MASD 2021, Assignment 1

Hand-in in groups of 2 or 3 before 14.9.2021 at 10:00
One submission per group
Remember to include the names of all group memebers

Exercise 1 (Writing proofs). Apart from submitting your answers to all three exercises, submit your answer to this Exercise 1 separately as it is to be commented by another group. You will receive Exercise 1 for comments shortly after the hand-in deadline on September 8-th. Submit your comments as soon as possible but not later than 21.9.2021 at 10:00.

In this exercise, you will practice writing proofs. Remember that the proofs in this exercise (and any other proofs) must satisfy:

- Clearly stated assumptions (if there are any),
- Clearly stated claims (what do you want to prove),
- Clear logical arguments leading from assumptions to claims.

Your proofs will get peer-feedback from fellow students before they are corrected by your TA. *If you do not wish your fellow students to know your identity, do not put your names on the PDF that will be forwarded to another group.*

A function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, has a limit L when x approaches $c \in D$ if for **every** number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

a) Suppose that the above implication holds for **some** fixed $\epsilon_0 > 0$. Prove that it holds for all $\epsilon \geq \epsilon_0$.

We need some definition before the remaining questions of this exercise can be formulated. A function: $f : D \to \mathbb{R}$ is **one-to-one** if for every pair of values $x_1, x_2 \in D$

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Let R_f denote the range of a one-to-one function $f : D \to R$. f has a unique inverse function $f^{-1} : R_f \to D$ such that $f(f^{-1}(y)) = y$ for all $y \in R_f$. You can read more about inverse functions in section 1.5 of the textbook.

A function $f : D \to \mathbb{R}$ is **increasing** if

$$x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$$

for every pair of values $x_1, x_2 \in I$. f is **strictly increasing** if

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

for every pair of values $x_1, x_2 \in I$. **Decreasing** and **strictly decreasing** functions are defined in analogous way.

Assume for the rest of this exercise that D is an open interval $]a, b[\subseteq \mathbb{R}$. Let $f : D \to \mathbb{R}$ be a strictly increasing and continuous function. It is intuitively clear that the range R_f of f is an open interval and the it is one-to-one (no need to prove that in this exercise). Hence f has the inverse function f^{-1}. Prove that

b) f^{-1} is strictly increasing.
c) f^{-1} is continuous.

Deliverables. The proofs.
Solution:

a) Suppose that for some $\epsilon_0 > 0$, there is a number $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon_0$$

But then for every $\epsilon \geq \epsilon_0$, the same δ implies that $|f(x) - L| < \epsilon$. This ensures that when determining the limits, one does not need to show the implication for all $\epsilon > 0$. It is enough to show it for ϵ in an arbitrarily small open interval $]0, \epsilon_0[$, $\epsilon_0 > 0$.

b) Let $y_1, y_2 \in R_f$ and $y_1 < y_2$. Since $y_1, y_2 \in R_f$, there exist x_1 and x_2 such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is strictly increasing, x_1 and x_2 are unique. Since $y_1 < y_2$ and f is strictly increasing, then $x_1 < x_2$. But $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$ which shows that $f^{-1}(y_1) < f^{-1}(y_2)$. Hence, f^{-1} is strictly increasing.

c) To show that f^{-1} is continuous in the open interval R_f, we need to show that it is continuous in every point $y_0 \in R_f$. Let $x_0 \in D$ be such that $y_0 = f(x_0)$. We have to show that

$$\lim_{y \to y_0} f^{-1}(y) = x_0$$

or equivalently that

$$\forall \epsilon > 0, \exists \delta > 0, \forall y : |y - y_0| < \delta \implies |f^{-1}(y) - x_0| < \epsilon$$

Let $\epsilon_0 > 0$ be such that $x_0 - \epsilon_0 \in D$ and $x_0 + \epsilon_0 \in D$. Since D is an open interval, such ϵ_0 always exists. By the result in a) we can assume that $\epsilon \in]0, \epsilon_0[$. We therefore know that

$$x_0 - \epsilon < x_0 < x_0 + \epsilon$$

Since f is strictly increasing in D, we have

$$f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$$

Let

$$\delta = \min\{f(x_0) - f(x_0 - \epsilon), f(x_0 + \epsilon) - f(x_0)\}$$

So

$$\delta \leq f(x_0) - f(x_0 - \epsilon) \iff f(x_0 - \epsilon) \leq f(x_0) - \delta$$

and

$$\delta \leq f(x_0 + \epsilon) - f(x_0) \iff f(x_0) + \delta \leq f(x_0 + \epsilon)$$

If we now take y such that

$$f(x_0) - \delta < y < f(x_0) + \delta$$

then

$$f(x_0 - \epsilon) < y < f(x_0 + \epsilon)$$

Since f^{-1} is strictly increasing on R_f, we get

$$f^{-1}(f(x_0 - \epsilon)) < f^{-1}(y) < f^{-1}(f(x_0 + \epsilon)) \iff x_0 - \epsilon < f^{-1}(y) < x_0 + \epsilon$$

Exercise 2 (Limits and continuity). Consider the function

$$f(x) = \begin{cases} x^2 & \text{when } x < 0, \\ x & \text{when } x \in [0, 2], \\ 5 & \text{when } x > 2, \end{cases}$$

a) In the supplied Jupyter notebook template `A1template.ipynb`, plot the function $f(x)$ on the interval $x \in [-5, 5]$, and based on your plot, decide if there are points $a \in [-5, 5]$ where f is not continuous? If yes, for which points it is not?
Include the plot and your claimed non-continuous points in your report.

b) Prove that your observations from a) are correct. That is:
   - Prove that f is continuous at all $a \in [-5, 5]$ where you claim that it is, *and*
   - Prove that f is not continuous at those points $a \in [-5, 5]$ where you claim that it is not.

   That is, you should have a proof of either continuity or non-continuity for every $a \in [-5, 5]$. Your proofs can be using any results mentioned Section 2.5 on the textbook.

Deliverables. Please submit a) the filled-out Jupyter template, and include the plot and the non-continuous points in your report; b) The proofs, following the same guidelines as in Exercise 1.

Solution:

b) $f(x)$ is continuous on $]-\infty, 0[$ since it is a polynomium (see Section 2.5). $f(x)$ is continuous at 0 since $\lim_{x\to 0^-} x^2 = \lim_{x\to 0^+} x = 0$. $f(x)$ is continuous on $]0, 2[$ since it is a polynomium (see Section 2.5). $f(x)$ is not continuous at 2 since $\lim_{x\to 2^-} x = 2$ while $\lim_{x\to 2^+} 5 = 5$ (continuity requires that these two limits are equal). $f(x)$ er continuous on $[5, \infty[$ as it then is a constant function.

Exercise 3 (Limits and area of a disk). Let $n \in \mathbb{N}$, $n \geq 3$, denote the number of sides of a regular polygon P_n inscribed in a disk C with radius r and center O.

a) As $n \to \infty$, the area S_n of P_n approximates the area of C. We know that it is πr^2. Prove that $\lim_{n\to\infty} S_n = \pi r^2$. Hint: You may need that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Deliverables. The proof.

Solution:

- Let $A_1, A_2, ..., A_n$ denote the corners of P_n in clockwise order. P_n can be split into n identical triangles. Each has two consecutive corners A_i and A_{i+1} of P_n (index modulo n). All n triangles have O as their third corner.
- Interior angles of these n triangles $\Delta A_i A_{i+1} O$ (index modulo n) at O are $\frac{2\pi}{n}$. Let C_i be the midpoint between A_i and A_{i+1}. $\Delta A_i C_i O$ and $\Delta O A_{i+1} C$ are right triangles and each of them has an area that is half of the area of $\Delta A_i A_{i+1} O$. Their interior angles at O are $\frac{\pi}{n}$.
- Let $h = |OC_i| = r\cos\left(\frac{\pi}{n}\right)$ and $a = |A_i C_i| = r\sin\left(\frac{\pi}{n}\right)$.
- $S_n = 2n * \frac{1}{2}ah = nr^2 \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$
- $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[nr^2 \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)\right] = \lim_{n\to\infty} \left[\pi r^2 \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \cos\left(\frac{\pi}{n}\right)\right] = \lim_{n\to\infty} \pi r^2 * \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} * \lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \pi r^2$ since $\lim_{n\to\infty} \pi r^2 = \pi r^2$, $\lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 1$ and $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = 1$.