Amath753 Advanced PDEs

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This note is based on the book "Partial Differential Equations" by Lawrence C. Evans and the course AMATH753 in Winter 2025.

Contents

1 Preliminaries

See more in AMATH731-Functional Analysis Notes from Prof. Giang Tran, and my PMATH651-Measure Theory Notes.

1.1 Introduction

Definition 1.1. We will use the following notations:

- C means a positive constant.
- $U \subset \mathbb{R}^n$ is open.
- If $u : U \to \mathbb{R}$ is a function, we write $u(x) := u(x^1, \ldots, x^n)$ for $x = (x^1, \ldots, x^n) \in U$.
- A function u is **smooth** if $u \in C^\infty(U)$.
- For $1 \leq i \leq n$, we write $\partial_i u := u_{x^i} := u_i := D_i u := \frac{\partial}{\partial x^i} u := \frac{\partial u}{\partial x^i}$.
- Let $\alpha = (\alpha_1 \ldots, \alpha_n) \in \mathbb{N}^n$, we let $|\alpha| := \sum_{i=1}^n \alpha_i$, and
$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n}} = \partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n} u.$$
- If $k \in \mathbb{N}$, we let $D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}$
- When $k = 1$, we write $Du := D_x u := (u_{x^1}, \ldots, u_{x^n})^T = \nabla u$ to be the **gradient**.
- When $k = 2$, we write $D^2 u := \begin{pmatrix} u_{x^1,x^1} & \cdots & u_{x^1,x^n} \\ \vdots & & \vdots \\ u_{x^n,x^1} & \cdots & u_{x^n,x^n} \end{pmatrix}$ to be the **Hessian** matrix.
- $\Delta u := \sum_{i=1}^n u_{x^i,x^i} = div\ Du = tr(D^2 u)$ is the **Laplacian** of u.

Example 1.1.1. Consider a body $U \subset \mathbb{R}^3$ and let $U_0 \subseteq U$ with boundary ∂U_0, which does not change over time.
The Conservation of Energy states that the rate of change of total energy in U_0 is the inflow of heat through the boundaries plus heat produced by the source in U_0.
Let $e(x,t) \in \mathbb{R}$ be the density of internal energy, then the total energy is $\int_{U_0} e dx$.
Let $j(x,t) \in \mathbb{R}^3$ be the heat flux (vector pointing at the direction that heat is flowing).
Let n denote the exterior unit normal on ∂U_0.
The net outflow of the heat through ∂U_0 is $\int_{\partial U_0} j \cdot n ds$.
Let $p(x,t) \in \mathbb{R}$ be the power density of the source. Heat production in U_0 is $\int_{U_0} p dx$.
Thus we have
$$\frac{d}{dx} \int_{U_0} e dx = - \int_{\partial U_0} j \cdot n ds + \int_{U_0} p dx.$$
By divergence theorem, we have $\int_{\partial U_0} j \cdot n ds = \int_{U_0} div\ j dx$.
Thus we have
$$\int_{U_0} (\partial_t e + div\ j - p) dx = 0.$$
Since U_0 is arbitrary, we must have
$$\partial_t e + div\ j - p = 0.$$
Assume that e depends linearly on temperature T as $e = e_0 + \sigma u$, where e_0 is a constant reference internal energy, and $u = T - T_0$, where T_0 is a constant reference temperature, and σ is the specific heat capacity.
A generalized form of Fourier's law states that:

- Heat flow is proportional to the temperature gradient.
- Heat is transformed by convection with heat flux be, where $b(x,t) \in \mathbb{R}^3$ is a given convection velocity.

Namely, $j = -aDu + be$, where $a(x)$ is a known heat conductivity.
Thus we have

$$\sigma \partial_t u + div(b\sigma u) - div(aDu) = p - div(be_0).$$

Definition 1.2. We consider the operator

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij} u_{x^i}\right)_{x^j} + \sum_{i=1}^{n} b^i u_{x^i} + cu,$$

for given coefficients a^{ij}, b^i, c.

- The second-order elliptic boundary-value problems are $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$
- The second-order parabolic boundary-value problems are $\begin{cases} u_t + Lu = f & x \in U, t \in (0,T] \\ u = 0 & \text{on } \partial U, t \in (0,T] \\ u = u_0 & \text{on } \partial U, t = 0 \end{cases}$

Example 1.1.2. Some special cases are

- Laplace equation: $-\Delta u = 0$
- Poisson's equation: $-\Delta u = f$
- Heat equation: $u_t - \Delta u = 0$

1.2 Metric Spaces and Complete Spaces

Definition 1.3. A **metric space** is a set X that has a (distance) metric:

$$\begin{aligned} d(\cdot,\cdot) : X \times X \to \mathbb{R}, \text{ such that } \forall x, y, z \in X \\ d(x,x) = 0 \\ \forall x \neq y, d(x,y) > 0 \\ d(x,y) = d(y,x) \\ d(x,z) \geq d(x,y) + d(y,z) \end{aligned}$$

Definition 1.4. Given a metric space (X,d), a sequence $(x_n)_{n=1}^{\infty}$ in X has a **limit point** $x \in X$ if $\lim\limits_{n\to\infty} d(x,x_n) = 0$. In this case, we say $(x_n)_{n=1}^{\infty}$ is a **convergent sequence**, and write $x = \lim_{n\to\infty} x_n$.

Definition 1.5. A sequence $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** in a metric space (X,d) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m,n \in \mathbb{N},\ m,n > N \implies d(x_m,x_n) < \epsilon.$$

Definition 1.6. A metric space X is **complete** if every Cauchy sequence $(x_i)_{i=1}^{\infty}$ converges to a limit point in X. i.e. $\exists x \in X, \lim\limits_{i\to\infty} x_i = x$.

Proposition 1.1. *Let (X,d) be a metric space, then every convergent sequence is Cauchy.*

Proposition 1.2. *Let (X,d) be a metric space. If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a convergent subsequence such that $\lim_{k\to\infty} x_{n_k} = x \in X$, then $\lim_{n\to\infty} x_n = x$.*

1.2.1 Compactness

Remark. See the definition of compactness and more in Section 2.9 of AMATH731 Notes from Prof. Tran.

Definition 1.7. Let (X, d) be a metric space. A set $S \subseteq X$ is **sequentially compact** if every sequence $(x_i)_{i=1}^{\infty}$ in S has a convergent subsequence whose limit is in S. Namely, $\exists x \in S$, such that $x = \lim_{j\to\infty} x_{i_j}$ for some choice of i_j's.

Remark. In metric spaces, sequentially compact and compact are equivalent, so we will just use this as the definition for compactness.

Definition 1.8. Let (X, d) be a metric space. A set $S \subseteq X$ is **relatively compact**, or **pre-compact** if its closure $\overline{S}$ is compact in X.

Proposition 1.3. *Let (X, d) be a metric space, then $S \subseteq X$ is relatively compact iff for any sequence $(x_n)_{n=1}^{\infty} \subseteq S$, it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$, such that $x_{n_k} \to x$ for some $x \in X$.*

1.3 Banach Spaces

Definition 1.9. A **normed vector space** is a vector space $(X, ||\cdot||)$ that has an norm (length):

$$\begin{aligned} ||\cdot|| : X \to \mathbb{R},\ &\text{such that } \forall x, y \in X, a \in \mathbb{C} \\ &||a \cdot x|| = |a|||x|| \\ &||x + y|| \le ||x|| + ||y|| \\ &||x|| \ge 0 \\ &||x|| = 0 \iff x = 0. \end{aligned}$$

Proposition 1.4. *For every **normed space** with $||\cdot||$, there is a metric $d(x, y) = ||x - y||$.*

Proof.

$$\begin{aligned} d(x,x) &= ||x - x|| = ||0|| = 0 \\ \forall x \ne y, d(x,y) &= ||x - y|| > 0 \\ d(x,y) &= ||x - y|| = ||-(y - x)|| = |-1|||y - x|| = ||y - x|| = d(y,x) \\ d(x,z) &= ||x - z|| = ||x - y + y - z|| \ge ||x - y|| + ||y - z|| = d(x,y) + d(y,z) \end{aligned}$$

Thus $d(x, y) = ||x - y||$ is a metric. ☐

Definition 1.10. A normed space is called a **Banach space** if it is complete.

Definition 1.11. Let $(X, ||\cdot||)$ be a Banach space, a subset $A \subseteq X$ is **dense** in X if the closure $\bar{A} = X$.

Definition 1.12. A Banach space is **separable** if there is a dense countable subset of it.

1.4 Hilbert Spaces

Definition 1.13. An **inner product space** is a vector space H that has an inner product: $\langle\cdot, -\rangle : H \times H \to \mathbb{C}$, such that $\forall u, v, w \in H, a, b \in \mathbb{C}$, it satisfies

1. conjugate symmetry; i.e. $\langle v, w\rangle = \overline{\langle w, v\rangle}$,
2. linearity in the second argument; i.e. $\langle v, au + bw\rangle = a\langle v, u\rangle + b\langle v, w\rangle$, and
3. positive definiteness; i.e. if $v \ne 0$, we must have $\langle v, v\rangle > 0$.

Remark. The conventional mathematical definition of an inner product is linear in the first argument. We are using the current definition to make the "bra-ket" notation easier to understand. Also, notice that the conjugate symmetry implies $\langle v, v\rangle = \overline{\langle v, v\rangle} \in \mathbb{R}$, and the linearity implies $\langle 0, v\rangle = 0$ for any $v \in H$.

Lemma 1.5. *For every inner product space with $\langle \cdot, - \rangle$, and $x, y \in H$, we have*

$$\langle x, y \rangle + \langle y, x \rangle = 2\Re(\langle x, y \rangle),$$

which is twice the real part of $\langle x, y \rangle$.

Proof.

$$\langle x, y \rangle + \langle y, x \rangle = \langle x, y \rangle + \overline{\langle x, y \rangle} = 2\Re(\langle x, y \rangle).$$

$\square$

Proposition 1.6. *For every inner product space with $\langle \cdot, - \rangle$, there is a norm $||x|| = \sqrt{\langle x, x \rangle}$.*

Proof. Consider any $x \in H, a \in \mathbb{C}$,

$$\begin{aligned}
||a \cdot x|| &= \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2}\sqrt{\langle x, x \rangle} = |a|||x|| \\
||x + y||^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \\
&= ||x||^2 + ||y||^2 + 2\Re(\langle x, y \rangle) \\
&\leq ||x||^2 + ||y||^2 + 2|\langle x, y \rangle| \\
&\leq ||x||^2 + ||y||^2 + 2||x||||y|| \\
&\leq (||x|| + ||y||)^2 \\
\forall x \neq 0, ||x|| &= \sqrt{\langle x, x \rangle} > 0 \\
||0|| &= \sqrt{\langle 0, 0 \rangle} = 0
\end{aligned}$$

Thus $||x|| = \sqrt{\langle x, x \rangle}$ is a norm. $\square$

Corollary 1.7. *For every inner product space, there is a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$*

Theorem 1.8 (Cauchy-Schwarz)**.** *For every inner product space H,*

$$\forall u, v \in H, |\langle u, v \rangle| \leq ||u||||v||.$$

In particular, when $||u|| \neq 0, ||u||^2||v||^2 - |\langle u, v \rangle|^2 = ||z||^2$, where $z := ||u||v - \frac{\langle u, v \rangle}{||u||}u$.

Proof. Notice that this is trivially true and equality holds to be zero when $u = 0$.
Now we assume $||u|| \neq 0$, then

$$\begin{aligned}
||z||^2 &= \langle z, z \rangle \\
&= \left\langle ||u||v - \frac{\langle u, v \rangle}{||u||}u, ||u||v - \frac{\langle u, v \rangle}{||u||}u \right\rangle \\
&= ||u||^2 \langle v, v \rangle - \langle u, v \rangle \langle u, v \rangle - \overline{\langle u, v \rangle} \langle v, u \rangle + \frac{\langle v, u \rangle \langle u, v \rangle}{\cancel{||u||^2}} \cancelto{||u||^2}{\langle u, u \rangle} \\
&= ||u||^2||v||^2 - |\langle u, v \rangle|^2 - \cancel{|\langle v, u \rangle|^2} + \cancel{|\langle v, u \rangle|^2} \\
&= ||u||^2||v||^2 - |\langle u, v \rangle|^2.
\end{aligned}$$

$\square$

Proposition 1.9. *If $\forall v, \langle v, u \rangle = 0$, then $u = 0$.*

Proposition 1.10. *For an Inner product space $H, \forall y, x = \lim_{i \to \infty} x_i \in H$, we have*

$$\langle x, y \rangle = \lim_{i \to \infty} \langle x_i, y \rangle.$$

Proof. Given any $\epsilon > 0$, let $\epsilon_0 = \frac{\epsilon}{||y||}$.
Since $x = \lim_{i\to\infty} x_i$, we can find $N > 0$, such that $\forall n > N, ||x - x_n|| < \epsilon_0$,
thus $|\langle x, y\rangle - \langle x_n, y\rangle| = |\langle x - x_n, y\rangle| \leq ||x - x_n||||y|| < \epsilon_0||y|| = \epsilon$ □

Corollary 1.11. *For an Inner product space $H, \forall y, x = \lim\limits_{i\to\infty} x_i \in H$, we have $\langle y, x\rangle = \lim\limits_{i\to\infty} \langle y, x_i\rangle$.*

Definition 1.14. An inner product space $\mathcal{H}$ is called a Hilbert space if it is complete.

Definition 1.15. Let H be an inner product space. Two vectors $u, v \in H$ are called **orthogonal** if $\langle u, v\rangle = 0$.

Definition 1.16. Let H be an inner product space. A set $\{e_i\}_{i\in I} \subseteq H$ is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j\rangle = \delta_{ij}.$$

Definition 1.17. Let H be an inner product space. An orthonormal set $\{e_i\}_{i\in\mathbb{N}} \subseteq H$ is called a **maximal orthonormal set / orthonormal basis / total orthonormal set** if

$$H = \overline{Span(\{e_1, e_2, \ldots\})}.$$

Theorem 1.12. *Let $\mathcal{H}$ be a Hilbert space, and $\{e_i\}_{i\in\mathbb{N}} \subseteq \mathcal{H}$ be an orthonormal set, then TFAE:*

1. *$\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis*
2. *If $\forall i \in \mathbb{N}, \langle x, e_i\rangle = 0$, then $x = 0$.*
3. *$\forall x \in \mathcal{H}, x = \sum_{i\in\mathbb{N}} \langle e_i, x\rangle e_i$. (Fourier series)*
4. *$\forall x \in \mathcal{H}, ||x||^2 = \sum_{i\in\mathbb{N}} |\langle e_i, x\rangle|^2$. (Parseval Identity)*

Theorem 1.13. *$\mathcal{H}$ is a separable Hilbert space, if and only if there is a maximal orthonormal set in $\mathcal{H}$. Moreover, in this case, every maximal orthonormal set is at most countable.*

Definition 1.18. Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$, the subspace **orthogonal** to S is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v\rangle = 0, \forall u \in S\}.$$

Lemma 1.14. *Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$, we always have $S^\perp$ is a subspace of $\mathcal{H}$.*

Definition 1.19. Let V be a vector space, and $U, W \subseteq V$ be two subspaces, we say $V = U \oplus W$, if $\forall v \in V$, it can be uniquely written as $v = u + w$, where $u \in U, w \in W$.

Theorem 1.15. *Let $\mathcal{H}$ be a Hilbert space, if $S \subseteq \mathcal{H}$ is a closed subspace, then*

$$\mathcal{H} = S \oplus S^\perp.$$

1.5 Bounded linear operators

Definition 1.20. Let X, Y be vector spaces, $A : X \to Y$ is a linear operator if $\forall c \in \mathbb{R}, u, v \in X$,

$$A(u + cv) = Au + cAv.$$

Definition 1.21. Let X, Y be normed spaces, the **operator norm** of a linear operator $A : X \to Y$ is

$$||A|| := \sup_{||u||_X \leq 1} ||Au||_Y = \sup_{||u||_X = 1} ||Au||_Y = \sup_{u\neq 0\in X} \frac{||Au||_Y}{||x||_X}.$$

Definition 1.22. Let X, Y be normed spaces, a linear operator $A : X \to Y$ is **bounded** if $||A|| < \infty$.

Definition 1.23. Let X, Y be normed spaces, we denote

$$B(X, Y) := \{A : X \to Y | A \text{ is bounded linear opeartor}\}.$$

Theorem 1.16. *The set $B(X, Y)$ is a normed linear space with the operator norm.*

Proposition 1.17. *Let X, Y, Z be normed spaces, if $A : X \to Y, B : Y \to Z$ are both linear bounded operators, then so is $B \circ A$, with*

$$||B \circ A|| \leq ||B||||A||.$$

Theorem 1.18. *Let X, Y be normed spaces, a linear operator $A : X \to Y$ is bounded if and only if it is continuous.*

Definition 1.24. Let X, Y be normed spaces, a linear operator $A : X \to Y$ is **closed** if $\forall u_k \to u$ in X and $Au_k \to v$ in Y, we have $Au = v$.

Theorem 1.19. *(closed graph) Let X, Y be Banach spaces, if a linear operator $A : X \to Y$ is closed, it is also bounded.*

Theorem 1.20. *(Bounded inverse Theorem) Let X, Y be normed spaces, if a bounded linear operator $A : X \to Y$ is bijective, then A^{-1} is continuous and bounded as well.*

Proposition 1.21. *Let Y be a Banach space, S be a dense subset of a normed space X. For any bounded linear operator $E : S \to Y$, we can extend it to $\tilde{E} : X \to Y$, such that $\tilde{E}$ is also bounded and linear, with $\left|\left|\tilde{E}\right|\right| = ||E||$, and $\tilde{E}|_S = E$.*

Proof. Consider any $x \in X$.
Since S is dense in X, We know $\forall m \in \mathbb{N}^+, \exists x_m \in S$, such that $||x - x_m||_X \leq \frac{1}{m}$.
Since E is linear on S, we have that

$$\begin{aligned} ||Ex_m - Ex_l||_Y &= ||E(x_m - x_l)||_Y \\ &\leq ||E||||x_m - x_l||_X \\ &= ||E||||(x_m - x) + (x - x_l)||_X \\ &\leq ||E||||x - x_m||_X + ||E||||x - x_l||_X \\ &\leq ||E||\left(\frac{1}{m} + \frac{1}{l}\right). \end{aligned}$$

Thus given any $\epsilon > 0$, for any $m, l \geq \lceil \frac{2\epsilon}{||E||} \rceil$, we can make $||Ex_m - Ex_l||_Y < \epsilon$. Thus $(Ex_m)_{m=1}^{\infty}$ is a Cauchy sequence in Y.
Since Y is a Banach space, $\exists y^* \in Y$, such that $Ex_m \to y^*$ in Y.
We claim that y^* is independent of choice of the sequence $(x_m)_{m=1}^{\infty}$.
Indeed, consider any other sequence $(v_m)_{m=1}^{\infty} \subseteq C^{\infty}(\bar{x})$, such that $\forall m \in \mathbb{N}^+, ||x - x_m||_X \leq \frac{1}{m}$,

$$\begin{aligned} ||y^* - Ev_m||_Y &\leq ||y^* - Ex_m||_Y + ||Ex_m - Ev_m||_Y \\ &\leq ||y^* - Ex_m||_Y + ||E||||x_m - v_m||_X \\ &\leq ||y^* - Ex_m||_Y + ||E||||x_m - x||_X + ||E||||x - v_m||_X. \end{aligned}$$

Since all three terms on the right go to 0 when $m \to \infty$, we have that $Ev_m \to y^*$ in Y.
Thus we can uniquely define $\tilde{E}x := y^*$. In addition,

$$\begin{aligned} \left|\left|\tilde{E}x\right|\right|_Y &= \left|\left|\lim_{m\to\infty} Ex_m\right|\right|_Y \\ &= \lim_{m\to\infty} ||Ex_m||_Y \\ &\leq \lim_{m\to\infty} ||E||||x_m||_X \\ &= ||E||\left|\left|\lim_{m\to\infty} x_m\right|\right|_X \\ &= ||E||||x||_X. \end{aligned}$$

Thus $\left|\left|\tilde{E}\right|\right| = ||E||$. □

1.5.1 Compact Operators

Definition 1.25. Let X, Y be metric spaces, a linear operator $A : X \to Y$ is **compact** if for each bounded subset $S \subseteq X$, we have its image $A(S)$ is pre-compact in Y.

Proposition 1.22. *Let X, Y be metric spaces, a linear operator $A : X \to Y$ is compact if and only if A is bounded, and each bounded sequence $(x_n)_{n=1}^{\infty} \subseteq X$ has some subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $(Ax_{n_k})_{k=1}^{\infty}$ converges to some $y \in Y$.*

Definition 1.26. Let X, Y be Banach spaces and $X \subseteq Y$, then we say X is **compactly embedded** in Y, denoted

$$X \subset\subset Y$$

if the inclusion map $i : X \hookrightarrow Y;\ x \mapsto x$ is compact.
Namely, $\exists C > 0,$ such that $\forall x \in X, ||x||_Y \leq C||x||_X$, and each bounded sequence $(x_n)_{n=1}^{\infty} \subseteq X$ having some subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to some $y \in Y$.

Proposition 1.23. *Let X, Y, Z be Banach spaces and $X \subset\subset Y$, if an operator $T : Z \to X$ is bounded, then $\tilde{T} := i \circ T : Z \to Y$ is compact.*

Proof. Consider any bounded set $S \subseteq Z$, such that $\forall z \in S, ||z||_Z \leq M$.
We have $||Tz||_X \leq ||T||||z||_M \leq M||T|| < \infty$, and thus $T(S)$ is bounded in X.
Yet i is compact, and thus $i(T(S))$ is pre-compact.
This shows $\tilde{T}(S) = (i \circ T)(S)$ is pre-compact for any bounded set $S \subseteq Z$.
Thus $\tilde{T}$ is compact. ∎

Theorem 1.24 (Spectral theorem for compact operators)**.** *Let $K : \mathcal{H} \to \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space $\mathcal{H}$, then*

1. $0 \in \operatorname{Spec}(K)$.
2. $\operatorname{Spec}(K) \setminus \{0\} = \operatorname{Spec}_p(K) \setminus \{0\}$.
3. $\operatorname{Spec}(K) \setminus \{0\}$ *is finite, or* $\operatorname{Spec}(K) \setminus \{0\} = (\lambda_k)_{k=1}^{\infty}$ *such that* $\lim_{k\to\infty} \lambda_k = 0$.

1.5.2 Dual Space

Definition 1.27. Let X be a normed space over $\mathbb{F}$, a **functional** is an operator that maps into $\mathbb{F}$.

Definition 1.28. Let X be a normed space over $\mathbb{F}$, the **dual space** of X is the collection of bounded linear functionals on X, denoted

$$X^* := B(X, \mathbb{F}).$$

Definition 1.29. Let X be a normed space, if $v \in X, u^* \in X^*$, we can write $\langle u^*|v\rangle_{X^*,X} := u^*(v)$ as the action of u^* on v.

Definition 1.30. Let X be a normed space, the **dual norm** is defined to be

$$||u^*||_{X^*} := \sup_{||u||\leq 1} \left|\langle u^*|u\rangle_{X^*,X}\right|.$$

Definition 1.31. A Banach space X is **reflexive** if $(X^*)^* \simeq X$. Namely, $\forall u^{**} \in (X^*)^*, \exists! u \in X$ such that

$$\forall v^* \in X^*, \langle u^{**}|v^*\rangle_{(X^*)^*,X^*} = \langle v^*|u\rangle_{X^*,X}.$$

Theorem 1.25. *(Riesz-Frechet Representation theorem)*
Let $\mathcal{H}$ be a Hilbert space, then for each $u^* \in \mathcal{H}, \exists! u \in \mathcal{H}$, such that $\forall v \in \mathcal{H}, \langle u^*|v\rangle_{\mathcal{H}^*,\mathcal{H}} = \langle u, v\rangle_{\mathcal{H}}$, and $||u^*||_{\mathcal{H}^*} = ||u||_{\mathcal{H}}$.

Corollary 1.26. *Every Hilbert space is reflexive.*

Corollary 1.27. *Let $\mathcal{H}$ be a Hilbert space, then $\mathcal{H} \cong^* \mathcal{H}$, where the map $\Phi : \mathcal{H} \to \mathcal{H}^*;\ u \mapsto \langle u, \cdot\rangle_{\mathcal{H}}$ is the* ***canonical bijective isometric antilinear isomorphism****.*

Remark. We thus abuse the notation, and denote canonical bijective isometric antilinear isomorphism by $u^\dagger := \Phi(u)\ \forall u \in \mathcal{H}$, and $(u^*)^\dagger := \Phi^{-1}(u^*)\ \forall u^* \in \mathcal{H}^*$. Notice that by definition

$$(u^\dagger)^\dagger = u, ((u^*)^\dagger)^\dagger = u^*\ \forall u \in \mathcal{H}, u^* \in \mathcal{H}^*.$$

We might further abuse the notation, and write

$$\langle u|v\rangle := \langle u, v\rangle = \left\langle u^\dagger \middle| v\right\rangle =: \left\langle u^\dagger, v\right\rangle$$

interchangeably instead of $\left\langle u^\dagger \middle| v\right\rangle_{\mathcal{H}^*,\mathcal{H}}$ or $\langle u, v\rangle_{\mathcal{H}}$ when the context is clear.

Definition 1.32. Let X be a Banach Space, we say $(u_k)_{k=1}^\infty \subset X$ converges to $u \in X$ weakly, denoted $u_k \rightharpoonup u$, if

$$\forall v^* \in X^*, \langle v^*|u_k\rangle \to \langle v^*|u\rangle$$

as real numbers.

Proposition 1.28. *Let X be a Banach Space, $(u_k)_{k=1}^\infty \subset X$ be a sequence, then*

1. *If $u_k \to u$, we always have $u_k \rightharpoonup u$.*
2. *If $u_k \rightharpoonup u$, we have that u is unique.*
3. *If $u_k \rightharpoonup u$, we have $(u_k)_{k=1}^\infty$ is bounded.*
4. *If $u_k \rightharpoonup u$, every subsequence $(u_{k_j})_{j=1}^\infty$ also converges weakly to u.*

Proof. See A5Q1 for 1. □

Theorem 1.29 (Weakly compact for reflexive Banach Space)**.** *Let X be a reflexive Banach Space, and $(u_k)_{k=1}^\infty \subset X$ be a bounded sequence, then $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.*

Proposition 1.30. *Let $\mathcal{H}$ be a Hilbert space, then $u_k \rightharpoonup u$ if and only if $\forall v \in \mathcal{H}, \langle v, u_k\rangle \to \langle v, u\rangle$ as real numbers.*

Proof. Suppose $u_k \rightharpoonup u$.
Notice that for all $v \in \mathcal{H}$, we have that $v^\dagger \in \mathcal{H}^*$, and thus $\langle v, u_k\rangle = \left\langle v^\dagger \middle| u_k\right\rangle \to \left\langle v^\dagger \middle| u\right\rangle = \langle v, u\rangle$.
Now suppose $\forall v \in \mathcal{H}, \langle v, u_k\rangle \to \langle v, u\rangle$.
Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 1.25, there is some $f^\dagger \in \mathcal{H}$, such that

$$\left\langle f \middle| u_{k_j}\right\rangle = \left\langle f^\dagger, u_{k_j}\right\rangle \to \left\langle f^\dagger, u\right\rangle = \langle f|u\rangle.$$

Thus, $u_{k_j} \rightharpoonup u$. □

Proposition 1.31. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Tu_k \rightharpoonup \mathcal{H}_2$.*

Proof. Let $y_k := Tu_k, y := Tu \in \mathcal{H}_2$.
Consider any $g \in \mathcal{H}_2^*$, we define $f := g \circ K \in \mathcal{H}_1^*$.
Since $u_k \rightharpoonup u$, we must have

$$\lim_{k\to\infty} f(u_k) = f(u)$$
$$\lim_{k\to\infty} g(Ku_k) = g(Ku)$$
$$\lim_{k\to\infty} g(y_k) = g(y).$$

We thus have $y_k \rightharpoonup y$. □

Proposition 1.32. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $K : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator, and $(u_k)_{k=1}^{\infty} \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Ku_k \to \mathcal{H}_2$.*

Proof. Let $y_k := Ku_k, y := Ku \in \mathcal{H}_2$.
Since K is compact, it is bounded, so $y_k \rightharpoonup y$.
Now suppose for contradiction $\lim_{k\to\infty} ||y_k - y|| \neq 0$.
Then there is some $\epsilon > 0$ and a subsequence $(u_{k_j})_{j=1}^{\infty}$ such that $\forall j \geq 1, \left|\left|y_{k_j} - y\right|\right| \geq \epsilon$.
Since $u_k \rightharpoonup u \in \mathcal{H}$, we have $(u_k)_{k=1}^{\infty}$ is bounded, and thus $(u_{k_j})_{j=1}^{\infty}$ is bounded.
Since K is compact, there is some further subsequence $(u_{k_{j_m}})_{m=1}^{\infty}$ such that $\lim_{m\to\infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$.
Thus $Ku_{k_{j_m}} \rightharpoonup \tilde{y}$. Since weak convergence, we must have $\tilde{y} = y$.
Thus $\lim_{m\to\infty} Ku_{k_{j_m}} = y$, which is a contradiction. □

1.5.3 Adjoint Operator

Definition 1.33. Let X, Y be normed spaces, the **dual operator** of a linear operator $A : X \to Y$ is

$$A^* : Y^* \to X^*;\ f \mapsto f \circ A.$$

Proposition 1.33. *Let X, Y, Z be normed spaces, $S \in B(X,Y), T \in B(Y,Z)$, then $(S \circ T)^* = T^* \circ S^*$.*

Proof. Consider any $f \in Z^*$, and any $x \in X$, we have

$$\begin{aligned}(T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx)\\ &= (f)(S(T(x)))\\ &= (f \circ (S \circ T))(x)\\ &= (S \circ T)^*(f)(x).\end{aligned}$$

Thus $(T^* \circ S^*)(f) = (S \circ T)^*(f)$. □

Definition 1.34. Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, the **Hilbert adjoint operator** of T is $T^\dagger : \mathcal{H} \to \mathcal{H}$ such that $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \forall x, y \in \mathcal{H}$.

Theorem 1.34. *Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, $T^\dagger$ always exists, and is given by $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$, where $\Phi : \mathcal{H} \to \mathcal{H}^*;\ u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism, and T^* is the dual operator of T. In addition, $T^\dagger$ is also a bounded linear operator, with $\left|\left|T^\dagger\right|\right| = ||T||$, and $(T^\dagger)^\dagger = T$.*

Proof. $\forall y \in \mathcal{H}$, we have that

$$\begin{aligned}\left\langle T^\dagger x, y \right\rangle &= \left\langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \right\rangle\\ &= ((T^* \circ \Phi)(x))(y)\\ &= (\Phi(x))(Ty)\\ &= \langle x, Ty \rangle.\end{aligned}$$

Now consider any $x, y, z \in \mathcal{H}, c \in \mathbb{C}$, we have that

$$\begin{aligned}\left\langle T^\dagger(x + cz), y \right\rangle &= \langle x + cz, Ty \rangle\\ &= \langle x, Ty \rangle + \bar{c}\langle z, Ty \rangle\\ &= \left\langle T^\dagger x, y \right\rangle + \bar{c}\left\langle T^\dagger z, y \right\rangle\\ &= \left\langle T^\dagger x + cT^\dagger z, y \right\rangle.\end{aligned}$$

Since this holds for any $y \in \mathcal{H}$, we have that $T^\dagger(x+cz) = T^\dagger x + cT^\dagger z$, and thus $T^\dagger$ is linear.
Now given any $x \in \mathcal{H}$, we have that

$$\begin{aligned}
\left|\left|T^\dagger x\right|\right|^2 &= \langle T^\dagger x, T^\dagger x\rangle \\
&= \langle x, TT^\dagger x\rangle \\
&\le ||x||\left|\left|TT^\dagger x\right|\right| \\
&\le ||x||||T||\left|\left|T^\dagger x\right|\right| \\
&\implies \\
\left|\left|T^\dagger x\right|\right| &\le ||x||||T|| \\
&\implies \\
\left|\left|T^\dagger\right|\right| &= \sup_{x\neq 0\in\mathcal{H}} \frac{\left|\left|T^\dagger x\right|\right|}{||x||} \\
&\le \sup_{x\neq 0\in\mathcal{H}} \frac{||x||||T||}{||x||} \\
&= ||T||.
\end{aligned}$$

Thus $T^\dagger$ is also a bounded linear operator.
Now $\forall x, y \in \mathcal{H}$, $\langle x, T^\dagger y\rangle = \overline{\langle T^\dagger y, x\rangle} = \overline{\langle y, Tx\rangle} = \langle Tx, y\rangle$.
Thus $(T^\dagger)^\dagger = T$. □

Remark. $\forall x, y \in \mathcal{H}$, $\langle (Tx)^\dagger | y\rangle = \langle Tx, y\rangle = \langle x, T^\dagger y\rangle = \langle x^\dagger | T^\dagger y\rangle$. We thus abuse the notation, and write $(Tx)^\dagger = \langle x|T^\dagger$

Definition 1.35. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is **delf-adjoint** if $T^\dagger = T$.

Theorem 1.35. *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then $K^\dagger$ is also compact.*

Proof. $K^\dagger$ is bounded by 1.22.
Let $(u_k)_{k=1}^\infty$ be any bounded sequence in $\mathcal{H}$.
By 1.29, we have that $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.
Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 1.25, there is some $f^\dagger \in \mathcal{H}$, such that

$$\begin{aligned}
\langle f | K^\dagger(u_{k_j} - u)\rangle &= \langle f^\dagger, K^\dagger(u_{k_j} - u)\rangle \\
&= \langle Kf^\dagger, u_{k_j} - u\rangle \\
&= \langle Kf^\dagger, u\rangle - \langle Kf^\dagger, u\rangle \qquad \to 0,
\end{aligned}$$

since $u_{k_j} \rightharpoonup u$ and by 1.30.
Since $\langle f | K^\dagger(u_{k_j} - u)\rangle \to 0 = \langle f|0\rangle$ for any $f \in \mathcal{H}^*$, we have that $K^\dagger(u_{k_j} - u) \rightharpoonup 0$.
By 1.32, we have that $KK^\dagger(u - u_{k_j}) \to 0$.

$$\begin{aligned}
\left|\left|K^\dagger u - K^\dagger u_{k_j}\right|\right|^2 &= \langle K^\dagger u - K^\dagger u_{k_j}, K^\dagger u - K^\dagger u_{k_j}\rangle \\
&= \langle K^\dagger(u - u_{k_j}), K^\dagger(u - u_{k_j})\rangle \\
&= \langle KK^\dagger(u - u_{k_j}), u - u_{k_j}\rangle \\
&\le \left|\left|KK^\dagger(u - u_{k_j})\right|\right|\left|\left|u - u_{k_j}\right|\right| \\
&\to 0.
\end{aligned}$$

Thus $K^\dagger u_{k_j} \to K^\dagger u \in \mathcal{H}$,
Since $(u_k)_{k=1}^\infty$ is any bounded sequence, we have that $K^\dagger$ is compact by 1.22. □

Theorem 1.36. *(Fredholm's alternative)*
Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then

1. $\operatorname{Ker}(I-K)$ *is finite dimensional.*

2. $\operatorname{Im}(I-K)$ *is closed.*

3. $\operatorname{Im}(I-K) = \operatorname{Ker}(I-K^\dagger)^\perp$.

4. $\dim(\operatorname{Ker}(I-K)) = \dim(\operatorname{Ker}(I-K^\dagger))$.

5. $\operatorname{Ker}(I-K) = \{0\} \iff \operatorname{Im}(I-K) = \mathcal{H}$.

Corollary 1.37. *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then exactly one of the following holds:*

1. $\forall v \in \mathcal{H},\ \exists! u \in \mathcal{H},$ *such that* $(I-K)u = v$.

2. $\exists u \neq 0 \in \mathcal{H},$ *such that* $(I-K)u = 0$.

Proof. When $\operatorname{Ker}(I-K) = \{0\}$, we have that $I-K$ is injective, and $\operatorname{Im}(I-K) = \mathcal{H}$.
Thus $\forall v \in \mathcal{H},\ \exists! u \in \mathcal{H},$ such that $(I-K)u = v$.
On the other hand, if 1. is true, we have that $(I-K)$ is surjective, so $\operatorname{Im}(I-K) = \mathcal{H}$, so $\operatorname{Ker}(I-K) = \{0\}$.
Thus $\operatorname{Ker}(I-K) = \{0\} \iff$ 1..
We also have that $\operatorname{Ker}(I-K) \neq \{0\} \iff \exists u \neq 0 \in \operatorname{Ker}(I-K) \iff$ 2.. □

Theorem 1.38. *(Spectral theorem / Hilbert-Schmidt Theorem)*
Let $T : \mathcal{H} \to \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space $\mathcal{H}$, and $n = \dim(\Im(T)) \in \mathbb{N} \cap \{\infty\}$, then

1. *There exists orthonormal eigenvectors $(\phi_k)_{k=1}^n \subset \mathcal{H}$ and eigenvalues $(\lambda_k)_{k=1}^n \subset \mathbb{R}$ such that $|\lambda_1| \geq |\lambda_2| \geq \dots$, and*

$$T\phi_k = \lambda_k \phi_k, \lambda_k \neq 0,\ \forall 1 \leq k \leq n,$$

$$\forall v \in \mathcal{H}, Tv = \sum_{k=1}^{n} \lambda_k \langle \phi_k, v \rangle \phi_k = \sum_{k=1}^{n} \langle \phi_k, Tv \rangle \phi_k.$$

2. *If $n = \infty$, then $\lim_{k\to\infty} \lambda_k = 0$, and $(\phi_k)_{k=1}^\infty$ is an orthonormal set for $\mathcal{H}$ iff 0 is not an eigenvalue for T.*

1.6 Function Spaces

1.6.1 Continuous functions

Definition 1.36. $u : U \to \mathbb{R}$ is **continuous** at $x \in U$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in U, ||x-y|| < \delta \implies |u(x) - u(y) < \epsilon|.$$

A function u is continuous if it is continuous at all $x \in U$.

- $C(U) := \{u : U \to \mathbb{R} : u \text{ is continuous}\}$
- $C^k(U) := \{u : U \to \mathbb{R} : u \text{ is k-times continuously differentiable}\}$
- $C^\infty(U) := \{u : U \to \mathbb{R} : u \text{ has continuous derivatives of all orders}\}$

Definition 1.37. $u : U \to \mathbb{R}$ is **uniformly continuous** if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in U, ||x-y|| < \delta \implies |u(x) - u(y) < \epsilon|.$$

- $C(\bar{U}) := \{u : U \to \mathbb{R} : u \text{ is uniformly continuous on bounded subsets of } U\}$
- $C^k(\bar{U}) := \{u : U \to \mathbb{R} : \forall |\alpha| \leq k, D^\alpha u \text{ is uniformly continuous on bounded subsets of } U, \}$
- If $u \in C^k(\bar{U})$, then we can extend $D^\alpha u$ continuously to $\bar{U}$.

Definition 1.38. The support of $u : U \to \mathbb{R}$ is

$$\text{Supp}(u) := \overline{\{x \in U : u(x) \neq 0\}}.$$

Definition 1.39. $u : U \to \mathbb{R}$ has compact support if Supp(u) is a compact subset of U.

Definition 1.40. We denote the functions in $C(U)$ and $C^k(U)$ with compact support by $C_c(U), C_c^k(U)$.

Definition 1.41. Consider a sequence of functions $\{u_m\}_1^\infty$ with $u_m : U \to \mathbb{R}$ and a function $u : U \to \mathbb{R}$, we have

- $u_m \to u$ point-wise on U if

$$\forall x \in U,\ \delta > 0,\ \exists M \in \mathbb{N}, \quad \text{such that } m > M \implies |u_m(x) - u(x)| < \delta.$$

- $u_m \to u$ uniformly on U if

$$\forall \delta > 0,\ \exists M \in \mathbb{N}, \quad \text{such that } \forall x \in U,\ m > M \implies |u_m(x) - u(x)| < \delta.$$

Definition 1.42. $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** if $\forall \epsilon > 0, \exists \delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),

$$\sum_{k=1}^n (b_k - a_k) < \delta \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

1.6.2 Lebesgue Spaces

See more in my Measure Theory Notes.

Definition 1.43. We denote the Lebesgue measure by λ on $\mathbb{R}^n$. We denote $\int_A f d\lambda$ by $\int_A f(x)dx$ for any measurable set $A \subseteq \mathbb{R}^n$.

Definition 1.44. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, we define

$$\mathcal{L}^1(\Omega) := \left\{ f : \Omega \to \mathbb{R} \middle| \int_\Omega |f(x)| dx < \infty \right\}.$$

Definition 1.45. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, and $1 \le p < \infty$ we define

$$\mathcal{L}^p(\Omega) := \left\{ f : \Omega \to \mathbb{R} | f^p \in L^1(\Omega) \right\} = \left\{ f : \Omega \to \mathbb{R} \middle| \int_\Omega |f(x)|^p dx < \infty \right\}.$$

In addition, we define the norm

$$||f||_p := \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Definition 1.46. The **essential supremum** of a function $u : U \to \mathbb{R}$ is

$$\operatorname{ess\,sup} f := \inf \left\{ M \in \mathbb{R} : |\{x : f(x) > M\}| = 0 \right\}.$$

Definition 1.47. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, we define

$$\mathcal{L}^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} | \operatorname{ess\,sup} f < \infty \right\}.$$

In addition, we define the norm

$$||f||_\infty := \operatorname{ess\,sup} f.$$

Definition 1.48. Two measurable functions $f, g : \Omega \to \mathbb{R}$ are said to be equal almost everywhere if $\{x \in \Omega : f(x) \neq g(x)\}$ has measure zero.

Proposition 1.39. *For any $1 \le p \le \infty$, we have $||f-g||_p = 0 \iff f = g$ almost everywhere.*

Definition 1.49. For any $1 \le p \le \infty$, if we identify $f, g \in \mathcal{L}^p(\Omega)$ by $f \sim g \iff f = g$ almost everywhere, we get the quotient space

$$L^p := \mathcal{L}^p/_\sim = \{[f] : f \in L^p(\Omega)\}$$

to be the collection of all equivalence classes of functions in $\mathcal{L}^p$.

Theorem 1.40. *(completeness of L^p)*
For any $1 \le p \le \infty$, we have the space $(L^p, ||\cdot||_p)$ is a Banach space, where $||[f]||_p := ||f||_p$ for any representative $f \in [f]$. One can check this norm is well-defined.

Theorem 1.41. *For any $1 \le p < \infty$,*

- *$C_c(U)$ is dense in $L^p(U)$.*
- *$C(\bar{U})$ is dense in $L^p(U)$.*

Definition 1.50. Let $U, V \subseteq \mathbb{R}^n$ be open, we say that V is **compactly contained** in U if $V \subseteq \bar{V} \subseteq U$, and $\bar{V}$ is compact. We write this as $V \subset\subset U$.

Definition 1.51. The **locally summable spaces** are

$$L^p_{loc}(U) := \{f : U \to \mathbb{R} : \forall V \subset\subset U, u \in L^p(V)\}.$$

Definition 1.52. We say some property holds for $L^p_{loc}(U)$, if it holds $\forall L^p(V)$ such that $V \subset\subset U$.
For instance, let $(f_n)_{n=1}^\infty \subseteq L^p_{loc}(U)$, then $f_n \to f$ in $L^p_{loc}(U)$ if $f_n \to f$ in $L^p(V)$, $\forall V \subset\subset U$.

Proposition 1.42. *For any $1 \le p \le \infty$, we have*

$$L^p(U) \subseteq L^1_{loc}(U).$$

Example 1.6.1. Let $u(x) = \frac{1}{x}$ on $U = (0,1)$.
We have $\int_0^1 |u| dx = \infty$, and thus $u \notin L^1(U)$. However, $u \in L^1_{loc}(U)$.

Theorem 1.43. *(Holder's Inequality)*
Assume $1 \le p, q \le \infty, \frac{1}{p} + \frac{1}{q} = 1$.
If $u \in L^p(U), v \in L^q(U)$, we have

$$\int_U |uv| dx \le ||u||_p ||v||_q.$$

For $a, b \in \mathbb{R}^n$, we have

$$\left|\sum_{k=1}^n a_k b_k\right| \le \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \left(\sum_{k=1}^n |b_k|^q\right)^{1/q}$$

Theorem 1.44. *(Minkowski's Inequality)*
Assume $1 \le p \le \infty$.
Let $u, v \in L^p(U)$, we have

$$||u+v||_p \le ||u||_p + ||v||_p.$$

For $a, b \in \mathbb{R}^n$, we have

$$\left(\sum_{k=1}^n |a_k + b_k|^p\right)^{1/p} \le \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p\right)^{1/p}$$

Theorem 1.45. *(Lebesgue Monotone Convergence)*
Let $f_n : X \to [0, \infty]$ be measurable functions with $0 \le f_1 \le f_2 \le \cdots \le \infty$. Let $f(x) := \lim_{n\to\infty} f_n(x)$, then $f : X \to [0,\infty]$ is measurable, and

$$\lim_{n\to\infty} \int_X f_n dx = \int_X f dx.$$

Theorem 1.46. *(Lebesgue Dominated Convergence)*
Let $f_n : X \to \mathbb{C}$ be measurable functions, defined almost everywhere on X, such that $f(x) := \lim_{n\to\infty} f_n(x)$ is defined almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X,\mu)$, such that for almost everywhere $x \in X, \forall n \in \mathbb{N},\ |f_n(x)| \leq g(x)$, then $f \in \mathcal{L}^1(X,\mu)$, and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n\to\infty} \int_X |f - f_n| d\mu = 0.$$

Theorem 1.47. *We have that*

$$L^q(U) \simeq L^p(U)^*,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and the isometric isomorphism $L^q(U) \overset{\sim}{\to} L^p(U)^*;\ u \mapsto u^*$ is defined to be

$$\forall v \in L^p(U), \langle u^* | v \rangle := \int_U uvdx.$$

Remark. We will abuse the notation, and write $\langle u|v\rangle := \int_U uvdx$ with $u \in L^q(U)$ instead of $u^* \in L^p(U)^*$.

Corollary 1.48. *In particular, $L^2(U) \simeq L^2(U)^*$, with the isometric isomorphism $L^2(U) \to L^2(U)^*;\ u \mapsto u^*$ is defined to be*

$$\forall v \in L^2(U), \langle u^* | v \rangle = \int_U uvdx = \langle u, v \rangle_{L^2(U)}.$$

Definition 1.53. For $f : U \to \mathbb{R}^m$, we define

$$||f||_{L^p(U)} := \Big|\Big| ||f||_p \Big|\Big|_{L^p(U)}.$$

2 Sobolev Spaces

This section follows Chapter 5 in Evan's book.

2.1 Holder Spaces

Definition 2.1. For $u : U \to \mathbb{R}$ be bounded and continuous, we write

$$||u||_{C(\bar{U})} := \sup_{x \in U} |u(x)|.$$

Definition 2.2. A function $u : U \to \mathbb{R}$ is **Holder continuous** with $0 < \gamma \leq 1$ if

$$\exists C, \text{ such that } \forall x, y \in U, \ |u(x) - u(y)| \leq C||x - y||^{\gamma}.$$

Definition 2.3. The γ^{th}**-Holder semi-norm** of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{x,y \in U, x \neq y} \left(\frac{|u(x) - u(y)|}{||x - y||^{\gamma}} \right).$$

The γ^{th}**-Holder norm** of $u : U \to \mathbb{R}$ is

$$||u||_{C^{0,\gamma}(\bar{U})} := [u]_{C^{0,\gamma}(\bar{U})} + ||u||_{C(\bar{U})}.$$

Definition 2.4. For $k \in \mathbb{N}, u \in C^k(\bar{U})$ we define

$$||u||_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} ||D^{\alpha} u||_{C(\bar{U})} + \sum_{|\alpha| = k} [D^{\alpha} u]_{C^{0,\gamma}(\bar{U})}.$$

The **Holder Space** is

$$C^{k,\gamma}(\bar{U}) := \left\{ u \in C^k(\bar{U}) : ||u||_{C^{k,\gamma}(\bar{U})} < \infty \right\}.$$

Theorem 2.1.

$$\left(C^{k,\gamma}(\bar{U}), ||\cdot||_{C^{k,\gamma}(\bar{U})} \right)$$

is a Banach Space.

2.2 Convolution and Mollification

Definition 2.5. For $f, g : \mathbb{R}^n \to \mathbb{R}$, we define the **convolution** $f * g : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ to be

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y) g(y) dy.$$

Proposition 2.2. *For $f, g : \mathbb{R}^n \to \mathbb{R}$, we have $f * g = g * f$.*

Proof. Take $z := x - y$, we have $y = x - z$, and $d(z^i) = -d(y^i)$. We have that for any $x \in \mathbb{R}$,

$$\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}^n} f(x - y) g(y) dy \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x - y) g(y) d(y^1) \cdots d(y^n) \\
&= (-1)^n \int_{\infty}^{-\infty} \cdots \int_{\infty}^{-\infty} f(z) g(x - z) d(z^1) \cdots d(z^n) \\
&= (-1)^n (-1)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z) g(x - z) dz \\
&= \int_{\mathbb{R}^n} f(z) g(x - z) dz \\
&= (g * f)(x).
\end{aligned}$$

$\square$

Proposition 2.3.

$$\text{Supp}(f * g) \subseteq \text{Supp}(f) + \text{Supp}(g).$$

Proof. Let $f^x(y) := f(x-y)$, we have $f * g(x) = \int_{\mathbb{R}^n} f^x(y)g(y)dy$.
Suppose $\text{Supp}(f^x) \cap \text{Supp}(g) = \emptyset$, then we have $(f * g)(x) = 0$.
In addition,

$$\begin{aligned} &\text{Supp}(f^x) \cap \text{Supp}(g) \neq \emptyset \\ \iff &\exists y, x-y \in \text{Supp}(f), y \in \text{Supp}(g) \\ \iff &x \in \text{Supp}(f) + \text{Supp}(g). \end{aligned}$$

Thus $\text{Supp}(f * g) \subseteq \{x \in \mathbb{R}^n : \text{Supp}(f^x) \cap \text{Supp}(g) \neq \emptyset\} = \text{Supp}(f) + \text{Supp}(g)$. □

Proposition 2.4 (Young's Convolution Inequality)**.** *Let $f \in L^1(\mathbb{R}^n), g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, then for a.e.$x \in \mathbb{R}^n$, the function $f(x-y)g(y)$ is integrable. Thus $f * g : \mathbb{R}^n \to \mathbb{R}$ is well-defineda.e.. In addition, $f * g \in L^p(\mathbb{R}^n)$, and*

$$||f * g||_{L^p(\mathbb{R}^n)} \leq ||f||_{L^1(\mathbb{R}^n)} ||g||_{L^p(\mathbb{R}^n)}.$$

Definition 2.6.

$$\bar{B}(x,r) = \{y \in \mathbb{R}^n : ||x-y|| \leq r\}$$

is the closed ball around x of radius r, and

$$B(x,r) = \{y \in \mathbb{R}^n : ||x-y|| < r\}$$

is the closed ball around x of radius r.

Definition 2.7. For $\epsilon > 0$,

$$U_\epsilon := \{x \in U : \text{dist}(x, \partial u) > \epsilon\}.$$

Remark. This definition does not require U to be bounded.

Definition 2.8. The **standard mollifier** $\eta(x) \in C^\infty(\mathbb{R}^n)$ is defined as

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|-1}\right), & |x| < 1 \\ 0, & o.w. \end{cases},$$

with C such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.
For each $\epsilon > 0$,

$$\eta_\epsilon := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

Proposition 2.5. $\forall \epsilon > 0$*, we have*

1. $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$,
2. $\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1$,
3. $\text{Supp}(\eta_\epsilon) \subseteq \bar{B}(0,\epsilon)$.

Definition 2.9. Let $f \in L^1_{loc}(U), \epsilon > 0$, its **mollification** $f^\epsilon : U_\epsilon \to \mathbb{R}$ is defined as

$$f^\epsilon(x) := \eta_\epsilon * f := \int_U \eta_\epsilon(x-y)f(y)dy = \int_{\bar{B}(0,\epsilon)} f(x-z)\eta_\epsilon(z)dz.$$

Remark. When $U \subsetneq \mathbb{R}^n$, the mollification $\eta_\epsilon * f$ is not using the formal definition of convolution, but we will soon see the abuse of notation makes sense.

Proposition 2.6. *Let f^ϵ be defined as above, if we zero-extend f outside of U to be*

$$\bar{f}(x) := \begin{cases} f(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases},$$

we have $\forall\ x \in U_\epsilon$,

$$\begin{aligned}(\eta_\epsilon * \bar{f})(x) &= \int_{\mathbb{R}^n} \eta_\epsilon(x-y)\bar{f}(y)dy \\ &= \int_U \eta_\epsilon(x-y)f(y)dy \\ &= f^\epsilon(x)\end{aligned}$$

Theorem 2.7. *Let f^ϵ be defined as above, we have:*

1. $f^\epsilon \in C^\infty(U_\epsilon)$,
2. $D^\alpha(f^\epsilon) = (D^\alpha \eta_\epsilon) * f$ *on* U_ϵ,
3. $f^\epsilon \to f$ *a.e., as* $\epsilon \to 0$,
4. *If* $f \in C(U)$, *we have* $f^\epsilon \to f$ *uniformly on compact subsets of* U,
5. *If* $1 \le p < \infty, f \in L^p_{loc}(U)$, *we have* $f^\epsilon \to f$ *in* $L^p_{loc}(U)$. *Namely,* $f^\epsilon \to f$ *in* $L^p(V)$, $\forall V \subset\subset U$.
6. $\mathrm{Supp}(f^\epsilon) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(\eta_\epsilon) = \mathrm{Supp}(f) + \bar{B}(0, \epsilon)$.

Proposition 2.8. *Let $1 \le p \le \infty$. Let $u \in L^p(U)$, then for any $\epsilon > 0$, $U_\epsilon \supseteq V \supseteq \mathrm{Supp}(u^\epsilon)$, we have*

$$||u^\epsilon||_{L^p(V)} \le ||u||_{L^p(V)}.$$

Proof. Notice that $u^\epsilon \in C^\infty(U_\epsilon) \subseteq L^1(U_\epsilon)$.
If we zero-extend u outside of U to be $\bar{u}(x) := \begin{cases} u(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$, we have

$$\begin{aligned}||\bar{u}||_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\bar{u}(x)|^p dx \\ &= \int_U |u(x)|^p dx + 0 \\ &= ||u||^p_{L^p(U)} \\ &< \infty.\end{aligned}$$

Thus, $\bar{u} \in L^p(\mathbb{R}^n)$.
By 2.4, we have

$$\begin{aligned}||u^\epsilon||_{L^p(V)} &= ||\eta_\epsilon * \bar{u}||_{L^p(V)} \\ &\le ||\eta_\epsilon * \bar{u}||_{L^p(\mathbb{R}^n)} \\ &\le ||\eta_\epsilon||_{L^1(\mathbb{R}^n)} ||\bar{u}||_{L^p(\mathbb{R}^n)} \\ &= \left(\int_{\mathbb{R}^n} |\eta_\epsilon(x)| dx\right) ||u||_{L^p(V)} \\ &= ||u||_{L^p(V)}.\end{aligned}$$

$\square$

2.3 Weak derivative and Sobolev Spaces

Theorem 2.9. *For $u \in C^k(U), \phi \in C_c^\infty(U), |\alpha| = k$, integration by parts gives:*

$$\int_U uD^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx.$$

Definition 2.10. Suppose $u, v \in L^1_{loc}(U)$, then v is the $\alpha^{th}-$**weak derivative** of u if

$$\forall \phi \in C_c^\infty(U), \int_U uD^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx.$$

If v exists, we say that $D^\alpha u = v$ in the weak sense. Otherwise, u does not possess a α^{th} weak derivative.

Theorem 2.10. *Suppose $v \in L^1_{loc}(U)$ be such that*

$$\forall \phi \in C_c^\infty(U), \int_U \phi v dx = 0,$$

we must have $v = 0$ a.e..

Proof. By 2.7, we have $v^\epsilon \to v$ a.e., as $\epsilon \to 0$.
Noe pick any such $y \in U$ where $v^\epsilon(y) \to v(y)$. Since U is open, we can find $r > 0$, such that $\bar{B}(y, r) \subset U$.
Now we define the function $\phi_{y,\epsilon}(x) := \eta_\epsilon(y - x)$ for each $\epsilon \in (0, r)$.
Since $\text{Supp}(\eta_\epsilon) \subseteq \bar{B}(0, \epsilon)$, we have $\text{Supp}(\phi_{y,\epsilon}) \subseteq \bar{B}(y, \epsilon) \subset U$ is compactly contained in U.
Also, $\phi_{y,\epsilon} \in C^\infty(\mathbb{R}^n) \subset C^\infty(U)$.
This shows that $\phi_{y,\epsilon} \in C_c^\infty(U)$.
Now we have

$$\begin{aligned} 0 &= \int_U \phi_{y,\epsilon} v dx \\ &= \int_U \eta_\epsilon(y - x) v(x) dx \\ &= v^\epsilon(y). \end{aligned}$$

Since this holds for all $\epsilon \in (0, r)$, we must have $v(y) = \lim_{\epsilon \to 0} v^\epsilon(y) = 0$.
Since this holds for a.e. $y \in U$, we have that $v = 0$a.e.. □

Proposition 2.11. *If $D^\alpha u$ exists, it is uniquely defined up to a set of measure zero.*

Proof. Suppose $v, \tilde{v}$ are both $D^\alpha u$, then $\forall \phi \in C_c^\infty(U)$,

$$\int_U uD^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx = (-1)^{|\alpha|} \int_U \tilde{v}\phi dx.$$

Thus $\forall \phi \in C_c^\infty(U), \int_U (v - \tilde{v})\phi dx = 0$.
By the previous theorem, we have that $v = \tilde{v}$a.e.. □

Definition 2.11. Let $k \in \mathbb{N}, 1 \le p \le \infty, u \in L^1_{loc}(U)$, suppose $D^\alpha u$ exists in the weak sense for each $|\alpha| \le k$.
The **Sobolev norm** is

$$||u||_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \le k} ||D^\alpha u||^p_{L^p(U)}\right)^{1/p}, & 1 \le p < \infty \\ \sum_{|\alpha| \le k} \text{ess sup}_{x \in U} |D^\alpha u(x)| \simeq \max_{|\alpha| \le k} ||D^\alpha u||_{L^\infty(U)}, & p = \infty \end{cases}.$$

Definition 2.12. For $k = 1$, we write

$$||Du||^p_{L^p(U)} := \int_U ||Du||^p_p dx = \int_U \sum_{i=1}^n |\partial_i u|^p dx = \sum_{i=1}^n ||\partial_i u||^p_{L^p(U)}$$

for $1 \le p < \infty$, and

$$||Du||_{L^\infty(U)} := \operatorname*{ess\,sup}_{x\in U} ||Du(x)||_1 = \operatorname*{ess\,sup}_{x\in U} \sum_{i=1}^n |\partial_i u(x)| = \sum_{i=1}^n ||\partial_i u||_{L^\infty(U)}$$

for $p = \infty$.
In this case,

$$||u||_{W^{1,p}(U)} = \begin{cases} \left(||u||^p_{L^p(U)} + ||Du||^p_{L^p(U)}\right)^{1/p} & 1 \le p < \infty \\ ||u||_{L^\infty(U)} + ||Du||_{L^\infty(U)} & p = \infty. \end{cases}.$$

Proposition 2.12. *Let $k \in \mathbb{N}, 1 \le p \le \infty, u \in L^1_{loc}(U)$, we have*

$$\forall |\alpha| \le k, ||u||_{W^{k,p}(U)} \ge ||D^\alpha u||_{L^p(U)}.$$

Definition 2.13. The Sobolev space is defined as

$$W^{k,p}(U) := \left\{ v \in L^1_{loc}(U) : ||v||_{W^{k,p}(U)} < \infty \right\}.$$

Definition 2.14.

$$H^k(U) := W^{k,2}(U).$$

Remark.

$$W^{0,1}(U) = H^0(U) = L^2(U).$$

Definition 2.15. Let $(u_m)_{m=1}^\infty, u \in W^{k,p}(U)$, then

- $u_m \to u$ in $W^{k,p}(U)$ if $\lim_{m\to\infty} ||u_m - u||_{W^{k,p}(U)} = 0$.
- $u_m \to u$ in $W^{k,p}_{loc}(U)$ if $u_m \to u$ in $W^{k,p}(V)$ for all $V \subset\subset U$.

Definition 2.16.

$$W^{k,p}_0(U) = \overline{C^\infty_c(U)} = \left\{ u \in W^{k,p}(U) : \exists (u_m)_{m=1}^\infty \subset C^\infty_c(U) \text{ such that } u_m \to u \text{ in } W^{k,p}(U) \right\}.$$

$$H^k_0(U) = W^{k,2}_0.$$

Remark. $W^{k,p}_0(U)$ are those $u \in W^{k,p}(U)$ such that $D^\alpha u = 0$ on ∂U.

Theorem 2.13. *Assume $u, v \in W^{k,p}(U), |\alpha| \le k$, then*

1. *$D^\alpha u \in W^{k-|\alpha|,p}(U)$.*
2. *$D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u, \forall \alpha, \beta$ such that $|\alpha| + |\beta| \le k$.*
3. *$\lambda u + v \in W^{k,p}(U), D^\alpha(\lambda u + v) = \lambda D^\alpha u + D^\alpha v, \forall \lambda \in \mathbb{R}$.*
4. *$\forall V \subseteq U$ be open, $u \in W^{k,p}(U)$.*

Proof. 1. This is by definition.

2. Consider any $\phi \in C^\infty_c(U)$, we have

$$\begin{aligned} \int_U D^\alpha(D^\beta u)\phi dx &= (-1)^{|\alpha|} \int_U D^\beta u D^\alpha \phi dx \\ &= (-1)^{|\alpha|}(-1)^{|\beta|} \int_U u D^\beta(D^\alpha \phi) dx \\ &= (-1)^{|\alpha+\beta|} \int_U u D^{\alpha+\beta}\phi dx \\ &= \int_U D^{\alpha+\beta} u \phi dx. \end{aligned}$$

Thus $D^{\alpha+\beta}u = D^\alpha(D^\beta u)$. Similarly, $D^{\alpha+\beta}u = D^\beta(D^\alpha u)$.

3. See A2.
4. See A2.

$\square$

Proposition 2.14 (Leibniz rule for weak derivatives)**.** *Assume $u \in W^{k,p}(U), |\alpha| \le k$. If $\xi \in C_c^\infty(U), \xi u \in W^{k,p}(U)$, and the Leibniz formula holds:*

$$D^\alpha(\xi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u,$$

where $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, and $\alpha! := \alpha_1! \cdots \alpha_n!$.

Proof. We have $\forall \phi \in C_c^\infty(U), \int_U \xi u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha(\xi u) \phi dx$.
We prove by induction:
The base case is $|\alpha| = 1$, we have by Leibniz rule on regular derivatives:

$$\begin{aligned} D^\alpha(\xi\phi) &= \xi D^\alpha \phi + \phi D^\alpha \xi \\ \int_U \xi u D^\alpha \phi dx &= \int_U u(D^\alpha(\xi\phi) - \phi D^\alpha \xi) dx \\ &= \int_U u D^\alpha(\xi\phi) dx - \int_U u\phi D^\alpha \xi dx \\ &= -\int_U \xi\phi D^\alpha u dx - \int_U u\phi D^\alpha \xi dx \\ &= -\int_U \phi(\xi D^\alpha u + u D^\alpha \xi) dx. \end{aligned}$$

Since this hold for any $\phi \in C_c^\infty(U)$, we have

$$\xi D^\alpha u + u D^\alpha \xi = D^\alpha(u\xi).$$

Now suppose $l < k$ and the result holds $\forall |\beta| \le l$.

Consider any $|\alpha| = l+1$, we have $\alpha = \beta + \gamma$ where $|\beta| = l, |\gamma| = 1$.

$$\begin{aligned}
\int_U \xi u D^\alpha \phi dx &= \int_U \xi u D^\beta (D^\gamma \phi) dx \\
&= (-1)^{|\beta|} \int_U D^\beta (\xi u) D^\gamma \phi dx \\
&= (-1)^{|\beta|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\beta-\sigma} u D^\gamma \phi dx \\
&= (-1)^{|\beta|} (-1)^{|\gamma|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\gamma (D^\sigma \xi D^{\beta-\sigma} u) \phi dx \\
&= (-1)^{|\beta+\gamma|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left(D^\sigma \xi D^\gamma D^{\beta-\sigma} u + D^{\beta-\sigma} u D^\gamma D^\sigma \xi \right) \phi dx \\
&= (-1)^{|\alpha|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left(D^\sigma \xi D^{\gamma+\beta-\sigma} u + D^{\beta-\sigma} u D^{\gamma+\sigma} \xi \right) \phi dx \\
&= (-1)^{|\alpha|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left(D^\sigma \xi D^{\alpha-\sigma} u + D^{\alpha-(\gamma+\sigma)} u D^{\gamma+\sigma} \xi \right) \phi dx \\
&= (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u + \sum_{\rho \le \alpha, \rho_j \ge 1} \binom{\beta}{\rho - \gamma} D^{\alpha-\rho} u D^\rho \xi \right) \phi dx \\
&= (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \le \alpha} \left(\mathbb{1}_{\sigma_j \le \alpha_j - 1} \binom{\beta}{\sigma} + \mathbb{1}_{\sigma_j \ge 1} \binom{\beta}{\sigma - \gamma} \right) D^\sigma \xi D^{\alpha-\sigma} u \right) \phi dx,
\end{aligned}$$

where $\gamma_i = \delta_{ij}$.
Now consider any $\sigma \le \alpha$.
If $\sigma_j = 0$, we have

$$\begin{aligned}
\binom{\beta}{\sigma} &= \frac{\beta!}{\sigma!(\beta-\sigma)!} \\
&= \frac{\beta!(\beta_j+1)}{\sigma!(\beta-\sigma)!(\beta_j+\sigma_j+1)} \\
&= \frac{(\beta+\gamma)!}{\sigma!(\beta-\sigma+\gamma)!} \\
&= \frac{\alpha!}{\sigma!(\alpha-\sigma)!} \\
&= \binom{\alpha}{\sigma}.
\end{aligned}$$

If $\sigma_j = \alpha_j$, we have

$$\begin{aligned}
\binom{\beta}{\sigma-\gamma} &= \frac{\beta!}{(\sigma-\gamma)!(\beta-\sigma+\gamma)!} \\
&= \frac{\beta!\alpha_j}{\alpha_j(\sigma-\gamma)!(\alpha-\sigma)!} \\
&= \frac{\beta!(\beta_j+1)}{\sigma_j(\sigma-\gamma)!(\alpha-\sigma)!} \\
&= \frac{(\beta+\gamma)!}{(\sigma-\gamma+\gamma)!(\alpha-\sigma)!} \\
&= \frac{\alpha!}{\sigma!(\alpha-\sigma)!} \\
&= \binom{\alpha}{\sigma}.
\end{aligned}$$

If $1 \leq \sigma_j \leq \alpha_j - 1$, we have

$$\begin{aligned}
\binom{\beta}{\sigma} + \binom{\beta}{\sigma-\gamma} &= \frac{\beta!}{\sigma!(\beta-\sigma)!} + \frac{\beta!}{(\sigma-\gamma)!(\beta-\sigma+\gamma)!} \\
&= \frac{\beta!(\beta_j-\sigma_j+1)}{\sigma!(\beta-\sigma)!(\beta_j-\sigma_j+1)} + \frac{\beta!\sigma_j}{\sigma_j(\sigma-\gamma)!(\beta-\sigma+\gamma)!} \\
&= \frac{\beta!(\beta_j-\sigma_j+1)+\beta!\sigma_j}{\sigma!(\beta-\sigma+\gamma)!} \\
&= \frac{\beta!(\beta_j+1)}{\sigma!(\beta-\sigma+\gamma)!} \\
&= \frac{(\beta+\gamma)!}{\sigma!(\beta-\sigma+\gamma)!} \\
&= \frac{\alpha!}{\sigma!(\alpha-\sigma)!} \\
&= \binom{\alpha}{\sigma}.
\end{aligned}$$

Thus we can see that

$$\int_U \xi u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u \right) \phi dx.$$

Since ϕ is arbitrary, we have that

$$D^\alpha(\xi u) = \sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u.$$

Inductively, we can prove this for any $|\alpha| = n \geq 1$. $\square$

Theorem 2.15. *$\left(W^{k,p}(U), ||\cdot||_{W^{k,p}(U)}\right)$ is a Banach space for $k \in \mathbb{N}, 1 \leq p \leq \infty$.*

Proof. See A2 for the proof of $||\cdot||_{W^{1,\infty}(U)}$ is a norm.
Now for $1 \leq p < \infty$, we want to check:

1. If $||u||_{W^{k,p}(U)} = 0$, then $||u||_{L^p(U)} = 0$, and thus $u = 0$a.e. on U.
2. If $u = 0$ a.e. on U, then $\forall \phi \in C_c^\infty(U)$, we have

$$\int_U D^\alpha u \phi dx = (-1)^{|\alpha|} \int_U u D^\alpha \phi dx = 0.$$

Thus $D^\alpha u = 0$ a.e. for any $|\alpha| \leq k$.
Thus $||u||_{W^{k,p}(U)} = 0$.

3. Let $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
||\lambda u||_{W^{k,p}(U)} &= \left(\sum_{|\alpha| \leq k} ||D^\alpha (\lambda u)||^p_{L^p(U)} \right)^{1/p} \\
&= \left(\sum_{|\alpha| \leq k} ||\lambda D^\alpha (u)||^p_{L^p(U)} \right)^{1/p} \\
&= \left(\sum_{|\alpha| \leq k} |\lambda|^p ||D^\alpha (u)||^p_{L^p(U)} \right)^{1/p} \\
&= |\lambda| \left(\sum_{|\alpha| \leq k} ||D^\alpha (u)||^p_{L^p(U)} \right)^{1/p} \\
&= |\lambda| ||u||_{W^{k,p}(U)}.
\end{aligned}$$

4. Consider any $u, v \in W^{k,p}(U)$,

$$\begin{aligned}
||u + v||_{W^{k,p}(U)} &= \left(\sum_{|\alpha| \leq k} ||D^\alpha (u + v)||^p_{L^p(U)} \right)^{1/p} \\
&\leq \left(\sum_{|\alpha| \leq k} \left(||D^\alpha u||_{L^p(U)} + ||D^\alpha v||_{L^p(U)} \right)^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha| \leq k} ||D^\alpha u||^p_{L^p(U)} \right)^{1/p} + \left(\sum_{|\alpha| \leq k} ||D^\alpha u||^p_{L^p(U)} \right)^{1/p} \\
&= ||u||_{W^{k,p}(U)} + ||v||_{W^{k,p}(U)}.
\end{aligned}$$

Thus $||\cdot||_{W^{k,p}(U)}$ is a norm.
Consider any Cauchy sequence $(u_m)_{m=1}^\infty$.
Given any $\epsilon > 0, \exists N \geq 1,$ such that $\forall n, m \geq N, ||u_m - u_n||_{W^{k,p}(U)} < \epsilon$.
Consider any $|\alpha| \leq k$, we have

$$||D^\alpha u_m - D^\alpha u_n||_{L^p(U)} = ||u||_{W^{k,p}(U)} \geq ||D^\alpha (u_m - u_n)||_{L^p(U)} \leq ||u_m - u_n||_{W^{k,p}(U)} < \epsilon.$$

Thus $(D^\alpha u_n)_{n=1}^\infty$ must be a Cauchy sequence in $(L^p(U), ||\cdot||_{L^p(U)})$ for any $|\alpha| \leq k$.
Since $(L^p(U), ||\cdot||_{L^p(U)})$ is complete, there must be some

$$u_\alpha \in L^p(U) \text{ such that } \lim_{n \to \infty} ||u_\alpha - D^\alpha u_n||_{L^p(U)} = 0.$$

In particular, we have $u \in L^p(U),$ such that $\lim_{n \to \infty} ||u - u_n||_{L^p(U)} = 0$.
Now consider any $|\alpha| \leq k$.

Given any $\phi \in C_c^\infty(U)$, we have

$$\begin{aligned}
\left|\int_U uD^\alpha \phi dx - \int_U u_n D^\alpha \phi dx\right| &= \left|\int_U (u-u_n)D^\alpha \phi dx\right| \\
&\leq \int_U |(u-u_n)D^\alpha \phi| dx \\
&\leq ||u-u_n||_{L^p(U)} ||D^\alpha \phi||_{L^{\frac{p}{p-1}}(U)}, \\
\left|\int_U u_\alpha \phi dx - \int_U D^\alpha u_n \phi dx\right| &= \left|\int_U (u_\alpha - D^\alpha u_n)\phi dx\right| \\
&\leq \int_U |(u_\alpha - D^\alpha u_n)\phi| dx \\
&\leq ||u_\alpha - D^\alpha u_n||_{L^p(U)} ||\phi||_{L^{\frac{p}{p-1}}(U)}.
\end{aligned}$$

Since $u_n \to u, D^\alpha u_n \to u_\alpha$ in $L^p(U)$, and $||\phi||_{L^{\frac{p}{p-1}}(U)}, ||D^\alpha \phi||_{L^{\frac{p}{p-1}}(U)} < \infty$, those two limits converges to 0. Thus we have

$$\begin{aligned}
\int_U uD^\alpha \phi dx &= \lim_{n\to\infty} \int_U u_n D^\alpha \phi dx \\
&= \lim_{n\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_n \phi dx \\
&= (-1)^{|\alpha|} \int_U u_\alpha \phi dx.
\end{aligned}$$

Since this is true for any $\phi \in C_c^\infty(U)$, we have that $D^\alpha u = u_\alpha = \lim_{n\to\infty} D^\alpha u_n$ in $L^p(U)$. Since this is true for any $|\alpha| \leq k$, we have that $u_n \to u$ in $W^{k,p}(U)$. ◻

Proposition 2.16. *For any $1 \leq s \leq r < \infty, k \geq 1$, and bounded U, we have some constant $C := |U|^{\frac{1}{s}-\frac{1}{r}} m^{\frac{1}{s}-\frac{1}{r}}$, where $m = |\{\beta \in \mathbb{N}^n : |\beta| \leq k\}|$, such that*

$$\forall u \in W^{k,r}(U),\ ||u||_{W^{k,s}(U)} \leq C||u||_{W^{k,r}(U)}, u \in W^{k,s}(U).$$

Proof. We have

$$\begin{aligned}
||u||^s_{W^{1,s}(U)} &= \sum_{|\beta|\leq 1} ||D^\beta u||^s_{L^s(U)} \\
&\leq \sum_{|\beta|\leq 1} \left(|U|^{\frac{1}{s}-\frac{1}{r}} ||D^\beta u||_{L^r(U)}\right)^s \\
&= \left(|U|^{\frac{1}{s}-\frac{1}{r}}\right)^s \sum_{|\beta|\leq 1} ||D^\beta u||^{r*\frac{s}{r}}_{L^r(U)} \\
&\leq \left(|U|^{\frac{1}{s}-\frac{1}{r}}\right)^s m^{1-\frac{s}{r}} \left(\sum_{|\beta|\leq 1} ||D^\beta u||^r_{L^r(U)}\right)^{\frac{s}{r}} \\
&= \left(|U|^{\frac{1}{s}-\frac{1}{r}}\right)^s m^{1-\frac{s}{r}} \left(||D^\alpha u||^r_{W^{1,r}(U)}\right)^{\frac{s}{r}} \\
&= \left(|U|^{\frac{1}{s}-\frac{1}{r}}\right)^s m^{1-\frac{s}{r}} ||D^\alpha u||^s_{W^{1,r}(U)} \\
&\implies \\
||u||_{W^{1,s}(U)} &\leq |U|^{\frac{1}{s}-\frac{1}{r}} m^{\frac{1}{s}-\frac{1}{r}} ||D^\alpha u||_{W^{1,r}(U)}.
\end{aligned}$$

◻

Proposition 2.17. *For any $u \in W^{k,p}(U)$, and $|\alpha| \le k, \epsilon > 0$, we have that*

$$D^\alpha u^\epsilon|_{U_\epsilon} = (\eta_\epsilon * D^\alpha u)|_{U_\epsilon}.$$

Proof. Fix any $x \in U_\epsilon$, we have

$$\begin{aligned} D^\alpha u^\epsilon(x) &= D^\alpha(\eta_\epsilon * u)(x) \\ &= (D^\alpha \eta_\epsilon * u)(x) \qquad\qquad 2.7 \\ &= \int_U D^\alpha \eta_\epsilon(x-y)u(y)dy, \end{aligned}$$

Consider $\eta_{\epsilon,x}(y) := \eta_\epsilon(x-y)$, we can see $\forall i \in [n],\ \partial_i \eta_{\epsilon,x}(y) = -\partial_i \eta_\epsilon(x-y)$, thus we have

$$\begin{aligned} D^\alpha u^\epsilon(x) &= \int_U D^\alpha \eta_\epsilon(x-y)u(y)dy \\ &= (-1)^{|\alpha|} \int_U D^\alpha \eta_{\epsilon,x}(y)u(y)dy \\ &= \int_U \eta_{\epsilon,x}(y) D^\alpha u(y)dy \\ &= \int_U \eta_\epsilon(x-y) D^\alpha u(y)dy \\ &= (\eta_\epsilon * D^\alpha u)(x). \end{aligned}$$

Since this holds for any $x \in U_\epsilon$, we proved our result. □

Corollary 2.18. *Let $1 \le p \le \infty, k \ge 1$. Let $u \in W^{k,p}(U)$, then for any $\epsilon > 0$, $U_\epsilon \subseteq V \supseteq \mathrm{Supp}(u) + \bar{B}(0,\epsilon)$, we have that*

$$||u^\epsilon||_{W^{k,p}(V)} \le ||u||_{W^{k,p}(V)}.$$

Proof. By 2.17, $\forall |\alpha| \le k$, we have $D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$ on the entire U_ϵ and thus on V.
Since $\forall |\alpha| \le k,\ \mathrm{Supp}(D^\alpha u) \subseteq \mathrm{Supp}(u)$, we have $\mathrm{Supp}(\eta_\epsilon * D^\alpha u) \subseteq \mathrm{Supp}(u) + \bar{B}(0,\epsilon) \subseteq V$. By 2.8,

$$\begin{aligned} ||u^\epsilon||^p_{W^{k,p}(V)} &= \sum_{|\alpha| \le k} ||D^\alpha u^\epsilon||_{L^p(V)} \\ &= \sum_{|\alpha| \le k} ||\eta_\epsilon * D^\alpha u||_{L^p(V)} \\ &\le \sum_{|\alpha| \le k} ||D^\alpha u||_{L^p(V)} \\ &= ||u||^p_{W^{k,p}(V)}. \end{aligned}$$

□

2.4 Smooth Approximation

Theorem 2.19. *(Local Smooth Approximation)*
Let $1 \le p < \infty, k \ge 1$. Suppose U is open, and $u \in W^{k,p}(U)$, we have that

1. $\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$,
2. $u^\epsilon \to u$ *in* $W^{k,p}_{loc}(U)$ *as* $\epsilon \to 0$.

Proof. $\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$ by 2.7.1.
Given any $V \subset\subset U$, we can find some $\epsilon_V > 0$ such that $V \subset\subset U^{\epsilon_V}$.
Consider any $|\alpha| \le k$.
We have $D^\alpha u \in L^p(U) \subseteq L^p_{loc}(U)$.

By 2.7.5, we have that $\eta_\epsilon * D^\alpha u \to D^\alpha u$ in $L^p_{loc}(U)$ as $\epsilon \to 0$, and thus $\eta_\epsilon * D^\alpha u \to D^\alpha u$ in $L^p(V)$.
In addition, by 2.17, $\forall \epsilon > 0,\ D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$ in U^ϵ.
Now $\forall 0 < \epsilon < \epsilon_V,\ V \subset\subset U^{\epsilon_V} \subseteq U^\epsilon$, and thus $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$ in V.
Thus $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$.
Since this is true $\forall |\alpha| \leq k$, we have $u^\epsilon \to u$ in $W^{k,p}(V)$.
Since this holds for any $V \subset\subset U$, $u^\epsilon \to u$ in $W^{k,p}_{loc}(U)$. □

Corollary 2.20. *Suppose U is open, and $u \in W^{k,p}(U)$ is compactly supported in U, then $u \in W^{k,p}_0(U)$.*

Proof. Since $\text{Supp}(u) \subset U$ is compact, we must have $r := \frac{1}{2}\operatorname{dist}(\text{Supp}(u), \partial U) > 0$.
For $n \in \mathbb{N}^+$, let $u_n := u^{\frac{r}{n}}$.
We have that $u_n \to u$ in $W^{k,p}_{loc}(U)$ as $n \to \infty$.
Let $W := \overline{\text{Supp}(u) + \bar{B}(0, r/2)} \subset U$. Notice that it is compact, and $\forall n \in \mathbb{N}^+, \text{Supp}(u_m) \subseteq \text{Supp}(u) + \bar{B}(0, \frac{r}{n}) \subseteq W$, which means $u_m \in C^\infty_c(U)$.
Now there is some $W \subset V \subset\subset U$, so $u_m \to u$ in $W^{k,p}(V)$.
In addition,

$$\begin{aligned}
||u - u_m||^p_{W^{k,p}(U)} &= \int_U \sum_{|\alpha| \leq k} |D^\alpha(u - u_m)|^p dx \\
&= \int_V \sum_{|\alpha| \leq k} |D^\alpha(u - u_m)|^p dx \\
&= ||u - u_m||^p_{W^{k,p}(V)}.
\end{aligned}$$

Thus $\lim_{m\to\infty} ||u - u_m||_{W^{k,p}(V)} = \lim_{m\to\infty} ||u - u_m||_{W^{k,p}(U)} = 0$.
Since each $u_m \in C^\infty_c(U)$, we have $u \in \overline{C^\infty_c(U)} = W^{k,p}_0(U)$. □

Theorem 2.21. *(Meyer-Serrin)*
Let $1 \leq p < \infty, k \geq 1$. Suppose U is open and bounded, and $u \in W^{k,p}(U)$. There exists $(u_m)_{m=1}^\infty \subseteq C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.

Proof. Let $\delta > 0$ be given.
Let $U_i := U_{\frac{1}{i}} = \left\{x \in U : \operatorname{dist}(x, \partial u) > \frac{1}{i}\right\}$ for $i \in \mathbb{N}^+$.
We have $U_1 \subseteq \bar{U_1} \subseteq U_2 \subseteq \bar{U_2} \subseteq U_3 \subseteq \cdots U$.
Indeed, for some $x \in \bar{U_i}$, we know that $\forall y \in \partial U, ||x - y|| \geq \frac{1}{i} > \frac{1}{i+1} \implies x \in U_{i+1}$.
Since U is open, for any $x \in U$, we can find some $i \geq 1,$ such that $B(x, \frac{1}{i}) \subseteq U$, which means $\operatorname{dist}(x, \partial U) \geq \frac{1}{i}$, and thus $x \in \bar{U_i} \subseteq U_{i+1}$. Thus we have $U = \bigcup_{i=1}^\infty U_i$.
Let $V_i := U_{i+3} \setminus \bar{U}_{i+1}$ for $i \in \mathbb{N}^+$. Since U is bounded, we can choose $V_0 \subset\subset U$ with $V_0 \supset \bar{U_2}$, we claim that $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$.
It is easy to see $\bigcup_{i=0}^n V_i \subseteq U_{i+3}$. For the other direction, we will prove by induction.
The base case $n = 1$, we can see that $V_0 \cup V_1 \supset \bar{U_2} \cup (U_4 \setminus \bar{U_2}) = U_4$.
Now suppose $n > 1$, and it holds for $n - 1$, we have that

$$\begin{aligned}
\bigcup_{i=0}^n V_i &= \left(\bigcup_{i=0}^{n-1} V_i\right) \cup (V_n) \\
&= U_{n-1+3} \cup \left(U_{n+3} \setminus \bar{U}_{n+1}\right) \\
&\supset U_{n+2} \cup (U_{n+3} \setminus U_{n+2}) \\
&= U_{n+3}.
\end{aligned}$$

By induction, we have that $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$.
Notice that $\forall x \in U = \bigcup_{n=1}^\infty U_n, \exists n \geq 1,$ such that $x \in U_n \subseteq U_{n+3} \subseteq \bigcup_{i=0}^n V_i \implies \exists i \geq 0,$ such that $x \in V_i$.
Thus

$$U = \bigcup_{i=0}^\infty V_i.$$

Now let $W_i := U_{i+4} \setminus \bar{U}_i$ for $i \in \mathbb{N}^+$.
Since each $U_{i+4} \subseteq \bar{U}_{i+4} \subseteq U_{i+5} \subseteq U$, we also have $U_{i+4} \subset\subset U$ and thus

$$W_i \subset\subset U.$$

Notice that $\forall x, y \in U$,

$$\begin{aligned}
\operatorname{dist}(x, \partial U) &= \inf \{||z-x|| : z \in \partial U\} \\
&= \inf \{||z-y+y-x|| : z \in \partial U\} \\
&\leq \inf \{||z-y|| + ||y-x|| : z \in \partial U\} \\
&= \inf \{||z-y|| : z \in \partial U\} + ||y-x|| \\
&= \operatorname{dist}(y, \partial U) + ||y-x||.
\end{aligned}$$

Similarly, $\operatorname{dist}(y, \partial U) \leq \operatorname{dist}(x, \partial U) + ||y-x||$. Thus we have

$$\operatorname{dist}(y, \partial U) - ||y-x|| \leq \operatorname{dist}(x, \partial U) \leq \operatorname{dist}(y, \partial U) + ||y-x||.$$

Consider any $0 < \epsilon < \frac{1}{i+3} - \frac{1}{i+4} < \frac{1}{i} - \frac{1}{i+1}$, we have that

$$\begin{aligned}
x \in \bar{B}(0, \epsilon) + V_i &\implies \exists y \in U_{i+3} \setminus \bar{U}_{i+1} \text{ such that } ||x-y|| \leq \epsilon \\
&\implies \exists y \in U \text{ such that } \frac{1}{i+3} < \operatorname{dist}(y, \partial U) < \frac{1}{i+1}, ||x-y|| \leq \epsilon \\
&\implies \exists y \in U \text{ such that } \frac{1}{i+3} - ||x-y|| < \operatorname{dist}(x, \partial U) < \frac{1}{i+1} + ||x-y||, ||x-y|| \leq \epsilon \\
&\implies \frac{1}{i+3} - \epsilon < \operatorname{dist}(x, \partial U) < \frac{1}{i+1} + \epsilon \\
&\implies \frac{1}{i+4} < \operatorname{dist}(x, \partial U) < \frac{1}{i} \\
&\implies x \in W_i.
\end{aligned}$$

Thus we have

$$\forall 0 < \epsilon < \frac{1}{i+3}, \ \bar{B}(0, \epsilon) + V_i \subseteq W_i.$$

Finally, since $V_0 \subset\subset U$, we can choose $V_0 \subset\subset W_0 \subset\subset U$, such that $V_0 + B(0, \epsilon_0'') \subseteq W_i$ for some $\epsilon_0' > 0$.
Let $(\zeta_i)_{i=0}^{\infty}$ be a **smooth partition of unity** such that

$$\forall x \in U \ \sum_{i=0}^{\infty} \zeta_i(x) = 1, \ \forall i \geq 0, \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^{\infty}(U), \\ \operatorname{Supp}(\zeta_i) \subseteq V_i. \end{cases}$$

Notice that $\forall u \in W^{k,p}(U)$, we have $\zeta_i u \in W^{k,p}(U)$ as well. Moreover, $\operatorname{Supp}(\zeta_i u) \subseteq V_i$.
Let $u_i^\epsilon := \eta_\epsilon * (\zeta_i u) \ \forall \epsilon > 0$.
By previous theorem, we have that $u_i^\epsilon \to \zeta_i u$ in $W_{loc}^{k,p}(U)$.
Thus for $W_i \subset\subset U$, we can find $\epsilon_i' > 0$ such that $\forall \epsilon < \epsilon_i', \ ||u_i^\epsilon - \zeta_i u||_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}$.
Now pick $\epsilon_0 = \min(\epsilon_0'', \epsilon_0'), \forall i \in \mathbb{N}^+, \epsilon_i = \frac{1}{2} \min\left(\frac{1}{i+3} - \frac{1}{i+4}, \epsilon_i'\right) > 0$.
We have that by 2.7,

$$\operatorname{Supp}(u_i^{\epsilon_i}) \subseteq \operatorname{Supp}(\eta_{\epsilon_i}) + \operatorname{Supp}(\zeta_i u) \subseteq \bar{B}(0, \epsilon_i) + V_i \subseteq W_i,$$

and

$$||u_i^{\epsilon_i} - \zeta_i u||_{W^{k,p}(U)} = ||u_i^{\epsilon_i} - \zeta_i u||_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}.$$

Now let $v := \sum_{i=0}^{\infty} u_i^{\epsilon_i}$.
Notice that $\forall x \in U, \exists V \subset\subset U_{\epsilon_i}$ be open, such that $x \in V$. Since $V \cap W_i \neq \emptyset$ for only finitely many i, and

$\text{Supp}(u_i^{\epsilon_i}) \subseteq W_i$, we must have $v = \sum_{i=0}^{k} u_i^{\epsilon_i}$ on V for some finite k.
In addition, by 2.7, each $u_i^{\epsilon_i} \in C^\infty(U_{\epsilon_i})$, thus infinitely differentiable at x.
Thus $v = \sum_{i=0}^{k} u_i^{\epsilon_i}$ is infinitely differentiable at x.
Since $x \in U$ is arbitrary, we have that $v \in C^\infty(U)$.
In addition,

$$\forall x \in U,\ u(x) = \sum_{i=0}^{\infty} \zeta_i(x)u(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x)$$

by definition of partition of unity. Thus

$$u(x) - v(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x) - \sum_{i=0}^{\infty} u_i^{\epsilon_i}(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x) - u_i^{\epsilon_i}(x) = \sum_{i=0}^{\infty} (\zeta_i u - u_i^{\epsilon_i})(x).$$

Since this holds for all $x \in U$, we have that

$$u - v = \sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i}.$$

Now we have

$$\begin{aligned}
||v - u||_{W^{k,p}(U)} &= \left|\left|\sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i}\right|\right|_{W^{k,p}(U)} \\
&\leq \sum_{i=0}^{\infty} ||\zeta_i u - u_i^{\epsilon_i}||_{W^{k,p}(U)} \\
&< \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} \\
&= \delta.
\end{aligned}$$

□

Definition 2.17. Let $U \subseteq \mathbb{R}^n$ be open and bounded, then ∂U is C^k if $\forall z \in \partial U, \exists r > 0, \gamma \in C^k(\mathbb{R}^{n-1})$, such that

$$U \cap \bar{B}(z, r) = \left\{x \in B(z, r) : x^n > \gamma(x^1, \ldots, x^{n-1})\right\}.$$

Theorem 2.22. *Let U be bounded, and ∂U is C^1, then $\forall u \in W^{k,p}(U)$ for $1 \leq p < \infty$, there exists functions $u_m \in C^\infty(\bar{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.*

Proof. See 5.3.3 in Eavan's book. □

2.5 Extensions

Proposition 2.23. *Let $U \subseteq \mathbb{R}^n$ be open and bounded, with ∂U be C^k. Then $\forall z \in \partial U, \exists r > 0, \Phi \in C^k(B(z, r), \mathbb{R}^n)$ a diffeomorphism, such that $\Phi(\partial U \cap B(z, r))$ is in a flat hyperplane, and $\det(D\Phi) = \det(D\Psi) = 1$, for $\Psi := \Phi^{-1}$.*

Proof. Let

$$\Phi^i(x) := x^i \ \forall i \in [n-1], \Phi^n(x) := x^n - \gamma(x^1, \ldots, x^{n-1}),$$

and let

$$\Phi^i(y) := y^i \ \forall i \in [n-1], \Phi^n(y) := y^n - \gamma(y^1, \ldots, y^{n-1}).$$

□

Theorem 2.24. *(Sobolev Norm Equivalence Under Diffeomorphism)*
Let $W \subseteq \mathbb{R}^n$ and Φ is a $C^1(W)$ diffeomorphism, i.e., it has inverse $\Psi \in C^1(W)$. Let $v := u \circ \Psi$, then

$$\exists C_0, C_1 \text{ such that } C_0||u||_{W^{1,p}(W)} \leq ||v||_{W^{1,p}(\Phi(W))} \leq C_1||u||_{W^{1,p}(W)}.$$

Lemma 2.25. *Let $1 \le p < \infty$. Assume U is bounded, with ∂U be C^1. Let V be open and bounded, with $U \subset\subset V$, then there exists a bounded linear operator $E : C^1\big(\bar{U}\big) \to W^{1,p}(\mathbb{R}^n)$, such that $\forall u \in C^1\big(\bar{U}\big)$:*

1. *$Eu = u$ in U,*
2. *$\mathrm{Supp}(Eu) \subseteq V$,*
3. *$\exists C > 0,$ such that $||Eu||_{W^{1,p}(\mathbb{R}^n)} \le C||u||_{W^{1,p}(U)}$.*

Proof. Fix $z \in \partial U$.
In addition, we assume ∂U is flat around z on the plane $\{x^n = 0\}$.
Then there exists an open ball $B := B(z, r)$, such that

$$B^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B^- := B \cap \{x^n \le 0\} \subseteq \mathbb{R}^n \setminus U.$$

Let $\bar{u}_z(x) := \begin{cases} u(x) & x \in B^+ \\ -3u(x^1, \ldots, x^{n-1}, -x^n) + 4u(x^1, \ldots, x^{n-1}, -\frac{1}{2}x^n) & x \in B^-. \end{cases}$
Then we claim $\bar{u}_z \in C^1(B)$.
Indeed, let $u^- := \bar{u}_z|_{B^-}, u^+ := \bar{u}_z|_{B^+}$.

$$\begin{aligned} u^-|_{x^n=0} &= -3 + 4u|_{x^n=0} \\ &= u|_{x^n=0} \\ &= u^+|_{x^n=0}; \\ &\forall i \in [n-1], \\ \partial_i u^-|_{x^n=0} &= -3\partial_i u|_{x^n=0} + 4\partial_i u|_{x^n=0} \\ &= \partial_i u|_{x^n=0} \\ &= \partial_i u^+|_{x^n=0} \\ \partial_n u^-|_{x^n=0} &= 3\partial_n u|_{x^n=0} - 4\frac{1}{2}\partial_n u|_{x^n=0} \\ &= u|_{x^n=0} \\ &= \partial_n u^+|_{x^n=0}. \end{aligned}$$

Thus $\bar{u} \in C^1(B)$.
By A2, we have

$$\exists C > 0, \text{ such that } ||\bar{u}_z||_{W^{1,p}(B)} \le C||u||_{W^{1,p}(B^+)}.$$

Now suppose ∂U is not flat around z, we can find $r_1 > 0, \Phi \in C^1(B(z, r_1), \mathbb{R}^n)$, such that $\Phi(\partial U \cap B(z, r_1))$ is in a flat hyperplane, WLOG $\{y_n = 0\}$, and $\det(D\Phi) = \det(D\Psi) = 1$, for $\Psi := \Phi^{-1}$.
Notice that we can find $B(z, r_2) \subset\subset V$ since V is open and $z \in \bar{U} \subseteq V$.
By setting $r = \min(r_1, r_2) > 0$, we can WLOG work with $B(z, r) \subset\subset V$.
Let $z' := \Phi(z), v := u \circ \Psi \in C^1(\Phi(\bar{U})) = C^1(\overline{\Phi(U)})$.
Since $\Phi(B(z, r))$ is open, we can choose some open ball $B := B(z', r') \subseteq \Phi(B(z, r))$. Let $W_z := \Psi(B)$.
Since $\Phi(\partial U \cap B(z, r))$ is in the plane $\{y_n = 0\}$, we have

$$B^+ := B \cap \{y^n \ge 0\} = \Phi(W_z \cap \bar{U}), B^- := B \cap \{y^n \le 0\} = \Phi(W_z \setminus U).$$

Now we can extend v form B^+ to B with

$$||\bar{v}||_{W^{1,p}(B)} \le C||v||_{W^{1,p}(B^+)}.$$

Now let $\bar{u}_z := \bar{v} \circ \Psi$, we have that $B = \Phi(W_z), \bar{v} = \bar{u}_z \circ \Phi$, and by 2.24, we have

$$\begin{aligned} ||\bar{u}_z||_{W^{1,p}(W_z)} &\le C_1||\bar{v}||_{W^{1,p}(\Phi(W_z))} \\ &= C_1||\bar{v}||_{W^{1,p}(B)} \\ &\le C_2||v||_{W^{1,p}(B^+)} \\ &\le C_3||u||_{W^{1,p}(\Psi(B^+))} \\ &\le C_3||u||_{W^{1,p}(U)} \end{aligned}$$

Notice that $\forall z, \Phi(z) \in B \implies z \in W_z$, thus $\{W_z\}_{z\in\partial U}$ forms an open cover for ∂U.
Since ∂U is compact, we can find a finite subcover $\{W_i\}_{i=1}^N$.
Notice that $\left(\bar{U} \setminus \bigcup_{i=1}^N W_i\right) \subset U$ is closed, and U is bounded, so we can find $\left(\bar{U} \setminus \bigcup_{i=1}^N W_i\right) \subseteq W_0 \subset\subset U$.
We then have $\bigcup_{i=0}^N W_i = U$.
Now let $(\zeta_i)_{i=0}^N$ be a partition of unity subordinate to W_i, such that

$$\forall x \in U \ \sum_{i=0}^N \zeta_i(x) = 1, \ \forall i \geq 0, \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^\infty(\mathbb{R}^n), \\ \mathrm{Supp}(\zeta_i) \subseteq W_i. \end{cases}$$

Let $\bar{u} := \sum_{i=0}^N \zeta_i \bar{u}_i$, with $\bar{u}_0 := u$. We have that

$$\begin{aligned} ||\bar{u}||_{W^{1,p}(\mathbb{R}^n)} &\leq \sum_{i=0}^N ||\zeta_i \bar{u}_i||_{W^{1,p}(\mathbb{R}^n)} \\ &= \sum_{i=0}^N ||\zeta_i \bar{u}_i||_{W^{1,p}(W_i)} \\ &\leq C_4 \sum_{i=0}^N ||\bar{u}_i||_{W^{1,p}(W_i)} \\ &= C_5 ||u||_{W^{1,p}(W_i)}, \end{aligned}$$

since each term is bounded, and we have a finite sum.
We thus define $Eu := \bar{u}$.
We can check that E is linear and bounded. □

Theorem 2.26. *Let $1 \leq p < \infty$. Assume U is bounded, with ∂U be C^1. Let V be open and bounded, with $U \subset\subset V$, then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$, such that $\forall u \in W^{1,p}(U)$:*

1. *$Eu = u$a.e. in U,*
2. *$\mathrm{Supp}(Eu) \subseteq V$,*
3. *$\exists C > 0$, such that $||Eu||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)}$.*

Proof. By 2.22, we know $C^\infty\left(\bar{U}\right) \subseteq C^1\left(\bar{U}\right)$ is dense in $W^{1,p}(U)$, and thus $C^1\left(\bar{U}\right)$ is also dense in $W^{1,p}(U)$.
By 1.21, we can extend the result in the above lemma to get $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$.
In addition, since $Eu = \lim_{m\to\infty} Eu_m$ for some $u_m \to u$ in $W^{1,p}(U)$, we also have $Eu = \lim_{m\to\infty} Eu_m = Eu = \lim_{m\to\infty} u_m = u$, a.e..
Also, $\mathrm{Supp}(Eu) \subseteq \bigcup_{m=1}^\infty \mathrm{Supp}(Eu_m) \subseteq V$. □

2.6 Traces

Proposition 2.27. *(Young's inequality)*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \forall a, b > 0, \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.28. *Let U be bounded, and ∂U is C^1, and $1 \leq p < \infty$. Then there exists a bounded linear operator $T : C^1\left(\bar{U}\right) \to L^p(\partial U)$; $u \mapsto u|_{\partial U}$ and a constant $C > 0$, such that*

$$\forall u \in C^1\left(\bar{U}\right), ||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(U)}.$$

Proof. Consider $z \in \partial U$.
Assume ∂U is flat near z in the hyperplane $\{x^n = 0\}$.
Then there exists an open ball $B_z := B(z, r)$, such that

$$B_z^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B_z^- := B \cap \{x^n \leq 0\} \subseteq \mathbb{R}^n \setminus U.$$

Since u is C^1 and thus continuous, WLOG, we can take r small enough, such that u does not change sign in B_z. Namely, $|u| = u \operatorname{sgn}(u(z))$ in B_z.
Let $\hat{B}_z := B(z, \frac{r}{2})$, and let $\xi \in C_c^\infty(B_z)$ such that $\xi \geq 0$ in B_z, and $\xi = 1$ in $\hat{B}_z$.
Let $\Gamma_z := \hat{B}_z \cap \partial U$, then we have $\operatorname{Supp}(\xi u) \subseteq B_z^+$, and $\xi u = u$ on T.
Let $x' := (x^1, \ldots, x^{n-1})$, by Fundamental Theorem of Calculus, we have

$$\int_0^\infty (\xi |u|^p)(x', t) dt = -(\xi |u|^p)(x', 0).$$

In addition, we have

$$\begin{aligned}
||u||_{L^p(\Gamma_z)}^p &= \int_{\Gamma_z} |u|^p (x', 0) dx' \\
&\leq \int_{\mathbb{R}^{n-1}} (\xi |u|^p)(x', 0) dx' \\
&= -\int_0^\infty \int_{\mathbb{R}^{n-1}} (\xi |u|^p)(x', t) dx' dt \\
&= -\int_{B_z^+} (\xi |u|^p)(x) dx \\
&= -\int_{B_z^+} \xi_{x_n} |u|^p + \xi p |u|^{p-1} (\operatorname{sgn} u(z)) u_{x_n} dx \\
&\leq \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi p |u|^{p-1} |u_{x_n}| dx \\
&\leq \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi p \left(\frac{(|u|^{p-1})^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{|u_{x_n}|^p}{p} \right) dx \\
&= \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi (p-1) |u|^p + \xi |u_{x_n}|^p dx \\
&\leq \int_{B_z^+} (|\xi_{x_n}| + \xi (p-1)) |u|^p + \xi |Du|^p dx.
\end{aligned}$$

Since $\xi \in C_c^\infty(B_z)$, by EVT, $|\xi_{x_n}|, \xi$ are all bounded. Thus $\exists C > 0,$ such that $|\xi_{x_n}| + \xi(p-1), \xi \leq C$ in B_z.
Thus

$$||u||_{L^p(\Gamma_z)}^p \leq \int_{B_z^+} C|u|^p + C|Du|^p dx = C||u||_{W^{1,p}(B_z^+)}^p \leq C||u||_{W^{1,p}(U)}^p.$$

Now if ∂U is not flat near z, we can find a C^1 diffeomorphism Φ to make it flat. We still have

$$||u||_{L^p(\Gamma_z)} \leq C||u||_{W^{1,p}(U)},$$

by the equivalence of Sobolev norms under diffeomorphism 2.24.
Since $\{B_z\}_{z \in \partial U}$ form an open cover for ∂U, and ∂U is compact, we can find a finite subcover $\{B_i : x_i \in \partial U\}_{i=1}^N$, and their corresponding Γ_i.
For each $i \in [N]$, we have that

$$||u||_{L^p(\Gamma_i)} \leq C_i ||u||_{W^{1,p}(U)}^p.$$

We have that

$$\begin{aligned}
||Tu||^p_{L^p(\partial U)} &= \int_{\partial U} |u|^p dx \\
&\leq \sum_{i=1}^N \int_{\Gamma_i} |u|^p dx \\
&= \sum_{i=1}^N ||u||^p_{L^p(\Gamma_i)} \\
&\leq \sum_{i=1}^N C_i ||u||^p_{W^{1,p}(U)} \\
&= C||u||^p_{W^{1,p}(U)},
\end{aligned}$$

by taking $C := \sum_{i=1}^N C_i$. ☐

Theorem 2.29. *Let U be bounded, and ∂U is C^1, and $1 \leq p < \infty$. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ and a constant $C > 0$, such that*

$$\forall u \in W^{1,p}(U) \cap C(\bar{U}), Tu = u|_{\partial U},$$

and

$$\forall u \in W^{1,p}(U), ||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(U)}.$$

Proof. By 2.22, we know $C^\infty(\bar{U}) \subseteq C^1(\bar{U})$ is dense in $W^{1,p}(U)$, and thus $C^1(\bar{U})$ is also dense in $W^{1,p}(U)$. By 1.21, we can extend the result in the above lemma to get $T : W^{1,p}(U) \to L^p(\partial U)$. ☐

Theorem 2.30. *Let U be bounded, and ∂U is C^1, then for any $u \in W^{1,p}(U)$, we have that*

$$u \in W^{1,p}_0(U) \iff Tu = 0 \text{ on } \partial U.$$

2.7 Weak and Normal Derivatives

Proposition 2.31. *If $u, v \in C(U)$ are both continuous, and $u = v$ a.e., then $\forall x \in U, u(x) = v(x)$.*

Proof. Consider any $x \in U$.
Since U is open, we can find some $r > 0$, such that $B(x, r) \subseteq U$.
For any $i \geq \lceil \frac{1}{r} \rceil$, we must have some $x_i \in B(x, \frac{1}{i}) \subseteq U$, such that $u(x_i) = v(x_i)$.
Otherwise $\{x \in U : u(x) \neq v(x)\} \supseteq B(x, \frac{1}{i}) \cap U = B(x, \frac{1}{i})$ does not have measure 0.
Thus $\lim_{i\to\infty} x_i = x$.
Since u, v are both continuous, we have that

$$\begin{aligned}
u(x) &= u\Big(\lim_{i\to\infty} x_i\Big) \\
&= \lim_{i\to\infty} u(x_i) \\
&= \lim_{i\to\infty} v(x_i) \\
&= v(\lim_{i\to\infty} x_i) \\
&= v(x).
\end{aligned}$$

This is true for any $x \in U$, which completes the proof. ☐

Remark. For the following part in this subsection, we will use D^α to denote the α^{th} normal derivative of u, and $\bar{D}^\alpha$ to be the α^{th} weak derivative of u to avoid confusion.

Proposition 2.32. *Given any $\alpha \in \mathbb{N}^n$. $\forall u$ such that its α^{th} normal derivative $D^\alpha u$ exists and is continuous, and any $v = u$ a.e., we have that $D^\alpha u$ is an α^{th} weak derivative of v. Namely, $\bar{D}^\alpha v = D^\alpha u$ a.e..*

Proof. Consider any $\phi \in C_c^\infty(U)$, we have that

$$\begin{aligned}\int_U vD^\alpha \phi dx &= \int_U uD^\alpha \phi dx \\ &= (-1)^{|\alpha|} \int_U D^\alpha u \phi dx,\end{aligned}$$

where the second equality follows from integration by part over some $\mathrm{Supp}(\phi) \subseteq V \subseteq U$ with Lipschitz boundary. $\square$

Definition 2.18. A domain U is **path-connected** if $\forall x, y$, there is some continuous path $\gamma : [0,1] \to U$, such that $\gamma(0) = x, \gamma(1) = y$.

Proposition 2.33. *Let $U \subseteq \mathbb{R}^n$ be open and connected, and $1 \leq p \leq \infty$, $u \in W^{1,p}(U)$, then*

$$\bar{D}u = 0 \text{ a.e.} \iff u \text{ is a constant a.e.}.$$

Proof. Suppose u is a constant a.e., then it means u has a version $\tilde{u}(x) = C,\ \forall\ x \in U$ for some constant C. Clearly $\tilde{u}$ is differentiable, and has continuous normal derivative $D\tilde{u}(x) = 0,\ \forall x \in U$.
By 2.32, and since $\tilde{u} = u$ a.e., we have that $D\tilde{u}$ is an weak derivative of v. Since the weak derivative is unique a.e., we must have $\bar{D}u = 0$ a.e..
On the other hand, suppose $\bar{D}u = 0$ a.e..
We know that by 2.17, for any $\epsilon > 0, x \in U_\epsilon$, and any direction $i \in [n]$,

$$\begin{aligned}(\partial_i u^\epsilon)(x) &= (\eta_\epsilon * \partial_i u)(x) \\ &= \int_U \eta_\epsilon(x)\partial_i u(y)dy \\ &= \int_U \eta_\epsilon(x)0dy \\ &= 0,\end{aligned}$$

since $\partial_i u(y) = 0$ for a.e. $y \in U$.
Since this holds for all $i \in [n]$, we must have $\forall x \in U,\ Du^\epsilon = 0$.
Notice that by 2.7, $u^\epsilon \in C^\infty(U_\epsilon)$, which by normal calculus means that $\forall x \in U_\epsilon,\ u_\epsilon(x) = C_\epsilon$ for some constant C_ϵ that does not depend on x.
Again by 2.7, $u^\epsilon \to u$ a.e. as $\epsilon \to 0$. Pick any such x with $u^\epsilon(x) \to u(x)$.
Notice that we can find $\delta > 0$, such that $\forall \epsilon \in (0, \delta), x \in U_\epsilon$.
Thus we have $u(x) = \lim_{\epsilon\to 0} u_\epsilon(x) = \lim_{\epsilon\to 0} C_\epsilon$.
Since $\lim_{\epsilon\to 0} C_\epsilon$ converges, we can call it $C := \lim_{\epsilon\to 0} C_\epsilon$, which is a constant that is independent of x, ϵ.
Now any such x satisfies $u(x) = \lim_{\epsilon\to 0} C_\epsilon = C$, and they are by choice a.e.. $\square$

Lemma 2.34. *Consider $1 \leq p \leq \infty$, and $U = (a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n$ be an open rectangle. Let $1 \leq i \leq n$, suppose $u \in W^{1,p}(U)$ has a continuous representative $u^* \in C(U)$, and its i^{th} weak derivative $\bar{\partial}_i u$ has a continuous representative $(\bar{\partial}_i u)^* \in C(U)$, then the regular i^{th} partial derivative*

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x)\ \forall x \in U$$

exists and is continuous.

Proof. Pick some $s \in (a_i, b_i)$, let $S := \left\{x \in U : x^i = s\right\}$ be the slice of U.
By FTC, there is a unique v, defined by

$$v(x^1, \ldots, x^n) := u^*(x^1, \ldots, x^{i-1}, s, x^{i+1}, \ldots x^n) + \int_s^{x^i} (\bar{\partial}_i u)^*(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n)dt,$$

such that $v|_S = u^*|_S$, and the i^{th} normal partial derivative

$$\partial_i v(x) = (\bar{\partial}_i u)^*(x)\ \forall x \in U.$$

We notice that $\bar{\partial}_i v = \partial_i v$a.e. by 2.32.
Thus the weak derivative $\bar{\partial}_i(u^* - v) = \bar{\partial}_i(u^*) - \bar{\partial}_i v = \bar{\partial}_i u - \partial_i v = \bar{\partial}_i u - (\bar{\partial}_i u)^* = 0$a.e..
Fix any $(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots x^n)$, and denote $w : (a_i, b_i) \to \mathbb{R}$ by

$$w(t) := (u^* - v)(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n).$$

We have that $\bar{D}w = \bar{\partial}_i(u^* - v) = 0$ with respect to $t \in (a_i, b_i)$a.e..
By 2.33, $w(t) = C$a.e. $t \in (a_i, b_i)$ form some constant C, since (a_i, b_i) is clearly connected.
Notice that w is continuous, since both u^*, v are continuous on the x^i direction.
Since both w, C are continuous, we have $\forall t \in (a_i, b_i), w(t) = C$.
Since $v|_S = u^*|_S$, we must have $C = w(s) = 0$ and thus

$$\forall t \in (a_i, b_i), u^*(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n) = v(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n).$$

Since this holds for all $(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots x^n)$, we must have $u^*(x) = v(x)\ \forall x \in U$.
By construction of v, we have that

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x)\ \forall x \in U.$$

□

Lemma 2.35. *Consider $1 \le p \le \infty$, and $U \subseteq \mathbb{R}^n$ be open. If $u \in W^{1,p}(U)$ has a continuous representative $u^* \in C(U)$, and its weak derivative $\partial_i u$ has a continuous representative $(\bar{\partial}_i u)^* \in C(U)$, then the regular i^{th} partial derivative*

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x)\ \forall x \in U$$

exists and is continuous.

Proof. Notice that any open $U \subseteq \mathbb{R}^n$ can be written as $\bigcup_{j=1}^{\infty} R_j$, where each R_j is an open rectangle.
Fix any $x \in U$, there must be some $R_j \ni x$.
By previous lemma, $\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x)$.
Since this holds for any $x \in U$, we have our result. □

Proposition 2.36. *Consider $1 \le p \le \infty, k \ge 0$, and $U \subseteq \mathbb{R}^n$ be open. If $u \in W^{k,p}(U)$ has a continuous representative $u^* \in C(U)$, and all of its weak derivatives $D^\alpha u$ have continuous representatives $(\bar{D}^\alpha u)^* \in C(U)$ for any $|\alpha| \le k$, then*

$$u^* \in C^k(U),\ D^\alpha(u^*)(x) = (\bar{D}^\alpha u)^*(x)\ \forall x \in U, \forall |\alpha| \le k.$$

Proof. We will use induction on k.
The base case is $k = 0$.
Since $|\alpha| = 0$, we trivially have $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$.
Now, suppose this holds for $k - 1$.
Consider any $u \in W^{k,p}(U)$.
If $|\alpha| = 0$, we trivially have $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$ as before.
Now consider any $|\gamma| = 1$. We know $\gamma = e_i$ for some $1 \le i \le n$.
By previous lemma, we have that

$$D^\gamma(u^*)(x) = \partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) = (\bar{D}^\gamma u)^*(x)\ \forall x \in U.$$

Notice that $\bar{D}^\gamma u \in W^{k-1,p}(U)$, and all of its weak derivatives $\bar{D}^\beta \bar{D}^\gamma u = \bar{D}^{\beta+\gamma} u$ have continuous representatives $(\bar{D}^{\beta+\gamma} u)^* \in C(U)$ for any $|\beta| \le k - 1$. By the induction hypothesis, we have that

$$D^\beta((\bar{D}^\gamma u)^*)(x) = (\bar{D}^{\beta+\gamma} u)^*(x)\ \forall x \in U, \forall |\beta| \le k - 1.$$

For any $1 \leq |\alpha| \leq k$, we can have $\alpha = \beta + \gamma$, where $|\beta| \leq k-1, |\gamma| = 1$.
Now we have $\forall x \in U$,

$$\begin{aligned}(\bar{D}^\alpha u)^*(x) &= (\bar{D}^{\beta+\gamma} u)^*(x) \\ &= D^\beta((\bar{D}^\gamma u)^*)(x) \\ &= D^\beta(D^\gamma(u^*))(x) \\ &= D^{\beta+\gamma}(u^*)(x) \\ &= D^\alpha(u^*)(x).\end{aligned}$$

We have thus proven the result for any $|\alpha| \leq k$.
Since all of its α^{th} derivatives exists and are continuous, we further have that $u^* \in C^k(U)$. ☐

Theorem 2.37 (Differentiability almost everywhere)**.** *(Theorem 5.8.5 in Eavan's)*
Consider $n \leq p \leq \infty$, and $U \subseteq \mathbb{R}^n$ be open. Assume $u \in W^{1,p}_{loc}(U)$, then u is differentiable a.e. in U, and its gradient $Du(x)$ equals its weak gradient $\bar{D}u(x)$ for a.e. $x \in U$.

2.8 Sobolev Inequalities

Definition 2.19. For $1 \leq p < n$, the **Sobolev conjugate** of p is $p^* := \frac{np}{n-p}$, with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Theorem 2.38. *(Gagliardo–Nirenberg–Sobolev)*
Let $1 \leq p < n$, then

$$\exists C > 0, \text{ such that } \forall u \in C^1_c(\mathbb{R}^n),\ ||u||_{L^{p^*}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)}.$$

Corollary 2.39. *Let $1 \leq p < n$, and $U \subseteq \mathbb{R}^n$, then*

$$\exists C > 0, \text{ such that } \forall u \in C^1_c(U),\ ||u||_{L^{p^*}(U)} \leq C||Du||_{L^p(U)}.$$

Proof. By Gagliardo–Nirenberg–Sobolev's Inequality, there is some $C > 0$, such that

$$\forall v \in C^1_c(\mathbb{R}^n), ||v||_{L^{p^*}(\mathbb{R}^n)} \leq C||Dv||_{L^p(\mathbb{R}^n)}.$$

Notice that for each $u \in C^1_c(U)$, we can extend it by $v(x) := \begin{cases} u(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$.
Notice that $\text{Supp}(v) \subseteq U$, and $v = u$ on U.
Thus $||v||_{L^{p^*}(\mathbb{R}^n)} = ||v||_{L^{p^*}(U)} = ||u||_{L^{p^*}(U)}$, and $||Dv||_{L^p(\mathbb{R}^n)} = ||Du||_{L^p(U)}$.
In addition, we have that $\lim_{x \to \partial U} D^\alpha u(x) = 0 = \lim_{x \to \partial U} D^\alpha 0, \forall |\alpha| \leq 1$.
Thus this extension is smooth. i.e. $v \in C^1_c(\mathbb{R}^n)$.
We thus have

$$||u||_{L^{p^*}(U)} = ||v||_{L^{p^*}(\mathbb{R}^n)} \leq C||Dv||_{L^p(\mathbb{R}^n)} = C||Du||_{L^p(U)}.$$

☐

Theorem 2.40. *($W^{1,p}$ embedding into L^{p^*}, with $1 \leq p < n$)*
Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded. If ∂U is C^1, then

$$\exists C > 0, \text{ such that } \forall u \in W^{1,p}(U),\ ||u||_{L^{p^*}(U)} \leq C||u||_{W^{1,p}(U)}.$$

In addition, since U is bounded, $\forall q \in [1, p^*]$, we have

$$\exists C > 0, \forall u \in W^{1,p}(U),\ ||u||_{L^q(U)} \leq C||u||_{W^{1,p}(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q1. ☐

Theorem 2.41. *(Poincaré's Inequality)*
Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded, then

$$\forall q \in [1, p^*], \exists C \geq 0, \text{ such that } \forall u \in W_0^{1,p}(U), \ ||u||_{L^q(U)} \leq C||Du||_{L^p(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q2. □

Corollary 2.42. *Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded, then $||Du||_{L^p(U)}$ and $||u||_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$.*

Theorem 2.43. *Let $1 \leq p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, then*

$$\exists C \geq 0, \text{ such that } \forall u \in W_0^{1,p}(U), \ ||u||_{L^p(U)} \leq C||Du||_{L^p(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q2. □

Corollary 2.44. *Let $1 \leq p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, then $||Du||_{L^p(U)}$ and $||u||_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$.*

Proof. See A3Q2. □

Theorem 2.45. *($W^{1,p}(U)$ embedding into $C^{0,\gamma}(\bar{U})$, with $n < p \leq \infty$, Morrey's)*
Let $n < p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1 Then there is some constant $C \geq 0$ such that

$$\forall u \in W^{1,p}(U), \exists \tilde{u} \in C^{0,\gamma}(\bar{U}), \text{ such that } ||\tilde{u}||_{C^{0,\gamma}(\bar{U})} \leq C||u||_{W^{1,p}(U)},$$

where $\gamma := 1 - \frac{n}{p}$, and $\tilde{u} \in [u]$ is a representative of the equivalence class $[u] \in W^{1,p}(U)$.

Remark. If $p = \infty$, then $\gamma = 1$, and u^* is Lipschitz.

Theorem 2.46 (Sobolev Inequalities)**.** *Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1. Let $u \in W^{k,p}(U)$, we have*

1. *If $k < \frac{n}{p}$, we define q by $\frac{1}{q} := \frac{1}{p} - \frac{k}{n}$, then*

$$||u||_{L^q(U)} \leq C||u||_{W^{k,p}(U)}.$$

2. *If $k > \frac{n}{p}$, we define $t := k - \lfloor \frac{n}{p} \rfloor - 1$, then we have a representative $\tilde{u} \in C^{t,\gamma}(\bar{U})$, such that*

$$||u^*||_{C^{t,\gamma}(\bar{U})} \leq C||u||_{W^{k,p}(U)},$$

where $\gamma = \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}$ if $\frac{n}{p} \notin \mathbb{Z}$, and γ can be any integer if $\frac{n}{p} \in \mathbb{Z}$.

Proof. See Theorem 5.6-6 of Evans and A3Q3. □

2.9 Compactness

Definition 2.20. Let $(f_k)_{k=1}^{\infty}$ be a sequence of real-valued functions on $\mathbb{R}^n$. It is **uniformly bounded** if

$$\exists M > 0, \text{ such that } |f_k(x)| \leq M, \ \forall k \in \mathbb{N}^+, x \in \mathbb{R}^n$$

Definition 2.21. Let $(f_k)_{k=1}^{\infty}$ be a sequence of real-valued functions on $\mathbb{R}^n$. It is **equicontinuous** if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x, y \in \mathbb{R}^n, \ ||x - y|| < \delta \implies |f_k(x) - f_k(y)| < \epsilon, \ \forall k \in \mathbb{N}^+$$

Theorem 2.47. *(Arzela-Ascoli Compact criterion)*
Let $(f_k)_{k=1}^{\infty}$ be a sequence of real-valued functions on $\mathbb{R}^n$ such that it is uniformly bounded and equicontinuous, then there exists a subsequence $(f_{k_j})_{j=1}^{\infty}$ and a continuous function f such that $f_{k_j} \to f$ uniformly on compact subsets of $\mathbb{R}^n$.

Proposition 2.48. *(interpolation) Assume* $1 \leq s \leq r \leq t \leq \infty$, *and* $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$ *with* $0 \leq \theta \leq 1$. *Suppose* $u \in L^s(U) \cap L^t(U)$, *then* $u \in L^r(U)$ *and*

$$||u||_{L^r(U)} \leq ||u||^{\theta}_{L^s(U)} ||u||^{1-\theta}_{L^t(U)}.$$

Proof. See AMATH731 A2. □

Lemma 2.49. *Let* $V \subseteq \mathbb{R}^n$ *be open and bounded. Let* $1 \leq p < n$, *and* $(u_m)_{m=1}^{\infty} \subset W^{1,p}(\mathbb{R}^n)$ *be any bounded sequence with* $\mathrm{Supp}(u_m) \subseteq V$. *For* $u_m^{\epsilon} := \eta_\epsilon * u_m$, *we have that for each* $\epsilon > 0$, *there exists a subsequence* $(u_{m_j}^{\epsilon})_{j=1}^{\infty}$ *that converges in* $L^q(V)$.

Proof.

Claim 2.49.1. *The sequence* $(u_m^{\epsilon})_{m=1}^{\infty}$ *is uniformly bounded.*

Proof. Since $(u_m)_{m=1}^{\infty}$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, ||\hat{u}_m||_{W^{1,p}(\mathbb{R}^n)} \leq M$.
Consider any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
|u_m^{\epsilon}(x)| &= \left| \int_{\mathbb{R}^n} \eta_\epsilon(x-y) u_m(y) dy \right| \\
&\leq ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\
&\leq ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |u_m(y)| dy \\
&= ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} ||u_m||_{L^1(\mathbb{R}^n)} \\
&= ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} ||u_m||_{L^p(\mathbb{R}^n)} \\
&= ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} ||u_m||_{W^{1,p}(\mathbb{R}^n)} \\
&\leq ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
&= \frac{1}{\epsilon^n} ||\eta||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
&\leq \frac{C}{\epsilon^n} |V|^{1-\frac{1}{p}} M.
\end{aligned}$$

Since $\frac{C}{\epsilon^n}|V|^{1-\frac{1}{p}} M < \infty$ is independent of m, we have that the sequence $(u_m^{\epsilon})_{m=1}^{\infty}$ is uniformly bounded. □

Claim 2.49.2. *The sequence* $(u_m^{\epsilon})_{m=1}^{\infty}$ *is equicontinuous.*

Proof. Since $(u_m)_{m=1}^{\infty}$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, ||\hat{u}_m||_{W^{1,p}(\mathbb{R}^n)} \leq M$.
By 2.7.2, we have that $\partial_i u_m^{\epsilon} = (\partial_i \eta_\epsilon) * u_m$.
Thus for any $x \in \mathbb{R}^n, 1 \leq i \leq n$, we have

$$\begin{aligned}
|\partial_i u_m^{\epsilon}(x)| &= \left| \int_{\mathbb{R}^n} (\partial_i \eta_\epsilon)(x-y) u_m(y) dy \right| \\
&\leq ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\
&\leq ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} ||u_m||_{L^1(\mathbb{R}^n)} \\
&\leq ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
||Du_m^{\epsilon}(x)||_1 &\leq \sum_{i=1}^{n} ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
&= ||D\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
&= ||D\eta_\epsilon||_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M \\
&< \infty.
\end{aligned}$$

Since $||D\eta_\epsilon||_{L^\infty(B(0,\epsilon))}|V|^{1-\frac{1}{p}}M$ is independent of x, m, we have that

$$C := \sup_{m\geq 1} ||Du_m^\epsilon||_{L^\infty(U)} \leq ||D\eta_\epsilon||_{L^\infty(B(0,\epsilon))}|V|^{1-\frac{1}{p}}M < \infty.$$

Since $u_m^\epsilon \in C_c^\infty(\mathbb{R}^n)$ by 2.7.1, we have each u_m^ϵ is Lipschitz with Lipschitz-constant C.
Given any $\delta > 0$, we can let $\delta_0 = \frac{\delta}{C}$.
Thus $\forall x, y \in \mathbb{R}^n,$ such that $||x - y|| < \delta_0$, we have

$$|u_m^\epsilon(x) - u_m^\epsilon(y)| \leq C||x - y|| < \delta, \ \forall m \in \mathbb{N}^+.$$

Thus the sequence $(u_m^\epsilon)_{m=1}^\infty$ is equicontinuous. □

By the above two lemmas and Arzela-Ascoli Compact criterion 2.47, we know for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ and a continuous function u^ϵ such that $u_{m_j}^\epsilon \to f$ uniformly on compact subsets of $\mathbb{R}^n$.
Since V is bounded, $\bar{V}$ is compact, we have that $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges uniformly on $\bar{V}$.
Thus $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges in $L^\infty(V)$.
Thus $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges in $L^q(V)$. □

Lemma 2.50. *Let $V \subseteq \mathbb{R}^n$ be open and bounded, such that ∂V is C^1. Let $1 \leq p < n$, and $(u_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$ be any bounded sequence with $\mathrm{Supp}(u_m) \subseteq V$. For $u_m^\epsilon := \eta_\epsilon * u_m$, we have that $u_m^\epsilon \to u_m$ uniformly in $L^q(V)$ as $\epsilon \to 0$.*

Proof. By taking V' to be $V + B(0,1)$ and WLOG consider $\epsilon < 1$, we assume the support of u_m^ϵ is in V.
Since $(u_m)_{m=1}^\infty$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, ||u_m||_{W^{1,p}(\mathbb{R}^n)} = ||u_m||_{W^{1,p}(V)} \leq M$.

Claim 2.50.1. *If u_m are smooth, then $||u_m^\epsilon - u_m||_{L^1(V)} \leq \epsilon|V|^{1-\frac{1}{p}}M$ for any $\epsilon > 0$.*

Proof.

$$\begin{aligned}
u_m^\epsilon(x) - u_m(x) &= (\eta_\epsilon * u_m)(x) - u_m(x) \\
&= \int_{B(0,\epsilon)} \eta_\epsilon(y)u_m(x-y)dy - u_m(x)\int_{B(0,\epsilon)} \eta_\epsilon(y)dy \\
&= \int_{B(0,\epsilon)} \eta_\epsilon(y)(u_m(x-y) - u_m(x))dy
\end{aligned}$$

Let $z := \frac{y}{\epsilon}$, we have $dy = \epsilon^n dz$. Recall $\eta_\epsilon = \frac{1}{\epsilon^n}\eta\left(\frac{y}{\epsilon}\right)$. We thus have

$$\begin{aligned}
u_m^\epsilon(x) - u_m(x) &= \int_{B(0,\epsilon)} \eta_\epsilon(y)(u_m(x-y) - u_m(x))dy \\
&= \int_{B(0,1)} \frac{\eta(z)}{\epsilon^n} - (u_m(x-\epsilon z)u_m(x))(\epsilon^n dz) \\
&= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))dy \\
&= \int_{B(0,1)} \eta(y)\int_0^1 \frac{d}{dt}u_m(x-\epsilon yt)dtdy \\
&= \int_{B(0,1)} \eta(y)\int_0^1 Du_m(x-\epsilon yt)\cdot(-\epsilon y)dtdy \\
|u_m^\epsilon(x) - u_m(x)| &\leq \int_{B(0,1)} \eta(y)\int_0^1 |Du_m(x-\epsilon yt)\cdot(-\epsilon y)|dtdy \\
&= \epsilon\int_{B(0,1)} \eta(y)\int_0^1 |Du_m(x-\epsilon yt)\cdot y|dtdy \\
&\leq \epsilon\int_{B(0,1)} \eta(y)\int_0^1 ||Du_m(x-\epsilon yt)||_1 dtdy,
\end{aligned}$$

since $||y||_2 < \epsilon < 1$. Thus

$$\begin{aligned}
||u_m^\epsilon - u_m||_{L^1(V)} &= \int_V |u_m^\epsilon(x) - u_m(x)| dx \\
&\leq \int_V \epsilon \int_{B(0,1)} \eta(y) \int_0^1 ||Du_m(x - \epsilon yt)||_1 dtdydx \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V ||Du_m(x - \epsilon yt)||_1 dxdtdy \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} ||Du_m(x - \epsilon yt)||_1 dxdtdy \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} ||Du_m(z)||_1 dzdtdy \\
&= \epsilon \left(\int_{B(0,\epsilon)} \eta(y) dy \right) \left(\int_0^1 dt \right) \left(\int_{\mathbb{R}^n} ||Du_m(z)||_1 dz \right) \\
&= \epsilon \int_{\mathbb{R}^n} ||Du_m(z)||_1 dz \\
&= \epsilon \int_V ||Du_m(z)||_1 dz \\
&= \epsilon \sum_{i=1}^n ||\partial_i u_m||_{L^1(V)} \\
&\leq \epsilon \sum_{i=1}^n |V|^{1-\frac{1}{p}} ||\partial_i u_m||_{L^p(V)} \\
&\leq \epsilon |V|^{1-\frac{1}{p}} ||u_m||_{W^{1,p}(V)} \\
&\leq \epsilon |V|^{1-\frac{1}{p}} M.
\end{aligned}$$

Notice that this is true for any $\epsilon > 0$. □

Let $\delta > 0$ be given. Since $C^\infty(\bar{V})$ is dense in $W^{1,p}(V)$ by 2.22, we can find some $\bar{u_m} \in W^{1,p}(V)$, such that $||\bar{u}_m - u_m||_{W^{1,p}(V)} < \frac{\delta}{3|V|^{1-\frac{1}{p}}}$.
Notice that $\forall m, ||\bar{u}_m||_{W^{1,p}(V)} \leq M + \frac{\delta}{3|V|^{1-\frac{1}{p}}}$ is bounded. From the claim above, we can find

$$\epsilon_0 := \frac{\delta}{3\left(M + \frac{\delta}{3|V|^{1-\frac{1}{p}}}\right)|V|^{1-\frac{1}{p}}} > 0,$$

such that $\forall 0 < \epsilon < \epsilon_0$, we have

$$||\bar{u}_m^\epsilon - \bar{u}_m||_{L^1(V)} < \frac{\delta}{3}, \ \forall m \in \mathbb{N}^+.$$

Now $||u_m - \bar{u}_m||_{L^1(V)} \leq |V|^{1-\frac{1}{p}} ||u_m - \bar{u}_m||_{L^p(V)} \leq |V|^{1-\frac{1}{p}} ||u_m - \bar{u}_m||_{W^1(V)} < \frac{\delta}{3}$.
In addition, by 2.8, we have

$$\begin{aligned}
||u_m^\epsilon - \bar{u}_m^\epsilon||_{L^1(V)} &= ||\eta_\epsilon * u_m - \eta_\epsilon * \bar{u}_m||_{L^1(V)} \\
&= ||\eta_\epsilon * (u_m - \bar{u}_m)||_{L^1(V)} \\
&\leq ||u_m - \bar{u}_m||_{L^1(V)} \\
&< \frac{\delta}{3}.
\end{aligned}$$

Now we have

$$||u_m^\epsilon - u_m||_{L^1(V)} \leq ||u_m^\epsilon - \bar{u}_m^\epsilon||_{L^1(V)} + ||\bar{u}_m^\epsilon - \bar{u}_m||_{L^1(V)} + ||\bar{u}_m - u_m||_{L^1(V)} < \delta.$$

Notice that this holds for all $\epsilon < \epsilon_0, m \in \mathbb{N}^+$, where the choice of ϵ_0 does not depend on m, and thus $||u_m^\epsilon - u_m||_{L^1(V)} \to 0$ uniformly when $\epsilon \to 0$.
Now $1 \leq q \leq p^*$, by letting $s = 1, r = q, t = p^*$, we have

$$\begin{aligned}
||u_m^\epsilon - u_m||_{L^q(V)} &\leq ||u_m^\epsilon - u_m||_{L^1(V)}^\theta ||u_m^\epsilon - u_m||_{L^{p^*}(V)}^{1-\theta} && 2.48 \\
&\leq ||u_m^\epsilon - u_m||_{L^1(V)}^\theta C^{1-\theta} ||u_m^\epsilon - u_m||_{W^{1,p}(V)}^{1-\theta} && 2.46 \\
&\leq ||u_m^\epsilon - u_m||_{L^1(V)}^\theta C^{1-\theta} \Big(||u_m^\epsilon||_{W^{1,p}(V)} + ||u_m||_{W^{1,p}(V)} \Big)^{1-\theta} \\
&\leq ||u_m^\epsilon - u_m||_{L^1(V)}^\theta C^{1-\theta} \Big(2||u_m||_{W^{1,p}(V)} \Big)^{1-\theta} && 2.18 \\
&\leq ||u_m^\epsilon - u_m||_{L^1(V)}^\theta (2CM)^{1-\theta}.
\end{aligned}$$

Given any $\delta > 0$, since $||u_m^\epsilon - u_m||_{L^1(V)} \to 0$ uniformly when $\epsilon \to 0$, we can always find some $\epsilon_0 > 0$, such that

$$\forall \epsilon < \epsilon_0, m \in \mathbb{N}^+, \ ||u_m^\epsilon - u_m||_{L^1(V)} < \left(\frac{\delta}{(2CM)^{1-\theta}} \right)^{1/\theta}.$$

Now for any $m \in \mathbb{N}^+$, we have

$$||u_m^\epsilon - u_m||_{L^q(V)} \leq ||u_m^\epsilon - u_m||_{L^1(V)}^\theta (2CM)^{1-\theta} < \delta.$$

This proves that $u_m^\epsilon \to u_m$ uniformly in $L^q(V)$ as $\epsilon \to 0$. $\square$

Theorem 2.51 (Rellich-Kondrachov Compactness)**.** *Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1. Let $1 \leq p < n$, then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for any $1 \leq q < p^*$.

Proof. The continuous embedding is done before in 2.46.
Now consider any bounded sequence $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(U)$.
Thus there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, ||\hat{u}_m||_{W^{1,p}(U)} \leq M$.
By extension theorem, we may assume $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$, with $u_m|_U = \hat{u}_m$, and there is some V such that $U \subset\subset V$ and $\forall m \in \mathbb{N}^+, \mathrm{Supp}(u_m) \subseteq V$. In addition,

$$\sup ||u_m||_{W^{1,p}(\mathbb{R}^n)} = \sup ||u_m||_{W^{1,p}(V)} \leq \sup C ||\hat{u}_m||_{W^{1,p}(U)} \leq CM.$$

Thus $(u_m)_{m=1}^\infty$ is bounded.
WLOG, we can take V to have ∂V being C^1.
Let $u_m^\epsilon := \eta_\epsilon * u_m$.
By the above lemmas, we know that

1. for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$, and
2. $u_m^\epsilon \to u_m$ uniformly in $L^q(V)$ as $\epsilon \to 0$.

Now given any $\delta > 0$.
By 2, we can find some $\epsilon_0 > 0$, such that $\forall 0 < \epsilon < \epsilon_0$, we have $\forall m \in \mathbb{N}^+, ||u_m^\epsilon - u_m||_{L^q(V)} < \frac{\delta}{3}$.
Now fix some $0 < \epsilon < \epsilon_0$.
By 1, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$.

In particular, it is Cauchy, and we can find some $N \in \mathbb{N}^+$, such that $\forall i, j \geq N, \left|\left|u^\epsilon_{m_j} - u^\epsilon_{m_i}\right|\right|_{L^q(V)} < \frac{\delta}{3}$.
Now for any $i, j \geq N$, we have that

$$\begin{aligned}
\left|\left|u_{m_i} - u_{m_j}\right|\right|_{L^q(V)} &= \left|\left|u_{m_i} - u^\epsilon_{m_i} + u^\epsilon_{m_i} - u^\epsilon_{m_j} + u^\epsilon_{m_i} - u_{m_j}\right|\right|_{L^q(V)} \\
&\leq \left|\left|u_{m_i} - u^\epsilon_{m_i}\right|\right|_{L^q(V)} + \left|\left|u^\epsilon_{m_i} - u^\epsilon_{m_j}\right|\right|_{L^q(V)} + \left|\left|u^\epsilon_{m_i} - u_{m_j}\right|\right|_{L^q(V)} \\
&< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} \\
&= \delta.
\end{aligned}$$

Thus $(u_{m_j})_{j=1}^\infty$ is a Cauchy sequence in $L^q(V)$.
Since $L^q(V)$ is complete, there is some $u \in L^q(V)$, such that $\lim_{j\to\infty} \left|\left|u_{m_j} - u\right|\right|_{L^q(V)} = 0$.
Since $U \subseteq V$, we have that $\lim_{j\to\infty} \left|\left|u_{m_j} - u\right|\right|_{L^q(U)} = 0$.
Since $u + m|_U = \hat{u}_m$, we also have that $\lim_{j\to\infty} \left|\left|\hat{u}_{m_j} - u\right|\right|_{L^q(U)} = 0$.
Thus the subsequence $\hat{u}_{m_j}$ converges to some $u \in L^q(V) \subseteq L^q(U)$.
Since $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(U)$ is any bounded sequence, we have that any bounded subset of $W^{1,p}(U)$ is relative compact in $L^q(U)$. □

Theorem 2.52. *Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1. Let $1 \leq p \leq \infty$, then*

$$W^{1,p}(U) \subset\subset L^p(U).$$

Theorem 2.53. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $1 \leq p \leq \infty$, then*

$$W^{1,p}_0(U) \subset\subset L^p(U).$$

2.10 Poincare Inequalities

Definition 2.22. For a bounded domain $U \subset \mathbb{R}^n$, we denote the average of u over U by

$$(u)U := \frac{1}{|U|}\int_U udx.$$

Theorem 2.54. *(Poincaré–Wirtinger's Inequality)*
Let $U \subset \mathbb{R}^n$ be open, bounded, and connected, such that ∂U is C^1. For any $1 \leq p \leq \infty$, $\exists C > 0$, such that

$$\forall u \in W^{1,p}(U),\ ||u - (u)_U||_{L^p(U)} \leq C||Du||_{L^p(U)}.$$

Proof. Suppose for contradiction it is not true.
Then $\forall k \in \mathbb{N}, \exists u_k \in W^{1,p}(U)$, such that $||u_k - (u_k)_U||_{L^p(U)} > k||Du||_{L^p(U)}$.
Let $v_k := \frac{u_k - (u_k)_U}{||u_k - (u_k)_U||_{L^p(U)}}$.
Notice that

$$\forall k \in \mathbb{N}^+,\ ||v_k||_{L^p(U)} = 1, (v_k)_U = 0, Dv_k = \frac{Du_k}{||u_k - (u_k)_U||_{L^p(U)}}.$$

Thus $||Dv_k||_{L^p(U)} = \frac{||Du_k||_{L^p(U)}}{||u_k - (u_k)_U||_{L^p(U)}} < \frac{1}{k}$.
Which means $||v_k||^p_{W^{1,p}(U)} = ||v_k||^p_{L^p(U)} + ||Dv_k||^p_{L^p(U)} < 1 + \frac{1}{k^p} \leq 2$.
Since this is true for any $k \in \mathbb{N}^+$, we have that $(v_k)_{k=1}^\infty$ is bounded in $W^{1,p}(U)^p$.
Since $W^{1,p}(U)^p \subset\subset L^p(U)$, there is a subsequence $(v_{k_j})_{j=1}^\infty$ and some $v \in L^p(U)$, such that

$$\lim_{j\to\infty} \left|\left|v_{k_j} - v\right|\right|_{L^p(U)} = 0.$$

Now consider any $1 \le i \le k$, and any $\phi \in C_c^\infty(U)$.

$$\begin{aligned}\left|\left|v_{k_j}\partial_i\phi - v\partial_i\phi\right|\right|_{L^p(U)}^p &= \int_U \left|v_{k_j}\partial_i\phi - v\partial_i\phi\right|^p dx \\ &= \int_U |\partial_i\phi|^p \left|v_{k_j} - v\right|^p dx \\ &\le ||\partial_i\phi||_{L^\infty(U)}^p \left|\left|v_{k_j} - v\right|\right|_{L^p(U)}^p.\end{aligned}$$

Since $\phi \in C_c^\infty(U)$, we have that $||\partial_i\phi||_{L^\infty(U)}^p$ is bounded by some $M > 0$.
Since $\lim_{j\to\infty} \left|\left|v_{k_j} - v\right|\right|_{L^p(U)} = 0$, we also have $\lim_{j\to\infty} \left|\left|v_{k_j}\partial_i\phi - v\partial_i\phi\right|\right|_{L^p(U)}^p = 0$.
In addition, $\lim_{j\to\infty} \left|\left|v_{k_j}\partial_i\phi - v\partial_i\phi\right|\right|_{L^1(U)} \le \lim_{j\to\infty} |U|^{1-\frac{1}{p}} \left|\left|v_{k_j}\partial_i\phi - v\partial_i\phi\right|\right|_{L^p(U)} = 0$.
We have

$$\begin{aligned}&\lim_{j\to\infty}\int_U \left|v_{k_j}\partial_i\phi - v\partial_i\phi\right| dx = 0 \\ \implies &\lim_{j\to\infty}\int_U (v_{k_j}\partial_i\phi - v\partial_i\phi) dx = 0 \\ \implies &\lim_{j\to\infty}\int_U v_{k_j}\partial_i\phi dx = \lim_{j\to\infty}\int_U v\partial_i\phi dx \\ \implies & -\lim_{j\to\infty}\int_U \partial_i v_{k_j}\phi dx = \int_U v\partial_i\phi dx.\end{aligned}$$

Yet

$$\begin{aligned}\left|\lim_{j\to\infty}\int_U \partial_i v_{k_j}\phi dx\right| &\le \lim_{j\to\infty}\int_U \left|\partial_i v_{k_j}\phi\right| dx \\ &\le \lim_{j\to\infty} \left|\left|\partial_i v_{k_j}\right|\right|_{L^p(U)} ||\phi||_{L^{\frac{p}{p-1}}(U)} \\ &\le \lim_{j\to\infty} \left|\left|Dv_{k_j}\right|\right|_{L^p(U)} ||\phi||_{L^{\frac{p}{p-1}}(U)} \\ &\le \lim_{j\to\infty} \frac{1}{k_j} ||\phi||_{L^{\frac{p}{p-1}}(U)} \\ &= 0,\end{aligned}$$

since $\phi \in C_c^\infty(U)$ and U is bounded, which implies $||\phi||_{L^{\frac{p}{p-1}}(U)} < \infty$. Thus

$$\int_U v\partial_i\phi dx = -\lim_{j\to\infty}\int_U \partial_i v_{k_j}\phi dx = 0 = -\int_U 0\phi dx.$$

Since this holds for any $\phi \in C_c^\infty(U)$, we must have $\partial_i v = 0$a.e. for any $1 \le i \le n$.
Thus $v \in W^{1,p}(U)$, with $Dv = 0$a.e..
Since U is connected, v is a constant by 2.33.
Since $(v)_U = 0$, we must have $v = 0$a.e..
However, this contradicts with $||v||_{L^p(U)} = 1$. □

2.11 H^{-1} Spaces

Definition 2.23. The dual space to $H_0^1(U)$ is $H^{-1}(U)$.

Theorem 2.55. *Consider any $f \in H^{-1}(U)$.*

1. *There is a tuple $(f^0, \dots, f^n)$ of functions in $L^2(U)$, such that*

$$\forall v \in H_0^1(H),\ \langle f|v\rangle_{H^{-1}(H),H_0^1(H)} = \left\langle f^0, v\right\rangle_{L^2(U)} + \sum_{i=1}^n \left\langle f^i, \partial_i v\right\rangle_{L^2(U)}.$$

   In this case, we write $f = f^0 - \sum_{i=1}^n f^i_{x^i}$.

2.

$$||f||_{H^{-1}(U)} = \inf\left\{\left(\sum_{i=0}^{n}\left|\left|f^i\right|\right|_{L^2(U)}^2\right)^{\frac{1}{2}} : (f^i)_{i=0}^n \textit{ satisfies } 1.\right\}.$$

Proof. 1. Let $f \in H^{-1}(U)$, by the Riesz-Frechet Representation theorem1.25, $\exists! u \in H_0^1(U)$, such that

$$\forall v \in H_0^1(U), \langle f|v\rangle_{H^{-1}(U),H_0^1(U)} = \langle u, v\rangle_{H_0^1(U)},$$

and $||f||_{H_0^{-1}(U)} = ||u||_{H_0^1(U)}$.
Let $f^0 = u, \forall 1 \le n, f^i := \partial_i u$. we have

$$\begin{aligned}\left\langle f^0, v\right\rangle_{L^2(U)} + \sum_{i=1}^{n}\left\langle f^i, \partial_i v\right\rangle_{L^2(U)} &= \langle u, v\rangle_{L^2(U)} + \sum_{i=1}^{n}\langle \partial_i u, \partial_i v\rangle_{L^2(U)}\\ &= \langle u, v\rangle_{H_0^1(U)}\\ &= \langle f|v\rangle_{H^{-1}(U),H_0^1(U)}.\end{aligned}$$

2. Consider any $f \in H^{-1}(U)$, from 1, we know that there is such $f^0 = u, \forall 1 \le n, f^i := \partial_i u$, satisfying 1, with

$$||f||_{H_0^{-1}(U)} = ||u||_{H_0^1(U)} = \left(\sum_{i=0}^{n}\left|\left|f^i\right|\right|_{L^2(U)}^2\right)^{\frac{1}{2}} \ge \inf\left\{\left(\sum_{i=0}^{n}\left|\left|g^i\right|\right|_{L^2(U)}^2\right)^{\frac{1}{2}} : (g^i)_{i=0}^n \text{ satisfies } 1.\right\}.$$

Now consider any $g^0, \ldots, g^n \in L^2(U)$, such that they satisfies

$$\langle f|v\rangle = \left\langle g^0, v\right\rangle_{L^2(U)} + \sum_{i=1}^{n}\left\langle g^i, v\right\rangle_{L^2(U)}.$$

For any $v \in H_0^1(U)$, we have

$$\begin{aligned}|\langle f|v\rangle| &= \left|\left\langle g^0, v\right\rangle_{L^2(U)} + \sum_{i=1}^{n}\left\langle g^i, \partial_i v\right\rangle_{L^2(U)}\right|\\ &\le \left|\left\langle g^0, v\right\rangle_{L^2(U)}\right| + \sum_{i=1}^{n}\left|\left\langle g^i, \partial_i v\right\rangle_{L^2(U)}\right|\\ &\le \left|\left|g^0\right|\right|_{L^2(U)}||v||_{L^2(U)} + \sum_{i=1}^{n}\left|\left|g^i\right|\right|_{L^2(U)}||\partial_i v||_{L^2(U)}\\ &\le \left(\sum_{i=0}^{n}\left|\left|g^i\right|\right|_{L^2(U)}^2\right)^{\frac{1}{2}}\left(||v||_{L^2(U)}^2 + \sum_{i=1}^{n}||\partial_i v||_{L^2(U)}^2\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=0}^{n}\left|\left|g^i\right|\right|_{L^2(U)}^2\right)^{\frac{1}{2}}||v||_{H_0^1(U)}.\end{aligned}$$

Thus we know

$$||f||_{H_0^{-1}(U)} = \sup_{v \in H_0^1(U), v \neq 0}\frac{|\langle f|v\rangle|}{||v||_{H_0^1(U)}} \le \inf\left\{\left(\sum_{i=0}^{n}\left|\left|g^i\right|\right|_{L^2(U)}^2\right)^{\frac{1}{2}} : (g^i)_{i=0}^n \text{ satisfies } 1.\right\}.$$

□

Corollary 2.56. *For any $v^* \in L^2(U)^* \subset L(L^2(U), \mathbb{R}) \subset L(H_0^1(U), \mathbb{R})$, with v^* identified with $v \in L^2(U)$, and any $u \in H_0^1(U) \subseteq L^2(U)$, we have*

$$\langle v^*|u\rangle_{H^{-1}(U),H_0^1(U)} = \langle v^*|u\rangle_{L^2(U)^*,L^2(U)} = \langle v, u\rangle_{L^2(U)}.$$

In addition, $v^* \in H^{-1}(U)$, and has a representation $(v, 0, \dots, 0)$ as in above theorem, with

$$||v^*||_{H^{-1}(U)} \leq ||v||_{L^2(U)}.$$

Proof. The first equality is by definition and the second equality is by 1.48.
Thus, for any $||u||_{H_0^1(U)} = 1$, we have that

$$\begin{aligned}
\left|\langle v^*|u\rangle_{H^{-1}(U),H_0^1(U)}\right| &= \left|\langle v, u\rangle_{L^2(U)}\right| \\
&\leq ||v||_{L^2(U)}||u||_{L^2(U)} \\
&\leq ||v||_{L^2(U)}||u||_{H_0^1(U)} \\
&= ||v||_{L^2(U)}.
\end{aligned}$$

Since this holds for any unitary $u \in H_0^1(U)$, we have that

$$||v^*||_{H^{-1}(U)} = \sup_{||u||_{H_0^1(U)}=1} \left|\langle v^*|u\rangle_{H^{-1}(U),H_0^1(U)}\right| \leq ||v||_{L^2(U)} < \infty,$$

which proves $v^* \in H^{-1}(U)$.
In addition, $\langle u, v\rangle_{L^2(U)} + \sum_{i=1}^n \langle 0, \partial_i v\rangle_{L^2(U)} = \langle u, v\rangle_{L^2(U)} = \langle v^*|u\rangle_{H^{-1}(U),H_0^1(U)}$. ☐

Corollary 2.57. *$\forall v \in H_0^1(U) \subset L^2(U)$, we have $v^* := \langle v, \cdot\rangle_{L^2(U)} \in H^{-1}(U)$, with*

$$||v^*||_{H^{-1}(U)} \leq ||v||_{L^2(U)} \leq ||v||_{H_0^1(U)}.$$

In other words, if we identify v with v^*, then $H_0^1(U) \subset L^2(U) \subset H^{-1}(U)$ are continuous embeddings.

Proof. Since $v \in H_0^1(U) \subseteq L^2(U)$, by the above corollary, $v^* \in H^{-1}(U)$, and has

$$||v^*||_{H^{-1}(U)} \leq ||v||_{L^2(U)} \leq ||v||_{H_0^1(U)}.$$

☐

2.12 Difference Quotients

Definition 2.24. Let $U \subset \mathbb{R}^n$ be open, $u \in L^1_{loc}(U), V \subset\subset U$, then for $|h| \in (0, \mathrm{dist}(V, \partial U)), x \in V$, we define:

1. For $i \in [n]$, $u_i^h(x) := u(x + he_i)$
2. For $i \in [n]$, the i^{th} **difference quotient** of size h at x is
$$D_i^h u(x) = \frac{u_i^h(x) - u(x)}{h} = \frac{u(x + he_i) - u(x)}{h}.$$
3. $$D^h u(x) := (D_1^h u(x), \dots, D_n^h u(x)).$$

Proposition 2.58. *Let $U \subset \mathbb{R}^n$ be open, $u \in L^1_{loc}(U)$, then $\forall i \in [n], |h| > 0$, we have*

$$\mathrm{Supp}(D_i^h u) \subseteq \mathrm{Supp}(u) + \bar{B}(0, |h|).$$

Thus,

$$\mathrm{Supp}(D^h u) \subseteq \mathrm{Supp}(u) + \bar{B}(0, |h|).$$

Proposition 2.59. *Let $U \subset \mathbb{R}^n$ be open, $u, v \in L^1_{loc}(U), V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \operatorname{dist}(V, \partial U))$, we have*

$$D_i^h(uv) = v_i^h D_i^h u + u D_i^h v.$$

Proof. We have

$$\begin{aligned}
v_i^h D_i^h u + u D_i^h v &= v_i^h \frac{u_i^h - u}{h} + u \frac{v_i^h - v}{h} \\
&= \frac{v_i^h u_i^h - v_i^h u + u v_i^h - uv}{h} \\
&= \frac{v_i^h u_i^h - uv}{h} \\
&= \frac{(uv)_i^h - uv}{h} \\
&= D_i^h(uv).
\end{aligned}$$

□

Proposition 2.60. *Let $U \subset \mathbb{R}^n$ be open, $u, v \in L^1_{loc}(U), \operatorname{Supp}(u) \subset V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \frac{1}{3}\operatorname{dist}(V, \partial U))$, we have*

$$\int_U v D_i^{-h} u dx = -\int_U u D_i^h v dx.$$

Proof. Notice that $\operatorname{Supp}(D_i^h u) \subseteq \operatorname{Supp}(u) + \bar{B}(0, |h|) \subseteq V + \bar{B}(0, |h|) \subseteq \overline{V + B(0, |h|)}$.
Since $\operatorname{dist}(\overline{V + B(0, |h|)}, \partial U) \geq 2|h|$, we can find $\overline{V + B(0, |h|)} \subset W \subset\subset V$, with $|h| < \operatorname{dist}(W, \partial U)$, where $D_i^{-h} u$ is well-defined in W.
In addition, $\operatorname{Supp}(u) \subset V \subset W$, so we can view the integrals as over W, by extending $D_i^{-h} u$ to be zero outside of W.

$$\begin{aligned}
\int_W v D_i^{-h} u dx &= \int_V v D_i^{-h} u dx \\
&= \int_V v(x) \frac{u(x - he_i) - u(x)}{-h} dx \\
&= -\int_V \frac{v(x)u(x - he_i) - v(x)u(x)}{h} dx \\
&= -\left(\int_V \frac{v(x - he_i + he_i)u(x - he_i)}{h} dx - \int_V \frac{v(x)u(x)}{h} dx \right) \\
&= -\left(\int_{V - he_i} \frac{v(y + he_i)u(y)}{h} dy - \int_V \frac{v(x)u(x)}{h} dx \right) \\
&= -\left(\int_W \frac{v_i^h(y)u(y)}{h} dy - \int_W \frac{v(x)u(x)}{h} dx \right) \\
&= -\int_W \frac{v_i^h(x)u(x) - v(x)u(x)}{h} dx \\
&= -\int_W u \frac{v_i^h - v}{h} dx \\
&= -\int_W u D_i^h v dx.
\end{aligned}$$

□

Proposition 2.61. *Let $U \subset \mathbb{R}^n$ be open, $u, D^\alpha u \in L^p_{loc}(U), V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \operatorname{dist}(V, \partial U))$, we have*

$$D^\alpha(u_i^h) = (D^\alpha u)_i^h, \ D^\alpha(D_i^h u) = D_i^h(D^\alpha u) \text{ in } V.$$

In addition, if $u \in W^{k,p}(U)$, we have $u_i^h, D_i^h u \in W^{k,p}(V)$.

Proof. Given any $i \in [n], |h| \in (0, \text{dist}(V, \partial U))$.
$\forall \phi \in C_c^\infty(V)$, we have $\phi_i^{-h} \in C_c^\infty(V + he_i) \subseteq C_c^\infty(U)$, with $D^\alpha \phi(x) = D^\alpha \phi_i^{-h}(x + he_i)$.

$$\begin{aligned}
\int_V u_i^h(x) D^\alpha \phi(x) dx &= \int_V u(x + he_i) D^\alpha \phi_i^{-h}(x + he_i) dx \\
&= \int_{V + he_i} u(y) D^\alpha \phi_i^{-h}(y) dy \\
&= \int_U u(y) D^\alpha \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_U D^\alpha u(y) \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_{V + he_i} D^\alpha u(y) \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_V D^\alpha u(x + he_i) \phi_i^{-h}(x + he_i) dx \\
&= (-1)^{|\alpha|} \int_V (D^\alpha u)_i^h(x) \phi(x) dx.
\end{aligned}$$

Since this holds for all $\phi \in C_c^\infty(V)$, we must have $D^\alpha(u_i^h) = (D^\alpha u)_i^h$.
In addition,

$$\begin{aligned}
D^\alpha(D_i^h u) &= D^\alpha\left(\frac{u_i^h - u}{h}\right) \\
&= \frac{D^\alpha(u_i^h) - D^\alpha u}{h} \\
&= \frac{(D^\alpha u)_i^h - D^\alpha u}{h} \\
&= D_i^h(D^\alpha u).
\end{aligned}$$

Now suppose $u \in W^{k,p}(U)$.

$$\begin{aligned}
\left|\left|u_i^h\right|\right|_{W^{k,p}(V)}^p &= \int_V \sum_{|\alpha| \le k} \left|(D^\alpha(u_i^h))(x)\right|^p dx \\
&= \int_V \sum_{|\alpha| \le k} \left|(D^\alpha u)_i^h(x)\right|^p dx \\
&= \int_V \sum_{|\alpha| \le k} |D^\alpha u(x + he_i)|^p dx \\
&= \int_{V + he_i} \sum_{|\alpha| \le k} |D^\alpha u(y)|^p dy \\
&\le \int_U \sum_{|\alpha| \le k} |D^\alpha u(y)|^p dy \\
&= ||u||_{W^{k,p}(U)}^p.
\end{aligned}$$

Thus $u_i^h \in W^{k,p}(V)$.
Clearly $u \in W^{k,p}(V)$, so a linear combination of them $D_i^h u \in W^{k,p}(V)$. $\square$

Theorem 2.62. *Let $U \subset \mathbb{R}^n$ be open, we have:*

1. *For $p \in [1, \infty)$, and $\forall V \subset\subset U, \exists C > 0$, such that*

$$\left|\left|D^h u\right|\right|_{L^p(V)} \le C||Du||_{L^p(U)}, \ \forall u \in W^{1,p}(U), \forall |h| \in (0, \text{dist}(V, \partial U)).$$

2. *For* $p \in (1,\infty), V \subset\subset U, u \in L^p(V)$, *if* $\exists C, \delta > 0$, *such that* $\left|\left|D^h u\right|\right|_{L^p(V)} \leq C,\ \forall |h| \in (0,\delta)$, *then*

$$u \in W^{1,p}(V),\ ||Du||_{L^p(V)} \leq C.$$

Theorem 2.63. *Let* $U \subset \mathbb{R}^n$ *be open and bounded, with* ∂U *being* C^1, *then* $u : U \to \mathbb{R}$ *is Lipschitz continuous if and only if* $u \in W^{1,\infty}(U)$.

3 Elliptic PDEs

3.1 Weak Solutions

We will consider the model problem: $U \in \mathbb{R}^n$ be open and bounded, with some $f : U \to \mathbb{R}$ be given. We want to find $u : \bar{U} \to \mathbb{R}$, such that $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$.

Definition 3.1. A **second order differential operator** is

$$Lu := -\sum_{i,j=1}^{n} \partial_j (a^{ij}(x)\partial_i u) + \sum_{i=1}^{n} b^i(x)\partial_i u + c(x)u.$$

Definition 3.2. A **symmetric (uniformly) elliptic second order differential operator** is an L such that $a^{ij} = a^{ji}$, and $\exists \theta > 0,$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta ||\xi||_2^2 \text{ for a.e. } x \in U, \forall \xi \in \mathbb{R}^n.$$

Remark. The above definition is equivalent to saying $A(x) = (a^{ij}(x)) \in \mathbb{R}^{n*n}$ is symmetric positive definite for a.e. $x \in U$, with a uniform positive lower bound $\theta > 0$ on their eigenvalues.

Example 3.1.1. If we take $a^{ij} = C\delta_{ij}$, we have $Lu = -C\Delta u + b \cdot Du + cu$.

Definition 3.3. The **bilinear form associated with** L is given by:

$$B[u, v] := \int_U \left(\sum_{i,j=1}^{n} a^{ij}\partial_i u \partial_j v + \sum_{i=1}^{n} b^i \partial_i u v + cuv \right) dx, \ \forall u, v \in H_0^1(U).$$

Definition 3.4. Consider $f = f^0 - \sum_{i=1}^{n} f^i_{x^i} \in H^{-1}(U)$ as in 2.55.
$u \in H_0^1(U)$ is called a **weak solution** to the BVP $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if u satisfies the **weak formulation:**

$$\forall v \in H_0^1(U), B[u, v] = \langle f|v \rangle = \left\langle f^0, v \right\rangle_{L^2(U)} + \sum_{i=1}^{n} \left\langle f^i, \partial_i v \right\rangle_{L^2(U)}.$$

Definition 3.5. For $f \in L^2(U)$, we have the special case:
$u \in H_0^1(U)$ is called a **weak solution** to the BVP $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if u satisfies the **weak formulation:**

$$\forall v \in H_0^1(U), B[u, v] = \langle f, v \rangle_{L^2(U)}.$$

Proposition 3.1. *If a classical solution u exists, i.e u is smooth, and $Lu = f, u|_{\partial U} = 0$, then u is always a weak solution.*

Proof. Firstly consider any $v \in C_c^\infty(U)$, we have

$$\begin{aligned}
\langle f|v\rangle &= \langle Lu, v\rangle \\
&= \int_U Luvdx \\
&= \int_U \left(-\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u) + \sum_{i=1}^n b^i \partial_i u + cu \right) vdx \\
&= -\sum_{i,j=1}^n \int_U \partial_j(a^{ij}\partial_i u)vdx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= -\sum_{i,j=1}^n \int_{\partial U} a^{ij}\partial_i u\cancelto{0}{v} e_i \cdot \hat{n}dx + \sum_{i,j=1}^n \int_U a^{ij}\partial_i u \partial_j vdx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= \int_U \sum_{i,j=1}^n a^{ij}\partial_i u \partial_j vdx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= \int_U \left(\sum_{i,j=1}^n a^{ij}\partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= B[u,v].
\end{aligned}$$

Since $H_0^1(U) = \overline{C_c^\infty(v)}$, this holds for any $v \in H_0^1(U)$. □

3.2 Existence of weak solution

3.2.1 First Existence Theorem

Theorem 3.2. *(Lax-Milgram)*
Consider a real Hilbert space $\mathcal{H}$ with $\langle\cdot,\cdot\rangle$ and action $\langle\cdot|\cdot\rangle_{\mathcal{H}^*,\mathcal{H}}$. Assume $B : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ is a bilinear form such that $\exists a, b > 0$ such that $\forall u, v \in \mathcal{H}$,

$$|B[u,v]| \le a||u||||v||$$
$$B[u,u] \ge b||u||^2.$$

Then $\forall f \in \mathcal{H}^*, \exists! u \in \mathcal{H}$ such that $\forall v \in \mathcal{H},\ B[u,v] = \langle f|v\rangle$.

Proof. For each $u \in \mathcal{H}$, we define the operator $T_u : v \mapsto B[u,v]$.
$|T_u v| = |B[u,v]| \le a||u||||v||$, and thus $||T_u||_{\mathcal{H}^*} \le a||u|| < \infty$ is bounded. Thus $T_u \in \mathcal{H}^*$.
By Riesz-Frechet Representation theorem 1.25, we have that $\exists! w \in \mathcal{H}$, such that $\forall v \in \mathcal{H}, T_u v = \langle w, v\rangle_{\mathcal{H}}$, and $||T_u||_{\mathcal{H}^*} = ||w||_{\mathcal{H}}$.
Now define $A : \mathcal{H} \to \mathcal{H}$ by $u \mapsto w$ in the above setting, such that $\forall v \in \mathcal{H}, \langle Au, v\rangle = B[u,v]$.

Claim 3.2.1. *For any $u \in \mathcal{H}$, we have that*

$$b||u|| \le ||Au|| \le a||u||.$$

Proof. We have
$$||Au||^2 = \langle Au, Au\rangle = B[u, Au] \le a||u||||Au||.$$
If $||Au|| = 0$, clearly $||Au|| \le a||u||$.
Otherwise we can divide both side by $||Au||$, and get $||Au|| \le a||u||$.
On the other hand, we have
$$b||u||^2 \le B[u,u] = \langle Au, u\rangle \le ||Au||||u||.$$
If $||u|| = 0$, clearly $b||u|| \le ||Au||$.
Otherwise we can divide both side by $||u||$, and get $b||u|| \le ||Au||$. □

Claim 3.2.2. *We have $A \in \mathcal{H}^*$.*

Proof. For any $u_1, u_2, v \in \mathcal{H}, c \in \mathbb{R}$, we have that

$$\begin{aligned} \langle A(u_1 + cu_2), v\rangle &= B[u_1 + cu_2, v] \\ &= B[u_1, v] + cB[u_2, v] \\ &= \langle Au_1, v\rangle + c\langle Au_2, v\rangle \\ &= \langle Au_1 + cAu_2, v\rangle. \end{aligned}$$

Since this holds for all $v \in \mathcal{H}$, we have $A(u_1 + cu_2) = Au_1 + cAu_2$, and thus A is linear.
In addition, we have

$$||A||_{\mathcal{H}^*} = \sup_{u\in\mathcal{H}, u\neq 0} \frac{||Au||}{||u||} \leq \sup_{u\in\mathcal{H}, u\neq 0} \frac{a||u||}{||u||} = a < \infty.$$

This shows A is bounded, and thus $A \in \mathcal{H}^*$. ∎

Claim 3.2.3. *A is bijective.*

Proof. Suppose $Au = 0$, we have that

$$b||u|| \leq ||Au|| = 0,$$

which means that $u = 0$. Thus A is injective.
Consider any sequence $(y_j)_{j=1}^{\infty} \subset \mathrm{Im}(A)$, such that $\lim_{j\to\infty} y_j = y \in \mathcal{H}$.
We can find $(x_j)_{j=1}^{\infty} \subset \mathcal{H}$, such that $\forall j \geq 1, Ax_j = y_j$.
Since $(y_j)_{j=1}^{\infty}$ is convergent and thus Cauchy, given any $\epsilon > 0$, we can find some $N \geq 1$, such that $\forall i, j \geq N, ||y_j - y_i|| < b\epsilon$.
Now

$$\begin{aligned} ||x_j - x_i|| &\leq \frac{1}{b}||A(x_j - x_i)|| \\ &= \frac{1}{b}||Ax_j - Ax_i|| \\ &= \frac{1}{b}||y_j - y_i|| \\ &< \frac{1}{b}b\epsilon \\ &< \epsilon. \end{aligned}$$

Thus $(x_j)_{j=1}^{\infty}$ is Cauchy.
Since $\mathcal{H}$ is complete, there is some $x \in \mathcal{H}$, such that $\lim_{j\to\infty} x_j = x$.
Since A is bounded and thus continuous, we have that

$$\begin{aligned} Ax &= A\left(\lim_{j\to\infty} x_j\right) \\ &= \lim_{j\to\infty} Ax_j \\ &= \lim_{j\to\infty} y_j \\ &= y. \end{aligned}$$

Thus $y \in \mathrm{Im}(A)$.
This proves that $\mathrm{Im}(A)$ is closed.
Since A is linear, $\mathrm{Im}(A)$ is a closed subspace of $\mathcal{H}$, and thus $\mathcal{H} = \mathrm{Im}(A) \oplus \mathrm{Im}(A)^{\perp}$.
Consider any $w \in \mathrm{Im}(A)^{\perp}$, we must have

$$b||w||^2 \leq B[w, w] = \langle Aw, w\rangle = 0.$$

Thus $\mathrm{Im}(A)^{\perp} = \{0\}$, and thus $\mathrm{Im}(A) = \mathcal{H}$.
Thus A is surjective. ∎

Now by the Bounded inverse Theorem, A^{-1} exists and is bounded.
By Riesz-Frechet Representation theorem1.25, given any $f \in \mathcal{H}^*$, we have

$$\exists! w \in \mathcal{H}, \text{ such that } \langle f|v\rangle = \langle w, v\rangle \ \forall v \in \mathcal{H}.$$

Let $u = A^{-1}w$, we have that

$$\forall v \in \mathcal{H}, \ B[u,v] = \langle Au, v\rangle = \langle w, v\rangle = \langle f|v\rangle.$$

This proves the existence.
Now suppose there is some $\hat{u}$ such that $\forall v \in \mathcal{H}, \ B[\hat{u}, v] = \langle f|v\rangle = B[u,v]$.
We must have $B[u-\hat{u}, v] = 0, \ \forall v \in \mathcal{H}$. Thus

$$b||u-\hat{u}|| \leq B[u-\hat{u}, u-\hat{u}] = 0,$$

and thus $\hat{u} = u$ is unique. ∎

Proposition 3.3 (Cauchy's inequality). *For any $a, b, \epsilon > 0$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

Theorem 3.4 (Energy estimates). *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and $a^{ij}, b^i, c \in L^\infty(U)$, such that (a^{ij}) is symmetric positive definite. For the bilinear form defined in 3.3, there exists constants $\alpha, \beta > 0, \gamma \geq 0$, such that $\forall u, v \in H_0^1(U)$,*

$$|B[u,v]| \leq \alpha ||u||_{H^1(U)} ||v||_{H^1(U)} \tag{1}$$

$$\beta ||u||^2_{H^1(U)} \leq B[u,u] + \gamma ||u||^2_{L^2(U)}. \tag{2}$$

Proof. We have

$$\begin{aligned}
|B[u,v]| &= \left| \int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cuv \right) dx \right| \\
&\leq \sum_{i,j=1}^n \left|\left|a^{ij}\right|\right|_{L^\infty(U)} \int_U |\partial_i u||\partial_j v| dx + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} \int_U |\partial_i u||v| dx + ||c||_{L^\infty(U)} \int_U |u||v| dx \\
&\leq \sum_{i,j=1}^n \left|\left|a^{ij}\right|\right|_{L^\infty(U)} ||\partial_i u||_{L^2(U)} ||\partial_j v||_{L^2(U)} + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} ||\partial_i u||_{L^2(U)} ||v||_{L^2(U)} \\
&\quad + ||c||_{L^\infty(U)} ||u||_{L^2(U)} ||v||_{L^2(U)} \\
&\leq \sum_{i,j=1}^n \left|\left|a^{ij}\right|\right|_{L^\infty(U)} ||u||_{H^1(U)} ||v||_{H^1(U)} + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} ||u||_{H^1(U)} ||v||_{H^1(U)} \\
&\quad + ||c||_{L^\infty(U)} ||u||_{H^1(U)} ||v||_{H^1(U)} \\
&= \left(\sum_{i,j=1}^n \left|\left|a^{ij}\right|\right|_{L^\infty(U)} + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} + ||c||_{L^\infty(U)} \right) ||u||_{H^1(U)} ||v||_{H^1(U)}.
\end{aligned}$$

Taking $\alpha := \sum_{i,j=1}^n \left|\left|a^{ij}\right|\right|_{L^\infty(U)} + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} + ||c||_{L^\infty(U)}$, we notice that $\alpha \geq 0$, and $\alpha = 0 \implies \forall i, j, \ a^{ij} = 0$, which contradicts (a_{ij}) is positive definite. Thus $\alpha > 0$, and $|B[u,v]| \leq \alpha ||u||_{H^1(U)} ||v||_{H^1(U)}$.
On the other hand, consider $\xi = Du \in \mathbb{R}^n$.
We have that

$$\theta ||Du||_2^2 \leq \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u.$$

Thus

$$\begin{aligned}
\theta||Du||^2_{L^2(U)} &= \theta\int_U ||Du||_2^2 dx \\
&\le \int_U \sum_{i,j=1}^n a^{ij}\partial_i u \partial_j u dx \\
&= B[u,u] - \int_U \left(\sum_{i=1}^n b^i \partial_i u u + cuu\right) dx \\
&\le B[u,u] + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} ||\partial_i u||_{L^2(U)} ||u||_{L^2(U)} + ||c||_{L^\infty(U)} ||u||^2_{L^2(U)} \\
&\le B[u,u] + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} \left(\epsilon||\partial_i u||^2_{L^2(U)} + \frac{1}{4\epsilon}||u||^2_{L^2(U)}\right) + ||c||_{L^\infty(U)} ||u||^2_{L^2(U)} \\
&= B[u,u] + \epsilon\sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} ||\partial_i u||^2_{L^2(U)} + \left(\frac{1}{4\epsilon}\sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} + ||c||_{L^\infty(U)}\right) ||u||^2_{L^2(U)} \\
&\le B[u,u] + \epsilon\sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} ||Du||^2_{L^2(U)} + \left(\frac{1}{4\epsilon}\sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} + ||c||_{L^\infty(U)}\right) ||u||^2_{L^2(U)}.
\end{aligned}$$

If $\sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} = 0$, pick any $\epsilon > 0$.
Otherwise choose $\epsilon := \frac{\theta}{2\sum_{i=1}^n ||b^i||_{L^\infty(U)}} > 0$, and $\gamma := \frac{1}{4\epsilon}\sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} + ||c||_{L^\infty(U)}$, we have

$$\frac{\theta}{2}||Du||^2_{L^2(U)} \le B[u,u] + \gamma||u||^2_{L^2(U)}.$$

Since $||Du||_{L^p(U)}$ and $||u||_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$ by 2.42, we have that

$$\exists C > 0, \text{ such that } \forall u \in H_0^1(U),\ ||u||^2_{H^1(U)} \le C||Du||^2_{L^p(U)}.$$

Taking $\beta := \frac{\theta}{2C} > 0$, we have

$$\beta||u||^2_{H^1(U)} \le \frac{\theta}{2}||Du||^2_{L^2(U)} \le B[u,u] + \gamma||u||^2_{L^2(U)}.$$

□

Definition 3.6. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\mu \in \mathbb{R}$, we define the operator L_μ by

$$L_\mu u := Lu + \mu u.$$

We define the bilinear form associated to L_μ to be B_μ.

Proposition 3.5.

$$B_\mu[u,v] = B[u,v] + \int_U \mu uv dx = B[u,v] + \mu\langle u, v\rangle_{L^2(U)}.$$

Theorem 3.6. *(First Existence Theorem)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \ge 0$ be the same as in Energy Estimate 3.4. For any $\mu \ge \gamma$ and $\forall f \in H^{-1}(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP: $\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

Proof. By Energy estimates, we have that $\forall u, v \in H_0^1(U)$,

$$\begin{aligned}
|B_\mu[u,v]| &\le |B[u,v]| + \mu\left|\langle u,v\rangle_{L^2(U)}\right| \\
&\le \alpha||u||_{H^1(U)}||v||_{H^1(U)} + \mu||u||_{L^2(U)}||v||_{L^2(U)} \\
&\le (\alpha+\mu)||u||_{H^1(U)}||v||_{H^1(U)} \\
B_\mu[u,u] &= B[u,u] + \mu\langle u,u\rangle_{L^2(U)} \\
&= B[u,u] + \mu||u||_{L^2(U)} \\
&\ge \beta||u||^2_{H^1(U)} + (\mu-\gamma)||u||^2_{L^2(U)} \\
&\ge \beta||u||^2_{H^1(U)}.
\end{aligned}$$

By Lax–Milgram Theorem, for any $f \in H^{-1}(U)$, there is a unique $u \in H_0^1(U)$, such that

$$\forall v \in H_0^1(U), B_\mu[u,v] = \langle f|v\rangle.$$

□

Corollary 3.7. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \ge 0$ be the same as in Energy Estimate 3.4. For any $\mu \ge \gamma$ and $\forall f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP:* $\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

3.2.2 More Existence Theorems

Definition 3.7. Consider $Lu := -\sum_{i,j=1}^n \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^n b^i(x)\partial_i u + c(x)u$, we define its **formal adjoint**

$$L^\dagger v := -\sum_{i,j=1}^n \partial_i(a^{ij}(x)\partial_j v) + \sum_{i=1}^n b^i(x)\partial_i v + c(x)v.$$

For $f \in H^{-1}(U)$, the **adjoint problem** is $\begin{cases} L^\dagger v = f & \text{in } U, \\ v = 0 & \text{on } \partial U \end{cases}$, and the bilinear form associated with it is $B^*[u,v]$.
Notice that $v \in H_0^1(U)$ is a weak solution of the adjoint problem if v satisfies $\forall u \in H_0^1(U), B^*[u,v] = \langle f|u\rangle$.

Proposition 3.8.

$$B^*[u,v] := B[v,u].$$

Remark. Since L is not bounded, $L^\dagger$ is not its usual adjoint operator. However, when u, v are both smooth, we have that $\langle Lu, v\rangle_{L^2(U)} = B[u,v] = B^*[v,u] = \langle v, L^\dagger u\rangle$.

Definition 3.8. For $\mu \in \mathbb{R}$, we can similarly define $L^\dagger_\mu u := L^\dagger u + \mu u$, and the bilinear form associated with it is $B^*_\mu[u,v]$.

Proposition 3.9.

$$B^*_\mu[u,v] = B^*[u,v] + \mu\langle u,v\rangle_{L^2(U)} = B[v,u] + \mu\langle v,u\rangle_{L^2(U)} = B_\mu[v,u].$$

Proposition 3.10. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \ge 0$ be the same as in Energy Estimate 3.4. For any $\mu \ge \gamma$ and $\forall f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP:* $\begin{cases} L^\dagger u + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ *Namely,*

$$\exists! u \in H_0^1(U), \text{ such that } \forall v \in H_0^1(U), B^*_\mu[u,v] = \langle f, v\rangle_{L^2(U)}.$$

Proof. For $\alpha, \beta > 0, \gamma \geq 0$ from Energy Estimate 3.4, we have that $\forall u, v \in H_0^1(U)$,

$$\begin{aligned} |B^*[v,u]| &= |B[u,v]| \quad (3) \\ &\leq \alpha ||u||_{H^1(U)} ||v||_{H^1(U)} \quad (4) \\ \beta ||u||^2_{H^1(U)} &\leq B[u,u] + \gamma ||u||^2_{L^2(U)} \quad (5) \\ &= B^*[u,u] + \gamma ||u||^2_{L^2(U)}. \quad (6) \end{aligned}$$

Thus B and B^* have the same energy estimate. By First Existence Theorem 3.6, we have the result. □

Definition 3.9. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$, we define $L_\mu^{-1} : L^2(U) \to H_0^1(U)$ by $f \mapsto u$, where u is the unique solution to

$$\forall v \in H_0^1(U), B_\mu[u,v] = \langle f, v \rangle_{L^2(U)}$$

given by the First Existence Theorem 3.6.
We can also define $(L_\mu^\dagger)^{-1} : L^2(U) \to H_0^1(U)$ by $f \mapsto u$, where u is the unique solution to

$$\forall v \in H_0^1(U), B^*_\mu[u,v] = \langle f, v \rangle_{L^2(U)}.$$

Remark. We notice that by definition $B_\mu[L_\mu^{-1} f, v] = \langle f, v \rangle_{L^2(U)}, \forall v \in H_0^1(U), \forall f \in L^2(U), \forall \mu \geq \gamma$.

Lemma 3.11. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. Then for any $\mu \geq \gamma$, if we let $K = \mu L_\mu^{-1}$, we have that $K : L^2(U) \to H_0^1(U) \subseteq L^2(U)$ is compact.*

Proof. Consider any $g \in L^2(U)$, we have that

$$\begin{aligned} \beta \left|\left| L_\mu^{-1} g \right|\right|^2_{H^1(U)} &\leq B[L_\mu^{-1} g, L_\mu^{-1} g] + \gamma \left|\left| L_\mu^{-1} g \right|\right|^2_{L^2(U)} \\ &\leq B[L_\mu^{-1} g, L_\mu^{-1} g] + \mu \left|\left| L_\mu^{-1} g \right|\right|^2_{L^2(U)} \\ &= B_\mu[L_\mu^{-1} g, L_\mu^{-1} g] \\ &= \left\langle g, L_\mu^{-1} g \right\rangle_{L^2(U)} \\ &\leq ||g||_{L^2(U)} \left|\left| L_\mu^{-1} g \right|\right|_{L^2(U)} \\ &\leq ||g||_{L^2(U)} \left|\left| L_\mu^{-1} g \right|\right|_{H^1(U)} \\ &\implies \\ \left|\left| L_\mu^{-1} g \right|\right|_{H^1(U)} &\leq \frac{1}{\beta} ||g||_{L^2(U)} \\ &\implies \\ ||Kg||_{H^1(U)} &\leq \frac{\mu}{\beta} ||g||_{L^2(U)}. \end{aligned}$$

Thus, $K : L^2(U) \to H_0^1(U)$ is bounded.
Since $H_0^1(U) \subset\subset L^2(U)$, by 1.23, we have that $K : L^2(U) \to L^2(U)$ is compact. □

Lemma 3.12. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $f \in L^2(U)$, if we let $h := L_\gamma^{-1} f, K = \gamma L_\gamma^{-1}$, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ if and only if u solves $(I - K)u = h$.*

Proof. We will firstly show that u solves $\forall v \in H_0^1(U),\ B_\gamma[u,v] = \langle f+\gamma u, v\rangle_{L^2(U)}$, if and only if u solves $u = L_\gamma^{-1}(f+\gamma u)$.
Suppose $\forall v \in H_0^1(U),\ B_\gamma[u,v] = \langle f+\gamma u, v\rangle_{L^2(U)}$.
we have that $u' := L_\gamma^{-1}(f+\gamma u) \in H_0^1(U)$ is the unique solution, such that

$$B_\gamma[u',v] = \langle f+\gamma u, v\rangle,\ \forall v \in H_0^1(U).$$

Thus $u = u' = L_\gamma^{-1}(f+\gamma u)$.
On the other hand, suppose $u = L_\gamma^{-1}(f+\gamma u)$, then we have that

$$\forall v \in H_0^1(U),\ B_\gamma[u,v] = B_\gamma[L_\gamma^{-1}(f+\gamma u), v] = \langle f+\gamma u, v\rangle_{L^2(U)}.$$

Thus, $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if and only if
u solves $\forall v \in H_0^1(U),\ B[u,v] = \langle f, v\rangle_{L^2(U)}$, if and only if
u solves $\forall v \in H_0^1(U),\ B[u,v] + \gamma\langle u, v\rangle_{L^2(U)} = \langle f, v\rangle_{L^2(U)} + \gamma\langle u, v\rangle_{L^2(U)}$, if and only if
u solves $\forall v \in H_0^1(U),\ B_\gamma[u,v] = \langle f+\gamma u, v\rangle_{L^2(U)}$, if and only if
u solves $u = L_\gamma^{-1}(f+\gamma u)$, if and only if
u solves $u = L_\gamma^{-1} f + \gamma L_\gamma^{-1} u$, if and only if
u solves $Iu = h + Ku$, if and only if
u solves $(I-K)u = h$. □

Theorem 3.13. *(Second Existence Theorem)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator.

1. *Precisely one of the following must be true:*

   (a) *$\forall f \in L^2(U), \exists! u \in H_0^1(U)$, a unique weak solution to* $\begin{cases} Lu = f & \textit{in } U, \\ u = 0 & \textit{on } \partial U. \end{cases}$

   (b) *There is a weak solution $u \neq 0 \in H_0^1(U)$ to the homogeneous problem* $\begin{cases} Lu = 0, & \textit{in } U, \\ u = 0, & \textit{on } \partial U. \end{cases}$

2. *Let $N \subset H_0^1(U)$ be the solution space of weak solutions to* $\begin{cases} Lu = 0, & \textit{in } U, \\ u = 0, & \textit{on } \partial U \end{cases}$*, and let $N^* \subset H_0^1(U)$ be the solution space of weak solutions to* $\begin{cases} L^\dagger u = 0, & \textit{in } U, \\ u = 0, & \textit{on } \partial U \end{cases}$*, then* $\dim(N) = \dim(N^*) < \infty$.

3. *The problem* $\begin{cases} Lu = f & \textit{in } U, \\ u = 0 & \textit{on } \partial U. \end{cases}$ *has a weak solution if and only if $f \in (N^*)^\perp \subseteq L^2(U)$.*

Proof. Take $\mu = \gamma$.
From the above lemma, we know that for any $f \in L^2(U)$, if we let $K = \gamma L_\gamma^{-1}$, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ if and only if u solves $(I-K)u = L_\gamma^{-1} f$.
We also have shown that $K : L^2(U) \to H_0^1(U) \subseteq L^2(U)$ is compact.

1. By 1.36, we have that exactly one of the following holds:

   (a) $\forall v \in L^2(U),\ \exists! u \in L^2(U),$ such that $(I-K)u = v$.
   In this case, for any $f \in L^2(U),\ \exists! u \in L^2(U),$ such that $(I-K)u = L_\gamma^{-1} f$.
   In addition, since $L_\gamma^{-1} f \in H_0^1(U), Ku = \gamma L_\gamma^{-1} u \in H_0^1(U)$, we must have $u = L_\gamma^{-1} f + Ku \in H_0^1(U)$.
   Thus u is the unique weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$.

(b) $\exists u \neq 0 \in L^2(U),$ such that $(I-K)u = 0 = L_\gamma^{-1}0$.
Similarly, we can see that $u = Ku = \gamma L_\gamma^{-1} u \in H_0^1(U)$.
Thus u is a non-trivial solution to $\begin{cases} Lu = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$.

2. By the above lemma, $N = \mathrm{Ker}(I-K)$.
By 1.36, we have that $\dim(N) = \dim(\mathrm{Ker}(I-K^\dagger)) < \infty$. Let $L_\gamma^\dagger u := L^\dagger u + \gamma u$.
Consider any $g, h \in L^2(U)$, we have that

$$\begin{aligned}\left\langle h, K^\dagger g\right\rangle &= \langle Kh, g\rangle \\ &= \langle g, Kh\rangle_{L^2(U)} \\ &= \gamma\left\langle g, L_\gamma^{-1}h\right\rangle_{L^2(U)} \\ &= \gamma B_\gamma^*[(L_\gamma^\dagger)^{-1}g, L_\gamma^{-1}h] \\ &= \gamma B_\gamma[L_\gamma^{-1}h, (L_\gamma^\dagger)^{-1}g] \\ &= \gamma\left\langle h, (L_\gamma^\dagger)^{-1}g\right\rangle_{L^2(U)} \\ &= \left\langle h, \gamma(L_\gamma^\dagger)^{-1}g\right\rangle_{L^2(U)}.\end{aligned}$$

Since this holds for all $g, h \in L^2(U)$, we have that $K^\dagger = \gamma(L_\gamma^\dagger)^{-1}$.
By the above lemma, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} L^\dagger u = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if and only if u solves $(I-K^\dagger)u = 0$, if and only if $u \in \mathrm{Ker}(I-K^\dagger)$.
Thus $N^* = \mathrm{Ker}(I-K^\dagger)$.

3. (a) $\gamma = 0$.
Notice that $K = 0$, and thus $N^* = \ker(I-K^\dagger) = \ker(I) = \{0\}$.
Thus $(N^*)^\perp = L^2(U)$.
In addition, $N = \ker(I-K) = \ker(I) = \{0\}$, so we must be in case (a).
Thus $\forall f \in (N^*)^\perp$, the problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a (unique) weak solution.
The other direction is trivial since $(N^*)^\perp = L^2(U)$ is the whole space.

(b) $\gamma \neq 0$.
By 1.36, we have that $\mathrm{Im}(I-K) = \mathrm{Ker}(I-K^\dagger)^\perp$.
By the above lemma, we have that the problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a weak solution, if and only if,
there is some u that solves $(I-K)u = L_\gamma^{-1}f$, if and only if,
$L_\gamma^{-1}f \in \mathrm{Im}(I-K) = \mathrm{Ker}(I-K^\dagger)^\perp$, if and only if,
$\forall v \in \mathrm{Ker}(I-K^\dagger) = N^*$,

$$\begin{aligned}\left\langle L_\gamma^{-1}f, v\right\rangle &= 0 \\ \frac{1}{\gamma}\langle Kf, v\rangle &= 0 \\ \frac{1}{\gamma}\left\langle f, K^\dagger v\right\rangle &= 0 \\ \frac{1}{\gamma}\left\langle f, K^\dagger v + (I-K^\dagger)v\right\rangle &= 0 \\ \frac{1}{\gamma}\langle f, v\rangle &= 0 \\ \langle f, v\rangle &= 0,\end{aligned}$$

if and only if $f \in (N^*)^{\perp}$.

$\square$

Definition 3.10. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. The **spectrum** of L is defined to be

$$\Sigma := \mathbb{R} \setminus \left\{ \lambda \in \mathbb{R} : \forall f \in L^2(U), \exists! u \in H_0^1(U), \text{ such that } \forall v \in H_0^1(U), B_{-\lambda}[u, v] = \langle f, v \rangle_{L^2(U)} \right\}.$$

Proposition 3.14. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let Σ be the spectrum of L.*

1. *$\lambda \notin \Sigma$ if and only if* $\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ *has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$.*

2. *$\lambda \in \Sigma$ if and only if* $\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ *has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$.*

Proof. 1. This is by definition.

2. By Second Existence Theorem 3.13 on $L_{-\lambda}$.

$\square$

Lemma 3.15. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4, and Σ be the spectrum of L, we always have $\Sigma \subseteq (-\gamma, \infty)$.*

Proof. If $\lambda \leq -\gamma$, we have that $-\lambda \geq \gamma$, and by First Existence Theorem 3.6, we have that the problem has a unique weak solution, and thus $\lambda \notin \Sigma$. $\square$

Theorem 3.16. *(Third existence theorem)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let Σ be the spectrum of L.

1. *Σ is at most countable.*

2. *If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$ can be arranged in non-decreasing sequence with $\lim_{k\to\infty} \lambda_k = \infty$.*

Proof. Let $\gamma' \geq 0$ be the same as in Energy Estimate 3.4, we have $\Sigma \subseteq (-\gamma', \infty) \subseteq (-\gamma, \infty)$ for any $\gamma \geq \gamma'$. We will take some $\gamma > 0$, and consider $\lambda > -\gamma$.
$\lambda \in \Sigma$, if and only if $\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$,
if and only if $\begin{cases} Lu + \gamma u = (\lambda + \gamma) u, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$.
Suppose $\lambda \in \Sigma$, then let $g = (\lambda + \gamma) u$. By First Existence Theorem 3.6, there is a unique weak solution $(L_\gamma)^{-1}((\lambda + \gamma) u) = \frac{\lambda+\gamma}{\gamma} K u$ to the problem $\begin{cases} Lu + \gamma u = g, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$.
Since $u \neq 0 \in H_0^1(U)$ is a weak solution to the problem, we have

$$u = \frac{\lambda + \gamma}{\gamma} K u.$$

Thus $u \neq 0 \in L^2(U)$ is an eigen-vector for K, with corresponding eigenvalue $\frac{\gamma}{\lambda+\gamma}$.
Notice that $\frac{\gamma}{\lambda+\gamma} > 0$, since $\gamma > 0, \lambda > -\gamma$, and thus $\frac{\gamma}{\lambda+\gamma} \in \mathrm{Spec}_p(K) \setminus (\infty, 0]$.
Since this holds for any $\lambda \in \Sigma$, we have $\left\{ \frac{\gamma}{\lambda+\gamma} : \lambda \in \Sigma \right\} \subseteq \mathrm{Spec}_p(K) \setminus (\infty, 0]$.

On the other hand, $\forall \mu \in \mathrm{Spec}_p(K) \setminus \{0\}$, we have that $\lambda' := \frac{\gamma(1-\mu)}{\mu} = -\gamma + \frac{\gamma}{\mu}$ satisfies $\mu = \frac{\gamma}{\lambda'+\gamma}$.
Pick any eigen-vector $u \neq 0 \in L^2(U)$ corresponds to μ, we have that $\frac{\gamma}{\lambda+\gamma} u = Ku$.
Thus $u = (L_\gamma)^{-1}((\lambda'+\gamma)u) \neq 0 \in H_0^1(U)$ is a weak solution to the problem $\begin{cases} Lu + \gamma u = (\lambda'+\gamma)u, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$.
If $\lambda' > -\gamma \iff \frac{\gamma}{\mu} > 0 \iff \mu > 0$, we have that $\lambda' \in \Sigma$.
Thus, we have $\left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \mathrm{Spec}_p(K) \setminus (\infty, 0] \right\} \subseteq \Sigma$.
We have shown that

$$\Sigma = \left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \mathrm{Spec}_p(K) \setminus (\infty, 0] \right\}.$$

Since K is compact, by the Spectral theorem 1.24, we have that either

1. $\mathrm{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^N$ is finite, which means $\Sigma \subseteq \left(\lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^N$ is finite.
2. $\mathrm{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^\infty$ is countable, and $\lim_{k\to\infty} \mu_k = 0$, which means that $\Sigma \subseteq \left(\lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^\infty$ is at most countable.
   In addition, if Σ is infinite, it must be $(\lambda_{k_j})_{j=1}^\infty \subseteq (\lambda_k)_{k=1}^\infty$.
   $$\lim_{k\to\infty} |\lambda_k| = \lim_{k\to\infty} \left| \frac{\gamma(1-\mu_k)}{\mu_k} \right| = \lim_{k\to\infty} \left| \frac{\gamma}{\mu_k} \right| = \infty.$$
   Thus $\lim_{k\to\infty} \left|\lambda_{k_j}\right| = \lim_{k\to\infty} |\lambda_k| = \infty$.
   Since we have $\forall j, \lambda_{k_j} > -\gamma$, we must have $\lim_{j\to\infty} \lambda_{k_j} = \infty$.

□

Theorem 3.17. *(Boundedness of inverse)*
Let Σ be the spectrum of L, and $\lambda \notin \Sigma$. Then there is a constant $C > 0$ such that for all $f \in L^2(U)$ and the unique weak solution $u \in H_0^1(U)$ to $\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$, we always have

$$||u||_{L^2(U)} \leq C||f||_{L^2(U)}.$$

Proof. Consider any $\lambda \notin \Sigma$.
Suppose for contradiction, we can find $(\tilde{u}_k)_{k=1}^\infty \subset H_0^1(U), (\tilde{f}_k)_{k=1}^\infty \subset L^2(U)$, such that $\forall k \geq 1$,

$$\begin{cases} L\tilde{u_k} = \lambda \tilde{u_k} + \tilde{f_k} & \text{in } U, \\ \tilde{u_k} = 0, & \text{on } \partial U \end{cases}$$

and

$$||\bar{u}||_{L^2(U)} > k\left|\left|\bar{f}\right|\right|_{L^2(U)}.$$

Let $u_k := \frac{\tilde{u}_k}{||\tilde{u}_k||_{L^2(U)}}, f_k := \frac{\tilde{f}_k}{||\tilde{u}_k||_{L^2(U)}}$.
Notice that $\forall k \geq 1, ||u_k||_{L^2(U)} = 1$, and $||f_k||_{L^2(U)} = \frac{||\tilde{f}_k||_{L^2(U)}}{||\tilde{u}_k||_{L^2(U)}} < \frac{1}{k}$.
In addition, $\forall v \in H_0^1(U)$,

$$\begin{aligned} B[u_k, v] &= \frac{1}{||\tilde{u}_k||_{L^2(U)}} B[\tilde{u}_k, v] \\ &= \frac{1}{||\tilde{u}_k||_{L^2(U)}} \left(\left\langle \tilde{f}_k, v \right\rangle_{L^2(U)} + \lambda \langle \tilde{u}_k, v \rangle_{L^2(U)} \right) \\ &= \left\langle \frac{\tilde{f}_k}{||\tilde{u}_k||_{L^2(U)}}, v \right\rangle_{L^2(U)} + \lambda \left\langle \frac{\tilde{u}_k}{||\tilde{u}_k||_{L^2(U)}}, v \right\rangle_{L^2(U)} \\ &= \langle f_k, v \rangle_{L^2(U)} + \lambda \langle u_k, v \rangle_{L^2(U)}. \end{aligned}$$

By Energy Estimate 3.4, we have that

$$\begin{aligned}
\beta||u_k||^2_{H^1(U)} &\le B[u_k, u_k] + \gamma||u_k||^2_{L^2(U)} \\
&= \langle f_k, u_k\rangle_{L^2(U)} + \lambda\langle u_k, u_k\rangle_{L^2(U)} + \gamma||u_k||^2_{L^2(U)} \\
&\le ||f_k||_{L^2(U)}||u_k||_{L^2(U)} + (\lambda+\gamma)||u_k||^2_{L^2(U)} \\
&= ||f_k||_{L^2(U)} + \lambda + \gamma \\
&< \lambda + \gamma + \frac{1}{k} \\
&\le \lambda + \gamma + 1. \\
||u_k||_{H^1(U)} &\le \sqrt{\frac{\lambda+\gamma+1}{\beta}}
\end{aligned}$$

Thus $(u_k)_{k=1}^{\infty}$ is a bounded sequence in $H_0^1(U)$.
Since $H_0^1(U)$ is a Hilbert space, and thus reflexive, by 1.29, there $\exists (u_{k_j})_{j=1}^{\infty}$ a subsequence, and $u \in H_0^1(U)$, such that $u_{k_j} \rightharpoonup u$.
Also, since $H_0^1(U) \subset\subset L^2(U)$, by 1.32, we have that $u_{k_j} \to u$ in $L^2(U)$. Thus,

$$||u||_{L^2(U)} = \lim_{j\to\infty} \left|\left|u_{k_j}\right|\right|_{L^2(U)} = 1.$$

Now consider any $v \in H_0^1(U)$, we have that the map $w \mapsto B[w, v]$ is a linear bounded operator, so by weak convergence of $(u_{k_j})_{j=1}^{\infty}$, we have that

$$\begin{aligned}
B[u, v] &= \lim_{j\to\infty} B[u_{k_j}, v] \\
&= \lim_{j\to\infty} \Big(\langle f_{k_j}, v\rangle_{L^2(U)} + \lambda\langle u_{k_j}, v\rangle_{L^2(U)}\Big) \\
&= \lim_{j\to\infty} \langle f_{k_j}, v\rangle_{L^2(U)} + \lambda\left\langle \lim_{j\to\infty} u_{k_j}, v\right\rangle_{L^2(U)} \\
&\le \lim_{j\to\infty} \left|\left|f_{k_j}\right|\right|_{L^2(U)}||v||_{L^2(U)} + \lambda\langle u, v\rangle_{L^2(U)} \\
&\le \lim_{j\to\infty} \frac{1}{k_j}||v||_{L^2(U)} + \lambda\langle u, v\rangle_{L^2(U)} \\
&= \lambda\langle u, v\rangle_{L^2(U)}.
\end{aligned}$$

Namely, $\hat{u} = u$ satisfies $\forall v \in H_0^1(U),\ B_{-\lambda}[\hat{u}, v] = 0 = \langle 0, v\rangle_{L^2(0)}$.
Yet since $\lambda \notin \Sigma$, by definition, we know there is a unique $\hat{u}$ that satisfies the above condition.
Clearly $\hat{u} = 0$ satisfies, so by the uniqueness of weak solution, $u = 0$.
This contradicts with $||u||_{L^2(U)} = 1$. □

3.3 Regularity

Theorem 3.18. *(Interior H^2 regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(U), b^i, c \in L^\infty(U), \forall i, j \in [n]$. $\forall V \subset\subset U, \exists C > 0$, such that for all $f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U, namely,

$$\forall v \in H_0^1(U), B[u, v] = \langle f, v\rangle_{L^2(U)},$$

then

$$||u||_{H^2(V)} \le C\Big(||f||_{L^2(U)} + ||u||_{L^2(U)}\Big).$$

Thus $u \in H^2_{loc}(U)$.

Proof. Let $V \subset\subset U$ be given.
The idea is to choose a particular v, then repeatedly bound all $\left|\left|D_k^h u\right|\right|$ from the product rule by $||Du||$. The only leftover term will be either $D_k^h(Du)$, or part of $\langle f, v\rangle_{L^2(U)}$ or $||u||_H^1(U)$. We thus achieve a bound on $\left|\left|D_k^h(Du)\right|\right|$, which allows us to say $u \in H^2_{loc}(U)$.

1. We now fix some $f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U. We have $\forall v \in H_0^1(U)$,

$$B[u,v] = \langle f, v\rangle_{L^2(U)}$$
$$\int_U \left(\sum_{i,j=1}^n a^{ij}\partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cuv\right) dx = \int_U fv dx$$
$$\int_U \left(\sum_{i,j=1}^n a^{ij}\partial_i u \partial_j v\right) dx = \int_U \tilde{f} v dx$$
$$\sum_{i,j=1}^n \int_U (a^{ij}\partial_i u \partial_j v) dx = \left\langle \tilde{f}, v\right\rangle_{L^2(U)},$$

   where $\tilde{f} := f - \sum_{i=1}^n b^i \partial_i u - cu \in L^2(U)$, since $f, \partial_i u, u \in L^2(U), b^i, c \in L^\infty(U)$.

2. Since $V \subset\subset U$, we can choose some $V \subset\subset W \subset\subset U$, and $\zeta \in C_c^\infty(U)$ such that $V < \zeta < W$.
Choose $|h| > 0$ such that $\operatorname{dist}(V, \partial U) > 8|h|, \operatorname{dist}(W, \partial U) > 6|h|$.
WLOG, we assume $h > 0$.
Fix some $k \in [n]$.
Let $Z := U_{2h} := \{x \in U : \operatorname{dist}(x, \partial U) > 2h\}$ be open.
Since U is bounded, we have that $Z \subset\subset U, \operatorname{dist}(Z, \partial U) = 2h > |h|$.
Let $v(x) := \begin{cases} -D_k^{-h}(\zeta^2 D_k^h u)(x) & x \in Z \\ 0 & x \in U \setminus Z \end{cases}$.

   Remark. For $x \in V$, we have that

$$\begin{aligned} v(x) &= -D_k^{-h}(D_k^h u)(x) \\ &= -D_k^{-h}\left(\frac{u(x+he_k) - u(x)}{h}\right) \\ &= -\frac{\frac{u(x+he_k-he_k)-u(x-he_k)}{h} - \frac{u(x+he_k)-u(x)}{h}}{h} \\ &= -\frac{2u(x) - u(x+he_k) - u(x-he_k)}{h^2} \\ &= \frac{u(x+he_k) - 2u(x) + u(x-he_k)}{h^2}, \end{aligned}$$

   which is an approximation to $\partial_k^2 u$ if u is smooth.

   Since $u \in H^1(U)$, we have $D_k^h u \in H^1(Z)$.
Since $\operatorname{Supp}(\zeta) \subset W \subset\subset Z$ is compact, we have $\zeta \in C_c^\infty(Z)$, so $\zeta^2 D_k^h u \in H^1(Z)$.
Since $U_{4h} \subset\subset Z, \operatorname{dist}(U_{4h}, \partial Z) = 2h > |h|$, we have that $v \in H^1(U_{4h})$.
In addition, $\operatorname{Supp}(v) \subset \operatorname{Supp}(\zeta^2 D_k^h u) + \bar{B}(0,h) \subseteq W + \bar{B}(0,h) \subseteq U_{6h} + \bar{B}(0,h) \subset U_{4h}$.
Since $v \in H^1(U_{4h})$ and $\operatorname{Supp}(v) \subset U_{4h}$, we must have $v \in H_0^1(U)$.

3. Now we have

$$
\begin{aligned}
\sum_{i,j=1}^{n}\int_U \big(a^{ij}\partial_i u\partial_j v\big)dx &= \sum_{i,j=1}^{n}\int_Z \big(a^{ij}\partial_i u\partial_j v\big)dx \\
&= sum_{i,j=1}^{n}\int_Z \big(a^{ij}\partial_i u\partial_j\big(-D_k^{-h}(\zeta^2 D_k^h u)\big)\big)dx \\
&= -\sum_{i,j=1}^{n}\int_Z \big(a^{ij}\partial_i u D_k^{-h}\big(\partial_j(\zeta^2 D_k^h u)\big)\big)dx \\
&= \sum_{i,j=1}^{n}\int_Z D_k^h(a^{ij}\partial_i u)\big(\partial_j(\zeta^2 D_k^h u)\big)dx \\
&= \sum_{i,j=1}^{n}\int_Z \big(D_k^h(a^{ij})\partial_i u + a^{ij}D_k^h(\partial_i u)\big)\big(\partial_j(\zeta^2)D_k^h u + \zeta^2\partial_j(D_k^h u)\big)dx \\
&= A_1 + A_2 + A_3 + A_4,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &:= \sum_{i,j=1}^{n}\int_Z a^{ij}D_k^h(\partial_i u)\zeta^2\partial_j(D_k^h u)dx, \\
A_2 &:= \sum_{i,j=1}^{n}\int_Z a^{ij}D_k^h(\partial_i u)\partial_j(\zeta^2)D_k^h u dx \\
&= \sum_{i,j=1}^{n}\int_Z a^{ij}D_k^h(\partial_i u)2\zeta(\partial_j\zeta)D_k^h u dx \\
A_3 &:= \sum_{i,j=1}^{n}\int_Z D_k^h(a^{ij})\partial_i u\zeta^2\partial_j(D_k^h u)dx \\
A_4 &:= \sum_{i,j=1}^{n}\int_Z D_k^h(a^{ij})\partial_i u\partial_j(\zeta^2)D_k^h u dx \\
&= \sum_{i,j=1}^{n}\int_Z D_k^h(a^{ij})\partial_i u 2\zeta(\partial_j\zeta)D_k^h u dx
\end{aligned}
$$

Now we will examine each term.

$$
\begin{aligned}
A_1 &= \sum_{i,j=1}^{n}\int_Z a^{ij}D_k^h(\partial_i u)\zeta^2\partial_j(D_k^h u)dx \\
&= \int_Z \zeta^2\sum_{i,j=1}^{n} a^{ij}\partial_i(D_k^h u)\partial_j(D_k^h u)dx \\
&\geq \int_Z \zeta^2\theta\big|\big|D(D_k^h u)\big|\big|_2^2 dx \\
&= \theta\int_Z \zeta^2\big|\big|D(D_k^h u)\big|\big|_2^2 dx.
\end{aligned}
$$

We also have

$$
\begin{aligned}
|A_2| &\leq \sum_{i,j=1}^{n} \int_Z \left|a^{ij} D_k^h(\partial_i u) 2\zeta(\partial_j \zeta) D_k^h u\right| dx \\
&\leq \sum_{i,j=1}^{n} \int_Z \left|\left|a^{ij}\right|\right|_{L^\infty(U)} ||\partial_j \zeta||_{L^\infty(U)} \left|D_k^h(\partial_i u) 2\zeta D_k^h u\right| dx \\
&= 2 \sum_{i,j=1}^{n} \left|\left|a^{ij}\right|\right|_{L^\infty(U)} ||\partial_j \zeta||_{L^\infty(U)} \int_Z \left|D_k^h(\partial_i u) \zeta D_k^h u\right| dx \\
&\leq 2 \sum_{i,j=1}^{n} \left|\left|a^{ij}\right|\right|_{L^\infty(U)} ||\partial_j \zeta||_{L^\infty(U)} \int_Z \epsilon \left|D_k^h(\partial_i u)\right|^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx \\
&= C_1 \int_Z \epsilon \left|D_k^h(\partial_i u)\right|^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx \\
&= C_1 \int_Z \epsilon \left|\partial_i(D_k^h u)\right|^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx \\
&\leq C_1 \int_Z \epsilon \left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx,
\end{aligned}
$$

since $a^{ij} \in L^\infty(U)$, and $\zeta \in C^c(U)$, we have $C_1 := 2\sum_{i,j=1}^{n} \left|\left|a^{ij}\right|\right|_{L^\infty(U)} ||\partial_j \zeta||_{L^\infty(U)} \in (0, \infty)$. Similarly,

$$
\begin{aligned}
|A_3| &\leq \sum_{i,j=1}^{n} \int_Z \left|D_k^h(a^{ij}) \partial_i u \zeta^2 \partial_j(D_k^h u)\right| dx \\
&\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z \left|\partial_i u \partial_j(D_k^h u)\right| \zeta^2 dx \\
&\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z ||Du||_2 \left|\left|D(D_k^h u)\right|\right|_2 \zeta^2 dx \\
&\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z ||Du||_2 \left|\left|D(D_k^h u)\right|\right|_2 \zeta dx \\
&\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z \frac{1}{4\epsilon} ||Du||_2^2 + \epsilon \zeta^2 \left|\left|D(D_k^h u)\right|\right|_2^2 dx \\
&= C_2 \int_Z \frac{1}{4\epsilon} ||Du||_2^2 + \epsilon \zeta^2 \left|\left|D(D_k^h u)\right|\right|_2^2 dx,
\end{aligned}
$$

where

$$
\begin{aligned}
C_2 &:= \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \\
&\leq \frac{1}{h} \sum_{i,j=1}^{n} \left(\left|\left|a^{ij}\right|\right|_{L^\infty(Z)} + \left|\left|a^{ij}\right|\right|_{L^\infty(Z+he_k)}\right) \\
&\leq \frac{1}{h} \sum_{i,j=1}^{n} \left(\left|\left|a^{ij}\right|\right|_{L^\infty(U)} + \left|\left|a^{ij}\right|\right|_{L^\infty(U)}\right) \\
&\in (0, \infty).
\end{aligned}
$$

Lastly,

$$\begin{aligned}|A_4| &\leq \sum_{i,j=1}^{n} \int_Z \left|D_k^h(a^{ij})\partial_i u 2\zeta(\partial_j\zeta)D_k^h u\right| dx \\ &\leq 2\sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} ||\partial_j\zeta||_{L^\infty(U)} \int_Z \left|\partial_i u D_k^h u\right| dx \\ &\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} ||\partial_j\zeta||_{L^\infty(U)} \int_Z |\partial_i u|^2 + \left|D_k^h u\right|^2 dx \\ &\leq C_3 \int_Z ||Du||_2^2 + \left|D_k^h u\right|^2 dx,\end{aligned}$$

where $C_3 := \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} ||\partial_j\zeta||_{L^\infty(U)} \in (0,\infty)y$ as argued before.
Now

$$\begin{aligned}&|A_2 + A_3 + A_4| \\ &\leq |A_1| + |A_2| + |A_3| \\ &\leq \int_Z \epsilon C_1 \left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 + \frac{C_1}{4\epsilon}\left|D_k^h u\right|^2 + \frac{C_2}{4\epsilon}||Du||_2^2 + C_2\epsilon\zeta^2\left|\left|D(D_k^h u)\right|\right|_2^2 + C_3||Du||_2^2 + C_3\left|D_k^h u\right|^2 dx \\ &= \int_Z (C_1+C_2)\epsilon\left|\left|D(D_k^h u)\right|\right|_2^2\zeta^2 + \left(\frac{C_1}{4\epsilon}+C_3\right)\left|D_k^h u\right|^2 + \left(\frac{C_2}{4\epsilon}+C_3\right)||Du||_2^2 dx \\ &\leq \int_Z (C_1+C_2)\epsilon\left|\left|D(D_k^h u)\right|\right|_2^2\zeta^2 + \left(\frac{C_1}{4\epsilon}+C_3\right)\left|\left|D^h u\right|\right|_2^2 + \left(\frac{C_2}{4\epsilon}+C_3\right)||Du||_2^2 dx \\ &= (C_1+C_2)\epsilon\int_Z \left|\left|D(D_k^h u)\right|\right|_2^2\zeta^2 dx + \left(\frac{C_1}{4\epsilon}+C_3\right)\left|\left|D^h u\right|\right|_{L^2(U)}^2 + \left(\frac{C_2}{4\epsilon}+C_3\right)||Du||_{L^2(U)}^2.\end{aligned}$$

We know there $\exists C_4 > 0$, such that

$$\left|\left|D^h u\right|\right|_{L^2(Z)} \leq C_4||Du||_{L^2(U)}, \forall |h| \in (0, \operatorname{dist}(Z,\partial U)), \forall u \in H_0^1(U).$$

Thus

$$|A_2 + A_3 + A_4| \leq (C_1+C_2)\epsilon\int_Z \left|\left|D(D_k^h u)\right|\right|_2^2\zeta^2 dx + \left(\frac{C_2}{4\epsilon}+C_3+\left(\frac{C_1}{4\epsilon}+C_3\right)C_4^2\right)||Du||_{L^2(U)}^2.$$

Taking $\epsilon := \frac{\theta}{2(C_1+C_2)}, C_5(\epsilon) := \frac{C_2}{4\epsilon} + C_3 + \left(\frac{C_1}{4\epsilon}+C_3\right)C_4^2 \in (0,\infty)$, we have

$$\begin{aligned}\sum_{i,j=1}^{n}\int_U \left(a^{ij}\partial_i u \partial_j v\right) dx &= A_1 + A_2 + A_3 + A_4 \\ &\geq A_1 - |A_2 + A_3 + A_4| \\ &\geq \theta\int_Z \zeta^2\left|\left|D(D_k^h u)\right|\right|_2^2 dx - \frac{\theta}{2}\int_Z \left|\left|D(D_k^h u)\right|\right|_2^2\zeta^2 dx - C_5||Du||_{L^2(U)}^2 \\ &= \frac{\theta}{2}\int_Z \left|\left|D(D_k^h u)\right|\right|_2^2\zeta^2 dx - C_5||Du||_{L^2(U)}^2.\end{aligned}$$

4. On the other hand,

$$\begin{aligned}
\left|\left\langle \tilde{f}, v\right\rangle_{L^2(U)}\right| &= \int_U \left|f - \sum_{i=1}^n b^i \partial_i u - cu\right| |v| dx \\
&= \int_U \left(|f| + \sum_{i=1}^n \left|b^i \partial_i u\right| + |cu|\right)|v| dx \\
&\le \int_U \left(|f| + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} |\partial_i u| + ||c||_{L^\infty(U)} |u|\right)|v| dx \\
&= \int_U |f||v| dx + \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)} \int_U |\partial_i u||v| dx + ||c||_{L^\infty(U)} \int_U |u||v| dx \\
&\le C_6 \left(\int_U |f||v| dx + \int_U |\partial_i u||v| dx + \int_U |u||v| dx\right) \\
&\le C_6 \left(\int_U \frac{1}{4\epsilon}|f|^2 + \epsilon |v|^2 dx + \int_U \frac{1}{4\epsilon}|\partial_i u|^2 + \epsilon |v|^2 dx + \int_U \frac{1}{4\epsilon}|u|^2 + \epsilon |v|^2 dx\right) \\
&\le C_6 \int_U \frac{1}{4\epsilon}\left(|f|^2 + |\partial_i u|^2 + |u|^2\right) + 3\epsilon |v|^2 dx \\
&\le \frac{C_6}{4\epsilon} \int_U |f|^2 + ||Du||_2^2 + |u|^2 dx + 3C_6 \epsilon \int_U |v|^2 dx,
\end{aligned}$$

where $C_6 := \max\left(1, \sum_{i=1}^n \left|\left|b^i\right|\right|_{L^\infty(U)}, ||c||_{L^\infty(U)}\right) \in (0, \infty)$.
We have shown in step 2 that $\zeta^2 D_k^h u \in H^1(Z), \mathrm{Supp}(\zeta^2 D_k^h u) \subset Z \subset\subset U$, thus $\zeta^2 D_k^h u \in H_0^1(U)$.

$$\begin{aligned}
\int_U |v|^2 dx &= \int_Z |v|^2 dx \\
&= \int_Z \left|-D_k^{-h}(\zeta^2 D_k^h u)\right|^2 dx \\
&\le \int_Z \left|D^{-h}(\zeta^2 D_k^h u)\right|^2 dx \\
&\le C_4^2 \int_U \left|D(\zeta^2 D_k^h u)\right|^2 dx \\
&= C_4^2 \int_W \left|D(\zeta^2 D_k^h u)\right|^2 dx \\
&= C_4^2 \int_W \left|D(\zeta^2) D_k^h u + D(D_k^h u)\zeta^2\right|^2 dx \\
&\le 2C_4^2 \int_W D(\zeta^2)^2 \left|D_k^h u\right|^2 + \left|D(D_k^h u)\right|^2 \zeta^4 dx \\
&\le 2C_4^2 \int_W \left|\left|D(\zeta^2)^2\right|\right|_{L^\infty(U)} \left|D_k^h u\right|^2 + \left|D(D_k^h u)\right|^2 \zeta^2 dx \\
&\le 2C_4^2 \left|\left|D(\zeta^2)^2\right|\right|_{L^\infty(U)} \int_W \left|D_k^h u\right|^2 dx + 2C_4^2 \int_W \left|D(D_k^h u)\right|^2 \zeta^2 dx \\
&\le 2C_4^4 \left|\left|D(\zeta^2)^2\right|\right|_{L^\infty(U)} \int_U ||Du||_2^2 dx + 2C_4^2 \int_U \left|D(D_k^h u)\right|^2 \zeta^2 dx \\
&\le C_7 \int_U ||Du||_2^2 + \left|D(D_k^h u)\right|^2 \zeta^2 dx,
\end{aligned}$$

where $C_7 := 2C_4^2 \max\left(C_4^2 \big|\big|D(\zeta^2)^2\big|\big|_{L^\infty(U)}, 1\right) \in (0, \infty)$. Thus we have

$$\begin{aligned}
\left|\left\langle \tilde{f}, v\right\rangle_{L^2(U)}\right| &\leq \frac{C_6}{4\epsilon}\int_U |f|^2 + ||Du||_2^2 + |u|^2 dx + 3C_6\epsilon\int_U |v|^2 dx \\
&\leq \frac{C_6}{4\epsilon}\int_U |f|^2 + ||Du||_2^2 + |u|^2 dx + 3C_6C_7\epsilon\int_U ||Du||_2^2 + \left|D(D_k^h u)\right|^2\zeta^2 dx \\
&\leq \left(\frac{C_6}{4\epsilon} + 3C_6C_7\epsilon\right)\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) + 3C_6C_7\epsilon\int_U \left|D(D_k^h u)\right|^2\zeta^2 dx.
\end{aligned}$$

5. Taking $\epsilon := \frac{\theta}{12C_6C_7} > 0, C_8 := \frac{C_6}{4\epsilon} + 3C_6C_7\epsilon > 0$, we have

$$\begin{aligned}
\sum_{i,j=1}^n \int_U (a^{ij}\partial_i u \partial_j v)dx &= \left\langle \tilde{f}, v\right\rangle_{L^2(U)} \\
&\leq \left|\left\langle \tilde{f}, v\right\rangle_{L^2(U)}\right| \\
&\leq C_8\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) + \frac{\theta}{4}\int_U \left|D(D_k^h u)\right|^2\zeta^2 dx \\
&= C_8\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) + \frac{\theta}{4}\int_Z \left|D(D_k^h u)\right|^2\zeta^2 dx \\
\sum_{i,j=1}^n \int_U (a^{ij}\partial_i u \partial_j v)dx &\geq \frac{\theta}{2}\int_Z \big|\big|D(D_k^h u)\big|\big|_2^2\zeta^2 dx - C_5||Du||_{L^2(U)}^2 \\
\frac{\theta}{4}\int_Z \big|\big|D(D_k^h u)\big|\big|_2^2\zeta^2 dx &\leq (C_5 + C_8)\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) \\
\frac{\theta}{4}\int_V \big|\big|D(D_k^h u)\big|\big|_2^2 dx &\leq (C_5 + C_8)\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) \\
\int_V \big|\big|D_k^h(Du)\big|\big|_2^2 dx &\leq C_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right),
\end{aligned}$$

where $C_9 := \frac{4(C_5+C_8)}{\theta} \in (0, \infty)$.
Notice that for all $j \in [n]$, we have $\partial_j u \in L^2(U)$, and

$$\int_V \big|\big|D_k^h(\partial_j u)\big|\big|_2^2 dx \leq \int_V \big|\big|D_k^h(Du)\big|\big|_2^2 dx \leq C_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right),$$

and this holds for all $k \in [n]$. Thus,

$$\begin{aligned}
\big|\big|D^h(\partial_j u)\big|\big|_{L^2(V)}^2 &= \int_V \big|\big|D^h(\partial_j u)\big|\big|_2^2 dx \\
&= \int_V \sum_{k=1}^n \big|\big|D_k^h(\partial_j u)\big|\big|_2^2 dx \\
&= \sum_{k=1}^n \int_V \big|\big|D_k^h(\partial_j u)\big|\big|_2^2 dx \\
&\leq \sum_{k=1}^n C_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right) \\
&= nC_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right). \\
\big|\big|D^h(\partial_j u)\big|\big|_{L^2(V)} &\leq \sqrt{nC_9}\left(||f||_{L^2(U)} + ||u||_{H^1(U)}\right) \\
&< \infty.
\end{aligned}$$

Since this holds for all $|h| > 0$ such that $\operatorname{dist}(V, \partial U) > 8|h|, \operatorname{dist}(W, \partial U) > 6|h|$, we have $\partial_j u \in H^1(U)$, with

$$||D(\partial_j u)||_{L^2(V)} \leq \sqrt{nC_9}\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big).$$

Since this holds for all $j \in [n]$, we have $u \in H^2(V)$, and

$$\begin{aligned}
||D^2 u||^2_{L^2(V)} &= \int_V ||D^2 u||_2^2 dx \\
&= \int_V \sum_{j=1}^n ||\partial_j (Du)||_2^2 dx \\
&= \sum_{j=1}^n \int_V ||D(\partial_j u)||_2^2 dx \\
&\leq \sum_{j=1}^n nC_9 \Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2 \\
&= n^2 C_9 \Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2 \\
&\Longrightarrow \\
||u||^2_{H^2(V)} &= ||D^2 u||^2_{L^2(V)} + ||u||^2_{H^1(V)} \\
&\leq n^2 C_9 \Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2 + ||u||^2_{H^1(V)} \\
&\leq (n^2 C_9 + 1)\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2.
\end{aligned}$$

Thus we have found $C := \sqrt{n^2 C_9 + 1} \in (0, \infty)$, such that $||u||_{H^2(V)} \leq C\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)$. Since V is arbitrary, we have that $u \in H^2_{loc}(U)$.

6. Notice that the above estimate holds as long as $V \subset\subset U$ and $u \in H^1(U)$. Since $u \in H^1(W)$, we can find some constant C', such that $||u||^2_{H^2(V)} \leq C'\Big(||f||_{L^2(W)} + ||u||_{H^1(W)}\Big)$.
   Now consider $v := \xi^2 u \in H^1_0(U)$, we can find $||Du||_{L^2(W)} \leq C''||u||_{L^2(U)}$ for some $C'' > 0$.
   Plugging in will give us
   $$||u||^2_{H^2(V)} \leq C\Big(||f||_{L^2(U)} + ||u||_{L^2(U)}\Big).$$

□

Definition 3.11. If $Lu(x) = f(x)$ a.e.$x \in U$, we say u is a **strong solution** to the problem $Lu = f$ in U.

Corollary 3.19. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(U), b^i, c \in L^\infty(U), \forall i, j \in [n]$. If $f \in L^2(U)$, and $u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then u is a strong solution.*

Proof. We have that $u \in H^2_{loc}(U)$.
Consider any $V \subset\subset U$, since $a^{ij} \in C^1$, we have $a^{ij} u \in H^2(V)$.

Consider any $v \in C_c^\infty(V)$, we must have

$$\begin{aligned}
\langle f, v \rangle_{L^2(V)} &= B[u,v] \\
&= \int_V \left(\sum_{i,j=1}^n a^{ij}\partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cuv \right) dx \\
&= \int_V \left(\sum_{i,j=1}^n a^{ij}\partial_j(\partial_i u) v + \sum_{i=1}^n b^i \partial_i u v + cuv \right) dx \\
&= \int_V \left(\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u) + \sum_{i=1}^n b^i \partial_i u + cu \right) v dx \\
&= \int_V (Lu) v dx \\
&= \langle Lu, v \rangle_{L^2(V)}.
\end{aligned}$$

Since this holds for all $v \in C_c^\infty(V)$, we must have $Lu(x) = f(x)$a.e. $x \in V$.
Since this hold for all $V \subset\subset U$, we have that $Lu(x) = f(x)$a.e. $x \in U$. $\square$

Theorem 3.20. *(Higher Interior regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(U), \forall i,j \in [n]$ for some $m \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in H_{loc}^{m+2}(U)$. In addition, $\forall V \subset\subset U, \exists C > 0$, such that $\forall f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U, we have

$$||u||_{H^{m+2}(U)} \leq C\Big(||f||_{H^m(U)} + ||u||_{L^2(U)}\Big).$$

Corollary 3.21. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(U), \forall i,j \in [n]$ for some $m > \frac{n}{2} - 2 \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in C^l(U)$, where $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$.*

Theorem 3.22. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^\infty(U), \forall i,j \in [n]$. If $f \in C^\infty(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in C^\infty(U)$.*

Theorem 3.23. *(Boundary H^2 regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^2, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U), \forall i,j \in [n]$. Then $\exists C > 0$, such that $\forall f \in L^2(U)$ and $u \in H_0^1(U)$ being a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$, we have

$$||u||_{H^2(U)} \leq C\Big(||f||_{L^2(U)} + ||u||_{L^2(U)}\Big),$$

and thus $u \in H^2(U)$.

Proof. 1. First prove the case if the boundary is locally flat:

$$U = B(0,1) \cap \{x : x^n > 0\}, V = B(0, \frac{1}{2}) \cap \{x : x^n > 0\}.$$

Similar to the proof of Interior H^2 regularity, we first use difference quotients to obtain a bound for derivatives that are not normal to the flat boundary:

$$\sum_{k,l=1, k+l<2n}^n ||\partial_k \partial_l u||_{L^2(V)} \leq C\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big),$$

where we can transform $||u||_{H^1(U)}$ to $||u||_{L^2(U)}$.
For the derivative that is normal to the flat boundary $\partial_n\partial_n$, we write the PDE in non divergence form, and use ellipticity to note that $a^{nn} > \theta > 0$ to find:

$$|\partial_n\partial_n| \leq C\left(\sum_{k,l=1,k+l<2n}^{n} |\partial_k\partial_l u| + ||Du||_2 + |u| + |f|\right) \text{a.e.} x \in U.$$

Thus

$$||\partial_n\partial_n||_{L^2(U)} \leq C\left(\sum_{k,l=1,k+l<2n}^{n} ||\partial_k\partial_l u||_{L^2(U)} + ||Du||_{L^2(U)} + ||u||_{L^2(U)} + ||f||_{L^2(U)}\right).$$

This leads to

$$||u||_{H^2(V)} \leq C\Big(||f||_{L^2(U)} + ||u||_{L^2(U)}\Big)$$

2. Take any $x_0 \in \partial U$, let $y = \Phi(x)$ be a C^2 straightening map on $B(x_0, r)$ with a C^2 inverse $x = \Psi(y)$. Pick some small enough s, such that

$$U' = B(0,s) \cap \{y : y^n > 0\} \subseteq \Phi(U \cap B(x_0, r)), V' = B(0, \frac{1}{2}s) \cap \{y : y^n > 0\}.$$

We check the weak formulation is well-defined on U' and that L' satisfies the assumptions of L. Apply step 1 to get

$$||u'||_{H^2(V')} \leq C\Big(||f'||_{L^2(U')} + ||u'||_{L^2(U')}\Big).$$

Transform back using Ψ.

3. Use compactness to find $V_1, \ldots, V_N$ to cover ∂U. Find $V_0 \subset\subset U$ such that $U = \bigcup_{i=0}^{N} V_i$.
Use interior result on V_0.
Combine them together.

□

Remark. When the solution is unique, we can throw away the $||u||_{L^2(U)}$ by boundedness of inverse in the last section.

Theorem 3.24. *(Higher boundary regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^{m+2}, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \forall i, j \in [n]$. Then $\exists C > 0$, such that $\forall f \in H^m(U)$ and $u \in H_0^1(U)$ being a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$, we have

$$||u||_{H^{m+2}(U)} \leq C\Big(||f||_{H^m(U)} + ||u||_{L^2(U)}\Big),$$

and thus $u \in H^{m+2}(U)$.

Corollary 3.25. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^{m+2}, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \forall i, j \in [n]$ for some $m > \frac{n}{2} - 2 \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in C^l(U)$, where $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$.*

Theorem 3.26. *(Infinite differentiability up to the boundary)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^∞, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^\infty(\bar{U}), \forall i, j \in [n]$. Then $\forall f \in H^\infty(U)$ and $u \in H_0^1(U)$ being a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$, we have $u \in C^\infty(\bar{U})$.

4 Parabolic PDEs

4.1 Spaces Involving Time

4.1.1 Bochner Spaces

See more about Bochner Spaces in my Measure Theory Notes.

Definition 4.1. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, a function $u : [0,T] \to X$ is **continuous** at a point $t \in (0,T)$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall s,t \in [0,T], |s-t| < \delta \implies ||u(s) - u(t)|| < \epsilon.$$

A function u is continuous if it is continuous at all $t \in (0,1)$.
$||u||_{C([0,T];X)} := \sup_{t\in(0,T)} ||u(t)||$.

Theorem 4.1. *$(C([0,T];X), ||u||_{C([0,T];X)})$ is a Banach Space.*

See the definition of Bochner integrable functions in notes of Measure Theory. We will still consider the Lebesgue measure on $[0,T]$.

Theorem 4.2 (Bochner)**.** *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, a strongly measurable function $f : [0,T] \to X$ is Bochner integrable if and only if $t \mapsto ||f(t)||_X$ is integrable. In this case,*

$$\left|\left|\int_0^T f(t)dt\right|\right|_X \leq \int_0^T ||f(t)||_X dt,$$
$$\forall u^* \in X^*, \left\langle u^* \middle| \int_0^T f(t)dt \right\rangle = \int_0^T \langle u^* | f(t) \rangle dt.$$

Theorem 4.3. *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, then Dominated Convergence Theorem, Holder's Inequality, and Minkowski's Inequality still work with the Bochner integral.*

Theorem 4.4. *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, then for any Bochner integrable $f : [0,T] \to X$, we have $\int_s^t f(\tau)d\tau$ is continuous in both $s,t \in [0,T]$.*

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

Definition 4.2. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, and $1 \leq p < \infty$, we define

$$\mathcal{L}^p([0,T];X) := \left\{ f : [0,T] \to X \middle| f \text{ is measurable}, \int_X ||f||_X^p d\mu < \infty \right\}.$$

In addition, we define

$$||f||_{\mathcal{L}^p([0,T];X)} := \left(\int_X ||f||_X^p d\mu \right)^{\frac{1}{p}}.$$

Definition 4.3. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, $(B, ||\cdot||)$ be a Banach Space, we define

$$\mathcal{L}^\infty([0,T];X) := \{ f : X \to B | f \text{ is measurable}, \operatorname{ess\,sup} ||f||_X < \infty \}.$$

In addition, we define

$$||f||_{\mathcal{L}^\infty([0,T];X)} := \operatorname{ess\,sup} ||f||_B.$$

Definition 4.4. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space. For any $p \in [1,\infty]$, we define

$$L^p([0,T];X) := \mathcal{L}^p([0,T];X)/N,$$

where $N := \{f : X \to B | f \text{ is measurable}, f = 0 \ \mu-\text{a.e.}\}$. Namely, $[f] \in L^p([0,T];X)$ is the equivalence class of all $g = f$ μ-a.e. for $f \in \mathcal{L}^p([0,T];X)$.
In addition, we define

$$||[f]||_{L^p([0,T];X)} := ||f||_{\mathcal{L}^\infty([0,T];X)}$$

for any representative f.

Theorem 4.5 (Fischer-Riesz-Bochner)**.** *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space. For all $1 \leq p \leq \infty$, we have that $\left(L^p([0,T]; X), ||\cdot||_{L^p([0,T];X)}\right)$ is a Banach Space.*

Similarly, we can also define $L^p_{loc}(0,T;X), W^{k,p}(0,T;X), H^k(0,T;X)$ and weak derivatives where the test functions are $\phi \in C_c^\infty(0,T;\mathbb{R})$.
We can similarly define the mollification of $f \in L^1_{loc}(0,T;X)$ to be

$$f^\epsilon := \eta_\epsilon * f : (\epsilon, T-\epsilon) \to X;\ t \mapsto \int_{t-\epsilon}^{t+\epsilon} \eta_\epsilon(t-\tau) f(\tau) d\tau.$$

Similarly, we have

Theorem 4.6. *Let f^ϵ be defined as above, we have:*

1. *$f^\epsilon \in C^\infty((\epsilon, T-\epsilon); X)$,*
2. *$\partial_t^k(f^\epsilon) = (\partial_t^k \eta_\epsilon) * f$ on $(\epsilon, T-\epsilon)$,*
3. *$f^\epsilon \to f$ a.e. $t \in (0,T)$, as $\epsilon \to 0$,*
4. *If $f \in C(0,T;X)$, we have $f^\epsilon \to f$ uniformly on compact subsets of U,*
5. *If $1 \leq p < \infty, f \in L^p_{loc}(0,T;X)$, we have $f^\epsilon \to f$ in $L^p_{loc}(0,T;X)$. Namely, $f^\epsilon \to f$ in $L^p(V)$, $\forall V \subset\subset (0,T)$.*

Theorem 4.7. *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, $p \in [1, \infty]$, and $u \in W^{1,p}(0,T;X)$, then*

1. *$u(t) = u(s) + \int_s^t u'(\tau) d\tau$ for a.e. $0 \leq s \leq t \leq T$.*
2. *There is a representative $\tilde{u} \in C([0,T], X)$ of u. In particular, $\tilde{u}(t) = \tilde{u}(s) + \int_s^t u'(\tau) d\tau$ for any $0 \leq s \leq t \leq T$.*
3. *$\exists C > 0$ such that $\forall u \in W^{1,p}(0,T;X), \sup_{t\in[0,T]} ||u(t)||_X \leq C||u||_{W^{1,p}(0,T;X)}$.*

Proof. We will prove for $p \in [1, \infty)$.

1. Let $u^\epsilon := \eta_\epsilon * u$, we have that $u^\epsilon \in C^\infty((\epsilon, T-\epsilon); X)$, and $\partial_t(u^\epsilon) = (\partial_t \eta_\epsilon) * u$ on $(\epsilon, T-\epsilon)$.
   We also have $f^\epsilon(t) \to f(t)$ a.e. $t \in (0,T)$.
   Similar to 2.17, we can show that $\partial_t(u^\epsilon) = \eta_\epsilon * \partial_t u = (\partial_t u)^\epsilon$ on $(\epsilon, T-\epsilon)$.
   Since $u \in W^{1,p}(0,T;X)$, we know that $\partial_t u \in L^p_{loc}(0,T;X)$, so $(\partial_t u)^\epsilon \to \partial_t u$ in $L^p_{loc}(0,T;X)$.
   Since $|(0,T)| = T < \infty$, we have that $\partial_t(u^\epsilon) \to \partial_t u$ in $L^1_{loc}(0,T;X)$, which means

   $$\forall [s,t] \subset (0,T),\ \lim_{\epsilon\to 0} \int_s^t ||(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)||_X d\tau = 0.$$

   We have that $\left|\left|\int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau\right|\right|_X \leq \int_s^t ||(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)||_X d\tau$ for any fixed $[s,t] \subset (0,T)$ and $\epsilon < \min(s, T-t)$. Thus

   $$\lim_{\epsilon\to 0} \left|\left|\int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau\right|\right|_X = 0$$

   for any $[s,t] \subset (0,T)$.
   Now $u^\epsilon(t) = u^\epsilon(s) + \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau$ for any $[s,t] \subset (\epsilon, T-\epsilon)$ by FTC, since $u^\epsilon \in C^\infty((\epsilon, T-\epsilon); X)$.

We have

$$\begin{aligned}
&\left|\left|-u(t)+u(s)+\int_s^t(\partial_t u)(\tau)d\tau\right|\right|_X \\
=&\left|\left|u^\epsilon(t)-u(t)-u^\epsilon(s)+u(s)-\int_s^t(\partial_t(u^\epsilon))(\tau)d\tau+\int_s^t(\partial_t u)(\tau)d\tau\right|\right|_X \\
\leq&||u^\epsilon(t)-u(t)||_X+||u^\epsilon(s)-u(s)||_X+\left|\left|\int_s^t(\partial_t(u^\epsilon))(\tau)d\tau-\int_s^t(\partial_t u)(\tau)d\tau\right|\right|_X \\
\leq&||u^\epsilon(t)-u(t)||_X+||u^\epsilon(s)-u(s)||_X+\left|\left|\int_s^t(\partial_t(u^\epsilon))(\tau)-(\partial_t u)(\tau)d\tau\right|\right|_X
\end{aligned}$$

for any s, t, ϵ such that $[s,t]\subset(\epsilon, T-\epsilon)$.
Since each term goes to 0 as $\epsilon\to 0$ for a.e. $0\leq s\leq t\leq T$, we must have

$$\left|\left|-u(t)+u(s)+\int_s^t(\partial_t u)(\tau)d\tau\right|\right|_X=0$$

for a.e. $0\leq s\leq t\leq T$.
We thus have

$$u(t)=u(s)+\int_s^t(\partial_t u)(\tau)d\tau$$

for a.e. $0\leq s\leq t\leq T$.

2. Fix any representative for u.
Notice that the set N where the above property does not hold has measure 0.
Now fix some point $s\in[0,T]\setminus N$, we define

$$\tilde{u}(t):=\begin{cases}u(s)-\int_t^s u'(\tau)d\tau & t<s\\ u(s)+\int_s^t u'(\tau)d\tau & t\geq s\end{cases}.$$

For any $t\in[0,T]\setminus N$, we have that

$$u(t):=u(s)+\int_s^t u'(\tau)d\tau=\tilde{u}(t)$$

if $t\geq s$, and

$$u(s)=u(t)+\int_t^s u'(\tau)d\tau\implies u(t)=u(s)-\int_t^s u'(\tau)d\tau=u(s)$$

if $t<s$.
Thus $\tilde{u}=u$ a.e. $t\in[0,T]$, which means $\tilde{u}$ is a representative of u.
In addition, $\tilde{u}(t)$ is continuous since $\int_t^s u'(\tau)d\tau$ and $\int_s^t u'(\tau)d\tau$ are both continuous in t, and

$$\lim_{t\to s^-}\tilde{u}(t)=\lim_{t\to s^-}\left(\tilde{u}(t)u(s)-\int_t^s u'(\tau)d\tau\right)=u(s)=u(s)+\int_s^s u'(\tau)d\tau=\tilde{u}(s).$$

3. See A5Q2.

□

Proposition 4.8. *Suppose $\mathcal{H}$ is a Hilbert Space, and $u, v\in C^1(0,T;\mathcal{H})$, then we have*

$$\forall t\in[0,T],\ \frac{d}{dt}\langle u(t),v(t)\rangle_{\mathcal{H}}=\langle u'(t),v(t)\rangle_{\mathcal{H}}+\langle v'(t),u(t)\rangle_{\mathcal{H}},$$

where $u'(t):=\lim_{h\to 0}\frac{u(t+h)-u(t)}{t}$ is the normal derivative in X.

Proof. We have

$$\begin{aligned}
\frac{d}{dt}\langle u(t), v(t)\rangle_{\mathcal{H}} &= \lim_{h\to 0}\frac{\langle u(t+h), v(t+h)\rangle_{\mathcal{H}} - \langle u(t), v(t)\rangle_{\mathcal{H}}}{h} \\
&= \lim_{h\to 0}\frac{\langle u(t+h), v(t+h)\rangle_{\mathcal{H}} - \langle u(t+h), v(t)\rangle_{\mathcal{H}} + \langle u(t+h), v(t)\rangle_{\mathcal{H}} - \langle u(t), v(t)\rangle_{\mathcal{H}}}{h} \\
&= \lim_{h\to 0}\frac{\langle u(t+h), v(t+h)\rangle_{\mathcal{H}} - \langle u(t+h), v(t)\rangle_{\mathcal{H}}}{h} + \lim_{h\to 0}\frac{\langle u(t+h), v(t)\rangle_{\mathcal{H}} - \langle u(t), v(t)\rangle_{\mathcal{H}}}{h} \\
&= \lim_{h\to 0}\frac{\langle u(t+h), v(t+h) - v(t)\rangle_{\mathcal{H}}}{h} + \lim_{h\to 0}\frac{\langle u(t+h) - u(t), v(t)\rangle_{\mathcal{H}}}{h} \\
&= \left\langle \lim_{h\to 0} u(t+h), \lim_{h\to 0}\frac{v(t+h)-v(t)}{h}\right\rangle_{\mathcal{H}} + \left\langle \lim_{h\to 0}\frac{u(t+h)-u(t)}{h}, v(t)\right\rangle_{\mathcal{H}} \\
&= \langle u(t), v'(t)\rangle_{\mathcal{H}} + \langle u'(t), v(t)\rangle_{\mathcal{H}}.
\end{aligned}$$

□

4.1.2 Sobolev Spaces In Time

Now we consider the cases where X might be any of the tuple $H_0^1(U) \subset L^2(U) \cong L^2(U)^* \subset H^{-1}(U)$, and see what is the relationship between the time weak derivatives in each space.

Lemma 4.9. *Suppose $\boldsymbol{u}, \boldsymbol{u}' \in L^1\big(0, T; H_0^1(U)\big)$, then we must have $\boldsymbol{u}'$ is also the time weak derivative of $\boldsymbol{u}$ in $L^1\big(0, T; L^2(U)\big)$.*

Proof. Firstly, for each $t \in [0, T]$, we have $\mathbf{u}(t), \mathbf{u}'(t) \in H_0^1(U) \subset L^2(U)$, so $\mathbf{u}, \mathbf{u}'$ are indeed functions $[0, T] \to L^2(U)$.
In addition,

$$\begin{aligned}
\int_0^T ||\mathbf{u}(t)||_{L^2(U)} dt &\le \int_0^T ||\mathbf{u}(t)||_{H^2(U)} dt \\
&= ||\mathbf{u}||_{L^1\left(0,T;H_0^1(U)\right)} \\
&< \infty.
\end{aligned}$$

Similarly for $\int_0^T ||\mathbf{u}'(t)||_{L^2(U)} dt < \infty$.
Thus $\mathbf{u}, \mathbf{u}' \in L^1\big(0, T; L^2(U)\big)$.
Now $\forall \phi \in C_c^\infty(0, T)$, we have

$$\begin{aligned}
0 &\le \left|\left|\int_0^T \phi'(t)\mathbf{u}(t)dt + \int_0^T \phi(t)\mathbf{u}'(t)dt\right|\right|_{L^2(U)} \\
&\le \left|\left|\int_0^T \phi'(t)\mathbf{u}(t)dt + \int_0^T \phi(t)\mathbf{u}'(t)dt\right|\right|_{H_0^1(U)} \\
&= 0.
\end{aligned}$$

Thus $\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \phi(t)\mathbf{u}'(t)dt$ in $L^2(U)$ for any $\phi \in C_c^\infty(0, T)$, which shows that $\mathbf{u}'$ is also the time weak derivative of $\mathbf{u}$ in $L^1\big(0, T; L^2(U)\big)$. □

Lemma 4.10. *Let $\boldsymbol{u} \in L^1\big(0, T; H_0^1(U)\big), \boldsymbol{v} \in L^1\big(0, T; H^{-1}(U)\big)$, we have $\boldsymbol{v} = (\boldsymbol{u}^*)' \iff$*

$$\forall \phi \in C_c^\infty(0, T), w \in H_0^1(U), \ \int_0^T \phi'(t)\langle \boldsymbol{u}(t), w\rangle_{L^2(U)} dt = -\int_0^T \phi(t)\langle \boldsymbol{v}(t)|w\rangle_{H^{-1}(U), H_0^1(U)} dt,$$

where $\boldsymbol{u}^*(t) := \langle \boldsymbol{u}(t), \cdot\rangle_{L^2(U)} \in H^{-1}(U)$ as usual.

Proof. $(\mathbf{u}^*)' = \mathbf{v}$ by definition means

$$\forall \phi \in C_c^\infty(0,T),\ \int_0^T \phi'(t)\mathbf{u}^*(t)dt = -\int_0^T \phi(t)\mathbf{v}(t)dt$$

in $H^{-1}(U)$.
Consider any $w \in H_0^1(U)$, we have $\langle \cdot | w \rangle_{H^{-1}(U),H_0^1(U)} \in \big(H^{-1}(U)\big)^*$.
By Bochner's Theorem 4.2 and linearity of duality pairing, we have

$$\begin{aligned}
\left\langle \int_0^T \phi'(t)\mathbf{u}^*(t)dt \middle| w \right\rangle_{H^{-1}(U),H_0^1(U)} &= \left\langle -\int_0^T \phi(t)\mathbf{v}(t)dt \middle| w \right\rangle_{H^{-1}(U),H_0^1(U)} \\
\int_0^T \langle \phi'(t)\mathbf{u}^*(t)|w\rangle_{H^{-1}(U),H_0^1(U)} dt &= -\int_0^T \langle \phi(t)\mathbf{v}(t)|w\rangle_{H^{-1}(U),H_0^1(U)} dt \\
\int_0^T \phi'(t)\langle \mathbf{u}^*(t)|w\rangle_{H^{-1}(U),H_0^1(U)} dt &= -\int_0^T \phi(t)\langle \mathbf{v}(t)|w\rangle_{H^{-1}(U),H_0^1(U)} dt \\
\int_0^T \phi'(t)\langle \mathbf{u}(t), w\rangle_{L^2(U)} dt &= -\int_0^T \phi(t)\langle \mathbf{v}(t)|w\rangle_{H^{-1}(U),H_0^1(U)} dt.
\end{aligned}$$

$\square$

Lemma 4.11. *Suppose* $\boldsymbol{u} \in L^1\big(0,T;H_0^1(U)\big)$, *and* $\boldsymbol{u}'$ *is its time weak derivative in* $L^1\big(0,T;L^2(U)\big)$, *then we must have the action function*

$$(\boldsymbol{u}')^* := t \mapsto \langle \boldsymbol{u}(t), \cdot \rangle_{L^2(U)}$$

is the time weak derivative of $\boldsymbol{u}^* := t \mapsto \langle \boldsymbol{u}(t), \cdot \rangle_{L^2(U)}$ *in* $L^1\big(0,T;H^{-1}(U)\big)$. *Namely,*

$$(\boldsymbol{u}')^* = (\boldsymbol{u}^*)'.$$

Proof. Consider any $\phi \in C_c^\infty(0,T)$, by definition of weak derivative, we have

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \phi(t)\mathbf{u}'(t)dt.$$

in $L^2(U)$.
Now for any $w \in H_0^1(U) \subset L^2(U)$, we have $\langle \cdot, w \rangle_{L^2(U)} \in \big(L^2(U)\big)^*$.
By Bochner's Theorem 4.2 and linearity of the inner product, we have

$$\begin{aligned}
\left\langle \int_0^T \phi'(t)\mathbf{u}(t)dt, w \right\rangle_{L^2(U)} &= \left\langle -\int_0^T \phi(t)\mathbf{u}'(t)dt, w \right\rangle_{L^2(U)} \\
\int_0^T \langle \phi'(t)\mathbf{u}(t), w\rangle_{L^2(U)} dt &= -\int_0^T \langle \phi(t)\mathbf{u}'(t), w\rangle_{L^2(U)} \\
\int_0^T \phi'(t)\langle \mathbf{u}(t), w\rangle_{L^2(U)} dt &= -\int_0^T \phi(t)\langle \mathbf{u}'(t), w\rangle_{L^2(U)} dt \\
\int_0^T \phi'(t)\langle \mathbf{u}(t), w\rangle_{L^2(U)} dt &= -\int_0^T \phi(t)\langle (\mathbf{u}')^*(t)|w\rangle_{H^{-1}(U),H_0^1(U)} dt.
\end{aligned}$$

Thus $(\mathbf{u}')^* = (\mathbf{u}^*)'$ from the above lemma. $\square$

Corollary 4.12. *Suppose* $\boldsymbol{u}, \boldsymbol{u}' \in L^1\big(0,T;H_0^1(U)\big)$, *then we must have the action function*

$$(\boldsymbol{u}')^* := t \mapsto \langle \boldsymbol{u}(t), \cdot \rangle_{L^2(U)}$$

is the time weak derivative of $\boldsymbol{u}^* := t \mapsto \langle \boldsymbol{u}(t), \cdot \rangle_{L^2(U)}$ *in* $L^1\big(0,T;H^{-1}(U)\big)$.

Remark. Recall $H_0^1(U) \subset L^2(U) \cong L^2(U)^* \subset H^{-1}(U)$, and we can identify $\mathbf{u}(t) \in H_0^1(U)$ with $\mathbf{u}^*(t) := \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)} \in H^{-1}(U)$. The above lemmas allow us to further abuse this notation and identify $\mathbf{u}'$ with $(\mathbf{u}')^* = (\mathbf{u}^*)'$.

Definition 4.5. Suppose $\mathbf{u} \in L^1\big(0, T; H_0^1(U)\big)$, we abuse the notation and denote

$$\mathbf{u}' := \mathbf{v} \in L^1\big(0, T; H^{-1}(U)\big)$$

to be the time weak derivative of $\mathbf{u}$, if $\mathbf{v} = (\mathbf{u}^*)'$ is the time weak derivative of the action function

$$\mathbf{u}^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$$

in $L^1\big(0, T; H^{-1}(U)\big)$.

Remark. This is a further extension of the original definition of the weak derivative, since $\mathbf{u}'$ may not exist in $L^1\big(0, T; H_0^1(U)\big)$ even if such a $\mathbf{v} = (\mathbf{u}^*)'$ exists in $L^1\big(0, T; H^{-1}(U)\big)$ or even $L^1\big(0, T; L^2(U)^*\big)$.

We also have the following results:

Theorem 4.13. *The dual space of $L^2\big(0, T; H_0^1(U)\big)$ is $L^2\big(0, T; H^{-1}(U)\big)$, and the dual space of $L^2\big(0, T; H^{-1}(U)\big)$ is $L^2\big(0, T; H_0^1(U)\big)$. The contraction map is defined to be $\forall \boldsymbol{u} \in L^2\big(0, T; H_0^1(U)\big), \boldsymbol{v} \in L^2\big(0, T; H^{-1}(U)\big)$,*

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle_{L^2(0,T;H_0^1(U)), L^2(0,T;H^{-1}(U))} := \langle \boldsymbol{v} | \boldsymbol{u} \rangle_{L^2(0,T;H^{-1}(U)), L^2(0,T;H_0^1(U))} := \int_0^T \langle \boldsymbol{v}(t) | \boldsymbol{u}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt.$$

Proof. We can quickly show one side of inclusion:

$$\begin{aligned}
\int_0^T \langle \mathbf{v}(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt &\leq \int_0^T ||\mathbf{v}(t)||_{H^{-1}(U)} ||\mathbf{u}(t)||_{H_0^1(U)} dt \\
&\leq \left(\int_0^T ||\mathbf{v}(t)||_{H^{-1}(U)}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T ||\mathbf{u}(t)||_{H_0^1(U)}^2 dt \right)^{\frac{1}{2}} \\
&= ||u||_{L^2(0,T;H_0^1(U))} ||v||_{L^2(0,T;H^{-1}(U))},
\end{aligned}$$

which shows $L^2\big(0, T; H_0^1(U)\big) \subseteq L^2\big(0, T; H^{-1}(U)\big)^*$, and $L^2\big(0, T; H^{-1}(U)\big) \subseteq L^2\big(0, T; H_0^1(U)\big)^*$. ☐

Theorem 4.14 (Royden-Fitzpatrick)**.** *$f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if there is a Lebesgue integrable function g, such that $\forall x \in [a, b]$, $f(x) = f(a) + \int_a^x g(t) dt$. In this case, f is differentiable a.e., and $f'(x) = g(x)$ for a.e. $x \in [a, b]$.*

Lemma 4.15. *Suppose $\boldsymbol{u} \in L^1\big(0, T; H_0^1(U)\big), \boldsymbol{u}' \in L^1\big(0, T; H^{-1}(U)\big)$, then $\forall \epsilon > 0$,*

$$\boldsymbol{u}'^{\epsilon} = \Big(t \mapsto \langle (\boldsymbol{u}^{\epsilon})'(t), \cdot \rangle_{L^2(U)} \Big),$$

where $(\boldsymbol{u}^{\epsilon})'$ is the normal derivative of $\boldsymbol{u}^{\epsilon}$ in $C^{\infty}\big(\epsilon, T - \epsilon; H_0^1(U)\big)$.

Proof. Since $\mathbf{u}'$ is the time weak derivative of $\mathbf{u}^*$ in $L^2\big((0, T); H^{-1}(U)\big)$, we can show that

$$\mathbf{u}'^{\epsilon} = \eta_{\epsilon} * \mathbf{u}' = (\eta_{\epsilon} * \mathbf{u}^*)'$$

in $L^2\big((0, T); H^{-1}(U)\big)$ similarly as in 2.17.

Also, one can see that $\forall t \in [\epsilon, T-\epsilon], v \in H_0^1(U)$,

$$\begin{aligned}
\langle(\eta_\epsilon * \mathbf{u}^*)(t)|v\rangle_{H^{-1}(U),H_0^1(U)} &= \left\langle \int_0^T \eta_\epsilon(t-\tau)\mathbf{u}^*(\tau)d\tau \middle| v \right\rangle_{H^{-1}(U),H_0^1(U)} \\
&= \int_0^T \eta_\epsilon(t-\tau)\langle \mathbf{u}^*(\tau)|v\rangle_{H^{-1}(U),H_0^1(U)} d\tau \\
&= \int_0^T \eta_\epsilon(t-\tau)\langle \mathbf{u}(\tau), v\rangle_{L^2(U)} d\tau \\
&= \left\langle \int_0^T \eta_\epsilon(t-\tau)\mathbf{u}(\tau)d\tau, v \right\rangle_{L^2(U)} \\
&= \langle \mathbf{u}^\epsilon(t), v\rangle_{L^2(U)}.
\end{aligned}$$

Since $(\mathbf{u}^\epsilon)'$ exists as the weak derivative of $\mathbf{u}^\epsilon$, we have

$$(\eta_\epsilon * \mathbf{u}^*)' = \Big(t \mapsto \langle(\mathbf{u}^\epsilon)'(t), \cdot\rangle_{L^2(U)}\Big)$$

in $L^2\big((0,T); H^{-1}(U)\big)$. ◻

Theorem 4.16. *Suppose $\boldsymbol{u} \in L^2\big(0,T; H_0^1(U)\big), \boldsymbol{u}' \in L^2\big(0,T; H^{-1}(U)\big)$, then*

1. *There is a representative $\tilde{\boldsymbol{u}} \in C\big([0,T]; L^2(U)\big)$ of $\boldsymbol{u}$.*
2. *For any $\boldsymbol{v} \in L^2\big(0,T; H_0^1(U)\big), \boldsymbol{v}' \in L^2\big(0,T; H^{-1}(U)\big)$, the mapping $t \mapsto \langle \tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{v}}(t)\rangle_{L^2(U)}$ is absolutely continuous, and for a.e. $t \in [0,T]$, we have*

$$\frac{d}{dt}\langle \tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{v}}(t)\rangle_{L^2(U)} = \langle \boldsymbol{u}'(t)|\boldsymbol{v}(t)\rangle_{H^{-1}(U),H_0^1(U)} + \langle \boldsymbol{v}'(t)|\boldsymbol{u}(t)\rangle_{H^{-1}(U),H_0^1(U)}.$$

3. *The mapping $t \mapsto ||\tilde{\boldsymbol{u}}(t)||^2_{L^2(U)}$ is absolutely continuous, and for a.e. $t \in [0,T]$, we have*

$$\frac{d}{dt}||\boldsymbol{u}(t)||^2_{L^2(U)} = 2\langle \boldsymbol{u}'(t)|\boldsymbol{u}(t)\rangle_{H^{-1}(U),H_0^1(U)}.$$

4. *$\exists C > 0$, such that $\forall \boldsymbol{u} \in L^2\big(0,T; H_0^1(U)\big), \boldsymbol{u}' \in L^2\big(0,T; H^{-1}(U)\big)$,*

$$\sup_{t\in[0,T]} ||\tilde{\boldsymbol{u}}(t)||_{L^2(U)} \le C\Big(||\boldsymbol{u}||_{L^2(0,T;H_0^1(U))} + ||\boldsymbol{u}'||_{L^2(0,T;H^{-1}(U))}\Big),$$

where the constant C only depends on T.

Proof. 1. We can extend $\mathbf{u}$ to $[-\sigma, T+\sigma]$ for an $\delta > 0$ by reflection and cut off as done in 2.26. Now for any $\epsilon, \delta \in (0,\sigma)$, we can define $\mathbf{u}^\epsilon := \eta_\epsilon * \mathbf{u}, \mathbf{u}^\delta := \eta_\delta * \mathbf{u}$, both well-defined on $[0,T]$. By 4.6, $\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty\big([0,T]; H_0^1(U)\big)$, so we have

$$\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty\big([0,T]; H_0^1(U)\big) \subset C^\infty\big([0,T]; L^2(U)\big).$$

Now for any $t \in [0,T]$, we have that

$$\begin{aligned}
&\frac{d}{dt}\big|\big|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\big|\big|_{L^2(U)} \\
&= \frac{d}{dt}\big\langle \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t), u^\epsilon(t) - \mathbf{u}^\delta(t)\big\rangle_{L^2(U)} \\
&= \big\langle (\mathbf{u}^\epsilon - \mathbf{u}^\delta)'(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\big\rangle_{L^2(U)} + \big\langle \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t), (\mathbf{u}^\epsilon - \mathbf{u}^\delta)'(t)\big\rangle_{L^2(U)} \\
&= 2\big\langle (\mathbf{u}^\epsilon)'(t) - (\mathbf{u}^\delta)'(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\big\rangle_{L^2(U)},
\end{aligned}$$

where $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)'$ are the normal derivatives as functions $[0,T] \to L^2(U)$.
Also, since $\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty\big([0,T]; H_0^1(U)\big)$, we have that their weak derivatives exist in $L^2\big((0,T); H_0^1(U)\big) \subset L^1\big((0,T); H_0^1(U)\big)$, and by above lemma, are also weak derivatives in $L^1\big((0,T); L^2(U)\big)$.
Since any weak derivative is a.e. equal to the normal derivative if the latter exists, we will just use $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)'$ to represent the weak derivatives in $L^2\big((0,T); H_0^1(U)\big)$, and the above equality still holds for a.e. $t \in [0,T]$.
Integrating both sides on any $[s,t] \subseteq [0,T]$, we get

$$\begin{aligned}
&\big|\big|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\big|\big|_{L^2(U)} - \big|\big|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\big|\big|_{L^2(U)} \\
=& \int_s^t \frac{d}{d\tau}\big|\big|\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\big|\big|_{L^2(U)} d\tau \\
=& \int_s^t 2\big\langle(\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau), \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\big\rangle_{L^2(U)} d\tau \\
=& \int_s^t 2\big\langle(\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau)\big|\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\big\rangle_{H^{-1}(U), H_0^1(U)} d\tau \\
\leq& \int_s^t 2\big|\big|(\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau)\big|\big|_{H^{-1}(U)}\big|\big|\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\big|\big|_{H_0^1(U)} d\tau \\
\leq& \int_s^t \big|\big|(\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau)\big|\big|^2_{H^{-1}(U)} + \big|\big|\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\big|\big|^2_{H_0^1(U)} d\tau \\
=& \big|\big|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\big|\big|^2_{L^2(0,T;H^{-1}(U))} + \big|\big|\mathbf{u}^\epsilon - \mathbf{u}^\delta\big|\big|^2_{L^2\left(0,T;H_0^1(U)\right)},
\end{aligned}$$

where we again identify $(\mathbf{u}^\epsilon)'(\tau), (\mathbf{u}^\delta)'(\tau) \in H_0^1(U)$ with $\langle(\mathbf{u}^\epsilon)'(\tau), \cdot\rangle_{L^2(U)}, \big\langle(\mathbf{u}^\delta)'(\tau), \cdot\big\rangle_{L^2(U)} \in H^{-1}(U)$ as usual.
By 4.6, we have $\mathbf{u}^\epsilon, \mathbf{u}^\delta \to \mathbf{u}$ in $L^2\big((0,T); H_0^1(U)\big)$, so

$$\begin{aligned}
\big|\big|\mathbf{u}^\epsilon - \mathbf{u}^\delta\big|\big|_{L^2\left(0,T;H_0^1(U)\right)} &= \big|\big|\mathbf{u}^\epsilon - \mathbf{u} + \mathbf{u} - \mathbf{u}^\delta\big|\big|_{L^2\left(0,T;H_0^1(U)\right)} \\
&\leq ||\mathbf{u}^\epsilon - \mathbf{u}||_{L^2\left(0,T;H_0^1(U)\right)} + \big|\big|\mathbf{u} - \mathbf{u}^\delta\big|\big|_{L^2\left(0,T;H_0^1(U)\right)} \\
&\to 0
\end{aligned}$$

as $\epsilon, \delta \to 0$.
Since $\mathbf{u}'$ is the time weak derivative of $\mathbf{u}^*$ in $L^2\big((0,T); H^{-1}(U)\big)$, we can again show that $(\eta_\epsilon * \mathbf{u}^*)' = \eta_\epsilon * \mathbf{u}'$ in $L^2\big((0,T); H^{-1}(U)\big)$.
By 4.6 and lemma, we have $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)' \to \mathbf{u}'$ in $L^2\big(0,T; H^{-1}(U)\big)$.
Similar as above, we have $\big|\big|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\big|\big|_{L^2(0,T;H^{-1}(U))} \to 0$ as $\epsilon, \delta \to 0$.
In addition, for a.e. $s \in [0,T]$, we must have $\big|\big|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\big|\big|_{L^2(U)} \to 0$.
Pick any of these s, we have

$$\big|\big|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\big|\big|_{L^2(U)} \leq \big|\big|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\big|\big|_{L^2(U)} + \big|\big|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\big|\big|^2_{L^2(0,T;H^{-1}(U))} + \big|\big|\mathbf{u}^\epsilon - \mathbf{u}^\delta\big|\big|^2_{L^2\left(0,T;H_0^1(U)\right)}.$$

Since this holds for any $t \in [0,T]$, we have

$$\begin{aligned}
&\big|\big|\mathbf{u}^\epsilon - \mathbf{u}^\delta\big|\big|_{C([0,T];L^2(U))} \\
=& \sup_{t \in [0,T]; L^2(U)} \big|\big|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\big|\big|_{L^2(U)} \\
\leq& \big|\big|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\big|\big|_{L^2(U)} + \big|\big|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\big|\big|^2_{L^2(0,T;H^{-1}(U))} + \big|\big|\mathbf{u}^\epsilon - \mathbf{u}^\delta\big|\big|^2_{L^2\left(0,T;H_0^1(U)\right)} \\
\to& 0.
\end{aligned}$$

as $\epsilon, \delta \to 0$, since each term goes to 0.
This shows that $\mathbf{u}^\epsilon$ is a Cauchy sequence in $C\big([0,T]; L^2(U)\big)$, and since it is a Banach Space, there

must be some

$$\tilde{\mathbf{u}} := \lim_{\epsilon\to 0} \mathbf{u}^\epsilon \in C\big([0,T];L^2(U)\big).$$

Now since for a.e. $t\in[0,T]$, $\mathbf{u}(t) = \lim_{\epsilon\to0}\mathbf{u}^\epsilon(t)$, and $\forall t\in[0,T]$, $\tilde{\mathbf{u}}(t) = \lim_{\epsilon\to0}\mathbf{u}^\epsilon(t)$, we have that $\tilde{\mathbf{u}} = \mathbf{u}$ for a.e. $t\in[0,T]$ is a representative of $\mathbf{u}$.

2. Similar as above, we can show that for a.e. $t\in[0,T]$, we have

$$\frac{d}{dt}\langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t)\rangle_{L^2(U)} = \langle (\mathbf{u}^\epsilon)'(t), \mathbf{v}^\epsilon(t)\rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(t), \mathbf{u}^\epsilon(t)\rangle_{L^2(U)}.$$

Integrating over any $(s,t)\subset[0,T]$ gives

$$\langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t)\rangle_{L^2(U)} = \langle \mathbf{u}^\epsilon(s), \mathbf{v}^\epsilon(s)\rangle_{L^2(U)} + \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau)\rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau)\rangle_{L^2(U)} d\tau.$$

Now

$$\begin{aligned}
&\left|\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau)\rangle_{L^2(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau\right| \\
&= \left|\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau)|\mathbf{v}^\epsilon(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau\right| \\
&\le \left|\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau)|\mathbf{v}^\epsilon(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau - \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau\right| \\
&\quad + \left|\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau\right| \\
&= \left|\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau)|\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau\right| + \left|\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau\right| \\
&\le \int_s^t ||(\mathbf{u}^\epsilon)'(\tau)||_{H^{-1}(U)}||\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)||_{H_0^1(U)} d\tau + \int_s^t ||(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)||_{H^{-1}(U)}||\mathbf{v}(\tau)||_{H_0^1(U)} d\tau \\
&\le \sqrt{\int_s^t ||(\mathbf{u}^\epsilon)'(\tau)||^2_{H^{-1}(U)} d\tau}\sqrt{\int_s^t ||\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)||^2_{H_0^1(U)} d\tau} \\
&\quad + \sqrt{\int_s^t ||(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)||^2_{H^{-1}(U)} d\tau}\sqrt{\int_s^t ||\mathbf{v}(\tau)||^2_{H_0^1(U)} d\tau} \\
&\le \sqrt{\int_0^T ||(\mathbf{u}^\epsilon)'(\tau)||^2_{H^{-1}(U)} d\tau}\sqrt{\int_0^T ||\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)||^2_{H_0^1(U)} d\tau} \\
&\quad + \sqrt{\int_0^T ||(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)||^2_{H^{-1}(U)} d\tau}\sqrt{\int_0^T ||\mathbf{v}(\tau)||^2_{H_0^1(U)} d\tau} \\
&= ||(\mathbf{u}^\epsilon)'||_{L^2(0,T;H^{-1}(U))}||\mathbf{v}^\epsilon - \mathbf{v}||_{L^2\left(0,T;H_0^1(U)\right)} + ||(\mathbf{u}^\epsilon)' - \mathbf{u}'||_{L^2(0,T;H^{-1}(U))}||\mathbf{v}||_{L^2\left(0,T;H_0^1(U)\right)},
\end{aligned}$$

by Holder's Inequality.
Notice that

$$||(\mathbf{u}^\epsilon)'||_{L^2(0,T;H^{-1}(U))} \to ||\mathbf{u}'||_{L^2(0,T;H^{-1}(U))} < \infty, ||(\mathbf{u}^\epsilon)' - \mathbf{u}'||_{L^2(0,T;H^{-1}(U))} \to 0,$$

since we have shown $(\mathbf{u}^\epsilon)' \to \mathbf{u}'$ in $L^2\big(0,T;H^{-1}(U)\big)$.
Also,

$$||\mathbf{v}^\epsilon - \mathbf{v}||_{L^2\left(0,T;H_0^1(U)\right)} \to 0,$$

since $\mathbf{v}^\epsilon \to \mathbf{v}$ in $L^2\big(0,T;H_0^1(U)\big)$.
Thus, $\lim_{\epsilon\to0}\left|\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau)\rangle_{L^2(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau\right| = 0$, which means

$$\lim_{\epsilon\to0}\int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau)\rangle_{L^2(U)} d\tau = \int_s^t \langle \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau.$$

Similarly, we also have

$$\lim_{\epsilon\to 0}\int_s^t \langle(\mathbf{v}^\epsilon)'(\tau),\mathbf{u}^\epsilon(\tau)\rangle_{L^2(U)}d\tau = \int_s^t \langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}d\tau.$$

Taking the limit of $\epsilon\to 0$, we have

$$\begin{aligned}
&\langle\tilde{\mathbf{u}}(t),\tilde{\mathbf{v}}(t)\rangle_{L^2(U)}\\
&=\lim_{\epsilon\to 0}\langle\mathbf{u}^\epsilon(t),\mathbf{v}^\epsilon(t)\rangle_{L^2(U)}\\
&=\lim_{\epsilon\to 0}\left(\langle\mathbf{u}^\epsilon(s),\mathbf{v}^\epsilon(s)\rangle_{L^2(U)}+\int_s^t \langle(\mathbf{u}^\epsilon)'(\tau),\mathbf{v}^\epsilon(\tau)\rangle_{L^2(U)}+\langle(\mathbf{v}^\epsilon)'(\tau),\mathbf{u}^\epsilon(\tau)\rangle_{L^2(U)}d\tau\right)\\
&=\lim_{\epsilon\to 0}\langle\mathbf{u}^\epsilon(s),\mathbf{v}^\epsilon(s)\rangle_{L^2(U)}+\lim_{\epsilon\to 0}\int_s^t \langle(\mathbf{u}^\epsilon)'(\tau),\mathbf{v}^\epsilon(\tau)\rangle_{L^2(U)}d\tau+\lim_{\epsilon\to 0}\int_s^t \langle(\mathbf{v}^\epsilon)'(\tau),\mathbf{u}^\epsilon(\tau)\rangle_{L^2(U)}d\tau\\
&=\langle\tilde{\mathbf{u}}(s),\tilde{\mathbf{v}}(s)\rangle_{L^2(U)}+\int_s^t \langle\mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}d\tau+\int_s^t \langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}d\tau\\
&=\langle\tilde{\mathbf{u}}(s),\tilde{\mathbf{v}}(s)\rangle_{L^2(U)}+\int_s^t \Big(\langle\mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}+\langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}\Big)d\tau.
\end{aligned}$$

In particular, if we take $s=0$, we have

$$\langle\tilde{\mathbf{u}}(t),\tilde{\mathbf{v}}(t)\rangle_{L^2(U)}=\langle\tilde{\mathbf{u}}(0),\tilde{\mathbf{v}}(0)\rangle_{L^2(U)}+\int_0^t \Big(\langle\mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}+\langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}\Big)d\tau.$$

Also, by Holder's Inequality,

$$\begin{aligned}
&\int_0^T \Big|\langle\mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}+\langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}\Big|d\tau\\
&\le\int_0^T \Big|\langle\mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}\Big|d\tau+\int_0^t \Big|\langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}\Big|d\tau\\
&\le\int_0^T ||\mathbf{u}'(\tau)||_{H^{-1}(U)}||\mathbf{v}(\tau)||_{H_0^1(U)}d\tau+\int_0^T ||\mathbf{v}'(\tau)||_{H^{-1}(U)}||\mathbf{u}(\tau)||_{H_0^1(U)}d\tau\\
&\le\sqrt{\int_0^T ||\mathbf{u}'(\tau)||^2_{H^{-1}(U)}d\tau}\sqrt{\int_0^T ||\mathbf{v}(\tau)||^2_{H_0^1(U)}d\tau}+\sqrt{\int_0^T ||\mathbf{v}'(\tau)||^2_{H^{-1}(U)}d\tau}\sqrt{\int_0^T ||\mathbf{u}(\tau)||^2_{H_0^1(U)}d\tau}\\
&=||\mathbf{u}'||_{L^2(0,T;H^{-1}(U))}||\mathbf{v}||_{L^2\left(0,T;H_0^1(U)\right)}+||\mathbf{v}'||_{L^2(0,T;H^{-1}(U))}||\mathbf{u}||_{L^2\left(0,T;H_0^1(U)\right)}\\
&<\infty.
\end{aligned}$$

We have shown that $\langle\mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}+\langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}$ is Lebesgue integrable.
By Royden-Fitzpatrick's Theorem, we have that $t\mapsto\langle\tilde{\mathbf{u}}(t),\tilde{\mathbf{v}}(t)\rangle_{L^2(U)}$ is absolutely continuous, and for a.e. $t\in[0,T]$,

$$\frac{d}{dt}\langle\tilde{\mathbf{u}}(t),\tilde{\mathbf{v}}(t)\rangle_{L^2(U)}=\langle\mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}+\langle\mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)}.$$

Since $\tilde{\mathbf{u}}(t)=\mathbf{u}(t)$ for a.e. $t\in[0,T]$, we have the result.

3. Take $\mathbf{v}=\mathbf{u}$ in 2.

4. Integrate $||\tilde{\mathbf{u}}(t)||_{L^2(U)} = ||\tilde{\mathbf{u}}(s)||_{L^2(U)} + \int_s^t 2\langle \mathbf{u}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau$ for $0 \le s \le T$, we have

$$\begin{aligned}
T||\tilde{\mathbf{u}}(t)||^2_{L^2(U)} &= \int_0^T ||\mathbf{u}(t)||^2_{L^2(U)} ds \\
&= \int_0^T ||\tilde{\mathbf{u}}(s)||^2_{L^2(U)} ds + 2\int_0^T \int_s^t \langle \mathbf{u}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} d\tau ds \\
&\le \int_0^T ||\tilde{\mathbf{u}}(s)||^2_{H_0^1(U)} ds + 2\int_0^T \int_s^t ||\mathbf{u}'(\tau)||_{H^{-1}(U)} ||\mathbf{u}(\tau)||_{H_0^1(U)} d\tau ds \\
&\le ||\mathbf{u}||^2_{L^2\left(0,T;H_0^1(U)\right)} + 2\int_0^T \int_0^T ||\mathbf{u}'(\tau)||_{H^{-1}(U)} ||\mathbf{u}(\tau)||_{H_0^1(U)} d\tau ds \\
&= ||\mathbf{u}||^2_{L^2\left(0,T;H_0^1(U)\right)} + 2T\int_0^T ||\mathbf{u}'(\tau)||_{H^{-1}(U)} ||\mathbf{u}(\tau)||_{H_0^1(U)} d\tau \\
&\le ||\mathbf{u}||^2_{L^2\left(0,T;H_0^1(U)\right)} + T\int_0^T ||\mathbf{u}'(\tau)||^2_{H^{-1}(U)} + ||\mathbf{u}(\tau)||^2_{H_0^1(U)} d\tau \\
&= (T+1)||\mathbf{u}||^2_{L^2\left(0,T;H_0^1(U)\right)} + T||\mathbf{u}'||^2_{L^2(0,T;H^{-1}(U))} \\
||\tilde{\mathbf{u}}(t)||^2_{L^2(U)} &\le \frac{T+1}{T}||\mathbf{u}||^2_{L^2\left(0,T;H_0^1(U)\right)} + ||\mathbf{u}'||^2_{L^2(0,T;H^{-1}(U))} \\
&\le C^2||\mathbf{u}||^2_{L^2\left(0,T;H_0^1(U)\right)} + C^2||\mathbf{u}'||^2_{L^2(0,T;H^{-1}(U))} \\
&\le C^2||\mathbf{u}||^2_{L^2\left(0,T;H_0^1(U)\right)} + C^2||\mathbf{u}'||^2_{L^2(0,T;H^{-1}(U))} + 2C^2||\mathbf{u}||_{L^2\left(0,T;H_0^1(U)\right)}||\mathbf{u}'||_{L^2(0,T;H^{-1}(U))} \\
&= \Big(C||\mathbf{u}||_{L^2\left(0,T;H_0^1(U)\right)} + C||\mathbf{u}'||_{L^2(0,T;H^{-1}(U))}\Big)^2 \\
||\tilde{\mathbf{u}}(t)||^2_{L^2(U)} &\le C\Big(||\mathbf{u}||_{L^2\left(0,T;H_0^1(U)\right)} + ||\mathbf{u}'||_{L^2(0,T;H^{-1}(U))}\Big),
\end{aligned}$$

where we take $C^2 := \max\left(1, \frac{T+1}{T}\right) > 1$, which is independent of $\mathbf{u}$ and $\mathbf{u}'$.
Since this holds for any $t \in [0,T]$, we have

$$\sup_{t\in[0,T]} ||\tilde{\mathbf{u}}(t)||_{L^2(U)} \le C\Big(||\mathbf{u}||_{L^2\left(0,T;H_0^1(U)\right)} + ||\mathbf{u}'||_{L^2(0,T;H^{-1}(U))}\Big).$$

□

4.2 Second Order Parabolic Equations

Definition 4.6. Let $U \subseteq \mathbb{R}^n$ be open and bounded, we define $U_T := U \times (0,T]$ for $T > 0$.

Definition 4.7. An **initial boundary value problem** is: given $f : U_T \to \mathbb{R}, g : U \to \mathbb{R}$, we want to find $u(x,t) : \bar{U_T} \to \mathbb{R}$, such that

$$\begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases},$$

where

$$Lu := -\sum_{i,j=1}^n \partial_j(a^{ij}(\cdot,t)\partial_i u) + \sum_{i=1}^n b^i(\cdot,t)\partial_i u + c(\cdot,t)u$$

for some $a^{ij}, b^i, c : U_T \to \mathbb{R}$.
We say that the partial differential operator $\partial_t + L$ is an **symmetric (uniformly) parabolic second order differential operator** if $a^{ij} = a^{ji}$, and $\exists\theta > 0,$ such that

$$\sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \ge \theta||\xi||_2^2, \ \forall (x,t) \in U_T, \xi \in \mathbb{R}^n.$$

Definition 4.8. The **parabolic assumptions** are:

1. $U \subseteq \mathbb{R}^n$ is bounded and open
2. $T > 0$
3. $a^{ij}, b^i, c \in L^\infty(U_T)$
4. $f \in L^2(U_T), g \in L^2(U)$
5. $\partial_t + L$ is a symmetric (uniformly) parabolic second order differential operator.

Definition 4.9. Given a function $u : U_T \to \mathbb{R}$, we want to consider $\mathbf{u} : t \mapsto u(\cdot, t)$, for any $t \in [0, T]$.

Proposition 4.17. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, $T > 0$, then $f \in L^2(U_T) \iff \boldsymbol{f} \in L^2\big(0, T; L^2(U)\big)$.*

Proof. We have

$$\begin{aligned} ||\mathbf{f}||^2_{L^2(0,T;L^2(U))} &= \int_0^T ||\mathbf{f}(t)||^2_{L^2(U)} dt \\ &= \int_0^T \int_U |f(x,t)|^2 dx dt \\ &= ||f||_{L^2(U_T)}. \end{aligned}$$

□

Definition 4.10. $\mathbf{u} \in L^2\big(0, T; H_0^1(U)\big)$, identified with its continuous representative $\tilde{\mathbf{u}} \in C\big([0,T]; L^2(U)\big)$ as in 4.16.1, with the time weak derivative $\mathbf{u}' \in L^2\big(0, T; H^{-1}(U)\big)$, is a **weak solution** of the IBVP if

$$\begin{aligned} \forall v \in H_0^1(U),\ \langle \mathbf{u}'(t)|v\rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t] = \langle \mathbf{f}(t), v\rangle_{L^2(U)},\ \text{a.e. } t \in [0,T], \\ \mathbf{u}(0) = g, \end{aligned}$$

where bilinear form associated to the above problem is

$$\forall w, v \in H_0^1(U),\ B[w, v; t] := \int_U \left(\sum_{i,j=1}^n a^{ij}(\cdot, t)\partial_i w \partial_j v + \sum_{i=1}^n b^i(\cdot, t)\partial_i w v + c(\cdot, t) w v \right) dx.$$

4.3 Galerkin Method

Definition 4.11. Let $(w_k)_{k=1}^\infty$ be an orthogonal basis of $H_0^1(U)$, and also an orthonormal basis of $L^2(U)$. For $m \in \mathbb{N}^+$, we define $V_m := \operatorname{Span}\left(\{w_j\}_{j=1}^m\right) \subset H_0^1(U)$ be a subspace. A function $\mathbf{u}_m := t \mapsto \sum_{k=1}^m d_m^k(t) w_k$ is a **weak solution of the problem in** V_m if $\forall v \in V_m$,

$$\begin{aligned} \left\langle \sum_{k=1}^m {d_m^k}'(t) w_k, v \right\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] = \langle \mathbf{f}(t), v\rangle_{L^2(U)},\ \text{for a.e. } t \in [0,T], \\ \langle \mathbf{u}_m(0), v\rangle_{L^2(U)} = \langle g, v\rangle_{L^2(U)}. \end{aligned}$$

Definition 4.12. We define the **ODE system associated to the problem** to be: $\forall j \in [m]$,

1. $d_m^j : [0,T] \to \mathbb{R}$ is absolutely continuous.
2. For a.e. $t \in [0,T]$, ${d_m^j}'(t) = -\sum_{k=1}^m e_k^j(t) d_m^k(t) + f^j(t)$
3. $d_m^j(0) = \langle g, w_j\rangle_{L^2(U)}$,

where $e_k^j(t) := B[w_k, w_j; t]$, $f^j(t) := \langle \mathbf{f}(t), w_j\rangle_{L^2(U)}$.

Proposition 4.18. *$\boldsymbol{u}_m(t) = \sum_{k=1}^m d_m^k(t)w_k$ is a weak solution in V_m if and only if $\vec{d}_m$ is a solution to the ODE system.*

Proof. Since $(w_k)_{k=1}^\infty$ is an orthonormal basis of V_m in $\langle \cdot, \cdot \rangle_{L^2(U)}$, we have

$$\left\langle \sum_{k=1}^m {d_m^k}'(t)w_k, v \right\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] = \langle \mathbf{f}(t), v \rangle, \qquad \forall v \in V_m$$

$$\iff$$

$$\left\langle \sum_{k=1}^m {d_m^k}'(t)w_k, v \right\rangle_{L^2(U)} + B\left[\sum_{k=1}^m d_m^k(t)w_k, v; t\right] = \langle \mathbf{f}(t), v \rangle, \qquad \forall v \in V_m$$

$$\iff$$

$$\left\langle \sum_{k=1}^m {d_m^k}'(t)w_k, w_j \right\rangle_{L^2(U)} + B\left[\sum_{k=1}^m d_m^k(t)w_k, w_j; t\right] = \langle \mathbf{f}(t), w_j \rangle, \qquad \forall j \in [m]$$

$$\iff$$

$$\sum_{k=1}^m {d_m^k}'(t)\langle w_k, w_j \rangle_{L^2(U)} + \sum_{k=1}^m d_m^k(t) B[w_k, w_j; t] = \langle \mathbf{f}(t), w_j \rangle, \qquad \forall j \in [m]$$

$$\iff$$

$$\sum_{k=1}^m {d_m^k}'(t)\delta_k^j + \sum_{k=1}^m d_m^k(t)e_k^j(t) = f^j(t), \qquad \forall j \in [m]$$

$$\iff$$

$${d_m^j}'(t) + \sum_{k=1}^m d_m^k(t)e_k^j(t) = f^j(t), \qquad \forall j \in [m].$$

On the other hand,

$$\langle \mathbf{u}_m(0), v \rangle_{L^2(U)} = \langle g, v \rangle_{L^2(U)}, \qquad v \in V_m$$

$$\iff$$

$$\langle \mathbf{u}_m(0), w_j \rangle_{L^2(U)} = \langle g, w_j \rangle_{L^2(U)}, \qquad \forall j \in [m]$$

$$\iff$$

$$\left\langle \sum_{k=1}^m d_m^k(0)w_k, w_j \right\rangle_{L^2(U)} = \langle g, w_j \rangle_{L^2(U)}, \qquad \forall j \in [m]$$

$$\iff$$

$$\sum_{k=1}^m d_m^k(0)\langle w_k, w_j \rangle_{L^2(U)} = \langle g, w_j \rangle_{L^2(U)}, \qquad \forall j \in [m]$$

$$\iff$$

$$\sum_{k=1}^m d_m^k(0)\delta_k^j = \langle g, w_j \rangle_{L^2(U)}, \qquad \forall j \in [m]$$

$$\iff$$

$$d_m^j(0) = \langle g, w_j \rangle_{L^2(U)}, \qquad \forall j \in [m].$$

□

Theorem 4.19. *Since f^i, e_j^k are locally integrable, there is a unique absolutely continuous solution $\vec{d}_m$ to the ODE system.*

Corollary 4.20. *For each $m \in \mathbb{N}^+$, there is a unique weak solution $\boldsymbol{u}_m$ of the form $t \mapsto \sum_{k=1}^{m} d_m^k(t) w_k$ of the problem in V_m.*

Proposition 4.21. *The weak solution $\boldsymbol{u}_m$ also satisfies $\forall v \in V_m$,*

$$\langle \boldsymbol{u}_m'(t), v\rangle_{L^2(U)} + B[\boldsymbol{u}_m(t), v; t] = \langle \boldsymbol{f}(t), v\rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T],$$
$$\langle \boldsymbol{u}_m(0), v\rangle_{L^2(U)} = \langle g, v\rangle_{L^2(U)}.$$

In addition,

$$||\boldsymbol{u}_m(0)||_{L^2(U)} \leq ||g||_{L^2(U)},$$

and

$$\lim_{m\to\infty} \boldsymbol{u}_m(0) = g$$

in $L^2(U)$.

Proof. Consider any $\phi \in C_c^\infty(0, T)$. By linearity of the Bochner integral, we have

$$\begin{aligned}
\int_0^T \phi'(t)\mathbf{u}_m(t)dt &= \int_0^T \left(\phi'(t)\sum_{k=1}^m d_m^k(t)w_k\right)dt \\
&= \sum_{k=1}^m \left(\int_0^T \phi'(t)d_m^k(t)w_k dt\right) \\
&= \sum_{k=1}^m \left(\int_0^T \phi'(t)d_m^k(t)dt\right)w_k \\
&= \sum_{k=1}^m \left(\int_0^T \phi(t){d_m^k}'(t)dt\right)w_k \\
&= \int_0^T \phi(t)\left(\sum_{k=1}^m {d_m^k}'(t)w_k\right)dt.
\end{aligned}$$

Thus it has a weak derivative $\mathbf{u}_m'(t) = \sum_{k=1}^m {d_m^k}'(t)w_k$.
We then plug that in the definition of weak solution of the problem in V_m.
On the other hand, since $\mathbf{u}_m(0) \in V_m$, we have that

$$\begin{aligned}
||\mathbf{u}_m(0)||_{L^2(U)}^2 &= \langle \mathbf{u}_m(0), \mathbf{u}_m(0)\rangle_{L^2(U)} \\
&= \langle g, \mathbf{u}_m(0)\rangle_{L^2(U)} \\
&\leq ||g||_{L^2(U)}||\mathbf{u}_m(0)||_{L^2(U)}.
\end{aligned}$$

Since $||\mathbf{u}_m(0)||_{L^2(U)} \geq 0$, we have $||\mathbf{u}_m(0)||_{L^2(U)} \leq ||g||_{L^2(U)}$.
Also, since $(w_k)_{k=1}^\infty$ is an orthonormal basis of $L^2(U)$, we have

$$\begin{aligned}
\lim_{m\to\infty} \mathbf{u}_m(0) &= \lim_{m\to\infty} \sum_{k=1}^m d_m^k(t)w_k \\
&= \lim_{m\to\infty} \sum_{k=1}^m \langle g, w_j\rangle_{L^2(U)} w_k \\
&= g.
\end{aligned}$$

□

Proposition 4.22. *The weak solution $\boldsymbol{u}_m \in L^2\big(0, T; H_0^1(U)\big)$.*

Proof.

$$\begin{aligned}
\int_0^T ||\mathbf{u}_m||^2_{H^1_0(U)} dt &= \int_0^T \left|\left|\sum_{k=1}^m d^k_m(t) w_k\right|\right|^2_{H^1_0(U)} dt \\
&= \int_0^T \left\langle \sum_{k=1}^m d^k_m(t) w_k, \sum_{j=1}^m d^j_m(t) w_j \right\rangle_{H^1_0(U)} dt \\
&= \int_0^T \sum_{k=1}^m \sum_{j=1}^m d^k_m(t) d^j_m(t) \langle w_k, w_j\rangle_{H^1_0(U)} dt \\
&= \int_0^T \sum_{k=1}^m {d^k_m}^2(t) ||w_k||^2_{H^1_0(U)} dt \\
&= \sum_{k=1}^m ||w_k||^2_{H^1_0(U)} \int_0^T {d^k_m}^2(t) dt \\
&= \sum_{k=1}^m ||w_k||^2_{H^1_0(U)} ||d^k_m||^2_{L^2(0,T)} \\
&\leq \infty,
\end{aligned}$$

since each $w_k \in H^1_0(U)$, and each d^k_m is absolutely continuous (thus continuous, thus in $L^2(0,T)$). $\square$

Proposition 4.23. *The weak solution* $\boldsymbol{u}_m \in C\big([0,T]; L^2(U)\big)$.

Proof. Given any $\epsilon > 0$.
For any $1 \leq k \leq m$, since d^k_m is is absolutely continuous (thus continuous), we can find $\delta_k > 0,$ such that

$$\forall s,t \in [0,T],\ |t-s| < \delta_k \implies \left|d^k_m(t) - d^k_m(s)\right| < \frac{\epsilon}{\sqrt{m}}.$$

Now take $\delta := \min_{k\in[m]} \delta_k > 0$, we have that $\forall s,t \in [0,T],$ such that $|t-s| < \delta_k$,

$$\begin{aligned}
||\mathbf{u}_m(t) - \mathbf{u}_m(s)||^2_{L^2(U)} &= \left|\left|\sum_{k=1}^m {d^k_m}'(t) w_k - \sum_{k=1}^m {d^k_m}'(s) w_k\right|\right|^2_{L^2(U)} \\
&= \left|\left|\sum_{k=1}^m \Big({d^k_m}'(t) - {d^k_m}'(s)\Big) w_k\right|\right|^2_{L^2(U)} \\
&= \sum_{k=1}^m \Big({d^k_m}'(t) - {d^k_m}'(s)\Big)^2 \\
&< \sum_{k=1}^m \frac{\epsilon^2}{m} \\
&= \epsilon^2,
\end{aligned}$$

since $(w_k)_{k=1}^\infty$ is an orthornormal basis for ${L^2(U)}^2$.
Thus $||\mathbf{u}_m(t) - \mathbf{u}_m(s)||_{L^2(U)} < \epsilon$ and since this works for any $\epsilon > 0$, we have that $\mathbf{u}_m \in C\big([0,T]; L^2(U)\big)$. $\square$

Notice that the above two propositions says that $\mathbf{u}_m$ is the representative $\tilde{u_m}$ as in 4.16, and we thus have the following result:

Corollary 4.24. *The weak solution* $\boldsymbol{u}_m$ *satisfies that the mapping* $t \mapsto ||\boldsymbol{u}_m(t)||^2_{L^2(U)}$ *is absolutely continuous, and for a.e.* $t \in [0,T]$, *we have*

$$\frac{d}{dt}||\boldsymbol{u}_m(t)||^2_{L^2(U)} = 2\langle \boldsymbol{u}'_m(t)|\boldsymbol{u}_m(t)\rangle_{H^{-1}(U),H^1_0(U)} = 2\langle \boldsymbol{u}'_m(t), \boldsymbol{u}_m(t)\rangle_{L^2(U)}.$$

Theorem 4.25 (Gronwall's inequality)**.** *Let $\eta : [0,T] \to \mathbb{R}$ be nonnegative and absolutely continuous, ϕ, ψ both nonnegative summable functions. If*

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t) \text{ a.e. } t \in [0,T],$$

then

$$\eta(t) \le \exp\left(\int_0^t \phi(s)ds\right)\left(\eta(0) + \int_0^t \psi(s)ds\right), \ \forall t \in [0,T].$$

Theorem 4.26 (Energy Estimate)**.** *Let $U \subseteq \mathbb{R}^n$ be bounded and open, $T > 0$, $a^{ij}, b^i, c \in L^\infty(U_T)$, and $\partial_t + L$ be a symmetric (uniformly) parabolic second order differential operator. There exists $C > 0$ that only depends on U, T, L, such that $\forall f \in L^2(U_T),\ g \in L^2(U),\ m \in \mathbb{N}^+$,*

$$\sup_{0\le t\le T} ||\boldsymbol{u}_m(t)||_{L^2(U)} + ||\boldsymbol{u}_m||_{L^2\left(0,T;H_0^1(U)\right)} + ||\boldsymbol{u}_m'||_{L^2(0,T;H^{-1}(U))} \le C\Big(||\boldsymbol{f}||_{L^2(0,T;L^2(U))} + ||g||_{L^2(U)}\Big),$$

where $\boldsymbol{u}_m$ are the weak solutions in V_m as in above.

Proof. We will bound each term on the left hand side.

1. Consider any $m \in \mathbb{N}^+$, we have that the $\mathbf{u}_m$ satisfies $\forall v \in V_m$,

$$\begin{aligned}\langle \mathbf{u}_m'(t), v\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v\rangle_{L^2(U)}, \text{ for a.e. } t \in [0,T],\\ \langle \mathbf{u}_m(0), v\rangle_{L^2(U)} &= \langle g, v\rangle_{L^2(U)},\end{aligned}$$

   In particular, $\mathbf{u}_m(t) = \sum_{k=1}^m d_m^k(t) w_k \in V_m$.
   Thus for a.e. $t \in [0,T]$, we have

$$\langle \mathbf{u}_m'(t), \mathbf{u}_m(t)\rangle_{L^2(U)} + B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] = \langle \mathbf{f}(t), \mathbf{u}_m(t)\rangle_{L^2(U)}.$$

   By a similar proof as in 3.4, there exists constants $\beta > 0, \gamma \ge 0$ that only depends on U and the coefficients of L, such that $\forall u \in H_0^1(U)$, and for a.e. $t \in [0,T]$,

$$\beta||u||_{H^1(U)}^2 \le B[u,u;t] + \gamma||u||_{L^2(U)}^2.$$

   We thus have

$$\begin{aligned}\langle \mathbf{u}_m'(t), \mathbf{u}_m(t)\rangle_{L^2(U)} + B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] &\ge \langle \mathbf{u}_m'(t), \mathbf{u}_m(t)\rangle_{L^2(U)} + \beta||\mathbf{u}_m(t)||_{H^1(U)}^2 - \gamma||\mathbf{u}_m(t)||_{L^2(U)}^2\\ &= \frac{1}{2}\frac{d}{dt}||\mathbf{u}_m(t)||_{L^2(U)}^2 + \beta||\mathbf{u}_m(t)||_{H^1(U)}^2 - \gamma||\mathbf{u}_m(t)||_{L^2(U)}^2\\ \langle \mathbf{f}(t), \mathbf{u}_m(t)\rangle_{L^2(U)} &\le ||\mathbf{f}(t)||_{L^2(U)}||\mathbf{u}_m(t)||_{L^2(U)}\\ &\le \frac{1}{2}||\mathbf{f}(t)||_{L^2(U)}^2 + \frac{1}{2}||\mathbf{u}_m(t)||_{L^2(U)}^2\\ \frac{d}{dt}||\mathbf{u}_m(t)||_{L^2(U)}^2 + 2\beta||\mathbf{u}_m(t)||_{H^1(U)}^2 &\le ||\mathbf{f}(t)||_{L^2(U)}^2 + ||\mathbf{u}_m(t)||_{L^2(U)}^2 + 2\gamma||\mathbf{u}_m(t)||_{L^2(U)}^2\\ &= ||\mathbf{f}(t)||_{L^2(U)}^2 + (1+2\gamma)||\mathbf{u}_m(t)||_{L^2(U)}^2.\end{aligned}$$

   Notice that this inequality hold for any $f \in L^2(U_T),\ g \in L^2(U),\ m \in \mathbb{N}^+$, and the corresponding weak solutions $\mathbf{u}_m$ in V_m.

2. Since $2\beta||\mathbf{u}_m(t)||_{H^1(U)}^2 \ge 0$, we have

$$\frac{d}{dt}||\mathbf{u}_m(t)||_{L^2(U)}^2 \le ||\mathbf{f}(t)||_{L^2(U)}^2 + (1+2\gamma)||\mathbf{u}_m(t)||_{L^2(U)}^2.$$

Take $\eta(t) := ||\mathbf{u}_m(t)||^2_{L^2(U)}$, which is nonnegative and absolutely continuous.
Also, take $\psi(t) := ||\mathbf{f}(t)||^2_{L^2(U)}$, $\phi(t) := 1 + 2\gamma$, which are both nonnegative and summable.
By Gronwall's inequality, we have that $\forall t \in [0, T]$,

$$\begin{aligned}
||\mathbf{u}_m(t)||^2_{L^2(U)} &\leq \exp\left(\int_0^t (1+2\gamma)ds\right)\left(||\mathbf{u}_m(0)||^2_{L^2(U)} + \int_0^t ||\mathbf{f}(s)||^2_{L^2(U)}ds\right) \\
&= \exp(t(1+2\gamma))\left(||\mathbf{u}_m(0)||^2_{L^2(U)} + \int_0^t ||\mathbf{f}(s)||^2_{L^2(U)}ds\right) \\
&\leq \exp(T(1+2\gamma))\left(||g||^2_{L^2(U)} + \int_0^T ||\mathbf{f}(s)||^2_{L^2(U)}ds\right) \\
&= C_1\Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big) \\
&\leq C_1\Big(||g||_{L^2(U)} + ||\mathbf{f}||_{L^2(0,T;L^2(U))}\Big)^2,
\end{aligned}$$

where we take $C_1 := \exp(T(1+2\gamma)) > 0$ that only depends on T, γ.
We thus have shown that $\forall t \in [0,T], ||\mathbf{u}_m(t)||_{L^2(U)} \leq \sqrt{C_1}\Big(||g||_{L^2(U)} + ||\mathbf{f}||_{L^2(0,T;L^2(U))}\Big)$.
Thus,

$$\sup_{0\leq t\leq T} ||\mathbf{u}_m(t)||_{L^2(U)} \leq \sqrt{C_1}\Big(||g||_{L^2(U)} + ||\mathbf{f}||_{L^2(0,T;L^2(U))}\Big),$$

which bounds the first term in what we want.

3. From step 1, we have that for a.e. $t \in [0,T]$,

$$\frac{d}{dt}||\mathbf{u}_m(t)||^2_{L^2(U)} + 2\beta||\mathbf{u}_m(t)||^2_{H^1(U)} \leq ||\mathbf{f}(t)||^2_{L^2(U)} + (1+2\gamma)||\mathbf{u}_m(t)||^2_{L^2(U)}.$$

Integrating over $[0,T]$ gives

$$\int_0^T \frac{d}{dt}||\mathbf{u}_m(t)||^2_{L^2(U)}dt + 2\beta\int_0^T ||\mathbf{u}_m(t)||^2_{H^1(U)}dt \leq \int_0^T ||\mathbf{f}(t)||^2_{L^2(U)}dt + (2\gamma+1)\int_0^T ||\mathbf{u}_m(t)||^2_{L^2(U)}dt$$

$$||\mathbf{u}_m(T)||^2_{L^2(U)} - ||\mathbf{u}_m(0)||^2_{L^2(U)} + 2\beta||\mathbf{u}_m||^2_{L^2\left(0,T;H^1_0(U)\right)} \leq ||\mathbf{f}||^2_{L^2(0,T;L^2(U))} + (2\gamma+1)\int_0^T ||\mathbf{u}_m(t)||^2_{L^2(U)}dt.$$

Notice that $||\mathbf{u}_m(T)||^2_{L^2(U)} \geq 0$, and in step 2, we have shown that for a.e. $t \in [0,T]$,

$$||\mathbf{u}_m(t)||^2_{L^2(U)} \leq C_1\Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big).$$

$$\begin{aligned}
2\beta||\mathbf{u}_m||^2_{L^2\left(0,T;H^1_0(U)\right)} &\leq ||\mathbf{f}||^2_{L^2(0,T;L^2(U))} + (2\gamma+1)\int_0^T ||\mathbf{u}_m(t)||^2_{L^2(U)}dt + ||\mathbf{u}_m(0)||^2_{L^2(U)} \\
2\beta||\mathbf{u}_m||^2_{L^2(0,T;H^1(U))} &\leq ||\mathbf{f}||^2_{L^2(0,T;L^2(U))} + (2\gamma+1)\int_0^T C_1\Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big)dt + ||g||^2_{L^2(U)} \\
&= ||\mathbf{f}||^2_{L^2(0,T;L^2(U))} + (2\gamma+1)TC_1\Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big) + ||g||^2_{L^2(U)} \\
&= ((2\gamma+1)TC_1 + 1)\Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big) \\
||\mathbf{u}_m||^2_{L^2\left(0,T;H^1_0(U)\right)} &\leq C_2\Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big) \\
&\leq C_2\Big(||g||_{L^2(U)} + ||\mathbf{f}||_{L^2(0,T;L^2(U))}\Big)^2,
\end{aligned}$$

where $C_2 := \frac{(2\gamma+1)TC_1+1}{\beta} > 0$ only depends on β, γ, T.
We thus have bounded the second term

$$||\mathbf{u}_m||_{L^2\left(0,T;H^1_0(U)\right)} \leq \sqrt{C_2}\Big(||g||_{L^2(U)} + ||\mathbf{f}||_{L^2(0,T;L^2(U))}\Big).$$

4. Now fix any function $v \in H_0^1(U)$ with $||v||_{H_0^1(U)} = 1$, write $v = \sum_{j=1}^{\infty} \hat{v}_j w_j = v_1 + v_2$, where $v_1 := \sum_{j=1}^{m} \hat{v}_j w_j \in V_m, v_2 := \sum_{j=m+1}^{\infty} \hat{v}_j w_j \in V_m^{\perp}$ is the unique decomposition of v in $H_0^1(U) = V_m \oplus V_m^{\perp}$.
Notice that $||v||^2_{H^1(U)} = ||v_1||^2_{H^1(U)} + ||v_2||^2_{H^1(U)}$.
Thus $||v_1||_{H^1(U)} \leq 1$.
Since $v_1 \in V_m$, we have for a.e. $t \in [0,T]$, $\langle \mathbf{u}'_m(t), v_1\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v_1; t] = \langle \mathbf{f}(t), v_1\rangle_{L^2(U)}$.
Now since $\mathbf{u}'_m(t) \in V_m$, we have $\langle \mathbf{u}'_m(t), v_2\rangle_{L^2(U)} = 0$, so

$$\begin{aligned}\langle \mathbf{u}'_m(t)|v\rangle_{H^{-1}(U),H_0^1(U)} &= \langle \mathbf{u}'_m(t), v\rangle_{L^2(U)}\\ &= \langle \mathbf{u}'_m(t), v_1\rangle_{L^2(U)} + \langle \mathbf{u}'_m(t), v_2\rangle_{L^2(U)}\\ &= \langle \mathbf{u}'_m(t), v_1\rangle_{L^2(U)}\\ &= \langle \mathbf{f}(t), v_1\rangle_{L^2(U)} - B[\mathbf{u}_m(t), v_1; t].\end{aligned}$$

Again by a similar proof as in 3.4, there exists constants $\alpha > 0$ that only depends on U and the coefficients of L, such that $\forall u, v \in H_0^1(U)$, and for a.e. $t \in [0,T]$,

$$|B[u, v; t]| \leq \alpha ||u||_{H^1(U)} ||v||_{H^1(U)}.$$

We thus have

$$\begin{aligned}\langle \mathbf{u}'_m(t)|v\rangle_{H^{-1}(U),H_0^1(U)} &= \langle \mathbf{f}(t), v_1\rangle_{L^2(U)} - B[\mathbf{u}_m(t), v_1; t]\\ &\leq ||\mathbf{f}(t)||_{L^2(U)} ||v_1||_{L^2(U)} + \alpha ||\mathbf{u}_m(t)||_{H^1(U)} ||v_1||_{H^1(U)}\\ &\leq ||\mathbf{f}(t)||_{L^2(U)} ||v_1||_{H^1(U)} + \alpha ||\mathbf{u}_m(t)||_{H^1(U)} ||v_1||_{H^1(U)}\\ &\leq ||\mathbf{f}(t)||_{L^2(U)} + \alpha ||\mathbf{u}_m(t)||_{H^1(U)}.\end{aligned}$$

Since this holds for any $v \in H_0^1(U)$ with $||v||_{H_0^1(U)} = 1$, we have

$$||\mathbf{u}'_m(t)||_{H^{-1}(U)} = \sup_{v \in H_0^1(U) \text{ such that } ||v||_{H_0^1(U)}=1} \langle \mathbf{u}'_m(t)|v\rangle_{H^{-1}(U),H_0^1(U)} \leq ||\mathbf{f}(t)||_{L^2(U)} + \alpha ||\mathbf{u}_m(t)||_{H^1(U)}.$$

Squaring and integrating this over $[0,T]$, we have

$$\begin{aligned}\int_0^T ||\mathbf{u}'_m(t)||^2_{H^{-1}(U)} dt &\leq \int_0^T \Big(||\mathbf{f}(t)||_{L^2(U)} + \alpha ||\mathbf{u}_m(t)||_{H^1(U)}\Big)^2 dt\\ ||\mathbf{u}'_m||^2_{L^2(0,T;H^{-1}(U))} &\leq \int_0^T 2\Big(||\mathbf{f}(t)||^2_{L^2(U)} + \alpha^2 ||\mathbf{u}_m(t)||^2_{H^1(U)}\Big) dt\\ &\leq 2\int_0^T ||\mathbf{f}(t)||^2_{L^2(U)} dt + 2\alpha^2 \int_0^T ||\mathbf{u}_m(t)||^2_{H^1(U)} dt\\ &= 2||\mathbf{f}||^2_{L^2(0,T;L^2(U))} + 2\alpha^2 ||\mathbf{u}_m||^2_{L^2\left(0,T;H_0^1(U)\right)}\\ &\leq 2||\mathbf{f}||^2_{L^2(0,T;L^2(U))} + 2\alpha^2 C_2 \Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big)\\ &\leq C_3 \Big(||g||^2_{L^2(U)} + ||\mathbf{f}||^2_{L^2(0,T;L^2(U))}\Big)\\ &\leq C_3 \Big(||g||_{L^2(U)} + ||\mathbf{f}||_{L^2(0,T;L^2(U))}\Big)^2,\end{aligned}$$

where we take $C_3 := 2\alpha^2 C_2 + 1 > 0$ that only depends on β, γ, α, T.
We thus have bounded the third term $||\mathbf{u}'_m||_{L^2(0,T;H^{-1}(U))} \leq \sqrt{C_3}\Big(||g||_{L^2(U)} + ||\mathbf{f}||_{L^2(0,T;L^2(U))}\Big)$.

Now let us take $C := \sqrt{\max\{C_1, C_2, C_3\}} > 0$, which only depends on U, T, L, we have that

$$\sup_{0 \leq t \leq T} ||\mathbf{u}_m(t)||_{L^2(U)} + ||\mathbf{u}_m||_{L^2\left(0,T;H_0^1(U)\right)} + ||\mathbf{u}'_m||_{L^2(0,T;H^{-1}(U))} \leq C\Big(||\mathbf{f}||_{L^2(0,T;L^2(U))} + ||g||_{L^2(U)}\Big).$$

$\square$

Theorem 4.27. *There is a weak solution to the IBVP, namely, $\exists \boldsymbol{u} \in L^2\big(0,T;H_0^1(H)\big)$, identified with its continuous representative $\tilde{\mathbf{u}} \in C\big([0,T];L^2(U)\big)$, such that*

$$\langle \boldsymbol{u}'(t)|v\rangle + B[\boldsymbol{u}(t),v;t] = \langle \boldsymbol{f}(t),v\rangle_{L^2(U)},\ \forall v \in H_0^1(U),\ a.e.\ t\in[0,T]$$
$$\boldsymbol{u}(0) = g$$

Proof. By energy estimate, we have that $(\mathbf{u}_m)_{m=1}^\infty$ is bounded in $L^2\big(0,T;H_0^1(U)\big)$.
By 1.29, there is a subsequence $(\mathbf{u}_{m_j})_{j=1}^\infty$ and $\mathbf{u} \in L^2\big(0,T;H_0^1(U)\big)$ such that $\mathbf{u}_{m_j} \rightharpoonup \mathbf{u}$. WLOG, we will consider its continuous representative $\mathbf{u} \in C\big(0,T;L^2(U)\big)$ by 4.16.1.
Similarly, $(\mathbf{u}_m')_{m=1}^\infty$ is bounded in $L^2\big(0,T;H^{-1}(U)\big)$, so is $(\mathbf{u}_{m_j}')_{j=1}^\infty$. Thus there is a subsequence $(\mathbf{u}_{m_{j_l}}')_{l=1}^\infty$ and $\mathbf{w} \in L^2\big(0,T;H^{-1}(U)\big)$ such that $\mathbf{u}_{m_{j_l}}' \rightharpoonup \mathbf{w}$.
Since $\mathbf{u}_{m_j} \rightharpoonup \mathbf{u}$, we must have $\mathbf{u}_{m_{j_l}} \rightharpoonup \mathbf{u}$ as well. By A5Q3, $\mathbf{w} = \mathbf{u}'$.
Now we would like to show that $\mathbf{u}$ is indeed a a weak solution to the IBVP.
Consider $\mathbf{v} \in C^1\big([0,T];H_0^1(U)\big)$ of the form $\mathbf{v}(t) = \sum_{k=1}^N d^k(t)w_k$, where $N>0$ is an integer, $(d^k(t))_{k=1}^N$ are smooth functions, and $(w_k)_{k=1}^\infty$ be a basis as before.
We can show that these $\mathbf{v}$ are dense in $L^2\big(0,T;H_0^1(U)\big)$.
For any such $\mathbf{v}$, if we choose any $m \geq N$, we have the weak solution $\vec{u}_m$ in V_m satisfies

$$\langle \mathbf{u}_m'(t)|w_k\rangle_{H^{-1}(U),H_0^1(U)} + B[\mathbf{u}_m(t),w_k;t] = \langle \mathbf{f}(t),w_k\rangle_{L^2(U)},\ \forall k\in[m],\ \text{for a.e. } t\in[0,T].$$

Multiplying by $d^k(t)$ and summing over $k \in [N]$, we have that

$$\langle \mathbf{u}_m'(t)|\mathbf{v}(t)\rangle + B[\mathbf{u}_m(t),\mathbf{v}(t);t] = \langle \mathbf{f}(t),\mathbf{v}(t)\rangle_{L^2(U)},\ \text{for a.e. } t\in[0,T].$$

Integrating over $t\in[0,T]$, we have

$$\int_0^T \langle \mathbf{u}_m'(t)|\mathbf{v}(t)\rangle_{H^{-1}(U),H_0^1(U)}dt + \int_0^T B[\mathbf{u}_m(t),\mathbf{v}(t);t]dt = \int_0^T \langle \mathbf{f}(t),\mathbf{v}(t)\rangle_{L^2(U)}dt.$$

Since $\mathbf{v} \in C^1\big([0,T];H_0^1(U)\big) \subset L^2\big(0,T;H_0^1(U)\big) \cong \big(L^2\big(0,T;H^{-1}(U)\big)\big)^*$, and $\mathbf{u}_{m_{j_l}}' \rightharpoonup \mathbf{v}$, we have

$$\int_0^T \Big\langle \mathbf{u}_{m_{j_l}}'(t)\Big|\mathbf{v}(t)\Big\rangle_{H^{-1}(U),H_0^1(U)}dt \to \int_0^T \langle \mathbf{u}'(t)|\mathbf{v}(t)\rangle_{H^{-1}(U),H_0^1(U)}dt.$$

Also, if we consider the operator $T_v : \mathbf{w} \mapsto \int_0^T B[\mathbf{w}(t),\mathbf{v}(t);t]dt$ for any $\mathbf{w} \in L^2\big(0,T;H_0^1(U)\big)$, we can see that

$$\begin{aligned}
|T_v\mathbf{w}| &= \left|\int_0^T B[\mathbf{w}_m(t),\mathbf{v}(t);t]dt\right| \\
&\leq \int_0^T |B[\mathbf{w}_m(t),\mathbf{v}(t);t]|dt \\
&\leq \int_0^T \alpha||\mathbf{w}_m(t)||_{H^1(U)}||\mathbf{v}(t)||_{H^1(U)}dt \\
&\leq \alpha\left(\int_0^T ||\mathbf{w}_m(t)||_{H^1(U)}^2\right)^{\frac{1}{2}}\left(\int_0^T ||\mathbf{v}(t)||_{H^1(U)}^2\right)^{\frac{1}{2}} \\
&= \alpha||\mathbf{w}_m||_{L^2(H^1(U))}||\mathbf{v}||_{L^2(H^1(U))}.
\end{aligned}$$

For some $\alpha>0$ that only depends on U,L.
Thus $||T_v||_{(L^2(0,T;H_0^1(U)))^*} \leq \alpha||\mathbf{v}||_{L^2(H^1(U))}$, so $T_v \in \big(L^2\big(0,T;H_0^1(U)\big)\big)^*$.
Since $\mathbf{u}_{m_{j_l}} \rightharpoonup \mathbf{u}$, we have that

$$\int_0^T B[\mathbf{u}_{m_{j_l}}(t),\mathbf{v}(t);t]dt \to \int_0^T B[\mathbf{u}(t),\mathbf{v}(t);t]dt.$$

We now have

$$\begin{aligned}
\int_0^T \langle \mathbf{f}(t), \mathbf{v}(t)\rangle_{L^2(U)} dt &= \lim_{l\to\infty} \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t)\rangle_{L^2(U)} dt \\
&= \lim_{l\to\infty} \left(\int_0^T \left\langle \mathbf{u}'_{m_{j_l}}(t) \middle| \mathbf{v}(t) \right\rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \right) \\
&= \lim_{l\to\infty} \int_0^T \left\langle \mathbf{u}'_{m_{j_l}}(t) \middle| \mathbf{v}(t) \right\rangle_{H^{-1}(U), H_0^1(U)} dt + \lim_{l\to\infty} \int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \\
&= \int_0^T \langle \mathbf{u}'(t) | \mathbf{v}(t)\rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt
\end{aligned}$$

Since such $\mathbf{v}$ are dense in $L^2\big(0,T; H_0^1(U)\big)$ and both sides of the above equality are continuous, we can extend it so that $\forall \mathbf{v} \in L^2\big(0,T; H_0^1(U)\big)$,

$$\int_0^T \langle \mathbf{f}(t), \mathbf{v}(t)\rangle_{L^2(U)} dt = \int_0^T \langle \mathbf{u}'(t) | \mathbf{v}(t)\rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt.$$

Now consider any $v \in H_0^1(U)$ and any $\phi \in C_c^\infty(0,T)$, we always have $v\phi \in C\big(0,T; H_0^1(U)\big) \subset L^2\big(0,T; H_0^1(U)\big)$. Thus we have

$$\begin{aligned}
\int_0^T \langle \mathbf{f}(t), v\phi(t)\rangle_{L^2(U)} dt &= \int_0^T \langle \mathbf{u}'(t) | v\phi(t)\rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), v\phi(t); t] dt \\
\int_0^T \phi(t) \langle \mathbf{f}(t), v\rangle_{L^2(U)} dt &= \int_0^T \phi(t) \Big(\langle \mathbf{u}'(t) | v\rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t] \Big) dt
\end{aligned}$$

Since this works for all $\phi \in C_c^\infty(0,T)$, we must have for a.e. $t \in [0,T]$,

$$\langle \mathbf{f}(t), v\rangle_{L^2(U)} = \langle \mathbf{u}'(t) | v\rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t].$$

Now consider any $\mathbf{v} \in C^1\big(0,T; H_0^1(U)\big)$ such that $\mathbf{v}(T) = 0$.
By IBP 4.16.2, we have

$$\begin{aligned}
0 &= \langle \mathbf{u}(T), \mathbf{v}(T)\rangle_{L^2(U)} \\
&= \langle \mathbf{u}(0), \mathbf{v}(0)\rangle_{L^2(U)} + \int_0^T \Big(\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau)\rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau)\rangle_{H^{-1}(U), H_0^1(U)} \Big) d\tau \\
0 &= \left\langle \mathbf{u}_{m_{j_l}}(T), \mathbf{v}(T) \right\rangle_{L^2(U)} \\
&= \left\langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \right\rangle_{L^2(U)} + \int_0^T \left(\left\langle \mathbf{u}'_{m_{j_l}}(\tau) \middle| \mathbf{v}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} + \left\langle \mathbf{v}'(\tau) \middle| \mathbf{u}_{m_{j_l}}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau.
\end{aligned}$$

Since $\mathbf{v} \in L^2\big(0,T; H_0^1(U)\big) \cong L^2\big(0,T; H^{-1}(U)\big)^*$ and $\mathbf{u}'_{m_{j_l}} \rightharpoonup \mathbf{u}'$ in $L^2\big(0,T; H^{-1}(U)\big)$, we have

$$\lim_{l\to\infty} \int_0^T \left\langle \mathbf{u}'_{m_{j_l}}(\tau) \middle| \mathbf{v}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} d\tau = \int_0^T \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau)\rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Similarly,

$$\lim_{l\to\infty} \int_0^T \left\langle \mathbf{v}'(\tau) \middle| \mathbf{u}_{m_{j_l}}(\tau) \right\rangle_{H^{-1}(U), H_0^1(U)} d\tau = \int_0^T \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau)\rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Also, since $\lim_{m\to\infty} \mathbf{u}_m(0) = g$ in $L^2(U)$, we have that the subsequence $\lim_{l\to\infty} \mathbf{u}_{m_{j_l}}(0) = g$ as well.
Thus

$$\lim_{l\to\infty} \left\langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \right\rangle_{L^2(U)} = \langle \mathbf{u}(T), \mathbf{v}(T)\rangle_{L^2(U)}.$$

Thus we have

$$
\begin{aligned}
0 &= \langle \mathbf{u}(0), \mathbf{v}(0)\rangle_{L^2(U)} + \int_0^T \Big(\langle \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} + \langle \mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} \Big) d\tau \\
&= \lim_{l\to\infty} \left(\Big\langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \Big\rangle_{L^2(U)} + \int_0^T \left(\Big\langle \mathbf{u}'_{m_{j_l}}(\tau) \Big| \mathbf{v}(\tau) \Big\rangle_{H^{-1}(U),H_0^1(U)} + \Big\langle \mathbf{v}'(\tau) \Big| \mathbf{u}_{m_{j_l}}(\tau) \Big\rangle_{H^{-1}(U),H_0^1(U)} \right) d\tau \right) \\
&= \langle g, \mathbf{v}(0)\rangle_{L^2(U)} + \int_0^T \Big(\langle \mathbf{u}'(\tau)|\mathbf{v}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} + \langle \mathbf{v}'(\tau)|\mathbf{u}(\tau)\rangle_{H^{-1}(U),H_0^1(U)} \Big) d\tau.
\end{aligned}
$$

We thus have $\langle \mathbf{u}(0), \mathbf{v}(0)\rangle_{L^2(U)} = \langle g, \mathbf{v}(0)\rangle_{L^2(U)}$ for any such $\mathbf{v}$.
Notice that for any $v \in H_0^1(U)$, we can simply let $\mathbf{v}(t) := \frac{T-t}{T}v$, which satisfies the requirement, and $\mathbf{v}(0) = v$.
Thus $\langle \mathbf{u}(0), v\rangle_{L^2(U)} = \langle g, v\rangle_{L^2(U)}$ for any $v \in H_0^1(U)$.
Since $H_0^1(U)$ is dense in $L^2(U)$, we have that $\langle \mathbf{u}(0), v\rangle_{L^2(U)} = \langle g, v\rangle_{L^2(U)}$ for any $v \in L^2(U)$.
This proves

$$\mathbf{u}(0) = g.$$

□

Theorem 4.28. *A weak solution to our IBVP is unique.*

Proof. Assume $\mathbf{u}_1, \mathbf{u}_2$ are both weak solutions to our IBVP.
Then $\forall v \in H_0^1(U)$, for a.e. $t \in [0,T]$,

$$\langle \mathbf{u}_1'(t)|v\rangle + B[\mathbf{u}_1, v; t] = \langle \mathbf{u}_2'(t)|v\rangle + B[\mathbf{u}_2, v; t] = \langle \mathbf{f}(t), v\rangle_{L^2(U)},$$

and

$$\mathbf{u}_1(0) = g = \mathbf{u}_2(0).$$

Let $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$, we have that

$$\langle \mathbf{u}'(t)|v\rangle + B[\mathbf{u}, v; t] = 0,\ \forall v \in H_0^1(U), \text{for a.e. } t \in [0,T],$$

and

$$\mathbf{u}(0) = 0.$$

Choosing $v = \mathbf{u}(t) \in H_0^1(U)$, we have that

$$\langle \mathbf{u}'(t)|\mathbf{u}(t)\rangle + B[\mathbf{u}(t), \mathbf{u}(t); t] = 0.$$

By 4.16.3, we have that for a.e. $t \in [0,T]$,

$$
\begin{aligned}
\frac{d}{dt}||\mathbf{u}(t)||^2_{L^2(U)} &= 2\langle \mathbf{u}'(t)|\mathbf{u}(t)\rangle \\
&= -2B[\mathbf{u}(t), \mathbf{u}(t); t] \\
&\leq 2\gamma||\mathbf{u}(t)||^2_{L^2(U)} - 2\beta||\mathbf{u}(t)||^2_{H^1(U)} \\
&\leq 2\gamma||\mathbf{u}(t)||^2_{L^2(U)},
\end{aligned}
$$

where $\gamma \geq 0, \beta > 0$ are constants similar in 3.4.
Take $\eta(t) := ||\mathbf{u}(t)||^2_{L^2(U)}$, by Gronwall's inequality, we have that $\forall t \in [0,T]$,

$$||\mathbf{u}(t)||^2_{L^2(U)} \leq \exp(2t)||\mathbf{u}(0)||^2_{L^2(U)} = 0.$$

Thus $\mathbf{u}(t) = 0$ and so $\mathbf{u}_1 = \mathbf{u}_2$ is unique. □