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1 Continuity

1.1 Properties of Continuous Functions

If f and g are continuous at a and c is a constant. then the following functions are also continuous at a:

$$f+g$$
$$f-g$$
$$cf$$
$$fg$$
$$\frac{f}{g}\text{if g(a)} \neq 0$$

Any polynomial is continuous everywhere, that is it is continuous in the space of real numbers.

Any rational function is continuous whereever it is defined - it is continuous on its domain. The same goes for root functions, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.

If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.
In other words

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

If g is cont. at a and f is continuous at $g(a)$, then the composite function $f(g(x))$ is cont. at a.

1.2 The Intermediate Value Theorem

Suppose that f is cont. on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$

If a function f is differentiable at a, then f is cont. at a.

2 Differentiation

2.1 Easy Conversions

f	f′
c	0
kx	k
ln(x)	$\frac{1}{x}$, x > 0
ln(-x)	$\frac{1}{x}$, x < 0
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
x^a	ax^{a-1}
a^x	a$^x ln(a)$
e^x	e^x
e^{kx}	ke^{kx}
ln(x)	$\frac{1}{x}$
sin(x)	cos(x)
cos(x)	-sin(x)
tan(x)	1+tan(x)^2

2.2 Maple

2.2.1 DiffTutor

To open the `DiffTutor` use the following commands in Maple:

```
with(Student[Calculus1]):
DiffTutor()
```

2.3 Differentiation Rules

If c is a constant, and f and g are differentiable functions then

2.3.1 The Constant Multiple Law

$$(cf(x))' = c \cdot f'(x)$$

2.3.2 The Sum Rule

$$(f(x) + g(x))' = f'(x) + g'(x)$$

2.3.3 The Difference Rule

$$(f(x) - g(x))' = f'(x) - g'(x)$$

2.3.4 The Product Rule

$$(f(x) \cdot g(x))' = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

2.3.5 The Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g'(x)^2}$$

2.3.6 The Chain Rule

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

2.4 Linear Approximation

2.4.1 Linearization of f

If we want to get the linear function $L(x)$ for a tangent in the point $(a, f(a)$ we can use the following formula:

$$L(x) = f(a) + f'(a)(x - a)$$

or for two variables at the point $(a, b, f(a, b))$:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

2.5 Partial Derivatives

2.5.1 Basics

$$f_x(a, b) = g'(a) \text{ where } g(x) = f(x, b)$$

Alternatively:
To find f_x regard y as a constant and differentiate $f(x, y)$ with respect to x.
To find f_y regard x as a constant and differentiate $f(x, y)$ with respect to ý.

2.5.2 Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

2.5.3 Chain Rule with Partial Derivatives

Suppose that $z = f(x, y)$ is a diff. function of x and y, where $x = g(t)$ and $y = h(t)$ are both diff. functions of t. Then z is a diff. function of t and

$$z' = f_x(x, y) \cdot g'(t) + f_y(x, y) \cdot h'(t)$$

2.5.4 Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a diff. function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are diff. functions of s and t. Then

$$f_s(x, y) = f_x(x, y) \cdot g_s(s, t) + f_y(x, y) \cdot h_s(s, t)$$

and

$$f_t(x, y) = f_x(x, y) \cdot g_t(s, t) + f_y(x, y) \cdot h_t(s, t)$$

2.5.5 Chain Rule (Generalized)

Suppose that u is a diff. function of the n variables $x_1, x_2, ..., x_n$ and each x_j is a diff. function of the m variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and

$$u_{t_i}(x_1, ..., x_n) = u_{x_1} \cdot (x_1)_{t_i} + u_{x_2} \cdot (x_2)_{t_i} + ... + u_{x_n} \cdot (x_n)_{t_i}$$

For all $i = 1, 2, ..., m$

2.6 Implicit Differentiation

2.6.1 Base Case

Suppose that an equation of the form $F(x, y) = 0$ defines y as a diff. function of x, that is $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If f is diff. we have that

$$\frac{dy}{dx} = f'(x) = -\frac{F_x}{F_y}$$

2.6.2 Case 2

Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y))$ for all (x, y) in the domain of f. If F and f are diff. then we have that

$$z_x = -\frac{F_x}{F_z}$$

and

$$z_y = -\frac{F_z}{F_z}$$

2.7 Gradient Vector

If f is a function of two variables x and y, the the gradient of f is a vector function $\bigtriangledown f$ defined by

$$\bigtriangledown f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix}$$

We can express the directional derivative in the direction of a unit vector u as the scalar projection of the gradient vector onto u:

$$D_u f(x, y) = \bigtriangledown f(x, y) \cdot u$$

Or generalized:

$$\bigtriangledown f(x_1, x_2, ..., x_n) = \begin{pmatrix} f_{x_1}(x_1, x_2, ..., x_n) \\ f_{x_2}(x_1, x_2, ..., x_n) \\ ... \\ f_{x_n}(x_1, x_2, ..., x_n) \end{pmatrix}$$

$$D_u f(x_1, x_2, ..., x_n) = \bigtriangledown f(x_1, x_2, ..., x_n) \cdot u$$

2.7.1 Maximizing the Directional Derivative

Suppose f is a diff. function of two or three variables. The maximum value of the directional derivative $D_u f(x)$ is $| \bigtriangledown f(x)|$ and it occurs when u has the same direction as the gradient vector $\bigtriangledown f(x)$.

3 X-TREME Values

3.1 The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:
1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values is the absolute maximum, the smallest is the absolute minimum.

3.2 Rolle's Theorem

Let f be continuous at the closed interval $[a, b]$ and diff. at (a, b), and let $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$.

3.3 The Mean Theorem

Let f be continuous at the closed interval $[a, b]$ and diff. at (a, b). Then there is a number c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently

$$f(b) - f(a) = f'(c)(b - a)$$

3.4 The Increasing/Decreasing Test

If $f'(x) > 0$ on an interval, then f is increasing on that interval.
If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

3.5 The First Derivative Test

Suppose c is a critical number of a continuous function f,
if f' changes from positive to negative at c, then f as a local maximum at c.
if f' changes from negative to positive at c, then f as a local minimum at c.
if f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

3.6 Concavity Test

If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.
If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

3.7 The Second Derivative Test

Suppose f'' is continuous near c.
If $f'(c) = 0$ and $f''(c) > 0$, then f has a local maximum at c.
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local minimum at c.

3.8 The Second Derivatives Test (Multi Variable)

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (so (a, b) is a critical point of f). Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
If $D < 0$, then (a, b) is a saddle point of f.

4 Integration

4.1 Easy Conversions

f	f′
0	c
k	kx + c
x	$\frac{1}{2}x^2 + c$
$\frac{1}{x}$, x > 0	ln(x) + c
$\frac{1}{x}$, x < 0	ln(-x) + c
$\sqrt{x}$	$\frac{2}{3}x\sqrt{x}$ + c
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + c$
x^a	$\frac{1}{a+1}x^{a+1}$ + c
a^x	$\frac{1}{ln(a)}a^x$ + c
e^x	e$^x + c$
e^{kx}	$\frac{1}{k}e^{kx}$ + c
ln(x)	xln(x) - x + c
cos(x)	sin(x) + c
sin(x)	-cos(x) + c
tan(x)	-ln(cos(x)) + c

4.2 Maple

4.2.1 IntTutor

To open the `IntTutor` use the following commands in Maple:

```
with(Student[Calculus1]):
IntTutor()
```

4.3 Indefinite Integration Laws

Assume f and g are continuous functions and c is a constant. Then

4.3.1 Constant Multiple Law

$$\int cf(x)dx = c\int f(x)dx$$

4.3.2 Sum Law

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

4.3.3 Difference Law

$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$$

4.4 Definite Integration Laws

Assume f and g are continuous functions and c is a constant. Then

4.4.1 Constant Law

$$\int_a^b c dx = c(b-a)$$

4.4.2 Sum Law

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

4.4.3 Constant Multiple Law

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

4.4.4 Difference Law

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

4.5 Comparison Properties of the Integral

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

And for improper integrals: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.
If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

4.6 Substitution Rule

If $u = g(x)$ is a diff. function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

4.7 Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or laternatively:
Let $u = f(x)$ and $v = g(x)$. Then the differentials $du = f'(x)dx$ and $dv = g'(x)dx$, so, by the substitution rule, the formula for integration by parts becomes

$$\int udv = uv - \int vdu$$

4.8 Misc. Integrals

$\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

5 Limits

5.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist. Then

5.1.1 Sum Law

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

5.1.2 Difference Law

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

5.1.3 Contant Multiple Law

$$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$$

5.1.4 Product Law

$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

5.1.5 Quotient Law

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0$$

5.1.6 Power Law

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$$

5.1.7 Root Law

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

5.1.8 Special Laws

$$\lim_{x\to a} c = c$$
$$\lim_{x\to a} x = a$$
$$\lim_{x\to a} x^n = a^n \text{ where } n \text{ is a positive integer}$$
$$\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a} \text{ where } n \text{ is a positive integer}$$

(if n is even, we assume that $a > 0$)

5.2 Direct Substitution Property

if f is a polynomial or a rational function and a is in the domain of f then

$$\lim_{x\to a} f(x) = f(a)$$

also

$$f(x) = g(x) \text{ when } x \neq a \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} g(x)\text{, provided the limit exists}$$

5.3 The Squeeze Theorem

$$f(x) \leq g(x) \leq h(x) \text{ when } x \text{ is near } a \land \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \Rightarrow \lim_{x \to a} g(x) = L$$

5.4 Limits at Infinity

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

and

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = \lim_{x \to \infty} x^{-r} = 0$$

$$\lim_{x \to -\infty} e^x = 0$$

5.5 Misc. Formulas for Limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x \to 0^+} \ln(x) = -\infty$$

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0 \forall x \in R$$

6 Probability

6.1 Axioms

1. For any event $A, P(A) \geq 0$.
2. Probability of the sample splace S is $P(S) = 1$.
3. If $A_1, A_2, A_3, ...$ are disjoint events, then
$$P(A_1 \cup A_2 \cup A_3 \cup ...) = P(A_1) + P(A_2) + ...$$
4. In a finite sample space S, where all outcomes are equally likely, the probability of any event A can be found by
$$P(A) = \frac{|A|}{|S|}$$

6.2 Conditional Probability

If A and B are two events in a sample space S, then the conditional probability of A given B is defined as
$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{when } P(B) > 0$$

6.2.1 Chain Rule for Conditional Probability

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1)...P(A_n|A_{n-1}, A_{n-2}, ..., A_1)$$

6.3 Law of Total Probability

If $B_1, B_2, ...$ is a partition of the sample space S, then for any event A we have
$$P(A) = \sum_i P((A \cap B_i) = \sum_i P(A|B_i)P(B_i)$$

6.4 Bayes' Rule

For any two events A and B, where $P(A) \neq 0$, we have
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

If $B_1, B_2, ...$ form a partition of the sample space S, and A is any event with $P(A) \neq 0$, we have
$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}$$

6.5 Sampling

6.5.1 Ordered Sampling with Replacement

Chosing k objects from a set with n elements:
$$n^k$$

6.5.2 Ordered Sampling without Replacement

Chosing k objects from a set with n elements:
$$P_k^n = P_{n,k} = P(n,k) = nPk = \frac{n!}{(n-k)!}, \text{for } 0 \leq k \leq n$$

6.5.3 Unordered Sampling without Replacement (The Binomial Coefficient)

Chosing k objects from a set with n elements:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{for } 0 \leq k \leq n$$

The total number of ways to divide n distinct objects into two groups A and B such that group A consists of k objects and group B consists of $n-k$ objects is $\binom{n}{k}$.

6.5.4 Binomial Formula

For n independent Bernoulli trials (tests that can either be success or failure) where each trial has success probability p, the probability of k successes is given by

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

6.5.5 Unordered Sampling with Replacement

The total number of distinct k samples from an n-element set such that repetition is allowed and ordering does not matter is the same as the number of distinct solutions to the equation

$$x_1 + x_2 + ... + x_n = k, x_i \in \{0, 1, 2, 3, ...\}$$

This number is equal to

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

6.6 Random Variables

6.6.1 Probability Mass Function (PMF)

The PMF has the following properties:

1. $0 \leq P_X(x) \leq 1$ for all x
2. $\sum_{x \in R_X} P_X(x) = 1$
3. for any set $A \subset R_X, P(X \in A) = \sum_{x \in A} P_X(x)$

6.6.2 Binomial Random Variables as a Sum of Bernoulli Random Variables

If $X_1, X_2, ..., X_n$ are independent $Bernoulli(p)$ random variables, then the random variable X defined by $X = X_1 + X_2 + ... + X_n$ has a $Binomial(n, p)$ distribution.

6.6.3 Probability Density Function (PDF)

Consider a random variable X with an absolutely continuous CDF $F_X(x)$. The function $f_X(x)$ defined by

$$f_X(x) = F'_X(x)$$

if f is diff. at x, is called the probability density function of X.
Consider a continuous random variable X with PDF $f_X(x)$. We have

1. $f_X(x) \geq 0 \forall x \in R$
2. $\int_{\infty}^{\infty} f_X(u) du = 1$

3. $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)du$
4. More generally, for a set $A, P(X \in A) = \int_A f_X(u)du$

6.7 Expected Values

6.7.1 EX of Bernoulli

$$EX = p$$

6.7.2 EX of Geometric

$$EX = \frac{1}{p}$$

6.7.3 EX of Poisson

$$EX = \lambda$$

6.7.4 EX of Binomial

$$EX = np$$

6.7.5 EX of Pascal

$$EX = \frac{m}{p}$$

6.7.6 EX linearity

We have

1. $E[aX + b] = aEX + b, \forall a, b \in R$
2. $E[X_1 + X_2 + ... + X_n = EX_1 + EX_2 + ...EX_n$, for any set of random variables $X_1, X_2, ..., X_n$.

6.7.7 Law of the unconscious statistician (LOTUS)

X is discrete:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k)P_X(x_k)$$

X is continuous:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$$

6.7.8 Misc. EX

$$E[sin(X)] = \frac{\sqrt{2}+1}{5}$$

6.8 Variance

The variance of a random variable X, with mean $EX = \mu_X$, is defined as

$$Var(X) = E[(X - \mu_X)^2]$$

6.8.1 Computional Formula for the Variance

X is discrete:

$$Var(X) = E[X^2] - [EX]^2$$

X is continuous:

$$Var(X) = EX^2 - (EX)^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx - \mu_X^2$$

6.8.2 Non-Linearity

For a random variable X and real numbers a and b,

$$Var(aX + b) = a^2 Var(X)$$

6.8.3 Sum of Independent Random Variables

If $X_1, X_2, ..., X_n$ are independent random variables and $X = X_1 + X_2 + ... + X_n$, then

$$Var(X) = Var(X_1) + Var(X_2) + ... + Var(X_n)$$

6.8.4 Var. of Binomial

$$Var(X) = np(1 - p)$$

6.8.5 Var. of Bernoulli

$$Var(X) = p(1 - p)$$

6.9 Standard Deviation

The standard deviation of a random variable X is defined as

$$SD(X) = \sigma_X = \sqrt{Var(X)}$$

7 Sequences

7.1 Limit Laws for Sequences

Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then:

7.1.1 Sum Law

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

7.1.2 Difference Law

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

7.1.3 Constant Multiple Law

$$\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n$$

7.1.4 Product Law

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

7.1.5 Quotient Law

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

7.1.6 Power Law

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^2 \text{ if } p > 0 \text{ and } a_n > 0$$

7.1.7 The Squeeze Theorem for Sequences

$$a_n \leq b_n \leq c_n \forall n \geq n_0 \wedge \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \Rightarrow \lim_{n\to\infty} b_n = L$$

7.1.8 Absolute Law?

$$\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$

7.1.9 Applying Continuous Functions to a Convergent Sequence Law???

If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

7.1.10 The Other Power Law

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

8 Series

8.1 Convergent Series Laws

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n + b_n)$, and

8.1.1 Constant Multiple Law

$$\sum_{n=1}^{\infty} ca_n = c \cdot \sum_{n=1}^{\infty} a_n$$

8.1.2 Sum Law

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

8.1.3 Difference Law

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

8.2 Convergence and Divergence

If the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.
This is *not* a bi-implication, meaning $\lim\limits_{n\to\infty} a_n = 0$ does not imply $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

8.2.1 Test for Divergence

If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum\limits_{n} = 1^{\infty} a_n$ is divergent.

8.3 The Geometric Series

The geometric series

$$a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

has the partial sum s_n

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

the sum

$$s = \frac{a}{1 - r}$$

and is convergent if $|r| < 1$
If $|r| \geq 1$, the geometric series is divergent.

8.4 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$$

where x is a variable and the c_n's are constants called the coefficients of the series.
The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ...$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

8.4.1 Power Series in $(x - a)$

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ...$$

is called a power series in $(x - a)$ or a power series centered at a or a power series about a.
For such a series there are only three possibilities:

1. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. R is called the radius of convergence of the power series.
2. The series converges only when $x = a$. $(R = 0)$
3. The series converges for all x. $(R = \infty)$

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f is defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is diff. (and therefore continuous) on the interval $(a - R, a + R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... = \sum_{n=1}^{\infty} (x-a)^{n-1}$$

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + ... = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radius of convergence of the power series in both equations are R.

8.5 Taylor and Maclaurin Series

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then the coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see if f has a power series expansion at a, then it must be of the following form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

also called the Taylor series of the function f at a (or about a or centered at a).
If $a = 0$ we get

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

This series is called the Maclaurin series.

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

8.5.1 Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)}|x-a|^{n+1} \text{ for } |x-a| \leq d$$

8.5.2 The Binomial Series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

8.5.3 Taylor Formulas

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \forall x \in R$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$
$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$
$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad R = 1$$
$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1$$
$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad R = 1$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad R = 1$$

8.6 Misc. Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

9 Sets

9.1 Laws

9.1.1 De Morgan's Law

For any sets $A_1, A_2, ..., A_n$, we have

$$(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c \cap$$

and

$$(A_1 \cap A_2 \cap ... \cap A_n)^c = A_1^c \cup A_2^c \cup ... \cup A_n^c$$

9.1.2 Distributive Law

For any sets A, B, and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

9.2 The Inclusion-Exclusion Principle

For two finite sets A and B we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three finite sets A, B, and C we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Generally, for n finite sets $A_1, A_2, ..., A_n$, we can write

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - ... + (-1)^{n+1} |A_1 \cap ... \cap A_n|$$

10 Sums

10.1 Sum Laws

10.1.1 Constant Multiple Law

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

10.1.2 Sum Law (lol)

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

10.1.3 Difference Law

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

10.2 Sums Of Power

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

11 Misc.

11.1 [kvadratsætningerne, men på engelsk]

$$(a+b)^2 = a^2 + b^2 + 2ab$$
$$(a-b)^2 = a^2 + b^2 - 2ab$$
$$(a+b)(a-b) = a^2 - b^2$$

11.2 Average Velocity

The average velocity can be found using the following equation:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

11.3 Instantaneous Velocity

The instantaneous velocity can be found by differentiating the function used for finding the position and inserting the specified time.

Fx. the instantaneous at 5 seconds for a car which position is given by the formula $p(t) = t^2$ is going to be $p'(t) = 2t$ with $t = 5$ giving us:

$$\text{instantaneous velocity} = p'(t) = 2 \cdot 5 = 10$$

11.4 Newton's Method

Used to find the solution to equations of the format $f(x) = 0$. Chose an approximation such that $f(x_1) = 0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If x_n falls outside of the domain of f, the method has failed, and a better approximation of x_1 should be chosen.

12 Definitions

12.1 Continuity

12.1.1 Single Variable

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

A function is not continues if it is not continues at its entire domain, but it can still be continuous in an interval.

12.1.2 Two Variables

A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

We say that f is continuous on D if f is continuous at every point (a,b) in D.

12.1.3 Three or More Variables

if f is defined on a subset D of R^n, then $\lim_{x \to a} f(x) = L$ means that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$x \in D \wedge 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

12.2 Level Curve

The Level Curve of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

12.3 Limits

12.3.1 The Intuitive Definition of a limit

Suppose $f(x)$ is defined when x is near the number a:

$$\lim_{x \to a} f(x) = L$$

(The limit of $f(x)$, as x approaches a equals L)

12.3.2 Precise Definition of a Limit

$\forall \Sigma > 0 \; \exists \delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \Sigma$$

12.3.3 Two-Sided Limit

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

(For a limit to exist, it must be approached from both left and right, else it is considered a one-sided limit)

12.3.4 Vertically Asymptote

The vertical line $x = a$ is called vertically asymptote if one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty$$

$$\lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

12.3.5 Limit of a function of two variables

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$(x, y) \in D \land 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon$$

12.3.6 Limit of a function with two variables does not exist

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y) \to (a,b)} f(x, y)$ does not exist.
Alternatively, there is not limit if $f(a, y) \neq f(x, b)$ (see Stewart p. 953).

12.4 Probability

12.4.1 Notations

1. $P(A \cap B) = P(A \text{ and } B) = P(A, B)$
2. $P(A \cup B) = P(A \text{ or } B)$

12.4.2 Independence

Two events A and B are independent if

$$P(A \cap B = P(A)P(B)$$

or

$$P(A|B) = P(A)$$

Three events A, B, and C are independent if all the following conditions hold:

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cap C) = P(A)P(C)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

If $A_1, A_2, ...A_n$ are independent then

$$P(A_1 \cup A_2 \cup ... \cup A_n) = 1 - (1 - P(A_1))(1 - P(A_2))...(1 - P(A_n))$$

12.4.3 Conditional Independence

Two events A and B are conditionally independent given an event C with $P(C) > 0$ if

$$P(A \cap B|C) = P(A|C)P(B|C)$$

or

$$P(A|B, C) = P(A|C)$$

12.4.4 Random Variables

A random variable X is a function from the sample space to the real numbers

$$X : S \to R$$

The range of a random variable X, shown by $Range(X)$ or R_X, is the set of possible values of X.

12.4.5 Discrete Random Variables

X is a discrete random variable, if its range is countable.

12.4.6 Probability Mass Function (PMF)

Let X be a discrete random variable with range $R_X = \{x_1, x_2, ...\}$ (finite or countable infinite). The function

$$P_X(x_k) = P(X = x_k), \text{ for } k = 1, 2, 3, ...$$

is called the probability mass function of X.

12.4.7 Bernoulli Random Variable

A random variable X is said to be a Bernoulli random variable with parameter p, shown as $X \sim Bernoulli(p)$, if its PMF is given by

$$P_X(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & otherwise \end{cases}$$

where $0 < p < 1$.

12.4.8 Geometric Random Variable

A random variable X is said to be a geometric random variable with parameter p, shown as $X \sim Geometric(p)$, if its PMF is given by

$$P_X(k) = \begin{cases} p(1-p)^{k-1} & k = 1, 2, 3, ... \\ 0 & otherwise \end{cases}$$

where $0 < p < 1$.

12.4.9 Binomial Random Variable

A random variable X is said to be a binomial random variable with parameters n and p, shown as $X \sim Binomial(n, p)$, if its PMF is given by

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 1, 2, 3, ..., n \\ 0 & otherwise \end{cases}$$

where $0 < p < 1$.

12.4.10 Negative Binomial/ Pascal Random Variable

A random variable X is said to be a Pascal random variable with parameters m and p, shown as $X \sim Pascal(m, p)$, if its PMF is given by

$$P_X(k) = \begin{cases} \binom{k-1}{m-1} p^m (1-p)^{k-m} & k = m, m+1, m+2, m+3, ... \\ 0 & otherwise \end{cases}$$

where $0 < p < 1$.

12.4.11 Hypergeometric Random Variable

A random variable X is said to be a Hypergeometric random variable with parameters b, r and k, shown as $X \sim Hypergeometric(b, r, k)$, if its range is $R_X = \{max(0, k-r), max(0, k-r)+1, ...min(k, b)\}$, and its PMF is given by

$$P_X(x) = \begin{cases} \dfrac{\binom{b}{x}\binom{r}{k-x}}{\binom{b+r}{k}} & x \in R_X \\ 0 & otherwise \end{cases}$$

where $0 < p < 1$.

12.4.12 Poisson Random Variable

A random variable X is said to be a Poisson random variable with parameter λ, shown as $X \sim Poisson(\lambda)$, if its range is $R_X = \{0, 1, 2, 3, ...\}$, and its PMF is given by

$$P_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & k \in R_X \\ 0 otherwise \end{cases}$$

12.4.13 Cumulative Distribution Function (CDF)

The cumulative distribution function of a random variable X is defined as

$$F_X(x) = P(X \leq x), \forall x \in R$$

(The sum of all probabilities up to and including x)

For all $a \leq b$, we have

$$P(a \leq X \leq b) = F(b) - F(a)$$

12.4.14 Expected Value/Mean/Average

Let X be a discrete random variable with range $R_X = \{x_1, x_2, ...\}$ (finite or countably infinite). The expected value of X, denoted by EX is defined as

$$EX = E[X] = E(X) = \mu_X = \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k)$$

If X is continuous we have that

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx$$

12.4.15 Functions of Random Variables

If X is a random variable, then $Y = g(X)$ is also a random variable.

12.5 Sequences

12.5.1 Intuitive Definition of a Limit of a Sequence

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by making n sufficiently large.
If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise we say the sequence diverges (or is divergent).

12.5.2 Precise Definition of a Limit of a Sequence

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding integer N such that

$$n > N \Rightarrow |a_n - L| < \epsilon$$

12.5.3 Monotonic, Increasing, or Decreasing Sequences

A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1} \forall n \geq 1$.
A sequence $\{a_n\}$ is called decreasing if $a_n > a_{n+1} \forall n \geq 1$.
If neither is the case, the sequence is called monotonic.

12.5.4 Bounded Sequences

A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \forall n \geq 1$$

A sequence is bounded below if there is a number m such that

$$m \leq a_n \forall n \geq 1$$

If a sequence is bounded above and below, then it is called a bounded sequence.

12.5.5 Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent. In particular, a sequence that is increasing and bounded above converges , and a sequence that is decreasing and bounded below converges.

12.6 Series

12.6.1 nth Partial Sum

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + ... + a_n$$

12.6.2 Sum

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$a_1 + a_2 + ... + a_n + ... = s \text{ or } \sum_{n=1}^{\infty} = s$$

The number s is called the sum of the series.
If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

12.7 Sets

12.7.1 Union

The union of two sets is a set containing all elements that are in A *or* B (possibly both), denoted by $\cup$.

12.7.2 Intersection

The intersection of two sets is a set containing all elements that are in A *and* B, denoted by $\cap$.

12.7.3 Complement

The complement of a set A, denoted by A^c is the set of all elements that are in the universal set S, but not in A.

12.7.4 Subtraction

$$A - B = A \cap B^c$$

12.7.5 Mutually Exclusive/Disjointed

Two sets are mutually exclusive or disjointed if they do not have any shared elements.

12.7.6 Partion

A collection of nonempty sets $A_1, A_2, A_3, ...$ is a partion of a set A if they are disjoint and their union is A

12.7.7 Cartesian Product

A cartesian product of two sets A and B, written as $A \times B$, is the set containing ordered pairs from A and B. That is, if $C = A \times B$, then each element of C is of the form (x, y), where $x \in A$ and $y \in B$:

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$$

Note that

$$A \times B \neq B \times A$$

as the sets are ordered.
Example:

$$A = \{1, 2, 3\} \text{ and } B\{H, T\}$$
$$A \times B = \{(1, H), (1, T), (2, H), (2, T), (3, H), (2, T)\}$$

12.7.8 Number of Elements

The number of elements in the set A is denoted by

$$|A|$$

12.7.9 Functions

$$f : A \to B$$

Is a way to write we have the function f which takes input from the domain A and outputs the co-domain B. The range of the function is the set containing all possible values of $f(x)$.

12.8 Tangent Plane

(See Stewart p. 975)
Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f(x_0, y_0)(y - y_0)$$

12.8.1 Tangent Plane to the Level Surface

(p. 1002)

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

12.9 X-TREME Values

12.9.1 Absolute X-TREME Values

Let c be a number in the domain D of a function f. Then $f(c)$ is the:
Absolute maximum value of f if $f(c) \geq f(x) \forall x \in D$.
Absolute minimum value of f if $f(c) \leq f(x) \forall x \in D$.

12.9.2 Local X-TREME Values

The number $f(c)$ is a:
Local maximum value of f if $f(c) \geq f(x)$ if x is near c.
Local minimum value of f if $f(c) \leq f(x)$ if x is near c.

12.9.3 Fermat's Theorem

If f has a local maximum or minimum at c, and if $f(c)$ exists, then $f'(c) = 0$.

12.9.4 Critical Number

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
If f has a local maximum or minimum at c, then c is a critical number of f.

12.9.5 Concaving

If the graph of f lies above all of its tangents on an interval I, then f is called concave upward on I. If the graph of f lies bellow all of its tangents on an interval I, then f is called concave downward on I.

12.9.6 Inflection Point

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or concave downward to concave upward at P.

12.9.7 X-TREME Values for Function with Multiple Variables

A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) and $f(a, b)$ is called a local maximum value.
A function of two variables has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) and $f(a, b)$ is called a local minimum value.

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exists there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.