MASD 2020, Assignment 4

Hand-in in groups of 2 or 3 before 29.9.2020 at 10.00
One submission per group
Remember to include the names of all group memebers

Exercise 1 (Sequences). a) Prove that if you have a convergent sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$, then the sequence $\{-a_n\}$ is also convergent and satisfies $\lim_{n\to\infty} -a_n = -a$.

b) Using the definition of a limit in one and higher dimensions, prove that if $\mathbf{a}_n \in \mathbb{R}^d$ is a sequence of vectors and $\mathbf{a} \in \mathbb{R}^d$ is a vector, then

$$\mathbf{a}_n \to \mathbf{a} \qquad \text{if and only if} \qquad (\mathbf{a}_n)_i \to \mathbf{a}_i \quad \text{for all } i = 1, \dots, d,$$

where $(\mathbf{a}_n)_i$ is the i^{th} coordinate of $\mathbf{a}_n$, and $\mathbf{a}_i$ is the i^{th} coordinate of $\mathbf{a}$.

Deliverables. The proofs.

Solution:

a) **Assumptions:** $\lim_{n\to\infty} a_n = a$. That is, there exists some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n > N$.

Claim: $\lim_{n\to\infty} -a_n = -a$. That is, for any $\varepsilon > 0$ there exists a corresponding $N \in \mathbb{N}$ such that

$$|-a_n - (-a)| < \varepsilon \quad \text{whenever } n > N.$$

Proof: Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exists an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n > N$. Now, since

$$|-a_n - (-a)| = |(-1)(a_n - a)| = |-1||a_n - a| = |a_n - a|,$$

we also have, whenever $n > N$, that $|a_n - a| < \varepsilon$, which was what we needed to show. (That is, we can use the same N for the sequence $\{-a_n\}$ as for the sequence $\{a_n\}$.)

Hence, the sequence $\{-a_n\}$ converges with limit $\lim_{n\to\infty} -a_n = -a$. □

b) **Claim:** We need to show that

$$\mathbf{a}_n \to \mathbf{a} \qquad \text{if and only if} \qquad (\mathbf{a}_n)_i \to \mathbf{a}_i \quad \text{for all } i = 1, \dots, d.$$

Note that "if and only if" is another way to say "$\Leftrightarrow$". That is, in order to prove the claim, then we need to prove:

$$\mathbf{a}_n \to \mathbf{a} \qquad \Rightarrow \qquad (\mathbf{a}_n)_i \to \mathbf{a}_i \quad \text{for all } i = 1, \dots, d,$$

and

$$(\mathbf{a}_n)_i \to \mathbf{a}_i \quad \text{for all } i = 1, \dots, d \qquad \Rightarrow \qquad \mathbf{a}_n \to \mathbf{a}.$$

Proof: First, we prove "$\Rightarrow$". In this case, we **assume** that $\mathbf{a}_n \to \mathbf{a}$, and **claim** that $(\mathbf{a}_n)_i \to \mathbf{a}_i$ for all $i = 1, \dots, d$.

Let $\varepsilon > 0$. By the definition of convergence, there exists some integer $N \in \mathbb{N}$ such that whenever $n > N$, we have $\|\mathbf{a}_n - \mathbf{a}\| < \varepsilon$. Now, let $i \in \{1, \dots, d\}$ be any coordinate index, and let and let $n > N$. We have

$$|(\mathbf{a}_n)_i - \mathbf{a}_i| = \sqrt{|(\mathbf{a}_n)_i - \mathbf{a}_i|^2} \leq \sqrt{\sum_{j=1}^{d} |(\mathbf{a}_n)_j - \mathbf{a}_j|^2} = \|\mathbf{a}_n - \mathbf{a}\| < \varepsilon,$$

where the inequality holds because

$$|(\mathbf{a}_n)_i - \mathbf{a}_i|^2 \leq \sum_{j=1}^{d} |(\mathbf{a}_n)_j - \mathbf{a}_j|^2,$$

as i is one of the $j = 1, \dots, d$, and $|(\mathbf{a}_n)_j - \mathbf{a}_j|^2 \geq 0$ for all $j = 1, \dots d$.
By the definition of convergence, we have now shown $(\mathbf{a}_n)_i \to \mathbf{a}_i$. Since i was arbitrarily chosen, this holds for all $i = 1, \dots, d$.

Next, we prove "$\Leftarrow$". In this case, we **assume** that $(\mathbf{a}_n)_i \to \mathbf{a}_i$ for all $i = 1, \dots, d$, and **claim** that $\mathbf{a}_n \to \mathbf{a}$.
Let $\varepsilon > 0$, and define another $\varepsilon' = \frac{\varepsilon}{\sqrt{d}} > 0$. Let $i \in \{1, \dots, d\}$. By the assumption and the definition of convergence, there exists some $N_i \in \mathbb{N}$ such that if $n > N_i$, then we have

$$|(\mathbf{a}_n)_i - \mathbf{a}_i| < \varepsilon' \Rightarrow |(\mathbf{a}_n)_i - \mathbf{a}_i|^2 = |(\mathbf{a}_n)_i - \mathbf{a}_i| \cdot |(\mathbf{a}_n)_i - \mathbf{a}_i| < \varepsilon'^2 = (\frac{\varepsilon}{\sqrt{d}})^2 = \frac{\varepsilon^2}{d}.$$

Since this holds for any $i \in \{1, \dots, d\}$, we can set $N = \max\{N_i | i = 1, \dots, d\}$. Now, if $n > N$, we also have $n > N_i$ for all $i = 1, \dots, d$. Thus, whenever $n > N$, we have

$$\|\mathbf{a}_n - \mathbf{a}\|^2 = \sqrt{\sum_{i=1}^{d} |(\mathbf{a}_n)_i - \mathbf{a}_i|^2} < \sqrt{\sum_{i=1}^{d} \varepsilon'^2} = \sqrt{d\varepsilon'^2} = \sqrt{d}\varepsilon' = \sqrt{d}\frac{\varepsilon}{\sqrt{d}} = \varepsilon.$$

But then, by the definition of convergence, we have $\mathbf{a}_n \to \mathbf{a}$.
We have now shown both directions, and are therefore done. □

Exercise 2 (Series.). Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Hint: Convert this problem to the well-known cake-eating problem: If there is a cake and an infinite line of cake-hungry people, and every person takes half of the piece that is left, the sequence c_n measuring the proportion of the cake left after n people converges to 0.
Help: If you have forgotten all about induction proofs, you can find a note on induction proofs from DMA in Absalon under Files/dma5noter.pdf.

Deliverables. The proof.
Solution:
Assumptions: There are no non-obvious assumptions here.
Claim: Let $n \in \mathbb{N}$. Then our claim is that series $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.
Proof: A series $\sum_{n=1}^{\infty} a_n$ is convergent and has the sum $\sum_{n=1}^{\infty} a_n = 1$ if the corresponding sequence

$$S_k = \sum_{n=1}^{k} a_n$$

is convergent and satisfies $\lim_{k\to\infty} S_k = 1$.

In this exercise, we have $a_n = \frac{1}{2^n}$, giving $S_k = \sum_{n=1}^{k} \frac{1}{2^n}$.

As hinted in the exercise, one way to prove the claim is by showing that for the sequence $S'_k = 1 - S_k$, we have $\lim_{k\to\infty} S'_k = 0$, in which case

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} (1 - S'_k) = \lim_{k\to\infty} 1 - \lim_{k\to\infty} S'_k = 1 - 0 = 1,$$

where the second and third equalites follow from the limit properties (sum rule and limit of constant). Thus, we are left to show that $\lim_{k\to\infty} S'_k = 0$.

As hinted in the exercise, we can do so by showing that $S'_k = \frac{1}{2^k}$. As showed in the lecture, we know that $\lim_{k\to\infty} \frac{1}{2^k} = 0$.

We shall prove this by induction on k. In the case $k = 1$, it is clear that

$$S'_k = 1 - \sum_{n=1}^{k=1} \frac{1}{2^n} = 1 - \frac{1}{2} = \frac{1}{2} = \frac{1}{2^k}.$$

Next, **assume** that the claim holds for $k = K - 1$, that is,

$$S'_{K-1} = 1 - \sum_{n=1}^{K-1} \frac{1}{2^n} = \frac{1}{2^{K-1}}.$$

Claim: We need to show that

$$S'_K = 1 - \sum_{n=1}^{K} \frac{1}{2^n} = \frac{1}{2^K}.$$

To **prove** this, we compute

$$S'_K = 1 - \sum_{n=1}^{K} \frac{1}{2^n} = 1 - \left(\sum_{n=1}^{K-1} \frac{1}{2^n} + \frac{1}{2^K}\right) = 1 - \sum_{n=1}^{K-1} \frac{1}{2^n} - \frac{1}{2^K} = \frac{1}{2^{K-1}} - \frac{1}{2^K} = \frac{1}{2^{K-1}}\left(1 - \frac{1}{2}\right) = \frac{1}{2^K}.$$

Here, the fourth equality used the inductive assumption.

Now, by induction, we have shown that $S'_k = \frac{1}{2^k}$ for all $k \in \mathbb{N}$. By the arguments above, the proof is complete.

Exercise 3 (Practicing integration). Solve the following definite and indefinite integrals. Solving an indefinite integral means writing down all its derivatives. Include (and explain) intermediate steps.

a) $\int 3x^2 dx$

b) $\int_{-1}^{1} x^{100} dx$

c) $\int_{1}^{8} \sqrt[3]{x} dx$

d) $\int_{0}^{1} (3 + x\sqrt{x}) dx$

e) $\int xe^{-x^2} dx$ using substitution $u = -x^2$

f) $\int \sqrt{x} \ln(x) dx$ using integration by parts with $u = \ln(x), dv = \sqrt{x} dx$

g) $\int (3x - 2)^{20} dx$

h) $\int x^2 e^{x^3} dx$

i) $\int s2^s ds$

j) $\int (\ln x)^2 dx$

Deliverables. Solutions and intermediate steps.

Solution:

In the following $C \in \mathbb{R}$.

a) $\int 3x^2 dx = 3 \int x^2 dx = x^3 + C$

b) $\int_{-1}^{1} x^{100} dx = \left[\frac{x^{101}}{101}\right]_{-1}^{1} = \frac{1^{101}}{101} - \frac{(-1)^{101}}{101} = \frac{1}{101} + \frac{1}{101} = \frac{2}{101}$

c) $\int_{1}^{8} \sqrt[3]{x} dx = \int_{1}^{8} x^{\frac{1}{3}} dx = \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_{1}^{8} = \frac{3}{4} 8^{\frac{4}{3}} - \frac{3}{4} 1^{\frac{4}{3}} = \frac{45}{4}$

d) $\int_0^1 (3+x\sqrt{x})dx = \int_0^1 3dx + \int_0^1 x^{\frac{3}{2}}dx = [3x]_0^1 + \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = (3-0) + (\frac{2}{5}*1 - \frac{2}{5}*0) = \frac{17}{5}$

e) We set $u = -x^2$ which gives $du = -2xdx$, so that $dx = \frac{-du}{2x}$ and

$$\int xe^{-x^2}dx = -\int e^u \frac{du}{2} = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

f) $u = \ln x$ $u' = \frac{1}{x}$, $v' = \sqrt{x}$, $v = \frac{2}{3}x^{\frac{3}{2}}$. When integrating by parts, we have $\int v'u = uv - \int vu'$. Hence

$$\int \sqrt{x}\ln(x)dx = \ln(x)\frac{2}{3}x^{\frac{3}{2}} - \int \frac{2}{3}x^{\frac{3}{2}} * \frac{1}{x}dx = \frac{2}{3}x^{\frac{3}{2}}\ln(x) - \frac{2}{3}\int x^{\frac{1}{2}}dx =$$

$$\frac{2}{3}x^{\frac{3}{2}}\ln(x) - \frac{2}{3}*\frac{2}{3}x^{\frac{3}{2}} + C = \frac{2}{3}x^{\frac{3}{2}}\ln(x) - \frac{4}{9}x^{\frac{3}{2}} + C$$

g) Here we use integration by substitution. Let $u = 3x-2$ which gives $du = 3dx$ or $dx = \frac{1}{3}du$. Then

$$\int (3x-2)^{20}dx = \int u^{20} * \frac{1}{3}du = \frac{1}{3}\int u^{20}du = \frac{1}{3}\frac{u^{21}}{21} + C = \frac{u^{21}}{63} + C = \frac{(3x-2)^{21}}{63} + C$$

h) Here we use integration by substitution. Let $u = x^3$ which gives $du = 3x^2dx$ or $x^2dx = \frac{du}{3}$.

$$\int x^2 e^{x^3}dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

i) Here we use integration by parts. We set $u = s$, $u' = 1$, $v' = 2^s$ and $v = \frac{2^s}{\ln 2}$. When integrating by parts, we have $\int v'u = uv - \int vu'$. Hence

$$\int s2^s ds = s\frac{2^s}{\ln 2} - \int 1 * \frac{2^s}{\ln 2}ds = \frac{s2^s}{\ln 2} - \frac{1}{\ln 2}\int 2^s ds = \frac{s2^s}{\ln 2}\frac{2^s}{\ln 2} - \frac{1}{\ln 2} + C = \frac{s2^s}{\ln 2} - \frac{2^s}{(\ln 2)^2} + C$$

j) Here we use integration by parts. We set $u = \ln x$, $u' = \frac{1}{x}$, $v' = \ln x$, $v = x\ln x - x$. When integrating by parts, we have $\int v'u = uv - \int vu'$. Hence

$$\int (\ln x)^2 dx = \ln(x) * (x\ln x - x) - \int (x\ln x - x) * \frac{1}{x}dx =$$

$$x(\ln x)^2 - x\ln x - \int (\ln x - 1)dx = x(\ln x)^2 - x\ln x - (x\ln x - x - x) + C = x(\ln x)^2 - 2x\ln x + 2x + C$$

Exercise 4 (Integration). a) Consider the following code.

```
import numpy as np
def magic(n):
    s = np.linspace(0, 3, n+1)[1:]
    return np.sum(3.0/n*s*np.exp(s*s))
```

The output magic(n) approximates an integral. Write down that integral and compute its value (you can round that number to an integer).
Hint1: `np.linspace(0, 3, n+1)[1:]` generates an array with the n entries $1\cdot\frac{3}{n}, 2\cdot\frac{3}{n}, \ldots, n\cdot\frac{3}{n}$.
Hint2: $\exp(9) \approx 8103.08$.

b) Compute the following integral:

$$\int_0^1 \int_{-1}^2 x\exp(xy)\,dx\,dy.$$

Hint: This exercise b) is tricky, so maybe you want to do the other exercises first.

Solution:

a) We have

$$\begin{aligned}\lim_{n\to\infty} \texttt{magic}(n) &= \int_0^3 x\exp(x^2)\,dx = \frac{1}{2}\exp(x^2)\Big|_0^3 \\ &= \frac{1}{2}(\exp(9)-\exp(0)) \approx \frac{1}{2}8102.08 = 4051.04 \approx 4051.\end{aligned}$$

b) Because $(x,y)\mapsto x\exp(xy)$ is continuous, we have

$$\begin{aligned}\int_0^1\int_{-1}^2 x\exp(xy)\,dx\,dy &= \int_{-1}^2\int_0^1 x\exp(xy)\,dy\,dx = \int_{-1}^2 \exp(xy)\Big|_0^1\,dx = \int_{-1}^2 \exp(x)-1\,dx \\ &= \exp(x)-x\Big|_{-1}^2 = \exp(2)-2-\exp(-1)-1 = \exp(2)-\exp(-1)-3\end{aligned}$$

Exercise 5 (**Numerical integration).** See Exercise4.ipynb. (Please submit the code as a separate file and as part of the latex document.)