Ejercicios ecuaciones diferenciales

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1. $\begin{cases} (4x^2 - 2y^2)\,dx = 2xy\,dy \\ y(1) = 3 \end{cases}$

$$(4x^2-2y^2)\,dx = 2xy\,dy \Rightarrow 4x^2-2y^2 = 2xy\,y' \Rightarrow y' = \frac{4x^2-2y^2}{2xy}$$ (mismo grado)

Podemos comprobarlo:

$$y' = \frac{f(x,y)}{g(x,y)} \qquad f(tx,ty) = t^2 f(x,y)$$

Homogénea

$$g(tx,ty) = 2\cdot(tx)\cdot(ty) = 2t^2xy = t^2\,2xy = t^2 g(x,y)$$
$$f(tx,ty) = 4(tx)^2 - 2(ty)^2 = t^2\cdot(4x^2-2y^2) = t^2 f(x,y)$$

Mismo grado

$$v = \frac{y}{x} \Rightarrow y = v\cdot x \Rightarrow y' = v'x + v$$

$$y' = \frac{4x^2-2y^2}{2xy} \Rightarrow v'x+v = \frac{4x^2-2\cdot(vx)^2}{2\cdot x(v\cdot x)} \Rightarrow \frac{4x^2-2v^2x^2}{2x^2\cdot v} = v'x+v \Rightarrow$$

$$v'x = \frac{2-v^2}{v} - v \Rightarrow v'x = \frac{2-2v^2}{v} \Rightarrow \frac{v\cdot v'}{2-2v^2} = \frac{1}{x} \Rightarrow$$

$$\frac{v\cdot v'}{2v^2-2} = -\frac{1}{x} \Rightarrow \frac{v\,v'}{v^2-1} = -\frac{2}{x} \Rightarrow \frac{1}{2}\int \frac{2\cdot v\cdot v'}{v^2-1}\,dv = -\int \frac{2}{x}\,dx \Rightarrow$$

$$\frac{1}{2}\log|v^2-1| = -2\log|x| + c \Rightarrow \frac{1}{2}\log\left|\left(\frac{y}{x}\right)^2-1\right| = -2\log|x|+c \Rightarrow$$

$$\Rightarrow \log\left(\left|\left(\frac{y}{x}\right)^2-1\right|\right)^{\frac{1}{2}} = \log(|x|)^{-2} + c \Rightarrow \log\sqrt{\left|\left(\frac{y}{x}\right)^2-1\right|} = \log\left|\frac{1}{x^2}\right| + c \Rightarrow$$

$$c = \log\left(\frac{\sqrt{\left|\left(\frac{y}{x}\right)^2-1\right|}}{x^2}\right)$$

• Condiciones: $y(1) = 3 \Rightarrow \log\left(\dfrac{\sqrt{\left(\frac{3}{1}\right)^2-1}}{1^2}\right) = c$

$\rightarrow c = \log\sqrt{8}$

$$\boxed{\log\left(\sqrt{\left|\left(\frac{y}{x}\right)^2-1\right|}\right) = \log\left|\frac{1}{x^2}\right| + \log\sqrt{8}}$$

2. $\begin{cases} y'' + 4y = -4 \operatorname{sen}(2x) \\ y(0) = -1 \quad ; \quad y'(0) = 4 \end{cases}$

• Ec. dif. homogénea asoc. $\Rightarrow$ $\boxed{y_h = C_1 \operatorname{sen}(2x) + C_2 \cos(2x) \qquad C_1, C_2 \in \mathbb{R}}$

$y'' + 4y = 0$

$P(x) = x^2 + 4 \Rightarrow x = \pm\sqrt{4} = \pm 2i \Rightarrow x = 0 \pm 2i$

Raíces : $0 \pm 2i$ mult. 1 $\quad e^{0x} \cdot \operatorname{sen}(2x) = \operatorname{sen}(2x)$; $e^{0x} \cdot \cos(2x) = \cos(2x)$

• Sol particular $\Rightarrow$ $\boxed{y_p(x) = x\cos(2x)}$

$y(x) = Ax\operatorname{sen}(2x) + Bx\cos(2x) \quad ; \quad y'(x) = A\operatorname{sen}(2x) + 2Ax\cos(2x) + B\cos(2x) - 2Bx\operatorname{sen}(2x)$

$\cancel{y'(x) = 2A\cos(2x) - 2B\operatorname{sen}(2x)}$

$y''(x) = 2A\cos(2x) + 2A\cos(2x) - 4Ax\operatorname{sen}(2x) - 2B\operatorname{sen}(2x) - 2B\operatorname{sen}(2x) - 4Bx\cos(2x)$

Imponemos solución :

$2A\cos(2x) + 2A\cos(2x) - \cancel{4Ax\operatorname{sen}(2x)} - 2B\operatorname{sen}(2x) - 2B\operatorname{sen}(2x) - \cancel{4Bx\cos(2x)} + \cancel{4Ax\operatorname{sen}(2x)} +$
$+ \cancel{4Bx\cos(2x)} = -4\operatorname{sen}(2x) \Rightarrow 4A\cos(2x) - 4B\operatorname{sen}(2x) = -4\operatorname{sen}(2x)$

$\begin{cases} -4B = -4 \Rightarrow \underline{B = 1} \\ 4A = 0 \Rightarrow \underline{A = 0} \end{cases}$

Sol general : $\boxed{y(x) = x\cos(2x) + C_1\operatorname{sen}(2x) + C_2\cos(2x) \qquad C_1, C_2 \in \mathbb{R}}$

• Condiciones : $-1 = y(0) = 0 \cdot \cos(2 \cdot 0) + C_1 \cdot \operatorname{sen}(2 \cdot 0) + C_2 \cdot \cos(2 \cdot 0) \Rightarrow \underline{C_2 = -1}$

$y'(x) = \cos(2x) - 2x\operatorname{sen}(2x) + 2C_1\cos(2x) - 2C_2\operatorname{sen}(2x)$

$y'(0) = 4 = \cos(2 \cdot 0) - 2 \cdot 0 \cdot \operatorname{sen}(2 \cdot 0) + 2 \cdot C_1 \cdot \cos(2 \cdot 0) - 2 \cdot C_2 \cdot \operatorname{sen}(2 \cdot 0)$

$4 = 1 + 2C_1 \Rightarrow \underline{C_1 = \frac{3}{2}}$

Sol. del problema de cond. iniciales : $\boxed{y(x) = x\cos(2x) + \frac{3}{2}\operatorname{sen}(2x) - \cos(2x)}$

3. $y' - 6y = 5e^{6x}y^4 \Rightarrow \alpha = 4 \Rightarrow z = y^{1-4} \Rightarrow z = y^{-3}$

$z' = -3y^{-4}\cdot y' \Rightarrow$

$\Rightarrow z' = -3y^{-4}\cdot(y^4 5e^{6x} + 6y) \Rightarrow z' = -18y^{-3} - 15e^{6x} \Rightarrow$

$\Rightarrow z' = -15e^{6x} - 18z \Rightarrow \underbrace{z' + 18z = -15e^{6x}}_{\text{Ec. dif. lin de 1}^{\text{er}}\text{ orden}}$

$\underbrace{z' + 18z = 0}_{\text{Ec. dif. lin homog. asoc.}} \Rightarrow z' = -18z \Rightarrow \frac{dz}{dx} = -18z \Rightarrow \frac{dz}{z} = -18\,dx \Rightarrow$

$\Rightarrow \int\frac{dz}{z} = \int -18\,dx \Rightarrow \log|z| = -18x + c \Rightarrow z = e^{-18x+c} \Rightarrow z = e^{-18x}\cdot \underbrace{e^c}_{K}$

$z = Ke^{-18x}$

consideramos $z = K(x)e^{-18x}$ e imponemos que sea sol. de la ec. dif. lineal.

$z' = K'(x)\cdot e^{-18x} + K(x)\cdot(-18e^{-18x}) \Rightarrow z' = K'(x)e^{-18x} - 18K(x)e^{-18x}$

$K'(x)e^{-18x} - \cancel{18K(x)e^{-18x}} + \cancel{18K(x)e^{-18x}} = -15e^{6x}$

$K'(x)e^{-18x} = -15e^{6x} \Rightarrow K'(x) = \frac{-15e^{6x}}{e^{-18x}} \Rightarrow K'(x) = -15e^{24x}$

$K(x) = \int -15e^{24x}\,dx = \frac{-15}{24}\int 24e^{24x}\,dx = \frac{-15}{24}e^{24x} + c, \quad c \in \mathbb{R}$

$K(x) = \frac{-15}{24}e^{24x} + c, \quad c \in \mathbb{R}$

Así: $z = \left(\frac{-15}{24}e^{24x} + c\right)\cdot e^{-18x} \Rightarrow z = \frac{-15}{24}e^{6x} + ce^{-18x}$

$z = y^{-3} \Rightarrow y^{-3} = \frac{-15}{24}e^{6x} + ce^{-18x} \Rightarrow \frac{1}{y^3} = \frac{-15}{24}e^{6x} + ce^{-18x}$

$y^3 = \frac{1}{\frac{-15}{24e^{6x}} + ce^{-18x}} \Rightarrow y^3 = \frac{1}{\frac{-5}{8e^{6x}} + ce^{-18x}} \Rightarrow \boxed{y = \sqrt[3]{\frac{1}{\frac{-5}{8e^{6x}} + ce^{-18x}}}}$

$y = \sqrt[3]{\frac{1}{\frac{-5+8ce^{-12x}}{8e^{6x}}}} \Rightarrow \boxed{y = \sqrt[3]{\frac{8e^{6x}}{-5+8ce^{-12x}}}}$

4. $\begin{cases} y'' + 6y' + 9y = 6e^{-3x} + 18 \\ y(0) = 2 \quad ; \quad y'(0) = 25 \end{cases}$

· Ec. dif. homog. asoc. ⤳ $\boxed{y_h(x) = C_1 e^{-3x} + C_2 x e^{-3x} \quad C_1, C_2 \in \mathbb{R}}$

$y'' + 6y' + 9y = 0$

$\Downarrow$

$P(x) = x^2 + 6x + 9$

$x = \dfrac{-6 \pm \sqrt{36-36}}{2} = \dfrac{-6 \pm 0}{2} = \begin{cases} -3 \\ -3 \end{cases}$

Raíces: -3 con mult 2. $\Rightarrow e^{-3x}, xe^{-3x}$

· Sol particular ⤳ $\boxed{y_p(x) = 3x^2 e^{-3x} + 2}$

$y(x) = Ax^2 e^{-3x} + B$

$y'(x) = 2Ax e^{-3x} - 3Ax^2 e^{-3x} = e^{-3x} \cdot (2Ax - 3Ax^2)$

$y''(x) = 2Ae^{-3x} - 6Axe^{-3x} - (6Axe^{-3x} - 9Ax^2 e^{-3x})$

$= 2Ae^{-3x} - 6Axe^{-3x} - 6Axe^{-3x} + 9Ax^2 e^{-3x}$

$= 2Ae^{-3x} - 12Axe^{-3x} + 9Ax^2 e^{-3x} = e^{-3x} \cdot (2A - 12Ax + 9Ax^2)$

Entonces:

$e^{-3x} \cdot (2A - 12Ax + 9Ax^2) + 6e^{-3x} \cdot (2Ax - 3Ax^2) + 9 \cdot (Ax^2 e^{-3x} + B) = 6e^{-3x} + 18$

$\cancel{e^{-3x}} \cdot (2A - \cancel{12Ax} + \cancel{9Ax^2}) + \cancel{e^{-3x}} \cdot (\cancel{12Ax} - \cancel{18Ax^2}) + (\cancel{9Ax^2} e^{-3x} + 9B) = 6e^{-3x} + 18$

$2Ae^{-3x} + 9B = 6e^{-3x} + 18 \Rightarrow \begin{cases} 2A = 6 \Rightarrow \underline{A = 3} \\ 9B = 18 \Rightarrow \underline{B = 2} \end{cases}$

Sol general: $\boxed{y(x) = 3x^2 e^{-3x} + 2 + C_1 e^{-3x} + C_2 x e^{-3x}}$

Condiciones: $2 = y(0) = 3 \cdot 0^2 \cdot e^{-3\cdot 0} + 2 + C_1 \cdot e^{-3\cdot 0} + C_2 \cdot 0 \cdot e^{-3\cdot 0}$

$2 = 2 + C_1 \Rightarrow C_1 = 0$

$y'(x) = 6xe^{-3x} - 9x^2 e^{-3x} - 3C_1 e^{-3x} + C_2 e^{-3x} - 3xC_2 e^{-3x}$

$25 = y'(0) = 6 \cdot 0 \cdot e^{-3\cdot 0} - 9 \cdot 0^2 \cdot e^{-3\cdot 0} - 3C_1 \cdot e^{-3\cdot 0} + C_2 \cdot e^{-3\cdot 0} - 3 \cdot 0 \cdot C_2 \cdot e^{-3\cdot 0}$

$25 = -3C_1 + C_2 \Rightarrow C_2 = 25$

Sol del problema de cond. iniciales: $\boxed{y(x) = 3x^2 e^{-3x} + 2 + 25xe^{-3x}}$

5. $\begin{cases} y' + xy = 3xe^{x^2} \\ y(0) = 2 \end{cases}$

$y' + xy = 0 \Rightarrow y' = -xy \Rightarrow \frac{dy}{dx} = -xy \Rightarrow \frac{dy}{y} = -x\,dx$

$\log y = \frac{-x^2}{2} + C \Rightarrow y = e^{\frac{-x^2}{2} + C} = e^{\frac{-x^2}{2}} \cdot \underbrace{e^{C}}_{K} \Rightarrow y = K \cdot e^{\frac{-x^2}{2}}$

• $y = K(x) \cdot e^{\frac{-x^2}{2}}$ e imponemos solución

$y' = k'(x) \cdot e^{\frac{-x^2}{2}} + k(x) \cdot \left(-x \cdot e^{\frac{-x^2}{2}}\right) \Rightarrow y' = k'(x) e^{\frac{-x^2}{2}} - x K(x) e^{-\frac{x^2}{2}}$

$\cancel{k'(x) e^{\frac{-x^2}{2}} = 3xe^{x^2}}$

$k'(x) e^{\frac{-x^2}{2}} - x K(x) \cdot e^{\frac{-x^2}{2}} + x \left(K(x) e^{-\frac{x^2}{2}}\right) = 3x e^{x^2}$

$k'(x) e^{\frac{-x^2}{2}} = 3x e^{x^2} \Rightarrow k'(x) = \frac{3xe^{x^2}}{e^{\frac{-x^2}{2}}} = 3xe^{\frac{3x^2}{2}}$

$K(x) = \int 3xe^{\frac{3x^2}{2}} = 3 \int xe^{\frac{3x^2}{2}} = e^{\frac{3x^2}{2}} + C \quad , C \in \mathbb{R}$

Sustituimos: $y = \left(e^{\frac{3x^2}{2}} + C\right) \cdot e^{\frac{-x^2}{2}} \Rightarrow \boxed{y = e^{x^2} + Ce^{-\frac{x^2}{2}}}$ Sol de la ec. dif.

Condiciones:

$2 = y(0) = e^{0^2} + C e^{\frac{-0^2}{2}} = 1 + C \Rightarrow 1 + C = 2 \Rightarrow \underline{C = 1}$

Sol. del prob. de cond. iniciales: $\boxed{y = e^{x^2} + e^{-\frac{x^2}{2}}}$

6. $\underbrace{(2xy - 3x^2y^2)}_{M(x,y)} + \underbrace{(x^2 - 2x^3y)}_{N(x,y)}\, y' = 0$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = 2x - 6x^2y \\[2ex] \dfrac{\partial N}{\partial x} = 2x - 6x^2y \end{array}\right\} \text{Como } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Es exacta} \Rightarrow \exists f \Big/ \begin{array}{l} \dfrac{\partial f}{\partial x} = M(x,y) \\[2ex] \dfrac{\partial f}{\partial y} = N(x,y) \end{array}$$

Como $\dfrac{\partial f}{\partial x} = M(x,y) \Rightarrow \dfrac{\partial f}{\partial x} = 2xy - 3x^2y^2 \Rightarrow$

$$\Rightarrow f = \int (2xy - 3x^2y^2)\,dx = \int 2xy\,dx - \int 3x^2y^2\,dx =$$

$$= 2y\int x\,dx - 3y^2\int x^2\,dx = x^2y - x^3y^2 + K(y)$$ $\xrightarrow{\text{derivamos respecto de } y}$

$$\frac{\partial f}{\partial y} = x^2 - 2x^3y \Rightarrow \cancel{x^2 - 2x^3y} + K'(y) = \cancel{x^2 - 2x^3y}$$

$$K'(y) = 0 \Rightarrow K(y) = 0$$

$$f = x^2y - x^3y^2 \Rightarrow \boxed{x^2y - x^2y^2 = c}$$ define de forma implícita las sol. de las ec. dif.

7. $y' - \frac{2}{x+2} y = 2 \cdot (x+2)^3$

· Ec. dif. lin. homog. asoc.

$$y' - \frac{2}{x+2} y = 0 \Rightarrow \frac{dy}{dx} = \frac{2}{x+2} y \Rightarrow \frac{dy}{y} = \frac{2}{x+2} dx \Rightarrow$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2}{x+2} dx \Rightarrow \log |y| = 2 \log |x+2| + c$$

$$y = e^{2\log|x+2| + c} = e^{2\log|x+2|} \cdot \underbrace{e^c}_{k} = k \cdot \left(e^{\log|x+2|}\right)^2 =$$

$$= k \cdot (x+2)^2 \Rightarrow y = k \cdot (x+2)^2$$

Tomamos $y = k(x) \cdot (x+2)^2$ y sustituimos

$$y' = k'(x) \cdot (x+2)^2 + 2k(x) \cdot (x+2)$$

$$k'(x) \cdot (x+2)^2 + 2k(x) \cdot (x+2) - \frac{2}{\cancel{x+2}} \cdot \left(k(x) \cdot (x+2)^{\cancel{2}}\right) = 2(x+2)^3$$

$$k'(x) \cdot (x+2)^2 + \cancel{2k(x) \cdot (x+2)} - \cancel{2k(x) \cdot (x+2)} = 2(x+2)^3$$

$$k'(x) \cdot (x+2)^2 = 2(x+2)^3 \Rightarrow k'(x) = 2(x+2)$$

$$k(x) = \int 2(x+2)\,dx = 2\int x\,dx + 2\int 2\,dx = x^2 + 4x + c \quad , c \in \mathbb{R}$$

Finalizamos ...

$$\boxed{y = (x^2 + 4x + c) \cdot (x+2)^2 \quad , c \in \mathbb{R}}$$

8. $e^x y^2 + y \operatorname{sen} x + (3y e^x - 2\cos x) y' = 0$

$$\Downarrow$$

$$\underbrace{\mu(y) \cdot (e^x y^2 + y \operatorname{sen} x)}_{M(x,y)} + \underbrace{\mu(y) \cdot (3y e^x - 2\cos x) y'}_{N(x,y)} = 0$$

$$\frac{\delta M}{\delta Y} = \mu'(y) \cdot (e^x y^2 + y \operatorname{sen} x) + \mu(y) \cdot (2e^x y + \operatorname{sen} x)$$

$$\frac{\delta N}{\delta X} = \mu(y) \cdot (3y e^x + 2 \operatorname{sen} x) \qquad \text{Como } \frac{\delta M}{\delta Y} = \frac{\delta N}{\delta X} \Rightarrow$$

$$\Rightarrow \mu' \cdot (e^x y^2 + y \operatorname{sen} x) + \mu \cdot (2e^x y + \operatorname{sen} x) = \mu \cdot (3y e^x + 2 \operatorname{sen} x)$$

$$\mu \cdot (3y e^x + 2 \operatorname{sen} x) - \mu (2e^x y + \operatorname{sen} x) = \mu' \cdot (e^x y^2 + y \operatorname{sen} x)$$

$$\mu \cdot (3y e^x + 2 \operatorname{sen} x - 2e^x y - \operatorname{sen} x) = \mu' \cdot (e^x y^2 + y \operatorname{sen} x)$$

$$\mu \cdot (y e^x + \operatorname{sen} x) = \mu' \cdot (e^x y^2 + y \operatorname{sen} x)$$

$$\mu \cdot (y e^x + \operatorname{sen} x) = \frac{d\mu}{dy} \cdot (e^x y^2 + y \operatorname{sen} x)$$

$$\frac{y e^x \operatorname{sen} x}{e^x y^2 + y \operatorname{sen} x} dy = \frac{d\mu}{\mu} \Rightarrow \frac{\cancel{(y e^x + \operatorname{sen} x)}}{y\cancel{(y e^x + \operatorname{sen} x)}} dy = \frac{d\mu}{\mu} \Rightarrow \frac{1}{y} dy = \frac{d\mu}{\mu} \Rightarrow$$

$$\int \frac{1}{y} dy = \int \frac{d\mu}{\mu} \Rightarrow \log|y| + c = \log|\mu| \Rightarrow \mu = e^{\log y + c} = e^{\log y} \cdot e^c = k \cdot y \Rightarrow$$

$$\Rightarrow \mu = K(x) \cdot y \xrightarrow{\text{le damos el valor de 1}} \mu(y) = y$$

Así: $y(e^x y^2 + y \operatorname{sen} x) + y(3y e^x - 2\cos x) y' = 0$

$$\frac{\delta f}{\delta X} = y(e^x y^2 + y \operatorname{sen} x) \Rightarrow f = \int y \cdot (e^x y^2 + y \operatorname{sen} x)\, dx =$$

$$= \int e^x y^3\, dx + y^2 \int \operatorname{sen} x\, dx = y^3 \int e^x dx + y^2 \int \operatorname{sen} x\, dx =$$

$$= y^3 e^x - y^2 \cos x + K(y) \Rightarrow f = y^3 e^x - y^2 \cos x + K(y)$$

¿?

$$\frac{\partial f}{\partial y} = y(3ye^x - 2\cos x) \Rightarrow 3y^2e^x - 2y\cos x + k'(y) = y(3ye^x - 2\cos x)$$

$$k'(y) = y(3y e^x - 2\cos x) - 3y^2 e^x + 2y\cos x$$

$$k'(y) = 3y^2e^x - \cancel{2y\cos x} - \cancel{3y^2e^x} + \cancel{2y\cos x} \Rightarrow k'(y) = 0 \Rightarrow k(y) = 0$$

Esto significa que $k(y) = c$

$$f = y^3e^x - y^2\cos x + c$$

$\hookrightarrow$ $\boxed{y^3e^x - y^2\cos x + c = 0}$ define de forma implícita las sol de las ec. dif.

9. $y^{vii} + y^{vi} + 2y^{v} + 10 y^{iv} + 13 y''' + 5y'' = 0$

$$P(x) = x^7 + x^6 + 2x^5 + 10x^4 + 13x^3 + 5x^2$$
$$= x^2 \cdot (x^5 + x^4 + 2x^3 + 10x^2 + 13x + 5)$$

	1	1	2	10	13	5
-1		-1	0	-2	-8	-5
	1	0	2	8	5	$\boxed{0}$
-1		-1	1	-3	-5	
	1	-1	3	5	$\boxed{0}$	
-1		-1	2	-5		
	1	-2	5	$\boxed{0}$		

$$x^2 - 2x + 5 = 0$$
$$x = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$
$$x = \frac{2 \pm \sqrt{16} \cdot \sqrt{-1}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Raíces:

-1 con mult $3 \Rightarrow e^{-1x}, xe^{-1x}, x^2e^{-1x} \Rightarrow e^{-x}, xe^{-x}, x^2e^{-x}$

0 con mult $2 \Rightarrow e^{0x}, xe^{0x} \Rightarrow 1, x$

$1 \pm 2i \Rightarrow e^{x}\,\mathrm{sen}(2x)$, $e^{x}\cos(2x)$

Sol general: $\boxed{y(x) = C_1 + C_2 x + C_3 e^{-x} + C_4 x e^{-x} + C_5 x^2 e^{-x} + C_6 e^{x}\,\mathrm{sen}(2x) + C_7 e^{x}\cos(2x)}$

$C_1, C_2, C_3, C_4, C_5, C_6, C_7 \in \mathbb{R}$