MASD noter til eksamen

HOLD 2

Introduktion

Dette dokument er noter til MASD eksamen. På dette link, ligger der en mappe med maplefiler, med formler til at lave flere af typeopgaverne.
`https://github.com/mikkelwillen/Masd_Eksamen`

Writing Proofs

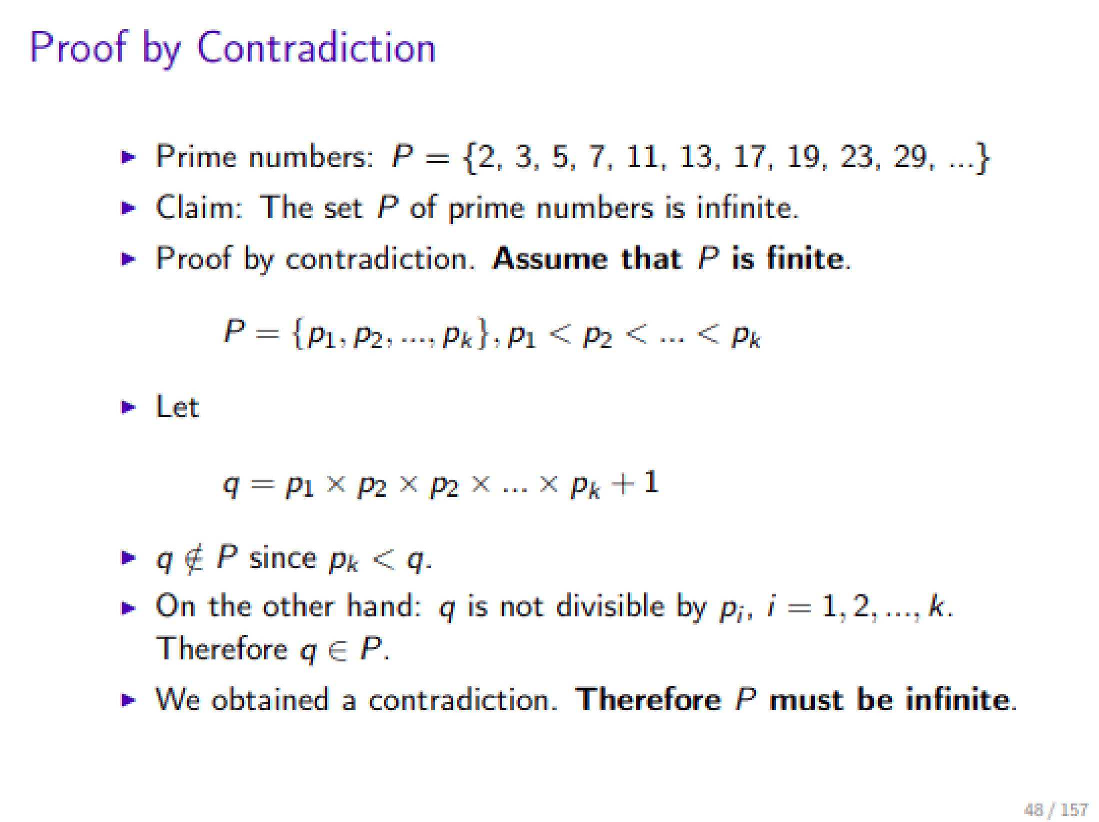
A proof needs to have certain contents to be categorised as a proof. These contents are listed below:

- Assumptions
- Claim to prove
- Series og logical arguments. Note that these arguments does not necessarily have to be in the form of algebra.

When writing a proof it is important that the statements are clear and not creating any ambiguities. The proof should be done when the claim is proved by the logical arguments.

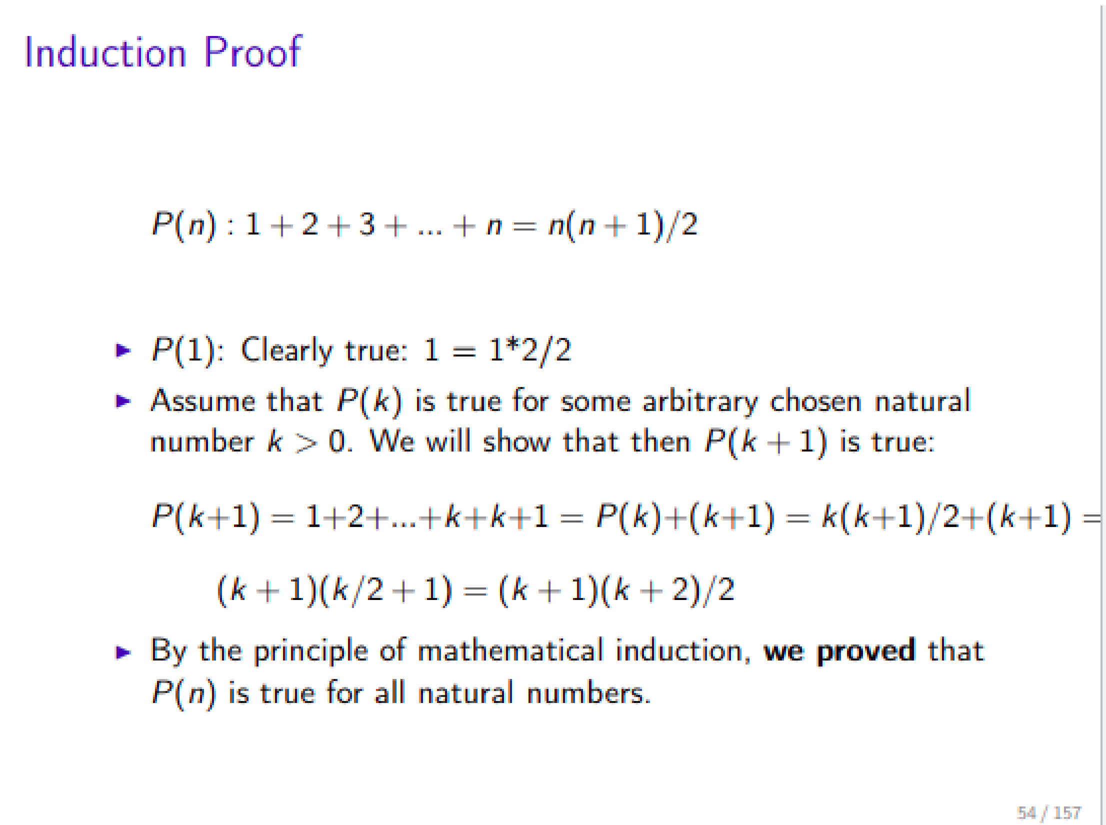
Proof by Contradiction

A proof by contradiction is when you are trying to prove the opposite of the original claim. It is important in a proof by contradiction that if the contradicting claim is false the original claim MUST be true.
When you have shown that the contradicting claim is false, you have proven the original claim, and the proof by contradiction is done.
Example:



Proof by Induction

The proof is usually used when you want to prove a function over a series' result. A proof by induction consists of 2 steps after the claim. The base step, and the induction step.
The base case is proving the claim for one case. This is usually for 0 or 1.
The induction case is proving the claims' f(n+1) is equal to the next step in the claim.
When the induction case is completed, the proof is done.
Example:



Limits

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

This definition descibes that you can find a distance between L and $f(x)$ that is arbitrarily small by choosing an x that is close to a. This $f(x)$ is $\lim_{x\to a} f(x)$.
If you can not find this value, or $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ then $\lim_{x\to a}$ does not exist.

3 Precise Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

4 Precise Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Figure 1: page 95 section 2.3

Continuity

$\boxed{1}$ **Definition** A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

$\boxed{2}$ **Definition** A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Vertical asymptote

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Critical Points

Rolle's Theorum

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.

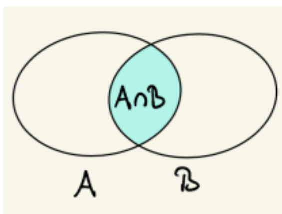
2. f is differentiable on the open interval (a, b).

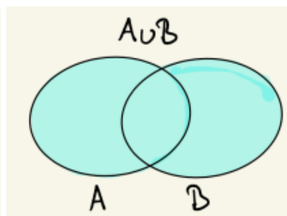
3. $f(a) = f(b)$

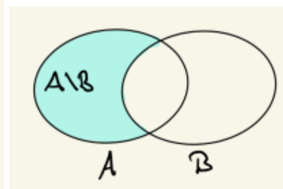
Then there is a number c in (a, b) such that $f'(c) = 0$.

Basic set theory

Et set A er en samling af x elementer. Vi skriver $x \in A$, hvis x er en del af A og skriver $x \notin A$, hvis x ikke er et element i A. Et tomt set bliver beskrevet med ø. Vi skriver $A \subset B$, hvis alle elementer i A også er elementer i B.
$\mathbb{R}$ = alle reelle tal
$\mathbb{N}$ = positive heltal
$\mathbb{Z}$ = alle heltal
Intersection: $A \cap B$
Union: $A \cup B$
Complement: $A\ B = A - B$







Cardinaliteten eller størrelsen af et set bliver beskrevet ved $|A|$. Dette beskriver antallet af elementer i settet. Hvis settet har uendeligt mange elementer, skriver vi $|A| = \infty$. I dette tilfælde er der to muligheder. $|A|$ kan være tælleligt uendeligt, hvis det kan beskrives ved $\mathbb{N}$ og utælleligt uendeligt, hvis det ikke kan.
Eksempler:
$A_1 = \{x \in \mathbb{N} : x \geq 10\}, \ \ |A_1| = 10$
$A_2 = \{ø\}, \ \ |A_2| = 0$
$A_3 = \{x : x \in \mathbb{N}\}, \ \ |A_3| = \infty$ og kan tælles
$A_4 = \{x : x \subset \mathbb{N}\}, \ \ |A_4| = \infty$ og kan ikke tælles

Probability spaces

Sættet S er vores sample space, og A er vores events, som er subsets af S. $A \subset S$.
$\mathbb{P}$ er probabilty og $\mathbb{P}(A)$ er sandsynligheden for eventet A. Der gælder 3 regler for $\mathbb{P}$:

1. Total probability: $\mathbb{P}(S) = 1$.
2. Non-negativity: $\mathbb{P}(A) \geq 0$.
3. Additivity:

Sampling types

Ordered sampling with replacement

Formlen for ordered sampling with replacement er

$$P = n^k$$

hvor n er hvor mange muligheder vi har at vælge imellem og k er hvor mange vi skal vælge.
Eksempel:
Hvis vi har en skål med n bolde, som er nummereret. Vi vælger en tilfældig bold fra skålen og skriver tallet ned. Derefter putter vi bolden tilbage. Vi gør dette k gange, og sandsynligheden er derfor n^k.
Hvis vi skal udregne chancen for et sekvens af bolde, vil sandsynligheden være

$$P = n^{-k} = \frac{1}{n^k}$$

Partial Derivatives - Limits and Continuity

Directional Derivatives

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

3 Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Gradient

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Vis at limits ikke eksisterer

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

Eksempel

Hvis $f(x,y) = \frac{xy}{x^2+y^2}$, findes $\lim_{(x,y)\to(0,0)}$?
Hvis $y = 0$, så vil $f(x,0) = \frac{0}{x^2} = 0$ og derfor

$$f(x,y) \to 0 \quad \text{når} \quad (x,y) \to (0,0) \;\; \text{langs med x-aksen}$$

Hvis $x = 0$, så vil $f(0, y) = \frac{0}{y^2} = 0$ og derfor

$$f(x, y) \to 0 \quad \text{når} \quad (x, y) \to (0, 0) \ \text{ langs med y-aksen}$$

Selvom vi har fået identiske limits langs med de to akser, beviser det ikke, at limit er 0. Vi vil nå gå i mod $(0, 0)$ langs med en anden linje, lad os sige $y = x$, for alle $x \neq 0$:

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

Derfor vil

$$f(x, y) \to 0 \quad \text{når} \quad (x, y) \to (0, 0) \ \text{ langs med} x = y$$

Eftersom vi har fået forskellige limits langs med forskellige linjer, findes limit ikke.

The Squeeze Theorem

Definition for Squeeze Theorem:

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Dette theorem siger, hvis $g(x)$ er klemt mellem $f(x)$ og $h(x)$ tæt på a, og hvis f og h har den samme limit L ved a, så bliver g nødt til at have det samme limit L ved a.

Eksempel

Vis at $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Først skal vi lige lægge mærke til, at vi ikke na skrive limit som produktet af limits $\lim_{x \to 0} x^2$ og $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$, da den sidste ikke eksisterer (vi kan ikke dividere med 0).

Vi kan dog finde limit ved brug af Squeeze Theorem. For at anvende Squeeze Theorem, skal vi finde en funktion f som er mindre end $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ og en funktion større end g. Funktionerne f og h skal også gå mod 0, når $x \to 0$. For at gøre dette, skal vi bruge vores viden om sinus funktioner. Siden sinus af et vilkårligt tal ligger mellem -1 og 1, kan vi skrive

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Dette vil være sandt, ligemeget hvilket positivt tal vi ganger på. Vi ved $x^2 \geq 0$ for alle x, and derfor vi kan derfor gange det på alle sider af ovenstående.

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Vi ved

$$\lim_{x\to 0} x^2 = 0, \quad \lim_{x\to 0} -x^2 = 0$$

Hvis vi tager $f(x) = -x^2$, $g(x) = x^2 sin\left(\frac{1}{x}\right)$ og $h(x) = x^2$ i Squeeze Theorem, vi kan se

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

Eksempel med 2 variable

Beregn $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$
Vi starter med at finde et udtryk der er mindre end denne funktion for $x \geq 0$. y^4 vil altid give et positivt tal, og derfor vil $0 \leq \frac{xy^4}{x^4+y^4}$.
Vi finder nu et udtryk, som er større ende denne funktion. Det gør vi ved at gange $\frac{1}{y^4}$ på i tæller og nævner:

$$\frac{xy^4 \cdot \frac{1}{y^4}}{(x^4+y^4)\cdot\frac{1}{y^4}} = \frac{x}{\frac{x^4}{y^4}+1}$$

da x^4 og y^4 altid vil give et positivt tal, dividere vi med et tal som er større end eller lig med 1, og vi kan derfor skrive:

$$0 \leq \frac{xy^4}{x^4+y^4} \leq x$$

Vi kan nu finde limits for de to funktioner.

$$\lim_{(x,y)\to(0,0)} 0 = 0, \quad \lim_{(x,y)\to(0,0)} x = 0$$

Vi ved derfor at

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} = 0$$

Continuity

Definition:

6 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say that f is **continuous on** D if f is continuous at every point (a, b) in D.

Eksempel

Hvor er funktion $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ kontinuer?
Funktionen f er ikke kontinuer i $(0,0)$, da den ikke er defineret der.

Implicit derivatives

Eksempel

Brug implicit differentation til at finde den partielle afledte af $\partial z/\partial x$ og $\partial z/\partial y$ for

$$x^2 - y^2 + z^2 - 2z = 4$$

Vi starter med at finde $\partial z/\partial x$

$$2x + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0$$

Det løser vi i maple og får

$$\frac{\partial z}{\partial x} = -\frac{x}{z-1}$$

Så finder vi $\partial z/\partial y$

$$-2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0$$

Det løser vi i maple og får

$$\frac{\partial z}{\partial y} = \frac{y}{z-1}$$

Appendix

Vejledende løsninger eksamen 2020

Problem 1 (Function Limits, Continuity)

Consider the function $f(x) = \frac{3x^2-5x-2}{5x^2-20}$

a)

Is f defined for all $x \in \mathbb{R}$? If not, specify for which $x \in \mathbb{R}$ the function f is not defined.

Solution

Nej. f er ikke defineret i $x = 2 \vee x = -2$ da tælleren vil give 0.

b)

Determine $\lim_{x\to a} f(x)$ for any $a \in \mathbb{R}, \quad |a| \neq 2$.

Solution

Vi sætter a ind i funktionen i stedet for x og simplificere og får

$$\lim_{x\to a} f(x) = \frac{3a^2-5a-2}{5a^2-20} = \frac{3a+1}{5a+10}$$

c)

Determine $\lim_{x\to 2} f(x)$ or decide that is does not exist. Note: LimitTutor kan bruges til dette.

$$\begin{aligned}
\lim_{x\to 2} \frac{3x^2-5x-2}{5x^2-20} &= \lim_{x\to 2} \frac{1}{5}\frac{3x+1}{x+2} && \text{Faktoriseringsreglen er blevet brugt} \\
&= \frac{1}{5}\lim_{x\to 2} \frac{3x+1}{x+2} && \text{Konstantreglen er blevet brugt} \\
&= \frac{1}{5}\frac{\lim_{x\to 2}(3x+1)}{\lim_{x\to 2}(x+2)} && \text{Kvocientreglen er blevet brugt} \\
&= \frac{1}{5}\frac{\lim_{x\to 2} 3x + \lim_{x\to 2} 1}{4} && \text{Sumreglen er blevet brugt} \\
&= \frac{1}{5}\cdot\frac{7}{4} && \text{Identitetsreglen er blevet brugt} \\
&= \frac{7}{20}
\end{aligned}$$

d)

Determine $\lim_{x\to -2} f(x)$ or decide that is does not exist.

Solution

Nævneren kommer til at gå imod 20, mens tælleren kommer til at gå imod 0. Når man kigger på $\lim_{x\to-2} f(x)$ fra positiv og negativ retning, kan vi se at

$$\lim_{x\to-2^-} f(x) = -\infty, \quad \lim_{x\to-2^+} f(x) = \infty$$

$\lim_{x\to-2}$ eksisterer derfor ikke.

e)

Is f continuous for $x \in \mathbb{R}$?

Solution

Funktionen f er ikke kontinuer, da $f(2)$ og $f(-2)$ ikke er defineret og derfor ikke er kontinuer i disse punkter.

f)

We now restrict the domain of f to the open interval $]-2,2[$. Is restricted f differentiable everywhere in $]-2,2[$?

Solution

Da tælleren aldrig bliver nul da hverken $f(-2)$ eller $f(2)$ er med, vil funktionen være differentiabel i alle $x \in]-2,2[$

Problem 2 (Functions of 2 and more variables)

Consider the funktion $f(x,y,z) = (x+y)^2+(y+z)^2+(z+x)^2$ defined for all $x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}$.

a)

What is the gradient vector of f at the point (2, -1, 2)?

Solution

$\nabla(f(x,y,z)) = (f_x, f_y, f_z)$ ved $(2,-1,2)$. Vi skal derfor bestemme de pariale afledte f_x, f_y, f_z, som er:

$$f_x(x,y,z) = 4x+2y+2z$$
$$f_y(x,y,x) = 4y+2x+2z$$
$$f_z(x,y,z) = 4z+2x+2y$$

Hvis vi sætter det ind får vi:

$$f_x(2,-1,2) = 4\cdot 2 + 2\cdot(-1) + 2\cdot 2 = 10$$
$$f_y(2,-1,2) = 4\cdot(-1) + 2\cdot 2 + 2\cdot 2 = 4$$
$$f_z(2,-1,2) = 4\cdot 2 + 2\cdot(-1) + 2\cdot 2 = 10$$

og

$$\nabla(f(2,-1,2)) = \langle 10, 4, 10\rangle$$

b)

What is the directinal derivative of f in the gradient vector direction at the point $(2, -1, 2)$?

Solution

Vi starter med at normalisere vektoren $\langle 10, 4, 10 \rangle$ (kan gøres i maple) og får

$$u = \left\langle \frac{10}{6\sqrt{6}}, \frac{4}{6\sqrt{6}}, \frac{10}{6\sqrt{6}} \right\rangle$$

hvilket vektoren i retningen af gradienten. Denne ganger vi på $\nabla f(2, -1, 2)$

$$D_u(f(2, -1, 2)) = \nabla f(2, -1, 2) \cdot u = \frac{108}{3\sqrt{6}} = \frac{36}{\sqrt{6}} = 6\sqrt{6}$$

c)

What are the critical points of f?

Solution

$f_x(x, y, z) = 0,\ f_y(x, y, z) = 0,\ f_z(x, y, z) = 0$ har kun en løsning, nemlig når $x = y = z = 0$. Siden f er defineret over det hele, er der kun et kritisk punkt $(0, 0, 0)$.

d)

Classify each critical point as a local minimum, local maximum og neither.

Solution

Siden f kun har positive værdier og $f(0, 0, 0) = 0$, er det kritiske punkt et globalt minimum.

Problem 3 (Taylor and Maclaurin series)

Consider the function

$$f(x) = (1 + x)^s$$

where s is an arbitrary real number other than 0.

a)

What is the Maclaurin series for $f(x)$?

Solution

Vi starter med at finde værdier for $f(0), f'(0)$ osv, indtil vi kan se et mønster.

$$f(0) = 1$$

$$f'(x) = s(1+x)^{s-1}, \;\; f'(0) = s$$
$$f''(x) = s(s-1)(1+x)^{s-2}, \;\; f''(0) = s(s-1)$$
$$f'''(x) = s(s-1)(s-2)(1+x)^{s-3}, \;\; f'''(0) = s(s-1)(s-2)$$

Generelt

$$f^{(x)}(x) = s(s-1)(s-2)\dots(s-n+1)(1+x)^{s-n}, \;\; f^{(n)}(0) = s(s-1)(s-2)\dots(s-n+1)$$

Maclaurin serien er derfor

$$1 + \frac{s}{1!}x + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!}x^3 + \dots$$

b)

What is the radius of the convergence of this Maclaurin series?

Solution

Vi har

$$a_{n+1} = \frac{s(s-1)\dots(s-n)}{(n+1)!}x^{n+1}$$

og

$$a_n = \frac{s(s-1)\dots(s-n+1)}{n!}x^n$$

Vi skriver formlen fra side 774 i bogen op og simplificere

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{s(s-1)\dots(s-n)}{(n+1)!} \cdot \frac{n!}{s(s-1)\dots(s-n+1)} \cdot \frac{x^{n+1}}{x^n}\right| = \left|\frac{s-n}{n+1}x\right| = \left|\frac{\frac{s}{n}-1}{1+\frac{1}{n}}x\right|$$

Vi kan nu tage limit hvor x går mod uendelig af dette

$$\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{s}{n}-1}{1+\frac{1}{n}}x\right| = |x|$$

Ved ratio testen kan vi se, at serien konvergere for $x \in]-1,1[$ og at radius for konvergering er 1.

c)

When s is a positiv integer then it can be shown that f is equal to the sum of its Maclaurin series for $x \in]-1,1[$. Suppose that you are asked to approximate $f(0.07)$ for any positiv integer s. Explain (without doing any calculations) how would you do it.

Solution

Vi tager den n'te grad af Taylor polynomiet $T_n(x)$ af f for en lav værdi af n og derefter finder $T_n(0.07)$. Større $n, n \leq s$ skal bruges for at finde en bedre approksimation.

Problem 4 (Integration)

Remember to justify all non-trivial details. In particular, clearly state if you use the substitution rule or integration by parts.

a)

Determine $\int x(x-1)(x-2)dx$ and $\int_0^1 x(x-1)(x-2)dx$
Note: Mange af disse slags problem kan blive løst med IntTutor i maple (der ligger en maplefil på github, hvor det er sat op).

Solution

$$
\begin{aligned}
\int (x(x-1)(x-2))dx &= \int (x^3 - 3x^2 + 2x)dx && \text{Omskrivningsreglen er blevet brugt} \\
&= \int x^3 dx + \int -3x^2 dx + \int 2x dx && \text{Sumreglen er blevet brugt} \\
&= \frac{1}{4}x^4 + \int -3x^2 dx + \int 2x dx && \text{Potensreglen er blevet brugt} \\
&= \frac{1}{4}x^4 + -3\int x^2 dx + \int 2x dx && \text{Konstantreglen er blevet brugt} \\
&= \frac{1}{4}x^4 - x^3 dx + \int 2x dx && \text{Potensreglen er blevet brugt} \\
&= \frac{1}{4}x^4 - x + 2\int x dx && \text{Konstantreglen er blevet brugt} \\
&= \frac{1}{4}x^4 - x^3 + x^2 + K && \text{Potensreglen er blevet brugt}
\end{aligned}
$$

For at finde $\int_0^1 (x(x-1)(x-2)$ kan vi bruge det forrige integral, og sætte ind. Vi får

$$\int_0^1 = [\frac{1}{4}x^4 - x^3 + x^2]_0^1 = \frac{1}{4} \cdot 1 - 1 + 1 - \frac{1}{4} \cdot 0 - 0 + 0 = \frac{1}{4}$$

b)

Determine $\int \frac{x-3}{x^2-6x-5}dx$

Solution

$$\begin{aligned}
\int \left(\frac{x-3}{x^2-6x-5}\right) dx &= \int \frac{1}{2u} du && \text{Substitutionsreglen er blevet brugt} \\
&= \frac{1}{2}\int \frac{1}{u} du && \text{Konstantreglen er blevet brugt} \\
&= \frac{1}{2} \cdot \ln(u) && \text{Potensreglen er blevet brugt} \\
&= \frac{1}{2} \cdot \ln(x^2-6x-5) + K && \text{Sætter u ind igen}
\end{aligned}$$

c)

Determine $\int (x^{10} \cdot ln(x))dx$

Solution

$$\begin{aligned}
\int x^{10} \cdot ln(x) dx &= \frac{1}{11}x^{11} \cdot ln(x) - \int \frac{1}{11}x^{10} dx && \text{Delereglen er blevet brugt} \\
&= \frac{1}{11}x^{11} \cdot ln(x) - \frac{1}{11}\int x^{10} dx && \text{Kontantsreglen er blevet brugt} \\
&= \frac{1}{11}x^{11} \cdot ln(x) - \frac{1}{121}x^{11} + K && \text{Potensreglen er blevet brugt}
\end{aligned}$$

Problem 5 (Combinatorics)

We sample n balls enumareted with the numbers $1, 2, \dots, n$ one by one fra a hat without replacement, i.e. we blindly pick a ball in each of n rounds.

a)

Provide an appropriate model for the probability space underlying this experiment

Solution

$\Omega = S_n$ er permutationen af numrene $\{1, 2, \dots, n\}$ med en ligelig fordelt distribution.

b)

What is the probability of at least one ball having the number on it as the round it was picked in?

Solution

Vi starter med at definerer $A_i = \{\omega \in S_n : \ \omega_i = i\}$ og bruger inklusion-eksklusion princippet til at se

$$\begin{aligned}
\mathbb{D}\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{\emptyset \neq i \subset 1,2,\ldots,n} (-1)^{|i|+1} \mathbb{P}(\cap_i A_i) \\
&= \sum_{\emptyset \neq i \subset 1,2,\ldots,n} (-1)^{|I|+1} \frac{(n-|i|)!}{n!} \\
&= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{(n-k)!}{n!} \\
&= 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\end{aligned}$$

c)

What is the probability p_n taht the balls with the numbers 1, 2 are picked consecutively at some point during our experiment, i.e. wihtout any other balls being picked in between them? What does p_n converge to as $n \to \infty$.

Solution

Vi lader A_i være det event, hvor 1, 2 bliver valg i runderne i, i + 1, så vil

$$p_n = \mathbb{P}(\text{Vi ser 1, 2 efter hinanden}) = \mathbb{P}\left(\bigcup_{i=1}^{n-1} A_i\right) = \sum_{i=1}^{n-1} \mathbb{P}(A_i) = \sum_{i=1}^{n-1} \frac{(n-2)!}{n!} = \frac{n-1)!}{n!} = \frac{1}{n}$$

Vi kan nu tage limit af dette.

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Problem 6 (Random variables and distributions)

a)

Let (X, Y) be a 2-dimensionel Gaussian random vector with

$$\mathbb{E}X = 0, \ \ \mathbb{E}Y = 0, \ \ Var\ X = 2, \ \ Var\ Y = 4, \ \ Cov(X,Y) = 1$$

What is the joint distriubution of $X + Y$ and $X - Y$?

Solution

Eftersom (X + Y, X - Y) er en lineær transformation af en centreret Gaussian vector, er det også en centreret Gaussian vector. Derfor kan vi betemme the kovariante matrice.

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = 2 + 4 + 2 = 8$$

$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y) = 2 + 4 - 2 = 4$$

$$Cov(X+Y, X-Y) = Var(X) - Var(Y) = 2 - 4 = -2$$

Derfor er $(X+Y, X-Y) \sim N(0, A)$ med

$$A = \begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix}$$

b)

Let X_1, X_2 be independent random variables with distribution Exp(1). Determine the probability density function (pdf) of $X_1 + X_2$ and $\min\{X_1, X_2\}$.

Solution

Her har vi $\alpha = 1$. We beregner $X_{\min} = \min_{i=1}^{2} X_i$ som

$$\begin{aligned}
\mathbb{P}(X_{\min} \le x) &= 1 - \mathbb{P}(X_{\min} > x) \\
&= 1 - \mathbb{P}(X_1 > 0 and X_2 > 0) \\
&= 1 - \mathbb{P}(X_1 > 0)^2 \\
&= 1 - (1 - F_{X_1}(x))^2 \\
&= 1 - \mathrm{e}^{-2ax}
\end{aligned}$$

Derfor er $X_{\min} \sim \mathrm{Exp}(2\alpha)$. Nu bruger vi foldningsformelen til at bestemme pdf for $X_1 + X_2$ og får

$$\rho_{X_1+X_2}(x) = \alpha^2 \int_0^x e^{-\alpha y} e^{-\alpha(x-y)} dy \mathbb{1}_{(0,\infty)}(x) = \alpha^2 x e^{-\alpha x} \mathbb{1}_{(0,\infty)}(x)$$

Problem 7 (Expectation and variance)

Compute the following (each item is a separate problem):

a)

Expectation $\mathbb{E}X$ and variance $VarX$ of a random variable X with $X = 2Y + 1$, where $Y$is uniformly distributed on $[0, 1]$.

Solution

Vi har

$$\mathbb{E}Y = \int_0^1 y dy = \frac{1}{2}, \mathbb{E}Y^2 = \int_0^1 y^2 dy = \frac{1}{3}, Var(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \frac{1}{12}$$

Derfor er

$$\mathbb{E}X = 2\mathbb{E}Y + 1 = 2, \;\; Var(Y) = Var(2X) = 4Var(X) = \frac{1}{3}$$

b)

Variance $Var(X)$ of the random variable $X = e^{-tY}$ for $t > 0$, where $Y \sim Exp(1)$.

Solution

Her er $\alpha = 1$. Vi beregner det forventede ved brug af pdf'en for en exponential distribureret tilfældig variabel

$$\mathbb{E}X = \mathbb{E}e^{-tY} = \alpha \int_0^\infty e^{-ty} e^{-\alpha y} dy = \frac{\alpha}{t + a} = \frac{1}{1 + t}$$

Hvis vi laver samme beregning med $2t$ i stedet for t får vi $\mathbb{E}X^2 = \frac{\alpha}{2t+\alpha}$ og vi har derfor

$$Var(X) = \frac{\alpha}{2t + \alpha} - \frac{\alpha^2}{(t + \alpha)^2} = \frac{t^2}{(1 + 2t)(1 + t)^2}$$

c)

Expectation $\mathbb{E}X$ of a random variable X that has cumulative distribution function (cdf)

$$F_X(x) = 1 - e^{-\alpha \lfloor x \rfloor}$$

for any $x \geq 0$, where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ is rounding down and $\alpha > 0$ is a constant.

Solution

Fra cdf'en kan vi see at

$$\mathbb{P}(X = x) = e^{\alpha(x-1)} - e^{-\alpha x} = e^{-\alpha x}(e^\alpha - 1)$$

for et hvert $x \in \mathbb{N}$. Vi kan derfor beregne

$$\begin{aligned}
\mathbb{E}X &= \sum_{x \in \mathbb{N}} x \mathbb{P}(X = x) \\
&= (e^\alpha - 1) \sum_{x \in \mathbb{N}} x e^{-\alpha x} \\
&= -(e^\alpha - 1) \frac{\mathrm{d}}{\mathrm{d}\alpha} \sum_{x \in \mathbb{N}} e^{-\alpha x} \\
&= -(e^\alpha - 1) \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{1}{1 - e^{-\alpha}} \\
&= \frac{(e^\alpha - 1)e^{-\alpha}}{(1 - e^{-\alpha})^2} \\
&= \frac{1}{1 - e^{-\alpha}}
\end{aligned}$$

Tables

Contents:

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$\boxed{1} \qquad F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\boxed{2} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$\boxed{1}$ Table of Indefini e Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Sums of Powers

$\boxed{5}$ $$\sum_{i=1}^{n} 1 = n$$

$\boxed{6}$ $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$\boxed{7}$ $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{8}$ $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Properties of Sums

$$\boxed{9} \qquad \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$$

$$\boxed{10} \qquad \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\boxed{11} \qquad \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$\boxed{7}$ **Theorem** The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

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