Phys484: Quantum Theory

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1 Deferential Geometry

1.1 Topology and manifold

Definition 1. Given any set X, a **topology** is a pair $(X, \mathcal{S}), \mathcal{S} \subseteq \mathcal{P}(X)$ that satisfies:

1. $\emptyset \in \mathcal{S}$
2. If $\forall \alpha, S_\alpha \in \mathcal{S}$, then $\bigcup_\alpha S_\alpha \in \mathcal{S}$
3. If $S_1, \ldots, S_n \in \mathcal{S}$, then $\bigcap_{i=1}^n S_i \in \mathcal{S}$

Definition 2. Given any set X, and a topology $(X, \mathcal{S})$, the elements in $\mathcal{S}$ are called open.

Definition 3. Given any set X, a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ is said to be a basis of the topology $\mathcal{S}$ if

1. $X = \bigcup_{B \in \mathcal{B}}$
2. If $B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2$, then $\exists B_x \in \mathcal{B}, x \in B_x \subseteq B_1 \cap B_2$
3. $\mathcal{S}$ is the collection of all unions of the elements of $\mathcal{B}$.

Definition 4. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$ is called a **cover** for X if $X \subseteq \bigcup_{\alpha \in A} U_\alpha$. A cover is called an open cover if every U_α is open in the topology of X.

Definition 5. Given two sets X, Y, and there corresponding topology $\mathcal{S}, \mathcal{T}$, a map $f : X \to Y$ is **continuous** if $\forall T \in \mathcal{T}, f^{-1}(T) \in \mathcal{S}$. Namely, for any open set in Y, its preimage of f is also open in X.

Definition 6. Given two sets X, Y, and their corresponding topology $\mathcal{S}, \mathcal{T}$, a continuous map $f : X \to Y$ is a **homeomorphism** if it is invertible, and its inverse function is also continuous.

Remark. A homeomorphism is a map that preserves the topology structure between two sets.

Definition 7. An **atlas** $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_\alpha$ is a collection of **local charts** (U_α, ϕ_α), where each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto its image $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, and that $\bigcup_\alpha U_\alpha = X$.

Definition 8. A smooth atlas is an atlas such that $\forall (U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in \mathcal{A}$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ is C^∞ smooth.

Definition 9. A **smooth manifold** $M = (\mathcal{S}, \mathcal{A})$ is a 2nd-countable Hausdorff topology with a smooth atlas. The dimension n of the manifold is the dimension of $\mathbb{R}^n$ in the atlas $\mathcal{A}$.

1.2 Smooth functions and Diffeomorphism

Definition 10. Let M,N be smooth manifolds of dimension m,n, we say a function $F : M \to N$ is **smooth** at point $p \in M$ if and only if there are local charts (U_α, ϕ_α) for M and (V_β, ψ_β) for N, such that:

1. $p \in U_\alpha$
2. $F(p) \in V_\beta$

3. $U_\alpha \cap F^{-1}(V_\alpha) \subseteq M$ is open
4. The **coordinate representation** $\hat{F} := \psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to \mathbb{R}^m$ is smooth at $\phi_\alpha(p) \in \mathbb{R}^n$

Proposition 1.1. *If F is continuous, then 3 is always met.*

Proposition 1.2. *If 4 is met for some coordinate maps, then it is always true for any other coordinate maps. Namely, the smoothness is independent of choices of coordinate maps.*

Definition 11. If F is smooth at every $p \in M$, we say that F is a smooth function.

Definition 12. F is a **diffeomorphism** if F is invertible, and that both F, F^{-1} are smooth.

Proposition 1.3. *A diffeomorphism is always a homeomorphism.*

1.3 Tangent space and derivatives

Definition 13. Let $C^\infty(M)$ be the real vector space of smooth functions from $M \to \mathbb{R}$, a **derivation** or **tangent vector** at $p \in M$ is an $\mathbb{R}-$linear map $D : C^\infty(M) \to \mathbb{R}$ satisfying the **Leibniz condition**: $\forall f, g \in C^\infty(M), D(fg) = f(p)D(g) + g(p)D(f)$

Definition 14. The **tangent space** to M at $p \in M$, T_pM, is the set of all tangent vectors at p.

Proposition 1.4. *Let $D \in T_pM$, if $\forall x \in M, f(x) = c$ is a constant function, then $D(f) = 0$*

Proposition 1.5. *If $\exists U \ni p$ be open, and $\forall x \in U, f(x) = 0$, then $\forall D \in T_pM, D(f) = 0$*

Proposition 1.6. *If $f, g \in C^\infty(M)$ and $f = g$ on some open $U \ni p$, then $\forall D \in T_pM, D(f) = D(g)$*

Proposition 1.7. *$\{\partial_1|_p, \dots, \partial_n|_p\}$ is a basis for $T_p\mathbb{R}^n$.*

Definition 15. Given a smooth function $F : M \to N$, the **differential** or **derivative** of F at $p \in M$ is the map $dF_p : T_pM \to T_{F(p)}N$, given by $\forall D \in T_pM, f \in C^\infty(N), dF_p(D)(f) := D(f \circ F)$

Proposition 1.8. *Let M, N, R be smooth manifolds and $F : M \to N, G : N \to R$ are smooth maps, then for any $p \in M$, we have:*

1. *$dF_p : T_pM \to T_{F(p)}N$ is linear*
2. *$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G(F(p))}R$*
3. *$d(Id_M)_p : T_pM \to T_pM$ is an identity isomorphism.*
4. *If F is a diffeomorphism, then dF_p is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$*

Corollary 1.9. *Let M be a n-dimensional smooth manifold, for any $p \in M$, and any local chart (U, ϕ) containing p, we have $T_pM \cong T_pU \cong T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n$, and the n-dimensional vector space T_pM has a basis of $\left\{\Upsilon_j|_p := \partial_j|_p := \frac{\partial}{\partial x^j}|_p := (di_p \circ (d\phi_p)^{-1})(\frac{\partial}{\partial x^j}|_{\phi(p)}) = (di_p \circ (d\phi_p)^{-1})(\partial_j|_{\phi(p)})\right\}_j$.*

Proposition 1.10. *Given $f \in C^\infty(M), \Upsilon_j|_p(f) = \frac{\partial(f\circ\phi^{-1})}{\partial x^j}(\phi(p))$*

Corollary 1.11. *Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M, then*

$$\Upsilon_j|_p(x^i) = \frac{\partial x^i}{\partial x^j}(p) = \frac{\partial(x^i \circ \phi^{-1})}{\partial x^j}(\phi(p)) = \delta^i_j$$

Theorem 1.12. *Let $F : M \to N$ be smooth, (U, ϕ) and (V, ψ) be local charts for M and N, such that $p \in U, F(p) \in V$, if we choose the basis $\left\{\Upsilon_j|_p\right\}_j, \left\{\Upsilon_i|_{F(p)}\right\}_i$ associated to (U, ϕ) and (V, ψ) , we have that $[dF_p]_{ij} = \frac{\partial \hat{F}^i}{\partial x^j}(\phi(p))$. Namely, $dF_p(\Upsilon_j|_p) = \sum_{i=1}^{\dim(M)} \frac{\partial \hat{F}^i}{\partial x^j}(\phi(p))\Upsilon_i|_{F(p)} \in T_{F(p)}N$, where $\hat{F} = \psi \circ F \circ \phi^{-1}$ is the coordinate representation of F.*

Definition 16. The **tangent bundle** $TM := \bigsqcup_{p\in M} T_pM$.

Proposition 1.13. *If M is a n-dimension smooth manifold, then TM is a 2n-dimension smooth manifold.*

1.4 Vector field

Definition 17. A **vector field** is a smooth function $\mathbf{v} : M \to TM$, such that $\forall p, \mathbf{v}_p := \mathbf{v}(p) \in T_pM$

Definition 18. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M, then we can always write $\mathbf{v}_p = \sum_{i=1}^n \mathbf{v}^i(p)\Upsilon_i|_p$, since $\left\{\Upsilon_i|_p = \frac{\partial}{\partial x^i}|_p\right\}$ is a basis for T_pM. The functions $\mathbf{v}^i : M \to \mathbb{R}$ are called the **component functions**.

Definition 19. Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M, the **partial derivatives** $\Upsilon_i = \partial_i : U \to TM$ is given by $\Upsilon_i(p) = \Upsilon_i|_p$, where $\left\{\Upsilon_i|_p = \frac{\partial}{\partial x^i}|_p\right\}$ is a basis for T_pM associated to $(U, \phi = (x^1, \dots, x^n))$. One can check that $\Upsilon_i \in \mathfrak{X}(M)$

Definition 20. $\mathfrak{X}(M)$ is the set of all vector fields.

Definition 21. Given a smooth function $f \in C^\infty(M)$, and a vector field $\mathbf{v} \in \mathfrak{X}(M)$, we define $f\mathbf{v} := f \cdot \mathbf{v} \in \mathfrak{X}(M)$ to be $(f\mathbf{v})(p) := f(p)\mathbf{v}_p \in T_pM$

Remark. We can write any vector field $\mathbf{v} \in \mathfrak{X}(M)$ as

$$\mathbf{v} = \sum_{i=1}^n \mathbf{v}^i \Upsilon_i = \sum_{i=1}^n \mathbf{v}^i \partial_i$$

Definition 22. Given a smooth function $f \in C^\infty(M)$, and a vector field $\mathbf{v} \in \mathfrak{X}(M)$, we define $\mathbf{v}(f) \in C^\infty(M)$ to be $\mathbf{v}(f)(p) := \mathbf{v}_p(f) \in \mathbb{R}$. Thus we can view a vector field $\mathbf{v}$ as a function $C^\infty(M) \to C^\infty(M)$ as well.

1.5 Lie Bracket

Definition 23. Given two vector fields $\mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$, the **Lie Bracket** is $[\mathbf{v}, \mathbf{w}] := \mathbf{v} \circ \mathbf{w} - \mathbf{w} \circ \mathbf{v}$

Proposition 1.14. $[\boldsymbol{v}, \boldsymbol{w}] \in \mathfrak{X}(M)$ *is a vector field.*

Proposition 1.15. $[\boldsymbol{v}, \boldsymbol{w}]$ *is bilinear.*

Proposition 1.16. $[\boldsymbol{v}, \boldsymbol{w}] = -[\boldsymbol{w}, \boldsymbol{v}]$ *is anti-symmetric*

Proposition 1.17. $[\boldsymbol{v}, \boldsymbol{w}]$ *satisfies the Jacobian Identity:* $[\boldsymbol{u}, [\boldsymbol{v}, \boldsymbol{w}]] + [\boldsymbol{w}, [\boldsymbol{u}, \boldsymbol{v}]] + [\boldsymbol{v}, [\boldsymbol{w}, \boldsymbol{u}]] = 0$

Proposition 1.18. *For any* $f, g \in C^\infty(M), \boldsymbol{v}, \boldsymbol{u} \in \mathfrak{X}(M)$*, we have* $[f\boldsymbol{v}, g\boldsymbol{u}] = fg\,[\boldsymbol{v}, \boldsymbol{u}] + f(\boldsymbol{v}g)\boldsymbol{u} - g(\boldsymbol{u}f)\boldsymbol{v}$

Proposition 1.19. *Let* $(U, \phi = (x^1, \dots, x^n))$ *be a coordinate chart for M, then for any two vector fields* $\boldsymbol{v} = \sum_{i=1}^n \boldsymbol{v}^i\Upsilon_i, \boldsymbol{u} = \sum_{i=1}^n \boldsymbol{u}^j\Upsilon_j \in \mathfrak{X}(M)$*, we have* $[\boldsymbol{u}, \boldsymbol{v}] = \sum_{j=1}^n \left(\boldsymbol{v}\boldsymbol{u}^j - \boldsymbol{u}\boldsymbol{v}^j\right)\Upsilon_j$

1.6 Curve and Flow

Definition 24. Let $J \subseteq \mathbb{R}$ be open, a smooth map $\gamma : J \to M$ is called a **smooth curve** in M. Given $t_0 \in J$, let $\frac{d}{dt}|_{t_0}$ be the coordinate basis in $T_{t_0}J \cong T_{t_0}\mathbb{R}$. The **velocity** of γ at t_0 is $\gamma'(t_0) := d\gamma_{t_0}(\frac{d}{dt}|_{t_0}) \in T_{\gamma(t_0)}\mathbb{R}^n$

Proposition 1.20. *Let* $(U, \phi = (x^1, \dots, x^n))$ *be a coordinate chart for M, and a curve* $\gamma : J \to M$*. If we let* $\phi \circ \gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$*, then we have* $\gamma'(t_0) = d\gamma_{t_0}(\frac{d}{dt}|_{t_0}) = \sum_{i=1}^n \dot{\gamma}^i(t_0)\Upsilon_i|_{\gamma(t_0)} \in T_{\gamma(t_0)}M$

Definition 25. Let $\mathbf{v} \in \mathfrak{X}(M)$, an **integral curve** of $\mathbf{v}$ is a curve $\gamma : J \to M$ such that $\forall t \in J, \gamma'(t) = \mathbf{v}_t$. If $0 \in J$, then $\gamma(0)$ is called the **starting point** of γ

Definition 26. A **smooth global flow** on M is a smooth map $\Theta : \mathbb{R} \times M \to M$, such that $\forall s, t \in \mathbb{R}, p \in M, \Theta(0, p) = p, \Theta(t, \Theta(s, p)) = \Theta(t + s, p)$

Remark. A global flow can be thought of as a table to show where something should be after t time and starting from p.

Definition 27. For $p \in M$, we have $\Theta^{(p)} : \mathbb{R} \to M$ is a curve given by $\Theta^{(p)}(t) := \Theta(t, p)$

Definition 28. For $t \in \mathbb{R}$, we have $\Theta_t : \mathbb{R} \to M$ is a smooth map given by $\Theta_t(p) := \Theta(t, p)$

Proposition 1.21. *Given a global flow* $\Theta : \mathbb{R} \times M \to M$*, we define* $\boldsymbol{v} : M \to TM$ *by* $\boldsymbol{v}_p := {\Theta^{(p)}}'(0) \in T_pM$*, then* $\boldsymbol{v} \in \mathfrak{X}(M)$ *is a vector field, and* $\Theta^{(p)}$ *is an integral curve for* $\boldsymbol{v}$*.*

Definition 29. The $\mathbf{v} \in \mathfrak{X}(M)$ in the above proposition is called **infinitesimal generator** for the flow Θ.

Remark. The infinitesimal generator tells us how should something move at every point in M.

1.7 One-form and Co-vector fields

Definition 30. Let V be a vector space, a **co-vector** on V is a linear map $f : V \to \mathbb{R}$. The set of all co-vectors is called the **dual space** V^*.

Definition 31. A **contraction** $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is the evaluation $\langle f, v \rangle := f(v)$

Proposition 1.22. *Given a basis $\{E_1, \ldots, E_n\}$ for a finite-dimensional V, let $E^1, \ldots, E^n \in V^*$ be defined by $\left\langle E^i, E_j \right\rangle := E^i(E_j) = \delta_{ij}$, then $\left\{E^i\right\}$ is a basis for V^*, called the* ***dual basis****.*

Definition 32. Let V, W be vector spaces, $A : V \to W$ be a linear map. The **dual map** $A^* : W^* \to V^*$ is defined by $\forall f \in W^*, v \in V, \langle A^*(f), v \rangle = A^*(f)(v) := f(A(v)) = \langle f, A(v) \rangle$

Definition 33. Let T_pM be the tangent space to M at p, then the **cotangent space** to M at p is the dual space of T_pM, denoted by T_p^*M. The elements in T_p^*M are called **co-vectors**.

Definition 34. $T^*M := \bigsqcup_{p \in M} T_p^*M$ is called the **cotangent bundle**.

Definition 35. A **one-form** or **co-vector field** is a smooth map $\omega : M \to T^*M$ such that $\forall p \in M, \omega_p := \omega(p) \in T_p^*M$

Definition 36. $\mathfrak{X}^*(M)$ is the set of all one-forms on M.

Proposition 1.23. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, we have that for each $p \in U, \{\Upsilon_1|_p, \ldots, \Upsilon_n|_p\}$ is a basis for T_pM, the dual basis for T_p^*M is $\left\{\Upsilon^1|_p, \ldots, \Upsilon^n|_p\right\}$, such that $\left\langle \Upsilon^i, \Upsilon_j \right\rangle = \delta_{ij}$. Thus $\forall \omega_p \in T_p^*M, \omega_p = \sum_{i=1}^n \omega_i(p) \Upsilon^i|_p$ uniquely. And $\omega_i(p)$ can be get by $\omega_i(p) = \langle \omega, \Upsilon_i \rangle$*

Definition 37. The **coordinate co-vector field** is the map $\Upsilon^i : U \to T^*M$ by $\Upsilon^i(p) := \Upsilon^i|_p$. One can check that $\Upsilon^i \in \mathfrak{X}^*(M)$ is a co-vector field.

Definition 38. Given a smooth function $f \in C^\infty(M)$, and a co-vector field $\omega \in \mathfrak{X}^*(M)$, we define $f\omega := f \cdot \omega \in \mathfrak{X}^*(M)$ to be $(f\omega)(p) := f(p)\omega_p \in T_p^*M$

Corollary 1.24. *We can thus write any co-vector field $\omega \in \mathfrak{X}^*(M)$ as*

$$\omega = \sum_{i=1}^n \omega_i \Upsilon^i$$

Definition 39. Given any $\omega \in \mathfrak{X}^*(M), \mathbf{v} \in \mathfrak{X}(M)$, we can define $\langle \omega, \mathbf{v} \rangle := \omega(\mathbf{v})$ by $\langle \omega, \mathbf{v} \rangle (p) := \langle \omega_p, \mathbf{v}_p \rangle$.

Proposition 1.25. *Given any $\omega = \sum_{i=1}^n \omega_i \Upsilon^i \in \mathfrak{X}^*(M), \boldsymbol{v} = \sum_{i=1}^n \boldsymbol{v}^i \Upsilon_i \in \mathfrak{X}(M)$,*

$$\langle \omega, \boldsymbol{v} \rangle = \sum_{i=1}^n \omega_i \boldsymbol{v}^i$$

Definition 40. Let $f \in C^\infty(M)$, the **differential** of f is $df \in \mathfrak{X}^*(M)$, such that $\forall p \in M, D \in T_pM, (df)_p(D) := Df \in \mathbb{R}$. Thus we have a function $d : C^\infty(M) \to \mathfrak{X}^*(M)$

Proposition 1.26. *Given a vector field $\boldsymbol{v}$, we have*

$$\langle df, \boldsymbol{v} \rangle |_p = \langle df_p, \boldsymbol{v}_p \rangle = \boldsymbol{v}_p(f)$$

Proposition 1.27.

$$\Upsilon^j = dx^j, \Upsilon_j = \frac{\partial}{\partial x^j} = \partial_j$$

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Corollary 1.28. *Let $(U, \phi = (x^1, \ldots, x^n))$ be a coordinate chart for M, then for any $f \in C^\infty(M)$, we have*

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i = (\partial_i f) dx^i$$

1.8 Tensors

Definition 41. A (r,s) tensor, or a **r-variant-s-covariant-tensor** is an element from $(V)^{\otimes r} \otimes (V^*)^{\otimes s}$.

Definition 42. A (r,s) tensor $\mathcal{T} = \sum_{(i_1,\dots,i_r,j_1,\dots,j_s)\in[n]^{r+s}} \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} E_{i_1} \otimes \cdots \otimes E_{i_r} \otimes E^{j_1} \otimes \cdots \otimes E^{j_s}$ can be viewed as a map $(V^*)^r \times V^s \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{T}(\omega^1,\dots,\omega^r,v_1,\dots,v_s) &:= \sum_{(i_1,\dots,i_r,j_1,\dots,j_s)\in[n]^{r+s}} \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} \omega^1(E_{i_1})\cdots\omega^r(E_{i_r})E^{j_1}(v_1)\cdots E^{j_s}(v_s)\\ &= \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} \left\langle \omega^1, E_{i_1}\right\rangle \cdots \left\langle \omega^r, E_{i_r}\right\rangle \left\langle E^{j_1}, v_1\right\rangle \cdots \left\langle E^{j_s}, v_s\right\rangle\end{aligned}$$

Example 1.8.1. A vector is a (1,0) tensor.

Example 1.8.2. A co-vector is a (0,1) tensor.

Example 1.8.3. A real inner product is a (0,2) tensor.

Example 1.8.4. The determinant of a $n \times n$ real matrix is a (0,n) tensor as a function on the column/row vectors.

1.9 Alternating Tensor and wedge product

Definition 43. A covariant k-tensor is **symmetric** if $\forall 1 \le i < j \le k, v_1,\dots,v_k \in V$,
$\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k) = \alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k)$.
It is **alternating** or **anti-symmetric** if $\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k) = -\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k)$

Definition 44. Given a permutation $\sigma \in S_k$, **sign** of it is $Sgn(\sigma) := \begin{cases} 1 & \text{permutation is got by even transposition} \\ -1 & \text{permutation is got by odd transposition} \end{cases}$

Definition 45. Given a tensor $\alpha \in T^k(V^*)$, the **alternation** of it is $Alt(\alpha) \in T^k(V^*)$, defined by $Alt(\alpha)(v_1,\dots,v_k) := \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})$

Proposition 1.29. $\forall \alpha \in T^k(V^*), Alt(\alpha) \in \Lambda^k(V^*)$

Proposition 1.30. $\forall \alpha \in \Lambda^k(V^*), Alt(\alpha) = \alpha$

Definition 46. The **wedge product** of two covariant tensors u, v is defined to be $u \wedge v := u \otimes v - v \otimes u$

Proposition 1.31. $\forall s, s \wedge s = 0$

Proposition 1.32. *For* $u_1,\dots,u_k$, *we have* $u_1 \wedge \cdots \wedge u_k = k!Alt(u_1 \otimes \cdots \otimes u_k) = \sum_{\sigma\in S_k} sgn(\sigma)v_{\sigma(1)},\dots,v_{\sigma(k)}$

Proposition 1.33. *Let* $\left\{E^1,\dots,E^n\right\}$ *be a dual basis for* V^*, *and* $I = (i_1,\dots,i_k) \in [n]^k$, *then we define the* ***elementary alternating tensor***

$$E^I = E^{i_1} \wedge \cdots \wedge E^{i_k}$$

Proposition 1.34. *Let* $I, J \in [n]^k$ *be both increasing, then* $E^I \wedge E^J = (-1)^{|I|\cdot|J|} E^J \wedge E^I$

Definition 47. Let $I = (i_1,\dots,i_k), J = (j_1,\dots,j_k) \in [n]^k$, then
$$\delta^I_J = \begin{cases} 0 & \text{if I or J have repeated indices, or I is not any permutation of J} \\ 1 & \text{if I is an even permutation of J} \\ -1 & \text{if I is an odd permutation of J} \end{cases}$$

Proposition 1.35. *If* $I = (i_1,\dots,i_k)$ *have any repeated indices, i.e.* $\exists 1 \le l \ne m \le k, i_l = i_m$, *then* $E^I = 0$

Proposition 1.36. *If* $J = (i_{\sigma(1)},\dots,i_{\sigma(k)})$, *then* $E^J = sgn(\sigma)E^I$

Proposition 1.37. *Let* $I = (i_1,\dots,i_k), J = (j_1,\dots,j_k) \in [n]^k$, *then* $E^I(E_{j_1},\dots,E_{j_k}) = \delta^I_J$

Proposition 1.38. *Let* $\dim(V) = n, k \in \mathbb{N}^+$, *if* $K > n$, *then* $\Lambda^k(V^*) = \{0\}$, *otherwise* $\dim(\Lambda^k(V^*)) = \binom{n}{k}$, *and a basis is given by* $\mathcal{E}^k := \left\{E^I | I \text{ is increasing multi-index of length } k\right\}$

1.10 Tensor fields or k-form

Definition 48. Given a smooth manifold M, a **covariant k-tensor field on M** or a **k-form** is a smooth map $A : M \to T^k T^* M$, s.t. $A_p := A(p) \in T^K(T_p^* M)$. The set of all k-forms is $\Gamma(T^K T_p^* M)$

Proposition 1.39. *$\Gamma(T^K T_p^* M)$ is an infinite dimensional vector space.*

Proposition 1.40. *Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M, then for covariant k-tensor field $A \in \Gamma(T^K T_p^* M)$, it can be written as $A|_U = \sum_{(i_1,\dots,i_k)\in[n]^k} A_{i_1,\dots,i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$*

Definition 49. An **alternating covariant k-tensor field** or **alternating k-form** on M is a smooth map $A : M \to \bigsqcup_{p\in M} \Lambda^k(T_p^* M)$ such that $\forall p \in M, A_p := A(p) \in \Lambda^k(T_p^* M)$. The set of all alternating k-forms on M is $\Omega^k(M)$

Definition 50. $\Omega^0(M) := c^\infty(M)$

Definition 51. $\wedge : \Omega^k(M)\times\Omega^l(M) \to \Omega^{k+l}(M)$ is defined to be $(\zeta\wedge\eta)_p := \zeta_p\wedge\eta_p$, where $\zeta_p \in \Lambda^k(T_p^* M), \eta_p \in \Lambda^l(T_p^* M)$. For $k = 0, f \in C^\infty(M), f \wedge \eta := f\eta$

Proposition 1.41. *Let $(U, \phi = (x^1, \dots, x^n))$ be a coordinate chart for M, then for an alternating covariant k-tensor field $\omega \in \Omega^k(M)$, it can be written as $\omega|_U = \sum_{(i_1<\cdots<i_k)\in[n]^k} \omega_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $\omega_{i_1,\dots,i_k} = \omega(\partial_{i_1}, \dots, \partial_{i_k})$, and $dx^{i_1} \wedge \cdots \wedge dx^{i_k}(\partial_{j_1}, \dots, \partial_{j_k}) = \delta^I_J$*

1.11 Push-forward and Pull-back

Definition 52. For $f \in \Omega^0(N) = C^\infty(N)$, we have the **pull-back** of f by a smooth map $F : M \to N$ is

$$F^* f = f \circ F \in C^\infty(M)$$

.

Definition 53. The **push-forward** of a vector field $\mathbf{v} \in \mathfrak{X}(M)$ by a diffeomorphism $F : M \to N$ is the unique vector field $F_*\mathbf{v} \in \mathfrak{X}(N)$, defined by

$$\forall q \in N, F_*\mathbf{v}_q := dF_{F^{-1}(q)}(\mathbf{v}_{F^{-1}(q)})$$

Proposition 1.42. *For any diffeomorphism $F : M \to N$, vector field $\boldsymbol{v} \in \mathfrak{X}(N), f \in c^\infty(N)$, we always have*

$$\langle \boldsymbol{v}, F^* f\rangle_p = \langle F_*\boldsymbol{v}, f\rangle_{F(p)}$$

Definition 54. Given a smooth map $F : M \to N$, and a co-vector field $\omega \in \mathfrak{X}(N)$, the **pull-back of a 1-form** ω by F is $F^*\omega \in \mathfrak{X}^*(M)$ defined by $\langle F^*\omega, \mathbf{v}\rangle|_p = \langle \omega, F_*\mathbf{v}\rangle|_{F(p)}$

Proposition 1.43. *Given $u \in C^\infty(M)$*

1. *$F^*uF^*\omega = (u \circ F)F^*\omega = F^*(u\omega) \in \mathfrak{X}^*(M)$*
2. *$F^*(du) = d(u \circ F) = d(F^*u) \in \mathfrak{X}^*(M)$*

Definition 55. Given a smooth map $F : M \to N, p \in M, \alpha \in T^k(T^*_{F(p)}N)$, the **pull-back of a covariant k-tensor** α by F at p is $dF_p^*(\alpha) \in T^k(T_p^* M)$, defined by $dF_p^*(\alpha)(v_1, \dots, v_k) = \alpha(dF_p(v_1), \dots, dF_p(v_k)) \in C^\infty(N)$. This way we obtain a linear map $dF_p^* : T^k(T^*_{F(p)}N) \to T^k(T_p^* M)$

Definition 56. Let $A \in \Gamma(T^kT^*N)$ be a k-form, the **pull-back of a k-form** A by a smooth map $F : M \to N$ is $F^*A \in \Gamma(T^kT^*M)$ defined by $(F^*A)_p := dF_p^*(A_{F(p)})$. This way we get a $F^* : \Gamma(T^kT^*N) \to \Gamma(T^kT^*M)$

Definition 57. The **push-forward of a contra-variant k-tensor field V** $\in \Lambda(T^kTM)$ by a diffeomorphism $F : M \to N$ is $F^*(\mathbf{V}) \in \Lambda(T^kTN)$, defined by $\forall A \in \Gamma(T^{k-1}T^*N), p \in M, \langle \mathbf{V}, F^*A\rangle|_p = \langle(F_*\mathbf{V}), A\rangle|_{F(p)}$

Proposition 1.44. *For any* $\forall A \in \Gamma(T^kT^*N)$, $\boldsymbol{V} \in \Lambda(T^kTM), p \in M, \langle F^*A, \boldsymbol{V}\rangle|_p = \langle A, (F_*\boldsymbol{V})\rangle|_{F(p)}$

Definition 58. We define the **pull-back of a contra-variant k-tensor field V** $\in \Gamma(T^kTN)$ by a diffeomorphism $F : M \to N$ to be $F^*(\mathbf{V}) := F_*^{-1}(\mathbf{V}) \in \Lambda(T^kTM)$, which is the push-forward of $\mathbf{V}$ by $F^{-1} : N \to N$

Definition 59. The **pull-back of an (r,s)-tensor field** $\mathcal{T}$ is by taking the pullback on the r-contra-variant field and the s-co-variant part respectively.

1.12 Lie Derivative

Definition 60. Let $\mathbf{v}, \mathbf{w} \in \mathfrak{X}(M)$, the **Lie derivative** of $\mathbf{w}$ with respect to $\mathbf{v}$ is a map $\mathcal{L}_\mathbf{v}\mathbf{w} : M \to TM$ defined by $\mathcal{L}_\mathbf{v}\mathbf{w}_p = \frac{d}{dt}|_{t=0}(d(\Theta_{-t})_{\Theta_t(p)}\mathbf{w}_{\Theta_t(p)})$, where Θ is the flow generated by $\mathbf{v}$.

Lemma 1.45. $\mathcal{L}_{\boldsymbol{v}}\boldsymbol{w} \in \mathfrak{X}(M)$ *is a vector field.*

Theorem 1.46. *If* $\boldsymbol{v}, \boldsymbol{w} \in \mathfrak{X}(M)$*, then* $\mathcal{L}_{\boldsymbol{v}}\boldsymbol{w} = [\boldsymbol{v}, \boldsymbol{w}]$

Remark. $d(\Theta_{-t})_{\Theta_t(p)}\mathbf{w}_{\Theta_t(p)} = d(\Theta_{-t})_{(\Theta_{-t})^{-1}(p)}\mathbf{w}_{(\Theta_{-t})^{-1}(p)} = (\Theta_{-t})_*\mathbf{w}_p = \Theta_t^*\mathbf{w}_p$, thus we can write $\mathcal{L}_\mathbf{v}\mathbf{w}_p = \frac{d}{dt}|_{t=0}\Theta_t^*\mathbf{w}_p$

Definition 61. We can generalize the **Lie derivative** to act on any (r,s)-tensor field $\mathcal{T}$ by

$$\mathcal{L}_\mathbf{v}\mathcal{T}_p := \frac{d}{dt}|_{t=0}(\Theta_t^*\mathcal{T})|_p := \lim_{t\to 0}\frac{\Theta_t^*\mathcal{T}_{\Theta_t(p)} - \mathcal{T}_p}{t},$$

which is still a (r,s)-tensor. As before, Θ is the flow generated by $\mathbf{v}$.

Proposition 1.47. $\mathcal{L}_{\boldsymbol{v}}(\mathcal{T} \otimes \mathcal{S}) = \mathcal{L}_{\boldsymbol{v}}\mathcal{T} \otimes \mathcal{S} + \mathcal{T} \otimes \mathcal{L}_{\boldsymbol{v}}\mathcal{S}$

Proposition 1.48. $\mathcal{L}_{\boldsymbol{v}}(\langle \mathcal{T}, \mathcal{S}\rangle) = \langle \mathcal{L}_{\boldsymbol{v}}\mathcal{T}, \mathcal{S}\rangle + \langle \mathcal{T}, \mathcal{L}_{\boldsymbol{v}}\mathcal{S}\rangle$

Proposition 1.49. *For* $f \in C^\infty(M), \mathcal{L}_{\boldsymbol{v}}(f) = \boldsymbol{v}(f)$

Proposition 1.50. *Consider a 1-form* $\sigma \in \mathfrak{X}^*(M), \mathcal{L}_{\boldsymbol{v}}\sigma = (\boldsymbol{v}^\mu\partial_\mu\sigma_\nu + \sigma_\mu\partial_\nu\boldsymbol{v}^\mu)dx^\nu$

Proposition 1.51. *In general, for any (r,s)-tensor* $\mathcal{T} = \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s}\partial_{i_r} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$*, we have*

$$(\mathcal{L}_{\boldsymbol{v}}\mathcal{T})^{i_1,\dots,i_r}_{j_1,\dots,j_s} = \left(\boldsymbol{v}^k\partial_k\mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_s} - \mathcal{T}^{k,i_2,\dots,i_r}_{j_1,\dots,j_s}\partial_k\boldsymbol{v}^{i_1} - \cdots - \mathcal{T}^{i_1,\dots,i_{r-1},k}_{j_1,\dots,j_s}\partial_k\boldsymbol{v}^{i_r} + \mathcal{T}^{i_1,\dots,i_r}_{k,j_2,\dots,j_s}\partial_{j_1}\boldsymbol{v}^k + \cdots + \mathcal{T}^{i_1,\dots,i_r}_{j_1,\dots,j_{s-1},k}\partial_{j_s}\boldsymbol{v}^k\right)$$

2 Exterior derivative

Definition 62. Let $\omega = \sum_{(i_1<\cdots<i_k)\in[n]^k}\omega_{i_1,\dots,i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Omega^k(\mathbb{R})$, the **exterior derivative** of ω is $\mathbf{d}\omega := \sum_{(i_1<\cdots<i_k)\in[n]^k} d(\omega_{i_1,\dots,i_k})\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \frac{\partial\omega_{i_1,\dots,i_k}}{\partial x^j}dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Omega^{k+1}(\mathbb{R})$

Proposition 2.1. *If* $f \in C^\infty(U) = \Omega^0(U)$*, then we have* $\boldsymbol{d}f = df \in \Omega^1(U)$

Proposition 2.2. $\boldsymbol{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ *is linear*

Proposition 2.3. $\boldsymbol{d}\circ\boldsymbol{d} : \Omega^k(U) \to \Omega^{k+2}(U)$ *is the zero map.*

Proposition 2.4. $\forall\omega \in \Omega^k(U), \eta \in \Omega^l(U)$*, we have* $\boldsymbol{d}(\omega\wedge\eta) = \boldsymbol{d}\omega\wedge\eta + (-1)^k\omega\wedge\boldsymbol{d}\eta$

Proposition 2.5. *For any smooth map* $F : U \to V$*, and any alternating k-form* $\omega \in \Omega^k(V)$*, we always have* $\boldsymbol{d}(F^*\omega) = F^*(\boldsymbol{d}\omega)$

Theorem 2.6. *For any smooth manifold M and* $k \in \mathbb{N}$*, there is a unique linear map* $\boldsymbol{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ *such that*

1. *If* $f \in C^\infty(M) = \Omega^0(M)$*, then we have* $\boldsymbol{d}f = df \in \Omega^1(M)$
2. $\boldsymbol{d}\circ\boldsymbol{d} : \Omega^k(M) \to \Omega^{k+2}(M)$ *is the zero map.*
3. $\forall\omega \in \Omega^k(M), \eta \in \Omega^l(M)$*, we have* $\boldsymbol{d}(\omega\wedge\eta) = \boldsymbol{d}\omega\wedge\eta + (-1)^k\omega\wedge\boldsymbol{d}\eta$

Theorem 2.7. *Cartan identity:* $\mathcal{L}_{\boldsymbol{v}}\mathcal{T} = \langle\boldsymbol{d}\mathcal{T}, \boldsymbol{v}\rangle + \boldsymbol{d}\langle\mathcal{T}, \boldsymbol{v}\rangle$

3 Mechanics

3.1 Legendre transform

Consider a physical system with n degree of freedom, described by a configuration space Q of n dimensions. Let $q^\alpha \in \mathbb{R}, \alpha = 1, \ldots, n$ to be some generalized coordinates for some point $q \in Q$. The trajectory of the system is a function $s \in \mathbb{R} \to q(s) \in Q$. Note the possible tangent vectors $\dot{q} = \frac{d}{ds}q(s)$ spans the tangent vector space T_qQ. The **velocity phase space** TQ is the tangent bundle $\bigsqcup_{q \in Q} T_qQ$.
Note that a chart $\phi : Q \to \mathbb{R}^n$ induces a chart $\phi' : TQ \to \mathbb{R}^{2n}$ by $\forall \dot{q} \in T_qQ, \phi'(\dot{q}) = (q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$

Definition 63. Let V be a vector space and $l : V \to \mathbb{R}$ be a convex function, the **Legendre transform** of l is a new function $h : V^* \to \mathbb{R}$, defined by $\forall p \in V^*, h(p) := \max_{v \in V} f(p, v)$, where $f(p, v) := p_a v^a - l(v)$

Proposition 3.1. *The value of v to maximize* $f(p, v)$ *is when* $\frac{\partial f}{\partial v^a} = p_a - \frac{\partial l}{\partial v^a} = 0$. *By convexity, we can always get such a maximum value, call it* $v(p)$

Corollary 3.2. $h(p) = f(p, v(p))$

Definition 64. For $l(v) = e^v, h(p) = p\log(p) - p$

3.2 Hamilton's equations

Definition 65. Consider a system with non-singular Lagrangian $L(q, \dot{q}, t)$. We define the **Hamiltonian** of the system as the Legendre transform of the Lagrangian with respect to the generalized velocity $\dot{q}$. Namely, with $p_a = \frac{\partial L}{\partial \dot{q}^a}, H(q, p, t) = \max_{\dot{q}} [p_a \dot{q}^a - L(q, \dot{q}, t)]$

Proposition 3.3. $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \frac{\partial H}{\partial q^a} = -\dot{p}_a, \frac{\partial H}{\partial p^a} = \dot{q}_a$