

# Amath753 Advanced PDEs

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# 1 Preliminaries

See more in AMATH731-Functional Analysis Notes from Prof. Giang Tran, and my PMATH651-Measure Theory and PMATH753-Functional Analysis Notes.

## 1.1 Introduction

**Definition 1.1.** We will use the following notations:

- $C$  means a positive constant.
- $U \subset \mathbb{R}^n$  is open.
- If  $u : U \rightarrow \mathbb{R}$  is a function, we write  $u(x) := u(x^1, \dots, x^n)$  for  $x = (x^1, \dots, x^n) \in U$ .
- A function  $u$  is **smooth** if  $u \in C^\infty(U)$ .
- For  $1 \leq i \leq n$ , we write  $\partial_i u := u_{x^i} := u_i := D_i u := \frac{\partial}{\partial x^i} u := \frac{\partial u}{\partial x^i}$ .
- Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , we let  $|\alpha| := \sum_{i=1}^n \alpha_i$ , and

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x^1^{\alpha_1} \cdots \partial x^n^{\alpha_n}} = \partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n} u.$$

- If  $k \in \mathbb{N}$ , we let  $D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}$
- When  $k = 1$ , we write  $Du := D_x u := (u_{x^1}, \dots, u_{x^n})^T = \nabla u$  to be the **gradient**.
- When  $k = 2$ , we write  $D^2 u := \begin{pmatrix} u_{x^1,x^1} & \cdots & u_{x^1,x^n} \\ \vdots & & \vdots \\ u_{x^n,x^1} & \cdots & u_{x^n,x^n} \end{pmatrix}$  to be the **Hessian** matrix.
- $\Delta u := \sum_{i=1}^n u_{x^i,x^i} = \operatorname{div} Du = \operatorname{tr}(D^2 u)$  is the **Laplacian** of  $u$ .

**Example 1.1.1.** Consider a body  $U \subset \mathbb{R}^3$  and let  $U_0 \subseteq U$  with boundary  $\partial U_0$ , which does not change over time.

The Conservation of Energy states that the rate of change of total energy in  $U_0$  is the inflow of heat through the boundaries plus heat produced by the source in  $U_0$ .

Let  $e(x, t) \in \mathbb{R}$  be the density of internal energy, then the total energy is  $\int_{U_0} e dx$ .

Let  $j(x, t) \in \mathbb{R}^3$  be the heat flux (vector pointing at the direction that heat is flowing).

Let  $n$  denote the exterior unit normal on  $\partial U_0$ .

The net outflow of the heat through  $\partial U_0$  is  $\int_{\partial U_0} j \cdot n ds$ .

Let  $p(x, t) \in \mathbb{R}$  be the power density of the source. Heat production in  $U_0$  is  $\int_{U_0} pdx$ .

Thus we have

$$\frac{d}{dx} \int_{U_0} e dx = - \int_{\partial U_0} j \cdot n ds + \int_{U_0} pdx.$$

By divergence theorem, we have  $\int_{\partial U_0} j \cdot n ds = \int_{U_0} \operatorname{div} j dx$ .

Thus we have

$$\int_{U_0} (\partial_t e + \operatorname{div} j - p) dx = 0.$$

Since  $U_0$  is arbitrary, we must have

$$\partial_t e + \operatorname{div} j - p = 0.$$

Assume that  $e$  depends linearly on temperature  $T$  as  $e = e_0 + \sigma u$ , where  $e_0$  is a constant reference internal energy, and  $u = T - T_0$ , where  $T_0$  is a constant reference temperature, and  $\sigma$  is the specific heat capacity.

A generalized form of Fourier's law states that:

- Heat flow is proportional to the temperature gradient.
- Heat is transformed by convection with heat flux  $be$ , where  $b(x, t) \in \mathbb{R}^3$  is a given convection velocity.

Namely,  $j = -aDu + be$ , where  $a(x)$  is a known heat conductivity.  
Thus we have

$$\sigma\partial_t u + \operatorname{div}(b\sigma u) - \operatorname{div}(aDu) = p - \operatorname{div}(be_0).$$

**Definition 1.2.** We consider the operator

$$Lu := - \sum_{i,j=1}^n (a^{ij}u_{x^i})_{x^j} + \sum_{i=1}^n b^i u_{x^i} + cu,$$

for given coefficients  $a^{ij}, b^i, c$ .

- The second-order elliptic boundary-value problems are  $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$
- The second-order parabolic boundary-value problems are  $\begin{cases} u_t + Lu = f & x \in U, t \in (0, T] \\ u = 0 & \text{on } \partial U, t \in (0, T] \\ u = u_0 & \text{on } \partial U, t = 0 \end{cases}$

**Example 1.1.2.** Some special cases are

- Laplace equation:  $-\Delta u = 0$
- Poisson's equation:  $-\Delta u = f$
- Heat equation:  $u_t - \Delta u = 0$

## 1.2 Metric Spaces and Complete Spaces

**Definition 1.3.** A **metric space** is a set  $X$  that has a (distance) metric:

$$\begin{aligned} d(\cdot, \cdot) : X \times X &\rightarrow \mathbb{R}, \text{ such that } \forall x, y, z \in X \\ d(x, x) &= 0 \\ \forall x \neq y, d(x, y) &> 0 \\ d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

**Definition 1.4.** Given a metric space  $(X, d)$ , a sequence  $(x_n)_{n=1}^\infty$  in  $X$  has a **limit point**  $x \in X$  if  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ . In this case, we say  $(x_n)_{n=1}^\infty$  is a **convergent sequence**, and write  $x = \lim_{n \rightarrow \infty} x_n$ .

**Definition 1.5.** A sequence  $(x_n)_{n=1}^\infty$  is a **Cauchy sequence** in a metric space  $(X, d)$  if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m, n \in \mathbb{N}, m, n > N \implies d(x_m, x_n) < \epsilon.$$

**Definition 1.6.** A metric space  $X$  is **complete** if every Cauchy sequence  $(x_i)_{i=1}^\infty$  converges to a limit point in  $X$ . i.e.  $\exists x \in X, \lim_{i \rightarrow \infty} x_i = x$ .

**Proposition 1.1.** Let  $(X, d)$  be a metric space, then every convergent sequence is Cauchy.

**Proposition 1.2.** Let  $(X, d)$  be a metric space. If  $(x_n)_{n=1}^\infty$  is a Cauchy sequence and has a convergent subsequence such that  $\lim_{k \rightarrow \infty} x_{n_k} = x \in X$ , then  $\lim_{n \rightarrow \infty} x_n = x$ .

### 1.2.1 Compactness

*Remark.* See the definition of compactness and more in Section 2.9 of AMATH731 Notes from Prof. Tran.

**Definition 1.7.** Let  $(X, d)$  be a metric space. A set  $S \subseteq X$  is **sequentially compact** if every sequence  $(x_i)_{i=1}^{\infty}$  in  $S$  has a convergent subsequence whose limit is in  $S$ . Namely,  $\exists x \in S$ , such that  $x = \lim_{j \rightarrow \infty} x_{i_j}$  for some choice of  $i_j$ 's.

*Remark.* In metric spaces, sequentially compact and compact are equivalent, so we will just use this as the definition for compactness.

**Definition 1.8.** Let  $(X, d)$  be a metric space. A set  $S \subseteq X$  is **relatively compact**, or **pre-compact** if its closure  $\bar{S}$  is compact in  $X$ .

**Proposition 1.3.** Let  $(X, d)$  be a metric space, then  $S \subseteq X$  is relatively compact iff for any sequence  $(x_n)_{n=1}^{\infty} \subseteq S$ , it has a convergent subsequence  $(x_{n_k})_{k=1}^{\infty}$ , such that  $x_{n_k} \rightarrow x$  for some  $x \in X$ .

## 1.3 Banach Spaces

**Definition 1.9.** A **normed vector space** is a vector space  $(X, \|\cdot\|)$  over a field  $\mathbb{F}$  endowed with a norm (length) function:  $\|\cdot\| : X \rightarrow [0, \infty)$ , such that  $\forall x, y \in X, a \in \mathbb{F}$ , it satisfies

1. subadditivity (triangular inequality); i.e.  $\|x + y\| \leq \|x\| + \|y\|$ ,
2. absolute homogeneity; i.e.  $\|a \cdot x\| = |a| \|x\|$ , and
3. positive definiteness; i.e. if  $x \neq 0$ , we must have  $\|x\| > 0$ .

**Proposition 1.4.** For every **normed space** with  $\|\cdot\|$ , there is a metric  $d(x, y) = \|x - y\|$ .

*Proof.*

$$\begin{aligned} d(x, x) &= \|x - x\| = \|0\| = 0 \\ \forall x \neq y, d(x, y) &= \|x - y\| > 0 \\ d(x, y) &= \|x - y\| = \|-(y - x)\| = |-1| \|y - x\| = \|y - x\| = d(y, x) \\ d(x, z) &= \|x - z\| = \|x - y + y - z\| \geq \|x - y\| + \|y - z\| = d(x, y) + d(y, z) \end{aligned}$$

Thus  $d(x, y) = \|x - y\|$  is a metric. □

**Definition 1.10.** A normed space is called a **Banach space** if it is complete.

**Definition 1.11.** Let  $(X, \|\cdot\|)$  be a Banach space, a subset  $A \subseteq X$  is **dense** in  $X$  if the closure  $\bar{A} = X$ .

**Definition 1.12.** A Banach space is **separable** if there is a dense countable subset of it.

## 1.4 Hilbert Spaces

**Definition 1.13.** An **inner product space** is a vector space  $H$  that has an inner product:  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$ , such that  $\forall u, v, w \in H, a, b \in \mathbb{C}$ , it satisfies

1. conjugate symmetry; i.e.  $\langle v, w \rangle = \overline{\langle w, v \rangle}$ ,
2. linearity in the first argument; i.e.  $\langle au + bw, v \rangle = a\langle u, v \rangle + b\langle w, v \rangle$ , and
3. positive definiteness; i.e.  $\langle v, v \rangle \geq 0$ , and if  $v \neq 0$ , we must have  $\langle v, v \rangle > 0$ .

**Lemma 1.5.** For every inner product space with  $\langle \cdot, - \rangle$ , and  $x, y \in H$ , we have

$$\langle x, y \rangle + \langle y, x \rangle = 2\Re(\langle x, y \rangle) = 2\Re(\langle y, x \rangle),$$

which is twice the real part of  $\langle x, y \rangle$ . Similarly,

$$\langle x, y \rangle - \langle y, x \rangle = 2\Im(\langle x, y \rangle) = -2\Im(\langle y, x \rangle),$$

which is twice the imaginary part of  $\langle x, y \rangle$ .

Also, we have

$$\langle x, y \rangle \langle y, x \rangle = |\langle x, y \rangle|^2.$$

*Proof.*

$$\begin{aligned} \langle x, y \rangle + \langle y, x \rangle &= \langle x, y \rangle + \overline{\langle x, y \rangle} \\ &= 2\Re(\langle x, y \rangle) \\ \langle x, y \rangle - \langle y, x \rangle &= \langle x, y \rangle - \overline{\langle x, y \rangle} \\ &= 2\Im(\langle x, y \rangle) \\ \langle x, y \rangle \langle y, x \rangle &= \langle x, y \rangle \langle y, x \rangle \\ &= |\langle x, y \rangle|^2 \end{aligned}$$

□

**Theorem 1.6** (Cauchy-Schwarz). For every inner product space  $H$ ,

$$\forall u, v \in H, |\langle u, v \rangle| \leq \|u\| \|v\|,$$

where we define  $\|x\| = \sqrt{\langle x, x \rangle}$  or any  $x \in H$ .

In particular, when  $\|u\| \neq 0$ ,  $\|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 = \|z\|^2$ , where  $z := \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u$ .

*Proof.* Notice that this is trivially true and equality holds to be zero when  $u = 0$ .

Now we assume  $u \neq 0$ , then  $\|u\| = \sqrt{\langle u, u \rangle} > 0$ .

$$\begin{aligned} \|z\|^2 &= \langle z, z \rangle \\ &= \left\langle \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u, \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u \right\rangle \\ &= \|u\|^2 \langle v, v \rangle - \langle u, v \rangle \langle u, v \rangle - \overline{\langle u, v \rangle} \langle v, u \rangle + \frac{\langle v, u \rangle \langle u, v \rangle}{\|u\|^2} \underbrace{\langle u, u \rangle}_{\|u\|^2} \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 - |\langle v, u \rangle|^2 + |\langle v, u \rangle|^2 \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2. \end{aligned}$$

Now  $\|z\|^2 = \langle z, z \rangle \geq 0$ , we have the result.

□

**Proposition 1.7.** For every inner product space with  $\langle \cdot, - \rangle$ , there is a norm  $\|x\| = \sqrt{\langle x, x \rangle}$ .

*Proof.* Consider any  $x \in H, a \in \mathbb{C}$ ,

$$\begin{aligned}
\|a \cdot x\| &= \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2} \sqrt{\langle x, x \rangle} = |a| \|x\| \\
\forall x \neq 0, \|x\| &= \sqrt{\langle x, x \rangle} > 0 \\
\|0\| &= \sqrt{\langle 0, 0 \rangle} = 0 \\
\|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \\
&= \|x\|^2 + \|y\|^2 + 2\Re(\langle x, y \rangle) \\
&\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y \rangle| \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\
&\leq (\|x\| + \|y\|)^2.
\end{aligned}$$

Thus  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm.  $\square$

**Corollary 1.8.** For every inner product space, there is a metric  $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$

**Proposition 1.9.** If  $\forall v, \langle v, u \rangle = 0$ , then  $u = 0$ .

**Proposition 1.10.** For an Inner product space  $H$ ,  $\forall y, x = \lim_{i \rightarrow \infty} x_i \in H$ , we have

$$\langle x, y \rangle = \lim_{i \rightarrow \infty} \langle x_i, y \rangle.$$

*Proof.* Given any  $\epsilon > 0$ , let  $\epsilon_0 = \frac{\epsilon}{\|y\|}$ .

Since  $x = \lim_{i \rightarrow \infty} x_i$ , we can find  $N > 0$ , such that  $\forall n > N, \|x - x_n\| < \epsilon_0$ , thus  $|\langle x, y \rangle - \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq \|x - x_n\|\|y\| < \epsilon_0\|y\| = \epsilon$   $\square$

**Corollary 1.11.** For an Inner product space  $H$ ,  $\forall y, x = \lim_{i \rightarrow \infty} x_i \in H$ , we have  $\langle y, x \rangle = \lim_{i \rightarrow \infty} \langle y, x_i \rangle$ .

**Definition 1.14.** An inner product space  $\mathcal{H}$  is called a **Hilbert space** if it is complete.

**Definition 1.15.** Let  $H$  be an inner product space. Two vectors  $u, v \in H$  are called **orthogonal** if  $\langle u, v \rangle = 0$ .

**Definition 1.16.** Let  $H$  be an inner product space. A set  $\{e_i\}_{i \in I} \subseteq H$  is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij}.$$

**Definition 1.17.** Let  $H$  be an inner product space. A set  $\{e_i\}_{i \in I} \subseteq H$  is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij} := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

**Theorem 1.12** (generalized Fourier series). Let  $\mathcal{H}$  be a Hilbert space, and  $\{e_i\}_{i \in I} \subseteq \mathcal{H}$  be an orthonormal set, then the following are equivalent:

1.  $\{e_i\}_{i \in I}$  is an orthonormal basis.
2. If  $\forall i \in I, \langle x, e_i \rangle = 0$ , then  $x = 0$ .
3.  $\forall x \in \mathcal{H}, x = \sum_{i \in I} \langle e_i, x \rangle e_i$ . (Fourier series)
4.  $\forall x \in \mathcal{H}, \|x\|^2 = \sum_{i \in I} |\langle e_i, x \rangle|^2$ . (Parseval Identity)

**Theorem 1.13.**  $\mathcal{H}$  is a separable Hilbert Space if and only if there is a maximal orthonormal set in  $\mathcal{H}$ . Moreover, in this case, every maximal orthonormal set is at most countable.

**Definition 1.18.** Let  $\mathcal{H}$  be a Hilbert space.  $S \subseteq \mathcal{H}$ , the subspace **orthogonal** to  $S$  is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v \rangle = 0, \forall v \in S\}.$$

**Lemma 1.14.** Let  $\mathcal{H}$  be a Hilbert space,  $S \subseteq \mathcal{H}$ , we always have  $S^\perp$  is a subspace of  $\mathcal{H}$ .

**Definition 1.19.** Let  $V$  be a vector space, and  $U, W \subseteq V$  be two subspaces, we say  $V = U \oplus W$ , if  $V = U + W = \{v + w : v \in U, w \in W\}$ , and  $V \cap W = \{0\}$ .

**Proposition 1.15.** Let  $V$  be a vector space, and  $U, W \subseteq V$  be two subspaces.  $V = U \oplus W$ , if and only if  $\forall v \in V$ , it can be uniquely written as  $v = u + w$ , where  $u \in U, w \in W$ .

**Theorem 1.16.** Let  $\mathcal{H}$  be a Hilbert space, if  $S \subseteq \mathcal{H}$  is a subspace, then

$$\mathcal{H} = \bar{S} \oplus S^\perp.$$

## 1.5 Bounded linear operators

**Definition 1.20.** Let  $X, Y$  be vector spaces,  $A : X \rightarrow Y$  is a linear operator if  $\forall c \in \mathbb{R}, u, v \in X$ ,

$$A(u + cv) = Au + cAv.$$

**Definition 1.21.** Let  $X, Y$  be normed spaces, the **operator norm** of a linear operator  $A : X \rightarrow Y$  is

$$\|A\| := \sup_{\|u\|_X \leq 1} \|Au\|_Y = \sup_{\|u\|_X = 1} \|Au\|_Y = \sup_{u \neq 0 \in X} \frac{\|Au\|_Y}{\|u\|_X}.$$

**Definition 1.22.** Let  $X, Y$  be normed spaces, a linear operator  $A : X \rightarrow Y$  is **bounded** if  $\|A\| < \infty$ .

**Definition 1.23.** Let  $X, Y$  be normed spaces, we denote

$$B(X, Y) := \{A : X \rightarrow Y | A \text{ is bounded linear operator}\}.$$

**Theorem 1.17.** The set  $B(X, Y)$  is a normed linear space with the operator norm.

**Proposition 1.18.** Let  $X, Y, Z$  be normed spaces, if  $A : X \rightarrow Y, B : Y \rightarrow Z$  are both linear bounded operators, then so is  $B \circ A$ , with

$$\|B \circ A\| \leq \|B\| \|A\|.$$

**Theorem 1.19.** Let  $X, Y$  be normed spaces, a linear operator  $A : X \rightarrow Y$  is bounded if and only if it is continuous.

**Definition 1.24.** Let  $X, Y$  be normed spaces, a linear operator  $A : X \rightarrow Y$  is **closed** if  $\forall u_k \rightarrow u$  in  $X$  and  $Au_k \rightarrow v$  in  $Y$ , we have  $Au = v$ .

**Theorem 1.20.** (closed graph) Let  $X, Y$  be Banach spaces, if a linear operator  $A : X \rightarrow Y$  is closed, it is also bounded.

**Theorem 1.21.** (Bounded inverse Theorem) Let  $X, Y$  be normed spaces, if a bounded linear operator  $A : X \rightarrow Y$  is bijective, then  $A^{-1}$  is continuous and bounded as well.

**Proposition 1.22.** Let  $Y$  be a Banach space,  $S$  be a dense subset of a normed space  $X$ . For any bounded linear operator  $E : S \rightarrow Y$ , we can extend it to  $\tilde{E} : X \rightarrow Y$ , such that  $\tilde{E}$  is also bounded and linear, with  $\|\tilde{E}\| = \|E\|$ , and  $\tilde{E}|_S = E$ .

*Proof.* Consider any  $x \in X$ .

Since  $S$  is dense in  $X$ , We know  $\forall m \in \mathbb{N}^+, \exists x_m \in S$ , such that  $\|x - x_m\|_X \leq \frac{1}{m}$ .

Since  $E$  is linear on  $S$ , we have that

$$\begin{aligned}\|Ex_m - Ex_l\|_Y &= \|E(x_m - x_l)\|_Y \\ &\leq \|E\|\|x_m - x_l\|_X \\ &= \|E\|\|(x_m - x) + (x - x_l)\|_X \\ &\leq \|E\|\|x - x_m\|_X + \|E\|\|x - x_l\|_X \\ &\leq \|E\|\left(\frac{1}{m} + \frac{1}{l}\right).\end{aligned}$$

Thus given any  $\epsilon > 0$ , for any  $m, l \geq \lceil \frac{2\epsilon}{\|E\|} \rceil$ , we can make  $\|Ex_m - Ex_l\|_Y < \epsilon$ . Thus  $(Ex_m)_{m=1}^\infty$  is a Cauchy sequence in  $Y$ .

Since  $Y$  is a Banach space,  $\exists y^* \in Y$ , such that  $Ex_m \rightarrow y^*$  in  $Y$ .

We claim that  $y^*$  is independent of choice of the sequence  $(x_m)_{m=1}^\infty$ .

Indeed, consider any other sequence  $(v_m)_{m=1}^\infty \subseteq C^\infty(\bar{x})$ , such that  $\forall m \in \mathbb{N}^+, \|x - x_m\|_X \leq \frac{1}{m}$ ,

$$\begin{aligned}\|y^* - Ev_m\|_Y &\leq \|y^* - Ex_m\|_Y + \|Ex_m - Ev_m\|_Y \\ &\leq \|y^* - Ex_m\|_Y + \|E\|\|x_m - v_m\|_X \\ &\leq \|y^* - Ex_m\|_Y + \|E\|\|x_m - x\|_X + \|E\|\|x - v_m\|_X.\end{aligned}$$

Since all three terms on the right go to 0 when  $m \rightarrow \infty$ , we have that  $Ev_m \rightarrow y^*$  in  $Y$ .

Thus we can uniquely define  $\tilde{E}x := y^*$ . In addition,

$$\begin{aligned}\|\tilde{E}x\|_Y &= \left\| \lim_{m \rightarrow \infty} Ex_m \right\|_Y \\ &= \lim_{m \rightarrow \infty} \|Ex_m\|_Y \\ &\leq \lim_{m \rightarrow \infty} \|E\|\|x_m\|_X \\ &= \|E\| \left\| \lim_{m \rightarrow \infty} x_m \right\|_X \\ &= \|E\|\|x\|_X.\end{aligned}$$

Thus  $\|\tilde{E}\| = \|E\|$ . □

### 1.5.1 Dual Space

**Definition 1.25.** Let  $X$  be a normed space over  $\mathbb{F}$ , a **functional** is an operator that maps into  $\mathbb{F}$ .

**Definition 1.26.** Let  $X$  be a normed space over  $\mathbb{F}$ , the **dual space** of  $X$  is the collection of bounded linear functionals on  $X$ , denoted

$$X^* := B(X, \mathbb{F}).$$

**Definition 1.27.** Let  $X$  be a normed space over  $\mathbb{F}$ , and a subspace  $Y \subseteq X^*$  we define the **duality pairing** to be  $\langle \cdot | \cdot \rangle_{X,Y} : X \times Y \rightarrow \mathbb{F}$  to be  $x \in X, \phi \in Y$ , we can write  $\langle x | \phi \rangle_{X,Y} := \phi(x)$  as the **action** of  $\phi$  on  $x$ . We may also write  $\langle \cdot | \cdot \rangle_{Y,X} : Y \times X \rightarrow \mathbb{F}$  in view of  $X \subseteq Y^{**}$  to be  $\langle \phi | x \rangle_{Y,X} := \langle \phi | x^{**} \rangle_{Y,X} = \phi(x)$ .

**Definition 1.28.** Let  $X$  be a normed space, the **dual norm** is defined to be

$$\|u^*\|_{X^*} := \sup_{\|u\| \leq 1} \left| \langle u^* | u \rangle_{X^*, X} \right|.$$

**Definition 1.29.** A Banach space  $X$  is **reflexive** if  $i(X) = X^{**}$ , where  $i : X \hookrightarrow X^{**}$  is the canonical linear isometric embedding by  $i(x) := x^{**}$ ,  $x^{**}(\phi) := \phi(x)$ .

**Theorem 1.23** (Riesz-Frechet Representation theorem). *Let  $\mathcal{H}$  be a Hilbert space, then for each  $u^* \in \mathcal{H}$ ,  $\exists! u \in \mathcal{H}$ , such that  $\forall v \in \mathcal{H}, \langle u^* | v \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$ , and  $\|u^*\|_{\mathcal{H}^*} = \|u\|_{\mathcal{H}}$ .*

**Definition 1.30.** Let  $X$  be a vector space over  $\mathbb{F}$ . We define

$$\bar{X} := \{\bar{x} : x \in X\}$$

to be the **conjugate vector space**. The scalar multiplication is given by  $\lambda \bar{x} = \overline{\lambda x}$ .

**Proposition 1.24.** *Let  $X$  be a normed space over  $\mathbb{F}$ . The map  $X \rightarrow \bar{X}$  by  $x \mapsto \bar{x}$  is an isometric antilinear bijection.*

**Corollary 1.25.** *Let  $\mathcal{H}$  be a Hilbert space, then  $\bar{\mathcal{H}} \cong \mathcal{H}^*$ , where the map  $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*; u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$  is the canonical bijective isometric antilinear isomorphism.*

**Corollary 1.26.** *Every Hilbert space is reflexive.*

*Remark.* We thus abuse the notation, and denote canonical bijective isometric antilinear isomorphism by  $u^\dagger := \Phi(u) \forall u \in \mathcal{H}$ , and  $(u^*)^\dagger := \Phi^{-1}(u^*) \forall u^* \in \mathcal{H}^*$ . Notice that by definition

$$\forall u \in \mathcal{H}, u^* \in \mathcal{H}^*, (u^\dagger)^\dagger = u, ((u^*)^\dagger)^\dagger = u^*.$$

We might further abuse the notation, and write

$$\langle u | v \rangle := \langle u, v \rangle = \langle u^\dagger | v \rangle =: \langle u^\dagger, v \rangle$$

interchangeably instead of  $\langle u^\dagger | v \rangle_{\mathcal{H}^*, \mathcal{H}}$  or  $\langle u, v \rangle_{\mathcal{H}}$  when the context is clear.

**Definition 1.31.** Let  $X$  be a Banach Space, we say  $(u_k)_{k=1}^\infty \subset X$  converges to  $u \in X$  weakly, denoted  $u_k \rightharpoonup u$ , if

$$\forall v^* \in X^*, \langle v^* | u_k \rangle \rightarrow \langle v^* | u \rangle$$

as real numbers.

**Proposition 1.27.** *Let  $X$  be a Banach Space,  $(u_k)_{k=1}^\infty \subset X$  be a sequence, then*

1. *If  $u_k \rightarrow u$ , we always have  $u_k \rightharpoonup u$ .*
2. *If  $u_k \rightharpoonup u$ , we have that  $u$  is unique.*
3. *If  $u_k \rightharpoonup u$ , we have  $(u_k)_{k=1}^\infty$  is bounded.*
4. *If  $u_k \rightharpoonup u$ , every subsequence  $(u_{k_j})_{j=1}^\infty$  also converges weakly to  $u$ .*

*Proof.* See A5Q1 for 1. □

**Theorem 1.28** (Weakly compact for reflexive Banach Space). *Let  $X$  be a reflexive Banach Space, and  $(u_k)_{k=1}^\infty \subset X$  be a bounded sequence, then  $\exists (u_{k_j})_{j=1}^\infty$  a subsequence, and  $u \in X$ , such that  $u_{k_j} \rightharpoonup u$ .*

**Proposition 1.29.** *Let  $\mathcal{H}$  be a Hilbert space, then  $u_k \rightharpoonup u$  if and only if  $\forall v \in \mathcal{H}, \langle v, u_k \rangle \rightarrow \langle v, u \rangle$  as real numbers.*

*Proof.* Suppose  $u_k \rightharpoonup u$ .

Notice that for all  $v \in \mathcal{H}$ , we have that  $v^\dagger \in \mathcal{H}^*$ , and thus  $\langle v, u_k \rangle = \langle v^\dagger | u_k \rangle \rightarrow \langle v^\dagger | u \rangle = \langle v, u \rangle$ .

Now suppose  $\forall v \in \mathcal{H}, \langle v, u_k \rangle \rightarrow \langle v, u \rangle$ .

Notice that for any  $f \in \mathcal{H}^*$ , by Riesz-Frechet Representation theorem 1.23, there is some  $f^\dagger \in \mathcal{H}$ , such that

$$\langle f | u_{k_j} \rangle = \langle f^\dagger, u_{k_j} \rangle \rightarrow \langle f^\dagger, u \rangle = \langle f | u \rangle.$$

Thus,  $u_{k_j} \rightharpoonup u$ . □

**Proposition 1.30.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces,  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded operator, and  $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$  be a sequence. If  $u_k \rightharpoonup u \in \mathcal{H}_1$ , then  $Tu_k \rightharpoonup Tu \in \mathcal{H}_2$ .

*Proof.* Let  $y_k := Tu_k, y := Tu \in \mathcal{H}_2$ .

Consider any  $g \in \mathcal{H}_2^*$ , we define  $f := g \circ K \in \mathcal{H}_1^*$ .

Since  $u_k \rightharpoonup u$ , we must have

$$\begin{aligned}\lim_{k \rightarrow \infty} f(u_k) &= f(u) \\ \lim_{k \rightarrow \infty} g(Ku_k) &= g(Ku) \\ \lim_{k \rightarrow \infty} g(y_k) &= g(y).\end{aligned}$$

We thus have  $y_k \rightharpoonup y$ .  $\square$

**Proposition 1.31.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces,  $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a compact operator, and  $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$  be a sequence. If  $u_k \rightharpoonup u \in \mathcal{H}_1$ , then  $Ku_k \rightharpoonup Ku \in \mathcal{H}_2$ .

*Proof.* Let  $y_k := Ku_k, y := Ku \in \mathcal{H}_2$ .

Since  $K$  is compact, it is bounded, so  $y_k \rightharpoonup y$ .

Now suppose for contradiction  $\lim_{k \rightarrow \infty} \|y_k - y\| \neq 0$ .

Then there is some  $\epsilon > 0$  and a subsequence  $(u_{k_j})_{j=1}^\infty$  such that  $\forall j \geq 1, \|y_{k_j} - y\| \geq \epsilon$ .

Since  $u_k \rightharpoonup u \in \mathcal{H}$ , we have  $(u_k)_{k=1}^\infty$  is bounded, and thus  $(u_{k_j})_{j=1}^\infty$  is bounded.

Since  $K$  is compact, there is some further subsequence  $(u_{k_{j_m}})_{m=1}^\infty$  such that  $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$ .

Thus  $Ku_{k_{j_m}} \rightharpoonup \tilde{y}$ . Since weak convergence, we must have  $\tilde{y} = y$ .

Thus  $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = y$ , which is a contradiction.  $\square$

### 1.5.2 Adjoint Operator

**Definition 1.32.** Let  $X, Y$  be normed spaces, the **dual operator** of a linear operator  $A : X \rightarrow Y$  is

$$A^* : Y^* \rightarrow X^*; f \mapsto f \circ A.$$

**Proposition 1.32.** Let  $X, Y, Z$  be normed spaces,  $S \in B(X, Y), T \in B(Y, Z)$ , then  $(S \circ T)^* = T^* \circ S^*$ .

*Proof.* Consider any  $f \in Z^*$ , and any  $x \in X$ , we have

$$\begin{aligned}(T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx) \\ &= (f)(S(T(x))) \\ &= (f \circ (S \circ T))(x) \\ &= (S \circ T)^*(f)(x).\end{aligned}$$

Thus  $(T^* \circ S^*)(f) = (S \circ T)^*(f)$ .  $\square$

**Definition 1.33.** Let  $\mathcal{H}$  be a Hilbert space, and  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a bounded linear operator, the **Hilbert adjoint operator** of  $T$  is  $T^\dagger : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \forall x, y \in \mathcal{H}$ .

**Theorem 1.33.** Let  $\mathcal{H}$  be a Hilbert space, and  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a bounded linear operator,  $T^\dagger$  always exists, and is given by  $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$ , where  $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$ ;  $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$  is the canonical bijective isometric antilinear isomorphism, and  $T^*$  is the dual operator of  $T$ . In addition,  $T^\dagger$  is also a bounded linear operator, with  $\|T^\dagger\| = \|T\|$ , and  $(T^\dagger)^\dagger = T$ .

*Proof.*  $\forall y \in \mathcal{H}$ , we have that

$$\begin{aligned}\langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\ &= ((T^* \circ \Phi)(x))(y) \\ &= (\Phi(x))(Ty) \\ &= \langle x, Ty \rangle.\end{aligned}$$

Now consider any  $x, y, z \in \mathcal{H}, c \in \mathbb{C}$ , we have that

$$\begin{aligned}\langle T^\dagger(x + cz), y \rangle &= \langle x + cz, Ty \rangle \\ &= \langle x, Ty \rangle + \bar{c}\langle z, Ty \rangle \\ &= \langle T^\dagger x, y \rangle + \bar{c}\langle T^\dagger z, y \rangle \\ &= \langle T^\dagger x + cT^\dagger z, y \rangle.\end{aligned}$$

Since this holds for any  $y \in \mathcal{H}$ , we have that  $T^\dagger(x + cz) = T^\dagger x + cT^\dagger z$ , and thus  $T^\dagger$  is linear.  
Now given any  $x \in \mathcal{H}$ , we have that

$$\begin{aligned}\|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\ &= \langle x, TT^\dagger x \rangle \\ &\leq \|x\|\|TT^\dagger x\| \\ &\leq \|x\|\|T\|\|T^\dagger x\| \\ &\implies \\ \|T^\dagger x\| &\leq \|x\|\|T\| \\ &\implies \\ \|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\ &\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{\|x\|\|T\|}{\|x\|} \\ &= \|T\|.\end{aligned}$$

Thus  $T^\dagger$  is also a bounded linear operator.

Now  $\forall x, y \in \mathcal{H}$ ,  $\langle x, T^\dagger y \rangle = \overline{\langle T^\dagger y, x \rangle} = \overline{\langle y, Tx \rangle} = \langle Tx, y \rangle$ .

Thus  $(T^\dagger)^\dagger = T$ .  $\square$

*Remark.*  $\forall x, y \in \mathcal{H}$ ,  $\langle (Tx)^\dagger | y \rangle = \langle Tx, y \rangle = \langle x, T^\dagger y \rangle = \langle x^\dagger | T^\dagger y \rangle$ . We thus abuse the notation, and write  $(Tx)^\dagger = \langle x | T^\dagger$

**Definition 1.34.** A bounded linear operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  is **self-adjoint** if  $T^\dagger = T$ .

### 1.5.3 Compact Operators

**Definition 1.35.** Let  $X, Y$  be metric spaces, a linear operator  $A : X \rightarrow Y$  is **compact** if for each bounded subset  $S \subseteq X$ , we have its image  $A(S)$  is pre-compact in  $Y$ .

**Proposition 1.34.** Let  $X, Y$  be metric spaces, a linear operator  $A : X \rightarrow Y$  is compact if and only if  $A$  is bounded, and each bounded sequence  $(x_n)_{n=1}^\infty \subseteq X$  has some subsequence  $(x_{n_k})_{k=1}^\infty$  such that  $(Ax_{n_k})_{k=1}^\infty$  converges to some  $y \in Y$ .

**Definition 1.36.** Let  $X, Y$  be Banach spaces and  $X \subseteq Y$ , then we say  $X$  is **compactly embedded** in  $Y$ , denoted

$$X \subset\subset Y$$

if the inclusion map  $i : X \hookrightarrow Y$ ;  $x \mapsto x$  is compact.

Namely,  $\exists C > 0$ , such that  $\forall x \in X$ ,  $\|x\|_Y \leq C\|x\|_X$ , and each bounded sequence  $(x_n)_{n=1}^\infty \subseteq X$  having some subsequence  $(x_{n_k})_{k=1}^\infty$  that converges to some  $y \in Y$ .

**Proposition 1.35.** Let  $X, Y, Z$  be Banach spaces and  $X \subset\subset Y$ , if an operator  $T : Z \rightarrow X$  is bounded, then  $\tilde{T} := i \circ T : Z \rightarrow Y$  is compact.

*Proof.* Consider any bounded set  $S \subseteq Z$ , such that  $\forall z \in S, \|z\|_Z \leq M$ . We have  $\|Tz\|_X \leq \|T\|\|z\|_M \leq M\|T\| < \infty$ , and thus  $T(S)$  is bounded in  $X$ . Yet  $i$  is compact, and thus  $i(T(S))$  is pre-compact. This shows  $\tilde{T}(S) = (i \circ T)(S)$  is pre-compact for any bounded set  $S \subseteq Z$ . Thus  $\tilde{T}$  is compact.  $\square$

**Theorem 1.36.** Let  $\mathcal{H}$  be a Hilbert space, and  $K : \mathcal{H} \rightarrow \mathcal{H}$  be a compact linear operator, then  $K^\dagger$  is also compact.

*Proof.*  $K^\dagger$  is bounded by 1.34.

Let  $(u_k)_{k=1}^\infty$  be any bounded sequence in  $\mathcal{H}$ .

By 1.28, we have that  $\exists (u_{k_j})_{j=1}^\infty$  a subsequence, and  $u \in X$ , such that  $u_{k_j} \rightharpoonup u$ .

Notice that for any  $f \in \mathcal{H}^*$ , by Riesz-Frechet Representation theorem 1.23, there is some  $f^\dagger \in \mathcal{H}$ , such that

$$\begin{aligned} \langle f | K^\dagger(u_{k_j} - u) \rangle &= \langle f^\dagger, K^\dagger(u_{k_j} - u) \rangle \\ &= \langle Kf^\dagger, u_{k_j} - u \rangle \\ &= \langle Kf^\dagger, u \rangle - \langle Kf^\dagger, u \rangle \rightarrow 0, \end{aligned}$$

since  $u_{k_j} \rightharpoonup u$  and by 1.29.

Since  $\langle f | K^\dagger(u_{k_j} - u) \rangle \rightarrow 0 = \langle f | 0 \rangle$  for any  $f \in \mathcal{H}^*$ , we have that  $K^\dagger(u_{k_j} - u) \rightharpoonup 0$ .

By 1.31, we have that  $KK^\dagger(u - u_{k_j}) \rightarrow 0$ .

$$\begin{aligned} \|K^\dagger u - K^\dagger u_{k_j}\|^2 &= \langle K^\dagger u - K^\dagger u_{k_j}, K^\dagger u - K^\dagger u_{k_j} \rangle \\ &= \langle K^\dagger(u - u_{k_j}), K^\dagger(u - u_{k_j}) \rangle \\ &= \langle KK^\dagger(u - u_{k_j}), u - u_{k_j} \rangle \\ &\leq \|KK^\dagger(u - u_{k_j})\| \|u - u_{k_j}\| \\ &\rightarrow 0. \end{aligned}$$

Thus  $K^\dagger u_{k_j} \rightarrow K^\dagger u \in \mathcal{H}$ ,

Since  $(u_k)_{k=1}^\infty$  is any bounded sequence, we have that  $K^\dagger$  is compact by 1.34.  $\square$

**Theorem 1.37** (Fredholm alternative). Let  $\mathcal{H}$  be a Hilbert space, and  $K : \mathcal{H} \rightarrow \mathcal{H}$  be a compact linear operator, then

1.  $\text{Ker}(I - K)$  is finite dimensional.
2.  $\text{Im}(I - K)$  is closed.
3.  $\text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$ .
4.  $\dim(\text{Ker}(I - K)) = \dim(\text{Ker}(I - K^\dagger))$ .
5.  $\text{Ker}(I - K) = \{0\} \iff \text{Im}(I - K) = \mathcal{H}$ .

**Corollary 1.38.** Let  $\mathcal{H}$  be a Hilbert space, and  $K : \mathcal{H} \rightarrow \mathcal{H}$  be a compact linear operator, then exactly one of the following holds:

1.  $\forall v \in \mathcal{H}, \exists! u \in \mathcal{H}$ , such that  $(I - K)u = v$ .
2.  $\exists u \neq 0 \in \mathcal{H}$ , such that  $(I - K)u = 0$ .

*Proof.* When  $\text{Ker}(I - K) = \{0\}$ , we have that  $I - K$  is injective, and  $\text{Im}(I - K) = \mathcal{H}$ .

Thus  $\forall v \in \mathcal{H}, \exists! u \in \mathcal{H}$ , such that  $(I - K)u = v$ .

On the other hand, if 1. is true, we have that  $(I - K)$  is surjective, so  $\text{Im}(I - K) = \mathcal{H}$ , so  $\text{Ker}(I - K) = \{0\}$ . Thus  $\text{Ker}(I - K) = \{0\} \iff 1.$

We also have that  $\text{Ker}(I - K) \neq \{0\} \iff \exists u \neq 0 \in \text{Ker}(I - K) \iff 2..$   $\square$

**Theorem 1.39** (Structure theorem for compact operators). *Let  $K : X \rightarrow X$  be a compact operator on an infinite-dimensional Banach space  $X$ , then*

1.  $0 \in \text{Spec}(K)$ .
2.  $\text{Spec}(K) \setminus \{0\} = \text{Spec}_p(K) \setminus \{0\}$ .
3.  $\text{Spec}(K) \setminus \{0\}$  is finite, or  $\text{Spec}(K) \setminus \{0\} = (\lambda_k)_{k=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} \lambda_k = 0$ .

**Theorem 1.40** (Spectral theorem / Hilbert-Schmidt Theorem). *Let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a compact linear self-adjoint operator on a countably infinite-dimensional complex Hilbert space  $\mathcal{H}$ , then*

1. There exists orthonormal eigenvectors  $(\phi_k)_{k=1}^{\infty} \subset \mathcal{H}$  and eigenvalues  $(\lambda_k)_{k=1}^{\infty} \subset \mathbb{R}$  such that  $|\lambda_1| \geq |\lambda_2| \geq \dots$ , and

$$T\phi_k = \lambda_k \phi_k, \lambda_k \neq 0, \forall k \in \mathbb{N}^+,$$

$$\forall v \in \mathcal{H}, T v = \sum_{k=1}^{\infty} \lambda_k \langle \phi_k, v \rangle \phi_k = \sum_{k=1}^{\infty} \langle \phi_k, T v \rangle \phi_k.$$

2.  $\lim_{k \rightarrow \infty} \lambda_k = 0$ ,
3.  $(\phi_k)_{k=1}^{\infty}$  is an orthonormal basis for  $\mathcal{H}$  iff 0 is not an eigenvalue for  $T$ .

## 1.6 Function Spaces

### 1.6.1 Continuous functions

**Definition 1.37.**  $u : U \rightarrow \mathbb{R}$  is **continuous** at  $x \in U$  if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in U, \|x - y\| < \delta \implies |u(x) - u(y)| < \epsilon.$$

A function  $u$  is continuous if it is continuous at all  $x \in U$ .

- $C(U) := \{u : U \rightarrow \mathbb{R} : u \text{ is continuous}\}$
- $C^k(U) := \{u : U \rightarrow \mathbb{R} : u \text{ is } k\text{-times continuously differentiable}\}$
- $C^{\infty}(U) := \{u : U \rightarrow \mathbb{R} : u \text{ has continuous derivatives of all orders}\}$

**Definition 1.38.**  $u : U \rightarrow \mathbb{R}$  is **uniformly continuous** if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in U, \|x - y\| < \delta \implies |u(x) - u(y)| < \epsilon.$$

- $C(\bar{U}) := \{u : U \rightarrow \mathbb{R} : u \text{ is uniformly continuous on bounded subsets of } U\}$
- $C^k(\bar{U}) := \{u : U \rightarrow \mathbb{R} : \forall |\alpha| \leq k, D^{\alpha}u \text{ is uniformly continuous on bounded subsets of } U\}$
- If  $u \in C^k(\bar{U})$ , then we can extend  $D^{\alpha}u$  continuously to  $\bar{U}$ .

**Definition 1.39.** The support of  $u : U \rightarrow \mathbb{R}$  is

$$\text{Supp}(u) := \overline{\{x \in U : u(x) \neq 0\}}.$$

**Definition 1.40.**  $u : U \rightarrow \mathbb{R}$  has compact support if  $\text{Supp}(u)$  is a compact subset of  $U$ .

**Definition 1.41.** We denote the functions in  $C(U)$  and  $C^k(U)$  with compact support by  $C_c(U), C_c^k(U)$ .

**Definition 1.42.** Let  $U \subseteq \mathbb{R}^n$  be Lebesgue measurable, we define the set of **test functions**

$$C_c^{\infty}(U) := \{\phi \in C^{\infty}(U) : \text{Supp}(\phi) \subseteq U \text{ and is compact}\},$$

where

$$\text{Supp}(\phi) := \overline{\{x \in U : \phi(x) \neq 0\}}.$$

**Definition 1.43.** Consider a sequence of functions  $\{u_m\}_1^\infty$  with  $u_m : U \rightarrow \mathbb{R}$  and a function  $u : U \rightarrow \mathbb{R}$ , we have

- $u_m \rightarrow u$  point-wise on  $U$  if

$$\forall x \in U, \delta > 0, \exists M \in \mathbb{N}, \text{ such that } m > M \implies |u_m(x) - u(x)| < \delta.$$

- $u_m \rightarrow u$  uniformly on  $U$  if

$$\forall \delta > 0, \exists M \in \mathbb{N}, \text{ such that } \forall x \in U, m > M \implies |u_m(x) - u(x)| < \delta.$$

**Definition 1.44.**  $f : [a, b] \rightarrow \mathbb{R}$  is **absolutely continuous** if  $\forall \epsilon > 0, \exists \delta > 0$  such that for every finite disjoint collection  $\{(a_k, b_k)\}_{k=1}^n$  of open intervals in  $(a, b)$ ,

$$\sum_{k=1}^n (b_k - a_k) < \delta \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

### 1.6.2 Lebesgue Spaces

See more in my Measure Theory Notes.

**Definition 1.45.** We denote the Lebesgue measure by  $\lambda$  on  $\mathbb{R}^n$ . We denote  $\int_A f d\lambda$  by  $\int_A f(x) dx$  for any measurable set  $A \subseteq \mathbb{R}^n$ .

**Definition 1.46.** Let  $\Omega \subseteq \mathbb{R}^n$  be Lebesgue measurable, we define

$$\mathcal{L}^1(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} \mid \int_\Omega |f(x)| dx < \infty \right\}.$$

**Definition 1.47.** Let  $\Omega \subseteq \mathbb{R}^n$  be Lebesgue measurable, and  $1 \leq p < \infty$  we define

$$\mathcal{L}^p(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} \mid f^p \in L^1(\Omega) \right\} = \left\{ f : \Omega \rightarrow \mathbb{R} \mid \int_\Omega |f(x)|^p dx < \infty \right\}.$$

In addition, we define the norm

$$\|f\|_p := \left( \int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

**Definition 1.48.** The **essential supremum** of a function  $u : U \rightarrow \mathbb{R}$  is

$$\text{ess sup } f := \inf \{M \in \mathbb{R} : |\{x : f(x) > M\}| = 0\}.$$

**Definition 1.49.** Let  $\Omega \subseteq \mathbb{R}^n$  be Lebesgue measurable, we define

$$\mathcal{L}^\infty(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid \text{ess sup } f < \infty\}.$$

In addition, we define the norm

$$\|f\|_\infty := \text{ess sup } f.$$

**Definition 1.50.** Two measurable functions  $f, g : \Omega \rightarrow \mathbb{R}$  are said to be equal almost everywhere if  $\{x \in \Omega : f(x) \neq g(x)\}$  has measure zero.

**Proposition 1.41.** For any  $1 \leq p \leq \infty$ , we have  $\|f - g\|_p = 0 \iff f = g$  almost everywhere.

**Definition 1.51.** For any  $1 \leq p \leq \infty$ , if we identify  $f, g \in \mathcal{L}^p(\Omega)$  by  $f \sim g \iff f = g$  almost everywhere, we get the quotient space

$$L^p := \mathcal{L}^p / \sim = \{[f] : f \in L^p(\Omega)\}$$

to be the collection of all equivalence classes of functions in  $\mathcal{L}^p$ .

**Theorem 1.42** (completeness of  $L^p$ ). *For any  $1 \leq p \leq \infty$ , we have the space  $(L^p, \|\cdot\|_p)$  is a Banach space, where  $\|[f]\|_p := \|f\|_p$  for any representative  $f \in [f]$ . One can check this norm is well-defined.*

**Theorem 1.43.** *For any  $1 \leq p < \infty$ ,*

- $C_c(U)$  is dense in  $L^p(U)$ .
- $C(\bar{U})$  is dense in  $L^p(U)$ .

**Definition 1.52.** Let  $U, V \subseteq \mathbb{R}^n$  be open, we say that  $V$  is **compactly contained** in  $U$  if  $V \subseteq \bar{V} \subseteq U$ , and  $\bar{V}$  is compact. We write this as  $V \subset\subset U$ .

**Definition 1.53.** The **locally summable spaces** are

$$L_{loc}^p(U) := \{u : U \rightarrow \mathbb{R} : \forall V \subset\subset U, u \in L^p(V)\}.$$

**Definition 1.54.** We say some property holds for  $L_{loc}^p(U)$ , if it holds  $\forall L^p(V)$  such that  $V \subset\subset U$ . For instance, let  $(f_n)_{n=1}^\infty \subseteq L_{loc}^p(U)$ , then  $f_n \rightarrow f$  in  $L_{loc}^p(U)$  if  $f_n \rightarrow f$  in  $L^p(V)$ ,  $\forall V \subset\subset U$ .

**Proposition 1.44.** *For any  $1 \leq p \leq \infty$ , we have*

$$L^p(U) \subseteq L_{loc}^1(U).$$

**Example 1.6.1.** Let  $u(x) = \frac{1}{x}$  on  $U = (0, 1)$ .

We have  $\int_0^1 |u| dx = \infty$ , and thus  $u \notin L^1(U)$ . However,  $u \in L_{loc}^1(U)$ .

**Theorem 1.45** (Holder's Inequality). *Assume  $1 \leq p, q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .*

*If  $u \in L^p(U), v \in L^q(U)$ , we have*

$$\int_U |uv| dx \leq \|u\|_p \|v\|_q.$$

For  $a, b \in \mathbb{R}^n$ , we have

$$\left| \sum_{k=1}^n a_k b_k \right| \leq \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} \left( \sum_{k=1}^n |b_k|^q \right)^{1/q}$$

**Theorem 1.46** (Minkowski's Inequality). *Assume  $1 \leq p \leq \infty$ .*

*Let  $u, v \in L^p(U)$ , we have*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

For  $a, b \in \mathbb{R}^n$ , we have

$$\left( \sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} + \left( \sum_{k=1}^n |b_k|^p \right)^{1/p}$$

**Theorem 1.47** (Lebesgue Monotone Convergence). *Let  $f_n : X \rightarrow [0, \infty]$  be measurable functions with  $0 \leq f_1 \leq f_2 \leq \dots \leq \infty$ . Let  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ , then  $f : X \rightarrow [0, \infty]$  is measurable, and*

$$\lim_{n \rightarrow \infty} \int_X f_n dx = \int_X f dx.$$

**Theorem 1.48** (Lebesgue Dominated Convergence). *Let  $f_n : X \rightarrow \mathbb{C}$  be measurable functions, defined almost everywhere on  $X$ , such that  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$  is defined almost everywhere for  $x \in X$ . If there is  $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$ , such that for almost everywhere  $x \in X, \forall n \in \mathbb{N}$ ,  $|f_n(x)| \leq g(x)$ , then  $f \in \mathcal{L}^1(X, \mu)$ , and*

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = 0.$$

**Theorem 1.49.** *We have that*

$$L^q(U) \cong L^p(U)^*,$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ , and the isometric isomorphism  $L^q(U) \xrightarrow{\sim} L^p(U)^*$ ;  $u \mapsto u^*$  is defined to be

$$\forall v \in L^p(U), \langle u^* | v \rangle := \int_U uv dx.$$

*Remark.* We will abuse the notation, and write  $\langle u | v \rangle := \int_U uv dx$  with  $u \in L^q(U)$  instead of  $u^* \in L^p(U)^*$ .

**Corollary 1.50.** *In particular,  $L^2(U) \cong L^2(U)^*$ , with the isometric isomorphism  $L^2(U) \rightarrow L^2(U)^*$ ;  $u \mapsto u^*$  is defined to be*

$$\forall v \in L^2(U), \langle u^* | v \rangle = \int_U uv dx = \langle u, v \rangle_{L^2(U)}.$$

**Definition 1.55.** For  $f : U \rightarrow \mathbb{R}^m$ , we define

$$\|f\|_{L^p(U)} := \left\| \|f\|_p \right\|_{L^p(U)}.$$

## 2 Sobolev Spaces

This section follows Chapter 5 in Evan's book.

### 2.1 Holder Spaces

**Definition 2.1.** For  $u : U \rightarrow \mathbb{R}$  be bounded and continuous, we write

$$\|u\|_{C(\bar{U})} := \sup_{x \in \bar{U}} |u(x)|.$$

**Definition 2.2.** A function  $u : U \rightarrow \mathbb{R}$  is **Holder continuous** with  $0 < \gamma \leq 1$  if

$$\exists C, \text{ such that } \forall x, y \in U, |u(x) - u(y)| \leq C \|x - y\|^\gamma.$$

**Definition 2.3.** The  $\gamma^{th}$ -**Holder semi-norm** of  $u : U \rightarrow \mathbb{R}$  is

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{x,y \in \bar{U}, x \neq y} \left( \frac{|u(x) - u(y)|}{\|x - y\|^\gamma} \right).$$

The  $\gamma^{th}$ -**Holder norm** of  $u : U \rightarrow \mathbb{R}$  is

$$\|u\|_{C^{0,\gamma}(\bar{U})} := [u]_{C^{0,\gamma}(\bar{U})} + \|u\|_{C(\bar{U})}.$$

**Definition 2.4.** For  $k \in \mathbb{N}, u \in C^k(\bar{U})$  we define

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}.$$

The **Holder Space** is

$$C^{k,\gamma}(\bar{U}) := \left\{ u \in C^k(\bar{U}) : \|u\|_{C^{k,\gamma}(\bar{U})} < \infty \right\}.$$

**Theorem 2.1.**

$$(C^{k,\gamma}(\bar{U}), \|\cdot\|_{C^{k,\gamma}(\bar{U})})$$

is a Banach Space.

### 2.2 Convolution and Mollification

**Definition 2.5.** For  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ , we define the **convolution**  $f * g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$  to be

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$

**Proposition 2.2.** For  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have  $f * g = g * f$ .

*Proof.* Take  $z := x - y$ , we have  $y = x - z$ , and  $d(z^i) = -d(y^i)$ . We have that for any  $x \in \mathbb{R}$ ,

$$\begin{aligned} (f * g)(x) &= \int_{\mathbb{R}^n} f(x - y)g(y)dy \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x - y)g(y)d(y^1) \cdots d(y^n) \\ &= (-1)^n \int_{\infty}^{-\infty} \cdots \int_{\infty}^{-\infty} f(z)g(x - z)d(z^1) \cdots d(z^n) \\ &= (-1)^n (-1)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z)g(x - z)dz \\ &= \int_{\mathbb{R}^n} f(z)g(x - z)dz \\ &= (g * f)(x). \end{aligned}$$

□

**Proposition 2.3.**

$$\text{Supp}(f * g) \subseteq \text{Supp}(f) + \text{Supp}(g).$$

*Proof.* Let  $f^x(y) := f(x - y)$ , we have  $f * g(x) = \int_{\mathbb{R}^n} f^x(y)g(y)dy$ .

Suppose  $\text{Supp}(f^x) \cap \text{Supp}(g) = \emptyset$ , then we have  $(f * g)(x) = 0$ .

In addition,

$$\begin{aligned} & \text{Supp}(f^x) \cap \text{Supp}(g) \neq \emptyset \\ \iff & \exists y, x - y \in \text{Supp}(f), y \in \text{Supp}(g) \\ \iff & x \in \text{Supp}(f) + \text{Supp}(g). \end{aligned}$$

Thus  $\text{Supp}(f * g) \subseteq \{x \in \mathbb{R}^n : \text{Supp}(f^x) \cap \text{Supp}(g) \neq \emptyset\} = \text{Supp}(f) + \text{Supp}(g)$ .  $\square$

**Proposition 2.4** (Young's Convolution Inequality). *Let  $f \in L^1(\mathbb{R}^n), g \in L^p(\mathbb{R}^n)$  with  $1 \leq p \leq \infty$ , then for a.e.  $x \in \mathbb{R}^n$ , the function  $f(x - y)g(y)$  is integrable. Thus  $f * g : \mathbb{R}^n \rightarrow \mathbb{R}$  is well-defined a.e.. In addition,  $f * g \in L^p(\mathbb{R}^n)$ , and*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

**Definition 2.6.**

$$\bar{B}(x, r) = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$$

is the closed ball around  $x$  of radius  $r$ , and

$$B(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$$

is the closed ball around  $x$  of radius  $r$ .

**Definition 2.7.** For  $\epsilon > 0$ ,

$$U_\epsilon := \{x \in U : \text{dist}(x, \partial U) > \epsilon\}.$$

*Remark.* This definition does not require  $U$  to be bounded.

**Definition 2.8.** The standard mollifier  $\eta(x) \in C^\infty(\mathbb{R}^n)$  is defined as

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^{-1}}\right), & |x| < 1 \\ 0, & \text{o.w.} \end{cases}$$

with  $C$  such that  $\int_{\mathbb{R}^n} \eta(x)dx = 1$ .

For each  $\epsilon > 0$ ,

$$\eta_\epsilon := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$

**Proposition 2.5.**  $\forall \epsilon > 0$ , we have

1.  $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$ ,
2.  $\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1$ ,
3.  $\text{Supp}(\eta_\epsilon) \subseteq \bar{B}(0, \epsilon)$ .

**Definition 2.9.** Let  $f \in L^1_{loc}(U), \epsilon > 0$ , its **mollification**  $f^\epsilon : U_\epsilon \rightarrow \mathbb{R}$  is defined as

$$f^\epsilon(x) := \eta_\epsilon * f := \int_U \eta_\epsilon(x - y)f(y)dy = \int_{\bar{B}(0, \epsilon)} f(x - z)\eta_\epsilon(z)dz.$$

*Remark.* When  $U \subsetneq \mathbb{R}^n$ , the mollification  $\eta_\epsilon * f$  is not using the formal definition of convolution, but we will soon see the abuse of notation makes sense.

**Proposition 2.6.** Let  $f^\epsilon$  be defined as above, if we zero-extend  $f$  outside of  $U$  to be

$$\bar{f}(x) := \begin{cases} f(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases},$$

we have  $\forall x \in U_\epsilon$ ,

$$\begin{aligned} (\eta_\epsilon * \bar{f})(x) &= \int_{\mathbb{R}^n} \eta_\epsilon(x-y) \bar{f}(y) dy \\ &= \int_U \eta_\epsilon(x-y) f(y) dy \\ &= f^\epsilon(x) \end{aligned}$$

**Theorem 2.7** (C.5.7 in Evans). Let  $f^\epsilon$  be defined as above, we have:

1.  $f^\epsilon \in C^\infty(U_\epsilon)$ ,
2.  $D^\alpha(f^\epsilon) = (D^\alpha \eta_\epsilon) * f$  on  $U_\epsilon$ ,
3.  $f^\epsilon \rightarrow f$  a.e., as  $\epsilon \rightarrow 0$ ,
4. If  $f \in C(U)$ , we have  $f^\epsilon \rightarrow f$  uniformly on compact subsets of  $U$ ,
5. If  $1 \leq p < \infty$ ,  $f \in L_{loc}^p(U)$ , we have  $f^\epsilon \rightarrow f$  in  $L_{loc}^p(U)$ . Namely,  $f^\epsilon \rightarrow f$  in  $L^p(V)$ ,  $\forall V \subset \subset U$ .
6.  $\text{Supp}(f^\epsilon) \subseteq \text{Supp}(f) + \text{Supp}(\eta_\epsilon) = \text{Supp}(f) + \bar{B}(0, \epsilon)$ .

**Proposition 2.8.** Let  $1 \leq p \leq \infty$ . Let  $u \in L^p(U)$ , then for any  $\epsilon > 0$ ,  $U_\epsilon \supseteq V \supseteq \text{Supp}(u^\epsilon)$ , we have

$$\|u^\epsilon\|_{L^p(V)} \leq \|u\|_{L^p(V)}.$$

*Proof.* Notice that  $u^\epsilon \in C^\infty(U_\epsilon) \subseteq L^1(U_\epsilon)$ .

If we zero-extend  $u$  outside of  $U$  to be  $\bar{u}(x) := \begin{cases} u(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$ , we have

$$\begin{aligned} \|\bar{u}\|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\bar{u}(x)|^p dx \\ &= \int_U |u(x)|^p dx + 0 \\ &= \|u\|_{L^p(U)}^p \\ &< \infty. \end{aligned}$$

Thus,  $\bar{u} \in L^p(\mathbb{R}^n)$ .

By 2.4, we have

$$\begin{aligned} \|u^\epsilon\|_{L^p(V)} &= \|\eta_\epsilon * \bar{u}\|_{L^p(V)} \\ &\leq \|\eta_\epsilon * \bar{u}\|_{L^p(\mathbb{R}^n)} \\ &\leq \|\eta_\epsilon\|_{L^1(\mathbb{R}^n)} \|\bar{u}\|_{L^p(\mathbb{R}^n)} \\ &= \left( \int_{\mathbb{R}^n} |\eta_\epsilon(x)| dx \right) \|u\|_{L^p(V)} \\ &= \|u\|_{L^p(V)}. \end{aligned}$$

□

## 2.3 Weak Derivative and Sobolev Spaces

### 2.3.1 Weak Derivatives

**Theorem 2.9.** For  $u \in C^k(U)$ ,  $\phi \in C_c^\infty(U)$ ,  $|\alpha| = k$ , integration by parts gives:

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx.$$

**Definition 2.10.** Let  $U \subseteq \mathbb{R}^n$  be Lebesgue measurable, and  $\alpha \in \mathbb{N}^n$  is an  $n$  tuple. For  $u, v \in L_{loc}^1(U)$ , we say  $v$  is the  $\alpha^{th}$ -weak derivative of  $u$  if

$$\forall \phi \in C_c^\infty(U), \int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx,$$

where  $D^\alpha \phi := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} \phi$ , and  $|\alpha| := \sum_{i=1}^n \alpha_i$ .

If such a  $v$  exists, we say that  $D^\alpha u = v$  in the weak sense. Otherwise,  $u$  does not possess a  $\alpha^{th}$  weak derivative.

**Theorem 2.10.** Suppose  $v \in L_{loc}^1(U)$  be such that

$$\forall \phi \in C_c^\infty(U), \int_U \phi v dx = 0,$$

we must have  $v = 0$  a.e..

*Proof.* By 2.7, we have  $v^\epsilon \rightarrow v$  a.e., as  $\epsilon \rightarrow 0$ .

Now pick any such  $y \in U$  where  $v^\epsilon(y) \rightarrow v(y)$ . Since  $U$  is open, we can find  $r > 0$ , such that  $\bar{B}(y, r) \subset U$ .

Now we define the function  $\phi_{y,\epsilon}(x) := \eta_\epsilon(y - x)$  for each  $\epsilon \in (0, r)$ .

Since  $\text{Supp}(\eta_\epsilon) \subseteq \bar{B}(0, \epsilon)$ , we have  $\text{Supp}(\phi_{y,\epsilon}) \subseteq \bar{B}(y, \epsilon) \subset U$  is compactly contained in  $U$ .

Also,  $\phi_{y,\epsilon} \in C^\infty(\mathbb{R}^n) \subset C^\infty(U)$ .

This shows that  $\phi_{y,\epsilon} \in C_c^\infty(U)$ .

Now we have

$$\begin{aligned} 0 &= \int_U \phi_{y,\epsilon} v dx \\ &= \int_U \eta_\epsilon(y - x) v(x) dx \\ &= v^\epsilon(y). \end{aligned}$$

Since this holds for all  $\epsilon \in (0, r)$ , we must have  $v(y) = \lim_{\epsilon \rightarrow 0} v^\epsilon(y) = 0$ .

Since this holds for a.e.  $y \in U$ , we have that  $v = 0$  a.e.. □

**Proposition 2.11.** If a weak derivative  $D^\alpha u$  exists, it is uniquely defined up to a set of measure zero.

*Proof.* Suppose  $v, \tilde{v}$  are both  $D^\alpha u$ , then  $\forall \phi \in C_c^\infty(U)$ ,

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi dx.$$

Thus  $\forall \phi \in C_c^\infty(U)$ ,  $\int_U (v - \tilde{v}) \phi dx = 0$ .

By the previous theorem, we have that  $v = \tilde{v}$  a.e.. □

**Proposition 2.12.** If  $u, v \in C(U)$  are both continuous, and  $u = v$  a.e., then  $\forall x \in U, u(x) = v(x)$ .

*Proof.* Consider any  $x \in U$ .

Since  $U$  is open, we can find some  $r > 0$ , such that  $B(x, r) \subseteq U$ .

For any  $i \geq \lceil \frac{1}{r} \rceil$ , we must have some  $x_i \in B(x, \frac{1}{i}) \subseteq U$ , such that  $u(x_i) = v(x_i)$ .

Otherwise  $\{x \in U : u(x) \neq v(x)\} \supseteq B(x, \frac{1}{i}) \cap U = B(x, \frac{1}{i})$  does not have measure 0.

Thus  $\lim_{i \rightarrow \infty} x_i = x$ .

Since  $u, v$  are both continuous, we have that

$$\begin{aligned} u(x) &= u\left(\lim_{i \rightarrow \infty} x_i\right) \\ &= \lim_{i \rightarrow \infty} u(x_i) \\ &= \lim_{i \rightarrow \infty} v(x_i) \\ &= v\left(\lim_{i \rightarrow \infty} x_i\right) \\ &= v(x). \end{aligned}$$

This is true for any  $x \in U$ , which completes the proof.  $\square$

**Proposition 2.13** (Continuous normal derivative is weak derivative). *Given any  $\forall u \in C^k(U)$ , and any  $|\alpha| \leq k$ . Namely, its  $\alpha^{th}$  normal derivative  $D^\alpha u$  exists and is continuous. For any  $v = u$  a.e., we have that  $D^\alpha u$  is an  $\alpha^{th}$  weak derivative of  $v$ .*

*Proof.* Consider any  $\phi \in C_c^\infty(U)$ , we have that

$$\begin{aligned} \int_U v D^\alpha \phi dx &= \int_U u D^\alpha \phi dx \\ &= (-1)^{|\alpha|} \int_U D^\alpha u \phi dx, \end{aligned}$$

where the second equality follows from integration by parts over some  $\text{Supp}(\phi) \subseteq V \subseteq U$  with Lipschitz boundary.  $\square$

**Definition 2.11.** Let  $k \in \mathbb{N}$ . We define

$$W^k(U) := \{u \in L^1_{loc}(U) : \forall |\alpha| \leq k, \exists v \in L^1_{loc}(U) \text{ such that } D^\alpha u = v \text{ in the weak sense}\}.$$

Namely, it is the set of functions in  $L^1_{loc}(U)$ , whose  $\alpha^{th}$ -weak derivative exists for each  $|\alpha| \leq k$ .

**Theorem 2.14.** Assume  $u, v \in W^k(U), |\alpha| \leq k$ , then

1.  $D^\alpha u \in W^{k-|\alpha|}(U)$ .
2.  $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u, \forall \alpha, \beta \text{ such that } |\alpha| + |\beta| \leq k$ .
3.  $\lambda u + v \in W^k(U), D^\alpha(\lambda u + v) = \lambda D^\alpha u + D^\alpha v, \forall \lambda \in \mathbb{R}$ .
4.  $\forall V \subseteq U \text{ be open}, u \in W^k(V)$ .

*Proof.* 1. This is by definition.

2. Consider any  $\phi \in C_c^\infty(U)$ , we have

$$\begin{aligned} \int_U D^\alpha(D^\beta u) \phi dx &= (-1)^{|\alpha|} \int_U D^\beta u D^\alpha \phi dx \\ &= (-1)^{|\alpha|} (-1)^{|\beta|} \int_U u D^\beta(D^\alpha \phi) dx \\ &= (-1)^{|\alpha+\beta|} \int_U u D^{\alpha+\beta} \phi dx \\ &= \int_U D^{\alpha+\beta} u \phi dx. \end{aligned}$$

Thus  $D^{\alpha+\beta} u = D^\alpha(D^\beta u)$ . Similarly,  $D^{\alpha+\beta} u = D^\beta(D^\alpha u)$ .  $\square$

**Proposition 2.15** (Leibniz rule for weak derivatives). *Assume  $u \in W^k(U)$ ,  $|\alpha| \leq k$ . If  $\xi \in C_c^\infty(U)$ ,  $\xi u \in W^k(U)$ , and the Leibniz formula holds:*

$$D^\alpha(\xi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u,$$

where  $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$ , and  $\alpha! := \alpha_1! \cdots \alpha_n!$ .

*Proof.* We have  $\forall \phi \in C_c^\infty(U)$ ,  $\int_U \xi u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha(\xi u) \phi dx$ .

We prove by induction:

The base case is  $|\alpha| = 1$ , we have by Leibniz rule on regular derivatives:

$$\begin{aligned} D^\alpha(\xi \phi) &= \xi D^\alpha \phi + \phi D^\alpha \xi \\ \int_U \xi u D^\alpha \phi dx &= \int_U u(D^\alpha(\xi \phi) - \phi D^\alpha \xi) dx \\ &= \int_U u D^\alpha(\xi \phi) dx - \int_U u \phi D^\alpha \xi dx \\ &= - \int_U \xi \phi D^\alpha u dx - \int_U u \phi D^\alpha \xi dx \\ &= - \int_U \phi(\xi D^\alpha u + u D^\alpha \xi) dx. \end{aligned}$$

Since this hold for any  $\phi \in C_c^\infty(U)$ , we have

$$\xi D^\alpha u + u D^\alpha \xi = D^\alpha(u \xi).$$

Now suppose  $l < k$  and the result holds  $\forall |\beta| \leq l$ .

Consider any  $|\alpha| = l+1$ , we have  $\alpha = \beta + \gamma$  where  $|\beta| = l$ ,  $|\gamma| = 1$ .

$$\begin{aligned} \int_U \xi u D^\alpha \phi dx &= \int_U \xi u D^\beta(D^\gamma \phi) dx \\ &= (-1)^{|\beta|} \int_U D^\beta(\xi u) D^\gamma \phi dx \\ &= (-1)^{|\beta|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\beta-\sigma} u D^\gamma \phi dx \\ &= (-1)^{|\beta|} (-1)^{|\gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\gamma(D^\sigma \xi D^{\beta-\sigma} u) \phi dx \\ &= (-1)^{|\beta+\gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} (D^\sigma \xi D^\gamma D^{\beta-\sigma} u + D^{\beta-\sigma} u D^\gamma D^\sigma \xi) \phi dx \\ &= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} (D^\sigma \xi D^{\gamma+\beta-\sigma} u + D^{\beta-\sigma} u D^{\gamma+\sigma} \xi) \phi dx \\ &= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} (D^\sigma \xi D^{\alpha-\sigma} u + D^{\alpha-(\gamma+\sigma)} u D^{\gamma+\sigma} \xi) \phi dx \\ &= (-1)^{|\alpha|} \int_U \left( \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u + \sum_{\rho \leq \alpha, \rho_j \geq 1} \binom{\beta}{\rho-\gamma} D^{\alpha-\rho} u D^\rho \xi \right) \phi dx \\ &= (-1)^{|\alpha|} \int_U \left( \sum_{\sigma \leq \alpha} \left( \mathbb{1}_{\sigma_j \leq \alpha_j-1} \binom{\beta}{\sigma} + \mathbb{1}_{\sigma_j \geq 1} \binom{\beta}{\sigma-\gamma} \right) D^\sigma \xi D^{\alpha-\sigma} u \right) \phi dx, \end{aligned}$$

where  $\gamma_i = \delta_{ij}$ .

Now consider any  $\sigma \leq \alpha$ .

If  $\sigma_j = 0$ , we have

$$\begin{aligned} \binom{\beta}{\sigma} &= \frac{\beta!}{\sigma!(\beta-\sigma)!} \\ &= \frac{\beta!(\beta_j+1)}{\sigma!(\beta-\sigma)!(\beta_j+\sigma_j+1)} \\ &= \frac{(\beta+\gamma)!}{\sigma!(\beta-\sigma+\gamma)!} \\ &= \frac{\alpha!}{\sigma!(\alpha-\sigma)!} \\ &= \binom{\alpha}{\sigma}. \end{aligned}$$

If  $\sigma_j = \alpha_j$ , we have

$$\begin{aligned} \binom{\beta}{\sigma-\gamma} &= \frac{\beta!}{(\sigma-\gamma)!(\beta-\sigma+\gamma)!} \\ &= \frac{\beta!\alpha_j}{\alpha_j(\sigma-\gamma)!(\alpha-\sigma)!} \\ &= \frac{\beta!(\beta_j+1)}{\sigma_j(\sigma-\gamma)!(\alpha-\sigma)!} \\ &= \frac{(\beta+\gamma)!}{(\sigma-\gamma+\gamma)!(\alpha-\sigma)!} \\ &= \frac{\alpha!}{\sigma!(\alpha-\sigma)!} \\ &= \binom{\alpha}{\sigma}. \end{aligned}$$

If  $1 \leq \sigma_j \leq \alpha_j - 1$ , we have

$$\begin{aligned} \binom{\beta}{\sigma} + \binom{\beta}{\sigma-\gamma} &= \frac{\beta!}{\sigma!(\beta-\sigma)!} + \frac{\beta!}{(\sigma-\gamma)!(\beta-\sigma+\gamma)!} \\ &= \frac{\beta!(\beta_j-\sigma_j+1)}{\sigma!(\beta-\sigma)!(\beta_j-\sigma_j+1)} + \frac{\beta!\sigma_j}{\sigma_j(\sigma-\gamma)!(\beta-\sigma+\gamma)!} \\ &= \frac{\beta!(\beta_j-\sigma_j+1) + \beta!\sigma_j}{\sigma!(\beta-\sigma+\gamma)!} \\ &= \frac{\beta!(\beta_j+1)}{\sigma!(\beta-\sigma+\gamma)!} \\ &= \frac{(\beta+\gamma)!}{\sigma!(\beta-\sigma+\gamma)!} \\ &= \frac{\alpha!}{\sigma!(\alpha-\sigma)!} \\ &= \binom{\alpha}{\sigma}. \end{aligned}$$

Thus we can see that

$$\int_U \xi u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U \left( \sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u \right) \phi dx.$$

Since  $\phi$  is arbitrary, we have that

$$D^\alpha(\xi u) = \sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u.$$

Inductively, we can prove this for any  $|\alpha| = n \geq 1$ .  $\square$

**Proposition 2.16.** For  $u, v \in L^1_{loc}(U)$ , suppose  $v|_V = D^\alpha(u|_V)$  on every  $V \subset\subset U$ , then  $v$  is the  $\alpha^{th}$  weak derivative of  $u$  on globally. i.e.  $v = D^\alpha u$ .

*Proof.* Consider any  $\phi \in C_c(U)$ , we have that  $\text{Supp}(\phi) \subset\subset U$ , so we can find a  $\text{Supp}(\phi) \subset\subset V \subset\subset U$ .

Now

$$\int_U \phi v dx = \int_V \phi v dx = (-1)^{|\alpha|} \int_V D^\alpha \phi v dx = (-1)^{|\alpha|} \int_U D^\alpha \phi v dx,$$

since  $v|_V = D^\alpha(u|_V)$  and  $\phi$  is constantly 0 outside of  $V$ .  $\square$

### 2.3.2 Sobolev Spaces

**Definition 2.12.** Let  $k \in \mathbb{N}, 1 \leq p \leq \infty, u \in W^{k,p}(U)$ . The **Sobolev norm** of  $u$  is

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(U)}^p \right)^{1/p}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \text{ess sup}_{x \in U} |D^\alpha u(x)| \cong \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(U)}, & p = \infty \end{cases}.$$

**Definition 2.13.** For  $k = 1$ , we write

$$\|Du\|_{L^p(U)}^p := \int_U \|Du\|_p^p dx = \int_U \sum_{i=1}^n |\partial_i u|^p dx = \sum_{i=1}^n \|\partial_i u\|_{L^p(U)}^p$$

for  $1 \leq p < \infty$ , and

$$\|Du\|_{L^\infty(U)} := \text{ess sup}_{x \in U} \|Du(x)\|_1 = \text{ess sup}_{x \in U} \sum_{i=1}^n |\partial_i u(x)| = \sum_{i=1}^n \|\partial_i u\|_{L^\infty(U)}$$

for  $p = \infty$ .

In this case,

$$\|u\|_{W^{1,p}(U)} = \begin{cases} \left( \|u\|_{L^p(U)}^p + \|Du\|_{L^p(U)}^p \right)^{1/p} & 1 \leq p < \infty \\ \|u\|_{L^\infty(U)} + \|Du\|_{L^\infty(U)} & p = \infty. \end{cases}$$

**Proposition 2.17.** Let  $k \in \mathbb{N}, 1 \leq p \leq \infty, u \in L^1_{loc}(U)$ , we have

$$\forall |\alpha| \leq k, \|u\|_{W^{k,p}(U)} \geq \|D^\alpha u\|_{L^p(U)}.$$

**Definition 2.14.** The **Sobolev space** is defined as

$$W^{k,p}(U) := \left\{ v \in L^1_{loc}(U) : \|v\|_{W^{k,p}(U)} < \infty \right\}.$$

**Proposition 2.18.** Let  $k \in \mathbb{N}, 1 \leq p \leq \infty, u \in L^1_{loc}(U)$ , then  $u \in W^{k,p}(U)$  if and only if

$$\forall |\alpha| \leq k, D^\alpha u \in L^p(U),$$

where  $D^\alpha u \in L^p(U)$  is the  $\alpha^{th}$  weak derivative of  $u$ .

**Definition 2.15.** In the special case where  $p = 2$ , we write

$$H^k(U) := W^{k,2}(U).$$

*Remark.*

$$W^{0,1}(U) = H^0(U) = L^2(U).$$

**Definition 2.16.** Let  $(u_m)_{m=1}^\infty, u \in W^{k,p}(U)$ , then

- $u_m \rightarrow u$  in  $W^{k,p}(U)$  if  $\lim_{m \rightarrow \infty} \|u_m - u\|_{W^{k,p}(U)} = 0$ .
- $u_m \rightarrow u$  in  $W_{loc}^{k,p}(U)$  if  $u_m \rightarrow u$  in  $W^{k,p}(V)$  for all  $V \subset\subset U$ .

**Theorem 2.19.** Assume  $u, v \in W^{k,p}(U), |\alpha| \leq k$ , then

1.  $D^\alpha u \in W^{k-|\alpha|,p}(U)$ .
2.  $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u, \forall \alpha, \beta$  such that  $|\alpha| + |\beta| \leq k$ .
3.  $\lambda u + v \in W^{k,p}(U), D^\alpha(\lambda u + v) = \lambda D^\alpha u + D^\alpha v, \forall \lambda \in \mathbb{R}$ .
4.  $\forall V \subseteq U$  be open,  $u \in W^{k,p}(V)$ .
5. Assume  $u \in W^{k,p}(U), |\alpha| \leq k$ . If  $\xi \in C_c^\infty(U), \xi u \in W^{k,p}(U)$ , and the Leibniz formula holds:

$$D^\alpha(\xi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u,$$

where  $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$ , and  $\alpha! := \alpha_1! \cdots \alpha_n!$ .

**Theorem 2.20.**  $(W^{k,p}(U), \|\cdot\|_{W^{k,p}(U)})$  is a Banach space for  $k \in \mathbb{N}, 1 \leq p \leq \infty$ .

*Proof.* See A2 for the proof of  $\|\cdot\|_{W^{1,\infty}(U)}$  is a norm.

Now for  $1 \leq p < \infty$ , we want to check:

1. If  $\|u\|_{W^{k,p}(U)} = 0$ , then  $\|u\|_{L^p(U)} = 0$ , and thus  $u = 0$  a.e. on  $U$ .
2. If  $u = 0$  a.e. on  $U$ , then  $\forall \phi \in C_c^\infty(U)$ , we have

$$\int_U D^\alpha u \phi dx = (-1)^{|\alpha|} \int_U u D^\alpha \phi dx = 0.$$

Thus  $D^\alpha u = 0$  a.e. for any  $|\alpha| \leq k$ .

Thus  $\|u\|_{W^{k,p}(U)} = 0$ .

3. Let  $\lambda \in \mathbb{R}$ , we have

$$\begin{aligned} \|\lambda u\|_{W^{k,p}(U)} &= \left( \sum_{|\alpha| \leq k} \|D^\alpha(\lambda u)\|_{L^p(U)}^p \right)^{1/p} \\ &= \left( \sum_{|\alpha| \leq k} \|\lambda D^\alpha(u)\|_{L^p(U)}^p \right)^{1/p} \\ &= \left( \sum_{|\alpha| \leq k} |\lambda|^p \|D^\alpha(u)\|_{L^p(U)}^p \right)^{1/p} \\ &= |\lambda| \left( \sum_{|\alpha| \leq k} \|D^\alpha(u)\|_{L^p(U)}^p \right)^{1/p} \\ &= |\lambda| \|u\|_{W^{k,p}(U)}. \end{aligned}$$

4. Consider any  $u, v \in W^{k,p}(U)$ ,

$$\begin{aligned} \|u + v\|_{W^{k,p}(U)} &= \left( \sum_{|\alpha| \leq k} \|D^\alpha(u + v)\|_{L^p(U)}^p \right)^{1/p} \\ &\leq \left( \sum_{|\alpha| \leq k} \left( \|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)} \right)^p \right)^{1/p} \\ &\leq \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(U)}^p \right)^{1/p} + \left( \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p(U)}^p \right)^{1/p} \\ &= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}. \end{aligned}$$

Thus  $\|\cdot\|_{W^{k,p}(U)}$  is a norm.

Consider any Cauchy sequence  $(u_m)_{m=1}^\infty$ .

Given any  $\epsilon > 0$ ,  $\exists N \geq 1$ , such that  $\forall n, m \geq N$ ,  $\|u_m - u_n\|_{W^{k,p}(U)} < \epsilon$ .

Consider any  $|\alpha| \leq k$ , we have

$$\|D^\alpha u_m - D^\alpha u_n\|_{L^p(U)} = \|u\|_{W^{k,p}(U)} \geq \|D^\alpha(u_m - u_n)\|_{L^p(U)} \leq \|u_m - u_n\|_{W^{k,p}(U)} < \epsilon.$$

Thus  $(D^\alpha u_n)_{n=1}^\infty$  must be a Cauchy sequence in  $(L^p(U), \|\cdot\|_{L^p(U)})$  for any  $|\alpha| \leq k$ .

Since  $(L^p(U), \|\cdot\|_{L^p(U)})$  is complete, there must be some

$$u_\alpha \in L^p(U) \text{ such that } \lim_{n \rightarrow \infty} \|u_\alpha - D^\alpha u_n\|_{L^p(U)} = 0.$$

In particular, we have  $u \in L^p(U)$ , such that  $\lim_{n \rightarrow \infty} \|u - u_n\|_{L^p(U)} = 0$ .

Now consider any  $|\alpha| \leq k$ .

Given any  $\phi \in C_c^\infty(U)$ , we have

$$\begin{aligned} \left| \int_U u D^\alpha \phi dx - \int_U u_n D^\alpha \phi dx \right| &= \left| \int_U (u - u_n) D^\alpha \phi dx \right| \\ &\leq \int_U |(u - u_n) D^\alpha \phi| dx \\ &\leq \|u - u_n\|_{L^p(U)} \|D^\alpha \phi\|_{L^{\frac{p}{p-1}}(U)}, \\ \left| \int_U u_\alpha \phi dx - \int_U D^\alpha u_n \phi dx \right| &= \left| \int_U (u_\alpha - D^\alpha u_n) \phi dx \right| \\ &\leq \int_U |(u_\alpha - D^\alpha u_n) \phi| dx \\ &\leq \|u_\alpha - D^\alpha u_n\|_{L^p(U)} \|\phi\|_{L^{\frac{p}{p-1}}(U)}. \end{aligned}$$

Since  $u_n \rightarrow u$ ,  $D^\alpha u_n \rightarrow u_\alpha$  in  $L^p(U)$ , and  $\|\phi\|_{L^{\frac{p}{p-1}}(U)}, \|D^\alpha \phi\|_{L^{\frac{p}{p-1}}(U)} < \infty$ , those two limits converges to 0.

Thus we have

$$\begin{aligned} \int_U u D^\alpha \phi dx &= \lim_{n \rightarrow \infty} \int_U u_n D^\alpha \phi dx \\ &= \lim_{n \rightarrow \infty} (-1)^{|\alpha|} \int_U D^\alpha u_n \phi dx \\ &= (-1)^{|\alpha|} \int_U u_\alpha \phi dx. \end{aligned}$$

Since this is true for any  $\phi \in C_c^\infty(U)$ , we have that  $D^\alpha u = u_\alpha = \lim_{n \rightarrow \infty} D^\alpha u_n$  in  $L^p(U)$ .  
Since this is true for any  $|\alpha| \leq k$ , we have that  $u_n \rightarrow u$  in  $W^{k,p}(U)$ .  $\square$

**Proposition 2.21.** For any  $1 \leq s \leq r < \infty, k \geq 1$ , and bounded  $U$ , we have some constant  $C := |U|^{\frac{1}{s} - \frac{1}{r}} m^{\frac{1}{s} - \frac{1}{r}}$ , where  $m = |\{\beta \in \mathbb{N}^n : |\beta| \leq k\}|$ , such that

$$\forall u \in W^{k,r}(U), \|u\|_{W^{k,s}(U)} \leq C \|u\|_{W^{k,r}(U)}.$$

Thus,  $W^{k,r}(U) \subseteq W^{k,s}(U)$ .

*Proof.* We have

$$\begin{aligned} \|u\|_{W^{1,s}(U)}^s &= \sum_{|\beta| \leq 1} \|D^\beta u\|_{L^s(U)}^s \\ &\leq \sum_{|\beta| \leq 1} \left( |U|^{\frac{1}{s} - \frac{1}{r}} \|D^\beta u\|_{L^r(U)} \right)^s \\ &= \left( |U|^{\frac{1}{s} - \frac{1}{r}} \right)^s \sum_{|\beta| \leq 1} \|D^\beta u\|_{L^r(U)}^{r \cdot \frac{s}{r}} \\ &\leq \left( |U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \left( \sum_{|\beta| \leq 1} \|D^\beta u\|_{L^r(U)}^r \right)^{\frac{s}{r}} \\ &= \left( |U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \left( \|D^\alpha u\|_{W^{1,r}(U)}^r \right)^{\frac{s}{r}} \\ &= \left( |U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \|D^\alpha u\|_{W^{1,r}(U)}^s \\ &\implies \|u\|_{W^{1,s}(U)} \leq |U|^{\frac{1}{s} - \frac{1}{r}} m^{\frac{1}{s} - \frac{1}{r}} \|D^\alpha u\|_{W^{1,r}(U)}. \end{aligned}$$

□

**Definition 2.17.**

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)} = \{u \in W^{k,p}(U) : \exists (u_m)_{m=1}^\infty \subset C_c^\infty(U) \text{ such that } u_m \rightarrow u \text{ in } W^{k,p}(U)\}.$$

$$H_0^k(U) = W_0^{k,2}.$$

*Remark.*  $W_0^{k,p}(U)$  are those  $u \in W^{k,p}(U)$  such that  $D^\alpha u = 0$  on  $\partial U$  for all  $|\alpha| \leq k$ .

## 2.4 Smooth Approximation

**Proposition 2.22** (weak derivative commute with mollification). Let  $1 \leq p \leq \infty, k \geq 1$ . For any  $u \in W^{k,p}(U)$ , and  $|\alpha| \leq k, \epsilon > 0$ , we have that

$$D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$$

on  $U_\epsilon$ .

*Proof.* Fix any  $x \in U_\epsilon$ , we have

$$\begin{aligned} D^\alpha u^\epsilon(x) &= D^\alpha(\eta_\epsilon * u)(x) \\ &= (D^\alpha \eta_\epsilon * u)(x) \quad 2.7 \\ &= \int_U D^\alpha \eta_\epsilon(x-y) u(y) dy, \end{aligned}$$

Consider  $\eta_{\epsilon,x}(y) := \eta_\epsilon(x - y)$ , we can see  $\forall i \in [n]$ ,  $\partial_i \eta_{\epsilon,x}(y) = -\partial_i \eta_\epsilon(x - y)$ , thus we have

$$\begin{aligned} D^\alpha u^\epsilon(x) &= \int_U D^\alpha \eta_\epsilon(x - y) u(y) dy \\ &= (-1)^{|\alpha|} \int_U D^\alpha \eta_{\epsilon,x}(y) u(y) dy \\ &= \int_U \eta_{\epsilon,x}(y) D^\alpha u(y) dy \\ &= \int_U \eta_\epsilon(x - y) D^\alpha u(y) dy \\ &= (\eta_\epsilon * D^\alpha u)(x). \end{aligned}$$

Since this holds for any  $x \in U_\epsilon$ , we proved our result.  $\square$

**Corollary 2.23.** Let  $1 \leq p < \infty, k \geq 1$ . Let  $u \in W^{k,p}(U)$ , then for any  $\epsilon > 0$ ,  $U_\epsilon \supseteq V \supseteq \text{Supp}(u) + \bar{B}(0, \epsilon)$ , we have that

$$\|u^\epsilon\|_{W^{k,p}(V)} \leq \|u\|_{W^{k,p}(V)}.$$

*Proof.* By 2.22,  $\forall |\alpha| \leq k$ , we have  $D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$  on the entire  $U_\epsilon$  and thus on  $V$ .

Since  $\forall |\alpha| \leq k$ ,  $\text{Supp}(D^\alpha u) \subseteq \text{Supp}(u)$ , we have  $\text{Supp}(\eta_\epsilon * D^\alpha u) \subseteq \text{Supp}(u) + \bar{B}(0, \epsilon) \subseteq V$ . By 2.8,

$$\begin{aligned} \|u^\epsilon\|_{W^{k,p}(V)}^p &= \sum_{|\alpha| \leq k} \|D^\alpha u^\epsilon\|_{L^p(V)} \\ &= \sum_{|\alpha| \leq k} \|\eta_\epsilon * D^\alpha u\|_{L^p(V)} \\ &\leq \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(V)} \\ &= \|u\|_{W^{k,p}(V)}^p. \end{aligned}$$

$\square$

**Theorem 2.24** (Local Smooth Approximation). Let  $1 \leq p < \infty, k \geq 1$ . Suppose  $U$  is open, and  $u \in W^{k,p}(U)$ , we have that

1.  $\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$ ,
2.  $u^\epsilon \rightarrow u$  in  $W_{loc}^{k,p}(U)$  as  $\epsilon \rightarrow 0$ .

*Proof.*  $\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$  by 2.7.1.

Given any  $V \subset\subset U$ , we can find some  $\epsilon_V > 0$  such that  $V \subset\subset U^{\epsilon_V}$ .

Consider any  $|\alpha| \leq k$ .

We have  $D^\alpha u \in L^p(U) \subseteq L_{loc}^p(U)$ .

By 2.7.5, we have that  $\eta_\epsilon * D^\alpha u \rightarrow D^\alpha u$  in  $L_{loc}^p(U)$  as  $\epsilon \rightarrow 0$ , and thus  $\eta_\epsilon * D^\alpha u \rightarrow D^\alpha u$  in  $L^p(V)$ .

In addition, by 2.22,  $\forall \epsilon > 0$ ,  $D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$  in  $U^\epsilon$ .

Now  $\forall 0 < \epsilon < \epsilon_V$ ,  $V \subset\subset U^{\epsilon_V} \subseteq U^\epsilon$ , and thus  $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$  in  $V$ .

Thus  $D^\alpha u^\epsilon \rightarrow D^\alpha u$  in  $L^p(V)$  as  $\epsilon \rightarrow 0$ .

Since this is true  $\forall |\alpha| \leq k$ , we have  $u^\epsilon \rightarrow u$  in  $W^{k,p}(V)$ .

Since this holds for any  $V \subset\subset U$ ,  $u^\epsilon \rightarrow u$  in  $W_{loc}^{k,p}(U)$ .  $\square$

**Corollary 2.25.** Suppose  $U$  is open, and  $u \in W^{k,p}(U)$  is compactly supported in  $U$ , then  $u \in W_0^{k,p}(U)$ .

*Proof.* Since  $\text{Supp}(u) \subset U$  is compact, we must have  $r := \frac{1}{2} \text{dist}(\text{Supp}(u), \partial U) > 0$ .

For  $n \in \mathbb{N}^+$ , let  $u_n := u^{\frac{r}{n}}$ .

We have that  $u_n \rightarrow u$  in  $W_{loc}^{k,p}(U)$  as  $n \rightarrow \infty$ .

Let  $W := \overline{\text{Supp}(u) + \bar{B}(0, r/2)} \subset U$ . Notice that it is compact, and  $\forall n \in \mathbb{N}^+, \text{Supp}(u_n) \subseteq \text{Supp}(u) +$

$\bar{B}(0, \frac{r}{n}) \subseteq W$ , which means  $u_m \in C_c^\infty(U)$ .

Now there is some  $W \subset V \subset\subset U$ , so  $u_m \rightarrow u$  in  $W^{k,p}(V)$ .

In addition,

$$\begin{aligned}\|u - u_m\|_{W^{k,p}(U)}^p &= \int_U \sum_{|\alpha| \leq k} |D^\alpha(u - u_m)|^p dx \\ &= \int_V \sum_{|\alpha| \leq k} |D^\alpha(u - u_m)|^p dx \\ &= \|u - u_m\|_{W^{k,p}(V)}^p.\end{aligned}$$

Thus  $\lim_{m \rightarrow \infty} \|u - u_m\|_{W^{k,p}(V)} = \lim_{m \rightarrow \infty} \|u - u_m\|_{W^{k,p}(U)} = 0$ .

Since each  $u_m \in C_c^\infty(U)$ , we have  $u \in \overline{C_c^\infty(U)} = W_0^{k,p}(U)$ .  $\square$

**Theorem 2.26** (Meyer-Serrin). *Let  $1 \leq p < \infty, k \geq 1$ . Suppose  $U$  is open and bounded, and  $u \in W^{k,p}(U)$ . There exists  $(u_m)_{m=1}^\infty \subseteq C^\infty(U) \cap W^{k,p}(U)$  such that  $u_m \rightarrow u$  in  $W^{k,p}(U)$ .*

*Proof.* Let  $\delta > 0$  be given.

Let  $U_i := U_{\frac{1}{i}} = \{x \in U : \text{dist}(x, \partial U) > \frac{1}{i}\}$  for  $i \in \mathbb{N}^+$ .

We have  $U_1 \subseteq \bar{U}_1 \subseteq U_2 \subseteq \bar{U}_2 \subseteq U_3 \subseteq \dots$ .

Indeed, for some  $x \in \bar{U}_i$ , we know that  $\forall y \in \partial U, \|x - y\| \geq \frac{1}{i} > \frac{1}{i+1} \implies x \in U_{i+1}$ .

Since  $U$  is open, for any  $x \in U$ , we can find some  $i \geq 1$ , such that  $B(x, \frac{1}{i}) \subseteq U$ , which means  $\text{dist}(x, \partial U) \geq \frac{1}{i}$ , and thus  $x \in \bar{U}_i \subseteq U_{i+1}$ . Thus we have  $U = \bigcup_{i=1}^\infty U_i$ .

Let  $V_i := U_{i+3} \setminus U_{i+1}^-$  for  $i \in \mathbb{N}^+$ . Since  $U$  is bounded, we can choose  $V_0 \subset\subset U$  with  $V_0 \supset \bar{U}_2$ , we claim that  $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$ .

It is easy to see  $\bigcup_{i=0}^n V_i \subseteq U_{i+3}$ . For the other direction, we will prove by induction.

The base case  $n = 1$ , we can see that  $V_0 \cup V_1 \supset \bar{U}_2 \cup (U_4 \setminus \bar{U}_2) = U_4$ .

Now suppose  $n > 1$ , and it holds for  $n - 1$ , we have that

$$\begin{aligned}\bigcup_{i=0}^n V_i &= \left( \bigcup_{i=0}^{n-1} V_i \right) \cup (V_n) \\ &= U_{n-1+3} \cup (U_{n+3} \setminus U_{n+1}^-) \\ &\supset U_{n+2} \cup (U_{n+3} \setminus U_{n+2}) \\ &= U_{n+3}.\end{aligned}$$

By induction, we have that  $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$ .

Notice that  $\forall x \in U = \bigcup_{n=1}^\infty U_n, \exists n \geq 1$ , such that  $x \in U_n \subseteq U_{n+3} \subseteq \bigcup_{i=0}^n V_i \implies \exists i \geq 0$ , such that  $x \in V_i$ . Thus

$$U = \bigcup_{i=0}^\infty V_i.$$

Now let  $W_i := U_{i+4} \setminus \bar{U}_i$  for  $i \in \mathbb{N}^+$ .

Since each  $U_{i+4} \subseteq \bar{U}_{i+4} \subseteq U_{i+5} \subseteq U$ , we also have  $U_{i+4} \subset\subset U$  and thus

$$W_i \subset\subset U.$$

Notice that  $\forall x, y \in U$ ,

$$\begin{aligned}\text{dist}(x, \partial U) &= \inf \{\|z - x\| : z \in \partial U\} \\ &= \inf \{\|z - y + y - x\| : z \in \partial U\} \\ &\leq \inf \{\|z - y\| + \|y - x\| : z \in \partial U\} \\ &= \inf \{\|z - y\| : z \in \partial U\} + \|y - x\| \\ &= \text{dist}(y, \partial U) + \|y - x\|.\end{aligned}$$

Similarly,  $\text{dist}(y, \partial U) \leq \text{dist}(x, \partial U) + \|y - x\|$ . Thus we have

$$\text{dist}(y, \partial U) - \|y - x\| \leq \text{dist}(x, \partial U) \leq \text{dist}(y, \partial U) + \|y - x\|.$$

Consider any  $0 < \epsilon < \frac{1}{i+3} - \frac{1}{i+4} < \frac{1}{i} - \frac{1}{i+1}$ , we have that

$$\begin{aligned} x \in \bar{B}(0, \epsilon) + V_i &\implies \exists y \in U_{i+3} \setminus U_{i+1}^- \text{ such that } \|x - y\| \leq \epsilon \\ &\implies \exists y \in U \text{ such that } \frac{1}{i+3} < \text{dist}(y, \partial U) < \frac{1}{i+1}, \|x - y\| \leq \epsilon \\ &\implies \exists y \in U \text{ such that } \frac{1}{i+3} - \|x - y\| < \text{dist}(x, \partial U) < \frac{1}{i+1} + \|x - y\|, \|x - y\| \leq \epsilon \\ &\implies \frac{1}{i+3} - \epsilon < \text{dist}(x, \partial U) < \frac{1}{i+1} + \epsilon \\ &\implies \frac{1}{i+4} < \text{dist}(x, \partial U) < \frac{1}{i} \\ &\implies x \in W_i. \end{aligned}$$

Thus we have

$$\forall 0 < \epsilon < \frac{1}{i+3}, \bar{B}(0, \epsilon) + V_i \subseteq W_i.$$

Finally, since  $V_0 \subset\subset U$ , we can choose  $V_0 \subset\subset W_0 \subset\subset U$ , such that  $V_0 + B(0, \epsilon_0'') \subseteq W_i$  for some  $\epsilon_0' > 0$ . Let  $(\zeta_i)_{i=0}^\infty$  be a **smooth partition of unity** such that

$$\forall x \in U \sum_{i=0}^{\infty} \zeta_i(x) = 1, \quad \forall i \geq 0, \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^\infty(U), \\ \text{Supp}(\zeta_i) \subseteq V_i. \end{cases}$$

Notice that  $\forall u \in W^{k,p}(U)$ , we have  $\zeta_i u \in W^{k,p}(U)$  as well. Moreover,  $\text{Supp}(\zeta_i u) \subseteq V_i$ .

Let  $u_i^\epsilon := \eta_\epsilon * (\zeta_i u) \forall \epsilon > 0$ .

By previous theorem, we have that  $u_i^\epsilon \rightarrow \zeta_i u$  in  $W_{loc}^{k,p}(U)$ .

Thus for  $W_i \subset\subset U$ , we can find  $\epsilon_i' > 0$  such that  $\forall \epsilon < \epsilon_i'$ ,  $\|u_i^\epsilon - \zeta_i u\|_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}$ .

Now pick  $\epsilon_0 = \min(\epsilon_0'', \epsilon_0')$ ,  $\forall i \in \mathbb{N}^+$ ,  $\epsilon_i = \frac{1}{2} \min\left(\frac{1}{i+3} - \frac{1}{i+4}, \epsilon_i'\right) > 0$ .

We have that by 2.7,

$$\text{Supp}(u_i^{\epsilon_i}) \subseteq \text{Supp}(\eta_{\epsilon_i}) + \text{Supp}(\zeta_i u) \subseteq \bar{B}(0, \epsilon_i) + V_i \subseteq W_i,$$

and

$$\|u_i^{\epsilon_i} - \zeta_i u\|_{W^{k,p}(U)} = \|u_i^{\epsilon_i} - \zeta_i u\|_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}.$$

Now let  $v := \sum_{i=0}^{\infty} u_i^{\epsilon_i}$ .

Notice that  $\forall x \in U, \exists V \subset\subset U_{\epsilon_i}$  be open, such that  $x \in V$ . Since  $V \cap W_i \neq \emptyset$  for only finitely many  $i$ , and  $\text{Supp}(u_i^{\epsilon_i}) \subseteq W_i$ , we must have  $v = \sum_{i=0}^k u_i^{\epsilon_i}$  on  $V$  for some finite  $k$ .

In addition, by 2.7, each  $u_i^{\epsilon_i} \in C^\infty(U_{\epsilon_i})$ , thus infinitely differentiable at  $x$ .

Thus  $v = \sum_{i=0}^k u_i^{\epsilon_i}$  is infinitely differentiable at  $x$ .

Since  $x \in U$  is arbitrary, we have that  $v \in C^\infty(U)$ .

In addition,

$$\forall x \in U, u(x) = \sum_{i=0}^{\infty} \zeta_i(x) u(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x)$$

by definition of partition of unity. Thus

$$u(x) - v(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x) - \sum_{i=0}^{\infty} u_i^{\epsilon_i}(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x) - u_i^{\epsilon_i}(x) = \sum_{i=0}^{\infty} (\zeta_i u - u_i^{\epsilon_i})(x).$$

Since this holds for all  $x \in U$ , we have that

$$u - v = \sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i}.$$

Now we have

$$\begin{aligned} \|v - u\|_{W^{k,p}(U)} &= \left\| \sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i} \right\|_{W^{k,p}(U)} \\ &\leq \sum_{i=0}^{\infty} \|\zeta_i u - u_i^{\epsilon_i}\|_{W^{k,p}(U)} \\ &< \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} \\ &= \delta. \end{aligned}$$

□

**Definition 2.18.** Let  $U \subseteq \mathbb{R}^n$  be open and bounded, then  $\partial U$  is  $C^k$  if  $\forall z \in \partial U, \exists r > 0, \gamma \in C^k(\mathbb{R}^{n-1})$ , such that

$$U \cap \bar{B}(z, r) = \{x \in B(z, r) : x^n > \gamma(x^1, \dots, x^{n-1})\}.$$

**Theorem 2.27.** Let  $U$  be bounded, and  $\partial U$  is  $C^1$ , then  $\forall u \in W^{k,p}(U)$  for  $1 \leq p < \infty$ , there exists functions  $u_m \in C^\infty(\bar{U})$  such that  $u_m \rightarrow u$  in  $W^{k,p}(U)$ .

*Proof.* See 5.3.3 in Eavan's book. □

## 2.5 Extensions

**Proposition 2.28.** Let  $U \subseteq \mathbb{R}^n$  be open and bounded, with  $\partial U$  be  $C^k$ . Then  $\forall z \in \partial U, \exists r > 0, \Phi \in C^k(B(z, r), \mathbb{R}^n)$  a diffeomorphism, such that  $\Phi(\partial U \cap B(z, r))$  is in a flat hyperplane, and  $\det(D\Phi) = \det(D\Psi) = 1$ , for  $\Psi := \Phi^{-1}$ .

*Proof.* Let

$$\Phi^i(x) := x^i \quad \forall i \in [n-1], \quad \Phi^n(x) := x^n - \gamma(x^1, \dots, x^{n-1}),$$

and let

$$\Psi^i(y) := y^i \quad \forall i \in [n-1], \quad \Psi^n(y) := y^n + \gamma(y^1, \dots, y^{n-1}).$$

□

**Theorem 2.29** (Sobolev Norm Equivalence Under Diffeomorphism). Let  $W \subseteq \mathbb{R}^n$  and  $\Phi$  is a  $C^1(W)$  diffeomorphism, i.e., it has inverse  $\Psi \in C^1(W)$ . Let  $v := u \circ \Psi$ , then

$$\exists C_0, C_1 \text{ such that } C_0 \|u\|_{W^{1,p}(W)} \leq \|v\|_{W^{1,p}(\Phi(W))} \leq C_1 \|u\|_{W^{1,p}(W)}.$$

**Lemma 2.30.** Let  $1 \leq p < \infty$ . Assume  $U$  is bounded, with  $\partial U$  be  $C^1$ . Let  $V$  be open and bounded, with  $U \subset\subset V$ , then there exists a bounded linear operator  $E : C^1(\bar{U}) \rightarrow W^{1,p}(\mathbb{R}^n)$ , such that  $\forall u \in C^1(\bar{U})$ :

1.  $Eu = u$  in  $U$ ,
2.  $\text{Supp}(Eu) \subseteq V$ ,
3.  $\exists C > 0$ , such that  $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)}$ .

*Proof.* Fix  $z \in \partial U$ .

In addition, we assume  $\partial U$  is flat around  $z$  on the plane  $\{x^n = 0\}$ .

Then there exists an open ball  $B := B(z, r)$ , such that

$$B^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B^- := B \cap \{x^n \leq 0\} \subseteq \mathbb{R}^n \setminus U.$$

$$\text{Let } \bar{u}_z(x) := \begin{cases} u(x) & x \in B^+ \\ -3u(x^1, \dots, x^{n-1}, -x^n) + 4u(x^1, \dots, x^{n-1}, -\frac{1}{2}x^n) & x \in B^- \end{cases}$$

Then we claim  $\bar{u}_z \in C^1(B)$ .

Indeed, let  $u^- := \bar{u}_z|_{B^-}, u^+ := \bar{u}_z|_{B^+}$ .

$$\begin{aligned} u^-|_{x^n=0} &= -3 + 4u|_{x^n=0} \\ &= u|_{x^n=0} \\ &= u^+|_{x^n=0}; \\ \forall i \in [n-1], \quad & \\ \partial_i u^-|_{x^n=0} &= -3\partial_i u|_{x^n=0} + 4\partial_i u|_{x^n=0} \\ &= \partial_i u|_{x^n=0} \\ &= \partial_i u^+|_{x^n=0} \\ \partial_n u^-|_{x^n=0} &= 3\partial_n u|_{x^n=0} - 4\frac{1}{2}\partial_n u|_{x^n=0} \\ &= u|_{x^n=0} \\ &= \partial_n u^+|_{x^n=0}. \end{aligned}$$

Thus  $\bar{u} \in C^1(B)$ .

By A2, we have

$$\exists C > 0, \text{ such that } \|\bar{u}_z\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}.$$

Now suppose  $\partial U$  is not flat around  $z$ , we can find  $r_1 > 0, \Phi \in C^1(B(z, r_1), \mathbb{R}^n)$ , such that  $\Phi(\partial U \cap B(z, r_1))$  is in a flat hyperplane, WLOG  $\{y_n = 0\}$ , and  $\det(D\Phi) = \det(D\Psi) = 1$ , for  $\Psi := \Phi^{-1}$ .

Notice that we can find  $B(z, r_2) \subset\subset V$  since  $V$  is open and  $z \in \bar{U} \subseteq V$ .

By setting  $r = \min(r_1, r_2) > 0$ , we can WLOG work with  $B(z, r) \subset\subset V$ .

Let  $z' := \Phi(z), v := u \circ \Psi \in C^1(\Phi(\bar{U})) = C^1(\overline{\Phi(U)})$ .

Since  $\Phi(B(z, r))$  is open, we can choose some open ball  $B := B(z', r') \subseteq \Phi(B(z, r))$ . Let  $W_z := \Psi(B)$ .

Since  $\Phi(\partial U \cap B(z, r))$  is in the plane  $\{y_n = 0\}$ , we have

$$B^+ := B \cap \{y^n \geq 0\} = \Phi(W_z \cap \bar{U}), B^- := B \cap \{y^n \leq 0\} = \Phi(W_z \setminus U).$$

Now we can extend  $v$  from  $B^+$  to  $B$  with

$$\|\bar{v}\|_{W^{1,p}(B)} \leq C\|v\|_{W^{1,p}(B^+)}.$$

Now let  $\bar{u}_z := \bar{v} \circ \Psi$ , we have that  $B = \Phi(W_z), \bar{v} = \bar{u}_z \circ \Phi$ , and by 2.29, we have

$$\begin{aligned} \|\bar{u}_z\|_{W^{1,p}(W_z)} &\leq C_1\|\bar{v}\|_{W^{1,p}(\Phi(W_z))} \\ &= C_1\|\bar{v}\|_{W^{1,p}(B)} \\ &\leq C_2\|v\|_{W^{1,p}(B^+)} \\ &\leq C_3\|u\|_{W^{1,p}(\Psi(B^+))} \\ &\leq C_3\|u\|_{W^{1,p}(U)} \end{aligned}$$

Notice that  $\forall z, \Phi(z) \in B \implies z \in W_z$ , thus  $\{W_z\}_{z \in \partial U}$  forms an open cover for  $\partial U$ .

Since  $\partial U$  is compact, we can find a finite subcover  $\{W_i\}_{i=1}^N$ .

Notice that  $(\bar{U} \setminus \bigcup_{i=1}^N W_i) \subset U$  is closed, and  $U$  is bounded, so we can find  $(\bar{U} \setminus \bigcup_{i=1}^N W_i) \subseteq W_0 \subset\subset U$ . We then have  $\bigcup_{i=0}^N W_i = U$ . Now let  $(\zeta_i)_{i=0}^N$  be a partition of unity subordinate to  $W_i$ , such that

$$\forall x \in U \quad \sum_{i=0}^N \zeta_i(x) = 1, \quad \forall i \geq 0, \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^\infty(\mathbb{R}^n), \\ \text{Supp}(\zeta_i) \subseteq W_i. \end{cases}$$

Let  $\bar{u} := \sum_{i=0}^N \zeta_i \bar{u}_i$ , with  $\bar{u}_0 := u$ . We have that

$$\begin{aligned} \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} &\leq \sum_{i=0}^N \|\zeta_i \bar{u}_i\|_{W^{1,p}(\mathbb{R}^n)} \\ &= \sum_{i=0}^N \|\zeta_i \bar{u}_i\|_{W^{1,p}(W_i)} \\ &\leq C_4 \sum_{i=0}^N \|\bar{u}_i\|_{W^{1,p}(W_i)} \\ &= C_5 \|u\|_{W^{1,p}(W_i)}, \end{aligned}$$

since each term is bounded, and we have a finite sum.

We thus define  $Eu := \bar{u}$ .

We can check that  $E$  is linear and bounded.  $\square$

**Theorem 2.31** (Extension). *Let  $1 \leq p < \infty$ . Assume  $U$  is bounded, with  $\partial U$  be  $C^1$ . Let  $V$  be open and bounded, with  $U \subset\subset V$ , then there exists a bounded linear operator  $E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$ , such that  $\forall u \in W^{1,p}(U)$  :*

1.  $Eu = u$  a.e. in  $U$ ,
2.  $\text{Supp}(Eu) \subseteq V$ ,
3.  $\exists C > 0$ , such that  $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)}$ .

*Proof.* By 2.27, we know  $C^\infty(\bar{U}) \subseteq C^1(\bar{U})$  is dense in  $W^{1,p}(U)$ , and thus  $C^1(\bar{U})$  is also dense in  $W^{1,p}(U)$ . By 1.22, we can extend the result in the above lemma to get  $E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$ .

In addition, since  $Eu = \lim_{m \rightarrow \infty} Eu_m$  for some  $u_m \rightarrow u$  in  $W^{1,p}(U)$ , we also have  $Eu = \lim_{m \rightarrow \infty} Eu_m = Eu = \lim_{m \rightarrow \infty} u_m = u$ , a.e..

Also,  $\text{Supp}(Eu) \subseteq \bigcup_{m=1}^\infty \text{Supp}(Eu_m) \subseteq V$ .  $\square$

## 2.6 Traces

**Proposition 2.32** (Young's inequality).

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b > 0, \frac{1}{p} + \frac{1}{q} = 1.$$

**Lemma 2.33.** *Let  $U$  be bounded, and  $\partial U$  is  $C^1$ , and  $1 \leq p < \infty$ . Then there exists a bounded linear operator  $T : C^1(\bar{U}) \rightarrow L^p(\partial U)$ ;  $u \mapsto u|_{\partial U}$  and a constant  $C > 0$ , such that*

$$\forall u \in C^1(\bar{U}), \|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)}.$$

*Proof.* Consider  $z \in \partial U$ .

Assume  $\partial U$  is flat near  $z$  in the hyperplane  $\{x^n = 0\}$ .

Then there exists an open ball  $B_z := B(z, r)$ , such that

$$B_z^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B_z^- := B \cap \{x^n \leq 0\} \subseteq \mathbb{R}^n \setminus U.$$

Since  $u$  is  $C^1$  and thus continuous, WLOG, we can take  $r$  small enough, such that  $u$  does not change sign in  $B_z$ . Namely,  $|u| = u \operatorname{sgn}(u(z))$  in  $B_z$ .

Let  $\hat{B}_z := B(z, \frac{r}{2})$ , and let  $\xi \in C_c^\infty(B_z)$  such that  $\xi \geq 0$  in  $B_z$ , and  $\xi = 1$  in  $\hat{B}_z$ .

Let  $\Gamma_z := \hat{B}_z \cap \partial U$ , then we have  $\operatorname{Supp}(\xi u) \subseteq B_z^+$ , and  $\xi u = u$  on  $T$ .

Let  $x' := (x^1, \dots, x^{n-1})$ , by Fundamental Theorem of Calculus, we have

$$\int_0^\infty (\xi|u|^p)(x', t) dt = -(\xi|u|^p)(x', 0).$$

In addition, we have

$$\begin{aligned} \|u\|_{L^p(\Gamma_z)}^p &= \int_{\Gamma_z} |u|^p(x', 0) dx' \\ &\leq \int_{\mathbb{R}^{n-1}} (\xi|u|^p)(x', 0) dx' \\ &= - \int_0^\infty \int_{\mathbb{R}^{n-1}} (\xi|u|^p)(x', t) dx' dt \\ &= - \int_{B_z^+} (\xi|u|^p)(x) dx \\ &= - \int_{B_z^+} \xi_{x_n} |u|^p + \xi p |u|^{p-1} (\operatorname{sgn} u(z)) u_{x_n} dx \\ &\leq \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi p |u|^{p-1} |u_{x_n}| dx \\ &\leq \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi p \left( \frac{(|u|^{p-1})^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{|u_{x_n}|^p}{p} \right) dx \\ &= \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi(p-1) |u|^p + \xi |u_{x_n}|^p dx \\ &\leq \int_{B_z^+} (|\xi_{x_n}| + \xi(p-1)) |u|^p + \xi |Du|^p dx. \end{aligned}$$

Since  $\xi \in C_c^\infty(B_z)$ , by EVT,  $|\xi_{x_n}|, \xi$  are all bounded. Thus  $\exists C > 0$ , such that  $|\xi_{x_n}| + \xi(p-1), \xi \leq C$  in  $B_z$ . Thus

$$\|u\|_{L^p(\Gamma_z)}^p \leq \int_{B_z^+} C |u|^p + C |Du|^p dx = C \|u\|_{W^{1,p}(B_z^+)}^p \leq C \|u\|_{W^{1,p}(U)}^p.$$

Now if  $\partial U$  is not flat near  $z$ , we can find a  $C^1$  diffeomorphism  $\Phi$  to make it flat. We still have

$$\|u\|_{L^p(\Gamma_z)} \leq C \|u\|_{W^{1,p}(U)},$$

by the equivalence of Sobolev norms under diffeomorphism 2.29.

Since  $\{B_z\}_{z \in \partial U}$  form an open cover for  $\partial U$ , and  $\partial U$  is compact, we can find a finite subcover  $\{B_i : x_i \in \partial U\}_{i=1}^N$ , and their corresponding  $\Gamma_i$ .

For each  $i \in [N]$ , we have that

$$\|u\|_{L^p(\Gamma_i)} \leq C_i \|u\|_{W^{1,p}(U)}^p.$$

We have that

$$\begin{aligned}
\|Tu\|_{L^p(\partial U)}^p &= \int_{\partial U} |u|^p dx \\
&\leq \sum_{i=1}^N \int_{\Gamma_i} |u|^p dx \\
&= \sum_{i=1}^N \|u\|_{L^p(\Gamma_i)}^p \\
&\leq \sum_{i=1}^N C_i \|u\|_{W^{1,p}(U)}^p \\
&= C \|u\|_{W^{1,p}(U)}^p,
\end{aligned}$$

by taking  $C := \sum_{i=1}^N C_i$ .  $\square$

**Theorem 2.34.** *Let  $U$  be bounded, and  $\partial U$  is  $C^1$ , and  $1 \leq p < \infty$ . Then there exists a bounded linear operator  $T : W^{1,p}(U) \rightarrow L^p(\partial U)$  and a constant  $C > 0$ , such that*

$$\forall u \in W^{1,p}(U) \cap C(\bar{U}), Tu = u|_{\partial U},$$

and

$$\forall u \in W^{1,p}(U), \|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)}.$$

*Proof.* By 2.27, we know  $C^\infty(\bar{U}) \subseteq C^1(\bar{U})$  is dense in  $W^{1,p}(U)$ , and thus  $C^1(\bar{U})$  is also dense in  $W^{1,p}(U)$ . By 1.22, we can extend the result in the above lemma to get  $T : W^{1,p}(U) \rightarrow L^p(\partial U)$ .  $\square$

**Theorem 2.35.** *Let  $U$  be bounded, and  $\partial U$  is  $C^1$ , then for any  $u \in W^{1,p}(U)$ , we have that*

$$u \in W_0^{1,p}(U) \iff Tu = 0 \text{ on } \partial U.$$

## 2.7 Weak and Normal Derivatives

*Remark.* For the following part in this subsection, we will use  $D^\alpha$  to denote the  $\alpha^{th}$  normal derivative of  $u$ , and  $\bar{D}^\alpha$  to be the  $\alpha^{th}$  weak derivative of  $u$  to avoid confusion.

**Definition 2.19.** A domain  $U$  is **path-connected** if  $\forall x, y$ , there is some continuous path  $\gamma : [0, 1] \rightarrow U$ , such that  $\gamma(0) = x, \gamma(1) = y$ .

**Proposition 2.36.** *Let  $U \subseteq \mathbb{R}^n$  be open and connected, and  $1 \leq p \leq \infty$ ,  $u \in W^{1,p}(U)$ , then*

$$\bar{D}u = 0 \text{ a.e.} \iff u \text{ is a constant a.e.}$$

*Proof.* Suppose  $u$  is a constant a.e., then it means  $u$  has a version  $\tilde{u}(x) = C, \forall x \in U$  for some constant  $C$ . Clearly  $\tilde{u}$  is differentiable, and has continuous normal derivative  $D\tilde{u}(x) = 0, \forall x \in U$ . By 2.13, and since  $\tilde{u} = u$  a.e., we have that  $D\tilde{u}$  is a weak derivative of  $v$ . Since the weak derivative is unique a.e., we must have  $\bar{D}u = 0$  a.e..

On the other hand, suppose  $\bar{D}u = 0$  a.e.. We know that by 2.22, for any  $\epsilon > 0, x \in U_\epsilon$ , and any direction  $i \in [n]$ ,

$$\begin{aligned}
(\partial_i u^\epsilon)(x) &= (\eta_\epsilon * \partial_i u)(x) \\
&= \int_U \eta_\epsilon(x) \bar{\partial}_i u(y) dy \\
&= 0,
\end{aligned}$$

since  $\bar{\partial}_i u(y) = 0$  for a.e.  $y \in U$ . Since this holds for all  $i \in [n]$ , we must have  $Du^\epsilon(x) = 0$ . Thus, we have  $Du^\epsilon(x) = 0$  in  $U_\epsilon$ . Notice that by 2.7,  $u^\epsilon \in C^\infty(U_\epsilon)$ , which by normal calculus means that  $\forall x \in U_\epsilon$ ,  $u_\epsilon(x) = C_\epsilon$  for some constant  $C_\epsilon$  that does not depend on  $x$ .

Again by 2.7,  $u^\epsilon \rightarrow u$  a.e. as  $\epsilon \rightarrow 0$ . Pick any such  $x$  with  $u^\epsilon(x) \rightarrow u(x)$ . Notice that we can find  $\delta > 0$ , such that  $\forall \epsilon \in (0, \delta), x \in U_\epsilon$ . Thus we have  $u(x) = \lim_{\epsilon \rightarrow 0} u_\epsilon(x) = \lim_{\epsilon \rightarrow 0} C_\epsilon$ . Since  $\lim_{\epsilon \rightarrow 0} C_\epsilon$  converges, we can call it  $C := \lim_{\epsilon \rightarrow 0} C_\epsilon$ , which is a constant that is independent of  $x, \epsilon$ .

Now any such  $x$  satisfies  $u(x) = \lim_{\epsilon \rightarrow 0} C_\epsilon = C$ , and they are by choice a.e..  $\square$

**Lemma 2.37.** Consider  $1 \leq p \leq \infty$ , and  $U = (a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n$  be an open rectangle. Let  $1 \leq i \leq n$ , suppose  $u \in W^{1,p}(U)$  has a continuous representative  $u^* \in C(U)$ , and its  $i^{th}$  weak derivative  $\bar{\partial}_i u$  has a continuous representative  $(\bar{\partial}_i u)^* \in C(U)$ , then the regular  $i^{th}$  partial derivative

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U$$

exists and is continuous.

*Proof.* Pick some  $s \in (a_i, b_i)$ , let  $S := \{x \in U : x^i = s\}$  be the slice of  $U$ . By FTC, there is a unique  $v$ , defined by

$$v(x^1, \dots, x^n) := u^*(x^1, \dots, x^{i-1}, s, x^{i+1}, \dots, x^n) + \int_s^{x^i} (\bar{\partial}_i u)^*(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n) dt,$$

such that  $v|_S = u^*|_S$ , and the  $i^{th}$  normal partial derivative

$$\partial_i v(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U.$$

We notice that  $\bar{\partial}_i v = \partial_i v$ a.e. by 2.13.

Thus the weak derivative  $\bar{\partial}_i(u^* - v) = \bar{\partial}_i(u^*) - \bar{\partial}_i v = \bar{\partial}_i u - \partial_i v = \bar{\partial}_i u - (\bar{\partial}_i u)^* = 0$ a.e.. Fix any  $(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$ , and denote  $w : (a_i, b_i) \rightarrow \mathbb{R}$  by

$$w(t) := (u^* - v)(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n).$$

We have that  $\bar{D}w = \bar{\partial}_i(u^* - v) = 0$  with respect to  $t \in (a_i, b_i)$ a.e..

By 2.36,  $w(t) = C$ a.e.  $t \in (a_i, b_i)$  form some constant  $C$ , since  $(a_i, b_i)$  is clearly connected.

Notice that  $w$  is continuous, since both  $u^*, v$  are continuous on the  $x^i$  direction.

Since both  $w, C$  are continuous, we have  $\forall t \in (a_i, b_i), w(t) = C$ .

Since  $v|_S = u^*|_S$ , we must have  $C = w(s) = 0$  and thus

$$\forall t \in (a_i, b_i), u^*(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n) = v(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n).$$

Since this holds for all  $(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$ , we must have  $u^*(x) = v(x) \quad \forall x \in U$ .

By construction of  $v$ , we have that

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U.$$

$\square$

**Lemma 2.38.** Consider  $1 \leq p \leq \infty$ , and  $U \subseteq \mathbb{R}^n$  be open. If  $u \in W^{1,p}(U)$  has a continuous representative  $u^* \in C(U)$ , and its weak derivative  $\bar{\partial}_i u$  has a continuous representative  $(\bar{\partial}_i u)^* \in C(U)$ , then the regular  $i^{th}$  partial derivative

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U$$

exists and is continuous.

*Proof.* Notice that any open  $U \subseteq \mathbb{R}^n$  can be written as  $\bigcup_{j=1}^{\infty} R_j$ , where each  $R_j$  is an open rectangle.

Fix any  $x \in U$ , there must be some  $R_j \ni x$ .

By previous lemma,  $\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x)$ .

Since this holds for any  $x \in U$ , we have our result.  $\square$

**Proposition 2.39.** Consider  $1 \leq p \leq \infty$ ,  $k \geq 0$ , and  $U \subseteq \mathbb{R}^n$  be open. If  $u \in W^{k,p}(U)$  has a continuous representative  $u^* \in C(U)$ , and all of its weak derivatives  $D^\alpha u$  have continuous representatives  $(\bar{D}^\alpha u)^* \in C(U)$  for any  $|\alpha| \leq k$ , then

$$u^* \in C^k(U), D^\alpha(u^*)(x) = (\bar{D}^\alpha u)^*(x) \quad \forall x \in U, \forall |\alpha| \leq k.$$

*Proof.* We will use induction on  $k$ .

The base case is  $k = 0$ .

Since  $|\alpha| = 0$ , we trivially have  $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$ .

Now, suppose this holds for  $k - 1$ .

Consider any  $u \in W^{k,p}(U)$ .

If  $|\alpha| = 0$ , we trivially have  $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$  as before.

Now consider any  $|\gamma| = 1$ . We know  $\gamma = e_i$  for some  $1 \leq i \leq n$ .

By previous lemma, we have that

$$D^\gamma(u^*)(x) = \partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) = (\bar{D}^\gamma u)^*(x) \quad \forall x \in U.$$

Notice that  $\bar{D}^\gamma u \in W^{k-1,p}(U)$ , and all of its weak derivatives  $\bar{D}^\beta \bar{D}^\gamma u = \bar{D}^{\beta+\gamma} u$  have continuous representatives  $(\bar{D}^{\beta+\gamma} u)^* \in C(U)$  for any  $|\beta| \leq k - 1$ . By the induction hypothesis, we have that

$$D^\beta((\bar{D}^\gamma u)^*)(x) = (\bar{D}^{\beta+\gamma} u)^*(x) \quad \forall x \in U, \forall |\beta| \leq k - 1.$$

For any  $1 \leq |\alpha| \leq k$ , we can have  $\alpha = \beta + \gamma$ , where  $|\beta| \leq k - 1, |\gamma| = 1$ .

Now we have  $\forall x \in U$ ,

$$\begin{aligned} (\bar{D}^\alpha u)^*(x) &= (\bar{D}^{\beta+\gamma} u)^*(x) \\ &= D^\beta((\bar{D}^\gamma u)^*)(x) \\ &= D^\beta(D^\gamma(u^*))(x) \\ &= D^{\beta+\gamma}(u^*)(x) \\ &= D^\alpha(u^*)(x). \end{aligned}$$

We have thus proven the result for any  $|\alpha| \leq k$ .

Since all of its  $\alpha^{th}$  derivatives exists and are continuous, we further have that  $u^* \in C^k(U)$ . □

**Theorem 2.40** (Differentiability almost everywhere). (*Theorem 5.8.5 in Eavan's*)

Consider  $n < p \leq \infty$ , and  $U \subseteq \mathbb{R}^n$  be open. Assume  $u \in W_{loc}^{1,p}(U)$ , then  $u$  is differentiable a.e. in  $U$ , and its gradient  $Du(x)$  equals its weak gradient  $\bar{D}u(x)$  for a.e.  $x \in U$ .

## 2.8 Sobolev Inequalities

**Definition 2.20.** For  $1 \leq p < n$ , the **Sobolev conjugate** of  $p$  is  $p^* := \frac{np}{n-p}$ , with  $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ .

**Theorem 2.41** (Gagliardo–Nirenberg–Sobolev). Let  $1 \leq p < n$ , then

$$\exists C > 0, \text{ such that } \forall u \in C_c^1(\mathbb{R}^n), \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}.$$

**Corollary 2.42.** Let  $1 \leq p < n$ , and  $U \subseteq \mathbb{R}^n$ , then

$$\exists C > 0, \text{ such that } \forall u \in C_c^1(U), \|u\|_{L^{p^*}(U)} \leq C \|Du\|_{L^p(U)}.$$

*Proof.* By Gagliardo–Nirenberg–Sobolev's Inequality, there is some  $C > 0$ , such that

$$\forall v \in C_c^1(\mathbb{R}^n), \|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Dv\|_{L^p(\mathbb{R}^n)}.$$

Notice that for each  $u \in C_c^1(U)$ , we can extend it by  $v(x) := \begin{cases} u(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$ .

Notice that  $\text{Supp}(v) \subseteq U$ , and  $v = u$  on  $U$ .

Thus  $\|v\|_{L^{p^*}(\mathbb{R}^n)} = \|v\|_{L^{p^*}(U)} = \|u\|_{L^{p^*}(U)}$ , and  $\|Dv\|_{L^p(\mathbb{R}^n)} = \|Du\|_{L^p(U)}$ . In addition, we have that  $\lim_{x \rightarrow \partial U} D^\alpha u(x) = 0 = \lim_{x \rightarrow \partial U} D^\alpha 0, \forall |\alpha| \leq 1$ . Thus this extension is smooth. i.e.  $v \in C_c^1(\mathbb{R}^n)$ .

We thus have

$$\|u\|_{L^{p^*}(U)} = \|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Dv\|_{L^p(\mathbb{R}^n)} = C\|Du\|_{L^p(U)}.$$

□

**Theorem 2.43** ( $W^{1,p}$  embedding into  $L^{p^*}$ , with  $1 \leq p < n$ ). Let  $1 \leq p < n$ ,  $U \subseteq \mathbb{R}^n$  be open and bounded. If  $\partial U$  is  $C^1$ , then

$$\exists C > 0, \text{ such that } \forall u \in W^{1,p}(U), \|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

In addition, since  $U$  is bounded,  $\forall q \in [1, p^*]$ , we have

$$\exists C > 0, \forall u \in W^{1,p}(U), \|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

*Proof.* See Theorem 5.6-2 of Evans and A3Q1. □

**Theorem 2.44** (Poincaré's Inequality). Let  $1 \leq p < n$ ,  $U \subseteq \mathbb{R}^n$  be open and bounded, then

$$\forall q \in [1, p^*], \exists C \geq 0, \text{ such that } \forall u \in W_0^{1,p}(U), \|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}.$$

*Proof.* See Theorem 5.6-2 of Evans and A3Q2. □

**Corollary 2.45.** Let  $1 \leq p < n$ ,  $U \subseteq \mathbb{R}^n$  be open and bounded, then  $\|Du\|_{L^p(U)}$  and  $\|u\|_{W^{1,p}(U)}$  are equivalent norms on  $W_0^{1,p}(U)$ .

**Theorem 2.46.** Let  $1 \leq p \leq \infty$ ,  $U \subseteq \mathbb{R}^n$  be open and bounded, then

$$\exists C \geq 0, \text{ such that } \forall u \in W_0^{1,p}(U), \|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}.$$

*Proof.* See Theorem 5.6-2 of Evans and A3Q2. □

**Corollary 2.47.** Let  $1 \leq p \leq \infty$ ,  $U \subseteq \mathbb{R}^n$  be open and bounded, then  $\|Du\|_{L^p(U)}$  and  $\|u\|_{W^{1,p}(U)}$  are equivalent norms on  $W_0^{1,p}(U)$ .

*Proof.* See A3Q2. □

**Theorem 2.48.** ( $W^{1,p}(U)$  embedding into  $C^{0,\gamma}(\bar{U})$ , with  $n < p \leq \infty$ , Morrey's)

Let  $n < p \leq \infty$ ,  $U \subseteq \mathbb{R}^n$  be open and bounded, such that  $\partial U$  is  $C^1$ . Then there is some constant  $C \geq 0$  such that

$$\forall u \in W^{1,p}(U), \exists \tilde{u} \in C^{0,\gamma}(\bar{U}), \text{ such that } \|\tilde{u}\|_{C^{0,\gamma}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)},$$

where  $\gamma := 1 - \frac{n}{p}$ , and  $\tilde{u} \in [u]$  is a representative of the equivalence class  $[u] \in W^{1,p}(U)$ .

*Remark.* If  $p = \infty$ , then  $\gamma = 1$ , and  $u^*$  is Lipschitz.

**Theorem 2.49** (Sobolev Inequalities). Let  $U \subseteq \mathbb{R}^n$  be open and bounded, such that  $\partial U$  is  $C^1$ . Let  $u \in W^{k,p}(U)$ , we have

1. If  $k < \frac{n}{p}$ , we define  $q$  by  $\frac{1}{q} := \frac{1}{p} - \frac{k}{n}$ , then

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}.$$

2. If  $k > \frac{n}{p}$ , we define  $t := k - \lfloor \frac{n}{p} \rfloor - 1$ , then we have a representative  $\tilde{u} \in C^{t,\gamma}(\bar{U})$ , such that

$$\|u^*\|_{C^{t,\gamma}(\bar{U})} \leq C\|u\|_{W^{k,p}(U)},$$

where  $\gamma = \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}$  if  $\frac{n}{p} \notin \mathbb{Z}$ , and  $\gamma$  can be any integer if  $\frac{n}{p} \in \mathbb{Z}$ .

*Proof.* See Theorem 5.6-6 of Evans and A3Q3. □

## 2.9 Compactness

**Definition 2.21.** Let  $(f_k)_{k=1}^\infty$  be a sequence of real-valued functions on  $\mathbb{R}^n$ . It is **uniformly bounded** if

$$\exists M > 0, \text{ such that } |f_k(x)| \leq M, \forall k \in \mathbb{N}^+, x \in \mathbb{R}^n$$

**Definition 2.22.** Let  $(f_k)_{k=1}^\infty$  be a sequence of real-valued functions on  $\mathbb{R}^n$ . It is **equicontinuous** if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x, y \in \mathbb{R}^n, \|x - y\| < \delta \implies |f_k(x) - f_k(y)| < \epsilon, \forall k \in \mathbb{N}^+$$

**Theorem 2.50.** (Arzela-Ascoli Compact criterion)

Let  $(f_k)_{k=1}^\infty$  be a sequence of real-valued functions on  $\mathbb{R}^n$  such that it is uniformly bounded and equicontinuous, then there exists a subsequence  $(f_{k_j})_{j=1}^\infty$  and a continuous function  $f$  such that  $f_{k_j} \rightarrow f$  uniformly on compact subsets of  $\mathbb{R}^n$ .

**Proposition 2.51.** (interpolation) Assume  $1 \leq s \leq r \leq t \leq \infty$ , and  $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$  with  $0 \leq \theta \leq 1$ . Suppose  $u \in L^s(U) \cap L^t(U)$ , then  $u \in L^r(U)$  and

$$\|u\|_{L^r(U)} \leq \|u\|_{L^s(U)}^\theta \|u\|_{L^t(U)}^{1-\theta}.$$

*Proof.* See AMATH731 A2. □

**Lemma 2.52.** Let  $V \subseteq \mathbb{R}^n$  be open and bounded. Let  $1 \leq p < n$ , and  $(u_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$  be any bounded sequence with  $\text{Supp}(u_m) \subseteq V$ . For  $u_m^\epsilon := \eta_\epsilon * u_m$ , we have that for each  $\epsilon > 0$ , there exists a subsequence  $(u_{m_j}^\epsilon)_{j=1}^\infty$  that converges in  $L^q(V)$ .

*Proof.*

**Claim 2.52.1.** The sequence  $(u_m^\epsilon)_{m=1}^\infty$  is uniformly bounded.

*Proof.* Since  $(u_m)_{m=1}^\infty$  is bounded, there is some  $M > 0$ , such that  $\forall m \in \mathbb{N}^+, \|u_m\|_{W^{1,p}(\mathbb{R}^n)} \leq M$ . Consider any  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} |u_m^\epsilon(x)| &= \left| \int_{\mathbb{R}^n} \eta_\epsilon(x-y) u_m(y) dy \right| \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |u_m(y)| dy \\ &= \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(\mathbb{R}^n)} \\ &= \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} \|u_m\|_{L^p(\mathbb{R}^n)} \\ &= \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} \|u_m\|_{W^{1,p}(\mathbb{R}^n)} \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &= \frac{1}{\epsilon^n} \|\eta\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &\leq \frac{C}{\epsilon^n} |V|^{1-\frac{1}{p}} M. \end{aligned}$$

Since  $\frac{C}{\epsilon^n} |V|^{1-\frac{1}{p}} M < \infty$  is independent of  $m$ , we have that the sequence  $(u_m^\epsilon)_{m=1}^\infty$  is uniformly bounded. □

**Claim 2.52.2.** The sequence  $(u_m^\epsilon)_{m=1}^\infty$  is equicontinuous.

*Proof.* Since  $(u_m)_{m=1}^\infty$  is bounded, there is some  $M > 0$ , such that  $\forall m \in \mathbb{N}^+, \|u_m\|_{W^{1,p}(\mathbb{R}^n)} \leq M$ . By 2.7.2, we have that  $\partial_i u_m^\epsilon = (\partial_i \eta_\epsilon) * u_m$ . Thus for any  $x \in \mathbb{R}^n, 1 \leq i \leq n$ , we have

$$\begin{aligned} |\partial_i u_m^\epsilon(x)| &= \left| \int_{\mathbb{R}^n} (\partial_i \eta_\epsilon)(x-y) u_m(y) dy \right| \\ &\leq \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\ &\leq \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(\mathbb{R}^n)} \\ &\leq \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ \|Du_m^\epsilon(x)\|_1 &\leq \sum_{i=1}^n \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &= \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &= \|D\eta_\epsilon\|_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M \\ &< \infty. \end{aligned}$$

Since  $\|D\eta_\epsilon\|_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M$  is independent of  $x, m$ , we have that

$$C := \sup_{m \geq 1} \|Du_m^\epsilon\|_{L^\infty(U)} \leq \|D\eta_\epsilon\|_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M < \infty.$$

Since  $u_m^\epsilon \in C_c^\infty(\mathbb{R}^n)$  by 2.7.1, we have each  $u_m^\epsilon$  is Lipschitz with Lipschitz-constant  $C$ .

Given any  $\delta > 0$ , we can let  $\delta_0 = \frac{\delta}{C}$ .

Thus  $\forall x, y \in \mathbb{R}^n$ , such that  $\|x - y\| < \delta_0$ , we have

$$|u_m^\epsilon(x) - u_m^\epsilon(y)| \leq C\|x - y\| < \delta, \quad \forall m \in \mathbb{N}^+.$$

Thus the sequence  $(u_m^\epsilon)_{m=1}^\infty$  is equicontinuous. □

By the above two lemmas and Arzela-Ascoli Compact criterion 2.50, we know for each  $\epsilon > 0$ , there exists a subsequence  $(u_{m_j}^\epsilon)_{j=1}^\infty$  and a continuous function  $u^\epsilon$  such that  $u_{m_j}^\epsilon \rightarrow f$  uniformly on compact subsets of  $\mathbb{R}^n$ .

Since  $V$  is bounded,  $\bar{V}$  is compact, we have that  $(u_{m_j}^\epsilon)_{j=1}^\infty$  converges uniformly on  $\bar{V}$ .

Thus  $(u_{m_j}^\epsilon)_{j=1}^\infty$  converges in  $L^\infty(V)$ .

Thus  $(u_{m_j}^\epsilon)_{j=1}^\infty$  converges in  $L^q(V)$ . □

**Lemma 2.53.** Let  $V \subseteq \mathbb{R}^n$  be open and bounded, such that  $\partial V$  is  $C^1$ . Let  $1 \leq p < n$ , and  $(u_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$  be any bounded sequence with  $\text{Supp}(u_m) \subseteq V$ . For  $u_m^\epsilon := \eta_\epsilon * u_m$ , we have that  $u_m^\epsilon \rightarrow u_m$  uniformly in  $L^q(V)$  as  $\epsilon \rightarrow 0$ .

*Proof.* By taking  $V'$  to be  $V + B(0, 1)$  and WLOG consider  $\epsilon < 1$ , we assume the support of  $u_m^\epsilon$  is in  $V$ . Since  $(u_m)_{m=1}^\infty$  is bounded, there is some  $M > 0$ , such that  $\forall m \in \mathbb{N}^+, \|u_m\|_{W^{1,p}(\mathbb{R}^n)} = \|u_m\|_{W^{1,p}(V)} \leq M$ .

**Claim 2.53.1.** If  $u_m$  are smooth, then  $\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon |V|^{1-\frac{1}{p}} M$  for any  $\epsilon > 0$ .

*Proof.*

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= (\eta_\epsilon * u_m)(x) - u_m(x) \\ &= \int_{B(0,\epsilon)} \eta_\epsilon(y) u_m(x-y) dy - u_m(x) \int_{B(0,\epsilon)} \eta_\epsilon(y) dy \\ &= \int_{B(0,\epsilon)} \eta_\epsilon(y) (u_m(x-y) - u_m(x)) dy \end{aligned}$$

Let  $z := \frac{y}{\epsilon}$ , we have  $dy = \epsilon^n dz$ . Recall  $\eta_\epsilon = \frac{1}{\epsilon^n} \eta(\frac{y}{\epsilon})$ . We thus have

$$\begin{aligned}
u_m^\epsilon(x) - u_m(x) &= \int_{B(0,\epsilon)} \eta_\epsilon(y)(u_m(x-y) - u_m(x))dy \\
&= \int_{B(0,1)} \frac{\eta(z)}{\epsilon^n} - (u_m(x-\epsilon z)u_m(x))(\epsilon^n dz) \\
&= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))dy \\
&= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x-\epsilon yt) dt dy \\
&= \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\epsilon yt) \cdot (-\epsilon y) dt dy \\
|u_m^\epsilon(x) - u_m(x)| &\leq \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x-\epsilon yt) \cdot (-\epsilon y)| dt dy \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x-\epsilon yt) \cdot y| dt dy \\
&\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \|Du_m(x-\epsilon yt)\|_1 dt dy,
\end{aligned}$$

since  $\|y\|_2 < \epsilon < 1$ . Thus

$$\begin{aligned}
\|u_m^\epsilon - u_m\|_{L^1(V)} &= \int_V |u_m^\epsilon(x) - u_m(x)| dx \\
&\leq \int_V \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \|Du_m(x-\epsilon yt)\|_1 dt dy dx \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V \|Du_m(x-\epsilon yt)\|_1 dx dt dy \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} \|Du_m(x-\epsilon yt)\|_1 dx dt dy \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} \|Du_m(z)\|_1 dz dt dy \\
&= \epsilon \left( \int_{B(0,\epsilon)} \eta(y) dy \right) \left( \int_0^1 dt \right) \left( \int_{\mathbb{R}^n} \|Du_m(z)\|_1 dz \right) \\
&= \epsilon \int_{\mathbb{R}^n} \|Du_m(z)\|_1 dz \\
&= \epsilon \int_V \|Du_m(z)\|_1 dz \\
&= \epsilon \sum_{i=1}^n \|\partial_i u_m\|_{L^1(V)} \\
&\leq \epsilon \sum_{i=1}^n |V|^{1-\frac{1}{p}} \|\partial_i u_m\|_{L^p(V)} \\
&\leq \epsilon |V|^{1-\frac{1}{p}} \|u_m\|_{W^{1,p}(V)} \\
&\leq \epsilon |V|^{1-\frac{1}{p}} M.
\end{aligned}$$

Notice that this is true for any  $\epsilon > 0$ . □

Let  $\delta > 0$  be given. Since  $C^\infty(\bar{V})$  is dense in  $W^{1,p}(V)$  by 2.27, we can find some  $u_m^- \in W^{1,p}(V)$ , such that  $\|\bar{u}_m - u_m\|_{W^{1,p}(V)} < \frac{\delta}{3|V|^{1-\frac{1}{p}}}$ .

Notice that  $\forall m, \|\bar{u}_m\|_{W^{1,p}(V)} \leq M + \frac{\delta}{3|V|^{1-\frac{1}{p}}}$  is bounded. From the claim above, we can find

$$\epsilon_0 := \frac{\delta}{3\left(M + \frac{\delta}{3|V|^{1-\frac{1}{p}}}\right)|V|^{1-\frac{1}{p}}} > 0,$$

such that  $\forall 0 < \epsilon < \epsilon_0$ , we have

$$\|\bar{u}_m^\epsilon - \bar{u}_m\|_{L^1(V)} < \frac{\delta}{3}, \quad \forall m \in \mathbb{N}^+.$$

Now  $\|u_m - \bar{u}_m\|_{L^1(V)} \leq |V|^{1-\frac{1}{p}} \|u_m - \bar{u}_m\|_{L^p(V)} \leq |V|^{1-\frac{1}{p}} \|u_m - \bar{u}_m\|_{W^{1,p}(V)} < \frac{\delta}{3}$ . In addition, by 2.8, we have

$$\begin{aligned} \|u_m^\epsilon - \bar{u}_m^\epsilon\|_{L^1(V)} &= \|\eta_\epsilon * u_m - \eta_\epsilon * \bar{u}_m\|_{L^1(V)} \\ &= \|\eta_\epsilon * (u_m - \bar{u}_m)\|_{L^1(V)} \\ &\leq \|u_m - \bar{u}_m\|_{L^1(V)} \\ &< \frac{\delta}{3}. \end{aligned}$$

Now we have

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \|u_m^\epsilon - \bar{u}_m^\epsilon\|_{L^1(V)} + \|\bar{u}_m^\epsilon - \bar{u}_m\|_{L^1(V)} + \|\bar{u}_m - u_m\|_{L^1(V)} < \delta.$$

Notice that this holds for all  $\epsilon < \epsilon_0, m \in \mathbb{N}^+$ , where the choice of  $\epsilon_0$  does not depend on  $m$ , and thus  $\|u_m^\epsilon - u_m\|_{L^1(V)} \rightarrow 0$  uniformly when  $\epsilon \rightarrow 0$ .

Now  $1 \leq q \leq p^*$ , by letting  $s = 1, r = q, t = p^*$ , we have

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta} \tag{2.51}$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta C^{1-\theta} \|u_m^\epsilon - u_m\|_{W^{1,p}(V)}^{1-\theta} \tag{2.49}$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta C^{1-\theta} \left( \|u_m^\epsilon\|_{W^{1,p}(V)} + \|u_m\|_{W^{1,p}(V)} \right)^{1-\theta} \tag{2.23}$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta C^{1-\theta} \left( 2\|u_m\|_{W^{1,p}(V)} \right)^{1-\theta}$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta (2CM)^{1-\theta}.$$

Given any  $\delta > 0$ , since  $\|u_m^\epsilon - u_m\|_{L^1(V)} \rightarrow 0$  uniformly when  $\epsilon \rightarrow 0$ , we can always find some  $\epsilon_0 > 0$ , such that

$$\forall \epsilon < \epsilon_0, m \in \mathbb{N}^+, \|u_m^\epsilon - u_m\|_{L^1(V)} < \left( \frac{\delta}{(2CM)^{1-\theta}} \right)^{1/\theta}.$$

Now for any  $m \in \mathbb{N}^+$ , we have

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta (2CM)^{1-\theta} < \delta.$$

This proves that  $u_m^\epsilon \rightarrow u_m$  uniformly in  $L^q(V)$  as  $\epsilon \rightarrow 0$ .  $\square$

**Theorem 2.54** (Rellich-Kondrachov Compactness). *Let  $U \subseteq \mathbb{R}^n$  be open and bounded, such that  $\partial U$  is  $C^1$ . Let  $1 \leq p < n$ , then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for any  $1 \leq q < p^*$ .

*Proof.* The continuous embedding is done before in 2.49.

Now consider any bounded sequence  $(\hat{u}_m)_{m=1}^{\infty} \subset W^{1,p}(U)$ .

Thus there is some  $M > 0$ , such that  $\forall m \in \mathbb{N}^+$ ,  $\|\hat{u}_m\|_{W^{1,p}(U)} \leq M$ .

By extension theorem, we may assume  $(\hat{u}_m)_{m=1}^{\infty} \subset W^{1,p}(\mathbb{R}^n)$ , with  $u_m|_U = \hat{u}_m$ , and there is some  $V$  such that  $U \subset\subset V$  and  $\forall m \in \mathbb{N}^+$ ,  $\text{Supp}(u_m) \subseteq V$ . In addition,

$$\sup \|u_m\|_{W^{1,p}(\mathbb{R}^n)} = \sup \|u_m\|_{W^{1,p}(V)} \leq \sup C \|\hat{u}_m\|_{W^{1,p}(U)} \leq CM.$$

Thus  $(u_m)_{m=1}^{\infty}$  is bounded.

WLOG, we can take  $V$  to have  $\partial V$  being  $C^1$ .

Let  $u_m^\epsilon := \eta_\epsilon * u_m$ .

By the above lemmas, we know that

1. for each  $\epsilon > 0$ , there exists a subsequence  $(u_{m_j}^\epsilon)_{j=1}^{\infty}$  that converges in  $L^q(V)$ , and
2.  $u_m^\epsilon \rightarrow u_m$  uniformly in  $L^q(V)$  as  $\epsilon \rightarrow 0$ .

Now given any  $\delta > 0$ .

By 2, we can find some  $\epsilon_0 > 0$ , such that  $\forall 0 < \epsilon < \epsilon_0$ , we have  $\forall m \in \mathbb{N}^+$ ,  $\|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{3}$ .

Now fix some  $0 < \epsilon < \epsilon_0$ .

By 1, there exists a subsequence  $(u_{m_j}^\epsilon)_{j=1}^{\infty}$  that converges in  $L^q(V)$ .

In particular, it is Cauchy, and we can find some  $N \in \mathbb{N}^+$ , such that  $\forall i, j \geq N$ ,  $\|u_{m_j}^\epsilon - u_{m_i}^\epsilon\|_{L^q(V)} < \frac{\delta}{3}$ .

Now for any  $i, j \geq N$ , we have that

$$\begin{aligned} \|u_{m_i} - u_{m_j}\|_{L^q(V)} &= \left\| u_{m_i} - u_{m_i}^\epsilon + u_{m_i}^\epsilon - u_{m_j}^\epsilon + u_{m_j}^\epsilon - u_{m_j} \right\|_{L^q(V)} \\ &\leq \|u_{m_i} - u_{m_i}^\epsilon\|_{L^q(V)} + \|u_{m_i}^\epsilon - u_{m_j}^\epsilon\|_{L^q(V)} + \|u_{m_j}^\epsilon - u_{m_j}\|_{L^q(V)} \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} \\ &= \delta. \end{aligned}$$

Thus  $(u_{m_j})_{j=1}^{\infty}$  is a Cauchy sequence in  $L^q(V)$ .

Since  $L^q(V)$  is complete, there is some  $u \in L^q(V)$ , such that  $\lim_{j \rightarrow \infty} \|u_{m_j} - u\|_{L^q(V)} = 0$ .

Since  $U \subseteq V$ , we have that  $\lim_{j \rightarrow \infty} \|u_{m_j} - u\|_{L^q(U)} = 0$ .

Since  $u + m|_U = \hat{u}_m$ , we also have that  $\lim_{j \rightarrow \infty} \|\hat{u}_{m_j} - u\|_{L^q(U)} = 0$ .

Thus the subsequence  $\hat{u}_{m_j}$  converges to some  $u \in L^q(V) \subseteq L^q(U)$ .

Since  $(\hat{u}_m)_{m=1}^{\infty} \subset W^{1,p}(U)$  is any bounded sequence, we have that any bounded subset of  $W^{1,p}(U)$  is relative compact in  $L^q(U)$ .  $\square$

**Theorem 2.55.** Let  $U \subseteq \mathbb{R}^n$  be open and bounded, such that  $\partial U$  is  $C^1$ . Let  $1 \leq p \leq \infty$ , then

$$W^{1,p}(U) \subset\subset L^p(U).$$

**Theorem 2.56.** Let  $U \subseteq \mathbb{R}^n$  be open and bounded. Let  $1 \leq p \leq \infty$ , then

$$W_0^{1,p}(U) \subset\subset L^p(U).$$

## 2.10 Poincare Inequalities

**Definition 2.23.** For a bounded domain  $U \subset \mathbb{R}^n$ , we denote the average of  $u$  over  $U$  by

$$(u)_U := \frac{1}{|U|} \int_U u dx.$$

**Theorem 2.57** (Poincaré–Wirtinger’s Inequality). *Let  $U \subset \mathbb{R}^n$  be open, bounded, and connected, such that  $\partial U$  is  $C^1$ . For any  $1 \leq p \leq \infty$ ,  $\exists C > 0$ , such that*

$$\forall u \in W^{1,p}(U), \|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}.$$

*Proof.* Suppose for contradiction it is not true.

Then  $\forall k \in \mathbb{N}, \exists u_k \in W^{1,p}(U)$ , such that  $\|u_k - (u_k)_U\|_{L^p(U)} > k\|Du_k\|_{L^p(U)}$ .

Let  $v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}$ .

Notice that

$$\forall k \in \mathbb{N}^+, \|v_k\|_{L^p(U)} = 1, (v_k)_U = 0, Dv_k = \frac{Du_k}{\|u_k - (u_k)_U\|_{L^p(U)}}.$$

Thus  $\|Dv_k\|_{L^p(U)} = \frac{\|Du_k\|_{L^p(U)}}{\|u_k - (u_k)_U\|_{L^p(U)}} < \frac{1}{k}$ .

Which means  $\|v_k\|_{W^{1,p}(U)}^p = \|v_k\|_{L^p(U)}^p + \|Dv_k\|_{L^p(U)}^p < 1 + \frac{1}{k^p} \leq 2$ .

Since this is true for any  $k \in \mathbb{N}^+$ , we have that  $(v_k)_{k=1}^\infty$  is bounded in  $W^{1,p}(U)^p$ .

Since  $W^{1,p}(U) \subset\subset L^p(U)$ , there is a subsequence  $(v_{k_j})_{j=1}^\infty$  and some  $v \in L^p(U)$ , such that

$$\lim_{j \rightarrow \infty} \|v_{k_j} - v\|_{L^p(U)} = 0.$$

Now consider any  $1 \leq i \leq k$ , and any  $\phi \in C_c^\infty(U)$ .

$$\begin{aligned} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^p(U)}^p &= \int_U |v_{k_j} \partial_i \phi - v \partial_i \phi|^p dx \\ &= \int_U |\partial_i \phi|^p |v_{k_j} - v|^p dx \\ &\leq \|\partial_i \phi\|_{L^\infty(U)}^p \|v_{k_j} - v\|_{L^p(U)}^p. \end{aligned}$$

Since  $\phi \in C_c^\infty(U)$ , we have that  $\|\partial_i \phi\|_{L^\infty(U)}^p$  is bounded by some  $M > 0$ .

Since  $\lim_{j \rightarrow \infty} \|v_{k_j} - v\|_{L^p(U)} = 0$ , we also have  $\lim_{j \rightarrow \infty} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^p(U)}^p = 0$ .

In addition,  $\lim_{j \rightarrow \infty} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^1(U)} \leq \lim_{j \rightarrow \infty} |U|^{1-\frac{1}{p}} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^p(U)} = 0$ .

We have

$$\begin{aligned} &\lim_{j \rightarrow \infty} \int_U |v_{k_j} \partial_i \phi - v \partial_i \phi| dx = 0 \\ \implies &\lim_{j \rightarrow \infty} \int_U (v_{k_j} \partial_i \phi - v \partial_i \phi) dx = 0 \\ \implies &\lim_{j \rightarrow \infty} \int_U v_{k_j} \partial_i \phi dx = \lim_{j \rightarrow \infty} \int_U v \partial_i \phi dx \\ \implies &-\lim_{j \rightarrow \infty} \int_U \partial_i v_{k_j} \phi dx = \int_U v \partial_i \phi dx. \end{aligned}$$

Yet

$$\begin{aligned} \left| \lim_{j \rightarrow \infty} \int_U \partial_i v_{k_j} \phi dx \right| &\leq \lim_{j \rightarrow \infty} \int_U |\partial_i v_{k_j} \phi| dx \\ &\leq \lim_{j \rightarrow \infty} \|\partial_i v_{k_j}\|_{L^p(U)} \|\phi\|_{L^{\frac{p}{p-1}}(U)} \\ &\leq \lim_{j \rightarrow \infty} \|Dv_{k_j}\|_{L^p(U)} \|\phi\|_{L^{\frac{p}{p-1}}(U)} \\ &\leq \lim_{j \rightarrow \infty} \frac{1}{k_j} \|\phi\|_{L^{\frac{p}{p-1}}(U)} \\ &= 0, \end{aligned}$$

since  $\phi \in C_c^\infty(U)$  and  $U$  is bounded, which implies  $\|\phi\|_{L^{\frac{p}{p-1}}(U)} < \infty$ . Thus

$$\int_U v \partial_i \phi dx = - \lim_{j \rightarrow \infty} \int_U \partial_i v_{k_j} \phi dx = 0 = - \int_U 0 \phi dx.$$

Since this holds for any  $\phi \in C_c^\infty(U)$ , we must have  $\partial_i v = 0$ a.e. for any  $1 \leq i \leq n$ .

Thus  $v \in W^{1,p}(U)$ , with  $Dv = 0$ a.e..

Since  $U$  is connected,  $v$  is a constant by 2.36.

Since  $(v)_U = 0$ , we must have  $v = 0$ a.e..

However, this contradicts with  $\|v\|_{L^p(U)} = 1$ .  $\square$

## 2.11 $H^{-1}$ Spaces

**Definition 2.24.** The dual space to  $H_0^1(U)$  is  $H^{-1}(U)$ .

**Theorem 2.58.** Consider any  $f \in H^{-1}(U)$ .

1. There is a tuple  $(f^0, \dots, f^n)$  of functions in  $L^2(U)$ , such that

$$\forall v \in H_0^1(H), \langle f|v \rangle_{H^{-1}(H), H_0^1(H)} = \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f^i, \partial_i v \rangle_{L^2(U)}.$$

In this case, we write  $f = f^0 - \sum_{i=1}^n f_x^i$ .

2.

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \left( \sum_{i=0}^n \|f^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} : (f^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

*Proof.* 1. Let  $f \in H^{-1}(U)$ , by the Riesz-Frechet Representation theorem 1.23,  $\exists! u \in H_0^1(U)$ , such that

$$\forall v \in H_0^1(U), \langle f|v \rangle_{H^{-1}(U), H_0^1(U)} = \langle u, v \rangle_{H_0^1(U)},$$

and  $\|f\|_{H_0^{-1}(U)} = \|u\|_{H_0^1(U)}$ .

Let  $f^0 = u, \forall 1 \leq n, f^i := \partial_i u$ . we have

$$\begin{aligned} \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f^i, \partial_i v \rangle_{L^2(U)} &= \langle u, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle \partial_i u, \partial_i v \rangle_{L^2(U)} \\ &= \langle u, v \rangle_{H_0^1(U)} \\ &= \langle f|v \rangle_{H^{-1}(U), H_0^1(U)}. \end{aligned}$$

2. Consider any  $f \in H^{-1}(U)$ , from 1, we know that there is such  $f^0 = u, \forall 1 \leq n, f^i := \partial_i u$ , satisfying 1, with

$$\|f\|_{H_0^{-1}(U)} = \|u\|_{H_0^1(U)} = \left( \sum_{i=0}^n \|f^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \geq \inf \left\{ \left( \sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} : (g^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

Now consider any  $g^0, \dots, g^n \in L^2(U)$ , such that they satisfies

$$\langle f|v \rangle = \langle g^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle g^i, v \rangle_{L^2(U)}.$$

For any  $v \in H_0^1(U)$ , we have

$$\begin{aligned}
|\langle f|v \rangle| &= \left| \langle g^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle g^i, \partial_i v \rangle_{L^2(U)} \right| \\
&\leq \left| \langle g^0, v \rangle_{L^2(U)} \right| + \sum_{i=1}^n \left| \langle g^i, \partial_i v \rangle_{L^2(U)} \right| \\
&\leq \|g^0\|_{L^2(U)} \|v\|_{L^2(U)} + \sum_{i=1}^n \|g^i\|_{L^2(U)} \|\partial_i v\|_{L^2(U)} \\
&\leq \left( \sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \left( \|v\|_{L^2(U)}^2 + \sum_{i=1}^n \|\partial_i v\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \\
&= \left( \sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \|v\|_{H_0^1(U)}.
\end{aligned}$$

Thus we know

$$\|f\|_{H_0^{-1}(U)} = \sup_{v \in H_0^1(U), v \neq 0} \frac{|\langle f|v \rangle|}{\|v\|_{H_0^1(U)}} \leq \inf \left\{ \left( \sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} : (g^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

□

**Corollary 2.59.** For any  $v^* \in L^2(U)^* \subset L(L^2(U), \mathbb{R}) \subset L(H_0^1(U), \mathbb{R})$ , with  $v^*$  identified with  $v \in L^2(U)$ , and any  $u \in H_0^1(U) \subseteq L^2(U)$ , we have

$$\langle v^*|u \rangle_{H^{-1}(U), H_0^1(U)} = \langle v^*|u \rangle_{L^2(U)^*, L^2(U)} = \langle v, u \rangle_{L^2(U)}.$$

In addition,  $v^* \in H^{-1}(U)$ , and has a representation  $(v, 0, \dots, 0)$  as in above theorem, with

$$\|v^*\|_{H^{-1}(U)} \leq \|v\|_{L^2(U)}.$$

*Proof.* The first equality is by definition and the second equality is by 1.50.

Thus, for any  $\|u\|_{H_0^1(U)} = 1$ , we have that

$$\begin{aligned}
|\langle v^*|u \rangle_{H^{-1}(U), H_0^1(U)}| &= |\langle v, u \rangle_{L^2(U)}| \\
&\leq \|v\|_{L^2(U)} \|u\|_{L^2(U)} \\
&\leq \|v\|_{L^2(U)} \|u\|_{H_0^1(U)} \\
&= \|v\|_{L^2(U)}.
\end{aligned}$$

Since this holds for any unitary  $u \in H_0^1(U)$ , we have that

$$\|v^*\|_{H^{-1}(U)} = \sup_{\|u\|_{H_0^1(U)}=1} |\langle v^*|u \rangle_{H^{-1}(U), H_0^1(U)}| \leq \|v\|_{L^2(U)} < \infty,$$

which proves  $v^* \in H^{-1}(U)$ .

In addition,  $\langle u, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle 0, \partial_i v \rangle_{L^2(U)} = \langle u, v \rangle_{L^2(U)} = \langle v^*|u \rangle_{H^{-1}(U), H_0^1(U)}$ . □

**Corollary 2.60.**  $\forall v \in H_0^1(U) \subset L^2(U)$ , we have  $v^* := \langle v, \cdot \rangle_{L^2(U)} \in H^{-1}(U)$ , with

$$\|v^*\|_{H^{-1}(U)} \leq \|v\|_{L^2(U)} \leq \|v\|_{H_0^1(U)}.$$

In other words, if we identify  $v$  with  $v^*$ , then  $H_0^1(U) \subset L^2(U) \subset H^{-1}(U)$  are continuous embeddings.

*Proof.* Since  $v \in H_0^1(U) \subseteq L^2(U)$ , by the above corollary,  $v^* \in H^{-1}(U)$ , and has

$$\|v^*\|_{H^{-1}(U)} \leq \|v\|_{L^2(U)} \leq \|v\|_{H_0^1(U)}.$$

□

## 2.12 Difference Quotients

**Definition 2.25.** Let  $U \subset \mathbb{R}^n$  be open,  $u \in L_{loc}^1(U)$ ,  $V \subset\subset U$ , then for  $|h| \in (0, \text{dist}(V, \partial U))$ ,  $x \in V$ , we define:

1. For  $i \in [n]$ ,  $u_i^h(x) := u(x + he_i)$
2. For  $i \in [n]$ , the  $i^{th}$  **difference quotient** of size  $h$  at  $x$  is

$$D_i^h u(x) = \frac{u_i^h(x) - u(x)}{h} = \frac{u(x + he_i) - u(x)}{h}.$$

3.

$$D^h u(x) := (D_1^h u(x), \dots, D_n^h u(x)).$$

**Proposition 2.61.** Let  $U \subset \mathbb{R}^n$  be open,  $u \in L_{loc}^1(U)$ , then  $\forall i \in [n], |h| > 0$ , we have

$$\text{Supp}(D_i^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|).$$

Thus,

$$\text{Supp}(D^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|).$$

**Proposition 2.62.** Let  $U \subset \mathbb{R}^n$  be open,  $u, v \in L_{loc}^1(U)$ ,  $V \subset\subset U$ , then  $\forall i \in [n], |h| \in (0, \text{dist}(V, \partial U))$ , we have

$$D_i^h(uv) = v_i^h D_i^h u + u D_i^h v.$$

*Proof.* We have

$$\begin{aligned} v_i^h D_i^h u + u D_i^h v &= v_i^h \frac{u_i^h - u}{h} + u \frac{v_i^h - v}{h} \\ &= \frac{v_i^h u_i^h - v_i^h u + u v_i^h - u v}{h} \\ &= \frac{v_i^h u_i^h - u v}{h} \\ &= \frac{(uv)_i^h - u v}{h} \\ &= D_i^h(uv). \end{aligned}$$

□

**Proposition 2.63.** Let  $U \subset \mathbb{R}^n$  be open,  $u, v \in L_{loc}^1(U)$ ,  $\text{Supp}(u) \subset V \subset\subset U$ , then  $\forall i \in [n], |h| \in (0, \frac{1}{3} \text{dist}(V, \partial U))$ , we have

$$\int_U v D_i^{-h} u dx = - \int_U u D_i^h v dx.$$

*Proof.* Notice that  $\text{Supp}(D_i^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|) \subseteq V + \bar{B}(0, |h|) \subseteq \overline{V + B(0, |h|)}$ .

Since  $\text{dist}(\overline{V + B(0, |h|)}, \partial U) \geq 2|h|$ , we can find  $\overline{V + B(0, |h|)} \subset W \subset\subset V$ , with  $|h| < \text{dist}(W, \partial U)$ , where  $D_i^{-h} u$  is well-defined in  $W$ .

In addition,  $\text{Supp}(u) \subset V \subset W$ , so we can view the integrals as over  $W$ , by extending  $D_i^{-h} u$  to be zero

outside of  $W$ .

$$\begin{aligned}
\int_W v D_i^{-h} u dx &= \int_V v D_i^{-h} u dx \\
&= \int_V v(x) \frac{u(x - he_i) - u(x)}{-h} dx \\
&= - \int_V \frac{v(x)u(x - he_i) - v(x)u(x)}{h} dx \\
&= - \left( \int_V \frac{v(x - he_i + he_i)u(x - he_i)}{h} dx - \int_V \frac{v(x)u(x)}{h} dx \right) \\
&= - \left( \int_{V - he_i} \frac{v(y + he_i)u(y)}{h} dy - \int_V \frac{v(x)u(x)}{h} dx \right) \\
&= - \left( \int_W \frac{v_i^h(y)u(y)}{h} dy - \int_W \frac{v(x)u(x)}{h} dx \right) \\
&= - \int_W \frac{v_i^h(x)u(x) - v(x)u(x)}{h} dx \\
&= - \int_W u \frac{v_i^h - v}{h} dx \\
&= - \int_W u D_i^h v dx.
\end{aligned}$$

□

**Proposition 2.64.** Let  $U \subset \mathbb{R}^n$  be open,  $u, D^\alpha u \in L_{loc}^p(U)$ ,  $V \subset\subset U$ , then  $\forall i \in [n], |h| \in (0, \text{dist}(V, \partial U))$ , we have

$$D^\alpha(u_i^h) = (D^\alpha u)_i^h, \quad D^\alpha(D_i^h u) = D_i^h(D^\alpha u) \text{ in } V.$$

In addition, if  $u \in W^{k,p}(U)$ , we have  $u_i^h, D_i^h u \in W^{k,p}(V)$ .

*Proof.* Given any  $i \in [n], |h| \in (0, \text{dist}(V, \partial U))$ .

$\forall \phi \in C_c^\infty(V)$ , we have  $\phi_i^{-h} \in C_c^\infty(V + he_i) \subseteq C_c^\infty(U)$ , with  $D^\alpha \phi(x) = D^\alpha \phi_i^{-h}(x + he_i)$ .

$$\begin{aligned}
\int_V u_i^h(x) D^\alpha \phi(x) dx &= \int_V u(x + he_i) D^\alpha \phi_i^{-h}(x + he_i) dx \\
&= \int_{V + he_i} u(y) D^\alpha \phi_i^{-h}(y) dy \\
&= \int_U u(y) D^\alpha \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_U D^\alpha u(y) \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_{V + he_i} D^\alpha u(y) \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_V D^\alpha u(x + he_i) \phi_i^{-h}(x + he_i) dx \\
&= (-1)^{|\alpha|} \int_V (D^\alpha u)_i^h(x) \phi(x) dx.
\end{aligned}$$

Since this holds for all  $\phi \in C_c^\infty(V)$ , we must have  $D^\alpha(u_i^h) = (D^\alpha u)_i^h$ .

In addition,

$$\begin{aligned}
D^\alpha(D_i^h u) &= D^\alpha \left( \frac{u_i^h - u}{h} \right) \\
&= \frac{D^\alpha(u_i^h) - D^\alpha u}{h} \\
&= \frac{(D^\alpha u)_i^h - D^\alpha u}{h} \\
&= D_i^h(D^\alpha u).
\end{aligned}$$

Now suppose  $u \in W^{k,p}(U)$ .

$$\begin{aligned}
\|u_i^h\|_{W^{k,p}(V)}^p &= \int_V \sum_{|\alpha| \leq k} |(D^\alpha(u_i^h))(x)|^p dx \\
&= \int_V \sum_{|\alpha| \leq k} |(D^\alpha u)_i^h(x)|^p dx \\
&= \int_V \sum_{|\alpha| \leq k} |D^\alpha u(x + he_i)|^p dx \\
&= \int_{V+he_i} \sum_{|\alpha| \leq k} |D^\alpha u(y)|^p dy \\
&\leq \int_U \sum_{|\alpha| \leq k} |D^\alpha u(y)|^p dy \\
&= \|u\|_{W^{k,p}(U)}^p.
\end{aligned}$$

Thus  $u_i^h \in W^{k,p}(V)$ .

Clearly  $u \in W^{k,p}(V)$ , so a linear combination of them  $D_i^h u \in W^{k,p}(V)$ .  $\square$

**Theorem 2.65.** Let  $U \subset \mathbb{R}^n$  be open, we have:

1. For  $p \in [1, \infty)$ , and  $\forall V \subset\subset U, \exists C > 0$ , such that

$$\|D^h u\|_{L^p(V)} \leq C \|Du\|_{L^p(U)}, \quad \forall u \in W^{1,p}(U), \forall |h| \in (0, \text{dist}(V, \partial U)).$$

2. For  $p \in (1, \infty)$ ,  $V \subset\subset U, u \in L^p(V)$ , if  $\exists C, \delta > 0$ , such that  $\|D^h u\|_{L^p(V)} \leq C$ ,  $\forall |h| \in (0, \delta)$ , then

$$u \in W^{1,p}(V), \quad \|Du\|_{L^p(V)} \leq C.$$

**Theorem 2.66.** Let  $U \subset \mathbb{R}^n$  be open and bounded, with  $\partial U$  being  $C^1$ , then  $u : U \rightarrow \mathbb{R}$  is Lipschitz continuous if and only if  $u \in W^{1,\infty}(U)$ .

### 3 Elliptic PDEs

#### 3.1 Weak Solutions

We will consider the model problem:  $U \in \mathbb{R}^n$  be open and bounded, with some  $f : U \rightarrow \mathbb{R}$  be given. We want to find  $u : \bar{U} \rightarrow \mathbb{R}$ , such that  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ .

**Definition 3.1.** A second order differential operator is

$$Lu := - \sum_{i,j=1}^n \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^n b^i(x)\partial_i u + c(x)u.$$

**Definition 3.2.** A symmetric (uniformly) elliptic second order differential operator is an  $L$  such that  $a^{ij} = a^{ji}$ , and  $\exists \theta > 0$ , such that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta \|\xi\|_2^2 \text{ for a.e. } x \in U, \forall \xi \in \mathbb{R}^n.$$

*Remark.* The above definition is equivalent to saying  $A(x) = (a^{ij}(x)) \in \mathbb{R}^{n \times n}$  is symmetric positive definite for a.e.  $x \in U$ , with a uniform positive lower bound  $\theta > 0$  on their eigenvalues.

**Example 3.1.1.** If we take  $a^{ij} = C\delta_{ij}$ , we have  $Lu = -C\Delta u + b \cdot Du + cu$ .

**Definition 3.3.** The bilinear form associated with  $L$  is given by:

$$B[u, v] := \int_U \left( \sum_{i,j=1}^n a^{ij}\partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cuv \right) dx, \quad \forall u, v \in H_0^1(U).$$

**Definition 3.4.** Consider  $f = f^0 - \sum_{i=1}^n f_x^i \in H^{-1}(U)$  as in 2.58.

$u \in H_0^1(U)$  is called a **weak solution** to the BVP  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ , if  $u$  satisfies the **weak formulation**:

$$\forall v \in H_0^1(U), B[u, v] = \langle f | v \rangle = \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f_x^i, \partial_i v \rangle_{L^2(U)}.$$

**Definition 3.5.** For  $f \in L^2(U)$ , we have the special case:

$u \in H_0^1(U)$  is called a **weak solution** to the BVP  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ , if  $u$  satisfies the **weak formulation**:

$$\forall v \in H_0^1(U), B[u, v] = \langle f, v \rangle_{L^2(U)}.$$

**Definition 3.6.** If  $u \in W^2(U)$  satisfies  $Lu(x) = f(x)$  a.e.  $x \in U$ , we say  $u$  is a **strong solution** to the problem  $Lu = f$  in  $U$ .

**Proposition 3.1.** If a classical solution  $u$  exists, i.e  $u$  is smooth, and  $Lu = f, u|_{\partial U} = 0$ , then  $u$  is always a strong solution.

**Proposition 3.2.** If a strong solution  $u$  exists, then  $u$  is always a weak solution.

*Proof.* Firstly consider any  $v \in C_c^\infty(U)$ , we have

$$\begin{aligned}
\langle f|v \rangle &= \langle Lu, v \rangle \\
&= \int_U L u v dx \\
&= \int_U \left( - \sum_{i,j=1}^n \partial_j(a^{ij} \partial_i u) + \sum_{i=1}^n b^i \partial_i u + cu \right) v dx \\
&= - \sum_{i,j=1}^n \int_U \partial_j(a^{ij} \partial_i u) v dx + \int_U \left( \sum_{i=1}^n b^i \partial_i u v + cu v \right) dx \\
&= \int_U \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v dx + \int_U \left( \sum_{i=1}^n b^i \partial_i u v + cu v \right) dx \\
&= \int_U \left( \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cu v \right) dx \\
&= B[u, v].
\end{aligned}$$

Since  $H_0^1(U) = \overline{C_c^\infty(U)}$ , this holds for any  $v \in H_0^1(U)$ .  $\square$

## 3.2 Existence of weak solution

### 3.2.1 First Existence Theorem

**Theorem 3.3** (Lax-Milgram). *Consider a real Hilbert space  $\mathcal{H}$  with  $\langle \cdot, \cdot \rangle$  and action  $\langle \cdot | \cdot \rangle_{\mathcal{H}^*, \mathcal{H}}$ . Assume  $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  is a bilinear form such that  $\exists a, b > 0$  such that  $\forall u, v \in \mathcal{H}$ ,*

$$\begin{aligned}
|B[u, v]| &\leq a \|u\| \|v\| \\
B[u, u] &\geq b \|u\|^2.
\end{aligned}$$

*Then  $\forall f \in \mathcal{H}^*, \exists! u \in \mathcal{H}$  such that  $\forall v \in \mathcal{H}, B[u, v] = \langle f | v \rangle$ .*

*Proof.* For each  $u \in \mathcal{H}$ , we define the linear functional  $T_u : v \mapsto B[u, v]$ .

We have that  $|T_u v| = |B[u, v]| \leq a \|u\| \|v\|$ , so  $\|T_u\|_{\mathcal{H}^*} \leq a \|u\| < \infty$  is bounded. Thus  $T_u \in \mathcal{H}^*$ . By the Riesz-Frechet Representation theorem 1.23,  $\exists! w \in \mathcal{H}$ , such that

$$\forall v \in \mathcal{H}, T_u v = \langle w, v \rangle_{\mathcal{H}},$$

with  $\|T_u\|_{\mathcal{H}^*} = \|w\|_{\mathcal{H}}$ .

Now define  $A : \mathcal{H} \rightarrow \mathcal{H}$  by  $u \mapsto w$  in the above setting, such that  $\forall v \in \mathcal{H}, \langle Au, v \rangle = B[u, v]$ .

**Claim 3.3.1.** *For any  $u \in \mathcal{H}$ , we have that*

$$b \|u\| \leq \|Au\| \leq a \|u\|.$$

*Proof.* We have

$$\|Au\|^2 = \langle Au, Au \rangle = B[u, Au] \leq a \|u\| \|Au\|.$$

If  $\|Au\| = 0$ , clearly  $\|Au\| \leq a \|u\|$ .

Otherwise we can divide both side by  $\|Au\|$ , and get  $\|Au\| \leq a \|u\|$ .

On the other hand, we have

$$b \|u\|^2 \leq B[u, u] = \langle Au, u \rangle \leq \|Au\| \|u\|.$$

If  $\|u\| = 0$ , clearly  $b \|u\| \leq \|Au\|$ .

Otherwise we can divide both side by  $\|u\|$ , and get  $b \|u\| \leq \|Au\|$ .  $\square$

**Claim 3.3.2.** We have  $A \in B(\mathcal{H})$ .

*Proof.* For any  $u_1, u_2, v \in \mathcal{H}, c \in \mathbb{R}$ , we have that

$$\begin{aligned}\langle A(u_1 + cu_2), v \rangle &= B[u_1 + cu_2, v] \\ &= B[u_1, v] + cB[u_2, v] \\ &= \langle Au_1, v \rangle + c\langle Au_2, v \rangle \\ &= \langle Au_1 + cAu_2, v \rangle.\end{aligned}$$

Since this holds for all  $v \in \mathcal{H}$ , we have  $A(u_1 + cu_2) = Au_1 + cAu_2$ , and thus  $A$  is linear.  
In addition, we have

$$\|A\|_{B(\mathcal{H})} = \sup_{u \in \mathcal{H}, u \neq 0} \frac{\|Au\|}{\|u\|} \leq \sup_{u \in \mathcal{H}, u \neq 0} \frac{a\|u\|}{\|u\|} = a < \infty.$$

This shows  $A$  is bounded, and thus  $A \in B(\mathcal{H})$ .  $\square$

**Claim 3.3.3.**  $A$  is bijective.

*Proof.* Suppose  $Au = 0$ , we have that

$$b\|u\| \leq \|Au\| = 0,$$

which means that  $u = 0$ . Thus  $A$  is injective.

Consider any sequence  $(y_j)_{j=1}^{\infty} \subset \text{Im}(A)$ , such that  $\lim_{j \rightarrow \infty} y_j = y \in \mathcal{H}$ .

We can find  $(x_j)_{j=1}^{\infty} \subset \mathcal{H}$ , such that  $\forall j \geq 1, Ax_j = y_j$ .

Since  $(y_j)_{j=1}^{\infty}$  is convergent and thus Cauchy, given any  $\epsilon > 0$ , we can find some  $N \geq 1$ , such that  $\forall i, j \geq N, \|y_j - y_i\| < b\epsilon$ .

Now

$$\begin{aligned}\|x_j - x_i\| &\leq \frac{1}{b} \|A(x_j - x_i)\| \\ &= \frac{1}{b} \|Ax_j - Ax_i\| \\ &= \frac{1}{b} \|y_j - y_i\| \\ &< \frac{1}{b} b\epsilon \\ &< \epsilon.\end{aligned}$$

Thus  $(x_j)_{j=1}^{\infty}$  is Cauchy.

Since  $\mathcal{H}$  is complete, there is some  $x \in \mathcal{H}$ , such that  $\lim_{j \rightarrow \infty} x_j = x$ .

Since  $A$  is bounded and thus continuous, we have that

$$\begin{aligned}Ax &= A\left(\lim_{j \rightarrow \infty} x_j\right) \\ &= \lim_{j \rightarrow \infty} Ax_j \\ &= \lim_{j \rightarrow \infty} y_j \\ &= y.\end{aligned}$$

Thus  $y \in \text{Im}(A)$ .

This proves that  $\text{Im}(A)$  is closed.

Since  $A$  is linear,  $\text{Im}(A)$  is a closed subspace of  $\mathcal{H}$ , and thus  $\mathcal{H} = \text{Im}(A) \oplus \text{Im}(A)^{\perp}$ .

Consider any  $w \in \text{Im}(A)^{\perp}$ , we must have

$$b\|w\|^2 \leq B[w, w] = \langle Aw, w \rangle = 0.$$

Thus  $\text{Im}(A)^{\perp} = \{0\}$ , and thus  $\text{Im}(A) = \mathcal{H}$ .

Thus  $A$  is surjective.  $\square$

Now by the Bounded Inverse Theorem,  $A^{-1}$  exists and is bounded.  
By Riesz-Frechet Representation theorem 1.23, given any  $f \in \mathcal{H}^*$ , we have

$$\exists! w \in \mathcal{H}, \text{ such that } \langle f|v \rangle = \langle w, v \rangle \quad \forall v \in \mathcal{H}.$$

Let  $u = A^{-1}w$ , we have that

$$\forall v \in \mathcal{H}, \quad B[u, v] = \langle Au, v \rangle = \langle w, v \rangle = \langle f|v \rangle.$$

This proves the existence.

Now suppose there is some  $\hat{u}$  such that  $\forall v \in \mathcal{H}, \quad B[\hat{u}, v] = \langle f|v \rangle = B[u, v]$ .  
We must have  $B[u - \hat{u}, v] = 0, \quad \forall v \in \mathcal{H}$ . Thus

$$b\|u - \hat{u}\| \leq B[u - \hat{u}, u - \hat{u}] = 0,$$

and thus  $\hat{u} = u$  is unique.  $\square$

**Proposition 3.4** (Cauchy's inequality). *For any  $a, b, \epsilon > 0$ , we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

**Theorem 3.5** (Energy estimates). *Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $a^{ij}, b^i, c \in L^\infty(U)$ , such that  $(a^{ij})$  is symmetric positive definite. For the bilinear form defined in 3.3, there exists constants  $\alpha, \beta > 0, \gamma \geq 0$ , such that  $\forall u, v \in H_0^1(U)$ ,*

$$|B[u, v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} \tag{1}$$

$$\beta \|u\|_{H^1(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2. \tag{2}$$

*Proof.* We have

$$\begin{aligned} |B[u, v]| &= \left| \int_U \left( \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx \right| \\ &\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \int_U |\partial_i u| |\partial_j v| dx + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \int_U |\partial_i u| |v| dx + \|c\|_{L^\infty(U)} \int_U |u| |v| dx \\ &\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|\partial_j v\|_{L^2(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|v\|_{L^2(U)} \\ &\quad + \|c\|_{L^\infty(U)} \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\ &\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|u\|_{H^1(U)} \|v\|_{H^1(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|u\|_{H^1(U)} \|v\|_{H^1(U)} \\ &\quad + \|c\|_{L^\infty(U)} \|u\|_{H^1(U)} \|v\|_{H^1(U)} \\ &= \left( \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right) \|u\|_{H^1(U)} \|v\|_{H^1(U)}. \end{aligned}$$

Taking  $\alpha := \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)}$ , we notice that  $\alpha \geq 0$ , and  $\alpha = 0 \implies \forall i, j, \quad a^{ij} = 0$ , which contradicts  $(a_{ij})$  is positive definite. Thus  $\alpha > 0$ , and  $|B[u, v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}$ .  
On the other hand, consider  $\xi = Du \in \mathbb{R}^n$ .

We have that

$$\theta \|Du\|_2^2 \leq \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u.$$

Thus

$$\begin{aligned}
\theta \|Du\|_{L^2(U)}^2 &= \theta \int_U \|Du\|_2^2 dx \\
&\leq \int_U \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u dx \\
&= B[u, u] - \int_U \left( \sum_{i=1}^n b^i \partial_i u u + c u u \right) dx \\
&\leq B[u, u] + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|u\|_{L^2(U)} + \|c\|_{L^\infty(U)} \|u\|_{L^2(U)}^2 \\
&\leq B[u, u] + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \left( \epsilon \|\partial_i u\|_{L^2(U)}^2 + \frac{1}{4\epsilon} \|u\|_{L^2(U)}^2 \right) + \|c\|_{L^\infty(U)} \|u\|_{L^2(U)}^2 \\
&= B[u, u] + \epsilon \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)}^2 + \left( \frac{1}{4\epsilon} \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right) \|u\|_{L^2(U)}^2 \\
&\leq B[u, u] + \epsilon \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|Du\|_{L^2(U)}^2 + \left( \frac{1}{4\epsilon} \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right) \|u\|_{L^2(U)}^2.
\end{aligned}$$

If  $\sum_{i=1}^n \|b^i\|_{L^\infty(U)} = 0$ , pick any  $\epsilon > 0$ .

Otherwise choose  $\epsilon := \frac{\theta}{2 \sum_{i=1}^n \|b^i\|_{L^\infty(U)}} > 0$ , and  $\gamma := \frac{1}{4\epsilon} \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)}$ , we have

$$\frac{\theta}{2} \|Du\|_{L^2(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2.$$

Since  $\|Du\|_{L^p(U)}$  and  $\|u\|_{W^{1,p}(U)}$  are equivalent norms on  $W_0^{1,p}(U)$  by 2.45, we have that

$$\exists C > 0, \text{ such that } \forall u \in H_0^1(U), \|u\|_{H^1(U)}^2 \leq C \|Du\|_{L^p(U)}^2.$$

Taking  $\beta := \frac{\theta}{2C} > 0$ , we have

$$\beta \|u\|_{H^1(U)}^2 \leq \frac{\theta}{2} \|Du\|_{L^2(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2.$$

□

**Definition 3.7.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\mu \in \mathbb{R}$ , we define the operator  $L_\mu$  by

$$L_\mu u := Lu + \mu u.$$

We define the bilinear form associated to  $L_\mu$  to be  $B_\mu$ .

**Proposition 3.6.**

$$B_\mu[u, v] = B[u, v] + \int_U \mu u v dx = B[u, v] + \mu \langle u, v \rangle_{L^2(U)}.$$

**Theorem 3.7** (First Existence Theorem). *Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\gamma \geq 0$  be the same as in Energy Estimate 3.5. For any  $\mu \geq \gamma$  and  $\forall f \in H^{-1}(U)$ , there is a unique weak solution  $u \in H_0^1(U)$  of the BVP:  $\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$*

*Proof.* By Energy estimates, we have that  $\forall u, v \in H_0^1(U)$ ,

$$\begin{aligned} |B_\mu[u, v]| &\leq |B[u, v]| + \mu |\langle u, v \rangle_{L^2(U)}| \\ &\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} + \mu \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\ &\leq (\alpha + \mu) \|u\|_{H^1(U)} \|v\|_{H^1(U)} \\ B_\mu[u, u] &= B[u, u] + \mu \langle u, u \rangle_{L^2(U)} \\ &= B[u, u] + \mu \|u\|_{L^2(U)} \\ &\geq \beta \|u\|_{H^1(U)}^2 + (\mu - \gamma) \|u\|_{L^2(U)}^2 \\ &\geq \beta \|u\|_{H^1(U)}^2. \end{aligned}$$

By Lax–Milgram Theorem, for any  $f \in H^{-1}(U)$ , there is a unique  $u \in H_0^1(U)$ , such that

$$\forall v \in H_0^1(U), B_\mu[u, v] = \langle f | v \rangle.$$

□

**Corollary 3.8.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\gamma \geq 0$  be the same as in Energy Estimate 3.5. For any  $\mu \geq \gamma$  and  $\forall f \in L^2(U)$ , there is a unique weak solution  $u \in H_0^1(U)$  of the BVP:  $\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

### 3.2.2 More Existence Theorems

**Definition 3.8.** Consider  $Lu := -\sum_{i,j=1}^n \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^n b^i(x)\partial_i u + c(x)u$ , we define its **formal adjoint**

$$L^\dagger v := -\sum_{i,j=1}^n \partial_i(a^{ij}(x)\partial_j v) + \sum_{i=1}^n b^i(x)\partial_i v + c(x)v.$$

For  $f \in H^{-1}(U)$ , the **adjoint problem** is  $\begin{cases} L^\dagger v = f & \text{in } U, \\ v = 0 & \text{on } \partial U \end{cases}$ , and the bilinear form associated with it is  $B^*[u, v]$ .

Notice that  $v \in H_0^1(U)$  is a weak solution of the adjoint problem if  $v$  satisfies  $\forall u \in H_0^1(U), B^*[u, v] = \langle f | u \rangle$ .

**Proposition 3.9.**

$$B^*[u, v] := B[v, u].$$

*Remark.* Since  $L$  is not bounded,  $L^\dagger$  is not its usual adjoint operator. However, when  $u, v$  are both smooth, we have that  $\langle Lu, v \rangle_{L^2(U)} = B[u, v] = B^*[v, u] = \langle v, L^\dagger u \rangle$ .

**Definition 3.9.** For  $\mu \in \mathbb{R}$ , we can similarly define  $L_\mu^\dagger u := L^\dagger u + \mu u$ , and the bilinear form associated with it is  $B_\mu^*[u, v]$ .

**Proposition 3.10.**

$$B_\mu^*[u, v] = B^*[u, v] + \mu \langle u, v \rangle_{L^2(U)} = B[v, u] + \mu \langle v, u \rangle_{L^2(U)} = B_\mu[u, v].$$

**Proposition 3.11.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\gamma \geq 0$  be the same as in Energy Estimate 3.5. For any  $\mu \geq \gamma$  and  $\forall f \in L^2(U)$ ,

there is a unique weak solution  $u \in H_0^1(U)$  of the BVP:  $\begin{cases} L^\dagger u + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ . Namely,

$$\exists! u \in H_0^1(U), \text{ such that } \forall v \in H_0^1(U), B_\mu^*[u, v] = \langle f, v \rangle_{L^2(U)}.$$

*Proof.* For  $\alpha, \beta > 0, \gamma \geq 0$  from Energy Estimate 3.5, we have that  $\forall u, v \in H_0^1(U)$ ,

$$|B^*[v, u]| = |B[u, v]| \quad (3)$$

$$\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} \quad (4)$$

$$\beta \|u\|_{H^1(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2 \quad (5)$$

$$= B^*[u, u] + \gamma \|u\|_{L^2(U)}^2. \quad (6)$$

Thus  $B$  and  $B^*$  have the same energy estimate. By First Existence Theorem 3.7, we have the result.  $\square$

**Definition 3.10.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\gamma \geq 0$  be the same as in Energy Estimate 3.5. For any  $\mu \geq \gamma$ , we define  $L_\mu^{-1} : L^2(U) \rightarrow H_0^1(U)$  by  $f \mapsto u$ , where  $u$  is the unique solution to

$$\forall v \in H_0^1(U), B_\mu[u, v] = \langle f, v \rangle_{L^2(U)}$$

given by the First Existence Theorem 3.7.

We can also define  $(L_\mu^+)^{-1} : L^2(U) \rightarrow H_0^1(U)$  by  $f \mapsto u$ , where  $u$  is the unique solution to

$$\forall v \in H_0^1(U), B_\mu^*[u, v] = \langle f, v \rangle_{L^2(U)}.$$

*Remark.* We notice that by definition  $B_\mu[L_\mu^{-1}f, v] = \langle f, v \rangle_{L^2(U)}, \forall v \in H_0^1(U), \forall f \in L^2(U), \forall \mu \geq \gamma$ .

**Lemma 3.12.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\gamma \geq 0$  be the same as in Energy Estimate 3.5. Then for any  $\mu \geq \gamma$ , if we let  $K = \mu L_\mu^{-1}$ , we have that  $K : L^2(U) \rightarrow H_0^1(U) \subseteq L^2(U)$  is compact.

*Proof.* Consider any  $g \in L^2(U)$ , we have that

$$\begin{aligned} \beta \|L_\mu^{-1}g\|_{H^1(U)}^2 &\leq B[L_\mu^{-1}g, L_\mu^{-1}g] + \gamma \|L_\mu^{-1}g\|_{L^2(U)}^2 \\ &\leq B[L_\mu^{-1}g, L_\mu^{-1}g] + \mu \|L_\mu^{-1}g\|_{L^2(U)}^2 \\ &= B_\mu[L_\mu^{-1}g, L_\mu^{-1}g] \\ &= \langle g, L_\mu^{-1}g \rangle_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|L_\mu^{-1}g\|_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|L_\mu^{-1}g\|_{H^1(U)} \\ &\implies \\ \|L_\mu^{-1}g\|_{H^1(U)} &\leq \frac{1}{\beta} \|g\|_{L^2(U)} \\ &\implies \\ \|Kg\|_{H^1(U)} &\leq \frac{\mu}{\beta} \|g\|_{L^2(U)}. \end{aligned}$$

Thus,  $K : L^2(U) \rightarrow H_0^1(U)$  is bounded.

Since  $H_0^1(U) \subset\subset L^2(U)$ , by 1.35, we have that  $K : L^2(U) \rightarrow L^2(U)$  is compact.  $\square$

**Lemma 3.13.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\gamma \geq 0$  be the same as in Energy Estimate 3.5. For any  $f \in L^2(U)$ , if we let  $h := L_\gamma^{-1}f, K = \gamma L_\gamma^{-1}$ , we have that  $u \in H_0^1(U)$  is a weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$  if and only if  $u$  solves  $(I - K)u = h$ .

*Proof.* We will firstly show that  $u$  solves  $\forall v \in H_0^1(U)$ ,  $B_\gamma[u, v] = \langle f + \gamma u, v \rangle_{L^2(U)}$ , if and only if  $u$  solves  $u = L_\gamma^{-1}(f + \gamma u)$ .

Suppose  $\forall v \in H_0^1(U)$ ,  $B_\gamma[u, v] = \langle f + \gamma u, v \rangle_{L^2(U)}$ .

we have that  $u' := L_\gamma^{-1}(f + \gamma u) \in H_0^1(U)$  is the unique solution, such that

$$B_\gamma[u', v] = \langle f + \gamma u, v \rangle, \quad \forall v \in H_0^1(U).$$

Thus  $u = u' = L_\gamma^{-1}(f + \gamma u)$ .

On the other hand, suppose  $u = L_\gamma^{-1}(f + \gamma u)$ , then we have that

$$\forall v \in H_0^1(U), \quad B_\gamma[u, v] = B_\gamma[L_\gamma^{-1}(f + \gamma u), v] = \langle f + \gamma u, v \rangle_{L^2(U)}.$$

Thus,  $u \in H_0^1(U)$  is a weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ , if and only if

$u$  solves  $\forall v \in H_0^1(U)$ ,  $B[u, v] = \langle f, v \rangle_{L^2(U)}$ , if and only if

$u$  solves  $\forall v \in H_0^1(U)$ ,  $B[u, v] + \gamma \langle u, v \rangle_{L^2(U)} = \langle f, v \rangle_{L^2(U)} + \gamma \langle u, v \rangle_{L^2(U)}$ , if and only if

$u$  solves  $\forall v \in H_0^1(U)$ ,  $B_\gamma[u, v] = \langle f + \gamma u, v \rangle_{L^2(U)}$ , if and only if

$u$  solves  $u = L_\gamma^{-1}(f + \gamma u)$ , if and only if

$u$  solves  $u = L_\gamma^{-1}f + \gamma L_\gamma^{-1}u$ , if and only if

$u$  solves  $Iu = h + Ku$ , if and only if

$u$  solves  $(I - K)u = h$ . □

### Theorem 3.14. (Second Existence Theorem)

Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator.

1. Precisely one of the following must be true:

(a)  $\forall f \in L^2(U), \exists! u \in H_0^1(U)$ , a unique weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

(b) There is a weak solution  $u \neq 0 \in H_0^1(U)$  to the homogeneous problem  $\begin{cases} Lu = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U. \end{cases}$

2. Let  $N \subset H_0^1(U)$  be the solution space of weak solutions to  $\begin{cases} Lu = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ , and let  $N^* \subset H_0^1(U)$  be the solution space of weak solutions to  $\begin{cases} L^\dagger u = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ , then  $\dim(N) = \dim(N^*) < \infty$ .

3. The problem  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$  has a weak solution if and only if  $f \in (N^*)^\perp \subseteq L^2(U)$ .

*Proof.* Take  $\mu = \gamma$ .

From the above lemma, we know that for any  $f \in L^2(U)$ , if we let  $K = \gamma L_\gamma^{-1}$ , we have that  $u \in H_0^1(U)$  is a weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$  if and only if  $u$  solves  $(I - K)u = L_\gamma^{-1}f$ .

We also have shown that  $K : L^2(U) \rightarrow H_0^1(U) \subseteq L^2(U)$  is compact.

1. By 1.37, we have that exactly one of the following holds:

(a)  $\forall v \in L^2(U), \exists! u \in L^2(U)$ , such that  $(I - K)u = v$ .

In this case, for any  $f \in L^2(U), \exists! u \in L^2(U)$ , such that  $(I - K)u = L_\gamma^{-1}f$ .

In addition, since  $L_\gamma^{-1}f \in H_0^1(U), Ku = \gamma L_\gamma^{-1}u \in H_0^1(U)$ , we must have  $u = L_\gamma^{-1}f + Ku \in H_0^1(U)$ .

Thus  $u$  is the unique weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

- (b)  $\exists u \neq 0 \in L^2(U)$ , such that  $(I - K)u = 0 = L_\gamma^{-1}0$ .  
 Similarly, we can see that  $u = Ku = \gamma L_\gamma^{-1}u \in H_0^1(U)$ .

Thus  $u$  is a non-trivial solution to  $\begin{cases} Lu = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

2. By the above lemma,  $N = \text{Ker}(I - K)$ .

By 1.37, we have that  $\dim(N) = \dim(\text{Ker}(I - K^\dagger)) < \infty$ . Let  $L_\gamma^\dagger u := L^\dagger u + \gamma u$ . Consider any  $g, h \in L^2(U)$ , we have that

$$\begin{aligned} \langle h, K^\dagger g \rangle &= \langle Kh, g \rangle \\ &= \langle g, Kh \rangle_{L^2(U)} \\ &= \gamma \langle g, L_\gamma^{-1}h \rangle_{L^2(U)} \\ &= \gamma B_\gamma^*[(L_\gamma^\dagger)^{-1}g, L_\gamma^{-1}h] \\ &= \gamma B_\gamma[L_\gamma^{-1}h, (L_\gamma^\dagger)^{-1}g] \\ &= \gamma \langle h, (L_\gamma^\dagger)^{-1}g \rangle_{L^2(U)} \\ &= \langle h, \gamma(L_\gamma^\dagger)^{-1}g \rangle_{L^2(U)}. \end{aligned}$$

Since this holds for all  $g, h \in L^2(U)$ , we have that  $K^\dagger = \gamma(L_\gamma^\dagger)^{-1}$ .

By the above lemma, we have that  $u \in H_0^1(U)$  is a weak solution to  $\begin{cases} L^\dagger u = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$ , if and only if  $u$  solves  $(I - K^\dagger)u = 0$ , if and only if  $u \in \text{Ker}(I - K^\dagger)$ .  
 Thus  $N^* = \text{Ker}(I - K^\dagger)$ .

3. (a)  $\gamma = 0$ .

Notice that  $K = 0$ , and thus  $N^* = \text{ker}(I - K^\dagger) = \text{ker}(I) = \{0\}$ .

Thus  $(N^*)^\perp = L^2(U)$ .

In addition,  $N = \text{ker}(I - K) = \text{ker}(I) = \{0\}$ , so we must be in case (a).

Thus  $\forall f \in (N^*)^\perp$ , the problem  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$  has a (unique) weak solution.

The other direction is trivial since  $(N^*)^\perp = L^2(U)$  is the whole space.

- (b)  $\gamma \neq 0$ .

By 1.37, we have that  $\text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$ .

By the above lemma, we have that the problem  $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$  has a weak solution, if and only if,

there is some  $u$  that solves  $(I - K)u = L_\gamma^{-1}f$ , if and only if,

$L_\gamma^{-1}f \in \text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$ , if and only if,

$\forall v \in \text{Ker}(I - K^\dagger) = N^*$ ,

$$\begin{aligned} \langle L_\gamma^{-1}f, v \rangle &= 0 \\ \frac{1}{\gamma} \langle Kf, v \rangle &= 0 \\ \frac{1}{\gamma} \langle f, K^\dagger v \rangle &= 0 \\ \frac{1}{\gamma} \langle f, K^\dagger v + (I - K^\dagger)v \rangle &= 0 \\ \frac{1}{\gamma} \langle f, v \rangle &= 0 \\ \langle f, v \rangle &= 0, \end{aligned}$$

if and only if  $f \in (N^*)^\perp$ .

□

**Definition 3.11.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. The **spectrum** of  $L$  is defined to be

$$\Sigma := \mathbb{R} \setminus \left\{ \lambda \in \mathbb{R} : \forall f \in L^2(U), \exists! u \in H_0^1(U), \text{ such that } \forall v \in H_0^1(U), B_{-\lambda}[u, v] = \langle f, v \rangle_{L^2(U)} \right\}.$$

**Proposition 3.15.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\Sigma$  be the spectrum of  $L$ .

1.  $\lambda \notin \Sigma$  if and only if  $\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$  has a unique weak solution  $u \in H_0^1(U)$  for each  $f \in L^2(U)$ .
2.  $\lambda \in \Sigma$  if and only if  $\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$  has a non-trivial weak solution  $u \neq 0 \in H_0^1(U)$ .

*Proof.* 1. This is by definition.

2. By Second Existence Theorem 3.14 on  $L_{-\lambda}$ .

□

**Lemma 3.16.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\gamma \geq 0$  be the same as in Energy Estimate 3.5, and  $\Sigma$  be the spectrum of  $L$ , we always have  $\Sigma \subseteq (-\gamma, \infty)$ .

*Proof.* If  $\lambda \leq -\gamma$ , we have that  $-\lambda \geq \gamma$ , and by First Existence Theorem 3.7, we have that the problem has a unique weak solution, and thus  $\lambda \notin \Sigma$ . □

**Theorem 3.17. (Third existence theorem)**

Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator. Let  $\Sigma$  be the spectrum of  $L$ .

1.  $\Sigma$  is at most countable.

2. If  $\Sigma$  is infinite, then  $\Sigma = \{\lambda_k\}_{k=1}^\infty$  can be arranged in non-decreasing sequence with  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ .

*Proof.* Let  $\gamma' \geq 0$  be the same as in Energy Estimate 3.5, we have  $\Sigma \subseteq (-\gamma', \infty) \subseteq (-\gamma, \infty)$  for any  $\gamma \geq \gamma'$ . We will take some  $\gamma > 0$ , and consider  $\lambda > -\gamma$ .

$\lambda \in \Sigma$ , if and only if  $\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$  has a non-trivial weak solution  $u \neq 0 \in H_0^1(U)$ ,

if and only if  $\begin{cases} Lu + \gamma u = (\lambda + \gamma)u, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$  has a non-trivial weak solution  $u \neq 0 \in H_0^1(U)$ .

Suppose  $\lambda \in \Sigma$ , then let  $g = (\lambda + \gamma)u$ . By First Existence Theorem 3.7, there is a unique weak solution

$(L_\gamma)^{-1}((\lambda + \gamma)u) = \frac{\lambda + \gamma}{\gamma} Ku$  to the problem  $\begin{cases} Lu + \gamma u = g, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ .

Since  $u \neq 0 \in H_0^1(U)$  is a weak solution to the problem, we have

$$u = \frac{\lambda + \gamma}{\gamma} Ku.$$

Thus  $u \neq 0 \in L^2(U)$  is an eigen-vector for  $K$ , with corresponding eigenvalue  $\frac{\gamma}{\lambda + \gamma}$ .

Notice that  $\frac{\gamma}{\lambda + \gamma} > 0$ , since  $\gamma > 0, \lambda > -\gamma$ , and thus  $\frac{\gamma}{\lambda + \gamma} \in \text{Spec}_p(K) \setminus (\infty, 0]$ .

Since this holds for any  $\lambda \in \Sigma$ , we have  $\left\{ \frac{\gamma}{\lambda + \gamma} : \lambda \in \Sigma \right\} \subseteq \text{Spec}_p(K) \setminus (\infty, 0]$ .

On the other hand,  $\forall \mu \in \text{Spec}_p(K) \setminus \{0\}$ , we have that  $\lambda' := \frac{\gamma(1-\mu)}{\mu} = -\gamma + \frac{\gamma}{\mu}$  satisfies  $\mu = \frac{\gamma}{\lambda'+\gamma}$ . Pick any eigen-vector  $u \neq 0 \in L^2(U)$  corresponds to  $\mu$ , we have that  $\frac{\gamma}{\lambda'+\gamma}u = Ku$ .

Thus  $u = (L_\gamma)^{-1}((\lambda' + \gamma)u) \neq 0 \in H_0^1(U)$  is a weak solution to the problem  $\begin{cases} Lu + \gamma u = (\lambda' + \gamma)u, & \text{in } U, \\ u = 0, & \text{on } \partial U. \end{cases}$

If  $\lambda' > -\gamma \iff \frac{\gamma}{\mu} > 0 \iff \mu > 0$ , we have that  $\lambda' \in \Sigma$ .

Thus, we have  $\left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \text{Spec}_p(K) \setminus (\infty, 0] \right\} \subseteq \Sigma$ .

We have shown that

$$\Sigma = \left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \text{Spec}_p(K) \setminus (\infty, 0] \right\}.$$

Since  $K$  is compact, by the Spectral theorem 1.39, we have that either

1.  $\text{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^N$  is finite, which means  $\Sigma \subseteq \left( \lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^N$  is finite.
2.  $\text{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^\infty$  is countable, and  $\lim_{k \rightarrow \infty} \mu_k = 0$ , which means that  $\Sigma \subseteq \left( \lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^\infty$  is at most countable.

In addition, if  $\Sigma$  is infinite, it must be  $(\lambda_{k_j})_{j=1}^\infty \subseteq (\lambda_k)_{k=1}^\infty$ .

$$\lim_{k \rightarrow \infty} |\lambda_k| = \lim_{k \rightarrow \infty} \left| \frac{\gamma(1-\mu_k)}{\mu_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\gamma}{\mu_k} \right| = \infty.$$

Thus  $\lim_{k \rightarrow \infty} |\lambda_{k_j}| = \lim_{k \rightarrow \infty} |\lambda_k| = \infty$ .

Since we have  $\forall j, \lambda_{k_j} > -\gamma$ , we must have  $\lim_{j \rightarrow \infty} \lambda_{k_j} = \infty$ .

□

### Theorem 3.18. (Boundedness of inverse)

Let  $\Sigma$  be the spectrum of  $L$ , and  $\lambda \notin \Sigma$ . Then there is a constant  $C > 0$  such that for all  $f \in L^2(U)$  and the unique weak solution  $u \in H_0^1(U)$  to  $\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$  we always have

$$\|u\|_{L^2(U)} \leq C \|f\|_{L^2(U)}.$$

*Proof.* Consider any  $\lambda \notin \Sigma$ .

Suppose for contradiction, we can find  $(\tilde{u}_k)_{k=1}^\infty \subset H_0^1(U), (\tilde{f}_k)_{k=1}^\infty \subset L^2(U)$ , such that  $\forall k \geq 1$ ,

$$\begin{cases} L\tilde{u}_k = \lambda \tilde{u}_k + \tilde{f}_k & \text{in } U, \\ \tilde{u}_k = 0, & \text{on } \partial U \end{cases}$$

and

$$\|\bar{u}\|_{L^2(U)} > k \|\bar{f}\|_{L^2(U)}.$$

Let  $u_k := \frac{\tilde{u}_k}{\|\tilde{u}_k\|_{L^2(U)}}, f_k := \frac{\tilde{f}_k}{\|\tilde{u}_k\|_{L^2(U)}}$ .

Notice that  $\forall k \geq 1, \|u_k\|_{L^2(U)} = 1$ , and  $\|f_k\|_{L^2(U)} = \frac{\|\tilde{f}_k\|_{L^2(U)}}{\|\tilde{u}_k\|_{L^2(U)}} < \frac{1}{k}$ .

In addition,  $\forall v \in H_0^1(U)$ ,

$$\begin{aligned} B[u_k, v] &= \frac{1}{\|\tilde{u}_k\|_{L^2(U)}} B[\tilde{u}_k, v] \\ &= \frac{1}{\|\tilde{u}_k\|_{L^2(U)}} \left( \langle \tilde{f}_k, v \rangle_{L^2(U)} + \lambda \langle \tilde{u}_k, v \rangle_{L^2(U)} \right) \\ &= \left\langle \frac{\tilde{f}_k}{\|\tilde{u}_k\|_{L^2(U)}}, v \right\rangle_{L^2(U)} + \lambda \left\langle \frac{\tilde{u}_k}{\|\tilde{u}_k\|_{L^2(U)}}, v \right\rangle_{L^2(U)} \\ &= \langle f_k, v \rangle_{L^2(U)} + \lambda \langle u_k, v \rangle_{L^2(U)}. \end{aligned}$$

By Energy Estimate 3.5, we have that

$$\begin{aligned}
\beta \|u_k\|_{H^1(U)}^2 &\leq B[u_k, u_k] + \gamma \|u_k\|_{L^2(U)}^2 \\
&= \langle f_k, u_k \rangle_{L^2(U)} + \lambda \langle u_k, u_k \rangle_{L^2(U)} + \gamma \|u_k\|_{L^2(U)}^2 \\
&\leq \|f_k\|_{L^2(U)} \|u_k\|_{L^2(U)} + (\lambda + \gamma) \|u_k\|_{L^2(U)}^2 \\
&= \|f_k\|_{L^2(U)} + \lambda + \gamma \\
&< \lambda + \gamma + \frac{1}{k} \\
&\leq \lambda + \gamma + 1. \\
\|u_k\|_{H^1(U)} &\leq \sqrt{\frac{\lambda + \gamma + 1}{\beta}}
\end{aligned}$$

Thus  $(u_k)_{k=1}^\infty$  is a bounded sequence in  $H_0^1(U)$ .

Since  $H_0^1(U)$  is a Hilbert space, and thus reflexive, by 1.28, there  $\exists (u_{k_j})_{j=1}^\infty$  a subsequence, and  $u \in H_0^1(U)$ , such that  $u_{k_j} \rightharpoonup u$ .

Also, since  $H_0^1(U) \subset\subset L^2(U)$ , by 1.31, we have that  $u_{k_j} \rightarrow u$  in  $L^2(U)$ . Thus,

$$\|u\|_{L^2(U)} = \lim_{j \rightarrow \infty} \|u_{k_j}\|_{L^2(U)} = 1.$$

Now consider any  $v \in H_0^1(U)$ , we have that the map  $w \mapsto B[w, v]$  is a linear bounded operator, so by weak convergence of  $(u_{k_j})_{j=1}^\infty$ , we have that

$$\begin{aligned}
B[u, v] &= \lim_{j \rightarrow \infty} B[u_{k_j}, v] \\
&= \lim_{j \rightarrow \infty} \left( \langle f_{k_j}, v \rangle_{L^2(U)} + \lambda \langle u_{k_j}, v \rangle_{L^2(U)} \right) \\
&= \lim_{j \rightarrow \infty} \langle f_{k_j}, v \rangle_{L^2(U)} + \lambda \left\langle \lim_{j \rightarrow \infty} u_{k_j}, v \right\rangle_{L^2(U)} \\
&\leq \lim_{j \rightarrow \infty} \|f_{k_j}\|_{L^2(U)} \|v\|_{L^2(U)} + \lambda \langle u, v \rangle_{L^2(U)} \\
&\leq \lim_{j \rightarrow \infty} \frac{1}{k_j} \|v\|_{L^2(U)} + \lambda \langle u, v \rangle_{L^2(U)} \\
&= \lambda \langle u, v \rangle_{L^2(U)}.
\end{aligned}$$

Namely,  $\hat{u} = u$  satisfies  $\forall v \in H_0^1(U)$ ,  $B_{-\lambda}[\hat{u}, v] = 0 = \langle 0, v \rangle_{L^2(0)}$ .

Yet since  $\lambda \notin \Sigma$ , by definition, we know there is a unique  $\hat{u}$  that satisfies the above condition.

Clearly  $\hat{u} = 0$  satisfies, so by the uniqueness of weak solution,  $u = 0$ .

This contradicts with  $\|u\|_{L^2(U)} = 1$ . □

### 3.3 Regularity

**Theorem 3.19.** (*Interior  $H^2$  regularity*)

Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij} \in C^1(U)$ ,  $b^i, c \in L^\infty(U)$ ,  $\forall i, j \in [n]$ .  $\forall V \subset\subset U, \exists C > 0$ , such that for all  $f \in L^2(U)$ , and  $u \in H^1(U)$  being a weak solution to  $Lu = f$  in  $U$ , namely,

$$\forall v \in H_0^1(U), B[u, v] = \langle f, v \rangle_{L^2(U)},$$

then

$$\|u\|_{H^2(V)} \leq C \left( \|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right).$$

Thus  $u \in H_{loc}^2(U)$ .

*Proof.* Let  $V \subset\subset U$  be given.

The idea is to choose a particular  $v$ , then repeatedly bound all  $\|D_k^h u\|$  from the product rule by  $\|Du\|$ . The only leftover term will be either  $D_k^h(Du)$ , or part of  $\langle f, v \rangle_{L^2(U)}$  or  $\|u\|_H^1(U)$ . We thus achieve a bound on  $\|D_k^h(Du)\|$ , which allows us to say  $u \in H_{loc}^2(U)$ .

1. We now fix some  $f \in L^2(U)$ , and  $u \in H^1(U)$  being a weak solution to  $Lu = f$  in  $U$ . We have  $\forall v \in H_0^1(U)$ ,

$$\begin{aligned} B[u, v] &= \langle f, v \rangle_{L^2(U)} \\ \int_U \left( \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx &= \int_U f v dx \\ \int_U \left( \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v \right) dx &= \int_U \tilde{f} v dx \\ \sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= \langle \tilde{f}, v \rangle_{L^2(U)}, \end{aligned}$$

where  $\tilde{f} := f - \sum_{i=1}^n b^i \partial_i u - cu \in L^2(U)$ , since  $f, \partial_i u, u \in L^2(U), b^i, c \in L^\infty(U)$ .

2. Since  $V \subset\subset U$ , we can choose some  $V \subset\subset W \subset\subset U$ , and  $\zeta \in C_c^\infty(U)$  such that  $V < \zeta < W$ .

Choose  $|h| > 0$  such that  $\text{dist}(V, \partial U) > 8|h|, \text{dist}(W, \partial U) > 6|h|$ .

WLOG, we assume  $h > 0$ .

Fix some  $k \in [n]$ .

Let  $Z := U_{2h} := \{x \in U : \text{dist}(x, \partial U) > 2h\}$  be open.

Since  $U$  is bounded, we have that  $Z \subset\subset U, \text{dist}(Z, \partial U) = 2h > |h|$ .

$$\text{Let } v(x) := \begin{cases} -D_k^{-h}(\zeta^2 D_k^h u)(x) & x \in Z \\ 0 & x \in U \setminus Z. \end{cases}$$

*Remark.* For  $x \in V$ , we have that

$$\begin{aligned} v(x) &= -D_k^{-h}(D_k^h u)(x) \\ &= -D_k^{-h}\left(\frac{u(x+he_k) - u(x)}{h}\right) \\ &= -\frac{\frac{u(x+he_k) - u(x-he_k)}{h} - \frac{u(x+he_k) - u(x)}{h}}{h} \\ &= -\frac{2u(x) - u(x+he_k) - u(x-he_k)}{h^2} \\ &= \frac{u(x+he_k) - 2u(x) + u(x-he_k)}{h^2}, \end{aligned}$$

which is an approximation to  $\partial_k^2 u$  if  $u$  is smooth.

Since  $u \in H^1(U)$ , we have  $D_k^h u \in H^1(Z)$ .

Since  $\text{Supp}(\zeta) \subset W \subset\subset Z$  is compact, we have  $\zeta \in C_c^\infty(Z)$ , so  $\zeta^2 D_k^h u \in H^1(Z)$ .

Since  $U_{4h} \subset\subset Z, \text{dist}(U_{4h}, \partial Z) = 2h > |h|$ , we have that  $v \in H^1(U_{4h})$ .

In addition,  $\text{Supp}(v) \subset \text{Supp}(\zeta^2 D_k^h u) + \bar{B}(0, h) \subseteq W + \bar{B}(0, h) \subseteq U_{6h} + \bar{B}(0, h) \subset U_{4h}$ .

Since  $v \in H^1(U_{4h})$  and  $\text{Supp}(v) \subset U_{4h}$ , we must have  $v \in H_0^1(U)$ .

3. Now we have

$$\begin{aligned}
\sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= \sum_{i,j=1}^n \int_Z (a^{ij} \partial_i u \partial_j v) dx \\
&= \text{sum}_{i,j=1}^n \int_Z (a^{ij} \partial_i u \partial_j (-D_k^{-h}(\zeta^2 D_k^h u))) dx \\
&= - \sum_{i,j=1}^n \int_Z (a^{ij} \partial_i u D_k^{-h}(\partial_j(\zeta^2 D_k^h u))) dx \\
&= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij} \partial_i u)(\partial_j(\zeta^2 D_k^h u)) dx \\
&= \sum_{i,j=1}^n \int_Z (D_k^h(a^{ij}) \partial_i u + a^{ij} D_k^h(\partial_i u)) (\partial_j(\zeta^2 D_k^h u) + \zeta^2 \partial_j(D_k^h u)) dx \\
&= A_1 + A_2 + A_3 + A_4,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &:= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) \zeta^2 \partial_j(D_k^h u) dx, \\
A_2 &:= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) \partial_j(\zeta^2) D_k^h u dx \\
&= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) 2\zeta(\partial_j \zeta) D_k^h u dx \\
A_3 &:= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij}) \partial_i u \zeta^2 \partial_j(D_k^h u) dx \\
A_4 &:= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij}) \partial_i u \partial_j(\zeta^2) D_k^h u dx \\
&= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij}) \partial_i u 2\zeta(\partial_j \zeta) D_k^h u dx
\end{aligned}$$

Now we will examine each term.

$$\begin{aligned}
A_1 &= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) \zeta^2 \partial_j(D_k^h u) dx \\
&= \int_Z \zeta^2 \sum_{i,j=1}^n a^{ij} \partial_i(D_k^h u) \partial_j(D_k^h u) dx \\
&\geq \int_Z \zeta^2 \theta \|D(D_k^h u)\|_2^2 dx \\
&= \theta \int_Z \zeta^2 \|D(D_k^h u)\|_2^2 dx.
\end{aligned}$$

We also have

$$\begin{aligned}
|A_2| &\leq \sum_{i,j=1}^n \int_Z |a^{ij} D_k^h(\partial_i u) 2\zeta(\partial_j \zeta) D_k^h u| dx \\
&\leq \sum_{i,j=1}^n \int_Z \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} |D_k^h(\partial_i u) 2\zeta D_k^h u| dx \\
&= 2 \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z |D_k^h(\partial_i u) \zeta D_k^h u| dx \\
&\leq 2 \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z \epsilon |D_k^h(\partial_i u)|^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx \\
&= C_1 \int_Z \epsilon |D_k^h(\partial_i u)|^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx \\
&= C_1 \int_Z \epsilon |\partial_i(D_k^h u)|^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx \\
&\leq C_1 \int_Z \epsilon \|D(D_k^h u)\|_2^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx,
\end{aligned}$$

since  $a^{ij} \in L^\infty(U)$ , and  $\zeta \in C^c(U)$ , we have  $C_1 := 2 \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} \in (0, \infty)$ . Similarly,

$$\begin{aligned}
|A_3| &\leq \sum_{i,j=1}^n \int_Z |D_k^h(a^{ij}) \partial_i u \zeta^2 \partial_j(D_k^h u)| dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z |\partial_i u \partial_j(D_k^h u)| \zeta^2 dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z \|Du\|_2 \|D(D_k^h u)\|_2 \zeta^2 dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z \|Du\|_2 \|D(D_k^h u)\|_2 \zeta dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z \frac{1}{4\epsilon} \|Du\|_2^2 + \epsilon \zeta^2 \|D(D_k^h u)\|_2^2 dx \\
&= C_2 \int_Z \frac{1}{4\epsilon} \|Du\|_2^2 + \epsilon \zeta^2 \|D(D_k^h u)\|_2^2 dx,
\end{aligned}$$

where

$$\begin{aligned}
C_2 &:= \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \\
&\leq \frac{1}{h} \sum_{i,j=1}^n \left( \|a^{ij}\|_{L^\infty(Z)} + \|a^{ij}\|_{L^\infty(Z+he_k)} \right) \\
&\leq \frac{1}{h} \sum_{i,j=1}^n \left( \|a^{ij}\|_{L^\infty(U)} + \|a^{ij}\|_{L^\infty(U)} \right) \\
&\in (0, \infty).
\end{aligned}$$

Lastly,

$$\begin{aligned}
|A_4| &\leq \sum_{i,j=1}^n \int_Z |D_k^h(a^{ij}) \partial_i u 2\zeta(\partial_j \zeta) D_k^h u| dx \\
&\leq 2 \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z |\partial_i u D_k^h u| dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z |\partial_i u|^2 + |D_k^h u|^2 dx \\
&\leq C_3 \int_Z \|Du\|_2^2 + |D_k^h u|^2 dx,
\end{aligned}$$

where  $C_3 := \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \|\partial_j \zeta\|_{L^\infty(U)} \in (0, \infty) y$  as argued before.

Now

$$\begin{aligned}
&|A_2 + A_3 + A_4| \\
&\leq |A_1| + |A_2| + |A_3| \\
&\leq \int_Z \epsilon C_1 \|D(D_k^h u)\|_2^2 \zeta^2 + \frac{C_1}{4\epsilon} |D_k^h u|^2 + \frac{C_2}{4\epsilon} \|Du\|_2^2 + C_2 \epsilon \zeta^2 \|D(D_k^h u)\|_2^2 + C_3 \|Du\|_2^2 + C_3 |D_k^h u|^2 dx \\
&= \int_Z (C_1 + C_2) \epsilon \|D(D_k^h u)\|_2^2 \zeta^2 + \left( \frac{C_1}{4\epsilon} + C_3 \right) |D_k^h u|^2 + \left( \frac{C_2}{4\epsilon} + C_3 \right) \|Du\|_2^2 dx \\
&\leq \int_Z (C_1 + C_2) \epsilon \|D(D_k^h u)\|_2^2 \zeta^2 + \left( \frac{C_1}{4\epsilon} + C_3 \right) \|D^h u\|_2^2 + \left( \frac{C_2}{4\epsilon} + C_3 \right) \|Du\|_2^2 dx \\
&= (C_1 + C_2) \epsilon \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx + \left( \frac{C_1}{4\epsilon} + C_3 \right) \|D^h u\|_{L^2(U)}^2 + \left( \frac{C_2}{4\epsilon} + C_3 \right) \|Du\|_{L^2(U)}^2.
\end{aligned}$$

We know there  $\exists C_4 > 0$ , such that

$$\|D^h u\|_{L^2(Z)} \leq C_4 \|Du\|_{L^2(U)}, \forall |h| \in (0, \text{dist}(Z, \partial U)), \forall u \in H_0^1(U).$$

Thus

$$|A_2 + A_3 + A_4| \leq (C_1 + C_2) \epsilon \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx + \left( \frac{C_2}{4\epsilon} + C_3 + \left( \frac{C_1}{4\epsilon} + C_3 \right) C_4^2 \right) \|Du\|_{L^2(U)}^2.$$

Taking  $\epsilon := \frac{\theta}{2(C_1 + C_2)}$ ,  $C_5(\epsilon) := \frac{C_2}{4\epsilon} + C_3 + \left( \frac{C_1}{4\epsilon} + C_3 \right) C_4^2 \in (0, \infty)$ , we have

$$\begin{aligned}
\sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= A_1 + A_2 + A_3 + A_4 \\
&\geq A_1 - |A_2 + A_3 + A_4| \\
&\geq \theta \int_Z \zeta^2 \|D(D_k^h u)\|_2^2 dx - \frac{\theta}{2} \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx - C_5 \|Du\|_{L^2(U)}^2 \\
&= \frac{\theta}{2} \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx - C_5 \|Du\|_{L^2(U)}^2.
\end{aligned}$$

4. On the other hand,

$$\begin{aligned}
\left| \langle \tilde{f}, v \rangle_{L^2(U)} \right| &= \int_U \left| f - \sum_{i=1}^n b^i \partial_i u - cu \right| |v| dx \\
&= \int_U \left( |f| + \sum_{i=1}^n |b^i \partial_i u| + |cu| \right) |v| dx \\
&\leq \int_U \left( |f| + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} |\partial_i u| + \|c\|_{L^\infty(U)} |u| \right) |v| dx \\
&= \int_U |f| |v| dx + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \int_U |\partial_i u| |v| dx + \|c\|_{L^\infty(U)} \int_U |u| |v| dx \\
&\leq C_6 \left( \int_U |f| |v| dx + \int_U |\partial_i u| |v| dx + \int_U |u| |v| dx \right) \\
&\leq C_6 \left( \int_U \frac{1}{4\epsilon} |f|^2 + \epsilon |v|^2 dx + \int_U \frac{1}{4\epsilon} |\partial_i u|^2 + \epsilon |v|^2 dx + \int_U \frac{1}{4\epsilon} |u|^2 + \epsilon |v|^2 dx \right) \\
&\leq C_6 \int_U \frac{1}{4\epsilon} (|f|^2 + |\partial_i u|^2 + |u|^2) + 3\epsilon |v|^2 dx \\
&\leq \frac{C_6}{4\epsilon} \int_U |f|^2 + \|Du\|_2^2 + |u|^2 dx + 3C_6 \epsilon \int_U |v|^2 dx,
\end{aligned}$$

where  $C_6 := \max \left( 1, \sum_{i=1}^n \|b^i\|_{L^\infty(U)}, \|c\|_{L^\infty(U)} \right) \in (0, \infty)$ .

We have shown in step 2 that  $\zeta^2 D_k^h u \in H^1(Z)$ ,  $\text{Supp}(\zeta^2 D_k^h u) \subset Z \subset U$ , thus  $\zeta^2 D_k^h u \in H_0^1(U)$ .

$$\begin{aligned}
\int_U |v|^2 dx &= \int_Z |v|^2 dx \\
&= \int_Z |-D_k^{-h}(\zeta^2 D_k^h u)|^2 dx \\
&\leq \int_Z |D^{-h}(\zeta^2 D_k^h u)|^2 dx \\
&\leq C_4^2 \int_U |D(\zeta^2 D_k^h u)|^2 dx \\
&= C_4^2 \int_W |D(\zeta^2 D_k^h u)|^2 dx \\
&= C_4^2 \int_W |D(\zeta^2) D_k^h u + D(D_k^h u) \zeta^2|^2 dx \\
&\leq 2C_4^2 \int_W |D(\zeta^2)|^2 |D_k^h u|^2 + |D(D_k^h u)|^2 \zeta^4 dx \\
&\leq 2C_4^2 \int_W \|D(\zeta^2)\|_{L^\infty(U)}^2 |D_k^h u|^2 + |D(D_k^h u)|^2 \zeta^2 dx \\
&\leq 2C_4^2 \|D(\zeta^2)\|_{L^\infty(U)}^2 \int_W |D_k^h u|^2 dx + 2C_4^2 \int_W |D(D_k^h u)|^2 \zeta^2 dx \\
&\leq 2C_4^4 \|D(\zeta^2)\|_{L^\infty(U)}^2 \int_U \|Du\|_2^2 dx + 2C_4^2 \int_U |D(D_k^h u)|^2 \zeta^2 dx \\
&\leq C_7 \int_U \|Du\|_2^2 + |D(D_k^h u)|^2 \zeta^2 dx,
\end{aligned}$$

where  $C_7 := 2C_4^2 \max(C_4^2 \|D(\zeta^2)^2\|_{L^\infty(U)}, 1) \in (0, \infty)$ . Thus we have

$$\begin{aligned} \left| \langle \tilde{f}, v \rangle_{L^2(U)} \right| &\leq \frac{C_6}{4\epsilon} \int_U |f|^2 + \|Du\|_2^2 + |u|^2 dx + 3C_6\epsilon \int_U |v|^2 dx \\ &\leq \frac{C_6}{4\epsilon} \int_U |f|^2 + \|Du\|_2^2 + |u|^2 dx + 3C_6C_7\epsilon \int_U \|Du\|_2^2 + |D(D_k^h u)|^2 \zeta^2 dx \\ &\leq \left( \frac{C_6}{4\epsilon} + 3C_6C_7\epsilon \right) \left( \|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) + 3C_6C_7\epsilon \int_U |D(D_k^h u)|^2 \zeta^2 dx. \end{aligned}$$

5. Taking  $\epsilon := \frac{\theta}{12C_6C_7} > 0$ ,  $C_8 := \frac{C_6}{4\epsilon} + 3C_6C_7\epsilon > 0$ , we have

$$\begin{aligned} \sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= \langle \tilde{f}, v \rangle_{L^2(U)} \\ &\leq \left| \langle \tilde{f}, v \rangle_{L^2(U)} \right| \\ &\leq C_8 \left( \|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) + \frac{\theta}{4} \int_U |D(D_k^h u)|^2 \zeta^2 dx \\ &= C_8 \left( \|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) + \frac{\theta}{4} \int_Z |D(D_k^h u)|^2 \zeta^2 dx \\ \sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &\geq \frac{\theta}{2} \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx - C_5 \|Du\|_{L^2(U)}^2 \\ \frac{\theta}{4} \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx &\leq (C_5 + C_8) \left( \|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) \\ \frac{\theta}{4} \int_V \|D(D_k^h u)\|_2^2 dx &\leq (C_5 + C_8) \left( \|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) \\ \int_V \|D_k^h(Du)\|_2^2 dx &\leq C_9 \left( \|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right), \end{aligned}$$

where  $C_9 := \frac{4(C_5 + C_8)}{\theta} \in (0, \infty)$ .

Notice that for all  $j \in [n]$ , we have  $\partial_j u \in L^2(U)$ , and

$$\int_V \|D_k^h(\partial_j u)\|_2^2 dx \leq \int_V \|D_k^h(Du)\|_2^2 dx \leq C_9 \left( \|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right),$$

and this holds for all  $k \in [n]$ . Thus,

$$\begin{aligned} \|D^h(\partial_j u)\|_{L^2(V)}^2 &= \int_V \|D_k^h(\partial_j u)\|_2^2 dx \\ &= \int_V \sum_{k=1}^n \|D_k^h(\partial_j u)\|_2^2 dx \\ &= \sum_{k=1}^n \int_V \|D_k^h(\partial_j u)\|_2^2 dx \\ &\leq \sum_{k=1}^n C_9 \left( \|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right) \\ &= nC_9 \left( \|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right). \\ \|D^h(\partial_j u)\|_{L^2(V)} &\leq \sqrt{nC_9} \left( \|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right) \\ &< \infty. \end{aligned}$$

Since this holds for all  $|h| > 0$  such that  $\text{dist}(V, \partial U) > 8|h|, \text{dist}(W, \partial U) > 6|h|$ , we have  $\partial_j u \in H^1(U)$ , with

$$\|D(\partial_j u)\|_{L^2(V)} \leq \sqrt{nC_9} (\|f\|_{L^2(U)} + \|u\|_{H^1(U)}).$$

Since this holds for all  $j \in [n]$ , we have  $u \in H^2(V)$ , and

$$\begin{aligned} \|D^2 u\|_{L^2(V)}^2 &= \int_V \|D^2 u\|_2^2 dx \\ &= \int_V \sum_{j=1}^n \|\partial_j(Du)\|_2^2 dx \\ &= \sum_{j=1}^n \int_V \|D(\partial_j u)\|_2^2 dx \\ &\leq \sum_{j=1}^n nC_9 (\|f\|_{L^2(U)} + \|u\|_{H^1(U)})^2 \\ &= n^2 C_9 (\|f\|_{L^2(U)} + \|u\|_{H^1(U)})^2 \\ &\implies \\ \|u\|_{H^2(V)}^2 &= \|D^2 u\|_{L^2(V)}^2 + \|u\|_{H^1(V)}^2 \\ &\leq n^2 C_9 (\|f\|_{L^2(U)} + \|u\|_{H^1(U)})^2 + \|u\|_{H^1(V)}^2 \\ &\leq (n^2 C_9 + 1) (\|f\|_{L^2(U)} + \|u\|_{H^1(U)})^2. \end{aligned}$$

Thus we have found  $C := \sqrt{n^2 C_9 + 1} \in (0, \infty)$ , such that  $\|u\|_{H^2(V)} \leq C (\|f\|_{L^2(U)} + \|u\|_{H^1(U)})$ . Since  $V$  is arbitrary, we have that  $u \in H_{loc}^2(U)$ .

6. Notice that the above estimate holds as long as  $V \subset\subset U$  and  $u \in H^1(U)$ . Since  $u \in H^1(W)$ , we can find some constant  $C'$ , such that  $\|u\|_{H^2(V)}^2 \leq C' (\|f\|_{L^2(W)} + \|u\|_{H^1(W)})$ .

Now consider  $v := \xi^2 u \in H_0^1(U)$ , we can find  $\|Du\|_{L^2(W)} \leq C'' \|u\|_{L^2(U)}$  for some  $C'' > 0$ .

Plugging in will give us

$$\|u\|_{H^2(V)}^2 \leq C (\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

□

**Corollary 3.20.** *Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij} \in C^1(U), b^i, c \in L^\infty(U), \forall i, j \in [n]$ . If  $f \in L^2(U)$ , and  $u \in H^1(U)$  is a weak solution to  $Lu = f$  in  $U$ , then  $u$  is a strong solution.*

*Proof.* We have that  $u \in H_{loc}^2(U)$ .

Consider any  $V \subset\subset U$ , since  $a^{ij} \in C^1$ , we have  $a^{ij}u \in H^2(V)$ .

Consider any  $v \in C_c^\infty(V)$ , we must have

$$\begin{aligned}
\langle f, v \rangle_{L^2(V)} &= B[u, v] \\
&= \int_V \left( \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx \\
&= \int_V \left( \sum_{i,j=1}^n a^{ij} \partial_j (\partial_i u) v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx \\
&= \int_V \left( \sum_{i,j=1}^n \partial_j (a^{ij} \partial_i u) + \sum_{i=1}^n b^i \partial_i u + c u \right) v dx \\
&= \int_V (Lu) v dx \\
&= \langle Lu, v \rangle_{L^2(V)}.
\end{aligned}$$

Since this holds for all  $v \in C_c^\infty(V)$ , we must have  $Lu(x) = f(x)$  a.e.  $x \in V$ .

Since this hold for all  $V \subset\subset U$ , we have that  $Lu(x) = f(x)$  a.e.  $x \in U$ .  $\square$

**Theorem 3.21.** (Higher Interior regularity)

Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij}, b^i, c \in C^{m+1}(U)$ ,  $\forall i, j \in [n]$  for some  $m \in \mathbb{N}$ . If  $f \in H^m(U)$ ,  $u \in H^1(U)$  is a weak solution to  $Lu = f$  in  $U$ , then  $u \in H_{loc}^{m+2}(U)$ . In addition,  $\forall V \subset\subset U, \exists C > 0$ , such that  $\forall f \in L^2(U)$ , and  $u \in H^1(U)$  being a weak solution to  $Lu = f$  in  $U$ , we have

$$\|u\|_{H^{m+2}(U)} \leq C \left( \|f\|_{H^m(U)} + \|u\|_{L^2(U)} \right).$$

**Corollary 3.22.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij}, b^i, c \in C^{m+1}(U)$ ,  $\forall i, j \in [n]$  for some  $m > \frac{n}{2} - 2 \in \mathbb{N}$ . If  $f \in H^m(U)$ ,  $u \in H^1(U)$  is a weak solution to  $Lu = f$  in  $U$ , then  $u \in C^l(U)$ , where  $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$ .

**Theorem 3.23.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij}, b^i, c \in C^\infty(U)$ ,  $\forall i, j \in [n]$ . If  $f \in C^\infty(U)$ ,  $u \in H^1(U)$  is a weak solution to  $Lu = f$  in  $U$ , then  $u \in C^\infty(U)$ .

**Theorem 3.24.** (Boundary  $H^2$  regularity)

Let  $U \subseteq \mathbb{R}^n$  be bounded and open, with  $\partial U$  being  $C^2$ , and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij} \in C^1(\bar{U})$ ,  $b^i, c \in L^\infty(U)$ ,  $\forall i, j \in [n]$ . Then  $\exists C > 0$ , such that  $\forall f \in L^2(U)$  and  $u \in H_0^1(U)$  being a weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$  we have

$$\|u\|_{H^2(U)} \leq C \left( \|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right),$$

and thus  $u \in H^2(U)$ .

*Proof.* 1. First prove the case if the boundary is locally flat:

$$U = B(0, 1) \cap \{x : x^n > 0\}, V = B(0, \frac{1}{2}) \cap \{x : x^n > 0\}.$$

Similar to the proof of Interior  $H^2$  regularity, we first use difference quotients to obtain a bound for derivatives that are not normal to the flat boundary:

$$\sum_{k,l=1, k+l < 2n}^n \|\partial_k \partial_l u\|_{L^2(V)} \leq C \left( \|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right),$$

where we can transform  $\|u\|_{H^1(U)}$  to  $\|u\|_{L^2(U)}$ .

For the derivative that is normal to the flat boundary  $\partial_n \partial_n$ , we write the PDE in non divergence form, and use ellipticity to note that  $a^{nn} > \theta > 0$  to find:

$$|\partial_n \partial_n| \leq C \left( \sum_{k,l=1, k+l < 2n}^n |\partial_k \partial_l u| + \|Du\|_2 + |u| + |f| \right) \text{a.e. } x \in U.$$

Thus

$$\|\partial_n \partial_n\|_{L^2(U)} \leq C \left( \sum_{k,l=1, k+l < 2n}^n \|\partial_k \partial_l u\|_{L^2(U)} + \|Du\|_{L^2(U)} + \|u\|_{L^2(U)} + \|f\|_{L^2(U)} \right).$$

This leads to

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

2. Take any  $x_0 \in \partial U$ , let  $y = \Phi(x)$  be a  $C^2$  straightening map on  $B(x_0, r)$  with a  $C^2$  inverse  $x = \Psi(y)$ . Pick some small enough  $s$ , such that

$$U' = B(0, s) \cap \{y : y^n > 0\} \subseteq \Phi(U \cap B(x_0, r)), V' = B(0, \frac{1}{2}s) \cap \{y : y^n > 0\}.$$

We check the weak formulation is well-defined on  $U'$  and that  $L'$  satisfies the assumptions of  $L$ .

Apply step 1 to get

$$\|u'\|_{H^2(V')} \leq C(\|f'\|_{L^2(U')} + \|u'\|_{L^2(U')}).$$

Transform back using  $\Psi$ .

3. Use compactness to find  $V_1, \dots, V_N$  to cover  $\partial U$ . Find  $V_0 \subset\subset U$  such that  $U = \bigcup_{i=0}^N V_i$ .  
Use interior result on  $V_0$ .  
Combine them together.

□

*Remark.* When the solution is unique, we can throw away the  $\|u\|_{L^2(U)}$  by boundedness of inverse in the last section.

**Theorem 3.25. (Higher boundary regularity)**

Let  $U \subseteq \mathbb{R}^n$  be bounded and open, with  $\partial U$  being  $C^{m+2}$ , and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij}, b^i, c \in C^{m+1}(\bar{U})$ ,  $\forall i, j \in [n]$ . Then  $\exists C > 0$ , such that  $\forall f \in H^m(U)$  and  $u \in H_0^1(U)$  being a weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ , we have

$$\|u\|_{H^{m+2}(U)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}),$$

and thus  $u \in H^{m+2}(U)$ .

**Corollary 3.26.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open, with  $\partial U$  being  $C^{m+2}$ , and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij}, b^i, c \in C^{m+1}(\bar{U})$ ,  $\forall i, j \in [n]$  for some  $m > \frac{n}{2} - 2 \in \mathbb{N}$ . If  $f \in H^m(U)$ ,  $u \in H^1(U)$  is a weak solution to  $Lu = f$  in  $U$ , then  $u \in C^l(U)$ , where  $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$ .

**Theorem 3.27. (Infinite differentiability up to the boundary)**

Let  $U \subseteq \mathbb{R}^n$  be bounded and open, with  $\partial U$  being  $C^\infty$ , and  $L$  be a symmetric (uniformly) elliptic second order differential operator, with  $a^{ij}, b^i, c \in C^\infty(\bar{U})$ ,  $\forall i, j \in [n]$ . Then  $\forall f \in H^\infty(U)$  and  $u \in H_0^1(U)$  being a weak solution to  $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ , we have  $u \in C^\infty(\bar{U})$ .

## 4 Parabolic PDEs

### 4.1 Spaces Involving Time

#### 4.1.1 Bochner Spaces

See more about Bochner Spaces in my Measure Theory Notes.

**Definition 4.1.** Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space, a function  $u : [0, T] \rightarrow X$  is **continuous** at a point  $t \in (0, T)$  if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall s, t \in [0, T], |s - t| < \delta \implies \|u(s) - u(t)\| < \epsilon.$$

A function  $u$  is continuous if it is continuous at all  $t \in (0, 1)$ .

$$\|u\|_{C([0, T]; X)} := \sup_{t \in (0, T)} \|u(t)\|.$$

**Theorem 4.1.**  $(C([0, T]; X), \|u\|_{C([0, T]; X)})$  is a Banach Space.

See the definition of Bochner integrable functions in notes of Measure Theory. We will still consider the Lebesgue measure on  $[0, T]$ .

**Theorem 4.2** (Bochner). *Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space, a strongly measurable function  $f : [0, T] \rightarrow X$  is Bochner integrable if and only if  $t \mapsto \|f(t)\|_X$  is integrable. In this case,*

$$\begin{aligned} \left\| \int_0^T f(t) dt \right\|_X &\leq \int_0^T \|f(t)\|_X dt, \\ \forall u^* \in X^*, \left\langle u^* \left| \int_0^T f(t) dt \right. \right\rangle &= \int_0^T \langle u^* | f(t) \rangle dt. \end{aligned}$$

**Theorem 4.3.** Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space, then Dominated Convergence Theorem, Holder's Inequality, and Minkowski's Inequality still work with the Bochner integral.

**Theorem 4.4.** Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space, then for any Bochner integrable  $f : [0, T] \rightarrow X$ , we have  $\int_s^t f(\tau) d\tau$  is continuous in both  $s, t \in [0, T]$ .

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

**Definition 4.2.** Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space, and  $1 \leq p < \infty$ , we define

$$\mathcal{L}^p([0, T]; X) := \left\{ f : [0, T] \rightarrow X \mid f \text{ is measurable, } \int_X \|f\|_X^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p([0, T]; X)} := \left( \int_X \|f\|_X^p d\mu \right)^{\frac{1}{p}}.$$

**Definition 4.3.** Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space,  $(B, \|\cdot\|)$  be a Banach Space, we define

$$\mathcal{L}^\infty([0, T]; X) := \{ f : X \rightarrow B \mid f \text{ is measurable, } \text{ess sup } \|f\|_X < \infty \}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty([0, T]; X)} := \text{ess sup } \|f\|_B.$$

**Definition 4.4.** Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space. For any  $p \in [1, \infty]$ , we define

$$L^p([0, T]; X) := \mathcal{L}^p([0, T]; X)/N,$$

where  $N := \{ f : X \rightarrow B \mid f \text{ is measurable, } f = 0 \text{ } \mu - \text{a.e.} \}$ . Namely,  $[f] \in L^p([0, T]; X)$  is the equivalence class of all  $g = f \text{ } \mu\text{-a.e.}$  for  $f \in \mathcal{L}^p([0, T]; X)$ .

In addition, we define

$$\|[f]\|_{L^p([0, T]; X)} := \|f\|_{\mathcal{L}^\infty([0, T]; X)}$$

for any representative  $f$ .

**Theorem 4.5** (Fischer-Riesz-Bochner). *Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space. For all  $1 \leq p \leq \infty$ , we have that  $(L^p([0, T]; X), \|\cdot\|_{L^p([0, T]; X)})$  is a Banach Space.*

Similarly, we can also define  $L_{loc}^p(0, T; X)$ ,  $W^{k,p}(0, T; X)$ ,  $H^k(0, T; X)$  and weak derivatives where the test functions are  $\phi \in C_c^\infty(0, T; \mathbb{R})$ .

We can similarly define the mollification of  $f \in L_{loc}^1(0, T; X)$  to be

$$f^\epsilon := \eta_\epsilon * f : (\epsilon, T - \epsilon) \rightarrow X; t \mapsto \int_{t-\epsilon}^{t+\epsilon} \eta_\epsilon(t - \tau) f(\tau) d\tau.$$

Similarly, we have

**Theorem 4.6.** *Let  $f^\epsilon$  be defined as above, we have:*

1.  $f^\epsilon \in C^\infty((\epsilon, T - \epsilon); X)$ ,
2.  $\partial_t^k(f^\epsilon) = (\partial_t^k \eta_\epsilon) * f$  on  $(\epsilon, T - \epsilon)$ ,
3.  $f^\epsilon \rightarrow f$  a.e.  $t \in (0, T)$ , as  $\epsilon \rightarrow 0$ ,
4. If  $f \in C(0, T; X)$ , we have  $f^\epsilon \rightarrow f$  uniformly on compact subsets of  $U$ ,
5. If  $1 \leq p < \infty$ ,  $f \in L_{loc}^p(0, T; X)$ , we have  $f^\epsilon \rightarrow f$  in  $L_{loc}^p(0, T; X)$ . Namely,  $f^\epsilon \rightarrow f$  in  $L^p(V)$ ,  $\forall V \subset \subset (0, T)$ .

**Theorem 4.7.** *Let  $T > 0$  and  $(X, \|\cdot\|)$  be a Banach Space,  $p \in [1, \infty]$ , and  $u \in W^{1,p}(0, T; X)$ , then*

1.  $u(t) = u(s) + \int_s^t u'(\tau) d\tau$  for a.e.  $0 \leq s \leq t \leq T$ .
2. There is a representative  $\tilde{u} \in C([0, T], X)$  of  $u$ . In particular,  $\tilde{u}(t) = \tilde{u}(s) + \int_s^t u'(\tau) d\tau$  for any  $0 \leq s \leq t \leq T$ .
3.  $\exists C > 0$  such that  $\forall u \in W^{1,p}(0, T; X)$ ,  $\sup_{t \in [0, T]} \|u(t)\|_X \leq C \|u\|_{W^{1,p}(0, T; X)}$ .

*Proof.* We will prove for  $p \in [1, \infty)$ .

1. Let  $u^\epsilon := \eta_\epsilon * u$ , we have that  $u^\epsilon \in C^\infty((\epsilon, T - \epsilon); X)$ , and  $\partial_t(u^\epsilon) = (\partial_t \eta_\epsilon) * u$  on  $(\epsilon, T - \epsilon)$ . We also have  $f^\epsilon(t) \rightarrow f(t)$  a.e.  $t \in (0, T)$ . Similar to 2.22, we can show that  $\partial_t(u^\epsilon) = \eta_\epsilon * \partial_t u = (\partial_t u)^\epsilon$  on  $(\epsilon, T - \epsilon)$ . Since  $u \in W^{1,p}(0, T; X)$ , we know that  $\partial_t u \in L_{loc}^p(0, T; X)$ , so  $(\partial_t u)^\epsilon \rightarrow \partial_t u$  in  $L_{loc}^p(0, T; X)$ . Since  $|(0, T)| = T < \infty$ , we have that  $\partial_t(u^\epsilon) \rightarrow \partial_t u$  in  $L_{loc}^1(0, T; X)$ , which means

$$\forall [s, t] \subset (0, T), \lim_{\epsilon \rightarrow 0} \int_s^t \|(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)\|_X d\tau = 0.$$

We have that  $\left\| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right\|_X \leq \int_s^t \|(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)\|_X d\tau$  for any fixed  $[s, t] \subset (0, T)$  and  $\epsilon < \min(s, T - t)$ . Thus

$$\lim_{\epsilon \rightarrow 0} \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right\|_X = 0$$

for any  $[s, t] \subset (0, T)$ .

Now  $u^\epsilon(t) = u^\epsilon(s) + \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau$  for any  $[s, t] \subset (\epsilon, T - \epsilon)$  by FTC, since  $u^\epsilon \in C^\infty((\epsilon, T - \epsilon); X)$ .

We have

$$\begin{aligned}
& \left\| -u(t) + u(s) + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X \\
&= \left\| u^\epsilon(t) - u(t) - u^\epsilon(s) + u(s) - \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X \\
&\leq \|u^\epsilon(t) - u(t)\|_X + \|u^\epsilon(s) - u(s)\|_X + \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau - \int_s^t (\partial_t u)(\tau) d\tau \right\|_X \\
&\leq \|u^\epsilon(t) - u(t)\|_X + \|u^\epsilon(s) - u(s)\|_X + \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right\|_X
\end{aligned}$$

for any  $s, t, \epsilon$  such that  $[s, t] \subset (\epsilon, T - \epsilon)$ .

Since each term goes to 0 as  $\epsilon \rightarrow 0$  for a.e.  $0 \leq s \leq t \leq T$ , we must have

$$\left\| -u(t) + u(s) + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X = 0$$

for a.e.  $0 \leq s \leq t \leq T$ .

We thus have

$$u(t) = u(s) + \int_s^t (\partial_t u)(\tau) d\tau$$

for a.e.  $0 \leq s \leq t \leq T$ .

2. Fix any representative for  $u$ .

Notice that the set  $N$  where the above property does not hold has measure 0.

Now fix some point  $s \in [0, T] \setminus N$ , we define

$$\tilde{u}(t) := \begin{cases} u(s) - \int_t^s u'(\tau) d\tau & t < s \\ u(s) + \int_s^t u'(\tau) d\tau & t \geq s \end{cases}$$

For any  $t \in [0, T] \setminus N$ , we have that

$$u(t) := u(s) + \int_s^t u'(\tau) d\tau = \tilde{u}(t)$$

if  $t \geq s$ , and

$$u(s) = u(t) + \int_t^s u'(\tau) d\tau \implies u(t) = u(s) - \int_t^s u'(\tau) d\tau = u(s)$$

if  $t < s$ .

Thus  $\tilde{u} = u$  a.e.  $t \in [0, T]$ , which means  $\tilde{u}$  is a representative of  $u$ .

In addition,  $\tilde{u}(t)$  is continuous since  $\int_t^s u'(\tau) d\tau$  and  $\int_s^t u'(\tau) d\tau$  are both continuous in  $t$ , and

$$\lim_{t \rightarrow s^-} \tilde{u}(t) = \lim_{t \rightarrow s^-} \left( \tilde{u}(t)u(s) - \int_t^s u'(\tau) d\tau \right) = u(s) = u(s) + \int_s^s u'(\tau) d\tau = \tilde{u}(s).$$

3. See A5Q2.

□

**Proposition 4.8.** Suppose  $\mathcal{H}$  is a Hilbert Space, and  $u, v \in C^1(0, T; \mathcal{H})$ , then we have

$$\forall t \in [0, T], \quad \frac{d}{dt} \langle u(t), v(t) \rangle_{\mathcal{H}} = \langle u'(t), v(t) \rangle_{\mathcal{H}} + \langle v'(t), u(t) \rangle_{\mathcal{H}},$$

where  $u'(t) := \lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h}$  is the normal derivative in  $X$ .

*Proof.* We have

$$\begin{aligned}
\frac{d}{dt} \langle u(t), v(t) \rangle_{\mathcal{H}} &= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) \rangle_{\mathcal{H}} - \langle u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) \rangle_{\mathcal{H}} - \langle u(t+h), v(t) \rangle_{\mathcal{H}} + \langle u(t+h), v(t) \rangle_{\mathcal{H}} - \langle u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) \rangle_{\mathcal{H}} - \langle u(t+h), v(t) \rangle_{\mathcal{H}}}{h} + \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t) \rangle_{\mathcal{H}} - \langle u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) - v(t) \rangle_{\mathcal{H}}}{h} + \lim_{h \rightarrow 0} \frac{\langle u(t+h) - u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \left\langle \lim_{h \rightarrow 0} u(t+h), \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h} \right\rangle_{\mathcal{H}} + \left\langle \lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h}, v(t) \right\rangle_{\mathcal{H}} \\
&= \langle u(t), v'(t) \rangle_{\mathcal{H}} + \langle u'(t), v(t) \rangle_{\mathcal{H}}.
\end{aligned}$$

□

#### 4.1.2 Sobolev Spaces In Time

Now we consider the cases where  $X$  might be any of the tuple  $H_0^1(U) \subset L^2(U) \cong L^2(U)^* \subset H^{-1}(U)$ , and see what is the relationship between the time weak derivatives in each space.

**Lemma 4.9.** Suppose  $\mathbf{u}, \mathbf{u}' \in L^1(0, T; H_0^1(U))$ , then we must have  $\mathbf{u}'$  is also the time weak derivative of  $\mathbf{u}$  in  $L^1(0, T; L^2(U))$ .

*Proof.* Firstly, for each  $t \in [0, T]$ , we have  $\mathbf{u}(t), \mathbf{u}'(t) \in H_0^1(U) \subset L^2(U)$ , so  $\mathbf{u}, \mathbf{u}'$  are indeed functions  $[0, T] \rightarrow L^2(U)$ .

In addition,

$$\begin{aligned}
\int_0^T \|\mathbf{u}(t)\|_{L^2(U)} dt &\leq \int_0^T \|\mathbf{u}(t)\|_{H^2(U)} dt \\
&= \|\mathbf{u}\|_{L^1(0, T; H_0^1(U))} \\
&< \infty.
\end{aligned}$$

Similarly for  $\int_0^T \|\mathbf{u}'(t)\|_{L^2(U)} dt < \infty$ .

Thus  $\mathbf{u}, \mathbf{u}' \in L^1(0, T; L^2(U))$ .

Now  $\forall \phi \in C_c^\infty(0, T)$ , we have

$$\begin{aligned}
0 &\leq \left\| \int_0^T \phi'(t) \mathbf{u}(t) dt + \int_0^T \phi(t) \mathbf{u}'(t) dt \right\|_{L^2(U)} \\
&\leq \left\| \int_0^T \phi'(t) \mathbf{u}(t) dt + \int_0^T \phi(t) \mathbf{u}'(t) dt \right\|_{H_0^1(U)} \\
&= 0.
\end{aligned}$$

Thus  $\int_0^T \phi'(t) \mathbf{u}(t) dt = - \int_0^T \phi(t) \mathbf{u}'(t) dt$  in  $L^2(U)$  for any  $\phi \in C_c^\infty(0, T)$ , which shows that  $\mathbf{u}'$  is also the time weak derivative of  $\mathbf{u}$  in  $L^1(0, T; L^2(U))$ . □

**Lemma 4.10.** Let  $\mathbf{u} \in L^1(0, T; H_0^1(U))$ ,  $\mathbf{v} \in L^1(0, T; H^{-1}(U))$ , we have  $\mathbf{v} = (\mathbf{u}^*)'$  ⇔

$$\forall \phi \in C_c^\infty(0, T), w \in H_0^1(U), \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt = - \int_0^T \phi(t) \langle \mathbf{v}(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt,$$

where  $\mathbf{u}^*(t) := \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)} \in H^{-1}(U)$  as usual.

*Proof.*  $(\mathbf{u}^*)' = \mathbf{v}$  by definition means

$$\forall \phi \in C_c^\infty(0, T), \int_0^T \phi'(t) \mathbf{u}^*(t) dt = - \int_0^T \phi(t) \mathbf{v}(t) dt$$

in  $H^{-1}(U)$ .

Consider any  $w \in H_0^1(U)$ , we have  $\langle \cdot | w \rangle_{H^{-1}(U), H_0^1(U)} \in (H^{-1}(U))^*$ .

By Bochner's Theorem 4.2 and linearity of the duality pairing, we have

$$\begin{aligned} \left\langle \int_0^T \phi'(t) \mathbf{u}^*(t) dt \middle| w \right\rangle_{H^{-1}(U), H_0^1(U)} &= \left\langle - \int_0^T \phi(t) \mathbf{v}(t) dt \middle| w \right\rangle_{H^{-1}(U), H_0^1(U)} \\ \int_0^T \langle \phi'(t) \mathbf{u}^*(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt &= - \int_0^T \langle \phi(t) \mathbf{v}(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}^*(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt &= - \int_0^T \phi(t) \langle \mathbf{v}(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \phi(t) \langle \mathbf{v}(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt. \end{aligned}$$

□

**Lemma 4.11.** Suppose  $\mathbf{u} \in L^1(0, T; H_0^1(U))$ , and  $\mathbf{u}'$  is its time weak derivative in  $L^1(0, T; L^2(U))$ , then we must have the action function

$$(\mathbf{u}')^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$$

is the time weak derivative of  $\mathbf{u}^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$  in  $L^1(0, T; H^{-1}(U))$ . Namely,

$$(\mathbf{u}')^* = (\mathbf{u}^*)'.$$

*Proof.* Consider any  $\phi \in C_c^\infty(0, T)$ , by definition of weak derivative, we have

$$\int_0^T \phi'(t) \mathbf{u}(t) dt = - \int_0^T \phi(t) \mathbf{u}'(t) dt.$$

in  $L^2(U)$ .

Now for any  $w \in H_0^1(U) \subset L^2(U)$ , we have  $\langle \cdot, w \rangle_{L^2(U)} \in (L^2(U))^*$ .

By Bochner's Theorem 4.2 and linearity of the inner product, we have

$$\begin{aligned} \left\langle \int_0^T \phi'(t) \mathbf{u}(t) dt, w \right\rangle_{L^2(U)} &= \left\langle - \int_0^T \phi(t) \mathbf{u}'(t) dt, w \right\rangle_{L^2(U)} \\ \int_0^T \langle \phi'(t) \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \langle \phi(t) \mathbf{u}'(t), w \rangle_{L^2(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \phi(t) \langle \mathbf{u}'(t), w \rangle_{L^2(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \phi(t) \langle (\mathbf{u}')^*(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt. \end{aligned}$$

Thus  $(\mathbf{u}')^* = (\mathbf{u}^*)'$  from the above lemma. □

**Corollary 4.12.** Suppose  $\mathbf{u}, \mathbf{u}' \in L^1(0, T; H_0^1(U))$ , then we must have the action function

$$(\mathbf{u}')^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$$

is the time weak derivative of  $\mathbf{u}^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$  in  $L^1(0, T; H^{-1}(U))$ .

*Remark.* Recall  $H_0^1(U) \subset L^2(U) \cong L^2(U)^* \subset H^{-1}(U)$ , and we can identify  $\mathbf{u}(t) \in H_0^1(U)$  with  $\mathbf{u}^*(t) := \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)} \in H^{-1}(U)$ . The above lemmas allow us to further abuse this notation and identify  $\mathbf{u}'$  with  $(\mathbf{u}')^* = (\mathbf{u}^*)'$ .

**Definition 4.5.** Suppose  $\mathbf{u} \in L^1(0, T; H_0^1(U))$ , we abuse the notation and denote

$$\mathbf{u}' := \mathbf{v} \in L^1(0, T; H^{-1}(U))$$

to be the time weak derivative of  $\mathbf{u}$ , if  $\mathbf{v} = (\mathbf{u}^*)'$  is the time weak derivative of the action function

$$\mathbf{u}^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$$

in  $L^1(0, T; H^{-1}(U))$ .

*Remark.* This is a further extension of the original definition of the weak derivative, since  $\mathbf{u}'$  may not exist in  $L^1(0, T; H_0^1(U))$  even if such a  $\mathbf{v} = (\mathbf{u}^*)'$  exists in  $L^1(0, T; H^{-1}(U))$  or even  $L^1(0, T; L^2(U)^*)$ .

We also have the following results:

**Theorem 4.13.** The dual space of  $L^2(0, T; H_0^1(U))$  is  $L^2(0, T; H^{-1}(U))$ , and the dual space of  $L^2(0, T; H^{-1}(U))$  is  $L^2(0, T; H_0^1(U))$ . The contraction map is defined to be  $\forall \mathbf{u} \in L^2(0, T; H_0^1(U)), \mathbf{v} \in L^2(0, T; H^{-1}(U))$ ,

$$\langle \mathbf{u} | \mathbf{v} \rangle_{L^2(0, T; H_0^1(U)), L^2(0, T; H^{-1}(U))} := \langle \mathbf{v} | \mathbf{u} \rangle_{L^2(0, T; H^{-1}(U)), L^2(0, T; H_0^1(U))} := \int_0^T \langle \mathbf{v}(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt.$$

*Proof.* We can quickly show one side of inclusion:

$$\begin{aligned} \int_0^T \langle \mathbf{v}(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt &\leq \int_0^T \|\mathbf{v}(t)\|_{H^{-1}(U)} \|\mathbf{u}(t)\|_{H_0^1(U)} dt \\ &\leq \left( \int_0^T \|\mathbf{v}(t)\|_{H^{-1}(U)}^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \|\mathbf{u}(t)\|_{H_0^1(U)}^2 dt \right)^{\frac{1}{2}} \\ &= \|\mathbf{v}\|_{L^2(0, T; H^{-1}(U))} \|\mathbf{u}\|_{L^2(0, T; H_0^1(U))}, \end{aligned}$$

which shows  $L^2(0, T; H_0^1(U)) \subseteq L^2(0, T; H^{-1}(U))^*$ , and  $L^2(0, T; H^{-1}(U)) \subseteq L^2(0, T; H_0^1(U))^*$ .  $\square$

**Theorem 4.14** (Fundamental Theorem of Lebesgue Integral Calculus).  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous if and only if there is a Lebesgue integrable function  $g$ , such that  $\forall x \in [a, b]$ ,  $f(x) = f(a) + \int_a^x g(t) dt$ . In this case,  $f$  is differentiable a.e., and  $f'(x) = g(x)$  for a.e.  $x \in [a, b]$ . (Theorem 6.10 and Cor 6.12 of Real-Analysis by Royden)

**Lemma 4.15.** Suppose  $\mathbf{u} \in L^1(0, T; H_0^1(U))$ ,  $\mathbf{u}' \in L^1(0, T; H^{-1}(U))$ , then  $\forall \epsilon > 0$ ,

$$\mathbf{u}'^\epsilon = \left( t \mapsto \langle (\mathbf{u}^\epsilon)'(t), \cdot \rangle_{L^2(U)} \right),$$

where  $(\mathbf{u}^\epsilon)'$  is the normal derivative of  $\mathbf{u}^\epsilon$  in  $C^\infty(\epsilon, T - \epsilon; H_0^1(U))$ .

*Proof.* Since  $\mathbf{u}'$  is the time weak derivative of  $\mathbf{u}^*$  in  $L^2((0, T); H^{-1}(U))$ , we can show that

$$\mathbf{u}'^\epsilon = \eta_\epsilon * \mathbf{u}' = (\eta_\epsilon * \mathbf{u}^*)'$$

in  $L^2((0, T); H^{-1}(U))$  similarly as in 2.22.

Also, one can see that  $\forall t \in [\epsilon, T - \epsilon], v \in H_0^1(U)$ ,

$$\begin{aligned}
\langle (\eta_\epsilon * \mathbf{u}^*)(t) | v \rangle_{H^{-1}(U), H_0^1(U)} &= \left\langle \int_0^T \eta_\epsilon(t - \tau) \mathbf{u}^*(\tau) d\tau \Big| v \right\rangle_{H^{-1}(U), H_0^1(U)} \\
&= \int_0^T \eta_\epsilon(t - \tau) \langle \mathbf{u}^*(\tau) | v \rangle_{H^{-1}(U), H_0^1(U)} d\tau \\
&= \int_0^T \eta_\epsilon(t - \tau) \langle \mathbf{u}(\tau), v \rangle_{L^2(U)} d\tau \\
&= \left\langle \int_0^T \eta_\epsilon(t - \tau) \mathbf{u}(\tau) d\tau, v \right\rangle_{L^2(U)} \\
&= \langle \mathbf{u}^\epsilon(t), v \rangle_{L^2(U)}.
\end{aligned}$$

Since  $(\mathbf{u}^\epsilon)'$  exists as the weak derivative of  $\mathbf{u}^\epsilon$ , we have

$$(\eta_\epsilon * \mathbf{u}^*)' = \left( t \mapsto \langle (\mathbf{u}^\epsilon)'(t), \cdot \rangle_{L^2(U)} \right)$$

in  $L^2((0, T); H^{-1}(U))$ .  $\square$

**Theorem 4.16.** Suppose  $\mathbf{u} \in L^2(0, T; H_0^1(U))$ ,  $\mathbf{u}' \in L^2(0, T; H^{-1}(U))$ , then

1. There is a representative  $\tilde{\mathbf{u}} \in C([0, T]; L^2(U))$  of  $\mathbf{u}$ .
  2. For any  $\mathbf{v} \in L^2(0, T; H_0^1(U))$ ,  $\mathbf{v}' \in L^2(0, T; H^{-1}(U))$ , the mapping  $t \mapsto \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)}$  is absolutely continuous, and for a.e.  $t \in [0, T]$ , we have
- $$\frac{d}{dt} \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} = \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)}.$$
3. The mapping  $t \mapsto \|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2$  is absolutely continuous, and for a.e.  $t \in [0, T]$ , we have
- $$\frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(U)}^2 = 2 \langle \mathbf{u}'(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)}.$$

4.  $\exists C > 0$ , such that  $\forall \mathbf{u} \in L^2(0, T; H_0^1(U))$ ,  $\mathbf{u}' \in L^2(0, T; H^{-1}(U))$ ,

$$\sup_{t \in [0, T]} \|\tilde{\mathbf{u}}(t)\|_{L^2(U)} \leq C \left( \|\mathbf{u}\|_{L^2(0, T; H_0^1(U))} + \|\mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} \right),$$

where the constant  $C$  only depends on  $T$ .

*Proof.* 1. We can extend  $\mathbf{u}$  to  $[-\sigma, T + \sigma]$  for an  $\delta > 0$  by reflection and cut off as done in 2.31. Now for any  $\epsilon, \delta \in (0, \sigma)$ , we can define  $\mathbf{u}^\epsilon := \eta_\epsilon * \mathbf{u}$ ,  $\mathbf{u}^\delta := \eta_\delta * \mathbf{u}$ , both well-defined on  $[0, T]$ . By 4.6,  $\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty([0, T]; H_0^1(U))$ , so we have

$$\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty([0, T]; H_0^1(U)) \subset C^\infty([0, T]; L^2(U)).$$

Now for any  $t \in [0, T]$ , we have that

$$\begin{aligned}
&\frac{d}{dt} \|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\|_{L^2(U)} \\
&= \frac{d}{dt} \langle \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \rangle_{L^2(U)} \\
&= \langle (\mathbf{u}^\epsilon - \mathbf{u}^\delta)'(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \rangle_{L^2(U)} + \langle \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t), (\mathbf{u}^\epsilon - \mathbf{u}^\delta)'(t) \rangle_{L^2(U)} \\
&= 2 \langle (\mathbf{u}^\epsilon)'(t) - (\mathbf{u}^\delta)'(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \rangle_{L^2(U)},
\end{aligned}$$

where  $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)'$  are the normal derivatives as functions  $[0, T] \rightarrow L^2(U)$ .

Also, since  $\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty([0, T]; H_0^1(U))$ , we have that their weak derivatives exist in  $L^2((0, T); H_0^1(U)) \subset L^1((0, T); H_0^1(U))$ , and by above lemma, are also weak derivatives in  $L^1((0, T); L^2(U))$ .

Since any weak derivative is a.e. equal to the normal derivative if the latter exists, we will just use  $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)'$  to represent the weak derivatives in  $L^2((0, T); H_0^1(U))$ , and the above equality still holds for a.e.  $t \in [0, T]$ .

Integrating both sides on any  $[s, t] \subseteq [0, T]$ , we get

$$\begin{aligned} & \| \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \|_{L^2(U)} - \| \mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s) \|_{L^2(U)} \\ &= \int_s^t \frac{d}{d\tau} \| \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) \|_{L^2(U)} d\tau \\ &= \int_s^t 2 \langle (\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau), \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) \rangle_{L^2(U)} d\tau \\ &= \int_s^t 2 \langle (\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau), \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \\ &\leq \int_s^t 2 \| (\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau) \|_{H^{-1}(U)} \| \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) \|_{H_0^1(U)} d\tau \\ &\leq \int_s^t \| (\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau) \|_{H^{-1}(U)}^2 + \| \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) \|_{H_0^1(U)}^2 d\tau \\ &= \| (\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)' \|_{L^2(0, T; H^{-1}(U))}^2 + \| \mathbf{u}^\epsilon - \mathbf{u}^\delta \|_{L^2(0, T; H_0^1(U))}^2, \end{aligned}$$

where we again identify  $(\mathbf{u}^\epsilon)'(\tau), (\mathbf{u}^\delta)'(\tau) \in H_0^1(U)$  with  $\langle (\mathbf{u}^\epsilon)'(\tau), \cdot \rangle_{L^2(U)}, \langle (\mathbf{u}^\delta)'(\tau), \cdot \rangle_{L^2(U)} \in H^{-1}(U)$  as usual.

By 4.6, we have  $\mathbf{u}^\epsilon, \mathbf{u}^\delta \rightarrow \mathbf{u}$  in  $L^2((0, T); H_0^1(U))$ , so

$$\begin{aligned} \| \mathbf{u}^\epsilon - \mathbf{u}^\delta \|_{L^2(0, T; H_0^1(U))} &= \| \mathbf{u}^\epsilon - \mathbf{u} + \mathbf{u} - \mathbf{u}^\delta \|_{L^2(0, T; H_0^1(U))} \\ &\leq \| \mathbf{u}^\epsilon - \mathbf{u} \|_{L^2(0, T; H_0^1(U))} + \| \mathbf{u} - \mathbf{u}^\delta \|_{L^2(0, T; H_0^1(U))} \\ &\rightarrow 0 \end{aligned}$$

as  $\epsilon, \delta \rightarrow 0$ .

Since  $\mathbf{u}'$  is the time weak derivative of  $\mathbf{u}^*$  in  $L^2((0, T); H^{-1}(U))$ , we can again show that  $(\eta_\epsilon * \mathbf{u}^*)' = \eta_\epsilon * \mathbf{u}'$  in  $L^2((0, T); H^{-1}(U))$ .

By 4.6 and lemma, we have  $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)' \rightarrow \mathbf{u}'$  in  $L^2(0, T; H^{-1}(U))$ .

Similar as above, we have  $\| (\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)' \|_{L^2(0, T; H^{-1}(U))} \rightarrow 0$  as  $\epsilon, \delta \rightarrow 0$ .

In addition, for a.e.  $s \in [0, T]$ , we must have  $\| \mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s) \|_{L^2(U)} \rightarrow 0$ .

Pick any of these  $s$ , we have

$$\| \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \|_{L^2(U)} \leq \| \mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s) \|_{L^2(U)} + \| (\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)' \|_{L^2(0, T; H^{-1}(U))}^2 + \| \mathbf{u}^\epsilon - \mathbf{u}^\delta \|_{L^2(0, T; H_0^1(U))}^2.$$

Since this holds for any  $t \in [0, T]$ , we have

$$\begin{aligned} & \| \mathbf{u}^\epsilon - \mathbf{u}^\delta \|_{C([0, T]; L^2(U))} \\ &= \sup_{t \in [0, T]; L^2(U)} \| \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \|_{L^2(U)} \\ &\leq \| \mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s) \|_{L^2(U)} + \| (\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)' \|_{L^2(0, T; H^{-1}(U))}^2 + \| \mathbf{u}^\epsilon - \mathbf{u}^\delta \|_{L^2(0, T; H_0^1(U))}^2 \\ &\rightarrow 0. \end{aligned}$$

as  $\epsilon, \delta \rightarrow 0$ , since each term goes to 0.

This shows that  $\mathbf{u}^\epsilon$  is a Cauchy sequence in  $C([0, T]; L^2(U))$ , and since it is a Banach Space, there

must be some

$$\tilde{\mathbf{u}} := \lim_{\epsilon \rightarrow 0} \mathbf{u}^\epsilon \in C([0, T]; L^2(U)).$$

Now since for a.e.  $t \in [0, T]$ ,  $\mathbf{u}(t) = \lim_{\epsilon \rightarrow 0} \mathbf{u}^\epsilon(t)$ , and  $\forall t \in [0, T]$ ,  $\tilde{\mathbf{u}}(t) = \lim_{\epsilon \rightarrow 0} \mathbf{u}^\epsilon(t)$ , we have that  $\tilde{\mathbf{u}} = \mathbf{u}$  for a.e.  $t \in [0, T]$  is a representative of  $\mathbf{u}$ .

2. Similar as above, we can show that for a.e.  $t \in [0, T]$ , we have

$$\frac{d}{dt} \langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} = \langle (\mathbf{u}^\epsilon)'(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(t), \mathbf{u}^\epsilon(t) \rangle_{L^2(U)}.$$

Integrating over any  $(s, t) \subset [0, T]$  gives

$$\langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} = \langle \mathbf{u}^\epsilon(s), \mathbf{v}^\epsilon(s) \rangle_{L^2(U)} + \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau.$$

Now

$$\begin{aligned} & \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &= \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) | \mathbf{v}^\epsilon(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &\leq \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) | \mathbf{v}^\epsilon(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau - \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &\quad + \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &= \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) | \mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| + \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &\leq \int_s^t \|(\mathbf{u}^\epsilon)'(\tau)\|_{H^{-1}(U)} \|\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)\|_{H_0^1(U)} d\tau + \int_s^t \|(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{v}(\tau)\|_{H_0^1(U)} d\tau \\ &\leq \sqrt{\int_s^t \|(\mathbf{u}^\epsilon)'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_s^t \|\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &\quad + \sqrt{\int_s^t \|(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_s^t \|\mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &\leq \sqrt{\int_0^T \|(\mathbf{u}^\epsilon)'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \|\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &\quad + \sqrt{\int_0^T \|(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \|\mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &= \|(\mathbf{u}^\epsilon)'\|_{L^2(0, T; H^{-1}(U))} \|\mathbf{v}^\epsilon - \mathbf{v}\|_{L^2(0, T; H_0^1(U))} + \|(\mathbf{u}^\epsilon)' - \mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} \|\mathbf{v}\|_{L^2(0, T; H_0^1(U))}, \end{aligned}$$

by Holder's Inequality.

Notice that

$$\|(\mathbf{u}^\epsilon)'\|_{L^2(0, T; H^{-1}(U))} \rightarrow \|\mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} < \infty, \|(\mathbf{u}^\epsilon)' - \mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} \rightarrow 0,$$

since we have shown  $(\mathbf{u}^\epsilon)' \rightarrow \mathbf{u}'$  in  $L^2(0, T; H^{-1}(U))$ .

Also,

$$\|\mathbf{v}^\epsilon - \mathbf{v}\|_{L^2(0, T; H_0^1(U))} \rightarrow 0,$$

since  $\mathbf{v}^\epsilon \rightarrow \mathbf{v}$  in  $L^2(0, T; H_0^1(U))$ .

Thus,  $\lim_{\epsilon \rightarrow 0} \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| = 0$ , which means

$$\lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau = \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Similarly, we also have

$$\lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau = \int_s^t \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Taking the limit of  $\epsilon \rightarrow 0$ , we have

$$\begin{aligned} & \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} \\ &= \lim_{\epsilon \rightarrow 0} \langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} \\ &= \lim_{\epsilon \rightarrow 0} \left( \langle \mathbf{u}^\epsilon(s), \mathbf{v}^\epsilon(s) \rangle_{L^2(U)} + \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau \right) \\ &= \lim_{\epsilon \rightarrow 0} \langle \mathbf{u}^\epsilon(s), \mathbf{v}^\epsilon(s) \rangle_{L^2(U)} + \lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau + \lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau \\ &= \langle \tilde{\mathbf{u}}(s), \tilde{\mathbf{v}}(s) \rangle_{L^2(U)} + \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau + \int_s^t \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \\ &= \langle \tilde{\mathbf{u}}(s), \tilde{\mathbf{v}}(s) \rangle_{L^2(U)} + \int_s^t \left( \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau. \end{aligned}$$

In particular, if we take  $s = 0$ , we have

$$\langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} = \langle \tilde{\mathbf{u}}(0), \tilde{\mathbf{v}}(0) \rangle_{L^2(U)} + \int_0^t \left( \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau.$$

Also, by Holder's Inequality,

$$\begin{aligned} & \int_0^T \left| \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right| d\tau \\ &\leq \int_0^T \left| \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right| d\tau + \int_0^T \left| \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right| d\tau \\ &\leq \int_0^T \| \mathbf{u}'(\tau) \|_{H^{-1}(U)} \| \mathbf{v}(\tau) \|_{H_0^1(U)} d\tau + \int_0^T \| \mathbf{v}'(\tau) \|_{H^{-1}(U)} \| \mathbf{u}(\tau) \|_{H_0^1(U)} d\tau \\ &\leq \sqrt{\int_0^T \| \mathbf{u}'(\tau) \|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \| \mathbf{v}(\tau) \|_{H_0^1(U)}^2 d\tau} + \sqrt{\int_0^T \| \mathbf{v}'(\tau) \|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \| \mathbf{u}(\tau) \|_{H_0^1(U)}^2 d\tau} \\ &= \| \mathbf{u}' \|_{L^2(0,T;H^{-1}(U))} \| \mathbf{v} \|_{L^2(0,T;H_0^1(U))} + \| \mathbf{v}' \|_{L^2(0,T;H^{-1}(U))} \| \mathbf{u} \|_{L^2(0,T;H_0^1(U))} \\ &< \infty. \end{aligned}$$

We have shown that  $\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)}$  is Lebesgue integrable. By Royden-Fitzpatrick's Theorem, we have that  $t \mapsto \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)}$  is absolutely continuous, and for a.e.  $t \in [0, T]$ ,

$$\frac{d}{dt} \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} = \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)}.$$

Since  $\tilde{\mathbf{u}}(t) = \mathbf{u}(t)$  for a.e.  $t \in [0, T]$ , we have the result.

3. Take  $\mathbf{v} = \mathbf{u}$  in 2.

4. Integrate  $\|\tilde{\mathbf{u}}(t)\|_{L^2(U)} = \|\tilde{\mathbf{u}}(s)\|_{L^2(U)} + \int_s^t 2\langle \mathbf{u}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau$  for  $0 \leq s \leq T$ , we have

$$\begin{aligned}
T\|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2 &= \int_0^T \|\mathbf{u}(t)\|_{L^2(U)}^2 dt \\
&= \int_0^T \|\tilde{\mathbf{u}}(s)\|_{L^2(U)}^2 ds + 2 \int_0^T \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau ds \\
&\leq \int_0^T \|\tilde{\mathbf{u}}(s)\|_{H_0^1(U)}^2 ds + 2 \int_0^T \int_s^t \|\mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}(\tau)\|_{H_0^1(U)} d\tau ds \\
&\leq \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + 2 \int_0^T \int_0^T \|\mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}(\tau)\|_{H_0^1(U)} d\tau ds \\
&= \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + 2T \int_0^T \|\mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}(\tau)\|_{H_0^1(U)} d\tau \\
&\leq \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + T \int_0^T \|\mathbf{u}'(\tau)\|_{H^{-1}(U)}^2 + \|\mathbf{u}(\tau)\|_{H_0^1(U)}^2 d\tau \\
&= (T+1)\|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + T\|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 \\
\|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2 &\leq \frac{T+1}{T} \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 \\
&\leq C^2 \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + C^2 \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 \\
&\leq C^2 \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + C^2 \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 + 2C^2 \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \\
&= \left( C\|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} + C\|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \right)^2 \\
\|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2 &\leq C \left( \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \right),
\end{aligned}$$

where we take  $C^2 := \max(1, \frac{T+1}{T}) > 1$ , which is independent of  $\mathbf{u}$  and  $\mathbf{u}'$ . Since this holds for any  $t \in [0, T]$ , we have

$$\sup_{t \in [0, T]} \|\tilde{\mathbf{u}}(t)\|_{L^2(U)} \leq C \left( \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \right).$$

□

## 4.2 Second Order Parabolic Equations

**Definition 4.6.** Let  $U \subseteq \mathbb{R}^n$  be open and bounded, we define  $U_T := U \times (0, T]$  for  $T > 0$ .

**Definition 4.7.** An **initial boundary value problem** is: given  $f : U_T \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}$ , we want to find  $u(x, t) : \bar{U}_T \rightarrow \mathbb{R}$ , such that

$$\begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases},$$

where

$$Lu := - \sum_{i,j=1}^n \partial_j (a^{ij}(\cdot, t) \partial_i u) + \sum_{i=1}^n b^i(\cdot, t) \partial_i u + c(\cdot, t)u$$

for some  $a^{ij}, b^i, c : U_T \rightarrow \mathbb{R}$ .

We say that the partial differential operator  $\partial_t + L$  is an **symmetric (uniformly) parabolic second order differential operator** if  $a^{ij} = a^{ji}$ , and  $\exists \theta > 0$ , such that

$$\sum_{i,j=1}^n a^{ij}(x, t) \xi_i \xi_j \geq \theta \|\xi\|_2^2, \quad \forall (x, t) \in U_T, \xi \in \mathbb{R}^n.$$

**Definition 4.8.** The **parabolic assumptions** are:

1.  $U \subseteq \mathbb{R}^n$  is bounded and open
2.  $T > 0$
3.  $a^{ij}, b^i, c \in L^\infty(U_T)$
4.  $f \in L^2(U_T), g \in L^2(U)$
5.  $\partial_t + L$  is a symmetric (uniformly) parabolic second order differential operator.

**Definition 4.9.** Given a function  $u : U_T \rightarrow \mathbb{R}$ , we want to consider  $\mathbf{u} : t \mapsto u(\cdot, t)$ , for any  $t \in [0, T]$ .

**Proposition 4.17.** Let  $U \subseteq \mathbb{R}^n$  be bounded and open,  $T > 0$ , then  $f \in L^2(U_T) \iff \mathbf{f} \in L^2(0, T; L^2(U))$ .

*Proof.* We have

$$\begin{aligned} \|\mathbf{f}\|_{L^2(0, T; L^2(U))}^2 &= \int_0^T \|\mathbf{f}(t)\|_{L^2(U)}^2 dt \\ &= \int_0^T \int_U |f(x, t)|^2 dx dt \\ &= \|f\|_{L^2(U_T)}^2. \end{aligned}$$

□

**Definition 4.10.**  $\mathbf{u} \in L^2(0, T; H_0^1(U))$ , identified with its continuous representative  $\tilde{\mathbf{u}} \in C([0, T]; L^2(U))$  as in 4.16.1, with the time weak derivative  $\mathbf{u}' \in L^2(0, T; H^{-1}(U))$ , is a **weak solution** of the IBVP if

$$\begin{aligned} \forall v \in H_0^1(U), \quad &\langle \mathbf{u}'(t) | v \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t] = \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ a.e. } t \in [0, T], \\ &\mathbf{u}(0) = g, \end{aligned}$$

where bilinear form associated to the above problem is

$$\forall w, v \in H_0^1(U), \quad B[w, v; t] := \int_U \left( \sum_{i,j=1}^n a^{ij}(\cdot, t) \partial_i w \partial_j v + \sum_{i=1}^n b^i(\cdot, t) \partial_i w v + c(\cdot, t) w v \right) dx.$$

### 4.3 Galerkin Method

**Definition 4.11.** Let  $(w_k)_{k=1}^\infty$  be an orthogonal basis of  $H_0^1(U)$ , and also an orthonormal basis of  $L^2(U)$ . For  $m \in \mathbb{N}^+$ , we define  $V_m := \text{Span}(\{w_j\}_{j=1}^m) \subset H_0^1(U)$  be a subspace. A function  $\mathbf{u}_m := t \mapsto \sum_{k=1}^m d_m^k(t) w_k$  is a **weak solution of the problem in  $V_m$**  if  $\forall v \in V_m$ ,

$$\begin{aligned} \left\langle \sum_{k=1}^m d_m^k'(t) w_k, v \right\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T], \\ \langle \mathbf{u}_m(0), v \rangle_{L^2(U)} &= \langle g, v \rangle_{L^2(U)}. \end{aligned}$$

**Definition 4.12.** We define the **ODE system associated to the problem** to be:  $\forall j \in [m]$ ,

1.  $d_m^j : [0, T] \rightarrow \mathbb{R}$  is absolutely continuous.
2. For a.e.  $t \in [0, T]$ ,  $d_m^j'(t) = - \sum_{k=1}^m e_k^j(t) d_m^k(t) + f^j(t)$
3.  $d_m^j(0) = \langle g, w_j \rangle_{L^2(U)}$ ,

where  $e_k^j(t) := B[w_k, w_j; t]$ ,  $f^j(t) := \langle \mathbf{f}(t), w_j \rangle_{L^2(U)}$ .

**Proposition 4.18.**  $\mathbf{u}_m(t) = \sum_{k=1}^m d_m^k(t)w_k$  is a weak solution in  $V_m$  if and only if  $\vec{d}_m$  is a solution to the ODE system.

*Proof.* Since  $(w_k)_{k=1}^\infty$  is an orthonormal basis of  $V_m$  in  $\langle \cdot, \cdot \rangle_{L^2(U)}$ , we have

$$\begin{aligned}
& \left\langle \sum_{k=1}^m d_m^k'(t)w_k, v \right\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] = \langle \mathbf{f}(t), v \rangle, & \forall v \in V_m \\
& \iff \\
& \left\langle \sum_{k=1}^m d_m^k'(t)w_k, v \right\rangle_{L^2(U)} + B \left[ \sum_{k=1}^m d_m^k(t)w_k, v; t \right] = \langle \mathbf{f}(t), v \rangle, & \forall v \in V_m \\
& \iff \\
& \left\langle \sum_{k=1}^m d_m^k'(t)w_k, w_j \right\rangle_{L^2(U)} + B \left[ \sum_{k=1}^m d_m^k(t)w_k, w_j; t \right] = \langle \mathbf{f}(t), w_j \rangle, & \forall j \in [m] \\
& \iff \\
& \sum_{k=1}^m d_m^k'(t) \langle w_k, w_j \rangle_{L^2(U)} + \sum_{k=1}^m d_m^k(t) B[w_k, w_j; t] = \langle \mathbf{f}(t), w_j \rangle, & \forall j \in [m] \\
& \iff \\
& \sum_{k=1}^m d_m^k'(t) \delta_k^j + \sum_{k=1}^m d_m^k(t) e_k^j(t) = f^j(t), & \forall j \in [m] \\
& \iff \\
& d_m^j'(t) + \sum_{k=1}^m d_m^k(t) e_k^j(t) = f^j(t), & \forall j \in [m].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \langle \mathbf{u}_m(0), v \rangle_{L^2(U)} = \langle g, v \rangle_{L^2(U)}, & v \in V_m \\
& \iff \\
& \langle \mathbf{u}_m(0), w_j \rangle_{L^2(U)} = \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
& \iff \\
& \left\langle \sum_{k=1}^m d_m^k(0)w_k, w_j \right\rangle_{L^2(U)} = \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
& \iff \\
& \sum_{k=1}^m d_m^k(0) \langle w_k, w_j \rangle_{L^2(U)} = \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
& \iff \\
& \sum_{k=1}^m d_m^k(0) \delta_k^j = \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
& \iff \\
& d_m^j(0) = \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m].
\end{aligned}$$

□

**Theorem 4.19.** Since  $f^i, e_j^k$  are locally integrable, there is a unique absolutely continuous solution  $\vec{d}_m$  to the ODE system.

**Corollary 4.20.** For each  $m \in \mathbb{N}^+$ , there is a unique weak solution  $\mathbf{u}_m$  of the form  $t \mapsto \sum_{k=1}^m d_m^k(t)w_k$  of the problem in  $V_m$ .

**Proposition 4.21.** The weak solution  $\mathbf{u}_m$  also satisfies  $\forall v \in V_m$ ,

$$\begin{aligned}\langle \mathbf{u}'_m(t), v \rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T], \\ \langle \mathbf{u}_m(0), v \rangle_{L^2(U)} &= \langle g, v \rangle_{L^2(U)}.\end{aligned}$$

In addition,

$$\|\mathbf{u}_m(0)\|_{L^2(U)} \leq \|g\|_{L^2(U)},$$

and

$$\lim_{m \rightarrow \infty} \mathbf{u}_m(0) = g$$

in  $L^2(U)$ .

*Proof.* Consider any  $\phi \in C_c^\infty(0, T)$ . By linearity of the Bochner integral, we have

$$\begin{aligned}\int_0^T \phi'(t) \mathbf{u}_m(t) dt &= \int_0^T \left( \phi'(t) \sum_{k=1}^m d_m^k(t) w_k \right) dt \\ &= \sum_{k=1}^m \left( \int_0^T \phi'(t) d_m^k(t) w_k dt \right) \\ &= \sum_{k=1}^m \left( \int_0^T \phi'(t) d_m^k(t) dt \right) w_k \\ &= \sum_{k=1}^m \left( \int_0^T \phi(t) d_m^{k'}(t) dt \right) w_k \\ &= \int_0^T \phi(t) \left( \sum_{k=1}^m d_m^{k'}(t) w_k \right) dt.\end{aligned}$$

Thus it has a weak derivative  $\mathbf{u}'_m(t) = \sum_{k=1}^m d_m^{k'}(t) w_k$ .

We then plug that in the definition of weak solution of the problem in  $V_m$ .

On the other hand, since  $\mathbf{u}_m(0) \in V_m$ , we have that

$$\begin{aligned}\|\mathbf{u}_m(0)\|_{L^2(U)}^2 &= \langle \mathbf{u}_m(0), \mathbf{u}_m(0) \rangle_{L^2(U)} \\ &= \langle g, \mathbf{u}_m(0) \rangle_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|\mathbf{u}_m(0)\|_{L^2(U)}.\end{aligned}$$

Since  $\|\mathbf{u}_m(0)\|_{L^2(U)} \geq 0$ , we have  $\|\mathbf{u}_m(0)\|_{L^2(U)} \leq \|g\|_{L^2(U)}$ .

Also, since  $(w_k)_{k=1}^\infty$  is an orthonormal basis of  $L^2(U)$ , we have

$$\begin{aligned}\lim_{m \rightarrow \infty} \mathbf{u}_m(0) &= \lim_{m \rightarrow \infty} \sum_{k=1}^m d_m^k(t) w_k \\ &= \lim_{m \rightarrow \infty} \sum_{k=1}^m \langle g, w_k \rangle_{L^2(U)} w_k \\ &= g.\end{aligned}$$

□

**Proposition 4.22.** The weak solution  $\mathbf{u}_m \in L^2(0, T; H_0^1(U))$ .

*Proof.*

$$\begin{aligned}
\int_0^T \|\mathbf{u}_m\|_{H_0^1(U)}^2 dt &= \int_0^T \left\| \sum_{k=1}^m d_m^k(t) w_k \right\|_{H_0^1(U)}^2 dt \\
&= \int_0^T \left\langle \sum_{k=1}^m d_m^k(t) w_k, \sum_{j=1}^m d_m^j(t) w_j \right\rangle_{H_0^1(U)} dt \\
&= \int_0^T \sum_{k=1}^m \sum_{j=1}^m d_m^k(t) d_m^j(t) \langle w_k, w_j \rangle_{H_0^1(U)} dt \\
&= \int_0^T \sum_{k=1}^m \|d_m^k(t)\|^2 \|w_k\|_{H_0^1(U)}^2 dt \\
&= \sum_{k=1}^m \|w_k\|_{H_0^1(U)}^2 \int_0^T \|d_m^k(t)\|^2 dt \\
&= \sum_{k=1}^m \|w_k\|_{H_0^1(U)}^2 \|d_m^k\|_{L^2(0,T)}^2 \\
&\leq \infty,
\end{aligned}$$

since each  $w_k \in H_0^1(U)$ , and each  $d_m^k$  is absolutely continuous (thus continuous, thus in  $L^2(0,T)$ ).  $\square$

**Proposition 4.23.** *The weak solution  $\mathbf{u}_m \in C([0,T]; L^2(U))$ .*

*Proof.* Given any  $\epsilon > 0$ .

For any  $1 \leq k \leq m$ , since  $d_m^k$  is absolutely continuous (thus continuous), we can find  $\delta_k > 0$ , such that

$$\forall s, t \in [0, T], |t - s| < \delta_k \implies |d_m^k(t) - d_m^k(s)| < \frac{\epsilon}{\sqrt{m}}.$$

Now take  $\delta := \min_{k \in [m]} \delta_k > 0$ , we have that  $\forall s, t \in [0, T]$ , such that  $|t - s| < \delta$ ,

$$\begin{aligned}
\|\mathbf{u}_m(t) - \mathbf{u}_m(s)\|_{L^2(U)}^2 &= \left\| \sum_{k=1}^m d_m^k(t) w_k - \sum_{k=1}^m d_m^k(s) w_k \right\|_{L^2(U)}^2 \\
&= \left\| \sum_{k=1}^m (d_m^k(t) - d_m^k(s)) w_k \right\|_{L^2(U)}^2 \\
&= \sum_{k=1}^m (d_m^k(t) - d_m^k(s))^2 \\
&< \sum_{k=1}^m \frac{\epsilon^2}{m} \\
&= \epsilon^2,
\end{aligned}$$

since  $(w_k)_{k=1}^\infty$  is an orthonormal basis for  $L^2(U)$ .<sup>2</sup>

Thus  $\|\mathbf{u}_m(t) - \mathbf{u}_m(s)\|_{L^2(U)} < \epsilon$  and since this works for any  $\epsilon > 0$ , we have that  $\mathbf{u}_m \in C([0, T]; L^2(U))$ .  $\square$

Notice that the above two propositions says that  $\mathbf{u}_m$  is the representative  $\tilde{u}_m$  as in 4.16, and we thus have the following result:

**Corollary 4.24.** *The weak solution  $\mathbf{u}_m$  satisfies that the mapping  $t \mapsto \|\mathbf{u}_m(t)\|_{L^2(U)}^2$  is absolutely continuous, and for a.e.  $t \in [0, T]$ , we have*

$$\frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 = 2 \langle \mathbf{u}'_m(t) | \mathbf{u}_m(t) \rangle_{H^{-1}(U), H_0^1(U)} = 2 \langle \mathbf{u}'_m(t), \mathbf{u}_m(t) \rangle_{L^2(U)}.$$

**Theorem 4.25** (Gronwall's inequality). *Let  $\eta : [0, T] \rightarrow \mathbb{R}$  be nonnegative and absolutely continuous,  $\phi, \psi$  both nonnegative summable functions. If*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t) \text{ a.e. } t \in [0, T],$$

then

$$\eta(t) \leq \exp\left(\int_0^t \phi(s)ds\right)\left(\eta(0) + \int_0^t \psi(s)ds\right), \quad \forall t \in [0, T].$$

**Theorem 4.26** (Energy Estimate). *Let  $U \subseteq \mathbb{R}^n$  be bounded and open,  $T > 0$ ,  $a^{ij}, b^i, c \in L^\infty(U_T)$ , and  $\partial_t + L$  be a symmetric (uniformly) parabolic second order differential operator. There exists  $C > 0$  that only depends on  $U, T, L$ , such that  $\forall f \in L^2(U_T)$ ,  $g \in L^2(U)$ ,  $m \in \mathbb{N}^+$ ,*

$$\sup_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(U)} + \|\mathbf{u}_m\|_{L^2(0, T; H_0^1(U))} + \|\mathbf{u}'_m\|_{L^2(0, T; H^{-1}(U))} \leq C\left(\|f\|_{L^2(0, T; L^2(U))} + \|g\|_{L^2(U)}\right),$$

where  $\mathbf{u}_m$  are the weak solutions in  $V_m$  as in above.

*Proof.* We will bound each term on the left hand side.

1. Consider any  $m \in \mathbb{N}^+$ , we have that the  $\mathbf{u}_m$  satisfies  $\forall v \in V_m$ ,

$$\begin{aligned} \langle \mathbf{u}'_m(t), v \rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T], \\ \langle \mathbf{u}_m(0), v \rangle_{L^2(U)} &= \langle g, v \rangle_{L^2(U)}, \end{aligned}$$

In particular,  $\mathbf{u}_m(t) = \sum_{k=1}^m d_m^k(t)w_k \in V_m$ .

Thus for a.e.  $t \in [0, T]$ , we have

$$\langle \mathbf{u}'_m(t), \mathbf{u}_m(t) \rangle_{L^2(U)} + B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] = \langle \mathbf{f}(t), \mathbf{u}_m(t) \rangle_{L^2(U)}.$$

By a similar proof as in 3.5, there exists constants  $\beta > 0, \gamma \geq 0$  that only depends on  $U$  and the coefficients of  $L$ , such that  $\forall u \in H_0^1(U)$ , and for a.e.  $t \in [0, T]$ ,

$$\beta\|u\|_{H^1(U)}^2 \leq B[u, u; t] + \gamma\|u\|_{L^2(U)}^2.$$

We thus have

$$\begin{aligned} \langle \mathbf{u}'_m(t), \mathbf{u}_m(t) \rangle_{L^2(U)} + B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] &\geq \langle \mathbf{u}'_m(t), \mathbf{u}_m(t) \rangle_{L^2(U)} + \beta\|\mathbf{u}_m(t)\|_{H^1(U)}^2 - \gamma\|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ &= \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + \beta\|\mathbf{u}_m(t)\|_{H^1(U)}^2 - \gamma\|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ \langle \mathbf{f}(t), \mathbf{u}_m(t) \rangle_{L^2(U)} &\leq \|\mathbf{f}(t)\|_{L^2(U)} \|\mathbf{u}_m(t)\|_{L^2(U)} \\ &\leq \frac{1}{2} \|\mathbf{f}(t)\|_{L^2(U)}^2 + \frac{1}{2} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + 2\beta\|\mathbf{u}_m(t)\|_{H^1(U)}^2 &\leq \|\mathbf{f}(t)\|_{L^2(U)}^2 + \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + 2\gamma\|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ &= \|\mathbf{f}(t)\|_{L^2(U)}^2 + (1 + 2\gamma)\|\mathbf{u}_m(t)\|_{L^2(U)}^2. \end{aligned}$$

Notice that this inequality hold for any  $f \in L^2(U_T)$ ,  $g \in L^2(U)$ ,  $m \in \mathbb{N}^+$ , and the corresponding weak solutions  $\mathbf{u}_m$  in  $V_m$ .

2. Since  $2\beta\|\mathbf{u}_m(t)\|_{H^1(U)}^2 \geq 0$ , we have

$$\frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 \leq \|\mathbf{f}(t)\|_{L^2(U)}^2 + (1 + 2\gamma)\|\mathbf{u}_m(t)\|_{L^2(U)}^2.$$

Take  $\eta(t) := \|\mathbf{u}_m(t)\|_{L^2(U)}^2$ , which is nonnegative and absolutely continuous.

Also, take  $\psi(t) := \|\mathbf{f}(t)\|_{L^2(U)}^2$ ,  $\phi(t) := 1 + 2\gamma$ , which are both nonnegative and summable.

By Gronwall's inequality, we have that  $\forall t \in [0, T]$ ,

$$\begin{aligned} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 &\leq \exp\left(\int_0^t (1+2\gamma)ds\right) \left( \|\mathbf{u}_m(0)\|_{L^2(U)}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2(U)}^2 ds \right) \\ &= \exp(t(1+2\gamma)) \left( \|\mathbf{u}_m(0)\|_{L^2(U)}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2(U)}^2 ds \right) \\ &\leq \exp(T(1+2\gamma)) \left( \|g\|_{L^2(U)}^2 + \int_0^T \|\mathbf{f}(s)\|_{L^2(U)}^2 ds \right) \\ &= C_1 \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) \\ &\leq C_1 \left( \|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right)^2, \end{aligned}$$

where we take  $C_1 := \exp(T(1+2\gamma)) > 0$  that only depends on  $T, \gamma$ .

We thus have shown that  $\forall t \in [0, T]$ ,  $\|\mathbf{u}_m(t)\|_{L^2(U)} \leq \sqrt{C_1} \left( \|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right)$ . Thus,

$$\sup_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(U)} \leq \sqrt{C_1} \left( \|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right),$$

which bounds the first term in what we want.

3. From step 1, we have that for a.e.  $t \in [0, T]$ ,

$$\frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + 2\beta \|\mathbf{u}_m(t)\|_{H^1(U)}^2 \leq \|\mathbf{f}(t)\|_{L^2(U)}^2 + (1+2\gamma) \|\mathbf{u}_m(t)\|_{L^2(U)}^2.$$

Integrating over  $[0, T]$  gives

$$\begin{aligned} \int_0^T \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt + 2\beta \int_0^T \|\mathbf{u}_m(t)\|_{H^1(U)}^2 dt &\leq \int_0^T \|\mathbf{f}(t)\|_{L^2(U)}^2 dt + (2\gamma+1) \int_0^T \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt \\ \|\mathbf{u}_m(T)\|_{L^2(U)}^2 - \|\mathbf{u}_m(0)\|_{L^2(U)}^2 + 2\beta \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 &\leq \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma+1) \int_0^T \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt. \end{aligned}$$

Notice that  $\|\mathbf{u}_m(T)\|_{L^2(U)}^2 \geq 0$ , and in step 2, we have shown that for a.e.  $t \in [0, T]$ ,

$$\|\mathbf{u}_m(t)\|_{L^2(U)}^2 \leq C_1 \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right).$$

$$\begin{aligned} 2\beta \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 &\leq \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma+1) \int_0^T \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt + \|\mathbf{u}_m(0)\|_{L^2(U)}^2 \\ 2\beta \|\mathbf{u}_m\|_{L^2(0,T;H^1(U))}^2 &\leq \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma+1) \int_0^T C_1 \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) dt + \|g\|_{L^2(U)}^2 \\ &= \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma+1)TC_1 \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) + \|g\|_{L^2(U)}^2 \\ &= ((2\gamma+1)TC_1 + 1) \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) \\ \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 &\leq C_2 \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) \\ &\leq C_2 \left( \|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right)^2, \end{aligned}$$

where  $C_2 := \frac{(2\gamma+1)TC_1+1}{\beta} > 0$  only depends on  $\beta, \gamma, T$ .

We thus have bounded the second term

$$\|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))} \leq \sqrt{C_2} \left( \|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right).$$

4. Now fix any function  $v \in H_0^1(U)$  with  $\|v\|_{H_0^1(U)} = 1$ , write  $v = \sum_{j=1}^{\infty} \hat{v}_j w_j = v_1 + v_2$ , where  $v_1 := \sum_{j=1}^m \hat{v}_j w_j \in V_m$ ,  $v_2 := \sum_{j=m+1}^{\infty} \hat{v}_j w_j \in V_m^\perp$  is the unique decomposition of  $v$  in  $H_0^1(U) = V_m \oplus V_m^\perp$ . Notice that  $\|v\|_{H^1(U)}^2 = \|v_1\|_{H^1(U)}^2 + \|v_2\|_{H^1(U)}^2$ . Thus  $\|v_1\|_{H^1(U)} \leq 1$ .

Since  $v_1 \in V_m$ , we have for a.e.  $t \in [0, T]$ ,  $\langle \mathbf{u}'_m(t), v_1 \rangle_{L^2(U)} + B[\mathbf{u}_m(t), v_1; t] = \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)}$ .

Now since  $\mathbf{u}'_m(t) \in V_m$ , we have  $\langle \mathbf{u}'_m(t), v_2 \rangle_{L^2(U)} = 0$ , so

$$\begin{aligned} \langle \mathbf{u}'_m(t) | v \rangle_{H^{-1}(U), H_0^1(U)} &= \langle \mathbf{u}'_m(t), v \rangle_{L^2(U)} \\ &= \langle \mathbf{u}'_m(t), v_1 \rangle_{L^2(U)} + \langle \mathbf{u}'_m(t), v_2 \rangle_{L^2(U)} \\ &= \langle \mathbf{u}'_m(t), v_1 \rangle_{L^2(U)} \\ &= \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)} - B[\mathbf{u}_m(t), v_1; t]. \end{aligned}$$

Again by a similar proof as in 3.5, there exists constants  $\alpha > 0$  that only depends on  $U$  and the coefficients of  $L$ , such that  $\forall u, v \in H_0^1(U)$ , and for a.e.  $t \in [0, T]$ ,

$$|B[u, v; t]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}.$$

We thus have

$$\begin{aligned} \langle \mathbf{u}'_m(t) | v \rangle_{H^{-1}(U), H_0^1(U)} &= \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)} - B[\mathbf{u}_m(t), v_1; t] \\ &\leq \|\mathbf{f}(t)\|_{L^2(U)} \|v_1\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \|v_1\|_{H^1(U)} \\ &\leq \|\mathbf{f}(t)\|_{L^2(U)} \|v_1\|_{H^1(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \|v_1\|_{H^1(U)} \\ &\leq \|\mathbf{f}(t)\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)}. \end{aligned}$$

Since this holds for any  $v \in H_0^1(U)$  with  $\|v\|_{H_0^1(U)} = 1$ , we have

$$\|\mathbf{u}'_m(t)\|_{H^{-1}(U)} = \sup_{v \in H_0^1(U) \text{ such that } \|v\|_{H_0^1(U)}=1} \langle \mathbf{u}'_m(t) | v \rangle_{H^{-1}(U), H_0^1(U)} \leq \|\mathbf{f}(t)\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)}.$$

Squaring and integrating this over  $[0, T]$ , we have

$$\begin{aligned} \int_0^T \|\mathbf{u}'_m(t)\|_{H^{-1}(U)}^2 dt &\leq \int_0^T \left( \|\mathbf{f}(t)\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \right)^2 dt \\ \|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))}^2 &\leq \int_0^T 2 \left( \|\mathbf{f}(t)\|_{L^2(U)}^2 + \alpha^2 \|\mathbf{u}_m(t)\|_{H^1(U)}^2 \right) dt \\ &\leq 2 \int_0^T \|\mathbf{f}(t)\|_{L^2(U)}^2 dt + 2\alpha^2 \int_0^T \|\mathbf{u}_m(t)\|_{H^1(U)}^2 dt \\ &= 2\|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + 2\alpha^2 \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 \\ &\leq 2\|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + 2\alpha^2 C_2 \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) \\ &\leq C_3 \left( \|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) \\ &\leq C_3 \left( \|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right)^2, \end{aligned}$$

where we take  $C_3 := 2\alpha^2 C_2 + 1 > 0$  that only depends on  $\beta, \gamma, \alpha, T$ .

We thus have bounded the third term  $\|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))} \leq \sqrt{C_3} \left( \|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right)$ .

Now let us take  $C := \sqrt{\max\{C_1, C_2, C_3\}} > 0$ , which only depends on  $U, T, L$ , we have that

$$\sup_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(U)} + \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))} \leq C \left( \|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)} \right).$$

□

**Theorem 4.27.** *There is a weak solution to the IVP, namely,  $\exists \mathbf{u} \in L^2(0, T; H_0^1(H))$ , identified with its continuous representative  $\tilde{\mathbf{u}} \in C([0, T]; L^2(U))$ , such that*

$$\begin{aligned} \langle \mathbf{u}'(t)|v \rangle + B[\mathbf{u}(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \quad \forall v \in H_0^1(U), \text{ a.e. } t \in [0, T] \\ \mathbf{u}(0) &= g \end{aligned}$$

*Proof.* By energy estimate, we have that  $(\mathbf{u}_m)_{m=1}^\infty$  is bounded in  $L^2(0, T; H_0^1(U))$ .

By 1.28, there is a subsequence  $(\mathbf{u}_{m_j})_{j=1}^\infty$  and  $\mathbf{u} \in L^2(0, T; H_0^1(U))$  such that  $\mathbf{u}_{m_j} \rightharpoonup \mathbf{u}$ . WLOG, we will consider its continuous representative  $\mathbf{u} \in C(0, T; L^2(U))$  by 4.16.1.

Similarly,  $(\mathbf{u}'_m)_{m=1}^\infty$  is bounded in  $L^2(0, T; H^{-1}(U))$ , so is  $(\mathbf{u}'_{m_j})_{j=1}^\infty$ . Thus there is a subsequence  $(\mathbf{u}'_{m_{j_l}})_{l=1}^\infty$  and  $\mathbf{w} \in L^2(0, T; H^{-1}(U))$  such that  $\mathbf{u}'_{m_{j_l}} \rightharpoonup \mathbf{w}$ .

Since  $\mathbf{u}_{m_j} \rightharpoonup \mathbf{u}$ , we must have  $\mathbf{u}_{m_{j_l}} \rightharpoonup \mathbf{u}$  as well. By A5Q3,  $\mathbf{w} = \mathbf{u}'$ .

Now we would like to show that  $\mathbf{u}$  is indeed a weak solution to the IVP.

Consider  $\mathbf{v} \in C^1([0, T]; H_0^1(U))$  of the form  $\mathbf{v}(t) = \sum_{k=1}^N d^k(t) w_k$ , where  $N > 0$  is an integer,  $(d^k(t))_{k=1}^N$  are smooth functions, and  $(w_k)_{k=1}^\infty$  be a basis as before.

We can show that these  $\mathbf{v}$  are dense in  $L^2(0, T; H_0^1(U))$ .

For any such  $\mathbf{v}$ , if we choose any  $m \geq N$ , we have the weak solution  $\vec{u}_m$  in  $V_m$  satisfies

$$\langle \mathbf{u}'_m(t)|w_k \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}_m(t), w_k; t] = \langle \mathbf{f}(t), w_k \rangle_{L^2(U)}, \quad \forall k \in [m], \text{ for a.e. } t \in [0, T].$$

Multiplying by  $d^k(t)$  and summing over  $k \in [N]$ , we have that

$$\langle \mathbf{u}'_m(t)|\mathbf{v}(t) \rangle + B[\mathbf{u}_m(t), \mathbf{v}(t); t] = \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T].$$

Integrating over  $t \in [0, T]$ , we have

$$\int_0^T \langle \mathbf{u}'_m(t)|\mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}_m(t), \mathbf{v}(t); t] dt = \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt.$$

Since  $\mathbf{v} \in C^1([0, T]; H_0^1(U)) \subset L^2(0, T; H_0^1(U)) \cong (L^2(0, T; H^{-1}(U)))^*$ , and  $\mathbf{u}'_{m_{j_l}} \rightharpoonup \mathbf{v}$ , we have

$$\int_0^T \langle \mathbf{u}'_{m_{j_l}}(t)|\mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt \rightarrow \int_0^T \langle \mathbf{u}'(t)|\mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt.$$

Also, if we consider the operator  $T_v : \mathbf{w} \mapsto \int_0^T B[\mathbf{w}(t), \mathbf{v}(t); t] dt$  for any  $\mathbf{w} \in L^2(0, T; H_0^1(U))$ , we can see that

$$\begin{aligned} |T_v \mathbf{w}| &= \left| \int_0^T B[\mathbf{w}_m(t), \mathbf{v}(t); t] dt \right| \\ &\leq \int_0^T |B[\mathbf{w}_m(t), \mathbf{v}(t); t]| dt \\ &\leq \int_0^T \alpha \|\mathbf{w}_m(t)\|_{H^1(U)} \|\mathbf{v}(t)\|_{H^1(U)} dt \\ &\leq \alpha \left( \int_0^T \|\mathbf{w}_m(t)\|_{H^1(U)}^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \|\mathbf{v}(t)\|_{H^1(U)}^2 dt \right)^{\frac{1}{2}} \\ &= \alpha \|\mathbf{w}_m\|_{L^2(H^1(U))} \|\mathbf{v}\|_{L^2(H^1(U))}. \end{aligned}$$

For some  $\alpha > 0$  that only depends on  $U, L$ .

Thus  $\|T_v\|_{(L^2(0, T; H_0^1(U)))^*} \leq \alpha \|\mathbf{v}\|_{L^2(H^1(U))}$ , so  $T_v \in (L^2(0, T; H_0^1(U)))^*$ .

Since  $\mathbf{u}_{m_{j_l}} \rightharpoonup \mathbf{u}$ , we have that

$$\int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \rightarrow \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt.$$

We now have

$$\begin{aligned}
\int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt &= \lim_{l \rightarrow \infty} \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt \\
&= \lim_{l \rightarrow \infty} \left( \int_0^T \left\langle \mathbf{u}'_{m_{j_l}}(t) \Big| \mathbf{v}(t) \right\rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \right) \\
&= \lim_{l \rightarrow \infty} \int_0^T \left\langle \mathbf{u}'_{m_{j_l}}(t) \Big| \mathbf{v}(t) \right\rangle_{H^{-1}(U), H_0^1(U)} dt + \lim_{l \rightarrow \infty} \int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \\
&= \int_0^T \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt
\end{aligned}$$

Since such  $\mathbf{v}$  are dense in  $L^2(0, T; H_0^1(U))$  and both sides of the above equality are continuous, we can extend it so that  $\forall \mathbf{v} \in L^2(0, T; H_0^1(U))$ ,

$$\int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt = \int_0^T \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt.$$

Now consider any  $v \in H_0^1(U)$  and any  $\phi \in C_c^\infty(0, T)$ , we always have  $v\phi \in C(0, T; H_0^1(U)) \subset L^2(0, T; H_0^1(U))$ . Thus we have

$$\begin{aligned}
\int_0^T \langle \mathbf{f}(t), v\phi(t) \rangle_{L^2(U)} dt &= \int_0^T \langle \mathbf{u}'(t) | v\phi(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), v\phi(t); t] dt \\
\int_0^T \phi(t) \langle \mathbf{f}(t), v \rangle_{L^2(U)} dt &= \int_0^T \phi(t) \left( \langle \mathbf{u}'(t) | v \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t] \right) dt
\end{aligned}$$

Since this works for all  $\phi \in C_c^\infty(0, T)$ , we must have for a.e.  $t \in [0, T]$ ,

$$\langle \mathbf{f}(t), v \rangle_{L^2(U)} = \langle \mathbf{u}'(t) | v \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t].$$

Now consider any  $\mathbf{v} \in C^1(0, T; H_0^1(U))$  such that  $\mathbf{v}(T) = 0$ .

By IBP 4.16.2, we have

$$\begin{aligned}
0 &= \langle \mathbf{u}(T), \mathbf{v}(T) \rangle_{L^2(U)} \\
&= \langle \mathbf{u}(0), \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left( \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau \\
0 &= \left\langle \mathbf{u}_{m_{j_l}}(T), \mathbf{v}(T) \right\rangle_{L^2(U)} \\
&= \left\langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \right\rangle_{L^2(U)} + \int_0^T \left( \left\langle \mathbf{u}'_{m_{j_l}}(\tau) \Big| \mathbf{v}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} + \left\langle \mathbf{v}'(\tau) \Big| \mathbf{u}_{m_{j_l}}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau.
\end{aligned}$$

Since  $\mathbf{v} \in L^2(0, T; H_0^1(U)) \cong L^2(0, T; H^{-1}(U))^*$  and  $\mathbf{u}'_{m_{j_l}} \rightharpoonup \mathbf{u}'$  in  $L^2(0, T; H^{-1}(U))$ , we have

$$\lim_{l \rightarrow \infty} \int_0^T \left\langle \mathbf{u}'_{m_{j_l}}(\tau) \Big| \mathbf{v}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} d\tau = \int_0^T \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Similarly,

$$\lim_{l \rightarrow \infty} \int_0^T \left\langle \mathbf{v}'(\tau) \Big| \mathbf{u}_{m_{j_l}}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} d\tau = \int_0^T \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Also, since  $\lim_{m \rightarrow \infty} \mathbf{u}_m(0) = g$  in  $L^2(U)$ , we have that the subsequence  $\lim_{l \rightarrow \infty} \mathbf{u}_{m_{j_l}}(0) = g$  as well. Thus

$$\lim_{l \rightarrow \infty} \left\langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \right\rangle_{L^2(U)} = \langle \mathbf{u}(T), \mathbf{v}(T) \rangle_{L^2(U)}.$$

Thus we have

$$\begin{aligned}
0 &= \langle \mathbf{u}(0), \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left( \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau \\
&= \lim_{l \rightarrow \infty} \left( \left\langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \right\rangle_{L^2(U)} + \int_0^T \left( \left\langle \mathbf{u}'_{m_{j_l}}(\tau) | \mathbf{v}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} + \left\langle \mathbf{v}'(\tau) | \mathbf{u}_{m_{j_l}}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau \right) \\
&= \langle g, \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left( \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau.
\end{aligned}$$

We thus have  $\langle \mathbf{u}(0), \mathbf{v}(0) \rangle_{L^2(U)} = \langle g, \mathbf{v}(0) \rangle_{L^2(U)}$  for any such  $\mathbf{v}$ .

Notice that for any  $v \in H_0^1(U)$ , we can simply let  $\mathbf{v}(t) := \frac{T-t}{T}v$ , which satisfies the requirement, and  $\mathbf{v}(0) = v$ . Thus  $\langle \mathbf{u}(0), v \rangle_{L^2(U)} = \langle g, v \rangle_{L^2(U)}$  for any  $v \in H_0^1(U)$ .

Since  $H_0^1(U)$  is dense in  $L^2(U)$ , we have that  $\langle \mathbf{u}(0), v \rangle_{L^2(U)} = \langle g, v \rangle_{L^2(U)}$  for any  $v \in L^2(U)$ . This proves

$$\mathbf{u}(0) = g.$$

□

**Theorem 4.28.** *A weak solution to our IBVP is unique.*

*Proof.* Assume  $\mathbf{u}_1, \mathbf{u}_2$  are both weak solutions to our IBVP.

Then  $\forall v \in H_0^1(U)$ , for a.e.  $t \in [0, T]$ ,

$$\langle \mathbf{u}'_1(t) | v \rangle + B[\mathbf{u}_1, v; t] = \langle \mathbf{u}'_2(t) | v \rangle + B[\mathbf{u}_2, v; t] = \langle \mathbf{f}(t), v \rangle_{L^2(U)},$$

and

$$\mathbf{u}_1(0) = g = \mathbf{u}_2(0).$$

Let  $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$ , we have that

$$\langle \mathbf{u}'(t) | v \rangle + B[\mathbf{u}, v; t] = 0, \quad \forall v \in H_0^1(U), \text{ for a.e. } t \in [0, T],$$

and

$$\mathbf{u}(0) = 0.$$

Choosing  $v = \mathbf{u}(t) \in H_0^1(U)$ , we have that

$$\langle \mathbf{u}'(t) | \mathbf{u}(t) \rangle + B[\mathbf{u}(t), \mathbf{u}(t); t] = 0.$$

By 4.16.3, we have that for a.e.  $t \in [0, T]$ ,

$$\begin{aligned}
\frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(U)}^2 &= 2 \langle \mathbf{u}'(t) | \mathbf{u}(t) \rangle \\
&= -2B[\mathbf{u}(t), \mathbf{u}(t); t] \\
&\leq 2\gamma \|\mathbf{u}(t)\|_{L^2(U)}^2 - 2\beta \|\mathbf{u}(t)\|_{H^1(U)}^2 \\
&\leq 2\gamma \|\mathbf{u}(t)\|_{L^2(U)}^2,
\end{aligned}$$

where  $\gamma \geq 0, \beta > 0$  are constants similar in 3.5.

Take  $\eta(t) := \|\mathbf{u}(t)\|_{L^2(U)}^2$ , by Gronwall's inequality, we have that  $\forall t \in [0, T]$ ,

$$\|\mathbf{u}(t)\|_{L^2(U)}^2 \leq \exp(2t) \|\mathbf{u}(0)\|_{L^2(U)}^2 = 0.$$

Thus  $\mathbf{u}(t) = 0$  and so  $\mathbf{u}_1 = \mathbf{u}_2$  is unique. □