MASD

Lecture 2
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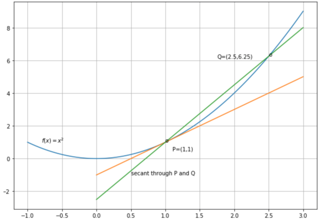
Objectives

After today's lecture, you should

- Know the definition of the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ and have an intuitive understanding of what it means (Sections 2.7-2.8).
- Know some differentiation rules (Sections 3.1 and 3.2). More rules and their use will follow next week.

Tangents Revisited



- When $x \to a$ then $Q = (x, f(x))$ approaches $P = (a, f(a))$ on the graph of f. The slope m_{PQ} of the secant line approaches the slope of the tangent line of f at P.
- The slope of the secant line is given by

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

- The slope of the tangent line of C at P is defined by

$$m_P = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Tangents Revisited

- Let $h = x - a$. Then m_{PQ} and m_P can be written as

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}$$

$$m_P = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Consider the function $f(x) = \frac{3}{x}$. What is the slope of the tangent at point $P = (3, 1)$?

$$m_P = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\frac{3-(3+h)}{3+h}}{h} =$$

$$\lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

Velocity Revisited

- What is the velocity of an object at a specific time?
- Assume that an object starts in the origo and moves along the x-axis. Its position (in meters) s at time t (in seconds) is given by $s(t) = 3t^2, t \geq 0$
- Average velocity of the object between times t_0 and t?

$$\text{av. velocity} = \frac{\text{distance travelled}}{\text{time elapsed}} = \frac{s(t) - s(t_0)}{t - t_0} = 3\frac{t^2 - t_0^2}{t - t_0}$$

- Instantaneous velocity at time t_0?

$$\text{inst. velocity} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h} =$$

$$\lim_{h \to 0} \frac{3(t_0 + h)^2 - 3t_0^2}{h} = \lim_{h \to 0} \frac{3t_0^2 + 6t_0 h + h^2 - 3t_0^2}{h} =$$

$$\lim_{h \to 0} (6t_0 + h) = \lim_{h \to 0} 6t_0 + \lim_{h \to 0} h = 6t_0 + 0 = 6t_0$$

Derivatives as Functions

- Given a function $f : D \to \mathbb{R}$, its *derivative function* (da. afledte funktion) f' is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  where x is a variable. This is a function not a number.
- The domain D' of f' is the set $\{x | f'(x) \text{ exists}\}$.
- Common alternative notations (assuming that $y = f(x)$):

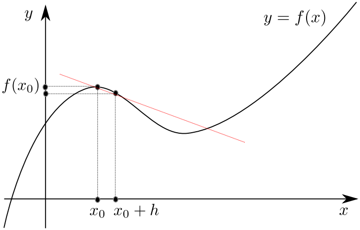
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

- Leibnitz notation: $\frac{dy}{dx}$

Differentiability

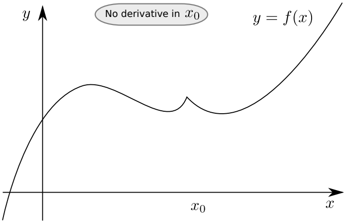
- The *derivative* of a function $f : \mathbb{R} \to \mathbb{R}$ at some x_0 in the domain of f is the limit

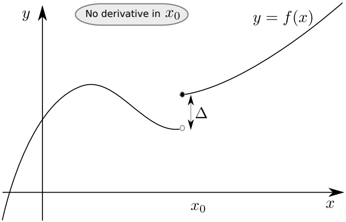
$$\frac{df}{dx}(x_0) = f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



- A function f is *differentiable at some* x_0 in the domain of f if $f'(a)$ exists.
- When the derivative f' of f exists in every point x_0 in an open interval $]a, b[\in \mathbb{R}$, then f is *differentiable* in $]a, b[$ (a could be $-\infty$ and/or b could be ∞).

Does a derivative always exist?





- If f is differentiable at x_0 then it is continuous in x_0.
- There are continuous functions that are not differentiable (e.g. $f(x) = |x|$ is not differentiable at 0).

Proof: f is differentiable in a $\implies$ f is continuous in a

We have to show that $\lim_{x\to a} f(x) = f(a)$.

$$\lim_{x\to a} f(x) = \lim_{x\to a}(f(x) + f(a) - f(a)) =$$

$$\lim_{x\to a} f(a) + \lim_{x\to a}[f(x) - f(a)] =$$

$$\lim_{x\to a} f(a) + \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}(x - a)\right] =$$

$$\lim_{x\to a} f(a) + \lim_{x\to a}\frac{f(x) - f(a)}{x - a}\lim_{x\to a}(x - a) =$$

$$\lim_{x\to a} f(a) + f'(a) * 0 = f(a)$$

Derivatives of Constant Functions

- **Constant Rule**: Let $f(x) = c$ where c is an arbitrary constant. Then $f'(x) = 0$. Let $a \in \mathbb{R}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{c - c}{h} = 0$$

Derivatives of Power Functions

- **Power Rule**: Let $f(x) = x^n$ where n is a positive integer. Then $f'(x) = nx^{n-1}$. Let $a \in \mathbb{R}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} =$$

$$\lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})}{x - a} =$$

$$\lim_{x \to a} (x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1}) =$$

$$a^{n-1} + a^{n-2}a + \ldots + aa^{n-2} + a^{n-1} = na^{n-1}$$

- Generalizes to n being any real number (proof in Section 3.6).

New Derivatives from Old Derivatives

- **Constant Multiple Rule**: Let $g(x) = cf(x)$ where c is a constant and f is a differentiable function.

  $$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} =$$

  $$\lim_{h \to 0} c\frac{f(x+h) - f(x)}{h} = c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = cf'(x)$$

  where the second last equality follows from one of the limit laws.

Sum Rule

- Let $F(x) = f(x) + g(x)$ where both f and g are differentiable.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(h)}{h} =$$

$$\lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} =$$

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] =$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} =$$

$$f'(x) + g'(x)$$

  where the second last equality follows from one of the limit laws.
- **Difference Rule**: similar.
- These rules can be used to differentiate any polynomial.

Differentiating Polynomials

- Let $f(x) = x^7 - 4x^5 + 13x^4 - x + 19$

  $f'(x) = 7x^6 - 4*5x^4 + 13*4x^3 - 1x^0 + 0 = 7x^6 - 20x^4 + 52x^3 - 1$

Exponential Functions

- Let $f(x) = b^x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} =$$

$$\lim_{h\to 0} \frac{b^x b^h - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h\to 0} \frac{b^h - 1}{h} = b^x f'(0)$$

- This shows that if $f(x) = b^x$ is differentiable at 0 then it is differentiable everywhere.
- If $f(x) = 2^x$ then $f'(x) = f'(0)2^x$ and $f'(0)$=0.693...
- If $f(x) = 3^x$ then $f'(x) = f'(0)3^x$ and $f'(0)$=1.099...
- Determine constant b such that the exponential function $f(x) = b^x$ satisfies $f'(0) = 1$. This happens for b=2.71828... Then $f'(x) = f(x)$. This special constant has its own letter e (**e**xponential or natural number).

Product Rule (da. Produktreglen)

- Let $F(x) = f(x)g(x)$ where both f and g are differentiable. Then

$$F'(x) = f(x)g'(x) + f'(x)g(x)$$

- Let $F(x) = x^3 \cos x$. Hence, $f(x) = x^3$ and $g(x) = \cos x$

$$F'(x) = x^3(-\sin x) + 3x^2 \cos x = -x^3 \sin x + 3x^2 \cos x$$

Product Rule - Algebraic Proof

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} =$$

$$\lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0} \frac{g(x)[f(x+h) - f(x)]}{h} =$$

$$\lim_{h\to 0} \left[f(x+h) \frac{g(x+h) - g(x)}{h} \right] + \lim_{h\to 0} \left[g(x) \frac{f(x+h) - f(x)}{h} \right] =$$

$$\lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

$$f(x)g'(x) + g(x)f'(x)$$

Quotient Rule (da. Kvotientreglen)

- Let $F(x) = \frac{f(x)}{g(x)}$ where both f and g are differentiable.

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

- Let $F(x) = \frac{2-x^2}{2x^3+x+3}$. Hence $f(x) = 2 - x^2$ and $g(x) = 2x^3 + x + 3$.

$$F'(x) = \frac{(2x^3 + x + 3)(-2x) - (2 - x^2)(6x^2 + 1)}{(2x^3 + x + 3)^2} =$$

$$\frac{2x^4 - 13x^2 - 6x - 2}{(2x^3 + x + 3)^2}$$

- Low Dee High Minus High Dee Low, Over the Square of What is Below

Quotient Rule - Algebraic Proof

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} =$$

$$\lim_{h \to 0} \left\{ \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right\} =$$

$$\lim_{h \to 0} \left\{ \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right\} =$$

$$\lim_{h \to 0} \left\{ \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \right\} =$$

$$\lim_{h \to 0} \left\{ \frac{1}{g(x+h)g(x)} \left[\frac{f(x+h)g(x) - f(x)g(x)}{h} + \frac{f(x)g(x) - f(x)g(x+h)}{h} \right] \right\} =$$

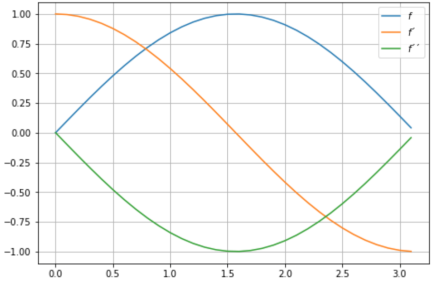
$$\lim_{h \to 0} \left\{ \frac{1}{g(x+h)g(x)} \left[g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h} \right] \right\} =$$

$$\lim_{h \to 0} \frac{1}{g(x+h)g(x)} \left[\lim_{h \to 0} g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right] =$$

$$\frac{g(x)f'(x) - f(x)g'(x)}{\lim_{h \to 0} g(x+h) \lim_{h \to 0} g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Higher Order Derivatives

- f'' is the *second order* derivative of f and the derivative of f'. It can be interpreted as the rate of change of the slope of the curve defined by f.
- If f is the distance covered by a moving object as the function of time, f' is its instantaneous velocity as the function of time (or the rate of position change), and f'' is its instantaneous accelaration as the function of time (or the rate of velocity change).



Summary

By now, you should be familiar with

- the definition of function limits,
- the most familiar rules for function limits, and be able to use them for deriving limits of functions,
- the definition of continuity and proving continuity of functions,
- the definition of the derivative of a function $f : \mathbb{R} \to \mathbb{R}$,
- basic differentiation rules.