PMATH343 A5

YIXING GU

Question 1

Consider any bases $\{v_i\}_1^n \subseteq \mathcal{H}_1, \{w_j\}_1^m \subseteq \mathcal{K}_1, \{u_k\}_1^o \subseteq \mathcal{H}_2, \{t_l\}_1^p \subseteq \mathcal{K}_2$, we know that $\{v_i \otimes w_j\}_{ij} \subseteq \mathcal{H}_1 \otimes \mathcal{K}_1, \{u_k \otimes t_l\}_{kl} \subseteq \mathcal{H}_2 \otimes \mathcal{K}_2$ are also bases.
By definition of adjoint operators,

$$\begin{aligned}
\langle v_i \otimes w_j, (S \otimes T)^*(u_k \otimes t_l)\rangle &= \langle (S \otimes T)(v_i \otimes w_j), u_k \otimes t_l\rangle \\
&= \langle Sv_i \otimes Tw_j, u_k \otimes t_l\rangle \\
&= \langle Sv_i, u_k\rangle \langle Tw_j, t_l\rangle \\
&= \langle v_i, S^*u_k\rangle \langle w_j, T^*t_l\rangle \\
&= \langle v_i \otimes w_j, S^*u_k \otimes T^*t_l\rangle \\
&= \langle v_i \otimes w_j, (S^* \otimes T^*)(u_k \otimes t_l)\rangle
\end{aligned}$$

Since $\{v_i \otimes w_j\}_{ij} \subseteq \mathcal{H}_1 \otimes \mathcal{K}_1$ is a basis, $\forall x \in \mathcal{H}_1 \otimes \mathcal{K}_1, \langle x, (S \otimes T)^*(u_k \otimes t_l)\rangle = \langle x, (S^* \otimes T^*)(u_k \otimes t_l)\rangle$
We have seen that this is true iff $(S \otimes T)^*(u_k \otimes t_l) = (S^* \otimes T^*)(u_k \otimes t_l)$.
Since $\{u_k \otimes t_l\}_{kl} \subseteq \mathcal{H}_2 \otimes \mathcal{K}_2$ is a basis, $\forall y \in \mathcal{H}_2 \otimes \mathcal{K}_2, (S \otimes T)^*y = (S^* \otimes T^*)y$
Thus $(S \otimes T)^* = S^* \otimes T^*$
Suppose S and T are unitary,
$(S \otimes T)(S \otimes T)^* = (S \otimes T)(S^* \otimes T^*) = SS^* \otimes TT^* = \mathbf{1} \otimes \mathbf{1} = \mathbf{1}$
Thus $S \otimes T$ is unitary as well.

Question 2

Lemma: Let $\mathcal{B} = \{v_i\}_1^n \subseteq \mathcal{H}, \mathcal{C} = \{w_j\}_1^m \subseteq \mathcal{K}$ be orthornormal bases of finite dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}$, respectively. Let $S \in Lin(\mathcal{H}, \mathcal{K}), A = [S]_{\mathcal{B}',\mathcal{B}}$, and $\phi : Lin(\mathcal{H}, \mathcal{K}) \to \mathcal{H} \otimes \mathcal{K}^*$ be the Frobenius Reciprocity natural isomorphism, then $\phi(S) = \sum_{ij} A_{ji} |w_j\rangle \otimes \langle v_i|$.

Proof. Notice that since the bases are orthonormal, $v_i = |v_i\rangle, w_j = |v_j\rangle, v^i = \langle v_i|$

Since $w_j \otimes v^i$ form a basis for $\mathcal{H} \otimes \mathcal{K}^*$, we must have $\phi(S) = \sum_{ij} c_{ij} w_j \otimes v^i$

$$\begin{aligned}
A_{lm} &= ([Sv_m]_{\mathcal{B}'})_l \\
&= ([\phi^{-1}(\sum_{ij} c_{ij} w_j \otimes v^i)v_m]_{\mathcal{B}'})_l \\
&= ([\sum_{ij} c_{ij} \phi^{-1}(w_j \otimes v^i)v_m]_{\mathcal{B}'})_l \\
&= ([\sum_{ij} c_{ij} (w_j \otimes v^i)v_m]_{\mathcal{B}'})_l \\
&= ([\sum_{ij} c_{ij} v^i(v_m) w_j]_{\mathcal{B}'})_l \\
&= ([\sum_{ij} c_{ij} \delta_{im} w_j]_{\mathcal{B}'})_l \\
&= ([\sum_{j} c_{mj} w_j]_{\mathcal{B}'})_l \\
&= c_{ml}
\end{aligned}$$

Thus $\phi(S) = \sum_{ij} A_{ji} w_j \otimes v^i = \sum_{ij} A_{ji} |w_j\rangle \otimes \langle v_i|$ ∎

$$\begin{aligned}
\langle S, T\rangle_F &= \langle \phi(S), \phi(T)\rangle_{\mathcal{H}\otimes\mathcal{K}^*} \\
&= \left\langle \sum_{ij} A_{ji} |w_j\rangle \otimes \langle v_i|, \sum_{kl} B_{kl} |w_k\rangle \otimes \langle v_l| \right\rangle_{\mathcal{H}\otimes\mathcal{K}^*} \\
&= \sum_{ij} \bar{A_{ji}} \sum_{kl} B_{kl} \langle |w_j\rangle \otimes \langle v_i|, |w_k\rangle \otimes \langle v_l|\rangle_{\mathcal{H}\otimes\mathcal{K}^*} \\
&= \sum_{ij}\sum_{kl} \bar{A_{ji}} B_{kl} \langle |w_j\rangle, |w_k\rangle\rangle_{\mathcal{H}} \langle \langle v_i|, \langle v_l|\rangle_{\mathcal{K}^*} \\
&= \sum_{ij}\sum_{kl} \bar{A_{ji}} B_{kl} \langle w_j, w_k\rangle_{\mathcal{H}} \langle v_l, v_i\rangle_{\mathcal{K}} \\
&= \sum_{ij}\sum_{kl} \bar{A_{ji}} B_{kl} \delta_{jk}\delta_{li} \\
&= \sum_{ij} \bar{A_{ji}} B_{ji} \\
&= \sum_{ij} \bar{A_{ij}} B_{ij}
\end{aligned}$$

Question 3

(a) Let $\{v_i\}_i^n \subseteq \mathcal{H}_B$ be a basis, notice that by assumption $n = \dim(\mathcal{H}_B) \geq 2$.
Notice that if we fix $|\eta\rangle \in \mathcal{H}_A$, then

$$
\begin{aligned}
|\{[|\eta\rangle\,|\xi\rangle] : |\xi\rangle \in \mathcal{H}_B\}| &= |\{[|\xi\rangle] : |\xi\rangle \in \mathcal{H}_B\}| \\
&\text{since } [|\eta\rangle\,|\xi\rangle] = [|\eta\rangle\,|z\rangle] \iff \exists c \in \mathbb{C}, |\eta\rangle\,|\xi\rangle = c\,|\eta\rangle\,|z\rangle \iff |\xi\rangle = c\,|z\rangle \iff [|\xi\rangle] = [|z\rangle] \\
&= \left|\left\{\left[\sum_{i=1}^n a_i v_i\right] : (a_i)_1^n \in \mathbb{C}^n\right\}\right| \\
&\geq \left|\left\{\left[\sum_{i=1}^n a_i v_i\right] : (a_i)_1^n \in \mathbb{C}^n, a_i \neq 0\right\}\right| \\
&= \left|\left\{(a_i)_1^n \in \mathbb{C}^n : a_1 > 0 \in \mathbb{R}, \sum_{i=1}^n |a_i|^2 = 1\right\}\right| \\
&\text{by normalizing and taking the global phase so that } a_1 \text{ is fixed on positive real line} \\
&= \left|\left\{a_1 \in \mathbb{R}^+, (a_i)_2^n \in \mathbb{C}^n : a_1^2 = 1 - \sum_{i=2}^n |a_i|^2\right\}\right| \\
&\geq \left|\left\{a_1 \in \mathbb{R}^+ : a_1^2 \leq 1\right\}\right| \text{ there is at least one by taking } a_2 = \sqrt{1-a_1^2} \\
&= |\{a \in \mathbb{R} : 0 < a \leq 1\}|
\end{aligned}
$$

Thus infinite.

(b) Notice that the result state we want to get after locally measuring $\mathcal{H}_A$ is $\frac{\langle u_i, \phi_A\rangle}{|\langle u_i, \phi_A\rangle|}\,|u_i\rangle\,|\phi_B\rangle$
From a), even if we fix $|\eta\rangle = \frac{\langle u_i, \phi_A\rangle}{|\langle u_i, \phi_A\rangle|}\,|u_i\rangle$, there are still infinitely many outcome states (modular global phase) in the above form if we don't know $|\phi_B\rangle$.
However, fix any basis of $\mathcal{H}_A \otimes \mathcal{H}_B$, it is finite since both are finite dimensional, and the possible outcome of measuring against this basis must be one of the elements in this basis, modular global phase, thus only finitely many possible outcome states.

Question 4

let $\mathcal{B} = \{e_1, e_2\}$ denote the standard basis.

Easy to see that $[P^2]_{\mathcal{B},\mathcal{B}} = [P]_{\mathcal{B},\mathcal{B}}[P]_{\mathcal{B},\mathcal{B}} = [P]_{\mathcal{B},\mathcal{B}} = [P]^*_{\mathcal{B},\mathcal{B}} = [P^*]_{\mathcal{B},\mathcal{B}}$ all have matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, thus $P^2 = P = P^*$, and P is indeed a self-adjoint projection.

For the sake of simplicity, for a vector $v \in \mathbb{C}^2$, we will write the $[v]_{\mathcal{B}}$ indexing simply as v, and for a linear map T, we will write $[T]_{\mathcal{B},\mathcal{B}}$ as $[T]$.

Consider any $ce_2 = \begin{pmatrix} 0 \\ c \end{pmatrix} \in span\{e_2\}, P(ce_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$

Thus $span\{e_2\} \subseteq \ker(P)$

Consider any $c_1e_1 + c_2e_2 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{C}^2, P(c_1e_1 + c_2e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} = c_1e_1 \in span\{e_1\}$

Thus $Img(P) \subseteq span\{e_1\}$.

Since this works for all $c_1 \in \mathcal{C}$, thus given any $c_1e_1 \in span\{e_1\}$, if we take $c_2 = 0$, we have that $P(c_1e_1) = c_1e_1$, thus $span\{e_1\} \in Img(P) \implies Img(P) = span\{e_1\} = W$

Notice that $P(c_1e_1 + c_2e_2) = c_1e_1 = 0 \implies c_1 = 0 \implies c_1e_1 + c_2e_2 = c_2e_2 \in span\{e_2\}$

Thus $\ker(P) \subseteq span\{e_2\} \implies \ker(P) = span\{e_2\}$

Now consider any other projection Q, s.t. $\mathrm{Im}(Q) = W, \ker(Q) = span\{e_2\}$.

$[Q] = \begin{pmatrix} Q(e_1) & Q(e_2) \end{pmatrix} = \begin{pmatrix} ce_1 & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$

since $e_2 \in span\{e_2\} = \ker(Q), Q(e_1) \in \mathrm{Im}(Q) = span\{e_1\}$

For it to be a projection, we will need

$[Q^2] = [Q][Q] = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c^2 & 0 \\ 0 & 0 \end{pmatrix} = [Q] = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ Thus $c = 0$ or $c = 1$.

Since $c = 0$ will make Q the zero map, which is not a projection, thus $c = 1$

and $[Q] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = [P] \implies Q = P$ is unique.

Now consider P' to be the projection represented by $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

For any $c_1e_1 + c_2e_2 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{C}^2$,

$P'(c_1e_1 + c_2e_2) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ 0 \end{pmatrix} = (c_1 - c_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (c_1 - c_2)e1 \in span\{e_1\} = W$

Thus $Img(P') \subseteq W$

If we are given any $cw \in W$, pick $c_1 = c, c_2 = 0 \in \mathbb{C}$, we have $P'(c_1e_1) = (c_1 - 0)w = cw$

Thus $W \subseteq Img(P') \implies Img(P') = W$

And $P'(c_1e_1 + c_2e_2) = (c_1 - c_2)w = 0 \iff c_1 = c_2 \iff c_1e_1 + c_2e_2 = c_1(e_1 + e_2) \in span\{e_1 + e_2\}$.

Thus $\ker(P') = span\{e_1 + e_2\}$

In addition $[(P')^2] = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = [P']$.

Thus $P' = p'^2$ is indeed a projection.

Question 5

(a) We first show 1) implies 2)

Given $v = \sum_{i=1}^k v_i, w = \sum_{j=1}^k w_j \in \mathcal{H}$ be direct decomposition, where $v_i, w_i \in \mathcal{H}_i$.

By sesqui-linearity of inner product, $\langle v, w\rangle = \left\langle \sum_{i=1}^k v_i, \sum_{j=1}^k w_j \right\rangle = \sum_{ij} \langle v_i, w_j\rangle$.

If $i \neq j$, since $\mathcal{H}_i \perp \mathcal{H}_j$ by 1), we have $\langle v_i, w_j\rangle = 0$.

Thus $\langle v, w\rangle = \left\langle \sum_{i=1}^k v_i, \sum_{j=1}^k w_j \right\rangle = \sum_{i=1}^k \langle v_i, w_i\rangle$

(b) Then we will show 2) implies 1)

Given any $1 \leq i \neq j \leq k$, consider any $v \in H_i, w \in H_j$.

Since the direct sum decomposition is unique, we have $v = \sum_{l=1}^k v_l, w = \sum_{l=1}^k w_l$, where $v_i = v, w_j = w$ and all the others 0.

By 2), $\langle v, w\rangle = \sum_{l=1}^k \langle v_l, w_l\rangle = \sum_{l\neq i,j} \langle 0, 0\rangle + \langle v_i, 0\rangle + \langle 0, w_j\rangle = 0$ since $i \neq j$.

(c) 1) implies 5)

Pick any orthonormal basis $\mathcal{B}_i = \{v_{ij}\}_{j=1}^{n_i}$ for each $\mathcal{H}_i$ with $\dim(\mathcal{H}_i) = n_i$.

Since $\mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i$, for any $v \in \mathcal{H}$, there is a unique decomposition $v = \sum_{i=1}^k v_i$, where $v_i = \sum_{j=1}^{n_i} c_{ij} v_{ij} \in \mathcal{H}_i$.

Thus $v = \sum_{i=1}^k v_i = \sum_{i=1}^k \sum_{j=1}^{n_i} c_{ij} v_{ij}$

Thus $\mathcal{B} := \bigcup \mathcal{B}_i$ spans $\mathcal{H}$

In addition, for any $\sum_{i=1}^k \sum_{j=1}^{n_i} c_{ij} v_{ij} = 0$, the unique direct sum decomposition of 0 will give $0 = \sum_{i=1}^k v_i$, where each $v_i = \sum_{j=1}^{n_i} c_{ij} v_{ij} = 0 \in \mathcal{H}$, and since $\mathcal{B}_i$ is a basis, thus linearly independent, $c_{ij} = 0$ for all j.

Since this works for all i, we have that all $c_{ij} = 0$. Thus $\mathcal{B}$ is linearly independent.

Thus $\mathcal{B}$ is a basis for $\mathcal{H}$.

Consider any two $v_{i_1 j_1} \in \mathcal{H}_{i_1}, v_{i_2 j_2} \in \mathcal{H}_{i_2}$. If $i_i \neq i_2$, then $\langle v_{i_1 j_1}, v_{i_2 j_2}\rangle = 0$ since $H_{i_1} \perp H_{i_2}$.

If $i_1 = i_2 = i$, then since $\mathcal{B}_i$ is picked to be an orthonormal basis, $\langle v_{ij_1}, v_{ij_2}\rangle = \delta_{j_1,j_2}$

Thus $\langle v_{i_1 j_1}, v_{i_2 j_2}\rangle = \delta_{i_1,i_2}\delta_{j_1,j_2}$

Thus $\mathcal{B}$ is an orthonormal basis.

Consider any $v \in \mathcal{B}_i \cap \mathcal{B}_j, i \neq j$.

Then $v \in \mathcal{B}_i \subseteq \mathcal{H}_i, v \in \mathcal{B}_j \subseteq \mathcal{H}_j$.

By 1) $H_{i_1} \perp H_{i_2}$, thus $\langle v, v\rangle = 0 \implies v = 0$. However, we picked it to be in a basis, thus a contradiction, thus $\mathcal{B}_i \cap \mathcal{B}_j = \phi$

Thus we have proved 5)

(d) 5) implies 1)

Given any $v \in \mathcal{H}$,

Pick any orthonormal basis $\mathcal{B}_i = \{v_{ij}\}_{j=1}^{n_i}$ for each $\mathcal{H}_i$ with $\dim(\mathcal{H}_i) = n_i$.

Since $\mathcal{B} := \bigcup \mathcal{B}_i$ forms a basis for $\mathcal{H}$, and $i \neq j \implies \mathcal{B}_i \cap \mathcal{B}_j = \phi$, by theorem 11.1.1, $\mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i$.

And for any $v = \sum_{k=1}^{n_i} v_{ik} \in \mathcal{H}_i, w = \sum_{k=1}^{n_j} v_{jk} \in \mathcal{H}_j, i \neq j$,

$\langle v, w\rangle = \langle \sum_{k=1}^{n_i} v_{ik}, \sum_{l=1}^{n_j} v_{jl}\rangle = \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \langle v_{ik}, v_{jl}\rangle = \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \delta_{ij}\delta_{kl} = 0$ since $\mathcal{B}$ is an orthornormal basis.

Thus $\mathcal{H}_i \perp \mathcal{H}_j$

Question 6

$(|\xi_i\rangle\langle\xi_i|)^* = |\xi_i\rangle\langle\xi_i|\,, (|\xi_i\rangle\langle\xi_i|)^2 = |\xi_i\rangle\langle\xi_i|\,|\xi_i\rangle\langle\xi_i| = |\xi_i\rangle\langle\xi_i|$

Thus self-adjoint projections.

Notice that $|\xi_i\rangle\langle\xi_i|\,|\xi_j\rangle\langle\xi_j| = \langle\xi_i,\xi_j\rangle\,|\xi_i\rangle\langle\xi_j| = \delta_{ij}\,|\xi_i\rangle\langle\xi_j| = \delta_{ij}\,|\xi_i\rangle\langle\xi_i|$

Thus indeed piecewise-orthogonal projections.

$\forall v = \sum_{j=1}^n c_j\,|\xi_j\rangle\,,$

$(\sum_{i=1}^n |\xi_i\rangle\langle\xi_i|)(v) = (\sum_{i=1}^n |\xi_i\rangle\langle\xi_i|)(\sum_{j=1}^n c_j\,|\xi_j\rangle) = \sum_{i=1}^n\sum_{j=1}^n c_j\,|\xi_i\rangle\langle\xi_i|\,|\xi_j\rangle = \sum_{i=1}^n\sum_{j=1}^n c_j\delta_{ij}\,|\xi_i\rangle$
$= \sum_{i=1}^n c_i\,|\xi_i\rangle = v \implies \sum_{i=1}^n |\xi_i\rangle\langle\xi_i| = \mathbf{1}$

Thus this is a complete set of pairwise-orthogonal self-adjoint projections.

Question 7

From Problem 1, we know that $(P_i \otimes Q_j)^* = P_i^* \otimes Q_j^* = P_i \otimes Q_j$, since P_i and Q_j are self adjoint projections.

In addition, $(P_i \otimes Q_j)^2 = (P_i \otimes Q_j)(P_i \otimes Q_j) = P_i^2 \otimes Q_j^2 = P_i \otimes Q_j$.

Thus they are indeed self-adjoint projections.

Consider $(P_{i_1} \otimes Q_{j_1})(P_{i_2} \otimes Q_{j_2}) = P_{i_1}P_{i_2} \otimes Q_{j_1}Q_{j_2} = \delta_{i_1,i_2}P_{i_1} \otimes \delta_{j_1,j_2}Q_{j_1} = \delta_{i_1,i_2}\delta_{j_1,j_2}P_{i_1} \otimes Q_{j_1}$

Thus piece-wise orthonormal.

$\sum_{i,j} P_i \otimes Q_j = (\sum_i p_i) \otimes (\sum_j Q_j) = \mathbf{1}_A \otimes \mathbf{1}_B = \mathbf{1}_{A\otimes B}$

Thus indeed complete set of orthornormal self-adjoint projections.

By Axiom 6, we can define a projection measurement by this.

Question 8

Since $|\xi_i\rangle |\eta_j\rangle$ form an ONB for $\mathcal{H}_A \otimes \mathcal{H}_B$, given any $v \in \mathcal{H}_A \otimes \mathcal{H}_B$, it can be represented as $v = \sum_{ij} a_{ij} |\xi_i\rangle |\eta_j\rangle$

For projective measurement, we have the possible outcomes in $O_1 := \{1, \dots, n\} \times O_2 := \{1, \dots, m\}$

$$\begin{aligned}
P\left[(k,l) \in O_1 \times O_2 | v \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)\right] &= |P_{kl} |v\rangle|^2 \\
&= \left| (|\xi_k\rangle \langle \xi_k| \otimes |\eta_l\rangle \langle \eta_l|)(\sum_{ij} a_{ij} |\xi_i\rangle |\eta_j\rangle) \right|^2 \\
&= \left| \sum_{ij} a_{ij} (|\xi_k\rangle \langle \xi_k| |\xi_i\rangle) \otimes (|\eta_l\rangle \langle \eta_l| |\eta_j\rangle) \right|^2 \\
&= \left| \sum_{ij} a_{ij} \delta_{ki} |\xi_i\rangle \otimes \delta_{lj} |\eta_l\rangle \right|^2 \\
&= |a_{kl} |\xi_k\rangle |\eta_l\rangle|^2 \\
&= |a_{kl}|^2
\end{aligned}$$

And we are left in the state

$$\begin{aligned}
\frac{P_{kl} |v\rangle}{|P_{kl} |v\rangle|} &= \frac{(|\xi_k\rangle \langle \xi_k| \otimes |\eta_l\rangle \langle \eta_l|)(\sum_{ij} a_{ij} |\xi_i\rangle |\eta_j\rangle)}{\left|(|\xi_k\rangle \langle \xi_k| \otimes |\eta_l\rangle \langle \eta_l|)(\sum_{ij} a_{ij} |\xi_i\rangle |\eta_j\rangle)\right|} \\
&= \frac{a_{kl} |\xi_k\rangle |\eta_l\rangle}{|a_{kl} |\xi_k\rangle |\eta_l\rangle|} \\
&= \frac{a_{kl}}{|a_{kl}|} |\xi_k\rangle |\eta_l\rangle
\end{aligned}$$

Now consider the basis measurement.

First notice that the possible outcomes are exactly the same.

$$\begin{aligned}
P\left[(k,l) \in O_1 \times O_2 | v \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)\right] &= |\langle |\xi_k\rangle |\eta_l\rangle , v\rangle|^2 \\
&= \left| \left\langle |\xi_k\rangle |\eta_l\rangle , \sum_{ij} a_{ij} |\xi_i\rangle |\eta_j\rangle \right\rangle \right|^2 \\
&= |a_{kl}|^2
\end{aligned}$$

And we are left in state $\frac{\langle |\xi_k\rangle |\eta_l\rangle, v\rangle}{|\langle |\xi_k\rangle |\eta_l\rangle, v\rangle|} |\xi_k\rangle |\eta_l\rangle = \frac{a_{kl}}{|a_{kl}|} |\xi_k\rangle |\eta_l\rangle$

Thus they are the same.

Question 9

Lemma 1: Given any ONB $\left\{\xi_l^i\right\}_{l=1}^{n_i} \subseteq \mathcal{H}_i, P := \sum_{k=1}^{n_i} \left|\xi_k^i\right\rangle \left\langle\xi_k^i\right|$, then $\forall v \in \mathcal{H}_i, P(v) = v$

Proof. $\forall v \in \mathcal{H}_i$, we have $v = \sum_{l=1}^{n_i} a_l \left|\xi_l^i\right\rangle$

$P_i(v) = \sum_{k=1}^{n_i} \left|\xi_k^i\right\rangle \left\langle\xi_k^i\right| \sum_{l=1}^{n_i} a_l \left|\xi_l^i\right\rangle = \sum_{k=1}^{n_i} \sum_{l=1}^{n_i} a_l \left|\xi_k^i\right\rangle \left\langle\xi_k^i\right| \left|\xi_l^i\right\rangle = \sum_{k=1}^{n_i} \sum_{l=1}^{n_i} a_l \delta_{lk} \left|\xi_k^i\right\rangle$

$= \sum_{k=1}^{n_i} a_k \left|\xi_k^i\right\rangle = v$ □

Lemma 2: Given any ONB $\left\{\xi_l^i\right\}_{l=1}^{n_i} \subseteq \mathcal{H}_i, P := \sum_{k=1}^{n_i} \left|\xi_k^i\right\rangle \left\langle\xi_k^i\right|$, then $\forall w \in \mathcal{H}_j, i \neq j, P(w) = 0$

Proof. $P_i(w) = \sum_{k=1}^{n_i} \left|\xi_k^i\right\rangle \left\langle\xi_k^i\right| |w\rangle = 0$ since $w \in \mathcal{H}_j, \left|\xi_k^i\right\rangle \in \mathcal{H}_i, \mathcal{H}_i \perp \mathcal{H}_j$, the last is because this is a Hilbert space direct sum. □

Consider any other ONB $\left\{\left|\eta_l^i\right\rangle\right\}_{l=1}^{n_i} \subseteq \mathcal{H}_i$

Let $P_i = \sum_{k=1}^{n_i} \left|\xi_k^i\right\rangle \left\langle\xi_k^i\right|, P_i' = \sum_{k=1}^{n_i} \left|\eta_k^i\right\rangle \left\langle\eta_k^i\right|$.

For any $u \in \mathcal{H}$, we can write its unique direct sum decomposition $u = \sum_{k=1}^{n} v_k$.

Then $P_i(u) = P_i(\sum_{k=1}^{n} v_k) = \sum_{k=1}^{n} P_i(v_k) = \sum_{k \neq i} P_i(v_k) + P_i(v_i) = 0 + v_i = v_i$ by lemma 1 and 2.

Notice that since Lemma 1 and 2 does not care about choice of ONB, by the same way,

$P_i'(u) = P_i'(\sum_{k=1}^{n} v_k) = \sum_{k \neq i} P_i'(v_k) + P_i(v_i) = 0 + v_i = v_i$

Thus $P_i(u) = P_i'(u)$ for and u, thus the same.