PMATH 343 Winter 2023 HW4

Due: Friday March 10 2023 at 11:59pm on Crowdmark

Problem 0: Read Chapters 8 and 9.

Problem 1: Let V and W be vector spaces. Show that there is a unique linear isomorphism

$$\beta : V \otimes W \to W \otimes V$$

such that $\beta(v \otimes w) = w \otimes v$ on $V \otimes W$.
We call such β the *braiding isomorphism of vector spaces.*

Problem 2: *We shall now establish properties of the Frobenius Reciprocity natural isomorphism.*
For finite dimensional vector spaces V and W, let
$\phi_{V,W}^{-1} : \mathsf{Lin}(V, W) \to W \otimes V^*$.
For $T \in \mathsf{Lin}(V, W)$ and $S \in \mathsf{Lin}(U, V)$, show that

$$\phi_{U,W}^{-1}(T \circ S) = (T \otimes \mathbf{1}_{V^*}) \circ \phi_{U,V}^{-1}(S).$$

Hint: Provide a diagrammatic proof.

Problem 3: Let U, V, W be vector spaces, and recall that $\mathsf{Bil}(U, V; W)$ denotes the set of bilinear maps $U \times V \to W$.
a: Show that $\mathsf{Bil}(U, V; W) \subset \mathsf{Fun}(U \times V, W)$ –the set of all functions $U \times V \to W$– is a subspace, and hence $\mathsf{Bil}(U, V; W)$ is a vector space in its own right.
b: Show that the usual map $\mathsf{Bil}(U, V; W) \to \mathsf{Lin}(U \otimes V, W)$ given by the universal property of tensor products is an isomorphism.

Problem 4: *We shall see how to embed a vector space canonically into its double dual.*
Let V be a vector space. Recall that for $v \in V$, the function $\mathsf{ev}_v : V^* \to \mathbb{F}$ mapping $f \to f(v)$.
A: Show that ev_v is linear and thus belongs to $(V^*)^*$.
B: Show that $\mathsf{ev} : V \to V^{**}$ mapping $v \to \mathsf{ev}_v$ is linear.
C: Show that if V is finite dimensional then the map from B is an isomorphism.
Warning: keep en eye on the various different contractions/evaluation maps.

Problem 5: Let V, W be vector spaces.
A: Show that the map

$$
\begin{aligned}
m : \mathsf{Lin}(V, W) \times V &\to W \\
(T, v) &\mapsto T(v)
\end{aligned}
$$

is bilinear and hence descends to a linear map $\hat{m} : \mathsf{Lin}(V, W) \otimes V \to W$ given by $\hat{m}(T \otimes v) = T(v)$
B: Let ϕ be the Frobenius Reciprocity natural isomorphism from above, and let $\lambda : W \otimes \mathbb{F} \to W$ be the *unitor* that absorbs the scalar field; ie $\lambda(w \otimes \mathbf{1}_W) = w$ on W. Show that $\hat{m} = \lambda \circ (\mathbf{1} \otimes \mathsf{ev}_V) \circ (\phi^{-1}_{V,W} \otimes \mathbf{1}_V)$.
Here, $\mathsf{ev}_V : V^* \otimes V \to \mathbb{F}$ is given by $f \otimes v \mapsto f(v)$.
hint: Give a diagrammatic proof.

Problem 6: *The trace map is intrinsic.*
Let V be a finite dimensional vector space. Show that the definition of the trace on a linear map $T \in \mathsf{Lin}(V, V)$ is independent from the choice of basis.
hint: Use the change of basis formula for $[T]_{\mathcal{B},\mathcal{B}}$ and the identity $\mathsf{tr}(AB) = \mathsf{tr}(BA)$.

Problem 7: *String diagrammatic gymnastics.*
Show that the canonical isomorphism $\mathsf{Lin}(V, V) \to \mathsf{Lin}(V, V)^*$ maps $\mathbf{1}_V$ to tr. Provide also a diagrammatic proof.