PHY240 A1

1.1)

Take O as origin, x being // to AB, y ⊥ to x

No need to consider z in this question

$\theta = \angle AOB = \arccos\left(\frac{OA^2+OB^2-AB^2}{2\cdot OA\cdot OB}\right) = \arccos\left(\frac{2R^2-2d}{2R^2}\right)$

$= \arccos\left(1-\frac{d}{R^2}\right)$

Now we notice that due to symety of the ring over O,

the field will be 0 if the arc $\widehat{AB}$ is closed

Namely $\vec{E}+\vec{E}_{\widehat{AB}}=0 \Rightarrow \vec{E} = -\vec{E}_{\widehat{AB}}$

Consider the cylindcal system, where O is ther origin, $\hat{r}$ is $\frac{\vec{R}}{R}$,

and $\hat{\varphi}$ is counter-clockwise. φ is the angle from OC, where C is

mid point of AB, in $-\hat{y}$ direction

Then a point (r, φ) in Cylinder is $(-\sin\varphi\cdot r, -\cos\varphi\cdot r)$

in Cartesian system

Again Notice that due to the symmetry of the $\widehat{AB}$ arc

on the x-axis over O, there will be no x-component of $\vec{E}$

Namely, $\vec{E}_{AB} = \vec{E}_y$

Given any infidecimal arc $d\vec{r}$ on $\widehat{AB}$, which has $(R\varphi)$ as position,

and $dr\cdot\lambda$ as charge, $\vec{E}_{y(\varphi)} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{dr\lambda\cdot\cos\varphi}{R^2}\hat{y}$

Note $\lambda = \frac{q}{L} = \frac{q}{(2\pi-\theta)R}$, $dr = R\cdot d\varphi$

$\therefore \vec{E}_{AB} = \vec{E}_y = \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} \frac{1}{4\pi\varepsilon_0}\cdot\frac{\cos\varphi\cdot d\varphi\cdot \cancel{R}\lambda}{R^{\cancel{2}}}\hat{y} = \frac{\lambda\hat{y}}{4\pi\varepsilon_0 R}\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}}\cos\varphi\, d\varphi$

$= \frac{\lambda\hat{y}}{4\pi\varepsilon_0 R}\cdot\sin(\varphi)\Big|_{-\theta/2}^{\theta/2}$ Note $\sin(\theta/2) = \frac{d}{R}$, $\sin(-\theta/2) = \frac{-d}{R}$

$= \frac{\lambda}{4\pi\varepsilon_0 R}\cdot\frac{2d}{R}\hat{y} = 14.36\hat{y} \Rightarrow \vec{E} = -\vec{E}_{AB} = -14.36\hat{y}$

2) 1 cm is very small comparing to 1 m

Again, observe the symmetry on x-direction

The arc length of $\widehat{AB}$ can be approximated by length $\overline{AB} = 2d$

We can view $\widehat{AB}$ as a point sitting below O at R distance, carrying charge $\frac{q}{2\pi R} \cdot 2d = \frac{qd}{\pi R}$

Thus $\vec{E}_{AB} = \frac{qd}{\pi R \cdot R} \cdot \frac{1}{4\pi\varepsilon_0} = \frac{qd\,\hat{y}}{4\pi^2\varepsilon_0 R^2} = 14.32\hat{y}$

$\vec{E} = -14.32\hat{y}$

2.1) Consider Cartesian System, with origin at center of the shell, (x, y, z) be any three $\perp$ directions

Given any point (a,b,c) in this coordinate

Consider any ring of radius r', at the plane $z=h$, centered on z axis

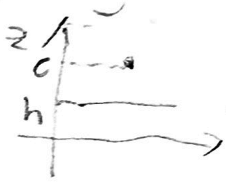
The field from any infinidecimal arc dr on the ring is given by

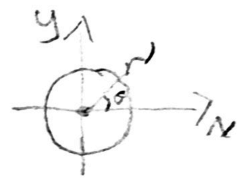
$$\vec{E}_{x(h,\theta)} = \frac{\lambda\, dr \cdot (a - r'\cos\theta)}{4\pi\varepsilon_0 \underbrace{\left((a-r'\cos\theta)^2 + (b-r'\sin\theta)^2 + (c-h)^2\right)^{3/2}}_{d^{3/2}}}\hat{x}$$

$$\vec{E}_{y(h,\theta)} = \frac{\lambda dr}{4\pi\varepsilon_0} \cdot \frac{b-\sin\theta}{d^{3/2}}\hat{y}$$

$$\vec{E}_{z(h,\theta)} = \frac{\lambda dr}{4\pi\varepsilon_0} \cdot \frac{c-h}{d^{3/2}}\hat{z}$$

where λ is charge per length





where θ = angle from $\hat{x}$ on x-y plane, $dr = r'd\theta$

$$\therefore \vec{E}_x^{h}(\theta) = \int_0^{2\pi} \vec{E}_x(h,\theta)d\theta = \frac{\lambda r'}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{a-\cos\theta}{d(\theta)^{3/2}}d\theta$$

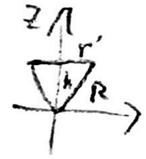
Notice that if $a=b=0$, namely the point is ON z-axis, then the x, y-components are all 0 due to symmetry, and we are left with

$$\vec{E}(h) = \vec{E}_z(h) = \frac{\lambda}{4\pi\varepsilon_0}\cdot\int_0^{2\pi}\frac{c-h}{\left[(r'\cos\theta)^2+(r'\sin\theta)^2+(c-h)^2\right]^{3/2}}\cdot r'd\theta\cdot\hat{z}$$

$$= \frac{\lambda}{\cancel{4\pi}_2\varepsilon_0}\cdot\frac{\cancel{2\pi} r'(c-h)}{(r'^2+(c-h)^2)^{3/2}}\hat{z}$$

Again Notice that if we break the shell into infinite rings $\perp$ z,

the radius r' at $z=h$ is $r' = \sqrt{R^2-h^2}$



$$\therefore \vec{E}(h) = \frac{\lambda}{2\varepsilon_0} \cdot \frac{\sqrt{R^2-h^2}\,(c-h)}{(R^2-h^2+(c-h)^2)^{3/2}}\hat{z}$$, for a point $(0,0,c)$, where $\lambda = \rho \cdot dh$

Then $$\vec{E} = \int_{-R}^{R} \vec{E}(h) = \frac{\rho}{2\varepsilon_0}\int_{-R}^{R} \frac{\sqrt{R^2-h^2}\,(c-h)}{(R^2-h^2+(c-h)^2)^{3/2}}\,dh \cdot \hat{z} = \frac{\rho}{2\varepsilon_0} \cdot \frac{2R^2}{c^2}\hat{z} = \frac{\rho R^2}{\varepsilon_0 c^2}\hat{z}$$

ρ is chareg per area

Note: $\rho = q/4\pi R^2$

$$\therefore \vec{E} = \frac{q}{4\pi \cancel{R^2}} \cdot \frac{\cancel{R^2}}{\varepsilon_0 c^2}\hat{z} = \frac{q}{4\pi\varepsilon_0 c^2}\hat{z}$$

Now, note that given any point, if we take $\hat{z}$ axis to be the line from center of shell to the point, the point is alway in the form $(0,0,c)$

$$\therefore \vec{E} = \frac{q}{4\pi\varepsilon_0 \cdot c^2}\hat{z}$$, where c is distance from center to point, and z-axis is ~~the~~ described as above

b) Given a point charge, the field $\vec{E} = \frac{q}{4\pi\varepsilon_0 d^2}\hat{d}$, where $\vec{d}$ is the line from charge to the point in question

Is same as from a)