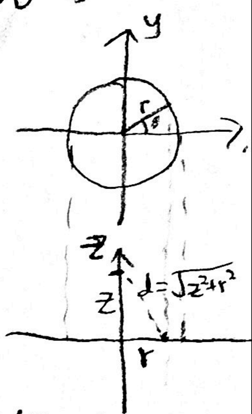
1.1

1) Note that for a uniform ring of radius r, on x-y plane, $\vec{E}$ at point $(0,0,z)$, $z>0$ is

$$\vec{E}(z,r) = \int_0^{2\pi} \frac{d\theta \cdot q}{2\pi}\cdot k \cdot \frac{z}{(z^2+r^2)^{3/2}}\cdot\hat{z} + 0$$

since symmetry on x-y plane



$$= \frac{kq}{2\pi}\cdot\frac{z}{(z^2+r^2)^{3/2}}\cdot\int_0^{2\pi} d\theta\,\hat{z} = \frac{kqz}{(z^2+r^2)^{3/2}}\cdot\hat{z}$$

Note that a disk is $\sum^{n\to\infty}$ of rings with radius $r: 0 \to a$, width dr all centered at the same point $(0,0,0)$. Each ring carry charge $q\cdot\frac{2\pi r}{\pi a}$

$$\therefore \vec{E}_{disk}(z) = \int_{r=0}^{a} \vec{E}_{ring}(z,r) = \int_0^a \frac{kzq}{(z^2+r^2)^{3/2}}\cdot\frac{\overbrace{2r\,dr}^{du}}{a^2}$$, take $u = z^2+r^2$, $du = 2r\,dr$

$$= \frac{kzq}{a^2}\left[\int\frac{du}{u^{3/2}}\right]_{r=0}^{a} = \frac{kz}{a^2}\cdot\left[-2\cdot u^{-1/2}\right]_{r=0}^{a}\hat{z}$$

$$= \frac{-2kzq}{a^2}\left(\frac{-1}{\sqrt{z^2+a^2}} - \frac{-1}{\sqrt{z^2}}\right)\hat{z}$$, WLOG $z>0$ (or just flip axis)

$$= \frac{2kzq}{a^2}\cdot\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+a^2}}\right)\hat{z} = \frac{2kq}{a^2}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)\hat{z}$$, where $k = \frac{1}{4\pi\varepsilon_0}$

$$= \frac{q}{2\pi\varepsilon_0 a^2}\cdot\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)\hat{z}$$

2) with $z \gg a$, the disk should just behave live a point charge

$\therefore \vec{E}(z) \approx \frac{q}{4\pi\varepsilon_0 z^2}$. Note for the equation in 1), when $z \gg a$, or $z \to 0$

$\vec{E}(z) \approx \frac{q}{2\pi\varepsilon_0 a^2}\left(1 - \frac{z}{\sqrt{z^2}}\right) \approx 0$

2.2.1. A sphere can be considered as $n \to \infty$ shells with radius $r = 0 \to R$ and width dr centered at origin = center of sphere

Thus $\vec{E}_{sp}(z) = \int \vec{E}_{sh} \overset{A_1}{=} \int \frac{Q_{sh}}{4\pi\varepsilon_0 z^2} \hat{z}$ $\leftarrow = \frac{4\cancel{\pi} r^2 \cdot dr}{\frac{4}{3}\cancel{\pi} R^3} q$

$$= \frac{1}{4\pi\varepsilon_0 z^2} \cdot \frac{3q}{R^3} \int_0^R r^2 \, dr \, \hat{z}$$

$$= \frac{3q}{4\pi\varepsilon_0 z^2 R^3} \cdot \left[\frac{1}{3} r^3\right]_0^R \hat{z}$$

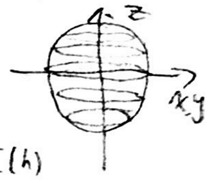
$$= \frac{3q}{4\pi\varepsilon_0 z^2 \cancel{R^3}} \cdot \frac{\cancel{R^3}}{\cancel{3}} \hat{z} = \frac{q}{4\pi\varepsilon_0 z^2} \hat{z}$$

for point $(0,0,z)$, where z-axis is center of sphere $\to$ point

2.2 A sphere is the combination of ∞ disks at $z=h$, and width dh

radius $a=\sqrt{R^2-h^2}$

charge $= \frac{\pi a^2\, dh}{\frac{4}{3}\pi R^3}\, q$



$$\therefore \vec{E}_{h<z} = \int_{h=-R}^{z} \frac{2k\cdot q3}{\cancel{4}_{2} R^3}\, dh \left(1-\frac{\overset{>0}{z-h}}{\sqrt{(z-h)^2+a^2}}\right)\hat{z}$$

$$= \frac{3kq}{2R^3}\int_{-R}^{z} 1-\frac{z-h}{\sqrt{(z-h)^2+a^2}}\,dh\,\hat{z} = \frac{3k}{2R^3}\left(z+R-\int_{-R}^{R}\frac{z-h}{\sqrt{(z-h)^2+R^2-h^2}}\,dh\right)\hat{z}$$

$I(h)$

$$= \frac{3kq}{2R^3}\left(R+z-[I]_{h=-R}^{z}\right)\hat{z}$$

$$\vec{E}_{h>z} = \frac{3kq}{2R^3}\int_{h=z} -1-\cdots\, dh$$

$$= \frac{3kq}{2R^3}\left(-R-(-z)\right)-[I]_{h=z}^{R}$$

$$\vec{E} = \vec{E}_{h<z}+\vec{E}_{h>z}$$

$$= \frac{3kq}{2R^3}\left(2z-[I]_{-R}^{R}\right)$$

$z^2-2zh+R^2=u$

$du=-2z\,dh$

$h=\frac{u-z^2-R^2}{-2z}$

Consider $I=\int \frac{z}{\sqrt{u}}\,dh - \int \frac{h}{\sqrt{u}}\,dh$

$I_1 + I_2$

$$I_1 = \int \frac{du}{\sqrt{u}}\cdot\frac{-1}{2} = -\frac{1}{2}\cdot 2u^{1/2} = -u^{1/2}$$

$$I_2 = \frac{-1}{2z}\int \frac{u-z^2-R^2}{2z}\cdot\frac{1}{\sqrt{u}}\,du = -\frac{1}{4z^2}\int u^{\frac12}-(z^2+R^2)\cdot u^{-\frac12}\,du$$

$$= -\frac{1}{4z^2}\,\frac{2}{3}u^{\frac32}+2\,\frac{z^2+R^2}{4z^2}\,u^{\frac12}$$

$$\therefore I = -u^{1/2}-\frac{u^{\frac32}}{6z^2}+\frac{z^2+R^2}{2z^2}u^{\frac12}$$

Note when $h=-R$, $u=z^2+2zR+R^2 = (z+R)^2$

R, $u=z^2-2zR+R^2 = (z-R)^2$

$$= -u^{1/2}\left(1+\frac{z^2-2zh+R^2}{6z^2}-\frac{3z^2+3R^2}{6z^2}\right)$$

$$= -u^{1/2}\cdot\left(1+\frac{-\cancel{2}(z^2+zh+R^2)}{3\cancel{6}z^2}\right)$$

$$\therefore [I]_{h=-R}^{R} = -(\overset{R>z}{R}-z)\left(1+\frac{z^2+zR+R^2}{3z^2}\right)+(R+z)\left(1-\frac{z^2-zR+R^2}{3z^2}\right)$$

$$= 2z+\frac{\cancel{Rz^2}+\cancel{zR^2}+\cancel{R^3}-z^3-\cancel{z^2R}-\cancel{R^2z}-\cancel{Rz^2}+\cancel{zR^2}-\cancel{R^3}-z^3+\cancel{z^2R}-\cancel{zR^2}}{3z^2}$$

$$= 2z+\frac{-2z^3}{3z^2} = 2z-\frac{2}{3}z$$

$$\therefore \vec{E} = \frac{3kq}{2R^3}\left(2z-\left(2z-\frac{2}{3}z\right)\right) = \frac{3kq}{2R^3}\cdot\frac{2}{3}z = \frac{kq}{R^3}z$$