Introduction to Quantum Information Processing

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1 Mutual Information - Solution

1. We expand the expression to get:

$$I(X;Y) = H(X) - H(X|Y) = -\sum_x p(x)\log_2 p(x) + \sum_{x,y} p(x,y)\log_2 p(x|y)$$

We know that a probability distribution of variable X can be computed from the mutual probability of X and Y by summing all the values in range for Y:

$$p(x) = \sum_y p(x,y)$$

Plugging this into the previous equation, we get:

$$I(X;Y) = \sum_{x,y} p(x,y)\log_2 p(x|y) - \sum_{x,y} p(x,y)\log_2 p(x) = \sum_{x,y} p(x,y)\log_2\left(\frac{p(x|y)}{p(x)}\right)$$

From logarithm rules and conditional probability definition:

$$\log_2\left(\frac{p(x|y)}{p(x)}\right) = -\log_2\left(\frac{p(x)p(y)}{p(x,y)}\right)$$

Plugging it, we get:

$$\boxed{I(X;Y) = -\sum_{x,y} p(x,y)\log_2\left(\frac{p(x)p(y)}{p(x,y)}\right)}$$

2. We can write $I(X;Y)$ using the logarithm-identity $\log_a x \cdot \log_b a = \log_b x$ as:

$$K \cdot I(X;Y) = -\sum_{x,y} p(x,y)\ln\left(\frac{p(x)p(y)}{p(x,y)}\right)$$

Where $K = \ln 2 > 0$. By using the identity $\ln t \leq t-1$ for $t > 0$ and knowing that $\frac{p(x)p(y)}{p(x,y)}$ is non-negative (because probability cannot be negative, obv.) we show:

$$-K \cdot I(X;Y) \leq \sum_{x,y} p(x,y)\left(\frac{p(x)p(y)}{p(x,y)} - 1\right) = \sum_{x,y}(p(x)p(y) - p(x,y))$$

$\ln t \leq t-1$ is equality **iff** $t = 1$.

Corollary 1.0.1. $-K \cdot I(X;Y) = \sum_{x,y}(p(x)p(y) - p(x,y))$ iff X, Y *are independent.* ***Proof:*** *If* X, Y *are independent* $\iff$ $p(x)p(y) = p(x,y)$ *for every pair of* x, y $\iff$ $\forall x, y : \frac{p(x)p(y)}{p(x,y)} = 1$ $\iff$

We expand the expression to get sums over probabilities (which we can reduce to 1):

$$-K \cdot I(X;Y) \leq \sum_x \left(p(x) \cdot \sum_y p(y) \right) - \sum_{x,y} p(x,y)$$

$$\Rightarrow I(X;Y) \geq K \cdot \sum_x p(x) - 1 = 0 \Rightarrow \boxed{I(X;Y) \geq 0}$$

From 1.0.1 and the above expansion we can conclude that $I(X;Y) = 0 \iff X, Y$ are independent.