

Introduction to Quantum Information Processing

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1.1

$$\text{tr}(A^\dagger) = \sum_{i=1}^n a_{ii}^* = (\sum_{i=1}^n a_{ii})^* = (\text{tr}(A))^*$$

1.2

$$\text{tr}(\alpha A + \beta B) = \sum_{i=1}^n \alpha A_{ii} + \beta B_{ii} = \alpha \sum_{i=1}^n A_{ii} + \beta \sum_{i=1}^n B_{ii} = \alpha \text{tr}(A) + \beta \text{tr}(B)$$

1.3

$$\text{tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n (\sum_{k=1}^n A_{ik} B_{ki}) = \sum_{i=1}^n (\sum_{k=1}^n B_{ki} A_{ik}) = \sum_{i=1}^n (BA)_{ii} = \text{tr}(BA)$$

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Throughout this part we mark eigenvalues with λ_i and eigenvectors with v_i . In order to find the eigenvalue we use the identities $\text{tr}(A) = \sum \lambda_i$ and $\det(A) = \prod \lambda_i$.

2.1

$$\begin{aligned} \text{tr}(H) &= \lambda_1 + \lambda_2 = 0 \\ \det(H) &= \lambda_1 \lambda_2 = -1 \end{aligned}$$

Solving for $\lambda_{1,2}$ yields:

$$\begin{aligned} \lambda_1 &= \pm 1 \\ \lambda_2 &= \mp 1 \end{aligned}$$

Because of the symmetry, we can just assume $\lambda_1 = 1$ and $\lambda_2 = -1$. To find the eigenvectors we solve for v_i :

$$(H - I\lambda_i)v_i = 0$$

TODO

3

3.1

3.1.1

σ_x is a symmetric and real matrix, and as such it is also Hermitian.

3.1.2

$$\sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i^* \\ -i^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$$

because $\sigma_y^\dagger = \sigma_y$, σ_y is Hermitian.

3.1.3

σ_z is a symmetric and real matrix, and as such it is also Hermitian.

3.1.4

$$H^\dagger = \frac{1}{\sqrt{2}} \star \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\dagger \stackrel{\text{real numbers, symmetric matrix}}{=} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

3.2

3.2.1

$$\begin{aligned} (\alpha A + \beta B)^\dagger &= (\alpha A)^\dagger + (\beta B)^\dagger && \text{(Linearity of conjugation)} \\ &= \alpha^* A^\dagger + \beta^* B^\dagger \\ &= \alpha A^\dagger + \beta B^\dagger && (\alpha, \beta \in \mathbb{R}) \\ &= \alpha A + \beta B && (A, B \text{ are Hermitian}) \end{aligned}$$

3.2.2

$$\begin{aligned} (\star) &= (\alpha - \beta) \langle v|u \rangle \\ &= \alpha \langle v|u \rangle - \beta \langle v|u \rangle \\ &= \alpha \langle u|v \rangle - \beta \langle v|u \rangle && (\langle v|u \rangle = \langle u|v \rangle) \\ &= \langle u|A|v \rangle - \beta \langle v|u \rangle && (A|v \rangle = \alpha|v \rangle) \\ &= \langle u|A|v \rangle - \langle v|A|u \rangle && (A|u \rangle = \beta|u \rangle) \\ &= 0 \end{aligned}$$

Because $\alpha \neq \beta$ and $(\star) = 0$, we must conclude that $\langle v|u \rangle = 0$ \square .

4

4.1

For Hermitian matrix $\alpha_i = \alpha_i^\dagger$, $\alpha_i \alpha_i^\dagger = \alpha_i^\dagger \alpha_i = I \iff \alpha_i^2 = I$.

- $\alpha_x \cdot \alpha_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I$
- $\alpha_y \cdot \alpha_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = I$
- $\alpha_z \cdot \alpha_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = I$
- $H \cdot H = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right)^2 = I$

4.2

Denote the i -th column of U by U_i . It holds that $U^\dagger U = I$, so from the definition of matrices multiplication we get:

$$(U_i)^\dagger U_j = I_{ij}$$

By definition,

$$I_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Putting it all together:

$$(U_i)^\dagger U_j = \langle U_i, U_j \rangle = \delta_{ij}$$

4.3

TODO - Orthonormality of column \rightarrow unitary matrix

4.4

Let λ be an eigenvalue with eigenvector $v \neq \vec{0}$ of unitary matrix U . Then,

$$\langle v, v \rangle = \langle Uv, Uv \rangle = \langle \lambda v, \lambda v \rangle = \lambda^* \lambda \langle v, v \rangle \Rightarrow \lambda^* \lambda = 1 \Rightarrow |\lambda| = 1$$

5

5.1

5.1.1

$$(\vec{a} + \vec{c}) \otimes \vec{b} = \begin{pmatrix} (a_1 + c_1)\vec{b} \\ (a_2 + c_2)\vec{b} \\ \vdots \\ (a_k + c_k)\vec{b} \end{pmatrix} = \begin{pmatrix} a_1\vec{b} \\ a_2\vec{b} \\ \vdots \\ a_k\vec{b} \end{pmatrix} + \begin{pmatrix} c_1\vec{b} \\ c_2\vec{b} \\ \vdots \\ c_k\vec{b} \end{pmatrix} = \vec{a} \otimes \vec{b} + \vec{c} \otimes \vec{b}$$

5.1.2

$$\vec{a} \otimes (\vec{b} + \vec{c}) = \begin{pmatrix} a_1(\vec{b} + \vec{c}) \\ a_2(\vec{b} + \vec{c}) \\ \vdots \\ a_k(\vec{b} + \vec{c}) \end{pmatrix} = \begin{pmatrix} a_1\vec{b} \\ a_2\vec{b} \\ \vdots \\ a_k\vec{b} \end{pmatrix} + \begin{pmatrix} a_1\vec{c} \\ a_2\vec{c} \\ \vdots \\ a_k\vec{c} \end{pmatrix} = \vec{a} \otimes \vec{b} + \vec{a} \otimes \vec{c}$$

5.1.3

$$\vec{a} \otimes (c \cdot \vec{b}) = \begin{pmatrix} a_1 \cdot c\vec{b} \\ a_2 \cdot c\vec{b} \\ \vdots \\ a_k \cdot c\vec{b} \end{pmatrix} = \underbrace{\begin{pmatrix} c \cdot a_1\vec{b} \\ c \cdot a_2\vec{b} \\ \vdots \\ c \cdot a_k\vec{b} \end{pmatrix}}_{(c \cdot \vec{a}) \otimes \vec{b}} = c \cdot \begin{pmatrix} a_1\vec{b} \\ a_2\vec{b} \\ \vdots \\ a_k\vec{b} \end{pmatrix} = c \cdot (\vec{a} \otimes \vec{b})$$

5.1.4

Let

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{C}^2$$

We can compute the following tensor products:

$$\vec{a} \otimes \vec{b} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

$$\vec{c} \otimes \vec{d} = \begin{pmatrix} c_1 d_1 \\ c_1 d_2 \\ c_2 d_1 \\ c_2 d_2 \end{pmatrix}$$

Plugging them into the inner product yields:

$$\begin{aligned}
 \langle \vec{a} \otimes \vec{b}, \vec{c} \otimes \vec{d} \rangle &= \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}^\dagger \cdot \begin{pmatrix} c_1 d_1 \\ c_1 d_2 \\ c_2 d_1 \\ c_2 d_2 \end{pmatrix} = a_1^* b_1^* c_1 d_1 + a_1^* b_2^* c_1 d_2 + a_2^* b_1^* c_2 d_1 + a_2^* b_2^* c_2 d_2 \\
 &= (a_1^* c_1)(b_1^* d_1) + (a_1^* c_1)(b_2^* d_2) + (a_2^* c_2)(b_1^* d_1) + (a_2^* c_2)(b_2^* d_2) \\
 &= (a_1^* c_1 + a_2^* c_2)(b_1^* d_1 + b_2^* d_2) \\
 &= \langle \vec{a}, \vec{c} \rangle \langle \vec{b}, \vec{d} \rangle \quad \square
 \end{aligned}$$

5.2

$$(\vec{a} \otimes \vec{b}) \otimes \vec{c} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 c_1 \\ a_1 b_1 c_2 \\ a_1 b_2 c_1 \\ a_1 b_2 c_2 \\ a_2 b_1 c_1 \\ a_2 b_1 c_2 \\ a_2 b_2 c_1 \\ a_2 b_2 c_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 c_1 \\ b_1 c_2 \\ b_2 c_1 \\ b_2 c_2 \end{pmatrix} = \vec{a} \otimes (\vec{b} \otimes \vec{c})$$

5.3

We prove only for matrices of size 2×2 . Let:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Now we can "explicitly" compute:

$$\begin{aligned}
 \text{tr}(A \otimes B) &= \text{tr} \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \\
 &= a_{11}b_{11} + a_{11}b_{22} + a_{22}b_{11} + a_{22}b_{22} \\
 &= (a_{11} + a_{22})(b_{11} + b_{22}) \\
 &= \text{tr}(A) \text{tr}(B)
 \end{aligned}$$

5.4

As we've seen in previous exercises, for 2 matrices of size 2×2 we get

$$A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

Denote the i -th column in this product by D_i . Without loss of generality, we show that $\langle D_1, D_2 \rangle = 1$ and that $\langle D_1, D_1 \rangle = 0$. The proof for the other cases is very similar.

$$\begin{aligned}
\langle D_1, D_2 \rangle &= a_{11}^2 b_{11} b_{12} + a_{11}^2 b_{21} b_{22} + a_{21}^2 b_{11} b_{12} + a_{21}^2 b_{21} b_{22} \\
&= a_{11}^2 (b_{11} b_{12} + b_{21} b_{22}) + a_{21}^2 (b_{11} b_{12} + b_{21} b_{22}) \\
&= (a_{11}^2 + a_{21}^2) \underbrace{\langle B_1, B_2 \rangle}_{=0} = 0
\end{aligned}$$

$$\begin{aligned}
\langle D_1, D_1 \rangle &= a_{11}^2 b_{11}^2 + a_{11}^2 b_{21}^2 + a_{21}^2 b_{11}^2 + a_{21}^2 b_{21}^2 \\
&= (a_{11}^2 + a_{21}^2) (b_{11}^2 + b_{21}^2) \\
&= \langle A_1, A_1 \rangle \langle B_1, B_1 \rangle = 1
\end{aligned}$$

5.5

Recall our previous notation:

$$A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

Also,

$$\vec{u} \otimes \vec{v} = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \end{pmatrix}$$

Thus we get:

$$\begin{aligned}
(A \otimes B) (\vec{u} \otimes \vec{v}) &= \begin{pmatrix} a_{11}b_{11}u_1v_1 + a_{11}b_{12}u_1v_2 + a_{12}b_{11}u_2v_1 + a_{12}b_{12}u_2v_2 \\ a_{11}b_{21}u_1v_1 + a_{11}b_{22}u_1v_2 + a_{12}b_{21}u_2v_1 + a_{12}b_{22}u_2v_2 \\ a_{21}b_{11}u_1v_1 + a_{21}b_{12}u_1v_2 + a_{22}b_{11}u_2v_1 + a_{22}b_{12}u_2v_2 \\ a_{21}b_{21}u_1v_1 + a_{21}b_{22}u_1v_2 + a_{22}b_{21}u_2v_1 + a_{22}b_{22}u_2v_2 \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}u_1 + a_{12}u_2)(b_{11}v_1 + b_{12}v_2) \\ (a_{11}u_1 + a_{12}u_2)(b_{21}v_1 + b_{22}v_2) \\ (a_{21}u_1 + a_{22}u_2)(b_{11}v_1 + b_{12}v_2) \\ (a_{21}u_1 + a_{22}u_2)(b_{21}v_1 + b_{22}v_2) \end{pmatrix} \\
&= \begin{pmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{pmatrix} \otimes \begin{pmatrix} b_{11}v_1 + b_{12}v_2 \\ b_{21}v_1 + b_{22}v_2 \end{pmatrix} \\
&= A\vec{u} \otimes B\vec{v}
\end{aligned}$$