

# Pmath753 Functional Analysis

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December 1, 2025

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# 1 Metric Spaces and Complete Spaces

**Definition 1.1.** A **metric space** is a set  $X$  that has a (distance) metric:

$$\begin{aligned} d(\cdot, \cdot) : X \times X &\rightarrow \mathbb{R}, \text{ such that } \forall x, y, z \in X \\ d(x, x) &= 0 \\ \forall x \neq y, d(x, y) &> 0 \\ d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

**Definition 1.2.** Given a metric space  $(X, d)$ , a sequence  $(x_n)_{n=1}^\infty$  in  $X$  has a **limit point**  $x \in X$  if  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ . In this case, we say  $(x_n)_{n=1}^\infty$  is a **convergent sequence**, and write  $x = \lim_{n \rightarrow \infty} x_n$ .

**Definition 1.3.** A sequence  $(x_n)_{n=1}^\infty$  is a **Cauchy sequence** in a metric space  $(X, d)$  if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m, n \in \mathbb{N}, m, n > N \implies d(x_m, x_n) < \epsilon.$$

**Definition 1.4.** A metric space  $X$  is **complete** if every Cauchy sequence  $(x_i)_{i=1}^\infty$  converges to a limit point in  $X$ . i.e.  $\exists x \in X, \lim_{i \rightarrow \infty} x_i = x$ .

**Proposition 1.1.** Let  $(X, d)$  be a metric space; then every convergent sequence is Cauchy.

**Proposition 1.2.** Let  $(X, d)$  be a metric space. Suppose  $(x_n)_{n=1}^\infty$  is a Cauchy sequence and has a convergent subsequence such that  $\lim_{k \rightarrow \infty} x_{n_k} = x \in X$ , then  $\lim_{n \rightarrow \infty} x_n = x$ .

## 2 Topology

See more in the notes of Pmath367 Topology by Professor S. New.

### 2.1 Topological Spaces

**Definition 2.1.** Let  $X \neq \emptyset$  be a set. A **topology** on  $X$  is a collection  $\mathcal{T} \subseteq \mathcal{P}(X) := 2^X =$  power set of  $X$ , satisfying

1.  $\emptyset, X \in \mathcal{T}$ ,
2.  $\mathcal{T}$  is closed under arbitrary union; namely,  $\forall \{A_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}, \bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$ , and
3.  $\mathcal{T}$  is closed under finite intersection; namely,  $\forall \{A_i\}_{i=1}^n \subseteq \mathcal{T}, \bigcup_{i=1}^n A_i \in \mathcal{T}$ .

Also,  $(X, \mathcal{T})$  is a **topological space** if  $\mathcal{T}$  is a topology on  $X$ .

**Definition 2.2.** For any  $S \subseteq \mathcal{P}(X)$ , we define the **topology generated by  $S$**  to be

$$\mathcal{T}_S := \langle S \rangle := \{\emptyset, X, \text{ all unions of finite intersections of elements of } S\},$$

which is the intersection of all topologies on  $X$  that contains  $S$ , and it is the smallest topology on  $X$  containing  $S$ .

**Proposition 2.1.** Let  $(X, d)$  be a metric space, then there is a **metric topology**  $\mathcal{T}_d$  that is generated by open balls.

**Definition 2.3.**  $(X, \leq)$  is a **partially ordered set (poset)** if  $\leq$  is

1. anti-symmetric:  $\forall x, y \in X$ , if  $x \leq y$  and  $y \leq x$ , we have  $x = y$ ,
2. reflexive:  $\forall x \in X, x \leq x$ , and

3. transitive:  $\forall x, y, z \in X$ , if  $x \leq y, y \leq z$ , we have  $x \leq z$ .

We can define  $\geq, <, >$  by

$$\begin{aligned} x \geq y &\iff y \leq x \\ x < y &\iff x \leq y \wedge x \neq y \\ x > y &\iff y < x. \end{aligned}$$

**Definition 2.4.**  $(X, \leq)$  is a **totally ordered set** if it is a partially ordered set such that  $\forall x, y \in X$ ,  $x \leq y$  or  $y \leq x$ .

**Proposition 2.2.**  $(X, \leq)$  is a totally ordered set if and only if  $<$  satisfies

1.  $\forall x, y \in X$ , exactly one of the following is true:  $x < y$ ,  $x = y$ ,  $y < x$ .
2.  $\forall x, y, z \in X$ , if  $x < y, y < z$ , we have  $x < z$ .

**Definition 2.5.** Let  $(X, \leq)$  be a totally ordered set, we can define for each  $a, b \in X$ ,

1.  $(-\infty, a) := \{x \in X : x < a\}$ ,
2.  $(a, \infty) := \{x \in X : a < x\}$ , and
3.  $(a, b) := (a, \infty) \cap (-\infty, b)$ .

**Proposition 2.3.** Let  $\mathcal{T}_{\leq}$  be the topology generated by all the sets above, then  $\mathcal{T}_{\leq}$  is a topology.

**Definition 2.6.** Let  $(X, \mathcal{T})$  be a topological space, then we say  $U \subseteq X$  is **open** if  $U \in \mathcal{T}$ . We say  $E \subseteq X$  is **closed** if  $E^c \in \mathcal{T}$  is open.

**Definition 2.7.** For  $E \subseteq X$ , the **closure** of  $E$  is

$$\bar{E} = \bigcap_{F \supseteq E: F \text{ is closed}} F,$$

which is the smallest closed set containing  $E$ .

**Definition 2.8.** For  $E \subseteq X$ , the **interior** of  $E$  is

$$E^o = \bigcup_{U \subseteq E: U \text{ is open}} U,$$

which is the largest open set contained in  $E$ .

**Proposition 2.4.** Closed sets are stable under finite unions and arbitrary intersections.

**Proposition 2.5.** For any set  $A$ ,

$$\bar{A} = ((A^c)^o)^c.$$

**Proposition 2.6.** For any set  $A$ ,

$$x \in \bar{A} \iff (\forall U \text{ open}, x \in U \implies U \cap A \neq \emptyset).$$

**Definition 2.9.** Let  $(X, \mathcal{T})$  be a topological space, a  $\mathcal{B} = \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{P}(X)$  is said to be a **basis/base** of the topology  $\mathcal{T}$  if it is a collection of open sets, and for every  $U \in \mathcal{T}$ , we have  $U = \bigcup_{\alpha \in J} U_\alpha$  for some  $J \subseteq I$ .

**Proposition 2.7.** Let  $(X, \mathcal{T})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{T}$  is a basis of  $\mathcal{T}$  if and only if

$$\forall x \in U \in \mathcal{T}, \exists U_\alpha \in \mathcal{B} \text{ such that } x \in U_\alpha \subseteq U.$$

**Proposition 2.8.** Let  $(X, \mathcal{T})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{P}(X)$  is a basis of  $\mathcal{T}$ , if and only if

1.  $X = \bigcup_{\alpha \in I} U_\alpha$ ,
2. For any  $U_1, U_2 \in \mathcal{B}$ ,  $x \in U_1 \cap U_2$ , we have  $\exists U_x \in \mathcal{B}, x \in U_x \subseteq U_1 \cap U_2$ ,
3.  $\mathcal{T} = \langle \mathcal{B} \rangle$  is the topology generated by  $\mathcal{B}$ .

**Example 2.1.1.** Let  $(X, d)$  be a metric space, then  $\{B(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\}$  is a base.

**Definition 2.10.** Let  $(X, \mathcal{T})$  be a topological space, a **subbase** is a collection of open sets  $S \subseteq \mathcal{T}$  such that

$$\left\{ X, \bigcap_{i=1}^n S_i : n \in \mathbb{N}^+, S_1, \dots, S_n \in S \right\}$$

forms a base for  $\mathcal{T}$ .

**Proposition 2.9.** Let  $(X, \mathcal{T})$  be a topological space, then any subbase  $S$  generates  $\mathcal{T}$ . Also, any set  $S$  such that  $\bigcup S = X$  is always a subbase for the topology  $\mathcal{T}_S$  generated by  $S$ .

**Definition 2.11.** Let  $(X, \mathcal{T})$  be a topological space. Given  $x \in X$ , a **neighbourhood** of  $x$  is a set  $V \ni x$ , such that  $\exists U \in \mathcal{T}$  with  $x \in U \subseteq V$ .

**Definition 2.12.** Let  $(X, \mathcal{T})$  be a topological space. Given  $x \in X$ , a **neighbourhood basis** of  $x$  is a set of open neighbourhoods  $\mathcal{B}_x \subset \mathcal{T}$ , such that for any (open) neighbourhood  $U$  of  $x$ , there is  $B \in \mathcal{B}_x$  such that  $x \in B \subseteq U$ .

**Proposition 2.10.** Let  $(X, \mathcal{T})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{T}$  is a basis of  $\mathcal{T}$  if and only if  $\forall x \in X$ ,  $\mathcal{B}$  is a neighbourhood basis of  $x$ .

**Definition 2.13.** We say  $S \subseteq X$  is **dense** in a topological space  $(X, \mathcal{T})$  if  $\forall \text{open } U \neq \emptyset, S \cap U \neq \emptyset$ .

**Proposition 2.11.**  $S$  is dense if and only if  $\bar{S} = X$ .

**Definition 2.14.** A topological space  $(X, \mathcal{T})$  is **separable** if there is a countable subset.

**Definition 2.15.** A topological space  $(X, \mathcal{T})$  is **first countable** if  $\forall x \in X$ , there is a countable open neighbourhood basis  $\{B_n\}_{n=1}^\infty \subset \mathcal{T}$  at  $x$ . Namely, for any neighbourhood  $U$  of  $x$ , there is  $n \in \mathbb{N}$  such that  $x \in B_n \subseteq U$ .

**Definition 2.16.** A topological space  $(X, \mathcal{T})$  is called **2nd countable** if it has a countable basis.

**Proposition 2.12.** Every metric space  $(X, d)$  are first countable.

**Proposition 2.13.** Every metric space  $(X, d)$  is 2nd countable if and only if  $X$  is countable.

**Proposition 2.14.** The discrete topology of  $X$  is separable if and only if  $|X|$  is at most countable.

**Definition 2.17** (Axiom of Choice). If  $X \neq \emptyset$ , then there is a choice function  $C : P(X) \setminus \{\emptyset\} \rightarrow X$  such that  $\forall A \subseteq X$ , if  $A \neq \emptyset$ , we have  $C(A) \in A$ .

**Proposition 2.15** (Axiom of Choice Equivalence). The Axiom of Choice is equivalent to: Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of non-empty sets, then

$$\prod_{\alpha \in A} X_\alpha := \left\{ f : A \rightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha) \in X_\alpha \right\} \neq \emptyset.$$

*Proof.* Suppose AOC holds, then taking  $X = \bigcup_{\alpha \in A} X_\alpha$ , we have the choice function  $C$ . Now take  $f(a) := C(X_a)$ .

On the other hand, suppose the latter holds, then consider  $\prod_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$ , which is non-empty. Consider any  $f \in \prod_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$ , then  $C(X_\alpha) := f(a)$  is a choice function.  $\square$

**Proposition 2.16.** A metric space  $(X, d)$  is separable if and only if it is 2nd countable.

*Proof.* If  $S = \{x_k\}_{k=1}^\infty$  is dense, then  $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$  is a countable base. Indeed, consider any  $x \in X$  with any open  $U \ni x$ , we know  $\exists r > 0$ , such that  $x \in B(x, r) \subset U$ . Also, there is  $x_k$  such that  $d(x, x_k) < \frac{r}{2}$ .

Now choose some  $r' \in \mathbb{Q}$  such that  $d(x, x_k) < r' < \frac{r}{2}$ , then  $x \in B(x_k, r') \subset B(x, r) \subset U$ .

Thus  $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$  is a base.

On the other hand, suppose  $X$  is second countable with a countable base  $\{U_n\}_{n=1}^\infty$ . WLOG,  $U_n \neq \emptyset$ .

Now for any  $n \in \mathbb{N}$ , pick  $x_n \in U_n$  by the axiom of countable choice. Let  $S = \{x_n\}_{n=1}^\infty$ , then we claim  $S$  is dense.

Indeed, for any open  $U \neq \emptyset$ , we can find some  $U_n \subset U$ . Thus  $x_n \in S \cap U$ .  $\square$

**Proposition 2.17.** *If  $\{\mathcal{T}_\alpha\}_{\alpha \in A}$  is a set of topologies on  $X$ ,*

1. *There is a weakest topology  $\tau := \langle \bigcup_{\alpha \in A} \mathcal{T}_\alpha \rangle$  that is stronger than each  $\mathcal{T}_\alpha$ .*
2. *There is a strongest topology  $\delta := \bigcap_{\alpha \in A} \mathcal{T}_\alpha$  that is weaker than each  $\mathcal{T}_\alpha$ .*

**Definition 2.18.** A topological space  $(X, \mathcal{S})$  is **Hausdorff** if

$$\forall x \neq y \in X, \exists S_x, S_y \in \mathcal{S}, \text{ such that } x \in S_x, y \in S_y, S_x \cap S_y = \emptyset.$$

**Proposition 2.18.** *Any space with its discrete topology is always Hausdorff.*

**Example 2.1.2.** Every metric space is Hausdorff.

**Proposition 2.19.** *Any space with more than one element with the trivial topology is not Hausdorff.*

**Example 2.1.3.** Consider  $X := (0, 1) \cup \{1^+, 1^-\}$ . Let  $(0, 1)$  have the usual open topology. Also, let  $(r, 1) \cup \{1^+\}$  and  $(r, 1) \cup \{1^-\}$  be open for any  $0 < r < 1$ . The topology generated by this basis will not be Hausdorff.

Indeed, consider  $1^+, 1^-$ , then for any  $U \ni 1^+, V \ni 1^-$ , we can find  $r_U, r_V \in (0, 1)$ , such that  $(r_U, 1) \cup \{1^+\} \subseteq U, (r_V, 1) \cup \{1^-\} \subseteq V$ . Yet  $(\max(r_U, r_V), 1) \subseteq U \cap V$ , which is not empty.

**Proposition 2.20.** *If  $X$  is Hausdorff, then for any  $x \in X$ , we have that  $\{x\}$  is closed.*

*Proof.* For any  $y \neq x$ , we can find open  $V_y \ni y, U_y \ni x$ , such that  $V_y \cap \{x\} \subseteq V_y \cap U_y = \emptyset$ . Thus  $X \setminus \{x\} = \bigcup_{y \in X} V_y$ , which is open.  $\square$

## 2.2 Continuous Functions

**Definition 2.19.** A function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$  is **continuous** if

$$\forall U \in \mathcal{S}, f^{-1}(U) \in \mathcal{T}.$$

Namely, the preimage of any open set is open.

**Definition 2.20.** A function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$  is **continuous at  $x \in X$**  if

$$\forall V \in \mathcal{S}, \text{ such that } f(x) \in V \exists U \in \mathcal{T}, \text{ such that } x \in U \subseteq f^{-1}(V).$$

**Proposition 2.21.** *Let  $f : X \rightarrow Y$  be a map between topological spaces. Then  $f$  is continuous (on  $X$ ) if and only if  $f$  is continuous at every point  $x \in X$ .*

*Proof.* ( $\implies$ ):

Assume  $f$  is continuous, then for every point  $x \in X$  and open  $V \ni f(x)$ , we have  $f^{-1}(V)$  is open. Clearly  $x \in f^{-1}(V)$ .

( $\impliedby$ ):

Assume  $f$  is continuous at every point  $x$ . Given any open  $V \in \mathcal{S}$ , and any point  $x \in f^{-1}(V)$ , we have  $f(x) \in V$ .

By assumption, there is open  $U_x$ , such that  $x \in U_x \subseteq f^{-1}(V)$ .

Now  $\bigcup_{x \in f^{-1}(V)} U_x$  is open, while  $\bigcup_{x \in f^{-1}(V)} U_x \supseteq \bigcup_{x \in f^{-1}(V)} \{x\} = f^{-1}(V) \supseteq \bigcup_{x \in f^{-1}(V)} U_x$ .

Thus  $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$  is open.  $\square$

**Definition 2.21.** A function  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$  is **open** if

$$\forall V \in \mathcal{T}, f(V) \in \mathcal{S}.$$

Namely, the image of any open set is open.

**Definition 2.22.** Given two sets  $X, Y$ , and their corresponding topology  $\mathcal{T}, \mathcal{S}$ , a continuous map  $f : X \rightarrow Y$  is a **homeomorphism** if it is bijective, and its inverse function is also continuous.

*Remark.* A homeomorphism is a map that preserves the topological structure between two sets.

**Definition 2.23.** Let  $C(X)$  be the collection of functions  $f : X \rightarrow \mathbb{C}$  that are continuous.

**Definition 2.24.** Let  $C(X, \mathbb{R})$  be the collection of functions  $f : X \rightarrow \mathbb{R}$  that are continuous.

**Definition 2.25.** Let  $C_b(X)$  be the collection of functions  $f \in C(X) : \|f\|_\infty < \infty$ .

**Definition 2.26.** Let  $C_b(X, \mathbb{R})$  be the collection of functions  $f \in C(X, \mathbb{R}) : \|f\|_\infty < \infty$ .

**Proposition 2.22.** If  $C(X)$  separates points, so do  $C_b(X), C_b(X, \mathbb{R})$ . Also,  $X$  is Hausdorff.

*Proof.* If  $x \neq y$ , we have  $f \in C(X)$  such that  $f(x) \neq f(y)$ .

WLOG,  $\Re(f(x)) \neq \Re(f(y))$ , and  $\Re(f(x)) < \Re(f(y))$ .

Now define  $g(z) := \min \{ \Re(f(y)), \max \{ \Re(f(x)), \Re(f(y)) \} \}$ , which is bounded and continuous. Also  $g(x) = f(x), g(y) = f(y)$ .

Thus  $C_b(X, \mathbb{R})$  separates the points.

Now if  $x \neq y$ , we can find  $f \in C(X)$ , such that  $|f(x) - f(y)| = r > 0$ .

Now let  $U := f^{-1}(B(f(x), \frac{r}{2})), V := f^{-1}(B(f(y), \frac{r}{2}))$ , which are both open. Also,  $x \in U, y \in V$ , and  $U \cap V = f^{-1}(\emptyset) = \emptyset$ .  $\square$

**Definition 2.27.** A topological space  $(X, \mathcal{T})$  is **normal** if for any disjoint closed sets  $A, B$ , we can find open  $U \supset A, V \supset B$  such that  $U \cap V = \emptyset$ .

**Theorem 2.23** (Urysohn's Lemma for normal spaces).  $(X, \mathcal{T})$  is normal if and only if for any disjoint closed sets  $A, B$ ,  $\exists f : X \rightarrow [0, 1]$  continuous, such that  $f|_A = 0, f|_B = 1$ .

**Corollary 2.24.** If  $X$  is normal and Hausdorff,  $C_b(X)$  separates the points.

## 2.3 Nets

**Definition 2.28.** Let  $(X, \mathcal{T})$  be a topological space, say a sequence  $(x_i)_{i=1}^\infty$  **converges to**  $x \in X$  if  $\forall$  open  $U \ni x$ ,  $\exists N \in \mathbb{N}$  such that  $\forall i \geq N, x_i \in U$ .

**Example 2.3.1.** Consider  $X = \mathbb{N} \times \mathbb{N}$ , and the projection  $\pi_1 : X \rightarrow \mathbb{N}$  by  $\pi_1(m, n) := m$ .

Let any  $U$  be open if  $(0, 0) \notin U$ , or if  $\{m \in \mathbb{N} : \pi_1^{-1}(m) \cap U \text{ is co-finite in } \{m\} \times \mathbb{N}\}$  is co-finite.

One can show this defines a topology on  $X$ , and it is Hausdorff.

Indeed, let  $X_0 := X \setminus \{(0, 0)\}$ . Consider any  $(m, n) \neq (m', n') \in X$ . If both are in  $X_0$  then  $\{(m, n)\}, \{(m', n')\}$  are open and disjoint.

If  $(m', n') = (0, 0)$ , then  $\{(m, n)\}, X \setminus \{(m, n)\}$  are open and disjoint.

This shows Hausdorff.

Also,  $\bar{X}_0 = X$ .

Indeed, consider any open  $U \ni (0, 0)$ , we must have that  $U \cap X_0 \neq \emptyset$ .

However, there is no sequence in  $X_0$  that converges to  $(0, 0)$ .

Indeed, assume for contradiction that there is such a convergent sequence  $(x_k)_{k=1}^\infty$  in  $X_0$ .

Write each  $x_k = (m^k, n^k)$ .

Suppose there is some  $M \in \mathbb{N}^+$ , such that  $\forall k \in \mathbb{N}^+, m_k \leq M$ .

Consider  $U := \{(m, n) : m > M, n \in \mathbb{N}\} \cup \{(0, 0)\}$ .

Now for each  $m \in \mathbb{N}$ ,

$$\pi_1^{-1}(m) \cap U = \begin{cases} (0, 0) & \text{if } m = 0 \\ \emptyset & \text{if } 0 < m \leq M \\ \{m\} \times \mathbb{N} & \text{if } m > M. \end{cases}$$

Thus for all  $m > M$ ,  $\pi^{-1}(m) \cap U$  is co-finite in  $\{m\} \times \mathbb{N}$ . This shows  $U$  is open.

Yet  $U \cap \{x_k\}_{k=1}^\infty = \emptyset$ .

Now suppose there is no such  $M$ , then we can find a subsequence  $m_{k_1} < m_{k_2} < m_{k_3} < \dots$ .

Now let  $U := X \setminus \{x_{k_i}\}_{i=1}^\infty$ .

For each  $m \in \mathbb{N}$ ,

$$\pi^{-1}(m) \cap U = \begin{cases} \{m\} \times \mathbb{N} & \text{if } m \notin \{m_{k_i}\}_{i=1}^\infty \\ \{m\} \times (\mathbb{N} \setminus n_{k_i}) & \text{if } m \in \{m_{k_i}\}_{i=1}^\infty. \end{cases}$$

Notice that there cannot be two  $n_{k_i} \neq n_{k_j}$  for any  $m$ , since  $k_i \neq k_j \implies m_{k_i} \neq m_{k_j}$ .

Thus all  $\pi^{-1}(m) \cap U$  is co-finite in  $\{m\} \times \mathbb{N}$ . This shows  $U$  is open.

Yet  $U \cap \{x_{k_i}\}_{i=1}^\infty = \emptyset$ .

Thus there cannot be any convergent sequence  $(x_k)_{k=1}^\infty$  in  $X_0$ .

*Remark.* The above example shows that sequences do not behave as we want in topological spaces.

**Definition 2.29.** An **upwards directed set** is a poset  $(\Lambda, \leq)$  such that if  $\lambda_1, \lambda_2 \in \Lambda$ ,  $\exists \lambda_0 \in \Lambda$  such that  $\lambda_1 \leq \lambda_0, \lambda_2 \leq \lambda_0$ .

**Definition 2.30.** For  $X \neq \emptyset$ , a **net** in  $X$  is a function  $j : \Lambda \rightarrow X$ , where  $(\Lambda, \leq)$  is an upwards directed set. Write  $x_\lambda := j(\lambda) \in X$ , and we can use  $(x_\lambda)_{\lambda \in \Lambda}$  to represent a net.

**Definition 2.31.** Let  $(X, \mathcal{T})$  be a topological space, say a net  $(x_\lambda)_{\lambda \in \Lambda}$  **converges to**  $x \in X$  if

$$\forall \text{open } U \ni x, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U.$$

In this case, we say  $x$  is a **limit** of the net, and write it as  $x = \lim_{\lambda \in \Lambda} x_\lambda$  or  $x_\lambda \rightarrow x$ .

**Proposition 2.25.** Let  $(X, \mathcal{T})$  be a topological space with a neighbourhood basis  $\mathcal{B}$  at  $x \in X$ , then a net  $(x_\lambda)_{\lambda \in \Lambda}$  converges to  $x$  if and only if

$$\forall U \in \mathcal{B} \text{ such that } x \in U, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U.$$

*Proof.* The forward direction is trivial.

Now assume  $\forall U \in \mathcal{B}$  such that  $x \in U$ ,  $\exists \lambda_0 \in \Lambda$  such that  $\forall \lambda \geq \lambda_0, x_\lambda \in U$ .

Given any open  $V \ni x$ , since  $\mathcal{B}$  is a neighbourhood basis, there is some  $U \in \mathcal{B}$ , such that  $x \in U \subseteq V$ .

Thus, there is  $\lambda_0 \in \Lambda$  such that  $\forall \lambda \geq \lambda_0, x_\lambda \in U \subseteq V$ . □

**Definition 2.32.** Given a net  $(x_\lambda)_{\lambda \in \Lambda}$ , then a **subnet** of it  $(y_\gamma)_{\gamma \in \Gamma}$  is given by an upwards directed set  $(\Gamma, \leq)$  and a function  $\phi : \Gamma \rightarrow \Lambda$  that is **cofinal**, which means  $\forall \lambda_0 \in \Lambda, \exists \gamma_0 \in \Gamma$ , such that  $\forall \gamma \geq \gamma_0, \phi(\gamma) \geq \lambda_0$ . Each  $y_\gamma$  is given by  $x_{\phi(\gamma)}$ .

**Example 2.3.2.** Notice that if we take  $(\mathbb{N}, \leq)$ , the net is just a sequence. To get a subsequence, we can take  $\Gamma = \mathbb{N}$ , and  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  to be any increasing function. The generated subnet will be a subsequence.

**Definition 2.33.** Let  $(X, \mathcal{T})$  be a topological space, and  $x \in X$ , define the **system of open neighbourhoods of  $x$**  to be  $\mathcal{O}(x) := \{U \in \mathcal{T} : x \in U\}$ .

**Proposition 2.26.**  $(\mathcal{O}(x), \supseteq)$  is an upwards directed set.

**Example 2.3.3.** For  $X = \mathbb{N} \times \mathbb{N}$  and  $X_0 = X \setminus \{(0, 0)\}$  as above, there is a net in  $X_0$  converging to  $(0, 0)$ .

Indeed, let us enumerate  $X_0 = \{x_k\}_{k=1}^\infty$  as  $(0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \dots$

Now  $\Lambda := \mathcal{O}((0, 0))$  is an upward directed set by containment.

Then for each  $U \in \Lambda$ , we can pick  $x_U := x_{k_U}$ , where  $k_U$  is the first  $k \in \mathbb{N}$  such that  $x_k \in U$ .

Claim:  $(x_U)_{U \in \Lambda}$  converges to  $(0, 0)$ .

Pick any  $U_0 \ni (0, 0)$ , then for all  $U \geq U_0$ , it is open and  $U \subseteq U_0$ . Thus, we must have  $x_U \in U \subseteq U_0$ .

Indeed,  $(x_U)_{U \in \Lambda}$  is a subnet of  $\{x_k\}_{k \in \mathbb{N}^+}$  by  $\phi(U) := k_U$ .

**Theorem 2.27.** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be two topological spaces, then

1. For any  $A \subseteq X$ , we have  $x \in \bar{A}$  if and only if  $\exists$  a net  $(x_\lambda)_{\lambda \in \Lambda}$  in  $A$ , such that  $x_\lambda \rightarrow x$ .



2.  $f : X \rightarrow Y$  is continuous if and only if for any net  $(x_\lambda)_{\lambda \in \Lambda} \subset X$  such that  $x_\lambda \rightarrow x \in X$ , we have  $f(x_\lambda) \rightarrow f(x)$ .

*Proof.* 1. Consider any  $x \in \bar{A}$ , then for any open  $U \ni x$ , we have  $U \cap A \neq \emptyset$ .

By the Axiom of Choice, we can have  $x_U \in U \cap A$  for each open neighbourhood  $U$ .

Consider the net  $(x_U)_{U \in \mathcal{O}(x)}$ .

Given any open  $U \ni x$ , if  $V \geq U$ , we must have  $V \subseteq U$ , and  $x_v \in V \subseteq U$ .

Thus  $x_U \rightarrow x$ .

On the other hand, for any net  $x_\lambda \rightarrow x$  in  $A$ , consider any open  $U \ni x$ , there is  $\lambda_0$ , such that

$$\forall \lambda \in \Lambda, \lambda_0 \leq \lambda \implies x_\lambda \in U \implies U \cap A \neq \emptyset.$$

Thus  $x \in \bar{A}$ .

2. Assume  $f$  is continuous, and  $x_\lambda \rightarrow x$ . Let  $V \in \mathcal{O}(f(x))$ , then  $U := f^{-1}(V)$  is open and  $x \in U$ .

Thus there is  $\lambda_0 \in \Lambda$ , such that  $\forall \lambda \geq \lambda_0, x_\lambda \in U$ .

Thus  $f(x_\lambda) \in V$ .

On the other hand, assume for any net  $(x_\lambda)_{\lambda \in \Lambda} \subset X$  such that  $x_\lambda \rightarrow x \in X$ , we have  $f(x_\lambda) \rightarrow f(x)$ .

For contradiction, suppose there is open  $V \in Y$ , with  $U := f^{-1}(V)$  is not open in  $X$ .

Then  $U^c$  is not closed, and  $U^c \neq \overline{U^c}$ .

Thus, there is  $x \in \overline{U^c} \setminus U^c = \overline{U^c} \cap U$ .

Since  $x \in \overline{U^c}$ , by 1., there is a net  $(x_\lambda)_{\lambda \in \Lambda}$  in  $U^c$ , such that  $x_\lambda \rightarrow x$ .

By assumption, we have  $f(x_\lambda) \rightarrow f(x)$ .

Since each  $f(x_\lambda)$  is in  $f(U^c) = V^c$ , by 1., we have that  $f(x) \in \overline{V^c} = V^c$  since  $V^c$  is closed ( $V$  is open).

However, since  $x \in U$ , we also have  $f(x) \in f(U) = V$ , which is a contradiction.  $\square$

## 2.4 Compactness

**Definition 2.34.** Let  $(X, \mathcal{T})$  be a topological space. A collection of subsets  $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$  is called a **cover** for  $X$  if  $X = \bigcup_{\alpha \in A} U_\alpha$ . A cover is called an open cover if every  $U_\alpha$  is open in  $\mathcal{T}$ .

**Definition 2.35.** Let  $(X, \mathcal{T})$  be a topological space. A collection  $\{C_\alpha\}_{\alpha \in A}$  of non-empty closed sets is a FIP-family if for any finite  $F \subseteq A$ ,  $\bigcap_{\alpha \in F} C_\alpha \neq \emptyset$ .  $X$  has the finite intersection property (FIP) if for all FIP-familie  $\{C_\alpha\}_{\alpha \in A}$ , we have  $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$ .

**Definition 2.36.** Let  $(X, \mathcal{T})$  be a topological space.  $X$  is **compact** if every open cover of  $X$  has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open, } X = \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } X = \bigcup_{i=1}^n U_{\alpha_i}.$$

**Definition 2.37.** Let  $(X, \mathcal{T})$  be a topological space. A collection of subsets  $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$  is called a **cover** for  $K \subseteq X$  in  $X$  if  $K = \bigcup_{\alpha \in A} U_\alpha$ . A cover in  $X$  is called an open cover in  $X$  if every  $U_\alpha$  is open in  $\mathcal{T}$ .

**Definition 2.38.** Let  $(X, \mathcal{T})$  be a topological space. A set  $K \subseteq X$  is **compact in  $X$**  if every open cover of  $K$  in  $X$  has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open, } K \subseteq \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } K \subseteq \bigcup_{i=1}^n U_{\alpha_i}.$$

**Proposition 2.28.** Let  $(X, \mathcal{T})$  be a topological space. A set  $K \subseteq X$  is compact under the subspace topology if and only if it is compact in  $X$ .

**Theorem 2.29.** Let  $(X, \mathcal{T})$  be a topological space, TFAE:

1.  $X$  is compact.

2.  $X$  has the finite intersection property.
3. For all nets  $(x_\lambda)_{\lambda \in \Lambda}$  in  $X$ , there is a convergent subnet.

*Proof.* (1)  $\implies$  (2).

For contradiction, suppose there is some FIP-family such that  $\bigcap_{\alpha \in A} C_\alpha = \emptyset$ .

We have  $X = \bigcup_{\alpha \in A} C_\alpha^c$ , which is an open cover for  $X$ .

Since  $X$  is compact, there is a finite  $F \subseteq A$ , such that  $X = \bigcup_{\alpha \in F} C_\alpha^c$ .

Thus  $\bigcap_{\alpha \in A} C_\alpha = \emptyset$ , which contradict  $\{C_\alpha\}_{\alpha \in A}$  being a FIP family.

(2)  $\implies$  (1).

Consider any open cover  $X = \bigcup_{\alpha \in A} U_\alpha$ , then  $\bigcap_{\alpha \in A} U_\alpha^c = \emptyset$ , and it is not a FIP-family.

Thus there is a finite  $F \subseteq A$ , such that  $\bigcap_{\alpha \in F} U_\alpha^c = \emptyset$ .

Thus  $X = \bigcup_{\alpha \in F} U_\alpha$  is a finite open cover.

(2)  $\implies$  (3).

Let  $(x_\lambda)_{\lambda \in \Lambda}$  be any net in  $X$ .

Define  $C_\lambda := \{x_\mu : \mu \geq \lambda\}$ . Notice that  $C_\lambda \neq \emptyset$  since  $x_\lambda \in C_\lambda$ .

We claim that  $\{C_\lambda\}_{\lambda \in \Lambda}$  is a FIP family.

The closeness is by definition.

Now fix any  $\lambda_1, \dots, \lambda_n \in \Lambda$ .

Since  $\Lambda$  is upwards directed, there is  $\lambda_0 \in \Lambda$ , such that  $\forall i \in [n], \lambda_i \leq \lambda_0$ .

Thus  $\bigcap_{i=1}^n C_{\lambda_i} \supseteq C_{\lambda_0} \neq \emptyset$ .

By FIP,  $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$ .

Pick any  $x \in \bigcap_{\lambda \in \Lambda} C_\lambda$ .

Let  $\Gamma := \Lambda \times \mathcal{O}(x)$  with the partial order  $(\lambda, U) \leq (\lambda', U')$  if  $\lambda \leq \lambda'$  and  $U \supseteq U'$ .

Fix  $(\lambda, U) \in \Gamma$ , we know that  $x \in C_\lambda = \{x_\mu : \mu \geq \lambda\}$ .

Thus  $U \cap \{x_\mu : \mu \geq \lambda\} \neq \emptyset$ .

By the Axiom of Choice, there is  $y_{(\lambda, U)} = x_{\phi(\lambda, U)} \in U \cap \{x_\mu : \mu \geq \lambda\}$ , where  $\phi(\lambda, U) := C(\{\mu \geq \lambda : x_\mu \in U\})$ .

For any  $\lambda_0 \in \Lambda$ , let  $\gamma_0 = (\lambda_0, X)$ , then for any  $\gamma = (\lambda, U) \geq \gamma_0$ , we have  $\phi(\gamma) \geq \lambda \geq \lambda_0$ .

Thus  $\phi$  is cofinal, and  $(y_\gamma)_{\gamma \in \Gamma}$  is a subnet.

In addition, given any  $U_0 \in \mathcal{O}(x)$ , we can pick any  $\lambda_0 \in \Lambda$ , and let  $\gamma_0 := (\lambda_0, U_0)$ .

Then for any  $(\lambda, U) \geq \gamma_0$ , we must have  $y_{(\lambda, U)} = x_{\phi(\lambda, U)} \in U \subseteq U_0$ .

(3)  $\implies$  (2).

Fix any FIP-family  $\{C_\alpha\}_{\alpha \in A}$  in  $X$ . Then for any finite  $F \subseteq A$ , by the Axiom of Choice, we can find  $x_F \in \bigcap_{\alpha \in F} C_\alpha$ .

Now consider the net  $(x_F)_{\text{finite } F \subseteq A}$ , where  $F_1 \leq F_2$  if  $F_1 \subseteq F_2$ .

By 3., there is a convergent subnet  $\phi : \Gamma \rightarrow \Lambda$ , such that  $x_{\phi(\gamma)} \rightarrow x \in X$ .

Now fix any  $\alpha \in A$ , then  $\{\alpha\} \in \Lambda$ .

Thus there is some  $\gamma_0 \in \Gamma$ , such that  $\forall \gamma \geq \gamma_0, \phi(\gamma) \supseteq \{\alpha\} \ni \alpha$ .

We have  $x_{\phi(\gamma)} \in \bigcap_{\beta \in \phi(\gamma)} C_\beta \subseteq C_\alpha$ .

Since this holds for all  $\gamma \geq \gamma_0$ , and  $x_{\phi(\gamma)} \rightarrow x$ , we have that  $x \in \bar{C}_\alpha = C_\alpha$ .

Since this holds for any  $\alpha \in A$ , we have that  $x \in \bigcap_{\alpha \in F} C_\alpha$ . Thus  $\bigcap_{\alpha \in F} C_\alpha \neq \emptyset$ .  $\square$

With the above theorem, we can identify the definition of compactness in a metric space via sequences with the definition of compactness with its metric topology.

**Proposition 2.30.** *Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be two topological spaces. If  $X$  is compact, and  $f : X \rightarrow Y$  is continuous, then  $f(X)$  is compact in  $Y$ .*

**Theorem 2.31.** *Let  $(X, \mathcal{T})$  be a topological space,*

1. *Suppose  $K$  is compact, then  $\forall F \subseteq K$  that is closed,  $F$  is also compact.*
2. *If  $X$  is Hausdorff, for any compact  $K \subseteq X, x \in X \setminus K, \exists$  open neighbourhood  $U$  of  $x$ , and open  $W \supset K$ , such that  $W \cap U = \emptyset$ .*

*Proof.* 1. Let  $(U_\alpha)_{\alpha \in A}$  be an open cover for  $F$ .

Since  $F$  is closed, then  $F^c$  is open. Thus  $\{F^c\} \cup \{U_\alpha\}_{\alpha \in A}$  is an open cover for  $K$ .

Thus there are  $U_{\alpha_1}, \dots, U_{\alpha_n}$ , such that  $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$ . Thus  $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$  has a finite subcover.

2. Consider any  $y \in K$ , there is some open neighborhoods  $U_y \ni x, W_y \ni y$ , such that  $U_y \cap W_y = \emptyset$ . Since  $K \subseteq \bigcup_{y \in K} W_y$  is compact, we have  $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$  for some  $y_1, \dots, y_n$ . Let  $U = \bigcap_{i=1}^n U_{y_i}$ , we have  $x \in U, K \subseteq W, U \cap W = \emptyset$  as required.

□

**Corollary 2.32.** *Let  $(X, \mathcal{T})$  be a Hausdorff topological space, then any compact set  $K$  is closed. In addition, for any closed  $F \subseteq X$ , we have  $F \cap K$  is compact.*

*Proof.* Suppose for contradiction that  $K$  is not closed, then there is some  $y \in \bar{K} \setminus K$ . Thus we can find open neighbourhood  $U$  of  $x$ , and open  $W \supset K$ , such that  $W \cap U = \emptyset$ . Now  $K \subset \bar{K} \setminus U \subsetneq \bar{K}$  is closed, which is a contradiction.

Since  $K$  is closed, so is  $F \cap K \subseteq K$ , and thus it is compact. □

**Definition 2.39.**  $X$  is **locally compact** if  $\forall x \in X$ , there is an open neighbourhood  $U_x \in \mathcal{O}(x)$  such that  $\bar{U}_x$  is compact.

**Example 2.4.1.**  $\mathbb{R}^n$  is locally compact by the Heinz-Borel theorem.

**Proposition 2.33.** *A Banach space  $(X, \|\cdot\|)$  is locally compact iff  $\dim(X) < \infty$ .*

**Lemma 2.34.** *Let  $(X, \mathcal{T})$  be a Hausdorff topological space, and  $(K_\alpha)_{\alpha \in A}$  be a collections of compact sets such that*

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

*We must have  $\alpha_1, \dots, \alpha_n \in A$ , such that*

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

*Proof.* Fix  $\alpha_1 \in A$ , then  $K_{\alpha_1} \subseteq \left( \bigcap_{\alpha \neq \alpha_1} K_\alpha \right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$  is compact and has an open cover.

Thus there must be  $\alpha_2, \dots, \alpha_n \in A$ , such that  $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = \left( \bigcap_{i=2}^n K_{\alpha_i} \right)^c$ .

Thus  $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$ . □

**Theorem 2.35.** *Let  $X$  be a Locally Compact Hausdorff space, and let  $K \subseteq U \subseteq X$  be such that  $K$  is compact, and  $U$  is open. Then there exists some open set  $V$  such that  $\bar{V}$  is compact, and*

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

*Proof.* Since  $X$  is a Locally Compact Hausdorff space, there are  $V_1, \dots, V_n$ , each with  $\bar{V}_i$  be compact, such that  $K \subseteq \bigcup_{i=1}^n V_i =: G$ . Note that  $\bar{G}$  is compact, and  $G$  is open.

If  $U = X$ , then  $G \subseteq U$ , and we are done.

Otherwise, let  $C := X \setminus U$  be non-empty and closed.

Consider any  $y \in C$ , we know that  $y \notin K$ . Since  $X$  is Hausdorff, we can find open  $W_y \supset K$ , and  $U_y \ni y$ , such that  $W_y \cap U_y = \emptyset$ . Then  $W_y \subseteq U_y^c$ , and thus  $\bar{W}_y \subseteq U_y^c$ , since  $U_y^c$  is closed. Yet  $y \notin U_y^c$ , thus  $y \notin \bar{W}_y$ .

Now consider the family  $\{C \cap \bar{W}_y \cap \bar{G}\}_{y \in C}$ . Notice that each  $C \cap \bar{W}_y \cap \bar{G}$  is compact, since  $C, \bar{W}_y$  are closed, and  $\bar{G}$  is compact.

Yet  $\bigcap_{y \in C} (C \cap \bar{W}_y \cap \bar{G}) = \emptyset$ .

Thus  $\exists y_1, \dots, y_n \in C$ , such that  $\bigcap_{i=1}^n (C \cap \bar{W}_{y_i} \cap \bar{G}) = \emptyset$ .

Now let  $V := G \cap \bigcap_{i=1}^n W_{y_i}$ .

Clearly  $V$  is open, and  $K \subseteq V$ .

In addition,  $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$ , yet the intersection of righthand side and  $C$  is empty, thus contained in  $C^c = U$ . □

## 2.5 Compactly Supported Continuous Functions

**Definition 2.40.** For  $f \in C(X)$ , the **support** of it is

$$\text{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

**Definition 2.41.** The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \text{Supp}(f) \text{ is compact}\}.$$

**Definition 2.42.**  $C_0(X)$  is the closure of  $C_c(X)$  in  $\|\cdot\|_\infty$ .

**Proposition 2.36.**  $C_0(X)$  is the set of all continuous functions that vanishes at  $\infty$ .  $(C_0(X), \|\cdot\|_\infty)$  is a Banach Space and a commutative  $C^*$ -algebra with the involution  $f^*(x) := \overline{f(x)}$ .

**Proposition 2.37.**  $f \in C_0(X)$  if and only if  $\forall \epsilon > 0, \exists K \subset\subset X$ , such that  $\forall x \in X \setminus K, |f(x)| < \epsilon$ .

**Theorem 2.38.** Any commutative  $C^*$ -algebra  $(A, \|\cdot\|)$  is isomorphic to  $C_0(X)$  for some unique Locally Compact Hausdorff  $X$ .

### 2.5.1 Partition of Unity

**Definition 2.43.** Let  $K$  be a compact set, and  $V$  be an open set of  $X$ . Let  $f \in C_c(X)$ . We say  $f < V$  if  $0 \leq f \leq 1$ , and  $\text{Supp}(f) \subseteq V$ . We say  $K < f$  if  $0 \leq f \leq 1$ , and  $f|_K = 1$ . We say  $K < f < V$  if  $K \subset V, K < f, f < V$ .

*Remark.*  $f$  is a “bump” function that approximates  $\chi_K$  when  $V$  shrinks towards  $K$ .

**Lemma 2.39** (Urysohn’s lemma for Locally Compact Hausdorff Space). *Let  $X$  be a Locally Compact Hausdorff space,  $K \subseteq V \subseteq X$  be such that  $K$  is compact, and  $V$  is open. Then there exists  $f \in C_c(V)$ , such that  $K < f < V$ .*

*Proof.* we want to construct a family of open sets  $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$ , such that  $\bar{V}_r$  is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for  $r < s$ .

By 2.35, we can find  $K \subset V_0 \subset \bar{V}_0 \subset V$ .

Pick an enumeration of  $r \in \mathbb{Q} \cap (0,1]$ , i.e.  $(r_n)_{n=1}^\infty$ . WLOG, we can let  $r_1 = 1$ .

By 2.35, we can find  $K \subset V_1 \subset \bar{V}_1 \subset V_0$ .

Suppose we have constructed the  $V_{r_i}$  for  $1 \leq i \leq n$ , such that  $\bar{V}_r$  is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for  $r < s \in \{r_i\}_{i=1}^n$ .

Let  $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$ .

Now by 2.35, we can find  $\bar{V}_t \subset V_{n+1} \subset V_{n+1}^- \subset V_s$ .

For any  $r < r_{n+1}$ , we have  $r \leq s$ , and thus  $V_{n+1} \subset V_{n+1}^- \subset V_s \subset \bar{V}_s \subseteq V_r$  by induction hypothesis, and similarly for any  $r > r_{n+1}$ .

Inductively, we can prove there is such a family.

Define  $f_r := r\chi_{V_r}$ , and  $g_r := r\chi_{\bar{V}_r^c} + \chi_{\bar{V}_r}$ , and  $f := \sup_r f_r, g := \inf_r g_r$ .

We can show that  $f, g$  are upper and lower continuous, respectively.

In addition,  $f, g$  are both 0 outside of  $V_1$ , and 1 on  $K$ .

Suppose there is some  $x \in X, r, s \in \mathbb{Q} \cap [0,1]$ , such that  $f_r(x) > g_s(x)$ . Then we must have  $f_r(x) > 0$ , and thus  $x \in V_r$  and  $1 \leq r = f_r(x)$ .

Thus  $1 > g_s(x)$ , and thus  $x \in \bar{V}_s^c$  and  $f_s = s$ .

Since  $r > s$ , we must have  $V_r \subset \bar{V}_r \subset V_s \subset \bar{V}_s$ , which is a contradiction to  $x \in V_r, x \notin \bar{V}_s$ .

Thus for any  $x \in X, r, s \in \mathbb{Q} \cap [0,1]$ , we must have  $f_r(x) \leq g_s(x)$ .

Thus we must have  $f(x) \leq g(x)$  for any  $x \in V$ .

Now suppose there is some  $x \in X$ , such that  $f(x) < g(x)$ .

There must be some rationals, such that  $f(x) < r < s < g(x)$ , since  $\mathbb{Q}$  is dense.

Thus  $\sup_r f_r(x) < r$ , and thus  $x \notin V_r$ .

Also,  $\inf_s g_s(x) > s$ , and thus  $x \in \bar{V}_s$ .

Yet  $r < s$ , we must have  $V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r$ , which is a contradiction.

Thus we must have  $f = g$ , and it forces  $f$  to be continuous.  $\square$

**Definition 2.44.** Let  $X$  be a Locally Compact Hausdorff space,  $K \subseteq X$  be compact, and some finite open cover  $\bigcup_{i=1}^n V_i \supseteq K$ .

A collection  $(h_i)_{i=1}^n \subset C_c(X)$  is called a **partition of unity** on  $K$  subordinate to  $(V_i)_{i=1}^n$  if

$$\begin{cases} \forall 1 \leq i \leq n, & h_i < V_i, \\ \forall x \in K, & \sum_{i=1}^n h_i(x) = 1. \end{cases}$$

**Theorem 2.40.** Let  $X$  be a Locally Compact Hausdorff space,  $K \subseteq X$  be compact, and some finite open cover  $\bigcup_{i=1}^n V_i \supseteq K$ , there always exists a partition of unity on  $K$  subordinated to  $(V_i)_{i=1}^n$ .

*Proof.* Since  $K$  is compact, we can find some open cover  $W_1, \dots, W_m$ , such that for all  $j$ , we have  $W_j \subset \bar{W}_j \subset V_{i(j)}$  for some  $1 \leq i(j) \leq n$ .

Let  $K_i := \bigcup_{1 \leq j \leq m \text{ such that } W_j \subset V_i} \bar{W}_j \subset V_i$ , which is compact.

By Urysohn's lemma, we can find  $K_i < g_i < V_i$ .

Now let  $h_1 := g_1$ , and in general,  $h_i := g_i \prod_{j < i} (1 - g_j)$ .

It is easy to check that  $0 \leq h_i \leq 1$ , and  $h_i \in C_c(X)$ .

In addition,  $\text{Supp}(h_i) \subseteq \text{Supp}(g_i) \subset V_i$ .

Thus  $h_i < V_i$ . Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have  $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_i)$ .

For any  $x \in K$ , there must be some  $i \in [n]$  such that  $x \in K_i$ , and thus  $g_i(x) = 1$ , and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_i(x)) = 1 - 0 = 1.$$

$\square$

## 2.6 Product Topology

**Definition 2.45.** Let  $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$  be a family of topological spaces. The **product topology** on  $\prod_\alpha X_\alpha$  is the topology generated by the sets

$$\left\{ U_\alpha \times \prod_{\beta \in A, \beta \neq \alpha} X_\beta \mid \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\} = \left\{ \pi_\alpha^{-1}(U_\alpha) \mid \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\},$$

where the **projection map onto**  $X_\alpha$  is  $\pi_\alpha : \prod_{\beta \in A} X_\beta \rightarrow X_\alpha$  by  $(x_\beta)_{\beta \in A} \mapsto x_\alpha$ .

**Proposition 2.41.** The product topology is the weakest topology in which each  $\pi_\alpha$  is continuous.

**Proposition 2.42.** A net  $(x_\lambda)_{\lambda \in \Lambda}$  in  $\prod_{\alpha \in A} X_\alpha$  converges to  $x$  if and only if  $\forall \alpha \in A$ ,  $\pi_\alpha(x_\lambda) \rightarrow \pi_\alpha(x)$  in  $X_\alpha$ .

*Proof.* See A1.  $\square$

**Theorem 2.43** (Tychonoff). *Let  $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$  be a family of compact topological spaces, then  $\prod_\alpha X_\alpha$  is compact under product topology.*

**Definition 2.46.** Let  $(P, \leq)$  be a partially ordered set. We call a totally ordered subset  $Q \subseteq P$  a **chain**.

**Definition 2.47.** Let  $(P, \leq)$  be a partially ordered set. We call  $\leq$  **inductive** if every chain  $Q \subseteq P$  has an upper bound.

**Definition 2.48.**  $\leq$  is called a **well-order** if it is a total order, and for every  $\emptyset \neq S \subseteq X$  has a minimal element.  $\exists x \in P$  such that  $\forall y \in P, y \leq x \implies x = y$ .

**Lemma 2.44** (Zorn's). *Every inductive partial order  $(P, \leq)$ , defined on a nonempty  $P$ , has a maximal element. Namely,  $\exists x \in P$  such that  $\forall y \in P, x \leq y \implies x = y$ .*

**Proposition 2.45.** *Every vector space  $V$  has a basis.*

*Proof.* Consider  $P = \{S \subset V \mid S \text{ is linearly independent}\}$ , with  $S \leq S' \iff S \subseteq S'$ .

Let  $Q$  be a chain in  $P$ , then it has an upper bound  $\tilde{S} = \bigcup_{S \in Q} S$ , which we can check is still linearly independent.

By Zorn's lemma, there is a maximal  $S \in P$ . □

**Theorem 2.46** (Well-Ordering Principle). *Every set  $X$  admits a well-ordering.*

**Theorem 2.47.** *The Following Are Equal:*

1. *Tychonoff's Theorem*
2. *Axiom of Choice*
3. *Zorn's Lemma*
4. *Well-Ordering Principle*

*Proof.* (1)  $\implies$  (2).

Let  $(X_\alpha)_{\alpha \in A}$  be a family of non-empty set.

Let  $Y_\alpha := \{p_\alpha\} \sqcup X_\alpha$  for some additional symbol  $p_\alpha$ .

Define the topology  $\mathcal{T}_\alpha := \{\emptyset, Y_\alpha, X_\alpha, \{p_\alpha\}\}$ .

Then  $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$  are all compact.

By Tychonoff's Theorem,  $\prod_\alpha Y_\alpha$  is also compact.

Now consider  $C_\alpha := X_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$ .

Since  $C_\alpha^c = \{p_\alpha\} \times \prod_{\beta \neq \alpha} Y_\beta$  is open, we have that  $C_\alpha$  is closed in the product topology.

Also,  $C_\alpha \neq \emptyset$ .

Now for any finite  $\bigcap_{i=1}^n C_{\alpha_i}$ , we have that  $(x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_n}, p_\alpha, \dots) \in \bigcap_{i=1}^n C_{\alpha_i}$ .

Thus  $(C_\alpha)_{\alpha \in A}$  is an FIP family.

Since  $\prod_\alpha Y_\alpha$  is compact, we have that  $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$ .

Now  $\prod_{\alpha \in A} X_\alpha = \bigcap_{\alpha \in A} C_\alpha$ , and we have seen that it being nonempty is equivalent as the Axiom of Choice.

(4)  $\implies$  (2).

Take  $C : \mathcal{P}(X) \setminus \emptyset \rightarrow X$  to be  $C(S) :=$  minimal element of  $S$ .

(3)  $\implies$  (2)

Let  $\{X_\alpha\}_{\alpha \in A}$  to be non-empty sets. Let  $X := \bigcup_{\alpha \in A} X_\alpha$ ,  $P = \{f_B : B \rightarrow X \mid B \subseteq A, f_B(\beta) \in X_\beta, \forall \beta \in B\}$ .

Clearly  $P \neq \emptyset$ .

Define the order by  $f_B \leq f_{B'} \iff B \subseteq B', f_{B'}|_B = f_B$ .

For any chain  $Q \subseteq P$ , define  $\tilde{B} = \bigcup_{B \in Q} B$ , and  $f_{\tilde{B}}(\beta) = f_B(\beta)$  for  $\beta \in B$  of any  $B$ .

We can check  $f_{\tilde{B}} \in P$  is an upper bound.

By Zorn's Lemma, there is a maximal element  $f_B \in P$ .

If  $B \subsetneq A$ , then we can extend the function to contain another point  $a \in A \setminus B$ , and send  $a$  to any  $x \in X_a$ , contradicting maximality.

Thus there is some  $f_A \in P$ , which we have seen is equivalent to the Axiom of Choice.

(4) + (2)  $\implies$  (3).

See Pmath432 A1.

(3) + (2)  $\implies$  (1).

For contradiction, suppose  $X = \prod_{\alpha \in A} X_\alpha$  is not compact.

Let  $\text{NFS} := \{\mathcal{C} \subseteq \mathcal{P}(X) : \mathcal{C} \text{ is a cover with no finite sub cover}\}$ .

Define  $\mathcal{C}_1 \leq \mathcal{C}_2 \iff \mathcal{C}_1 \subseteq \mathcal{C}_2$ .

Take any chain  $Q$  in NFS.

Let  $\mathcal{C}' := \bigcup_{\mathcal{C} \in Q} \mathcal{C}$ , which we can check is an open cover and an upper bound for the chain.

Indeed, suppose  $\mathcal{C}' \notin \text{NFS}$ , then  $\exists U_1, U_2, \dots, U_n \in \mathcal{C}'$  such that  $X = \bigcup_{i=1}^n U_i$ .

For all  $i \in [n]$ ,  $U_i \in \mathcal{C}_i$  for some  $\mathcal{C}_i \in Q$ .

Since  $Q$  is a chain, there is some  $i_0 \in [n]$  such that  $\forall i \in [n]$ ,  $\mathcal{C}_i \subseteq \mathcal{C}_{i_0}$ .

Thus  $\mathcal{C}_{i_0} \notin \text{NFS}$ , a contradiction.

Thus  $\mathcal{C} \in \text{NFS}$ .

By Zorn's Lemma, there is a maximal open cover  $\mathcal{C}_{\max}$  with no subcover.

Notice that if  $U \in \mathcal{C}_{\max}$ , and  $V \subseteq U$  is open, then  $V \in \mathcal{C}_{\max}$  as well, since any finite subcover of  $\{V\} \cup \mathcal{C}_{\max}$  can give a finite subcover of  $\mathcal{C}_{\max}$  by replacing  $V$  by  $U$ .

Also, if  $U_1, U_2 \in \mathcal{C}_{\max}$ , we must have  $U_1 \cup U_2 \in \mathcal{C}_{\max}$  as well.

Also, suppose  $V_1, \dots, V_n$  are open in  $X$ , such that  $\bigcap_{i \in [n]} V_i \in \mathcal{C}_{\max}$ , then  $\exists i_0 \in [n]$ , such that  $V_{i_0} \in \mathcal{C}_{\max}$ .

Indeed, suppose not, for any  $i \in [n]$ , there is a finite cover  $V_i \cup \bigcup_{j \in [N_i]} U_{i,j}$  for  $U_{i,j} \in \mathcal{C}_{\max}$ . We must have

$\left(\bigcap_{i \in [n]} V_i\right) \cup \bigcup_{i \in [n], j \in [N_i]} U_{i,j}$  is a finite sub-cover of  $\mathcal{C}_{\max}$ .

Now let  $W_\alpha := \{\text{open } U_\alpha \subseteq X_\alpha \mid \pi_\alpha^{-1}(U_\alpha) \in \mathcal{C}_{\max}\}$ .

For contradiction, suppose  $W_\alpha$  covers  $X_\alpha$ , then there is a finite subcover  $\{U_i\}_{i \in [n]}$  such that  $X_\alpha = \bigcup_{i \in [n]} U_i$ .

Thus  $X = \bigcup_{i=1}^n \pi_\alpha^{-1}(U_i)$ , which is a subcover of  $\mathcal{C}_{\max}$ .

Thus  $X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right) \neq \emptyset$ .

By the Axiom of Choice, there is  $x_\alpha \in X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right)$  for each  $\alpha$ .

Let  $x \in X$  be  $x(\alpha) = x_\alpha$ .

Since  $\mathcal{C}_{\max}$  is a cover for  $X$ , there is some open  $U \in \mathcal{C}_{\max}$  with  $x \in U$ .

Thus, there must be some  $x \in U_1 \times U_2 \times \dots \times U_n \times \prod_{\beta \in A \setminus \{\alpha_i : i \in [n]\}} = \bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$  for open  $U_i \in X_{\alpha_i}$ , since such sets forms a basis.

Thus,  $\bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{\max}$  as well.

Thus, there is some  $i_0 \in [n]$ , such that  $\pi_{\alpha_{i_0}}^{-1}(U_{i_0}) \in \mathcal{C}_{\max}$ , which means  $U_{i_0} \in W_{\alpha_{i_0}}$ .

However,  $x_{\alpha_{i_0}} \in U_{\alpha_{i_0}}$ , which is a contradiction to the choice of  $x_{\alpha_{i_0}} \notin \left(\bigcup_{U \in W_{\alpha_{i_0}}} U\right)$ . □

### 3 Banach Spaces

**Definition 3.1.** A **normed vector space** is a vector space  $(X, \|\cdot\|)$  over a field  $\mathbb{F}$  endowed with a norm (length) function:  $\|\cdot\| : X \rightarrow [0, \infty)$ , such that  $\forall x, y \in X, a \in \mathbb{F}$ , it satisfies

1. subadditivity (triangular inequality); i.e.  $\|x + y\| \leq \|x\| + \|y\|$ ,
2. absolute homogeneity; i.e.  $\|a \cdot x\| = |a| \|x\|$ , and
3. positive definiteness; i.e. if  $x \neq 0$ , we must have  $\|x\| > 0$ .

**Proposition 3.1.** For every **normed space** with  $\|\cdot\|$ , there is a metric  $d(x, y) = \|x - y\|$ .

*Proof.*

$$d(x, x) = \|x - x\| = \|0\| = 0$$

$$\forall x \neq y, d(x, y) = \|x - y\| > 0$$

$$d(x, y) = \|x - y\| = \|-(y - x)\| = |-1| \|y - x\| = \|y - x\| = d(y, x)$$

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \geq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

Thus  $d(x, y) = \|x - y\|$  is a metric. □

**Definition 3.2.** Let  $\|\cdot\|_1, \|\cdot\|_2$  be two norms on  $X$ , then they are called **equivalent** if there are  $C_1, C_2 > 0$ , such that

$$\forall x \in X, C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

**Definition 3.3.** A normed space is called a **Banach space** if it is complete.

**Proposition 3.2.** The Euclidean space  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , with the Euclidean norm  $\|x\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$  is a Banach space.

**Definition 3.4.** For  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , and  $p \in [1, \infty)$ , the  $\ell^p$  norm is  $\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ . For  $p = \infty$ , the  $\ell_\infty$  norm is  $\|x\|_\infty = \max_{i \in [n]} |x_i|$ .

**Proposition 3.3.**  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , with any  $\ell^p$  norm is a Banach space.

*Remark.* Notice that  $\forall n \in \mathbb{N}^+, \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty$ , so they are equivalent.

**Proposition 3.4.** If  $X$  is compact and Hausdorff, we have  $(C(X), \|\cdot\|_\infty)$  is a Banach Space.

*Proof.* The Extreme Value Theorem shows it is a normed space.

The convergence in  $\|\cdot\|_\infty$  is uniform convergence, and the uniform limit of continuous functions is continuous. Thus,  $(C(X), \|\cdot\|_\infty)$  is complete.  $\square$

**Proposition 3.5.** For a Locally Compact Hausdorff Space  $X$ ,  $(C_b(X), \|\cdot\|_\infty)$  and  $(C_0(X), \|\cdot\|_\infty)$  are both Banach Spaces.

**Example 3.0.1.**  $C_0(\mathbb{N}) = \{(x_n)_{n=1}^\infty : \lim_{n \rightarrow \infty} x_n = 0\}$  with the discrete topology.

**Proposition 3.6.**  $C^k([0, 1])$  is a Banach Space with  $\|f\|_{C^k([a, b])} := \sum_{i=1}^k \|f^{(i)}\|_\infty$ .

*Proof.* It is easy to check that this is a norm.

Now take any Cauchy sequence  $(f_n)_{n=1}^\infty$ , then for each  $i \in [k]$ , we have  $(f_n^{(i)})_{n=1}^\infty$  is Cauchy in  $C([a, b])$  as well.

Since  $[a, b]$  is compact and Hausdorff, we have  $C([a, b])$  is a Banach Space, so there is a  $g_i(x) := \lim_{n \rightarrow \infty} f_n^{(i)}(x)$ . In addition, since the convergence is uniform, we have  $g_i \in C([0, 1])$ .

By the Fundamental Theorem of Calculus, we have that for all  $i \in [k-1], x \in [0, 1]$ ,

$$f_n^{(i)}(x) = f_n^{(i)}(0) + \int_0^x f_n^{(i+1)}(t) dt.$$

Taking the limit of  $n \rightarrow \infty$ , we have that

$$g_i(x) = g_i(0) + \int_0^x g_{i+1}(t) dt,$$

which means  $g_i \in C^1([0, 1])$  with  $g'_i = g_{i+1}$ .

Thus  $g_0 \in C^k([0, 1])$ .  $\square$

### 3.1 Bounded linear operators

**Definition 3.5.** Let  $X, Y$  be vector spaces,  $T : X \rightarrow Y$  is a linear operator if  $\forall c \in \mathbb{R}, u, v \in X$ ,

$$T(u + cv) = Tu + cTv.$$

**Definition 3.6.** Let  $X, Y$  be linear normed spaces, the **operator norm** of a linear operator  $T : X \rightarrow Y$  is

$$\|T\| := \sup_{\|u\|_X \leq 1} \|Tu\|_Y = \sup_{\|u\|_X = 1} \|Tu\|_Y = \sup_{u \neq 0 \in X} \frac{\|Tu\|_Y}{\|u\|_X}.$$

**Definition 3.7.** Let  $X, Y$  be normed spaces, a linear operator  $T : X \rightarrow Y$  is **bounded** if  $\|A\| < \infty$ .



**Theorem 3.7.** Let  $X, Y$  be two normed linear spaces, let  $T : X \rightarrow Y$  be linear, then the following are equal:

1.  $T$  is continuous,
2.  $T$  is continuous at 0,
3.  $T$  is bounded,
4.  $T$  is uniformly continuous.

*Proof.* (4)  $\implies$  (1)  $\implies$  (2) trivially.

(3)  $\implies$  (4).

Suppose  $T$  is bounded, then

$$\begin{aligned}\|T(x) - T(y)\| &= \|T(x - y)\| \\ &\leq \|T\| \|x - y\|.\end{aligned}$$

Thus,  $T$  is  $\|T\|$  Lipschitz and so uniformly continuous.

(2)  $\implies$  (3).

Suppose  $T$  is continuous at 0, and suppose for contradiction that  $\|T\| = \infty$ .

There must be  $(x_n)_{n=1}^\infty$  in  $X$ , such that  $\|x_n\| \leq 1$ ,  $\|Tx_n\| \geq n^2$  for each  $n \geq 1$ .

Notice that  $\frac{x_n}{n} \rightarrow 0$ , but  $\|T(\frac{x_n}{n})\| = \frac{1}{n} \|Tx_n\| \geq n$  for each  $n$ .

Thus  $\lim_{n \rightarrow \infty} T(\frac{x_n}{n}) \neq 0 = T(0)$ , which contradicts that  $T$  is continuous at 0.  $\square$

**Proposition 3.8.** Let  $\|\cdot\|_1, \|\cdot\|_2$  be two norms on  $X$ , then they are equivalent if and only if they induce the same topology.

*Proof.* Assume that  $\|\cdot\|_1, \|\cdot\|_2$  are equivalent, then there are  $C_1, C_2 > 0$ , such that

$$\forall x \in X, C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1.$$

Consider the identity function  $id : (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$ , we see that

$$\|id\| = \sup_{x \neq 0 \in X} \frac{\|x\|_2}{\|x\|_1} \leq \sup_{x \neq 0 \in X} \frac{C_2 \|x\|_1}{\|x\|_1} = C_2,$$

and

$$\|id^{-1}\| = \sup_{x \neq 0 \in X} \frac{\|x\|_1}{\|x\|_2} \leq \sup_{x \neq 0 \in X} \frac{\|x\|_1}{C_1 \|x\|_1} = \frac{1}{C_1}.$$

Thus  $id$  is a homeomorphism.

On the other hand, suppose  $\|\cdot\|_1, \|\cdot\|_2$  induces the same topology, then  $id$  is a homeomorphism.

Thus,

$$\frac{1}{\|id\|} \|x\|_2 = \frac{1}{\|id\|} \|id(x)\|_2 \leq \|x\|_1 = \|id^{-1}(x)\|_1 \leq \|id^{-1}\| \|x\|_2.$$

$\square$

**Definition 3.8.** Let  $X, Y$  be normed spaces, we denote

$$B(X, Y) := \{T : X \rightarrow Y \mid T \text{ is a bounded linear operator}\}.$$

**Theorem 3.9.** The set  $B(X, Y)$  is a normed linear space with the operator norm.

**Proposition 3.10.** Let  $X, Y, Z$  be normed spaces, if  $T : X \rightarrow Y, S : Y \rightarrow Z$  are both linear bounded operators, then so is  $S \circ T$ , with

$$\|S \circ T\| \leq \|S\| \|T\|.$$

**Theorem 3.11.** Let  $X$  be a normed space, and  $Y$  be a Banach Space, then  $B(X, Y)$  is a Banach Space.

*Proof.* Let  $(T_n)_{n=1}^\infty$  be a Cauchy sequence in  $B(X, Y)$ .  
For any  $x \in X$ , we have that  $(T_n x)_{n=1}^\infty$  is Cauchy in  $Y$ .  
Indeed,  $\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\|$ .  
Since  $Y$  is complete, there must be a unique  $y = \lim_{n \rightarrow \infty} T_n x \in Y$ .  
Define  $Tx := \lim_{n \rightarrow \infty} T_n x$  for any  $x \in X$ .

Notice that  $T$  is linear.

Given  $\epsilon > 0$ , we know there must be some  $N \in \mathbb{N}$ , such that  $\forall m, n \geq N$ ,  $\|T_n - T_m\| < \epsilon$ .  
Consider any  $x \in X$ .

$$\begin{aligned} \|Tx\| &\leq \|(T - T_N)x\| + \|T_N x\| \\ &= \lim_{m \rightarrow \infty} \|(T_m - T_N)x\| + \|T_N x\| \\ &\leq \limsup_m \|T_m - T_N\| \|x\| + \|T_N\| \|x\| \\ &\leq \epsilon \|x\| + \|T_N\| \|x\|. \end{aligned}$$

Thus  $\|T\| \leq \epsilon + \|T_N\| < \infty$ .

This shows  $T \in B(X, Y)$ .

Again, for any  $x \in X$ ,  $n \geq N$ , we have

$$\begin{aligned} \|(T_n - T)x\| &= \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| \\ &\leq \limsup_m \|T_n - T_m\| \|x\| \\ &< \epsilon \|x\|. \end{aligned}$$

Thus  $\|T_n - T\| < \epsilon$  for any  $n \geq N$ , which shows  $\lim_{n \rightarrow \infty} T_n = T$  in  $B(X, Y)$  with the operator norm.  $\square$

**Definition 3.9.** Let  $X, Y$  be normed spaces. We say a linear operator  $T : X \rightarrow Y$  is **bounded below** if  $\exists m > 0$ , such that  $\forall x \in X$ ,  $\|Tx\| \geq m\|x\|$ .

**Definition 3.10.** Let  $X, Y$  be normed spaces. We say  $T : X \rightarrow Y$  is an **isomorphism between normed spaces** or  $T$  is **invertible** if  $T$  is bijective and  $T, T^{-1}$  are both bounded. i.e.  $T$  is a homeomorphism between  $X, Y$ .

**Theorem 3.12.** Let  $T : X \rightarrow Y$  be a linear operator between normed linear spaces  $X$  and  $Y$ . The operator  $T$  is invertible, namely,  $T$  has a bounded inverse  $T^{-1} : \text{Im}(T) \rightarrow X$ , if and only if  $T$  is bounded below. In this case,

$$\|T^{-1}\| = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

*Proof.* ( $\implies$ ) Suppose  $T$  has a bounded inverse operator. Let  $x \in X$  and  $y = Tx \in \text{Im}(T)$ . By the assumption,

$$\begin{aligned} \|x\| &= \|T^{-1}y\| \leq \|T^{-1}\| \|y\| = \|T^{-1}\| \|Tx\|, \\ \|Tx\| &\geq \frac{1}{\|T^{-1}\|} \|x\|. \end{aligned}$$

Since the inequality holds for all  $x \in X$ ,  $T$  is bounded below.

( $\impliedby$ ) Suppose  $T$  is bounded below. There exists  $m > 0$  such that  $\|Tx\| \geq m\|x\|$  for all  $x \in X$ . Let  $z \in \ker T$ . We have  $0 = \|Tz\| \geq m\|z\| \geq 0$ . Therefore,  $\|z\| = 0$ , which means  $z = 0$ . Thus  $\ker T = \{0\}$  and  $T$  is one-to-one. Thus,  $T$  is a bijection between  $X \longleftrightarrow \text{Im}(T)$ , which means  $T^{-1} : \text{Im}(T) \rightarrow X$  exists.

Finally, for any  $y \in \text{Im}(T)$ ,  $y = Tx$  for some  $x \in X$ . Then

$$\|T^{-1}y\| = \|x\| \stackrel{T \text{ bounded below}}{\leq} \frac{1}{m} \|Tx\| = \frac{1}{m} \|y\|.$$

Therefore  $T^{-1}$  is bounded. In particular,

$$\|T^{-1}\| = \sup_{y \in \text{Im}(T), y \neq 0} \frac{\|T^{-1}y\|}{\|y\|} = \sup_{x \neq 0} \frac{\|x\|}{\|Tx\|} = \frac{1}{\inf_{x \neq 0} \frac{\|Tx\|}{\|x\|}} = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

□

**Definition 3.11.** Let  $X, Y$  be normed spaces. We call  $T : X \rightarrow Y$  a **contraction** if  $\|T\| \leq 1$ .

**Definition 3.12.** Let  $X, Y$  be normed spaces. We call  $T : X \rightarrow Y$  an **isometry** if  $\forall x \in X, \|Tx\| = \|x\|$ .

**Proposition 3.13.** Let  $X, Y$  be normed spaces. If a linear operator  $T : X \rightarrow Y$  is a surjective isometry, it is an isometric isomorphism between normed spaces.

*Proof.*  $T$  is bounded since  $\|T\| = 1$ . Suppose  $T(x) = 0$ , we have  $\|Tx\|_Y = 0$ . Since  $T$  is an isometry and thus bounded below by 1,  $T$  has a bounded inverse  $T^{-1} : \text{Im}(T) \rightarrow X$  by theorem 3.12. Since  $T$  is surjective,  $\text{Im}(T) = Y$ . □

**Proposition 3.14.** Let  $Y$  be a Banach space,  $S$  be a dense subset of a normed space  $X$ . For any bounded linear operator  $E : S \rightarrow Y$ , we can extend it to  $\tilde{E} : X \rightarrow Y$ , such that  $\tilde{E}$  is also bounded and linear, with  $\|\tilde{E}\| = \|E\|$ , and  $\tilde{E}|_S = E$ .

*Proof.* Consider any  $x \in X$ .

Since  $S$  is dense in  $X$ , We know  $\forall m \in \mathbb{N}^+, \exists x_m \in S$ , such that  $\|x - x_m\|_X \leq \frac{1}{m}$ . Since  $E$  is linear on  $S$ , we have that

$$\begin{aligned} \|Ex_m - Ex_l\|_Y &= \|E(x_m - x_l)\|_Y \\ &\leq \|E\| \|x_m - x_l\|_X \\ &= \|E\| \|(x_m - x) + (x - x_l)\|_X \\ &\leq \|E\| \|x - x_m\|_X + \|E\| \|x - x_l\|_X \\ &\leq \|E\| \left( \frac{1}{m} + \frac{1}{l} \right). \end{aligned}$$

Thus given any  $\epsilon > 0$ , for any  $m, l \geq \lceil \frac{2\epsilon}{\|E\|} \rceil$ , we can make  $\|Ex_m - Ex_l\|_Y < \epsilon$ . Thus  $(Ex_m)_{m=1}^\infty$  is a Cauchy sequence in  $Y$ .

Since  $Y$  is a Banach space,  $\exists y^* \in Y$ , such that  $Ex_m \rightarrow y^*$  in  $Y$ .

We claim that  $y^*$  is independent of choice of the sequence  $(x_m)_{m=1}^\infty$ .

Indeed, consider any other sequence  $(v_m)_{m=1}^\infty \subset C^\infty(\bar{x})$ , such that  $\forall m \in \mathbb{N}^+, \|x - v_m\|_X \leq \frac{1}{m}$ ,

$$\begin{aligned} \|y^* - Ev_m\|_Y &\leq \|y^* - Ex_m\|_Y + \|Ex_m - Ev_m\|_Y \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - v_m\|_X \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - x\|_X + \|E\| \|x - v_m\|_X. \end{aligned}$$

Since all three terms on the right go to 0 when  $m \rightarrow \infty$ , we have that  $Ev_m \rightarrow y^*$  in  $Y$ .

Thus we can uniquely define  $\tilde{E}x := y^*$ . In addition,

$$\begin{aligned} \|\tilde{E}x\|_Y &= \left\| \lim_{m \rightarrow \infty} Ex_m \right\|_Y \\ &= \lim_{m \rightarrow \infty} \|Ex_m\|_Y \\ &\leq \lim_{m \rightarrow \infty} \|E\| \|x_m\|_X \\ &= \|E\| \left\| \lim_{m \rightarrow \infty} x_m \right\|_X \\ &= \|E\| \|x\|_X. \end{aligned}$$

Thus  $\|\tilde{E}\| = \|E\|$ . □

### 3.1.1 Dual Spaces

**Definition 3.13.** Let  $X$  be a normed space over  $\mathbb{F}$ , a **functional** is an operator that maps into  $\mathbb{F}$ .

**Definition 3.14.** Let  $X$  be a normed space over  $\mathbb{F}$ , the **dual space** of  $X$  is the collection of bounded linear functionals on  $X$ , denoted

$$X^* := B(X, \mathbb{F}) = \{\phi : X \rightarrow \mathbb{F} : \phi \text{ is linear and bounded}\}.$$

**Definition 3.15.** Let  $X$  be a normed space over  $\mathbb{F}$ , and a subspace  $Y \subseteq X^*$  we define the **duality pairing** to be  $\langle \cdot | \cdot \rangle_{X,Y} : X \times Y \rightarrow \mathbb{F}$  to be  $x \in X, \phi \in Y$ , we can write  $\langle x | \phi \rangle_{X,Y} := \phi(x)$  as the **action** of  $\phi$  on  $x$ .

**Definition 3.16.** Let  $X$  be a normed space, the **dual norm** is defined to be

$$\|u^*\|_{X^*} := \sup_{\|u\| \leq 1} |\langle u^* | u \rangle_{X^*, X}|.$$

**Example 3.1.1.** Consider  $C_0(\mathbb{N}) := \{(x_i)_{i=1}^\infty | \lim_{i \rightarrow \infty} x_i = 0\}$ , then  $C_0(\mathbb{N})^* \cong \ell^1(\mathbb{N})$ .

Indeed, let  $e_n := (m \mapsto \delta_{nm}) = (\delta_{nm})_{n=1}^\infty \in C_0(\mathbb{N})$ .

Given any  $\phi \in C_0(\mathbb{N})^*$ , we define  $a_n := \phi(e_n) \in \mathbb{F}$ .

We claim  $a = (a_n)_{n=1}^\infty$  completely determines  $\phi$ .

Indeed, consider any  $x \in C_0$ , let  $x^N := \sum_{n=1}^N x_n e_n$ .

We have  $\|x - x^N\|_{C_0} = 0$ , so

$$\begin{aligned} \phi(x) &= \lim_{N \rightarrow \infty} \phi(x^N) \\ &= \lim_{N \rightarrow \infty} \phi\left(\sum_{n=1}^N x_n e_n\right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n \phi(e_n) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n a_n. \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=1}^N |a_n| &= \sum_{n=1}^N a_n \operatorname{sgn}(a_n) \\ &= \sum_{n=1}^N \phi(\operatorname{sgn}(a_n) e_n) \\ &= \phi(y_N) \\ &\leq \|\phi\|_{C_0^*} \|y_N\|_{C_0}, \end{aligned}$$

where  $y_N = \sum_{i=1}^N \operatorname{sgn}(a_i) e_i$ .

Since  $\|y_N\| = 1$ , we have  $\sum_{n=1}^N |a_n| \leq \|\phi\|_{C_0^*}$  for any  $N$ .

Thus

$$\|a\|_1 = \sum_{n=1}^\infty |a_n| = \lim_{N \rightarrow \infty} \sum_{n=1}^N |a_n| \leq \|\phi\|_{C_0^*} < \infty.$$

Thus  $a \in \ell^1(\mathbb{N})$ .

Notice that  $\Phi : C_0^*(\mathbb{N}) \rightarrow \ell^1(\mathbb{N})$  by  $\phi \mapsto a$  is linear, contractive, and injective.

Also, given any  $a \in \ell^1(\mathbb{N})$ , we can set  $\phi_a(x) := \sum_{n=1}^{\infty} x_n a_n$ .  
Thus,

$$\begin{aligned} |\phi_n(x)| &\leq \sum_{n=1}^{\infty} |x_n| |a_n| \\ &\leq \|x\|_{\infty} \|a\|_1, \end{aligned}$$

which shows  $\|\phi_a\|_{C_0^*} \leq \|a\|_1$ .

Thus  $\phi_a \in C_0^*(\mathbb{F})$ . Notice that  $\Phi(\phi_a) = a$ .

This shows that  $\Phi$  is surjective, and it is actually an isometry, since  $\forall \phi \in C_0^*(\mathbb{F})$ , we have

$$\|\Phi(\phi)\|_1 \leq \|\phi\|_{C_0^*} = \|\Phi^{-1}(\Phi(\phi))\|_{C_0^*} \leq \|\Phi(\phi)\|_1.$$

**Example 3.1.2.** Consider  $C_0(\mathbb{N}) := \{(x_i)_{i=1}^{\infty} \mid \lim_{i \rightarrow \infty} x_i = 0\}$ , then

$$B(C_0(\mathbb{N})) = B(C_0(\mathbb{N}), C_0(\mathbb{N})) \cong \{(t_{ij})_{i,j \in \mathbb{N}} : \|(t_{ij})_{i,j \in \mathbb{N}}\| < \infty, \forall j \in \mathbb{N}, (t_{ij})_{i \in \mathbb{N}} \in C_0(\mathbb{N}),$$

which are infinite matrices whose rows are uniformly in  $\ell^1$ , and columns are in  $C_0(\mathbb{N})$ .

In addition, it is an isometry under

$$\|(t_{ij})_{i,j \in \mathbb{N}}\| := \sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1.$$

Indeed, let

$$e_n := (m \mapsto \delta_{nm}) \cong (\delta_{nm})_{n=1}^{\infty} \in C_0(\mathbb{N}), \delta_i := ((x_j)_{j=1}^{\infty} \mapsto x_i) \in C_0^*(\mathbb{N})$$

with  $\Phi(\delta_i) = (\delta_{ij})_{j=1}^{\infty} \in \ell^1(\mathbb{N})$  as in previous example.

Easy to check  $\|\delta_i\| = 1$  by the above example.

Consider any  $T \in B(C_0(\mathbb{N}))$ , define  $\phi_i := (\delta_i \circ T), t_{ij} := \phi_i(e_j)$ .

Notice that  $\phi_i : C_0(\mathbb{N}) \rightarrow \mathbb{F}$  is linear, and  $\|\phi_i\| \leq \|\delta_i\| \|T\| = \|T\| < \infty$ , so  $\phi_i \in C_0^*(\mathbb{N})$ .

Thus,  $(t_{ij})_{j=1}^{\infty} = (\phi_i(e_j))_{j=1}^{\infty} = \Phi(\phi_i) \in \ell^1(\mathbb{N})$  as in previous example, with  $\|\phi_i\| = \|(t_{ij})_{j=1}^{\infty}\|_1$ .

Since this hold for all  $i \in \mathbb{N}$ , we have  $\sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1 \leq \|T\|$ .

In addition,  $(t_{ij})_{i \in \mathbb{N}} = ((\delta_i \circ T)(e_j))_{i \in \mathbb{N}} = T e_j \in C_0(\mathbb{N})$ .

On the other hand, suppose we have such  $(t_{ij})_{i,j \in \mathbb{N}}$  with  $\sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1 < \infty$ , we can define  $\phi_i := \Phi^{-1}((t_{ij})_{j=1}^{\infty}) \in C_0^*(\mathbb{N})$  with  $\phi_i(x) = \sum_{j=1}^{\infty} x_j t_{ij}$  as in previous example.

Let  $Tx := \sum_{i=1}^{\infty} \phi_i(x) e_i$  for any  $x \in C_0(\mathbb{N})$ .

Clearly,  $T$  is linear, and we have  $(\delta_i \circ T)(x) = \delta_i(\sum_{j=1}^{\infty} \phi_j(x) e_j) = \phi_i(x)$ .

$$\begin{aligned} \|Tx\|_{\infty} &= \|(\phi_i(x))_{i=1}^{\infty}\|_{\infty} \\ &\leq \sup_{i \in \mathbb{N}} |\phi_i(x)| \\ &\leq \sup_{i \in \mathbb{N}} \|\phi_i\| \|x\| \\ \|T\| &\leq \sup_{i \in \mathbb{N}} \|\phi_i\| \\ &= \sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\| \\ &< \infty. \end{aligned}$$

Thus  $T \in B(C_0(\mathbb{N}), \ell_{\infty})$ .

Now we claim  $T(C_0(\mathbb{N})) = C_0(\mathbb{N})$ , which will mean  $T \in B(C_0(\mathbb{N}))$ .

Indeed, for any  $x = \sum_{n=1}^{\infty} x_n e_n \in C_0(\mathbb{N})$ , we have

$$\begin{aligned} Tx &= T \left( \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n e_n \right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n T(e_n). \end{aligned}$$

Since each

$$\begin{aligned} T(e_n) &= \sum_{i=1}^{\infty} \phi_i(x) e_i \\ &= \sum_{i=1}^{\infty} t_{in} e_i \\ &\in C_0(\mathbb{N}), \end{aligned}$$

and  $C_0(\mathbb{N})$  is closed, we have  $Tx \in C_0(\mathbb{N})$ .

In addition,  $(\delta_i \circ T)(e_j) = \phi_i(e_j) = \Phi^{-1}((t_{ik})_{k=1}^{\infty})(e_j) = t_{ij}$ .

Thus  $T \longleftrightarrow (t_{ij})_{i,j \in \mathbb{N}}$  is an isometric bijection.

**Example 3.1.3.** Consider the **Disk Algebra**

$$A(\mathbb{D}) := \left\{ f \in C(\mathbb{T}) : \forall n \in \mathbb{Z}^{--}, \hat{f}(n) = 0 \right\},$$

where  $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}$ ,  $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C}, |z| = 1\}$  is the unit circle, and

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt$$

is the  $n^{\text{th}}$  Fourier Transform of  $f$ .

Consider  $\phi_n : f \mapsto \hat{f}(n)$ , which is clearly in  $C^*(\mathbb{T})$ .

We notice that  $A(\mathbb{D}) = \bigcap_{n < 0} \ker(\phi_n)$  is closed in  $C(\mathbb{T})$ , since each kernel of a continuous functional is closed.

In fact, for  $f, g \in C(\mathbb{T})$ , we have  $\hat{f}g(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \hat{g}(n-k)$ .

Thus, for  $f, g \in A(\mathbb{D})$ , we have  $\hat{f}g(n) = \sum_{k \in \mathbb{N}} \hat{f}(k) \hat{g}(n-k) = 0$  for  $n < 0$ .

This shows  $A(\mathbb{D})$  is actually an Algebra.

Also,  $A(\mathbb{D})$  is exactly the set of  $f \in C(\mathbb{T})$  that admits an extension  $F \in C(\bar{\mathbb{D}})$  with  $F|_{\mathbb{D}}$  being analytic, with  $F(z) := \sum_{n=1}^{\infty} \hat{f}(n) z^n$ .

### 3.2 Quotient Spaces

**Definition 3.17.** Let  $X$  be a Banach Space, and  $Y \subseteq X$  be a closed subspace. The **quotient space** is  $X/Y := \{x + Y : x \in X\}$ , with the **quotient map**  $Q : X \rightarrow X/Y$  by  $Q(x) := [x] := x + Y = \{x + y : y \in Y\}$ .

**Proposition 3.15.** Let  $X$  be a Banach Space, and  $Y \subseteq X$  be a closed subspace.  $X/Y$  is always a vector space with  $[0] = Y$ ,  $[x] + [y] = [x + y]$ ,  $c[x] = [cx]$ .

**Proposition 3.16.** Let  $X$  be a Banach Space, and  $Y \subseteq X$  be a closed subspace.  $(X/Y, \|\cdot\|_{X/Y})$  is always a Banach space with  $\|[x]\|_{X/Y} := \inf_{y \in Y} \|x + y\|_X$ . In addition,  $Q$  is isometric if  $Y \subsetneq X$ .

*Proof.*  $\|[x]\|_{X/Y} = 0 \iff \inf_{y \in Y} \|x + y\|_X = 0 \iff x \in \bar{Y} \iff x \in Y$ .

Scaling is clear.

Also,

$$\begin{aligned} \|[x] + [z]\|_{X/Y} &= \inf_{y \in Y} \|x + y + z\|_X \\ &= \inf_{y_1, y_2 \in Y} \|x + y_1 + z + y_2\|_X \\ &\leq \inf_{y_1 \in Y} \|x + y_1\|_X + \inf_{y_2 \in Y} \|z + y_2\|_X \\ &= \|[x]\|_{X/Y} + \|[z]\|_{X/Y}. \end{aligned}$$

This shows  $(X/Y, \|\cdot\|_{X/Y})$  is a normed space.

We note that  $\|Qx\|_{X/Y} = \inf_{y \in Y} \|x + y\|_X \leq \|x + 0\|_X = \|x\|_X$ , so  $\|Q\| \leq 1$ .

Now consider any Cauchy sequence  $([x_n])_{n=1}^\infty$  in  $X/Y$ .

We can pick a subsequence  $([x_{n_i}])_{i=1}^\infty$  such that  $\|[x_{n_{i+1}}] - [x_{n_i}]\|_{X/Y} < 2^{-i}$ .

Pick  $z_1 \in X$  such that  $[z_1] = [x_{n_1}]$ .

Since  $\|[x_{n_2}] - [x_1]\|_{X/Y} = \inf_{y \in Y} \|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$ , there is  $y \in Y$ , such that  $\|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$ .

Take  $z_2 = x_{n_2} + y$ , we have  $\|z_2 - z_1\|_X < \frac{1}{2}$ .

Inductively, we can pick  $(z_i)_{i=1}^\infty$ , such that  $\|z_i - z_{i-1}\|_X < 2^{-i}$ .

We can check that this is a Cauchy sequence in  $X$ , so it has a limit  $z = \lim_{i \rightarrow \infty} z_i \in X$ .

Now for any  $i \in \mathbb{N}^+$ , we have

$$\begin{aligned} \|[x_{n_i}] - [z]\|_{X/Y} &= \|[z_i] - [z]\|_{X/Y} \\ &= \|Q(z_i) - Q(z)\|_{X/Y} \\ &= \|Q(z_i - z)\|_{X/Y} \\ &\leq \|Q\| \|z_i - z\| \\ &\rightarrow 0. \end{aligned}$$

Thus  $([x_{n_i}])_{i=1}^\infty \rightarrow [z]$  is a convergent subsequence, which mean  $([x_n])_{n=1}^\infty$  is convergent.

This shows  $(X/Y, \|\cdot\|_{X/Y})$  is a Banach space.

Now if  $Y \subsetneq X$ , then  $X/Y \neq \{0\}$ , there must be some  $[x] \in X/Y$  with  $\|[x]\|_{X/Y} = 1$ .

Thus, for all  $k \in \mathbb{N}^+$ , there is some  $y_k \in Y$ , such that  $\|x + y_k\|_X \leq 1 + \frac{1}{k}$ .

Now

$$\left\| Q\left(\frac{x + y_k}{\|x + y_k\|_X}\right) \right\|_{X/Y} = \frac{1}{\|x + y_k\|_X} \|Q(x + y_k)\| = \frac{\|[x]\|_{X/Y}}{\|x + y_k\|_X} \geq \frac{1}{1 + \frac{1}{k}}.$$

Since this is true for any  $k \in \mathbb{N}^+$ , taking the limit  $k \rightarrow \infty$ , we have  $\left\| Q\left(\frac{x + y_k}{\|x + y_k\|_X}\right) \right\|_{X/Y} \geq 1$ .

Yet  $\left\| \frac{x + y_k}{\|x + y_k\|_X} \right\|_X = 1$ , so  $\|Q\| \geq 1$ .

This shows  $\|Q\| = 1$ . □

**Example 3.2.1.** Consider a compact and Hausdorff  $X$ , and consider  $(C(X), \|\cdot\|_\infty)$ . Let  $E \subseteq X$  be closed, and  $I(E) := \{f \in C(X) : f|_E = 0\}$ .

One can check  $I(E)$  is closed (ideal), and  $C(X)/I(E) \cong C(E)$  with an isometric isomorphism  $\tilde{R} : [f] \mapsto f|_E$ .

We claim that  $\tilde{R}$  is well-defined.

Indeed, if  $[f] = [g]$ , we must have  $f - g \in I(E)$ , which means  $(f - g)|_E = 0$ .

Thus  $\tilde{R}([f]) = f|_E = g|_E = \tilde{R}([g])$ .

Clearly  $\tilde{R}$  is linear.

Also,  $\tilde{R}([f]) = 0 \implies f|_E = 0 \implies f \in I(E) \implies [f] = 0$ , so  $\tilde{R}$  is injective.

By Tietze's Theorem, given any  $g \in C(E)$ , we can extend it to  $f \in C(X)$ , such that  $f|_E = g$ . Thus,  $\tilde{R}$  is surjective.

Consider any  $f \in C(X), g \in I(E)$ , we have

$$\begin{aligned} \|\tilde{R}([f])\| &= \|f|_E\|_{C(E)} \\ &= \sup_{x \in E} |f(x)| \\ &= \sup_{x \in E} |(f + g)(x)| \\ &\leq \sup_{x \in X} |(f + g)(x)| \\ &= \|f + g\|_{C(X)}. \end{aligned}$$

Since this hold for all  $g \in I(E)$ , we have

$$\|\tilde{R}([f])\| \leq \inf_{g \in I(E)} \|f + g\|_{C(X)} = \|[f]\|.$$

Thus,  $\tilde{R}$  is a contraction.

Consider any  $f \in C(X)$ .

If  $f|_E = 0$ , we have  $\|[f]\| = \|[0]\| = 0 = \|f|_E\| = \|\tilde{R}([f])\|$ .

Now consider  $f|_E \neq 0$ .

Define the function  $k : \mathbb{C} \rightarrow \mathbb{C}$  by  $k(z) := \begin{cases} z, & |z| \leq \|f|_E\|_\infty \\ \frac{z}{|z|} \|f|_E\|_\infty, & |z| \geq \|f|_E\|_\infty, \end{cases}$  which is well-defined and continuous.

Let  $g := k \circ f \in C(X)$ .

For any  $x \in E$ , we have  $|f(x)| \leq \|f|_E\|_\infty$ , so  $g(x) = k(f(x)) = f(x)$ .

Thus  $g|_E = f|_E$ , and there is  $h \in I(E)$ , such that  $g = f + h$ .

$$\begin{aligned} \|[f]\| &= \inf_{h \in I(E)} \|f + h\|_{C(X)} \\ &\leq \|g\|_{C(X)} \\ &\leq \|k\| \\ &\leq \|f|_E\|_\infty \\ &= \|\tilde{R}(f)\|. \end{aligned}$$

This proves  $\tilde{R}$  is an isometry.

**Proposition 3.17.** *For any infinite dimensional Banach Space  $X$ , it does not have a countable basis. Also, the unit ball is not compact.*

*Proof.* See A2. □

### 3.3 Baire Category Theorem

**Definition 3.18.** Let  $(X, \mathcal{T})$  be a topological space, then  $A \subseteq X$  is called **nowhere dense** if  $(\bar{A})^\circ = \emptyset$ .

**Theorem 3.18** (Baire Category Theorem). *Let  $(X, d)$  be a complete metric space, then  $X$  cannot be written as a countable union of nowhere dense sets.*

**Corollary 3.19.** *Let  $\{U_i\}_{i=1}^\infty$  be a countable set of open dense sets, then  $\bigcap_{i=1}^\infty U_i$  is dense.*

#### 3.3.1 Banach-Steinhaus Theorem

**Definition 3.19.** Let  $X, Y$  be Banach spaces,  $\mathcal{S} \subseteq B(X, Y)$  is called **pointwise bounded** if  $\forall x \in X$ ,  $\mathcal{S}x$  is bounded. Namely,  $\exists k_x > 0$ , such that  $\forall S \in \mathcal{S}$ ,  $\|Sx\| \leq k_x$ .

**Theorem 3.20** (Banach-Steinhaus). *Let  $X, Y$  be Banach spaces, suppose  $\mathcal{S} \subseteq B(X, Y)$  is pointwise bounded, then  $\mathcal{S}$  is bounded in  $B(X, Y)$ . Namely,  $\sup_{S \in \mathcal{S}} \|S\| < \infty$ .*

*Proof.* For each  $x$ , let  $k_x > 0$  be such  $\forall S \in \mathcal{S}$ ,  $\|Sx\| \leq k_x$ .

For each  $n \in \mathbb{N}$ , let  $A_n := \{x \in X : k_x \leq n\}$ .

For any  $(x_k)_{k \in \mathbb{N}}$  in  $A_n$ , with  $x = \lim_{k \rightarrow \infty} x_k$ , we have

$$\|Sx\| = \lim_{k \rightarrow \infty} \|Sx_k\| \leq n.$$

Thus,  $x \in A_n$ , which shows  $A_n$  is closed.

Notice that  $X = \bigcup_{n \in \mathbb{N}} A_n$ , so by Baire Category Theorem, there is  $n_0 \in \mathbb{N}$ , such that  $(A_{n_0})^\circ \neq \emptyset$ .

Thus, there is  $x_0 \in A_{n_0}$ ,  $r > 0$ , such that  $\bar{B}(x_0, r) \subset B(x_0, 2r) \subseteq (A_{n_0})^\circ \subset A_{n_0} = A_{n_0}$ .

Now for any  $s \in \mathcal{S}$ ,  $y \in X$  such that  $\|y\| \leq 1$ , we have that

$$\begin{aligned} \|sy\| &= \left\| \frac{s(x_0) - s(x_0 - ry)}{r} \right\| \\ &\leq \frac{\|s(x_0)\| + \|s(x_0 - ry)\|}{r} \\ &\leq \frac{2n_0}{r}, \end{aligned}$$



since  $x_0, x_0 - ry \in \bar{B}(x_0, r) \subset A_{n_0}$ .

Thus,  $\|s\| = \sup_{y \in X \text{ such that } \|y\| \leq 1} \|sy\| \leq \frac{2n_0}{r}$ .

Since this holds for all  $s \in \mathcal{S}$ , we have  $\sup_{s \in \mathcal{S}} \|s\| = \frac{2n_0}{r} < \infty$ .  $\square$

**Corollary 3.21** (Limit of bounded operators). *Let  $X, Y$  be Banach spaces, consider  $(T_n)_{n=1}^\infty$  be a sequence of  $B(X, Y)$ . Suppose  $\forall x \in X$ ,  $(T_n x)_{n=1}^\infty$  is convergent, then  $\mathcal{S} := \{T_n : n \in \mathbb{N}\}$  is bounded. In addition, for  $Tx := \lim_{n \rightarrow \infty} T_n x$ , we have  $T \in B(X, Y)$ , and  $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$ .*

*Proof.* Since  $(T_n x)_{n=1}^\infty$  is convergent, it is bounded. This is equivalent to saying  $\mathcal{S}$  is pointwise bounded. By the Banach-Steinhaus Theorem,  $\mathcal{S}$  is bounded.

Now  $\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| = \sup_{n \in \mathbb{N}} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n\| \|x\|$ .

Since this holds for all  $x \in X$ , we have  $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$ .  $\square$

**Example 3.3.1.** Consider  $f \in C(\mathbb{T})$ , we can define the  $N^{\text{th}}$  partial sum of its Fourier series  $S_N(f)(e^{it}) := \sum_{n=-N}^N \hat{f}(n) e^{int}$ .

We note that  $S_N(f)$  does not necessarily converge to  $f$  in  $C(\mathbb{T})$ , nor pointwise.

Indeed, consider  $\phi_N(f) := S_N(f)(1) \in \mathbb{C}$ . Note that  $\phi_N$  is linear.

One can show that

$$\begin{aligned} \phi_N(f) &= \sum_{n=-N}^N \hat{f}(n) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \left( \sum_{n=-N}^N e^{-int} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) D_N(t) dt, \end{aligned}$$

where  $D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}$  is the **Dirichlet's Kernel**.

Actually,  $\exists C > 0$ , such that  $\|D_N\|_1 \geq C \log(N)$ .

Also,  $\|\phi_N\|_{C_*(\mathbb{T})} = \|D_N\|_1$ .

Thus,  $(\phi_N)_{N \in \mathbb{N}}$  is not bounded.

By Banach-Steinhaus,  $(\phi_N)_{N \in \mathbb{N}}$  is not pointwise bounded.

Thus, there is  $f \in C(\mathbb{T})$ , such that  $|\phi_N(f)| \rightarrow \infty$ .

Thus,  $S_N f$  does not converge to  $f$  at 1.

### 3.3.2 Open Mapping Theorem

**Theorem 3.22** (Open Mapping Theorem). *Let  $X, Y$  be Banach spaces, suppose  $T \in B(X, Y)$  is surjective, then it is **open**. i.e.  $\forall$  open  $U \subseteq X$ ,  $T(U) \subseteq Y$  is open.*

*Proof.* We have

$$\begin{aligned} Y &= T(X) \\ &= T\left(\bigcup_{n=1}^{\infty} B^X(0, n)\right) \\ &= T\left(\bigcup_{n=1}^{\infty} nB^X(0, 1)\right) \\ &= \bigcup_{n=1}^{\infty} nT(B^X(0, 1)). \end{aligned}$$

By the Baire Category Theorem, there is  $n_0$ , such that  $\left(n_0 \overline{T(B^X(0, 1))}\right)^o = \left(\overline{n_0 T(B^X(0, 1))}\right)^o \neq \emptyset$ .

Thus there is  $r_0 > 0, y_0 \in Y$ , such that  $B^Y(y_0, r_0) \subset n_0 \overline{T(B^X(0, 1))}$ .

Notice that  $B^Y(-y_0, r_0) \subset \overline{n_0 T(B^X(0, 1))}$  as well, and  $\overline{n_0 T(B^X(0, 1))}$  is convex.

Thus, for any  $y \in B^Y(0, r)$ , we have  $y = \frac{1}{2}(y - y_0) + \frac{1}{2}(y + y_0)$ , where  $y - y_0 \in B^Y(-y_0, r_0)$ ,  $y + y_0 \in B^Y(y_0, r_0)$ .

By convexity,  $y \in \overline{n_0 T(B^X(0, 1))}$ .

Thus  $B^Y(0, r_0) \subset \overline{n_0 T(B^X(0, 1))}$ .

Take  $r := \frac{r_0}{n_0}$ , we have  $B^Y(0, r) \subset \overline{T(B^X(0, 1))}$ .

Now we want to show  $\overline{T(B^X(0, 1))} \subset T(B^X(0, 2))$ .

Let  $y \in \overline{T(B^X(0, 1))}$ , there is  $x_1 \in B^X(0, 1)$ , such that  $\|y - Tx_1\| < \frac{r}{2}$ .

Let  $y_1 := y - Tx_1 \in B^Y(0, \frac{r}{2}) \subset \overline{T(B^X(0, \frac{1}{2}))}$ .

Thus there is  $x_2 \in B^X(0, \frac{1}{2})$ , such that  $\|y_1 - Tx_2\| = \|y - Tx_1 - Tx_2\| < \frac{r}{4}$ .

Recursively, we can find  $x_k \in B^X(0, \frac{1}{2^{k-1}})$ , such that  $\|y_k\| < \frac{r}{2^k}$  for  $y_k := y - Tx_1 - \dots - Tx_k$ .

Since  $\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k+1} = 2$ , and  $X$  is complete, we have  $x := \sum_{k=1}^{\infty} x_k$  converges in  $X$ , and  $x \in B^X(0, 2)$ .

Since  $T$  is continuous, we have  $y = \sum_{k=1}^{\infty} Tx_k = Tx \in T(B^X(0, 2))$ .

Thus  $B^Y(0, r) \subset \overline{T(B^X(0, 1))} \subset T(B^X(0, 2))$ .

Now for any open  $U \subseteq X$ , there is  $\epsilon > 0$ , such that  $x + \frac{\epsilon}{2}B^X(0, 2) = B^X(x, \epsilon) \subseteq U$ .

Thus  $B^Y(Tx, \frac{\epsilon}{2}r) = Tx + \frac{\epsilon}{2}B^Y(0, r) \subseteq Tx + \frac{\epsilon}{2}T(B^X(0, 2)) \subseteq T(U)$ .

Thus  $T(U)$  is open.  $\square$

**Theorem 3.23** (Banach Isomorphism Theorem). *Let  $X, Y$  be Banach spaces, suppose  $A \in B(X, Y)$  is bijective, then  $A^{-1}$  is continuous and bounded as well. Namely,  $A$  is invertible and is an isomorphism of  $X \cong Y$  as Banach Spaces.*

*Proof.*  $A$  is open by the open mapping theorem. Now for any open  $U \subseteq X$ , we have that  $(A^{-1})^{-1}(U) = A(U)$  is open. Thus  $A^{-1}$  is continuous.  $\square$

**Corollary 3.24.** *Let  $X, Y$  be Banach spaces, suppose  $T \in B(X, Y)$  is surjective, then  $X/\ker(T) \cong Y$  as Banach spaces with  $\tilde{T} : X/\ker(T) \rightarrow Y$  by  $\tilde{T}([x]) := T(x)$ .*

*Proof.* We can check that  $\tilde{T}$  is a well-defined bijection.

Also, for any  $x \in X, y \in \ker(T)$ , we have

$$\begin{aligned} \|\tilde{T}([x])\| &= \|T(x)\| \\ &= \|T(x + y)\| \\ &\leq \|T\| \|x + y\|. \end{aligned}$$

Since this holds for all  $y \in \ker(T)$ , we have

$$\|\tilde{T}([x])\| \leq \inf_{y \in \ker(T)} \|T\| \|x + y\| = \|T\| \|x\|.$$

Thus,  $\|\tilde{T}\| \leq \|T\|$ , which means  $\tilde{T} \in B(X/\ker(T), Y)$  is continuous.

By the Banach Isomorphism Theorem,  $\tilde{T}^{-1}$  is continuous as well.  $\square$

**Corollary 3.25.** *Suppose  $X$  is a vector space that is complete under two different norms  $\|\cdot\|_1, \|\cdot\|_2$ , and  $\exists C > 0$ , such that  $\forall x \in X, \|x\|_1 \leq C\|x\|_2$ , then the two norms are equivalent.*

*Proof.* Consider the map  $id : (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$ , we have that  $id$  is bijective and bounded by  $C$ . By the Banach Isomorphism Theorem,  $id^{-1}$  is bounded as well.

Thus  $\forall x \in X$ , we have  $\|x\|_2 \leq \|id^{-1}\| \|x\|_1$ .  $\square$

**Corollary 3.26.** *Let  $X$  be any finite-dimensional linear normed space over  $\mathbb{F}$ , then any two norms on  $X$  are equivalent and  $X$  is complete.*

*Proof.* Let  $\|\cdot\|$  be any norm on  $X$ , and let  $\|\cdot\|_2$  be the Euclidean norm on  $\mathbb{F}^n$ .

Pick any basis  $\{e_i\}_{i=1}^n$  for  $X$ .

Consider the function  $T : (\mathbb{F}^n, \|\cdot\|_2) \rightarrow (X, \|\cdot\|)$  by  $T(a) := \sum_{i=1}^n a^i e_i \in X$  for any  $a = (a^i)_{i=1}^n \in \mathbb{F}^n$ .

It is easy to check that  $T$  is linear and bijective. Also,

$$\begin{aligned} \|Ta\| &= \left\| \sum_{i=1}^n a^i e_i \right\| \\ &\leq \sum_{i=1}^n |a^i| \|e_i\| \\ &\leq \|a\|_2 \left( \sum_{i=1}^n \|e_i\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus,  $\|T\| \leq \alpha := \left( \sum_{i=1}^n \|e_i\|^2 \right)^{\frac{1}{2}}$ , so it is continuous.

Since  $S := \{a \in \mathbb{F}^n : \|a\|_2 = 1\}$  is closed and bounded thus compact in  $(\mathbb{F}^n, \|\cdot\|_2)$ , and  $\|\cdot\|$  is continuous, we have  $r := \inf_{a \in S} \|T(a)\|$  is achieved in  $S$  by the Extreme Value Theorem.

Since  $0 \notin S$ , and  $\|T(a)\| = 0 \implies T(a) = 0 \implies a = 0$ , we have  $r \neq 0$ .

Consider any  $a \neq 0 \in \mathbb{F}^n$  such that  $\|T(a)\| \leq r$ , we must have  $\frac{a}{\|a\|_2} \in S$ , so

$$\frac{1}{\|a\|_2} \|T(a)\| = \left\| T\left(\frac{a}{\|a\|_2}\right) \right\| \geq r.$$

Thus  $\|a\|_2 \leq 1$ .

This shows  $\forall x \in X$  with  $\|x\| \leq 1$ , we must have  $\|T^{-1}(x)\|_2 \leq \frac{1}{r}$ , which show that  $\|T^{-1}\| \leq \frac{1}{r} < \infty$  is bounded.

This shows that  $T : (\mathbb{F}^n, \|\cdot\|_2) \rightarrow (X, \|\cdot\|)$  is an homeomorphism for any  $\|\cdot\|$  on  $X$ , which means  $X$  is complete and all norms are equivalent.  $\square$

### 3.3.3 Closed Graph Theorem

**Definition 3.20.** Let  $X, Y$  be Banach spaces, we can consider  $X \oplus Y := \{(x, y) : x \in X, y \in Y\}$ , which is a vector space under component-wise addition and scale multiplication. For  $1 \leq p < \infty$ , we define

$$\|(x, y)\|_p := (\|x\|^p + \|y\|^p)^{\frac{1}{p}},$$

and

$$\|(x, y)\|_\infty := \max\{\|x\|, \|y\|\}.$$

**Definition 3.21.** Let  $X, Y$  be Banach spaces, and  $D \subseteq X$  is a subspace, the **graph** of a linear map  $T : D \rightarrow Y$  is

$$\mathcal{G}(T) := \{(x, Tx) : x \in D\}.$$

We say  $T$  is **closed** if  $\mathcal{G}(T)$  is closed in  $X \oplus_\infty Y$ .

**Theorem 3.27** (Closed Graph Theorem). *Let  $X, Y$  be Banach spaces, a linear map  $T : X \rightarrow Y$  is closed if and only if  $T \in B(X, Y)$ .*

*Proof.* ( $\implies$ ).

Consider the projection maps  $\pi_1(x, y) := x, \pi_2(x, y) := y$ , which are both continuous.

Since  $\mathcal{G}(T)$  is closed in  $X \oplus Y$ , it is a Banach Space.

Since  $T$  is defined on entire  $X$ , we have that  $\pi_1|_{\mathcal{G}(T)}$  is a continuous bijection.

By the Bounded Inverse Theorem,  $(\pi_1|_{\mathcal{G}(T)})^{-1} : X \rightarrow \mathcal{G}(T)$  is bounded.

Notice that  $T = \pi_2 \circ (\pi_1|_{\mathcal{G}(T)})^{-1}$ , which will also be bounded.

( $\impliedby$ ).

Consider any sequence  $((x_i, Tx_i))_{i \in \mathbb{N}}$  in  $\mathcal{G}(T)$ , such that  $(x_i, Tx_i) \rightarrow (x, y) \in X \oplus_\infty Y$ .

Thus  $x_i \rightarrow x \in X$  and  $Tx_i \rightarrow y \in Y$ .

Since  $T$  is continuous,  $Tx_i \rightarrow Tx$ , which means  $Tx = y$ , so  $(x, y) \in \mathcal{G}(T)$ .  $\square$

*Remark.* It is important that  $T$  is defined on the entire  $X$ .

**Example 3.3.2.** Consider  $X = Y = (C[0, 1], \|\cdot\|_1)$ , and  $D = C^1([0, 1])$ . Let  $T := \frac{d}{dx} : D \rightarrow C^1([0, 1])$ , which is unbounded but closed.

**Corollary 3.28.** Let  $X, Y$  be Banach spaces, then for any linear map  $T : X \rightarrow Y$   $T \in B(X, Y)$  if and only if  $\forall (x_n)_{n=1}^\infty$  in  $X$ , such that  $x_n \rightarrow 0$ ,  $Tx_n \rightarrow y \in Y$ , we have  $y = 0$ .

*Proof.* ( $\Rightarrow$ ) is easy.

( $\Leftarrow$ ).

Consider any  $((x_n, Tx_n))_{n=1}^\infty$  in  $\mathcal{G}(T)$ , such that  $x_n \rightarrow x \in X$ , and  $Tx_n \rightarrow y \in Y$ .

We have that  $(x - x_n) \rightarrow 0$ , and by linearity of  $T$ , we have  $T(x - x_n) = Tx - Tx_n \rightarrow Tx - y$ .

By assumption, we have  $y - Tx = 0$ , which means  $y = Tx$ .

Thus,  $(x, y) \in \mathcal{G}(T)$ , which means  $\mathcal{G}(T)$  is closed.  $\square$

### 3.4 Hahn-Banach Theorem

**Definition 3.22.** Let  $X$  be a normed linear space.  $p : X \rightarrow \mathbb{R}$  is called a **sublinear functional** if  $\forall t > 0, \forall x \in X$ ,  $p(tx) = tp(x)$ , and  $p(x + y) \leq p(x) + p(y)$ .

**Proposition 3.29.** Let  $X$  be a normed linear space, then  $\|\cdot\|$  is always a sublinear functional.

**Proposition 3.30.** Let  $X$  be a Banach space, then for any  $x \in X$ , the functional  $p_x : B(X) \rightarrow \mathbb{R}$  defined by  $p_x(T) := \|Tx\|$  is sublinear.

**Proposition 3.31.** Let  $X$  be a Banach space, then for any  $x \in X, \phi \in X^*$ , the functional  $p_{\phi, x} : B(X) \rightarrow \mathbb{R}$  defined by  $p_{\phi, x}(T) := |\phi(Tx)|$  is sublinear.

**Theorem 3.32** (Extension). Let  $X$  be a linear vector space over  $\mathbb{R}$ . Let  $M_0 \subseteq X$  be a linear subspace, and  $p : X \rightarrow \mathbb{R}$  be a sublinear functional, then for any linear  $f_0 : M_0 \rightarrow \mathbb{R}$  such that  $\forall x \in M_0, f_0(x) \leq p(x)$ , there is an extension  $f : X \rightarrow \mathbb{R}$ , such that  $f|_{M_0} = f_0$ , and  $\forall x \in X, f(x) \leq p(x)$ .

*Proof.* Consider

$$P := \{(M, f) | M_0 \subseteq M \subseteq X \text{ is a subspace; } f : M \rightarrow \mathbb{R} \text{ is linear, } f|_{M_0} = f_0; \forall x \in M, f(x) \leq p(x)\},$$

with the partial order  $(M, f) \leq (M', f')$  if  $M \subseteq M', f'|_M = f$ .

Consider any chain  $\{(M_\alpha, f_\alpha)\}_{\alpha \in A} \subset P$ .

Let  $M := \bigcup_{\alpha \in A} M_\alpha \subseteq X$ , and let  $f(x) := f_\alpha(x)$  for any  $M_\alpha \ni x$ .

We can check that  $f$  is well-defined and linear, satisfying the requirement.

Thus,  $(M, f) \in P$  is an upper bound for the chain.

By Zorn's lemma, there is a maximal element  $(M_1, f_1)$  of  $P$ .

Suppose for contradiction that  $M_1 \neq X$ , then there is some  $x \in X \setminus M_1$ .

Notice that if we take any  $m_1, m_2 \in M_1$ , we have that

$$\begin{aligned} f_1(m_1) + f_1(m_2) &= f_1(m_1 + m_2) \\ &\leq p(m_1 + m_2) \\ &\leq p(m_1 - x) + p(m_2 + x). \end{aligned}$$

Thus,  $f_1(m_1) - p(m_1 - x) \leq p(m_2 + x) - f_1(m_2)$ .

Since this holds for all  $m_1, m_2 \in M_1$ , we have

$$\sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \leq \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)).$$

Take any  $a \in [\sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)), \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2))]$ .

Let  $M := M_1 \oplus \text{Span}\{x\}$ . Since  $M_1$  is a subspace, this is a direct product, i.e.,  $\forall y \in M_1 \oplus \text{Span}\{x\}$ , there is some unique  $m \in M_1, t \in \mathbb{R}$ , such that  $y = m + tx$ .

Define  $f : M \rightarrow \mathbb{R}$  by  $f(m + tx) := f_1(m) + |t|a$  for any  $t \in \mathbb{R}$ .

We can easily check  $f|_{M_1} = f_1$  and that  $f$  is linear.

Suppose  $t > 0$ , we have

$$\begin{aligned}
f(m + tx) &= f_1(m) + |t|a \\
&= f_1(m) + ta \\
&\leq f_1(m) + t \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)) \\
&\leq f_1(m) + t \left( p\left(\frac{m}{t} + x\right) - f_1\left(\frac{m}{t}\right) \right) \\
&= f_1(m) + p\left(t\left(\frac{m}{t} + x\right)\right) - f_1\left(t\frac{m}{t}\right) \\
&= p(m + tx).
\end{aligned}$$

Similarly, if  $t \leq 0$ , we have

$$\begin{aligned}
f(m + tx) &= f_1(m) + |t|a \\
&= f_1(m) - ta \\
&\leq f_1(m) - t \sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \\
&\leq f_1(m) - t \left( f_1\left(\frac{m}{t}\right) - p\left(\frac{m}{t} - x\right) \right) \\
&= f_1(m) - f_1\left(t\frac{m}{t}\right) + p\left(t\left(\frac{m}{t} + x\right)\right) \\
&= p(m + tx).
\end{aligned}$$

This contradicts with the maximality of  $(M_1, f_1)$ .

Thus,  $M_1 = X$ , and  $f_1 : X \rightarrow \mathbb{R}$  is the desired extension.  $\square$

**Theorem 3.33** (Hahn-Banach). *Let  $X$  be a normed linear space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $M \subseteq X$  be a linear subspace, and linear  $\phi_0 : M \rightarrow \mathbb{F}$  with  $\|\phi_0\|_{M^*} < \infty$ , then there is a norm preserving extension  $\phi \in X^*$ , such that  $\phi|_M = \phi_0$ , and  $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$ .*

*Proof.* First, consider  $\mathbb{F} = \mathbb{R}$ .

We note that  $p : x \mapsto \|\phi_0\|_{M^*} \|x\|$  is sublinear, and  $\forall x \in M$ ,  $\phi_0(x) \leq \|\phi_0\|_{M^*} \|x\| = p(x)$ .

Thus, there is a linear extension  $\phi : X \rightarrow \mathbb{R}$ , such that  $\forall x \in X$ ,  $\phi(x) \leq p(x) = \|\phi_0\|_{M^*} \|x\|$ .

Also,  $-\phi(x) = \phi(-x) \leq \|\phi_0\|_{M^*} \|-x\| = \|\phi_0\|_{M^*} \|x\|$ .

Thus  $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$ .

Now suppose  $\mathbb{F} = \mathbb{C}$ .

Consider  $g_0 := \Re \phi_0 : M \rightarrow \mathbb{R}$ , which is  $\mathbb{R}$ -linear, and  $\|g_0\|_{M_{\mathbb{R}}^*} \leq \|\phi_0\|_{M^*}$ .

Using the real case, we can extend  $g_0$  to  $g \in X_{\mathbb{R}}^*$ .

Now define  $\phi : X \rightarrow \mathbb{C}$  by  $\phi(x) := g(x) + ig(-ix)$ .

We can see that  $\phi$  is  $\mathbb{R}$  linear.

Also,

$$\phi(ix) = g(ix) + ig(x) = i(g(x) - ig(ix)) = i(g(x) + ig(-ix)) = i\phi(x),$$

so  $\phi$  is  $\mathbb{C}$ -linear.

For any  $m \in M$ , we have that

$$\begin{aligned}
\phi(x) &= g(m) + ig(-im) \\
&= g_0(m) + ig_0(-im) \\
&= \Re(\phi_0(m)) + i\Re(\phi_0(-im)) \\
&= \Re(\phi_0(m)) + i\Re(-i\phi_0(m)) \\
&= \Re(\phi_0(m)) + i\Im(\phi_0(m)) \\
&= \phi_0(m).
\end{aligned}$$

Thus  $\phi|_M = \phi_0$ .

Now consider any  $x \in X$  with  $\|x\| \leq 1$ .

We have that

$$|\phi(x)| = \lambda \phi(x) = \phi(\lambda x) = g(\lambda x) + ig(-i\lambda x)$$

for some  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$ .

Since  $g$  is real-valued, and  $|\phi(x)| \in \mathbb{R}$ , we must have

$$|\phi(x)| = g(\lambda x) \leq \|g\|_{X^*} \|\lambda x\| = \|g_0\|_{M_{\mathbb{R}}^*} \|x\| \leq \|\phi_0\|_{M^*}.$$

Thus  $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$ .

Lastly, we always have

$$\begin{aligned} \|\phi_0\|_{M^*} &= \sup_{x \in M, \|x\| \leq 1} |\phi_0(x)| \\ &= \sup_{x \in M, \|x\| \leq 1} |\phi(x)| \\ &\leq \sup_{x \in X, \|x\| \leq 1} |\phi(x)| \\ &= \|\phi\|_{X^*}. \end{aligned}$$

□

**Corollary 3.34.** *Let  $X$  be a normed linear space over  $\mathbb{R}$ . Let  $M \subseteq X$  be a linear subspace, and linear  $\phi_0 : M \rightarrow \mathbb{F}$  with  $\|\phi_0\|_{M^*} < \infty$ , then for any  $x \in X, a \in \mathbb{R}$ , there is a Hahn-Banach extension  $\phi \in X^*$ , such that  $\phi|_M = \phi_0$ ,  $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$  and  $\phi(x) = a$  if and only if*

$$\begin{aligned} \sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) &\leq a \leq \inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|) \\ &= \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)). \end{aligned}$$

*Proof.* We first note that since  $M$  is a subspace,  $m \in M$  if and only if  $-m \in M$ , and

$$\begin{aligned} \phi_0(m) + \|\phi_0\|_{M^*} \|m - x\| &= -\phi_0(-m) + \|\phi_0\|_{M^*} \|(-m) - x\| \\ &= \|\phi_0\|_{M^*} \|(-m) + x\| - \phi_0(-m). \end{aligned}$$

Thus,

$$\inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|) = \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)).$$

Consider any Hahn-Banach extension  $\phi \in X^*$ , such that  $\phi|_M = \phi_0$ ,  $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$ .

For any  $x \in X$ , we must satisfy

$$\begin{aligned} |\phi_0(m) - \phi(x)| &= |\phi(m) - \phi(x)| \\ &= |\phi(m - x)| \\ &\leq \|\phi\|_{X^*} \|m - x\| \\ &= \|\phi_0\|_{M^*} \|m - x\|. \end{aligned}$$

Thus,

$$\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\| \leq \phi(x) \leq \phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|.$$

Since this holds for all  $m \in M$ , we have

$$\sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) \leq \phi(x) \leq \inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|).$$

On the other hand, suppose

$$\sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) \leq a \leq \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)).$$

Taking  $p : x \mapsto \|\phi_0\|_{M^*} \|x\|$  as in the proof of the Hahn-Banach Theorem, we have that

$$\sup_{m \in M} (\phi_0(m) - p(m - x)) \leq a \leq \inf_{m \in M} (p(m + x) - \phi_0(m)).$$

By the proof of the Hahn-Banach Theorem, we can always extend  $\phi_0$  on  $M \oplus \text{Span}\{x\}$  to

$$\tilde{\phi}(m + tx) := \phi_0(m) + |t|a$$

for any  $t \in \mathbb{R}$ . Also,

$$\tilde{\phi}(x) = \tilde{\phi}(0 + 1 \cdot x) = \phi_0(0) + 1 \cdot a = a.$$

Now we can take the Hahn-Banach extension of  $\tilde{\phi}$  to be  $\phi$ , which will satisfy  $\phi(x) = \tilde{\phi}(x) = a$ .  $\square$

**Corollary 3.35.** *Let  $X$  be a normed linear space. Given  $0 \neq x \in X$ , then there is  $\phi \in X^*$ , such that  $\|\phi\| = 1$ , and  $\phi(x) = \|x\|$ . In particular,  $\|x\|_X = \sup\{|\phi(x)| : \phi \in X^*, \|\phi\| = 1\}$ .*

*Proof.* Let  $M = \text{Span}\{x\}$ , and define  $\phi_0(\lambda x) := \lambda\|x\|$ .

Then we have  $\|\phi_0\| = 1$ , and by Hahn-Banach Theorem, there is  $\phi \in X^*$ , such that  $\|\phi\| = \|\phi_0\| = 1$ , and  $\phi(x) = \phi_0(x) = \|x\|$ .

In particular,  $\|x\| = \phi(x) = |\phi(x)|$ , so  $\|x\|_X \leq \sup\{|\psi(x)| : \psi \in X^*, \|\psi\| = 1\}$ .

Also, for any  $\psi \in X^*$ ,  $\|\psi\| = 1$ , we have  $|\psi(x)| \leq \|\psi\| \|x\| = \|x\|$ .  $\square$

**Corollary 3.36.** *Let  $X$  be a normed linear space, then  $X^*$  separates the points of  $X$ .*

*Proof.* For any  $x \neq y$ , we have  $x - y \neq 0$ , so there is  $\phi \in X^*$ , such that

$$\phi(x) - \phi(y) = \phi(x - y) = \|x - y\| \neq 0,$$

which separates  $x, y$ .  $\square$

**Corollary 3.37.** *Let  $X$  be a normed linear space, then there is a canonical linear isometric embedding  $i : X \hookrightarrow X^{**}$  by  $i(x) := \hat{x}$ ,  $\hat{x}(\phi) := \phi(x)$ .*

*Proof.* We have  $\|\hat{x}\| = \sup_{\|\phi\|=1} \|\hat{x}(\phi)\| = \sup_{\|\phi\|=1} |\phi(x)| = \|x\|$ .  $\square$

*Remark.* It is not necessarily that  $X \cong X^{**}$ . Indeed, if  $X = C_0(\mathbb{N})$ , we have that  $X^* \cong \ell^1(\mathbb{N})$ , and  $X^{**} \cong \ell_\infty(\mathbb{N})$ .

**Definition 3.23.** A Banach space  $X$  is **reflexive** if  $i(X) = X^{**}$ , where  $i : X \hookrightarrow X^{**}$  is the canonical linear isometric embedding by  $i(x) := \hat{x}$ ,  $\hat{x}(\phi) := \phi(x)$ .

**Corollary 3.38.** *Let  $X$  be a normed linear space. For any closed subspace  $M$ , and  $x \notin M$ , there is  $f \in X^*$ , such that  $\|f\| = 1$ ,  $f|_M = 0$ , and  $f(x) = \text{dist}(x, M)$ .*

*Proof.* Consider the quotient map  $Q : X \rightarrow Y$ , where  $Y := X/M$ . Since  $x \notin M$ , we have  $[x] \neq 0$ .

Let  $\phi \in Y^*$ , such that  $\|\phi\|_{Y^*} = 1$ , and  $\phi([x]) = \|[x]\|_Y = \inf_{m \in M} \|x + m\| = \text{dist}(x, M)$ .

Let  $f := \phi \circ Q$ , we have that  $f(x) = \text{dist}(x, M)$ , and  $\forall m \in M$ ,  $f(m) = \phi(m + M) = \phi(0) = 0$ .

Also,  $\|f\| \leq \|\phi\| \|Q\| \leq \|\phi\| = 1$ .  $\square$

**Definition 3.24.** Let  $X$  be a Banach Space. For  $Y \subseteq X$ , the **annihilator** of  $Y$  is

$$Y^\perp := \{\varphi \in X^* : \varphi(Y) = \{0\}\}.$$

For  $Z \subseteq X^*$ , the **preannihilator** of  $Z$  is

$$Z_\perp := \{x \in X : \hat{x}|_Z = 0\} = Z^\perp \cap X.$$

**Proposition 3.39.** *Let  $X$  be a Banach Space, then*

$$(Y^\perp)_\perp = \overline{\text{Span}(Y)}.$$

*Proof.* Let  $M := \overline{\text{Span}(Y)}$ .

For any  $y \in M$ ,  $f \in Y^\perp$ , we have  $\hat{y}(f) = f(y) = 0$ , so  $y \in (Y^\perp)^\perp$ . Thus,  $M \subseteq (Y^\perp)^\perp$ .

Suppose  $x \notin Y$ , then there is  $f \in X^*$  such that  $f|_M = 0$ , and  $f(x) = \text{dist}(x, M)$ . In particular,  $f \in Y^\perp$  and  $\hat{x}(f) = f(x) \neq 0$ .

Thus,  $x \notin (Y^\perp)^\perp$ .

This shows  $(Y^\perp)^\perp \subseteq M$ .

Thus,

$$M = (Y^\perp)^\perp.$$

□

## 4 Hilbert Spaces

**Definition 4.1.** Let  $X$  be a vector space over  $\mathbb{F}$ .  $M : X \times X \rightarrow \mathbb{F}$  is a **sesquilinear positive semidefinite form** on  $X$  if it satisfies

1. conjugate symmetry; i.e.  $M[x, y] = \overline{M[y, x]}$ .
2. linearity in the second argument; i.e.  $M[x, ay + z] = M[x, y] + aM[x, z]$ .
3. positive semi-definiteness; i.e.  $M[x, x] \geq 0$ .

**Proposition 4.1.** Let  $X$  be a vector space over  $\mathbb{F}$ , with a sesquilinear positive semidefinite form  $M$ . For all  $x, y \in H$ , we have

1.  $M[x, y] + M[y, x] = 2\Re(M[x, y]) = 2\Re(M[y, x])$ , which is twice the real part of  $M[x, y]$ .
2.  $M[x, y] - M[y, x] = 2\Im(M[x, y]) = -2\Im(M[y, x])$ , which is twice the imaginary part of  $M[x, y]$ .
3.  $M[x, y]M[y, x] = |M[x, y]|^2 = |M[y, x]|^2$ .
4.  $M[x + y, x + y] = M[x, x] + 2\Re(M[y, x]) + M[y, y]$ .
5.  $M[x - y, x - y] = M[x, x] - 2\Re(M[y, x]) + M[y, y]$ .

*Proof.* 1.  $M[x, y] + M[y, x] = M[x, y] + \overline{M[x, y]} = 2\Re(M[y, x])$ .

2.  $M[x, y] - M[y, x] = M[x, y] - \overline{M[x, y]} = 2\Im(M[x, y])$ .

3.  $M[x, y]M[y, x] = M[x, y]\overline{M[x, y]} = |M[x, y]|^2$ .

4.

$$\begin{aligned} M[x + y, x + y] &= M[x + y, x] + M[x + y, y] \\ &= M[x, x] + M[y, x] + \overline{M[x, y]} + M[y, y] \\ &= M[x, x] + M[y, x] + \overline{M[y, x]} + M[y, y] \\ &= M[x, x] + 2\Re(M[y, x]) + M[y, y]. \end{aligned}$$

5. By 3., we have

$$\begin{aligned} M[x - y, x - y] &= M[x + (-y), x + (-y)] \\ &= M[x, x] + 2\Re(M[-y, x]) + M[-y, -y] \\ &= M[x, x] - 2\Re(M[y, x]) + M[y, y]. \end{aligned}$$

□

**Definition 4.2.** An **inner product space** is a vector space  $H$  that has an inner product:  $\langle -, \cdot \rangle : H \times H \rightarrow \mathbb{C}$ , such that  $\forall u, v, w \in H, a, b \in \mathbb{C}$ , it satisfies



1. conjugate symmetry; i.e.  $\langle v, w \rangle = \overline{\langle w, v \rangle}$ ,
2. linearity in the second argument; i.e.  $\langle v, au + bw \rangle = a\langle v, u \rangle + b\langle v, w \rangle$ , and
3. positive definiteness; i.e.  $\langle v, v \rangle \geq 0$ , and if  $v \neq 0$ , we must have  $\langle v, v \rangle > 0$ .

*Remark.* Notice that the inner product is a sesquilinear positive semidefinite form, with the additional requirement: if  $v \neq 0$ , we must have  $\langle v, v \rangle > 0$ .

**Proposition 4.2.** *For every inner product space with  $\langle -, \cdot \rangle$ , and for all  $x, y \in H$ , we have*

1.  $\langle x, y \rangle + \langle y, x \rangle = 2\Re(\langle x, y \rangle) = 2\Re(\langle y, x \rangle)$ .
2.  $\langle x, y \rangle - \langle y, x \rangle = 2\Im(\langle x, y \rangle) = -2\Im(\langle y, x \rangle)$ .
3.  $\langle x, y \rangle \langle y, x \rangle = |\langle x, y \rangle|^2$ .
4.  $\langle x + y, x + y \rangle = \langle x, x \rangle + 2\Re(\langle y, x \rangle) + \langle y, y \rangle$ .
5.  $\langle x - y, x - y \rangle = \langle x, x \rangle - 2\Re(\langle x, y \rangle) + \langle y, y \rangle$ .

**Theorem 4.3** (Cauchy-Schwarz). *For every inner product space  $H$ ,*

$$\forall u, v \in H, |\langle u, v \rangle| \leq \|u\| \|v\|,$$

where we define  $\|x\| = \sqrt{\langle x, x \rangle}$  or any  $x \in H$ .

In particular, when  $\|u\| \neq 0$ ,  $\|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 = \|z\|^2$ , where  $z := \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u$ .

*Proof.* Notice that this is trivially true and equality holds to be zero when  $u = 0$ . Now we assume  $u \neq 0$ , then  $\|u\| = \sqrt{\langle u, u \rangle} > 0$ .

$$\begin{aligned} \|z\|^2 &= \langle z, z \rangle \\ &= \left\langle \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u, \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u \right\rangle \\ &= \|u\|^2 \langle v, v \rangle - \langle u, v \rangle \langle u, v \rangle - \overline{\langle u, v \rangle} \langle v, u \rangle + \frac{\langle v, u \rangle \langle u, v \rangle}{\|u\|^2} \langle u, u \rangle \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 - \cancel{|\langle v, u \rangle|^2} + \cancel{|\langle v, u \rangle|^2} \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2. \end{aligned}$$

Now  $\|z\|^2 = \langle z, z \rangle \geq 0$ , we have the result. □

*Remark.* This proof does not use the positive definiteness of the inner product, so it also works for a sesquilinear positive semidefinite form  $M$ . i.e.

$$|M[x, y]| \leq M[x, x]M[y, y].$$

**Proposition 4.4.** *For every inner product space with  $\langle -, \cdot \rangle$ , there is a norm  $\|x\| = \sqrt{\langle x, x \rangle}$ .*

*Proof.* Consider any  $x \in H, a \in \mathbb{C}$ ,

$$\begin{aligned}
\|a \cdot x\| &= \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2} \sqrt{\langle x, x \rangle} = |a| \|x\| \\
\forall x \neq 0, \|x\| &= \sqrt{\langle x, x \rangle} > 0 \\
\|0\| &= \sqrt{\langle 0, 0 \rangle} = 0 \\
\|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \\
&= \|x\|^2 + \|y\|^2 + 2\Re(\langle x, y \rangle) \\
&\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y \rangle| \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\
&\leq (\|x\| + \|y\|)^2.
\end{aligned}$$

Thus  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm. □

**Corollary 4.5.** For every inner product space, there is a metric  $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$

**Proposition 4.6** (polarization identity). Let  $H$  be an inner product space over  $\mathbb{F}$ . The inner product is completely determined by the induced norm. Indeed, for any  $x, y \in H$ , we have

$$\langle x, y \rangle = \frac{1}{4} \sum_{\epsilon \in \{\pm i, \pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if  $\mathbb{F} = \mathbb{C}$ , and

$$\langle x, y \rangle = \frac{1}{2} \sum_{\epsilon \in \{\pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if  $\mathbb{F} = \mathbb{R}$ .

**Proposition 4.7.** If  $\forall v, \langle v, u \rangle = 0$ , then  $u = 0$ .

**Proposition 4.8.** For an Inner product space  $H, \forall y, x = \lim_{i \rightarrow \infty} x_i \in H$ , we have

$$\langle x, y \rangle = \lim_{i \rightarrow \infty} \langle x_i, y \rangle.$$

*Proof.* Given any  $\epsilon > 0$ , let  $\epsilon_0 = \frac{\epsilon}{\|y\|}$ .

Since  $x = \lim_{i \rightarrow \infty} x_i$ , we can find  $N > 0$ , such that  $\forall n > N, \|x - x_n\| < \epsilon_0$ ,

thus  $|\langle x, y \rangle - \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq \|x - x_n\| \|y\| < \epsilon_0 \|y\| = \epsilon$  □

**Corollary 4.9.** For an Inner product space  $H, \forall y, x = \lim_{i \rightarrow \infty} x_i \in H$ , we have  $\langle y, x \rangle = \lim_{i \rightarrow \infty} \langle y, x_i \rangle$ .

**Definition 4.3.** An inner product space  $\mathcal{H}$  is called a **Hilbert space** if it is complete.

**Definition 4.4.** Let  $H$  be an inner product space. Two vectors  $u, v \in H$  are called **orthogonal** if  $\langle u, v \rangle = 0$ .

**Definition 4.5.** Let  $H$  be an inner product space. A set  $\{e_i\}_{i \in I} \subseteq H$  is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij}.$$

**Definition 4.6.** Let  $H$  be an inner product space. An orthonormal set  $\{e_i\}_{i \in I} \subseteq H$  is called a **maximal orthonormal set** / **orthonormal basis** / **total orthonormal set** if  $\text{Span}(\{e_i\}_{i \in I})$  is dense in  $H$ . Namely,

$$H = \overline{\text{Span}(\{e_i\}_{i \in I})}.$$

**Theorem 4.10** (generalized Fourier series). Let  $\mathcal{H}$  be a Hilbert space, and  $\{e_i\}_{i \in \mathbb{N}} \subseteq \mathcal{H}$  be an orthonormal set, then TFAE:

1.  $\{e_i\}_{i \in \mathbb{N}}$  is an orthonormal basis
2. If  $\forall i \in \mathbb{N}, \langle x, e_i \rangle = 0$ , then  $x = 0$ .
3.  $\forall x \in \mathcal{H}, x = \sum_{i \in \mathbb{N}} \langle e_i, x \rangle e_i$ . (Fourier series)
4.  $\forall x \in \mathcal{H}, \|x\|^2 = \sum_{i \in \mathbb{N}} |\langle e_i, x \rangle|^2$ . (Parseval Identity)

**Theorem 4.11.**  $\mathcal{H}$  is a separable Hilbert space, if and only if there is a maximal orthonormal set in  $\mathcal{H}$ . Moreover, in this case, every maximal orthonormal set is at most countable.

**Definition 4.7.** Let  $\mathcal{H}$  be a Hilbert space.  $S \subseteq \mathcal{H}$ , the subspace **orthogonal** to  $S$  is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v \rangle = 0, \forall v \in S\}.$$

**Lemma 4.12.** Let  $\mathcal{H}$  be a Hilbert space,  $S \subseteq \mathcal{H}$ , we always have  $S^\perp$  is a subspace of  $\mathcal{H}$ .

**Definition 4.8.** Let  $V$  be a vector space, and  $U, W \subseteq V$  be two subspaces, we say  $V = U \oplus W$ , if  $V = U + W = \{v + w : v \in U, w \in W\}$ , and  $V \cap W = \{0\}$ .

**Proposition 4.13.** Let  $V$  be a vector space, and  $U, W \subseteq V$  be two subspaces.  $V = U \oplus W$ , if and only if  $\forall v \in V$ , it can be uniquely written as  $v = u + w$ , where  $u \in U, w \in W$ .

**Theorem 4.14.** Let  $\mathcal{H}$  be a Hilbert space, if  $S \subseteq \mathcal{H}$  is a subspace, then

$$\mathcal{H} = \bar{S} \oplus S^\perp.$$

**Theorem 4.15.** [Riesz-Frechet Representation theorem] Let  $\mathcal{H}$  be a Hilbert space, then for each  $u^* \in \mathcal{H}^*$ ,  $\exists! u \in \mathcal{H}$ , such that  $\forall v \in \mathcal{H}, \langle u^*, v \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$ , and  $\|u^*\|_{\mathcal{H}^*} = \|u\|_{\mathcal{H}}$ .

**Definition 4.9.** Let  $X$  be a vector space over  $\mathbb{F}$ . We define

$$\bar{X} := \{\bar{x} : x \in X\}$$

to be the **conjugate vector space**. The scalar multiplication is given by  $\lambda \bar{x} = \overline{\lambda x}$ .

**Proposition 4.16.** Let  $X$  be a normed space over  $\mathbb{F}$ . The map  $X \rightarrow \bar{X}$  by  $x \mapsto \bar{x}$  is an isometric antilinear bijection.

**Corollary 4.17.** Let  $\mathcal{H}$  be a Hilbert space, then  $\bar{\mathcal{H}} \cong \mathcal{H}^*$ , where the map  $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$ ;  $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$  is the **canonical bijective isometric antilinear isomorphism**.

**Corollary 4.18.** Every Hilbert space is reflexive.

**Definition 4.10.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces. We say  $U \in B(\mathcal{H}_1, \mathcal{H}_2)$  is **unitary** if it is an isometric isomorphism between Banach Spaces.

**Proposition 4.19.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces. If  $U$  is unitary, it preserves the inner product as well. Namely,

$$\forall x, y \in \mathcal{H}_1, \langle x, y \rangle = \langle Ux, Uy \rangle.$$

*Proof.* Since  $U$  is isometric, by the polarization identity proposition 4.6, it preserves the inner product as well.  $\square$

## 5 Locally Convex Topological Vector Spaces and Weak Topology

### 5.1 Locally Convex Topological Vector Spaces

**Definition 5.1.** Let  $X$  be a vector space over  $\mathbb{F}$ , a **semi-norm** is a map  $p : X \rightarrow [0, \infty)$  such that  $\forall t \in \mathbb{F}, x, y \in X$ ,

$$\begin{aligned} p(tx) &= |t|p(x), \\ p(x+y) &\leq p(x) + p(y). \end{aligned}$$

The null space of  $p$  is denoted  $N_p := \{x \in X : p(x) = 0\}$ .

*Remark.* Notice that  $p$  is a norm if and only if  $N_p = \{0\}$ .

**Definition 5.2.** A **Locally Convex Topological Vector Space** is a vector space  $X$  over  $\mathbb{F}$  with a family of semi-norms  $P = \{p_\alpha\}_{\alpha \in A}$ , such that  $\bigcap_{p \in P} \ker(p) = \{0\}$ .  $\mathcal{T}_P$  is the topology on  $X$  generated by the convex sets

$$U(x, p, r) := \{y \in X : p(y - x) < r\}$$

for  $x \in X, r > 0, p \in P$ .

**Definition 5.3.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, for  $r > 0, x_0 \in X$ , and a finite subset  $F \subseteq P$ , we define

$$U_{F,r}(x_0) := \{x \in X : \forall p \in F, p(x - x_0) < r\},$$

and  $U_{F,r} := U_{F,r}(0)$ .

**Proposition 5.1.** Each  $U_{F,r}(x_0)$  is a finite intersection of  $\bigcap_{p \in F} U(x, p, r)$ , so it is open. Also, each  $U_{F,r}(x_0) = x_0 + U_{F,r}$ .

**Proposition 5.2.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, then for any  $x_0 \in X$  the sets  $\{U_{F,r}(x_0)\}_{\text{finite } F \subseteq P, r > 0}$  form a neighbourhood basis at  $x_0$ .

*Proof.* Consider any open  $U \ni 0$ , then there are  $p_1, \dots, p_n \in P, r_1, \dots, r_n > 0, x_1, \dots, x_n \in X$ , such that

$$0 \in \bigcap_{i=1}^n U(x_i, p_i, r_i) \subseteq U.$$

Let  $r := \min_{i \in [n]} (r_i - p_i(x_i)) > 0$ , and let  $F := \{p_1, \dots, p_n\}$ .

Consider any  $x \in U_{F,r}$ , we have for any  $i \in [n]$ ,

$$\begin{aligned} p_i(x_i - x) &\leq p_i(x_i) + p_i(x) \\ &\leq p_i(x_i) + r \\ &\leq p_i(x_i) + r_i - p_i(x_i) \\ &= r_i. \end{aligned}$$

Thus  $x \in \bigcap_{i=1}^n U(x_i, p_i, r_i) \subseteq U$ .

This shows  $0 \in U_{F,r} \subseteq U$ .

Thus,  $\{U_{F,r}\}_{\text{finite } F \subseteq P, r > 0}$  form a neighbourhood basis at 0. By translation,  $\{U_{F,r}(x_0)\}_{\text{finite } F \subseteq P, r > 0}$  form a neighbourhood basis at  $x_0$ .  $\square$

**Proposition 5.3.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, it is Hausdorff.

*Proof.* Given any  $x \neq y \in X$ , then  $x - y \neq 0$ .

There is  $p \in P$  such that  $r = p(x - y) > 0$ .

Now  $U(x, p, \frac{r}{2}) \ni x$  and  $U(y, p, \frac{r}{2}) \ni y$  has empty intersection.  $\square$

**Proposition 5.4.** *Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, then the addition map  $A : X \times X \mapsto X$  and scalar multiplication map  $B : \mathbb{F} \times X \rightarrow X$  are continuous.*

*Proof.* Given any open set  $U$ , with  $x_0 + y_0 \in U$  for some  $x_0, y_0 \in X$ .

Since  $\{U_{F,r}(x_0 + y_0)\}_{\text{finite } F \subseteq P, r > 0}$  form a neighbourhood basis at  $x_0 + y_0$ , there is some finite  $F \subseteq P$  and  $r > 0$ , such that

$$U_{F,r}(x_0 + y_0) \subseteq U,$$

since they form a neighbourhood basis.

We claim that  $A^{-1}(U_{F,r}(x_0 + y_0)) \supseteq (x_0 + U_{F, \frac{r}{2}}) \times (y_0 + U_{F, \frac{r}{2}})$ , which is open.

Indeed, take any  $x \in x_0 + U_{F, \frac{r}{2}}$ ,  $y \in y_0 + U_{F, \frac{r}{2}}$  and  $p \in F$ , we have

$$\begin{aligned} p((x + y) - (x_0 + y_0)) &\leq p(x - x_0) + p(y - y_0) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

Thus, we have find

$$(x_0, y_0) \in (x_0 + U_{F, \frac{r}{2}}) \times (y_0 + U_{F, \frac{r}{2}}) \subseteq A^{-1}(U_{F,r}(x_0 + y_0)) \subseteq A^{-1}(U).$$

Since this hold for all  $(x_0, y_0) \in A^{-1}(U)$ , we have that  $A^{-1}(U)$  is open.  $\square$

**Proposition 5.5.** *Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, a net  $(x_\lambda)_{\lambda \in \Lambda}$  converges to  $x \in X$  if and only if  $\forall p \in P$ ,  $p(x - x_\lambda) \rightarrow 0$ .*

*Proof.*

$$(x_\lambda)_{\lambda \in \Lambda} \rightarrow x$$

if and only if

$$\forall \text{finite } F \subseteq P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U_{F,r}(x)$$

if and only if

$$\forall p \in P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U(x, p, r)$$

if and only if

$$\forall p \in P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, p(x_\lambda - x) < r$$

if and only if

$$\forall p \in P, p(x_\lambda - x) \rightarrow 0.$$

$\square$

**Proposition 5.6.** *Let  $(X, \|\cdot\|)$  be a normed vector space, then taking  $P := \{\|\cdot\|\}$ , we have a Locally Convex Topological Vector Space.*

## 5.2 Weak Topology

**Proposition 5.7.** *Let  $(X, \|\cdot\|)$  be a normed vector space, and  $Y \subseteq X^*$  be a linear subspace that separates the points, then  $P := \{p_\phi\}_{\phi \in Y}$  given by  $p_\phi(x) := |\phi(x)|$  gives a Locally Convex Topological Vector Space  $(X, \mathcal{T}_Y)$ , where  $\mathcal{T}_Y := \mathcal{T}_P$ .*

*Proof.* Clearly,  $\forall t \in \mathbb{F}$ ,  $p_\phi(tx) = |\phi(tx)| = |t\phi(x)| = |t||\phi(x)| = |t|p_\phi(x)$ .

Also,  $p_\phi(x + y) = |\phi(x + y)| = |\phi(x) + \phi(y)| \leq |\phi(x)| + |\phi(y)| = p_\phi(x) + p_\phi(y)$ .

Thus, each  $p_\phi$  is a semi-norm.

Suppose  $p_\phi(x) = 0$  for all  $\phi \in Y$ , then  $\phi(x) = 0$  for all  $\phi \in Y$ . Since  $Y$  separates the points, we must have  $x = 0$ .  $\square$

*Remark.* If  $Y$  is not a subspace but just a subset, we can WLOG take  $Y' = \text{Span}(Y)$ , which will generate the same topology.

**Definition 5.4.** Let  $(X, \|\cdot\|)$  be a normed vector space. The **weak topology on  $X$**   $\sigma(X, X^*)$  is  $(X, \mathcal{T}_{X^*})$ , where we take  $Y = X^*$ , which separates points by the Hahn-Banach Theorem. We say  $(x_\lambda)_{\lambda \in \Lambda}$  **converges weakly to**  $x \in X^*$  if it converges in the weak topology, denoted  $x_\lambda \rightharpoonup x$ .

Also, the **weak-\* topology on  $X^*$**   $\sigma(X^*, X)$  is  $(X^*, \mathcal{T}_X)$ , where we take  $Y = X \subseteq X^{**}$ . We say  $(\phi_\lambda)_{\lambda \in \Lambda}$  **converges weakly to**  $\phi \in X^*$  if it converges in the weak-\* topology, denoted  $\phi_\lambda \rightharpoonup \phi$ .

**Proposition 5.8.** Let  $(X, \|\cdot\|)$  be a normed vector space.  $x_\lambda \rightharpoonup x$  in  $X$  if and only if  $\forall \phi \in X^*$ ,  $\phi(x_\lambda) \rightarrow \phi(x)$ . Also,  $\phi_\lambda \rightharpoonup \phi$  in  $X^*$  if and only if  $\forall x \in X$ ,  $\phi_\lambda(x) \rightarrow \phi(x)$ .

**Proposition 5.9.** Let  $(X, \|\cdot\|)$  be a normed vector space, and  $Y \subseteq X^*$  be a linear subspace that separates the points, then if  $x_\lambda \rightarrow x$  in norm, it also converges in  $(X, \mathcal{T}_Y)$ .

**Proposition 5.10.** Let  $(X, \|\cdot\|)$  be a normed vector space,  $(u_k)_{k=1}^\infty \subset X$  be a sequence, then

1. If  $u_k \rightarrow u$ , we always have  $u_k \rightharpoonup u$ .
2. If  $u_k \rightharpoonup u$ , we have that  $u$  is unique.
3. If  $u_k \rightharpoonup u$ , we have  $(u_k)_{k=1}^\infty$  is bounded.
4. If  $u_k \rightharpoonup u$ , every subsequence  $(u_{k_j})_{j=1}^\infty$  also converges weakly to  $u$ .

**Theorem 5.11.** Let  $X$  be a reflexive Banach Space, and  $(u_k)_{k=1}^\infty \subset X$  be a bounded sequence, then  $\exists (u_{k_j})_{j=1}^\infty$  a subsequence, and  $u \in X$ , such that  $u_{k_j} \rightharpoonup u$ .

**Proposition 5.12.** Let  $\mathcal{H}$  be a Hilbert space, then  $u_k \rightharpoonup u$  if and only if  $\forall v \in \mathcal{H}$ ,  $\langle v, u_k \rangle \rightarrow \langle v, u \rangle$  as real numbers.

*Proof.* Suppose  $u_k \rightharpoonup u$ .

Notice that for all  $v \in \mathcal{H}$ , we have that  $v^\dagger \in \mathcal{H}^*$ , and thus  $\langle v, u_k \rangle = \langle v^\dagger | u_k \rangle \rightarrow \langle v^\dagger | u \rangle = \langle v, u \rangle$ .

Now suppose  $\forall v \in \mathcal{H}$ ,  $\langle v, u_k \rangle \rightarrow \langle v, u \rangle$ .

Notice that for any  $f \in \mathcal{H}^*$ , by Riesz-Frechet Representation theorem 4.15, there is some  $f^\dagger \in \mathcal{H}$ , such that

$$\langle f | u_{k_j} \rangle = \langle f^\dagger, u_{k_j} \rangle \rightarrow \langle f^\dagger, u \rangle = \langle f | u \rangle.$$

Thus,  $u_{k_j} \rightharpoonup u$ . □

**Proposition 5.13.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces,  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded operator, and  $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$  be a sequence. If  $u_k \rightharpoonup u \in \mathcal{H}_1$ , then  $Tu_k \rightharpoonup Tu \in \mathcal{H}_2$ .

*Proof.* Let  $y_k := Tu_k, y := Tu \in \mathcal{H}_2$ .

Consider any  $g \in \mathcal{H}_2^*$ , we define  $f := g \circ K \in \mathcal{H}_1^*$ .

Since  $u_k \rightharpoonup u$ , we must have

$$\begin{aligned} \lim_{k \rightarrow \infty} f(u_k) &= f(u) \\ \lim_{k \rightarrow \infty} g(Ku_k) &= g(Ku) \\ \lim_{k \rightarrow \infty} g(y_k) &= g(y). \end{aligned}$$

We thus have  $y_k \rightharpoonup y$ . □

**Proposition 5.14.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces,  $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a compact operator, and  $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$  be a sequence. If  $u_k \rightharpoonup u \in \mathcal{H}_1$ , then  $Ku_k \rightarrow Ku \in \mathcal{H}_2$ .

*Proof.* Let  $y_k := Ku_k, y := Ku \in \mathcal{H}_2$ .

Since  $K$  is compact, it is bounded, so  $y_k \rightharpoonup y$ .

Now suppose for contradiction  $\lim_{k \rightarrow \infty} \|y_k - y\| \neq 0$ .

Then there is some  $\epsilon > 0$  and a subsequence  $(u_{k_j})_{j=1}^\infty$  such that  $\forall j \geq 1, \|y_{k_j} - y\| \geq \epsilon$ .

Since  $u_k \rightharpoonup u \in \mathcal{H}$ , we have  $(u_k)_{k=1}^\infty$  is bounded, and thus  $(u_{k_j})_{j=1}^\infty$  is bounded.

Since  $K$  is compact, there is some further subsequence  $(u_{k_{j_m}})_{m=1}^\infty$  such that  $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$ .

Thus  $Ku_{k_{j_m}} \rightharpoonup \tilde{y}$ . Since weak convergence, we must have  $\tilde{y} = y$ .

Thus  $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = y$ , which is a contradiction. □

**Definition 5.5.** Let  $X, Y$  be two normed vector spaces, then the **weak operator topology** on  $B(X, Y)$  is induced by

$$P := \{p_{x, \phi}(T) : x \in X, \phi \in Y^*\},$$

where for all  $T \in B(X, Y)$ ,

$$p_{x, \phi}(T) := |\phi(T(x))|.$$

We say  $(T_\lambda)_{\lambda \in \Lambda}$  **converges weakly to**  $T \in B(X, Y)$  if it converges in the weak operator topology, denoted  $T_\lambda \rightharpoonup T$ .

*Remark.* Notice that these functions separate points by the Hahn-Banach Theorem. Indeed,  $T \neq S$  implies  $\exists x \in X$  such that  $Tx \neq Sx$ , which implies  $\exists \phi \in Y^*$  such that  $\phi(Tx) \neq \phi(Sx)$ .

**Proposition 5.15.** Let  $X, Y$  be two normed vector spaces, then  $T_\lambda \rightharpoonup T \in B(X, Y)$  if and only if  $\forall \phi \in Y^*, x \in X, \phi(T_\lambda x) \rightarrow \phi(Tx)$ .

**Definition 5.6.** Let  $X, Y$  be two normed vector spaces, then the **strong operator topology** is the topology induced by

$$P := \{p_x(T) := \|Tx\|_Y : x \in X\}.$$

**Proposition 5.16.** Let  $X, Y$  be two normed vector spaces, then  $T_\lambda \rightarrow T \in B(X, Y)$  strongly if and only if  $\forall x \in X, T_\lambda x \rightarrow Tx \in Y$ .

**Proposition 5.17.** Let  $X, Y$  be two normed vector spaces, then convergence in the operator norm implies convergence in strong operator topology, which implies convergence in weak operator topology.

**Proposition 5.18.** When  $X = Y = \mathcal{H}$  is a Hilbert space, then  $T_\lambda \rightharpoonup T$  if and only if

$$\forall \xi, \eta \in \mathcal{H}, \langle T_\lambda \xi, \eta \rangle \rightarrow \langle T \xi, \eta \rangle.$$

*Proof.* By the Riesz-Representation Theorem, for each  $\phi \in \mathcal{H}^*$ , there is unique  $\eta \in \mathcal{H}$  such that

$$\forall \xi \in \mathcal{H}, \phi(\xi) = \langle \xi, \eta \rangle.$$

Thus  $T_\lambda \rightharpoonup T$  if and only if  $\forall \phi \in \mathcal{H}^*, \xi \in \mathcal{H}, \phi(T_\lambda \xi) \rightarrow \phi(T \xi)$ , if and only if  $\forall \xi, \eta \in \mathcal{H}, \langle T_\lambda \xi, \eta \rangle \rightarrow \langle T \xi, \eta \rangle$ .  $\square$

### 5.3 Continuous Functions

**Theorem 5.19.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, then for a linear  $\phi : X \rightarrow \mathbb{F}$ , the following are equal:

1.  $\phi$  is continuous,
2.  $\phi$  is continuous at 0,
3.  $\ker(\phi)$  is closed,
4.  $\exists p_1, \dots, p_n \in P, \alpha_1, \dots, \alpha_n > 0$ , such that

$$\forall x \in X, |\phi(x)| \leq \sum_{i=1}^n \alpha_i p_i(x).$$

*Proof.* (1)  $\implies$  (2) is trivial.

(2)  $\implies$  (3):

Let  $x_\lambda \in \ker(\phi)$ , with  $x_\lambda \rightarrow x \in X$ .

For any  $\lambda$ , we have  $\phi(x) = \phi(x) - \phi(x_\lambda) = \phi(x - x_\lambda)$ .

Since  $x - x_\lambda \rightarrow 0$ , and  $\phi$  is continuous at 0, we have  $\phi(x - x_\lambda) \rightarrow 0$ , which means  $\phi(x) = 0$ . Namely,  $x \in \ker(\phi)$ .

Thus  $\ker(\phi)$  is closed.

(3)  $\implies$  (4):

Suppose  $\phi = 0$ , (4) is trivially true.

Otherwise, there is  $x_0 \in \ker(\Phi)^c$ . WLOG, by taking  $x'_0 := \frac{x_0}{\phi(x_0)}$ , we can assume  $\phi(x_0) = 1$ .

Since  $\ker(\phi)$  is closed,  $\ker(\Phi)^c$  is open.

There must be some finite  $F \subseteq P$  and  $r > 0$ , such that  $x_0 + U_{F,r} = U_{F,r}(x_0) \subseteq \ker(\phi)^c$ .

Thus,  $0 \notin \phi(x_0 + U_{F,r}) = \phi(x_0) + \Phi(U_{F,r}) = 1 + \Phi(U_{F,r})$ .

Namely,  $-1 \notin \Phi(U_{F,r})$ .

Thus  $\forall x \in U_{F,r}$ , we have  $\phi(x) \neq -1$ .

Take  $\{p_1, \dots, p_n\} = F$ , and  $\alpha_i = \frac{1}{r}$ .

Suppose for contradiction that there is some  $x \in X$  with  $|\phi(x)| > \sum_{i=1}^n \frac{1}{r} p_i(x)$ .

In particular,  $|\phi(x)| > \frac{1}{r} p_i(x)$  for all  $p_i \in F$ .

We must have some  $|\lambda|$  such that  $\phi(x) = \lambda|\phi(x)|$ .

Then for  $y := \frac{x}{-\lambda|\phi(x)|}$ , we have

$$\phi(y) = \phi\left(\frac{x}{-\lambda|\phi(x)|}\right) = \frac{\phi(x)}{-\lambda|\phi(x)|} = -1.$$

However,

$$p_i(y) = \left| \left( \frac{1}{-\lambda|\phi(x)|} \right) \right| p_i(x) = \frac{p_i(x)}{|\phi(x)|} < \frac{r|\phi(x)|}{|\phi(x)|} = r.$$

Thus,  $y \in U_{F,r}$ , which is a contradiction.

(4)  $\implies$  (1):

If  $x_\lambda \rightarrow x$ , then  $\forall p \in P$ ,  $p(x_\lambda - x) \rightarrow 0$ .

Thus,  $\sum_{i=1}^n \alpha_i p_i(x_\lambda - x) \rightarrow 0$ .

Thus,  $|\phi(x_\lambda) - \phi(x)| = |\phi(x_\lambda - x)| \rightarrow 0$ , which means  $\phi(x_\lambda) \rightarrow \phi(x)$ .

This shows  $\phi$  is continuous. □

**Definition 5.7.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, then the **continuous dual space**  $(X, \mathcal{T}_P)^*$  is the set of continuous linear functions  $\phi : X \rightarrow \mathbb{F}$ .

**Theorem 5.20.** Let  $(X, \|\cdot\|)$  be a normed vector space, and  $Y \subseteq X^*$  be a linear subspace that separates the points, then  $\phi \in (X, \mathcal{T}_Y)^*$  if and only if  $\phi \in \text{Span}(Y)$ .

*Proof.*  $\phi \in (X, \mathcal{T}_Y)^*$  if and only if  $\exists \phi_1, \dots, \phi_n \in Y, \alpha_i > 0$ , such that  $\forall x \in X$ ,  $|\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)|$ .

( $\Leftarrow$ ):

Suppose  $\phi = \sum_{i=1}^n c_i \phi_i$ , then taking  $\alpha_i := \max(|c_i|, 1)$ , we have

$$\begin{aligned} |\phi(x)| &= \left| \sum_{i=1}^n c_i \phi_i(x) \right| \\ &\leq \sum_{i=1}^n |c_i| |\phi_i(x)| \\ &\leq \sum_{i=1}^n \alpha_i |\phi_i(x)|. \end{aligned}$$

( $\implies$ ):

Suppose  $\{\phi_i\}_{i=1}^n$  is not linearly independent, WLOG,  $\phi_n = \sum_{i=1}^{n-1} c_i \phi_i$ , then

$$|\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)| \leq \sum_{i=1}^{n-1} (\alpha_i + \alpha_n |c_i|) |\phi_i(x)|.$$

Thus, we can assume  $\{\phi_i\}_{i=1}^n$  is linearly independent.

Consider  $T : X \rightarrow \mathbb{F}^n$  by  $T(x) := (\phi_1(x), \dots, \phi_n(x))$ .

Clearly  $\ker(T) = \bigcap_{i=1}^n \ker(\phi_i)$ .

Since  $\{\phi_i\}_{i=1}^n$  is linearly independent,  $T$  is surjective.



By a corollary of the Banach Isomorphism Theorem,  $X/\bigcap_{i=1}^n \ker(\phi_i) \cong \mathbb{F}^n$  by  $\hat{T} : \hat{x} \mapsto (\phi_1(x), \dots, \phi_n(x))$ . Since  $\ker(\phi) \supseteq \bigcap_{i=1}^n \ker(\phi_i)$ , we can define  $\tilde{\phi} : \mathbb{F}^n \rightarrow \mathbb{F}$  by

$$\tilde{\phi}(\hat{T}\hat{x}) := \phi(x).$$

This is well-defined, since  $\hat{x} = \hat{y} \implies x - y \in \bigcap_{i=1}^n \ker(\phi_i) \implies \phi(x - y) = 0 \implies \phi(x) = \phi(y)$ .

Thus for all  $i \in [n]$ , there is  $\beta_i := \hat{\phi}(e_i)$ , such that  $\tilde{\phi}(z_1, \dots, z_n) = \sum_{i=1}^n \beta_i z_i$ .

Thus,  $\phi(x) = \tilde{\phi}(\hat{T}\hat{x}) = \tilde{\phi}(Tx) = \sum_{i=1}^n \beta_i \phi_i(x)$ , which means  $\phi = \sum_{i=1}^n \beta_i \phi_i \in \text{Span}(\{\phi_i\}_{i=1}^n) \subseteq \text{Span}(Y)$ .  $\square$

**Corollary 5.21.** *Let  $(X, \|\cdot\|)$  be a normed vector space, then  $\sigma(X, X^*)^* = (X, \mathcal{T}_{X^*})^* = X^*$  and  $\sigma(X^*, X)^* = (X^*, \mathcal{T}_X)^* = X$ .*

*Remark.* A convergent net in weak-\* topology is not necessarily bounded.

**Example 5.3.1.** Consider  $X = \ell^1 = C_0^*$ , with the weak-\* topology  $\mathcal{T}_{C_0}$ .

For any finite  $F \subsetneq C_0 \subset (\ell^1)^*$ , we have that  $\bigcap_{\phi \in F} \ker(\phi) \neq \{0\}$ , so there is a  $x_F \neq 0 \in \bigcap_{\phi \in F} \ker(\phi)$ . By

taking  $\tilde{x}_F := \frac{|F|}{\|x_F\|} x_F$ , we can have  $\tilde{x}_F \in \bigcap_{\phi \in F} \ker(\phi)$ , with  $\|\tilde{x}_F\| = |F|$ .

Clearly  $(\tilde{x}_F)_{\text{finite } F \subsetneq C_0}$  is not bounded.

However, consider any finite  $F \subseteq C_0, r > 0$ , we can pick  $F_0 := F$ .

For all finite  $F' \supseteq F_0$ , we have that  $\forall \phi \in F, |\phi(\tilde{x}_{F'} - 0)| = 0 < r$ , which means  $\tilde{x}_{F'} \in U_{F,r}$ .

Thus  $\tilde{x}_F \rightarrow 0$ .

## 5.4 Geometric Hahn–Banach Theorems

**Definition 5.8.** Let  $X$  be a vector space, a set  $K \subseteq X$  is **convex** if

$$\forall x, y \in K, t \in (0, 1), (1 - t)x + ty \in K.$$

**Definition 5.9.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, and  $U \ni 0$  be a convex open set, the **Minkowski functional associated to  $U$**  is

$$p_U(x) := \inf \{t > 0 : x \in tU\}.$$

**Lemma 5.22.** *Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, and  $U \ni 0$  be a convex open set, then the Minkowski functional associated to  $U$  is always well-defined.*

*Proof.* Since  $0 \in U$ , which is open, there is finite  $F \subseteq P, r > 0$  such that  $U_{F,r} \subseteq U$ .

Thus, for  $t = \frac{r}{2 \max\{p(x) : p \in F\}} > 0$ , we have  $\forall p \in P, p(tx) = \frac{r}{2 \max\{p(x) : p \in F\}} p(x) \leq \frac{r}{2} < r$ , so  $tx \in U_{F,r} \subseteq U$ .

Thus  $\{t > 0 : x \in tU\} \neq \emptyset$ .  $\square$

**Theorem 5.23.** *Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, and  $U \ni 0$  be a convex open set, then the Minkowski functional  $p_U : X \rightarrow [0, \infty)$  is a sublinear functional, and  $U = \{x \in X : p_U(x) < 1\}$ .*

*Proof.* For any  $t > 0$ , we have  $x \in sU \iff tx \in tsU$ .

$$\begin{aligned} p_U(tx) &= \inf \{r > 0 : tx \in rU\} \\ &= \inf \{st > 0 : tx \in tsU\} \\ &= \inf \{st > 0 : x \in sU\} \\ &= t \inf \{s > 0 : x \in sU\} \\ &= tp_U(x). \end{aligned}$$

Also, consider any  $x, y \in X$  and any  $s, t > 0$  such that  $x \in sU, y \in tU$ , we have that  $\frac{x}{s}, \frac{y}{t} \in U$ .  
By the convexity of  $U$ ,

$$\begin{aligned} \frac{s}{s+t} \frac{x}{s} + \frac{t}{s+t} \frac{y}{t} &\in U \\ x+y &= \left( \frac{s}{s+t} \frac{x}{s} + \frac{t}{s+t} \frac{y}{t} \right) (s+t) \\ &\in (s+t)U. \end{aligned}$$

Thus,  $p_U(x+y) \leq s+t$ . Since this holds for any such  $s, t$ , we have

$$p_U(x+y) \leq p_U(x) + p_U(y).$$

This shows  $P_U$  is sublinear.

Suppose  $p_U(x) < 1$ , then there is  $t < 1$  such that  $x \in tU \implies \frac{x}{t} \in U \implies x = t\frac{x}{t} + (1-t)0 \in U$  by convexity of  $U$ .

Now for any  $x \in U$ , since  $t \mapsto tx$  is continuous, and  $1x = x \in U$  which is open, we have some  $\delta > 0$ , such that  $(1-\delta, 1+\delta)x \subseteq U$ .

Thus  $\forall t \in \left(\frac{1}{1+\delta}, 1\right)$ ,  $x \in tU$ , which means  $p_U(x) \leq \frac{1}{1+\delta} < 1$ .  $\square$

**Theorem 5.24** (First Separation). *Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space,  $A, B \subseteq X$  be disjoint convex sets. Suppose  $A$  is open, then  $\exists t \in \mathbb{R}$ ,  $\phi \in (X, \mathcal{T}_P)^*$ , such that*

$$\forall x \in A, y \in B, \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

*Namely,  $\phi(A), \phi(B)$  can be separated by a vertical line in  $\mathbb{C}$ .*

*Proof.* 1. We first assume  $\mathbb{F} = \mathbb{R}$ .

Fix  $x_0 \in A, y_0 \in B$ , let  $z_0 := y_0 - x_0 \neq 0$ .

Consider  $U := z_0 + A - B = \bigcup_{y \in B} (z_0 - y + A)$ , which is open and convex. Also,  $0 \in U$ .

Consider the Minkowski functional  $p_U : X \rightarrow [0, \infty)$ , which is sublinear.

Notice that  $z_0 \notin U$ , so  $p_U(z_0) \geq 1$ .

Let  $\phi_0 : \text{Span}\{z_0\} \rightarrow \mathbb{R}$  be  $\lambda z_0 \mapsto \lambda$ , which is linear.

In addition,  $\forall \lambda \geq 0$ , we have  $\phi_0(\lambda z_0) = \lambda \leq \lambda p_U(z_0) = p_U(\lambda z_0)$ .

Also,  $\phi_0(-\lambda z_0) = -\lambda \leq 0 \leq p_U(-\lambda z_0)$ .

Thus  $\phi_0 \leq P_U$  on  $\text{Span}\{z_0\}$ . By the extension theorem 3.32, there is a linear extension  $\phi : X \rightarrow \mathbb{R}$ , such that  $\phi \leq p_U$ .

Let  $\epsilon > 0$ .

Take any  $x \in \epsilon U \cap (-\epsilon U) \in \mathcal{O}(0)$ , which is open.

We have that  $\pm \frac{x}{\epsilon} \in U$ , so  $\phi(\pm \frac{x}{\epsilon}) \leq p_U(\pm \frac{x}{\epsilon}) < 1$ .

Thus,  $|\phi(x)| < \epsilon$ .

This shows that  $\phi$  is continuous.

Now take any  $x \in A, y \in B$ , we have that  $z_0 + x - y \in U$ , so  $1 + \phi(x) - \phi(y) = \phi(z_0 + x - y) \leq p_U(z_0 + x - y) < 1$ .

Thus,  $\phi(x) < \phi(y)$ .

Notice that since  $A$  is open, there is  $\epsilon_0 > 0$  such that  $\forall 0 < \epsilon < \epsilon_0$ ,  $(1 \pm \epsilon)x \in A$ . Thus,  $(1 \pm \epsilon)\phi(x) \in \phi(A)$ , which means  $\phi(A)$  is open.

Since  $A$  is convex,  $\phi(A)$  is also convex, thus connected. Thus,  $\phi(A) = (b, t)$  is an interval.

This shows  $\forall x \in A, y \in B$ ,  $\phi(x) < t \leq \phi(y)$ .

2. Now assume  $\mathbb{F} = \mathbb{C}$ .

Consider  $(X_{\mathbb{R}}, \mathcal{T}_P)$ , we have that  $A, B$  are still disjoint convex sets, and  $A$  is open.

Thus there is a  $\mathbb{R}$ -linear continuous map  $\psi \in (X_{\mathbb{R}}, \mathcal{T}_P)^*$ , such that  $\psi(A) < \psi(B)$ .

Now take  $\phi(x) := \psi(x) - i\psi(ix)$ .

$\square$

However, this is not always true when  $A$  is not open.

**Example 5.4.1.** Consider the weak topology  $(\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})$ , and  $A := \{x = (x_i)_{i=1}^\infty \in \ell^1(\mathbb{N}) : \sum_{i=1}^\infty x_i = 0\}$ ,  $B = \{\delta_1\}$ . They are disjoint convex sets.

However, for any  $\phi \in C_0(\mathbb{N}) = \text{Span}(C_0(\mathbb{N})) = (\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})^*$ , we have that  $\phi(A) \cap \phi(B) \neq \emptyset$ .

Indeed, consider any  $\phi = (a_1, a_2, \dots) \in C_0(\mathbb{N})$ , there is  $m \in \mathbb{N}$  such that  $a_m \neq 0$ .

Now consider  $(\delta_m - \delta_n)_{n=1}^\infty \subset A$ , we have that  $\phi(\delta_m - \delta_n) = a_m - a_n \rightarrow a_m \neq 0$ .

Since  $\ker(\phi)$  is closed,  $A$  is not contained in  $\ker(\phi)$ , which means  $\phi(A) = \mathbb{C}$ .

**Lemma 5.25.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, suppose compact  $K \subseteq$  open  $V \subseteq X$ , then there is an open convex neighbourhood  $U$  of 0, such that  $K + U \subseteq V$ .

*Proof.* For all  $x \in K$ , since  $x \in V$ , there is finite  $F_x \subseteq P$ , and  $r_x > 0$ , such that  $U_{F_x, 2r_x}(x) \subseteq V$ .

Since  $K \subseteq \bigcup_{x \in K} U_{F_x, r_x}(x)$  is compact, there is a finite subcover  $K \subseteq \bigcup_{i=1}^n U_{F_{x_i}, r_{x_i}}(x_i)$ .

Now let  $F := \bigcup_{i=1}^n F_{x_i}$ , which is finite, and  $r := \min_{i \in [n]} \{r_{x_i}\} > 0$ .

Let  $U := U_{F, r}$ .

For any  $z \in K + U$ , there is some  $x \in K$  such that  $z \in U_{F, r}(x)$ .

Also, since  $K \subseteq \bigcup_{i=1}^n U_{F_{x_i}, r_{x_i}}(x_i)$ , there is some  $i \in [n]$  such that  $x \in U_{F_{x_i}, r_{x_i}}(x_i)$ .

Thus for any  $p \in F_{x_i} \subseteq F$ , we have

$$\begin{aligned} p(z - x_i) &\leq p(z - x) + p(x - x_i) \\ &< r + r_{x_i} \\ &\leq 2r_{x_i}. \end{aligned}$$

Thus,  $z \in U_{F_{x_i}, 2r_{x_i}}(x_i) \subseteq V$ . □

**Theorem 5.26** (Second Separation). Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space,  $A, B \subseteq X$  be disjoint convex sets. Suppose  $A$  is compact,  $B$  is closed, then  $\exists t \in \mathbb{R}$ ,  $\phi \in (X, \mathcal{T}_P)^*$ , such that

$$\forall x \in A, y \in B, \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

Namely,  $\phi(A), \phi(B)$  can be separated by a vertical line in  $\mathbb{C}$ .

*Proof.* Since  $A$  is compact, and  $B^c$  is open, there is an open convex neighbourhood  $U$  of 0, such that  $A + U \subseteq B^c$ ; namely  $A + U \cap B = \emptyset$ .

By the first separation theorem, there is  $t \in \mathbb{R}$ ,  $\phi \in (X, \mathcal{T}_P)^*$ , such that

$$\sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

Since  $\phi$  is continuous and  $A$  is compact, by the Extreme Value Theorem, there is  $x_0 \in A$ , such that  $\Re(\phi(x_0)) = \sup_{x \in A} \Re(\phi(x))$ .

Notice that  $x_0 = x_0 + 0 \in A + U$ , so

$$\sup_{x \in A} \Re(\phi(x)) = \Re(\phi(x_0)) \leq \sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

□

**Corollary 5.27.** Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, then  $(X, \mathcal{T}_P)^*$  separates the points of  $X$ .

*Proof.* Given  $x \neq y \in X$ .

Take  $A := \{x\}$ , and  $B := \{y\}$ , which is closed. They are trivially convex and disjoint. □

**Definition 5.10.** Let  $(X, \mathcal{T})$  be a topological space, and  $A \subseteq X$ . The **convex hull** of  $A$  is

$$\text{conv}(A) := \left\{ x = \sum_{i=1}^n \lambda_i x_i \mid n \in \mathbb{N}, x_i \in A, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The **closed convex hull** of  $A$  is  $\overline{\text{conv}(A)}$ .

**Proposition 5.28.** *The convex hull is the smallest convex set containing  $A$ , and the closed convex hull is the smallest closed convex set containing  $A$ .*

**Definition 5.11.** Let  $(X, \mathcal{T})$  be a topological space, and  $A \subseteq X, \phi \in (X, \mathcal{T})$ . The **closed half square containing  $A$**  under  $\phi$  is

$$H_{\phi, A} := \{x \in X : \Re(\phi(x)) \leq \alpha_{\phi, A}\},$$

where  $\alpha_{\phi, A} := \sup_{x \in A} \Re(\phi(x))$ .

**Proposition 5.29.** *Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, then for all  $A \subseteq X$ , we have*

$$\overline{\text{conv}(A)} = \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}.$$

*Proof.* For any  $\phi \in (X, \mathcal{T}_P)^*$ , we have that  $H_{\phi, A}$  is convex and closed, and  $A \subseteq H_{\phi, A}$ , so  $\overline{\text{conv}(A)} \subseteq H_{\phi, A}$ . Thus,

$$\overline{\text{conv}(A)} \subseteq \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}.$$

On the other hand, suppose for contradiction that  $\bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \setminus \overline{\text{conv}(A)} \neq \emptyset$ .

Take any  $x \in \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \setminus \overline{\text{conv}(A)}$ , we have that  $\{x\}$  is compact, and  $\overline{\text{conv}(A)}$  is closed.

By the Second Separation Theorem, there is  $\phi \in (X, \mathcal{T}_P)^*$  such that  $\Re(\phi(x)) < \inf_{y \in \overline{\text{conv}(A)}} \Re(\phi(y))$ .

We thus have  $\Re(-\phi(x)) > \sup_{y \in \overline{\text{conv}(A)}} \Re(-\phi(y)) \geq \sup_{y \in A} \Re(-\phi(y))$ , so  $x \notin H_{-\phi, A}$ , which contradicts  $x \in \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}$ .  $\square$

**Corollary 5.30.** *Suppose  $X$  is a vector space with two Locally Convex Topologies  $\mathcal{T}_P, \mathcal{T}_{P'}$ . Suppose  $(X, \mathcal{T}_P)^* = (X, \mathcal{T}_{P'})^*$ , then  $(X, \mathcal{T}_P), (X, \mathcal{T}_{P'})$  have the same closed convex sets.*

*Proof.* Suppose  $A$  is convex and closed in  $(X, \mathcal{T}_P)$ , then

$$\begin{aligned} A &= \overline{\text{conv}(A)}^{\mathcal{T}_P} \\ &= \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \\ &= \bigcap_{\phi \in (X, \mathcal{T}_{P'})^*} H_{\phi, A} \\ &= \overline{\text{conv}(A)}^{\mathcal{T}_{P'}} \\ &= \overline{A}^{\mathcal{T}_{P'}}. \end{aligned}$$

Thus,  $A$  is convex and closed in  $(X, \mathcal{T}_{P'})$ .  $\square$

## 5.5 Weakly Closeness and Compactness

**Proposition 5.31.** *Let  $(X, \|\cdot\|)$  be a Normed Space. Suppose  $A \subseteq X$  is closed in  $(X, \mathcal{T}_Y)$ , where  $Y \subseteq X$  separates the points, then  $A$  is normed closed.*

*Proof.* Suppose a net  $(x_\lambda)_{\lambda \in \Lambda}$  in  $A$  converges to  $x \in X$ , we must have  $\phi(x_\lambda) \rightarrow \phi(x)$  for all  $\phi \in X^*$ .

Thus  $x_\lambda \rightarrow x$  in  $(X, \mathcal{T}_Y)$ , which means  $x \in A$ .  $\square$

The converse is not in general true.

**Example 5.5.1.** Consider  $(C[0, 1], \|\cdot\|_\infty)$ , with the dual space  $M([0, 1])$  (the complex Borel measures on  $[0, 1]$ ).

For  $n \geq 1$ , let  $f_n := \begin{cases} 1 - 2nx, & 0 \leq x \leq \frac{1}{2n} \\ 2nx - 1, & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 1, & x \geq \frac{1}{n} \end{cases}$ . We have that  $f_n \in C[0, 1]$ , and  $f_n(x) \rightarrow 1$  pointwise for all

$x \in [0, 1]$ .

By Lebesgue's Dominated Convergence Theorem, for any complex Borel measure  $\mu \in M([0, 1])$ , we have

$$\int_0^1 f_n d\mu \rightarrow \int_0^1 1 d\mu.$$

Thus,  $f_n \rightharpoonup f$ .

However, for any  $n \geq 1$ ,  $\|f_n - 1\|_n = 1$ .

**Proposition 5.32.** *Let  $(X, \|\cdot\|)$  be a Normed Space, then  $(X, \|\cdot\|)$  and  $(X, \mathcal{T}_{X^*})$  have the same closed convex set.*

*Proof.* This follows from that  $(X, \mathcal{T}_{X^*})^* = X^*$ . □

**Proposition 5.33.** *Let  $(X, \|\cdot\|)$  be a Normed Space, then the closed balls are weak-\* closed. Namely, the closed balls in  $(X^*, \|\cdot\|_{X^*})$  are closed in  $(X^*, \mathcal{T}_X)$ .*

*Proof.* For any  $\phi_0 \in X^*$ ,  $r > 0$ , then

$$\begin{aligned} \bar{B}^{(X^*, \|\cdot\|_{X^*})}(\phi_0, r) &= \{\phi \in X^* : \|\phi - \phi_0\|_{X^*} \leq r\} \\ &= \{\phi \in X^* : |(\phi - \phi_0)(x)| \leq r, \forall x \in X \text{ such that } \|x\| \leq 1\} \\ &= \bigcap_{x \in X \text{ such that } \|x\| \leq 1} \{\phi \in X^* : |\hat{x}(\phi - \phi_0)| \leq r\}, \end{aligned}$$

which is an intersection of weak-\* closed sets. □

**Theorem 5.34** (Goldstine). *Let  $(X, \|\cdot\|)$  be a Banach Space, then  $\bar{B}^{(X, \|\cdot\|)}(0, 1)$  is weak-\* dense in  $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$ , with the weak-\* topology  $(X^{**}, \mathcal{T}_{X^*})$ . In particular,  $X$  is weak-\* dense in  $X^{**}$ .*

*Proof.* If  $x_\lambda \rightharpoonup \psi \in X^{**}$ , with  $(x_\lambda)_{\lambda \in \Lambda} \subset \bar{B}^{(X, \|\cdot\|)}(0, 1)$ , for any  $\phi \in X^*$ , we have

$$\begin{aligned} |\psi(\phi)| &= \left| \lim_{\lambda} \hat{x}_\lambda(\phi) \right| \\ &= \lim_{\lambda} |\phi(x_\lambda)| \\ &\leq \|\phi\|. \end{aligned}$$

Thus,  $\|\psi\| \leq 1$  and  $\psi \in \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$ .

This shows  $\overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak} \subseteq \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$ .

Suppose for contradiction that there is  $\psi \in \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) \setminus \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$ .

Since  $\{\psi\}$  is convex and compact, and  $\overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$  is closed in  $(X^{**}, \mathcal{T}_{X^*})$ , by the second separation theorem, we can find  $\phi \in (X^{**}, \mathcal{T}_{X^*})^* = X^*$ , such that

$$\Re(\psi(\phi)) < \inf_{\xi \in \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}} \Re(\xi(\phi)).$$

We thus have

$$\begin{aligned} \|\phi\| &= \sup_{x \in \bar{B}^{(X, \|\cdot\|)}(0, 1)} \Re(-\phi(x)) \\ &\leq \sup_{\xi \in \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}} \Re(\xi(\phi)) \\ &< \Re(\psi(-\phi)) \\ &\leq \|\psi\| \|\phi\| \\ &\leq \|\phi\|, \end{aligned}$$

which is a contradiction.

Thus,

$$\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$$

□

**Theorem 5.35** (Banach-Alaoglu). *Let  $(X, \|\cdot\|)$  be a Normed Vector Space, then  $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$  is compact in the weak-\* topology.*

*Proof.* Consider the set

$$\begin{aligned} D &:= \{f : X \rightarrow \mathbb{F} \mid |f(x)| \leq \|x\|\} \\ &= \prod_{x \in X} D_x, \end{aligned}$$

where  $D_x := \{x \in \mathbb{F} : |z| \leq \|x\|\}$ .

By Tychonoff's Theorem,  $D$  is compact and Hausdorff in the product topology.

Also,  $\bar{B}^{(X^*, \|\cdot\|)}(0, 1) \subseteq D$ .

In addition,  $f_\lambda \rightarrow f$  in the product topology if and only if  $\forall x \in X, f_\lambda(x) \rightarrow f(x)$  if and only if  $f_\lambda \rightharpoonup f$  in weak-\* topology.

Thus, the weak-\* topology and the product topology are the same.

Since  $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$  is weak-\* closed, it is compact.  $\square$

**Corollary 5.36.** *Let  $(X, \|\cdot\|)$  be a Banach Space, then  $X$  is reflexive if and only if  $\bar{B}^{(X, \|\cdot\|)}(0, 1)$  is weakly compact in  $(X, \mathcal{T}_{X^*})$ .*

*Proof.* ( $\implies$ ) :

If  $X^{**} = X$ , then the weak-\* topology  $\mathcal{T}_{X^*}$  on  $X^{**}$  is the same as the weak topology  $\mathcal{T}_{X^*}$  on  $X$ .

Since  $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$  is weakly compact in  $(X^{**}, \mathcal{T}_{X^*})$  by the Banach-Alaoglu Theorem, we have that  $\bar{B}^{(X, \|\cdot\|)}(0, 1)$  is weakly compact in  $(X, \mathcal{T}_{X^*})$ .

( $\impliedby$ ) :

By the Goldstine Theorem, we have that  $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$ .

Since  $\bar{B}^{(X, \|\cdot\|)}(0, 1)$  is weakly compact in  $(X, \mathcal{T}_{X^*})$ , and  $i : (X, \mathcal{T}_{X^*}) \rightarrow (X^{**}, \mathcal{T}_{X^*})$  is continuous, it is also weakly compact in  $(X, \mathcal{T}_{X^*})$ . Thus,

$$\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak} = \bar{B}^{(X, \|\cdot\|)}(0, 1).$$

Thus,  $X = X^{**}$ .  $\square$

## 5.6 Extreme Points

**Definition 5.12.** Let  $X$  be a Vector Space, and let  $\emptyset \neq A \subseteq X$  be convex. A **face** of  $A$  is some convex  $\emptyset \neq F \subseteq A$ , such that for all  $t \in (0, 1), x, y \in A$ , if  $(1-t)x + ty \in F$ , then  $x, y \in F$ .

If a face  $F = \{z\}$ , then we call  $z$  an **extreme point** of  $A$ .

$\text{Ext}(A)$  is the set of extreme points of  $A$ .

**Proposition 5.37.** *Suppose  $F$  is a face for  $A$ , and  $F'$  is a face for  $F$ , then  $F'$  is also a face for  $A$ .*

*Proof.* Consider any  $x, y \in A, t \in (0, 1)$ , with  $(1-t)x + ty \in F' \subseteq F$ . Since  $F$  is a face of  $A$ ,  $x, y \in F$ . Since  $F'$  is a face of  $F$ ,  $x, y \in F'$ . Thus,  $F'$  is a face of  $A$ .  $\square$

**Example 5.6.1.** Let  $X = L^1([0, 1])$  with the Lebesgue measure, and consider  $A := \bar{B}^{(X, \|\cdot\|_1)}(0, 1)$ . For any  $f \in A$  such that  $\|f\|_{L^1([0, 1])} = a \neq 0$ , we can pick  $t_0 \in (0, 1)$ , such that  $\int_0^{t_0} |f| dx = \int_{t_0}^1 |f| dx = \frac{1}{2}a$ .

Now take  $g := 2f\chi_{[0, t_0]}, h := 2f\chi_{[t_0, 1]}$ , we have that  $f = \frac{1}{2}g + \frac{1}{2}h$ , and  $g, h \in A$ .

Thus,  $f \notin \text{Ext}(A)$ .

Thus,  $\bar{B}(0, 1)$  has no extreme points.

**Proposition 5.38.** *Let  $(X, \mathcal{T})$  be a Topological Vector Space, and  $\emptyset \neq K \subseteq X$  be convex and compact, the for any  $\phi \in (X, \mathcal{T})^*$ ,*

$$F_\phi := \arg \min_{x \in K} (\Re(\phi(x)))$$

*is always a closed face of  $K$ .*

*Proof.* Let  $\alpha_\phi := \inf_{x \in K} (\Re(\phi(x)))$ .

Since  $K$  is compact, and  $\phi$  is continuous,  $\alpha_\phi$  is achieved.

Thus  $F_\phi = \{x \in K : \Re(\phi(x)) = \alpha_\phi\} \neq \emptyset$ .

For any  $(x_\lambda)_{\lambda \in \Lambda}$  in  $F_\phi$ , such that  $x_\lambda \rightarrow x \in K$ , since  $\phi$  is continuous, we have that

$$\phi(x) = \lim_{\lambda} \phi(x_\lambda) = \lim_{\lambda} \alpha_\phi = \alpha_\phi.$$

Thus,  $x \in F_\phi$ , so  $F_\phi$  is closed.

For any  $x, y \in F_\phi, t \in (0, 1)$ , we have

$$\phi((1-t)x + ty) = (1-t)\phi(x) + t\phi(y) = (1-t)\alpha_\phi + t\alpha_\phi = \alpha_\phi.$$

Thus,  $(1-t)x + ty \in F_\phi$ , so  $F_\phi$  is convex.

For any  $x, y \in K, t \in (0, 1)$ , if  $\phi((1-t)x + ty) \in F_\phi$ , we must have

$$\begin{aligned} \alpha_\phi &= \phi((1-t)x + ty) \\ &= (1-t)\phi(x) + t\phi(y) \\ &\geq (1-t)\alpha_\phi + t\alpha_\phi \\ &= \alpha_\phi. \end{aligned}$$

This forces the inequality to be equality, and  $\phi(x) = \phi(y) = \alpha_\phi$ . Thus,  $x, y \in F_\phi$ , so  $F_\phi$  is a face of  $K$ .  $\square$

**Theorem 5.39** (Krein-Milman). *Let  $(X, \mathcal{T}_P)$  be a Locally Convex Topological Vector Space, and  $\emptyset \neq K \subseteq X$  be convex and compact, then*

$$K = \overline{\text{conv}(\text{Ext}(K))}.$$

*Proof.* Since  $(X, \mathcal{T}_P)$  is Hausdorff,  $K$  is compact means  $K$  is closed.

Thus  $K \supseteq \overline{\text{conv}(\text{Ext}(K))}$  since it is a closed convex set containing  $\text{Ext}(K)$ .

On the other hand, we firstly show that for any closed face  $\emptyset \neq F_0 \subseteq K$ , we have  $\text{Ext}(K) \cap F_0 \neq \emptyset$ .

Let  $\Lambda := \{F \subseteq F_0 : F \text{ is a closed face of } F_0\}$ , with the partial order  $F_1 \leq F_2$  if  $F_2 \subseteq F_1$ .

Let  $\mathcal{C} = \{F_\alpha\}_{\alpha \in A}$  be a chain in  $\Lambda$ .

Let  $F := \bigcap_{\alpha \in A} F_\alpha$ .

Since  $K$  is compact, by FIP,  $F \neq \emptyset$ , and it is closed and convex.

Also, if  $x, y \in F_0, t \in (0, 1)$ , and  $(1-t)x + ty \in F$ , we have  $(1-t)x + ty \in F_\alpha$  for some  $\alpha \in A$ .

Since  $F_\alpha$  is a face of  $F_0$ , we must have  $x, y \in F_\alpha \subseteq F$ .

Thus,  $F \in \Gamma$ , and it's clear that  $F$  is an upper bound for  $\mathcal{C}$ .

By Zorn's lemma, there is a maximal element  $F$  of  $\Gamma$ . Notice that it is also a face of  $K$ .

Suppose for contradiction, that there are  $x \neq y \in F$ , then by the second separation theorem, there is  $\phi \in (X, \mathcal{T}_P)^*$ , such that  $\Re(\phi(x)) \neq \Re(\phi(y))$ .

Now let  $F_\phi := \arg \min_{x \in F} (\Re(\phi(x)))$ .

Since  $F$  is a closed subset of compact  $K$ , it is compact. By the proposition,  $F_\phi$  is a closed face of  $F$ , and thus a closed face of  $F_0$ . Thus  $F_\phi \in \Gamma$ .

By maximality of  $F$ , we must have  $F_\phi = F$ , which means  $\phi(x) = \phi(y) = \min_{x \in F} (\Re(\phi(x)))$ , a contradiction with the choice of  $\phi$ .

Thus  $F$  only has one point  $x$ , so  $x \in \text{Ext}(K) \cap F_0 \neq \emptyset$ .

In particular, since  $K$  is a closed face for itself,  $\text{Ext}(K) \neq \emptyset$ .

Now suppose for contradiction that there is  $x_0 \in K \setminus B$ , where  $B := \overline{\text{conv}(\text{Ext}(K))}$ .

By the Second Separation Theorem, there is  $\phi \in (X, \mathcal{T}_P)^*, t \in \mathbb{R}$  such that

$$\Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

In particular,  $\min_{x \in K} \Re(\phi(x)) < \Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y))$ .

Thus,  $F_\phi \cap B = \emptyset$  for  $F_\phi := \arg \min_{x \in K} (\Re(\phi(x)))$ .

However,  $F_\phi$  is a closed face, so  $F_\phi \cap \text{Ext}(K) \neq \emptyset$ , thus a contradiction.  $\square$

**Corollary 5.40.** *Let  $(X, \|\cdot\|)$  be a Normed Vector Space, then  $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$  is the weak-\* closed convex hull of its extreme points.*

*Proof.* By the Banach-Alaoglu Theorem,  $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$  is compact in the weak-\* topology. The convexity is easy to see. Indeed, for any  $f, g \in \bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ ,  $t \in (0, 1)$ , we have  $\|tf + (1-t)g\| \leq t\|f\| + (1-t)\|g\| \leq t + (1-t) = 1$ , so  $tf + (1-t)g \in \bar{B}^{(X^*, \|\cdot\|)}(0, 1)$  and  $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$  is convex.  $\square$

**Corollary 5.41.**  $L^1([0, 1])$  with the Lebesgue measure is not a dual space.

### 5.6.1 Probability Measure

See more about the space of measures in my Pmath651 Measure Theory Notes.

**Definition 5.13.** Let  $X$  be a locally compact Hausdorff space, we define  $M(X) := \{\mu : \text{complex Radon measure}\}$ , and  $\|\mu\|_{M(X)} := |\mu|(X)$ , where  $|\mu|$  is the total variation.

**Theorem 5.42.**  $(M(X), \|\cdot\|_{M(X)}) \cong (C_0(X), \|\cdot\|_\infty)^*$  isometrically by  $\mu(f) := \int_X f d\mu$ .

**Definition 5.14.** The **probability measures** on  $X$  is

$$P(X) := \{\mu \in M(X) \mid \mu \geq 0, \mu(X) = 1\}.$$

The **Dirac measures** are  $\delta_x : f \mapsto f(x)$  for  $x \in X, f \in C(X)$ .

Notice that  $P(X)$  is clearly convex by linearity. Also, when  $X$  is compact,  $C(X) = C_c(X) = C_0(X)$ .

**Lemma 5.43.** *Let  $X$  be a compact Hausdorff space, suppose  $\mu \in P(X), x \in X$ , and  $\ker(\delta_x) \supseteq \ker(\mu)$ , then  $\mu = \delta_x$ .*

*Proof.* For any  $g \in C(X)$ , we have  $\mu(g - \mu(g)\mathbb{1}) = \mu(g) - \mu(g)\mu(\mathbb{1}) = 0$ , so  $g - \mu(g)\mathbb{1} \in \ker(\mu)$ .

Thus,  $\delta_x(g - \mu(g)\mathbb{1}) = 0$ , which means  $\delta_x(g) = \mu(g) \cdot 1 = \mu(g)$ .

Thus  $\mu = \delta_x$ .  $\square$

**Proposition 5.44.** *Let  $X$  be a compact Hausdorff space, then*

$$\text{Ext}(P(X)) = \{\delta_x : x \in X\}.$$

*Proof.* Let  $\mu \in \text{Ext}(P(X))$ .

Fix any  $0 \leq f < 1$  in  $C(X)$ .

Let  $\lambda := \mu(f) \in [0, 1]$ .

If  $0 < \lambda < 1$ , we can define  $\mu_1(g) := \frac{1}{\lambda}\mu(fg) = \frac{1}{\lambda}\int_X fg d\mu$ , and  $\mu_2(g) := \frac{1}{1-\lambda}\mu((1-f)g) = \frac{1}{1-\lambda}\int_X g(1-f)d\mu$ .

We can check that  $\mu_1, \mu_2 \in P(X)$ , and  $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ .

Since  $\mu \in \text{Ext}(P(X))$ , we must have  $\mu = \mu_1 = \mu_2$ .

Thus, for any  $g \in C(X)$ ,

$$\begin{aligned} \mu(g) &= \mu_1(g) \\ &= \frac{\mu(fg)}{\lambda} \\ &= \frac{\mu(fg)}{\mu(f)}. \end{aligned}$$



Thus,  $\mu(fg) = \mu(f)\mu(g)$ .

Now suppose  $\mu(f) = 0$ , we have

$$\begin{aligned}
0 &\leq |\mu(fg)| \\
&= \left| \int_X fg d\mu \right| \\
&\leq \int_X |fg| d\mu \\
&= \int_X f|g| d\mu \\
&\leq \|g\|_\infty \int_X f|g| d\mu \\
&= 0.
\end{aligned}$$

Which means  $\mu(fg) = 0 = \mu(f)\mu(g)$ .

Thus,  $\mu(fg) = \mu(f)\mu(g)$  for all  $f, g \in C(X)$  such that  $0 \leq f < 1$ .

Since  $\mu$  is linear, and  $\text{Span}\{f \in C(X) : 0 \leq f < 1\} = C(X)$ , we have that for all  $f, g \in C(X)$ ,

$$\mu(fg) = \mu(f)\mu(g).$$

We claim that  $\exists x \in X$ , such that  $\ker(\delta_x) \supsetneq \ker(\mu)$ .

Indeed, suppose for contradiction that  $\forall x \in X$ , there is  $f_x \in C(X)$ , such that  $f_x \in \ker(\mu) \setminus \ker(\delta_x)$ .

Namely,  $f_x(x) \neq 0, \mu(f_x) = 0$ .

Thus  $X = \bigcup_{x \in X} \{y : f_x(y) \neq 0\}$  is an open cover. Since  $X$  is compact, there is a finite subcover

$$X = \bigcup_{i=1}^n \{y : f_{x_i}(y) \neq 0\}.$$

Define  $f := \sum_{i=1}^n |f_{x_i}|^2 \in C(X)$ . Notice that  $f > 0$ , which means  $\frac{1}{f} \in C(X)$ .

Now we have

$$\begin{aligned}
1 &= \mu(\mathbb{1}) \\
&= \mu\left(\frac{f}{f}\right) \\
&= \mu(f)\mu\left(\frac{1}{f}\right) \\
&= \sum_{i=1}^n \mu(|f_{x_i}|^2) \mu\left(\frac{1}{f}\right) \\
&= \sum_{i=1}^n \mu(f_{x_i}) \mu(\overline{f_{x_i}}) \mu\left(\frac{1}{f}\right) \\
&= 0 \mu\left(\frac{1}{f}\right) \\
&= 0,
\end{aligned}$$

which is a contradiction.

This proves the claim.

By the lemma,

$$\text{Ext}(P(X)) \subseteq \{\delta_x : x \in X\}.$$

Now given any  $x \in X$ , suppose  $\delta_x = \lambda\mu + (1 - \lambda)\nu$  for some  $\mu, \nu \in P(X), \lambda \in (0, 1)$ . For any  $f \in C(X)$ , we

have

$$\begin{aligned}
|\delta_x(f)| &= |f(x)| \\
&= \delta_x(|f|) \\
&= \lambda\mu(|f|) + (1-\lambda)\nu(|f|) \\
&\geq \lambda\mu(|f|) \\
&= \lambda \int_X |f| d\mu \\
&= \lambda \left| \int_X f d\mu \right| \\
&\geq \lambda |\mu(f)|.
\end{aligned}$$

Suppose  $f \in \ker(\delta_x)$ , we will have  $\lambda|\mu(f)| \leq |\delta_x(f)| = 0$ , so  $\mu(f) = 0$ .

Thus  $\ker(\delta_x) \subseteq \ker(\mu)$ .

By lemma,  $\delta_x = \mu$ , so  $\delta_x = \lambda\delta_x + (1-\lambda)\nu$ , which make  $\nu = \delta_x$  as well.

Thus  $\delta_x \in \text{Ext}(P(X))$ . □

**Corollary 5.45.** *Let  $X$  be a compact Hausdorff space, then*

$$P(X) = \overline{\text{conv}\{\delta_x : x \in X\}}^{\text{weak}}.$$

*Proof.* For any  $\mu \in P(X)$ , we have  $\|\mu\| = |\mu|(X) = \mu(X) = 1$ , so

$$P(X) \subseteq \bar{B}^{(M(X), \|\cdot\|)}(0, 1).$$

By Banach-Alaoglu,  $\bar{B}^{(M(X), \|\cdot\|)}(0, 1)$  is weak-\* compact.

Suppose there are  $(\mu_\lambda)_{\lambda \in \Lambda}$  in  $P(X)$  with  $\mu_\lambda \rightharpoonup \mu \in M(X)$ , we have that

$$\mu(X) = \mu(\mathbb{1}) = \lim_{\lambda \in \Lambda} \mu_\lambda(\mathbb{1}) = 1,$$

since  $\mathbb{1} \in C(X)$ . We can also show  $\mu$  is a positive measure, so  $\mu \in P(X)$ .

This shows  $P(X)$  is weak-\* closed, so it is also weak-\* compact. The result follows from the Krein-Milman Theorem. □

*Remark.* When  $X$  is not compact,  $P(X)$  is not weak-\* closed.

**Example 5.6.2.** Consider  $\delta_n \in P(\mathbb{R})$ , then  $\delta_n \rightharpoonup 0$  in  $\bar{B}^{M(\mathbb{R})}(0, 1)^{\|\cdot\|}$ , but  $0 \notin P(\mathbb{R})$ .

Nevertheless, when  $X$  is a Locally Compact Hausdorff space, with a variation of the proof, we still have

$$\text{Ext}(P(X)) = \{\delta_x | x \in X\},$$

and

$$\text{Ext}\left(\bar{B}^{M(X)}(0, 1)^{\|\cdot\|}\right) = \{z\delta_x : x \in X, z \in \mathbb{C} \text{ such that } \|z\| = 1\}.$$

### 5.6.2 Stone-Weierstrass Theorem

**Theorem 5.46.** *Let  $X$  be a locally compact Hausdorff Space, and  $A \subseteq C_0(X, \mathbb{R})$ , such that*

1.  *$A$  is closed under  $\|\cdot\|_\infty$ ,*
2.  *$A$  is a sub-algebra of  $C_0(X, \mathbb{R})$ ,*
3.  *$A$  separates points (i.e.  $\forall x \neq y \in X, \exists f \in A$  such that  $f(x) \neq f(y)$ ),*
4.  *$A$  vanishes nowhere (i.e.  $\forall x \in X, \exists f \in A$ , such that  $f(x) \neq 0$ ),*

then  $A = C_0(X, \mathbb{R})$ .

*Proof.* Suppose for contradiction that  $A \subsetneq C_0(X, \mathbb{R})$ .

By 1. and a corollary of the Hahn-Banach theorem, there is  $\mu \neq 0$  in  $A^\perp = \{\mu \in M(X) : \mu(f) = 0 \ \forall f \in A\}$ . Thus,

$$K := \bar{B}^{(M(X), \|\cdot\|)}(0, 1) \cap A^\perp$$

is a non-empty weak-\* compact convex set. By the Krein-Milman theorem, there is  $\mu \in \text{Ext}(K)$ .

We claim that

$$\text{Supp}(\mu) := \{x \in X : \forall U \in \mathcal{O}(X), |\mu|(U) > 0\}$$

is a singleton  $\{x\}$  for some  $x \in X$ .

Suppose for contradiction that there are  $x \neq y \in \text{Supp}(\mu)$ . By 3., there is  $f \in A$  such that  $f(x) \neq f(y)$ . Notice that  $f$  is not  $|\mu|$ -a.e. constant since it is continuous.

Now find  $a, b > 0$  such that  $g(x) := a(f(x) + b)$  satisfies  $g(x) \geq 0$  and  $\int_X g d|\mu| = 1$ . Suppose that  $\|g\|_\infty \leq 1$ , then  $0 \leq g \leq 1$  a.e., which means

$$\begin{aligned} \int_X |1 - g| d|\mu| &= \int_X (1 - g) d|\mu| \\ &= |\mu|(X) - \int_X g d|\mu| \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Thus,  $1 - g = 0$   $|\mu|$ -a.e., which means  $g = 1$   $|\mu|$ -a.e. and it contradicts with  $f$  not being constant a.e.. Thus, we must have  $\|g\|_\infty > 1$ , so

$$\lambda := \frac{1}{\|g\|_\infty} \in (0, 1).$$

Let  $\mu_1, \mu_2 \in M(X)$  be as

$$\mu_1(h) := \mu(gh) = \int_X gh d\mu, \quad \mu_2(h) := \mu\left(\frac{1 - \lambda g}{1 - \lambda} h\right) = \int_X \frac{1 - \lambda g}{1 - \lambda} h d\mu.$$

Notice that  $\forall h \in A$ ,  $hg = a(f + b)h = afh + abh \in A$ , so  $\mu_1, \mu_2 \in A^\perp$ . We can check that

$$\begin{aligned} |\mu_1|(X) &= \int_X d|\mu_1| \\ &= \int_X g d|\mu| \\ &= 1, \\ |\mu_2|(X) &= \int_X d|\mu_2| \\ &= \int_X \frac{1 - \lambda g}{1 - \lambda} d|\mu| \\ &= \frac{1 - \lambda}{1 - \lambda} \\ &= 1. \end{aligned}$$

Thus,  $\mu_1, \mu_2 \in K$ , with  $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$ . Since  $\mu \in \text{Ext}(K)$ , we must have  $\mu_1 = \mu_2 = \mu$ , which means  $g = 1$   $|\mu|$ -a.e., a contradiction.

This proves that  $\text{Supp}(\mu) = \{x\}$  for some  $x \in X$ . Thus,  $\mu = z\delta_x$  for some  $z \in \mathbb{C}$ . Also,  $|z| = |\mu|(X) = 1$ . Since  $\mu \in A^\perp$ , for any  $h \in A$ , we must have  $zh(x) = \int_X h d\mu = \mu(h) = 0$  which means  $h(x) = 0$  and  $A$  vanishes at  $x$ .  $\square$

## 6 Spectral Theory

### 6.1 Adjoint Operators

#### 6.1.1 Adjoint Operators on Normed Spaces

**Definition 6.1.** Let  $X, Y$  be normed spaces, the **adjoint operator** or **dual operator** of a linear operator  $A : X \rightarrow Y$  is

$$A^* : Y^* \rightarrow X^*; \quad f \mapsto f \circ A,$$

namely,

$$\forall x \in X, f \in Y^*, \quad \langle x | A^* f \rangle := \langle Ax | f \rangle.$$

**Proposition 6.1.** Let  $X, Y, Z$  be normed spaces,  $S \in B(X, Y), T \in B(Y, Z)$ , then  $(S \circ T)^* = T^* \circ S^*$ .

*Proof.* Consider any  $f \in Z^*$ , and any  $x \in X$ , we have

$$\begin{aligned} (T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx) \\ &= (f)(S(T(x))) \\ &= (f \circ (S \circ T))(x) \\ &= (S \circ T)^*(f)(x). \end{aligned}$$

Thus  $(T^* \circ S^*)(f) = (S \circ T)^*(f)$ . □

**Proposition 6.2.** Let  $X, Y$  be normed spaces, and  $T \in B(X, Y)$ , then

1.  $\|T\| = \|T^*\|$ , so  $T^* \in B(Y^*, X^*)$ ;
2. The map  $T \mapsto T^*$  is linear;
3.  $T^*$  is weak-\* weak-\* continuous; namely, if a net  $(\phi_\lambda)_{\lambda \in \Lambda}$  converges to  $\phi \in Y$  in weak-\* topology on  $Y^*$ , we have that  $T^* \phi_\lambda$  converges to  $T^* \phi$  in weak-\* topology on  $X^*$ ;
4. For  $\mathbb{1}_X : x \mapsto x$ , we have  $(\mathbb{1}_X)^* = \mathbb{1}_{X^*}$ ;
5.  $T^{**} \in B(X^{**}, Y^{**})$  satisfies  $T^{**}|_X = T$ .

*Proof.* 1. We have

$$\begin{aligned} \|T^*\| &= \sup_{\phi \in Y^*, \|\phi\| \leq 1} \|T^* \phi\|_{X^*} \\ &= \sup_{\phi \in Y^*, \|\phi\| \leq 1} \sup_{x \in X, \|x\| \leq 1} |\langle x | T^* \phi \rangle| \\ &= \sup_{x \in X, \|x\| \leq 1} \sup_{\phi \in Y^*, \|\phi\| \leq 1} |\langle Tx | \phi \rangle| \\ &= \sup_{x \in X, \|x\| \leq 1} \|Tx\| \\ &= \|T\|. \end{aligned}$$

2. It is obvious by definition.

3. For net  $(\phi_\lambda)_{\lambda \in \Lambda}$  in  $Y^*$  that converges to  $\phi \in Y$  in weak-\* topology, and any  $x \in X$ , we have

$$\begin{aligned} \langle x | T^* \phi_\lambda \rangle &= \langle Tx | \phi_\lambda \rangle \\ &\rightarrow \langle Tx | \phi \rangle \\ &= \langle x | T^* \phi \rangle. \end{aligned}$$

Thus,  $T^* \phi_\lambda \rightarrow T^* \phi$  in weak-\* topology in  $X^*$ .

4. This is obvious.

5. Consider the canonical embeddings  $x \mapsto \hat{x}, y \mapsto \hat{y}$ , we want to show  $T^{**}\hat{x} = \widehat{Tx}$ . Indeed, for any  $\phi \in Y^*$

$$\begin{aligned}\langle \phi | T^{**}\hat{x} \rangle &= \langle T^*\phi | \hat{x} \rangle \\ &= \langle x | T^*\phi \rangle \\ &= \langle Tx | \phi \rangle \\ &= \langle \phi | \widehat{Tx} \rangle.\end{aligned}$$

□

**Theorem 6.3.** *Let  $X, Y$  be normed spaces, and  $S \in B(Y^*, X^*)$ , then there is  $T \in B(X, Y)$  such that  $T^* = S$  if and only if  $S$  is weak-\* weak-\* continuous.*

*Proof.* ( $\implies$ ) is by the previous proposition.

( $\impliedby$ ): Assume  $S$  is weak-\* weak-\* continuous.

Consider  $S^* \in B(X^{**}, Y^{**})$ , and let  $T := S^*|_X \in B(X, Y^{**})$ . Fix  $x \in X$ , let  $(\phi_\lambda)_{\lambda \in \Lambda}$  be a net that converges in weak-\* topology  $(Y^*, \mathcal{T}_Y)$ , then

$$\begin{aligned}\langle \phi_\lambda | S^*\hat{x} \rangle &= \langle S\phi_\lambda | \hat{x} \rangle \\ &\rightarrow \langle S\phi | \hat{x} \rangle \\ &= \langle \phi | S^*\hat{x} \rangle.\end{aligned}$$

Thus,  $S^*\hat{x}$  is continuous in weak-\* topology  $(Y^*, \mathcal{T}_Y)$ , so  $S^*\hat{x} \in Y$ . This shows we can redefine  $T \in B(X, Y)$ . Now for any  $x \in X, \phi \in Y^*$ , we have

$$\begin{aligned}\langle x | T^*\phi \rangle &= \langle Tx | \phi \rangle \\ &= \langle \phi | S^*\hat{x} \rangle \\ &= \langle S\phi | \hat{x} \rangle \\ &= \langle x | S\phi \rangle.\end{aligned}$$

□

**Theorem 6.4.** *Let  $X, Y$  be Banach spaces, then  $T : X \rightarrow Y$  is bounded if and only if  $T$  is weak-weak continuous.*

*Proof.* See A4.

□

### 6.1.2 Hilbert Adjoint Operators

**Definition 6.2.** Let  $\mathcal{H}$  be a Hilbert space, and  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a bounded linear operator, the **Hilbert adjoint operator** of  $T$  is  $T^\dagger : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \forall x, y \in \mathcal{H}$ .

**Proposition 6.5.** *Let  $\mathcal{H}$  be a Hilbert space, and  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a bounded linear operator, we have  $\langle Ty, x \rangle = \langle y, T^\dagger x \rangle \forall x, y \in \mathcal{H}$ . Thus,  $(T^\dagger)^\dagger = T$ .*

*Proof.*

$$\begin{aligned}\langle Ty, x \rangle &= \overline{\langle x, Ty \rangle} \\ &= \overline{\langle T^\dagger x, y \rangle} \\ &= \langle y, T^\dagger x \rangle.\end{aligned}$$

□

**Theorem 6.6.** Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$  be a bounded linear operator.  $T^\dagger$  always exists, and is given by  $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$ , where  $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$ ;  $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$  is the canonical bijective isometric antilinear isomorphism, and  $T^*$  is the dual operator of  $T$ . In addition,  $T^\dagger$  is also a bounded linear operator, with  $\|T^\dagger\| = \|T\|$ .

*Proof.*  $\forall y \in \mathcal{H}$ , we have that

$$\begin{aligned} \langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\ &= \langle y | (T^* \circ \Phi)(x) \rangle \\ &= \langle Ty | \Phi(x) \rangle \\ &= \langle x, Ty \rangle. \end{aligned}$$

Linearity follows from  $\Phi, \Phi^{-1}$  being anti-linear, and  $T^*$  being linear. We can also compute it directly from the definition. Indeed, consider any  $x, y, z \in \mathcal{H}, c \in \mathbb{C}$ , we have that

$$\begin{aligned} \langle T^\dagger(x + cz), y \rangle &= \langle x + cz, Ty \rangle \\ &= \langle x, Ty \rangle + \bar{c} \langle z, Ty \rangle \\ &= \langle T^\dagger x, y \rangle + \bar{c} \langle T^\dagger z, y \rangle \\ &= \langle T^\dagger x + cT^\dagger z, y \rangle. \end{aligned}$$

Since this holds for any  $y \in \mathcal{H}$ , we have that  $T^\dagger(x + cz) = T^\dagger x + cT^\dagger z$ , and thus  $T^\dagger$  is linear.

Also,

$$\|T^\dagger\| \leq \|\Phi^{-1}\| \|T\| \|\Phi\| = \|T\|.$$

We can also compute this directly from the definition. Indeed, given any  $x \in \mathcal{H}$ , we have that

$$\begin{aligned} \|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\ &= \langle x, TT^\dagger x \rangle \\ &\leq \|x\| \|TT^\dagger x\| \\ &\leq \|x\| \|T\| \|T^\dagger x\| \\ &\implies \\ \|T^\dagger x\| &\leq \|x\| \|T\| \\ &\implies \\ \|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\ &\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{\|x\| \|T\|}{\|x\|} \\ &= \|T\|. \end{aligned}$$

Thus  $T^\dagger$  is also a bounded linear operator, with

$$\|T\| = \|(T^\dagger)^\dagger\| \leq \|T^\dagger\| \leq \|T\|.$$

□

*Remark.*  $\forall x, y \in \mathcal{H}$ ,  $\langle (Tx)^\dagger | y \rangle = \langle Tx, y \rangle = \langle x, T^\dagger y \rangle = \langle x^\dagger | T^\dagger y \rangle$ . We thus abuse the notation, and write  $(Tx)^\dagger = \langle x | T^\dagger$

**Corollary 6.7.** Let  $\mathcal{H}$  be a Hilbert space, and  $S, T \in B(\mathcal{H})$  be bounded linear operators. We have that  $(S \circ T)^\dagger = T^\dagger \circ S^\dagger$ . Also, for any  $\alpha \in \mathbb{C}$ , we have  $(\alpha T + S)^\dagger = \bar{\alpha} T^\dagger + S^\dagger$ .

**Proposition 6.8.** Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$  be a bounded linear operator. We have

$$\|T\|^2 = \|TT^\dagger\| = \|T^\dagger T\|.$$

*Proof.* We have  $\|TT^\dagger\| \leq \|T\|\|T^\dagger\| = \|T\|^2$ .

Now for any  $x \in \mathcal{H}$ , we have

$$\begin{aligned}
\|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\
&= \langle x, TT^\dagger x \rangle \\
&\leq \|x\| \|TT^\dagger x\| \\
&\leq \|x\| \|TT^\dagger\| \|x\| \\
&= \|TT^\dagger\| \|x\|^2; \\
&\implies \\
\|T^\dagger x\| &\leq \sqrt{\|TT^\dagger\|} \|x\| \\
&\implies \\
\|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\
&\leq \sqrt{\|TT^\dagger\|} \\
&\implies \\
\|T^\dagger\|^2 &\leq \|TT^\dagger\|.
\end{aligned}$$

□

**Corollary 6.9.** *Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$  be a bounded linear operator. We have that  $T = 0$  if and only if  $T^\dagger T = 0$ .*

There is a deep analogy between operators  $B(\mathcal{H})$  and algebra of the form  $(\ell^\infty(X), \|\cdot\|_\infty)$  for a set  $X$ , with the pointwise multiplication and involution  $f^*(x) := \overline{f(x)}$ .

**Example 6.1.1.** Consider  $\pi : \ell^\infty(X) \rightarrow B(\ell^2(X))$  by  $f \mapsto M_f$ , where the **multiplication operator**  $M_f(\phi)(x) := f(x)\phi(x)$  for all  $\phi \in \ell^2(X), x \in X$ . We claim that it is a unital isometric homomorphism. Indeed, for any  $\phi \in \ell^2(X)$ , we have

$$\begin{aligned}
\pi(\mathbb{1})(\phi)(x) &= 1 \cdot \phi(x) \\
&= \phi(x),
\end{aligned}$$

so  $\pi(\mathbb{1})(\phi) = \phi$  and  $\pi(\mathbb{1}) = \mathbb{1}$ . Also, for any  $f, g \in \ell^\infty(X)$ ,

$$\begin{aligned}
(\pi(fg)(\phi))(x) &= (fg)(x)\phi(x) \\
&= f(x)g(x)\phi(x) \\
&= f(x)(\pi(g)(\phi))(x) \\
&= \pi(f)(\pi(g)(\phi))(x) \\
&= ((\pi(f) \circ \pi(g))(\phi))(x).
\end{aligned}$$

Thus,  $\pi(fg) = \pi(f) \circ \pi(g)$ . Also, for any  $\phi, \eta \in X$ , we have

$$\begin{aligned}
\langle \pi(f^*)(\phi), \eta \rangle &= \sum_{x \in X} \pi(f^*)(\phi)(x) \overline{\eta(x)} \\
&= \sum_{x \in X} f^*(x) \phi(x) \overline{\eta(x)} \\
&= \sum_{x \in X} \phi(x) \overline{f(x) \eta(x)} \\
&= \sum_{x \in X} \phi(x) \overline{\pi(f)(\eta)(x)} \\
&= \langle \phi, \pi(f)(\eta) \rangle \\
&= \langle \pi(f)^*(\phi), \eta \rangle.
\end{aligned}$$

Thus,  $\pi(f^*) = \pi(f)^*$ . Lastly, we have that

$$\begin{aligned}
\|\pi(f)(\phi)\|^2 &= \langle \pi(f)(\phi), \pi(f)(\phi) \rangle \\
&= \sum_{x \in X} |\pi(f)(\phi)(x)|^2 \\
&= \sum_{x \in X} |f(x)\phi(x)|^2 \\
&= \sum_{x \in X} |f(x)|^2 |\phi(x)|^2 \\
&\leq \|f\|_\infty^2 \left( \sum_{x \in X} |\phi(x)|^2 \right) \\
&= \|f\|_\infty^2 \|\phi\|^2.
\end{aligned}$$

Thus,  $\|\pi(f)\| \leq \|f\|_\infty$ . On the other hand,  $\delta_x \in \ell^2(X)$  for any  $x$ , and

$$\begin{aligned}
\|f\|_\infty^2 &= \sup_{x \in X} |f(x)|^2 \\
&= \sup_{x \in X} \langle \pi(f)(\delta_x), \pi(f)(\delta_x) \rangle \\
&= \sup_{x \in X} \|\pi(f)(\delta_x)\|^2 \\
&\leq \sup_{\phi \in \ell^2(X)} \|\pi(f)(\phi)\|^2 \\
&= \|\pi(f)\|^2.
\end{aligned}$$

Thus,  $\|f\|_\infty = \|\pi(f)\|$ .

Indeed, when we take  $X = \{1, \dots, n\}$ , then  $\ell^2(X) = \mathbb{C}^n$ , and  $\pi : \ell^\infty(X) \rightarrow B(\ell^2(X)) \cong M_n(\mathbb{C})$ , and  $\pi(\ell^\infty(X)) = D_n$ , the diagonal matrices.

**Proposition 6.10.** *Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$ . We have*

$$\ker(T^\dagger) = \text{Im}(T)^\perp.$$

*Proof.* For any  $x \in \ker(T^\dagger)$ ,  $Ty \in \text{Im}(T)$ , we have

$$\begin{aligned}
\langle x, Ty \rangle &= \langle T^\dagger x, y \rangle \\
&= \langle 0, y \rangle \\
&= 0.
\end{aligned}$$



Thus,  $x \in \text{Im}(T)^\perp$ , and we have  $\ker(T^\dagger) \subseteq \text{Im}(T)^\perp$ .

On the other hand, for any  $x \in \text{Im}(T)^\perp, y \in \mathcal{H}$ , we have

$$\begin{aligned}\langle T^\dagger x, y \rangle &= \langle x, Ty \rangle \\ &= 0.\end{aligned}$$

Thus,  $T^\dagger x = 0$ , so  $x \in \ker(T^\dagger)$ . We thus have  $\text{Im}(T)^\perp \subseteq \ker(T^\dagger)$ .  $\square$

**Proposition 6.11.** *Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$ . Suppose  $T^{-1} \in B(\mathcal{H})$  exists, then  $(T^\dagger)^{-1} \in B(\mathcal{H})$  exists, and*

$$(T^\dagger)^{-1} = (T^{-1})^\dagger.$$

*Proof.*

$$\begin{aligned}TT^{-1} &= \mathbb{1} = T^{-1}T \\ (TT^{-1})^\dagger &= \mathbb{1}^\dagger = (T^{-1}T)^\dagger \\ (T^{-1})^\dagger T^\dagger &= \mathbb{1} = T^\dagger (T^{-1})^\dagger.\end{aligned}$$

Thus,  $(T^\dagger)^{-1} = (T^{-1})^\dagger$ .  $\square$

**Proposition 6.12.** *Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$ . The following are equivalent:*

1.  *$T$  is invertible (i.e. an isomorphism between  $\mathcal{H}$  and  $\mathcal{H}$  as Banach spaces);*
2.  *$T^\dagger$  is invertible;*
3.  *$T$  is bounded below;*
4.  *$T^\dagger$  is bounded below;*
5. *Both  $\text{Im}(T), \text{Im}(T^\dagger)$  are closed, and  $T, T^\dagger$  are injective;*
6.  *$\mathcal{H} = \text{Im}(T) = \text{Im}(T^\dagger)$ ;*
7.  *$T$  is bijective;*
8.  *$T^\dagger$  is bijective.*

*Proof.* 1  $\implies$  2 is done above.

2  $\implies$  1. We have  $T = (T^\dagger)^\dagger$ , so applying the above result to  $T^\dagger$  gives 1.

1  $\implies$  3. This is by theorem 3.12.

2  $\implies$  4. This is by theorem 3.12.

3, 4  $\implies$  5. Let  $T$  be bounded below by  $c$ . The injectivity is from theorem 3.12. Also, consider any convergent sequence  $(Tx_n)_{n=1}^\infty$  in  $\text{Im}(T)$ . Since it is Cauchy, for any  $\epsilon > 0$ , there is  $N \geq 1$ , such that for all  $n, m \geq N$ ,

$$c\|x_n - x_m\| \leq \|Tx_n - Tx_m\| < c\epsilon.$$

Thus,  $(x_n)_{n=1}^\infty$  is Cauchy in  $\mathcal{H}$ , so there must be some  $x \in \mathcal{H}$  such that  $x_n \rightarrow x$ . Since  $T$  is continuous,  $Tx_n \rightarrow Tx \in \text{Im}(T)$ . This shows  $\text{Im}(T)$  is closed. Similarly,  $\text{Im}(T^\dagger)$  is closed and it is injective.

5  $\implies$  6. We have that  $\overline{\text{Im}(T)} \oplus \text{Im}(T)^\perp = \mathcal{H}$  by theorem 4.14. Since  $T^\dagger$  is injective, we have  $\text{Im}(T)^\perp = \ker(T^\dagger) = \{0\}$  by proposition 6.10. Thus,  $\overline{\text{Im}(T)} = \mathcal{H}$ . Since  $\text{Im}(T)$  is closed, we have  $\text{Im}(T) = \mathcal{H}$ . Similarly,  $\text{Im}(T^\dagger) \oplus \ker(T) = \mathcal{H}$ , so  $\text{Im}(T^\dagger) = \mathcal{H}$ .

6  $\implies$  7. We have that  $\ker(T) = \text{Im}(T^\dagger)^\perp = (\mathcal{H})^\perp = \{0\}$ , so  $T$  is injective. Also,  $T$  is surjective by assumption.

7  $\implies$  1. By the Banach Isomorphism theorem 3.23,  $T^{-1} \in B(\mathcal{H})$  is bounded.

6  $\implies$  8. We have that  $\ker(T^\dagger) = \text{Im}(T)^\perp = (\mathcal{H})^\perp = \{0\}$ , so  $T^\dagger$  is injective. Also,  $T^\dagger$  is surjective by assumption.

8  $\implies$  2. By the Banach Isomorphism Theorem,  $(T^\dagger)^{-1} \in B(\mathcal{H})$  is bounded.  $\square$

### 6.1.3 Normal, Self-adjoint and Positive Semidefinite Operators

**Definition 6.3.** Let  $\mathcal{H}$  be a Hilbert space. A bounded linear operator  $T \in B(\mathcal{H})$  is **normal** if  $T^\dagger T = TT^\dagger$ .

**Proposition 6.13.** Let  $\mathcal{H}$  be a Hilbert space. If  $T \in B(\mathcal{H})$  is normal, we have that for all  $c \in \mathbb{F}$ ,  $c\mathbb{I} - T$  is also normal.

*Proof.*

$$\begin{aligned} (c\mathbb{I} - T)^\dagger (c\mathbb{I} - T) &= (\bar{c}\mathbb{I} - T^\dagger)(c\mathbb{I} - T) \\ &= |c|^2 \mathbb{I} - \bar{c}T - cT^\dagger + T^\dagger T \\ &= |c|^2 \mathbb{I} - cT^\dagger - \bar{c}T + TT^\dagger, \\ (c\mathbb{I} - T)(c\mathbb{I} - T)^\dagger &= (c\mathbb{I} - T)(\bar{c}\mathbb{I} - T^\dagger) \\ &= |c|^2 \mathbb{I} - cT^\dagger - \bar{c}T + TT^\dagger \\ &= (c\mathbb{I} - T)^\dagger (c\mathbb{I} - T). \end{aligned}$$

□

**Proposition 6.14.** Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$  be a normal operator. For  $x \in \mathcal{H}$ , we have  $Tx = \lambda x$  if and only if  $T^\dagger x = \bar{\lambda}x$ .

*Proof.* Suppose  $Tx = \lambda x$ . Let  $N := \lambda\mathbb{I} - T$ , which is normal, and  $Nx = \lambda x - Tx = 0$ ,  $N^\dagger = \bar{\lambda}\mathbb{I} - T^\dagger$ . We have that

$$\begin{aligned} \|N^\dagger x\|^2 &= \langle N^\dagger x, N^\dagger x \rangle \\ &= \langle NN^\dagger x, x \rangle \\ &= \langle N^\dagger Nx, x \rangle \\ &= \langle N^\dagger 0, x \rangle \\ &= 0. \end{aligned}$$

Thus  $N^\dagger x = 0$ , which means  $\bar{\lambda}x - T^\dagger x = 0$ .

The converse is true by  $(T^\dagger)^\dagger = T$ .

□

**Definition 6.4.** Let  $\mathcal{H}$  be a Hilbert space. A bounded linear operator  $T \in B(\mathcal{H})$  is **self-adjoint** if  $T^\dagger = T$ . The set of all self-adjoint operators is  $B(\mathcal{H})_{sa}$ .

**Definition 6.5.** Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$ , we define

$$\Re(T) := \frac{T + T^\dagger}{2}, \quad \Im(T) := \frac{T - T^\dagger}{2i}.$$

**Proposition 6.15.** Let  $\mathcal{H}$  be a Hilbert space.  $B(\mathcal{H})_{sa}$  is a  $\mathbb{R}$  subspace of  $B(\mathcal{H})$ . We have  $\Re(T), \Im(T) \in B(\mathcal{H})_{sa}$ , and  $\Re(T) + i\Im(T) = T$ , so

$$B(\mathcal{H}) = B(\mathcal{H})_{sa} + iB(\mathcal{H})_{sa}.$$

**Proposition 6.16.** Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})_{sa}$ . We have  $\langle Tx, x \rangle \in \mathbb{R}$  for any  $x \in \mathcal{H}$ .

*Proof.* We have

$$\begin{aligned} \langle Tx, x \rangle &= \langle x, T^\dagger x \rangle \\ &= \langle x, Tx \rangle \\ &= \overline{\langle x, Tx \rangle}. \end{aligned}$$

□

**Proposition 6.17.** Let  $\mathcal{H}$  be a Hilbert space. Any self-adjoint operator  $T$  is normal.

*Proof.*  $T^\dagger T = TT = TT^\dagger$ . □

**Proposition 6.18.** Let  $\mathcal{H}$  be a Hilbert space, for any  $T \in B(\mathcal{H})$ , we have that  $T^\dagger T, TT^\dagger \in B(\mathcal{H})_{sa}$ .

*Proof.* We have  $(T^\dagger T)^\dagger = T^\dagger (T^\dagger)^\dagger = T^\dagger T$ , so it is self-adjoint. Similarly,  $(TT^\dagger)^\dagger = (T^\dagger)^\dagger T^\dagger = TT^\dagger$ . □

**Lemma 6.19.** Let  $\mathcal{H}$  be a Hilbert space, for any  $T \in B(\mathcal{H})_{sa}$ , we have that  $\|T^2\| = \|T\|^2$ .

*Proof.*  $\|T^2\| = \|TT^\dagger\| = \|T\|^2$ . □

**Proposition 6.20.** Let  $\mathcal{H}$  be a Hilbert space, for any normal  $T \in B(\mathcal{H})$ , we have that  $\|T^2\| = \|T\|^2$ .

*Proof.* Since  $T^\dagger T$  is self-adjoint, we have  $\|(T^\dagger T)^2\| = \|T^\dagger T\|^2$ . Thus,

$$\begin{aligned} \|T^2\| &= \|(T^2)^\dagger T^2\|^{\frac{1}{2}} \\ &= \|T^\dagger T^\dagger T T\|^{\frac{1}{2}} \\ &= \|T^\dagger T T^\dagger T\|^{\frac{1}{2}} \\ &= \|(T^\dagger T)^2\|^{\frac{1}{2}} \\ &= (\|T^\dagger T\|^2)^{\frac{1}{2}} \\ &= \|T^\dagger T\| \\ &= \|T\|^2. \end{aligned}$$

□

**Definition 6.6.** Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})_{sa}$ . We say  $T$  is **positive semidefinite**, written as  $T \geq 0$ , if

$$\forall x \in \mathcal{H}, \langle Tx, x \rangle \geq 0.$$

For  $S, T \in B(\mathcal{H})_{sa}$ , we say  $S \leq T$  if  $T - S \geq 0$ . We also define  $B(\mathcal{H})^+$  to be the set of all positive semidefinite operators.

**Proposition 6.21.** Let  $\mathcal{H}$  be a Hilbert space, for any  $T \in B(\mathcal{H})$ , we have that  $T^\dagger T \geq 0$ .

*Proof.* It is self-adjoint by above proposition. Consider any  $x \in \mathcal{H}$ , we have

$$\begin{aligned} \langle T^\dagger T x, x \rangle &= \langle T x, T x \rangle \\ &= \|T x\|^2 \\ &\geq 0. \end{aligned}$$

□

**Proposition 6.22.** Let  $\mathcal{H}$  be a Hilbert space, and  $T \geq 0$ . We have that  $M : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  by  $M[x, y] := \langle T x, y \rangle$  is a positive semidefinite form on  $\mathcal{H}$ .

*Proof.* 1. conjugate symmetry; i.e.  $M[x, y] = \langle T x, y \rangle = \langle x, T y \rangle = \overline{\langle T y, x \rangle} = \overline{M[y, x]}$ .

2. linearity in the second argument; i.e.  $M[x, ay + z] = \langle T x, ay + z \rangle = a \langle T x, y \rangle + \langle T x, z \rangle = M[x, y] + aM[x, z]$ .

3. positive semi-definiteness; i.e.  $M[x, x] = \langle T x, x \rangle \geq 0$ . □

**Proposition 6.23.** Let  $\mathcal{H}$  be a Hilbert space, and  $S, T \in B(\mathcal{H})_{sa}$ .

1. Suppose  $S \leq T$ , then  $A^\dagger S A \leq A^\dagger T A$  for all  $A \in B(\mathcal{H})$ .
2. Suppose  $0 \leq S \leq T$ , then  $\|S\| \leq \|T\|$ .
3.  $-\|T\|\mathbb{I} \leq T \leq \|T\|\mathbb{I}$ .
4. Suppose  $-C\mathbb{I} \leq T \leq C\mathbb{I}$  for some  $C \geq 0$ , then  $\|T\| \leq C$ .
5. Suppose  $0 \leq S$ , we have  $aS^n \geq 0$  for any non-negative  $a \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ .

*Proof.* 1. For any  $x \in \mathcal{H}$ , we have  $\langle (T - S)Ax, Ax \rangle \geq 0$ , since  $T - S \geq 0$ . Thus,  $\langle A^\dagger(T - S)Ax, x \rangle \geq 0$ , which means  $A^\dagger T A - A^\dagger S A = A^\dagger(T - S)A \geq 0$ .

2. Consider  $M[x, y] := \langle Sx, y \rangle$ , which defines a positive semidefinite form 4.1 on  $\mathcal{H}$ . By Cauchy-Schwartz theorem 4.3, we have

$$\begin{aligned} |\langle Sx, y \rangle|^2 &\leq \langle Sx, x \rangle \langle Sy, y \rangle \\ &\leq \langle Tx, x \rangle \langle Ty, y \rangle \\ &\leq \|T\| \|x\|^2 \|T\| \|y\|^2 \\ &\leq \|T\|^2 \|x\|^2 \|y\|^2. \end{aligned}$$

Thus  $|\langle Sx, y \rangle| \leq \|T\|$  for any  $\|x\|, \|y\| \leq 1$ , which means  $\|S\| \leq \|T\|$ .

3. Since  $|\langle Tx, x \rangle| \leq \|T\| \|x\|^2$ , we have

$$\begin{aligned} -\|T\| \|x\|^2 &\leq \langle Tx, x \rangle \leq \|T\| \|x\|^2 \\ -\|T\| \|x\| &\leq \langle Tx, x \rangle \leq \|T\| \|x\|. \end{aligned}$$

4. For any  $x, y \in \mathcal{H}$ , we have  $\langle T(x + y), x + y \rangle = \langle Tx, x \rangle + 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$  and  $\langle T(x - y), x - y \rangle = \langle Tx, x \rangle - 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$  by proposition 4.1, so

$$\langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle = 4\Re(\langle Ty, x \rangle).$$

Similarly,  $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ .

By assumption, we have

$$\langle T(x + y), x + y \rangle \leq \langle C\mathbb{I}(x + y), x + y \rangle = C\|x + y\|^2$$

since  $T \leq C\mathbb{I}$ , and

$$\langle T(x - y), x - y \rangle \geq \langle -C\mathbb{I}(x - y), x - y \rangle = -C\|x - y\|^2$$

since  $T \geq -C\mathbb{I}$ . Thus we have

$$\begin{aligned} \langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle &\leq C\|x + y\|^2 - (-C\|x - y\|^2) \\ &= C(\|x + y\|^2 + \|x - y\|^2) \\ 4\Re(\langle Ty, x \rangle) &\leq 2C(\|x\|^2 + \|y\|^2). \end{aligned}$$

Taking supreme over all  $\|x\|, \|y\| \leq 1$ , we have  $4\|T\| \leq 4C$ .

5. It is obviously true when  $n = 0, 1$ . Now assume  $n \geq 2$ , and it already holds for  $n - 2$ . Clearly  $aS^n$  is self-adjoint. For any  $x \in \mathcal{H}$ , we have

$$\begin{aligned} \langle aS^n x, x \rangle &= a \langle S^{n-2} Sx, x \rangle \\ &= a \langle S^{n-2} Sx, S^\dagger x \rangle \\ &= a \langle S^{n-2}(Sx), Sx \rangle \\ &\geq 0. \end{aligned}$$

By induction, it holds for all  $n \in \mathbb{N}$ .

□

**Theorem 6.24** (Dini's). *Let  $K$  be compact, and  $f \in C(K)$ . Suppose there are  $(g_n)_{n=1}^\infty \subseteq C(K)$  such that they are monotone increasing, and for all  $x \in K$ ,  $g_n(x) \nearrow f(x)$  pointwise, then  $g_n \rightarrow f$  uniformly.*

*Proof.* Let  $g_n := f - q_n$ , we have that  $g_n(x) \searrow 0$  monotonously pointwise. Given any  $\epsilon > 0$ , for each  $n \geq 1$ , define

$$U_n := \{x \in K : g_n(x) < \epsilon\},$$

which are open in  $K$ . We have that  $K = \bigcup_{n=1}^\infty U_n$  is an open cover. Since  $K$  is compact, there is  $N \geq 1$ , such that  $K = \bigcup_{n=1}^N U_n$ . Since  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_N$ , we have that  $K = U_N$ . For any  $n \geq N$ , we have that  $K = U_N \subseteq U_n \subseteq K$ , so  $K = U_n$ . This show that  $\forall x \in K$ ,

$$f(x) - q_n(x) = g_n(x) < \epsilon.$$

□

**Lemma 6.25.** *For  $f \in C([0, 1])$ , with  $f(t) := 1 - \sqrt{1-t}$ , there are polynomials  $(p_n)_{n=1}^\infty$ , whose coefficients are all non-negative, then  $f = \sum_{n=1}^\infty p_n$  uniformly.*

*Proof.* Let  $q_0(t) := 0$ , and  $q_n(t) := \frac{1}{2}(t + q_{n-1}(t)^2)$ . By induction,  $0 \leq q_n \leq 1$ , and  $q_n$  has non-negative coefficients. Also,  $q_n \leq q_{n+1}$ .

Define  $p_n := q_n - q_{n-1}$  for all  $n \geq 1$ .

$$\begin{aligned} p_n(q_n + q_{n-1}) &= (q_n - q_{n-1})(q_n + q_{n-1}) \\ &= q_n^2 - q_{n-1}^2 \\ &= 2q_{n+1} - t - (1q_n - t) \\ &= 2(q_{n+1} - q_n) \\ &= 2p_{n+1}. \end{aligned}$$

By induction, we have that all  $p_n$  have non-negative coefficient.

For any  $t \in [0, 1]$ , since  $q_n(t)$  is monotone increasing, there is some  $q(t) := \lim_{n \rightarrow \infty} q_n(t) \leq 1$ . Also,  $q(t) = \frac{1}{2}(t + q(t)^2)$ . Thus,  $q(t) = 1 - \sqrt{1-t} = f(t)$ . We now have

$$f(t) = \lim_{n \rightarrow \infty} q_n(t) = \sum_{n=1}^\infty p_n(t).$$

This convergence is uniform by Dini's Theorem. □

**Theorem 6.26** (Square Root Theorem). *Let  $\mathcal{H}$  be a Hilbert space, and  $0 \leq T$ , then there is a unique  $0 \leq S$ , such that  $S^2 = T$ . We write  $S := T^{\frac{1}{2}}$ , and it commutes with all operators that commute with  $T$ .*

*Proof.* Assume  $0 \leq T \leq \mathbb{I}$ . Consider  $S := \mathbb{I} - T$ . Note that  $0 \leq S \leq \mathbb{I}$ . Let  $p_n(t) := \sum_k \alpha_{n,k} t^k$  and  $q_n := \sum_{i=1}^n p_n$  be from the above lemma. For any  $n \geq 1$ , we have

$$\begin{aligned} \|p_n(S)\| &\leq \sum_k \alpha_{n,k} \|S\|^k \\ &\leq \sum_k \alpha_{n,k} \|\mathbb{I}\|^k \\ &= \sum_k \alpha_{n,k} 1^k \\ &= p_n(1). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n=1}^\infty \|p_n(S)\| &\leq \sum_{n=1}^\infty p_n(1) \\ &= f(1) \\ &= 1. \end{aligned}$$

Since  $B(\mathcal{H})$  is a Banach space, we can define

$$R := \sum_{n=1}^{\infty} p_n(S) = \lim_{n \rightarrow \infty} q_n(S)$$

by A2Q4. Also,  $\|R\| \leq 1$ , so  $R \leq \mathbb{I}$ . Also, since each coefficient in  $q_n$  is non-negative, we have that each  $q_n(S) \geq 0$ . Thus,  $R \geq 0$ . We now have

$$\begin{aligned} (\mathbb{I} - R)^2 &= (\mathbb{I} - \lim_{n \rightarrow \infty} q_n(S))^2 \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - q_n(S))^2 \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - 2q_n(S) + q_n(S)^2) \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - 2q_n(S) + 2q_{n+1}(S) - S) \\ &= \mathbb{I} - S - 2 \lim_{n \rightarrow \infty} q_n(S) + 2 \lim_{n \rightarrow \infty} q_{n+1}(S) \\ &= T - 2R + 2R \\ &= T. \end{aligned}$$

We can thus define  $T^{\frac{1}{2}} := \mathbb{I} - R$ .

For any  $A$  that commutes with  $T$ , we have

$$SA = A(\mathbb{I} - T) = A - AT = A - TA = (\mathbb{I} - T)A = SA.$$

Also, for any  $k \geq 1$ , we have

$$S^k A = S^{k-1} AS = S^{k-2} AS^2 = \dots = AS^k.$$

Thus,  $Aq_n(S) = q_n(S)A$ , so  $AR = RA$ . This shows

$$AT^{\frac{1}{2}} = A(\mathbb{I} - R) = A - AR = A - RA = (\mathbb{I} - R)A = T^{\frac{1}{2}}A.$$

Now for any general  $T$ , we can just take  $\tilde{T} := \frac{1}{\|T\|}T$ , so we have  $\tilde{T} \leq \frac{1}{\|T\|}\|T\|\mathbb{I} = \mathbb{I}$ . Now take  $T^{\frac{1}{2}} := \sqrt{\|T\|}\tilde{T}^{\frac{1}{2}}$ . Clearly,  $T^{\frac{1}{2}}$  is self-adjoint, and

$$T^{\frac{1}{2}}T^{\frac{1}{2}} = \sqrt{\|T\|}\tilde{T}^{\frac{1}{2}}\sqrt{\|T\|}\tilde{T}^{\frac{1}{2}} = \|T\|\tilde{T} = T.$$

Also, if  $AT = TA$ , we will have  $A\tilde{T} = \tilde{T}A$ , so  $A\tilde{T}^{\frac{1}{2}} = \tilde{T}^{\frac{1}{2}}A$  and  $AT^{\frac{1}{2}} = T^{\frac{1}{2}}A$ .

Suppose there is some  $A \geq 0$ , such that  $A^2 = T$ , we have  $AT = A^3 = TA$ . Since  $A$  commutes with  $T$ , it also commutes with  $T^{\frac{1}{2}}$ . We thus have

$$(T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A) = T - A^2 = 0.$$

Now let  $Y := \text{Im}(T^{\frac{1}{2}} + A)$ . For any  $y \in Y$ , we have  $(T^{\frac{1}{2}} - A)(y) = 0$ , since there will be  $x \in \mathcal{H}$ , such that  $(T^{\frac{1}{2}} - A)(x) = y$ , so

$$(T^{\frac{1}{2}} - A)(y) = (T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A)(x) = 0.$$

On the other hand, for any  $z \in Y^{\perp} = \ker((T^{\frac{1}{2}} + A)^{\dagger}) = \ker(T^{\frac{1}{2}} - A)$  by proposition 6.10, we have that

$$\begin{aligned} \|T^{\frac{1}{4}}z\|^2 &= \langle T^{\frac{1}{4}}z, T^{\frac{1}{4}}z \rangle \\ &= \langle z, T^{\frac{1}{4}}T^{\frac{1}{4}}z \rangle \\ &= \langle z, T^{\frac{1}{2}}z \rangle \\ &\leq \langle z, T^{\frac{1}{2}}z \rangle + \langle z, Az \rangle \\ &= \langle z, (T^{\frac{1}{2}} + A)z \rangle \\ &= 0. \end{aligned}$$

Thus,  $T^{\frac{1}{4}}z = 0$ , and  $T^{\frac{1}{2}}z = T^{\frac{1}{4}}T^{\frac{1}{4}}z = 0$ . Similarly,

$$\begin{aligned}\|A^{\frac{1}{2}}z\|^2 &= \langle A^{\frac{1}{2}}z, A^{\frac{1}{2}}z \rangle \\ &= \langle z, Az \rangle \\ &\leq \langle z, (A + T^{\frac{1}{2}})z \rangle \\ &= 0,\end{aligned}$$

so  $A^{\frac{1}{2}}z = 0$  and  $Az = 0$ . Thus,

$$(T^{\frac{1}{2}} - A)(z) = T^{\frac{1}{2}}z - Az = 0.$$

Now for any  $x \in \mathcal{H}$ , there is  $y \in Y$ , and  $z \in Y^\perp$ , such that  $x = y + z$ , so we have

$$(A - T^{\frac{1}{2}})(x) = (A - T^{\frac{1}{2}})(y) + (A - T^{\frac{1}{2}})(z) = 0.$$

This shows  $T^{\frac{1}{2}} = A$ , which means  $T^{\frac{1}{2}}$  is unique. □

**Corollary 6.27.** *Let  $\mathcal{H}$  be a Hilbert space. If  $S, T \geq 0$ , and  $ST = TS$ , then  $ST \geq 0$ .*

*Proof.* We have  $(ST)^\dagger = T^\dagger S^\dagger = TS = ST$ , so  $ST \in B(\mathcal{H})_{sa}$ .

Also, for any  $x \in \mathcal{H}$ , we have

$$\begin{aligned}\langle STx, x \rangle &= \langle S^{\frac{1}{2}}S^{\frac{1}{2}}Tx, x \rangle \\ &= \langle S^{\frac{1}{2}}Tx, (S^{\frac{1}{2}})^\dagger x \rangle \\ &= \langle TS^{\frac{1}{2}}x, S^{\frac{1}{2}}x \rangle \\ &\geq 0.\end{aligned}$$

□

**Corollary 6.28.** *Let  $\mathcal{H}$  be a Hilbert space. Suppose  $T \in B(\mathcal{H})_{sa}$ , and  $\langle Tx, x \rangle = 0$  for all  $x \in \mathcal{H}$ , then  $T = 0$ .*

*Proof.* For any  $x \in \mathcal{H}$ , we have  $\langle Tx, x \rangle = 0 \geq 0$ , so  $T \geq 0$ . In addition,

$$\begin{aligned}0 &= \langle Tx, x \rangle \\ &= \langle T^{\frac{1}{2}}T^{\frac{1}{2}}x, x \rangle \\ &= \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x \rangle \\ &= \|T^{\frac{1}{2}}x\|^2.\end{aligned}$$

Thus  $T^{\frac{1}{2}}x = 0$ , so  $Tx = T^{\frac{1}{2}}T^{\frac{1}{2}}x = T^{\frac{1}{2}}0 = 0$ . □

**Proposition 6.29.** *Let  $\mathcal{H}$  be a Hilbert space, and  $U \in B(\mathcal{H})$ .  $U$  is an isometry if and only if  $U^\dagger U = \mathbb{I}$ . Also, the following are equivalent:*

1.  $U$  is unitary,
2.  $U^\dagger = U^{-1}$ , and
3.  $U$  is a normal isometry.

*Proof.* See A5. □

## 6.2 Compact Operators

### 6.2.1 Compactness

See more in Real Analysis Pmath351 or Applied Functional Analysis Amath731.

**Definition 6.7.** Let  $(X, d)$  be a metric space. We say  $S \subseteq X$  is **sequentially compact** if for every bounded sequence  $(x_n)_{n=1}^\infty$ , there is a convergent subsequence  $x_{n_k} \rightarrow x \in S$ .

**Definition 6.8.** Let  $(X, d)$  be a metric space. We say  $S \subseteq X$  is **relatively compact** if for every bounded sequence  $(x_n)_{n=1}^\infty$ , there is a convergent subsequence  $x_{n_k} \rightarrow x \in X$ .

**Definition 6.9.** Let  $X$  be a metric space. We say  $S \subseteq X$  is **totally bounded** if for any  $\epsilon > 0$ , there exists a finite subset  $\{x_i\}_{i=1}^n$ , such that  $S \subseteq \bigcup_{i=1}^n B(a_i, \epsilon)$ . Such a subset is called an  $\epsilon$ -**net**.

**Definition 6.10.** Let  $X$  be a metric space. We say  $S \subseteq X$  is **bounded** if there exists a  $r > 0$ , such that for all  $x, y \in X$ ,  $d(x, y) < r$ .

**Theorem 6.30.** Let  $S$  be a subset of a metric space  $X$ . The following are equivalent:

1.  $S$  is compact.
2.  $S$  has the finite intersection property.
3.  $S$  is sequentially compact.
4. Every infinite subset  $A \subseteq S$  has a limit point.
5.  $S$  is complete and totally bounded.
6.  $S$  is relatively compact and closed.

**Corollary 6.31.** Let  $S$  be a subset of a metric space  $X$ , then  $S$  is relatively compact if and only if  $\bar{S}$  is compact.

**Corollary 6.32.** Let  $S$  be a subset of a complete space  $X$ , then  $S$  is relatively compact if and only if  $S$  is totally bounded.

**Proposition 6.33.** Let  $S$  be a subset of a metric space  $X$ .

1. Suppose  $S$  is relatively compact, then  $S$  is bounded.
2. Suppose  $S$  is totally bounded, then  $S$  is bounded.
3. Suppose  $S$  is compact, then  $S$  is closed and bounded.

**Theorem 6.34** (Heine-Borel). Let  $S$  be a subset of a finite dimensional normed vector space  $X$ , then  $S$  is compact if and only if  $S$  is closed and bounded.

### 6.2.2 Compact Operators

**Definition 6.11.** Let  $X, Y$  be normed vector spaces. A linear operator  $K : X \rightarrow Y$  is **compact** if  $\overline{K(B^X(0, 1))}$  is compact in  $Y$ . Let  $\mathcal{K}(X, Y)$  denote the set of compact operators.

**Proposition 6.35.** Let  $X, Y$  be normed vector spaces, and  $K : X \rightarrow Y$  be a linear operator, then the following are equivalent:

1.  $K$  is compact.
2.  $K(B^X(0, 1))$  is relative compact in  $Y$ .
3. For any bounded set  $S \subseteq X$ ,  $\overline{K(S)}$  is compact in  $Y$ .
4. For any bounded set  $S \subseteq X$ ,  $K(S)$  is relative compact in  $Y$ .



5. Each bounded sequence  $(x_n)_{n=1}^\infty \subseteq X$  has some subsequence  $(x_{n_k})_{k=1}^\infty$  such that  $(Kx_{n_k})_{k=1}^\infty$  converges to some  $y \in Y$ .

**Proposition 6.36.** Let  $X, Y$  be normed vector spaces. Suppose a linear operator  $K : X \rightarrow Y$  is compact, then it is always bounded. Namely,  $\mathcal{K}(X, Y) \subseteq B(X, Y)$ .

**Proposition 6.37.** Let  $X$  be an infinite-dimensional Banach space, then the identity map  $\mathbb{I} : X \rightarrow X$  is not compact.

*Proof.*  $\bar{B}^X(0, 1)$  is not compact by A2. □

**Proposition 6.38.** Let  $X$  be a Banach space. Suppose  $E = E^2 \in B(X)$  is compact, then  $E$  is finite rank.

*Proof.* See A5. □

**Definition 6.12.** Let  $X, Y$  be vector spaces. Let

$$\mathcal{F}(X, Y) : \{T \in B(X, Y) \mid \text{rank}(T) < \infty\}$$

denote the set of finite rank bounded linear operators.

**Proposition 6.39.** Let  $X, Y$  be normed vector spaces. Suppose a linear operator  $T \in B(X, Y)$  is finite-dimensional, then it is always compact. Namely,  $\mathcal{F}(X, Y) \subseteq \mathcal{K}(X, Y)$ .

*Proof.*  $\text{Im}(K) \subseteq Y$  is finite dimensional. For any bounded  $M \subseteq X$ , we have that  $K(M)$  is bounded. Since  $\overline{K(M)} \subseteq \text{Im}(T)$  is closed and bounded, it is compact in  $\text{Im}(T)$ , thus compact in  $Y$ . □

**Theorem 6.40.** Let  $X, Y$  be Banach Spaces.  $\mathcal{K}(X, Y)$  is norm-closed  $B(X) - B(Y)$  sub-bi-module of  $B(X, Y)$ . Namely, for any  $a \in B(Y), b \in B(X)$ , and  $K \in \mathcal{K}(X, Y)$ , we have  $aKb \in \mathcal{K}(X, Y)$ . Also, if  $(K_n)_{n=1}^\infty$  is a sequence of compact operators in  $\mathcal{K}(X, Y)$ , and  $K_n \rightarrow K$  in  $B(X, Y)$ , we have  $K \in \mathcal{K}(X, Y)$  as well.

*Proof.* Clearly  $\mathcal{K}(X, Y)$  is a vector space.

Since  $b \in B(X)$ , we have  $b(B^X(0, 1)) \subseteq B^X(0, \|b\|)$ , which is bounded. Since  $K$  is compact,  $\overline{K(b(B^X(0, 1)))}$  is compact. Thus,  $\overline{Kb(B^X(0, 1))} \subseteq \overline{K(B^X(0, \|b\|))}$  is compact since it is closed. Since  $a \in B(Y)$  is continuous, we have that  $a(\overline{Kb(B^X(0, 1))})$  is compact as well. Thus  $\overline{aKb(B^X(0, 1))} \subseteq \overline{a(\overline{Kb(B^X(0, 1))})} = \overline{a(Kb(B^X(0, 1)))}$  is compact, since it is closed.

This shows  $aTb$  is compact.

Now consider any sequence of compact operators  $(K_n)_{n=1}^\infty$  in  $\mathcal{K}(X, Y)$ , with  $K_n \rightarrow K$  in  $B(X, Y)$ . Given any  $\epsilon > 0$ , there is  $N \geq 1$  such that  $\|K - K_N\| < \frac{\epsilon}{3}$ . Since  $K_N$  is compact,  $\overline{K_N(B^X(0, 1))}$  is compact and thus totally bounded, so we can find  $\{x_1, \dots, x_m\} \subseteq B^X(0, 1)$ , such that  $\overline{K_N(B^X(0, 1))} \subseteq \bigcup_{i=1}^m B^X(x_i, \frac{\epsilon}{3})$ . Now for any  $x \in B^X(0, 1)$ , consider  $K_N x \in B^X(x_i, \frac{\epsilon}{3})$ , we have

$$\begin{aligned} \|Kx - Kx_i\| &\leq \|Kx - K_N x\| + \|K_N x - K_N x_i\| + \|K_N x_i - Kx_i\| \\ &< \|K - K_N\| \|x\| + \frac{\epsilon}{3} + \|K_N - K\| \|x_i\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

This shows that  $K(B^X(0, 1)) \subseteq \bigcup_{i=1}^m B^X(x_i, \epsilon)$ , so it is totally bounded and thus relatively compact. This proves  $K$  is compact, so  $\mathcal{K}(X, Y)$  is norm-closed. □

**Proposition 6.41.** Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces. We have

$$\overline{\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)} = \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

*Proof.* Let  $K \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ , and  $\epsilon > 0$ . We can find an  $\epsilon$ -net  $\{Kx_1, \dots, Kx_m\}$  for  $K(B^{\mathcal{H}_1}(0, 1))$ , since it is totally bounded.

Let  $K_0 := \text{Span}\{Kx_1, \dots, Kx_m\} \subseteq \mathcal{H}_2$ , and consider  $P : \mathcal{H}_2 \rightarrow K_0$ , the orthogonal projection onto  $K_0$ . For any  $x \in B^{\mathcal{H}_1}(0, 1)$ , we have

$$\begin{aligned} \|(K - PK)x\| &= \|(\mathbb{I} - P)Kx\| \\ &= \text{dist}(Kx, K_0) \\ &\leq \min_{i \in [m]} \|Kx - Kx_i\| \\ &< \epsilon. \end{aligned}$$

Thus,  $\|K - PK\| < \epsilon$ , where  $\text{rank}(PK) \leq \dim(K_0) \leq m < \infty$ .  $\square$

**Corollary 6.42.** *Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces. For  $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ , we have  $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$  if and only if  $T^\dagger \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ .*

*Proof.* By finite-dimensional approximation.  $\square$

**Example 6.2.1.** Consider  $f \in \ell^\infty(\mathbb{N})$ , and the multiplication operator  $M_f : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ . We have that  $M_f$  is compact if and only if  $f \in C_0(\mathbb{N})$ , namely,  $\lim_{i \rightarrow \infty} f(i) = 0$ .

Indeed, if  $f \in C_0(\mathbb{N})$ , we can let  $f_N(i) := \begin{cases} f(i), & i \leq N \\ 0, & \text{o.w.} \end{cases}$ . We have  $\|M_f - M_{f_N}\| = \|M_{f-f_N}\| = \|f - f_N\|_\infty \rightarrow 0$ .

On the other hand, if  $f \notin C_0(\mathbb{N})$ , there is  $\epsilon > 0$ , and  $i_1 < i_2 < i_3 < \dots$ , such that  $|f(i_k)| \geq \epsilon$ . Let  $K_0 := \overline{\text{Span}\{e_{i_k}\}} \subseteq \ell^2(\mathbb{N})$ , we have  $M_f(e_{i_k})(i) = \begin{cases} f(i), & i = i_k \\ 0, & \text{o.w.} \end{cases}$ , so  $M_f(e_{i_k}) = f(i_k)e_{i_k} \in K_0$ . Thus,  $K := M_f|_{K_0} : K_0 \rightarrow K_0$  is well defined. In addition,  $K$  is bounded below by  $\epsilon$ . By theorem 3.12, it has a bounded inverse  $K^{-1}$ .

Suppose for contradiction that  $M_f$  is compact, then  $\overline{K(B^{K_0}(0, 1))} \subseteq \overline{M_f(B^{\ell^2(\mathbb{N})}(0, 1))}$ . Since  $K^{-1}$  is bounded,  $K^{-1}\overline{K(B^{K_0}(0, 1))}$  is compact as well. Now

$$\overline{B^{K_0}(0, 1)} = \overline{K^{-1}K(B^{K_0}(0, 1))} \subseteq \overline{K^{-1}\overline{K(B^{K_0}(0, 1))}} = \overline{K^{-1}K(B^{K_0}(0, 1))}$$

is compact. Since  $K_0$  is infinite dimensional,  $\overline{B^{K_0}(0, 1)}$  cannot be compact, a contradiction.

### 6.2.3 Integral Operators

**Definition 6.13.** Let  $k \in L^2([0, 1] \times [0, 1])$ , the **integral operator** with **kernel**  $k$  is  $T_k \in B(L^2(0, 1))$ , with

$$(T_k f)(x) := \int_0^1 k(x, y)f(y)dy.$$

**Lemma 6.43.** *Let  $K \in L^2([0, 1] \times [0, 1])$ , the integral operator with kernel  $K$  is well-defined.*

*Proof.* Consider any  $f \in L^2(0, 1)$ ,

$$\begin{aligned} \|T_k f\|_2^2 &= \int_0^1 \left| \int_0^1 k(x, y)f(y)dy \right|^2 dx \\ &\leq \int_0^1 \left( \int_0^1 |k(x, y)|^2 dy \right) \|f\|_2^2 dx \\ &= \|f\|_2^2 \|k\|_2^2 \\ &< \infty. \end{aligned}$$

Thus,  $T_k f \in L^2(0, 1)$ .  $\square$

**Proposition 6.44.** *Let  $k \in L^2([0, 1] \times [0, 1])$ , then  $T_k$  is compact.*

*Proof.* Pick any orthonormal basis  $(e_i)_{i=1}^\infty$  for  $L^2(0, 1)$ , then  $e_{ij}(x, y) := e_i(x)e_j(y)$  forms an orthonormal basis for  $L^2([0, 1] \times [0, 1])$ .

Thus,  $k = \sum_{i,j=1}^\infty a^{ij} e_{ij}$ , where  $\|k\|_2^2 = \sum_{i,j=1}^\infty |a_{ij}|^2$ . Let  $k_N := \sum_{i,j=1}^N a^{ij} e_{ij}$ . Thus,

$$\|T_k - T_{k_N}\|^2 \leq \|k - k_N\|_2^2 = \sum_{i,j=N+1}^\infty |a_{ij}|^2 \rightarrow 0.$$

Now for any  $f \in L^2(0, 1)$ , we have

$$\begin{aligned} (T_{k_N} f)(x) &= \int_0^1 k_N(x, y) f(y) dy \\ &= \langle \bar{k}_N(x, \cdot), f \rangle \\ &= \left\langle \sum_{i,j=1}^N a^{ij} \bar{e}_{ij}(x, \cdot), f \right\rangle \\ &= \sum_{i,j=1}^N a^{ij} e_i(x) \langle \bar{e}_j, f \rangle \\ &= \sum_{i,j=1}^N a^{ij} \langle \bar{e}_j, f \rangle e_i(x) \\ T_{k_N} f &= \sum_{i,j=1}^N a_{ij} \langle \bar{e}_j, f \rangle e_i \\ &\in \text{Span} \{e_1, \dots, e_N\}. \end{aligned}$$

Thus,  $T_{k_N}$  is finite rank and thus compact, which means  $T_k$  is compact as well. □

**Proposition 6.45.** *Let  $k \in L^2([0, 1] \times [0, 1])$ , then*

$$(T_k^\dagger g)(x) = \int_0^1 k^*(x, y) g(y) dy,$$

where  $k^*(x, y) := \overline{k(y, x)}$ .

*Proof.* Consider any  $f, g \in L^2(0, 1)$ , let  $h(x) := \int_0^1 k^*(x, y) g(y) dy$ , we have

$$\begin{aligned} \langle V^* g, f \rangle &= \langle g, V f \rangle \\ &= \int_0^1 g(x) \overline{V f(x)} dx \\ &= \int_0^1 g(x) \overline{\int_0^1 k(x, y) f(y) dy} dx \\ &= \int_0^1 g(x) \int_0^1 \overline{k(x, y)} \bar{f}(y) dy dx \\ &= \int_0^1 \int_0^1 g(x) k^*(y, x) dx \bar{f}(y) dy \\ &= \int_0^1 h(y) \bar{f}(y) dy \\ &= \langle h, f \rangle. \end{aligned}$$

We can swap the order of integration by the Fubini Theorem and the fact that  $f, g \in L^2(0, 1)$ .

Thus we have  $(T_k^\dagger g)(x) = h(x) = \int_0^1 k^*(x, y) g(y) dy$ . □

**Example 6.2.2.** The **Volterra Operator** is  $V : L^2(0, 1) \rightarrow L^2(0, 1)$  by

$$(Vf)(x) := \int_0^x f(y)dy.$$

Its kernel is  $k(x, y) := \chi_{y \leq x}(x, y) := \begin{cases} 1, & y \leq x \\ 0, & \text{o.w.} \end{cases}$ . We have

$$\begin{aligned} \int_0^1 \int_0^1 |\chi_{y \leq x}(x, y)|^2 dx dy &= \int_0^1 \int_0^y 1 dx dy \\ &= \int_0^1 y dy \\ &= \frac{1}{2}, \end{aligned}$$

so  $k \in L^2([0, 1] \times [0, 1])$  by the Fubini Theorem. Also,

$$\begin{aligned} (T_k f)(x) &= \int_0^1 k(x, y) f(y) dy \\ &= \int_0^1 \chi_{y \leq x}(x, y) f(y) dy \\ &= \int_0^x f(y) dy \\ &= (Vf)(x). \end{aligned}$$

See more in A5Q6.

### 6.3 Spectrum

**Definition 6.14.** Let  $X$  be a Banach space, and  $T \in B(X)$ . The **spectrum** of  $T$  is

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda \mathbb{I} - T \text{ is not invertible.}\}.$$

The **point spectrum** of  $T$  is

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda \mathbb{I} - T \text{ is not injective.}\} = \{\lambda \in \mathbb{C} : \exists v \neq 0 \text{ such that } Tv = \lambda v\} = \{\text{eigenvalues of } T\}.$$

The **resolvent** of  $T$  is

$$\rho(T) := \mathbb{C} \setminus \sigma(T).$$

**Proposition 6.46.** When  $X$  is finite dimensional, then

$$\sigma(T) = \{\lambda : \lambda \mathbb{I} - T \text{ is not injective}\} = \sigma_p(T).$$

**Example 6.3.1.** Consider  $f(n) := \frac{1}{n} \in \ell^\infty$ , and the multiplication operator  $M_f$  we have that  $0 \in \sigma(M_f)$ , but not an eigenvalue.

**Example 6.3.2.** Let  $f \in C([0, 1])$ , and the multiplication operator  $M_f : L^2(0, 1) \rightarrow L^2(0, 1)$  by  $(M_f(\xi))(x) := f(x)\xi(x)$ , where we have  $M_f M_g = M_{fg}$ ,  $M_f^\dagger = M_{\bar{f}}$ ,  $\|M_f\| = \|f\|_\infty$ .

Suppose  $\lambda \notin \text{Im}(f)$ , then  $\lambda - f \neq 0$  on  $[0, 1]$ , so  $\frac{1}{\lambda - f} \in C([0, 1])$ . We thus have

$$M_{\frac{1}{\lambda - f}} = (M_{\lambda - f})^{-1} = (\lambda \mathbb{I} - M_f)^{-1}.$$

This shows  $\sigma(M_f) \subseteq \text{Im}(f)$ .

On the other hand, suppose  $\lambda = f(x_0)$  for some  $x_0 \in [0, 1]$ , we have that  $(\lambda - M_f)(\xi)(x) = (f(x_0) - f(x))\xi(x)$ . Now for any  $\epsilon > 0$ , since  $f$  is continuous, we have that

$$A_\epsilon := \{x : |f(x) - f(x_0)| < \epsilon\}$$

has positive measure. Let  $\xi_\epsilon := \frac{\chi_{A_\epsilon}}{\mu(A_\epsilon)^{\frac{1}{2}}}$ , with  $\|\xi_\epsilon\|_2 = 1$ , we have that

$$\begin{aligned} \|(\lambda \mathbb{I} - M_f)\xi_\epsilon\|_2^2 &= \int_{A_\epsilon} |\lambda - f(x)|^2 \frac{1}{\mu(A_\epsilon)} dx \\ &< \frac{1}{\mu(A_\epsilon)} \int_{A_\epsilon} \epsilon^2 dx \\ &= \epsilon^2. \end{aligned}$$

Thus,  $\lambda \mathbb{I} - M_f$  is not bounded below. By theorem 3.12,  $\lambda \mathbb{I} - M_f$  is not invertible, so  $\lambda \in \sigma(M_f)$ .

Thus,  $\sigma(M_f) = \text{Im}(f)$ .

However, such a  $M_f$  may have no eigenvalue! Let  $f(x) = x$ , if  $t$  is an eigenvalue with eigenvector  $\xi$ , we have  $x\xi(x) = t\xi(x)$  a.e., so  $(x - t)\xi(x) = 0$  a.e., which means  $\xi(x) = 0$  a.e..

**Theorem 6.47.** *Let  $X$  be a Banach space, and  $T \in B(X)$ . We have  $\sigma(T) \neq \emptyset$ , and it is always compact.*

*Proof.* For any  $|\lambda| > \|T\|$ , we have that  $\|\frac{T}{\lambda}\| < 1$ , so  $(\mathbb{I} - \frac{T}{\lambda})^{-1} = \sum_{k=0}^{\infty} (\frac{T}{\lambda})^k$ , and

$$(\lambda \mathbb{I} - T)^{-1} = \left( \lambda \left( \mathbb{I} - \frac{T}{\lambda} \right) \right)^{-1} = \lambda^{-1} \left( \mathbb{I} - \frac{T}{\lambda} \right)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}.$$

Thus,  $\lambda \in \rho(T)$ . This shows that  $\forall \lambda \in \sigma(T)$ ,  $|\lambda| \leq \|T\|$ , which means  $\sigma(T)$  is bounded.

Take any  $\phi \in B(\mathcal{H})^*$ , we can define  $f_\phi : \rho(T) \rightarrow \mathbb{C}$  by  $f_\phi(z) := \phi((z\mathbb{I} - T)^{-1})$ , for any  $z \in \rho(T) = \mathbb{C}$ . Fix  $z_0 \in \mathbb{C}$ , and pick  $|h| < \|(z_0\mathbb{I} - T)^{-1}\|^{-1}$ , we have  $\|h(z_0\mathbb{I} - T)^{-1}\| < 1$ , so  $(\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})^{-1} = \sum_{k=0}^{\infty} (h(z_0\mathbb{I} - T)^{-1})^k$ .

$$\begin{aligned} f_\phi(z_0 + h) &= \phi((z_0\mathbb{I} + h\mathbb{I} - T)^{-1}) \\ &= \phi(((\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})(z_0\mathbb{I} - T))^{-1}) \\ &= \phi\left((z_0\mathbb{I} - T)^{-1} \sum_{k=0}^{\infty} (h(z_0\mathbb{I} - T)^{-1})^k\right) \\ &= \sum_{k=0}^{\infty} h^k \phi((z_0\mathbb{I} - T)^{-1} ((z_0\mathbb{I} - T)^{-1})^k) \\ &= \sum_{k=0}^{\infty} h^k \phi((z_0\mathbb{I} - T)^{-1-k}), \end{aligned}$$

which is analytic. Now, suppose for contradiction that  $\sigma(T) = \emptyset$ , then  $f_\phi : \mathbb{C} \rightarrow \mathbb{C}$  is an entire function. By Liouville's theorem,  $f_\phi$  must be a constant function.

Also, we can see that when  $|z| \rightarrow \infty$ ,  $f_\phi(z) = \phi((z\mathbb{I} - T)^{-1}) \rightarrow 0$ . Thus,  $f_\phi$  must be constantly zero, for any  $\phi$ . However, by Hahn-Banach Theorem, there has to be some  $\phi \in B(\mathcal{H})^*$  that separates  $\{(z\mathbb{I} - T)^{-1}\} \subseteq B(\mathcal{H})$ , a contradiction.

This shows  $\sigma(T) \neq \emptyset$ .

In addition, the above argument shows that for any  $z \in \rho(T)$ ,  $(z_0\mathbb{I} + h\mathbb{I} - T)^{-1}$  is well-defined in a neighbourhood around it, so  $\rho(T)$  is open and  $\sigma(T)$  is closed.  $\square$

**Definition 6.15.** Let  $X$  be a Banach space, and  $T \in B(X)$ . The **spectral radius** of  $T$  is

$$\text{spr}(T) := \sup \{|\lambda| : \lambda \in \sigma(T)\}.$$

Notice that by the above theorem, we have  $\text{spr}(T) = \max \{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$ .

**Theorem 6.48** (Spectral mapping). *Let  $X$  be a Banach space, and  $T \in B(X)$ . For any polynomial  $p$ , we have that  $\sigma(p(T)) = p(\sigma(T))$ .*

*Proof.* Suppose  $p(x)$  is order  $n$ . Let  $q(x) = \lambda - p(x) = (\beta_1 - x) \cdots (\beta_n - x)$ , where  $\beta_i$ 's are the  $n$  roots of  $q$ .

Let  $\lambda \in \mathbb{C}$ , we have  $\lambda \in \sigma(p(T))$ , if and only if  $q(T) = \lambda - p(T) = (\beta_1 \mathbb{I} - T) \cdots (\beta_n \mathbb{I} - T)$  is not invertible, if and only if there is some  $\beta_i$ , such that  $\beta_i \mathbb{I} - T$  is not invertible, if and only if  $q(x)$  has a root  $\beta_i \in \sigma(T)$ , if and only if  $\lambda = p(\beta_i) \in p(\sigma(T))$ .  $\square$

**Theorem 6.49** (Spectral Radius Formula). *Let  $X$  be a Banach space, and  $T \in B(X)$ .*

$$\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}.$$

*Proof.* By the spectral mapping theorem,  $\sigma(T^n) = \sigma(T)^n$ . Thus,  $\|T^n\| \geq \text{spr}(T^n) = \text{spr}(T)^n$ , which means  $\text{spr}(T) \leq \|T^n\|^{\frac{1}{n}}$ . Since this holds for all  $n \in \mathbb{N}$ , we have

$$\text{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}.$$

On the other hand, consider any  $\phi \in B(X)^*$ , we have that  $f_\phi(\lambda) := \phi((\lambda \mathbb{I} - T)^{-1})$  is analytic on  $\rho(T)$  as in the proof of theorem 6.47. In addition, for  $|\lambda| > \|T\|$ , we have

$$f_\phi(\lambda) = \phi\left(\sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}\right) = \sum_{k=0}^{\infty} \frac{\phi(T^k)}{\lambda^{k+1}}.$$

Since there is no residue between the annulus  $\|T\|$  and  $\text{spr}(T)$ , from complex analysis, this Laurent series has to converge for any  $|\lambda| > \text{spr}(T)$ .

Thus, for any  $|\lambda| > \text{spr}(T)$ ,  $\frac{\phi(T^k)}{\lambda^{k+1}} = \phi\left(\frac{T^k}{\lambda^{k+1}}\right)$  has to be bounded. Since this holds for any  $\phi \in B(X)^*$ , by the Banach-Steinhaus Theorem 3.20,  $\left\{\frac{T^k}{\lambda^{k+1}} : k \geq 0\right\}$  has to be bounded in  $B(X)^{**}$ , and thus in  $B(X)$ . Namely, there is  $C_\lambda > 0$ , such that  $\left\|\frac{T^k}{\lambda^{k+1}}\right\| < C_\lambda$  for all  $k \geq 0$ . Namely,  $\|T^k\| \leq C_\lambda |\lambda|^{k+1}$ . Thus, we have

$$\|T^k\|^{\frac{1}{k}} \leq C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} \implies \limsup_k \|T^k\|^{\frac{1}{k}} \leq \lim_{k \rightarrow \infty} C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} = |\lambda|.$$

Since this holds for all  $|\lambda| > \text{spr}(T)$ , we have

$$\limsup_k \|T^k\|^{\frac{1}{k}} \leq \text{spr}(T).$$

Since  $\limsup_n \|T^n\|^{\frac{1}{n}} \leq \text{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}$ , we have  $\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$ .  $\square$

**Corollary 6.50.** *Let  $\mathcal{H}$  be a Hilbert space, and  $T \in B(\mathcal{H})$  be normal. We have  $\text{spr}(T) = \|T\|$ .*

*Proof.* By proposition 6.20, we have  $\|T^2\| = \|T\|^2$ , and inductively, we have  $\|T^{2^n}\| = \|T\|^{2^n}$ . Thus,

$$\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left\|T^{2^n}\right\|^{\frac{1}{2^n}} = \|T\|.$$

$\square$

**Example 6.3.3.** For the Volterra operator, we have that  $\|V^n\|^{\frac{1}{n}} \rightarrow 0$ , so  $\sigma(V) \subseteq \{0\}$ , which means  $\sigma(V) = \{0\}$ .

**Theorem 6.51.** [Fredholm alternative] *Let  $X$  be a Banach space, and  $K \in \mathcal{K}(X)$  be a compact linear operator. Suppose  $\lambda \neq 0$ , then  $\ker(\lambda \mathbb{I} - T)$  is finite dimensional and  $\text{Im}(\lambda \mathbb{I} - T)$  is finite co-dimensional. Also, either  $\lambda \in \rho(K)$ , or  $\lambda \in \sigma_p(K)$ .*

**Theorem 6.52** (Structure theorem for compact operators). *Let  $K \in \mathcal{K}(X)$  be a compact operator on an infinite-dimensional Banach space  $X$ , then*

1.  $0 \in \sigma(K)$ .
2.  $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$ .
3.  $\sigma(K) \setminus \{0\}$  is finite, or  $\sigma(K) \setminus \{0\} = (\lambda_k)_{k=1}^\infty$  such that  $\lim_{k \rightarrow \infty} \lambda_k = 0$ .

**Theorem 6.53** (Spectral theorem for normal operators). *Let  $\mathcal{H}$  be a Hilbert space, and  $K \in \mathcal{K}(\mathcal{H})$  be a compact normal operator, then there exist eigenvectors  $(\phi_j)_{j \in J}$ , which form an orthonormal basis for  $\mathcal{H}$ .*

*Proof.* By Zorn's Lemma, we can find a maximal set of orthonormal eigenvectors  $(\phi_j)_{j \in J}$  of  $K$ . Let  $M := \overline{\text{Span}\{\phi_j\}_{j \in J}}$ , and  $P : \mathcal{H} \rightarrow M$  be the projection onto  $M$ . Also, let  $\lambda_j \in \mathbb{C}$  be such that  $K\phi_j = \lambda_j\phi_j$ .

Consider any  $x \in \mathcal{H}$ , we have that

$$\begin{aligned}
 KPx &= K \sum_{j \in J} \langle \phi_j, x \rangle \phi_j \\
 &= \sum_{j \in J} \langle \phi_j, x \rangle K\phi_j \\
 &= \sum_{j \in J} \langle \phi_j, x \rangle \lambda_j \phi_j \\
 &= \sum_{j \in J} \langle \bar{\lambda}_j \phi_j, x \rangle \phi_j \\
 &= \sum_{j \in J} \langle K^\dagger \phi_j, x \rangle \phi_j \\
 &= \sum_{j \in J} \langle \phi_j, Kx \rangle \phi_j \\
 &= PKx.
 \end{aligned}$$

Thus,  $KP = PK$ .

Since  $\mathbb{I} - P \in B(\mathcal{H})_{sa}$ , we have that  $(\mathbb{I} - P)K$  is compact and normal. Suppose  $(\mathbb{I} - P)K \neq 0$ , then by corollary 6.50, there is  $\lambda \in \sigma((\mathbb{I} - P)K)$ , such that  $|\lambda| = \text{spr}((\mathbb{I} - P)K) = \|(\mathbb{I} - P)K\| > 0$ . Thus, by Fredholm alternative 6.51,  $\lambda \in \sigma_p((\mathbb{I} - P)K)$ , which means there is a  $\phi_\lambda \neq 0 \in \mathcal{H}$ , such that  $(\mathbb{I} - P)K\phi_\lambda = \lambda\phi_\lambda$ . WLOG, we may pick  $\|\phi_\lambda\| = 1$ . Notice that this means  $\phi_\lambda \in \text{Im}(I - P) = M^\perp = \ker(P)$ , so

$$\begin{aligned}
 \lambda\phi_\lambda &= (\mathbb{I} - P)K\phi_\lambda \\
 &= K\phi_\lambda - PK\phi_\lambda \\
 &= K\phi_\lambda - KP\phi_\lambda \\
 &= K\phi_\lambda - K(0) \\
 &= K\phi_\lambda,
 \end{aligned}$$

which means  $\phi_\lambda$  is an eigenvector of  $K$ . However,  $\phi_\lambda \in M^\perp$ , so  $(\phi_j)_{j \in J \cup \{\phi_\lambda\}} \supsetneq (\phi_j)_{j \in J}$  is still orthonormal, contradicting the maximality of  $(\phi_j)_{j \in J}$ .

This shows  $(\mathbb{I} - P)K = 0$ . Now suppose  $\mathbb{I} - P \neq 0$ , we have  $K(\mathbb{I} - P) = K = KP = K - PK = (\mathbb{I} - P)K = 0$ , so there is  $\phi \neq 0 \in \text{Im}(\mathbb{I} - P) = M^\perp$ , such that  $K\phi = 0 = 0\phi$ , so  $\phi$  is an eigenvector of  $K$ , which again contradicts the maximality of  $(\phi_j)_{j \in J}$ .

Thus,  $\mathbb{I} - P = 0$ , which means  $M = \text{Im}(P) = \text{Im}(\mathbb{I}) = \mathcal{H}$ . □

**Theorem 6.54** (Spectral theorem for infinite range normal operators). *Let  $\mathcal{H}$  be a Hilbert space, and  $K \in \mathcal{K}(\mathcal{H})$  be a compact normal operator with infinite range, then there exists countably many eigenvalues  $(\lambda_k)_{k=1}^\infty \subset \mathbb{C}$  such that  $\forall k \geq 1$ ,  $n_k := \dim(\ker(\lambda_k \mathbb{I} - K)) < \infty$ , and  $\sigma(K) = (\lambda_k)_{k=0}^\infty$  for  $\lambda_0 = 0 = \lim_{k \rightarrow \infty} \lambda_k$ . In addition, there are eigenvectors  $(\phi_{k_i})_{k \geq 0, i \in [n_k]}$ , which form an orthonormal basis for  $\mathcal{H}$ , and  $\forall k \geq 0$ ,  $i \in [n_k]$ ,  $K\phi_{k_i} = \lambda_k \phi_{k_i}$ .*

*Proof.* □