Pmath 651: Measure Theory

Roger Gu

December 1, 2025

This note is based on the course Pmath641 in Fall 2024 and Winter 2025.

It mainly serves for my personal learning purposes, and may not follow the official syllabus. I may include related topics or details that are not included in the course, and I may not cover all the topics taught in the course.

Please do not use them for any purposes other than learning, and please do not directly copy from them for your assignments and/or tests.

Contents

1 Introductions

1.1 Lebesgue Measure

Definition 1.1. Lebesgue outer measure of $A \in \mathbb{R}$ is $\lambda^*(A) := \inf\{\sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i\}$, where each $I_i \subseteq \mathbb{R}$ is an open interval.

Definition 1.2. If $\forall E \in \mathbb{R}, \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$, then A is Lebesgue measurable, and its Lebesgue measure is defined to be $\lambda(A) := \lambda^*(A)$

Proposition 1.1. $\forall a < b \in \mathbb{R}, \lambda((a,b)) = b - a$

Proposition 1.2. $\forall x \in \mathbb{R}, \lambda(x + A) = \lambda(A)$

Proposition 1.3. *If A_m are $\mathcal{L}-$measurable and pairwise disjoint ($A_m \cap A_n = \emptyset, \forall n \neq m$), then $m(\sqcup_{i\geq 1} A_i) = \sum_{i=1}^{\infty} m(A_i)$*

Proposition 1.4. *Every Riemann integrable function is Lebesgue integrable.*

2 Measure

2.1 Algebra of Sets

Definition 2.1. Let X be a set and $\mathcal{P}(X) := \{A | A \subseteq X\}$, then an algebra of subsets of X is $\mathcal{A} \subseteq \mathcal{P}(X)$, such that

1. $\emptyset \in \mathcal{A}$
2. If $E \in \mathcal{A}$, then $E^c := X \setminus E \in \mathcal{A}$
3. If $E_1, \dots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} E_i \in \mathcal{A}$

Definition 2.2. Let X be a set and $\mathcal{P}(X) := \{A | A \subseteq X\}$, then a $\sigma-$algebra of subsets of X is $\mathcal{M} \subseteq \mathcal{P}(X)$, such that

1. $\emptyset \in \mathcal{M}$
2. If $E \in \mathcal{M}$, then $E^c := X \setminus E \in \mathcal{M}$
3. If $E_1, E_2, \dots \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$

Definition 2.3. If $\mathcal{M}$ is $\sigma-$algebra, we call $(X, \mathcal{M})$ a measurable space, and a set $E \in \mathcal{M}$ is called $\mathcal{M}$-measurable.

Remark. Every $\sigma-$algebra is an algebra.

Proposition 2.1. *If $\mathcal{A}$ is an algebra, and $E_1, E_2 \in \mathcal{A}$, then $E_1 \cap E_2 \in \mathcal{A}$*

Proof. $(E_1 \cap E_2)^c = E_1^c \cup E_2^c$ is in $\mathcal{A}$ by 2,3. □

Proposition 2.2. *If $\mathcal{A}$ is an algebra, and $E, F \in \mathcal{A}$, then $E \setminus F = E \cap F^c \in \mathcal{A}$.*

Proposition 2.3. *If $\mathcal{A}$ is an algebra, and $E, F \in \mathcal{A}$, then $E \Delta F = (E \setminus F) \cup (F \setminus E) \in \mathcal{A}$.*

Proposition 2.4. *If $\mathcal{M}$ is $\sigma-$algebra, $E_i \in \mathcal{M}$, then we can define $F_i := E_i \setminus \bigcup_{j=1}^{i-1} E_i$, and $\bigcup_{i=1}^{\infty} E_i = \bigsqcup_{i=1}^{\infty} F_i$*

Proposition 2.5. *If $\mathcal{M}$ is $(\sigma-)$algebra, and $E \in \mathcal{M}$, then $A|_E := \{E \cap A | A \in \mathcal{M}\}$ is an $(\sigma-)$algebra.*

Example 2.1.1. $P(X)$ is $\sigma - algebra$, and $\{\emptyset, X\}$ is $\sigma - algebra$.

Example 2.1.2. $\mathcal{A} = \{E \subseteq X : |E| < \infty \vee |E^c| < \infty\}$ is an algebra. However, if X is infinite, then it is not a $\sigma - algebra$

Example 2.1.3. $\mathcal{M} = \{E \subseteq X : |E| \leq \mathcal{N}_0 \vee |E^c| \leq \mathcal{N}_0\}$ is a $\sigma-$algebra.

Example 2.1.4. Let $X = \mathbb{R}$, the collection of all finite union of sets in $\{\mathbb{R}, (-\infty, b], (a, b], (a, \infty) | a, b \in \mathbb{R}\}$ is an algebra but not $\sigma-$algebra.

Proposition 2.6. *Let $\{\mathcal{M}_\alpha\}_{\alpha \in I}$ is a collection of ($\sigma-$)algebras of X, then $\bigcap_{\alpha \in I} \mathcal{M}_\alpha$ is an ($\sigma-$)algebra*

Definition 2.4. Let $\mathcal{C}$ be a collection of subsets of X, then $\sigma(\mathcal{C}) := \bigcap \{\mathcal{M} : \sigma - alg, \mathcal{C} \subseteq \mathcal{M}\}$ is a $\sigma - algebra$ containing $\mathcal{C}$, and is called the $\sigma-$algebra generated by $\mathcal{C}$.

Definition 2.5. Let X be a topological space, and let $\mathcal{G}$ be the collection of all open sets of X, then the Borel algebra is $Bol_X := \sigma(\mathcal{G})$

2.2 Measures

Definition 2.6. A function $\mu : \mathcal{M} \to [0, \infty]$ is called a **positive measure** if it satisfies **countable additivity**. Namely, for any pairwise disjoint sets $E_1, E_2, \dots$ in $\mathcal{M}$, we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

We call $(X, \mathcal{M}, \mu)$ a **measure space**.

Remark. We can similarly define a complex or a signed measure to be a function $\mathcal{M} \to \mathbb{C}$ or $\mathcal{M} \to [-\infty, \infty]$ that satisfies countable additivity. (See later chapter for more about this.)

Definition 2.7. μ is finite if $\mu(X) < \infty$. μ is $\sigma-$finite if $X = \bigcup_{i=1}^{\infty} A_i$, where each $\mu(A_i) < \infty$. μ is semi-finite if $\forall E \in \mathcal{M}$, such that $\mu(E) \neq 0$, there is always $F \in \mathcal{M}, F \subseteq E, 0 < \mu(F) < \infty$

Remark. We will only work with positive measures where it satisfies $\exists A \in \mathcal{M}, \mu(A) < \infty$.

Example 2.2.1. For any X, we can define $\mu : \mathcal{P}(X) \to [0, \infty]$ by $\mu(A) := \begin{cases} |A|, & |A| < \infty \\ \infty, & \text{otherwise} \end{cases}$ is the **counting measure** on X

Example 2.2.2. For any set X and $x \in X$, we can define $\delta_x : \mathcal{P}(X) \to [0, \infty]$ by $\delta_x(A) := \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$ is the **point measure** or **Dirac measure** of x.

Example 2.2.3. Let $X = \mathbb{R}, \mathcal{M} = \mathcal{P}(X)$, let $x_1, x_2, \dots \in \mathbb{R}, a_1, a_2, \dots \geq 0$, then $\mu(E) := \sum_{i | x_i \in E} a_i$ is a measure.

Definition 2.8. A positive measure μ is a **probability measure** if $\mu(X) = 1$. In this case, $(X, \mathcal{M}, \mu)$ is called a probability space.

Proposition 2.7. $\mu(\emptyset) = 0$

Proof. Choose $A \in \mathcal{M}$ with finite measure, take $A_1 = A$, and $A_2 = A_3 = \cdots = \emptyset$.
Then $\mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset) = \mu(A) < \infty$, thus we must have $\mu(\emptyset) = 0$ □

Proposition 2.8 (Finite Additivity)**.** *If $E_1, E_2, \dots, E_n \in \mathcal{M}$, then $\mu(\bigsqcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$*

Proof. Take $E_{n+1} = E_{n+2} = \cdots = \emptyset$, then $\mu(\bigsqcup_{i=1}^{n} E_i) = \mu(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{n} \mu(E_i) + \sum_{i=n+1}^{\infty} \mu(E_i) = \sum_{i=1}^{n} \mu(E_i)$ □

Remark. This holds for complex measures as well.

Proposition 2.9 (Monotonicity)**.** *If $E, F \in \mathcal{M}, E \subseteq F$, then $\mu(E) \leq \mu(F)$*

Proof. We have $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$ □

Remark. This does not hold for complex measures.

Proposition 2.10 (Subadditivity)**.** *If $E_1, E_2, \dots \in \mathcal{M}$, then $\mu(\bigcup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu(E_i)$*

Proposition 2.11 (Continuity)**.** *If $E_1, E_2, \dots \in \mathcal{M}, E_n \subseteq E_{n+1}$, we have that $\mu(\bigcup_{i=1}^\infty E_i) = \lim_{n\to\infty} \mu(E_n)$. If $E_1, E_2, \dots \in \mathcal{M}, E_{n+1} \subseteq E_n, \mu(E_1) < \infty$ we have that $\mu(\bigcap_{i=1}^\infty E_i) = \lim_{n\to\infty} \mu(E_n)$.*

Proof. Let $E_0 = \emptyset$, then we can write $\bigcup_{i=1}^\infty E_i = \bigsqcup_{i=1}^\infty (E_i \setminus E_{i-1})$, and we have $E_n = \bigsqcup_{i=1}^n (E_i \setminus E_{i-1})$.

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^\infty E_i\right) &= \mu\left(\bigsqcup_{i=1}^\infty (E_i \setminus E_{i-1})\right) \\
&= \sum_{i=1}^\infty \mu(E_i \setminus E_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \mu(E_i \setminus E_{i-1}) \\
&= \lim_{n\to\infty} \mu(\bigsqcup_{i=1}^n (E_i \setminus E_{i-1})) \\
&= \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

For the second part, let $A = \bigcap_{i=1}^\infty E_i$.

$$\begin{aligned}
\mu(E_1 \setminus A) &= \mu(E_1 \cap A^c) \\
&= \mu\left(E_1 \cap \left(\bigcap_{i=1}^\infty E_i\right)^c\right) \\
&= \mu\left(E_1 \cap \bigcup_{i=1}^\infty E_i^c\right) \\
&= \mu\left(\bigcup_{i=1}^\infty E_1 \cap E_i^c\right) \\
&= \lim_{n\to\infty} \mu(E_1 \cap E_n^c) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n)
\end{aligned}$$

By finite additivity, we have that

$$\begin{aligned}
\mu(E_1 \setminus A) + \mu(A) &= \mu(E_1 \setminus A \sqcup A) \\
&= \mu(E_1) \\
&= \lim_{n\to\infty} \mu(E_1) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n \sqcup E_n) \\
&= \lim_{n\to\infty} (\mu(E_1 \setminus E_n) + \mu(E_n)) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n) + \lim_{n\to\infty} \mu(E_n) \\
&= \mu(E_1 \setminus A) + \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

Since $\mu(E_1 \setminus A) \leq \mu(E_1) < \infty$, we have $\mu(A) = \lim_{n\to\infty} \mu(E_n)$ □

Remark. This holds for complex measures as well. However, for the second property, it is essential for $\mu(E_1) < \infty$. Indeed, consider the following example:

Example 2.2.4. Let $X = \mathbb{N}, \mathcal{M} = \mathcal{P}(X), \mu$ be the counting measure. Let $A_n := \{i : i \geq n\}$. Notice that $A_1 \supseteq A_2 \supseteq A_3 \dots$ and $\lim_{n\to\infty} \mu(A_n) = \infty \neq 0 = \mu(\emptyset)$. However, $\bigcap_{n=1}^\infty A_n = \emptyset$

2.3 Measurable Function

Definition 2.9. If $(X, \mathcal{M}_1), (Y, \mathcal{M}_2)$ are measure spaces, then $f : X \to Y$ is a **measurable function** if $\forall B \in \mathcal{M}_2, f^{-1}(B) \in \mathcal{M}_1$.

Definition 2.10. If $(Y, \mathcal{T})$ is a topological space, we say a function $f : X \to Y$ is **Borel measurable** if it is measurable with respect to $\mathcal{M}_2 = Bol_{(Y,\mathcal{T})}$, the Borel σ-algebra.

Proposition 2.12. *For $(B_i) \subseteq Y$, we have*

1. $f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$
2. $f^{-1}(\bigcup_i B_i)) = \bigcup_i f^{-1}(B)$
3. $f^{-1}(\bigcap_i B_i)) = \bigcap_i f^{-1}(B)$.

Proposition 2.13. *If $(Y, \mathcal{T})$ is a topological space, a function $f : X \to Y$ is Borel measurable if and only if $\forall B \in \mathcal{T}$ open, $f^{-1}(B) \in \mathcal{M}_1$.*

Proposition 2.14. *For $f : X \to \mathbb{R}$, the following are equal:*

1. *f is (Borel) measurable*
2. *$\forall a, f^{-1}((-\infty, a))$ is measurable*
3. *$\forall a, f^{-1}((-\infty, a])$ is measurable*
4. *$\forall a, f^{-1}((a, \infty))$ is measurable*
5. *$\forall a, f^{-1}([a, \infty))$ is measurable*
6. *$\forall a < b, f^{-1}((a, b))$ is measurable*

Proposition 2.15. *If $f : X \to Y, g : Y \to Z$ are both measurable, then $f \circ g$ is also measurable.*

Corollary 2.16. *If $f : X \to \mathbb{C}$ is measurable, we have $u = Re(f), v = Im(f), z = |f|$ are all measurable.*

Theorem 2.17. *Let $(X, \mathcal{M})$ is a measurable space, and $u, v : X \to \mathbb{R}$ be measurable, and (Y, τ) is a topological space. If $\Phi : \mathbb{R}^2 \to Y$ is continuous, then $h : X \to Y; x \mapsto \Phi(u(x), v(x))$ is measurable.*

Proof. Let $f : X \to \mathbb{R}^2; x \mapsto (u(x), v(x))$, it suffices to check that f is measurable.
Notice that $Bol_{\mathbb{R}^2}$ is generated by open rectangles $R = (a, b) \times (c, d)$.
Yet $f^{-1}(R) = u^{-1}(a, b) \cap v^{-1}(c, d)$ is measurable. □

Corollary 2.18. *If $u, v : X \to \mathbb{R}$ are both measurable, we have $f := u + iv : X \to \mathbb{C}$ is also measurable.*

Proof. Choose $\Phi : \mathbb{R}^2 \to \mathbb{C}; (s, t) \mapsto s + it$. □

Corollary 2.19. *If $f, g : X \to \mathbb{R}$ are measurable, then we have $fg, f + g$ are both measurable.*

Proof. choose $\Phi : (s, t) \mapsto st$ or $\Phi : (s, t) \mapsto s + t$. □

Corollary 2.20. *If $f, g : X \to \mathbb{C}$ are measurable, then for any $\alpha \in \mathbb{C}$, we have $fg, f + g, \alpha f$ are all measurable.*

Proof. We write $f = u + iv, g = w + iz$. We have that u, v, z, z are all real-valued and measurable, so are $u + w, v + z$, and so are $(u + w) + i(v + z) = f + g$ and $(uw - vz) + i(vw + uz) = fg$.
For αf, it is obvious since $B \in Bol(\mathbb{C}) \iff \alpha B \in Bol(\mathbb{C})$ for $\alpha \neq 0$, and $0f = 0$ is measurable. □

Definition 2.11. For **extended real functions** $f : X \to [-\infty, \infty]$, it is measurable if $\forall \alpha \in \mathbb{R}, f^{-1}((\alpha, \infty]) \in \mathcal{M}$, or equivalently, $\forall \alpha \in \mathbb{R}, f^{-1}([-\infty, \alpha)) \in \mathcal{M}$.

Proposition 2.21. *If $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions $X \to [-\infty, \infty]$, we have*

$$g(x) := \sup_n f_n(x), \;\; h(x) := \limsup_{n\to\infty} f_n(x) = \inf_k \left(\sup_{n\geq k} f_n(x) \right)$$

are also measurable. Similarly for $\inf$ *and* $\liminf$*.*

Proof. Notice that

$$\begin{aligned} x \in g^{-1}((\alpha, \infty]) &\iff g(x) > \alpha \\ &\iff \exists f_n(x) > \alpha \\ &\iff x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty]), \end{aligned}$$

which is a union of measurable sets. Thus g is measurable. $\square$

Corollary 2.22. *If $f_n : X \to [-\infty, \infty]$ or $f_n : X \to \mathbb{C}$ are measurable functions, and $\forall x \in X, f(x) = \lim_{n\to\infty} f_n(x)$ exists, then f is measurable.*

Corollary 2.23. *If $f, g : X \to [-\infty, \infty]$ are both measurable, then $\max(f, g), \min(f, g)$ are measurable.*

Corollary 2.24. *If $f : X \to [-\infty, \infty]$ is measurable, then $f^+ := \max(f, 0), f^- := \max(-f, 0)$ are both measurable, with $f = f^+ - f^-$.*

Proposition 2.25. *If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.*

Proof. Let $\alpha \in \mathbb{R}$, we need to show that $\{x \in \mathbb{R} | f(x) > \alpha\}$ is a Borel set.
We may assume that f is non-decreasing, if not we take $f \to -f$. If $\{x \in \mathbb{R} | f(x) > \alpha\} \in \{\emptyset, \mathbb{R}\}$, we have nothing to prove.
Now if $\{x \in \mathbb{R} | f(x) > \alpha\} \notin \{\emptyset, \mathbb{R}\}$, we have that $\{x \in \mathbb{R} | f(x) \leq \alpha\}$ is not empty and bounded above since f is increasing. Let $x_0 := \sup \{x \in \mathbb{R} | f(x) \leq \alpha\}$. If $f(x_0) \leq \alpha, \{x \in \mathbb{R} | f(x) > \alpha\} = (x_0, \infty)$, otherwise $\{x \in \mathbb{R} | f(x) > \alpha\} = [x_0, \infty)$, both Borel. $\square$

2.4 Simple Functions

Definition 2.12. Let $(X, \mathcal{M})$ be a measurable space, a **characteristic function** for a subset $E \subseteq X$ is

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

Definition 2.13. Let $(X, \mathcal{M})$ be a measurable space, a function $\phi : X \to [-\infty, \infty]$ is **simple** if $\phi(X)$ is finite.

Proposition 2.26. *Let $(X, \mathcal{M})$ be a measurable space, for any simple function ϕ with $\phi(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we have*

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

where $E_i = \phi^{-1}(\{\alpha_i\})$ are pairwise disjoint. In this case, ϕ is measurable if and only if $\forall i, E_i \in \mathcal{M}$.

Lemma 2.27. *For any $\alpha \in \mathbb{R}, n \geq 1$, we have that*

$$\alpha - \frac{1}{2^n} < \frac{\lfloor 2^n \alpha \rfloor}{2^n} \leq \alpha$$

Proof.

$$\begin{aligned}\lfloor 2^n\alpha\rfloor &\le 2^n\alpha < \lfloor 2^n\alpha\rfloor + 1\\ 2^n\alpha - 1 &< \lfloor 2^n\alpha\rfloor\\ \alpha - \frac{1}{2^n} &< \frac{\lfloor 2^n\alpha\rfloor}{2^n}\\ \frac{\lfloor 2^n\alpha\rfloor}{2^n} &\le \alpha\end{aligned}$$

□

Lemma 2.28. *Consider $id : [0,\infty) \to [0,\infty), x \mapsto x$, then there are simple functions $s_n : [0,\infty] \to [0,\infty)$ such that each s_n is measurable and* $\begin{cases} 0 \le s_1 \le s_2 \le \cdots \le id, \\ \forall x \in X, \lim_{n\to\infty} s_n(x) = id(x), \\ \forall R > 0, s_n \to id \text{ uniformly on } [0,R] \end{cases}$

Proof. For $n \ge 1, t \in [0,\infty)$, let $s_n(t) := \begin{cases} \frac{\lfloor 2^n t\rfloor}{2^n}, & t \in [0,n] \\ n, & t > n \end{cases}$
Notice that s_n is simple. It is also measurable since it is monotone.
We also have that $0 \le s_1 \le s_2 \le \cdots \le f$, and by squeeze theorem, we have that

$$\lim_{n\to\infty} s_n(x) = x = id(x).$$

In addition, we can check that this convergence is uniform on any $[0,R]$. □

Theorem 2.29. *Let $f : X \to [0,\infty]$ be measurable, then there are simple functions $s_n : X \to [0,\infty)$ such that each s_n is measurable and* $\begin{cases} 0 \le s_1 \le s_2 \le \cdots \le f \\ \forall x \in X, \lim_{n\to\infty} s_n(x) = f(x) \\ \forall R > 0, s_n \to f \text{ uniformly on } E_R := \{x \in X : f(x) \le R\}. \end{cases}$

Proof. Notice that for any simple function s and any arbitrary measurable function f, we have that $s \circ f$ is simple. Thus it suffices to find s_n' that approximates $id : x \mapsto x$, which is done by the above lemma.
Let $s_n := s_n' \circ f$, they are measurable by result on compositions, and

$$0 \le s_1 \le \cdots \le f, \quad \lim_{n\to\infty} (s_n \circ f)(x) = f(x).$$

□

Corollary 2.30 (Simple function approximation)**.** *Let $f : X \to \mathbb{C}$ be measurable, then there are simple functions $s_n : X \to [0,\infty)$ such that each s_n is measurable and*

$$\begin{cases} 0 \le |s_1| \le |s_2| \le \cdots \le |f| \\ \forall x \in X, \lim_{n\to\infty} s_n(x) = f(x) \\ \forall R > 0, s_n \to f \text{ uniformly on } E_R := \{x \in X : |f(x)| \le R\}. \end{cases}$$

Corollary 2.31. *For $f, g : X \to [0.\infty]$ being measurable, we have that $f \cdot g$ is also measurable.*

Proof. One can check that for monotone non-decreasing $(a_n), (b_n) \subseteq [0,\infty)$ with $a_n \to a, b_n \to b$ for $a, b \in [0,\infty]$, then $a_n b_n \to ab$.
Approximate f with simple functions s_n, and g with simple functions t_n, then each of them is measurable, hence so is $s_n \cdot t_n$, hence so is $\lim_{n\to\infty} s_n t_n = fg$ □

3 Integration

3.1 Integration of non-negative functions

Definition 3.1. Let $(X, \mathcal{M}, \mu)$ be a measure space, $s : X \to [0, \infty)$ be a simple measurable function, with $s(X) = \{a_1, \ldots, a_n\}$, such that $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, where $A_i := s^{-1}(\{a_i\})$. For $A \in \mathcal{M}$, define the **integral** of s over A to be

$$\int_A s d\mu := \sum_{i=1}^{n} a_i \mu(A_i \cap A).$$

Definition 3.2. For $f : X \to [0, \infty]$ measurable, the **integral** of f over $A \in \mathcal{M}$ is

$$\int_A f d\mu := \sup \int_A s d\mu,$$

where the sup is taken over all measurable simple $s : X \to [0, \infty)$ such that $0 \leq s \leq f$.

Proposition 3.1. *Let $f, g : X \to [0, \infty]$ be measurable, then*

1. $f \leq g \implies \forall\ A \in \mathcal{M}, \int_A f d\mu \leq \int_A g d\mu$
2. *For any $A \subseteq B \in \mathcal{M}$, we have that $\int_A f d\mu \leq \int_B f d\mu$*
3. $\forall\ c \in [0, \infty), A \in \mathcal{M}$, *we have that $\int_A c f d\mu = c \int_A f d\mu$*
4. *If $\forall\ x \in X, f(x) = 0$, we have that $\forall\ A \in \mathcal{M}, \int_A f d\mu = 0$*
5. *If $\forall\ x \in A \in \mathcal{M}, f(x) = 0$, we have that $\int_A f d\mu = 0$*
6. *If $\mu(A) = 0$ for $A \in \mathcal{M}$, we have that $\int_A f d\mu = 0$*
7. $\int_A f d\mu = \int_X \mathcal{X}_A f d\mu$

Proposition 3.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $s : X \to [0, \infty)$ a measurable simple function. Then $\lambda : \mathcal{M} \to [0, \infty]$ defined by*

$$\lambda(A) := \int_A s d\mu$$

is a measure on (X, M)

Proof. Write $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, and let $C := \bigsqcup_{k=1}^{\infty} C_k$, then

$$\begin{aligned}
\lambda(C) &= \sum_{i=1}^{n} a_i \mu(Ai \cap C) \\
&= \sum_{i=1}^{n} a_i \mu(\bigsqcup_{k=1}^{\infty} (Ai \cap C_k)) \\
&= \sum_{i=1}^{n} a_i \sum_{k=1}^{\infty} \mu(A_i \cap C_k) \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{n} a_i \mu(A_i \cap C_k) \\
&= \sum_{k=1}^{\infty} \lambda(C_k)
\end{aligned}$$

Thus λ satisfies countable additivity, and in addition $\lambda(\emptyset) = \sum_{i=1}^{n} \cancelto{0}{a_i \mu(A_i \cap \emptyset)} = 0$. □

Corollary 3.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $s : X \to [0, \infty)$ a measurable simple function, with $C := \bigsqcup_{k=1}^{\infty} C_k$. Then we have*

$$\int_C s d\mu = \sum_{k=1}^{\infty} \int_{C_k} s d\mu.$$

Proof.

$$\begin{aligned}
\int_C s d\mu &= \lambda_s(C) \\
&= \lambda_s\left(\bigsqcup_{k=1}^{\infty} C_k\right) \\
&= \sum_{k=1}^{\infty} \lambda_s(C_k) \\
&= \sum_{k=1}^{\infty} \int_{C_k} s d\mu
\end{aligned}$$

□

Proposition 3.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $s, t : X \to [0, \infty)$ both be measurable simple functions, then*

$$\int_X (s + t) d\mu = \int_X s d\mu + \int_X t d\mu$$

Proof. Write $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}, t = \sum_{j=1}^{m} b_j \mathcal{X}_{B_j}$, and let $C_{ij} = A_i \cap B_j$, then C_{ij} are disjoint, and $\bigsqcup_{ij} C_{ij} = X$

$$\begin{aligned}
\int_{C_{ij}} (s + t) d\mu &= (a_i + b_j)\mu(C_{ij}) \\
&= a_i \mu(C_{ij}) + b_j \mu(C_{ij}) \\
&= \int_{C_{ij}} s d\mu + \int_{C_{ij}} t d\mu \\
\int_X (s + t) d\mu &= \int_{\bigsqcup_{ij} C_{ij}} (s + t) d\mu \\
&= \sum_{ij} \int_{C_{ij}} (s + t) d\mu \\
&= \sum_{ij} \int_{C_{ij}} s d\mu + \sum_{ij} \int_{C_{ij}} t d\mu \\
&= \int_X s d\mu + \int_X t d\mu
\end{aligned}$$

□

Theorem 3.5 (Lebesgue's Monotone Convergence)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to [0, \infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$. Let $f(x) := \lim_{n\to\infty} f_n(x)$, then $f : X \to [0, \infty]$ is measurable, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Proof. Since $f_n \leq f_{n+1}$, we have that $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$, so by monotone convergence theorem,

$$\alpha := \lim_{n\to\infty} \int_X f_n d\mu \in [0, \infty]$$

exists.
As a limit of measurable functions, f is measurable. Also, $\forall n, \int_X f_n d\mu \leq \int_X f d\mu$, and thus $\alpha \leq \int_X f d\mu$.
Consider any $s : X \to [0,\infty)$ be simple and measurable with $0 \leq s \leq f$, and consider any $0 < c < 1$.
For $n \geq 1$, let $A_n := \{x \in X : f_n(x) \geq cs(x)\}$.
Then $X = \bigcup_{n=1}^{\infty} A_n$ since f_n converges point-wise.
In addition, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$
Also each A_n is measurable, since $A_n = \{x \in X : (f_n - cs)(x) \geq 0\} = (f_n - cs)^{-1}([0,\infty])$, and $f_n - cs$ is measurable.
Since $\lambda_s : A \mapsto \int_A s d\mu$ is a measure, so by property of measures,

$$\int_X s d\mu = \lambda_s(X) = \lim_{n\to\infty} \lambda_s(A_n) = \lim_{n\to\infty} \int_{A_n} s d\mu.$$

In addition, we have

$$\begin{aligned}
\int_X f_n d\mu &\geq \int_{A_n} f_n d\mu \\
&\geq \int_{A_n} cs d\mu \\
&= c\int_{A_n} s\mu \\
\alpha &= \lim_{n\to\infty} \int_X f_n d\mu \\
&\geq \lim_{n\to\infty} c\int_{A_n} s\mu \\
&= c\int_X s d\mu.
\end{aligned}$$

Now take $c \to 1$, we have that $\alpha \geq \int_X s d\mu$.
Then take sup of all simple $s \leq f$, we have that $\alpha \geq \int_X f d\mu$. □

Corollary 3.6. *For a measure space $(X, \mathcal{M}, \mu), A \in \mathcal{M}$, let $f_n : X \to [0,\infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$. Let $f(x) := \lim_{n\to\infty} f_n(x)$. We can consider the restriction $(A, \mathcal{M}' := \{B \cap A : B \in \mathcal{M}\}, \mu|_{\mathcal{M}'})$, and we will have*

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu$$

Corollary 3.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to [0,\infty]$ be measurable. Let $s_n : X \to [0,\infty]$ be any measurable simple functions with $0 \leq s_1 \leq s_2 \leq \cdots \leq \infty$ with $f(x) = \lim_{n\to\infty} s_n(x)$. We have*

$$\lim_{n\to\infty} \int_X s_n d\mu = \int_X f d\mu.$$

Proposition 3.8 (finite additivity for positive functions)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g : X \to [0,\infty]$ be measurable functions, then*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Proof. Approximate f, g by simple functions s_n, t_n, such that $\lim_{n\to\infty} s_n(x) = f(x), \lim_{n\to\infty} t_n(x) = g(x)$ and $0 \leq s_1 \leq \cdots \leq f, 0 \leq t_1 \leq \cdots \leq g$.

Notice that $0 \leq s_1 + t_1 \leq \cdots \leq f + g$, and $\lim_{n\to\infty}(s_n + t_n)(x) = (f + g)(x)$. Thus

$$\begin{aligned}
\int_X (f+g)d\mu &= \lim_{n\to\infty} \int_X (s_n + t_n)d\mu \\
&= \lim_{n\to\infty} \left(\int_X s_n d\mu + \int_X st_n d\mu \right) \\
&= \lim_{n\to\infty} \int_X s_n d\mu + \lim_{n\to\infty} \int_X t_n d\mu \\
&= \int_X f d\mu + \int_X g d\mu.
\end{aligned}$$

□

Corollary 3.9 (countable additivity for positive functions)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to [0, \infty]$ be measurable functions. Then*

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

is measurable and

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n(x) d\mu.$$

Proof. Define $g_n(x) := \sum_{i=1}^{n} f_i(x)$, then $0 \leq g_1 \leq \cdots \leq f$ and $\lim_{n\to\infty} g_n = f$.
By previous proposition and induction,

$$\int_X g_n d\mu = \sum_{i=1}^{n} \int_X f_n d\mu.$$

By LMCT, we have

$$\begin{aligned}
\int_X f d\mu &= \int_X \lim_{n\to\infty} g_n d\mu \\
&= \lim_{n\to\infty} \int_X g_n d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int_X f_i d\mu \\
&= \sum_{n=1}^{\infty} \int_X f_n d\mu
\end{aligned}$$

□

Theorem 3.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f : X \to [0, \infty]$ a measurable function. Then $\lambda : \mathcal{M} \to [0, \infty]$ defined by*

$$\lambda(A) := \int_A f d\mu$$

is a measure on (X, M). Moreover, for some $g : X \to [0, \infty]$ such that fg is measurable, then

$$\int_X g d\lambda = \int_X g f d\mu.$$

Proof. Let $A = \bigsqcup_{n=1}^{\infty} A_n$ with A_n disjoint measurable subsets of X. We have that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$, and thus

$$\begin{aligned}\lambda(A) &= \int_X \chi_A f d\mu \\ &= \int_X \sum_{n=1}^{\infty} \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \int_X \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \lambda(A_n)\end{aligned}$$

Thus λ is a measure.
In addition, when $g = \chi_A$ for $A \in \mathcal{M}$, we have that $\int_X g d\lambda = \lambda(A) = \int_X \chi_A f d\mu = \int_X g f d\mu$. And thus simple functions, and thus all non-negative measurable functions by LMCT. □

3.2 Integration of real and complex functions

Definition 3.3. For $f : X \to \mathbb{R} \cup \{\pm\infty\}$, we define $f^+(x) := \max(f(x), 0), f^-(x) := \max(-f(x), 0)$, and thus $f = f^+ - f^-$, with both $f^+.f^- : X \to [0, \infty]$. We define

$$\int_X f d\mu := \int_X f^+ d\mu - \int_X f^- d\mu$$

when only one of the integrals is ∞.

Definition 3.4. Let $(X, \mathcal{M}, \mu)$ be a measure space, the define

$$\mathcal{L}^1(X, \mu) := \left\{ f : X \to \mathbb{C} \,\middle|\, \int_X |f| d\mu < \infty \right\}$$

be the **set of Lebesgue integrable functions**.

Definition 3.5. For $f = u + iv \in \mathcal{L}^1(X, \mu)$, where $u, v : X \to \mathbb{R}$, then the integral of f is defined as

$$\int_X f d\mu := \int_X u d\mu + i \int_X v d\mu.$$

Proposition 3.11. *The above integral is well-defined.*

Proof. u^+, u^-, v^+, v^- are measurable, and $0 \le u^+, u^- \le |u| \le |f|$, thus each integral is finite. □

Proposition 3.12. *For $f = u + iv \in \mathcal{L}^1(X, \mu)$, where $u, v : X \to \mathbb{R}$, we have*

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu.$$

Proof. By definition. □

Theorem 3.13. *$\forall f, g \in \mathcal{L}^1(X, \mu), \alpha \in \mathbb{C}$, we have that $\alpha f + g \in \mathcal{L}^1(X, \mu)$ and*

$$\int_X \alpha f + g d\mu = \alpha \int_X f d\mu + \int_X g d\mu.$$

Thus, $\mathcal{L}^1(X, \mu)$ is a vector space over $\mathbb{C}$.

Proof. Clearly $\alpha f + g$ is measurable. In addition,

$$\begin{aligned}\int_X |\alpha f + g| d\mu &\leq \int_X |\alpha f| + |g| d\mu \\ &= \int_X |\alpha||f| d\mu + \int_X |g| d\mu \\ &= |\alpha| \int_X |f| d\mu + \int_X |g| d\mu \\ &< \infty\end{aligned}$$

Now we check the addition: Consider ant $f = a + ib, g = c + id : X \to \mathbb{C}$, such that $a, b, c, d : X \to \mathbb{R}$.

$$\begin{aligned}(a+c)^+ - (a+c)^- &= a + c \\ &= (a^+ - a^-) + (c^+ - c^-) \\ &= (a^+ + c^+) - (a^- + c^-). \\ (a+c)^+ + (a^- + c^-) &= (a+c)^- + (a^+ + c^+),\end{aligned}$$

where both sides of the equality are sums of two non-negative functions. Thus we have

$$\begin{aligned}\int_X (a+c)^+ + (a^- + c^-) d\mu &= \int_X (a+c)^- + (a^+ + c^+) d\mu \\ \int_X (a+c)^+ d\mu + \int_X (a^- + c^-) d\mu &= \int_X (a+c)^- d\mu + \int_X (a^+ + c^+) d\mu \\ \int_X (a+c)^+ d\mu - \int_X (a+c)^- d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\ \int_X (a+c) d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\ &= \int_X a^+ d\mu + \int_X c^+ d\mu - \int_X a^- d\mu - \int_X c^- d\mu \\ &= \left(\int_X a^+ d\mu - \int_X a^- d\mu\right) + \left(\int_X c^+ d\mu - \int_X c^- d\mu\right) \\ &= \int_X a d\mu + \int_X c d\mu \\ \int_X (f+g) d\mu &= \int_X (a+c)\mu + i \int_X (b+d) d\mu \\ &= \int_X a d\mu + \int_X c d\mu + i \int_X b d\mu + i \int_X d d\mu \\ &= \left(\int_X a d\mu + i \int_X b d\mu\right) + \left(\int_X c d\mu + i \int_X d d\mu\right) \\ &= \int_X f d\mu + \int_X g d\mu.\end{aligned}$$

Now we check the scalar multiplication: $\forall \alpha \geq 0$, we have $\int_X \alpha f d\mu = \alpha \int_X f d\mu$ by definition.
We can also check for $\alpha = -1$ and $\alpha = i$, and conclude this holds for all $\alpha \in \mathbb{C}$. □

Theorem 3.14. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f \in \mathcal{L}^1(X, \mu)$, then*

$$\left|\int_X f d\mu\right| \leq \int_X |f| d\mu$$

Proof. Let $\alpha := \int_X f d\mu \in \mathbb{C}$, and let $\beta \in \mathbb{C}, |\beta| = 1$, such that $\alpha\beta = |\alpha|$. Take $u = Re(\beta f) : X \to \mathbb{R}$, note

$u \le |\beta f| = |f|$. Now

$$\begin{aligned}\left|\int_X f d\mu\right| &= |\alpha| \\ &= \beta\alpha \\ &= \beta \int_X f d\mu \\ &= \int_X \beta f d\mu \\ &= \int_X u d\mu \\ &\le \int_X |f| d\mu\end{aligned}$$

$\square$

3.3 Lebesgue Dominated Convergence Theorem

Lemma 3.15 (Fatou's)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to [0, \infty]$ be measurable functions. Then*

$$\int_X (\liminf_n f_n) d\mu \le \liminf_n \int_X f_n d\mu$$

Proof. Let $g_n(x) := \inf_{i \ge n} f_i(x)$, then $\liminf_n f_n(x) = \lim_{n\to\infty} g_n(x)$.
Also, $g_n \le f_n$, so $\int_X g_n d\mu \le \int_X f_n d\mu, \ \forall n \ge 1$.
Note g_n is measurable, and $0 \le g_1 \le g_2 \le \cdots$.
By LMCT,

$$\lim_{n\to\infty} \int_X g_n d\mu = \int_X (\lim_{n\to\infty} g_n) d\mu = \int_X (\liminf_n f_n) d\mu.$$

Since the left hand side converges,

$$\int_X (\liminf_n f_n) d\mu = \lim_{n\to\infty} \int_X g_n d\mu = \liminf_n \int_X g_n d\mu \le \liminf_n \int_X f_n d\mu.$$

$\square$

Theorem 3.16 (Lebesgue Dominated Convergence)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{C}$ be measurable functions such that $f(x) := \lim_{n\to\infty} f_n(x)$ exists $\forall x \in X$. If there is $0 \le g(x) \in \mathcal{L}^1(X, \mu)$, such that $\forall x \in X, \forall n \in \mathbb{N}, \ |f_n(x)| \le g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n\to\infty} \int_X |f - f_n| d\mu = 0.$$

Proof. Firstly, $\forall n \in \mathbb{N}, x \in X, \ |f_n(x)| \le g(x)$ implies that $\forall x \in X, \ |f(x)| \le g(x)$, and thus $\int_X |f| d\mu \le \int_X g d\mu$. This hows that $f \in \mathcal{L}^1(X, \mu)$.
Notice that $|f_n - f| \le |f_n| + |f| \le 2g$. Thus $2g - |f_n - f| \ge 0$. By Fatou's Lemma, we have that

$\int_X(\liminf(2g-|f_n-f|))d\mu \leq \liminf \int_X 2g-|f_n-f|d\mu$. Thus

$$\begin{aligned}
\int_X 2g d\mu &= \int_X (2g-\liminf(|f_n-f|))d\mu \\
&= \int_X (\liminf(2g-|f_n-f|))d\mu \\
&\leq \liminf \int_X 2g-|f_n-f|d\mu \\
&= \int_X 2gd\mu + \liminf(-\int_X |f_n-f|d\mu) \\
&= \int_X 2gd\mu - \limsup \int_X |f_n-f|d\mu \\
0 &\leq -\limsup \int_X |f_n-f|d\mu
\end{aligned}$$

Thus $0 \leq \liminf \int_X |f_n-f|d\mu \leq \limsup \int_X |f_n-f|d\mu \leq 0$, and thus $\lim_{n\to\infty}\int_X |f_n-f|d\mu = 0$.
Finally,

$$\begin{aligned}
\left|\lim_{n\to\infty}\int_X f_n d\mu - \int_X f d\mu\right| &= \left|\lim_{n\to\infty}\int_X (f_n-f)d\mu\right| \\
&\leq \lim_{n\to\infty}\int_X |f_n-f|d\mu \\
&= 0
\end{aligned}$$

□

3.4 Almost Everywhere

Definition 3.6. Let $(X,\mathcal{M},\mu)$ be a measure space, $A\in\mathcal{M}$, and $P=\{p(x)\}_{x\in A}$ be a family of logical statements, then we say the property P holds or is true μ-everywhere on A, if $\exists N\in\mathcal{M}$, such that $\mu(N)=0$ and $\forall x\in A\setminus N,\ p(x)=True$.

Definition 3.7. For measurable functions $f,g:X\to Y$, we say that $f=g$ μ-almost everywhere if

$$\mu(\{x\in X|f(x)\neq g(x)\})=0.$$

Remark. For some $A\in\mathcal{M}$, we have that $\mu(A\cap N)\leq\mu(N)=0$, and thus $\int_{A\cap N}(f-g)d\mu=0$. Thus $\int_A(f-g)d\mu=\int_{A\cap N}(f-g)d\mu+\int_{A\setminus N}(f-g)d\mu=0$.

Definition 3.8. Let $(f_n)_{n=1}^\infty$, we say $f_n\to f$ a.e. if $f_n(x)\to f(x)$ for $\mu-$ a.e.$x\in X$.

Proposition 3.17. *Let $(X,\mathcal{M},\mu)$ be a measure space.*

1. *If $f:X\to[0,\infty]$ is measurable, we have $f=0$ μ-a.e. $\iff$ $\int_E fd\mu=0\ \forall E\in\mathcal{M}$.*
2. *If $f\in\mathcal{L}^1(X,\mu)$ we have $f=0$ μ-a.e. $\iff$ $\int_E fd\mu=0\ \forall E\in\mathcal{M}$.*
3. *If $f\in\mathcal{L}^1(X,\mu)$, and $\left|\int_X fd\mu\right|=\int_X|f|d\mu$, there exist must a constant α such that $\alpha f=|f|$ almost everywhere.*

Proof. 1. Let $N=\{x\in X: f(x)>0\}$.
Suppose $f=0$ μ-a.e., then $\mu(N)=0$.
We have

$$\int_X fd\mu=\int_{X\setminus N}fd\mu+\int_N fd\mu=\int_{X\setminus N}0d\mu+0=0.$$

Thus $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$.
Now suppose $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$.
Let $A_n := \left\{x \in X : f(x) > \frac{1}{n}\right\}$, then we have

$$\frac{1}{n}\mu(A_n) = \int_{A_n} \frac{1}{n} d\mu \leq \int_{A_n} f d\mu = 0.$$

Thus $\mu(A_n) = 0$.
Notice that $N = \bigcup_{n=1}^{\infty} A_n, A_1 \subseteq A_2 \subseteq \cdots$, thus $\mu(N) = \lim_{n\to\infty} \mu(A_n) = 0$.

2. Suppose $f = 0$ μ-a.e., we have that $|f| = 0$ μ-a.e., thus $\left|\int_E f d\mu\right| \leq \int_E |f| d\mu = 0$.
Now suppose $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$.
Let $f = u + iv$, then we have $\int_E u d\mu = \int_E v d\mu = 0 \ \forall E \in \mathcal{M}$.
Let $u = u^+ - u^-$, and $E = \{x \in X : u(x) \geq 0\}$.

$$\begin{aligned}\int_X u^+ d\mu &= \int_{X\setminus E} u^+ d\mu + \int_E u^+ d\mu \\ &= \int_{X\setminus E} 0 d\mu + \int_E u d\mu \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Thus $u^+ = 0$ μa.e..
Similarly for u^-, and thus $u = 0$ $\mu -$ a.e..
Similarly for v, and thus $f = 0$ $\mu -$ a.e..

□

Theorem 3.18 (Lebesgue Dominated Convergence - almost everywhere)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{C}$ be measurable functions, defined $\mu-$almost everywhere on X, such that $f(x) := \lim_{n\to\infty} f_n(x)$ is defined $\mu-$almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that for $\mu-$almost everywhere $x \in X, \forall n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n\to\infty} \int_X |f - f_n| d\mu = 0.$$

Proof. Let N_n denote the zero measure set where f_n is not defined. Let N' denote the zero measure set where f is not defined. Then

$$N := N' \cup \{x \in X : \exists n \in \mathbb{N}, \text{ such that } |f_n(x) > g(x)|\} \cup \bigcup_{n=1}^{\infty} N_n$$

is measurable and has zero measure.
Define

$$h_n(x) := \begin{cases} f_n(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \ h(x) := \begin{cases} f(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \ g'(x) := \begin{cases} g(x) & x \in X \setminus N, \\ 0 & x \in N. \end{cases}$$

It is clear $\forall x \in X, h_n(x) \to h(x)$ point-wise, and dominated by $g'(x)$.
Since $g = g' \mu$-a.e. and thus $g' \in \mathcal{L}^1(X, \mu)$, by LDCT, we have

$$\lim_{n\to\infty} \int_X |f - f_n| d\mu = \lim_{n\to\infty} \int_X |g - g_n| d\mu = 0.$$

□

Theorem 3.19 (countable additivity). *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{C}$ be measurable functions, defined $\mu-$almost everywhere on X, such that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$. We have that $f(x) := \sum_{n=1}^{\infty} f_n(x)$ exists $\mu-$almost everywhere for $x \in X$, and that $f \in \mathcal{L}^1(X, \mu)$, and that*

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

Proof. For each n, let $D_n \subseteq X$ be the domain of f_n, then by assumption $\mu(X \setminus D_n) = 0$.
Let $D := \bigcap_{n=1}^{\infty} D_n, g(x) := \sum_{n=1}^{\infty} |f_n(x)|$. Note that $\mu(D^c) = \mu((\bigcap_{n=1}^{\infty} D_n)^c) = \mu(\bigcup_{n=1}^{\infty} D_n^c) = 0$.
Thus $g : X \to [0, \infty]$ is defined almost everywhere by Monotone Convergence Theorem.
By countable additivity of positive functions and assumption,

$$\int_X g d\mu = \sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty,$$

so $g \in \mathcal{L}^1$.
Let $A := \{x \in D : g(x) < \infty\}$, then we have $\mu(A^c) = 0$.
By definition, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ absolutely on A. Thus $f \in \mathcal{L}^1(A, \mathcal{M}|_A, \mu|_{\mathcal{M}|_A})$.
Let $h_n = \sum_{i=1}^{n} f_i$ on A, then $|h_n| \leq \sum_{i=1}^{n} |f_n| \leq g$. Also, we have that $h_n(x) \to f(x)$ for any $x \in A$, then by LDCT and linearity, we have

$$\int_A f d\mu = \lim_{n\to\infty} \int_A h_n d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_A f_i d\mu = \sum_{n=1}^{\infty} \int_A f_n d\mu.$$

Since $\mu(A^c) = 0$, we have that $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$. □

3.5 Complete Measure

Theorem 3.20. *Let $(X, \mathcal{M}, \mu)$ be a measure space, let*

$$\mathcal{M}^* := \{A \subseteq X : \exists B, C \in \mathcal{M}, \text{ such that } B \subseteq A \subseteq C, \mu(C \setminus B) = 0\}.$$

Define $\mu^*(A) = \mu(B) = \mu(C)$, then $\mathcal{M}^*$ is a σ-algebra, μ^* is a measure, and $(X, \mathcal{M}^*, \mu^*)$ is a measure space.

Proof. $X \in \mathcal{M}$, and $X \subseteq X \subseteq X$ and $\mu(X \setminus X) = 0$, thus $X \in \mathcal{M}^*$.
Let $A \in \mathcal{M}^*$, then there are $B, C \in \mathcal{M}$ such that $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$. Thus $B^c \supseteq A^c \supseteq C^c$, and $B^c, C^c \in \mathcal{M}$. In addition, $\mu(B^c \setminus C^c) = \mu(B^c \cap C) = \mu(C \setminus B) = 0$. Thus $A^c \in \mathcal{M}^*$.
Let $A_n \in \mathcal{M}^*$ and $A = \bigcup_n A_n$, and $B_n, C_n \in \mathcal{M}$ such that $B_n \subseteq A_n \subseteq C_n, \mu(C_n \setminus B_n) = 0$. Let $B = \bigcup_n B_n, C = \bigcup_n C_n \in \mathcal{M}$, thus $B \subseteq A \subseteq C$. Now $\mu(C \setminus B) = \mu(\bigcup_n C_n \setminus B) \leq \mu(\bigcup_n C_n \setminus B_n) \leq \sum_n \mu(C_n \setminus b_n) = 0$.
Thus $\mathcal{M}^*$ is a σ-algebra.
Now suppose $B, B', C, C' \in \mathcal{M}$, and $A, \in \mathcal{M}^*$, with $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$, and $B' \subseteq A \subseteq C', \mu(C' \setminus B') = 0$.
Thus $B \setminus B' \subseteq A \setminus B' \subseteq C' \setminus B'$, and thus $\mu(B \subset B') \leq \mu(C' \setminus B') = 0$. Thus $\mu(B) = \mu(B \cap B') + \mu(B \subset B') = \mu(B \cap B')$. Similarly, we can show that $\mu(B') = \mu(B \cap B')$. Thus $\mu^* : \mathcal{M}^* \to [0, \infty]$ is well-defined.
Consider A_n a sequence of disjoint sets in $\mathcal{M}^*$, and $B_n, C_n \in \mathcal{M}$ as above. We have $\mu^*(A) = \mu(B) = \mu(\bigsqcup_n b_n) = \sum_n \mu(B_n) = \sum_n \mu(A_n)$. □

Corollary 3.21. *$(X, \mathcal{M}^*, \mu^*)$ has the property that if $N \in \mathcal{M}^*$ has $\mu(N) = 0$, we always have*

$$\forall A \subseteq N, \ A \in \mathcal{M}^*, \mu^*(A) = 0.$$

Proof. Notice that $\forall A \subseteq N$, we have $\mu(N) = \mu(\emptyset) = 0$, with $\emptyset \subseteq A \subseteq N$, so $A \in \mathcal{M}^*, \mu^*(A) = 0$. □

Definition 3.9. $(X, \mathcal{M}^*, \mu^*)$ defined above is called the **completion** of $(X, \mathcal{M}, \mu)$. In addition, we say $(X, \mathcal{M}, \mu)$ is **complete** if $(X, \mathcal{M}, \mu) = (X, \mathcal{M}^*, \mu^*)$

Remark. If there is some $A \in \mathcal{M}$ such that $\mu(A^c) = 0$, then for any measurable $f : A \to Y$, we can extend it to X by $\forall x \in A^c,\ f(x) := 0$. Furthermore, if $(X, \mathcal{M}, \mu)$ is complete, we can extend f to whatever value we want. One can check that $f : X \to Y$ is measurable, and the integral $\int_X f d\mu$ does not depend on the extension.

Proposition 3.22. *If $(X, \mathcal{M}, \mu)$ is a complete measure, we always have that property P holds μ-a.e. iff*

$$\mu(\{x \in A : p(x) = False\}) = 0.$$

Proof. If P holds μ-a.e., there is $\exists N \in \mathcal{M}$, such that $\mu(N) = 0$ and $\forall x \in A \setminus N,\ p(x) = True$. Since $\{x \in A : p(x) = False\} \subseteq A \setminus (A \setminus N) = N$, we have $\mu(\{x \in A : p(x) = False\}) = 0$.
On the other hand, if $\mu(\{x \in A : p(x) = False\}) = 0$, we can just let $N := \mu(\{x \in A : p(x) = False\})$. Notice $\mu(N) = 0$, and $\forall x \in A \setminus N,\ p(x) = True$. □

Proposition 3.23. *Let μ be a complete measure on $(X, \mathcal{M})$, suppose that f is measurable, and $g = f$, a.e., then g is also measurable. Moreover, if (f_n) is a sequence of measurable functions, and $f_n \to f$, μ-a.e., we always have that f is also measurable.*

Proof. Suppose f is measurable, and we consider $D := \{x : X | f(x) \neq g(x)\}, \mu(D) = 0$.
Now let $B \subseteq \mathbb{R}$ be a Borel set, we need to show that $\{x \in X | g(x) \in B\} \in \mathcal{M}$.
Write $\{x \in X | g(x) \in B\} = (\{x \in X | g(x) \in B\} \cap D) \sqcup (\{x \in X | g(x) \in B\} \setminus D)$.
Since μ is complete, we have that $\{x \in X | g(x) \in B\} \cap D \in \mathcal{M}$ and has measure zero. Since f is measurable, we have that $f^{-1}(B) = \{x \in X | f(x) \in B\} \supseteq \{x \in X | f(x) = g(x) \in B\} = \{x \in X | g(x) \in B\} \setminus D$ is measurable.
Since μ is complete, we have that $\{x \in X | g(x) \in B\} \setminus D$ is measurable.
Thus $\{x \in X | g(x) \in B\} \in \mathcal{M}$ is measurable.
For the second part, consider $g = \limsup_{n \to \infty} f_n$. □

4 Construction of Measure

4.1 Caratheodory Theorem

Definition 4.1. Let X be a non-empty set, an **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\mu^*(\emptyset) = 0$
2. Monotone: $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$
3. Countable subadditive: For any $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$, we have that $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$

Proposition 4.1. *Let $\mathcal{C} \subseteq \mathcal{P}(x)$ with $\emptyset, X \in \mathcal{C}$. Let $\tilde{\mu} : \mathcal{C} \to [0, \infty]$ be a function such that $\tilde{\mu}(\emptyset) = 0$. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by $\mu^*(A) := \inf \{\sum_{i=1}^\infty \tilde{\mu}(C_i) : C_i \in \mathcal{C}, A \subseteq \bigcup_{i=1}^\infty C_i\}$. Then μ^* is an outer measure.*

Proof. Clearly $\mu^*(\emptyset) = 0$, since $\emptyset \in \mathcal{C}$.

In addition, $A \subseteq B \subseteq \bigcup_{i=1}^\infty C_i$ for any cover for B, and thus $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

Given any $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$. If $\sum_{n=1}^\infty \mu^*(A_n) = \infty$, then $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$ trivially.

Now assume that $\sum_{n=1}^\infty \mu^*(A_n) < \infty$. Consider any $\epsilon > 0$. For each $n \geq 0$, choose $(C_{n,i})_{i=1}^\infty \subset \mathcal{C}$, such that $A_n \subseteq \bigcup_{i=1}^\infty C_{n,i}$ and $\mu^*(A_n) \leq \sum_{i=1}^\infty \tilde{\mu}(C_{n,i}) < \mu^*(A_n) + \frac{\epsilon}{2^n}$. Thus $\bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty C_{n,i}$, so by construction of the outer measure,

$$\begin{aligned}
\mu^*\left(\bigcup_{n=1}^\infty A_n\right) &\leq \sum_{n=1}^\infty \sum_{i=1}^\infty \tilde{\mu}(C_{n,i}) \\
&\leq \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) \\
&\leq \sum_{n=1}^\infty \mu^*(A_n) + \epsilon.
\end{aligned}$$

Taking $\epsilon \to 0$, we have $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$. □

Remark. Notice that if $X = \mathbb{R}$, and we take $\mathcal{C} := \{(a, b] : a < b \in \mathbb{R}\}$ to be the collection of finite half open intervals, and $\mu((a, b])$ to be the length of the interval $b - a$, then the outer measure is the Lebesgue outer measure.

Definition 4.2. For an outer measure μ^*, we say $A \subseteq X$ is μ^***-measurable**, or satisfies the **Caratheodory condition** if

$$\forall E \subseteq X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Remark. To check $A \in \mathcal{M}$, it suffices to check for $\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(E \cap A^c)$, since $\mu^*(E) \leq \mu^*(A \cap E) + \mu^*(E \cap A^c)$ always holds by subadditivity of μ^*. Further, when $\mu^*(E) = \infty$, the inequality is always true, so it suffices to check

$$\forall E \subseteq X, \text{ such that } \mu^*(E) < \infty, \ \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Lemma 4.2. *Let μ^* be an outer measure on X, and $\mathcal{M}$ be the μ^*-measurable subsets of X, then $\mathcal{M}$ is an algebra, and $\mu := \mu^*|_{\mathcal{M}}$ has finite additivity.*

Proof. When $\mu^*(E) = \infty$, this holds trivially, and thus it suffices to only check for $\mu^*(E) < \infty$.

- Clearly $\emptyset \in \mathcal{M}$.
- $A \in \mathcal{M} \implies A^c \in \mathcal{M}$ since the condition is symmetric.
- Now consider $A, B \in \mathcal{M}$. For any $E \subseteq X$, we have

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\
&\geq \mu^*((E \cap A) \cup (E \cap A^c \cap B)) + \mu^*(E \cap A^c \cap B^c) \\
&= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).
\end{aligned}$$

  Thus $A \cup B \in \mathcal{M}$.

Thus $\mathcal{M}$ is an algebra.
To show finite additivity: Let $(A_i)_{i=1}^n$ be disjoint in $\mathcal{M}$, we will use induction on n.
Clearly, it is true for $n = 1$.
Now suppose it holds for n, let $B = \bigsqcup_{i=1}^n A_i$. Since $\mathcal{M}$ is an algebra, we have $B \in \mathcal{M}$. For any $E \subseteq X$,

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\
&= \mu^*\left(\bigsqcup_{i=1}^n (E \cap A_i)\right) + \mu^*(E \cap B^c) \\
&= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B^c).
\end{aligned}$$

Taking $E = \bigsqcup_{i=1}^{n+1} A_i$, we have

$$\begin{aligned}
\mu^*\left(\bigsqcup_{i=1}^{n+1} A_i\right) &= \sum_{i=1}^n \mu^*\left(\left(\bigsqcup_{i=1}^{n+1} A_i\right) \cap A_i\right) + \mu^*\left(\bigsqcup_{i=1}^{n+1} A_i \cap B^c\right) \\
&= \sum_{i=1}^n \mu^*(A_i) + \mu^*(A_{n+1}) \\
&= \sum_{i=1}^{n+1} \mu^*(A_i).
\end{aligned}$$

By induction, we have finite additivity for any $n \geq 1$. $\square$

Theorem 4.3 (Caratheodory)**.** *Let μ^* be an outer measure on X, and $\mathcal{M}$ be the μ^*-measurable subsets of X, then $\mathcal{M}$ is a σ-algebra, and $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure.*

Proof. Consider any $\{A_i\} \subset \mathcal{M}, B := \bigcup_{i=1}^{\infty} A_i$. By taking $\tilde{A}_n := A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ we can WLOG assume A_n are pair-wise disjoint, and $B = \bigsqcup_{i=1}^{\infty} A_i$.
For any $E \in X$, we have $\forall n \geq 1, \bigsqcup_{i=1}^{n} A_i \in \mathcal{M}$, and thus

$$\begin{aligned} \mu^*(E) &= \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)\right) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)^c\right) \\ &= \mu^*\left(\bigsqcup_{i=1}^{n} (E \cap A_i)\right) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)^c\right) \\ &= \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)^c\right) \\ &\geq \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{\infty} A_i\right)^c\right) \\ &= \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap B^c). \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} \mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c) \\ &\geq \mu^*\left(\bigsqcup_{i=1}^{\infty} (E \cap A_i)\right) + \mu^*(E \cap B^c) \\ &\geq \mu^*\left(E \cap \bigsqcup_{i=1}^{\infty} A_i\right) + \mu^*(E \cap B^c) \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\ &\geq \mu^*((E \cap B) \cup (E \cap B^c)) \\ &= \mu^*(E). \end{aligned}$$

Thus $B \in \mathcal{M}$, and thus $\mathcal{M}$ is a σ-algebra.
In addition, taking $E = B$, we have

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) + \mu^*(B \cap B^c) = \sum_{i=1}^{\infty} \mu^*(A_i) + \mu^*(\emptyset) = \sum_{i=1}^{\infty} \mu^*(A_i),$$

which shows countable additivity, and thus $\mu^*|_{\mathcal{M}}$ is a measure.
To show completeness, suppose $A \subseteq X$ such that $\mu^*(A) = 0$, then for any $E \subseteq X$, we have

$$\begin{aligned} \mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(A) + \mu^*(E) \\ &= \mu^*(E). \end{aligned}$$

Thus we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and thus $A \in \mathcal{M}$. $\square$

4.2 Premeasures

Definition 4.3. Recall an algebra of subsets of a set X is a family of subsets that is closed under complements, finite unions, and finite intersections, and contains the empty set.

Definition 4.4. A premeasure on an algebra of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ is a function $\tilde{\mu} : \mathcal{A} \to [0, \infty]$, such that $\tilde{\mu}$ is countably additive. Namely, if $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$ are disjoint, and $\bigsqcup_{i=1}^{\infty} A_i \subseteq \mathcal{A}$, then we have

$$\tilde{\mu}\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \tilde{\mu}(A_i).$$

Remark. If $\mathcal{A}$ is a σ-algebra, a premeasure on $\mathcal{A}$ is always a measure.

Theorem 4.4. *Let $\mathcal{A}$ be an algebra of subsets of X, and $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ be a premeasure. Apply the Caratheodory Theorem 4.3 to the outer measure μ^* by proposition 4.1 gives a complete measure space $(X, \mathcal{M}, \mu)$, such that $\mathcal{A} \subseteq \mathcal{M}$, and $\mu|_{\mathcal{A}} = \tilde{\mu}$.*

Proof. Choose any $A \in \mathcal{A}$. We have

$$\mu^*(A) := \inf\left\{\sum_{i=1}^{\infty} \tilde{\mu}(A_i) : A_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^{\infty} A_i\right\} \leq \tilde{\mu}(A).$$

Choose any $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$, such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$. Let $B_i = (A \cap A_i) \setminus \bigcup_{j=1}^{i-1} A_j$. Notice that $B_i \in \mathcal{A}$, and are pairwise disjoint, and $A = \bigsqcup_{i=1}^{\infty} B_i$. Since $\tilde{\mu}$ is a premeasure, $\tilde{\mu}(A) = \sum_{i=1}^{\infty} \tilde{\mu}(B_i) \leq \sum_{i=1}^{\infty} \tilde{\mu}(A_i)$. Since above holds for any $\bigcup_{i=1}^{\infty} A_i \supseteq A$, we can see that $\mu^*(A) \geq \tilde{\mu}(A)$, which forces

$$\mu^*(A) = \tilde{\mu}(A).$$

Now it remains to show $A \in \mathcal{M}$, which is the same as A is μ^*-measurable. Given any $E \subseteq X$ with $\mu^*(E) < \infty$. Fix any $\epsilon > 0$, there are $(E_i)_{i=1}^{\infty} \subseteq \mathcal{A}$, such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$, and

$$\sum_{i=1}^{\infty} \tilde{\mu}(E_i) < \mu^*(E) + \epsilon.$$

Notice that $E \cap A \subseteq \bigcup_{i=1}^{\infty} E_i \cap A$, and $E \cap A^c \subseteq \bigcup_{i=1}^{\infty} E_i \cap A^c$. Also, each $E_i \cap A, E_i \cap A^c \in \mathcal{A}$, since $\mathcal{A}$ is an algebra. Thus,

$$\begin{aligned}
\mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \sum_{i=1}^{\infty} \mu^*(E_i \cap A) + \sum_{i=1}^{\infty} \mu^*(E_i \cap A^c) \\
&= \sum_{i=1}^{\infty} \tilde{\mu}(E_i \cap A) + \sum_{i=1}^{\infty} \tilde{\mu}(E_i \cap A^c) \\
&= \sum_{i=1}^{\infty} \tilde{\mu}(E_i) \\
&< \mu^*(E) + \epsilon
\end{aligned}$$

Now take $\epsilon \to 0$, we have that

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E).$$

This shows that A is μ^*-measurable, which means $\mathcal{A} \subseteq \mathcal{M}$.

We have shown that $\tilde{\mu} = \mu^*|_A$, but we also know that $\mu = \mu^*|_{\mathcal{M}}$, and $\mathcal{A} \subseteq \mathcal{M}$, so $\mu|_{\mathcal{A}} = \tilde{\mu}$. ∎

Definition 4.5. A premeasure $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ on an algebra $\mathcal{A}$ for X is σ-finite if there are $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$, such that $\tilde{\mu}(A_i) < \infty$ and $\bigcup_{i=1}^{\infty} A_i = X$.

Proposition 4.5. *Let $\mathcal{A}$ be an algebra of sets on X. Let $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ be a premeasure, with the corresponding complete measure space $(X, \mathcal{M}, \mu)$ as in the above theorem. Suppose $(X, \mathcal{N}, \nu)$ is a measure space with $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M}$ and $\nu|_{\mathcal{A}} = \tilde{\mu}$. Then if $\tilde{\mu}$ is σ-finite, we have that*

$$\nu = \mu|_{\mathcal{N}},$$

so $\mu|_{\mathcal{N}}$ is the unique extension of $\tilde{\mu}$ to a measure on $\mathcal{N}$.

4.3 Lebesgue-Stieltjes Measures

Definition 4.6. Let μ be a Borel measure on $\mathbb{R}$, such that $\mu(\mathcal{K}) < \infty$ for any compact $\mathcal{K} \subseteq \mathbb{R}$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \begin{cases} \mu((0,x]) & \text{if } x \geq 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}$

Proposition 4.6. *F is monotone non-decreasing. i.e. If $b \geq a$, then $F(b) - F(a) \geq 0$.*

Proof. For $0 < a < b$, we have that $\mu((a,b]) = \mu((0,b] \setminus (0,a]) = \mu((0,b]) - \mu((0,a]) = F(b) - F(a)$.
For $0 \geq b > a$, we have that $\mu((a,b]) = \mu((a,0] \setminus (b,0]) = \mu((a,0]) - \mu((b,0]) = -F(a) - (-F(b)) = F(b) - F(a)$.
Similarly, we can check for $a < 0 \leq b$. □

Proposition 4.7. *F is right continuous.*

Proof. Fix $x \geq 0 \in \mathbb{R}$, and choose any sequence $(x_n) \subseteq \mathbb{R}$ such that $x_n \geq x_{n+1}$ and $x_n \to x$.
Since μ is a measure and $\mu((0,x_1]) \leq \mu([0,x_1]) < \infty$, we have

$$\begin{aligned} F(x) &= \mu((0,x]) \\ &= \lim_{n\to\infty} \mu((0,x_n]) \\ &= \lim_{n\to\infty} F(x_n). \end{aligned}$$

Similar proof for $x < 0 \in \mathbb{R}$. □

Example 4.3.1. If $\mu = \delta_c, \delta_c(A) = \begin{cases} 0 & \text{if } c \in A \\ 1 & \text{if } c \notin A \end{cases}$, then F is the (translated) Heaviside function.

Example 4.3.2. If μ is the Lebesgue measure, then F is the identity function $F(x) = x$.

Now given a right-continuous increasing function F, we want to construct a measure.

Proposition 4.8. *Let $\mathcal{A}$ be the collection of sets consisting of all the finite disjoint unions of half-open intervals $(a,b], -\infty \leq a \leq b \leq \infty$. Then $\mathcal{A}$ is an algebra of sets.*

Proof. Firstly notice that for any interval $(a,b] \in \mathcal{A}$, we have that $(a,b]^c = [-\infty,a] \cup (b,\infty] \in \mathcal{A}$.
Also, any finite union of such disjoint unions can be written as a disjoint union. $(a,b] \cup (c,d] = (a,c] \cup (c,b] \cup (b,d]$ for $c < b$. We can show any finite union by induction. □

Definition 4.7. Let $\mathcal{A}$ be the algebra of sets consisting of all the finite disjoint unions of half-open intervals $(a,b], -\infty \leq a \leq b \leq \infty$. Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous monotone non-decreasing function, we can extend F to $[-\infty,\infty]$ by $F(\pm\infty) := \lim_{x\to\pm\infty} F(x)$, which exists by MCT. Now define $\tilde{\mu}_F : \mathcal{A} \to [0,\infty]$ to be

$$\tilde{\mu}_F\left(\bigsqcup_{i=1}^{n} (a_i,b_i]\right) := \sum_{i=1}^{n} F(b_i) - F(a_i)$$

Lemma 4.9. *$\tilde{\mu}_F$ is a pre-measure*

Proof. We firstly show that $\tilde{\mu}_F$ is well defined.
Consider $I_i = (a_i,b_i]$ and $I = (a,b] = \bigsqcup_1^n I_i$
By reordering, we can WLOG assume that $a = a_1 < b_1 = a_2 < b_2 = \cdots < b_{n-1} = a_n < b_n = b$.
Let $a_{n+1} := b_n = b$, we have that

$$\begin{aligned} \tilde{\mu}_F\left(\bigsqcup_{i=1}^{n} (a_i,b_i]\right) &= \sum_{i=1}^{n} F(b_i) - F(a_i) \\ &= \sum_{i=1}^{\infty} F(a_{i+1}) - F(a_i) \\ &= F(a_{n+1}) - F(a_1) \\ &= F(b) - F(a) \\ &= \tilde{\mu}_F(I). \end{aligned}$$

Thus $\tilde{\mu}_F(I)$ does not depend on the decomposition of I. This extends to finite disjoint unions of half-open intervals. Hence $\tilde{\mu}_F$ is well-defined.
Monotone follows from the fact that F is increasing.
Consider pair-wise disjoint $(A_i)_{i=1}^{\infty} \in \mathcal{A}$, and $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$, we want to show $\tilde{\mu}_F(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \tilde{\mu}_F(A_i)$.
We first assume that each $A_i = (a_i, b_i]$ and $\bigsqcup_1^{\infty} A_i = (a, b]$
Notice that $\forall n \in \mathbb{N}$, we have that $\bigsqcup_{i=1}^{n}(a_i, b_i] \in \mathcal{A}$, and thus $(a, b] \setminus \bigsqcup_{i=1}^{n}(a_i, b_i] \in \mathcal{A}$.

$$\begin{aligned}
\tilde{\mu}_F((a,b]) &= \tilde{\mu}_F(\bigsqcup_{i=1}^{n}(a_i, b_i]) + \tilde{\mu}_F((a,b] \setminus \bigsqcup_{i=1}^{n}(a_i, b_i]) \\
&= \sum_{i=1}^{n} \tilde{\mu}_F((a_i, b_i]) + \tilde{\mu}_F((a,b] \setminus \bigsqcup_{i=1}^{n}(a_i, b_i]) \\
&\geq \sum_{i=1}^{n} \tilde{\mu}_F((a_i, b_i])
\end{aligned}$$

Thus $\tilde{\mu}_F((a,b]) \geq \sum_1^{\infty} \tilde{\mu}_F((a_i, b_i])$.
For the other direction, fix $\epsilon > 0$, by right continuity, $\exists \delta > 0$, such that $F(a+\delta) < F(a) + \epsilon, a + \delta < b$. Now suppose $b \neq \infty$, then $\exists \delta_i > 0, F(b_i + \delta_i) < F(b_i) + 2^{-i}\epsilon$.
Thus $[a_i + \delta, b_i] \subseteq (a_i, b_i + \delta_i)$, and thus $\{(a_i, b_i + \delta_i)\}$ is an open cover for $[a + \delta, b] = \bigcup_{i=1}^{\infty}[a_i + \delta, b_i]$. Since the closed interval is compact, there is a finite sub-cover $\left\{(a_{i_j}, b_{i_j} + \delta_{i_j})\right\}_{j=1}^{n}$. Then

$$\begin{aligned}
\sum_{j=1}^{n}(F(b_{i_j} + \delta_{i_j}) - F(a_{i_j})) &= \sum_{j=1}^{n} \tilde{\mu}_F((a_{i_j}, b_{i_j} + \delta_{i_j})) \\
&\geq \tilde{\mu}_F((a+\delta, b]) \\
&\geq F(b) - F(a+\delta)
\end{aligned}$$

since $\tilde{\mu}_F$ is monotone. Hence

$$\begin{aligned}
\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i, b_i]) &= \sum_{i=1}^{\infty}(F(b_i) - F(a_i)) \\
&\geq \sum_{j=1}^{n}(F(b_{i_j}) - F(a_{i_j})) \\
&\geq \sum_{j=1}^{n}(F(b_{i_j} + \delta_{i_j}) - 2^{-i_j}\epsilon - F(a_{i_j})) \\
&\geq F(b) - F(a+\delta) - \epsilon \\
&\geq F(b) - F(a) - 2\epsilon \\
&= \tilde{\mu}_F((a,b]) - 2\epsilon.
\end{aligned}$$

Take $\epsilon \to 0$, we have $\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i, b_i]) \geq \tilde{\mu}_F((a,b])$
When $b = \infty$, we have that $\forall N \geq a, \sum_{i=1}^{\infty} \tilde{\mu}_F((a_i, b_i]) \geq \tilde{\mu}_F((a, N]) = F(N) - F(a)$.
Hence $\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i, b_i]) \geq F(b) - F(a) = \lim_{N \to b} F(N) - F(a)$
Thus we have shown that $\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i, b_i]) = \tilde{\mu}_F((a,b])$
If $A = \bigsqcup_1^m (c_i, d_i]$, we can use finite additivity and the previous case. □

Theorem 4.10. *Let $F : \mathbb{R} \to \mathbb{R}$ is monotone non-decreasing and right-continuous, then there is a complete measure space $(\mathbb{R}, \mathcal{M}, \mu_F)$, which extends $\tilde{\mu}_F$ as $\mu_F|_{\mathcal{A}} = \tilde{\mu}_F$, and the $\sigma-$algebra $\mathcal{M}$ contains $Bor(\mathbb{R})$ and $\mu_F|_{Bor(\mathbb{R})}$ is the unique extension of $\tilde{\mu}_F$, i.e. $\mu_F((a,b]) = F(b) - F(a)$.*

Conversely, given a Borel measure μ on $\mathbb{R}$, such that $\forall K$ compact, $\mu(K) < \infty$, there is a (up to constant) unique non-decreasing right-continuous F with $\mu = \mu_F|_{B_{\mathbb{R}}}$.

Proof. By the previous lemma, $\tilde{\mu}_F$ is a premeasure, so applying Caratheodory gives a complete measure space $(\mathbb{R}, \mathcal{M}, \mu_F)$. We have seen that the σ-algebra generated by $\mathcal{A}$ is $Bor(\mathbb{R})$, so $Bor(\mathbb{R}) \subseteq \mathcal{M}$. The uniqueness follows from the fact that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}(n, n+2]$, $\tilde{\mu}_F((n, n+2]) = F(n+2) - F(n) < \infty$ and thus $\tilde{\mu}_F$ is σ-finite.

Conversely, let F be the function defined at the beginning fo this section. Then we know that $\mu = \mu_F|_{B_\mathbb{R}}$ since μ is σ-finite and agree with the premeasure $\tilde{\mu}_F$ on the algebra $\mathcal{M}$ of all finite disjoint unions of half open interval.

If $G : \mathbb{R} \to \mathbb{R}$ is another monotone non-decreasing right-continuous function, then $\mu_F = \bar{\mu_G} \implies \tilde{\mu}_F((a, b]) = \mu_G((a, b]) \implies F(b) - F(a) = G(b) - G(a)$ for any $a < b$. Thus $\forall a \in \mathbb{R}, F(x) - G(x) = c := F(0) - G(0)$, which is a constant. □

Example 4.3.3. The Lebesgue measure is got by taking $F(x) = x$.

Example 4.3.4. The Dirac measure

$$\delta_c(A) := \begin{cases} 1 & c \in A \\ 0 & c \notin A \end{cases}$$

is got by taking

$$F(x) = H_c(x) = \begin{cases} 1 & x \geq c \\ 0 & x < c \end{cases},$$

the Heaviside function.

Definition 4.8. A μ be a Borel measure on $\mathbb{R}$, such that $\mu(K) < \infty$ for any compact $K \subseteq \mathbb{R}$, is called a **Lebesgue-Stieltjes measure**. For any $F : \mathbb{R} \to \mathbb{R}$ that is monotone non-decreasing and right-continuous, and any Borel measure μ such that $\mu = \mu_F|_{\mathrm{Bor}_\mathbb{R}}$, we call μ the **Lebesgue-Stieltjes measure corresponding to F**.

Proposition 4.11. *Let μ be a Lebesgue-Stieltjes measure corresponding to some $F : \mathbb{R} \to \mathbb{R}$, then $\forall a \in \mathbb{R}$,*

$$\mu(\{a\}) = \mu\left(\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, a]\right) = \lim_{n\to\infty} \mu((a - \frac{1}{n}), a]) = F(a) - \lim_{n\to\infty} F(a - \frac{1}{n}) = F(a) - F(a^-).$$

Thus $\mu(\{a\}) > 0$ if and only if F has a jump discontinuity at a, since every discontinuity of a monotone non-decreasing function is a jump discontinuity.

Corollary 4.12. *If $F(x) = x$ is the identity function, every countable set has measure 0, by subadditivity and that $\forall a \in \mathbb{R}, \mu(\{a\}) = 0$*

Proposition 4.13. *A monotone non-decreasing function F can have at most countably many discontinuities.*

Proof. Choose countably many disjoint points $\{c_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Define a measure $\mu := \sum_{n\geq 1} \frac{1}{2^n}\delta_{c_n}$. This is a Borel measure with $\mu(\mathcal{K}) < \infty$ for any compact $\mathcal{K} \subseteq \mathbb{R}$. Thus μ is a Lebesgue-Stieltjes measure. Note $\mu(\{c_n\}) = \frac{1}{2^n} > 0$, thus each c_n is a jump discontinuity for the corresponding F. Thus F has countably many discontinuities.
In fact, no such F can have uncountably many discontinuities. □

Theorem 4.14. *Lebesgue measure $(\mathbb{R}, \mathcal{L}, \lambda)$ is translation-invariant, meaning*

$$\forall A \in \mathcal{L}, s \in \mathbb{R},\ \lambda(A + s) = \lambda(A).$$

Also,

$$\forall s > 0, A \in \mathcal{L},\ \lambda(sA) = s\lambda(A).$$

Proof. If $A \subseteq \mathbb{R}$ is open, then so is $A + s$. Similarly for closed sets. Hence for $A \in B_\mathbb{R}, A + s \in B_\mathbb{R}$. Define a new measure λ_s on $B_\mathbb{R}$ by $\lambda_s(A) = \lambda(A + s)$. Note that λ and λ_s correspond to the functions

$$F(x) = \begin{cases} \lambda((0, x]) & \text{if } x \geq 0 \\ -\lambda((x, 0]) & \text{if } x < 0 \end{cases},$$

$$G(x) = \begin{cases} \lambda_s((0,x]) & \text{if } x \geq 0 \\ -\lambda_s((x,0]) & \text{if } x < 0 \end{cases}.$$

Yet $\lambda((0,x]) = \lambda((s,x+s]) = \lambda_s((0,x+s])$, and thus $F = G$. Thus $\lambda_s|_{B_\mathbb{R}} = \lambda|_{B_\mathbb{R}}$. By uniqueness for σ-finite Caratheodoy Theorem, we have that they extends to $\lambda = \lambda_s$. $\square$

Definition 4.9. A point $c \in \mathbb{R}$ with $\mu(\{x\}) \neq 0$ is called an **atom** of μ.

Corollary 4.15. *Lebesgue-Stieltjes measures can have at most countably many atoms.*

Definition 4.10. Let X be a topological space, then a $\mathcal{G}_\delta$ set is a countable intersection of open subsets of X, and a $\mathcal{F}_\sigma$ set is a countable union of closed subsets.

Remark. $\mathcal{G}_\delta$ sets and $\mathcal{F}_\sigma$ sets are Borel sets.

Theorem 4.16 (regularity)**.** *Let μ be a Lebesgue-Stieltjes measure with outer measure μ_F^*, and $E \subseteq \mathbb{R}$, the following are equal:*

(1) E is μ-measurable

(2) $\forall \epsilon > 0$, there is some open $O \supseteq E, \mu_F^*(O \setminus E) < \epsilon$ (Outer regularity)

(3) $\forall \epsilon > 0$, there is some closed $C \subseteq E, \mu_F^*(E \setminus C) < \epsilon$ (Inner regularity)

(4) There is a $\mathcal{G}_\delta$ set $G \supseteq E, \mu_F^*(G \setminus E) = 0$

(5) There is a $\mathcal{F}_\sigma$ set $F \subseteq E, \mu_F^*(E \setminus F) = 0$

Proof. Notice that E is μ-measurable means that

$$\forall A \subseteq \mathbb{R}, \mu_F^*(A) = \mu_F^*(E \cup A) + \mu_F^*(E^c \cup A).$$

1. (1) implies (2):
   If E is μ-measurable,

$$\begin{aligned} \mu(E) &= \mu_F^*(E) \\ &= \inf_{B \supseteq E} \mu_F(B), \end{aligned}$$

   where $B = \bigcup_{i=1}^\infty (a_i, b_i] \supseteq E$.
   Firstly, assume that E is bounded, we have $\mu_F^*(B) < \mu(E) + \frac{\epsilon}{2}$ for some $B = \bigsqcup_{i=1}^\infty (a_i, b_i] \supseteq E$.
   Since F is right-continuous, we have that $\forall i, \exists c_i > b_i,$ such that $F(c_i) < F(b_i) + \frac{\epsilon}{2^{i+1}}$.
   Let $O := \bigcup_i (a_i, c_i) \supseteq B \supseteq E$.
   Since E is measurable, we have that $\mu_F^*(B) = \mu_F^*(B \cap E) + \mu_F^*(B \setminus E) = \mu(E) + \mu_F^*(B \setminus E)$, thus

$\mu_F^*(B \setminus E) < \frac{\epsilon}{2}$

$$\begin{aligned}
\mu_F^*(O \setminus B) &= \mu_F^*(\bigcup_i (a_i, c_i) \cap B^c) \\
&\leq \sum_i \mu_F^*((a_i, c_i) \cap B^c) \\
&\leq \sum_i \mu_F^*((b_i, c_i)) \\
&\leq \sum_i \mu_F^*((b_i, c_i]) \\
&= \sum_i F(c_i) - F(b_i) \\
&< \sum_i \frac{\epsilon}{2^{i+1}} \\
&= \frac{\epsilon}{2}. \\
\mu_F^*(O \setminus E) &\leq \mu_F^*(O \cap E^c \cap B) + \mu_F^*(O \cap E^c \setminus B) \\
&= \mu_F^*(B \setminus E) + \mu_F^*(O \setminus B) \\
&< \epsilon.
\end{aligned}$$

This proves the bounded case.
If E is not bounded, we let $E_n = E \cap (n-1, n], n \in \mathbb{Z}$, each is bounded, and we can have open $O_n \supseteq E_n, \mu_F(O_n \setminus E_n) < \frac{\epsilon}{2^{2|n|}+1}$, and take $O = \cup_{n \in \mathbb{Z}} O_n \supseteq A$.

2. (2) implies (4):
For each $n \geq 1$, take open $O_n \supseteq E, \mu_F(O_n \setminus E) < \frac{1}{n}$, and WLOG, take $O_n = O_n \cap O_{n-1}$ so that $O_n \supseteq O_{n-1}$.
Take $G := \bigcap_{n=1}^{\infty} O_n$, which is a $\mathcal{G}_\delta$ set.
We have that $\forall n \geq 1, \mu_F^*(G \setminus E) \leq \mu_F^*(U_n \setminus E) < \frac{1}{n}$.
Thus $\mu_F^*(G \setminus E) = 0$.

3. (4) implies (1):
$G \setminus E$ is measurable since it is a null set, and μ is complete. G is also measurable, thus $E = G \setminus (G \setminus E)$ is also measurable.

4. (1) implies (3):
E is μ-measurable, so is E^c.
By (2), there is some open $O \supseteq E^c$, such that $\mu_F^*(O \setminus E^c) < \epsilon$.
Notice that $C := O^c$ is closed, and $C \subseteq E$, and

$$\begin{aligned}
\mu_F^*(E \setminus C) &= \mu_F^*(E \cap C^c) \\
&= \mu_F^*((E^c)^c \cap O) \\
&= \mu_F^*(O \setminus E^c) \\
&< \epsilon.
\end{aligned}$$

5. (3) implies (5)
For each $n \geq 1$, take closed $C_n \subseteq E, \mu_F(E \setminus C_n) < \frac{1}{n}$, and WLOG, take $C_n = C_n \cap C_{n-1}$ so that $C_n \supseteq C_{n-1}$.
Take $F := \bigcup_{n=1}^{\infty} C_n$, which is a $\mathcal{F}_\sigma$ set.
We have that $\forall n \geq 1, \mu_F^*(E \setminus F) \leq \mu_F^*(E \setminus C_n) < \frac{1}{n}$.
Thus $\mu_F^*(E \setminus F) = 0$.

6. (5) implies (1)
$E \setminus F$ is a measurable set. F is also a measurable set, and thus so is $E = (E \setminus F) \cup F$.

$\square$

Corollary 4.17. *Let μ be a Lebesgue-Stieltjes measure, and A be μ-measurable, we have*

$$\mu(A) = \inf\{\mu(O) : O \supseteq E \text{ is open}\} = \sup\{\mu(C) : C \subseteq A \text{ is compact}\}$$

Proof. The first equality is (2).

For the second equality, if A is bounded, and $C \subseteq A$ is closed, then it is compact. We can use (3) to prove it.
If A is not bounded, let $A_n := A \cap [-n, n]$ for each $N \geq 1$. Thus,

$$\mu(A) = \sup_{n\geq 1} \mu(A_n) = \sup_{n\geq 1} \sup_{C\subseteq A_n \text{ is compact}} \mu(C).$$

$\square$

4.4 Littlewood's Three Principles

Recall Littlewood's Three Principles for Lebesgue Measure:

Theorem 4.18. *Littlewood's first Principle (regularity)*
Every measurable set is almost a finite union of intervals.

Theorem 4.19. *Littlewood's second Principle (Lusin's)*
Every measurable function is almost continuous.

Theorem 4.20. *Littlewood's third Principle (Egorov's)*
A point-wise convergent sequence of measurable functions is almost uniformly convergent.

Theorem 4.21 (Egorov's)**.** *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $f_n : X \to \mathbb{C}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ μ-almost everywhere. Then $\forall \epsilon > 0, \exists A \in \mathcal{M}$, such that $\mu(X \setminus A) < \epsilon$, and $f_n \to f$ uniformly on A.*

Proof. $f_n \to f$ uniformly on A means $\forall m \in \mathbb{N}^+, \exists N_m \geq 1$, such that

$$\forall x \in A, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m}.$$

Let $A_{mN} := \left\{x \in X : \forall n \geq N, |f_n(x) - f(x)| < \frac{1}{m}\right\} = \bigcap_{n\geq N} \left\{x \in X : |f_n(x) - f(x)| < \frac{1}{m}\right\}$, which is intersection of preimages of $(-\frac{2}{m}, \frac{1}{m})$ of measurable functions $f_n - f$, thus measurable.
Note $A_{m1} \subseteq A_{m2} \subseteq \cdots$, and $\bigcup_{n\geq 1} A_{m,n} = X \setminus N$ for some $N \in \mathcal{M}, \mu(N) = 0$ since $f_n \to f$ μ-a.e..

$$\begin{aligned}\mu(X) &= \mu(X \setminus N)\\ &= \mu(\bigcup_{n\geq 1} A_{mn})\\ &= \lim_{n\to\infty} \mu(A_{mn}).\end{aligned}$$

Since $\mu(X) < \infty$, there is $N_m \geq 1$ such that $\mu(A_{m,N_m}) > \mu(X) - \frac{\epsilon}{2^m}$ for any $\epsilon > 0$.
Thus $\mu(X \setminus A_{m,N_m}) < \frac{\epsilon}{2^m}$.
Letting $E := \bigcap_{m\geq 1} A_{m,N_m}$, we have that

$$\begin{aligned}\mu(X \setminus E) &= \mu(\bigcup_{m\geq 1} (X \setminus A_{m,N_m}))\\ &\leq \sum_{m\geq 1} \mu(X \setminus A_{m,N_m})\\ &< \epsilon.\end{aligned}$$

In addition,

$$\begin{aligned} E &= \bigcap_{m\geq 1} A_{m,N_m} \\ &= \left\{ x \in X : \forall m \geq 1, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m} \right\}. \end{aligned}$$

Thus $f_n \to f$ uniformly on E. $\square$

Theorem 4.22 (Lusin's)**.** *Let $f : [a,b] \to \mathbb{C}$ be a Lebesgue-Stieltjes measurable function. For any $\epsilon > 0$, there is a continuous function $g : [a,b] \to \mathbb{C}$ such that*

$$\mu(\{x \in [a,b] : f(x) \neq g(x)\}) < \epsilon.$$

Proof. Consider a simple function $s := \sum_{i=1}^m \alpha_i \chi_{E_i}$, where $\alpha_i \in \mathbb{C}$, and E_i are disjoint and Lebesgue measurable.
Notice that by the regularity theorem, for any $\delta > 0$, there are closed sets $A_i \subseteq E_i$, such that $\mu(E_i \setminus A_i) < \frac{\delta}{m}$ for each i. Thus, $\mu(\bigsqcup_{i=1}^m E_i \setminus A_i) = \mu((\bigsqcup_{i=1}^m E_i) \setminus (\bigsqcup_{i=1}^m A_i)) = \mu([a,b] \setminus \mathcal{K}) < \delta$, for $\mathcal{K} := \bigsqcup_{i=1}^m A_i$.
Notice that $\mathcal{K}$ is closed (thus compact since $[a,b]$ is bounded), and $s|_{\mathcal{K}}$ is continuous since s is locally constant. Indeed, $\forall x \in \mathcal{K}$, there is unique A_i such that $x \in A_i$.
Suppose for contradiction that $\forall \delta_0 > 0$, there is some $y \in (x - \delta_0, x + \delta_0) \cap A_j$ for some $j \neq i$. Let $\mathcal{K}' := \bigsqcup_{j=1, j\neq i}^m A_j$, which is closed. Now we have a sequence $y_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap \mathcal{K}'$. Notice that $y_n \to x$, and since $\mathcal{K}'$ is closed, $x \in \mathcal{K}'$, which is a contradiction.
Thus $\exists \delta_0 > 0$, such that $(x - \delta_0, x + \delta_0) \cap \mathcal{K} \subseteq A_i$; namely, s in constant on $(x - \delta_0, x + \delta_0) \cap \mathcal{K}$. Thus, $\forall \epsilon_0 > 0, y \in \mathcal{K}$, such that $|y - x| < \delta_0, |s(x) - s(y)| = 0 < \epsilon_0$, whcih shows s is continuous around x.
Now given any measurable f, we can choose simple functions $s_n : [a,b] \to \mathbb{C}$ converging point-wise to f. For each n, construct $\mathcal{K}_n$ as above such that $s_n|_{\mathcal{K}_n}$ is continuous and $\mu([a,b] \setminus \mathcal{K}_n) < \frac{\epsilon}{2^{n+1}}$.
Let $\mathcal{K}_0 = \bigcap_{n\geq 1} \mathcal{K}_n$, which is compact. For all n, we have that $s_n|_{\mathcal{K}_n}$ is continuous.
In addition, $\mu([a,b] \setminus \mathcal{K}_0) \leq \sum_{n=1}^\infty \mu([a,b] \setminus \mathcal{K}_n) < \epsilon/2$.
By Egorov's Theorem, there is a measurable $E \subseteq \mathcal{K}_0$, such that $\mu(\mathcal{K}_0 \setminus E) < \epsilon/4$ and $s_n \to f$ uniformly on E.
Applying the regularity theorem again, there is a compact $\mathcal{K} \subseteq E$ such that $\mu(E \setminus \mathcal{K}) < \epsilon/4$. Notice that $s_n \to f$ uniformly on $\mathcal{K}$. Thus $f|_{\mathcal{K}}$ is continuous.
Also, $\mu([a,b] \setminus \mathcal{K}) \leq \mu([a,b] \setminus \mathcal{K}_0) + \mu(\mathcal{K}_0 \setminus E) + \mu(E \setminus \mathcal{K}) = \epsilon$.
By Tietze's Theorem, we can extend $f|_{\mathcal{K}}$ to some continuous $g : [a,b] \to \mathbb{C}$. We thus have

$$\mu(\{x \in [a,b] : f(x) \neq g(x)\}) \leq \mu([a,b] \setminus \mathcal{K}) < \epsilon.$$

$\square$

5 Lebesgue Spaces

5.1 The First Lebesgue Space

Definition 5.1. Given some measure space $(X, \mathcal{M}, \mu)$, define

$$\mathcal{L}^1(\mu) := \mathcal{L}^1(X, \mathcal{M}, \mu) := \left\{ f : X \to \mathbb{C} \,\middle|\, f \text{ is measurable}, \int_X |f| d\mu < \infty \right\}.$$

Proposition 5.1. *$\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space.*

Proof. Clearly $\int_X |0| d\mu = 0$, so the zero function $0 \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, we have

$$\begin{aligned} \int_X |c \cdot f| d\mu &= \int_X |c||f| d\mu \\ &= |c| \int_X |f| d\mu \\ &< \infty. \end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Now for any $f, g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, we have

$$\begin{aligned}\int_X |f+g| d\mu &\leq \int_X |f| + |g| d\mu \\ &= \int_X |f| d\mu + \int_X |g| d\mu \\ &< \infty.\end{aligned}$$

Thus $f + g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Since the set of all functions $\{f : X \to \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a subspace of it. □

Definition 5.2. Let

$$N = \left\{ f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : \int_X |f| d\mu = 0 \right\} = \left\{ f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f = 0 \ \mu - a.e. \right\}.$$

Define

$$L^1(X, \mathcal{M}, \mu) := \mathcal{L}^1(X, \mathcal{M}, \mu)/N,$$

which is the quotient vector space of $\mathcal{L}^1(X, \mathcal{M}, \mu)$ mod N.

Remark. $[f] = \left\{ g \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f - g = 0 \ \mu - \text{a.e.} \right\} \in L^1(X, \mathcal{M}, \mu)$

Definition 5.3. $\|[f]\|_{L^1(X,\mathcal{M},\mu)} := \int_X |f| d\mu$ for any choice of representative $f \in [f]$.

When the context is clear, we might write $L^1(X, \mathcal{M}, \mu)$ as $L^1(\mu)$ or $L^1(X)$. We might also write $\|\cdot\|_{L^1(X,\mathcal{M},\mu)}$ as $\|\cdot\|_{L^1(\mu)}$, $\|\cdot\|_{L^1(\mu)}$, $\|\cdot\|_1$.

Lemma 5.2. *The above definition is well defined.*

Proof. Take any $g, f \in [f]$. Let $K = \{x \in X : f(x) \neq g(x)\}$, we have $\mu(K) = 0$.

$$\begin{aligned}\int_X |f| d\mu &= \int_{X \setminus K} |f| d\mu + \int_K |f| d\mu \\ &= \int_{X \setminus K} |f| d\mu \\ &= \int_{X \setminus K} |g| d\mu \\ &= \int_{X \setminus K} |g| d\mu + \int_K |g| d\mu \\ &= \int_X |g| d\mu\end{aligned}$$

□

Proposition 5.3. $\|\cdot\|_1$ *is a norm on* $L^1(X, \mathcal{M}, \mu)$.

Proof. Consider any $[f], [g] \in L^1(X, \mathcal{M}, \mu)$.

$$\begin{aligned}\|[f] + [g]\|_1 &= \|[f + g]\|_1 \\ &= \int_X |f + g| d\mu \\ &\leq \int_X |f| d\mu + \int_X |g| d\mu \\ &= \|[f]\|_1 + \|[g]\|_1\end{aligned}$$

For any $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\|\alpha[f]\|_1 &= \|[\alpha f]\|_1 \\
&= \int_X |\alpha f| d\mu \\
&= |\alpha| \int_X |f| d\mu \\
&= |\alpha| \|[f]\|_1
\end{aligned}$$

If $\|[f]\|_1 = 0$, we must have $f = 0$ $\mu - a.e.$. Thus $f \in N$, thus $[f] = [0] = 0$. $\square$

Theorem 5.4 (Fischer-Riesz)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $\left(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)}\right)$ is a Banach Space.*

Proof. Let $([f_n])_1^\infty$ be a Cauchy sequence in $L^1(X, \mathcal{M}, \mu)$. Then for each $k \in \mathbb{N}^+$, there is some $N_k \geq 1$, such that $\forall m, n \geq N_k, \|[f_m] - [f_n]\|_{L^1(\mu)} < \frac{1}{2^k}$.
WLOG, $\forall k, N_{k+1} \geq N_k$.
Thus $\left\|[f_{N_{k+1}}] - [f_{N_k}]\right\|_{L^1(\mu)} < \frac{1}{2^k}$.
Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} \left|f_{N_{j+1}} - f_{N_j}\right|$, where we fix f_n to be a representative of $[f_n]$.
Notice that $\forall k \geq 1$,

$$\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1} \left|f_{N_{j+1}} - f_{N_j}\right| \\
&= g_k
\end{aligned}$$

We have that $\int_X g_k d\mu = \int_X |f_{N_1}| d\mu + \sum_{j=1}^{n} \int_X \left|f_{N_{j+1}} - f_{N_j}\right| d\mu$.
Let $g = \lim_{k\to\infty} g_k = |f_{N_1}| + \sum_{j=1}^{\infty} \left|f_{N_{j+1}} - f_{N_j}\right|$.
By LMCT, we have that

$$\begin{aligned}
\int_X g d\mu &= \lim_{k\to\infty} \int_X g_k d\mu \\
&= \int_X |f_{N_1}| d\mu + \sum_{j=1}^{\infty} \int_X \left|f_{N_{j+1}} - f_{N_j}\right| d\mu \\
&= \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^{\infty} \left\|[f_{N_{j+1}} - f_{N_j}]\right\|_{L^1(\mu)} \\
&= \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^{\infty} \left\|[f_{N_{j+1}}] - [f_{N_j}]\right\|_{L^1(\mu)} \\
&< \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&< \infty.
\end{aligned}$$

Thus $g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Thus $N := \{x \in X : g(x) = \infty\}$ has measure 0.
This implies that $f_{N_1}(x) + \sum_{k=1}^{\infty} (f_{N_{k+1}}(x) - f_{N_k}(x))$ converges absolutely for $x \in X \setminus N$.

We can thus define

$$
\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty}(f_{N_{k+1}}(x) - f_{N_k}(x)) \\
&= \lim_{k\to\infty}\left(f_{N_1}(x) + \sum_{j=1}^{k}(f_{N_{j+1}}(x) - f_{N_j}(x))\right) \\
&= \lim_{k\to\infty} f_{N_{k+1}}(x) \\
&= \lim_{k\to\infty} f_{N_k}(x)
\end{aligned}
$$

for $x \in X \setminus N$.
We then extend f to X by $f|_N := 0$.
Then $|f| \leq g$, and thus $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Notice that $|f_{N_k}| \leq g_k \leq g$, thus $|f - f_{N_k}| \leq g + g = 2g$.
By LDCT,

$$
\begin{aligned}
\lim_{k\to\infty} \|[f_{N_k}] - [f]\|_{L^1(\mu)} &= \lim_{k\to\infty} \|[f_{N_k} - f]\|_{L^1(\mu)} \\
&= \lim_{k\to\infty} \int_X |f_{N_k} - f| d\mu \\
&= \int_X \lim_{k\to\infty} |f_{N_k} - f| d\mu \\
&= 0.
\end{aligned}
$$

Thus $\lim_{k\to\infty}[f_{N_k}]$ converges to $[f]$. Since this is a subsequence of the Cauchy sequence $([f_n])_1^\infty$, we have that $\lim_{n\to\infty}[f_n] = [f]$.
This shows that $(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)})$ is complete. □

Remark. When we write $f \in L^1(\mu)$, we will mean $[f] \in L^1(\mu)$, and let $f \in \mathcal{L}^1(\mu)$ be any representative of $[f]$ when the context is clear.

5.2 Convex functions

Definition 5.4. A function $\phi : U \to \mathbb{R}$ is **convex** if

$$\forall x, y \in U, \forall \lambda \in [0, 1],\ \phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y).$$

Theorem 5.5 (Jensen's Inequality)**.** *If ϕ is convex, we have $\forall x_1, \ldots, x_n \in U$, and $\forall 0 \leq \lambda_1, \ldots, \lambda_n \leq 1$ such that $\sum_{i=1}^n \lambda_i = 1$,*

$$\phi\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i \phi(x_i).$$

Proof. The base case is when $n = 1$, which is trivial.
Now suppose this holds for $n - 1 \in \mathbb{N}$.
Given any $\forall x_1, \ldots, x_n \in U$, and $0 \leq \lambda_1, \ldots, \lambda_n \leq 1$ such that $\sum_{i=1}^n \lambda_i = 1$. If $\lambda_n = 0$, we can reduce the sum to a $n - 1$ sum. If $\lambda_n = 1$, then the other λ_i must be all 0, and we can reduce the sum to only x_n.
Now suppose $0 < \lambda_1 < 1$. Notice that $\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} = \frac{\sum_{i=1}^{n-1}\lambda_i}{1-\lambda_n} = \frac{1-\lambda_n}{1-\lambda_n} = 1$.

We have that

$$\begin{aligned}
\phi\left(\sum_{i=1}^{n}\lambda_i x_i\right) &= \phi\left(\lambda_n x_n + (1-\lambda_n)\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}x_i\right) \\
&\le \lambda_n\phi(x_n) + (1-\lambda_n)\phi\left(\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}x_i\right) \\
&\le \lambda_n\phi(x_n) + (1-\lambda_n)\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}\phi(x_i) \\
&= \lambda_n\phi(x_n) + \sum_{i=1}^{n-1}\lambda_i\phi(x_i) \\
&= \sum_{i=1}^{n}\lambda_i\phi(x_i).
\end{aligned}$$

By induction, this is true for any $n \ge 1$. □

Theorem 5.6 (Arithmetic Mean Geometric Mean Inequality)**.** *Let $x_1, \dots, x_n \ge 0$, with $0 \le \lambda_1, \dots, \lambda_n \le 1$ such that $\sum_{i=1}^{n}\lambda_i = 1$. We have that*

$$\prod_{i=1}^{n} x_i^{\lambda_i} \le \sum_{i=1}^{n}\lambda_i x_i.$$

Proof. If any of $x_i = 0$, then the inequality is trivially true.
Now suppose $\forall i, x_i > 0$.
Notice that exp is convex, and we have

$$\begin{aligned}
\prod_{i=1}^{n} x_i^{\lambda_i} &= \exp\left(\sum_{i=1}^{n}\lambda_i\ln(x_i)\right) \\
&\le \sum_{i=1}^{n}\lambda_i\exp(\ln(x_i)) \\
&= \sum_{i=1}^{n}\lambda_i x_i.
\end{aligned}$$

□

Proposition 5.7. *Let $x_1, \dots, x_n \ge 0$, and $n \in \mathbb{N}^+, p \ge 1$, we have that*

$$\sum_{i=1}^{n} x_i^p \le \left(\sum_{i=1}^{n} x_i\right)^p \le n^{p-1}\sum_{i=1}^{n} x_i^p.$$

Proof. For $p \ge 1$, we have $(\cdot)^p$ is convex.

$$\begin{aligned}
\left(\sum_{i=1}^{n}\frac{1}{n}x_i\right)^p &\le \sum_{i=1}^{n}\frac{1}{n}x_i^p \\
\frac{1}{n^p}\left(\sum_{i=1}^{n}x_i\right)^p &\le \frac{1}{n}\sum_{i=1}^{n}x_i^p \\
\left(\sum_{i=1}^{n}x_i\right)^p &\le n^{p-1}\sum_{i=1}^{n}x_i^p.
\end{aligned}$$

This proves the second inequality.
Now when $n = 1$, we have the first inequality trivially.
Suppose the first inequality holds for $n \in \mathbb{N}^+$, we have

$$\begin{aligned}\left(\sum_{i=1}^{n+1} x_i\right)^p &= \left(\sum_{i=1}^{n} x_i + x_{n+1}\right)^p \\ &\geq \left(\sum_{i=1}^{n} x_i\right)^p + x_{n+1}^p \\ &\geq \left(\sum_{i=1}^{n} x_i^p\right) + x_{n+1}^p \\ &= \sum_{i=1}^{n+1} x_i^p.\end{aligned}$$

By induction, the first inequality is true for all $n \in \mathbb{N}^+$. □

5.3 L^p Spaces

Definition 5.5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \leq p < \infty$ we define

$$\mathcal{L}^p(\mu) := \left\{f : X \to \mathbb{C} \middle| f^p \in L^1(\mu)\right\} = \left\{f : X \to \mathbb{C} \middle| f \text{ is measurable}, \int_X |f|^p d\mu < \infty\right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p} := \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}.$$

Proposition 5.8. *$\mathcal{L}^p(\mu)$ is a vector space.*

Proof. Clearly $\int_X |0|^p d\mu = 0$, so the zero function $0 \in \mathcal{L}^p(\mu)$.
Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^p(\mu)$, we have

$$\begin{aligned}\int_X |c \cdot f|^p d\mu &= \int_X |c|^p |f|^p d\mu \\ &= |c|^p \int_X |f|^p d\mu \\ &< \infty.\end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^p(\mu)$.
Now for any $f, g \in \mathcal{L}^p(\mu)$, we have

$$\begin{aligned}\int_X |f+g|^p d\mu &\leq \int_X (|f| + |g|)^p d\mu \\ &\leq \int_X 2^{p-1}(|f|^p + |g|^p) d\mu \\ &= 2^{p-1}\left(\int_X |f|^p d\mu + \int_X |g|^p d\mu\right) \\ &< \infty.\end{aligned}$$

Thus $f + g \in \mathcal{L}^p(\mu)$.
Since the set of all functions $\{f : X \to \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^p(\mu)$ is a subspace of it. □

Definition 5.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, the **essential supremum** of a function $f : X \to \mathbb{R}$ is

$$\operatorname{ess\,sup} f := \inf \{M \in \mathbb{R} : \mu(\{x : f(x) > M\}) = 0\}.$$

Proposition 5.9. *For any* $\lambda \geq 0, f : X \to \mathbb{R}$, *we have*

$$\lambda(\operatorname{ess\,sup} f) = \operatorname{ess\,sup}(\lambda f).$$

Proof. It is easy to see this is true for $\lambda = 0$.
Now suppose $\lambda > 0$.

$$\begin{aligned}
\operatorname{ess\,sup}(\lambda f) &= \inf \{M \in \mathbb{R} : \mu(\{x : \lambda f(x) > M\}) = 0\} \\
&= \inf \left\{M \in \mathbb{R} : \mu\left(\left\{x : f(x) > \frac{M}{\lambda}\right\}\right) = 0\right\} \\
&= \inf \{\lambda \cdot N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0\} \\
&= \lambda \inf \{N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0\} \\
&= \lambda(\operatorname{ess\,sup} f).
\end{aligned}$$

□

Definition 5.7. Let $(X, \mathcal{M}, \mu)$ be a measure space, we define

$$\mathcal{L}^\infty(\mu) := \{f : X \to \mathbb{C} | f \text{ is measurable}, \operatorname{ess\,sup} |f| < \infty\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty} := \operatorname{ess\,sup} |f|.$$

Proposition 5.10. $\mathcal{L}^\infty(\mu)$ *is a vector space.*

Proof. Clearly $\operatorname{ess\,sup} 0 = 0$, so the zero function $0 \in \mathcal{L}^\infty(\mu)$.
Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^\infty(\mu)$, we have

$$\begin{aligned}
\|c \cdot f\|_{\mathcal{L}^\infty} &= \operatorname{ess\,sup} |c \cdot f| \\
&= \operatorname{ess\,sup} (|c| \cdot |f|) \\
&= |c| \operatorname{ess\,sup} |f| \\
&= |c| \|f\|_{\mathcal{L}^\infty} \\
&< \infty.
\end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^\infty(\mu)$.
Now for any $f, g \in \mathcal{L}^\infty(\mu)$.
Consider any $L, N \in \mathbb{R}$, such that $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$.
Thus $\mu(\{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}) = 0$.
Now for any $x \in X$, if $|f(x) + g(x)| > L + N$, we must have $|f(x)| + |g(x)| \geq |f(x) + g(x)| > L + N$.
Thus $|f(x)| > L$ or $|g(x)| > N$.
Since this holds for any $x \in X$, we have $\{x : |f(x) + g(x)| > N + L\} \subseteq \{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}$.
Thus $\mu(\{x : |f(x) + g(x)| > N + L\}) = 0$.
By definition, we have

$$\begin{aligned}
\|f + g\|_{\mathcal{L}^\infty} &= \operatorname{ess\,sup} |f + g| \\
&= \inf \{M \in \mathbb{R} : \mu(\{x : |f(x) + g(x)| > M\}) = 0\} \\
&\leq N + L.
\end{aligned}$$

Since this hold for any such $N, L \in \mathbb{R}$, such that $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$, we have

$$\begin{aligned}
\|f + g\|_{\mathcal{L}^\infty} &= \inf \{N + L : \mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0\} \\
&= \inf \{N : \mu(\{x : |f(x)| > N\}) = 0\} + \inf \{L : \mu(\{x : |g(x)| > L\}) = 0\} \\
&= \|f\|_{\mathcal{L}^\infty} + \|g\|_{\mathcal{L}^\infty} \\
&< \infty.
\end{aligned}$$

Thus $f+g \in \mathcal{L}^\infty(\mu)$.
Since the set of all functions $\{f : X \to \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^\infty(\mu)$ is a subspace of it. □

Proposition 5.11. *For any $1 \le p \le \infty$, we have $\|f-g\|_{\mathcal{L}^p} = 0 \iff f = g$ almost everywhere.*

Proof. For $1 \le p < \infty$,

$$\begin{aligned} & \|f-g\|_{\mathcal{L}^p} = 0 \\ \iff & \int_X |f-g|^p d\mu = 0 \\ \iff & |f-g|^p = 0 \text{ a.e.} \\ \iff & f-g = 0 \text{ a.e.} \\ \iff & f = g \text{ a.e..} \end{aligned}$$

For $p = \infty$,

$$\begin{aligned} & \|f-g\|_{\mathcal{L}^\infty} = 0 \\ \iff & \operatorname{ess\,sup} |f-g| = 0 \\ \iff & f-g = 0 \text{ a.e.} \\ \iff & f = g \text{ a.e..} \end{aligned}$$

□

Definition 5.8. For any $1 \le p \le \infty$, if we identify $f, g \in \mathcal{L}^p(\mu)$ by $f \sim g \iff f = g$ almost everywhere, we get the quotient vector space

$$L^p(\mu) := \mathcal{L}^p(\mu)/_\sim = \{[f] : f \in \mathcal{L}^p(\mu)\}$$

to be the collection of all equivalence classes of functions in $\mathcal{L}^p$.

Definition 5.9. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \le p \le \infty$ we define the norm

$$\|[f]\|_{L^p(\mu)} := \|f\|_{\mathcal{L}^p}$$

for any representative $f \in [f]$.

Lemma 5.12. *The above definition is well-defined.*

Remark. As before, when we write $f \in L^p(\mu)$, we will mean $[f] \in L^p(\mu)$, and let $f \in \mathcal{L}^p(\mu)$ be any representative of $[f]$ when the context is clear.

Theorem 5.13 (Holder's Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \le p \le \infty$. Suppose $\frac{1}{p} + \frac{1}{q} = 1$, then $\forall f \in L^p(\mu), g \in L^q(\mu),\ fg \in L^1(\mu)$ and*

$$\|fg\|_{L^1(\mu)} \le \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Proof. If $p = 1$, then $q = \infty$.
Now $|fg| = |f||g| \le |f|\|g\|_{L^\infty(\mu)}$.
Thus

$$\begin{aligned} \|fg\|_{L^1(\mu)} &= \int_X |fg| d\mu \\ &\le \int_X |f|\|g\|_{L^\infty(\mu)} d\mu \\ &= \|g\|_{L^\infty(\mu)} \int_X |f| d\mu \\ &= \|g\|_{L^\infty(\mu)} \|f\|_{L^1(\mu)}. \end{aligned}$$

Now suppose $1 < p < \infty$. We have $1 < q < \infty$.
If $\|f\|_{L^p(\mu)} = 0$ or $\|g\|_{L^q(\mu)} = 0$, then it is trivial, since this implies $f = 0$ a.e. or $g = 0$ a.e., which means $fg = 0$ a.e..
Now let $F := \frac{|f|}{\|f\|_{L^p(\mu)}}, G := \frac{|g|}{\|g\|_{L^q(\mu)}}$.
By the Arithmetic Mean Geometric Mean Inequality 5.6, we have that

$$\begin{aligned}
F(x)G(x) &= (F(x)^p)^{1/p}(G(x)^q)^{1/q} \\
&\le \frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q \\
\int_X FG d\mu &\le \frac{1}{p}\int_X F^p d\mu + \frac{1}{q}\int_X G^q d\mu \\
\frac{\|fg\|_{L^1(\mu)}}{\|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}} &\le \frac{1}{p}\int_X \frac{|f|^p}{\|f\|_{L^p(\mu)}^p} d\mu + \frac{1}{q}\int_X \frac{|g|^p}{\|g\|_{L^q(\mu)}^q} d\mu \\
&= \frac{1}{p}\frac{\|f\|_{L^p(\mu)}^p}{\|f\|_{L^p(\mu)}^p} + \frac{1}{q}\frac{\|g\|_{L^q(\mu)}^q}{\|g\|_{L^q(\mu)}^q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

Thus $\|fg\|_{L^1(\mu)} \le \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}$. □

Theorem 5.14 (Minkowski's Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \le p \le \infty$. For any $f, g \in L^p(\mu)$, we have*

$$\|f + g\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}.$$

Proof. We have proven for $p = 1$ and $p = \infty$.
Now suppose $p \in (1, \infty)$. Then $q = \frac{p}{p-1} \in (1, \infty)$.
Since $f, g \in L^p(\mu)$, we have $f + g \in L^p(\mu)$, so

$$\begin{aligned}
\left\| |f + g|^{p-1} \right\|_{L^q(\mu)}^q &= \int_X \left(|f + g|^{p-1} \right)^q d\mu \\
&= \int_X \left(|f + g|^{p-1} \right)^{\frac{p}{p-1}} d\mu \\
&= \int_X |f + g|^p d\mu \\
&= \|f + g\|_{L^p(\mu)}^p \\
&< \infty.
\end{aligned}$$

Thus $|f+g|^{p-1} \in L^q(\mu)$. By Holder's Inequality, we have

$$\begin{aligned}
\|f+g\|_{L^p(\mu)}^p &= \int_X |f+g|^p d\mu \\
&= \int_X |f+g||f+g|^{p-1} d\mu \\
&\leq \int_X (|f|+|g|)|f+g|^{p-1} d\mu \\
&\leq \int_X |f| \cdot |f+g|^{p-1} d\mu + \int_X |g| \cdot |f+g|^{p-1} d\mu \\
&\leq \|f\|_{L^p(\mu)} \left\| |f+g|^{p-1} \right\|_{L^q(\mu)} + \|g\|_{L^p(\mu)} \left\| |f+g|^{p-1} \right\|_{L^q(\mu)} \\
&= \left(\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \right) \left\| |f+g|^{p-1} \right\|_{L^q(\mu)} \\
&= \left(\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \right) \|f+g\|_{L^p(\mu)}^{p/q} \\
\|f+g\|_{L^p(\mu)}^{p-p/q} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\
\|f+g\|_{L^p(\mu)}^{p(1-1/q)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\
\|f+g\|_{L^p(\mu)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}.
\end{aligned}$$

□

Corollary 5.15. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$. We have $\|\cdot\|_{L^p(\mu)}$ is a norm over $L^p(\mu)$.*

Proof. The triangle Inequality is done by Minkowski's Inequality.
Consider any $\in L^p(\mu)$.
For any $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\|\alpha f\|_{L^p(\mu)}^p &= \int_X |\alpha f|^p d\mu \\
&= \int_X |\alpha|^p |f|^p d\mu \\
&= |\alpha|^p \int_X |f|^p d\mu \\
&= |\alpha|^p \|f\|_{L^p(\mu)}^p \\
&\Longrightarrow \\
\|\alpha f\|_{L^p(\mu)} &= |\alpha| \|f\|_{L^p(\mu)}.
\end{aligned}$$

In addition, $\|f\|_{L^p(\mu)} = 0$, if and only if $f = 0$ $\mu - a.e.$. □

Theorem 5.16 (Fischer-Riesz)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$. $\left(L^p(\mu), \|\cdot\|_{L^p(\mu)} \right)$ is a Banach Space.*

Proof.

1. We first consider $1 \leq p < \infty$.
   Let $(f_n)_1^\infty$ be a Cauchy sequence in $L^p(X, \mathcal{M}, \mu)$. Then for each $k \in \mathbb{N}^+$, there is some $N_k \geq 1$, such that $\forall m, n \geq N_k, \|f_m - f_n\|_{L^p(\mu)} < \frac{1}{2^k}$.
   WLOG, $\forall k, N_{k+1} \geq N_k$.
   Thus $\left\| f_{N_{k+1}} - f_{N_k} \right\|_{L^p(\mu)} < \frac{1}{2^k}$.
   Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} \left| f_{N_{j+1}} - f_{N_j} \right|$, where we fix f_n to be a representative of $[f_n]$.

Notice that $\forall k \geq 1$,

$$\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1} \left| f_{N_{j+1}} - f_{N_j} \right| \\
&= g_k \\
\|g_k\|_{L^p(\mu)} &= \left\| |f_{N_1}| + \sum_{j=1}^{k-1} \left| f_{N_{j+1}} - f_{N_j} \right| \right\|_{L^p(\mu)} \\
&\leq \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \left\| f_{N_{j+1}} - f_{N_j} \right\|_{L^p(\mu)}.
\end{aligned}$$

Let $g = \lim_{k\to\infty} g_k = |f_{N_1}| + \sum_{j=1}^{\infty} \left| f_{N_{j+1}} - f_{N_j} \right|$.
Notice that g_k are monotone increasing. By LMCT, we have that

$$\begin{aligned}
\|g\|_{L^p(\mu)} &= \int_X |g|^p d\mu \\
&= \int_X g^p d\mu \\
&= \int_X \lim_{k\to\infty} g_k^p d\mu \\
&= \lim_{k\to\infty} \int_X g_k^p d\mu \\
&= \lim_{k\to\infty} \|g_k\|_{L^p(\mu)} \\
&\leq \lim_{k\to\infty} \left(\|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \left\| f_{N_{j+1}} - f_{N_j} \right\|_{L^p(\mu)} \right) \\
&= \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \left\| f_{N_{j+1}} - f_{N_j} \right\|_{L^p(\mu)} \\
&< \|[f_{N_1}]\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&< \infty.
\end{aligned}$$

Thus $g \in \mathcal{L}^p(X, \mathcal{M}, \mu)$, which means $g^p \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ and $N := \{x \in X : g(x) = \infty\}$ has measure 0.
This implies that $f_{N_1}(x) + \sum_{k=1}^{\infty} (f_{N_{k+1}}(x) - f_{N_k}(x))$ converges absolutely for $x \in X \setminus N$.

We can thus define

$$
\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty}(f_{N_{k+1}}(x) - f_{N_k}(x)) \\
&= \lim_{k\to\infty}\left(f_{N_1}(x) + \sum_{j=1}^{k}(f_{N_{j+1}}(x) - f_{N_j}(x))\right) \\
&= \lim_{k\to\infty} f_{N_{k+1}}(x) \\
&= \lim_{k\to\infty} f_{N_k}(x)
\end{aligned}
$$

for $x \in X \setminus N$.
We then extend f to X by $f|_N := 0$.
Then $|f| \leq g \implies |f|^p \leq g^p$, and thus $f \in \mathcal{L}^p(X, \mathcal{M}, \mu)$.
Notice that $|f_{N_k}| \leq g_k \leq g$, thus $|f - f_{N_k}|^p \leq (g+g)^p = 2^p g^p$.
By LDCT,

$$
\begin{aligned}
\lim_{k\to\infty} \|f_{N_k} - f\|_{L^p(\mu)}^p &= \lim_{k\to\infty} \int_X |f_{N_k} - f|^p d\mu \\
&= \int_X \lim_{k\to\infty} |f_{N_k} - f|^p d\mu \\
&= 0.
\end{aligned}
$$

Thus $\lim_{k\to\infty} f_{N_k}$ converges to f.
Since this is a subsequence of the Cauchy sequence $(f_n)_1^\infty$, we have that $\lim_{n\to\infty} f_n = f$.
This shows that $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(\mu)})$ is complete.

2. Now consider $p = \infty$.
   Let $(f_n)_1^\infty$ be a Cauchy sequence in $L^\infty(X, \mathcal{M}, \mu)$. As before, we can take some subsequence $(f_{N_k})_{k=1}^\infty$ with $\left\|f_{N_{k+1}} - f_{N_k}\right\|_{L^p(\mu)} < \frac{1}{2^k}$.
   Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1}\left|f_{N_{j+1}} - f_{N_j}\right|$, where we fix f_n to be a representative of $[f_n]$.
   Notice that $\forall k \geq 1$,

$$
\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1}(f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left|f_{N_1} + \sum_{j=1}^{k-1}(f_{N_{j+1}} - f_{N_j})\right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1}\left|f_{N_{j+1}} - f_{N_j}\right| \\
&= g_k.
\end{aligned}
$$

□

Theorem 5.17 (Density of simple functions)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$. The simple functions*

$$S := \{\phi \in L^p(\mu) \,|\, \phi \text{ is simple, measurable}\}$$

are dense in $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(X,\mathcal{M},\mu)})$.

Proof. 1. First consider $1 \leq p < \infty$.
   Let $f \in L^p(X, \mathcal{M}, \mu), \exists (\phi_n)_{n=1}^\infty$ be simple and measurable functions, such that

$$f(x) = \lim_{n\to\infty} \phi_n(x), \text{a.e. } x \in X,$$

and

$$|\phi_1| \leq |\phi_1| \leq \cdots \leq |f|.$$

Thus

$$f^p(x) = \lim_{n\to\infty} \phi_n^p(x), \text{a.e. } x \in X,$$

and

$$|\phi_1|^p \leq |\phi_1|^p \leq \cdots \leq |f|^p.$$

Since $|f - \phi_n|^p \leq (2|f|)^p = 2^p|f|^p \in L^1(X, \mathcal{M}, \mu)$, by LDCT 3.18, we have that

$$\begin{aligned}
\lim_{n\to\infty} \|f - \phi_n\|^p_{L^p(X,\mathcal{M},\mu)} &= \lim_{n\to\infty} \int_X |f - \phi_n|^p d\mu \\
&= \int_X \lim_{n\to\infty} |f - \phi_n|^p d\mu \\
&= \int_X \lim_{n\to\infty} 0 d\mu \\
&= 0.
\end{aligned}$$

2. Now consider $p = \infty$.
Let $f \in L^p(X, \mathcal{M}, \mu)$, we know $\mu(N) = 0$ for $N := \left\{x \in X : |f(x)| > \|f\|_{L^\infty(X,\mathcal{M},\mu)}\right\}$.
Let $f' := f\chi_{N^c}$. We notice that f' is measurable and bounded, with $|f'| \leq \|f\|_{L^\infty(X,\mathcal{M},\mu)}, \forall x \in X$.
Thus we can find $(\phi_n)_{n=1}^\infty$ be simple and measurable functions, such that

$$f(x) = \lim_{n\to\infty} \phi_n(x), \text{a.e. } x \in X \text{ uniformly, and } |\phi_1| \leq |\phi_1| \leq \cdots \leq |f|.$$

Now

$$\begin{aligned}
\|f - \phi_n\|_{L^\infty(X,\mathcal{M},\mu)} &= \|f\chi_N + f' - \phi_n\|_{L^\infty(X,\mathcal{M},\mu)} \\
&\leq \|f\chi_N\|_{L^\infty(X,\mathcal{M},\mu)} + \|f' - \phi_n\|_{L^\infty(X,\mathcal{M},\mu)} \\
&= \|f' - \phi_n\|_{L^\infty(X,\mathcal{M},\mu)} \\
&= \operatorname*{ess\,sup}_{x\in X} |f'(x) - \phi(x)| \\
&\to 0.
\end{aligned}$$

□

Remark. For $1 \leq p < \infty$,

$$S = \text{Span}\{\chi_E \,|\mu(E) < \infty\} = \{\phi : X \to \mathbb{C} \,|\phi \text{ is simple, measurable}, \mu(\{x \in X|\phi(x) \neq 0\}) < \infty\}$$

Theorem 5.18 (Density of compactly supported continuous functions)**.** *Given some measure space $(X, \mathcal{M}, \mu)$, where μ is a Randon measure, then $C_c(X)$ is dense for $p < \infty$.*

Proof. Given any $\epsilon > 0$.
Consider any measurable E, such that $\mu(E) < \infty$.
By regularity, we can find some compact $K \subset E \subset V$ open , such that $\mu(V \setminus E) < \frac{\epsilon^p}{2^p}$.
Now we take the bump function $K < f < V$ by Urysohn's Lemma 6.9, where $f \in C_c(V) \subseteq C_c(X)$, and $f|_K = 1, f|_{V^c} = 0, 0 \leq f \leq 1$.

Now

$$\begin{aligned}\|\chi_E - f\|_{L^p(\mu)}^p &= \int_X |\chi_E - f|^p d\mu \\ &= \int_{V\setminus K} |\chi_E - f|^p d\mu \\ &\leq \int_{V\setminus K} 2^p d\mu \\ &= 2^p \mu(V \setminus K) \\ &< \epsilon^p.\end{aligned}$$

Thus $\chi_E \in \overline{C_c(X)}$.
Since $S = \text{Span}\{\chi_E \,|\mu(E) < \infty\}$ is dense in $L^p(\mu)$, so is $C_c(X)$. ☐

Remark. This is not true for $p = \infty$. For instance, consider $X = \mathbb{R}$ with Lebesgue measure, or $X = \mathbb{N}$ with counting measure.

Proposition 5.19 ($L^q(\mu) \subseteq L^p(\mu)^*$)**.** *Let $p \in [1,\infty]$, with $\frac{1}{p} + \frac{1}{q} = 1$. Let $g \in L^q(\mu)$, then $\Lambda_g \in L^p(\mu)^*$, where $\Lambda_g(f) = \int_X fgd\mu$. Moreover, $\forall p \in (1,\infty], \|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$. This also holds for $p = 1$ if μ is semi-finite.*

Proof. clearly Λ_g is linear.
By Holder's Inequality, we have

$$\begin{aligned}|\Lambda_g(f)| &= \left|\int_X fgd\mu\right| \\ &\leq \int_X |fg|d\mu \\ &\leq \|g\|_{L^q(\mu)}\|f\|_{L^p(\mu)}.\end{aligned}$$

Thus $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f\in L^p(\mu), f\neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \leq \|g\|_{L^q(\mu)} < \infty$.
Thus Λ_g is bounded and $\Lambda_g \in L^p(\mu)^*$.
We now want to show $\|\Lambda_g\|_{L^p(\mu)^*} \geq \|g\|_{L^q(\mu)}$.
If $\|g\|_{L^q(U)} = 0$, we have that $g = 0$a.e., and $\|\Lambda_g\|_{L^p(\mu)^*} = 0 = \|g\|_{L^q(U)}$.
Now consider $\|g\|_{L^q(U)} \neq 0$. It suffices to find some $\|f\|_{L^q(U)} = 1$, such that $\Lambda_g(f) \geq \|g\|_{L^q(U)}$.

1. $1 < p < \infty$.
   Notice that $p = \frac{1}{1-\frac{1}{q}} = \frac{1}{\frac{q-1}{q}} = \frac{q}{q-1}$.
   Let $f = \overline{\text{sgn}(g)} \frac{|g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}}$, we have that

$$\begin{aligned}\|f\|_{L^p(\mu)}^p &= \int |f|^p d\mu \\ &= \int \frac{|g|^q}{\|g\|_{L^q(\mu)}^q} d\mu \\ &= \frac{1}{\|g\|_{L^q(\mu)}^q} \int |g|^q d\mu \\ &= \frac{1}{\|g\|_{L^q(\mu)}^q} \|g\|_{L^q(U)}^q \\ &= 1,\end{aligned}$$

which means that $f \in L^p(\mu)$. In addition,

$$
\begin{aligned}
|\Lambda_g(f)| &= \left| \int_X fg d\mu \right| \\
&= \left| \int_X \overline{\operatorname{sgn}(g)} \frac{|g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}} g d\mu \right| \\
&= \frac{1}{\|g\|_{L^q(\mu)}^{q/p}} \left| \int_X |g|^{1+q/p} d\mu \right| \\
&= \frac{1}{\|g\|_{L^q(\mu)}^{q-1}} \left| \int_X |g|^q d\mu \right| \\
&= \|g\|_{L^q(U)}.
\end{aligned}
$$

Thus, $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$.

2. $p = \infty, q = 1$.
   Let $f = \overline{\operatorname{sgn}(g)} \in L^\infty(\mu)$. We have $\|f\|_{L^\infty(\mu)} = 1$. In addition,

$$
\Lambda_g(f) = \int_X \overline{\operatorname{sgn}(g)} g d\mu = \int_X |g| d\mu = \|g\|_{L^1(\mu)}.
$$

   Thus, $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$

3. $p = 1, q = \infty$, and μ is semi-finite.
   Choose $\epsilon \in (0, \|g\|_{L^\infty(\mu)})$.
   Let $A = \left\{ x \in X \middle| |g(x)| > \|g\|_{L^\infty(\mu)} - \epsilon \right\}$.
   Notice that $\mu(A) > 0$, otherwise $\|g\|_{L^\infty(\mu)} = \epsilon$.
   Since μ is semi-finite, we can find $E \in \mathcal{M}$, such that $0 < \mu(E) < \infty, E \subseteq A$.
   Let $f = \frac{\chi_E}{\mu(E)} \overline{\operatorname{sgn}(g)}$.
   Notice that

$$
\begin{aligned}
\|f\|_{L^1(\mu)} &= \int_X |f| d\mu \\
&= \int_X \left| \frac{\chi_E}{\mu(E)} \overline{\operatorname{sgn}(g)} \right| \\
&= \frac{1}{\mu(E)} \int_X \chi_E d\mu \\
&= 1.
\end{aligned}
$$

   Thus $f \in L^1(\mu)$. In addition,

$$
\begin{aligned}
\Lambda_g(f) &= \int_X fg d\mu \\
&= \int_X \frac{\chi_E}{\mu(E)} \overline{\operatorname{sgn}(g)} g d\mu \\
&= \int_E \frac{|g|}{\mu(E)} d\mu \\
&\geq \int_E \frac{\|g\|_{L^\infty(\mu)} - \epsilon}{\mu(E)} d\mu \\
&\geq \|g\|_{L^\infty(\mu)} - \epsilon.
\end{aligned}
$$

Since this holds for any $\epsilon > 0$, we have that

$$\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \sup_{\epsilon > 0} \|g\|_{L^\infty(\mu)} - \epsilon = \|g\|_{L^q(\mu)}.$$

We thus have $\|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$. □

Remark. In the above case, the map $g \mapsto \Lambda_g$ is isometric.

6 Borel Measures on Topological Spaces

6.1 Topological Spaces

See more in my notes for Pmath753 Functional Analysis or the notes of Pmath367 Topology by Professor S. New.

Definition 6.1. Let $X \neq \emptyset$ be a set. A **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X) := 2^X =$ power set of X satisfying

1. $\emptyset, X \in \mathcal{T}$,
2. $\mathcal{T}$ is closed under arbitrary union; namely, $\forall \{A_\alpha\}_{\alpha \in K} \subseteq \mathcal{T},\ \bigcup_{\alpha \in K} A_\alpha \in \mathcal{T}$, and
3. $\mathcal{T}$ is closed under finite intersection; namely, $\forall \{A_i\}_{i=1}^n \subseteq \mathcal{T},\ \bigcup_{i=1}^n A_i \in \mathcal{T}$.

Also, $(X, \mathcal{T})$ is a **topological space** if $\mathcal{T}$ is a topology on X.

Definition 6.2. Let $(X, \mathcal{T})$ be a topological space, then we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$. We say $E \subseteq X$ is **closed** if $E^c \in \mathcal{T}$ is open.

Definition 6.3. For $E \in X$, the **closure** of E is

$$\bar{E} = \bigcap_{F \supseteq E : F \text{ is closed}} F.$$

Definition 6.4. A set $K \subseteq X$ is **compact** if every open cover of K has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open}, K \subseteq \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n, \text{ such that } K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Definition 6.5. An (open) neighborhood of $x \in X$ is some

$$U_x \in \mathcal{T}, \text{ such that } x \in U_x.$$

Definition 6.6. X is **Hausdorff** if

$$\forall x \neq y \in X,\ \exists U_x, U_y \text{ open neighborhoods for } x, y, \text{ such that } U_x \cap U_y = \emptyset.$$

Example 6.1.1. Every metric space is Hausdorff.

Definition 6.7. X is **locally compact** if $\forall x \in X$, there is a neighborhood U_x such that $\overline{U_x}$ is compact.

Example 6.1.2. $\mathbb{R}^n$ are locally compact by Heinz-Borel theorem.

Proposition 6.1. *A Banach space* $(X, \|\cdot\|)$ *is locally compact iff* $\dim(X) < \infty$.

Theorem 6.2. *Let* $(X, \mathcal{T})$ *be a topological space,*

1. *Suppose* K *is compact, then* $\forall F \subseteq K$ *that is closed,* F *is also compact.*

2. *If X is Hausdorff, for any compact $K \subseteq X, x \in X \setminus K$, $\exists$ open neighborhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$.*

Proof. 1. Let $(U_\alpha)_{\alpha \in A}$ be an open cover for F.
Since F is closed, then F^c is open. Thus $\{F^c\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover for K.
Thus there are $U_{\alpha_1}, \dots, U_{\alpha_n}$, such that $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$. Thus $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ has a finite subcover.

2. Consider any $y \in K$, there is some open neighborhoods $U_y \ni x, W_y \ni y$, such that $U_y \cap W_y = \emptyset$.
Since $K \subseteq \bigcup_{y \in K} W_y$ is compact, we have $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$ for some $y_1, \dots, y_n$.
Let $U = \bigcap_{i=1}^n U_{y_i}$, we have $x \in U, K \subseteq W, U \cap W = \emptyset$ as required.

$\square$

Corollary 6.3. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, then any compact set K is closed. In addition, for any closed $F \subseteq X$, we have $F \cap K$ is compact.*

Proof. Suppose for contradiction that K is not closed, then there is some $y \in \bar{K} \setminus K$. Thus we can find open neighborhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$. Now $K \subset \bar{K} \setminus U \subsetneq \bar{K}$ is closed, which is a contradiction.
Since K is closed, so is $F \cap K \subseteq K$, and thus it is compact. $\square$

Lemma 6.4. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, and $(K_\alpha)_{\alpha \in A}$ be a collections of compact sets such that*

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

We must have $\alpha_1, \dots, \alpha_n \in A$, such that

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

Proof. Fix $\alpha_1 \in A$, then $K_{\alpha_0} \subseteq \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$ is compact and has an open cover.
Thus there must be $\alpha_2, \dots, \alpha_n \in A$, such that $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = \left(\bigcap_{i=2}^n K_{\alpha_i}\right)^c$.
Thus $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$. $\square$

Theorem 6.5. *Let X be a Locally Compact Hausdorff space, and let $K \subseteq U \subseteq X$ be such that K is compact, and U is open. Then there exists some open set V such that $\bar{V}$ is compact, and*

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof. Since X is a Locally Compact Hausdorff space, there are $V_1, \dots, V_n$, each with $\bar{V}_i$ be compact, such that $K \subseteq \bigcup_{i=1}^n V_i =: G$. Note that $\bar{G}$ is compact, and G is open.
If $U = X$, then $G \subseteq U$, and we are done.
Otherwise, let $C := X \setminus U$ be non-empty and closed.
Consider any $y \in C$, we know that $y \notin K$. Since X is Hausdorff, we can find open $W_y \supset K$, and $U_y \ni y$, such that $W_y \cap U_y = \emptyset$. Then $W_y \subseteq U_y^c$, and thus $\bar{W}_y \subseteq U_y^c$, since U_y^c is closed. Yet $y \notin U_y^c$, thus $y \notin \bar{W}_y$.
Now consider the family $\left\{C \cap \bar{W}_y \cap \bar{G}\right\}_{y \in C}$. Notice that each $C \cap \bar{W}_y \cap \bar{G}$ is compact, since $C, \bar{W}_y$ are closed, and $\bar{G}$ is compact.
Yet $\bigcap_{y \in C} \left(C \cap \bar{W}_y \cap \bar{G}\right) = \emptyset$.
Thus $\exists y_1, \dots, y_n \in C$, such that $\bigcap_{i=1}^n \left(C \cap \bar{W}_y \cap \bar{G}\right) = \emptyset$.
Now let $V := G \cap \bigcap_{i=1}^n W_{y_i}$.
Clearly V is open, and $K \subseteq V$.
In addition, $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$, yet the intersection of righthand side and C is empty, thus contained in $C^c = U$. $\square$

6.2 Compactly Supported Functions

Definition 6.8. Let $C(X)$ be the collection of functions $f : X \to \mathbb{C}$ that are continuous.

Proposition 6.6. *$C(X)$ is a $\mathbb{C}$ vector space, and also an Algebra over $\mathbb{C}$. It also admits a partial order by $f \geq g \iff \forall x \in X, f(x) \geq g(x)$.*

Definition 6.9. For $f \in C(X)$, the **support** of it is

$$\mathrm{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

Definition 6.10. The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \mathrm{Supp}(f) \text{ is compact}\}.$$

Proposition 6.7. *Suppose every compact set K is Borel-measurable, then $C_c(X)$ is a sub-algebra of $C(X)$.*

Proposition 6.8. *Suppose every compact set K is Borel-measurable, let $\mu : Bor(X) \to [0, \infty]$ be a Borel measure on X, such that $\forall K$ be compact, $\mu(K) < \infty$, then*

$$C_c(X) \subseteq L^1(\mu).$$

Proof. Given any $f \in C_c(X)$. Let $K = \mathrm{Supp}(f)$, then

$$\int_X |f| d\mu = \int_K |f| d\mu \leq \int_K \|f\|_\infty d\mu = \mu(K)\|f\|_\infty < \infty.$$

□

6.3 Partition of Unity

Definition 6.11. Let K be a compact set, and V be an open set of X. Let $f \in C_c(X)$. We say $f < V$ if $0 \leq f \leq 1$, and $\mathrm{Supp}(f) \subseteq V$. We say $K < f$ if $0 \leq f \leq 1$, and $f|_K = 1$. We say $K < f < V$ if $K \subset V, K < f, f < V$.

Remark. f is a "bump" function that approximates χ_K when V shrinks towards K.

Theorem 6.9 (Urysohn's Lemma)**.** *Let X be a Locally Compact Hausdorff space, $K \subseteq V \subseteq X$ be such that K is compact, and V is open. Then there exists $f \in C_c(V)$, such that $K < f < V$.*

Proof. we want to construct a family of open sets $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s$.
By 6.5, we can find $K \subset V_0 \subset \bar{V_0} \subset V$.
Pick an enumeration of $r \in \mathbb{Q} \cap (0, 1]$, i.e. $(r_n)_{n=1}^\infty$. WLOG, we can let $r_1 = 1$.
By 6.5, we can find $K \subset V_1 \subset \bar{V_1} \subset V_0$.
Suppose we have constructed the V_{r_i} for $1 \leq i \leq n$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s \in \{r_i\}_{i=1}^n$.
Let $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n.$
Now by 6.5, we can find $\bar{V_t} \subset V_{n+1} \subset \bar{V_{n+1}} \subset V_s$.
For any $r < r_{n+1}$, we have $r \leq s$, and thus $V_{n+1} \subset \bar{V_{n+1}} \subset V_s \subset \bar{V_s} \subseteq V_r$ by induction hypothesis, and similarly for any $r > r_{n+1}$.
Inductively, we can prove there is such a family.
Define $f_r := r\chi_{V_r}$, and $g_r := r\chi_{\bar{V_r}^c} + \chi_{\bar{V_r}}$, and $f := \sup_r f_r, g := \inf_r g_r$.
We can show that f, g are upper and lower continuous, respectively.

In addition, f, g are both 0 outside of V_1, and 1 on K.
Suppose there is some $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, such that $f_r(x) > g_s(x)$. Then we must have $f_r(x) > 0$, and thus $x \in V_r$ and $1 \leq r = f_r(x)$.
Thus $1 > g_s(x)$, and thus $x \in \bar{V_s}^c$ and $f_s = s$.
Since $r > s$, we must have $V_r \subset \bar{V_r} \subset V_s \subset \bar{V_s}$, which is a contradiction to $x \in V_r, x \notin \bar{V_s}$.
Thus for any $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, we must have $f_r(x) \leq g_s(x)$.
Thus we must have $f(x) \leq g(x)$ for any $x \in V$.
Now suppose there is some $x \in X$, such that $f(x) < g(x)$.
There must be some rationals, such that $f(x) < r < s < g(x)$, since $\mathbb{Q}$ is dense.
Thus $\sup_r f_r(x) < r$, and thus $x \notin V_r$.
Also, $\inf_s g_s(x) > s$, and thus $x \in \bar{V_s}$.
Yet $r < s$, we must have $V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r}$, which is a contradiction.
Thus we must have $f = g$, and it forces f to be continuous. ◻

Definition 6.12. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$.
A collection $(h_i)_{i=1}^n \subset C_c(X)$ is called a **partition of unity** on K subordinate to $(V_i)_{i=1}^n$ if

$$\begin{cases} \forall 1 \leq i \leq n, & h_i < V_i, \\ \forall x \in K, & \sum_{i=1}^n h(x) = 1. \end{cases}$$

Theorem 6.10. *Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$, there always exists a partition of unity on K subordinated to $(V_i)_{i=1}^n$.*

Proof. Since K is compact, we can find some open cover $W_1, \dots, W_m$, such that for all j, we have $W_j \subset \bar{W_j} \subset V_{i(j)}$ for some $1 \leq i(j) \leq n$.
Let $K_i := \bigcup_{1 \leq j \leq m \text{ such that } W_j \subset V_i} \bar{W_j} \subset V_i$, which is compact.
By Urysohn's lemma, we can find $K_i < g_i < V_i$.
Now let $h_1 := g_1$, and in general, $h_i := g_i \prod_{j<i}(1 - g_j)$.
It is easy to check that $0 \leq h_i \leq 1$, and $h_i \in C_c(X)$.
In addition, $\mathrm{Supp}(h_i) \subseteq \mathrm{Supp}(g_i) \subset V_i$.
Thus $h_i < V_i$. Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_n)$.
For any $x \in K$, there must be some $i \in [n]$ such that $x \in K_i$, and thus $g_i(x) = 1$, and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_n(x)) = 1 - 0 = 1.$$

◻

6.4 Linear Functional

Definition 6.13. Let X be a locally compact Hausdorff space. A **linear functional** on $C_c(X)$ is a linear map $\Lambda : C_c(X) \to \mathbb{R}$. A linear functional Λ is **positive** if $\Lambda(f) \geq 0$ for all $f \in C_c(X)$ such that $f \geq 0$.

Remark. If X is a compact Hausdorff space, $C_c(X) = C(X)$.

Proposition 6.11. *Let X be a compact Hausdorff space, then for a Borel measure μ on X,*

1. *If μ is finite, $\Lambda_\mu(f) := \int_X f d\mu$ is a positive linear functional.*
2. *If μ is finite, Λ_μ is bounded and hence continuous. Indeed, $\forall f \in C(X), |\Lambda_\mu(f)| \leq \mu(X)\|f\|_\infty$.*

3. *Λ_μ is a finite-value linear functional iff $\mu(X) < \infty$.*

Proof. 1. By properties fo integral and 6.8.

2. For $f \in C(X)$.

$$\begin{aligned} |\Lambda_\mu(f)| &= \left| \int_X f d\mu \right| \\ &\leq \int_X |f| d\mu \\ &\leq \int_X \|f\|_\infty d\mu \\ &= \mu(X)\|f\|_\infty. \end{aligned}$$

□

6.5 Radon Meaure

Definition 6.14. Let X be a topological space, $\mu : \mathrm{Bor}(X) \to [0, \infty]$ be a Borel measure on X. For $A \in \mathrm{Bor}(X)$, we say μ is **outer regular** if $\mu(A) = \inf\{\mu(U) : \text{open } U \supseteq A\}$. μ is **inner regular** if $\mu(A) = \sup\{\mu(K) : \text{compact } K \subseteq A\}$. μ is **regular** is it is inner and outer regular for any $A \in \mathrm{Bor}(X)$.

Definition 6.15. Let X be a topological space, $\mu : \mathrm{Bor}(X) \to [0, \infty]$ be a Borel measure on X. μ is a **Radon** measure if

1. $\forall$compact $K \subseteq X,\ \mu(K) < \infty$,
2. μ is outer regular on Borel sets,
3. μ is inner regular on open sets.

Remark. We have seen that Lebesgue-Stieltjes Measures are regular and Radon.

Proposition 6.12. *A finite Borel measure on a compact metric space is always regular (hence Radon).*

Proof. Let μ be a finite Borel measure on a compact metric space X. Let $S \subseteq Bol(X)$ on which μ is regular. If $C \subseteq X$ is closed, it is compact. Thus μ is inner regular for C. Since X is a metric space, $C = \bigcap_{n\geq 1} \left\{x \in X : d(x, C) < \frac{1}{n}\right\}$ is G_δ. By continuity from above of μ, it follows that μ is also outer-regular. Thus all the closed sets belong to S.
Since Borel sets are generated by closed sets, it suffices to show S is a σ-algebra.
For $A \in S, \epsilon > 0$, there is compact K and open U such that $K \subseteq A \subseteq U, \mu(U \setminus K) < \epsilon$. Then $U^c \subseteq A^c \subseteq K^c$, where U^c is compact, K^c is open. In addition,

$$\mu(K^c \setminus U^c) = \mu(K^c \cap U) = \mu(U \setminus K) < \epsilon.$$

Thus $A^c \in S$.
Consider $(A_i)_{i=1}^\infty \subseteq S, \epsilon > 0$. Choose compact $K_i \subseteq A_i$ and open $U_i \supseteq A_i$, such that $\mu(U_i \setminus K_i) < \epsilon/2^i$. Let $A = \bigcup_{i=1}^\infty A_i, C_n = \bigcup_{i=1}^n K_i, C = \bigcup_{i=1}^\infty K_i, U = \bigcup_{i=1}^\infty U_i$.
Thus C_n are closed, U is open, and $C_n \subseteq A \subseteq U$. By continuity and finiteness of μ, we have

$$\begin{aligned} \lim_{n\to\infty} \mu(U \setminus C_n) &= \mu(U \setminus C) \\ &\leq \sum_{i=1}^\infty \mu(U_i \setminus K_i) \\ &< \epsilon. \end{aligned}$$

Thus μ is regular on A, and thus $A \in S$, and thus S is closed under countable unions.
Thus $S = Bol(X)$. □

6.6 Extremely Disconnected Spaces

Definition 6.16. A compact space X is **extremely disconnected** if the closure of every open set is open.

Proposition 6.13. *If X is extremely disconnected, then there is a basis of clopen sets.*

Proposition 6.14. *If $A \subseteq X$ is clopen, then $\chi_A \in C(X)$.*

Proposition 6.15. *(Stone–Čech compactification) Let D be a discrete space (thus every subset is open). The* ***Stone–Čech compactification*** *is the unique compact (Hausdorff) space βD with the following universal properties:*

1. *$D \subseteq \beta D$ as topology inclusion.*
2. *For any compact K, and every continuous map $f : D \to K$, there is a unique continuous extension $\beta f : \beta D \to K$.*

Proposition 6.16. $\ell^\infty(D) \cong C(\beta D)$

Proposition 6.17. *βD is the set of ultrafilters on D.*

Proposition 6.18. *βD is extremely disconnected.*

6.7 Riesz-Markov-Kakutani

Lemma 6.19. *Let X be a Locally Compact Hausdorff space, $\Lambda : C(X) \to \mathbb{C}$ a positive linear functional. For any compact $K \subseteq X$, there is $C_K \geq 0$, such that for any f with $\operatorname{Supp}(f) \subseteq K$, we have*

$$|\Lambda(f)| \leq C_K \|f\|_\infty.$$

Proof. By Urysohn's Lemma 6.9, there is some $g \in C_c(X)$, such that $K < g$. Since $g \geq 0$, we have $C_K := \Lambda(g) \geq 0$.

Also, since $g|_K = 1$, we have $\|f\|_\infty g \pm f \geq 0$. Thus,

$$\Lambda(\|f\|_\infty g \pm f) \geq 0 \implies \|f\|_\infty \Lambda(g) \pm \Lambda(f) \geq 0,$$

which means

$$\pm\Lambda(f) \leq \|f\|_\infty \Lambda(g) = C_K \|f\|_\infty.$$

Thus,

$$|\Lambda(f)| \leq C_K \|f\|_\infty.$$

□

Theorem 6.20 (Riesz-Markov-Kakutani)**.** *Let X be a Locally Compact Hausdorff space, $\Lambda : C(X) \to \mathbb{F}$ a positive linear functional. There is a unique Radon measure μ on X, such that $\Lambda(f) = \Lambda_\mu(f) = \int_X f d\mu$. In addition,*

(a) $\forall U \subseteq X$ be open, we have $\mu(U) = \sup\{\Lambda(f) : f < U\}$.

(b) $\forall K \subseteq X$ be compact, we have $\mu(K) = \inf\{\Lambda(f) : K < f\}$.

Proof. 1. We first show the uniqueness. Let μ be any Radon measure such that $\Lambda(f) = \int_X f d\mu$.

Given any open U. Consider any compact K such that $K \subset U$. By Urysohn's lemma, we can find a function f, such that $K < f < U$. Thus $\chi_K \leq f \leq \chi_U \implies \mu(K) \leq \Lambda(f) \leq \mu(U)$. Thus by inner regularity, $\mu(U) = \sup\{\mu(K) : \text{compact } K \subseteq U\} = \sup\{\Lambda(f) : f < U\}$, which is uniquely determined by Λ.

For any Borel set A, we have that by outer regularity, $\mu(A) = \inf\{\mu(U) : \text{open } U \supseteq A\}$, which is uniquely determined.

This shows the uniqueness, now let us check existence.

2. We now construct μ by defining an outer measure and apply Caratheodory.

   Define $\mu^* : \mathcal{T} \to [0, \infty]$ by: for any open $U \subseteq X$,

   $$\mu^*(U) := \sup \{\Lambda(f) : f < U\}.$$

   Clearly for any open $V \supseteq U$, we have $\mu^*(U) \leq \mu^*(V)$. Thus we have $\mu^*(U) := \inf \{\mu^*(V) : \text{open } V \subseteq U\}$. Now we extend $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by

   $$\mu^*(A) := \inf \{\mu^*(U) : \text{open } U \supseteq A\}.$$

   Notice that

   $$\mu^*(A) = \inf \{\mu^*(U) : \text{open } U \supseteq A\} \geq \inf \left\{ \sum_{i=1}^{\infty} \mu^*(U_i) : \text{open } U_i, A \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$

   On the other hand, consider any open U_i, such that $A \subseteq \bigcup_{i=1}^{\infty} U_i$. Notice that $U := \bigcup_{i=1}^{\infty} U_i$ is open, and $U \supseteq A$. Pick any $f < U$, and we have $K := \operatorname{Supp}(f) \subseteq U = \bigcup_{i=1}^{\infty} U_i$ is compact. Thus, there is a finite subcover $\bigcup_{i=1}^{n} U_i \supset K$. By theorem 6.10, there is a partition of unity on K subordinated to $(U_i)_{i=1}^n$, $(h_i)_{i=1}^n \subseteq C_c(X)$, such that each $h_i < U_i$, and $\sum_{i=1}^n h_i = 1$. Now take $f_i := h_i f$ for all $i \in [n]$, and $f_i := 0$ for $i > n$. Clearly each $f_i < U_i$, so $\Lambda(f_i) \leq \mu^*(U_i)$. Also, $\sum_{i=1}^n f_i = f$ on $K = \operatorname{Supp}(f)$, and thus on X. Thus,

   $$\begin{aligned} \Lambda(f) &= \sum_{i=1}^{n} \Lambda(f_i) \\ &= \sum_{i=1}^{\infty} \Lambda(f_i) \\ &\leq \sum_{i=1}^{\infty} \mu^*(U_i). \end{aligned}$$

   Since this holds for all such $f < U$, we have that

   $$\mu^*(U) \leq \sum_{i=1}^{\infty} \mu^*(U_i).$$

   Since U is open, and $U \supseteq A$, we have $\mu^*(A) \leq \mu^*(U) \leq \sum_{i=1}^{\infty} \mu^*(U_i)$. Since this holds for all such open U_i, such that $A \subseteq \bigcup_{i=1}^{\infty} U_i$, we have

   $$\mu^*(A) \leq \inf \left\{ \sum_{i=1}^{\infty} \mu^*(U_i) : \text{open } U_i, A \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$

   This shows $\mu^*(A) = \inf \{\sum_{i=1}^{\infty} \mu^*(U_i) : \text{open } U_i, A \subseteq \bigcup_{i=1}^{\infty} U_i\}$, and clearly we have $\mu^*(\emptyset) = \Lambda(0) = 0$. By proposition 4.1, μ^* is an outer measure. Thus, there is a complete measure space $(X, \mathcal{M}, \mu)$ induced by the Caratheodory Theorem 4.3.

3. We now check any open set $O \in \mathcal{M}$, so μ is a Borel measure. Namely, for all $A \subseteq X$,

   $$\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A).$$

   First, suppose A is open, then $O \cap A$ is open. Fix any $\epsilon > 0$. By definition, there is $f < O \cap A$, such that

   $$\Lambda(f) \geq \mu^*(A \cap O) - \frac{1}{2}\epsilon.$$

Since $A \setminus \mathrm{Supp}(f)$ is open, there is $g < A \setminus \mathrm{Supp}(f) \subseteq A \setminus (A \cap O) = A \setminus O$, such that

$$\begin{aligned}\Lambda(g) &\geq \mu^*(A \setminus \mathrm{Supp}(f)) - \frac{1}{2}\epsilon \\ &\geq \mu^*(A \setminus O) - \frac{1}{2}\epsilon.\end{aligned}$$

Notice that $0 \leq f + g \leq 1$, since $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$, and$0 \leq f, g \leq 1$. Also, $\mathrm{Supp}(f) \cup \mathrm{Supp}(g) \subseteq (A \cap O) \cup (A \setminus O) = A$. Also, $f + g$ is continuous. This shows $f + g < A$. Thus,

$$\begin{aligned}\mu^*(A) &\geq \Lambda(f + g) \\ &= \Lambda(f) + \Lambda(g) \\ &\geq \mu^*(A \cap O) - \frac{1}{2}\epsilon + \mu^*(A \setminus O) - \frac{1}{2}\epsilon \\ &= \mu^*(O \cap A) + \mu^*(O^c \cap A) - \epsilon.\end{aligned}$$

Since this holds for all $\epsilon > 0$, we have that $\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A)$.

Now for any set A, fix any $\epsilon > 0$. We have that there is some open $U \supseteq A$, such that

$$\mu^*(U) \leq \mu^*(A) + \epsilon.$$

Since U is open, we have

$$\mu^*(U) = \mu^*(O \cap U) + \mu^*(O^c \cap U) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A).$$

Thus, $\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A) - \epsilon$. Since this holds for all $\epsilon > 0$, we have

$$\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A).$$

This shows that any open $O \in \mathcal{M}$. Since the Borel algebra is generated by the open sets, all Borel sets are in $\mathcal{M}$. Thus, μ (restricted to the Borel algebra) is a Borel Measure.

4. (a) is by definition of μ^*, and that any open U is measurable, so

$$\mu(U) = \mu^*(U) = \sup\{\Lambda(f) : f < U\}.$$

To check (b), we fix any compact K. Consider any function f such that $K < f$. For any $\epsilon > 0$, let $O_\epsilon := \{x \in X : f(x) > 1 - \epsilon\}$, which is open. Clearly $K \subseteq O_\epsilon$, so

$$\mu(K) \leq \mu(O_\epsilon) = \sup\{\Lambda(g) : g < O_\epsilon\}.$$

However, for any such g, we always have $\frac{f}{1-\epsilon} > 1 \geq g$ on O_ϵ, and $\frac{f}{1-\epsilon} \geq 0 = g$ outside of O_ϵ, so $\frac{f}{1-\epsilon} - g$ is positive. Thus, $\Lambda\left(\frac{f}{1-\epsilon}\right) \geq \Lambda(g)$. Since this holds for all such g,

$$\mu(K) \leq \sup\{\Lambda(g) : g < O_\epsilon\} \leq \Lambda\left(\frac{f}{1-\epsilon}\right) = \frac{1}{1-\epsilon}\Lambda(f).$$

Since this holds for all $\epsilon > 0$, we have $\mu(K) \leq \Lambda(f)$. Since this holds for all $K < f$, we have

$$\mu(K) \leq \inf\{\Lambda(f) : K < f\}.$$

On the other hand, for any open $U \supseteq K$, we can find some $K < f < U$. Notice that $\inf\{\Lambda(f) : K < f\} \leq \Lambda(f) \leq \mu(U)$. Since this holds for all open $U \supseteq K$, we have

$$\mu(K) = \inf\{\mu(U) : \text{open } U \supseteq K\} \geq \inf\{\Lambda(f) : K < f\}.$$

This shows (b).

5. We now check that μ is indeed a Radon measure.

   For any compact $K \subseteq X$, by Urysohn's lemma 6.9, there is some function f such that $K < f$. We thus have
$$\mu(K) \le \Lambda(f) < \infty.$$

   The outer regularity on Borel sets is by definition of μ^*, since
$$\mu(A) = \mu^*(A) = \inf\{\mu^*(U) : \text{open } U \supseteq A\} = \inf\{\mu(U) : \text{open } U \supseteq A\}.$$

   Now consider any open set U. For any $a < \mu(U)$, by (a), there is some $f < U$, such that $\Lambda(f) > a$. Let $K := \operatorname{Supp}(f) \subseteq U$, which is compact. For any function g such that $K < g$, we have that $g \ge 1 \ge f$ on K, and $g \ge 0 = f$ outside of f. Thus, $g - f \ge 0$, so $\Lambda(f) \le \Lambda(g)$. Since this holds for all g, by (b),
$$a < \Lambda(f) \le \inf\{\Lambda(g) : K < g\} \le \mu(K) \le \sup\{\mu(K) : \text{compact } K \subseteq U\}.$$

   Since this holds for all $a < \mu(U)$, we have
$$\mu(U) \le \sup\{\mu(K) : \text{compact } K \subseteq U\}.$$

   On the other hand, clearly $\sup\{\mu(K) : \text{compact } K \subseteq U\} \le \mu(U)$, so we have
$$\mu(U) = \sup\{\mu(K) : \text{compact } K \subseteq U\},$$

   which show outer regularity on open sets.

   Now we have shown that μ is Radon.

6. It is left to show for all $f \in C_c(U)$, $\Lambda(f) = \int_X f d\mu$. We first assume $\operatorname{Im}(f) \in [0,1]$. Fix some $N \ge 1$, let $K_0 := \operatorname{Supp}(f)$, and for each $j \in [N]$, let $K_j := \left\{x : f(x) \ge \frac{j}{N}\right\}$, each of which is closed since f is continuous. Since K_0 is compact, and $K_N \subseteq K_{N-1} \subseteq \cdots K_1 \subseteq K_0$, we have that all of them are compact. Now define
$$f_j(x) := \begin{cases} 0, & x \notin K_{j-1} \\ f(x) - \frac{j-1}{N}, & x \in K_{j-1} \setminus K_j \\ \frac{1}{N}, & x \in K_j. \end{cases}$$

   We have that for any $x \in K_0$, and let n be maximal such that $x \in K_n$, we have that
$$\begin{aligned} \sum_{j=1}^{N} f_j(x) &= \sum_{j=1}^{n} f_j(x) + f_{n+1}(x) \\ &= n\frac{1}{N} + f(x) - \frac{n+1-1}{N} \\ &= f(x). \end{aligned}$$

   Thus $f(x) = \sum_{j=1}^N f_j$. Also, for $x \in K_j$, we have $Nf_j(x) = 1 = \chi_{K_j}(x) = \chi_{K_{j-1}}(x)$, and for all $x \notin K_j$, we have $Nf_j(x) = 0 = \chi_{K_j}(x) = \chi_{K_{j-1}}(x)$. Lastly, for all $x \in K_{j-1} \setminus K_j$, we have $\frac{j-1}{N} \le f(x) < \frac{j}{N}$, so
$$\begin{gathered} Nf_j(x) = N\left(f(x) - \frac{j-1}{N}\right) \\ N\left(\frac{j-1}{N} - \frac{j-i}{N}\right) \le Nf_j(x) < N\left(\frac{j}{N} - \frac{j-1}{N}\right) \\ 0 \le Nf_j(x) < 1 \\ \chi_{K_j}(x) \le Nf_j(x) < \chi_{K_{j-1}}(x) \end{gathered}$$

   Thus,
$$\mu(K_j) = \int_X \chi_{K_j}(x) d\mu \le \int_X Nf_j(x) d\mu < \int_X \chi_{K_{j-1}}(x) d\mu = \mu(K_{j-1}).$$

Summing up over $j \in [N]$, we have

$$\begin{aligned}
\sum_{j=1}^{N} \mu(K_j) &\leq \sum_{j=1}^{N} \int_X N f_j d\mu < \sum_{j=1}^{N} \mu(K_{j-1}) \\
\sum_{j=1}^{N} \mu(K_j) &\leq N \int_X f d\mu < \sum_{j=0}^{N-1} \mu(K_j).
\end{aligned}$$

On the other hand, notice that we have $0 \leq N f_j < 1$. Also, each f_j is continuous, and $\operatorname{Supp}(f_j) = K_{j-1}$ is compact, so $f_j \in C_c(X)$. Clearly $K_j < N f_j$, so by (b), $\mu(K_j) \leq \Lambda(N f_j) = N\Lambda(f_j)$. Also, for any open $U \supset K_{j-1}$, we have $\operatorname{Supp}(f_j) = K_{j-1} \subseteq U$, so $N f_j < U$. By (a), $\mu(U) \geq \Lambda(N f_j) = N\Lambda(f_j)$. By outer regularity, $\mu(K_{j-1}) \geq N\Lambda(f_j)$. Thus,

$$\begin{aligned}
\sum_{j=1}^{N} \mu(K_j) &\leq \sum_{j=1}^{N} N\Lambda(f_j) < \sum_{j=1}^{N} \mu(K_{j-1}) \\
\sum_{j=1}^{N} \mu(K_j) &\leq N\Lambda(f) < \sum_{j=0}^{N-1} \mu(K_j).
\end{aligned}$$

This show that

$$\begin{aligned}
N\left|\Lambda(f) - \int_X f d\mu\right| &= \left|N\Lambda(f) - N\int_X f d\mu\right| \\
&\leq \sum_{j=0}^{N-1} \mu(K_j) - \sum_{j=1}^{N} \mu(K_j) \\
&= \mu(K_0) - \mu(K_N) \\
&= \mu(K_0) \\
&< \infty.
\end{aligned}$$

Since this holds for all $N \geq 1$, we must have

$$\Lambda(f) = \int_X f d\mu.$$

Now for any $f \in C_c(X)$ such that $\operatorname{Im}(f) \subseteq \mathbb{R}^+$, we can let $\tilde{f} := \frac{f}{\|f\|_\infty}$, which is still in $C_c(X)$, and $\operatorname{Im}(\tilde{f}) \in [0,1]$. Thus, the previous case applies. By linearity of Λ and the integral, we have

$$\Lambda(f) = \|f\|_\infty \Lambda(\tilde{f}) = \|f\|_\infty \int_X \tilde{f} d\mu = \int_X f d\mu.$$

Now for any $f \in C_c(X)$, such that $\operatorname{Im}(f) \subseteq \mathbb{R}$, we can break it into $f^+ := \max\{0, f\}, f^- := -\min\{0, f\}$, both nonnegative and still in $C_c(X)$. Since $f = f^+ - f^-$, by linearity, we have $\Lambda(f) = \int_X f d\mu$. Lastly, for any $f \in C_c(X)$, we have that $\Re(f), \Im(f) \in C_c(X)$ and are both real-valued, so again by linearity,

$$\Lambda(f) = \int_X f d\mu.$$

□

7 Signed and Complex measures

7.1 Signed measures

Recall that if $(X, \mathcal{M}, \mu)$ is a measure space, and $f : X \to [0, \infty)$ is measurable, then we can set a measure $\mu_f(A) := \int_X \chi_A f d\mu$, and we have $\int_X g d\mu_f = \int g f d\mu$.

Example 7.1.1. Consider the regular Lebesgue measure, and $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, then λ_f gives a probability measure with the standard distribution.

Now we want to generalize this to functions that are not non-negative.

Definition 7.1. Let $(X, \mathcal{M})$ be a measurable space. A function $\nu : \mathcal{M} \to [-\infty, \infty]$ is a **signed measure** if ν only takes at most one of $\pm\infty$ and satisfies countable additivity. Namely, for any pairwise disjoint sets $E_1, E_2, \dots$ in $\mathcal{M}$, we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Proposition 7.1. *Suppose $|\nu(\bigsqcup_{i=1}^{\infty} E_i)| < \infty$, then $\sum_{i=1}^{\infty} \nu(E_i)$ must converge absolutely, since we want $\nu(\bigsqcup_{i=1}^{\infty} E_i)$ to be invariant of the order of union.*

Proposition 7.2. *If $f \in \mathcal{L}^1(X, \mu)$, then $\mu_f(A) := \int_X \chi(A) f d\mu$ is a signed measure.*

Proposition 7.3. *If $f, g \geq 0$ is measurable, and $g \in \mathcal{L}^1(\mu)$, then $\nu(E) := \int_X \chi_E(f - g) d\mu$ is a signed measure.*

Definition 7.2. Suppose ν is a signed measure, then $E \in \mathcal{M}$ is

1. **null** for ν if $\forall F \subseteq E, \nu(F) = 0$.
2. **positive** for ν if $\forall F \subseteq E, \nu(F) \geq 0$.
3. **negative** for ν if $\forall F \subseteq E, \nu(F) \leq 0$.

Lemma 7.4. *Let $E \in \mathcal{M}$, if $0 < \nu(E) < \infty$, then $\exists A \subseteq E, A \in \mathcal{M}$ is positive, and $\nu(A) > 0$.*

Proof. Suppose E is positive, then we are done.
Suppose E is not positive, then $\inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\} < 0$.
Thus, $\frac{1}{2}\inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\} > \inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\}$, and we can choose $B_1 \subseteq E, B_1 \in \mathcal{M}$, such that

$$\nu(B_1) \leq \frac{1}{2}\inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\} \leq \max\left\{-1, \frac{1}{2}\inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\}\right\}.$$

Recursively choose $B_n \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i$ with $\nu(B_n) \leq \max\left\{-1, \frac{1}{2}\inf\left\{\nu(B) | B \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i, B \in \mathcal{M}\right\}\right\}$.
Now either this sequence terminates (then $A = E \setminus \bigsqcup_{i=1}^{n-1} B_i$ is positive), or we get an infinite sequence.
Set $A := E \setminus \bigsqcup_{i=1}^{\infty} B_i$.
We have $\nu(E) = \nu(A) + \sum_{i=1}^{\infty} \nu(B_i) < \infty$, thus $\sum_{i=1}^{\infty} \nu(B_i)$ converges absolutely.
Thus, $\nu(A) = \nu(E) - \sum_{i=1}^{\infty} \nu(B_i) > \nu(E) > 0$, since each $\nu(B_i) < 0$ by construction.
Notice that $\nu(B_n) \to 0^-$ since the sum converges, so if $B \subseteq A \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i$ has $\nu(B) < 0$, we must have $\nu(B) < 2\nu(B_n)$ for some large n. However, by construction,

$$\inf\left\{\nu(B) | B \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i, B \in \mathcal{M}\right\} \geq 2\nu(B_n),$$

which is a contradiction.
Thus A is positive. $\square$

Lemma 7.5. *If $(A_n)_{n=1}^{\infty}$ is a sequence of positive sets, then $A := \bigcup_{n=1}^{\infty} A_n$ is positive.*

Proof. Consider any $B \subseteq A, B \in \mathcal{M}$. Let $B_n := B \cap (A_n \setminus \bigcup_{i=1}^{n-1} A_i)$, then $(B_n)_{n=1}^{\infty} \subseteq \mathcal{M}$ are pairwise disjoint, with $B = \bigsqcup_{n=1}^{\infty} B_n$.
For any n, since A_n is positive, and $B_n \subseteq A_n$, we have $\nu(B_n) > 0$.
Thus $\nu(B) = \sum_{n=1}^{\infty} \nu(B_n) > 0$. $\square$

Theorem 7.6 (Hahn decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$, there are $P, N \in \mathcal{M}$ such that $X = P \sqcup N$, and P is positive, N is negative. Moreover, this is unique in the sense that if $X = P' \sqcup N'$ is another such decomposition, then the symmetric difference $P \Delta P'$ is null.*

Proof. Existence:
By taking $-\nu$ if necessary, we can WLOG assume ν never takes $+\infty$.
Let $m := \sup\{\nu(A) : A \text{ is positive}\} < \infty$.
Choose positive sets A_n such that $\nu(A_n) \to m$, and let $P := \bigcup_{n=1}^{\infty} A_n$.
Thus P is positive by lemma, so $\nu(P) \leq m$.
Notice that $\forall n,\ \nu(P) = \nu(A_n) + \nu(P \setminus A_n) \geq \nu(A_n)$, since $P \setminus A_n \subseteq P$ must have $\nu(P \setminus A_n) \geq 0$.
Taking the supremum over n, we have $\nu(P) \geq m$.
Thus $\nu(P) = m$.
Let $N := X \setminus P$.
Suppose N is not negative, $\exists E \subseteq N, E \in \mathcal{M}$, such that $\nu(E) > 0$. By lemma, there is positive $A \subseteq E$ with $\nu(A) > 0$. Then $P \sqcup A$ is measurable, positive, and $\nu(P \sqcup A) = \nu(P) + \nu(A) > m$, which is a contradiction.
Uniqueness:
Let

$$A := P \setminus P' = (X \setminus N) \setminus P' = X \cap N^c \cap (P')^c = N' \setminus N.$$

It is both positive and negative, thus null. Similarly, $B := P' \setminus P = N \setminus N'$ is null. Thus, $P \Delta P' = A \cup B$ is null. □

Definition 7.3. Let ν be a signed measure on $(X, \mathcal{M})$, and $P, N \in \mathcal{M}$ be as from Hahn decomposition, the **Jordan decomposition** of it is $\nu = \nu^+ - \nu^-$, where $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$ are positive measures.

Proposition 7.7. *Let ν be a signed measure on $(X, \mathcal{M})$, with Jordan decomposition $\nu = \nu^+ - \nu^-$, then for any other positive $\lambda_1, \lambda_2,$ such that $\nu = \lambda_1 - \lambda_2$, we have $\lambda_1 \geq \nu^+, \lambda_2 \geq \nu^-$.*

Proof. Let $X = P \sqcup N$ be the Hahn decomposition.
Consider any $E \in \mathcal{M}$, we have

$$\begin{aligned}
\lambda_1(E) &\geq \lambda_1(E \cap P) \\
&= \nu(E \cap P) + \lambda_2(E \cap P) \\
&\geq \nu(E \cap P) \\
&= \nu^+(E) \\
\lambda_2(E) &\geq \lambda_2(E \cap N) \\
&= -\nu(E \cap N) + \lambda_1(E \cap N) \\
&\geq -\nu(E \cap N) \\
&= \nu^-(E).
\end{aligned}$$

□

Definition 7.4. Let ν be a signed measure on $(X, \mathcal{M})$, with Jordan decomposition $\nu = \nu^+ - \nu^-$, the **total variation** is

$$|\nu| := \nu^+ + \nu^-.$$

Proposition 7.8. *Let $(X, \mathcal{M})$ be a measure space, ν be a signed measure, we have*

$$\forall E \in \mathcal{M}, |\nu|(E) = \sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}\right\}.$$

Proof. For any $E = \bigsqcup_{i=1}^{\infty} E_i \in \mathcal{M}$, we have $|\nu(E_i)| = |\nu^+(E_i) - \nu^-(E_i)| \leq \nu^+(E_i) + \nu^-(E_i)$, since both $\nu^+(E_i), \nu^-(E_i) \geq 0$ are positive measures. Thus,

$$\begin{aligned}\sum_{i=1}^{\infty} |\nu(E_i)| &\leq \sum_{i=1}^{\infty} \nu^+(E_i) + \nu^-(E_i) \\ &= \nu^+(E) + \nu^-(E) \\ &|\nu|(E).\end{aligned}$$

This shows

$$\sup \left\{ \sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M} \right\} \leq |\nu|(E).$$

Now for $E = (E \cap P) \sqcup (E \cap N)$, we have

$$\begin{aligned}|\nu(E \cap P)| + |\nu(E \cap N)| &= \nu(E \cap P) + \nu(E \cap N) \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E).\end{aligned}$$

This shows

$$\sup \left\{ \sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M} \right\} \geq |\nu|(E).$$

□

Definition 7.5. Let ν be a signed measure with its Jordan decomposition $\nu = \nu^+ - \nu^-$, we define

$$\int_X f d\nu := \int_X f d\nu^+ - \int_X f d\nu^-$$

for any measurable function $f : X \to \mathbb{C}$ which is integrable with respect to each $\nu^{\pm}$ and the subtraction makes sense.

7.2 Complex measures

Definition 7.6. Let $(X, \mathcal{M})$ be a measurable space, a **complex measure** is a function $\nu : \mathcal{M} \to \mathbb{C}$, such that it satisfies countable additivity. Namely, for any pairwise disjoint sets $E_1, E_2, \ldots$ in $\mathcal{M}$, we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Proposition 7.9. *Suppose $|\nu(\bigsqcup_{i=1}^{\infty} E_i)| < \infty$, then $\sum_{i=1}^{\infty} \nu(E_i)$ must converge absolutely, since we want $\nu(\bigsqcup_{i=1}^{\infty} E_i)$ to be invariant of the order of union.*

Proposition 7.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f \in \mathcal{L}^{\infty}(\mu)$ with $\|f\|_{L^{\infty}()} = 1$, then with $\nu(E) := \int_E f d\mu$, we have ν is a complex measure, and $\forall E \in \mathcal{M}$,*

$$\begin{aligned}\mu(E) &= \int_E d\mu \\ &\geq \int_E |f| d\mu \\ &\geq \left| \int_E h d\mu \right| \\ &\geq \left| \int_E d\nu \right| \\ &= |\nu(E)|.\end{aligned}$$

Definition 7.7. Let $(X, \mathcal{M})$ be a measurable space, the **total variation** of a complex measure μ is $|\mu| : \mathcal{M} \to [0, \infty]$, defined by

$$\forall E \in \mathcal{M}, |\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M} \right\}.$$

Proposition 7.11. *Let $(X, \mathcal{M})$ be a measurable space, μ be a complex measure, then $|\mu|$ is a positive measure.*

Proof.

1. $|\mu|(\emptyset) = 0$.
2. $|\mu|(E) \geq 0, \forall E \in \mathcal{M}$.
3. Fix $E = \bigsqcup_{i=1}^{\infty} E_i \in \mathcal{M},\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}$.
   Consider any $(A_j)_{j=1}^{\infty} \subset \mathcal{M}$ such that $E = \bigsqcup_{j=1}^{\infty} A_j$, then $A_j = A_j \cap E = \bigsqcup_{i=1}^{\infty} A_j \cap E_i$.

$$\begin{aligned} \sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)| \\ &\leq \sum_{i=1}^{\infty} |\mu| \left(\bigsqcup_{j=1}^{\infty} (A_j \cap E_i) \right) \\ &= \sum_{i=1}^{\infty} |\mu|(E_i). \end{aligned}$$

   We have that

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(A_j)| : E = \bigsqcup_{j=1}^{\infty} A_j,\ (A_j)_{j=1}^{\infty} \subset \mathcal{M} \right\} \leq \sum_{i=1}^{\infty} |\mu|(E_i).$$

   Now given any $\epsilon > 0$.
   $\forall i$, pick $t_i := |\mu|(E_i) - \frac{\epsilon}{2^i}$ and we can find $E_{ij} \in \mathcal{M}$, such that

$$E_i = \bigsqcup_{j=1}^{\infty} E_{ij},\ \sum_{j=1}^{\infty} |\mu(E_{ij})| > t_i.$$

   We have

$$\begin{aligned} |\mu|(E) &\geq \sum_{i,j=1}^{\infty} |\mu(E_{ij})| \\ &\geq \sum_{i=1}^{\infty} t_i \\ &= \sum_{i=1}^{\infty} |\mu|(E_i) - \epsilon. \end{aligned}$$

   Taking $\epsilon \to 0$, we have $|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(E_i)$.
   Thus $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(E_i)$.

$\square$

Lemma 7.12. *Let $\{z_1, \ldots, z_N\} \subset \mathbb{C}$, then $\exists S \subseteq [N]$, such that*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

Theorem 7.13. *Let $(X,\mathcal{M})$ be a measurable space, μ be a complex measure, then $|\mu|$ is a finite measure. Namely, $\forall E \in \mathcal{M},$ such that $|\mu|(E) < \infty$.*

Proof. Suppose $\exists E \in \mathcal{M},$ such that $|\mu|(E) = \infty$.
Let $B_0 = E$.
Let $t := \pi(1 + |\mu(E)|) \geq \pi$.
Then we can find a partition $E_i \in \mathcal{M}$, such that

$$E = \bigsqcup_{i=1}^{\infty} E_i, \quad \sum_{i=1}^{\infty} |\mu(E_{ij})| > t.$$

Thus there is some $N \in \mathbb{N},$ such that $\sum_{i=1}^{N} |\mu(E_{ij})| > t$.
By lemma, $\exists S \subseteq [N]$, such that

$$\begin{aligned}\left|\mu\left(\bigsqcup_{i\in S} E_i\right)\right| &= \left|\sum_{i\in S}\mu(E_i)\right| \\ &\geq \frac{1}{\pi}\sum_{i=1}^{N}|\mu(E_i)| \\ &> \frac{t}{\pi} \\ &\geq 1.\end{aligned}$$

Now let $A := \bigsqcup_{i\in S} E_i, B = E \setminus A$.
We have

$$\begin{aligned}|\mu(B)| &= |\mu(E) - \mu(A)| \\ &\geq |\mu(A)| - |\mu(E)| \\ &> \frac{t}{\pi} - |\mu(E)| \\ &= 1.\end{aligned}$$

Thus $E = A \sqcup B$, where $|\mu(A)| > 1, |\mu(B)| > 1$.
Since $|\mu|(A \sqcup B) = |\mu|(A) + |\mu|(B) = \infty$, at least one of $|\mu|(A), |\mu|(B)$ is ∞.
WLOG, say $|\mu|(B) = \infty$. We let $A_1 = A, B_1 = B$.
Now apply the above argument on $B_1 = A_2 \sqcup B_2$, where $|\mu(A_2)| > 1, |\mu(B_2)| > 1, |\mu|(B_2) = \infty$.
Repetitively, we construct disjoint $(A_k)_{k=1}^{\infty}$, such that $\forall i \geq 1, |\mu(A_i)| > 1$.
Notice that $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{M}$, and we have $\mu(\bigsqcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty}\mu(A_k)$ absolutely.
However, $\sum_{k=1}^{\infty}|\mu(A_k)| \geq \sum_{k=1}^{\infty} 1 = \infty$ diverges. ∎

Definition 7.8. Let ν be a complex measure on $(X,\mathcal{M})$, with its real-imaginary decomposition $\nu = \nu_{\Re} + i\nu_{\Im}$, where each $\nu_{\Re} := \Re(\nu), \nu_{\Im} := \Im(\nu)$ is a finite signed measure. The **real-imaginary Jordan decomposition** of ν is $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, where each of them is a finite positive measure from the Jordan decompositions of $\nu_{\Re}, \nu_{\Im}$.

Definition 7.9. Let ν be a complex measure with its real-imaginary Jordan decomposition $\nu = \nu_{\Re} + i\nu_{\Im} = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, we define

$$\int_X f d\nu := \int_X f d\nu_{\Re} + i\int_X f d\nu_{\Im} = \int_X f d\nu_1 - \int_X f d\nu_2 + i\int_X f d\nu_3 - i\int_X f d\nu_4$$

for any measurable function f which is integrable with respect to each composition.

8 Radon–Nikodym-Lebesgue Decomposition

Now we want to consider the converse: Given two measures ν, μ, when can we find a f, such that $\nu(E) = \int_E f d\mu$?

8.1 Absolutely Continuous Measures

Definition 8.1. Let μ, ν be two (complex or signed or positive) measures on $(X, \mathcal{M})$, we say ν is **absolutely continuous** with respect to μ if $\mu(A) = 0 \implies \nu(A) = 0$, and is written as $\nu \ll \mu$.

Example 8.1.1. Consider the counting measure μ, then for any other (non trivially infinite) measure ν, we always have $\nu \ll \mu$, since $\mu(E) = 0 \implies E = \emptyset$.

Proposition 8.1. *Let ν be a signed measure with its Jordan decomposition $\nu = \nu^+ + \nu^-$, we have that $\nu^+ \ll |\nu|,\ \nu^- \ll |\nu|$.*

Proof. Since both ν^+, ν^- are positive measures, $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0 \implies \nu^+(E) = \nu^-(E) = 0$. $\square$

Proposition 8.2. *Let ν be a complex measure with its real-imaginary Jordan decomposition, we have that $|\nu_{\Re}|, |\nu_{\Im}| \ll |\nu|$, so each $\nu_i \ll |\nu|$.*

Proof. For any $E = \bigsqcup_{i=1}^\infty E_i \in \mathcal{M}$, we have that $|\nu_{\Re}(E_i)|, |\nu_{\Im}(E_i)| \leq |\nu(E_i)|$, so

$$\begin{aligned}
|\nu_{\Re}|(E) &= \sup\left\{\sum_{i=1}^\infty |\nu_{\Re}(E_i)| : E = \bigsqcup_{i=1}^\infty E_i,\ (E_i)_{i=1}^\infty \subset \mathcal{M}\right\} \\
&\leq \sup\left\{\sum_{i=1}^\infty |\nu(E_i)| : E = \bigsqcup_{i=1}^\infty E_i,\ (E_i)_{i=1}^\infty \subset \mathcal{M}\right\} \\
&= |\nu|(E).
\end{aligned}$$

Thus, $|\nu|(E) = 0 \implies abs\nu_{\Re}(E) = 0$, so $|\nu_{\Re}| \ll |\nu|$. Similarly $|\nu_{\Im}| \ll |\nu|$. We can apply the previous result to get $\nu_1, \nu_2 \ll |\nu_{\Re}| \ll |\nu|, \nu_3, \nu_4 \ll |\nu_{\Im}| \ll |\nu|$. $\square$

8.2 Radon–Nikodym Derivatives

Definition 8.2. Let μ, ν be two (complex or signed or positive) measures on $(X, \mathcal{M})$. A function f is the **Radon–Nikodym derivative**, written as $\frac{d\nu}{d\mu}$, if

$$\forall E \in \mathcal{M},\ \nu(E) = \int_E f d\mu.$$

Proposition 8.3. *If the Radon–Nikodym derivative exists, it is unique μ-a.e..*

Proof. Suppose there are two f, h that satisfies $\forall E \in \mathcal{M},\ \nu(E) = \int_E f d\mu = \int_E h d\mu$, then $\int_E (f - h) d\mu = 0$. Thus $f = h$ μ-a.e.. $\square$

Proposition 8.4. *Suppose μ is a positive measure on a measurable space $(X, \mathcal{M})$, and ν is a (complex or signed or positive) measure such that $\frac{d\nu}{d\mu}$ exists, then $|\nu| \ll \mu$.*

Proof. Consider any $E \in \mathcal{M}$, suppose $\mu(E) = 0$, then $c = \int_E \frac{d\nu}{d\mu} d\mu = 0$. Thus, $\nu \ll \mu$.

Now consider any $E = \bigsqcup_{i=1}^\infty E_i$, since each $E_i \subseteq E$, we have $\mu(E) = 0 \implies \mu(E_i) = 0 \implies \nu(E_i) = 0$. Thus, $\sum_{i=1}^\infty |\nu(E_i)| = 0$. Since this holds for all decompositions,

$$|\nu|(E) = \sup\left\{\sum_{i=1}^\infty |\nu(E_i)| :,\ (E_i)_{i=1}^\infty \subset \mathcal{M}\right\} = 0.$$

$\square$

Proposition 8.5. *Suppose μ is a positive measure on a measurable space $(X, \mathcal{M})$, and ν is a (complex or signed or positive) measure such that $\frac{d\nu}{d\mu}$ exists, then for any g integrable with respect to ν, we have*

$$\int_X g d\nu = \int_X g \frac{d\nu}{d\mu} d\mu.$$

Thus, we can also abuse the notation and write $d\nu = \frac{d\nu}{d\mu} d\mu$ when μ is a positive measure.

Proof.

1. Assume μ, ν are both positive measure. First suppose $g = \chi_E$ for some $E \in \mathcal{M}$. Clearly,
$$\int_X g d\nu = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_X \chi_E \frac{d\nu}{d\mu} d\mu = \int_X g \frac{d\nu}{d\mu} d\mu.$$
By linearity, it holds for all simple functions.

   By Lebesgue's Monotone Convergence Theorem, this holds for all measurable $g : X \to [0, \infty]$.

   For $g : X \to [-\infty, \infty]$, write $g = g^+ - g^-$, where each function is $X \to [0, \infty]$. The result holds by linearity.

   For $g \in \mathcal{L}^1(X, \mu)$, write $g = \Re(g) + i\Im(g)$, where each function is $X \to \mathbb{R}$. The result holds by linearity.

2. Now suppose ν is a signed measure. Consider its Jordan decomposition $\nu = \nu^+ - \nu^-$ as in 7.3. We have that for any $E \in \mathcal{M}$,
$$\begin{aligned} \nu^+(E) &= \nu(E \cap P) \\ &= \int_{E \cap P} \frac{d\nu}{d\mu} d\mu \\ &= \int_E \chi_P \frac{d\nu}{d\mu} d\mu. \end{aligned}$$
Thus, $\int_X g d\nu^+ = \int_X g \chi_P \frac{d\nu}{d\mu} d\mu$. Similarly, for any $E \in \mathcal{M}$,
$$\begin{aligned} \nu^-(E) &= -\nu(E \cap N) \\ &= -\int_{E \cap N} \frac{d\nu}{d\mu} d\mu \\ &= \int_E -\chi_N \frac{d\nu}{d\mu} d\mu. \end{aligned}$$
Thus, $\int_X g d\nu^- = \int_X -g \chi_N \frac{d\nu}{d\mu} d\mu$. Now,
$$\begin{aligned} \int_X g d\nu &= \int_X g d\nu^+ - \int_X g d\nu^- \\ &= \int_X g \chi_P \frac{d\nu}{d\mu} d\mu - \int_X -g \chi_N \frac{d\nu}{d\mu} d\mu \\ &= \int_X g(\chi_P + \chi_N) \frac{d\nu}{d\mu} d\mu \\ &= \int_X g \frac{d\nu}{d\mu} d\mu. \end{aligned}$$

3. Now suppose ν is a complex measure. Consider its real-imaginary decompositions $\nu_\Re, \nu_\Im$, both signed measures. We have that for any $E \in \mathcal{M}$,
$$\begin{aligned} \nu_\Re(E) &= \Re(\nu(E)) \\ &= \Re\left(\int_E \frac{d\nu}{d\mu} d\mu \right) \\ &= \int_E \Re\left(\frac{d\nu}{d\mu} \right) d\mu. \end{aligned}$$

Thus, $\int_X g d\nu_{\Re} = \int_X g\Re\left(\frac{d\nu}{d\mu}\right)d\mu$. Similarly, $\int_X g d\nu_{\Im} = \int_X g\Im\left(\frac{d\nu}{d\mu}\right)d\mu$. We have

$$\begin{aligned}
\int_X g d\nu &= \int_X g d\nu_{\Re} + i\int_X g d\nu_{\Im} \\
&= \int_X g\Re\left(\frac{d\nu}{d\mu}\right)d\mu + i\int_X g\Im\left(\frac{d\nu}{d\mu}\right)d\mu \\
&= \int_X g\left(\Re\left(\frac{d\nu}{d\mu}\right) + i\Im\left(\frac{d\nu}{d\mu}\right)\right)d\mu \\
&= \int_X g\frac{d\nu}{d\mu}d\mu.
\end{aligned}$$

□

Proposition 8.6 (Chain Rule)**.** *Suppose μ is a positive measure on a measurable space $(X, \mathcal{M})$, and ν, λ are (complex or signed or positive) measures such that $\frac{d\nu}{d\mu}, \frac{d\lambda}{d\nu}$ exists, then $\frac{d\lambda}{d\mu}$ exists, and*

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}.$$

Proof. For any $E \in \mathcal{M}$, we have

$$\begin{aligned}
\lambda(E) &= \int_E \frac{d\lambda}{d\nu}d\nu \\
&= \int_X \chi_E\frac{d\lambda}{d\nu}d\nu \\
&= \int_X \chi_E\frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}d\mu \\
&= \int_E \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}d\mu.
\end{aligned}$$

□

8.3 Radon–Nikodym Theorem for Positive Measure

Lemma 8.7. *Let μ be a σ-finite measure on a measurable space $(X, \mathcal{M})$, then there is some $w \in L^1(\mu)$, such that $\forall x \in X, 0 < w(x) < 1$.*

Proof. Write $X = \bigcup_{n=1}^\infty E_n$, where $\forall n \geq 1, \mu(E_n) < \infty$.
Let $w_n := \frac{2^{-n}\chi_{E_n}}{1+\mu(E_n)}, w = \sum_{n=1}^\infty w_n$.
Notice that $0 < w_n(x) < 1$, and

$$\begin{aligned}
\int_X w d\mu &= \sum_{n=1}^\infty \int_X w_n d\mu \\
&= \sum_{n=1}^\infty \int_X \frac{2^{-n}\chi_{E_n}}{1+\mu(E_n)}d\mu \\
&\leq \sum_{n=1}^\infty \frac{2^{-n}\mu(E_n)}{1+\mu(E_n)} \\
&< \sum_{n=1}^\infty 2^{-n} \\
&< \infty.
\end{aligned}$$

□

Lemma 8.8. *Let μ be a σ-finite measure on a measurable space $(X, \mathcal{M})$, and $g \in L^1(\mu)$. Suppose $\forall E \in \mathcal{M}$ such that $\mu(E) > 0$, we have that*

$$\frac{1}{\nu(E)} \int_E g d\mu \in S$$

for some closed $S \subseteq \mathbb{C}$, then

$$g(x) \in S, a.e.\ x \in X.$$

Proof. Assume for contradiction that there is $E =: g^{-1}\big(\bar{B}(t,r)\big)$ such that $\mu(E) > 0, \bar{B}(t,r) \subseteq S^c$.
Then $A_E(g) := \frac{1}{\mu(E)} \int_E g d\mu \in S$, while

$$\begin{aligned}
|A_E(g) - t| &= \left| \frac{1}{\mu(E)} \int_E (g-t) d\mu \right| \\
&\leq \frac{1}{\mu(E)} \int_E |g - t| d\mu \\
&\leq \frac{1}{\mu(E)} \int_E r d\mu \\
&= r.
\end{aligned}$$

Thus $A_E(g) \in \bar{B}(t,r) \subseteq S^c$, which is a contradiction. $\square$

Theorem 8.9 (Radon–Nikodym for finite measures)**.** *Let μ be a σ-finite measure, and ν is a positive finite measure on a measurable space $(X, \mathcal{M})$. Suppose $\nu \ll \mu$, then $\exists h \in L^1(\mu) \cap \mathcal{L}^+$, such that $\forall E \in \mathcal{M}, \nu(E) = \int_E h d\mu$. Namely, $\frac{d\nu}{d\mu}$ exists in $L^1(\mu) \cap \mathcal{L}^+$. Moreover, $\frac{d\nu}{d\mu}$ is unique μ-a.e.*

Proof. (Von Neumann's proof).
Since μ is σ-finite, there is some $w \in L^1(\mu)$, such that $\forall x \in X, 0 < w(x) < 1$.
Define a new measure $d\lambda := d\nu + w d\mu$, namely, $\forall E \in \mathcal{M}, \lambda(E) := \nu(E) + \int_E w d\mu$.

Claim 8.9.1. *There is some measurable g such that $\forall x \in X, g(x) \in [0,1]$, and for any measurable $f \in L^2(\lambda)$, we have $\int_E f(1-g) d\nu = \int_E f g w d\mu$.*

Proof. Notice that $\int_X f d\lambda = \int_X f d\lambda + \int_X f w d\lambda$ for any measurable f. Consider any $f \in L^2(\lambda)$,

$$\begin{aligned}
\left| \int_X f d\nu \right| &\leq \int_X |f| d\nu \\
&= \int_X |f| d\lambda - \int_X |f| w d\mu \\
&\leq \int_X |f| d\lambda \\
&\leq \int_X |f| \cdot 1 d\lambda \\
&\leq \|f\|_{L^2(\lambda)} \lambda(X).
\end{aligned}$$

Notice that $\lambda(X) = \nu(X) + \int_X w d\mu < \infty$, so $\Lambda : f \mapsto \int_X f d\nu \in L^2(\lambda)^*$. Since $L^2(\lambda)$ is a Hilbert space, there is a unique $g \in L^2(\lambda)$, such that $\int_X f g d\lambda = \Lambda(f) = \int_X f d\nu,\ \forall f \in L^2(\lambda)$.
Now we know $\int_X f d\nu = \int_X f g d\lambda = \int_X f g d\nu + \int_X f g w d\mu$.

For any $E \in \mathcal{M}, f \in L^2(\lambda)$, we can take $\tilde{f} := f\chi_E \in L^2(\lambda)$, and we get

$$\begin{aligned}
\int_E f(1-g)d\nu &= \int_X \tilde{f}(1-g)d\nu \\
&= \int_X \tilde{f}d\nu - \int_X \tilde{f}gd\nu \\
&= \int_X \tilde{f}gwd\mu + \int_X \tilde{f}gd\nu - \int_X \tilde{f}gd\nu \\
&= \int_X \tilde{f}gwd\mu \\
&= \int_E fgwd\mu.
\end{aligned}$$

In addition, for any $E \in \mathcal{M}$, taking $f = 1$, we have that

$$\nu(E) = \int_E d\nu = \int_E gd\lambda.$$

Thus $0 \leq \int_E gd\lambda \leq \lambda(E)$. Thus $\forall E \in \mathcal{M},$ such that $\lambda(E) > 0$, we have

$$\frac{\int_E gd\lambda}{\lambda(E)} \in [0,1].$$

By the above lemma, we have that $g(x) \in [0,1], \lambda$-a.e. $x \in X$.
WLOG, we can redefine $g(x) = 0$ for any $g(x) \notin [0,1]$. □

Let $A := g^{-1}([0,1)), B := g^{-1}(\{1\})$. Let $f = \chi_B$, we have that

$$\begin{aligned}
\int_X \chi_B(1-g)d\nu &= \int_X \chi_B gwd\mu \\
\int_B (1-g)d\nu &= \int_B wd\mu \\
0 &= \int_B wd\mu.
\end{aligned}$$

Since $w > 0$, we must have $\mu(B) = 0$. Since $\nu \ll \mu, \nu(B) = 0$. Thus $\forall E \in \mathcal{M}, \nu(E) = \nu(E\cap A) + \nu(E \cap B) = \nu(E \cap A)$.

Now, let $f_n := \sum_{k=0}^n g^k$, we have that $f_n(1-g) = 1 - g^{n+1}$, so

$$\int_E (1-g^{n+1})d\nu = \int_E f_n(1-g)d\nu = \int_E f_n gwd\mu.$$

Notice that $1 - g^{n+1}(x) \to \begin{cases} 1, & x \in A, \\ 0, & x \in B. \end{cases}$ monotonically. In addition, $(f_n gw)(x)$ is increasing and bounded, so there is some $h(x) := \lim_{n\to\infty}(f_n gw)(x)$. Thus, by LMCT, we have

$$\begin{aligned}
\nu(E) &= \nu(A \cap E) \\
&= \int_{E\cap A} d\nu \\
&= \int_{E\cap A} \lim_{n\to\infty} (1-g^{n+1})d\nu \\
&= \lim_{n\to\infty} \int_{E\cap A} (1-g^{n+1})d\nu \\
&= \lim_{n\to\infty} \int_E f_n gwd\mu \\
&= \int_E hd\mu.
\end{aligned}$$

Since ν is finite, we have $h \in L^1(\mu)$. $\square$

Theorem 8.10. *[Radon–Nikodym] Let μ, ν be two σ-finite measures on a measurable space $(X, \mathcal{M})$. Suppose $\nu \ll \mu$, then $\exists h \in \mathcal{L}^+$, such that $\forall E \in \mathcal{M}, \nu(E) = \int_E h d\mu$. Namely, $\frac{d\nu}{d\mu}$ exists in $\mathcal{L}^+$. Moreover, $\frac{d\nu}{d\mu}$ is unique μ-a.e.*

Proof. Since ν is σ-finite, we have $X = \bigsqcup_{n=1}^{\infty} X_n$, where each $\nu(X_n)$ is finite.

We can apply the above theorem on $\nu_n(E) := \nu(E \cap X_n)$, which are finite measures, and let $h = \sum_{n=1}^{\infty} h_n$. h will be positive and measurable, but not in $L^1(\mu)$. Yet it is in $L^1(\mu|_{X_n})$ for all n. $\square$

Remark. The σ-finiteness is essential. Indeed, consider the following counterexample.

Example 8.3.1. Consider λ to be the Lebesgue measure on $(0, 1)$, and μ to be the counting measure, which is not σ-finite.
Although $\lambda \ll \mu$, it is impossible to find such an $h = \frac{d\lambda}{d\mu}$, because for any $E \in \mathcal{M}$, we will have

$$\begin{aligned} \lambda_a(E) &= \int_E h d\mu \\ &= \sum_{x \in E} h(x), \end{aligned}$$

which is not possible.

8.4 Signed and Complex Measures

Proposition 8.11 (Polar decomposition of signed measure)**.** *Let $\nu = \nu^+ - \nu^-$ be a signed measure on a measurable space $(X, \mathcal{M})$. There exists a measurable function $f : X \to \mathbb{R}$ with $|f| = 1$, and*

$$\forall E \in \mathcal{M}, \ \nu(E) = \int_E f d|\nu|.$$

Namely, $f = \frac{d\nu}{d|\nu|}$.

Proof. Let $X = P \sqcup N$ be the Hahn decomposition theorem 7.6. Define $f := \chi_P - \chi_N$, then clearly $|f| = 1$. In addition,

$$\begin{aligned} \int_E f d|\nu| &= \int_E \chi_P - \chi_N d|\nu| \\ &= \int_{E \cap P} d|\nu| - \int_{E \cap N} d|\nu| \\ &= |\nu|(E \cap P) - |\nu|(E \cap N) \\ &= \nu^+(E \cap P) - \nu^-(E \cap N) \\ &= \nu^+(E) - \nu^-(E) \\ &= \nu(E). \end{aligned}$$

$\square$

Corollary 8.12 (Radon–Nikodym for Signed Measure)**.** *Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. Suppose $|\nu|, \mu$ are σ-finite with $|\nu| \ll \mu$, then there is a measurable function $g : X \to [-\infty, \infty]$, with at most one of g^+, g^- takes ∞, such that $\forall E \in \mathcal{M}, \ \nu(E) = \int_E g d\mu$.*

Proof. By the Radon–Nikodym Theorem 8.10, there is a unique $h \in \mathcal{L}^+$, such that $h = \frac{d|\nu|}{d\mu}$. Now consider $g^+ := h\chi_P, g^- := h\chi_N$, we have that $g^+, g^- \geq 0$. Also, $\forall E \in \mathcal{M}$,

$$\begin{aligned}\nu^+(E) &= |\nu|(E \cap P)\\ &= \int_{E\cap P} hd\mu\\ &= \int_E h\chi_P d\mu\\ &= \int_E g^+ d\mu.\end{aligned}$$

Similarly, $\nu^-(E) = \int_E g^- d\mu$. Since only one of ν^+, ν^- takes infinity, g^+, g^- cannot be both infinity on a non-μ-null space.

Let $g := g^+ - g^-$, we have

$$\begin{aligned}\nu(E) &= \nu^+(E) - \nu^-(E)\\ &= \int_E g^+ d\mu - \int_E g^- d\mu\\ &= \int_E gd\mu.\end{aligned}$$

□

Corollary 8.13 (Polar decomposition of complex measures)**.** *Let ν be a complex measure on $(X, \mathcal{M})$. There is a unique measurable function h, such that $|h| = 1$ $|\nu|$-a.e., and $d\nu = hd|\nu|$. Also, for any any integrable function f, we have*

$$\int_X fd\nu = \int_X fhd|\nu|.$$

Proof. Write $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ with Jordan decomposition. Notice that $\nu_i \ll |\nu|$. Applying Randon-Nikodym, we have some $h_i \in \mathcal{L}^+$, where $\forall E \in \mathcal{M}, \nu_i(E) = \int_E h_i d|\nu|$.

Define $h := (h_1 - h_2 + ih_3 - ih_4)$. Since $\nu_i, |\nu|$ are all positive measures, for any measurable f, we have

$$\begin{aligned}\int_X fd\nu &= \int_X fd\nu_1 - \int_X fd\nu_2 + i\int_X fd\nu_3 - i\int_X fd\nu_4\\ &= \int_X fh_1 d|\nu| - \int_X fh_2 d|\nu| + i\int_X fh_3 d|\nu| - i\int_X fh_4 d|\nu|\\ &= \int_X f(h_1 - h_2 + ih_3 - ih_4)d|\nu|\\ &= \int_X fhd|\nu|.\end{aligned}$$

For any $E \in \mathcal{M}$, taking $f = \chi_E$, we have

$$\nu(E) = \int_X \chi_E d\nu = \int_X \chi_E hd|\nu| = \int_E hd|\nu|.$$

For any $E \in \mathcal{M}$, we have

$$1 \geq \frac{|\nu(E)|}{|\nu|(E)} = \left|\frac{1}{|\nu|(E)}\int_E hd|\nu|\right|.$$

Thus $|h(x)| \leq 1$ a.e..

For any $0 < r < 1$, consider $A_r := \{x \in X : |h(x)| < r\} = \bigsqcup_{i=1}^{\infty} E_i$.

We have

$$\begin{aligned}
\sum_{i=1}^{\infty} |\nu(E_i)| &= \sum_{i=1}^{\infty} \left| \int_{E_i} hd|\nu| \right| \\
&\leq \sum_{i=1}^{\infty} \left| \int_{E_i} rd|\nu| \right| \\
&= r \sum_{i=1}^{\infty} |\nu|(E_i) \\
&= r|\nu|(A_r).
\end{aligned}$$

Taking sup over all E_i, we have that $|\nu|(A_r) \leq r|\nu|(E_i)$. Since $r < 1$, we have $|\nu|(E_i) = 0$.
Thus $|h(x)| > 1$ a.e.$x \in X$ for all $0 < r < 1$.
Thus $|h| = 1$ a.e.. □

Corollary 8.14. *Suppose μ, ν are (positive or signed or complex) measures on a measurable space $(X, \mathcal{M})$, where $|\mu|$ is σ-finite, and $f = \frac{d\nu}{d\mu}$ exists, then we have $\frac{d|\nu|}{d|\mu|}$ exists and*

$$\frac{d|\nu|}{d|\mu|} = |f|.$$

Proof. Let h_1, h_2 be measurable functions such that $d\mu = h_1 d|\mu|, d\nu = h_2 d|\nu|$, with $|h_1| = |h_2| = 1$ from polar decomposition. Now, $\frac{d\nu}{d|\mu|} = \frac{d\nu}{d\mu}\frac{d\mu}{d|\mu|} = h_1 f$ exists by proposition 8.6 since $|\mu|$ is a positive measure. Thus, $|\nu| \ll |\mu|$ by proposition 8.4.

By Radon–Nikodym theorem 8.10, there is $h \in \mathcal{L}^+$, with $d|\nu| = hd|\mu|$.

$$\begin{aligned}
\int_E h_1 f d|\mu| &= \nu(E) \\
&= \int_E h_2 d|\nu| \\
&= \int_E hh_2 d|\mu|.
\end{aligned}$$

Thus $hh_2 = h_1 f$ $|\mu|$-a.e.. Since $h \in \mathcal{L}^+$, and $|h_1| = |h_2| = 1$, we must have $h = |f|$ $|\mu|$-a.e.. □

8.5 Lebesgue Decompositions

Definition 8.3. Two positive measures μ, ν on a measurable space $(X, \mathcal{M})$ are said to be **mutually singular**, written as $\mu \perp \nu$, if $X = A \sqcup B$, where A is μ-null and B is ν-null. Namely, for all measurable $E \subseteq A$, $\mu(E) = 0$, and for all measurable $E \subseteq B$, $\nu(E) = 0$.

Proposition 8.15. *Let ν be a signed measure on $(X, \mathcal{M})$, with Jordan decomposition $\nu = \nu^+ - \nu^-$, then $\nu^+ \perp \nu^-$.*

Theorem 8.16 (Lebesgue decomposition)**.** *Let μ, ν be two σ-finite measures on a measurable space $(X, \mathcal{M})$. There is a unique decomposition $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu$ be the absolutely continuous part, and $\nu_s \perp \mu$ be the singular part, both positive measures.*

Proof. Take $\lambda = \mu + \nu$, which is σ-finite, and $\mu, \nu \ll \lambda$. By the Radon-Nikodym theorem 8.10, $\exists f, g \in \mathcal{L}^+$, such that

$$\mu(E) = \int_E f d\lambda,\ \nu(E) = \int_E g d\lambda.$$

Let $A = f^{-1}((0, \infty]), B = f^{-1}(\{0\}), \nu_a(E) = \nu(E \cap A), \nu_s(E) = \nu(E \cap B)$. Since $X = A \sqcup B$, clearly $\nu = \nu_a + \nu_s$. We can see that for any measurable $E \subseteq A$,

$$\nu_s(E) = \nu(E \cap B) = \nu(\emptyset) = 0.$$

On the other hand, for any measurable $E \subseteq B$, $f = 0$ on E, so

$$\mu(E) = \int_E f d\lambda = 0.$$

This shows $\nu_s \perp \mu$.

In addition, suppose $\mu(E) = \int_E f d\lambda = \int_{E\cap A} f d\lambda = 0$, then we must have $\lambda(E \cap A) = 0$, since $f > 0$ on $E \cap A$. Thus,

$$\nu_a(E) = \nu(E \cap A) = \int_{E\cap A} g d\lambda = 0.$$

This shows $\nu_a \ll \mu$. ☐

Theorem 8.17 (Lebegue-Radon-Nikodym). *Let ν be a (complex or signed or positive) measure on a measurable space $(X, \mathcal{M})$ such that $|\nu|$ is σ-finite. If μ is a σ-finite measure on $(X, \mathcal{M})$, then ν decomposes uniquely as $\nu = \nu_a + \nu_s$, such that $\nu_a \ll \mu$ is the absolutely continuous part and $\nu_s \perp \mu$ is the singular part. Also, $d\nu_a = f d\mu$ for some $f \in \mathcal{L}^1(\mu)$.*

Proof. From the polar decomposition, we have $d\nu = hd|\nu|$ for some $|h| = 1$, and we can use Lebesgue decomposition to write $|\nu| = |\nu|_a + |\nu|_s$, where $|\nu|_a \ll \mu$, and $|\nu|_s \perp \mu$.

By the Randon-Nikodym theorem 8.10, $g = \frac{d|\nu|_a}{d\mu}$ exists, so we have $d\nu = hgd\mu + hd|\nu|_s$. Let $f := hg$, and $d\nu_a := fd\mu, d\nu_s := hd|\nu|_s$, and we have the result.

Now we want to uniqueness: Suppose $\nu = \nu_a + \nu_s = \nu'_a + \nu'_s$ are two decompositions as in the theorem, then $(\nu_a - \nu'_a) + (\nu_s - \nu'_s) = 0$ is the zero measure. Thus, $\mu' := \nu_a - \nu'_a = \nu'_s - \nu_s$, but $\nu_a - \nu'_a \ll \mu$, and $\nu'_s - \nu_s \perp \mu$. Thus, $\mu' = 0$, and $\nu_a = \nu'_a, \nu'_s = \nu_s$. ☐

9 Dual of Function Spaces

9.1 Dual of L^p Spaces

Theorem 9.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\frac{1}{p} + \frac{1}{q} = 1$ for $p \in (1, \infty)$, we have*

$$L^q(\mu) \cong L^p(\mu)^*,$$

where the isometric isomorphism $L^q(\mu) \xrightarrow{\sim} L^p(\mu)^*$; $g \mapsto \Lambda_g$ is defined to be

$$\forall f \in L^p(\mu), \Lambda_g(f) := \int_X fgdx.$$

In addition, the same is true for $p = 1$ if μ is σ-finite.

Proof. Let $1 \le p < \infty$.

1. By 5.19, we only need to show the subjectivity: $\forall \Lambda \in L^p(\mu)^*, \exists g \in L^q(\mu)$, such that $\Lambda = \Lambda_g$.

2. First we assume $\mu(X) < \infty$ is finite.
   Given any $\Lambda \in L^p(\mu)^*$.
   Consider the mapping $\nu : E \mapsto \Lambda(\chi_E)$ for any measurable $E \in \mathcal{M}$.
   This is well-defined since $\chi_E \in L^\infty(X) \subseteq L^p(X)$.
   Notice that $|\nu(E)| = |\Lambda(\chi_E)| \le \|\Lambda\|_{L^p(\mu)^*} \|\chi_E\|_{L^p(\mu)} < \infty$, thus ν is finite.
   We have $\nu(\emptyset) = \Lambda(0) = 0$ since Λ is linear.
   For any $B = \bigsqcup_{i=0}^\infty A_i$, with $A_i \subseteq U$ be measurable, we have $\chi_B = \sum_{i=0}^\infty \chi_{A_i}$ in $L^p(\mu)$.

Indeed,

$$\begin{aligned}
\left\| \chi_B - \sum_{i=0}^{N} \chi_{A_i} \right\|_{L^p(\mu)}^p &= \left\| \sum_{i=N+1}^{\infty} \chi_{A_i} \right\|_{L^p(\mu)}^p \\
&= \left\| \chi_{\bigsqcup_{i=N+1}^{\infty} A_i} \right\|_{L^p(\mu)}^p \\
&= \mu\left(\bigsqcup_{i=N+1}^{\infty} A_i \right)^p \\
&\to 0.
\end{aligned}$$

Notice that this fails when $p = \infty$!
Thus,

$$\begin{aligned}
\nu(B) &= \Lambda(\chi_B) \\
&= \Lambda\left(\sum_{i=0}^{\infty} \chi_{A_i} \right) \\
&= \Lambda\left(\lim_{n\to\infty} \sum_{i=0}^{n} \chi_{A_i} \right) \\
&= \lim_{n\to\infty} \Lambda\left(\sum_{i=0}^{n} \chi_{A_i} \right) && \text{continuity of } \Lambda \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \Lambda(\chi_{A_i}) && \text{linearity of } \Lambda \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \nu(A_i) \\
&= \sum_{i=0}^{\infty} \nu(A_i),
\end{aligned}$$

which shows countable additivity.

In addition,

$$\begin{aligned}
\sum_{i=0}^{\infty}|\nu(A_i)| &= \lim_{n\to\infty}\sum_{i=0}^{n}|\Lambda(\chi_{A_i})| \\
&\le \lim_{n\to\infty}\sum_{i=0}^{n}\|\Lambda\|_{L^p(\mu)^*}\|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*}\lim_{n\to\infty}\sum_{i=0}^{n}\|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*}\lim_{n\to\infty}\sum_{i=0}^{n}\mu(A_i)^{1/p} \\
&\le \|\Lambda\|_{L^p(\mu)^*}\left(\lim_{n\to\infty}\sum_{i=0}^{n}\mu(A_i)\right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*}\mu\left(\bigsqcup_{i=0}^{\infty}A_i\right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*}\mu(B)^{1/p} \\
&\le \|\Lambda\|_{L^p(\mu)^*}\mu(X)^{1/p} \\
&< \infty,
\end{aligned}$$

which converges absolutely.
Thus ν is a complex measure.
In addition, if $\mu(E)=0$, we have $\nu(E)=\Lambda(\chi_E)=\Lambda(0)=0$.
Thus, $\nu \ll \mu$.
By Lebesgue-Radon-Nikodym for complex measures, $\exists! g\in L^1(\mu)$, such that $\Lambda(\chi_E)=\nu(E)=\int_E g d\mu$.
By linearity, $\Lambda(f)=\int_X fg d\mu$ for all simple measurable f.
By uniform simple function approximation, we have $\Lambda(f)=\int_X fg d\mu$ for all $f\in L^\infty(\mu)$.
Indeed, given any $f\in L^\infty(\mu)$, we have a sequence of simple measurable functions $|f_1|\le|f_2|\le\cdots\le|f|$ that converges uniformly with $\|f-f_n\|_{L^\infty(\mu)}\to 0$.
Thus $\|f-f_n\|_{L^p(\mu)}\to 0$.
Thus $|\Lambda(f)-\Lambda(f_n)|\le\|\Lambda\|_{L^p(\mu)^*}\|f-f_n\|_{L^p(\mu)}\to 0$.

$$\begin{aligned}
\Lambda(f) &= \lim_{n\to\infty}\Lambda(f_n) \\
&= \lim_{n\to\infty}\int_X f_n g d\mu \\
&= \int_X \lim_{n\to\infty} f_n g d\mu \\
&= \int_X fg d\mu.
\end{aligned}$$

(a) $p=1, q=\infty$.
Consider ant $E\in\mathcal{M}$, such that $\mu(E)>0$.

We have

$$\begin{aligned}
\left|\frac{1}{\mu(E)}\int_E g d\mu\right| &= \left|\frac{1}{\mu(E)}\Lambda(\chi_E)\right| \\
&\le \frac{1}{\mu(E)}\|\Lambda\|_{L^1(\mu)^*}\|\chi_E\|_{L^1(\mu)} \\
&= \frac{1}{\mu(E)}\|\Lambda\|_{L^1(\mu)^*}\mu(E) \\
&= \|\Lambda\|_{L^1(\mu)^*}.
\end{aligned}$$

Thus $|g(x)| \le \|\Lambda\|_{L^1(\mu)^*}$ a.e..
Thus $g \in L^\infty(\mu)$. Since simple functions are dense in $L^p(\mu)$, we have $L^\infty(\mu)$ is dense in $L^p(\mu)$.
Since Λ, Λ_q are both bounded linear functionals, we have $\Lambda(f) = \int_X fg d\mu$ for all $f \in L^p(\mu)$.

(b) $p > 1$.
Let $E_n := \{x \in X : |g(x)| \le n\}$.
By LMCT, we have $\|g\|_q = \lim_{n\to\infty} \|\chi_{E_n} g\|_q$.
Let $f = \chi_{E_n}\overline{\operatorname{sgn}(g)}|g|^{q-1} \in L^\infty(\mu)$, we have

$$\begin{aligned}
\|f\|_{L^p(\mu)}^p &= \int_{E_n} |g|^{(q-1)p} d\mu \\
&= \int_{E_n} |g|^q d\mu \\
&= \|\chi_{E_n} g\|_{L^q()}^q \\
\|\chi_{E_n} g\|_q^q &= \int_{E_n} |g|^q d\mu \\
&= \int_X fg d\mu \\
&= |\Lambda(f)| \\
&\le \|\Lambda\|\|f\|_{L^p(\mu)} \\
&\Longrightarrow \\
\|g\chi_{E_n}\|_{L^q(\mu)}^{q-\frac{q}{p}} &\le \|\Lambda\| \\
&\Longrightarrow \\
\|g\|_{L^q(\mu)}^{q-\frac{q}{p}} &\le \|\Lambda\| \\
&< \infty.
\end{aligned}$$

Thus $g \in L^q(U)$.
Since Λ, Λ_q are both bounded linear functionals, we have $\Lambda(f) = \int_X fg d\mu$ for all $f \in L^p(\mu)$.

3. Now we assume that μ is σ-finite.
We have $X = \bigcup_{n=1}^\infty X_n, \forall n \ge 1,\ X_n \subset X_{n+1}, \mu(X_n) < \infty$.
We can get

$$\forall n \ge 1, g_n \in L^q(X_n,\mu), \text{ such that } \Lambda(f) = \int_X f g_n d\mu, \forall f \in L^p(X_n,\mu).$$

Notice that $L^p(X_n,\mu) \subset L^p(X_{n+1},\mu)$.
We thus have $\forall n > m, g_n|_{X_m} = g_m$.
Let $g : X \to \mathbb{C};\ x \mapsto g_n(x)$ for $x \in X_n$.
Then $g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} g\chi_n$ in $\|\cdot\|_{L^q(\mu)}$, and thus $g \in L^q(\mu)$.
In addition, for any $f \in L^p(\mu)$, we have $\lim_{n\to\infty} f\chi_{X_n} = f$ in $\|\cdot\|_{L^q(\mu)}$.

We have

$$\begin{aligned}\Lambda(f) &= \Lambda(\lim_{n\to\infty} f\chi_{X_n}) \\ &= \lim_{n\to\infty}\Lambda(f\chi_{X_n}) \\ &= \lim_{n\to\infty}\int_X f\chi_{X_n} g d\mu \\ &= \int_X f g d\mu.\end{aligned}$$

4. Now suppose μ is not necessarily σ-finite, but $p \in (1, \infty)$.
   $\forall E \subseteq X$ be σ-finite, we have

   $$g_E \in L^q(E,\mu), \text{ such that } \Lambda(f) = \int_X f g_E d\mu, \forall f \in L^p(E,\mu)$$

   .
   In addition, $\|g_E\|_{L^q(\mu)} \le \|\Lambda\|$.
   Let $M := \sup_{E \text{ is } \sigma\text{-finite}} \|g_E\|_{L^q(\mu)} \le \|\Lambda\|$.
   Choose $(E_n)_{n=1}^{\infty}$ such that $\|g_E\|_{L^q(\mu)} \to M$.
   Then $F := \bigcup_{n=1}^{\infty} E_n$ is σ-finite, and $\|g_F\|_{L^q(\mu)} = M$.
   In addition, for any σ-finite $A \supseteq F$, we have $A \setminus F$ is σ-finite as well. Thus $g_A = g_F + g_{A\setminus F}$.
   We have $g_{A\setminus F} = 0$a.e., which means $g_a = g_F$a.e..
   Let $g := g_F \in L^q(\mu)$.
   Given any $f \in L^p()$, let $A := \{x \in X : f(x) \neq 0\}$, which has to be σ-finite.
   Thus $\Lambda(f) = \int_X g_A f d\mu = \int_X g_X f d\mu = \int_X g f d\mu$.

□

Remark. This is in general not true for $p = \infty$.

Remark. If μ is not σ-finite, it might be the case where $L^1(\mu) = \{0\}$, while $L^\infty(\mu) \neq \{0\}$.

9.2 Complex Regular Measure Space

Definition 9.1. Let $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ be a complex Borel measure on a locally compact Hausdorff space X, with its Jordan decomposition. We say μ is a **complex Radon measure** or **complex regular measure** if all μ_i are finite Radon measures.

Proposition 9.2. *μ is a complex Radon measure if and only if $|\mu|$ is a Radon measure.*

Proof. It follows from $\mu_i \le |\mu| \le \mu_1 + \mu_2 + \mu_3 + \mu_4$. □

Definition 9.2. Let X be a locally compact Hausdorff space, we define $M(X) := \{\mu : \text{complex Radon measure}\}$, and $\|\mu\|_{M(X)} := |\mu|(X)$

Proposition 9.3. *$(M(X), \|\cdot\|_{M(X)})$ is a normed vector space over $\mathbb{C}$.*

Definition 9.3. $C_0(X)$ is the closure of $C_c(X)$ in $\|\cdot\|_\infty$.

Theorem 9.4 (Jordan Decomposition for $C_0(X,\mathbb{R})$)**.** *For any $\phi \in C_0(X,\mathbb{R})^*$, we have ϕ^+, ϕ^- positive bounded linear functionals, such that $\phi = \phi^+ - \phi^-$ on $C_0(X,\mathbb{R})$.*

Proof. For any $f \ge 0$, let $\phi^+(f) := \sup(\phi(g) : 0 \le g \le f)$.
Notice that if $c \ge 0$, we have $\phi^+(cf) = c\phi^+(f)$.
In addition, $\forall f_1, f_2 \ge 0 \in C_0(\mathbb{R})$, and any $0 \le g_1 \le f_1, 0 \le g_2 \le f_2$, we have $0 \le g_1 + g_2 \le f_1 + f_2$.
Thus $\phi(g_1) + \phi(g_2) = \phi(g_1 + g_2) \le \phi^+(f_1 + f_2)$. Since g_1, g_2 are arbitrary, we have $\phi^+(f_1) + \phi^+(f_2) \le \phi^+(f_1 + f_2)$.

On the other hand, if we take any $g \le f_1 + f_2$, and $g_1 := \min(g, f_1), g_2 := g - g_2$, we have $g_2 \le g - f_1 \le f_2$.
Thus $\phi(g) = \phi(g_1) + \phi(g_2) \le \phi^+(f_1) + \phi^+(f_2)$.
Since g is arbitrary, $\phi^+(f_1 + f_2) \le \phi^+(f_1) + \phi^+(f_2)$.
Thus,

$$\phi^+(f_1) + \phi^+(f_2) = \phi^+(f_1 + f_2).$$

Now extend ϕ^+ to $C_0(X, \mathbb{R})$ by $\phi^+(f) = \phi^+(f^+) - \phi^+(f^-)$.
This is well-defined. Indeed, if $f = g - h = f^+ - f^-$ for $g, h \ge 0$, we have $g + f^- = h + f^+$, and thus $\phi^+(g) + \phi^+(f^-) = \phi^+(h) + \phi^+(f^+)$.
We can also check that ϕ^+ is linear, and $|\phi^+(f)| \le \|\phi\|\|f\|$.
Thus ϕ^+ is a bounded positive linear functional.
We will then define $\phi^- := \phi^+ - \phi$, and check it is also a bounded positive linear functional. $\square$

Theorem 9.5. *Let $\Lambda \in C_0(X)^*$, then $\exists!$ complex Radon measure $\mu \in M(X)$, such that*

$$\forall f \in C_0(X), \Lambda(f) = \int_X f d\mu.$$

Moreover, $\|\Lambda\| = \|\mu\|_{M(X)} = |\mu|(X)$.

Proof. We first consider $C_0(X; \mathbb{R})$, which is a real Banach subspace of $C_0(X)$.
Let $\Psi := \Lambda|_{C_0(X;\mathbb{R})}$.
Now let $\Psi_1 := \Re(\Psi), \Psi_2 := \Im(\Psi)$, we have that $\Psi_1, \Psi_2 \in C_0(X, \mathbb{R})^*$ over $\mathbb{R}$, with $\|\Psi_i\|_{C_0(X,\mathbb{R})^*} \le \|\Lambda\|$.
In addition,

$$\begin{aligned}
\Lambda(f) &= \Lambda(\Re(f) + i\Im(f)) \\
&= \Lambda(\Re(f)) + i\Lambda(\Im(f)) \\
&= \Psi_1(\Re(f)) + i\Psi_2(\Re(f)) + i(\Psi_1(\Im(f)) + i\Psi_2(\Im(f)))
\end{aligned}$$

is uniquely determined by Ψ_1, Ψ_2.
Yet $\Psi_1 = \Psi_1^+ - \Psi_1^-, \Psi_2 = \Psi_2^+ - \Psi_2^-$, thus by Riesz-Markov-Kakutani, we have $\mu_i^\pm$ being finite Radon measures, such that $\Psi_i^\pm = \int_X f d\mu_i^\pm$.
Let $\mu := (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$, we have the result.
Now the uniqueness:
If $\Lambda = \Lambda_{\mu_1} = \Lambda_{\mu_2}$, we have $\forall f \in C_0(X)$,

$$\begin{aligned}
0 &= \int_X f d(\mu_1 - \mu_2) \\
&= \int_X f h d|\mu_1 - \mu_2|.
\end{aligned}$$

By density of $C_0(X)$, it is also true for all $f \in L^1(X)$, so $|\mu_1 - \mu_2| = 0$. $\square$

Corollary 9.6. *$(M(X), \|\cdot\|_{M(X)}) \cong C_0(X)^*$ isometrically.*

10 Product Measures

Definition 10.1. Let $(X_i)_{i\in I}$ be a collection of non-empty sets, we define the **product** of the sets to be

$$X := \prod_{i \in I} X_i := \{(x_i)_{i\in I} | \forall i \in I, x_i \in X_i\} = \left\{ f : I \to \bigsqcup_{i\in I} X_i | \forall i \in I, f(i) \in X_i \right\}$$

Definition 10.2. We have a **canonical coordinate projections** $\pi : X \to X_i$ by $(x_j)_{j\in I} \mapsto x_i$.

Definition 10.3. If $(X_i, \mathcal{M}_i)$ are measurable spaces, then the **product measurable** space is

$$\left(\prod_{i\in I} X_i, \bigotimes_{i\in I} \mathcal{M}_i\right),$$

where $\bigotimes_{i\in I} \mathcal{M}_i$ is the σ-algebra generated by the sets $\left\{\pi_i^{-1}(A) | i \in I, A \in \mathcal{M}_i\right\}$.

Remark. When I is finite, this is the same as tensor products generated by $A_1 \times A_2 \times \cdots \times A_n$.

Proposition 10.1. *Let $(X_i, d_i)_{i=1}^n$ be separable metric spaces, then*

$$\bigotimes_{i=1}^n Bor(X_i) = Bor\left(\prod_{i=1}^n X_i\right).$$

Proof. Given any open $U_i \subseteq X_i$, we must have $\pi_i^{-1}(U_i) \subseteq X$ is open. Thus $\bigotimes_{i=1}^n Bor(X_i) \subseteq Bor(\prod_{i=1}^n X_i)$.
On the other hand, each X_i is separable, so X is also separable. Thus X is second countable. If $(x_n)_{n=1}^\infty$ is a dense sequence in X, then

$$\left\{B_r(x_n) | n \in \mathbb{N}, r \in \mathbb{Q}^{++}\right\}$$

is a basis for the topology. Namely, every open set can be written as a countable union of these open balls.
Setting $x_n^i := \pi_i(x_n)$, we have that $B_r(x_n) = \prod_{i=1}^n B_r(x_N^i)$, which is a subset of $\bigotimes_{i=1}^n Bor(X_i)$.
Since every open set $U \subseteq X$ is a countable union of these sets, so $U \subseteq \bigotimes_{i=1}^n Bor(X_i)$.
Thus $\bigotimes_{i=1}^n Bor(X_i) \supseteq Bor(\prod_{i=1}^n X_i)$. □

Corollary 10.2.

$$Bor(\mathbb{R}^n) = \bigotimes_{i=1}^n Bor(\mathbb{R})$$

Proposition 10.3. *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be measure spaces. Let R be the collection of all finite unions of disjoint rectangles $A \times B$ with $A \in \mathcal{M}, B \in \mathcal{N}$. Then R is an algebra of subsets of $X \times Y$*

Proof.

$$(A \times B)^c = (A^c \times Y) \sqcup (A \times B^c)$$
$$(A_1 \times B_1) \cup (A_2 \times B_2) = (A_1 \times B_1) \sqcup ((A_2 \setminus A_1) \times B_2) \sqcup ((A_2 \setminus A_1) \times (B_1 \setminus B_2))$$

□

Proposition 10.4. *The σ-algebra generated by R is $\mathcal{M} \otimes \mathcal{N}$.*

Definition 10.4. We can define a function $\pi : R \to [0, \infty]$ by $\pi(\bigsqcup_{i=1}^n A_i \times B_i) := \sum_{i=1}^n \mu(A_i)\nu(B_i)$

Lemma 10.5. *π is a premeasure.*

Proof. Firstly, $\pi(\emptyset) = \mu(\emptyset) \times \nu(\emptyset) = 0$.
Secondly, we consider any $A \times B = \bigsqcup_{n\in\mathbb{N}}(A_n \times B_n) \subseteq R$, where $(A_n \times B_n)_{n\in\mathbb{N}} \subseteq R$.
Fix any $y \in Y$, we have that $\chi_A(x)\chi_B(y) = \sum_{n\in\mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y)$, which is a sum of non-negative measurable

functions on X. By LMCT, we have that

$$\begin{aligned}
\mu(A)\chi_B(y) &= \int_X \chi_A(x)d\mu\chi_B(y) \\
&= \int_X \chi_A(x)\chi_B(y)d\mu \\
&= \int_X \sum_{n\in\mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y)d\mu \\
&= \sum_{n\in\mathbb{N}} \int_X \chi_{A_n}(x)\chi_{B_n}(y)d\mu \\
&= \sum_{n\in\mathbb{N}} \int_X \chi_{A_n}(x)d\mu\chi_{B_n}(y) \\
&= \sum_{n\in\mathbb{N}} \mu(A_n)\chi_{B_n}(y).
\end{aligned}$$

In addition, $\sum_{n\in\mathbb{N}} \mu(A_n)\chi_{B_n}(y)$ is a sum of non-negative measurable functions on Y. By LMCT, we again have that

$$\begin{aligned}
\mu(A)\nu(B) &= \mu(A)\int_Y \chi_B(y)d\nu \\
&= \int_Y \mu(A)\chi_B(y)d\nu \\
&= \int_Y \sum_{n\in\mathbb{N}} \mu(A_n)\chi_{B_n}(y)d\nu \\
&= \sum_{n\in\mathbb{N}} \int_Y \mu(A_n)\chi_{B_n}(y)d\nu \\
&= \sum_{n\in\mathbb{N}} \mu(A_n)\int_Y \chi_{B_n}(y)d\nu \\
&= \sum_{n\in\mathbb{N}} \mu(A_n)\nu(B_n).
\end{aligned}$$

This will now extend to any $\bigsqcup_{n\in\mathbb{N}}(A_n \times B_n) \subseteq R$, by finite additivity. $\square$

Theorem 10.6. *There is a complete measure space $(X \times X, \overline{\mathcal{M}\otimes\mathcal{N}}, \mu\times\nu)$, such that $\mu\times\nu(A\times B) = \mu(A)\times\nu(B)$.*

Proof. Apply Caratheodory on the above lemma. $\square$

For the flowing, let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces.

Definition 10.5. Take R a s before, let $R_\sigma := \left\{\bigcup_{n\geq 1} A_n | A_n \in R\right\}, R_{\sigma\delta} := \left\{\bigcap_{n\geq 1} E_n | E_n \in R_\sigma\right\}$

Lemma 10.7. *If $E \in \overline{\mathcal{M}\otimes\mathcal{N}}$, with $\mu\times\nu(E) < \infty$, then $\exists G \in R_{\sigma\delta}$, such that $E \subseteq G, \mu\times\nu(G\setminus E) = 0$*

Proof. We have $\mu\times\nu(E) = \inf\left\{\sum_{i\geq 1}\mu\times\nu(A_i) | A_i \in R, E \subseteq \bigcup_{i\geq 1} A_i\right\}$.
Let $E_j := \bigcup_{i\geq 1} A_{ji} \supseteq E$, with $\mu\times\nu(E_j) < \mu\times\nu(E) + \frac{1}{j}$.
Notice that $E_j \in R_\sigma$ by construction.
Now take $G = \bigcup_{j\geq 1} E_j \in R_{\sigma\delta}$. Then we have that $E \subseteq G$, and $\forall j, \mu\times\nu(G) \leq \mu\times\nu(E_j) < \mu\times\nu(E) + \frac{1}{j}$.
Thus $\mu\times\nu(G) = \mu\times\nu(E)$. $\square$

Lemma 10.8. *Let $E \in R_{\sigma\delta}$, with $\mu\times\nu(E) < \infty$. Let $E_x = \{y \in Y | (x,y) \in E\}, E^y = \{x \in X | (x,y) \in E\}$. Define $g(x) := \nu(E_x), h(y) := \mu(E^y)$. Then we have g is non-negative and μ-measurable, $g \in \mathcal{L}^1(\mu), \int_X gd\mu = \mu\times\nu(E)$. Similarly, h is non-negative and ν-measurable, $h \in \mathcal{L}^1(\nu), \int_Y gd\nu = \mu\times\nu(E)$*

Proof. If $E = A \times B$, with $A \in \mathcal{M}, B \in \mathcal{N}$, then $E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$.
Then $g(x) = \nu(B)\chi_A$ is μ-measurable, and $g \geq 0$. Moreover,

$$\int_X g d\mu = \int_X \nu(B)\chi_A d\mu = \nu(B) \int_X \chi_A d\mu = \mu(A)\nu(B) = \mu \times \nu(A \times B).$$

Now suppose $E = \bigcup_{i\geq 1} A_i \times B_i \in R_\delta$, with $A_i \in \mathcal{M}, B_i \in \mathcal{N}$. WLOG, we can take $E = \bigsqcup_{i\geq 1} A_i \times B_i$.
Let $g_i(x) = \nu(B_i)\chi_{A_i}(x)$, we have $\sum_{i=1}^n g_i(x) = \sum_{i=1}^n \nu(B_i)\chi_{A_i}(x) = \begin{cases} \nu(B_i) = \nu(E_x) & \text{if } x \in A_i \\ 0 & \text{if } x \notin \bigsqcup_{i=1}^n A_n \end{cases}$.
Thus $g(x) = \sum_{i=1}^\infty g_i(x)$ is measurable. By LMCT, $g \in \mathcal{L}^1(\mu)$, and

$$\begin{aligned} \int_X g d\mu &= \sum_{i=1}^\infty \int_X g_i d\mu \\ &= \sum_{i=1}^\infty \int_X \nu(B_i)\chi_{A_i} d\mu \\ &= \sum_{i=1}^\infty \nu(B_i)\mu(A_i) \\ &= \sum_{i=1}^\infty \mu \times \nu(A_i \times B_i) \\ &= \mu \times \nu\left(\bigsqcup_{i=1}^\infty A_i \times B_i\right) \\ &= \mu \times \nu(E). \end{aligned}$$

Now take $E = \bigcap_{i\geq 1} E_i \in R_{\delta\sigma}$ with $E_i \in R_\delta$. WLOG, we can take $E_i \supseteq E_{i+1}$.
Notice that $(E_i)_x = \{y \in Y | (x,y) \in E_i\} \supseteq \{y \in Y | (x,y) \in E_{i+1}\} = (E_{i+1})_x \supseteq \cdots \supseteq E_x$.
Let $g_i(x) = \nu((E_i)_x) = \mu \times \nu(E_i)$, then we have $0 \leq g \leq \cdots \leq g_i \leq \cdots \leq g_1$.
In addition, $E_x = \bigcap_{i\geq 1}(E_i)_x$, and thus $g(x) = \lim_{i\to\infty} g_i(x)$ by continuity of ν.
Thus g is μ-measurable, and since g_i are all dominated by g_1, we can used LDCT to get

$$\begin{aligned} \int_X g d\mu &= \lim_{i\to\infty} \int_X g_i d\mu \\ &= \lim_{i\to\infty} \mu \times \nu(E_i) \\ &= \mu \times \nu(E). \end{aligned}$$

□

Lemma 10.9. *Let $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$ with $\mu \times \nu(E) = 0$, then for μ-a.e. $x \in X$, we have $\nu(E_x) = 0$; for ν-a.e. $y \in Y$, we have $\mu(E_y) = 0$.*

Proof. We have some $G \in R_{\sigma\delta}$, such that $E \subseteq G, \mu \times \nu(G \setminus E) = 0$.
Let $f(x) := \nu(G_x)$, we have $f \in \mathcal{L}^1(\mathcal{M})$ is nonnegative. Yet $\int_X f d\mu = 0$, and thus $f(x) = 0$ for μ-a.e. $x \in X$.
Since $E_X \subseteq G_X$, and that ν is complete, we have that $g(x) = \nu(E_x) = 0$ for μ-a.e. $x \in X$. □

Corollary 10.10. *Let $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$ with $\mu \times \nu(E) < \infty$, then E_x is ν-measurable, for μ-a.e. $x \in X$, and $g(x) = \nu(E_x)$ is μ-measurable, with $g \geq 0, g \in \mathcal{L}^1(\mathcal{M})$, and $\int_X g d\mu = \mu \times \nu(E)$.*

Theorem 10.11 (Fubini's)**.** *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces. Take $f \in \mathcal{L}^1(\mu \times \nu)$, then*

1. *For μ-a.e. $x \in X, f_x := f(x, \cdot) \in \mathcal{L}^1(\nu)$.*
2. *For ν-a.e. $y \in Y, f_y := f(\cdot, y) \in \mathcal{L}^1(\mu)$.*

3. $F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^1(\mu)$.
4. $G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^1(\nu)$.
5. $\int_{X\times Y} f d(\mu\times\nu) = \int_X \left(\int_Y f(x,y)d\nu\right)d\mu = \int_Y \left(\int_X f(x,y)d\mu\right)d\nu$

Proof. Notice that $f^1 \in \mathcal{L}^1$ means that $f = f_1 - f_2 + if_3 - if_4$, where $f_i \geq 0, f_i \in \mathcal{L}^1$.
We first show the theorem holds for $f \geq 0, f \in \mathcal{L}^1$. There are simple functions $0 \leq s_1 \leq \cdots \leq s_n \leq \cdots \leq f$, such that $f(x) = \lim_{n\to\infty} s_n(x)$.
Let $F_n(x) = \int_Y s_n(x,y) d\nu \geq 0$ be measurable and $\mathcal{L}^1$. We have that □

Theorem 10.12 (Tonelli's)**.** *Let $(X,\mathcal{M},\mu),(Y,\mathcal{N},\nu)$ be complete measure spaces. Take $f \in \mathcal{L}^+(\mu\times\nu)$, and $\mu\times\nu$ is σ-finite, then*

1. *For μ-a.e. $x \in X, f_x := f(x,\cdot) \in \mathcal{L}^+(\nu)$.*
2. *For ν-a.e. $y \in Y, f_y := f(\cdot,y) \in \mathcal{L}^+(\mu)$.*
3. *$F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^+(\mu)$.*
4. *$G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^+(\nu)$.*
5. *$\int_{X\times Y} f d(\mu\times\nu) = \int_X \left(\int_Y f(x,y)d\nu\right)d\mu = \int_Y \left(\int_X f(x,y)d\mu\right)d\nu$*

Proof. $\mu\times\nu$ is σ-finite, thus $\exists C_1 \subseteq C_2 \subseteq \cdots$, with $C_n \in \overline{\mathcal{M}\otimes\mathcal{N}}, X\times Y = \bigcup_{n=1}^\infty C_n$, and $\mu\times\nu(C_n) < \infty$. Let $f_n(x) = \max\{f(x), n\}\chi_{C_n}(x)$, we have $0 \leq f_n \leq n\chi_{C_n}$, and $f_n \in \mathcal{L}^+\cap\mathcal{L}^1(\mu\times\nu), \lim_{n\to\infty} f_n(x) = f(x)$.

$$\begin{aligned}\int f d\mu\times\nu &= \lim_{n\to\infty}\int f_n d\mu\times\nu \\ &= \lim_{n\to\infty}\int_X\int_Y f_n(x,y)d\nu d\mu \\ &=: \lim_{n\to\infty}\int_X F_n(x)d\mu.\end{aligned}$$

Then F_n are measurable, non-negative, and monotone increasing to $F(x) := \int_Y f(x,y)d\nu$. By LMCT, we have F is measurable, and

$$\begin{aligned}\int f d\mu\times\nu &= \lim_{n\to\infty}\int_X F_n(x)d\mu \\ &= \int_X F(x)d\mu \\ &= \int_X\int_Y f(x,y)d\nu d\mu\end{aligned}$$

□

Remark. If $f \in \mathcal{L}^1$, we get σ-finite by free on $C = \mathrm{Supp}(f)$ if we look at $C_n := \left\{(x,y) : |f(x,y)| \geq \frac{1}{n}\right\}$. Notice that $\mu\times\nu(C_n) \leq n\int|f|d\mu\times\nu < \infty$.

Example 10.0.1. Consider $X = Y = \mathbb{N}, \mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, with the counting measure m_c.
Consider $f(m,n) := \begin{cases} 1 & n = m \\ -1 & n = m+1 \\ 0 & o.w. \end{cases} \sim \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ \vdots & 0 & 1 & -1 & \cdots \\ \vdots & \vdots & \ddots & & \end{pmatrix}$.

However,

$$\begin{aligned}
\int_X \int_Y f(x,y) dm_c(n) dm_c(m) &= \sum_{m\geq 1} \sum_{n \geq 1} f(m,n) \\
&= \sum_{m\geq 1} 0 \\
&= 0, \\
\int_Y \int_X f(x,y) dm_c(m) dm_c(n) &= \sum_{n\geq 1} \sum_{m \geq 1} f(m,n) \\
&= 1 + \sum_{n\geq 2} 0 \\
&= 1.
\end{aligned}$$

This is because $f \notin \mathcal{L}^1$.

Example 10.0.2. Consider $X = Y = [0,1]$, and the Lebesgue measure.
Take $t_n = 1 - \frac{1}{n}$.
Define $g_n : [0,1] \to \mathbb{R}$ by starting at $\frac{2t_n}{3} + \frac{t_{n+1}}{3}$, linear and reach $\frac{t_{n+1}-t_n}{3}$ at mid point, and decrease linearly to 0 at $\frac{t_n}{3} + \frac{2t_{n+1}}{3}$, and 0 outside. We thus have $\int g_n(x)dx = 1$.
Define $f(x,y) = \sum_{i=1}^{\infty}(g_n(x) - g_{n+1}(x))g_n(y)$, where only one of these summands will be non-zero in each interval of x. Actually $f(x,y)$ is continuous $\forall (x,y) \neq (1,1)$.
However, $\int f(x,y)dx = g_n(y)$, and thus $\int\int f(x,y)dxdy = 1$, while $\int \int f(x,y)dydx = 0$.

Theorem 10.13. *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be (not necessarily complete) measure spaces. Then Fubini and Tonelli still apply with restriction to $\mathcal{M} \otimes \mathcal{N}$.*

10.1 Lebesgue Measure on $\mathbb{R}^n$

Lemma 10.14. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be Lebesgue measurable, then there is a G_δ set $G \subseteq \mathbb{R}^n$, such that $\lambda^n(G) = 0$, and $g = f\chi_{G^c}$ be Borel measurable, and $f = g$ λ^n-a.e..*

Proof. By writing $f = f_1 - f_2 + if_3 - if_4$ for $f_i \geq 0$, we can assume $f \geq 0$. We first consider $n = 1$.
Choose a dense subset $\{r_i\}_{i\in\mathbb{N}}$ of $[0,\infty)$. Let $A_i = f^{-1}([0, r_n])$. Since f is Lebesgue measurable, $A_n \in \mathcal{L}$. By regularity for the Lebesgue measure, there is an F_δ set $F_i \subseteq A_n$ and a null set $N_i = A_i \backslash F_i$. Let $N = \bigcup_{i\in\mathbb{N}} N_i$, then N is a null set.
Applying regularity again, there is a G_δ set $G \supseteq N$ such that $\lambda(G) = 0$.
Let $g = f\chi_{G^c}$. we have $g^{-1}([0,r_i)) = f^{-1}([0,r_i)) \cup G = A_i \cup G = (F_i \cup N_n) \cap G = F_i \cup G$, which is a union of two Borel sets, and thus Borel.
To verify that g is Borel, it surfaces to prove $g^{-1}([0,r))$ is Borel for all $r > 0$. By density of $\{r_i\}$, there is a sequence r_{n_k} such that $r_{n_k} \leq r$ and $r_{n_k} \to r$. Thus $\bigcup_{k\geq 1}[0, r_{n_k}) = [0,r)$, so $g^{-1}([0,r)) = \bigcup_{k\geq 1}([0, r_{n_k}))$ is a union of Borel sets, and thus Borel.
By construction, G is a null set and $f|_{G^c} = g|_{G^c}$, so $f = g$ λ-a.e..
Now suppose $n \geq 1$. For each i, let f_{x_i} be the function obtained by fixing all but the i^{th} variable x_i. From above we can find $G\delta$ set $G_i \subseteq \mathbb{R}$, such that $f_{x_i} = f_{x_i}\chi_{G_i^c}$ λ-a.e..
Let $G = (G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup (\mathbb{R} \times G_2 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup \cdots (\mathbb{R} \times \cdots \times \mathbb{R} \times G_n)$.
Then $G^c = G_1^c \times G_2^c \times \cdots \times G_n^c$. Let $G_1 = \bigcap_k U_{1k}$, then $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R} = \bigcap_k (U_{1k} \times \mathbb{R} \times \cdots \times \mathbb{R})$, where each is open, since $U_{1k}, \mathbb{R}$ are open. Thus $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}$ is a G_δ set. Thus G is a finite union of G_δ sets, which is G_δ. $\square$

Definition 10.6. For $A \in \mathcal{L}^n, X \in \mathbb{R}^n$, write the **translation** of A by x as $A + x = \{a + x : a \in A\}$.

Definition 10.7. Let GLn be the set of invertible $n \times n$ matrices.

Theorem 10.15. *Consider the Lebesgue measure λ^n in $\mathbb{R}^n$.*

1. *(translation) For $A \in \mathcal{L}^n$ and $x \in \mathbb{R}^n$, we have $A + x \in \mathcal{L}^n, \lambda^n(A + x) = \lambda^n(A)$.*

2. *(scaling) For $T \in GLn, f : \mathbb{R}^n \to \mathbb{C}$ be Lebesgue measurable, $f \circ T$ is Lebesgue measurable, and*

$$\int f d\lambda^n = |\det(T)| \int (f \circ T) d\lambda^n.$$

   In particular, for $A \in \mathcal{L}^n$, we have $\lambda^n(T(A)) = |\det(T)|\lambda^n(A)$.

3. *(rotation) For a unitary $U \in GLn$, we have*

$$\int (f \circ U) d\lambda^n = \int f d\lambda^n,$$

   and $\forall A \in \mathcal{L}^n, \lambda^n(U(A)) = \lambda(A)$.

Proof. 1.

2. Notice that $x \in T(A) \iff T^{-1}x \in A$, thus $\chi_{T(A)} = \chi_A \circ T^{-1}$. Thus

$$\begin{aligned}
\lambda^n(T(A)) &= \int \chi_{T(A)} d\lambda^n \\
&= \int \chi_A \circ T^{-1} d\lambda^n \\
&= \frac{1}{|\det(T^{-1})|} \int \chi_A d\lambda^n \\
&= |\det(T)|\lambda^n(A).
\end{aligned}$$

□

11 Convolutions and Fourier Transforms

Definition 11.1. For $y \in \mathbb{R}, f : \mathbb{R} \to \mathbb{C}$, we define the **translation** of f by y to be $L_y f(x) := f(x - y)$.

Proposition 11.1. *We have $L_y : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is linear, isometric, and $\forall f \in L^1(\mathbb{R})$, we have*

$$\lim_{y \to 0} \|L_y f - f\|_1 = 0.$$

Proof. If $f \in C_c(\mathbb{R})$, then it is uniformly continuous, so

$$\lim_{y \to 0} \|L_y f - f\|_\infty = 0.$$

Take compact $K \supseteq \mathrm{Supp}(f)$, we have that

$$\begin{aligned}
\|L_y f - f\|_1 &= \int_{K \cup (K+y)} |f(x - y) - f(x)| dx \\
&\leq \lambda(K \cup (K + y)) \|Lxf - f\|_\infty,
\end{aligned}$$

where the first term is bounded by $2\lambda(K) < \infty$, and the second term goes to 0.
Now since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, we have the result by triangle inequality. □

Theorem 11.2 (Young's Convolution Inequality)**.** *Consider $X = \mathbb{R}$, with Lebesgue measure λ. Let $f, g \in L^1(\mathbb{R})$, then for a.e. $x \in \mathbb{R}$, the function $y \mapsto f(x - y)g(y)$ is in $L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$, and the* ***convolution***

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y)g(y) dy$$

is also in $L^1(\mathbb{R})$. In addition, $\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$.

Proof. Consider the function $F : (x, y) \mapsto f(x-y)g(y)$, which is a measurable function on $\mathbb{R} \times \mathbb{R}$ (can show with approximation by $C_c(\mathbb{R})$ functions).
By Tonelli's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| d\lambda^2 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,y)| dx \right) dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y)||g(y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |f(x-y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \|f\|_{L^1(\mathbb{R})} dy \\
&= \|f\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} |g(y)| dy \\
&= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \\
&< \infty.
\end{aligned}$$

Thus, $F \in L^1(\mathbb{R}^2)$.
Now we apply Fubini's Theorem to F, and get $F_x(y) = f(x-y)g(y) \in L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$.
In addition,

$$\begin{aligned}
\|f * g\|_{L^1(U)} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y) dy \right| dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)g(y)| dy dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |F(x,y)| dy dx \\
&= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.
\end{aligned}$$

□

Corollary 11.3. $(L^1(\mathbb{R}), *)$ *defines a communicative associative algebra.*

Definition 11.2. Given $f \in L^1(\mathbb{R})$, its **Fourier Transform** is $\mathcal{F}(f) := \hat{f} : \mathbb{R} \to \mathbb{C}$, where

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-ix\omega} dx.$$

Lemma 11.4 (Riemann-Lebesgue)**.** *$\forall f \in L^1(\mathbb{R})$, we have $\hat{f} \in C_0(\mathbb{R})$, and $\left\|\hat{f}\right\|_\infty \leq \|f\|_{L^1(\mathbb{R})}$. Namely, $\mathcal{F}$ is a contraction map.*

Proof. Consider any sequence $(\omega_n)_{n=1}^\infty \subset \mathbb{R}$ that converges to $\omega \in \mathbb{R}$.
Let $h_n(x) := f(x)(e^{i\omega_n x} - e^{i\omega x})$, we have that $h_n \in L^1(\mathbb{R})$, $h_n(x) \to 0$ pointwise for a.e. $x \in \mathbb{R}$, and $|h_n| \leq |f| \left| e^{i\omega_n x} - e^{i\omega x} \right| \leq 2|f|$.
In addition,

$$\begin{aligned}
\hat{f}(\omega_n) - \hat{f}(\omega) &= \int_{\mathbb{R}} f(x)(e^{i\omega_n x} - e^{i\omega x}) dx \\
&= \int_{\mathbb{R}} h_n(x) dx
\end{aligned}$$

By LDCT, we have that $\lim_{n\to\infty}\left(\hat{f}(\omega_n)-\hat{f}(\omega)\right)=0$, so $\hat{f}$ is continuous.
In addition,

$$\begin{aligned}\left|\hat{f}(\omega)\right| &\leq \int_{\mathbb{R}} |f(x)||e^{ix\omega}|dx \\ &= \int_{\mathbb{R}} |f(x)|dx \\ &= \|f\|_{L^1(\mathbb{R})}.\end{aligned}$$

Now

$$\begin{aligned}\hat{f}(\omega) &= \int_{\mathbb{R}} f(x)e^{-ix\omega}dx \\ &= -\int_{\mathbb{R}} f(x)e^{-ix\omega+\pi i}dx \\ &= -\int_{\mathbb{R}} f(x)e^{-i\omega(x-\pi/\omega)}dx \\ &= -\int_{\mathbb{R}} f(z+\pi/\omega)e^{-i\omega z}dz \\ &= -\int_{\mathbb{R}} L_{-\pi/\omega}f(z)e^{-i\omega z}dz \\ 2\hat{f}(\omega) &= \int_{\mathbb{R}} f(x)e^{-ix\omega}dx - \int_{\mathbb{R}} L_{-\pi/\omega}f(z)e^{-i\omega z}dz \\ &= \int_{\mathbb{R}} (f-L_{-\pi/\omega}f)(x)e^{-i\omega x}dx \\ &= \mathcal{F}(f-L_{-\pi/\omega}f)(\omega) \\ 2\left|\hat{f}(\omega)\right| &\leq \left\|f-L_{-\pi/\omega}f\right\|_{L^1(\mathbb{R})},\end{aligned}$$

which goes to 0 when $\omega\to\infty$.
Thus, $\hat{f}\in C_0(\mathbb{R})$. ☐

Theorem 11.5 ($L^1(\mathbb{R})$ Inversion)**.** *If $f,\hat{f}\in L^1(\mathbb{R})$, we have*

$$f(x)=\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)e^{ix\omega}d\omega$$

for a.e. $x\in\mathbb{R}$.
In particular, such f must be almost everywhere equal to a continuous function.

Proof. Let $\lambda>0$, and $H_\lambda(\omega):=e^{-\lambda|\omega|}$.
Let

$$\begin{aligned}h_\lambda(x) &:= \frac{1}{2\pi}\int_{\mathbb{R}} H_\lambda(\omega)e^{ix\omega}d\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\omega-\lambda|\omega|}d\omega \\ &= \frac{\lambda}{\pi}\frac{1}{x^2+\lambda^2}.\end{aligned}$$

Fix $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
(f * h_\lambda)(x) &= \int_{\mathbb{R}} f(x-y)h_\lambda(y)dy \\
&= \int_{\mathbb{R}} f(x-y)\frac{1}{2\pi}\int_{\mathbb{R}} H_\lambda(\omega)e^{iy\omega}d\omega dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)H_\lambda(\omega)e^{iy\omega}d\omega dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)H_\lambda(\omega)e^{iy\omega}dy d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(z)H_\lambda(\omega)e^{i\omega(x-z)}dz d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} H_\lambda(\omega)\hat{f}(\omega)e^{i\omega(x)}dz d\omega.
\end{aligned}$$

One can show that $H_\lambda(\omega) = 1$ as $\lambda \to 0$, and $f * h_\lambda \to f$.
If $\hat{f} \in L^1(\mathbb{R})$, we can use DCT to get the result. □

Corollary 11.6. *If $f, g \in L^1(\mathbb{R})$, and $\mathcal{F}(f) = \mathcal{F}(g)$, we must have $\mathcal{F}(f-g) = 0 \in L^1(\mathbb{R})$. Thus, $f = g$ a.e.$x \in \mathbb{R}$.*

Remark. Not all $\hat{f} \in L^1(\mathbb{R})$.

Example 11.0.1. If $f = \chi_{[-1,1]}$, we have $\hat{f} = \frac{2\sin(\omega)}{\omega} \in C_0(\mathbb{R}) \setminus L^1(\mathbb{R})$.

Remark. We can equivalently define

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx,$$

then the inverse is given by

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i x\xi}d\xi.$$

Proposition 11.7. *For any $f, g \in L^1(\mathbb{R})$, we always have $\widehat{f * g} = \hat{f}\hat{g}$.*

Example 11.0.2. Consider the heat equation initial problem: $\begin{cases} \frac{\partial}{\partial t}u = \Delta u \\ u(\cdot, 0) = f. \end{cases}$ Taking the Fourier Transform with respect to x, we have $\begin{cases} \frac{\partial}{\partial t}\hat{u} = (2\pi i\xi)^2\hat{u} \\ \hat{u}(\cdot, 0) = \hat{f}, \end{cases}$ with the solution $\hat{u}(\xi, t) = e^{-4\pi^2\xi^2 t}\hat{f(\xi)}$.
Assuming we can apply the inverse formula, we have

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} \hat{u}(\xi, t)e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} e^{-4\pi^2\xi^2 t}\hat{f(\xi)}e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} e^{-4\pi^2\xi^2 t}\int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}dy e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} e^{-4\pi^2\xi^2 t}e^{-2\pi i y\xi}dy e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} f(y)\frac{1}{(4\pi t)^{\frac{1}{2}}}e^{-\frac{(x-y)^2}{4t}}d\xi \\
&= \int_{\mathbb{R}} f(y)H_t(x-y)d\xi \\
&= (H_t * f)(x),
\end{aligned}$$

where the **heat kernel** is $H_t(x) = \frac{1}{(4\pi t)^2} e^{-\frac{x^2}{4t}}$.
The heat kernel satisfies $\frac{\partial}{\partial t} H_t = \Delta H_t$, $\int_{\mathbb{R}} H_t(x)dx = 1$, and $\int_{|x|\geq \epsilon} H_t(x)dx \to 0$.

Definition 11.3. The **Schwartz class** S is the set of functions $f : \mathbb{R} \to \mathbb{C}$, such that

$$\exists C \geq 0, \text{ such that } \forall \alpha, \beta, \left| x^\alpha \frac{d}{dx^\beta} f \right| \leq C.$$

Proposition 11.8. *$C_c^\infty(\mathbb{R}) \subset S$.*

Proposition 11.9. *Suppose $f \in S$, then $\hat{f} \in S$.*

Theorem 11.10 (Plancherel)**.** *Suppose $f \in S$, then $\|f\|_{L^2(\mathbb{R})} = \left\|\hat{f}\right\|_{L^2(\mathbb{R})}$.*

Proposition 11.11. *S is dense in $L^2(\mathbb{R})$. Namely, $\bar{S} = L^2(\mathbb{R})$.*

Definition 11.4. For $f \in L^2(\mathbb{R})$, with $(f_i)_{i=1}^\infty$ in S such that $f_i \to f$, we define the **Fourier Transform** of f to be

$$\hat{f} := \lim_{i\to\infty} \hat{f}_i.$$

Lemma 11.12. *The above definition is well-defined.*

Proof. Given any $\epsilon > 0$.
Since $f_i \to f$ in $L^2(\mathbb{R})$, there is $N \geq 1$, such that $\forall i \geq N$, $\|f_i - f\|_{L^2(\mathbb{R})} < \epsilon/2$.
Thus, for any $i, j \geq N$,

$$\begin{aligned}\left\|\hat{f}_i - \hat{f}_j\right\|_{L^2(\mathbb{R})} &= \|f_i - f_j\|_{L^2(\mathbb{R})} \\ &\leq \|f_i - f\|_{L^2(\mathbb{R})} + \|f - f_j\|_{L^2(\mathbb{R})} \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon.\end{aligned}$$

Thus $(\hat{f}_i)_{i=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R})$, so $\lim_{i\to\infty} \hat{f}_i$ exists in $L^2(\mathbb{R})$. We can also see that this is independent of the choice of choice of the sequence. ☐

12 Bochner Spaces

Definition 12.1. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \to B$ is **weakly measurable** if $\forall \Lambda \in B^*,\ \Lambda \circ f : X \to \mathbb{C}$ is measurable.

Definition 12.2. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \to B$ is **Bochner measurable** or **strongly measurable** if $f(x) = g(x)$ for μ-a.e. $x \in X$, for some measurable g, with $\mathrm{Im}(g) \subseteq B$ being separable.

Proposition 12.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \to B$ is strongly measurable if $f(x) = \lim_{n\to\infty} f_n(x)$ for μ-a.e. $x \in X$, for some sequence of measurable functions f_n, each with countable range.*

Definition 12.3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $s : X \to [0, \infty)$ be a simple measurable function, with $s(X) = \{a_1, \ldots, a_n\} \subset B$, such that

$$s = \sum_{i=1}^n a_i \mathcal{X}_{A_i},$$

where $A_i := s^{-1}(\{a_i\})$. For $A \in \mathcal{M}$, we say s is integrable over A if $\forall a_i \neq 0, \mu(A_i \cap A) < \infty$, and define the **integral** of s over A to be

$$\int_A s d\mu := \sum_{i=1}^n a_i \mu(A_i \cap A).$$

Definition 12.4. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $f : X \to [0, \infty)$ be a measurable function. If there is a sequence of simple integrable functions $(s_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \int_A \|f - s_n\|_B d\mu = 0,$$

then we say f is **Bochner integrable**, and we define the **Bochner integral** to be

$$\int_A f d\mu := \lim_{n\to\infty} \int_A s_n d\mu.$$

Lemma 12.2. *The right hand side of the above definition always exists, and is independent of the choice of the sequence of simple integrable functions $(s_n)_{n=1}^{\infty}$. Thus, the above definition is well-defined.*

Theorem 12.3 (Bochner)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. A strongly measurable function $f : X \to B$ is Bochner integrable if and only if $x \mapsto \|f(x)\|_B$ is integrable. In this case, $\forall E \in \mathcal{M}$,*

$$\left\| \int_E f(x)dx \right\|_B \le \int_E \|f(x)\|_B dx,$$
$$\forall \Lambda \in B^*, \ \Lambda\left(\int_E f(x)dx \right) = \int_E \Lambda(f(x))dx.$$

Theorem 12.4 (Dominated Convergence Theorem for Bochner integral)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. Let $f_n : X \to \mathbb{C}$ be measurable functions, defined μ-a.e. on X, such that $f(x) := \lim_{n\to\infty} f_n(x)$ is defined $\mu-$almost everywhere for $x \in X$. If there is $0 \le g(x) \in \mathcal{L}^1(X, \mu)$, such that for μ-a.e. $x \in X, \forall n \in \mathbb{N}$, $\|f_n(x)\|_B \le g(x)$, then f is Bochner integrable, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n\to\infty} \int_X \|f - f_n\|_B d\mu = 0.$$

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

Definition 12.5. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $1 \le p < \infty$, we define

$$\mathcal{L}^p(\mu, B) := \left\{ f : X \to B \middle| f \text{ is measurable}, \int_X \|f\|_B^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p(\mu,B)} := \left(\int_X \|f\|_B^p d\mu \right)^{\frac{1}{p}}.$$

Definition 12.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, we define

$$\mathcal{L}^\infty(\mu, B) := \{ f : X \to B | f \text{ is measurable}, \operatorname{ess\,sup} \|f\|_B < \infty \}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty(\mu,B)} := \operatorname{ess\,sup} \|f\|_B.$$

Definition 12.7. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. For any $p \in [1, \infty]$, we define

$$L^p(\mu, B) := \mathcal{L}^p(\mu, B)/N,$$

where $N := \{ f : X \to B | f \text{ is measurable}, f = 0 \ \mu - \text{a.e.} \}$. Namely, $[f] \in L^p(\mu, B)$ is the equivalence class of all $g = f$ μ-a.e. for $f \in \mathcal{L}^p(\mu, B)$.
In addition, we define

$$\|[f]\|_{L^p(\mu,B)} := \|f\|_{\mathcal{L}^\infty(\mu,B)}$$

for any representative f.

Theorem 12.5 (Fischer-Riesz-Bochner)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. For all $1 \le p \le \infty$, we have that $\left(L^p(\mu, B), \|\cdot\|_{L^p(\mu,B)} \right)$ is a Banach Space.*