Date :

Assignment Sheet-1

* Problem 1:

⇒ Based on Bayesian theorem, It is possible to find that the both balls in the box are red. We can assume that,

$$P(RR|RRR) = \frac{P(RRR|RR)\,P(RR)}{P(RRR)}$$

We have two balls Red and White, so the possible outcomes are, $P(RR) = P(RW) = P(WR) = P(WW)$

$$= \frac{1}{4} = 0.25$$

Where, $P(RRR/RR) = 1$, Because the probabilities of drawing 3 Red ball out of two Red balls in the box is always 1.

$$P(RRR) = P(RRR|RW)\cdot P(RW) + P(RRR|WR)\cdot P(WR) + P(RRR|RR)\cdot P(RR)$$

$$= \left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}\right) + \left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}\right) + \left(1\times\frac{1}{4}\right)$$

$$= \frac{1}{32} + \frac{1}{32} + \frac{1}{4} = \frac{10}{32} = \frac{5}{16}$$

Where, $P(RRR/WW)\cdot P(WW) = 0$

$$\therefore P(RR/RRR) = \frac{P(RRR|RR)\,P(RR)}{P(RRR)} = \frac{1\times\frac{1}{4}}{\frac{5}{16}} = \frac{4}{5}.$$

Date :

* Problem 2:

1 ⇒ According to the Gaussian PDF,

$$N(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$p(x|\mu,\sigma^2) = \prod_{n=1}^{N} N(x_n|\mu,\sigma^2)$$

$$\Rightarrow \ln p(x|\mu,\sigma^2) = -\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2 - N/2 \ln\sigma^2 - N/2 \ln(2\pi)$$

So, $\frac{dll}{d\mu} = 0$, $\frac{dll}{d\sigma^2} = 0$

$$\frac{dll}{d\mu} = -N/2 \; dll/d\mu\left(\ln(2\pi\sigma^2)\right) - \frac{1}{2\sigma^2} \; dll/d\mu\left(\sum_{n=1}^{N}(x_n-\mu)^2\right)$$

$$= -\frac{1}{2\sigma^2} \; dll/d\mu\left(\sum_{n=1}^{N}(x_n-\mu)^2\right)$$

$$= -2 \times -\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)$$

$$= \sum_{n=1}^{N}\frac{x_n-\mu}{\sigma^2}$$

$$\therefore \sum_{n=1}^{N}\frac{x_n-\mu_{ml}}{\sigma^2} = 0$$

$$\Rightarrow \sum_{n=1}^{N} x_n - \mu_{ml} = 0$$

Date :

$$\Rightarrow \sum_{n=1}^{N} x_n - N\mu_{ml} = 0$$

$$\therefore \mu_{ml} = \frac{1}{N}\sum_{n=1}^{N} x_n$$

For σ^2 :-

$$dLL/d\sigma^2 = -N/2 \, dLL/d\sigma^2 \left(\ln(2\pi\sigma^2)\right) - dLL/d\sigma^2 \left(\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2\right)$$

$$= -N/2\left(\ln(2\pi)\right) - N/2 \, dLL/d\sigma^2 \left(\ln\sigma^2\right) - dLL/d\sigma^2\left(\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2\right)$$

$$= -N/2\sigma^2 + \frac{2}{2\sigma^4}\sum_{n=1}^{N}(x_n-\mu)^2$$

$$\therefore -\frac{N}{2\sigma_{ml}^2} + \frac{1}{2\sigma_{ml}^4}\sum_{n=1}^{N}(x_n-\mu)^2 = 0$$

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ml})^2.$$

Date :

*Problem 3:

⇒ Given that. $E[x] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x\, dx = \mu$

$$E[x^2] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x^2\, dx = \mu^2 + \sigma^2$$

According to the question,

When, $n=m$

$$E[x_n x_m] = \mu^2 + I_{nm}\sigma^2$$

$$= \mu^2 + \sigma^2 \quad [\text{by Comparing } I_{nm} = 1]$$

Again, when $n \neq m$

$$E[x_n x_m] = \mu^2 + I_{nm}\sigma^2$$

$$= \mu^2 \quad [\text{by Comparing } I_{nm} = 0]$$

With the knowledge that x_n in Identical and Independdly distributed and that it is also obeys the Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ is relatively Straightforward.

Date :

$$E[\mu_{ML}] = E\left[\frac{1}{N}\sum_{n=1}^{N} x_n\right]$$
$$= \frac{1}{N} E\left[\sum_{n=1}^{N} x_n\right]$$
$$= \frac{1}{N} \times N\, E[x_n]$$
$$= E[x_n]$$
$$= \mu.$$

We know that, when $n=m$

$$E[x_n x_m] = \mu^2 + I_{nm}\sigma^2$$
$$= \mu^2 + \sigma^2 \quad [\text{by Comparing } I_{nm} = 1]$$

Again, when $n \neq m$

$$E[x_n x_m] = \mu^2 + I_{nm}\sigma^2$$
$$= \mu^2 \quad [\text{by Comparing } I_{nm} = 0]$$

By maximizing the above-mentioned eqn with the respect to σ^2, we obtain the the maximum likelihood solution for the variance in the form

$$\sigma^2_{ML} = \frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2 \quad \dots (1)$$

Date :.................................

Considering the eq^{n} ① for $E[\sigma^2_{ML}]$. We can obtain

$$E[\sigma^2_{ML}] = E\left[\frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2\right]$$

$$= \frac{1}{N} E\left[\sum_{n=1}^{N}(x_n - \mu_{ML})^2\right]$$

$$= \frac{1}{N} E\left[\sum_{n=1}^{N}(x_n^2 - 2x_n\mu_{ML} + \mu^2_{ML})\right]$$

$$= \frac{1}{N} E\left[\sum_{n=1}^{N} x_n^2\right] - \frac{1}{N} E\left[\sum_{n=1}^{N} 2x_n\mu_{ML}\right] + \frac{1}{N} E\left[\sum_{n=1}^{N} \mu^2_{ML}\right]$$

$$= \mu^2 + \sigma^2 - 2/N\, E\left[\sum_{n=1}^{N} x_n \left(1/N \sum_{n=1}^{N} x_n\right)\right] + \frac{1}{N} \times N\, E[\mu^2_{ML}]$$

$$= \mu^2 + \sigma^2 - 2/N^2\, E\left[\sum_{n=1}^{N} x_n \left(\sum_{n=1}^{N} x_n\right)\right] + E[\mu^2_{ML}]$$

$$= \mu^2 + \sigma^2 - 2/N^2\, E\left[\sum_{n=1}^{N} x_n \left(\sum_{n=1}^{N} x_n\right)\right] + E\left[\left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)^2\right]$$

$$= \mu^2 + \sigma^2 - 2/N^2\, E\left[\left(\sum_{n=1}^{N} x_n\right)^2\right] + \frac{1}{N^2} E\left[\left(\sum_{n=1}^{N} x_n\right)^2\right]$$

$$= \mu^2 + \sigma^2 - \frac{1}{N^2} E\left[\left(\sum_{n=1}^{N} x_n\right)\right] - \frac{1}{N^2}\left[\left(\sum_{n=1}^{N} x_n\right)\right] + \frac{1}{N^2} E\left[\left(\sum_{n=1}^{N} x_n\right)^2\right]$$

$$= \mu^2 + \sigma^2 - \frac{1}{N^2} E\left[\left(\sum_{n=1}^{N} x_n\right)^2\right]$$

$$= \mu^2 + \sigma^2 - 1/N^2\left[N(N\mu^2 + \sigma^2)\right]$$ $\therefore$ Hence, $E[\sigma^2_{ML}] = \left(\frac{N-1}{N}\right)\sigma^2$

Date :

*Problem 4:

$$\Rightarrow \quad P(x) = \begin{cases} \lambda \exp(-\lambda x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Considering all the points.

$$P(\lambda | x_1 \ldots\ldots x_n) = \prod_{n=1}^{N} \lambda \exp(-\lambda x_n)$$

Simplifying the Right hand side We obtain,

$$= \lambda^N \exp\left(-\lambda \sum_{n=1}^{N} x_n\right)$$

Now,

$$\ln(P(\lambda | x_1, \ldots x_n)) = N \ln(\lambda) - \ln \lambda \sum_{n=1}^{N} x_n$$

$$\Rightarrow \frac{d \ln(P(\lambda | x_1 \ldots x_n))}{d\lambda} = N \times 1/\lambda - \sum_{n=1}^{N} x_n$$

$$\Rightarrow 0 = N/\lambda - \sum_{n=1}^{N} x_n$$

$$\Rightarrow N/\lambda = \sum_{n=1}^{N} x_n$$

$$\therefore \lambda_{ML} = \frac{N}{\sum_{n=1}^{N} x_n}$$

Date :

*Problem 5:

⇒ According to the question

$$x \sim N(\mu, \Sigma)$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

We know that,

$$N(x|\mu,\Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right]$$

The equation for the pdf is as follows,

$$P(x_1,x_2) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right] \quad \ldots\ldots \text{eqn} ①$$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2$$

$$= \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$\Rightarrow |\Sigma|^{1/2} = \sigma_1\sigma_2\sqrt{(1-\rho^2)}$$

$$\Rightarrow \Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$x-\mu = \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$\Rightarrow (x-\mu)^T = \begin{bmatrix}(x_1-\mu_1) & (x_2-\mu_2)\end{bmatrix}$$

$$\Rightarrow \Sigma^{-1}\cdot(x-\mu) = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{bmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2\end{bmatrix}\begin{bmatrix}x_1-\mu_1 \\ x_2-\mu_2\end{bmatrix}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{bmatrix}\sigma_2^2(x_1-\mu_1)-\rho\sigma_1\sigma_2(x_2-\mu_2) \\ -\rho\sigma_1\sigma_2(x_1-\mu_1)+\sigma_1^2(x_2-\mu_2)\end{bmatrix}$$

$$\Rightarrow (x-\mu)^T\Sigma^{-1}(x-\mu) = \begin{bmatrix}(x_1-\mu_1) & (x_2-\mu_2)\end{bmatrix}\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{bmatrix}\sigma_2^2(x_1-\mu_1)-\rho\sigma_1\sigma_2(x_2-\mu_2) \\ \sigma_1^2(x_2-\mu_2)-\rho\sigma_1\sigma_2(x_1-\mu_1)\end{bmatrix}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\Big[\sigma_2^2(x_1-\mu_1)^2-\rho\sigma_1\sigma_2(x_2-\mu_2)(x_1-\mu_1)+\sigma_1^2(x_2-\mu_2)^2-\rho\sigma_1\sigma_2(x_2-\mu_2)(x_1-\mu_1)\Big]$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\Big[\sigma_2^2(x_1-\mu_1)^2-2\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2)+\sigma_1^2(x_2-\mu_2)^2\Big]$$

$$= \frac{1}{(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}-2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]$$

Date :

from eq^n ① we can get that,

$$P(x_1, x_2) = \frac{1}{(2\pi)\sigma_1\sigma_2\sqrt{(1-\rho)^2}}$$

$$\exp\left[-\frac{1}{2(1-\rho)^2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(x_1-\mu_1)}{\sigma_1}\frac{(x_2-\mu_2)}{\sigma_2}\right)\right]$$

Date ....

*Problem 6:

⇒ By the properties of Gaussian distribution, we know that,

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ \& } \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

According precision matrix we also know that,

$$\Lambda = \begin{pmatrix} \Lambda_{aa} & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_{bb} \end{pmatrix}$$ which can be depicted as,

$$= \Sigma^{-1} - \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) \text{ expanded as}$$

$$-\frac{1}{2}(x_a - \mu_a)^T \Lambda_{aa}(x_a - \mu_a) - \frac{1}{2}(x_a - \mu_a)^T \Lambda_{ab}(x_b - \mu_b) - \frac{1}{2}(x_b - \mu_b)^T$$

$$\Lambda_{ba}(x_a - \mu_a) - \frac{1}{2}(x_b - \mu_b)^T \Lambda_{bb}(x_b - \mu_b)$$

The inverse of the covariance matrix $\Sigma_{a/b}^{-1}$ is represented by the second order term in x_2, where x_1 is constant and the mean executed from the first order in x_2, which is equal to $\Sigma_{a/b}^{-1} \mu_{a/b}$.

Date :

from the eq^{n}; $-\frac{1}{2}(x_b-\mu_b)^T\Lambda_{ba}(x_a-\mu_a)-\frac{1}{2}(x_b-\mu_b)^T\Lambda_{bb}(x_b-\mu_b)$

So, the second order term would be;

$-\frac{1}{2}(x_b)^T\Lambda_{bb}(x_b)$

$\therefore \Sigma_{2|1} = \Lambda_{22}^{-1}$

We can write the first order term,

$x_2^T\{\Lambda_{22}\mu_2-\Lambda_{21}(x_1-\mu_1)$

And Coefficient equal to $\Sigma_{2|1}^{-1}\mu_{2|1}$

then, $\mu_{2|1} = \Sigma_{2|1}\{\Lambda_{22}\mu_2-\Lambda_{21}(x_1-\mu_1)\}$

$$= \Lambda_{22}^{-1}\{\Lambda_{22}\mu_2-\Lambda_{21}(x_1-\mu_1)\}$$

$$= \mu_2-\Lambda_{22}^{-1}\Lambda_{21}(x_1-\mu_1)$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \sigma_1\sigma_2\begin{bmatrix} \sigma_1/\sigma_2 & \rho \\ \rho & \sigma_2/\sigma_1 \end{bmatrix}$$

$$\Lambda = \frac{1}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2}\begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$\Sigma_{2|1} = \Lambda_{22}^{-1} = \frac{\{\sigma_1^2\sigma_2^2(1-\rho^2)\}}{\sigma_1^2}$$

$$= \sigma_2^2(1-\rho^2)$$

$$\mu_{2|1} = \mu_2 - \Lambda_{22}^{-1}\Lambda_{21}(x_1-\mu_1)$$

$$= \mu_2 - \sigma_2^2(1-\rho^2)\frac{-\sigma_1\sigma_2\rho}{\sigma_1^2\sigma_2^2(1-\rho^2)}(x_1-\mu_1)$$

$$= \mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x_1-\mu_1)$$

therefore, $P(x_2|x_1) = \frac{1}{\sqrt{2\pi\Sigma_{2|1}}}\exp\left\{-\frac{1}{2}(x_2-\mu_{2|1})^T\Sigma_{2|1}^{-1}(x_2-\mu_{2|1})\right\}$

$$P(x_2|x_1) = \frac{1}{\sqrt{2\pi(1-\rho^2)}\,\sigma_2}\exp\left\{-\frac{1}{2}\left(x_2-\mu_2-\frac{\sigma_2\rho}{\sigma_1}(x_1-\mu_1)\right)^T\frac{1}{\sigma_2^2(1-\rho^2)}\left(x_2-\mu_2-\frac{\sigma_2\rho}{\sigma_1}(x_1-\mu_1)\right)\right\}$$

If $\sigma_1=\sigma_2=1$, then

$$P(x_2|x_1) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{1}{2}(x_2-\mu_2-\rho(x_1-\mu_1))^T\frac{1}{(1-\rho^2)}(x_2-\mu_2-\rho(x_1-\mu_1))\right\}$$

Date : ....

***Problem 7:**

(1) ⇒ We know that, Uniform distribution, $f(x) = \frac{1}{b-a}$

$$\Rightarrow E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{1}{b-a} \times 2 \left[b^2 - a^2 \right]$$

$$= \frac{b+a}{2}$$

$$= 4.$$

For Variance.

$$V(x) = E[x^2] - (E[x])^2$$

$$= E[x^2] - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{1}{b-a} \int_a^b x^2 dx - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2$$

Date :

$$= \frac{4(b^3-a^3)-3(b-a)(b+a)}{12(b-a)}$$

$$= \frac{4b^3-4a^3-3(b-a)(a^2+2ab+b^2)}{12(b-a)}$$

$$= \frac{4b^3-4a^3-3a^2b-6ab^2-3b^3+3a^3+6a^2b+3ab^2}{12(b-a)}$$

$$= \frac{b^3+3a^2b-3ab^2-a^3}{12(b-a)}$$

$$= \frac{(b-a)^3}{12(b-a)}$$

$$= \frac{(b-a)^2}{12}$$

Assignment 1

$\mathcal{X}$] b

⇒ Multivariate Gaussian

$$N(x|\mu,\Sigma)=\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$$

$$E[x]=\int N(x|\mu,\Sigma)\,x\,dx$$

$$E[x]=\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}\int\exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}x\,dx \quad \left[\begin{array}{l}\# z=x-\mu\\ \Rightarrow x=z+\mu\\ \therefore dx=dz\end{array}\right]$$

$$E[x]=\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}\int\exp\left\{-\frac{1}{2}z^T\Sigma^{-1}z\right\}(z+\mu)\,dz$$

$$=\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}\left[\underbrace{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}z^T\Sigma^{-1}z\right\}z\,dz}_{0}+\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}z^T\Sigma^{-1}z\right\}\mu\,dz\right]$$

\# 0 because the function is odd including the ratio $\int_{-\infty}^{\infty}$

$$=\frac{\mu}{(2\pi)^{D/2}|\Sigma|^{1/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\cdot z^T\cdot\Sigma^{-1}\cdot z\right)dz.$$

the reason for the normalized Multivariate Gaussian,

$$\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\cdot z^T\cdot\Sigma^{-1}\cdot z\right)dz=1$$

$$\therefore E[x]=\mu.$$

$$E[xx^T] = \mu\mu^T + \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \int \exp\left(-\tfrac{1}{2} z^T \Sigma^{-1} z\right) z z^T \, dz$$

$$\Rightarrow E[xx^T] = \mu\mu^T + \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \sum_{i=1}^{D}\sum_{j=1}^{D} u_i u_j^T \exp\left(-\sum_{k=1}^{D} \frac{y_k^2}{2\lambda_k}\right) y_i y_j \, dy$$

$$\Rightarrow E[xx^T] = \mu\mu^T + \sum_{i=1}^{D} u_i u_i \lambda_i$$

$$\therefore E[xx^T] = \mu\mu^T + \Sigma$$

By substracting the mean form $E[x x^T]$, we will be able to execute the covariance of x.

$$\therefore \operatorname{cov}[x] = E\left[(x - E[x])(x - E[x]^T)\right]$$

$$\therefore \operatorname{cov}[x] = \Sigma$$

There are D^2 second order moments generated by $E[x_i, x_j]$ by studying second order moments in Multivariate Gaussian, which can be driven as matrix $E[xx^T]$, can be formed as,

$$E[xx^T] = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \int \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right] x \cdot x^T dx$$

As mentioned before, $z = x-\mu$

$\therefore x = z+\mu$.

$$E[xx^T] = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \int \exp\left(-\frac{1}{2} z^T \Sigma^{-1} \cdot z\right)(z+\mu)(z+\mu)^T dz$$

$$= \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} z^T \Sigma^{-1} z\right)\left(zz^T + z\mu^T + \mu z + \mu\mu^T\right) dz$$

By the symmetry, the conditions μz^T and $\mu^T z$ will be disappeared where the condition $\mu\mu^T$ is constant and the leftover zz^T we need to deal with. Utilizing the eigenvector expansion of the covariance matrix, we can derive

$$z = \sum_{j=1}^{D} y_j u_j \quad \text{where } y_j = u_j z$$