Solutions to Homework Assignment 5

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Problem 1: We consider a two stage experiment. First we roll two dice. Then we determine the roll sum and toss that many coins.

1. Construct an appropriate probability space for the experiment.
2. Determine the probability of observing exactly three heads during the coin toss phase of the experiment.

Solution for Problem 1: We choose the probability space

$$\Omega = \bigcup_{k=2}^{12} \{0,1\}^k ,$$

where k indexes the roll sum of the two dice. To choose appropriate probabilities we first determine the probability that the row sum is k. For $k \leq 7$ the first die can show any value $k_1 \in \{1, \dots k-1\}$ and the second die is then fixed to show the value $k - k_1$. In this case

$$\mathbb{P}(\text{row sum equals } k) = \frac{k-1}{6^2}$$

For $k \geq 7$ the first die can show any value $k_1 \in \{k-6, \dots, 6\}$ and the second is fixed to show $k - k_1$ again. Thus, in this case

$$\mathbb{P}(\text{row sum equals } k) = \frac{13-k}{6^2} .$$

Together we have

$$\mathbb{P}(\text{row sum equals } k) = \frac{\min\{k-1, 13-k\}}{6^2} .$$

We fix the probability distribution on Ω to be given by

$$\mathbb{P}(\{(\omega_1, \dots, \omega_k)\}) := \frac{\min\{k-1, 13-k\}}{6^2} 2^{-k} .$$

Now we answer the question of how likely it is to get exactly three heads. For that we certainly need $k \geq 3$. For fixed $k \geq 3$ all $(\omega_1, \dots, \omega_k)$ have the same probability. Therefore we have to count how many $(\omega_1, \dots, \omega_k)$ there are for which $\omega_i = 1$ for three different i. This is the same question as 'How many subsets of $\{1, \dots, k\}$ are there of size 3'. The answer is $\binom{k}{3}$. We conclude

$$\mathbb{P}\big(\text{exactly three heads during the coin toss phase}\big) = \sum_{k=3}^{12} \binom{k}{3} \frac{\min\{k-1, 13-k\}}{6^2} 2^{-k} .$$

Problem 2: Count the number of ways of distributing n indistinguishable balls over k distinguishable boxes.

Hint: Think of configurations of the form $| \circ \circ \circ | | \circ \circ | \circ |$, where $\circ$ is a ball and $|$ is the wall of a box.

Solution for Problem 2: The outer walls are always there and we have $k-1$ inner walls. Otherwise every choice of $\circ$ and $|$ is a different configuration. Therefore, the question is how to choose $k-1$ out of $n+k-1$ possible positions for the inner walls. There are $\binom{n+k-1}{k-1}$ ways to do that.

Problem 3: We want to pair up all n students in our tutorial randomly. For that purpose everybody writes her/his name on a piece of paper. We collect all the names in a bag and then everybody picks one piece of paper blindly from the bag. How likely is it that this will work and nobody picks her/himself? What happens if n becomes very large (say when we use this method for all students of the university)?

Hint: Use the inclusion-exclusion principle.

Solution for Problem 3: We model the probability space by permutations of $\{1,\dots,n\}$, i.e. $\Omega = S_n$. Then $A_i = \{\sigma \in \Omega : \sigma(i) = i\}$ is the event that student i picks her/himself. The probability that this hapens for at least one student is

$$\begin{aligned}\mathbb{P}\big(\cup_{i=1}^n A_i\big) &= \sum_{k=1}^n (-1)^{k+1} \sum_{I:\#I=k} \mathbb{P}\big(\cap_{i\in I} A_i\big)\\ &= \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!}\\ &= \sum_{k=1}^n \frac{(-1)^{k+1}}{k!}\,.\end{aligned}$$

Thus,

$$\mathbb{P}\big(\text{nobody picks himself}\big) = 1 - \sum_{k=1}^n \frac{(-1)^{k+1}}{k!} = \sum_{k=0}^n \frac{(-1)^k}{k!} \to \mathrm{e}^{-1} \approx 0.368\,,$$

which is a quite reasonable probability.