

Introduction to Quantum Information Processing

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1 Mutual Information - Solution

1. We expand the expression to get:

$$I(X; Y) = H(X) - H(X|Y) = - \sum_x p(x) \log_2 p(x) + \sum_{x,y} p(x,y) \log_2 p(x|y)$$

We know that a probability distribution of variable X can be computed from the mutual probability of X and Y by summing all the values in range for Y :

$$p(x) = \sum_y p(x,y)$$

Plugging this into the previous equation, we get:

$$I(X; Y) = \sum_{x,y} p(x,y) \log_2 p(x|y) - \sum_{x,y} p(x,y) \log_2 p(x) = \sum_{x,y} p(x,y) \log_2 \left(\frac{p(x|y)}{p(x)} \right)$$

From logarithm rules and conditional probability definition:

$$\log_2 \left(\frac{p(x|y)}{p(x)} \right) = - \log_2 \left(\frac{p(x)p(y)}{p(x,y)} \right)$$

Plugging it, we get:

$$I(X; Y) = - \sum_{x,y} p(x,y) \log_2 \left(\frac{p(x)p(y)}{p(x,y)} \right)$$

2. We can write $I(X; Y)$ using the logarithm-identity $\log_a x \cdot \log_b a = \log_b x$ as:

$$K \cdot I(X; Y) = - \sum_{x,y} p(x,y) \ln \left(\frac{p(x)p(y)}{p(x,y)} \right)$$

Where $K = \ln 2 > 0$. By using the identity $\ln t \leq t - 1$ for $t > 0$ and knowing that $\frac{p(x)p(y)}{p(x,y)}$ is non-negative (because probability cannot be negative, obv.) we show:

$$-K \cdot I(X; Y) \leq \sum_{x,y} p(x,y) \left(\frac{p(x)p(y)}{p(x,y)} - 1 \right) = \sum_{x,y} (p(x)p(y) - p(x,y))$$

$\ln t \leq t - 1$ is equality **iff** $t = 1$.

Corollary 1.0.1. $-K \cdot I(X; Y) = \sum_{x,y} (p(x)p(y) - p(x,y))$ iff X, Y are independent. **Proof:** If X, Y are independent $\iff p(x)p(y) = p(x,y)$ for every pair of $x, y \iff \forall x, y : \frac{p(x)p(y)}{p(x,y)} = 1 \iff$

We expand the expression to get sums over probabilities (which we can reduce to 1):

$$\begin{aligned}
 -K \cdot I(X; Y) &\leq \sum_x \left(p(x) \cdot \sum_y p(y) \right) - \sum_{x,y} p(x, y) \\
 \Rightarrow I(X; Y) &\geq K \cdot \sum_x p(x) - 1 = 0 \Rightarrow \boxed{I(X; Y) \geq 0}
 \end{aligned}$$

From 1.0.1 and the above expansion we can conclude that $I(X; Y) = 0 \iff X, Y$ are independent.

2 Entropy and Mutual Information

1.

$$\begin{aligned}
 Y &= \begin{cases} 1, & \text{the keys are in the pocket} \\ 0, & \text{otherwise} \end{cases} \\
 X &= \begin{cases} i \in [1, 100], & \text{the keys are in the } i\text{-th location} \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

2.

	Y = 0	Y = 1	
X = 0	$p(X = 0, Y = 0) = 0$ $p(X = 0 Y = 0) = 0$ $p(Y = 0 X = 0) = 0$		$p(X = 0) = 0.99$ $p(X = 0, Y = 1) = 0.99$ $p(X = 0 Y = 1) = 1$ $p(Y = 1 X = 0) = 1$
X = i > 0	$i > 0$ $p(X = i, Y = 1) = 0$ $p(X = i Y = 1) = 0$ $p(Y = 1 X = i) = 0$		$p(X = i > 0) = 0.0001$ $i > 0$ $p(X = i, Y = 0) = 0.0001$ $p(X = i Y = 0) = 0.01$ $p(Y = 0 X = i) = 1$
	$p(Y = 0) = 0.01$	$p(Y = 1) = 0.99$	1

3.

$$\begin{aligned}
 H(X) &= - \sum_x p(X = x) \log_2 p(X = x) \\
 &= -p(X = 0) \log_2 p(X = 0) - 100 \cdot (X = i) \log_2 p(X = i) \\
 &= -0.99 \cdot \log_2 0.99 - 100 \cdot 0.0001 \log_2 0.0001 \\
 &= 0.044
 \end{aligned}$$

$$\begin{aligned}
 H(Y) &= - \sum_y p(Y = y) \log_2 p(Y = y) \\
 &= -p(Y = 0) \log_2 p(Y = 0) - (Y = 1) \log_2 p(Y = 1) \\
 &= -0.01 \cdot \log_2 0.01 - 0.99 \log_2 0.99 \\
 &= 0.024
 \end{aligned}$$

$$\begin{aligned}
H(X|Y=0) &= - \sum_x p(X|Y=0) \log_2 p(X|Y=0) \\
&= -p(X=0|Y=0) \log_2 p(X=0|Y=0) - 100 \cdot (X=i|Y=0) \log_2 p(X=i|Y=0) \\
&= -100 \cdot 0.01 \log_2 0.01 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
H(X|Y=1) &= - \sum_x p(X|Y=1) \log_2 p(X|Y=1) \\
&= -p(X=0|Y=1) \log_2 p(X=0|Y=1) - 100 \cdot (X=i|Y=1) \log_2 p(X=i|Y=1) \\
&= -1 \cdot 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
H(X|Y) &= - \sum_{x,y} p(X=x, Y=y) \log_2 p(X=x|Y=y) \\
&= -p(X=0, Y=0) \log_2 p(X=0|Y=0) - \sum_{i=1}^{100} p(X=i, Y=0) \log_2 p(X=i|Y=0) \\
&\quad - p(X=0, Y=1) \log_2 p(X=0|Y=1) - \sum_{i=1}^{100} p(X=i, Y=1) \log_2 p(X=i|Y=1) \\
&= 0 - 100 \cdot 0.0001 \log_2 0.01 - 0.99 \log_2 1 - 100 \cdot 0 \\
&= 0.02
\end{aligned}$$

$$\begin{aligned}
H(Y|X) &= - \sum_{x,y} p(X=x, Y=y) \log_2 p(Y=y|X=x) \\
&= -p(X=0, Y=0) \log_2 p(Y=0|X=0) - \sum_{i=1}^{100} p(X=i, Y=0) \log_2 p(Y=0|X=i) \\
&\quad - p(X=0, Y=1) \log_2 p(Y=1|X=0) - \sum_{i=1}^{100} p(X=i, Y=1) \log_2 p(Y=1|X=i) \\
&= 0 - 0 - 0 - 0 \\
&= 0
\end{aligned}$$

4.

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= 0.044 - 0.02 = 0.024
\end{aligned}$$

$$\begin{aligned}
I(Y;X) &= H(Y) - H(Y|X) \\
&= 0.024 - 0 = 0.024
\end{aligned}$$

5. When we learn that the keys are not in the pocket, the entropy raises: $H(X|Y=0) > H(X)$. It seems strange because by "learning" something about the system, we reduced our information, but it is only guaranteed that **on average** the entropy is reduced when learning. The amount of entropy $H(X)$ reduced on average when learning the value of random variable Y is $I(X;Y)$, and this value is always non-negative.

3 Poincare Sphere

1. • **Direction:** $\langle \phi | \phi' \rangle = 0 \Rightarrow |\phi\rangle$ and $|\phi'\rangle$ are on opposite sides of poincaré sphere

If $\langle \phi | \phi' \rangle = 0$ then:

$$\left(\cos \frac{\theta}{2} \quad e^{-i\phi} \sin \frac{\theta}{2} \right) \cdot \begin{pmatrix} \cos \frac{\theta'}{2} \\ e^{i\phi'} \sin \frac{\theta'}{2} \end{pmatrix} = 0$$

Unpacking this equation, we get:

$$(\star) = \cos \frac{\theta}{2} \cos \frac{\theta'}{2} + e^{i(\phi' - \phi)} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} = 0$$

In particular, $Im(\star) = 0$, thus $\phi \equiv \phi' \pmod{\pi}$. We are left with:

$$\cos \frac{\theta}{2} \cos \frac{\theta'}{2} \pm \sin \frac{\theta}{2} \sin \frac{\theta'}{2} = 0 \Rightarrow \sin \left(\frac{\pi}{2} + \frac{\theta}{2} \right) \cos \frac{\theta'}{2} = \pm \cos \left(\frac{\pi}{2} + \frac{\theta}{2} \right) \sin \frac{\theta'}{2}$$

Which we can simplify to

$$\tan \left(\frac{\pi}{2} + \frac{\theta}{2} \right) = \mp \tan \frac{\theta'}{2} \Rightarrow \pi + \theta = \mp \theta'$$

Thus we get:

$$\theta' = \begin{cases} \theta + \pi, & \phi' = \phi + \pi \\ -(\theta + \pi), & \phi' = \phi \end{cases}$$

Obviously, the second case is impossible because then θ' is negative. So we must conclude that:

$$\begin{aligned} \theta' &= \theta + \pi \\ \phi' &= \phi + \pi \end{aligned}$$

- **Direction:** $|\phi\rangle$ and $|\phi'\rangle$ on opposite direction $\Rightarrow \langle \phi | \phi' \rangle = 0$

Suppose $\phi = \pi + \phi'$ and $\theta = \pi + \theta'$. Recall (\star) from the previous direction of the proof, and plug in:

$$\begin{aligned} (\star) &= \cos \frac{\theta}{2} \cos \frac{\theta + \pi}{2} + e^{i(\pi + \phi - \phi)} \sin \frac{\theta}{2} \sin \frac{\theta + \pi}{2} \\ &= \cos \frac{\theta}{2} \cos \frac{\theta + \pi}{2} - \sin \frac{\theta}{2} \sin \frac{\theta + \pi}{2} \\ &= \cos \frac{\theta}{2} \sin \frac{\theta}{2} - \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 0 \end{aligned}$$

2. There is an infinite amount of ensembles, and exactly one ensembles that consists of 2 orthogonal states.
3. There is an infinite amount of ensembles, all of which are pairs of orthogonal states.

4 Measurements

1. • $Pr(|+\rangle) = |\langle + | \phi \rangle|^2 = \frac{1}{2} \cdot (\alpha + \beta)^2$
• $Pr(|\theta\rangle) = |\langle \theta | \phi \rangle|^2 = \alpha^2 \cos^2 \theta - 2\alpha\beta \cos \theta \sin \theta + \beta^2 \sin^2 \theta = 1 - \alpha\beta \sin 2\theta$
2. • $Pr(|10\rangle) = \langle 10 | p | 10 \rangle = p_2$
• $Pr(|00\rangle) = \langle 00 | p | 00 \rangle = p_0$

•

$$|-+\rangle = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \otimes \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

Thus,

$$Pr(|-+\rangle) = \frac{1}{4} \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} p0 & 0 & 0 & 0 \\ 0 & p1 & 0 & 0 \\ 0 & 0 & p2 & 0 \\ 0 & 0 & 0 & p3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \frac{p0 + p1 + p2 + p3}{4}$$

•

$$|1+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus,

$$Pr(|1+\rangle) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p0 & 0 & 0 & 0 \\ 0 & p1 & 0 & 0 \\ 0 & 0 & p2 & 0 \\ 0 & 0 & 0 & p3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \frac{p2 + p3}{2}$$

5 Double slit experiment

1.

$$\begin{aligned} \vec{E}_1(\vec{r}_1, t) + \vec{E}_2(\vec{r}_2, t) &= \frac{\vec{E}_0}{r_1} \cos(k \cdot r_1 - w \cdot r) + \frac{\vec{E}_0}{r_2} \cos(k \cdot r_2 - w \cdot r) \\ &\approx \frac{\vec{E}_0}{r} (\cos(k \cdot r_1 - w \cdot r) + \cos(k \cdot r_2 - w \cdot r)) \\ &= \boxed{2 \frac{\vec{E}_0}{r} \cos\left(\frac{k \cdot r_1 + k \cdot r_2 - 2w \cdot t}{2}\right) \cos\left(\frac{k(r_1 - r_2)}{2}\right)} \end{aligned}$$

2.

$$\begin{aligned} |\vec{E}_1(\vec{r}_1, t) + \vec{E}_2(\vec{r}_2, t)|^2 &= \left| 2 \frac{\vec{E}_0}{r} \cos\left(\frac{k \cdot r_1 + k \cdot r_2 - 2w \cdot t}{2}\right) \cos\left(\frac{k(r_1 - r_2)}{2}\right) \right|^2 \\ &= \frac{4E_0^2}{r^2} \left| \cos\left(\frac{k(r_1 - r_2)}{2}\right) \right|^2 \left| \cos\left(\frac{k \cdot r_1 + k \cdot r_2 - 2w \cdot t}{2}\right) \right|^2 \end{aligned}$$

3.

$$\begin{aligned} \langle |\vec{E}_1(\vec{r}_1, t) + \vec{E}_2(\vec{r}_2, t)|^2 \rangle &= \left\langle \frac{4E_0^2}{r^2} \left| \cos\left(\frac{k(r_1 - r_2)}{2}\right) \right|^2 \left| \cos\left(\frac{k \cdot r_1 + k \cdot r_2 - 2w \cdot t}{2}\right) \right|^2 \right\rangle \\ &= \frac{4E_0^2}{r^2} \left| \cos\left(\frac{k(r_1 - r_2)}{2}\right) \right|^2 \left\langle \left| \cos\left(\frac{k \cdot r_1 + k \cdot r_2 - 2w \cdot t}{2}\right) \right|^2 \right\rangle \\ &= \frac{4E_0^2}{r^2} \left| \cos\left(\frac{k(r_1 - r_2)}{2}\right) \right|^2 \frac{1}{2} \end{aligned}$$

4.

$$\boxed{I(\theta) = \frac{4E_0^2}{r^2} \left| \cos\left(\frac{k \cos \theta}{2}\right) \right|^2 \frac{1}{2}}$$