

## MASD 2021, Assignment 4

Hand-in in groups of 2 or 3 before 7.10.2021 at 10.00

One submission per group

Remember to include the names of all group members

- Exercise 1 (Sequences).** a) Prove that if you have a convergent sequence  $\{a_n\}$  with  $\lim_{n \rightarrow \infty} a_n = a$ , then the sequence  $\{-a_n\}$  is also convergent and satisfies  $\lim_{n \rightarrow \infty} -a_n = -a$ .
- b) Using the definition of a limit in one and higher dimensions, prove that if  $\mathbf{a}_n \in \mathbb{R}^d$  is a sequence of vectors and  $\mathbf{a} \in \mathbb{R}^d$  is a vector, then

$$\mathbf{a}_n \rightarrow \mathbf{a} \quad \text{if and only if} \quad (\mathbf{a}_n)_i \rightarrow \mathbf{a}_i \quad \text{for all } i = 1, \dots, d,$$

where  $(\mathbf{a}_n)_i$  is the  $i^{\text{th}}$  coordinate of  $\mathbf{a}_n$ , and  $\mathbf{a}_i$  is the  $i^{\text{th}}$  coordinate of  $\mathbf{a}$ .

*Deliverables.* The proofs.

**Solution:**

- a) **Assumptions:**  $\lim_{n \rightarrow \infty} a_n = a$ . That is, there exists some  $N \in \mathbb{N}$  such that  $|a_n - a| < \varepsilon$  whenever  $n > N$ .

**Claim:**  $\lim_{n \rightarrow \infty} -a_n = -a$ . That is, for any  $\varepsilon > 0$  there exists a corresponding  $N \in \mathbb{N}$  such that

$$|-a_n - (-a)| < \varepsilon \quad \text{whenever } n > N.$$

**Proof:** Let  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} a_n = a$ , there exists an  $N \in \mathbb{N}$  such that  $|a_n - a| < \varepsilon$  whenever  $n > N$ . Now, since

$$|-a_n - (-a)| = |(-1)(a_n - a)| = |-1||a_n - a| = |a_n - a|,$$

we also have, whenever  $n > N$ , that  $|a_n - a| < \varepsilon$ , which was what we needed to show. (That is, we can use the same  $N$  for the sequence  $\{-a_n\}$  as for the sequence  $\{a_n\}$ .)

Hence, the sequence  $\{-a_n\}$  converges with limit  $\lim_{n \rightarrow \infty} -a_n = -a$ .  $\square$

- b) **Claim:** We need to show that

$$\mathbf{a}_n \rightarrow \mathbf{a} \quad \text{if and only if} \quad (\mathbf{a}_n)_i \rightarrow \mathbf{a}_i \quad \text{for all } i = 1, \dots, d.$$

Note that "if and only if" is another way to say " $\Leftrightarrow$ ". That is, in order to prove the claim, then we need to prove:

$$\mathbf{a}_n \rightarrow \mathbf{a} \quad \Rightarrow \quad (\mathbf{a}_n)_i \rightarrow \mathbf{a}_i \quad \text{for all } i = 1, \dots, d,$$

and

$$(\mathbf{a}_n)_i \rightarrow \mathbf{a}_i \quad \text{for all } i = 1, \dots, d \quad \Rightarrow \quad \mathbf{a}_n \rightarrow \mathbf{a}.$$

**Proof:** First, we prove " $\Rightarrow$ ". In this case, we **assume** that  $\mathbf{a}_n \rightarrow \mathbf{a}$ , and **claim** that  $(\mathbf{a}_n)_i \rightarrow \mathbf{a}_i$  for all  $i = 1, \dots, d$ .

Let  $\varepsilon > 0$ . By the definition of convergence, there exists some integer  $N \in \mathbb{N}$  such that whenever  $n > N$ , we have  $\|\mathbf{a}_n - \mathbf{a}\| < \varepsilon$ . Now, let  $i \in \{1, \dots, d\}$  be any coordinate index, and let  $n > N$ . We have

$$|(\mathbf{a}_n)_i - \mathbf{a}_i| = \sqrt{|(\mathbf{a}_n)_i - \mathbf{a}_i|^2} \leq \sqrt{\sum_{j=1}^d |(\mathbf{a}_n)_j - \mathbf{a}_j|^2} = \|\mathbf{a}_n - \mathbf{a}\| < \varepsilon,$$

where the inequality holds because

$$|(\mathbf{a}_n)_i - \mathbf{a}_i|^2 \leq \sum_{j=1}^d |(\mathbf{a}_n)_j - \mathbf{a}_j|^2,$$

as  $i$  is one of the  $j = 1, \dots, d$ , and  $|(\mathbf{a}_n)_j - \mathbf{a}_j|^2 \geq 0$  for all  $j = 1, \dots, d$ .

By the definition of convergence, we have now shown  $(\mathbf{a}_n)_i \rightarrow \mathbf{a}_i$ . Since  $i$  was arbitrarily chosen, this holds for all  $i = 1, \dots, d$ .

Next, we prove " $\Leftarrow$ ". In this case, we **assume** that  $(\mathbf{a}_n)_i \rightarrow \mathbf{a}_i$  for all  $i = 1, \dots, d$ , and **claim** that  $\mathbf{a}_n \rightarrow \mathbf{a}$ .

Let  $\varepsilon > 0$ , and define another  $\varepsilon' = \frac{\varepsilon}{\sqrt{d}} > 0$ . Let  $i \in \{1, \dots, d\}$ . By the assumption and the definition of convergence, there exists some  $N_i \in \mathbb{N}$  such that if  $n > N_i$ , then we have

$$|(\mathbf{a}_n)_i - \mathbf{a}_i| < \varepsilon' \Rightarrow |(\mathbf{a}_n)_i - \mathbf{a}_i|^2 = |(\mathbf{a}_n)_i - \mathbf{a}_i| \cdot |(\mathbf{a}_n)_i - \mathbf{a}_i| < \varepsilon'^2 = \left(\frac{\varepsilon}{\sqrt{d}}\right)^2 = \frac{\varepsilon^2}{d}.$$

Since this holds for any  $i \in \{1, \dots, d\}$ , we can set  $N = \max\{N_i | i = 1, \dots, d\}$ . Now, if  $n > N$ , we also have  $n > N_i$  for all  $i = 1, \dots, d$ . Thus, whenever  $n > N$ , we have

$$\|\mathbf{a}_n - \mathbf{a}\|^2 = \sqrt{\sum_{i=1}^d |(\mathbf{a}_n)_i - \mathbf{a}_i|^2} < \sqrt{\sum_{i=1}^d \varepsilon'^2} = \sqrt{d\varepsilon'^2} = \sqrt{d}\varepsilon' = \sqrt{d}\frac{\varepsilon}{\sqrt{d}} = \varepsilon.$$

But then, by the definition of convergence, we have  $\mathbf{a}_n \rightarrow \mathbf{a}$ .

We have now shown both directions, and are therefore done.  $\square$

**Exercise 2 (Series.).** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

**Hint:** Convert this problem to the well-known cake-eating problem: If there is a cake and an infinite line of cake-hungry people, and every person takes half of the piece that is left, the sequence  $c_n$  measuring the proportion of the cake left after  $n$  people converges to 0.

**Help:** If you have forgotten all about induction proofs, you can find a note on induction proofs from DMA in Absalon under Files/dma5noter.pdf.

*Deliverables.* The proof.

**Solution:**

**Assumptions:** There are no non-obvious assumptions here.

**Claim:** Let  $n \in \mathbb{N}$ . Then our claim is that series  $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ .

**Proof:** A series  $\sum_{n=1}^{\infty} a_n$  is convergent and has the sum  $\sum_{n=1}^{\infty} a_n = 1$  if the corresponding sequence

$$S_k = \sum_{n=1}^k a_n$$

is convergent and satisfies  $\lim_{k \rightarrow \infty} S_k = 1$ .

In this exercise, we have  $a_n = \frac{1}{2^n}$ , giving  $S_k = \sum_{n=1}^k \frac{1}{2^n}$ .

As hinted in the exercise, one way to prove the claim is by showing that for the sequence  $S'_k = 1 - S_k$ , we have  $\lim_{k \rightarrow \infty} S'_k = 0$ , in which case

$$\lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} (1 - S'_k) = \lim_{k \rightarrow \infty} 1 - \lim_{k \rightarrow \infty} S'_k = 1 - 0 = 1,$$

where the second and third equalities follow from the limit properties (sum rule and limit of constant). Thus, we are left to show that  $\lim_{k \rightarrow \infty} S'_k = 0$ .

As hinted in the exercise, we can do so by showing that  $S'_k = \frac{1}{2^k}$ . As showed in the lecture, we know that  $\lim_{k \rightarrow \infty} \frac{1}{2^k} = 0$ .

We shall prove this by induction on  $k$ . In the case  $k = 1$ , it is clear that

$$S'_k = 1 - \sum_{n=1}^{k-1} \frac{1}{2^n} = 1 - \frac{1}{2} = \frac{1}{2} = \frac{1}{2^k}.$$

Next, **assume** that the claim holds for  $k = K - 1$ , that is,

$$S'_{K-1} = 1 - \sum_{n=1}^{K-1} \frac{1}{2^n} = \frac{1}{2^{K-1}}.$$

**Claim:** We need to show that

$$S'_K = 1 - \sum_{n=1}^K \frac{1}{2^n} = \frac{1}{2^K}.$$

To **prove** this, we compute

$$S'_K = 1 - \sum_{n=1}^K \frac{1}{2^n} = 1 - \left( \sum_{n=1}^{K-1} \frac{1}{2^n} + \frac{1}{2^K} \right) = 1 - \sum_{n=1}^{K-1} \frac{1}{2^n} - \frac{1}{2^K} = \frac{1}{2^{K-1}} - \frac{1}{2^K} = \frac{1}{2^{K-1}} \left( 1 - \frac{1}{2} \right) = \frac{1}{2^K}.$$

Here, the fourth equality used the inductive assumption.

Now, by induction, we have shown that  $S'_k = \frac{1}{2^k}$  for all  $k \in \mathbb{N}$ . By the arguments above, the proof is complete.

**Exercise 3 (Practicing integration).** Solve the following definite and indefinite integrals. Solving an indefinite integral means writing down all its derivatives. Include (and explain) intermediate steps.

- $\int 3x^2 dx$
- $\int_{-1}^1 x^{100} dx$
- $\int_1^8 \sqrt[3]{x} dx$
- $\int_0^1 (3 + x\sqrt{x}) dx$
- $\int x e^{-x^2} dx$  using substitution  $u = -x^2$
- $\int \sqrt{x} \ln(x) dx$  using integration by parts with  $u = \ln(x)$ ,  $dv = \sqrt{x} dx$
- $\int (3x - 2)^{20} dx$
- $\int x^2 e^{x^3} dx$
- $\int s 2^s ds$
- $\int (\ln x)^2 dx$

**Deliverables.** Solutions and intermediate steps.

**Solution:**

In the following  $C \in \mathbb{R}$ .

- $\int 3x^2 dx = 3 \int x^2 dx = x^3 + C$
- $\int_{-1}^1 x^{100} dx = \left[ \frac{x^{101}}{101} \right]_{-1}^1 = \frac{1^{101}}{101} - \frac{(-1)^{101}}{101} = \frac{1}{101} + \frac{1}{101} = \frac{2}{101}$
- $\int_1^8 \sqrt[3]{x} dx = \int_1^8 x^{\frac{1}{3}} dx = \left[ \frac{x^{\frac{4}{3}}}{\frac{4}{3}} \right]_1^8 = \frac{3}{4} 8^{\frac{4}{3}} - \frac{3}{4} 1^{\frac{4}{3}} = \frac{45}{4}$

d)  $\int_0^1 (3 + x\sqrt{x})dx = \int_0^1 3dx + \int_0^1 x^{\frac{3}{2}}dx = [3x]_0^1 + \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = (3 - 0) + (\frac{2}{5} * 1 - \frac{2}{5} * 0) = \frac{17}{5}$

e) We set  $u = -x^2$  which gives  $du = -2xdx$ , so that  $dx = \frac{-du}{2x}$  and

$$\int xe^{-x^2}dx = -\int e^u \frac{du}{2} = -\frac{1}{2} \int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

f)  $u = \ln x$   $u' = \frac{1}{x}$ ,  $v' = \sqrt{x}$ ,  $v = \frac{2}{3}x^{\frac{3}{2}}$ . When integrating by parts, we have  $\int v'u = uv - \int vu'$ . Hence

$$\begin{aligned} \int \sqrt{x} \ln(x) dx &= \ln(x) \frac{2}{3} x^{\frac{3}{2}} - \int \frac{2}{3} x^{\frac{3}{2}} * \frac{1}{x} dx = \frac{2}{3} x^{\frac{3}{2}} \ln(x) - \frac{2}{3} \int x^{\frac{1}{2}} dx = \\ &= \frac{2}{3} x^{\frac{3}{2}} \ln(x) - \frac{2}{3} * \frac{2}{3} x^{\frac{3}{2}} + C = \frac{2}{3} x^{\frac{3}{2}} \ln(x) - \frac{4}{9} x^{\frac{3}{2}} + C \end{aligned}$$

g) Here we use integration by substitution. Let  $u = 3x - 2$  which gives  $du = 3dx$  or  $dx = \frac{1}{3}du$ . Then

$$\int (3x - 2)^{20} dx = \int u^{20} * \frac{1}{3} du = \frac{1}{3} \int u^{20} du = \frac{1}{3} \frac{u^{21}}{21} + C = \frac{u^{21}}{63} + C = \frac{(3x - 2)^{21}}{63} + C$$

h) Here we use integration by substitution. Let  $u = x^3$  which gives  $du = 3x^2 dx$  or  $x^2 dx = \frac{du}{3}$ .

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C$$

i) Here we use integration by parts. We set  $u = s$ ,  $u' = 1$ ,  $v' = 2^s$  and  $v = \frac{2^s}{\ln 2}$ . When integrating by parts, we have  $\int v'u = uv - \int vu'$ . Hence

$$\int s 2^s ds = s \frac{2^s}{\ln 2} - \int 1 * \frac{2^s}{\ln 2} ds = \frac{s 2^s}{\ln 2} - \frac{1}{\ln 2} \int 2^s ds = \frac{s 2^s}{\ln 2} - \frac{1}{\ln 2} \frac{2^s}{\ln 2} + C = \frac{s 2^s}{\ln 2} - \frac{2^s}{(\ln 2)^2} + C$$

j) Here we use integration by parts. We set  $u = \ln x$ ,  $u' = \frac{1}{x}$ ,  $v' = \ln x$ ,  $v = x \ln x - x$ . When integrating by parts, we have  $\int v'u = uv - \int vu'$ . Hence

$$\begin{aligned} \int (\ln x)^2 dx &= \ln(x) * (x \ln x - x) - \int (x \ln x - x) * \frac{1}{x} dx = \\ &= x(\ln x)^2 - x \ln x - \int (\ln x - 1) dx = x(\ln x)^2 - x \ln x - (x \ln x - x - x) + C = x(\ln x)^2 - 2x \ln x + 2x + C \end{aligned}$$

**Exercise 4 (Integration).** a) Consider the following code.

```
import numpy as np
def magic(n):
    s = np.linspace(0, 3, n+1)[1:]
    return np.sum(3.0/n*s*np.exp(s*s))
```

The output `magic(n)` approximates an integral. Write down that integral and compute its value (you can round that number to an integer).

Hint1: `np.linspace(0, 3, n+1)[1:]` generates an array with the  $n$  entries  $1 \cdot \frac{3}{n}, 2 \cdot \frac{3}{n}, \dots, n \cdot \frac{3}{n}$ .

Hint2:  $\exp(9) \approx 8103.08$ .

b) Compute the following integral:

$$\int_0^1 \int_{-1}^2 x \exp(xy) dx dy.$$

Hint: This exercise b) is tricky, so maybe you want to do the other exercises first.

**Solution:**

a) We have

$$\begin{aligned}\lim_{n \rightarrow \infty} \text{magic}(n) &= \int_0^3 x \exp(x^2) dx = \frac{1}{2} \exp(x^2) \Big|_0^3 \\ &= \frac{1}{2} (\exp(9) - \exp(0)) \approx \frac{1}{2} 8102.08 = 4051.04 \approx 4051.\end{aligned}$$

b) Because  $(x, y) \mapsto x \exp(xy)$  is continuous, we have

$$\begin{aligned}\int_0^1 \int_{-1}^2 x \exp(xy) dx dy &= \int_{-1}^2 \int_0^1 x \exp(xy) dy dx = \int_{-1}^2 \exp(xy) \Big|_0^1 dx = \int_{-1}^2 \exp(x) - 1 dx \\ &= \exp(x) - x \Big|_{-1}^2 = \exp(2) - 2 - \exp(-1) - 1 = \exp(2) - \exp(-1) - 3\end{aligned}$$

**Exercise 5 (Numerical integration).** See Exercise4.ipynb. (Please submit the code as a separate file and as part of the latex document.)