MASD 2021, Assignment 1

Hand-in in groups of 2 or 3 before 14.9.2021 at 10:00

One submission per group
Remember to include the names of all group members

Exercise 1 (Writing proofs). Apart from submitting your answers to all three exercises, submit your answer to this Exercise 1 separately as it is to be commented by another group. You will receive Exercise 1 for comments shortly after the hand-in deadline on September 8-th. Submit your comments as soon as possible but not later than 21.9.2021 at 10:00.

In this exercise, you will practice writing proofs. Remember that the proofs in this exercise (and any other proofs) must satisfy:

- Clearly stated assumptions (if there are any),
- Clearly stated claims (what do you want to prove),
- Clear logical arguments leading from assumptions to claims.

Your proofs will get peer-feedback from fellow students before they are corrected by your TA. If you do not wish your fellow students to know your identity, do not put your names on the PDF that will be forwarded to another group.

A function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, has a limit L when x approaches $c \in D$ if for **every** number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

a) Suppose that the above implication holds for some fixed $\epsilon_0 > 0$. Prove that it holds for all $\epsilon \geq \epsilon_0$.

We need some definition before the remaining questions of this exercise can be formulated. A function: $f: D \to \mathbb{R}$ is **one-to-one** if for every pair of values $x_1, x_2 \in D$

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Let R_f denote the range of a one-to-one function $f: D \to R$. f has a unique inverse function $f^{-1}: R_f \to D$ such that $f(f^{-1}(y)) = y$ for all $y \in R_f$. You can read more about inverse functions in section 1.5 of the textbook.

A function $f: D \to \mathbb{R}$ is **increasing** if

$$x_1 \le x_2 \implies f(x_1) \le f(x_2)$$

for every pair of values $x_1, x_2 \in I$. f is **strictly increasing** if

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

for every pair of values $x_1, x_2 \in I$. Decreasing and strictly decreasing functions are defined in analogous way.

Assume for the rest of this exercise that D is an open interval $]a,b[\subseteq \mathbb{R}$. Let $f:D\to \mathbb{R}$ be a strictly increasing and continuous function. It is intuitively clear that the range R_f of f is an open interval and the it is one-to-one (no need to prove that in this exercise). Hence f has the inverse function f^{-1} . Prove that

- b) f^{-1} is strictly increasing.
- c) f^{-1} is continuous.

Deliverables. The proofs.

Exercise 2 (Limits and continuity). Consider the function

$$f(x) = \begin{cases} x^2 & \text{when } x < 0, \\ x & \text{when } x \in [0, 2], \\ 5 & \text{when } x > 2, \end{cases}$$

- a) In the supplied Jupyter notebook template Altemplate.ipynb, plot the function f(x) on the interval $x \in [-5, 5]$, and based on your plot, decide if there are points $a \in [-5, 5]$ where f is not continuous? If yes, for which points it is not?
 - Include the plot and your claimed non-continuous points in your report.
- b) Prove that your observations from a) are correct. That is:
 - Prove that f is continuous at all $a \in [-5, 5]$ where you claim that it is, and
 - Prove that f is not continuous at those points $a \in [-5, 5]$ where you claim that it is not.

That is, you should have a proof of either continuity or non-continuity for every $a \in [-5, 5]$. Your proofs can be using any results mentioned Section 2.5 on the textbook.

Deliverables. Please submit a) the filled-out Jupyter template, and include the plot and the non-continuous points in your report; b) The proofs, following the same guidelines as in Exercise 1.

Exercise 3 (Limits and area of a disk). Let $n \in \mathbb{N}$, $n \geq 3$, denote the number of sides of a regular polygon P_n inscribed in a disk C with radius r and center O.

a) As $n \to \infty$, the area S_n of P_n approximates the area of C. We know that it is πr^2 . Prove that $\lim_{n\to\infty} S_n = \pi r^2$. Hint: You may need that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Deliverables. The proof.