EECE 4353 Image Processing

Lecture Notes on Mathematical Morphology: Grayscale Images

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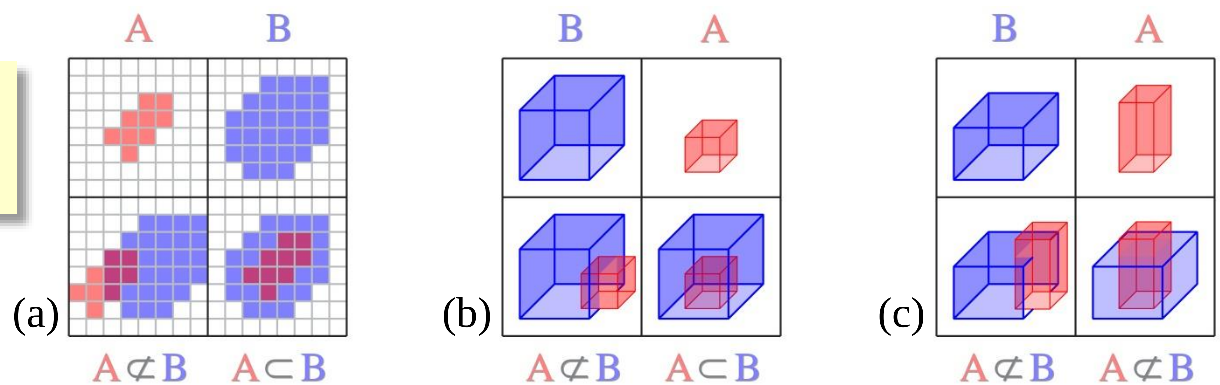
Grayscale Morphology

Grayscale morphology is a multidimensional generalization of the binary operations. Binary morphology is defined in terms of set-inclusion of pixel sets. So is the grayscale case, but the pixel sets are of higher dimension. In particular, standard $R \times C$, 1-band intensity images and the associated structuring elements are defined as 3-D solids wherein the 3rd axis is intensity and set-inclusion is volumetric.

set inclusion (explained on p. 11)



(a) binary, (b) & (c) grayscale

Extended Real Numbers

Let $\mathbb{R}$ represent the real numbers.

Define the *extended* real numbers, $\mathbb{R}^*$, as the real numbers plus two symbols, $-\infty$ and ∞ such that

$$-\infty < x < \infty,$$

for all numbers $x \in \mathbb{R}$.

That is if x is any real number, then ∞ is always greater than x and $-\infty$ is always less than x. Moreover,

$$x + \infty = \infty, \quad x - \infty = -\infty, \quad \infty - \infty = 0,$$

for all numbers $x \in \mathbb{R}$.

Real Images

In mathematical morphology a real image, **I**, is defined as a function that occupies a volume in a Euclidean vector space. **I** comprises a set, S_p, of coordinate vectors (or pixel locations), **p**, in an *n*-dimensional vector space $\mathbb{R}^n$. Associated with each **p** is a value from $\mathbb{R}^*$. The set of pixel locations together with their associated values form the image – a set in $\mathbb{R}^{n+1}$:

$$\mathbf{I} = \left\{ \left[\mathbf{p}, \mathbf{I}(\mathbf{p}) \right] \middle| \mathbf{p} \in S_p \subseteq \mathbb{R}^n,\ \mathbf{I}(\mathbf{p}) \in \mathbb{R}^* \right\}$$

Thus, a conventional, 1-band, $R \times C$ image is a 3D structure with $S_p \subset \mathbb{R}^2$ and $\mathbf{I}(\mathbf{p}) \in \mathbb{R}$. By convention in the literature of MM, $S_p \equiv \mathbb{R}^n$, a real image is defined over all of $\mathbb{R}^n$.

Support of an Image

The support of a real image, **I**, is

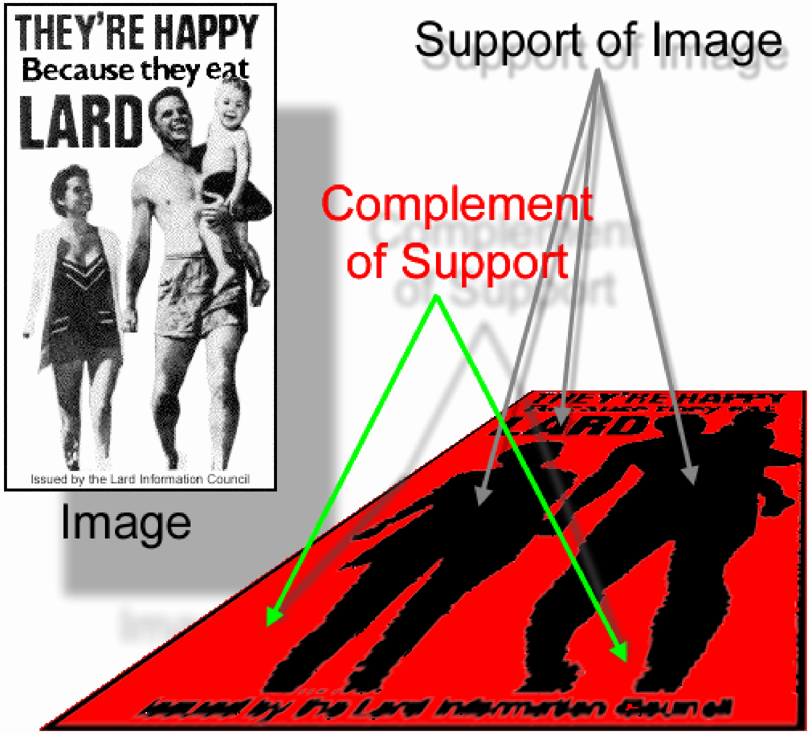
$$\text{supp}\left\{\mathbf{I}\right\} = \left\{\, \mathbf{p} \in \mathbb{R}^n \,\middle|\, \mathbf{I}\left(\mathbf{p}\right) \in \mathbb{R} \,\right\}.$$

That is, the support of a real image is the set pixel locations in $\mathbb{R}^n$ such that

$$\mathbf{I}(\mathbf{p}) \neq -\infty \quad \text{and} \quad \mathbf{I}(\mathbf{p}) \neq \infty.$$

The complement of the support is, therefore, the set of pixel locations in $\mathbb{R}^n$ where

$$\mathbf{I}(\mathbf{p}) = -\infty \quad \text{or} \quad \mathbf{I}(\mathbf{p}) = \infty.$$



Grayscale Images

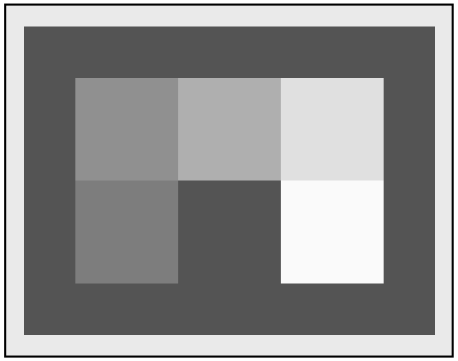
If over its support, **I** takes on more than one real value, then **I** is called *grayscale*.

The object commonly known as a black and white photograph is a grayscale image that has support in a rectangular subset of $\mathbb{R}^2$. Within that region, the image has gray values that vary between black and white. If the intensity of each pixel is plotted over the support plane, then

$$\mathbf{I} = \left\{ \mathbf{p}, \mathbf{I}\left(\mathbf{p}\right) \,\middle|\, \mathbf{p} \in \mathrm{supp}\left\{\mathbf{I}\right\} \right\}.$$

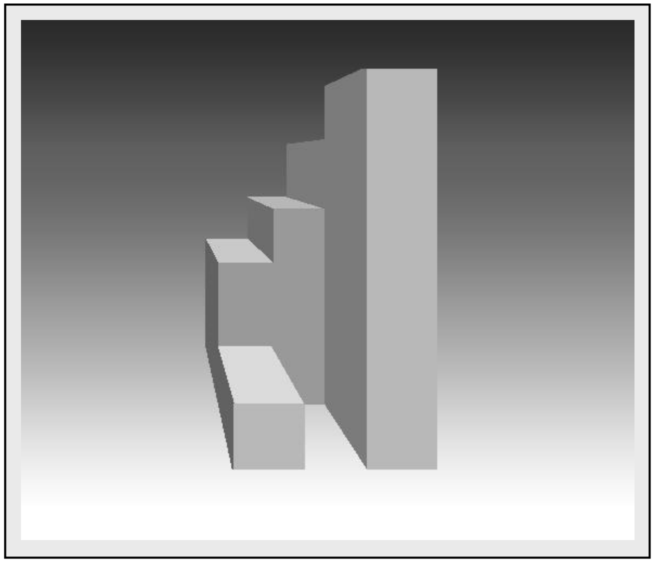
is a volume in $\mathbb{R}^3$. In the abstraction of MM we assume the image does exist outside the support rectangle, but that $\mathbf{I}(\mathbf{p}) = -\infty$ there.

Grayscale Images



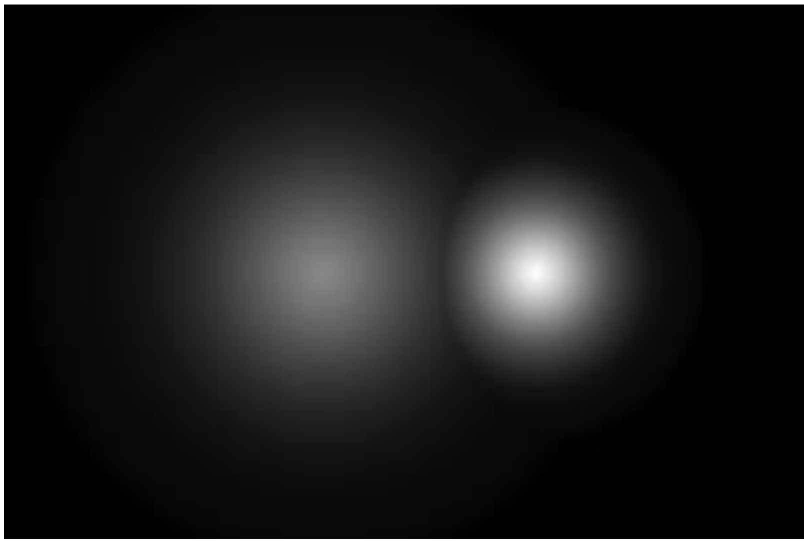
grayscale image

3D solid representation

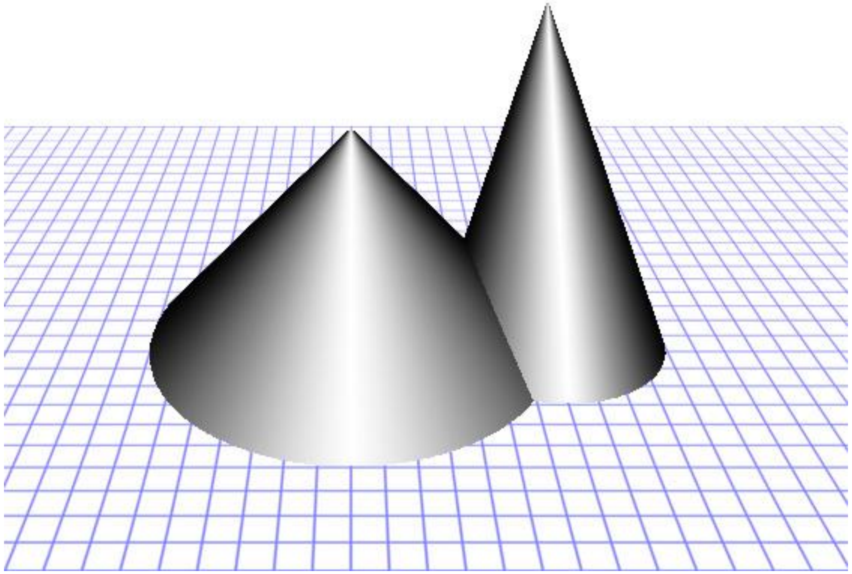


In MM, a 2D grayscale image is treated as a 3D solid in space – a landscape – whose height above the surface at a point is proportional to the brightness of the corresponding pixel.

Representation of Grayscale Images



image

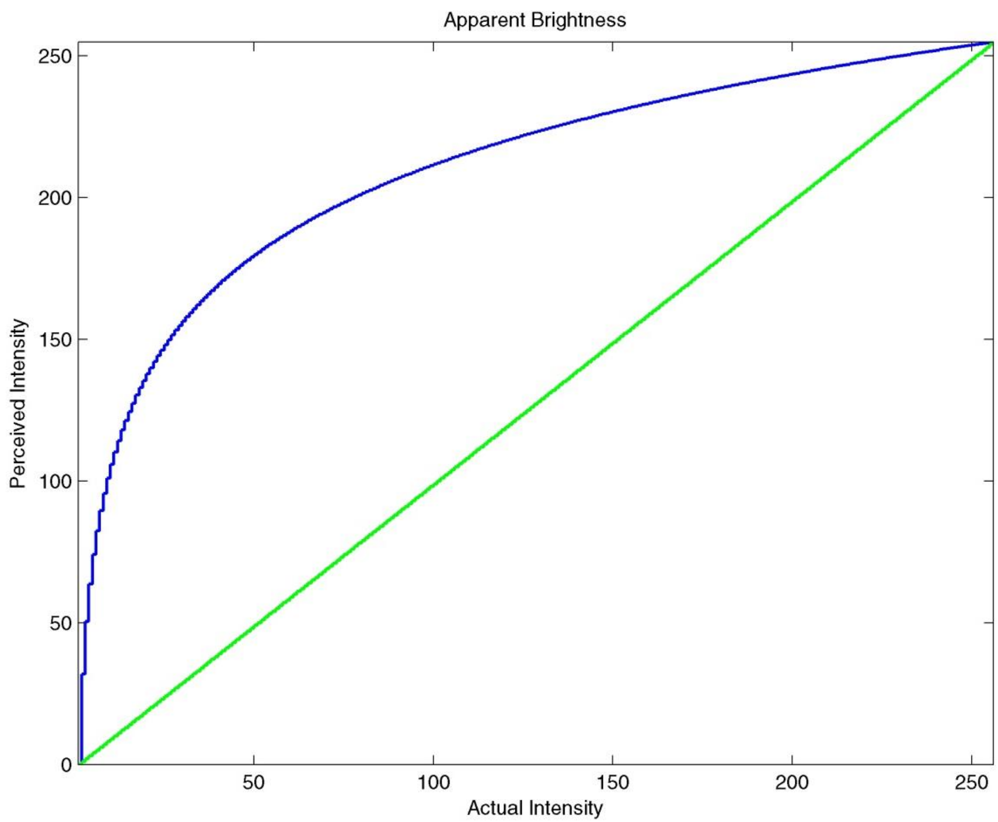


landscape

Example: grayscale cones

Aside: Brightness Perception

The previous slide demonstrates the Weber-Fechner relation. The linear slope of the intensity change is perceived as logarithmic.



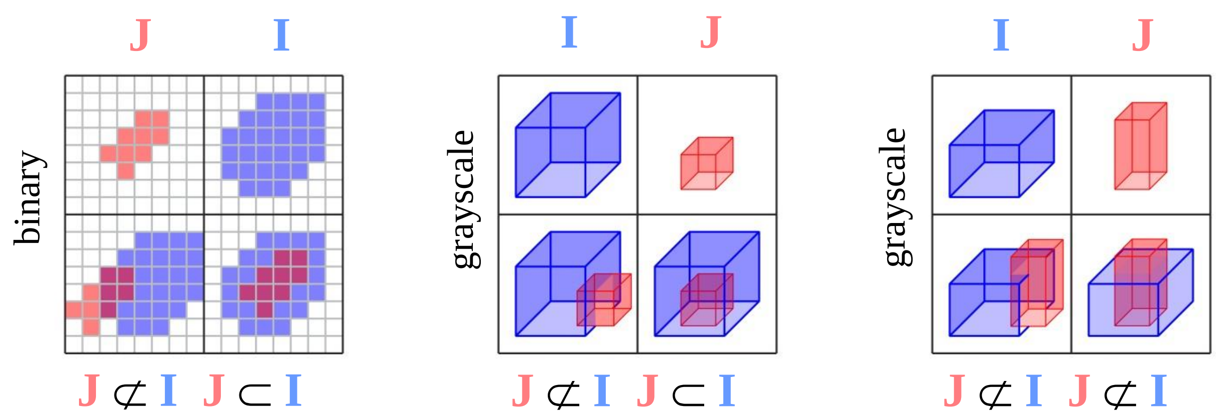
The green curve is the actual intensity; the blue curve is the perceived intensity.

Set Inclusion in Grayscale Images

In grayscale morphology, set inclusion depends on the implicit 3D structure of a 2D image. If $\mathbf{I}$ and $\mathbf{J}$ are grayscale images then

$$\mathbf{J} \subseteq \mathbf{I} \Leftrightarrow \operatorname{supp}(\mathbf{J}) \subseteq \operatorname{supp}(\mathbf{I}) \text{ AND } \left\{ \mathbf{J}(\mathbf{p}) \leq \mathbf{I}(\mathbf{p}) \,\middle|\, \mathbf{p} \in \operatorname{supp}(\mathbf{J}) \right\}.$$

That is $\mathbf{J} \subseteq \mathbf{I}$ if and only if the support of $\mathbf{J}$ is contained in that of $\mathbf{I}$ *and* the value of $\mathbf{J}$ is nowhere greater than the value of $\mathbf{I}$ on the support of $\mathbf{J}$.



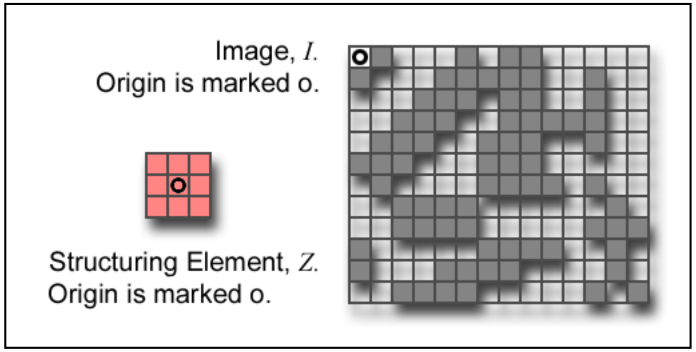
Recall: Binary Structuring Element (SE)

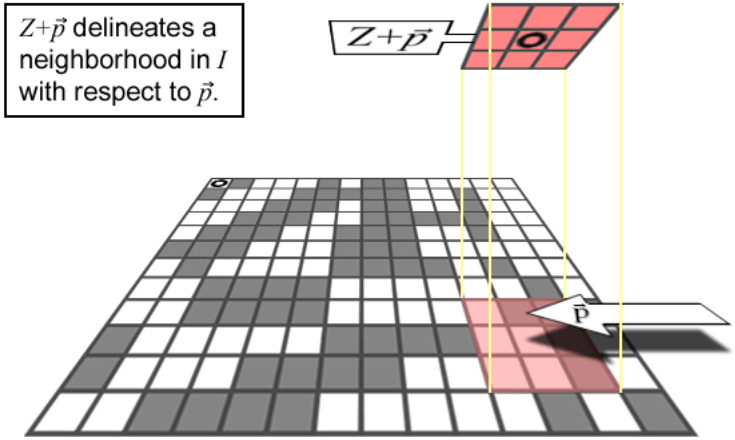
Let **I** be an image and **Z** a SE.

Z+**p** means that **Z** is moved so that its origin coincides with location **p** in S_p.

Z+**p** is the *translate* of **Z** to location **p** in S_p.

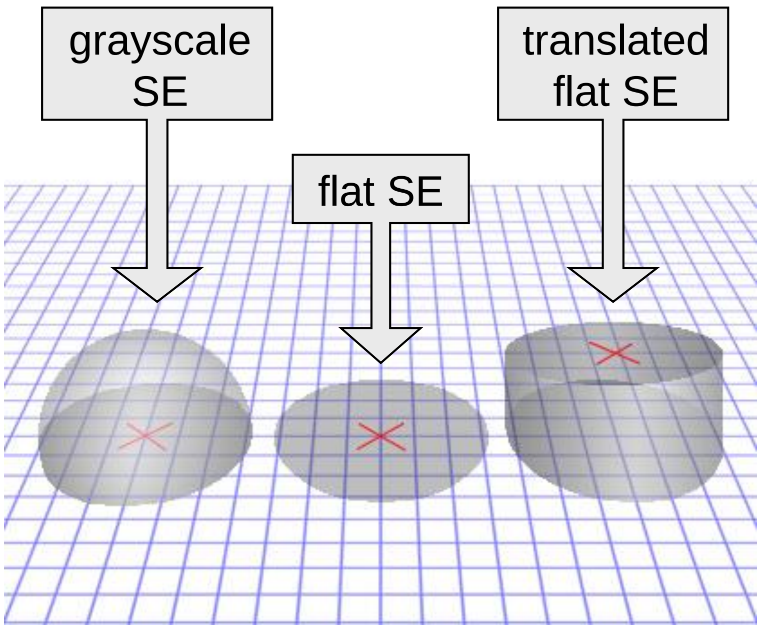
The set of locations in the image delineated by **Z**+**p** is called the **Z**-*neighborhood* of **p** in **I** denoted N{**I**,**Z**}(**p**).



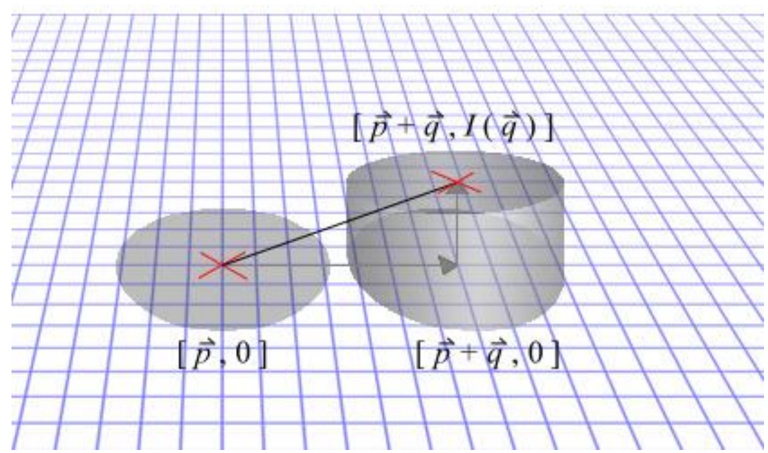


Grayscale Structuring Elements

A grayscale structuring element is a small image that delineates a volume at each pixel [**p** , **I**(**p**)] through out the image volume.

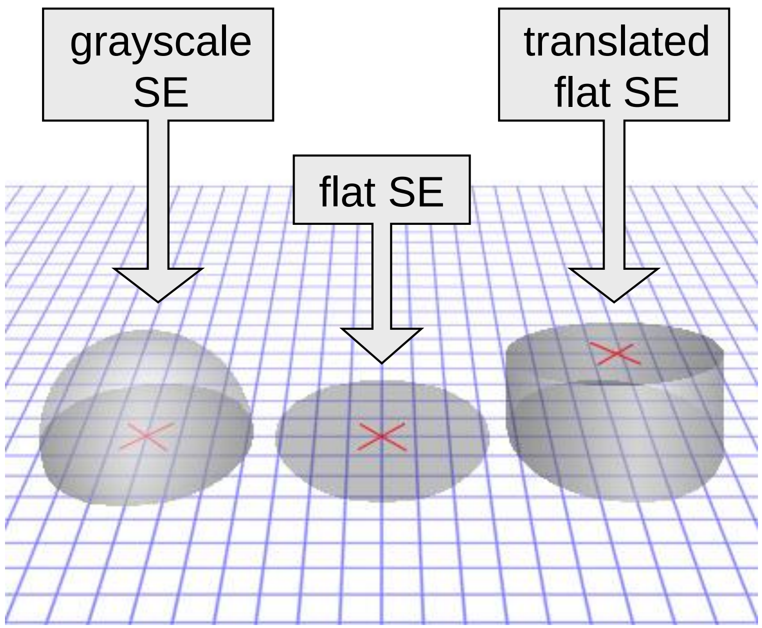


Translation of a flat SE on its support plane and in gray value.

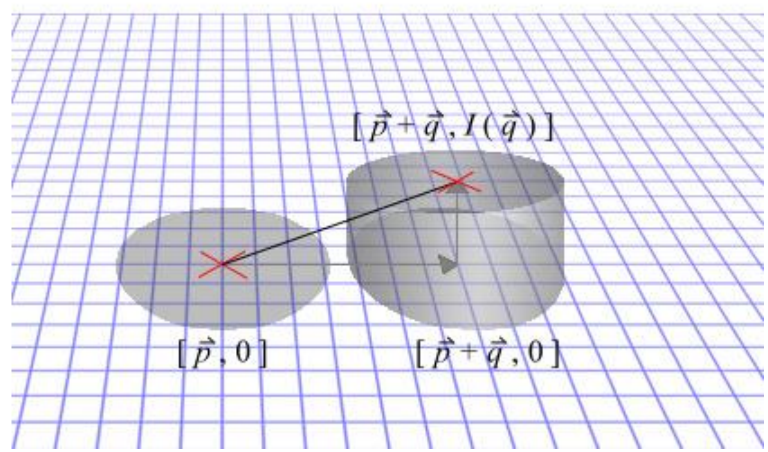


SE Translation: × marks the location of the structuring element origin.

Grayscale Structuring Elements



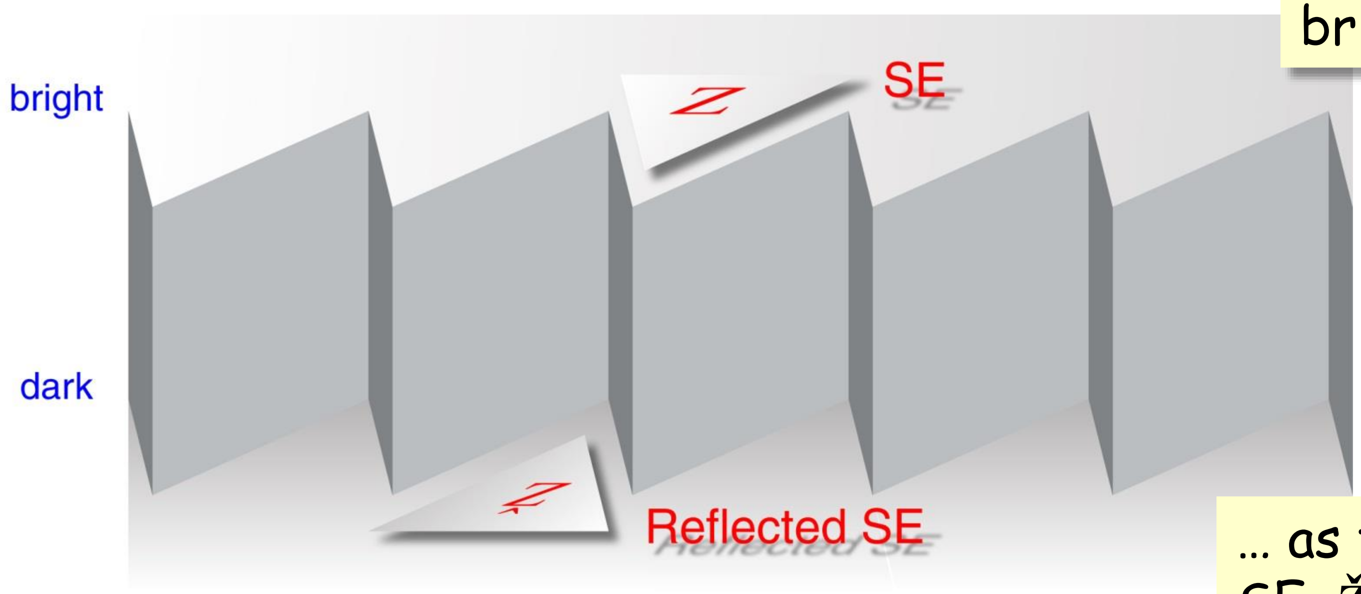
Translation of a flat SE on its support plane and in gray value.



If $\mathbf{Z} = [\ \mathbf{p}, \mathbf{Z}(\mathbf{p})\ ]$ is a structuring element and if $\mathbf{q} = [\ \mathbf{q}_s, q_g\ ]$ is a pixel [location, value] then $\mathbf{Z}+\mathbf{q} = [\ \mathbf{p}+\mathbf{q}_s, \mathbf{Z}(\mathbf{p})+q_g\ ]$ for all $\mathbf{p} \in \text{supp}\{\mathbf{Z}\}$.

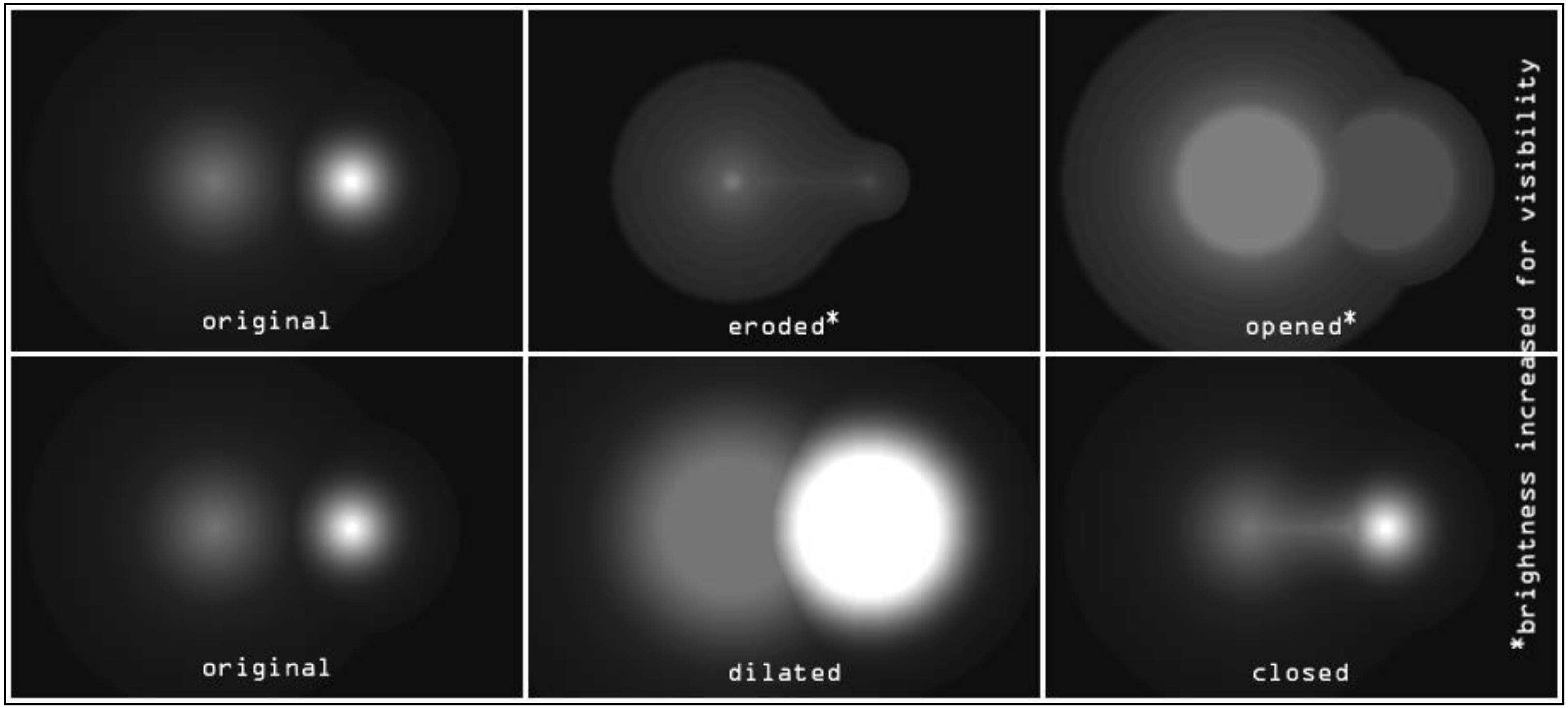
Reflected Structuring Elements

Note that the SE, Z, is to the bright regions…



… as the reflected SE, Ž, is to the dark regions.

Grayscale Morphology: Basic Operations



Dilation: General Definition

The *dilation* of image **I** by structuring element **Z** at coordinate $\mathbf{p}\in\mathbb{R}^n$ is defined by

$$[\mathbf{I}\oplus\mathbf{Z}](\mathbf{p}) = \max_{\mathbf{q}\in\mathrm{supp}(\breve{\mathbf{Z}}+\mathbf{p})}\left\{\mathbf{I}(\mathbf{q})+\mathbf{Z}(\mathbf{p}-\mathbf{q})\right\} = \max_{\mathbf{q}\in\mathrm{supp}(\breve{\mathbf{Z}}+\mathbf{p})}\left\{\mathbf{I}(\mathbf{q})-\breve{\mathbf{Z}}(\mathbf{q}-\mathbf{p})\right\}.$$

This can be computed as follows:

1. Translate $\breve{\mathbf{Z}}$ to **p**.
2. Trace out the $\breve{\mathbf{Z}}$ –neighborhood of **I** at **p**.
3. Let **p** be the origin of **I** temporarily during the operation
4. Compute the set of numbers

$$\mathfrak{D} = \left\{\mathbf{I}(\mathbf{q})+\mathbf{Z}(-\mathbf{q})\,\middle|\,\mathbf{q}\in\mathrm{supp}(\breve{\mathbf{Z}})\right\} = \left\{\mathbf{I}(\mathbf{q})-\breve{\mathbf{Z}}(\mathbf{q})\,\middle|\,\mathbf{q}\in\mathrm{supp}(\breve{\mathbf{Z}})\right\}.$$

5. The output value, $[\mathbf{I}\oplus\mathbf{Z}](\mathbf{p})$, is the maximum value in the set, $\mathfrak{D}$.

Fast Computation of Dilation

The fastest way to compute *grayscale* dilation is to use the translates-of-the-image definition of dilation. That is, use

$$\mathbf{J} = \mathbf{I} \oplus \mathbf{Z} = \max_{\mathbf{q} \in \mathrm{supp}\{\mathbf{Z}\}} \left\{ \left[\mathbf{I} + \mathbf{q} \right] + \mathbf{Z}\left(\mathbf{q} \right) \right\}.$$

Note that if **Z** is flat -- all its foreground elements are 0 -- then step (3) is unnecessary. Then it is a *maximum* filter.

That is,

(1) Make a copy of **I** for each foreground element, **q**, in **Z**.

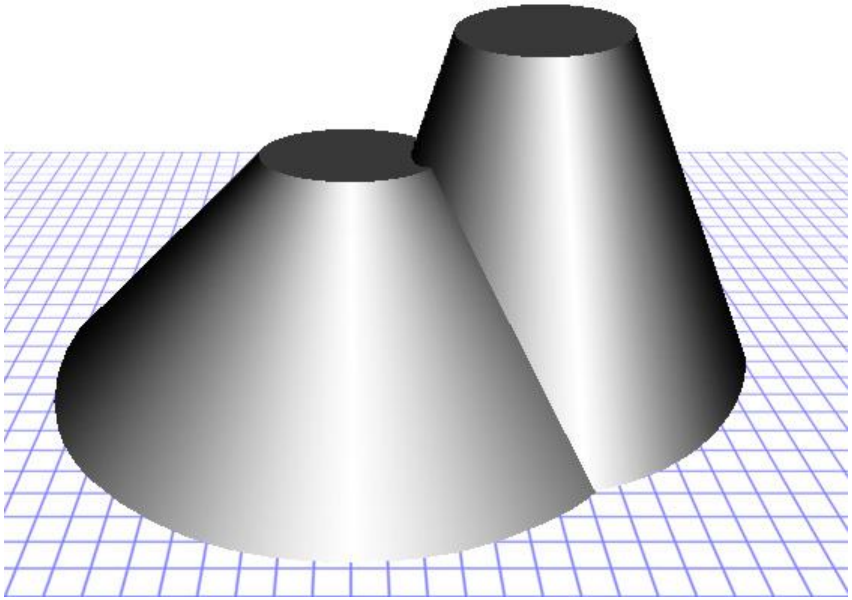
(2) Translate the **q**th copy so that its ULHC (origin) is at position **q** in **Z**.

(3) Add **Z**(**q**) to every pixel in the **q**th copy.

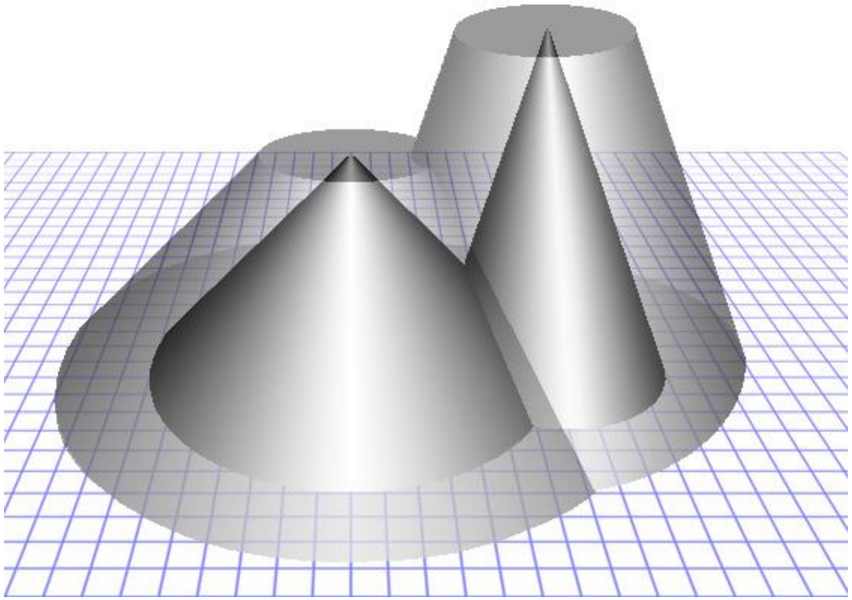
(4) Take the pixelwise maximum of the resultant stack of images.

(5) Copy out the result starting at the SE origin in the maximum image.

Grayscale Morphology: Dilation



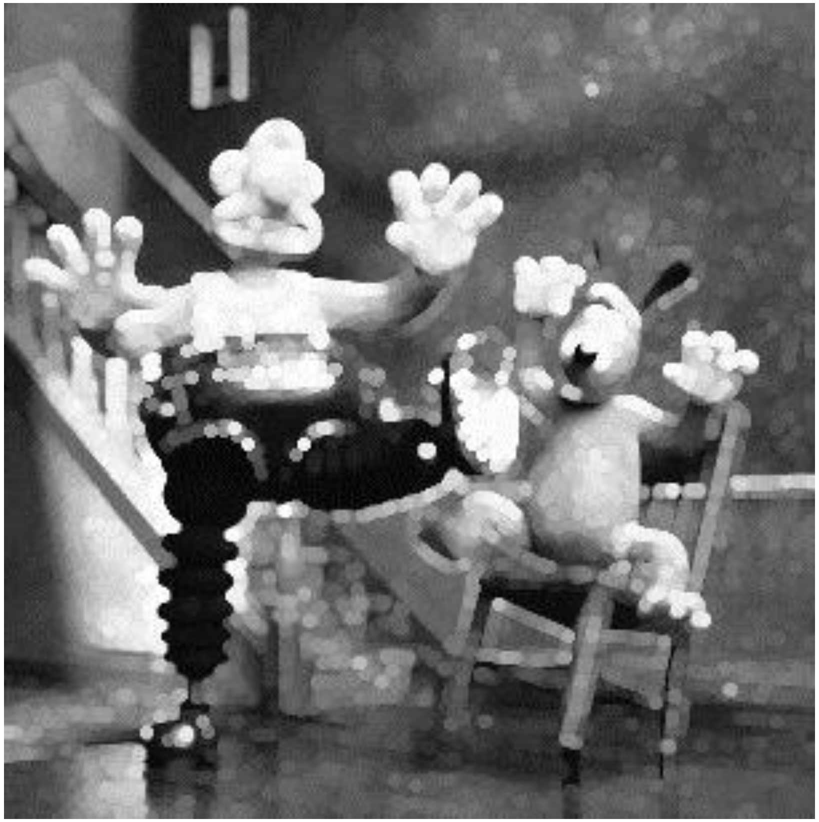
dilation



dilation over original

SE, **Z**, is a flat disk the size of the tops of the truncated cones.

Grayscale Morphology: Dilation



SE, **Z**, is a flat disk.

Erosion: General Definition

The *erosion* of image **I** by structuring element **Z** at coordinate $\mathbf{p}\in\mathbb{R}^n$ is defined by

$$[\mathbf{I}\ominus\mathbf{Z}](\mathbf{p}) = \min_{\mathbf{q}\in\mathrm{supp}(\mathbf{Z}+\mathbf{p})}\left\{\mathbf{I}(\mathbf{q})-\mathbf{Z}(\mathbf{q}-\mathbf{p})\right\}.$$

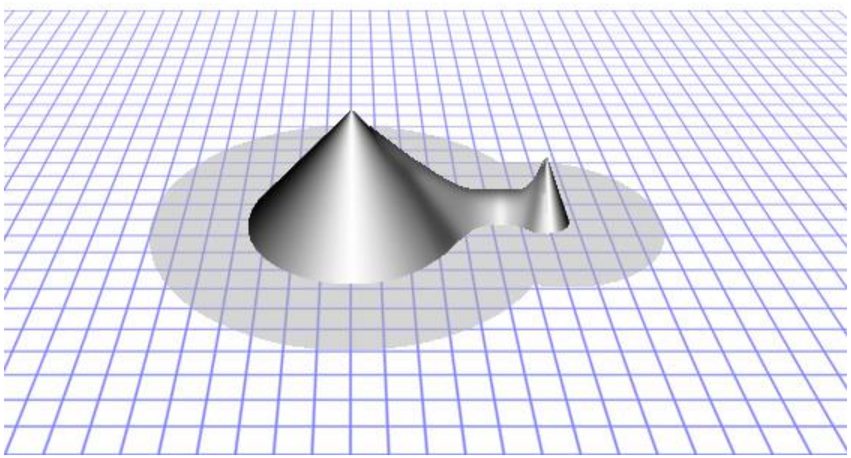
This can be computed as follows:

1. Translate **Z** to **p**.
2. Trace out the **Z** – neighborhood of **I** at **p**.
3. Let **p** be the origin of **I** temporarily during the operation
4. Compute the set of numbers

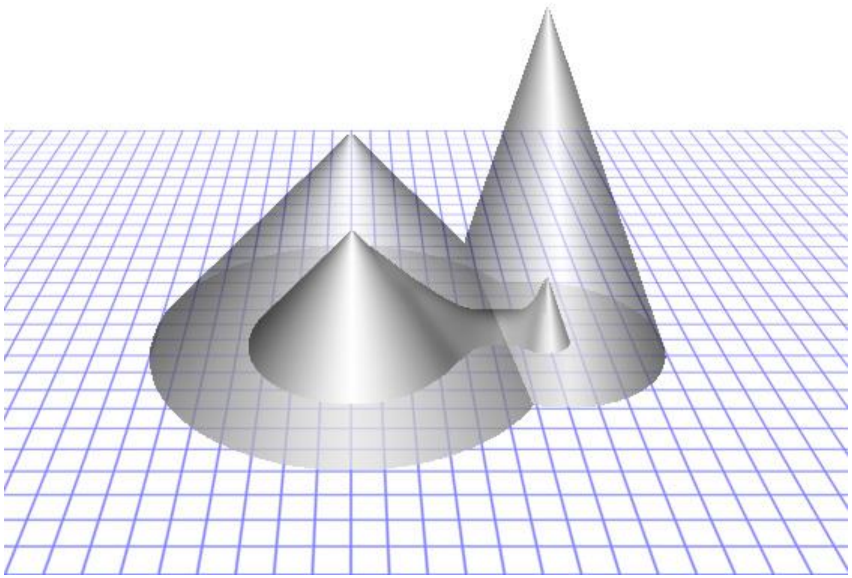
$$\mathcal{E} = \left\{\mathbf{I}(\mathbf{q})-\mathbf{Z}(\mathbf{q})\,\middle|\,\mathbf{q}\in\mathrm{supp}(\mathbf{Z})\right\}.$$

5. The output value, $[\mathbf{I}\ominus\mathbf{Z}](\mathbf{p})$, is the minimum value in the set, $\mathcal{E}$.

Grayscale Morphology: Erosion



erosion



erosion under original

SE, **Z**, is the same flat disk as used for the dilation on page 19.

Fast Computation of Erosion

The fastest way to *grayscale* erosion is to create a stack of images translated to minus the values of the reflected SE then take the pixelwise minimum:

$$\mathbf{J} = \mathbf{I} \ominus \mathbf{Z} = \min_{\mathbf{q} \in \operatorname{supp}\{\breve{\mathbf{Z}}\}} \left\{ \left[\mathbf{I} + \mathbf{q} \right] + \breve{\mathbf{Z}}(\mathbf{q}) \right\}.$$

$$\breve{\mathbf{Z}} = \left\{ -\mathbf{Z}(-\mathbf{q}) \,\middle|\, \mathbf{q} \in \mathbb{R}^2 \right\}$$

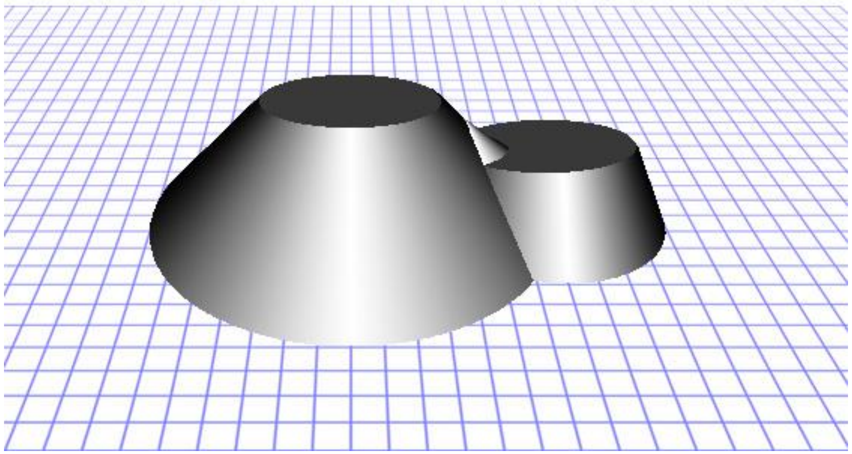
Note that if **Z** is symmetric and if all the foreground elements are 0, then **Ž**=**Z** and step (3) is unnecessary. Then it is a *minimum* filter

That is, (1) make a copy of **I** for each foreground element, **q**, in **Ž**. (Note that if **q** is a foreground element in **Ž** then **-q** is a foreground element in **Z**.) (2) Translate each copy so that its ULHC (origin) is at position **q** in **Ž** (or **-q** in **Z**). (3) Then add **Ž**(**q**) (or subtract **Z**(-**q**)) to every pixel in the **q**th copy. Finally, (4) take the pixelwise minimum of the resultant stack of images.

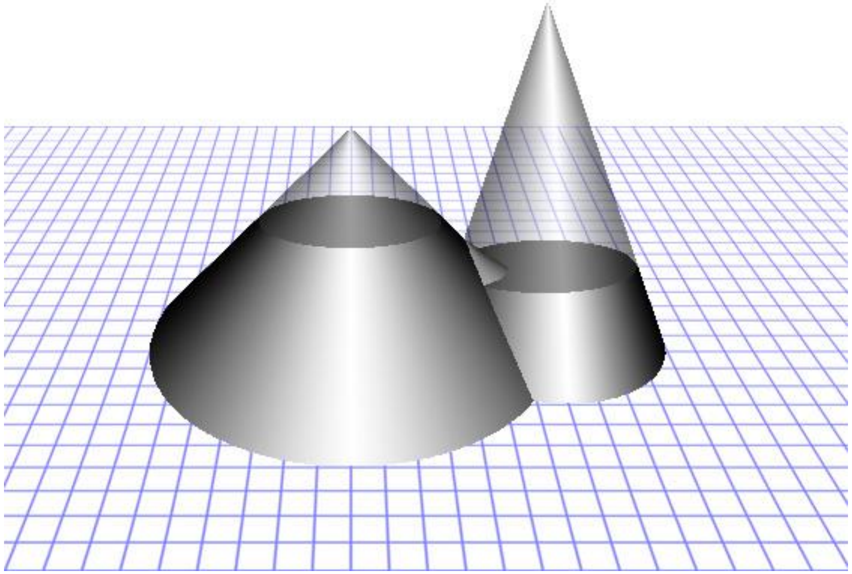
Grayscale Morphology: Erosion

SE, Z, is a flat disk.

Grayscale Morphology: Opening



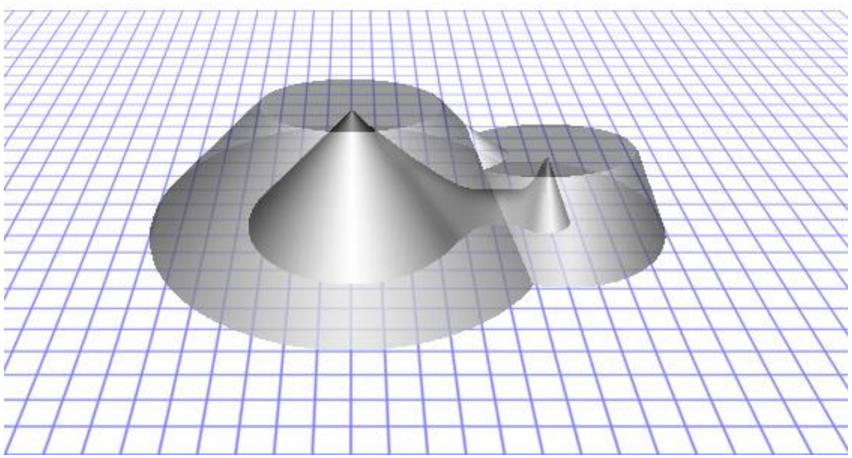
opening: erosion then dilation



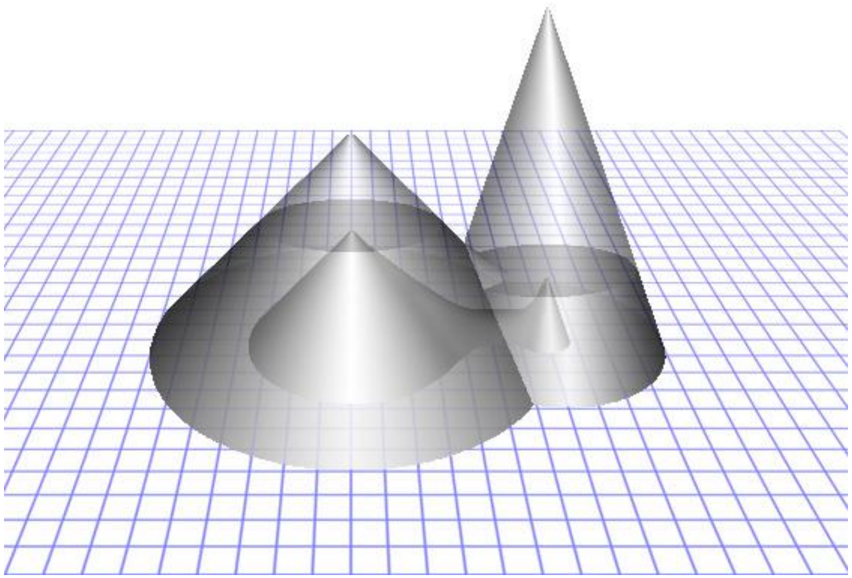
opened & original

SE, **Z**, is a flat disk the size of the tops of the truncated cones.

Grayscale Morphology: Opening



erosion & opening



erosion & opening & original

SE, Z, is a flat disk the size of the tops of the truncated cones.

Opening and Closing

Opening is erosion by **Z** followed by dilation by **Z**.

$$\mathbf{I} \circ \mathbf{Z} = \left(\mathbf{I} \ominus \mathbf{Z}\right) \oplus \mathbf{Z}$$

The opening is the best approximation of the image FG that can be made from copies of the SE, given that the opening is contained in the original. $\mathbf{I} \circ \mathbf{Z}$ contains no FG features that are smaller than the SE.

Closing is dilation by **Ž** followed by erosion by **Ž**.

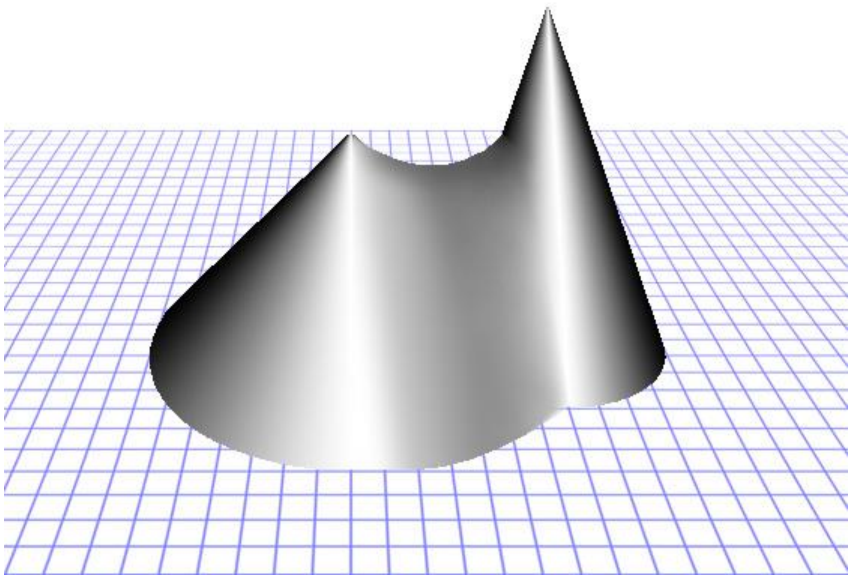
$$\mathbf{I} \bullet \mathbf{Z} = \left(\mathbf{I} \oplus \breve{\mathbf{Z}}\right) \ominus \breve{\mathbf{Z}}$$

The closing is the best approximation of the image BG that can be made from copies of the SE, given that the closing is contained in the image BG. $\mathbf{I} \bullet \mathbf{Z}$ contains no BG features that are smaller than the SE.

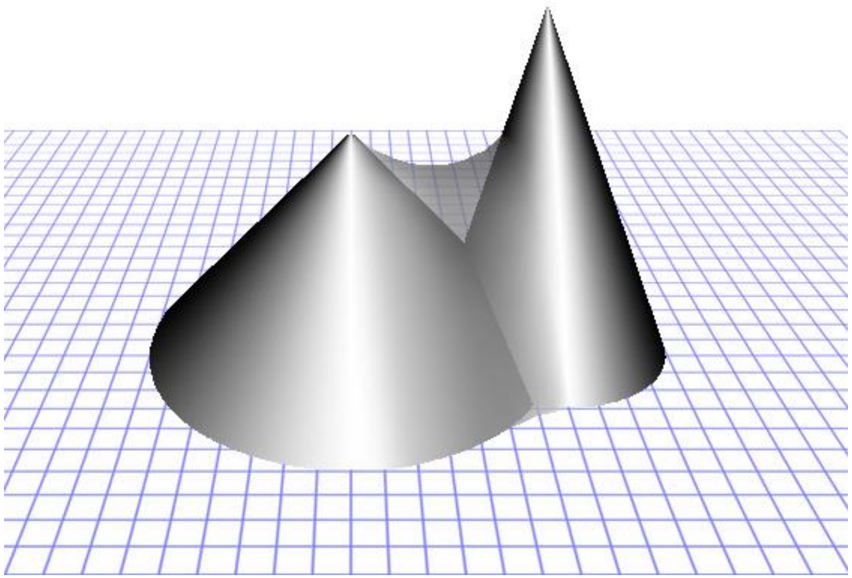
Grayscale Morphology: Opening

SE, Z, is a flat disk.

Grayscale Morphology: Closing



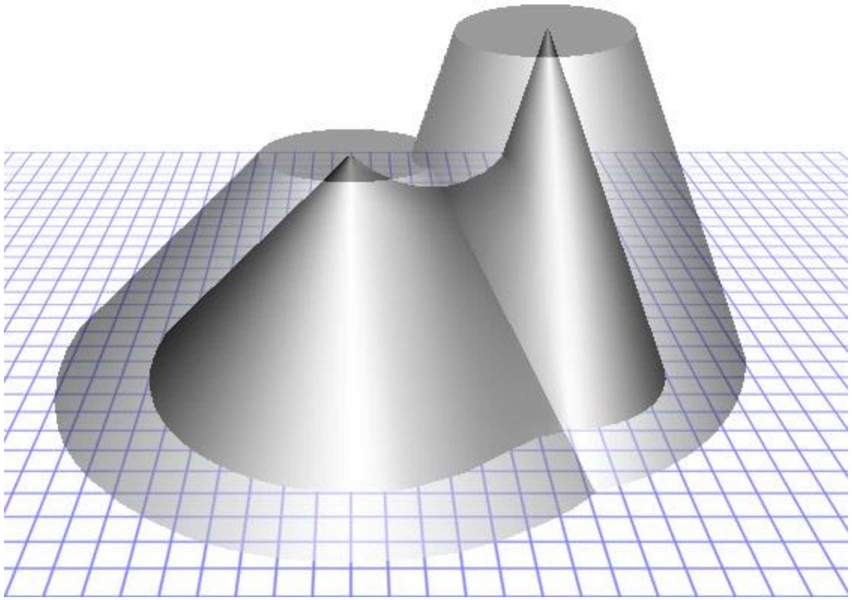
closing: dilation then erosion



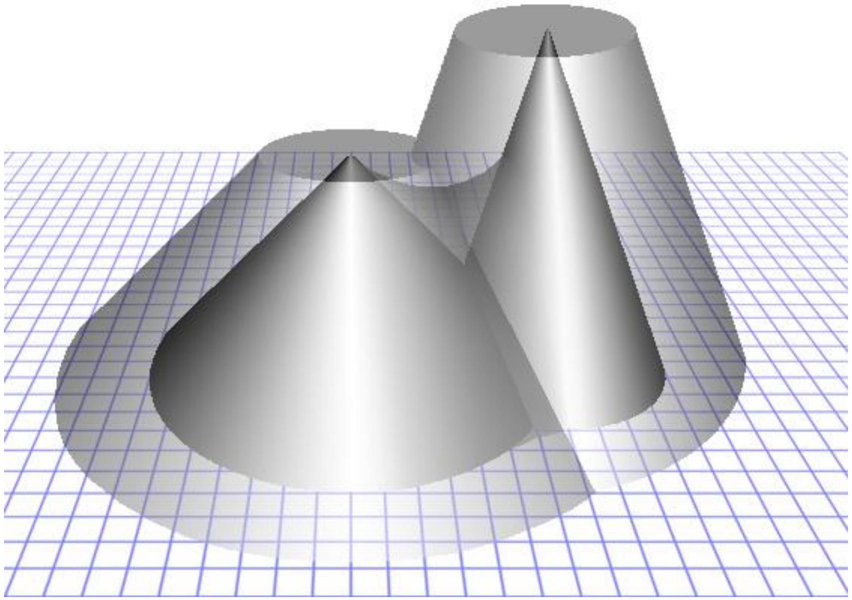
closing & original

SE, Z, is the same flat disk as used for the dilation on page 19.

Grayscale Morphology: Closing



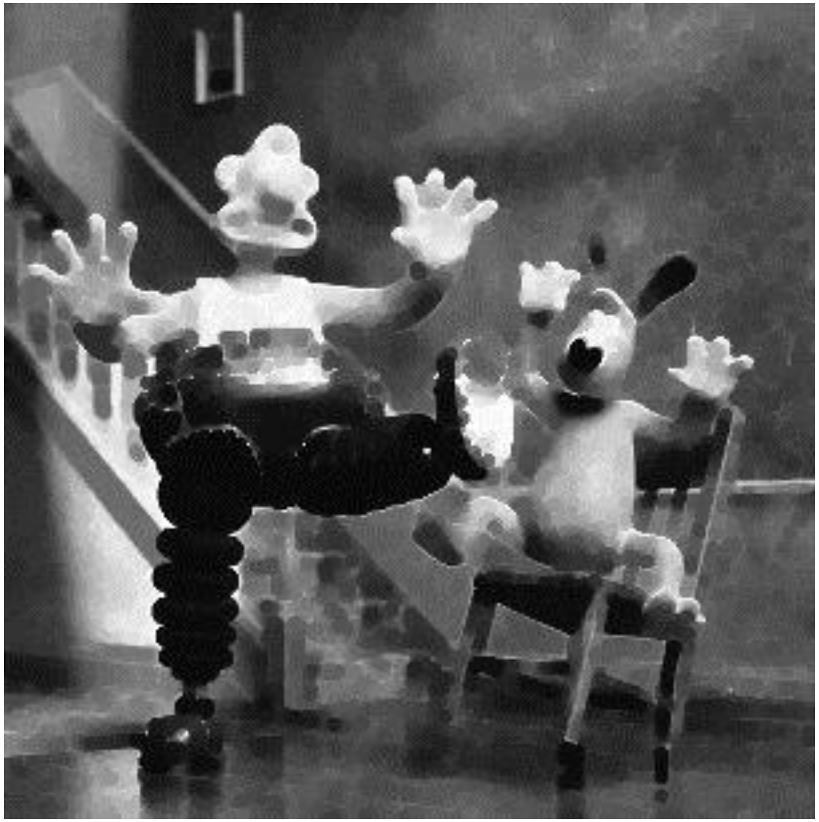
dilation over closing



dilation & closing & original

SE, Z, is a flat disk the size of the tops of the truncated cones.

Grayscale Morphology: Closing



SE, Z, is a flat disk.

Duality Relationships

Erosion in terms of dilation: $\mathbf{I} \ominus \mathbf{Z} = \left[\mathbf{I}^C \oplus \breve{\mathbf{Z}} \right]^C$

Dilation in terms of erosion: $\mathbf{I} \oplus \mathbf{Z} = \left[\mathbf{I}^C \ominus \breve{\mathbf{Z}} \right]^C$

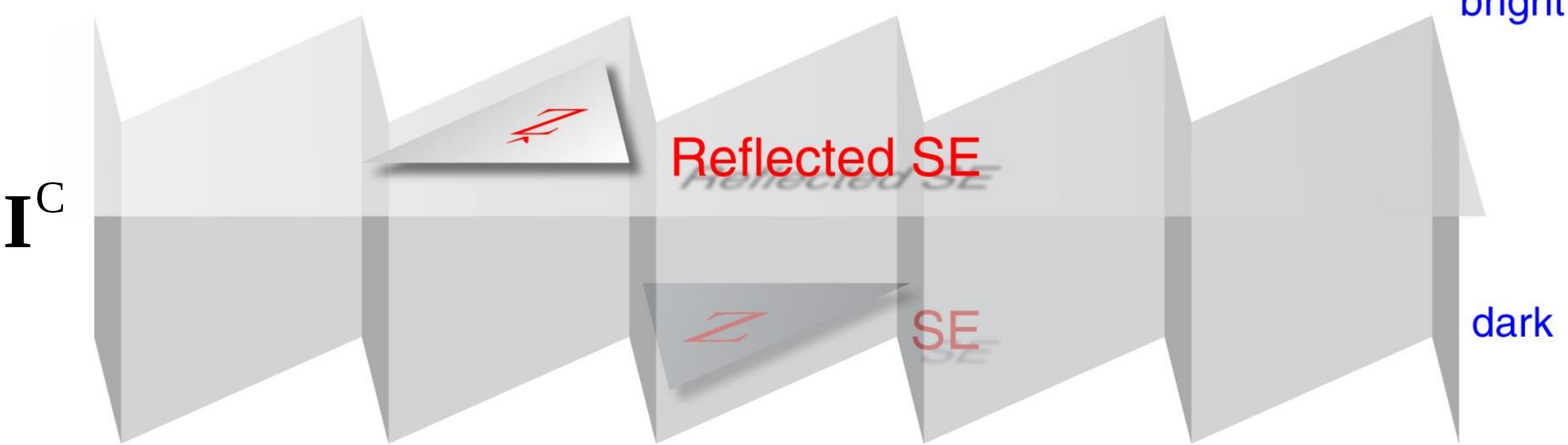
Opening in terms of closing: $\mathbf{I} \circ \mathbf{Z} = \left[\mathbf{I}^C \bullet \mathbf{Z} \right]^C$

Closing in terms of opening: $\mathbf{I} \bullet \mathbf{Z} = \left[\mathbf{I}^C \circ \mathbf{Z} \right]^C$

$\mathbf{I}^C$ is the complement of $\mathbf{I}$ and $\check{\mathbf{Z}}$ is the reflected SE.

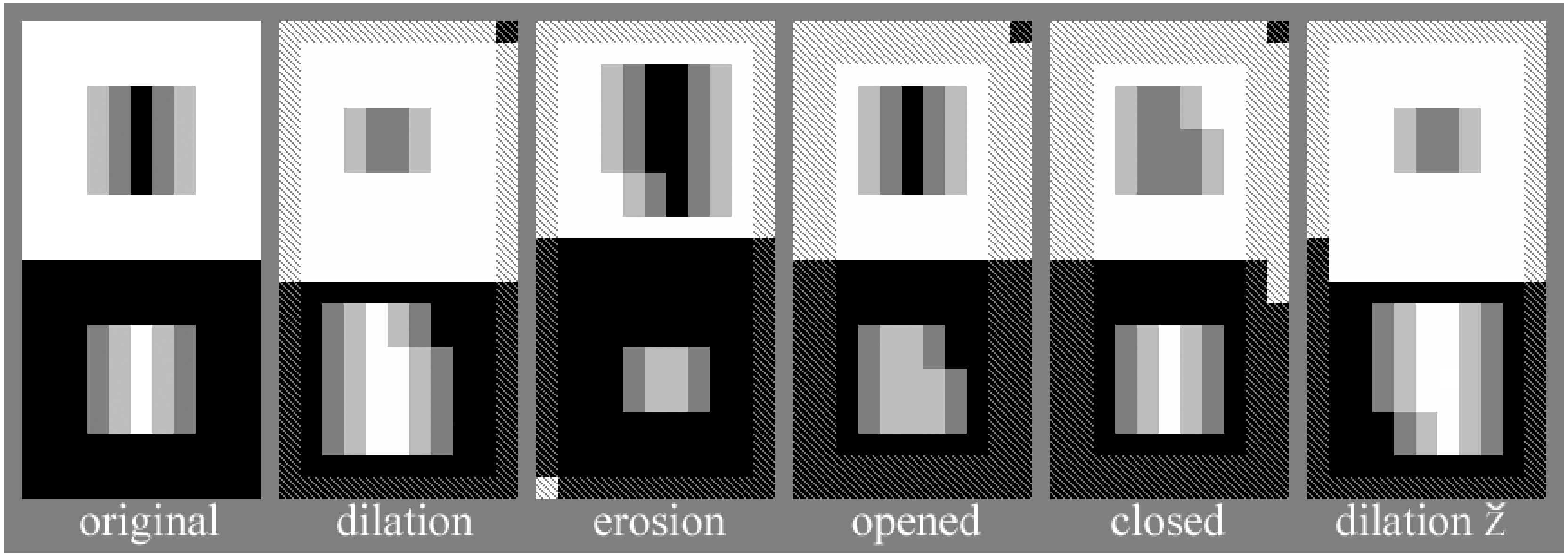
Duality Relationships

SE, Ž, operates on $\mathbf{I}^C$ as if it were Z operating on $\mathbf{I}$.



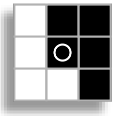
SE, Z, operates on $\mathbf{I}^C$ as if it were Ž operating on $\mathbf{I}$.

Gray Ops with Asymmetric SEs



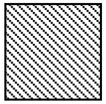
"L" shaped SE

O marks origin



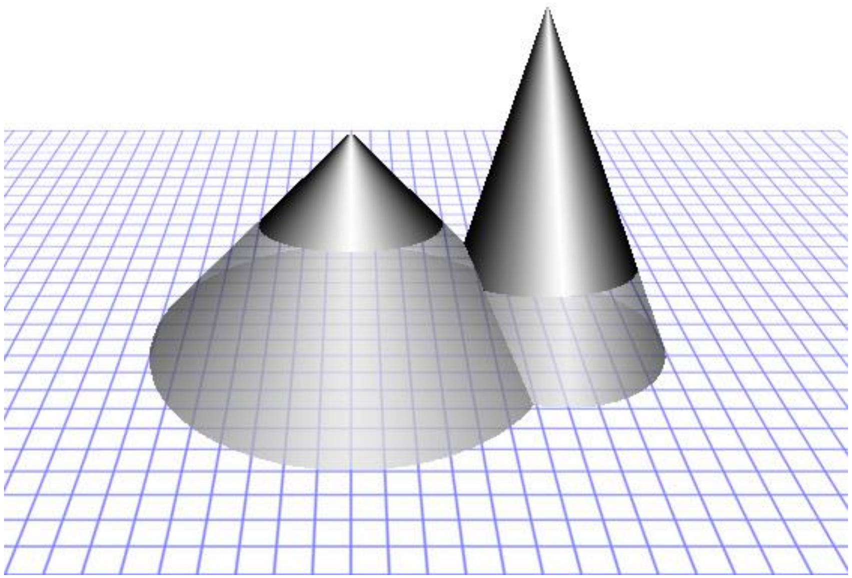
Foreground: white pixels

Background: black pixels

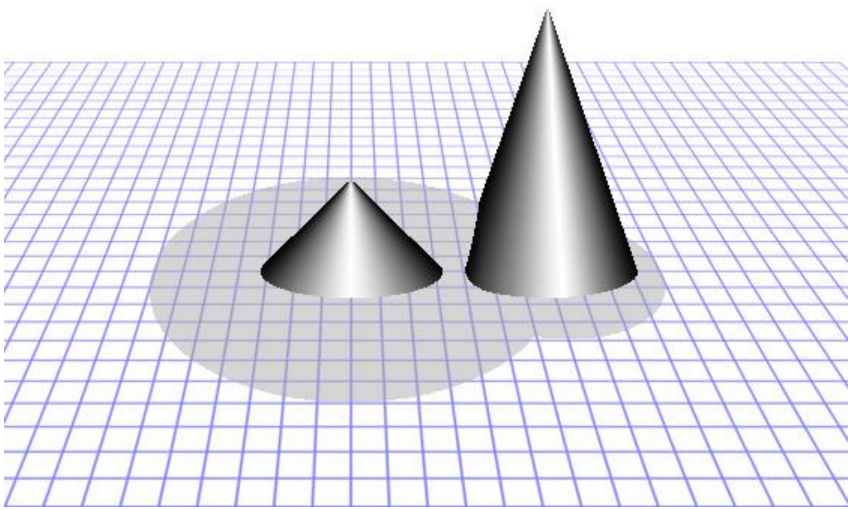


Cross-hatched pixels are indeterminate.

Grayscale Morphology: Tophat



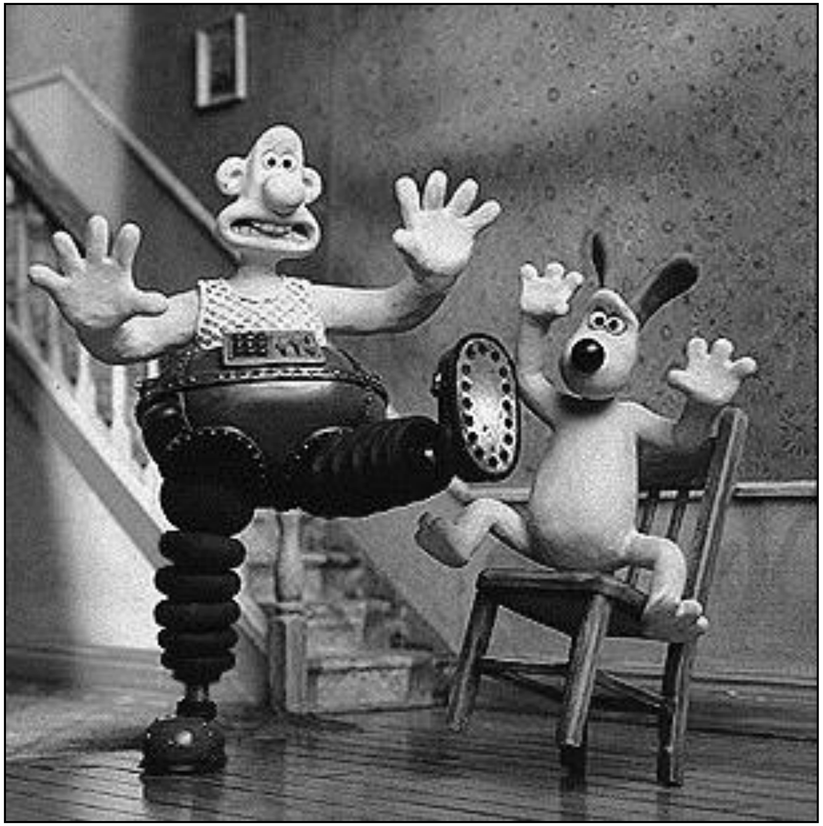
tophat + opened = original

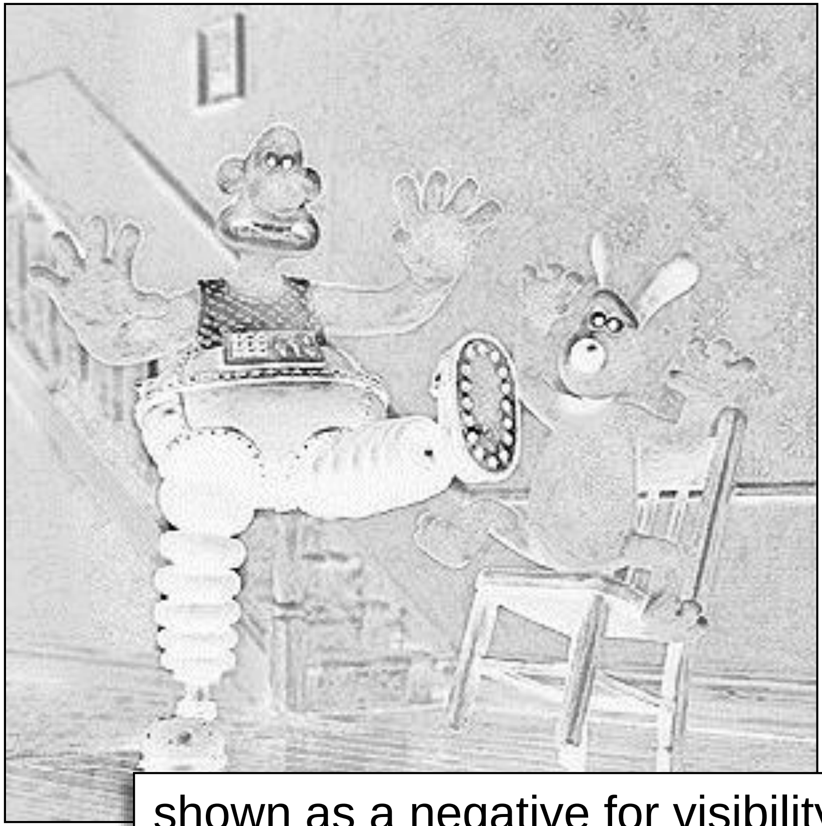


tophat: original - opening

SE, Z, is the same flat disk as used for the dilation on page 19.

Grayscale Morphology: Tophat

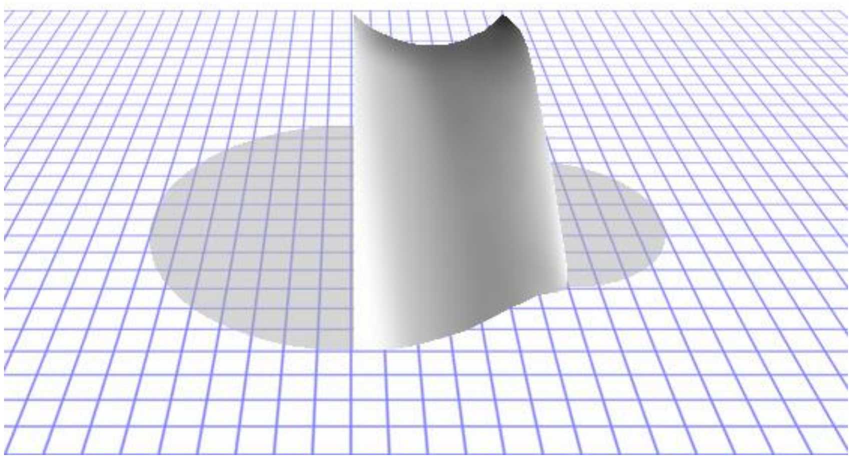




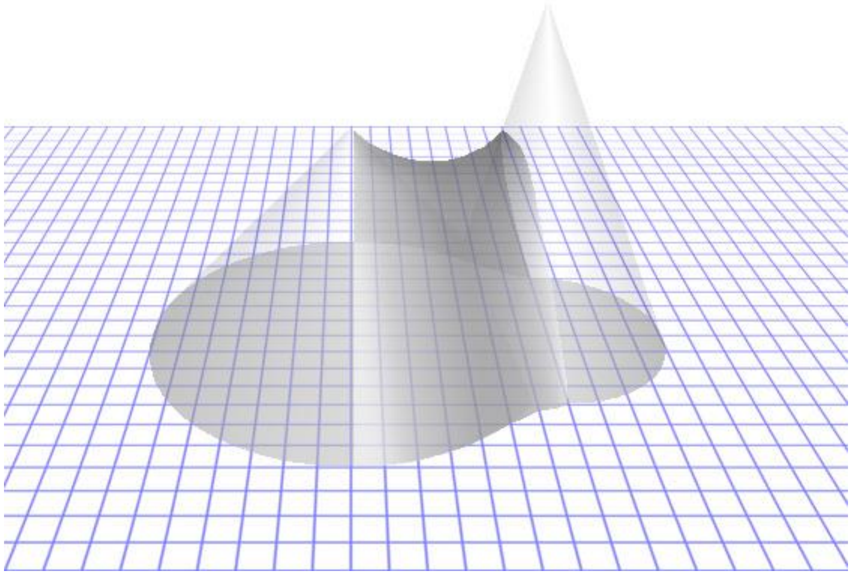
shown as a negative for visibility

SE, Z, is a flat disk.

Grayscale Morphology: Bothat



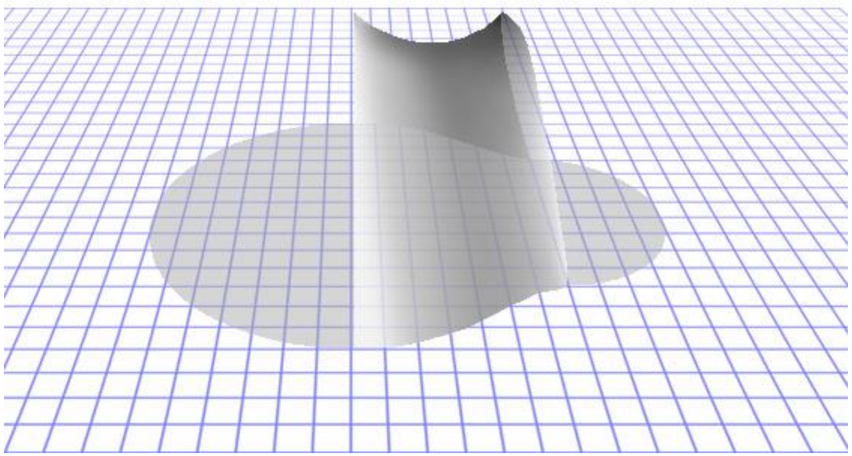
region added by dilation



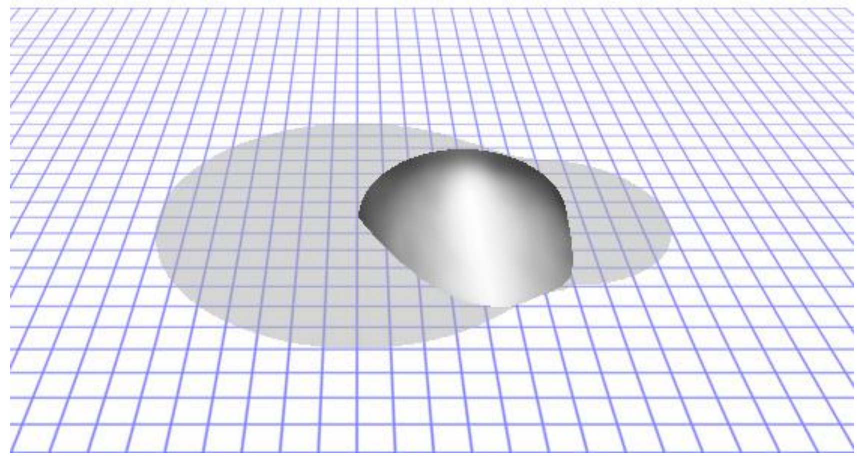
superimposed on original

SE, Z, is the same flat disk as used for the dilation on page 19.

Grayscale Morphology: Bothat



region added by dilation



Bothat: closing - original

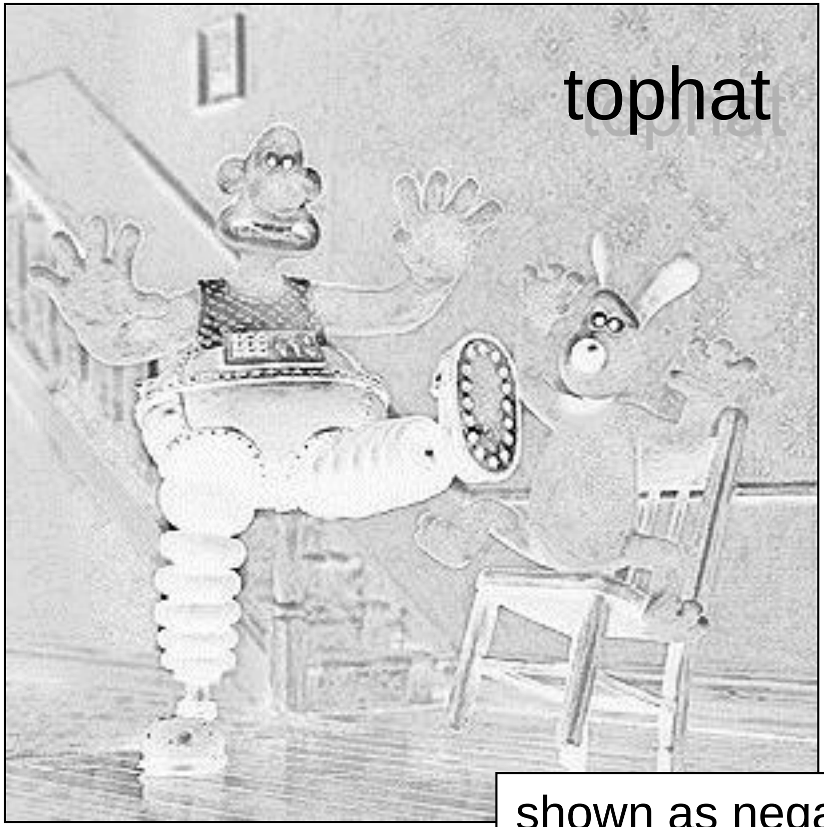
SE, Z, is the same flat disk as used for the dilation on page 19.

Grayscale Morphology: Bothat

shown as a negative for visibility

SE, Z, is a flat disk.

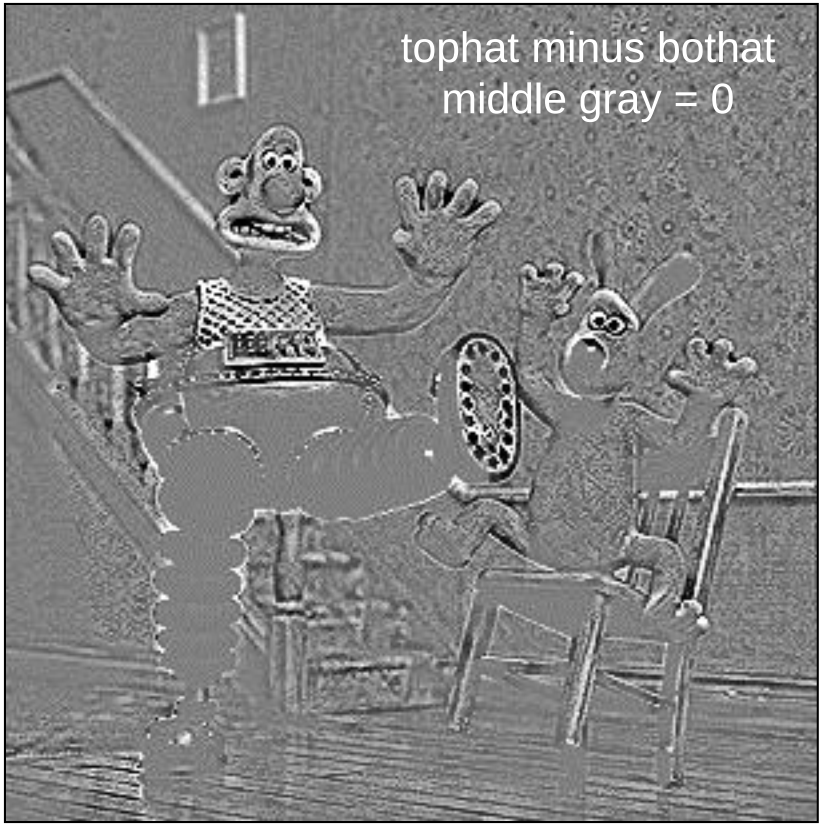
Grayscale Morphology: Tophat and Bothat



shown as negatives for visibility

SE, Z, is a flat disk.

Grayscale Morphology: Small Feature Detection



SE, Z, is a flat disk.

Algorithm for Grayscale Reconstruction

1. $\mathbf{J} = \mathbf{I} \circ \mathbf{Z}$, where $\mathbf{Z}$ is any SE.
2. $\mathbf{T} = \mathbf{J}$,
3. $\mathbf{J} = \mathbf{J} \oplus \mathbf{Z}_k$, where $k=4$ or $k=8$,
4. $\mathbf{J} = \min\{\mathbf{I}, \mathbf{J}\}$, [*pixelwise minimum of* $\mathbf{I}$ *and* $\mathbf{J}$.]
5. if $\mathbf{J} \neq \mathbf{T}$ then go to 2,
6. else stop; [$\mathbf{J}$ *is the reconstructed image.*]

This is the same as binary reconstruction but for grayscale images $\mathbf{J}(r,c) \in \mathbf{I}$ if and only if $\mathbf{J}(r,c) \leq \mathbf{I}(r,c)$.

Algorithm for Graysc

Usually a program for reconstruction will take both **J** and **I** as inputs. E.g,

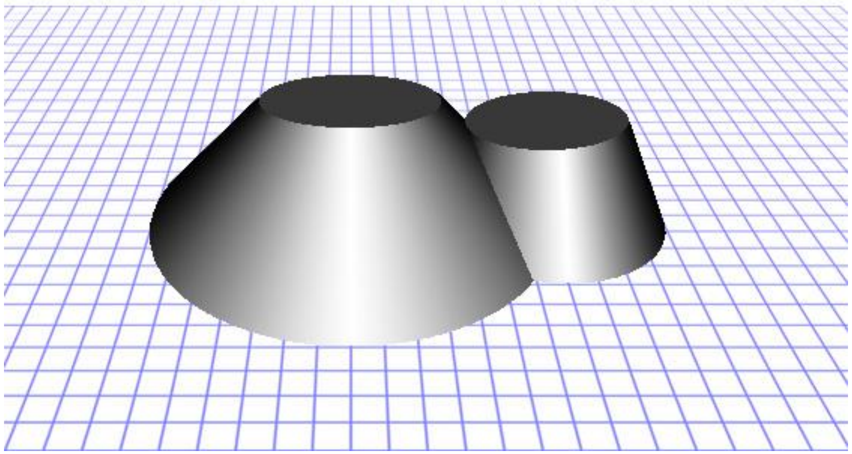
```
K = ReconGray(I,J,Z);
```

Then the algorithm starts at step 2.

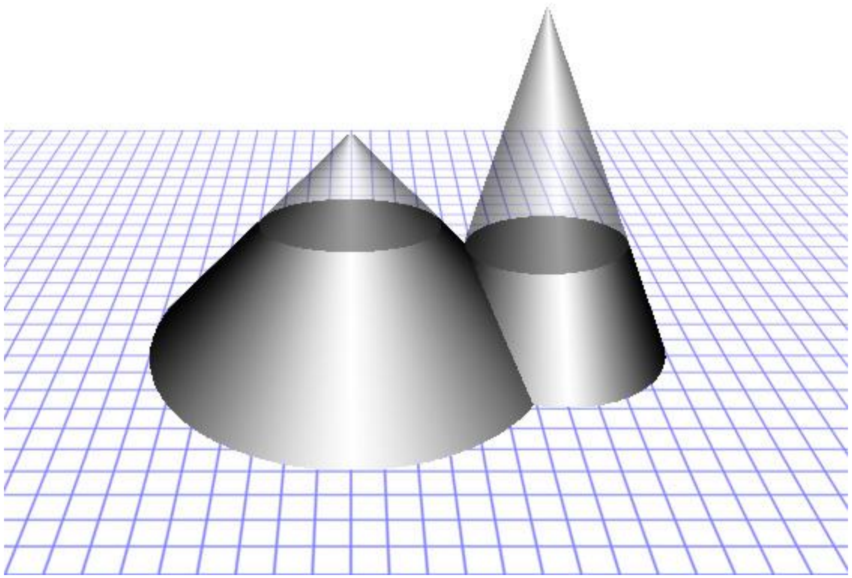
1. $\mathbf{J} = \mathbf{I} \circ \mathbf{Z}$, where $\mathbf{Z}$ i
2. $\mathbf{T} = \mathbf{J}$,
3. $\mathbf{J} = \mathbf{J} \oplus \mathbf{Z}_k$, where $k=4$ or $k=8$,
4. $\mathbf{J} = \min\{\mathbf{I}, \mathbf{J}\}$, [*pixelwise minimum of* **I** *and* **J**.]
5. if $\mathbf{J} \neq \mathbf{T}$ then go to 2,
6. else stop; [**J** *is the reconstructed image.*]

This is the same as binary reconstruction but for grayscale images $\mathbf{J}(r,c) \in \mathbf{I}$ if and only if $\mathbf{J}(r,c) \leq \mathbf{I}(r,c)$.

Grayscale Reconstruction



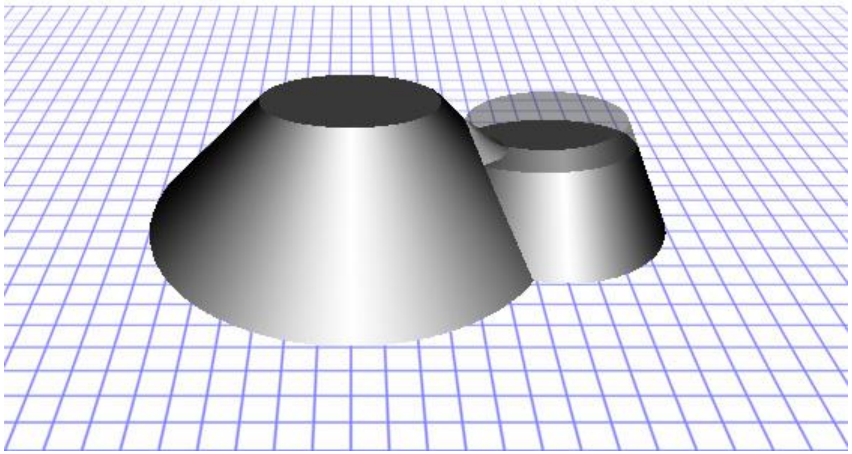
opened image



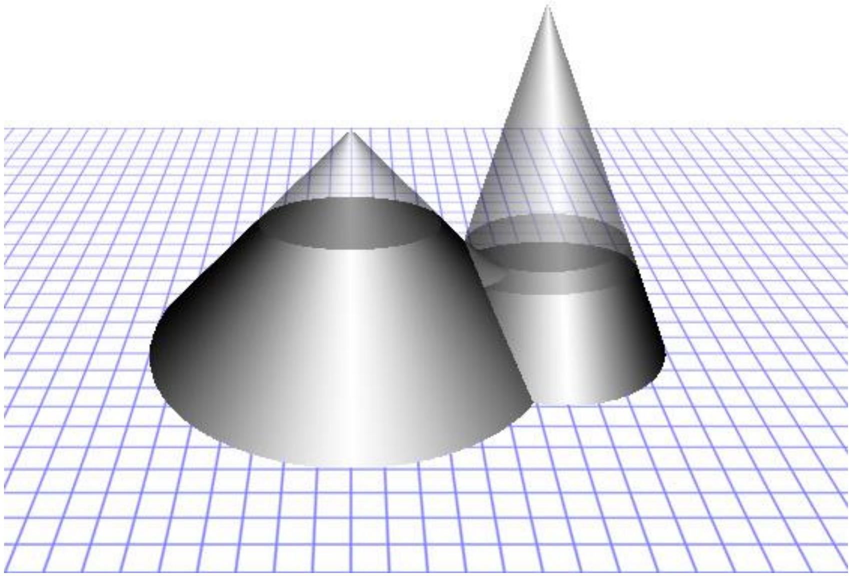
opened image & original

SE, Z, is a flat disk the size of the tops of the truncated cones.

Grayscale Reconstruction



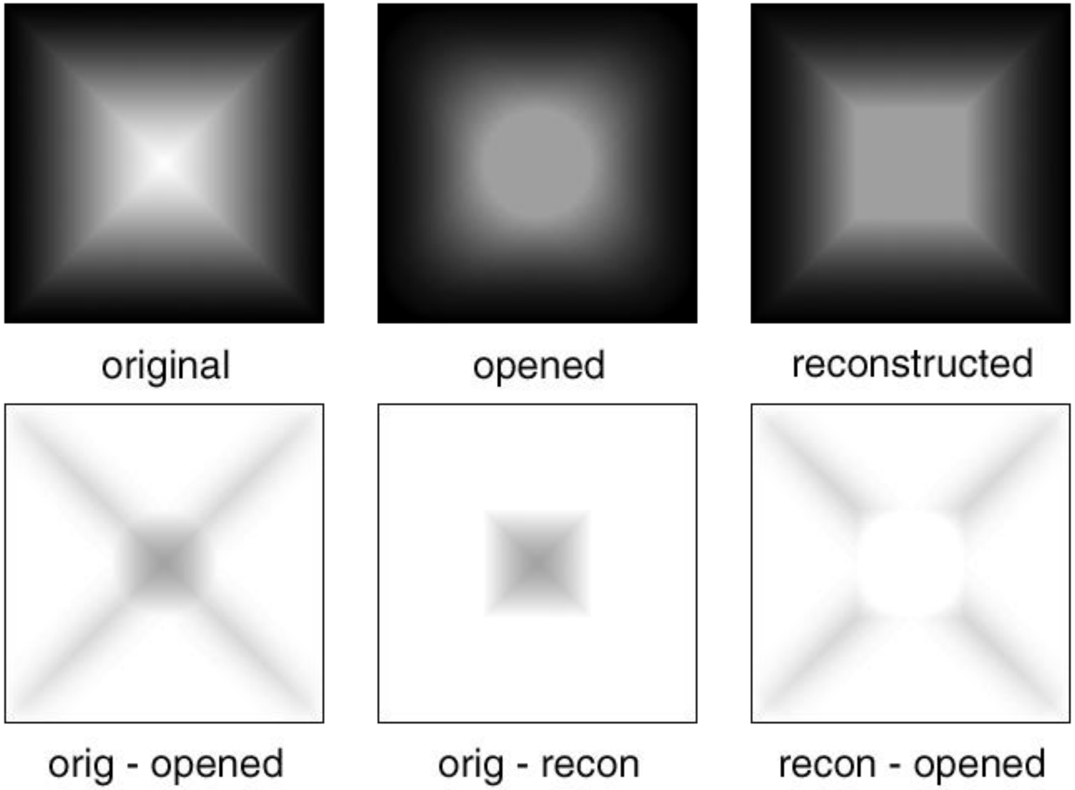
opened & recon. image



opened, recon., & original

SE, Z, is a flat disk the size of the tops of the truncated cones.

Grayscale Morphology: Reconstruction



SE, Z, is a flat disk.

Grayscale Reconstruction

SE, Z, is a flat disk.

Grayscale Reconstruction

SE, Z, is a flat disk.

Grayscale Reconstruction

SE, Z, is a flat disk.