

Pmath 451: Measure Theory

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1 Introductions

1.1 Lebesgue Measure

Definition 1.1. Lebesgue outer measure of $A \subseteq \mathbb{R}$ is $\lambda^*(A) := \inf \{ \sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i \}$, where each $I_i \subseteq \mathbb{R}$ is an open interval.

Definition 1.2. If $\forall E \in \mathbb{R}, \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$, then A is Lebesgue measurable, and its Lebesgue measure is defined to be $\lambda(A) := \lambda^*(A)$

Proposition 1.1. $\forall a < b \in \mathbb{R}, \lambda((a, b)) = b - a$

Proposition 1.2. $\forall x \in \mathbb{R}, \lambda(x + A) = \lambda(A)$

Proposition 1.3. If A_m are \mathcal{L} -measurable and pairwise disjoint ($A_m \cap A_n = \emptyset, \forall m \neq n$), then $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$

Proposition 1.4. Every Riemann integrable function is Lebesgue integrable.

2 Measure

2.1 Algebra of Sets

Definition 2.1. Let X be a set and $\mathcal{P}(X) := \{A | A \subseteq X\}$, then an algebra of subsets of X is $\mathcal{A} \subseteq \mathcal{P}(X)$, such that

1. $\emptyset \in \mathcal{A}$
2. If $E \in \mathcal{A}$, then $E^c := X \setminus E \in \mathcal{A}$
3. If $E_1, \dots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^n E_i \in \mathcal{A}$

Definition 2.2. Let X be a set and $\mathcal{P}(X) := \{A | A \subseteq X\}$, then a σ -algebra of subsets of X is $\mathcal{M} \subseteq \mathcal{P}(X)$, such that

1. $\emptyset \in \mathcal{M}$
2. If $E \in \mathcal{M}$, then $E^c := X \setminus E \in \mathcal{M}$
3. If $E_1, E_2, \dots \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$

Definition 2.3. If \mathcal{M} is σ -algebra, we call (X, \mathcal{M}) a measurable space, and a set $E \in \mathcal{M}$ is called \mathcal{M} -measurable.

Remark. Every σ -algebra is an algebra.

Proposition 2.1. If \mathcal{A} is an algebra, and $E_1, E_2 \in \mathcal{A}$, then $E_1 \cap E_2 \in \mathcal{A}$

Proof. $(E_1 \cap E_2)^c = E_1^c \cup E_2^c$ is in \mathcal{A} by 2,3. □

Proposition 2.2. If \mathcal{A} is an algebra, and $E, F \in \mathcal{A}$, then $E \setminus F = E \cap F^c \in \mathcal{A}$.

Proposition 2.3. If \mathcal{A} is an algebra, and $E, F \in \mathcal{A}$, then $E \Delta F = (E \setminus F) \cup (F \setminus E) \in \mathcal{A}$.

Proposition 2.4. If \mathcal{M} is σ -algebra, $E_i \in \mathcal{M}$, then we can define $F_i := E_i \setminus \bigcup_{j=1}^{i-1} E_j$, and $\bigcup_{i=1}^{\infty} E_i = \bigsqcup_{i=1}^{\infty} F_i$

Proposition 2.5. If \mathcal{M} is $(\sigma-)$ algebra, and $E \in \mathcal{M}$, then $A|_E := \{E \cap A | A \in \mathcal{M}\}$ is an $(\sigma-)$ algebra.

Example 2.1.1. $\mathcal{P}(X)$ is σ -algebra, and $\{\emptyset, X\}$ is σ -algebra.

Example 2.1.2. $\mathcal{A} = \{E \subseteq X : |E| < \infty \vee |E^c| < \infty\}$ is an algebra. However, if X is infinite, then it is not a σ -algebra

Example 2.1.3. $\mathcal{M} = \{E \subseteq X : |E| \leq \mathcal{N}_0 \vee |E^c| \leq \mathcal{N}_0\}$ is a σ -algebra.

Example 2.1.4. Let $X = \mathbb{R}$, the collection of all finite union of sets in $\{\mathbb{R}, (-\infty, b], (a, b], (a, \infty) | a, b \in \mathbb{R}\}$ is an algebra but not σ -algebra.

Proposition 2.6. Let $\{\mathcal{M}_\alpha\}_{\alpha \in I}$ is a collection of $(\sigma-)$ algebras of X , then $\bigcap_{\alpha \in I} \mathcal{M}_\alpha$ is an $(\sigma-)$ algebra

Definition 2.4. Let \mathcal{C} be a collection of subsets of X , then $\sigma(\mathcal{C}) := \bigcap \{\mathcal{M} : \sigma\text{-alg}, \mathcal{C} \subseteq \mathcal{M}\}$ is a σ -algebra containing \mathcal{C} , and is called the σ -algebra generated by \mathcal{C} .

Definition 2.5. Let X be a topological space, and let \mathcal{G} be the collection of all open sets of X , then the Borel algebra is $Bol_X := \sigma(\mathcal{G})$

2.2 Measures

Definition 2.6. A function $\mu : \mathcal{M} \rightarrow [0, \infty]$ is called a **positive measure** if:

1. $\mu(\emptyset) = 0$
2. If E_1, E_2, \dots are pairwise disjoint sets in \mathcal{M} , then $\mu(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$

We call (X, \mathcal{M}, μ) a **measure space**.

Definition 2.7. μ is finite if $\mu(X) < \infty$. μ is σ -finite if $X = \bigcup_{i=1}^{\infty} A_i$, where each $\mu(A_i) < \infty$. μ is semi-finite if $\forall E \in \mathcal{M}$, such that $\mu(E) \neq 0$, there is always $F \in \mathcal{M}, F \subseteq E, 0 < \mu(F) < \infty$

Definition 2.8. A **complex measure** is a function $\mu : \mathcal{M} \rightarrow \mathbb{C}$ that is countably additive, as above. Similarly, we can define a **real measure** $\mu : \mathcal{M} \rightarrow \mathbb{R}$

Remark. We will only work with positive measures where it satisfies $\exists A \in \mathcal{M}, \mu(A) < \infty$.

Example 2.2.1. For any X , we can define $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ by $\mu(A) := \begin{cases} |A|, & |A| < \infty \\ \infty, & \text{otherwise} \end{cases}$ is the **counting measure** on X

Example 2.2.2. For any set X and $x \in X$, we can define $\delta_x : \mathcal{P}(X) \rightarrow [0, \infty]$ by $\delta_x(A) := \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$ is the **point measure** or **Dirac measure** of x .

Example 2.2.3. Let $X = \mathbb{R}, \mathcal{M} = \mathcal{P}(X)$, let $x_1, x_2, \dots \in \mathbb{R}, a_1, a_2, \dots \geq 0$, then $\mu(E) := \sum_{i|x_i \in E} a_i$ is a measure.

Definition 2.9. A positive measure μ is a **probability measure** if $\mu(X) = 1$. In this case (X, \mathcal{M}, μ) is called a probability space.

Proposition 2.7. $\mu(\emptyset) = 0$

Proof. Choose $A \in \mathcal{M}$ with finite measure, take $A_1 = A$, and $A_2 = A_3 = \dots = \emptyset$. Then $\mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset) = \mu(A) < \infty$, thus we must have $\mu(\emptyset) = 0$ □

Remark. This holds for complex measures as well.

Proposition 2.8. *Finite Additivity:* If $E_1, E_2, \dots, E_n \in \mathcal{M}$, then $\mu(\bigsqcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$

Proof. Take $E_{n+1} = E_{n+2} = \dots = \emptyset$, then $\mu(\bigsqcup_{i=1}^n E_i) = \mu(\bigsqcup_{i=1}^\infty E_i) = \sum_{i=1}^n \mu(E_i) + \sum_{i=n+1}^\infty \mu(E_i) = \sum_{i=1}^n \mu(E_i)$ \square

Remark. This holds for complex measures as well.

Proposition 2.9. *Monotonicity:* If $E, F \in \mathcal{M}, E \subseteq F$, then $\mu(E) \leq \mu(F)$

Proof. We have $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$ \square

Remark. This does not hold for complex measures.

Proposition 2.10. *Subadditivity:* If $E_1, E_2, \dots \in \mathcal{M}$, then $\mu(\bigcup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu(E_i)$

Proposition 2.11. *Continuity:*

If $E_1, E_2, \dots \in \mathcal{M}, E_n \subseteq E_{n+1}$, we have that $\mu(\bigcup_{i=1}^\infty E_i) = \lim_{n \rightarrow \infty} \mu(E_n)$.

If $E_1, E_2, \dots \in \mathcal{M}, E_{n+1} \subseteq E_n, \mu(E_1) < \infty$ we have that $\mu(\bigcap_{i=1}^\infty E_i) = \lim_{n \rightarrow \infty} \mu(E_n)$.

Proof. Let $E_0 = \emptyset$, then we can write $\bigcup_{i=1}^\infty E_i = \bigsqcup_{i=1}^\infty (E_i \setminus E_{i-1})$, and we have $E_n = \bigsqcup_{i=1}^n (E_i \setminus E_{i-1})$.

$$\begin{aligned} \mu\left(\bigcup_{i=1}^\infty E_i\right) &= \mu\left(\bigsqcup_{i=1}^\infty (E_i \setminus E_{i-1})\right) \\ &= \sum_{i=1}^\infty \mu(E_i \setminus E_{i-1}) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i \setminus E_{i-1}) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigsqcup_{i=1}^n (E_i \setminus E_{i-1})\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

For the second part, let $A = \bigcap_{i=1}^\infty E_i$.

$$\begin{aligned} \mu(E_1 \setminus A) &= \mu(E_1 \cap A^c) \\ &= \mu\left(E_1 \cap \left(\bigcap_{i=1}^\infty E_i\right)^c\right) \\ &= \mu\left(E_1 \cap \bigcup_{i=1}^\infty E_i^c\right) \\ &= \mu\left(\bigcup_{i=1}^\infty E_1 \cap E_i^c\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_1 \cap E_n^c) \\ &= \lim_{n \rightarrow \infty} \mu(E_1 \setminus E_n) \end{aligned}$$

By finite additivity, we have that

$$\begin{aligned}
\mu(E_1 \setminus A) + \mu(A) &= \mu(E_1 \setminus A \sqcup A) \\
&= \mu(E_1) \\
&= \lim_{n \rightarrow \infty} \mu(E_1) \\
&= \lim_{n \rightarrow \infty} \mu(E_1 \setminus E_n \sqcup E_n) \\
&= \lim_{n \rightarrow \infty} (\mu(E_1 \setminus E_n) + \mu(E_n)) \\
&= \lim_{n \rightarrow \infty} \mu(E_1 \setminus E_n) + \lim_{n \rightarrow \infty} \mu(E_n) \\
&= \mu(E_1 \setminus A) + \lim_{n \rightarrow \infty} \mu(E_n)
\end{aligned}$$

Since $\mu(E_1 \setminus A) \leq \mu(E_1) < \infty$, we have $\mu(A) = \lim_{n \rightarrow \infty} \mu(E_n)$ \square

Remark. This holds for complex measures as well. However, for the second property, it is essential for $\mu(E_1) < \infty$. Indeed, consider the following example:

Example 2.2.4. Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(X)$, μ be the counting measure. Let $A_n := \{i : i \geq n\}$. Notice that $A_1 \supseteq A_2 \supseteq A_3 \dots$ and $\lim_{n \rightarrow \infty} \mu(A_n) = \infty \neq 0 = \mu(\emptyset)$. However, $\bigcap_{n=1}^{\infty} A_n = \emptyset$

2.3 Measurable Function

Definition 2.10. If $(X, \mathcal{M}_1), (Y, \mathcal{M}_2)$ are measure spaces, then $f : X \rightarrow Y$ is a **measurable function** if $\forall B \in \mathcal{M}_2, f^{-1}(B) \in \mathcal{M}_1$.

Definition 2.11. If (Y, τ) is a topological space, we say a function $f : X \rightarrow Y$ is **Borel measurable** if it is measurable with respect to $\mathcal{M}_2 = \text{Bol}_{(Y, \tau)}$, the Borel σ -algebra.

Proposition 2.12. For $(B_i) \subseteq Y$, we have

1. $f^{-1}(B^c) = (f^{-1}(B))^c$
2. $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$
3. $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$.

Proposition 2.13. If (Y, τ) is a topological space, a function $f : X \rightarrow Y$ is Borel measurable if and only if $\forall B$ open, $f^{-1}(B) \in \mathcal{M}_1$.

Proposition 2.14. For $f : X \rightarrow \mathbb{R}$, the following are equal:

1. f is measurable
2. $\forall a, f^{-1}(-\infty, a)$ is measurable
3. $\forall a, f^{-1}(-\infty, a]$ is measurable
4. $\forall a, f^{-1}(a, \infty)$ is measurable
5. $\forall a, f^{-1}[a, \infty)$ is measurable
6. $\forall a < b, f^{-1}(a, b)$ is measurable

Proposition 2.15. If $f : X \rightarrow Y, g : Y \rightarrow Z$ are both measurable, then $f \circ g$ is also measurable.

Corollary 2.16. If $f : X \rightarrow \mathbb{C}$ is measurable, we have $u = \text{Re}(f), v = \text{Im}(f), z = |f|$ are all measurable.

Theorem 2.17. Let (X, \mathcal{M}) is a measurable space, and $u, v : X \rightarrow \mathbb{R}$ be measurable, and (Y, τ) is a topological space. If $\Phi : \mathbb{R}^2 \rightarrow Y$ is continuous, then $h : X \rightarrow Y; x \mapsto \Phi(u(x), v(x))$ is measurable.

Proof. Let $f : X \rightarrow \mathbb{R}^2; x \mapsto (u(x), v(x))$, it suffices to check that f is measurable. Notice that $Bol_{\mathbb{R}^2}$ is generated by open rectangles $R = (a, b) \times (c, d)$. Yet $f^{-1}(R) = u^{-1}(a, b) \cap v^{-1}(c, d)$ is measurable. \square

Corollary 2.18. *If $u, v : X \rightarrow \mathbb{R}$ are both measurable, we have $f := u + iv : X \rightarrow \mathbb{C}$ is also measurable.*

Proof. Choose $\Phi : \mathbb{R}^2 \rightarrow \mathbb{C}; (s, t) \mapsto s + it$. \square

Corollary 2.19. *If $f, g : X \rightarrow \mathbb{R}$ are measurable, then we have $fg, f + g$ are both measurable.*

Proof. choose $\Phi : (s, t) \mapsto st$ or $\Phi : (s, t) \mapsto s + t$. \square

Corollary 2.20. *If $f, g : X \rightarrow \mathbb{C}$ are measurable, then for any $\alpha \in \mathbb{C}$, we have $fg, f + g, \alpha f$ are all measurable.*

Proof. We write $f = u + iv, g = w + iz$. We have that u, v, z, w are all real-valued and measurable, so are $u + w, v + z$, and so are $(u + w) + i(v + z) = f + g$ and $(uw - vz) + i(vw + uz) = fg$. For αf , it is obvious since $B \in Bol(\mathbb{C}) \iff \alpha B \in Bol(\mathbb{C})$ for $\alpha \neq 0$, and $0f = 0$ is measurable. \square

Definition 2.12. For **extended real functions** $f : X \rightarrow [-\infty, \infty]$, it is measurable if $\forall \alpha \in \mathbb{R}, f^{-1}((\alpha, \infty]) \in \mathcal{M}$, or equivalently, $\forall \alpha \in \mathbb{R}, f^{-1}([-\infty, \alpha)) \in \mathcal{M}$.

Proposition 2.21. *If $(f_n)_{n=1}^\infty$ is a sequence of measurable functions $X \rightarrow [-\infty, \infty]$, we have $g(x) := \sup_n f_n(x), h(x) := \limsup_{n \rightarrow \infty} f_n(x) = \inf_k (\sup_{n \geq k} f_n(x))$ are also measurable. Similarly for \inf and \liminf .*

Proof. Notice that $g^{-1}((\alpha, \infty]) = \bigcup_{n=1}^\infty f_n^{-1}((\alpha, \infty])$, which is a union of measurable sets. Thus g is measurable. \square

Corollary 2.22. *If $f_n : X \rightarrow [-\infty, \infty]$ or $f_n : X \rightarrow \mathbb{C}$ are measurable functions, and $\forall x \in X, f(x) = \lim_{n \rightarrow \infty} f_n(x)$ exists, then f is measurable.*

Corollary 2.23. *If $f, g : X \rightarrow [-\infty, \infty]$ are both measurable, then $\max(f, g), \min(f, g)$ are measurable.*

Corollary 2.24. *If $f : X \rightarrow [-\infty, \infty]$ is measurable, then $f^+ := \max(f, 0), f^- := \max(-f, 0)$ are both measurable, with $f = f^+ - f^-$.*

Proposition 2.25. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is monotone, then f is Borel measurable.*

Proof. Let $\alpha \in \mathbb{R}$, we need to show that $\{x \in \mathbb{R} | f(x) > \alpha\}$ is a Borel set.

We may assume that f is non-decreasing, if not we take $f \rightarrow -f$. If $\{x \in \mathbb{R} | f(x) > \alpha\} \in \{\emptyset, \mathbb{R}\}$, we have nothing to prove.

Now if $\{x \in \mathbb{R} | f(x) > \alpha\} \notin \{\emptyset, \mathbb{R}\}$, we have that $\{x \in \mathbb{R} | f(x) \leq \alpha\}$ is not empty and bounded above since f is increasing. Let $x_0 := \sup \{x \in \mathbb{R} | f(x) \leq \alpha\}$. If $f(x_0) \leq \alpha$, $\{x \in \mathbb{R} | f(x) > \alpha\} = (x_0, \infty)$, otherwise $\{x \in \mathbb{R} | f(x) > \alpha\} = [x_0, \infty)$, both Borel. \square

2.4 Simple Functions

Definition 2.13. Let (X, \mathcal{M}) be a measurable space, a **characteristic function** for a subset $E \subseteq X$ is

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

Definition 2.14. Let (X, \mathcal{M}) be a measurable space, a function $\phi : X \rightarrow [-\infty, \infty]$ is **simple** if $\phi(X)$ is finite.

Proposition 2.26. Let (X, \mathcal{M}) be a measurable space, for any simple function ϕ with $\phi(X) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, we have

$$\phi = \sum_{i=1}^n \alpha_i \chi_{E_i},$$

where $E_i = \phi^{-1}(\{\alpha_i\})$ are pairwise disjoint. In this case, ϕ is measurable if and only if $\forall i, E_i \in \mathcal{M}$.

Lemma 2.27. For any $\alpha \in \mathbb{R}, n \geq 1$, we have that

$$\alpha - \frac{1}{2^n} < \frac{\lfloor 2^n \alpha \rfloor}{2^n} \leq \alpha$$

Proof.

$$\begin{aligned} \lfloor 2^n \alpha \rfloor &\leq 2^n \alpha < \lfloor 2^n \alpha \rfloor + 1 \\ 2^n \alpha - 1 &< \lfloor 2^n \alpha \rfloor \\ \alpha - \frac{1}{2^n} &< \frac{\lfloor 2^n \alpha \rfloor}{2^n} \\ \frac{\lfloor 2^n \alpha \rfloor}{2^n} &\leq \alpha \end{aligned}$$

□

Lemma 2.28. Consider $id : [0, \infty) \rightarrow [0, \infty), x \mapsto x$, then there are simple functions $s_n : [0, \infty) \rightarrow [0, \infty)$

such that each s_n is measurable and $\begin{cases} 0 \leq s_1 \leq s_2 \leq \dots \leq id, \\ \forall x \in X, \lim_{n \rightarrow \infty} s_n(x) = id(x), \\ \forall R > 0, s_n \rightarrow id \text{ uniformly on } [0, R] \end{cases}$

Proof. For $n \geq 1, t \in [0, \infty)$, let $s_n(t) := \begin{cases} \frac{\lfloor 2^n t \rfloor}{2^n}, & t \in [0, n] \\ n, & t > n \end{cases}$

Notice that s_n is simple. It is also measurable since it is monotone.

We also have that $0 \leq s_1 \leq s_2 \leq \dots \leq f$, and by squeeze theorem, we have that

$$\lim_{n \rightarrow \infty} s_n(x) = x = id(x).$$

□

Theorem 2.29. Let $f : X \rightarrow [0, \infty]$ be measurable, then there are simple functions $s_n : X \rightarrow [0, \infty)$ such

that each s_n is measurable and $\begin{cases} 0 \leq s_1 \leq s_2 \leq \dots \leq f \\ \forall x \in X, \lim_{n \rightarrow \infty} s_n(x) = f(x) \\ \forall R > 0, s_n \rightarrow f \text{ uniformly on } E_R := \{x \in X : f(x) \leq R\}. \end{cases}$

Proof. Notice that for any simple function s and any arbitrary function f , we have that $s \circ f$ is simple. Thus it suffices to find s'_n that approximates $id : x \mapsto x$, which is done by the above lemma.

Let $s_n := s'_n \circ f$, they are measurable by result on compositions, and $0 \leq s_1 \leq \dots \leq f, \lim_{n \rightarrow \infty} (s_n \circ f)(x) = f(x)$ □

Corollary 2.30. Let $f : X \rightarrow \mathbb{C}$ be measurable, then there are simple functions $s_n : X \rightarrow [0, \infty)$ such that

each s_n is measurable and $\begin{cases} 0 \leq |s_1| \leq |s_2| \leq \dots \leq |f| \\ \forall x \in X, \lim_{n \rightarrow \infty} s_n(x) = f(x) \\ \forall R > 0, s_n \rightarrow f \text{ uniformly on } E_R := \{x \in X : |f(x)| \leq R\}. \end{cases}$

Corollary 2.31. For $f, g : X \rightarrow [0, \infty]$ being measurable, we have that $f \cdot g$ is also measurable.

Proof. One can check that for monotone non-decreasing $(a_n), (b_n) \subseteq [0, \infty)$ with $a_n \rightarrow a, b_n \rightarrow b$ for $a, b \in [0, \infty]$, then $a_n b_n \rightarrow ab$.

Approximate f with simple functions s_n , and g with simple functions t_n , then each of them is measurable, hence so is $s_n \cdot t_n$, hence so is $\lim_{n \rightarrow \infty} s_n t_n = fg$ □

3 Integration

3.1 Integration of non-negative functions

Definition 3.1. Let (X, \mathcal{M}, μ) be a measure space, $s : X \rightarrow [0, \infty)$ be a simple measurable function, with $s(X) = \{a_1, \dots, a_n\}$, such that $s = \sum_{i=1}^n a_i \mathcal{X}_{A_i}$, where $A_i := s^{-1}(\{a_i\})$. For $A \in \mathcal{M}$, define the **integral** of s over A to be

$$\int_A s d\mu := \sum_{i=1}^n a_i \mu(A_i \cap A).$$

Definition 3.2. For $f : X \rightarrow [0, \infty]$ measurable, the **integral** of f over $A \in \mathcal{M}$ is

$$\int_A f d\mu := \sup \int_A s d\mu,$$

where the sup is taken over all measurable simple $s : X \rightarrow [0, \infty)$ such that $0 \leq s \leq f$.

Proposition 3.1. Let $f, g : X \rightarrow [0, \infty]$ be measurable, then

1. $f \leq g \implies \forall A \in \mathcal{M}, \int_A f d\mu \leq \int_A g d\mu$
2. For any $A \subseteq B \in \mathcal{M}$, we have that $\int_A f d\mu \leq \int_B f d\mu$
3. $\forall c \in [0, \infty), A \in \mathcal{M}$, we have that $\int_A c f d\mu = c \int_A f d\mu$
4. If $\forall x \in X, f(x) = 0$, we have that $\forall A \in \mathcal{M}, \int_A f d\mu = 0$
5. If $\forall x \in A \in \mathcal{M}, f(x) = 0$, we have that $\int_A f d\mu = 0$
6. If $\mu(A) = 0$ for $A \in \mathcal{M}$, we have that $\int_A f d\mu = 0$
7. $\int_A f d\mu = \int_X \mathcal{X}_A f d\mu$

Proposition 3.2. Let (X, \mathcal{M}, μ) be a measure space, and $s : X \rightarrow [0, \infty)$ a measurable simple function. Then $\lambda : \mathcal{M} \rightarrow [0, \infty]$ defined by

$$\lambda(A) := \int_A s d\mu$$

is a measure on (X, \mathcal{M})

Proof. Write $s = \sum_{i=1}^n a_i \mathcal{X}_{A_i}$, and let $C := \bigsqcup_{k=1}^{\infty} C_k$, then

$$\begin{aligned} \lambda(C) &= \sum_{i=1}^n a_i \mu(A_i \cap C) \\ &= \sum_{i=1}^n a_i \mu\left(\bigsqcup_{k=1}^{\infty} (A_i \cap C_k)\right) \\ &= \sum_{i=1}^n a_i \sum_{k=1}^{\infty} \mu(A_i \cap C_k) \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^n a_i \mu(A_i \cap C_k) \\ &= \sum_{k=1}^{\infty} \lambda(C_k) \end{aligned}$$

Thus λ is a measure, and in addition $\lambda(\emptyset) = \sum_{i=1}^n a_i \mu(A_i \cap \emptyset) = 0$

□

Corollary 3.3. Let (X, \mathcal{M}, μ) be a measure space, and $s : X \rightarrow [0, \infty)$ a measurable simple function, with $C := \bigsqcup_{k=1}^{\infty} C_k$. Then we have

$$\int_C s d\mu = \sum_{k=1}^{\infty} \int_{C_k} s d\mu.$$

Proof.

$$\begin{aligned} \int_C s d\mu &= \lambda_s(C) \\ &= \lambda_s\left(\bigsqcup_{k=1}^{\infty} C_k\right) \\ &= \sum_{k=1}^{\infty} \lambda_s(C_k) \\ &= \sum_{k=1}^{\infty} \int_{C_k} s d\mu \end{aligned}$$

□

Proposition 3.4. Let (X, \mathcal{M}, μ) be a measure space, and $s, t : X \rightarrow [0, \infty)$ a measurable simple function. Then

$$\int_X (s + t) d\mu = \int_X s d\mu + \int_X t d\mu$$

Proof. Write $s = \sum_{i=1}^n a_i \chi_{A_i}$, $t = \sum_{j=1}^m b_j \chi_{B_j}$, and let $C_{ij} = A_i \cap B_j$, then C_{ij} are disjoint, and $\bigsqcup_{ij} C_{ij} = X$

$$\begin{aligned} \int_{C_{ij}} (s + t) d\mu &= (a_i + b_j) \mu(C_{ij}) \\ &= a_i \mu(C_{ij}) + b_j \mu(C_{ij}) \\ &= \int_{C_{ij}} s d\mu + \int_{C_{ij}} t d\mu \\ \int_X (s + t) d\mu &= \int_{\bigsqcup_{ij} C_{ij}} (s + t) d\mu \\ &= \sum_{ij} \int_{C_{ij}} (s + t) d\mu \\ &= \sum_{ij} \int_{C_{ij}} s d\mu + \sum_{ij} \int_{C_{ij}} t d\mu \\ &= \int_X s d\mu + \int_X t d\mu \end{aligned}$$

□

Theorem 3.5. (Lebesgue Monotone Convergence)

Let (X, \mathcal{M}, μ) be a measure space. Let $f_n : X \rightarrow [0, \infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \dots \leq \infty$. Let $f(x) := \lim_{n \rightarrow \infty} f_n(x)$, then $f : X \rightarrow [0, \infty]$ is measurable, and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. Since $f_n \leq f_{n+1}$, we have that $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$, so by monotone convergence theorem,

$$\alpha := \lim_{n \rightarrow \infty} \int_X f_n d\mu \in [0, \infty]$$

exists.

As a limit of measurable functions, f is measurable. Also, $\forall n, \int_X f_n d\mu \leq \int_X f d\mu$, and thus $\alpha \leq \int_X f d\mu$. Consider any $s : X \rightarrow [0, \infty)$ be simple and measurable with $0 \leq s \leq f$, and consider any $0 < c < 1$.

For $n \geq 1$, let $A_n := \{x \in X : f_n(x) \geq cs(x)\}$.

Then $X = \bigcup_{n=1}^{\infty} A_n$ since f_n converges point-wise.

In addition, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

Also each A_n is measurable, since $A_n = \{x \in X : (f_n - cs)(x) \geq 0\} = (f_n - cs)^{-1}([0, \infty])$, and $f_n - cs$ is measurable.

Since $\lambda_s : A \mapsto \int_A s d\mu$ is a measure, so by property of measures,

$$\int_X s d\mu = \lambda_s(X) = \lim_{n \rightarrow \infty} \lambda_s(A_n) = \lim_{n \rightarrow \infty} \int_{A_n} s d\mu.$$

In addition, we have

$$\begin{aligned} \int_X f_n d\mu &\geq \int_{A_n} f_n d\mu \\ &\geq \int_{A_n} cs d\mu \\ &= c \int_{A_n} s d\mu \\ \alpha &= \lim_{n \rightarrow \infty} \int_X f_n d\mu \\ &\geq \lim_{n \rightarrow \infty} c \int_{A_n} s d\mu \\ &= c \int_X s d\mu. \end{aligned}$$

Now take $c \rightarrow 1$, we have that $\alpha \geq \int_X s d\mu$.

Then take sup of all simple $s \leq f$, we have that $\alpha \geq \int_X f d\mu$. \square

Corollary 3.6. *For a measure space (X, \mathcal{M}, μ) , $A \in \mathcal{M}$, let $f_n : X \rightarrow [0, \infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \dots \leq \infty$. Let $f(x) := \lim_{n \rightarrow \infty} f_n(x)$. We can consider the restriction $(A, \mathcal{M}' := \{B \cap A : B \in \mathcal{M}\}, \mu|_{\mathcal{M}'})$, and we will have*

$$\lim_{n \rightarrow \infty} \int_A f_n d\mu = \int_A f d\mu$$

Corollary 3.7. *Let (X, \mathcal{M}, μ) be a measure space. Let $f : X \rightarrow [0, \infty]$ be measurable. Let $s_n : X \rightarrow [0, \infty]$ be any measurable simple functions with $0 \leq s_1 \leq s_2 \leq \dots \leq \infty$ with $f(x) = \lim_{n \rightarrow \infty} s_n(x)$. We have*

$$\lim_{n \rightarrow \infty} \int_X s_n d\mu = \int_X f d\mu.$$

Proposition 3.8. *(additivity)*

Let (X, \mathcal{M}, μ) be a measure space. Let $f, g : X \rightarrow [0, \infty]$ be measurable functions, then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Proof. Approximate f, g by simple functions s_n, t_n , such that $\lim_{n \rightarrow \infty} s_n(x) = f(x), \lim_{n \rightarrow \infty} t_n(x) = g(x)$ and $0 \leq s_1 \leq \dots \leq f, 0 \leq t_1 \leq \dots \leq g$.

Notice that $0 \leq s_1 + t_1 \leq \dots \leq f + g$, and $\lim_{n \rightarrow \infty} (s_n + t_n)(x) = (f + g)(x)$. Thus

$$\begin{aligned} \int_X (f + g) d\mu &= \lim_{n \rightarrow \infty} \int_X (s_n + t_n) d\mu \\ &= \lim_{n \rightarrow \infty} \left(\int_X s_n d\mu + \int_X t_n d\mu \right) \\ &= \lim_{n \rightarrow \infty} \int_X s_n d\mu + \lim_{n \rightarrow \infty} \int_X t_n d\mu \\ &= \int_X f d\mu + \int_X g d\mu. \end{aligned}$$

□

Corollary 3.9. (*countable additivity*)

Let (X, \mathcal{M}, μ) be a measure space. Let $f_n : X \rightarrow [0, \infty]$ be measurable functions. Then

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

is measurable and

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

Proof. Define $g_n(x) := \sum_{i=1}^n f_i(x)$, then $0 \leq g_1 \leq \dots \leq f$ and $\lim_{n \rightarrow \infty} g_n = f$. By previous proposition and induction,

$$\int_X g_n d\mu = \sum_{i=1}^n \int_X f_i d\mu.$$

By LMCT, we have

$$\begin{aligned} \int_X f d\mu &= \int_X \lim_{n \rightarrow \infty} g_n d\mu \\ &= \lim_{n \rightarrow \infty} \int_X g_n d\mu \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_X f_i d\mu \\ &= \sum_{n=1}^{\infty} \int_X f_n d\mu \end{aligned}$$

□

Theorem 3.10. Let (X, \mathcal{M}, μ) be a measure space, and $f : X \rightarrow [0, \infty]$ a measurable function. Then $\lambda : \mathcal{M} \rightarrow [0, \infty]$ defined by

$$\lambda(A) := \int_A f d\mu$$

is a measure on (X, \mathcal{M}) . Moreover, for some $g : X \rightarrow [0, \infty]$ such that fg is measurable, then

$$\int_X g d\lambda = \int_X g f d\mu.$$

Proof. Let $A = \bigsqcup_{n=1}^{\infty} A_n$ with A_n disjoint measurable subsets of X . We have that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$, and thus

$$\begin{aligned}\lambda(A) &= \int_X \chi_A f d\mu \\ &= \int_X \sum_{n=1}^{\infty} \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \int_X \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \lambda(A_n)\end{aligned}$$

Thus λ is a measure.

In addition, when $g = \chi_A$ for $A \in \mathcal{M}$, we have that $\int_X g d\lambda = 1 * \lambda(A) = \int_X \chi_A f d\mu = \int_X g f d\mu$. And thus simple functions, and thus all non-negative measurable functions by LMCT. \square

3.2 Integration of complex functions

Definition 3.3. Let (X, \mathcal{M}, μ) be a measure space, and $|f| : X \rightarrow [0, \infty)$ a measurable function. Let

$$\mathcal{L}^1(X, \mu) := \left\{ f : X \rightarrow \mathbb{C} \mid \int_X |f| d\mu < \infty \right\}$$

be the **set of Lebesgue integrable functions**. For $f \in \mathcal{L}^1(X, \mu)$, we define $\|f\|_1 := \int_X |f| d\mu$.

Definition 3.4. For $f = u + iv \in \mathcal{L}^1(X, \mu)$, where $u, v : X \rightarrow \mathbb{R}$, then the integral of f is defined as

$$\int_X f d\mu := \int_X u^+ d\mu - \int_X u^- d\mu + i \int_X v^+ d\mu - i \int_X v^- d\mu,$$

where $u^+(x) := \max(u(x), 0)$, $u^-(x) := \max(-u(x), 0)$, and thus $u = u^+ - u^-$. Similar definition for v^+, v^- .

Proposition 3.11. *The above integral is well-defined.*

Proof. u^+, u^-, v^+, v^- are measurable, and $0 \leq u^+, u^- \leq |u| \leq |f|$, thus the integral is finite. \square

Proposition 3.12. *For $f = u + iv \in \mathcal{L}^1(X, \mu)$, where $u, v : X \rightarrow \mathbb{R}$, we have*

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu.$$

Proof. By definition. \square

Theorem 3.13. $\forall f, g \in \mathcal{L}^1(X, \mu), \alpha \in \mathbb{C}$, we have that $\alpha f + g \in \mathcal{L}^1$ and

$$\int_X \alpha f + g d\mu = \alpha \int_X f d\mu + \int_X g d\mu$$

Proof. Clearly $\alpha f + g$ is measurable. In addition,

$$\begin{aligned}\int_X |\alpha f + g| d\mu &\leq \int_X |\alpha f| + |g| d\mu \\ &= \int_X |\alpha| |f| d\mu + \int_X |g| d\mu \\ &= |\alpha| \int_X |f| d\mu + \int_X |g| d\mu \\ &< \infty\end{aligned}$$

Now we check the addition: Consider $f = a + ib, g = c + id : X \rightarrow \mathbb{C}$, such that $a, b, c, d : X \rightarrow \mathbb{R}$.

$$\begin{aligned}
(a + c)^+ - (a + c)^- &= a + c \\
&= (a^+ - a^-) + (c^+ - c^-) \\
&= (a^+ + c^+) - (a^- + c^-). \\
(a + c)^+ + (a^- + c^-) &= (a + c)^- + (a^+ + c^+),
\end{aligned}$$

where both sides of the equality are sums of two non-negative functions. Thus we have

$$\begin{aligned}
\int_X (a + c)^+ + (a^- + c^-) d\mu &= \int_X (a + c)^- + (a^+ + c^+) d\mu \\
\int_X (a + c)^+ d\mu + \int_X (a^- + c^-) d\mu &= \int_X (a + c)^- d\mu + \int_X (a^+ + c^+) d\mu \\
\int_X (a + c)^+ d\mu - \int_X (a + c)^- d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\
\int_X (a + c) d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\
&= \int_X a^+ d\mu + \int_X c^+ d\mu - \int_X a^- d\mu - \int_X c^- d\mu \\
&= \left(\int_X a^+ d\mu - \int_X a^- d\mu \right) + \left(\int_X c^+ d\mu - \int_X c^- d\mu \right) \\
&= \int_X a d\mu + \int_X c d\mu \\
\int_X (f + g) d\mu &= \int_X (a + c) d\mu + i \int_X (b + d) d\mu \\
&= \int_X a d\mu + \int_X c d\mu + i \int_X b d\mu + i \int_X d d\mu \\
&= \left(\int_X a d\mu + i \int_X b d\mu \right) + \left(\int_X c d\mu + i \int_X d d\mu \right) \\
&= \int_X f d\mu + \int_X g d\mu.
\end{aligned}$$

Now we check the scalar multiplication: $\forall \alpha \geq 0$, we have $\int_X \alpha f d\mu = \alpha \int_X f d\mu$ by definition.

We can also check for $\alpha = -1$ and $\alpha = i$, and conclude this holds for all $\alpha \in \mathbb{C}$. □

Theorem 3.14. *Let (X, \mathcal{M}, μ) be a measure space, and $f \in \mathcal{L}^1(X, \mu)$, then*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$$

Proof. Let $\alpha := \int_X f d\mu \in \mathbb{C}$, and let $\beta \in \mathbb{C}, |\beta| = 1$, such that $\alpha\beta = |\alpha|$. Take $u = \operatorname{Re}(\beta f) : X \rightarrow \mathbb{R}$, note

$u \leq |\beta f| = |f|$. Now

$$\begin{aligned}
\left| \int_X f d\mu \right| &= |\alpha| \\
&= \beta \alpha \\
&= \beta \int_X f d\mu \\
&= \int_X \beta f d\mu \\
&= \int_X u d\mu \\
&\leq \int_X |f| d\mu
\end{aligned}$$

□

3.3 Lebesgue Dominated Convergence Theorem

Lemma 3.15. (*Fatou's*)

Let (X, \mathcal{M}, μ) be a measure space. Let $f_n : X \rightarrow [0, \infty]$ be measurable functions. Then

$$\int_X (\liminf f_n) d\mu \leq \liminf \int_X f_n d\mu$$

Proof. Let $g_n(x) := \inf_{i \geq n} f_i(x)$, then $\liminf f_n(x) = \lim g_n(x) = \sup g_n(x)$.

Also, $g_n \leq f_n$, so $\int_X g_n d\mu \leq \int_X f_n d\mu, \forall n \geq 1$.

Note g_n is measurable, and $0 \leq g_1 \leq g_2 \leq \dots$.

By LMCT, $\lim \int_X g_n d\mu = \int_X (\liminf f_n) d\mu$.

Since the left hand side converges, $\int_X (\liminf f_n) d\mu = \lim \int_X g_n d\mu = \liminf \int_X g_n d\mu \leq \liminf \int_X f_n d\mu$ □

Theorem 3.16. (*Lebesgue Dominated Convergence*)

Let (X, \mathcal{M}, μ) be a measure space. Let $f_n : X \rightarrow \mathbb{C}$ be measurable functions such that $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists $\forall x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that $\forall x \in X, \forall n \in \mathbb{N}, |f_n(x)| \leq g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = 0.$$

Proof. Firstly, $\forall n \in \mathbb{N}, x \in X, |f_n(x)| \leq g(x)$ implies that $\forall x \in X, |f(x)| \leq g(x)$, and thus $\int_X |f| d\mu \leq \int_X g d\mu$. This shows that $f \in \mathcal{L}^1(X, \mu)$.

Notice that $|f_n - f| \leq |f_n| + |f| \leq 2g$. Thus $2g - |f_n - f| \geq 0$. By Fatou's Lemma, we have that $\int_X (\liminf (2g - |f_n - f|)) d\mu \leq \liminf \int_X (2g - |f_n - f|) d\mu$. Thus

$$\begin{aligned}
\int_X 2g d\mu &= \int_X (2g - \liminf (|f_n - f|)) d\mu \\
&= \int_X (\liminf (2g - |f_n - f|)) d\mu \\
&\leq \liminf \int_X (2g - |f_n - f|) d\mu \\
&= \int_X 2g d\mu + \liminf \left(- \int_X |f_n - f| d\mu \right) \\
&= \int_X 2g d\mu - \limsup \int_X |f_n - f| d\mu \\
0 &\leq - \limsup \int_X |f_n - f| d\mu
\end{aligned}$$

Thus $0 \leq \liminf \int_X |f_n - f| d\mu \leq \limsup \int_X |f_n - f| d\mu \leq 0$, and thus $\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0$. Finally,

$$\begin{aligned} \left| \lim_{n \rightarrow \infty} \int_X f_n d\mu - \int_X f d\mu \right| &= \left| \lim_{n \rightarrow \infty} \int_X (f_n - f) d\mu \right| \\ &\leq \lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu \\ &= 0 \end{aligned}$$

□

3.4 Almost Everywhere

Definition 3.5. Let (X, \mathcal{M}, μ) be a measure space, $A \in \mathcal{M}$, and $P = \{p(x)\}_{x \in A}$ be a family of logical statements, then we say the property P holds or is true μ -everywhere on A , if $\exists N \in \mathcal{M}$, such that $\mu(N) = 0$ and $\forall x \in A \setminus N$, $p(x) = \text{True}$.

Definition 3.6. For measurable functions $f, g : X \rightarrow Y$, we say that $f = g$ μ -almost everywhere if

$$\mu(\{x \in X | f(x) \neq g(x)\}) = 0.$$

Remark. For some $A \in \mathcal{M}$, we have that $\mu(A \cap N) \leq \mu(N) = 0$, and thus $\int_{A \cap N} (f - g) d\mu = 0$. Thus $\int_A (f - g) d\mu = \int_{A \cap N} (f - g) d\mu + \int_{A \setminus N} (f - g) d\mu = 0$.

Definition 3.7. Let $(f_n)_{n=1}^\infty$, we say $f_n \rightarrow f$ a.e. if $f_n(x) \rightarrow f(x)$ for μ -a.e. $x \in X$.

Proposition 3.17. Let (X, \mathcal{M}, μ) be a measure space.

1. If $f : X \rightarrow [0, \infty]$ is measurable, then $f = 0$ μ -a.e. $\iff \int_E f d\mu = 0 \forall E \in \mathcal{M}$.
2. If $f \in \mathcal{L}^1(X, \mu)$ we have $f = 0$ μ -a.e. $\iff \int_E f d\mu = 0 \forall E \in \mathcal{M}$.
3. If $f \in \mathcal{L}^1(X, \mu)$, and $|\int_X f d\mu| = \int_X |f| d\mu$, there exist a constant α such that $\alpha f = |f|$ almost everywhere.

Proof. 1. Let $N = \{x \in X : f(x) > 0\}$.

Suppose $f = 0$ μ -a.e., then $\mu(N) = 0$.

We have

$$\int_X f d\mu = \int_{X \setminus N} f d\mu + \int_N f d\mu = \int_{X \setminus N} 0 d\mu + 0 = 0.$$

Thus $\int_E f d\mu = 0 \forall E \in \mathcal{M}$.

Now suppose $\int_E f d\mu = 0 \forall E \in \mathcal{M}$.

Let $A_n := \{x \in X : f(x) > \frac{1}{n}\}$, then we have

$$\frac{1}{n} \mu(A_n) = \int_{A_n} \frac{1}{n} d\mu \leq \int_{A_n} f d\mu = 0.$$

Thus $\mu(A_n) = 0$.

Notice that $N = \bigcup_{n=1}^\infty A_n$, $A_1 \subseteq A_2 \subseteq \dots$, thus $\mu(N) = \lim_{n \rightarrow \infty} \mu(A_n) = 0$.

2. Suppose $f = 0$ μ -a.e., we have that $|f| = 0$ μ -a.e., thus $|\int_E f d\mu| \leq \int_E |f| d\mu = 0$.

Now suppose $\int_E f d\mu = 0 \forall E \in \mathcal{M}$.

Let $f = u + iv$, then we have $\int_E u d\mu = \int_E v d\mu = 0 \forall E \in \mathcal{M}$.

Let $u = u^+ - u^-$, and $E = \{x \in X : u(x) \geq 0\}$.

$$\begin{aligned} \int_X u^+ d\mu &= \int_{X \setminus E} u^+ d\mu + \int_E u^+ d\mu \\ &= \int_{X \setminus E} 0 d\mu + \int_E u d\mu \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Thus $u^+ = 0$ μ -a.e..

Similarly for u^- , and thus $u = 0$ μ -a.e..

Similarly for v , and thus $f = 0$ μ -a.e..

□

Theorem 3.18 (Lebesgue Dominated Convergence - almost everywhere). *Let (X, \mathcal{M}, μ) be a measure space. Let $f_n : X \rightarrow \mathbb{C}$ be measurable functions, defined μ -almost everywhere on X , such that $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ is defined μ -almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that for μ -almost everywhere $x \in X, \forall n \in \mathbb{N}, |f_n(x)| \leq g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and*

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = 0.$$

Proof. Let N_n denote the zero measure set where f_n is not defined. Let N' denote the zero measure set where f is not defined. Then

$$N := N' \cup \{x \in X : \exists n \in \mathbb{N}, \text{ such that } |f_n(x)| > g(x)\} \cup \bigcup_{n=1}^{\infty} N_n$$

is measurable and has zero measure.

Define

$$h_n(x) := \begin{cases} f_n(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \quad h(x) := \begin{cases} f(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \quad g'(x) := \begin{cases} g(x) & x \in X \setminus N, \\ 0 & x \in N. \end{cases}$$

It is clear $\forall x \in X, h_n(x) \rightarrow h(x)$ point-wise, and dominated by $g'(x)$.

Since $g = g'$ μ -a.e. and thus $g' \in \mathcal{L}^1(X, \mu)$, by LDCT, we have

$$\lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = \lim_{n \rightarrow \infty} \int_X |g - g_n| d\mu = 0.$$

□

Theorem 3.19. *Let (X, \mathcal{M}, μ) be a measure space. Let $f_n : X \rightarrow \mathbb{C}$ be measurable functions, defined μ -almost everywhere on X , such that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$. We have that $f(x) := \sum_{n=1}^{\infty} f_n(x)$ exists μ -almost everywhere for $x \in X$, and that $f \in \mathcal{L}^1(X, \mu)$, and that*

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

Proof. For each n , let $D_n \subseteq X$ be the domain of f_n , then by assumption $\mu(X \setminus D_n) = 0$.

Let $D := \bigcap_{n=1}^{\infty} D_n, g(x) := \sum_{n=1}^{\infty} |f_n(x)|$. Note that $\mu(D^c) = \mu((\bigcap_{n=1}^{\infty} D_n)^c) = \mu(\bigcup_{n=1}^{\infty} D_n^c) = 0$.

Thus $g : X \rightarrow [0, \infty]$ is defined almost everywhere by Monotone Convergence Theorem.

By LMCT and assumption, $\int_X g d\mu$, and thus $g \in \mathcal{L}^1$.

Let $A := \{x \in D : g(x) < \infty\}$, then we have $\mu(A^c) = 0$.

By definition, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ absolutely on A . Thus $f \in \mathcal{L}^1(A, \mathcal{M}|_A, \mu|_{\mathcal{M}|_A})$.

Let $h_n = \sum_{i=1}^n f_i$ on A , then $|h_n| \leq \sum_{i=1}^n |f_i| \leq g$. Also, we have that $h_n(x) \rightarrow f(x)$ for any $x \in A$, then by LDCT, we have $\int_A f d\mu = \lim_{n \rightarrow \infty} \int_A h_n d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_A f_i d\mu = \sum_{n=1}^{\infty} \int_A f_n d\mu$.

Since $\mu(A^c) = 0$, we have that $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$

□

3.5 Complete Measure

Theorem 3.20. *Let (X, \mathcal{M}, μ) be a measure space, let*

$$\mathcal{M}^* := \{A \subseteq X : \exists B, C \in \mathcal{M}, \text{ such that } B \subseteq A \subseteq C, \mu(C \setminus B) = 0\}.$$

Define $\mu^(A) = \mu(B) = \mu(C)$, then \mathcal{M}^* is a σ -algebra, μ^* is a measure, and $(X, \mathcal{M}^*, \mu^*)$ is a measure space.*

Proof. $X \in \mathcal{M}$, and $X \subseteq X \subseteq X$ and $\mu(X \setminus X) = 0$, thus $X \in \mathcal{M}^*$.

Let $A \in \mathcal{M}^*$, then there are $B, C \in \mathcal{M}$ such that $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$. Thus $B^c \supseteq A^c \supseteq C^c$, and $B^c, C^c \in \mathcal{M}$. In addition, $\mu(B^c \setminus C^c) = \mu(B^c \cap C) = \mu(C \setminus B) = 0$. Thus $A^c \in \mathcal{M}^*$.

Let $A_n \in \mathcal{M}^*$ and $A = \bigcup_n A_n$, and $B_n, C_n \in \mathcal{M}$ such that $B_n \subseteq A_n \subseteq C_n, \mu(C_n \setminus B_n) = 0$. Let $B = \bigcup_n B_n, C = \bigcup_n C_n \in \mathcal{M}$, thus $B \subseteq A \subseteq C$. Now $\mu(C \setminus B) = \mu(\bigcup_n C_n \setminus B) \leq \mu(\bigcup_n C_n \setminus B_n) \leq \sum_n \mu(C_n \setminus B_n) = 0$.

Thus \mathcal{M}^* is a σ -algebra.

Now suppose $B, B', C, C' \in \mathcal{M}$, and $A \in \mathcal{M}^*$, with $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$, and $B' \subseteq A \subseteq C', \mu(C' \setminus B') = 0$.

Thus $B \setminus B' \subseteq A \setminus B' \subseteq C' \setminus B'$, and thus $\mu(B \setminus B') \leq \mu(C' \setminus B') = 0$. Thus $\mu(B) = \mu(B \cap B') + \mu(B \setminus B') = \mu(B \cap B')$. Similarly, we can show that $\mu(B') = \mu(B \cap B')$. Thus $\mu^* : \mathcal{M}^* \rightarrow [0, \infty]$ is well-defined.

Consider A_n a sequence of disjoint sets in \mathcal{M}^* , and $B_n, C_n \in \mathcal{M}$ as above. We have $\mu^*(A) = \mu(B) = \mu(\bigsqcup_n B_n) = \sum_n \mu(B_n) = \sum_n \mu(A_n)$. \square

Corollary 3.21. $(X, \mathcal{M}^*, \mu^*)$ has the property that if $N \in \mathcal{M}^*$ has $\mu(N) = 0$, we always have

$$\forall A \subseteq N, A \in \mathcal{M}^*, \mu^*(A) = 0.$$

Proof. Notice that $\forall A \subseteq N$, we have $\mu(N) = \mu(\emptyset) = 0$, with $\emptyset \subseteq A \subseteq N$, so $A \in \mathcal{M}^*, \mu^*(A) = 0$. \square

Definition 3.8. $(X, \mathcal{M}^*, \mu^*)$ defined above is called the **completion** of (X, \mathcal{M}, μ) . In addition, we say (X, \mathcal{M}, μ) is **complete** if $(X, \mathcal{M}, \mu) = (X, \mathcal{M}^*, \mu^*)$

Remark. If there is some $A \in \mathcal{M}$ such that $\mu(A^c) = 0$, then for any measurable $f : A \rightarrow Y$, we can extend it to X by $\forall x \in A^c, f(x) := 0$. Furthermore, if (X, \mathcal{M}, μ) is complete, we can extend f to whatever value we want. One can check that $f : X \rightarrow Y$ is measurable, and the integral $\int_X f d\mu$ does not depend on the extension.

Proposition 3.22. If (X, \mathcal{M}, μ) is a complete measure, we always have that property P holds μ -a.e. iff

$$\mu(\{x \in A : p(x) = \text{False}\}) = 0.$$

Proof. If P holds μ -a.e., there is $\exists N \in \mathcal{M}$, such that $\mu(N) = 0$ and $\forall x \in A \setminus N, p(x) = \text{True}$. Since $\{x \in A : p(x) = \text{False}\} \subseteq A \setminus (A \setminus N) = N$, we have $\mu(\{x \in A : p(x) = \text{False}\}) = 0$.

On the other hand, if $\mu(\{x \in A : p(x) = \text{False}\}) = 0$, we can just let $N := \{x \in A : p(x) = \text{False}\}$. Notice $\mu(N) = 0$, and $\forall x \in A \setminus N, p(x) = \text{True}$. \square

Proposition 3.23. Let μ be a complete measure on (X, \mathcal{M}) , suppose that f is measurable, and $g = f$, a.e., then g is also measurable. Moreover, if (f_n) is a sequence of measurable functions, and $f_n \rightarrow f$, μ -a.e., we always have that f is also measurable.

Proof. Suppose f is measurable, and we consider $D := \{x : X | f(x) \neq g(x)\}, \mu(D) = 0$.

Now let $B \subseteq \mathbb{R}$ be a Borel set, we need to show that $\{x \in X | g(x) \in B\} \in \mathcal{M}$.

Write $\{x \in X | g(x) \in B\} = (\{x \in X | g(x) \in B\} \cap D) \sqcup (\{x \in X | g(x) \in B\} \setminus D)$.

Since μ is complete, we have that $\{x \in X | g(x) \in B\} \cap D \in \mathcal{M}$ and has measure zero. Since f is measurable, we have that $f^{-1}(B) = \{x \in X | f(x) \in B\} \supseteq \{x \in X | f(x) = g(x) \in B\} = \{x \in X | g(x) \in B\} \setminus D$ is measurable.

Since μ is complete, we have that $\{x \in X | g(x) \in B\} \setminus D$ is measurable.

Thus $\{x \in X | g(x) \in B\} \in \mathcal{M}$ is measurable.

For the second part, consider $g = \limsup_{n \rightarrow \infty} f_n$. \square

4 Construction of Measure

4.1 Caratheodoy Theorem

Definition 4.1. Let X be a non-empty set, an **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. Monotone: $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$
3. Countable subadditive: For any $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$, we have that $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$

Proposition 4.1. Let $C \subseteq \mathcal{P}(X)$ with $\emptyset, X \in C$. Let $\mu : C \rightarrow [0, \infty]$ be a function such that $\mu(\emptyset) = 0$. Define $\mu^* : X \rightarrow [0, \infty]$ by $\mu^*(A) := \inf \{ \sum_{i=1}^\infty \mu(C_i) : C_i \in C, A \subseteq \bigcup_{i=1}^\infty C_i \}$. Then μ^* is an outer measure.

Proof. Clearly $\mu^*(\emptyset) = 0$ by taking $\forall i, C_i = \emptyset$.

In addition, $A \subseteq B \subseteq \bigcup_{i=1}^\infty C_i$ for any cover for B , and thus $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

Given any $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$. If $\sum_{n=1}^\infty \mu^*(A_n) = \infty$, then $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$ trivially.

Now assume that $\sum_{n=1}^\infty \mu^*(A_n) < \infty$.

Consider any $\epsilon > 0$.

For each $n \geq 0$, choose $(C_{n,i})_{i=1}^\infty \subseteq C$, such that $A_n \subseteq \bigcup_{i=1}^\infty C_{n,i}$ and $\mu^*(A_n) \leq \sum_{i=1}^\infty \mu(C_{n,i}) < \mu^*(A_n) + \frac{\epsilon}{2^n}$.

Thus $\bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty C_{n,i}$ and $\sum_{n=1}^\infty \sum_{i=1}^\infty \mu(C_{n,i}) \leq \sum_{n=1}^\infty (\mu^*(A_n) + \frac{\epsilon}{2^n}) \leq \sum_{n=1}^\infty \mu^*(A_n) + \epsilon$.

Take $\epsilon \rightarrow 0$ $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$ □

Definition 4.2. For an outer measure μ^* , we say $A \subseteq X$ is μ^* -measurable if

$$\forall E \subseteq X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Lemma 4.2. Let μ^* be an outer measure on X , and \mathcal{M} be the μ^* -measurable subsets of X , then \mathcal{M} is an algebra, and $\mu := \mu^*|_{\mathcal{M}}$ has finite additivity.

Proof. To check $A \in \mathcal{M}$, it suffices to check for $\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(E \cap A^c)$, since $\mu^*(E) \leq \mu^*(A \cap E) + \mu^*(E \cap A^c)$ holds by subadditivity of μ^* .

When $\mu^*(E) = \infty$, this holds trivially, and thus it suffices to only check for $\mu^*(E) < \infty$.

- Clearly $\emptyset \in \mathcal{M}$.
- $A \in \mathcal{M} \implies A^c \in \mathcal{M}$ since the condition is symmetric.
- Now consider $A, B \in \mathcal{M}$. For any $E \subseteq X$, we have

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*((E \cap A) \cup (E \cap A^c \cap B)) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c). \end{aligned}$$

Thus $A \cup B \in \mathcal{M}$.

Thus \mathcal{M} is an algebra.

To show finite additivity: Let $(A_i)_{i=1}^n$ be disjoint in \mathcal{M} , we will use induction on n .

Clearly it is true for $n = 1$.

Now suppose it holds for n , let $B = \bigsqcup_{i=1}^n A_i$. Since \mathcal{M} is an algebra, we have $B \in \mathcal{M}$. For any $E \subseteq X$,

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\ &= \mu^*\left(\bigsqcup_{i=1}^n (E \cap A_i)\right) + \mu^*(E \cap B^c) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B^c). \end{aligned}$$

Taking $E = \bigcup_{i=1}^{n+1} A_i$, we have

$$\begin{aligned}\mu^*\left(\bigcup_{i=1}^{n+1} A_i\right) &= \sum_{i=1}^n \mu^*\left(\left(\bigcup_{i=1}^{n+1} A_i\right) \cap A_i\right) + \mu^*\left(\bigcup_{i=1}^{n+1} A_i \cap B^c\right) \\ &= \sum_{i=1}^n \mu^*(A_i) + \mu^*(A_{n+1}) \\ &= \sum_{i=1}^{n+1} \mu^*(A_i).\end{aligned}$$

By induction, we have finite additivity for any $n \geq 1$. \square

Theorem 4.3. (Caratheodoy) Let μ^* be an outer measure on X , and \mathcal{M} be the μ^* -measurable subsets of X , then \mathcal{M} is a σ -algebra, and $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure.

Proof. Consider any $\{A_i\} \subset \mathcal{M}$, $B := \bigcup_{i=1}^{\infty} A_i$. By taking $\tilde{A}_n := A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ we can WLOG assume A_n are pair-wise disjoint, and $B = \bigcup_{i=1}^{\infty} A_i$.

For any $E \in X$, we have $\forall n \geq 1$, $\bigcup_{i=1}^n A_i \in \mathcal{M}$, and thus

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)^c\right) \\ &= \mu^*\left(\bigcup_{i=1}^n (E \cap A_i)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)^c\right) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)^c\right) \\ &\geq \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B^c).\end{aligned}$$

Taking $n \rightarrow \infty$, we have

$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c) \\ &\geq \mu^*\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right) + \mu^*(E \cap B^c) \\ &\geq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*(E \cap B^c) \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\ &\geq \mu^*((E \cap B) \cup (E \cap B^c)) \\ &= \mu^*(E).\end{aligned}$$

Thus $B \in \mathcal{M}$, and thus \mathcal{M} is a σ -algebra.

In addition, taking $E = B$, we have

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) + \mu^*(B \cap B^c) = \sum_{i=1}^{\infty} \mu^*(A_i) + \mu^*(\emptyset) = \sum_{i=1}^{\infty} \mu^*(A_i),$$

which shows countable additivity, and thus $\mu^*|_{\mathcal{M}}$ is a measure.

To show completeness, suppose $A \subseteq X$ such that $\mu^*(A) = 0$, then for any $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(A) + \mu^*(E) \\ &= \mu^*(E).\end{aligned}$$

Thus we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and thus $A \in \mathcal{M}$. \square

4.2 Premeasures

Definition 4.3. Recall an algebra of subsets of a set X is a family of subsets that is closed under complements, finite unions, and finite intersections, and contains the empty set.

Definition 4.4. A premeasure on an algebra of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$, such that μ is countably additive. Namely, if $(A_i)_{i=1}^\infty \subseteq \mathcal{A}$ are disjoint, and $\bigcup_{i=1}^\infty A_i \subseteq \mathcal{A}$, then we have

$$\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$$

Remark. If \mathcal{A} is a σ -algebra, a premeasure on \mathcal{A} is always a measure.

Theorem 4.4. Let \mathcal{A} be an algebra of subsets of X , and $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a premeasure. Apply Caratheodoy to the outer measure μ^* gives a complete measure space $(X, \mathcal{M}, \bar{\mu})$, such that $\mathcal{A} \subseteq \mathcal{M}$, and $\bar{\mu}|_{\mathcal{A}} = \mu$

Proof. We want to show

1. $\mathcal{A} \subseteq \mathcal{M}$.

Choose any $A \in \mathcal{A}$ and $(A_i)_{i=1}^\infty \subseteq \mathcal{A}$, such that $A \subseteq \bigcup_{i=1}^\infty A_i$.

Let $B_i = (A \cap A_i) \setminus \bigcup_{j=1}^{i-1} A_j$.

Notice that $B_i \in \mathcal{A}$, and are pairwise disjoint, and $A = \bigcup_{i=1}^\infty B_i$.

Since μ is a premeasure, $\mu(A) = \sum_{i=1}^\infty \mu(B_i) \leq \sum_{i=1}^\infty \mu(A_i)$.

Now $\mu^*(A) := \int \{\sum_{i=1}^\infty \mu(A_i) : A_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^\infty A_i\}$.

From above, we can see that $\mu^*(A) = \sum_{i=1}^\infty \mu(B_i) = \mu(A)$.

Now it remains to show $A \in \mathcal{M}$, which is the same as A is μ^* -measurable. Choose $E \subseteq X$ with $\mu^*(E) < \infty$ and $\epsilon > 0$.

There are $(E_i)_{i=1}^\infty \subseteq \mathcal{A}$, such that $E \subseteq \bigcup_{i=1}^\infty E_i$, and $\sum_{i=1}^\infty \mu(E_i) < \mu^*(E) + \epsilon$.

Then $E \cap A \subseteq \bigcup_{i=1}^\infty E_i \cap A$, and $E \cap A^c \subseteq \bigcup_{i=1}^\infty E_i \cap A^c$. Thus

$$\begin{aligned}\mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \sum_{i=1}^\infty \mu^*(E_i \cap A) + \sum_{i=1}^\infty \mu^*(E_i \cap A^c) \\ &= \sum_{i=1}^\infty \mu(E_i \cap A) + \sum_{i=1}^\infty \mu(E_i \cap A^c) \\ &= \sum_{i=1}^\infty \mu(E_i) \\ &< \mu^*(E) + \epsilon\end{aligned}$$

Now take $\epsilon \rightarrow 0$, we have that $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E)$. However, by subadditivity of μ^* , we have that $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and thus $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$

2. $\bar{\mu}|_{\mathcal{A}} = \mu$

We have shown that $\mu = \mu^*|_{\mathcal{A}}$, but we also know that $\bar{\mu} = \mu^*|_{\mathcal{M}}$, and $\mathcal{A} \subseteq \mathcal{M}$. \square

Definition 4.5. A premeasure $\mu : \mathcal{A} \rightarrow [0, \infty]$ on an algebra \mathcal{A} for X is σ -finite if there are $(A_i)_{i=1}^\infty \subseteq \mathcal{A}$, such that $\mu(A_i) < \infty$ and $\bigcup_{i=1}^\infty A_i = X$

Proposition 4.5. Let \mathcal{A} be an algebra of sets on X . Let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a premeasure, with the corresponding complete measure space $(X, \mathcal{M}, \bar{\mu})$ as in the above theorem. Suppose (X, \mathcal{N}, ν) is a measure space with $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M}$ and $\nu|_{\mathcal{A}} = \mu$. Then if μ is σ -finite, we have that

$$\nu = \bar{\mu}|_{\mathcal{N}},$$

so $\bar{\mu}|_{\mathcal{N}}$ is the unique extension of μ to a measure on \mathcal{N} .

4.3 Lebesgue-Stieltjes Measures

Definition 4.6. Let μ be a Borel measure on \mathbb{R} , such that $\mu(\mathcal{K}) < \infty$ for any compact $\mathcal{K} \subseteq \mathbb{R}$. Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by $F(x) = \begin{cases} \mu((0, x]) & \text{if } x \geq 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$

Proposition 4.6. F is monotone non-decreasing. i.e. If $b \geq a$, then $F(b) - F(a) \geq 0$.

Proof. For $0 < a < b$, we have that $\mu((a, b]) = \mu((0, b] \setminus (0, a]) = \mu((0, b]) - \mu((0, a]) = F(b) - F(a)$. For $0 \geq b > a$, we have that $\mu((a, b]) = \mu((a, 0] \setminus (b, 0]) = \mu((a, 0]) - \mu((b, 0]) = -F(a) - (-F(b)) = F(b) - F(a)$. Similarly, we can check for $a < 0 \leq b$. \square

Proposition 4.7. F is right continuous.

Proof. Fix $x \geq 0 \in \mathbb{R}$, and choose any sequence $(x_n) \subseteq \mathbb{R}$ such that $x_n \geq x_{n+1}$ and $x_n \rightarrow x$. Since μ is a measure and $\mu((0, x_1]) \leq \mu([0, x_1]) < \infty$, we have

$$\begin{aligned} F(x) &= \mu((0, x]) \\ &= \lim_{n \rightarrow \infty} \mu((0, x_n]) \\ &= \lim_{n \rightarrow \infty} F(x_n). \end{aligned}$$

Similar proof for $x < 0 \in \mathbb{R}$. \square

Example 4.3.1. If $\mu = \delta_c$, $\delta_0(A) = \begin{cases} 0 & \text{if } c \in A \\ 1 & \text{if } c \notin A \end{cases}$, then F is the (translated) Heaviside function.

Example 4.3.2. If μ is the Lebesgue measure, then F is the identity function $F(x) = x$.

Now given a right-continuous increasing function F , we want to construct a measure.

Proposition 4.8. Let \mathcal{A} be the collection of sets consisting of all the finite disjoint unions of half-open intervals $(a, b]$, $-\infty \leq a \leq b \leq \infty$. Then \mathcal{A} is an algebra of sets.

Proof. Firstly notice that for any interval $(a, b] \in \mathcal{A}$, we have that $(a, b]^c = [-\infty, a] \cup (b, \infty] \in \mathcal{A}$. Also, any finite union of such disjoint unions can be written as a disjoint union. $(a, b] \cup (c, d] = (a, c] \cup (c, b] \cup (b, d]$ for $c < b$. We can show any finite union by induction. \square

Definition 4.7. Let \mathcal{A} be the algebra of sets consisting of all the finite disjoint unions of half-open intervals $(a, b]$, $-\infty \leq a \leq b \leq \infty$. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a right-continuous monotone non-decreasing function, we can extend F to $[-\infty, \infty]$ by $F(\pm\infty) := \lim_{x \rightarrow \pm\infty} F(x)$, which exists by MCT. Now define $\mu_F : \mathcal{A} \rightarrow [0, \infty]$ to be

$$\mu_F\left(\bigsqcup_{i=1}^n (a_i, b_i]\right) := \sum_{i=1}^n F(b_i) - F(a_i)$$

Lemma 4.9. μ_F is a pre-measure

Proof. We firstly show that μ_F is well defined.

Consider $I_i = (a_i, b_i]$ and $I = (a, b] = \bigsqcup_{i=1}^n I_i$

By reordering, we can WLOG assume that $a = a_1 < b_1 = a_2 < b_2 = \dots < b_{n-1} = a_n < b_n = b$.

Let $a_{n+1} := b_n = b$, we have that

$$\begin{aligned}\mu_F\left(\bigsqcup_{i=1}^n (a_i, b_i]\right) &= \sum_{i=1}^n F(b_i) - F(a_i) \\ &= \sum_{i=1}^{\infty} F(a_{i+1}) - F(a_i) \\ &= F(a_{n+1}) - F(a_1) \\ &= F(b) - F(a) \\ &= \mu_F(I).\end{aligned}$$

Thus $\mu_F(I)$ does not depend on the decomposition of I . This extends to finite disjoint unions of half-open intervals. Hence μ_F is well-defined.

Monotone follows from the fact that F is increasing.

Consider pair-wise disjoint $(A_i)_{i=1}^{\infty} \in \mathcal{A}$, and $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$, we want to show $\mu_F(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_F(A_i)$.

We first assume that each $A_i = (a_i, b_i]$ and $\bigsqcup_{i=1}^{\infty} A_i = (a, b]$

Notice that $\forall n \in \mathbb{N}$, we have that $\bigsqcup_{i=1}^n (a_i, b_i] \in \mathcal{A}$, and thus $(a, b] \setminus \bigsqcup_{i=1}^n (a_i, b_i] \in \mathcal{A}$.

$$\begin{aligned}\mu_F((a, b]) &= \mu_F\left(\bigsqcup_{i=1}^n (a_i, b_i]\right) + \mu_F\left((a, b] \setminus \bigsqcup_{i=1}^n (a_i, b_i]\right) \\ &= \sum_{i=1}^n \mu_F((a_i, b_i]) + \mu_F\left((a, b] \setminus \bigsqcup_{i=1}^n (a_i, b_i]\right) \\ &\geq \sum_{i=1}^n \mu_F((a_i, b_i])\end{aligned}$$

Thus $\mu_F((a, b]) \geq \sum_{i=1}^{\infty} \mu_F((a_i, b_i])$.

For the other direction, fix $\epsilon > 0$, by right continuity, $\exists \delta > 0$, such that $F(a + \delta) < F(a) + \epsilon$, $a + \delta < b$. Now suppose $b \neq \infty$, then $\exists \delta_i > 0$, $F(b_i + \delta_i) < F(b_i) + 2^{-i}\epsilon$.

Thus $[a_i + \delta, b_i] \subseteq (a_i, b_i + \delta_i)$, and thus $\{(a_i, b_i + \delta_i)\}$ is an open cover for $[a + \delta, b] = \bigcup_{i=1}^{\infty} [a_i + \delta, b_i]$. Since the closed interval is compact, there is a finite sub-cover $\{(a_{i_j}, b_{i_j} + \delta_{i_j})\}_{j=1}^n$. Then

$$\begin{aligned}\sum_{j=1}^n (F(b_{i_j} + \delta_{i_j}) - F(a_{i_j})) &= \sum_{j=1}^n \mu_F((a_{i_j}, b_{i_j} + \delta_{i_j})) \\ &\geq \mu_F((a + \delta, b]) \\ &\geq F(b) - F(a + \delta)\end{aligned}$$

since μ_F is monotone. Hence

$$\begin{aligned}\sum_{i=1}^{\infty} \mu_F((a_i, b_i]) &= \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) \\ &\geq \sum_{j=1}^n (F(b_{i_j}) - F(a_{i_j})) \\ &\geq \sum_{j=1}^n (F(b_{i_j} + \delta_{i_j}) - 2^{-i_j}\epsilon - F(a_{i_j})) \\ &\geq F(b) - F(a + \delta) - \epsilon \\ &\geq F(b) - F(a) - 2\epsilon \\ &= \mu_F((a, b]) - 2\epsilon.\end{aligned}$$

Take $\epsilon \rightarrow 0$, we have $\sum_{i=1}^{\infty} \mu_F((a_i, b_i]) \geq \mu_F((a, b])$

When $b = \infty$, we have that $\forall N \geq a, \sum_{i=1}^{\infty} \mu_F((a_i, b_i]) \geq \mu_F((a, N]) = F(N) - F(a)$.

Hence $\sum_{i=1}^{\infty} \mu_F((a_i, b_i]) \geq F(b) - F(a) = \lim_{N \rightarrow b} F(N) - F(a)$

Thus we have shown that $\sum_{i=1}^{\infty} \mu_F((a_i, b_i]) = \mu_F((a, b])$

If $A = \bigsqcup_{i=1}^m (c_i, d_i]$, we can use finite additivity and the previous case. \square

Theorem 4.10. *If $F : \mathbb{R} \rightarrow \mathbb{R}$ is monotone non-decreasing and right-continuous, then there is a complete measure space $(\mathbb{R}, \mathcal{M}, \bar{\mu}_F)$, which extends μ_F as $\bar{\mu}_F|_{\mathcal{A}} = \mu_F$, the σ -algebra \mathcal{M} contains $\text{Bor}(\mathbb{R})$ and $\bar{\mu}_F|_{\text{Bor}(\mathbb{R})}$ is the unique extension of μ_F , i.e. $\bar{\mu}_F((a, b]) = F(b) - F(a)$.*

Conversely, given a Borel measure μ on \mathbb{R} , such that $\forall K$ compact, $\mu(K) < \infty$, there is a (up to constant) unique non-decreasing right-continuous F with $\mu = \bar{\mu}_F|_{B_{\mathbb{R}}}$.

Proof. By the previous lemma, μ_F is a premeasure, so applying Caratheodory gives a complete measure space $(\mathbb{R}, \mathcal{M}, \bar{\mu}_F)$. We have seen that the σ -algebra generated by \mathcal{A} is $\text{Bor}(\mathbb{R})$, so $\text{Bor}(\mathbb{R}) \subseteq \mathcal{M}$. The uniqueness follows from the fact that $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+2]$, $\mu_F((n, n+2]) = F(n+2) - F(n) < \infty$ and thus μ_F is σ -finite.

Conversely, let F be the function defined at the beginning of this section. Then we know that $\mu = \bar{\mu}_F|_{B_{\mathbb{R}}}$ since μ is σ -finite and agree with the premeasure μ_F on the algebra \mathcal{M} of all finite disjoint unions of half open interval.

If $G : \mathbb{R} \rightarrow \mathbb{R}$ is another monotone non-decreasing right-continuous function, then $\bar{\mu}_F = \bar{\mu}_G \implies \mu_F((a, b]) = \mu_G((a, b]) \implies F(b) - F(a) = G(b) - G(a)$ for any $a < b$. Thus $\forall a \in \mathbb{R}, F(x) - G(x) = c := F(0) - G(0)$, which is a constant. \square

Remark. The Lebesgue is got by taking $F(x) = x$.

Definition 4.8. A μ be a Borel measure on \mathbb{R} , such that $\mu(K) < \infty$ for any compact $K \subseteq \mathbb{R}$, is called a **Lebesgue-Stieltjes measure**. For any $F : \mathbb{R} \rightarrow \mathbb{R}$ that is monotone non-decreasing and right-continuous, and any Borel measure μ such that $\mu = \bar{\mu}_F|_{\text{Bor}(\mathbb{R})}$, we call μ the **Lebesgue-Stieltjes measure corresponding to F** .

Proposition 4.11. *Let μ be a Lebesgue-Stieltjes measure corresponding to some $F : \mathbb{R} \rightarrow \mathbb{R}$, then $\forall a \in \mathbb{R}$,*

$$\mu(\{a\}) = \mu\left(\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a]\right) = \lim_{n \rightarrow \infty} \mu((a - \frac{1}{n}, a]) = F(a) - \lim_{n \rightarrow \infty} F(a - \frac{1}{n}) = F(a) - F(a^-).$$

Thus $\mu(\{a\}) > 0$ if and only if F has a jump discontinuity at a , since every discontinuity of a monotone non-decreasing function is a jump discontinuity.

Corollary 4.12. *If $F(x) = x$ is the identity function, every countable set has measure 0, by subadditivity and that $\forall a \in \mathbb{R}, \mu(\{a\}) = 0$*

Proposition 4.13. *A monotone non-decreasing function F can have at most countably many discontinuities.*

Proof. Choose countably many disjoint points $\{c_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$. Define a measure $\mu := \sum_{n \geq 1} \frac{1}{2^n} \delta_{c_n}$. This is a Borel measure with $\mu(K) < \infty$ for any compact $K \subseteq \mathbb{R}$. Thus μ is a Lebesgue-Stieltjes measure. Note $\mu(\{c_n\}) = \frac{1}{2^n} > 0$, thus each c_n is a jump discontinuity for the corresponding F . Thus F has countably many discontinuities.

In fact, no such F can have uncountably many discontinuities. \square

Theorem 4.14. *Lebesgue measure $(\mathbb{R}, \mathcal{L}, \lambda)$ is translation-invariant, meaning*

$$\forall A \in \mathcal{L}, s \in \mathbb{R}, \lambda(A + s) = \lambda(A).$$

Also,

$$\forall s > 0, A \in \mathcal{L}, \lambda(sA) = s\lambda(A).$$

Proof. If $A \subseteq \mathbb{R}$ is open, then so is $A + s$. Similarly for closed sets. Hence for $A \in B_{\mathbb{R}}$, $A + s \in B_{\mathbb{R}}$. Define a new measure λ_s on $B_{\mathbb{R}}$ by $\lambda_s(A) = \lambda(A + s)$. Note that λ and λ_s correspond to the functions

$$F(x) = \begin{cases} \lambda((0, x]) & \text{if } x \geq 0 \\ -\lambda((x, 0]) & \text{if } x < 0 \end{cases},$$

$$G(x) = \begin{cases} \lambda_s((0, x]) & \text{if } x \geq 0 \\ -\lambda_s((x, 0]) & \text{if } x < 0 \end{cases}.$$

Yet $\lambda((0, x]) = \lambda((s, x + s]) = \lambda_s((0, x + s])$, and thus $F = G$. Thus $\lambda_s|_{B_{\mathbb{R}}} = \lambda|_{B_{\mathbb{R}}}$. By uniqueness for σ -finite Caratheodoy Theorem, we have that they extends to $\lambda = \lambda_s$. \square

Definition 4.9. A point $c \in \mathbb{R}$ with $\mu(\{x\}) \neq 0$ is called an **atom** of μ .

Corollary 4.15. *Lebesgue-Stieltjes measures can have at most countably many atoms.*

Definition 4.10. Let X be a topological space, then a \mathcal{G}_{δ} set is a countable intersection of open subsets of X , and a \mathcal{F}_{σ} set is a countable union of closed subsets.

Remark. \mathcal{G}_{δ} sets and \mathcal{F}_{σ} sets are Borel sets.

Theorem 4.16. *Let μ be a Lebesgue-Stieltjes measure with outer measure μ_F^* , and $E \subseteq \mathbb{R}$, the following are equal:*

- (1) E is μ -measurable
- (2) $\forall \epsilon > 0$, there is some open $O \supseteq E$, $\mu_F^*(O \setminus E) < \epsilon$ (Outer regularity)
- (3) $\forall \epsilon > 0$, there is some closed $C \subseteq E$, $\mu_F^*(E \setminus C) < \epsilon$ (Inner regularity)
- (4) There is a \mathcal{G}_{δ} set $G \supseteq E$, $\mu_F^*(G \setminus E) = 0$
- (5) There is a \mathcal{F}_{σ} set $F \subseteq E$, $\mu_F^*(E \setminus F) = 0$

Proof. Notice that E is μ -measurable means that

$$\forall A \subseteq \mathbb{R}, \mu_F^*(A) = \mu_F^*(E \cup A) + \mu_F^*(E^c \cup A).$$

1. (1) implies (2):

If E is μ -measurable,

$$\begin{aligned} \mu(E) &= \mu_F^*(E) \\ &= \inf_{B \supseteq E} \mu_F(B), \end{aligned}$$

where $B = \bigcup_{i=1}^{\infty} (a_i, b_i] \supseteq E$.

Firstly, assume that E is bounded, we have $\mu_F^*(B) < \mu(E) + \frac{\epsilon}{2}$ for some $B = \bigcup_{i=1}^{\infty} (a_i, b_i] \supseteq E$.

Since F is right-continuous, we have that $\forall i, \exists c_i > b_i$, such that $F(c_i) < F(b_i) + \frac{\epsilon}{2^{i+1}}$.

Let $O := \bigcup_i (a_i, c_i) \supseteq B \supseteq E$.

Since E is measurable, we have that $\mu_F^*(C) = \mu_F^*(B \cap E) + \mu_F^*(B \setminus E) = \mu(E) + \mu_F^*(B \setminus E)$, thus

$$\mu_F^*(B \setminus E) < \frac{\epsilon}{2}$$

$$\begin{aligned}
\mu_F^*(O \setminus B) &= \mu_F^*\left(\bigcup_i (a_i, c_i) \cap B^c\right) \\
&\leq \sum_i \mu_F^*((a_i, c_i) \cap B^c) \\
&\leq \sum_i \mu_F^*((b_i, c_i)) \\
&\leq \sum_i \mu_F^*((b_i, c_i]) \\
&= \sum_i F(c_i) - F(b_i) \\
&< \sum_i \frac{\epsilon}{2^{i+1}} \\
&= \frac{\epsilon}{2}. \\
\mu_F^*(O \setminus E) &\leq \mu_F^*(O \cap E^c \cap B) + \mu_F^*(O \cap E^c \setminus B) \\
&= \mu_F^*(B \setminus E) + \mu_F^*(O \setminus B) \\
&< \epsilon.
\end{aligned}$$

This proves the bounded case.

If E is not bounded, we let $E_n = E \cap (n-1, n]$, $n \in \mathbb{Z}$, each is bounded, and we can have open $O_n \supseteq E_n$, $\mu_F(O_n \setminus E_n) < \frac{\epsilon}{2^{|n|+1}}$, and take $O = \bigcup_{n \in \mathbb{Z}} O_n \supseteq A$.

2. (2) implies (4):

For each $n \geq 1$, take open $O_n \supseteq E$, $\mu_F(O_n \setminus E) < \frac{1}{n}$, and WLOG, take $O_n = O_n \cap O_{n-1}$ so that $O_n \supseteq O_{n-1}$.

Take $G := \bigcap_{n=1}^{\infty} O_n$, which is a \mathcal{G}_δ set.

We have that $\forall n \geq 1$, $\mu_F^*(G \setminus E) \leq \mu_F^*(O_n \setminus E) < \frac{1}{n}$.

Thus $\mu_F^*(G \setminus E) = 0$.

3. (4) implies (1):

$G \setminus E$ is measurable since it is a null set, and μ is complete. G is also measurable, thus $E = G \setminus (G \setminus E)$ is also measurable.

4. (1) implies (3):

E is μ -measurable, so is E^c .

By (2), there is some open $O \supseteq E^c$, such that $\mu_F^*(O \setminus E^c) < \epsilon$.

Notice that $C := O^c$ is closed, and $C \subseteq E$, and

$$\begin{aligned}
\mu_F^*(E \setminus C) &= \mu_F^*(E \cap C^c) \\
&= \mu_F^*((E^c)^c \cap O) \\
&= \mu_F^*(O \setminus E^c) \\
&< \epsilon.
\end{aligned}$$

5. (3) implies (5)

For each $n \geq 1$, take closed $C_n \subseteq E$, $\mu_F(E \setminus C_n) < \frac{1}{n}$, and WLOG, take $C_n = C_n \cap C_{n-1}$ so that $C_n \supseteq C_{n-1}$.

Take $F := \bigcup_{n=1}^{\infty} C_n$, which is a \mathcal{F}_σ set.

We have that $\forall n \geq 1$, $\mu_F^*(E \setminus F) \leq \mu_F^*(E \setminus C_n) < \frac{1}{n}$.

Thus $\mu_F^*(E \setminus F) = 0$.

6. (5) implies (1)

$E \setminus F$ is a measurable set. F is also a measurable set, and thus so is $E = (E \setminus F) \cup F$.

□

Corollary 4.17. *Let μ be a Lebesgue-Stieltjes measure, and A be μ -measurable, we have*

$$\mu(A) = \inf \{\mu(O) : O \supseteq A \text{ is open}\} = \sup \{\mu(C) : C \subseteq A \text{ is compact}\}$$

Proof. The first equality is (2).

For second equality, if A is bounded, and $C \subseteq A$ is closed, then it is compact. We can use (3) to prove it. If A is not bounded, let $A_n := A \cap [-n, n]$ for each $N \geq 1$. Thus

$$\mu(A) = \sup_{n \geq 1} \mu(A_n) = \sup_{n \geq 1} \sup_{C \subseteq A_n \text{ is compact}} \mu(C).$$

□

4.4 Littlewood's Three Principles

Recall Littlewood's Three Principles for Lebesgue Measure:

Theorem 4.18. *Littlewood's first Principle (regularity)
Every measurable set is almost a finite union of intervals.*

Theorem 4.19. *Littlewood's second Principle (Lusin's)
Every measurable function is almost continuous.*

Theorem 4.20. *Littlewood's third Principle (Egorov's)
A point-wise convergent sequence of measurable functions is almost uniformly convergent.*

Theorem 4.21 (Egorov's). *Let (X, \mathcal{M}, μ) be a finite measure space. Suppose $f_n : X \rightarrow \mathbb{C}$ is a sequence of measurable functions such that $f_n(x) \rightarrow f(x)$ μ -almost everywhere. Then $\forall \epsilon > 0, \exists A \in \mathcal{M}$, such that $\mu(X \setminus A) < \epsilon$, and $f_n \rightarrow f$ uniformly on A .*

Proof. $f_n \rightarrow f$ uniformly on A means $\forall m \in \mathbb{N}^+, \exists N_m \geq 1$, such that

$$\forall x \in A, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m}.$$

Let $A_{mN} := \{x \in X : \forall n \geq N, |f_n(x) - f(x)| < \frac{1}{m}\} = \bigcap_{n \geq N} \{x \in X : |f_n(x) - f(x)| < \frac{1}{m}\}$, which is measurable.

Note $A_{m1} \subseteq A_{m2} \subseteq \dots$, and $\bigcup_{n \geq 1} A_{m,n} = X \setminus N$ for some $N \in \mathcal{M}, \mu(N) = 0$ since $f_n \rightarrow f$ μ -a.e..

$$\begin{aligned} \mu(X) &= \mu(X \setminus N) \\ &= \mu\left(\bigcup_{n \geq 1} A_{mn}\right) \\ &= \lim_{n \rightarrow \infty} \mu(A_{mn}). \end{aligned}$$

Since $\mu(X) < \infty$, there is $N_m \geq 1$ such that $\mu(A_{m, N_m}) > \mu(X) - \frac{\epsilon}{2^m}$ for any $\epsilon > 0$.

Thus $\mu(X \setminus A_{m, N_m}) < \frac{\epsilon}{2^m}$.

Letting $E := \bigcap_{m \geq 1} A_{m, N_m}$, we have that

$$\begin{aligned} \mu(X \setminus E) &= \mu\left(\bigcup_{m \geq 1} (X \setminus A_{m, N_m})\right) \\ &\leq \sum_{m \geq 1} \mu(X \setminus A_{m, N_m}) \\ &< \epsilon. \end{aligned}$$

In addition,

$$\begin{aligned} E &= \bigcap_{m \geq 1} A_{m, N_m} \\ &= \left\{ x \in X : \forall m \geq 1, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m} \right\}. \end{aligned}$$

Thus $f_n \rightarrow f$ uniformly on E . \square

Theorem 4.22 (Lusin's). *Let $f : [a, b] \rightarrow \mathbb{C}$ be a Lebesgue-Stieltjes measurable function. For any $\epsilon > 0$, there is a continuous function $g : [a, b] \rightarrow \mathbb{C}$ such that*

$$\mu(\{x \in [a, b] : f(x) \neq g(x)\}) < \epsilon.$$

Proof. Consider a simple function $s := \sum_{i=1}^m \alpha_i \chi_{E_i}$, where $\alpha_i \in \mathbb{C}$, and E_i are disjoint and Lebesgue measurable.

Notice that by the regularity theorem, for any $\delta > 0$, there are compact sets $A_i \subseteq E_i$, such that $\mu(E_i \setminus A_i) < \frac{\delta}{m}$ for each i . Thus $\mu(\bigcup_{i=1}^m E_i \setminus A_i) < \delta$.

Let $\mathcal{K} := \bigcup_{i=1}^m A_i$. Notice that \mathcal{K} is compact, and $s|_{\mathcal{K}}$ is continuous since s is locally constant. Indeed, $\forall x \in \mathcal{K}$, there is unique A_i such that $x \in A_i$. Suppose $\forall \delta_0 > 0$, there is some $y \in (x - \delta_0, x + \delta_0) \cap A_j$ for some $j \neq i$. Let $\mathcal{K}' := \bigcup_{j=1, j \neq i}^m A_j$. Then we have a sequence $y_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap \mathcal{K}'$. Notice that $y_n \rightarrow x$, and since \mathcal{K}' is compact, $x \in \mathcal{K}'$, which is a contradiction. Thus $\exists \delta_0 > 0$, such that $(x - \delta_0, x + \delta_0) \cap \mathcal{K} \subseteq A_i$; namely, s is constant on $(x - \delta_0, x + \delta_0) \cap \mathcal{K}$. Thus $\forall \epsilon_0 > 0$, $y \in \mathcal{K}$, such that $|y - x| < \delta_0$, $|s(x) - s(y)| = 0 < \epsilon_0$.

Now given any measurable f , we can choose simple functions $s_n : [a, b] \rightarrow \mathbb{C}$ converging point-wise to f . For each n , construct \mathcal{K}_n as above such that $s_n|_{\mathcal{K}_n}$ is continuous and $\mu([a, b] \setminus \mathcal{K}_n) < \frac{\epsilon}{2^{n+1}}$.

Let $\mathcal{K}_0 = \bigcap \mathcal{K}_n$, which is compact. For all n , we have that $s_n|_{\mathcal{K}_n}$ is continuous.

In addition, $\mu([a, b] \setminus \mathcal{K}_0) \leq \sum_{n=1}^{\infty} \mu([a, b] \setminus \mathcal{K}_n) < \epsilon/2$.

By Egorov's Theorem, there is a measurable $E \subseteq \mathcal{K}_0$, such that $\mu(\mathcal{K}_0 \setminus E) < \epsilon/4$ and $s_n \rightarrow f$ uniformly on E .

Applying the regularity theorem again, there is a compact $\mathcal{K} \subseteq E$ such that $\mu(E \setminus \mathcal{K}) < \epsilon/4$. Notice that $s_n \rightarrow f$ uniformly on \mathcal{K} . Thus $f|_{\mathcal{K}}$ is continuous.

Also, $\mu([a, b] \setminus \mathcal{K}) \leq \mu([a, b] \setminus \mathcal{K}_0) + \mu(\mathcal{K}_0 \setminus E) + \mu(E \setminus \mathcal{K}) = \epsilon$.

By Tietze's Theorem, we can extend $f|_{\mathcal{K}}$ to some continuous $g : [a, b] \rightarrow \mathbb{C}$. We thus have

$$\mu(\{x \in [a, b] : f(x) \neq g(x)\}) \leq \mu([a, b] \setminus \mathcal{K}) < \epsilon$$

.

\square

5 Borel Measures on Topological Spaces

5.1 Topological Spaces

Definition 5.1. Let (X, \mathcal{T}) be a topological space, then we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$. We say $E \subseteq X$ is **closed** if $E^c \in \mathcal{T}$ is open.

Definition 5.2. For $E \subseteq X$, the **closure** of E is

$$\bar{E} = \bigcap_{F \supseteq E : F \text{ is closed}} F.$$

Definition 5.3. A set $K \subseteq X$ is **compact** if every open cover of K has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open, } K \subseteq \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Definition 5.4. An (open) neighborhood of $x \in X$ is some

$$U_x \in \mathcal{T}, \text{ such that } x \in U_x.$$

Definition 5.5. X is **Hausdorff** if

$$\forall x \neq y \in X, \exists U_x, U_y \text{ open neighborhoods for } x, y, \text{ such that } U_x \cap U_y = \emptyset.$$

Example 5.1.1. Every metric space is Hausdorff.

Definition 5.6. X is **locally compact** if $\forall x \in X$, there is a neighborhood U_x such that $\overline{U_x}$ is compact.

Example 5.1.2. \mathbb{R}^n are locally compact by Heinz-Borel theorem.

Proposition 5.1. A Banach space $(X, \|\cdot\|)$ is locally compact iff $\dim(X) < \infty$.

Theorem 5.2. Let (X, \mathcal{T}) be a topological space,

1. Suppose K is compact, then $\forall F \subseteq K$ that is closed, F is also compact.
2. If X is Hausdorff, for any compact $K \subseteq X, x \in X \setminus K$, \exists open neighborhood U of x , and open $W \supset K$, such that $W \cap U = \emptyset$.

Proof. 1. Let $(U_\alpha)_{\alpha \in A}$ be an open cover for F .

Since F is closed, then F^c is open. Thus $\{F^c\} \cup \bigcup_{\alpha \in A} U_\alpha$ is an open cover for K .

Thus there are $U_{\alpha_1}, \dots, U_{\alpha_n}$, such that $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$. Thus $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ has a finite subcover.

2. Consider any $y \in K$, there is some open neighborhoods $U_y \ni x, W_y \ni y$, such that $U_y \cap W_y = \emptyset$.

Since $K \subseteq \bigcup_{y \in K} W_y$ is compact, we have $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$ for some y_1, \dots, y_n .

Let $U = \bigcap_{i=1}^n U_{y_i}$, we have $x \in U, K \subseteq W, U \cap W = \emptyset$ as required. □

Corollary 5.3. Let (X, \mathcal{T}) be a Hausdorff topological space, then any compact set K is closed. In addition, for any closed $F \subseteq X$, we have $F \cap K$ is compact.

Proof. Suppose for contradiction that K is not closed, then there is some $y \in \bar{K} \setminus K$. Thus we can find open neighborhood U of x , and open $W \supset K$, such that $W \cap U = \emptyset$. Now $K \subset \bar{K} \setminus U \subsetneq \bar{K}$ is closed, which is a contradiction.

Since K is closed, so is $F \cap K \subseteq K$, and thus it is compact. □

Lemma 5.4. Let (X, \mathcal{T}) be a Hausdorff topological space, and $(K_\alpha)_{\alpha \in A}$ be a collections of compact sets such that

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

We must have $\alpha_1, \dots, \alpha_n \in A$, such that

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

Proof. Fix $\alpha_1 \in A$, then $K_{\alpha_1} \subseteq \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha \right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$ is compact and has an open cover.

Thus there must be $\alpha_2, \dots, \alpha_n \in A$, such that $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = \left(\bigcap_{i=2}^n K_{\alpha_i} \right)^c$.

Thus $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$. □

Theorem 5.5. Let X be a Locally Compact Hausdorff space, and let $K \subseteq U \subseteq X$ be such that K is compact, and U is open. Then there exists some open set V such that \bar{V} is compact, and

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof. Since X is a Locally Compact Hausdorff space, there are V_1, \dots, V_n , each with \bar{V}_i be compact, such that $K \subseteq \bigcup_{i=1}^n V_i =: G$. Note that \bar{G} is compact, and G is open.

If $U = X$, then $G \subseteq U$, and we are done.

Otherwise, let $C := X \setminus U$ be non-empty and closed.

Consider any $y \in C$, we know that $y \notin K$. Since X is Hausdorff, we can find open $W_y \supset K$, and $U_y \ni y$,

such that $W_y \cap U_y = \emptyset$. Then $W_y \subseteq U_y^c$, and thus $\bar{W}_y \subseteq U_y^c$, since U_y^c is closed. Yet $y \notin U_y^c$, thus $y \notin \bar{W}_y$. Now consider the family $\{C \cap \bar{W}_y \cap \bar{G}\}_{y \in C}$. Notice that each $C \cap \bar{W}_y \cap \bar{G}$ is compact, since C, \bar{W}_y are closed, and \bar{G} is compact.

Yet $\bigcap_{y \in C} (C \cap \bar{W}_y \cap \bar{G}) = \emptyset$.

Thus $\exists y_1, \dots, y_n \in C$, such that $\bigcap_{i=1}^n (C \cap \bar{W}_{y_i} \cap \bar{G}) = \emptyset$.

Now let $V := G \cap \bigcap_{i=1}^n W_{y_i}$.

Clearly V is open, and $K \subseteq V$.

In addition, $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$, yet the intersection of righthand side and C is empty, thus contained in $C^c = U$. \square

5.2 Compactly Supported Functions

Definition 5.7. Let $C(X)$ be the collections of functions $f : X \rightarrow \mathbb{C}$ that are continuous.

Proposition 5.6. $C(X)$ is a \mathbb{C} vector space, and also an Algebra over \mathbb{C} . It also admits a partial order by $f \geq g \iff \forall x \in X, f(x) \geq g(x)$.

Definition 5.8. For $f \in C(X)$, the **support** of it is

$$\text{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

Definition 5.9. The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \text{Supp}(f) \text{ is compact}\}.$$

Proposition 5.7. Suppose every compact set K is Borel-measurable, then $C_c(X)$ is a sub-algebra of $C(X)$.

Proposition 5.8. Suppose every compact set K is Borel-measurable, let $\mu : \text{Bol}(X) \rightarrow [0, \infty]$ be a Borel measure on X , such that $\forall K$ be compact, $\mu(K) < \infty$, then

$$C_c(X) \subseteq L^1(\mu).$$

Proof. Given any $f \in C_c(X)$. Let $K = \text{Supp}(f)$, then

$$\int_X |f| d\mu = \int_K |f| d\mu \leq \int_K \|f\|_\infty d\mu = \mu(K) \|f\|_\infty < \infty.$$

\square

5.3 Partition of Unity

Definition 5.10. Let K be a compact set, and V be an open set of X . Let $f \in C_c(X)$. We say $f < V$ if $0 \leq f \leq 1$, and $\text{Supp}(f) \subseteq V$. We say $K < f$ if $0 \leq f \leq 1$, and $f|_K = 1$. We say $K < f < V$ if $K \subset V, K < f, f < V$.

Remark. f is a “bump” function that approximates χ_K when V shrinks towards K .

Lemma 5.9 (Urysohn’s). Let X be a Locally Compact Hausdorff space, $K \subseteq V \subseteq X$ be such that K is compact, and V is open. Then there exists $f \in C_c(V)$, such that $K < f < V$.

Proof. we want to construct a family of open sets $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$, such that \bar{V}_r is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \dots \subset V_0 \subset \bar{V}_0 \subset V,$$

for $r < s$.

By 5.5, we can find $K \subset V_0 \subset \bar{V}_0 \subset V$.

Pick an enumeration of $r \in \mathbb{Q} \cap (0,1]$, i.e. $(r_n)_{n=1}^\infty$. WLOG, we can let $r_1 = 1$.

By 5.5, we can find $K \subset V_1 \subset \bar{V}_1 \subset V_0$.

Suppose we have constructed the V_{r_i} for $1 \leq i \leq n$, such that \bar{V}_r is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for $r < s \in \{r_i\}_{i=1}^n$.

Let $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$.

Now by 5.5, we can find $\bar{V}_t \subset V_{n+1} \subset \bar{V}_{n+1} \subset V_s$.

For any $r < r_{n+1}$, we have $r \leq s$, and thus $V_{n+1} \subset \bar{V}_{n+1} \subset V_s \subset \bar{V}_s \subseteq V_r$ by induction hypothesis, and similarly for any $r > r_{n+1}$.

Inductively, we can prove there is such a family.

Define $f_r := r\chi_{V_r}$, and $g_r := r\chi_{\bar{V}_r^c} + \chi_{\bar{V}_r}$, and $f := \sup_r f_r, g := \inf_r g_r$.

We can show that f, g are upper and lower continuous, respectively.

In addition, f, g are both 0 outside of V_1 , and 1 on K .

Suppose there is some $x \in X, r, s \in \mathbb{Q} \cap [0, 1]$, such that $f_r(x) > g_s(x)$. Then we must have $f_r(x) > 0$, and thus $x \in V_r$ and $1 \leq r = f_r(x)$.

Thus $1 > g_s(x)$, and thus $x \in \bar{V}_s^c$ and $f_s = s$.

Since $r > s$, we must have $V_r \subset \bar{V}_r \subset V_s \subset \bar{V}_s$, which is a contradiction to $x \in V_r, x \notin \bar{V}_s$.

Thus for any $x \in X, r, s \in \mathbb{Q} \cap [0, 1]$, we must have $f_r(x) \leq g_s(x)$.

Thus we must have $f(x) \leq g(x)$ for any $x \in V$.

Now suppose there is some $x \in X$, such that $f(x) < g(x)$.

There must be some rationals, such that $f(x) < r < s < g(x)$, since \mathbb{Q} is dense.

Thus $\sup_r f_r(x) < r$, and thus $x \notin V_r$.

Also, $\inf_s g_s(x) > s$, and thus $x \in \bar{V}_s$.

Yet $r < s$, we must have $V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r$, which is a contradiction.

Thus we must have $f = g$, and it forces f to be continuous. \square

Definition 5.11. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$.

A collection $(h_i)_{i=1}^n \subset C_c(X)$ is called a **partition of unity** on K subordinate to $(V_i)_{i=1}^n$ if

$$\begin{cases} \forall 1 \leq i \leq n, & h_i < V_i, \\ \forall x \in K, & \sum_{i=1}^n h(x) = 1. \end{cases}$$

Theorem 5.10. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$, there always exists a partition of unity on K subordinated to $(V_i)_{i=1}^n$.

Proof. Since K is compact, we can find some open cover W_1, \dots, W_m , such that for all j , we have $W_j \subset \bar{W}_j \subset V_{i(j)}$ for some $1 \leq i(j) \leq n$.

Let $K_i := \bigcup_{1 \leq j \leq m \text{ such that } W_j \subset V_i} \bar{W}_j \subset V_i$, which is compact.

By Urysohn's lemma, we can find $K_i < g_i < V_i$.

Now let $h_1 := g_1$, and in general, $h_i := g_i \prod_{j < i} (1 - g_j)$.

It is easy to check that $0 \leq h_i \leq 1$, and $h_i \in C_c(X)$.

In addition, $\text{Supp}(h_i) \subseteq \text{Supp}(g_i) \subset V_i$.

Thus $h_i < V_i$. Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_i)$.

For any $x \in K$, there must be some $i \in [n]$ such that $x \in K_i$, and thus $g_i(x) = 1$, and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_i(x)) = 1 - 0 = 1.$$

\square

5.4 Linear Functional

Definition 5.12. Let X be a compact Hausdorff space.

A linear functional on $C(X)$ is a linear map $\Lambda : C(X) \rightarrow \mathbb{C}$.

A linear functional Λ is **positive** if $\Lambda(f) \geq 0$ for all $f \in C(X)$ such that $f \geq 0$.

Proposition 5.11. Let X be a compact Hausdorff space, then for a Borel measure μ on X ,

1. If μ is finite, $\Lambda_\mu(f) := \int_X f d\mu$ is a positive linear functional.
2. If μ is finite, Λ_μ is bounded and hence continuous. Indeed, $\forall f \in C(X), |\Lambda_\mu(f)| \leq \mu(X) \|f\|_\infty$.
3. Λ_μ is a finite-value linear functional iff $\mu(X) < \infty$.

Proof. 1. By properties of integral and 5.8.

2. For $f \in C(X)$.

$$\begin{aligned} |\Lambda_\mu(f)| &= \left| \int_X f d\mu \right| \\ &\leq \int_X |f| d\mu \\ &\leq \int_X \|f\|_\infty d\mu \\ &= \mu(X) \|f\|_\infty. \end{aligned}$$

□

5.5 Radon Measure

Definition 5.13. Let X be a topological space, $\mu : \text{Bol}(X) \rightarrow [0, \infty]$ be a Borel measure on X . For $A \in \text{Bol}(X)$, μ is **outer regular** if $\mu(A) = \inf \{\mu(U) : \text{open } U \supseteq A\}$. μ is **inner regular** if $\mu(A) = \sup \{\mu(K) : \text{compact } K \subseteq A\}$. μ is **regular** if it is inner and outer regular for any $A \in \text{Bol}(X)$.

Definition 5.14. Let X be a topological space, $\mu : \text{Bol}(X) \rightarrow [0, \infty]$ be a Borel measure on X . μ is a **Radon measure** if

1. $\forall \text{compact } K \subseteq X, \mu(K) < \infty$,
2. μ is outer regular on Borel sets,
3. μ is inner regular on open sets.

Remark. We have seen that Lebesgue-Stieltjes Measures are regular and Radon.

Proposition 5.12. A finite Borel measure on a compact metric space is always regular (hence Radon).

Proof. Let μ be a finite Borel measure on a compact metric space X . Let $S \subseteq \text{Bol}(X)$ on which μ is regular. If $C \subseteq X$ is closed, it is compact. Thus μ is inner regular for C . Since X is a metric space, $C = \bigcap_{n \geq 1} \{x \in X : d(x, C^c) < \frac{1}{n}\}$ is G_δ . By continuity from above of μ , it follows that μ is also outer-regular. Thus all the closed sets belong to S .

Since Borel sets are generated by closed sets, it suffices to show S is a σ -algebra.

For $A \in S, \epsilon > 0$, there is compact K and open U such that $K \subseteq A \subseteq U, \mu(U \setminus K) < \epsilon$. Then $U^c \subseteq A^c \subseteq K^c$, where U^c is compact, K^c is open. In addition,

$$\mu(K^c \setminus U^c) = \mu(K^c \cap U) = \mu(U \setminus K) < \epsilon.$$

Thus $A^c \in S$.

Consider $(A_i)_{i=1}^\infty \subseteq S, \epsilon > 0$. Choose compact $K_i \subseteq A_i$ and open $U_i \supseteq A_i$, such that $\mu(U_i \setminus K_i) < \epsilon/2^i$. Let

$A = \bigcup_{i=1}^{\infty} A_i, C_n = \bigcup_{i=1}^n K_i, C = \bigcup_{i=1}^{\infty} K_i, U = \bigcup_{i=1}^{\infty} U_i.$

Thus C_n are closed, U is open, and $C_n \subseteq A \subseteq U$. By continuity and finiteness of μ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(U \setminus C_n) &= \mu(U \setminus C) \\ &\leq \sum_{i=1}^{\infty} \mu(U_i \setminus K_i) \\ &< \epsilon. \end{aligned}$$

Thus μ is regular on A , and thus $A \in S$, and thus S is closed under countable unions. Thus $S = \text{Bol}(X)$. □

5.6 Extremely Disconnected Spaces

Definition 5.15. A compact space X is **extremely disconnected** if the closure of every open set is open.

Proposition 5.13. *If X is extremely disconnected, then there is a basis of clopen sets.*

Proposition 5.14. *If $A \subseteq X$ is clopen, then $\chi_A \in C(X)$.*

Proposition 5.15. (Stone–Čech compactification) *Let D be a discrete space (thus every subset is open). The **Stone–Čech compactification** is the unique compact (Hausdorff) space βD with the following universal properties:*

1. $D \subseteq \beta D$ as topology inclusion.
2. For any compact K , and every continuous map $f : D \rightarrow K$, there is a unique continuous extension $\beta f : \beta D \rightarrow K$.

Proposition 5.16. $\ell^\infty(D) \simeq C(\beta D)$

Proposition 5.17. βD is the set of ultrafilters on D .

Proposition 5.18. βD is extremely disconnected.

5.7 Riesz-Markov-Kakutani

Theorem 5.19. (Riesz-Markov-Kakutani) *Let X be a Locally Compact Hausdorff space, $\Lambda : C(X) \rightarrow \mathbb{C}$ a positive linear functional. Then there is a unique Radon measure μ on X , such that $\Lambda = \Lambda_\mu = \int_X f d\mu$. In addition,*

1. $\forall U \subseteq X$ be open, we have $\mu(U) = \sup \{ \Lambda(f) : f < U \}.$
2. $\forall K \subseteq X$ be compact, we have $\mu(K) = \inf \{ \Lambda(f) : K < f \}.$

Proof. Define $\mu^* : \mathcal{T} \rightarrow [0, \infty]$ by: for any open $U \subseteq X, \mu^*(U) := \sup \{ \Lambda(f) : f < U \}.$

Clearly for any $V \supseteq U$, we have $\mu^*(U) \leq \mu^*(V)$.

Thus we have $\mu^*(U) := \inf \{ \mu^*(V) : \text{open } V \supseteq U \}.$

Now we extend $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ by $\mu^*(E) := \inf \{ \mu^*(U) : \text{open } U \supseteq E \}.$

One can check this is an outer measure, and the measure induced by Caratheodory is a Radon measure.

Now let us check the uniqueness.

Suppose there is some Radon measure μ , such that $\Lambda = \Lambda_\mu = \int_X f d\mu$.

Given any open U .

Consider any compact K such that $K \subset U$. By Urysohn's lemma, we can find a function f , such that $K < f < U$.

Then $\chi_K \leq f \leq \chi_U \implies \mu(K) \leq \Lambda(f) \leq \mu(U)$.

Thus by inner regularity, $\mu(U) = \sup \{ \mu(K) : \text{compact } K \subseteq U \} = \sup \{ \Lambda(f) : f < U \}$, which is uniquely determined.

For any Borel set E , we have that by outer regularity, $\mu(E) = \inf \{ \mu(U) : \text{open } U \supseteq E \}$, which is uniquely determined. □

6 Lebesgue Spaces

6.1 The First Lebesgue Space

Definition 6.1. Given some measure space (X, \mathcal{M}, μ) , define

$$\mathcal{L}^1(\mu) := \mathcal{L}^1(X, \mathcal{M}, \mu) := \left\{ f : X \rightarrow \mathbb{C} \mid f \text{ is measurable, } \int_X |f| d\mu < \infty \right\}.$$

Proposition 6.1. $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space.

Proof. Clearly $\int_X |0| d\mu = 0$, so the zero function $0 \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, we have

$$\begin{aligned} \int_X |c \cdot f| d\mu &= \int_X |c| |f| d\mu \\ &= |c| \int_X |f| d\mu \\ &< \infty. \end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.

Now for any $f, g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, we have

$$\begin{aligned} \int_X |f + g| d\mu &\leq \int_X |f| + |g| d\mu \\ &= \int_X |f| d\mu + \int_X |g| d\mu \\ &< \infty. \end{aligned}$$

Thus $f + g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.

Since the set of all functions $\{f : X \rightarrow \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a subspace of it. \square

Definition 6.2. Let

$$N = \left\{ f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : \int_X |f| d\mu = 0 \right\} = \left\{ f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f = 0 \text{ } \mu - \text{a.e.} \right\}.$$

Define

$$L^1(X, \mathcal{M}, \mu) := \mathcal{L}^1(X, \mathcal{M}, \mu) / N,$$

which is the quotient vector space of $\mathcal{L}^1(X, \mathcal{M}, \mu)$ mod N .

Remark. $[f] = \{g \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f - g = 0 \text{ } \mu - \text{a.e.}\} \in L^1(X, \mathcal{M}, \mu)$

Definition 6.3. $\|[f]\|_{L^1(X, \mathcal{M}, \mu)} := \int_X |f| d\mu$ for any choice of representative $f \in [f]$.

When the context is clear, we might write $L^1(X, \mathcal{M}, \mu)$ as $L^1(\mu)$ or $L^1(X)$. We might also write $\|\cdot\|_{L^1(X, \mathcal{M}, \mu)}$ as $\|\cdot\|_{L^1(\mu)}$, $\|\cdot\|_{L^1(\mu)}$, $\|\cdot\|_1$.

Lemma 6.2. The above definition is well defined.

Proof. Take any $g, f \in [f]$. Let $K = \{x \in X : f(x) \neq g(x)\}$, we have $\mu(K) = 0$.

$$\begin{aligned}
\int_X |f| d\mu &= \int_{X \setminus K} |f| d\mu + \int_K |f| d\mu \\
&= \int_{X \setminus K} |f| d\mu \\
&= \int_{X \setminus K} |g| d\mu \\
&= \int_{X \setminus K} |g| d\mu + \int_K |g| d\mu \\
&= \int_X |g| d\mu
\end{aligned}$$

□

Proposition 6.3. $\|\cdot\|_1$ is a norm on $L^1(X, \mathcal{M}, \mu)$.

Proof. Consider any $[f], [g] \in L^1(X, \mathcal{M}, \mu)$.

$$\begin{aligned}
\|[f] + [g]\|_1 &= \|[f + g]\|_1 \\
&= \int_X |f + g| d\mu \\
&\leq \int_X |f| d\mu + \int_X |g| d\mu \\
&= \|[f]\|_1 + \|[g]\|_1
\end{aligned}$$

For any $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\|\alpha[f]\|_1 &= \|[\alpha f]\|_1 \\
&= \int_X |\alpha f| d\mu \\
&= |\alpha| \int_X |f| d\mu \\
&= |\alpha| \|[f]\|_1
\end{aligned}$$

If $\|[f]\|_1 = 0$, we must have $f = 0$ μ -a.e.. Thus $f \in N$, thus $[f] = [0] = 0$.

□

Theorem 6.4. (Fischer-Riesz)

Let (X, \mathcal{M}, μ) be a measure space, $(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)})$ is a Banach Space.

Proof. Let $([f_n])_1^\infty$ be a Cauchy sequence in $L^1(X, \mathcal{M}, \mu)$. Then for each $k \in \mathbb{N}^+$, there is some $N_k \geq 1$, such that $\forall m, n \geq N_k, \|[f_m] - [f_n]\|_{L^1(\mu)} < \frac{1}{2^k}$.

WLOG, $\forall k, N_{k+1} \geq N_k$.

Thus $\|[f_{N_{k+1}}] - [f_{N_k}]\|_{L^1(\mu)} < \frac{1}{2^k}$.

Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}|$, where we fix f_n to be a representative of $[f_n]$.

Notice that $\forall k \geq 1$,

$$\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \\
&= g_k
\end{aligned}$$

We have that $\int_X g_k d\mu = \int_X |f_{N_1}| d\mu + \sum_{j=1}^n \int_X |f_{N_{j+1}} - f_{N_j}| d\mu$.

Let $g = \lim_{k \rightarrow \infty} g_k = |f_{N_1}| + \sum_{j=1}^{\infty} |f_{N_{j+1}} - f_{N_j}|$.

By LMCT, we have that

$$\begin{aligned}
\int_X g d\mu &= \lim_{k \rightarrow \infty} \int_X g_k d\mu \\
&= \int_X |f_{N_1}| d\mu + \sum_{j=1}^{\infty} \int_X |f_{N_{j+1}} - f_{N_j}| d\mu \\
&= \| [f_{N_1}] \|_{L^1(\mu)} + \sum_{j=1}^{\infty} \| [f_{N_{j+1}} - f_{N_j}] \|_{L^1(\mu)} \\
&= \| [f_{N_1}] \|_{L^1(\mu)} + \sum_{j=1}^{\infty} \| [f_{N_{j+1}}] - [f_{N_j}] \|_{L^1(\mu)} \\
&< \| [f_{N_1}] \|_{L^1(\mu)} + \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&< \infty.
\end{aligned}$$

Thus $g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Thus $N := \{x \in X : g(x) = \infty\}$ has measure 0.

This implies that $f_{N_1}(x) + \sum_{k=1}^{\infty} (f_{N_{k+1}}(x) - f_{N_k}(x))$ converges absolutely for $x \in X \setminus N$.

We can thus define

$$\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty} (f_{N_{j+1}}(x) - f_{N_j}(x)) \\
&= \lim_{k \rightarrow \infty} \left(f_{N_1}(x) + \sum_{j=1}^k (f_{N_{j+1}}(x) - f_{N_j}(x)) \right) \\
&= \lim_{k \rightarrow \infty} f_{N_{k+1}}(x) \\
&= \lim_{k \rightarrow \infty} f_{N_k}(x)
\end{aligned}$$

for $x \in X \setminus N$.

We then extend f to X by $f|_N := 0$.

Then $|f| \leq g$, and thus $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.

Notice that $|f_{N_k}| \leq g_k \leq g$, thus $|f - f_{N_k}| \leq g + g = 2g$.

By LDCT,

$$\begin{aligned}
\lim_{k \rightarrow \infty} \|[f_{N_k}] - [f]\|_{L^1(\mu)} &= \lim_{k \rightarrow \infty} \|[f_{N_k} - f]\|_{L^1(\mu)} \\
&= \lim_{k \rightarrow \infty} \int_X |f_{N_k} - f| d\mu \\
&= \int_X \lim_{k \rightarrow \infty} |f_{N_k} - f| d\mu \\
&= 0.
\end{aligned}$$

Thus $\lim_{k \rightarrow \infty} [f_{N_k}]$ converges to $[f]$. Since this is a subsequence of the Cauchy sequence $([f_n])_1^\infty$, we have that $\lim_{n \rightarrow \infty} [f_n] = [f]$.

This shows that $(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)})$ is complete. \square

Remark. When we write $f \in L^1(\mu)$, we will mean $[f] \in L^1(\mu)$, and let $f \in \mathcal{L}^1(\mu)$ be any representative of $[f]$ when the context is clear.

6.2 Convex functions

Definition 6.4. A function $\phi : U \rightarrow \mathbb{R}$ is **convex** if

$$\forall x, y \in U, \forall \lambda \in [0, 1], \phi(\lambda x + (1 - \lambda)y) \leq \lambda \phi(x) + (1 - \lambda)\phi(y).$$

Theorem 6.5 (Jensen's Inequality). *If ϕ is convex, we have $\forall x_1, \dots, x_n \in U$, and $\forall 0 \leq \lambda_1, \dots, \lambda_n \leq 1$ such that $\sum_{i=1}^n \lambda_i = 1$,*

$$\phi\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i \phi(x_i).$$

Proof. The base case is when $n = 1$, which is trivial.

Now suppose this holds for $n - 1 \in \mathbb{N}$.

Given any $\forall x_1, \dots, x_n \in U$, and $0 \leq \lambda_1, \dots, \lambda_n \leq 1$ such that $\sum_{i=1}^n \lambda_i = 1$. If $\lambda_n = 0$, we can reduce the sum to a $n - 1$ sum. If $\lambda_n = 1$, then the other λ_i must be all 0, and we can reduce the sum to only x_n .

Now suppose $0 < \lambda_1 < 1$. Notice that $\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} = \frac{\sum_{i=1}^{n-1} \lambda_i}{1 - \lambda_n} = \frac{1 - \lambda_n}{1 - \lambda_n} = 1$.

We have that

$$\begin{aligned}
\phi\left(\sum_{i=1}^n \lambda_i x_i\right) &= \phi\left(\lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i\right) \\
&\leq \lambda_n \phi(x_n) + (1 - \lambda_n) \phi\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i\right) \\
&\leq \lambda_n \phi(x_n) + (1 - \lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} \phi(x_i) \\
&= \lambda_n \phi(x_n) + \sum_{i=1}^{n-1} \lambda_i \phi(x_i) \\
&= \sum_{i=1}^n \lambda_i \phi(x_i).
\end{aligned}$$

By induction, this is true for any $n \geq 1$. \square

Theorem 6.6 (Arithmetic Mean Inequality). *Let $x_1, \dots, x_n \geq 0$, with $0 \leq \lambda_1, \dots, \lambda_n \leq 1$ such that $\sum_{i=1}^n \lambda_i = 1$. We have that*

$$\prod_{i=1}^n x_i^{\lambda_i} \leq \sum_{i=1}^n \lambda_i x_i.$$

Proof. If any of $x_i = 0$, then the inequality is trivially true.
 Now suppose $\forall i, x_i > 0$.

Notice that \exp is convex, and we have

$$\begin{aligned} \prod_{i=1}^n x_i^{\lambda_i} &= \exp \left(\sum_{i=1}^n \lambda_i \ln(x_i) \right) \\ &= \exp \left(\sum_{i=1}^n \lambda_i \ln(x_i) \right) \\ &\leq \sum_{i=1}^n \lambda_i \exp(\ln(x_i)) \\ &= \sum_{i=1}^n \lambda_i x_i. \end{aligned}$$

□

Proposition 6.7. Let $x_1, \dots, x_n \geq 0$, and $n \in \mathbb{N}^+, p \geq 1$, we have that

$$\sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p \leq n^{p-1} \sum_{i=1}^n x_i^p.$$

Proof. For $p \geq 1$, we have $(\cdot)^p$ is convex.

$$\begin{aligned} \left(\sum_{i=1}^n \frac{1}{n} x_i \right)^p &\leq \sum_{i=1}^n \frac{1}{n} x_i^p \\ \frac{1}{n^p} \left(\sum_{i=1}^n x_i \right)^p &\leq \frac{1}{n} \sum_{i=1}^n x_i^p \\ \left(\sum_{i=1}^n x_i \right)^p &\leq n^{p-1} \sum_{i=1}^n x_i^p. \end{aligned}$$

This proves the second inequality.

Now when $n = 1$, we have the first inequality trivially.

Suppose the first inequality holds for $n \in \mathbb{N}^+$, we have

$$\begin{aligned} \left(\sum_{i=1}^{n+1} x_i \right)^p &= \left(\sum_{i=1}^n x_i + x_{n+1} \right)^p \\ &\geq \left(\sum_{i=1}^n x_i \right)^p + x_{n+1}^p \\ &\geq \left(\sum_{i=1}^n x_i^p \right) + x_{n+1}^p \\ &= \sum_{i=1}^{n+1} x_i^p. \end{aligned}$$

By induction, the first inequality is true for all $n \in \mathbb{N}^+$.

□

6.3 L^p Spaces

Definition 6.5. Let (X, \mathcal{M}, μ) be a measure space, and $1 \leq p < \infty$ we define

$$\mathcal{L}^p(\mu) := \left\{ f : X \rightarrow \mathbb{C} \mid f^p \in L^1(\mu) \right\} = \left\{ f : X \rightarrow \mathbb{C} \mid f \text{ is measurable, } \int_X |f|^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p} := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

Proposition 6.8. $\mathcal{L}^p(\mu)$ is a vector space.

Proof. Clearly $\int_X |0|^p d\mu = 0$, so the zero function $0 \in \mathcal{L}^p(\mu)$.

Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^p(\mu)$, we have

$$\begin{aligned} \int_X |c \cdot f|^p d\mu &= \int_X |c|^p |f|^p d\mu \\ &= |c|^p \int_X |f|^p d\mu \\ &< \infty. \end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^p(\mu)$.

Now for any $f, g \in \mathcal{L}^p(\mu)$, we have

$$\begin{aligned} \int_X |f + g|^p d\mu &\leq \int_X (|f| + |g|)^p d\mu \\ &\leq \int_X 2^{p-1} (|f|^p + |g|^p) d\mu \\ &= 2^{p-1} \left(\int_X |f|^p d\mu + \int_X |g|^p d\mu \right) \\ &< \infty. \end{aligned}$$

Thus $f + g \in \mathcal{L}^p(\mu)$.

Since the set of all functions $\{f : X \rightarrow \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^p(\mu)$ is a subspace of it. \square

Definition 6.6. Let (X, \mathcal{M}, μ) be a measure space, the **essential supremum** of a function $f : X \rightarrow \mathbb{R}$ is

$$\text{ess sup } f := \inf \{M \in \mathbb{R} : \mu(\{x : f(x) > M\}) = 0\}.$$

Proposition 6.9. For any $\lambda \geq 0$, $f : X \rightarrow \mathbb{R}$, we have

$$\lambda(\text{ess sup } f) = \text{ess sup } (\lambda f).$$

Proof. It is easy to see this is true for $\lambda = 0$.

Now suppose $\lambda > 0$.

$$\begin{aligned} \text{ess sup } (\lambda f) &= \inf \{M \in \mathbb{R} : \mu(\{x : \lambda f(x) > M\}) = 0\} \\ &= \inf \left\{ M \in \mathbb{R} : \mu \left(\left\{ x : f(x) > \frac{M}{\lambda} \right\} \right) = 0 \right\} \\ &= \inf \{ \lambda \cdot N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0 \} \\ &= \lambda \inf \{ N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0 \} \\ &= \lambda(\text{ess sup } f). \end{aligned}$$

\square

Definition 6.7. Let (X, \mathcal{M}, μ) be a measure space, we define

$$\mathcal{L}^\infty(\mu) := \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable, } \text{ess sup } |f| < \infty\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty} := \text{ess sup } |f|.$$

Proposition 6.10. $\mathcal{L}^\infty(\mu)$ is a vector space.

Proof. Clearly $\text{ess sup } 0 = 0$, so the zero function $0 \in \mathcal{L}^\infty(\mu)$.

Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^\infty(\mu)$, we have

$$\begin{aligned} \|c \cdot f\|_{\mathcal{L}^\infty} &= \text{ess sup } |c \cdot f| \\ &= \text{ess sup } (|c| \cdot |f|) \\ &= |c| \text{ess sup } |f| \\ &= |c| \|f\|_{\mathcal{L}^\infty} \\ &< \infty. \end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^\infty(\mu)$.

Now for any $f, g \in \mathcal{L}^\infty(\mu)$.

Consider any $L, N \in \mathbb{R}$, such that $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$.

Thus $\mu(\{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}) = 0$.

Now for any $x \in X$, if $|f(x) + g(x)| > L + N$, we must have $|f(x)| + |g(x)| \geq |f(x) + g(x)| > L + N$.

Thus $|f(x)| > L$ or $|g(x)| > N$.

Since this holds for any $x \in X$, we have $\{x : |f(x) + g(x)| > L + N\} \subseteq \{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}$.

Thus $\mu(\{x : |f(x) + g(x)| > L + N\}) = 0$.

By definition, we have

$$\begin{aligned} \|f + g\|_{\mathcal{L}^\infty} &= \text{ess sup } |f + g| \\ &= \inf \{M \in \mathbb{R} : \mu(\{x : |f(x) + g(x)| > M\}) = 0\} \\ &\leq L + N. \end{aligned}$$

Since this hold for any such $N, L \in \mathbb{R}$, such that $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$, we have

$$\begin{aligned} \|f + g\|_{\mathcal{L}^\infty} &= \inf \{N + L : \mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0\} \\ &= \inf \{N : \mu(\{x : |f(x)| > N\}) = 0\} + \inf \{L : \mu(\{x : |g(x)| > L\}) = 0\} \\ &= \|f\|_{\mathcal{L}^\infty} + \|g\|_{\mathcal{L}^\infty} \\ &< \infty. \end{aligned}$$

Thus $f + g \in \mathcal{L}^\infty(\mu)$.

Since the set of all functions $\{f : X \rightarrow \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^\infty(\mu)$ is a subspace of it. \square

Proposition 6.11. For any $1 \leq p \leq \infty$, we have $\|f - g\|_{\mathcal{L}^p} = 0 \iff f = g$ almost everywhere.

Proof. For $1 \leq p < \infty$,

$$\begin{aligned} \|f - g\|_{\mathcal{L}^p} &= 0 \\ \iff \int_X |f - g|^p d\mu &= 0 \\ \iff |f - g|^p &= 0 \text{ a.e.} \\ \iff f - g &= 0 \text{ a.e.} \\ \iff f &= g \text{ a.e..} \end{aligned}$$

For $p = \infty$,

$$\begin{aligned} \|f - g\|_{\mathcal{L}^\infty} &= 0 \\ \iff \text{ess sup } |f - g| &= 0 \\ \iff f - g &= 0 \text{ a.e.} \\ \iff f &= g \text{ a.e..} \end{aligned}$$

\square

Definition 6.8. For any $1 \leq p \leq \infty$, if we identify $f, g \in \mathcal{L}^p(\mu)$ by $f \sim g \iff f = g$ almost everywhere, we get the quotient vector space

$$L^p(\mu) := \mathcal{L}^p(\mu)/\sim = \{[f] : f \in \mathcal{L}^p(\mu)\}$$

to be the collection of all equivalence classes of functions in \mathcal{L}^p .

Definition 6.9. Let (X, \mathcal{M}, μ) be a measure space, and $1 \leq p \leq \infty$ we define the norm

$$\|[f]\|_{L^p(\mu)} := \|f\|_{\mathcal{L}^p}$$

for any representative $f \in [f]$.

Lemma 6.12. *The above definition is well-defined.*

Remark. As before, when we write $f \in L^p(\mu)$, we will mean $[f] \in L^p(\mu)$, and let $f \in \mathcal{L}^p(\mu)$ be any representative of $[f]$ when the context is clear.

Theorem 6.13 (Holder's Inequality). *Let (X, \mathcal{M}, μ) be a measure space, $1 \leq p \leq \infty$. Suppose $\frac{1}{p} + \frac{1}{q} = 1$, then $\forall f \in L^p(\mu), g \in L^q(\mu), fg \in L^1(\mu)$ and*

$$\|fg\|_{L^1(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Proof. If $p = 1$, then $q = \infty$.

Now $|fg| = |f||g| \leq |f||g|_{L^\infty(\mu)}$.

Thus

$$\begin{aligned} \|fg\|_{L^1(\mu)} &= \int_X |fg| d\mu \\ &\leq \int_X |f| \|g\|_{L^\infty(\mu)} d\mu \\ &= \|g\|_{L^\infty(\mu)} \int_X |f| d\mu \\ &= \|g\|_{L^\infty(\mu)} \|f\|_{L^1(\mu)}. \end{aligned}$$

Now suppose $1 < p < \infty$. We have $1 < q < \infty$.

If $\|f\|_{L^p(\mu)} = 0$ or $\|g\|_{L^q(\mu)} = 0$, then it is trivial, since this implies $f = 0$ a.e. or $g = 0$ a.e., which means $fg = 0$ a.e..

Now let $F := \frac{|f|}{\|f\|_{L^p(\mu)}}, G := \frac{|g|}{\|g\|_{L^q(\mu)}}$.

By Arithmetic Mean Inequality, we have that

$$\begin{aligned} F(x)G(x) &= (F(x)^p)^{1/p} (G(x)^q)^{1/q} \\ &\leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q \\ \int_X FG d\mu &\leq \frac{1}{p} \int_X F^p d\mu + \frac{1}{q} \int_X G^q d\mu \\ \frac{\|fg\|_{L^1(\mu)}}{\|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}} &\leq \frac{1}{p} \int_X \frac{|f|^p}{\|f\|_{L^p(\mu)}^p} d\mu + \frac{1}{q} \int_X \frac{|g|^q}{\|g\|_{L^q(\mu)}^q} d\mu \\ &= \frac{1}{p} \frac{\|f\|_{L^p(\mu)}^p}{\|f\|_{L^p(\mu)}^p} + \frac{1}{q} \frac{\|g\|_{L^q(\mu)}^q}{\|g\|_{L^q(\mu)}^q} \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

Thus $\|fg\|_{L^1(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}$. □

Theorem 6.14 (Minkowski's Inequality). *Let (X, \mathcal{M}, μ) be a measure space, $1 \leq p \leq \infty$. For any $f, g \in L^p(\mu)$, we have*

$$\|f + g\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}.$$

Proof. We have proven for $p = 1$ and $p = \infty$.

Now suppose $p \in (1, \infty)$. Then $q = \frac{p}{p-1} \in (1, \infty)$.

Since $f, g \in L^p(\mu)$, we have $f + g \in L^p(\mu)$, so

$$\begin{aligned} \left\| |f + g|^{p-1} \right\|_{L^q(\mu)}^q &= \int_X (|f + g|^{p-1})^q d\mu \\ &= \int_X (|f + g|^{p-1})^{\frac{p}{p-1}} d\mu \\ &= \int_X |f + g|^p d\mu \\ &= \|f + g\|_{L^p(\mu)}^p \\ &< \infty. \end{aligned}$$

Thus $|f + g|^{p-1} \in L^q(\mu)$. By Holder's Inequality, we have

$$\begin{aligned} \|f + g\|_{L^p(\mu)}^p &= \int_X |f + g|^p d\mu \\ &= \int_X |f + g| |f + g|^{p-1} d\mu \\ &\leq \int_X (|f| + |g|) |f + g|^{p-1} d\mu \\ &\leq \int_X |f| \cdot |f + g|^{p-1} d\mu + \int_X |g| \cdot |f + g|^{p-1} d\mu \\ &\leq \|f\|_{L^p(\mu)} \left\| |f + g|^{p-1} \right\|_{L^q(\mu)} + \|g\|_{L^p(\mu)} \left\| |f + g|^{p-1} \right\|_{L^q(\mu)} \\ &= \left(\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \right) \left\| |f + g|^{p-1} \right\|_{L^q(\mu)} \\ &= \left(\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \right) \|f + g\|_{L^p(\mu)}^{p/q} \\ \|f + g\|_{L^p(\mu)}^{p-p/q} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\ \|f + g\|_{L^p(\mu)}^{p(1-1/q)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\ \|f + g\|_{L^p(\mu)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}. \end{aligned}$$

□

Corollary 6.15. *Let (X, \mathcal{M}, μ) be a measure space, $1 \leq p \leq \infty$. We have $\|\cdot\|_{L^p(\mu)}$ is a norm over $L^p(\mu)$.*

Proof. The triangle Inequality is done by Minkowski's Inequality.

Consider any $f \in L^p(\mu)$.

For any $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\|\alpha f\|_{L^p(\mu)}^p &= \int_X |\alpha f|^p d\mu \\
&= \int_X |\alpha|^p |f|^p d\mu \\
&= |\alpha|^p \int_X |f|^p d\mu \\
&= |\alpha|^p \|f\|_{L^p(\mu)}^p \\
&\implies \\
\|\alpha f\|_{L^p(\mu)} &= |\alpha| \|f\|_{L^p(\mu)}.
\end{aligned}$$

In addition, $\|f\|_{L^p(\mu)} = 0$, if and only if $f = 0$ μ -a.e.. \square

Theorem 6.16 (Fischer-Riesz). *Let (X, \mathcal{M}, μ) be a measure space, $1 \leq p \leq \infty$. $(L^p(\mu), \|\cdot\|_{L^p(\mu)})$ is a Banach Space.*

Proof. 1. We first consider $1 \leq p < \infty$.

Let $(f_n)_1^\infty$ be a Cauchy sequence in $L^p(X, \mathcal{M}, \mu)$. Then for each $k \in \mathbb{N}^+$, there is some $N_k \geq 1$, such that $\forall m, n \geq N_k, \|f_m - f_n\|_{L^p(\mu)} < \frac{1}{2^k}$.

WLOG, $\forall k, N_{k+1} \geq N_k$.

Thus $\|f_{N_{k+1}} - f_{N_k}\|_{L^p(\mu)} < \frac{1}{2^k}$.

Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}|$, where we fix f_n to be a representative of $[f_n]$.

Notice that $\forall k \geq 1$,

$$\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \\
&= g_k \\
\|g_k\|_{L^p(\mu)} &= \left\| |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \right\|_{L^p(\mu)} \\
&\leq \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \|f_{N_{j+1}} - f_{N_j}\|_{L^p(\mu)}.
\end{aligned}$$

Let $g = \lim_{k \rightarrow \infty} g_k = |f_{N_1}| + \sum_{j=1}^{\infty} |f_{N_{j+1}} - f_{N_j}|$.

Notice that g_k are monotone increasing. By LMCT, we have that

$$\begin{aligned}
\|g\|_{L^p(\mu)} &= \int_X |g|^p d\mu \\
&= \int_X g^p d\mu \\
&= \int_X \lim_{k \rightarrow \infty} g_k^p d\mu \\
&= \lim_{k \rightarrow \infty} \int_X g_k^p d\mu \\
&= \lim_{k \rightarrow \infty} \|g_k\|_{L^p(\mu)} \\
&\leq \lim_{k \rightarrow \infty} \left(\|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \|f_{N_{j+1}} - f_{N_j}\|_{L^p(\mu)} \right) \\
&= \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \|f_{N_{j+1}} - f_{N_j}\|_{L^p(\mu)} \\
&< \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&< \infty.
\end{aligned}$$

Thus $g \in \mathcal{L}^p(X, \mathcal{M}, \mu)$, which means $g^p \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ and $N := \{x \in X : g(x) = \infty\}$ has measure 0. This implies that $f_{N_1}(x) + \sum_{k=1}^{\infty} (f_{N_{k+1}}(x) - f_{N_k}(x))$ converges absolutely for $x \in X \setminus N$. We can thus define

$$\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty} (f_{N_{j+1}}(x) - f_{N_j}(x)) \\
&= \lim_{k \rightarrow \infty} \left(f_{N_1}(x) + \sum_{j=1}^k (f_{N_{j+1}}(x) - f_{N_j}(x)) \right) \\
&= \lim_{k \rightarrow \infty} f_{N_{k+1}}(x) \\
&= \lim_{k \rightarrow \infty} f_{N_k}(x)
\end{aligned}$$

for $x \in X \setminus N$.

We then extend f to X by $f|_N := 0$.

Then $|f| \leq g \implies |f|^p \leq g^p$, and thus $f \in \mathcal{L}^p(X, \mathcal{M}, \mu)$.

Notice that $|f_{N_k}| \leq g_k \leq g$, thus $|f - f_{N_k}|^p \leq (g + g)^p = 2^p g^p$.

By LDCT,

$$\begin{aligned}
\lim_{k \rightarrow \infty} \|f_{N_k} - f\|_{L^p(\mu)}^p &= \lim_{k \rightarrow \infty} \int_X |f_{N_k} - f|^p d\mu \\
&= \int_X \lim_{k \rightarrow \infty} |f_{N_k} - f|^p d\mu \\
&= 0.
\end{aligned}$$

Thus $\lim_{k \rightarrow \infty} f_{N_k}$ converges to f .

Since this is a subsequence of the Cauchy sequence $(f_n)_1^\infty$, we have that $\lim_{n \rightarrow \infty} f_n = f$.

This shows that $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(\mu)})$ is complete.

2. Now consider $p = \infty$.

Let $(f_n)_1^\infty$ be a Cauchy sequence in $L^\infty(X, \mathcal{M}, \mu)$. As before, we can take some subsequence $(f_{N_k})_{k=1}^\infty$

with $\|f_{N_{k+1}} - f_{N_k}\|_{L^p(\mu)} < \frac{1}{2^k}$.

Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}|$, where we fix f_n to be a representative of $[f_n]$. Notice that $\forall k \geq 1$,

$$\begin{aligned} f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\ |f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\ &\leq |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \\ &= g_k. \end{aligned}$$

□

Theorem 6.17. (*Density of simple functions*)

Let (X, \mathcal{M}, μ) be a measure space, $1 \leq p \leq \infty$. The simple functions

$$S := \{\phi \in L^p(\mu) \mid \phi \text{ is simple, measurable}\}$$

are dense in $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(X, \mathcal{M}, \mu)})$.

Proof. 1. First consider $1 \leq p < \infty$.

Let $f \in L^p(X, \mathcal{M}, \mu)$, $\exists(\phi_n)_{n=1}^\infty$ be simple and measurable functions, such that

$$f(x) = \lim_{n \rightarrow \infty} \phi_n(x), \text{ a.e. } x \in X,$$

and

$$|\phi_1| \leq |\phi_2| \leq \dots \leq |f|.$$

Thus

$$f^p(x) = \lim_{n \rightarrow \infty} \phi_n^p(x), \text{ a.e. } x \in X,$$

and

$$|\phi_1|^p \leq |\phi_2|^p \leq \dots \leq |f|^p.$$

Since $|f - \phi_n|^p \leq (2|f|)^p = 2^p |f|^p \in L^1(X, \mathcal{M}, \mu)$, by LDCT 11.4, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|f - \phi_n\|_{L^p(X, \mathcal{M}, \mu)}^p &= \lim_{n \rightarrow \infty} \int_X |f - \phi_n|^p d\mu \\ &= \int_X \lim_{n \rightarrow \infty} |f - \phi_n|^p d\mu \\ &= \int_X \lim_{n \rightarrow \infty} 0 d\mu \\ &= 0. \end{aligned}$$

2. Now consider $p = \infty$.

Let $f \in L^p(X, \mathcal{M}, \mu)$, we know $\mu(N) = 0$ for $N := \{x \in X : |f(x)| > \|f\|_{L^\infty(X, \mathcal{M}, \mu)}\}$.

Let $f' := f\chi_{N^c}$. We notice that f' is measurable and bounded, with $|f'| \leq \|f\|_{L^\infty(X, \mathcal{M}, \mu)}, \forall x \in X$.

Thus we can find $(\phi_n)_{n=1}^\infty$ be simple and measurable functions, such that

$$f(x) = \lim_{n \rightarrow \infty} \phi_n(x), \text{ a.e. } x \in X \text{ uniformly, and } |\phi_1| \leq |\phi_2| \leq \dots \leq |f|.$$

Now

$$\begin{aligned}
\|f - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} &= \|f\chi_N + f' - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} \\
&\leq \|f\chi_N\|_{L^\infty(X, \mathcal{M}, \mu)} + \|f' - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} \\
&= \|f' - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} \\
&= \operatorname{ess\,sup}_{x \in X} |f'(x) - \phi(x)| \\
&\rightarrow 0.
\end{aligned}$$

□

Remark. For $1 \leq p < \infty$,

$$S = \operatorname{Span} \{\chi_E \mid \mu(E) < \infty\} = \{\phi : X \rightarrow \mathbb{C} \mid \phi \text{ is simple, measurable, } \mu(\{x \in X \mid \phi(x) \neq 0\}) < \infty\}$$

Theorem 6.18. (*Density of compactly supported continuous functions*)

Given some measure space (X, \mathcal{M}, μ) , where μ is a Radon measure, then $C_c(X)$ is dense for $p < \infty$.

Proof. Given any $\epsilon > 0$.

Consider any measurable E , such that $\mu(E) < \infty$.

By regularity, we can find some compact $K \subset E \subset V$ open, such that $\mu(V \setminus E) < \frac{\epsilon^p}{2^p}$.

Now we take the bump function $K \subset f \subset V$ by Urysohn's Lemma 5.9, where $f \in C_c(V) \subseteq C_c(X)$, and $f|_K = 1, f|_{V^c} = 0, 0 \leq f \leq 1$.

Now

$$\begin{aligned}
\|\chi_E - f\|_{L^p(\mu)}^p &= \int_X |\chi_E - f|^p d\mu \\
&= \int_{V \setminus K} |\chi_E - f|^p d\mu \\
&\leq \int_{V \setminus K} 2^p d\mu \\
&= 2^p \mu(V \setminus K) \\
&< \epsilon^p.
\end{aligned}$$

Thus $\chi_E \in \overline{C_c(X)}$.

Since $S = \operatorname{Span} \{\chi_E \mid \mu(E) < \infty\}$ is dense in $L^p(\mu)$, so is $C_c(X)$. □

Remark. This is not true for $p = \infty$. For instance, consider $X = \mathbb{R}$ with Lebesgue measure, or $X = \mathbb{N}$ with counting measure.

Proposition 6.19. ($L^q(\mu) \subseteq L^p(\mu)^*$)

Let $p \in [1, \infty]$, with $\frac{1}{p} + \frac{1}{q} = 1$. Let $g \in L^q(\mu)$, then $\Lambda_g \in L^p(\mu)^*$, where $\Lambda_g(f) = \int_X fg d\mu$. Moreover, $\forall p \in (1, \infty], \|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$. This also holds for $p = 1$ if μ is semi-finite.

Proof. clearly Λ_g is linear.

By Holder's Inequality, we have

$$\begin{aligned}
|\Lambda_g(f)| &= \left| \int_X fg d\mu \right| \\
&\leq \int_X |fg| d\mu \\
&\leq \|g\|_{L^q(\mu)} \|f\|_{L^p(\mu)}.
\end{aligned}$$

Thus $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \leq \|g\|_{L^q(\mu)} < \infty$.

Thus Λ_g is bounded and $\Lambda_g \in L^p(\mu)^*$.

We now want to show $\|\Lambda_g\|_{L^p(\mu)^*} \geq \|g\|_{L^q(\mu)}$.

If $\|g\|_{L^q(U)} = 0$, we have that $g = 0$ a.e., and $\|\Lambda_g\|_{L^p(\mu)^*} = 0 = \|g\|_{L^q(U)}$.

Now consider $\|g\|_{L^q(U)} \neq 0$. It suffices to find some $\|f\|_{L^q(U)} = 1$, such that $\Lambda_g(f) \geq \|g\|_{L^q(U)}$.

1. $1 < p < \infty$.

Notice that $p = \frac{1}{1-\frac{1}{q}} = \frac{1}{\frac{q-1}{q}} = \frac{q}{q-1}$.

Let $f = \frac{\overline{\text{sgn}(g)} |g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}}$, we have that

$$\begin{aligned} \|f\|_{L^p(\mu)}^p &= \int |f|^p d\mu \\ &= \int \frac{|g|^q}{\|g\|_{L^q(\mu)}^q} d\mu \\ &= \frac{1}{\|g\|_{L^q(\mu)}^q} \int |g|^q d\mu \\ &= \frac{1}{\|g\|_{L^q(\mu)}^q} \|g\|_{L^q(U)}^q \\ &= 1, \end{aligned}$$

which means that $f \in L^p(\mu)$. In addition,

$$\begin{aligned} |\Lambda_g(f)| &= \left| \int_X f g d\mu \right| \\ &= \left| \int_X \overline{\text{sgn}(g)} \frac{|g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}} g d\mu \right| \\ &= \frac{1}{\|g\|_{L^q(\mu)}^{q/p}} \left| \int_X |g|^{1+q/p} d\mu \right| \\ &= \frac{1}{\|g\|_{L^q(\mu)}^{q-1}} \left| \int_X |g|^q d\mu \right| \\ &= \|g\|_{L^q(U)}. \end{aligned}$$

Thus, $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$.

2. $p = \infty, q = 1$.

Let $f = \text{sgn}(g) \in L^\infty(\mu)$. We have $\|f\|_{L^\infty(\mu)} = 1$. In addition,

$$\Lambda_g(f) = \int_X \overline{\text{sgn}(g)} g d\mu = \int_X |g| d\mu = \|g\|_{L^1(\mu)}.$$

Thus, $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$

3. $p = 1, q = \infty$, and μ is semi-finite.

Choose $\epsilon \in (0, \|g\|_{L^\infty(\mu)})$.

Let $A = \{x \in X \mid |g(x)| > \|g\|_{L^\infty(\mu)} - \epsilon\}$.

Notice that $\mu(A) > 0$, otherwise $\|g\|_{L^\infty(\mu)} = \epsilon$.

Since μ is semi-finite, we can find $E \in \mathcal{M}$, such that $0 < \mu(E) < \infty, E \subseteq A$.

Let $f = \frac{\chi_E}{\mu(E)} \text{sgn}(g)$.

Notice that

$$\begin{aligned}
\|f\|_{L^1(\mu)} &= \int_X |f| d\mu \\
&= \int_X \left| \frac{\chi_E}{\mu(E)} \overline{\text{sgn}(g)} \right| d\mu \\
&= \frac{1}{\mu(E)} \int_X \chi_E d\mu \\
&= 1.
\end{aligned}$$

Thus $f \in L^1(\mu)$. In addition,

$$\begin{aligned}
\Lambda_g(f) &= \int_X f g d\mu \\
&= \int_X \frac{\chi_E}{\mu(E)} \overline{\text{sgn}(g)} g d\mu \\
&= \int_E \frac{|g|}{\mu(E)} d\mu \\
&\geq \int_E \frac{\|g\|_{L^\infty(\mu)} - \epsilon}{\mu(E)} d\mu \\
&\geq \|g\|_{L^\infty(\mu)} - \epsilon.
\end{aligned}$$

Since this holds for any $\epsilon > 0$, we have that

$$\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \sup_{\epsilon > 0} \|g\|_{L^\infty(\mu)} - \epsilon = \|g\|_{L^q(\mu)}.$$

We thus have $\|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$. □

Remark. In the above case, the map $g \mapsto \Lambda_g$ is isometric.

7 Complex measures

7.1 Signed measures

Recall that if (X, \mathcal{M}, μ) is a measure space, and $f : X \rightarrow [0, \infty)$ is measurable, then we can set a measure $\mu_f(A) := \int_X \chi_A f d\mu$, and we have $\int_X g d\mu_f = \int g f d\mu$.

Example 7.1.1. Consider the regular Lebesgue measure, and $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, then λ_f gives a probability measure with the standard distribution.

We want to generalize this to functions that are not non-negative.

Definition 7.1. Let (X, \mathcal{M}) be a measurable space. A function $\nu : \mathcal{M} \rightarrow [-\infty, \infty]$ is a **signed measure** if

$$\begin{aligned}
\nu(\emptyset) &= 0 \\
\nu\left(\bigsqcup_{i=1}^{\infty} E_i\right) &= \sum_{i=1}^{\infty} \nu(E_i),
\end{aligned}$$

and ν only takes at most one of $\pm\infty$.

Proposition 7.1. If $|\nu(\bigsqcup_{i=1}^{\infty} E_i)| < \infty$, then $\sum_{i=1}^{\infty} \nu(E_i)$ must converge absolutely, since we want $\nu(\bigsqcup_{i=1}^{\infty} E_i)$ to be invariant of the order of union.

Proposition 7.2. If $f \in \mathcal{L}^1(X, \mu)$, then $\mu_f := \int_X \chi(A) f d\mu$ is a signed measure.

Proposition 7.3. If $f, g \geq 0$ is measurable, and $g \in \mathcal{L}^1(\mu)$, then $\nu(E) := \int_X \chi_E(f - g) d\mu$ is a signed measure.

Definition 7.2. Suppose ν is a signed measure, then $E \in \mathcal{M}$ is

1. **null** for ν if $\forall F \subseteq E, \nu(F) = 0$.
2. **positive** for ν if $\forall F \subseteq E, \nu(F) \geq 0$.
3. **negative** for ν if $\forall F \subseteq E, \nu(F) \leq 0$.

Lemma 7.4. Let $E \in \mathcal{M}$, if $0 < \nu(E) < \infty$, then $\exists A \subseteq E, A \in \mathcal{M}$ is positive, and $\nu(A) > 0$.

Proof. Choose $B_1 \subseteq E, B_1 \in \mathcal{M}$, such that $\nu(B_1) \leq \max \left\{ -1, \frac{1}{2} \inf \{ \nu(B) | B \subseteq E, B \in \mathcal{M} \} \right\}$.

Recursively choose $B_n \subseteq E \setminus \bigcup_{i=1}^{n-1} B_i$ with $\nu(B_n) \leq \max \left\{ -1, \frac{1}{2} \inf \left\{ \nu(B) | B \subseteq E \setminus \bigcup_{i=1}^{n-1} B_i, B \in \mathcal{M} \right\} \right\}$.

Now either this sequence terminates (then $A = E \setminus \bigcup_{i=1}^{\infty} B_i$ is positive), or we get an infinite sequence.

Set $A := E \setminus \bigcup_{i=1}^{\infty} B_i$.

We have $\nu(E) = \nu(A) + \sum_{i=1}^{\infty} \nu(B_i) < \infty$, thus $\sum_{i=1}^{\infty} \nu(B_i)$ converges absolutely.

So $\nu(A) = \nu(E) - \sum_{i=1}^{\infty} \nu(B_i) > \nu(E) > 0$.

Notice that $\nu(B_n) \rightarrow 0^-$, so if $B \subseteq A \subseteq E \setminus \bigcup_{i=1}^{n-1} B_i$ has $\nu(B) < 0$, we must have $\nu(B) < 2\nu(B_n)$ for some large n . But

$$\inf \left\{ \nu(B) | B \subseteq E \setminus \bigcup_{i=1}^{n-1} B_i, B \in \mathcal{M} \right\} > 2\nu(B_n),$$

which is a contradiction.

Thus A is positive. □

Lemma 7.5. If $(A_n)_{n=1}^{\infty}$ is a sequence of positive sets, then $A := \bigcup_{n=1}^{\infty} A_n$ is positive.

Proof. If $B \subseteq A, B \in \mathcal{M}$, let $B_n = B \cap (A_n \setminus \bigcup_{i=1}^{n-1} A_i)$. Then $(B_n)_{n=1}^{\infty} \subseteq \mathcal{M}$ are pairwise disjoint, with $B = \bigcup_{n=1}^{\infty} B_n$.

For any n , since A_n is positive, we have $\nu(B_n) > 0$.

Thus $\nu(B) = \sum_{n=1}^{\infty} \nu(B_n) > 0$. □

Theorem 7.6. (Hahn decomposition) Let ν be a signed measure on (X, \mathcal{M}) , there are $P, N \in \mathcal{M}$ such that $X = P \sqcup N$, and P is positive, N is negative. Moreover, this is unique in the sense that if $X = P' \sqcup N'$ is another such decomposition, then the symmetric difference $P \Delta P'$ is null.

Proof. Existence:

By taking $-\nu$ if necessary, we can WLOG assume ν never takes $+\infty$.

Let $m := \sup \{ \nu(A) : A \text{ is positive} \} < \infty$.

Choose positive sets A_n such that $\nu(A_n) \rightarrow m$, and let $P := \bigcup_{n=1}^{\infty} A_n$.

Thus P is positive by lemma, and $\forall n, \nu(P) = \nu(A_n) + \nu(P \setminus A_n) \geq \nu(A_n)$. Thus $\nu(P) = m$.

Let $N := X \setminus P$.

Suppose N is not negative, $\exists E \subseteq N, E \in \mathcal{M}$, such that $\nu(E) > 0$. By lemma, there is positive $A \subseteq E$ with $\nu(A) > 0$. Then $P \sqcup A$ is measurable, positive, and $\nu(P \sqcup A) = \nu(P) + \nu(A) > m$, which is a contradiction.

Uniqueness:

Let $A := P \setminus P' = N' \setminus N$ is both positive and negative, thus null. Similarly $B := P' \setminus P = N \setminus N'$ is null.

Thus $P \Delta P' = A \cup B$ is null. □

Definition 7.3. Let ν be a signed measure on (X, \mathcal{M}) , and $P, N \in \mathcal{M}$ be as from Hahn decomposition, the **Jordan decomposition** of it is $\nu = \nu^+ - \nu^-$, where $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$ are positive measures.

Definition 7.4. Let ν be a signed measure on (X, \mathcal{M}) , with Jordan decomposition $\nu = \nu^+ - \nu^-$, the total variation $|\nu| = \nu^+ + \nu^-$.

Corollary 7.7. Let $\nu = \nu_+ - \nu_-$ be a signed measure with its Jordan decomposition, there exists a measurable function $f : X \rightarrow \mathbb{R}$ with $|f| = 1$, and

$$\nu(E) = \int_E f d|\nu|,$$

where $|\nu| = \nu_+ + \nu_-$.

Proof. Let $X = P \sqcup N$ be the Hahn decomposition. Define $f(x) = \chi_P - \chi_N$, then clearly $|f| = 1$. In addition,

$$\begin{aligned} \int_E f d|\nu| &= \int_E \chi_P - \chi_N d|\nu| \\ &= \int_{E \cap P} d|\nu| - \int_{E \cap N} d|\nu| \\ &= \nu(E) \end{aligned}$$

□

Corollary 7.8. Let $\nu = \nu_+ - \nu_-$ be a signed measure with its Jordan decomposition. If $|\nu|, \mu$ are σ -finite with $|\nu| \ll \mu$, then $\exists g = g_+ + g_- : X \rightarrow \mathbb{R} \cup \pm\infty$ measurable function, with at most one of g_+, g_- take ∞ , such that $\nu(E) = \int_E g d\mu$.

Proposition 7.9. Let ν be a signed measure on (X, \mathcal{M}) , with Jordan decomposition $\nu = \nu^+ - \nu^-$, then for any other positive λ_1, λ_2 , such that $\nu = \lambda_1 - \lambda_2$, we have $\lambda_1 \geq \nu^+, \lambda_2 \geq \nu^-$.

Proof. Let $X = P \sqcup N$ be the Hahn decomposition.

Consider any $E \in \mathcal{M}$, we have

$$\begin{aligned} \lambda_1(E) &\geq \lambda_1(E \cap P) \\ &= \nu(E \cap P) + \lambda_2(E \cap P) \\ &\geq \nu(E \cap P) \\ &= \nu^+(E) \\ \lambda_2(E) &\geq \lambda_2(E \cap N) \\ &= -\nu(E \cap N) + \lambda_1(E \cap N) \\ &\geq -\nu(E \cap N) \\ &= \nu^-(E). \end{aligned}$$

□

7.2 Complex measures

Definition 7.5. Let (X, \mathcal{M}) be a measurable space, a **complex measure** is a function $\nu : \mathcal{M} \rightarrow \mathbb{C}$, such that

1. $\nu(\emptyset) = 0$
2. $\nu(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$ absolutely.

Remark. The absolute convergence is important since we want $\nu(\bigsqcup_{i=1}^{\infty} E_i)$ to be invariant of the order of union.

Example 7.2.1. Let (X, \mathcal{M}, μ) be a measure space, and $f \in \mathcal{L}^\infty(\mu)$ with $\|f\|_{L^\infty(\mu)} = 1$, then with $\nu(E) := \int_E f d\mu$, we have ν is a complex measure, and $\forall E \in \mathcal{M}$,

$$\begin{aligned}\mu(E) &= \int_E d\mu \\ &\geq \int_E |f| d\mu \\ &\geq \left| \int_E f d\mu \right| \\ &\geq \left| \int_E d\nu \right| \\ &= |\nu(E)|.\end{aligned}$$

Definition 7.6. Let (X, \mathcal{M}, μ) be a measure space, the **total variation** of a complex measure μ is $|\mu| : \mathcal{M} \rightarrow [0, \infty]$, defined by

$$\forall E \in \mathcal{M}, |\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i, (E_i)_{i=1}^{\infty} \subset \mathcal{M} \right\}.$$

Proposition 7.10. Let (X, \mathcal{M}, μ) be a measure space, μ be a complex measure, then $|\mu|$ is a positive measure.

Proof. 1. $|\mu|(\emptyset) = 0$.

2. $|\mu|(E) \geq 0, \forall E \in \mathcal{M}$.

3. Fix $E = \bigsqcup_{i=1}^{\infty} E_i \in \mathcal{M}$, $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$.

Consider any $(A_j)_{j=1}^{\infty} \subset \mathcal{M}$ such that $E = \bigsqcup_{j=1}^{\infty} A_j$, then $A_j = A_j \cap E = \bigsqcup_{i=1}^{\infty} A_j \cap E_i$.

$$\begin{aligned}\sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)| \\ &\leq \sum_{i=1}^{\infty} |\mu| \left(\bigsqcup_{j=1}^{\infty} (A_j \cap E_i) \right) \\ &= \sum_{i=1}^{\infty} |\mu|(E_i).\end{aligned}$$

We have that

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(A_j)| : E = \bigsqcup_{j=1}^{\infty} A_j, (A_j)_{j=1}^{\infty} \subset \mathcal{M} \right\} \leq \sum_{i=1}^{\infty} |\mu|(E_i).$$

Now given any $\epsilon > 0$.

$\forall i$, pick $t_i := |\mu|(E_i) - \frac{\epsilon}{2^i}$ and we can find $E_{ij} \in \mathcal{M}$, such that

$$E_i = \bigsqcup_{j=1}^{\infty} E_{ij}, \quad \sum_{j=1}^{\infty} |\mu(E_{ij})| > t_i.$$

We have

$$\begin{aligned}
|\mu|(E) &\geq \sum_{i,j=1}^{\infty} |\mu(E_{ij})| \\
&\geq \sum_{i=1}^{\infty} t_i \\
&= \sum_{i=1}^{\infty} |\mu|(E_i) - \epsilon.
\end{aligned}$$

Taking $\epsilon \rightarrow 0$, we have $|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(E_i)$.
Thus $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(E_i)$.

□

Lemma 7.11. *Let $\{z_1, \dots, z_N\} \subset \mathbb{C}$, then $\exists S \subseteq [N]$, such that*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^N |z_k|.$$

Theorem 7.12. *Let (X, \mathcal{M}, μ) be a measure space, μ be a complex measure, then $|\mu|$ is a finite measure. Namely, $\forall E \in \mathcal{M}$, such that $|\mu|(E) < \infty$.*

Proof. Suppose $\exists E \in \mathcal{M}$, such that $|\mu|(E) = \infty$.

Let $B_0 = E$.

Let $t := \pi(1 + |\mu(E)|) \geq \pi$.

Then we can find a partition $E_i \in \mathcal{M}$, such that

$$E = \bigsqcup_{i=1}^{\infty} E_i, \quad \sum_{i=1}^{\infty} |\mu(E_{ij})| > t.$$

Thus there is some $N \in \mathbb{N}$, such that $\sum_{i=1}^N |\mu(E_{ij})| > t$.

By lemma, $\exists S \subseteq [N]$, such that

$$\begin{aligned}
\left| \mu \left(\bigsqcup_{i \in S} E_i \right) \right| &= \left| \sum_{i \in S} \mu(E_i) \right| \\
&\geq \frac{1}{\pi} \sum_{i=1}^N |\mu(E_i)| \\
&> \frac{t}{\pi} \\
&\geq 1.
\end{aligned}$$

Now let $A := \bigsqcup_{i \in S} E_i$, $B = E \setminus A$.

We have

$$\begin{aligned}
|\mu(B)| &= |\mu(E) - \mu(A)| \\
&\geq |\mu(A)| - |\mu(E)| \\
&> \frac{t}{\pi} - |\mu(E)| \\
&= 1.
\end{aligned}$$

Thus $E = A \sqcup B$, where $|\mu(A)| > 1$, $|\mu(B)| > 1$.

Since $|\mu|(A \sqcup B) = |\mu|(A) + |\mu|(B) = \infty$, at least one of $|\mu|(A)$, $|\mu|(B)$ is ∞ .

WLOG, say $|\mu|(B) = \infty$. We let $A_1 = A, B_1 = B$.

Now apply the above argument on $B_1 = A_2 \sqcup B_2$, where $|\mu(A_2)| > 1, |\mu(B_2)| > 1, |\mu|(B_2) = \infty$.

Repetitively, we construct disjoint $(A_k)_{k=1}^\infty$, such that $\forall i \geq 1, |\mu(A_i)| > 1$.

Notice that $\bigsqcup_{k=1}^\infty A_k \in \mathcal{M}$, and we have $\mu(\bigsqcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \mu(A_k)$ absolutely.

However, $\sum_{k=1}^\infty |\mu(A_k)| \geq \sum_{k=1}^\infty 1 = \infty$ diverges. \square

Definition 7.7. Let μ, ν be two complex measures on (X, \mathcal{M}) . A function f is the **Radon–Nikodym derivative**, written as $\frac{d\nu}{d\mu}$, if $\forall E \in \mathcal{M}, \nu(E) = \int_E f d\mu$. We can also abuse the notation and write $d\nu = f d\mu$.

Proposition 7.13. *If the Radon–Nikodym derivative exists, it is unique μ -a.e..*

Proof. Suppose there are two f, h that satisfies $\forall E \in \mathcal{M}, \nu(E) = \int_E f d\mu = \int_E h d\mu$, then $\int_E (f - h) d\mu = 0$. Thus $f = h$ μ -a.e.. \square

7.3 Radon–Nikodym–Lebesgue Decomposition

Definition 7.8. Let μ, ν be two measures on (X, \mathcal{M}) , we say ν is **dominated** by μ if $\mu(A) = 0 \implies \nu(A) = 0$, and is written as $\nu \ll \mu$.

Example 7.3.1. Consider the counting measure μ , then for any other measure $\nu, \nu \ll \mu$, since $\mu(E) = 0 \implies E = \emptyset$.

Lemma 7.14. *Let μ be a σ -finite measure on (X, \mathcal{M}) , then there is some $w \in L^1(\mu)$, such that $\forall x \in X, 0 < w(x) < 1$.*

Proof. Write $X = \bigcup_{n=1}^\infty E_n$, where $\forall n \geq 1, \mu(E_n) < \infty$.

Let $w_n := \frac{2^{-n} \chi_{E_n}}{1 + \mu(E_n)}, w = \sum_{n=1}^\infty w_n$.

Notice that $0 < w_n(x) < 1$, and

$$\begin{aligned} \int_X w d\mu &= \sum_{n=1}^\infty \int_X w_n d\mu \\ &= \sum_{n=1}^\infty \int_X \frac{2^{-n} \chi_{E_n}}{1 + \mu(E_n)} d\mu \\ &\leq \sum_{n=1}^\infty \frac{2^{-n} \mu(E_n)}{1 + \mu(E_n)} \\ &< \sum_{n=1}^\infty 2^{-n} \\ &< \infty. \end{aligned}$$

\square

Lemma 7.15. *Let μ be a σ -finite measure on (X, \mathcal{M}) , and $g \in L^1(\mu)$. Suppose $\forall E \in \mathcal{M}$ such that $\mu(E) > 0$, we have that*

$$\frac{1}{\nu(E)} \int_E g d\mu \in S$$

for some closed $S \subseteq \mathbb{C}$, then

$$g(x) \in S, \text{ a.e. } x \in X.$$

Proof. Assume for contradiction that there is $E =: g^{-1}(\bar{B}(t, r))$ such that $\mu(E) > 0$, $\bar{B}(t, r) \subseteq S^c$. Then $A_E(g) := \frac{1}{\mu(E)} \int_E g d\mu \in S$, while

$$\begin{aligned} |A_E(g) - t| &= \left| \frac{1}{\mu(E)} \int_E (g - t) d\mu \right| \\ &\leq \frac{1}{\mu(E)} \int_E |g - t| d\mu \\ &\leq \frac{1}{\mu(E)} \int_E r d\mu \\ &= r. \end{aligned}$$

Thus $A_E(g) \in \bar{B}(t, r) \subseteq S^c$, which is a contradiction. \square

Theorem 7.16. (*Radon–Nikodym for finite measure*) Let μ be a σ -finite measure, and ν is a positive finite measure on (X, \mathcal{M}) . Suppose $\nu \ll \mu$, then $\exists h \in L^1(\mu) \cap \mathcal{L}^+$, such that $\forall E \in \mathcal{M}, \nu(E) = \int_E h d\mu$. Moreover, h is unique μ -a.e.

Proof. (Von Neumann's proof).

Since μ is σ -finite, there is some $w \in L^1(\mu)$, such that $\forall x \in X, 0 < w(x) < 1$.

Define a new measure $d\lambda := d\nu + w d\mu$, namely, $\forall E \in \mathcal{M}, \lambda(E) := \nu(E) + \int_E w d\mu$.

Claim 7.16.1. *There is some measurable g such that $\forall x \in X, g(x) \in [0, 1]$, and for any measurable $f \in L^2(\lambda)$, we have $\int_E f(1 - g) d\nu = \int_E f g w d\mu$.*

Proof. Notice that $\int_X f d\lambda = \int_X f d\nu + \int_X f w d\mu$ for any measurable f . Consider any $f \in L^2(\lambda)$,

$$\begin{aligned} \left| \int_X f d\nu \right| &\leq \int_X |f| d\nu \\ &= \int_X |f| d\lambda - \int_X |f| w d\mu \\ &\leq \int_X |f| d\lambda \\ &\leq \int_X |f| \cdot 1 d\lambda \\ &\leq \|f\|_{L^2(\lambda)} \lambda(X). \end{aligned}$$

Notice that $\lambda(X) = \nu(X) + \int_X w d\mu < \infty$, so $\Lambda : f \mapsto \int_X f d\nu \in L^2(\lambda)^*$.

Since $L^2(\lambda)$ is a Hilbert space, there is a unique $g \in L^2(\lambda)$, such that $\int_X f g d\lambda = \Lambda(f) = \int_X f d\nu, \forall f \in L^2(\lambda)$.

Now we know $\int_X f d\nu = \int_X f g d\lambda = \int_X f g d\nu + \int_X f g w d\mu$.

For any $E \in \mathcal{M}, f \in L^2(\lambda)$, we can take $\tilde{f} := f \chi_E \in L^2(\lambda)$, and we get

$$\begin{aligned} \int_E f(1 - g) d\nu &= \int_X \tilde{f}(1 - g) d\nu \\ &= \int_X \tilde{f} d\nu - \int_X \tilde{f} g d\nu \\ &= \int_X \tilde{f} g w d\mu + \int_X \tilde{f} g d\nu - \int_X \tilde{f} g d\nu \\ &= \int_X \tilde{f} g w d\mu \\ &= \int_E f g w d\mu. \end{aligned}$$

In addition, for any $E \in \mathcal{M}$, taking $f = 1$, we have that

$$\nu(E) = \int_E d\nu = \int_E g d\lambda.$$

Thus $0 \leq \int_E g d\lambda \leq \lambda(E)$.

Thus $\forall E \in \mathcal{M}$, such that $\lambda(E) > 0$, we have

$$\frac{\int_E g d\lambda}{\lambda(E)} \in [0, 1].$$

By the above lemma, we have that $g(x) \in [0, 1]$, λ -a.e. $x \in X$.

WLOG, we can redefine $g(x) = 0$ for any $g(x) \notin [0, 1]$. □

Let $A := g^{-1}([0, 1))$, $B := g^{-1}(\{1\})$.

Let $f = \chi_B$, we have that

$$\begin{aligned} \int_X \chi_B(1 - g) d\nu &= \int_X \chi_B g w d\mu \\ \int_B (1 - g) d\nu &= \int_B w d\mu \\ 0 &= \int_B w d\mu. \end{aligned}$$

Since $w > 0$, we must have $\mu(B) = 0$. Since $\nu \ll \mu$, $\nu(B) = 0$

Thus $\forall E \in \mathcal{M}$, $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A)$.

Now, let $f_n := \sum_{k=0}^n g^k$, we have that $f_n(1 - g) = 1 - g^{n+1}$, so

$$\int_E (1 - g^{n+1}) d\nu = \int_E f_n(1 - g) d\nu = \int_E f_n g w d\mu.$$

Notice that $1 - g^{n+1}(x) \rightarrow \begin{cases} 1, & x \in A, \\ 0, & x \in B. \end{cases}$ monotonically.

In addition, $(f_n g w)(x)$ is increasing and bounded.

So there is some $h(x) := \lim_{n \rightarrow \infty} (f_n g w)(x)$.

Thus, by LMCT, we have

$$\begin{aligned} \nu(E) &= \nu(A \cap E) \\ &= \int_{E \cap A} d\nu \\ &= \int_{E \cap A} \lim_{n \rightarrow \infty} (1 - g^{n+1}) d\nu \\ &= \lim_{n \rightarrow \infty} \int_{E \cap A} (1 - g^{n+1}) d\nu \\ &= \lim_{n \rightarrow \infty} \int_E f_n g w d\mu \\ &= \int_E h d\mu. \end{aligned}$$

Since ν is finite, we have $h \in L^1(\mu)$. □

Theorem 7.17. (Radon-Nikodym) Let μ, ν be two σ -finite measures on (X, \mathcal{M}) . Suppose $\nu \ll \mu$, then $\exists f \in \mathcal{L}^+$, such that $\forall E \in \mathcal{M}$, $\nu(E) = \int_E f d\mu$. Moreover, f is unique μ -a.e.

Proof. Since ν is σ -finite, we have $X = \bigsqcup_{n=1}^{\infty} X_n$, where each $\nu(X_n)$ is finite.

We can apply the above theorem on $\nu_n(E) := \nu(E \cap X_n)$, which are finite measures, and let $h = \sum_{n=1}^{\infty} h_n$. h will be positive and measurable, but not in $L^1(\mu)$. Yet it is in $L^1(\mu|_{X_n})$ for all n . □

Remark. The σ -finiteness is essential. Indeed, consider the following counter example.

Example 7.3.2. Consider λ to be the Lebesgue measure on $(0, 1)$, and μ to be the counting measure, which is not σ -finite.

Although $\lambda \ll \mu$, it is impossible to find such an $h = \frac{d\lambda}{d\mu}$, because for any $E \in \mathcal{M}$, we will have

$$\begin{aligned}\lambda_a(E) &= \int_E h d\mu \\ &= \sum_{x \in E} h(x),\end{aligned}$$

which is not possible.

Definition 7.9. Two measures μ, ν on (X, \mathcal{M}) are said to be **mutually singular**, written as $\mu \perp \nu$, if $X = A \sqcup B$, where A is μ -null and B is ν -null.

Theorem 7.18. (*Lebesgue decomposition*) Let μ, ν be two σ -finite measures on (X, \mathcal{M}) . There is a unique decomposition $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu, \nu_s \perp \mu$, both positive measures.

Proof. Take $\lambda = \mu + \nu$, which is σ -finite, and $\mu, \nu \ll \lambda$. By Radon-Nikodym, $\exists f, g \in \mathcal{L}^+$, such that

$$\mu(E) = \int_E f d\lambda, \nu(E) = \int_E g d\lambda.$$

Let $A = f^{-1}((0, \infty]), B = f^{-1}(\{0\}), \nu_a(E) = \nu(E \cap A), \nu_s(E) = \nu(E \cap B)$.

Since $X = A \sqcup B$, clearly $\nu = \nu_a + \nu_s$.

We can see that ν_s is A -null.

On the other hand, $\forall E \subseteq B, \mu(E) = \int_E f d\lambda = 0$, so μ is B -null.

This shows $\nu_s \perp \mu$.

In addition, suppose $\mu(E) = \int_E f d\lambda = 0$, then we must have $\lambda(E \cap A) = 0$, which implies $\nu_a(E) = \nu(E \cap A) = \int_{E \cap A} g d\lambda = 0$. Thus $\nu_a \ll \mu$. \square

Theorem 7.19. (*Lebesgue-Radon-Nikodym for complex measures*)

1. Let ν be a complex measure on (X, \mathcal{M}) . There is a unique finite measure $|\nu|$ and a measurable function h , such that $|h| = 1$ $|\nu|$ -a.e., and $d\nu = h d|\nu|$.
2. If μ is a σ -finite measure on (X, \mathcal{M}) , then ν decomposes uniquely as $\nu = \nu_a + \nu_s$, such that $\nu_a \ll \mu$ is the absolutely continuous part and $\nu_s \perp \mu$ is the singular part. Also, $d\nu_a = f d\mu$ for some $f \in \mathcal{L}^1(\mu)$.

Proof. 1. Write $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ with Jordan decomposition. Define $\mu := \nu_1 + \nu_2 + \nu_3 + \nu_4$, which is a positive finite measure.

Notice that $\nu_i \ll \mu$ since they have disjoint supports. Applying Radon-Nikodym, we have some $f_i \in \mathcal{L}^+$, where $\forall E \in \mathcal{M}, \nu_i(E) = \int_E f_i d\mu$. Thus

$$\nu(E) = \int_E (f_1 - f_2 + if_3 - if_4) d\mu.$$

Now we define $|\nu|$ by

$$|\nu|(E) := \int_E |f_1 - f_2 + if_3 - if_4| d\mu.$$

Define $h := \text{sgn}(f_1 - f_2 + if_3 - if_4)$. Notice that $|h| = 1$ unless $f_1 - f_2 + if_3 - if_4 = 0$, which is a null set since $\nu(A) = \int_A (f_1 - f_2 + if_3 - if_4) d\mu = 0$ for $A := \{x \in X : (f_1 - f_2 + if_3 - if_4)(x) = 0\}$.

Notice that by definition $d|\nu| = \bar{h}(f_1 - f_2 + if_3 - if_4) d\mu = \bar{h} d\nu$. Thus

$$\int_E h d|\nu| = \int_E h \bar{h} d\nu = \nu(E).$$

In addition, $|\nu|(X) = \int_X |f_1 - f_2 + if_3 - if_4| d\mu \leq \|f_1\|_1 + \|f_2\|_1 + \|f_3\|_1 + \|f_4\|_1 < \infty$.

In addition, it is unique up to a ν -null set.

2. From 1, we have $d\nu = hd|\nu|$, and we can use Lebesgue decomposition to write $|\nu| = |\nu|_a + |\nu|_s$.
 Let $f = \frac{d|\nu|}{d\mu}$ be the Radon-Nikodym derivative, we have $d\nu = hfd\mu + hd|\nu|_s$.
 Let $d\nu_a := hfd\mu, d\nu_s := hd|\nu|_s$, and we have the result.
 Now we want to uniqueness:
 Suppose $\nu = \nu_a + \nu_s = \nu'_a + \nu'_s$ are two decompositions as in the theorem.
 Then $(\nu_a - \nu'_a) + (\nu_s - \nu'_s) = 0$ is the zero measure.
 Thus, $\mu' := \nu_a - \nu'_a = \nu'_s - \nu_s$, where $\nu_a - \nu'_a \ll \mu$, and $\nu'_s - \nu_s \perp \mu$.
 Thus, $\mu' = 0$, and $\nu_a = \nu'_a, \nu'_s = \nu_s$.

□

Corollary 7.20. (Polar decomposition of complex measures)

Let ν be a complex measure on (X, \mathcal{M}) . There is a unique measurable function h , such that $|h| = 1$ $|\nu|$ -a.e., and $d\nu = h d|\nu|$.

Proof.

$$1 \geq \frac{|\nu(E)|}{|\nu|(E)} = \left| \frac{1}{|\nu|(E)} \int_E h d|\nu| \right|.$$

Thus $|h(x)| \leq 1$.

For any $0 < r < 1$, consider $A_r := \{x \in X : |h(x)| < r\} = \bigsqcup_{i=1}^{\infty} E_i$.

We have

$$\begin{aligned} \sum_{i=1}^{\infty} |\nu(E_i)| &= \sum_{i=1}^{\infty} \left| \int_{E_i} h d|\nu| \right| \\ &\leq \sum_{i=1}^{\infty} \left| \int_{E_i} r d|\nu| \right| \\ &= r \sum_{i=1}^{\infty} |\nu|(E_i) \\ &= r |\nu|(A_r). \end{aligned}$$

Taking sup over all E_i , we have that $|\nu|(E_i) \leq r |\nu|(E_i)$. Since $r < 1$, we have $|\nu|(E_i) = 0$.

Thus $|h(x)| > 1$ a.e. $x \in X$ for all $0 < r < 1$.

Thus $|h| = 1$ a.e..

□

Corollary 7.21. If $d\lambda = g d\mu$, then we have $h|\lambda| = |g| d\mu$.

8 Dual of Function Spaces

8.1 Dual of L^p Spaces

Theorem 8.1. Let (X, \mathcal{M}, μ) be a measure space, and $\frac{1}{p} + \frac{1}{q} = 1$ for $p \in (1, \infty)$, we have

$$L^q(\mu) \simeq L^p(\mu)^*,$$

where the isometric isomorphism $L^q(\mu) \xrightarrow{\sim} L^p(\mu)^*$; $g \mapsto \Lambda_g$ is defined to be

$$\forall f \in L^p(\mu), \Lambda_g(f) := \int_X fg dx.$$

In addition, the same is true for $p = 1$ if μ is σ -finite.

Proof. Let $1 \leq p < \infty$.

1. By 6.19, we only need to show the subjectivity: $\forall \Lambda \in L^p(\mu)^*, \exists g \in L^q(\mu)$, such that $\Lambda = \Lambda_g$.

2. First we assume $\mu(X) < \infty$ is finite.

Given any $\Lambda \in L^p(\mu)^*$.

Consider the mapping $\nu : E \mapsto \Lambda(\chi_E)$ for any measurable $E \in \mathcal{M}$.

This is well-defined since $\chi_E \in L^\infty(X) \subseteq L^p(X)$.

Notice that $|\nu(E)| = |\Lambda(\chi_E)| \leq \|\Lambda\|_{L^p(\mu)^*} \|\chi_E\|_{L^p(\mu)} < \infty$, thus ν is finite.

We have $\nu(\emptyset) = \Lambda(0) = 0$ since Λ is linear.

For any $B = \bigsqcup_{i=0}^\infty A_i$, with $A_i \subseteq U$ be measurable, we have $\chi_B = \sum_{i=0}^\infty \chi_{A_i}$ in $L^p(\mu)$.

Indeed,

$$\begin{aligned} \left\| \chi_B - \sum_{i=0}^N \chi_{A_i} \right\|_{L^p(\mu)}^p &= \left\| \sum_{i=N+1}^\infty \chi_{A_i} \right\|_{L^p(\mu)}^p \\ &= \left\| \chi_{\bigsqcup_{i=N+1}^\infty A_i} \right\|_{L^p(\mu)}^p \\ &= \mu \left(\bigsqcup_{i=N+1}^\infty A_i \right)^p \\ &\rightarrow 0. \end{aligned}$$

Notice that this fails when $p = \infty$!

Thus,

$$\begin{aligned} \nu(B) &= \Lambda(\chi_B) \\ &= \Lambda \left(\sum_{i=0}^\infty \chi_{A_i} \right) \\ &= \Lambda \left(\lim_{n \rightarrow \infty} \sum_{i=0}^n \chi_{A_i} \right) \\ &= \lim_{n \rightarrow \infty} \Lambda \left(\sum_{i=0}^n \chi_{A_i} \right) && \text{continuity of } \Lambda \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^n \Lambda(\chi_{A_i}) && \text{linearity of } \Lambda \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^n \nu(A_i) \\ &= \sum_{i=0}^\infty \nu(A_i), \end{aligned}$$

which shows countable additivity.

In addition,

$$\begin{aligned}
\sum_{i=0}^{\infty} |\nu(A_i)| &= \lim_{n \rightarrow \infty} \sum_{i=0}^n |\Lambda(\chi_{A_i})| \\
&\leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \|\Lambda\|_{L^p(\mu)^*} \|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*} \lim_{n \rightarrow \infty} \sum_{i=0}^n \|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*} \lim_{n \rightarrow \infty} \sum_{i=0}^n \mu(A_i)^{1/p} \\
&\leq \|\Lambda\|_{L^p(\mu)^*} \left(\lim_{n \rightarrow \infty} \sum_{i=0}^n \mu(A_i) \right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*} \mu \left(\bigsqcup_{i=0}^{\infty} A_i \right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*} \mu(B)^{1/p} \\
&\leq \|\Lambda\|_{L^p(\mu)^*} \mu(X)^{1/p} \\
&< \infty,
\end{aligned}$$

which converges absolutely.

Thus ν is a complex measure.

In addition, if $\mu(E) = 0$, we have $\nu(E) = \Lambda(\chi_E) = \Lambda(0) = 0$.

Thus, $\nu \ll \mu$.

By Radon-Nikodym, $\exists! g \in L^1(\mu)$, such that $\Lambda(\chi_E) = \nu(E) = \int_E g d\mu$.

By linearity, $\Lambda(f) = \int_X f g d\mu$ for all simple measurable f .

By uniform step function approximation, we have $\Lambda(f) = \int_X f g d\mu$ for all $f \in L^\infty(\mu)$.

Indeed, given any $f \in L^\infty(\mu)$, we have a sequence of simple measurable functions $\|f - f_n\|_{L^\infty(\mu)} \rightarrow 0$.

Thus $\|f - f_n\|_{L^p(\mu)} \rightarrow 0$.

Thus $|\Lambda(f) - \Lambda(f_n)| \leq \|\Lambda\|_{L^p(\mu)^*} \|f - f_n\|_{L^p(\mu)} \rightarrow 0$.

$$\begin{aligned}
\Lambda(f) &= \lim_{n \rightarrow \infty} \Lambda(f_n) \\
&= \lim_{n \rightarrow \infty} \int_X f_n g d\mu \\
&= \int_X \lim_{n \rightarrow \infty} f_n g d\mu \\
&= \int_X f g d\mu.
\end{aligned}$$

(a) $p = 1, q = \infty$.

Consider any $E \in \mathcal{M}$, such that $\mu(E) > 0$.

We have

$$\begin{aligned}
\left| \frac{1}{\mu(E)} \int_E g d\mu \right| &= \left| \frac{1}{\mu(E)} \Lambda(\chi_E) \right| \\
&\leq \frac{1}{\mu(E)} \|\Lambda\|_{L^1(\mu)^*} \|\chi_E\|_{L^1(\mu)} \\
&= \frac{1}{\mu(E)} \|\Lambda\|_{L^1(\mu)^*} \mu(E) \\
&= \|\Lambda\|_{L^1(\mu)^*}.
\end{aligned}$$

Thus $|g(x)| \leq \|\Lambda\|_{L^1(\mu)^*}$ a.e..

Thus $g \in L^\infty(\mu)$. Since simple functions are dense in $L^p(\mu)$, we have $L^\infty(\mu)$ is dense in $L^p(\mu)$.

Since Λ, Λ_q are both bounded linear functionals, we have $\Lambda(f) = \int_X f g d\mu$ for all $f \in L^p(\mu)$.

(b) $p > 1$.

Let $E_n := \{x \in X : |g(x)| \leq n\}$.

By LMCT, we have $\|g\|_q = \lim_{n \rightarrow \infty} \|\chi_{E_n} g\|$.

Let $f = \chi_{E_n} \overline{\text{sgn}(g)} |g|^{q-1} \in L^\infty(\mu)$, we have

$$\begin{aligned} \|f\|_{L^p(\mu)}^p &= \int_{E_n} |g|^{(q-1)p} d\mu \\ &= \int_{E_n} |g|^q d\mu \\ &= \|\chi_{E_n} g\|_{L^q(\mu)}^q \\ \|\chi_{E_n} g\|_q^q &= \int_{E_n} |g|^q d\mu \\ &= \int_X f g d\mu \\ &= |\Lambda(f)| \\ &\leq \|\Lambda\| \|f\|_{L^p(\mu)} \\ &\implies \\ \|g \chi_{E_n}\|_{L^q(\mu)}^{q-\frac{q}{p}} &\leq \|\Lambda\| \\ &\implies \\ \|g\|_{L^q(\mu)}^{q-\frac{q}{p}} &\leq \|\Lambda\| \\ &< \infty. \end{aligned}$$

Thus $g \in L^q(\mu)$.

Since Λ, Λ_q are both bounded linear functionals, we have $\Lambda(f) = \int_X f g d\mu$ for all $f \in L^p(\mu)$.

3. Now we assume that μ is σ -finite.

We have $X = \bigcup_{n=1}^\infty X_n, \forall n \geq 1, X_n \subset X_{n+1}, \mu(X_n) < \infty$.

We can get

$$\forall n \geq 1, g_n \in L^q(X_n, \mu), \text{ such that } \Lambda(f) = \int_X f g_n d\mu, \forall f \in L^p(X_n, \mu).$$

Notice that $L^p(X_n, \mu) \subset L^p(X_{n+1}, \mu)$.

We thus have $\forall n > m, g_n|_{X_m} = g_m$.

Let $g : X \rightarrow \mathbb{C}; x \mapsto g_n(x)$ for $x \in X_n$.

Then $g = \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} g \chi_{X_n}$ in $\|\cdot\|_{L^q(\mu)}$, and thus $g \in L^q(\mu)$.

In addition, for any $f \in L^p(\mu)$, we have $\lim_{n \rightarrow \infty} f \chi_{X_n} = f$ in $\|\cdot\|_{L^q(\mu)}$.

We have

$$\begin{aligned} \Lambda(f) &= \Lambda(\lim_{n \rightarrow \infty} f \chi_{X_n}) \\ &= \lim_{n \rightarrow \infty} \Lambda(f \chi_{X_n}) \\ &= \lim_{n \rightarrow \infty} \int_X f \chi_{X_n} g d\mu \\ &= \int_X f g d\mu. \end{aligned}$$

4. Now suppose μ is not necessarily σ -finite, but $p \in (1, \infty)$.
 $\forall E \subseteq X$ be σ -finite, we have

$$g_E \in L^q(E, \mu), \text{ such that } \Lambda(f) = \int_X f g_E d\mu, \forall f \in L^p(E, \mu)$$

In addition, $\|g_E\|_{L^q(\mu)} \leq \|\Lambda\|$.

Let $M := \sup_{E \text{ is } \sigma\text{-finite}} \|g_E\|_{L^q(\mu)} \leq \|\Lambda\|$.

Choose $(E_n)_{n=1}^\infty$ such that $\|g_{E_n}\|_{L^q(\mu)} \rightarrow M$.

Then $F := \bigcup_{n=1}^\infty E_n$ is σ -finite, and $\|g_F\|_{L^q(\mu)} = M$.

In addition, for any σ -finite $A \supseteq F$, we have $A \setminus F$ is σ -finite as well. Thus $g_A = g_F + g_{A \setminus F}$.

We have $g_{A \setminus F} = 0$ a.e., which means $g_A = g_F$ a.e..

Let $g := g_F \in L^q(\mu)$.

Given any $f \in L^p()$, let $A := \{x \in X : f(x) \neq 0\}$, which has to be σ -finite.

Thus $\Lambda(f) = \int_X g_A f d\mu = \int_X g f d\mu = \int_X g f d\mu$.

□

Remark. This is in general not true for $p = \infty$.

Remark. If μ is not σ -finite, it might be the case where $L^1(\mu) = \{0\}$, while $L^\infty(\mu) \neq \{0\}$.

8.2 Complex Regular Measure Space

Definition 8.1. Let $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ be a complex Borel measure on a locally compact Hausdorff space X , with its Jordan decomposition. We say μ is a **complex Radon measure** or **complex regular measure** if all μ_i are finite Radon measures.

Proposition 8.2. μ is a complex Radon measure if and only if $|\mu|$ is a Radon measure.

Proof. It follows $\mu_i \leq |\mu| \leq \mu_1 + \mu_2 + \mu_3 + \mu_4$. □

Definition 8.2. We define $M(X) := \{\mu : \text{complex Radon measure}\}$, and $\|\mu\|_{M(X)} := |\mu|(X)$

Proposition 8.3. $(M(X), \|\cdot\|_{M(X)})$ is a normed vector space over \mathbb{C} .

Definition 8.3. $C_0(X)$ is the closure of $C_c(X)$ in $\|\cdot\|_\infty$.

Definition 8.4. Let μ be a complex measure with $d\mu = h d|\mu|$, we define $\int_X f d\mu := \int_X f h d|\mu|$.

Theorem 8.4. (Jordan Decomposition for $C_0(X, \mathbb{R})$)

For any $\phi \in C_0(\mathbb{R})^*$, we have ϕ^+, ϕ^- positive bounded linear functionals, such that $\phi = \phi^+ - \phi^-$ on $C_0(\mathbb{R})$.

Proof. For any $f \geq 0$, let $\phi^+(f) := \sup\{\phi(g) : 0 \leq g \leq f\}$.

Notice that if $c \geq 0$, we have $\phi^+(cf) = c\phi^+(f)$.

In addition, $\forall f_1, f_2 \geq 0 \in C_0(\mathbb{R})$, and any $0 \leq g_1 \leq f_1, 0 \leq g_2 \leq f_2$, we have $0 \leq g_1 + g_2 \leq f_1 + f_2$.

Thus $\phi(g_1) + \phi(g_2) = \phi(g_1 + g_2) \leq \phi^+(f_1 + f_2)$. Since g_1, g_2 are arbitrary, we have $\phi^+(f_1) + \phi^+(f_2) \leq \phi^+(f_1 + f_2)$.

On the other hand, if we take any $g \leq f_1 + f_2$, and $g_1 := \min(g, f_1), g_2 := g - g_1$, we have $g_2 \leq g - f_1 \leq f_2$.

Thus $\phi(g) = \phi(g_1) + \phi(g_2) \leq \phi^+(f_1) + \phi^+(f_2)$.

Since g is arbitrary, $\phi^+(f_1 + f_2) \leq \phi^+(f_1) + \phi^+(f_2)$.

Thus,

$$\phi^+(f_1) + \phi^+(f_2) = \phi^+(f_1 + f_2).$$

Now extend ϕ^+ to $C_0(X, \mathbb{R})$ by $\phi^+(f) = \phi^+(f^+) - \phi^+(f^-)$.

This is well-defined. Indeed, if $f = g - h = f^+ - f^-$ for $g, h \geq 0$, we have $g + f^- = h + f^+$, and thus $\phi^+(g) + \phi^+(f^-) = \phi^+(h) + \phi^+(f^+)$.

We can also check that ϕ^+ is linear, and $|\phi^+(f)| \leq \|\phi\| \|f\|$.

Thus ϕ^+ is a bounded positive linear functional.

We will then define $\phi^- := \phi^+ - \phi$, and check it is also a bounded positive linear functional. □

Theorem 8.5. Let $\Lambda \in C_0(X)^*$, then $\exists!$ complex Radon measure $\mu \in M(X)$, such that

$$\forall f \in C_0(X), \Lambda(f) = \int_X f d\mu.$$

Moreover, $\|\Lambda\| = \|\mu\|_{M(X)} = |\mu|(X)$.

Proof. We first consider $C_0(X; \mathbb{R})$, which is a real Banach subspace of $C_0(X)$.

Let $\Psi := \Lambda|_{C_0(X; \mathbb{R})}$.

Now let $\Psi_1 := \Re(\Psi)$, $\Psi_2 := \Im(\Psi)$, we have that $\Psi_1, \Psi_2 \in C_0(X, \mathbb{R})^*$ over \mathbb{R} , with $\|\Psi_i\|_{C_0(X, \mathbb{R})^*} \leq \|\Lambda\|$. In addition,

$$\begin{aligned} \Lambda(f) &= \Lambda(\Re(f) + i\Im(f)) \\ &= \Lambda(\Re(f)) + i\Lambda(\Im(f)) \\ &= \Psi_1(\Re(f)) + i\Psi_2(\Re(f)) + i(\Psi_1(\Im(f)) + i\Psi_2(\Im(f))) \end{aligned}$$

is uniquely determined by Ψ_1, Ψ_2 .

Yet $\Psi_1 = \Psi_1^+ - \Psi_1^-$, $\Psi_2 = \Psi_2^+ - \Psi_2^-$, thus by Riesz-Markov-Kakutani, we have μ_i^\pm being finite Radon measures, such that $\Psi_i^\pm = \int_X f d\mu_i^\pm$.

Let $\mu := (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$, we have the result.

Now the uniqueness:

If $\Lambda = \Lambda_{\mu_1} = \Lambda_{\mu_2}$, we have $\forall f \in C_0(X)$,

$$\begin{aligned} 0 &= \int_X f d(\mu_1 - \mu_2) \\ &= \int_X f h d|\mu_1 - \mu_2|. \end{aligned}$$

By density of $C_0(X)$, it is also true for all $f \in L^1(X)$, so $|\mu_1 - \mu_2| = 0$. □

Corollary 8.6. $(M(X), \|\cdot\|_{M(X)}) \simeq C_0(X)^*$ isometrically.

9 Product Measures

Definition 9.1. Let $(X_i)_{i \in I}$ be a collection of non-empty sets, we define the **product** of the sets to be

$$X := \prod_{i \in I} X_i := \{(x_i)_{i \in I} | \forall i \in I, x_i \in X_i\} = \left\{ f : I \rightarrow \bigsqcup_{i \in I} X_i | \forall i \in I, f(i) \in X_i \right\}$$

Definition 9.2. We have a **canonical coordinate projections** $\pi : X \rightarrow X_i$ by $(x_j)_{j \in I} \mapsto x_i$.

Definition 9.3. If (X_i, \mathcal{M}_i) are measurable spaces, then the **product measurable space** is

$$\left(\prod_{i \in I} X_i, \bigotimes_{i \in I} \mathcal{M}_i \right),$$

where $\bigotimes_{i \in I} \mathcal{M}_i$ is the σ -algebra generated by the sets $\{\pi_i^{-1}(A) | i \in I, A \in \mathcal{M}_i\}$.

Remark. When I is finite, this is the same as tensor products generated by $A_1 \times A_2 \times \cdots \times A_n$.

Proposition 9.1. Let $(X_i, d_i)_{i=1}^n$ be separable metric spaces, then

$$\bigotimes_{i=1}^n \text{Bor}(X_i) = \text{Bor}\left(\prod_{i=1}^n X_i\right).$$

Proof. Given any open $U_i \subseteq X_i$, we must have $\pi_i^{-1}(U_i) \subseteq X$ is open. Thus $\bigotimes_{i=1}^n \text{Bor}(X_i) \subseteq \text{Bor}(\prod_{i=1}^n X_i)$. On the other hand, each X_i is separable, so X is also separable. Thus X is second countable. If $(x_n)_{n=1}^\infty$ is a dense sequence in X , then

$$\{B_r(x_n) | n \in \mathbb{N}, r \in \mathbb{Q}^{++}\}$$

is a basis for the topology. Namely, every open set can be written as a countable union of these open balls.

Setting $x_n^i := \pi_i(x_n)$, we have that $B_r(x_n) = \prod_{i=1}^n B_r(x_n^i)$, which is a subset of $\bigotimes_{i=1}^n \text{Bor}(X_i)$.

Since every open set $U \subseteq X$ is a countable union of these sets, so $U \subseteq \bigotimes_{i=1}^n \text{Bor}(X_i)$.

Thus $\bigotimes_{i=1}^n \text{Bor}(X_i) \supseteq \text{Bor}(\prod_{i=1}^n X_i)$. □

Corollary 9.2.

$$\text{Bor}(\mathbb{R}^n) = \bigotimes_{i=1}^n \text{Bor}(\mathbb{R})$$

Proposition 9.3. Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be measure spaces. Let R be the collection of all finite unions of disjoint rectangles $A \times B$ with $A \in \mathcal{M}, B \in \mathcal{N}$. Then R is an algebra of subsets of $X \times Y$

Proof.

$$\begin{aligned} (A \times B)^c &= (A^c \times Y) \sqcup (A \times B^c) \\ (A_1 \times B_1) \cup (A_2 \times B_2) &= (A_1 \times B_1) \sqcup ((A_2 \setminus A_1) \times B_2) \sqcup ((A_2 \setminus A_1) \times (B_1 \setminus B_2)) \end{aligned}$$

□

Proposition 9.4. The σ -algebra generated by R is $\mathcal{M} \otimes \mathcal{N}$.

Definition 9.4. We can define a function $\pi : R \rightarrow [0, \infty]$ by $\pi(\bigsqcup_{i=1}^n A_i \times B_i) := \sum_{i=1}^n \mu(A_i)\nu(B_i)$

Lemma 9.5. π is a premeasure.

Proof. Firstly, $\pi(\emptyset) = \mu(\emptyset) \times \nu(\emptyset) = 0$.

Secondly, we consider any $A \times B = \bigsqcup_{n \in \mathbb{N}} (A_n \times B_n) \subseteq R$, where $(A_n \times B_n)_{n \in \mathbb{N}} \subseteq R$.

Fix any $y \in Y$, we have that $\chi_A(x)\chi_B(y) = \sum_{n \in \mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y)$, which is a sum of non-negative measurable functions on X . By LMCT, we have that

$$\begin{aligned} \mu(A)\chi_B(y) &= \int_X \chi_A(x) d\mu \chi_B(y) \\ &= \int_X \chi_A(x) \chi_B(y) d\mu \\ &= \int_X \sum_{n \in \mathbb{N}} \chi_{A_n}(x) \chi_{B_n}(y) d\mu \\ &= \sum_{n \in \mathbb{N}} \int_X \chi_{A_n}(x) \chi_{B_n}(y) d\mu \\ &= \sum_{n \in \mathbb{N}} \int_X \chi_{A_n}(x) d\mu \chi_{B_n}(y) \\ &= \sum_{n \in \mathbb{N}} \mu(A_n) \chi_{B_n}(y). \end{aligned}$$

In addition, $\sum_{n \in \mathbb{N}} \mu(A_n) \chi_{B_n}(y)$ is a sum of non-negative measurable functions on Y . By LMCT, we again

have that

$$\begin{aligned}
\mu(A)\nu(B) &= \mu(A) \int_Y \chi_B(y) d\nu \\
&= \int_Y \mu(A) \chi_B(y) d\nu \\
&= \int_Y \sum_{n \in \mathbb{N}} \mu(A_n) \chi_{B_n}(y) d\nu \\
&= \sum_{n \in \mathbb{N}} \int_Y \mu(A_n) \chi_{B_n}(y) d\nu \\
&= \sum_{n \in \mathbb{N}} \mu(A_n) \int_Y \chi_{B_n}(y) d\nu \\
&= \sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n).
\end{aligned}$$

This will now extend to any $\bigsqcup_{n \in \mathbb{N}} (A_n \times B_n) \subseteq R$, by finite additivity. \square

Theorem 9.6. *There is a complete measure space $(X \times X, \overline{\mathcal{M} \otimes \mathcal{N}}, \mu \times \nu)$, such that $\mu \times \nu(A \times B) = \mu(A) \times \nu(B)$.*

Proof. Apply Caratheodory on the above lemma. \square

For the flowing, let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces.

Definition 9.5. Take R as before, let $R_\sigma := \left\{ \bigcup_{n \geq 1} A_n \mid A_n \in R \right\}, R_{\sigma\delta} := \left\{ \bigcap_{n \geq 1} E_n \mid E_n \in R_\sigma \right\}$

Lemma 9.7. *If $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$, with $\mu \times \nu(E) < \infty$, then $\exists G \in R_{\sigma\delta}$, such that $E \subseteq G, \mu \times \nu(G \setminus E) = 0$*

Proof. We have $\mu \times \nu(E) = \inf \left\{ \sum_{i \geq 1} \mu \times \nu(A_i) \mid A_i \in R, E \subseteq \bigcup_{i \geq 1} A_i \right\}$.

Let $E_j := \bigcup_{i \geq 1} A_{ji} \supseteq E$, with $\mu \times \nu(E_j) < \mu \times \nu(E) + \frac{1}{j}$.

Notice that $E_j \in R_\sigma$ by construction.

Now take $G = \bigcap_{j \geq 1} E_j \in R_{\sigma\delta}$. Then we have that $E \subseteq G$, and $\forall j, \mu \times \nu(G) \leq \mu \times \nu(E_j) < \mu \times \nu(E) + \frac{1}{j}$. Thus $\mu \times \nu(G) = \mu \times \nu(E)$. \square

Lemma 9.8. *Let $E \in R_{\sigma\delta}$, with $\mu \times \nu(E) < \infty$. Let $E_x = \{y \in Y \mid (x, y) \in E\}, E^y = \{x \in X \mid (x, y) \in E\}$. Define $g(x) := \nu(E_x), h(y) := \mu(E^y)$. Then we have g is non-negative and μ -measurable, $g \in \mathcal{L}^1(\mu), \int_X g d\mu = \mu \times \nu(E)$. Similarly, h is non-negative and ν -measurable, $h \in \mathcal{L}^1(\nu), \int_Y g d\nu = \mu \times \nu(E)$*

Proof. If $E = A \times B$, with $A \in \mathcal{M}, B \in \mathcal{N}$, then $E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$.

Then $g(x) = \nu(B)\chi_A$ is μ -measurable, and $g \geq 0$. Moreover,

$$\int_X g d\mu = \int_X \nu(B)\chi_A d\mu = \nu(B) \int_X \chi_A d\mu = \mu(A)\nu(B) = \mu \times \nu(A \times B).$$

Now suppose $E = \bigcup_{i \geq 1} A_i \times B_i \in R_\delta$, with $A_i \in \mathcal{M}, B_i \in \mathcal{N}$. WLOG, we can take $E = \bigsqcup_{i \geq 1} A_i \times B_i$.

Let $g_i(x) = \nu(B_i)\chi_{A_i}(x)$, we have $\sum_{i=1}^n g_i(x) = \sum_{i=1}^n \nu(B_i)\chi_{A_i}(x) = \begin{cases} \nu(B_i) = \nu(E_x) & \text{if } x \in A_i \\ 0 & \text{if } x \notin \bigsqcup_{i=1}^n A_i \end{cases}$.

Thus $g(x) = \sum_{i=1}^{\infty} g_i(x)$ is measurable. By LMCT, $g \in \mathcal{L}^1(\mu)$, and

$$\begin{aligned}
\int_X g d\mu &= \sum_{i=1}^{\infty} \int_X g_i d\mu \\
&= \sum_{i=1}^{\infty} \int_X \nu(B_i) \chi_{A_i} d\mu \\
&= \sum_{i=1}^{\infty} \nu(B_i) \mu(A_i) \\
&= \sum_{i=1}^{\infty} \mu \times \nu(A_i \times B_i) \\
&= \mu \times \nu \left(\bigsqcup_{i=1}^{\infty} A_i \times B_i \right) \\
&= \mu \times \nu(E).
\end{aligned}$$

Now take $E = \bigcap_{i \geq 1} E_i \in R_{\delta\sigma}$ with $E_i \in R_{\delta}$. WLOG, we can take $E_i \supseteq E_{i+1}$. Notice that $(E_i)_x = \{y \in Y \mid (x, y) \in E_i\} \supseteq \{y \in Y \mid (x, y) \in E_{i+1}\} = (E_{i+1})_x \supseteq \cdots \supseteq E_x$. Let $g_i(x) = \nu((E_i)_x) = \mu \times \nu(E_i)$, then we have $0 \leq g \leq \cdots \leq g_i \leq \cdots \leq g_1$. In addition, $E_x = \bigcap_{i \geq 1} (E_i)_x$, and thus $g(x) = \lim_{i \rightarrow \infty} g_i(x)$ by continuity of ν . Thus g is μ -measurable, and since g_i are all dominated by g_1 , we can use LDCT to get

$$\begin{aligned}
\int_X g d\mu &= \lim_{i \rightarrow \infty} \int_X g_i d\mu \\
&= \lim_{i \rightarrow \infty} \mu \times \nu(E_i) \\
&= \mu \times \nu(E).
\end{aligned}$$

□

Lemma 9.9. Let $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$ with $\mu \times \nu(E) = 0$, then for μ -a.e. $x \in X$, we have $\nu(E_x) = 0$; for ν -a.e. $y \in Y$, we have $\mu(E_y) = 0$.

Proof. We have some $G \in R_{\sigma\delta}$, such that $E \subseteq G, \mu \times \nu(G \setminus E) = 0$. Let $f(x) := \nu(G_x)$, we have $f \in \mathcal{L}^1(\mathcal{M})$ is nonnegative. Yet $\int_X f d\mu = 0$, and thus $f(x) = 0$ for μ -a.e. $x \in X$. Since $E_x \subseteq G_x$, and that ν is complete, we have that $g(x) = \nu(E_x) = 0$ for μ -a.e. $x \in X$. □

Corollary 9.10. Let $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$ with $\mu \times \nu(E) < \infty$, then E_x is ν -measurable, for μ -a.e. $x \in X$, and $g(x) = \nu(E_x)$ is μ -measurable, with $g \geq 0, g \in \mathcal{L}^1(\mathcal{M})$, and $\int_X g d\mu = \mu \times \nu(E)$.

Theorem 9.11. (Fubini's) Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces. Take $f \in \mathcal{L}^1(\mu \times \nu)$, then

1. For μ -a.e. $x \in X, f_x := f(x, \cdot) \in \mathcal{L}^1(\nu)$.
2. For ν -a.e. $y \in Y, f_y := f(\cdot, y) \in \mathcal{L}^1(\mu)$.
3. $F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^1(\mu)$.
4. $G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^1(\nu)$.
5. $\int_{X \times Y} f d(\mu \times \nu) = \int_X (F(x)) d\mu = \int_Y (G(y)) d\nu$

Proof. Notice that $f^1 \in \mathcal{L}^1$ means that $f = f_1 - f_2 + if_3 - if_4$, where $f_i \geq 0, f_i \in \mathcal{L}^1$. We first show the theorem holds for $f \geq 0, f \in \mathcal{L}^1$. There are simple functions $0 \leq s_1 \leq \cdots \leq s_n \leq \cdots \leq f$, such that $f(x) = \lim_{n \rightarrow \infty} s_n(x)$. Let $F_n(x) = \int_Y s_n(x, y) d\nu \geq 0$ be measurable and \mathcal{L}^1 . We have that □

Theorem 9.12. (Tonelli's) Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces. Take $f \in \mathcal{L}^+(\mu \times \nu)$, and $\mu \times \nu$ is σ -finite, then

1. For μ -a.e. $x \in X$, $f_x := f(x, \cdot) \in \mathcal{L}^+(\nu)$.
2. For ν -a.e. $y \in Y$, $f_y := f(\cdot, y) \in \mathcal{L}^+(\mu)$.
3. $F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^+(\mu)$.
4. $G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^+(\nu)$.
5. $\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) d\nu \right) d\mu = \int_Y \left(\int_X f(x, y) d\mu \right) d\nu$

Proof. $\mu \times \nu$ is σ -finite, thus $\exists C_1 \subseteq C_2 \subseteq \dots$, with $C_n \in \overline{\mathcal{M} \otimes \mathcal{N}}$, $X \times Y = \bigcup_{n=1}^{\infty} C_n$, and $\mu \times \nu(C_n) < \infty$. Let $f_n(x) = \max\{f(x), n\} \chi_{C_n}(x)$, we have $0 \leq f_n \leq n \chi_{C_n}$, and $f_n \in \mathcal{L}^+ \cap \mathcal{L}^1(\mu \times \nu)$, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

$$\begin{aligned} \int f d\mu \times \nu &= \lim_{n \rightarrow \infty} \int f_n d\mu \times \nu \\ &= \lim_{n \rightarrow \infty} \int_X \int_Y f_n(x, y) d\nu d\mu \\ &=: \lim_{n \rightarrow \infty} \int_X F_n(x) d\mu. \end{aligned}$$

Then F_n are measurable, non-negative, and monotone increasing to $F(x) := \int_Y f(x, y) d\nu$. By LMCT, we have F is measurable, and

$$\begin{aligned} \int f d\mu \times \nu &= \lim_{n \rightarrow \infty} \int_X F_n(x) d\mu \\ &= \int_X F(x) d\mu \\ &= \int_X \int_Y f(x, y) d\nu d\mu \end{aligned}$$

□

Remark. If $f \in \mathcal{L}^1$, we get σ -finite by free on $C = \text{Supp}(f)$ if we look at $C_n := \{(x, y) : |f(x, y)| \geq \frac{1}{n}\}$. Notice that $\mu \times \nu(C_n) \leq n \int |f| d\mu \times \nu < \infty$.

Example 9.0.1. Consider $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, with the counting measure m_c .

$$\text{Consider } f(m, n) := \begin{cases} 1 & n = m \\ -1 & n = m + 1 \\ 0 & \text{o.w.} \end{cases} \sim \begin{pmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ \vdots & 0 & 1 & -1 & \dots \\ \vdots & \vdots & \ddots & & \end{pmatrix}.$$

However,

$$\begin{aligned} \int_X \int_Y f(x, y) dm_c(n) dm_c(m) &= \sum_{m \geq 1} \sum_{n \geq 1} f(m, n) \\ &= \sum_{m \geq 1} 0 \\ &= 0, \\ \int_Y \int_X f(x, y) dm_c(m) dm_c(n) &= \sum_{n \geq 1} \sum_{m \geq 1} f(m, n) \\ &= 1 + \sum_{n \geq 2} 0 \\ &= 1. \end{aligned}$$

This is because $f \notin \mathcal{L}^1$.

Example 9.0.2. Consider $X = Y = [0, 1]$, and the Lebesgue measure.

Take $t_n = 1 - \frac{1}{n}$.

Define $g_n : [0, 1] \rightarrow \mathbb{R}$ by starting at $\frac{2t_n}{3} + \frac{t_{n+1}}{3}$, linear and reach $\frac{t_{n+1}-t_n}{3}$ at mid point, and decrease linearly to 0 at $\frac{t_n}{3} + \frac{2t_{n+1}}{3}$, and 0 outside. We thus have $\int g_n(x)dx = 1$.

Define $f(x, y) = \sum_{i=1}^{\infty} (g_n(x) - g_{n+1}(x))g_n(y)$, where only one of these summands will be non-zero in each interval of x . Actually $f(x, y)$ is continuous $\forall (x, y) \neq (1, 1)$.

However, $\int f(x, y)dx = g_n(y)$, and thus $\int \int f(x, y)dxdy = 1$, while $\int \int f(x, y)dydx = 0$.

Theorem 9.13. Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be (not necessarily complete) measure spaces. Then Fubini and Tonelli still apply with restriction to $\mathcal{M} \otimes \mathcal{N}$.

9.1 Lebesgue Measure on \mathbb{R}^n

Lemma 9.14. Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be Lebesgue measurable, then there is a G_δ set $G \subseteq \mathbb{R}^n$, such that $\lambda^n(G) = 0$, and $g = f\chi_{G^c}$ be Borel measurable, and $f = g$ λ^n -a.e..

Proof. By writing $f = f_1 - f_2 + if_3 - if_4$ for $f_i \geq 0$, we can assume $f \geq 0$. We first consider $n = 1$.

Choose a dense subset $\{r_i\}_{i \in \mathbb{N}}$ of $[0, \infty)$. Let $A_i = f^{-1}([0, r_n])$. Since f is Lebesgue measurable, $A_n \in \mathcal{L}$. By regularity for the Lebesgue measure, there is an F_δ set $F_i \subseteq A_n$ and a null set $N_i = A_i \setminus F_i$. Let $N = \bigcup_{i \in \mathbb{N}} N_i$, then N is a null set.

Applying regularity again, there is a G_δ set $G \supseteq N$ such that $\lambda(G) = 0$.

Let $g = f\chi_{G^c}$. we have $g^{-1}([0, r_i]) = f^{-1}([0, r_i]) \cup G = A_i \cup G = (F_i \cup N_n) \cap G = F_i \cup G$, which is a union of two Borel sets, and thus Borel.

To verify that g is Borel, it surfaces to prove $g^{-1}([0, r])$ is Borel for all $r > 0$. By density of $\{r_i\}$, there is a sequence r_{n_k} such that $r_{n_k} \leq r$ and $r_{n_k} \rightarrow r$. Thus $\bigcup_{k \geq 1} [0, r_{n_k}) = [0, r)$, so $g^{-1}([0, r)) = \bigcup_{k \geq 1} g^{-1}([0, r_{n_k}))$ is a union of Borel sets, and thus Borel.

By construction, G is a null set and $f|_{G^c} = g|_{G^c}$, so $f = g$ λ -a.e..

Now suppose $n \geq 1$. For each i , let f_{x_i} be the function obtained by fixing all but the i^{th} variable x_i . From above we can find G_δ set $G_i \subseteq \mathbb{R}$, such that $f_{x_i} = f_{x_i}\chi_{G_i^c}$ λ -a.e..

Let $G = (G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup (\mathbb{R} \times G_2 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup \cdots (\mathbb{R} \times \cdots \times \mathbb{R} \times G_n)$.

Then $G^c = G_1^c \times G_2^c \times \cdots \times G_n^c$. Let $G_1 = \bigcap_k U_{1k}$, then $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R} = \bigcap_k (U_{1k} \times \mathbb{R} \times \cdots \times \mathbb{R})$, where each is open, since U_{1k}, \mathbb{R} are open. Thus $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}$ is a G_δ set. Thus G is a finite union of G_δ sets, which is G_δ . \square

Definition 9.6. For $A \in \mathcal{L}^n, X \in \mathbb{R}^n$, write the **translation** of A by x as $A + x = \{a + x : a \in A\}$.

Definition 9.7. Let GLn be the set of invertible $n \times n$ matrices.

Theorem 9.15. Consider the Lebesgue measure λ^n in \mathbb{R}^n .

1. (translation) For $A \in \mathcal{L}^n$ and $x \in \mathbb{R}^n$, we have $A + x \in \mathcal{L}^n, \lambda^n(A + x) = \lambda^n(A)$.
2. (scaling) For $T \in GLn, f : \mathbb{R}^n \rightarrow \mathbb{C}$ be Lebesgue measurable, $f \circ T$ is Lebesgue measurable, and

$$\int f d\lambda^n = |\det(T)| \int (f \circ T) d\lambda^n.$$

In particular, for $A \in \mathcal{L}^n$, we have $\lambda^n(T(A)) = |\det(T)|\lambda^n(A)$.

3. (rotation) For a unitary $U \in GLn$, we have

$$\int (f \circ U) d\lambda^n = \int f d\lambda^n,$$

and $\forall A \in \mathcal{L}^n, \lambda^n(U(A)) = \lambda^n(A)$.

Proof. 1.

2. Notice that $x \in T(A) \iff T^{-1}x \in A$, thus $\chi_{T(A)} = \chi_A \circ T^{-1}$. Thus

$$\begin{aligned}\lambda^n(T(A)) &= \int \chi_{T(A)} d\lambda^n \\ &= \int \chi_A \circ T^{-1} d\lambda^n \\ &= \frac{1}{|\det(T^{-1})|} \int \chi_A d\lambda^n \\ &= |\det(T)| \lambda^n(A).\end{aligned}$$

□

10 Convolutions and Fourier Transforms

Definition 10.1. For $y \in \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{C}$, we define the **translation** of f by y to be $L_y f(x) := f(x - y)$.

Proposition 10.1. We have $L_y : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ is linear, isometric, and $\forall f \in L^1(\mathbb{R})$, we have

$$\lim_{y \rightarrow 0} \|L_y f - f\|_1 = 0.$$

Proof. If $f \in C_c(\mathbb{R})$, then it is uniformly continuous, so

$$\lim_{y \rightarrow 0} \|L_y f - f\|_\infty = 0.$$

Take compact $K \supseteq \text{Supp}(f)$, we have that

$$\begin{aligned}\|L_y f - f\|_1 &= \int_{K \cup (K+y)} |f(x - y) - f(x)| dx \\ &\leq \lambda(K \cup (K + y)) \|L_y f - f\|_\infty,\end{aligned}$$

where the first term is bounded by $2\lambda(K) < \infty$, and the second term goes to 0.

Now since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, we have the result by triangle inequality. □

Theorem 10.2. (*Young's Convolution Inequality*)

Consider $X = \mathbb{R}$, with Lebesgue measure λ . Let $f, g \in L^1(\mathbb{R})$, then for a.e. $x \in \mathbb{R}$, the function $y \mapsto f(x - y)g(y)$ is in $L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$, and the **convolution**

$$f * g(x) := \int_{\mathbb{R}} f(x - y)g(y)dy$$

is also in $L^1(\mathbb{R})$. In addition, $\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$.

Proof. Consider the function $F : (x, y) \mapsto f(x - y)g(y)$, which is a measurable function on $\mathbb{R} \times \mathbb{R}$ (can show with approximation by $C_c(\mathbb{R})$ functions).

By Tonelli's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| d\lambda^2 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x, y)| dx \right) dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x - y)| |g(y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |f(x - y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \|f\|_{L^1(\mathbb{R})} dy \\
&= \|f\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} |g(y)| dy \\
&= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \\
&< \infty.
\end{aligned}$$

Thus, $F \in L^1(\mathbb{R}^2)$.

Now we apply Fubini's Theorem to F , and get $F_x(y) = f(x - y)g(y) \in L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$. In addition,

$$\begin{aligned}
\|f * g\|_{L^1(U)} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x - y)g(y) dy \right| dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x - y)g(y)| dy dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |F(x, y)| dy dx \\
&= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.
\end{aligned}$$

□

Corollary 10.3. $(L^1(\mathbb{R}), *)$ defines a commutative associative algebra.

Definition 10.2. Given $f \in L^1(\mathbb{R})$, its **Fourier Transform** is $\mathcal{F}(f) := \hat{f} : \mathbb{R} \rightarrow \mathbb{C}$, where $\hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$.

Lemma 10.4 (Riemann-Lebesgue). $\forall f \in L^1(\mathbb{R})$, we have $\hat{f} \in C_0(\mathbb{R})$, and $\|\hat{f}\|_{\infty} \leq \|f\|_{L^1(\mathbb{R})}$. Namely, \mathcal{F} is a contraction map.

Proof. Consider any sequence $(\omega_n)_{n=1}^{\infty} \subset \mathbb{R}$ that converges to $\omega \in \mathbb{R}$.

Let $h_n(x) := f(x)(e^{i\omega_n x} - e^{i\omega x})$, we have that $h_n \in L^1(\mathbb{R})$, $h_n(x) \rightarrow 0$ pointwise for a.e. $x \in \mathbb{R}$, and $|h_n| \leq |f| |e^{i\omega_n x} - e^{i\omega x}| \leq 2|f|$.

In addition,

$$\begin{aligned}
\hat{f}(\omega_n) - \hat{f}(\omega) &= \int_{\mathbb{R}} f(x)(e^{i\omega_n x} - e^{i\omega x}) dx \\
&= \int_{\mathbb{R}} h_n(x) dx
\end{aligned}$$

By LDCT, we have that $\lim_{n \rightarrow \infty} (\hat{f}(\omega_n) - \hat{f}(\omega)) = 0$, so \hat{f} is continuous.

In addition,

$$\begin{aligned}
|\hat{f}(\omega)| &\leq \int_{\mathbb{R}} |f(x)| |e^{i\omega x}| dx \\
&= \int_{\mathbb{R}} |f(x)| dx \\
&= \|f\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Now

$$\begin{aligned}
\hat{f}(\omega) &= \int_{\mathbb{R}} f(x) e^{-ix\omega} dx \\
&= - \int_{\mathbb{R}} f(x) e^{-ix\omega + \pi i} dx \\
&= - \int_{\mathbb{R}} f(x) e^{-i\omega(x - \pi/\omega)} dx \\
&= - \int_{\mathbb{R}} f(z + \pi/\omega) e^{-i\omega z} dz \\
&= - \int_{\mathbb{R}} L_{-\pi/\omega} f(z) e^{-i\omega z} dz \\
2\hat{f}(\omega) &= \int_{\mathbb{R}} f(x) e^{-ix\omega} dx - \int_{\mathbb{R}} L_{-\pi/\omega} f(z) e^{-i\omega z} dz \\
&= \int_{\mathbb{R}} (f - L_{-\pi/\omega} f)(x) e^{-i\omega x} dx \\
&= \mathcal{F}(f - L_{-\pi/\omega} f)(\omega) \\
2|\hat{f}(\omega)| &\leq \|f - L_{-\pi/\omega} f\|_{L^1(\mathbb{R})},
\end{aligned}$$

which goes to 0 when $\omega \rightarrow \infty$.

Thus, $\hat{f} \in C_0(\mathbb{R})$. □

Theorem 10.5. ($L^1(\mathbb{R})$ Inversion)

If $f, \hat{f} \in L^1(\mathbb{R})$, we have

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{ix\omega} d\omega$$

for a.e. $x \in \mathbb{R}$.

In particular, such f must be almost everywhere equal to a continuous function.

Proof. Let $\lambda > 0$, and $H_\lambda(\omega) := e^{-\lambda|\omega|}$.

Let

$$\begin{aligned}
h_\lambda(x) &:= \frac{1}{2\pi} \int_{\mathbb{R}} H_\lambda(\omega) e^{ix\omega} d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\omega - \lambda|\omega|} d\omega \\
&= \frac{\lambda}{\pi} \frac{1}{x^2 + \lambda^2}.
\end{aligned}$$

Fix $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
(f * h_\lambda)(x) &= \int_{\mathbb{R}} f(x - y) h_\lambda(y) dy \\
&= \int_{\mathbb{R}} f(x - y) \frac{1}{2\pi} \int_{\mathbb{R}} H_\lambda(\omega) e^{iy\omega} d\omega dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x - y) H_\lambda(\omega) e^{iy\omega} d\omega dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x - y) H_\lambda(\omega) e^{iy\omega} dy d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) H_\lambda(\omega) e^{i\omega(x - z)} dz d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} H_\lambda(\omega) \hat{f}(\omega) e^{i\omega(x)} dz d\omega.
\end{aligned}$$

Notice that $H_\lambda(\omega) = 1$ as $\lambda \rightarrow 0$, and $f * h_\lambda \rightarrow f$.

If $\hat{f} \in L^1(\mathbb{R})$, we can use DCT to get the result. \square

Corollary 10.6. *If $f, g \in L^1(\mathbb{R})$, and $\mathcal{F}(f) = \mathcal{F}(g)$, we must have $\mathcal{F}(f - g) = 0 \in L^1(\mathbb{R})$. Thus, $f = g$ a.e. $x \in \mathbb{R}$.*

Remark. Not all $\hat{f} \in L^1(\mathbb{R})$.

Example 10.0.1. If $f = \chi_{[-1,1]}$, we have $\hat{f} = \frac{2\sin(\omega)}{\omega} \in C_0(\mathbb{R}) \setminus L^1(\mathbb{R})$.

11 Bochner Spaces

Definition 11.1. Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \rightarrow B$ is **weakly measurable** if $\forall \Lambda \in B^*$, $\Lambda \circ f : X \rightarrow \mathbb{C}$ is measurable.

Definition 11.2. Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \rightarrow B$ is **Bochner measurable** or **strongly measurable** if $f(x) = g(x)$ for μ -a.e. $x \in X$, for some measurable g , with $\text{Im}(g) \subseteq B$ being separable.

Proposition 11.1. *Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \rightarrow B$ is strongly measurable if $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for μ -a.e. $x \in X$, for some sequence of measurable functions f_n , each with countable range.*

Definition 11.3. Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $s : X \rightarrow [0, \infty)$ be a simple measurable function, with $s(X) = \{a_1, \dots, a_n\} \subset B$, such that

$$s = \sum_{i=1}^n a_i \chi_{A_i},$$

where $A_i := s^{-1}(\{a_i\})$. For $A \in \mathcal{M}$, we say s is integrable over A if $\forall a_i \neq 0, \mu(A_i \cap A) < \infty$, and define the **integral** of s over A to be

$$\int_A s d\mu := \sum_{i=1}^n a_i \mu(A_i \cap A).$$

Definition 11.4. Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $f : X \rightarrow [0, \infty)$ be a measurable function. If there is a sequence of simple integrable functions $(s_n)_{n=1}^\infty$ such that

$$\lim_{n \rightarrow \infty} \int_A |f - s_n| d\mu = 0,$$

then we say f is **Bochner integrable**, and we define the **Bochner integral** to be

$$\int_A f d\mu := \lim_{n \rightarrow \infty} \int_A s_n d\mu.$$

Lemma 11.2. *The right hand side of the above definition always exists, and is independent of the choice of the sequence of simple integrable functions $(s_n)_{n=1}^\infty$. Thus, the above definition is well-defined.*

Theorem 11.3 (Bochner). *Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space. A strongly measurable function $f : X \rightarrow B$ is Bochner integrable if and only if $x \mapsto \|f(x)\|_B$ is integrable. In this case, $\forall E \in \mathcal{M}$,*

$$\left\| \int_E f(x) dx \right\|_B \leq \int_E \|f(x)\|_B dx,$$

$$\forall \Lambda \in B^*, \quad \Lambda \left(\int_E f(x) dx \right) = \int_E \Lambda(f(x)) dx.$$

Theorem 11.4 (Dominated Convergence Theorem for Bochner integral). *Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space. Let $f_n : X \rightarrow \mathbb{C}$ be measurable functions, defined μ -a.e. on X , such that $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ is defined μ -almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that for μ -a.e. $x \in X, \forall n \in \mathbb{N}, \|f_n(x)\|_B \leq g(x)$, then f is Bochner integrable, and*

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n \rightarrow \infty} \int_X \|f - f_n\|_B d\mu = 0.$$

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

Definition 11.5. Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $1 \leq p < \infty$, we define

$$\mathcal{L}^p(\mu, B) := \left\{ f : X \rightarrow B \mid f \text{ is measurable, } \int_X \|f\|_B^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p(\mu, B)} := \left(\int_X \|f\|_B^p d\mu \right)^{\frac{1}{p}}.$$

Definition 11.6. Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space, we define

$$\mathcal{L}^\infty(\mu, B) := \{ f : X \rightarrow B \mid f \text{ is measurable, } \text{ess sup } \|f\|_B < \infty \}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty(\mu, B)} := \text{ess sup } \|f\|_B.$$

Definition 11.7. Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space. For any $p \in [1, \infty]$, we define

$$L^p(\mu, B) := \mathcal{L}^p(\mu, B)/N,$$

where $N := \{ f : X \rightarrow B \mid f \text{ is measurable, } f = 0 \text{ } \mu\text{-a.e.} \}$. Namely, $[f] \in L^p(\mu, B)$ is the equivalence class of all $g = f$ μ -a.e. for $f \in \mathcal{L}^p(\mu, B)$.

In addition, we define

$$\|[f]\|_{L^p(\mu, B)} := \|f\|_{\mathcal{L}^p(\mu, B)}$$

for any representative f .

Theorem 11.5 (Fischer-Riesz-Bochner). *Let (X, \mathcal{M}, μ) be a measure space, $(B, \|\cdot\|)$ be a Banach Space. For all $1 \leq p \leq \infty$, we have that $(L^p(\mu, B), \|\cdot\|_{L^p(\mu, B)})$ is a Banach Space.*