PMATH 343 Winter 2023 HW5

Due: Friday March 24 2023 at 11:59pm on Crowdmark

Problem 0: Read Chapters 10, 11, 12 ...

Problem 1: Let H_1, H_2, K_1, K_2 be finite-dimensional Hilbert spaces, and $S \in \mathsf{Lin}(H_1, H_2)$ and $T \in \mathsf{Lin}(K_1, K_2)$ be linear maps. Use the definition of adjoint operators to show that

$$(S \otimes T)^* = S^* \otimes T^*.$$

Furthermore, if S and T are unitaries, conclude that $(S \otimes T)$ is a unitary.

Problem 2: Let H and K be finite-dimensional Hilbert spaces, with orthonormal bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Let $S, T \in \mathsf{Lin}(H, K)$ with associated matrices $A := [S]_{\mathcal{B}',\mathcal{B}}$ and $B := [T]_{\mathcal{B}',\mathcal{B}}$. Show that

$$\langle S|\ T\rangle_F = \sum_{i,j} \overline{A_{ij}} B_{ij}.$$

What is the corresponding expression for $\langle S|\ S\rangle_F$?

Problem 3: Let H_A and H_B be finite-dimensional Hilbert spaces, and assume $\mathsf{dim}(H_B) \geq 2$. Given $|\psi\rangle \in H_A \otimes H_B$, let $[|\psi\rangle]$ denote the equivalence class of $|\psi\rangle$ modulo global phase.

(A): Show that for any fixed non-zero $|\eta\rangle \in H_A$, the set

$$\{[|\eta\rangle|\xi\rangle] : |\xi\rangle \in H_B\} \subset H_A \otimes H_B$$

is infinite.

(B): Conclude that even if we restrict the input to product states, it's impossible for measurement with respect to a basis to give all the possible output states for the 'local' measurement process. (See page 1-7.)

Problem 4: Let $W = \mathsf{span}\{e_1\} \subset \mathbb{C}^2$. Then $\mathbb{C}^2 = W \oplus \mathsf{span}\{e_2\}$ (as vector spaces). Show there is a unique self-adjoint projection P with $\mathsf{Im}(P) = W$ and $\mathsf{Ker}(P) = \mathsf{span}\{e_2\}$ whose matrix is given by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Moreover, since we can also decompose $\mathbb{C}^2 \cong W \oplus \mathsf{span}\{e_1 + e_2\}$, show there is a projection P' with $\mathsf{Im}(P') = W$, $\mathsf{Ker}(P') = \mathsf{span}\{e_1 + e_2\}$, and find the matrix of P' wrt the canonical basis.

Problem 5: Prove equivalence of (1), (2) and (5) in Theorem 11.2.2.

Problem 6: Let $\{|\xi_i\rangle\}_{i=1}^n \subset H$ be an ONB for a Hilbert space H. Show that $\{|\xi_i\rangle\langle\xi_i|\}_{i=1}^n$ is a complete set of pairwise-orthogonal self-adjoint projections.

Problem 7: Let $\{P_i\}_{i\in\mathcal{O}_A} \subset H_B$ be a projective measurement on the finite-dimensional Hilbert space H_A, and let $\{Q_j\}_{j\in\mathcal{O}_B} \subset H_B$ be a projective measurement on the finite-dimensional Hilbert space H_B. Show that the joint measurement $\{P_i \otimes Q_j\}_{i.j}$ is a projective measurement.

Problem 8: Let $\{|\xi_i\rangle\}_1^n \subset H_A$ and $\{|\eta_j\rangle\}_1^m \subset H_B$ be ONB for the Hilbert spaces H_A and H_B, respectively. Show that the projective measurement

$$\{|\xi_i\rangle\langle\xi_i| \otimes |\eta_j\rangle\langle\eta_j|\}_{i,j}$$

is equivalent to measuring wrt the ONB $\{|\xi_i\rangle|\eta_j\rangle\}_{i,j} \subset H_A \otimes H_B$.

Problem 9: Let $H \cong \bigoplus_1^n H_i$ be a Hilbert space direct sum, where H is finite-dimensional. For $i = 1,\dots n$, let $\{|\xi_\ell^i\rangle\}_{\ell=1}^{n_i} \subset H_i$ be an ONB. Show that the projection onto H_i given by $P_i = \sum_{\ell=1}^{n_i} |\xi_\ell\rangle\langle\xi_\ell|$ does not depend on the chosen ONB for H_i.