EECE 4353 Image Processing

Lecture Notes: Sampling and Aliasing

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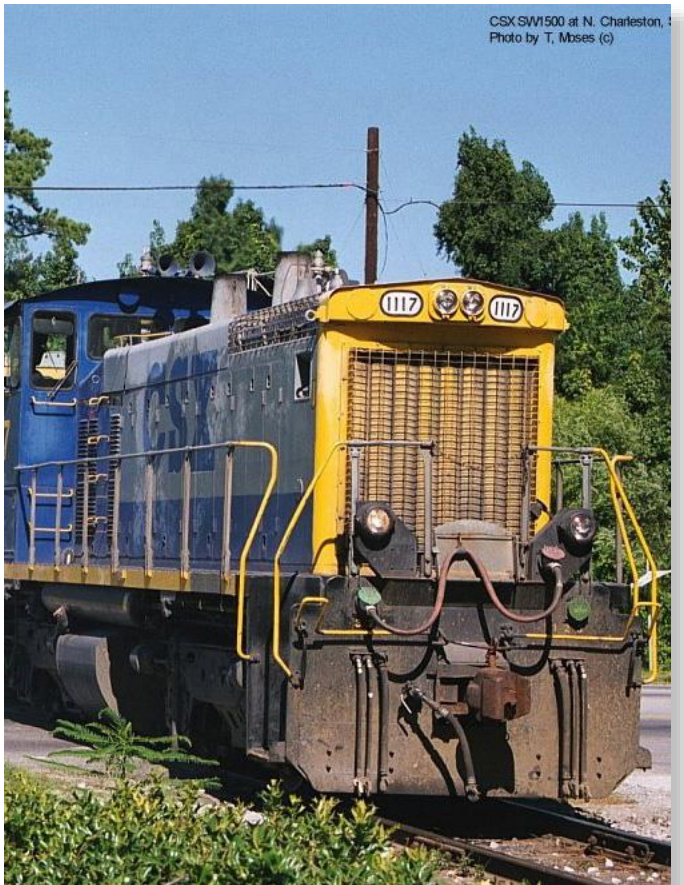
Warning:

Image Resampling Can Lead to *the Jaggies*!



The jaggies!

Image: T. Moses, CSX SW1500 at N. Charleston, SC, Sept. 1999

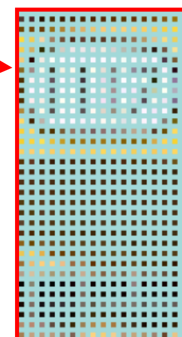
Downsampling (Decimation)


```
>> J = I(1:2:R,1:2:C,:);
```

E.g.: every 2nd pixel in every 2nd row

Downsampling (Decimation)

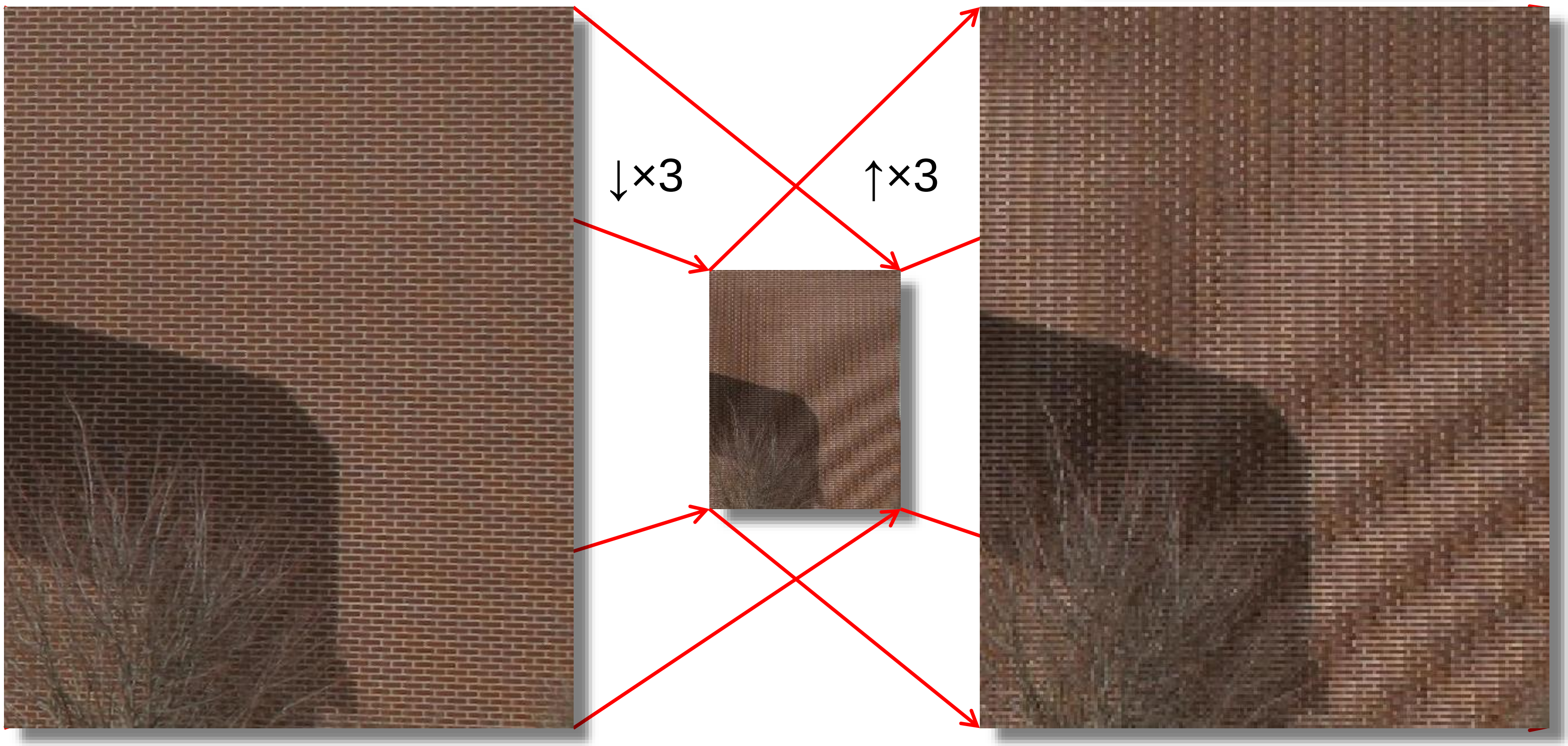

```
>> J = I(1:2:R,1:2:C,:);
```



E.g.: every 2nd pixel in every 2nd row

Jaggies and Moire

Another fine mess:
Moire distortion.

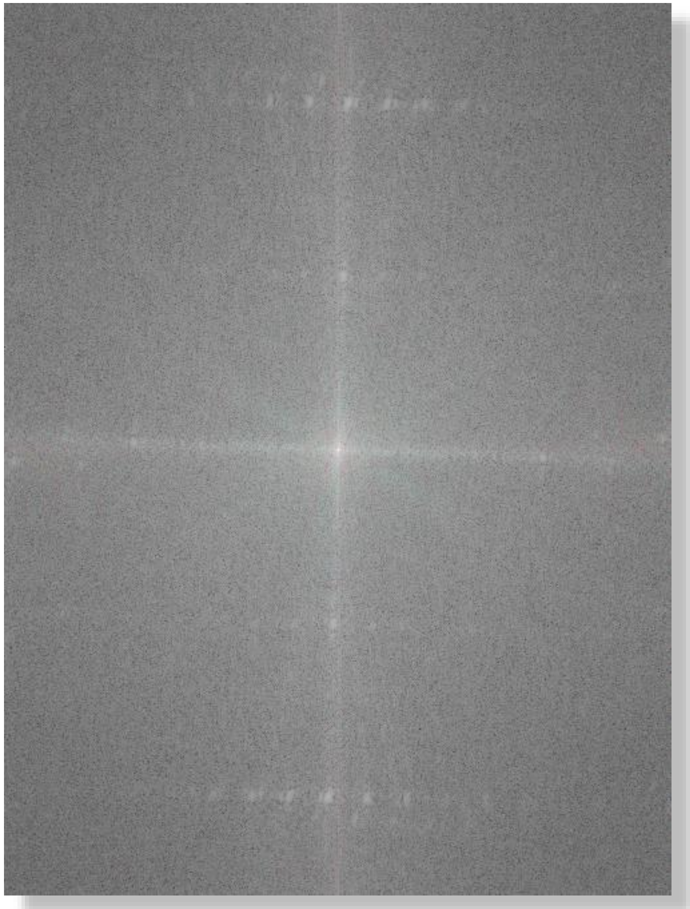


original image | decimated by 3× | upsampled by 3×

Images: C. Burnett, Wikimedia Commons,
http://en.wikipedia.org/wiki/File:Moire_pattern_of_bricks.jpg

Power Spectrum from Discrete Fourier Transform

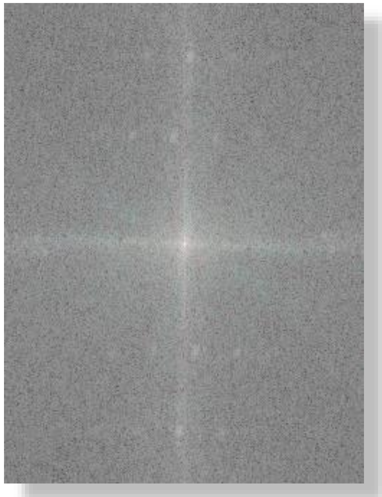
DFT



Power Spectrum from Discrete Fourier Transform

decimated image

DFT



power spectrum

The DFT of an image is the same size as the image.

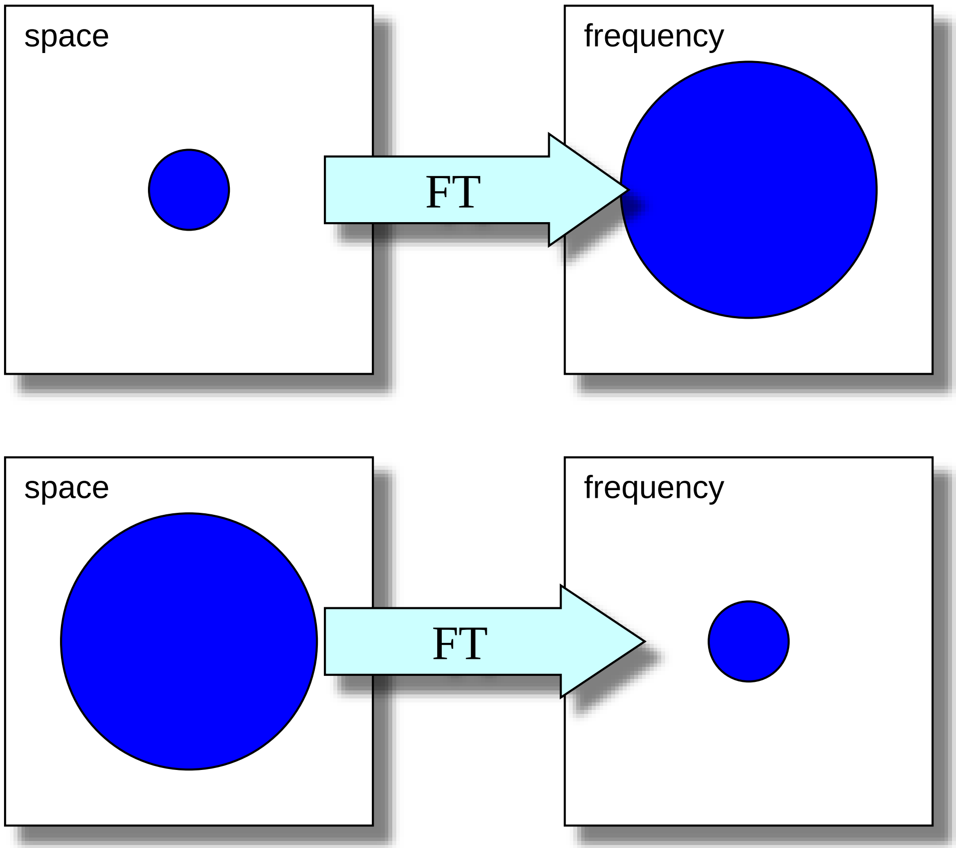
The Scaling Property of the FT

If $$\mathfrak{F}\{\mathbf{I}\}(v,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{I}(r,c)\, e^{-i2\pi(uc+vr)}\, dc\, dr,$$

then $$\mathbf{I}\left(\frac{r}{a},\frac{c}{b}\right) \Leftrightarrow |ab|\,\mathfrak{F}\{\mathbf{I}\}(av,bu).$$

This implies that if an image is reduced in size, its features in the spatial domain become smaller and its features in the frequency domain become larger.

The Uncertainty Relation



If ΔxΔy is the extent of the object in space and if ΔxΔy is its extent in frequency then

$$\Delta x \Delta y \cdot \Delta u \Delta v \geq \frac{1}{16\pi^2}$$

A small object in space has a large frequency extent and vice-versa.

Effect of Decimation on the DFT of an Image

1. Decimation of an $R \times C$ image, **I**, by a factor of n results in an $\lfloor R/n \rfloor \times \lfloor C/n \rfloor$ image, **J**.
2. The DFT of image **J** is the same size as **J**.
3. The uncertainty relation implies that the FT of **J** should be $(R\text{-}1) \times (C\text{-}1) < n\lfloor R/n \rfloor \times n\lfloor C/n \rfloor \leq R \times C$.

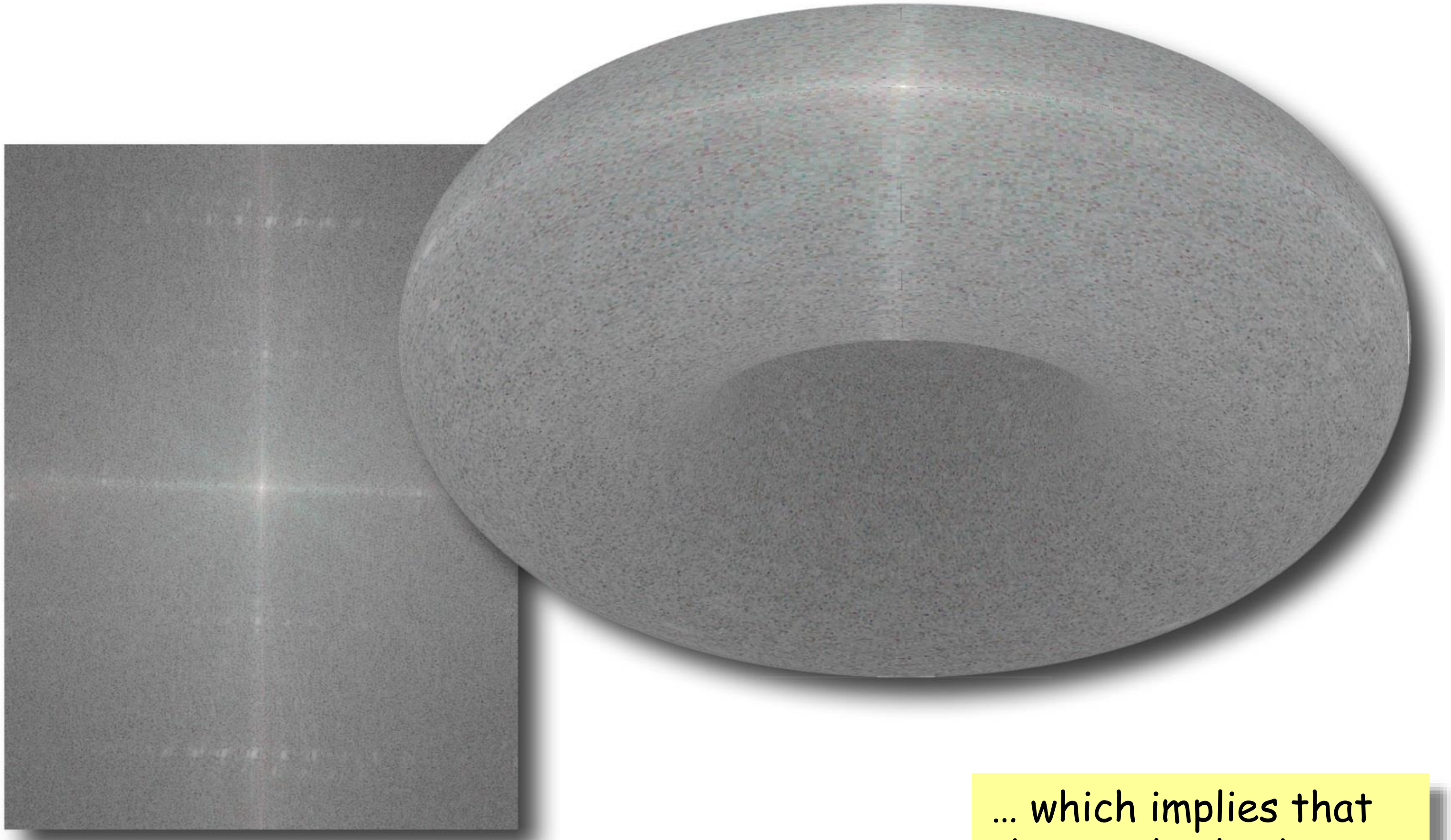
Q: How can these 3 facts be true simultaneously?

A: The FT of **J** *folds over* or *aliases* itself on the DFT of **J** because the DFT is defined on a torus.

Discrete FT is on a Torus

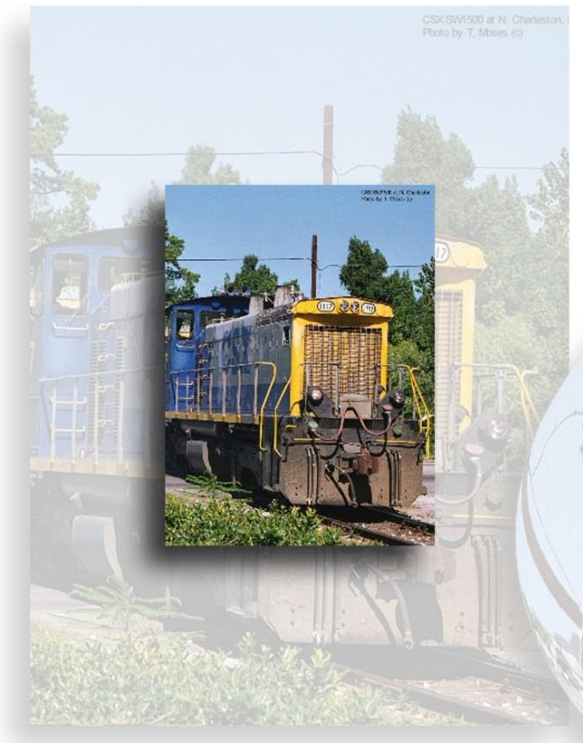
Recall: the DFT of a digital image assumes the image has a toroidal topology…

Discrete Fourier Transform is on a Torus



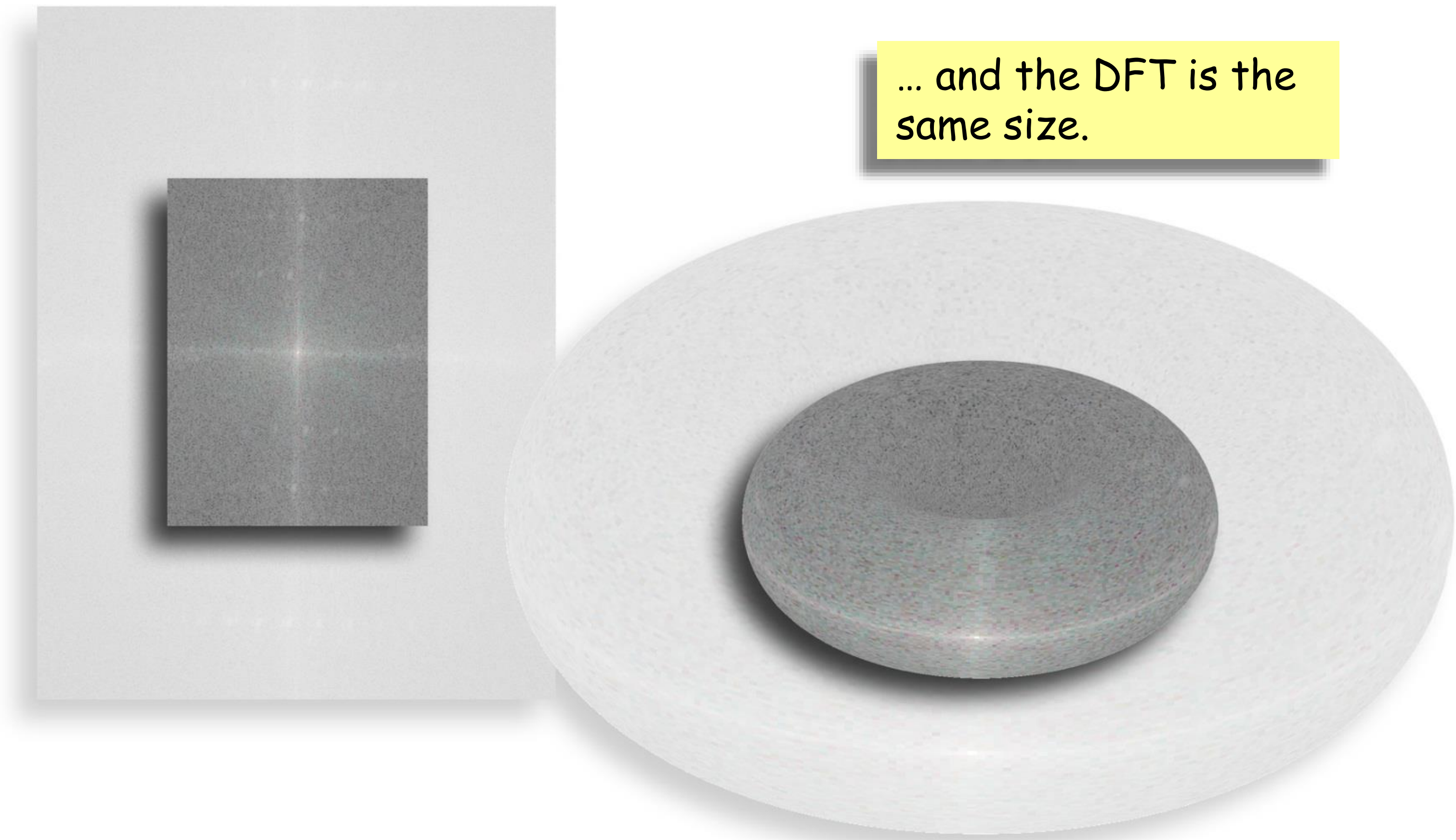
… which implies that the result also has a toroidal topology.

Discrete FT is on a Torus



If the image is made smaller the torus is likewise …

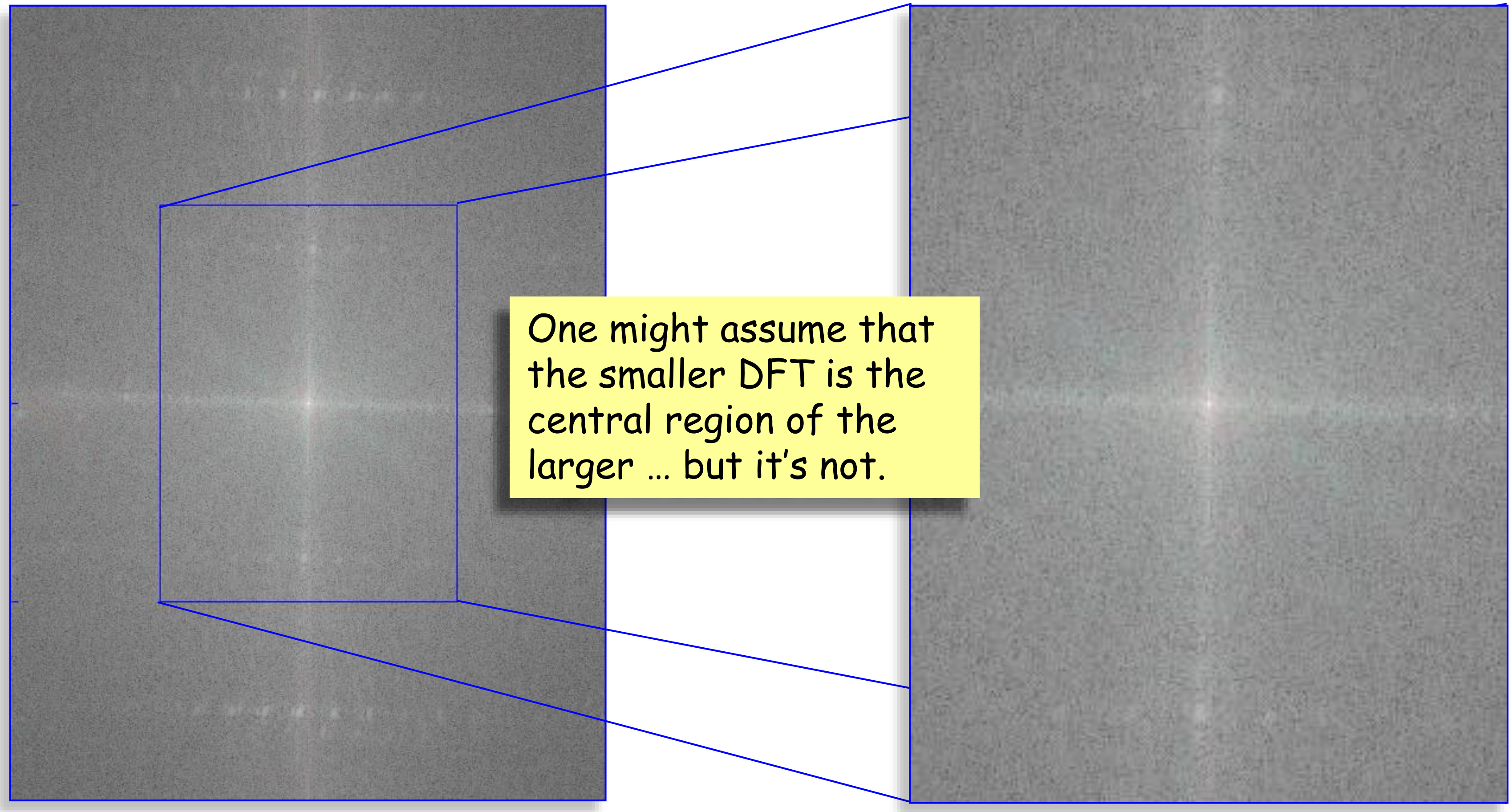
Discrete FT is on a Torus



… and the DFT is the same size.

Ideal PS of 2× Decimated Image

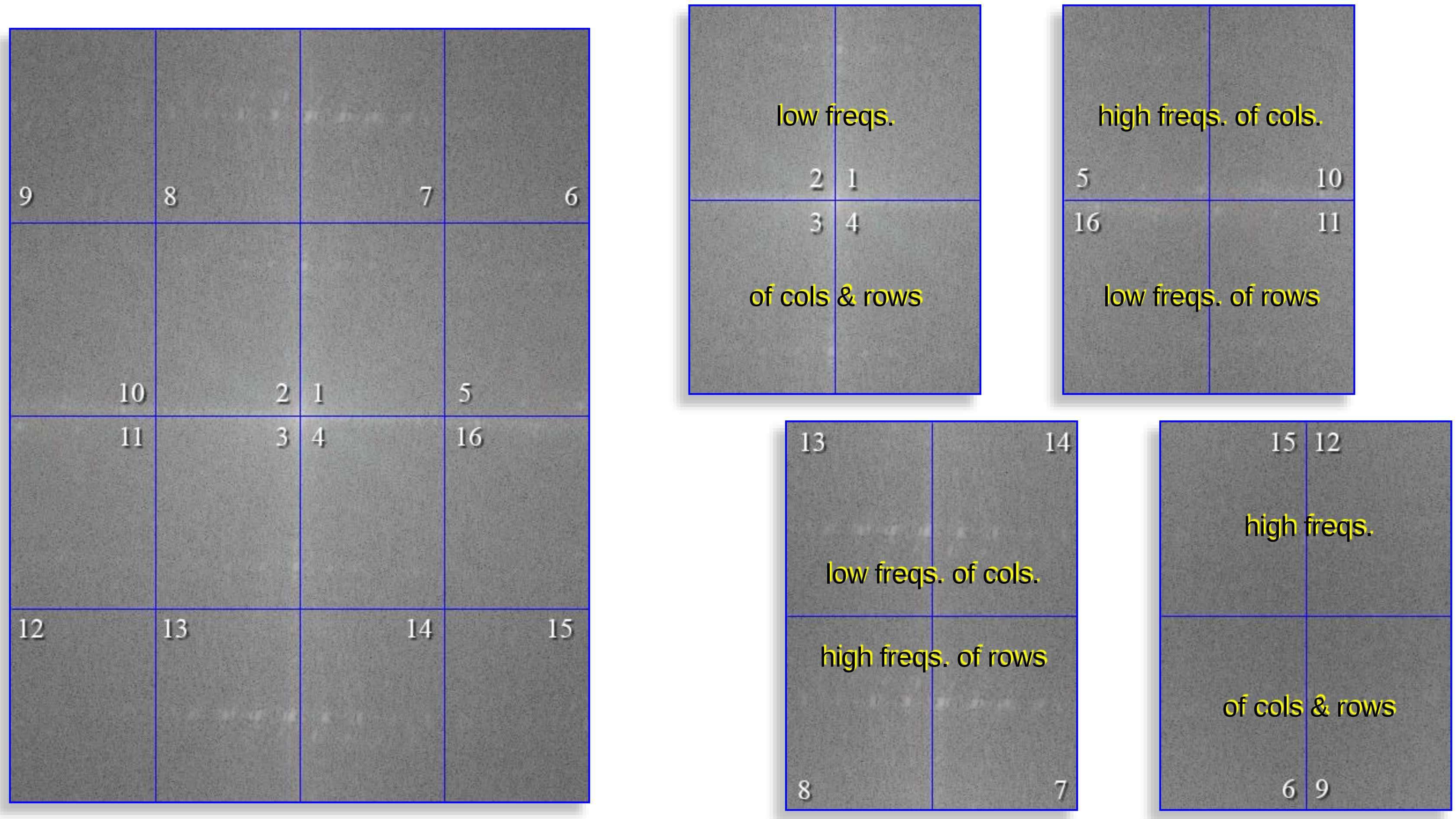
zoomed × 2



One might assume that the smaller DFT is the central region of the larger … but it's not.

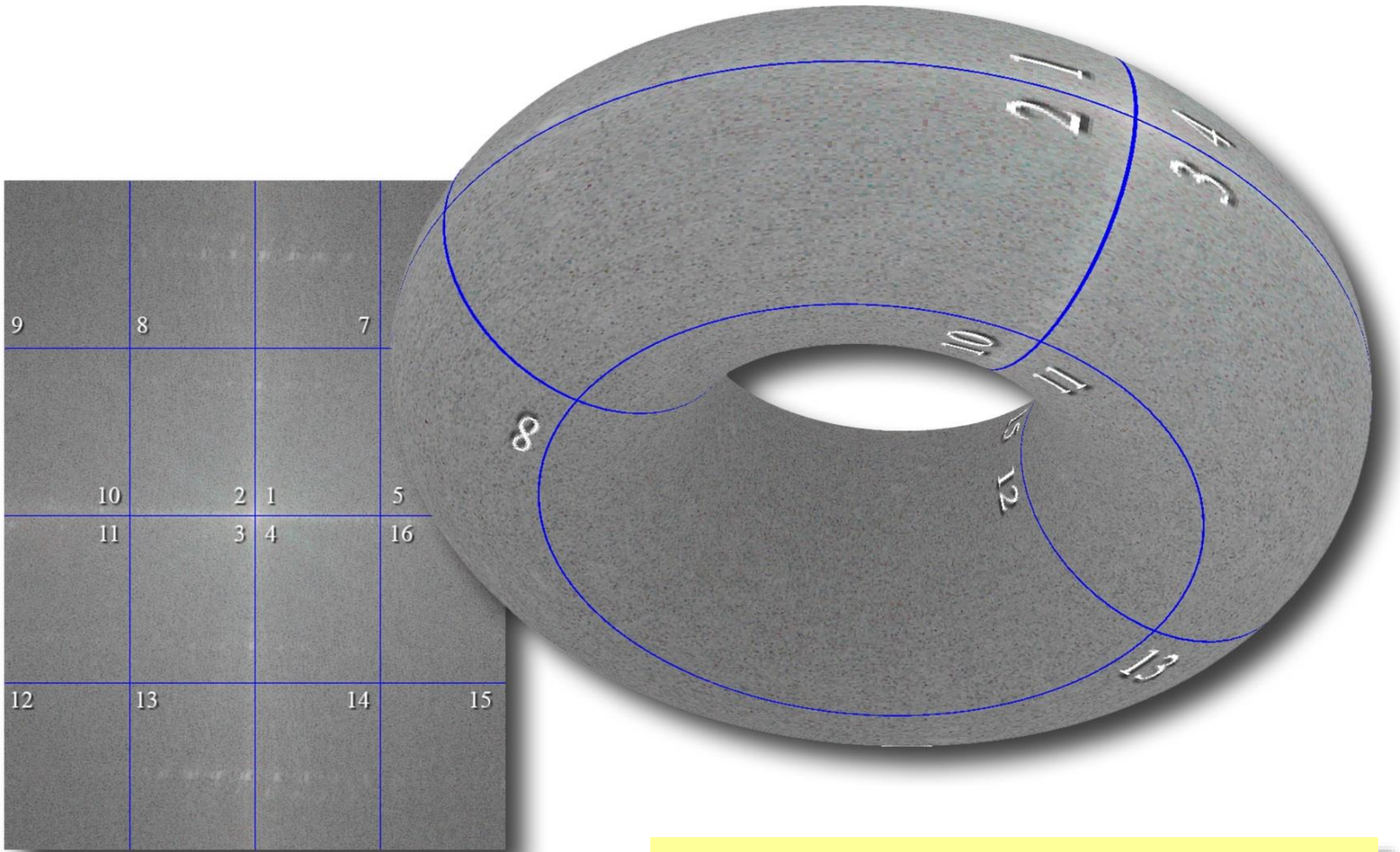
All of the larger one is present in the smaller one.

To Make Actual PS of $2\times$Decimated Image:



Partition $\mathfrak{F}\{\mathbf{I}\}$ into these four parts, …

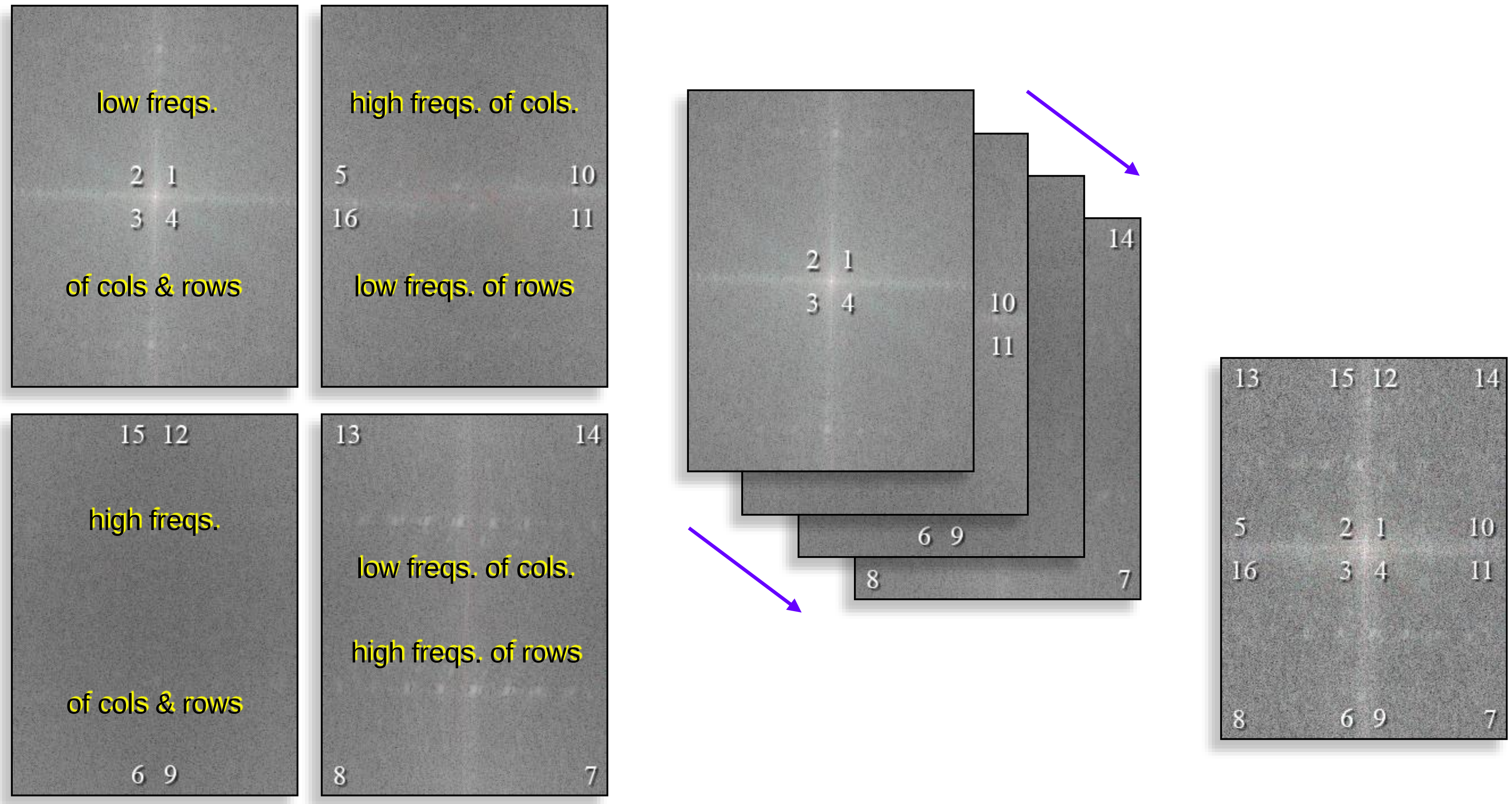
Actual PS of 2× Dec:



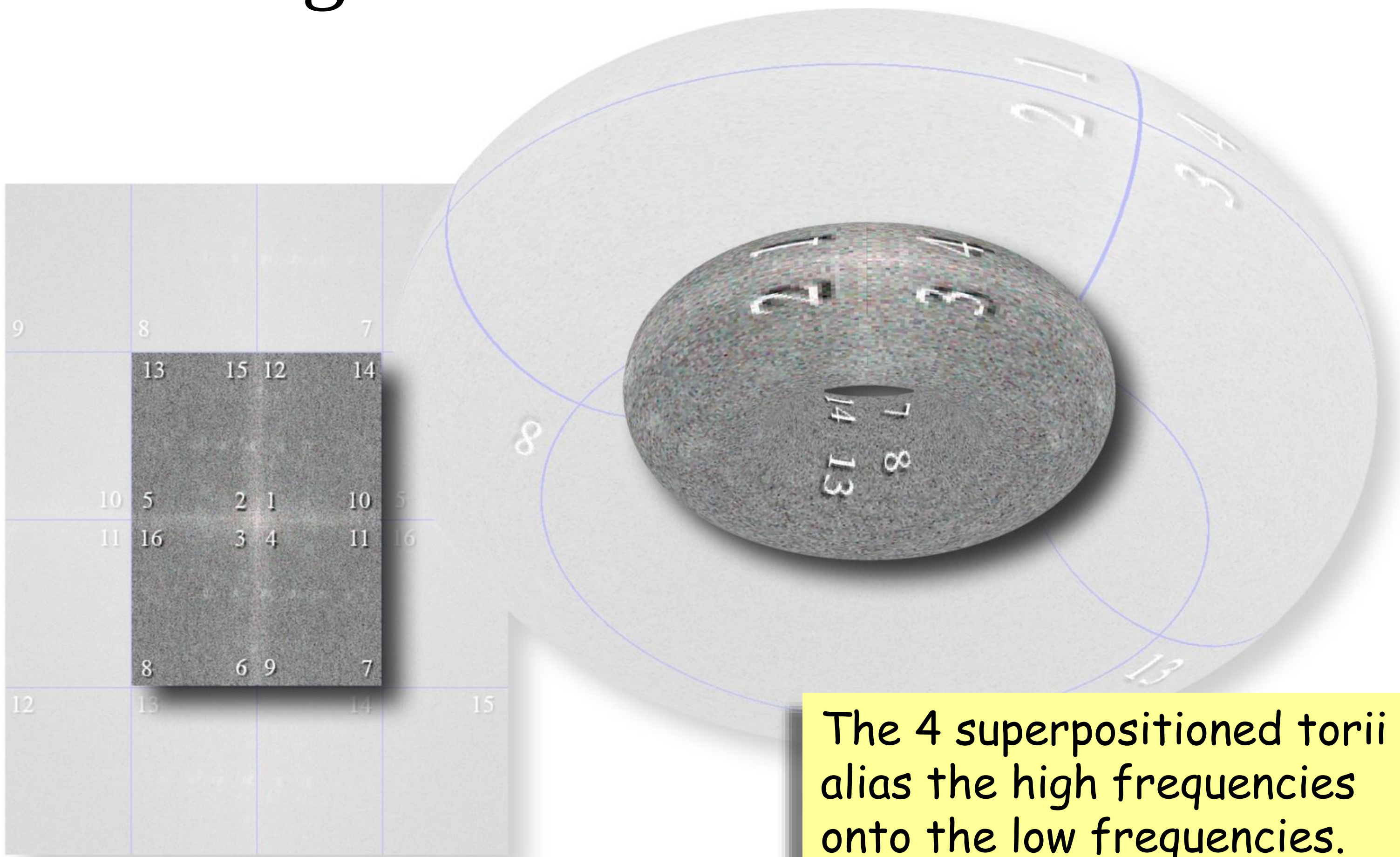
… four parts that are contiguous on the full-sized DFT torus.

Each of the 4 PS regions forms a torus. The 4 torii are superpositioned onto 1.

Actual PS of 2× Decimated Image

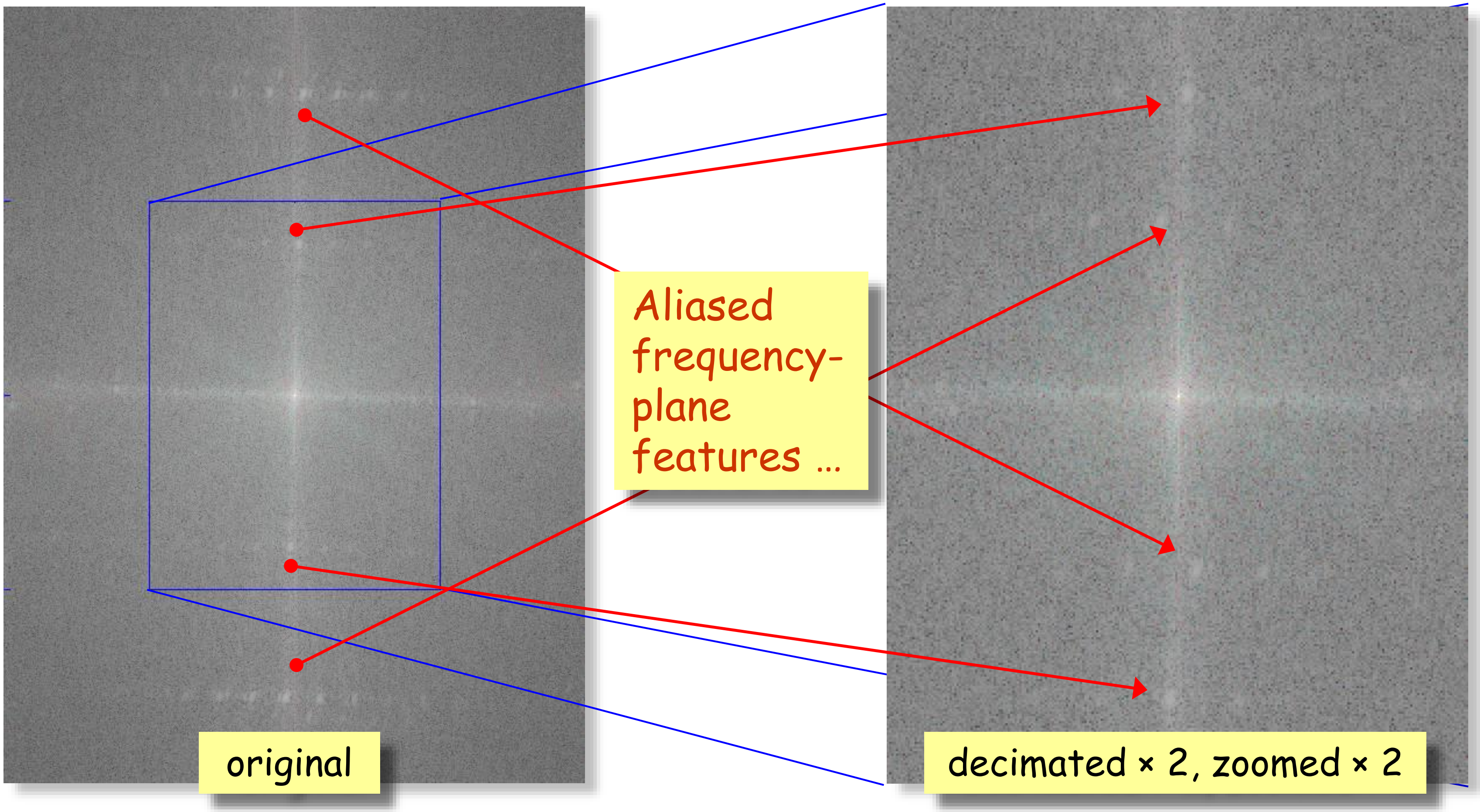


DFT Aliasing on the Torus



The 4 superpositioned torii alias the high frequencies onto the low frequencies.

PS of Original and PS of 2× Decimated Image



… lead to jaggies in the decimated image.

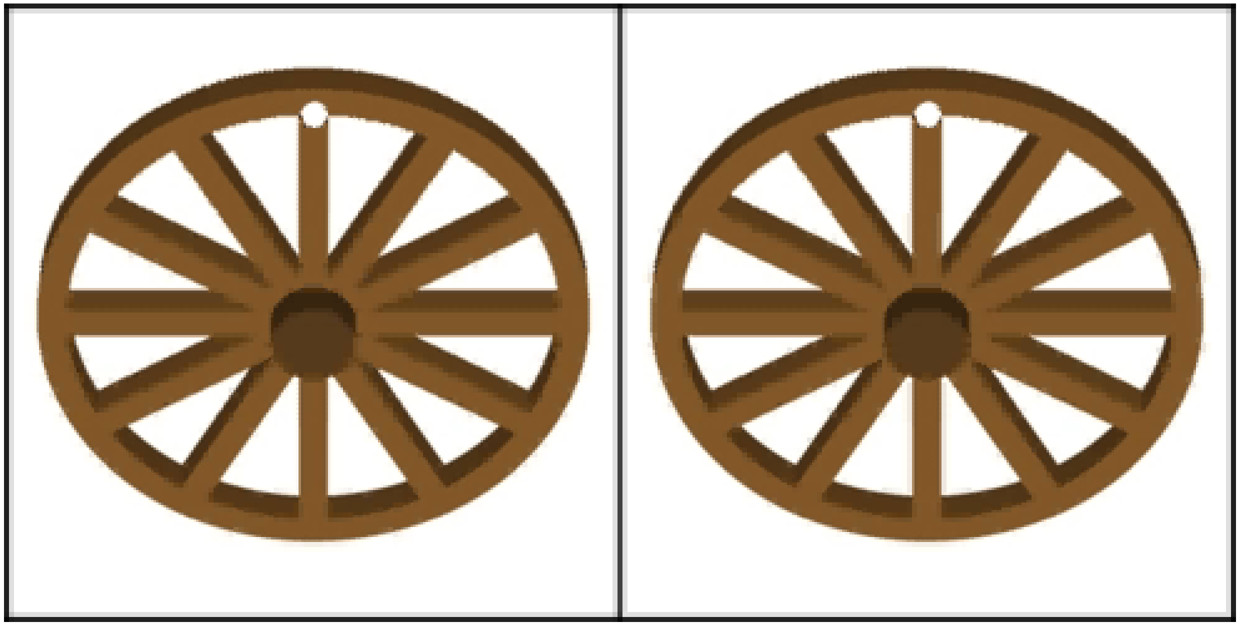
Original Image and 2× Decimated Image

original

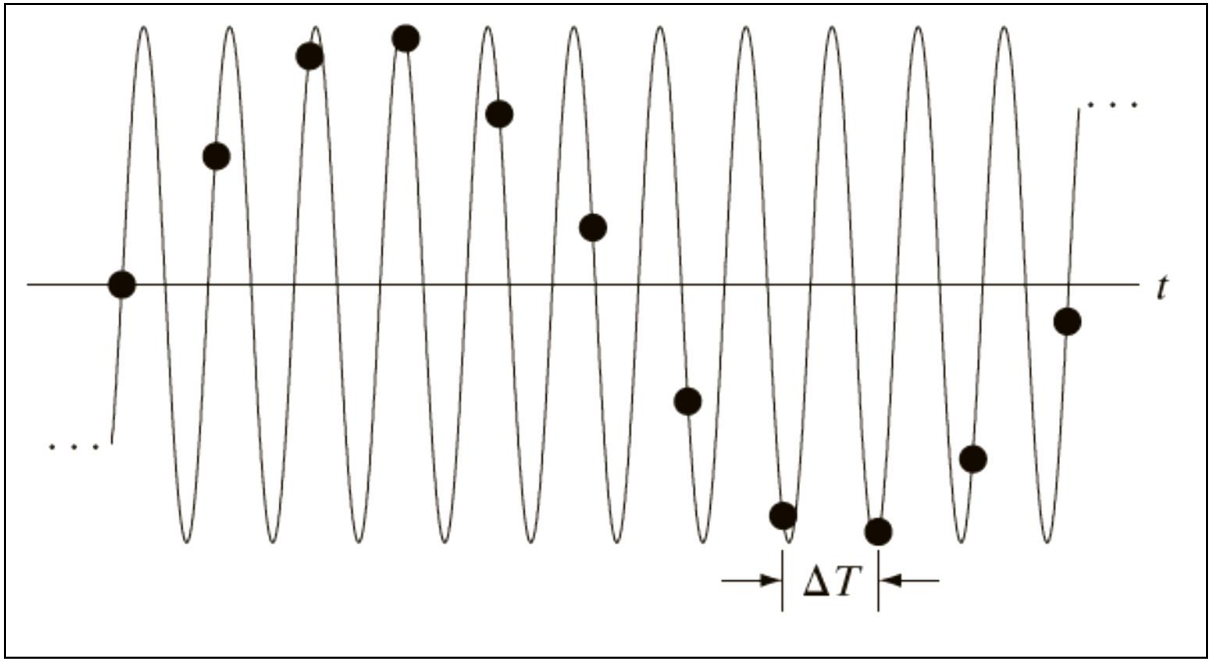
decimated × 2, zoomed × 2

First Alternative Explanation of Aliasing

The aliasing phenomenon can be described in terms of the so-called “wagon wheel” effect. The name comes from the appearance in a 24 frame per second movie of a wagon wheel rotating slightly faster than 12 frames per second. It appears to be rotating slowly backward.



Signal Aliasing



An under-sampled sinusoid is a sinusoid of a different frequency.

Figure from George Bebis, Univ. of Nevada, Reno, CS474, Image Processing and Interpretation, Lecture 4.3.

Nyquist-Shannon & Whittaker-Shannon Sampling

If a function [signal or image] x(t) contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced 1/(2B) seconds apart. In other words, a band-limited function can be perfectly reconstructed from a countable sequence of samples if the band limit, B, is no greater than half the sampling rate (samples per second).[1]

Given a sequence of real numbers, x[n], the continuous function:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t - nT}{T}\right)$$

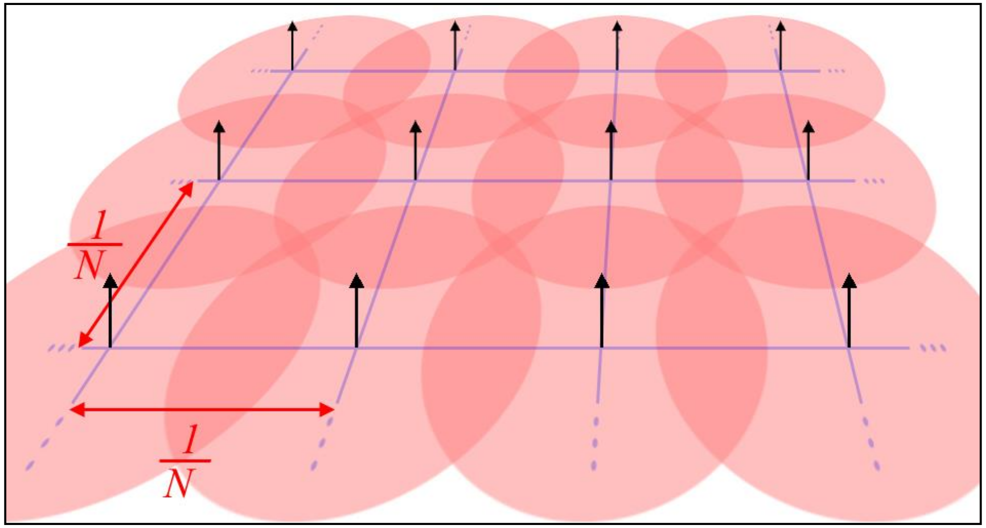
has a Fourier transform, $X(f)$, whose non-zero values are confined to the region: $|f| \leq 1/2T$. When parameter T has units of seconds, the band limit, $1/2T$, has units of cycles/sec (hertz). When the x[n] sequence represents time samples, at interval T, of a continuous function, the quantity $f_s = 1/T$ is known as the sample rate, and $f_s/2$ is the corresponding Nyquist frequency. When the sampled function has a bandlimit, B, less than the Nyquist frequency, $x(t)$ is a perfect reconstruction of the original function.[2]

[1] http://en.wikipedia.org/wiki/Nyquist–Shannon_sampling_theorem

[2] http://en.wikipedia.org/wiki/Whittaker–Shannon_interpolation_formula

Second Alternative Explanation of Aliasing

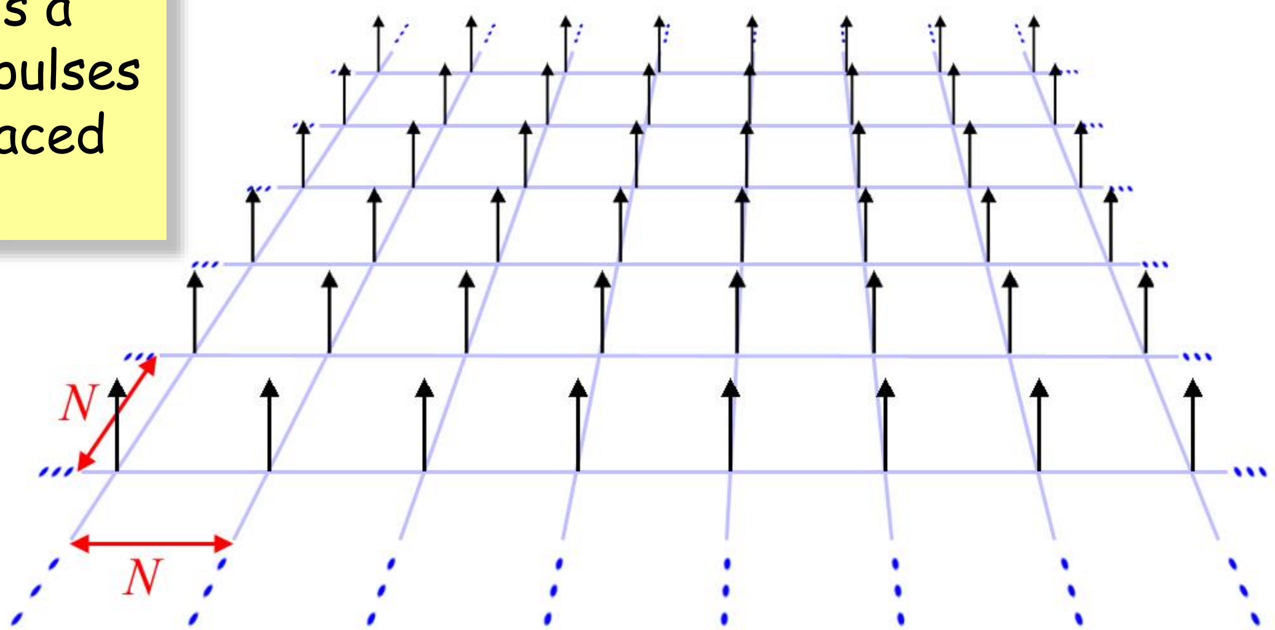
The aliasing phenomenon can also be described in terms of the convolution property of the Fourier Transform. In fact, this is the way it is usually explained.



The Sampling Function

The sampling function is a set of impulses evenly spaced on a grid.

$$\text{samp}_N(r,c) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(r - jN)\,\delta(c - kN)$$



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The Sampling of an Image

An image is sampled by multiplying it by the sampling function

$$\mathbf{I}(r,c)\cdot\ \mathrm{samp}_N(r,c)$$

$$\mathrm{samp}_N(r,c)=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(r-jN)\,\delta(c-kN)$$

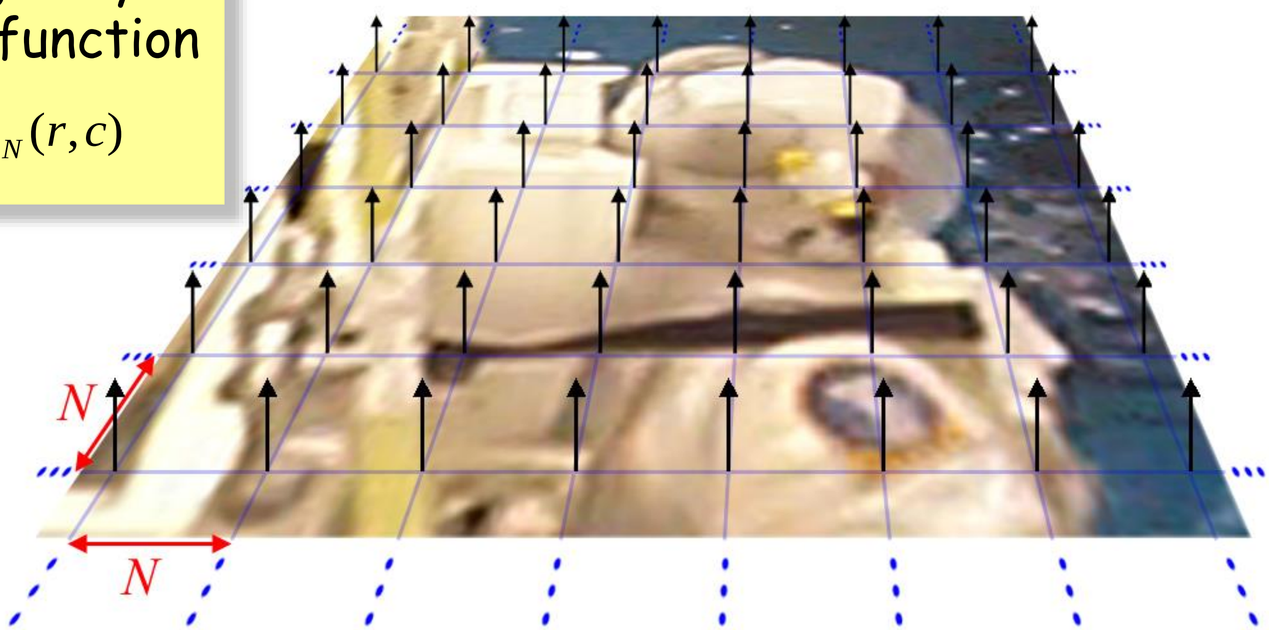
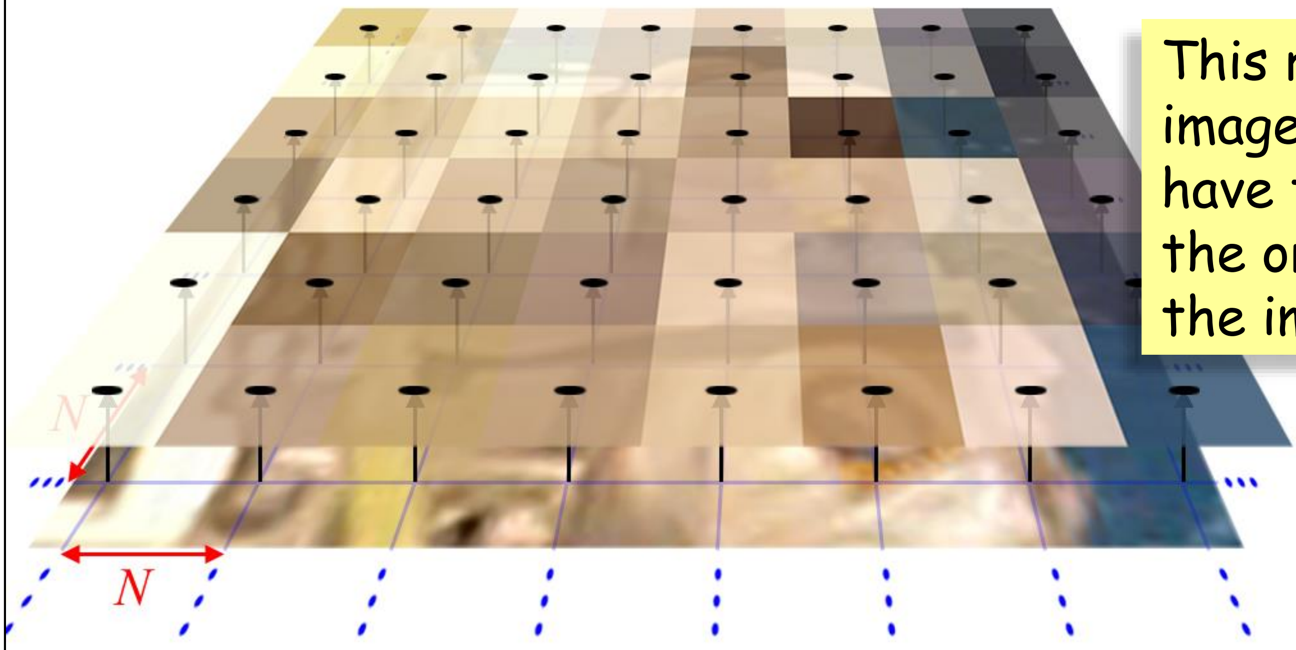


Image Sampling in the Spatial Domain

$$\operatorname{samp}_N\{\mathbf{I}\}(r,c)=\mathbf{I}(r,c)\cdot\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(r-jN)\cdot\delta(c-kN)$$



This results in an image whose pixels have the values of the original image at the impulse locations.

Convolution Property of the Fourier Transform

Let functions $f(r,c)$ and $g(r,c)$ have Fourier Transforms $F(u,v)$ and $G(u,v)$. Then,

$$\mathfrak{F}\{f * g\} = F \cdot G.$$

Moreover,

$$\mathfrak{F}\{f \cdot g\} = F * G.$$

$*$ represents convolution

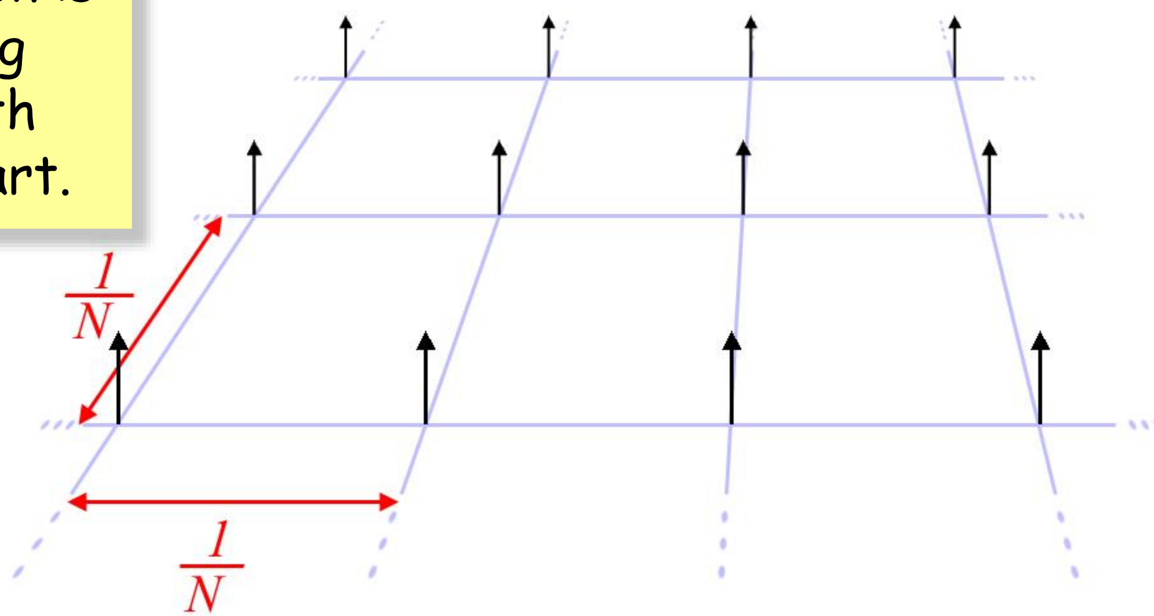
$\cdot$ represents pointwise multiplication

The Fourier Transform of a product equals the convolution of the Fourier Transforms. Similarly, the Fourier Transform of a convolution is the product of the Fourier Transforms

The Fourier Transform of the Sampling Function

The Fourier Transform of the sampling function is another sampling function but with impulses $1/N$ apart.

$$\text{samp}_{1/N}(u,v) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta\left(u-\frac{j}{N}\right)\delta\left(v-\frac{k}{N}\right)$$



Cf. slide 23

Convolution by an Impulse

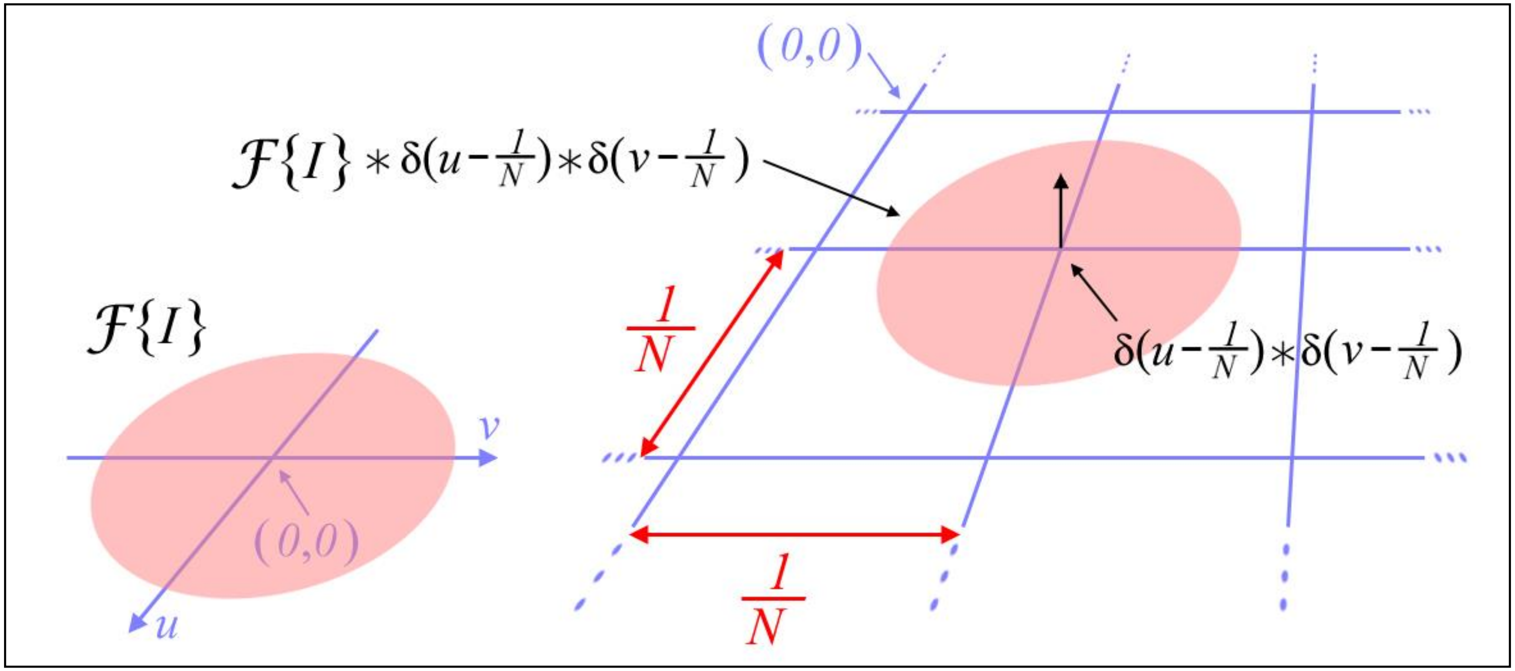
An *impulse* is a digital image, that has a single pixel with value 1; all others have value zero. An impulse at location (ρ, χ) is represented by:

$$\delta(r-\rho, c-\chi) = \begin{cases} 1, & \text{if } \; r=\rho \;\text{ and }\; c=\chi \\ 0, & \text{otherwise} \end{cases}$$

If an image is convolved with an impulse at location (ρ, χ), the image is shifted in location down by ρ pixels and to the right by χ pixels.

$$\left[\mathbf{I} * \delta(r-\rho, c-\chi)\right](r,c) = \mathbf{I}(r-\rho, c-\chi).$$

Convolution by an Impulse

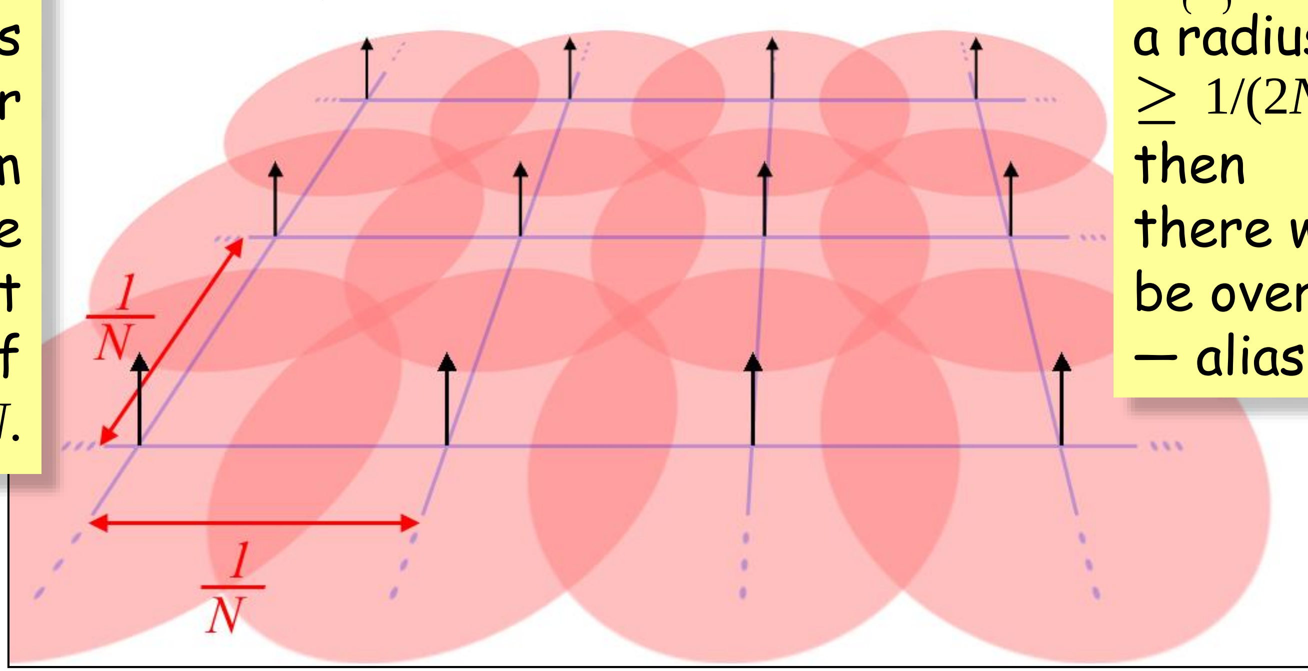


The convolution of any function with a delta function translates the function to the location of the impulse.

The Fourier Transform of a Sampled Image

The sampling of image $\mathbf{I}$ causes its Fourier Transform $\mathfrak{F}\{\mathbf{I}\}$ to be repeated at intervals of $1/N$.

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\mathfrak{F}\{\mathbf{I}\}(u,v)*\delta\left(u-\frac{j}{N}\right)*\delta\left(v-\frac{k}{N}\right)$$

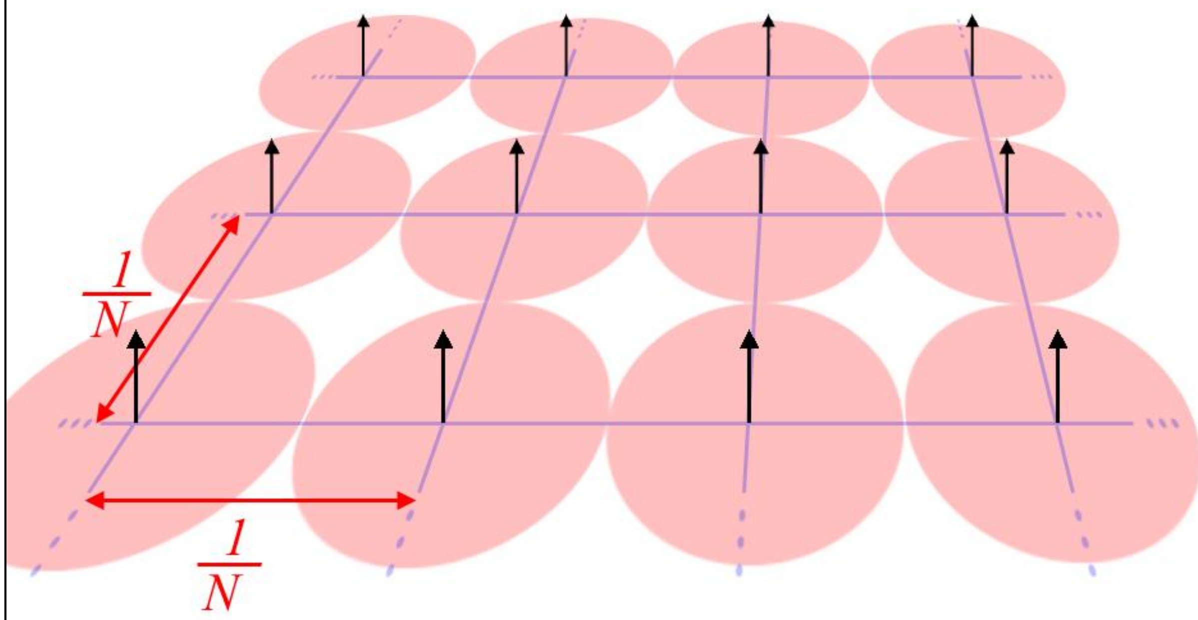


If the support of $\mathfrak{F}\{\mathbf{I}\}$ has a radius $\geq 1/(2N)$ then there will be overlap — aliasing

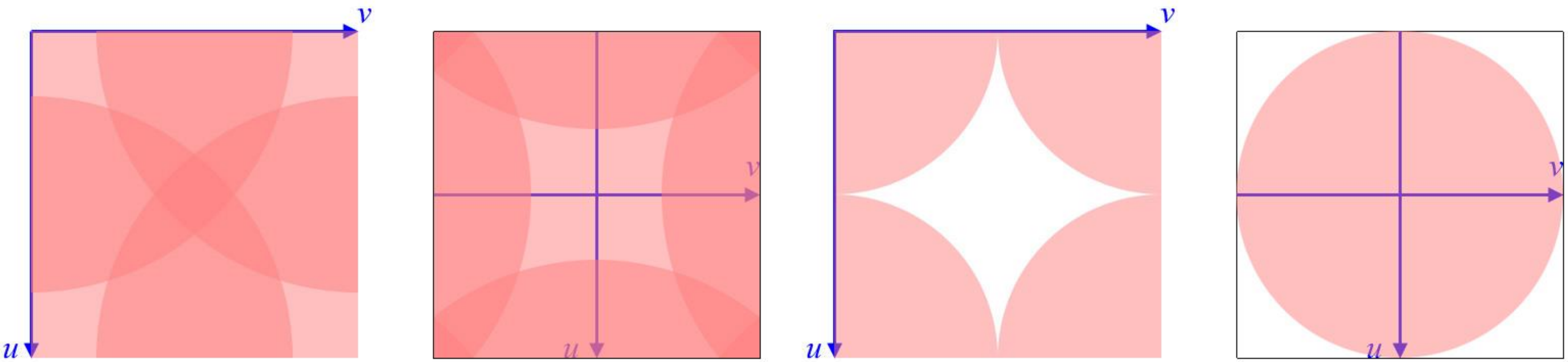
The FT of a Sampled Image with No Aliasing

For aliasing not to occur the frequency support of $\mathbf{I}$ must have a radius of $< 1/(2N)$

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\mathfrak{F}\{\mathbf{I}\}(u,v)*\delta\left(u-\frac{j}{N}\right)*\delta\left(v-\frac{k}{N}\right)$$



Part of the Transform Computed by the FFT

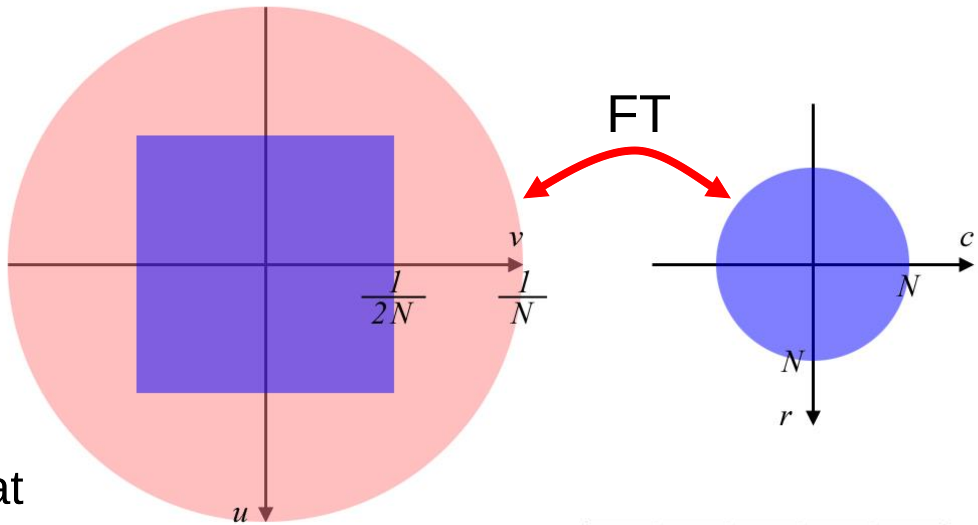


as computed (aliasing) | origin shifted (aliasing) | as computed (no aliasing) | origin shifted (no aliasing)

Although the FT of the sampled image continues on indefinitely in theory, only one copy of the complete pattern is required for processing. The Fast Fourier Transform (FFT) algorithm computes the transform with the origin in the upper left. Usually the transform is displayed with the origin in the center.

Filtering Before Downsampling to Prevent Aliasing

To sample $\mathbf{I}(r,c)$ every N pixels multiply $\mathfrak{I}(u,v) = \mathfrak{F}\{\mathbf{I}\}$ by a window $\mathbf{H}(u,v)$ such that



$$\mathbf{H}\left(u,v\right) = 0 \text{ for } u > \frac{1}{2N} \text{ or } v > \frac{1}{2N}$$

That is equivalent to convolving $\mathbf{I}(r,c)$ with a spatial filter of *width* $2N$.

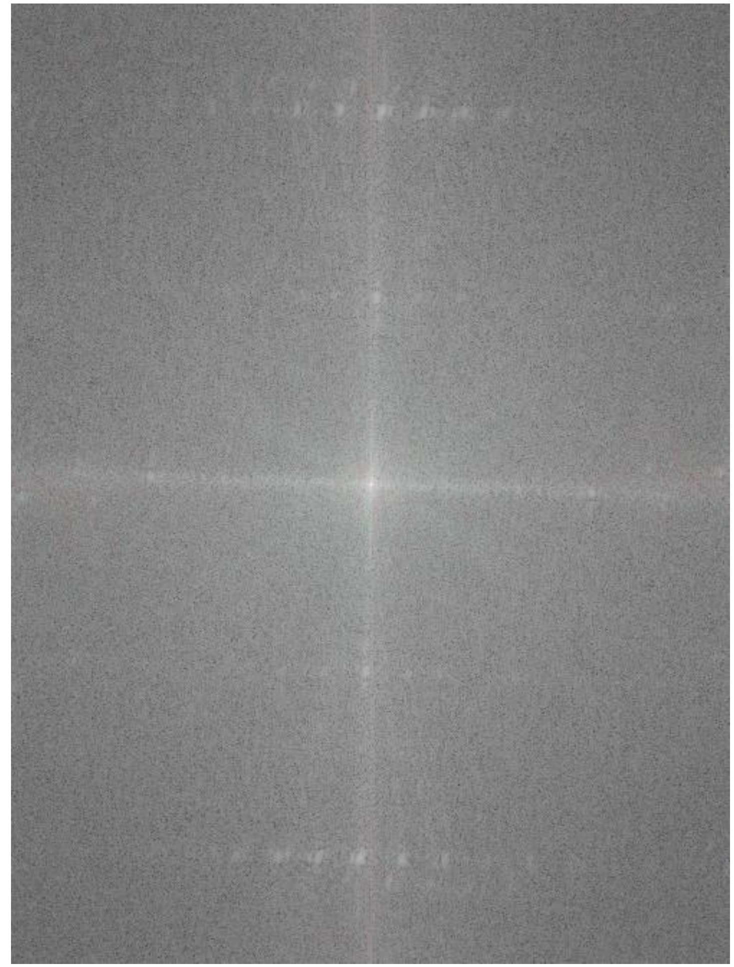
h_{11}	h_{12}	h_{13}	h_{14}	h_{15}
h_{21}	h_{22}	h_{23}	h_{24}	h_{25}
h_{31}	h_{32}	h_{33}	h_{34}	h_{35}
h_{41}	h_{42}	h_{43}	h_{44}	h_{45}
h_{51}	h_{52}	h_{53}	h_{54}	h_{55}

Thus, to down-sample $\mathbf{I}(r,c)$ by a factor of 2, use a spatial filter of width 4. Since this is midway between 3 and 5, use a 5x5 convolution mask.

If the image is a bit blurry then a 3x3 will usually suffice

Filtering to Prevent Aliasing

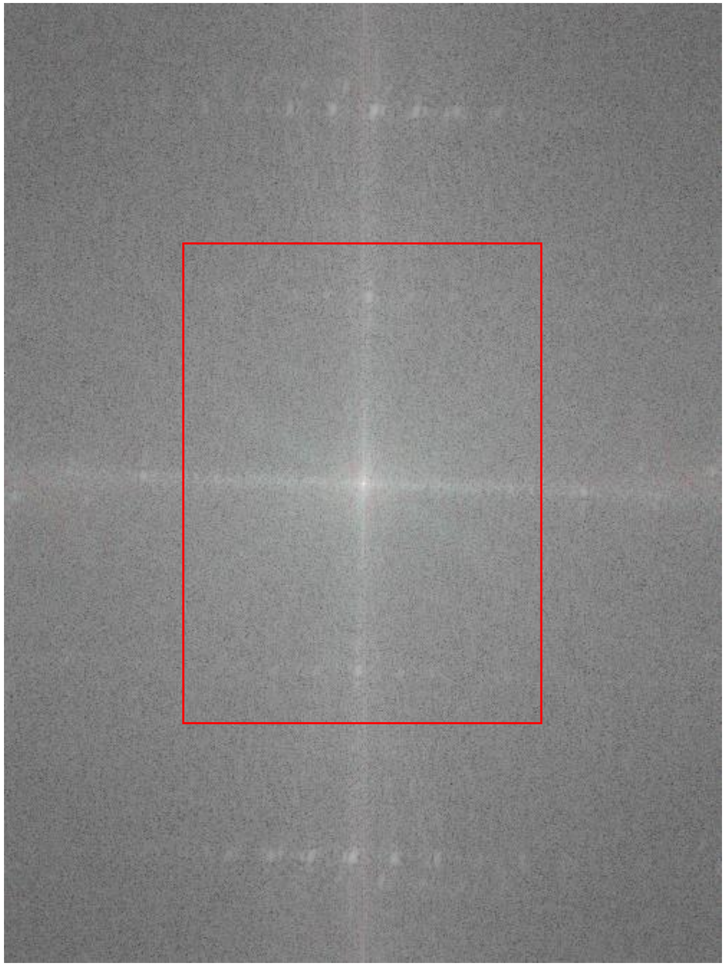
original image



power spectrum

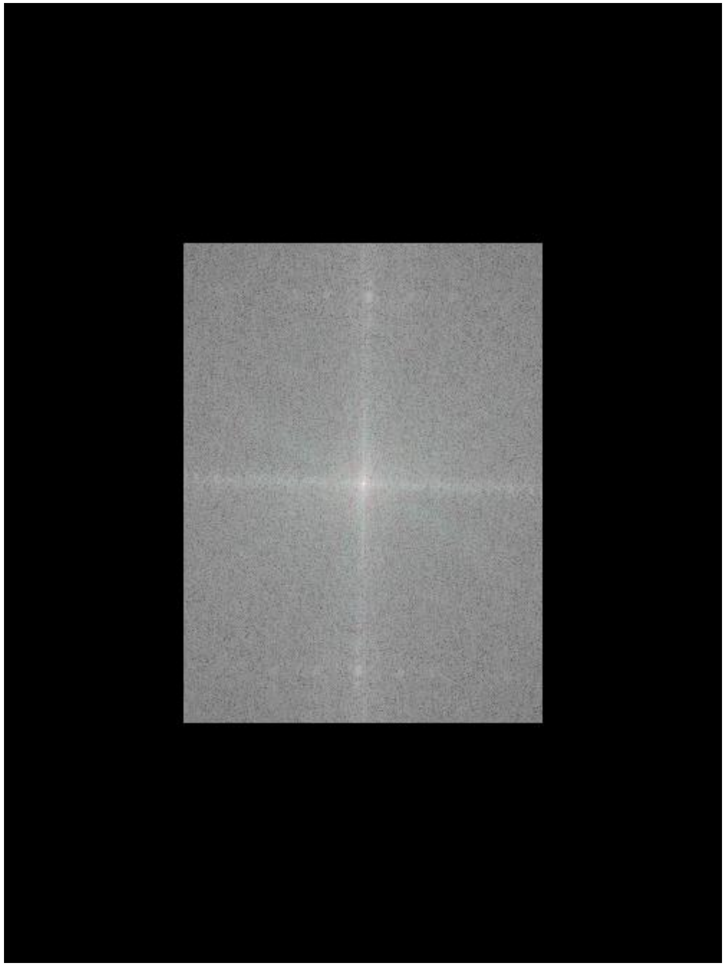
Filtering to Prevent Aliasing

to decimate original by 2 …



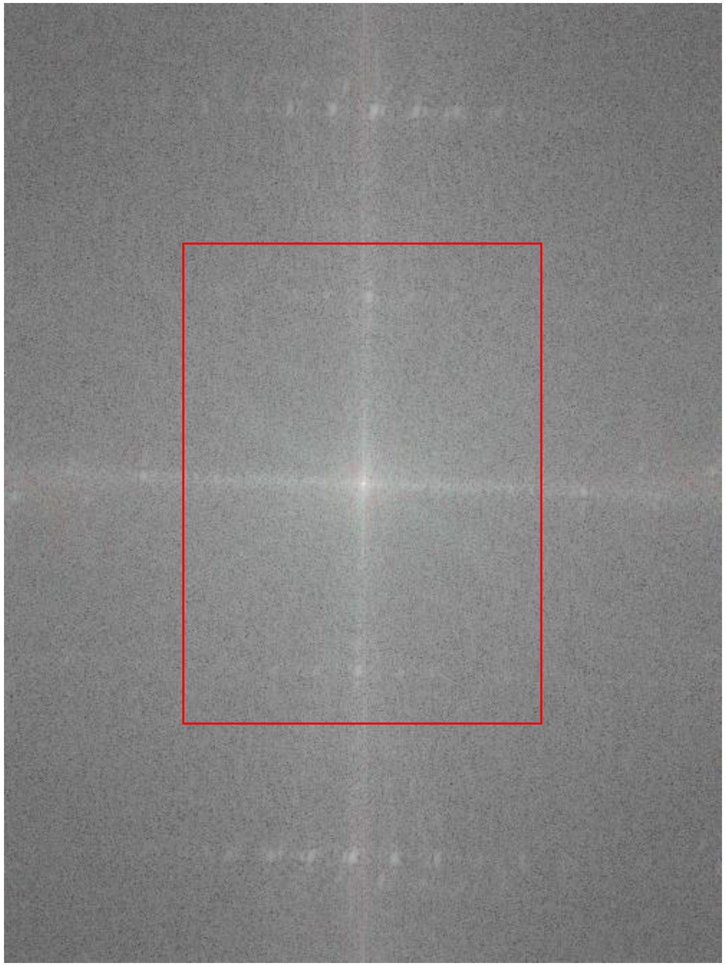
… zero outside red square.

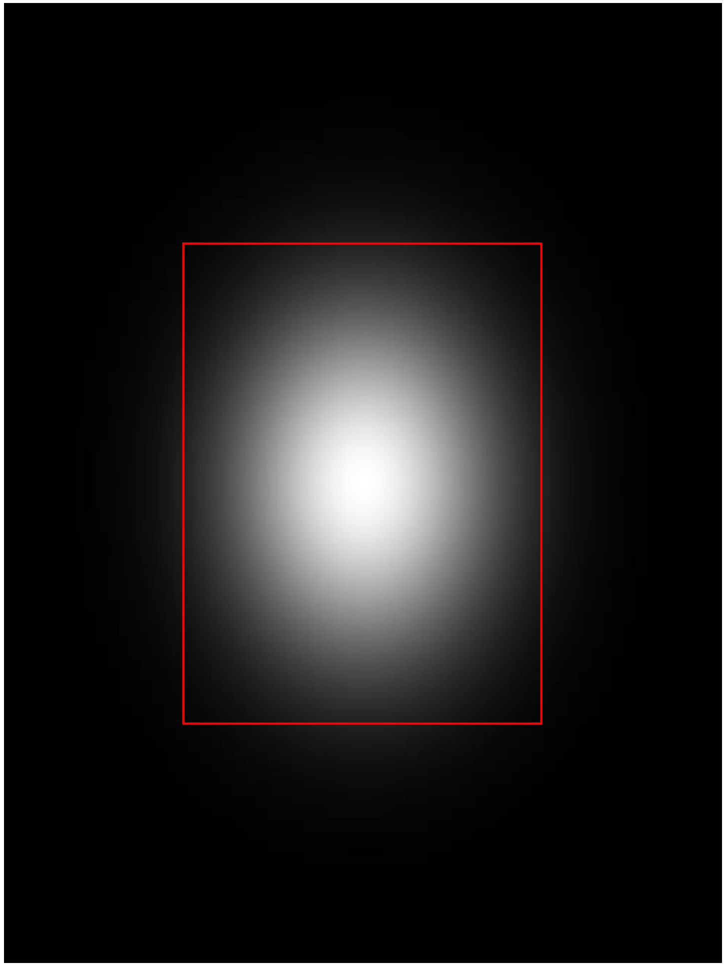
Notice the shift & ringing in the result



ideal filter

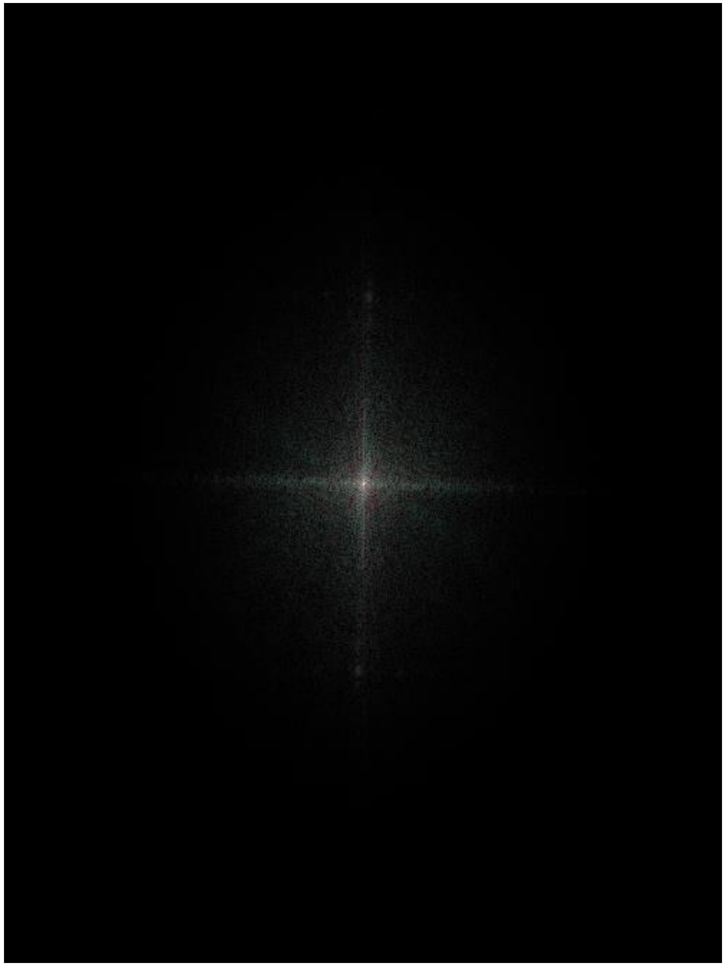
Filtering to Prevent Aliasing





To reduce ringing try multiplying the FT by a Gaussian w/ $(\sigma_v,\sigma_u)=(¼R, ¼C)$

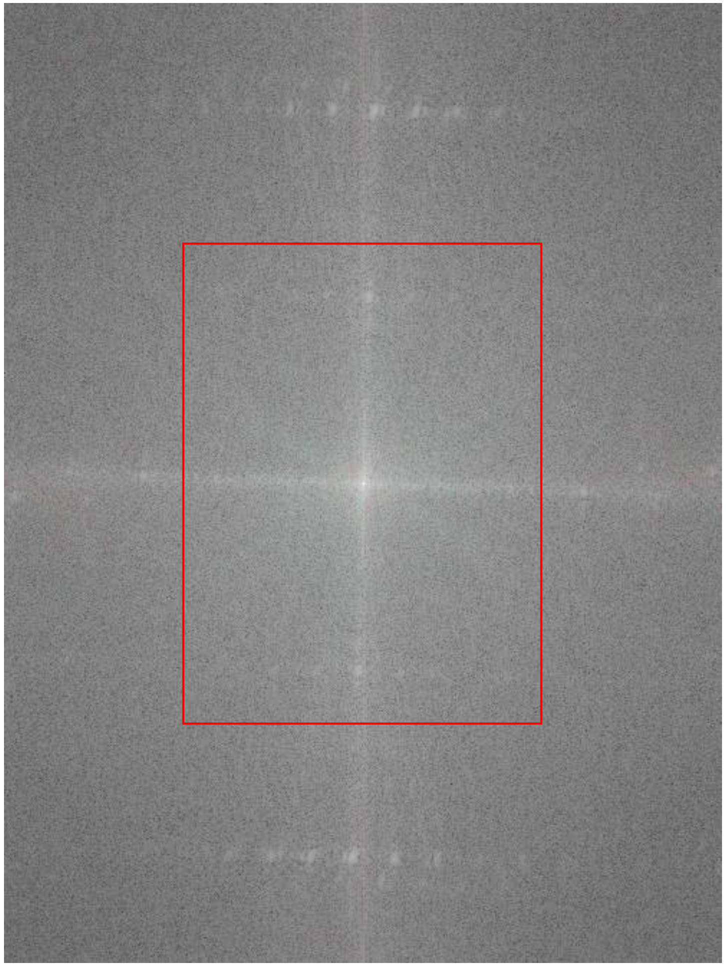
Filtering to Prevent Aliasing



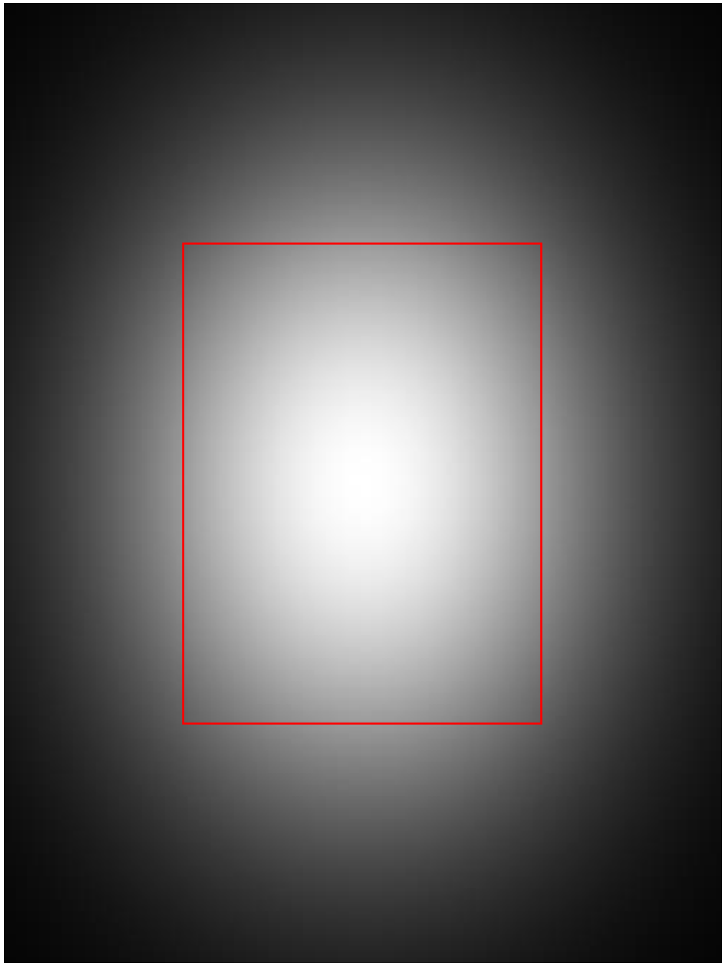
PS of FT × Gaussian

ringing is reduced but result is blurry

Filtering to Prevent Aliasing

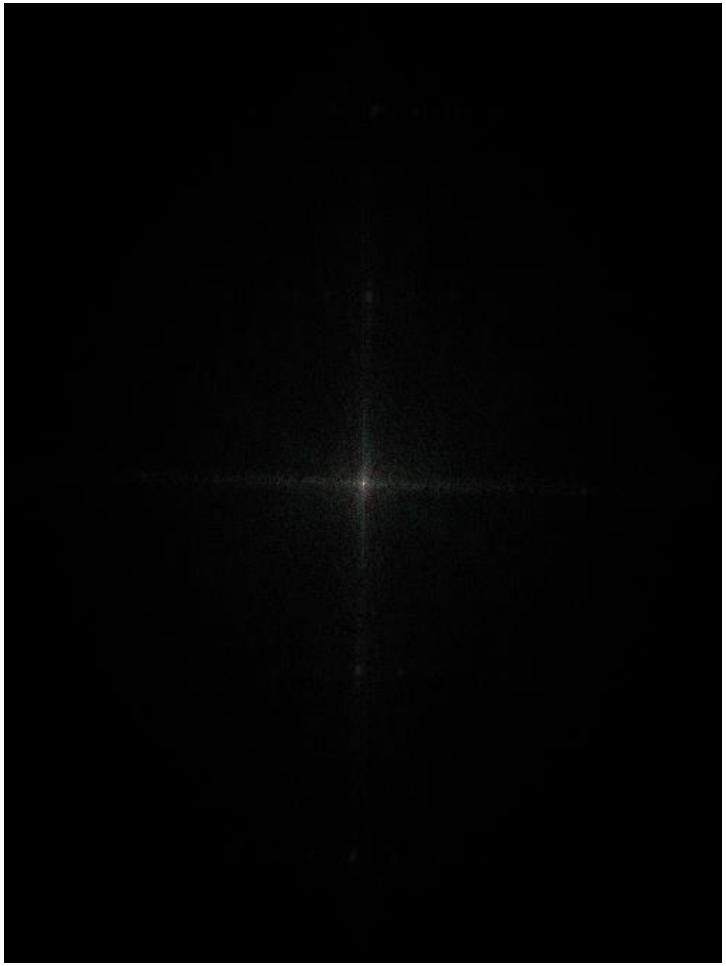


try multiplying the FT



by a Gaussian w/ $(\sigma_v,\sigma_u)=(½R, ½C)$

Filtering to Prevent Aliasing



PS of FT × Gaussian

some aliasing but result is less blurry

Filtering to Prevent Aliasing

ideal filtered

original

Filtering to Prevent Aliasing

original

Gaussian w/ $(\sigma_v,\sigma_u)=(\frac{1}{4}R, \frac{1}{4}C)$

Gaussian w/ $(\sigma v, \sigma u) = (\frac{1}{2}R, \frac{1}{2}C)$

original

original

decimated by factor of 2

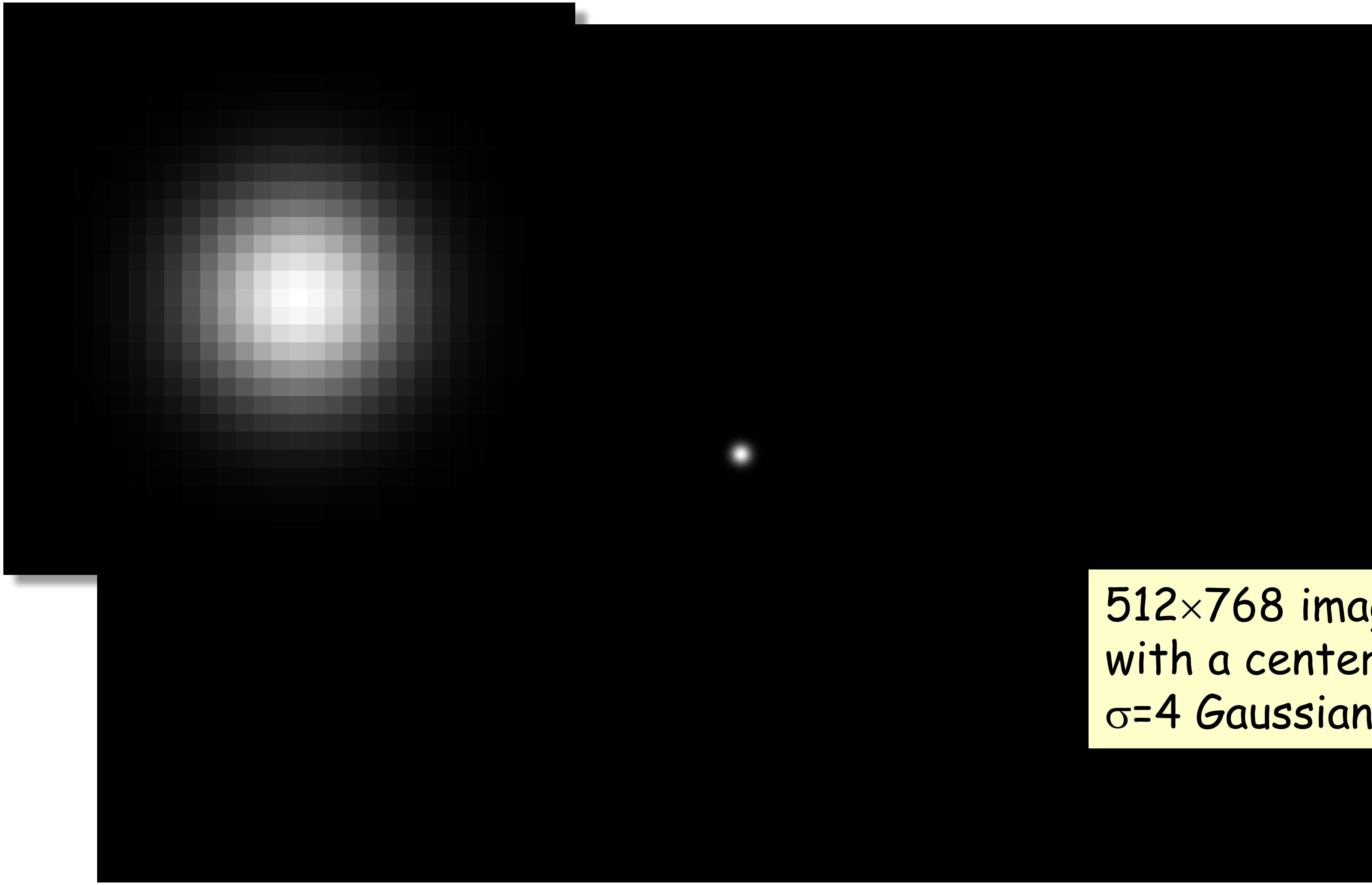
Reconstruction of a Downsampled Image

A band-limited image that was downsampled by a factor of two can be reconstructed – upsampled – using the Fourier Transform.

Simply copy the FT of the downsampled image into the center of a complex-valued image of all zeros that has twice the dimensions of the downsampled image. Then take the inverse FT.

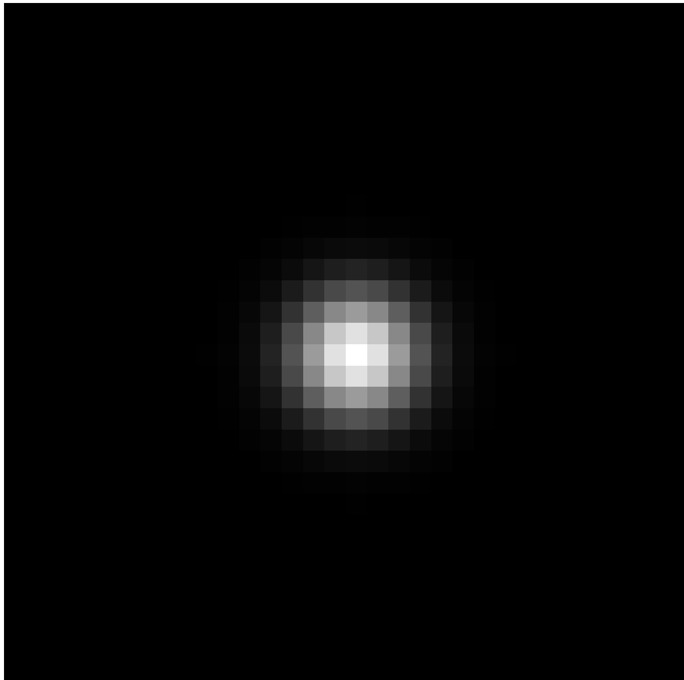
This follows from the FT that results from multiplying the image with a "checkerboard" sampling grid. The following example with a Gaussian distribution demonstrates this.

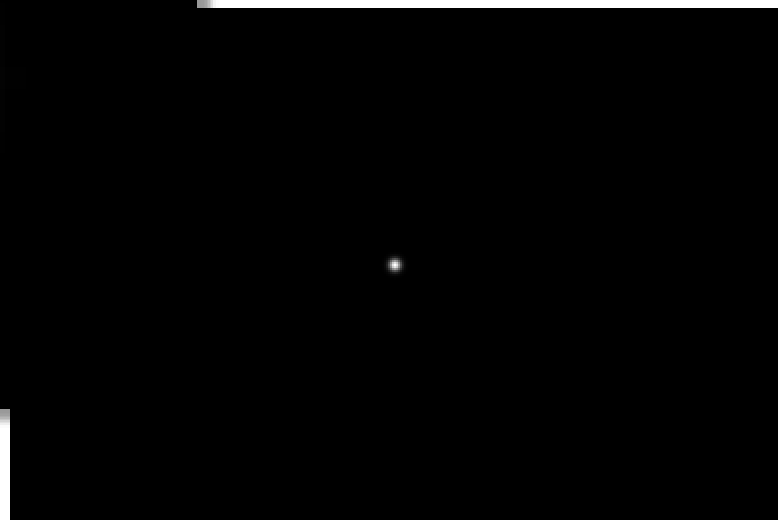
Reconstruction



512×768 image with a centered, σ=4 Gaussian.

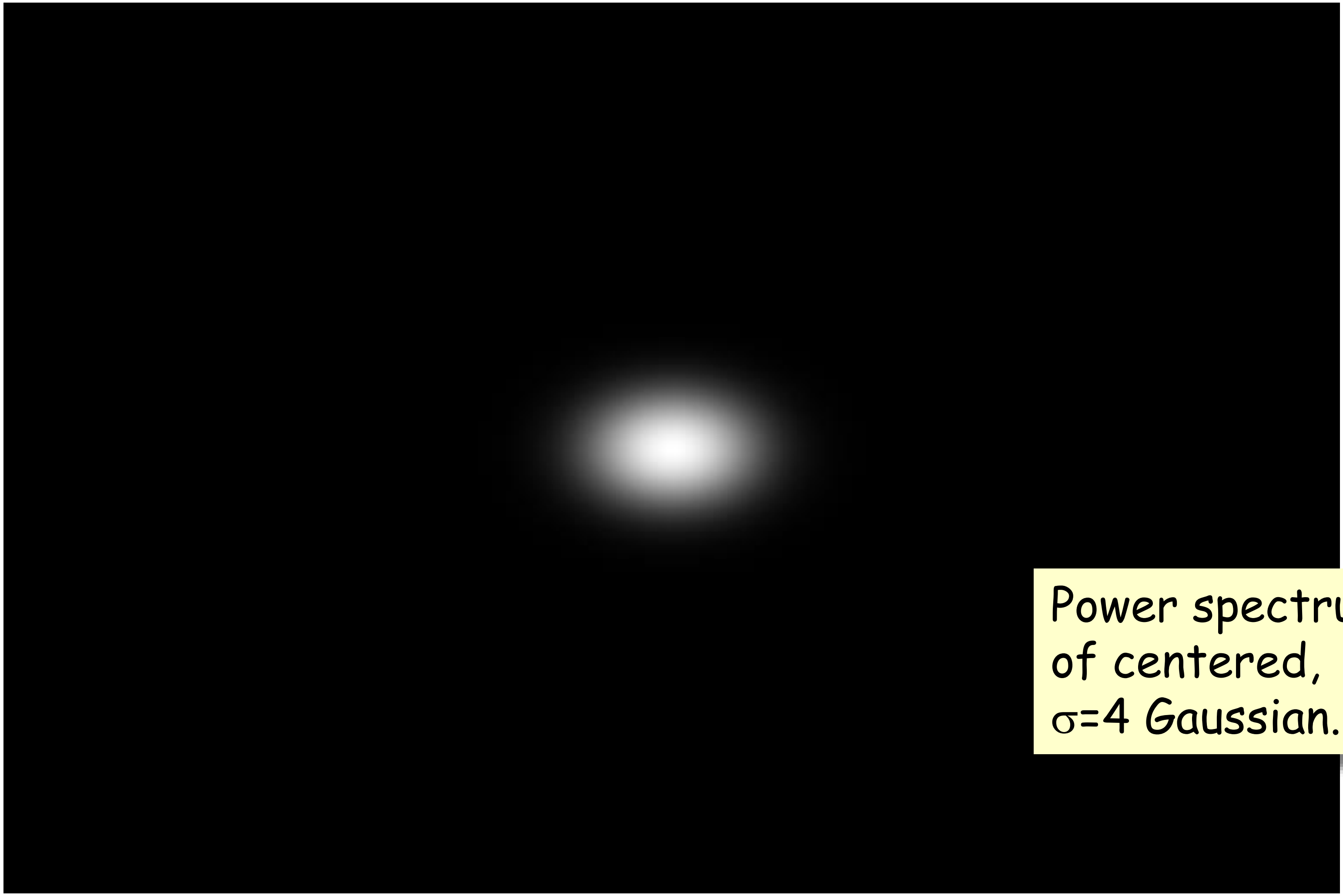
Reconstruction





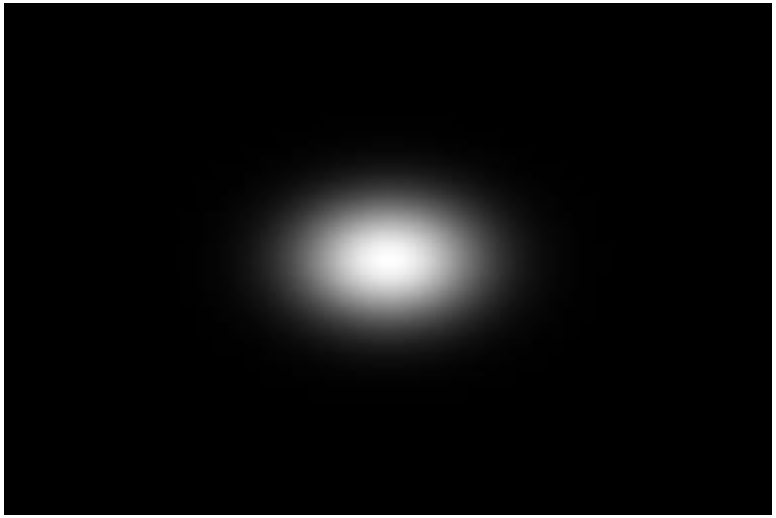
256×384 image of downsampled (1:2:512,1:2:768) σ=4 Gaussian.

Reconstruction



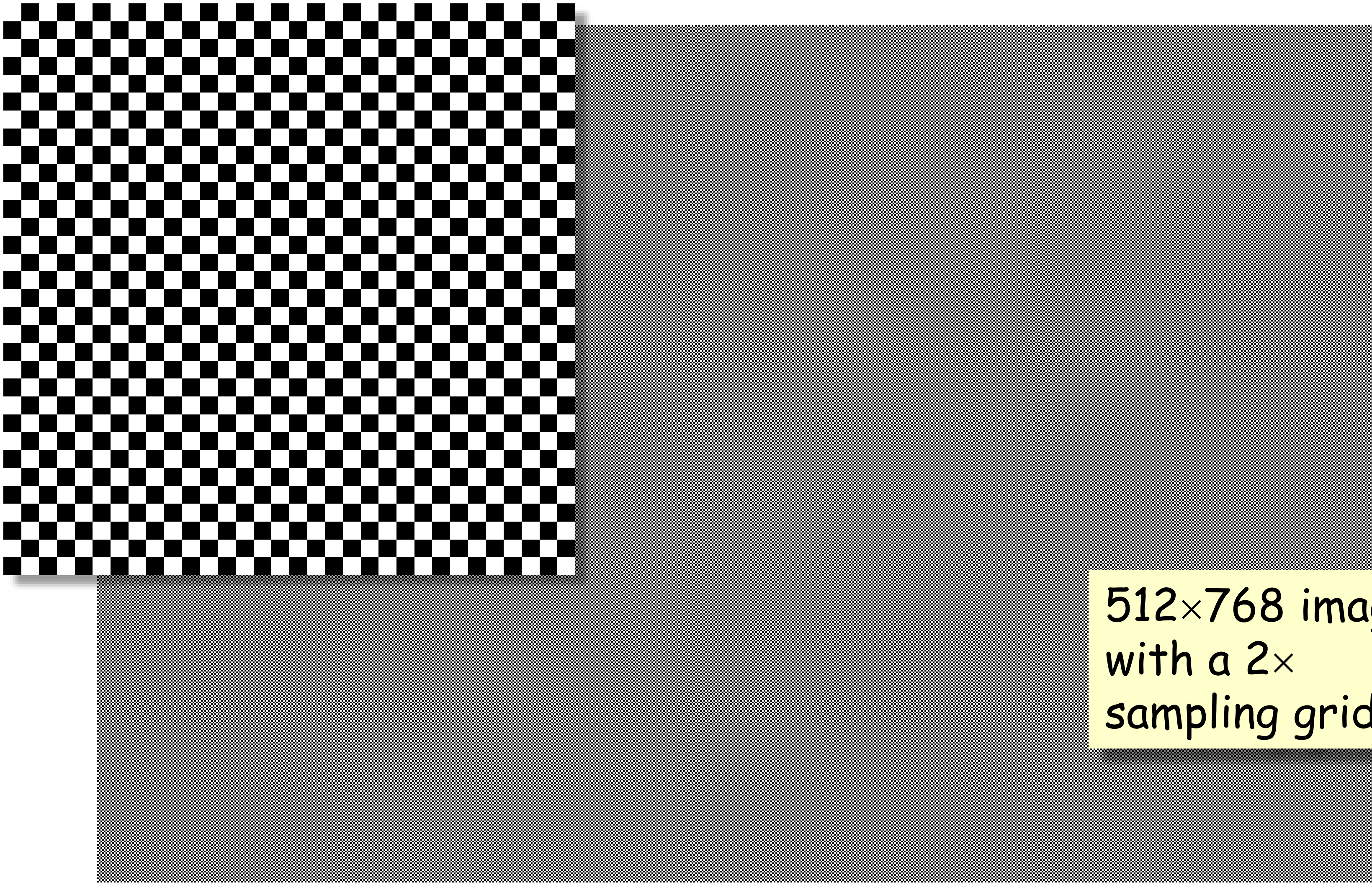
Power spectrum of centered, σ=4 Gaussian.

Reconstruction



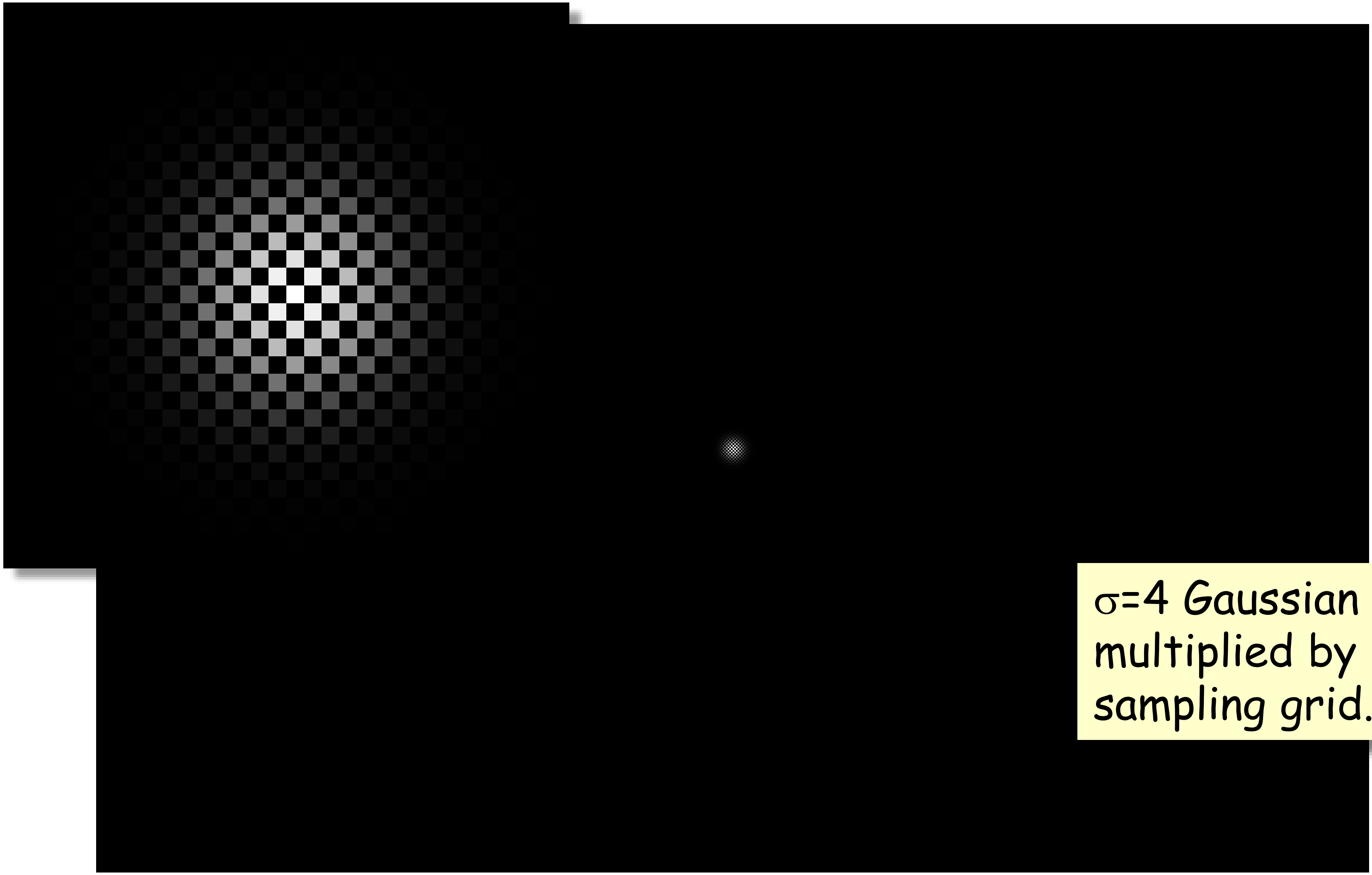
Power spectrum of downsampled σ=4 Gaussian.

Reconstruction



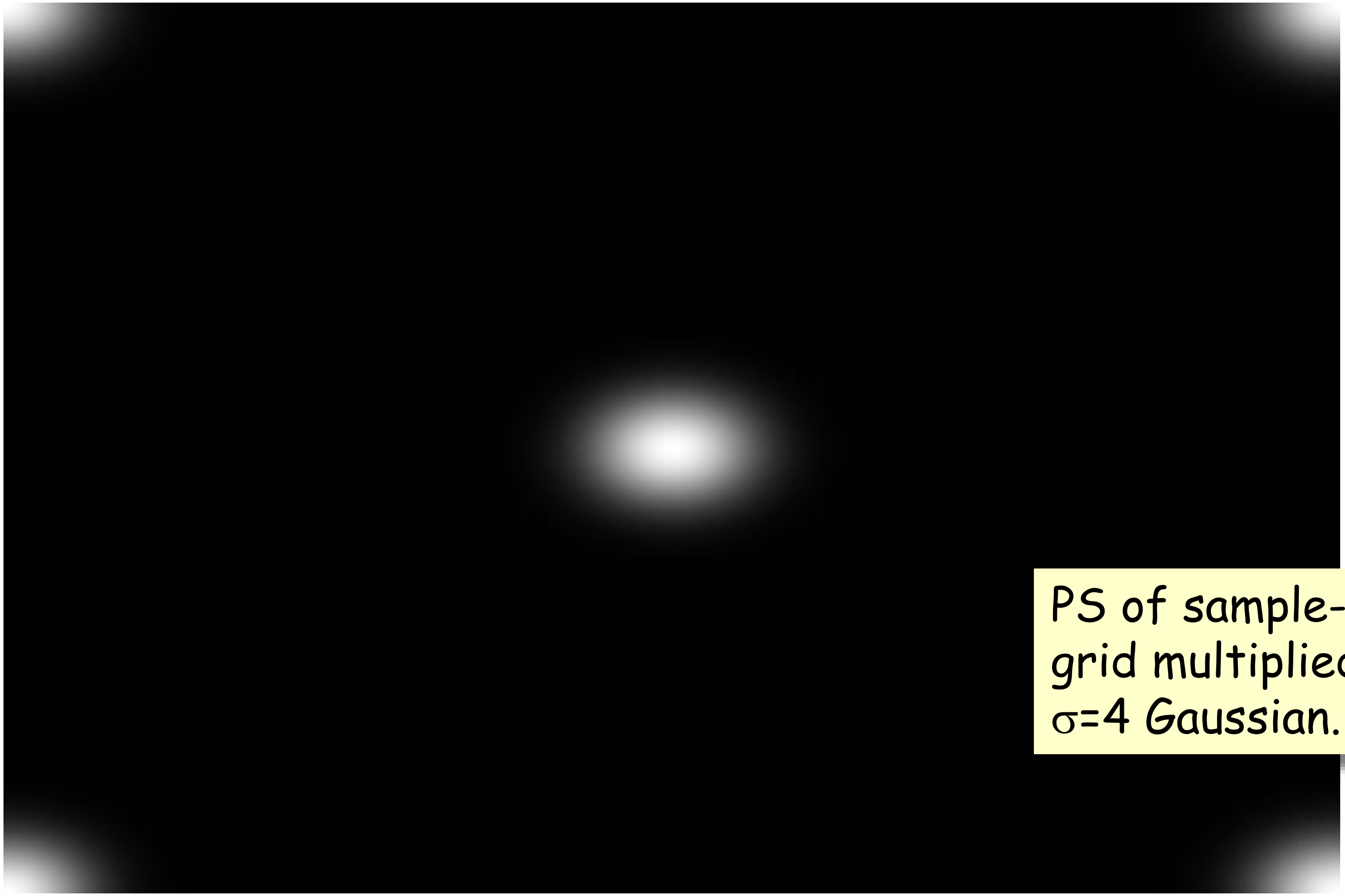
512×768 image with a 2× sampling grid

Reconstruction



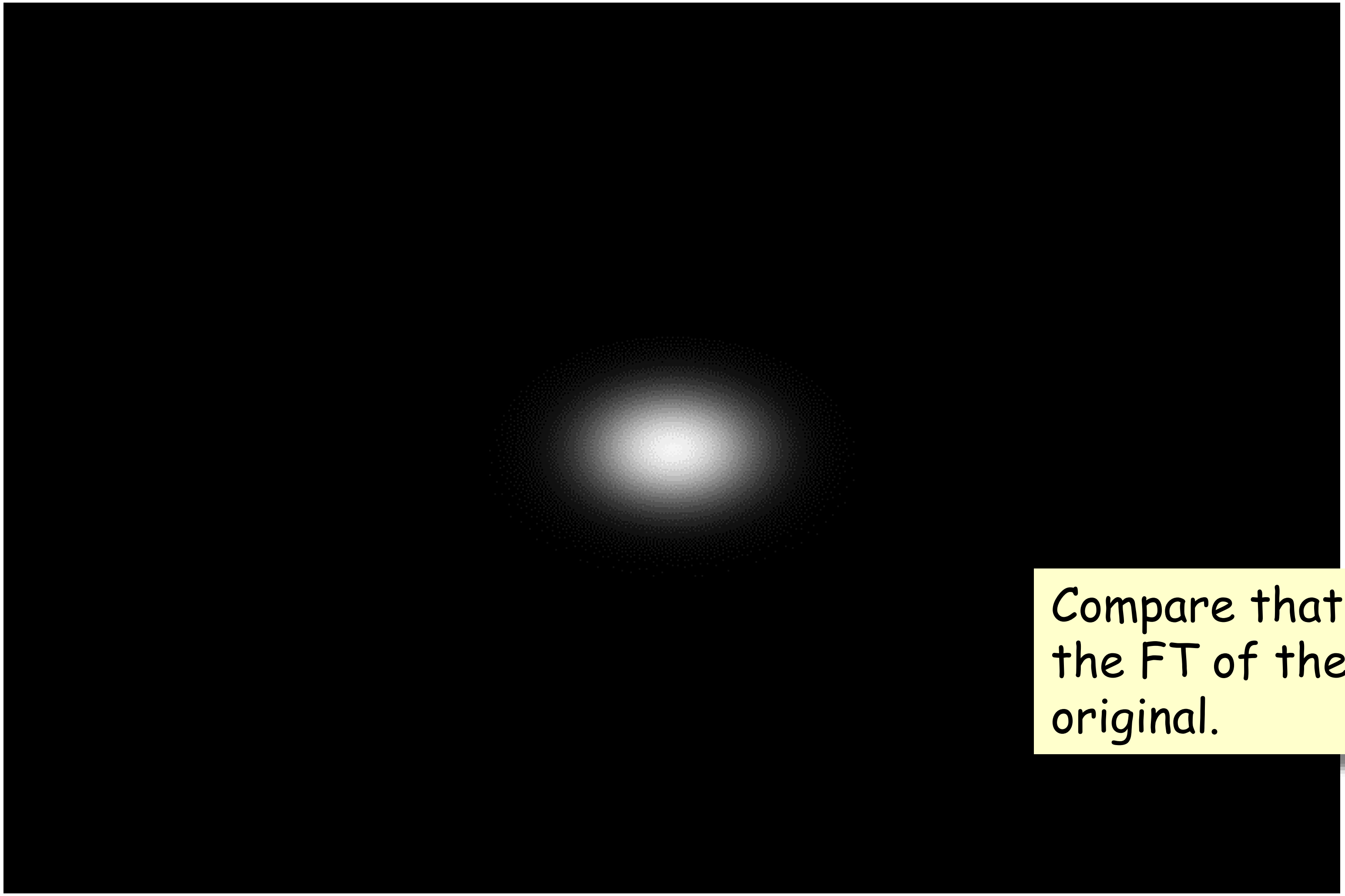
σ=4 Gaussian multiplied by sampling grid.

Reconstruction



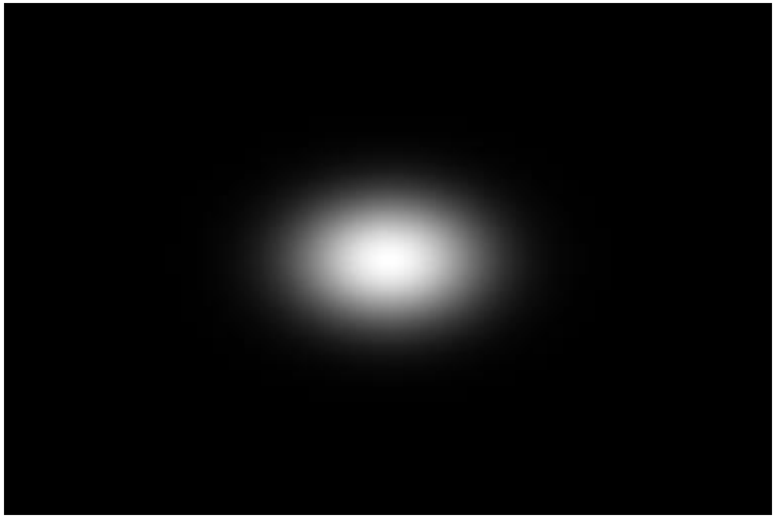
PS of sample-grid multiplied, σ=4 Gaussian.

Reconstruction



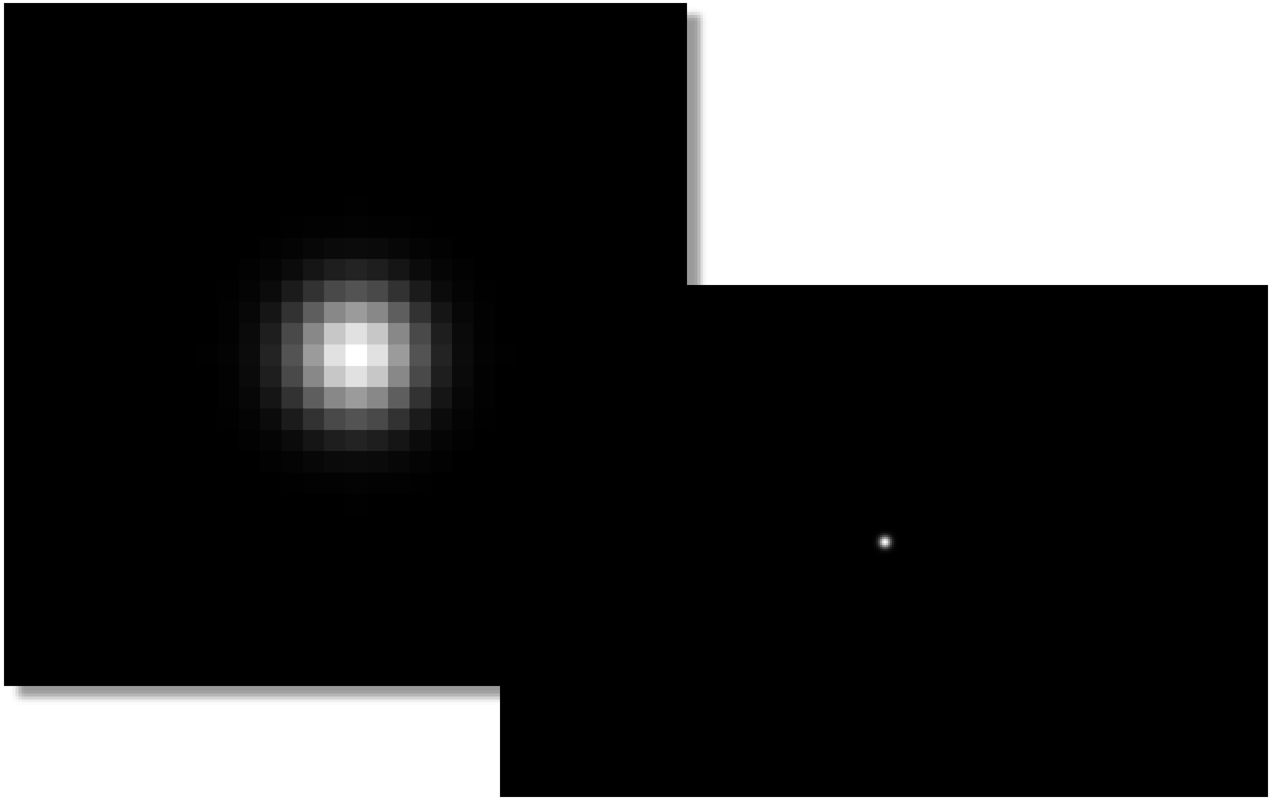
Compare that to the FT of the original.

Reconstruction



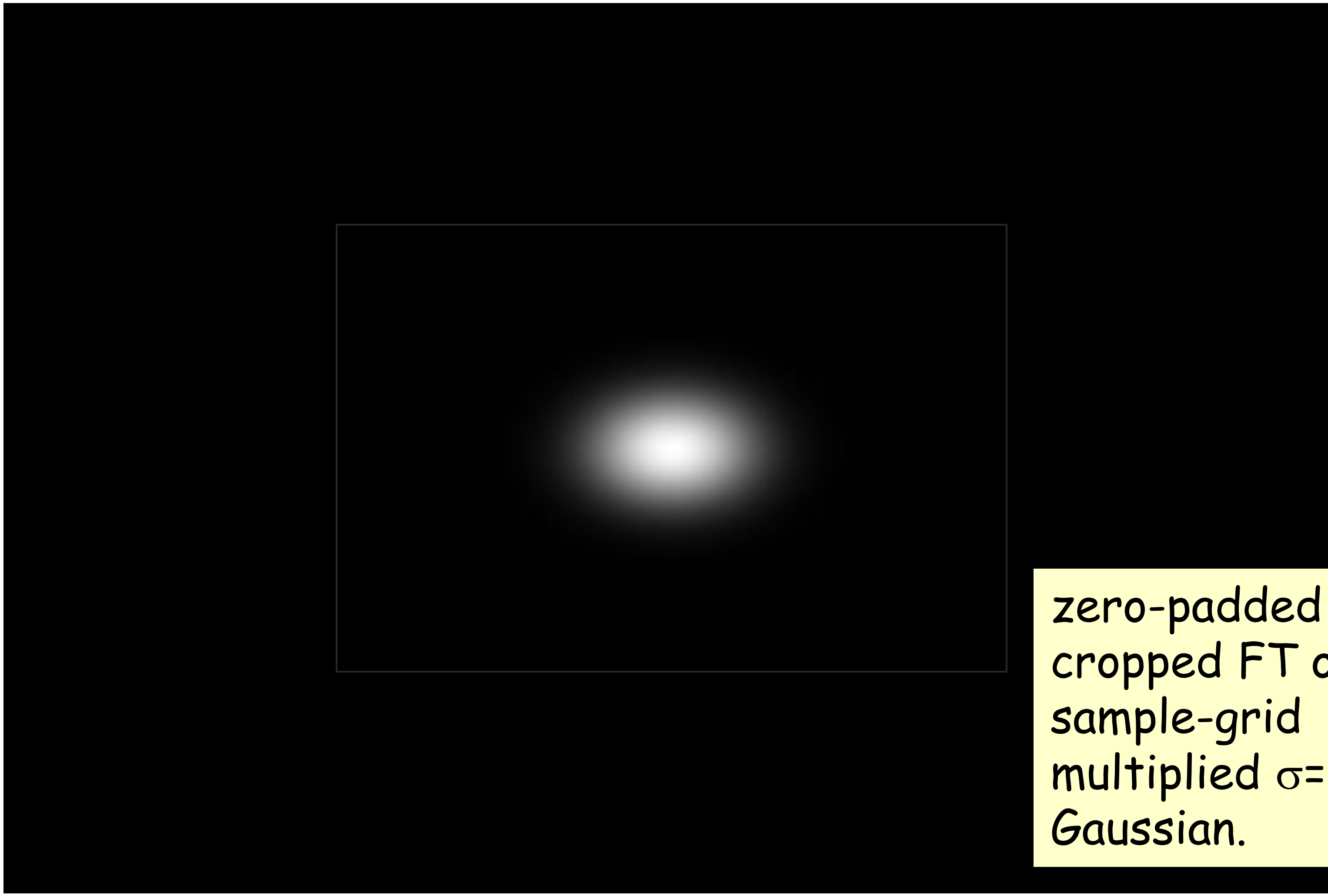
PS from cropped FT of sample-grid multiplied σ=4 Gaussian.

Reconstruction



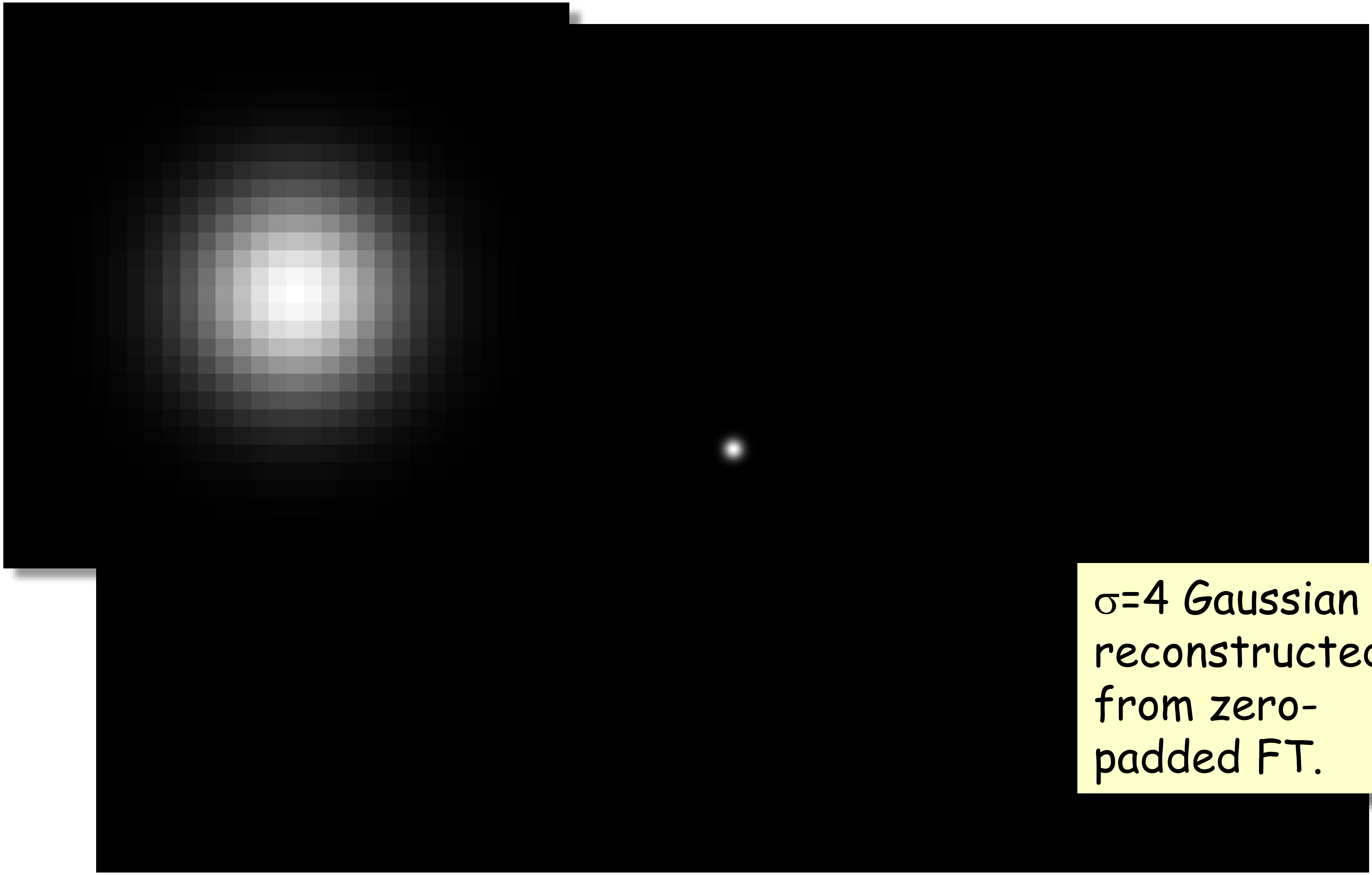
IFT of cropped sample-grid mpy'd FT of σ=4 Gaussian.

Reconstruction



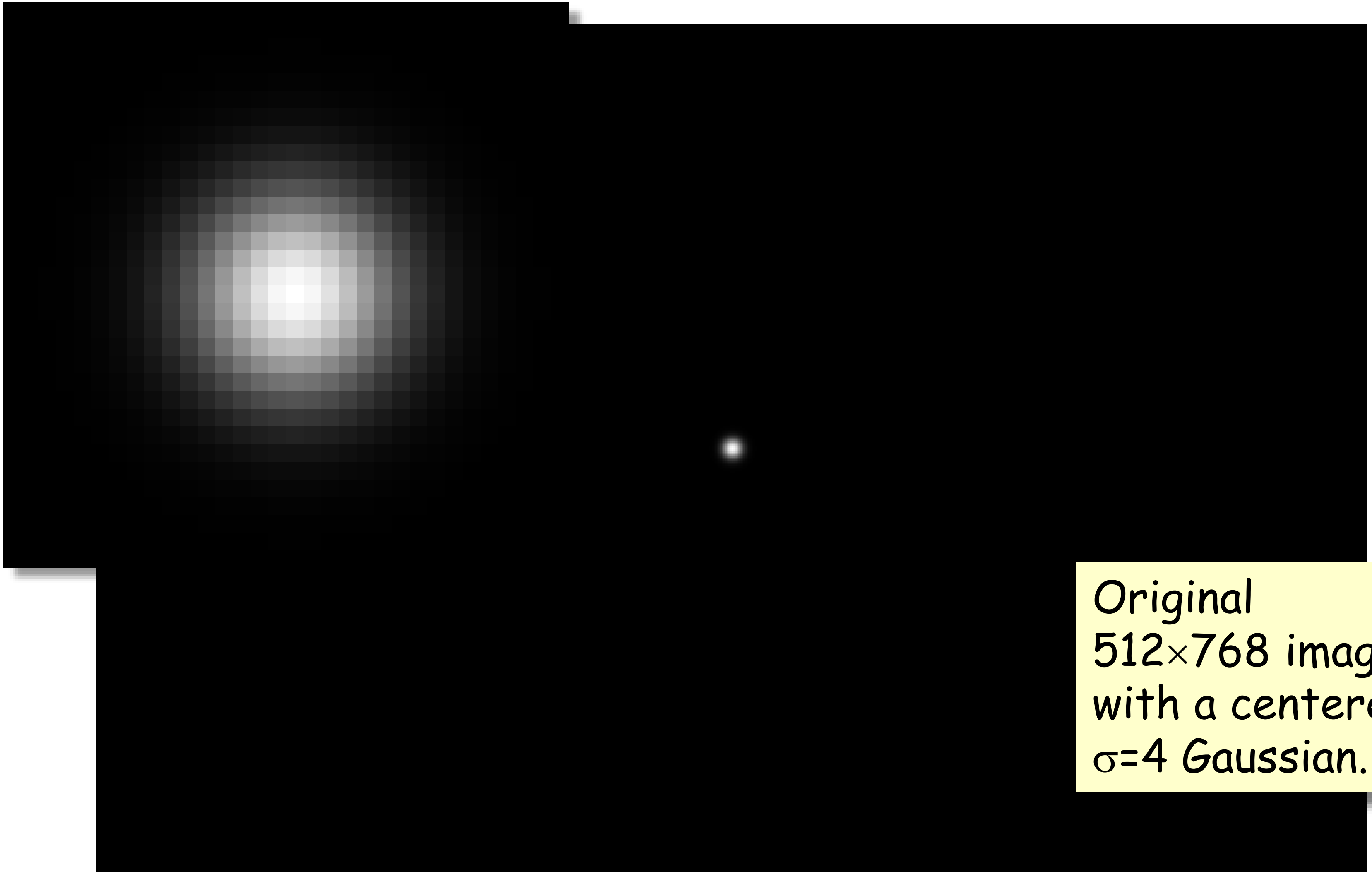
zero-padded cropped FT of sample-grid multiplied σ=4 Gaussian.

Reconstruction



σ=4 Gaussian reconstructed from zero-padded FT.

Reconstruction



Original 512×768 image with a centered, σ=4 Gaussian.

