Pmath465: Smooth Manifold

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1 Topology and manifold

Definition 1. Given any set X, a **topology** is a pair $(X, \mathcal{S}), \mathcal{S} \subseteq \mathcal{P}(X)$ that satisfies:

1. $\emptyset \in \mathcal{S}$
2. If $\forall \alpha, S_\alpha \in \mathcal{S}$, then $\bigcup_\alpha S_\alpha \in \mathcal{S}$
3. If $S_1, \ldots, S_n \in \mathcal{S}$, then $\bigcap_{i=1}^n S_i \in \mathcal{S}$

Definition 2. Given any set X, and a topology $(X, \mathcal{S})$, the elements in $\mathcal{S}$ are called open.

Definition 3. Given any set X, a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ is said to be a basis of the topology $\mathcal{S}$ if

1. $X = \bigcup_{B \in \mathcal{B}}$
2. If $B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2$, then $\exists B_x \in \mathcal{B}, x \in B_x \subseteq B_1 \cap B_2$
3. $\mathcal{S}$ is the collection of all unions of the elements of $\mathcal{B}$.

Definition 4. A topology space $\mathcal{S}$ is called **2nd countable** if it has a countable basis.

Definition 5. A topology space is **Hausdorff** if $\forall x \neq y \in \mathcal{S}, \exists S_x, S_y \in \mathcal{S}, x \in S_x, y \in S_y, S_x \cap S_y = \emptyset$

Definition 6. If X is a topology space, and $Y \subseteq X$, then the subspace topology on Y is obtained by $U \subseteq Y$ is open if and only if $\exists V \subseteq X$ that is open, and $U = V \cap Y$

Proposition 1.1. *If $Y \subseteq X$ with subspace topology, then if X is 2nd countable or Hausdorff, so is Y.*

Definition 7. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$ is called a **cover** for X if $X \subseteq \bigcup_{\alpha \in A} U_\alpha$. A cover is called an open cover if every U_α is open in the topology of X.

Definition 8. A collection of subsets $\mathcal{X}$ is called **locally finite** if $\forall x \in X, \exists S_x \in \mathcal{S}$ an open neighborhood, such that S_x only intersects with finitely many elements in $\mathcal{X}$.

Definition 9. Given two sets X, Y, and there corresponding topology $\mathcal{S}, \mathcal{T}$, a map $f : X \to Y$ is **continuous** if $\forall T \in \mathcal{T}, f^{-1}(T) \in \mathcal{S}$. Namely, for any open set in Y, its preimage of f is also open in X.

Definition 10. Given two sets X, Y, and their corresponding topology $\mathcal{S}, \mathcal{T}$, a continuous map $f : X \to Y$ is a **homeomorphism** if it is invertible, and its inverse function is also continuous.

Remark. A homeomorphism is a map that preserves the topology structure between two sets.

Definition 11. An **atlas** $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_\alpha$ is a collection of **local charts** (U_α, ϕ_α), where each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, and that $\bigcup_\alpha U_\alpha = X$.

Definition 12. A smooth atlas is an atlas such that $\forall (U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in \mathcal{A}$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ is C^∞ smooth.

Definition 13. A **smooth manifold** $M = (\mathcal{S}, \mathcal{A})$ is a 2nd-countable Hausdorff topology with a smooth atlas. The dimension n of the manifold is the dimension of $\mathbb{R}^n$ in the atlas $\mathcal{A}$.

2 Smooth functions and Diffeomorphism

Definition 14. Let M,N be smooth manifolds of dimension m,n, we say a function $F : M \to N$ is **smooth** at point $p \in M$ if and only if there are local charts (U_α, ϕ_α) for M and (V_β, ψ_β) for N, such that:

1. $p \in U_\alpha$
2. $F(p) \in V_\beta$
3. $U_\alpha \cap F^{-1}(V_\alpha) \subseteq M$ is open
4. The **coordinate representation** $\hat{F} := \psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to \mathbb{R}^m$ is smooth at $\phi_\alpha(p) \in \mathbb{R}^n$

Proposition 2.1. *If F is continuous, then 3 is always met.*

Proposition 2.2. *If 4 is met for some coordinate maps, then it is always true for any other coordinate maps. Namely, the smoothness is independent of choices of coordinate maps.*

Definition 15. If F is smooth at every $p \in M$, we say that F is a smooth function.

Definition 16. F is a **diffeomorphism** if F is invertible, and that both F, F^{-1} are smooth.

Proposition 2.3. *A diffeomorphism is always a homeomorphism.*

3 Bump functions

Definition 17. Given a function $f : M \to \mathbb{R}$, the **support** is $\mathrm{Supp}(f) := \overline{\{x \in M | f(x) \neq 0\}}$

Definition 18. Let M be a smooth manifold, $\mathcal{X} = \{X_\alpha\}$ is an open cover of M. A smooth **partition of unity** is $\{\psi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$, such that:

1. $0 \leq \psi_\alpha(x) \leq 1, \forall \alpha \in A, x \in M$
2. $\forall \alpha, \mathrm{Supp}(\psi_\alpha) \subseteq X_\alpha$
3. $\{\mathrm{Supp}(\psi_\alpha)\}_{\alpha \in A}$ is locally finite.
4. $\forall x \in M, \sum_{\alpha \in A} \psi_\alpha(x) = 1$

Theorem 3.1. *A partition of unity always exists for any smooth manifold M and any open cover $\mathcal{X}$ of M.*

Proposition 3.2. *Bump function*
Let $A \subseteq U \subseteq M$, where M is a smooth manifold, A is a closed set, and U is an open set. There exists a smooth map $\psi : M \to \mathbb{R}$, such that

1. $\forall x \in M, 0 \leq \psi(x) \leq 1$
2. $\forall x \in A, \psi(x) = 1$
3. $\mathrm{Supp}(\psi) \subseteq U$

Definition 19. If A is closed, then a function $f : A \to \mathbb{R}^k$ is **smooth** if $\forall x \in A, \exists W_x \subseteq M, F_x : W_x \to \mathbb{R}^k$, such that W_x is open and $x \in W_x$, and $\forall p \in W_x \cap A, F_x(p) = f(p), F_x \in C^\infty(W_x)$

Lemma 3.3. *Extension Lemma:*
Let M be a smooth manifold, and $A \subseteq M$ be a closed subset. Let $f : A \to \mathbb{R}^k$ be smooth, then $\forall U \supseteq A$ be open, $\exists \tilde{f} : M \to \mathbb{R}^k$, such that $\tilde{f}$ is smooth, and $\tilde{f}|_A = f, \mathrm{Supp}(\tilde{f}) \subseteq U$

4 Tangent space and derivatives

Definition 20. Let $C^\infty(M)$ be the real vector space of smooth functions from $M \to \mathbb{R}$, a **derivation** or **tangent vector** at $p \in M$ is an $\mathbb{R}-$linear map $D : C^\infty(M) \to \mathbb{R}$ satisfying the **Leibniz condition**: $\forall f, g \in C^\infty(M), D(fg) = f(p)D(g) + g(p)D(f)$

Definition 21. The **tangent space** to M at $p \in M$, T_pM, is the set of all tangent vectors at p.

Proposition 4.1. *T_pM is a real vector space where $\forall X, Y \in T_pM, c \in \mathbb{R}, f \in C^\infty(M), (X + Y)(f) := X(f) + Y(f), (cX)(f) := c \cdot X(f), (-X)(f) := -X(f), 0(f) := 0$*

Proposition 4.2. *Let $D \in T_pM$, if $\forall x \in M, f(x) = c$ is a constant function, then $D(f) = 0$*

Proposition 4.3. *If $\exists U \ni p$ be open, and $\forall x \in U, f(x) = 0$, then $\forall D \in T_pM, D(f) = 0$*

Proposition 4.4. *If $f, g \in C^\infty(M)$ and $f = g$ on some open $U \ni p$, then $\forall D \in T_pM, D(f) = D(g)$*

Definition 22. For $p, v = (v^1, \ldots v^n) \in \mathbb{R}^n$, then the **directional derivative** of $f \in C^\infty(\mathbb{R}^n)$ in the direction of v is $D_v(f) := \left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}|_p\right)(f) := \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p)$

Proposition 4.5. *The directional derivative is a derivation.*

Proposition 4.6. *$\forall p \in \mathbb{R}^n$, the map $L : \mathbb{R}^n \to T_p\mathbb{R}^n$, where $L(v) := D_v$ is a vector space isomorphism.*

Corollary 4.7. *$\{\partial_1|_p, \ldots, \partial_n|_p\}$ is a basis for $T_p\mathbb{R}^n$.*

Definition 23. Given a smooth function $F : M \to N$, the **differential** or **derivative** of F at $p \in M$ is the map $dF_p : T_pM \to T_{F(p)}N$, given by $\forall D \in T_pM, f \in C^\infty(N), dF_p(D)(f) := D(f \circ F)$

Remark. One can check that $dF_p(D) \in T_{F(p)}N$ for any $D \in T_pM$, and that $f \circ F \in C^\infty(M)$, so the derivative of F above is well-defined.

Proposition 4.8. *Let M, N, R be smooth manifolds and $F : M \to N, G : N \to R$ are smooth maps, then for any $p \in M$, we have:*

1. *$dF_p : T_pM \to T_{F(p)}N$ is linear*
2. *$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G(F(p))}R$*
3. *$d(Id_M)_p : T_pM \to T_pM$ is an identity isomorphism.*
4. *If F is a diffeomorphism, then dF_p is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$*

Proposition 4.9. *Let M be a smooth manifold, $U \subseteq M$ be open, and $i : U \to M$ be the inclusion map, then $\forall p \in U, di_p : T_pU \to T_pM$ is an isomorphism.*

Corollary 4.10. *Let M be a n-dimensional smooth manifold, for any $p \in M$, and any local chart (U, ϕ) containing p, we have $T_pM \cong T_pU \cong T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n$, and the n-dimensional vector space T_pM has a basis of $\left\{\Upsilon_j|_p := \partial_j|_p := \frac{\partial}{\partial x^j}|_p := (di_p \circ (d\phi_p)^{-1})(\frac{\partial}{\partial x^j}|_{\phi(p)}) = (di_p \circ (d\phi_p)^{-1})(\partial_j|_{\phi(p)})\right\}_j$.*

Proposition 4.11.

$$\forall f \in C^\infty(M), \partial_j|_p(f) = \Upsilon_j|_p(f) = \frac{\partial(f \circ \phi^{-1})}{\partial x^j}(\phi(p))$$

Proof.

$$\begin{aligned}
\Upsilon_j|_p(f) &= (di_p \circ (d\phi_p)^{-1})(\partial_j|_{\phi(p)})(f) \\
&= (d\phi_p)^{-1}(\partial_j|_{\phi(p)})(f \circ \mathbf{i}) \\
&= d(\phi^{-1})_{\phi(p)}(\partial_j|_{\phi(p)})(f \circ \mathbf{i}) \\
&= (\partial_j|_{\phi(p)})(f \circ \mathbf{i} \circ \phi^{-1})(\phi(p)) \\
&= \frac{\partial(f \circ \phi^{-1})}{\partial x^j}(\phi(p))
\end{aligned}$$

□

Corollary 4.12. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, then*

$$\partial_j|_p(x^i) = \frac{\partial x^i}{\partial x^j}(p) = \frac{\partial(x^i \circ \phi^{-1})}{\partial x^j}(\phi(p)) = \delta^i_j$$

Theorem 4.13. *Let $F : M \to N$ be smooth, (U, ϕ) and (V, ψ) be local charts for M and N, such that $p \in U, F(p) \in V$, if we choose the basis $\{\partial_j|_p\}_j, \{\partial_i|_{F(p)}\}_i$ associated to (U, ϕ) and (V, ψ) , we have that $[dF_p]_{ij} = \frac{\partial \hat{F}^i}{\partial x^j}(\phi(p))$. Namely, $dF_p(\partial_j|_p) = \sum_{i=1}^{\dim(M)} \frac{\partial \hat{F}^i}{\partial x^j}(\phi(p))\partial_i|_{F(p)} \in T_{F(p)}N$, where $\hat{F} = \psi \circ F \circ \phi^{-1}$ is the coordinate representation of F.*

Definition 24. The **tangent bundle** $TM := \bigsqcup_{p \in M} T_pM$.

Proposition 4.14. *If M is a n-dimension smooth manifold, then TM is a 2n-dimension smooth manifold.*

5 Vector field

Definition 25. A **vector field** is a smooth function $\mathbf{v} : M \to TM$, such that $\forall p, \mathbf{v}_p := \mathbf{v}(p) \in T_pM$

Definition 26. Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, then we can always write $\mathbf{v}_p = \sum_{i=1}^n \mathbf{v}^i(p)\Upsilon_i|_p$, since $\left\{\Upsilon_i|_p = \frac{\partial}{\partial x^i}|_p\right\}$ is a basis for T_pM. The functions $\mathbf{v}^i : M \to \mathbb{R}$ are called the **component functions**.

Definition 27. Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, the **partial derivatives** $\Upsilon_i = \partial_i : U \to TM$ is given by $\Upsilon_i(p) = \Upsilon_i|_p$, where $\left\{\Upsilon_i|_p = \frac{\partial}{\partial x^i}|_p\right\}$ is a basis for T_pM associated to $(U, \phi = (x^1, \ldots, x^n))$. One can check that $\Upsilon_i \in \mathfrak{X}(M)$

Proposition 5.1. *Given $p \in M, \boldsymbol{u} \in T_pM$, and some open $U \subseteq M$ that contains p, there is a vector field $\boldsymbol{v}$ on M, such that $\boldsymbol{v}|_p = \boldsymbol{u}$, and $\mathrm{Supp}(\boldsymbol{v}) \subseteq U$.*

Definition 28. $\mathfrak{X}(M)$ is the set of all vector fields.

Proposition 5.2. *$\mathfrak{X}(M)$ is a real vector space.*

Definition 29. Given a smooth function $f \in C^\infty(M)$, and a vector field $\mathbf{v} \in \mathfrak{X}(M)$, we define $f\mathbf{v} := f \cdot \mathbf{v} \in \mathfrak{X}(M)$ to be $(f\mathbf{v})(p) := f(p)\mathbf{v}_p \in T_pM$

Proposition 5.3. *The above definition of $\cdot : C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ gives a $C^\infty(M)-$module of the vector fields.*

Corollary 5.4. *We can thus write any vector field $\boldsymbol{v} \in \mathfrak{X}(M)$ as*

$$\boldsymbol{v} = \sum_{i=1}^n \boldsymbol{v}^i \Upsilon_i$$

In particular, let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, we have $\boldsymbol{v}|U = \sum_{i=1}^n \boldsymbol{v}^i \partial_i$

Theorem 5.5. ***Canonical form of vector field***
Let $\boldsymbol{v} \in \mathfrak{X}(M)$, and $V_p \neq 0$ for some $p \in M$, then there is a coordinate chart $(U, \phi = (x^1, \ldots, x^n))$ for M, such that $p \in U, \boldsymbol{v}|_U = \frac{\partial}{\partial x^1} = \Upsilon_1$

Definition 30. Given a smooth function $f \in C^\infty(M)$, and a vector field $\mathbf{v} \in \mathfrak{X}(M)$, we define $\mathbf{v}(f) \in C^\infty(M)$ to be $\mathbf{v}(f)(p) := \mathbf{v}_p(f) \in \mathbb{R}$. Thus we can view a vector field $\mathbf{v}$ as a function $C^\infty(M) \to C^\infty(M)$ as well.

Proposition 5.6. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, given any function $f \in C^\infty(M)$, we have $\partial_j(f) \circ \phi^{-1} = \frac{\partial f \circ \phi^{-1}}{\partial x^i}$*

Proof. Consider any $p \in U, \phi(p) \in \phi(U) \subseteq \mathbb{R}^n$

$$\begin{aligned}(\partial_j(f) \circ \phi^{-1})(\phi(p)) &= \partial_j(f)(p) \\ &= \partial_j|_p(f) \\ &= \frac{\partial f \circ \phi^{-1}}{\partial x^i}(\phi(p))\end{aligned}$$

□

Proposition 5.7. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, given any function $f \in C^\infty(M)$, we have*

$$\partial_i(\partial_j(f)) \circ \phi^{-1} = \frac{\partial^2(f \circ \phi^{-1})}{\partial x^j x^i}$$

Proof. Consider any $p \in U$,

$$\begin{aligned}\partial_i(\partial_j(f))|_p &= \frac{\partial(\partial_j(f) \circ \phi^{-1})}{\partial x^j}(\phi(p)) \\ &= \frac{\partial \frac{\partial f \circ \phi^{-1}}{\partial x^i}}{\partial x^j}(\phi(p)) \\ &= \frac{\partial^2(f \circ \phi^{-1})}{\partial x^j x^i}(\phi(p)) \\ &= \frac{\partial^2(f \circ \phi^{-1})}{\partial x^i x^j}(\phi(p)) \\ &= \partial_j(\partial_i(f))|_p\end{aligned}$$

□

Corollary 5.8. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, given any function $f \in C^\infty(M)$,*

$$\partial_i(\partial_j(f)) = \partial_j(\partial_i(f))$$

Proof. Consider any $p \in U, \partial_i(\partial_j(f))|_p = \frac{\partial^2(f \circ \phi^{-1})}{\partial x^j x^i}(\phi(p)) = \frac{\partial^2(f \circ \phi^{-1})}{\partial x^i x^j}(\phi(p)) = \partial_j(\partial_i(f))|_p$ □

6 Lie Bracket

Definition 31. Given two vector fields $\mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$, the **Lie Bracket** is $[\mathbf{v}, \mathbf{w}] := \mathbf{v} \circ \mathbf{w} - \mathbf{w} \circ \mathbf{v}$

Proposition 6.1. $[\boldsymbol{v}, \boldsymbol{w}] \in \mathfrak{X}(M)$ *is a vector field.*

Proposition 6.2. $[\boldsymbol{v}, \boldsymbol{w}]$ *is bilinear.*

Proposition 6.3. $[\boldsymbol{v}, \boldsymbol{w}] = -[\boldsymbol{w}, \boldsymbol{v}]$ *is anti-symmetric*

Proposition 6.4. $[\boldsymbol{v}, \boldsymbol{w}]$ *satisfies the Jacobian Identity:* $[\boldsymbol{u}, [\boldsymbol{v}, \boldsymbol{w}]] + [\boldsymbol{w}, [\boldsymbol{u}, \boldsymbol{v}]] + [\boldsymbol{v}, [\boldsymbol{w}, \boldsymbol{u}]] = 0$

Proposition 6.5. *For any $f, g \in C^\infty(M), \boldsymbol{v}, \boldsymbol{u} \in \mathfrak{X}(M)$, we have* $[f\boldsymbol{v}, g\boldsymbol{u}] = fg[\boldsymbol{v}, \boldsymbol{u}] + f(\boldsymbol{v}g)\boldsymbol{u} - g(\boldsymbol{u}f)\boldsymbol{v}$

Proposition 6.6. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, then for any two vector fields $\boldsymbol{v} = \sum_{i=1}^n \boldsymbol{v}^i \partial_i, \boldsymbol{u} = \sum_{j=1}^n \boldsymbol{u}^j \partial_j \in \mathfrak{X}(M)$, we have*

$$[\boldsymbol{u}, \boldsymbol{v}] = \sum_{j=1}^n \left(\boldsymbol{u}\boldsymbol{v}^j - \boldsymbol{v}\boldsymbol{u}^j\right) \partial_j = \left(\boldsymbol{u}^i \partial_i \boldsymbol{v}^j - \boldsymbol{v}^i \partial_i \boldsymbol{u}^j\right) \partial_j$$

Proof. Consider any function $f \in C^\infty(M)$

$$\begin{aligned}
[\mathbf{u},\mathbf{v}](f) &= \mathbf{u}^j\partial_j(\mathbf{v}^i\partial_i(f)) - \mathbf{v}^i\partial_i(\mathbf{u}^j\partial_j(f)) \\
&= \mathbf{u}^j(\mathbf{v}^i\partial_j(\partial_i(f)) + \partial_j(\mathbf{v}^i)\partial_i(f)) - \mathbf{v}^i(\mathbf{u}^j\partial_i(\partial_j(f)) + \partial_i(\mathbf{u}^j)\partial_j(f)) \\
&= \mathbf{u}^j\mathbf{v}^i\partial_j(\partial_i(f)) + \mathbf{u}^j\partial_j(\mathbf{v}^i)\partial_i(f) - \mathbf{v}^i\mathbf{u}^j\partial_j(\partial_i(f)) - \mathbf{v}^i\partial_i(\mathbf{u}^j)\partial_j(f) \\
&= \mathbf{u}^j\partial_j(\mathbf{v}^k)\partial_k(f) - \mathbf{v}^i\partial_i(\mathbf{u}^k)\partial_k(f) \\
&= (\mathbf{u}^j\partial_j\mathbf{v}^k - \mathbf{v}^i\partial_i\mathbf{u}^k)\partial_k(f) \\
&= ((\mathbf{u}^i\partial_i\mathbf{v}^k - \mathbf{v}^i\partial_i\mathbf{u}^k)\partial_k)(f) \\
[\mathbf{u},\mathbf{v}] &= (\mathbf{u}^i\partial_i\mathbf{v}^k - \mathbf{v}^i\partial_i\mathbf{u}^k)\partial_k \\
&= ((\mathbf{u}^i\partial_i)\mathbf{v}^k - (\mathbf{v}^i\partial_i)\mathbf{u}^k)\partial_k \\
&= (\mathbf{u}\mathbf{v}^k - \mathbf{v}\mathbf{u}^k)\partial_k
\end{aligned}$$

□

7 Curve and Flow

Definition 32. Let $J \subseteq \mathbb{R}$ be open, a smooth map $\gamma : J \to M$ is called a **smooth curve** in M. Given $t_0 \in J$, let $\frac{d}{dt}|_{t_0}$ be the coordinate basis in $T_{t_0}J \cong T_{t_0}\mathbb{R}$. The **velocity** of γ at t_0 is $\gamma'(t_0) := d\gamma_{t_0}(\frac{d}{dt}|_{t_0}) \in T_{\gamma(t_0)}\mathbb{R}^n$

Proposition 7.1. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, and a curve $\gamma : J \to M$. If we let $\phi \circ \gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$, then we have $\gamma'(t_0) = d\gamma_{t_0}(\frac{d}{dt}|_{t_0}) = \sum_{i=1}^n \dot{\gamma}^i(t_0)\partial_i|_{\gamma(t_0)} \in T_{\gamma(t_0)}M$*

Definition 33. Let $\mathbf{v} \in \mathfrak{X}(M)$, an **integral curve** of $\mathbf{v}$ is a curve $\gamma : J \to M$ such that $\forall t \in J, \gamma'(t) = \mathbf{v}_t$. If $0 \in J$, then $\gamma(0)$ is called the **starting point** of γ

Proposition 7.2. *An integral curve should satisfy $\forall t \in J, \sum_{i=1}^n \dot{\gamma}^i(t)\Upsilon_i|_{\gamma(t_0)} = \gamma'(t) = \boldsymbol{v}_t = \sum_{i=1}^n \boldsymbol{v}^i(\gamma(t))\Upsilon_i|_{\gamma(t_0)}$, thus $\forall i, \dot{\gamma}^i(t) = v^i \circ \phi^{-1}(\gamma^1(t), \ldots, \gamma^n(t))$ gives an ODE.*

Proposition 7.3. *$\forall \boldsymbol{v} \in \mathfrak{X}(M), p \in M, \exists \epsilon > 0, \gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve of $\boldsymbol{v}$ with starting point p.*

Definition 34. A **smooth global flow** on M is a smooth map $\Theta : \mathbb{R} \times M \to M$, such that $\forall s, t \in \mathbb{R}, p \in M, \Theta(0, p) = p, \Theta(t, \Theta(s, p)) = \Theta(t + s, p)$

Remark. A global flow can be thought of as a table to show where something should be after t time and starting from p.

Definition 35. For $p \in M$, we have $\Theta^{(p)} : \mathbb{R} \to M$ is a curve given by $\Theta^{(p)}(t) := \Theta(t, p)$

Definition 36. For $t \in \mathbb{R}$, we have $\Theta_t : \mathbb{R} \to M$ is a smooth map given by $\Theta_t(p) := \Theta(t, p)$

Proposition 7.4. *Given a global flow $\Theta : \mathbb{R} \times M \to M$, we define $\boldsymbol{v} : M \to TM$ by $\boldsymbol{v}_p := {\Theta^{(p)}}'(0) \in T_pM$, then $\boldsymbol{v} \in \mathfrak{X}(M)$ is a vector field, and $\Theta^{(p)}$ is an integral curve for $\boldsymbol{v}$.*

Definition 37. The $\mathbf{v} \in \mathfrak{X}(M)$ in the above proposition is called **infinitesimal generator** for the flow Θ.

Remark. The infinitesimal generator tells us how should something move at every point in M.

Definition 38. Let $D \subseteq \mathbb{R} \times M$ be open, such that $\forall p \in M, D^{(p)} := \{t \in \mathbb{R} | (t, p) \in D\}$ is an open interval containing 0. A **local flow** on M is a smooth map $\Theta : D \to M$, such that $\forall p \in M, s \in D^{(p)}, t \in D^{(\Theta(s,p))}$, s.t. $t + s \in D^{(p)}, \Theta(0, p) = p, \Theta(t, \Theta(s, p)) = \Theta(t + s, p)$

Definition 39. $\Theta^{(p)} : D^{(p)} \to M$ is a curve given by $\Theta^{(p)}(t) := \Theta(t, p)$. And for $t \in \mathbb{R}, M_t := \{p \in M | (t, p) \in D\}$, then $\Theta_t : M_t \to M$ is a smooth map given by $\Theta_t(p) := \Theta(t, p)$

Proposition 7.5. *A local flow also has an infinitesimal generator as before.*

Definition 40. An integral curve is **maximal** if it cannot be extended to an integral curve with a greater domain. A local flow is **maximal** if it cannot be extended to a local flow with a greater domain D.

Theorem 7.6. ***Fundamental theorem of flows:***
For any $\boldsymbol{v} \in \mathfrak{X}(M)$, there is a unique maximal local flow $\Theta : D \to M$, such that $\boldsymbol{v}$ is the generator of Θ. Moreover,

1. *$\forall p \in M$, $\Theta^{(p)} : D^{(p)} \to M$ is the unique maximal integral curve of $\boldsymbol{v}$ starting at p.*
2. *If $s \in D^{(p)}$, then $D^{(\Theta(s,p))} = D^{(p)} - s := \left\{t - s : t \in D^{(p)}\right\}$*
3. *$\forall t \in \mathbb{R}, M_t$ is an open subset of M, and $\Theta_t : M_t \to M_{-t}$ is a diffeomorphism with its inverse $\Theta_t^{-1} = \Theta_{-t}$*

8 One-form and Co-vector fields

Definition 41. Let V be a vector space, a **co-vector** on V is a linear map $f : V \to \mathbb{R}$. The set of all co-vectors is called the **dual space** V^*.

Definition 42. A **contraction** $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is the evaluation $\langle f, v \rangle := f(v)$

Proposition 8.1. *Given a basis $\{E_1, \dots, E_n\}$ for a finite-dimensional V, let $E^1, \dots, E^n \in V^*$ be defined by $\left\langle E^i, E_j \right\rangle := E^i(E_j) = \delta_{ij}$, then $\left\{E^i\right\}$ is a basis for V^*, called the* ***dual basis****.*

Definition 43. Let V, W be vector spaces, $A : V \to W$ be a linear map. The **dual map** $A^* : W^* \to V^*$ is defined by $\forall f \in W^*, v \in V, \langle A^*(f), v \rangle = A^*(f)(v) := f(A(v)) = \langle f, A(v) \rangle$

Proposition 8.2. *If $B : V \to W, A : W \to U$ are linear maps, then $(A \circ B)^* = B^* \circ A^*$*

Proposition 8.3. *$(Id_V)^*$ is the identity map for V^**

Proposition 8.4. *If $A : V \to W$ is an isomorphism, then $(A^*)^{-1} = (A^{-1})^*$*

Proposition 8.5. *Let V and W be finite-dimensional vector spaces of dimensions n and m, $A : V \to W$ be a linear map, and $[A]$ is the matrix with respect to basis $\{v_i\}_1^n, \{w_j\}_1^m$ for V, W, then $[A^*] = [A]^\dagger$*

Definition 44. Let T_pM be the tangent space to M at p, then the **cotangent space** to M at p is the dual space of T_pM, denoted by T_p^*M. The elements in T_p^*M are called **co-vectors**.

Definition 45. $T^*M := \bigsqcup_{p \in M} T_p^*M$ is called the **cotangent bundle**.

Proposition 8.6. *If M is an n-dimensional smooth manifold, then T^*M is a 2n-dimensional smooth manifold.*

Definition 46. A **one-form** or **co-vector field** is a smooth map $\omega : M \to T^*M$ such that $\forall p \in M, \omega_p := \omega(p) \in T_p^*M$

Definition 47. $\mathfrak{X}^*(M)$ is the set of all one-forms on M.

Proposition 8.7. *Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M, we have that for each $p \in U, \{\Upsilon_1|_p, \dots, \Upsilon_n|_p\}$ is a basis for T_pM, the dual basis for T_p^*M is $\left\{\Upsilon^1|_p, \dots, \Upsilon^n|_p\right\}$, such that $\left\langle \Upsilon^i, \Upsilon_j \right\rangle = \delta_{ij}$. Thus $\forall \omega_p \in T_p^*M, \omega_p = \sum_{i=1}^n \omega_i(p)\Upsilon^i|_p$ uniquely. And $\omega_i(p)$ can be get by $\omega_i(p) = \langle \omega, \Upsilon_i \rangle$*

Definition 48. The **coordinate co-vector field** is the map $\Upsilon^i : U \to T^*M$ by $\Upsilon^i(p) := \Upsilon^i|_p$. One can check that $\Upsilon^i \in \mathfrak{X}^*(M)$ is a co-vector field.

Definition 49. Given a smooth function $f \in C^\infty(M)$, and a co-vector field $\omega \in \mathfrak{X}^*(M)$, we define $f\omega := f \cdot \omega \in \mathfrak{X}^*(M)$ to be $(f\omega)(p) := f(p)\omega_p \in T_p^*M$

Proposition 8.8. *The above definition of $\cdot : C^\infty(M) \times \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$ gives a $C^\infty(M)-$module of the co-vector fields.*

Corollary 8.9. *We can thus write any co-vector field* $\omega \in \mathfrak{X}^*(M)$ *as* $\sum_{i=1}^n \omega_i \Upsilon^i$

Definition 50. Given any $\omega \in \mathfrak{X}^*(M), \mathbf{v} \in \mathfrak{X}(M)$, we can define $\langle \omega, \mathbf{v} \rangle := \omega(\mathbf{v}) : M \to \mathbb{R}$ by $\langle \omega, \mathbf{v} \rangle (p) := \langle \omega_p, \mathbf{v}_p \rangle$.

Proposition 8.10. *Given any* $\omega = \sum_{i=1}^n \omega_i \Upsilon^i \in \mathfrak{X}^*(M), \boldsymbol{v} = \sum_{i=1}^n \boldsymbol{v}^i \Upsilon_i \in \mathfrak{X}(M)$,

$$\langle \omega, \boldsymbol{v} \rangle = \sum_{i=1}^n \omega_i \boldsymbol{v}^i$$

Definition 51. Let $f \in C^\infty(M)$, the **differential** of f is $df \in \mathfrak{X}^*(M)$, such that $\forall p \in M, D \in T_pM, (df)_p(D) := Df \in \mathbb{R}$. Thus we have a function $d : C^\infty(M) \to \mathfrak{X}^*(M)$

Proposition 8.11. *Given a vector field* $\boldsymbol{v}$, *we have* $\langle df, \boldsymbol{v} \rangle |_p = \langle df_p, \boldsymbol{v}_p \rangle = \boldsymbol{v}_p(f)$

Proposition 8.12. *Notice that* $T_{f(p)}\mathbb{R} \cong \mathbb{R}$, *and if we identify them canonically, we have that* $(df)_p \in T_p^*M : T_pM \to \mathbb{R} \cong df_p : T_pM \to T_{f(p)}\mathbb{R}$. *Namely,* $\forall D \in T_pM, df_p(D) = (df)_p(D) \frac{d}{dt}|_{f(p)} \in T_{f(p)}\mathbb{R}$

Proposition 8.13. $\forall p \in M$, *fix a basis* $\{\Upsilon_i|_p\}_1^n$ *for* T_pM, *then for any* $f \in C^\infty(M)$, *we have*

$$df = \sum_{i=1}^n \Upsilon_i(f) \Upsilon^i$$

Proof.

$$\begin{aligned}
\forall D &= \sum_i D^i \Upsilon_i \in \mathfrak{X}(M), \\
df_p(D_p) &= \sum_{i=1}^n \Upsilon_i|_p(f) \Upsilon^i|_p (\sum_j D^j(p) \Upsilon_j|_p) \\
&= \sum_{i,j=1}^n D^j(p) \Upsilon_i|_p(f) \Upsilon^i|_p(\Upsilon_j|_p) \\
&= \sum_{i,j=1}^n D^j(p) \Upsilon_i|_p(f) \delta_j^i \\
&= \sum_{i=1}^n D^i(p) \Upsilon_i|_p(f) \\
&= D(f)
\end{aligned}$$

□

Corollary 8.14. *Let* $(U, \phi = (x^1, \ldots, x^n))$ *be a coordinate chart for M, with the induced basis* $\{\Upsilon_i = \partial_i\}$, *we have* $d(x^j)|_U = \sum_{i=1}^n \Upsilon_i(x^j) \Upsilon^i = \sum_{i=1}^n \delta_{ij} \Upsilon^i = \Upsilon^j$. *Thus we can write* Υ^j *as* dx^j, *and* Υ_j *as* $\frac{\partial}{\partial x^j}$ *or* ∂_j.

Corollary 8.15. *Let* $(U, \phi = (x^1, \ldots, x^n))$ *be a coordinate chart for M, then for any* $f \in C^\infty(M)$, *we have*

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i = (\partial_i f) dx^i$$

Corollary 8.16.

$$dx^i \partial_j = \delta_j^i$$

9 Tensors

Definition 52. Let V be a n-dimensional vector space, then $T^k(V^*) := V^* \otimes V^* \otimes \cdots \otimes V^* = (V^*)^{\otimes k}$ is a vector space of dimension n^k. An element in $T^k(V^*)$ is called a **covariant k-tensor** on V.

Definition 53. More generally, a (r, s) tensor, or a **r-variant-s-covariant-tensor** is an element from $(V)^{\otimes r} \otimes (V^*)^{\otimes s}$.

Proposition 9.1. *Let V be n-dimensional vector space, and $\{E_1, \dots, E_n\}$ be a basis for V, then $\left\{E^{i_1} \otimes \cdots \otimes E^{i_k}\right\}_{(i_1,\dots,i_k)\in[n]^k}$ is a basis for $T^k(V^*)$, thus any $\alpha \in T^k(V^*)$ can be uniquely written as $\sum_{(i_1,\dots,i_k)\in[n]^k} \alpha_{i_1,\dots,i_k} E^{i_1} \otimes \cdots \otimes E^{i_k}$*

Definition 54. We can thus view $\alpha \in T^k(V^*)$ as a function $V^k \to \mathbb{R}$ defined by
$\alpha(v_1, \dots, v_n) := \sum_{(i_1,\dots,i_k)\in[n]^k} \alpha_{i_1,\dots,i_k} E^{i_1}(v_1) \cdots E^{i_k}(v_k) = \alpha_{i_1,\dots,i_k} \left\langle E^{i_1}, v_1\right\rangle \cdots \left\langle E^{i_k}, v_k\right\rangle$

Definition 55. More generally, a (r, s) tensor $\mathcal{T} = \displaystyle\sum_{(i_1,\dots,i_r,j_1,\dots,j_s)\in[n]^{r+s}} \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} E_{i_1} \otimes \cdots \otimes E_{i_r} \otimes E^{j_1} \otimes \cdots \otimes E^{j_s}$ can be viewed as a map $(V^*)^r \times V^s \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{T}(\omega^1, \dots, \omega^r, v_1, \dots, v_s) &:= \sum_{(i_1,\dots,i_r,j_1,\dots,j_s)\in[n]^{r+s}} \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} \omega^1(E_{i_1}) \cdots \omega^r(E_{i_r}) E^{j_1}(v_1) \cdots E^{j_s}(v_s) \\ &= \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} \left\langle \omega^1, E_{i_1}\right\rangle \cdots \left\langle \omega^r, E_{i_r}\right\rangle \left\langle E^{j_1}, v_1\right\rangle \cdots \left\langle E^{j_s}, v_s\right\rangle\end{aligned}$$

Definition 56. Fix a basis $\{E_i\}$, the **components** of a (r,s) tensor $\mathcal{T}$ is $\mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s}$

Definition 57. A **contraction** on the first indices of a (r,s) tensor $\mathcal{T}$ is a (r-1,s-1) tensor $con\mathcal{T} := \mathcal{T}^{i,i_2,\dots,i_r}_{i,j_2,\dots,j_s,j} \Upsilon_{i_2} \otimes \cdots \otimes \Upsilon_{i_r} \otimes \Upsilon_{j_2} \otimes \cdots \otimes \Upsilon_{j_s}$

Proposition 9.2. *The map defined by the tensors is multi-linear.*

Example 9.0.1. A vector is a (1,0) tensor.

Example 9.0.2. A co-vector is a (0,1) tensor.

Example 9.0.3. A real inner product is a (0,2) tensor.

Example 9.0.4. The determinant of a $n \times n$ real matrix is a (0,n) tensor as a function on the column/row vectors.

10 Alternating Tensor and wedge product

Definition 58. A covariant k-tensor is **symmetric** if $\forall 1 \le i < j \le k, v_1, \dots, v_k \in V$,
$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$.
It is **alternating** or **anti-symmetric** if $\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$

Definition 59. The set of alternating covariant k-tensors is $\Lambda^k(V^*) \subseteq T^k(V^*)$

Definition 60. The **permutation group** S_k is the set of all permutations of $[k] := \{1, \dots, k\}$

Definition 61. Given a permutation $\sigma \in S_k$, **sign** of it is $Sgn(\sigma) := \begin{cases} 1 & \text{permutation is got by even transposition} \\ -1 & \text{permutation is got by odd transposition} \end{cases}$

Definition 62. Given a tensor $\alpha \in T^k(V^*)$, the **alternation** of it is $Alt(\alpha) \in T^k(V^*)$, defined by $Alt(\alpha)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

Proposition 10.1. $\forall \alpha \in T^k(V^*), Alt(\alpha) \in \Lambda^k(V^*)$

Proposition 10.2. $\forall \alpha \in \Lambda^k(V^*), Alt(\alpha) = \alpha$

Definition 63. More generally, given a (r, t+k) tensor of components $T^{i_1,\dots,i_r}_{j_1,\dots,j_{t+k}}$, we define the **symmetrization** of it to be

$$T^{i_1,\dots,i_r}_{j_1,\dots,j_t,(j_{t+1},\dots,j_{t+k})} := \frac{1}{k!}\sum_{\sigma\in S_k} T^{i_1,\dots,i_r}_{j_1,\dots,j_t,j_{t+\sigma(1)},\dots,j_{t+\sigma(k)}}$$

And the **Antisymmetrization** of it to be

$$T^{i_1,\dots,i_r}_{j_1,\dots,j_t,[j_{t+1},\dots,j_{t+k}]} := \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma) T^{i_1,\dots,i_r}_{j_1,\dots,j_t,j_{t+\sigma(1)},\dots,j_{t+\sigma(k)}}$$

Definition 64. An ordered k-tuple $I = (i_1,\dots,i_k) \in [n]^k$ is called a **multi-index of length k**.

Definition 65. Let $\{E^1,\dots,E^n\}$ be a dual basis for V^*, and $I = (i_1,\dots,i_k) \in [n]^k$, then we define the **elementary alternating tensor** $E^I \in \Lambda^k(V^*)$ by $E^I(v_1,\dots,v_k) := \det \begin{pmatrix} E^{i_1}(v_1) & \dots & E^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ E^{i_k}(v_1) & \dots & E^{i_k}(v_k) \end{pmatrix}$

Definition 66. Let $I = (i_1,\dots,i_k), J = (j_1,\dots,j_k) \in [n]^k$, then $\delta^I_J := \det \begin{pmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \dots & \delta^{i_k}_{j_k} \end{pmatrix}$

Proposition 10.3. $\delta^I_J = \begin{cases} 0 & \textit{if } I \textit{ or } J \textit{ have repeated indices, or } I \textit{ is not any permutation of } J \\ 1 & \textit{if } I \textit{ is an even permutation of } J \\ -1 & \textit{if } I \textit{ is an odd permutation of } J \end{cases}$

Proposition 10.4. *If $I = (i_1,\dots,i_k)$ have any repeated indices, i.e. $\exists 1 \le l \ne m \le k, i_l = i_m$, then $E^I = 0$*

Proposition 10.5. *If $J = (i_{\sigma(1)},\dots,i_{\sigma(k)})$, then $E^J = sgn(\sigma)E^I$*

Proposition 10.6. *Let $I = (i_1,\dots,i_k), J = (j_1,\dots,j_k) \in [n]^k$, then $E^I(E_{j_1},\dots,E_{j_k}) = \delta^I_J$*

Definition 67. $I = (i_1,\dots,i_k) \in [n]^k$ is increasing if $i_1 < i_2 < \dots < i_k$

Proposition 10.7. *Let $\dim(V) = n, k \in \mathbb{N}^+$, if $K > n$, then $\Lambda^k(V^*) = \{0\}$, otherwise $\dim(\Lambda^k(V^*)) = \binom{n}{k}$, and a basis is given by $\mathcal{E}^k := \{E^I | I$ is increasing multi-index of length $k\}$*

Definition 68. The **wedge product** of two covariant tensors u, v is defined to be $u \wedge v := u \otimes v - v \otimes u$

Proposition 10.8. $\forall s, s \wedge s = 0$

Proposition 10.9. *For $u_1,\dots,u_k$, we have $u_1 \wedge \dots \wedge u_k = k! Alt(u_1 \otimes \dots \otimes u_k) = \sum_{\sigma\in S_k} sgn(\sigma) v_{\sigma(1)},\dots,v_{\sigma(k)}$*

Proposition 10.10. $E^I = E^{i_1} \wedge \dots \wedge E^{i_k}$

Proposition 10.11. *Let $I, J \in [n]^k$ be both increasing, then $E^I \wedge E^J = (-1)^{|I|\cdot|J|} E^J \wedge E^I$*

11 Tensor fields or k-form

Definition 69. The bundle of all covariant k-tensors on M is $T^kT^*M := \bigsqcup_{p\in M} T^K(T^*_pM)$

Proposition 11.1. *T^kT^*M is a smooth manifold of dimension n^{k+1}*

Definition 70. Given a smooth manifold M, a **covariant k-tensor field on M** or a **k-form** is a smooth map $A : M \to T^kT^*M$, s.t. $A_p := A(p) \in T^K(T^*_pM)$. The set of all k-forms is $\Gamma(T^KT^*M)$

Proposition 11.2. *$\Gamma(T^KT^*M)$ is an infinite dimensional vector space.*

Proposition 11.3. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, then for covariant k-tensor field $A \in \Gamma(T^K T^* M)$, it can be written as $A|_U = \sum_{(i_1,\ldots,i_k)\in[n]^k} A_{i_1,\ldots,i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$*

Definition 71. An **alternating covariant k-tensor field** or **alternating k-form** on M is a smooth map $A : M \to \bigsqcup_{p\in M} \Lambda^k(T_p^* M)$ such that $\forall p \in M, A_p := A(p) \in \Lambda^k(T_p^* M)$. The set of all alternating k-forms on M is $\Omega^k(M)$

Definition 72. $\Omega^0(M) := c^\infty(M)$

Definition 73. $\wedge : \Omega^k(M)\times\Omega^l(M) \to \Omega^{k+l}(M)$ is defined to be $(\zeta\wedge\eta)_p := \zeta_p\wedge\eta_p$, where $\zeta_p \in \Lambda^k(T_p^* M), \eta_p \in \Lambda^l(T_p^* M)$. For $k = 0, f \in C^\infty(M), f \wedge \eta := f\eta$

Proposition 11.4. *$\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$ is an anti-commutative algebra.*

Proposition 11.5. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, then for an alternating covariant k-tensor field $\omega \in \Omega^k(M)$, it can be written as $\omega|_U = \sum_{(i_1<\cdots<i_k)\in[n]^k} \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $\omega_{i_1,\ldots,i_k} = \omega(\partial_{i_1}, \ldots, \partial_{i_k})$, and $dx^{i_1} \wedge \cdots \wedge dx^{i_k}(\partial_{j_1}, \ldots, \partial_{j_k}) = \delta^I_J$*

Definition 74. In general, fix any p, the collection of all (r, s) tensor at p is $(T_p)^r_s M := (T_p^* M)^{\otimes r} \otimes (T_p M)^{\otimes s}$, and the collection of all (r, s) tensor fields is $T^r_s M$

12 Push-forward and Pull-back

Definition 75. For $f \in \Omega^0(N) = C^\infty(N)$, we have the **pull-back** of f by a smooth map $F : M \to N$ is

$$F^* f := f \circ F \in C^\infty(M)$$

.

Remark. Notice that the definition of pull-back between smooth function spaces coincides with the definition of dual maps for linear function spaces.

Definition 76. Let M, N be smooth manifolds and $F : M \to N$ be a smooth function, then two vector fields $\mathbf{v} \in \mathfrak{X}(M), \mathbf{u} \in \mathfrak{X}(N)$ are **F-related** if $\forall p \in M, dF_p(\mathbf{v}_p) = \mathbf{u}_{F(p)}$

Proposition 12.1. *If $F : M \to N$ is a diffeomorphism, for every vector field $\boldsymbol{v} \in \mathfrak{X}(M)$, there is a unique vector field $F_*\boldsymbol{v} \in \mathfrak{X}(N)$ that is F-related to $\boldsymbol{v}$.*

Definition 77. The **push-forward** of a vector field $\mathbf{v} \in \mathfrak{X}(M)$ by a diffeomorphism $F : M \to N$ is the unique vector field $F_*\mathbf{v} \in \mathfrak{X}(N)$ in the previous proposition, defined by

$$\forall q \in N, F_*\mathbf{v}_q := dF_{F^{-1}(q)}(\mathbf{v}_{F^{-1}(q)})$$

Proposition 12.2. *For any diffeomorphism $F : M \to N$, vector field $\boldsymbol{v} \in \mathfrak{X}(N), f \in c^\infty(N)$, we always have $\boldsymbol{v}(F^* f) = \boldsymbol{v}(f \circ F) = ((F_*\boldsymbol{v})f) \circ F \in C^\infty(M)$*

Definition 78. Let $F : M \to N$ be a smooth map, the **cotangent map** of F is $dF_p^* : T^*_{F(p)}N \to T_p^* M$, which is the dual map of $dF_p : T_p M \to T_{F(p)}N$

Proposition 12.3. *Given any $\omega_{F(p)} \in T^*_{F(p)}N, v_p \in T_p M$, we have $\langle dF_p^*(\omega_{F(p)}), v_p\rangle = \langle \omega_{F(p)}, dF_p(v_p)\rangle$*

Definition 79. Given a smooth map $F : M \to N$, and a co-vector field $\omega \in \mathfrak{X}(N)$, the **pull-back of a 1-form** ω by F is $F^*\omega \in \mathfrak{X}^*(M)$ defined by $(F^*\omega)_p := dF_p^*(\omega_{F(p)}) \in T_p^* M$

Proposition 12.4. *Consider any $D \in T_p M, (F^*\omega)_p(D) = \langle dF_p^*(\omega_{F(p)}), D\rangle = \langle \omega_{F(p)}, dF_p(D)\rangle$*

Proposition 12.5. *For any smooth $F : M \to N$, co-vector field $\omega \in \mathfrak{X}^*(N), u \in c^\infty(N), \boldsymbol{v} \in \mathfrak{X}(M), p \in M$,*

1. *$\langle F^*\omega, \boldsymbol{v}\rangle|_p = \langle \omega, F_*\boldsymbol{v}\rangle|_{F(p)}$*

2. $F^*uF^*\omega = (u\circ F)F^*\omega = F^*(u\omega) \in \mathfrak{X}^*(M)$

3. $F^*(du) = d(u\circ F) = d(F^*u) \in \mathfrak{X}^*(M)$

Proof. 1. $\langle F^*\omega, \mathbf{v}\rangle|_p = \langle F^*\omega_p, \mathbf{v}_p\rangle = \left\langle dF_p^*(\omega_{F(p)}), \mathbf{v}_p\right\rangle = \left\langle \omega_{F(p)}, dF_p(\mathbf{v}_p)\right\rangle = \left\langle \omega_{F(p)}, F_*\mathbf{v}_{F(p)}\right\rangle = \langle \omega, F_*\mathbf{v}\rangle|_{F(p)}$ □

Definition 80. Given a smooth map $F: M\to N, p\in M, \alpha\in T^k(T^*_{F(p)}N)$, the **pull-back of a covariant k-tensor** α by F at p is $dF_p^*(\alpha)\in T^k(T_p^*M)$, defined by $dF_p^*(\alpha)(v_1,\dots,v_k) = \alpha(dF_p(v_1),\dots,dF_p(v_k)) \in C^\infty(N)$. This way we obtain a linear map $dF_p^*: T^k(T^*_{F(p)}N)\to T^k(T_p^*M)$

Definition 81. Let $A\in\Gamma(T^kT^*N)$ be a k-form, the **pull-back of a k-form** A by a smooth map $F: M\to N$ is $F^*A\in\Gamma(T^kT^*M)$ defined by $(F^*A)_p := dF_p^*(A_{F(p)})$. This way we get a $F^*:\Gamma(T^kT^*N)\to\Gamma(T^kT^*M)$

Definition 82. The **push-forward of a contra-variant k-tensor field V** $\in\Lambda(T^kTM)$ by a diffeomorphism $F: M\to N$ is $F^*(\mathbf{V})\in\Lambda(T^kTN)$, defined by $\forall A\in\Gamma(T^{k-1}T^*N), p\in M, \langle\mathbf{V}, F^*A\rangle|_p = \langle(F_*\mathbf{V}), A\rangle|_{F(p)}$

Proposition 12.6. *For any* $\forall A\in\Gamma(T^kT^*N)$, $\boldsymbol{V}\in\Lambda(T^kTM), p\in M, \langle F^*A, \boldsymbol{V}\rangle|_p = \langle A, (F_*\boldsymbol{V})\rangle|_{F(p)}$

Definition 83. We define the **pull-back of a contra-variant k-tensor field V** $\in\Gamma(T^kTN)$ by a diffeomorphism $F: M\to N$ to be $F^*(\mathbf{V}) := F_*^{-1}(\mathbf{V})\in\Lambda(T^kTM)$, which is the push-forward of **V** by $F^{-1}: N\to N$

Definition 84. The **pull-back of an (r,s)-tensor field** $\mathcal{T}$ is by taking the pullback on the r-contra-variant field and the s-co-variant part respectively.

Definition 85. Let $A\in\Omega^k(N)$, the **pull-back of an alternating k-form** A by a smooth map $F: M\to N$ is $F^*A\in\Omega^k(M)$ defined by $(F^*A)_p := dF_p^*(A_{F(p)})$. This way we get a $F^*:\Omega^k(N)\to\Omega^k(M)$

Proposition 12.7. $F^*:\Omega^k(N)\to\Omega^k(M)$ *is a linear map.*

Proposition 12.8. *Given any two alternating k-covariant tensor fields* $A, B\in\Omega^k(N)$, *we have* $F^*(A\wedge B) = F^*(A)\wedge F^*(B)$

Proposition 12.9. $F^*(\sum_{I increasing} A_I dy^{i_1}\wedge\dots\wedge dy^{i_k}) = \sum_{I increasing}(A_I\circ F)d(y^{i_1}\circ F)\wedge\dots\wedge d(y^{i_k}\circ F)$

Example 12.0.1. Consider $M = U = \mathbb{R}^2\setminus\{(x,0)|x\le 0\}$ be the open set of $\mathbb{R}^2$ minus the negative x axis, with a polar coordinate local map $\phi(r\cos(\theta), r\sin(\theta)) = (r,\theta)$ on U, let this be M. We then consider $N = U$ to be with a Cartesian coordinate $\psi = (x(a,b) = a, y(a,b) = b): N\to\mathbb{R}^2$. Let $Id: U\to U$ be the identity map. We thus have $x\circ Id\circ\phi^{-1}, y\circ Id\circ\phi^{-1}:\mathbb{R}^2\to\mathbb{R}^2, x\circ Id\circ\phi^{-1}(r,\theta) = r\cos(\theta), y\circ Id\circ\phi^{-1}(r,\theta) = r\sin(\theta)$.

$$\begin{aligned}
Id^*(dx\wedge dy) &= Id^*(\mathbf{1}dx\wedge dy),\ \text{in which } \mathbf{1}(x,y) = 1: N\to\mathbb{R}\\
&= (\mathbf{1}\circ Id)\cdot d(x\circ Id)\wedge d(y\circ Id),\ \text{in which } x\circ Id, y\circ Id: U\to\mathbb{R},\\
&= d(x\circ Id)\wedge d(x\circ Id)\\
&= \left(\frac{\partial x\circ Id\circ\phi^{-1}}{\partial\theta}d\theta + \frac{\partial y\circ Id\circ\phi^{-1}}{\partial r}dr\right)\wedge\left(\frac{\partial x\circ Id\circ\phi^{-1}}{\partial\theta}d\theta + \frac{\partial y\circ Id\circ\phi^{-1}}{\partial r}dr\right)\\
&= \left(\frac{\partial r\cos\theta}{\partial\theta}d\theta + \frac{\partial r\cos\theta}{\partial r}dr\right)\wedge\left(\frac{\partial r\sin\theta}{\partial\theta}d\theta + \frac{\partial r\sin\theta}{\partial r}dr\right)\\
&= (-r\sin\theta d\theta + \cos\theta dr)\wedge(r\cos\theta d\theta + \sin\theta dr)\\
&= -r\sin^2\theta d\theta\wedge dr + r\cos^2\theta dr\wedge d\theta,\ \text{since } dr\wedge dr = d\theta\wedge d\theta = 0\\
&= r\sin^2\theta dr\wedge d\theta + r\cos^2\theta dr\wedge d\theta\\
&= rdr\wedge d\theta
\end{aligned}$$

Thus there is a canonical mapping between $dx\wedge dy\in\Omega^2(M)$ and $rdr\wedge d\theta\in\Omega^2(N)$

13 Lie Derivative

Definition 86. Let $\mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$, the **Lie derivative** of $\mathbf{w}$ with respect to $\mathbf{v}$ is a map $\mathcal{L}_{\mathbf{v}}\mathbf{w} : M \to TM$ defined by $\mathcal{L}_{\mathbf{v}}\mathbf{w}_p = \frac{d}{dt}|_{t=0}(d(\Theta_{-t})_{\Theta_t(p)}\mathbf{w}_{\Theta_t(p)})$, where Θ is the flow generated by $\mathbf{v}$.

Lemma 13.1. *$\mathcal{L}_{\boldsymbol{v}}\boldsymbol{w} \in \mathfrak{X}(M)$ is a vector field.*

Theorem 13.2. *If $\boldsymbol{v}, \boldsymbol{w} \in \mathfrak{X}(M)$, then $\mathcal{L}_{\boldsymbol{v}}\boldsymbol{w} = [\boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{v}^\nu\partial_\nu\boldsymbol{w}^\mu - \boldsymbol{w}^\nu\partial_\nu\boldsymbol{v}^\mu)\partial_\mu$*

Remark. $d(\Theta_{-t})_{\Theta_t(p)}\mathbf{w}_{\Theta_t(p)} = d(\Theta_{-t})_{(\Theta_{-t})^{-1}(p)}\mathbf{w}_{(\Theta_{-t})^{-1}(p)} = (\Theta_{-t})_*\mathbf{w}_p = \Theta_t^*\mathbf{w}_p$, thus we can write $\mathcal{L}_{\mathbf{v}}\mathbf{w}_p = \frac{d}{dt}|_{t=0}\Theta_t^*\mathbf{w}_p$

Definition 87. We can generalize the **Lie derivative** to act on any (r,s)-tensor field $\mathcal{T}$ by

$$\mathcal{L}_{\mathbf{v}}\mathcal{T}_p := \frac{d}{dt}|_{t=0}(\Theta_t^*\mathcal{T})|_p := \lim_{t\to 0}\frac{\Theta_t^*\mathcal{T}_{\Theta_t(p)} - \mathcal{T}_p}{t},$$

which is still a (r,s)-tensor. As before, Θ is the flow generated by $\mathbf{v}$.

Proposition 13.3. *$\mathcal{L}_{\boldsymbol{v}}(\mathcal{T}\otimes\mathcal{S}) = \mathcal{L}_{\boldsymbol{v}}\mathcal{T}\otimes\mathcal{S} + \mathcal{T}\otimes\mathcal{L}_{\boldsymbol{v}}\mathcal{S}$*

Proposition 13.4. *$\mathcal{L}_{\boldsymbol{v}}(\langle\mathcal{T},\mathcal{S}\rangle) = \langle\mathcal{L}_{\boldsymbol{v}}\mathcal{T},\mathcal{S}\rangle + \langle\mathcal{T},\mathcal{L}_{\boldsymbol{v}}\mathcal{S}\rangle$*

Proposition 13.5. *For $f \in C^\infty(M), \mathcal{L}_{\boldsymbol{v}}(f) = \boldsymbol{v}(f) = \langle df, \boldsymbol{v}\rangle = \boldsymbol{v}^\mu\partial_\mu f$*

Proof. $\mathcal{L}_{\mathbf{v}}(f)_p = \frac{d}{dt}|_{t=0}(\Theta_t^* f)|_p = \frac{d}{dt}|_{t=0}(f\circ\Theta_t)|_p = d(\Theta_t)_p(\frac{d}{dt}|_{t=0})(f) = (\Theta_t'(0))(f) = \mathbf{v}f$ □

Proposition 13.6. *Consider a 1-form $\sigma \in \mathfrak{X}^*(M), \mathcal{L}_{\boldsymbol{v}}\sigma = (\boldsymbol{v}^\mu\partial_\mu\sigma_\nu + \sigma_\mu\partial_\nu\boldsymbol{v}^\mu)dx^\nu$*

Proof.

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}}\langle\sigma,\mathbf{x}\rangle &= \langle\sigma,\mathcal{L}_{\mathbf{v}}\mathbf{x}\rangle + \langle\mathcal{L}_{\mathbf{v}}\sigma,\mathbf{x}\rangle\\
\mathbf{v}^\mu\partial_\mu(\sigma_\nu\mathbf{x}^\nu) &= \sigma_\mu\mathbf{v}^\nu\partial_\nu\mathbf{x}^\mu - \sigma_\mu\mathbf{x}^\nu\partial_\nu\mathbf{v}^\mu + \langle\mathcal{L}_{\mathbf{v}}\sigma,\mathbf{x}\rangle\\
\mathbf{v}^\mu(\partial_\mu\sigma_\nu)\mathbf{x}^\nu + \mathbf{v}^\mu\sigma_\nu(\partial_\mu\mathbf{x}^\nu) &= \sigma_\mu\mathbf{v}^\nu\partial_\nu\mathbf{x}^\mu - \sigma_\mu\mathbf{x}^\nu\partial_\nu\mathbf{v}^\mu + \langle\mathcal{L}_{\mathbf{v}}\sigma,\mathbf{x}\rangle\\
\mathbf{v}^\mu(\partial_\mu\sigma_\nu)\mathbf{x}^\nu + \sigma_\mu\mathbf{x}^\nu\partial_\nu\mathbf{v}^\mu &= \langle\mathcal{L}_{\mathbf{v}}\sigma,\mathbf{x}\rangle\\
(\mathbf{v}^\mu\partial_\mu\sigma_\nu + \sigma_\mu\partial_\nu\mathbf{v}^\mu)\mathbf{x}^\nu &= (\mathcal{L}_{\mathbf{v}}\sigma)_\nu\mathbf{x}^\nu
\end{aligned}$$

□

Proposition 13.7. *In general, for any (r,s)-tensor $\mathcal{T} = \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s}\partial_{i_r}\otimes\cdots\otimes\partial_{i_r}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}$, we have*

$$(\mathcal{L}_{\boldsymbol{v}}\mathcal{T})^{i_1,\dots,i_r}_{j_1,\dots,j_s} = \Big(\boldsymbol{v}^k\partial_k\mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} - \mathcal{T}^{k,i_2,\dots,i_r}_{j_1,\dots,j_s}\partial_k\boldsymbol{v}^{i_1} - \cdots - \mathcal{T}^{i_1,\dots,i_{r-1},k}_{j_1,\dots,j_s}\partial_k\boldsymbol{v}^{i_r} + \mathcal{T}^{i_1,\dots,i_r}_{k,j_2,\dots,j_s}\partial_{j_1}\boldsymbol{v}^k + \cdots + \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_{s-1},k}\partial_{j_s}\boldsymbol{v}^k\Big)$$

14 Exterior derivative

Definition 88. Let $\omega = \sum_{(i_1<\cdots<i_k)\in[n]^k}\omega_{i_1,\dots,i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Omega^k(\mathbb{R})$, the **exterior derivative** of ω is $\mathbf{d}\omega := \sum_{(i_1<\cdots<i_k)\in[n]^k} d(\omega_{i_1,\dots,i_k})\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \frac{\partial\omega_{i_1,\dots,i_k}}{\partial x^j}dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Omega^{k+1}(\mathbb{R})$

Proposition 14.1. *If $f \in C^\infty(U) = \Omega^0(U)$, then we have $\boldsymbol{d}f = df \in \Omega^1(U)$*

Proposition 14.2. *$\boldsymbol{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ is linear*

Proposition 14.3. *$\boldsymbol{d}\circ\boldsymbol{d} : \Omega^k(U) \to \Omega^{k+2}(U)$ is the zero map.*

Proposition 14.4. *$\forall\omega \in \Omega^k(U), \eta \in \Omega^l(U)$, we have $\boldsymbol{d}(\omega\wedge\eta) = \boldsymbol{d}\omega\wedge\eta + (-1)^k\omega\wedge\boldsymbol{d}\eta$*

Proposition 14.5. *For any smooth map $F : U \to V$, and any alternating k-form $\omega \in \Omega^k(V)$, we always have $\boldsymbol{d}(F^*\omega) = F^*(\boldsymbol{d}\omega)$*

Theorem 14.6. *For any smooth manifold M and $k \in \mathbb{N}$, there is a unique linear map $\boldsymbol{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ such that*

1. *If $f \in C^\infty(M) = \Omega^0(M)$, then we have $\boldsymbol{d}f = df \in \Omega^1(M)$*
2. *$\boldsymbol{d} \circ \boldsymbol{d} : \Omega^k(M) \to \Omega^{k+2}(M)$ is the zero map.*
3. *$\forall \omega \in \Omega^k(M), \eta \in \Omega^l(M)$, we have $\boldsymbol{d}(\omega \wedge \eta) = \boldsymbol{d}\omega \wedge \eta + (-1)^k \omega \wedge \boldsymbol{d}\eta$*

Remark. Notice that we can generalize the exterior derivative to work on any tensor $\mathcal{T}$.

Theorem 14.7. *Cartan identity: $\mathcal{L}_{\boldsymbol{v}}\mathcal{T} = \langle \boldsymbol{d}\mathcal{T}, \boldsymbol{v} \rangle + \boldsymbol{d} \langle \mathcal{T}, \boldsymbol{v} \rangle$*

15 Affine connection and Covariant derivative

Definition 89. $\forall p \in M$, fix a basis $\{\Upsilon_i|_p\}$ for T_pM, the **covariant derivative** ∇ is a function that sends a (r, s) tensor $\mathcal{T}$ with coordinates $\mathcal{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}$ to a $(r, s+1)$ tensor $\nabla\mathcal{T}$, in addition, we define $\nabla_j\mathcal{T}$ to be a (r, s) tensor with coordinate $(\nabla_j\mathcal{T})^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} := (\nabla\mathcal{T})^{i_1,\ldots,i_r}_{j_1,\ldots,j_s,j}$. Namely,

$$\nabla\mathcal{T} = (\nabla_j\mathcal{T})^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\Upsilon_{i_1} \otimes \cdots \otimes \Upsilon_{i_r} \otimes \Upsilon^{j_1} \otimes \cdots \otimes \Upsilon^{j_s} \otimes \Upsilon^j = (\nabla_j\mathcal{T}) \otimes \Upsilon^j$$

In addition, the **covariant derivative** should satisfy the following properties:

1. (Additivity) $\nabla(\mathcal{T} + \mathcal{A}) = \nabla\mathcal{T} + \nabla\mathcal{A}$
2. (Leibniz) $\nabla(\mathcal{T} \otimes \mathcal{A}) = \nabla\mathcal{T} \otimes \mathcal{A} + \mathcal{T} \otimes \nabla\mathcal{A}$
3. (Commutes with contraction) $con(\nabla\mathcal{T}) = \nabla con(\mathcal{T})$, namely,
$(\nabla\mathcal{T})^{i,i_2,\ldots,i_r}_{i,j_2,\ldots,j_s,j}\Upsilon_{i_2} \otimes \cdots \otimes \Upsilon_{i_r} \otimes \Upsilon_{j_2} \otimes \cdots \otimes \Upsilon_{j_s} \otimes \Upsilon_j = \nabla\left(\mathcal{T}^{i,i_2,\ldots,i_r}_{i,j_2,\ldots,j_s,j}\Upsilon_{i_2} \otimes \cdots \otimes \Upsilon_{i_r} \otimes \Upsilon_{j_2} \otimes \cdots \otimes \Upsilon_{j_s}\right)$
4. $\forall f \in C^\infty(M) = \Omega^0(M), \nabla f = df \in TM^*$

Proposition 15.1. $\forall v \in \mathfrak{X}(M), \omega \in \mathfrak{X}^*(M), \nabla \langle \omega, v \rangle = \langle \nabla\omega, v \rangle + \langle \omega, \nabla v \rangle$

Proof.

$$\begin{aligned}
\nabla \langle \omega, v \rangle &= \nabla con(v \otimes \omega) \\
&= con(\nabla(v \otimes \omega)) \\
&= con(\nabla v \otimes \omega + v \otimes \nabla\omega) \\
&= con(\nabla v \otimes \omega) + con(v \otimes \nabla\omega) \\
&= \langle \omega, \nabla v \rangle + \langle \nabla\omega, v \rangle
\end{aligned}$$

□

Definition 90. The **directional covariant derivative** of $\mathcal{T}$ in the direction of $\mathbf{v} \in \mathfrak{X}(M)$ is a (r,s) tensor

$$\nabla_{\mathbf{v}}\mathcal{T} := \langle \nabla\mathcal{T}, \mathbf{v} \rangle = \mathbf{v}^j\nabla_j\mathcal{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\Upsilon_{i_r} \otimes \cdots \otimes \Upsilon_{i_r} \otimes \Upsilon^{j_1} \otimes \cdots \otimes \Upsilon^{j_s} = \mathbf{v}^j\nabla_j\mathcal{T}$$

Proposition 15.2. *For any basis $\{\Upsilon_i\}$ and tensor $\mathcal{T}$, we have $\nabla_{\Upsilon_a}\mathcal{T} = \delta^i_a\nabla_i\mathcal{T} = \nabla_a\mathcal{T}$*

Proposition 15.3. *$\nabla_{\boldsymbol{v}}\mathcal{T}$ is linear in $\boldsymbol{v}$, namely, $\forall a \in \mathbb{F}, \boldsymbol{v}, \boldsymbol{w} \in \mathfrak{X}(M), \mathcal{T} \in T^r_sM, \nabla_{\boldsymbol{v}+a\boldsymbol{w}}\mathcal{T} = \nabla_{\boldsymbol{v}}\mathcal{T} + a\nabla_{\boldsymbol{w}}\mathcal{T}$*

Definition 91. For any covariant derivative connection ∇ and any basis $\{\Upsilon_i\}$, $\Gamma^a_{bc} := (\nabla\Upsilon_b)^a_c = \langle \Upsilon^a, \nabla_{\Upsilon_c}\Upsilon_b \rangle$

Proposition 15.4. $\nabla\Upsilon_b = \Gamma^a{}_{bc}\Upsilon_a\Upsilon^c$

Proposition 15.5. $\Gamma^a_{bc}\Upsilon_a = \nabla_{\Upsilon_c}\Upsilon_b = \nabla_c\Upsilon_b$

Proposition 15.6. *Consider a change of basis* $\Upsilon'_{a'} = \Lambda_{a'}{}^{a}\Upsilon_a, \Upsilon'^{a'} = \tilde{\Lambda}^{a'}_{a}\Upsilon^a$*, where* $\tilde{\Lambda}^{a'}_{a} = (\Lambda^{-1})_a{}^{a'}$*, then we have* $\Gamma'^{a'}{}_{b'c'} = \tilde{\Lambda}^{a'}_{a}\Lambda_{b'}{}^{b}\Lambda_{c'}{}^{c} + \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c}\Upsilon_c(\Lambda_{b'}{}^{a})$

Proof.

$$
\begin{aligned}
\Gamma'^{a'}{}_{b'c'} &= \left\langle \Upsilon'^{a'}, \nabla_{\Upsilon'_{c'}}\Upsilon'_{b'} \right\rangle \\
&= \left\langle \tilde{\Lambda}^{a'}_{a}\Upsilon^a, \nabla_{\Lambda_{c'}{}^{c}\Upsilon_c}\Upsilon'_{b'} \right\rangle \\
&= \tilde{\Lambda}^{a'}_{a} \left\langle \Upsilon^a, \Lambda_{c'}{}^{c} \left(\nabla\Upsilon'_{b'}\right)^d_c \Upsilon_d \right\rangle \\
&= \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c} \left(\nabla\Upsilon'_{b'}\right)^a_c \\
&= \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c} \left(\nabla\left(\Lambda_{b'}{}^{b}\Upsilon_b\right)\right)^a_c \\
&= \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c} \left(\nabla\left(\Lambda_{b'}{}^{b}\right)\Upsilon_b + \Lambda_{b'}{}^{b}\nabla\Upsilon_b\right)^a_c \\
&= \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c} \left(d\left(\Lambda_{b'}{}^{b}\right)\Upsilon_b\right)^a_c + \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c}\Lambda_{b'}{}^{b}\left(\nabla\Upsilon_b\right)^a_c \\
&= \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c} \left(\Upsilon_d\left(\Lambda_{b'}{}^{b}\right)\Upsilon^d\Upsilon_b\right)^a_c + \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c}\Lambda_{b'}{}^{b}\Gamma^a_{bc} \\
&= \tilde{\Lambda}^{a'}_{a}\Lambda_{c'}{}^{c}\Upsilon_c\left(\Lambda_{b'}{}^{a}\right) + \tilde{\Lambda}^{a'}_{a}\Lambda_{b'}{}^{b}\Lambda_{c'}{}^{c}\Gamma^a_{bc}
\end{aligned}
$$

□

Proposition 15.7. $\forall \boldsymbol{v} \in \mathfrak{X}(M), \nabla_a v^b = \Upsilon_a(v^b) + \Gamma^b{}_{ca}v^c$

Proof.

$$
\begin{aligned}
\nabla\mathbf{v} &= \nabla\left(v^c\Upsilon_c\right) \\
&= \nabla\left(v^c\right)\Upsilon_c + v^c\nabla\Upsilon_c \\
&= d(v^c)\Upsilon_c + v^c\Gamma^d{}_{ce}\Upsilon_d\Upsilon^e \\
\nabla_a v^b &= \left(\nabla\mathbf{v}\right)^b{}_a \\
&= \nabla\mathbf{v}(\Upsilon^b, \Upsilon_a) \\
&= \left\langle \Upsilon_c, \Upsilon^b \right\rangle \left\langle d(v^c), \Upsilon_a \right\rangle + v^c\Gamma^b{}_{ca} \\
&= \Upsilon_a(v^b) + v^c\Gamma^b{}_{ca}
\end{aligned}
$$

□

Proposition 15.8. $\nabla\Upsilon^c = -\Gamma^c{}_{ba}\Upsilon^b\Upsilon^a$

Proof.

$$
\begin{aligned}
&\forall v \in \mathcal{X}(M), \\
&\quad \left\langle \nabla\Upsilon^c, v \right\rangle = \nabla\left\langle \Upsilon^c, v \right\rangle - \left\langle \Upsilon^c, \nabla v \right\rangle \\
&\quad\quad = \nabla(v_c) - d(v^c) - v^b\Gamma^c{}_{be}\Upsilon^e \\
&\quad\quad = d(v_c) - d(v^c) - v^b\Gamma^c{}_{be}\Upsilon^e \\
&\quad\quad = -\Gamma^c{}_{ba}v^b\Upsilon^a \\
&\quad\quad = \left\langle -\Gamma^c{}_{ba}\Upsilon^b\Upsilon^a, v \right\rangle
\end{aligned}
$$

□

Proposition 15.9. $\forall \omega \in \mathcal{X}^*(M), \nabla_a\omega_b = \Upsilon_a(\omega_b) - \Gamma^c{}_{ba}\omega_c$

Proof.

$$\begin{aligned}
\nabla\omega &= \nabla\left(\omega_c\Upsilon^c\right) \\
&= \nabla\left(\omega_c\right)\Upsilon^c + \omega_c\nabla\left(\Upsilon^c\right) \\
&= d\omega_c\Upsilon^c - \omega_c\Gamma^c{}_{ba}\Upsilon^b\Upsilon^a \\
\nabla_a\omega_b &= \nabla\omega(\Upsilon_b,\Upsilon_a) \\
&= d\omega_c(\Upsilon_a)\Upsilon^c(\Upsilon_b) - \omega_c\Gamma^c{}_{ba} \\
&= \Upsilon_a(\omega_b) - \omega_c\Gamma^c{}_{ba}
\end{aligned}$$

□

Definition 92. An affine connection is **symmetric (or torsion-free)** if and only if it act asymmetrically on functions: $\nabla_a\nabla_b f = \nabla_b\nabla_a f$

Lemma 15.10. *Consider any* $f \in C^\infty(M), d(\Upsilon_a(f)) = (\nabla^2 f)_a + \Gamma^b{}_{ac}\Upsilon_b(f)\Upsilon^c$

Proof.

$$\begin{aligned}
d(\Upsilon_a(f)) &= \nabla(\Upsilon_a(f)) \\
&= \nabla(\langle df, \Upsilon_a\rangle) \\
&= \langle\nabla df, \Upsilon_a\rangle + \langle df, \nabla\Upsilon_a\rangle \\
&= \left\langle\nabla^2 f, \Upsilon_a\right\rangle + \left\langle df, \Gamma^b{}_{ac}\Upsilon_b\Upsilon^c\right\rangle \\
&= (\nabla^2 f)_a + \Gamma^b{}_{ac}\Upsilon_b(f)\Upsilon^c
\end{aligned}$$

□

Proposition 15.11. *If the affine connection is symmetric, then* $2\Gamma^a{}_{[bc]}\Upsilon_a = [\Upsilon_c,\Upsilon_b]$ *for any basis* $\{\Upsilon_i\}$

Proof. Pick any function $f \in C^\infty(M)$,

$$\begin{aligned}
\Upsilon_c\circ\Upsilon_b(f) &= \langle f(\Upsilon_b(f)), \Upsilon_c\rangle \\
&= \left\langle(\nabla^2 f)_b + \Gamma^a{}_{be}\Upsilon_a(f)\Upsilon^e, \Upsilon_c\right\rangle \\
&= (\nabla^2 f)_{bc} + \Gamma^a{}_{bc}\Upsilon_a(f) \\
[\Upsilon_c,\Upsilon_b](f) &= \Upsilon_c\circ\Upsilon_b(f) - \Upsilon_b\circ\Upsilon_c(f) \\
&= (\nabla^2 f)_{bc} + \Gamma^a{}_{bc}\Upsilon_a(f) - (\nabla^2 f)_{cb} - \Gamma^a{}_{cb}\Upsilon_a(f) \\
&= \nabla_b\nabla_c f - \nabla_c\nabla_b f + 2\Gamma^a{}_{[bc]}\Upsilon_a(f) \\
&= 2\Gamma^a{}_{[bc]}\Upsilon_a(f)
\end{aligned}$$

□

Corollary 15.12. *If the affine connection is symmetric, and let* $(U, \phi = (x^1,\dots,x^n))$ *be a coordinate chart for* M, *we have*

$$\Gamma^\mu{}_{\gamma\nu} = \Gamma^\mu{}_{\gamma\nu}$$

Proof. $2\Gamma^\mu{}_{[\gamma\nu]}\partial_\mu = [\partial_\nu,\partial_\gamma] = 0$ □