MASD 2021, Assignment 4

Hand-in in groups of 2 or 3 before 7.10.2021 at 10.00
One submission per group
Remember to include the names of all group memebers

Exercise 1 (Sequences). a) Prove that if you have a convergent sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$, then the sequence $\{-a_n\}$ is also convergent and satisfies $\lim_{n\to\infty} -a_n = -a$.

b) Using the definition of a limit in one and higher dimensions, prove that if $\mathbf{a}_n \in \mathbb{R}^d$ is a sequence of vectors and $\mathbf{a} \in \mathbb{R}^d$ is a vector, then

$$\mathbf{a}_n \to \mathbf{a} \qquad \text{if and only if} \qquad (\mathbf{a}_n)_i \to \mathbf{a}_i \quad \text{for all } i = 1, \ldots, d,$$

where $(\mathbf{a}_n)_i$ is the i^{th} coordinate of $\mathbf{a}_n$, and $\mathbf{a}_i$ is the i^{th} coordinate of $\mathbf{a}$.

Deliverables. The proofs.

Exercise 2 (Series.). Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Hint: Convert this problem to the well-known cake-eating problem: If there is a cake and an infinite line of cake-hungry people, and every person takes half of the piece that is left, the sequence c_n measuring the proportion of the cake left after n people converges to 0.

Help: If you have forgotten all about induction proofs, you can find a note on induction proofs from DMA in Absalon under Files/dma5noter.pdf.

Deliverables. The proof.

Exercise 3 (Practicing integration). Solve the following definite and indefinite integrals. Solving an indefinite integral means writing down all its derivatives. Include (and explain) intermediate steps.

a) $\int 3x^2 dx$

b) $\int_{-1}^{1} x^{100} dx$

c) $\int_1^8 \sqrt[3]{x} dx$

d) $\int_0^1 (3 + x\sqrt{x}) dx$

e) $\int xe^{-x^2} dx$ using substitution $u = -x^2$

f) $\int \sqrt{x} \ln(x) dx$ using integration by parts with $u = \ln(x), dv = \sqrt{x} dx$

g) $\int (3x - 2)^{20} dx$

h) $\int x^2 e^{x^3} dx$

i) $\int s2^s ds$

j) $\int (\ln x)^2 dx$

Deliverables. Solutions and intermediate steps.

Exercise 4 (Integration). a) Consider the following code.

```
import numpy as np
def magic(n):
    s = np.linspace(0, 3, n+1)[1:]
    return np.sum(3.0/n*s*np.exp(s*s))
```

The output `magic`(n) approximates an integral. Write down that integral and compute its value (you can round that number to an integer).
Hint1: `np.linspace(0, 3, n+1)[1:]` generates an array with the n entries $1 \cdot \frac{3}{n}, 2 \cdot \frac{3}{n}, \ldots, n \cdot \frac{3}{n}$.
Hint2: $\exp(9) \approx 8103.08$.

b) Compute the following integral:

$$\int_0^1 \int_{-1}^2 x \exp(xy)\, dx\, dy.$$

Hint: This exercise b) is tricky, so maybe you want to do the other exercises first.

Exercise 5 (Numerical integration). See Exercise4.ipynb. (Please submit the code as a separate file and as part of the latex document.)