Introduction to Quantum Information Processing

Roei Rosenzweig 313590937,
Roey Maor 205798440

May 28, 2017

1 Harmonic Quantum Oscillator - Solution

1.

$$a + a^{\dagger} = 2 \cdot X \cdot \sqrt{\frac{m\omega}{2\hbar}} \Rightarrow X = \sqrt{\frac{2\hbar}{m\omega}} \frac{a + a^{\dagger}}{2}$$

$$a - a^{\dagger} = 2 \cdot \frac{i}{m\omega} P \cdot \sqrt{\frac{m\omega}{2\hbar}} \Rightarrow P = \sqrt{\frac{2\hbar}{m\omega}} \frac{m\omega}{i} \frac{a - a^{\dagger}}{2}$$

2. We begin by calculating an expression for the operator $aa^{\dagger}$, by using $[X, P] = i\hbar$:

$$\begin{aligned} aa^{\dagger} &= \frac{m\omega}{2\hbar} \left(X^2 + \frac{i}{m\omega} PX - \frac{i}{m\omega} XP + \frac{1}{m^2\omega^2} P^2 \right) \\ &= \frac{m\omega}{2\hbar} \left(X^2 - i\hbar \frac{i}{m\omega} + \frac{1}{m^2\omega^2} P^2 \right) \\ &= \frac{m\omega}{2\hbar} \left(X^2 + \frac{1}{m^2\omega^2} P^2 \right) + \frac{m\omega}{2\hbar} \frac{h}{m\omega} = C + \frac{1}{2} \end{aligned}$$

From symmetry, we get that $a^{\dagger}a = C - \frac{1}{2} = N$, thus $aa^{\dagger} = N + 1$. Now we can calculate the hamiltonian:

$$\begin{aligned} \mathcal{H} &= \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 = \frac{1}{2m} \left(\sqrt{\frac{2\hbar}{m\omega}} \frac{m\omega}{i} \frac{a - a^{\dagger}}{2} \right)^2 + \frac{1}{2} m\omega^2 \left(\sqrt{\frac{2\hbar}{m\omega}} \frac{a + a^{\dagger}}{2} \right)^2 \\ &= \frac{1}{2m} \frac{2\hbar}{m\omega} \frac{m^2\omega^2}{-1} \frac{a^2 - (N+1) - N + \left(a^{\dagger}\right)^2}{4} + \frac{1}{2} m\omega^2 \frac{2\hbar}{m\omega} \frac{a^2 + (N+1) + N + \left(a^{\dagger}\right)^2}{4} \\ &= -\omega\hbar \frac{a^2 - (N+1) - N + \left(a^{\dagger}\right)^2}{4} + \omega\hbar \frac{a^2 + (N+1) + N + \left(a^{\dagger}\right)^2}{4} \\ &= \hbar\omega \frac{4N + 2}{4} = \hbar\omega \left(N + \frac{1}{2} \right) \end{aligned}$$

3. for $n > 0$:

$$N|n\rangle = a^{\dagger}a|n\rangle = \sqrt{n}a^{\dagger}|n-1\rangle = \sqrt{n}^2|n\rangle = n \cdot |n\rangle$$

for $n = 0$:

$$\begin{aligned} a^{\dagger}|0\rangle &= \sqrt{1}|1\rangle \\ a|1\rangle &= \sqrt{1}|0\rangle \\ &\Downarrow \\ aa^{\dagger}|0\rangle &= a|1\rangle = |0\rangle = (N+1)|0\rangle \Rightarrow N|0\rangle = 0 \cdot |0\rangle \end{aligned}$$

4. Proof by induction. we claim that $|n\rangle = \frac{1}{\sqrt{n!}} \cdot \left(a^\dagger\right)^n \cdot |0\rangle$ for $n \geq 0$.

   - base for $n = 0$ we get $|0\rangle = \frac{1}{\sqrt{0!}} \cdot \left(a^\dagger\right)^0 \cdot |0\rangle = |0\rangle$.
   - step assume $|n\rangle = \frac{1}{\sqrt{n!}} \cdot \left(a^\dagger\right)^n \cdot |0\rangle$ for some $n \geq 0$:

$$|n+1\rangle = \frac{1}{\sqrt{n+1}} a^\dagger |n\rangle = \frac{1}{\sqrt{n+1}} a^\dagger \cdot \frac{1}{\sqrt{n!}} \cdot \left(a^\dagger\right)^n \cdot |0\rangle = \frac{1}{\sqrt{n+1!}} \cdot \left(a^\dagger\right)^{n+1} \cdot |0\rangle$$

2 State Classification - Solution

1. First, lets compute:

$$\frac{|00\rangle + |{+}{+}\rangle + |{-}{-}\rangle}{\alpha} = \frac{1}{a}\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right) = \frac{1}{\alpha}\begin{pmatrix}2\\0\\0\\3\end{pmatrix}$$

   This state is pure (by definition - Its a vector in Hilbert space), and it is **entangled** - it can't be written as a tensor product of 2 vectors. Separable states must have the property $\alpha_{01}\alpha 10 = \alpha 00\alpha 11$, and in our case $0 \cdot 0 \neq 3 \cdot 2$. The norm of a pure state must be 1, and we can use this to calculate α:

$$\begin{aligned} \frac{9}{\alpha^2} + \frac{4}{\alpha^2} &= 1 \\ 13 &= \alpha \end{aligned}$$