$a(t) = Cu^2(t),$ $v(t) = \sqrt{2Gh(t)},$ $n(t) = a(t)v(t).$ $q(t) = Ah(t).$ $\frac{dq(t)}{dt} = m(t) - n(t).$

2.1. **(T)** Compute the outflow of the tank as a function of its level $h(t)$ and the control input $u(t)$.

$a(t) = Cu^2(t),$ $v(t) = \sqrt{2Gh(t)},$ $n(t) = a(t)v(t).$

$$n(t) = a(t)\,v(t) = C\,u^2(t) \cdot \sqrt{2G\,h(t)}$$

2.2. **(T)** Determine the nonlinear differential equation that models the evolution of the liquid level $h(t)$ as a function of the control input $u(t)$ and the inflow $m(t)$. Write it also in the form

$$\frac{dh(t)}{dt} = f\left(h(t), u(t), m(t)\right). \qquad (6)$$

$q(t) = Ah(t).$ $\frac{dq(t)}{dt} = m(t) - n(t).$

$$\boxed{n(t) = C\,u^2(t) \cdot \sqrt{2G\,h(t)} \quad (\text{Ex } 2.1)}$$

$$\frac{dq(t)}{dt} = \frac{dA\,h(t)}{dt} \Leftrightarrow A\frac{dh(t)}{dt} = m(t) - n(t) \Leftrightarrow$$

$$\Leftrightarrow \frac{dh(t)}{dt} = \frac{m(t) - n(t)}{A} \Leftrightarrow \frac{dh(t)}{dt} = \frac{m(t) - C\,u^2(t) \cdot \sqrt{2G\,h(t)}}{A}$$

$$f(h(t), u(t), m(t)) = \frac{m(t) - C\,u^2(t) \cdot \sqrt{2G\,h(t)}}{A}$$

2.3. **(T)** Suppose that the tank is operated around an equilibrium point determined by a constant inflow M_{eq} and a constant level H_{eq}. Determine the corresponding control input, at equilibrium, U_{eq}.

$$\boxed{\frac{dh(t)}{dt} = \frac{m(t) - C\,u^2(t) \cdot \sqrt{2G\,h(t)}}{A} \quad (\text{Ex } 2.2)}$$

Se h(t) é constante, então dh(t)/dt = 0. Logo ficamos com a seguinte expressão:

$$0 = \frac{M_{eq} - C\,U_{eq}^2 \cdot \sqrt{2G\,H_{eq}}}{A} \Leftrightarrow U_{eq}^2 = \frac{M_{eq}}{C \cdot \sqrt{2G\,H_{eq}}} \Leftrightarrow$$

$$U_{eq} = \sqrt{\frac{M_{eq}}{C \cdot \sqrt{2G\,H_{eq}}}}$$

Consider now incremental variables around the equilibrium point, i.e., let $h(t) := H_{eq} + x(t)$, $u(t) = U_{eq} + \mu(t)$, and $m(t) = M_{eq} + d(t)$, where $x(t)$ corresponds to small deviations of the liquid level around the equilibrium level H_{eq}, $\mu(t)$ corresponds to small deviations of the control input around the equilibrium input U_{eq}, and $d(t)$ corresponds to small deviations of the inflow around the equilibrium inflow M_{eq}.

2.4. **(T)** The dynamical system

$$\dot{x} = a_1 x$$

with

$$a_1 := \left.\frac{\partial f}{\partial h}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})}$$

approximately describes the behavior of (6) near the equilibrium point (H_{eq}, U_{eq}, M_{eq}) when $\mu(t) = d(t) = 0$ for all $t \in \mathbb{R}$. Compute a_1 as a function of the system parameters and of M_{eq} and H_{eq}.

$$f(H_{eq}, U_{eq}, M_{eq}) = 0$$

Sistema linearizado:

$$\dot{x}(t) = \underline{\partial f}\Big| \cdot \overbrace{\delta h}^{x(t)} + \underline{\partial f}\Big| \cdot \overbrace{\delta u}^{\mu(t)} + \underline{\partial f}\Big| \cdot \overbrace{\delta m}^{d(t)} \Rightarrow$$

$$\dot{x}(t) = \frac{\partial f}{\partial h}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \overbrace{\delta h}^{x(t)} + \frac{\partial f}{\partial u}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \overbrace{\delta u}^{\mu(t)} + \frac{\partial f}{\partial m}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \overbrace{\delta m}^{d(t)} \Rightarrow$$

$$\underset{\substack{d(t)=0\\ \mu(t)=0}}{\Rightarrow} \dot{x}(t) = \frac{\partial f}{\partial h}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot x(t) = a_1 \cdot x(t)$$

$$\boxed{f(h(t), u(t), m(t)) = \frac{m(t) - C\,u^2(t)\cdot\sqrt{2Gh(t)}}{A} \quad \text{(Ex 2.2)}}$$

$$\frac{\partial f}{\partial h}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} = -\frac{C\,U_{eq}^2}{A}\cdot\frac{\partial\left(\sqrt{2Gh(t)}\right)}{\partial h} \Rightarrow -\frac{C\,U_{eq}^2}{A}\cdot\frac{G}{\sqrt{2GH_{eq}}}$$

C.A

$$(2Gh(t))^{\frac{1}{2}} = \frac{\cancel{2}\cdot G}{\cancel{2}\sqrt{2Gh(t)}}$$

$$\boxed{U_{eq}^2 = \frac{M_{eq}}{C\cdot\sqrt{2GH_{eq}}} \quad \text{(Ex 2.3)}}$$

$$\frac{\partial h(t)}{\partial h} = \dot{x} = a_1 x$$

Substituindo fica:

$$-\frac{\cancel{C}}{A}\cdot\frac{M_{eq}}{\cancel{C}\cdot\sqrt{2GH_{eq}}}\cdot\frac{G}{\sqrt{2GH_{eq}}} = -\frac{M_{eq}\cdot\cancel{G}}{A\cdot 2\cancel{G}\cdot H_{eq}} = -\frac{M_{eq}}{2\cdot A\cdot H_{eq}}$$

Logo $a_1 = -\dfrac{M_{eq}}{2\cdot A\cdot H_{eq}}$

2.5. **(T)** The dynamical system

$$\dot{x} = a_1 x + a_2 \mu \tag{7}$$

with

$$a_2 := \frac{\partial f}{\partial u}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})}$$

approximately describes the behavior of (6) near the equilibrium point (H_{eq}, U_{eq}, M_{eq}) when $d(t) = 0$ for all $t \in \mathbb{R}$. Compute a_2 as a function of the system parameters and of M_{eq} and U_{eq}.

Sistema linearizado:

$$\dot{x}(t) = \frac{\partial f}{\partial h}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \overbrace{\delta h}^{x(t)} + \frac{\partial f}{\partial u}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \overbrace{\delta u}^{\mu(t)} + \frac{\partial f}{\partial m}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \overbrace{\delta m}^{d(t)} \Rightarrow$$

$d(t) = 0$

$$\Rightarrow \dot{x}(t) = \frac{\partial f}{\partial h}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot x(t) + \frac{\partial f}{\partial u}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \mu(t) = a_1 x(t) + a_2 \mu(t)$$

$$\boxed{f(h(t), u(t), m(t)) = \frac{m(t) - C\,u^2(t)\cdot\sqrt{2Gh(t)}}{A} \quad \text{(Ex 2.2)}}$$

$$\frac{\partial f}{\partial u}\Big|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} = -\frac{C\cdot 2U_{eq}\cdot\sqrt{2GH_{eq}}}{A}$$

$$\left.\frac{\partial f}{\partial u}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} = -\frac{C\cdot 2U_{eq}\cdot\sqrt{2GH_{eq}}}{A}$$

$$\boxed{U_{eq}^2 = \frac{M_{eq}}{C\cdot\sqrt{2GH_{eq}}} \quad (\text{Ex } 2.3)} \Leftrightarrow \frac{U^2_{eq}}{M_{eq}}\cdot C = \frac{1}{\sqrt{2GH_{eq}}} \Leftrightarrow$$

$$\Leftrightarrow \frac{M_{eq}}{U^2_{eq}\cdot C} = \sqrt{2GH_{eq}}$$

Substituindo $-\frac{\cancel{C}\cdot 2\cancel{U_{eq}}}{A}\cdot\frac{M_{eq}}{U^{\cancel{2}}_{eq}\cdot\cancel{C}} = -\frac{2M_{eq}}{AU_{eq}}$

Logo $a_2 = -\frac{2M_{eq}}{AU_{eq}}$

2.6. **(T)** Show that the transfer function that describes the linearized system with input $\mu(t)$ and output $x(t)$, for $d(t)=0$, can be written as

$$G_1(s) = K_1\frac{p}{s+p}.$$

Determine the constants K_1 and p.

$$\boxed{a_1 = -\frac{M_{eq}}{2\cdot A\cdot H_{eq}} \quad (\text{Ex } 2.4)} \qquad \boxed{a_2 = -\frac{2M_{eq}}{AU_{eq}} \quad (\text{Ex } 2.5)}$$

$$\dot{x}(t) = a_1x(t) + a_2\mu(t) \Leftrightarrow$$

$$\Leftrightarrow \dot{x}(t) - a_1x(t) = a_2\mu(t) \Leftrightarrow$$

$$\Leftrightarrow sX(s) - a_1X(s) = a_2U(s) \Leftrightarrow$$

$$\Leftrightarrow \frac{X(s)}{U(s)} = \frac{a_2}{s-a_1} \Leftrightarrow G_1(s) = \underbrace{\frac{a_2}{-a_1}}_{K}\cdot\frac{\overset{p}{-a_1}}{s-a_1} \Leftrightarrow$$

$$\Leftrightarrow G_1(s) = \left(\frac{\frac{M_{eq}}{2AH_{eq}}}{s+\frac{M_{eq}}{2AH_{eq}}}\right)\times\left(\frac{-\frac{2\cancel{M_{eq}}}{\cancel{A}U_{eq}}}{\frac{\cancel{M_{eq}}}{2\cancel{A}H_{eq}}}\right) \Leftrightarrow$$

$$\Leftrightarrow G_1(s) = \left(\frac{\frac{M_{eq}}{2AH_{eq}}}{s+\frac{M_{eq}}{2AH_{eq}}}\right)\times\left(-\frac{2\cdot 2H_{eq}}{U_{eq}}\right)$$

Logo $p = -a_1 = +\frac{M_{eq}}{2\cdot A\cdot H_{eq}}$ e $K_1 = \frac{-4H_{eq}}{U_{eq}}$

2.7. **(T)** The dynamical system

$$\dot{x} = a_1 x + a_3 d \tag{8}$$

with

$$a_3 := \left.\frac{\partial f}{\partial m}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})}$$

approximately describes the behavior of (6) near the equilibrium point (H_{eq}, U_{eq}, M_{eq}) when $\mu(t) = 0$ for all $t \in \mathbb{R}$. Compute a_3 in function of the system parameters and of M_{eq} and U_{eq}.

Sistema linearizado:

$$\dot{x}(t) = \left.\frac{\partial f}{\partial h}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \underbrace{\delta h}_{x(t)} + \left.\frac{\partial f}{\partial u}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \underbrace{\delta u}_{\mu(t)} + \left.\frac{\partial f}{\partial m}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot \underbrace{\delta m}_{d(t)} \Rightarrow$$

$\mu(t) = 0$

$$\Rightarrow \dot{x}(t) = \left.\frac{\partial f}{\partial h}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot x(t) + \left.\frac{\partial f}{\partial m}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} \cdot d(t) = a_1 x(t) + a_3 d(t)$$

$$\boxed{f(h(t), u(t), m(t)) = \frac{m(t) - C\,u^2(t) \cdot \sqrt{2 G h(t)}}{A} \quad (\text{Ex } 2.2)}$$

$$\left.\frac{\partial f}{\partial m}\right|_{(h,u,m)=(H_{eq},U_{eq},M_{eq})} = \frac{1}{A}$$

2.8. **(T)** Show that the transfer function that describes the linearized system with input $d(t)$ and output $x(t)$, for $r(t) = 0$, can be written as

$$G_2(s) = K_2 \frac{p}{s+p}.$$

Determine K_2.

$$\dot{x}(t) = a_1 x(t) + a_3 d(t) \Leftrightarrow \dot{x}(t) - a_1 x(t) = a_3 d(t) \Leftrightarrow$$

$$\Leftrightarrow s X(s) - a_1 X(s) = a_3 D(s) \Leftrightarrow \frac{X(s)}{D(s)} = \frac{a_3}{s - a_1} \Leftrightarrow$$

$$\Leftrightarrow G_2(s) = \frac{a_3}{-a_1} \cdot \frac{-a_1}{s - a_1}$$

$$\boxed{a_1 = -\frac{M_{eq}}{2 \cdot A \cdot H_{eq}} \quad (\text{Ex } 2.4)} \qquad \boxed{a_3 = \frac{1}{A} \quad (\text{Ex } 2.7)}$$

$$K_2 = \frac{a_3}{-a_1} \Leftrightarrow K_2 = \frac{2 \cdot \cancel{A} \cdot H_{eq}}{\cancel{A} \cdot M_{eq}} = \frac{2 \cdot H_{eq}}{M_{eq}}$$

$$p = -a_1 = \frac{M_{eq}}{2 \cdot A \cdot H_{eq}}$$

2.9. **(T)** Derive the linear differential equation

$$\frac{dx(t)}{dt} = g(x(t), \mu(t), d(t))$$

that approximately describes the system operating close to the equilibrium point.

$$a_1 = -\frac{M_{eq}}{2 \cdot A \cdot H_{eq}} \quad (\text{Ex } 2.4)$$

$$a_2 = -\frac{2 M_{eq}}{A\, U_{eq}} \quad (\text{Ex } 2.5)$$

$$a_3 = \frac{1}{A} \quad (\text{Ex } 2.7)$$

$$\dot{x}(t) = a_1 x(t) + a_2 \mu(t) + a_3 d(t) \Leftrightarrow$$

$$\Leftrightarrow \dot{x}(t) = -\frac{M_{eq}}{2 \cdot A \cdot H_{eq}} x(t) - \frac{2 M_{eq}}{A\, U_{eq}} \mu(t) + \frac{1}{A} d(t)$$

3.1. (T) Considering $d(t) = 0$, determine the transfer function of the closed-loop system with input $r(t)$ and output $x(t)$, i.e., compute

$$G_{clr}(s) = \left.\frac{X(s)}{R(s)}\right|_{D(s)=0}.$$

as a function of $K_P, K_1, p \in \mathbb{R}$.

$$G_1(s) = \frac{p}{s+p} \times K_1 \quad (\text{Ex } 2.6)$$

$K(s) = -K_P, \quad K_P \in \mathbb{R},$

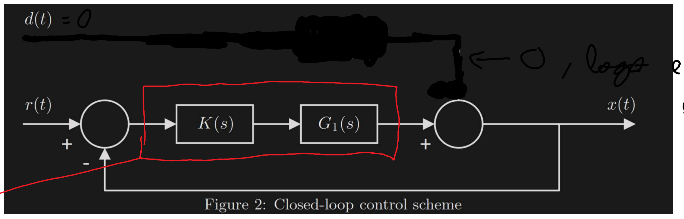
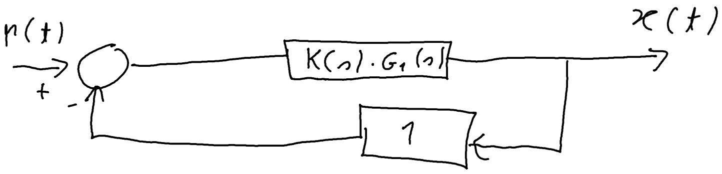


Figure 2: Closed-loop control scheme

← 0, logo é como se não existisse ramo de cima

$K(s) \cdot G_1(s)$, logo o circuito equivalente é



$$\left.\frac{X(s)}{R(s)}\right|_{D(s)=0} = \frac{K(s) \cdot G_1(s)}{1 + K(s) \cdot G_1} = \frac{-K_p \cdot \frac{p}{s+p} \cdot K_1}{1 - K_p \cdot \frac{p}{s+p} \cdot K_1}$$

$$\text{Logo} \quad G_{clr}(s) = \frac{-K_p \cdot \frac{p}{s+p} \cdot K_1}{1 - K_p \cdot \frac{p}{s+p} \cdot K_1} = \frac{-K_p \cdot p \cdot K_1}{s + p - K_p \cdot p \cdot K_1}$$

3.2. (T) Compute the static gain and the time constant of $G_{clr}(s)$ as a function of $K_P, K_1, p \in \mathbb{R}$.

Ganho estático, logo $s = 0$

$$G_{clr}(0) = \frac{-K_p \cdot K_1}{1 - K_p \cdot K_1}$$

$$G_{clr}(s) = \frac{G_{clr}(0)}{1 + s\tau} \Leftrightarrow 1 + s\tau = \frac{G_{clr}(0)}{G_{clr}(s)} \Leftrightarrow \tau = \left(\frac{G_{clr}(0)}{G_{clr}(s)} - 1\right) \cdot \frac{1}{s} \Rightarrow$$

$$G_{cln}(s) = \frac{G_{cln}(0)}{1+s\tau} \Leftrightarrow 1+s\tau = \frac{G_{cln}(0)}{G_{cln}(s)} \Leftrightarrow \tau = \left(\frac{G_{cln}(0)}{G_{cln}(s)} - 1\right)\cdot\frac{1}{s} \Rightarrow$$

$$\Rightarrow \tau = \left(\frac{-\cancel{K_p}\cdot\cancel{K_1}}{1-K_p\cdot K_1}\cdot\frac{s+p-K_p\cdot p\cdot K_1}{-\cancel{K_p}\cdot p\cdot\cancel{K_1}} - 1\right)\cdot\frac{1}{s} \Leftrightarrow \tau = \left(\frac{s+p(1-K_p\cdot K_1)}{(1-K_p\cdot K_1)\,p} - 1\right)\cdot\frac{1}{s} \Leftrightarrow$$

$$\Leftrightarrow \tau = \left(\frac{s}{(1-K_p\cdot K_1)\,p} + 1 - 1\right)\cdot\frac{1}{s} \Leftrightarrow \tau = \frac{1}{(1-K_p\cdot K_1)\cdot p}$$

3.3. (T) Considering $r(t) = 0$, determine the transfer function of the closed-loop system with input $d(t)$ and output $x(t)$, i.e., compute

$$G_{cld}(s) = \left.\frac{X(s)}{D(s)}\right|_{R(s)=0}.$$

as a function of $K_P, K_1, p \in \mathbb{R}$.

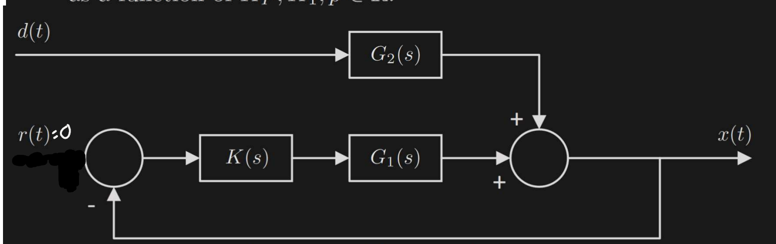
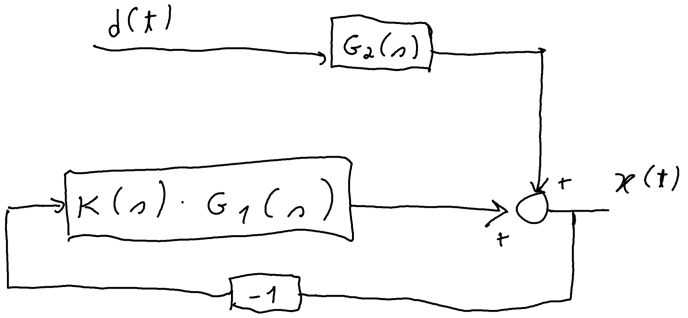


Figure 2: Closed-loop control scheme



$$X(s) = D(s)\,G_2(s) - K(s)\cdot G_1(s)\cdot X(s) \Leftrightarrow$$

$$\Leftrightarrow X(s)\,(1+K(s)\cdot G_1(s)) = D(s)\,G_2(s) \Leftrightarrow$$

$$\Leftrightarrow \frac{X(s)}{D(s)} = \frac{G_2(s)}{1+K(s)\cdot G_1(s)} \Leftrightarrow$$

$$\Leftrightarrow G_{cld}(s) = \frac{K_2\cdot\frac{p}{s+p}}{1-K_p\cdot K_1\frac{p}{s+p}}$$

C.A

$$\boxed{p = \frac{M_{eq}}{2\cdot A\cdot H_{eq}} \quad (\text{Ex } 2.8)} \qquad \boxed{K_2 = \frac{2\cdot\cancel{A}\cdot H_{eq}}{\cancel{A}\cdot M_{eq}} = \frac{2\cdot H_{eq}}{M_{eq}} \quad (\text{Ex } 2.8)}$$

$$K_2 = \frac{1}{pA}$$

$$G_{cld}(s) = \frac{\frac{1}{(s+p)A}}{1-K_p\cdot K_1\frac{p}{\quad}} \Leftrightarrow G_{cld}(s) = \frac{\frac{1}{A}}{\quad}$$

$$G_{cld}(s) = \frac{\frac{1}{(s+p)A}}{1 - K_p \cdot K_1 \frac{p}{s+p}} \Leftrightarrow G_{cld}(s) = \frac{\frac{1}{A}}{s+p-K_p \cdot K_1 \cdot p}$$

Logo $G_{cld}(s) = \dfrac{1}{A \cdot (s + p(1 - K_p \cdot K_1))}$

3.4. **(T)** Compute the static gain and the time constant of $G_{cld}(s)$ as a function of $K_P, K_1, p \in \mathbb{R}$.

$$s = 0 \Rightarrow G_{cld}(0) = \frac{\frac{1}{pA}}{1 - K_p \cdot K_1} \Leftrightarrow G_{cld}(0) = \frac{1}{pA \cdot (1 - K_p \cdot K_1)}$$

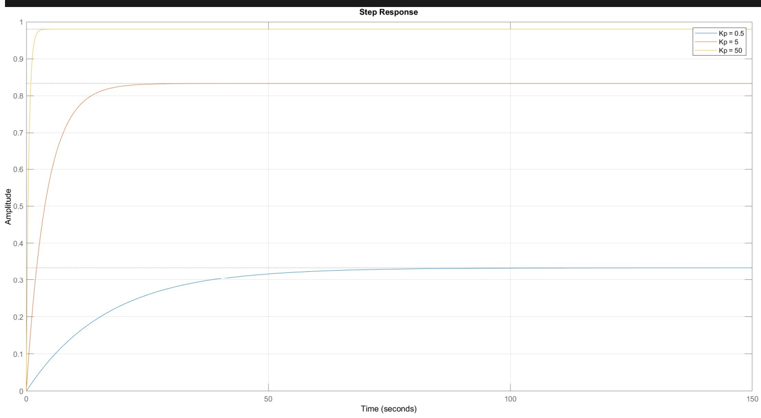
$$G_{cld}(s) = \frac{G_{cld}(0)}{1 + s\tau} \Leftrightarrow 1 + s\tau = \frac{G_{cld}(0)}{G_{cld}(s)} \Leftrightarrow \tau = \left(\frac{G_{cld}(0)}{G_{cld}(s)} - 1\right) \cdot \frac{1}{s} \Rightarrow$$

$$\Rightarrow \tau = \left(\frac{\cancel{A} \cdot (s + p(1 - K_p \cdot K_1))}{p\cancel{A} \cdot (1 - K_p \cdot K_1)} - 1\right) \frac{1}{s} \Leftrightarrow$$

$$\Leftrightarrow \tau = \left(\frac{s}{p \cdot (1 - K_p \cdot K_1)} + 1 - 1\right) \frac{1}{s} \Leftrightarrow \tau = \frac{1}{p \cdot (1 - K_p \cdot K_1)}$$

3.5. **(L)** Simulate and plot the response of the closed-loop system when $r(t)$ is a unit step and $d(t)$ is zero, for three different gains: i) $K_P = 0.5$; ii) $K_P = 5$; and iii) $K_P = 50$. Discuss the reference following properties of the closed-loop system with proportional control.

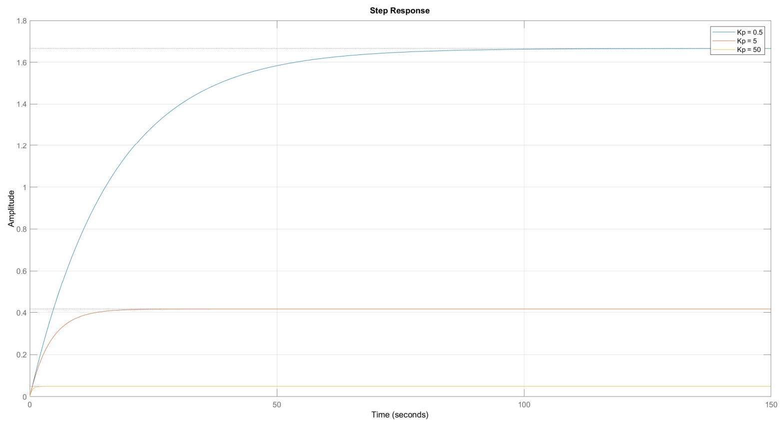
Suggestion: Recall question 3.3.




```
Máximo para Kp = 0.5:    0.333291
Máximo para Kp = 5:      0.833228
Máximo para Kp = 50:     0.980269
```

3.6. **(L)** Simulate and plot the response of the closed-loop system when $d(t)$ is a unit step and $r(t)$ is zero for three different gains: i) $K_P = 0.5$; ii) $K_P = 5$; and iii) $K_P = 50$. Discuss the disturbance rejection properties of the closed-loop system with proportional control.

Suggestion: Recall question 3.5.




```
Máximo para Kp = 0.5:    1.666457
Máximo para Kp = 5:      0.416614
Máximo para Kp = 50:     0.049013
```