

Pmath465: Smooth Manifold

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1 Topology and manifold

Definition 1. Given any set X , a **topology** is a pair (X, \mathcal{S}) , $\mathcal{S} \subseteq \mathcal{P}(X)$ that satisfies:

1. $\emptyset \in \mathcal{S}$
2. If $\forall \alpha, S_\alpha \in \mathcal{S}$, then $\bigcup_\alpha S_\alpha \in \mathcal{S}$
3. If $S_1, \dots, S_n \in \mathcal{S}$, then $\bigcap_{i=1}^n S_i \in \mathcal{S}$

Definition 2. Given any set X , and a topology (X, \mathcal{S}) , the elements in \mathcal{S} are called open.

Definition 3. Given any set X , a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ is said to be a basis of the topology \mathcal{S} if

1. $X = \bigcup_{B \in \mathcal{B}} B$
2. If $B_1, B_2 \in \mathcal{B}$, $x \in B_1 \cap B_2$, then $\exists B_x \in \mathcal{B}$, $x \in B_x \subseteq B_1 \cap B_2$
3. \mathcal{S} is the collection of all unions of the elements of \mathcal{B} .

Definition 4. A topology space \mathcal{S} is called **2nd countable** if it has a countable basis.

Definition 5. A topology space is **Hausdorff** if $\forall x \neq y \in \mathcal{S}$, $\exists S_x, S_y \in \mathcal{S}$, $x \in S_x$, $y \in S_y$, $S_x \cap S_y = \emptyset$

Definition 6. If X is a topology space, and $Y \subseteq X$, then the subspace topology on Y is obtained by $U \subseteq Y$ is open if and only if $\exists V \subseteq X$ that is open, and $U = V \cap Y$

Proposition 1.1. *If $Y \subseteq X$ with subspace topology, then if X is 2nd countable or Hausdorff, so is Y .*

Definition 7. A collection of subsets $\mathcal{C} = \{U_\alpha \subseteq X\}_{\alpha \in A}$ is called a **cover** for X if $X \subseteq \bigcup_{\alpha \in A} U_\alpha$. A cover is called an open cover if every U_α is open in the topology of X .

Definition 8. A collection of subsets \mathcal{X} is called **locally finite** if $\forall x \in X$, $\exists S_x \in \mathcal{S}$ an open neighborhood, such that S_x only intersects with finitely many elements in \mathcal{X} .

Definition 9. Given two sets X, Y , and their corresponding topology \mathcal{S}, \mathcal{T} , a map $f : X \rightarrow Y$ is **continuous** if $\forall T \in \mathcal{T}$, $f^{-1}(T) \in \mathcal{S}$. Namely, for any open set in Y , its preimage of f is also open in X .

Definition 10. Given two sets X, Y , and their corresponding topology \mathcal{S}, \mathcal{T} , a continuous map $f : X \rightarrow Y$ is a **homeomorphism** if it is invertible, and its inverse function is also continuous.

Remark. A homeomorphism is a map that preserves the topology structure between two sets.

Definition 11. An **atlas** $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_\alpha$ is a collection of **local charts** (U_α, ϕ_α) , where each $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ is a homeomorphism onto its image $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, and that $\bigcup_\alpha U_\alpha = X$.

Definition 12. A smooth atlas is an atlas such that $\forall (U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in \mathcal{A}$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}^n$ is C^∞ smooth.

Definition 13. A **smooth manifold** $M = (S, \mathcal{A})$ is a 2nd-countable Hausdorff topology with a smooth atlas. The dimension n of the manifold is the dimension of \mathbb{R}^n in the atlas \mathcal{A} .

2 Smooth functions and Diffeomorphism

Definition 14. Let M, N be smooth manifolds of dimension m, n , we say a function $F : M \rightarrow N$ is **smooth** at point $p \in M$ if and only if there are local charts (U_α, ϕ_α) for M and (V_β, ψ_β) for N , such that:

1. $p \in U_\alpha$
2. $F(p) \in V_\beta$
3. $U_\alpha \cap F^{-1}(V_\beta) \subseteq M$ is open
4. The **coordinate representation** $\hat{F} := \psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \rightarrow \mathbb{R}^n$ is smooth at $\phi_\alpha(p) \in \mathbb{R}^m$

Proposition 2.1. If F is continuous, then 3 is always met.

Proposition 2.2. If 4 is met for some coordinate maps, then it is always true for any other coordinate maps. Namely, the smoothness is independent of choices of coordinate maps.

Definition 15. If F is smooth at every $p \in M$, we say that F is a smooth function.

Definition 16. F is a **diffeomorphism** if F is invertible, and that both F, F^{-1} are smooth.

Proposition 2.3. A diffeomorphism is always a homeomorphism.

3 Bump functions

Definition 17. Given a function $f : M \rightarrow \mathbb{R}$, the **support** is $\text{Supp}(f) := \overline{\{x \in M \mid f(x) \neq 0\}}$

Definition 18. Let M be a smooth manifold, $\mathcal{X} = \{X_\alpha\}$ is an open cover of M . A smooth **partition of unity** is $\{\psi_\alpha : M \rightarrow \mathbb{R}\}_{\alpha \in A}$, such that:

1. $0 \leq \psi_\alpha(x) \leq 1, \forall \alpha \in A, x \in M$
2. $\forall \alpha, \text{Supp}(\psi_\alpha) \subseteq X_\alpha$
3. $\{\text{Supp}(\psi_\alpha)\}_{\alpha \in A}$ is locally finite.
4. $\forall x \in M, \sum_{\alpha \in A} \psi_\alpha(x) = 1$

Theorem 3.1. A partition of unity always exists for any smooth manifold M and any open cover \mathcal{X} of M .

Proposition 3.2. Bump function

Let $A \subseteq U \subseteq M$, where M is a smooth manifold, A is a closed set, and U is an open set. There exists a smooth map $\psi : M \rightarrow \mathbb{R}$, such that

1. $\forall x \in M, 0 \leq \psi(x) \leq 1$
2. $\forall x \in A, \psi(x) = 1$
3. $\text{Supp}(\psi) \subseteq U$

Definition 19. If A is closed, then a function $f : A \rightarrow \mathbb{R}^k$ is **smooth** if $\forall x \in A, \exists W_x \subseteq M, F_x : W_x \rightarrow \mathbb{R}^k$, such that W_x is open and $x \in W_x$, and $\forall p \in W_x \cap A, F_x(p) = f(p), F_x \in C^\infty(W_x)$

Lemma 3.3. Extension Lemma:

Let M be a smooth manifold, and $A \subseteq M$ be a closed subset. Let $f : A \rightarrow \mathbb{R}^k$ be smooth, then $\forall U \supseteq A$ be open, $\exists \tilde{f} : M \rightarrow \mathbb{R}^k$, such that \tilde{f} is smooth, and $\tilde{f}|_A = f, \text{Supp}(\tilde{f}) \subseteq U$

4 Tangent space and derivatives

Definition 20. Let $C^\infty(M)$ be the real vector space of smooth functions from $M \rightarrow \mathbb{R}$, a **derivation** or **tangent vector** at $p \in M$ is an \mathbb{R} -linear map $D : C^\infty(M) \rightarrow \mathbb{R}$ satisfying the **Leibniz condition**: $\forall f, g \in C^\infty(M), D(fg) = f(p)D(g) + g(p)D(f)$

Definition 21. The **tangent space** to M at $p \in M$, $T_p M$, is the set of all tangent vectors at p .

Proposition 4.1. $T_p M$ is a real vector space where $\forall X, Y \in T_p M, c \in \mathbb{R}, f \in C^\infty(M), (X + Y)(f) := X(f) + Y(f), (cX)(f) := c \cdot X(f), (-X)(f) := -X(f), 0(f) := 0$

Proposition 4.2. Let $D \in T_p M$, if $\forall x \in M, f(x) = c$ is a constant function, then $D(f) = 0$

Proposition 4.3. If $\exists U \ni p$ be open, and $\forall x \in U, f(x) = 0$, then $\forall D \in T_p M, D(f) = 0$

Proposition 4.4. If $f, g \in C^\infty(M)$ and $f = g$ on some open $U \ni p$, then $\forall D \in T_p M, D(f) = D(g)$

Definition 22. For $p, v = (v^1, \dots, v^n) \in \mathbb{R}^n$, then the **directional derivative** of $f \in C^\infty(\mathbb{R}^n)$ in the direction of v is $D_v(f) := (\sum_{i=1}^n v^i \frac{\partial}{\partial x^i} |_p)(f) := \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p)$

Proposition 4.5. The directional derivative is a derivation.

Proposition 4.6. $\forall p \in \mathbb{R}^n$, the map $L : \mathbb{R}^n \rightarrow T_p \mathbb{R}^n$, where $L(v) := D_v$ is a vector space isomorphism.

Corollary 4.7. $\{\partial_1|_p, \dots, \partial_n|_p\}$ is a basis for $T_p \mathbb{R}^n$.

Definition 23. Given a smooth function $F : M \rightarrow N$, the **differential** or **derivative** of F at $p \in M$ is the map $dF_p : T_p M \rightarrow T_{F(p)} N$, given by $\forall D \in T_p M, f \in C^\infty(N), dF_p(D)(f) := D(f \circ F)$

Remark. One can check that $dF_p(D) \in T_{F(p)} N$ for any $D \in T_p M$, and that $f \circ F \in C^\infty(M)$, so the derivative of F above is well-defined.

Proposition 4.8. Let M, N, R be smooth manifolds and $F : M \rightarrow N, G : N \rightarrow R$ are smooth maps, then for any $p \in M$, we have:

1. $dF_p : T_p M \rightarrow T_{F(p)} N$ is linear
2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p M \rightarrow T_{G(F(p))} R$
3. $d(Id_M)_p : T_p M \rightarrow T_p M$ is an identity isomorphism.
4. If F is a diffeomorphism, then dF_p is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

Proposition 4.9. Let M be a smooth manifold, $U \subseteq M$ be open, and $i : U \rightarrow M$ be the inclusion map, then $\forall p \in U, di_p : T_p U \rightarrow T_p M$ is an isomorphism.

Corollary 4.10. Let M be a n -dimensional smooth manifold, for any $p \in M$, and any local chart (U, ϕ) containing p , we have $T_p M \cong T_p U \cong T_{\phi(p)} \phi(U) \cong T_{\phi(p)} \mathbb{R}^n \cong \mathbb{R}^n$, and the n -dimensional vector space $T_p M$ has a basis of $\{\Upsilon_j|_p := \partial_j|_p := \frac{\partial}{\partial x^j}|_p := (di_p \circ (d\phi_p)^{-1})(\frac{\partial}{\partial x^j}|_{\phi(p)}) = (di_p \circ (d\phi_p)^{-1})(\partial_j|_{\phi(p)})\}_j$.

Proposition 4.11. Given $f \in C^\infty(M), \Upsilon_j|_p(f) = \frac{\partial(f \circ \phi^{-1})}{\partial x^j}(\phi(p))$

Proof.

$$\begin{aligned} \Upsilon_j|_p(f) &= (di_p \circ (d\phi_p)^{-1})(\partial_j|_{\phi(p)})(f) \\ &= (d\phi_p)^{-1}(\partial_j|_{\phi(p)})(f \circ \mathbf{i}) \\ &= d(\phi^{-1})_{\phi(p)}(\partial_j|_{\phi(p)})(f \circ \mathbf{i}) \\ &= (\partial_j|_{\phi(p)})(f \circ \mathbf{i} \circ \phi^{-1})(\phi(p)) \\ &= \frac{\partial(f \circ \phi^{-1})}{\partial x^j}(\phi(p)) \end{aligned}$$

□

Corollary 4.12. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , then

$$\Upsilon_j|_p(x^i) = \frac{\partial x^i}{\partial x^j}(p) = \frac{\partial(x^i \circ \phi^{-1})}{\partial x^j}(\phi(p)) = \delta_j^i$$

Theorem 4.13. Let $F : M \rightarrow N$ be smooth, (U, ϕ) and (V, ψ) be local charts for M and N , such that $p \in U, F(p) \in V$, if we choose the basis $\{\Upsilon_j|_p\}_j, \{\Upsilon_i|_{F(p)}\}_i$ associated to (U, ϕ) and (V, ψ) , we have that $[dF_p]_{ij} = \frac{\partial \hat{F}^i}{\partial x^j}(\phi(p))$. Namely, $dF_p(\Upsilon_j|_p) = \sum_{i=1}^{\dim(M)} \frac{\partial \hat{F}^i}{\partial x^j}(\phi(p)) \Upsilon_i|_{F(p)} \in T_{F(p)}N$, where $\hat{F} = \psi \circ F \circ \phi^{-1}$ is the coordinate representation of F .

Definition 24. The **tangent bundle** $TM := \bigsqcup_{p \in M} T_p M$.

Proposition 4.14. If M is a n -dimension smooth manifold, then TM is a $2n$ -dimension smooth manifold.

5 Vector field

Definition 25. A **vector field** is a smooth function $\mathbf{v} : M \rightarrow TM$, such that $\forall p, \mathbf{v}_p := \mathbf{v}(p) \in T_p M$

Definition 26. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , then we can always write $\mathbf{v}_p = \sum_{i=1}^n \mathbf{v}^i(p) \Upsilon_i|_p$, since $\{\Upsilon_i|_p = \frac{\partial}{\partial x^i}|_p\}$ is a basis for $T_p M$. The functions $\mathbf{v}^i : M \rightarrow \mathbb{R}$ are called the **component functions**.

Definition 27. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , the **partial derivatives** $\Upsilon_i = \partial_i : U \rightarrow TM$ is given by $\Upsilon_i(p) = \Upsilon_i|_p$, where $\{\Upsilon_i|_p = \frac{\partial}{\partial x^i}|_p\}$ is a basis for $T_p M$ associated to $(U, \phi = (x^1, \dots, x^n))$. One can check that $\Upsilon_i \in \mathfrak{X}(M)$

Proposition 5.1. Given $p \in M, \mathbf{u} \in T_p M$, and some open $U \subseteq M$ that contains p , there is a vector field \mathbf{v} on M , such that $\mathbf{v}|_p = \mathbf{u}$, and $\text{Supp}(\mathbf{v}) \subseteq U$.

Definition 28. $\mathfrak{X}(M)$ is the set of all vector fields.

Proposition 5.2. $\mathfrak{X}(M)$ is a real vector space.

Definition 29. Given a smooth function $f \in C^\infty(M)$, and a vector field $\mathbf{v} \in \mathfrak{X}(M)$, we define $f\mathbf{v} := f \cdot \mathbf{v} \in \mathfrak{X}(M)$ to be $(f\mathbf{v})(p) := f(p)\mathbf{v}_p \in T_p M$

Proposition 5.3. The above definition of $\cdot : C^\infty(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ gives a $C^\infty(M)$ -module of the vector fields.

Corollary 5.4. We can thus write any vector field $\mathbf{v} \in \mathfrak{X}(M)$ as $\sum_{i=1}^n \mathbf{v}^i \Upsilon_i$

Theorem 5.5. Canonical form of vector field

Let $\mathbf{v} \in \mathfrak{X}(M)$, and $V_p \neq 0$ for some $p \in M$, then there is a coordinate chart $(U, \phi = (x^1, \dots, x^n))$ for M , such that $p \in U, \mathbf{v}|_U = \frac{\partial}{\partial x^1} = \Upsilon_1$

Definition 30. Given a smooth function $f \in C^\infty(M)$, and a vector field $\mathbf{v} \in \mathfrak{X}(M)$, we define $\mathbf{v}(f) \in C^\infty(M)$ to be $\mathbf{v}(f)(p) := \mathbf{v}_p(f) \in \mathbb{R}$. Thus we can view a vector field \mathbf{v} as a function $C^\infty(M) \rightarrow C^\infty(M)$ as well.

6 Lie Bracket

Definition 31. Given two vector fields $\mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$, the **Lie Bracket** is $[\mathbf{v}, \mathbf{w}] := \mathbf{v} \circ \mathbf{w} - \mathbf{w} \circ \mathbf{v}$

Proposition 6.1. $[\mathbf{v}, \mathbf{w}] \in \mathfrak{X}(M)$ is a vector field.

Proposition 6.2. $[\mathbf{v}, \mathbf{w}]$ is bilinear.

Proposition 6.3. $[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$ is anti-symmetric

Proposition 6.4. $[\mathbf{v}, \mathbf{w}]$ satisfies the Jacobian Identity: $[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0$

Proposition 6.5. For any $f, g \in C^\infty(M), \mathbf{v}, \mathbf{u} \in \mathfrak{X}(M)$, we have $[f\mathbf{v}, g\mathbf{u}] = fg[\mathbf{v}, \mathbf{u}] + f(\mathbf{v}g)\mathbf{u} - g(\mathbf{u}f)\mathbf{v}$

Proposition 6.6. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , then for any two vector fields $\mathbf{v} = \sum_{i=1}^n \mathbf{v}^i \Upsilon_i, \mathbf{u} = \sum_{i=1}^n \mathbf{u}^i \Upsilon_i \in \mathfrak{X}(M)$, we have $[\mathbf{u}, \mathbf{v}] = \sum_{j=1}^n (\mathbf{u}^j \mathbf{v}^j - \mathbf{v}^j \mathbf{u}^j) \Upsilon_j$

7 Curve and Flow

Definition 32. Let $J \subseteq \mathbb{R}$ be open, a smooth map $\gamma : J \rightarrow M$ is called a **smooth curve** in M . Given $t_0 \in J$, let $\frac{d}{dt}|_{t_0}$ be the coordinate basis in $T_{t_0}J \cong T_{t_0}\mathbb{R}$. The **velocity** of γ at t_0 is $\gamma'(t_0) := d\gamma_{t_0}(\frac{d}{dt}|_{t_0}) \in T_{\gamma(t_0)}\mathbb{R}^n$

Proposition 7.1. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , and a curve $\gamma : J \rightarrow M$. If we let $\phi \circ \gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$, then we have $\gamma'(t_0) = d\gamma_{t_0}(\frac{d}{dt}|_{t_0}) = \sum_{i=1}^n \dot{\gamma}^i(t_0) \Upsilon_i|_{\gamma(t_0)} \in T_{\gamma(t_0)}M$

Definition 33. Let $\mathbf{v} \in \mathfrak{X}(M)$, an **integral curve** of \mathbf{v} is a curve $\gamma : J \rightarrow M$ such that $\forall t \in J, \gamma'(t) = \mathbf{v}_t$. If $0 \in J$, then $\gamma(0)$ is called the **starting point** of γ

Proposition 7.2. An integral curve should satisfy $\forall t \in J, \sum_{i=1}^n \dot{\gamma}^i(t) \Upsilon_i|_{\gamma(t_0)} = \gamma'(t) = \mathbf{v}_t = \sum_{i=1}^n \mathbf{v}^i(\gamma(t)) \Upsilon_i|_{\gamma(t_0)}$, thus $\forall i, \dot{\gamma}^i(t) = \mathbf{v}^i \circ \phi^{-1}(\gamma^1(t), \dots, \gamma^n(t))$ gives an ODE.

Proposition 7.3. $\forall \mathbf{v} \in \mathfrak{X}(M), p \in M, \exists \epsilon > 0, \gamma : (-\epsilon, \epsilon) \rightarrow M$ that is an integral curve of \mathbf{v} with starting point p .

Definition 34. A **smooth global flow** on M is a smooth map $\Theta : \mathbb{R} \times M \rightarrow M$, such that $\forall s, t \in \mathbb{R}, p \in M, \Theta(0, p) = p, \Theta(t, \Theta(s, p)) = \Theta(t + s, p)$

Remark. A global flow can be thought of as a table to show where something should be after t time and starting from p .

Definition 35. For $p \in M$, we have $\Theta^{(p)} : \mathbb{R} \rightarrow M$ is a curve given by $\Theta^{(p)}(t) := \Theta(t, p)$

Definition 36. For $t \in \mathbb{R}$, we have $\Theta_t : \mathbb{R} \rightarrow M$ is a smooth map given by $\Theta_t(p) := \Theta(t, p)$

Proposition 7.4. Given a global flow $\Theta : \mathbb{R} \times M \rightarrow M$, we define $\mathbf{v} : M \rightarrow TM$ by $\mathbf{v}_p := \Theta^{(p)'}(0) \in T_pM$, then $\mathbf{v} \in \mathfrak{X}(M)$ is a vector field, and $\Theta^{(p)}$ is an integral curve for \mathbf{v} .

Definition 37. The $\mathbf{v} \in \mathfrak{X}(M)$ in the above proposition is called **infinitesimal generator** for the flow Θ .

Remark. The infinitesimal generator tells us how should something move at every point in M .

Definition 38. Let $D \subseteq \mathbb{R} \times M$ be open, such that $\forall p \in M, D^{(p)} := \{t \in \mathbb{R} | (t, p) \in D\}$ is an open interval containing 0. A **local flow** on M is a smooth map $\Theta : D \rightarrow M$, such that $\forall p \in M, s \in D^{(p)}, t \in D^{(\Theta(s, p))}$, s.t. $t + s \in D^{(p)}, \Theta(0, p) = p, \Theta(t, \Theta(s, p)) = \Theta(t + s, p)$

Definition 39. $\Theta^{(p)} : D^{(p)} \rightarrow M$ is a curve given by $\Theta^{(p)}(t) := \Theta(t, p)$. And for $t \in \mathbb{R}, M_t := \{p \in M | (t, p) \in D\}$, then $\Theta_t : M_t \rightarrow M$ is a smooth map given by $\Theta_t(p) := \Theta(t, p)$

Proposition 7.5. A local flow also has an infinitesimal generator as before.

Definition 40. An integral curve is **maximal** if it cannot be extended to an integral curve with a greater domain. A local flow is **maximal** if it cannot be extended to a local flow with a greater domain D .

Theorem 7.6. Fundamental theorem of flows:

For any $\mathbf{v} \in \mathfrak{X}(M)$, there is a unique maximal local flow $\Theta : D \rightarrow M$, such that \mathbf{v} is the generator of Θ . Moreover,

1. $\forall p \in M, \Theta^{(p)} : D^{(p)} \rightarrow M$ is the unique maximal integral curve of \mathbf{v} starting at p .
2. If $s \in D^{(p)}$, then $D^{(\Theta(s, p))} = D^{(p)} - s := \{t - s : t \in D^{(p)}\}$
3. $\forall t \in \mathbb{R}, M_t$ is an open subset of M , and $\Theta_t : M_t \rightarrow M_{-t}$ is a diffeomorphism with its inverse $\Theta_t^{-1} = \Theta_{-t}$

8 One-form and Co-vector fields

Definition 41. Let V be a vector space, a **co-vector** on V is a linear map $f : V \rightarrow \mathbb{R}$. The set of all co-vectors is called the **dual space** V^* .

Definition 42. A **contraction** $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{R}$ is the evaluation $\langle f, v \rangle := f(v)$

Proposition 8.1. Given a basis $\{E_1, \dots, E_n\}$ for a finite-dimensional V , let $E^1, \dots, E^n \in V^*$ be defined by $\langle E^i, E_j \rangle := E^i(E_j) = \delta_{ij}$, then $\{E^i\}$ is a basis for V^* , called the **dual basis**.

Definition 43. Let V, W be vector spaces, $A : V \rightarrow W$ be a linear map. The **dual map** $A^* : W^* \rightarrow V^*$ is defined by $\forall f \in W^*, v \in V, \langle A^*(f), v \rangle = A^*(f)(v) := f(A(v)) = \langle f, A(v) \rangle$

Proposition 8.2. If $B : V \rightarrow W, A : W \rightarrow U$ are linear maps, then $(A \circ B)^* = B^* \circ A^*$

Proposition 8.3. $(Id_V)^*$ is the identity map for V^*

Proposition 8.4. If $A : V \rightarrow W$ is an isomorphism, then $(A^*)^{-1} = (A^{-1})^*$

Proposition 8.5. Let V and W be finite-dimensional vector spaces of dimensions n and m , $A : V \rightarrow W$ be a linear map, and $[A]$ is the matrix with respect to basis $\{v_i\}_1^n, \{w_j\}_1^m$ for V, W , then $[A^*] = [A]^\dagger$

Definition 44. Let $T_p M$ be the tangent space to M at p , then the **cotangent space** to M at p is the dual space of $T_p M$, denoted by $T_p^* M$. The elements in $T_p^* M$ are called **co-vectors**.

Definition 45. $T^* M := \bigsqcup_{p \in M} T_p^* M$ is called the **cotangent bundle**.

Proposition 8.6. If M is an n -dimensional smooth manifold, then $T^* M$ is a $2n$ -dimensional smooth manifold.

Definition 46. A **one-form** or **co-vector field** is a smooth map $\omega : M \rightarrow T^* M$ such that $\forall p \in M, \omega_p := \omega(p) \in T_p^* M$

Definition 47. $\mathfrak{X}^*(M)$ is the set of all one-forms on M .

Proposition 8.7. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , we have that for each $p \in U, \{\Upsilon_1|_p, \dots, \Upsilon_n|_p\}$ is a basis for $T_p M$, the dual basis for $T_p^* M$ is $\{\Upsilon^1|_p, \dots, \Upsilon^n|_p\}$, such that $\langle \Upsilon^i, \Upsilon_j \rangle = \delta_{ij}$. Thus $\forall \omega_p \in T_p^* M, \omega_p = \sum_{i=1}^n \omega_i(p) \Upsilon^i|_p$ uniquely. And $\omega_i(p)$ can be get by $\omega_i(p) = \langle \omega, \Upsilon_i \rangle$

Definition 48. The **coordinate co-vector field** is the map $\Upsilon^i : U \rightarrow T^* M$ by $\Upsilon^i(p) := \Upsilon^i|_p$. One can check that $\Upsilon^i \in \mathfrak{X}^*(M)$ is a co-vector field.

Definition 49. Given a smooth function $f \in C^\infty(M)$, and a co-vector field $\omega \in \mathfrak{X}^*(M)$, we define $f\omega := f \cdot \omega \in \mathfrak{X}^*(M)$ to be $(f\omega)(p) := f(p)\omega_p \in T_p^* M$

Proposition 8.8. The above definition of $\cdot : C^\infty(M) \times \mathfrak{X}^*(M) \rightarrow \mathfrak{X}^*(M)$ gives a $C^\infty(M)$ -module of the co-vector fields.

Corollary 8.9. We can thus write any co-vector field $\omega \in \mathfrak{X}^*(M)$ as $\sum_{i=1}^n \omega_i \Upsilon^i$

Definition 50. Given any $\omega \in \mathfrak{X}^*(M), \mathbf{v} \in \mathfrak{X}(M)$, we can define $\langle \omega, \mathbf{v} \rangle : M \rightarrow \mathbb{R}$ by $\langle \omega, \mathbf{v} \rangle(p) := \langle \omega_p, \mathbf{v}_p \rangle$.

Proposition 8.10. Given any $\omega = \sum_{i=1}^n \omega_i \Upsilon^i \in \mathfrak{X}^*(M), \mathbf{v} = \sum_{i=1}^n v^i \Upsilon_i \in \mathfrak{X}(M), \langle \omega, \mathbf{v} \rangle = \sum_{i=1}^n \omega_i v^i$

Definition 51. Let $f \in C^\infty(M)$, the **differential** of f is $df \in \mathfrak{X}^*(M)$, such that $\forall p \in M, D \in T_p M, (df)_p(D) := Df \in \mathbb{R}$. Thus we have a function $d : C^\infty(M) \rightarrow \mathfrak{X}^*(M)$

Proposition 8.11. Given a vector field \mathbf{v} , we have $\langle df, \mathbf{v} \rangle|_p = \langle df_p, \mathbf{v}_p \rangle = \mathbf{v}_p(f)$

Proposition 8.12. Notice that $T_{f(p)} \mathbb{R} \cong \mathbb{R}$, and if we identify them canonically, we have that $(df)_p \in T_p^* M : T_p M \rightarrow \mathbb{R} \cong df_p : T_p M \rightarrow T_{f(p)} \mathbb{R}$. Namely, $\forall D \in T_p M, df_p(D) = (df)_p(D) \frac{d}{dt}|_{f(p)} \in T_{f(p)} \mathbb{R}$

Proposition 8.13. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , then for any $f \in C^\infty(M)$, we have $df|_U = \sum_{i=1}^n \Upsilon_i(f) \Upsilon^i$

Corollary 8.14. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , we have $d(x^j)|_U = \sum_{i=1}^n \Upsilon_i(x^j) \Upsilon^i = \sum_{i=1}^n \delta_{ij} \Upsilon^i = \Upsilon^j$. Thus we can write Υ^j as dx^j , and Υ_j as $\frac{\partial}{\partial x^j}$ or ∂_j .

Corollary 8.15. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , then for any $f \in C^\infty(M)$, we have

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i = (\partial_i f) dx^i$$

9 Tensors

Definition 52. Let V be a n -dimensional vector space, then $T^k(V^*) := V^* \otimes V^* \otimes \dots \otimes V^* = (V^*)^{\otimes k}$ is a vector space of dimension n^k . An element in $T^k(V^*)$ is called a **covariant k-tensor** on V .

Definition 53. More generally, a (r, s) tensor, or a **r-variant-s-covariant-tensor** is an element from $(V)^{\otimes r} \otimes (V^*)^{\otimes s}$.

Proposition 9.1. Let V be n -dimensional vector space, and $\{E_1, \dots, E_n\}$ be a basis for V , then $\{E^{i_1} \otimes \dots \otimes E^{i_k}\}_{(i_1, \dots, i_k) \in [n]^k}$ is a basis for $T^k(V^*)$, thus any $\alpha \in T^k(V^*)$ can be uniquely written as $\sum_{(i_1, \dots, i_k) \in [n]^k} \alpha_{i_1, \dots, i_k} E^{i_1} \otimes \dots \otimes E^{i_k}$

Definition 54. We can thus view $\alpha \in T^k(V^*)$ as a function $V^k \rightarrow \mathbb{R}$ defined by $\alpha(v_1, \dots, v_n) := \sum_{(i_1, \dots, i_k) \in [n]^k} \alpha_{i_1, \dots, i_k} E^{i_1}(v_1) \dots E^{i_k}(v_k) = \alpha_{i_1, \dots, i_k} \langle E^{i_1}, v_1 \rangle \dots \langle E^{i_k}, v_k \rangle$

Definition 55. More generally, a (r, s) tensor $\mathcal{T} = \sum_{(i_1, \dots, i_r, j_1, \dots, j_s) \in [n]^{r+s}} \mathcal{T}_{j_1, \dots, j_s}^{i_1, \dots, i_r} E_{i_1} \otimes \dots \otimes E_{i_r} \otimes E^{j_1} \otimes \dots \otimes E^{j_s}$ can be viewed as a map $(V^*)^r \times V^s \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{T}(\omega^1, \dots, \omega^r, v_1, \dots, v_s) &:= \sum_{(i_1, \dots, i_r, j_1, \dots, j_s) \in [n]^{r+s}} \mathcal{T}_{j_1, \dots, j_s}^{i_1, \dots, i_r} \omega^1(E_{i_1}) \dots \omega^r(E_{i_r}) E^{j_1}(v_1) \dots E^{j_s}(v_s) \\ &= \mathcal{T}_{j_1, \dots, j_s}^{i_1, \dots, i_r} \langle \omega^1, E_{i_1} \rangle \dots \langle \omega^r, E_{i_r} \rangle \langle E^{j_1}, v_1 \rangle \dots \langle E^{j_s}, v_s \rangle \end{aligned}$$

Proposition 9.2. The map defined by the tensors is multi-linear.

Example 9.0.1. A vector is a $(1,0)$ tensor.

Example 9.0.2. A co-vector is a $(0,1)$ tensor.

Example 9.0.3. A real inner product is a $(0,2)$ tensor.

Example 9.0.4. The determinant of a $n \times n$ real matrix is a $(0,n)$ tensor as a function on the column/row vectors.

10 Alternating Tensor and wedge product

Definition 56. A covariant k -tensor is **symmetric** if $\forall 1 \leq i < j \leq k, v_1, \dots, v_k \in V$,

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

It is **alternating** or **anti-symmetric** if $\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$

Definition 57. The set of alternating covariant k -tensors is $\Lambda^k(V^*) \subseteq T^k(V^*)$

Definition 58. The **permutation group** S_k is the set of all permutations of $[k] := \{1, \dots, k\}$

Definition 59. Given a permutation $\sigma \in S_k$, **sign** of it is $Sgn(\sigma) := \begin{cases} 1 & \text{permutation is got by even transposition} \\ -1 & \text{permutation is got by odd transposition} \end{cases}$

Definition 60. Given a tensor $\alpha \in T^k(V^*)$, the **alternation** of it is $Alt(\alpha) \in T^k(V^*)$, defined by $Alt(\alpha)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

Proposition 10.1. $\forall \alpha \in T^k(V^*), Alt(\alpha) \in \Lambda^k(V^*)$

Proposition 10.2. $\forall \alpha \in \Lambda^k(V^*), Alt(\alpha) = \alpha$

Definition 61. An ordered k-tuple $I = (i_1, \dots, i_k) \in [n]^k$ is called a **multi-index of length k**.

Definition 62. Let $\{E^1, \dots, E^n\}$ be a dual basis for V^* , and $I = (i_1, \dots, i_k) \in [n]^k$, then we define the **elementary alternating tensor** $E^I \in \Lambda^k(V^*)$ by $E^I(v_1, \dots, v_k) := \det \begin{pmatrix} E^{i_1}(v_1) & \dots & E^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ E^{i_k}(v_1) & \dots & E^{i_k}(v_k) \end{pmatrix}$

Definition 63. Let $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k) \in [n]^k$, then $\delta_J^I := \det \begin{pmatrix} \delta_{j_1}^{i_1} & \dots & \delta_{j_k}^{i_1} \\ \vdots & \ddots & \vdots \\ \delta_{j_1}^{i_k} & \dots & \delta_{j_k}^{i_k} \end{pmatrix}$

Proposition 10.3. $\delta_J^I = \begin{cases} 0 & \text{if } I \text{ or } J \text{ have repeated indices, or } I \text{ is not any permutation of } J \\ 1 & \text{if } I \text{ is an even permutation of } J \\ -1 & \text{if } I \text{ is an odd permutation of } J \end{cases}$

Proposition 10.4. If $I = (i_1, \dots, i_k)$ have any repeated indices, i.e. $\exists 1 \leq l \neq m \leq k, i_l = i_m$, then $E^I = 0$

Proposition 10.5. If $J = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$, then $E^J = sgn(\sigma)E^I$

Proposition 10.6. Let $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k) \in [n]^k$, then $E^I(E_{j_1}, \dots, E_{j_k}) = \delta_J^I$

Definition 64. $I = (i_1, \dots, i_k) \in [n]^k$ is increasing if $i_1 < i_2 < \dots < i_k$

Proposition 10.7. Let $\dim(V) = n, k \in \mathbb{N}^+$, if $K > n$, then $\Lambda^k(V^*) = \{0\}$, otherwise $\dim(\Lambda^k(V^*)) = \binom{n}{k}$, and a basis is given by $\mathcal{E}^k := \{E^I | I \text{ is increasing multi-index of length } k\}$

Definition 65. The **wedge product** of two covariant tensors u, v is defined to be $u \wedge v := u \otimes v - v \otimes u$

Proposition 10.8. $\forall s, s \wedge s = 0$

Proposition 10.9. For u_1, \dots, u_k , we have $u_1 \wedge \dots \wedge u_k = k! Alt(u_1 \otimes \dots \otimes u_k) = \sum_{\sigma \in S_k} sgn(\sigma) u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(k)}$

Proposition 10.10. $E^I = E^{i_1} \wedge \dots \wedge E^{i_k}$

Proposition 10.11. Let $I, J \in [n]^k$ be both increasing, then $E^I \wedge E^J = (-1)^{|I| \cdot |J|} E^J \wedge E^I$

11 Tensor fields or k-form

Definition 66. The bundle of all covariant k-tensors on M is $T^k T^* M := \bigsqcup_{p \in M} T^k(T_p^* M)$

Proposition 11.1. $T^k T^* M$ is a smooth manifold of dimension n^{k+1}

Definition 67. Given a smooth manifold M, a **covariant k-tensor field on M** or a **k-form** is a smooth map $A : M \rightarrow T^k T^* M$, s.t. $A_p := A(p) \in T^k(T_p^* M)$. The set of all k-forms is $\Gamma(T^k T^* M)$

Proposition 11.2. $\Gamma(T^k T_p^* M)$ is an infinite dimensional vector space.

Proposition 11.3. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M, then for covariant k-tensor field $A \in \Gamma(T^k T_p^* M)$, it can be written as $A|_U = \sum_{(i_1, \dots, i_k) \in [n]^k} A_{i_1, \dots, i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$

Definition 68. An **alternating covariant k-tensor field** or **alternating k-form** on M is a smooth map $A : M \rightarrow \bigsqcup_{p \in M} \Lambda^k(T_p^*M)$ such that $\forall p \in M, A_p := A(p) \in \Lambda^k(T_p^*M)$. The set of all alternating k-forms on M is $\Omega^k(M)$

Definition 69. $\Omega^0(M) := C^\infty(M)$

Definition 70. $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ is defined to be $(\zeta \wedge \eta)_p := \zeta_p \wedge \eta_p$, where $\zeta_p \in \Lambda^k(T_p^*M), \eta_p \in \Lambda^l(T_p^*M)$. For $k = 0, f \in C^\infty(M), f \wedge \eta := f\eta$

Proposition 11.4. $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$ is an anti-commutative algebra.

Proposition 11.5. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M , then for an alternating covariant k-tensor field $\omega \in \Omega^k(M)$, it can be written as $\omega|_U = \sum_{(i_1 < \dots < i_k) \in [n]^k} \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$, where $\omega_{i_1, \dots, i_k} = \omega(\partial_{i_1}, \dots, \partial_{i_k})$, and $dx^{i_1} \wedge \dots \wedge dx^{i_k}(\partial_{j_1}, \dots, \partial_{j_k}) = \delta_J^I$

12 Push-forward and Pull-back

Definition 71. For $f \in \Omega^0(N) = C^\infty(N)$, we have the **pull-back** of f by a smooth map $F : M \rightarrow N$ is $F^*f = f \circ F \in C^\infty(M)$.

Remark. Notice that the definition of pull-back between smooth function spaces coincides with the definition of dual maps for linear function spaces.

Definition 72. Let M, N be smooth manifolds and $F : M \rightarrow N$ be a smooth function, then two vector fields $\mathbf{v} \in \mathfrak{X}(M), \mathbf{u} \in \mathfrak{X}(N)$ are **F-related** if $\forall p \in M, dF_p(\mathbf{v}_p) = \mathbf{u}_{F(p)}$

Proposition 12.1. If $F : M \rightarrow N$ is a diffeomorphism, for every vector field $\mathbf{v} \in \mathfrak{X}(M)$, there is a unique vector field $F_*\mathbf{v} \in \mathfrak{X}(N)$ that is F -related to \mathbf{v} .

Definition 73. The **push-forward** of a vector field $\mathbf{v} \in \mathfrak{X}(M)$ by a diffeomorphism $F : M \rightarrow N$ is the unique vector field $F_*\mathbf{v} \in \mathfrak{X}(N)$ in the previous proposition, defined by $\forall q \in N, F_*\mathbf{v}_q := dF_{F^{-1}(q)}(\mathbf{v}_{F^{-1}(q)})$

Proposition 12.2. For any diffeomorphism $F : M \rightarrow N$, vector field $\mathbf{v} \in \mathfrak{X}(N), f \in C^\infty(N)$, we always have $\mathbf{v}(F^*f) = (F_*\mathbf{v})(f) \circ F \in C^\infty(M)$

Definition 74. Let $F : M \rightarrow N$ be a smooth map, the **cotangent map** of F is $dF_p^* : T_{F(p)}^*N \rightarrow T_p^*M$, which is the dual map of $dF_p : T_pM \rightarrow T_{F(p)}N$

Proposition 12.3. Given any $\omega_{F(p)} \in T_{F(p)}^*N, v_p \in T_pM$, we have $\langle dF_p^*(\omega_{F(p)}), v_p \rangle = \langle \omega_{F(p)}, dF_p(v_p) \rangle$

Definition 75. Given a smooth map $F : M \rightarrow N$, and a co-vector field $\omega \in \mathfrak{X}(N)$, the **pull-back of a 1-form** ω by F is $F^*\omega \in \mathfrak{X}^*(M)$ defined by $(F^*\omega)_p := dF_p^*(\omega_{F(p)}) \in T_p^*M$

Proposition 12.4. Consider any $D \in T_pM, (F^*\omega)_p(D) = \langle dF_p^*(\omega_{F(p)}), D \rangle = \langle \omega_{F(p)}, dF_p(D) \rangle$

Proposition 12.5. For any smooth $F : M \rightarrow N$, co-vector field $\omega \in \mathfrak{X}^*(N), u \in C^\infty(N), \mathbf{v} \in \mathfrak{X}(M), p \in M$,

1. $\langle F^*\omega, \mathbf{v} \rangle|_p = \langle \omega, F_*\mathbf{v} \rangle|_{F(p)}$
2. $F^*u F^*\omega = (u \circ F) F^*\omega = F^*(u\omega) \in \mathfrak{X}^*(M)$
3. $F^*(du) = d(u \circ F) = d(F^*u) \in \mathfrak{X}^*(M)$

Proof. 1. $\langle F^*\omega, \mathbf{v} \rangle|_p = \langle F^*\omega_p, \mathbf{v}_p \rangle = \langle dF_p^*(\omega_{F(p)}), \mathbf{v}_p \rangle = \langle \omega_{F(p)}, dF_p(\mathbf{v}_p) \rangle = \langle \omega_{F(p)}, F_*\mathbf{v}_{F(p)} \rangle = \langle \omega, F_*\mathbf{v} \rangle|_{F(p)}$ \square

Definition 76. Given a smooth map $F : M \rightarrow N, p \in M, \alpha \in T^k(T_{F(p)}^*N)$, the **pull-back of a covariant k-tensor** α by F at p is $dF_p^*(\alpha) \in T^k(T_p^*M)$, defined by $dF_p^*(\alpha)(v_1, \dots, v_k) = \alpha(dF_p(v_1), \dots, dF_p(v_k)) \in C^\infty(N)$. This way we obtain a linear map $dF_p^* : T^k(T_{F(p)}^*N) \rightarrow T^k(T_p^*M)$

Definition 77. Let $A \in \Gamma(T^k T^* N)$ be a k-form, the **pull-back of a k-form** A by a smooth map $F : M \rightarrow N$ is $F^* A \in \Gamma(T^k T^* M)$ defined by $(F^* A)_p := dF_p^*(A_{F(p)})$. This way we get a $F^* : \Gamma(T^k T^* N) \rightarrow \Gamma(T^k T^* M)$

Definition 78. The **push-forward of a contra-variant k-tensor field** $\mathbf{V} \in \Lambda(T^k TM)$ by a diffeomorphism $F : M \rightarrow N$ is $F^*(\mathbf{V}) \in \Lambda(T^k TN)$, defined by $\forall A \in \Gamma(T^{k-1} T^* N), p \in M, \langle \mathbf{V}, F^* A \rangle|_p = \langle (F_* \mathbf{V}), A \rangle|_{F(p)}$

Proposition 12.6. For any $\forall A \in \Gamma(T^k T^* N), \mathbf{V} \in \Lambda(T^k TM), p \in M, \langle F^* A, \mathbf{V} \rangle|_p = \langle A, (F_* \mathbf{V}) \rangle|_{F(p)}$

Definition 79. We define the **pull-back of a contra-variant k-tensor field** $\mathbf{V} \in \Gamma(T^k TN)$ by a diffeomorphism $F : M \rightarrow N$ to be $F^*(\mathbf{V}) := F_*^{-1}(\mathbf{V}) \in \Lambda(T^k TM)$, which is the push-forward of \mathbf{V} by $F^{-1} : N \rightarrow M$

Definition 80. The **pull-back of an (r,s)-tensor field** \mathcal{T} is by taking the pullback on the r-contra-variant field and the s-co-variant part respectively.

Definition 81. Let $A \in \Omega^k(N)$, the **pull-back of an alternating k-form** A by a smooth map $F : M \rightarrow N$ is $F^* A \in \Omega^k(M)$ defined by $(F^* A)_p := dF_p^*(A_{F(p)})$. This way we get a $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$

Proposition 12.7. $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$ is a linear map.

Proposition 12.8. Given any two alternating k-covariant tensor fields $A, B \in \Omega^k(N)$, we have $F^*(A \wedge B) = F^*(A) \wedge F^*(B)$

Proposition 12.9. $F^*(\sum_{I \text{ increasing}} A_I dy^{i_1} \wedge \dots \wedge dy^{i_k}) = \sum_{I \text{ increasing}} (A_I \circ F) d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F)$

Example 12.0.1. Consider $M = U = \mathbb{R}^2 \setminus \{(x, 0) | x \leq 0\}$ be the open set of \mathbb{R}^2 minus the negative x axis, with a polar coordinate local map $\phi(r \cos(\theta), r \sin(\theta)) = (r, \theta)$ on U , let this be M . We then consider $N = U$ to be with a Cartesian coordinate $\psi = (x(a, b), y(a, b)) = (a, b) : N \rightarrow \mathbb{R}^2$. Let $Id : U \rightarrow U$ be the identity map. We thus have $x \circ Id \circ \phi^{-1}, y \circ Id \circ \phi^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, x \circ Id \circ \phi^{-1}(r, \theta) = r \cos(\theta), y \circ Id \circ \phi^{-1}(r, \theta) = r \sin(\theta)$.

$$\begin{aligned}
Id^*(dx \wedge dy) &= Id^*(\mathbf{1} dx \wedge dy), \text{ in which } \mathbf{1}(x, y) = 1 : N \rightarrow \mathbb{R} \\
&= (\mathbf{1} \circ Id) \cdot d(x \circ Id) \wedge d(y \circ Id), \text{ in which } x \circ Id, y \circ Id : U \rightarrow \mathbb{R}, \\
&= d(x \circ Id) \wedge d(y \circ Id) \\
&= \left(\frac{\partial x \circ Id \circ \phi^{-1}}{\partial \theta} d\theta + \frac{\partial y \circ Id \circ \phi^{-1}}{\partial r} dr \right) \wedge \left(\frac{\partial x \circ Id \circ \phi^{-1}}{\partial \theta} d\theta + \frac{\partial y \circ Id \circ \phi^{-1}}{\partial r} dr \right) \\
&= \left(\frac{\partial r \cos \theta}{\partial \theta} d\theta + \frac{\partial r \cos \theta}{\partial r} dr \right) \wedge \left(\frac{\partial r \sin \theta}{\partial \theta} d\theta + \frac{\partial r \sin \theta}{\partial r} dr \right) \\
&= (-r \sin \theta d\theta + \cos \theta dr) \wedge (r \cos \theta d\theta + \sin \theta dr) \\
&= -r \sin^2 \theta d\theta \wedge dr + r \cos^2 \theta dr \wedge d\theta, \text{ since } dr \wedge dr = d\theta \wedge d\theta = 0 \\
&= r \sin^2 \theta dr \wedge d\theta + r \cos^2 \theta dr \wedge d\theta \\
&= r dr \wedge d\theta
\end{aligned}$$

Thus there is a canonical mapping between $dx \wedge dy \in \Omega^2(M)$ and $r dr \wedge d\theta \in \Omega^2(N)$

13 Lie Derivative

Definition 82. Let $\mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$, the **Lie derivative** of \mathbf{w} with respect to \mathbf{v} is a map $\mathcal{L}_{\mathbf{v}} \mathbf{w} : M \rightarrow TM$ defined by $\mathcal{L}_{\mathbf{v}} \mathbf{w}_p = \frac{d}{dt}|_{t=0} (d(\Theta_{-t})_{\Theta_t(p)} \mathbf{w}_{\Theta_t(p)})$, where Θ is the flow generated by \mathbf{v} .

Lemma 13.1. $\mathcal{L}_{\mathbf{v}} \mathbf{w} \in \mathfrak{X}(M)$ is a vector field.

Theorem 13.2. If $\mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$, then $\mathcal{L}_{\mathbf{v}} \mathbf{w} = [\mathbf{v}, \mathbf{w}]$

Remark. $d(\Theta_{-t})_{\Theta_t(p)} \mathbf{w}_{\Theta_t(p)} = d(\Theta_{-t})_{(\Theta_{-t})^{-1}(p)} \mathbf{w}_{(\Theta_{-t})^{-1}(p)} = (\Theta_{-t})_* \mathbf{w}_p = \Theta_t^* \mathbf{w}_p$, thus we can write $\mathcal{L}_{\mathbf{v}} \mathbf{w}_p = \frac{d}{dt}|_{t=0} \Theta_t^* \mathbf{w}_p$

Definition 83. We can generalize the **Lie derivative** to act on any (r,s)-tensor field \mathcal{T} by

$$\mathcal{L}_{\mathbf{v}}\mathcal{T}_p := \frac{d}{dt}\big|_{t=0}(\Theta_t^*\mathcal{T})|_p := \lim_{t \rightarrow 0} \frac{\Theta_t^*\mathcal{T}_{\Theta_t(p)} - \mathcal{T}_p}{t},$$

which is still a (r,s)-tensor. As before, Θ is the flow generated by \mathbf{v} .

Proposition 13.3. $\mathcal{L}_{\mathbf{v}}(\mathcal{T} \otimes \mathcal{S}) = \mathcal{L}_{\mathbf{v}}\mathcal{T} \otimes \mathcal{S} + \mathcal{T} \otimes \mathcal{L}_{\mathbf{v}}\mathcal{S}$

Proposition 13.4. $\mathcal{L}_{\mathbf{v}}(\langle \mathcal{T}, \mathcal{S} \rangle) = \langle \mathcal{L}_{\mathbf{v}}\mathcal{T}, \mathcal{S} \rangle + \langle \mathcal{T}, \mathcal{L}_{\mathbf{v}}\mathcal{S} \rangle$

Proposition 13.5. For $f \in C^\infty(M)$, $\mathcal{L}_{\mathbf{v}}(f) = \mathbf{v}(f) = \langle df, \mathbf{v} \rangle$

Proof. $\mathcal{L}_{\mathbf{v}}(f)_p = \frac{d}{dt}\big|_{t=0}(\Theta_t^*f)|_p = \frac{d}{dt}\big|_{t=0}(f \circ \Theta_t)|_p = d(\Theta_t)_p(\frac{d}{dt}\big|_{t=0})(f) = (\Theta_t'(0))(f) = \mathbf{v}f$ □

14 Exterior derivative

Definition 84. Let $\omega = \sum_{(i_1 < \dots < i_k) \in [n]^k} \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(\mathbb{R})$, the **exterior derivative** of ω is $d\omega := \sum_{(i_1 < \dots < i_k) \in [n]^k} d(\omega_{i_1, \dots, i_k}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{\partial \omega_{i_1, \dots, i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^{k+1}(\mathbb{R})$

Proposition 14.1. If $f \in C^\infty(U) = \Omega^0(U)$, then we have $\mathbf{d}f = df \in \Omega^1(U)$

Proposition 14.2. $\mathbf{d} : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$ is linear

Proposition 14.3. $\mathbf{d} \circ \mathbf{d} : \Omega^k(U) \rightarrow \Omega^{k+2}(U)$ is the zero map.

Proposition 14.4. $\forall \omega \in \Omega^k(U), \eta \in \Omega^l(U)$, we have $\mathbf{d}(\omega \wedge \eta) = \mathbf{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathbf{d}\eta$

Proposition 14.5. For any smooth map $F : U \rightarrow V$, and any alternating k -form $\omega \in \Omega^k(V)$, we always have $\mathbf{d}(F^*\omega) = F^*(\mathbf{d}\omega)$

Theorem 14.6. For any smooth manifold M and $k \in \mathbb{N}$, there is a unique linear map $\mathbf{d} : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ such that

1. If $f \in C^\infty(M) = \Omega^0(M)$, then we have $\mathbf{d}f = df \in \Omega^1(M)$
2. $\mathbf{d} \circ \mathbf{d} : \Omega^k(M) \rightarrow \Omega^{k+2}(M)$ is the zero map.
3. $\forall \omega \in \Omega^k(M), \eta \in \Omega^l(M)$, we have $\mathbf{d}(\omega \wedge \eta) = \mathbf{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathbf{d}\eta$