MASD

Lecture 3
14.09.2021

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Objectives

We cover sections 3.4-3.6 and 3.10

- Differentiation of composite functions: Chain rule
- Implicit differentiation
- Logarithmic differentiation

Chain Rule (da. Kædereglen)

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- Intuition: Let $u = g(x)$ and $y = f(u)$. If u grows p times as fast as x and y grows q times as fast as u then y grows pq times as fast as x.

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- Leibniz notation

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Chain Rule - Example I

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  $3(4x^5 - 7x^3 + 14x^2 - 5)^2(20x^4 - 21x^2 + 28x)$

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- Let $y = \sqrt{u}$ and $u = 3x^2 - 7x + 12$
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- $\frac{du}{dx} = \frac{d}{dx}(3x^2 - 7x + 12) = 6x - 7$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(6x - 7) = \frac{6x - 7}{2\sqrt{3x^2 - 7x + 12}}$$

Chain Rule - Example III

- Let $F(x) = \sin(4x)$. This is a composite function.

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- Let $y = \sin(u)$ and $u = 4x$
- $\frac{dy}{du} = \frac{d}{du}(sin(u)) = \cos(u)$
- $\frac{du}{dx} = \frac{d}{dx}(4x) = 4$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos(u) * 4 = 4\cos(4x)$$

Chain Rule - Example IV

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- $\frac{du}{dx} = \frac{d}{dx}(\cos(x)) = -\sin(x)$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(-\sin(x)) = -3\cos^2(x)\sin(x)$

Chain Rule - Example V

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- $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$
- $\frac{du}{dx} = \frac{d}{dx}(-x) = -1$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(-1) = -e^{-x}$

Chain Rule - Example VI

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- Let $F(x) = e^{4x^3-6x+1}$. This is a composite function.
- Let $y = e^u$ and $u = 4x^3 - 6x + 1$
- $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$
- $\frac{du}{dx} = \frac{d}{dx}(4x^3 - 6x + 1) = 12x^2 - 6$

Chain Rule - Example VI

- Let $F(x) = e^{4x^3-6x+1}$. This is a composite function.
- Let $y = e^u$ and $u = 4x^3 - 6x + 1$
- $\frac{dy}{du} = \frac{d}{du}(e^u) = e^u$
- $\frac{du}{dx} = \frac{d}{dx}(4x^3 - 6x + 1) = 12x^2 - 6$
- $F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(12x^2 - 6) = (12x^2 - 6)e^{4x^3-6x+1}$

Chain Rule - Proof Attempt

- Definition of the derivative of $F = f \circ g$ at some a where g is differentiable at a and f is differentiable at $g(a)$:

$$F'(a) = (f \circ g)'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

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- If $g(x) \neq g(a)$ for any x near a, then

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \to a} \left[\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} \right] =$$

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- Unfortunately, there are functions such that $g(x) - g(a) = 0$ when x is arbitrarily close to a. For example $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ when x is close to 0.

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- Define $\epsilon(0) = 0$. Then $\lim_{\Delta x \to 0} \epsilon(\Delta x) = \epsilon(0)$ and ϵ is a continuous function of Δx
- $\Delta y = f'(a)\Delta x + \epsilon(\Delta x)\Delta x$ where $\epsilon(\Delta x) \to 0$ as $\Delta x \to 0$ for any function f differentiable in a.

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- $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $g(a)$.
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- $\Delta y = f'(b)\Delta u + \epsilon_2(\Delta u)\Delta u = [f'(b) + \epsilon_2(\Delta u)]\Delta u$ where $\epsilon_2(\Delta u) \to 0$ as $\Delta u \to 0$ (previous slide).

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- $\Delta y = [f'(b) + \epsilon_2(\Delta u)][g'(a) + \epsilon_1(\Delta x)]\Delta x$.
- $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}[f'(b) + \epsilon_2(\Delta u)][g'(a) + \epsilon_1(\Delta x)] = f'(b)g'(a) = f'(g(a))g'(a)$.

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- For example, $x^2 + y^2 = 25$ is an equation defining a circle with the center in origo and with radius 5.
- $x^2 + y^2 = 25$ implicitly defines 2 functions

  $$g_1(x) = \sqrt{25 - x^2} \text{ and } g_2(x) = -\sqrt{25 - x^2}$$

  on the closed interval $[-5, 5]$. g_1 defines the function whose graph is the upper half-circle while g_2 defines the function whose graph is the lower half-circle.

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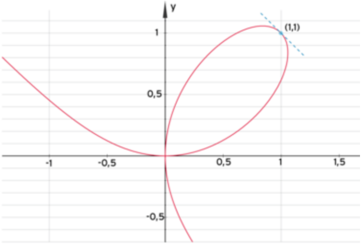
  on the closed interval $[-5, 5]$. g_1 defines the function whose graph is the upper half-circle while g_2 defines the function whose graph is the lower half-circle.
- We could find slopes at any point on this circle by determining the derivative of either g_1 or g_2.

Implicit Differentiation

- For the equation $x^3 + y^3 = 2xy$ it is however not that easy to identify implicitly defined functions with x as independent variable.



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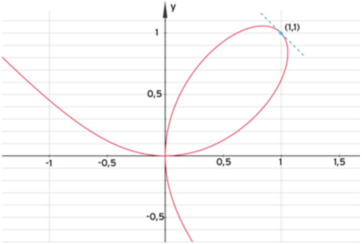

- But such equations define curves. One can still determine derivatives (slopes of tangents) of implicitly defined differentable functions without bothering about the functions itself.

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- Differentiate both sides with respect to x (applying the chain rule to y^2) and solve w.r.t. $\frac{dy}{dx}$.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) \iff \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = 0 \iff$$

$$2x+\frac{d}{dy}(y^2)\frac{dy}{dx} = 0 \iff 2x+2y\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = -\frac{x}{y}$$

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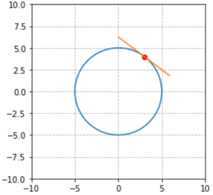
- At point $(3,4)$, $x = 3$ and $y = 4$ and $\frac{dy}{dx} = -\frac{3}{4}$



Implicit Differentiation - Example II: $(x-y)^2 = x + y - 1$

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$$\frac{d}{dx}[(x-y)^2] = \frac{d}{dx}[x+y-1] \iff$$

$$2(x-y)(1-\frac{dy}{dx}) = 1 + \frac{dy}{dx} \iff$$

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$$2x - 2y - 2(x-y)\frac{dy}{dx} = 1 + \frac{dy}{dx} \iff$$

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$$2x-2y-2(x-y)\frac{dy}{dx} = 1+\frac{dy}{dx} \iff$$

$$2x-2y-1 = (2x-2y+1)\frac{dy}{dx} \iff$$

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$$2x - 2y - 2(x-y)\frac{dy}{dx} = 1 + \frac{dy}{dx} \iff$$

$$2x - 2y - 1 = (2x - 2y + 1)\frac{dy}{dx} \iff$$

$$\frac{dx}{dy} = \frac{2x-2y-1}{2x-2y+1}$$

Implicit Differentiation - Example III: $x^3 + y^3 = 2xy$

- Changed notation.

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- Point $(1, 1)$ satisfies the equation and therefore the derivative y' (slope of the tangent) in that point is

$$y' = \frac{2 * 1 - 3 * 1^2}{3 * 1^2 - 2 * 1} = -1$$

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- Solving for y' and back substituting

$$y' = \sqrt{\frac{x-1}{x^4+1}} \left(\frac{1}{2x-2} - \frac{2x^3}{x^4+1} \right)$$

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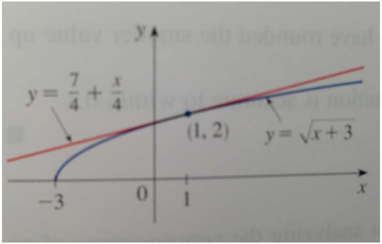
Linear Approximation

- Zooming in toward a point on the graph of a differentiable function, it looks more and more like its tangent line.



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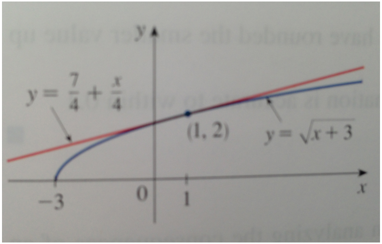

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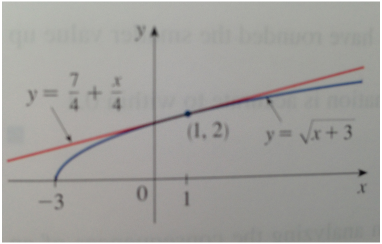

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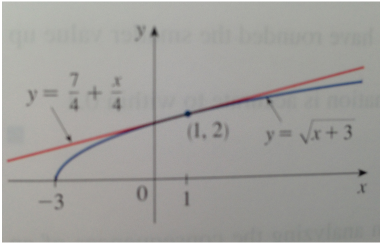

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- If the permitted error is < 0.1, then $x \in]-1.1, 3.9[$

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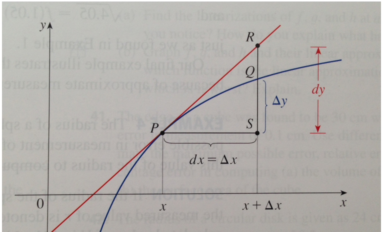
- Let $P = (x, f(x))$ and $Q = (x + \Delta x, f(x + \Delta x))$ and let $dx = \Delta x$. Δy represents how much the curve falls or rises when x changes by $\Delta x = dx$ while dy represents how much the tangent line falls or rises by the same change of x.



Summary

You should after this lecture be familar with:

- Chain rule
- Implicit differentiation
- Logarithmic differentiation