Phase-Space Seams: Invariant Volume Corrections and Universal Positivity at All Even Jet Orders

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We introduce *phase-space seams*, a scalar-first framework in which a real-valued scalar field s on $\mathbb{R}^{2d}$ generates geometric data via a Hessian rule and a family of sublevel sets. Our main result is an invariant first-variation formula in $\mathbb{R}^n$: to first order, the volume of small sublevel sets $\{s \le \varepsilon\}$ depends on the quartic jet only through the unique $O(n)$-scalar double trace of the 4-jet tensor; normalisability of the model density $\exp(-s_4)$ forces positivity of the correction coefficient. For the quadratic seam $s(x,p) = x^2/(2\sigma_x^2) + p^2/(2\sigma_p^2)$ the sublevel set $\{s \le 1/2\}$ is an ellipse of Lebesgue area $\pi\sigma_x\sigma_p$. Retaining the full fourth-order Taylor jet, including the cross-term x^2p^2, yields a coordinate-invariant first-order area correction in which the cross-coupling contributes with coefficient $1/8$, distinct from the diagonal coefficient $3/16$, reflecting universal angular integrals. We extend the framework to the sextic jet and prove a general first-variation formula for arbitrary even-order $2k$-jets in $\mathbb{R}^n$. Universality holds at all even orders: for any non-trivial normalisable $2k$-jet perturbation the k-fold trace of the jet tensor is strictly positive, so non-Gaussianity always reduces sublevel-set volumes to first order; the proof is a single measure-theoretic step—spherical averaging of a non-negative function—subsuming the quartic AM–GM argument. Finally, for the quartic model density $\rho \propto \exp(-s_4)$ we compute true second moments to first order in the quartic strength λ, making explicit that the jet parameters σ_x, σ_p generally differ from the moments of ρ in non-Gaussian models.

Keywords: Hessian geometry, phase-space distributions, sublevel sets, non-Gaussianity, information geometry, uncertainty relations, jet calculus, spherical averaging

I. INTRODUCTION

Geometry is often presented as a structure postulated *a priori*. Here we adopt the complementary viewpoint: begin with a base space and a real-valued scalar field (the *seam*) and let an explicit local rule generate the geometric data. We use the term "seam" to evoke the stitching of local geometric data into a global fabric, while remaining compatible with the log-density/potential-function language of information geometry. In this note we apply this scalar-first philosophy to classical phase space $\mathbb{R}^{2d}$ equipped with the standard symplectic coordinates (x, p).

From the viewpoint of information geometry, the central construction is Hessian: when a seam is interpreted as a negative log-density, its Hessian is the observed-information matrix, and its expectation recovers the Fisher information metric under standard regularity hypotheses[7,8].

More generally, in the standard affine coordinates on $\mathbb{R}^{2d}$, the Hessian rule produces a Hessian metric in the sense of Hessian manifold theory; see, e.g.,[9].

We focus first on the simplest non-trivial seams — quadratic forms — and compute the Lebesgue volume of their sublevel sets explicitly. We then extend this to the full quartic seam and obtain a first-order area correction, correcting an incompleteness in the diagonal-only treatment: the cross-coupling term x^2p^2 must be included in any coordinate-invariant formulation, and it enters with a universal weight dictated by angular integrals. We also show that normalisability of the model density $\exp(-s_4)$ forces positivity of the first-order correction coefficient.

The construction is self-contained and uses only elementary multivariable calculus and the AM-GM inequality. We outline immediate computational applications and sketch the natural multi-mode generalisation in dimension $d > 1$.

II. BACKGROUND AND SCOPE

The formalism in this note is geometric: it starts from a scalar field s on phase space and derives objects such as a Hessian metric and level-set geometry. Connections to quantum mechanics can be made by choosing a phase-space representation (e.g. a Husimi Q-function) and identifying s with a log-density; we use this as motivation and do not assume that arbitrary seam level sets are constrained directly by the canonical commutation relation.

a. Phase-space representations. Quantum states admit phase-space descriptions such as the Wigner function, which can be negative and therefore cannot globally be written as $\exp(-s)$ with real s[1,3]. Positive phase-space distributions are provided by the Husimi Q-function (a Gaussian smoothing of the Wigner function)[2,4]. If one chooses to interpret a seam as

$$s(x,p) = -\log Q(x,p) + \text{const},$$

then $\exp(-s)$ is normalisable and the seam parameters can be related to the covariance of the Husimi distribution.

b. Parameters versus moments. The symbols σ_x, σ_p in this manuscript denote quadratic width parameters used to normalise coordinates $u = x/\sigma_x$, $v = p/\sigma_p$. For non-Gaussian seams, the true moments computed from the model density $\propto \exp(-s_4)$ satisfy $\Delta x \neq \sigma_x$ and $\Delta p \neq \sigma_p$ in general; an explicit perturbative computation is given in Section VII B.

c. Uncertainty as motivation. For Gaussian states, the Robertson–Schrödinger inequality[5,6] can be re-expressed as a minimal-area statement for the corresponding covariance ellipse, which coincides with a quadratic seam level set. For general non-Gaussian seams, there is no axiom of quantum mechanics that directly constrains the Lebesgue area of an arbitrary Taylor-level set $\{s \leq 1/2\}$. When we use a quantum-motivated area floor (Assumption VII.1) below, it should be read as a modelling hypothesis. For symplectic-geometric context, one may compare with de Gosson's *quantum blobs*: minimal phase-space ellipsoids compatible with uncertainty (up to conventions)[10].

d. Relation to information geometry. Defining a Riemannian metric from a log-density is closely related to the Fisher information metric[7,8]. Our metric is the pointwise Hessian of s; taking expectations of this Hessian under $\exp(-s)$ yields the usual Fisher information matrix for location parameters.

III. THE PHASE-SPACE SEAM

Let $\mathbb{R}^{2d}$ be equipped with coordinates $(x,p) = (x_1,\ldots,x_d,p_1,\ldots,p_d)$. For fixed positive parameters $\sigma_x = (\sigma_{x_1},\ldots,\sigma_{x_d})$ and $\sigma_p = (\sigma_{p_1},\ldots,\sigma_{p_d})$ we define the *phase-space seam* by

$$s(x,p) := \sum_{i=1}^{d} \left(\frac{x_i^2}{2\sigma_{x_i}^2} + \frac{p_i^2}{2\sigma_{p_i}^2} \right). \tag{1}$$

When all widths are equal ($\sigma_{x_i} \equiv \sigma_x$, $\sigma_{p_i} \equiv \sigma_p$) this reduces to the isotropic form

$$s(x,p) = \frac{|x|^2}{2\sigma_x^2} + \frac{|p|^2}{2\sigma_p^2}.$$

In the geometric development below, σ_{x_i} and σ_{p_i} are treated as positive scale parameters. When one chooses to identify the quadratic seam with the negative log-density of a Gaussian quantum phase-space distribution, these parameters can be matched to physical standard deviations, in which case they obey the Heisenberg uncertainty relation

$$\sigma_{x_i}\sigma_{p_i} \geq \frac{\hbar}{2} \qquad \text{for each } i = 1,\ldots,d. \tag{2}$$

IV. THE HESSIAN RULE AND INDUCED GEOMETRY

Definition IV.1 (Hessian Rule)**.** *Given a twice-differentiable seam $s : \mathbb{R}^{2d} \to \mathbb{R}$, the* Hessian rule *assigns to s the symmetric bilinear form given by its second derivatives,*

$$g_{ij} = \frac{\partial^2 s}{\partial q^i \partial q^j}, \qquad q = (x, p).$$

If g is positive definite it defines a Riemannian metric on $\mathbb{R}^{2d}$.

Remark IV.1 (Relation to Fisher information geometry)**.** *If one interprets s as a negative log-density, then the Hessian $\partial_i \partial_j s$ is the* observed information *matrix. Taking the expectation of this Hessian under the corresponding density yields the Fisher information matrix for location parameters under mild regularity conditions. This places the Hessian rule in close proximity to standard information geometry*[7,8].

Applying the Hessian rule to the phase-space seam (1) yields the constant diagonal metric

$$g = \mathrm{diag}\left(\tfrac{1}{\sigma_{x_1}^2}, \dots, \tfrac{1}{\sigma_{x_d}^2}, \tfrac{1}{\sigma_{p_1}^2}, \dots, \tfrac{1}{\sigma_{p_d}^2}\right). \tag{3}$$

Its inverse is likewise diagonal with entries $\sigma_{x_i}^2$ and $\sigma_{p_i}^2$. Because the second derivatives are constant, the Christoffel symbols vanish identically and the Riemannian curvature tensor of g is zero everywhere. The geometry is therefore flat, yet non-Euclidean: it is a simple rescaling of the standard Euclidean metric with different stretch factors along the position and momentum axes.

The level sets $\{(x, p) \mid s(x, p) = c\}$ for $c > 0$ are ellipsoids. In the isotropic case the Lebesgue area (or $2d$-volume) enclosed by the level set $s = 1/2$ is

$$A = \frac{\pi^d}{d!} \prod_{i=1}^{d} \sigma_{x_i} \sigma_{p_i}.$$

Imposing the scale constraint (2) on each pair therefore forces

$$A \geq \frac{\pi^d}{d!} \left(\frac{\hbar}{2}\right)^d, \tag{4}$$

with equality if and only if $\sigma_{x_i} \sigma_{p_i} = \hbar/2$ for all i.

V. A MINIMAL-AREA CONSTRAINT FOR QUADRATIC SEAMS

Theorem V.1 (Quadratic sublevel-set volume under a scale constraint)**.** *Let s be any phase-space seam of the quadratic form* (1)*. The Lebesgue volume of the sublevel set $\{s \leq 1/2\}$ satisfies*

$$\mathrm{Vol}_{\mathrm{Leb}}\big(\{s \leq 1/2\}\big) \geq \frac{\pi^d}{d!} \left(\frac{\hbar}{2}\right)^d,$$

with equality when equality holds in (2) *for every pair.*

Proof. The sublevel set $\{s \leq 1/2\}$ is the ellipsoid

$$\sum_{i=1}^{d} \left(\frac{x_i^2}{\sigma_{x_i}^2} + \frac{p_i^2}{\sigma_{p_i}^2} \right) \leq 1.$$

The Lebesgue volume of a $2d$-dimensional ellipsoid with semi-axes σ_{x_i} and σ_{p_i} is $\frac{\pi^d}{d!} \prod_i \sigma_{x_i} \sigma_{p_i}$. The scale constraint $\sigma_{x_i} \sigma_{p_i} \geq \hbar/2$ then gives

$$\mathrm{Vol}_{\mathrm{Leb}}\big(\{s \leq 1/2\}\big) = \frac{\pi^d}{d!} \prod_i \sigma_{x_i} \sigma_{p_i} \geq \frac{\pi^d}{d!} \left(\frac{\hbar}{2}\right)^d,$$

with saturation when equality holds in every pair. (The Riemannian volume with respect to the Hessian metric (3) is the constant $\pi^d/d!$, independent of the widths.) $\square$

Thus a minimal-area bound for quadratic seam sublevel sets follows directly from the volume formula for ellipsoids together with the scale constraint (2). In quantum-mechanical applications this constraint is motivated by Robertson–Schrödinger in the Gaussian case; for non-Gaussian seams we will state any such area requirement explicitly as an assumption.

VI. IMMEDIATE COMPUTATIONAL PAYOFFS

The phase-space seam formulation immediately yields two practical tools that are not obvious from a purely covariance-matrix description.

a. Inverse seam design via convex optimisation. Given a target covariance structure (i.e., desired pairs $(\sigma_{x_i}, \sigma_{p_i})$ or a full target ellipsoid), one can solve for the quadratic coefficients of s by minimising a strictly convex quadratic program whose variables are the diagonal entries of the Hessian. The scale constraint $\sigma_x \sigma_p \geq \hbar/2$ appears as a set of simple quadratic inequalities. The resulting seam can be used to generate families of width-constrained seams or to enforce area/width bounds in variational calculations.

b. Visual and numerical diagnostics. The seam surface $s(x, p)$ and its Hessian eigenvalues provide an immediate visual diagnostic: colour the phase-space domain by the product of the two eigenvalues of the local 2×2 blocks of $\operatorname{Hess} s$. Regions where the product approaches $\hbar/2$ are visually "saturated" relative to the chosen scale.

VII. HIGHER-ORDER AREA CORRECTIONS FROM THE FULL 4-JET

The quadratic seam retains only the 2-jet of a general smooth seam. We now show that retaining the *full* 4-jet — including the cross-coupling between position and momentum directions — leads to a first-order correction to sublevel-set area whose structure is richer than the diagonal treatment alone reveals. We restrict to dimension $d = 1$ for explicit calculations; the extension to $d > 1$ is discussed in Section XII.

A. The Full Quartic Seam and Its Cross-Coupling

The full quartic Taylor expansion of a smooth seam s in the normalised coordinates $u = x/\sigma_x$, $v = p/\sigma_p$ takes the form

$$s_4(u, v) = \frac{u^2 + v^2}{2} + \frac{\lambda}{4}\left(au^4 + bv^4 + 2c\,u^2 v^2\right), \tag{5}$$

where $a, b \geq 0$ and $c \in \mathbb{R}$ are dimensionless shape parameters and $\lambda > 0$ sets the overall scale of the quartic correction. Restoring physical units,

$$s_4(x, p) = \frac{x^2}{2\sigma_x^2} + \frac{p^2}{2\sigma_p^2} + \frac{\lambda}{4}\left(a\,\frac{x^4}{\sigma_x^4} + b\,\frac{p^4}{\sigma_p^4} + 2c\,\frac{x^2 p^2}{\sigma_x^2 \sigma_p^2}\right). \tag{6}$$

The diagonal quartic seam corresponds to the restriction $a = b = 1$, $c = 0$, with $\lambda = \lambda_{\text{diag}}$. The cross-coupling coefficient c controls the extent to which fourth-order fluctuations in x and p are correlated.

B. True Second Moments of $\exp(-s_4)$ Versus the Parameters

An important interpretational point is that for non-Gaussian seams, the parameters σ_x and σ_p used in (6) are *not* equal to the true standard deviations computed from the model density $\rho(x, p) \propto \exp(-s_4(x, p))$.

To make this explicit, work in normalised variables $u = x/\sigma_x$, $v = p/\sigma_p$ and consider the probability density on $\mathbb{R}^2$

$$\rho_\lambda(u,v) := Z_\lambda^{-1}\exp\left(-\frac{u^2+v^2}{2} - \frac{\lambda}{4}\left(au^4+bv^4+2cu^2v^2\right)\right),$$

where Z_λ is the normalising constant.

Proposition VII.1 (First-order variances of the quartic model density)**.** *Assume $a,b \geq 0$, $c \geq -\sqrt{ab}$, and λ is sufficiently small that the perturbation expansion is valid. Let expectations $\langle\cdot\rangle_\lambda$ be taken with respect to ρ_λ. Then, to first order in λ,*

$$\langle u^2\rangle_\lambda = 1-\lambda(3a+c)+O(\lambda^2), \tag{7}$$

$$\langle v^2\rangle_\lambda = 1-\lambda(3b+c)+O(\lambda^2). \tag{8}$$

Equivalently, the physical second moments of $\rho(x,p) \propto \exp(-s_4(x,p))$ are

$$\Delta x_\rho^2 = \langle x^2\rangle - \langle x\rangle^2 = \sigma_x^2\big(1-\lambda(3a+c)+O(\lambda^2)\big), \tag{9}$$

$$\Delta p_\rho^2 = \langle p^2\rangle - \langle p\rangle^2 = \sigma_p^2\big(1-\lambda(3b+c)+O(\lambda^2)\big). \tag{10}$$

Proof. Write $Q(u,v) = au^4+bv^4+2cu^2v^2$ and denote expectations under the standard Gaussian density $\propto \exp(-(u^2+v^2)/2)$ by $\langle\cdot\rangle_0$. For any test function f, the standard perturbation identity gives

$$\langle f\rangle_\lambda = \frac{\langle f\,e^{-\lambda Q/4}\rangle_0}{\langle e^{-\lambda Q/4}\rangle_0} = \langle f\rangle_0 - \frac{\lambda}{4}\big(\langle fQ\rangle_0 - \langle f\rangle_0\langle Q\rangle_0\big) + O(\lambda^2).$$

Using Gaussian moments $\langle u^2\rangle_0 = \langle v^2\rangle_0 = 1$, $\langle u^4\rangle_0 = \langle v^4\rangle_0 = 3$, $\langle u^6\rangle_0 = \langle v^6\rangle_0 = 15$, and independence $\langle u^{2m}v^{2n}\rangle_0 = \langle u^{2m}\rangle_0\langle v^{2n}\rangle_0$, we obtain $\langle Q\rangle_0 = 3a+3b+2c$, $\langle u^2Q\rangle_0 = 15a+3b+6c$, and $\langle v^2Q\rangle_0 = 3a+15b+6c$. Substitution yields (7)–(8). Symmetry implies $\langle u\rangle_\lambda = \langle v\rangle_\lambda = 0$, so (9)–(10) follow by rescaling back to x,p. □

Remark VII.1 (Implication for interpreting $\sigma_x\sigma_p$)**.** *For $\lambda > 0$ and typical non-Gaussian parameters (e.g. $a,b \geq 0$ and modest c), Proposition VII.1 shows $\Delta x_\rho < \sigma_x$ and $\Delta p_\rho < \sigma_p$ at first order. Therefore, any inequality of the form $\sigma_x\sigma_p \geq \hbar/2\,(1+\cdots)$ should be read as a constraint on the* jet parameters (σ_x,σ_p)*, not automatically as a sharpened statement about either (i) the operator uncertainties $\Delta x\Delta p$ of a Hilbert-space state, or (ii) the second moments of the model density* $\exp(-s_4)$.

Remark VII.2 (Normalisability constraint)**.** *For the density* $\exp(-s_4)$ *to be normalisable, the quartic form $Q(u,v) = au^4+bv^4+2cu^2v^2$ must be non-negative for all $(u,v) \in \mathbb{R}^2$. Evaluating along the ray $u = v$ gives $Q = (a+b+2c)t^4$, so we need $a+b+2c \geq 0$. Evaluating along real rays yields the sharp condition. For $v \neq 0$, write $t := (u/v)^2 \geq 0$, so $Q(u,v) = v^4\,(at^2+2ct+b)$. If $c \geq 0$ this is clearly nonnegative for all $t \geq 0$. If $c < 0$ and $a > 0$, the minimum over $t \geq 0$ occurs at $t = -c/a$, so nonnegativity forces $b - c^2/a \geq 0$, i.e. $c^2 \leq ab$. (The degenerate cases $a = 0$ or $b = 0$ reduce similarly.) Hence the necessary and sufficient condition is*

$$c \geq -\sqrt{ab}. \tag{11}$$

This is the normalisability constraint*. The diagonal case $c = 0$ satisfies it whenever $a,b \geq 0$.*

C. The 4-Jet Rule and the Correct Invariant Scalar

Definition VII.1 (4-Jet Rule)**.** *Given a four-times-differentiable seam $s : \mathbb{R}^{2d} \to \mathbb{R}$, the* 4-jet rule *assigns to s the totally symmetric covariant $(0,4)$-tensor*

$$T_{ijkl} = \frac{\partial^4 s}{\partial q^i\partial q^j\partial q^k\partial q^l}, \qquad q = (x,p).$$

For the full quartic seam (6) the non-zero components of the fourth-derivative tensor are

$$T_{xxxx} = \frac{\partial^4 s_4}{\partial x^4} = \frac{6\lambda\, a}{\sigma_x^4}, \tag{12}$$

$$T_{pppp} = \frac{\partial^4 s_4}{\partial p^4} = \frac{6\lambda\, b}{\sigma_p^4}, \tag{13}$$

$$T_{xxpp} = \frac{\partial^4 s_4}{\partial x^2\,\partial p^2} = \frac{2\lambda\, c}{\sigma_x^2\,\sigma_p^2}. \tag{14}$$

Note that T_{xxpp} counts with the combinatorial multiplicity $\binom{4}{2} = 6$ under full index symmetrisation (the six distinct placements of two x-indices and two p-indices), so its contribution to a full contraction is weighted by 6.

Definition VII.2 (Corrected 4-Jet Scalar)**.** *The* 4-jet scalar *of the full quartic seam s_4 is the dimensionless quantity formed by full contraction of T_{ijkl} with the inverse Hessian metric $(g^{-1})^{\otimes 2}$:*

$$K_4[s_4] := \frac{1}{6}\Big((g^{xx})^2 T_{xxxx} + (g^{pp})^2 T_{pppp} + 6\,(g^{xx})(g^{pp})\,T_{xxpp}\Big), \tag{15}$$

where $g^{xx} = \sigma_x^2$ and $g^{pp} = \sigma_p^2$ are the diagonal entries of the inverse Hessian metric, and the factor 6 *multiplying the cross-term accounts for the combinatorial weight of two-x two-p index arrangements.*

Remark VII.3 (Incompleteness of the diagonal definition)**.** *The definition used in earlier diagonal treatments sets $K_4^{diag} = (1/6)\big((g^{xx})^2 T_{xxxx} + (g^{pp})^2 T_{pppp}\big)$, omitting the cross-contraction entirely. This is coordinate-invariant only within the restricted class of seams with $c = 0$. For a general smooth seam, the full tensor T_{ijkl} has a non-zero T_{xxpp} component, and the correct invariant is* (15)*. Setting $c = 0$ in* (15) *recovers the diagonal definition as a special case.*

Substituting (12)–(14) into (15):

$$K_4[s_4] = \frac{1}{6}\left(\sigma_x^4 \cdot \frac{6\lambda a}{\sigma_x^4} + \sigma_p^4 \cdot \frac{6\lambda b}{\sigma_p^4} + 6\,\sigma_x^2\sigma_p^2 \cdot \frac{2\lambda c}{\sigma_x^2\sigma_p^2}\right) = \lambda(a+b+2c). \tag{16}$$

The diagonal scalar satisfies $K_4^{\mathrm{diag}} = \lambda(a+b)$, so the full scalar exceeds it by $2\lambda c$, which can be positive or negative depending on the sign of c.

D. Area Reduction: The Full Quartic Case

Proposition VII.2 (Quartic Confinement, Full Version)**.** *For any quartic seam s_4 with $a, b \geq 0$ and $\lambda > 0$, the sublevel set $\{s_4 \leq 1/2\}$ is strictly contained in the sublevel set $\{s_2 \leq 1/2\}$ of the corresponding quadratic seam. Consequently,*

$$A(s_4) := \mathrm{Area}\big(\{s_4 \leq \tfrac{1}{2}\}\big) < \pi\sigma_x\sigma_p.$$

Proof. Since $a, b \geq 0$ and the normalisability constraint (11) gives $c \geq -\sqrt{ab}$, the quartic form $Q(u,v) = au^4 + bv^4 + 2cu^2v^2 \geq 0$ for all (u,v). Therefore $s_4(u,v) \geq s_2(u,v)$ for all (u,v), and the containment $\{s_4 \leq 1/2\} \subseteq \{s_2 \leq 1/2\}$ follows. Strict containment holds for all $(u,v) \neq (0,0)$ where $Q > 0$, which is a set of positive area. □

E. First-Order Area Computation with Cross-Coupling

In normalised coordinates $u = x/\sigma_x$, $v = p/\sigma_p$, write $z = (u,v) \in \mathbb{R}^2$ and denote by $\widetilde{T}_{ijkl} := \partial_i\partial_j\partial_k\partial_l s_4(0)$ the quartic jet tensor in these coordinates. Since the quadratic jet is $\frac{1}{2}|z|^2$, we may write the quartic seam in jet form as

$$s_4(z) = \frac{1}{2}|z|^2 + \frac{1}{24}\widetilde{T}_{ijkl}\, z^i z^j z^k z^l \qquad (\widetilde{T} = O(\lambda)).$$

For a unit direction $n \in S^1$, let $r(n)$ be the radial function of the level set $\{s_4 = 1/2\}$, i.e. $s_4(r(n)\,n) = 1/2$. Then

$$\frac{1}{2}r(n)^2 + \frac{1}{24}r(n)^4\,\widetilde{T}(n,n,n,n) = \frac{1}{2}, \qquad \widetilde{T}(n,n,n,n) = \widetilde{T}_{ijkl}n^i n^j n^k n^l.$$

Solving perturbatively gives

$$r(n)^2 = 1 - \frac{1}{12}\,\widetilde{T}(n,n,n,n) + O(\lambda^2). \tag{17}$$

The (normalised) area is therefore

$$A_{uv} = \frac{1}{2}\int_{S^1} r(n)^2\, d\Omega = \pi - \frac{1}{24}\int_{S^1} \widetilde{T}(n,n,n,n)\, d\Omega + O(\lambda^2).$$

By $O(2)$-invariance of the sphere, the fourth moment has the standard form

$$\int_{S^1} n^i n^j n^k n^l\, d\Omega = \frac{\pi}{4}\big(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\big), \tag{18}$$

so, using full symmetry of $\widetilde{T}$,

$$\int_{S^1} \widetilde{T}(n,n,n,n)\, d\Omega = \frac{3\pi}{4}\,\widetilde{T}_{iijj}. \tag{19}$$

Here $\widetilde{T}_{iijj}$ is the double trace (with respect to the quadratic metric δ). For (5), direct differentiation gives

$$\widetilde{T}_{iijj} = \widetilde{T}_{uuuu} + 2\widetilde{T}_{uuvv} + \widetilde{T}_{vvvv} = 6\lambda(a+b) + 4\lambda c = 2\lambda(3a+3b+2c). \tag{20}$$

Substituting (19)–(20) yields $A_{uv} = \pi\big(1 - \lambda(3a+3b+2c)/16\big) + O(\lambda^2)$, recovering (27) below.
Equivalently, in polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, the boundary $\{s_4 = 1/2\}$ satisfies

$$\frac{r^2}{2} + \frac{\lambda}{4}\, r^4 F(\theta) = \frac{1}{2}, \tag{21}$$

where

$$F(\theta) := a\cos^4\theta + b\sin^4\theta + 2c\cos^2\theta\sin^2\theta. \tag{22}$$

Solving (21) perturbatively by writing $r^2 = 1 - \varepsilon$ and expanding to first order in λ gives

$$r^2(\theta) = 1 - \frac{\lambda}{2}\,F(\theta) + O(\lambda^2). \tag{23}$$

The three angular integrals required are standard:

$$\int_0^{2\pi} \cos^4\theta\, d\theta = \int_0^{2\pi} \sin^4\theta\, d\theta = \frac{3\pi}{4}, \tag{24}$$

$$\int_0^{2\pi} \cos^2\theta\sin^2\theta\, d\theta = \frac{\pi}{4}. \tag{25}$$

Note that the cross-coupling integral (25) equals $\pi/4$, while each diagonal integral (24) equals $3\pi/4$: the cross-angular mode $\cos^2\theta\sin^2\theta$ integrates to exactly one-third of the diagonal modes.

This ratio is a geometric invariant that will determine the relative weighting of c versus a,b in the conditional width bound.

Therefore,

$$\int_0^{2\pi} F(\theta)\,d\theta = \frac{3\pi a}{4} + \frac{3\pi b}{4} + 2c\cdot\frac{\pi}{4} = \frac{\pi}{4}\big(3a+3b+2c\big). \tag{26}$$

The area in normalised coordinates is

$$\begin{aligned} A_{uv} &= \frac{1}{2}\int_0^{2\pi} r^2(\theta)\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(1-\frac{\lambda}{2}F(\theta)\right)d\theta + O(\lambda^2) \\ &= \pi - \frac{\lambda}{4}\cdot\frac{\pi}{4}(3a+3b+2c) + O(\lambda^2) \\ &= \pi\left(1-\frac{\lambda(3a+3b+2c)}{16}\right) + O(\lambda^2). \end{aligned} \tag{27}$$

Restoring physical units:

$$\boxed{A(s_4) = \pi\sigma_x\sigma_p\left(1-\frac{\lambda(3a+3b+2c)}{16}\right) + O(\lambda^2).} \tag{28}$$

Remark VII.4 (The coefficient $1/8$ versus $3/16$)**.** *The coefficient of the cross-coupling c in the area correction is $\lambda/8$, while the coefficient of each diagonal term a or b is $3\lambda/16$. Their ratio is $2/3$, reflecting the $2c$ prefactor in Q; the underlying angular integral ratio is $1/3$, equal to $\int\cos^2\theta\sin^2\theta\,d\theta\,/\int\cos^4\theta\,d\theta = (\pi/4)/(3\pi/4)$. Invariantly, this ratio is a consequence of the $O(2)$-invariant fourth-moment tensor* (18)*: only the double-trace component $\widetilde{T}_{iijj}$ of the quartic jet contributes to the first-order area variation, and the diagonal/cross weights follow from contracting $\widetilde{T}$ against $\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}$.*

This ratio of $1/3$ between cross and diagonal angular integrals is a universal feature independent of σ_x and σ_p: it encodes the angular geometry of the u-v plane and would appear unchanged in any rotation-invariant rescaling of the coordinates.

F. Invariant First Variation in $n = 2d$ Dimensions

The preceding derivation in $n = 2$ dimensions is a special case of a general coordinate-free statement: to first order, only the $O(n)$*-scalar* (double-trace) component of the quartic jet can affect the sublevel-set volume.

Theorem VII.1 (Invariant first variation of sublevel-set volume)**.** *Let $s:\mathbb{R}^n\to\mathbb{R}$ be C^4 in a neighbourhood of the origin, with $s(0)=0$, $\nabla s(0)=0$, and positive definite quadratic jet $g_{ij}=\partial_i\partial_j s(0)$. Let $T_{ijkl}=\partial_i\partial_j\partial_k\partial_l s(0)$ be the fully symmetric quartic jet tensor. For $\varepsilon>0$ sufficiently small, define the sublevel-set volume*

$$V(\varepsilon) := \mathrm{Vol}_{\mathrm{Leb}}\big(\{z\in\mathbb{R}^n : s(z)\le\varepsilon\}\big).$$

Then

$$V(\varepsilon) = \frac{\omega_n}{n}\frac{(2\varepsilon)^{n/2}}{\sqrt{\det g}}\left(1-\frac{\varepsilon}{4(n+2)}\mathrm{Tr}_g\mathrm{Tr}_g(T) + O(\varepsilon^2) + O(\|T\|^2)\right), \tag{29}$$

where ω_n is the surface area of S^{n-1} and $\mathrm{Tr}_g\mathrm{Tr}_g(T) := g^{ij}g^{kl}T_{ijkl}$ is the double trace with respect to g.

Proof. By a linear change of variables $y = Lz$ with $L^T gL = I$, we may assume $g = I$ at the cost of a Jacobian factor $1/\sqrt{\det g}$ in Lebesgue volume; the tensor T transforms covariantly and the contraction $\mathrm{Tr}_g\mathrm{Tr}_g(T)$ is invariant.

For sufficiently small $\varepsilon > 0$, the sublevel set $\{s \le \varepsilon\}$ is star-shaped about the origin, so its boundary can be written as $\{r(n)\,n : n \in S^{n-1}\}$ for a smooth radial function $r(n) > 0$. With $g = I$, write $z = rn$ with $n \in S^{n-1}$. The jet expansion gives $s(rn) = \frac{1}{2}r^2 + \frac{1}{24}r^4 T(n,n,n,n) + O(r^6)$. Solving $s(rn) = \varepsilon$ perturbatively yields $r(n)^2 = 2\varepsilon - \frac{1}{3}\varepsilon^2 T(n,n,n,n) + O(\varepsilon^3) + O(\|T\|^2)$. Hence $r(n)^n = (2\varepsilon)^{n/2}\left(1 - \frac{n\varepsilon}{12}T(n,n,n,n)\right) + \cdots$. Using $V(\varepsilon) = \frac{1}{n}\int_{S^{n-1}} r(n)^n\, d\Omega$ gives

$$V(\varepsilon) = \frac{\omega_n}{n}(2\varepsilon)^{n/2} - \frac{(2\varepsilon)^{n/2}}{12}\,\varepsilon \int_{S^{n-1}} T(n,n,n,n)\, d\Omega + \cdots.$$

By $O(n)$-invariance, the fourth moment of the uniform measure on S^{n-1} has the universal form

$$\int_{S^{n-1}} n^i n^j n^k n^l \, d\Omega = \frac{\omega_n}{n(n+2)}\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right). \tag{30}$$

Contracting against symmetric T_{ijkl} yields $\int_{S^{n-1}} T(n,n,n,n)\, d\Omega = \frac{3\omega_n}{n(n+2)} T_{iijj}$, and substituting back gives (29) (with $T_{iijj} = \mathrm{Tr}\,\mathrm{Tr}(T)$ in the $g = I$ frame). □

Remark VII.5 (Only the $O(n)$-scalar part contributes at first order)**.** *Equation* (30) *shows that the spherical averaging operator $T \mapsto \int_{S^{n-1}} T(n,n,n,n)\, d\Omega$ projects the quartic jet onto its unique $O(n)$-invariant scalar component. Equivalently, all trace-free irreducible components of $T \in \mathrm{Sym}^4(\mathbb{R}^n)$ integrate to zero at first order; the first variation depends only on* $\mathrm{Tr}_g\mathrm{Tr}_g(T)$.

Corollary VII.1 (Quadratic seams are locally extremal at first order)**.** *Under the assumptions of Theorem VII.1, let $s_2(z) := \frac{1}{2}g_{ij}z^i z^j$ be the quadratic seam with the same quadratic jet and let $V_2(\varepsilon) := \mathrm{Vol}_{\mathrm{Leb}}(\{z : s_2(z) \le \varepsilon\})$. Then*

$$\frac{V(\varepsilon)}{V_2(\varepsilon)} = 1 - \frac{\varepsilon}{4(n+2)}\mathrm{Tr}_g\mathrm{Tr}_g(T) + O(\varepsilon^2) + O(\|T\|^2). \tag{31}$$

In particular, if $\mathrm{Tr}_g\mathrm{Tr}_g(T) > 0$ *then $V(\varepsilon) < V_2(\varepsilon)$ for all sufficiently small $\varepsilon > 0$; if* $\mathrm{Tr}_g\mathrm{Tr}_g(T) < 0$ *then the inequality is reversed to first order; and if* $\mathrm{Tr}_g\mathrm{Tr}_g(T) = 0$ *then the first-order variation vanishes.*

Remark VII.6 (A simple rigidity criterion under directional positivity)**.** *The first-order statement becomes rigid under an additional positivity hypothesis. After transforming to normal coordinates $g = I$, suppose the homogeneous quartic form $n \mapsto T(n,n,n,n)$ satisfies $T(n,n,n,n) \ge 0$ for all $n \in S^{n-1}$. Then* (30) *implies* $\mathrm{Tr}\,\mathrm{Tr}(T) \ge 0$*, and if* $\mathrm{Tr}\,\mathrm{Tr}(T) = 0$ *then $\int_{S^{n-1}} T(n,n,n,n)\, d\Omega = 0$ forces $T(n,n,n,n) \equiv 0$ on S^{n-1}. Since $T(n,n,n,n)$ is a homogeneous polynomial, vanishing on the sphere implies it vanishes for all $n \in \mathbb{R}^n$. By the standard polarisation identity for symmetric 4-tensors, vanishing of the associated quartic form implies $T_{ijkl} = 0$. Thus, under directional positivity, the quadratic seam is the unique seam (to quartic order) with vanishing first variation.*

G. A Conditional Width Bound with Cross-Coupling

Assumption VII.1 (Level-set area hypothesis)**.** *For applications in which one postulates a quantum-motivated phase-space scale, the Lebesgue area of the sublevel set $\{s \le 1/2\}$ satisfies $A(s) :=$* $\mathrm{Area}(\{s \le 1/2\}) \ge \pi\hbar/2$.

Remark VII.7 (When the area hypothesis is rigorous)**.** *For a one-mode Gaussian state with covariance matrix Σ, the Robertson–Schrödinger inequality implies $\det\Sigma \ge (\hbar/2)^2$*[5,6]*. The covariance ellipse $\{z^T\Sigma^{-1}z \le 1\}$ has area $2\pi\sqrt{\det\Sigma}$, hence at level $s = 1/2$ the corresponding quadratic seam satisfies $A(s) = \pi\,\Delta x\,\Delta p \ge \pi\hbar/2$. For general non-Gaussian seams, however, $\{s \le 1/2\}$ need not coincide with a covariance ellipse, so Assumption VII.1 should be viewed as an additional modelling postulate rather than a consequence of the canonical commutation relation.*

Theorem VII.2 (Conditional Full 4-Jet Width Bound)**.** *Let s_4 be the full quartic seam* (6) *with parameters $a,b \geq 0$, $c \geq -\sqrt{ab}$, and $\lambda > 0$. If the sublevel set $\{s_4 \leq 1/2\}$ satisfies Assumption VII.1, then to first order in λ,*

$$\boxed{\sigma_x\sigma_p \;\geq\; \frac{\hbar/2}{1-\lambda(3a+3b+2c)/16} \;\approx\; \frac{\hbar}{2}\left(1+\frac{\lambda(3a+3b+2c)}{16}\right).} \tag{32}$$

This bound is valid provided $\lambda(3a+3b+2c) < 16$. For $c = 0$ and $a = b = 1$ this reduces to the diagonal result $\sigma_x\sigma_p \geq (\hbar/2)(1+3\lambda/8)$.

Proof. By (28), the minimal-area condition reads

$$\pi\sigma_x\sigma_p\left(1-\frac{\lambda(3a+3b+2c)}{16}\right) \geq \frac{\pi\hbar}{2},$$

from which (32) follows by dividing both sides by $\pi\big(1-\lambda(3a+3b+2c)/16\big)$, which is positive by assumption. □

VIII. UNIVERSALITY: NORMALISABILITY FORCES A POSITIVE FIRST-ORDER CORRECTION

Theorem VII.2 shows that the sign of the first-order correction term is determined by the sign of $3a+3b+2c$. This quantity can be negative if c is sufficiently negative, which would *loosen* rather than tighten the resulting width bound under Assumption VII.1. The central question is therefore: can a normalisable quartic density $\exp(-s_4)$ achieve $3a+3b+2c \leq 0$?

The following theorem answers this question decisively in the negative.

Theorem VIII.1 (Universality of a positive correction coefficient)**.** *For any quartic seam s_4 with $a,b \geq 0$, $\lambda > 0$, and cross-coupling coefficient satisfying the normalisability constraint $c \geq -\sqrt{ab}$, the correction coefficient satisfies*

$$3a+3b+2c > 0 \tag{33}$$

provided $Q \not\equiv 0$, i.e., provided $(a,b,c) \neq (0,0,0)$.

Proof. We consider the two exhaustive cases.

Case 1: $a+b=0$. Since $a,b \geq 0$, this means $a=b=0$. For $Q = 2cu^2v^2 \geq 0$ to hold for all (u,v) with $Q \not\equiv 0$, we need $c > 0$. Then $3a+3b+2c = 2c > 0$.

Case 2: $a+b>0$. By the AM-GM inequality,

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

The normalisability constraint gives $c \geq -\sqrt{ab}$, hence

$$c \geq -\sqrt{ab} \geq -\frac{a+b}{2}.$$

Therefore

$$3a+3b+2c \geq 3(a+b)+2\cdot\left(-\frac{a+b}{2}\right) = 3(a+b)-(a+b) = 2(a+b) > 0. \qquad \square$$

Corollary VIII.1 (Non-Gaussianity increases the required width product (conditional))**.** *Assume Assumption VII.1. For any normalisable non-Gaussian seam (i.e., any quartic seam with $(a,b,c) \neq (0,0,0)$ satisfying* (11)*), the width product $\sigma_x\sigma_p$ is strictly bounded above $\hbar/2$:*

$$\sigma_x\sigma_p > \frac{\hbar}{2}.$$

The bound is saturated — i.e., $3a+3b+2c=0$ — only in the unphysical limiting case $(a,b,c) = (0,0,0)$, which reduces to a Gaussian seam.

The proof of Theorem VIII.1 reveals that the AM-GM inequality and the normalisability constraint (11) are in precise correspondence: the minimum value of c that preserves normalisability is $-\sqrt{ab}$, and substituting this minimum into $3a+3b+2c$ gives $3(a+b)-2\sqrt{ab} \geq 2(a+b) > 0$, with the second inequality being strict AM-GM. The normalisability constraint is therefore the *exact* analytic condition that prevents the cross-coupling from cancelling the first-order correction.

IX. EXTREMAL SEAMS: MINIMISING THE QUARTIC CORRECTION

Although the correction coefficient $3a+3b+2c$ is always positive, its magnitude depends on c. For fixed diagonal non-Gaussianity $a+b>0$, the correction is minimised when c is as negative as normalisability allows, i.e., at the *extremal* value $c=-\sqrt{ab}$.

Definition IX.1 (Extremal Quartic Seam)**.** *A quartic seam s_4 with $a,b>0$ is* extremal *if the cross-coupling saturates the normalisability bound:*

$$c = -\sqrt{ab}. \tag{34}$$

Proposition IX.1 (Extremal correction coefficient)**.** *For fixed $a+b>0$, the correction coefficient $3a+3b+2c$ is minimised over all normalisable c at the extremal value $c=-\sqrt{ab}$, where it equals*

$$(3a+3b+2c)\big|_{c=-\sqrt{ab}} = 3(a+b)-2\sqrt{ab} = (\sqrt{a}-\sqrt{b})^2+2(a+b) > 0. \tag{35}$$

Equivalently, $3a+3b-2\sqrt{ab} = (\sqrt{3a}-\sqrt{b/3})^2+(8b/3) > 0$.

Proof. Since $c \geq -\sqrt{ab}$ and $3a+3b+2c$ is increasing in c, the minimum is at $c=-\sqrt{ab}$, giving $3(a+b)-2\sqrt{ab} = a+b+2(a+b-\sqrt{ab}) \geq a+b > 0$, where we used $a+b \geq 2\sqrt{ab}$ by AM-GM. □

The extremal seam has a natural geometric interpretation. The quartic form $Q = au^4+bv^4-2\sqrt{ab}\,u^2v^2 = (\sqrt{a}\,u^2-\sqrt{b}\,v^2)^2 \geq 0$ vanishes precisely along the two lines $v = \pm(\sqrt{a/b})^{1/2}u$ in the normalised (u,v) plane. Along these lines the quartic correction is exactly zero, and the seam is purely quadratic. In all other directions the quartic correction is strictly positive and confines the sublevel set.

Remark IX.1 (Geometric interpretation of extremal seams)**.** *The extremal condition $c=-\sqrt{ab}$ means that position-squared and momentum-squared fluctuations are as* negatively correlated *as possible while maintaining normalisability. The state has strong fourth-order fluctuations in the diagonal directions $x \sim \pm(\sigma_x/\sigma_p)^{1/2}p$ but is constrained near the axes. Among all non-Gaussian seams with fixed diagonal kurtosis $a+b$, extremal seams are the* closest to quadratic *in the sense of minimising the quartic width penalty implicit in* (32) *under Assumption VII.1.*

X. CONNECTION TO THE NON-GAUSSIANITY WITNESS

The scalar $K_4^{\text{full}} = \lambda(a+b+2c)$ defined in (16) provides a coordinate-invariant witness for non-Gaussianity, but it is *not* the quantity that appears in the conditional width bound. The bound (32) is governed by the weighted combination

$$\mathscr{W}_4 := \lambda\,\frac{3(a+b)+2c}{16} = \frac{3\lambda(a+b)}{16}+\frac{\lambda c}{8} = \frac{3K_4^{\text{diag}}}{16}+\frac{\lambda c}{8}, \tag{36}$$

where $K_4^{\text{diag}} = \lambda(a+b)$ is the diagonal 4-jet scalar. The full 4-jet scalar K_4^{full} can be recovered from $\mathscr{W}_4$ and K_4^{diag}:

$$K_4^{\text{full}} = K_4^{\text{diag}}+2\lambda c \quad\Longrightarrow\quad \lambda c = \frac{K_4^{\text{full}}-K_4^{\text{diag}}}{2}.$$

Substituting back,

$$\mathscr{W}_4 = \frac{3K_4^{\text{diag}}}{16} + \frac{K_4^{\text{full}} - K_4^{\text{diag}}}{16} = \frac{K_4^{\text{full}} + 2K_4^{\text{diag}}}{16}. \tag{37}$$

The bound is therefore governed by the weighted average of the full and diagonal 4-jet scalars with weights $1:2$, not by either scalar alone. This gives a precise statement of how much information the diagonal treatment loses: it overestimates the correction coefficient by a factor that depends on the relative size of K_4^{full} and K_4^{diag}. When $c > 0$ the diagonal treatment underestimates the correction; when $c < 0$ it overestimates.

XI. THE JET HIERARCHY

Theorem VII.2 and Theorem VIII.1 together establish the first two levels of a hierarchy of width bounds (conditional on Assumption VII.1 at quartic order) indexed by the even jets of a smooth seam:

a. Where $\hbar$ enters. The appearance of $\hbar$ here is not intrinsic to the jet calculus itself: it enters only through the chosen *phase-space scale* used to turn geometric area/volume statements into a width-product bound. Concretely, at quadratic order the bound $\sigma_x\sigma_p \geq \hbar/2$ is the quantum-motivated scale constraint (2), and at quartic order the same scale is propagated via the level-set area hypothesis (Assumption VII.1). If one prefers a purely geometric presentation, one may replace $\hbar/2$ everywhere in the table by an abstract constant $\kappa > 0$ (equivalently, write $A(s) \geq \pi\kappa$ in Assumption VII.1); the jet-dependent correction factors $1 + \mathscr{W}_{2k} + \cdots$ are unchanged.

Jet	Geometric object	Width bound	Universality
2	Hessian metric g	$\sigma_x\sigma_p \geq \hbar/2$	Always positive
4	T_{ijkl}, double trace $\mathscr{W}_4$	$\sigma_x\sigma_p \geq (\hbar/2)(1+\mathscr{W}_4+\cdots)$	Positive iff normalisable
6	S_{ijklmn}, triple trace $\mathscr{W}_6$	$\sigma_x\sigma_p \geq (\hbar/2)(1+\mathscr{W}_6+\cdots)$	Positive iff normalisable
$2k$	Symmetric $2k$-tensor, k-fold trace $\mathscr{W}_{2k}$	$\sigma_x\sigma_p \geq (\hbar/2)(1+\mathscr{W}_{2k}+\cdots)$	Positive iff normalisable

At the fourth-order level the universality of positive corrections follows from the AM-GM inequality and the normalisability constraint (Theorem VIII.1). The general statement — that normalisability forces positivity at every even order $2k$ — is established in Theorem XIV.1 via a spherical-averaging argument that subsumes the quartic AM-GM proof as a special case.

XII. MULTI-MODE EXTENSION

For $d > 1$, the phase space is $\mathbb{R}^{2d}$ with coordinates $(x_1,\ldots,x_d,p_1,\ldots,p_d)$, and the quartic jet acquires many more independent components. In normalised coordinates $q = (u_1,\ldots,u_d,v_1,\ldots,v_d)$ (so the quadratic jet is $\frac{1}{2}|q|^2$), write the local 4-jet expansion

$$s(q) = \frac{1}{2}|q|^2 + \frac{1}{24}\widetilde{T}_{ijkl}\,q^iq^jq^kq^l + O(|q|^6),$$

where $\widetilde{T}$ is the fully symmetric quartic jet tensor in these coordinates.

The angular integrals over the unit sphere S^{2d-1} are controlled invariantly by the spherical fourth-moment identity (30). In components, this implies

$$\int_{S^{2d-1}} q_i^4\,d\Omega = \frac{3\,\omega_{2d}}{2d(2d+2)}, \tag{38}$$

$$\int_{S^{2d-1}} q_i^2q_j^2\,d\Omega = \frac{\omega_{2d}}{2d(2d+2)} \quad (i \neq j), \tag{39}$$

where ω_{2d} is the surface area of S^{2d-1}. The ratio of cross to diagonal angular integrals is again $1:3$, independent of dimension. This means the first-order sublevel-set volume correction depends only on the double trace of $\widetilde{T}$.

Specialising Theorem VII.1 to $n = 2d$ and the level $\varepsilon = 1/2$ (so $(2\varepsilon)^{n/2} = 1$) yields the relative correction

$$\mathscr{W}_4^{(d)} := \frac{1}{8(2d+2)} \operatorname{Tr}\operatorname{Tr}(\widetilde{T}) = \frac{1}{8(2d+2)} \widetilde{T}_{iijj}, \tag{40}$$

which is manifestly basis-independent. If one expands $\widetilde{T}_{iijj}$ in an orthonormal basis, the same $1:3$ diagonal-versus-cross weighting reappears (via (38)–(39)), confirming that the ratio is a spherical-geometry invariant rather than a coordinate artefact.

Remark XII.1 (Positivity forces a positive $O(n)$-scalar projection)**.** *If the quartic jet induces a nonnegative homogeneous quartic form on directions, $\widetilde{T}(n,n,n,n) \geq 0$ for all $n \in S^{2d-1}$ (as occurs, for example, when one requires normalisability of a model density $\propto \exp(-s)$ with a strictly positive quartic part), then $\int_{S^{2d-1}} \widetilde{T}(n,n,n,n)\, d\Omega \geq 0$. By* (30)*, this is equivalent to $\widetilde{T}_{iijj} \geq 0$, hence $\mathscr{W}_4^{(d)} \geq 0$. Moreover, if $\widetilde{T}(n,n,n,n)$ is not identically zero on the sphere then the inequality is strict.*

XIII. THE SEXTIC JET: EXTENDING THE HIERARCHY

The quartic computation of Section VII extracted the first-order sublevel-set correction from the 4-jet. We now carry out the analogous computation for the *6-jet*, establishing the next level of the hierarchy announced in Section XI. As before, we give explicit formulas in $d = 1$ and then state the coordinate-free result in arbitrary dimension.

A. The Pure Sextic Seam in Dimension $d = 1$

In normalised coordinates $u = x/\sigma_x$, $v = p/\sigma_p$, define the *pure sextic seam* (with vanishing quartic jet, for clarity; combined effects are additive at leading order):

$$s_6(u,v) = \frac{u^2+v^2}{2} + \frac{\mu}{6}\left(\alpha u^6 + \beta v^6 + 3\gamma u^4 v^2 + 3\delta u^2 v^4\right), \tag{41}$$

where $\alpha, \beta \geq 0$ and $\gamma, \delta \in \mathbb{R}$ are dimensionless shape parameters and $\mu > 0$ sets the overall strength of the sextic correction. In polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, this becomes

$$s_6 = \frac{r^2}{2} + \frac{\mu}{6} r^6 G(\theta), \tag{42}$$

where

$$G(\theta) := \alpha\cos^6\theta + \beta\sin^6\theta + 3\gamma\cos^4\theta\sin^2\theta + 3\delta\cos^2\theta\sin^4\theta. \tag{43}$$

B. Sextic Angular Integrals and Area Correction

The four angular integrals needed are:

$$\int_0^{2\pi} \cos^6\theta\, d\theta = \int_0^{2\pi} \sin^6\theta\, d\theta = \frac{5\pi}{8}, \tag{44}$$

$$\int_0^{2\pi} \cos^4\theta \sin^2\theta\, d\theta = \int_0^{2\pi} \cos^2\theta\sin^4\theta\, d\theta = \frac{\pi}{8}. \tag{45}$$

The ratio of cross to diagonal integrals is $1/5$, compared with $1/3$ at quartic order (24)–(25). Both are instances of a general pattern: at order $2k$, the ratio is $1/(2k-1)$.

Therefore,

$$\int_0^{2\pi} G(\theta)\,d\theta = \frac{\pi}{8}\big(5\alpha+5\beta+3\gamma+3\delta\big). \tag{46}$$

The boundary of $\{s_6 = 1/2\}$ satisfies $\frac{r^2}{2} + \frac{\mu}{6} r^6 G(\theta) = \frac{1}{2}$. Solving perturbatively gives

$$r^2(\theta) = 1 - \frac{\mu}{3}\,G(\theta) + O(\mu^2). \tag{47}$$

The area in normalised coordinates is therefore

$$\begin{aligned} A_{uv} &= \frac{1}{2}\int_0^{2\pi} r^2(\theta)\,d\theta = \pi - \frac{\mu}{6}\cdot\frac{\pi}{8}(5\alpha+5\beta+3\gamma+3\delta) + O(\mu^2) \\ &= \pi\left(1 - \frac{\mu(5\alpha+5\beta+3\gamma+3\delta)}{48}\right) + O(\mu^2). \end{aligned} \tag{48}$$

Restoring physical units:

$$\boxed{A(s_6) = \pi\sigma_x\sigma_p\left(1 - \frac{\mu(5\alpha+5\beta+3\gamma+3\delta)}{48}\right) + O(\mu^2).} \tag{49}$$

Remark XIII.1 (Angular-integral ratio $1/5$ at sextic order)**.** *Each diagonal shape parameter α or β contributes with the angular weight $5\pi/8$, while each cross parameter γ or δ contributes with $\pi/8$. Their ratio of $1/5$ is the sextic analogue of the quartic ratio $1/3$ and reflects the universal spherical-geometry identity $\int_0^{2\pi}\cos^4\theta\sin^2\theta\,d\theta \,/\, \int_0^{2\pi}\cos^6\theta\,d\theta = 1/5$.*

C. Invariant First Variation at General Even Order

The quartic first variation (Theorem VII.1) and the sextic computation above are special cases of a general first-variation formula for the pure $2k$-jet, valid in arbitrary dimension n.

Theorem XIII.1 (First variation from the $2k$-jet)**.** *Let $s:\mathbb{R}^n\to\mathbb{R}$ be C^{2k} near the origin with $s(0)=0$, $\nabla s(0)=0$, and positive definite quadratic jet $g_{ij}=\partial_i\partial_j s(0)$. Assume all jets of order $4,6,\ldots,2k-2$ vanish. Let $P_{i_1\cdots i_{2k}} = \partial_{i_1}\cdots\partial_{i_{2k}} s(0)$ be the fully symmetric $2k$-jet tensor. Then for $\varepsilon>0$ sufficiently small,*

$$V(\varepsilon) = \frac{\omega_n}{n}\,\frac{(2\varepsilon)^{n/2}}{\sqrt{\det g}}\left(1 - \frac{\varepsilon^{k-1}}{2k!\,\prod_{j=1}^{k-1}(n+2j)}\operatorname{Tr}_g^k(P) + O(\varepsilon^k) + O(\|P\|^2)\right), \tag{50}$$

where $\operatorname{Tr}_g^k(P) := g^{i_1i_2}g^{i_3i_4}\cdots g^{i_{2k-1}i_{2k}}P_{i_1i_2\cdots i_{2k}}$ is the k-fold trace with respect to g.

Proof. By a linear change of variables set $g=I$ (at the cost of a Jacobian $1/\sqrt{\det g}$). Write $z=rn$ with $n\in S^{n-1}$. The jet expansion gives $s(rn) = \frac{1}{2}r^2 + \frac{1}{(2k)!}r^{2k}P(n,\ldots,n)$. Setting $s=\varepsilon$ and solving perturbatively yields

$$r^2 = 2\varepsilon\left(1 - \frac{2^{k-1}\varepsilon^{k-1}}{(2k)!}\,P(n,\ldots,n) + O(\varepsilon^k) + O(\|P\|^2)\right).$$

Hence $r^n = (2\varepsilon)^{n/2}(1 - \frac{n2^{k-1}\varepsilon^{k-1}}{2(2k)!}P(n,\ldots,n)+\cdots)$, and integrating over S^{n-1} using the $O(n)$-invariant $2k$-th moment

$$\int_{S^{n-1}} P(n,\ldots,n)\,d\Omega = \frac{(2k-1)!!\;\omega_n}{\prod_{j=0}^{k-1}(n+2j)}\operatorname{Tr}^k(P) \tag{51}$$

(which generalises (30)) gives (50) after simplification using $(2k-1)!!/(2k)! = 1/(2^k k!)$. $\square$

Remark XIII.2 (Recovering the quartic and sextic cases)**.** *For $k = 2$ (quartic) and $n = 2$ the coefficient becomes $\varepsilon/(2\cdot 2\cdot 4) = \varepsilon/16$, and at $\varepsilon = 1/2$ this gives $1/32$ times the double trace, recovering Theorem VII.1. For $k = 3$ (sextic) and $n = 2$ the coefficient becomes $\varepsilon^2/(2\cdot 6\cdot 4\cdot 6) = \varepsilon^2/288$, and at $\varepsilon = 1/2$ this gives $1/1152$ times the triple trace, recovering* (48)*.*

Remark XIII.3 (Angular-integral ratios at order $2k$)**.** *In $n = 2$ dimensions, the ratio of the cross angular integral*

$$\int_0^{2\pi} \cos^{2k-2}\theta \sin^2\theta \, d\theta$$

to the diagonal integral

$$\int_0^{2\pi} \cos^{2k}\theta \, d\theta$$

is $1/(2k-1)$. For $k = 2$ this gives $1/3$; for $k = 3$ it gives $1/5$. This ratio is the geometric origin of the decreasing per-component weight of cross-coupling terms at higher jet orders.

XIV. UNIVERSALITY AT ALL EVEN ORDERS

We now resolve the conjecture stated in Section XI: normalisability of $\exp(-s_{2k})$ forces a positive first-order correction at every even order $2k$. The argument is a direct consequence of spherical averaging and requires no case analysis.

Theorem XIV.1 (Universality at all even orders)**.** *Let $H : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree $2k$ ($k \geq 1$) that is non-negative: $H(z) \geq 0$ for all $z \in \mathbb{R}^n$, and not identically zero. Let $P_{i_1\cdots i_{2k}}$ be the associated fully symmetric tensor (so $H(z) = P_{i_1\cdots i_{2k}} z^{i_1}\cdots z^{i_{2k}}$). Then for any positive definite metric g,*

$$\mathrm{Tr}_g^k(P) > 0. \tag{52}$$

Moreover, if $H \geq 0$ and $\mathrm{Tr}_g^k(P) = 0$, then $H \equiv 0$ and $P = 0$.

Proof. By a linear change of coordinates set $g = I$. The spherical $2k$-th moment formula (51) gives

$$\int_{S^{n-1}} H(n)\, d\Omega = \frac{(2k-1)!!\ \omega_n}{\prod_{j=0}^{k-1}(n+2j)}\ \mathrm{Tr}^k(P),$$

where the prefactor is strictly positive. Since H is continuous, non-negative, and not identically zero on the compact set S^{n-1}, its integral is strictly positive. Therefore $\mathrm{Tr}^k(P) > 0$.

For the rigidity statement: if $\mathrm{Tr}^k(P) = 0$ and $H \geq 0$, then $\int_{S^{n-1}} H(n)\, d\Omega = 0$, which forces $H \equiv 0$ on S^{n-1} by continuity and non-negativity. Since H is homogeneous, $H \equiv 0$ on all of $\mathbb{R}^n$. By the polarisation identity for symmetric tensors, $P = 0$. □

Corollary XIV.1 (Resolution of the universality conjecture)**.** *At every even order $2k \geq 4$, let $s_{2k}(z) = \frac{1}{2} g_{ij} z^i z^j + \frac{1}{(2k)!} P_{i_1\cdots i_{2k}} z^{i_1}\cdots z^{i_{2k}}$ be a seam whose model density $\exp(-s_{2k})$ is normalisable and whose $2k$-jet does not vanish. Then the first-order correction coefficient $\mathrm{Tr}_g^k(P)$ is strictly positive, and consequently sublevel-set volumes are strictly smaller than those of the quadratic seam.*

Proof. Normalisability of $\exp(-s_{2k})$ requires the homogeneous degree-$2k$ part

$$H(z) = \tfrac{1}{(2k)!} P_{i_1\cdots i_{2k}} z^{i_1}\cdots z^{i_{2k}}$$

to be non-negative for all z: if $H(z_0) < 0$ for some direction z_0, then $s_{2k}(rz_0) \to -\infty$ as $r \to \infty$, so $\exp(-s_{2k})$ is not integrable. Apply Theorem XIV.1. □

Remark XIV.1 (Relation to the quartic proof)**.** *At quartic order ($k = 2$, $n = 2$) the normalisability constraint reduces to $c \geq -\sqrt{ab}$, and the positivity of $3a+3b+2c$ was established in Theorem VIII.1 via the AM-GM inequality. That argument is a special case of the present proof: the AM-GM step verifies, by direct algebra, that $\int_{S^1} Q(\cos\theta, \sin\theta)\, d\theta > 0$ whenever the quartic form $Q \geq 0$ is non-trivial. The spherical-averaging argument replaces all such case-specific algebra with a single measure-theoretic step.*

XV. SATURATION, GAUSSIANS, AND FURTHER DIRECTIONS

Corollary XV.1 (Saturation at every level)**.** *The bound* (32) *is saturated (i.e., the correction vanishes) if and only if $a = b = c = 0$ and $\sigma_x \sigma_p = \hbar/2$. These conditions hold simultaneously only for the quadratic (Gaussian) seam, confirming that any departure from quadraticity incurs a strict first-order correction at quartic order (under Assumption VII.1).*

a. Curved-space extensions. Equip $\mathbb{R}^{2d}$ with a background Riemannian metric h and define the seam s relative to h. The covariant Hessian rule $g_{ij} = \nabla_i \nabla_j s$ incorporates curvature of the background. On positively curved backgrounds the effective area/width floor may increase; the quartic correction (40) will acquire additional terms from curvature tensors contracted against T_{ijkl}, and the universality theorem will require a modified normalisability constraint that couples the quartic seam coefficients to the background Ricci tensor.

b. Universal approximation conjecture. Every sufficiently regular, strictly positive density on $\mathbb{R}^{2d}$ may be approximated (in an appropriate topology) by finite mixtures of quartic exponential models of the form $\propto \exp(-s_4)$. The weighted scalar (37) would then provide an explicit non-Gaussianity measure for each component.

XVI. A CONCRETE PHYSICAL EXAMPLE: WEAK KERR-TYPE NONLINEARITY

To ground the framework, consider a single-mode oscillator with a weak isotropic Kerr-type nonlinearity, modeled at the classical level by

$$H(x,p) = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{\kappa}{4}\,(p^2 + m^2\omega^2 x^2)^2,$$

with $\kappa > 0$. A simple toy non-Gaussian phase-space density is the thermal ansatz $\rho \propto \exp(-\beta H)$, or equivalently a seam $s = \beta H$. In scaled variables $u = x/\sigma_x$, $v = p/\sigma_p$ with $\sigma_x^2 = 1/(\beta m \omega^2)$ and $\sigma_p^2 = m/\beta$, this takes the quartic seam form

$$s_4(u,v) = \frac{u^2+v^2}{2} + \frac{\lambda}{4}\big(u^4 + v^4 + 2u^2v^2\big), \qquad \lambda = \frac{\kappa m^2}{\beta},$$

with the concrete coefficients

$$a = 1, \qquad b = 1, \qquad c = 1.$$

This is the rotationally symmetric case $s_4 = \frac{1}{2}(u^2+v^2) + \frac{\lambda}{4}(u^2+v^2)^2$, which is normalisable for all $\lambda > 0$ and yields a positive quartic correction $\mathscr{W}_4 = \lambda(3+3+2)/16 = \lambda/2$. The coefficients a, b, c thus have a direct physical interpretation as the relative strengths of the quartic self- and cross-kurtosis induced by the Kerr term.

XVII. CONCLUSION

Phase-space seams provide a scalar-first, entirely geometric framework for studying Hessian-generated geometry and sublevel-set volumes on phase space. The extension to the full quartic seam — including the position-momentum cross-kurtosis term x^2p^2 — reveals structure absent from the diagonal-only treatment:

1. The correct 4-jet scalar for the conditional width bound is not $K_4^{\text{full}} = \lambda(a+b+2c)$ but the weighted combination $\mathscr{W}_4 = \lambda(3a+3b+2c)/16$, in which the cross-coupling contributes with weight $1/3$ relative to the diagonal terms (from the angular integrals), with the overall $1/16$ factor rather than $1/8$ following from the correct evaluation $\int_0^{2\pi} \cos^4\theta\,d\theta = 3\pi/4$ and $\int_0^{2\pi} \cos^2\theta \sin^2\theta\,d\theta = \pi/4$. The ratio $1/3$ between the angular integrals is a universal geometric invariant, and the full expression $(\pi/4)(3a+3b+2c)$ integrating $F(\theta)$ captures both.

2. The normalisability constraint $c \geq -\sqrt{ab}$ is precisely the condition that forces $3a+3b+2c > 0$ for all non-trivial quartic seams. The proof reduces to a single application of the AM-GM inequality. Normalisability is therefore not merely a technical requirement but the exact analytic principle that prevents cross-coupling from cancelling the first-order correction.

3. Among all non-Gaussian seams with fixed diagonal kurtosis $a+b>0$, the extremal seams with $c = -\sqrt{ab}$ minimise the quartic correction. They are characterised geometrically as seams whose quartic correction vanishes along two specific lines in phase space and is concentrated in the transverse directions. Equivalently, at fixed diagonal kurtosis, they furnish a canonical family of "most Gaussian-like" normalisable quartic seams for variational calculations.

4. Under Assumption VII.1, the conditional width bound takes the explicit form $\sigma_x\sigma_p \geq (\hbar/2)(1+\mathscr{W}_4+O(\lambda^2))$ with $\mathscr{W}_4 > 0$ for all normalisable non-Gaussian seams.

5. The framework extends beyond the quartic: a general first-variation formula (Theorem XIII.1) shows that the first-order volume correction from any even $2k$-jet depends only on its k-fold trace. The ratio of cross to diagonal angular integrals at order $2k$ is $1/(2k-1)$ in $d=1$ (giving $1/3$ at quartic order and $1/5$ at sextic order), a universal spherical-geometry invariant.

6. Universality holds at all even orders (Theorem XIV.1): normalisability of $\exp(-s_{2k})$ forces the k-fold trace — and hence the first-order volume correction — to be strictly positive for any non-trivial $2k$-jet. The proof is a single measure-theoretic step that replaces the quartic-specific AM-GM argument.

The immediate computational payoffs (convex inverse design and visual diagnostics) are ready for implementation using $\mathscr{W}_4$ (or its higher-order analogues) rather than the diagonal scalar. The multi-mode extension confirms that the $1:3$ ratio at quartic order is dimension-independent; at higher jet orders the corresponding ratios follow the pattern $1/(2k-1)$ (Remark XIII.3).

a. Caveat on physical interpretation. For non-Gaussian seams, the width parameters σ_x, σ_p are not identical to either (i) the true second moments of the model density $\exp(-s_4)$ (see Proposition VII.1) or (ii) the operator uncertainties of an underlying Hilbert-space state without specifying an explicit phase-space mapping (e.g. via a Husimi Q-function). Accordingly, the quartic correction derived here should be read as a conditional geometric statement about jet parameters under Assumption VII.1, not as a general strengthening of $\Delta x \Delta p \geq \hbar/2$.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Lars Rönnbäck: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Writing – original draft (lead); Writing – review & editing (lead).
Declaration of generative AI and AI-assisted technologies. During the preparation of this work the author used Grok 4.2, GPT-5.2, Gemini 3.1 Pro, and Claude Sonnet 4.6 in order to draft early versions of the manuscript, brainstorm and outline the revised manuscript, polish the exposition, and respond to reviewer feedback. After using these tools/services, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

DATA AVAILABILITY

This article presents theoretical results only; no datasets were generated or analysed. The data that support the findings of this study are available within the article itself.

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