Solution. to Problem 1:

(a)

$$\begin{aligned}P(X \leq 2, Y > 1) &= P(X = 1, Y = 2) + P(X = 2, Y = 2)\\ &= \frac{1}{12} + 0 = \frac{1}{12}.\end{aligned}$$

(b)

$$P_X(x) = \sum_{y \in R_Y} P(X = x, Y = y).$$

$$P_X(x) = \begin{cases} \frac{1}{3} + \frac{1}{12} = \frac{5}{12} & \text{for } x = 1 \\ \frac{1}{6} + 0 = \frac{1}{6} & \text{for } x = 2 \\ \frac{1}{12} + \frac{1}{3} = \frac{5}{12} & \text{for } x = 4 \end{cases}$$

So:

$$P_X(x) = \begin{cases} \frac{5}{12} & x = 1 \\ \frac{1}{6} & x = 2 \\ \frac{5}{12} & x = 4 \end{cases}$$

$$P_Y(y) = \sum_{x \in R_X} P(X = x, Y = y).$$

$$P_Y(y) = \begin{cases} \frac{1}{3} + \frac{1}{6} + \frac{1}{12} = \frac{7}{12} & \text{for } y = 1 \\ \frac{1}{12} + 0 + \frac{1}{3} = \frac{5}{12} & \text{for } y = 2 \end{cases}$$

So:

$$P_Y(y) = \begin{cases} \frac{7}{12} & y = 1 \\ \\ \frac{5}{12} & y = 2 \end{cases}$$

(c)

$$P(Y = 2|X = 1) = \frac{P(Y = 2, X = 1)}{P(X = 1)} = \frac{\frac{1}{12}}{\frac{5}{12}} = \frac{1}{5}.$$

(d) Using the results of the previous part, we observe that:

$P(Y = 2|X = 1) = \frac{1}{5} \neq P(Y = 2) = \frac{5}{12}$.

So, we conclude that the two variables are not independent.

Solution. to Problem 3

We choose each coin with probability 0.5. We call the regular coin "coin1" and the biased coin "coin2."

Let X be a Bernoulli random variable associated with the first chosen coin toss. We can pick the first coin "coin1" or second coin "coin2" with equal probability 0.5. Thus, we can use the law of total probability:

$$
\begin{aligned}
P(X=1) &= P(\text{coin1})P(H|\text{coin 1}) + P(\text{coin2})P(H|\text{coin 2}) \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{7}{12}.
\end{aligned}
$$

$$
\begin{aligned}
P(X=0) &= P(\text{coin1})P(T|\text{coin 1}) + P(\text{coin2})P(T|\text{coin 2}) \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{12}.
\end{aligned}
$$

Let Y be a Bernoulli random variable associated with the second chosen coin toss. We can pick the first coin "coin1" or second coin "coin2" with equal probability 0.5.

$$
\begin{aligned}
P(Y=1) &= P(\text{coin1})P(H|\text{coin 1}) + P(\text{coin2})P(H|\text{coin 2}) \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{7}{12}.
\end{aligned}
$$

$$
\begin{aligned}
P(Y=0) &= P(\text{coin1})P(T|\text{coin 1}) + P(\text{coin2})P(T|\text{coin 2}) \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{12}.
\end{aligned}
$$

$$\begin{aligned}
P(X=0,Y=0) &= P(\text{first coin} = \text{coin1})P(T|\text{coin }1)P(T|\text{coin }2) \\
&\quad + P(\text{first coin} = \text{coin2})P(T|\text{coin }1)P(T|\text{coin }2) \\
&= P(T|\text{coin }1)P(T|\text{coin }2) \\
&= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.
\end{aligned}$$

$$\begin{aligned}
P(X=0,Y=1) &= P(\text{first coin} = \text{coin1})P(T|\text{coin }1)P(H|\text{coin }2) \\
&\quad + P(\text{first coin} = \text{coin2})P(T|\text{coin }2)P(H|\text{coin }1) \\
&= \frac{1}{2} \times \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

$$\begin{aligned}
P(X=1,Y=0) &= P(\text{first coin} = \text{coin1})P(H|\text{coin }1)P(T|\text{coin }2) \\
&\quad + P(\text{first coin} = \text{coin2})P(H|\text{coin }2)P(T|\text{coin }1) \\
&= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

$$\begin{aligned}
P(X=1,Y=1) &= P(\text{first coin} = \text{coin1})P(H|\text{coin }1)P(H|\text{coin }2) \\
&\quad + P(\text{first coin} = \text{coin2})P(H|\text{coin }1)P(H|\text{coin }2) \\
&= P(H|\text{coin }1)P(H|\text{coin }2) \\
&= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}.
\end{aligned}$$

Table 5.2 summarizes the joint PMF of X and Y.

Table 5.2: Joint PMF of X and Y

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$
$X=1$	$\frac{1}{4}$	$\frac{1}{3}$

By comparing joint PMFs and marginal PMFs, we conclude that the two variables are not independent.

For example:

$$
\begin{aligned}
P(X = 0) &= \frac{5}{12} \\
P(Y = 1) &= \frac{7}{12} \\
P(X = 0, Y = 1) &= \frac{1}{4} \neq P(X = 0) \times P(Y = 1).
\end{aligned}
$$

Solution. to Problem 9

(a) Note that here

$$R_{XY} = C = \{(x,y) | x,y \in \mathbb{Z}, x^2 + |y| \leq 2\}.$$

Thus, the joint PMF is given by

$$P_{XY}(x,y) = \begin{cases} \frac{1}{11} & (x,y) \in C \\ 0 & \text{otherwise} \end{cases}$$

To find the marginal PMF of Y, $P_Y(j)$, we use

$$P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y), \qquad \text{for any } y \in R_Y$$

Thus,

$$\begin{aligned}
P_Y(-2) &= P_{XY}(0,-2) = \frac{1}{11}, \\
P_Y(-1) &= P_{XY}(0,-1) + P_{XY}(-1,-1) + P_{XY}(1,-1) = \frac{3}{11}, \\
P_Y(0) &= P_{XY}(0,0) + P_{XY}(1,0) + P_{XY}(-1,0) = \frac{3}{11}, \\
P_Y(1) &= P_{XY}(0,1) + P_{XY}(-1,1) + P_{XY}(1,1) = \frac{3}{11}, \\
P_Y(2) &= P_{XY}(0,2) = \frac{1}{11}.
\end{aligned}$$

Similarly, we can find

$$P_X(i) = \begin{cases} \frac{3}{11} & \text{for } i = -1, 1 \\ \frac{5}{11} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) We have

$$\begin{aligned}
E[XY^2] &= \sum_{i,j \in R_{XY}} ij^2 P_{XY}(i,j) \\
&= \frac{1}{11} \sum_{i,j \in R_{XY}} ij^2 \\
&= 0
\end{aligned}$$

Solution to Problem 15

Solution: To find EY, we cannot directly use the linearity of expectation because N is random but, conditioned on $N = n$, we can use linearity and

find $E[Y|N = n]$; so, we use the law of iterated expectations:

$$\begin{aligned}
EY &= E[E[Y|N]] && \text{(law of iterated expectations)} \\
&= E\left[E\left[\sum_{i=1}^{N} X_i|N\right]\right] \\
&= E\left[\sum_{i=1}^{N} E[X_i|N]\right] && \text{(linearity of expectation)} \\
&= E\left[\sum_{i=1}^{N} E[X_i]\right] && \text{(X_i's and N are indpendent)} \\
&= E[NE[X]] && \text{(since $EX_i = EX$s)} \\
&= E[X]E[N] && \text{(since EX is not random).} \\
EY &= E[X]E[N] \\
EY &= \frac{1}{\lambda} \cdot \beta \\
EY &= \frac{\beta}{\lambda}
\end{aligned}$$

To find $\mathrm{Var}(Y)$, we use the law of total variance:

$$\begin{aligned}
\mathrm{Var}(Y) &= E(\mathrm{Var}(Y|N)) + \mathrm{Var}(E[Y|N]) \\
&= E(\mathrm{Var}(Y|N)) + \mathrm{Var}(NEX) && \text{(as above)} \\
&= E(\mathrm{Var}(Y|N)) + (EX)^2\mathrm{Var}(N).
\end{aligned}$$

To find $E(\mathrm{Var}(Y|N))$ note that, given $N = n$, Y is the sum of n independent random variables. As we discussed before, for n independent random variables, the variance of the sum is equal to sum of the variances. We can write

$$\begin{aligned}
\mathrm{Var}(Y|N) &= \sum_{i=1}^{N} \mathrm{Var}(X_i|N) \\
&= \sum_{i=1}^{N} \mathrm{Var}(X_i) && \text{(since X_i's are independent of N)} \\
&= N\mathrm{Var}(X).
\end{aligned}$$

Thus, we have

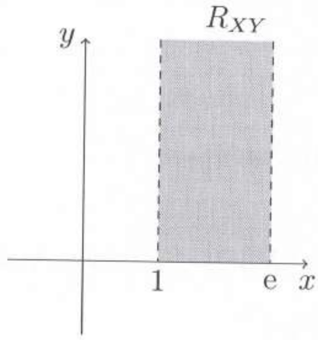
$$E(\mathrm{Var}(Y|N)) = EN\mathrm{Var}(X).$$

We obtain

$$\begin{aligned}
\mathrm{Var}(Y) &= ENVar(X) + (EX)^2\mathrm{Var}(N). \\
\mathrm{Var}(Y) &= \beta(\frac{1}{\lambda})^2 + (\frac{1}{\lambda})^2\beta. \\
&= \left(\frac{2\beta}{\lambda^2}\right)
\end{aligned}$$

Solution to Problem 17

(a) We have:



for $1 < x < e$:

$$\begin{aligned} f_X(x) &= \int_0^\infty e^{-xy} dy \\ &= -\frac{1}{x} e^{-xy} \Big|_0^\infty \\ &= \frac{1}{x} \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{1}{x} & 1 \leq x \leq e \\ 0 & \text{otherwise} \end{cases}$$

for $0 < y$

$$\begin{aligned} f_Y(y) &= \int_1^e e^{-xy} dx \\ &= \frac{1}{y}(e^{-y} - e^{-ey}) \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{y}(e^{-y} - e^{-ey}) & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} P(0 \leq Y \leq 1, 1 \leq X \leq \sqrt{e}) &= \int_{x=1}^{\sqrt{e}} \int_{y=0}^{1} e^{-xy} dy dx \\ &= \frac{1}{2} - \int_1^{\sqrt{e}} \frac{1}{x} e^{-x} dx \end{aligned}$$

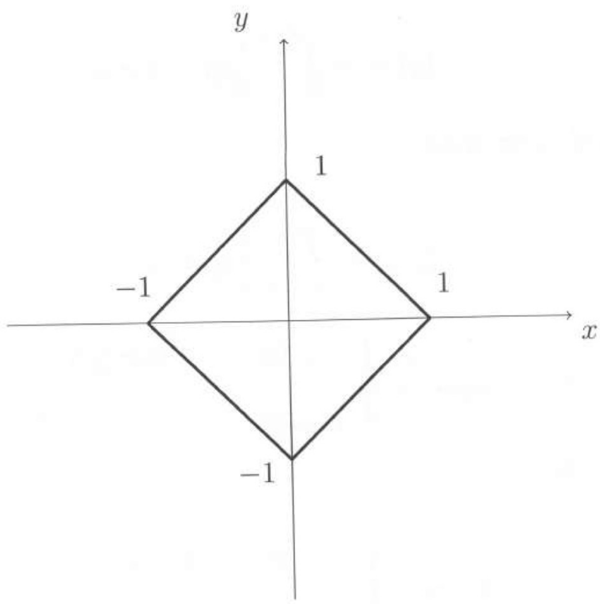
Solution to Problem 23a)

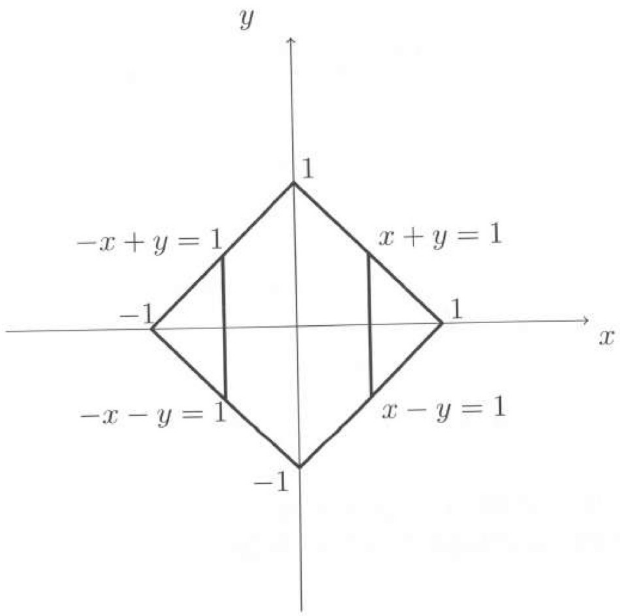
(a) We have:

$$1 = \int_E \int c\,dx\,dy = c(\text{area of E}) = c\sqrt{2}\cdot\sqrt{2} = 2c$$
$$\rightarrow c = \frac{1}{2}$$

(b)

For $0 \leq x \leq 1$, we have:





$$f_X(x) = \int_{x-1}^{1-x} \frac{1}{2} dy = 1 - x$$

For $-1 \leq x \leq 0$, we have:

$$f_X(x) = \int_{-x-1}^{1+x} \frac{1}{2} dy = 1 + x$$

$$f_X(x) = \begin{cases} 1 - |x| & -1 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Similarly, we find:

$$f_Y(y) = \begin{cases} 1 - |y| & -1 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

Solution. to Problem 27

First note that since $R_X = R_Y = [0, 1]$, we conclude $R_Z = [0, \infty)$. We first find the CDF of Z.

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(\frac{X}{Y} \leq z) \\ &= P(X \leq zY) \quad \text{(Since } Y \geq 0\text{)} \\ &= \int_0^1 P(X \leq zY | Y = y) f_Y(y) dy \quad \text{(Law of total prob)} \\ &= \int_0^1 P(X \leq zy) dy \quad \text{(Since } X \text{ and } Y \text{ are indep)} \end{aligned}$$

Note:

$$P(X \le zy) = \begin{cases} 1 & \text{if } y > \frac{1}{z} \\ zy & \text{if } y \le \frac{1}{z} \end{cases}$$

Consider two cases:

(a) If $0 \le z \le 1$, then $P(X \le zy) = zy$ for all $0 \le y \le 1$

Thus:

$$F_Z(z) = \int_0^1 (zy)dy = \frac{1}{2}zy^2\big|_0^1 = \frac{1}{2}z$$

(b) If $z > 1$, then

$$\begin{aligned} F_Z(z) &= \int_0^{\frac{1}{z}} zydy + \int_{\frac{1}{z}}^1 1dy \\ &= [\frac{1}{2}zy^2]_0^{\frac{1}{z}} + [y]_{\frac{1}{z}}^1 \\ &= \frac{1}{2z} + 1 - \frac{1}{z} = 1 - \frac{1}{2z} \end{aligned}$$

$$F_Z(z) = \begin{cases} \frac{1}{2}z & 0 \le z \le 1 \\ 1 - \frac{1}{2z} & z \ge 1 \\ 0 & z < 0 \end{cases}$$

Note that $F_Z(z)$ is a continuous function.

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{1}{2} & 0 \le z \le 1 \\ \frac{1}{2z^2} & z \ge 1 \\ 0 & \text{else} \end{cases}$$

Solution. to Problem 37
$X \sim N(1,4)$; $Y \sim N(1,1)$:
$\rho(X,Y) = 0$ and X , Y are jointly normal. Therefore X and Y are independent.

(a) $W = X + 2Y$ Therefore:
$W \sim N(3, 4+4) = N(3,8)$

$$P(W > 4) = 1 - \Phi(\frac{4-3}{\sqrt{8}}) = 1 - \Phi(\frac{1}{\sqrt{8}})$$

(b)

$$\begin{aligned} E[X^2Y^2] &= EX^2 \cdot EY^2 \quad \text{Since } X \text{ and } Y \text{ are independent.} \\ &= (4+1)\cdot(1+1) = 10 \end{aligned}$$