MASD

Lecture 6
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Objectives

After today's lecture, you should become familar with:

- Sequences and their limits
- Series and their sums
- The definition of Taylor expansions for functions of one and multiple variables, and have an intuitive understanding of them
- We cover sections 11.1-2 and 11.8-10

Sequences

- A *sequence* is an ordered list of real numbers $a_1, a_2, \ldots, a_n, \ldots$. It can either be *finite* or *countably infinite* (indexed by integers).
- Some examples of infinite sequences:
  - $1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots$
  - $1, 2, 3, \ldots, n, \ldots$
  - $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$
  - $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$
  - $1, -1, 1, -1, 1, -1, \ldots$
- Various notations: $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$, $\{a_1, a_2, \ldots\}$, (a_n), $(a_n)_{n=1}^{\infty}$.
- We focus on countably infinite sequences.

What are sequences useful for?

- Occur in nature.
- Useful in various fields of mathematics to study functions, spaces and other mathematical structures. For example, they are the basis for series, important in differential equations.
- Useful in computer science, for example in connection with time and space analysis of algorithms.
- Interesting by their own right, for example prime numbers.

Convergence

- A sequence $\{a_n\}$ has a *limit* at (or *converges to*) $a \in \mathbb{R}$ if for every $\varepsilon > 0$, the sequence eventually gets closer to a than ε and *stays there*.

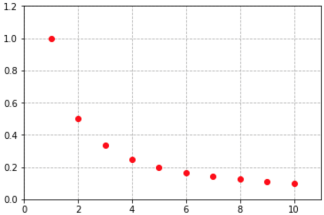
- Formal definition:
  For every $\varepsilon > 0$ there exists an integer $n_0 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n > n_0$. We write

  $$\lim_{n\to\infty} a_n = a, \text{ or } a_n \to a.$$

- An equivalent statement to $|a_n - a| < \varepsilon$ for all $n > n_0$ is

  $$a_n \in]a - \varepsilon, a + \varepsilon[\text{ for all } n > n_0.$$

Show that $\lim_{n\to\infty} \frac{1}{n} = 0$



- $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \cdots \qquad a_n = \frac{1}{n}$
- Let an arbitrary $\varepsilon > 0$ be given.
- Is there an integer n_0 such that $|a_n - 0| < \varepsilon$ for all $n > n_0$?
- Let n_0 be the smallest integer greater or equal to $\frac{1}{\varepsilon}$.
- $n_0 \geq \frac{1}{\varepsilon} \iff \varepsilon \geq \frac{1}{n_0} = a_{n_0}$.
- $|\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{n_0} \leq \varepsilon$ for all $n > n_0$.

Show that $\lim_{n\to\infty} \frac{1}{2^n} = 0$

- $\left\{\frac{1}{2^n}\right\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1026}, \cdots$
- Let an arbitrary $\varepsilon > 0$ be given.
- Is there an integer n_0 such that $|\frac{1}{2^n} - 0| < \varepsilon$, $\forall n > n_0$.
- Assume for a moment that $\frac{1}{2^n} < \frac{1}{n}$, $\forall n \in \mathbb{N}$.
- Since $\frac{1}{n} \to 0$ as $n \to \infty$, $\exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$: $|\frac{1}{n} - 0| < \varepsilon$.

$$|\frac{1}{2^n} - 0| = |\frac{1}{2^n}| = \frac{1}{2^n} < \frac{1}{n} = |\frac{1}{n}| = |\frac{1}{n} - 0| < \varepsilon$$

- We now need to show that $\frac{1}{2^n} < \frac{1}{n}$, $\forall n \in \mathbb{N}$.
- By induction. Base case, $n = 1$. Then $\frac{1}{2^1} = \frac{1}{2} < 1 = \frac{1}{1}$.
- Assume that the claim holds for $n-1$, $n \geq 2$. Show for n.

$$\frac{1}{2^{n-1}} < \frac{1}{n-1} \iff n - 1 < 2^{n-1}$$

$$n = (n-1) + 1 \leq (n-1) + (n-1) < 2^{n-1} + 2^{n-1} = 2^n$$

Some definitions and properties

- A sequence $\{a_n\}$ is *bounded from above* if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
- A sequence $\{a_n\}$ is *bounded from below* if there is a number m such that $a_n \geq m$ for all $n \geq 1$.
- A sequence is *bounded* if it is bounded from above and from below.
- Every convergent sequence is bounded.
- Every bounded monotonic sequence is convergent
- Every infinite subsequence of a convergent sequence is itself convergent.
- How would you prove now that $\lim_{n\to\infty} \frac{1}{2^n} = 0$?

A sequence can have at most one limit

- Consider a sequence $\{a_n\}_{n=1}^{\infty}$ and assume that it has two limits a' and a'', $a' \neq a''$. We are looking for a contradiction.
- Let $\epsilon = |a'' - a'|/2$. Hence, $\epsilon > 0$.
- Since $a_n \to a'$ for $n \to \infty$, there exists $n_0' \in \mathbb{N}$ such that

$$n \geq n_0' \implies |a' - a_n| < \epsilon$$

- Since $a_n \to a''$ for $n \to \infty$, there exists $n_0'' \in \mathbb{N}$ such that

$$n \geq n_0'' \implies |a'' - a_n| < \epsilon$$

- Pick $n_0 = \max\{n_0', n_0''\}$. For any $n \geq n_0$, we have

$$|a''-a'| = |a''-a_n+a_n-a'| \leq |a''-a_n|+|a_n-a'| < 2\epsilon = |a''-a'|$$

a contradiction.

Limit Laws

If $a_n \to a$ and $b_n \to b$ and $c \in \mathbb{R}$ is a constant, then

- $(a_n + b_n) \to a + b$
- $(a_n - b_n) \to a - b$
- $ca_n \to ca$
- $c \to c$
- $a_n b_n \to ab$
- $\frac{a_n}{b_n} \to \frac{a}{b}$ if $b \neq 0$

If $a_n \to a$ and $b_n \to b$ then $(a_n + b_n) \to (a + b)$

Let an arbitrary $\epsilon > 0$ be given. We need to show that there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow |(a_n + b_n) - (a + b)| < \epsilon.$$

Proof.

Let $\epsilon > 0$ be given. Hence, $\frac{\epsilon}{2} > 0$. By the assumptions, there exist $n_a, n_b \in \mathbb{N}$ such that

- $n > n_a \Rightarrow |a_n - a| < \frac{\epsilon}{2}$
- $n > n_b \Rightarrow |b_n - b| < \frac{\epsilon}{2}$

Set $n_0 = \max\{n_a, n_b\}$. Now, if $n > n_0$, we have

- $n > n_0 \geq n_a \Rightarrow |a_n - a| < \frac{\epsilon}{2}$
- $n > n_0 \geq n_b \Rightarrow |b_n - b| < \frac{\epsilon}{2}$

so

$$n > n_0 \Rightarrow |(a_n+b_n)-(a+b)| = |(a_n-a)+(b_n-b)| \underbrace{\leq}_{(*)} \underbrace{|a_n - a|}_{<\epsilon/2} + \underbrace{|b_n - b|}_{<\epsilon/2} < 2\frac{\epsilon}{2} = \epsilon,$$

Divergence to $+\infty$ and $-\infty$

- A sequence $\{a_n\}$ diverges to ∞ if for every $M \in \mathbb{R}$ there is $n_0 \in \mathbb{N}$ such that $a_n > M$ if $n \geq n_0$.
- More formally:

  $$\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N} : n \geq \mathbb{N} \implies a_n > M$$

  We write $a_n \to \infty$ for $n \to \infty$.
- $n \to \infty$ for $n \to \infty$
- If $a_n \to \infty$ then $\frac{1}{a_n} \to 0$, assuming $a_n \neq 0$ for all $n \in \mathbb{N}$

Series

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.
- Define *partial sums* $s_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + ... + a_n$ for all $n \in \mathbb{N}$.
- Define an infinite sequence $\{s_n\}_{n=1}^{\infty}$, also called *series*.
- The limit $\lim_{n\to\infty} s_n$ of this series is denoted by

$$\sum_{n=1}^{\infty} a_n$$

- A series *converges* if its limit exists and is a finite number. Otherwise, a series *diverges*.

Why are series interesting?

- Series can be viewed as functions. They arise f.ex. as solutions to differential equations and when analyzing algorithms.
- Consider the following game. You toss a dice until 1 or 6 comes up. You lose if 1 comes up, you win if 6 comes up. By symmetry, the probability of winning is $1/2$. But you can look at it differently. The probability that you win after exactly n tosses is $(\frac{4}{6})^{n-1} * \frac{1}{6}$. Therefore the probability of winning within the first n tosses is

  $$\left[\sum_{i=1}^{n-1} \left(\frac{4}{6}\right)^{i-1}\right] * \frac{1}{6}$$

  and the probability of winning is

  $$\left[\sum_{i=1}^{\infty} \left(\frac{4}{6}\right)^{i-1}\right] * \frac{1}{6}$$

  which converges to $\frac{1}{2}$. There are other probability applications where you cannot do without series.

$\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$

$$s_{2^1} = 1 + \frac{1}{2}$$

$$s_{2^2} = s_{2^1} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_{2^3} = s_{2^2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq 1 + \frac{2}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{3}{2}$$

$$s_{2^4} = s_{2^3} + \frac{1}{9} + ... + \frac{1}{16} \geq 1 + \frac{3}{2} + \frac{1}{16} + ... + \frac{1}{16} = 1 + \frac{4}{2}$$

$$s_{2^5} \geq 1 + \frac{5}{2}$$

$$s_{2^6} \geq 1 + \frac{6}{2}$$

$s_{2^k} \geq 1 + \frac{k}{2}$

Proof by induction.
Base case $k = 1$: Obvious.
Assume true for any $k - 1$, $k \geq 2$, show for k.

$$s_{2^k} = s_{2^{k-1}} + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + ... + \frac{1}{2^{k-1}+2^k}$$

$$\geq 1 + \frac{k-1}{2} + \frac{1}{2^k} + \frac{1}{2^k} + ... + \frac{1}{2^k}$$

$$= 1 + \frac{k-1}{2} + \frac{2^{k-1}}{2^k}$$

$$= 1 + \frac{k-1}{2} + \frac{1}{2}$$

$$= 1 + \frac{k}{2}$$

Conclusion?

$\sum_{n=1}^{\infty} a_n$ is convergent $\implies \lim_{n\to\infty} a_n = 0$

- Let $s_n = a_1 + a_2 + ... + a_n$.
- $a_n = s_n - s_{n-1}$
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Assume it converges to some $s \in \mathbb{R}$
- Since $n-1 \to \infty$ as $n \to \infty$, we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = s$.
- Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Some Definitions and Convergence Tests

- $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Ratio Test

- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ then $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.
- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$ then the test is inconclusive.

Alternating Series Test

- Alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

satisfying
  - $a_n > 0$ for all n
  - $a_{n+1} \leq a_n$
  - $\lim_{n\to\infty} a_n = 0$

are convergent.

Power Series

- Infinite sum

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\dots$$

  where x is a **variable**, a is a constant, and $c_0, c_1, c_2, c_3, \dots$ are constant *coefficients*, is called a *power series centered at a*.
- If $a = 0$ and $c_0 = c_1 = c_2 = c_3 = \dots 1$ then the power series becomes the *geometric series*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

  which converges to $\frac{1}{1-x}$ when $|x| < 1$. Otherwise it diverges (p. 742).

Radius of Convergence

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ one of the following 3 cases can occur:

- It converges for $x = a$ only. Radius of convergence is 0.
- It converges for all $x \in \mathbb{R}$. Radius of convergence is ∞.
- It converges for all $x \in]a-R, a+R[$ for some $R > 0$. Radius of convergence is R.

Functions as Power Series

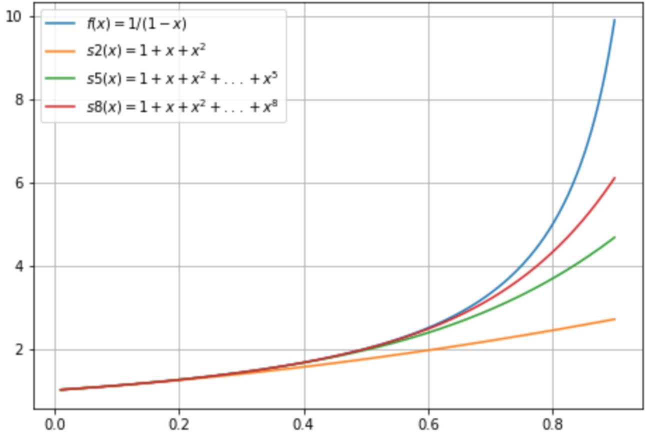
- Consider the function $f :]-1, 1[\to \mathbb{R}$ defined by $f(x) = \frac{1}{1-x}$.
- Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in]-1, 1[$, we can also write

  $$f(x) = \sum_{n=0}^{\infty} x^n$$

  for $x \in]-1, 1[$.
- Any function $f :]a - R, a + R[\to \mathbb{R}$ which is equal to the sum of power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ in $]a - R, a + R[$ can be viewed as defined by such series.
- Why is it useful to represent functions by power series?
  - Integration of complicated functions.
  - Approximation of functions by polynomials.

Approximation of Functions

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, x \in]-1, 1[$$



Example of a Function as a Power Series

- Can $\frac{1}{1+x^2}$ be represented as a power series?

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$$

- This is a geometric series (substitute $-x^2$ by r).
- This power series converges for $|-x^2| < 1$. This is the same as $x^2 < 1$ or $|x| < 1$.

Differentiation of Power Series

- If a power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$ then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on $]a-R, a+R[$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_2(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

with the same radius of convergence.

Getting Power Series by Differentiation

- We have seen that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \text{ for } x \in]-1,1[$$

- Differentiating both sides gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1)x^n \qquad \text{for } x \in]-1,1[$$

Which Functions Have Power Series Representation

- So far we have found power series representations for some functions.
- Which functions have power series representation?
- How can such representation be found?

Power Series of Functions

- Consider a function f and **assume** that it can be represented by a power series:

  $$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots \quad x \in]a-R, a+R[$$

- Plugging $x = a$ gives $f(a) = c_0$. Differentiating both sides gives

  $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \quad x \in]a-R, a+R[$$

- Plugging $x = a$ gives $f'(a) = c_1$. Differentiating both sides gives

  $$f''(x) = 2c_2 + 2\times 3c_3(x-a) + 3\times 4c_4(x-a)^2 + \dots \quad x \in]a-R, a+R[$$

- Plugging $x = a$ gives $f''(a) = 2c_2$. Differentiating both sides gives

  $$f'''(x) = 2\times 3c_3 + 2\times 3\times 4c_4(x-a) + \dots \quad x \in]a-R, a+R[$$

Power Series of Functions

- $f^{(n)}(a) = 1 \times 2 \times 3 \times 4 \ldots \times nc_n = n!c_n$
- Hence $c_n = \frac{f^{(n)}(a)}{n!}$
- Conclusion? If f has a power series representation

$$\sum_{n=0}^{\infty} c_n(x-a)^n, \qquad |x-a| < R$$

  then the coefficients are given by $c_n = \frac{f^{(n)}(a)}{n!}$
- Plugging the formula for c_n into the power series yields

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

- This power series is called the *Taylor series* for f at a.
- Functions that are not equal to the sums of their Taylor series **do exist**.

Maclaurin Series

- Plugging $a = 0$ into Taylor series for f yields Maclaurin series for f.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

- We have shown that if a function f has a power series representation, then it must be a Taylor series.
- But which functions are actually equal to the sums of their Taylor series? And for which values of x?

When is a Function Equal to the Sum of its Taylor Series?

- f must have derivatives of all orders.
- Let the partial sum and remainder of Taylor series for f be

$$T_n(x) = \sum_{i=0}^{n} \underbrace{\frac{f^{(i)}(a)}{i!}}_{c_i}(x-a)^i, \qquad R_n(x) = f(x) - T_n(x)$$

- If

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of Taylor series on the interval $|x-a| < R$.

Natural Exponential Function $f(x) = e^x$

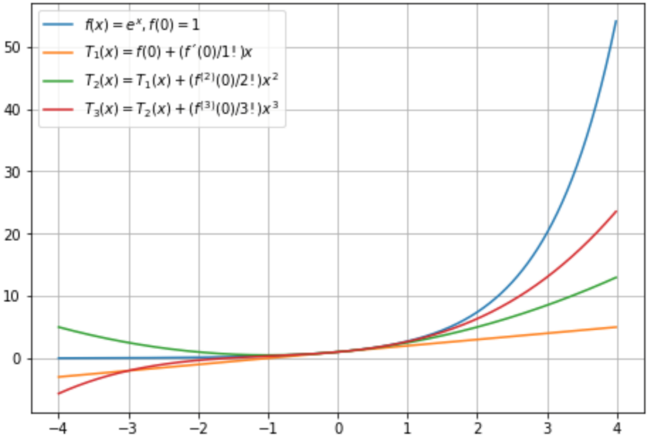
- $f^{(n)}(x) = e^x$ for all natural numbers n.
- Is e^x equal to the sum of its Maclaurin series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$?
- $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i = \sum_{i=0}^{n} \frac{e^0}{i!} x^i = \sum_{i=0}^{n} \frac{x^i}{i!}$
- $R_n(x) = f(x) - T_n(x)$
- It can be shown that $\lim_{n\to\infty} R_n(x) = 0$ for all real values of x.
- As a consequence e^x is equal to the sum of its Maclaurin series, i.e.,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all real values of x. In particular, for $x = 1$:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Approximation of $f(x) = e^x$ around $a = 0$



Natural Exponential Function $f(x) = e^x$

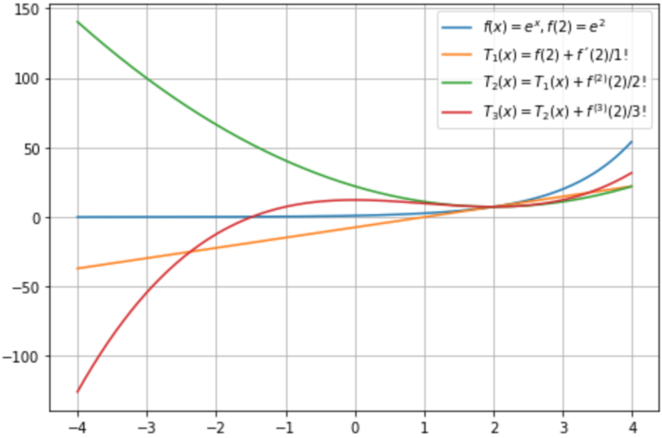
- $f^{(n)}(2) = e^2$ for all natural numbers n
- Is e^x equal to the sum of its Taylor series at a = 2?
- $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(2)}{i!}(x-2)^i = \sum_{i=0}^{n} \frac{e^2}{i!}(x-2)^i$
- $R_n(x) = f(x) - T_n(x)$
- It can be shown that $\lim_{n\to\infty} R_n(x) = 0$ for all real values of x.
- As a consequence e^x is equal to the sum of its Taylor series at 2, i.e.,

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

for all real values of x. In particular, for $x = 1$:

$$e = \sum_{n=0}^{\infty}(-1)^n \frac{e^2}{n!} = e^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^2(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots)$$

Approximation of $f(x) = e^x$ around $a = 2$



Summary

By now, you should be ready to work on your own with:

- Sequences and their limits
- Series and their sums
- Power series
- Taylor expansions and Newton's method