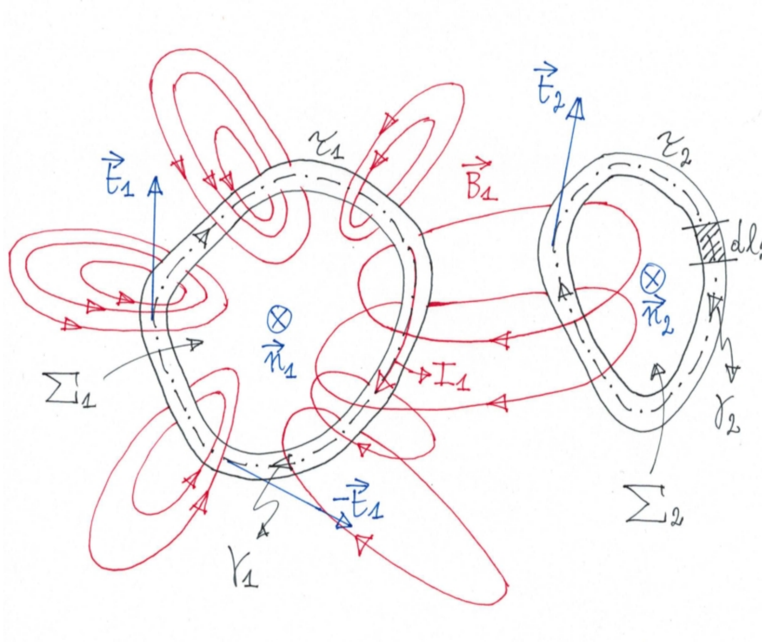
Lectures

PHYS-242



Electromagnetism I

Dr. Matteo Mariantoni

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Chapter 1

Electrostatic field in vacuum

1.1 Electric Interactions: Basic Examples

The semantic root of the word "electric" is from Greek *elektron*, which means amber. In fact, Thales of Miletus, Theophrastus (the successor to Aristotle in the Peripatetic school), and other Greek philosophers discovered in approximately 600 BC that after rubbing a piece of amber by means of a wool cloth, the amber was able to lift light materials such as little pieces of paper. In this state, the amber is said to be electrically charged. The forces acting in this simple experiment are mostly the gravitational force on the piece of paper and the electric force between paper and amber. It is remarkable that the electric force is stronger than the gravitational force on the paper due to the mass of the entire planet Earth. It is worth mentioning that the electric force between the electron and proton that make an hydrogen atom is approximately 36 orders of magnitude stronger than the gravitational force between them.

Little progress happened on the nature of electric interactions until the 16th century, when William Gilbert managed to distinguish for the first time between electric and magnetic phenomena. One hundred years later, Otto von Guericke invented the first electric generator based on friction. By the beginning of the 18th century, Charles François de Cisternay du Fay found that different materials react differently when rubbed by a cloth. For example, by rubbing a small piece of amber and a small piece of glass, he was able to show that the two materials repulse each other when placed close to each other (without touching). By blowing a powder made of minium and sulfur in proximity of the two pieces of amber and glass, du Fay was able to show that the minium (of red color) gets attracted by the amber and the sulfur (of yellow color) by the glass. We can thus distinguish between two types of electric charge, "yellow" and "red" charge, which are more commonly referred to as positive and negative charge, respectively. Figure 1.1 shows all possible attractions and interactions between charges of different sign.

1.1.1 Insulators and Conductors

Metals and other materials with metallic properties not always can lift light materials, even when rubbed with a wool cloth. However, if a piece of metal is attached to a plastic handle (or, more in general, a handle made from any of the materials discussed above, such as amber or glass), it can be electrically charged. It is worth mentioning

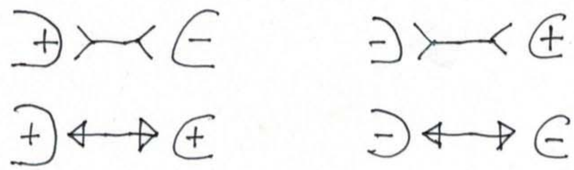


Figure 1.1: Attractive and repulsive interactions.

that as soon as a charged metal is touched by another piece of metal or even by the hand of the experimenter, it looses all its charge almost immediately.

We can thus distinguish between "plastic" type materials and "metallic" ones. The former can maintain an electrical charge, while the latter allow the charge to move freely. Plastic materials (including amber and glass) are thus called *insulators* and metals *conductors*.

It is worth noting that uncharged conductors are said to be in a neutral state, where positive and negative charges balance each other. We will come back to this topic later in the course.

1.2 Electric Pendulum

The electric pendulum is made by a light sphere of glass or wood or pith, covered by a layer of, e.g., aluminum foil. The sphere is attached to a fixed medium (e.g., the ceiling of a laboratory) by means of a silk thread. Figure 1.2 shows an example of electric pendulum. The pendulum is strongly attracted by a previously charged body, for example a rubbed amber or glass rod. This effect clearly shows the electric charge state of the rod.

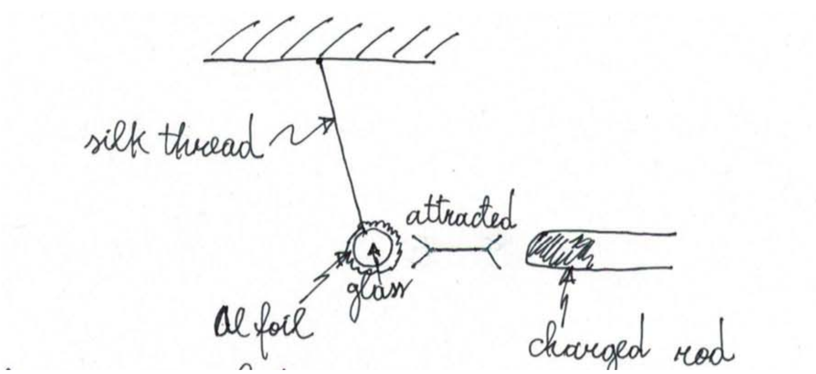


Figure 1.2: An example of electric pendulum.

The working principle of the electric pendulum is based on the phenomenon of electrostatic induction. In this course, we will only explain qualitatively this complex phenomenon. It would be possible to perform a more rigorous analysis by means of Poisson equation. However, this nontrivial task goes beyond the purpose of this course.

Consider a charged piece of glass A and an uncharged metallic object C (i.e., in the neutral state). In the beginning, the two objects are far from each other, as shown in Fig. 1.3, separated by a large distance d. When A and C are positioned close to each other (without touching, though), the positive charge on the glass *induces* a charge of the opposite sign, thus negative, on the closest part of the metallic object. On the

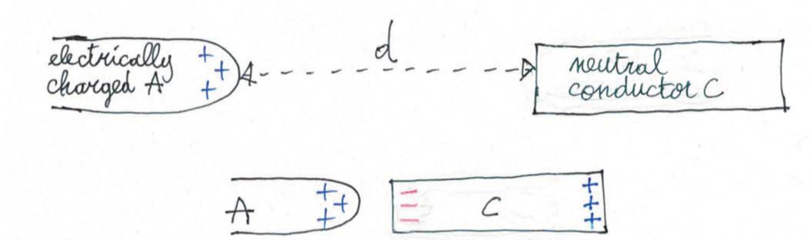


Figure 1.3: Qualitatively explanation of electrostatic induction.

contrary, the farthest part of the metallic object gets charged with an equal positive charge. This phenomenon can readily be revealed by blowing a minium-sulfur powder in proximity of the two objects. Remarkably, as soon as the objects are separated again, the metallic object becomes one more time neutral. In the neutral state the positive and negative charge cancel each other, while during the induction they get separated. This phenomenon can also take place for insulating materials, in which case is called *polarization*.

The sphere of the electric pendulum is attracted by a charged rod because of the induced charge on its surface: Charges of opposite sign attract each other.

1.3 Electric Charge

Operative definition of electric charge: Two electrically charged bodies A and B have equal electric charges, with equal sign (positive or negative), when they generate equal forces (in magnitude and sign; not necessarily in direction) on a third body C fixed in the reference frame of A and B. Experimentally, one finds that if the body C is substituted by a new body C$'$, the forces generated by A and B remain the same. This is called independence from the *test* body.

Definition of point-like bodies: Two bodies A and B are defined to be point-like bodies, if the distance between them is much larger than their dimensions.

In the definition of electric charge, all the bodies involved in the definition are considered to be point-like bodies.

1.3.1 Summation of Electric Charges

(i) Consider a body U with a positive or negative charge. Define U to be the unit charge.

(ii) Consider a body C charged with a charge of the same sign of U and placed at a distance r from a third body R (see Fig. 1.4). The force acting on R due to C is $\vec{F}$.

(iii) Finally, consider many bodies U_i each with unit charge. Assume to place enough of them next to each other so to measure exactly the same force as in point (ii) on a body R at distance r from the U_i's (see Fig. 1.5). The number of unit charges U_i defines the electric charge of C. Positive and negative charges are assumed to act with equal forces, but opposite sign on the same R.

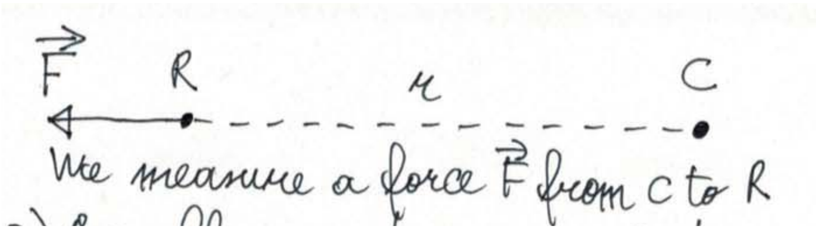


Figure 1.4: Force on R due to C.

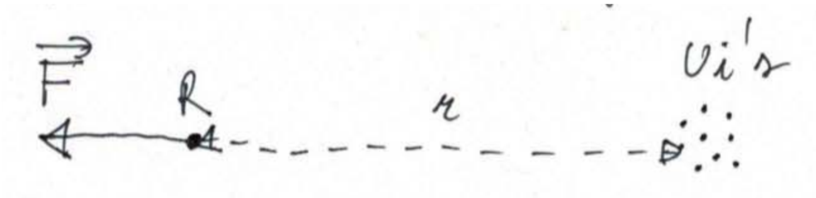


Figure 1.5: Force on R due to an ensemble of U_i.

1.4 Coulomb's Law

The force between two charges q_1 and q_2 is:

(i) Directed along a straight line between q_1 and q_2;

(ii) proportional to

$$\frac{|q_1||q_2|}{r^2}\,; \tag{1.1}$$

(iii) positive for charges of equal sign and negative for charges of opposite sign.

In summary, the force due to a charge q_1 on a charge q_2 can be written as

$$\begin{aligned}\vec{F}_{21} &= K\frac{q_1 q_2}{r^2}\vec{u}_{12} \\ &= -\vec{F}_{12}\,.\end{aligned} \tag{1.2}$$

Note that the minus sign in front of $\vec{F}_{12}$ is due to the action/reaction principle.

From point (iii), if

$$\frac{q_1}{|q_1|} = \frac{q_2}{|q_2|}\,,$$

the Coulomb force is repulsive. If, instead,

$$\frac{q_1}{|q_1|} = -\frac{q_2}{|q_2|}\,,$$

the force is attractive.

In the SI system, the unit for charge is

$$[\text{charge}] = \text{C (coulomb)}\,. \tag{1.3}$$

In addition, it was experimentally determined that the constant

$$K \simeq 8.988 \times 10^9 \mathrm{kg\,m^3 s^{-2} C^{-2}}\,. \tag{1.4}$$

As it will appear clear later, it can be shown that

$$K = \frac{1}{4\pi\epsilon_0}\,, \tag{1.5}$$

where $\epsilon_0 \simeq 8.854 \times 10^{-12}\,\mathrm{kg^{-1}\,m^{-3}\,s^2\,C^2}$ is called the vacuum dielectric constant. Note that 1 C is a huge charge and typical values, particularly in electronics, are on the order of 1μC or less.

1.5 Superposition Principle

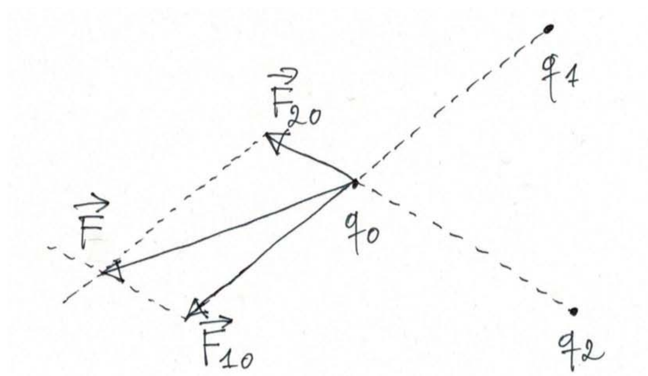


Figure 1.6: Summing forces according to the superposition principle.

Consider two point-like bodies with charges q_1 and q_2, respectively. Consider then a test point-like body with charge q_0 located at some distance from both q_1 and q_2. It can be experimentally shown that the total force due to both q_1 and q_2 on q_0 is equivalent to the vector sum of the forces generated by each charge (i.e., q_1 or q_2) as if they were independently acting on q_0. Each of this two forces must obey Coulomb's law, Eq. 1.2. In other words, we can think of calculating the force due to Coulomb's law between q_1 and q_0 while "switching" off the charge q_2, $\vec{F}_{10}$, and then the force between q_2 and q_0 while switching off the charge q_1, $\vec{F}_{20}$, and, finally, summing up the two forces by means of the parallelogram rule (see Fig. 1.6). It is obvious that the argument can be extended to the case of many charges.

It is important to realize that this principle is different from charge summation, which was given by definition, and Coulomb's law itself, which is valid only between two charges. The superposition principle is a new principle, which, in combination with Coulomb's law, makes it possible to solve all problems in electrostatics.

1.5.1 Example

Consider six point-like bodies each with a positive charge $q/6$ and located at the vertexes of a hexagon. The hexagon is enclosed by a circle of radius r. A test charge Q (also positive) is located on the circle's axis at a distance d from its center (see Fig. 1.7).

Calculate:

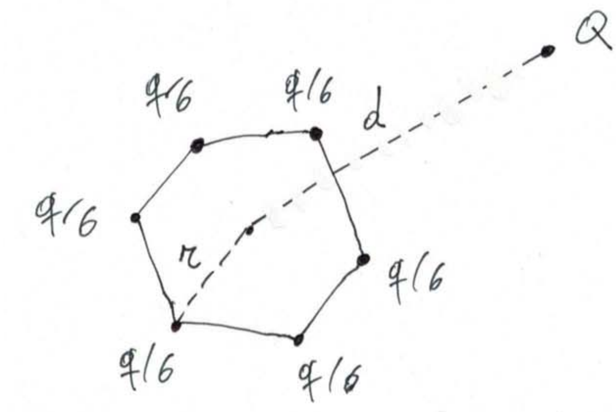


Figure 1.7: Six point-like source charges $q/6$ located at the vertexes of a hexagon.

- The total force $\vec{F}_{\mathrm{t}}$ acting on Q due to the six source charges.
- Study the case for $d \gg r$.

(I) *Reference frame and associated coordinate system.*

The walls of the laboratory where the charges are imagined to be located is the natural choice of a reference frame. For this as well as most of the problems we will encounter in electrostatics, such walls can be safely regarded to be an inertial reference frame. We thus only need to choose an opportune coordinate system within this frame. Noting that the six charges form three pairs, where each pair is contained within a plane passing through the center of the hexagon and the test charge Q, we can restrict ourselves to one such planes and therein consider a Cartesian coordinate system Oxy, as indicated in Figure 1.8. The unit vectors of this system are $\vec{u}_x$ and $\vec{u}_y$, respectively.

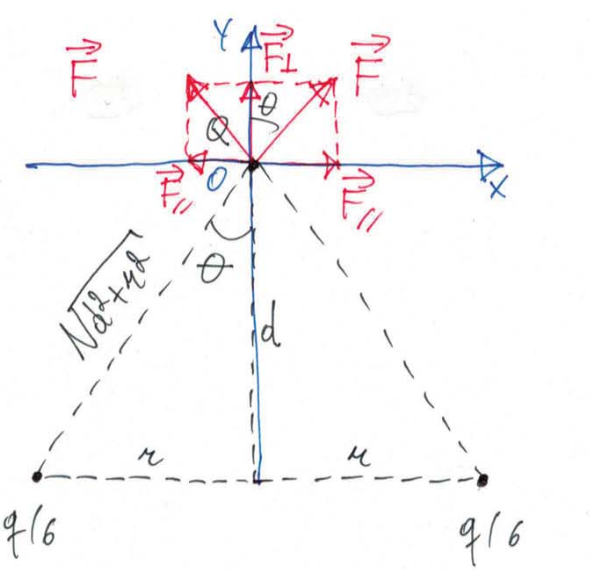


Figure 1.8: Cartesian coordinate system and forces due to a pair of charges.

(II) *Forces.*

All the forces acting on Q due to one pair of source charges with charge $q/6$ are shown in Fig. 1.8. Each charge $q/6$ acts with a force that can be decomposed into a parallel and a normal component, $\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp} = \vec{F}_x + \vec{F}_y$.

(III) *Degrees of Freedom (DOFs).*

It is always worth attempting to qualitatively figure out what the DOFs for the system under analysis are. The DOFs are the possible direction within a given coordinate system along which a particle (in our case, the test charge Q) can move under the action of the total force acting on it. In other words, the DOFs are all the possible components of the total acceleration vector $\vec{a}$ acting on Q (we remind that from second Newton's law $\vec{F} = m\vec{a}$).

In the problem at hand, we can easily see that the parallel components of the forces due to each pair of charges acting on Q are opposite of each other and, thus, cancel each other. This means that the particle Q can only be acted upon by a force (and, thus, an acceleration) along the normal direction (y axis). Hence, the only DOF is in this case y.

(IV) *Quantitative Solution.*

We are dealing with multiple charges. We thus need to make use of both the superposition principle and Coulomb's law.

The total force due to the three pairs of charges with charge $q/6$ is given by

$$\begin{aligned} \vec{F}_{\mathrm{t}} &= 3\left(\vec{F}_x - \vec{F}_x + \vec{F}_y + \vec{F}_y\right) \\ &= 6\frac{1}{4\pi\epsilon_0}\frac{qQ}{6\,(d^2+r^2)}\cos\theta\,\vec{u}_y\,, \end{aligned} \tag{1.6}$$

where $\sqrt{d^2+r^2}$ is the distance between each of the source charges with charge $q/6$ and Q calculated from Pythagoras' theorem and

$$\cos\theta = \frac{d}{\sqrt{d^2+r^2}}\,. \tag{1.7}$$

Therefore, the total force on Q reads

$$\vec{F}_{\mathrm{t}} = \frac{1}{4\pi\epsilon_0}qQ\frac{d}{(d^2+r^2)^{3/2}}\vec{u}_y\,. \tag{1.8}$$

When $d \gg r$,

$$\vec{F}_{\mathrm{t}} \approx \frac{1}{4\pi\epsilon_0}\frac{qQ}{d^2} \tag{1.9}$$

and the six charges on the hexagon appear to the far away charge Q as if they were a single charge q located at the center of the hexagon.

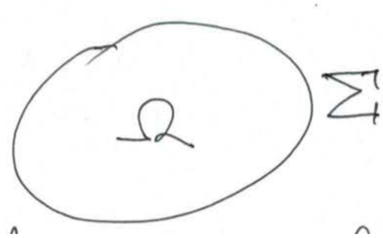


Figure 1.9: Charge conservation principle.

1.6 Charge Conservation Principle

Consider a region of the three-dimensional (3D) Eucledian space Ω bounded by a closed surface Σ, as in Fig. 1.9. If this is an isolated region of space, the total charge given by the algebraic sum of all positive and negative charges in the region is constant in time. From experiments, it also follows that if

$$\Delta Q = Q(t_2) - Q(t_1) \neq 0\,, \tag{1.10}$$

implies that some charged bodies passed through Σ between the initial and final time instants of the experiment, t_1 and t_2. Hence, the region confined within the boundary Σ is not isolated.

1.7 Electrostatic Field

We consider two point-like charges q_1 and q_0 fixed in vacuum and we imagine to measure the force acting on q_0 keeping q_1 fixed at point Q while moving q_0 to any point P in the 3D Euclidean space. This procedure allows us to determine the vector field $\vec{F}_0(P)$ due to the force $\vec{F}_0$ acting on q_0, where q_0 is at the generic point P, due to q_1 at point Q. The field $\vec{F}_0(P)$ is defined in the entire space.

Since $\vec{F}_0$ is given by Coulomb's law, the field is said to be a *central field*. We will come back to this definition later in this chapter.

- By substituting q_0 with q_0', according to Coulomb's law we obtain a new vector field
$$\vec{F}_0'(P) = \frac{q_0'}{q_0}\,\vec{F}_0(P) \quad . \tag{1.11}$$

- $\vec{E}(P) \rightarrow$ vector field acting on the *unitary positive charge* due to q_1,
$$\vec{F}_0(P) = q_0\vec{E}(P) \quad , \tag{1.12}$$
where
$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\,\frac{q_1}{r_{\mathrm{QP}}^2}\,\vec{u}_{\mathrm{QP}} \quad . \tag{1.13}$$

- With respect to a Cartesian coordinate system, as shown in Fig. 1.10,

$$\vec{u}_{10} = \vec{u}_{QP} = \frac{\vec{r}_P - \vec{r}_Q}{|\vec{r}_P - \vec{r}_Q|} = \frac{(x_P - x_Q)\vec{u}_x + (y_P - y_Q)\vec{u}_y + (z_P - z_Q)\vec{u}_z}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{1/2}}$$

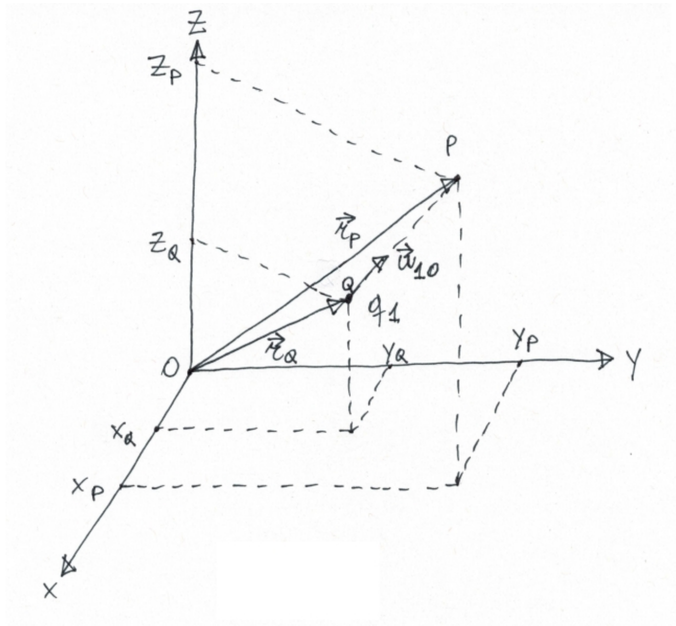


Figure 1.10

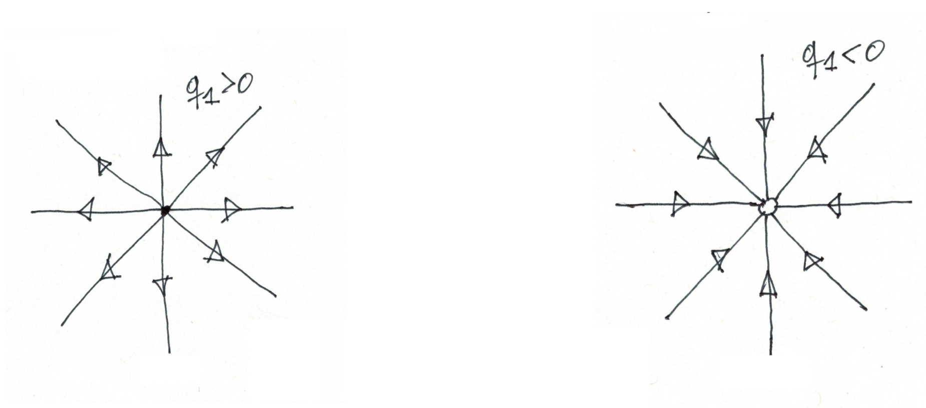
thus

$$E_x = \frac{q_1}{4\pi\epsilon_0} \frac{x_P - x_Q}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{3/2}} \tag{1.14a}$$

$$E_y = \frac{q_1}{4\pi\epsilon_0} \frac{y_P - y_Q}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{3/2}} \tag{1.14b}$$

$$E_z = \frac{q_1}{4\pi\epsilon_0} \frac{y_P - y_Q}{[(z_P - z_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{3/2}} \tag{1.14c}$$

- Geometric representation of $\vec{E}$ by means of field lines



(a) (b)

Figure 1.11: Electrostatic field due to single source charges.

In the 3D space, the field has a central symmetry, which is natural due to the homogeneity and isotropy of space in the inertial reference frame where the source charge q_1 is fixed.

- Extension from one source-charge q_1 to N source charges.

  Given N charges in vacuum, $q_1, q_2, \ldots, q_N$, fixed at points $Q_1, Q_2, \ldots, Q_N$ in an inertial reference frame, the force due to the N charges on a test charge q_0 at point P is

$$\begin{aligned}\vec{F}_0(P) &= \sum_{k=1}^{N} \vec{F}_{PQ_k} \\ &= \frac{q_0}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{q_k}{r^2_{Q_kP}} \vec{u}_{Q_kP} \quad , \\ &= q_0 \vec{E}(P) \quad ,\end{aligned} \tag{1.15}$$

  where

$$\begin{aligned}\vec{E}(P) &= \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{q_k}{r^2_{Q_kP}} \vec{u}_{Q_kP} \\ &= \sum_{k=1}^{N} \vec{E}_k(P) \quad .\end{aligned} \tag{1.16}$$

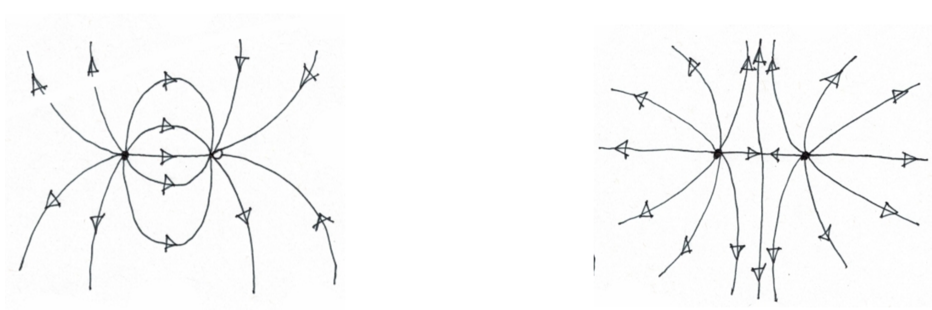
  Note, that $\vec{E}_k(P)$ is the electric field due to one source-charge q_k fixed at point Q_k.

  Equation (1.16) results from:

  (1) Coulomb's law.

  (2) Superposition principle.

- Geometric representation for multiple charges



(a) (b)

Figure 1.12: Electrostatic field due to pairs of source charges.

- If the charge distribution is unknown, Eq. (1.16) cannot be used. In this case

$$\vec{E}(P) = \frac{\vec{F}_0(P)}{q_0} \quad . \tag{1.17}$$

1) If all source charges remained fixed, Eq. (1.17) would be independent from the test charge q_0.
2) However, in reality the presence of q_0 disturbs the distribution of the source charges. Equation (1.17) would thus give a different result compared to Eq. (1.16), where the distribution of all source charges is known.
3) The disturbance due to q_0 on the source charges increases with the value of q_0. Hence, the operative definition of $\vec{E}$ is

$$\vec{E}(P) = \lim_{q_0 \to 0^+} \frac{\vec{F}_0(P)}{q_0} \quad . \tag{1.18}$$

Note that $\lim q_0 \to 0^+$ is a macroscopic limit. It is enough that the value of the test charge q_0 is much smaller than the value of each source charge. Also note that due to charge quantization $q_0 \neq 0$.

1.8 Continuous charge distributions

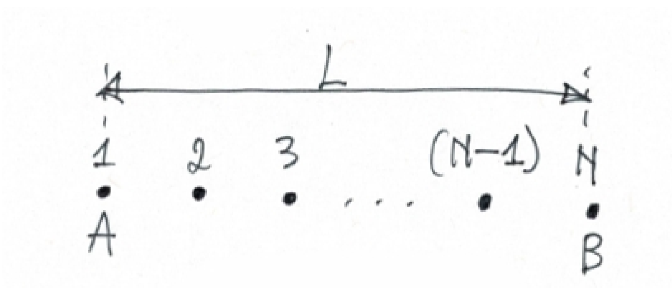


Figure 1.13: From a discrete to a continuous charge distribution.

Figure 1.13 shows a distribution of N point-like charges q along a segment AB. The charges are considered to be equal and equally spaced.

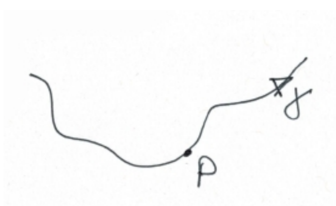
- Suppose to increase the number of charges N, while reducing their value q such that $Q = qN = \text{const}$. This means the inter-charge distance $\delta = L/(N-1)$ diminishes.

- In the $\lim N \to \infty$ (note that charge is quantized and, thus, in reality a single charge cannot be smaller than the electron charge), the charges are distributed continuously on AB. The charge on an infinitesimal element dx on AB is

$$dq = \frac{Q}{L} dx = \lambda dx \quad , \tag{1.19}$$

where λ is the linear charge density.

- Charges distributed on a generic line γ (AB was a straight line):

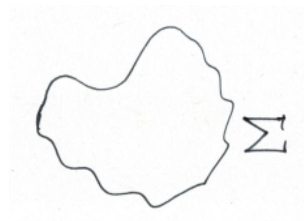


$$\begin{aligned} \lambda(P) &= \left.\frac{dq}{dl}\right|_P \quad , \\ [\lambda] &= \mathrm{m}^{-1} \quad . \end{aligned} \tag{1.20}$$

By simple line integration, the total charge Q on γ is

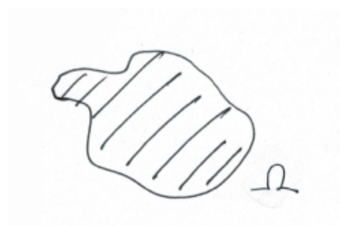
$$Q = \int_\gamma dl \lambda(P) \quad . \tag{1.21}$$

- Surface charge density:



$$\begin{aligned} \sigma(P) &= \left.\frac{dq}{dS}\right|_P \quad , \qquad (1.22) \\ [\sigma] &= \mathrm{m}^{-2} \quad . \end{aligned}$$

- Volume charge density:



$$\begin{aligned} \rho(P) &= \left.\frac{dq}{dV}\right|_P \quad , \qquad (1.23) \\ [\rho] &= \mathrm{m}^{-3} \quad . \end{aligned}$$

- In summary:

1) Line $\qquad \gamma \Rightarrow Q = \int\limits_{\gamma} dl\lambda(P) \quad .$

2) Surface $\qquad \Sigma \Rightarrow Q = \iint\limits_{\Sigma} dS\sigma(P) \quad . \qquad (1.24)$

3) Volume $\qquad \Omega \Rightarrow Q = \iiint\limits_{\Omega} dV\rho(P) \quad . \qquad (1.25)$

1.8.1 Physical meaning of a continuous charge distribution

A body of volume Ω is made by an Avogadro number of electrons and protons acting as point-like charges. These charges are separated by very large distances compared to their dimensions and are characterized by an incessant motion. If we were to use

Eq. (1.23), we would obtain a function changing abruptly from point to point, both in space and time. This function would be almost always zero, other than in the points where there is an actual charge, where it would assume very large values.

- Eq. 1.25 is defined for
$$\rho = \lim_{\Delta V \to 0^+} \frac{\Delta q}{\Delta V} \quad .$$
- Assume a $\Delta V \simeq 1\,\mathrm{cm}^3$ and reduce ΔV. The value of ρ changes until ΔV becomes small enough ($\simeq 10^{-3}$ mm^3), at which point ρ stabilizes.
- If ΔV becomes too small, charge quantization starts to play a role.
- ΔV must be chosen small enough for ρ to stabilize, but not too small for charge quantization to be important (in addition, note that ρ should be considered a time average $\bar{\rho}$ over a short enough, but not too short time interval).

  This is the definition of *physics infinitesimal*.

- **Example - Circular Loop of Charge:**

(1) A positive charge q is distributed along a ring of radius r, with linear density $\lambda = q/(2\pi r)$.
A positive point-like charge q_0 is located at point P at a distance y from the center of the ring, on the ring axis.
 1) Calculate the force $\vec{F}$ generated by q on q_0.
 2) Calculate for which value of y $\vec{F}$ is maximum.

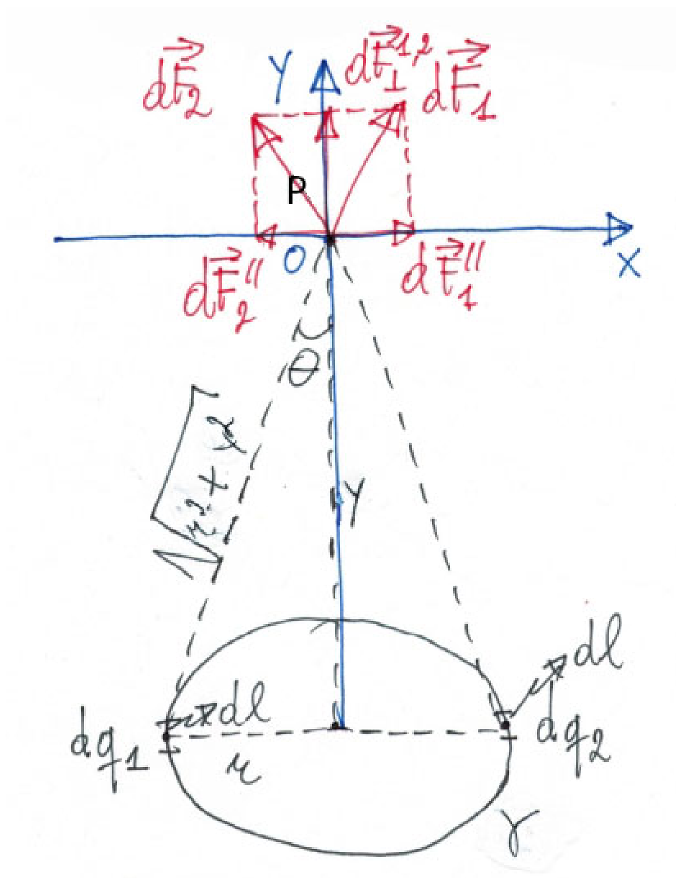


Figure 1.14: Circular loop of charge.

(2) *Coordinate system*: see Fig. 1.14. Note that we could have equally chosen a cylindrical coordinate system $Orz\varphi$. This would have been useful if we wanted to show in detail the line integral along the ring. This integral is however trivial and we will not delve into details in these notes.

(3) *Indicate all forces*: The infinitesimal forces due to two opposite infinitesimal charge elements dq_1 and dq_2, which behave as point-like charges, are indicated in the figure. Note that in this case there are infinite pairs of opposite charges, as opposed to the only three pairs in the problem of the hexagon. However, the argument for the hexagon that opposite charges generate an effective field only along the perpendicular direction remains valid.

(4) *DOF*: Because of the above argument, it appears clear that the only DOF is y. In the figure this direction is called the perpendicular direction, as opposed to the x direction, called parallel.

(5) Results:

1) The force $d\vec{F}$ generated by an infinitesimal charge element

$$dq = \lambda dl$$

has a magnitude

$$dF = \frac{1}{4\pi\epsilon_0}\,\frac{\lambda\, dl\, q_0}{r^2+y^2} \quad .$$

The force $d\vec{F}$ can be decomoposed in two components, $d\vec{F}''$ and $d\vec{F}^{\perp}$:

$$d\vec{F} = d\vec{F}'' + d\vec{F}^{\perp} \quad .$$

For each element dq and the element symmetrically opposed to it, the $d\vec{F}''$ components cancel each other. Thus, each dq generates only an effective $dF^{\perp}$

$$dF^{\perp} = dF\cos\theta \quad .$$

By integrating over the ring, the total force $\vec{F}$ is given by

$$\vec{F} = \int_{\gamma} dF^{\perp}\vec{u}_y = \int_{\gamma} dF\cos\theta\vec{u}_y \quad ,$$

where

$$\begin{aligned}
\cos\theta &= \frac{y}{\sqrt{r^2+y^2}} \quad . \\
\Rightarrow F &= \int_{\gamma} \frac{1}{4\pi\epsilon_0}\,\frac{\lambda\, dl\, q_0}{r^2+y^2}\,\frac{y}{\sqrt{r^2+y^2}} \\
&= \frac{1}{4\pi\epsilon_0}\,\frac{\lambda q_0 y}{(r^2+y^2)^{3/2}}\int_{\gamma} dl \\
&= \frac{1}{4\pi\epsilon_0}\,\frac{q q_0 y}{(r^2+y^2)^{3/2}} \quad .
\end{aligned} \tag{1.26}$$

The force is repulsive. Due to the symmetry of the problem with respect to the plane of the ring at $y = 0$, we can study the problem for $y \geq 0$. We first observe that for $y = 0$, $F = 0$, and for $y \to +\infty$, $F \to 0$. Since $F > 0$ for finite (nonzero) values of y, F should have at least one maximum for $y \geq 0$. The maximum can be found by

$$\begin{aligned}
\frac{d}{dy}F &= \frac{qq_0}{4\pi\epsilon_0}\left[\frac{1}{(r^2+y^2)^{3/2}} - \frac{3}{2}\,y\,\frac{2y}{(r^2+y^2)^{5/2}}\right] \quad , \\
&= 0 \quad , \\
&\Rightarrow \quad \frac{1}{(r^2+y^2)^{5/2}}\,(y^2+r^2-3y^2) = 0 \\
&\Rightarrow \quad y = \frac{r}{\sqrt{2}} \quad .
\end{aligned}$$

1.9 Electric Field Generated by a Generic Charge Distribution

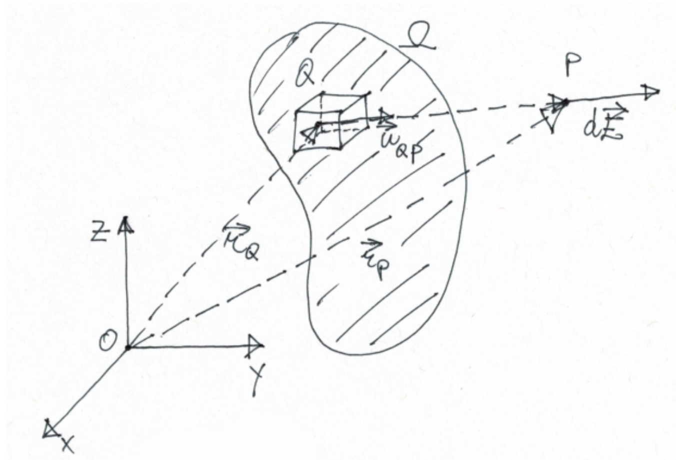


Figure 1.15

The charge dq in dV is

$$dq = \rho(Q)dV \quad . \tag{1.27}$$

dq can be considered to be point-like and, thus, it generates a field $d\vec{E}$ at a *field-point* P:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r_{PQ}^2}\vec{u}_{QP} = \frac{1}{4\pi\epsilon_0}\frac{\rho(Q)}{r_{PQ}^2}dV\,\vec{u}_{QP} \quad , \tag{1.28}$$

where $r_{PQ} = |\vec{r}_P - \vec{r}_Q|$. By means of the superposition principle:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\iiint\limits_{\Omega} dV\,\frac{\rho(Q)}{r_{PQ}^2}\vec{u}_{QP} \quad . \tag{1.29}$$

The integral in Eq. (1.29) is a vector integral extended to the entire region Ω where the volume charge density $\rho(Q)$ is defined.

In Cartesian coordinates,

$$\begin{aligned}\vec{E}(x_P, y_P, z_P) \quad = \quad & \frac{1}{4\pi\epsilon_0}\iiint\limits_{\Omega}\rho(x_Q, y_Q, z_Q) \\ & \frac{(x_P - x_Q)\vec{u_x} + (y_P - y_Q)\vec{u_y} + (z_P - z_Q)\vec{u_z}}{[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{3/2}} \\ & dx_Q dy_Q dz_Q \quad .\end{aligned} \tag{1.30}$$

For linear and surface charge densities we obtain, respectively:

$$\vec{E}(P) \quad = \quad \frac{1}{4\pi\epsilon_0}\int\limits_{\gamma} dl\,\frac{\lambda(Q)}{r_{PQ}^2}\vec{u}_{QP} \quad , \tag{1.31}$$

$$\vec{E}(P) \quad = \quad \frac{1}{4\pi\epsilon_0}\iint\limits_{\Sigma} dS\,\frac{\sigma(Q)}{r_{PQ}^2}\vec{u}_{QP} \quad . \tag{1.32}$$

- It is easy to verify that the electric field $\vec{E}$ generated by a finite distribution of charges tends to 0 for $r_{PQ} \to \infty$, at least as $1/r_{PQ}^2$.

- **Example 2 - Infinite Straight Line of Charge:**

  (1) Consider a uniform linear charge distribution with constant density λ on an infinite straight line γ (see Fig. 1.16).

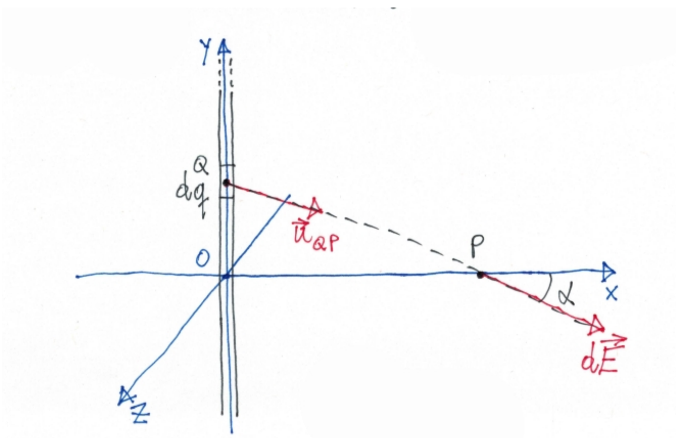


Figure 1.16

Calculate the electric field generated by λ at a generic point P that does not belong to γ.

  (2) *Coordinate system*:

    1) Symmetry argument. The charge distribution on γ is indistinguishable from any half plane originating from γ. This is because any observer fixed in space would see the same charge distribution upon rotating γ (rotation symmetry). To solve the problem without loosing generality, we can thus consider just one half plane, the half plane containing P.

    2) An $Oxyz$ Cartesian coordinate system for such a half plane is shown in Fig. 1.16. Note that we could have chosen also a cylindrical coordinate system $Orz\varphi$ with z-axis along the line γ.

  (3) *Indicate all fields*:

  The infinitesimal field $d\vec{E}$ at P generated by an infinitesimal charge element dq at point Q on γ is shown in Fig. 1.16.

  (4) *DOF*:

    1) Symmetry argument. From Fig. 1.16, for each infinitesimal charge dq at point $+|y|$ correspond an equal charge dq at point $-|y|$. Hence, the total electric field at point P will only have an E_x component. This is because the line is infinite and, thus, characterized by a translation symmetry.

2) DOF $= x$.

(5) Results

Using Eq. (1.31) with $dl = dy$, we obtain

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \, \frac{\lambda}{x^2 + y^2} \, \vec{u}_{QP} \quad . \tag{1.33}$$

Due to the symmetry argument in point (4), the only component of interest of $\vec{E}$ is E_x :

$$\begin{aligned} E_x &= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \, \frac{1}{x^2 + y^2} \cos\alpha \\ &= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \, \frac{x}{(x^2 + y^2)^{3/2}} \quad . \end{aligned}$$

The integral can be solved by means of a trigonometric substitution:

$$\frac{y}{x} = \tan\alpha \quad , \tag{1.34}$$

from which

$$dy = \frac{x}{\cos^2\alpha} \, d\alpha \quad . \tag{1.35}$$

With this substitution, $(x^2 + y^2)^{3/2} = x^3 / \cos^3\alpha$ and the limits of integration become $-\infty \to -\pi/2$ and $+\infty \to +\pi/2$. Thus,

$$\begin{aligned} E_x &= \frac{\lambda}{4\pi\epsilon_0} \int_{-\pi/2}^{+\pi/2} d\alpha \, \frac{\cos\alpha}{x} \\ &= \frac{\lambda}{4\pi\epsilon_0} \, \frac{2}{x} \quad . \end{aligned}$$

Finally,

$$\vec{E}(P) = \frac{\lambda}{2\pi\epsilon_0 x} \, \vec{u}_x \quad . \tag{1.36}$$

The field diverges in correspondence of any point on γ.

- **Example 3 - Infinite Sheet (Plane) of Charge:**

(1) Consider a charge distribution with constant surface density σ on an infinite plane π.

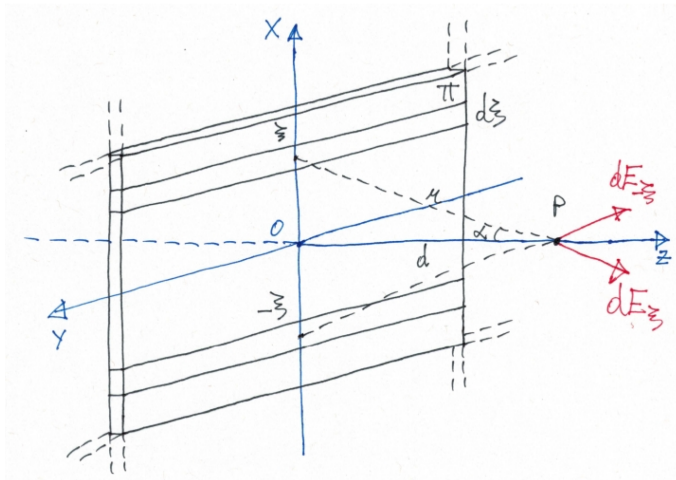


Figure 1.17

Calculate the electric field generated by σ at a generic point P at distance d from π.

(2) *Coordinate system*:

An $Oxyz$ Cartesian coordinate system is shown in Fig. 1.17.

(3) *Indicate all fields*:

From the results in example #2, it follows that each line ξ generates an electric field directed as the line between ξ and P and perpendicular to ξ in the half plane containing ξ and P (see Fig. 1.17).

(4) *DOF*:

1) Symmetry argument. The charge distribution σ on π is indistinguishable from any point P on the left or right of π. This is because any observer fixed in space would see the same charge distribution σ upon translating π vertically and/or horizontally (translation symmetry). We thus expect the total electric field $\vec{E}$ at P to only have a component along the z-axis.

2) DOF $= z$.

(5) Results [1]

$$\begin{aligned} d\vec{E}_\xi &= -\frac{\sigma d\xi}{2\pi\epsilon_0 r}\sin\alpha\vec{u}_y \\ &\quad+\frac{\sigma d\xi}{2\pi\epsilon_0 r}\cos\alpha\vec{u}_z \quad , \end{aligned}$$

[1] Do not confuse the distance d with the differentials $d\xi$, dx, and $d\alpha$.

$$
\begin{aligned}
d\vec{E}_{-\xi} &= +\frac{\sigma d\xi}{2\pi\epsilon_0 r}\sin\alpha\vec{u}_y \\
&\quad +\frac{\sigma d\xi}{2\pi\epsilon_0 r}\cos\alpha\vec{u}_z \quad .
\end{aligned}
$$

The y components cancel each other (as expected). Thus,

$$
d\vec{E} = 2\,\frac{\sigma d\xi}{2\pi\epsilon_0 r}\cos\alpha\vec{u}_z \quad . \tag{1.37}
$$

Integration from 0 to $+\infty$; note that $d\xi = dx$, $x = d\tan\alpha$, and $dx = d\,d\alpha/\cos^2\alpha$. Thus,

$$
\begin{aligned}
\vec{E} &= \frac{\sigma d}{\pi\epsilon_0}\int_0^{+\infty} d\xi\,\frac{1}{d^2+\xi^2}\,\vec{u}_z = \frac{\sigma d}{\pi\epsilon_0}\int_0^{+\pi/2} d\alpha\,\frac{d}{\cos^2\alpha}\,\frac{1}{d^2+d^2\tan^2\alpha}\,\vec{u}_z \\
&= \frac{\sigma d}{\pi\epsilon_0}\int_0^{+\pi/2} d\alpha\,\frac{d}{\cos^2\alpha}\,\frac{1}{d^2\left(1+\dfrac{\sin^2\alpha}{\cos^2\alpha}\right)}\,\vec{u}_z \\
&= \frac{\sigma}{2\epsilon_0}\,\vec{u}_z \quad .
\end{aligned}
$$

The total electric field is independent from the distance between P and π.

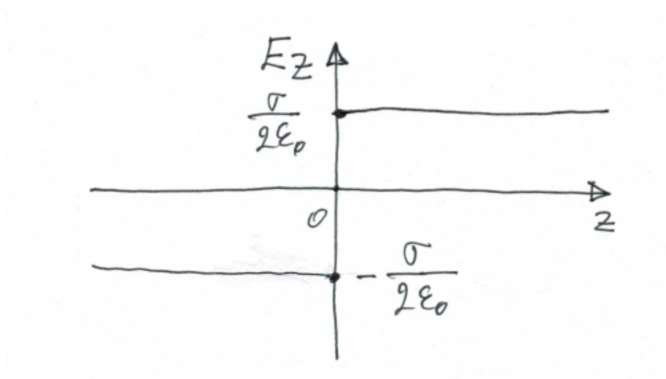


Figure 1.18

Figure 1.18 shows that E_z has a first kind discontinuity for $z = 0$:

$$
[E_z] = \frac{\sigma}{\epsilon_0} \quad .
$$

1.10 Gauss' Theorem

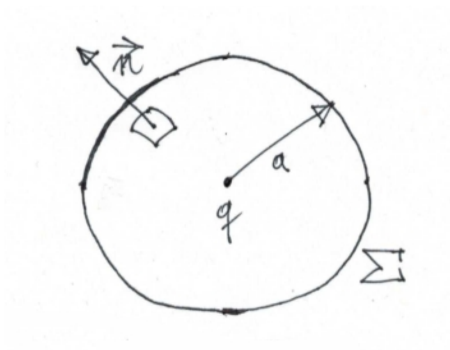


Figure 1.19

Consider a charge q in the center of a fictitious sphere with surface Σ. $\vec{n}$ is the normal unit vector at each point on Σ, directed outward. The flux Φ_Σ of the electric field through Σ is:

$$\Phi_\Sigma = \oint\!\!\!\oint_\Sigma \vec{E} \cdot \vec{n}\, dS \quad .$$

From Coulomb's law the electric field $\vec{E}$ generated by q is constant on Σ and is directed along $\vec{n}$. Thus,

$$\Phi_\Sigma = E 4\pi a^2 \quad .$$

Also from Coulomb's law and for the parameters in Fig. 1.19, we find

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{a^2}$$

and so

$$\Phi_\Sigma = \frac{q}{\epsilon_0} \quad . \tag{1.38}$$

- Gauss' theorem for a point-like charge - The flux of the electric field through any closed surface containing a charge is proportional to the value q of the charge, independently from the shape and area of the surface.
  Proof.

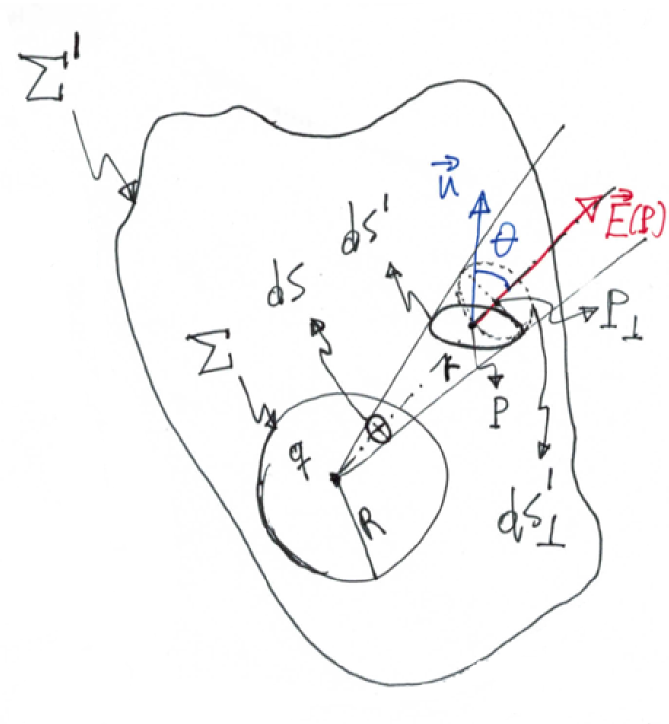


Figure 1.20

We need to show that the flux through Σ_1 in the same as the flux through Σ. The infinitesimal cone with vertex in q defines an infinitesimal surface dS on Σ and dS' on Σ'.
The infinitesimal surface $dS'_\perp$ is defined by the cone with vertex in q, on the surface of the sphere centered in q and containing the point $P_\perp$. Hence,

$$\frac{dS'_\perp}{dS} = \frac{r^2}{R^2} \quad , \tag{1.39}$$

where r is the distance between q and $P_\perp$ and R is the radius of Σ. From Fig. 1.20, it appears clear that

$$dS'_\perp = dS' \cos\theta \quad ,$$

which, using Eq. (1.39), gives:

$$dS' = dS \, \frac{r^2}{R^2} \, \frac{1}{\cos\theta} \quad .$$

We can now calculate the infinitesimal flux of $\vec{E}$ through dS':

$$d\Phi' = \vec{E}(P) \cdot \vec{n} dS' = E dS' \cos\theta \quad .$$

By using again Coulomb's law, we finally obtain

$$\begin{aligned} d\Phi' &= \frac{1}{4\pi\epsilon_0} \, \frac{q}{r^2} \, dS \, \frac{r^2}{R^2} \, \frac{1}{\cos\theta} \cos\theta \\ &= \frac{1}{4\pi\epsilon_0} \, \frac{q}{R^2} \, dS = d\Phi \quad . \end{aligned}$$

- By repeating this argument for each infinitesimal cone with vertex in q and summing, one obtains that the flux of $\vec{E}$ through the closed surface Σ' is the same as that through Σ.

- Corollary - The flux of $\vec{E}$ through any closed surface Σ that does not contain any charge q is zero. Note that the flux is zero, but the field can be nonzero at a given point on the surface. This is the case for an electrostatic dipole, which has zero total charge.

- Extension to the case of a generic charge distribution. If a closed surface Σ contains N point-like charges q_1, q_2, $\ldots$, q_n,

$$\vec{E} = \sum_{k=1}^{N} \vec{E}_k \quad .$$

From Gauss' theorem:

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n} dS = \sum_{k=1}^{N} \oiint_{\Sigma} \vec{E}_k \cdot \vec{n} dS = \frac{1}{\epsilon_0} \sum_{k=1}^{N} q_k \quad . \tag{1.40}$$

Similarly, for a continuous charge distribution inside a closed surface:

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n} dS = \frac{1}{\epsilon_0} q_{\mathrm{t}} \quad . \tag{1.41}$$

where q_{t} is the total charge inside Σ. In general:

$$q_{\mathrm{t}} = \iiint_{\Omega} dV \rho \quad , \tag{1.42}$$

where ρ is the volume charge density in the region of space Ω enclosed by Σ.

1.11 Line Integral of a Vector Field on an Oriented Curve

Consider a vector field $\vec{A}$ defined in a domain Ω, and a curve γ contained in Ω with extremes M and N (see Fig. 1.21). The curve γ is oriented from M to N. Consider the points $P_0, P_1, \ldots, P_n$ on γ. Each segment $P_k P_{k+1}$ is associated with a vector $\vec{l}_k$. The direction of $\vec{l}_k$ is along the line $P_k P_{k+1}$ with same orientation as γ; the magnitude of $\vec{l}_k$ is the length l_k of segment $P_k P_{k+1}$. Consider the values $\vec{A}_0, \vec{A}_1, \ldots, \vec{A}_n$ of $\vec{A}$ at each point $P_0, P_1, \ldots, P_n$, respectively. Given

$$\vec{A}_k \cdot \vec{l}_k \ , \qquad k = 0, 1, \cdots, n \qquad ,$$

we define

$$T_n = \sum_{k=0}^{n-1} \vec{A}_k \cdot \vec{l}_k \quad .$$

If the $\lim n \to \infty (\Rightarrow l_k \to 0)$ of T_n converges to a finite value, this value is called the line integral of $\vec{A}$ on γ:

$$T_\gamma = \lim_{n\to\infty} T_n = \int_\gamma \vec{A} \cdot d\vec{l} \quad . \tag{1.43}$$

In the case of a closed curve γ, Eq. (1.43) becomes:

$$C_\gamma = \oint_\gamma \vec{A} \cdot d\vec{l} \quad . \tag{1.44}$$

Defining $d\vec{l} \equiv \vec{t} dl$, where $\vec{t}$ is the tangent unit vector of γ at the point where $\vec{A}$ is considered and dl is the magnitude of the infinitesimal vector dl, we can write:

$$T_\gamma = \int_\gamma \vec{A} \cdot \vec{t} dl \quad \text{and} \quad C_\gamma = \oint_\gamma \vec{A} \cdot \vec{t} dl \quad . \tag{1.45}$$

In a Cartesian coordinate system:

$$\vec{A} \cdot d\vec{l} = A_x(x,y,z)dx + A_y(x,y,z)dy + A_z(x,y,z)dz \quad , \tag{1.46}$$

where dx, dy, and dz are the components of $d\vec{l}$. Thus,

$$\begin{aligned} T_\gamma &= \int_\gamma (A_x dx + A_y dy + A_z dz) \quad , \\ C_\gamma &= \oint_\gamma (A_x dx + A_y dy + A_z dz) \quad . \end{aligned} \tag{1.47}$$

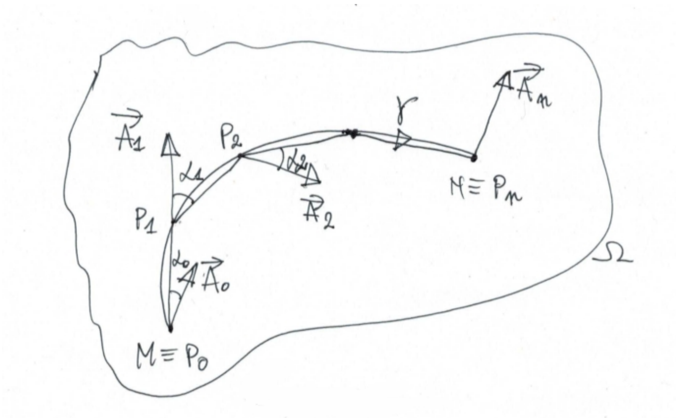


Figure 1.21

Assume

$$\int_{\gamma'} \vec{A} \cdot d\vec{l} = \int_{\gamma''} \vec{A} \cdot d\vec{l} \tag{1.48}$$

for each pair of curves γ' and γ'' in Ω with common extremes M and N. Equation (1.48) should be equivalent to

$$\oint_{\gamma} \vec{A} \cdot d\vec{l} = 0 \tag{1.49}$$

for each closed curve $\gamma \in \Omega$. To prove this last statement consider a vector field $\vec{A}$ defined in Ω for which Eq. (1.48) is true. Consider also a generic curve γ in Ω as shown in Fig. 1.22.

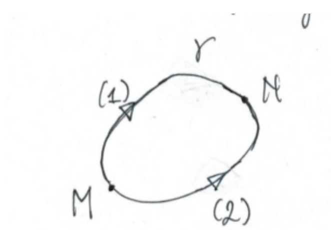


Figure 1.22

Consider two points M and N on the closed line γ. The two points divide γ into two arcs (1) and (2), both oriented from M to N (see Fig. 1.22). If it is true that

$$\int_{(1)} \vec{A} \cdot d\vec{l} = \int_{(2)} \vec{A} \cdot d\vec{l} \quad , \tag{1.50}$$

then

$$\oint_{\gamma} \vec{A} \cdot d\vec{l} = \int_{(1)} \vec{A} \cdot d\vec{l} - \int_{(2)} \vec{A} \cdot d\vec{l} \quad , \tag{1.51}$$

and thus

$$\oint_{\gamma} \vec{A} \cdot d\vec{l} = 0 \quad . \tag{1.52}$$

Similarly, it is possible to show that if $\vec{A}$ satisfies Eq. (1.49), it must be also satisfy Eq. (1.48). Therefore, Eqs. (1.48) and (1.49) are equivalent. A field $\vec{A}$ that satisfies Eq. (1.48) or Eq. (1.49) is defined to be an *irrotational* field. Note that an irrotational field is not always a conservative field. An irrotational field is conservative when defined in a star or simply connected domain (basically a domain without holes). A conservative field, however, is always irrotational. We will come back to the concept of conservative fields when introducing the concept of electrostatic scalar potential.

Consider a vector field $\vec{A}$ defined in a domain Ω. Assume the field is directed along a straight line from a center O to each point P in Ω (central field; see Fig. 1.23). Furthermore, assume $\vec{A}$ has a spherical symmetry, i.e., it depends only on the distance r between P and O,

$$\vec{A}(P) = f(r)\vec{u}_r \quad ,$$

where $\vec{u}_r$ is the radial unit vector and $f(r)$ a function describing how $\vec{A}$ varies with r.

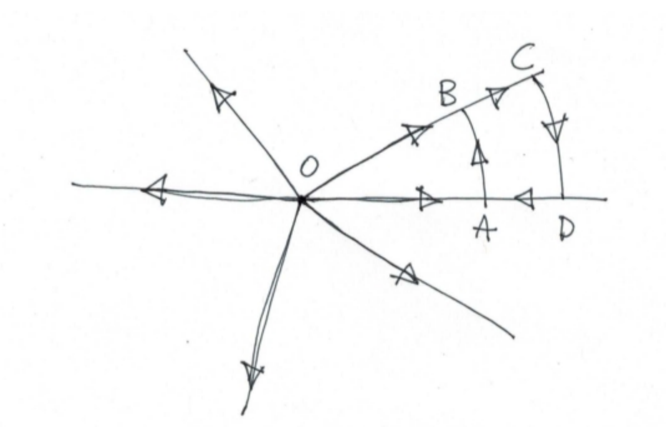


Figure 1.23

Consider the closed line $ABCDA$ in Fig. 1.23. The segments AB and CD are arcs of circles with radius r_1 and r_2, respectively. Hence,

$$\oint_{ABCDA} \vec{A} \cdot d\vec{l} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \quad .$$

The terms relative to AB and CD are zero because on those arcs $\vec{A}$ and $\vec{t}$ are orthogonal. The term relative to BC

$$\int_{BC} \vec{A} \cdot d\vec{l} = \int_{BC} f(r)\vec{u}_r \cdot d\vec{l} = \int_{r_1}^{r_2} f(r)dr \quad .$$

Similarly, for DA:[2]

$$\int_{DA} \vec{A} \cdot d\vec{l} = \int_{DA} f(r)\vec{u}_r \cdot d\vec{l} = -\int_{r_1}^{r_2} f(r)dr \quad .$$

Therefore,

$$\oint_{ABCDA} \vec{A} \cdot d\vec{l} = 0 \quad . \tag{1.53}$$

The result of Eq. (1.53) can be readily generalized to any arbitrary oriented curve in a central field $\vec{A}$, as the counterclockwise line γ shown in Fig. 1.24. In fact, the line γ can be regarded as the summation of infinite oriented lines, two of which are shown in the zoomed in region of the figure. These lines are the counterclockwise oriented lines $\gamma' = ABCDA$ and $\gamma'' = DCEFD$. The circulation of each of these lines must be zero because each of them must obey the result of Eq. (1.53). The oriented summation of the lines correspond to the line γ, which, thus, must also fulfill Eq. (1.53). Note that the oriented summation of the line segments DC and CD gives no line as the segments are equal, but with opposite orientation (same for all pairs of segments for the infinite lines into which γ can be decomposed).

As a consequence, any central field must be an irrotational field.

[2] Note that the length of DA, $\overline{DA}$, must be positive. Hence, if we integrate as:

$$\int_{r_2}^{r_1} -\vec{u}_r \cdot (-\vec{u}_r \, dr) = r_1 - r_2 < 0 \quad .$$

Therefore, we must choose $(+\vec{u}_r \cdot dr)$ for our $d\vec{l}$ when integrating from r_2 (larger) to r_1 (smaller).

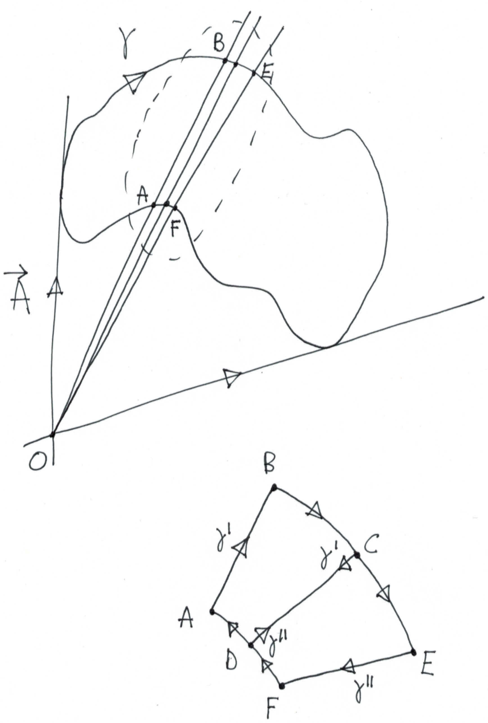


Figure 1.24

1.12 Irrotational Property of the Electrostatic Field

For a single point-like charge q_1,

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\,\frac{q_1}{r_{01}^2}\,\vec{u}_{10} \quad . \tag{1.54}$$

According to Eq. (1.54), the vector field $\vec{E}$ is of central type with spherical symmetry. As shown at the end of Sec. 1.11, this means that $\vec{E}$ is an irrotational field. In fact, for any closed line γ in the domain of definition of $\vec{E}$,

$$\oint_{\gamma} \vec{E} \cdot \vec{t}dl = 0 \quad . \tag{1.55}$$

Note that this result also applies for lines passing through the source charge, as long as the integral is intended as the Cauchy principal value. The physical meaning of Eq. (1.55) is clear from the definition of $\vec{E}$. In fact, the line integral is equal to the work generated by the field forces to move a unitary positive charge along the line. Equation (1.55) states that such a work along any closed line γ is zero. In other words, if the field forces generate a positive work along one segment of the line (i.e., the charge "falls" in the field along this segment), in the remaining part of the line they must generate a negative work (i.e., the charge must "rise" in the field along that part of the line).

As shown in Sec. 1.11, Eq. (1.55) can be expressed in an equivalent form. Consider two generic lines γ_1 and γ_2 with same orientation and common extremes A and B (the orientation, for example, goes from A to B), then

$$\int_{\gamma_1} \vec{E} \cdot \vec{t}dl = \int_{\gamma_2} \vec{E} \cdot \vec{t}dl \quad . \tag{1.56}$$

In other words, the field integral generated by a point-like source charge is independent from the line on which is calculated. Such an integral only depends on the extremes A and B.

The extension of Eqs. (1.55) and (1.56) to the case of a generic source-charge distributions is trivial. This can be realized by means of the superposition principle. Consider the electrostatic field generated by N point-like charges $q_0, q_1, \ldots, q_N$. In this case,

$$\oint_{\gamma} \vec{E} \cdot \vec{t}dl = \oint_{\gamma} \vec{E}_1 \cdot \vec{t}dl + \oint_{\gamma} \vec{E}_2 \cdot \vec{t}dl + \cdots + \oint_{\gamma} \vec{E}_N \cdot \vec{t}dl \quad , \tag{1.57}$$

where $\vec{E}_1, \vec{E}_2, \ldots, \vec{E}_N$ are the fields generated by $q_1, q_2, \ldots, q_N$, respectively. From Eq. (1.55) it follows that each of the integrals in Eq. (1.57) is zero, hence

$$\oint_{\gamma} \vec{E} \cdot \vec{t}dl = 0 \quad .$$

This result can readily be extended to the case of a continuous charge distribution. We can thus state the irrotational property of the electrostatic field:

The line integral of the electrostatic field generated by a generic distribution of source charges along a closed line is equal to zero.

It is worth mentioning that Gauss' theorem and the irrotational property of the electrostatic field are perfectly equivalent to Coulomb's law and the superposition principle (the only caveat being that Gauss' theorem and the irrotational property must be defined specifying the field behavior at infinite).

1.13 Symmetry Arguments = Irrotational Property

(1) Consider again an infinite straight line γ with a linear charge distribution λ uniformly distributed along the line (λ can be positive or negative).

Calculate the electric field $\vec{E}$ generated by λ at any generic point in space, P (see Fig. 1.25). Do not use Coulomb's law and the superposition principle.

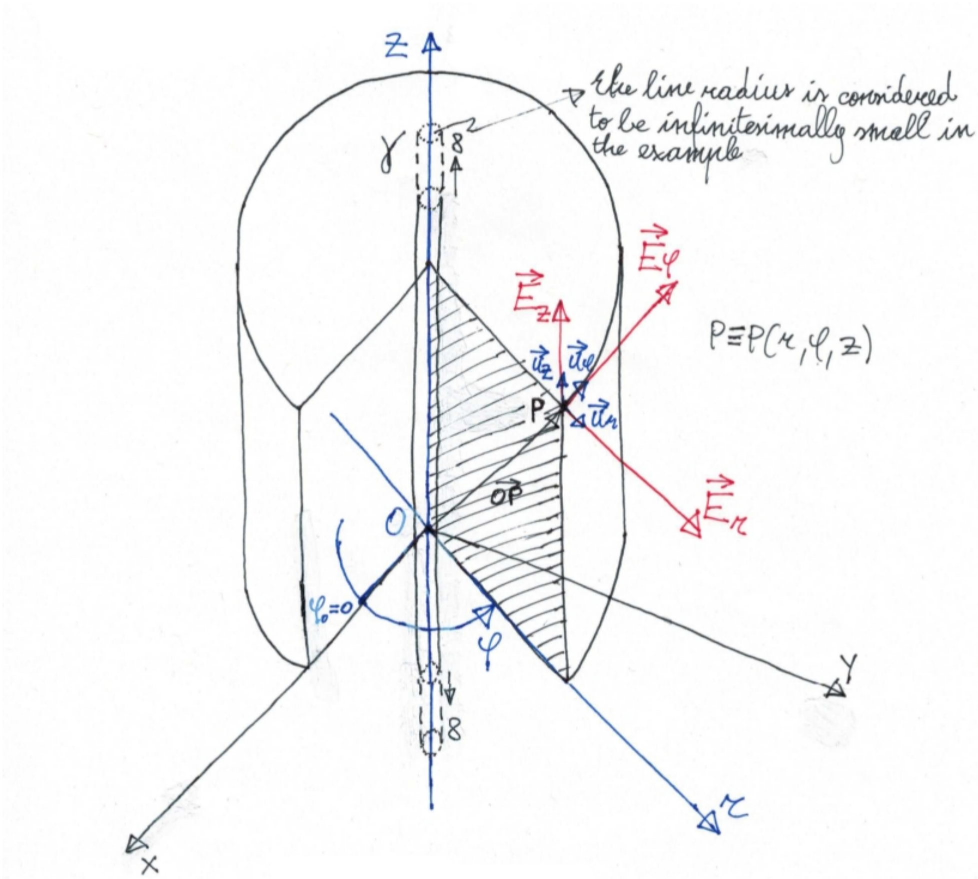


Figure 1.25

(2) *Coordinate system:*

The line γ can be thought as a cylinder of infinitesimal radius. As a consequence, the natural choice for the coordinate system is a cylindrical system $Or\varphi z$. Such a system is shown in Fig. 1.25 together with a condition $\varphi_0 = 0$ defining the zero of the φ coordinate. In this system, a generic point in space, P, is defined as

$$P \equiv P(r, \varphi, z) \quad \text{and} \quad \overrightarrow{OP} = r\vec{u}_r + \varphi\vec{u}_\varphi + z\vec{u}_z \quad ,$$

where $\vec{u}_r$, $\vec{u}_\varphi$, and $\vec{u}_z$ are the unit vectors for the $Or\varphi z$ system and r, φ, and z the magnitude of each spatial component of vector $\overrightarrow{OP}$.

(3) *Indicate all components of the electric field $\vec{E}$:*

The three components of the electric field $\vec{E}$ with respect to the $Or\varphi z$ coordinate system at a generic point in space, P, are shown in Fig. 1.25:

$$\begin{aligned} \vec{E}(P) = \vec{E} &= \vec{E}_r + \vec{E}_\varphi + \vec{E}_z \\ &= E_r \vec{u}_r + E_\varphi \vec{u}_\varphi + E_z \vec{u}_z \quad , \end{aligned} \tag{1.58}$$

where E_r, E_φ, and E_z are the magnitude of the components $\vec{E}_r$, $\vec{E}_\varphi$, and $\vec{E}_z$, respectively. Note that $\vec{E}_r$ is called the radial component, $\vec{E}_\varphi$ the tangent component, and $\vec{E}_z$ the vertical component of $\vec{E}$.

(4) *DOF:*

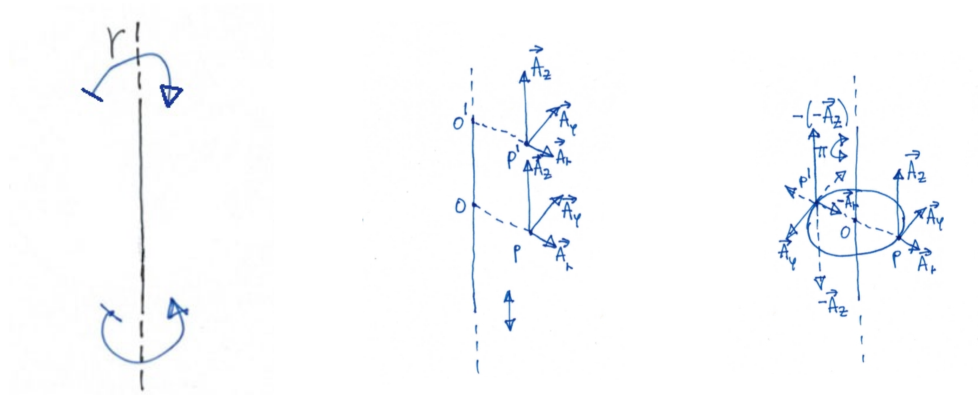
- Symmetry arguments.

We will use three symmetry arguments to gain as much information as possible on the components of $\vec{E}$. The symmetry arguments refer to the spatial properties of γ, regardless of any field or coordinate system.

(a) Rotation symmetry.
γ remains unchanged after a rotation of any arbitrary angle about its own axis (the rotation can be clockwise or counter-clockwise; see Fig. 1.26a).

(b) Translation symmetry.
γ remains unchanged after a translation of any arbitrary displacement (upward or downward) along its own axis (see Fig. 1.26b).

(c) Reflection symmetry.
γ remains unchanged after a π rotation about any point O' on γ (the rotation can be clockwise or counter-clockwise (see Fig. 1.26c).



(a) (b) (c)

Figure 1.26

In the reminder of this section, we will use combinations of these three symmetry arguments.

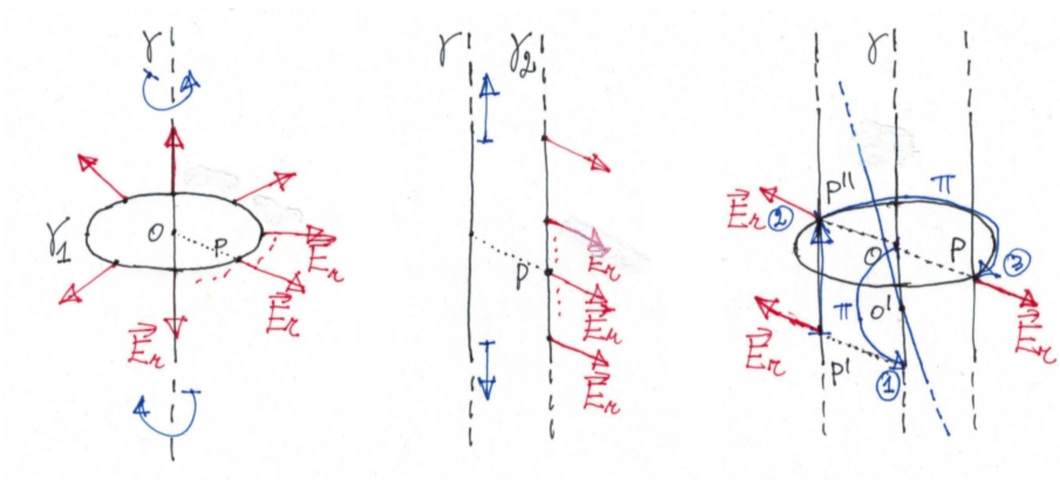
1) Radial component $\vec{E}_r$ of the electric field $\vec{E}$.

In general, $\vec{E}_r$ can be directed inward or outward at different points in space. The magnitude $\vec{E}_r$ can be different at any point.

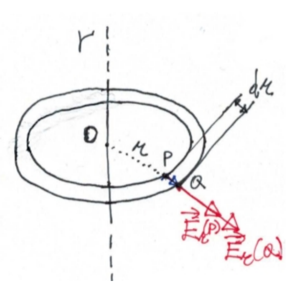
Consider a circle γ_1 lying on a plane perpendicular to γ passing through P and with center on γ, O (see Fig. 1.27a). We assume $\vec{E}_r$ to be directed outward in P (this will depend on the sign of λ). Hereafter, when performing a transformation on γ (e.g., a rotation about its axis, a translation along its axis, or a reflection about one of its points), the line γ, the component $\vec{E}_r$, and its application point P are considered to be a single rigid body. Consequently, any transformation on γ applies directly to $\vec{E}_r$ and P. The same assumption will be used for any charge distribution, transformation, electric field $\vec{E}$, and its application point P in the following. Due to the rotation symmetry of γ, the component $\vec{E}_r$ must be the same (direction, magnitude, and sign) at each point of γ_1 (see Fig. 1.27a).

Consider an infinite straight line γ_2 passing through P and parallel to γ (see Fig. 1.27b). Due to the translation symmetry of γ, $\vec{E}_r$ must be the same at each point on γ_2.

We now must perform a consistency check using the reflection symmetry of γ. Con-



(a) (b) (c)



(d)

Figure 1.27: (c) ①: Rotation of γ, P, and $\vec{E}_r$ by π about O' to P'. ②: Upward translation from P' to P''. ③: Rotation by π about γ from P'' back to P. Note that we could use $O' \equiv O$ for ①. We used O' for generality.

sider $\vec{E}_r$ at point P and perform a π rotation of γ, $\vec{E}_r$, and P about a point O' on γ different from O. After the transformation, P becomes P' and $\vec{E}_r$ remains the same: γ, P, and $\vec{E}_r$ become γ, P', and $\vec{E}_r$. Point P' is on a different plane perpendicular to γ compared to P. However, due to the translation symmetry of γ, P' and $\vec{E}_r$ at P' can be shifted upward to the same plane of P, thus obtaining a new point P'' with $\vec{E}_r$ applied to it. By rotating $\vec{E}_r$ at P'' by an angle π (clockwise or counterclockwise) about the axis of γ, we consistently obtain again $\vec{E}_r$ at P. A component $\vec{E}_r$ with $E_r \neq 0$ can thus exist. We also expect E_r to go to zero at infinite.

We expect $\vec{E}_r$ to be a continuous vector function away from γ. As a consequence, if $\vec{E}_r$ at a point P on a plane perpendicular to γ with distance r from the intersection point O between the plane and γ is directed outward, $\vec{E}_r$ at a point Q on the same plane, but with distance $r + dr$ from O should also be directed outward. By extension, $\vec{E}_r$ should be directed outward, everywhere in space. This consideration is consistent with the uniformity of λ: If $\lambda > 0$, $\vec{E}_r$ will be directed outward (remember the electric field is the vector field associated with the force acting on a positive test charge) and if $\lambda < 0$, $\vec{E}_r$ will be directed inward.

In summary, the symmetries of the line γ with λ allow a radial component $\vec{E}_r = E_r\vec{u}_r$ of $\vec{E}$ to exist. The component must be directed everywhere in space outward for $\lambda > 0$ and inward for $\lambda < 0$. E_r must go to zero at infinite.

2) Tangent component $\vec{E}_\varphi$ of the electric field $\vec{E}$.

In general, $\vec{E}_\varphi$ can be directed clockwise or counter-clockwise at different points in space. The magnitude E_φ can be different at any point.

Consider a circle γ_1 lying on a plane perpendicular to γ passing through P with

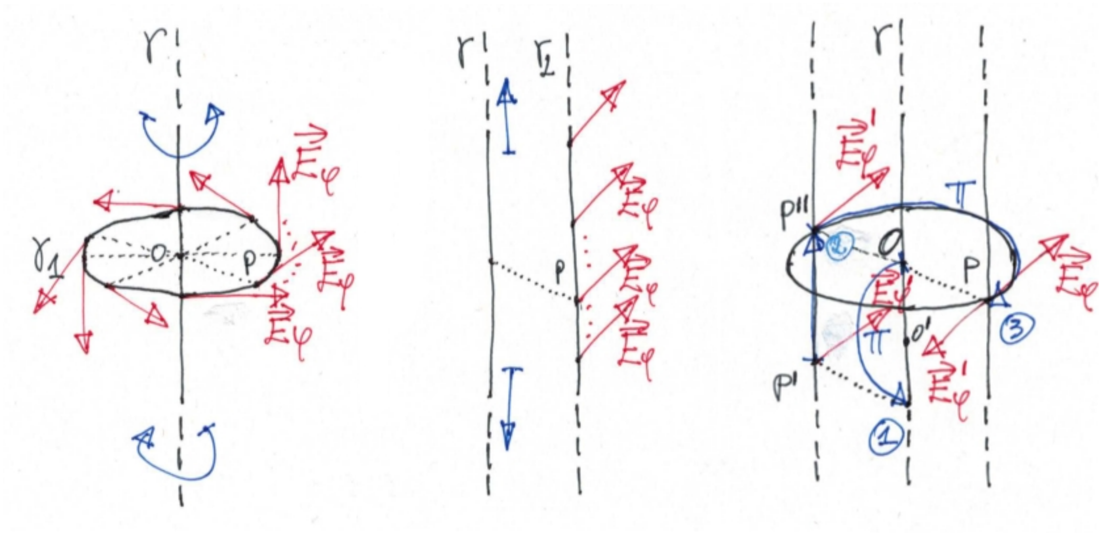


Figure 1.28: (c) ①: Rotation of γ, P, and $\vec{E}_\varphi$ by π about O' to γ, P', and $\vec{E}'_\varphi$. ②: Upward translation form γ, P', and $\vec{E}'_\varphi$ to γ, P'', and $\vec{E}'_\varphi$. ③: Rotation by π about γ from γ, P'', and $\vec{E}'_\varphi$ to γ, P, and $\vec{E}'_\varphi \neq \vec{E}_\varphi$.

centre on γ, O (see Fig. 1.28a). We assume $\vec{E}_\varphi$ to be directed counter-clockwise in P. Due to the rotation symmetry of γ, the component $\vec{E}_\varphi$ must be the same (direction, magnitude, and sign) at each point of γ_1 (see Fig. 1.28a).

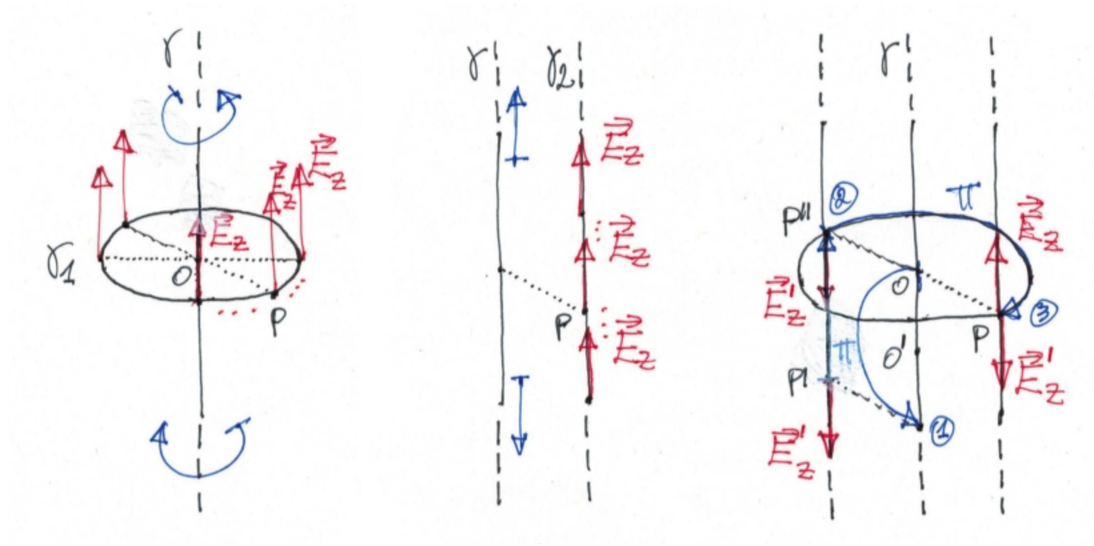
Consider an infinite straight line γ_2 passing through P and parallel to γ (see Fig. 1.28b). Due to the translation symmetry of γ, $\vec{E}_\varphi$ must be the same at each point on γ_2.

Therefore, $\vec{E}_\varphi$ must be the same at each point on the lateral surface of any cylinder with central axis on γ. In general, $\vec{E}_\varphi$ can be different on different cylinders.

We now must perform a consistency check using the reflection symmetry of γ. Consider $\vec{E}_\varphi$ at P and perform a counterclockwise rotation of γ, P, and $\vec{E}_\varphi$ at P about the point O' on γ, different from O. After the transformation γ remains the same, P becomes P', and $\vec{E}_\varphi$ becomes a vector $\vec{E}'_\varphi$ with same magnitude as $\vec{E}_\varphi$, but directed clockwise (i.e., opposite of $\vec{E}_\varphi$) and applied to P' (see Fig. 1.28c). Point P' is on a different plane perpendicular to γ compared to P. However, due to the translation symmetry of γ, P' and $\vec{E}'_\varphi$ can be shifted upward to the same plane of P, thus obtaining a new point P'' with $\vec{E}'_\varphi$ applied to it. By rotating γ, P'', and $\vec{E}'_\varphi$ at P'' by an angle π clockwise about the axis of γ, we must obtain the original component $\vec{E}_\varphi$ at P. However, we obtain $\vec{E}'_\varphi$ at P, which is inconsistent with $\vec{E}_\varphi$ at P since $\vec{E}_\varphi$ and $\vec{E}'_\varphi$ have opposite directions. In order to reconcile this inconsistency with the symmetry properties of γ, it must be $E_\varphi = E'_\varphi = 0$.

In summary, the symmetries of the line γ with λ do not allow a tangent component $\vec{E}_\varphi = E_\varphi \vec{u}_\varphi$ of $\vec{E}$ to exist. The component must be zero everywhere in space. Note that, in this case, we do not need to consider any further argument on the radial dependence of $\vec{E}_\varphi$ (i.e., an argument similar to that of Fig. 1.27d, but for $\vec{E}_\varphi$). This is because $\vec{E}_\varphi = \vec{0}$ everywhere. Also note that this funding is consistent with the zero-field condition at infinite.

3) Vertical component $\vec{E}_z$ of the electric field $\vec{E}$. In general, $\vec{E}_z$ can be directed upward or downward at different points in space. The magnitude E_z can be different at any point.



(a) (b) (c)

Figure 1.29: (c) ①: Rotation of γ, P, and $\vec{E}_z$ by π about O' to γ, P', and $\vec{E}'_z$. ②: Upward translation from γ, P', and $\vec{E}'_z$ to γ, P'', and $\vec{E}'_z$. ③: Rotation by π about γ from γ, P'', and $\vec{E}'_z$ to γ, P, and $\vec{E}'_z \neq \vec{E}_z$.

Consider a circle γ_1 lying on a plane perpendicular to γ passing through P and with centre on γ, O (see Fig. 1.29a). We assume $\vec{E}_z$ to be directed upward in P. Due to the rotation symmetry of γ, the component $\vec{E}_z$ must be the same (direction, magnitude, and sign) at each point of γ_1 (see Fig. 1.29a).

Consider an infinite straight line γ_2 passing through P and parallel to γ (see Fig. 1.29b). Due to the translation symmetry of γ, $\vec{E}_z$ must be the same at each point γ_2.

Therefore, as in the case of $\vec{E}_\varphi$, $\vec{E}_z$ must be the same at each point on the lateral surface of any cylinder with central axis on γ. In general, $\vec{E}_z$ can be different on different cylinders.

We now must perform a consistency check using the reflection symmetry of γ. Consider $\vec{E}_z$ at P and perform a counterclockwise rotation of γ, P, and $\vec{E}_z$ at P about a point O' on γ, different from O. After the transformation, γ remains the same, P becomes P', and $\vec{E}_z$ becomes a vector $\vec{E}'_z$ with same magnitude as $\vec{E}_z$, E_z, but directed downward (i.e., opposite of $\vec{E}_z$) and applied to P' (see Fig. 1.29c). Point P' is on a different plane perpendicular to γ compared to P. However, due to the translation symmetry of γ, P' and $\vec{E}'_z$ at P' can be shifted upward to the same plane of P, thus obtaining a new point P'' with $\vec{E}'_z$ applied to it. By rotating γ, P'', and $\vec{E}'_z$ at P'' by an angle π clockwise about the axis of γ, we must obtain the original component $\vec{E}_z$ at P. However, we obtain $\vec{E}'_z$ at P, which is inconsistent with $\vec{E}_z$ at P since $\vec{E}_z$ and $\vec{E}'_z$ have opposite directions. In order to reconcile this inconsistency with the symmetry properties of γ, it must be $E_z = E'_z = 0$.

In summary, the symmetries of the line γ with λ do not allow for a vertical component $\vec{E}_z = E_z \vec{u}_z$ of $\vec{E}$. Note that, also in this case, we do not need to consider any further argument on the radial dependence of $\vec{E}_z$ (i.e., an argument similar to that of Fig. 1.27d, but for $\vec{E}_z$). This is because $\vec{E}_z = \vec{0}$ everywhere. Also note that this funding is consistent with the zero-field condition at infinite.

4) DOF $= r$.

5) Result:

Considering that $\vec{E}_\varphi = \vec{E}_z = \vec{0}$ and $\vec{E}_r$ must be directed either inward or outward everywhere in space, we can resort to Gauss' theorem to calculate E_r.

Due to the rotation and translation symmetry of γ, $\vec{E}_r$ must be the same at each point of the lateral surface of any infinite cylinder with axis on γ. In order to calculate E_r everywhere in space, we thus consider an arbitrary closed cylindrical surface Σ of axis γ and passing through P. The lateral surface of Σ is S with height h and its bases are S_1 and S_2 (see Fig. 1.30). Gauss' theorem for surface Σ reads:

$$\begin{aligned} \oiint_{\Sigma} \vec{E} \cdot \vec{n} dS &= \iint_{S} \vec{E} \cdot \vec{n} dS + \iint_{S_1} \vec{E} \cdot \vec{n} dS + \iint_{S_2} \vec{E} \cdot \vec{n} dS \\ &= \frac{q_h}{\epsilon_0} \quad , \end{aligned} \tag{1.59}$$

where q_h is the charge associated with the portion of γ of length h contained in Σ. The only non-zero component of $\vec{E}$, $\vec{E}_r$, is tangent to both surface S_1 and S_2 and normal to surface S. As a consequence,

$$\iint_{S_1} \vec{E} \cdot \vec{n} dS = \iint_{S_2} \vec{E} \cdot \vec{n} dS = 0$$

and

$$\begin{aligned} \oiint_{\Sigma} \vec{E} \cdot n dS &= \iint_{S} \vec{E}_r \cdot \vec{n} dS = \iint_{S} E_r \vec{u}_r \cdot \vec{u}_r dS \\ &= \iint_{S} E_r dS = E_r \iint_{S} dS \\ &= E_r 2\pi r h = \frac{q_h}{\epsilon_0} \quad . \end{aligned}$$

In the above equation, we have expressed both $\vec{E}_r$ and $\vec{n}$ according to the coordinate system chosen in point (2), $\vec{E}_r = E_r \vec{u}_r$ and $\vec{n} = \vec{u}_r$. In addition, we have assumed the radius of S_1 and S_2 to be r (see Fig. 1.30).

Due to the arbitrary choice of Σ, we can thus conclude that

$$E_r = \frac{q_h}{2\pi\epsilon_0 r h} = \frac{\lambda}{2\pi\epsilon_0 r}$$

and, thus,

$$\vec{E}_r = \frac{\lambda}{2\pi\epsilon_0 r}\,\vec{u}_r \quad . \tag{1.60}$$

The result of Eq. (1.60) is the same as that obtained in Eq. (1.36).
In summary, the electric field $\vec{E}$ generated by a linear, uniform charge distribution λ on an infinite straight line γ at any generic point P in space is:

$$\vec{E}(P) = \vec{E}_r + \vec{E}_\varphi + \vec{E}_z \quad ,$$

with

$$\begin{cases} \vec{E}_r &= \dfrac{\lambda}{2\pi\epsilon_0 r}\,\vec{u}_r \\ \vec{E}_\varphi &= \vec{0} \\ \vec{E}_z &= \vec{0} \end{cases} \quad ,$$

where r is the minimum distance between P and γ. As expected, each component of $\vec{E}$ goes to zero at any point P at infinite distance from γ:

$$\lim_{P\to\infty} \vec{E}(P) = \lim_{r\to+\infty} \vec{E}_r(r) = \vec{0} \quad .$$

- Discussion:
Without resorting to any symmetry argument, the sole use of Gauss' theorem for an arbitrary closed cylindrical surface Σ would not suffice to calculate any of the components of the electric field $\vec{E}$ ($\vec{E}_r$, $\vec{E}_\varphi$, and $\vec{E}_z$) generated by a linear charge distribution λ on an infinite straight line γ at any generic point P in space.

In fact, without the knowledge that $\vec{E}_r$ must be the same at each point on the lateral surface of any infinite cylinder with axis on γ, it would be impossible to calculate $\displaystyle\iint_S \vec{E}_r \cdot \vec{n}dS$ because it would not be possible to take E_r outside the sign of integral. The fact that $\vec{E}_\varphi$ does not contribute to the total flux of $\vec{E}$ through Σ because $\vec{E}_\varphi$ is perpendicular to $\vec{n}$ on S, S_1, and S_2, gives no information on $\vec{E}_\varphi$. According to Gauss' theorem $\vec{E}_\varphi$ can be non-zero everywhere in space without adding any contribution to the flux through Σ (note that this depends on the choice of the specific surface Σ). At last, without any knowledge on $\vec{E}_z$, it would also be impossible to calculate $\displaystyle\iint_{S_1,S_2} \vec{E}_z \cdot \vec{n}dS$. Hence, the symmetry arguments are a necessary condition to apply Gauss' theorem to Σ.

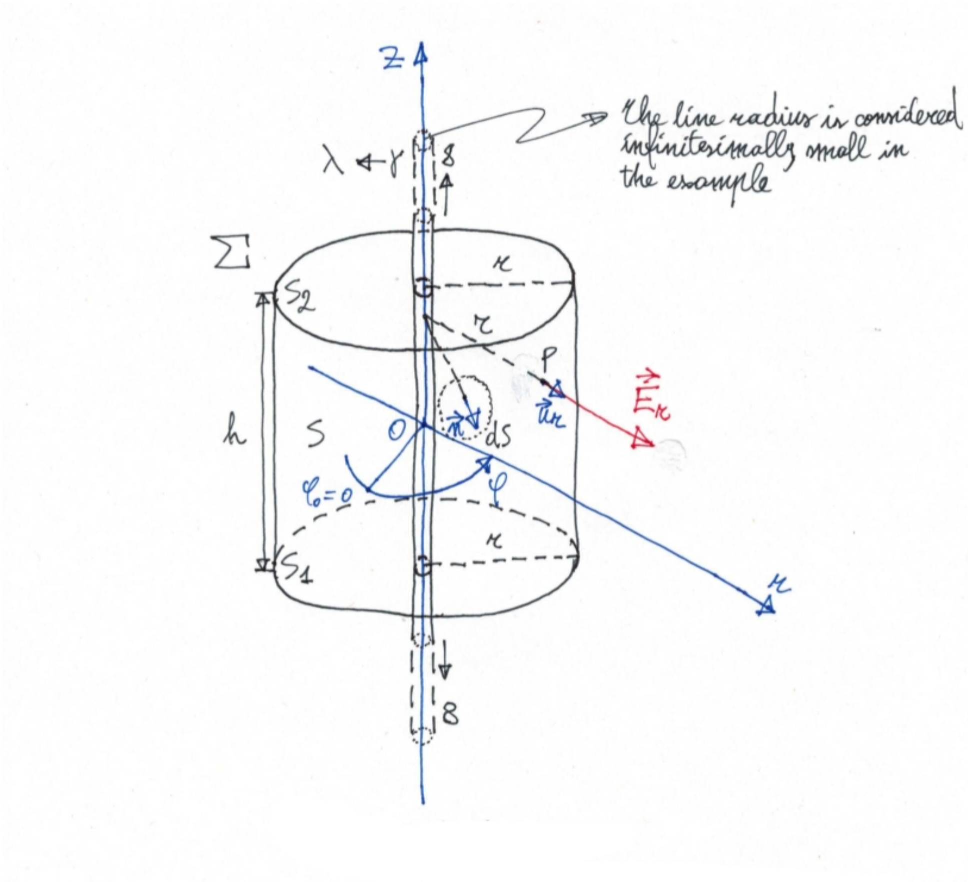

z
The line radius is considered infinitesimally small in the example
λ
∞
Σ
S_2
r
r
P
$\vec{E}_r$
$\vec{u}_r$
$\vec{n}$
dS
O
h
S
$\varphi_0 = 0$
φ
r
S_1
r
∞

Figure 1.30

Assume now we do not trust any symmetry argument to find properties of the components of $\vec{E}$. Instead, we intend to solely resort to, (a), the irrotational property of the electric field $\vec{E}$ and, (b), Gauss' theorem to calculate all components of $\vec{E}$. We conjecture (we will prove this later) that properties (a), (b), and (c) are equivalent to Coulomb's law and the superposition principle. Hence, it must be possible to solve the present example without using any symmetry argument.

- Without symmetry arguments.

(a) Irrotational property of $\vec{E}$.
For any given closed, oriented curve γ,

$$\oint_{\gamma} \vec{E} \cdot \vec{t} dl = 0 \quad ,$$

where $\vec{t}$ is the unit vector tangent to γ and directed as γ; dl is an infinitesimal line element along γ.

(b) Gauss' theorem.
For any given closed surface Σ,

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n} dS = \frac{q}{\epsilon_0} \quad ,$$

where $\vec{n}$ is the unit vector normal to dS and directed outward with respect to Σ, q is the total charge contained in Σ, and ϵ_0 the dielectric constant of vacuum; dS is an infinitesimal surface element on Σ.

Tangent component $\vec{E}_\varphi$ of the electric field $\vec{E}$.

In general, $\vec{E}_\varphi$ can be directed clockwise or counterclockwise at different points in space. The magnitude E_φ can be different at any point.

- Plane $Or\varphi$ perpendicular to γ.

Consider a closed curve γ_5 lying on a plane perpendicular to γ and oriented as shown in Fig. 1.31. The curve is assumed to contain γ and to be a continuous piecewise curve composed by four open curves: Line segment AB, arc BC, line segment CD, and arc DA. Arcs BC and DA are assumed to be portions of a circle γ_6 (oriented counterclockwise) lying on the plane perpendicular to γ, with radius R and center on γ. Line segments AB can be decomposed into a pair of line segments AA' and $B'B$ and line segment CD into a pair of line segments CC' and $D'D$. The line segments AA', $B'B$, CC', and $D'D$ are radii of γ_6 , with $\overline{AA'} = \overline{B'B} = \overline{CC'} = \overline{D'D} = |R|$. It is finally assumed that the aperture of the circular sectors associated with arcs BC and DA is equal to an infinitesimal angle $d\varphi$. Consequently, 6.0pt$\overset{\frown}{\mathrm{AB}}$ = 6.0pt$\overset{\frown}{\mathrm{CD}}$ $= |R| d\varphi$.

Due to the irrotational property of $\vec{E}$:

$$\oint_{\gamma_5} \vec{E} \cdot d\vec{l} = \int_{AA'} + \int_{B'B} + \int_{BC} + \int_{CC'} + \int_{D'D} + \int_{DA} = 0 \quad . \tag{1.61}$$

The only components of $\vec{E}$ contributing to the line integral of Eq. (1.61) are (all the other components of $\vec{E}$ are perpendicular to their corresponding elements $d\vec{l}$ and, so, they do not contribute to the line integral of Eq. (1.61)):

$$
\begin{aligned}
\text{on } AA' \ , \ \vec{E}_r\Big|_{AA'} &= \vec{E}_r(P_{AA'}) \text{ directed toward } (\lambda < 0) \text{ or opposite } (\lambda > 0) \text{ of } \gamma \\
d\vec{l}\Big|_{AA'} & \qquad\qquad\quad \text{directed toward } \gamma \\
\text{on } B'B \ , \ \vec{E}_r\Big|_{B'B} &= \vec{E}_r(P_{B'B}) \text{ directed toward or opposite of } \gamma \\
d\vec{l}\Big|_{B'B} & \qquad\qquad\quad \text{directed opposite of } \gamma \\
\text{on } BC \ , \ \vec{E}_\varphi\Big|_{BC} &= -E_{\varphi 1}\vec{u}_\varphi \ \ \text{directed clockwise} \\
d\vec{l}\Big|_{BC} &= R d\varphi \vec{u}_\varphi \\
\text{on } CC' \ , \ \vec{E}_r\Big|_{CC'} &= \vec{E}_r(P_{CC'}) \text{ directed toward or opposite of } \gamma \\
d\vec{l}\Big|_{CC'} & \qquad\qquad\quad \text{directed toward } \gamma \\
\text{on } D'D \ , \ \vec{E}_r\Big|_{D'D} &= \vec{E}_r(P_{D'D}) \text{ directed toward or opposite of } \gamma \\
d\vec{l}\Big|_{D'D} & \qquad\qquad\quad \text{directed toward } \gamma \\
\text{on } DA \ , \ \vec{E}_\varphi\Big|_{DA} &= E_{\varphi 2}\vec{u}_\varphi \ \ \ \text{directed counterclockwise} \\
d\vec{l}\Big|_{DA} &= R d\varphi \vec{u}_\varphi
\end{aligned}
$$

The line integrals along line segments $AA', B'B, CC'$, and $D'D$ can be readily calculated using Eq. (1.60). The only caveat is that those segments include a point on line γ, where Eq. (1.60) diverges to infinite. Thus, the line integrals are improper integrals and their Cauchy principle values must be calculated (to avoid the singularity). This can easily be done by choosing the limit of integration not to coincide with γ, but to be infinitesimally close to it, as a distance δ. We find,

$$
\begin{aligned}
\int_{AA'} + \int_{B'B} + \int_{CC'} + \int_{D'D} &= \lim_{\delta \to 0^+} \frac{\lambda}{2\pi\epsilon_0} \left(\int_R^\delta \frac{1}{r}\, dr + \int_\delta^R \frac{1}{r}\, dr + \int_R^\delta \frac{1}{r}\, dr + \int_\delta^R \frac{1}{r}\, dr \right) \\
&= \lim_{\delta \to 0^+} \frac{\lambda}{2\pi\epsilon_0} \left(\ln\frac{\delta}{R} + \ln\frac{R}{\delta} + \ln\frac{\delta}{R} + \ln\frac{R}{\delta} \right) = 0 \,. \qquad (1.62)
\end{aligned}
$$

Hence, from Eq. (1.61)

$$
\oint_{\gamma_5} \vec{E} \cdot d\vec{l} = \int_{BC} + \int_{DA} \qquad (1.63)
$$

$$
= -\int_{\varphi_1 + d\varphi}^{\varphi_1} E_{\varphi 1}\vec{u}_\varphi \cdot \vec{u}_\varphi R d\varphi + \int_{\varphi_2}^{\varphi_2 + d\varphi} E_{\varphi 2}\vec{u}_\varphi \cdot \vec{u}_\varphi R d\varphi = 0 \qquad (1.64)
$$

which corresponds to the condition

$$E_{\varphi 1} = -E_{\varphi 2} \quad . \tag{1.65}$$

This condition means that $\vec{E}_\varphi$ must be the same on arcs BC and DA, i.e., it must have the same magnitude and be directed counterclockwise.

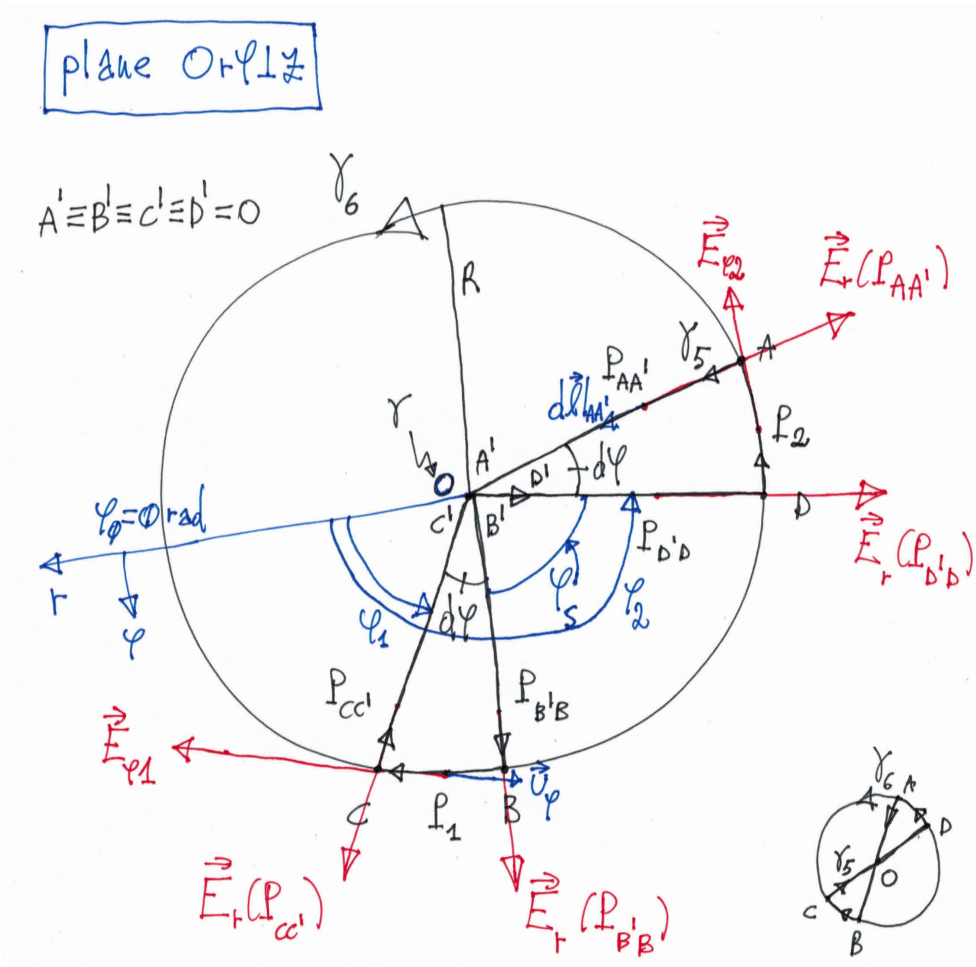


Figure 1.31

Note that the angular separation φ_s between the circular sectors BC and DA can be chosen arbitrarily. We can therefore conclude that the component $\vec{E}_\varphi$ must be the same at each point on any circle with center on γ.

This result, though, does not provide enough information on $\vec{E}_\varphi$ to fully solve problem. We must resort to one more curve, curve γ_6 and use the irrotational property. This gives:

$$\oint_{\gamma_6} \vec{E} \cdot d\vec{l} = \int_0^{2\pi} E_\varphi \vec{u}_\varphi \cdot \vec{u}_\varphi R d\varphi \tag{1.66}$$

$$= \quad E_\varphi \int_0^{2\pi} R d\varphi = 2\pi E_\varphi = 0 \quad , \tag{1.67}$$

from which it finally follows that $E_\varphi = 0$ at each point on any circle with center on γ. This means that $E_\varphi = 0$ at every point in space.

1.14 Electrostatic Field Properties in Local Form

Thus far, Gauss' theorem and the irrotational property of the electrostatic field $\vec{E}$ have been expressed in so-called integral form:

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n}\, dS = \frac{1}{\epsilon_0} q_{\mathrm{t}} \quad , \tag{1.68}$$

where q_{t} is the total charge within a volume bounded by a closed surface Σ, $\vec{E}$ the electrostatic field at each point on Σ, $\vec{n}$ the normal unit vector to an infinitesimal surface element on Σ, dS, and ϵ_0 the dialectic constant of vacuum;

$$\oint_{\gamma} \vec{E} \cdot \vec{t}\, dl = 0 \quad , \tag{1.69}$$

where γ is an arbitrary oriented closed curve, $\vec{E}$ the electrostatic field at each point on γ, and $\vec{t}$ the tangent unit vector to an infinitesimal curve element on γ, dl. All curves and surfaces are in the 3D Eucleadian space.

Equations (1.68) and (1.69) apply to any arbitrary small surface Σ and curve γ. Hence, they can be written in local form.

1.14.1 Gauss' Theorem in Local Form

Local form means that a law is written for just a neighborhood of a generic point P in space.

Case 1

Consider a charge distribution $\rho(Q)$ at any point Q of a region Ω in the 3D Eucledian space. We assume the region Ω to be compact in space (i.e., to be closed and bounded) with a piecewise smooth boundary Σ. We can write Eq. (1.68) as

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n}\, dS = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho\, dV \quad , \tag{1.70}$$

where dV is an infinitesimal volume element in Ω. Assuming $\vec{E}$ to be continuously differentiable in a vector sense (i.e., each component of $\vec{E}$ in a given inertial reference frame and with respect to a specific coordinate system is continuous with continuous first derivatives, $\vec{E} \in C^1(\Omega)$) in the neighborhood of any point in Ω, from the divergence theorem we obtain

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n}\, dS = \iiint_{\Omega} \vec{\nabla} \cdot \vec{E}\, dV$$

and, from Eq. (1.70),

$$\iiint_{\Omega} \vec{\nabla} \cdot \vec{E}\, dV = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho\, dV \quad .$$

Due to the arbitrariness of Ω, it finally follows

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \quad . \tag{1.71}$$

It is obvious that in points with no charge, $\rho = 0$,

$$\vec{\nabla} \cdot \vec{E} = 0 \quad .$$

In this sense, the volume charge density ρ appears as a source for the vector field $\vec{E}$.

In order to find the result of Eq. (1.71), we made use of the divergence (also known as Gauss-Ostrogradsky's) theorem. We now intend to set out to find the necessary and sufficient conditions on any given charge distribution for the result of Eq. (1.71) to hold. In fact, when trying to solve an electrostatic problem we are given the charge distribution and we are asked to find $\vec{E}$. However, in order to apply the divergence theorem we made use of the fact the $\vec{E}$ is continuously differentiable in a vector sense even though, *a priori*, we did not know anything about $\vec{E}$.

– **Direct statement.** It is a necessary condition for ρ to be a continuous and limited function in Ω for $\vec{E} \in C^1(\Omega)$ and, thus, for Eq. (1.71) to hold.

We assume that $\vec{E} \in C^1$ in the neighborhood of any point in Ω. With respect to a Cartesian coordinate system this means that

$$\begin{aligned}\vec{E} &= E_x(x,y,z)\vec{u}_x \\ &\quad +E_y(x,y,z)\vec{u}_y \\ &\quad +E_z(x,y,z)\vec{u}_z \quad ,\end{aligned}$$

with

$$\begin{aligned}&E_x \in C^\circ(\Omega)\\ &E_y \in C^\circ(\Omega)\\ &E_z \in C^\circ(\Omega)\end{aligned}$$

and

$$\left\{\begin{array}{l} \frac{\partial}{\partial_x}E_x, \quad \frac{\partial}{\partial_y}E_x, \quad \frac{\partial}{\partial_z}E_x \in C^\circ(\Omega) \\ \frac{\partial}{\partial_x}E_y, \quad \frac{\partial}{\partial_y}E_y, \quad \frac{\partial}{\partial_z}E_y \in C^\circ(\Omega) \\ \frac{\partial}{\partial_x}E_z, \quad \frac{\partial}{\partial_y}E_z, \quad \frac{\partial}{\partial_z}E_z \in C^\circ(\Omega) \end{array}\right. \quad . \tag{1.72}$$

We can thus write

$$\mathrm{div}\vec{E} = \vec{\nabla} \cdot \vec{E} = \frac{\partial}{\partial_x}E_x + \frac{\partial}{\partial_y}E_y + \frac{\partial}{\partial_z}E_z \quad .$$

Note that, for the divergence to be well defined, only a subset of the conditions (1.72) must be satisfied. Under those conditions, it follows that $\vec{\nabla} \cdot \vec{E} \in C^{\circ}(\Omega)$ (i.e., the divergence is a continuous scalar function). Due to Eq. (1.71), it must then be

$$\rho \in C^{\circ}(\Omega) \quad , \tag{1.73}$$

therefore proving the direct statement. Thus, by only inspecting the charge distribution ρ it is possible to assure whether at least one of the derivatives of the components of $\vec{E}$ with respect to its own variable is discontinuous.

– **In summary.** We conjecture (examples will confirm this conjecture) that it is a necessary and sufficient condition for $\rho \in C^0(\Omega)$ and be limited in Ω for $\vec{E} \in C^1(\Omega)$ and be limited in Ω. This means both $\vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \times \vec{E}$ are well defined. Particular care must be taken every time ρ presents any sort of discontinuity.

1.15 The Irrotational Property in Local Form.

Case 1

Consider a charge distribution $\rho(P) \in C^0(\Omega)$ and limited in Ω. According to our conjecture, we can thus assume $\vec{E} \in C^1(\Omega)$ and limited in Ω. Due to the irrotational property of $\vec{E}$, for any closed curve γ

$$\oint_\gamma \vec{E} \cdot \vec{t}\, dl = 0 \quad , \tag{1.74}$$

Since $\vec{E} \in C^1(\Omega)$ and is limited in Ω, we can apply Stokes' theorem:

$$\oint_\gamma \vec{E} \cdot \vec{t}\, dl = \oiint_\Sigma \vec{\nabla} \times \vec{E} \cdot \vec{n}\, dS \quad , \tag{1.75}$$

where Σ is any surface having γ as a boundary and oriented consistently with γ. Imposing condition (1.74) in Eq. (1.75), we obtain

$$\oiint_\Sigma \vec{\nabla} \times \vec{E} \cdot \vec{n}\, dS = 0 \quad .$$

Due to the arbitrary choice of γ, it finally follows

$$\vec{\nabla} \times \vec{E} = \vec{0} \quad . \tag{1.76}$$

Equations (1.71) and (1.76), which are valid in case 1 [$\rho \in C^0(\Omega)$ and limited in Ω], are Gauss' theorem and the irrotational property of $\vec{E}$ in local form. Since $\vec{E} \in C^1(\Omega)$ and is limited in Ω, the local form is actually a differential form in this case.

1.16 Gauss' theorem and the irrotational property of $\vec{E}$.

Case 2

Consider a continuous and limited charge distribution σ at any point of a surface Σ in the 3D Eucledian space. We assume the region Σ to be smooth. The total charge q_{t} in a volume Ω entirely containing Σ is

$$q_{\mathrm{t}} = \iiint\limits_{\Omega} dxdydz\, \sigma(\vec{r}_0(x,y,z))\delta(\vec{r}-\vec{r}_0(z,y,z)) \quad , \tag{1.77}$$

where the integral has been expressed in cartesian coordinates (note that σ has units Cm^{-2}), δ is the delta-Dirac function, $\vec{r}=\vec{r}(x,y,z)$ is any point in Ω, and $\vec{r}_0(x,y,z)$ is a point on the surface Σ. Due to the presence of the δ-Dirac function, Eq. (1.77) clearly shows that the charge distribution σ represents a discontinuity in the 3D space. Following our conjecture, in this case we should not resort to Eqs. (1.71) and (1.76) to express Gauss' theorem and the irrotational property of $\vec{E}$ in local form. Instead, we should use Eqs.(1.68) and (1.69) for a special "infinitesimal" surface and line, respectively. In fact, those equations are valid even for discontinuous charge distributions.

– *Gauss' theorem.*

Consider an infinitesimal closed surface of cylindrical type, Σ_c. Since we want to study the properties of $\vec{E}$ infinitesimally close to Σ, both above and below Σ, the bases S_1 and S_2 of the cylinder Σ_c are assumed to be above and below Σ, respectively. The two bases are assumed to be circles with equal radius dr, from which the infinitesimal nature of the bases. The area of the two bases is thus

$$S_1 = S_2 = \pi dr^2 \quad . \tag{1.78}$$

The lateral surface S_ℓ of cylinder Σ_c is assumed to have area

$$S_\ell = 2\pi dr \cdot h \quad , \tag{1.79}$$

where h is the height of Σ_c. Curve Σ_c is typically referred to as a "coin-type" curve. In order to study $\vec{E}$ infinitesimally close to Σ, we will need to consider the limit for h going to zero. Figure 5.1 shows a coin-type curve Σ_c in the vicinity of Σ. The center of Σ_c is a generic point P on Σ. The surface Σ is oriented in such a way that the normal unity vector $\vec{n}$ at P is directed upward with respect to Σ.

We can thus write Eq. (1.70) for Σ_c as

$$\begin{aligned}\oiint\limits_{\Sigma_c} \vec{E}\cdot\vec{n}dS &= \iint\limits_{S_1} \vec{E}_1\cdot\vec{n}dS + \iint\limits_{S_2} \vec{E}_2\cdot\vec{n}_2 dS + \iint\limits_{S_\ell} \vec{E}_\ell\cdot\vec{n}_\ell dS \\ &= \iint\limits_{\Omega_c} dV\,\sigma(\vec{r}_0)\delta(\vec{r}-\vec{r}_0) = \iint\limits_{S} dS\,\sigma \quad , \end{aligned}\tag{1.80}$$

where $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_\ell$ are the electric fields on surfaces S_1, S_2, and S_ℓ, respectively, Ω_c the volume of Σ_c, and S the portion of Σ included in Σ_c. The last two integrals in Eq. (1.80) are different ways to calculate the total charge in Σ_c. Due to the infinitesimal nature of S_1 and S_2, $\vec{E}_1$ and $\vec{E}_2$ can be assumed to be constant on S_1 and S_2. No assumption is made on $\vec{E}_\ell$ which, in general, can vary arbitrarily on S_ℓ. The last integral of Eq. (1.80) can be written as

$$\iint\limits_{S_\ell} \vec{E}_\ell \bullet \vec{n}_\ell dS = \iint\limits_{S_\ell} \vec{E}_\ell \bullet \vec{n}_\ell \cdot 2\pi dr \cdot h \quad .$$

Note that $\vec{n}_\ell$ is here the normal unity vector at each point on S_ℓ. It is easy to convince oneself that

$$\lim_{h\to 0} \iint\limits_{S_\ell} \vec{E}_\ell \bullet \vec{n}_\ell \cdot 2\pi dr \cdot h = 0 \quad .$$

In this limit we can thus disregard the contribution of $\vec{E}_\ell$ to the integral (1.80). In the same limit, it follows that $\vec{n}_1 = \vec{n}$ and $\vec{n}_2 = -\vec{n}$. Inserting Eq. (1.78) in the integral (1.80), the latter reads

$$\begin{aligned} \lim_{h\to 0} \oiint\limits_{\Sigma_C} \vec{E} \bullet \vec{n} dS &= \vec{n} \bullet \vec{E}_1 \cdot \pi dr^2 - \vec{n} \bullet \vec{E}_2 \cdot \pi dr^2 \\ &= \frac{1}{\epsilon_0}\sigma \cdot \pi dr^2 \quad , \end{aligned} \tag{1.81}$$

where we have assumed areas S_1 and S_2 to be equal to the area S of the portion of Σ included in Σ_c. This is a reasonable assumption considering that S_1 and S_2 are infinitesimally small and arbitrary close to Σ due to the limit on h. Therefore, we can write the integral (1.80) as

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) = \frac{1}{\epsilon_0}\sigma \quad . \tag{1.82}$$

Condition (1.82) shows that the normal components of the electric field $\vec{E}$ infinitesimally above and below a continuous and limited charge distribution σ on a smooth surface Σ present a discontinuity of the first kind. The electric field components remain limited. The difference between the normal components E_{n_1} and E_{n_2} at a generic point P on Σ, on each side of Σ, is proportional to the surface charge distribution evaluated at P:

$$E_{n_1}(P) - E_{n_2}(P) = \frac{1}{\epsilon_0}\sigma(P) \quad .$$

It is worth noting that if the charge was distributed with a volume density ρ inside the coin-type surface Σ_c, instead of a surface density σ, the total charge in Σ_c

would reduce to zero in the limit for h going to zero. It is the surface nature of the charge distribution to assure the charge inside Σ_c is not affected by the limit.

Condition (1.82) implies that the $\vec{\nabla} \cdot \vec{E}$ at any point on Σ is not well defined, at least in the usual sense. This confirms it is not possible to use Eq. (1.71) in this case. Due to the discontinuity of $\vec{E}$ at any point on Σ, also the $\vec{\nabla} \times \vec{E}$ at any point on Σ is not well defined, at least in the usual sense. Hence, it is impossible to use Eq. (1.76) in this case. $\vec{E}$ must be at least continuous for $\vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \times \vec{E}$ to be well defined.

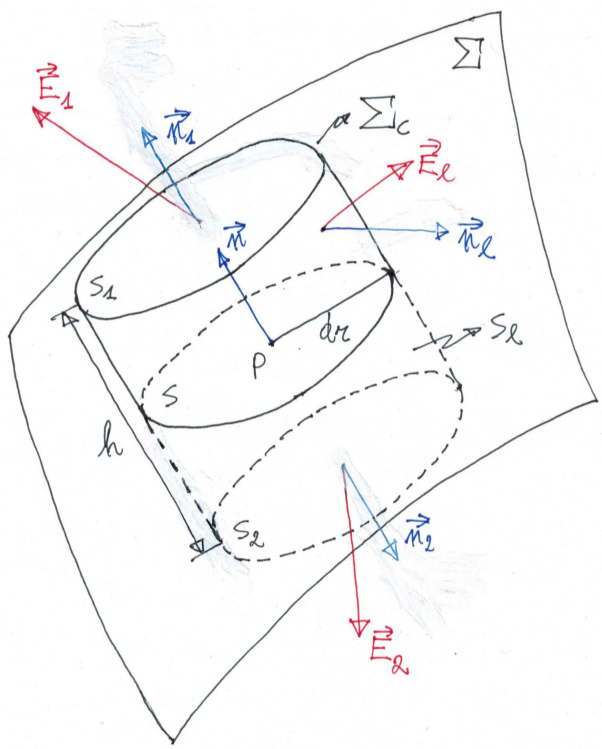


Figure 1.32

– *Irrotational property of $\vec{E}$.*

Consider an infinitesimal closed line of rectangular type, $\gamma_c = ABCDA$, oriented clockwise. Since we want to study the properties of $\vec{E}$ infinitesimally close to Σ, both above and below Σ, the sides AB and CD are assumed to be above and below Σ, respectively. These two sides are assumed to be line segments with equal length $\overline{AB} = \overline{CD} = dl$, from witch the infinitesimal nature of the sides. The sides BC and DA that intersect Σ are assumed to be line segments with equal length $\overline{BC} = \overline{DA} = h$. Curve γ_c is typically referred to as a "cut-type" curve. As for the coin-type curve, in order to study $\vec{E}$ infinitesimally close to Σ, we will need to consider the limit for h going to zero. Figure 5.2 shows a cut-type curve γ_c in the vicinity of Σ. The centre of γ_c is a generic point P on Σ. The tangent unit vector $\vec{E}$ at P on Σ is oriented as shown in Fig. 1.33.

We can thus write Eq. (1.33) for γ_c as

$$\begin{aligned}\oint_{\gamma_c}\vec{E}\cdot\vec{t}dl &= \int_{AB}\vec{E}_1\cdot\vec{t}_1 dl+\int_{BC}\vec{E}_{BC}\cdot\vec{t}_{BC}dl\\ &\quad+\int_{CD}\vec{E}_2\cdot\vec{t}_2 dl+\int_{DA}\vec{E}_{DA}\cdot\vec{t}_{DA}dl\\ &= 0 \quad ,\end{aligned}\tag{1.83}$$

where $\vec{E}_1$, $\vec{E}_{BC}$, $\vec{E}_2$, and $\vec{E}_{DA}$ are the electric fields on lines AB, BC, CD, and DA, respectively, and $\vec{t}_1$, $\vec{t}_{BC}$, $\vec{t}_2$, and $\vec{t}_{DA}$ the corresponding tangent unit vectors. Due to the infinitesimal nature of AB and CD, $\vec{E}_1$ and $\vec{E}_2$ can be asumed to constant on AB and CD, respectively. No assumptions are made on $\vec{E}_{BC}$ and $\vec{E}_{DA}$ which, in general, can vary arbitrarily on BC and DA, respectively. In the limit for h going to zero, it follows that $\vec{t}_1=\vec{t}$ and $\vec{t}_2=-\vec{t}$ and the integral (1.83) can be rewritten as

$$\begin{aligned}\lim_{h\to 0}\oint_{\gamma_c}\vec{E}\cdot\vec{t}dl &= \vec{E}_1\cdot\vec{t}dl+\lim_{h\to 0}\cancelto{\emptyset}{\int_{BC}\vec{E}_{BC}}\cdot\vec{t}_{BC}dl\\ &\quad-\vec{E}_2\cdot\vec{t}dl+\lim_{h\to 0}\cancelto{\emptyset}{\int_{DA}\vec{E}_{DA}}\cdot\vec{t}_{DA}dl\\ &= \vec{t}\cdot(\vec{E}_1-\vec{E}_2)=0 \quad ,\end{aligned}\tag{1.84}$$

where we have used the commutative property of the scalar product, $\vec{t}$ is the tangent unit vector at P, and we have assumed the contribution from the line integrals on BC and DA to be zero as the line lengths go to zero. The tangent components of the electric field $\vec{E}$ infinitesimally above and below a continuous and limited charge distribution σ on a smooth surface Σ are continuous and limited:

$$E_{t_1}(P)=E_{t_2}(P) \quad .$$

Case 3

Consider a compact region Ω in the 3D Eucledian space. Assume a continuous and limited change distribution ρ in a sub-region Ω_1 (also compact) of Ω and zero charge in the remaining part of Ω, $\Omega_2=\Omega-\Omega_1$. The charge distribution presents a discontinuity at the boundary between Ω_1 and Ω_2. Following again our guidelines on the use of the differential forms Eqs. (1.71) and (1.76), we shall resort again to coin- and cut-type curves to obtain Gauss' theorem and the irrotational property of $\vec{E}$ in local form (this case can be generalized to two arbitrary densities ρ_1 and ρ_2 in Ω_1 and Ω_2 continuous and limited in each sub-region and discontinuous on the boundary).

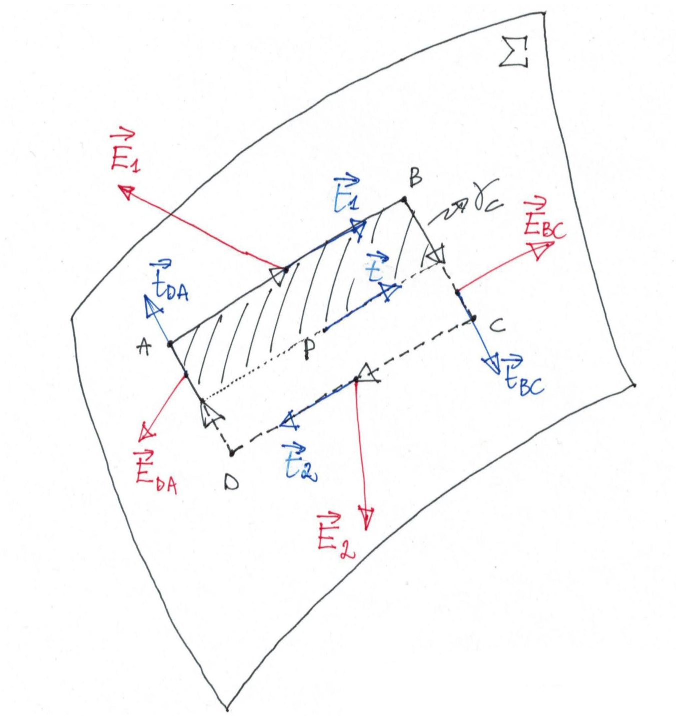


Figure 1.33

– *Gauss' theorem.*

Σ represents the boundary between Ω_1 and Ω_2. We can thus use the same coin-type curve to determine the integral (1.70). This reads:

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n} dS = \iint_{S_1} \vec{E}_1 \cdot \vec{n}_1 dS + \iint_{S_2} \vec{E}_2 \cdot \vec{n} dS + \iint_{S_\ell} \vec{E}_\ell \cdot \vec{n}_\ell dS = \iiint_{\Omega_c} dV \rho \quad . \tag{1.85}$$

In the limit of small h, ρ can be assumed to be constant in the region of Ω_1 contained in Ω_c. Hence,

$$\iiint_{\Omega_c} dV \rho = \rho \cdot \pi dr^2 \cdot \frac{1}{2} h \quad .$$

Under theses conditions

$$\lim_{h \to 0} \iiint_{\Omega_c} dV \rho = \lim_{h \to 0} \frac{1}{2} \rho \pi dr^2 \cdot h = 0 \quad . \tag{1.86}$$

Note that, as for the case of components $\vec{E}_{BC}$ and $\vec{E}_{DA}$ in Fig. 1.33, we would have obtained the same result even without assuming ρ to be constant. In fact, as soon as the volume of Ω_c goes to zero, there should be zero charge due to a volume charge distribution ρ. This was not the case for a surface charge distribution because of the presence of the δ-Dirac in the integral (1.80). Due to condition (1.86), in the limit for h going to zero, integral (1.85) reads:

$$\vec{n} \cdot (\vec{E}_1 - \vec{E}_2) = 0 \quad . \tag{1.87}$$

In case 3, the normal components of $\vec{E}$ on each side of Σ (note that Σ is smooth because Ω_1 and Ω_2 are both compact) are continuous:

$$E_{n_1}(P) = E_{n_2}(P) \quad .$$

The condition of Eq. (1.87), however, does not provide any knowledge on the behaviour of the derivatives of $\vec{E}$ on each side of Σ, infinitesimally close to it. We therefore recommend to use a cut-type curve to obtain the local form for the irrotational property of $\vec{E}$.

– *Irrotational property of $\vec{E}$.*

  By resorting to the cut-type curve of Fig. 1.33 for case 3, we obtain the very same result of condition (1.78):

$$\vec{t} \cdot (\vec{E}_1 - \vec{E}_2) = 0 \quad . \tag{1.88}$$

Case 4

Consider a continuous and limited linear charge distribution λ on a smooth line γ in the 3D Eucledian space. Consider a point P_0 on γ with distance r between P_0 and any point P in the 3D space. When P is very close to P_0, the line γ appears to be a straight line. Hence, the distance d can be approximated by the radial distance r between P and P_0.
Without entering in details.

– *Gauss' theorem.*

  It can be shown that

$$\lim_{r\to 0} E_n = \frac{\lambda}{2\pi\epsilon_0 r} \quad , \tag{1.89}$$

  where E_n is the magnitude of the normal component of $\vec{E}$ at P_0 on γ, $\vec{E}_n = E_n\vec{u}_r = E_n\vec{n}$. This result should not surprise as for P very close to $\vec{P}_0$ (eventually infinitesimally close), γ appears as an infinite straight line. Condition (1.89) implies that the normal component of $\vec{E}$ at any point on γ diverges to infinity asymptotically as $1/r$.

– *Irrotational property of $\vec{E}$.*

  It can be shown that

$$\lim_{r\to 0} E_t = \frac{\lambda'}{2\pi\epsilon_0} \ln r \quad , \tag{1.90}$$

where λ' is the derivation of λ along γ and E_t is the magnitude of the tangent component of $\vec{E}$ at P_0 on γ, $\vec{E}_t = E_t\vec{t}$. This result does not contradict the fact that for an infinite straight line the component $\vec{E}_z = \vec{E}_t = \vec{0}$. In fact, in the case of an infinite straight line $\lambda' = 0$. More in general, from conditions (1.89) and (1.90) it follows that

$$\lim_{P \to P_0} \frac{E_t}{E_n} = 0 \tag{1.91}$$

because E_n goes to infinity faster ($\sim 1/r$) than $E_t(\sim \ln r)$. This means that in very close proximity (in principle, infinitesimally close) to any line γ with λ, the electric field $\vec{E}$ is perpendicular to γ.

Case 5

Consider an isolated point-like charge. In this case the electric fields diverges as $1/r^2$ in very close proximity of the charge, as expected from Coulomb's law.

– All other cases must be considered individually.
  In summary, for cases 1-5 the local form of Gauss' theorem and of the irrotational property of $\vec{E}$ are not differential forms. These special local forms represents boundary conditions for $\vec{E}$. For cases 2 and 3 the boundary conditions are continuity or discontinuity conditions of the first kind. For cases 4 and 5 they are asymptotic conditions.

1.17 The Divergence Theorem in Detail

The physical origin of the result Eq. (1.71), i.e., Gauss theorem in differential form, can be better appreciated by resorting to infinitesimal boxes.

Consider a cube (or box) Ω_{box} with infinitesimal dimensions in the 3D Euclidean space. With respect to a Cartesian coordinate system $Oxyz$, the corner of Ω_{box} closest to O is a generic point $P(x,y,z)$ and the edges are parallel to the axes of the coordinate system. The length of the box is dx along the x direction, dy along the y direction, and dz along the z direction, where all lengths are assumed to be equal. Such an infinitesimal cube is shown in Fig. 1.34 (note that any infinitesimal rectangular parallelepiped would work as well). The infinitesimal volume of Ω_{box} is thus

$$dV = dxdydz \quad . \tag{1.92}$$

The face of Ω_{box} parallel to the yz plane and containing $P(x,y,z)$ is called A_1 in the figure. Similarly, the face parallel to the yz plane and containing $P(x+dx,y,z)$ is called A_2.

We assume Ω_{box} to be part of a larger region Ω characterized by a volume charge distribution with continuous and limited density ρ.

The electric field at P is

$$\vec{E} = E_x(x,y,z)\vec{u}_x + E_y(x,y,z)\vec{u}_y + E_z(x,y,z)\vec{u}_z \quad , \tag{1.93}$$

where E_x, E_y, and E_z are three generic functions of the coordinates x, y, and z.

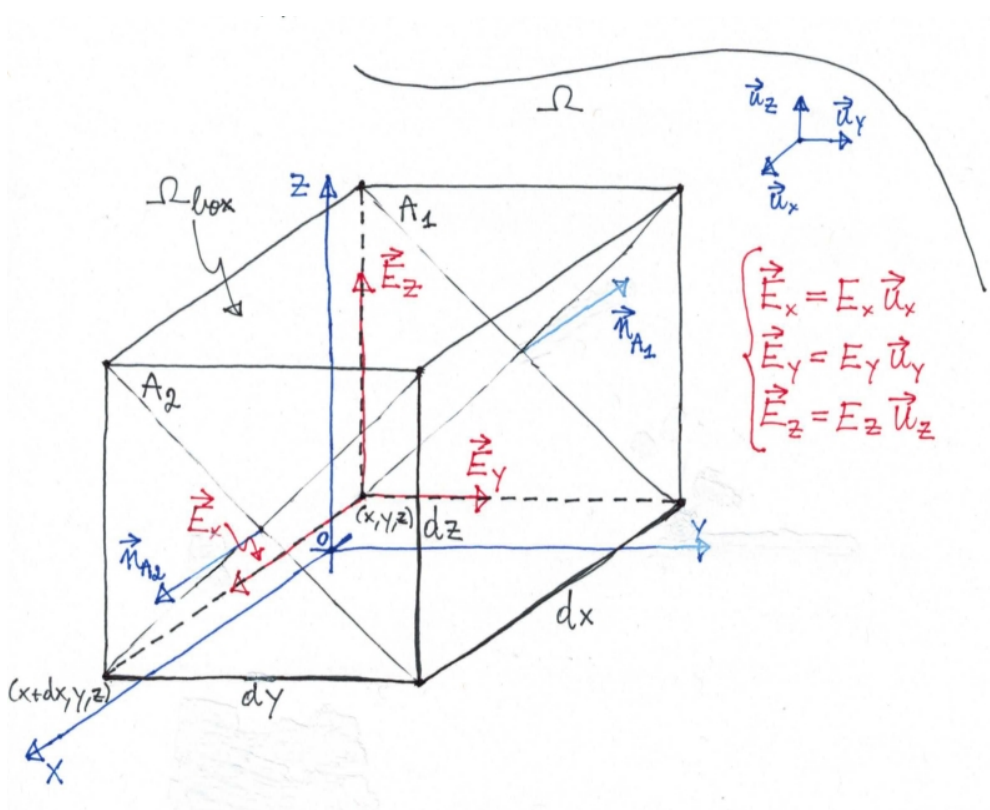


Figure 1.34

The flux of $\vec{E}$ through A_1 is given by

$$\Phi_{A_1} = \iint_{A_1} \vec{E} \bullet \vec{n}_{A_1} dS \quad , \tag{1.94}$$

where $\vec{n}_{A_1} = -\vec{u}_x$ and $dS = dydz$. Due to the infinitesimal dimensions of A_1, we can assume $\vec{E}$ to be constant on A_1 and equal to $\vec{E}$ at $P(x, y, z)$. Hence, the integral (1.94) can be written as

$$\begin{aligned} \Phi_{A_1} &= \big(E_x^{A_1}\vec{u}_x + E_y^{A_1}\vec{u}_y + E_z^{A_1}\vec{u}_z\big) \bullet \big(-\vec{u}_x\big) \cdot dydz \\ &= -E_x^{A_1}\big(x, y, z\big) \cdot dydz \quad . \end{aligned} \tag{1.95}$$

Similarly, the flux of $\vec{E}$ through A_2 is given by

$$\begin{aligned} \Phi_{A_2} &= \iint_{A_2} \vec{E} \bullet \vec{n}_{A_2} dS \\ &= \big(E_x^{A_1}\vec{u}_x + E_y^{A_1}\vec{u}_y + E_z^{A_1}\vec{u}_z\big) \bullet \vec{u}_x \cdot dydz \\ &= E_x^{A_1}\big(x, y, z\big) \cdot dydz \quad , \end{aligned} \tag{1.96}$$

where $\vec{n}_{A_2} = \vec{u}_x$ and we have assumed $\vec{E}$ to be constant on A_2 and equal to $\vec{E}$ at $P(x, y, z)$.

From integrals (1.95) and (1.96) it follows that the total flux of $\vec{E}$ through A_1 and A_2 is

$$\begin{aligned} \Phi^{A_1} + \Phi^{A_2} &= -E_x^{A_1} \cdot dydz + E_x^{A_1} \cdot dydz \\ &= 0 \quad . \end{aligned}$$

This is because we have assumed $\vec{E}$ to be equal on A_1 and A_2. However, the components E_x at $P(x, y, z)$ and $P(x + dx, y, z)$ are, in general, slightly different. In fact, we can expand E_x in Taylor series around $P(x, y, z)$ and obtain

$$E_x(x + dx, y, z) \;=\; E_x(x, y, z) + \frac{\partial}{\partial x}\, E_x(x, y, z) \cdot dx + o\big(dx^2\big) \quad .$$

Due to the infinitesimal nature of dx, all terms of equal or higher order than dx^2, $o(dx^2)$, can safely be neglected because they are infinitesimals of higher order. For the same reason the derivative can be calculated in $P(x, y, z)$ instead of $P(x, y + dy/2, z + dz/2)$. Thus, E_x on A_2, which is constant on A_2, is given by

$$E_x^{A_2} = E_x^{A_1}\vec{u}_x + \frac{\partial}{\partial x}\, E_x^{A_1} \cdot dx \cdot \vec{u}_x \quad . \tag{1.97}$$

Under this new approximation, the total flux of $\vec{E}$ through A_1 and A_2 is thus

$$\begin{aligned} \Phi^{A_1} + \Phi^{A_2} &= -\cancel{E_x^{A_1}} \cdot dydz + \cancel{E_x^{A_1}} \cdot dydz + \frac{\partial}{\partial x}\, E_x^{A_1} \cdot dx \cdot dydz \\ &= \frac{\partial}{\partial x}\, E_x^{A_1} \cdot dV = \frac{\partial}{\partial x}\, E_x\big(x, y, z\big) \cdot dV \quad . \end{aligned} \tag{1.98}$$

Similarly, we can calculate the total flux of $\vec{E}$ through A_3 and A_4 (see Fig. 1.35). In this case, we assume E_y (the only component of $\vec{E}$ to survive the scalar product in the flux calculation) to be constant on A_3 and equal to $\vec{E}$ at $P(x, y, z)$,

$$\vec{E}_y^{A_3} = E_y\big(x, y, z\big)\vec{u}_y = E_y^{A_3}\vec{u}_y \quad .$$

On A_4, we expand $E_y^{A_3}$ in Taylor series and obtain

$$\vec{E}_y^{A_4} = E_y^{A_3}\vec{u}_y + \frac{\partial}{\partial y} E_y^{A_3} \cdot dy\vec{u}_y \quad , \tag{1.99}$$

which we assume to be constant on A_4.

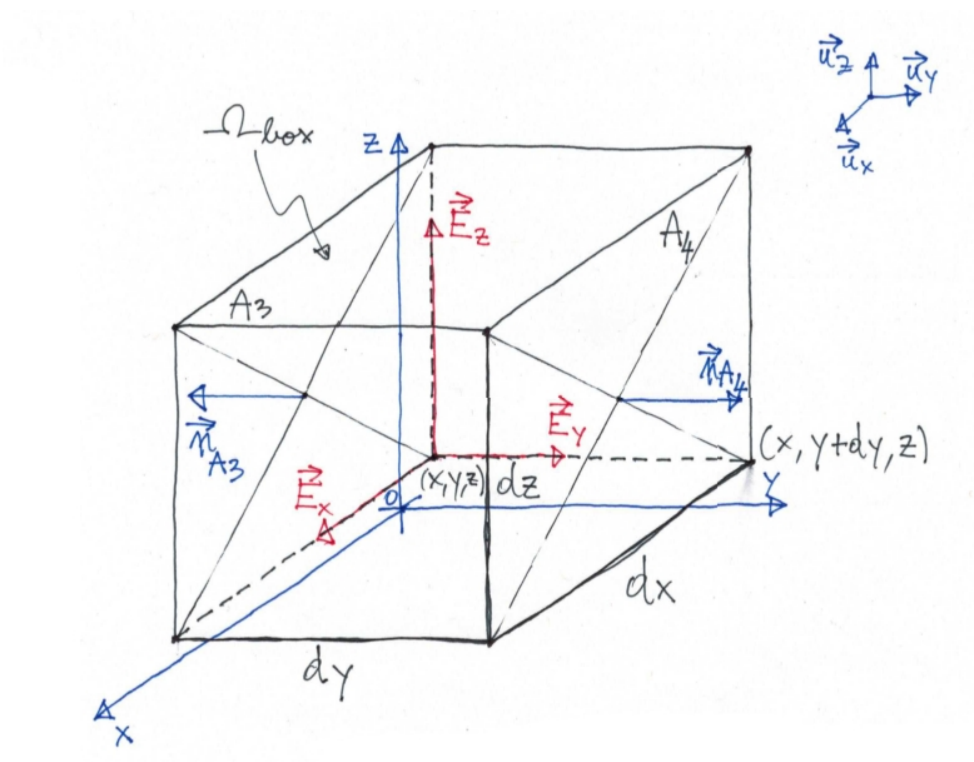


Figure 1.35

Following the notation in Fig. 1.35, $\vec{n}_{A_3} = -\vec{u}_y$ and $\vec{n}_{A_4} = \vec{u}_y$. The total flux through A_3 and A_4 is thus

$$\begin{aligned}
\Phi^{A_3} + \Phi^{A_4} &= \iint\limits_{A_3} \vec{E} \cdot \vec{n}_{A_3} dS + \iint\limits_{A_4} \vec{E} \cdot \vec{n}_{A_4} dS \\
&= -\cancel{E_y^{A_3}} \cdot dxdz + \cancel{E_y^{A_3}} \cdot dxdz + \frac{\partial}{\partial y} E_y^{A_3} \cdot dy \cdot dxdz \\
&= \frac{\partial}{\partial y} E_y^{A_3} \cdot dV = \frac{\partial}{\partial y} E_y\big(x, y, z\big) \cdot dV \quad .
\end{aligned} \tag{1.100}$$

Without entering in details, the total flux of $\vec{E}$ through the remaining two faces of Ω_{box}, A_5 and A_6 (parallel to the xy plane) is

$$\Phi^{A_5} + \Phi^{A_6} = \frac{\partial}{\partial z} E_z\big(x, y, z\big) \cdot dV \quad . \tag{1.101}$$

According to Gauss' theorem in integral form, calling Σ the outer surface of Ω_{box}, we can conclude that the total flux of $\vec{E}$ through Σ is

$$\begin{aligned}
\Phi^{\Sigma} &= \oiint_{\Sigma} \vec{E} \bullet \vec{n} dS \\
&= \big(\Phi^{A_1} + \Phi^{A_2}\big) + \big(\Phi^{A_3} + \Phi^{A_4}\big) + \big(\Phi^{A_5} + \Phi^{A_6}\big) \\
&= \left[\frac{\partial}{\partial x} E_x\big(x, y, z\big) + \frac{\partial}{\partial y} E_y\big(x, y, z\big) + \frac{\partial}{\partial z} E_z\big(x, y, z\big)\right] \cdot dV \\
&= (\vec{\nabla} \bullet \vec{E}) \cdot dV \\
&= \frac{1}{\epsilon_0} \iiint_{\Omega_{\text{box}}} \rho \cdot dV = \frac{1}{\epsilon_0} \rho dV \quad ,
\end{aligned}$$

where we have assumed ρ to be constant in Ω_{box} because of the infinitesimal dimensions of Ω_{box}. Hence,

$$\vec{\nabla} \bullet \vec{E} = \frac{1}{\epsilon_0} \rho \quad , \tag{1.102}$$

which is Gauss' theorem in differential form. By means of the infinitesimal box Ω_{box} we have thus rigorously proven Gauss' theorem in differential form.

1.18 Electrostatic Field $\vec{E}$ Generated by Point-Like Charges

(1) Consider two point-like charges q_1 and q_2. The first charge is located at point $Q = Q_1$ and the second at point $Q = Q_2$. The distance $\overline{Q_1Q_2} = d$.

Calculate $\vec{E}$ at point $P = O$ at the centre of line segments Q_1Q_2 and at any point $P = O'$ on the axis of Q_1Q_2 passing through O (see Fig. 1.36).

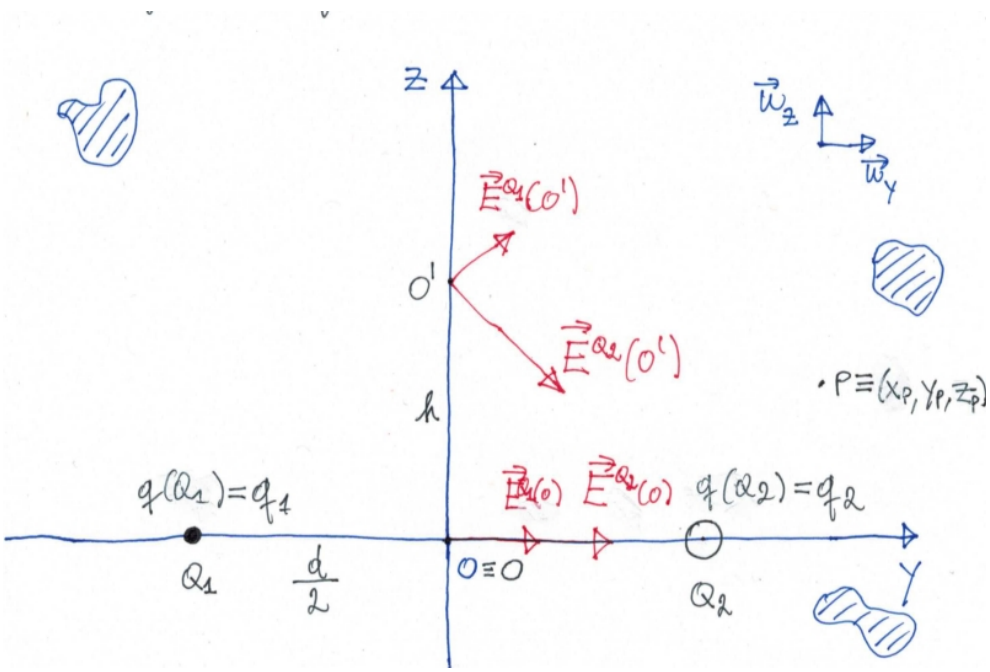


Figure 1.36

(2) *Reference frame and coordinate system:*

The two charges are considered with respect to the inertial reference frame (hatched blue region) shown in Fig. 1.36. The two charges belong to a single plane, where their position can be represented with respect to the Cartesian coordinates system Oyz shown in Fig. 1.36. The x axis, perpendicular to the plane of the two charges, is not indicated. The origin O has been chosen to coincide with the middle point of Q_1Q_2. Hence, O' is on the z axis. The coordinates of the various points are:

$$\begin{cases} Q_1 \equiv (0\ ,\ -\dfrac{d}{2}\ ,\ 0) \\ O \equiv (0\ ,\ 0\ ,\ 0) \\ Q_2 \equiv (0\ ,\ +\dfrac{d}{2}\ ,\ 0) \\ O' \equiv (0\ ,\ 0\ ,\ +h) \end{cases} . \tag{1.103}$$

The field $\vec{E}$ at any point P can be written in components as

$$\vec{E}(P) = E_x(P)\vec{u}_x + E_y(P)\vec{u}_y + E_z(P)\vec{u}_z \quad .$$

(3) *Indicate all fields $\vec{E}$:*

A point-like charge q at point Q, $q(Q)$, generates at any point P a field according to Coulomb's law:

$$\vec{E}^Q(P) = \frac{1}{4\pi\epsilon_0}\,\frac{q(Q)}{r_{PQ}^2}\,\vec{u}_{QP} \quad , \tag{1.104}$$

where r_{PQ} is the distance between Q and P and $\vec{u}_{QP}$ the corresponding unit vector oriented from Q to P. The fields generated according to Eq. (1.104) by the charges q_1 and q_2 have been indicated in Fig. 1.36, both at O and O'.

(4) *DOF:*

Due to Eq. (1.104), the total field at O and O' must be on the yz plane and, thus, cannot have an x component. In this case, the DOF are the y and z components of $\vec{E}$.

(5) Results:

- Point O.
The field generated by $q(Q_1)$ at O, $\vec{E}^{Q_1}(O)$, can be calculated by writing Eq. (1.104) for Q_1 and O in Cartesian coordinates:

$$\begin{aligned} \vec{E}^{Q_1}(O) &= E_x^{Q_1}(O)\vec{u}_x + E_y^{Q_1}(O)\vec{u}_y + E_z^{Q_1}(O)\vec{u}_z \\ &= \frac{q_1}{4\pi\epsilon_0}\left(\frac{x_O - x_{Q_1}}{[(x_O - x_{Q_1})^2 + (y_O - y_{Q_1})^2 + (z_O - z_{Q_1})^2]^{3/2}}\,\vec{u}_x\right. \\ &\quad + \frac{y_O - y_{Q_1}}{[(x_O - x_{Q_1})^2 + (y_O - y_{Q_1})^2 + (z_O - z_{Q_1})^2]^{3/2}}\,\vec{u}_y \\ &\quad \left. + \frac{z_O - z_{Q_1}}{[(x_O - x_{Q_1})^2 + (y_O - y_{Q_1})^2 + (z_O - z_{Q_1})^2]^{3/2}}\,\vec{u}_z\right) \quad . \end{aligned} \tag{1.105}$$

Inserting conditions (1.103) in Eq. (1.105) we obtain

$$\begin{aligned} \vec{E}^{Q_1}(O) &= E_y^{Q_1}(O)\vec{u}_y = \frac{q_1}{4\pi\epsilon_0}\,\frac{\dfrac{d}{2}}{\left(\dfrac{d}{2}\right)^3}\,\vec{u}_y \\ &= \frac{q_1}{\pi\epsilon_0}\,\frac{1}{d^2}\,\vec{u}_y \quad . \end{aligned} \tag{1.106}$$

Similarly,

$$\begin{aligned}\vec{E}^{Q_2}(O) &= E_y^{Q_2}(O)\vec{u}_y = \frac{q_2}{4\pi\epsilon_0}\,\frac{y_O - y_{Q_2}}{\left[(y_O - y_{Q_2})^2\right]^{3/2}}\,\vec{u}_y \\ &= -\,\frac{q_2}{\pi\epsilon_0}\,\frac{1}{d^2}\,\vec{u}_y \quad . \end{aligned} \tag{1.107}$$

By means of the superposition principle we can thus calculate the total field at O due to both q_1 and q_2:

$$\begin{aligned}\vec{E}(O) &= \vec{E}^{Q_1}(O) + \vec{E}^{Q_2}(O) \\ &= \frac{q_1 - q_2}{\pi\epsilon_0}\,\frac{1}{d^2}\,\vec{u}_y \quad . \end{aligned} \tag{1.108}$$

As expected from Coulomb's law, the field in O due to q_1 and q_2 has only a y component.

- Point O'.
The field generated by $q(Q_1)$ at O', $\vec{E}^{Q_1}(O')$, can be calculated by writing Eq. (1.105) for Q_1 and O':

$$\vec{E}^{Q_1}(O') = \frac{q_1}{4\pi\epsilon_0}\left(\frac{\dfrac{d}{2}}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_y + \frac{h}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_z\right) \quad . \tag{1.109}$$

Similarly,

$$\vec{E}^{Q_2}(O') = \frac{q_2}{4\pi\epsilon_0}\left(\frac{-\dfrac{d}{2}}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_y + \frac{h}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_z\right) \quad . \tag{1.110}$$

As before, by means of the superposition principle we can calculate the total field at O' due to both q_1 and q_2:

$$\begin{aligned}\vec{E}(O') &= \vec{E}^{Q_1}(O') + \vec{E}^{Q_2}(O') \\ &= \frac{1}{4\pi\epsilon_0}\,\frac{q_1 - q_2}{2}\,\frac{d}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_y \\ &\quad + \frac{1}{4\pi\epsilon_0}\,(q_1 + q_2)\,\frac{h}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_z \quad . \end{aligned} \tag{1.111}$$

Imposing $h = 0$ in Eq. (1.111) we obtain again the result of Eq. (1.108), as expected.

- Any point arbitrary close to Q_1 and Q_2.
For point Q_1 the total field is:

$$\vec{E}(P \sim Q_1) \simeq \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r^2}\,\vec{u}_{r_1} - \frac{q_2}{d^2}\,\vec{u}_y\right) \quad ,$$

where r is an arbitrary small distance between Q_1 and P in the radial direction, $\vec{u}_{r_1}$ is the unit vector in the radial direction at Q_1, and the distance between Q_1 and Q_2 has been approximated to be d (see Fig. 1.37).

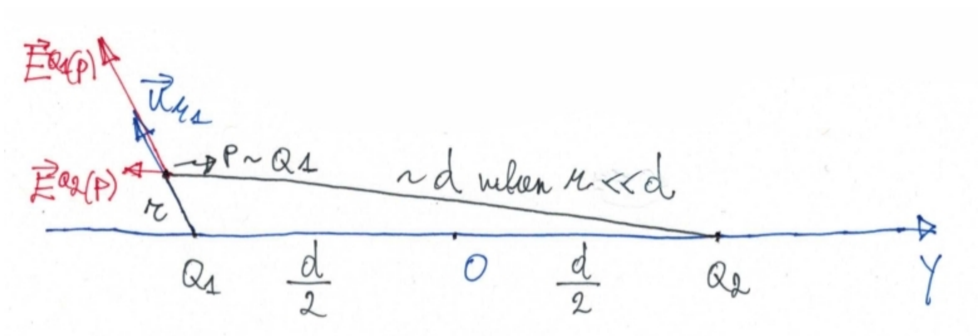


Figure 1.37

When $r << d$, the contribution to the field from Q_2 can be neglected:

$$\vec{E}(P \sim Q_1) \simeq \frac{1}{4\pi\epsilon_0}\,\frac{q_1}{r^2}\,\vec{u}_{r_1} \quad . \tag{1.112}$$

Similarly,

$$\vec{E}(P \sim Q_2) \simeq \frac{1}{4\pi\epsilon_0}\,\frac{q_2}{r^2}\,\vec{u}_{r_2} \quad . \tag{1.113}$$

where $\vec{u}_{r_2}$ is the unit vector in the radial direction at Q_2.

- Any point P on the y axis, $P \equiv (O, y_P, O)$.

$$\vec{E}(O, y_P, O) = \frac{1}{4\pi\epsilon_0}\left(q_1\,\frac{(y_P - y_{Q_1})}{|y_P - y_{Q_1}|^3} + q_2\,\frac{(y_P - y_{Q_2})}{|y_P - y_{Q_2}|^3}\right)\vec{u}_y \quad . \tag{1.114}$$

Case $q_1 = +q$ and $q_2 = -q$ (balanced charges).
From Eqs.(1.108), (1.111), (1.112), (1.113), and (1.114):

$$
\begin{cases}
\vec{E}(O) & = \dfrac{2q}{\pi\epsilon_0}\,\dfrac{1}{d^2}\,\vec{u}_y \\
\vec{E}(O') & = \dfrac{q}{4\pi\epsilon_0}\,\dfrac{d}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_y \\
\vec{E}(\sim Q_1) & \simeq \dfrac{q}{4\pi\epsilon_0}\,\dfrac{1}{r^2}\,\vec{u}_{r_1} \\
\vec{E}(\sim Q_2) & \simeq -\dfrac{q}{4\pi\epsilon_0}\,\dfrac{1}{r^2}\,\vec{u}_{r_2} \\
\vec{E}(O, y_P, O) & = \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{(y_P - y_{Q_1})}{|y_P - y_{Q_1}|^3} + \dfrac{(y_{Q_2} - y_P)}{|y_P - y_{Q_2}|^3}\right)\vec{u}_y
\end{cases} . \tag{1.115}
$$

The knowledge of Eqs. (1.115) allow us to draw the $\vec{E}$ field lines for case $q_1 = +q$ and $q_2 = -q$ (see Fig.1.38).

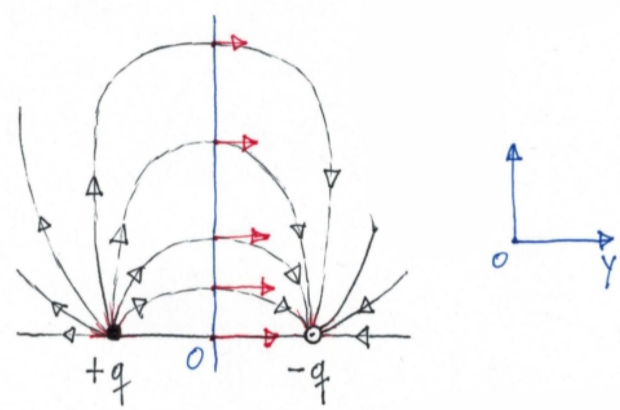


Figure 1.38

The $\vec{E}$ field lines in Fig. 1.38 are symmetric with respect to the z axis. They are also symmetric with respect to the y axis. This is why we decided not to draw them on the negative side of z.

The knowledge of $\vec{E}$ at O, any point O', and in the close vicinity of Q_1 and Q_2 (indicated in red in the figure) gives information on the tangent behaviour of the $\vec{E}$ field lines at those points. It thus helps draw the $\vec{E}$ field lines everywhere on the yz plane, under the reasonable assumption $\vec{E}$ to be smooth in the regions connecting the various points where we have calculated $\vec{E}$.

It is useful to consider $\vec{E}(O, y_P, O)$ when both $|y_P - y_{Q_1}|$, $|y_P - y_{Q_2}| >> d$. In this case, we can assume $|y_P - y_{Q_1}| \simeq |y_P - y_{Q_2}|$ and, similarly, $|y_P - y_{Q_1}| \simeq y_P$ and $|y_{Q_2} - y_P| \simeq -y_P$. Hence, Eq. (1.114) for $q_1 = +q$ and $q_2 = -q$ reads

$$
\begin{aligned}
\vec{E}(O, y_P, O) &\simeq \frac{q}{4\pi\epsilon_0}\left(\frac{y_P}{|y_P|^3} - \frac{y_P}{|y_P|^3}\right)\vec{u}_y \\
&= \vec{0} \quad .
\end{aligned}
$$

On the y axes, the far field on the positive and negative side of O is zero. The two charges appear as a single charge $Q = q - q = 0$. While it is in general true that the field at infinity for any charge distribution has to go to zero, it is not true that the far field (far, but not infinity) even for a charge distribution of total zero charge (neutral charge distribution, as in our case) has to be zero everywhere in space. In our case, it is true that the far field goes to zero in the y axis.

Case $q_1 = +q$ and $q_2 = -2q$ (unbalanced charges).
From Eqs. (1.108), (1.111), (1.112), 1.113) and (1.114):

$$
\begin{cases}
\vec{E}(O) &= \dfrac{3q}{\pi\epsilon_0}\,\dfrac{1}{d^2}\,\vec{u}_y \\[2ex]
\vec{E}(O') &= \dfrac{3q}{8\pi\epsilon_0}\,\dfrac{d}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_y - \dfrac{q}{4\pi\epsilon_0}\,\dfrac{h}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_z \\[2ex]
\vec{E}(\sim Q_1) &\simeq \dfrac{q}{4\pi\epsilon_0}\,\dfrac{1}{r^2}\,\vec{u}_{r_1} \\[2ex]
\vec{E}(\sim Q_2) &\simeq \dfrac{-2q}{4\pi\epsilon_0}\,\dfrac{1}{r^2}\,\vec{u}_{r_2} \\[2ex]
\vec{E}(O, y_P, O) &= \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{(y_P - y_{Q_1})}{|y_P - y_{Q_1}|} + 2\,\dfrac{(y_{Q_2} - y_P)}{|y_P - y_{Q_2}|^3}\right)\vec{u}_y
\end{cases}
\tag{1.116}
$$

The knowledge of Eq. (1.116) allow us to draw the $\vec{E}$ field lines for case $q_1 = +q$ and $q_2 = -2q$ (see Fig. 1.39).

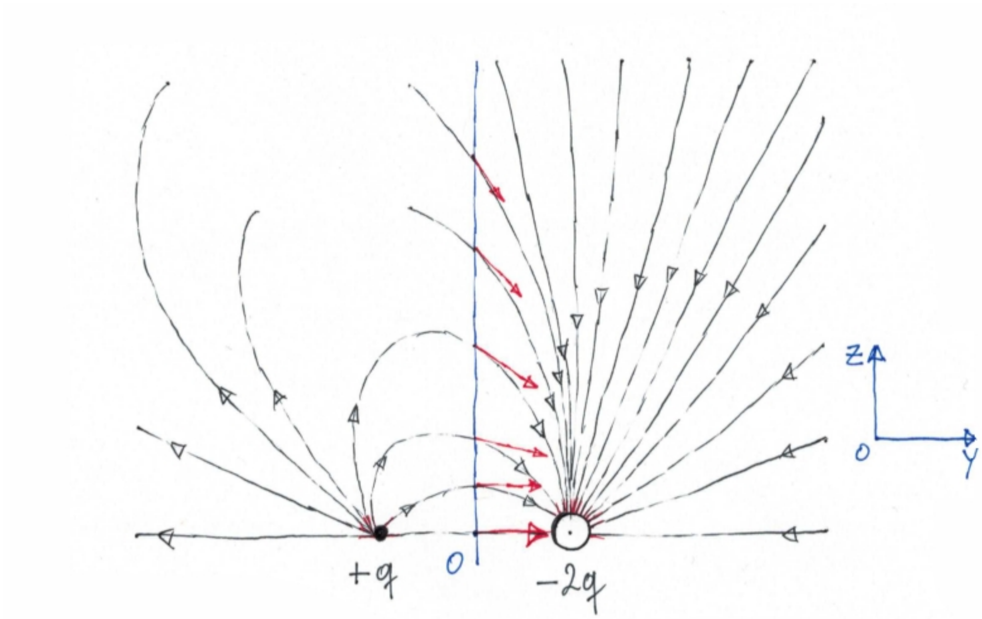


Figure 1.39

While again the $\vec{E}$ field lines in Fig. 1.39 are symmetric with respect to the y axis, they are asymmetric with respect to the z axis. This is expected from the unbalance of the charges.
We note that the y and z components of $\vec{E}(O')$ are proportional, by the same scaling factor, to

$$\begin{cases} \vec{E}_y(O') & \sim & \dfrac{3}{2}\, d\vec{u}_y \\ \vec{E}_z(O') & \sim & -h\vec{u}_z \end{cases}$$

The magnitude of these two components is the same when $h = \dfrac{3}{2}d$. This is the fifth red vector from the bottom on the z axis in Fig. 1.39. In this case, the vector displays an angle of 45° with respect to z. The angle is smaller further from 0 and larger closer too. In particular, the far field on z for $h >> d$ is given by

$$\begin{aligned} \vec{E}_{\mathrm{far}}(O') & \simeq & \frac{q}{4\pi\epsilon_0}\,\frac{3}{2}\,\frac{d}{h^3}\,\vec{u}_y \\ & & -\frac{q}{4\pi\epsilon_0}\,\frac{1}{h^2}\,\vec{u}_z \quad . \end{aligned}$$

This means the y component of the far field is much smaller than the z component. In other words, the far field tends to point downwards to the two charges, along z. This should not surprise because far enough from the two charges, they appear as a single charge $Q = q - 2q = -q$, which attracts field lines to it. Note that the y

component of the far field ($\sim 1/h^3$) does not go to zero, it is just smaller than the z component. Of course, the field at infinity ($h \to \infty$) goes to zero.
In this case, the far field on y is given by

$$
\begin{aligned}
\vec{E}(0, y_p, 0) &\simeq \frac{q}{4\pi\epsilon_0}\left(\frac{y_P}{|y_P|^3} - 2\,\frac{y_P}{|y_P|^3}\right)\vec{u}_y \\
&= -\frac{q}{4\pi\epsilon_0}\,\frac{y_P}{|y_P|^3}\,\vec{u}_y \quad .
\end{aligned}
$$

As expected the two charges behave as a single charge $Q = +q - 2q = -q$.

Case $q_1 = +q$ and $q_2 = +q$ (equal charges).
From Eqs. (1.108), (1.111), (1.112), (1.113), and (1.114):

$$
\begin{cases}
\vec{E}(O) &= \vec{0} \\
\vec{E}(O') &= \dfrac{2q}{4\pi\epsilon_0}\,\dfrac{h}{\left[\left(\dfrac{d}{2}\right)^2 + h^2\right]^{3/2}}\,\vec{u}_z \\
\vec{E}(\sim Q_1) &\simeq \dfrac{q}{4\pi\epsilon_0}\,\dfrac{1}{r^2}\,\vec{u}_{r_1} \\
\vec{E}(\sim Q_2) &\simeq \dfrac{q}{4\pi\epsilon_0}\,\dfrac{1}{r^2}\,\vec{u}_{r_2} \\
\vec{E}(O, y_P, O) &= \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{(y_P - y_{Q_1})}{|y_P - y_{Q_1}|^3} + \dfrac{(y_P - y_{Q_2})}{|y_P - y_{Q_2}|^3}\right)\vec{u}_y
\end{cases} \quad . \tag{1.117}
$$

Figure 1.40 shows the $\vec{E}$ field lines in this case

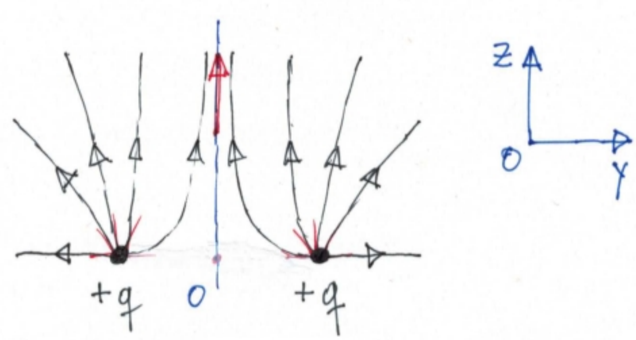


Figure 1.40

In this case, $\vec{E}(O')$ points always upwards along z and $\vec{E}(O, y_P, O)$ far from the charges is

$$\vec{E}_{\text{far}}(O, y_P, O) \simeq \frac{2q}{4\pi\epsilon_0}\frac{y_P}{|y_P|^3}\vec{u}_y \quad ,$$

which is the field of a single charge $Q = 2q$.

The **case for** $q_1 = -q$ **and** $q_2 = -q$ is analogous to the one just analyzed.
Note that the case of two charges q_1 and q_2 at a distance d is referred to as electric dipole. We will come back to the electric dipole extensively in the next few lectures.

1.19 Physics History Inverted: Coulomb's Theorem and the Superposition Property of $\vec{E}$ Derived from Gauss' Law and the Irrotational Principle of $\vec{E}$.

In this section, we will pretend to change the course of history by assuming that Johann Carl Friedrich Gauss, along with helper Michael Faraday, takes the place of Charles-Augustin de Coulomb and vice versa. This requires the first two gentlemen to perform a jump back in the past by approximately five decades and, correspondingly, Mr. Coulomb to go back to the future by the same time (see Fig. 1.41).

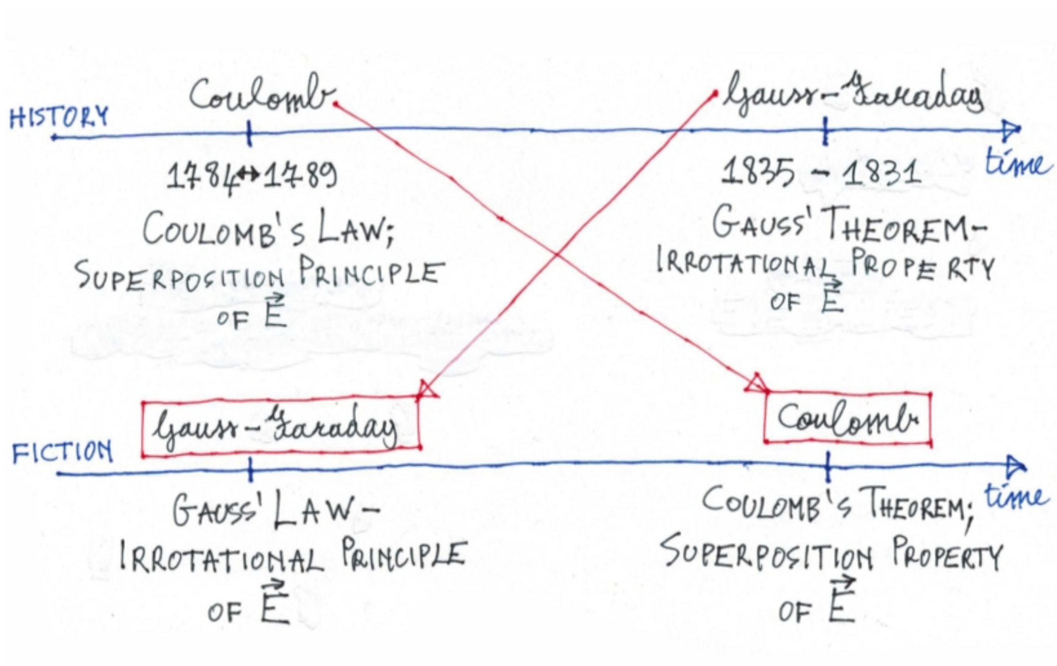


Figure 1.41

Before proceeding further with our fiction and its consequences on the way electrostatics would have been explained, it is worth giving a few definitions.

1.19.1 Laws vs. Theorems in Physics.

Laws. A physic's law is a (fundamental) principle that is demonstrated by means of empirical evidence.

- *Caveat 1.* Note that in theoretical physics a principle can be given without any experimental evidence. However, at a later time, such principles must be confirmed by experiments to be considered legitimate.

- *Caveat 2.* In the literature, the words *law* and *principle* are typically assumed to have the same meaning (e.g., the superposition "principle").

The concept of law in physics is analogous to that of axiom or postulate in mathematics, the main difference being the former is corroborated by experiments.

Theorems. A physics' theorem is a statement (or property) that can be mathematically demonstrated by means solely of physics' laws. Corollaries can be demonstrated using other physics' theorems.

– *Caveat 1.* In the literature, the words *theorem* and *property* are typically assumed to have the same meaning (e.g., Gauss' "theorem" or the irrotational "property" of $\vec{E}$).

1.19.2 Gauss and Faraday.

Assume that Gauss went back in time, replacing Coulomb. However, instead of performing the torsion balance experiment (i.e., Coulomb's historical experiment), Gauss decided to perform an experiment based on charged spheres (i.e., our fictional experiment), the apparatus of which is shown in Fig. 1.42.

The apparatus comprises an external metallic sphere Σ_{ext}, with radius $R_{\text{ext}} \simeq 1.5\,\text{m}$, containing a smaller metallic sphere Σ_{int}. Both spheres are centered in the same point O. Σ_{int} contains a device $\mathscr{E}$ able to reveal the passage of any even extremely small charge that goes through it. An image of the screen of $\mathscr{E}$ is deflected by means of a mirror B into the axis of a microscope M that, finally, allows an observer to monitor $\mathscr{E}$. A little hole through Σ_{int} and Σ_{ext} makes it possible to monitor $\mathscr{E}$ without significantly perturbing the experiment. Two thin conducting wires f_1 and f_2 connect Σ_{ext} to Σ_{int} through $\mathscr{E}$ (f_1 does not touch Σ_{int}). The observer applies a charge Q to Σ_{ext} while constantly monitoring $\mathscr{E}$. If a field $\vec{E} \neq \vec{0}$ was present inside Σ_{ext}, a fraction of the charge Q, q, would pass through $\mathscr{E}$ and, thus, would be detected by the observer. This is because a non-zero electrostatic field $\vec{E}$ would generate a force $\vec{F} = q\vec{E} = m_q\vec{a}$ (where m_q is the mass of q) that would move the charge from Σ_{ext} to Σ_{int}. However, repeating the experiment multiple times shows no charge passage through $\mathscr{E}$. As a consequence, the observer (in this case Gauss) can safely conclude that $\vec{E} = \vec{0}$ inside Σ_{ext}.

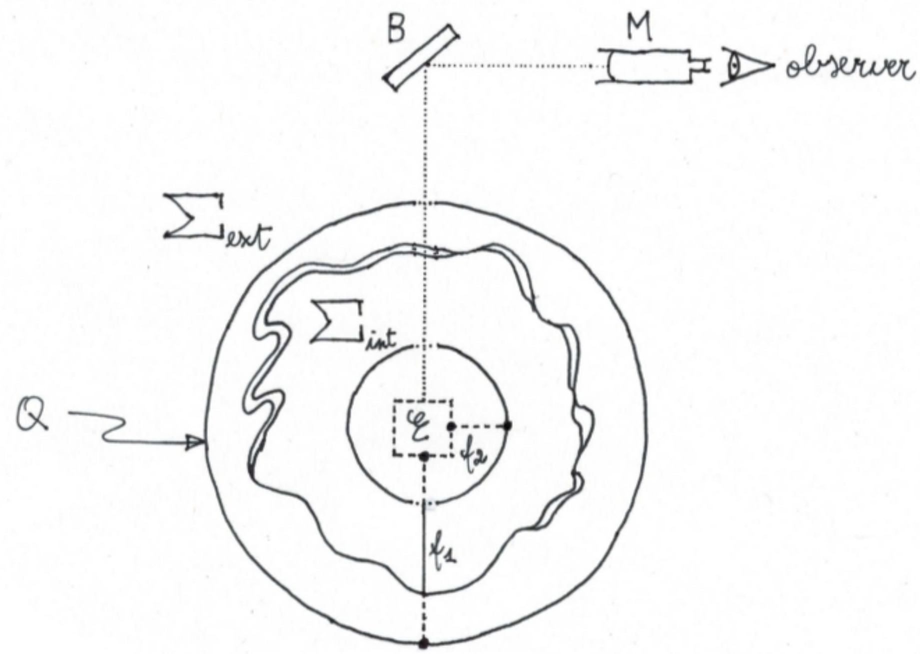


Figure 1.42

The apparatus of Fig. 1.42 can be simplified as shown in Fig. 1.43a. A metallic hollow sphere is separated from the ground by means of an insulating support. A charge Q, e.g., a positive charge, is applied to the sphere. A little metallic ball attached to an insulating handle is lowered through a small aperture into the sphere until it touches the interior wall of the sphere. If the little ball was initially discharged, after extracting it from the sphere it will remain discharged. This can easily be shown by connecting the ball to an electrometer: The pointer of the electrometer will indicate zero before and after lowering the ball into the sphere. Once again, this shows that $\vec{E} = \vec{0}$ inside a charged sphere.

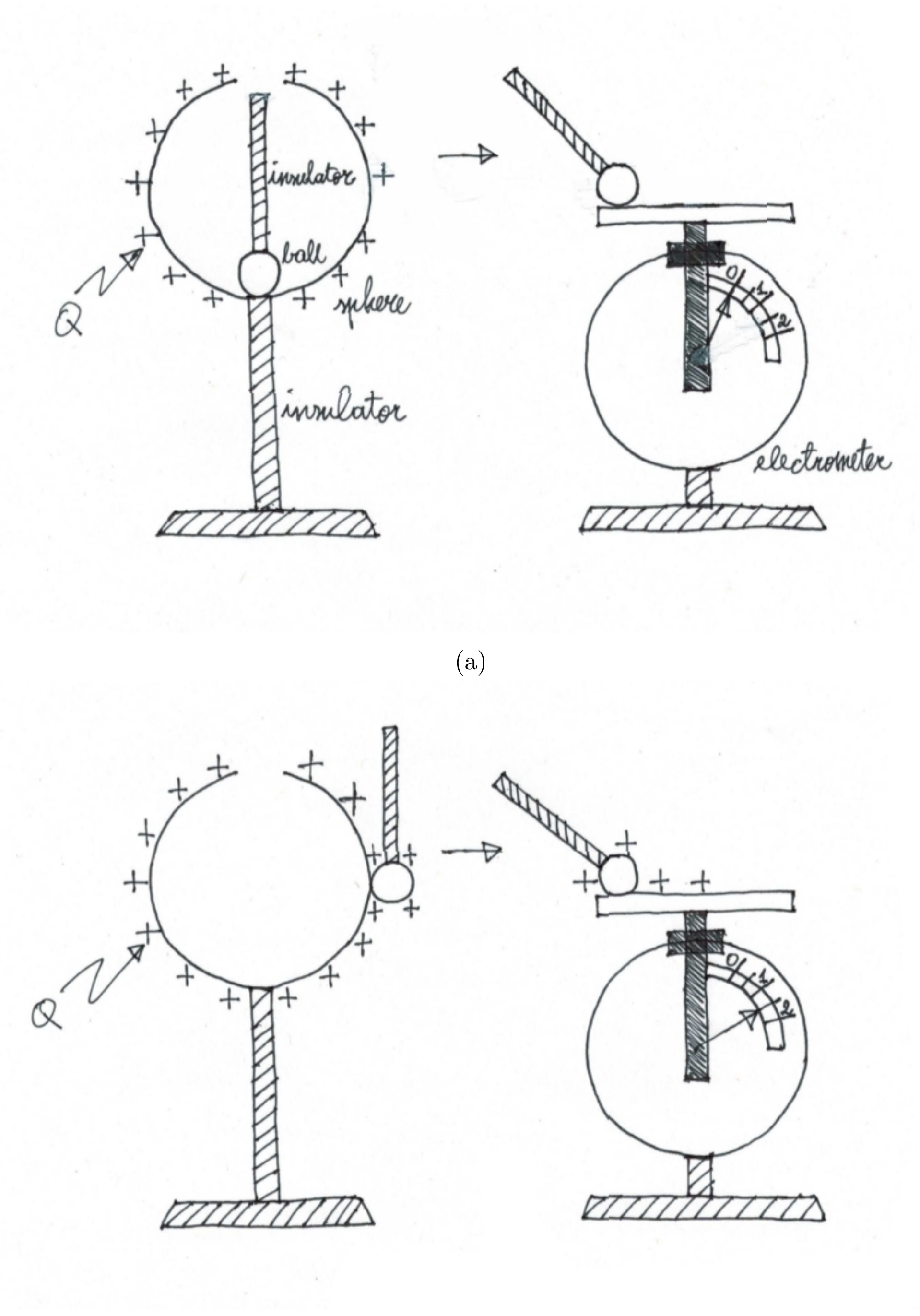
If, however, the ball touches the exterior of the sphere, connecting the ball afterwards to the electrometer will now show that the ball has been charged: The pointer of the electrometer will move from zero (see Fig. 1.43b). This new experiment shows that outside a charged sphere there is a non-zero electrostatic field $\vec{E}$, which moves a fraction of Q from the sphere to the ball. Note that we will study more in detail the behavior of metallic objects (e.g., spheres) in presence of an electrostatic field $\vec{E}$ in the lectures on conductors.

By performing experiments analogous to those described above, Gauss would have been eventually able to state that "the flux of the electrostatic field $\vec{E}$ through any arbitrary surface is proportional by $1/\epsilon_0$ to the total charge contained within the surface." In this form, Gauss' statement would have appeared to be a fundamental principle demonstrated by empirical evidence rather than a theorem, and, thus, we would have referred to it as "Gauss' law" instead of Gauss' theorem:

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n} dS = \frac{1}{\epsilon_0} Q \quad , \tag{1.118}$$

where, as always, Σ is an arbitrary surface containing a total charge Q and $\vec{n}dS$ an infinitesimal surface element on Σ oriented according to the unit vector $\vec{n}$, which is normal to dS and points outwards with respect to Σ.

– Faraday's help.
Assume that also Faraday went back in time, at the same time as Gauss and, at that time, performed a set of field-line experiments. In one such experiments, Faraday considered a charged sphere similar to that depicted in Fig. 1.43a. He then sprayed all around the sphere a charged powder that allowed him to reveal the field lines of the electrostatic field $\vec{E}$ generated by the charged sphere. He found the field lines to be directed

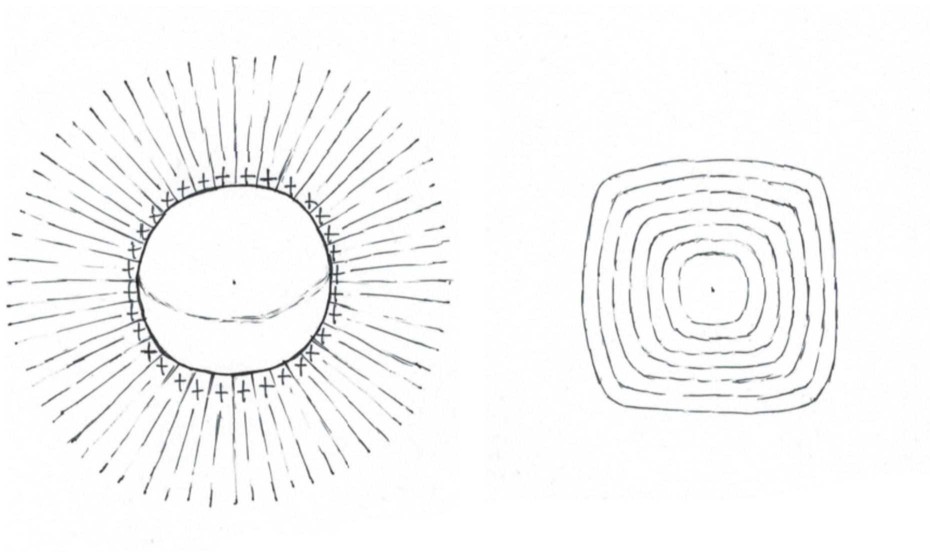


(a)

(b)

Figure 1.43

radially with respect to the centre of the sphere (see Fig. 1.44a). Faraday repeated many field-line experiments for different charge distributions, never obtaining a field-line configuration as that shown in Fig. 1.44b.



(a) (b)

Figure 1.44

From these findings, Faraday would have been able to deduce an important law of electrostatics, i.e., the irrotational principle of $\vec{E}$. With the help of a mathematician (it is well known that Faraday was a brilliant physicist, but not a strong mathematician), Faraday would have summarized his experimental results as:

$$\oint_{\gamma} \vec{E} \cdot \vec{t} dl = 0 \quad , \tag{1.119}$$

where γ is any arbitrary closed line oriented according to the unit vector $\vec{t}$ tangent to γ at each point on γ and $d\ell$ is the magnitude of an infinitesimal line element on γ. What in history happens to be a property (or theorem) that can be demonstrated from the characteristic of $\vec{E}$ to be a central field, in our fiction becomes a fundamental law demonstrated by empirical evidence.

By means of field-line-type experiments, Faraday would have also been able to show that, given a limited charge distribution in a finite region of space, the field lines of $\vec{E}$ become fainter and fainter the further from the charge. In other words, Faraday would have shown the zero-field principle of $\vec{E}$ at infinity.

1.19.3 Coulomb.

Following our fictional time lime, a few decades after the experimental findings of Gauss and Faraday, Coulomb came along. Instead of performing experiments and deducing laws, in our fiction Coulomb demonstrated theorems.

With empirical laws (1.118) and (1.119) in hand, Coulomb set out to find the electrostatic field $\vec{E}$ generated by a (point-like) charge q in the 3D Euclidean space.

Figure 1.45 shows the reference frame and associated coordinate system chosen by Coulomb to solve this problem. The coordinate system is a spherical system $Or\theta\varphi$ and the reference frame (not shown) is assumed to be inertial.

The figure also shows the vector components of $\vec{E}$ at a generic point $P(r, \theta, \varphi)$, $\vec{E}_r$, $\vec{E}_\theta$, and $\vec{E}_\varphi$.

Since Coulomb a priori has no knowledge on $\vec{E}$, he assumes three DOF: r, θ, and φ.

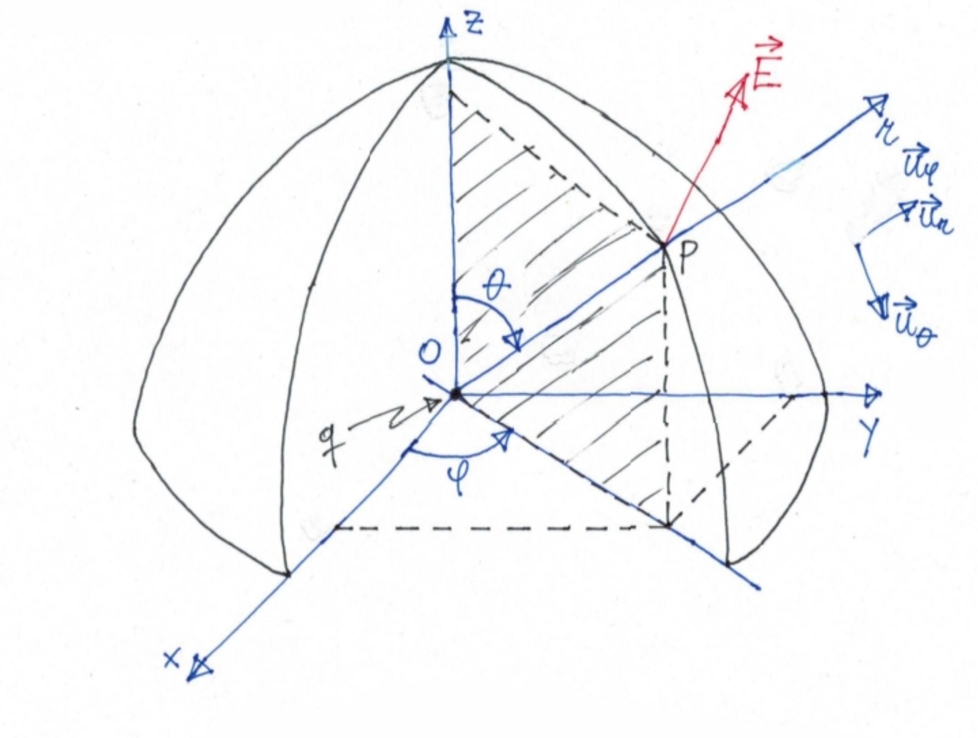


Figure 1.45

– **Coulomb** decides to begin his derivation of $\vec{E}$ with $\vec{E}_\theta$. He decides to resort to both Eqs. (1.118) and 1.119) and symmetry arguments to find $\vec{E}_\theta$. Coulomb strategy is to take advantage of the problem symmetries as much as possible and then, a posteriori, confirm those arguments by means of Eqs. (1.118) and (1.119).

A point-like charge distribution q is characterized by a rotation symmetry with respect to both θ and φ in the $Or\theta\varphi$ system: The charge appears the same to an observer at P for any value of θ and φ (see Fig. 1.46a).

Assume a non-zero $\vec{E}_\theta$ at point P (see Fig. 1.46b). Because of the rotation symmetry with respect to θ, $\vec{E}_\theta$ mist be the same at each point on the closed line (meridian circle) γ_θ in Fig. 1.46b. This is obviously true also for point P', which belongs to both γ_θ and the closed line (parallel circle) γ_φ. By rotating $\vec{E}_\theta$ at P' by an angle π clockwise (or counter-clockwise) along γ_φ, because of the rotation symmetry with respect to φ we obtain a component $\vec{E}_\theta$ at P with the same magnitude of the initial $\vec{E}_\theta$ at P, but opposite direction. The only case when this is possible is if $E_\theta = 0$ at P. Due to the arbitrary choice of P, Coulomb can conclude $\vec{E}_\theta = \vec{0}$ everywhere in space.

Coulomb can confirm this finding by assuming a non-zero $\vec{E}_\theta$ at each point on γ_θ. Because of the rotation symmetry with respect to θ we keep assuming $\vec{E}_\theta$ to be equal at each point on γ_θ (manifestly, this argument is relaxed compared to the argument based on the π rotation along γ_φ that led us to conclude $\vec{E}_\theta = \vec{0}$ everywhere in space). Under these assumption, Coulomb can invoke the law of Eq. (1.119) for the case of closed line γ_θ oriented as indicated in Fig. 1.46b. The only component of $\vec{E}$ contributing to the integral (1.119) is $\vec{E}_\theta$ because the only one directed as $\vec{t} = \vec{u}_\theta$. Hence, Coulomb can write

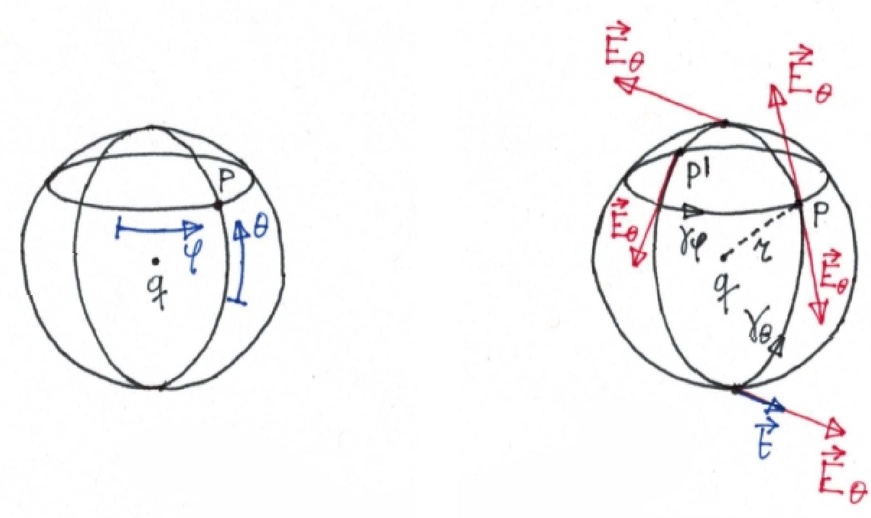
$$\begin{aligned}\oint_{\gamma_\theta} \vec{E} \bullet \vec{t} d\ell &= \int_0^{2\pi} E_\theta \vec{u}_\theta \bullet \vec{u}_\theta \cdot r d\theta \\ &= 2\pi r \cdot E_\theta = 0 \quad . \end{aligned} \tag{1.120}$$

Assuming $r \neq 0$ (i.e., for any P different from the point where q is located), the result of Eq. (1.120) is valid iff $E_\theta = 0$.

Coulomb can now safely conclude that $\vec{E}_\theta = \vec{0}$ everywhere in space.

It is worth noting that the use of symmetry arguments combined with Eqs. (1.118) and (1.119) in the present problem is similar to that of a charged sphere. In fact, a point-like charge distribution can be viewed as the limit $R \to 0^+$ of a volume charge distribution with constant density on a sphere of radius R.

– By means of arguments analogous to those used for $\vec{E}_\theta$, Coulomb would have also concluded that $\vec{E}_\varphi = \vec{0}$ everywhere in space (the reader can try to show it.)



(a) (b)

Figure 1.46

– The only component of $\vec{E}$ Coulomb must consider is, thus, $\vec{E}_r$.

Because of the rotation symmetry with respect to θ and φ, $\vec{E}_r$ must be the same at each point on the meridian and parallel circles passing through a generic point $P(r, \theta, \varphi)$, as shown in Fig. 1.47a. In particular, given a non-zero $\vec{E}_r$ at P a rotation of $\{P, \vec{E}_r\}$

by an angle θ along γ_θ leads to vector $\vec{E}_r$ at P'. A rotation of $\{P', \vec{E}_r\}$ by an angle π clockwise (or counter-clockwise) along γ_φ leads to vector $\vec{E}_r$ at P, consistently with the initial choice of $\vec{E}_r$ (see Fig. 1.47a). Due to the arbitrary choice of P, Coulomb can conclude that there can be a non-zero component $\vec{E}_r$ and that such a component must be the same at each point of a generic sphere with radius r and centered in q.

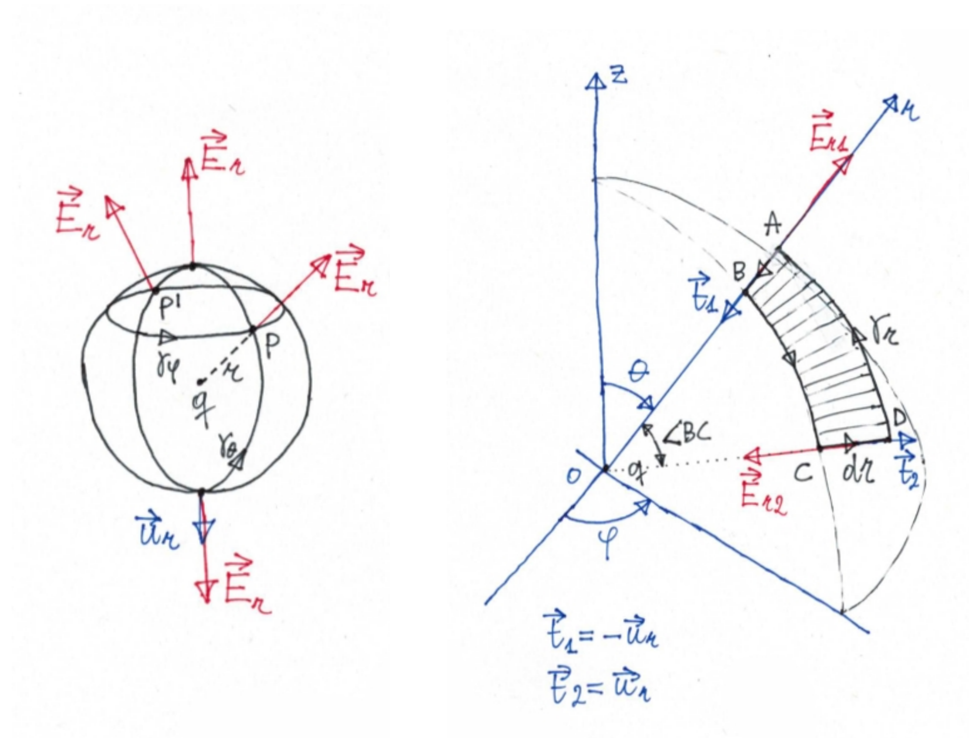
Coulomb can confirm this argument by considering a closed line γ_r as shown in Fig. 1.47b. The line is oriented counter-clockwise and comprises four parts, line segments AB and CD and arcs BC and DA. Segments AB and CD are directed radially with respect to q and arcs BC and DA belong to two circles both centered in q. The radial distance between the arcs is assumed to be infinitesimally small, i.e., $\overline{AB} = \overline{CD} = dr$. Note that, following our caveats in chapter 7 on lines comprising infinitesimal distances, the geometry of γ_r has been chosen consistently with the rotation symmetry of the given charge distribution q. Coulomb can now invoke Eq. (1.119) and calculate the line integral of $\vec{E}$ on γ_r. The only component of $\vec{E}$ contributing to the integral on AB and CD is $\vec{E}_r$. In addition, $\vec{E}_r$ is assumed to be constant on AB and CD because of segments of infinitesimal length. We define $\vec{E}_r$ on AB to be $\vec{E}_{r_1}$ and on CD to be $\vec{E}_{r_2}$. The only component contributing on BC and DA is $\vec{E}_\theta$, which, as we know, is zero everywhere. Hence, the integral (1.119) reads

$$
\begin{aligned}
\oint_{\gamma_r} \vec{E} \bullet \vec{t} d\ell &= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \\
&= \int_{r+dr}^{r} E_{r_1} \vec{u}_r \bullet \vec{u}_r \cdot dr + \int_{r}^{r+dr} -E_{r_2} \vec{u}_r \bullet \vec{u}_r \cdot dr \\
&= E_{r_1} \big(\cancel{r} - \cancel{r} - dr \big) - E_{r_2} \big(\cancel{r} + dr - \cancel{r} \big) \\
&= -E_{r_1} dr - E_{r_2} dr = 0 \quad ,
\end{aligned}
$$

from which it follows that

$$
E_{r_2} = -E_{r_1} \quad . \tag{1.121}
$$

Because of condition (1.121) and due to the arbitrary choice of angles θ and φ and of the aperture $\angle BC (= \angle DA)$, Coulomb can safely conclude that $\vec{E}_r$ must be the same at each point on a sphere with radius r and centered in q.



(a) (b)

Figure 1.47

By extensive use of symmetry arguments and, most importantly, of Eq. (1.119), i.e., the irrotational principle of $\vec{E}$, Coulomb has already found that $\vec{E}$ must be directed radially (central field) and E_r must be constant on each sphere centered in q. In order to find any possible functional dependence of E_r on r, θ, and φ, Coulomb must now invoke Gauss' law (Eq. (1.118). The first closed surface considered by Coulomb is surface Σ'_r in Fig. 1.48a. Since Σ'_r contains elements with infinitesimal dimensions, Coulomb must assume that its geometry is consistent with the rotation symmetry of the charge distribution. For this reason, Σ'_r is constructed as the semi-cone obtained from the intersection of a cone with apex q and infinitesimal aperture $d\theta$, with two concentric spheres centered in q and radius r and r', respectively. Thus, the semi-cone comprises infinitesimal surface dS and dS' as bases, and a lateral surface S'_ℓ. Because of the infinitesimal dimensions of the two bases, we assume $\vec{E}$ to be constant on each of them and we define it to be $\vec{E}_r$ on dS and $\vec{E'}_r$ on dS'. Since $\vec{E}_\theta = \vec{E}_\varphi = \vec{0}$ everywhere, $\vec{E}_r$ and $\vec{E'}_r$ are the only components of $\vec{E}$ contributing to the integral (1.118) on dS and dS', respectively. In particular, consistently with our choice in Fig. 1.48a, $\vec{E}_r$ is directed opposite with respect to the unit normal vector $\vec{n}$ on dS, and $\vec{E'}_r$ is directed as the unit normal vector $\vec{n'}$ on dS'. The only components of $\vec{E}$ contributing to the integral (1.118) on S'_ℓ would be $\vec{E}_\theta$ and $\vec{E}_\varphi$ ($\vec{E}_r$ is always orthogonal to the normal vector at each point on S'_ℓ), which, however, are both zero everywhere. Lastly, the area of the infinitesimal spheric elements dS and dS' can be approximated with the area of the disks obtained by intersecting two planes, both perpendicular to r, with the spheres with radius r and

r' centered in q, respectively. Each intersection is a circumference corresponding to the outer border of dS and dS', respectively (see Fig. 1.48a, inset). Making a negligible projection error, we can thus write:

$$\begin{aligned} dS &= \pi\Big(r\sin\frac{d\theta}{2}\Big)^2 \simeq \pi\Big(r\,\frac{d\theta}{2}\Big)^2 \\ &= \frac{\pi}{4}\,r^2\cdot\Big(d\theta\Big)^2 \end{aligned} \tag{1.122}$$

and

$$\begin{aligned} dS' &= \pi\Big(r'\sin\frac{d\theta}{2}\Big)^2 \simeq \Big(r'\,\frac{d\theta}{2}\Big)^2 \\ &= \frac{\pi}{4}\,r'^2\cdot\Big(d\theta\Big)^2 \quad . \end{aligned} \tag{1.123}$$

Under these assumption and due to the absence of charge within Σ'_r, Coulomb (we) can calculate the integral (1.118) as

$$\begin{aligned} \oiint_{\Sigma'_r} \vec{E}\bullet\vec{n}dS &= \iint_{dS} + \iint_{S'_\ell} + \iint_{dS'} \\ &= \iint_{dS} E_r\vec{u}_r \bullet \big(-\vec{u}_r\big)\cdot dS + \iint_{dS'} E'_r\vec{u}_r \bullet \vec{u}_r\cdot dS' \\ &= -E_r\frac{\cancel{\pi}}{\cancel{4}}r^2(\cancel{d\theta})^2 + E'_r\frac{\cancel{\pi}}{\cancel{4}}r'^2\big(\cancel{d\theta}\big)^2 \\ &= 0 \quad . \end{aligned} \tag{1.124}$$

This result is equivalent to

$$\frac{E'_r}{E_r} = \frac{r^2}{r'^2} = \frac{1/r'^2}{1/r^2} \quad . \tag{1.125}$$

The condition (1.125) clearly shows that E_r only depends on r (and E'_r on r') and, because $r < r'$, $E'_r < E_r$.

Coulomb can obtain a more conclusive functional dependence of E_r by considering the sphere with radius r and centered in q, Σ_r, as the closed surface in integral (1.118). Since E_r is constant at each point on Σ_r and assuming it to be directed outward (see Fig. 1.48a; here, also $\vec{n}$ is outward) the integral reads:

$$\begin{aligned} \oiint_{\Sigma_r} \vec{E}\bullet\vec{n}dS &= \oiint_{\Sigma_r} E_r\vec{u}_r \bullet \vec{u}_r\cdot dS \\ &= E_r \oiint_{\Sigma_r} dS \\ &= E_r\cdot 4\pi r^2 = \frac{1}{\epsilon_0}q \quad , \end{aligned} \tag{1.126}$$

from which it follows that

$$E_r = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}$$

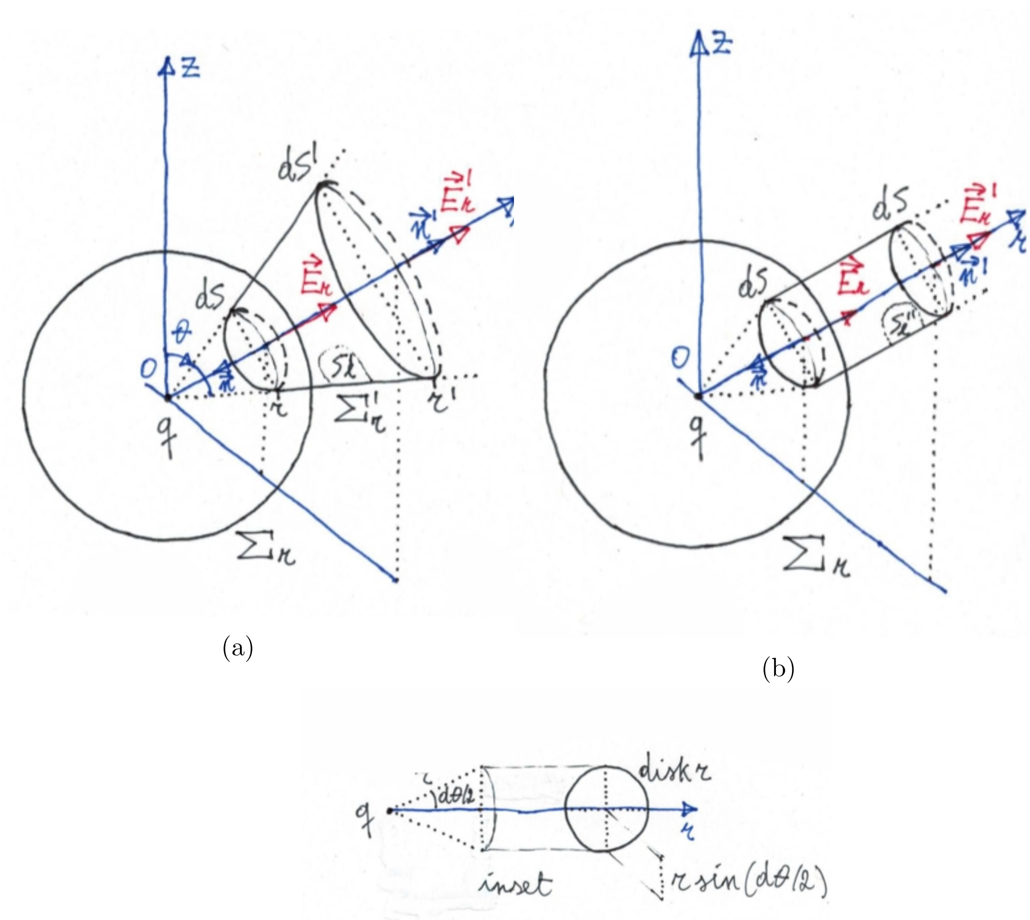
or, more in general,

$$\vec{E}_r = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}\,\vec{u}_r \quad . \tag{1.127}$$

From Eq. (1.127), given a charge q' at point P_2 in space, the force acting on q' due to q at point P_1 is

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\,\frac{qq'}{r_{12}^2}\,\vec{u}_{12} \quad , \tag{1.128}$$

where r_{12} is the absolute value of the distance between points P_1 and P_2 and $\vec{u}_{12}$ the unit vector indicating the direction between P_1 and P_2. Starting from Gauss' law and the irrotational principle of $\vec{E}$ (Eqs. (1.118) and (1.119)), Coulomb was able to demonstrate Eq. (1.128): Coulomb's theorem.



(a)

(b)

Figure 1.48

– Coulomb decided then to study two point-like charges q_1 and q_2 at a very small (negligible) distance from each other, but not on top of one another. Coulomb assumed the two charges to be in the neighbourhood of a point P in space.

By means of Gauss' law, Coulomb could readily calculate the total flux due to the electrostatic field of q_1 and q_2 through a sphere Σ_r with radius r and centered in P:

$$\Phi_{\text{tot}}^{\Sigma_r} = \frac{1}{\epsilon_0}\left(q_1 + q_2\right) \quad . \tag{1.129}$$

By means of his own theorem (e.g., in the form of Eq. (1.127)), Coulomb was also able to calculate directly (i.e., using the general definition of flux) the flux due to the electrostatic field $\vec{E}_1$ of q_1 and, separately, due to $\vec{E}_2$ of q_2 through Σ_r:

$$\begin{aligned}
\Phi_1^{\Sigma_r} &= \oiint_{\Sigma_r} \vec{E}_1 \bullet \vec{n} dS \\
&= \frac{1}{4\pi\epsilon_0} q_1 \frac{1}{\cancel{r^2}} \cancel{r^2} \vec{u}_r \bullet \vec{u}_r \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \cdot d\theta \\
&= \frac{1}{\epsilon_0} q_1
\end{aligned} \tag{1.130a}$$

and

$$\Phi_2^{\Sigma_r} = \oiint_{\Sigma_r} \vec{E}_2 \bullet \vec{n} dS = \frac{1}{\epsilon_0} q_2 \quad , \tag{1.130b}$$

where both $\vec{E}_1$ and $\vec{E}_2$ and $\vec{n}$ were assumed to be directed outward on Σ_r. While each of the two fluxes (1.130a) and (1.130b) is different form the total flux (1.129), their sum is equal to it:

$$\Phi_1^{\Sigma_r} + \Phi_2^{\Sigma_r} = \frac{1}{\epsilon_0}\left(q_1 + q_2\right) = \Phi_{\text{tot}}^{\Sigma_r} \quad .$$

In order for this result to be valid, a superposition property of $\vec{E}$ must exist, so that

$$\begin{aligned}
\Phi_{\text{tot}}^{\Sigma_r} &= \frac{1}{\epsilon_0}\left(q_1 + q_2\right) \\
&= \frac{1}{4\pi\epsilon_0}\left(q_1 + q_2\right) \frac{1}{r^2} r^2 \vec{u} \bullet \vec{u}_r \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \\
&= \oiint_{\Sigma_r} \left(\vec{E}_1 + \vec{E}_2\right) \bullet \vec{n} dS \quad ,
\end{aligned}$$

where the total field generated by q_1 and q_2 is $\vec{E}_{\text{tot}} = \vec{E}_1 + \vec{E}_2$. It is easy to prove that the superposition property of $\vec{E}$ is valid for any arbitrary distribution of N point-like charges and, by extension, can be applied to arbitrary continuous charge distributions.

In this way, Coulomb would have been able to demonstrate the superposition property of $\vec{E}$, which happens to be "built-in" in Gauss' law.

– Our historical and fictional journey demonstrates the equivalence between Coulomb's law and the superposition principle of $\vec{E}$ on one hand and Gauss' theorem and the irrotational property of $\vec{E}$ (with the addition of the zero-field condition at infinity) on the other hand.

1.19.4 Back to history.

According to Feynman, it was Benjamin Franklin to first notice that the electrostatic field inside a charged metallic sphere is zero. He probably used an apparatus similar to that sketched in Fig. 1.43. When he mentioned his finding to Priestley, the latter conjectured that it might had to do with an inverse square law. In fact, at the time it was known that a spherical shell of matter produces a zero gravitational field inside the shell (and, according to Newton's law, the gravitational field depends on the inverse distance squared). This happened 18 years before Coulomb's torsion balance experiment.

Six years after Franklin's discovery, Cavendish performed a more refined experiment, conceptually very similar to that sketched in Fig. 1.42. He found that upon charging an outer metallic sphere, 1/60 of the applied charge transferred to an inner sphere connected to the outer one by means of a conducting wire. As it turns out, Cavendish did not publish his null experiment, which approximately 100 years later, was resumed and further refined by Maxwell. Maxwell found that only 1/21600 of the applied charge was transferred to the inner sphere. It was not until 1936 that Plimpton and Lawton, by means of the apparatus sketched in Fig. 1.42, were able to measure a charge transfer of $1/10^9$. As a consequence, the experiment of Plimpton and Lawton confirms Gauss' theorem and shows that the exponent in Coulomb's law must be 2 within one part in a billion. Further refinements of this result were obtained using different experimental methods in the early 70's (see e.g., Williams, Faller, and Hill, 1971).

Chapter 2

The Electrostatic Potential

2.1 The electrostatic potential.

In the next series of chapters, we move from a vector quantity, the electrostatic field $\vec{E}$, to a scalar quantity, the electrostatic potential V. The field and potential, which are mathematically related to each other, make it possible to study the electrostatic interaction from different, ultimately equivalent prospectives:

The electrostatic potential V makes it possible to simplify calculations and is very useful solving problems where, a priori, the charge distribution is unknown.

2.1.1 The potential of a point-like charge.

Consider a point-like charge distribution q located at point q located at point Q in space. In a spherical coordinate system, a generic point P in space is represented by 3 numbers: r, θ, and φ. Because of its central and symmetric structure, the field $\vec{E}$ generated by q can be represented by the gradient of a scalar function. In particular, given r the distance between P and Q (r is an absolute value) and C an arbitrary constant, by defining a scalar function $V(P)$ as

$$V(P) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r} + C \quad , \tag{2.1}$$

it follows that

$$\begin{aligned}\vec{E}(P) &= \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}\,\vec{u}_{QP} \\ &= -\,\mathrm{grad}\,V(P) = -\nabla V(P) \quad .\end{aligned} \tag{2.2}$$

Hence, each function of the type (2.1) represents a scalar potential of the field generated by a point-like charge (see also tutorial 2 on the scalar potential).

Because of the arbitrary choice of C in definition (2.1), C can be chosen in such a way that $V = 0$ at an arbitrary point P of the field, so long $P \neq Q$. For example, if we want $V(P_0) = 0$ for a generic $P_0 \neq Q$, it is sufficient to impose

$$C = -\frac{1}{4\pi\epsilon_0}\,\frac{q}{r_0} \quad ,$$

where r_0 is the distance between Q and P_0.

In particular, among all potentials given by (2.1), only one verifies the zero-potential condition at infinity

$$\lim_{r\to\infty} V(r) = 0 \quad . \tag{2.3}$$

This is the potential for $C = 0$. Unless specific reasons occur, it is common to define precisely this function as the potential of the field. This potential is called Green function corresponding to normal conditions at infinity. It is simple to verify that V has the same value at each point on a generic sphere centred in Q. The sphere is thus called equipotential surface of the field.

Figure 2.1 shows the equipotential surfaces for a point-like charge q in 2D. Note that the equipotentials are perpendicular to the field lines. We can thus add another rule on how to sketch field lines!

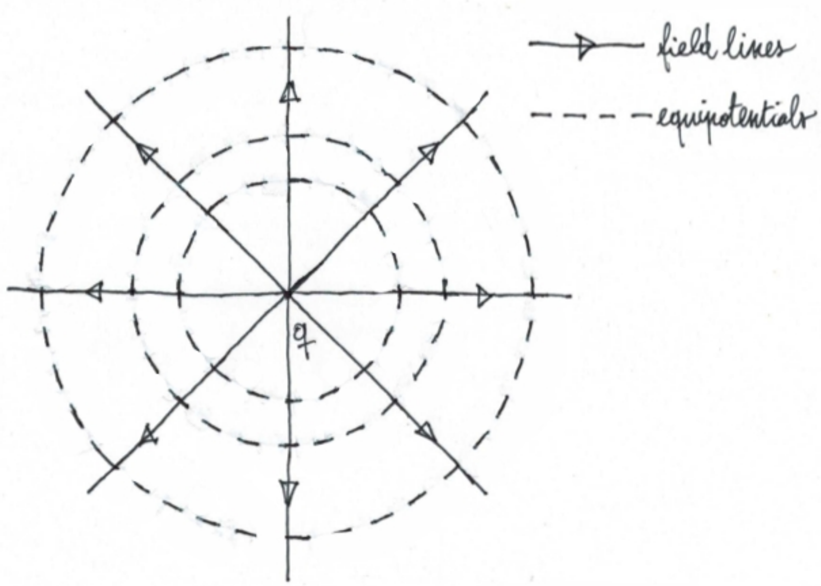


Figure 2.1

2.1.2 Work of the Field Forces.

We will now attempt to attach a deeper physical meaning to the concept of potential.

Referring again to the simple case of a point-like charge distribution q at point Q, we imagine to introduce a charge q_0 in the field and, by applying a suitable external force to it, to move it from point P_1 to point P_2 along a given line γ. We assume the external force on q_0 to be at each point on γ equal and opposite to the force on q_0 due to q. To ease the intuition, we can assume the external force to be slightly larger than the field force, so that it can overcome it without, however, giving a noticeable acceleration to q_0. Under these conditions, the charge q_0 moves extremely slowly (in the limit, with zero velocity and acceleration) through a continuous series of equilibrium steps with zero kinetic energy. In this sense, we perform a quasi-static transformation, so that the work of the external force is equal and contrary to that of the field force.

The work W_γ of the field force to move q_0 from P_1 to P_2 on γ is

$$W_\gamma = \int\limits_\gamma \vec{F} \cdot \vec{t}d\ell = q_0 \int\limits_\gamma \vec{E} \cdot \vec{t}d\ell \quad , \tag{2.4}$$

where $\vec{t}$ is the unit tangent vector to the oriented curve γ and $d\ell$ the infinitesimal arc element on γ.

By substituting (2.2) in (2.4), we obtain

$$W_\gamma = -q_0 \int\limits_\gamma \vec{\nabla} V \cdot \vec{t}d\ell \tag{2.5}$$

and finally, using the properties of the gradient (see tutorial 2),

$$W_\gamma = q_0 \Big[V\big(P_1\big) - V\big(P_2\big) \Big] \quad , \tag{2.6}$$

where V is any of the infinite potential functions of the field. It is worth noting that (2.6) is independent from the choice C.

In summary, chosen any of the field potentials with arbitrary C, the difference $V(P_1) - V(P_2)$ for two generic points P_1 and P_2 (different from Q) is equal to the work from P_1 to P_2 along any line connecting them. In fact, due to the properties of the gradient that express the irrotational property of $\vec{E}$, the integral (2.5) depends only on the limiting point of γ, P_1 and P_2. The work W_γ can be positive or negative (or, of course, zero), depending whether the charge displacement is along the field force or against it.

From (2.6), when $q_0 = 1C$ is possible to define the physical unit for the potential difference. In the SI this is called "volt" (V): Between two points there exists a potential difference (or drop) of 1 V, when work of the field force to move a charge of 1 C between the two points is 1 J. From this definition, it is also possible to obtain the most commonly used unit for $\vec{E}$, which is V/m (and its multiples, e.g., kV/cm).

2.2 Potential of a Generic Charge Distribution.

Consider a generic distribution of point-like charges q_1, q_2, ..., q_n. Each charge generates a field $\vec{E}_1$, $\vec{E}_2$, ..., $\vec{E}_n$. Because of the superposition principle of $\vec{E}$, it follows that

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n \quad ,$$

where $\vec{E}$ is the total field generated by the distribution.

Each field $\vec{E}_1$, $\vec{E}_2$, ..., $\vec{E}_n$ can be associated with a potential V_1, V_2, ..., V_n, respectively. As a consequence,

$$\vec{E} = -\vec{\nabla} V_1 - \vec{\nabla} V_2 + \cdots - \vec{\nabla} V_n \quad .$$

Because of the linearity of the gradient, it follows that

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\big(V_1 + V_2 + \cdots + V_n\big) \\ &= -\vec{\nabla} V \quad , \end{aligned} \tag{2.7}$$

where

$$V = V_1 + V_2 + \cdots + V_n \quad . \tag{2.8}$$

In other words, the field generated by a generic distribution of point-like charges (in limited number) is characterized by a potential, which is equal to the sum of the potentials associated with each point-like charge. This can be interpreted as the superposition principle of the electrostatic potential V.

In the case of n point-like charges, the potential of Eq. (2.1) at a generic point P becomes

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \left(\frac{q_k}{r_k} + C_k \right) \quad , \tag{2.9}$$

where r_k is the distance (absolute value) between the point Q_k at which the charge q_k is localized and the generic point P where the potential is evaluated and C_k is the arbitrary constant associated with the generic partial potential generated by q_k. For a limited number of charges, the constant C_k can be chosen arbitrary. Thus,

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{r_k} + C \quad , \tag{2.10}$$

with

$$C = \sum_{k=1}^{n} C_k \quad .$$

In particular, we can choose $C = 0$ from which it follows that $V(P)$ goes to zero at infinity. The zero-potential at infinity condition may become ambiguous when the field is not centrally symmetric. Hereafter, we will intend that $V(P)$ goes to zero at infinity when for arbitrary $\epsilon > 0$ is possible to determine a spherical surface Σ such that on the points of the surface and on those outside the surface

$$\left| V(P) \right|_{\Sigma} < \epsilon \quad .$$

It is easy to show that for $C = 0$, the potential $V(P)$ satisfies this condition.

If, instead, we want $V(P)$ to become zero at a point P_0 (different form any of the points Q_k where the charges q_k are localized), it is enough to choose C such that

$$V(P_0) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{r_{k_0}} + C = 0 \quad , \tag{2.11}$$

where, in this case, r_{k_0} indicates the distance between any point Q_k and P_0. Obviously, the same result is obtained by choosing the partial potential contributions from each charge such that they each are zero at P_0:

$$\frac{1}{4\pi\epsilon_0} \frac{q_k}{r_{k_0}} + C_k = 0 \quad .$$

We now consider a charge distribution with volume density ρ (in general variable from point to point) in a limited region of space Ω. We can divide this charge distribution in infinitesimal regions $d\tau$, each occupied by an infinitesimal charge

$$dq = \rho d\tau \quad .$$

The potential generated by the infinitesimal charge at point Q, $\rho(Q)d\tau$, at the generic point P can be written as

$$dV\big(P\big) = \frac{1}{4\pi\epsilon_0}\,\frac{\rho(Q)}{r_{QP}}\,d\tau \quad , \tag{2.12}$$

where r_{QP} is the distance between Q and P and the arbitrary constant in the definition of potential was assumed to be zero.

The superposition principle allows us to obtain the potential of a continuous charge distribution by integrating (2.12) over the entire region Ω where the charges are distributed:

$$V\big(P\big) = \frac{1}{4\pi\epsilon_0}\iiint\limits_{\Omega}\frac{\rho(Q)}{r_{QP}}\,d\tau \quad . \tag{2.13}$$

In a Cartesian coordinate system, the integral (2.13) becomes

$$\begin{aligned} V\big(x_P, y_P, z_P\big) &= \frac{1}{4\pi\epsilon_0}\iiint\limits_{\Omega}\frac{\rho\big(x_Q, y_Q, z_Q\big)}{\Big[\big(x_P - x_Q\big)^2 + \big(y_P - y_Q\big)^2 + \big(z_P - z_Q\big)^2\Big]^{1/2}} \\ &\quad \cdot dx_Q\, dy_Q\, dz_Q \quad . \end{aligned} \tag{2.14}$$

The scalar function $V(P)$ is characterized by interesting analytical and physical properties. First, given a charge distribution with continuous and limited volume density ρ, $V(P)$ is continuous and limited everywhere in space (including where the charge is distributed). Consider a point P outside the region Ω where the charge is distributed. In this case, the integrand function in (2.13) is well defined, continuous and limited, at each point in τ because ρ is by assumption continuous and limited in Ω and $r > 0$ for each P outside Ω. Hence, we can safely conclude that the integral in (2.13) is continuous and limited for each P outside Ω. The reader should prove that the integral in (2.13) is well defined, continuous and limited, even for each P inside Ω.

Second, it is easy to convince oneself that if the charge distribution is contained in a limited region of space, the potential V goes to zero at infinity (we will come back to this point when deriving the multiple expansion of the electrostatic potential). From this property of the potential it follows the analogous property for the field.

Last, also in the case of continuous distributions the work of the field forces to move a unitary positive charge from point P_1 to point P_2 is equal to the potential difference $V(P_1) - V(P_2)$ and, again, the equipotential surfaces are orthogonal to the field lines.

In the case of surface and linear charge distributions, we obtain

$$V\big(P\big) = \frac{1}{4\pi\epsilon_0}\iint\limits_{\Sigma}\frac{\sigma(Q)}{r_{QP}}\,dS \tag{2.15}$$

and

$$V\big(P\big) = \frac{1}{4\pi\epsilon_0}\int\limits_{\gamma}\frac{\lambda(Q)}{r_{QP}}\,d\ell \quad , \tag{2.16}$$

respectively.

Without demonstration, in the case of surface charge distributions the potential is continuous and limited even at the points where the charge is defined. In the case of linear charge distributions, however, the potential diverges logarithmically at the points where the charge is defined.

It is worth mentioning that calculating the potential generated by a known charge distribution reduces to a single integral of a scalar function. Thus, it is much easier to calculate the potential than the field. In addition, once V is known it is straightforward to calculate $\vec{E}$ from (2.7), i.e., by means of simple derivatives.

2.3 The Electrostatic Potential of Charged Rings, Shells, and Spheres.

In this section, we will review a few relevant examples of electrostatic potential. In the first two examples, we will calculate the potential directly from its definition. In the last example, we will calculate it from the knowledge of the electrostatic field.

2.3.1 The Electrostatic Potential of a Charged Ring on the Ring's Axis.

Consider a linear, uniform charge distribution with constant density λ on a circle γ with radius a (see Fig. 2.2). We want to calculate the electrostatic potential V on the axis of γ.

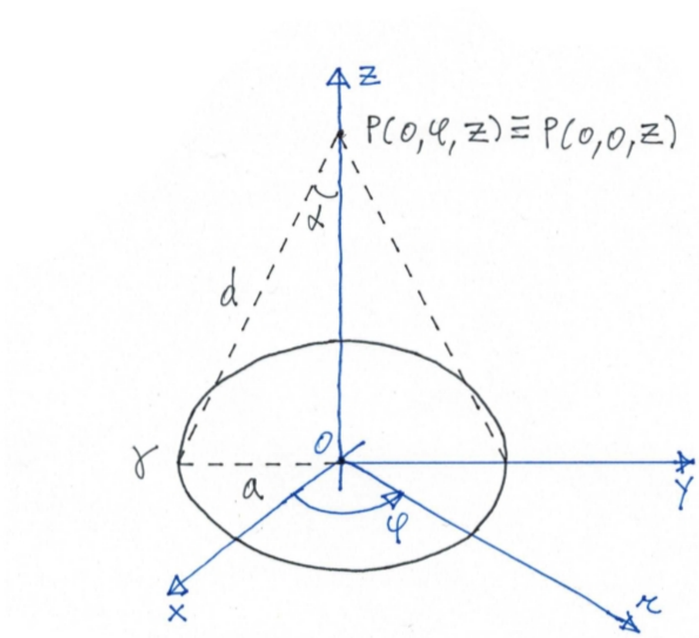


Figure 2.2

A cylindrical and a Cartesian coordinate systems are shown in Fig. 2.2. The axis z, common to both systems, coincides with the axis of γ.

In the case of a linear charge distribution, the infinitesimal potential due to an infinitesimal charge element $\lambda d\ell$ is given by:

$$\begin{aligned} dV &= \frac{1}{4\pi\epsilon_0}\frac{\lambda d\ell}{d} \\ &= \frac{1}{4\pi\epsilon_0}\frac{\lambda d\ell}{\dfrac{z}{\cos\alpha}} \quad , \end{aligned} \tag{2.17}$$

where $d\ell$ is an infinitesimal element on γ, d is the distance between $d\ell$ and P, which can be expressed as a function of the angle α. By integrating (2.17) along the entire circle γ, we obtain:

$$V = \frac{1}{4\pi\epsilon_0}\oint_\gamma \frac{\lambda d\ell}{d} \quad .$$

Noting that d is constant for each $d\ell$, we find:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\,\frac{\lambda\cos\alpha}{z}\oint_\gamma d\ell \\ &= \frac{1}{4\pi\epsilon_0}\,\frac{\lambda}{\cancel{z}}\,\frac{\cancel{z}}{\sqrt{a^2+z^2}}\int_0^{2\pi} a\cdot d\varphi \\ &= \frac{1}{4\cancel{\pi}\epsilon_0}\,\frac{\lambda}{\sqrt{a^2+z^2}}\,2\cancel{\pi}a \\ &= \frac{\lambda a}{2\epsilon_0\sqrt{a^2+z^2}} \quad . \end{aligned} \tag{2.18}$$

By means of definition (2.2), we can now easily find the electrostatic field at point P:

$$\begin{aligned} \vec{E} &= -\vec{\nabla}V \\ &= -\Big(-\frac{1}{2}\Big)\frac{\lambda a}{2\epsilon_0}\,\frac{1}{\big(a^2+z^2\big)^{3/2}}\,2z\,\frac{\pi}{\pi} \\ &= \frac{1}{4\pi\epsilon_0}\,\frac{qz}{\big(a^2+z^2\big)^{3/2}}\,\vec{u}_z \quad . \end{aligned}$$

$$\begin{aligned} \vec{\nabla}V &= \frac{\partial}{\partial_x}V\big(z\big)\cdot\vec{u}_x + \frac{\partial}{\partial_y}V\big(z\big)\cdot\vec{u}_y + \frac{\partial}{\partial_z}V\big(z\big)\cdot\vec{u}_z \\ &= \frac{\partial}{\partial_z}V\big(z\big)\cdot\vec{u}_z \quad . \end{aligned}$$

2.3.2 The Electrostatic Potential of a Charged Shell at any Point P in Space.

Consider a surface, uniform charge distribution with constant density σ on a shell Σ with radius R (see Fig. 2.3). Calculate V at any point P in space.

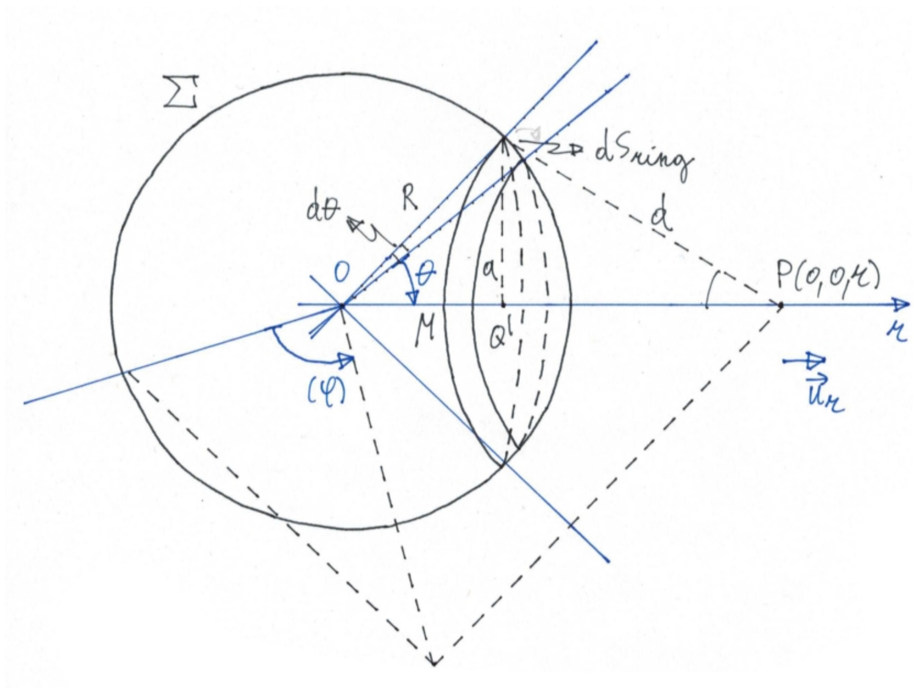


Figure 2.3

The spherical coordinate system used to solve this problem is shown in Fig. 2.3.

The potential at P due to the ring structure dS_{ring} indicated in the figure is given by:

$$dV = \frac{1}{4\pi\epsilon_0}\,\frac{\sigma dS_{\text{ring}}}{d} \quad ,$$

where

$$\begin{aligned} dS_{\text{ring}} &= 2\pi a R \cdot d\theta \\ &= 2\pi R^2 \sin\theta \cdot d\theta \end{aligned}$$

and

$$d = \left(R^2 + r^2 - 2\,R\,r\cos\theta\right)^{1/2} \quad ,$$

where we used Carnot's theorem for triangle $QQ'P$. Thus,

$$dV = \frac{1}{4\pi\epsilon_0}\,\frac{\sigma\,2\,\pi R^2 \sin\theta \cdot d\theta}{\left(R^2 + r^2 - 2\,R\,r\cos\theta\right)^{1/2}} \quad . \tag{2.19}$$

By integrating on the entire shell Σ, i.e., for $\theta \in \big[0, \pi\big]$, we obtain:

$$\begin{aligned}
V\big(0,0,r\big) &= \frac{1}{4\pi\epsilon_0} \oiint_{\Sigma} \frac{\sigma 2\pi R^2 \sin\theta}{\big(R^2 + r^2 - 2\,R\,r\cos\theta\big)^{1/2}} \cdot d\theta \\
&= \frac{1}{4\pi\epsilon_0} \int_0^{\pi} \frac{\sigma 2\pi R^2 \sin\theta}{\big(R^2 + r^2 - 2\,R\,r\cos\theta\big)^{1/2}} \cdot d\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{R}{r} \sigma 2\pi \Big[\big(R^2 + r^2 - 2\,R\,r\cos\theta\big)^{1/2}\Big]_0^{\pi} \\
&= \frac{\sigma R}{2\epsilon_0 r} \Big[\big(R^2 + r^2 - 2\,R\,r\cos\theta\big)^{1/2}\Big]_0^{\pi} \\
&= \begin{cases} \dfrac{\sigma R}{2\epsilon_0 r} \Big[\big(R + r\big) - \big(R - r\big)\Big] \ , & r \in \big[0, R\big] \\[2ex] \dfrac{\sigma R}{2\epsilon_0 r} \Big[\big(R + r\big) - \big(r - R\big)\Big] \ , & r \in \big[R, +\infty\big] \end{cases}
\end{aligned} \tag{2.20}$$

Noting that the total charge on Σ is given by:

$$q_\Sigma = 4\pi R^2 \sigma \quad ,$$

we finally obtain:

$$V\big(0,0,r\big) = \begin{cases} \dfrac{q_\Sigma}{4\pi\epsilon_0 R} \ , & r \in \big[0, R\big] \\[2ex] \dfrac{q_\Sigma}{4\pi\epsilon_0 r} \ , & r \in \big(R, +\infty\big) \end{cases} \tag{2.21}$$

Note that by rotating the axis r by arbitrary angles θ and φ around O, Eq. (2.21) makes it possible to calculate V at any point in space P. Equation (2.21) will always be the solution.

Also note that V is continuous at $r = R$:

$$\begin{aligned}
\lim_{r \to R^-} V\big(r\big) &= \frac{q_\Sigma}{4\pi\epsilon_0 R} \\
&= \lim_{r \to R^+} V\big(r\big) \quad .
\end{aligned}$$

In addition,

$$\lim_{r \to +\infty} V\big(r\big) = 0 \quad .$$

Finally, we note that the potential outside the shell is equivalent to that of a point-like charge q_Σ centred in Σ. Figure 2.4 shows a plot of $V(r)$.

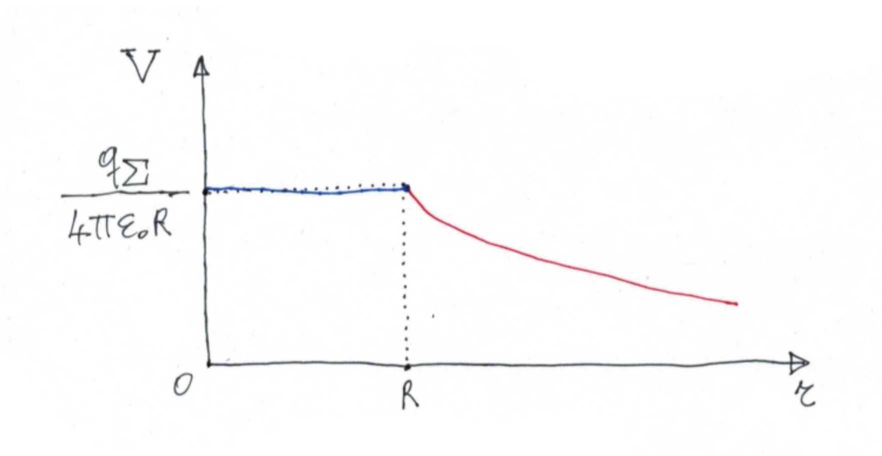


Figure 2.4

Again by means of definition (2.2), we can find $\vec{E}(P)$:

$$
\begin{aligned}
\vec{E} &= -\vec{\nabla} V \\
&= -\frac{\partial}{\partial_r} V\big(r\big) \cdot \vec{u}_r - \frac{1}{r}\frac{\partial}{\partial_\theta} V\big(r\big) \cdot \vec{u}_\theta \\
&\quad -\frac{1}{r\sin\theta}\frac{\partial}{\partial_\varphi} V\big(r\big) \cdot \vec{u}\varphi \\
&= -\frac{\partial}{\partial_r} V \cdot \vec{u}_r \\
&= \begin{cases} 0\ , & r \in \big[0, R\big] \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q_\Sigma}{r^2}\vec{u}_r . & r \in \big(R, +\infty\big) \end{cases}
\end{aligned}
$$

2.3.3 The Electrostatic Potential of a Charged Sphere with Known Field $\vec{E}$ at any Point in Space.

The field at any point in space for a charged full sphere with radius R is given by:

$$
\begin{cases} \vec{E} = \dfrac{\rho_0}{3\epsilon_0} r \cdot \vec{u}_r & , \ r \in \big[0, R\big] \\ \vec{E} = \dfrac{q_{\Sigma'_R}}{4\pi\epsilon_0 r^2} \cdot \vec{u}_r & , \ r \in \big(R, +\infty\big) \end{cases}
$$

where r is the distance from the centre of the sphere on an arbitrary axis passing through the centre and on the positive side of the axis,

$$
q_{\Sigma'_R} = \frac{4}{3}\pi R^3 \rho_0
$$

is the total charge contained within Σ, and ρ_0 is a constant charge density. As always the arbitrary choice of the axis r allows us to study only the case $r > 0$, without loosing generality.

We can now use definition (2.2) and obtain V from $\vec{E}$ by component integration.

Case $r \in \big[0, R\big]$

$$
\begin{aligned}
-\vec{\nabla} V &= -\frac{\partial}{\partial_r} V \cdot \vec{u}_r \\
&= \frac{\rho_0}{3\epsilon_0} r \cdot \vec{u}_r \quad ,
\end{aligned}
$$

from which it follows that

$$
dV = -\frac{\rho_0}{3\epsilon_0} r \cdot dr
$$

and so

$$
\begin{aligned}
V &= -\frac{\rho_0}{3\epsilon_0} \int\limits_0^r r \cdot dr \\
&= -\frac{\rho_0}{6\epsilon_0} r^2 \qquad\qquad , \; r \in \big[0, R\big] \quad . \qquad (2.22)
\end{aligned}
$$

Case $r \in \big(R, +\infty\big)$

$$
\begin{aligned}
-\vec{\nabla} V &= -\frac{\partial}{\partial_r} V \cdot \vec{u}_r \\
&= \frac{q_{\Sigma'_R}}{4\pi\epsilon_0 r^2} \cdot \vec{u}_r \quad ,
\end{aligned}
$$

from which it follows that

$$
dV = -\frac{q_{\Sigma'_R}}{4\pi\epsilon_0} \int\limits_R^r \frac{1}{r^2} \cdot dr
$$

and so

$$
V = \frac{q_{\Sigma'_R}}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{R} \right) \; , \; r \in \big(R, +\infty\big) \quad . \qquad (2.23)
$$

We remind that V is defined up to an arbitrary constant. In the case of (2.23), we want to choose a constant C such that

$$
\lim_{r \to +\infty} \left[\frac{q_{\Sigma'_R}}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{R} \right) + C \right] = 0 \quad .
$$

This normal condition at infinity corresponds to

$$
C = \frac{1}{4\pi\epsilon_0} \frac{q_{\Sigma'_R}}{R} \quad . \qquad (2.24)
$$

We can thus offset the potential of (2.23) by C and find

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\, q_{\Sigma'_R}\left(\frac{1}{r}-\frac{1}{R}\right)+C \\ &= \frac{1}{4\pi\epsilon_0}\,\frac{q_{\Sigma'_R}}{r} \qquad , \; r\in\left(R,+\infty\right) \quad . \end{aligned} \tag{2.25}$$

This potential is equal to that of a point-like charge $q_{\Sigma'_R}$ at the centre of the sphere. In particular,

$$\lim_{r\to R^+} V = \frac{1}{4\pi\epsilon_0}\,\frac{q_{\Sigma'_R}}{R} \quad .$$

In the case of (2.22), we can choose an arbitrary constant C' such that

$$\lim_{r\to R^-}\left(-\frac{\rho_0}{6\epsilon_0}\, r^2 + C'\right) = \frac{1}{4\pi\epsilon_0}\,\frac{q_{\Sigma'_R}}{R} \quad ,$$

in other words, a constant that allows us to connect continuously the potential inside and outside the sphere. This limit gives

$$-\frac{\rho_0}{6\epsilon_0}\, R^2 + C' = \frac{1}{4\pi\epsilon_0}\,\frac{q_{\Sigma'_R}}{R}$$

from which

$$\begin{aligned} C' &= \frac{1}{4\pi\epsilon_0}\,\frac{q_{\Sigma'_R}}{R} + \frac{1}{\cancel{6}\epsilon_0}\,\frac{\cancel{3}}{4\pi}\,\frac{1}{R^{\cancel{3}}}\,\cancel{R^2}\, q_{\Sigma'_R} \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{q_{\Sigma'_R}}{R}+\frac{q_{\Sigma'_R}}{2R}\right) \\ &= \frac{1}{4\pi\epsilon_0}\,\frac{3q_{\Sigma'_R}}{2R} \quad . \end{aligned} \tag{2.26}$$

We thus find

$$\begin{aligned} V &= -\frac{\rho_0}{6\epsilon_0}\, r^2 + \frac{1}{4\pi\epsilon_0}\,\frac{3q_{\Sigma'_R}}{2R} \\ &= -\frac{1}{\cancel{6}\epsilon_0}\,\frac{\cancel{3}}{4\pi}\,\frac{1}{R^3}\, q_{\Sigma'_R} r^2 + \frac{1}{4\pi\epsilon_0}\,\frac{3q_{\Sigma'_R}}{2R} \\ &= \frac{1}{4\pi\epsilon_0}\,\frac{q_{\Sigma'_R}}{2R}\left(3-\frac{r^2}{R^2}\right) \, , \; r\in\left[0,R\right] \quad . \end{aligned} \tag{2.27}$$

In particular,

$$V\left(0\right) = \frac{1}{4\pi\epsilon_0}\,\frac{3}{2}\,\frac{q_{\Sigma'_R}}{R} \quad .$$

Figure 2.5 shows a plot of $V(r)$ for the charged sphere.

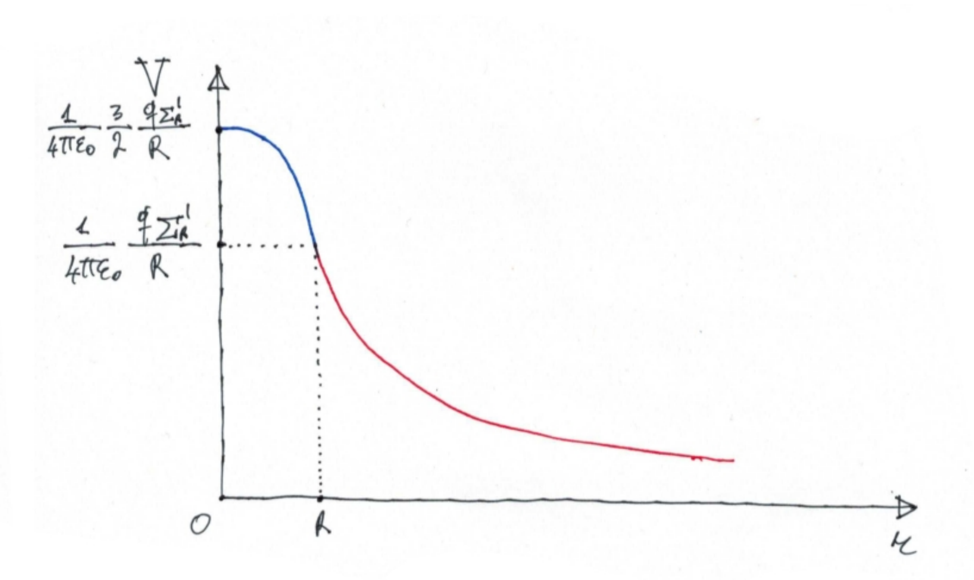

V
$\frac{1}{4\pi\varepsilon_0}\frac{3}{2}\frac{q\Sigma_R^1}{R}$
$\frac{1}{4\pi\varepsilon_0}\frac{q\Sigma_{iR}^1}{R}$
O
R
r

Figure 2.5

2.4 Poisson and Laplace Equations.

Given a charge distribution ρ in a region Ω of space, at each point where $\vec{E}$ is continuously differentiable

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \ , \tag{2.28}$$

$$\vec{\nabla} \times \vec{E} = \vec{0} \ . \tag{2.29}$$

By inserting

$$\vec{E} = - \text{ grad } V = \vec{\nabla} V$$

into Eq. (2.29), we find

$$\begin{aligned} \vec{\nabla} \times \big(- \text{ grad } V \big) &= \vec{\nabla} \times \big(- \vec{\nabla} V \big) \\ &= -\vec{\nabla} \times \big(\vec{\nabla} V \big) \\ &= -\vec{\nabla} \times \vec{\nabla} V = 0 \end{aligned}$$

because the vector $\vec{\nabla} V$, which is the formal product of V times the scalar V, is "parallel" to $\vec{\nabla}$. Another more physical way to see this is by noting that the vector field $\vec{\nabla} V$, having a potential, must be irrotational (see tutorial 3). In other words, (2.29) is identically verified.

Inserting $\vec{E} = -\vec{\nabla} V$ into (2.28), however, gives

$$\begin{aligned} \vec{\nabla} \cdot \big(- \text{ grad } V \big) &= \vec{\nabla} \cdot \big(- \vec{\nabla} V \big) \\ &= -\vec{\nabla} \cdot \big(\vec{\nabla} V \big) \\ &= -\vec{\nabla} \cdot \vec{\nabla} V \\ &= -\vec{\nabla}^2 V = \frac{1}{\epsilon_0}\rho \end{aligned}$$

and, thus,

$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0}\rho \quad , \tag{2.30}$$

where $\vec{\nabla}^2$ is called the Laplacian (see tutorial 3). The (2.30) is named Poisson equation.

In a region of space without any charge distribution (2.30) becomes

$$\vec{\nabla}^2 V = 0 \quad , \tag{2.31}$$

which is called Laplace equation.

Both Poisson and Laplace equations are partial differential equations. Their analytical form depends on the coordinate system chosen to represent the Laplacian operator.

Note that Poisson (and Laplace) equation includes both fundamental laws of electrostatic, i.e., Eqs. (2.28) and (2.29). In other words, given a charge distribution $\rho(Q)$, if we were able to find a function $V(P)$ that satisfies (2.30) at each point P, this would certainly be a potential of the field. From $\vec{E} = -\vec{\nabla}V$ we would then immediately obtain $\vec{E}(P)$. Also note that given a function $V(P)$ that satisfies (2.30), any other function

$$V^* = V + C \quad ,$$

where C is an arbitrary constant, is also a solution of (2.30). In fact,

$$\vec{\nabla}^2 V^* = \vec{\nabla}^2 V$$

at each point.

Remarkably, from all discussed so far, assume a charge density $\rho(Q)$ is known for each point Q. Further assume that ρ is different from zero only in a limited region τ, i.e., the charge does not extend to infinity.

2.5 Energy of a System of Point-Like Charges.

In this section, we intend to define and evaluate the electrostatic energy associated with a system of point-like charges in vacuum.

We first consider the simple case of two charges q_1 and q_2 in vacuum. The charges are located at points Q_1 and Q_2, respectively, at a distance r_{12} from each other (see Fig. 2.6). From the previous sections, we know that if q_1 is maintained fixed at Q_1 while q_2 is moved from a point Q_2' to Q_2 by means of a quasi-static transformation, the work generated by the field forces to perform the displacement is independent from the path followed and is given by

$$W = q_2\Big[V_1\big(Q_2'\big) - V_1\big(Q_2\big)\Big] \quad .$$

This means that when modifying a given initial configuration of two charges by displacing one or both of the charges, the work generated by the field forces depends only on the final charge configuration and not on the specific way the displacement brought from one to the other configuration. This is conceptually similar to the definition in thermodynamics of the internal energy of a physical system as a state function. In fact, also in the electrostatic case we can define an energy that depends only on the "state" of the system (i.e., on the configuration of the two charges) and that, with its variations, gives the work generated by the system when going from one to another configuration.

To uniquely define the energy associated with the state of a system we must choose a reference state that, by convention, has zero energy. In the electrostatic case it seems natural to choose such a state as the state where the two charges do not interact, i.e., a state where the charges are located at a large distance from each other (in the limiting case, at infinite distance).

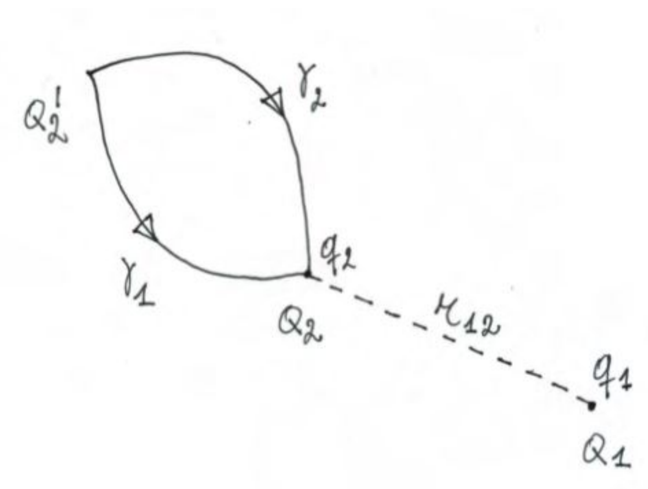


Figure 2.6

At this point, it is worth noting that until new we considered point-like charges, without taking into account the work necessary to "build" them. In other words, we did not take into account the work required to concentrate in a point in space a finite charge, which is initially distributed at infinite. With this preamble in mind, the energy associated with any charge configuration is uniquely defined and is opposite to the work generated by the field forces (i.e., the work due to external forces opposing internal forces) to move the charges under consideration from infinity to the configuration being studied. Indicating with U_e the electrostatic energy associated with the system, we find

$$\begin{aligned} U_e &= -W \\ &= q_2\Big[V_1\big(Q_2\big) - V_1\big(Q_2' \to +\infty\big)\Big] \\ &= q_2 V_1\big(Q_2\big) \\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} \quad , \end{aligned} \tag{2.32}$$

where we assumed normal conditions at infinity. Note that when the two charges have opposite sign, they tend to bound to each other and, thus, their energy is negative. The energy (2.32) can readily be generalized to the case of any system of point-like charges. Given, for example, three charges q_1, q_2, and q_3 initially at infinity. We move q_1 to point Q_1. Since at this stage the charges do not interact, this movements requires zero work. We then move q_2 to Q_2, while keeping q_1 fixed at Q_1. At this stage, q_2 "feels" the interaction with q_1 and, thus, the work done against the interaction forces is given by:

$$U_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} \quad . \tag{2.33a}$$

At last, we move q_3 to Q_3, while keeping both q_1 and q_2 fixed at Q_1 and Q_2, respectively. At this stage, q_3 feels the interaction with both q_1 and q_2 and, thus, we must perform a work

$$U_3 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}} \quad , \tag{2.33b}$$

where r_{13} and r_{23} are the distances between q_1 and q_2 and q_2 and q_3, respectively. We can thus conclude that the energy associated with the given configuration of three charges q_1, q_2, and q_3 at points Q_2, Q_2, and Q_3 is given by:

$$U_e = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) \quad , \tag{2.34}$$

i.e., the sum of works (2.33a) and (2.33b). It is worth noting that the energy V_e is independent from the specific way in which the charges have been moved from infinity to the final configuration.

The energy (2.34) can be rewritten in a form easier to generalize noting that, for each value of the indices, $r_{ij} = r_{ji}$. Hence,

$$\begin{aligned} U_e &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{2}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_1}{r_{21}} + \frac{q_3 q_1}{r_{31}} + \frac{q_3 q_2}{r_{32}}\right)\right] \\ &= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{3}\sum_{\substack{j=1 \\ (j\neq i)}}^{3}\frac{q_i q_j}{r_{ij}} \quad , \end{aligned} \tag{2.35}$$

where, as always, r_{ij} is the distance between q_i and q_j. In summary, the energy associated with a system of N point-like charges q_1 q_2, ..., q_N located at points Q_1, Q_2, ..., Q_N at distances r_{ij} from each other is given by:

$$U_e = \frac{1}{2}\,\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\(j\neq i)}}^{N}\frac{q_i q_j}{r_{ij}} \quad . \tag{2.36}$$

This result can be written in a different form using the concept of potential. In fact, in (2.36) appears a term

$$V_i = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\(j\neq i)}}^{N}\frac{q_j}{r_{ij}} \quad ,$$

which corresponds to the value of the potential generated at point P_i from all charges, excluded q_1. We can thus rewrite (2.36) as

$$U_e = \frac{1}{2}\sum_{i=1}^{N} q_i V_i \quad . \tag{2.37}$$

The potential V_i to be used in each term of the sum (2.37) is that produced by all charges excluded the i-th. In fact, if we attempted to calculate V_i taking into account also for the charge q_i, we would find meaningless results because the contribution to the potential of a point-like charge diverges at the point occupied by the charge itself.

We finally observe that, if we wanted to choose by convention a non-zero value of the potential at infinity, the work W would be given by $V(P) - V_\infty$ (still having the same value as before) and (2.37) would assume the equivalent form

$$\begin{aligned} U_e &= \frac{1}{2}\sum_{i=1}^{N} q_i\big(V_i - V_\infty\big) \\ &= \frac{1}{2}\sum_{i=1}^{N} q_i V_i - \frac{1}{2}V_\infty\sum_{i=1}^{N} q_i \quad . \end{aligned}$$

We now seek an expression for the electrostatic energy associated with a continuous charge distribution with volume density ρ (in general, variable from point to point). We assume to reach the desired charge configuration starting from a situation where the entire charge is divided in many small point-like charges dq, which can be treated as infinitesimal, located at a very large distance from each other so that we can neglect their interactions. We then assume to bring one be one the charges dq to the various points, distributing them in the volume elements $d\tau$, until reaching the desired value of ρ at each point. As always, we assume that the charges have moved very slowly, so that we can neglect their kinetic energy.

Indicating with dU_e the infinitesimal energy required to move the infinitesimal increment if charge dq' to the generic point P, when the potential at P has the value

$V'(P)$, we obtain

$$\begin{aligned} dU_e &= dq' \cdot V'(P) \\ &= d\rho' \cdot d\tau V'(P) \quad , \end{aligned} \tag{2.38}$$

where dS' is the infinitesimal increment of density at P and $d\tau$ the infinitesimal volume element centred at P.

Since the work required to build the charge distribution is independent from the order and mode in which the various contributions are brought together, we can assume to move the various parts of the system to the final value of their electrical charge simultaneously. In other words, we assume that at each instant of time during the charging process, the charge density ρ' at each point is the same fraction α of the final density ρ:

$$\rho' = \alpha\rho \quad .$$

The infinitesimal increment of charge density is, thus,

$$d\rho' = \rho \cdot d\alpha$$

and (2.38) can be rewritten as

$$dU_e = \rho(P)V'(P) \cdot d\alpha d\tau \quad . \tag{2.39}$$

Being all charges equal to the same function α of their final value, also the potential $V'(P)$ will be a fraction α of the final value $V(P)$:

$$V'(P) = \alpha V(P) \quad ,$$

and, thus, (2.39) becomes

$$dU_e = \alpha\rho(P)V(P) \cdot d\alpha d\tau \quad . \tag{2.40}$$

The total electrostatic energy is obtained by summing all terms of the type (2.40) and by varying α from 0 to 1:

$$\begin{aligned} U_e &= \int_0^1 \alpha \Big(\iiint_\tau \rho V \cdot d\tau \Big) \cdot d\alpha \\ &= \frac{1}{2} \iiint_\tau \rho V \cdot d\tau \quad . \end{aligned} \tag{2.41}$$

This result is of the same form as (2.37), with the difference that the operation of integration on a continuum has substituted the operation of sum. Note that the integral in (2.41) can be extended to any arbitrarily large region that contains τ, so long $\rho = 0$ outside τ.

Note that in (2.41) there is no need to specify how to calculate the potential. In fact, it is not necessary to exclude the contribution of the charge localized in the point where the potential is considered, because such a contribution is infinitesimally small.

Similar to (2.41), we can write the electrostatic energy associated with a surface or linear charge distributions.

$$U_e = \frac{1}{2} \iint\limits_{\Sigma} \sigma V \cdot dS \tag{2.42}$$

and

$$U_e = \frac{1}{2} \int\limits_{\gamma} \lambda V \cdot dl \quad , \tag{2.43}$$

respectively. Note that (2.43) has physical meaning only when the "wire" on which the charge is distributed has a small diameter, but not zero.

2.6 Energy density of an electrostatic field.

Consider a volume charge distribution with density ρ in a domain τ. The electrostatic energy associated with such a system is given by

$$U_e = \frac{1}{2} \iiint_\tau \rho V \cdot d\tau \quad .$$

Remembering that

$$\vec{\nabla} \bullet \vec{E} = \frac{1}{\epsilon_0} \rho \quad ,$$

we have

$$\rho V = \epsilon_0 V \vec{\nabla} \bullet \vec{E} \quad .$$

By means of the vector identity

$$\vec{\nabla} \bullet \left(f\vec{A}\right) = f\vec{\nabla} \bullet \vec{A} + \vec{A} \bullet \text{ grad } f \quad ,$$

where f and $\vec{A}$ are a generic scalar and vector field, respectively, we have

$$\begin{aligned} \rho V &= \epsilon_0 V \vec{\nabla} \bullet \vec{E} \\ &= \epsilon_0 \vec{\nabla} \bullet \left(V\vec{E}\right) - \epsilon_0 \vec{E} \bullet \text{ grad } V \quad . \end{aligned}$$

It is also

$$\vec{E} = - \text{ grad } V$$

and, thus,

$$\begin{aligned} \rho V &= \epsilon_0 \vec{\nabla} \bullet \left(V\vec{E}\right) + \epsilon_0 \vec{E} \bullet \vec{E} \\ &= \epsilon_0 \vec{\nabla} \bullet \left(V\vec{E}\right) + \epsilon_0 E^2 \quad . \end{aligned}$$

Equation (2.41) then becomes

$$U_e = \frac{1}{2} \epsilon_0 \iiint_\tau \vec{\nabla} \bullet \left(V\vec{E}\right) \cdot d\tau + \frac{1}{2} \iiint_\tau \epsilon_0 E^2 \cdot d\tau \quad .$$

From the divergence theorem it follows that

$$\iiint_\tau \vec{\nabla} \bullet \left(V\vec{E}\right) \cdot d\tau = \oiint_\Sigma V\vec{E} \bullet \vec{n} dS \quad ,$$

where $Sigma$ is the closed surface enclosing the domain τ where the charges are located. Furthermore, if the charge distribution producing the field is localized entirely in a finite region of space (all at a finite distance from a given point of reference), the domain τ and the surface Σ, which we assume to be, e.g., spherical, can be chosen so large to result in a zero surface integral $\big[$remember that the integral (2.41) can be extended to any region containing τ, so long $\rho = 0$ outside $\tau\big]$. In fact, in the limit the radius r of

Σ goes to infinity, the potential V goes to zero at least as r^{-1} and the field E as r^{-2}. Hence, the product EV goes to zero at least as r^{-3}, while the area of Σ increases as r^2. As a consequence, the surface integral goes to zero at least as r^{-1} and the electrostatic energy U_e can be written as

$$U_e = \frac{1}{2} \iiint\limits_{r_\infty} \epsilon_0 E^2 \cdot d\tau \quad , \tag{2.44}$$

so long the volume integral is extended to the entire space (indicated by r_∞). This result shows that the energy U_e can be calculated even without knowing the charge distribution, as long as the field distribution is known everywhere. Moreover, (2.44) can be formally interpreted stating that the energy U_e is distributed in the entire space with a volume energy density (or a specific energy, per unit volume)

$$\begin{aligned} u_e &= \frac{dU_e}{d\tau} \\ &= \frac{1}{2}\,\epsilon_0 E^2 \quad . \end{aligned} \tag{2.45}$$

In this way, we can attach another physical meaning to the field, which appears to the entity where the electrostatic energy of the system of charges is “stored”.

As a final remark, it is worth stressing the limitation of applicability of (2.44). Because of the way it has been derived (using, among others, the relation $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$), (2.44) cannot be used in the case of point-like linearly distributed charges. For example, using (2.44) in the case of a point-like charge, we have

$$E = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r} \quad ,$$

which, integrating over spherical shells gives

$$U_e = \lim_{r\to\infty} \frac{1}{2} \int\limits_0^r \epsilon_0 \left(\frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2} \right)^2 4\pi r^2 \cdot dr \to \infty \quad .$$

This result is obviously in contrast with the fact that the energy U_e, being defined as energy of interaction, must be zero in the case of a single point-like charge at finite distance.

2.7 Electrostatic Potential of a Double Infinite Layer.

We intend to study two parallel infinite planes at distance d from each other. Assuming the left plane to be uniformly charged with a positive surface charge density $+\sigma$ and the plane on the right with a negative charge density $-\sigma$, we found that

$$\begin{cases} \vec{E} = \vec{0}\ , & z \in \big(-\infty, 0\big) \cup \big(d, +\infty\big) \\ \vec{E} = \dfrac{\sigma}{\epsilon_0} \cdot \vec{u}_z\ , & z \in \big(0, d\big) \end{cases} \tag{2.46}$$

where z is an axis normal to both planes.

It is then easy to find the potential from

$$\begin{cases} \vec{\nabla} V = 0\ , & z \in \big(-\infty, 0\big) \cup \big(d, +\infty\big) \\ \vec{\nabla} V = -\dfrac{\sigma}{\epsilon_0} \cdot \vec{u}_z\ , & z \in \big(0, d\big) \end{cases} \tag{2.47}$$

which results in

$$\begin{cases} V = C_1\ , & z \in \big(-\infty, 0\big] \\ V = \displaystyle\int_0^z -\frac{\sigma}{\epsilon_0} \cdot dz = -\frac{\sigma}{\epsilon_0} z + C_2\ , & z \in \big[0, d\big] \\ V = C_3\,. & z \in \big[d, +\infty\big) \end{cases} \tag{2.48}$$

We can choose C_1 (e.g., $C_1 = 0$) arbitrarily. Then, when $z \to 0^+$ we find $V(z \to 0^+) = C_2 = V(z \to 0^-) = C_1$; so, $C_1 = C_2$. Finally, $C_3 = V(z \to d^-) = -\dfrac{\sigma}{\epsilon_0} d + C_1$, as shown in Fig. 2.7 (the potential must be continuous from $-\infty$ to $+\infty$).

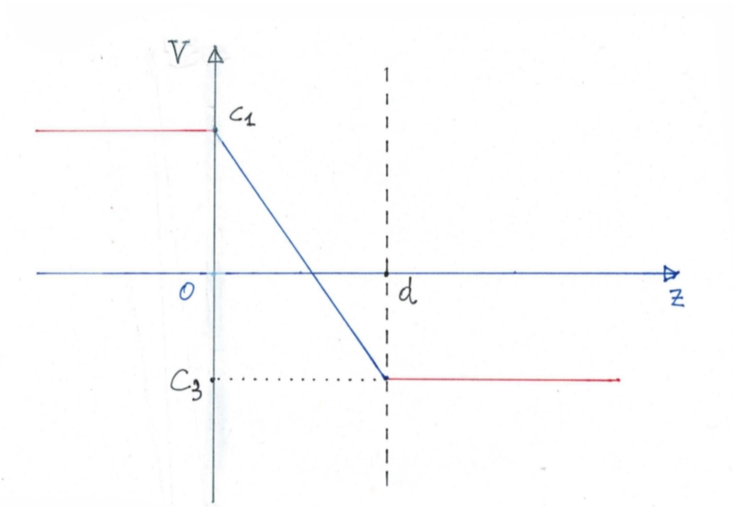


Figure 2.7

Independently from the choice of the arbitrary constants, however, the potential difference between the two planes is given by [mid equation in (2.48)]

$$V\big(d\big) - V\big(0\big) = -\frac{\sigma}{\epsilon_0} d \quad . \tag{2.49}$$

This result shows an interesting property of the double layer. Imagine to decrease the distance d between the two layers, while increasing the value of the surface charge density σ so that $\sigma d = \text{const}$. Under these conditions, the potential difference between the two layers remains constant. It is then possible to conceive a scenario where the distance $d \to 0$ and the potential V has a true discontinuity in correspondence of the distribution of charges. This property is typical for double layers and it is true even if the charge distribution is non-planar.

Another way to calculate the potential difference between two generic points P_2 and P_1 on the left and right plane, respectively, is by means of the definition of work,

$$W_\gamma = q_0 \int_\gamma \vec{E} \bullet \vec{t} d\ell = -q_0 \big[V\big(P_2\big) - V\big(P_1\big)\big] \quad . \tag{2.50}$$

It is quite clear that given a generic surface Σ, with $\vec{n}(P)$ the unit vector normal to Σ at a point P and $\vec{t}(P)$ the unit vector normal to $\vec{n}$ at P, if the field $\vec{E}(P)$ is normal to Σ at each P, then

$$\vec{E}\big(P\big) \bullet \vec{t} = 0$$

or

$$-\vec{\nabla} V\big(P\big) = 0$$

which means $V\big(P\big) = \text{const}$ at each P on Σ. This is the definition of level surface for the scalar field $V(P)$, in this case called an equipotential surface.

In the case of the double layer, from (2.46) it is clear that each of the two infinite planes is an equipotential surface. We can thus calculate the integral (2.50) on any line γ between any generic point P_2 on the left plane and any generic point P_1 on the right plane. For simplicity of calculations, we can choose the oriented line segment normal to both planes and directed from the negative (right) to the positive (left) planes (see Fig. 2.8). It results that $\vec{E}$ and $\vec{t}$ are parallel. Note that the relative sign between $\vec{E}$ and $\vec{t}$ is not necessarily negative $\big[$see text after Eq. (2.51)$\big]$. Hence, for the field (2.46),

$$\begin{aligned} -\big[V\big(P_2\big) - V\big(P_1\big)\big] &= \int_{P_1}^{P_2} \vec{E} \bullet \vec{t} \cdot d\ell \\ &= \int_{P_1}^{P_2} \frac{\sigma}{\epsilon_0} \cdot \vec{u}_z \bullet \vec{t} \cdot d\ell \\ &= \frac{\sigma}{\epsilon_0} \int_{P_1}^{P_2} \vec{u}_z \bullet \vec{t} \cdot d\ell \end{aligned} \tag{2.51}$$

In (2.51), we are integrating from a point on the right plane, P_1 at $z = d$, to a point on the left plane, P_2 at $z = 0$. Since $d > 0$, this means the line integral is defined from a point at "larger" distance to one at "smaller" distance. Therefore, particular care must be taken in choosing $\vec{t} \cdot d\ell$. The length of the straight line segment between

P_1 and P_2 along which we are integrating must be positive, $d > 0$. Assume we choose $\vec{t} \cdot d\ell = -\vec{u}_z \cdot dz$. The length P_1P_2 can be calculated as

$$\begin{aligned} \overline{P_1P_2} &= \int_d^0 -\vec{u}_z \bullet \left(-\vec{u}_z \cdot dz\right) \\ &= -\int_0^d dz = -d < 0 \quad . \end{aligned}$$

Thus, the correct choice is $\vec{t} \cdot d\ell = \vec{u}_z \cdot dz$. The integral (2.51) then becomes

$$\begin{aligned} V\left(P_1\right) - V\left(P_2\right) &= \frac{\sigma}{\epsilon_0} \int_d^0 \vec{u}_z \bullet \vec{u}_z \cdot dz \\ &= -\frac{\sigma}{\epsilon_0} d \end{aligned}$$

or

$$V\left(d\right) - V\left(0\right) = -\frac{\sigma}{\epsilon_0} d \quad ,$$

which is equal to (2.49).

Note that this result applies also to the case of a parallel plate capacitor under the condition $d << \sqrt{A}$, where d is the distance between the two plates and A the area of each plate (each plate is considered to be of finite size).

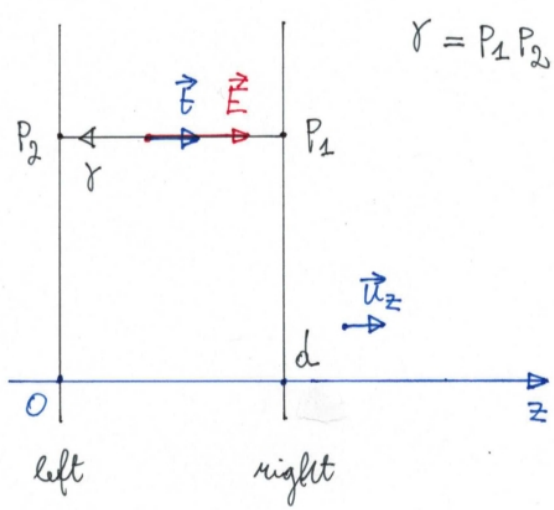


Figure 2.8

Figure 2.8 shows how to calculate the line integral (2.51). Note again that when integrating from P_1 to P_2, $\vec{t}$ and $\vec{E}$ are parallel and with the same sign.

2.8 The Potential Generated by Unlimited Charge Distributions.

We know from (2.9) and (2.10) that the potential generated by a system of N point-like charges in vacuum can be obtained by superimposing the potentials generated by each charge,

$$V\big(P\big) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\Big(\frac{q_k}{r_k} + C_k\Big) \quad , \tag{2.52}$$

where C_ks are N arbitrary constants. After choosing the constants, in the case of a limited number N of charges, we have

$$C = \sum_{k=1}^{N} C_k$$

and, thus,

$$V\big(P\big) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{q_k}{r_k} + C \quad . \tag{2.53}$$

It is always possible to choose a value of C such that V becomes zero at a given point (as long as the point is different from the points occupied by the charges). In particular, for $C = 0$ the potential becomes zero at infinity.

The scenario is quite different in the case of an unlimited number N of charges. In this case, in fact, it is not possible to choose the C_ks in a completely arbitrary manner. Before delving into the details why this is the case, it is worth considering a few examples.

2.8.1 The Potential of a Single Infinite Layer.

We know that the field generated by an infinite single layer (or infinite plane) with respect to a Cartesian coordinate system, where the axis z is normal to the plane, is given by

$$\begin{cases} \vec{E} = -\dfrac{\sigma}{2\epsilon_0}\cdot\vec{u}_z \quad , \quad z\in\big(-\infty, 0\big] & \text{(2.54a)} \\ \vec{E} = \dfrac{\sigma}{2\epsilon_0}\cdot\vec{u}_z \quad . \quad z\in\big[0, +\infty\big) & \text{(2.54b)} \end{cases}$$

It is then straightforward to obtain V at any point in space by integration, from $\vec{E} = -\operatorname{grad} V$. For $z\in(-\infty, 0]$, we use field (2.54a) and write

$$\vec{E} = -\frac{\sigma}{2\epsilon_0}\cdot\vec{u}_z = -\frac{d}{dz}V\big(z\big)\cdot\vec{u}_z$$

from which

$$dV\big(z\big) = \frac{\sigma}{2\epsilon_0}\cdot dz$$

and finally

$$\begin{aligned}\int_0^z dV(z) &= V(z) - V(O) \\ &= \frac{\sigma}{2\epsilon_0}\int_0^z dz = \frac{\sigma}{2\epsilon_0}\, z \quad .\end{aligned}$$

Calling $V(O) = C_0$, with C_0 an arbitrary constant, we can write

$$V(z) = \frac{\sigma}{2\epsilon_0}\, z + C_0 \quad . \tag{2.55a}$$

Similarly, for $z \in [0, +\infty)$, we use the field (2.54b) and write

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \cdot \vec{u}_z = -\frac{d}{dz} V(z) \cdot \vec{u}_z$$

from which

$$dV(z) = -\frac{\sigma}{2\epsilon_0} \cdot dz$$

and finally

$$\begin{aligned}\int_0^z dV(z) &= V(z) - V(O) \\ &= -\frac{\sigma}{2\epsilon_0}\int_0^z dz = -\frac{\sigma}{2\epsilon_0}\, z \quad .\end{aligned}$$

Since $V(O) = C_0$, we can write

$$V(z) = -\frac{\sigma}{2\epsilon_0}\, z + C_0 \quad . \tag{2.55b}$$

Figure 2.9 shows a plot of the potential for $z \in (-\infty, +\infty)$.

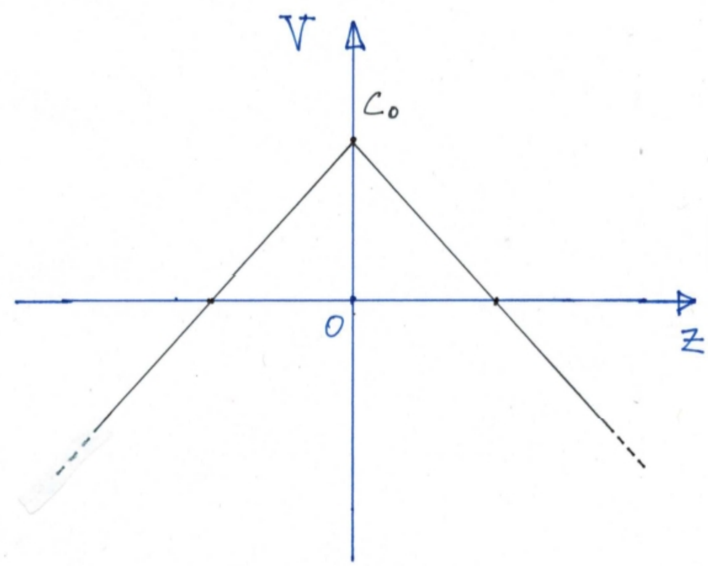


Figure 2.9

2.8.2 The Potential of a Charged Disk on the Disk's Axis.

We already studied the potential of a charged ring on the ring's axis. The potential is given by Eq. (2.18). By generalizing the notation used in that equation and calling the radius of the ring r instead of a, we have

$$V(z) = \frac{\lambda}{2\epsilon_0} \frac{r}{\sqrt{r^2+z^2}} \quad .$$

From the definition of linear charge density,

$$\lambda = \frac{q}{2\pi r} \quad ,$$

where q is the total charge on the ring, it follows that

$$V(z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2+z^2}} \quad . \tag{2.56}$$

With the result (2.56) in hand, we can easily calculate the potential of a disk uniformly charged with charge q on the disk's axis.

The surface charge density associated with the disk is constant and given by

$$\sigma = \frac{q}{\pi R^2} \quad , \tag{2.57}$$

where R is the radius of the disk.

(a) *Coordinate system.*
The coordinate system chosen to solve the problem is a cylindrical $Or\varphi z$, where O coincides with the centre of the disk and z with the disk's axis (see Fig. 2.10).

(b) *Results.*
In order to solve the problem and calculate the potential, we can divide the disk in circular crowns with radii r and $(r + dr)$, respectively, as shown in Fig. 2.10.

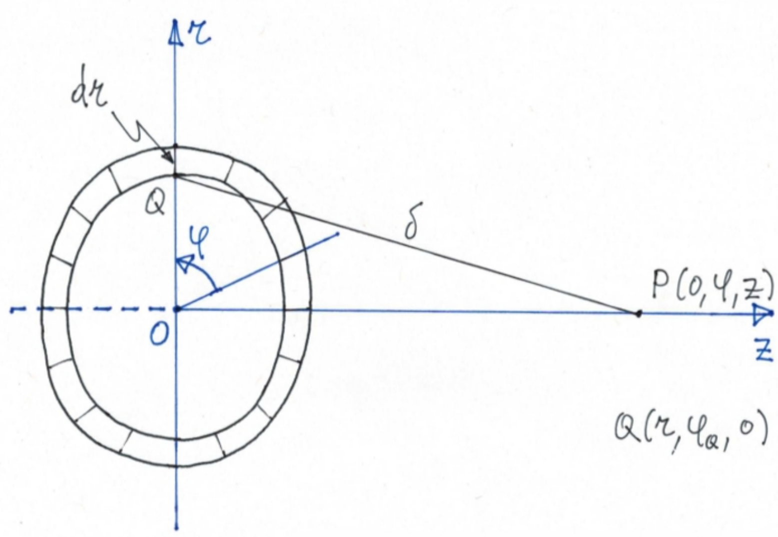


Figure 2.10

Each circular crown represents an infinitesimal ring-type structure with potential given by Eq. (2.56),

$$dV(z) = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{r^2 + z^2}} \quad ,$$

where the infinitesimal charge dq is given by the surface charge density (2.57) times the area of the circular crown,

$$dq = \sigma \cdot 2\pi r \cdot dr \quad .$$

Hence,

$$\begin{aligned} dV(z) &= \frac{1}{4\pi\epsilon_0} \frac{\sigma \cdot 2\pi r \cdot dr}{\sqrt{r^2 + z^2}} \\ &= \frac{\sigma}{2\epsilon_0} \frac{r \cdot dr}{\sqrt{r^2 + z^2}} \quad . \end{aligned} \tag{2.58}$$

By integration for $r \in [0, R]$,

$$V(z) = \frac{\sigma}{2\epsilon_0} \int_0^R \frac{r \cdot dr}{\sqrt{r^2 + z^2}} \quad . \tag{2.59}$$

Noting the triangle OQP in Fig. 2.10, we immediately find the substitution that simplifies the integral (2.59),

$$\delta^2 = r^2 + z^2 \quad . \tag{2.60a}$$

By differentiating δ^2 we then obtain

$$\cancel{2}\delta \cdot d\delta = \cancel{2}r \cdot dr \quad . \tag{2.60b}$$

Since the disk is characterized by a rotation symmetry with respect to φ and by a reflection symmetry (π rotation) with respect to the r axis (i.e., the disk is symmetric with respect to a change of sign in the z coordinate), without loosing generality we can consider the case $z \geq 0$ only. The limits of integration can then be found geometrically by inspecting the triangle OQP. When $r = 0$, $\delta = z$ and when $r = R$, $\delta = \sqrt{R^2 + z^2}$. Hence,

$$\delta^- = z \quad , \tag{2.60c}$$

$$\delta^+ = \sqrt{R^2 + z^2} \quad . \tag{2.60d}$$

By substituting (2.60a)-(2.60d) into (2.59) we find

$$\begin{aligned} V(z) &= \frac{\sigma}{2\epsilon_0} \int_{\delta^-}^{\delta+} \frac{\cancel{\delta} \cdot d\delta}{\cancel{\delta}} \\ &= \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2 + z^2} - z \right) \\ &= \frac{q}{2\pi\epsilon_0} \frac{\sqrt{R^2 + z^2} - z}{R^2} \quad , \end{aligned} \tag{2.61}$$

where we used (2.57). This solution is valid for $z \geq 0$. The solution for $z \in (-\infty, +\infty)$ must be written as

$$V(z) \, \frac{q}{2\pi\epsilon_0} \, \frac{\sqrt{R^2 + z^2} + |z|}{r^2} \quad . \tag{2.62}$$

The absolute value on z can be better understood resorting to the substitution (2.60a).

In fact, for $r = 0$

$$\delta^2 = z^2$$

and so

$$\delta = \sqrt{z^2} = z \geq 0 \quad .$$

For $z < 0$, this is possible only taking $\delta^- = |z|$, from which (2.62).

2.8.3 The Potential of a Single Infinite Layer Revisited.

A uniformly charged infinite plane can be divided in circular crowns with radii r and $(r + dr)$, respectively, exactly as in the case of the disk. In this case, however, the radius r instead of being limited to R can extend all the way to infinity (see Fig. 2.11).

With respect to an $Or\varphi z$ cylindrical system, we consider the origin O to coincide with the centre of the crowns making up the infinite plane. The potential at a generic point P is thus given by (2.58), this time integrated for $r \in [0, +\infty)$, i.e., $R \to +\infty$. For $z \in (-\infty, +\infty)$ and using the substitution (2.60a)-(2.60d), we obtain

$$\begin{aligned} V(z) &= \frac{\sigma}{2\epsilon_0} \int\limits_0^{+\infty} \frac{r \cdot dr}{\sqrt{r^2 + z^2}} \\ &= \frac{\sigma}{2\epsilon_0} \int\limits_{|z|}^{+\infty} d\delta \\ &= +\frac{\sigma}{|\sigma|} \infty \quad . \end{aligned} \tag{2.63}$$

The potential (2.63) diverges and can be either $-\infty$ or $+\infty$ depending on the sign of σ. The potential of Eq. (2.63) is in stark contrast with that of Eqs. (2.55a) and (2.55b). However, they both seem to be legitimate potential for a uniformly charged infinite plane.

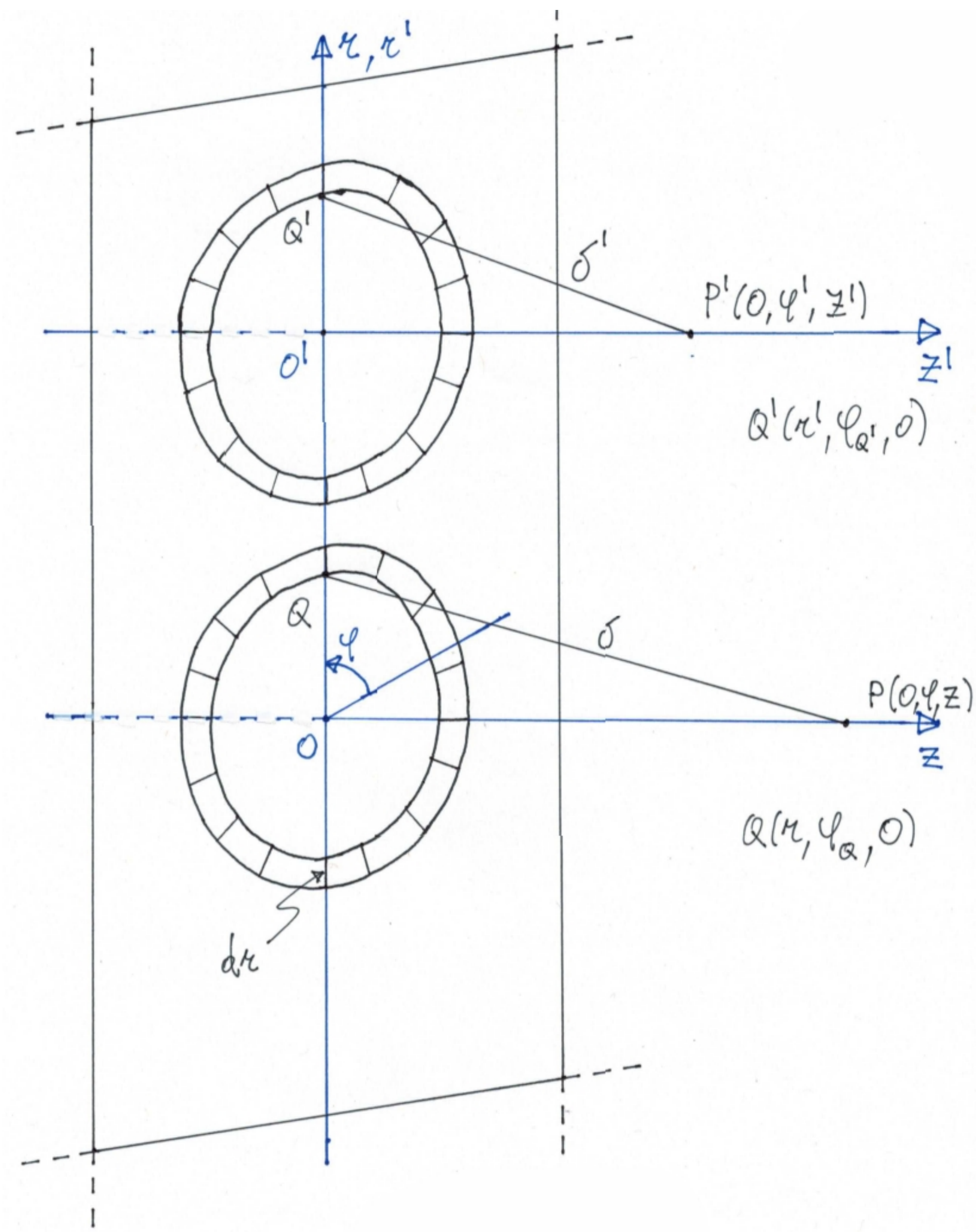


Figure 2.11

Chapter 3

The Electrostatic Field in Presence of Conductors in Vacuum.

3.1 Conductors and Insulators.

So far, we consider physical scenarios where we calculated the field given a charge distribution. It often happens, however, that the charge distribution associated with a field is a priori unknown and must be determined simultaneously with the field.

Material objects are made by an ensemble of positive and negative charges blended so that, in conditions of electrical neutrality, the total charge in any "physics infinitesimal" of an object is zero.

However, depending on the physical nature of the object, some of the object's charges are free to move within the object. As a consequence, when a neutral body (object) is immersed in an electrostatic field, charges with opposite sign, being acted upon by forces with opposite direction, they move away from each other. Such a separation destroys the local neutrality of the body and, point-by-point, allows for charge distributions. Note that, if no extra charges are externally added to the body, the body remains globally neutral. The so induced charge distribution contributes to the overall electrostatic field, which, on its own, modifies the local charge distribution in the body. Hence, the main difficulty of this problem: the electrostatic field is generated by both the external charge distributions and by the induced charges, which, on their own, depend on the resulting field.

It is clear that if the distribution of the induced charges was a priori known, the resulting field could be calculated as in the previous lectures. We could simply substitute the body with the distribution of the induced charges and calculate the field generated by all charges (internal and external to the body) as if they were acting in vacuum. (Note that, if it was possible to fix all charges in a body to their respective position, noting would happen by placing the body itself in an electrostatic field. In fact, due to the immobility of the charges the local electrostatic neutrality of the body would not be perturbed. Independently from the physical nature of the body, the body would be completely transparent for the electrostatic field. In other words, the field would be generated solely by the external charges, as if they were acting in vacuum and the body did not exists.)

However, it is impossible to do so because the charge distribution in a body is the result of a situation of equilibrium between the electrostatic forces acting on the charges and the reaction of the constraints to which the charges are subject to in the body. We can conclude saying that, a priori, it is impossible to give nor the induced charge distribution in the body, neither the resulting field. It is necessary to determine, instead, both the induced charge and the resulting field simultaneously.

The next sections will describe the phenomena that take place in conducting bodies in vacuum (or in air at normal conditions) in presence of an electrostatic field. A few methods to solve some of the problems outlined above will also be introduced.

In the beginning of this notes, we qualitatively defined insulators (also called dielectrics) as the bodies capable to store electric charge for long times and conductors at the bodies that allow a quick charge dispersion. This is only a qualitative distinction. A quantitative study needs the introduction of the concepts of conduction and polarization. In this course, we will limit the discussion to conduction only.

Conductors, in particular metals, are objects with an Avogadro number of charged particles (on the order of one per atom) free to move within the object. Those free charges (in conducting metals these are electrons, in electrolytic solutions positive and negative ions, in ionized gases positive ions and electrons, ...), within the entire region occupied by the body continuously hit each other or the "fixed" particles. In normal conditions, however, they will hardly exit the body. Those free particles can be thought as trapped in a potential well the walls of which coincide with the body's surface (as a liquid in a gravitational field is contained by a glass' walls). In other words, the external surface of the conductor behaves as if it were a charged double layer, which determines a sharp discontinuity of potential when going from inside to outside the body. It is exactly this discontinuity to determine the value of the potential barrier that prevents electrons to escape the body under normal conditions.

Hereafter, when we will talk about the potential of a conductor, we will intend the potential difference between the interior of the body and its external surface (i.e., without considering the aforementioned potential well).

Dielectrics, instead, are objects where charged particles are strongly attached to the atoms and molecules to which they belong. Under the action of an external electrostatic field, those particles can only move by a very small distance compared to their equilibrium position and remain attached to the corresponding atoms and molecules.

It is clear that this distinction between conductors and dielectrics refers to an ideal scenario. In reality, also in an insulator there are unbounded charged particles that are free to move under the action of an electrostatic field. Those particles, however, are in a much smaller number than in conductors and their "mobility" is much smaller. As we will see in a few lectures when studying electrical conduction, the attitude to conduct electricity, which distinguishes conductors from insulators, can be measured by means of a suitable physical quantity called electrical conductivity. For a typical insulator (such as glass, rubber or plastic), the conductivity is approximately 10^{20} times smaller than in a good conductor (such as copper, silver or aluminium).

3.2 Electrostatic Equilibrium in Homogeneous Conductors.

Consider a homogeneous and isotherm conductor. Homogeneous means that is all made of the same material and isotherm that is all at the same temperature (i.e., the conductor is assumed to be at the thermodynamic equilibrium at each point). These two conditions are required to exclude the presence of electromotive fields. The conductor is said to be in electrostatic equilibrium when no macroscopic motion of charges takes place within the conductor. This is possible if there are no forces acting on the free charges and, thus, the total macroscopic electrostatic field is zero at each point inside the conductor. To illustrate this statement, consider an initially neutral body that is then located in an external electrostatic field $\vec{E}_{\rm ex}$ generated by a set of charge distributions outside the conductor. Under the effect of $\vec{E}_{\rm ex}$, the free charges in the conductor move, thus, perturbing the conductor local neutrality. Even though the conductor remains overall neutral (because no charge has been added to or removed from the body), a distribution of charge with both negative and positive sign is found in it. Such a distribution generates a reaction field $\vec{E}_r$ that is superimposed to $\vec{E}_{\rm ex}$, generating a resulting field

$$\vec{E} = \vec{E}_{\rm ex} + \vec{E}_r \quad .$$

The dynamics evolves until $\vec{E}_r$ counter-balances the action of $\vec{E}_{\rm ex}$ at each point inside the conductor. Under these conditions,

$$\vec{E} = \vec{E}_{\rm ex} + \vec{E}_r = \vec{0}$$

and the free charges, not being acted upon by any acceleration, stop at a final configuration of electrostatic equilibrium. We can now ask the question whether the conductor always reaches a situation of electrostatic equilibrium or it is possible that this does not happen. It is easy to be convinced that, at least in normal conditions, a state of equilibrium must always to be reached. In fact, if this was not the case, the charges would continue to move indefinitely within the conductor, continuously bouncing against the fixed particles and, thus, transferring to them part of the kinetic energy gained under the action of the total field. This process cannot continue indefinitely if the conductor is acted upon by an electrostatic field because the field is irrotational. This problem is similar to that of a little ball bouncing off a floor under the action of the gravitational field. The ball cannot continue bouncing indefinitely because of the energy transferred to the floor (or any other medium in the proximity of the ball). An exception is that of special type of materials that, when cooled to very law temperatures (on the order of a few k) become superconducting. For these materials, even if the free electrons continually hit against the fixed lattice ions, they do not transfer energy to them because of perfect elastic scattering. Once a macroscopic movement of electrons is imitated, it will continue for even decades. In a normal conductor at room temperature, we can be sure that this does not happen and a state of equilibrium is rapidly reached (in a good conductor the time required to reach equilibrium, called the relaxation time, is on the order of $10^{-18} - 10^{-19}$s). Similar conclusions are reached also when an external charge is added to the conductor. The charge will distribute on the conductor such that, at equilibrium, the electrostatic field inside the conductor is zero everywhere.

In summary, from the macroscopic point of view a conductor in macroscopic equilibrium is a region inside which $\vec{E} = \vec{0}$ at each point. As a consequence, because of Gauss' theorem

$$\rho = 0$$

everywhere inside the conductor.

Thus, it is impossible for charges to localize at any point inside a conductor in electrostatic equilibrium.

The scenario is quite different at the external surface of a conductor. In fact, at the interface between the conductor and the space outside, forces due to the potential barrier previously discussed can balance the forces produced by the electrostatic field, thus preventing charges from leaving the conductor. In equilibrium, a charge with density σ (in general, variable from point to point) distributes on the surface of the conductor so that $\vec{E} = \vec{0}$ at each point inside the conductor. Outside the conductor, the action of σ superimposes to that of all other external charge distributions, resulting in an electrostatic field which is the same as if all surface charges on the conductor were "frozen" (in equilibrium) and the conductor was removed altogether. The presence of a surface charge distribution with density σ on the surface Σ of the conductor implies a discontinuity of the first kind of the normal component of $\vec{E}$ at each point on Σ (note that this is true even if the double layer that determines the characteristic potential barrier of the conductor is superimposed to σ).

Referring, for example, to Fig. 3.1, we find

$$\big[E_n\big] = \frac{\sigma}{\epsilon_0} \quad , \tag{3.1}$$

where $[E_n]$ indicates the difference between the values of the component of $\vec{E}$ normal to Σ for each pair of points with one point slightly above and one slightly below the surface. The volume of the conductor is associated with a charge density ρ, which, by means of Gauss' theorem, we have shown to be zero at each point in the conductor. The surface of the conductor is, instead, associated with a non-zero charge density σ. It is thus clear that the conductor behaves as if it was a surface charge distribution with density σ, leading directly to case 2,

$$\vec{n} \,\boldsymbol{\cdot}\, \big(\vec{E}_1 - \vec{E}_2\big) = \frac{\sigma}{\epsilon_0} \quad . \tag{3.2}$$

Following the notation in Fig. 3.1, $\vec{E}_2$ is the field inside the conductor that we know must be zero,

$$\vec{E}_2 = \vec{0}$$

at each point in the conductor. The unit vector $\vec{n}$ is normal to Σ at a generic point P_0 on Σ and points outside the conductor. The field $\vec{E}_1 = \vec{E}$ is the field outside the conductor. This field is initially evaluated at a point $P \neq P_0$ outside the conductor. In the limit that P approaches P_0 from outside the conductor, $\vec{E}$ and $\vec{n}$ are parallel and (3.2) becomes

$$\lim_{P \to P_0^+} E_n\big(P\big) = \frac{\sigma\big(P_0\big)}{\epsilon_0} \quad , \tag{3.3}$$

where $\sigma\big(P_0\big)$ is the value of σ at P_0. Since, in general, the surface density σ varies from point to point on the surface of the conductor, also E_n varies from point to point on Σ.

Similarly, from condition (1.84) we find

$$\lim_{P \to P_0^+} E_t\big(P\big) = 0 \quad ,$$

which means the field $\vec{E}$ does not have any tangent component at each point on Σ.

In summary, given a conductor in electrostatic equilibrium in vacuum, the electrostatic field at each point in proximity of the conductor surface (outside the conductor) is normal to the surface. The statement just proven goes under the name of Coulomb's theorem.

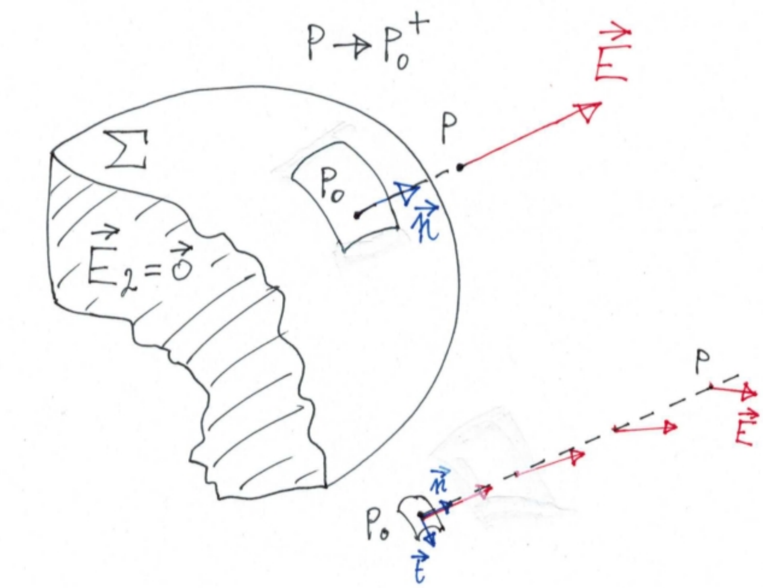


Figure 3.1

Coulomb's theorem can also be demonstrated from the concept of potential. In fact, since $\vec{E} = \vec{0}$ inside the conductor, it follows that $\vec{\nabla} V = \vec{0}$ and, thus, $V = const.$ inside the conductor. An argument similar to the Cauchy principle value for $\vec{E}$ in a volume Ω allows us to show that V is continuous and limited even in correspondence of a surface charge distribution, so ling the associated charge density σ is continuous and limited at each point on the surface (we remind that $\vec{E}$ is not continuous in correspondence of surface charge distributions).

Consider a surface Σ in the 3D Euclidean space, charges with variable density σ. At any point P on Σ,

$$V\big(P\big) = \frac{1}{4\pi\epsilon_0} \iint\limits_{\Sigma} \frac{\sigma\big(Q\big)}{r_{QP}} \cdot dS \quad , \tag{3.4}$$

where Q is a generic point on Σ. When $P = Q$, $r_{QP} = 0$ and the integrand in (3.4) is singular (diverges to infinity). We can thus define an improper integral, where we remove by surgery a circular neighbourhood Σ_n, with radius δ_n, of point P and assume Σ to be a circle with radius $R\big(>\delta_n\big)$,

$$\iint\limits_{\Sigma} \frac{\sigma}{r}\cdot dS \rightarrow \lim_{\delta_n\to 0} \iint\limits_{\Sigma-\Sigma_n} \frac{\sigma}{r}\cdot dS \quad , \tag{3.5}$$

where we defined $r_{QP}=r$. Assuming $\sigma\in C^{\circ}(\Sigma)$ and therein limited and Σ_n to be a series of circles with radii δ_n decreasing with larger n, we can integrate (3.5) in a polar coordinate system $Or\varphi$ centred in P,

$$\begin{aligned}\iint\limits_{\Sigma-\Sigma_n} \frac{\sigma}{r}\cdot dS &= \int_0^{2\pi} d\varphi \int_{\delta_n}^{R} \cancel{r}\cdot dr\cdot\frac{\sigma}{\cancel{r}} \\ &= 2\pi\sigma\big(R-\delta_n\big) \quad .\end{aligned} \tag{3.6}$$

In the limit

$$\lim_{\substack{n\to+\infty \\ (\delta_n\to 0^+)}} 2\pi\sigma\big(R-\delta_n\big) = 2\pi\sigma R \quad , \tag{3.7}$$

which is clearly continuous and limited on Σ, thus proving our thesis.

We are now in a position to safely state that, the potential is constant both inside a conductor and on the conductor's surface Σ, $V=V_\Sigma$. By definition, the conductor's surface Σ is an equipotential surface. As a consequence, the work of the field forces to move a test charge q_0 between two points P_1 and P_2 on Σ along any oriented line (from P_1 to P_2) also entirely on Σ, γ_Σ, is

$$\begin{aligned} W_{\gamma_\Sigma} &= q_0 \int\limits_{\gamma_\Sigma} \vec{E}\bullet\vec{t}\cdot d\ell \\ &= q_0\big[V\big(P_1\big)-V\big(P_2\big)\big] \\ &= q_0\big[V_\Sigma - V_\Sigma\big] = 0 \quad .\end{aligned} \tag{3.8}$$

Equation (3.8) is verified only if $E=0$ or $\vec{E}$ is normal to $\vec{t}$ at each point on γ_Σ. Assuming $E\neq 0$, it means that $\vec{E}$ must be normal to the conductor's surface Σ at each point on Σ. We can thus write

$$E_n\big(P\big) = -\frac{\partial}{\partial_n} V\big(P\big) \quad , \tag{3.9}$$

where (∂/∂_n) indicates the derivative outside the conductor at point P along the normal $\vec{n}(P)$ on Σ.

All discussed so far continues to apply even if a charge q is present on the conductor (q has to be considered here as an "extra" charge on the conductor). In this case, the charge q redistributes on the surface of the conductor with a surface density σ, in general variable from point to point. The value of q can readily be calculated as

$$q = \iint\limits_{\Sigma} \sigma\cdot dS = \epsilon_0 \oiint\limits_{\Sigma'} E_n\cdot dS \quad , \tag{3.10}$$

where $\Sigma' \supseteq \Sigma$ is a closed surface including the entire conductor (without any other charge except for those localized on the conductor itself).

Consider a charged conductor and assume to know the charge distribution σ on Σ (the surface of the conductor). Furthermore, consider a generic point P on σ and an infinitesimal surface element dS centred at P. We intend to calculate the force acting on $\sigma \cdot dS$. The field $\vec{E}$ on the outer surface Σ^+ of the conductor is normal to the surface with absolute value given by

$$E = \frac{\sigma}{\epsilon_0} \quad . \tag{3.11}$$

At first glance, we are tempted to conclude that the force on dS is

$$dF = \sigma \cdot dS \cdot E = \frac{\sigma^2}{\epsilon_0} \cdot dS \quad . \tag{3.12}$$

As it turns out, this result is incorrect. In fact, the correct value of the force is half the value given by (3.12). This is due to the general definition of electrostatic field, according to which, in order to calculate the force on a charge at P, the charge at P must ideally be removed while maintaining all the other charges fixed and calculating the field generated by those charges. In our case, the value of $\vec{E}$ that must be multiplied by $\sigma \cdot dS$ is not that given by (3.11), which is generated by all charges on Σ including $\sigma \cdot dS$ itself, but that obtained from (3.11) by subtracting the contribution due to $\sigma \cdot dS$. In order to calculate the contribution of $\sigma \cdot dS$ to the field, we can consider two points P_1 and P_2 in proximity of P, on one and the other side of Σ, respectively. We indicate the field at P_1 and P_2 generated by all charges on Σ except $\sigma \cdot dS$ as $\vec{\vec{E}}_1$ and $\vec{\vec{E}}_2$, respectively. Moreover, we indicate the field at P_1 and P_2 generated by $\sigma \cdot dS$ as $\vec{E}_1$ and $\vec{E}_2$, respectively (see Fig. 3.2). The field inside the conductor is zero, while outside it must verify Coulomb's theorem, hence

$$\begin{cases} \vec{\vec{E}}_1 + \vec{E}_1 = \vec{0} \quad , & (3.13a) \\ \vec{\vec{E}}_2 + \vec{E}_2 = \vec{E} = \dfrac{\sigma}{\epsilon_0}\vec{n} \quad , & (3.13b) \end{cases}$$

where $\vec{n}$ is the unit vector normal to Σ and directed outside the surface. The fields $\vec{E}_1$ and $\vec{E}_2$ are the same as the fields on two sides of an infinite layer. Thus, they are both directed along $\vec{n}$, they have the same absolute value, but opposite sign. The fields $\vec{\vec{E}}_1$ and $\vec{\vec{E}}_2$ are essentially evaluated at the same point and, thus, they are the same. In summary,

$$\begin{cases} \vec{E}_2 = -\vec{E}_1 \quad , & (3.14a) \\ \vec{\vec{E}}_2 = \vec{\vec{E}}_1 = \vec{\vec{E}} \quad . & (3.14b) \end{cases}$$

By summing each side of (3.13a) with each side of (3.13b), we obtain

$$2\vec{\vec{E}} = \vec{E} = \frac{\sigma}{\epsilon_0}\vec{n} \quad . \tag{3.15}$$

In order to calculate the force acting on the charge $\sigma \cdot dS$ is sufficient to multiply such a charge by $\vec{\tilde{E}}$,

$$\begin{aligned} d\vec{F} &= \sigma \cdot dS \cdot \vec{\tilde{E}} = \sigma \cdot dS \cdot \frac{\vec{E}}{2} \\ &= \frac{\sigma^2}{2\epsilon_0} \cdot dS \cdot \vec{n} \quad . \end{aligned} \tag{3.16}$$

The force acting on the charge $\sigma \cdot dS$ on the conductor is thus normal to Σ, directed outside the surface regardless from the sign of σ [the density is squared in (3.16)], and proportional to the area dS.

The forces acting on Σ thus behave similarly to those deriving from a pressure

$$p_e = \frac{\sigma^2}{2\epsilon_0} \quad , \tag{3.17}$$

which has units force per unit area and is called electrostatic pressure. We can thus imagine the electrostatic equilibrium on the surface of a conductor as due to the pressure of a fictitious fluid that fills the conductor and tends to expel its charges, in equilibrium with restoring forces on the conductor surface that prevents the charges from escaping. The restoring forces are due to the potential barrier at the conductor surface.

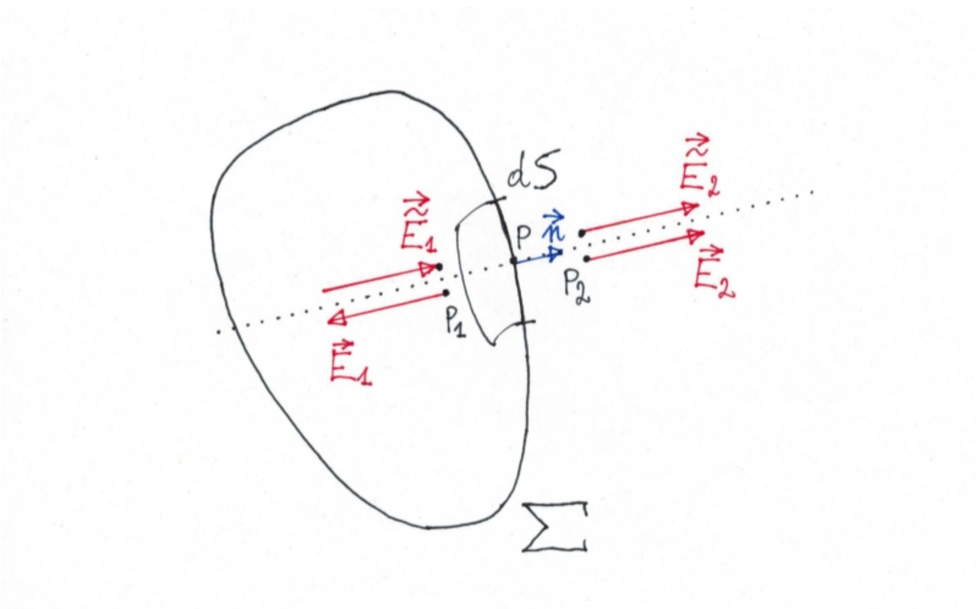


Figure 3.2

3.3 Field Calculation in Presence of Conductors.

Consider an initially neutral conductor to which a charge q is applied (q is assumed to be finite). This charge distributes on the conductor surface Σ with density σ (variable from point to point) such that the field inside the conductors is zero (at each point). We assume σ produces a potential V associated with a field $\vec{E}$. We indicate the internal and external regions of the conductor as Ω^- and Ω^+, respectively, as shown in Fig. 3.3.

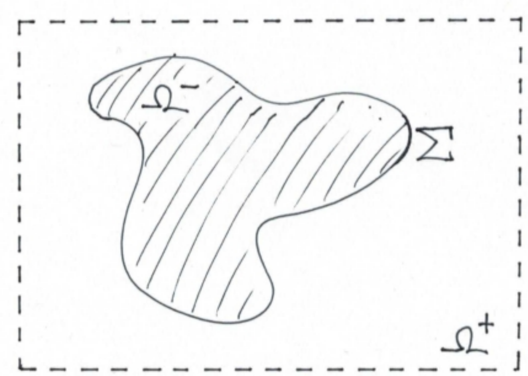


Figure 3.3

The potential function V must satisfy the following conditions:

a) It must be continuous at each point in Ω^+, including the frontier Σ.

b) At each point in Ω^+, where $\rho = 0$, it must satisfy Laplace equation

$$\vec{\nabla}^2 V = 0 \quad .$$

c) It must have a constant value at each point in Ω^-, including each point on Σ.

d) It must verify the condition

$$-\oiint_{\Sigma^+} \frac{\partial}{\partial_n} V \cdot dS = \frac{1}{\epsilon_0} q \quad .$$

e) It must verify the condition

$$\lim_{P \to \infty} V\big(P\big) = 0 \quad .$$

Condition a) is due to the property of the potential to be a continuous function even in correspondence to surface charge distributions. As always, the presence of the potential barrier due to the double layer at the conductor surface is not accounted for. Condition b) accounts for the absence of any charges outside the conductor. About condition c), it is worth mentioning that, a priori, we only know that V has to be constant on Σ, but its value is unknown. In fact, we are not allowed to arbitrary choose a value because condition e) must be fulfilled. As for condition d), the integral refers to the outer conductor surface Σ^+, since the normal derivative of V is discontinuous at each point on Σ. At last, condition e) is due to the fact that the charge q distributed with density σ entirely resides at a finite region of space. It is clear that this is true because the considered conductor is assumed to be limited in space. Obviously, we could have chosen any non-zero constant value at infinity by modifying condition e). We remind that condition e) means that for any $\epsilon > 0$, it is possible to find a sphere (enclosing the conductor) outside of which V has values smaller than ϵ (regular condition at infinity).

3.3.1 Conducting Sphere.

Consider a conducting sphere in vacuum, with radius R and centre O. A charge q is applied to the sphere.

With respect to a spherical coordinate system $Or\theta\varphi$, the sphere is characterized by a rotation symmetry with respect to both θ and φ. Hence, the potential V, which we intend to calculate, depends only on the distance r from the centre O of the sphere. Form condition b), outside the conducting sphere for $r \in (R, +\infty)$, we can write Laplace equation for V in spherical coordinates as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}V\right) = 0 \quad . \tag{3.18}$$

In this case, Laplace equation is a simple ordinary differential equation

$$\frac{1}{r^2}2r\frac{d}{dr}V + \frac{1}{r^2}r^2\frac{d^2}{dr^2}V = \frac{d^2}{dr^2}V + \frac{2}{r}\frac{d}{dr}V = 0 \quad . \tag{3.19}$$

It can be simply verified that a general solution to (3.19) is

$$V(r) = \frac{A}{r} + B \quad , \tag{3.20}$$

where A and B are arbitrary constants. Conditions a) and c) can easily be verified. In fact, by arbitrarily setting A and B, $V(r) \in C^\circ(r \in [R, +\infty))$ and

$$V(R) = \frac{A}{R} + B \quad , \tag{3.21}$$

which means $V(r)$ is constant at each point on Σ (and, thus, at each point inside the sphere). Condition e) is readily verified by imposing $B = 0$,

$$V(r) = \frac{A}{r} \quad . \tag{3.22}$$

At each point on Σ^+

$$\frac{\partial}{\partial_n}V\Big|_{\Sigma^+} = \frac{\partial}{\partial_r}V\Big|_{r=R^+} = -\frac{A}{R^2} \quad . \tag{3.23}$$

From condition d) we then obtain

$$-\oiint_{\Sigma+} \frac{\partial}{\partial_n} V \cdot dS = \frac{A}{R^2} 4\pi R^2 = \frac{1}{\epsilon_0} q \quad , \tag{3.24}$$

from which

$$A = \frac{q}{4\pi\epsilon_0} \quad . \tag{3.25}$$

The solution for V is thus

$$\begin{cases} V\big(r\big) = \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{R} \ , \quad r \in \big[0, R\big] & \text{(3.26a)} \\ V\big(r\big) = \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{r} \quad . \quad r \in \big[R, +\infty\big] & \text{(3.26b)} \end{cases}$$

As expected, for $r > R$ the potential corresponds to that of a point-like charge q located at O.

3.4 The Field in Hollow Conductors: Electrostatic Shields.

Consider a conductor Γ characterized by an internal cavity Ω as shown in Fig. 3.4. A charge q is applied to the conductor. At first glance, we might expect that q distributes both on the external surface Σ_{ext} and on the internal surface Σ_{int} of the conductor. We will prove that, however, the charge only distributes on Σ_{ext}.

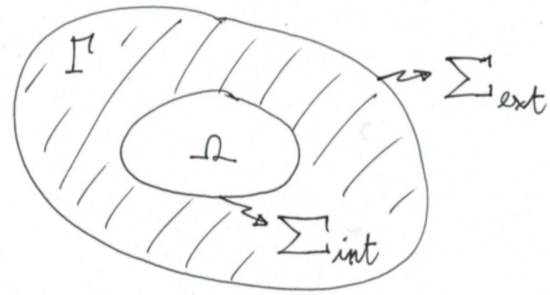


Figure 3.4

This can be shown by considering the potential function $V(P)$ solution to the problem. Due to the absence of any charges inside the inner cavity Ω, at each point in the region Ω bounded by the surface Σ_{int} the electrostatic potential $V(P)$ must satisfy Laplace equation,

$$\vec{\nabla}^2 V = 0 \quad .$$

In addition, $V(P)$ must have a constant value V_0 at each point in the conductor, including each point on Σ_{int}. In summary,

$$V\big(P\big) \in C^{\circ}\big(\Omega \cup \Sigma_{\text{int}}(\cup\Gamma)\big) \quad , \tag{3.27a}$$

must satisfy

$$\vec{\nabla}^2 V = 0 \, , \qquad \forall P \in \Omega \tag{3.27b}$$

and

$$V\big(P\big) = V_0 \, . \quad \forall P \in \Sigma_{\text{int}}\,(\cup\Gamma) \tag{3.27c}$$

Mathematically, this type of partial differential equation (PDE) problem is called an *internal Dirichlet problem* for a constant value of the potential on the frontier (boundary) of the region Ω. In brief, this problem can be summarized as

$$\begin{cases} \vec{\nabla}^2 V\big(P\big) & = 0 \quad , \quad \forall P \in \Omega \\ V\big(P\big)\Big|_{\Sigma_{\text{int}}} & = V_0 \quad . \end{cases}$$

under the assumption that $V(P) \in C^{\circ}(\Omega \cup \Sigma_{\text{int}}(\cup \Gamma))$. By definition, the function $V(P)$ is harmonic in Ω because it satisfies Laplace equation in Ω (see vector calculus). From Green's first identity,

$$\begin{aligned}
\oiint\limits_{\Sigma} f \frac{\partial}{\partial n} g \, dA &= \oiint\limits_{\Sigma} f \left(\vec{\nabla} g\right) \cdot \vec{n} dA \\
&= \iiint\limits_{\Omega} \left(\vec{\nabla} f\right) \cdot \left(\vec{\nabla} g\right) dV \\
&\quad + \iiint\limits_{\Omega} f\left(\vec{\nabla}^2 g\right) dV \quad ,
\end{aligned} \tag{3.28}$$

where f and g are harmonic functions defined in a region Ω of the 3D Euclidean space with frontier Σ and $\partial g/\partial n$ is the directional derivative of $g(P)$ with respect to a normal unit vector $\vec{n}$ at a generic point P on Σ directed outside the surface (for the external Dirichlet problem) and inside the surface (for the internal Dirichlet problem). In the case $f = g = V$ we find

$$\begin{aligned}
\oiint\limits_{\Sigma_{\text{int}}} V\left(P\right) \frac{\partial}{\partial n} V\left(P\right) dA &= \iiint\limits_{\Omega} \left|\left|\vec{\nabla} V\left(P\right)\right|\right|^2 dV \\
&\quad + \iiint\limits_{\Omega} V\left(P\right) \vec{\nabla}^2 V \, dV \\
&= \iiint\limits_{\Omega} \left|\left|\vec{\nabla} V\left(P\right)\right|\right|^2 dV \\
&\quad + \iiint\limits_{\Omega} V\left(P\right) \times 0 \times dV \\
&= \iiint\limits_{\Omega} \left|\left|\vec{\nabla} V\left(P\right)\right|\right|^2 dV \quad ,
\end{aligned} \tag{3.29}$$

where we used (3.27b) to calculate the second volume integral. Under the boundary condition (3.27c), (3.29) becomes

$$\begin{aligned}
\oiint\limits_{\Sigma_{\text{int}}} V\left(P\right) \times 0 \times dA &= 0 \\
&= \iiint\limits_{\Omega} \left|\left|\vec{\nabla} V\left(P\right)\right|\right|^2 dV \quad ,
\end{aligned}$$

(note that $\left|\left|\vec{\nabla} V\left(P\right)\right|\right|^2$ has absolutely nothing to do with the scalar laplacian) from which it follows that

$$\vec{\nabla} V\left(P\right)\Big|_{\Omega} = 0 \, .$$

This implies that

$$V\left(P\right)\Big|_{\Omega} = V_0 \,. \tag{3.30}$$

Because of the continuity of $V(P)$, condition (3.27a), it has to be

$$V\left(P\right)\Big|_{\Omega\cup\Sigma_{\text{int}}} = V_0 \,. \tag{3.31}$$

Since the conductor is in microscopic electrostatic equilibrium, it has to be

$$\vec{E}\left(P\right)\Big|_{\Gamma} = -\vec{\nabla}V\left(P\right)\Big|_{\Gamma} = \vec{0} \tag{3.32}$$

and, thus,

$$V\left(P\right)\Big|_{\Gamma} = V_1 \,, \tag{3.33}$$

but since $V(P)$ must be a continuous function,

$$V\left(P\right)\Big|_{\Omega\cup\Sigma_{\text{int}}\cup\Gamma} = V_1 = V_0 \,. \tag{3.34}$$

Finally, from (3.34)

$$\vec{E}\left(P\right)\Big|_{\Omega\cup\Sigma_{\text{int}}\cup\Gamma} = -\vec{\nabla}V\left(P\right)\Big|_{\Omega\cup\Sigma_{\text{int}}\cup\Gamma} = \vec{0} \quad . \tag{3.35}$$

In particular, the normal component of $\vec{E}$ in the neighborhood of each point on $Q \in \Sigma_{\text{int}}$ is zero. From Coulomb's theorem, calling any point on $\Sigma_{\textbf{int}}$ as Q,

$$\lim_{P\to Q^+} \vec{E}(P) \,\cdot\, \vec{n} = \frac{\sigma(Q)}{\epsilon_0} = 0 \;\;,\;\; \forall Q \in \Sigma_{\textbf{int}} \;\; ; \tag{3.36}$$

the surface charge density σ must be zero everywhere on Σ_{int}. Hence, the entire applied charge must distribute only on the external surface Σ_{ext}.

The key of this demonstration is in that Green's first identity allows us to show that $\vec{E}$ must be zero *inside* the volume Ω and, thus, there cannot be any charge distribution on $\Sigma_{\textbf{int}}$, which would otherwise generate a nonzero electrostatic field! It is important to realize that the key word here is *inside*; i.e., the mere boundary condition (3.27c) is not sufficient to demonstrate that $\vec{E} = \vec{0}$ inside Ω. You can think of a conducting sphere: The electrostatic potential is constant on the sphere's surface, but the electrostatic field is not zero immediately outside such a surface; in fact, if there is a charge q on the sphere, the field is definitely not zero immediately outside the sphere, signifying the presence of a source charge on the sphere. All these examples more clearly show that Coulomb's theorem has to be intended in the limit approaching a surface, not right on the surface.

A similar argument makes it possible to show that any charge distribution outside the conductor, i.e., not applied to the conductor directly but existing somewhere away from it, does not produce any field inside the cavity. In other words, a hollow conductor behaves as an electrostatic shield with respect to the cavity.

Note that, in reality, practical electrostatic shields are not completely closed hollow conductors (very hard to implement experimentally). Instead, they consist of metallic grids called Faraday cages. These are lattices of wires with smaller or larger unit cells, depending on the applications. In fact, it is possible to show (it is rather intuitive) that the fields due to charge distributions external to a cage "penetrate" through the cage cells only to a distance of a few cell's linear lengths. For example, a room the walls, floor, and ceiling of which incorporate a Faraday cage are electrostatically shielded ambients. Certain laboratories feature Faraday cages in order not to perturb experiments due to electromagnetic interference (EMI).

The internal Dirichlet problem (3.27a)-(3.27c) is based on both Gauss' theorem and the irrotational property of $\vec{E}$ in local form. In fact, Laplace equation (3.27b) was obtained from the differential form of both Maxwell's equations for the electrostatic field. Conclusion similar to (3.34) and (3.35) can be found from the integral form of Maxwell's equations.

Consider Fig. 3.5, representing again a hollow conductor Γ with an inner cavity Ω. A charge q is applied to the external surface Σ_{ext} of Γ. In order to show that $\vec{E} = \vec{0}$ at each point in $\Omega \cup \Sigma_{\text{int}}$, where Σ_{int} is the internal surface of Γ, we can use Gauss' theorem in integral form by defining a Gaussian surface Σ that encloses entirely $\Omega \cup \Sigma_{\text{int}}$ and, at the same time is entirely contained within the conducting region Γ. Under these conditions (see Fig. 3.5),

$$\Phi_\Sigma = \oiint_{\Sigma} \vec{E} \bullet \vec{n} \cdot dS = 0 = \frac{1}{\epsilon_0} q_\Sigma \quad , \tag{3.37}$$

where we used the condition $\vec{E} = \vec{0}$ at each point in Γ. The total flux through Σ is zero and, as a consequence, the total charge inside Σ must be zero, $q_\Sigma = 0$. Since the inner cavity Ω is assumed to be empty, this result also implies that the total charge on Σ_{int} must be zero, $q\big|_{\Sigma_{\text{int}}} = 0$.

In general, the result of Gauss' theorem of Eq. (3.37) only allows us to conclude that there must be equal amount of positive and negative charge on Σ_{int} for the total charge $q\big|_{\Sigma_{\text{int}}} = 0$. In other words, there could be a positive surface charge distribution on one area of Σ_{int} and a negative one on some different area, as shown in Fig. 3.5. Citing Feynman, "Such a thing cannot be ruled out by Gauss' law $\big[$theorem$\big]$." One more time, Gauss' theorem alone is insufficient to fully characterize the electrostatic field; the irrotational property of $\vec{E}$ must also be invoked.

Before proceeding further, we note that in simple case of a spherical conducting shell, only by means of (3.37) we could reach the final conclusion that there must be no charge at any point inside the shell, including its internal surface (note that the conducting shell is amongst the simplest types of hollow conductors).

This result can be obtained from symmetry arguments. In fact, because of the rotation symmetry of the shell with respect to both meridian and parallel lines, the asymmetric accumulation of, e.g., a positive surface charge on one part of the shell inner surface and, thus, of a corresponding negative surface charge on another part must be excluded. The only charge distribution on the inner surface that satisfies the symmetries is a zero charge distribution. This, however, could not be true for a hollow conductor of arbitrary geometry, for which symmetry arguments do not apply.

The physical process that takes place on the internal surface Σ_{int} of an arbitrary hollow conductor is that all equal and opposite charges on Σ_{int} move on the surface until they meet and cancel each other completely.

This can be shown by means of the irrotational property of $\vec{E}$. As shown in Fig. 3.5, assume a positive surface charge is distributed on one area of Σ_{int}. From (3.37) we know that there must be an equal negative surface charge distributed on some other area of Σ_{int}. Since, again, Ω is an empty space (i.e., it does not contain any free charges), in Ω there will be a field $\vec{E}$ with field lines directed from the positive charges to the negative charges (as in the case of the electrostatic dipole; note that, technically, $\vec{E}$ is in $\Omega \cup \Sigma_{\text{int}}$). We can now use the irrotational property of $\vec{E}$ in integral form by defining a closed line γ that crosses $\Omega \cup \Sigma_{\text{int}}$ along the field line between one positive charge and one negative charge and that returns to the initial point though Γ (see Fig. 3.5). Under these conditions,

$$\begin{aligned}\oint\limits_{\gamma} \vec{E} \cdot \vec{t} \cdot d\ell &= \int\limits_{\Omega \cup \Sigma_{\text{int}}} E \cdot d\ell + \int\limits_{\gamma_\Gamma} O \cdot d\ell \\ &\neq 0 \quad , \end{aligned} \tag{3.38}$$

where $\gamma_{\Omega \cup \Sigma_{\text{int}}}$ is the portion of γ in $\Omega \cup \Sigma_{\text{int}}$ and we used the condition $\vec{E} = \vec{0}$ at each point in Γ. The scalar product in the first integral simply becomes $E \cdot d\ell$ because of the definition of field line! The result (3.38) contradicts the irrotational property. The only case when the property remains valid is for $\vec{E} = \vec{0}$ at each point in $\Omega \cup \Sigma_{\text{int}}$. We thus obtain the same result as (3.35).

This also means there can be no charges on Σ_{int}. Note that this is true for an empty Ω.

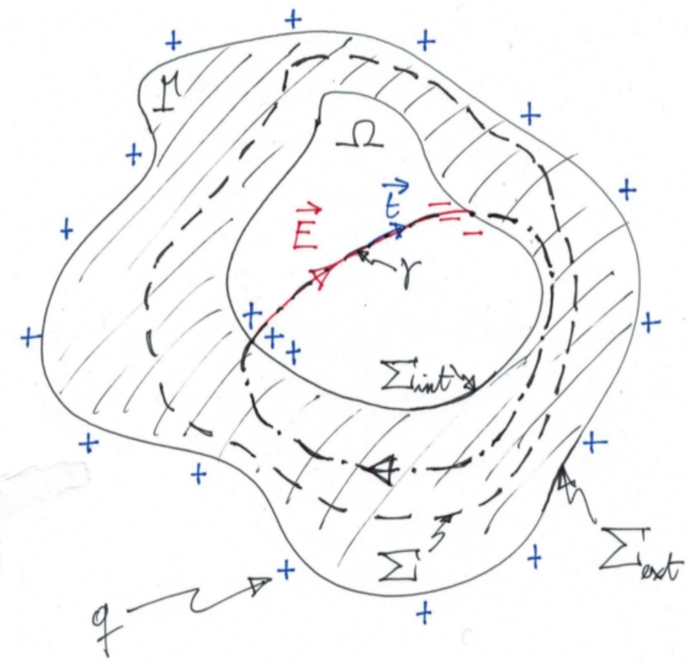


Figure 3.5

An important property of electrostatic shields is that the behavior of the shield is symmetrical, i.e., any charge distribution inside Ω does not affect the external region to Γ. At first glance, this might seam wrong. In fact, if we introduce inside the region Ω of the conductor Γ in Fig. 3.5 a charged body C, e.g., a charged conductor (see Fig. 3.6), it is easy to verify that around C a field $\vec{E}$ exists. Necessarily, the field lines associated with $\vec{E}$ must go from the outer surface of C (assuming is positively charged) to the inner surface of Γ, Σ_{int}. We are induced to think that the charge distribution on Σ_{int} must be equal and opposite than that on C. This can easily be demonstrated by using Gauss's theorem for the usual Gaussian surface Σ indicated in Fig. 3.6 (dashed black line). The flux of the electrostatic field through Σ is zero (again because the field is zero at each point in Γ). Hence, the charge contained within Σ is zero and, as a consequence, a charge equal and opposite to that on C must distribute on Σ_{int}. In addition, because of the charge conservation principle, the initially neutral conductor Γ must remain neutral even after the introduction of C in Ω. This is possible only if a charge equal and opposite to that on Σ_{int} distributes on Σ_{ext} (this charge is thus equal to that on C in Ω). The charge on Σ_{ext} generates a field outside the cavity. It would, however, be wrong to think that the charged body C inside the cavity affects electrically the region outside. In fact, what actually alters the electrostatic scenario outside the cavity is the introduction in Ω of a charged body. This event, however, does not take place entirely and solely inside the cavity. The external world must take part to the event by giving away some charge to C! In other words, during this operation the external world is in communication with Ω.

Consider instead what happens when C is already inside the cavity and we only change its position within Ω, or we divide it into two parts, or, in general, we perform any other physical operation that implies the total charge conservation inside the cavity. These operations have no influence on the external world. In other words, the surface charge distribution on Σ_{ext} does not change when the conditions inside the cavity are modified. If, for example, we imagine to change the position of C in Ω, the charge on Σ_{ext} remains unchanged. As a consequence, the calculation of V outside Σ_{ext} can be carried out by means of an external Dirichlet problem regardless from the position of C in Ω.

In summary, the charge on Σ_{ext} and the field outside Γ, for the same geometry and position of Γ and for the same external conditions (e.g., other nearby charges, other nearby conductors, etc.), are totally independent from what takes place in Ω: The electrostatic shield works both ways, isolating the interior from external influences and vice versa.

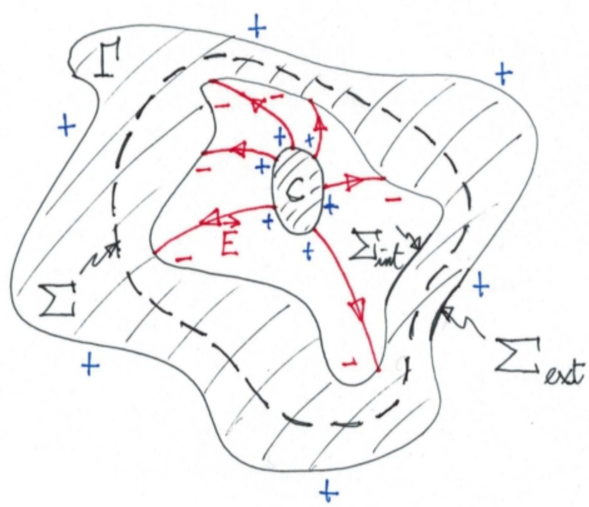


Figure 3.6

Now that we understand the concept of electrostatic shield, when you will fly next time over the Atlantic ocean on an A380 in the middle of a beautiful thunderstorm, you can reassure your shaking neighbour that there is nothing to worry about: Because of an internal Dirichlet problem (or, if you want, Gauss' theorem and the irrotational property), the aircraft's frame will make a Faraday cage that will prevent you to get any electric shock and that will protect the avionics. He might still tell you to get lost, but you can sit back, relax, and enjoy the ride.

3.5 Capacitance.

Consider the case of a conducting sphere. We decide to call

$$V_0(r) = +\frac{A}{r} \quad , \tag{3.39}$$

i.e., the potential that assumes a value

$$V_0 = +\frac{A}{R} \tag{3.40}$$

at each point on Σ. It is easy to calculate the charge q_0 as

$$q_0 = -\epsilon_0 \oint\!\!\!\oint_{\Sigma^+} \frac{\partial}{\partial_r} V_0(r) \cdot dS \quad . \tag{3.41}$$

This charge corresponds to the charge that would be globally present on the conductor when its potential were V_0 as in (3.40).

Imagine to associate a new potential V_1 to the conductor. It can be verified that

$$V_1(r) = \frac{V_1}{V_0} V_0(r) \quad , \tag{3.42}$$

when

$$V_1(r) = +\frac{\widetilde{A}}{r} \quad , \tag{3.43}$$

$$V_1 = +\frac{\widetilde{A}}{R} \quad , \tag{3.44}$$

and V_0 and $V_0(r)$ are given by (3.39) and (3.40), respectively. In fact,

$$\begin{aligned} V_1(r) &= +\frac{\widetilde{A}}{\cancel{R}}\left(\cancel{+}\frac{\cancel{R}}{\cancel{A}}\right)\left(\cancel{+}\frac{\cancel{A}}{r}\right) \\ &= +\frac{\widetilde{A}}{r} \quad . \end{aligned}$$

As it turns out, (3.42) is valid in general for all harmonic functions, i.e., continuously derivable function that satisfy Laplace equations (not only for the simple case of the conducting sphere).

The condition (3.42) means that when the solution to the problem is known for a given value of the potential on the conductor, it is straightforward to obtain the solution for any other value of the potential.

The condition (3.42) also shows that the equipotential surfaces associated with one solution are equipotentials for any other solution (even if, of course, the value of the potential on each surface must be suitable rescaled from case to case). This also means that the field lines are the same for different values of the potential at the conductor. This is a direct consequence of the linearity of Laplace equation.

Note that from the knowledge of $V_1(r)$, we can readily calculate the charge q_1 as

$$\begin{aligned} q_1 &= -\epsilon_0 \oiint_{\Sigma^+} \frac{\partial}{\partial_r} V_1(r) \cdot dS \\ &= -\epsilon_0 \frac{V_1}{V_0} \oiint_{\Sigma^+} \frac{\partial}{\partial_r} V_0(r) \cdot dS \\ &= \frac{V_1}{V_0} q_0 \quad . \end{aligned} \tag{3.45}$$

Thus,

$$\frac{q_1}{V_1} = \frac{q_0}{V_0} \quad . \tag{3.46}$$

From the arbitrariness of q_1, V_1, q_0, and V_0 it follows that the ratio between the charged applied to the conductors and its corresponding potential is a constant, independently from the value of q and V,

$$\frac{q}{V} = C \quad . \tag{3.47}$$

The constant of proportionality, C, is called the capacitance of the isolated conductor. The capacitance depends only on the conductor geometry. In the case of the conducting sphere (isolated in space), from (3.39) and (3.41)

$$\begin{aligned} \frac{q}{V} &= \frac{q\big|_{\Sigma^+}}{V\big|_{\Sigma}} = \frac{q_0}{V_0} \\ &= \frac{R}{A}\left[-\epsilon_0 \oiint_{\Sigma^+} \frac{\partial}{\partial_r} V(r) \cdot dS \right] \\ &= \frac{R}{\cancel{A}}\left[\cancel{-}\epsilon_0 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \cdot d\theta \left(\cancel{-} \frac{\cancel{A}}{\cancel{R^2}} \right) R^2 \right] \\ &= 4\pi\epsilon_0 R \quad . \end{aligned} \tag{3.48}$$

For $R = 1$ m, $C \simeq 10^{-10}\mathrm{F} = 1000$ pF, where SI units of C are

$$[C] = \frac{[q]}{[V]} = \frac{C}{V} = F \text{ (farad)} \quad . \tag{3.49}$$

3.5.1 System of Two Conductors: Capacitors.

By definition, a capacitor is a physical system characterized by two conductors facing each other and separated by an insulating material (or vacuum). The conductors are called the walls of the capacitor and are charged so that the charge on one wall is equal and opposite than the charge on the other wall. The walls are often made by plates, the linear dimensions of which are much larger than the separation between them.

These simple devices make it possible to create very strong electrostatic fields in limited regions of space and, thus, to store large quantities of electrostatic energy ($u_e \sim$

E^2). As it will be shown later, this phenomenon is a consequence of this simple fact: Called A and B the walls of a capacitor with charges $+q$ and $-q$, respectively, the potential difference between the walls is proportional to q and given by

$$V_A - V_B = \frac{q}{C} \quad . \tag{3.50}$$

In (3.50), the coefficient C is an intrinsic characteristic of the capacitor, again called capacitance. This coefficient has units $\big[$farad (F)$\big]$. Note that, the latter can always be regarded as the wall of a capacitor whose second wall is an imaginary equipotential surface at infinity (in reality, e.g., the walls of a lab).

Relation (3.50) can be determined experimentally for each type of capacitor. However, we will try to deduce it from dimple theoretical arguments.

We start by considering a general physical system made by two conductor Γ_1 and Γ_2 (see Fig. 3.7) with arbitrary shape, isolated from each other, and with respective charges q_1 and q_2. Assume V_1 and V_2 are the potentials for the two conductors. Defining the surface of each conductor as Σ_1 and Σ_2, and the space between them Ω, the function V must:

a) be continuous in $\Sigma_1 \cup \Omega \cup \Sigma_2$;

b) verify Laplace equation in Ω;

c) assume the constant values V_1 and V_2 on Σ_1 and Σ_2, respectively:

d) be zero at infinity.

The function V is thus solution of an external Dirichlet problem with multiple frontier $\Sigma_1 \cup \Sigma_2$. It can be shown that this problem has a unique solution.

We now indicate with $V_{10}(P)$ and $V_{01}(P)$ the solutions corresponding to the boundary conditions

c')

$$\begin{cases} V = 1 & \text{on } \Sigma_1 \quad , \\ V = 0 & \text{on } \Sigma_2 \quad ; \end{cases}$$

c")

$$\begin{cases} V = 0 & \text{on } \Sigma_1 \quad , \\ V = 1 & \text{on } \Sigma_2 \quad , \quad \text{respectively} \end{cases} \quad .$$

This means that both functions $V_{10}(P)$ and $V_{01}(P)$ satisfy conditions a)-d) and, in addition, V_{10} satisfies c') and V_{01} satisfies c"). Note that for dimensional reasons it is useful to assign the dimension of a potential to the constants V_1 and V_2 and assume the function $V_{10}(P)$ and $V_{01}(P)A$ to be adimensional.

Consider now the function

$$V(P) = V_1 \cdot V_{10}(P) + V_2 \cdot V_{01}(P) \quad , \tag{3.51}$$

where V_1 and V_2 are arbitrary constants. At each point in Ω we have

$$\vec{\nabla}^2 V = V_1 \cdot \vec{\nabla}^2 V_{10} + V_2 \cdot \vec{\nabla}^2 V_{01} \quad . \tag{3.52}$$

Since

$$\vec{\nabla}^2 V_{10} = \vec{\nabla}^2 V_{01} = 0 \quad , \tag{3.53}$$

it must be

$$\vec{\nabla}^2 V = 0 \quad . \tag{3.54}$$

In addition, as V_{10} and V_{01} go to zero at infinite, so does V. Moreover, the function V_{10} and V_{01} are both continuous in $\Sigma_1 \cup \Omega \cup \Sigma_2$ and so is V. At last, at each point on Σ_1

$$\begin{aligned} V\Big|_{\Sigma_1} &= V_1 \cdot V_{10}\Big|_{\Sigma_1} + V_2 \cdot V_{01}\Big|_{\Sigma_1} \\ &= V_1 \cdot 1 + V_2 \cdot 0 = V_1 \quad , \end{aligned} \tag{3.55a}$$

and at each point on Σ_2

$$\begin{aligned} V\Big|_{\Sigma_2} &= V_1 \cdot V_{10}\Big|_{\Sigma_2} + V_2 \cdot V_{01}\Big|_{\Sigma_2} \\ &= V_1 \cdot 0 + V_2 \cdot 1 = V_2 \quad . \end{aligned} \tag{3.55b}$$

Since the function V also satisfies to condition c) is the solution of our problem. Obviously, the knowledge of V is based on the knowledge of $V_{10}(P)$ and $V_{01}(P)$, which we assume to be known for now. Those functions depend only on the geometry of the problem, i.e., from the shape of surfaces Σ_1 and Σ_2. This is the case because the values of these functions on Σ_1 and Σ_2 must satisfy condition c') and c").

The charges on Γ_1 and Γ_2 are

$$\begin{aligned} q_1 &= -\epsilon_0 \iint\limits_{\Sigma_1} \frac{\partial}{\partial_n} V \cdot dS \\ &= -\epsilon_0 V_1 \iint\limits_{\Sigma_1} \frac{\partial}{\partial_n} V_{10} \cdot dS - \epsilon_0 V_2 \iint\limits_{\Sigma_1} \frac{\partial}{\partial_n} V_{01} \cdot dS \quad , \end{aligned} \tag{3.56a}$$

$$\begin{aligned} q_2 &= -\epsilon_0 \iint\limits_{\Sigma_2} \frac{\partial}{\partial_n} V \cdot dS \\ &= -\epsilon_0 V_1 \iint\limits_{\Sigma_1} \frac{\partial}{\partial_n} V_{10} \cdot dS - \epsilon_0 V_2 \iint\limits_{\Sigma_2} \frac{\partial}{\partial_n} V_{01} \cdot dS \quad . \end{aligned} \tag{3.56b}$$

By defining

$$C_{11} = -\epsilon_0 \iint\limits_{\Sigma_1} \frac{\partial}{\partial_n} V_{10} \cdot dS \quad , \tag{3.57a}$$

$$C_{12} = -\epsilon_0 \iint\limits_{\Sigma_1} \frac{\partial}{\partial_n} V_{01} \cdot dS \quad , \tag{3.57b}$$

$$C_{21} = -\epsilon_0 \iint\limits_{\Sigma_2} \frac{\partial}{\partial_n} V_{10} \cdot dS \quad , \tag{3.57c}$$

$$C_{22} = -\epsilon_0 \iint\limits_{\Sigma_2} \frac{\partial}{\partial_n} V_{01} \cdot dS \quad , \tag{3.57d}$$

we have

$$\begin{cases} q_1 = C_{11}V_1 + C_{12}V_2 \\ q_2 = C_{21}V_1 + C_{22}V_2 \quad , \end{cases} \tag{3.58}$$

or in matrix form

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad . \tag{3.59}$$

The so defined matrix is called the capacitance matrix of the considered physical system. The elements of this matrix depend only on the geometry of the system. It is possible to show (we will not do it here) that the capacitance matrix is symmetric, i.e.,

$$C_{12} = C_{21} \quad . \tag{3.60}$$

At last, from (3.58) and (3.59) we find

$$C_{11} = \left.\frac{q_1}{V_1}\right|_{V_2=0} \quad , \quad C_{12} = \left.\frac{q_1}{V_2}\right|_{V_1=0} = C_{21} \quad , \quad C_{22} = \left.\frac{q_2}{V_2}\right|_{V_1=0} \quad . \tag{3.61}$$

In order to calculate, for example, the coefficient C_{11}, we can think to "ground" the conductor Γ_2, while keeping Γ_1 at the potential V_1. Once the function $V_{10}(P)$ is calculated, the calculation (or measurement) of the charge q_1 on Γ_1 gives the value of the ratio q_1/V_1. A similar process applies to the remaining capacitance coefficients.

Note that, in real life, the operation of grounding is realized by connecting a body Γ to the Earth by means of a conducting wire (typically copper) ending in a sharp tip that is pushed into the dirt, deep into the ground. In this way, Γ becomes a small appendix to the huge conducting sphere which is the Earth. As a consequence, Γ must assume the same potential of the Earth. Independently from the charges that can be applied to the Earth (so long they are not huge), the Earth can be assumed to be a conducting sphere isolated in vacuum with a very large capacitance C_{T} ($\simeq$ 700 to 800μF). The Earth potential V_{T}, calculated by imposing the potential at infinite to be zero, is still very small,

$$V_{\mathrm{T}} = \frac{q_{\mathrm{T}}}{C_{\mathrm{T}}} \quad .$$

Within a good approximation, V_{T} can be assumed to be zero, even under the assumption that the potential is zero at infinite.

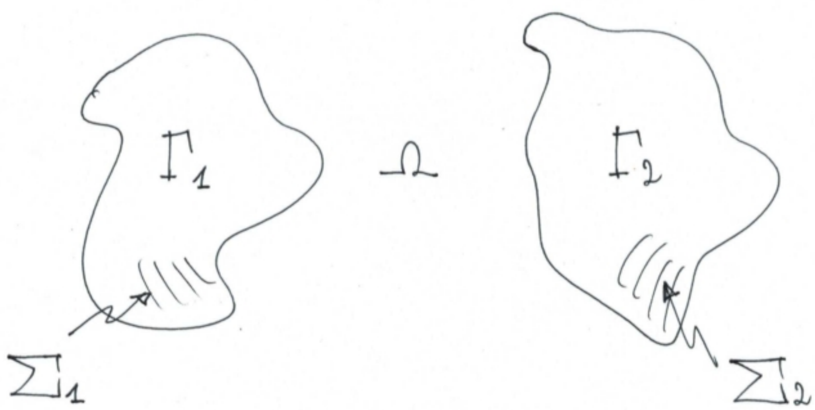


Figure 3.7

Note that (3.58) and (3.59) can be inverted with respect to V_1 and V_2. Considering q_1 and q_2 to be known, we find

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad . \tag{3.62}$$

where we indicated with $||C_{ij}||^{-1}$ the inverse of matrix $||C_{ij}||$. Calling

$$b_{11} = \frac{C_{22}}{|C|} \quad , \quad b_{12} = -\frac{C_{21}}{|C|} = b_{21} \quad , \quad b_{22} = \frac{C_{11}}{|C|} \quad ,$$

where $|C|$ indicates the determinant of matrix $||C_{ij}||$, we have

$$\begin{cases} V_1 = b_{11}q_1 + b_{12}q_2 \\ V_2 = b_{21}q_1 + b_{22}q_2 \quad , \end{cases} \tag{3.63}$$

or

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad . \tag{3.64}$$

We now specialize our discussion to the particular case (specific to capacitors) where the two conductors are charged so that

$$q_1 = -q_2 = q \quad . \tag{3.65}$$

From (3.58) it follows that

$$q = C_{11}V_1 + C_{12}V_2 = -C_{21}V_1 - C_{22}V_2$$

and then

$$\frac{V_1}{V_2} = -\frac{C_{22} + C_{12}}{C_{11} + C_{21}} \quad . \tag{3.66}$$

So, for the charges on the two conductors to be equal in absolute value and opposite in sign, V_1 and V_2 cannot be arbitrary chosen: They must satisfy condition (3.66), which means their ratio fixed when the geometry of the two bodies is given.

By writing V_1 and V_2 as a function of V_1/V_2 and $(V_1 - V_2)$ be means of the identities

$$V_1 = \left(\frac{V_1}{V_2}\right)\frac{V_1 - V_2}{\dfrac{V_1}{V_2} - 1}$$

and

$$V_2 = \frac{V_1 - V_2}{\dfrac{V_1}{V_2} - 1}$$

and substituting these expressions in (3.58), we find

$$q = C_{11}\,\frac{\alpha}{\alpha - 1}\,\nabla V + C_{12}\,\frac{1}{\alpha - 1}\,\nabla V \quad ,$$

where

$$\alpha = -\,\frac{C_{22} + C_{12}}{C_{11} + C_{21}}$$

and $\nabla V = V_1 - V_2$ $\big[$note that we have used only the top relation in (3.58). We could have equally used the second$\big]$. A simple calculation shows that

$$\begin{aligned}\alpha - 1 &= -\left(\frac{C_{22} + C_{12}}{C_{11} + C_{21} + 1}\right)\\ &= -\,\frac{C_{22} + C_{12} + C_{11} + C_{21}}{C_{11} + C_{21}}\\ &= -\,\frac{\beta}{C_{11} + C_{21}} \quad ,\end{aligned}$$

where $\beta = (C_{11} + C_{12} + C_{21} + C_{22})$. Hence,

$$\begin{aligned}q &= \left(C_{11}\,\frac{C_{22} + C_{12}}{\cancel{C_{11} + C_{21}}}\,\frac{\cancel{C_{11} + C_{21}}}{\beta} - C_{12}\,\frac{C_{11} + C_{21}}{\beta}\right)\nabla V\\ &= \frac{1}{\beta}\big(C_{11}C_{22} + \cancel{C_{11}C_{12}} - \cancel{C_{12}C_{11}} - C_{12}C_{21}\big)\nabla V\\ &= C\big(V_1 - V_2\big) \quad ,\end{aligned} \tag{3.67}$$

where

$$C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}} \quad . \tag{3.68}$$

Thus, the capacity of a capacitor depends only on the system geometry, i.e., on the shape and distance of its walls (note that if the space in between the walls is filled with a dielectric, C also depends on the physical properties of the system).

It is worth noting that if the capacitor's walls are not charged such that $q_1 = -q_2$, the only knowledge of the potential difference between them is insufficient to determine the charge on them. In fact, Eqs. (3.58) show that V_1 and V_2 must be separately given, not only the difference ∇V.

An example of important physical interest is shown in Fig. 3.8. In this case, one of the two walls encloses the other wall completely. When a charge q is applied to the internal wall Γ_1, the field lines in the space in between the two walls is uniquely defined, independently from the value of the charge applied to Γ_2, even if this is different from $-q$. This is true because of the properties of electrostatic shields. In the case of Fig. 3.8, the charge q on Γ_1 induces a charge $-q$ on the internal surface Σ_2^- of the wall Γ_2. In this case, a complete induction is created between Γ_1 and Γ_2: All field lines generating from Γ_1 end on Σ_2^-. In such a capacitor, the capacity defined as the ratio between q and the potential difference between the walls is independent from the total charge on Γ_2 and is not affected by the presence of external sources. A complete induction capacitor is considered an ideal capacitor. In the practical implementation of capacitor one has to try to make a capacitor as close as possible to a complete induction capacitor.

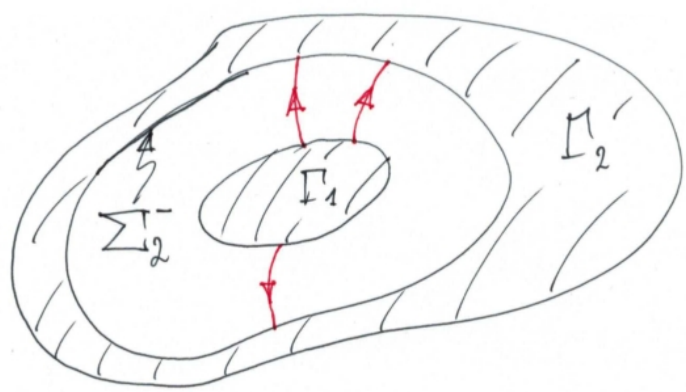
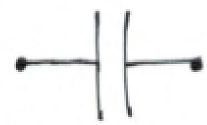


Figure 3.8

3.5.2 Parallel Plate Capacitor.

Consider a capacitor made by two parallel walls (plates) of area S and separated by a distance d. We want to calculate its capacitance. Note that referring to the geometrical sphere of this specific type of capacitor, it is common to indicate graphically any generic capacitor with the symbol



If the distance d between the plates is very small compared to the linear dimensions of the plates, $d << \sqrt{S}$, we can reduce the problem to that two infinite parallel plates at distance d (see Fig. 3.9). Hence, we want to attack the problem using Laplace equation and opportune boundary conditions.

As always, we choose a Cartesian coordinate system where the z axis is normal to the plates and the x and y axis are on the left wall. Due to the translation symmetry of both walls with respect to x and y, the potential can only depend on z. It must thus satisfy Laplace equation in the form

$$\vec{\nabla}^2 V(z) = \frac{d^2}{dz^2} V(z) = 0 \quad , \tag{3.69}$$

for $z \in \big((-\infty, +\infty) - \{0, d\}\big)$. Note that, while each of the two walls is assumed to be thin compare to the distance d, we must assume them to have a finite thickness t in order to properly solve the boundary conditions problem associated with (3.69). As shown in the inset of Fig. 3.9, the left wall is characterized by an outer surface Σ_1^- on the left side and Σ_1^+ on the right side. Since the plate is conducting with finite dimensions t and S, no charge can distribute at any point inside the plate. The total charge on the plate is $+q$. Because of charge conservation, there must be a charge $+q/2$ on Σ_1^- and $+q/2$ on Σ_1^+ (we neglect any surface charge that accumulates on the other four lateral surfaces of width t). Similarly the right wall has outer surfaces Σ_2^- and Σ_2^+ at a distance t from each other (again, these surface do not include the negligible outer surfaces of width t). In this case, the total charge on the plate is $-q$. Because of charge conservation, a charge $-q/2$ distributes on Σ_2^- and an equal charge $-q/2$ distributes on Σ_2^+.

We can solve the problem by finding the potential of each wall alone and, then, obtain the total potential by means of the superposition principle.

We begin with the potential of the left wall. In this case, the unique solution to Eq. (3.69) is

$$V_1(z) = A_1^- z + B_1^- \quad , \tag{3.70a}$$

for $z \in (-\infty, 0)$ and with A_1^- and B_1^- arbitrary constants. The constant A_1^- can be found from Coulomb's theorem applied to Σ_1^-,

$$-\iint\limits_{\Sigma_1^-} \frac{\partial}{\partial_{n_1^-}} V_1 \cdot dS = +\frac{1}{\epsilon_0}\,\frac{q}{2} \quad . \tag{3.71a}$$

From Fig. 3.9, we have $\vec{n}_1^- = -\vec{n} = -\vec{u}_z$. Hence, (3.71a) reads

$$
\begin{aligned}
+\iint\limits_{\Sigma_1^-} \frac{\partial}{\partial_z} V_1(z) \cdot dS &= A_1^- S \\
&= +\frac{1}{\epsilon_0}\frac{q}{2} \quad ,
\end{aligned}
$$

from which

$$
A_1^- = \frac{q}{2\epsilon_0 S} \quad . \tag{15.22a'}
$$

Hence,

$$
V_1(z) = \frac{q}{2\epsilon_0 S}\, z + \beta_1^- \tag{15.21a'}
$$

For $z \in (0, +\infty)$, the solution is

$$
V_1(z) = A_1^+ z + \beta_1^+ \quad , \tag{15.21b}
$$

with A_1^+ and B_1^+ arbitrary constants. As before, the constant A_1^+ can be found from

$$
-\iint\limits_{\Sigma_1^+} \frac{\partial}{\partial_{n_1^+}} V_1 \cdot dS = +\frac{1}{\epsilon_0}\frac{q}{2} \quad , \tag{3.71b}
$$

where, in this case, $\vec{n}_1^+ = +\vec{n} = +\vec{u}_z$ and, thus,

$$
\begin{aligned}
-\iint\limits_{\Sigma_1^+} \frac{\partial}{\partial_z} V_1(z) \cdot dS &= A_1^+ S \\
&= +\frac{1}{\epsilon_0}\frac{q}{2} \quad ,
\end{aligned}
$$

from which

$$
A_1^+ = -\frac{q}{2\epsilon_0 S} \quad . \tag{15.22b'}
$$

Hence,

$$
V_1(z) = -\frac{q}{2\epsilon_0 S}\, z + B_1^+ \quad . \tag{15.21b'}
$$

For the right wall, the unique solution to Eq. (3.69) is

$$
V_2(z) = A_2^- z + B_2^- \quad , \tag{15.21c}
$$

for $z \in (-\infty, d)$ and with A_2^- and B_2^- arbitrary constants. From Coulomb's theorem

$$
-\iint\limits_{\Sigma_2^-} \frac{\partial}{\partial_{n_2^-}} V_2 \cdot dS = -\frac{1}{\epsilon_0}\frac{q}{2} \quad , \tag{3.71c}
$$

where $\vec{n}_2^- = -\vec{n} = -\vec{u}_z$ and, thus,

$$
\begin{aligned}
+\iint\limits_{\Sigma_2^-} \frac{\partial}{\partial_z} V_2(z) \cdot dS &= A_2^- S \\
&= -\frac{1}{\epsilon_0}\frac{q}{2} \quad ,
\end{aligned}
$$

from which

$$A_2^- = -\frac{q}{2\epsilon_0 S} \quad . \tag{15.22c'}$$

Hence,

$$V_2\big(z\big) = -\frac{q}{2\epsilon_0 S}\, z + B_2^- \quad . \tag{15.21c'}$$

Finally, for $z \in (d, +\infty)$, the solution is

$$V_2\big(z\big) = A_2^+ z + B_2^+ \quad , \tag{15.21d}$$

with A_1^+ and B_1^+ arbitrary constant. From Coulomb's theorem

$$-\iint\limits_{\Sigma_2^+} \frac{\partial}{\partial_{n_2^+}} V_2 \cdot dS = -\frac{1}{\epsilon_0}\,\frac{q}{2} \quad , \tag{3.71d}$$

where $\vec{n}_2^+ = +\vec{n} = +\vec{u}_z$ and, thus,

$$\begin{aligned} -\iint\limits_{\Sigma_2^+} \frac{\partial}{\partial_z} V_2\big(z\big) \cdot dS &= -A_2^+ S \\ &= -\frac{1}{\epsilon_0}\,\frac{q}{2} \quad , \end{aligned}$$

from which

$$A_2^+ = \frac{q}{2\epsilon_0 S} \quad . \tag{15.22d'}$$

Hence,

$$V_2\big(z\big) = \frac{q}{2\epsilon_0 S}\, z + B_2^+ \quad . \tag{15.21d'}$$

By means of the superposition principle, the total potential in the region $z \in (-\infty, 0)$ is given by

$$\begin{aligned} V\big(z\big) &= V_1\big(z\big) + V_2\big(z\big) \\ &= \cancel{\frac{q}{2\epsilon_0 S}}\, z + B_1^- - \cancel{\frac{q}{2\epsilon_0 S}}\, z + B_2^- \\ &= B_1^- + B_2^- \quad , \end{aligned} \tag{3.72a}$$

where we superimposed (15.21a') with (15.21c').

The potential at each point on the left wall must be equal to a constant V_1,

$$V\big(z\big)\Big|_{\Sigma_1} = V\big(0\big) = B_1^- + B_2^- = V_1 \quad . \tag{3.73}$$

Hence, for $z \in (-\infty, 0]$,

$$V\big(z\big) = V_1 \quad . \tag{3.74a}$$

By superimposing (15.21b') with (15.21c'), we find the total potential for $z \in (0, d)$,

$$\begin{aligned} V\big(z\big) &= -\frac{q}{2\epsilon_0 S}\, z + B_1^+ - \frac{q}{2\epsilon_0 S}\, z + B_2^- \\ &= -\frac{q}{\epsilon_0 S}\, z + \big(B_1^+ + B_2^-\big) \quad . \end{aligned}$$

On the left wall,

$$V\big(z\big)\Big|_{\Sigma_1} = V\big(0\big) = B_1^+ + B_2^- = V_1 \quad . \tag{3.75}$$

The potential at each point on the right wall must be equal to a constant V_2,

$$V\big(z\big)\Big|_{\Sigma_2} = V\big(d\big) = -\frac{q}{\epsilon_0 S}\, d + V_1 = V_2 \quad , \tag{3.76}$$

from which

$$V_2 = V_1 - \frac{q}{\epsilon_0 S}\, d \quad .$$

From this condition follows that

$$V_2 < V_1 \quad .$$

Hence, for $z \in [0, d]$,

$$V\big(z\big) = -\frac{q}{\epsilon_0 S}\, z + V_1 \quad , \tag{15.25b}$$

with a non arbitrary choice of V_2,

$$V_2 = V_1 - \frac{q}{\epsilon_0 S}\, d \quad . \tag{15.25b'}$$

By superimposing (15.21b') with (15.21d'), we finally find the total potential for $z \in (d, +\infty)$,

$$\begin{aligned} V_{\cancel{2}}\big(z\big) &= -\cancel{\frac{q}{2\epsilon_0 S}}\, z + B_1^+ + \cancel{\frac{q}{2\epsilon_0 S}}\, z + B_2^+ \\ &= B_1^+ + B_2^+ \quad . \end{aligned}$$

On the right wall,

$$V\big(z\big)\Big|_{\Sigma_2} = V\big(d\big) = B_1^+ + B_2^+ = V_2 \quad . \tag{3.77}$$

Hence, for $z \in [d, +\infty)$,

$$V\big(z\big) = V_2 \quad . \tag{15.25c}$$

In summary, we obtain the same result as in lecture 12, Eq. (2.48),

$$\begin{cases} V\big(z\big) = V_1 \quad , & z(-\infty, 0] \\ V\big(z\big) = -\dfrac{q}{\epsilon_0 S}\, z + V_1 \quad , & z \in [0, d] \\ V\big(z\big) = V_2 = V_1 - \dfrac{q}{\epsilon_0 S}\, s \quad , & z \in [d, +\infty) \end{cases} \tag{3.78}$$

where, in (2.48), $\sigma = q/S$ and $C_1 = V_1$ and $C_3 = V_2$.

The capacitance of the parallel plate capacitor can readily be found from its definition

$$C = \frac{q}{V_1 - V_2} = \frac{q}{\dfrac{qd}{\epsilon_0 S}} = \epsilon_0 \frac{S}{d} \quad . \tag{3.79}$$

Note that we could have calculated the capacitance without a complete knowledge of all arbitrary constants. In fact, for $z \in [0, d]$ the total potential is given by the superposition of (15.21b) and (15.21c),

$$V(z) = \left(A_1^+ + A_2^-\right)z + \left(B_1^+ + B_2^-\right) \quad . \tag{3.80}$$

From (3.80),

$$\begin{aligned} V_1 - V_2 &= V(0) - V(d) \\ &= \cancel{\left(B_1^+ + B_2^-\right)} - \left(A_1^+ + A_2^-\right)d - \cancel{\left(B_1^+ + B_2^-\right)} \\ &= -\left(A_1^+ + A_2^-\right)d \quad . \end{aligned} \tag{3.81}$$

The charge $+q$ on Σ_1 can be found from (15.22b') and (15.22c'), which, thus, are the only boundary conditions to be imposed:

$$\begin{aligned} \frac{q}{2} &= -\epsilon_0 S A_1^+ \quad , \\ \frac{q}{2} &= -\epsilon_0 S A_2^- \end{aligned}$$

and, thus,

$$q = -\left(A_1^+ + A_2^-\right)\epsilon_0 S \quad . \tag{3.82}$$

At last,

$$C = \frac{\cancel{-\left(A_1^+ + A_2^-\right)}\epsilon_0 S}{\cancel{-\left(A_1^+ + A_2^-\right)}d} = \epsilon_0 \frac{S}{d} \quad .$$

Note that the potential difference in the definition of capacitance is always calculated from the positive to the negative wall.

It is worth performing a dimensional check on the capacitance:

$$[C] = \frac{F}{m}\frac{m^2}{m} = F \quad .$$

The bottom part of Fig. 3.9 shows the potential of each individual wall as well as the total potential of the capacitor obtained by graphically superimposing the individual potentials. Note that, as expected, the overall potential of each individual wall is continuous at the wall. For simplicity, we have chosen $q = \epsilon_0 = S = d = B_1^- = B_2^- = B_1^+ = V_2 = 1$, $V_1 = 2$, and $B_2^+ = 0$.

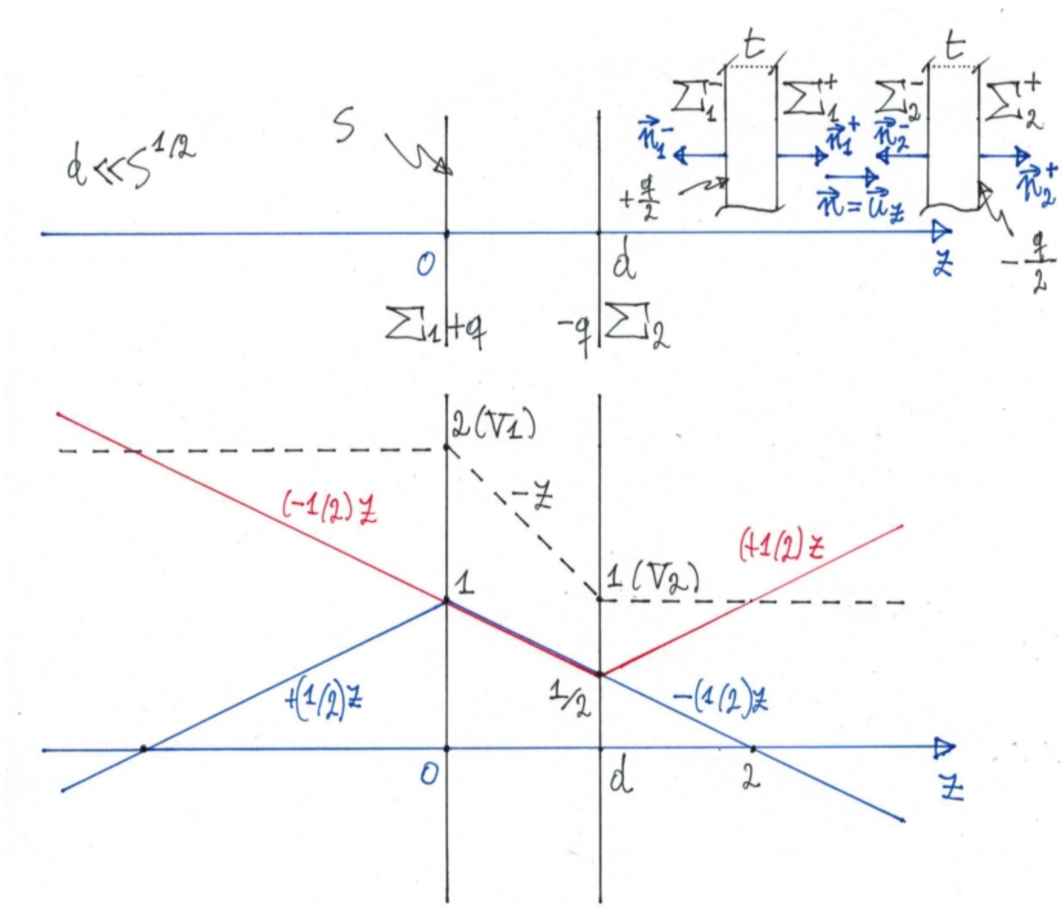


Figure 3.9

3.5.3 Electrostatic Energy of a System of Charged Conductors.

From lecture 10, the electrostatic energy associated with a surface charge distribution with density σ is given by

$$U_e = \frac{1}{2} \iint_{\Sigma} \sigma V \cdot dS \quad , \tag{3.83}$$

where Σ is the region where the charge is distributed.

In a system of n conductors, the electric charge is distributed on the conductors' surface. Naming Σ_i the surface of the i-th conductor Γ_i, (3.83) reads

$$U_e = \frac{1}{2} \sum_{i=1}^{n} \iint_{\Sigma_i} \sigma_i V_i \cdot dS \quad , \tag{3.84}$$

where σ_i is the density associated with the i-th conductor. Note that V_i must be constant on the surface of each conductor. Hence, we can take it outside the sign of integral. In addition, the total charge on Γ_i is given by

$$q_i = \iint_{\Sigma_i} \sigma_i \cdot dS \quad . \tag{3.85}$$

Thus,

$$U_e = \frac{1}{2}\sum_{i=1}^{n} q_i V_i \quad . \tag{3.86}$$

This is the wanted expression for the electrostatic energy of a system of n conductors.

In the particular case of a capacitor, the electrostatic energy is given by,

$$\begin{aligned} U_e &= \frac{1}{2}\,qV_1 - \frac{1}{2}\,aV_2 = \frac{1}{2}\,qV \\ &= \frac{1}{2}\,CV^2 = \frac{1}{2}\,\frac{q^2}{C} \quad , \end{aligned} \tag{3.87}$$

where q is the capacitor charge, V the potential difference between its walls, and C its capacitance.

Chapter 4

Electric Current

4.1 Introduction

So far, we only considered the interaction between charges that are at rest with respect to an inertial reference system. In this chapter, we will study interactions between charges in motion. These are at the basis of the magnetic phenomena known since antiquity.

Before analyzing the relationship between electric and magnetic phenomena and enumerating the corresponding general laws, we must verify whether the definition of electric charge given in electrostatic (see PHYS 242, Chapter 1) is still valid to describe electrodynamics phenomena.

Consider a point-like charge q_0, fixed at point $P_0 \equiv (x_0, y_0, x_0)$ in an inertial reference frame k (see Fig. 4.1). In the same frame, consider a point-like test body C in a generic motion along a path γ. We measure the force $\vec{F}$ generated by q_0 on C, when C is at a generic point $P \equiv (x, y, z)$ on γ. Note that, in this case, we cannot readily find the force on q_0 as in electrostatic. In fact, the action-reaction principle is not valid here. This is because when C is at P on γ at time t, q_0 "feels" the action due to C at a previous instant $(t - dt)$. The action due to C at t will reach q_0 an instant later, under the assumption that the transmission velocity of the interaction is finite.

We then substitute C with another charged point-like body C' in motion along a generic path γ', also going through P. We now measure the force $\vec{F}'$ at P generated by q_0 on C'. By definition, the bodies C and C' have the same charge $q = q'$, when $\vec{F} = \vec{F}'$. It would be possible to verify experimentally that, if $\vec{F} = \vec{F}'$ at P, they are equal at each point in space common to the two paths γ and γ'. Experience shows that two bodies of equal charge at rest (according to the definition given in PHYS 242, Chapter 1) keep having the same charge when each of them moves in k according to the new definition just given. The old definition of equal charge is, thus, a special case of the new definition.

Assuming that the additive property of charge continues to be valid also in electrodynamics (see PHYS 242, Chapter 1 for the additive property), we can define the physical quantity electric charge also for moving bodies. Moreover, it is possible to verify experimentally that if a pair of charges is in a ratio α, they remain in the same ratio even when they move. Hence, it is natural to associate to bodies in motion the same value of charge they had when they were at rest. In this sense, electric charges are characterized by a relativistic invariance property.

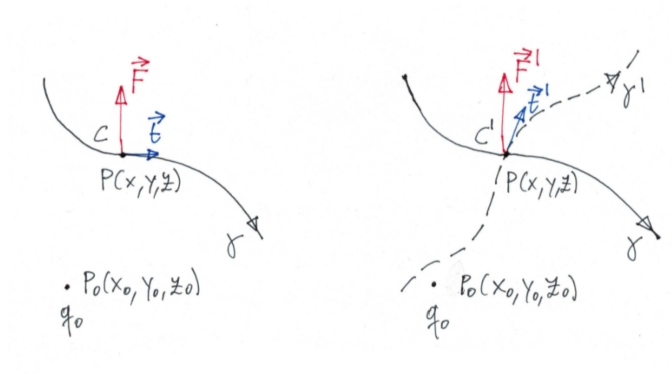


Figure 4.1

4.2 Interactions between Moving Charges

In PHYS 242, Chapter 1, we have seen that the force generated by a set of source charges, fixed in an inertial reference frame on a test charge q, also fixed in the same frame, can be written as

$$\vec{F} = q\vec{E} \quad , \tag{4.1}$$

where $\vec{E}$ is a vector field whose structure depends only on the distribution of the source charges.

Imagine now to perform an experiment as sketched in Fig. 4.1, where the source charge q_0 is fixed in the considered reference frame, while the test charge q moves according to a generic law. Such an experiment would show that the force generated by q_0 on q, when q goes through the generic point P, is the same force we would find if q was fixed at P. In summary, if the source charges are fixed, the motion of the test charge does not affect the forces acting on it due to the source charges.

The last statement is not valid, in general, when the source charges move in space. With this respect, it is worth briefly revisiting a well-known question in classical mechanics, on how to define and measure the force acting on a moving point-like particle, when the force depends on the velocity of the particle (with given mass m). In this case, since the force cannot be defined from its static effects (e.g., by means of a dynamometer), by definition we write

$$\vec{F} = \frac{d}{dt}\vec{p} \quad .$$

In other words, the force acting on the particle is defined as the time derivative of the linear momentum $\vec{p}$ of the particle.

From experiments, it is found that the force acting on a test charge q going through point P at time t and with velocity $\vec{v}$ in an inertial reference frame is given by

$$\vec{F} = q\big[\vec{E}\big(P,t\big) + \vec{v} \times \vec{B}\big(P,t\big)\big] \tag{4.2}$$

where $\vec{E}(P,t)$ and $\vec{B}(P,t)$ are vector fields called electric and magnetic field, respectively (note that $\vec{B}$ is also called magnetic induction).

We note that:

a) The force $\vec{F}$ acting on the test charge is proportional to the value of the charge itself (as in the electrostatic case).

b) If at time t the test charge is fixed at point P, the source charges generate on the test charge a force

$$\vec{F} = q\vec{E} \quad ,$$

as it can be shown from (4.2) by imposing $\vec{v} = \vec{0}$. This is the same force obtained in the electrostatic case.

However, the value of $\vec{E}(P,t)$ generated by a distribution of moving source charges (considered at time t in a given reference frame) is, in general, different from the value the same charges would generate if they were fixed at their position at the same instant t. For this reason, the vector $\vec{E}$, which was called electrostatic field in PHYS 242, is now called electric field. As for the static case, to measure the value of $\vec{E}$ (electric field) at a generic point P, at time t, is enough to measure the force that acts on a positive, unitary test charge, fixed in the given reference frame, at that point and time,

$$\vec{F}\big(P,t\big) = 1\,\vec{E}\big(P,t\big) \quad . \tag{4.3}$$

As for the influence that the positive test charge q has on the source charges, we can repeat the same arguments given in PHYS 242 for the static case: Also in the dynamic case we should define the field by means of a limit process for q going to zero. From (4.3) it appears that the so defined field (for a test charge at rest) is exactly the field that gives the first term in (4.2) of the force acting on the test charge when this moves with velocity $\vec{v}$.

c) The forces acting on a test charge moving at velocity $\vec{v}$ are the electric force $q\vec{E}$ as well as the magnetic force $q\vec{v} \times \vec{B}$. The latter can be operatively defined as the difference between the total force acting on the test charge q and the electric force,

$$q\vec{v} \times \vec{B} = \vec{F} - q\vec{E} \quad . \tag{4.4}$$

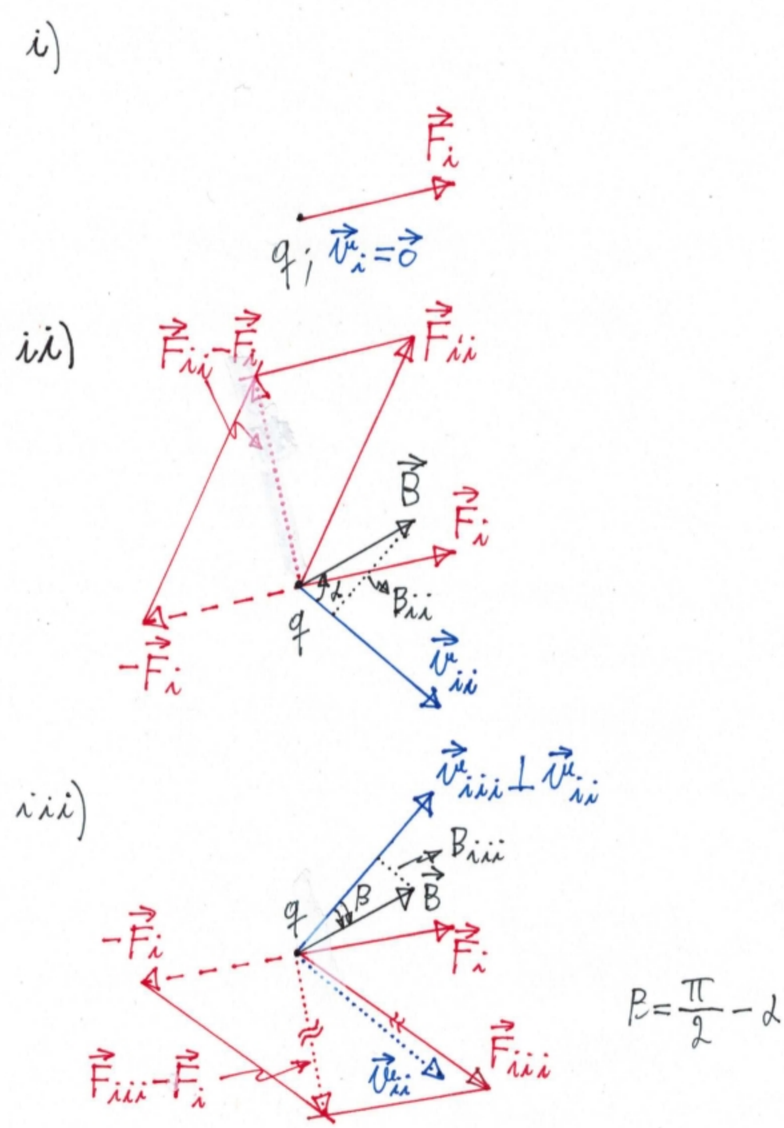


Figure 4.2

Once q, $\vec{v}$, and $\vec{F}$ are known, (4.4) allows us to define and measure the magnetic field $\vec{B}$ at point P and time t. More precisely, the operative procedure to define $\vec{B}$ (and $\vec{E}$) is the following (see Fig. 4.2):

i) A first force measurement is realized with q at rest ($\vec{v}_i = \vec{0}$). This measurement makes it possible to determine the field $\vec{E} = \vec{F}_i/q$.

ii) A second force measurement is realized with q moving at an arbitrary (but known) velocity $\vec{v}_{ii}$: This measurement gives the value $\vec{F}_{ii} = \vec{F}_i + q\vec{v}_{ii} \times \vec{B}$. Hence, we find

$$\vec{v}_{ii} \times \vec{B} = \frac{\vec{F}_{ii} - \vec{F}_i}{q} \quad . \tag{4.4'}$$

This result shows that $\vec{B}$ lies on a plane normal to vector $(\vec{F}_{ii} - \vec{F}_i)/q$. Thus, (4.4′) is insufficient to fully determine the induction vector, of which only gives the component on the aforementioned plane in the direction normal to $\vec{v}_{ii}$ (we basically find $B \sin \alpha$).

iii) A third force measurement is then realized with q moving at velocity $\vec{v}_{iii}$, where the velocity vector lies on the plane normal to $(\vec{F}_{ii} - \vec{F}_i)/q$ and is normal to $\vec{v}_{ii}$. This measurement finally gives the component of $\vec{B}$ in the direction of $\vec{v}_{ii}$ ($\vec{B}$ must be unique; we basically find $B \sin \beta$).

The dimensions of $\vec{B}$ (or, more correctly, of its magnitude B) can be readily found from (4.2),

$$\big[v\big]\big[B\big] = \big[E\big] \quad .$$

Hence,

$$\big[B\big] = \frac{V}{m}\frac{s}{m} = \frac{V s}{m^2} = T \quad .$$

This unit is called *tesla*. One *tesla* is the value of the field B that generates the force of 1 N on a charge 1 C, moving with a velocity of 1 ms^{-1}.

The magnetic force is proportional to the velocity of the test charge and is directed normally to $\vec{v}$. This means that such a force cannot perform any work on the charge.

It is worth noting that $\vec{B}$ depends only on the dynamics (motion) of the source charges. In particular, $\vec{B} = \vec{0}$ when the source charges are fixed. In this case, the only force acting on the test charge is the electric force. While it would be possible to show why and how moving charges generate a magnetic field, at this stage we prefer to accept the existence of $\vec{B}$ as an experimental fact. We will come back to this issue when studying the relationship between the electric and magnetic field in different frames of reference.

We could now proceed to the calculation of the fields (electric and magnetic) generated by a given distribution of moving source charges in a reference frame. Following this path, we would first calculate the fields generated by a single charge, and, then, by means of the superposition principle, we could move to the more complex case of fields generated by generic charge distributions. This approach would have the benefit to build on the logical process followed in electrostatics. However, the major downside of this approach is in that, in most applications, we will need to deal with the collective motion of an ensemble of charges (a current), rather than of a single charge: This can be particularly challenging from a mathematical point of view. We will thus follow a traditional approach, starting from the description of the fields associated with currents, to only later arrive at a description of the field generated by a single charge. Remarkably, this approach starts from stating Maxwell's equations for the magnetic field as empirical laws, and derive an equation similar to Coulomb's law (but for the magnetic field) as a theorem. In essence, we are following a dual approach to that followed in electrostatics.

4.3 The Electric Current

Consider two conducting objects A and B, at rest in a given inertial reference frame. Assume that A is charged with a positive charge q_A and that B is neutral. Initially, both objects are in macroscopic electrostatic equilibrium. Imagine to perturb the equilibrium condition by connecting A and B by means of a conducting wire. In order to be able to observe the resulting physical phenomena for a reasonable time window, it is possible to use a wool thread opportunely soaked in water (the water makes the thread slightly conducting; in fact, a bare wool thread would be, per se, an insulator). By simultaneously measuring the electric charge on A and B, it would be possible to show that q_A gradually diminishes, while q_B (initially zero) gradually increases. Because of the charge conservation principle, at each time t during the discharging / charging process

$$q_A(t) + q_B(t) = q = \text{const} \quad . \tag{4.5}$$

Qualitatively, it is possible to explain this phenomenon by thinking that the charge flows from A to B along the wet wool thread. Such a flow, which characterizes each point of the thread at the same time, is called electric current. From (4.5), we obtain by derivation

$$\frac{d}{dt} q_A(t) = -\frac{d}{dt} q_B(t) \tag{4.6}$$

for each time t. Given a cross-section S of the wool thread at any generic point on the thread, we imagine to measure directly the quantity of charge that goes through S in a certain time window. In other words, with a stopwatch at hand, we imagine to "count" all the charges that go through S in a fixed direction (e.g., form A to B), from the moment the connection between A and B is established. We thus obtain a function $q(t)$ that, for each time t measured by the stopwatch, gives the total quantity of charge that went through S from time $t = 0$ to t. Under the conditions assumed for such an experiment, it would result

$$q(t) = q_A(0) - q_A(t) \quad . \tag{4.7}$$

We are now in a position to define a new physical quantity, called the electric current intensity, by means of the relation

$$i(t) = \frac{d}{dt} q(t) \quad . \tag{4.8}$$

This is a scalar quantity that indicates the quantity of charge going through S in the unit time, for each time t (S, in general, is the cross-section of a conducting wire). The dimension of i,

$$[i] = \frac{[q]}{[t]} = \frac{C}{s} \quad ,$$

is called ampére (A) in the SI system. One A, thus, corresponds to a passage through S of 1 Cs^{-1}. In reality, in the SI system is preferred to define the unit of current independently from the *coulomb*. In fact, the unit of charge is defined from the unit of current. In any case, a current of 1 A still corresponds to the passage of 1 Cs^{-1}.

In the definition of $q(t)$, the passage of charges through S is assumed to be counted with respect to a given direction of reference. Depending on whether the charges go through S along the given direction or opposite to it, they must be summed or subtracted to the previous value of q. In summary, at each time t when $i > 0$, the charges (positive) go through S along the chosen direction; when, instead, $i < 0$, the charges (positive) go through S in the opposite direction.

So far, we assumed the current is generated by a flow of positive charges. If negative charges also contribute to the flow, the definition of current intensity must be modified by defining the function $q(t)$ as the net charge going through S in the chosen direction. This means that, given a direction of reference, when a positive or negative charge goes through S in the chosen direction, its values must be added in an algebraic sense to the previous value of q. If the value of q is positive, the value of i increases, if it is negative, it diminishes.

For example, if a positive and a negative charge (with equal absolute value) go through S together, in the same direction, it is equivalent to a neutral particle going through S: Such a particle does not contribute to the current.

Experiments confirm that, for all electric and magnetic effects, the motion of a positive charge is equivalent to that of a negative charge in the opposite direction.

4.4 Different Types of Current

A metallic conductor can be thought as made by a structure of positive charges fixed at the vertices of a lattice and by a "cloud" of electrons free to move in a chaotic manner, continuously colliding against each other and the positive charges.

In conditions of electrostatic equilibrium, the current intensity through a generic section S of the conductor is zero. In fact, due to the chaotic nature of the thermal motion of electrons, the same number of electrons goes through S in one direction and in the opposite direction during the same physics infinitesimal time window Δt.

As a consequence, from a macroscopic time scale (which is the scale of interest in the definition of current intensity) the function $q(t)$ is always zero. Hence, also its time derivative is zero.

The scenario is quite different if the free electrons acquire a motion in a specific direction due to suitable forces acting on them. Such a motion, drift, must be superimposed to the background chaotic thermal motion, diffusion. In the electrostatic case, the center of charge of a cloud of n electrons occupies, on average, always the same position (on a macroscopic time scale). In the dynamic case, instead, a component with constant velocity (even if small compare to the thermal component) is superimposed to the thermal motion. This velocity changes the average position of the free electrons in time. To get an idea of the order of magnitude of such drift velocities, it is worth mentioning that the electron velocity due to thermal motion in a metallic conductor at room temperature is on order of $10^5\, ms^{-1}$, whereas the typical drift velocity is on the order of fractions of millimeter per second. These types of drift current are called conduction currents. Due to the extremely small mass of the free electrons compared to the fixed ions, the current flows without a corresponding mass flow (at least at the macroscopic level). Moreover, a current flow in a conductor does not imply the presence of an electric net charge on it. In fact, in a physics infinitesimal volume of conductor the positive (fixed) charges are, on average, as many as the negative (moving) charges. As a consequence, the total charge keep on being zero, even if the volume is characterized by a continuous flow of negative charges.

Besides conduction currents, there are many other types of currents that depend on the physical properties of the media where they exist. An important example is that of convection currents, which typically take place in liquids and gases. Such currents correspond to macroscopic movements (of fluid-dynamics type) within the liquid or gas, where both negative and positive charges move in different ways. For example, the movement of large masses of air during a thunderstorm gives rise to convection currents. Other types of currents are those associated with the concept of holes in semiconductor devices.

At last, it is worth mentioning that in phenomena that vary with time, besides conduction and convection currents there are also so called displacement currents. These currents can take place in insulators or even in vacuum (in absence of charges of any type). We will study these currents in PHYS 342.

4.5 Current Density and Continuity Equation

Consider a region Ω in a conductor carrying a current i. Assume that ρ is the volume density of the moving charges only and $\vec{v}$ their velocity at point P and time t (see Fig. 4.3). Note that the velocity here considered is the average velocity vector of the microscopic velocities of the free carriers (electrons) in the neighbourhood of P. This average coincides, by definition, with the oriented drift velocity superimposed to the chaotic one. Given an infinitesimal surface element dS and fixed arbitrarily the direction (inward or outward) of the normal unit vector $\vec{n}$ to dS, we want to evaluate the charge going through dS in the $\vec{n}$ direction during an infinitesimal time window dt. During this time interval, the charge going through dS is all (and only) the charge that was initially at a distance smaller or equal to $v\,dt$ with respect to any point on dS. In other words, the moving free charge carriers going through dS are those contained in an oblique cylinder with base dS and height $\vec{v} \cdot \vec{n}\,dt$. Such a charge is given by

$$dq = \rho\vec{v} \cdot \vec{n}\,dt\,dS \quad . \tag{4.9}$$

By defining

$$\vec{J} = \rho\vec{v} \quad , \tag{4.10}$$

Eq. (4.9) can be rewritten as

$$dq = \vec{J} \cdot \vec{n}\,dt\,dS = J_n\,dt\,dS \quad , \tag{4.11}$$

where J_n is the component of vector $\vec{J}$ along the normal $\vec{n}$ at dS.

The vector $\vec{J}$ defined by (4.10) is called current volume density. The SI units of $\vec{J}$ are

$$\left[J\right] = \frac{C}{m^3}\,\frac{m}{s} = \frac{A}{m^2}$$

and its component along a generic oriented direction $\vec{n}$ gives [as shown by (4.11)] the quantity of charge that goes through a unitary surface (normal to $\vec{n}$) in the unit time.

Given a generic open surface Σ, it is possible to calculate (with respect to an arbitrary chosen direction) the current intensity that goes through Σ, i.e., the quantity of charge that goes through it in the unit time.

From (4.11), we find

$$i = \iint_{\Sigma} \rho\vec{v} \cdot \vec{n}\, dS = \iint_{\Sigma} \vec{J} \cdot \vec{n}\, dS \quad . \tag{4.12}$$

Thus, the current intensity i that goes through a generic oriented surface Σ is given by the flux through Σ of the current density vector $\vec{J}$. Note that, in this way, the definition of current intensity (previously only given for filiform conductors) is now extended to the generic case of conductors with any geometry.

Equation (4.10) gives the current density when the free charge carriers are all of the same sign. More in general, (4.10) can be written as

$$\vec{J} = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- \quad , \tag{4.10$'$}$$

where ρ_+ and $\vec{v}_+$ are the volume charge density and velocity for the positive carriers and ρ_- and $\vec{v}_-$ for the negative carriers.

In summary, the phenomenon of charge transport associated with a current is described in local form by a vector quantity (the current density) and in integral form by a scalar quantity (the current intensity).

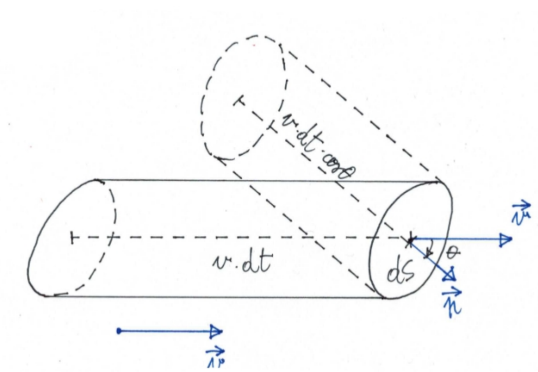


Figure 4.3

Assume now the conductor where the current flow takes place is a conducting plate S (see Fig. 4.4). In this case, we can define at each generic point P on S a vector called surface current density $\vec{J}_{\text{s}}$,

$$\vec{J}_{\text{s}} = \sigma\vec{v} \quad . \tag{4.13}$$

where σ is the surface charge density that goes through point P with velocity $\vec{v}$. The dimensions of $\vec{J}_{\text{s}}$ are obviously those of a current per unit length and, thus, the corresponding units are Am^{-1}. The current intensity that goes through the plate is

$$i = \int_{\gamma} \vec{J}_{\text{s}} \cdot \vec{n}\, d\ell \quad , \tag{4.14}$$

where γ is the input border of S (see Fig. 4.4) and $\vec{n}$ is the normal unit vector at a point on γ ($\vec{n}$ is tangent to S).

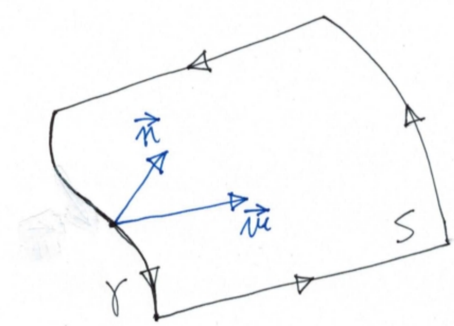


Figure 4.4

Charge transport fulfills the charge conservation principle. Given any closed surface Σ, the quantity of charge that goes through it in a generic time interval corresponds to the change of charge contained in the volume enclosed by Σ. The quantity of charge that goes outside Σ in the unit time is

$$i = \oiint_{\Sigma} \vec{J} \cdot \vec{n}\, dS \quad . \tag{4.15}$$

To this must correspond a change of the total charge contained within Σ. Indicating this charge as q, we find

$$\oiint_{\Sigma} \vec{J} \cdot \vec{n}\, dS = -\frac{d}{dt} q \quad . \tag{4.16}$$

The negative sign shows that, in the case of positive charges going out of Σ (i.e., the first term is positive), we find a reduction in q, i.e., $dq/dt < 0$.

Indicating the charge density in the volume Ω enclosed by Σ as ρ, we have

$$q = \iiint_{\Omega} \rho\, dV$$

and, thus,

$$\oiint_{\Sigma} \vec{J} \cdot \vec{n}\, dS = -\frac{d}{dt} \iiint_{\Omega} \rho\, dV \quad . \tag{4.17}$$

Since the volume Ω is time invariant, the operation of derivative with respect to time and integration in the volume are independent from each other. Hence, the sign of derivative and integral can be exchanged. The charge density ρ is a function of both the point and time.

Thus,

$$\frac{d}{dt}\iiint_{\Omega} \rho\, dV = \iiint_{\Omega} \frac{\partial}{\partial t}\rho\, dV$$

and, thus,

$$\oiint_{\Sigma} \vec{J} \cdot n\, dS = -\iiint_{\Omega} \frac{\partial}{\partial t}\rho\, dV \quad . \tag{4.18}$$

By means of the divergence theorem for the first term of (4.18), we find

$$\iiint_{\Omega} \vec{\nabla} \cdot \vec{J}\, dV = -\iiint_{\Omega} \frac{\partial}{\partial t}\rho\, dV \quad . \tag{4.19}$$

From the arbitrary choice of Σ and, thus, Ω, (4.19) implies the integral functions must be equal. Thus, at each point and time, we have

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial}{\partial t}\rho(\vec{r};\, t) \quad . \tag{4.20}$$

This relation, called the continuity equation, represents the charge conservation principle in local (differential) form. Equation (4.16) and (4.18) represents the principle in integral form.

4.6 Stationary Current

A local property of a physical system that is independent of time at each point of the system is defined a stationary property. If all the properties of a system are stationary, the system itself is called a stationary system. Consider, for example, a moving liquid and assume to be at a point P fixed in the frame of reference with respect to which the liquid is moving. We then examine the velocities of the liquid masses going through P at following time instants. This is called the Eulerian frame of reference. It can happen that the liquid velocity is constant at point P. When this happens at each point where the liquid flows, the flow is said to be stationary. At different time instants, the velocity measured at the same point P refers to the motion of different liquid particles. If we were to follow one of these particles during its motion, we would find that, even under stationary conditions, the particle velocity varies, in general, along the followed path and, thus, it varies in time. The latter is called Lagrangian frame of reference.

An electric current is stationary if at each point of the medium where it flows both the charge and current density are stationary (steady currents). In this case, from Eq. (4.20) it follows that

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{4.21}$$

at each point and time.

Thus, under stationary conditions the field associated with the current density vector is said to be solenoidal. Given any closed surface Σ in the field, it results

$$\oiint_{\Sigma} \vec{J} \cdot \vec{n}\, dS = 0 \quad . \tag{4.22}$$

This means that the quantity of charge that globally goes out of Σ in the unit time is zero. The same amount of charge goes into Σ as, in the same time interval it goes out. If this was not the case, we would find a time varying quantity of charge within the region enclosed by Σ. This would obviously contradict the stationary condition (which we assumed to be valid).

An important consequence of (4.22) is that the current flowing in a conductor has the same value through each conductor section. In other words, the conductor represents a flux tube (see tutorial 2) for the field associated with vector $\vec{J}$. This can be demonstrated by considering a piece of conductor with a certain current enclosed by a surface Σ (see Fig. 4.5). The considered closed surface comprises two generic conductor sections S_1 and S_2 and a lateral surface S. From (4.22), we find

$$\iint_{S_1} \vec{J} \cdot \vec{n}_1 \, dS + \iint_{S_2} \vec{J} \cdot \vec{n}_2 \, dS + \iint_{S} \vec{J} \cdot \vec{n} \, dS = 0 \quad , \tag{4.23}$$

where $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}$ are the normal unit vectors to S_1, S_2, and S, respectively. Moreover, given a coin-type surface δS so that the bottom of the surface resides within the conductor and the top outside, in correspondence to a point on S (see Fig. 4.6), it results

$$\oiint_{\delta S} \vec{J} \cdot \vec{n}' \, dS = 0 \quad . \tag{4.24}$$

In the usual limit for the height h of δS (as in lecture 5, case 2) that goes to zero, we can conclude that

$$J_{ni} \, dS + J_{n\sigma} \, dS = 0 \quad , \tag{4.24$'$}$$

where J_{ni} and $J_{n\sigma}$ are the normal components of $\vec{J}$ to the bases of δS immediately inside and outside S, respectively. We hypothesized that a current flows inside the conductor, but not outside. Hence, it must be $J_{n\sigma} = 0$ and, from (4.24′), also $J_{ni} = 0$. As a consequence, the vector $\vec{J}$ is tangent to the lateral surface of the conductor. Using this result in (4.23), we obtain

$$\iint_{S_1} \vec{J} \cdot \vec{n}_1 \, dS = - \iint_{S_2} \vec{J} \cdot \vec{n}_2 \, dS \quad . \tag{4.25}$$

If the direction chosen for $\vec{n}_1$ is consistent with that chosen for $\vec{n}_2$ (e.g., both outward with respect to the corresponding bases), (4.25) reads

$$\iint_{S_1} \vec{J} \cdot \vec{n}_2 \, dS = \iint_{S_2} \vec{J} \cdot \vec{n}_2 \, dS \quad . \tag{4.26}$$

This proves the conductor is a flux tube for $\vec{J}$.

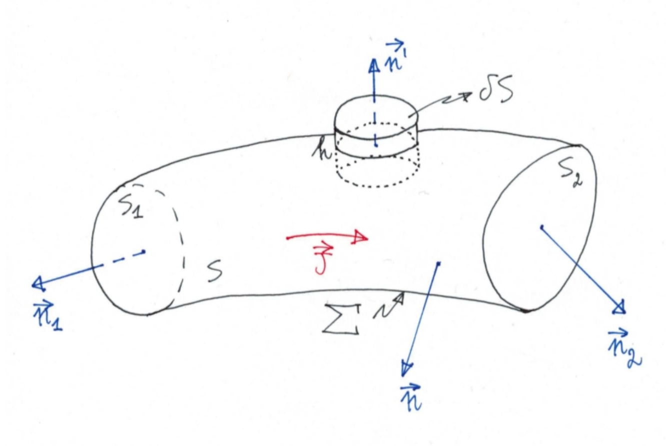


Figure 4.5

Another important consequence of (4.22) is that, given a current flowing in a medium divided by a surface S into two regions with different physical properties, the components of $\vec{J}$ normal to S have the same value on both "skins" of S, at each point on S:

$$J_{n_1} = \lim_{P_1 \to P_0^-} J_n = \lim_{P_2 \to P_0^+} J_n = Jn_2 \quad . \tag{4.27}$$

This result can easily be shown by considering again an elementary closed coin-type surface centred at P_0 on S and applying to it (4.22). Note that, however, the tangent components of $\vec{J}$ can show, in this case, a discontinuity.

In summary, the charge conservation principle in the case of stationary current guarantees the field of vector $\vec{J}$ is solenoidal. In integral form, this can be expressed stating that the flux of $\vec{J}$ out of any closed surface Σ in the field is zero,

$$\oiint_{\Sigma} \vec{J} \cdot \vec{n}\, dS = 0 \quad .$$

In local form (at each regular point in the medium; differential form), this corresponds to

$$\vec{\nabla} \cdot \vec{J} = 0 \quad .$$

Finally, in correspondence to discontinuity points between media with different physical characteristics,

$$J_{n_1} = J_{n_2} \qquad \text{(local form)} \quad .$$

4.7 The Electric Field in Conductors with a Stationary Current

So far, we treated the phenomenon of electric current from a "cinematic" point of view. That is, we introduced without, however, showing the causes of the phenomenon. We will now attack this problem, considering for the moment only stationary currants.

It is evident that if in a conductor the charge free carriers move in a specific direction, opportune forces must act on them. These forces can have different origins. For, now we will study the case of electric forces.

In the lectures on conductors, we saw that in a conductor in electrostatic equilibrium, the electrostatic field is zero at each point inside the conductor. If the conductor is not in equilibrium, i.e., a current flows through it, a non-zero electric field is present in the conductor. Such a field, acting on the free charges, makes them move generating the current. The question to ask is then what are the laws describing the behaviour of the electric field in a conductor with stationary current.

Experiments show that such laws are the same for electrostatic phenomena, both inside and outside the conductor. In fact, even in presence of a stationary current, the electric field obeys the two following fundamental theorems:

$$\oiint_{\Sigma} \vec{E} \cdot \vec{n}\, dS = \frac{1}{\epsilon_0} q \quad , \qquad \text{(Gauss' theorem)} \tag{4.28}$$

$$\oint_{\gamma} \vec{E} \cdot \vec{t}\, d\ell = 0 \quad , \qquad \text{(irrotational property)} \tag{4.29}$$

where Σ and γ are a closed surface and oriented line, respectively.

In summary, a conductor with stationary current is characterized by two vector fields, both stationary: The electric field $\vec{E}$ and the current density field $\vec{J}$. These fields obey (4.28) and (4.29), and (4.22), respectively. Such laws, however, (or theorems) do not show the relationship between $\vec{E}$ and $\vec{J}$. This relationship depends on the physical properties of thee conductor and, thus, it cannot be given independently from them.

The relation

$$\vec{J} = f\left(\vec{E}\right) \quad , \tag{4.30}$$

characteristic of each conductor, is called constitutive relation. The constitutive relation must be determined case by case experimentally.

For certain conductors at constant temperature, Eq. (4.30) is a linear relation,

$$\vec{J} = g\vec{E} \quad , \tag{4.31}$$

which is called Ohm's law (in local form). If the medium under consideration is isotropic, (4.31) becomes a simple proportionality relation with g a scalar constant (which, in general, depends on the point). In this case, $\vec{E}$ and $\vec{J}$ are parallel and have the same direction. If, instead, the medium is not isotropic, (4.31), written in the components of the two vectors, has typically the form

$$\left\{ \begin{array}{lcl} J_x &=& g_{xx}E_x + g_{xy}E_y + g_{xz}E_z \quad , \\ J_y &=& g_{yx}E_x + g_{yy}E_y + g_{yz}E_z \quad , \\ J_z &=& g_{zx}E_x + g_{zy}E_y + g_{zz}E_z \quad , \end{array} \right. \tag{4.31$'$}$$

where, in general, g_{ij} depends on the point. Crystalline bodies (which, in general, are anisotropic, at least in the case of single crystals), different directions are not equivalent. The most common conductors are metals, which crystallize in systems with very high symmetry (body-centred cubic, face-centred cubic, ...). Within a good approximation, metals can be assumed to be isotropic, even in the form of single crystals. In most cases, metals are macroscopically isotropic even if the constituent single crystals are not, because macroscopic properties correspond to measurement on ensembles of many single crystals that are randomly distributed.

The quantity g in (4.31) is called electric conductivity and depends only on the physical nature of the conductor. The inverse quantity $\mu = 1/g$ is called the electric resistivity. The conductivity is often indicated by the symbol σ (or γ), while the resistivity by ρ. The conductivity is measured in $AV^{-1}m^{-1}$ and the resistivity in VmA^{-1}. We will come back to these units later.

Relation (4.31) cannot be used for all conductors. With very good approximation, it can be used for metals and metallic alloys and, typically, can be used for electrolytic solutions. However, it cannot be used in the case of ionized gases. For many substances, (4.31) can nevertheless be used at least in some limited range of values of E and J. In fact, even for non-ohmic media, such as insulators, it is possible to define an electric resistivity. The materials with lowest resistivity are metals. In the case of copper, for example, $\mu = 1.69 \times 10^{-8} VmA^{-1}$ at room temperature. A typical value of μ for an insulator (e.g., quartz) is on the order of $10^{17} VmA^{-1}$ (i.e., approximately 10^{25} times larger than copper).

The conductivity of a medium depends on the physical conditions in which the medium is prepared, i.e., on temperature, pressure, and state of matter (i.e., solid, liquid or gas). A fundamental phenomenon is them the Joule effect, i.e., the well-known fact that a conductor with current becomes hotter (this is the case of light bulbs or electric heaters, for example).

The heat generated because of the Joule effect is due to the transformation of the work produced by the electric forces. In fact, during their motion the charge free carries interact with the surrounding atoms and molecules. During such interactions they loose part of the kinetic energy they acquire from the electric field between collisions. The energy lost in the collisions becomes heat. In order to calculate the specific power (i.e., the power per unit volume) dissipated in the conductor, we can consider a flux tube of vector $\vec{J}$, of length $d\ell$ and section dS. The charge that goes through dS in the time dt is

$$dq = \vec{J} \cdot \vec{n}\, dS\, dt \quad , \tag{4.32}$$

where $\vec{n}$ is the unit vector normal to dS. The work produced by the electric field $\vec{E}$ to move dq along $d\vec{\ell}$ (note that $d\vec{\ell}$ is directed as $\vec{J}$ and $d\vec{\ell}/|d\ell| = \vec{J}/|J|$) is

$$dW = \vec{E} \cdot d\vec{\ell}\, dq = \left(\vec{E} \cdot d\vec{\ell}\right)\left(\vec{J} \cdot \vec{n}\, dS\, dt\right) \quad . \tag{4.33}$$

Since $\vec{J}$ and $d\vec{\ell}$ have the same direction and sign, we can exchange the position of $d\vec{\ell}$ and $\vec{J}$. We can also divide by dt and obtain

$$\frac{dW}{dt} = \left(\vec{E} \cdot \vec{J}\right)\left(d\vec{\ell} \cdot \vec{n}\right) dS = \vec{E} \cdot \vec{J}\, dV \quad , \tag{4.34}$$

where dV is the infinitesimal volume of the piece of flux tube considered. Thus, the specific power (power for unit volume) dissipated at point P is

$$P_V = \frac{dW}{dt\, dV} = \vec{E} \cdot \vec{J} \quad . \tag{4.35}$$

If Ohm's law can be used for the conductor, we find

$$P_V = EJ = \mu J^2 = gE^2 \quad . \tag{4.36}$$

4.8 The Electromotive Force Acting in a Circuit with Current

In order to maintain a stationary current in a conductor a work must be produced. The work is continuously transformed into heat. This work can be locally produced by an electric field acting on the charge free carriers. It is thus natural to ask the question whether a stationary current in a conductor can be maintained exclusively by an electric field that obeys Gauss' theorem $\left[(4.28)\right]$ and the irrotational property $\left[(4.29)\right]$. The answer is negative. First of all, it is worth noting that a conductor where a current flow takes place (in stationary conditions) must be part of an electric circuit, i.e., a closed path along which current can flow. In fact, under stationary conditions the vector field $\vec{J}$ is solenoidal. In other words, $\vec{J}$ is characterized by closed field lines. Note that this last statement is not generally correct. In fact, the field lines of a solenoidal field can be open, but yet cover a surface in the ergodic sense. In any case, such lines do not have a beginning or end. This is the property used in the present argument. Hence, our hypothesis does not diminish the generality of the argument. Note that reader might encounter ergodic solenoidal fields in the study of magneto-hydrostatic. While it is true that an irrotational field can maintain a current in a part of a conductor, it cannot maintain it in a closed circuit. In fact, being a conservative field, it produces zero work along any closed line γ,

$$\oint_\gamma \vec{E} \cdot \vec{t}\, d\ell = 0 \quad . \tag{4.37}$$

Thus, a conservative field cannot produce the required work to maintain a current in the entire circuit.

In the case of ohmic conductors, Eq. (4.37) would read

$$\mu \oint_\gamma \vec{J} \cdot \vec{t}\, d\ell = 0 \quad , \tag{4.38}$$

which is an impossible result if $\mu \neq 0$. This is because γ is a field line of $\vec{J}$ and, thus, the circulation of $\vec{J}$ can be zero only if $\vec{J} = \vec{0}$ at each point in the circuit. Note that, while in (4.37) γ can be any closed line, in (4.38) we have chosen γ to be a vector field line of $\vec{J}$: This is a special case of closed line and, thus, (4.37) remains valid.

Thus, in order to maintain a stationary current in a closed circuit it is necessary that forces of different nature (chemical, mechanical, thermic, ...), superimposed to the electric conservative forces, act on the charge free carriers. These forces must be able to produce a non-zero work along a closed path. Note that particular care must be taken when distinguishing between electric and non-electric forces. In fact, at the microscopic level all interactions between atoms, electrons, and nuclei are, indeed, of electric nature. Such a distinction, thus, makes sense only at the macroscopic level. In a macroscopic context, electric forces are only those that are generated by globally non-zero electric charge distributions in macroscopic volumes.

As a consequence, the conducting circuits must be of the type shown in Fig. 4.6. The figure schematically shows two regions with different physical characteristics. Region C is a simple conductor (e.g., metallic), whereas region G (comprised between parallel faces) is where the non-electric phenomena take place. These phenomena generate the required non-conservative field. In other words, the non-electric force field $\vec{F}_m$ acting on the charge free carriers is supposed to exist only in the region G. The electric field $\vec{E}$, instead, is present in both regions and also in the space outside the conductor. In G takes place the superposition of the forces originated by $\vec{E}$ and the non-electric forces $\vec{F}_m$.

Given a closed line γ only partially located in G (see Fig. 4.6), it must be

$$\oint_\gamma \vec{F}_m \cdot \vec{t}\, d\ell \neq 0 \quad . \tag{4.39}$$

In fact, since $\vec{F}_m = \vec{0}$ outside G, it has to be

$$\oint_\gamma \vec{F}_m \cdot \vec{t}\, d\ell = \int_{\gamma_G} \vec{F}_m \cdot \vec{t}\, d\ell \neq 0 \quad , \tag{4.39$'$}$$

where γ_G indicates the part of γ within G. Assuming γ (and, thus, γ_G) to be oriented clockwise and F_m to be uniform and directed as in Fig. 4.6 inside G, we find

$$\oint_\gamma \vec{F}_m \cdot \vec{t}\, d\ell = F_m \ell_G \quad , \tag{4.40}$$

where l_G is the distance between the two of G.

In order to describe the non-conservative forces $\vec{F}_m$ at any point, it is useful to define a vector field $\vec{E}_m$ with the same dimensions of the electric field,

$$\vec{E}_m = \frac{\vec{F}_m}{q} \quad , \tag{4.41}$$

where q is the charge of the carrier that moves in the conductor. The vector $\vec{E}_m$ is called the electromotive field. Despite the fact that has the units of a force divided by a charge, its physical properties are totally different from those of an electric field. In particular, not only $\vec{E}_m$ is non-conservative, but also it depends, in general, on the value of q. In fact, is not necessarily true that $\vec{F}_m$ is proportional to the value of the charge.

Here, we will not delve into the details of the physical origin of $\vec{E}_m$. Intuitively, we can imagine that the carriers entering G are pushed through G by a microscopic hand with force F_m.

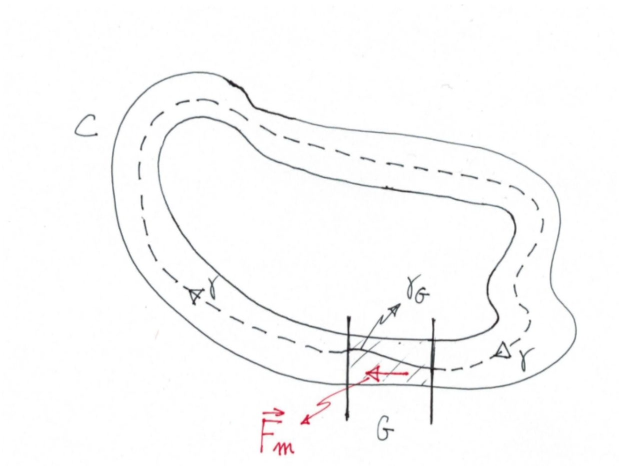


Figure 4.6

The component of the electric circuits where the forces $\vec{F}_m$ act are called sources. The sources provide the work required to make the current flow.

From (4.39) follows that

$$\mathcal{E} = \oint_{\gamma} \vec{E}_m \cdot \vec{t}\, d\ell \neq 0 \quad , \tag{4.42}$$

where $\mathcal{E}$ is the so called electromotive force (emf) acting along the line γ. The fact that $\mathcal{E}$ is called a force is far from being ideal. In fact, $\mathcal{E}$ has the dimensions of a work divided by a charge and, thus, has the SI units of a volt. $\mathcal{E}$ is a scalar quantity, the sign of which depends on the arbitrary orientation of the line γ. $\mathcal{E}$ represents the ratio W/q between the work of the electromotive field when moving the charge q along the line γ (we remind γ is a closed line) and the charge itself. From (4.39′) it obviously follows that

$$\mathcal{E} = \oint_{\gamma} \vec{E}_m \cdot \vec{t}\, d\ell = \int_{\gamma_G} \vec{E}_m \cdot \vec{t}\, d\ell \quad . \tag{4.42′}$$

We now intend to study in detail the working principle of the circuit, considering first the simple case of an source, i.e., a source disconnected from any external conductor that closes the circuit. The source is characterized by two terminals, A and B, that make it possible to eventually connect the source to an external conductor and, thus, to close the circuit. Assume we know the field distribution of the electromotive field $\vec{E}_m$ at each point in the source G. For example, assume $\vec{E}_m$ is uniformly distributed in G. In the case of an open circuit, there can be no current flowing in the source (as always, we assume a stationary current). Hence, under stationary conditions at each point in G it must be $\vec{J} = \vec{0}$. This implies that the total force acting on the charge carriers (sum of $\vec{F}_m$ and $q\vec{E}$) is zero at each point in G,

$$\vec{E}_m + \vec{E} = \vec{0} \quad . \tag{4.43}$$

In open circuit, the electromotive field $\vec{E}_m$ is balanced point by point by an electrostatic field $\vec{E}$. In order to explain the origin of $\vec{E}$ it is enough to notice that, because of the action of $\vec{E}_m$, the free charges in G start moving and migrate until is possible, i.e., till the limiting surface of the source (in particular, to the terminals A or B). On such a surface, which is originally neutral, a surface charge distribution takes place. This charge distribution generates a reaction electrostatic field $\vec{E}_r$. The process continues until the electrostatic field equilibrates the electromotive field point by point. Figure 4.7 shows a qualitative representation of the charge distribution on the surface of the source and of the field lines of the electrostatic field $\vec{E}$ generated by those charges, both inside and outside the source.

Referring to Fig. 4.8, given any line γ_{AB}^{ext} that connects the terminals A and B, outside the source, we have

$$\int_{\gamma_{AB}^{\text{ext}}} \vec{E} \cdot \vec{t}\, d\ell = V^\circ(A) - V^\circ(B) \tag{4.44}$$

because $\vec{E}$ is conservative. This means there is a potential difference $V_{AB}^\circ = V^\circ(A) - V^\circ(B)$ at the terminals (or, better, between them). Furthermore, because of the very definition of $\vec{E}$,

$$\int_{\gamma_{AB}^{\text{int}}} \vec{E} \cdot \vec{t}\, d\ell = V^\circ(A) - V^\circ(B) \quad , \tag{4.45}$$

where γ_{AB}^{int} is any line connecting A and B, inside the source. In addition, from (4.43) at each point of γ_{AB}^{int} it must be

$$\vec{E}_m = -\vec{E} \quad .$$

Hence,

$$\begin{aligned} V_{AB}^0 &= \int_{\gamma_{AB}^{\text{int}}} \vec{E} \cdot \vec{t}\, d\ell = -\int_{\gamma_{AB}^{\text{int}}} \vec{E}_m \cdot \vec{t}\, d\ell \\ &= \oint_{\gamma} \vec{E}_m \cdot \vec{t}\, d\ell = \mathcal{E} \quad , \end{aligned} \tag{4.46}$$

where γ is any closed line coinciding with γ_{AB}^{int} at each point inside G, but with opposite direction compared to γ_{AB}^{int}. As a consequence, in the case of an open circuit the potential difference between the source terminals coincides with the emf. Note that, if the electromotive field id uniform, the value of the emf does not depend on the considered line, so long the line crosses the region G only one time. If, instead, the electromotive field is non-uniform in G, the emf can depend on the specific chosen line. In these cases, even when the source is open, there can be currents flowing in its interior.

Referring to Fig. 4.9, assume now to connect the terminals A and B by means of an external conductor C. Due to the action of the electric field $\vec{E}$, the charge free carriers in C starts moving, generating a current that modifies the charge distribution at the interface between the entire system (which now also includes C) and the external environment. Once the stationary regime has been reached, the free charges (supposed to be positive, for simplicity), "fall" along C from the point at higher potential (in

Fig. 4.9, point A) to that at lower potential (B). The opposite, instead, takes place inside the source, where $\vec{E}_m$ moves the charges against the forces due to the electrostatic field. The electromotive field causes the free charges to rise from the point at a lower potential to that at a higher one. If the conducting material in the source is linear, at each point in the source it must be

$$\vec{J} = g\big(\vec{E} + \vec{E}_m\big) \quad , \tag{4.47}$$

where g is the electric conductivity inside the source. Note that (4.47) represents an extension of Ohm's law (in local form) to the case where both the electric field $\vec{E}$ and the electromotive field $\vec{E}_m$ are simultaneously present.

In the external conductor C, instead, it must be

$$\vec{J} = g_C\vec{E} \quad , \tag{4.48}$$

where g_C is the conductivity of C.

In stationary conditions, the electromotive field $\vec{E}_m$ is larger (in absolute value) than the electric field $\vec{E}$ inside the source, so that

$$\vec{E}_m + \vec{E} = \mu\vec{J} \quad . \tag{4.49}$$

The fact that $\vec{E}_m$ is larger (in absolute value) than $\vec{E}$ inside G depends on the potential reduction imposed by the external connection between terminals A and B. In the limiting case of infinite conductivity for C, the external connection would result in a short circuit, resulting in a zero potential difference between A and B and, thus, a zero field $\vec{E}$ inside the source.

The potential difference between the terminals A and B is given by

$$\begin{aligned} V_{AB} &= V\big(A\big) - V\big(B\big) \\ &= \int\limits_{\gamma_{AB}^{\text{int}}} \vec{E} \cdot \vec{t}\,d\ell \quad , \end{aligned} \tag{4.50}$$

where we have now removed the superscript in the potential function, V_0, that was used to indicate an open circuit situation. in (4.50), γ_{AB}^{int} indicates any line connecting A and B inside G. From (4.49), we have

$$\vec{E} = \mu\vec{J} - \vec{E}_m \quad , \tag{4.51}$$

and, thus,

$$\begin{aligned} V_{AB} = &= \int\limits_{\gamma_{AB}^{\text{int}}} \mu\vec{J} \cdot \vec{t}\,d\ell - \int\limits_{\gamma_{AB}^{\text{int}}} \vec{E}_m \cdot \vec{t}\,d\ell \\ &= \mathcal{E} - \int\limits_{\gamma_{AB}^{\text{int}}} \mu\big(-\vec{J}\big) \cdot \vec{t}\,d\ell \quad . \end{aligned} \tag{4.52}$$

Hence, in the case of a closed circuit, the potential difference between the source terminals does not coincide anymore with the emf. Referring to the directions indicated in Fig. 4.9, we have $-\vec{J} \cdot \vec{t} > 0$ i.e., the potential difference is smaller than the emf.

Before concluding this lecture, it is worth mentioning that the point by point balancing between $\vec{E}_m$ and $\vec{E}$ described after (4.43) is only valid at each point inside G. In fact, on the outer skin of G the action of the electromotive field is balanced (at least as far as the normal component to the surface of G is concerned) by the double layer forces that do not allow the charge carriers to exit the body.

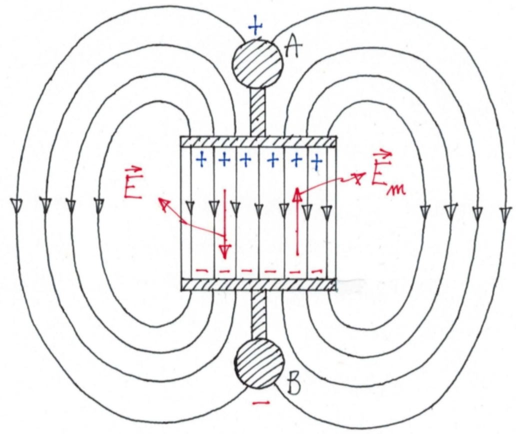


Figure 4.7

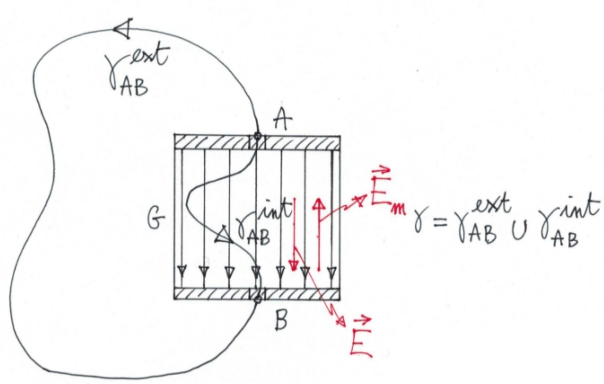


Figure 4.8

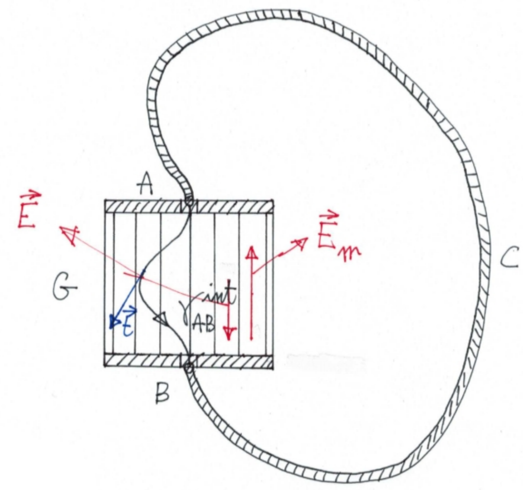


Figure 4.9

4.9 Ohm's and Joule's Law in Integral Form

We will show that when two equipotential regions (electrodes) between which a potential difference V exists are connected by means of a linear and isotropic conductor, a current I flows in the conductor such that

$$V = RI \tag{4.53}$$

where the proportionality constant R is called the electric resistance of the conductor. Equation (4.53) is Ohm's law in integral form. The resistance depends on the position of the electrodes. The unit measure of R in the SI is the ohm (Ω), which represents the resistance of a conductor where a current of 1 A flows and that is characterized by a potential difference of 1 V.

Ohm's law can also be written as

$$I - GV \quad , \tag{4.54}$$

where $G = 1/R$ is called the electric conductance of a conductor. The unit of measure of G in the SI is the siemens (S) or $1/\Omega$. Hence, the unit of resistivity and conductivity can be written as Ωm and Sm^{-1}, respectively.

We will now show that (4.53) and (4.54) can be derived from

$$\vec{J} = g\vec{E}$$

and, thus, are valid only for linear conductors. Given a conductor, consider a filiform flux tube for vector $\vec{J}$ (see Fig. 4.10). The filiform assumption means that each section S of the tube is so small that both vectors $\vec{J}$ and $\vec{E}$ can be assumed to be constant on each section (in general, $\vec{J}$ and $\vec{E}$ vary between different sections). By indicating with ℓ a curvilinear coordinate (abscissa) along the flux tube axis and considered a cross-section S_K of the tube, with S_K orthogonal to $\vec{J}$, we have

$$I_K = \vec{J} \cdot \vec{t} S_K \quad , \tag{4.55}$$

where $\vec{t}$ is the unit vector tangent to the flux tube axis in correspondence of S_K. Since $\vec{t}$ is directed as $\vec{J}$, (4.55) reads

$$I_K = J S_K \quad , \tag{4.55$'$}$$

where J is the magnitude of $\vec{J}$. As a consequence,

$$\begin{aligned} V &= \int_A^B \vec{E} \cdot \vec{t} d\ell = \int_A^B \mu \vec{J} \cdot \vec{t} d\ell \\ &= \int_A^B \mu \frac{I_K}{S_K} d\ell \quad . \end{aligned} \tag{4.56}$$

The area S_K varies, in general, with ℓ. Similarly, if the conductor is assumed to be non homogeneous, the value of the resistivity μ also varies with ℓ. The value of I_K, instead, is constant because of the very definition of flux tube. Hence,

$$V = I_K \int_A^B \frac{\mu}{S_K} d\ell \quad . \tag{4.57}$$

The quantity

$$R_K = \int_A^B \frac{\mu}{S_K} d\ell \tag{4.58}$$

is the resistance of the k-th flux tube. We can thus write

$$V_K = V = R_K I_K \quad . \tag{4.59}$$

The same considerations can be repeated for each of the filiform flux tubes the entire conductor can be thought to be made of. If I is current through the entire cross-section of the conductor, we have

$$I = \sum_K I_K \quad , \tag{4.60}$$

from which

$$I = \sum_K \frac{V}{R_K} = V \sum_K \frac{1}{R_K} \quad . \tag{4.61}$$

Note that if the cross-section of each filiform flux tube is infinitesimally small, the symbol of sum must be substituted by that of integral. In the following, for simplicity, we will keep using the sum symbol. Finally, by indicating

$$\frac{1}{R} = \sum_K \frac{1}{R_K} \tag{4.62}$$

we find

$$I = \frac{V}{R} \quad . \tag{4.63}$$

The result clearly shows that Ohm's law in integral form is a direct result of $\vec{J} = g\vec{E}$.

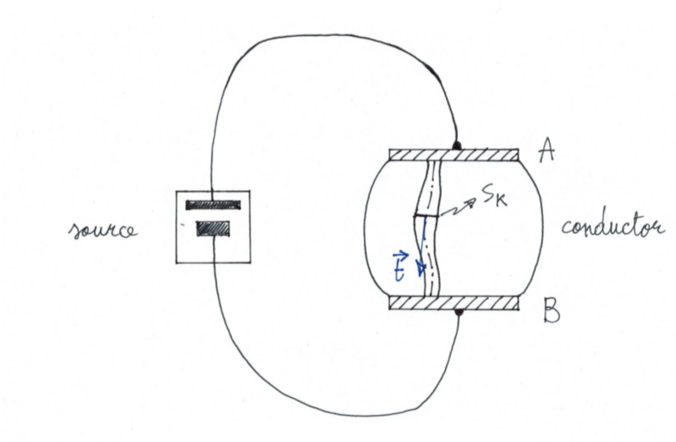


Figure 4.10

Joule's law in integral form can also be demonstrated by resorting to the flux tubes shown in Fig. 4.10. In fact, the power transformed into heat due to the Joule effect on an infinitesimal segment $d\ell$ of the k-th flux tube (still considered to be filiform) is given by

$$dP_K = \mu J^2 S_K \, d\ell \quad , \tag{4.64}$$

which, remembering that

$$J = \frac{I_K}{S_K} \quad ,$$

can be written as

$$dP_K = \mu \frac{I_K^2}{S_K^2} S_K \, d\ell = \mu \frac{I_K^2}{S_K} \, d\ell \quad . \tag{4.65}$$

Therefore, the total power P_K dissipated on the K-the flux tube is

$$P_K \int_A^B \mu \frac{I_K^2}{S_K} \, d\ell = R_K I_K^2 \quad , \tag{4.66}$$

where R_K is given be (4.58).

Since $V = R_K I_K$, (4.66) reads

$$P_K = V I_K \tag{4.67}$$

and, thus, the total power dissipated in the entire conductor is given by

$$\begin{aligned} P = \sum_K P_K &= \sum_K V I_K = V \sum_K I_K \\ &= V I \quad . \end{aligned} \tag{4.68}$$

Joule's law follows from (4.68),

$$P = VI = RI^2 = \frac{V^2}{R} \quad , \tag{4.69}$$

where we used Ohm's law (4.53) or (4.63).

Referring to Fig. 4.9, we now consider again the case of an electric circuit comprising an emf source $\mathcal{E}$ closed on a conductor AB. We remind that, in this case, the electromotive field $\vec{E}_m$ prevails on $\vec{E}$ and, thus, a current with density $\vec{J}$ directed as $\vec{E}_m$ is generated. The vector $\vec{J}$ is thus directed opposite of the tangent unit vector $\vec{t}$. This means that the scalar quantity in (4.52)

$$-\vec{J} \cdot \vec{t}$$

must be actually positive. In particular,

$$-\vec{J} \cdot \vec{t} S - I \quad , \tag{4.70}$$

where S is the source cross-section. Equation (4.52) can thus be written as

$$\begin{aligned} V_{AB} &= \mathcal{E} - \int\limits_{\gamma_{AB}^{\text{int}}} \mu\big(-\vec{J}\big) \cdot \vec{t} d\ell \\ &= \mathcal{E} - I \int\limits_{\gamma_{AB}^{\text{int}}} \frac{\mu}{S} \, d\ell \quad , \end{aligned} \tag{4.71}$$

where we assumed $\vec{J}$ to be uniform inside the source and we called I the total current flowing in the source.

By defining

$$R_g = \int\limits_{\gamma_{AB}^{\text{int}}} \frac{\mu}{S} \, d\ell \quad , \tag{4.72}$$

we have

$$V_{AB} = \mathcal{E} - R_g I \quad . \tag{4.73}$$

By multiplying both of this equation by I, we then obtain

$$\mathcal{E} I = V_{AB} I + R_g I^2 \quad , \tag{4.74}$$

which represents the energetic budget of the entire circuit. In fact, from the very definition of emf, it appears that $\mathcal{E} I$ is the power generated by the source, i.e., the work per unit time produced by the electromotive field acting inside the source. From (4.68), the term $V_{AB} I$ represents the power dissipated in the conductor AB due to the Joule effect and $R_g I^2$ the power dissipated inside the source. In this case, the power generated by the electromotive field inside the source against the electric field forces is transformed entirely into heat (due to the Joule effect) in the conductor and the source itself.

4.10 Resistance of Conductors with Current

We will now study two interesting examples of conductors with currents. In particular, we will calculate the resistance of such conductors starting from the potential equations.

4.10.1 Cylindrical Conductor with Longitudinal Current

Consider a cylindrical conductor C, homogeneous, with a generic cross-section, and with resistivity μ. The two bases of the cylinder (parallel to each other and normal to the axis of C) are connected to two planar electrodes S_1 and S_2 (with negligible resistivity with respect to μ). A potential difference is applied to S_1 and S_2. Figure 4.11 shows a schematic of the problem.

With respect to a Cartesian coordinate system with the z axis coinciding with the conductor axis, the potential function V is independent from x and y due to the rotation symmetry of the conductor about the z axis. At each point in C we thus have

$$\frac{d^2}{dz^2} V\left(z\right) = 0 \quad . \tag{4.75}$$

The general solution to this equation is

$$V\left(z\right) = A_z + B \quad , \tag{4.76}$$

where A and B are arbitrary constants. The electric field $\vec{E}$ is thus uniform and directed along z. In absolute value

$$|E| = \left| \frac{d}{dz} V\left(z\right) \right| = |A| \quad . \tag{4.77}$$

As a consequence, the current density $\vec{J}$ is also uniform and directed along z. The absolute value of $\vec{J}$ is given by

$$|J| = \frac{|E|}{\mu} = \frac{|A|}{\mu} \quad . \tag{4.78}$$

Thus, the current I through the conductor is given by

$$|I| = |J| S = \frac{|A|}{\mu} S \quad , \tag{4.79}$$

where S is the area of the normal cross-section of the conductor (orthogonal to z).

From (4.76), the potential difference between the conductor electrodes is

$$\Delta V = |V_1 - V_2| = |A| L \quad , \tag{4.80}$$

where V_1 and V_2 are the potentials at S_1 and S_2, respectively, and L the length of C.

By definition, the resistance is given by

$$R = \frac{\Delta V}{|I|} = \frac{\cancel{|A|} L}{\cancel{|A|} \dfrac{S}{\mu}} = \mu \frac{L}{S} \quad , \tag{4.81}$$

which is a very useful result to calculate the resistance of filiform conductors (i.e., conductors with longitudinal dimensions L much larger of the transversal dimension characteristic of the normal cross-section S, $L \gg \sqrt{S}$).

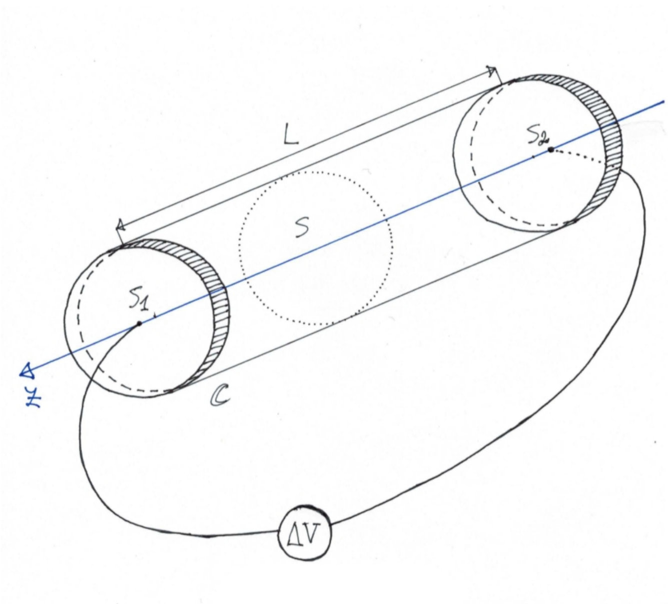


Figure 4.11

4.10.2 Hollow Cylindrical Conductor with Radial Current

Consider a hollow cylindrical conductor $\widetilde{C}$, with a circular cross-section with internal radius R_1 and external radius R_2, and with resistivity μ. Assume the internal and external surfaces, Σ_1 and Σ_2, respectively, are two electrodes (with resistivity much smaller that μ) between which a potential difference V is applied. Figure 4.55 shows a schematic of the problem.

Given a cylindrical coordinate system $Or\varphi z$ with the z axis coinciding with $\widetilde{C}$ axis, for symmetry reasons (translation symmetry with respect to z and rotation symmetry with respect to φ), the potential V only depends on r.

Thus, Laplace equation in cylindrical coordinates is

$$\frac{d^2}{dr^2} V(r) + \frac{1}{r}\frac{d}{dr} V(r) = 0 \quad . \tag{4.82}$$

By defining

$$\varphi(r) = \frac{d}{dr} V(r) \quad , \tag{4.83}$$

equation (4.82) reads

$$\frac{d}{dr}\varphi(r) + \frac{1}{r}\varphi(r) = 0 \quad , \tag{4.84}$$

the general solution of which is

$$\varphi(r) = \frac{A}{r} \quad , \tag{4.85}$$

where A is an arbitrary constant. By substituting (4.85) into (4.83), we finally find the general solution of (4.82),

$$V(r) = A \ln r + B \quad , \tag{4.86}$$

where A and B are arbitrary constants and r in the natural logarithm should be considered as normalized over 1m (for obvious dimensions reasons).

The electric field is also directed radially and has absolute value

$$|E| = \left| \frac{d}{dr} V(r) \right| = \frac{|A|}{r} \quad . \tag{4.87}$$

The current density $\vec{J}$ is also directed radially, with absolute value

$$|J| = \frac{1}{\mu} |E| = \frac{1}{\mu} \frac{|A|}{r} \quad . \tag{4.88}$$

The total current that goes through any cylindrical surface with radius r, comprised between Σ_1 and Σ_2, and coaxial with Σ_1 and Σ_2 is given by

$$\begin{aligned} |I| = 2\pi r L |J| &= 2\pi \cancel{r} L \frac{1}{\mu} \frac{|A|}{\cancel{r}} \\ &= \frac{1}{\mu} 2\pi L |A| \quad , \end{aligned} \tag{4.89}$$

where L is the conductor length.

From (4.86), the potential difference between the electrodes is

$$|V| = |A| \ln \left(\frac{R_2}{R_1} \right) \quad . \tag{4.90}$$

Under these conditions, the conductor resistance is given by

$$\begin{aligned} R = \frac{|V|}{|I|} &= \frac{\cancel{|A|} \ln \left(\frac{R_2}{R_1} \right)}{\frac{1}{\mu} 2\pi L \cancel{|A|}} \\ &= \mu \frac{\ln \left(\frac{R_2}{R_1} \right)}{2\pi L} \quad , \end{aligned} \tag{4.91}$$

which is a very useful expression when, for example, the so called insulating resistance of a coaxial cable needs to be calculated.

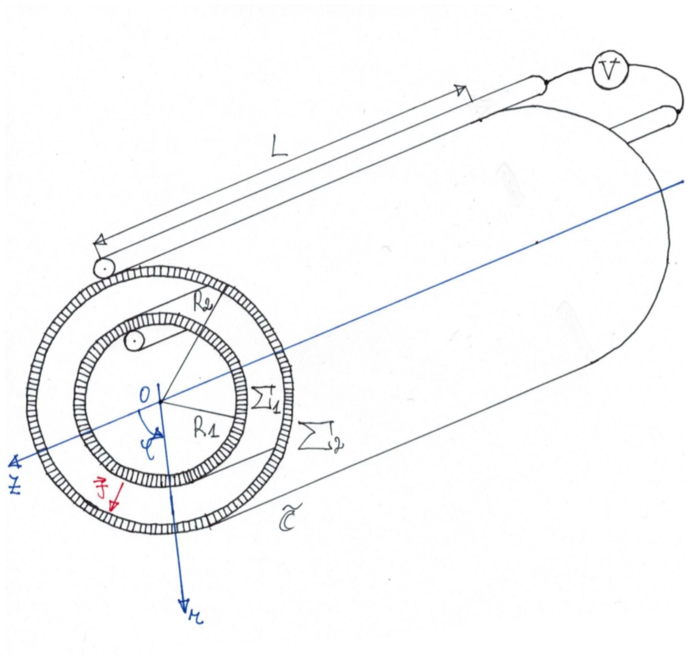


Figure 4.12

Chapter 5

The Magnetostatic Field in Vacuum

5.1 Introduction

By placing two electric circuits close to each other, when stationary currents flow through both circuits, mutually attractive or repulsive forces act between the circuits. Such forces disappear as soon as (at least) one of the two currents is turned off. The forces between the two circuits are called ponderomotive forces, i.e., forces able to act on the conductor as any other mechanical force, e.g., the weight. The first experiments on those forces were due to H. C. Oersted, J. B. Biot, and F. Savart.

The phenomenon of ponderomotive forces in circuits can be qualitatively explained by thinking that when a conductor is characterized by a current, the motion of the charges generates a magnetic field in the surrounding space. In presence of a second conductor, in proximity of the first conductor, and also characterized by a current, the moving charges in each of the two circuits are acted upon be a force of magnetic origin

$$\vec{F} = q\vec{v} \times \vec{B} \quad , \tag{5.1}$$

where $\vec{v}$ is the velocity corresponding to the oriented motion that characterizes the current. Note that, when averaged over a physics infinitesimal time interval, the force acting on a charge is not affected by the term due to the thermal motion.

The moving charge free carriers in the conductor are acted upon both by the forces due to the electric field in the conductor and by those due to the magnetic field. The electric field acts both on the positive ions of the crystalline lattice and on the negative free electrons. However, since the ions are fixed, the electric field acts effectively only on the negative electrons, making them move. In this sense, the electric field cannot have any effect on the entire conductor because, in each macroscopic portion of the conductor, the conductor is neutral.

Given a reference frame, the forces of magnetic origin (if the conductor is at rest) act only on the free charges, normally to the direction of their motion. However, such forces are (typically) insufficient to tear the electrons off the conductor due to the very strong Coulomb-type attraction forces that bind them to the positive fixed ions. The resulting macroscopic effect of the magnetic forces is, thus, a ponderomotive force acting on the entire conductor.

5.2 The Fundamental Laws of Magnetostatic

When presenting the laws of the magnetic field generated by stationary currents, in analogy to the electrostatic case, we could proceed starting from the law of the magnetic field generated in vacuum by an elementary current (e.g., by a circular loop). We could then extend this law to the general case of a distribution of currents by means of a superposition principle. As it turns out, this would be a not so easy task. We will thus present the fundamental laws directly (in integral form). From these laws, we will then find the magnetic field generated by a generic distribution of stationary currents. In the electrostatic case, this would correspond to present directly Gauss' theorem and the irrotational property and, then, obtain from them the field for a generic distribution of charges (see lecture 8).

The first law of magnetostatic states that the magnetic field generated by any distribution of currents must be solenoidal. In other words, indicating with $\vec{B}$ the magnetic field and with Σ any closed surface contained within the field, it must be

$$\oiint_{\Sigma} \vec{B} \cdot \vec{n}\, dS = 0 \quad , \tag{5.2}$$

where $\vec{n}$ is the normal unit vector associated with Σ. The difference to the electrostatic case is clear. In that case, instead of (5.2), Gauss' theorem would apply (according to which the flux of $\vec{E}$ through a closed surface is proportional to the sum of all charges contained within the surface). In the magnetic case, the flux of $\vec{B}$ is always zero: Magnetic charges from where or into which vector line start or end do not exist. The vector lines of $\vec{B}$ do not have any beginning or end (i.e., they are closed) or do extend to infinity.

The law (5.2) is equivalent to

$$\iint_{S_1} \vec{B} \cdot \vec{n}_1\, dS = \iint_{S_2} \vec{B} \cdot \vec{n}_2\, dS \quad , \tag{5.3}$$

where S_1 and S_2 are two generic (open) surfaces with common border γ and normal unit vector $\vec{n}_1$ and $\vec{n}_2$ (oriented consistently), respectively. Since the flux of the magnetic field depends only on γ, in the magnetic case the flux is usually referred to as the linked flux with respect to a closed line γ. In this case, it is unnecessary to specify the surface corresponding to γ. The dimensions of the flux of $\vec{B}$ are those of a magnetic field times a surface, $[\Phi] = [B][S]$. The corresponding SI unit measure is, thus, Tm^2 which is called the *weber* (Wb).

The second law of magnetostatic (also called the circulation law or Ampère's law, or Ampère's circuital law) is related to the circulation of the magnetic field. Differently from the electrostatic field, the magnetostatic field is rotational. Given a generic closed, oriented line γ in vacuum, contained within a magnetic field, it must be

$$\oint_{\gamma} \vec{B} \cdot \vec{t}\, d\ell = \mu_0 I \quad , \tag{5.4}$$

where μ_0 is the vacuum permeability constant and $\vec{t}$ a tangent unit vector to γ. By definition, the SI value of the vacuum permeability is $\mu_0 = 4\pi\, 10^{-7}$ Hm^{-1}. The term I in (5.4) indicates the sum of all currents linked with γ. It is worth reminding that two closed lines are linked to each other if it is impossible to disconnect them without cutting one of them. A more formal (but less intuitive) definition is that two lines are linked if at least one open surface exists, having one of the two lines as a border and only ??? one intersection point with the other line. The aforementioned sum for I must be intended in a algebraic sense. All currents flowing consistently with the orientation on γ are positive, all the others are negative. Note that the current flow is consistent with the line orientation if the right-hand rule is fulfilled. An example is in order. Consider Fig. 5.1, where two circuits with current I_1 and I_2, respectively, are linked with a closed, oriented line γ. The electric circuits form two closed loops γ_1 and γ_2. According to the figure, the orientation of I_1 on γ_1 has been chosen consistently with the orientation on γ, whereas the orientation of I_2 on γ_2 has been chosen with opposite orientation with respect to γ. In fact, when I_1 crosses the plane surface enclosed by γ, the direction of I_1 is equal to that of $\vec{n}$ for γ. On the contrary, when I_2 crosses the same surface, the direction of I_2 is the opposite of $\vec{n}$. In summary,

$$\oint_\gamma \vec{B} \cdot \vec{t}\, d\ell = \mu_0 (I_1 - I_1) \quad . \tag{5.5}$$

If the closed line γ does not link any current (or it links currents the sum of which is zero), the circulation of the magnetic field is zero. If γ coincides with a vector line, the circulation is certainly non-zero. In fact, since the scalar product $\vec{B} \cdot \vec{t}$ has always the same sign, the circulation is the sum of contributions all with the same sign. Hence, each vector line of the magnetic field is linked at least with one current.

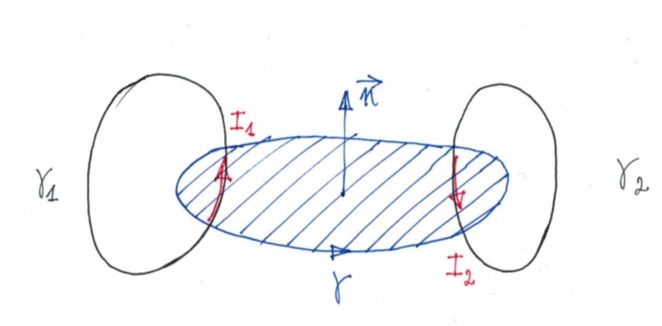


Figure 5.1

Consider two equally oriented open lines γ_1 and γ_2 with equal limits A and B. It must be

$$\int_{A\gamma_1 B} \vec{B} \cdot \vec{t}\, d\ell - \int_{A\gamma_2 B} \vec{B} \cdot \vec{t}\, d\ell = \mu_0 I \quad , \tag{5.6}$$

where I indicates the algebraic sum of all currents linked with the closed line $A\gamma_1 B(-\gamma_2)A$. If such a curve does not link any currents (or it links currents the sum of which is zero), we have

$$\int_{A\gamma_1 B} \vec{B} \cdot \vec{t}\, d\ell = \int_{A\gamma_2 B} \vec{B} \cdot \vec{t}\, d\ell \quad . \tag{5.7}$$

In summary, given internally connected regions of space where there are no currents, it is possible to find there a potential function for the magnetic field. This is because in such regions the circulation of $\vec{B}$ is zero. In presence of currents, this condition is in general not valid.

It is now worth giving an example why the regions must be internally connected. In Fig. 5.2, the point O is the trace of a current I perpendicular to the page. If we imagine that in the plane of the page (and in all planes parallel to it) there is a hole having a closed line γ_1 as a border that surrounds O, the magnetic field is considered in a region that does not contain the current that generates it. Since the domain where the field is considered does not include the region delimited by γ_1, each closed line entirely contained in the domain does not link any currents contained in the domain. Nevertheless, the field circulation around a closed line that contains point O is different from zero. In fact, considering the field outside γ_1 only change the values of $\vec{B}$ in that region. This difficulty can easily be resolved by noting that, once the hole around I is open, the remaining region of the considered plane is not internally connected. Hence, the reason for the internal connection assumption.

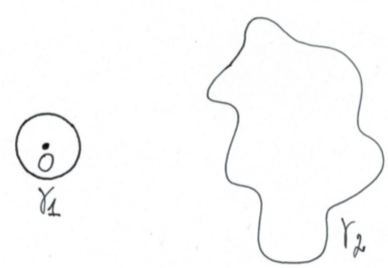


Figure 5.2

Consider one more example. In Fig. 5.3, point O still is the trace of a current I perpendicular to the page. A line of γ_1 type starts at infinite, goes around point O, and goes back to infinite. The region of plane Ω_2, which is obtained by excluding the region Ω_1 (hatched area) included inside γ_1, is internally connected. As a consequence, in this region the magnetic field has a potential (because the current I cannot be linked to any line in Ω_2). Referring to Fig. 5.3, the radius a of the circle that encloses point O can be chosen to be arbitrary small (so long $a \neq 0$); similarly, the distance δ between the two rays that delimit the rest of the region Ω_1 can also be chosen to be arbitrarily small. Thus, the region Ω_2 can be extended to include almost the entire plane (page). Hence, in contrast to our previous statement, it would seem the magnetic field has a (scalar) potential everywhere on the plane (and on all planes parallel to it). In reality, considering two points facing each other as, e.g., P and P', and indicating the potential of the field as φ, from the definition of potential we have

$$\varphi(P) - \varphi(P') = \int_{PAA'P'} \vec{B} \cdot \vec{t}\, d\ell \quad . \tag{5.8}$$

When $\delta \to 0$, the line $PAA'P'$ tends to close around O and, thus, the field line integral tends to the value $\mu_0 I$. As a consequence, when we try to extend the region where $\vec{B}$, has a potential, until the entire domain is included, the potential shows an abrupt jump through the infinitesimal distance δ, where the jump is equal to $\mu_0 I$. Therefore, we must consider a potential the values of which differ by a finite quantity between infinitesimally closed points (such as P and P'). In order to remove such a discontinuity, we should consider multivalued potential functions, i.e., potential functions that assume distinct values at the same point.

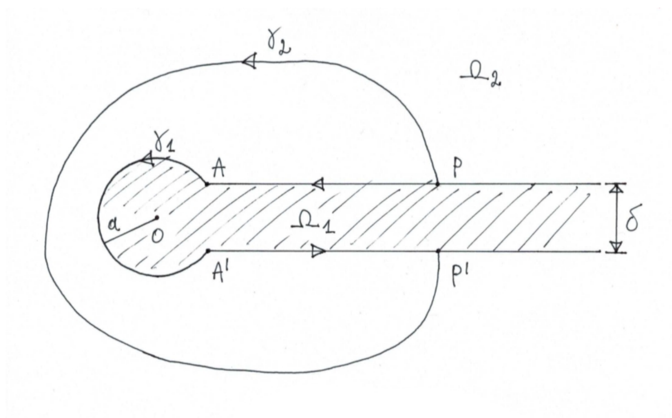


Figure 5.3

It is worth mentioning that the laws of magnetostatic (5.1) and (5.4) contain a superposition principle, according to which the magnetic field generated in vacuum by a set of currents is equal to the vector sum of the fields generated by each current (considered alone). In fact, the second term of (5.4) is the sum of the currents generating the field. As shown in lecture 8, this is analogous to the superposition principle for the electrostatic fields contained in Gauss' theorem. At last, we observe that also the magnetic field generated by spatially limited current distributions goes to zero at infinite (we will come back to this point when discussing the multipole expansion of the vector field).

It is useful to conclude this section with a table that compares the laws of electrostatics and magnetostatics, showing the dual aspect if these laws (Table 5.1).

	Electrostatics	Magnetostatics
Gauss' theorem vs. solenoidal property of $\vec{B}$	$\oiint\limits_{\Sigma} \vec{E} \cdot \vec{n}\, linedS = \dfrac{1}{\epsilon_1}\, q$	$\oiint\limits_{\Sigma} \vec{B} \cdot \vec{n}\, dS = 0$
irrotational property of $\vec{E}$ vs. Ampère's law	$\oint\limits_{\gamma} \vec{E} \cdot \vec{t}\, d\ell = 0$	$\oint\limits_{\gamma} \vec{B} \cdot \vec{t}\, d\ell = \mu_0 I$

Table 5.1

5.3 Magnetic Field Generated by Simple Current Distribution

The first problem to be studied is the magnetostatic dual of the electrostatic field generated by a uniformly charged infinite straight line γ (see lecture 4). Arguments similar to those used in lecture 4 will thus be used.

5.3.1 Infinite Straight Line with Stationary Current

Figure 5.4 shows an infinite straight filiform conductor (line) γ with a stationary current I directed upward along the line.

As in lecture 4, the natural choice for the coordinate system is a cylindrical system $Or\varphi z$, where the z axis coincides with γ and has same orientation as I (see Fig. 5.4).

The magnetostatic vector $\vec{B}$ at a generic point P is also shown in the figure,

$$\vec{B} = B_r\,\vec{u}_r + \vec{B}_\varphi\,\vec{u}_\varphi + \vec{B}_z\,\vec{u}_z \quad . \tag{5.9}$$

We first attempt to solve the problem only by means of symmetry argument (as in Lecture 4). The symmetries characteristic of this problem are:

(a) *Rotation symmetry.*

This is exactly as in lecture 4 and is schematically shown in Fig. 5.5a.

(b) *Translation symmetry.*

Also as in lecture 4 (shown in Fig. 5.5b).

(c) *Anti-reflection symmetry.*

The presence of a current I directed upward breaks the reflection symmetry encountered in lecture 4. This means that, given a vector $\vec{B}$ at a point P, $\{\vec{B}, P\}$, upon rotating γ by a clockwise or counter-clockwise π rotation about a pivot O' (O' can be any point on γ), the field obtained at the rotated point P' must be changed in sign, $\{-\vec{B}, P'\}$. Figure 5.5c shows schematically this type of symmetry.

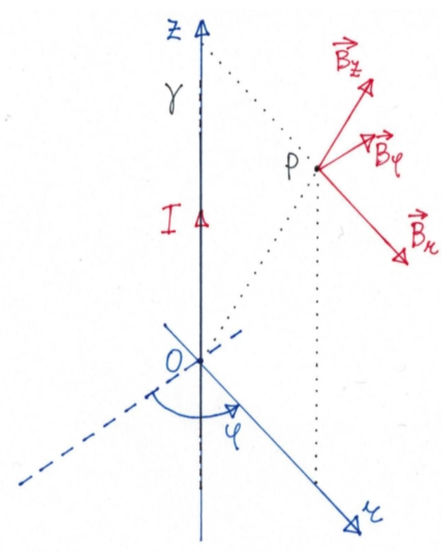
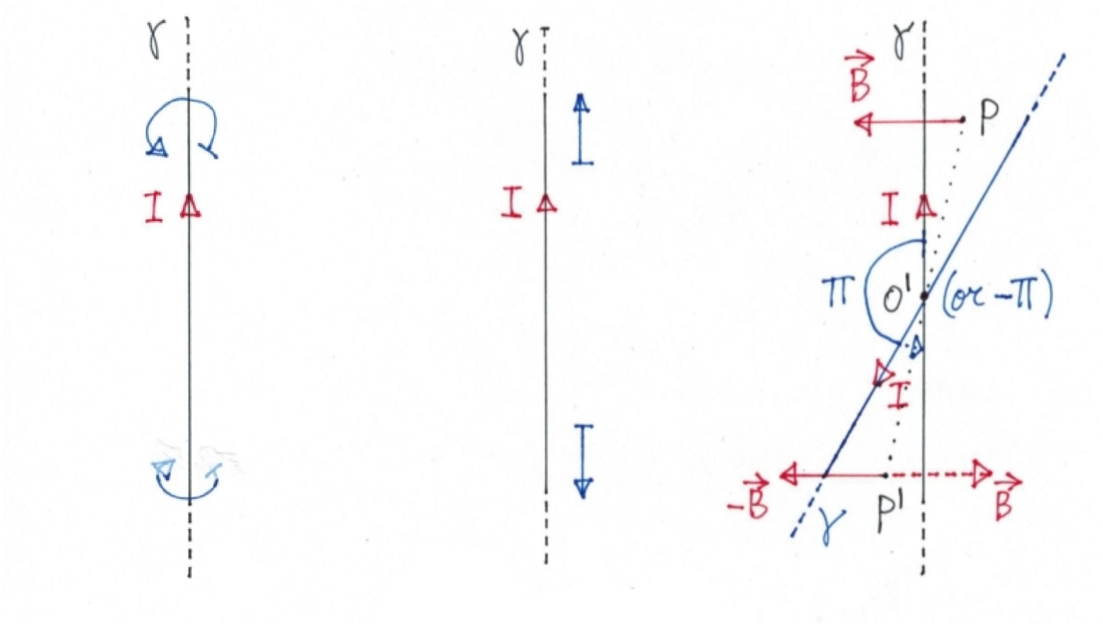


Figure 5.4



(a) (b) (c)

Figure 5.5

As in lecture 4, we will now consider each component of $\vec{B}$ and try to find as much information as possible on it by means of the three aforementioned symmetries.

1) *Radial component* $\vec{B}_r$.

A generic component $\vec{B}_r$ (e.g., pointing outward) at a point P, $\{\vec{B}_r, P\}$, is shown in Fig. 5.6a. Due to the rotation symmetry, if such a component exists, it must be the same at each point on a circle with surface normal to γ and centre O_1 at the intersection between such a surface and γ. The circle is called γ_1 in Fig. 5.6a.

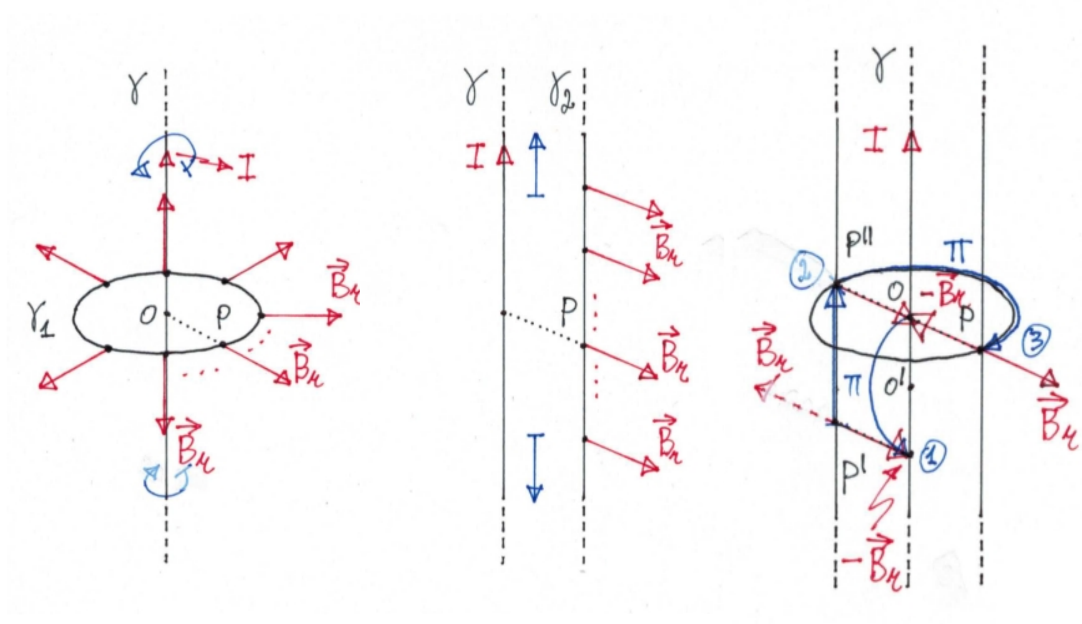
Note that, due to the homogeneity and isotropy of space, if $\vec{B}_r$ points outward at P, it must point outward also at $Q = P + dP$. This is the very same argument as for $\vec{E}_r$, as shown in Fig. **??** in lecture 4.

Due to the translation symmetry, $\vec{B}_r$ must also be the same at each point on a line γ_2 parallel to γ, as shown in Fig. 5.6b.

This means that $\vec{B}_r$ must be the same at each point on any cylinder coaxial with γ and must repeat equally on each plane normal to γ.

We can finally make use of the anti-reflection symmetry to perform a consistency check. Consider $\vec{B}_r$ at P, as in Fig. 5.6c, $\{\vec{B}_r, P\}$. Rotate γ as well as $\{\vec{B}_r, P\}$ as a rigid body by an angle π clockwise (or counter-clockwise) about the pivot O'. At the new point P', we have the new pair $\{\vec{B}_r, P'\}$ due to the anti-reflection property. Translate $-\vec{B}_r$ at P' upward to P'', so to obtain $\{\vec{B}_r, P''\}$. At last, rotate $\vec{B}_r$ at P'' by an angle π clockwise (or counter-clockwise) along γ_1 back to P. The new pair

$\{-\vec{B}_r, P\}$ clearly shows that the only possibility for $\vec{B}_r$ to be compatible with the symmetries of the problem is for $\vec{B}_r = \vec{0}$.



(a) (b) (c)

Figure 5.6

2) *Tangent component* $\vec{B}_\varphi$.

A generic component $\vec{B}_\varphi$ at a point P, $\{\vec{B}_\varphi, P\}$, is shown in Fig. 5.7a. Due to the rotation symmetry, $\vec{B}_\varphi$ must be the same at each point on the generic circle γ_1.

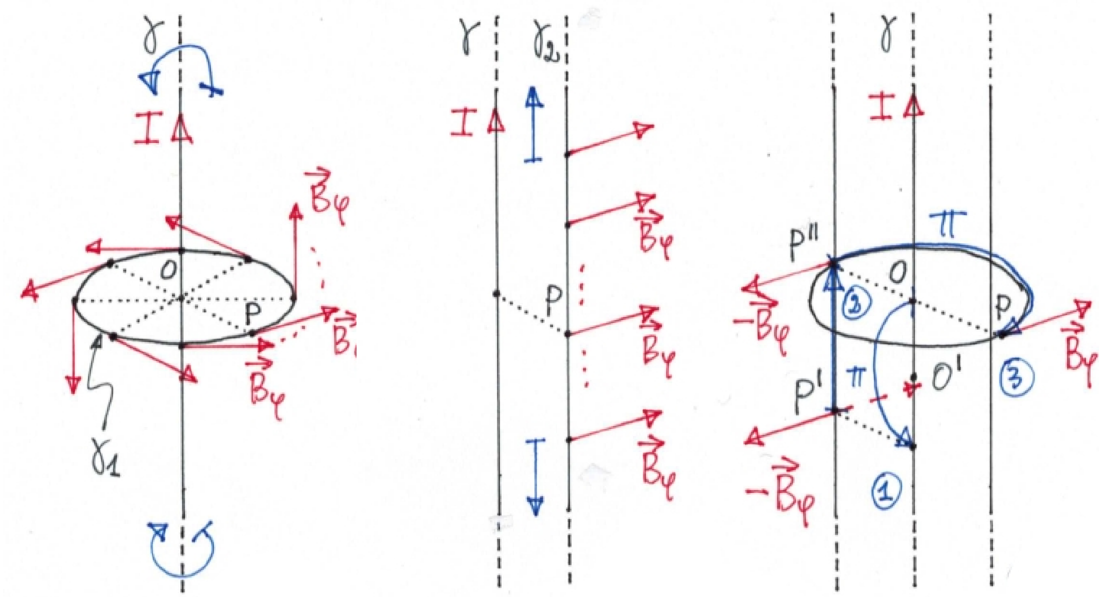
Due to the translation symmetry, $\vec{B}_\varphi$ must also be the same at each point on the generic line γ_2, as in Fig. 5.7b.

Hence, $\vec{B}_\varphi$ must be the same at each point on any cylinder coaxial with γ and must repeat equally on each plane normal to γ.

Finally, the anti-reflection symmetry allows us to perform a consistency check. Consider $\vec{B}_\varphi$ at P, as in Fig. 5.7c, $\{\vec{B}_\varphi, P\}$. Rotate γ as well as $\{\vec{B}_\varphi, P\}$ as a rigid body by an angle π counter-clockwise (or clockwise) about the pivot O'. At the new point P', we have the new pair $\{-\vec{B}_\varphi, P'\}$ due to the anti-reflection property. Translate $-\vec{B}_\varphi$ upward from P' to P'', so to obtain $\{-\vec{B}_\varphi, P''\}$. At last, rotate $-\vec{B}_\varphi$ at P'' by an angle π clockwise (or counter-clockwise) along γ_1 back to P. The new pair $\{-\vec{B}_\varphi, P\}$ is consistent with the initial pair $\{\vec{B}_\varphi, P\}$, showing that $\vec{B}_\varphi$ can exist.

3) *Vertical component* $\vec{B}_z$.

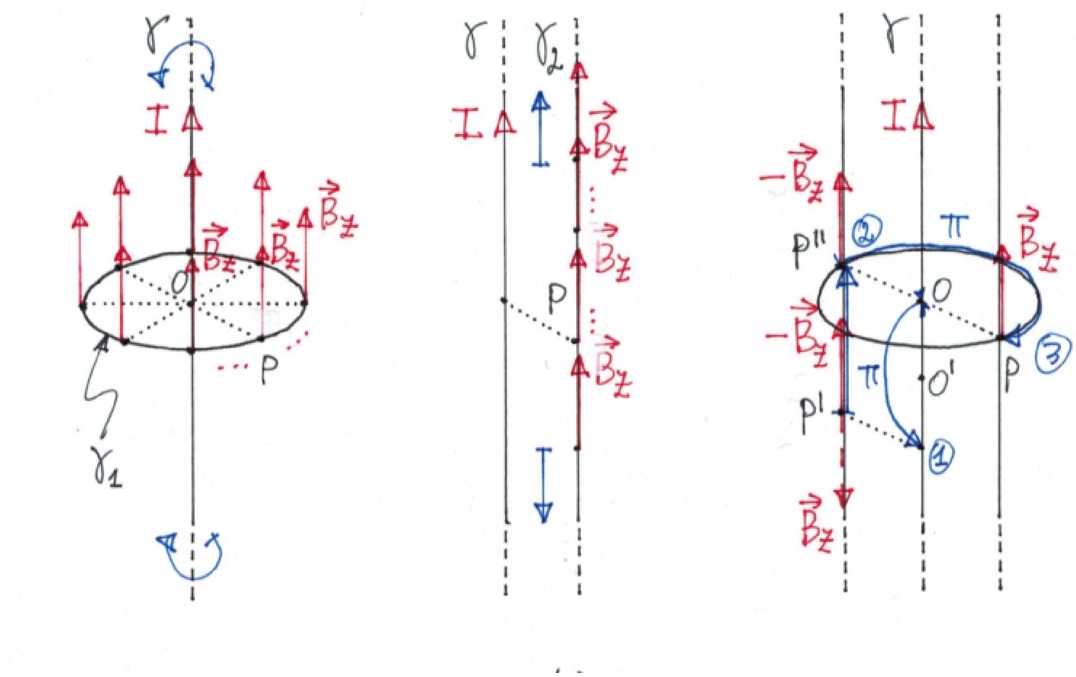
A component $\vec{B}_z$ at P, $\{\vec{B}_z, P\}$, is shown in Fig. 5.8a. Due to the rotation symmetry, $\vec{B}_z$ must be the same at each point on γ_1.



(a) (b) (c)

Figure 5.7

Due to the translation symmetry, $\vec{B}_z$ must be the same at each point on γ_2, as in Fig. 5.8b. The anti-reflection symmetry allows us to perform the usual consistency check. Consider $\vec{B}_z$ at P, as in Fig. 5.8c, $\{\vec{B}_z, P\}$. Rotate γ as well as $\{\vec{B}_z, P\}$ as a rigid body by an angle π counter-clockwise (or clockwise) about the pivot O'. At P', we have $\{-\vec{B}_z, P'\}$ due to the anti-reflection property. Translate $-\vec{B}_z$ upward P' to P'', so to obtain $\{-\vec{B}_z, P''\}$. At last, rotate $-\vec{B}_z$ at



(a) (b) (c)

Figure 5.8

P'' by an angle π counter-clockwise (or clockwise) along γ_1 back to P. The pair $\{-\vec{B}_z, P\}$ is consistent with the initial pair $\{\vec{B}_z, P\}$. Hence, a component $\vec{B}_z$ could exist.

We can now resort to the fundamental laws of magnetostatic, Eqs. (5.2) and (5.4), to find the magnitude of components $\vec{B}_\varphi$ and $\vec{B}_z$.

Figure 5.9 shows a closed oriented line γ_1 as well as a cylindrical surface Σ_1 coaxial with γ, completely closed by the bases S_1 and S_2 and the lateral surface S, and with height h. The radius of γ_1 and of a cross-section of Σ_1 normal to γ (e.g., γ) is r. The closed line γ_1 and surface Σ_1 will be used to calculate the circulation and flux of $\vec{B}$, respectively.

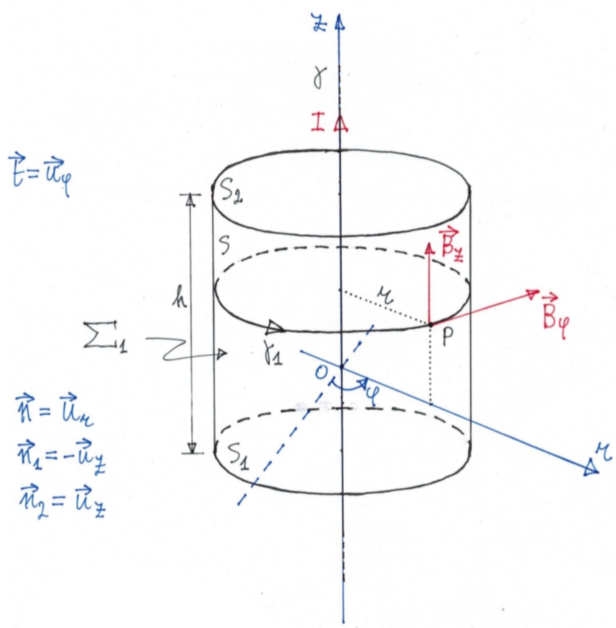


Figure 5.9

A cylindrical coordinate system $Or\varphi z$ as well as the components $\vec{B}_\varphi$ and $\vec{B}_z$ are also shown in Fig. 5.9.

From (5.4), and considering that $\vec{t} = \vec{u}_\varphi$, we find

$$\begin{aligned}\oint_{\gamma_1} \vec{B} \cdot \vec{t}\, d\ell &= \oint_{\gamma_1} B_\varphi \vec{u}_\varphi \cdot \vec{u}_\varphi\, d\varphi\, r \\ &= B_\varphi r \int_0^{2\pi} d\varphi \\ &= 2\pi r B_\varphi = \mu_0 I \quad . \end{aligned} \tag{5.10}$$

Note that the oriented of γ_1 (counter-clockwise) has been chosen consistently with direction (upward) of the linked current I, hence the positive sign in (5.10). From (5.10), it then follows that

$$B_\varphi = \mu_0 \frac{I}{2\pi r} \tag{5.11}$$

and, finally,

$$\vec{B}_\varphi = \mu_0 \frac{I}{2\pi r}\, \vec{u}_\varphi \quad . \tag{5.12}$$

Note that, while B_φ diverges in close proximity of the conductor,

$$\lim_{r\to+\infty} B_\varphi = \lim_{r\to+\infty} \mu_0 \frac{I}{2\pi r} = 0 \quad . \tag{5.13}$$

5.4 The Laws of Magnetostatics in Local Form

As in the electrostatic case, we now present the laws of magnetostatics in local form.

We begin considering the case of current in a stationary regime and distributed with a finite volume density $\vec{J}$.

Case 1

Consider a 3D region Ω in the Euclidean space that is enclosed by a closed surface Σ. Assume $\vec{B} \in C^1(\Omega)$, i.e., the magnetostatic field is continuously differentiable in a vector sense in Ω, and therein limited.

First law of magnetostatics

The integral form, Eq. (5.2), can be rewritten by means of the divergence theorem as

$$\iiint_{\Omega} \vec{\nabla} \cdot \vec{B}\, d\tau = 0 \quad . \tag{5.14}$$

Due to the arbitrariness of Σ and, thus, Ω, at each point where the field $\vec{B}$ is defined, it must then be

$$\vec{\nabla} \cdot \vec{B} = 0 \quad . \tag{5.15}$$

This is the local (differential) form of Eq. (5.15). Note that Eq. (5.15) is valid also in the special case $\vec{J} = \vec{0}$ and, in general, means that $\vec{B}$ (under stationary conditions) is a solenoidal field.

Second law of magnetostatics (Ampère's law)

In Ω, consider an open surface Σ_γ, the border of which is an oriented and closed line γ linking a current I.

The integral form, Eq. (5.2), can be rewritten by means of Stokes' theorem $\big[$we remind we are assuming $\vec{B} \in C^1(\Omega)$ $\big]$ as

$$\oint_{\gamma} \vec{B} \cdot \vec{t}\, d\ell = \iint_{\Sigma_\gamma} \vec{\nabla} \times \vec{B} \cdot \vec{n}\, dS \quad , \tag{5.16}$$

where $\vec{t}\, d\ell$ is an infinitesimal element of γ ($\vec{t}$ is directed consistently with the orientation of γ) and $\vec{n} dS$ an infinitesimal element of Σ_γ ($\vec{n}$ is the normal unit vector to Σ_γ). As always, $d\ell$ and dS have the dimensions of a length and surface, respectively.

Equation (5.16) is only the first term of (5.4). By noting that the current I linked with γ in the second term of (5.4) can be written as the flux through Σ_γ of the current volume density $\vec{J}$, we have

$$\begin{aligned} \iint_{\Sigma_\gamma} \vec{\nabla} \times \vec{B} \cdot \vec{n}\, dS &= \mu_0 I \\ &= \mu_0 \iint_{\Sigma_\gamma} \vec{J} \cdot \vec{n}\, dS \quad . \end{aligned} \tag{5.17}$$

Due to the arbitrariness of γ and, thus, Σ_γ, at each point where $\vec{B}$ is defined, it must then be

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad . \tag{5.18}$$

This is the local (differential) form of Eq. (5.4), which is valid also for $\vec{J} = \vec{0}$ and that, in general, means $\vec{B}$ is a rotational field. The current associated with $\vec{J}$ that fulfil (5.18) are called the vertices of $\vec{B}$.

Case 2

Consider now a surface Σ in the 3D Euclidean space characterized by a stationary current with a surface density $\vec{J}$s. We are in presence of a surface current discontinuity in space. In analogy to Case 2 in lecture 5, we conjecture the impossibility to well-define both divergence and curl in the neighbourhood of Σ.

We will thus resort to the integral forms (5.2) and (5.4) for a coin-type surface and cut-type line, respectively. Such curves were already defined in lecture 5.

First law of magnetostatics

Figure 5.10 shows the coin-type surface Σ_C to be used in (5.2). The cross-section between Σ_C and Σ is considered to be an infinitesimal surface dS. Similarly, the two bases of Σ_C, S_1 and S_2, are assumed to be infinitesimal surfaces dS_1 and dS_2, respectively. The normal unit vectors to dS_1, dS, and dS_2 are $\vec{n}_1$, $\vec{n}$, and $\vec{n}_2$, respectively. The height of Σ_C is h and its lateral surface S_ℓ, with normal unit vector $\vec{n}_\ell$. The centre of Σ_C is point P on surface Σ (P is also the centre of dS). Assuming the fields $\vec{B}_1$ and $\vec{B}_2$ to be constant at each point on §$_1$ and S_2, and assuming a generic field $\vec{B}_\ell$ on S_ℓ (see Fig. 5.10), Eq. (5.2) can be written as

$$\begin{aligned}\oiint_{\Sigma_C} \vec{B} \cdot \vec{n}\, dS &= \iint_{S_1} \vec{B}_1 \cdot \vec{n}_1\, dS + \iint_{S_2} \vec{B}_2 \cdot \vec{n}_2\, dS \\ &+ \iint_{S_\ell} \vec{B}_\ell \cdot \vec{n}_\ell\, dS = 0 \quad .\end{aligned} \tag{5.19}$$

For a small h, the diameter dr of Σ_C can be assumed to be constant and, thus,

$$\iint_{S_\ell} \vec{B}_\ell \cdot \vec{n}_\ell\, dS = \vec{B}_\ell \cdot \vec{n}_\ell\, 2\pi\, dr\, h \quad . \tag{5.20}$$

Hence,

$$\lim_{h \to 0} \vec{B}_\ell \cdot \vec{n}_\ell\, 2\pi\, dr\, h = 0 \tag{5.21}$$

and

$$\lim_{h \to 0} \vec{n}_1 = \vec{n} = -\lim_{h \to 0} \vec{n}_2 \quad . \tag{5.22}$$

Thus, (5.19) becomes

$$\begin{aligned}
\oiint_{\Sigma_C} \vec{B} \cdot \vec{n}\, dS &= \iint_{S_1} \vec{B}_1 \cdot \vec{n}\, dS - \iint_{S_2} \vec{B}_2 \cdot \vec{n}\, dS \\
&= \vec{B}_1 \cdot \vec{n}\, dS_1 - \vec{B}_2 \cdot \vec{n}\, dS_2 \\
&= \vec{B}_1 \cdot \vec{n}\, \cancel{dS} - \vec{B}_2 \cdot \vec{n}\, \cancel{dS} = 0 \quad ,
\end{aligned} \tag{5.23}$$

where we assumed $dS_1 = dS = dS_2$ in the limit for $h \to 0$. From (5.23) we obtain the condition

$$\vec{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \quad , \tag{5.24}$$

which is the first law of magnetostatics in local form and states that the normal components of $\vec{B}$ just above and below Σ in the vicinity of P are continuous,

$$B_{n_1}(P) - B_{n_2}(P) = 0 \quad . \tag{5.24'}$$

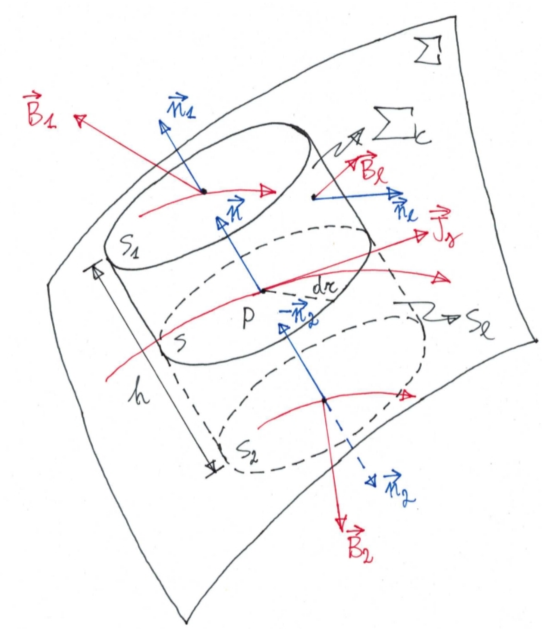


Figure 5.10

Second law of magnetostatics (Ampère's law)

Figures 5.11a and 5.11b show the current surface density vector $\vec{J}$s at a generic point P on Σ. As indicated in the figures, in general, $\vec{J}$s is characterized by two components, one along the tangent unit vector $\vec{t}'$, $\vec{J}'$s, and another along the tangent unit vector $\vec{t}''$, $\vec{J}''$s.

As a consequence, in this case we must consider two different cut-type curves, γ'_C and γ''_C, along which compute the circulation (5.4).

Figure 5.11a shows the first of two such curves, γ'_C. This is a rectangular curve consisting of four straight line segments AB, BC, CD, and DA. The curve is oriented clockwise with respect to the chosen direction for $\vec{J}'s$. The centre of the rectangular surface enclosed by γ'_C is at a generic point P on Σ. The component $\vec{J}'s$ is normal to such a surface at P, $\vec{J}'s = J's\,\vec{t}' = J's(P)\,\vec{t}'$. We assume $\overline{AB} = d\ell'_1$, $\overline{CD} = d\ell'_2$, and $\overline{BC} = \overline{DA} = h$. Because of the infinitesimal dimensions of lines AB and CD, the fields $\vec{B}_1$ and $\vec{B}_2$ can be assumed to be constant on AB and CD, respectively. No assumptions are made on $\vec{B}_{BC}$ and $\vec{B}_{DA}$, which can take any value on lines BC and DA, respectively. The unit vector tangent to each edge of γ'_C are $\vec{t}''_1$, $\vec{t}_{BC}$, $\vec{t}''_2$, and $\vec{t}_{DA}$, respectively. Finally, the normal unit vector to Σ at P is $\vec{n}$.

Ampère's law along γ'_C reads

$$\oint\limits_{\gamma'_C} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{AB} + \int\limits_{BC} + \int\limits_{CD} + \int\limits_{DA} = \mu_0 I's \quad . \tag{5.25}$$

From the definition of current intensity for a surface current distribution $\vec{J}s$, we have

$$\begin{aligned} I'_s &= \int\limits_{\gamma'_s} \vec{J}_s \cdot \vec{t}'\, d\ell = \int\limits_{\gamma' s} \left(\vec{J}'_s + \vec{J}''_s\right) \cdot \vec{t}'\, d\ell \\ &= \int\limits_{\gamma'_s} \left(J'_s\,\vec{t}' + J''_s\,\vec{t}''\right) \cdot \vec{t}'\, d\ell \\ &= J'_s\, d\ell' \quad , \end{aligned} \tag{5.26}$$

where γ'_s is the cross-section between the rectangular surface enclosed by γ'_C and Σ, and $d\ell'$ its length.

The integral along AB is given by

$$\int\limits_{AB} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{AB} \vec{B}_1 \cdot \vec{t}''_1\, d\ell = \vec{B}_1 \cdot \vec{t}''_1\, d\ell'_1 \quad . \tag{5.27a}$$

The integral along BC is given by

$$\int\limits_{BC} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{BC} \vec{B}_{BC} \cdot \vec{t}_{BC}\, d\ell \quad . \tag{5.27b}$$

The integral along CD is given by

$$\int\limits_{CD} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{CD} \vec{B}_2 \cdot \vec{t}''_2\, d\ell = \vec{B}_2 \cdot \vec{t}''_2\, d\ell'_2 \quad . \tag{5.27c}$$

Finally, the integral along DA is given by

$$\int\limits_{DA} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{DA} \vec{B}_{DA} \cdot \vec{t}_{DA}\, d\ell \quad . \tag{5.27d}$$

In order to study $\vec{B}$ on the positive page (i.e., just above) and on the negative page (i.e., just below) of Σ at P, we must consider the limit for $h \to 0$. In this case,

$$\lim_{h\to 0} d\ell'_1 = d\ell' = \lim_{h\to 0} d\ell'_2 \quad ; \tag{5.28}$$

$$\lim_{h\to 0} BC = 0 = \lim_{h\to 0} DA \quad ; \tag{5.29}$$

$$\lim_{h\to 0} \vec{t}''_1 = \vec{t}'' = -\lim_{h\to 0} \vec{t}''_2 \quad . \tag{5.30}$$

Under these conditions, using (5.28) and (5.30) in (5.27a) gives

$$\lim_{h\to 0} \int\limits_{AB} = \vec{B}_1 \cdot \vec{t}''\, d\ell' = B''_{t_1}\, d\ell' \tag{5.27a$'$}$$

Using (5.29) in (5.27b) gives

$$\lim_{h\to 0} \int\limits_{BC} = 0 \tag{5.27b$'$}$$

because the length of integration is zero. Using (5.28) and (5.30) in (5.27c) gives

$$\lim_{h\to 0} \int\limits_{CD} = -\vec{B}_2 \cdot \vec{t}''\, d\ell' = -B''_{t_2}\, d\ell' \quad . \tag{5.27c$'$}$$

Finally, using (5.29) in (5.27d) gives

$$\lim_{h\to 0} \int\limits_{DA} = 0 \quad , \tag{5.27d$'$}$$

as for (5.27b$'$). In the limit for $h \to 0$, (5.25) reads

$$\begin{aligned} \lim_{h\to 0} \oint\limits_{\gamma'_C} \vec{B} \cdot \vec{t} \cdot d\ell \;&=\; B''_{t_1}\, \cancel{d\ell'} - B''_{t_2}\, \cancel{d\ell'} \\ &=\; \mu_0 J'_s\, \cancel{d\ell'} \quad , \end{aligned} \tag{5.31}$$

where we used (5.26). From (5.31) we find

$$B''_{t_1} - B''_{t_2} = \left[B''_t\right] = \mu_0 J'_s \quad , \tag{5.32}$$

which shows a discontinuity of the first kind for the tangent component of the field along $\vec{t}''$ at each point in close proximity of Σ.

Figure 5.11b shows the second cut-type curve, γ''_C. This is exactly the same curve as γ'_C, but rotated by an angle $\pi/2$ about $\vec{n}$ so that the rectangular surface enclosed by γ''_C is now normal to the direction of $\vec{t}''$. For consistency with the sign in (5.30), however, γ''_C must be oriented counter-clockwise with respect to the chosen direction of $\vec{J}''_C$ (see Fig. 5.11b). For clarity of notation, the four vertices of γ''_C are called E, F,

G, and H. The centre of the rectangular surface enclosed by γ_C'' is still at P, and the component $\vec{J}_s''$ is normal to such a surface at P, as indicated in the figure, $\vec{J}_s'' = J_s''(P)\,\vec{t}''$. However, due to the orientation of the line, the normal unit vector to the rectangular surface is now $-\vec{t}''$. We further assume $\overline{EF} = d\ell_1''$, $\overline{GH} = d\ell_2''$, and $\overline{FG} = \overline{HE} = h$. Moreover, we assume $\vec{B}_1$ and $\vec{B}_2$ to be constant on EF and GH, respectively, and $\vec{B}_{FG}$ and $\vec{B}_{HE}$ to be totally generic fields on FG and HE. Finally, the unit vectors tangent to each edge of γ_C'' are $\vec{t}_1'$, $\vec{t}_{FG}$, $\vec{t}_2'$, and $\vec{t}_{HE}$, respectively.

Ampère's law along γ_C'' reads

$$\oint\limits_{\gamma_C''} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{EF} + \int\limits_{FG} + \int\limits_{GH} + \int\limits_{HE} = -\mu_0 I_s'' \quad , \tag{5.33}$$

where the minus sign is due to the fact that $\vec{J}_s''$ has opposite direction of $-\vec{t}''$ at P. From the definition of current intensity we have

$$I_s'' = \int\limits_{\gamma_s''} \vec{J}_s \cdot \vec{t}''\, d\ell = J_s''\, d\ell'' \quad , \tag{5.34}$$

where γ_s'' is the cross-section between the rectangular surface enclosed by γ_C'' and Σ, and $d\ell''$ its length.

The integral along EF is given by

$$\int\limits_{EF} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{EF} \vec{B}_1 \cdot \vec{t}_1'\, d\ell = \vec{B}_1 \cdot \vec{t}_1'\, d\ell_1'' \quad . \tag{5.35a}$$

The integral along FG is given by

$$\int\limits_{FG} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{FG} \vec{B}_{FG} \cdot \vec{t}_{FG}\, d\ell \quad . \tag{5.35b}$$

The integral along GH is given by

$$\int\limits_{GH} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{GH} \vec{B}_2 \cdot \vec{t}_2'\, d\ell = \vec{B}_2 \cdot \vec{t}_2'\, d\ell_2'' \quad . \tag{5.35c}$$

Finally, the integral along HE is given by

$$\int\limits_{HE} \vec{B} \cdot \vec{t}\, d\ell = \int\limits_{HE} \vec{B}_{HE} \cdot \vec{t}_{HE}\, d\ell \quad . \tag{5.35d}$$

In the limit for $h \to 0$,

$$\lim_{h\to 0} d\ell_1'' = d\ell'' = \lim_{h\to 0} d\ell_2'' \quad ; \tag{5.36}$$

$$\lim_{h\to 0} FG = 0 = \lim_{h\to 0} HE \quad ; \tag{5.37}$$

$$\lim_{h\to 0} \vec{t}_1' = \vec{t}' = -\lim_{h\to 0} t_2' \quad ; \tag{5.38}$$

Under these conditions,

$$\lim_{h\to 0}\int_{EF} = B'_{t_1}\, d\ell'' \quad ; \tag{5.35a$'$}$$

$$\lim_{h\to 0}\int_{FG} = 0 \quad ; \tag{5.35b$'$}$$

$$\lim_{h\to 0}\int_{GH} = -B'_{t_2}\, d\ell'' \quad ; \tag{5.35c$'$}$$

$$\lim_{h\to 0}\int_{HE} = 0 \quad . \tag{5.35d$'$}$$

In the limit for $h \to 0$, (5.33) reads

$$\begin{aligned}\lim_{h\to 0}\oint_{\gamma''_C} \vec{B} \cdot \vec{t}\, d\ell &= B'_{t_1}\, \cancel{d\ell''} - B'_{t_2}\, \cancel{d\ell''} \\ &= -\mu_0 J''_s\, \cancel{d\ell''} \quad ,\end{aligned} \tag{5.39}$$

where we used (5.34). From (5.39) we find

$$B'_{t_1} - B'_{t_2} = -\mu_0 J''_s \quad , \tag{5.40}$$

which shows a discontinuity of the first kind for the tangent component of the field along $\vec{t}'$ at each point in close proximity of Σ. Maintaining the same notation as in (5.32), (5.40) reads

$$B'_{t_1} - B'_{t_2} = \left[B'_t\right] = -\mu_0 J''_s \quad . \tag{5.41}$$

The discontinuities (5.32) and (5.41) clearly show that neither the divergence not the curl are well defined in a standard sense at each point on Σ.

Noting that $\vec{t}''$, $\vec{t}'$, and $\vec{n}$ are the unit vectors of a Cartesian coordinate system, (5.32) and (5.41) can be combined together as

$$\vec{n} \times \left[\vec{B}\,\right] = \mu_0 \vec{J}_s \quad , \tag{5.42}$$

where $\vec{n} = (0, 0, 1)$, $[\vec{B}] = [(B''_t, B'_t, B_n)] = [(B''_{t_1} - B''_{t_2}, B'_{t_1} - B'_{t_2}, B_{n_1} - B_{n_2})]$, and $\vec{J}_s = (J''_s, J'_s, 0)$.

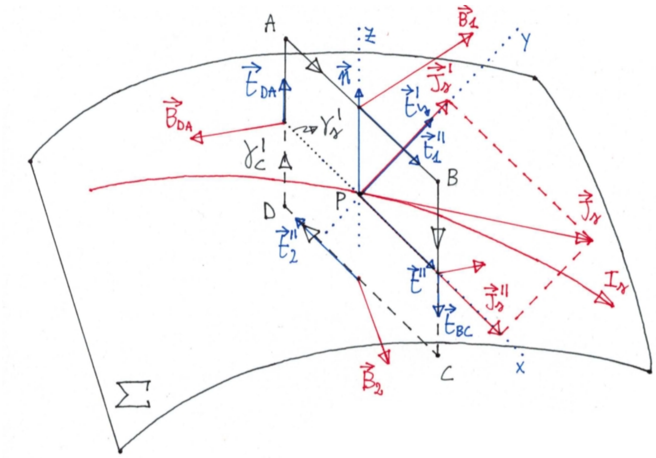
In fact,

$$\begin{aligned}\vec{n} \times \vec{B} &= \begin{vmatrix} \vec{t}'' & \vec{t}' & \vec{n} \\ 0 & 0 & 1 \\ B''_t & B'_t & B_n \end{vmatrix} . \\ &= B''_t\, \vec{t}' - B'_t\, \vec{t}'' = \mu_0\left(J'_s\, \vec{t}' + J''_s\, \vec{t}''\right) \quad ,\end{aligned}$$

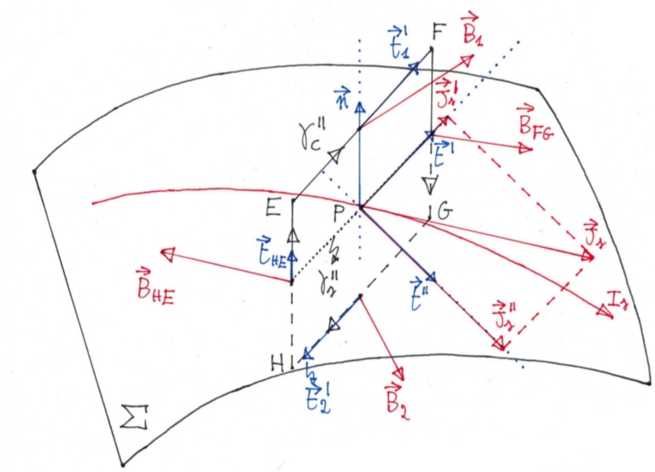
from which

$$\begin{cases} B''_t = B''_{t_1} - B'_{t_2} = \mu_0 J'_s \\ \\ B'_t = B'_{t_1} - B'_{t_2} = -\mu J''_s \end{cases} ,$$

which are equivalent to (5.32) and (5.41), respectively.



(a)



(b)

Figure 5.11

Case 4

Following the analogy with lecture 5, we finally briefly consider Case 4, the field $\vec{B}$ in proximity of a generic filiform conductor with a stationary current. Note that there is no Case 3 for $\vec{B}$.

Without giving a demonstration, in close proximity of the aforementioned filiform conductor, $\vec{B}$ tends to have the same behaviour as for an infinite straight filiform conductor. In particular,

$$\lim_{r \to 0^+} B_t = \mu_0 \frac{I}{2\pi r} \tag{5.43}$$

and

$$\lim_{r\to 0^+} \frac{B_n}{B_t} = 0 \quad , \tag{5.44}$$

where r is the (positive) radial distance from a generic point P in space and a point Q on the conductor, with P in the neighbourhood of Q. In addition, B_t and B_n are the tangent and normal components of $\vec{B}$ at P and I is the current flowing in the conductor.

5.5 Fields with Vector Potential

Necessary and sufficient condition for a generic vector field $\vec{F}$ to be expressed as the curl of another vector field $\vec{A}$ is that $\vec{F}$ is solenoidal,

$$\vec{\nabla} \cdot \vec{F} = 0 \quad . \tag{5.45}$$

The condition must be valid at each point where $\vec{F}$ is defined.

It is easy to demonstrate that the condition must be necessary. In fact, if

$$\vec{F} = \vec{\nabla} \times \vec{A} \quad , \tag{5.46}$$

it must also be

$$\vec{\nabla} \cdot \vec{F} = 0 \quad . \tag{5.47}$$

This is because

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0 \quad . \tag{5.48}$$

The last equality can be shown by writing the components of the curl and divergence of a vector field in Cartesian coordinates,

$$\begin{aligned}
\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) &= \frac{\partial}{\partial_x}\left(\vec{\nabla} \times \vec{A}\right)_x + \frac{\partial}{\partial_y}\left(\vec{\nabla} \times \vec{A}\right)_y + \frac{\partial}{\partial_z}\left(\vec{\nabla} \times \vec{A}\right)_z \\
&= \frac{\partial}{\partial_x}\left(\frac{\partial}{\partial_y} A_z - \frac{\partial}{\partial_z} A_y\right) + \frac{\partial}{\partial_y}\left(\frac{\partial}{\partial_z} A_x - \frac{\partial}{\partial_x} A_z\right) \\
&\quad + \frac{\partial}{\partial_z}\left(\frac{\partial}{\partial_x} A_y - \frac{\partial}{\partial_y} A_x\right) \\
&= \frac{\partial^2}{\partial_x\partial_y} A_z - \frac{\partial^2}{\partial_x\partial_z} A_y + \frac{\partial^2}{\partial_y\partial_z} A_x - \frac{\partial^2}{\partial_y\partial_x} A_z \\
&\quad + \frac{\partial^2}{\partial_z\partial_x} A_y - \frac{\partial^2}{\partial_z\partial_y} A_x = 0 \quad ,
\end{aligned} \tag{5.49}$$

where we assumed all derivatives to be continuous.

To demonstrate that the condition must also be sufficient is a non-trivial task and, thus, we will not show it in these lectures.

By definition, each vector field $\vec{A}$ that satisfies (5.46) and, thus, (5.48) is called a vector potential of the vector field $\vec{F}$.

Moreover, given an arbitrary scalar field Φ, we have by definition

$$\vec{\nabla} \times \left(\vec{\nabla}\,\Phi\right) = 0 \quad , \tag{5.50}$$

because Phi is the scalar potential of an irrotational field. As a consequence, if $\vec{A}$ is a vector potential of $\vec{F}$, any other vector field $\vec{\tilde{A}}$ such that

$$\vec{\tilde{A}} = \vec{A} + \vec{\nabla}\left(\Phi\right) \tag{5.51}$$

is also a vector potential of $\vec{F}$ (in (5.51), Φ is an arbitrary scalar field). In fact,

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\tilde{A}}\right) &= \vec{\nabla} \times \left(\vec{A} + \vec{\nabla}\,\Phi\right) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \left(\vec{\nabla}\,\Phi\right) \\ &= \vec{\nabla} \times \vec{A} \quad , \end{aligned} \tag{5.52}$$

where we made use of (5.50).

5.6 Vector Potential of $\vec{B}$ and Laplace's Elementary Theorem

As shown in Sec. 5.5, every solenoidal vector field has a vector potential. As a consequence, $\vec{B}$ has a vector potential. By calling $\vec{A}$ one arbitrary potential of $\vec{B}$, we have

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad . \tag{5.53}$$

Given an arbitrary scalar field φ, if $\vec{A}$ is a vector potential of $\vec{B}$, each vector field

$$\vec{A'} = \vec{A} + \vec{\nabla}\,\varphi \tag{5.54}$$

is a vector potential of $\vec{B}$. This means that each magnetic field $\vec{B}$ has infinite vector potentials. From (5.54), the knowledge of one vector potential implies the knowledge of all of them.

By substituting (5.53) into (5.18) we readily obtain

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 \vec{J} \quad . \tag{5.55}$$

From the definition of vector Laplacian

$$\vec{\nabla}^2 \vec{A} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) \quad , \tag{5.56}$$

We obtain

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J} \quad . \tag{5.57}$$

Among the infinite vector potentials of $\vec{B}$ is always possible to chose one which is solenoidal. In fact, given a generic vector potential $\vec{A}_0$, which is non-solenoidal, is sufficient to consider a scalar function ψ such that at each point of the domain of definition of $\vec{A}_0$.

$$\vec{\nabla}^2 \psi = -\vec{\nabla} \cdot \vec{A}_0 \tag{5.58}$$

In this case, the vector potential defined by

$$\vec{A} = \vec{A}_0 + \vec{\nabla}\,\psi \tag{5.59}$$

is solenoidal by definition.

Choosing $\vec{A}$ such that

$$\vec{\nabla} \cdot \vec{A} = 0 \quad , \tag{5.60}$$

(5.57) can be rewritten as

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \quad . \tag{5.61}$$

Each solution of this equation, so long solenoidal, is a vector potential of $\vec{B}$ generated by the given currents $\vec{J}$. In Cartesian coordinates, by projecting (5.61) onto the coordinate axes, we obtain the equivalent system

$$\begin{cases} \vec{\nabla}^2 A_x = -\mu_0 J_x \\ \vec{\nabla}^2 A_y = -\mu_0 J_y \\ \vec{\nabla}^2 A_z = -\mu_0 J_z \end{cases} \quad . \tag{5.61$'$}$$

Thus, looking for a solenoidal vector potential associated to a given current distribution is equivalent to look for the solution of a system of three scalar equations, each of which is similar to a Poisson equation. Under the assumption that the three functions A_x, A_y, and A_z go to zero at infinite, the solution of each equation in (5.61′) has the form of a Colombian integral (as in the electrostatic case),

$$\begin{cases} A_x(P) = \dfrac{\mu_0}{4\pi} \iiint\limits_\tau \dfrac{J_x(Q)}{r_{QP}}\, d\tau & (5.62a) \\ A_y(P) = \dfrac{\mu_0}{4\pi} \iiint\limits_\tau \dfrac{J_y(Q)}{r_{QP}}\, d\tau & (5.62b) \\ A_z(P) = \dfrac{\mu_0}{4\pi} \iiint\limits_\tau \dfrac{J_z(Q)}{r_{QP}}\, d\tau & (5.62c) \end{cases}$$

where the components A_x, A_y, and A_z are defined at the genetic field-point P, whereas J_x, J_y, and J_z are defined at the generic source-point Q and r_{QP} is the absolute value of the distance between P and Q. The volume integral is defined on the entire region τ where the currents are distributed.

Equation (5.62a)-(5.62c) can be summarized in a single vector relation that makes it possible to directly calculate the vector potential $\vec{A}$ from the knowledge of $\vec{J}$,

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \iiint\limits_\tau \frac{\vec{J}(Q)}{r_{QP}}\, d\tau \quad , \tag{5.62a$'$ - 5.62c$'$}$$

so long the distribution of $\vec{J}$ is known at each point in τ. Once the vector potential $\vec{A}$ is known, $\vec{B}$ can readily be calculated from

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad .$$

In this way, at least in principle, the fundamental problem of magnetostatics is solved. This consists in finding the magnetic field generated by a given distribution of currents flowing in conductors in vacuum. The only difference with the electrostatic case is in that the solution is more complicated due to the vector nature of the integrand in (5.62a′ - 5.62c′).

By substituting (5.53) into (5.62a′ - 5.62c′), we obtain

$$\vec{B}(P) = \frac{\mu_0}{4\pi}\,\vec{\nabla}_P \times \iiint_\tau \frac{\vec{J}(Q)}{r_{QP}}\,d\tau \quad , \tag{5.63}$$

where the sub-script P shows that the operation of curl must act of the field-point P (and not on the source-point Q). Due to the linearity of the operator curl, we have

$$\vec{B}(P) = \frac{\mu_0}{4\pi}\iiint_\tau \vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}}\,d\tau \quad . \tag{5.64}$$

From the known vector relation

$$\vec{\nabla}_P \times \frac{\vec{J}(Q)}{r_{QP}} = \frac{1}{r_{QP}}\,\vec{\nabla}_P \times \vec{J} - \vec{J} \times \vec{\nabla}_P\,\frac{1}{r_{QP}}$$

and being $\vec{\nabla}_P \times \vec{J} = \vec{0}$ since $\vec{J}$ is not a function of P, we finally obtain

$$\vec{B}(P) = -\,\frac{\mu_0}{4\pi}\iiint_\tau \vec{J}(Q) \times \vec{\nabla}_P\,\frac{1}{r_{QP}}\,d\tau \quad . \tag{5.65}$$

Moreover,

$$\vec{\nabla}_P\,\frac{1}{r_{QP}} = -\,\frac{1}{r_{QP}^2}\,\vec{u}_{QP} = -\,\frac{\vec{r}_{QP}}{r_{QP}^3} \tag{5.66}$$

and, so,

$$\vec{B}(P) = \frac{\mu_0}{4\pi}\iiint_\tau \frac{\vec{J}(Q) \times \vec{r}_{QP}}{r_{QP}^3}\,d\tau \quad . \tag{5.67}$$

This expression for $\vec{B}$ is exactly the same as (5.63). However, it makes it possible to more easily obtain a general expression for the magnetic field of a given distribution of filiform currents. For example, consider a filiform circuit with transversal cross-section Σ and characterized by a stationary current I. Due to the filiform nature of the conductor, Σ can be assumed to be infinitesimal, dS (see Fig. 5.12). Thus, the volume associated with an infinitesimal element $d\ell$ of the circuit is

$$d\tau = dS\,d\ell \quad . \tag{5.68}$$

From (5.67) we then have

$$
\begin{aligned}
\vec{B}(P) &= \frac{\mu_0}{4\pi} \iiint\limits_{\tau} \frac{\vec{J}(Q) \times \vec{r}_{QP}}{r_{QP}^3}\, d\tau \\
&= \frac{\mu_0}{4\pi} \iiint\limits_{\tau} \frac{\vec{J}(Q)\, dS \times \vec{r}_{QP}}{r_{QP}^3}\, d\ell \\
&= \frac{\mu_0}{4\pi}\, I \oint\limits_{\gamma} \frac{\vec{t} \times \vec{r}_{QP}}{r_{QP}^3}\, d\ell \quad ,
\end{aligned}
\tag{5.69}
$$

where

$$
I = \iint\limits_{\Sigma} J(Q)\, dS \quad , \tag{5.70}
$$

Σ is oriented so that $\vec{t}$ is its normal unit vector, γ is the conductor's axis (incidentally, $\vec{t}$ is tangent at each point on γ), and τ is the region where the conductor is defined and the current is distributed [note that

$$
\vec{J}(Q) = J(Q)\, \vec{t} \quad .
$$

Thus,

$$
\begin{aligned}
I &= \iint\limits_{\Sigma} \vec{J}(Q) \cdot \vec{n}\, dS = \iint\limits_{\Sigma} J(Q)\, \vec{t} \cdot \vec{t}\, dS \\
&= \iint\limits_{\Sigma} J(Q)\, dS \quad . \qquad]
\end{aligned}
$$

Assuming the entire filiform conductor with axis γ to be divided into infinitesimally small elements $d\ell$, from (5.69) the infinitesimal contribution to the field $\vec{B}$ can be calculated as

$$
d\vec{B} = \frac{\mu_0}{4\pi}\, I\, d\ell\, \frac{\vec{t} \times \vec{r}_{QP}}{r_{QP}^3} \quad , \tag{5.71}
$$

which is known as Laplace's elementary equation. It is obvious that this equation does not have independent value from (5.69). In fact, it has no physical meaning to consider an "infinitesimal current element" $I\, d\ell$ regardless from the closed circuit to which it must belong. However, it is remarkable the resemblance with the equation that gives the infinitesimal contribution to the electrostatic field $\vec{E}$ due to an infinitesimal charge dq at point P,

$$
d\vec{E} = \frac{1}{4\pi\epsilon_0}\, dq\, \frac{\vec{r}_{QP}}{r_{QP}^3} \quad .
$$

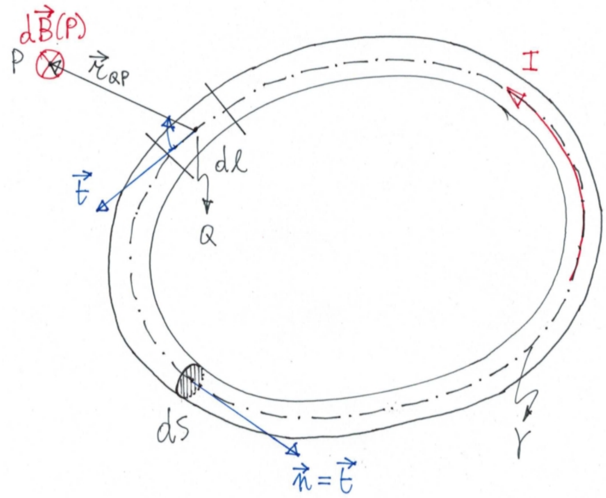


Figure 5.12

5.7 Coefficients of Self and Mutual Inductance of Circuits

Consider a quasi filiform circuit occupying a region of space τ and with a stationary current I. The current is generated by an emf not shown in Fig. 5.13, which only shows the conductive part of the circuit. The magnetostatic field $\vec{B}$ can be calculated at each point inside and outside the conductor τ from

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \tag{5.72}$$

where

$$\vec{A} = \frac{\mu_0}{4\pi} \iiint_{\tau} \frac{\vec{J}}{r}\, d\tau \quad . \tag{5.73}$$

In this equation, $\vec{J}$ is the known volume current density in τ and r the absolute value of the distance between the point where $\vec{A}$ is evaluated and that where $\vec{J}$ is defined inside τ (it is clear the latter varies and, thus, the necessity of the volume integral).

By indicating with γ the closed line representing the longitudinal axis of the conductor and with Φ_γ the flux of the magnetic field generated by I and linked with γ, we have

$$\Phi_\gamma = \iint_{\Sigma_\gamma} \vec{B} \cdot \vec{n}\, dS \quad . \tag{5.74}$$

In this equation, Σ_γ is any open surface having γ as border and oriented such that the positive direction of the normal unit vector $\vec{n}$ (to Σ_γ) and the orientation on γ used to evaluate I are consistent with the right-hand rule. From Stoke's theorem and the properties of $\vec{A}$, we obtain

$$\Phi_\gamma = \iint_{\Sigma_\gamma} \left(\vec{\nabla} \times \vec{A}\right) \cdot \vec{n}\, dS = \oint_{\gamma} \vec{A} \cdot \vec{t}\, d\ell \quad . \tag{5.75}$$

Thus, if the distribution of $\vec{J}$ in the conductor is known, (5.73) gives $\vec{A}$ and (5.75) gives Φ_γ.

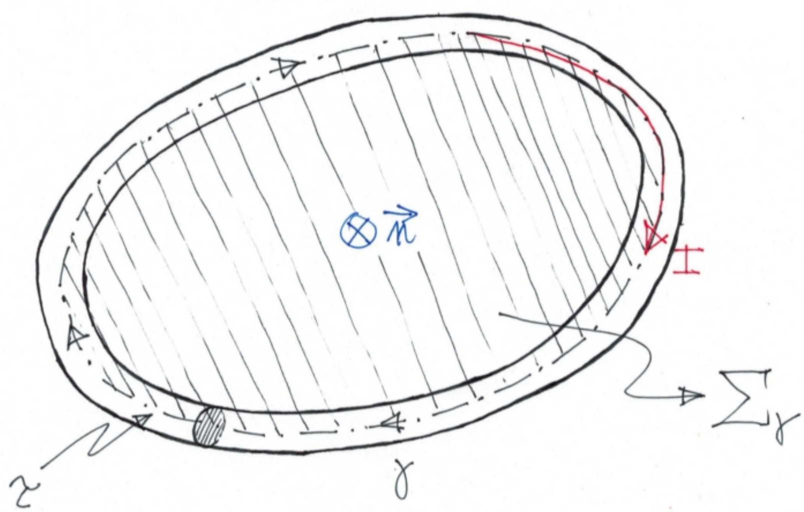


Figure 5.13

By means of a new emf source, we change the value of $\vec{J}$ at each point inside the circuit until we obtain a new current distribution

$$\vec{J'} = K\vec{J} \quad , \tag{5.76}$$

where K is an arbitrary constant. Under these conditions we have

$$I' = KI \quad . \tag{5.77}$$

From (5.73) and (5.72), also $\vec{A}$ and $\vec{B}$ vary linearly,

$$\vec{A'} = K\vec{A} \tag{5.78}$$

and

$$\vec{B'} = K\vec{B} \quad . \tag{5.79}$$

As a consequence,

$$\Phi'_\gamma = K\Phi_\gamma \quad , \tag{5.80}$$

where Φ'_Φ is the new value of the flux linked with γ. By defining as Φ_{1_γ} the value of Φ_γ corresponding to a unitary current flowing in the circuit, we have

$$\Phi_\gamma = \Phi_{1_\gamma} \frac{I}{1} = \frac{\Phi_{1_\gamma}}{1} I \quad , \tag{5.81}$$

where the "one" in the denominator is the value of the unitary current, $1A$. By defining the ratio Φ_{1_γ} as a coefficient L, we can write

$$\Phi_\gamma = LI \quad , \tag{5.82}$$

where L is called the self-inductance coefficient of the circuit with respect to γ. Note that this definition is possible because of (5.80). In fact, $K = I/1$ in (5.81).

The dimensions of L are those of a flux over a current (magnetic flux); hence, its SI unit is Wb/A, which is called "henry" (H).

Because of its very definition, L must be nonnegative. In fact, when I is positive (as, e.g., in the case of Fig. 5.13), the vector lines of the magnetic field point to the same direction of $\vec{n}$ with respect to γ. As a consequence, also Φ_γ must be positive. Similarly, when I is negative, also Φ_γ is negative. In both cases, the ratio Φ_γ/I is positive (or zero).

The flux Φ_γ has been defined with respect to a closed line γ representing the longitudinal axis of the circuit. We could repeat all the arguments for a different line γ', also closed and entirely contained within τ. If the length of τ is much larger than its diameter, the fluxes linked with γ and γ' and generated by the same current I are almost identical. Hence, the value of the inductance L calculated with respect to γ' would be approximately the same as that calculated with respect to γ. Note that, assuming the conductor to be rigorously filiform (instead of quasi filiform) i.e., with zero diameter, would lead to a meaningless definition of L. In this case, in fact, even in presence of a finite current, the field $\vec{B}$ would diverge to infinite when approaching the conductor. As a consequence, the flux of $\vec{B}$ lined with the circuit would also be infinite and, so, the ratio Φ_γ/I as well. As a matter of fact, assuming that a finite current flows in a filiform circuit does not have physical meaning.

Consider now two quasi filiform circuit τ_1 and τ_2, as shown in Fig. 5.14. The two circuits, which are close to each other, but do not touch each other, form a circuit system. Assume a stationary current I_1 (the sign of which is evaluated with respect to an arbitrary reference direction) flows in τ_1, whereas no current flows in τ_2. We choose arbitrary a positive direction on γ_2, which is the longitudinal axis of τ_2. The generic surface Σ_2 that has γ_2 as a border is oriented such that its normal unit vector $\vec{n}_2$ follows the usual right-hand rule. We intend to determine the flux due to the magnetic field generated by I_1 and linked with γ_2. Following an argument similar to that used in the self-inductance case, we find

$$\Phi_{21} = M_{21} I_1 \quad . \tag{5.83}$$

In this equation, Φ_{21} is the flux generated by I_1 and linked with γ_2 and M_{21} a proportionality factor with units of an inductance (i.e., measurable in H as L). This factor is called the mutual-inductance coefficient of the first circuit on the second. Note that sometimes L and M are called coefficient of self (or outo) and mutual (or cross) induction.

Depending on the chosen orientation of the circuits, M_{21} can be larger or smaller than zero. In Fig. 5.14, for example, the vector lines of the field $\vec{B}$ generated by a positive current I_1 are such that at each point on the surface Σ_2, $\vec{B}$ and $\vec{n}_2$ have opposite direction. Hence, $\Phi_{21} < 0$. In this case, the coefficient M_{21} is negative, $M_{21} < 0$. Note that the vector lines linked with both circuits are the only lines to contribute to the flux of mutual induction (e.g., Φ_{21}). If, while maintaining the same orientation for the first circuit, we invert the orientation of the second (and, thus, the positive direction of the normal $\vec{n}_2$ to Σ_2), the flux Φ_{21} generated by the same current I_1 would be positive.

In this case, $M_{21} > 0$.

Consider now a case, symmetric of the previous one, where current flows only in τ_2. In this case,

$$\Phi_{12} = M_{12} I_2 \quad , \tag{5.84}$$

where Φ_{12} is the flux generated by I_2 and linked with τ_1 and M_{12} is the mutual inductance of the second circuit on the first.

It is easy to verify that

$$M_{21} = M_{12} \quad . \tag{5.85}$$

It is thus possible to define a single mutual inductance $M = M_{21} = M_{12}$ between the two circuits.

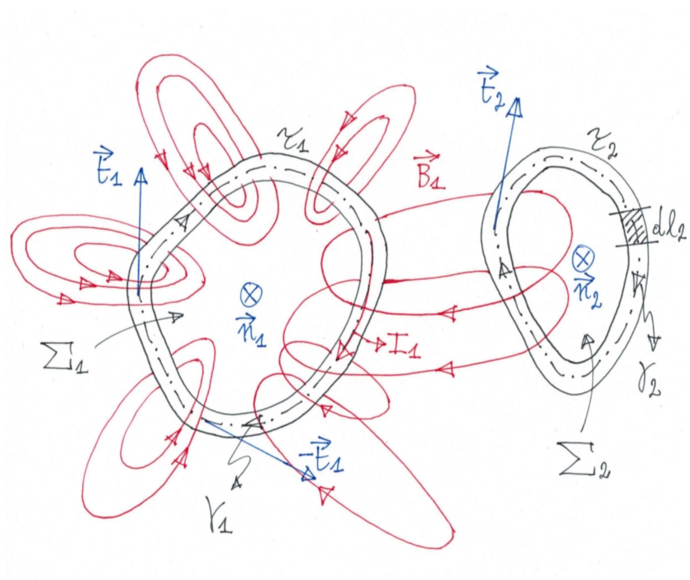


Figure 5.14

We now move to a more rigorous calculation of M_{21} and M_{12}.

Firstly, we attempt to explain the origin of the vector lines of the magnetostatic field $\vec{B}_1$ in Fig. 5.14. In order to sketch the lines qualitatively, we will resort to both Laplace's theorem and the solenoidal property of $\vec{B}$. Figure 5.74 shows the quasi filiform circuit that occupies the region of space τ_1. As in Fig. 5.14, γ_1 is one of the possible longitudinal axes of τ_1. As shown in the figure, γ_1 is a closed line oriented clockwise. Due to the quasi filiform nature of the circuit, we can assume all longitudinal axes of τ_1 to coincide with γ_1 (this is correct up to an infinitesimal distance). Under these conditions, the unit tangent vector at each point Q on γ_1, Q_1, is $\vec{t}(Q_1)$. Among all possible open surfaces having γ_1 as a border, Σ_1 is the surface belonging to the plane on which τ_1 is assumed to lie on (in Fig. 5.15 such a plane is the page where circuit is drawn). Due to the chosen orientation of γ_1, the normal unit vector $\vec{n}_1$ to Σ_1 must point inside the page (right-hand rule).

The field $\vec{B}_1$ is generated by a stationary current I_1, which is assumed to be positive and, thus, oriented clockwise, consistently with the orientation of γ_1. As a consequence, each point Q_1 is a source point for $\vec{B}_1$.

For simplicity, in Fig. 5.15 the circuit τ_2 is not shown. Only the longitudinal axis γ_2 is reported. As for γ_1, the open surface Σ_2 having γ_2 as a border belongs to the plane on which τ_2 is assumed to lie. Note that, for simplicity, we also assume τ_1 and τ_2 to lie on the same plane. This means that Σ_1 and Σ_2 belong to the same plane.

As shown in Fig. 5.15, we now consider two infinitesimal current element $I_1\, d\ell_1$ and $I_1\, d\ell_1'$ at the source points Q_1 and Q_1', respectively. The tangent unit vectors at these source points are $\vec{t}_1$ and $\vec{t}'$, respectively. We intend to calculate the infinitesimal fields due to the elements at Q_1 and Q_1' at two different field points P_1 and P_2. The point P_1 is one point of Σ_1 and P_2 of Σ_2. Such fields can be calculated by means of Laplace's theorem $\big[$Eq. (5.71)$\big]$. Since we are interested in the qualitative behaviour of the field lines of $\vec{B}_1$, the only part of (5.71 we need to consider is

$$\vec{t} \times \vec{r}_{QP} \quad , \tag{5.86}$$

which gives the direction of $\vec{B}$ (or $d\vec{B}$). The distance vector from Q_1 and Q_1' to P_1 and P_2 are indicated in Fig. 5.15 along with the tangent unit vectors at Q_1 and $\vec{Q}_1'$. By folding the latter onto the former (right-hand rule), we obtain the directions of the four infinitesimal fields $d\vec{B}_{Q_1}(P_1)$, $d\vec{B}_{Q_1}(P_2)$, $d\vec{B}_{Q_1'}(P_1)$, and $d\vec{B}_{Q_1'}(P_2)$. It is clear that all four fields are normal to the plane of Σ_1 and Σ_2. However, both $d\vec{B}_{Q_1}(P_1)$ and $d\vec{B}_{Q_1'}(P_1)$ point into the plane, whereas both $d\vec{B}_{Q_1}(P_2)$ and $d\vec{B}_{Q_1'}(P_2)$ point away from the plane. The argument is valid for any source point on γ_1. Hence, by means of the superposition principle for $\vec{B}$, $\vec{B}_{Q_1}(P_1)$, where here Q_1 is any source point, points into the plane and $\vec{B}_{Q_1}(P_2)$ points away from the plane.

Assuming $\vec{B}_1$ is a continuous function (there is no reason to assume the contrary since we are considering $\vec{B}_1$ in a region outside two quasi filiform conductors, and there are no surface current densities in the region), due to the solenoidal property of $\vec{B}_1$, the field lines must form closed loops. The same lines must go into the page at P_1 and out of the page at P_2. The lines sketched in Fig. 5.15 (and 5.14) fulfil both these conditions and, thus, represent possible field lines for $\vec{B}_1$.

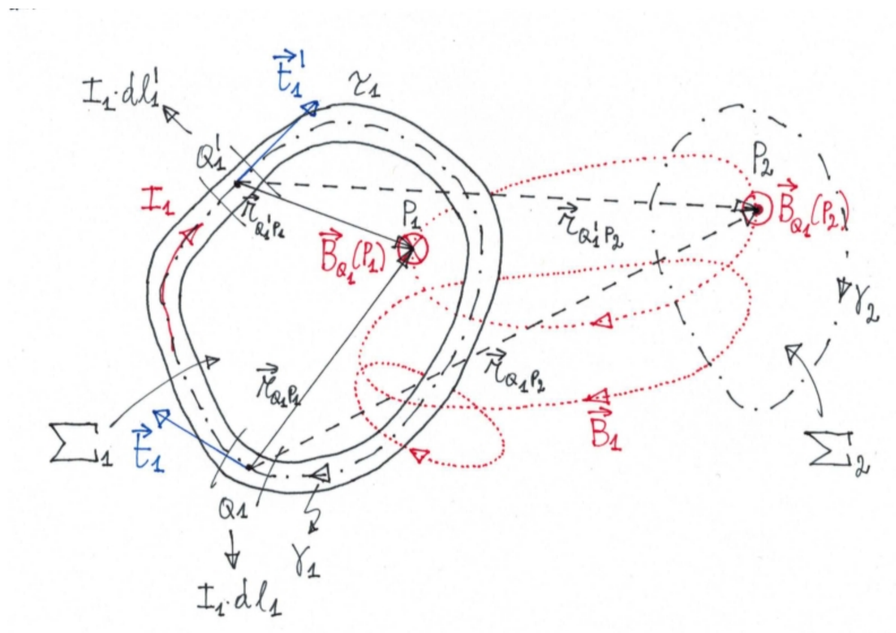


Figure 5.15

Now that we understand how the lines of $\vec{B}_1$ are linked to γ_2 (and, thus, τ_2), we could calculate M_{21} from the knowledge of $\vec{B}_1$ and the flux that $\vec{B}_1$ threads through γ_2. However, we prefer to resort to the distribution of the vector potential $\vec{A}_1$ generated by I_1. The vector potential $\vec{A}_1$ at any point in space P can be calculated from (5.14) as

$$\vec{A}_1(P) = \frac{\mu_0}{4\pi} \iiint\limits_{\tau_1} \frac{\vec{J}_1}{r}\, d\tau \quad , \tag{5.87}$$

where $\vec{J}_1$, which is the electric current volume density associated with I_1, is evaluated at each source point Q_1 on γ_1 (we remind τ_1 is quasi filiform; we can thus assume a point Q_1 inside τ_1 to be a point on γ_1 up to an infinitesimal distance). In (5.87), $r = ||\vec{r}_{Q_1P}||$ is the absolute value (i.e., the Euclidean norm) of the distance between the generic field point P and Q_1. Under stationary conditions, the quasi filiform conductor is a flux tube for $\vec{J}_1$. Hence, $I_1 = J_1\, dS$, where dS is the cross-section of τ_1. We can thus rewrite (5.87) as

$$\vec{A}_1(P) = \frac{\mu_0 I_1}{4\pi} \oint\limits_{\gamma_1} \frac{\vec{t}_1}{r}\, d\ell_1 \quad , \tag{5.88}$$

where $\vec{t}_1$ is the tangent unit vector at each point Q_1 and γ_1 and also the direction of J_1, $\vec{J}_1 = J_1 \vec{t}_1$ and $d\ell_1$ an infinitesimal element on γ_1. Figure 5.75 shows the longitudinal axes γ_1 and γ_2 of τ_1 and τ_2, respectively. It also shows the source points Q_1, a generic field point P, and the special field points P_2 on γ_2.

From Stokes theorem, we know that [see Eq. (5.75)]

$$\Phi_{21} = \oint\limits_{\gamma_2} \vec{A}_1 \cdot \vec{t}_2\, d\ell_2 \quad , \tag{5.89}$$

where $\vec{t}_2$ is a tangent unit vector at any field point P_2 on γ_2 and $d\ell_2$ an infinitesimal element on γ_2. Note that, (5.89) confirms the flux threaded by field $\vec{B}_1$ (or any other field $\vec{B}$, in general) depends only on the border γ_2 and not on the specific open surface Σ_{γ_2} that has γ_2 as a border.

We can combine (5.88) and (5.89) to obtain

$$
\begin{aligned}
M_{21} = \frac{\Phi_{21}}{I_1} &= \frac{\mu_0}{4\pi} \oint_{\gamma_2} \left(\oint_{\gamma_1} \frac{\vec{t}_1}{r} \, d\ell_1 \right) \cdot \vec{t}_2 \, d\ell_2 \\
&= \frac{\mu_0}{4\pi} \oint_{\gamma_2} \oint_{\gamma_1} \frac{\vec{t}_1 \cdot \vec{t}_2}{r} \, d\ell_1 \, d\ell_2 \quad . \qquad (5.90)
\end{aligned}
$$

while (5.90) is an elegant mathematical expression, when calculating M_{21} we recommend to first evaluate $\vec{A}_1$ at any point in space and, then, calculate its circulation along γ_2. The compactness of (5.90) usually leads to errors.

It is worth noting that the quantity r in the denominator of the integrand in (5.90) or (5.88) is always different from zero. This is clear from Fig. 5.16, from which it appears that any source point Q_1 on γ_1 will always be different from any field point P_2 on γ_2. Hence, in the case of the mutual inductance M_{21}, the integrand function is always well defined because γ_1 and γ_2 are different, and the integral (5.90) is always well defined.

A procedure similar to the one that led to integral (5.90) makes it possible to calculate the mutual inductance M_{12}. In this case, we set $I_1 = 0$ and calculate the flux Φ_{12} that a positive test current I_2 (also stationary) generates in Σ_1. Using the same notation as in (5.90), we obtain

$$
\begin{aligned}
M_{12} = \frac{\Phi_{12}}{I_2} &= \frac{\mu_0}{4\pi} \oint_{\gamma_1} \left(\oint_{\gamma_2} \frac{\vec{t}_2}{r} \, d\ell_2 \right) \cdot \vec{t}_1 \, d\ell_1 \\
&= \frac{\mu_0}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\vec{t}_2 \cdot \vec{t}_1}{r} \, d\ell_2 \, d\ell_1 \quad . \qquad (5.91)
\end{aligned}
$$

Because of the linearity of integration and for the commutative property of the scalar product, it is evident that M_{21} given by (5.90) is equal to M_{12} given by (5.91), $M_{21} = M_{12}$. For this reason, the mutual-inductance coefficient is typically simply called $M (= M_{12} = M_{21})$.

The result (5.90) and (5.91) are called Neumann integrals. The Neumann integrals are valid only under the assumption that the flux (either Φ_{21} or Φ_{12}) is proportional to the current that generates it. In this case, the Neumann integrals are very useful tools (typically used in numerical integration software) for the calculation of the mutual inductance between two circuits. In the case of ferromagnetic materials (not treated in these lectures), for example, there is no proportionality between flux and current. Hence, the integrals (5.90) and (5.91) cannot be used.

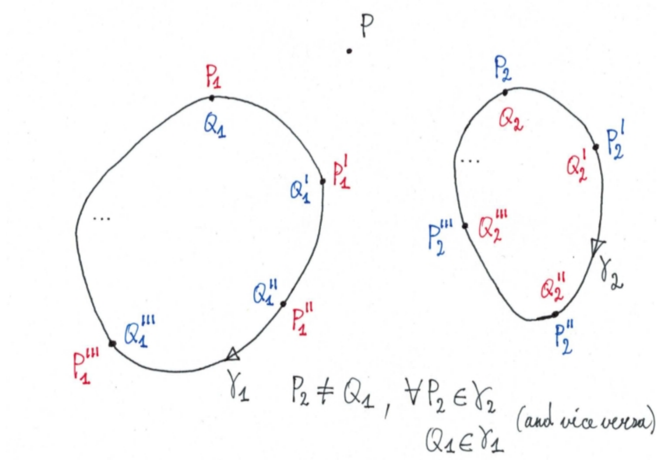


Figure 5.16

It is obvious that, similarly to (5.90), also the integral (5.91) is always well defined because its integrand is well defined. This is clear from Fig. 5.16: Source and field points are always different.

In the light of this observation, the question arises whether it is possible or not to define a Neumann-type integral for the calculation of the self inductance of a given quasi filiform circuit. The answer is more involved than for the simple case of the mutual inductance. Figure 5.76 shows a closed quasi filiform conductor similar to that sketched in Fig. 5.13. The conductor has cross-section dS and is characterized by a stationary (positive) test current I. The circuit is assumed to lie on a plane belonging to the page on which is drawn. One possible longitudinal axis is γ. This is a closed loop oriented, e.g., clockwise and bordering an open surface Σ_γ on the circuit plane. At first glance, we might think of defining a Neumann integral of type

$$\begin{aligned} L = L_\gamma = \frac{\Phi_\gamma}{I} &= \frac{\mu_0}{4\pi} \oint_\gamma \left(\oint_\gamma \frac{\vec{t}}{r}\, d\ell \right) \cdot \vec{t}\, d\ell \\ &= \frac{\mu_0}{4\pi} \oint_\gamma \oint_\gamma \frac{\vec{t} \cdot \vec{t}}{r}\, d\ell\, d\ell \\ &= \frac{\mu_0}{4\pi} \oint_\gamma \oint_\gamma \frac{1}{r}\, d\ell\, d\ell \quad , \end{aligned} \tag{5.92}$$

where $\vec{t}$ is a tangent unit vector to γ and $d\ell$ an infinitesimal element on γ. It is clear that for each source point Q on γ, the integrand in (5.92) will present a singularity (diverge) every time a field point P also on γ coincides with Q. Hence, the integrand function in (5.92), in general, is not well defined. The problem can be circumvented noting that the given conductor is quasi filiform. As a consequence, we can calculate the inner integral in (5.92) on the longitudinal axis γ and the outer integral on a different axis γ' at an average distance $\delta\ell$ from γ. We can thus define an improper Neumann-type integral

$$\begin{aligned} L_\gamma &= \lim_{\delta\ell\to 0} \frac{\mu_0}{4\pi} \oint_{\gamma'} \left(\oint_{\gamma} \frac{\vec{t}'}{r}\, d\ell \right) \cdot \vec{t}'\, d\ell' \\ &= \frac{\mu_0}{4\pi} \lim_{\delta\ell\to 0} \oint_{\gamma'} \oint_{\gamma} \frac{\vec{t} \cdot \vec{t}'}{r}\, d\ell\, d\ell' \quad , \end{aligned} \tag{5.93}$$

where $\vec{t}'\, d\ell'$ is an oriented infinitesimal element on γ', which, consistently with γ, is also a closed loop oriented clockwise (see Fig. 5.17). If the limit (5.93) converges, the integral is the Cauchy's value of (5.92). Due to the quasi filiform nature of the conductor, it is clear that any pair of axes of type γ and γ' will give the same result (because all the axes of the conductor are the same up to infinitesimal distances). The reader can try to calculate the self inductance of a circular loop by means of (5.93). This is not an easy task.

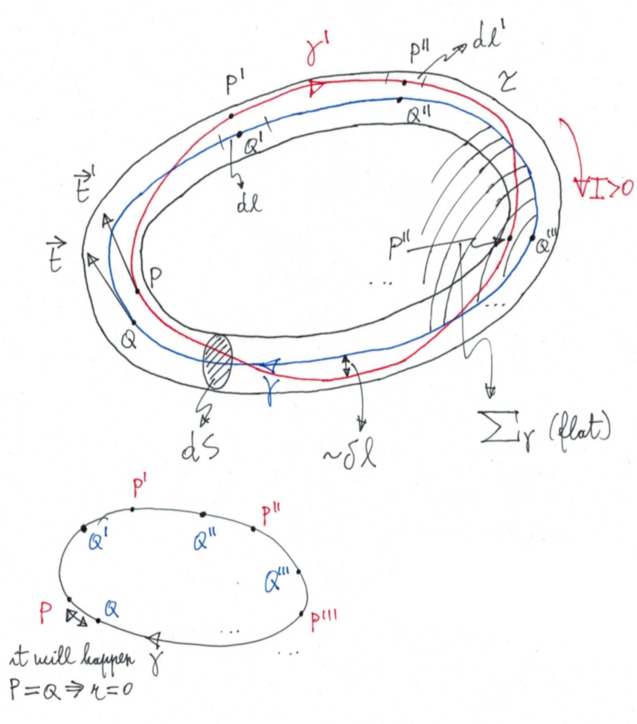


Figure 5.17

- Extra note on the vector lines of field $\vec{B}_1$.

As shown in Fig. 5.15, $\vec{B}_{Q_1}(P_1)$ points into the plane where the circuit γ_1 lies. On the contrary, $\vec{B}_{Q_1}(P_2)$ points away from the same plane. By definition, the vector lines of $\vec{B}_{Q_1}$, for simplicity indicated as $\vec{B}_1$ in the figure, must be tangent to $\vec{B}_1(P_1)$ and $\vec{B}_1(P_2)$, respectively. Figure 5.77 shows a prospective view of the same circuit. In general, the vector lines of $\vec{B}_1$ could follow the pattern in Fig. 5.18, while still being tangent to $\vec{B}_1(P_1)$ and $\vec{B}_1(P_2)$ at points P_1 and P_2.

However, because of Maiezza's conjecture, the vector lines of $\vec{B}_1$ will tend to show the minimum number of inflection points. As a consequence, the actual vector lines of $\vec{B}_1$ are those shown in Fig. 5.15, which are also confirmed by experiments.

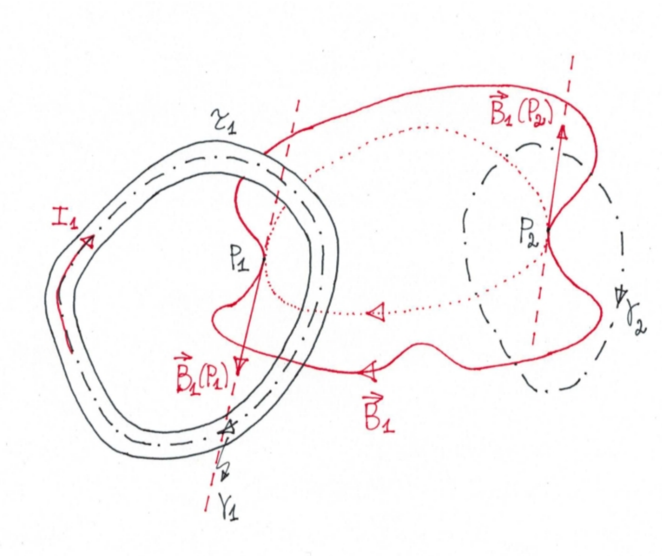


Figure 5.18

5.8 Selected examples on self and mutual inductance

We will now study two useful circuits and calculate their self and mutual inductance.

5.8.1 Toroidal Solenoid

Consider a conducting hollow toroid with circular cross-section and revolution symmetry axis γ_z, as shown in Fig. 5.19. Given a generic cross-section S of such a conductor, determined by one semiplane originating from γ_z, assume the vector lines of thee electric volume current density $\vec{J}$ are all contained within one such meridian semiplane.

The figure also shows a cylindrical coordinate system $Or\varphi z$, with centre O in the middle point of the torus, and z axis coinciding with γ_z. The central longitudinal axis of the torus, i.e., the longitudinal axis passing through the centre of each cross-section, is named γ. The distance of γ to the vertical axis γ_z is the radius r of a circle with centre O. By varying r is possible to select slightly different longitudinal axes.

Assuming the cross-section S has a small thickness compare to the maximum length $2\pi r$ of the torus, we

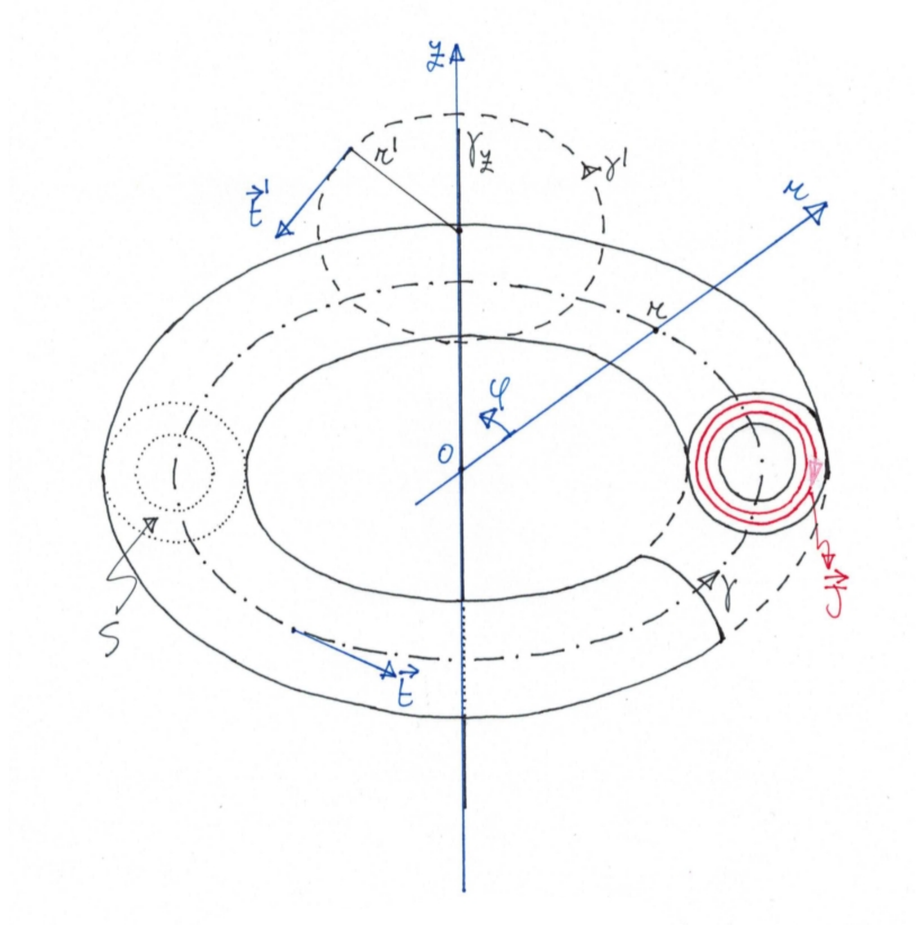


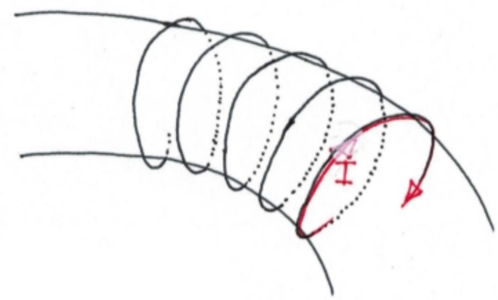
Figure 5.19

can assume the system under consideration to be made by a single filiform conductor winded around a medium (e.g., plastic) with shape of the given torus. Figure 5.20a shows an example of such a construction for a portion of the torus. Figures 5.20c show a completely generic cross-section and a simple (and common) rectangular cross-section for the torus. As it turns out, the distribution of $\vec{B}$ inside the torus does not depend on the specific torus' cross-section (see below). In order to better approximate the origin (solid) conducting toroid, the total number of loops of the filiform conductor should be as large as possible. In addition, the loop distribution should be as uniform as possible so that each loop is approximately entirely confined within one of the aforementioned meridian semiplanes. Such a system is called toroidal solenoid. Most inductors are realized as toroidal solenoid. It is clear that the filiform wire must be eventually connected to an external emf source in order to force a current I into the coil.

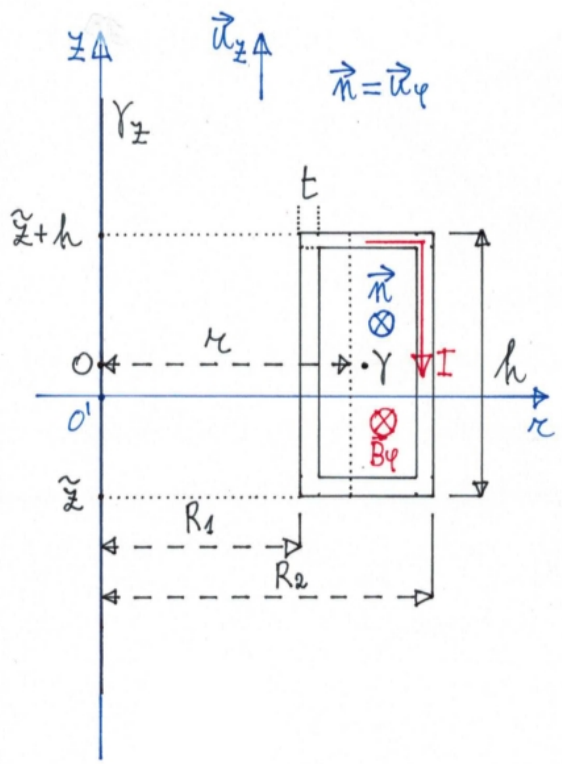
The field $\vec{B}$ generated by the toroidal solenoid at any point in space can be found by means of symmetry arguments and Ampère's law.

A toroidal solenoid with current I is characterized by two symmetries:

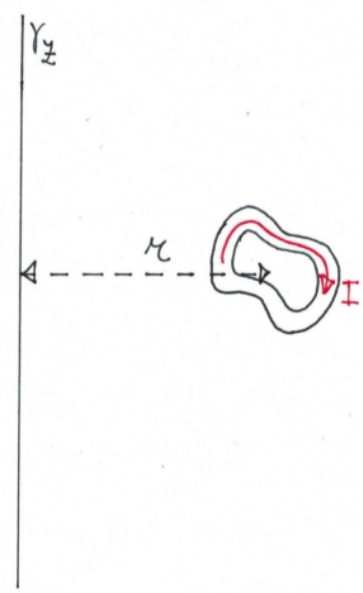
(a) Rotation symmetry with respect to the revolution



(a)



(b)



(c)

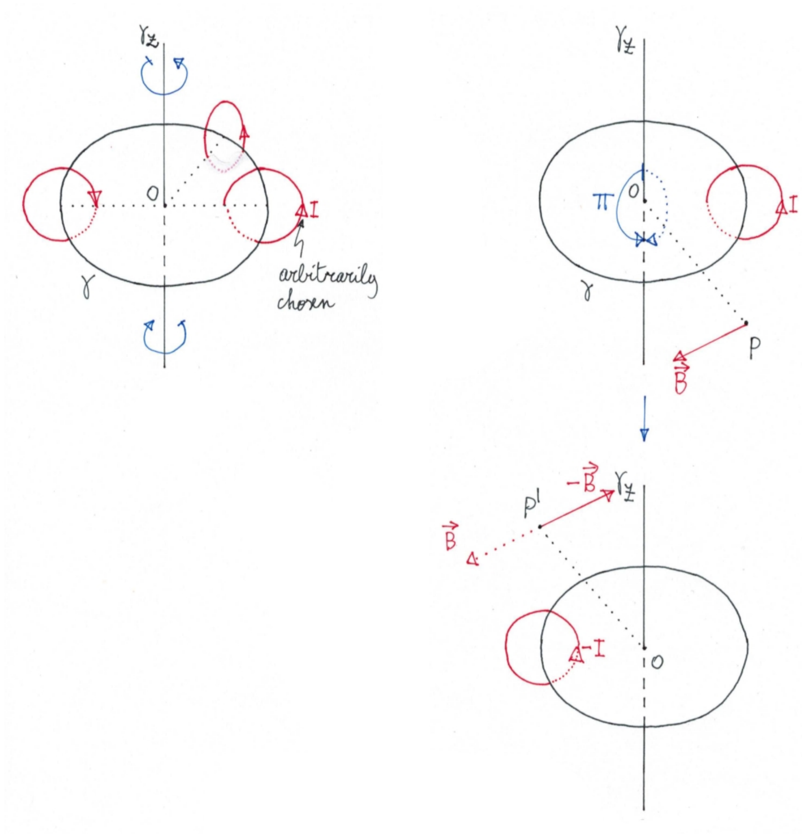
Figure 5.20

axis γ_z. The solenoidal and current direction remains the same upon rotating the system by any angle clockwise or counter-clockwise about γ_z. Figure 5.21a shows this type of symmetry. In the figure, the solenoid is represented by its central longitudinal axis γ (for simplicity). Three loops with the same current I flowing consistently on them are also shown.

(b) Anti-reflection symmetry with respect to the centre O. The top panel in Fig. 5.21b be shown the usual solenoid γ and one loop with current I flowing counter-clockwise.

In addition, the figure shows a generic field $\vec{B}$ at any point in space (inside or outside the solenoid).

By considering the entire system as a rigid body (γ, I, $\vec{B}$, and P), a reflection by an angle π clockwise or counter-clockwise about O leads to a new system that differs from the initial one for the orientation of I. Hence, for consistency the direction of $\vec{B}$ at P' must be inverted. Under these conditions the symmetry can still be used.



(a) (b)

Figure 5.21

We can now study the three components of field $\vec{B}$ at any point P in space. With respect to the $Or\varphi z$ coordinate system indicated in Fig. 5.19, the three components are $\vec{B}_r$, $\vec{B}_\varphi$, and $\vec{B}_z$.

1) Radial component $\vec{B}_r$.

   Figure 5.22a illustrates the rotation symmetry argument. As always, the toroidal solenoid is represented by its longitudinal axis γ. For simplicity, no current loop is shown and point P is chosen to be on γ (i.e., inside the solenoid). The vertical axis γ_z and centre O of the solenoid are also shown.

   Because of the rotation symmetry, given $\vec{B}_r$ at P, $\vec{B}_r$ must be the same (i.e., radially directed-inward or outward- and with constant magnitude) at each point P on γ, or any other circle centred in O.

Figure 5.22b illustrates the anti-reflection symmetry argument.

In step 1, a pair $\{\vec{B}_r, P\}$ is rotated, e.g., by an angle π counter-clockwise about O. Because of the anti-reflection symmetry, the sign of $\vec{B}_r$ in the new pair at P' must be changed, obtaining $\{-\vec{B}_r, P'\}$. In step 2, a rotation by an angle π clockwise about γ_z leads to the final pair $\{-\vec{B}_r, P\}$, which is obviously inconsistent with the original pair $\{\vec{B}_r, P\}$, as clearly shown by the figure. As a consequence, $B_r = 0$ at each point on any line γ with generic radius r. Therefore, $\vec{B}_r = \vec{0}$ at each point in space.

2) Tangent component $\vec{B}_\varphi$.

Figure 5.23a illustrates again the rotation symmetry argument. It is evident that, if a component $\vec{B}_\varphi$ exists, it must be the same (i.e., tangentially directed and with constant magnitude) at each point P on γ.

Figure 5.23b illustrates the anti-reflection symmetry argument. As before, step 1 brings a pair $\{\vec{B}_\varphi, P\}$ to a new pair $\{-\vec{B}_\varphi, P'\}$ (anti-reflection). Step 2 brings the pair $\{-\vec{B}_r, P'\}$ to $\{-\vec{B}_r, P\}$ (rotation), which is consistent with the initial pair at P (see figure). As a consequence, a non-zero component $\vec{B}_\varphi$ can exist and, it it does, it must be the same (i.e., tangentially directed and with constant magnitude) at each point on any line γ.

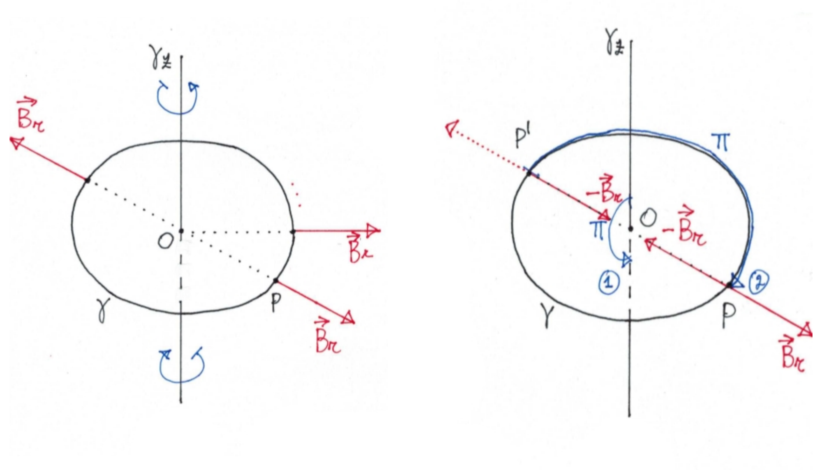
3) Vertical component $\vec{B}_z$.

Figures 5.24a and 5.24b illustrate the usual rotation and anti-reflection symmetry arguments.

Because of the rotation symmetry, if $\vec{B}_z$ exists, it must be the same (i.e., vertically directed and with constant magnitude) at each point on γ. It is clear that, in general, $\vec{B}_z$ components on lines γ with different radii be different in magnitude.

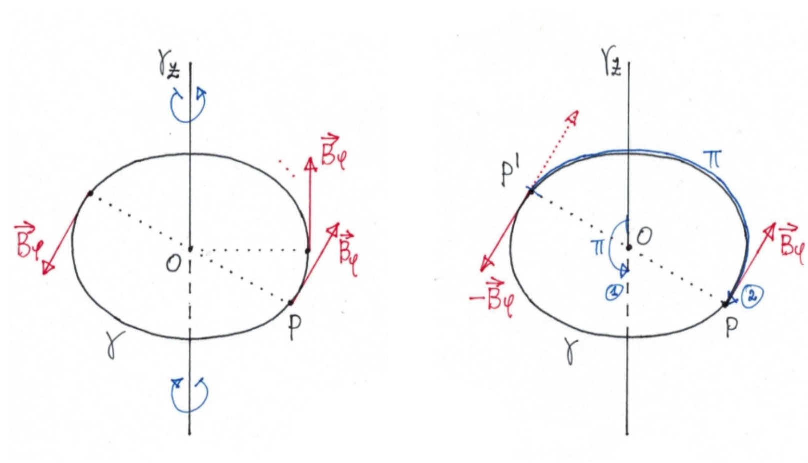
Because of the anti-reflection symmetry, in step 1, a pair $\{\vec{B}_z, P\}$ goes to $\{-\vec{B}_r, P'\}$ and then, in step 2, the pair $\{-\vec{B}_r, P'\}$ goes to $\{-\vec{B}_r, P\}$, which is consistent with the initial pair (see figure). As a consequence, from simple symmetry arguments, a non-zero component $\vec{B}_z$ can exist.

However, experimental evidence shows that $\vec{B}_z = \vec{0}$ at each point in space. While the symmetry arguments are insufficient to prove this fact, it would be possible to show that $\vec{B}_z = \vec{0}$ everywhere by resorting to Laplace's theorem. Due to the complexity of the toroidal solenoid, this is a nontrivial task and, thus, we will not show it in these notes.



(a) (b)

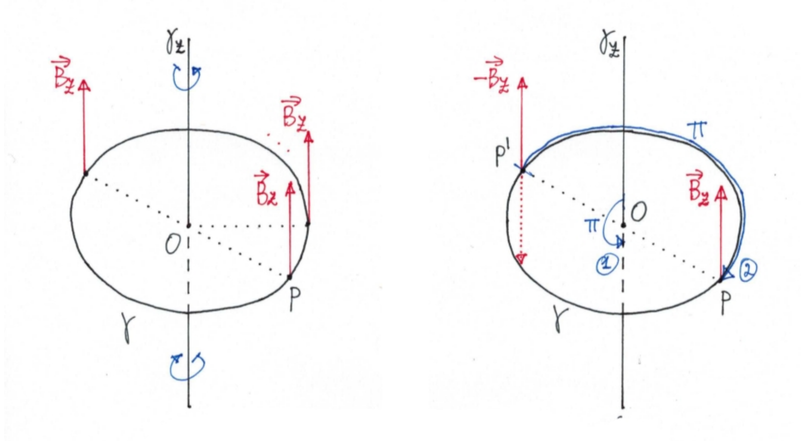
Figure 5.22



(a) (b)

Figure 5.23

Note that the absence of translation symmetry implies that equal components (i.e., same direction and constant magnitude) on a line γ are different from the corresponding component on a line γ' with centre in O, radius r (equal to the radius of γ), but on a plane parallel to the one where γ lies on. in other words, we cannot state that a component can be the same at each point on the lateral surface of a cylinder with axis γ_z.



(a) (b)

Figure 5.24

It is also clear that the magnitude of a given component is, in general, different on circles with different radii.

Finally, because of the homogeneity and isotropy of the 3D Euclidean space (excluding the region where the loops of the solenoidal are defined; we remind that the conductor forming the solenoid is assumed to be filiform and, thus, such a region is negligibly small), if one component of $\vec{B}$ has a specific direction at a point P, it will maintain that direction in the neighbourhood of P, $P + dP$ (see lecture 4, Fig. **??**).

We can now calculate the component $\vec{B}_\varphi$ of the field $\vec{B}$ generated by the toroidal solenoid, at any point P in space, by resorting to Ampèr's law,

$$\oint_{\gamma_a} \vec{B} \cdot \vec{t} d\ell = \mu_0 I_\ell \quad , \tag{5.94}$$

where γ_a is a closed oriented line in the region where $\vec{B}$ and I_ℓ the current (stationary) linked with γ_a.

- Region outside the toroidal solenoid.

In this case, $\gamma_a = \gamma'$, where γ' is a closed line lying on a plane parallel to the plane on which lies γ (see Fig. 5.19), with centre O' on the γ_z axis, generic radius r', and oriented counter-clockwise. The line γ' lies in a region of space entirely outside the toroidal solenoid.

Because of the very construction of the solenoid, it is always possible to find a transformation (involving translations, rotations, and positive or negative contractions) that reduces γ' to a single point, without ever "cutting" the solenoid. Such a transformation is called a zero homotopy. A direct consequence of the zero homotopy is that the current linked with γ' is always zero, $I_\ell = 0$.

By defining $\vec{t}'$ the unit tangent vector to γ' and $d\ell'$ an infinitesimal element on γ'

(see Fig. 5.19), we obtain

$$\begin{aligned}\oint_{\gamma'} \vec{B} \cdot \vec{t}'\, d\ell' &= \oint_{\gamma'} \vec{B}_\varphi \cdot \vec{u}_\varphi\, d\ell' \\ &= \oint_{\gamma'} B_\varphi\, \vec{u}_\varphi \cdot \vec{u}_\varphi\, r'\, d\varphi \\ &= B_\varphi r' \int_0^{2\pi} d\varphi \\ &= 2\pi r' B_\varphi = 0 \quad ,\end{aligned} \tag{5.95}$$

where we used the fact that $\vec{t}' = \vec{u}_\varphi$ with respect to the $Or\varphi z$ coordinate system indicated in Fig. 5.19 and $d\varphi$ is an infinitesimal angle. Assuming $r' \neq 0$ (or, at most, a very small, yet nonzero, quantity), Eq. (5.95) is fulfilled iff $\vec{B}_\varphi = 0$. The only region where B_φ must not necessarily be zero for (5.95) to be fulfilled is on the γ_z axis, where $r' = 0$. In this case, (5.95) is valid even if $B_\varphi \neq 0$. However, the absence of any source current and of any conducting or insulating material in correspondence of γ_z implies that if $B_\varphi = 0$ in the neighbourhood of γ_z (i.e., in a cylindrical region with vertical central axis γ_z and infinitesimal radius), it will be zero also at each point on γ_z. Hence, $\vec{B}_\varphi = \vec{0}$ at each point outside the solenoid.

- Region inside the toroidal solenoid.

In this case, $\gamma_a = \gamma$, where the radius r can vary so long γ remains confined within the inner region of the solenoid, $\vec{t} = \vec{u}_\gamma$ (as before), $d\ell = r\, d\varphi$, and

$$I_\ell = NI \quad , \tag{5.96}$$

where I is the current flowing in one loop of the solenoid. Since I flows clockwise, it is consistent with $\vec{u}_z$ when crossing the surface with border γ on the plane of γ. Hence, $I_\ell > 0$ and

$$\begin{aligned}\oint_{\gamma} \vec{B}_\varphi \cdot \vec{u}_\varphi\, r d\varphi &= r \int_0^{2\pi} B_\varphi\, \vec{u}_\varphi \cdot \vec{u}_\varphi\, d\varphi \\ &= r B_\varphi \int_0^{2\pi} d\varphi \\ &= 2\pi r B_\varphi = \mu_0 N I \quad .\end{aligned} \tag{5.97}$$

From (5.97) we finally obtain

$$B_\varphi = \frac{\mu_0}{2\pi}\, \frac{NI}{r} \quad , \tag{5.98}$$

where r is always different from zero.

Note that, inside the toroidal solenoid, the field does not depend on the specific conductor cross-section, which, e.g., can be of the type shown in Figs. 5.20b or 5.20c.

The field depends only on the total current NI and the distance r from γ_z. Furthermore, the field id identical to that of a single infinite straight conductor, directed along γ_z and with current intensity NI.

In summary,

$$\begin{cases} \vec{B} = \vec{0} & \text{, outside solenoid} \quad (5.99a) \\ \vec{B} = \dfrac{\mu_0}{2\pi}\,\dfrac{NI}{r}\,\vec{u}_\varphi & \text{. inside solenoid} \quad (5.99b) \end{cases}$$

- **Self inductance of a toroidal solenoid.**

Consider a toroidal solenoid with a rectangular cross-section, as shown in Fig. 5.20b. The internal radius of the solenoid is R_1 and the external radius R_2. The solenoid comprises a total number N of loops (or windings), which are uniformly distributed along its longitudinal central axis γ. The lateral dimensions of one winding are $(R_2 - R_1)\,h$. The distance between a line parallel to h (the black dashes line inside the rectangle in Fig. 5.20b; note that we here assume the thickness t of the rectangle to be zero) and the vertical central axis γ_z is r.

In order to calculate the self inductance of such a solenoid, we need to calculate the flux linked with the closed line (in reality it must be open at one point for connection to a suitable source) that represents the entire solenoid coil. By definition, we would then need to calculate the surface integral of the field $\vec{B}$ generated by the coil through a generic surface having the coil as a border. This calculation is rather complicated due to the complex geometric shape (helicoidal) of the coil itself. However, we can estimate the total flux Φ linked with the entire coil as the sum of the partial fluxes linked with each single loop, Φ_ℓ, where the loops are considered to be closed lines (in our case, rectangular). This is a good approximation that, as it turns out from comparison with experiments, leads to a negligible error. In addition, there is a bijective correspondence between each point on two different surfaces bordered by a loop and the field $\vec{B}$ is equal at corresponding points due to (5.20b). As a consequence, all partial fluxes Φ_ℓ are equal to each other and, thus,

$$\Phi = N\Phi_\ell \quad . \tag{5.100}$$

The partial flux Φ_ℓ can readily be calculated from (5.99b for the case of Fig. 5.20b. We obtain

$$\begin{aligned} \Phi_\ell &= \iint\limits_{\text{rect.}} \vec{B} \cdot \vec{n}\,dS = \iint\limits_{\text{rect.}} B_\varphi\,\vec{u}_\varphi \cdot \vec{u}_\varphi\,dS \\ &= \iint\limits_{\text{rect.}} B_\varphi\,dr\,dz \\ &= \int\limits_{R_1}^{R_2}\int\limits_{\tilde{z}}^{\tilde{z}+h} \frac{\mu_0}{2\pi}\,\frac{NI}{r}\,dr\,dz \\ &= \frac{\mu_0}{2\pi}\,NI\Big[\ln r\Big]_{R_1}^{R_2}\Big[z\Big]_{\tilde{z}}^{\tilde{z}+h} \end{aligned}$$

$$= \frac{\mu_0}{2\pi} N I h \ln \frac{R_2}{R_1} \quad . \tag{5.101}$$

In this integral, "rect." refers to one rectangular loop of the solenoid, $\vec{n}$ is the normal unit vector to the surface defined by the loop (see Fig. 5.20b), I is positive test current oriented clockwise consistently with the clockwise orientation of the loop. Hence, with respect to the $O'r\varphi z$ cylindrical coordinate system indicated in the figure (note that we arbitrarily chose O', which can coincide with the centre O of the toroid), $\vec{n} = \vec{u}_\varphi$ pointing into the page. This means that the flux given by (5.101) is positive, $\Phi_\ell > 0$. Finally, we have chosen the vertical coordinate of the bottom edge of the rectangle to be an arbitrary value $\tilde{z}$. Also note that, from (5.99b) $\vec{B} = \vec{B}_\varphi$ points into the page (hence, $\Phi_\ell > 0$).

By definition, the self inductance of the entire solenoid is given by

$$L = \frac{\Phi}{I} = \frac{N\Phi_\ell}{I} = \frac{\mu_0}{2\pi} N^2 h \ln \frac{R_2}{R_1} \quad , \tag{5.102}$$

where we used both (5.100) and (5.101). As expected, the dimension of L are those of a flux over a current, giving the units of henry,

$$\big[L\big] = \frac{H}{\cancel{m}} \cancel{m} = H \quad .$$

- **Self inductance of a toroidal solenoid.**

Mutual inductance between a straight infinite line and a toroidal solenoid.

Consider a system of two circuits, comprising a toroidal solenoid with central vertical axis γ_z and rectangular loops and a straight infinite filiform conductor on the γ_z axis (see Fig. 5.25. A positive test current I_1 flows in the straight infinite line, pointing upward. Since we intend to calculate the mutual inductance between the two circuits and since $M_{12} = M_{21}$ (where the subscripts refer to the straight line, circuit 1, and the solenoid circuit 2), we can either calculate M_{12} or M_{21}. We decide to calculate M_{21}. We remind that the mutual inductance can be positive or negative, depending on the orientation of the circuits. As an example, we choose the orientation on each rectangular loop of the solenoid to be counter-clockwise and switch off any current on the solenoid, $I_2 = 0$ (we want to calculate M_{21}). With these conventions, the field $\vec{B}_1$ generated by the infinite straight line is tangentially directed on each circle with axis γ_z, points counter-clockwise, and is characterized by a constant magnitude at each point of the lateral surface of a cylinder coaxial with γ_z (see Fig. 5.25). The figure also shows a cylindrical coordinate system $Or\varphi z$, which is consistent with the chosen directions on circuit 1. We remind the reader to lecture 19 for more details on the field $\vec{B}$ generated by an infinite straight line. Note that, by default we are under stationary conditions. The figure shows also two rectangular cross-sections of the solenoid, each of which is characterized by an internal radius R_1 and an external radius R_2 and a height h. The surface defined by each rectangular loop has area $(R_2 - R_1)\,h$ and its border line is oriented counter-clockwise, so that its normal unit vector $\vec{n}_2 = -\vec{u}_\varphi$.

Hence, the flux due to $\vec{B}_1$ through one rectangular loop is given by

$$\begin{aligned}
\Phi_{\ell_{21}} &= \iint\limits_{\text{rect.}_2} \vec{B}_1 \cdot \vec{n}_2 \, dS = \iint\limits_{\text{rect}_2} B_{\varphi 1} \, \vec{u}_\varphi \cdot \left(- \vec{u}_\varphi \right) dS \\
&= - \frac{\mu_0}{2\pi} \, I_1 h \int\limits_{R_1}^{R_2} \frac{1}{r} \, dr \\
&= - \frac{\mu_0}{2\pi} \, I_1 h \ln \frac{R_2}{R_1} \quad ,
\end{aligned} \tag{5.103}$$

where we used the result of lecture 19 for $\vec{B}_1$,

$$\vec{B}_1 = \vec{B}_{\varphi_1} = B_{\varphi_1} \, \vec{u}_\varphi = \frac{\mu_0}{2\pi} \, \frac{I_1}{r} \quad .$$

An argument similar to that used to calculate the self inductance of a toroidal solenoid makes it possible to find the total flux due to $\vec{B}_1$ and lined with the solenoid,

$$\begin{aligned}
\Phi_{21} &= N \Phi_{\ell_{21}} \\
&= - \frac{\mu_0}{2\pi} \, N I_1 h \ln \frac{R_2}{R_1} \quad ,
\end{aligned} \tag{5.104}$$

where N is the total number of loops in the solenoid. From the definition of mutual inductance we finally obtain

$$M_{21} = \frac{\Phi_{21}}{I_1} = - \frac{\mu_0}{2\pi} \, N h \ln \frac{R_2}{R_1} \quad . \tag{5.105}$$

The coefficient is in this case negative and its unit are ???

The same result could be obtained by means of a Neumann integral, but it would be much more complicated.

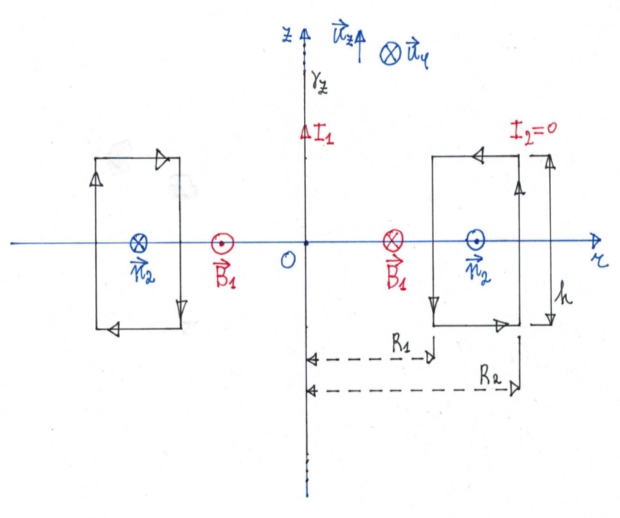


Figure 5.25

5.8.2 Infinite Straight Solenoid

A particularly interesting limiting case of a toroidal solenoid is when the radius r of the central longitudinal axis γ of the toroidal solenoid of Fig. 5.19 tends to infinite. In this case, the toroidal solenoid transforms into an infinite straight solenoid.

As always, when dealing with indefinite (infinite) structures, it is opportune to resort to "density". Therefore, given the total number of loops N of the original (finite) solenoidal, we can define the loop density as the number of loops in the coil per unit length,

$$n = \frac{N}{\ell} = \frac{N}{2\pi r} \quad . \tag{5.106}$$

The magnitude of the field $\vec{B}$ inside the solenoid, which is tangent to any line of γ type, is thus given by $\big[$(5.20b)$\big]$

$$\begin{aligned} B &= \frac{\mu_0}{2\pi} N \frac{I}{r} \\ &= \frac{\mu_0}{2\pi} n\ell \frac{I}{r} = \mu_0 n I \frac{2\pi r}{2\pi r} \quad , \end{aligned} \tag{5.107}$$

where we used (5.106) and, as always, I is positive test current flowing in one loop of the solenoid.

For an infinite straight solenoid it must then be

$$B = \lim_{r \to +\infty} \mu_0 n I \frac{\cancel{2\pi r}}{\cancel{2\pi r}} = \mu_0 n I \quad . \tag{5.108}$$

In this case, the field $\vec{B}$ is uniform inside the straight solenoid (i.e., it does not depend on r), directed along the solenoid central longitudinal axis, and with sign dictated by the right-hand rule (the field $\vec{B}$ is the thumb and the current I the rest of the hand). As in the case of the toroidal solenoid, the field $\vec{B}$ is zero outside the solenoid.

Under the sole assumption that the vector lines of $\vec{B}$ are parallel to the longitudinal axis of the solenoid, it is possible to reach the result (5.108) by means of Ampèr's theorem. The line $\widetilde{\gamma}$ used to calculate the circulation of $\vec{B}$ is shown in Fig. 5.26. The figure shows a lateral cross-section through the central longitudinal axis of the solenoid. Only a portion of the infinite (straight) solenoid is sketched. The dots and crosses indicate the flow direction of the current I in the solenoid. The line $\widetilde{\gamma}$ comprises four edges, AB, BC, CD, and DA and is oriented counter-clockwise. The edges, which are all straight line segments, are positioned so, that AB resides outside the solenoid, CD inside, and BC and DA are normal to the solenoid longitudinal axis. Because of the orientation of $\widetilde{\gamma}$, the normal unit vector to the surface defined by $\widetilde{\gamma}$ points away from the plane on which $\widetilde{\gamma}$ lies (right-hand rule). Hence, the current linked with $\widetilde{\gamma}$ has a positive sign and is equal to NI, where N is total number of loops of the solenoid linked with $\widetilde{\gamma}$. There is zero field outside the solenoid, no field component along BC and DA; assuming $\vec{B} = B\,\vec{t}_{CD}$ inside the solenoid along CD, where $\vec{t}_{CD}$ is the unit vector tangent to CD, and assuming $\overline{CD} = \ell$, the circulation of $\vec{B}$ along $\widetilde{\gamma}$ reads

$$\begin{aligned}\oint_{\widetilde{\gamma}} \vec{B} \cdot \vec{t}\, d\ell &= \int_{CD} B\, \vec{t}_{CD} \cdot \vec{t}_{CD}\, d\ell \\ &= B\, \overline{CD} = B\, \ell = \mu_0 N I \quad . \end{aligned} \tag{5.109}$$

From (5.109) we finally obtain again

$$B = \mu_0 \frac{N}{\ell} I = \mu_0 n I$$

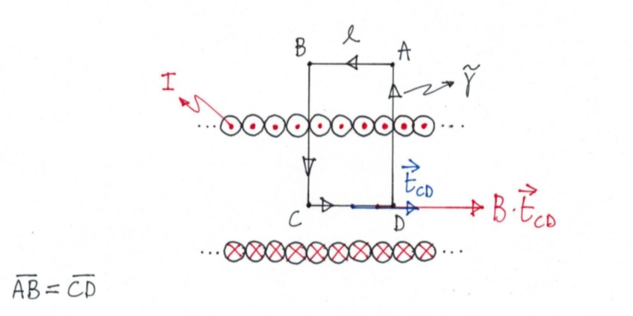


Figure 5.26

- **Self inductance of a "long" solenoid.**

Consider a straight solenoid the length of which is much longer than its transversal dimension, $l \gg \sqrt{S}$, where S is the area of one loop of the solenoid. If the loop density is big enough and the windings are uniform enough, the field $\vec{B}$ inside the solenoid can be approximated by (5.108), with direction along the solenoid longitude axis, and sign given by the right-hand rule $\big[$depending on the current orientation; in (5.108), we choose $I > 0\,\big]$.

Proceeding as for the toroidal solenoid, due to the uniformity of $\vec{B}$ inside the solenoid, the partial fluxes through each loop, Φ_ℓ, are all equal and, thus, the total flux through the entire solenoid can be approximated by

$$\Phi = N \Phi_\ell \tag{5.110}$$

where N is the total number of loops in the solenoid and

$$\Phi_\ell = BS = \mu_0 n I S = \mu_0 \frac{N}{\ell} I S \quad . \tag{5.111}$$

Therefore,

$$\Phi = \mu_0 \frac{N^2}{\ell} I S = \mu_0 \left(\frac{N}{\ell} \right)^2 I \ell S = \mu_0 n^2 I \ell S \quad . \tag{5.112}$$

From the definition of self inductance

$$L = \frac{\Phi}{I} = \mu_0 \frac{N^2}{\ell} S = \mu_0 n^2 \ell S \quad , \tag{5.113}$$

which has units

$$\left[L\right] = Hm^{-1}m^{-1}m^2 = H \quad .$$

Note that from (5.113) is also possible to calculate the inductance per unit length as

$$\widetilde{L} = \frac{L}{\ell} = \mu_0 n^2 S \quad , \tag{5.114}$$

which is very useful when the solenoid length is unknown. Finally note that any long solenoid must be connected to an emf source by means of two leads. In typical applications the length of the leads is negligible compared to that of the solenoid itself. Hence, the self inductance given by (5.113) remains a good approximation.