# MASD

Lecture 3
14.09.2021

Pawel Winter

# Objectives

We cover sections 3.4-3.6 and 3.10

- Differentiation of composite functions: Chain rule
- Implicit differentiation
- Logarithmic differentiation

# Chain Rule (da. Kædereglen)

- Assume that the function $g$ is differentiable at $x$ and the function $f$ is differentiable at $g(x)$.
- Consider a composite function $F = f \circ g$ defined by $F(x) = f(g(x))$.
- $F$ is differentiable at $x$ and $F'(x) = f'(g(x))g'(x)$.
- Intuition: Let $u = g(x)$ and $y = f(u)$. If $u$ grows $p$ times as fast as $x$ and $y$ grows $q$ times as fast as $u$ then $y$ grows $pq$ times as fast as $x$.
- Leibnitz notation

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

# Chain Rule - Example I

- Let $F(x) = f(g(x)) = (4x^5 - 7x^3 + 14x^2 - 5)^3$. This is a composite function.
- Let $u = g(x) = 4x^5 - 7x^3 + 14x^2 - 5$
- Let $y = f(g(x)) = f(u) = u^3$
- $f'(g(x)) = f'(u) = \frac{dy}{du} = \frac{d}{du}(u^3) = 3u^2$
- $g'(x) = \frac{du}{dx} = \frac{d}{dx}(4x^5 - 7x^3 + 14x^2 - 5) = 20x^4 - 21x^2 + 28x$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(20x^4 - 21x^2 + 28x) =$

  $3(4x^5 - 7x^3 + 14x^2 - 5)^2(20x^4 - 21x^2 + 28x)$

# Chain Rule - Example II

- Let $F(x) = \sqrt{3x^2 - 7x + 12}$. This is a composite function.
- Let $y = \sqrt{u}$ and $u = 3x^2 - 7x + 12$
- $\frac{dy}{du} = \frac{d}{du}(\sqrt{u}) = \frac{d}{du}(u^{\frac{1}{2}}) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$
- $\frac{du}{dx} = \frac{d}{dx}(3x^2 - 7x + 12) = 6x - 7$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(6x - 7) = \frac{6x - 7}{2\sqrt{3x^2 - 7x + 12}}$$

# Chain Rule - Example III

- Let $F(x) = \sin(4x)$. This is a composite function.
- Let $y = \sin(u)$ and $u = 4x$
- $\frac{dy}{du} = \frac{d}{du}(sin(u)) = \cos(u)$
- $\frac{du}{dx} = \frac{d}{dx}(4x) = 4$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos(u) * 4 = 4\cos(4x)$$

# Chain Rule - Example IV

- Let $F(x) = \cos^3(x)$. This is a composite function.
- Let $y = u^3$ and $u = \cos(x)$
- $\frac{dy}{du} = \frac{d}{du}(u^3) = 3u^2$
- $\frac{du}{dx} = \frac{d}{dx}(\cos(x)) = -\sin(x)$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(-\sin(x)) = -3\cos^2(x)\sin(x)$

# Chain Rule - Example V

- Let $F(x) = e^{-x}$. This is a composite function.
- Let $y = e^u$ and $u = -x$
- $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$
- $\frac{du}{dx} = \frac{d}{dx}(-x) = -1$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(-1) = -e^{-x}$

# Chain Rule - Example VI

- Let $F(x) = e^{4x^3-6x+1}$. This is a composite function.
- Let $y = e^u$ and $u = 4x^3 - 6x + 1$
- $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$
- $\frac{du}{dx} = \frac{d}{dx}(4x^3 - 6x + 1) = 12x^2 - 6$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(12x^2 - 6) = (12x^2 - 6)e^{4x^3-6x+1}$

# Chain Rule - Proof Attempt

- Definition of the derivative of $F = f \circ g$ at some $a$ where $g$ is differentiable at $a$ and $f$ is differentiable at $g(a)$:

$$F'(a) = (f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

- If $g(x) \neq g(a)$ for any $x$ near $a$, then

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \to a} \left[ \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} \right] =$$

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = f'(g(a))g'(a)$$

- Unfortunately, there are functions such that $g(x) - g(a) = 0$ when $x$ is arbitrarily close to $a$. For example $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ when $x$ is close to 0.

## Proving Chain Rule

- $y = f(x)$ is assumed to be differentiable in $a$.
- As $x$ changes from $a + \Delta x$ to $a$, the increment $\Delta y$ of $y$ is $f(a + \Delta x) - f(a)$.
- $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$
- Define $\epsilon(\Delta x) = \frac{\Delta y}{\Delta x} - f'(a)$ for $\Delta x \neq 0$.
- $\lim_{\Delta x \to 0} \epsilon(\Delta x) = \lim_{\Delta x \to 0}(\frac{\Delta y}{\Delta x} - f'(a)) = f'(a) - f'(a) = 0$
- Define $\epsilon(0) = 0$. Then $\lim_{\Delta x \to 0} \epsilon(\Delta x) = \epsilon(0)$ and $\epsilon$ is a continuous function of $\Delta x$
- $\Delta y = f'(a)\Delta x + \epsilon(\Delta x)\Delta x$ where $\epsilon(\Delta x) \to 0$ as $\Delta x \to 0$ for any function $f$ differentiable in $a$.

# Proving Chain Rule

- $u = g(x)$ is differentiable at $a$ and $y = f(u)$ is differentiable at $g(a)$.
- $\Delta u = g'(a)\Delta x + \epsilon_1(\Delta x)\Delta x = [g'(a) + \epsilon_1(\Delta x)]\Delta x$ where $\epsilon_1(\Delta x) \to 0$ as $\Delta x \to 0$ (previous slide).
- $\Delta y = f'(b)\Delta u + \epsilon_2(\Delta u)\Delta u = [f'(b) + \epsilon_2(\Delta u)]\Delta u$ where $\epsilon_2(\Delta u) \to 0$ as $\Delta u \to 0$ (previous slide).
- $\Delta y = [f'(b) + \epsilon_2(\Delta u)][g'(a) + \epsilon_1(\Delta x)]\Delta x$.
- $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}[f'(b) + \epsilon_2(\Delta u)][g'(a) + \epsilon_1(\Delta x)] = f'(b)g'(a) = f'(g(a))g'(a)$.

# Implicit Differentiation - Application of Chain Rule

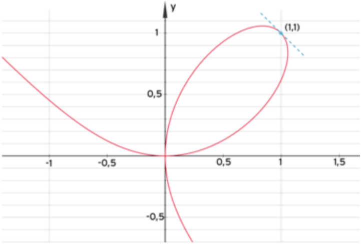
- Equations with two variables $x$ and $y$ cannot always be formulated by functions with $x$ as the only independent variable.
- For example, $x^2 + y^2 = 25$ is an equation defining a circle with the center in origo and with radius 5.
- $x^2 + y^2 = 25$ implicitly defines 2 functions

  $$g_1(x) = \sqrt{25 - x^2} \text{ and } g_2(x) = -\sqrt{25 - x^2}$$

  on the closed interval $[-5, 5]$. $g_1$ defines the function whose graph is the upper half-circle while $g_2$ defines the function whose graph is the lower half-circle.
- We could find slopes at any point on this circle by determining the derivative of either $g_1$ or $g_2$.

# Implicit Differentiation

- For the equation $x^3 + y^3 = 2xy$ it is however not that easy to identify implicitly defined functions with $x$ as independent variable.




- But such equations define curves. One can still determine derivatives (slopes of tangents) of implicitly defined differentable functions without bothering about the functions itself.

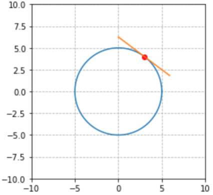
# Implicit Differentiation - Example I: $x^2 + y^2 = 25$

- Consider $y$ as an (unknown) function of $x$.
- Differentiate both sides with respect to $x$ (applying the chain rule to $y^2$) and solve w.r.t. $\frac{dy}{dx}$.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) \iff \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = 0 \iff$$

$$2x+\frac{d}{dy}(y^2)\frac{dy}{dx} = 0 \iff 2x+2y\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = -\frac{x}{y}$$

- At point $(3,4)$, $x=3$ and $y=4$ and $\frac{dy}{dx} = -\frac{3}{4}$



# Implicit Differentiation - Example II: $(x-y)^2 = x + y - 1$

$$\frac{d}{dx}[(x-y)^2] = \frac{d}{dx}[x+y-1] \iff$$

$$2(x-y)(1-\frac{dy}{dx}) = 1 + \frac{dy}{dx} \iff$$

$$2x - 2y - 2(x-y)\frac{dy}{dx} = 1 + \frac{dy}{dx} \iff$$

$$2x - 2y - 1 = (2x - 2y + 1)\frac{dy}{dx} \iff$$

$$\frac{dx}{dy} = \frac{2x-2y-1}{2x-2y+1}$$

# Implicit Differentiation - Example III: $x^3 + y^3 = 2xy$

- Changed notation.
- Regard $y$ as a function of $x$. Differentiate both sides with respect to $x$ (applying chain rule to $y^3$ on the left side and product rule to the right side).

  $$3x^2 + 3y^2y' = 2xy' + 2y$$

  where $y' = \frac{dy}{dx}$.
- Solve for $y'$

  $$3y^2y' - 2xy' = 2y - 3x^2 \iff (3y^2 - 2x)y' = 2y - 3x^2 \iff$$

  $$y' = \frac{2y - 3x^2}{3y^2 - 2x} \text{ if } 3y^2 - 2x \neq 0$$

- Point $(1,1)$ satisfies the equation and therefore the derivative $y'$ (slope of the tangent) in that point is

  $$y' = \frac{2 * 1 - 3 * 1^2}{3 * 1^2 - 2 * 1} = -1$$

# Logarithmic Differentiation - Example I

- Determine $f'(x)$ for $f(x) = \sqrt{\frac{x-1}{x^4+1}}, x > 1$
- Let $y = \sqrt{\frac{x-1}{x^4+1}}$
- Then

$$\ln y = \ln \sqrt{\frac{x-1}{x^4+1}} = \frac{1}{2} \ln \frac{x-1}{x^4+1} = \frac{1}{2} \ln(x-1) - \frac{1}{2} \ln(x^4+1)$$

- Implicit differentiation and chain rule give

$$y' \frac{1}{y} = \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x^4+1} 4x^3 = \frac{1}{2x-2} - \frac{2x^3}{x^4+1}$$

- Solving for $y'$ and back substituting

$$y' = \sqrt{\frac{x-1}{x^4+1}} \left( \frac{1}{2x-2} - \frac{2x^3}{x^4+1} \right)$$

# Logarithmic Differentiation - Example II

- Determine $f'(x)$ for $f(x) = x^x$, $x > 0$
- $\frac{d}{dx}(a^b) = 0$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(b^x) = b^x \ln b$
- $\frac{d}{dx}(x^x)$?

$$y = x^x \iff \ln(y) = \ln(x^x) = x\ln(x)$$

- Apply implicit differentiation

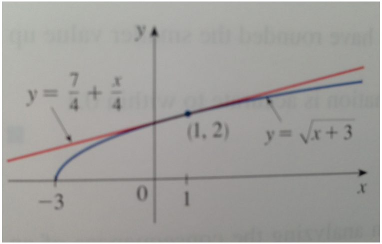
$$\frac{1}{y}y' = 1\ln(x) + x\frac{1}{x} = \ln(x) + 1$$

- Back substitute

$$y' = x^x(\ln(x) + 1)$$

# Linear Approximation

- Zooming in toward a point on the graph of a differentiable function, it looks more and more like its tangent line.




- Tangent lines are therefore good approximation of function values near the tangent point.
- Equation of the tangent line of a differentable function at $a$ is $y = f(a) + f'(a)(x - a)$
- The linear function given by $L(x) = f(a) + f'(a)(x - a)$ is called the *linearization* of $f$ at $a$.

# Linear Approximation - Example

- Let $f(x) = \sqrt{x+3} = (x+3)^{\frac{1}{2}}$
- $f'(x) = \frac{1}{2}(x+3)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+3}}$
- $f(1) = 2$ and $f'(1) = \frac{1}{4}$
- The linearization $L(x)$ of $f$ at $a = 1$ becomes

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}$$

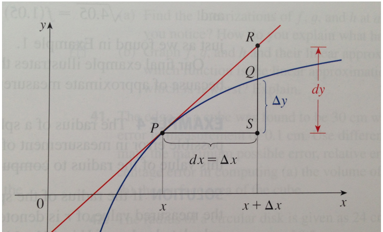
- Suppose that we want to approximate $f(x)$ around $x$ so that the error is $< 0.5$. Which values of $x$ will satisfy that?

$$\left|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)\right| < 0.5 \iff \sqrt{x+3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x+3} + 0.5$$

- These two inequalities imply that the error is $< 0.5$ if $x \in ]-2.66, 8.66[$
- If the permitted error is $< 0.1$, then $x \in ]-1.1, 3.9[$

# Differentials

- Differential provide another way of viewing linear approximations
- Let $f$ be a differentiable function and let $y = f(x)$
- Let $dx$ denote an independent variable that can be given the value of any real number.
- The *differential dy* is then the dependent variable given by $dy = f'(x)dx$.
- Let $P = (x, f(x))$ and $Q = (x + \Delta x, f(x + \Delta x))$ and let $dx = \Delta x$. $\Delta y$ represents how much the curve falls or rises when $x$ changes by $\Delta x = dx$ while $dy$ represents how much the tangent line falls or rises by the same change of $x$.



# Summary

You should after this lecture be familar with:

- Chain rule
- Implicit differentiation
- Logarithmic differentiation