we denote by $\mathsf{DC}_\kappa\mathsf{(X)}$ the principle saying that for each relation $R \subseteq X \times X$, such that for each $x \in X$ there is a $y \in X$ where $(x, y) \in R$, there exists a sequence $(x_\beta)_{\beta<\kappa}$ such that,

$$(x_\beta, x_{\beta+1}) \in R$$

for each $\beta < \kappa$. We denote by DC the principle stating that for all X, $\mathsf{DC}_\omega\mathsf{(X)}$ holds. We also have the following lemma:

Lemma 3. *Let $f : X \to X'$ and $g : Y \to Y'$ be injections. Then: $\mathsf{AC_{X'}(Y')}$ implies $AC_X(Y)$*

We will use the above lemmas freely in the sequel.

3 Countable games below ω^2

The case of the determinacy of finite games has been known for a long time, tracing its roots to the work of Zermelo [Zer13], though the original proof was less general than what we have below (see [SW01] for a detailed account):

Theorem 4. *[Zer13] Let $A \subseteq \omega^n$. Then $G(A)$ is determined. Thus,* $\mathsf{AD^n}$ *holds.*

Proof
See [Mac21, Theorem 2.1] for a proof of this statement.□

The case for ω is inconsistent with ZFC, motivating it as an alternative to AC – something which was realised from its introduction by Mycielski and Steinhaus [MS62]. This can be seen with a slightly more precise fact:

Theorem 5. *Assume* AD*. Then $|\omega_1| \nleq |\omega^\omega|$.*

Proof
This was originally shown in [MS64]. A proof can be found in [Kan09, Proposition 27.11].□

As an easy corollary, we have that $\mathsf{AD^n}$ cannot imply AD, since the former are theorems of ZF, and thus hold in any model of ZFC. Moreover, Theorem 5 can be used to obtain an upper bound on the consistency of AD^α. The following was also shown by Mycielski in the same paper as cited above:

Corollary 6. AD^{ω_1} *is inconsistent with* ZF*.*

Proof
Assume towards a contradiction that AD^{ω_1} holds. Then consider the following game: first, Player I fixes a given $\alpha < \omega_1$, and for α many steps, he plays a fixed element, say 1; during that time Player II's movements do not matter. Then after α many moves, say at position x_α, Player I begins playing only 0, whilst Player II plays an injective enumeration of α in ω many moves from that position onwards. We say that Player II wins if she is able to produce such an enumeration.
Now, by determinacy, we have that one of the two players must have a winning strategy. But it must be Player II who has a winning strategy, since all ordinals considered are countable. So using the strategy τ we can construct an injective enumeration of ω_1 into $\mathbb{R}$: we simply look at the response of Player II to Player I playing ordinal α. This is injective, since given different order types, the resulting elements of $\mathbb{R}$ will likewise be distinct. So this is a contradiction to Theorem 5.□

Turning upwards, we can see that AD implies the determinacy of all countable games of length α below the next limit ordinal. We take special note of this proof, since the core of its argument can be used to prove similar implications for higher ordinals:

Lemma 7. *For every n,* $\mathsf{AD} \implies \mathsf{AD}^{\omega+\mathsf{n}}$.

Proof
The proof is by induction. For the successor case, assume $\mathsf{AD}^{\omega+\mathsf{n}}$, and let

$X \subseteq \omega^{\omega+(n+1)}$. For each $x \in \omega^n$, consider the following payoff set:

$$A_x \coloneqq \{y : x^\frown y \in X\}$$

Note that by Zermelo's theorem, A_x is determined. Also define:

$$X' \coloneqq \{x \in \omega^n : A_x \text{ is won by PI }\}$$

By hypothesis, X' is determined. Note that if either player has a winning strategy in A_x for some x, we can in fact choose such a strategy: recall that strategies for this game are of the form $\sigma_x : \omega^{<n} \to \omega$; so ordering $\omega^{<n}$ lexicographically, we have a well-order. We can then use that well-order together with the well-order of ω to pick the least strategy for PI. For each such x, let σ_x be one such strategy.
First assume that PI has a winning strategy τ in X'. We construct a winning strategy for him by letting him play according to τ, and then, for each y played by PII, letting $x = \tau * y$, let PI play according go σ_x. By definition we see that this is winning for PI. If instead PII has a winning strategy, we note that by determinacy of X':

$$\mathbb{R} - X' = \{x \in \omega^n : A_x \text{ is won by PII}\}$$

So the above arguments follow in the same way for PII.□

Thus, we obtain (2) in Theorem 1. $\mathsf{AD}^{\omega+\omega}$, turns out to provably be distinct from AD. To see that, we begin by noting that a play of a game on $\omega+\omega$ can be identified with a pair of real numbers (x, y).

Definition 8. *Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be any relation. We say that $R^* \subseteq R$ is a **uniformisation** of R if $dom(R) = dom(R^*)$, and R^* is a function.*
*We say that $\mathbb{R}$ has the **Uniformisation Principle** (shortened to* Unif*) if every relation on $\mathbb{R} \times \mathbb{R}$ can be uniformised.*

Lemma 9. *The following are equivalent:*

1. $\mathsf{AD}^{\omega+\omega}$
2. $\mathsf{AD} + \mathsf{Unif}$
3. $\mathsf{AD} + \mathsf{AC}_{\mathbb{R}}(\mathbb{R})$

Proof
First we prove that (1) implies (2). Let R be any subset of $\mathbb{R} \times \mathbb{R}$. Consider the following game G over $\omega+\omega$: given an element $x \in \omega^{\omega+\omega}$, write $x = (x_1, x_2)$, where x_1 and x_2 are ω-length sequences. Then Player I wins a play of G if, $(x_1)_I$ is in the domain of R, but $(x_2)_{II}$ is not in the range of R. By determinacy, one of the two players has a strategy. But notice that Player I cannot have a winning strategy, since any element in the domain of R has a corresponding element z in the range. So PII has a winning strategy τ. We can then use τ to define the

uniformisation of R.
The fact that (2) implies (3) is straightforward. To see that (3) implies (1), take $A \subseteq \omega^{\omega+\omega}$. Similarly to Lemma 7, for each $x \in \omega^\omega$, consider the following set:

$$A_x \coloneqq \{y : x^\frown y \in A\}$$

And once more, define:

$$A' \coloneqq \{x \in \omega^\omega : A_x \text{ is won by PI }\}$$

We have that each A' is determined by assuming AD. Note moreover that using $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$ we can pick winning strategies for either player since any strategy $\sigma : \omega^{<\omega} \to \omega$ can be coded as a real number. We can then conclude the argument the same way as in Lemma 7.□

In the same spirit as lemma 7, we also have the following:

Lemma 10. *For every n, $\mathsf{AD}^{\omega+\omega} \implies \mathsf{AD}^{\omega\cdot\mathsf{n}}$*

Proof
By Lemma 9, we have that $\mathsf{AD}^{\omega+\omega}$ implies $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$. The argument follows by induction on n, using the same technique of splitting the game into two parts. In more detail, assume that $\mathsf{AD}^{\omega\cdot\mathsf{n}}$ holds for $n \geq 2$. Consider $G(A)$ where $A \subseteq \omega^{\omega\cdot(n+1)}$. Let $A_x = \{y : x^\frown y \in A\}$ and $A' = \{x \in \omega^{\omega\cdot n} : A_x$ is won by $PI\}$. We have that A' is determined by the induction hypothesis. By $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$, as in Lemma 9, we can pick a strategy for either player. We then conclude the argument the same way as before.□

Having these equivalences, we can see that AD^α for $\omega \cdot 2 \leq \alpha < \omega^2$ is provably equivalent to a weak choice principle on the reals. Solovay showed that we can separate such a principle from AD, using $\mathsf{L}(\mathbb{R})$ and an argument on ordinal definability:

Theorem 11. *Assume $\mathsf{V} = \mathsf{L}(\mathbb{R})$, and there is no well-ordering of the reals. Then Unif fails.*

Proof
A proof can be found in [Kan09, Proposition 27.16].□

4 Countable games above ω^2 and below ω_1

Once we get to ω^2, a play of such games looks like an ω-sequence of ω-sequences - something which, if we think about the latter sequences as reals, looks remarkably like an element of $\mathbb{R}^\omega$. The following theorem of Blass, credited by him to Mycielski, validates this intuition.

Theorem 12. *([Bla75]) Over ZF, $\mathsf{AD}_{\mathbb{R}}$ is equivalent to AD^{ω^2}.*

In fact, as noted by Blass himself, by making minor modifications to the original proof of the above theorem this generalises to all countable limit ordinals.

Theorem 13. *Let α be a countable limit ordinal. Then $\mathsf{AD}^{\alpha}_{\mathbb{R}}$ is equivalent to $\mathsf{AD}^{\omega\cdot\alpha}$.*

As mentioned in [LR01], given how natural this equivalence appears, one might reasonably expect that such a principle could be separated from $\mathsf{AD}^{\omega+\omega}$. However, under DC, all of these principles are shown to be equivalent. To see why, it will be useful to import another technical concept from descriptive set theory, which helps us to gauge the power of these principles.

Definition 14. *Let $A \subseteq \mathbb{R}$ be a subset. Suppose that for some ordinal δ, there is a tree T on $\omega \times \delta$ such that $p[T] = A$. Then we say that A is a **Suslin subset**. Scales denotes the following statement: for all subsets $A \subseteq \mathbb{R}$, A is Suslin.*

For a justification of the name Scales, see [Mos09]. The following theorem is due to Martin and Woodin, and was proved in 1980's, although it was published only in 2020:

Theorem 15. *[Mar21, Theorem 10] Assume $\mathsf{AD} + \mathsf{AC}_\omega + \mathsf{Scales}$. Then for every $\alpha < \omega_1$, AD^{α}.*

In the paper [Mar21], Martin mentions that Woodin has found a proof which removes the hypothesis AC_ω, and gives a brief argument communicated to him by Woodin, on page 96. Nevertheless, this result remains unpublished, which hinders further analysis.
The key lemma that Martin uses to establish (3) in Theorem 1, however, is the following result, again by Woodin:

Theorem 16. *(Woodin, unpublished) Assume $\mathsf{ZF}+\mathsf{DC}$. Then $\mathsf{AD}+\mathsf{Unif}$ implies Scales.*

Proof
See [Larng, Theorem 12.3.1] for a proof of this result.□

Putting this together with Martin's and Blass' theorems, we get the following, which is exactly (3) in the main result:

Corollary 17. *Assume $\mathsf{ZF} + \mathsf{DC}$. Then for every $\omega \cdot 2 \leq \alpha < \omega_1$, we have AD^{α} if and only if $\mathsf{AD}_{\mathbb{R}}$.*

Proof
By Lemma 9, $\mathsf{AD}^{\omega+\omega}$ implies Unif; putting together Woodin and Martin's results, this implies AD^{α} for every α. So letting $\alpha = \omega^2$ and using Blass' theorem, we have $\mathsf{AD}_{\mathbb{R}}$. The other direction uses the same argument, noting that $\mathsf{AD}_{\mathbb{R}}$ implies $\mathsf{AD}^{\omega+\omega}$ by Blass' theorem.□

Despite the assumptions we state here, however, Martin quotes Theorem 16 [Mar21, Theorem 4], under the assumptions of ZF. When appealing to Woodin's proof, he mentions Corollary 17 again under ZF. There is further confusion in the matter, since in the initial discussion of the paper [Mar21], Martin comments that Woodin has found a proof of Theorem 15 not using countable choice. On the status of this theorem we cannot say much, since Woodin's proof, as told by Martin. remains unpublished.
This is unusual, since Theorem 16 appears everywhere else we could find under the same base theory as we have stated it: in [Kan09, Theorem 32.23], [Woo10, Theorem 9.12], [Sol21, Theorem 13.1], [Ket10, Corollary 5.12], [Larng, Theorem 12.3.1], and [LR01, Theorem 2], we see the proof being underscored as a result of $\mathsf{ZF} + \mathsf{DC}$. In so far as we know, the only available proof of this result lies in as of yet unpublished work of Larson [Larng]. In the latter work, looking at the proof, it is not clear to us precisely how much choice is being used given the length and difficulty of the result, though Larson comments on multiple uses of DC throughout. As such, Martin's claim that Corollary 17 can be derived from ZF does not seem to be supported by the literature.

5 The Status of Choice in Determinacy of Long Games

The discussion in the previous section raises some questions about the best base theory in which one can obtain (3) in Theorem 1. Given the current status of these results, and the possibility that DC or AC_ω could be substantial hypothesis in establishing them, there seems to be a possibility for various determinacy principles above ω^2, that would continue to get stronger, or for $\omega + \omega$ to be separated from ω^2, in the absence of DC; alternatively, the relevant results might be strengthened to not depend on any choice whatsoever.

A final possibility would be that DC or AC_ω are in fact integral to the theory, in so far as they are equivalent to some of the results of Martin and Woodin we have been discussing. In this section we will show this is not the case.

First, we point out that both DC and AC_ω are known to be independent of $\mathsf{ZF} + \mathsf{AD}$, as well as $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$. This is obtained by two results of Kechris and Woodin. We also note that the first is a refinement of a result by Solovay [Sol21, Corollary 12.7].

Theorem 18. $\mathsf{Con}(\mathsf{ZF} + \mathsf{AD}_\mathbb{R}) \implies \mathsf{Con}(\mathsf{ZF} + \mathsf{AD}_\mathbb{R} + \neg\mathsf{AC}_\omega)$.

Theorem 19. *Assume* $\mathsf{ZF}+\mathsf{AD}$ *and* $\mathsf{V} = \mathsf{L}(\mathbb{R})$. *Then* DC *holds. Thus,* $\mathsf{Con}(\mathsf{ZF}+\mathsf{AD}) \implies \mathsf{Con}(\mathsf{ZF} + \mathsf{AD} + \mathsf{DC})$.

For a reference without proof, see [Kan09, Theorem 30.29, Theorem 30.30]. We will come back to these questions in the final section. A consequence of

these results is that DC appears as a legitimate hypothesis in proving results in contexts of Determinacy.
What we wish to show in turn is that neither of the implications

$$\mathsf{AD} + \mathsf{Unif} \implies \mathsf{Scales}$$
$$\mathsf{AD} + \mathsf{Scales} \implies \mathsf{AD}^{\alpha}$$

can imply DC. As it turns out, this is simply a consequence of a well-known fact from the theory of forcing:

Theorem 20. *If* ZFC *is consistent, then there is a model of* $\mathsf{ZF} + \neg\mathsf{DC} +$ "*The reals are wellordered*".

Proof
See [Kar12, pp.9] for a proof of this fact.□

Corollary 21. *Assuming* $\mathsf{Con}(\mathsf{ZF})$*, we have* $\mathsf{Con}(\mathsf{ZF} + \neg\mathsf{AD} + \neg\mathsf{AC}_{\omega})$*. Thus the following hold:*

- $\mathsf{ZF} \nvdash (\mathsf{AD} + \mathsf{Unif} \implies \mathsf{Scales}) \implies \mathsf{DC}$
- $\mathsf{ZF} \nvdash (\mathsf{AD} + \mathsf{Scales} \implies \forall\alpha < \omega_1, \mathsf{AD}^{\alpha}) \implies \mathsf{DC}$

Proof
By Theorem 5, we know that if there is an injection from ω_1 to the reals, then AD fails. By Theorem 20, we have that assuming the consistency of ZF, we obtain the consistency of $\mathsf{ZF} + \neg\mathsf{AD} + \neg\mathsf{AC}_{\omega}$. So we have a model M such that:

$$M \vDash \mathsf{AD} + \mathsf{Unif} \implies \mathsf{Scales}$$

Since this holds vacuously. Moreover, because $\neg\mathsf{AC}_{\omega} \implies \neg\mathsf{DC}$, we have that:

$$M \nvDash (\mathsf{AD} + \mathsf{Unif} \implies \mathsf{Scales}) \implies \mathsf{DC}$$

And similarly for the second statement.□

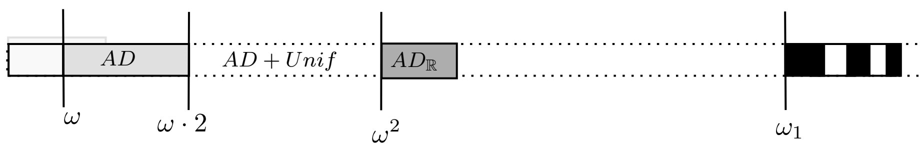
We note that the results in Corollary 21 do not tell us in any way whether any of the two implications – that is, $\mathsf{AD} + \mathsf{Unif} \implies \mathsf{Scales}$ and $\mathsf{AD} + \mathsf{Scales} \implies \forall\alpha < \omega_1 \mathsf{AD}^{\alpha}$ – is a theorem of ZF. What it does make clear is that the present status of such results cannot reduce to the role of DC. This is in some sense to be expected, since we are comparing very global principles - such as DC and AC_{ω} - with more local principles, which talk about properties of the reals. Nevertheless, it makes it clear that further questions about the status of countable length games can be fruitfully answered.

6 Moving Forward

In this section, we connect the issues discussed in the previous section, on the status of choice principles in establishing Theorem 1, with some open problems

related to the inner model theory of Determinacy.
To start, let us summarise the picture we have found so far. Visualising this in a diagram, akin to what we find in [LR01], in the context of $\mathsf{ZF} + \mathsf{DC}$, the equivalence of countable length determinacy principles looks as follows:

Alternatively, in light of the results of the previous sections, in the absence of any choice, we have the following:



As noted in the last section, the situation cannot already be settled: either there is a proof of Theorem 1 in ZF without either countable or dependent choice, or some of these principles must come apart in ZF. We will first look at some problems that would imply the former. To start we need some definitions.

Definition 22. *Let* AD^+ *denote the following principle:*

- *(1)* AD
- *(2)* $\mathsf{DC}_{\mathbb{R}}$
- *(3) Every subset of the reals is* ∞*-Borel*
- *(4) Ordinal determinacy: for every* $\gamma < \Theta$*, and every continuous function* $\pi : \lambda^\omega \to \mathbb{R}$*, where the former has the discrete topology on each factor, then for each* $A \subseteq \mathbb{R}$*,* $\pi^{-1}[A]$ *is determined.*

For a definition of ∞-Borel, and a general discussion of these principles, check [Larng]. This principle was introduced by Woodin, as an attractive principle from a descriptive set theoretic point of view. As mentioned by Larsson, it is currently unknown whether AD implies any of the above properties. It is likewise unknown whether $\mathsf{AD}_{\mathbb{R}}$ implies Ordinal determinacy. However, the following, due to Woodin, is known, and its proof can again be found in Larsson's book:

Theorem 23. *[Larng, [Theorem 0.1.2] Assume* AD*+Ordinal determinacy+*Unif*. Then* Scales *holds.*

Denote by # the result "$\mathsf{AD} + \mathsf{Unif} \implies \mathsf{Scales}$", and by ## the result "$\mathsf{AD} + \mathsf{Scales} \implies \mathsf{AD}^\alpha$". As we noted in Theorem 16, Woodin proved # under

the hypothesis of $\mathsf{ZF}+\mathsf{DC}$. Likewise, Theorem 15 can be stated as saying that ## has a proof in the context of $\mathsf{ZF}+\mathsf{AC}_\omega$, and according to Martin, Woodin has a proof in the context of ZF. Finally, notice that in light of all our results # and ## together suffice to settle point (3) in Theorem 1.
In light of Theorem 23, # is also a straightforward consequence of some hypotheses on AD^+

Corollary 24. *Assume* ZF*. Then:*

1. *Assuming* $\mathsf{AD}=\mathsf{AD}^+$*, we have #; further assuming* AC_ω *we have ##.*
2. *Assuming* $\mathsf{AD}_\mathbb{R}$ *implies* AD^+ *and* AC_ω *we have ##.*

Proof

For point (1), note that $\mathsf{AD}=\mathsf{AD}^+$ means that ordinal determinacy already follows from determinacy. So in light of Theorem 23 this implies Scales.
Now assuming $\mathsf{AD}^{\omega+\omega}$, we have by Lemma 9 that $\mathsf{AD}+\mathsf{Unif}$ holds. Thus by the present hypothesis, this gives us $\mathsf{AD}+\mathsf{Scales}$. Assuming AC_ω by Theorem 15 gives us AD^α for all α.
As for point (2), note that if $\mathsf{AD}_\mathbb{R}$ implies AD^+, since by Blass' theorem $\mathsf{AD}_\mathbb{R}$ implies $\mathsf{AD}+\mathsf{Unif}$, by the same arguments as above, we have AD^α for all α.□

The expectations about Theorem 1 were not always aligned in this direction. Indeed, Solovay seemed to think that stronger principles than AD^{ω_2} should exist, and made reference to this in [Sol78, Page 183], by conjecturing that $\mathsf{AD}^{\omega^3}\vdash \mathsf{Con}(\mathsf{AD}^{\omega^2}+\Theta$ is regular). Indeed, this would, by Gödel's theorem, give us a proof that AD^{ω^2} and AD^{ω^3} are distinct, by giving us a model of:

$$\mathsf{ZF}+\neg\mathsf{DC}+\neg\mathsf{AD}^{\omega_3}+\mathsf{AD}^{\omega_2}$$

Still in keeping with this possibility, but looking to smaller principles, one could try to separate $\mathsf{AD}^{\omega\cdot 2}$ from $\mathsf{AD}_\mathbb{R}$. The former is, as we mentioned, equivalent to $\mathsf{AD}+\mathsf{Unif}$. So we would be looking for models of

$$\mathsf{ZF}+\mathsf{AD}+\mathsf{Unif}+\neg\mathsf{AD}_\mathbb{R}$$

Here, we would be playing with the original intuition that $\mathsf{AD}+\mathsf{Unif}$ is strictly weaker than $\mathsf{AD}_\mathbb{R}$ (see [LR01] or [Kan09]), and that in fact it is choice that provides enough strength to equate the two principles.
Nevertheless, the current inner model theory of $\mathsf{AD}_\mathbb{R}$ seems to not have models of the right shape to produce such either analysis. One development that could be exploited in this direction lies in a model where AD^+ and $\neg\mathsf{AD}_\mathbb{R}$ hold, known by a result of Woodin [Wod06]. It seems to be open whether a similar model could be made where additionally Unif holds.□

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