Phys364 - Math Physics

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1 Vector spaces

Definition 1. A vector space V over $\mathbb{F}$ has two operations $+, \cdot$ that follows the axioms:

$$\begin{aligned}
&\forall x, y, z \in V, \forall a, b \in \mathbb{F} \\
&(x+y)+z = x+(y+z) \\
&x+y = y+x \\
\exists \mathbf{0} \in V, \forall x \in V, &x+\mathbf{0} = x \\
\exists -x \in M, &x+(-x) = \mathbf{0} \\
&(ab)\cdot x = a\cdot(b\cdot x) \\
1 \in \mathbb{F}, &1\cdot x = x \\
&(a+b)\cdot x = a\cdot x + b\cdot x \\
&a\cdot(x+y) = a\cdot x + a\cdot y
\end{aligned}$$

Definition 2. (Einstein Notation) $a^i b_i := \sum_i a^i b_i$ when it means summing over all i.

1.1 Change of Basis

Proposition 1.1. *For a finite dimensional vector space, consider two basis $\mathcal{B} = \{v_i\}_1^n, \mathcal{C} = \{u_j\}_1^n$, s.t. $v_i = \sum_j {\Lambda_i}^j u_j$, then the matrix Λ is invertible with inverse $\tilde{\Lambda}$, and $u_j = \sum_i {(\tilde{\Lambda})_j}^i v_i$.*

Proof. Since $\mathcal{B} = \{v_i\}_1^n$ is a basis, we can have some constants ${c_j}^i$s.t. $u_j = {c_j}^i v_i$.
Thus $\delta_i^k v_k = v_i = {\Lambda_i}^j u_j = {\Lambda_i}^j {c_j}^k v_k$.
Since $\mathcal{B} = \{v_i\}_1^n$ is a basis, thus admits a unique decomposition, thus $\delta_i^k = {\Lambda_i}^j {c_j}^k$, which means $\Lambda C = \mathbf{1}$.
Thus Λ is invertible, and C is its inverse $\tilde{\Lambda}$ □

Proposition 1.2. *For a finite dimensional vector space, consider two basis $\mathcal{B} = \{v_i\}_1^n, \mathcal{C} = \{u_j\}_1^n$, s.t. $v_i = \sum_j {\Lambda_i}^j u_j$, then for any $w = \sum_i (w')^i v_i = \sum_j w^j u_j \in \mathcal{V}, (w')^i = {(\tilde{\Lambda})_j}^i w^j, w^j = \sum_i {\Lambda_i}^j (w')^i$*

Proof. $w^j u_j = w = (w')^i v_i = (w')^i {\Lambda_i}^j u_j$, thus $w^j = (w')^i {\Lambda_i}^j$, since $\mathcal{C} = \{u_j\}_1^n$ is a basis and admits a unique decomposition.
Similarly, $(w')^i v_i = w = w^j u_j = w^j {(\tilde{\Lambda})_j}^i v_i \implies (w')^i = w^j {(\tilde{\Lambda})_j}^i$ □

1.2 Inner Product Space

Definition 3. An inner product space is a vector space $\mathcal{H}$ that has an inner product:

$$\begin{aligned}
\langle -, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \text{s.t. } &\forall |v\rangle, |w\rangle \in \mathcal{H}, a, b \in \mathbb{C} \\
&\langle u, av+bw \rangle = a\langle u, v\rangle + b\langle u, w\rangle \\
&\langle v, w\rangle^* = \langle w, v\rangle \\
\forall v \neq 0, &\langle v, v\rangle > 0 \\
&\langle 0, v\rangle = 0
\end{aligned}$$

Remark. Note that if we have an inner product space $\mathcal{H}$, we can define $\langle\psi| \in \mathcal{H}^*$ to be $\langle\psi, \cdot\rangle$ and thus $\forall \left|\phi\right\rangle \in \mathcal{H}, \langle\psi| \left(\left|\phi\right\rangle\right) = \langle\psi, \phi\rangle$

Proposition 1.3. *If $\forall\psi, \langle\psi, \phi\rangle = 0$, then $\left|\phi\right\rangle = 0$*

1.3 Normed Space

Definition 4. A normed vector space is a vector space $\mathcal{V}$ that has an norm (length):

$$\begin{aligned} ||\cdot|| : \mathcal{V} \to \mathbb{R}, \text{s.t. } \forall x, y \in \mathcal{V}, a \in \mathbb{C} \\ ||a \cdot x|| = |a| \, ||x|| \\ ||x + y|| \leq ||x|| + ||y|| \\ \forall x \neq 0, ||x|| > 0 \\ ||0|| = 0 \end{aligned}$$

Proposition 1.4. *For every inner product space with $\langle -, \cdot\rangle$, there is a norm $||x|| = \sqrt{\langle x, x\rangle}$.*

Proof.

$$\begin{aligned} ||a \cdot x|| &= \sqrt{\langle ax, ax\rangle} = \sqrt{a^* a \langle x, x\rangle} = \sqrt{|a|^2}\sqrt{\langle x, x\rangle} = |a| \, ||x|| \\ ||x + y||^2 &= \langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle \\ &\leq ||x||^2 + ||y||^2 + 2 \, ||x|| \, ||y|| \leq (||x|| + ||y||)^2 \\ \forall x \neq 0, ||x|| &= \sqrt{\langle x, x\rangle} > 0 \\ ||0|| &= \sqrt{\langle 0, 0\rangle} = 0 \end{aligned}$$

Thus $||x|| = \sqrt{\langle x, x\rangle}$ is a norm. □

Theorem 1.5. *Cauchy-Schwarz: For every inner product space $\mathcal{H}, \forall \left|\psi\right\rangle, \left|\phi\right\rangle \in \mathcal{H}, |\langle\psi|\phi\rangle| \leq ||\psi|| \, ||\phi||$. In particular, when $V := ||\phi|| \neq 0, ||\psi||^2 \, ||\phi||^2 - |\langle\psi|\phi\rangle|^2 = ||z||^2$, where $z := V \left|\psi\right\rangle - \frac{\langle\psi|\phi\rangle}{V} \left|\phi\right\rangle$*

Proof. Notice that this is trivially true and equality holds to be zero when $\left|\phi\right\rangle = 0$
Now we assume $V \neq 0$, then

$$\begin{aligned} ||z||^2 &= \langle z, z\rangle \\ &= \left\langle V \left|\psi\right\rangle - \frac{\langle\psi|\phi\rangle}{V} \left|\phi\right\rangle, V \left|\psi\right\rangle - \frac{\langle\psi|\phi\rangle}{V} \left|\phi\right\rangle \right\rangle \\ &= V^2 \langle\psi, \psi\rangle - \langle\psi|\phi\rangle \langle\psi, \phi\rangle - \langle\phi|\psi\rangle \langle\phi, \psi\rangle + \frac{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}{\cancel{V^2}} \cancelto{V^2}{\langle\phi, \phi\rangle} \\ &= V^2 \, ||\psi||^2 - | \langle\psi, \phi\rangle |^2 - \cancel{| \langle\psi, \phi\rangle |^2} + \cancel{| \langle\psi, \phi\rangle |^2} \\ &= ||\phi||^2 \, ||\psi||^2 - | \langle\psi, \phi\rangle |^2 \end{aligned}$$

□

1.4 Metric Space

Definition 5. A metric space is a vector space $\mathcal{V}$ that has a (distance) metric:

$$\begin{aligned} d(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}, \text{s.t. } \forall x, y, z \in \mathcal{V} \\ d(x, x) = 0 \\ \forall x \neq y, d(x, y) > 0 \\ d(x, y) = d(y, x) \\ d(x, z) \geq d(x, y) + d(y, z) \end{aligned}$$

Proposition 1.6. *For every normed space with $||\cdot||$, there is a metric $d(x,y) = ||x-y||$.*

Proof.

$$\begin{aligned} d(x,x) &= ||x-x|| = ||0|| = 0 \\ \forall x \neq y, d(x,y) &= ||x-y|| > 0 \\ d(x,y) &= ||x-y|| = ||-(y-x)|| = |-1|\,||y-x|| = ||y-x|| = d(y,x) \\ d(x,z) &= ||x-z|| = ||x-y+y-z|| \geq ||x-y|| + ||y-z|| = d(x,y) + d(y,z) \end{aligned}$$

Thus $d(x,y) = ||x-y||$ is a metric. □

Corollary 1.7. *For every inner product space, there is a metric $d(x,y) = \sqrt{\langle x-y, x-y\rangle}$*

1.5 Dual Space

Definition 6. Given a vector space $\mathcal{V}$, we can define its "dual" vector space $\mathcal{V}^* = \left\{f : Lin(\mathcal{H}, \mathbb{C}) \cap C^0(\mathcal{V}, \mathbb{C})\right\}$ where the dual vectors f are continuous linear maps $\mathcal{V} \to \mathbb{C}$

Definition 7. For an inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}$, we can define its **Kronecker dual basis** $\mathcal{B}^* = \left\{v^i\right\}$, where $\forall i, j \leq dim(\mathcal{H}), \left\langle v^i\right| (v_j) := \delta_{ij}$.

Proposition 1.8. *For a finite-dimensional inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}_{i=1}^n$, its Kronecker dual basis $\mathcal{B}^* = \left\{v^i\right\}_{i=1}^n$ is a basis for $\mathcal{H}^*$.*

Proposition 1.9. *For a finite-dimensional inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}_{i=1}^n$, and $f = \sum_i f_i v^i$, we have $f_i = f(v_i) \in \mathbb{F}$.*

Proof. $f(v_j) = (\sum_i f_i v^i)(v_j) = \sum_i f_i v^i(v_j) = \sum_i f_i \delta^i_j = f_j$ □

Proposition 1.10. *For a finite-dimensional inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}_{i=1}^n$, and $w = \sum_i (w')^i v_i$, we have $(w')^i = v^i(w) \in \mathbb{F}$.*

Proof. $v^j(w) = v^j(\sum_i (w')^i v_i) = \sum_i (w')^i v^j(v_i) = \sum_i (w')^i \delta^i_j = w^j$ □

Definition 8. For an inner product space $\mathcal{H}$ and $|v\rangle := v \in \mathcal{H}$, we can define its **Dirac dual** $\langle v|$, where $\forall w \in \mathcal{H}, \langle v| (w) := \langle v, w\rangle$.

Definition 9. Given a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for $\mathcal{H}$, we can define $g_{ij} := \langle v_i, v_j\rangle$ with respect to this basis.

Proposition 1.11. *The $n \times n$ matrix $\boldsymbol{g}$ is hermitian and positive-definite.*

Definition 10. Since $\mathbf{g}$ is hermitian and positive-definite, we can consider its inverse $\mathbf{g}^{-1}$, which is also hermitian and positive definite. We denote its i, k entry as g^{ik}. Namely, $g^{ik} g_{kj} = \delta^i_j$

Proposition 1.12. *For a finite-dimensional inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}_{i=1}^n$, then for any $w = \sum_i (w')^i v_i \in \mathcal{H}$, we have that $\langle w| = w_i v^i$, where $w_i = \sum_j \bar{w^j} g_{ji}$*

Proof. $w_i = \langle w| (v_i) = \langle w, v_i\rangle = \left\langle \sum_j w^j \vec{v_j}, v_i \right\rangle = \sum_j \bar{w^j} \langle v_j, v_i\rangle = \sum_j \bar{w^j} g_{ji}$ □

Proposition 1.13. *For a finite-dimensional inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}_{i=1}^n$, then we have $\langle v_i| = \sum_j g_{ij} v^j$, and $v^i = \sum_i g^{ij} \langle v_j|$.*

Proof. Notice that $v_k = \delta^i_k v_i$, thus $\langle v_k| = \delta^j_k g_{ji} v^i = g_{ki} v^i$. Thus $\sum_i g^{ij} \langle v_j| = g^{ij} g_{jk} v^k = \delta^i_k v^k = v^i$ □

Proposition 1.14. *For a finite-dimensional inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}_{i=1}^n$, its Dirac dual basis $\mathcal{B}^* = \{\langle v_i|\}_{i=1}^n$ is a basis for $\mathcal{H}^*$.*

Proposition 1.15. *For a finite-dimensional inner product space $\mathcal{H}$ with basis $\mathcal{B} = \{v_i\}_{i=1}^n$, then for any $f = \sum_i f_i v^i \in \mathcal{H}^*$, there is $u = \sum_j f^j \vec{v_j} \in \mathcal{H}$, s.t. $f = \langle u|$, $f^j = g^{ji} \bar{f}_i$.*

Proof. $\forall w \in \mathcal{H}, f(w) = \sum_i f_i v^i(w) = (f_i g^{ij} \langle v_j|)(w) = f_i g^{ij} \langle v_j, w\rangle = \left\langle \bar{f}_i \bar{g^{ij}} v_j, w \right\rangle = \left\langle \bar{f}_i g^{ji} v_j, w\right\rangle$ $\square$

Remark. Indeed, we can check that $f^j = g^{ji}\bar{f}_i = g^{ji} \overline{\overline{f^k g_{ki}}} = g^{ji} f^k g_{ik} = f^k \delta^j_k = f^j$

Proposition 1.16. *There is an anti-linear isomorphism between finite-dimensional inner product space* $\mathcal{H}$ *and* $\mathcal{H}^*$ *of* $|v\rangle \leftrightarrow \langle v|$*, such that* $\forall a, b \in \mathbb{F}, v, u \in \mathcal{H}, \langle av + bu| = \bar{a}\langle v| + \bar{b}\langle u|$

Definition 11. We define the inner product for $\mathcal{H}^*$ to be $\langle f_w, f_u\rangle_{\mathcal{H}^*} := \langle u, w\rangle_{\mathcal{H}}$, where $f_w = \langle w|, f_u = \langle u|$

Proposition 1.17. *For a finite-dimensional inner product space* $\mathcal{H}$ *with basis* $\mathcal{B} = \{v_i\}_{i=1}^n$*, for any* $w = \sum_i (w')^i v_i, u = \sum u^i v_i \in \mathcal{H}$*, we have that* $\langle f_u, f_w\rangle_{\mathcal{H}^*} = \langle w, u\rangle_{\mathcal{H}} = (\bar{w'})^i u^j g_{ij} = w_j u^j = w_j \bar{u_i} g^{ji}$*. One can check that* $\langle -, \cdot\rangle_{\mathcal{H}^*}$ *is indeed an inner product.*

Proof. $\langle w, u\rangle = \left\langle (w')^i v_i, u^j \vec{v_j}\right\rangle = (\bar{w'})^i u^j \langle v_i, v_j\rangle = (\bar{w'})^i u^j g_{ij} = (\bar{w'})^i g_{ij} u^j = w_j u^j = w_j \bar{u_i} g^{ji}$ $\square$

Proposition 1.18. $\left\langle v^j, v^i\right\rangle = \delta^j_k \delta^i_l g^{lk} = g^{ij}$

1.6 n-dimensional Real Space

Definition 12. The standard basis for $\mathbb{R}^n$ is $\{e_i\}_{i=0}^n$, where $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)^T$ is the vector with all entries are zero except the i^{th} entry. Namely, ${e_i}^{(j)} = \delta^j_i$

Definition 13. We will consider the inner product on $\mathbb{R}^n$ that is generated by the $\{e_i\}_{i=0}^n$ basis. Namely, $\langle e_i, e_j\rangle := \delta_{ij}$, thus for any $v = \sum_i v^i e_i, u = \sum_j u^j e_j \in \mathbb{R}^n$, we define $\langle v, u\rangle := \sum_{ij} v^i u^j \delta_{ij} = \sum_i v^i u^i$

Definition 14. the **dot product** of $v = \sum_i v^i e_i, u = \sum_j u^j e_j \in \mathbb{R}^n$ is defined to be $v \cdot u := \sum_i v^i u^i \in \mathbb{R}$

Remark. The dot product is equivalent to the standard inner product in $\mathbb{R}^n$.

Definition 15. The **Levi-Civita symbol** $\epsilon_{i_1,\ldots,i_k} := \begin{cases} 1 & \text{if there is an even permutation } \pi, \text{s.t. } \forall j \in [k], i_j = \pi(j) \\ -1 & \text{if there is an odd permutation } \pi, \text{s.t. } \forall j \in [k], i_j = \pi(j) \\ 0 & \text{if } (i_1, \ldots, i_k) \text{is not a permutation of } [k] \end{cases}$

Definition 16. The **cross product** of $v = \sum_i v^i e_i, u = \sum_j u^j e_j \in \mathbb{R}^n$ is defined to be $v \times u := \sum_{ijk} v^i u^j \epsilon_{ijk} e_k \in \mathbb{R}^n$

Proposition 1.19. *For any* $v = \sum_i v^i e_i, u = \sum_j u^j e_j, w = \sum_j w^k e_k \in \mathbb{R}^n$*, we have* $(v \times u) \cdot w = v^i u^j \epsilon_{ijk} e_k \cdots w^l e_l = v^i u^j w^k \epsilon_{ijk}$

Proposition 1.20. *For any* $v = \sum_i v^i e_i, u = \sum_j u^j e_j \in \mathbb{R}^n, (v \times u) \cdot v = (v \times u) \cdot u = 0$

Proof. $(v \times u) \cdot v = \sum_{ijk} v^i u^j v^k \epsilon_{ijk} = \sum_j (\sum_{i<k} (v^i u^j v^k \epsilon_{ijk} + v^k u^j v^i \epsilon_{kji}) + v^i u^j v^i \epsilon_{iji}) = \sum_j (\sum_{i<k} (v^i u^j v^k (\epsilon_{ijk} + \epsilon_{kji})) + v^i u^j v^i \cdot 0) = \sum_j (\sum_{i<k} v^i u^j v^k \cdot 0) = 0$ $\square$

Proposition 1.21. *For any* $v = \sum_i v^i e_i, u = \sum_j u^j e_j, w = \sum_j w^k e_k \in \mathbb{R}^n$*, we have* $(v \times u) \times w = v^i u^j \epsilon_{ijk} e_k \times w^l e_l = v^i u^j w^l \epsilon_{ijk} \epsilon_{klh} e_h$

Proposition 1.22. *(Leibniz formula) For a* $n \times n$ *matrix* A*, its determinant is* $\sum_{\pi \in S_n} sgn(\pi) A_{1,\pi(1)} \cdots A_{n,\pi(n)} = \sum_{i_1 \ldots i_n} \epsilon_{i_1 \ldots i_n} A_{1,i_1} \cdots A_{n,i_n}$

2 Coordinates

2.1 Coordinate System

Definition 17. A coordinate system for $\mathbb{R}^n$ is a homeomorphism (continuous and bijective map onto its image) $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ by $\Psi(p) := (y^1(p), \ldots, y^n(p))$, where each y^j as a function from $\mathbb{R}^n \to \mathbb{R}$, which takes p and outputs the j^{th} entry of $\Psi(p)$. Notice that we can also treat $\mathbf{p}(y^1, \ldots, y^n)$ as the inverse function of Ψ, which is defined by $\mathbf{p}(\Psi(p)) = \mathbf{p}(y^1(p), \ldots, y^n(p) := p$. For shorthand, we write $\Psi(p) = (y_1, \ldots, y_n)$ to denote this coordinate map.

Definition 18. Cartesian coordinate for $\mathbb{R}^n$ is the identity coordinate map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ by $\Psi(p) := (x^1, \ldots, x^n)$, where $p = x^1 e_1 + \cdots + x^n e_n = (x^1, \ldots, x^n)$.

Definition 19. For the sake of simplicity, we define $\frac{a}{0} := \begin{cases} \infty & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -\infty & \text{if } a < 0 \end{cases}$

Definition 20. Polar coordinate for 2D is a coordinate map $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Psi(p) := (\rho, \phi)$, where for $p = x^1 e_1 + x^2 e_2 = (x^1, x^2), \rho = ||p|| = \sqrt{(x^1)^2 + (x^2)^2}, \phi = \arctan \frac{x^2}{x^1}$. Thus $x^1 = \rho \cos\phi, x^2 = \rho \sin\phi$.

Definition 21. Cylindrical coordinate is a coordinate map $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Psi(p) := (\rho, \phi, z)$, where $p = (x^1, x^2, x^3), \rho = \sqrt{(x^1)^2 + (x^2)^2}, \phi = \arctan \frac{x_2}{x_1}, z = x^3$, and thus $x^1 = \rho\cos\phi, x^2 = \rho \sin\phi, x^3 = z$

Definition 22. Spherical coordinate is a coordinate map $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi(p) := (r, \theta, \phi)$, where $p = (x^1, x^2, x^3), r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \theta = \arccos \frac{x^3}{r}, \phi = \arctan \frac{x^2}{x^1}$, and thus $x^1 = r \sin\theta\cos\phi, x^2 = r\sin\theta\sin\phi, x^3 = r\cos\theta$

2.2 Tangent Vector and Tangent Space

Definition 23. Given a coordinate $\Psi = (y^1, \ldots, y^n) : \mathbb{R}^n \to \mathbb{R}^n$, with $\mathbf{p} := \Psi^{-1}$ the **tangent vector to the j^{th} coordinate line at a point** $p \in \mathbb{R}^n$ is $\vec{v_j}|_p := \frac{\partial}{\partial y^j}|_{\Psi(p)}\mathbf{p}(y^1, \ldots, y^n)$. To distinguish the points in the original $\mathbb{R}^n$ and the tangent vectors, we will use p to represent a point in $\mathbb{R}^n$ (without the vector sign), and $\vec{v}|_p$ (with the vector sign) to mean a tangent vector at p.

Proposition 2.1. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, and the standard representation $\mathbf{p} = \sum_i x^i e_i$, we have that $\vec{v_j}|_p = \sum_i \frac{\partial x^i}{\partial y^j}|_{\Psi(p)}\vec{e_i}|_p$.*

Proof. $\vec{v_j}|_p = \frac{\partial}{\partial y^j}|_{\Psi(p)}\mathbf{p}(y^1, \ldots, y^n) = \frac{\partial}{\partial x^j}|_{\Psi(p)} \sum_i x^i \vec{e_i}(y^1, \ldots, y^n) = \sum_i \frac{\partial}{\partial x^j}|_{\Psi(p)} x^i(y^1, \ldots, y^n)\vec{e_i} = \sum_i \frac{\partial x^i}{\partial y^j}|_{\Psi(p)}\vec{e_i}$ □

Definition 24. A **vector field** is a map $\mathbb{R}^n \to \mathbb{R}^n, p \mapsto \vec{v_\rho}|_p$.

Remark. We can thus write the previous prop as $\vec{v_j} = \sum_i \frac{\partial x^i}{\partial y^j}\vec{e_i}$, but we need to recall that neither side is a vector, but a function that depends on the local p, namely, a vector field.

Proposition 2.2. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, $\{\vec{v_j}|_p\}_{j=1}^n$ forms a basis at p.*

Definition 25. Let $C^\infty(\mathbb{R}^n)$ be the real vector space of smooth functions from $\mathbb{R}^n \to \mathbb{R}$, a **derivation** at $p \in M$ is an $\mathbb{R}-$linear map $D : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the **Leibniz condition**: $\forall f, g \in C^\infty(\mathbb{R}^n), D(fg) = f(p)D(g) + g(p)D(f)$

Definition 26. The **tangent space** to $\mathbb{R}^n$ at $p \in \mathbb{R}^n, T_p\mathbb{R}^n$, is the set of all derivations at p.

Proposition 2.3. *There is a natural isomorphism D_p between $\mathbb{R}^n$ and $T_p\mathbb{R}^n$ by $\vec{u} = \sum_i u^i \vec{e_i}|_p \leftrightarrow \sum_i u^i \frac{\partial}{\partial x^i}|_p =: D_u|_p$, thus $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ is a basis for $T_p\mathbb{R}^n$*

Proof. Consider the following function $g : T_p\mathbb{R}^n \to \mathbb{R}^n, D_u|_p \mapsto \sum_i D_u|_p(x^i)\vec{e_i}$.
Notice that g and D_p are both well-defined and linear.
For any $\vec{u} = \sum_i u^i \vec{e_i} \in \mathbb{R}^n, g(D(\vec{u})) = g(u^i \frac{\partial}{\partial x^i}|_p) = u^i \frac{\partial}{\partial x^i}|_p(x^j)e_j = u^i \frac{\partial x^j}{\partial x^i}|_p e_j = u^i \delta^i_j e_j = u^i \vec{e_i} = \vec{u}$.
Thus $g = D_p^{-1}$ and thus an isomorphism. $\square$

Definition 27. Given a coordinate $\Psi = (y^1, \ldots, y^n) : \mathbb{R}^n \to \mathbb{R}^n$, with $\mathbf{p} := \Psi^{-1}$, the j^{th} partial at a point $p \in \mathbb{R}^n$ is $\partial_j|_p$, defined by $\forall f \in C^1(\mathbb{R}^n), \partial_j|_p(f) := \frac{\partial(f \circ \mathbf{p})}{\partial y^j}|_{(y^1(p),\ldots,y^n(p))} \in T_p\mathbb{R}^n$

Proposition 2.4. *Given a coordinate $\Psi = (y^1, \ldots, y^n) : \mathbb{R}^n \to \mathbb{R}^n$, the natural isomorphism D_p gives that $\vec{v_j} \leftrightarrow \partial_j|_p$, namely, $\partial_j|_p = \sum_i \frac{\partial x^i}{\partial y^j}|_{\Psi(p)} \frac{\partial}{\partial x^i}|_p$*

Proof. For any $f \in C^1(\mathbb{R}^n)$,
$\partial_j|_p(f) = \frac{\partial(f \circ \mathbf{p})}{\partial y^j}|_{(y^1(p),\ldots,y^n(p))} = \sum_{i=1}^n \frac{\partial f}{\partial x^i}|_{\hat{r}(y^1(p),\ldots,y^n(p))} \frac{\partial x^i}{\partial y^j}|_{(y^1(p),\ldots,y^n(p))} = \sum_{i=1}^n \frac{\partial f}{\partial x^i}|_p \frac{\partial x^i}{\partial y^j}|_{\Psi(p)}$ $\square$

Corollary 2.5. *Given a coordinate $\Psi = (y^1, \ldots, y^n) : \mathbb{R}^n \to \mathbb{R}^n$, for any $\vec{u} = \sum_j u^j \vec{v_j} \in \mathbb{R}^n$, the natural isomorphism D_p gives $D_u|_p = \sum_j u^j \partial_j|_p \in T_p\mathbb{R}^n$*

2.2.1 Cartesian Coordinate

Example 2.2.1. Pick the Cartesian coordinate for $\mathbb{R}^n$, we have that $\vec{v_j}|_p = \sum_i \frac{\partial}{\partial x^j}|_{\Psi(p)} x^i \vec{e_i}|_p = \sum_i \delta^i_j \vec{e_i}|_p = \vec{e_j}|_p$

2.2.2 Polar Coordinate

Example 2.2.2. Pick the Polar coordinate for $\mathbb{R}^2$, we have that
$\vec{v_\rho}|_p = \sum_i \frac{\partial}{\partial y^j}|_{\Psi(p)} x^i \vec{e_i}|_p = \frac{\partial \rho \cos\phi}{\partial \rho}|_{\Psi(p)} \vec{e_1}|_p + \frac{\partial \rho \sin\phi}{\partial \rho}|_{\Psi(p)} \vec{e_2}|_p = \cos\phi(p)\vec{e_1}|_p + \sin\phi(p)\vec{e_2}|_p$
$\vec{v_\phi}|_p = \sum_i \frac{\partial}{\partial y^j}|_{\Psi(p)} x^i \vec{e_i}|_p = \frac{\partial \rho \cos\phi}{\partial \phi}|_{\Psi(p)} \vec{e_1}|_p + \frac{\partial \rho \sin\phi}{\partial \phi}|_{\Psi(p)} \vec{e_2}|_p = -\rho(p)\sin\phi(p)\vec{e_1}|_p + \rho(p)\cos\phi(p)\vec{e_2}|_p$.

Remark. We can write the previous example as $\vec{v_\rho} = \cos\phi\vec{e_1} + \sin\phi\vec{e_2}, \vec{v_\phi} = -\rho\sin\phi\vec{e_1} + \rho\cos\phi\vec{e_2}$, but we need to recall that it is not a vector, but a vector field.

2.2.3 Cylindrical Coordinate

Example 2.2.3. Pick the Cylindrical coordinate for $\mathbb{R}^3$, we have that
$\vec{v_\rho} = \frac{\partial \rho\cos\phi}{\partial \rho}\vec{e_1} + \frac{\partial \rho\sin\phi}{\partial \rho}\vec{e_2} + \frac{\partial z}{\partial \rho}\vec{e_3} = \cos\phi\vec{e_1} + \sin\phi\vec{e_2}$
$\vec{v_\phi} = \frac{\partial \rho\cos\phi}{\partial \phi}\vec{e_1} + \frac{\partial \rho\sin\phi}{\partial \phi}\vec{e_2} + \frac{\partial z}{\partial \phi}\vec{e_3} = -\rho\sin\phi\vec{e_1} + \rho\cos\phi\vec{e_2}$
$\vec{v_z} = \frac{\partial \rho\cos\phi}{\partial z}\vec{e_1} + \frac{\partial \rho\sin\phi}{\partial z}\vec{e_2} + \frac{\partial z}{\partial z}\vec{e_3} = \vec{e_3}$

2.2.4 Spherical Coordinate

Example 2.2.4. Pick the Spherical coordinate for $\mathbb{R}^3$, we have that
$\vec{v_r} = \frac{\partial r\sin\theta\cos\phi}{\partial r}\vec{e_1} + \frac{\partial r\sin\theta\sin\phi}{\partial r}\vec{e_2} + \frac{\partial r\cos\theta}{\partial r}\vec{e_3} = \sin\theta\cos\phi\vec{e_1} + \sin\theta\sin\phi\vec{e_2} + \cos\theta\vec{e_3}$
$\vec{v_\phi} = \frac{\partial r\sin\theta\cos\phi}{\partial \phi}\vec{e_1} + \frac{\partial r\sin\theta\sin\phi}{\partial \phi}\vec{e_2} + \frac{\partial r\cos\theta}{\partial \phi}\vec{e_3} = -r\sin\theta\sin\phi\vec{e_1} + r\sin\theta\cos\phi\vec{e_2}$
$\vec{v_\theta} = \frac{\partial r\sin\theta\cos\phi}{\partial \theta}\vec{e_1} + \frac{\partial r\sin\theta\sin\phi}{\partial \theta}\vec{e_2} + \frac{\partial r\cos\theta}{\partial \theta}\vec{e_3} = r\cos\theta\cos\phi\vec{e_1} + r\cos\theta\sin\phi\vec{e_2} - r\sin\theta\vec{e_3}$

2.3 Riemannian metric tensor and Jacobian

Definition 28. Given a coordinate $\Psi = (y^1, \ldots, y^n) : \mathbb{R}^n \to \mathbb{R}^n$, we have that $\{\vec{v_j}\}_{j=1}^n$ forms a basis for $\mathbb{R}^n$, or equivalently, $\{\partial_j\}_{j=1}^n$ forms a basis for $T_p\mathbb{R}^n$, we can thus consider $g_{ij}|_p := \langle \vec{v_i}|_p, \vec{v_j}|_p \rangle$ with respect to the basis $\{\vec{v_j}|_p\}_{j=1}^n$. Similar as before, we can consider g_{ij} to be a function from $\mathbb{R}^n \to \mathbb{R}$, which we call a **scalar field**, that maps $p \mapsto g_{ij}|_p$. The scalar field is $g_{ij} = \langle \vec{v_i}, \vec{v_j} \rangle = \langle \partial_i, \partial_j \rangle$, which is called a **Riemannian metric tensor**.

Definition 29. Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, and the standard representation $\mathbf{p} = \sum_i x^i e_i$, the **Jacobian** of it is a map from $p \in \mathbb{R}^n$ to an associated $n \times n$ matrix, where the (i, j) entry is given by $J^i{}_j|_p = \frac{\partial x^i}{\partial y^j}|_{\Psi(p)}$

Proposition 2.6. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, and the standard representation $\mathbf{p} = \sum_i x^i e_i$, we have that $\vec{v_j}|_p = \sum_i J^i{}_j|_p \vec{e_i}|_p$ or $\vec{v_j} = \sum_i J^i{}_j \vec{e_i}$*

Proposition 2.7. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, and the standard representation $\mathbf{p} = \sum_i x^i e_i$, we have $(J^{-1})^i{}_j|_p = \frac{\partial y^i}{\partial x^j}|_p$, or $(J^{-1})^i{}_j = \frac{\partial y^i}{\partial x^j}$*

Proof. $JJ^{-1}{}^i{}_k|_p = J^i{}_j|_p (J^{-1})^j{}_k|_p = \frac{\partial x^i}{\partial y^j}|_{\Psi(p)} \frac{\partial y^j}{\partial x^k}{}_p = \frac{\partial x^i}{\partial x^k}{}_p = \delta_{ik}$ □

Proposition 2.8. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, and the standard representation $\mathbf{p} = \sum_i x^i e_i$, by the Change of Basis 1.2, we have that $\forall \vec{w} = \sum_i w^i \vec{e_i} \in \mathbb{R}^n, \vec{w} = \sum_j (w')^j \vec{v_j}, (w')^j = (J^{-1})^j{}_i w^i$ in the $\{v_j\}_1^n$ basis. In addition, $w^i = J^i{}_j (w')^j, \vec{e_i} = (J^{-1})^j{}_i \vec{v_j}$.*

Proposition 2.9. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, and the standard representation $\mathbf{p} = \sum_i x^i e_i$, we have $g_{ij} = (J^\dagger J)_{ij}$. Namely, $\boldsymbol{g} = J^\dagger J$*

Proof. $g_{ij} = \langle v_i, v_j \rangle = \langle J^k{}_i e_k, J^l{}_j e_l \rangle = \overline{J^k{}_i} J^l{}_j \delta_{kl} = \overline{J^l{}_i} J^l{}_j = (J^\dagger)^i{}_l J^l{}_j = (J^\dagger J)_{ij}$ □

Definition 30. Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, any $p \in \mathbb{R}^n$, with the tangent vector basis $\{\vec{v_j}\}$, we define $g|_p := \det(\mathbf{g}|_p)$. As a scalar field, $g := \det(\mathbf{g})$

Proposition 2.10. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, with the tangent vector basis $\{\vec{v_j}\}$, we have $g = \det(\boldsymbol{g}) = \det(J^\dagger J) = \det(J^\dagger)\det(J) = |\det(J)|^2$*

2.3.1 Cartesian Coordinate

Example 2.3.1. Pick the Cartesian coordinate for $\mathbb{R}^n$, we have that $g_{ij} = \langle \vec{e_i}, \vec{e_j} \rangle = \delta_{ij}, g = 1$

2.3.2 Polar Coordinate

Example 2.3.2. Pick the Polar coordinate for $\mathbb{R}^2$, we have that
$g_{\rho\rho}|_p = \langle \vec{v_\rho}|_p, \vec{v_\rho}|_p \rangle = \cos^2\phi(p) + \sin^2\phi(p) = 1$
$g_{\rho\phi}|_p = \langle \vec{v_\rho}|_p, \vec{v_\phi}|_p \rangle = \cos\phi(p)(-\rho(p)\sin\phi(p)) + \sin\phi(p)\rho(p)\cos\phi(p) = 0$
$g_{\phi\rho}|_p = \overline{g_{\rho\phi}|_p} = 0$
$g_{\phi\phi}|_p = \langle \vec{v_\phi}|_p, \vec{v_\phi}|_p \rangle = (-\rho(p)\sin\phi(p))^2 + (\rho(p)\cos\phi(p))^2 = \rho^2(p)$

Remark. We can consider the scalar field g_{ij}, and write the above result as $g_{\rho\rho} = 1, g_{\rho\phi} = 0 = g_{\phi\rho}, g_{\phi\phi} = \rho^2, g = \det(\mathbf{g}) = \rho^2$

Example 2.3.3. Pick the Polar coordinate for $\mathbb{R}^2$, we have that
$J^1{}_\rho|_{(x^1,x^2)} = \frac{\partial x^1}{\partial \rho}|_{(\rho,\phi)} = \frac{\partial \rho\cos\phi}{\partial \rho}|_{(\rho\phi)} = \cos\phi|_{(\rho,\phi)}$
$J^2{}_\rho|_{(x^1,x^2)} = \frac{\partial x^2}{\partial \rho}|_{(\rho\phi)} = \frac{\partial \rho\sin\phi}{\partial \rho}|_{(\rho,\phi)} = \sin\phi|_{(\rho,\phi)}$
$J^1{}_\phi|_{(x^1,x^2)} = \frac{\partial x^1}{\partial \phi}|_{(\rho,\phi)} = \frac{\partial \rho\cos\phi}{\partial \phi}|_{(\rho,\phi)} = -\rho\sin\phi|_{(\rho,\phi)}$
$J^2{}_\phi|_{(x^1,x^2)} = \frac{\partial x^2}{\partial \phi}|_{(\rho\phi)} = \frac{\partial \rho\sin\phi}{\partial \phi}|_{(\rho,\phi)} = \rho\cos\phi|_{(\rho,\phi)}$
Thus $J|_{(x^1,x^2)} = \begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix} |_{(\rho,\phi)}$
Similarly, we have $J^{-1}|_{(x^1,x^2)} = \begin{pmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} \\ -\frac{x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} \end{pmatrix} |_{(x^1,x^2)}$
We can check that $\begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix} \begin{pmatrix} \frac{x^1}{\sqrt{(x^1)^2+(x^2)^2}} & \frac{x^2}{\sqrt{(x^1)^2+(x^2)^2}} \\ -\frac{x^2}{(x^1)^2+(x^2)^2} & \frac{x^1}{(x^1)^2+(x^2)^2} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix} \begin{pmatrix} \frac{\rho\cos\phi}{\rho} & \frac{\rho\sin\phi}{\rho} \\ -\frac{\rho\sin\phi}{\rho^2} & \frac{\rho\cos\phi}{\rho^2} \end{pmatrix}$

$$= \begin{pmatrix}\cos\phi & -\rho\sin\phi\\ \sin\phi & \rho\cos\phi\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi\\ -\frac{\sin\phi}{\rho} & \frac{\cos\phi}{\rho}\end{pmatrix} = \begin{pmatrix}\cos\phi\cos\phi+\rho\sin\phi\frac{\sin\phi}{\rho} & \cos\phi\sin\phi-\rho\sin\phi\frac{\cos\phi}{\rho}\\ \sin\phi\cos\phi-\rho\cos\phi\frac{\sin\phi}{\rho} & \sin\phi\sin\phi+\rho\cos\phi\frac{\cos\phi}{\rho}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

And also $J^\dagger J = \begin{pmatrix}\cos\phi & \sin\phi\\ -\rho\sin\phi & \rho\cos\phi\end{pmatrix}\begin{pmatrix}\cos\phi & -\rho\sin\phi\\ \sin\phi & \rho\cos\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \rho^2\end{pmatrix} = \mathbf{g}$

2.3.3 Cylindrical Coordinate

Example 2.3.4. Pick the Cylindrical coordinate for $\mathbb{R}^3$, we have that
$g_{\rho\rho} = 1, g_{\rho\phi} = g_{\phi\rho} = 0, g_{\phi\phi} = \rho^2, g_{zz} = 1, g_{\rho z} = g_{z\rho} = g_{\phi z} = g_{z\phi} = 0$
$\mathbf{g}_{\rho,\phi,z} = \begin{pmatrix}1 & 0 & 0\\ 0 & \rho^2 & 0\\ 0 & 0 & 1\end{pmatrix}, (\mathbf{g}^{-1})^{\rho,\phi,z} = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{\rho^2} & 0\\ 0 & 0 & 1\end{pmatrix}, g = \det(\mathbf{g}) = \rho^2$

2.3.4 Spherical Coordinate

Example 2.3.5. Pick the Spherical coordinate for $\mathbb{R}^3$, we have that
$g_{rr} = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1,$
$g_{\theta\theta} = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2\cos^2\theta + r^2\sin^2\theta = r^2,$
$g_{\phi\phi} = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2(\theta),$
$g_{r\theta} = r\cos\theta\cos\phi\sin\theta\cos\phi + r\cos\theta\sin\phi\sin\theta\sin\phi - r\sin\theta\cos\theta = r\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi - 1) = 0,$
$g_{\theta r} = g_{\theta\phi} = g_{\phi_\theta} = g_{\phi r} = g_{r\phi} = 0$
$\mathbf{g}_{r,\theta,\phi} = \begin{pmatrix}1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta\end{pmatrix} (\mathbf{g}^{-1})^{r,\theta,\phi} = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{r^2} & 0\\ 0 & 0 & \frac{1}{r^2\sin^2\theta}\end{pmatrix}, g = \det(\mathbf{g}) = r^4\sin^2\theta$

2.4 Differential and Gradient

Definition 31. The **total differential** of a function $f \in C^1(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ along a tangent vector $\vec{u}$ is

$$df_p(\vec{u}) := \lim_{\delta\to 0}\frac{f(p+\delta\vec{u}) - f(p)}{\delta}$$

Proposition 2.11. *Consider the standard representation* $\boldsymbol{p} = \sum_i x^i e_i$*, then* $\forall p \in \mathbb{R}^n, \vec{u} = \sum_i u^i\vec{e_i}$,

$$df_p(\vec{u}) = \sum_i u^i\frac{\partial f}{\partial x^i}|_p = \left(u^i\frac{\partial}{\partial x^i}|_p\right)(f)$$

Proof. Consider $f_p : \mathbb{R}^n \to \mathbb{R}, q \mapsto f(p+q), g : \mathbb{R} \to \mathbb{R}^n, t \mapsto t\vec{u}$.
Notice that f_p is just translation of f, thus $\frac{\partial f_p}{\partial x^i}|_q = \frac{\partial f}{\partial x^i}|_{p+q}$. In addition, $g^i(t) = tu^i$

$$\begin{aligned}
df_p(\vec{u}) &= \lim_{\delta\to 0}\frac{f(p+\delta\vec{u}) - f(p)}{\delta}\\
&= \frac{d}{d\delta}f_p(\delta\vec{u})|_{\delta=0}\\
&= \frac{d}{d\delta}(f_p\circ g)(\delta)|_0\\
&= \sum_i\frac{\partial f_p}{\partial x^i}|_{g(0)}\frac{\partial g^i}{\partial\delta}|_0\\
&= \frac{\partial f}{\partial x^i}|_{g(0)+p}\frac{\partial\delta u^i}{\partial\delta}|_0\\
&= \frac{\partial f}{\partial x^i}|_p u^i
\end{aligned}$$

∎

Corollary 2.12. *As a scalar field, we have* $df(\vec{u}) = \sum_i u^i \frac{\partial f}{\partial x^i} = \left(u^i \frac{\partial}{\partial x^i}\right)(f)$

Remark. Notice that we have shown that for any p, there is a natural isomorphism 2.3 between $\mathbb{R}^n$ and $T_p\mathbb{R}^n$, with $\vec{u} = u^i\vec{e_i} \mapsto D_u|_p = u^i \frac{\partial}{\partial e^i}|_p$, thus we can see $df_p \in T_p^*\mathbb{R}^n$, by defining $\forall D_u|_p \in T_p\mathbb{R}^n, df_p(D_u|_p) := D_u|_p(f) \in \mathbb{R}$. From the previous proposition, we see that the two definitions match by $df_p(D_u|_p) = df_p(\vec{u})$.

Proposition 2.13. *Consider any coordinate* $\Psi = (y^1, \ldots, y^n)$ *for* $\mathbb{R}^n$, *any* $p \in \mathbb{R}^n$, *with the tangent vector basis* $\{\vec{v_j}\}$, *then for any* $\vec{u} = \sum_i u^j \vec{v_j}$, *we have that* $df_p(\vec{u}) = \sum_i u^j \partial_j|_p(f)$, *or as scalar fields,* $df(\vec{u}) = \sum_j u^j \partial_j(f)$

Proof. $df_p(\vec{u}) = df_p(D_u|_p) = D_u|_p(f) = \sum_j u^j \partial_j|_p(f)$ by the previous result 2.5. ☐

Corollary 2.14. *Consider any coordinate* $\Psi = (y^1, \ldots, y^n)$ *for* $\mathbb{R}^n$, *any* $p \in \mathbb{R}^n$, *with the tangent vector basis* $\{\vec{v_j}\}$, *then for any differentiable function* $f =: \mathbb{R}^n \to \mathbb{R}$, *we have that* $df(\vec{v_j}) = \partial_j(f)$

Proposition 2.15. *Consider any coordinate* $\Psi = (y^1, \ldots, y^n)$ *for* $\mathbb{R}^n$, *any* $p \in \mathbb{R}^n$, *with the tangent vector basis* $\{\vec{v_j}\}$, *then for any* $j, k \in [n]$, *we have* $dy^j(\partial_k) = \delta^j_k$, *thus* $\left\{dy^j|_p\right\}_1^n$ *form the Kronecker dual basis7 of* $\{\partial_j|_p\}_1^n$ *in* $T_p^*\mathbb{R}^n$

Proof. $dy^j(\partial_k) = dy^j(\delta^l_k \partial_l) = \delta^l_k \partial_l(y^j) = \delta^l_k \partial^j_l = \delta^j_k$ ☐

Definition 32. The **gradient** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field define to be

$$\vec{\nabla} f := \sum_i \frac{\partial f}{\partial x^i} \vec{e_i} \in \mathbb{R}^n$$

Proposition 2.16. *For any* $\vec{u} = \sum_i u^i \vec{e_i}$, *we have* $df(\vec{u}) = \sum_i u^i \frac{\partial f}{\partial x^i} = \left\langle \vec{\nabla} f, \vec{u} \right\rangle$

Proposition 2.17. *Consider any coordinate* $\Psi = (y^1, \ldots, y^n)$ *for* $\mathbb{R}^n$, *any* $p \in \mathbb{R}^n$, *with the tangent vector basis* $\{\vec{v_j}\}$, *then we have that* $\vec{\nabla} f := \sum_j (\nabla f)^j \vec{v_j}$, *where* $(\nabla f)^j = g^{lj}\partial_l(f)$.

Proof. For any $\vec{u} = \sum_i u^j \vec{v_j}$, we have that $u^j \partial_j(f) = df(\vec{u}) = \langle \nabla f, \vec{u} \rangle = (\nabla f)^j u^k g_{jk}$.
In particular, let's take $\vec{u} = v_l = \sum_i \delta^j_l v_j$, which gives $\partial_l(f) = \delta^j_l \partial_j(f) = (\nabla f)^j \delta^k_l g_{jk} = (\nabla f)^j g_{jl}$
Thus $g^{lk}\partial_l(f) = (\nabla f)^j g_{jl} g^{lk} = (\nabla f)^j \delta^k_j = (\nabla f)^k$ ☐

Remark. Notice that we can again consider the natural isomorphism $D_p : \mathbb{R}^n \to T_p\mathbb{R}^n$, where $\vec{\nabla} f := \sum_i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i} \in T_p\mathbb{R}^n$, and the previous propositions give that $df = \left\langle \vec{\nabla} f \right| = (\nabla f)_j dy^j$ in the $\left\{dx^j\right\}_1^n$ basis, where $(\nabla f)_j = \partial_j(f) = (\nabla f)^l g_{lj}$

2.4.1 Cartesian Coordinate

Example 2.4.1. Pick the Cartesian coordinate for $\mathbb{R}^n$, we have that $\vec{\nabla} f = \partial_a(f)\vec{e_a}$

2.4.2 Cylindrical coordinate

Example 2.4.2. Pick the Cylindrical coordinate for $\mathbb{R}^3$, we have that
$\nabla f^\rho = \partial_\rho(f) g^{\rho\rho} + \partial_\phi(f) g^{\phi\rho} + \partial_z(f) g^{z\rho} = \partial_\rho(f); \nabla f^\phi = \frac{1}{\rho^2}\partial_\phi(f); \nabla f^z = \partial_z(f)$

2.4.3 Spherical Coordinate

Example 2.4.3. Pick the Spherical coordinate for $\mathbb{R}^3$, we have that
$\nabla f^r = \partial_r(f); \nabla f^\phi = \frac{1}{r^2 \sin^2 \theta}\partial_\phi(f); \nabla f^\theta = \frac{1}{r^2}\partial_\theta(f)$

2.5 Divergence

2.5.1 3D-Divergence

Definition 33. Given some $\vec{u} = \sum_i u^i(p)\vec{e_i}$, be some vector field in $\mathbb{R}^3$, we define the **divergence** $\vec{\nabla} \cdot \vec{u}|_p := \lim_{V\to 0} \frac{1}{|V|} \iint_{S(V)} \vec{u} \cdot \hat{n} dS$, where V is some volume around $p \in \mathbb{R}^3$

Lemma 2.18. *Consider any coordinate $\Psi = (y^1, y^2, y^3)$ for $\mathbb{R}^3$, with the tangent vector basis $\{\vec{v_j}\}$, we have that $(v_{\vec{\pi(1)}} \times v_{\vec{\pi(2)}}) \cdot v_{\vec{\pi(3)}} = sgn(\pi) \cdot \sqrt{g}$ for any permutation $\pi \in S_3$.*

Proof.

$$\begin{aligned}
(v_{\vec{\pi(1)}} \times v_{\vec{\pi(2)}}) \cdot v_{\vec{\pi(3)}} &= \sum_{i_1,i_2,i_3} {v_{\vec{\pi(1)}}}^{i_1} {v_{\vec{\pi(2)}}}^{i_2} {v_{\vec{\pi(3)}}}^{i_3} \epsilon_{i_1 i_2 i_3} && 1.19 \\
&= \sum_{i_1,i_2,i_3} (J^{i_1})_{\pi(1)} (J^{i_2})_{\pi(2)} (J^{i_3})_{\pi(3)} \epsilon_{i_1 i_2 i_3} && \vec{v_l} = (J^i)_l \vec{e_i} \\
&= \sum_{i'_1,i'_2,i'_3} (J^{i'_1})_1 (J^{i'_2})_2 (J^{i'_3})_3 \epsilon_{i_1 i_2 i_3} && i'_1 = i_{\pi^{-1}(1)}, i'_2 = i_{\pi^{-1}(2)}, i'_2 = i_{\pi^{-1}(3)} \\
&= \sum_{i'_1,i'_2,i'_3} (J^{i'_1})_1 (J^{i'_2})_2 (J^{i'_3})_3 \epsilon_{i'_1 i'_2 i'_3} sgn(\pi) \\
&= sgn(\pi) \sum_{i'_1,i'_2,i'_3} (J^{i'_1})_1 (J^{i'_2})_2 (J^{i'_3})_3 \epsilon_{i'_1 i'_2 i'_3} \\
&= sgn(\pi) \cdot \det(J^T) && 1.22
\end{aligned}$$

However, by prop2.10, we have that $g = |\det(J)|^2 = |\det(J^T)|^2$.
Since $\det(J) \in \mathbb{R}$, we have that $(v_{\vec{\pi(1)}} \times v_{\vec{\pi(2)}}) \cdot v_{\vec{\pi(3)}} = sgn(\pi) \cdot \sqrt{g}$. □

Proposition 2.19. *Consider any coordinate $\Psi = (y^1, y^2, y^3)$ for $\mathbb{R}^3$, and $\vec{u} = \sum_j u^j \vec{v_j}$, we have that $(\vec{\nabla} \cdot \vec{u})|_p = \frac{1}{\sqrt{g}|_p} \sum_j \partial_j \left(\sqrt{g} u^j\right)|_p$, where the scalar field $g = \det(\boldsymbol{g})$ is as defined before. And as scalar fields, $\vec{\nabla} \cdot \vec{u} = \frac{1}{\sqrt{g}} \sum_j \partial_j \left(\sqrt{g} u^j\right)$*

Proof. Consider some volume $V = 2\delta^1 \vec{v_1}|_p \times 2\delta^2 \vec{v_1}|_p \times 2\delta^3 \vec{v_3}|_p$, centered at p. For the sake of convenience, we will just write them as $\vec{v_1}, \vec{v_2}, \vec{v_3}$
Consider the two faces composed by $\vec{v_1}, \vec{v_2}$, the unit normal vector is $\hat{n} := \pm \frac{\vec{v_1} \times \vec{v_2}}{||\vec{v_1} \times \vec{v_2}||}$.
Thus $\vec{u} \cdot \hat{n} = \pm \sum_k \frac{u^k}{||\vec{v_1} \times \vec{v_2}||} (\vec{v_k} \cdot (\vec{v_1} \times \vec{v_2}))$. By Prop 1.20, we know that $\vec{v_1} \cdot (\vec{v_1} \times \vec{v_2}) = \vec{v_2} \cdot (\vec{v_1} \times \vec{v_2}) = 0$.
Thus $\vec{u} \cdot \hat{n} = \pm \frac{u^3}{||\vec{v_1} \times \vec{v_2}||} (\vec{v_3} \cdot (\vec{v_1} \times \vec{v_2})) = \pm \frac{u^3}{||\vec{v_1} \times \vec{v_2}||} \sqrt{g}$ from the previous lemma.
Notice that $u^3, \sqrt{g}$ are scalar fields depending on the point q in this face, and $\vec{v_1}, \vec{v_2}$ are fixed at p.
Thus the total sum of flux from the top face is
$I_1 := \lim_{\delta^1,\delta^2,\delta^3 \to 0} \frac{1}{|V|} \iint_{S_1} \frac{1}{||\vec{v_1} \times \vec{v_2}||} u^3 \sqrt{g} dS = \frac{1}{||\vec{v_1} \times \vec{v_2}||} \lim_{\delta^1,\delta^2,\delta^3 \to 0} \frac{1}{|V|} \iint_{S_1} u^3 \sqrt{g} dS$.
When $\delta^1, \delta^2 \to 0$, then this face just converges to the point $q = p + \delta^3 \vec{v_3}$, and notice that the value of δ^1, δ^2 does not affect $\frac{|V|}{|S_1|} = 2\delta^3 ||\vec{v_3}|| \cos(\theta(\vec{v_3}, \hat{n})) = 2\delta^3 ||\vec{v_3}|| \frac{||\vec{v_3} \cdot \hat{n}||}{||\vec{v_3}|| ||\hat{n}||} = 2\delta^3 \left|\left| \vec{v_3} \cdot \frac{\vec{v_1} \times \vec{v_2}}{||\vec{v_1} \times \vec{v_2}||} \right|\right| = \frac{2\delta^3}{||\vec{v_1} \times \vec{v_2}||} \sqrt{g}|_p$

$$\begin{aligned}
I_1 &= \frac{1}{||\vec{v_1}\times\vec{v_2}||}\lim_{\delta^1,\delta^2,\delta^3\to 0}\frac{1}{|V|}\iint_{S_1}(u^3\sqrt{g})|_q dS \\
&= \frac{1}{||\vec{v_1}\times\vec{v_2}||}\lim_{\delta^1,\delta^2,\delta^3\to 0}\frac{1}{|V|}(u^3\sqrt{g})|_q\iint_{S_1} dS \\
&= \frac{1}{||\vec{v_1}\times\vec{v_2}||}\lim_{\delta^1,\delta^2,\delta^3\to 0}\frac{|S_1|}{|V|}(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}} \\
&= \frac{1}{||\vec{v_1}\times\vec{v_2}||}\lim_{\delta^3\to 0}\frac{||\vec{v_1}\times\vec{v_2}||}{2\delta^3\sqrt{g}|_p}(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}} \\
&= \frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}}}{2\delta^3}
\end{aligned}$$

Similarly, we have that the total sum of flux from the bottom face is
$I_2 := \lim_{\delta^1,\delta^2,\delta^3\to 0}\frac{1}{|V|}\iint_{S_2}\frac{1}{||\vec{v_1}\times\vec{v_2}||}u^3\sqrt{g}dS = -\frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p-\delta^3\vec{v_3}}}{2\delta^3}$
And we have

$$\begin{aligned}
I_1+I_2 &= \frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}}}{2\delta^3} - \frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p-\delta^3\vec{v_3}}}{2\delta^3} \\
&= \frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}}-(u^3\sqrt{g})|_{p-\delta^3\vec{v_3}}}{2\delta^3}|_p \\
&= \frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}}-(u^3\sqrt{g})|_p+(u^3\sqrt{g})|_p-(u^3\sqrt{g})|_{p-\delta^3\vec{v_3}}}{2\delta^3} \\
&= \frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\left(\frac{(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}}-(u^3\sqrt{g})|_p}{2\delta^3}+\frac{(u^3\sqrt{g})|_p-(u^3\sqrt{g})|_{p-\delta^3\vec{v_3}}}{2\delta^3}\right) \\
&= \frac{1}{\sqrt{g}|_p}\lim_{\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p+\delta^3\vec{v_3}}-(u^3\sqrt{g})|_p}{2\delta^3}+\frac{1}{\sqrt{g}|_p}\lim_{-\delta^3\to 0}\frac{(u^3\sqrt{g})|_{p+(-\delta^3)\vec{v_3}}-(u^3\sqrt{g})|_p}{2(-\delta^3)} \\
&= \frac{1}{2\sqrt{g}|_p}d(u^3\sqrt{g})|_p(\vec{v_3})+\frac{1}{2\sqrt{g}|_p}d(u^3\sqrt{g})|_p(\vec{v_3}) \\
&= \frac{1}{\sqrt{g}|_p}d(u^3\sqrt{g})|_p(\vec{v_3}) \\
&= \frac{1}{\sqrt{g}|_p}\partial_3|_p(u^3\sqrt{g})
\end{aligned}$$

Similar proof for the other two pair of faces, and thus we get
$(\vec{\nabla}\cdot\vec{u})|_p = \lim_{V\to 0}\frac{1}{|V|}\iint_{S(V)}\vec{u}\cdot\hat{n}dS = I_1+I_2+I_3+I_4+I_5+I_6 = \frac{1}{\sqrt{g}|_p}\sum_j\partial_j|_p\left(\sqrt{g}u^j\right)$ □

2.5.2 Cartesian Coordinate

Example 2.5.1. Pick the Cartesian coordinate for $\mathbb{R}^3$, and $\vec{u}=\sum_j u^j\vec{v_j}$, we have that $\vec{\nabla}\cdot\vec{u}=\sum_j\partial_j\left(u^j\right)$

2.5.3 General Divergence

Definition 34. Given some $\vec{u}=\sum_i u^i\vec{e_i}$, be some vector field in $\mathbb{R}^n$, we define the **divergence** of ti to be $\vec{\nabla}\cdot\vec{u}:=\sum_i\partial_i\left(u^i\right)$

Remark. Notice that by the previous example, this new general definition of divergence matched the previous definition of 3D divergence in $\mathbb{R}^3$.

Proposition 2.20. *Consider any coordinate $\Psi=(y^1,\ldots,y^n)$ for $\mathbb{R}^n$, and $\vec{u}=\sum_j u^j\vec{v_j}$, we have that $\vec{\nabla}\cdot\vec{u}=\frac{1}{\sqrt{g}}\sum_j\partial_j\left(\sqrt{g}u^j\right)$*

Proof.

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\partial_j\left(\sqrt{g}u^j\right) &= \frac{1}{\sqrt{g}}\left(J^i{}_j\frac{\partial}{\partial x^i}\right)\left(\sqrt{g}u^j\right) \\
&= \frac{J^i{}_j}{\sqrt{g}}\left(\frac{\partial\sqrt{g}}{\partial x^i}u^j + \frac{\partial u^j}{\partial x^i}\sqrt{g}\right) \\
&= J^i{}_j\frac{\partial u^j}{\partial x^i} + \frac{J^i{}_j}{\sqrt{g}}u^j\frac{\partial\sqrt{g}}{\partial x^i} \\
&= J^i{}_j\frac{\partial(J^{-1})^j{}_{i'}w^{i'}}{\partial x^i} + \frac{J^i{}_j}{\sqrt{g}}(J^{-1})^j{}_{i'}w^{i'}\frac{\partial\sqrt{g}}{\partial x^i} \\
&= J^i{}_j w^{i'}\frac{\partial(J^{-1})^j{}_{i'}}{\partial x^i} + J^i{}_j(J^{-1})^j{}_{i'}\frac{\partial w^{i'}}{\partial x^i} + \frac{\delta^i{}_{i'}}{\sqrt{g}}w^{i'}\frac{\partial\sqrt{g}}{\partial x^i} \\
&= J^i{}_j w^{i'}\frac{\partial(J^{-1})^j{}_{i'}}{\partial x^i} + \frac{\partial w^i}{\partial x^i} + \frac{w^i}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i} \\
&= \frac{\partial x^i}{\partial y^j}w^{i'}\frac{\partial^2 y^j}{\partial x^i x^{i'}} + \frac{\partial w^i}{\partial x^i} + \frac{w^i}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i} \\
&= w^{i'}\frac{\partial^2 y^j}{\partial y^j x^{i'}} + \frac{\partial w^i}{\partial x^i} + \frac{w^i}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i} \\
&= w^{i'}\frac{\partial}{\partial x^{i'}}\cancelto{1}{\frac{\partial y^j}{\partial y^j}} + \frac{\partial w^i}{\partial x^i} + \frac{w^i}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i} \\
&= \frac{\partial w^i}{\partial x^i} + \frac{w^i}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i} \\
&\text{similarly, the last term cancels to zero} \\
&= \frac{\partial w^i}{\partial x^i} \\
&= \vec{\nabla}\cdot\vec{u}
\end{aligned}
$$

□

2.5.4 Cylindrical coordinate

Example 2.5.2. Pick the Cylindrical coordinate for $\mathbb{R}^3$, and $\vec{u} = \sum_j u^j\vec{v_j}$, we have that

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{u} &= \frac{1}{\rho}\left(\partial_\rho\left(\rho u^\rho\right) + \partial_\phi\left(\rho u^\phi\right) + \partial_z\left(\rho u^z\right)\right) \\
&= \frac{1}{\rho}\left(\rho\partial_\rho\left(u^\rho\right) + u^\rho + \rho\partial_\phi\left(u^\phi\right) + \rho\partial_z\left(u^z\right)\right) \\
&= \frac{1}{\rho}\partial_\rho\left(\rho u^\rho\right) + \partial_\phi\left(u^\phi\right) + \partial_z\left(u^z\right) \\
&= \partial_\rho\left(u^\rho\right) + \partial_\phi\left(u^\phi\right) + \partial_z\left(u^z\right) + \frac{u^\rho}{\rho}
\end{aligned}
$$

2.5.5 Spherical Coordinate

Example 2.5.3. Pick the Spherical coordinate for $\mathbb{R}^3$, and $\vec{u} = \sum_j u^j \vec{v_j}$, we have that

$$\begin{aligned}
\vec{\nabla} \cdot \vec{u} &= \frac{1}{r^2 \sin\theta} \left(\partial_r \left(r^2 \sin\theta u^r \right) + \partial_\phi \left(r^2 \sin\theta u^\phi \right) + \partial_\theta \left(r^2 \sin\theta u^\theta \right) \right) \\
&= \frac{1}{r^2 \sin\theta} \left(r^2 \sin\theta \partial_r \left(u^r \right) + 2r \sin\theta u^r + r^2 \sin\theta \partial_\phi \left(u^\phi \right) + r^2 \sin\theta \partial_\theta \left(u^z \right) + r^2 \cos\theta u^\theta \right) \\
&= \frac{1}{r^2} \partial_r \left(r^2 u^r \right) + \partial_\phi \left(u^\phi \right) + \frac{1}{\sin\theta} \partial_\theta \left(\sin\theta u^\theta \right) \\
&= \partial_r \left(u^r \right) + \partial_\phi \left(u^\phi \right) + \partial_\theta \left(u^\theta \right) + \frac{2}{r} u^r + \cot\theta u^\theta
\end{aligned}$$

2.6 Laplacian

Definition 35. The **Laplacian** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field $\Delta f := \nabla^2 f := \vec{\nabla} \cdot \vec{\nabla} f$

Proposition 2.21. *Consider any coordinate $\Psi = (y^1, \ldots, y^n)$ for $\mathbb{R}^n$, and a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we have that $\nabla^2 f = \sum_j \frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g} g^{lj} \partial_l(f) \right)$*

Proof. $\nabla^2 f := \vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla} \cdot g^{lj} \partial_l(f) \vec{v_j} = \frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g} g^{lj} \partial_l(f) \right)$ □

2.6.1 Cartesian Coordinate

Example 2.6.1. Pick the Cartesian coordinate for $\mathbb{R}^n$, we have that
$\nabla^2 f = \sum_j \frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g} g^{lj} \partial_l(f) \right) = \sum_j \frac{\partial}{\partial x^j} \left(\delta^{lj} \frac{\partial f}{\partial x^l} \right) = \sum_i \frac{\partial^2 f}{\partial (x^i)^2}$

2.6.2 Cylindrical coordinate

Example 2.6.2. Pick the Cylindrical coordinate for $\mathbb{R}^3$, we have that

$$\begin{aligned}
\nabla^2 f &= \sum_j \frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g} g^{lj} \partial_l(f) \right) \\
&= \frac{1}{\rho} \left(\partial_\rho \left(\rho \partial_\rho(f) \right) + \partial_\phi \left(\rho \frac{1}{\rho^2} \partial_\phi(f) \right) + \partial_z \left(\rho \partial_z(f) \right) \right) \\
&= \frac{1}{\rho} \partial_\rho \left(\rho \partial_\rho(f) \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

2.6.3 Spherical Coordinate

Example 2.6.3. Pick the Spherical coordinate for $\mathbb{R}^3$, we have that

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r^2} \partial_r \left(r^2 \frac{\partial f}{\partial r} \right) + \partial_\phi \left(\frac{1}{r^2 \sin^2\theta} \frac{\partial f}{\partial \phi} \right) + \frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \frac{1}{r^2} \frac{\partial f}{\partial \theta} \right) \\
&= \frac{1}{r^2} \partial_r \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2\theta} \left(\frac{\partial^2 f}{\partial \phi^2} \right) + \frac{1}{r^2 \sin\theta} \partial_\theta \left(\sin\theta \frac{\partial f}{\partial \theta} \right)
\end{aligned}$$

Notice that $\frac{1}{r} \frac{\partial^2 (rf)}{\partial r^2} = \frac{1}{r} \left(\partial_r (r \frac{\partial f}{\partial r} + f) \right) = \frac{1}{r} \left(r \frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r} + \frac{\partial f}{\partial r} \right) = \frac{1}{r^2} \left(r^2 \frac{\partial^2 f}{\partial r^2} + 2r \frac{\partial f}{\partial r} \right) = \frac{1}{r^2} \partial_r \left(r^2 \frac{\partial f}{\partial r} \right)$

$$\nabla^2 f = \frac{1}{r} \frac{\partial^2 (rf)}{\partial r^2} + \frac{1}{r^2 \sin^2\theta} \left(\frac{\partial^2 f}{\partial \phi^2} \right) + \frac{1}{r^2 \sin\theta} \partial_\theta \left(\sin\theta \frac{\partial f}{\partial \theta} \right)$$