Generating Geometries from Scalar Seams via Interpretive Rules

Lars Rönnbäck

Stockholm University
lars.ronnback@anchormodeling.com

Abstract

This paper introduces a framework for generating geometric structures. We propose that the geometry of a space U can be derived from a scalar function $s : U \to \mathbb{R}$, termed the 'seam', when interpreted through a well-defined 'Rule' $\mathcal{R}$. The Rule dictates how the seam determines geometric properties, primarily the distance function D on U. We formalize the minimal requirements for such Rules and argue that the triplet $(U, s, \mathcal{R})$ provides a potentially unifying approach to describing diverse geometric spaces, including pseudo-Riemannian and non-standard structures. The framework emphasizes the generative power of the seam function when coupled with a specific interpretive Rule.

I. Introduction

Standard mathematical frameworks describe geometry through structures like metric spaces (X, D) or Riemannian manifolds (M, g), where the distance function D or the metric tensor g is postulated as fundamental. This paper explores an alternative approach where geometric structure is not postulated directly, but rather generated from a more primitive object: a scalar field defined on a base space.

We propose that a geometry can arise from the combination of a base set U, potentially possessing some inherent structure (e.g., topological, differential, order), and a scalar function $s : U \to \mathbb{R}$, which we term the *seam*. The seam encodes information that, when interpreted through a specific mechanism, "stitches" the space together, defining its geometric properties. Crucially, the interpretation requires a third component: a well-defined *Rule*, denoted $\mathcal{R}$, which maps the seam s and the structure of U to a specific geometric realization, typically characterized by a distance function $D : U \times U \to \mathbb{R}$.

The central idea is that the triplet $(U, s, \mathcal{R})$ forms a complete system for generating a geometry. This perspective offers potential advantages:

- **Generativity:** Simple seam functions s might generate complex and varied geometries under a fixed Rule $\mathcal{R}$.
- **Unification:** Different geometries (e.g., Euclidean, Minkowski, spherical, hyperbolic) might be realized by varying s within a single framework $(U, \mathcal{R})$.
- **Foundation:** It connects geometry directly to scalar fields, which are ubiquitous in physical theories.
- **Flexibility:** Allows for the exploration of geometries on spaces U that are not standard manifolds, or arising from non-smooth seams s, depending on the nature of the Rule $\mathcal{R}$.

This paper will first formalize the components of this framework, establishing the minimal requirements for a valid Rule $\mathcal{R}$. We will then discuss different classes of Rules and illustrate the framework by exploring the geometries generated by specific choices of $(U, s, \mathcal{R})$, demonstrating the capability to reproduce known geometries and potentially generate novel ones.

II. Framework: Seams and Rules

Let U be a base set, representing the points of the space under consideration. U may possess some inherent structure $\mathcal{T}_U$, such as being a product space $X_1 \times \cdots \times X_n$, where components X_i might be equipped with standard structures (e.g., order relation for $\mathbb{N}$, differentiable structure for $\mathbb{R}$).

i. Seam Definition

Definition i.1 (Seam)**.** *A* seam *on U is a function $s : U \to \mathbb{R}$. Depending on the Rule used, the seam s may be required to belong to a specific class of functions $\mathcal{S}(U)$ (e.g., continuous, differentiable, convex).*

ii. Rule Definition

Definition ii.1 (Rule)**.** *A* Rule *$\mathcal{R}$ is a well-defined procedure that, given the base set U with its inherent structure $\mathcal{T}_U$ and an admissible seam function $s \in \mathcal{S}(U)$, generates a distance function $D : U \times U \to \mathbb{R}$, denoted $D = \mathcal{R}(s; U, \mathcal{T}_U)$.*

The concept of deriving geometry from scalar fields has precedents in physics, particularly in Kaluza-Klein theory [20, 21] and scalar-tensor theories [19].

For a Rule $\mathcal{R}$ to be considered geometrically meaningful in the context of defining distances, it must satisfy certain minimal requirements related to the properties of the generated function D.

Requirement ii.1.1 (Minimal Requirements for a Rule)**.** *A Rule $\mathcal{R}$ must ensure that for any admissible seam $s \in \mathcal{S}(U)$, the resulting $D = \mathcal{R}(s; U, \mathcal{T}_U)$ is a pseudo-metric on U, satisfying:*

(M1) Non-negativity: $D(u, v) \geq 0$ for all $u, v \in U$.

(M2) Identity: $D(u, u) = 0$ for all $u \in U$.

(M3) Symmetry: $D(u, v) = D(v, u)$ for all $u, v \in U$.

(M4) Triangle Inequality: $D(u, w) \leq D(u, v) + D(v, w)$ for all $u, v, w \in U$.

The procedure must be unambiguous and D must generally depend on s.

The possibility that $D(u, v) = 0$ for $u \neq v$ is allowed by the pseudo-metric definition, accommodating potentially degenerate geometries where distinct points are identified. A stricter requirement might demand that $\mathcal{R}$ generates a true metric (where $D(u, v) = 0 \iff u = v$) for certain classes of non-degenerate seams s.

The choice of Rule $\mathcal{R}$ embodies the principle by which the seam s translates into geometric distance. Different rules correspond to different interpretations of the seam's meaning. Examples include:

- Rules deriving a metric tensor g from second derivatives of s (e.g., the Hessian Rule $\mathcal{R}_{\text{Hessian}}$ where $g_{ij} = \partial^2 s / \partial x^i \partial x^j$).
- Rules interpreting s as defining a conformal factor relative to a background metric (e.g., the Conformal Rule $\mathcal{R}_{\text{Conf}}$ where $g_{ij} = e^{2s} \delta_{ij}$).
- Rules using the magnitude of the gradient of s to scale an isotropic metric (e.g., the Gradient Rule $\mathcal{R}_{\text{Grad}}$ where $g_{ij} = |\nabla s|^2 \delta_{ij}$).
- Rules based on graph approximations, where edge weights are derived from s, and D is the limiting shortest path distance.

These specific rules will be explored in subsequent sections.

The outcome of applying a Rule $\mathcal{R}$ to (U, s) is the pseudo-metric space (U, D), which represents the geometry generated by the triplet $(U, s, \mathcal{R})$.

III. Examples of Rules and Generated Geometries

In this section, we will explore specific Rules $\mathcal{R}$ and examine the geometries (U, D) they generate for various choices of U and s. We focus first on rules applicable when U possesses a differentiable structure.

I. The Hessian Rule ($\mathcal{R}_{\text{Hessian}}$)

Perhaps the most direct way to generate a tensor structure from a scalar field in a differentiable setting is by considering its second derivatives.

Definition i.1 (Hessian Rule)**.** *Let U be a differentiable manifold equipped with local coordinates $\{x^i\}$. Let the class of admissible seams $\mathcal{S}(U)$ be the set of twice continuously differentiable functions (C^2) $s : U \to \mathbb{R}$. The* Hessian Rule*, denoted $\mathcal{R}_{Hessian}$, generates a symmetric tensor field g of type (0,2) with components in local coordinates given by:*

$$g_{ij}(u) = \frac{\partial^2 s}{\partial x^i \partial x^j}(u) \qquad \text{(III.1)}$$

If this tensor g is non-degenerate (i.e., $\det(g) \neq 0$*) and has a constant signature, it defines a pseudo-Riemannian metric on U. If g is also positive definite (*$\det(g) > 0$ *and appropriate signature), it defines a Riemannian metric. The distance function $D = \mathcal{R}_{Hessian}(s; U)$ is then defined as the standard geodesic distance associated with the metric g.*

The use of Hessian structures to define metrics has been extensively studied in information geometry [1, 8].

Note that this rule does not automatically guarantee a positive definite (Riemannian) metric or even a non-degenerate one; the properties of the generated tensor g depend entirely on the choice of the seam function s. If g is degenerate or changes signature, the resulting geometry may be non-standard or only defined piecewise. However, g always defines a symmetric bilinear form on each tangent space, and the framework can still be explored. Let's examine some outcomes.

i.1 Example: Euclidean Space

Let $U = \mathbb{R}^n$ with standard Cartesian coordinates $(x^1, \dots, x^n)$. Consider the quadratic seam:

$$s(x^1, \dots, x^n) = \frac{1}{2} \sum_{i=1}^{n} (x^i)^2$$

Applying the Hessian rule (III.1):

$$g_{ij} = \frac{\partial^2}{\partial x^i \partial x^j} \left(\frac{1}{2} \sum_{k=1}^{n} (x^k)^2 \right) = \delta_{ij}$$

This is the standard Euclidean metric tensor. The generated geometry (U, D) is n-dimensional Euclidean space $\mathbb{E}^n$.

i.2 Example: Minkowski Spacetime

Let $U = \mathbb{R}^4$ with coordinates (x^0, x^1, x^2, x^3). Consider the seam representing the squared interval:

$$s(x^0, x^1, x^2, x^3) = \frac{1}{2} \left[(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \right] \qquad \text{(III.2)}$$

Applying the Hessian rule (III.1), using the $(+, -, -, -)$ signature convention:

$$g_{\mu\nu} = \eta_{\mu\nu}$$

This yields the Minkowski metric tensor. The generated geometry is 4-dimensional Minkowski spacetime.

i.3 Example: Flat 2D Lorentzian Space

Let $U = \mathbb{R}^2$ with coordinates (x, y). Consider the seam $s(x, y) = xy$. Applying the Hessian rule (III.1):

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This corresponds to a flat 2D Lorentzian geometry.

i.4 Example: Curved Riemannian Geometry

Let $U = \mathbb{R}^2$ with coordinates (x, y). Consider the seam $s(x, y) = \frac{1}{4}(x^2 + y^2)^2$. The Hessian rule yields the metric:

$$g = \begin{pmatrix} 3x^2 + y^2 & 2xy \\ 2xy & x^2 + 3y^2 \end{pmatrix}$$

This metric is positive definite for $(x, y) \neq (0, 0)$ and indicates a curved Riemannian geometry.

i.5 Example: Spherically Symmetric Geometries

Let $U = \mathbb{R}^3$ and consider a spherically symmetric seam $s = f(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. The Hessian rule generates the metric:

$$ds^2 = f''(r)dr^2 + \frac{f'(r)}{r}(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$$

For this to match the standard form $ds^2 = A(r)dr^2 + r^2 B(r)(d\Omega^2)$, we need $A(r) = f''(r)$ and $B(r) = f'(r)/r$. This implies $B'(r) = (rf''(r) - f'(r))/r^2$, so $rB'(r) = f''(r) - f'(r)/r = A(r) - B(r)$, or $A(r) = B(r) + rB'(r)$. This is a constraint satisfied by flat space ($A = 1, B = 1 \implies s = r^2/2$) but not generally by other important metrics like Schwarzschild.

i.6 Example: Changing Signature

Let $U = \mathbb{R}^2$ with coordinates (x, y). Consider the seam $s(x, y) = \cos(x) + \frac{1}{2}y^2$. The Hessian rule yields $g = \begin{pmatrix} -\cos(x) & 0 \\ 0 & 1 \end{pmatrix}$. The geometry changes type depending on the sign of $\cos(x)$.

i.7 Summary for Hessian Rule

The Hessian Rule $\mathcal{R}_{\text{Hessian}}$ provides a direct link from a scalar potential s (C^2) to a metric tensor g. It naturally generates flat Euclidean and Minkowski spaces. It can generate curved geometries, both Riemannian and pseudo-Riemannian. However, it does not guarantee positive definiteness and seems unable to generate certain important geometries like those with constant non-zero curvature directly on $\mathbb{R}^n$. It defines a specific class of geometries whose metric tensor is the Hessian of a potential.

II. The Conformal Rule ($\mathcal{R}_{\text{Conf}}$)

Another natural way to relate a scalar field s to a metric g is by using s to define a local scaling, or conformal factor, relative to some pre-existing background metric on U.

Definition ii.1 (Conformal Rule)**.** *Let U be a differentiable manifold equipped with a background pseudo-Riemannian metric h_{ij}. Typically, for $U = \mathbb{R}^n$, h_{ij} is taken to be the standard Euclidean metric δ_{ij}. (It should be noted that this background metric h_{ij} is conceptually flexible; it could potentially be derived from another seam s_0 using a different rule, such as $\mathcal{R}_{Hessian}$.) Let the class of admissible seams $\mathcal{S}(U)$ be the set of sufficiently smooth (e.g., continuous or differentiable) functions $s : U \to \mathbb{R}$. The* Conformal Rule*, denoted $\mathcal{R}_{Conf}$, generates a pseudo-Riemannian metric tensor g given by:*

$$g_{ij}(u) = e^{2s(u)} h_{ij}(u) \tag{III.3}$$

The distance function $D = \mathcal{R}_{Conf}(s; U)$ is then defined as the standard geodesic distance associated with the metric g.

Conformal transformations and their geometric properties are well-established in differential geometry [24, 25].
In this rule, the seam s directly controls the local "stretching" factor e^s applied to the background metric h. If h is positive definite (Riemannian), then g will also be positive definite. The resulting geometry (U, g) is conformally equivalent to the background geometry (U, h).

ii.1 Example: Euclidean Space

Let $U = \mathbb{R}^n$ with $h_{ij} = \delta_{ij}$. To generate $g_{ij} = \delta_{ij}$, we need $e^{2s} = 1$, implying the trivial seam $s = 0$.

ii.2 Example: Spherical Geometry

Let $U = \mathbb{R}^2$ with $h_{ij} = \delta_{ij}$. The metric of a sphere of radius R in stereographic coordinates $ds^2 = \frac{4R^4}{(R^2+x^2+y^2)^2}(dx^2 + dy^2)$ is generated by the seam:

$$s(x, y) = \ln(2R^2) - \ln(R^2 + x^2 + y^2)$$

via the Conformal Rule.

ii.3 Example: Hyperbolic Geometry (Upper Half-Plane)

Let $U = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with $h_{ij} = \delta_{ij}$. The hyperbolic metric $ds^2 = \frac{R^2}{y^2}(dx^2 + dy^2)$ is generated by the seam:

$$s(x, y) = \ln(R) - \ln(y)$$

via the Conformal Rule.

ii.4 Example: Flat, Conformally Distorted Plane

Let $U = \mathbb{R}^2$ with $h_{ij} = \delta_{ij}$. Consider $s(x, y) = xy$. The Conformal Rule yields:

$$g = \begin{pmatrix} e^{2xy} & 0 \\ 0 & e^{2xy} \end{pmatrix}$$

This metric is intrinsically flat ($K = 0$) but not globally Euclidean.

ii.5 Summary for Conformal Rule

The Conformal Rule $\mathcal{R}_{\text{Conf}}$ interprets s as controlling the logarithm of a local scale factor applied to a background metric h. Its strengths are generating conformally flat geometries like spheres and hyperbolic spaces (when $h = \delta$), and preserving metric type. Its limitation is that it can only generate geometries conformally equivalent to the background. (Notably, the background h itself could be the result of applying another rule, like $\mathcal{R}_{\text{Hessian}}$, to a different seam, allowing for the generation of metrics conformal to non-flat Hessian geometries.) It cannot change metric signature (e.g., Euclidean to Lorentzian) with real s.

III. The Gradient Rule ($\mathcal{R}_{\text{Grad}}$)

This rule uses the magnitude of the gradient of the seam function s to define an isotropic scaling factor relative to a background metric.

Definition iii.1 (Gradient Rule)**.** *Let U be a differentiable manifold with local coordinates $\{x^i\}$ and a background Euclidean metric $h_{ij} = \delta_{ij}$ in these coordinates. Let $\mathcal{S}(U)$ be the set of C^1 functions $s : U \to \mathbb{R}$. The* Gradient Rule, *$\mathcal{R}_{Grad}$, defines a metric tensor g with components:*

$$g_{ij}(u) = |\nabla s(u)|^2 \delta_{ij} \tag{III.4}$$

where $|\nabla s|^2 = \sum_{k=1}^{n} (\partial s/\partial x^k)^2$ is the squared magnitude of the gradient of s with respect to the background Euclidean metric. If $\nabla s(u) \neq 0$ for all u in a region, then g is a Riemannian metric in that region, and $D = \mathcal{R}_{Grad}(s; U)$ is the geodesic distance under g.

The interpretation of gradient magnitude as a geometric scaling factor connects to level set methods and eikonal equations [3].

This Rule interprets s as a potential whose gradient's magnitude directly scales an isotropic metric (conformally Euclidean). It requires only first derivatives of s, unlike the Hessian Rule which requires second derivatives.

iii.1 Example: Euclidean Space

Let $U = \mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$ and background δ_{ij}. Consider the linear seam:

$$s(x^1, \ldots, x^n) = x^1$$

The gradient is $\nabla s = (1, 0, \ldots, 0)$. The squared magnitude is $|\nabla s|^2 = 1^2 = 1$. Applying the Gradient Rule (III.4):

$$g_{ij} = (1)\delta_{ij} = \delta_{ij}$$

This generates the standard Euclidean metric. Any seam $s = \sum a_k x^k + c$ with $\sum a_k^2 = 1$ would also yield the Euclidean metric.

iii.2 Example: Radially Scaled Space

Let $U = \mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$ and background δ_{ij}. Consider the quadratic seam:

$$s(x^1, \ldots, x^n) = \frac{1}{2} \sum_{k=1}^{n} (x^k)^2 = \frac{1}{2} r^2$$

The gradient is $\nabla s = (x^1, x^2, \ldots, x^n)$. The squared magnitude is $|\nabla s|^2 = \sum_{k=1}^{n} (x^k)^2 = r^2$. Applying the Gradient Rule:

$$g_{ij} = r^2 \delta_{ij}$$

This generates a metric $ds^2 = r^2((dx^1)^2 + \cdots + (dx^n)^2) = r^2 d\mathbf{x}^2$. This is a conformally flat metric. It is Riemannian for $r \neq 0$ but becomes degenerate ($g_{ij} = 0$) at the origin $r = 0$, where $\nabla s = 0$. Geodesics in this space are related to circles passing through the origin in the underlying Euclidean space.

iii.3 Example: Constant Seam

Let $U = \mathbb{R}^n$. Consider a constant seam $s(u) = c$. The gradient is $\nabla s = (0, \ldots, 0)$. The squared magnitude is $|\nabla s|^2 = 0$. Applying the Gradient Rule:

$$g_{ij} = (0)\delta_{ij} = 0$$

This generates a completely degenerate tensor $g = 0$. The associated distance $D(u, v)$ would be 0 for all u, v, violating the metric property unless U is a single point. This highlights the importance of the condition $\nabla s \neq 0$.

iii.4 Relation to Conformal Rule

The Gradient Rule $g_{ij} = |\nabla s|^2 \delta_{ij}$ always produces a metric conformally related to the background Euclidean metric δ_{ij}. Comparing with the Conformal Rule $g_{ij} = e^{2\tilde{s}}\delta_{ij}$, we see that the Gradient Rule generates the same geometry as the Conformal Rule if we choose the conformal seam $\tilde{s}$ such that:

$$e^{2\tilde{s}} = |\nabla s|^2$$

This requires $|\nabla s|^2 > 0$, and gives $\tilde{s} = \ln(|\nabla s|)$. Therefore, any geometry generated by $\mathcal{R}_{\text{Grad}}$ (where $\nabla s \neq 0$) can also be generated by $\mathcal{R}_{\text{Conf}}$. However, the converse is not true: not every positive conformal factor $F(u) = e^{2\tilde{s}(u)}$ can be expressed as the squared magnitude of a gradient $|\nabla s(u)|^2$ for some C^1 function s. For example, the spherical metric factor $F = 4R^4/(R^2 + r^2)^2$ cannot be written as $|\nabla s|^2$ globally on $\mathbb{R}^2$. Thus, $\mathcal{R}_{\text{Grad}}$ generates a specific subset of conformally Euclidean geometries.

iii.5 Summary for Gradient Rule

The Gradient Rule $\mathcal{R}_{\text{Grad}}$ interprets the seam s via the magnitude of its gradient, requiring only C^1 smoothness.

- It generates an isotropic metric $g_{ij} = |\nabla s|^2 \delta_{ij}$, which is always conformally Euclidean.
- It can reproduce Euclidean space ($s = x^1$).
- It generates non-trivial conformally flat spaces (e.g., $s = r^2/2$ gives $g_{ij} = r^2 \delta_{ij}$).
- Requires $\nabla s \neq 0$ for the metric to be Riemannian (non-degenerate). Where $\nabla s = 0$, the geometry degenerates.
- Since $|\nabla s|^2 \geq 0$ and δ_{ij} is positive definite, it can only generate Riemannian or degenerate metrics, not pseudo-Riemannian metrics with mixed signatures (like Minkowski) from a Euclidean background.
- It generates a subset of the geometries accessible via the Conformal Rule $\mathcal{R}_{\text{Conf}}$, specifically those where the conformal factor can be written as $|\nabla s|^2$.

This rule offers an alternative mechanism based on first derivatives, leading naturally to isotropic scaling.

IV. Graph-Based Rules ($\mathcal{R}_{\text{Graph}}$)

The Hessian, Conformal, and Gradient rules rely fundamentally on the differentiable structure of the base space U. To extend the framework to spaces involving discrete components, such as lattices ($\mathbb{N}^n$) or mixed spaces ($\mathbb{N} \times \mathbb{R}$), a different approach is needed. Graph-based rules offer a natural pathway by interpreting U as a set of vertices and using the seam s to define connection costs (edge weights).

Definition iv.1 (Graph Rule Framework)**.** *Let U be a base set, potentially composed of discrete and/or continuous components. A* Graph-Based Rule, *$\mathcal{R}_{Graph}$, generates a distance function D through the following conceptual steps:*

1. ***Graph Structure:*** *Define an underlying graph structure $G = (U, E)$ on the base set U. This involves specifying the vertices (points in U) and edges E, representing allowed "connections" or adjacencies. For continuous or hybrid spaces, this might involve discretization or defining infinitesimal connections.*

2. ***Weight Assignment:*** *Use the seam function $s : U \to \mathbb{R}$ to assign a non-negative weight $w(e)$ or cost to each edge $e \in E$. This is the core interpretive step for graph rules. For continuous spaces, this translates to defining a cost density or local metric element ds.*

3. ***Distance Calculation:*** *Define the distance $D(u, v)$ between any two points $u, v \in U$ as the infimum of the total weight/cost along all possible paths connecting u and v. For discrete graphs, this is the standard shortest path distance. For continuous/hybrid spaces, this involves integrating the cost density along paths.*

The discrete metric structures generated by graph rules have been studied extensively in spectral graph theory [4] and discrete differential geometry [23].

iv.1 Candidate Rule 1: Cost from Seam Difference ($\mathcal{R}_{\text{Graph-}\Delta s}$)

A simple rule for discrete graphs defines the weight of an edge $e = \{u, v\}$ based on the difference in seam values at its endpoints:

$$w(e) = w(u, v) = |s(v) - s(u)|$$

If s is constant, all edge weights are 0, leading to $D(u, v) = 0$ within connected components. Variations could include adding a base cost: $w(u, v) = \epsilon + |s(v) - s(u)|$.

iv.2 Example: Weighted Lattice ($\mathbb{N} \times \mathbb{N}$)

Let $U = \mathbb{N} \times \mathbb{N}$ with edges between neighbors (i, j) and (i', j') if $|i - i'| + |j - j'| = 1$. Let $s(i, j) = i + j$. Then $w((i, j), (i + 1, j)) = |(i + 1 + j) - (i + j)| = 1$, and $w((i, j), (i, j + 1)) = |(i + j + 1) - (i + j)| = 1$. This recovers the standard Manhattan distance $D((i, j), (k, l)) = |i - k| + |j - l|$. If $s(i, j) = (i + j)^2$, weights become non-uniform, e.g., $w((i, j), (i + 1, j)) = |(i + 1 + j)^2 - (i + j)^2| = 2(i + j) + 1$.

iv.3 Example: Hybrid Space ($\mathbb{N} \times \mathbb{R}$)

Let $U = \mathbb{N} \times \mathbb{R}$. Define graph structure via adjacency: (i, x) is connected to $(i + 1, x)$ and $(i - 1, x)$. Within layer i, points (i, x) and (i, y) are connected infinitesimally. Let $s(i, x) = i^2 + f(x)$. A path cost could combine discrete jump costs $w((i, x), (i + 1, x)) = |s(i + 1, x) - s(i, x)| = |(i + 1)^2 - i^2| = |2i + 1|$ with continuous path integration using a local metric derived from s, e.g., $ds_i = \sqrt{(\partial s/\partial x)^2 dx^2} = |f'(x)|dx$. Defining the overall distance D rigorously requires careful treatment of mixed path types.

iv.4 Candidate Rule 2: Conformal-Inspired Cost ($\mathcal{R}_{\text{Graph-Exp s}}$)

Inspired by the Conformal Rule, we might define edge weights based on the average scale factor between nodes. For an edge $e = \{u, v\}$ and a base length ϵ_e:

$$w(e) = \epsilon_e \left(\frac{e^{s(u)} + e^{s(v)}}{2} \right)$$

Or in the continuous limit, a local metric $ds^2 = e^{2s(x)} dx^2$.

iv.5 Example: Standard Lattice ($\mathbb{N} \times \mathbb{N}$)

Using $\mathcal{R}_{\text{Graph-Exp s}}$ with $s(i, j) = 0$ and $\epsilon_e = 1$ for all edges gives $w(e) = 1(\frac{1+1}{2}) = 1$, recovering standard graph distance (Manhattan). If $s(i, j)$ is non-constant, the edge weights vary, creating a weighted graph.

iv.6 Summary for Graph Rules

Graph-Based Rules provide a mechanism to generate geometry on non-differentiable or mixed spaces. The core challenge lies in defining the edge/connection structure E and the weighting function $w(e; s)$ appropriately. Rule $\mathcal{R}_{\text{Graph-}\Delta s}$ uses seam differences, sensitive to gradients. Rule $\mathcal{R}_{\text{Graph-Exp s}}$ uses average seam exponentials, analogous to conformal scaling. These rules allow generating complex discrete or hybrid geometries from scalar seams but require careful formulation for consistency, especially for continuous limits or hybrid structures.

IV. Mathematical Properties of Rules

In this section, we delve deeper into the mathematical properties of the geometries generated by the proposed rules, focusing on the conditions under which they produce valid (pseudo-)metric spaces and characterizing the specific classes of geometries they generate. We assume U possesses the necessary structure (e.g., differentiability, graph structure) required by each rule.

i. Satisfaction of Pseudo-Metric Axioms

A fundamental requirement (Requirement ii.1.1) is that any valid rule $\mathcal{R}$ must generate a distance function $D = \mathcal{R}(s; U)$ that satisfies the pseudo-metric axioms (Non-negativity M1, Identity M2, Symmetry M3, Triangle Inequality M4).

Proposition IV.1 (Metric Properties of Differentiable Rules)**.** *Let $\mathcal{R}$ be $\mathcal{R}_{Hessian}$, $\mathcal{R}_{Conf}$, or $\mathcal{R}_{Grad}$, generating a tensor g from an admissible seam s on a differentiable manifold U.*

1. *If g is a Riemannian metric on a connected domain $\Omega \subseteq U$, the associated geodesic distance $D(u,v) = \inf\{\int \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}dt \mid \gamma : u \to v\}$ defines a true metric on Ω [24, 25].*

2. *If g is pseudo-Riemannian, the geodesic distance (defined appropriately, e.g., for timelike or spacelike paths) satisfies M1-M3. The triangle inequality M4 holds under specific conditions related to causal structure and path types [5, 6]. Degeneracies ($D(u,v) = 0$ for $u \neq v$) can occur, particularly for null-separated points.*

Proof Sketch. (1) Follows from standard properties of Riemannian distance functions [24, 25]. (2) Requires careful definition of distance in pseudo-Riemannian settings, often focusing on proper time/length along specific path types [5, 6]. Degeneracy for null paths is inherent. Symmetry follows if the Lagrangian $\sqrt{|g_{ij}\dot{\gamma}^i\dot{\gamma}^j|}$ is symmetric under time reversal. □

Proposition IV.2 (Metric Properties of Graph Rules)**.** *Let $\mathcal{R}_{Graph}$ generate edge weights $w(u,v)$ for adjacent nodes u,v in a graph $G = (U,E)$ based on a seam $s : U \to \mathbb{R}$. If the weight function satisfies $w(u,v) \geq 0$ (non-negativity) and $w(u,v) = w(v,u)$ (symmetry) for all edges $\{u,v\} \in E$, then the shortest path distance $D(u,v) = \inf\{\sum_{e\in\gamma} w(e) \mid \gamma$ path from u to $v\}$ defines a pseudo-metric on the connected components of G. If $w(u,v) > 0$ for all edges, D is a true metric on each component.*

Proof. This is a standard result in graph theory [4]. M1 and M2 are immediate. M3 follows from weight symmetry. M4 follows because concatenating optimal paths $u \to v$ and $v \to w$ yields a path $u \to w$, whose length is an upper bound for the shortest path $D(u,w)$. The specific rules proposed, $w = |\Delta s|$ and $w = (e^{s(u)} + e^{s(v)})/2$, satisfy non-negativity and symmetry (assuming s is real). □

For hybrid spaces ($\mathbb{N} \times \mathbb{R}$), rigorously proving the metric properties requires a careful definition of the path integral and infimum over combined discrete jumps and continuous segments, which is beyond the scope of this initial presentation but constitutes an important direction for future work.

ii. Equivalence Classes of Seams

Different seams may generate the same geometry under a given rule. We denote this equivalence by $s_1 \sim_{\mathcal{R}} s_2$.

Proposition IV.3 (Seam Equivalence)**.** *Let U be $\mathbb{R}^n$ with standard coordinates and $h = \delta$ where applicable.*

1. *$\mathcal{R}_{Hessian}$: $s_1 \sim_{\mathcal{R}_{Hess}} s_2 \iff s_1 - s_2$ is an affine function, i.e., $s_1(x) = s_2(x) + a \cdot x + b$ for some vector $a \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$.*

2. *$\mathcal{R}_{Conf}$: $s_1 \sim_{\mathcal{R}_{Conf}} s_2 \iff s_1(u) = s_2(u)$ for all $u \in U$.*

3. *$\mathcal{R}_{Grad}$: $s_1 \sim_{\mathcal{R}_{Grad}} s_2 \iff |\nabla s_1(u)|^2 = |\nabla s_2(u)|^2$ for all $u \in U$. This holds if s_1 and s_2 are different solutions to the same Eikonal*

equation $|\nabla s|^2 = F(u)$. (E.g., $s_1 = x^1$ and $s_2 = x^2$ both yield $|\nabla s|^2 = 1$ and $g = \delta$).

4. *$\mathcal{R}_{Graph\text{-}\Delta s}$ (discrete graph): $s_1 \sim_{\mathcal{R}_{Graph\text{-}\Delta s}} s_2 \iff s_1(u) = s_2(u) + c$ for some constant c (assuming identical adjacency and graph structure).*

5. *$\mathcal{R}_{Graph\text{-}Exp\ s}$ (discrete graph): $s_1 \sim_{\mathcal{R}_{Graph\text{-}Exp\ s}} s_2 \iff s_1(u) = s_2(u)$ for all $u \in U$.*

Proof. (1) $\partial_i\partial_j s_1 = \partial_i\partial_j s_2 \iff \partial_i\partial_j(s_1 - s_2) = 0$. Integrating twice yields $s_1 - s_2$ is affine. (2) $e^{2s_1}h_{ij} = e^{2s_2}h_{ij} \implies e^{2s_1} = e^{2s_2} \implies s_1 = s_2$ (assuming h_{ij} is non-degenerate). (3) $|\nabla s_1|^2\delta_{ij} = |\nabla s_2|^2\delta_{ij} \implies |\nabla s_1|^2 = |\nabla s_2|^2$. (4) For an edge $\{u, v\}$, $|s_1(v) - s_1(u)| = |s_2(v) - s_2(u)|$. If $s_1 = s_2 + c$, $|(s_2(v) + c) - (s_2(u) + c)| = |s_2(v) - s_2(u)|$. (5) For an edge $\{u, v\}$, $(e^{s_1(u)} + e^{s_1(v)})/2 = (e^{s_2(u)} + e^{s_2(v)})/2$. For this to hold for all edges in a connected graph generally requires $s_1(u) = s_2(u)$ for all u. □

III. Characterization of Generated Geometries

We can characterize the specific classes of geometries generated by each rule.

iii.1 Hessian Rule Geometries

The Hessian rule $g_{ij} = \partial_i\partial_j s$ generates *Hessian metrics*.

- **Metric Type:** g is Riemannian if s is strictly convex, positive semi-definite if s is convex, and pseudo-Riemannian if the Hessian matrix $(\partial_i\partial_j s)$ has the appropriate signature. Degeneracy occurs where $\det(\mathrm{Hess}(s)) = 0$ [7].
- **Geometric Class:** These metrics are central to Information Geometry and Affine Differential Geometry [1, 8]. Dually flat spaces in Information Geometry often possess metrics derived from the Hessian of a convex potential function (divergence) [1].
- **Integrability:** A given tensor g_{ij} can be locally written as a Hessian, $g_{ij} = \partial_i\partial_j s$, if and only if certain integrability conditions related to its curvature are met. For instance, if g is flat ($R_{ijkl} = 0$), it must be constant to be a Hessian globally on $\mathbb{R}^n$. If g is derived from a Kähler potential s in complex geometry ($g_{i\bar{j}} = \partial_i\partial_{\bar{j}}s$), this imposes specific curvature properties [12].

iii.2 Conformal Rule Geometries

The rule $g_{ij} = e^{2s}h_{ij}$ generates geometries *conformally equivalent* to the background (U, h).

- **Metric Type:** The signature of g is the same as the signature of h (since $e^{2s} > 0$). Riemannian remains Riemannian.
- **Geometric Class:** If $h = \delta$ is the flat Euclidean metric on $U \subseteq \mathbb{R}^n$, then $\mathcal{R}_{\mathrm{Conf}}$ generates *conformally flat* geometries. For $n \geq 3$, this is equivalent to the vanishing of the Weyl tensor $W_{ijkl} = 0$. For $n = 2$, all metrics are conformally flat [2].
- **Flexibility of Background:** While often assuming $h = \delta$, the background h could itself be generated by another rule. If $h_{ij} = (\mathcal{R}_{\mathrm{Hessian}}(s_0))_{ij} = \partial_i\partial_j s_0$, the resulting metric $g_{ij} = e^{2s}(\partial_i\partial_j s_0)$ is conformal to a Hessian metric, potentially representing a richer geometric class dependent on two seams, s and s_0.
- **Curvature:** The curvature (e.g., scalar curvature $\tilde{R}$ of $g = e^{2s}h$) is related to the curvature R of h and derivatives of s via known transformation laws, e.g., $\tilde{R} = e^{-2s}(R - 2(n-1)\Delta_h s - (n-1)(n-2)|\nabla s|_h^2)$ for Riemannian h [9].

iii.3 Gradient Rule Geometries

The rule $g_{ij} = |\nabla s|^2\delta_{ij}$ (relative to a background δ_{ij}) generates *isotropic, conformally flat* geometries.

- **Metric Type:** Since $|\nabla s|^2 \geq 0$, g is always Riemannian (positive definite) where $\nabla s \neq 0$ and degenerate ($g = 0$) where $\nabla s = 0$. It cannot generate pseudo-Riemannian geometries from a Riemannian background via a real seam s.
- **Geometric Class:** These metrics are a subset of conformally flat metrics. The conformal factor $F(u) = |\nabla s(u)|^2$ must be the

squared magnitude of a gradient. Not all positive functions $F(u)$ can be written this way [3]. Thus, $\mathcal{R}_{\text{Grad}}$ is strictly less general than $\mathcal{R}_{\text{Conf}}$ for generating conformally flat geometries. For example, the spherical metric factor $F = 4R^4/(R^2 + r^2)^2$ is not $|\nabla s|^2$ for any smooth s globally on $\mathbb{R}^2$.

- **Degeneracy:** Geodesics and distances are ill-defined at critical points of s where $\nabla s = 0$. The geometry collapses locally.

iii.4 Graph Rule Geometries

The geometries generated by $\mathcal{R}_{\text{Graph}}$ are weighted graphs where distances are shortest path lengths.

- **Geometric Class:** These are discrete metric spaces. Their large-scale geometry depends heavily on the choice of s and the weighting rule.
- **Examples:** $\mathcal{R}_{\text{Graph-Exp s}}$ with $s = 0$ recovers standard graph distances (e.g., Manhattan distance on $\mathbb{Z}^n$ with 2n-connectivity if edge lengths are 1). Non-zero s leads to non-homogeneous weighted graphs.
- **Continuum Limit:** Understanding the geometry generated by graph rules on increasingly fine discretizations of a manifold U, and how it relates to the differential rules, requires careful analysis using tools like Gromov-Hausdorff convergence or discrete exterior calculus [10, 11]. This is an active research area.
- **Hybrid Spaces:** Defining and analyzing the geometry of hybrid spaces like $\mathbb{N} \times \mathbb{R}$ under these rules requires rigorous formulation of path lengths combining discrete and continuous costs, ensuring the resulting distance satisfies metric properties, particularly the triangle inequality.

V. Discussion

The framework presented in this paper proposes that geometric structure, typically characterized by a distance function D, can be generated from the interaction between a base space U, a scalar seam function $s : U \to \mathbb{R}$, and an interpretive Rule $\mathcal{R}$. The exploration of different Rules $\mathcal{R}$ reveals distinct ways in which the seam s can be interpreted to "stitch" the space U together.

The Hessian Rule ($\mathcal{R}_{\text{Hessian}}$), $g_{ij} = \partial^2 s/\partial x^i \partial x^j$, treats the seam s as a potential function requiring C^2 smoothness. Its second derivatives directly define the metric tensor components. This rule naturally generates flat Euclidean and Minkowski geometries from simple quadratic seams, demonstrating a capacity to unify Riemannian and pseudo-Riemannian structures within a single mechanism. However, it does not guarantee a positive definite or even non-degenerate metric and appears unable to generate fundamental geometries with constant non-zero curvature directly on $\mathbb{R}^n$.

In contrast, the Conformal Rule ($\mathcal{R}_{\text{Conf}}$), $g_{ij} = e^{2s} h_{ij}$, interprets s as the logarithm of a local scaling factor applied to a background metric h_{ij}. This rule naturally generates conformally flat geometries, readily producing spherical and hyperbolic spaces by choosing the appropriate logarithmic seam function relative to a flat background. It preserves the metric type (e.g., Riemannian remains Riemannian) but is limited to geometries conformally equivalent to the background.

The Gradient Rule ($\mathcal{R}_{\text{Grad}}$), $g_{ij} = |\nabla s|^2 \delta_{ij}$, offers a third perspective for differentiable manifolds, using the first derivatives of s. It requires only C^1 smoothness and interprets the squared magnitude of the seam's gradient as an isotropic scaling factor for a background Euclidean metric δ_{ij}. Like the Conformal Rule, it generates conformally Euclidean geometries and cannot produce pseudo-Riemannian signatures from a Euclidean background. However, it is more restrictive than the Conformal Rule, as the scaling factor must be expressible as $|\nabla s|^2$. This rule highlights a direct link between the rate of change of the seam and the local metric scale, yielding degeneracy where the seam is stationary ($\nabla s = 0$).

The comparison between applying $\mathcal{R}_{\text{Hessian}}$ and $\mathcal{R}_{\text{Conf}}$ to the same seam $s = xy$ highlighted the critical role of the Rule: the former produced flat Lorentzian spacetime, while the latter produced a flat, conformally warped Riemannian plane. The Gradient Rule applied to $s = xy$ would yield $g_{ij} = (x^2 + y^2)\delta_{ij} = r^2\delta_{ij}$, the same outcome as $\mathcal{R}_{\text{Grad}}$ applied to $s = r^2/2$, demonstrating non-uniqueness of the seam for a given geometry under this rule and again producing a different geometry than the other two rules for $s = xy$.

Furthermore, the framework inherently allows for the composition of rules. For instance, the background metric h_{ij} required by the Conformal Rule need not be a fixed, predefined structure like δ_{ij}. It could itself be the geometric structure generated by applying a different rule, say $\mathcal{R}_{\text{Hessian}}$, to an underlying seam s_0. The resulting geometry $g_{ij} = e^{2s}(\partial_i\partial_j s_0)$ would then depend on two seams, s and s_0, interpreted sequentially. While this moves beyond the simplicity of the basic $(U, s, \mathcal{R})$ triplet focused upon in this introductory work, it highlights the potential for generating highly complex geometries from nested scalar field interpretations.

Graph-Based Rules ($\mathcal{R}_{\text{Graph}}$) offer a pathway to handle discrete or mixed base spaces. By defining adjacency and deriving edge weights or local costs from s, they generate geometry through shortest path computations. Defining universally applicable and consistent rules for deriving edge weights remains a challenge, particularly for mixed spaces. Simple rules based on seam differences ($w = |\Delta s|$) or inspired by conformal scaling ($w \approx e^s \epsilon$) illustrate different possibilities. Recent work connecting discrete and continuous notions of curvature [17] and geometric learning approaches [16] suggests promising avenues for further development of these hybrid frameworks.

The necessity of specifying a Rule $\mathcal{R}$ alongside (U, s) is perhaps the most crucial outcome. The Rule embodies the physical or mathematical principle translating the scalar information s into geometric relations D. While the search for a single, universal Rule encompassing all scenarios might be difficult, the framework allows for the study and comparison of different physically or mathematically motivated Rules. The choice of Rule dictates the interpretation of the seam and the class of geometries that can be generated.

This framework potentially offers a novel perspective by focusing on the generative capacity of scalar fields. If a suitable Rule can be identified for a particular context, the seam s becomes a powerful tool for parameterizing and exploring a landscape of geometries. Open questions remain regarding the rigorous formulation of graph rules for continuous limits, the handling of non-smooth seams, the geometric interpretation of non-positive definite or degenerate metrics arising from rules like $\mathcal{R}_{\text{Hessian}}$ and $\mathcal{R}_{\text{Grad}}$, and the potential physical meaning of the seam s under different rules.

VI. Conclusion

The framework $(U, s, \mathcal{R})$ generates geometries from a scalar seam s on a base space U via a Rule $\mathcal{R}$, producing a distance D. Requiring D to be a pseudo-metric, we showed $\mathcal{R}_{\text{Hessian}}$ yields Euclidean and Minkowski spaces from quadratic seams using second derivatives, $\mathcal{R}_{\text{Conf}}$ produces conformally flat spaces like spheres and hyperbolic planes using e^{2s} scaling, $\mathcal{R}_{\text{Grad}}$ generates isotropic conformally flat metrics using $|\nabla s|^2$ scaling from first derivatives, and $\mathcal{R}_{\text{Graph}}$ handles discrete structures via seam-derived costs. The triplet unifies diverse geometries within a scalar-driven approach, with the Rule fundamentally shaping the interpretation of the seam and the resulting geometry. Further study could expand its scope to novel spaces and physical interpretations. The framework's potential applications extend beyond pure mathematics to areas like quantum geometry [18] and information geometry [15].

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