Introduction to Quantum Information Processing

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1 Mutual Information - Solution

1. We expand the expression to get:

$$I(X;Y) = H(X) - H(X|Y) = -\sum_x p(x)\log_2 p(x) + \sum_{x,y} p(x,y)\log_2 p(x|y)$$

We know that a probability distribution of variable X can be computed from the mutual probability of X and Y by summing all the values in range for Y:

$$p(x) = \sum_y p(x,y)$$

Plugging this into the previous equation, we get:

$$I(X;Y) = \sum_{x,y} p(x,y)\log_2 p(x|y) - \sum_{x,y} p(x,y)\log_2 p(x) = \sum_{x,y} p(x,y)\log_2\left(\frac{p(x|y)}{p(x)}\right)$$

From logarithm rules and conditional probability definition:

$$\log_2\left(\frac{p(x|y)}{p(x)}\right) = -\log_2\left(\frac{p(x)p(y)}{p(x,y)}\right)$$

Plugging it, we get:

$$\boxed{I(X;Y) = -\sum_{x,y} p(x,y)\log_2\left(\frac{p(x)p(y)}{p(x,y)}\right)}$$

2. We can write $I(X;Y)$ using the logarithm-identity $\log_a x \cdot \log_b a = \log_b x$ as:

$$K \cdot I(X;Y) = -\sum_{x,y} p(x,y)\ln\left(\frac{p(x)p(y)}{p(x,y)}\right)$$

Where $K = \ln 2 > 0$. By using the identity $\ln t \leq t - 1$ for $t > 0$ and knowing that $\frac{p(x)p(y)}{p(x,y)}$ is non-negative (because probability cannot be negative, obv.) we show:

$$-K \cdot I(X;Y) \leq \sum_{x,y} p(x,y)\left(\frac{p(x)p(y)}{p(x,y)} - 1\right) = \sum_{x,y}(p(x)p(y) - p(x,y))$$

$\ln t \leq t - 1$ is equality **iff** $t = 1$.

Corollary 1.0.1. $-K \cdot I(X;Y) = \sum_{x,y}(p(x)p(y) - p(x,y))$ iff X, Y *are independent.* ***Proof:*** *If X, Y are independent* $\iff$ *$p(x)p(y) = p(x,y)$ for every pair of x, y* $\iff$ $\forall x, y : \frac{p(x)p(y)}{p(x,y)} = 1$ $\iff$

We expand the expression to get sums over probabilities (which we can reduce to 1):

$$-K \cdot I(X;Y) \leq \sum_x \left(p(x) \cdot \sum_y p(y) \right) - \sum_{x,y} p(x,y)$$

$$\Rightarrow I(X;Y) \geq K \cdot \sum_x p(x) - 1 = 0 \Rightarrow \boxed{I(X;Y) \geq 0}$$

From 1.0.1 and the above expansion we can conclude that $I(X;Y) = 0 \iff X, Y$ are independent.

2 Entropy and Mutual Information

1.

$$\begin{aligned} Y &= \begin{cases} 1, & \text{the keys are in the pocket} \\ 0, & otherwise \end{cases} \\ X &= \begin{cases} i \in [1,100], & \text{the keys are in the i-th location} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

2.

	$Y=0$	$Y=1$	
$X=0$	$p(X=0,Y=0) = 0$ $p(X=0\|Y=0) = 0$ $p(Y=0\|X=0) = 0$	$p(X=0,Y=1) = 0.99$ $p(X=0\|Y=1) = 1$ $p(Y=1\|X=0) = 1$	$p(X=0) = 0.99$
$X=i>0$	$i>0$ $p(X=i,Y=1) = 0$ $p(X=i\|Y=1) = 0$ $p(Y=1\|X=i) = 0$	$i>0$ $p(X=i,Y=0) = 0.0001$ $p(X=i\|Y=0) = 0.01$ $p(Y=0\|X=i) = 1$	$p(X=i>0) = 0.0001$
	$p(Y=0) = 0.01$	$p(Y=1) = 0.99$	1

3.

$$\begin{aligned} H(X) &= -\sum_x p(X=x)\log_2 p(X=x) \\ &= -p(X=0)\log_2 p(X=0) - 100 \cdot (X=i)\log_2 p(X=i) \\ &= -0.99 \cdot \log_2 0.99 - 100 \cdot 0.0001 \log_2 0.0001 \\ &= 0.044 \end{aligned}$$

$$\begin{aligned} H(Y) &= -\sum_y p(Y=y)\log_2 p(Y=y) \\ &= -p(Y=0)\log_2 p(Y=0) - \cdot (Y=1)\log_2 p(Y=1) \\ &= -0.01 \cdot \log_2 0.01 - 0.99 \log_2 0.99 \\ &= 0.024 \end{aligned}$$

$$
\begin{aligned}
H(X|Y=0) &= -\sum_x p(X|Y=0)\log_2 p(X|Y=0) \\
&= -p(X=0|Y=0)\log_2 p(X=0|Y=0) - 100\cdot(X=i|Y=0)\log_2 p(X=i|Y=0) \\
&= -100\cdot 0.01\log_2 0.01 \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
H(X|Y=1) &= -\sum_x p(X|Y=1)\log_2 p(X|Y=1) \\
&= -p(X=0|Y=1)\log_2 p(X=0|Y=1) - 100\cdot(X=i|Y=1)\log_2 p(X=i|Y=1) \\
&= -1\cdot 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
H(X|Y) &= -\sum_{x,y} p(X=x,Y=y)\log_2 p(X=x|Y=y) \\
&= -p(X=0,Y=0)\log_2 p(X=0|Y=0) - \sum_{i=1}^{100} p(X=i,Y=0)\log_2 p(X=i|Y=0) \\
&\quad - p(X=0,Y=1)\log_2 p(X=0|Y=1) - \sum_{i=1}^{100} p(X=i,Y=1)\log_2 p(X=i|Y=1) \\
&= 0 - 100\cdot 0.0001\log_2 0.01 - 0.99\log_2 1 - 100\cdot 0 \\
&= 0.02
\end{aligned}
$$

$$
\begin{aligned}
H(Y|X) &= -\sum_{x,y} p(X=x,Y=y)\log_2 p(Y=y|X=x) \\
&= -p(X=0,Y=0)\log_2 p(Y=0|X=0) - \sum_{i=1}^{100} p(X=i,Y=0)\log_2 p(Y=0|X=i) \\
&\quad - p(X=0,Y=1)\log_2 p(Y=1|X=0) - \sum_{i=1}^{100} p(Y=1,X=i)\log_2 p(Y=1|X=i) \\
&= 0-0-0-0 \\
&= 0
\end{aligned}
$$

4.

$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= 0.044 - 0.02 = 0.024
\end{aligned}
$$

$$
\begin{aligned}
I(Y;X) &= H(Y) - H(Y|X) \\
&= 0.024 - 0 = 0.024
\end{aligned}
$$

5. When we learn that the keys are not in the pocket, the entropy raises: $H(X|Y=0) > H(X)$. It seems strange because by "learning" something about the system, we reduced our information, but it is only guaranteed that **on average** the entropy be reduced when learning. The amount of entropy $H(X)$ reduced on average when learning the value of random variable Y is $I(X;Y)$, and this value is always non-negative.

3 Poincare Sphere

1. - **Direction: $\langle\phi|\phi'\rangle = 0 \Rightarrow |\phi\rangle$ and $|\phi'\rangle$ are on opposite sides of poincaré sphere**

   If $\langle\phi|\phi'\rangle = 0$ then:

$$\begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{pmatrix} \cdot \begin{pmatrix}\cos\frac{\theta'}{2} \\ e^{i\phi'}\sin\frac{\theta'}{2}\end{pmatrix} = 0$$

   Unpacking this equation, we get:

$$(\star) = \cos\frac{\theta}{2}\cos\frac{\theta'}{2} + e^{i(\phi'-\phi)}\sin\frac{\theta}{2}\sin\frac{\theta'}{2} = 0$$

   In particular, $Im(\star) = 0$, thus $\phi \equiv \phi' (\mod \pi)$. We are left with:

$$\cos\frac{\theta}{2}\cos\frac{\theta'}{2} \pm \sin\frac{\theta}{2}\sin\frac{\theta'}{2} = 0 \Rightarrow \sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)cos\frac{\theta'}{2} = \pm\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\sin\frac{\theta'}{2}$$

   Which we can simplify to

$$\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = \mp\tan\frac{\theta'}{2} \Rightarrow \pi+\theta = \mp\theta'$$

   Thus we get:

$$\theta' = \begin{cases}\theta+\pi, & \phi' = \phi+\pi \\ -(\theta+\pi), & \phi' = \phi\end{cases}$$

   Obviously, the second case is impossible because then θ' is negative. So we must conclude that:

$$\theta' = \theta+\pi$$
$$\phi' = \phi+\pi$$

   - **Direction: $|\phi\rangle$ and $|\phi'\rangle$ on opposite direction $\Rightarrow \langle\phi|\phi'\rangle = 0$**
   Suppose $\phi = \pi+\phi'$ and $\theta = \pi+\theta'$. Recall $(\star)$ from the previous direction of the proof, and plug in:

$$\begin{aligned}(\star) &= \cos\frac{\theta}{2}\cos\frac{\theta+\pi}{2} + e^{i(\pi+\phi-\phi)}\sin\frac{\theta}{2}\sin\frac{\theta+\pi}{2} \\ &= \cos\frac{\theta}{2}\cos\frac{\theta+\pi}{2} - \sin\frac{\theta}{2}\sin\frac{\theta+\pi}{2} \\ &= \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0\end{aligned}$$

2. There is an infinite amount of ensembles, and exactly one ensembles that consists of 2 orthogonal states.

3. There is an infinite amount of ensembles, all of which are pairs of orthogonal states.

4 Measurements

1. - $Pr(|+\rangle) = |\langle+|\phi\rangle|^2 = \frac{1}{2}\cdot(\alpha+\beta)^2$
   - $Pr(|\theta\rangle) = |\langle\theta|\phi\rangle|^2 = \alpha^2\cos^2\theta - 2\alpha\beta\cos\theta\sin\theta + \beta^2\sin^2\theta = 1 - \alpha\beta\sin 2\theta$

2. - $Pr(|10\rangle) = \langle 10|p|10\rangle = p_2$
   - $Pr(|00\rangle) = \langle 00|p|00\rangle = p_0$

- $$|-+\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right)\otimes\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$$

  Thus,

  $$Pr(|-+\rangle) = \frac{1}{4}\begin{pmatrix}1 & 1 & -1 & -1\end{pmatrix}\begin{pmatrix}p0 & 0 & 0 & 0\\0 & p1 & 0 & 0\\0 & 0 & p2 & 0\\0 & 0 & 0 & p3\end{pmatrix}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = \frac{p0+p1+p2+p3}{4}$$

- $$|1+\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

  Thus,

  $$Pr(|1+\rangle) = \frac{1}{2}\begin{pmatrix}0 & 0 & 1 & 1\end{pmatrix}\begin{pmatrix}p0 & 0 & 0 & 0\\0 & p1 & 0 & 0\\0 & 0 & p2 & 0\\0 & 0 & 0 & p3\end{pmatrix}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \frac{p2+p3}{2}$$

5 Double slit experiment

1. $$\begin{aligned}\vec{E_1}(\vec{r_1},t)+\vec{E_2}(\vec{r_2},t) &= \frac{\vec{E_0}}{r_1}\cos\left(k\cdot r_1 - w\cdot r\right)+\frac{\vec{E_0}}{r_2}\cos\left(k\cdot r_2 - w\cdot r\right)\\ &\approx \frac{\vec{E_0}}{r}\left(\cos\left(k\cdot r_1 - w\cdot r\right)+\cos\left(k\cdot r_2 - w\cdot r\right)\right)\\ &= \boxed{2\frac{\vec{E_0}}{r}\cos\left(\frac{k\cdot r_1 + k\cdot r_2 - 2w\cdot t}{2}\right)\cos\left(\frac{k(r_1-r_2)}{2}\right)}\end{aligned}$$

2. $$\begin{aligned}|\vec{E_1}(\vec{r_1},t)+\vec{E_2}(\vec{r_2},t)|^2 &= \left|2\frac{\vec{E_0}}{r}\cos\left(\frac{k\cdot r_1 + k\cdot r_2 - 2w\cdot t}{2}\right)\cos\left(\frac{k(r_1-r_2)}{2}\right)\right|^2\\ &= \frac{4E_0^2}{r^2}\left|\cos\left(\frac{k(r_1-r_2)}{2}\right)\right|^2\left|\cos\left(\frac{k\cdot r_1 + k\cdot r_2 - 2w\cdot t}{2}\right)\right|^2\end{aligned}$$

3. $$\begin{aligned}\left\langle|\vec{E_1}(\vec{r_1},t)+\vec{E_2}(\vec{r_2},t)|^2\right\rangle &= \left\langle\frac{4E_0^2}{r^2}\left|\cos\left(\frac{k(r_1-r_2)}{2}\right)\right|^2\left|\cos\left(\frac{k\cdot r_1 + k\cdot r_2 - 2w\cdot t}{2}\right)\right|^2\right\rangle\\ &= \frac{4E_0^2}{r^2}\left|\cos\left(\frac{k(r_1-r_2)}{2}\right)\right|^2\left\langle\left|\cos\left(\frac{k\cdot r_1 + k\cdot r_2 - 2w\cdot t}{2}\right)\right|^2\right\rangle\\ &= \frac{4E_0^2}{r^2}\left|\cos\left(\frac{k(r_1-r_2)}{2}\right)\right|^2\frac{1}{2}\end{aligned}$$

4. $$\boxed{I(\theta) = \frac{4E_0^2}{r^2}\left|\cos\left(\frac{k\cos\theta}{2}\right)\right|^2\frac{1}{2}}$$