Introduction to Quantum Information Processing

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1.1

$tr(A^\dagger) = \sum_{i=1}^{n} a_{ii}^* = (\sum_{i=1}^{n} a_{ii})^* = (tr(A))^*$

1.2

$tr(\alpha A + \beta B) = \sum_{i=1}^{n} \alpha A_{ii} + \beta B_{ii} = \alpha \sum_{i=1}^{n} A_{ii} + \beta \sum_{i=1}^{n} B_{ii} = \alpha tr(A) + \beta tr(B)$

1.3

$tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} (\sum_{k=1}^{n} A_{ik} B_{ki}) = \sum_{i=1}^{n} (\sum_{k=1}^{n} B_{ki} A_{ik}) = \sum_{i=1}^{n} (BA)_{ii} = tr(BA)$

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Throughout this part we mark eigenvalues with λ_i and eigenvectors with v_i. In order to find the eigenvalue we use the identities $tr(A) = \sum \lambda_i$ and $det(A) = \prod \lambda_i$.

2.1

$$\begin{array}{rcrcr} tr(H) & = & \lambda_1 + \lambda_2 & = & 0 \\ det(H) & = & \lambda_1 \lambda_2 & = & -1 \end{array}$$

Solving for $\lambda_{1,2}$ yields:

$$\begin{array}{rcl} \lambda_1 & = & \pm 1 \\ \lambda_2 & = & \mp 1 \end{array}$$

Because of the symmetry, we can just assume $\lambda_1 = 1$ and $\lambda_2 = -1$. To find the eigenvectors we solve for v_i:

$$(H - I\lambda_i)v_i = 0$$

TODO

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3.1

3.1.1

σ_x is a symmetric and real matrix, and as such it is also Hermitian.

3.1.2

$$\sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i^* \\ -i^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$$

because $\sigma_y^\dagger = \sigma_y$, σ_y is Hermitian.

3.1.3

σ_z is a symmetric and real matrix, and as such it is also Hermitian.

3.1.4

$H^\dagger = \frac{1}{\sqrt{2}}^\star \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\dagger \overset{\text{real numbers, symmetric matrix}}{=} \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$

3.2

3.2.1

$$
\begin{aligned}
&(\alpha A + \beta B)^\dagger \\
&\qquad = (\alpha A)^\dagger + (\beta B)^\dagger && \text{(Linearity of conjugation)} \\
&\qquad = \alpha^* A^\dagger + \beta^* B^\dagger \\
&\qquad = \alpha A^\dagger + \beta B^\dagger && (\alpha, \beta \in \mathbb{R}) \\
&\qquad = \alpha A + \beta B && (A, B \text{ are Hermitian})
\end{aligned}
$$

3.2.2

$$
\begin{aligned}
(\star) &= (\alpha - \beta) \langle v|u\rangle \\
&= \alpha\langle v|u\rangle - \beta\langle v|u\rangle \\
&= \alpha\langle u|v\rangle - \beta\langle v|u\rangle && (\langle v|u\rangle = \langle u|v\rangle) \\
&= \langle u|A|v\rangle - \beta\langle v|u\rangle && (A|v\rangle = \alpha|v\rangle) \\
&= \langle u|A|v\rangle - \langle v|A|u\rangle && (A|u\rangle = \beta|u\rangle) \\
&= 0
\end{aligned}
$$

Because $\alpha \neq \beta$ and $(\star) = 0$, we must conclude that $\langle v|u\rangle = 0 \quad \square$.

4

4.1

For Hermitian matrix $\alpha_i = \alpha_i^\dagger$, $\alpha_i \alpha_i^\dagger = \alpha_i^\dagger \alpha_i = I \iff \alpha_i^2 = I$.

- $\alpha_x \cdot \alpha_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I$
- $\alpha_y \cdot \alpha_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = I$
- $\alpha_z \cdot \alpha_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = I$
- $H \cdot H = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right)^2 = I$

4.2

Denote the i-th column of U by U_i. It holds that $U^\dagger U = I$, so from the definition of matrices multiplication we get:

$$(U_i)^\dagger U_j = I_{ij}$$

By definition,

$$I_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$$

Putting it all together:

$$(U_i)^\dagger U_j = \langle U_i, U_j \rangle = \delta_{ij}$$

4.3

TODO - Orthonormality of column → unitary matrix

4.4

Let λ be an eigenvalue with eigenvector $v \neq \vec{0}$ of unitary matrix U. Then,

$$\langle v, v \rangle = \langle Uv, Uv \rangle = \langle \lambda v, \lambda v \rangle = \lambda^\star \lambda \langle v, v \rangle \Rightarrow \lambda^\star \lambda = 1 \Rightarrow |\lambda| = 1$$

5

5.1

5.1.1

$$(\vec{a} + \vec{c}) \otimes \vec{b} \; = \begin{pmatrix} (a_1 + c_1)\vec{b} \\ (a_2 + c_2)\vec{b} \\ \vdots \\ (a_k + c_k)\vec{b} \end{pmatrix} = \begin{pmatrix} a_1\vec{b} \\ a_2\vec{b} \\ \vdots \\ a_k\vec{b} \end{pmatrix} + \begin{pmatrix} c_1\vec{b} \\ c_2\vec{b} \\ \vdots \\ c_k\vec{b} \end{pmatrix} = \vec{a} \otimes \vec{b} + \vec{c} \otimes \vec{b}$$

5.1.2

$$\vec{a} \otimes (\vec{b} + \vec{c}) \; = \begin{pmatrix} a_1(\vec{b} + \vec{c}) \\ a_2(\vec{b} + \vec{c}) \\ \vdots \\ a_k(\vec{b} + \vec{c}) \end{pmatrix} = \begin{pmatrix} a_1\vec{b} \\ a_2\vec{b} \\ \vdots \\ a_k\vec{b} \end{pmatrix} + \begin{pmatrix} a_1\vec{c} \\ a_2\vec{c} \\ \vdots \\ a_k\vec{c} \end{pmatrix} = \vec{a} \otimes \vec{b} + \vec{a} \otimes \vec{c}$$

5.1.3

$$\vec{a} \otimes \left(c \cdot \vec{b}\right) \; = \begin{pmatrix} a_1 \cdot c\vec{b} \\ a_2 \cdot c\vec{b} \\ \vdots \\ a_k \cdot c\vec{b} \end{pmatrix} = \underbrace{\begin{pmatrix} c \cdot a_1\vec{b} \\ c \cdot a_2\vec{b} \\ \vdots \\ c \cdot a_k\vec{b} \end{pmatrix}}_{(c \cdot \vec{a}) \otimes \vec{b}} = c \cdot \begin{pmatrix} a_1\vec{b} \\ a_2\vec{b} \\ \vdots \\ a_k\vec{b} \end{pmatrix} = c \cdot \left(\vec{a} \otimes \vec{b}\right)$$

5.1.4

Let

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{C}^2$$

We can compute the following tensor products:

$$\vec{a} \otimes \vec{b} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

$$\vec{c} \otimes \vec{d} = \begin{pmatrix} c_1 d_1 \\ c_1 d_2 \\ c_2 d_1 \\ c_2 d_2 \end{pmatrix}$$

Plugging them into the inner product yields:

$$\left\langle \vec{a}\otimes\vec{b}, \vec{c}\otimes\vec{d}\right\rangle = \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{pmatrix}^\dagger \cdot \begin{pmatrix} c_1d_1 \\ c_1d_2 \\ c_2d_1 \\ c_2d_2 \end{pmatrix} = a_1^\star b_1^\star c_1 d_1 + a_1^\star b_2^\star c_1 d_2 + a_2^\star b_1^\star c_2 d_1 + a_2^\star b_2^\star c_2 d_2$$
$$= (a_1^\star c_1)(b_1^\star d_1) + (a_1^\star c_1)(b_2^\star d_2) + (a_2^\star c_2)(b_1^\star d_1) + (a_2^\star c_2)(b_2^\star d_2)$$
$$= (a_1^\star c_1 + a_2^\star c_2)(b_1^\star d_1 + b_2^\star d_2)$$
$$= \langle \vec{a}, \vec{c}\rangle \left\langle \vec{b}, \vec{d}\right\rangle \quad \square$$

5.2

$$\left(\vec{a}\otimes\vec{b}\right)\otimes\vec{c} = \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{pmatrix} \otimes \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1b_1c_1 \\ a_1b_1c_2 \\ a_1b_2c_1 \\ a_1b_2c_2 \\ a_2b_1c_1 \\ a_2b_1c_2 \\ a_2b_2c_1 \\ a_2b_2c_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1c_1 \\ b_1c_2 \\ b_2c_1 \\ b_2c_2 \end{pmatrix} = \vec{a}\otimes\left(\vec{b}\otimes\vec{c}\right)$$

5.3

We prove only for matrices of size 2×2. Let:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Now we can "explicitly" compute:

$$tr\,(A\otimes B) = tr \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$
$$= a_{11}b_{11} + a_{11}b_{22} + a_{22}b_{11} + a_{22}b_{22}$$
$$= (a_{11}+a_{22})\,(b_{11}+b_{22})$$
$$= tr\,(A)\,tr\,(B)$$

5.4

As we've seen in previous exercises, for 2 matrices of size 2×2 we get

$$A\otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

Denote the i-th column in this product by D_1. Without loss of generality, we show that $\langle D_1, D_2\rangle = 1$ and that $\langle D_1, D_1\rangle = 0$. The proof for the other cases is very similar.

$$\begin{aligned}\langle D_1, D_2\rangle &= a_{11}^2 b_{11}b_{12} + a_{11}^2 b_{21}b_{22} + a_{21}^2 b_{11}b_{12} + a_{21}^2 b_{21}b_{22} \\ &= a_{11}^2\left(b_{11}b_{12} + b_{21}b_{22}\right) + a_{21}^2\left(b_{11}b_{12} + b_{21}b_{22}\right) \\ &= \left(a_{11}^2 + a_{21}^2\right)\underbrace{\langle B_1, B_2\rangle}_{=0} = 0\end{aligned}$$

$$\begin{aligned}\langle D_1, D_1\rangle &= a_{11}^2 b_{11}^2 + a_{11}^2 b_{21}^2 + a_{21}^2 b_{11}^2 + a_{21}^2 b_{21}^2 \\ &= \left(a_{11}^2 + a_{21}^2\right)\left(b_{11}^2 + b_{21}^2\right) \\ &= \langle A_1, A_1\rangle\langle B_1, B_1\rangle = 1\end{aligned}$$

5.5

Recall our previous notation:

$$A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22}\end{pmatrix}$$

Also,

$$\vec{u} \otimes \vec{v} = \begin{pmatrix} u_1v_1 \\ u_1v_2 \\ u_2v_1 \\ u_2v_2\end{pmatrix}$$

Thus we get:

$$\begin{aligned}\left(A \otimes B\right)\left(\vec{u} \otimes \vec{v}\right) &= \begin{pmatrix} a_{11}b_{11}u_1v_1 + a_{11}b_{12}u_1v_2 + a_{12}b_{11}u_2v_1 + a_{12}b_{12}u_2v_2 \\ a_{11}b_{21}u_1v_1 + a_{11}b_{22}u_1v_2 + a_{12}b_{21}u_2v_1 + a_{12}b_{22}u_2v_2 \\ a_{21}b_{11}u_1v_1 + a_{21}b_{12}u_1v_2 + a_{22}b_{11}u_2v_1 + a_{22}b_{12}u_2v_2 \\ a_{21}b_{21}u_1v_1 + a_{21}b_{22}u_1v_2 + a_{22}b_{21}u_2v_1 + a_{22}b_{22}u_2v_2\end{pmatrix} \\ &= \begin{pmatrix} (a_{11}u_1 + a_{12}u_2)(b_{11}v_1 + b_{12}v_2) \\ (a_{11}u_1 + a_{12}u_2)(b_{21}v_1 + b_{22}v_2) \\ (a_{21}u_1 + a_{22}u_2)(b_{11}v_1 + b_{12}v_2) \\ (a_{21}u_1 + a_{22}u_2)(b_{21}v_1 + b_{22}v_2)\end{pmatrix} \\ &= \begin{pmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2\end{pmatrix} \otimes \begin{pmatrix} b_{11}v_1 + b_{12}v_2 \\ b_{21}v_1 + b_{22}v_2\end{pmatrix} \\ &= A\vec{u} \otimes B\vec{v}\end{aligned}$$