PMATH 343 Winter 2023 HW6

Due: Friday April 7th 2023 at 11:59pm on Crowdmark

Problem 0: Read Chapters 12, 13, 14.

Problem 1: Show that

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are observables on $H = \mathbb{C}^2$ and find their spectral projections.

Problem 2: Let $T : H \to H$ be linear, and let $\mathcal{B} = \{\xi_i\}_1^n$ be an ONB for the Hilbert space H. Show that

$$H \times H \to \mathbb{C}, (\eta, \xi) \mapsto \langle \eta | T\xi \rangle$$

defines a sesquilinear form with associated matrix $[T]_{\mathcal{B},\mathcal{B}}$, and that this form is Hermitian if an only if T is self-adjoint.

Problem 3: Let X and Y be observables on a finite dimensional Hilbert space H.

(A): Show that $i[X, Y] := i(XY - YX)$ is a real-valued observable with zero trace.

(B): Show that if Z is a real-valued observable with zero trace, then there is some state $|\psi\rangle$ with $\langle \psi | Z | \psi \rangle$.

(C): Conclude there is some state $|\psi\rangle$ with $\langle \psi | [X, Y] | \psi \rangle$.

Problem 4: Let Z and X be the operators on $\mathbb{C}^2$ from Problem 1.

(A): Show that the anti-commutator $\{X, Z\} := XZ + ZX = 0$.

(B): Show that $\sigma_X^2 \sigma_y^2 = 0$ on a state $|\psi\rangle \in \mathbb{C}^2$ if and only if $|\psi\rangle$ is an eigenvalue of X or Y.

(C): Show that $\sigma_X^2 + \sigma_Z^2 \geq 1$ for all states $|\psi\rangle \in \mathbb{C}^2$.

Problem 5: Let $T \in \mathsf{Lin}(H)$ be a diagonalizable operator on the finite Hilbert space H. Show that if $T = \sum_{i=1}^m \lambda_i P_i$ is its spectral decomposition, then $T^n = \sum_{i=1}^m \lambda_i^n P_i$ gives the spectral decomposition of T^n.

Problem 6: (A): Show that if $T : H \to H$ is linear on the finite dimensional Hilbert space H, then $\langle \psi | T | \psi \rangle = 0$ for all $|\psi\rangle \in S(H)$ if and only if $T = 0$.

(B): Suppose X and Y are observables on a Hilbert space H with commutator $[X, Y] := XY - YX = 0$. Show there is some state $|\psi\rangle \in S(H)$ such that the variances $\sigma^2_{X,\psi}$ and $\sigma^2_{Y,\psi}$ are both non-zero.
(*hint:* Uncertainty Principle)

Problem 7: Suppose $P = \{P_i\}_{i \in \mathcal{O}_1}$ and $Q = \{Q_j\}_{j \in \mathcal{O}_2}$ are projective measurements on a Hilbert space H. Show that P and Q are compatible if and only if whenever we measure in P, then measure in Q, and then measure in P again, the first measurement in P always gives the same result as the last measurement in P.

Problem 8: In the context of nonlocal games, suppoce Alice and Bob are allowed to use randomness to pick their answer, so instead of having functions a and b giving their respective answers, we instead have a pair of probability distributions $a(i, \lambda)$ over $\mathcal{O}_A$ and $b(j, \lambda)$ over $\mathcal{O}_B$. Show that Alice and Bob cannot do better thatn $\omega_c(G)$ with such a strategy.

Problem 9: Let $\{A_i\}_{i \in \mathcal{O}}$ be a generalized measurement with outcome set $\mathcal{O}$ on a finite dimensional Hilbert space H. Show that $\sum_{i \in \mathcal{O}} \|A_i|\psi\rangle\|^2 = 1$ for all states $|\psi\rangle \in H$ if and only if $\sum_{i \in \mathcal{O}} A_i A_i^* = \mathbf{1}$.