

# Introduction to Quantum Information Processing

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## 1 Mutual Information - Solution

1. We expand the expression to get:

$$I(X; Y) = H(X) - H(X|Y) = - \sum_x p(x) \log_2 p(x) + \sum_{x,y} p(x,y) \log_2 p(x|y)$$

We know that a probability distribution of variable  $X$  can be computed from the mutual probability of  $X$  and  $Y$  by summing all the values in range for  $Y$ :

$$p(x) = \sum_y p(x,y)$$

Plugging this into the previous equation, we get:

$$I(X; Y) = \sum_{x,y} p(x,y) \log_2 p(x|y) - \sum_{x,y} p(x,y) \log_2 p(x) = \sum_{x,y} p(x,y) \log_2 \left( \frac{p(x|y)}{p(x)} \right)$$

From logarithm rules and conditional probability definition:

$$\log_2 \left( \frac{p(x|y)}{p(x)} \right) = - \log_2 \left( \frac{p(x)p(y)}{p(x,y)} \right)$$

Plugging it, we get:

$$I(X; Y) = - \sum_{x,y} p(x,y) \log_2 \left( \frac{p(x)p(y)}{p(x,y)} \right)$$

2. We can write  $I(X; Y)$  using the logarithm-identity  $\log_a x \cdot \log_b a = \log_b x$  as:

$$K \cdot I(X; Y) = - \sum_{x,y} p(x,y) \ln \left( \frac{p(x)p(y)}{p(x,y)} \right)$$

Where  $K = \ln 2 > 0$ . By using the identity  $\ln t \leq t - 1$  for  $t > 0$  and knowing that  $\frac{p(x)p(y)}{p(x,y)}$  is non-negative (because probability cannot be negative, obv.) we show:

$$-K \cdot I(X; Y) \leq \sum_{x,y} p(x,y) \left( \frac{p(x)p(y)}{p(x,y)} - 1 \right) = \sum_{x,y} (p(x)p(y) - p(x,y))$$

$\ln t \leq t - 1$  is equality **iff**  $t = 1$ .

**Corollary 1.0.1.**  $-K \cdot I(X; Y) = \sum_{x,y} (p(x)p(y) - p(x,y))$  iff  $X, Y$  are independent. **Proof:** If  $X, Y$  are independent  $\iff p(x)p(y) = p(x,y)$  for every pair of  $x, y \iff \forall x, y : \frac{p(x)p(y)}{p(x,y)} = 1 \iff$

We expand the expression to get sums over probabilities (which we can reduce to 1):

$$\begin{aligned}
 -K \cdot I(X; Y) &\leq \sum_x \left( p(x) \cdot \sum_y p(y) \right) - \sum_{x,y} p(x, y) \\
 \Rightarrow I(X; Y) &\geq K \cdot \sum_x p(x) - 1 = 0 \Rightarrow \boxed{I(X; Y) \geq 0}
 \end{aligned}$$

From 1.0.1 and the above expansion we can conclude that  $I(X; Y) = 0 \iff X, Y$  are independent.

## 2 Entropy and Mutual Information

1.

$$\begin{aligned}
 Y &= \begin{cases} 1, & \text{the keys are in the pocket} \\ 0, & \text{otherwise} \end{cases} \\
 X &= \begin{cases} i \in [1, 100], & \text{the keys are in the } i\text{-th location} \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

2.

	Y = 0	Y = 1	
X = 0	$p(X = 0, Y = 0) = 0$ $p(X = 0 Y = 0) = 0$ $p(Y = 0 X = 0) = 0$		$p(X = 0, Y = 1) = 0.99$ $p(X = 0 Y = 1) = 1$ $p(Y = 1 X = 0) = 1$
X = i > 0	$i > 0$ $p(X = i, Y = 1) = 0$ $p(X = i Y = 1) = 0$ $p(Y = 1 X = i) = 0$		$i > 0$ $p(X = i, Y = 0) = 0.0001$ $p(X = i Y = 0) = 0.01$ $p(Y = 0 X = i) = 1$
	$p(Y = 0) = 0.01$	$p(Y = 1) = 0.99$	1

3.

$$\begin{aligned}
 H(X) &= - \sum_x p(X = x) \log_2 p(X = x) \\
 &= -p(X = 0) \log_2 p(X = 0) - 100 \cdot (X = i) \log_2 p(X = i) \\
 &= -0.99 \cdot \log_2 0.99 - 100 \cdot 0.0001 \log_2 0.0001 \\
 &= 0.044
 \end{aligned}$$

$$\begin{aligned}
 H(Y) &= - \sum_y p(Y = y) \log_2 p(Y = y) \\
 &= -p(Y = 0) \log_2 p(Y = 0) - (Y = 1) \log_2 p(Y = 1) \\
 &= -0.01 \cdot \log_2 0.01 - 0.99 \log_2 0.99 \\
 &= 0.024
 \end{aligned}$$

$$\begin{aligned}
H(X|Y=0) &= - \sum_x p(X|Y=0) \log_2 p(X|Y=0) \\
&= -p(X=0|Y=0) \log_2 p(X=0|Y=0) - 100 \cdot (X=i|Y=0) \log_2 p(X=i|Y=0) \\
&= -100 \cdot 0.01 \log_2 0.01 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
H(X|Y=1) &= - \sum_x p(X|Y=1) \log_2 p(X|Y=1) \\
&= -p(X=0|Y=1) \log_2 p(X=0|Y=1) - 100 \cdot (X=i|Y=1) \log_2 p(X=i|Y=1) \\
&= -1 \cdot 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
H(X|Y) &= - \sum_{x,y} p(X=x, Y=y) \log_2 p(X=x|Y=y) \\
&= -p(X=0, Y=0) \log_2 p(X=0|Y=0) - \sum_{i=1}^{100} p(X=i, Y=0) \log_2 p(X=i|Y=0) \\
&\quad - p(X=0, Y=1) \log_2 p(X=0|Y=1) - \sum_{i=1}^{100} p(X=i, Y=1) \log_2 p(X=i|Y=1) \\
&= 0 - 100 \cdot 0.0001 \log_2 0.01 - 0.99 \log_2 1 - 100 \cdot 0 \\
&= 0.02
\end{aligned}$$

$$\begin{aligned}
H(Y|X) &= - \sum_{x,y} p(X=x, Y=y) \log_2 p(Y=y|X=x) \\
&= -p(X=0, Y=0) \log_2 p(Y=0|X=0) - \sum_{i=1}^{100} p(X=i, Y=0) \log_2 p(Y=0|X=i) \\
&\quad - p(X=0, Y=1) \log_2 p(Y=1|X=0) - \sum_{i=1}^{100} p(X=i, Y=1) \log_2 p(Y=1|X=i) \\
&= 0 - 0 - 0 - 0 \\
&= 0
\end{aligned}$$

4.

### 3 Poincare Sphere

1. • **Direction:**  $\langle \phi | \phi' \rangle = 0 \Rightarrow |\phi\rangle$  and  $|\phi'\rangle$  are on opposite sides of poincaré sphere

If  $\langle \phi | \phi' \rangle = 0$  then:

$$\left( \cos \frac{\theta}{2} \quad e^{i\phi} \sin \frac{\theta}{2} \right) \cdot \begin{pmatrix} \cos \frac{\theta'}{2} \\ e^{i\phi'} \sin \frac{\theta'}{2} \end{pmatrix} = 0$$

Unpacking this equation, we get:

$$(\star) = \cos \frac{\theta}{2} \cos \frac{\theta'}{2} + e^{i(\phi+\phi')} \sin \frac{\theta}{2} \sin \frac{\theta'}{2} = 0$$

In particular,  $Im(\star) = 0$ , thus  $\phi \equiv -\phi' \pmod{2\pi}$  (The states are in opposite directions w.r.t.  $\phi$  and  $\phi'$ ). We are left with:

$$\cos \frac{\theta}{2} \cos \frac{\theta'}{2} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2} = 0 \Rightarrow \sin \left( \frac{\pi}{2} + \frac{\theta}{2} \right) \cos \frac{\theta'}{2} = \cos \left( \frac{\pi}{2} + \frac{\theta}{2} \right) \sin \frac{\theta'}{2}$$

Which we can simplify to

$$\tan \left( \frac{\pi}{2} + \frac{\theta}{2} \right) = \tan \frac{\theta'}{2} \Rightarrow \pi + \theta = \theta'$$