MASD

Lecture 8
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Objectives

- Substitution rule (based on the chain rule)
- Integration by parts (based on pruduct rule).
- Approximating integrals
- Improper integrals
- Double integrals

Integration by Substitution - Indefinite Integrals

- Determine $\int (3x^4+5)^5 x^3 dx$
- Set $u = g(x) = 3x^4 + 5$. Then $g'(x) = 12x^3$ which is 12 times the second term in the product. Let $f(u) = u^5$.

$$\int (3x^4+5)^5 x^3 dx = \int f(g(x)) \frac{g'(x)}{12} dx = \frac{1}{12} \int f(g(x)) g'(x) dx =$$

$$\frac{1}{12} \int f(u) du = \frac{1}{12} \int u^5 du = \frac{1}{12} \frac{u^6}{6} = \frac{1}{72}(3x^4+5)^6 + C$$

- Verification:

$$\frac{d}{dx}\left[\frac{1}{72}(3x^4+5)^6 + C\right] = \frac{6}{72}(3x^4+5)^5 12x^3 = (3x^4+5)^5 x^3$$

Integration by Substitution - Definite Integrals

- Determine $\int_0^1 (3x^4+5)^5 x^3 dx$
- Set $u = 3x^4 + 5$. Then $g'(x) = 12x^3$ which is 12 times the second term in the product. Let $f(u) = u^5$.

$$\int_0^1 (3x^4+5)^5 x^3 dx = \int_0^1 f(g(x)) \frac{g'(x)}{12} dx = \frac{1}{12} \int_0^1 f(g(x)) g'(x) dx =$$

$$\frac{1}{12} \int_5^8 f(u) du = \frac{1}{12} \int_5^8 u^5 du = \frac{1}{12} \left(\frac{1}{6} 8^6 - \frac{1}{6} 5^6 \right) = \frac{8^6 - 5^6}{72}$$

Why Does Substitution Works?

- Let $u = g(x)$ be a differentiable function whose value range is an interval I. Let f be continuous on I.
- Claim: $\int f(g(x))g'(x)dx = \int f(u)du$.
- This follows directly from the chain rule for differentiation:
  - Let F be an antiderivative of f (i.e., $F'(x) = f(x)$ for all $x \in I$).
  - Chain rule: $\frac{d}{dx}[F[g(x)] = F'(g(x))g'(x)$

$$\int f(g(x))g'(x)dx = \int F'(g(x))g'(x)dx = \int \frac{d}{dx}[F[g(x)] =$$

$$F(g(x)) + C = F(u) + C = \int F'(u)du = \int f(u)du$$

Integration by Substitution - Example I

- Determine $\int (x^2 + a^2) x dx$.
- Solution 1:

$$\int (x^2 + a^2) x dx = \int \left(x^3 + a^2 x\right) dx = \int x^3 dx + \int a^2 x dx = \frac{1}{4}x^4 + \frac{1}{2}a^2 x^2 + C$$

- Solution 2: Let $u = g(x) = x^2 + a^2$. Then $g'(x) = 2x$ which is 2 times the second term in the product. Let $f(u) = u$.

$$\int (x^2 + a^2) x dx = \int f(g(x)) \frac{g'(x)}{2} dx = \frac{1}{2} \int f(g(x)) g'(x) dx =$$

$$\frac{1}{2} \int u du = \frac{1}{4} u^2 + C' = \frac{1}{4}(x^2 + a^2)^2 + C' = \frac{1}{4}x^4 + \frac{1}{2}a^2 x^2 + C''$$

since $\frac{1}{4}a^4$ is a constant hidden in C''.

Integration by Substitution - Example II

- Determine $\int \frac{x}{(x^2+a^2)^n} dx$, $a \neq 0$.
- Let $u = g(x) = x^2 + a^2$. $g'(x) = 2x$ which differs from the numerator x by factor 2. Let $f(u) = \frac{1}{u^n}$
- For $n \neq 1$, we get by the substitution rule:

$$\int \frac{x}{(x^2+a^2)^n} dx = \int f(g(x)) \frac{g'(x)}{2} dx = \frac{1}{2} \int f(g(x)) g'(x) dx =$$

$$\frac{1}{2} \int \frac{1}{u^n} du = \frac{1}{2} \int u^{-n} du = \frac{1}{2} \frac{u^{-n+1}}{-n+1} + C = \frac{-1}{2(n-1)u^{n-1}} + C$$

- Substituting back:

$$\int \frac{x}{(x^2+a^2)^n} dx = \frac{-1}{2(n-1)(x^2+a^2)^{n-1}} + C \text{ for } a \neq 0, n \neq 1$$

- for $n = 1$, we get

$$\int \frac{x}{x^2+a^2} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln u + C = \frac{1}{2} \ln(x^2+a^2) + C$$

Integration by Substitution - Example III

- Determine $\int \cos(2x)dx$.
- Let $u = g(x) = 2x$. $g'(x) = 2$ which differs from the second term (1) by factor 2. Let $f(u) = \cos(u)$

$$\int \cos(2x)dx = \int f(g(x))\frac{g'(x)}{2}dx = \frac{1}{2}\int f(g(x))g'(x)dx =$$

$$\frac{1}{2}\int \cos(u)du = \frac{1}{2}sin(u) + C = \frac{1}{2}\sin(2x) + C$$

Integration by Substitution - Example IV

- Determine $\int xe^{-x^2}dx$.
- Let $u = g(x) = -x^2$. $g'(x) = -2x$ which differs from the first term (x) by constant factor -2. Let $f(u) = e^u$.

$$\int xe^{-x^2}dx = \int \frac{g'(x)}{-2}f(g(x))dx = -\frac{1}{2}\int f(g(x))g'(x)dx =$$

$$-\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = \frac{1}{2}e^{-x^2} + C$$

Integration by Substitution - Example V

- Determine $\int x^2\sqrt{x^3+1}dx$.
- Let $u = x^3 + 1$. $g'(x) = 3x^2$ which differs from the first term (x^2) by constant factor 3. Let $f(u) = \sqrt{u} = u^{\frac{1}{2}}$.

$$\int x^2\sqrt{x^3+1}dx = \int \frac{g'(x)}{3} f(g(x))dx = \frac{1}{3}\int f(g(x))g'(x)dx =$$

$$\frac{1}{3}\int u^{\frac{1}{2}}du = \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{9}u^{\frac{3}{2}} + C = \frac{2}{9}(x^3+1)^{\frac{3}{2}} + C$$

Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

- Let $u = f(x)$ and $dv = g'(x)dx$.
- $du = f'(x)dx$ and $v = g(x)$.
- Formula for integration by parts can then be written as

$$\int udv = uv - \int vdu$$

Integration by Parts - Example I

$$\int udv = uv - \int vdu$$

- Determine $\int xcos(x)dx$.
- Not among basic integration formulas.
- Integration by substitution not applicable.
- Rule: Choose u so that the integration of dv is possible to obtain v and the derivative of u becomes simpler.
- Let $u = x$ and $dv = \cos(x)dx$. Then $v = \int cos(x)dx = \sin(x)$ and $du = dx$.

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + cos(x) + C$$

Integration by Parts - Example II

$$\int u dv = uv - \int v du$$

- Determine $\int xe^x dx$.
- Not among basic integration formulas.
- Integration by substitution not applicable.
- Rule: Choose u so that the integration of dv is possible to obtain v and the derivative of u becomes simpler.
- Let $u = x$ and $dv = e^x dx$. Then $v = \int e^x dx = e^x$ and $du = dx$.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x - 1) + C$$

Integration by Parts - Example III

$$\int u dv = uv - \int v du$$

- Determine $\int \ln x dx$.
- Not among basic integration formulas.
- Integration by substitution not applicable.
- Rule: Choose u so that the integration of dv is possible to obtain v and the derivative of u becomes simpler.
- Let $u = \ln x$ and $dv = 1dx$. Then $v = \int dx = x$ and $du = \frac{1}{x} dx$

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C$$

Integration by Parts - Example IV

$$\int u dv = uv - \int v du$$

- Determine $\int x^3 \ln x dx$.
- Not among basic integration formulas.
- Integration by substitution not applicable.
- Rule: Choose u so that the integration of dv is possible to obtain v and the derivative of u becomes simpler.
- Let $u = \ln x$ and $dv = x^3 dx$. Then $v = \int x^3 dx = \frac{x^4}{4}$ and $du = \frac{1}{x} dx$

$$\int x^3 \ln x dx = (\ln x)\frac{x^4}{4} - \int \frac{x^4}{4}\frac{1}{x} dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx =$$

$$\frac{x^4}{4}\ln x - \frac{1}{4}\frac{x^4}{4} + C = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

Integration by Parts - Example V

$$\int u dv = uv - \int v du$$

- Determine $\int x^2 \sin(2x) dx$.
- Not among basic integration formulas.
- Integration by substitution not applicable.
- Rule: Choose u so that the integration of dv is possible to obtain v and the derivative of u becomes simpler.
- Let $u = x^2$ and $dv = \sin(2x)dx$. Then
  $v = \int \sin(2x)dx = -\frac{\cos(2x)}{2}$ and $du = 2xdx$

$$\int x^2 \sin(2x)dx = x^2 \left(-\frac{cos(2x)}{2}\right) - \int -\frac{\cos(2x)}{2} 2xdx = -\frac{x^2 \cos(2x)}{2} + \int x \cos(2x)dx$$

$$\int x \cos(2x)dx = x\frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2} dx = x\frac{\sin(2x)}{2} + \frac{\cos 2x}{4}$$

$$\int x^2 \sin(2x)dx = -\frac{x^2 \cos(2x)}{2} + x\frac{\sin(2x)}{2} + \frac{\cos 2x}{4} + C$$

Integration by Parts - Example VI

$$\int u dv = uv - \int v du$$

- Determine $\int e^x \cos(x) dx$. Not among basic integration formulas. Integration by substitution not applicable.
- Rule: Choose u so that the integration of dv is possible and the derivative of u becomes simpler.
- Let $u = e^x$, $dv = \cos(x)dx$. Then $v = \int \cos(x) dx = \sin(x)$ and $du = e^x dx$

$$\int e^x \cos(x) dx = e^x \sin(x) - \int \sin(x) e^x dx$$

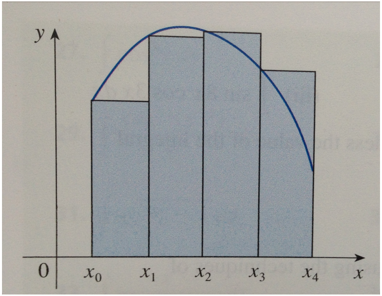
- Let $u = e^x$, $dv = \sin(x)dx$. Then $v = \int \sin(x) dx = -\cos(x)$ and $du = e^x dx$.

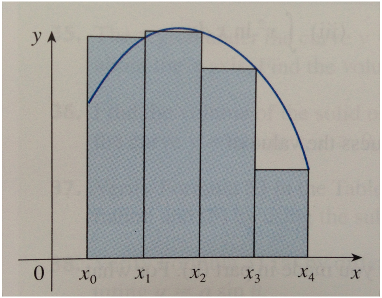
$$\int e^x \sin(x) = -e^x \cos(x) + \int \cos(x) e^x dx$$

$$\int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$$

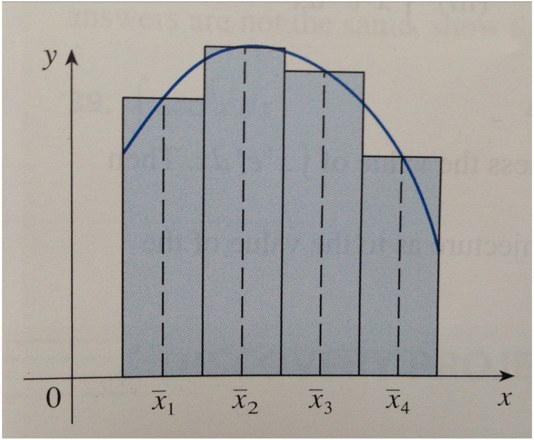
$$\int e^x \cos(x) dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + C$$

Approximate Integration - Left and Right Endpoints

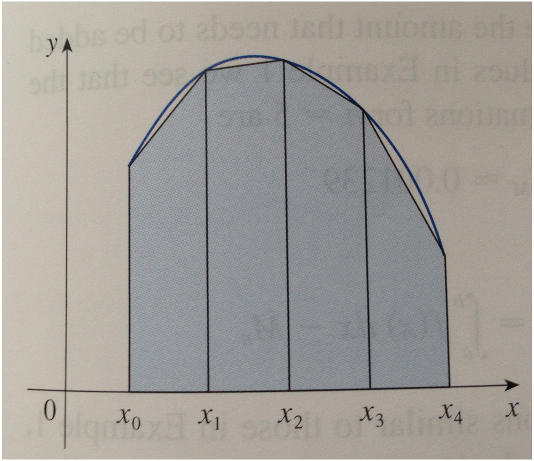




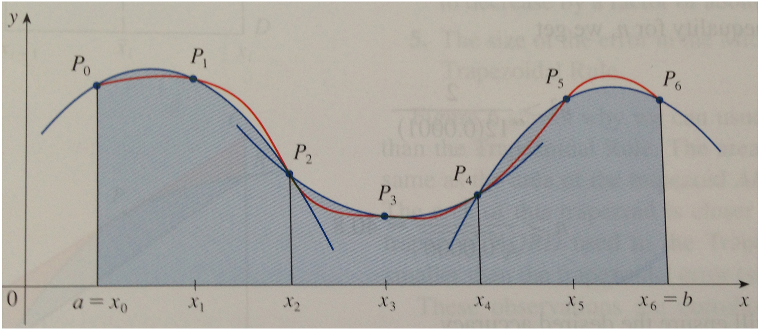
Approximate Integration - Middle Points



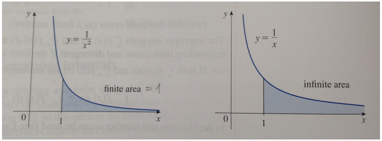
Approximate Integration - Trapezoidal Rule

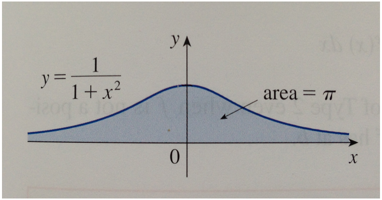


Approximate Integration - Simpson's Rule



Improper Integrals





- Assume that $\int_a^t f(x)dx$ exists for every $t \geq a$.

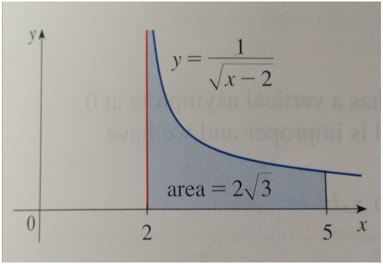
$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

provided that the limits exist.

Improper Integrals - Discontinuous Integrands

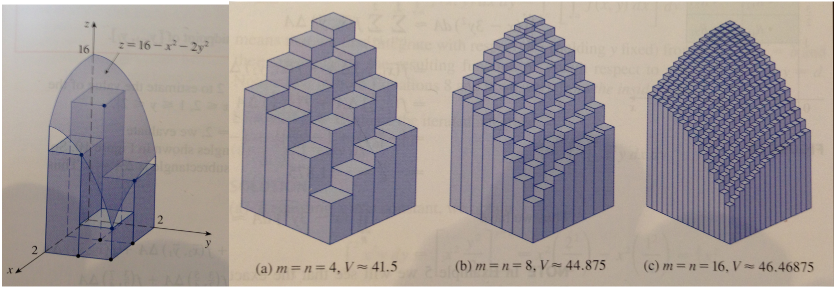


- Assume that f is positive, continuous on $]a, b]$, discontinuous at a.

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided that the limit exists. Similar if f is continuous on $[a, b[$ and discontinuous at b.

Double Integrals



- The *double integral* of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

provided that the limit exists.

Iterated Integrals

- It can be difficult to evaluate double integrals directly from their definition.
- *Iterated integrals* make it easier.

$$\iint_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_a^b \int_c^d f(x,y)dydx$$

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Summary

- Substitution rule
- Integration by parts
- Approximate integrals
- Improper integrals
- Double integrals

- Function limits, continuity
- Derivatives for functions with 1 variable, basic rules, chain rule
- Partial and directional derivatives, gradient, gradient descent, Newton's method
- Sequences, series, their limits, power series, Taylor and Maclaurin series
- Integration, Fundamental Theorem of Calculus (1 and 2), Substitution rule, Integration by Parts
- Exam preparation: Read relevant sections, *understand* theory and examples. Do not memorize - you can always go back to the book. Sections not covered in the lectures probably less important. Examples and exercises should make it easier to understand material