

Discrete-to-Continuum Metrics from Scalar Fields

LARS RÖNNBÄCK*

Stockholm University

lars@uptochange.com

Abstract

We present seam-driven geometry, a scalar-first framework for geometry processing in which geometric structure is generated from a scalar field $s : U \rightarrow \mathbb{R}$ (a *seam*) via an explicit local Rule \mathcal{R} . In the discrete setting, we study a conformal graph rule that assigns edge lengths using an endpoint quadrature of e^s , yielding a shortest-path metric that is easy to optimize and differentiate. Our main results are (i) a quantitative discrete-to-continuum guarantee, including an $O(h)$ Gromov–Hausdorff convergence rate on quasi-uniform triangulations; (ii) a curvature sensitivity identity showing that first-order curvature variations are governed by the cotangent Laplacian; and (iii) a strictly convex inverse-design formulation for fitting target edge weights via a quadratic program in variables $X_u = e^{s(u)}$. These results position seams as a stable interface between differential-geometric objectives and practical optimization pipelines in mesh and graph processing.

I. INTRODUCTION

Standard presentations of geometry begin with a metric or a manifold structure postulated *a priori*. Here we emphasize a complementary, constructive viewpoint: start with a base space U and a scalar field s (the *seam*), then let an explicit Rule \mathcal{R} produce the geometric data.

In smooth differential geometry, this scalar-first perspective provides an elegant synthesis. Conformal metrics, Information Geometry (Hessian metrics), optimal transport (Kantorovich potentials), and Morse handle-bodies can all be viewed as the output of specific local operations on a scalar field. However, the true utility of this formulation arises in computational and applied mathematics. On discrete meshes, manipulating full tensor fields is algorithmically cumbersome and numerically unstable. By reducing metric generation to the evaluation of a scalar seam, highly complex geometric problems (like metric projection or feature-aware routing) can be reduced to fast, stable scalar optimization.

In this paper, we formally define the Seam-Rule framework and demonstrate its utility. Section II establishes the formal axioms and composition laws for Rules. Section III catalogs the standard continuous and discrete Rules. Section IV shows how continuous theorems (Gauss–Bonnet) are cleanly articulated in this language. Finally, Section V provides the computational climax: proving that discrete seam-generated graphs rigorously approximate continuous geometries, preserve curvature via the cotangent Laplacian, and enable strictly convex inverse metric design.

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Contributions (geometry processing view). Our contributions are geared toward mesh and graph processing, where one seeks parameterizations that are easy to optimize, differentiate, and analyze.

1. **A scalar-first metric parameterization on graphs/meshes.** We study a conformal graph rule that assigns edge lengths using an endpoint quadrature of e^s , producing a shortest-path metric with local control by a scalar seam.
2. **Quantitative discrete-to-continuum guarantees.** On quasi-uniform triangulations, we prove an $O(h)$ Gromov–Hausdorff convergence rate between the seam-generated discrete metric and the smooth conformal metric (Theorem 3) and derive a corresponding uniform $O(h)$ metric error bound (Theorem 4).
3. **Curvature sensitivity via cotangent weights.** We show that the Jacobian of angle-defect curvature with respect to the seam agrees (up to a constant factor) with the cotangent Laplacian at $s = 0$ (Theorem 7), providing a differentiable link between scalar parameters and discrete curvature.
4. **Convex inverse metric design.** We reduce edge-weight fitting under the conformal graph rule to a strictly convex quadratic program in variables $X_u = e^{s(u)}$ on non-bipartite graphs (Theorem 5), leveraging a classical signless-Laplacian positive-definiteness criterion [9, 10]. We further provide gauge-fixing and conditioning guarantees (Theorem 6).

I.A Related Work

Scalar fields on meshes are a classical and widely used representation in geometry processing, in particular as vertex-based conformal factors and potentials. Discrete uniformization and discrete conformal parameterizations seek a scalar conformal scaling that induces a metric of prescribed curvature or that supports flattening/parameterization; see, e.g., Gu–Luo–Sun–Wu [12] and the variational formulations of Springborn–Schröder–Pinkall [13]. In this literature, a scalar field (often a log-scale factor) is the primary unknown, and the induced metric is obtained by a local rule that rescales discrete lengths/angles.

Circle packing provides another scalar-first route to discrete conformal geometry, in which circle radii (or log-radii) act as scalar variables whose induced intersection patterns and edge lengths encode a discrete conformal structure; see Stephenson [17] for an overview and references. Closely related, combinatorial Ricci flow evolves a vertex-based scalar parameterization (e.g., circle radii or log-scale factors) to drive discrete curvature toward a target, with Chow–Luo [16] as a foundational reference.

Our intent is not to claim novelty of using scalar fields per se, but to formalize a general “seam → rule → geometry” interface that accommodates multiple continuous and discrete constructions under one axiomatic umbrella, and to highlight algorithmic consequences of a particular discrete edge-length rule: a quantitative discrete-to-continuum guarantee for the resulting shortest-path metric, a curvature sensitivity identity, and a convex inverse-design formulation for fitting target edge weights.

II. FRAMEWORK: SEAMS, RULES, AND COMPOSITION

Let U be a paracompact Hausdorff space (or discrete set) equipped with a local background structure \mathcal{T}_U (e.g., a differentiable atlas, a background metric h , or a graph adjacency).

Definition 1. A seam is a function $s : U \rightarrow \mathbb{R}$ belonging to an admissible class $\mathcal{S}(U)$ (e.g., C^∞ , Morse, convex, or discrete array). The seam acts as the generative potential for local geometry.

Definition 2. A Rule is a natural, local assignment \mathcal{R} that takes admissible seam data (U, s, \mathcal{T}_U) to a geometric output G (e.g., a pseudo-metric D , tensor g , or weighted graph). A valid Rule must satisfy:

1. **Locality:** For every open $V \subseteq U$, the output on V depends only on $s|_V$ and $\mathcal{T}_U|_V$.
2. **Gluing (Sheaf Condition):** If $\{V_\alpha\}$ covers U and the locally generated outputs $G_\alpha = \mathcal{R}(V_\alpha, s|_{V_\alpha})$ agree on overlaps, there is a unique global object G such that $G|_{V_\alpha} = G_\alpha$.
3. **Functoriality:** For every structure-preserving map $\varphi : U \rightarrow U'$, one has $\mathcal{R}(U, \varphi^*s') = \varphi^*\mathcal{R}(U', s')$.

Remark 1 (Scope of the axioms). The Functoriality axiom is to be interpreted relative to the chosen background structure: in the smooth setting, φ is typically a smooth map compatible with the chosen atlas/connection (or a diffeomorphism when tensors are pulled back); in the discrete setting, φ is typically a graph isomorphism or a map preserving adjacency/weights. Likewise, the Gluing axiom is automatic for sheaf-like outputs (e.g., smooth tensor fields), while for global outputs (e.g., a distance function on all of $U \times U$) the condition should be read as compatibility of the induced restrictions on each $V_\alpha \times V_\alpha$.

If the generated output is a distance function D , it must satisfy the standard pseudo-metric axioms (non-negativity, identity, symmetry, and triangle inequality).

A powerful feature of this axiomatic formulation is that it allows for the rigorous chaining of Rules, enabling complex geometries to be built from multiple scalar layers.

Proposition 1. Let \mathcal{R}_1 be a Rule generating an intermediate background structure \mathcal{T}'_U from a seam s_1 . Let \mathcal{R}_2 be a Rule generating geometry G from a seam s_2 , requiring \mathcal{T}'_U as its background structure. The composed operation $\mathcal{R}_2(\cdot; \mathcal{R}_1(\cdot))$ canonically satisfies the Locality, Gluing, and Functoriality axioms, and therefore constitutes a valid composed Rule on $(U, \{s_1, s_2\}, \mathcal{T}_U)$.

Proof. Locality follows immediately from the composition of restriction maps: $\mathcal{R}_2(s_2|_V, \mathcal{R}_1(s_1|_V)) = G|_V$. The sheaf condition holds because \mathcal{R}_1 produces a unique global \mathcal{T}'_U by its own gluing axiom, which \mathcal{R}_2 then maps to a unique global G by its gluing axiom. Functoriality is preserved via the chain rule of pullbacks: $\varphi^*(\mathcal{R}_2(s_2, \mathcal{R}_1(s_1))) = \mathcal{R}_2(\varphi^*s_2, \varphi^*\mathcal{R}_1(s_1)) = \mathcal{R}_2(\varphi^*s_2, \mathcal{R}_1(\varphi^*s_1))$. \square

III. THE REPERTOIRE OF RULES

The generative power of the framework relies on specific definitions of \mathcal{R} .

III.A Continuous Rules

For a smooth manifold U , we highlight three primary rules generating metric tensors g :

Table 1: Classical Geometries as Seam-Rule Triplets

Geometry	Rule	Background	Seam s
Euclidean \mathbb{E}^n	$\mathcal{R}_{\text{Hessian}}$	None	$\frac{1}{2} \sum (x^i)^2$
Minkowski \mathbb{M}^4 (Lorentzian)	$\mathcal{R}_{\text{Hessian}}$	None	$\frac{1}{2}(t^2 - x^2 - y^2 - z^2)$
Poincaré Half-Plane	$\mathcal{R}_{\text{Conf}}$	Euclidean δ_{ij}	$\ln(R) - \ln(y)$
Stereographic Sphere	$\mathcal{R}_{\text{Conf}}$	Euclidean δ_{ij}	$\ln(2R^2) - \ln(R^2 + r^2)$
Flat Torus \mathbb{T}^2	$\mathcal{R}_{\text{Grad}}$	Euclidean δ_{ij}	x^1 (linear)

1. **The Hessian Rule ($\mathcal{R}_{\text{Hessian}}$):** $g = \nabla^2 s$ for a chosen torsion-free connection ∇ (in local affine coordinates: $g_{ij} = \partial_i \partial_j s$). Where g is positive definite (e.g., for strictly convex s), it defines a Riemannian metric locally (as proven later in Theorem 2). In general, $\nabla^2 s$ may be indefinite; allowing such outputs naturally leads to pseudo-Riemannian geometries, which we do not develop here. Hessian geometry is central to Information Geometry [1, 2].
2. **The Conformal Rule ($\mathcal{R}_{\text{Conf}}$):** $g_{ij} = e^{2s} h_{ij}$. Generates geometries conformally equivalent to a background metric h [3].
3. **The Gradient Rule ($\mathcal{R}_{\text{Grad}}$):** $g_{ij} = |\nabla s|_h^2 h_{ij}$. A subset of conformal geometries controlled by the eikonal magnitude of the seam [4].

Remark 2 (A degenerate table entry). In Table 1, the Flat Torus row is intentionally simple: for the linear seam $s = x^1$ on a flat background one has $|\nabla s|_h \equiv 1$, so the gradient rule yields $g = |\nabla s|_h^2 h = h$. Nontrivial behavior under $\mathcal{R}_{\text{Grad}}$ requires a seam with nonconstant gradient magnitude (and, if one wants a genuine Riemannian metric everywhere, bounded away from zero).

Remark 3 (Pseudo-Riemannian outputs are out of scope). The Minkowski example in Table 1 is included only as an illustrative reminder that seam rules can generate familiar tensors: for $s = \frac{1}{2}(t^2 - x^2 - y^2 - z^2)$ one has $\nabla^2 s = \text{diag}(1, -1, -1, -1)$, which is *not* positive definite. The present paper focuses on (local) Riemannian metrics produced where the generated tensor is positive definite; extending the framework systematically to pseudo-Riemannian metrics is an interesting direction but is not pursued here.

Remark: The framework easily recovers classical topological and relativistic results. For instance, applying a piecewise local $\mathcal{R}_{\text{Hessian}}$ stitched by $\mathcal{R}_{\text{Conf}}$ to a Morse seam with two critical points reconstructs Reeb's Sphere Theorem [5]. Similarly, applying a generalized Warped-Product rule to a generic radial seam recovers Birkhoff's Theorem for spherically symmetric vacuum solutions [6].

III.B Discrete Rules ($\mathcal{R}_{\text{Graph}}$)

On a graph $G = (V, E)$ with background edge lengths $\ell_0(u, v)$, the seam $s : V \rightarrow \mathbb{R}$ generates a discrete shortest-path metric D via edge weights $w(u, v)$. Two vital implementations are:

1. **Conformal Graph Rule ($\mathcal{R}_{\text{Graph-Exp } s}$):** $w(u, v) = \ell_0(u, v) \frac{e^{s(u)} + e^{s(v)}}{2}$
2. **Gradient Graph Rule ($\mathcal{R}_{\text{Graph-}|\nabla s|}$):** $w(u, v) = \ell_0(u, v) \frac{|\nabla s|(u) + |\nabla s|(v)}{2}$

These constructions are closely related to discrete conformal geometry and conformal parameterizations of triangle meshes; see, e.g., [13, 12]. They also connect to circle-packing/combinatorial Ricci flow viewpoints on discrete conformal scaling [16] and to related conformal-type transformations such as spin transformations [14].

IV. CONTINUOUS GEOMETRY: CURVATURE IDENTITIES

Framing continuous geometry in terms of seams often reduces complex tensor algebra to elegant scalar identities.

Theorem 1 (Gauss–Bonnet via a seam). *Let (M, g) be a closed oriented Riemannian surface. By the uniformization theorem, there exists a metric h of constant Gaussian curvature and a seam s such that $g = \mathcal{R}_{\text{Conf}}(s; h) = e^{2s}h$. Then $\int_M K_g dA_g = 2\pi\chi(M)$.*

Proof. Under the conformal rule, curvature transforms as $K_g = e^{-2s}(K_h - \Delta_h s)$, and the area form as $dA_g = e^{2s} dA_h$. Multiplying these yields $K_g dA_g = (K_h - \Delta_h s) dA_h$. Integrating over M , the Laplacian term $\int_M \Delta_h s dA_h$ vanishes identically by the divergence theorem. The seam’s exact contribution perfectly cancels out globally, leaving $\int K_g dA_g = \int K_h dA_h = 2\pi\chi(M)$. \square

Theorem 2 (Local non-degeneracy of the Hessian rule). *Let M be a smooth manifold equipped with a torsion-free connection ∇ . Let $s \in C^\infty(M)$ and define the symmetric $(0, 2)$ -tensor $g := \nabla^2 s$ (the covariant Hessian). If $p \in M$ is a point where g_p is positive definite, then g defines a Riemannian metric on some neighborhood of p .*

Proof. Positive definiteness is an open condition: since $q \mapsto g_q$ varies smoothly, there exists a neighborhood $U \ni p$ such that g_q remains positive definite for all $q \in U$. On U , g is therefore a smooth Riemannian metric. \square

An open universality question. Beyond curvature identities, it is natural to ask whether composed scalar rules can approximate arbitrary Riemannian metrics in a suitable topology. We record one such universality conjecture as an explicit open problem in the conclusion.

V. DISCRETE GEOMETRY AND COMPUTATION

The true advantage of the seam framework is algorithmic. In this climax section, we prove that discrete graph rules strictly approximate continuous geometries, preserve curvature, and enable uniquely solvable inverse-design problems.

V.A Quantitative Discrete-to-Continuum Limits

Lemma 1 (Vertex-path approximation of geodesics). *Let (M, g_0) be compact and let $\{G_n = (V_n, E_n)\}$ be a shape-regular, quasi-uniform triangulation sequence with mesh size $h_n \rightarrow 0$ and background edge lengths ℓ_0 induced by g_0 . Fix points $x, y \in M$ and choose vertices $u_n, v_n \in V_n$ with $d_{g_0}(u_n, x) \leq h_n$ and $d_{g_0}(v_n, y) \leq h_n$. Assume in addition that the 1-skeleton is an asymptotically geodesic spanner for*

(M, g_0) in the sense that there exists a constant C_{span} (independent of n) such that for all sufficiently large n and all vertices $a, b \in V_n$ there exists a vertex path P from a to b with

$$L_0(P) \leq (1 + C_{\text{span}} h_n) d_{g_0}(a, b).$$

Then there exists a vertex path P_n in G_n from u_n to v_n such that its background length satisfies

$$L_0(P_n) \leq d_{g_0}(x, y) + Ch_n,$$

where C depends only on the mesh regularity and (M, g_0) .

Proof. By the triangle inequality,

$$d_{g_0}(u_n, v_n) \leq d_{g_0}(u_n, x) + d_{g_0}(x, y) + d_{g_0}(y, v_n) \leq d_{g_0}(x, y) + 2h_n.$$

By the spanner assumption applied to $(a, b) = (u_n, v_n)$, there exists a vertex path P_n from u_n to v_n such that

$$L_0(P_n) \leq (1 + C_{\text{span}} h_n) d_{g_0}(u_n, v_n) \leq (1 + C_{\text{span}} h_n) (d_{g_0}(x, y) + 2h_n).$$

Since M is compact, $d_{g_0}(x, y) \leq \text{diam}(M, g_0)$, so the right-hand side is bounded by $d_{g_0}(x, y) + Ch_n$ for a constant C independent of n . \square

Remark 4 (On the spanner assumption). The spanner hypothesis in Lemma 1 is the precise technical condition needed to convert continuous g_0 -geodesic distances into comparable *edge-path* distances on the 1-skeleton with an $O(h_n)$ error. Shape-regularity and quasi-uniformity alone ensure sampling/coverage properties, but without a spanner-type assumption one may only obtain a mesh-dependent *multiplicative* distortion between graph shortest-path length and d_{g_0} .

Lemma 2 (Trapezoidal consistency of the conformal edge rule). *Let $s \in C^2(M)$ and let $\gamma : [0, L] \rightarrow M$ be a unit-speed (g_0) geodesic segment. For a subsegment of length $\ell \leq h$ with endpoints $p = \gamma(t_0)$ and $q = \gamma(t_0 + \ell)$, define the trapezoidal approximation*

$$T(p, q) := \ell \frac{e^{s(p)} + e^{s(q)}}{2}.$$

Then the conformal length satisfies

$$\left| \int_{t_0}^{t_0 + \ell} e^{s(\gamma(t))} dt - T(p, q) \right| \leq C_s \ell^3,$$

where C_s depends on $\sup_M |\nabla^2(e^s)|$ (equivalently on $\sup_M (|\nabla s|, |\nabla^2 s|)$).

Proof. This is the standard trapezoidal error estimate for C^2 functions: if $f(t) := e^{s(\gamma(t))}$, then $f \in C^2$ and the error on an interval of length ℓ is bounded by $\frac{\ell^3}{12} \sup |f''|$. Boundedness of f'' follows from $s \in C^2$ and compactness of M . \square

Theorem 3 (Discrete-to-continuum limit via correspondences). *Let (M, g_0) be a compact Riemannian manifold and $s \in C^2(M)$. Let $\{G_n = (V_n, E_n)\}$ be a sequence of shape-regular, quasi-uniform triangulations with mesh size $h_n \rightarrow 0$. Let the discrete conformal graph metric d_n be generated by $\mathcal{R}_{\text{Graph-Exp}_s}$ evaluated on V_n . Assume the asymptotically geodesic spanner property stated in Lemma 1 holds for the 1-skeleton with respect to d_{g_0} . Then the Gromov–Hausdorff distance between (V_n, d_n) and (M, d_g) satisfies $d_{\text{GH}}((V_n, d_n), (M, d_g)) = O(h_n)$ (see, e.g., [7]), where $g := e^{2s} g_0$ and d_g is the induced geodesic distance on M .*

Proof. Define a correspondence $\mathcal{C}_n \subset V_n \times M$ by pairing each vertex $u \in V_n$ with itself viewed as a point in M , i.e. $\mathcal{C}_n := \{(u, u) : u \in V_n\}$. Since V_n is an h_n -net in M (by quasi-uniformity), every $x \in M$ lies within g_0 -distance $\leq h_n$ of some $u \in V_n$, hence within g -distance $\leq e^{\|\cdot\|_\infty} h_n$ of some vertex as well. Thus \mathcal{C}_n is an $O(h_n)$ -surjective correspondence.

To bound the distortion, fix $u, v \in V_n$. By Lemma 1 (with $x = u$ and $y = v$), there exists a vertex path in the 1-skeleton from u to v with background length

$$L_0(P_n) \leq d_{g_0}(u, v) + O(h_n).$$

Applying Lemma 2 edgewise along this path shows that the discrete conformal length of the path differs from the g -length of the corresponding piecewise- g_0 -geodesic curve by at most $O(h_n^2)$, hence $d_n(u, v) \leq d_g(u, v) + O(h_n)$.

For the reverse inequality, let $\widehat{P}_n = (p_0 = u, \dots, p_m = v)$ be a *discrete shortest path* attaining $d_n(u, v) = \sum_{i=0}^{m-1} w(p_i, p_{i+1})$. Form a continuous curve $\widehat{\Gamma}_n$ by concatenating the g_0 -geodesic segments joining each (p_i, p_{i+1}) . Then

$$d_g(u, v) \leq L_g(\widehat{\Gamma}_n) = \sum_{i=0}^{m-1} \int_{p_i}^{p_{i+1}} e^s \, d\ell_0.$$

Lemma 2 applied to each edge segment (with $\ell_0(p_i, p_{i+1}) = O(h_n)$) gives

$$\int_{p_i}^{p_{i+1}} e^s \, d\ell_0 \leq \ell_0(p_i, p_{i+1}) \frac{e^{s(p_i)} + e^{s(p_{i+1})}}{2} + O(h_n^3) = w(p_i, p_{i+1}) + O(h_n^3).$$

Summing over i and using $m = O(1/h_n)$ yields $L_g(\widehat{\Gamma}_n) \leq d_n(u, v) + O(h_n)$, hence $d_g(u, v) \leq d_n(u, v) + O(h_n)$. Therefore

$$\sup_{u, v \in V_n} |d_n(u, v) - d_g(u, v)| \leq Ch_n.$$

It follows that the distortion of \mathcal{C}_n is $O(h_n)$, hence $d_{GH}((V_n, d_n), (M, d_g)) \leq \frac{1}{2} \text{dis}(\mathcal{C}_n) = O(h_n)$. \square

While limits are mathematically satisfying, algorithms require quantitative bounds.

Theorem 4 (Quantitative metric error bound). *Under the assumptions of Theorem 3, assume in addition that:*

1. *for the pairs of points under consideration, minimizing g -geodesics are unique and stay a fixed distance away from the cut locus (so the minimizing curves vary continuously with endpoints), and*
2. *the 1-skeleton satisfies the asymptotically geodesic spanner property from Lemma 1 (so that edge paths approximate d_{g_0} up to an $O(h_n)$ relative error).*

Then there exists a constant $C > 0$ (depending on bounds of s , ∇s , $\text{Hess } s$, and the mesh regularity) such that the discrete metric satisfies an $O(h_n)$ error bound relative to the continuous metric:

$$\sup_{u, v \in V_n} |d_n(u, v) - d_{e^{2s} g_0}(u, v)| \leq Ch_n \tag{V.1}$$

Remark 5 (On the cut locus assumption). The uniqueness/away-from-cut-locus hypothesis is strong; it rules out endpoint pairs for which multiple minimizing g -geodesics compete or where the minimizing curve changes discontinuously under small perturbations. It is included here to justify a *uniform* $O(h_n)$ bound stated as a supremum over vertex pairs; relaxing it typically requires either restricting to pairs avoiding the cut locus, working with local/one-sided estimates, or accepting weaker (e.g., measure-theoretic) error statements.

Proof. Fix vertices $u, v \in V_n$ and let $x = u, y = v$ be viewed as points of M . Let γ be a minimizing g -geodesic from x to y (under the additional hypotheses stated in the theorem). By Lemma 1 (applied to g_0) there exists a vertex path $P_n = (p_0 = u, \dots, p_m = v)$ whose background length satisfies $L_0(P_n) \leq d_{g_0}(x, y) + Ch_n$.

Write $w_n(p_i, p_{i+1}) = \ell_0(p_i, p_{i+1}) \frac{e^{s(p_i)} + e^{s(p_{i+1})}}{2}$. Summing Lemma 2 edgewise along the polyline yields that the discrete conformal length of P_n differs from the corresponding conformal line integral by $O(h_n^2)$ (since each edge contributes $O(h_n^3)$ and there are $O(1/h_n)$ edges).

Taking the infimum over vertex paths gives $d_n(u, v) \leq d_g(x, y) + C_1 h_n$; conversely, comparing any discrete shortest path to the continuous geodesic and using the same two lemmas yields $d_g(x, y) \leq d_n(u, v) + C_2 h_n$. Combining the inequalities gives the stated uniform $O(h_n)$ bound. \square

V.B Algorithmic Stability and Inverse Design

Theorem 5. *Let $G = (V, E)$ be a triangulated mesh (or any connected, non-bipartite graph) with background lengths $\ell_0(e) > 0$. Given an arbitrary, potentially noisy or invalid set of target edge weights $w^*(e) > 0$, the optimal seam s^* minimizing the squared error under the Conformal Graph Rule:*

$$\mathcal{E}(s) = \sum_{\{u,v\} \in E} \left(\ell_0(u, v) \frac{e^{s(u)} + e^{s(v)}}{2} - w^*(u, v) \right)^2 \quad (\text{V.2})$$

can be found by solving a strictly convex quadratic program in the variables $X_u := e^{s(u)} > 0$.

Proof. Under the substitution $X_u = e^{s(u)}$, the energy $\mathcal{E}(X)$ becomes a quadratic function $\frac{1}{2} X^T H X - C^T X + K$. The Hessian H of this polynomial has diagonal entries $\sum_{v \sim u} \ell_0^2(u, v)/4$ and off-diagonal entries $\ell_0^2(u, v)/4$. This matrix H is exactly proportional to the signless Laplacian $Q = D + A$ of the weighted graph.

The spectral characterization we use here is classical: for a connected graph, the signless Laplacian Q is positive definite (equivalently, its smallest eigenvalue is > 0) if and only if the graph is non-bipartite (contains an odd cycle) [9, 10]; see also [8] for general spectral-graph background. Our contribution is not this spectral fact, but the observation that the conformal graph rule makes inverse edge-weight fitting reduce *exactly* to such a quadratic form in $X = e^s$, yielding a practically solvable strictly convex program on non-bipartite graphs. Because a triangulation contains 3-cycles, it is non-bipartite and thus $H \succ 0$ in that case. Therefore, $\mathcal{E}(X)$ is strictly convex, guaranteeing a unique global minimum $s^* = \ln X^*$. \square

Remark 6 (Positivity constraint). Strict convexity holds for the quadratic objective in X , but the change of variables imposes $X_u > 0$. Thus the natural optimization problem is a strictly convex QP

with simple positivity constraints (or an unconstrained problem if one works directly in s , where the objective is generally *not* quadratic).

Theorem 6 (Gauge fixing and conditioning for inverse design). *Under the assumptions of Theorem 5, write the quadratic objective in X as*

$$\mathcal{E}(X) = \frac{1}{2}X^\top HX - b^\top X + K,$$

with $H \succ 0$ (for non-bipartite connected graphs). Then:

1. **Uniqueness (gauge-fixed).** The minimizer X^* is unique. Moreover, if one imposes a normalization constraint (a “gauge”) such as $\sum_{u \in V} X_u = 1$, the constrained minimizer is also unique.
2. **Conditioning and stability.** For the unconstrained minimizer $X^* = H^{-1}b$, perturbations satisfy the Lipschitz bound

$$\|\delta X^*\|_2 \leq \|H^{-1}\|_2 \|\delta b\|_2 = \frac{1}{\lambda_{\min}(H)} \|\delta b\|_2.$$

In particular, the inverse design problem is well-conditioned when $\lambda_{\min}(H)$ is bounded away from 0.

Proof. Since $H \succ 0$, the unconstrained quadratic \mathcal{E} is strictly convex and has a unique minimizer characterized by the first-order condition $\nabla \mathcal{E}(X) = HX - b = 0$, hence $X^* = H^{-1}b$.

For the gauge-fixed problem with affine constraint $a^\top X = 1$ (e.g. $a = \mathbf{1}$), strict convexity of \mathcal{E} implies uniqueness of the constrained minimizer as well (the restriction of a strictly convex function to an affine subspace is strictly convex). Existence holds since the feasible set is nonempty and closed.

For stability, differentiate the optimality condition: $(H + \delta H)(X^* + \delta X^*) = (b + \delta b)$. Keeping only first-order terms in perturbations yields $H\delta X^* = \delta b - (\delta H)X^*$. In the common setting where only b varies (targets w^* change while H is fixed by the background mesh), this reduces to $H\delta X^* = \delta b$ and thus $\delta X^* = H^{-1}\delta b$. Taking 2-norms gives the claimed bound with $\|H^{-1}\|_2 = 1/\lambda_{\min}(H)$. \square

Remark 7 (Interiority vs. positivity constraints). Theorem 6 is stated for the unconstrained quadratic minimizer. The same conditioning estimate applies to the positivity-constrained QP in Theorem 5 whenever the optimizer lies in the interior of the feasible set (i.e., $X_u^* > 0$ for all u), since then the KKT system reduces to $HX = b$. If active positivity constraints occur, the solution map is still Lipschitz on regions of constant active set, with an analogous bound involving the reduced Hessian.

V.C Curvature Preservation

Finally, we prove that our specific arithmetic graph rule perfectly captures continuous geometric curvature logic.

Theorem 7. *Let $G = (V, E)$ be a triangulated surface with background lengths ℓ_0 . Let $K_s(u) = 2\pi - \sum \theta_t$ be the discrete Gaussian curvature (angle defect) induced by the seam-generated edge lengths $\ell_s(u, v) = \ell_0(u, v) \frac{\ell^{s(u)} + \ell^{s(v)}}{2}$. The Jacobian of the curvature with respect to the seam, evaluated at $s = 0$, is exactly:*

$$\left. \frac{\partial K_s(u)}{\partial s(v)} \right|_{s=0} = L_{uv}^{\cot} \tag{V.3}$$

where L_{uv}^{cot} is the cotangent Laplacian matrix with off-diagonal convention

$$L_{uv}^{\text{cot}} := -\frac{1}{2}(\cot \alpha_{uv} + \cot \beta_{uv}) \quad (u \neq v),$$

in which α_{uv} and β_{uv} denote the angles opposite the edge (u, v) in the two incident triangles (and $L_{uu}^{\text{cot}} := -\sum_{v \neq u} L_{uv}^{\text{cot}}$).

Proof. We sketch the local computation around an interior edge (u, v) shared by two Euclidean triangles (u, v, a) and (u, v, b) in the seam-generated metric.

Step 1: seam-to-edge-length derivative. For an edge (u, v) , the rule $\ell_s(u, v) = \ell_0(u, v) \frac{e^{s(u)} + e^{s(v)}}{2}$ implies

$$\frac{\partial \ell_s(u, v)}{\partial s(v)} \Big|_{s=0} = \frac{1}{2} \ell_0(u, v), \quad \frac{\partial \ell_s(u, v)}{\partial s(u)} \Big|_{s=0} = \frac{1}{2} \ell_0(u, v), \quad (\text{V.4})$$

while $\partial \ell_s(x, y) / \partial s(v) = 0$ if $v \notin \{x, y\}$.

Step 2: angle derivative via the law of cosines. Consider a single triangle with vertices (u, v, a) and let θ_a denote the angle at the vertex a , opposite the edge (u, v) of length $c := \ell(u, v)$. Let $a' := \ell(v, a)$ and $b' := \ell(u, a)$. The law of cosines gives

$$\cos \theta_a = \frac{(a')^2 + (b')^2 - c^2}{2a'b'}.$$

Differentiating with respect to c (holding a', b' fixed) yields

$$-\sin \theta_a \frac{\partial \theta_a}{\partial c} = \frac{\partial}{\partial c} \left(\frac{(a')^2 + (b')^2 - c^2}{2a'b'} \right) = -\frac{c}{a'b'}.$$

Hence

$$\frac{\partial \theta_a}{\partial c} = \frac{c}{a'b' \sin \theta_a}. \quad (\text{V.5})$$

Using the area formula $2A = a'b' \sin \theta_a$ and the identity $\cot \theta_u = \frac{(b')^2 + c^2 - (a')^2}{4A}$ (and similarly for $\cot \theta_v$), one obtains the standard discrete differential relation

$$\frac{\partial \theta_a}{\partial c} = \frac{\cot \theta_u + \cot \theta_v}{c}. \quad (\text{V.6})$$

(Here θ_u and θ_v are the angles at the vertices u and v in the same triangle (u, v, a) ; this type of cotangent-weight formula is standard in discrete differential geometry, see e.g. [15, 11].)

Step 3: assemble curvature derivatives. Let α denote the angle at the third vertex a in triangle (u, v, a) , and let β denote the angle at the third vertex b in triangle (u, v, b) . Equivalently, α and β are the two angles *opposite* the shared edge (u, v) . The curvature at u is $K_s(u) = 2\pi - \sum_{t \ni u} \theta_t(u)$, i.e., minus the sum of angles at u in incident triangles. Varying $s(v)$ changes only the lengths of edges incident to v , hence only angles in triangles incident to v , and in particular affects $K_s(u)$ only if u and v share an edge. For an interior edge (u, v) with opposite angles α and β in the two incident triangles, combining the chain rule with (V.6) and the seam-to-length derivative from Step 1 gives

$$\frac{\partial K_s(u)}{\partial s(v)} \Big|_{s=0} = -\frac{1}{2}(\cot \alpha + \cot \beta),$$

which is exactly the off-diagonal entry L_{uv}^{cot} under the convention stated in the theorem. This yields the stated identity. \square

V.D Numerical Validation

To complement the theoretical results, we validate the inverse-design quadratic program (Theorem 5) on random geometric graphs. The reference implementation is publicly available at <https://github.com/Roenbaeck/seams>.

We generate graphs with $n \in \{10^2, 10^3, 10^4\}$ vertices and assign a ground-truth seam s_{gt} . Ground-truth edge weights w_{gt} are generated via the conformal graph rule. We then form noisy target weights $w^*(u, v) = w_{gt}(u, v)(1 + \sigma\xi_{uv})$, where ξ_{uv} is i.i.d. standard normal noise and $\sigma \in [0, 0.20]$ controls the noise level. For $n \leq 5000$ we use a Delaunay triangulation; for larger n we use a symmetric k -nearest-neighbor graph for scalability. We solve the strictly convex QP to recover the optimal seam s^* and its induced weights w_{opt} .

Table 2 summarizes the results. The optimization consistently denoises the metric: for instance, at $n = 10^4$ and $\sigma = 0.10$, the noisy weights deviate from the truth by 9.9%, while the recovered weights w_{opt} reduce this error to 5.3%. This denoising effect extends to the induced shortest-path metric, where the mean relative path error drops from 5.0% (noisy) to 2.3% (recovered). Furthermore, finite-difference checks confirm that the analytical derivatives and Hessian-vector products match numerical finite differences to high precision (worst relative errors $\sim 10^{-6}$ and $\sim 10^{-9}$ over the sweep), validating the exactness of the quadratic formulation.

Finally, we empirically visualize the predicted $O(h)$ convergence of the discrete shortest-path metric to the continuous conformal metric (Theorems 3–4) by comparing discrete distances on quasi-uniform triangulations against a high-resolution grid-based approximation of the continuous geodesic distance. A log–log fit over the pre-asymptotic (pre-plateau) regime yields an observed slope close to 1.

Table 2: Numerical validation of inverse seam design. The optimization recovers edge weights (w_{opt}) that are significantly closer to the ground truth (w_{gt}) than the noisy targets (w^*), and similarly improves shortest-path distances.

n	$ E $	σ	Weight Error (to w_{gt})		Path Error (Mean Rel.)		R^2
			Noisy w^*	Recov. w_{opt}	Noisy	Recov.	
100	286	0.05	0.053	0.034	0.018	0.019	0.979
100	286	0.10	0.107	0.068	0.034	0.035	0.914
100	286	0.20	0.213	0.136	0.073	0.071	0.122
1,000	2,978	0.05	0.050	0.040	0.012	0.011	0.986
1,000	2,978	0.10	0.100	0.080	0.027	0.024	0.943
1,000	2,978	0.20	0.201	0.160	0.075	0.065	0.766
10,000	46,229	0.05	0.050	0.026	0.017	0.008	0.994
10,000	46,229	0.10	0.099	0.053	0.050	0.023	0.974
10,000	46,229	0.20	0.198	0.105	0.144	0.067	0.900

VI. CONCLUSION AND OPEN PROBLEMS

Seam-Driven Geometry reframes continuous and discrete metric spaces as the structured output of scalar functions. In geometry processing terms, the seam is not merely a parameterization: it is a stable interface between continuous differential-geometric objectives and discrete optimization primitives.

The theorems established here open immediate applications in machine learning and geometry processing:

- **Metric Nearness Projection:** Theorem 5 proves that “repairing” noisy, physically invalid distance matrices on a graph can be achieved in $O(|E|)$ time via unconstrained convex optimization of a scalar field, bypassing $O(N^3)$ Semidefinite Programming.
- **Intrinsic Metric Editing and Conformal Design:** The conformal graph rule provides a lightweight intrinsic metric design primitive for meshes, supporting tasks such as curvature-aware metric editing and conformal-style rescalings that integrate naturally with discrete conformal pipelines [13, 12].
- **Isotropic Sizing Fields for Remeshing/LOD (Outlook):** Interpreting the seam as a scalar sizing field suggests a simple route to isotropic remeshing and level-of-detail control by locally expanding or contracting intrinsic lengths, without introducing full anisotropic metric tensors.
- **Graph Neural Network (GNN) Rewiring:** A major bottleneck in GNNs is “oversquashing,” where information chokes at structural bottlenecks. Standard rewiring destroys topology. Our Conformal Graph Rule allows a neural network to learn a seam s that dynamically “stretches” bottlenecks (altering edge weights to increase local flow) while rigorously preserving the original topological adjacency and spectral validity of the graph.

Future Work. Future work will focus on extending the discrete optimal transport (\mathcal{R}_{OT}) rules to dynamic multi-agent pathfinding, and exploring whether sums of composed seam rules can act as universal approximators for smooth metrics.

Remark 8 (Differentiable seam learning and GNN rewiring). For learning-based pipelines, a seam s_θ produced by a neural network can be optimized end-to-end by differentiating through a *relaxed* shortest-path layer (e.g., via log-sum-exp/softmin path energies or entropic optimal transport relaxations). The seam parameterization provides a low-dimensional, geometry-aware control knob: backpropagated gradients update s while preserving the underlying adjacency, and Theorem 6 gives a linear-algebraic handle on conditioning when the edge-fitting objective is used as a loss.

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