PMATH343 A2

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Question 1

Consider the map $T : V \times W \to W \otimes V, T(v, w) := w \otimes v$.
Notice that the tensor product is bi-linear by Lemma 8.2.2.
Thus $T(au + v, w) = w \otimes (au + v) = a(w \otimes u) + w \otimes v = aT(u, w) + T(v, w)$,
$T(v, au + w) = (au + w) \otimes v = a(u \otimes v) + w \otimes v = aT(v, u) + T(v, w)$, thus T is bi-linear as well.
By the universal property, there exists a unique linear map $\beta : V \otimes W \to W \otimes V, \text{s.t. } \beta(v \otimes w) = T(v, w) = w \otimes v$
Now take any two basis $\mathcal{B} = \{w_i\} \subseteq W, \mathcal{C} = \{v_j\} \subseteq V$, then we know that $\mathcal{D} = \mathcal{B} \otimes \mathcal{C} = \{w_i \otimes v_j\}_{ij}$ is a basis of $W \otimes V$, and $\mathcal{D}' = \mathcal{C} \otimes \mathcal{B} = \{v_j \otimes w_i\}_{ij}$ is a basis of $V \otimes W$.
Since $\beta(\mathcal{D}') = \mathcal{D}$, it is an isomorphism.
In addition, consider any other linear map $S : V \otimes W \to W \otimes V$ such that $S(v \otimes w) = w \otimes v$, and any $\phi = \sum_{ij} a_{ij} v_j \otimes w_i \in V \otimes W$,
$\beta(\phi) = \beta(\sum_{ij} a_{ij} v_j \otimes w_i) = \sum_{ij} a_{ij} \beta(v_j \otimes w_i) = \sum_{ij} a_{ij} w_i \otimes v_j = \sum_{ij} a_{ij} S(v_j \otimes w_i) = S(\sum_{ij} a_{ij} v_j \otimes w_i) = S(\phi)$, thus $S = \beta$ is unique.

Question 2

(a) Consider any bases $\mathcal{B} = \{v_i\}_1^n \subseteq V, \mathcal{C} = \{w_j\}_1^m \subseteq W, \mathcal{D} = \{u_k\}_1^l \subseteq U$.

We know that $\{u^k\}_1^l$ is a basis for U^*, $\{w_j \otimes u^k\}_{k,j}$ is a basis for $W \otimes U^*$, and $\{v_i \otimes u^k\}_{i,k}$ is a basis for $V \otimes U^*$

Thus $\phi_{U,V}^{-1}(S) = \sum_{i,k} a_{ik} v_i \otimes u^k \in V \otimes U^*$

$(T \otimes \mathbf{1}_{U^*}) \circ \phi_{U,V}(S) = (T \otimes \mathbf{1}_{U^*}) \sum_{i,k} a_{ik} v_i \otimes u^k = \sum_{i,k} a_{ik} (T \otimes \mathbf{1}_{v^*}) v_i \otimes u^k = \sum_{i,k} a_{ik} T(v_i) \otimes u^k$

Consider any $u = \sum_{h=1}^l c_h u_h \in U$,

$\phi_{U,W}((T \otimes \mathbf{1}_{U^*}) \circ \phi_{U,V}(S))(u) = \phi_{U,W}(\sum_{i,k} a_{ik} T(v_i) \otimes u^k)(u) = \sum_{i,k} a_{ik} \phi_{U,W}(T(v_i) \otimes u^k)(u)$

$= \sum_{i,k} a_{ik} u^k(u) T(v_i) = \sum_{i,k} a_{ik} u^k(\sum_{h=1}^l c_h u_h) T(v_i) = \sum_{i,k,h} a_{ik} c_h u^k u_h T(v_i)$

$= \sum_{i,k,h} a_{ik} c_h \delta_{hk} T(v_i) = \sum_{i,k} a_{ik} c_k T(v_i)$

And $T \circ S(u) = T \circ \phi_{U,W}(\phi_{U,V}^{-1}(S))(u) = T(\phi_{U,W}(\sum_{i,k} a_{ik} v_i \otimes u^k)(u))$

$= T(\sum_{i,k} a_{ik} \phi_{U,W}(v_i \otimes u^k)(u)) = T(\sum_{i,k} a_{ik} u^k(u) v_i) = T(\sum_{i,k} a_{ik} u^k(\sum_{h=1}^l c_h u_h) v_i)$

$= T(\sum_{i,k} a_{ikh} c_h u^k(u_h) v_i) = T(\sum_{i,k,h} a_{ik} c_h \delta_{kh} v_i) = T(\sum_{i,k} a_{ik} c_k v_i) = \sum_{i,k} a_{ik} c_k T(v_i)$

Thus $T \circ S = \phi_{U,W}((T \otimes \mathbf{1}_{U^*}) \circ \phi_{U,V}(S)) \implies$

$\phi_{U,W}^{-1}(T \circ S) = (T \otimes \mathbf{1}_{U^*}) \circ \phi_{U,V}(S)$ since $\phi_{U,W}$ is isomorphic thus injective.

Question 3

(a) Firstly, notice that the zero function is bi-linear, thus in $Bil(U, V; W)$, thus non-empty.
Consider any two bi-linear functions $f, g \in Bil(U, V; W), s, a \in \mathcal{F}, u, w \in u, v, x \in v$
$(sf + g)(au + w, v) = sf(au + w, v) + g(au + w, v)$
$= asf(u, v) + sf(w, v) + ag(u, v) + g(w, v)$ since f and g are both bilinear
$= a(sf(u, v) + g(u, v)) + (sf(w, v) + g(w, v)) = a(sf + g)(u, v) + (sf + g)(w, v)$
$(sf + g)(u, av + x) = sf(u, av + x) + g(u, av + x)$
$= asf(u, v) + sf(u, x) + ag(u, v) + g(u, x)$ since f and g are both bilinear
$= a(sf(u, v) + g(u, v)) + (sf(u, x) + g(u, x)) = a(sf + g)(u, v) + (sf + g)(u, x)$
Thus $sf + g \in Bil(U, V; W)$
Thus $Bil(U, V; W)$ is a subspace of $Fun(U \times V, W)$

(b) By the universal property, we know that for each $f \in Bil(U, V; W)$,
$\exists! \phi(f) \in Lin(U \otimes V, W), \text{s.t. } \forall u \in U, v \in V, f(u, v) = \phi(f)(u \otimes v)$
Consider any $f, g \in Bil(U, V; W), \forall u \in U, v \in V, s \in \mathcal{F}$
$(s\phi(f) + \phi(g))(u \otimes v) = s\phi(f)(u \otimes v) + \phi(g)(u \otimes v) = sf(u, v) + g(u, v) = (sf + g)(u, v) = \phi(sf + g)(u \otimes v)$
Since they agree on every element in a spanning set of $U \otimes V, \phi(sf + g) = s\phi(f) + \phi(g)$,
Thus ϕ is linear.
Given any $g \in Lin(U \otimes V, W)$, consider the map $h : U \times V \to W, h(u, v) := g(u \otimes v)$
$\forall a \in \mathcal{F}, u, w \in u, v, x \in v,$
$h(au + w, v) = g(au + w \otimes v) = g(au \otimes v + w \otimes v)$ since tensor products are bilinear
$= ag(u \otimes v) + g(w \otimes v)$ since g is linear
$= ah(u, v) + h(w, v)$
$h(u, av + x) = g(u \otimes av + x) = g(u \otimes v + u \otimes x)$ since tensor products are bilinear
$= ag(u \otimes v) + g(u \otimes x)$ since g is linear
$= ah(u, v) + h(u, x)$
Thus $h \in Bil(U, V; W)$.
In addition, $\phi(h)(u \otimes v) = h(u, v) = g(u \otimes v)$
$\implies \phi(h) = g$ Since they agree on every element in a spanning set
Thus ϕ is surjective.
Now consider any $f, g \in Bil(U, V; W), \text{s.t. } \phi(f) = \phi(g), \forall u \in U, v \in V$
$f(u, v) = \phi(f)(u \otimes v) = \phi(g)(u \otimes v) = g(u, v) \implies f = g$
Thus ϕ is injective.
This concludes that ϕ is an isomorphism.

Question 4

(a) Given any $f, g \in V^*, s \in \mathcal{F}, ev_v(sf+g) = (sf+g)(v) = sf(v) + g(v) = s \cdot ev_v(f) + ev_v(g)$
Thus $ev_v \in (V^*)^*$

(b) Consider any $v, w \in V, c \in \mathcal{F}, f \in V^*, ev_{cv+w}(f) = f(cv+w)$
$= cf(v) + f(w)$ since f is linear
$= c \cdot ev_v(f) + ev_w(f) = (c \cdot ev_v + ev_w)(f) \implies ev_{cv+w} = c \cdot ev_v + ev_w$
Thus $ev(v) := ev_v$ is linear.

(c) Let $\mathcal{B} = \{v_i\}_1^n$ be a basis for V,
Given any $v = \sum_{i=1}^n a_i v_i \in V,$ s.t. $ev_v = 0 \in (V^*)^*$
$\forall f \in V^*, ev_v(f) = 0$
Since ev is linear by b), $ev_v = ev(v) = ev(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i ev(v_i) = \sum_{i=1}^n a_i ev_{v_i}$
Thus $ev_v(f) = (\sum_{i=1}^n a_i ev_{v_i})(f) = \sum_{i=1}^n a_i (ev_{v_i}(f)) = \sum_{i=1}^n a_i f(v_i) = 0$
In particular, the $v^j \in V^*$ give that $\sum_{i=1}^n a_i v^j(v_i) = a_j = 0$.
Since this works for all j, we have $v = \sum_{i=1}^n a_i v_i = 0$
Thus ev is injective.
Since $\dim(V) = \dim(V^*) = \dim((V^*)^*)$, ev is an isomorphism.

Question 5

(a) $\forall f, g \in Lin(V, W), v, u \in V, c, s \in \mathcal{F}$,
$m(sf + g, v) = (sf + g)(v) = sf(v) + g(v) = sm(f, v) + m(g, v)$
$m(f, cv + u) = f(cv + u) = cf(v) + f(u) = cm(f, v) + m(f, u)$
Thus m is bilinear, and by the universal property, there is a unique linear map
$\hat{m} : Lin(V, W) \otimes V, \text{s.t. } \hat{m}(T \otimes v) = m(T, v) = T(v)$

(b) Pick some basis $\{v_i\} \subseteq V, \{w_j\} \subseteq W$, then $\{w_j \otimes v^i\}_{ij}$ form a basis for $W \otimes V^*$
Consider any $T \in Lin(V, W), v = \sum_k c_k v_k \in V, \phi_{V,W}^{-1}(T) = \sum_{ij} a_{ij} w_j \otimes v^i \in W \otimes V^*$
$\lambda \circ (\mathbf{1} \otimes ev_v) \circ (\phi_{V,W}^{-1} \otimes \mathbf{1}_V)(T \otimes v) = \lambda \circ (\mathbf{1} \otimes ev_v)(\sum_{ij} a_{ij} w_j \otimes v^i \otimes v)$
$= \sum_{ij} a_{ij} \lambda \circ (\mathbf{1} \otimes ev_v)(w_j \otimes v^i \otimes v) = \sum_{ij} a_{ij} \lambda(w_j \otimes v^i(v)) = \sum_{ij} a_{ij} \lambda \circ (\mathbf{1} \otimes ev_v)(w_j \otimes v^i \otimes v)$
$= \sum_{ij} a_{ij} \lambda(w_j \otimes v^i(\sum_k c_k v_k)) = \sum_{ij} a_{ij} \lambda(w_j \otimes \sum_k c_k v^i(v_k)) = \sum_{ij} a_{ij} \lambda(w_j \otimes c_i)$
$= \sum_{ij} a_{ij} c_i w_j$
$\hat{m}(T \otimes v) = T(v) = T(\sum_k c_k v_k) = \sum_k c_k T(v_k) = \sum_k c_k \phi_{V,W} \circ \phi_{V,W}^{-1}(T))(v_k)$
$= \sum_k c_k \phi_{V,W}(\sum_{ij} a_{ij} w_j \otimes v^i)(v_k) = \sum_{ij} \sum_k c_k a_{ij} \phi_{V,W}(w_j \otimes v^i)(v_k)$
$= \sum_{ij} \sum_k c_k a_{ij} v^i(v_k) w_j$
$= \sum_{ij} c_i a_{ij} w_j$
Thus $\hat{m} = \lambda \circ (\mathbf{1} \otimes ev_v) \circ (\phi_{V,W}^{-1} \otimes \mathbf{1}_V)$

Question 6

Consider any two bases $\mathcal{B} = \{v_i\}_1^n, \mathcal{C} = \{w_j\}_1^n$ of V,

$\forall T \in Lin(V,V), [T]_{\mathcal{C},\mathcal{C}} = [T]_{\mathcal{C},\mathcal{B}}[1]_{\mathcal{B},\mathcal{C}} = [1]_{\mathcal{C},\mathcal{B}}[T]_{\mathcal{B},\mathcal{B}}[1]_{\mathcal{B},\mathcal{C}} = M[T]_{\mathcal{B},\mathcal{B}}M^{-1}$,

where we have seen $M := [1]_{\mathcal{C},\mathcal{B}} \implies M^{-1} = [1]_{\mathcal{B},\mathcal{C}}$

$tr([T]_{\mathcal{C},\mathcal{C}}) = tr(M[T]_{\mathcal{B},\mathcal{B}}M^{-1}) = tr([T]_{\mathcal{B},\mathcal{B}}M^{-1}M) = tr([T]_{\mathcal{B},\mathcal{B}}M^{-1}M) = tr([T]_{\mathcal{B},\mathcal{B}}I) = tr([T]_{\mathcal{B},\mathcal{B}})$

Thus the definition of trace on T is independent of the choice of basis.

Question 7

Notice that this map is given by $\tilde{\phi} \circ \nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V}$, where $\phi_{V,V} : Lin(V,V) \to V \otimes V^*$ is the Frobenius Reciprocity natural isomorphism, $\beta : V \otimes V^* \to V^* \otimes V$ is the swap isomorphism, $ev : V \to (V^*)^*$ be the double duel embedding in Q4, $\nu : V^* \otimes (V^*)^* \to (V \otimes V^*)^*$ be the isomorphism for "taking out duel", and $\tilde{\phi} : (V \otimes V^*)^* \to Lin(V,V)^*$ be the isomorphism induced by ϕ, s.t. $\forall f \in (V \otimes V^*)^*, \tilde{\phi}(f) = f \circ \phi_{V,V}$

For any orthornormal basis $\mathcal{B} = \{v_i\} \subseteq V$, notice that $\left\{v^i\right\} \subseteq V^*, \left\{v_i \otimes v^j\right\} \subseteq V \otimes V^*$ are bases.

Thus for any $T \in Lin(V,V), \phi_{V,V}(T) = \sum_{ij} a_{ij} v_i \otimes v^j \in V \otimes V^*$

$(\tilde{\phi} \circ \nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V})(\mathbf{1}_V)(T) = (\nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V}(\mathbf{1}_V))(\phi_{V,V}(T))$

$= (\nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V}(\mathbf{1}_V))(\sum_{ij} a_{ij} v_i \otimes v^j) = \sum_{ij} a_{ij} (\nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V}(\mathbf{1}_V))(v_i \otimes v^j)$

On the other hand, let $\phi_{V,V}(\mathbf{1}_V) = \sum_{ij} b_{ij} v_i \otimes v^j \in V \otimes V^*$

$v_k = \mathbf{1}_V(v_k) = \phi^{-1}_{V,V}(\sum_{ij} b_{ij} v_i \otimes v^j)(v_k) = \sum_{ij} b_{ij} \phi^{-1}_{V,V}(v_i \otimes v^j)(v_k) = \sum_{ij} b_{ij} v^j(v_k) v_i$

$= \sum_{ij} b_{ij} \delta_{jk} v_i = \sum_i b_{ik} v_i \implies b_{ik} = \delta_{ik}$ since $\{v_i\}$ is a basis.

$\nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V}(\mathbf{1}_V) = \nu \circ (\mathbf{1} \otimes ev) \circ \beta(\sum_{ij} b_{ij} v_i \otimes v^j) = \nu \circ (\mathbf{1} \otimes ev)(\sum_{ij} b_{ij} \beta(v_i \otimes v^j))$

$= \nu \circ (\mathbf{1} \otimes ev)(\sum_{ij} b_{ij} v^j \otimes v_i) = \nu(\sum_{ij} b_{ij} (\mathbf{1} \otimes ev)(v^j \otimes v_i)) = \nu(\sum_{ij} b_{ij} v^j \otimes ev_{v_i}) = \sum_{ij} b_{ij} \nu(v^j \otimes ev_{v_i})$

$= \sum_{ij} \delta_{ij} \nu(v^j \otimes ev_{v_i}) = \sum_i \nu(v^i \otimes ev_{v_i})$

Thus $(\tilde{\phi} \circ \nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V})(\mathbf{1}_V)(T) = \sum_{ij} a_{ij} (\nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V}(\mathbf{1}_V))(v_i \otimes v^j)$

$= \sum_{ij} a_{ij} \sum_k \nu(v^k \otimes ev_{v_k})(v_i \otimes v^j) = \sum_{ij} a_{ij} \sum_k v^k(v_i) ev_{v_k}(v^j) = \sum_{ij} a_{ij} \sum_k \delta_{ki}\delta_{kj} = \boxed{\sum_i a_{ii}}$

$tr(T) = (ev_V \circ \beta \circ \phi_{V,V})(T) = (ev_V \circ \beta)(\sum_{ij} a_{ij} v_i \otimes v^j) = ev_V(\sum_{ij} a_{ij} \beta(v_i \otimes v^j)) = ev_V(\sum_{ij} a_{ij} v^j \otimes v_i)$

$= \sum_{ij} a_{ij} ev_V(v^j \otimes v_i) = \sum_{ij} a_{ij} v^j(v_i) = \sum_{ij} a_{ij} \delta_{ij} = \boxed{\sum_i a_{ii}}$

Since this works for any T, $tr = (\tilde{\phi} \circ \nu \circ (\mathbf{1} \otimes ev) \circ \beta \circ \phi_{V,V})(\mathbf{1}_V)$