Solutions to MASD (Probability Part) Assignment 6

Problem 1 (Conditional Probabilities):

1. Let B be an event with positive probability, $\mathbb{P}(B) > 0$, on some probability space $(S, \mathbb{P})$ and define $\widehat{\mathbb{P}}(A) := \mathbb{P}(A|B)$ for any event $A \subset S$. Show that $\widehat{\mathbb{P}}$ is also a probability distribution on S.

2. We are presented with a hat with $B = 2$ blue and $R = 3$ red balls in it. In the first round we blindly pick a ball from the hat. Then we return to the hat the ball we picked and add another one of the same colour (increasing the number of balls in the hat by 1). In the second round we proceed in the same way, but we add two balls of the picked colour (increasing the number of balls in the hat by 2 compared to before the second round). Afterwards we pick a ball blindly from the hat. What is the probability that this last ball is red?

   Hint: Use the law of total probability.

Solution to Problem 1: For 1: We check the two defining properties of probability distributions. We start with the non-negativity, which is obvious by the non-negativity of $\mathbb{P}$. Then we have

$$\mathbb{P}(S|B) = \frac{\mathbb{P}(S \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1\,.$$

To see additivity we pick a sequence $A_1, A_2, \dots$ of mutually exclusive events and compute

$$\mathbb{P}(\cup_i A_i|B) = \frac{\mathbb{P}(\cup_i (A_i \cap B))}{\mathbb{P}(B)} = \sum_i \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_i \mathbb{P}(A_i|B)\,,$$

where we used the additivity of $\mathbb{P}$ and that $A_1 \cap B, A_2 \cap B, \dots$ are mutually exclusive.

For 2: Let X_1, X_2 and X_3 random variables with values in $\{b, r\}$, indicating the colours blue and red, that model our picks in each of the three rounds. We are asked the probability $\mathbb{P}(X_3 = r)$. To compute it we use the law of total probability and compute

$$\begin{aligned}
\mathbb{P}(X_3 = r) &= \mathbb{P}(X_3 = r|X_2 = r, X_1 = r)\mathbb{P}(X_2 = r, X_1 = r) \\
&\quad + \mathbb{P}(X_3 = r|X_2 = r, X_1 = b)\mathbb{P}(X_2 = r, X_1 = b) \\
&\quad + \mathbb{P}(X_3 = r|X_2 = b, X_1 = r)\mathbb{P}(X_2 = b, X_1 = r) \\
&\quad + \mathbb{P}(X_3 = r|X_2 = b, X_1 = b)\mathbb{P}(X_2 = b, X_1 = b) \\
&= \mathbb{P}(X_3 = r|X_2 = r, X_1 = r)\mathbb{P}(X_2 = r|X_1 = r)\mathbb{P}(X_1 = r) \\
&\quad + \mathbb{P}(X_3 = r|X_2 = r, X_1 = b)\mathbb{P}(X_2 = r|X_1 = b)\mathbb{P}(X_1 = b) \\
&\quad + \mathbb{P}(X_3 = r|X_2 = b, X_1 = r)\mathbb{P}(X_2 = b|X_1 = r)\mathbb{P}(X_1 = r) \\
&\quad + \mathbb{P}(X_3 = r|X_2 = b, X_1 = b)\mathbb{P}(X_2 = b|X_1 = b)\mathbb{P}(X_1 = b)\,.
\end{aligned}$$

Now we simply compute all of these probabilities. This is simple since we given the past picks we know the number of balls of each colour in the hat. We find

$$\begin{aligned}
\mathbb{P}(X_1 = b) &= \frac{B}{R+B}\,,\\
\mathbb{P}(X_1 = r) &= \frac{R}{R+B}\,,\\
\mathbb{P}(X_2 = b|X_1 = b) &= \frac{B+1}{R+B+1}\,,\\
\mathbb{P}(X_2 = b|X_1 = r) &= \frac{B}{R+B+1}\,,\\
\mathbb{P}(X_2 = r|X_1 = b) &= \frac{R}{R+B+1}\,,\\
\mathbb{P}(X_2 = r|X_1 = r) &= \frac{R+1}{R+B+1}\,,\\
\mathbb{P}(X_3 = r|X_1 = b, X_2 = b) &= \frac{R}{R+B+3}\,,\\
\mathbb{P}(X_3 = r|X_1 = b, X_2 = r) &= \frac{R+2}{R+B+3}\,,\\
\mathbb{P}(X_3 = r|X_1 = r, X_2 = b) &= \frac{R+1}{R+B+3}\,,\\
\mathbb{P}(X_3 = r|X_1 = r, X_2 = r) &= \frac{R+3}{R+B+3}\,.
\end{aligned}$$

Inserting these gives the result

$$\begin{aligned}
\mathbb{P}(X_3 = r) = &\frac{R+3}{R+B+3}\frac{R+1}{R+B+1}\frac{R}{R+B} + \frac{R+1}{R+B+3}\frac{R}{R+B+1}\frac{B}{R+B}\\
&+ \frac{R+2}{R+B+3}\frac{B}{R+B+1}\frac{R}{R+B} + \frac{R}{R+B+3}\frac{B+1}{R+B+1}\frac{B}{R+B} = \frac{R}{R+B} = \frac{3}{5}\,.
\end{aligned}$$

Problem 2 (Discrete and Continuous Random Variables): Determine and draw the cdf of the following random variable X:

1. Let X be a roll of the unfair dice from Problem 1.2 on Assignment sheet 5.
2. Let X be a continuous random variable with PDF $f_X(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$.

Determine the PDF (for continuous random variables) or PMF (for discrete random variables) for the distribution of the following random variable X.

3. Let X have CDF $F_X(x) = 4x$ for $x \in [0, 1/4]$.
4. Let X have CDF $F_X(x) = 1 - \frac{1}{\lfloor x \rfloor^k}$ for any $x \geq 1$, where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ is rounding down and $k \in \mathbb{N}$.

Solution to Problem 2 For 1: We have

$$\mathbb{P}_1(X = i) := p_i\,, \qquad \text{where } p_1 := \frac{107}{1000}\,,\ p_2 := \frac{195}{1000}\,,\ p_3 := \frac{52}{1000}\,,\ p_4 := \frac{492}{1000}\,,\ p_5 := \frac{112}{1000}\,,\ p_6 := \frac{42}{1000}\,.$$

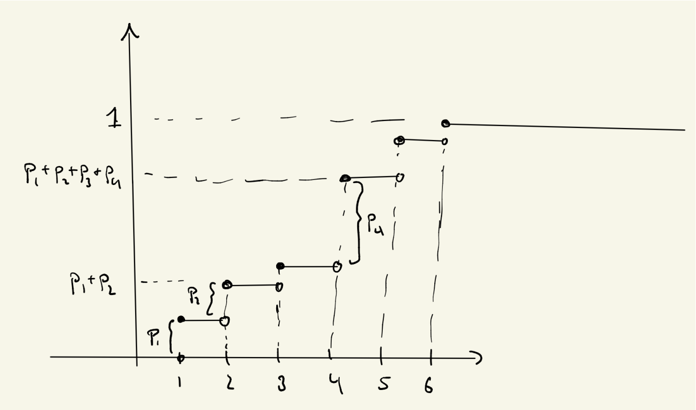
for $i = 1, 2, 3, 4, 5, 6$. The CDF F_X has jumps at positions $x = 1, 2, 3, 4, 5, 6$, otherwise it is constant. In particular,

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{i=1}^{x} \mathbb{P}(X = i) = \sum_{i=1}^{x} p_i\,,$$

for $x = 1, 2, 3, 4, 5, 6$. Strictly to the left of $x = 1$ the CDF is zero since dice rolls do not have such values. To the right of $x = 6$ the cdf is equal to 1 since the dice takes a value between $-\infty$ and 6 with probability 1. Altogether we get

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \sum_{i=1}^{\lfloor x \rfloor} p_i & \text{if } x \in [1, 6] \\ 1 & \text{if } x > 6 \end{cases},$$

where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ is rounding down.



For 2: To determine F_X we have to compute the probability

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(y)\mathrm{d}y = \frac{1}{2}\int_{-\infty}^{x} \mathrm{e}^{-|y|}\mathrm{d}y$$

for any $x \in \mathbb{R}$. It is best to distinguish the cases $x < 0$ and $x > 0$. Let us start with $x < 0$. Then

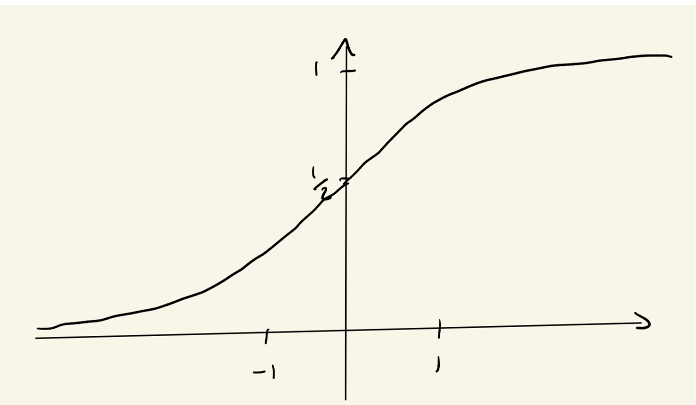
$$\mathbb{P}(X \leq x) = \frac{1}{2}\int_{-\infty}^{x} \mathrm{e}^{-|y|}\mathrm{d}y = \frac{1}{2}\int_{-\infty}^{x} \mathrm{e}^{y}\mathrm{d}y = \frac{1}{2}\mathrm{e}^{y}|_{-\infty}^{x} = \frac{\mathrm{e}^{x}}{2}.$$

For $x > 0$ we find

$$\mathbb{P}(X \leq x) = \mathbb{P}(X \leq 0) + \mathbb{P}(X \in (0, x)) = \frac{1}{2} + \frac{1}{2}\int_{0}^{x} \mathrm{e}^{-|y|}\mathrm{d}y = \frac{1}{2} + \frac{1}{2}\int_{0}^{x} \mathrm{e}^{-y}\mathrm{d}y = \frac{1}{2} + \frac{1 - \mathrm{e}^{-x}}{2}.$$

Altogether we found

$$F_X(x) = \begin{cases} \frac{\mathrm{e}^{x}}{2} & \text{if } x \leq 0 \\ \frac{1}{2} + \frac{1-\mathrm{e}^{-x}}{2} & \text{if } x > 0 \end{cases}.$$



For 3: The random variable X must be continuous. Since $F_X(1/4) = 1$ we conclude from monotonicity of F_X and F_X having values in $[0, 1]$ that $F_X(x) = 1$ for all $x > 1/4$. Similarly, we conclude from

$F_X(0) = 0$ that $F_X(x) = 0$ for all $x < 0$. In particular, the PDF f_X must vanish on $\mathbb{R} \setminus [0, 1/4]$. Finally for $x \in [0, 1/4]$ we compute $f_X(x) = F_X'(x) = 4$. Thus

$$f_X(x) = 4\mathbb{1}_{[0,1/4]}(x)\,,$$

i.e. X is uniformly distributed on $[0, 1/4]$.

For 4: Since $F_X(1) = 0$ we see that X can only take values greater or equal to 1. Furthermore, $F_X(x)$ has jumps at every $x \in \mathbb{N}$ and is otherwise constant. We conclude that X must be discrete with values in $\mathbb{N}$. Now we compute the PMF $P_X : \mathbb{N} \to [0, 1]$ of X by computing the jump heights

$$P_X(x) = \mathbb{P}(X = x) = F_X(x) - \lim_{y \uparrow x} F_X(y) = F_X(x) - F_X(x-1) = \frac{1}{(x-1)^k} - \frac{1}{x^k}\,.$$

Problem 3 (Expectation and Variance): Determine the expectation and variance of the following random variable X:

1. Let X be a roll of the unfair dice from Problem 1.2 on Assignment sheet 5.
2. Let $X = 4Y + 3$, where $Y \sim \text{Bernoulli}(p)$
3. Let $X \sim \text{Binomial}(n, p)$.
4. Let $X = \log Y$, where Y is uniformly distributed on $[0, 1]$.
5. Let X have CDF $F_X(x) = 1 - \mathrm{e}^{-\alpha x}$ for $x \geq 0$ with some constant $\alpha > 0$.

Solution to Problem 3

For 1: We use the notation from Problem 2.1 on this sheet. Then

$$\mathbb{E}X = 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + 4 \cdot p_4 + 5 \cdot p_5 + 6 \cdot p_6 = 3.433\,.$$

For the variance we first compute the second moment

$$\mathbb{E}X^2 = 1^2 \cdot p_1 + 2^2 \cdot p_2 + 3^2 \cdot p_3 + 4^2 \cdot p_4 + 5^2 \cdot p_5 + 6^2 \cdot p_6 = 13.539\,.$$

Then

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 1.753511\,.$$

For 2: First we compute expectation and variance of Y (or we look it up). We get

$$\mathbb{E}Y = 1 \cdot \mathbb{P}(Y = 1) = p\,,$$

and

$$\operatorname{Var} Y = (0 - p)^2\mathbb{P}(Y = 0) + (1 - p)^2\mathbb{P}(Y = 1) = p^2(1 - p) + (1 - p)^2 p = p(1 - p)\,.$$

Now we compute the expectation of X using linearity of the expectation

$$\mathbb{E}X = \mathbb{E}(4Y + 3) = 4\mathbb{E}Y + 3 = 4p + 3\,,$$

and the variance using its transformation rules

$$\operatorname{Var}(4Y + 3) = 4^2 \operatorname{Var} Y = 16p(1 - p)\,.$$

For 3: We determined the expectation in the lecture. It is $\mathbb{E}X = np$. Now we compute the second moment,

$$\begin{aligned}
\mathbb{E}X^2 &= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{k\,n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
&\overset{(1)}{=} np \sum_{k=0}^{n-1} \frac{(k+1)\,(n-1)!}{k!(n-k-1)!} p^k (1-p)^{n-k-1} \\
&\overset{(2)}{=} np \sum_{k=0}^{n-1} \frac{k\,(n-1)!}{k!(n-k-1)!} p^k (1-p)^{n-k-1} + np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \\
&\overset{(3)}{=} np \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k-1)!} p^k (1-p)^{n-k-1} + np \\
&\overset{(4)}{=} n(n-1)p^2 \sum_{k=0}^{n-2} \binom{n-2}{k} p^k (1-p)^{n-k-2} + np \\
&\overset{(5)}{=} n(n-1)p^2 + np\,.
\end{aligned}$$

Here, in (1) we changed the summation index $k \to k+1$. In (2) we pulled the sums apart. In (3) we used the binomial theorem to see that the last sum equals 1. In (4) we again changed the summation index $k \to k+1$ and in (5) we used again the binomial theorem. Finally, we compute the variance

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)\,.$$

Another simple way to see this result is to use that X has the same distribution as $\sum_{i=1}^{n} X_i$ where X_i are independent and Bernoulli(p)-distributed random variables. Therefore by the Bienaymé formula we get

$$\operatorname{Var} X = \operatorname{Var} \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \operatorname{Var} X_i = \sum_{i=1}^{n} p(1-p) = np(1-p)\,.$$

For 4: We use the formula $\mathbb{E}f(Y) = \int f(x)\rho_Y(x)\mathrm{d}x$ with $f(x) = \log x$ to compute

$$\mathbb{E}X = \int_0^1 \log x \,\mathrm{d}x = x \log x|_0^1 - \int_0^1 \mathrm{d}x = -1\,,$$

where integration by parts was used.

For 5: Since the cdf of X does not have jumps and is zero on negative numbers (since $F_X(0) = 0$ and F_X is monotonously increasing with non-negative values) we suspect that X must be a continuous random variable. To compute its pdf we differentiate F_X. Thus,

$$f_X(x) = F_X'(x) = \alpha\,\mathrm{e}^{-\alpha x},$$

i.e. $X \sim \operatorname{Exp}(\alpha)$ (as defined in the lecture). Now we compute the first two moments. The expectation is

$$\mathbb{E}\,X \overset{(1)}{=} \int_0^\infty x\,\alpha\,\mathrm{e}^{-\alpha x}\mathrm{d}x \overset{(2)}{=} -\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} \int_0^\infty \mathrm{e}^{-\alpha x}\mathrm{d}x \overset{(3)}{=} -\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{\alpha} \overset{(4)}{=} \frac{1}{\alpha}\,.$$

In (1) we used the formula for computing expectations of continuous random variables. In (2) we use a useful trick that gets rid of the factor x in the integrand, namely to write $x\mathrm{e}^{-\alpha x} = -\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{e}^{-\alpha x}$ and then to pull the derivative out of the integral. Instead of that we could have simply used integration by parts, but this trick is still useful to keep in mind. In (3) we integrated the exponential

$$\int_0^\infty \mathrm{e}^{-\alpha x} \mathrm{d}x = -\frac{1}{\alpha} \mathrm{e}^{-\alpha x}|_0^\infty = -\frac{1}{\alpha}(0-1) = \frac{1}{\alpha} \,.$$

In (5) we differentiate $\frac{\mathrm{d}}{\mathrm{d}\alpha}\alpha^{-1} = -\alpha^{-2}$.

Now that we have computed the expectation we will determine the variance. For that we compute the second moment

$$\mathbb{E}\, X^2 = \int_0^\infty x^2\, \alpha\, \mathrm{e}^{-\alpha x} \mathrm{d}x = \alpha \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \int_0^\infty \mathrm{e}^{-\alpha x} \mathrm{d}x = \alpha \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \frac{1}{\alpha} = \frac{2}{\alpha^2} \,.$$

Note that we used the same trick with differentiating the exponent and in this way the calculation did not become any longer (except that we have to take two derivatives at the end now). We could have used integration by part twice instead, but that is a slightly longer and messier calculation. Now we compute the variance

$$\mathrm{Var}(\mathrm{X}) = \mathbb{E}\,\mathrm{X}^2 - (\mathbb{E}\,\mathrm{X})^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2} \,.$$