MASD 2019, Written Exam

Dept. of Computer Science, Univ. of Copenhagen

Jonas Peters and Pawel Winter

7.11.2019

Exam: This 4 hours written exam consists of seven exercises, each related to respective topics covered in each of the seven weeks of lectures. Exercises are equally weighted. Start with those that you think would be easiest. Preferably, use separate sheets of paper for each exercise.
Open book. You are permitted to use all kinds of aid materials including electronic devices, provided that you do not access internet and do not communicate with others.
Justify your statements. In particular, you may refer to the textbooks used in the course, to slides (if available on absalon) and to the essentials and assignments given during the course. It is for example permitted to justify a statement by writing that it derives trivially from a result in the textbook (if this is the case). Always provide precise locations. References to other books or any other sources will not be accepted.

Exercise 1 (Sequences and Series). This exercise consists of two independent parts 1A and 1B.

1A. Limits of Sequences. Consider the infinite sequence $\{a_n\}$ defined by

$$a_n = b_n - c_n$$

where

$$b_n = \sqrt{4n^2 + 3n - 2}, \qquad c_n = 2n$$

Determine $\lim_{n\to\infty} a_n$. Hint: Use that $b_n - c_n = \frac{b_n^2 - c_n^2}{b_n + c_n}$.

1B. Power Series. Consider the power series

$$\sum_{n=1}^{\infty} nx^n$$

a) Does it converge for $x = 1$? No proof is required.
b) Does it converge for $x = -1$? No proof is required.
c) Use the Ratio Test (Stewart, p. 739 or slides, Lecture 6, p. 4) and the answers to the above two subquestions to determine the interval of convergence of $\sum_{n=1}^{\infty} nx^n$.

Exercise 2 (Differentiation). a) Draw the graph of a differentiable function $f\colon (-1, 5) \to \mathbb{R}$ for which the following holds:
- $f'(0) = f'(2) = 0$, and
- $f'(x) < 0$ for all $x \in (0, 2)$, and
- $f'(x) > 0$ for all $x \in (-1, 0) \cup (2, 5)$, and
- $f''(1) = f''(3) = 0$.

b) Compute

$$\frac{d}{dx}\left(\frac{\sin(x)}{x^2+1} + 42\right)$$

c) Prove that the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = |x - 3|$ is not differentiable at $x = 3$.

Exercise 3 (Optimization). This exercise consists of two independent parts 3A and 3B.

3A. Local Maxima and Minima. Consider the following function of two variables:

$$f(x, y) = 3x^3 + 3x^2y - y^3 - 15x$$

a) Determine the partial derivatives $f_x(x, y)$ and $f_y(x, y)$.
b) The critical points of f are $A = (1, 1), B = (-1, -1), C = (\sqrt{5}, -\sqrt{5})$ and $D = (-\sqrt{5}, \sqrt{5})$. They are solutions to a system of two equations. What are these equations? Note that you are not required to solve this system of two equations.
c) Determine the second partial derivatives $f_{xx}(x, y)$, $f_{xy}(x, y)$ and $f_{yy}(x, y)$ and use Second Derivatives Test to classify $A = (1, 1)$ and $C = (\sqrt{5}, -\sqrt{5})$ as local maximum, local minimum or neither.

3B. Newton's Method. Consider the continuous function

$$f(x) = x^3 - 6x + 2 = 0$$

a) Argue that the equation $f(x) = 0$ has a root between 0 and 1.
b) Apply Newton's method for finding the root. Start at $x_1 = \frac{1}{3}$. Derive the value of x_2 obtained after the first iteration.

Exercise 4 (Integration). a) Consider the following code.

```
import numpy as np
def magic(n):
    s = np.linspace(0, 3, n+1)[1:]
    return np.sum(3.0/n*s*np.exp(s*s))
```

The output `magic`(n) approximates an integral. Write down that integral and compute its value (you can round that number to an integer).
Hint1: `np.linspace(0, 3, n+1)[1:]` generates an array with the n entries $1 \cdot \frac{3}{n}, 2 \cdot \frac{3}{n}, \ldots, n \cdot \frac{3}{n}$.
Hint2: $\exp(9) \approx 8103.08$.

b) Compute the following integral:

$$\int_0^1 \int_{-1}^2 x \exp(xy)\, dx\, dy.$$

Hint: This exercise b) is tricky, so maybe you want to do the other exercises first.

Exercise 5 (Expectations and Variances). a) In roulette (assuming a so-called single-zero wheel), a single number from $\{0, 1, 2, 3, \ldots, 36\}$ comes up, where each outcome is equally likely. In the casino 'Giveme Money' you play the game 'Super 7' that has the following rules.

- The casino has six roulette tables. At each of the six roulette tables, one number comes up (these six numbers are assumed to be independent). You then count the number of tables, where '7' came up and call that number of tables X.
- Your score Y equals $Y := X + 8$.

a) X has a binomial distribution, i.e., $X \sim Bin(n, \theta)$. Write down the two parameters n and θ.
b) Compute the probability that your score equals 8 or 9.
c) Compute $\mathbb{E}Y$.
d) Compute $\text{var}(Y)$.
e) Compute $\text{cov}(X, Y)$.

Hint: If $X \sim Bin(n, \theta)$, then $\mathbb{E}X = n\theta$ and $\text{var}(X) = n\theta(1 - \theta)$.

Exercise 6 (Joint Distributions). Assume that you have invested money in two stocks: A and B. Let X and Y denote the money you earn in 2020 from A and B, respectively. Assume X, Y has the joint pmf

$p_{X,Y}(x,y)$	$y=-1$	$y=1$	$y=4$
$x=-1$	0.05	0.15	a
$x=1$	0.3	0	0.1

a) Fill in the value for a (no proof required).
b) Compute the expected money you make: $\mathbb{E}[X+Y]$.
c) Prove that X and Y are not independent.
d) Compute the conditional pmf of X given $Y=-1$.
e) Write down a joint pmf for a random vector (V, W), such that $p_{V,W}$ is different from $p_{X,Y}$, but satisfies $p_V = p_X$, $p_W = p_Y$ and V and W are independent (no proofs required).

Exercise 7 (Continuous Distributions). Let (X, Y) be a random vector with joint probability density function (joint pdf)

$$f(x,y) = \begin{cases} 6y^{\frac{1}{2}} & \text{if } 0 < x < 1 \text{ and } 0 < y < x^2 \\ 0 & \text{otherwise.} \end{cases}$$

a) Draw the area $B \subseteq \mathbb{R}^2$ for which f is strictly larger than zero.
b) Prove that the marginal pdf g of X equals

$$g(x) = \begin{cases} 4x^3 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

c) Compute the cumulative distribution function (cdf) F_X of X.
d) Consider the following code

```
import numpy as np
def func(n):
    s = 0
    for i in range(n):
        (x,y) = (np.random.uniform(0,3), np.random.uniform(0,3))
        if(np.square(x) + np.square(y)) <= 1:
            s += 1
    return(s/n)
```

The output `func`(n) approximates a probability, in particular for large n. Write down the probability and compute its value.