

Pmath753 Functional Analysis

Roger Gu

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1 Metric Spaces and Complete Spaces

Definition 1.1. A **metric space** is a set X that has a (distance) metric:

$$\begin{aligned} d(\cdot, \cdot) : X \times X &\rightarrow \mathbb{R}, \text{ such that } \forall x, y, z \in X \\ d(x, x) &= 0 \\ \forall x \neq y, d(x, y) &> 0 \\ d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

Definition 1.2. Given a metric space (X, d) , a sequence $(x_n)_{n=1}^\infty$ in X has a **limit point** $x \in X$ if $\lim_{n \rightarrow \infty} d(x, x_n) = 0$. In this case, we say $(x_n)_{n=1}^\infty$ is a **convergent sequence**, and write $x = \lim_{n \rightarrow \infty} x_n$.

Definition 1.3. A sequence $(x_n)_{n=1}^\infty$ is a **Cauchy sequence** in a metric space (X, d) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m, n \in \mathbb{N}, m, n > N \implies d(x_m, x_n) < \epsilon.$$

Definition 1.4. A metric space X is **complete** if every Cauchy sequence $(x_i)_{i=1}^\infty$ converges to a limit point in X . i.e. $\exists x \in X, \lim_{i \rightarrow \infty} x_i = x$.

Proposition 1.1. Let (X, d) be a metric space; then every convergent sequence is Cauchy.

Proposition 1.2. Let (X, d) be a metric space. Suppose $(x_n)_{n=1}^\infty$ is a Cauchy sequence and has a convergent subsequence such that $\lim_{k \rightarrow \infty} x_{n_k} = x \in X$, then $\lim_{n \rightarrow \infty} x_n = x$.

2 Topology

See more in the notes of Pmath367 Topology by Professor S. New.

2.1 Topological Spaces

Definition 2.1. Let $X \neq \emptyset$ be a set. A **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X) := 2^X =$ power set of X , satisfying

1. $\emptyset, X \in \mathcal{T}$,
2. \mathcal{T} is closed under arbitrary union; namely, $\forall \{A_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}, \bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$, and
3. \mathcal{T} is closed under finite intersection; namely, $\forall \{A_i\}_{i=1}^n \subseteq \mathcal{T}, \bigcap_{i=1}^n A_i \in \mathcal{T}$.

Also, (X, \mathcal{T}) is a **topological space** if \mathcal{T} is a topology on X .

Definition 2.2. For any $S \subseteq \mathcal{P}(X)$, we define the **topology generated by S** to be

$$\mathcal{T}_S := \langle S \rangle := \{\emptyset, X, \text{ all unions of finite intersections of elements of } S\},$$

which is the intersection of all topologies on X that contains S , and it is the smallest topology on X containing S .

Proposition 2.1. Let (X, d) be a metric space, then there is a **metric topology** \mathcal{T}_d that is generated by open balls.

Definition 2.3. (X, \leq) is a **partially ordered set (poset)** if \leq is

1. anti-symmetric: $\forall x, y \in X$, if $x \leq y$ and $y \leq x$, we have $x = y$,
2. reflexive: $\forall x \in X, x \leq x$, and

3. transitive: $\forall x, y, z \in X$, if $x \leq y, y \leq z$, we have $x \leq z$.

We can define $\geq, <, >$ by

$$\begin{aligned} x \geq y &\iff y \leq x \\ x < y &\iff x \leq y \wedge x \neq y \\ x > y &\iff y < x. \end{aligned}$$

Definition 2.4. (X, \leq) is a **totally ordered set** if it is a partially ordered set such that $\forall x, y \in X$, $x \leq y$ or $y \leq x$.

Proposition 2.2. (X, \leq) is a totally ordered set if and only if $<$ satisfies

1. $\forall x, y \in X$, exactly one of the following is true: $x < y$, $x = y$, $y < x$.
2. $\forall x, y, z \in X$, if $x < y, y < z$, we have $x < z$.

Definition 2.5. Let (X, \leq) be a totally ordered set, we can define for each $a, b \in X$,

1. $(-\infty, a) := \{x \in X : x < a\}$,
2. $(a, \infty) := \{x \in X : a < x\}$, and
3. $(a, b) := (a, \infty) \cap (-\infty, b)$.

Proposition 2.3. Let \mathcal{T}_{\leq} be the topology generated by all the sets above, then \mathcal{T}_{\leq} is a topology.

Definition 2.6. Let (X, \mathcal{T}) be a topological space, then we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$. We say $E \subseteq X$ is **closed** if $E^c \in \mathcal{T}$ is open.

Definition 2.7. For $E \subseteq X$, the **closure** of E is

$$\bar{E} = \bigcap_{F \supseteq E: F \text{ is closed}} F,$$

which is the smallest closed set containing E .

Definition 2.8. For $E \subseteq X$, the **interior** of E is

$$E^o = \bigcup_{U \subseteq E: U \text{ is open}} U,$$

which is the largest open set contained in E .

Proposition 2.4. Closed sets are stable under finite unions and arbitrary intersections.

Proposition 2.5. For any set A ,

$$\bar{A} = ((A^c)^o)^c.$$

Proposition 2.6. For any set A ,

$$x \in \bar{A} \iff (\forall U \text{ open}, x \in U \implies U \cap A \neq \emptyset).$$

Definition 2.9. Let (X, \mathcal{T}) be a topological space, a $\mathcal{B} = \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{P}(X)$ is said to be a **basis/base** of the topology \mathcal{T} if it is a collection of open sets, and for every $U \in \mathcal{T}$, we have $U = \bigcup_{\alpha \in J} U_\alpha$ for some $J \subseteq I$.

Proposition 2.7. Let (X, \mathcal{T}) be a topological space, $\mathcal{B} \subseteq \mathcal{T}$ is a basis of \mathcal{T} if and only if

$$\forall x \in U \in \mathcal{T}, \exists U_\alpha \in \mathcal{B} \text{ such that } x \in U_\alpha \subseteq U.$$

Proposition 2.8. Let (X, \mathcal{T}) be a topological space, $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis of \mathcal{T} , if and only if

1. $X = \bigcup_{\alpha \in I} U_\alpha$,
2. For any $U_1, U_2 \in \mathcal{B}$, $x \in U_1 \cap U_2$, we have $\exists U_x \in \mathcal{B}$, $x \in U_x \subseteq U_1 \cap U_2$,
3. $\mathcal{T} = \langle \mathcal{B} \rangle$ is the topology generated by \mathcal{B} .

Example 2.1.1. Let (X, d) be a metric space, then $\{B(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\}$ is a base.

Definition 2.10. Let (X, \mathcal{T}) be a topological space, a **subbase** is a collection of open sets $S \subseteq \mathcal{T}$ such that

$$\left\{ X, \bigcap_{i=1}^n S_i : n \in \mathbb{N}^+, S_1, \dots, S_n \in S \right\}$$

forms a base for \mathcal{T} .

Proposition 2.9. Let (X, \mathcal{T}) be a topological space, then any subbase S generates \mathcal{T} . Also, any set S such that $\bigcup S = X$ is always a subbase for the topology \mathcal{T}_S generated by S .

Definition 2.11. Let (X, \mathcal{T}) be a topological space. Given $x \in X$, a **neighbourhood** of x is a set $V \ni x$, such that $\exists U \in \mathcal{T}$ with $x \in U \subseteq V$.

Definition 2.12. Let (X, \mathcal{T}) be a topological space. Given $x \in X$, a **neighbourhood basis** of x is a set of open neighbourhoods $\mathcal{B}_x \subset \mathcal{T}$, such that for any (open) neighbourhood U of x , there is $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$.

Proposition 2.10. Let (X, \mathcal{T}) be a topological space, $\mathcal{B} \subseteq \mathcal{T}$ is a basis of \mathcal{T} if and only if $\forall x \in X$, \mathcal{B} is a neighbourhood basis of x .

Definition 2.13. We say $S \subseteq X$ is **dense** in a topological space (X, \mathcal{T}) if $\forall \text{open } U \neq \emptyset$, $S \cap U \neq \emptyset$.

Proposition 2.11. S is dense if and only if $\bar{S} = X$.

Definition 2.14. A topological space (X, \mathcal{T}) is **separable** if there is a countable subset.

Definition 2.15. A topological space (X, \mathcal{T}) is **first countable** if $\forall x \in X$, there is a countable open neighbourhood basis $\{B_n\}_{n=1}^\infty \subset \mathcal{T}$ at x . Namely, for any neighbourhood U of x , there is $n \in \mathbb{N}$ such that $x \in B_n \subseteq U$.

Definition 2.16. A topological space (X, \mathcal{T}) is called **2nd countable** if it has a countable basis.

Proposition 2.12. Every metric space (X, d) are first countable.

Proposition 2.13. Every metric space (X, d) is 2nd countable if and only if X is countable.

Proposition 2.14. The discrete topology of X is separable if and only if $|X|$ is at most countable.

Definition 2.17 (Axiom of Choice). If $X \neq \emptyset$, then there is a choice function $C : P(X) \setminus \{\emptyset\} \rightarrow X$ such that $\forall A \subseteq X$, if $A \neq \emptyset$, we have $C(A) \in A$.

Proposition 2.15 (Axiom of Choice Equivalence). The Axiom of Choice is equivalent to: Let $\{X_\alpha\}_{\alpha \in A}$ be a family of non-empty sets, then

$$\prod_{\alpha \in A} X_\alpha := \left\{ f : A \rightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha) \in X_\alpha \right\} \neq \emptyset.$$

Proof. Suppose AOC holds, then taking $X = \bigcup_{\alpha \in A} X_\alpha$, we have the choice function C . Now take $f(a) := C(X_a)$.

On the other hand, suppose the latter holds, then consider $\prod_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$, which is non-empty. Consider any $f \in \prod_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$, then $C(X_\alpha) := f(a)$ is a choice function. \square

Proposition 2.16. A metric space (X, d) is separable if and only if it is 2nd countable.

Proof. If $S = \{x_k\}_{k=1}^\infty$ is dense, then $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a countable base. Indeed, consider any $x \in X$ with any open $U \ni x$, we know $\exists r > 0$, such that $x \in B(x, r) \subset U$. Also, there is x_k such that $d(x, x_k) < \frac{r}{2}$.

Now choose some $r' \in \mathbb{Q}$ such that $d(x, x_k) < r' < \frac{r}{2}$, then $x \in B(x_k, r') \subset B(x, r) \subset U$.

Thus $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a base.

On the other hand, suppose X is second countable with a countable base $\{U_n\}_{n=1}^\infty$. WLOG, $U_n \neq \emptyset$.

Now for any $n \in \mathbb{N}$, pick $x_n \in U_n$ by the axiom of countable choice. Let $S = \{x_n\}_{n=1}^\infty$, then we claim S is dense.

Indeed, for any open $U \neq \emptyset$, we can find some $U_n \subset U$. Thus $x_n \in S \cap U$. \square

Proposition 2.17. *If $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ is a set of topologies on X ,*

1. *There is a weakest topology $\tau := \langle \bigcup_{\alpha \in A} \mathcal{T}_\alpha \rangle$ that is stronger than each \mathcal{T}_α .*
2. *There is a strongest topology $\delta := \bigcap_{\alpha \in A} \mathcal{T}_\alpha$ that is weaker than each \mathcal{T}_α .*

Definition 2.18. A topological space (X, \mathcal{S}) is **Hausdorff** if

$$\forall x \neq y \in X, \exists S_x, S_y \in \mathcal{S}, \text{ such that } x \in S_x, y \in S_y, S_x \cap S_y = \emptyset.$$

Proposition 2.18. *Any space with its discrete topology is always Hausdorff.*

Example 2.1.2. Every metric space is Hausdorff.

Proposition 2.19. *Any space with more than one element with the trivial topology is not Hausdorff.*

Example 2.1.3. Consider $X := (0, 1) \cup \{1^+, 1^-\}$. Let $(0, 1)$ have the usual open topology. Also, let $(r, 1) \cup \{1^+\}$ and $(r, 1) \cup \{1^-\}$ be open for any $0 < r < 1$. The topology generated by this basis will not be Hausdorff.

Indeed, consider $1^+, 1^-$, then for any $U \ni 1^+, V \ni 1^-$, we can find $r_U, r_V \in (0, 1)$, such that $(r_U, 1) \cup \{1^+\} \subseteq U, (r_V, 1) \cup \{1^-\} \subseteq V$. Yet $(\max(r_U, r_V), 1) \subseteq U \cap V$, which is not empty.

Proposition 2.20. *If X is Hausdorff, then for any $x \in X$, we have that $\{x\}$ is closed.*

Proof. For any $y \neq x$, we can find open $V_y \ni y, U_y \ni x$, such that $V_y \cap \{x\} \subseteq V_y \cap U_y = \emptyset$. Thus $X \setminus \{x\} = \bigcup_{y \in X} V_y$, which is open. \square

2.2 Continuous Functions

Definition 2.19. A function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ is **continuous** if

$$\forall U \in \mathcal{S}, f^{-1}(U) \in \mathcal{T}.$$

Namely, the preimage of any open set is open.

Definition 2.20. A function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ is **continuous at $x \in X$** if

$$\forall V \in \mathcal{S}, \text{ such that } f(x) \in V \exists U \in \mathcal{T}, \text{ such that } x \in U \subseteq f^{-1}(V).$$

Proposition 2.21. *Let $f : X \rightarrow Y$ be a map between topological spaces. Then f is continuous (on X) if and only if f is continuous at every point $x \in X$.*

Proof. (\implies):

Assume f is continuous, then for every point $x \in X$ and open $V \ni f(x)$, we have $f^{-1}(V)$ is open. Clearly $x \in f^{-1}(V)$.

(\impliedby):

Assume f is continuous at every point x . Given any open $V \in \mathcal{S}$, and any point $x \in f^{-1}(V)$, we have $f(x) \in V$.

By assumption, there is open U_x , such that $x \in U_x \subseteq f^{-1}(V)$.

Now $\bigcup_{x \in f^{-1}(V)} U_x$ is open, while $\bigcup_{x \in f^{-1}(V)} U_x \supseteq \bigcup_{x \in f^{-1}(V)} \{x\} = f^{-1}(V) \supseteq \bigcup_{x \in f^{-1}(V)} U_x$.

Thus $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is open. \square

Definition 2.21. A function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ is **open** if

$$\forall V \in \mathcal{T}, f(V) \in \mathcal{S}.$$

Namely, the image of any open set is open.

Definition 2.22. Given two sets X, Y , and their corresponding topology \mathcal{T}, \mathcal{S} , a continuous map $f : X \rightarrow Y$ is a **homeomorphism** if it is bijective, and its inverse function is also continuous.

Remark. A homeomorphism is a map that preserves the topological structure between two sets.

Definition 2.23. Let $C(X)$ be the collection of functions $f : X \rightarrow \mathbb{C}$ that are continuous.

Definition 2.24. Let $C(X, \mathbb{R})$ be the collection of functions $f : X \rightarrow \mathbb{R}$ that are continuous.

Definition 2.25. Let $C_b(X)$ be the collection of functions $f \in C(X) : \|f\|_\infty < \infty$.

Definition 2.26. Let $C_b(X, \mathbb{R})$ be the collection of functions $f \in C(X, \mathbb{R}) : \|f\|_\infty < \infty$.

Proposition 2.22. If $C(X)$ separates points, so do $C_b(X), C_b(X, \mathbb{R})$. Also, X is Hausdorff.

Proof. If $x \neq y$, we have $f \in C(X)$ such that $f(x) \neq f(y)$.

WLOG, $\Re(f(x)) \neq \Re(f(y))$, and $\Re(f(x)) < \Re(f(y))$.

Now define $g(z) := \min \{ \Re(f(y)), \max \{ \Re(f(x)), \Re(f(y)) \} \}$, which is bounded and continuous. Also $g(x) = f(x), g(y) = f(y)$.

Thus $C_b(X, \mathbb{R})$ separates the points.

Now if $x \neq y$, we can find $f \in C(X)$, such that $|f(x) - f(y)| = r > 0$.

Now let $U := f^{-1}(B(f(x), \frac{r}{2})), V := f^{-1}(B(f(y), \frac{r}{2}))$, which are both open. Also, $x \in U, y \in V$, and $U \cap V = f^{-1}(\emptyset) = \emptyset$. \square

Definition 2.27. A topological space (X, \mathcal{T}) is **normal** if for any disjoint closed sets A, B , we can find open $U \supset A, V \supset B$ such that $U \cap V = \emptyset$.

Theorem 2.23 (Urysohn's Lemma for normal spaces). (X, \mathcal{T}) is normal if and only if for any disjoint closed sets A, B , $\exists f : X \rightarrow [0, 1]$ continuous, such that $f|_A = 0, f|_B = 1$.

Corollary 2.24. If X is normal and Hausdorff, $C_b(X)$ separates the points.

2.3 Nets

Definition 2.28. Let (X, \mathcal{T}) be a topological space, say a sequence $(x_i)_{i=1}^\infty$ **converges to** $x \in X$ if \forall open $U \ni x$, $\exists N \in \mathbb{N}$ such that $\forall i \geq N, x_i \in U$.

Example 2.3.1. Consider $X = \mathbb{N} \times \mathbb{N}$, and the projection $\pi_1 : X \rightarrow \mathbb{N}$ by $\pi_1(m, n) := m$.

Let any U be open if $(0, 0) \notin U$, or if $\{m \in \mathbb{N} : \pi_1^{-1}(m) \cap U \text{ is co-finite in } \{m\} \times \mathbb{N}\}$ is co-finite.

One can show this defines a topology on X , and it is Hausdorff.

Indeed, let $X_0 := X \setminus \{(0, 0)\}$. Consider any $(m, n) \neq (m', n') \in X$. If both are in X_0 then $\{(m, n)\}, \{(m', n')\}$ are open and disjoint.

If $(m', n') = (0, 0)$, then $\{(m, n)\}, X \setminus \{(m, n)\}$ are open and disjoint.

This shows Hausdorff.

Also, $\bar{X}_0 = X$.

Indeed, consider any open $U \ni (0, 0)$, we must have that $U \cap X_0 \neq \emptyset$.

However, there is no sequence in X_0 that converges to $(0, 0)$.

Indeed, assume for contradiction that there is such a convergent sequence $(x_k)_{k=1}^\infty$ in X_0 .

Write each $x_k = (m^k, n^k)$.

Suppose there is some $M \in \mathbb{N}^+$, such that $\forall k \in \mathbb{N}^+, m_k \leq M$.

Consider $U := \{(m, n) : m > M, n \in \mathbb{N}\} \cup \{(0, 0)\}$.

Now for each $m \in \mathbb{N}$,

$$\pi_1^{-1}(m) \cap U = \begin{cases} (0, 0) & \text{if } m = 0 \\ \emptyset & \text{if } 0 < m \leq M \\ \{m\} \times \mathbb{N} & \text{if } m > M. \end{cases}$$

Thus for all $m > M$, $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.

Yet $U \cap \{x_k\}_{k=1}^\infty = \emptyset$.

Now suppose there is no such M , then we can find a subsequence $m_{k_1} < m_{k_2} < m_{k_3} < \dots$.

Now let $U := X \setminus \{x_{k_i}\}_{i=1}^\infty$.

For each $m \in \mathbb{N}$,

$$\pi^{-1}(m) \cap U = \begin{cases} \{m\} \times \mathbb{N} & \text{if } m \notin \{m_{k_i}\}_{i=1}^\infty \\ \{m\} \times (\mathbb{N} \setminus n_{k_i}) & \text{if } m \in \{m_{k_i}\}_{i=1}^\infty. \end{cases}$$

Notice that there cannot be two $n_{k_i} \neq n_{k_j}$ for any m , since $k_i \neq k_j \implies m_{k_i} \neq m_{k_j}$.

Thus all $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.

Yet $U \cap \{x_{k_i}\}_{i=1}^\infty = \emptyset$.

Thus there cannot be any convergent sequence $(x_k)_{k=1}^\infty$ in X_0 .

Remark. The above example shows that sequences do not behave as we want in topological spaces.

Definition 2.29. An **upwards directed set** is a poset (Λ, \leq) such that if $\lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_0 \in \Lambda$ such that $\lambda_1 \leq \lambda_0, \lambda_2 \leq \lambda_0$.

Definition 2.30. For $X \neq \emptyset$, a **net** in X is a function $j : \Lambda \rightarrow X$, where (Λ, \leq) is an upwards directed set. Write $x_\lambda := j(\lambda) \in X$, and we can use $(x_\lambda)_{\lambda \in \Lambda}$ to represent a net.

Definition 2.31. Let (X, \mathcal{T}) be a topological space, say a net $(x_\lambda)_{\lambda \in \Lambda}$ **converges to** $x \in X$ if

$$\forall \text{open } U \ni x, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U.$$

In this case, we say x is a **limit** of the net, and write it as $x = \lim_{\lambda \in \Lambda} x_\lambda$ or $x_\lambda \rightarrow x$.

Proposition 2.25. Let (X, \mathcal{T}) be a topological space with a neighbourhood basis \mathcal{B} at $x \in X$, then a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to x if and only if

$$\forall U \in \mathcal{B} \text{ such that } x \in U, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U.$$

Proof. The forward direction is trivial.

Now assume $\forall U \in \mathcal{B}$ such that $x \in U$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0, x_\lambda \in U$.

Given any open $V \ni x$, since \mathcal{B} is a neighbourhood basis, there is some $U \in \mathcal{B}$, such that $x \in U \subseteq V$.

Thus, there is $\lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0, x_\lambda \in U \subseteq V$. □

Definition 2.32. Given a net $(x_\lambda)_{\lambda \in \Lambda}$, then a **subnet** of it $(y_\gamma)_{\gamma \in \Gamma}$ is given by an upwards directed set (Γ, \leq) and a function $\phi : \Gamma \rightarrow \Lambda$ that is **cofinal**, which means $\forall \lambda_0 \in \Lambda, \exists \gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0, \phi(\gamma) \geq \lambda_0$. Each y_γ is given by $x_{\phi(\gamma)}$.

Example 2.3.2. Notice that if we take (\mathbb{N}, \leq) , the net is just a sequence. To get a subsequence, we can take $\Gamma = \mathbb{N}$, and $\phi : \mathbb{N} \rightarrow \mathbb{N}$ to be any increasing function. The generated subnet will be a subsequence.

Definition 2.33. Let (X, \mathcal{T}) be a topological space, and $x \in X$, define the **system of open neighbourhoods of x** to be $\mathcal{O}(x) := \{U \in \mathcal{T} : x \in U\}$.

Proposition 2.26. $(\mathcal{O}(x), \supseteq)$ is an upwards directed set.

Example 2.3.3. For $X = \mathbb{N} \times \mathbb{N}$ and $X_0 = X \setminus \{(0, 0)\}$ as above, there is a net in X_0 converging to $(0, 0)$.

Indeed, let us enumerate $X_0 = \{x_k\}_{k=1}^\infty$ as $(0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \dots$

Now $\Lambda := \mathcal{O}((0, 0))$ is an upward directed set by containment.

Then for each $U \in \Lambda$, we can pick $x_U := x_{k_U}$, where k_U is the first $k \in \mathbb{N}$ such that $x_k \in U$.

Claim: $(x_U)_{U \in \Lambda}$ converges to $(0, 0)$.

Pick any $U_0 \ni (0, 0)$, then for all $U \geq U_0$, it is open and $U \subseteq U_0$. Thus, we must have $x_U \in U \subseteq U_0$.

Indeed, $(x_U)_{U \in \Lambda}$ is a subnet of $\{x_k\}_{k \in \mathbb{N}^+}$ by $\phi(U) := k_U$.

Theorem 2.27. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces, then

1. For any $A \subseteq X$, we have $x \in \bar{A}$ if and only if \exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in A , such that $x_\lambda \rightarrow x$.

2. $f : X \rightarrow Y$ is continuous if and only if for any net $(x_\lambda)_{\lambda \in \Lambda} \subset X$ such that $x_\lambda \rightarrow x \in X$, we have $f(x_\lambda) \rightarrow f(x)$.

Proof. 1. Consider any $x \in \bar{A}$, then for any open $U \ni x$, we have $U \cap A \neq \emptyset$.

By the Axiom of Choice, we can have $x_U \in U \cap A$ for each open neighbourhood U .

Consider the net $(x_U)_{U \in \mathcal{O}(x)}$.

Given any open $U \ni x$, if $V \geq U$, we must have $V \subseteq U$, and $x_v \in V \subseteq U$.

Thus $x_U \rightarrow x$.

On the other hand, for any net $x_\lambda \rightarrow x$ in A , consider any open $U \ni x$, there is λ_0 , such that

$$\forall \lambda \in \Lambda, \lambda_0 \leq \lambda \implies x_\lambda \in U \implies U \cap A \neq \emptyset.$$

Thus $x \in \bar{A}$.

2. Assume f is continuous, and $x_\lambda \rightarrow x$. Let $V \in \mathcal{O}(f(x))$, then $U := f^{-1}(V)$ is open and $x \in U$.

Thus there is $\lambda_0 \in \Lambda$, such that $\forall \lambda \geq \lambda_0, x_\lambda \in U$.

Thus $f(x_\lambda) \in V$.

On the other hand, assume for any net $(x_\lambda)_{\lambda \in \Lambda} \subset X$ such that $x_\lambda \rightarrow x \in X$, we have $f(x_\lambda) \rightarrow f(x)$.

For contradiction, suppose there is open $V \in Y$, with $U := f^{-1}(V)$ is not open in X .

Then U^c is not closed, and $U^c \neq \overline{U^c}$.

Thus, there is $x \in \overline{U^c} \setminus U^c = \overline{U^c} \cap U$.

Since $x \in \overline{U^c}$, by 1., there is a net $(x_\lambda)_{\lambda \in \Lambda}$ in U^c , such that $x_\lambda \rightarrow x$.

By assumption, we have $f(x_\lambda) \rightarrow f(x)$.

Since each $f(x_\lambda)$ is in $f(U^c) = V^c$, by 1., we have that $f(x) \in \overline{V^c} = V^c$ since V^c is closed (V is open).

However, since $x \in U$, we also have $f(x) \in f(U) = V$, which is a contradiction. \square

2.4 Compactness

Definition 2.34. Let (X, \mathcal{T}) be a topological space. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$ is called a **cover** for X if $X = \bigcup_{\alpha \in A} U_\alpha$. A cover is called an open cover if every U_α is open in \mathcal{T} .

Definition 2.35. Let (X, \mathcal{T}) be a topological space. A collection $\{C_\alpha\}_{\alpha \in A}$ of non-empty closed sets is a FIP-family if for any finite $F \subseteq A$, $\bigcap_{\alpha \in F} C_\alpha \neq \emptyset$. X has the finite intersection property (FIP) if for all FIP-familie $\{C_\alpha\}_{\alpha \in A}$, we have $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$.

Definition 2.36. Let (X, \mathcal{T}) be a topological space. X is **compact** if every open cover of X has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open, } X = \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } X = \bigcup_{i=1}^n U_{\alpha_i}.$$

Definition 2.37. Let (X, \mathcal{T}) be a topological space. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$ is called a **cover** for $K \subseteq X$ in X if $K = \bigcup_{\alpha \in A} U_\alpha$. A cover in X is called an open cover in X if every U_α is open in \mathcal{T} .

Definition 2.38. Let (X, \mathcal{T}) be a topological space. A set $K \subseteq X$ is **compact in X** if every open cover of K in X has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open, } K \subseteq \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } K \subseteq \bigcup_{i=1}^n U_{\alpha_i}.$$

Proposition 2.28. Let (X, \mathcal{T}) be a topological space. A set $K \subseteq X$ is compact under the subspace topology if and only if it is compact in X .

Theorem 2.29. Let (X, \mathcal{T}) be a topological space, TFAE:

1. X is compact.

2. X has the finite intersection property.
3. For all nets $(x_\lambda)_{\lambda \in \Lambda}$ in X , there is a convergent subnet.

Proof. (1) \implies (2).

For contradiction, suppose there is some FIP-family such that $\bigcap_{\alpha \in A} C_\alpha = \emptyset$.

We have $X = \bigcup_{\alpha \in A} C_\alpha^c$, which is an open cover for X .

Since X is compact, there is a finite $F \subseteq A$, such that $X = \bigcup_{\alpha \in F} C_\alpha^c$.

Thus $\bigcap_{\alpha \in A} C_\alpha = \emptyset$, which contradict $\{C_\alpha\}_{\alpha \in A}$ being a FIP family.

(2) \implies (1).

Consider any open cover $X = \bigcup_{\alpha \in A} U_\alpha$, then $\bigcap_{\alpha \in A} U_\alpha^c = \emptyset$, and it is not a FIP-family.

Thus there is a finite $F \subseteq A$, such that $\bigcap_{\alpha \in F} U_\alpha^c = \emptyset$.

Thus $X = \bigcup_{\alpha \in F} U_\alpha$ is a finite open cover.

(2) \implies (3).

Let $(x_\lambda)_{\lambda \in \Lambda}$ be any net in X .

Define $C_\lambda := \{x_\mu : \mu \geq \lambda\}$. Notice that $C_\lambda \neq \emptyset$ since $x_\lambda \in C_\lambda$.

We claim that $\{C_\lambda\}_{\lambda \in \Lambda}$ is a FIP family.

The closeness is by definition.

Now fix any $\lambda_1, \dots, \lambda_n \in \Lambda$.

Since Λ is upwards directed, there is $\lambda_0 \in \Lambda$, such that $\forall i \in [n], \lambda_i \leq \lambda_0$.

Thus $\bigcap_{i=1}^n C_{\lambda_i} \supseteq C_{\lambda_0} \neq \emptyset$.

By FIP, $\bigcap_{\lambda \in \Lambda} C_{\lambda_i} \neq \emptyset$.

Pick any $x \in \bigcap_{\lambda \in \Lambda} C_{\lambda_i}$.

Let $\Gamma := \Lambda \times \mathcal{O}(x)$ with the partial order $(\lambda, U) \leq (\lambda', U')$ if $\lambda \leq \lambda'$ and $U \supseteq U'$.

Fix $(\lambda, U) \in \Gamma$, we know that $x \in C_\lambda = \{x_\mu : \mu \geq \lambda\}$.

Thus $U \cap \{x_\mu : \mu \geq \lambda\} \neq \emptyset$.

By the Axiom of Choice, there is $y_{(\lambda, U)} = x_{\phi(\lambda, U)} \in U \cap \{x_\mu : \mu \geq \lambda\}$, where $\phi(\lambda, U) := C(\{\mu \geq \lambda : x_\mu \in U\})$.

For any $\lambda_0 \in \Lambda$, let $\gamma_0 = (\lambda_0, X)$, then for any $\gamma = (\lambda, U) \geq \gamma_0$, we have $\phi(\gamma) \geq \lambda \geq \lambda_0$.

Thus ϕ is cofinal, and $(y_\gamma)_{\gamma \in \Gamma}$ is a subnet.

In addition, given any $U_0 \in \mathcal{O}(x)$, we can pick any $\lambda_0 \in \Lambda$, and let $\gamma_0 := (\lambda_0, U_0)$.

Then for any $(\lambda, U) \geq \gamma_0$, we must have $y_{(\lambda, U)} = x_{\phi(\lambda, U)} \in U \subseteq U_0$.

(3) \implies (2).

Fix any FIP-family $\{C_\alpha\}_{\alpha \in A}$ in X . Then for any finite $F \subseteq A$, by the Axiom of Choice, we can find $x_F \in \bigcap_{\alpha \in F} C_\alpha$.

Now consider the net $(x_F)_{\text{finite } F \subseteq A}$, where $F_1 \leq F_2$ if $F_1 \subseteq F_2$.

By 3., there is a convergent subnet $\phi : \Gamma \rightarrow \Lambda$, such that $x_{\phi(\gamma)} \rightarrow x \in X$.

Now fix any $\alpha \in A$, then $\{\alpha\} \in \Lambda$.

Thus there is some $\gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0, \phi(\gamma) \supseteq \{\alpha\} \ni \alpha$.

We have $x_{\phi(\gamma)} \in \bigcap_{\beta \in \phi(\gamma)} C_\beta \subseteq C_\alpha$.

Since this holds for all $\gamma \geq \gamma_0$, and $x_{\phi(\gamma)} \rightarrow x$, we have that $x \in \bar{C}_\alpha = C_\alpha$.

Since this holds for any $\alpha \in A$, we have that $x \in \bigcap_{\alpha \in F} C_\alpha$. Thus $\bigcap_{\alpha \in F} C_\alpha \neq \emptyset$. \square

With the above theorem, we can identify the definition of compactness in a metric space via sequences with the definition of compactness with its metric topology.

Proposition 2.30. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces. If X is compact, and $f : X \rightarrow Y$ is continuous, then $f(X)$ is compact in Y .*

Theorem 2.31. *Let (X, \mathcal{T}) be a topological space,*

1. *Suppose K is compact, then $\forall F \subseteq K$ that is closed, F is also compact.*
2. *If X is Hausdorff, for any compact $K \subseteq X, x \in X \setminus K, \exists$ open neighbourhood U of x , and open $W \supset K$, such that $W \cap U = \emptyset$.*

Proof. 1. Let $(U_\alpha)_{\alpha \in A}$ be an open cover for F .

Since F is closed, then F^c is open. Thus $\{F^c\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover for K .

Thus there are $U_{\alpha_1}, \dots, U_{\alpha_n}$, such that $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$. Thus $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ has a finite subcover.

2. Consider any $y \in K$, there is some open neighborhoods $U_y \ni x, W_y \ni y$, such that $U_y \cap W_y = \emptyset$. Since $K \subseteq \bigcup_{y \in K} W_y$ is compact, we have $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$ for some y_1, \dots, y_n . Let $U = \bigcap_{i=1}^n U_{y_i}$, we have $x \in U, K \subseteq W, U \cap W = \emptyset$ as required.

□

Corollary 2.32. *Let (X, \mathcal{T}) be a Hausdorff topological space, then any compact set K is closed. In addition, for any closed $F \subseteq X$, we have $F \cap K$ is compact.*

Proof. Suppose for contradiction that K is not closed, then there is some $y \in \bar{K} \setminus K$. Thus we can find open neighbourhood U of x , and open $W \supset K$, such that $W \cap U = \emptyset$. Now $K \subset \bar{K} \setminus U \subsetneq \bar{K}$ is closed, which is a contradiction.

Since K is closed, so is $F \cap K \subseteq K$, and thus it is compact. □

Definition 2.39. X is **locally compact** if $\forall x \in X$, there is an open neighbourhood $U_x \in \mathcal{O}(x)$ such that \bar{U}_x is compact.

Example 2.4.1. \mathbb{R}^n is locally compact by the Heinz-Borel theorem.

Proposition 2.33. *A Banach space $(X, \|\cdot\|)$ is locally compact iff $\dim(X) < \infty$.*

Lemma 2.34. *Let (X, \mathcal{T}) be a Hausdorff topological space, and $(K_\alpha)_{\alpha \in A}$ be a collections of compact sets such that*

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

We must have $\alpha_1, \dots, \alpha_n \in A$, such that

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

Proof. Fix $\alpha_1 \in A$, then $K_{\alpha_1} \subseteq \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$ is compact and has an open cover.

Thus there must be $\alpha_2, \dots, \alpha_n \in A$, such that $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = \left(\bigcap_{i=2}^n K_{\alpha_i}\right)^c$.

Thus $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$. □

Theorem 2.35. *Let X be a Locally Compact Hausdorff space, and let $K \subseteq U \subseteq X$ be such that K is compact, and U is open. Then there exists some open set V such that \bar{V} is compact, and*

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof. Since X is a Locally Compact Hausdorff space, there are V_1, \dots, V_n , each with \bar{V}_i be compact, such that $K \subseteq \bigcup_{i=1}^n V_i =: G$. Note that \bar{G} is compact, and G is open.

If $U = X$, then $G \subseteq U$, and we are done.

Otherwise, let $C := X \setminus U$ be non-empty and closed.

Consider any $y \in C$, we know that $y \notin K$. Since X is Hausdorff, we can find open $W_y \supset K$, and $U_y \ni y$, such that $W_y \cap U_y = \emptyset$. Then $W_y \subseteq U_y^c$, and thus $\bar{W}_y \subseteq U_y^c$, since U_y^c is closed. Yet $y \notin U_y^c$, thus $y \notin \bar{W}_y$.

Now consider the family $\{C \cap \bar{W}_y \cap \bar{G}\}_{y \in C}$. Notice that each $C \cap \bar{W}_y \cap \bar{G}$ is compact, since C, \bar{W}_y are closed, and \bar{G} is compact.

Yet $\bigcap_{y \in C} (C \cap \bar{W}_y \cap \bar{G}) = \emptyset$.

Thus $\exists y_1, \dots, y_n \in C$, such that $\bigcap_{i=1}^n (C \cap \bar{W}_{y_i} \cap \bar{G}) = \emptyset$.

Now let $V := G \cap \bigcap_{i=1}^n W_{y_i}$.

Clearly V is open, and $K \subseteq V$.

In addition, $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$, yet the intersection of righthand side and C is empty, thus contained in $C^c = U$. □

2.5 Compactly Supported Continuous Functions

Definition 2.40. For $f \in C(X)$, the **support** of it is

$$\text{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

Definition 2.41. The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \text{Supp}(f) \text{ is compact}\}.$$

Definition 2.42. $C_0(X)$ is the closure of $C_c(X)$ in $\|\cdot\|_\infty$.

Proposition 2.36. $C_0(X)$ is the set of all continuous functions that vanishes at ∞ . $(C_0(X), \|\cdot\|_\infty)$ is a Banach Space and a commutative C^* -algebra with the involution $f^*(x) := \overline{f(x)}$.

Proposition 2.37. $f \in C_0(X)$ if and only if $\forall \epsilon > 0, \exists K \subset\subset X$, such that $\forall x \in X \setminus K, |f(x)| < \epsilon$.

Theorem 2.38. Any commutative C^* -algebra $(A, \|\cdot\|)$ is isomorphic to $C_0(X)$ for some unique Locally Compact Hausdorff X .

2.5.1 Partition of Unity

Definition 2.43. Let K be a compact set, and V be an open set of X . Let $f \in C_c(X)$. We say $f < V$ if $0 \leq f \leq 1$, and $\text{Supp}(f) \subseteq V$. We say $K < f$ if $0 \leq f \leq 1$, and $f|_K = 1$. We say $K < f < V$ if $K \subset V, K < f, f < V$.

Remark. f is a “bump” function that approximates χ_K when V shrinks towards K .

Lemma 2.39 (Urysohn’s lemma for Locally Compact Hausdorff Space). *Let X be a Locally Compact Hausdorff space, $K \subseteq V \subseteq X$ be such that K is compact, and V is open. Then there exists $f \in C_c(V)$, such that $K < f < V$.*

Proof. we want to construct a family of open sets $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$, such that \bar{V}_r is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for $r < s$.

By 2.35, we can find $K \subset V_0 \subset \bar{V}_0 \subset V$.

Pick an enumeration of $r \in \mathbb{Q} \cap (0,1]$, i.e. $(r_n)_{n=1}^\infty$. WLOG, we can let $r_1 = 1$.

By 2.35, we can find $K \subset V_1 \subset \bar{V}_1 \subset V_0$.

Suppose we have constructed the V_{r_i} for $1 \leq i \leq n$, such that \bar{V}_r is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for $r < s \in \{r_i\}_{i=1}^n$.

Let $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$.

Now by 2.35, we can find $\bar{V}_t \subset V_{n+1} \subset V_{n+1}^- \subset V_s$.

For any $r < r_{n+1}$, we have $r \leq s$, and thus $V_{n+1} \subset V_{n+1}^- \subset V_s \subset \bar{V}_s \subseteq V_r$ by induction hypothesis, and similarly for any $r > r_{n+1}$.

Inductively, we can prove there is such a family.

Define $f_r := r\chi_{V_r}$, and $g_r := r\chi_{\bar{V}_r^c} + \chi_{\bar{V}_r}$, and $f := \sup_r f_r, g := \inf_r g_r$.

We can show that f, g are upper and lower continuous, respectively.

In addition, f, g are both 0 outside of V_1 , and 1 on K .

Suppose there is some $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, such that $f_r(x) > g_s(x)$. Then we must have $f_r(x) > 0$, and thus $x \in V_r$ and $1 \leq r = f_r(x)$.

Thus $1 > g_s(x)$, and thus $x \in \bar{V}_s^c$ and $f_s = s$.

Since $r > s$, we must have $V_r \subset \bar{V}_r \subset V_s \subset \bar{V}_s$, which is a contradiction to $x \in V_r, x \notin \bar{V}_s$.

Thus for any $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, we must have $f_r(x) \leq g_s(x)$.

Thus we must have $f(x) \leq g(x)$ for any $x \in V$.

Now suppose there is some $x \in X$, such that $f(x) < g(x)$.

There must be some rationals, such that $f(x) < r < s < g(x)$, since \mathbb{Q} is dense.

Thus $\sup_r f_r(x) < r$, and thus $x \notin V_r$.

Also, $\inf_s g_s(x) > s$, and thus $x \in \bar{V}_s$.

Yet $r < s$, we must have $V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r$, which is a contradiction.

Thus we must have $f = g$, and it forces f to be continuous. \square

Definition 2.44. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$.

A collection $(h_i)_{i=1}^n \subset C_c(X)$ is called a **partition of unity** on K subordinate to $(V_i)_{i=1}^n$ if

$$\begin{cases} \forall 1 \leq i \leq n, & h_i < V_i, \\ \forall x \in K, & \sum_{i=1}^n h_i(x) = 1. \end{cases}$$

Theorem 2.40. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$, there always exists a partition of unity on K subordinated to $(V_i)_{i=1}^n$.

Proof. Since K is compact, we can find some open cover W_1, \dots, W_m , such that for all j , we have $W_j \subset \bar{W}_j \subset V_{i(j)}$ for some $1 \leq i(j) \leq n$.

Let $K_i := \bigcup_{1 \leq j \leq m \text{ such that } W_j \subset V_i} \bar{W}_j \subset V_i$, which is compact.

By Urysohn's lemma, we can find $K_i < g_i < V_i$.

Now let $h_1 := g_1$, and in general, $h_i := g_i \prod_{j < i} (1 - g_j)$.

It is easy to check that $0 \leq h_i \leq 1$, and $h_i \in C_c(X)$.

In addition, $\text{Supp}(h_i) \subseteq \text{Supp}(g_i) \subset V_i$.

Thus $h_i < V_i$. Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_i)$.

For any $x \in K$, there must be some $i \in [n]$ such that $x \in K_i$, and thus $g_i(x) = 1$, and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_i(x)) = 1 - 0 = 1.$$

\square

2.6 Product Topology

Definition 2.45. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of topological spaces. The **product topology** on $\prod_\alpha X_\alpha$ is the topology generated by the sets

$$\left\{ U_\alpha \times \prod_{\beta \in A, \beta \neq \alpha} X_\beta \mid \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\} = \left\{ \pi_\alpha^{-1}(U_\alpha) \mid \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\},$$

where the **projection map onto** X_α is $\pi_\alpha : \prod_{\beta \in A} X_\beta \rightarrow X_\alpha$ by $(x_\beta)_{\beta \in A} \mapsto x_\alpha$.

Proposition 2.41. The product topology is the weakest topology in which each π_α is continuous.

Proposition 2.42. A net $(x_\lambda)_{\lambda \in \Lambda}$ in $\prod_{\alpha \in A} X_\alpha$ converges to x if and only if $\forall \alpha \in A$, $\pi_\alpha(x_\lambda) \rightarrow \pi_\alpha(x)$ in X_α .

Proof. See A1. \square

Theorem 2.43 (Tychonoff). *Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of compact topological spaces, then $\prod_\alpha X_\alpha$ is compact under product topology.*

Definition 2.46. Let (P, \leq) be a partially ordered set. We call a totally ordered subset $Q \subseteq P$ a **chain**.

Definition 2.47. Let (P, \leq) be a partially ordered set. We call \leq **inductive** if every chain $Q \subseteq P$ has an upper bound.

Definition 2.48. \leq is called a **well-order** if it is a total order, and for every $\emptyset \neq S \subseteq X$ has a minimal element. $\exists x \in P$ such that $\forall y \in P, y \leq x \implies x = y$.

Lemma 2.44 (Zorn's). *Every inductive partial order (P, \leq) , defined on a nonempty P , has a maximal element. Namely, $\exists x \in P$ such that $\forall y \in P, x \leq y \implies x = y$.*

Proposition 2.45. *Every vector space V has a basis.*

Proof. Consider $P = \{S \subset V | S \text{ is linearly independent}\}$, with $S \leq S' \iff S \subseteq S'$.

Let Q be a chain in P , then it has an upper bound $\tilde{S} = \bigcup_{S \in Q} S$, which we can check is still linearly independent.

By Zorn's lemma, there is a maximal $S \in P$. □

Theorem 2.46 (Well-Ordering Principle). *Every set X admits a well-ordering.*

Theorem 2.47. *The Following Are Equal:*

1. *Tychonoff's Theorem*
2. *Axiom of Choice*
3. *Zorn's Lemma*
4. *Well-Ordering Principle*

Proof. (1) \implies (2).

Let $(X_\alpha)_{\alpha \in A}$ be a family of non-empty set.

Let $Y_\alpha := \{p_\alpha\} \sqcup X_\alpha$ for some additional symbol p_α .

Define the topology $\mathcal{T}_\alpha := \{\emptyset, Y_\alpha, X_\alpha, \{p_\alpha\}\}$.

Then $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ are all compact.

By Tychonoff's Theorem, $\prod_\alpha Y_\alpha$ is also compact.

Now consider $C_\alpha := X_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$.

Since $C_\alpha^c = \{p_\alpha\} \times \prod_{\beta \neq \alpha} Y_\beta$ is open, we have that C_α is closed in the product topology.

Also, $C_\alpha \neq \emptyset$.

Now for any finite $\bigcap_{i=1}^n C_{\alpha_i}$, we have that $(x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_n}, p_\alpha, \dots) \in \bigcap_{i=1}^n C_{\alpha_i}$.

Thus $(C_\alpha)_{\alpha \in A}$ is an FIP family.

Since $\prod_\alpha Y_\alpha$ is compact, we have that $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$.

Now $\prod_{\alpha \in A} X_\alpha = \bigcap_{\alpha \in A} C_\alpha$, and we have seen that it being nonempty is equivalent as the Axiom of Choice.

(4) \implies (2).

Take $C : \mathcal{P}(X) \setminus \emptyset \rightarrow X$ to be $C(S) :=$ minimal element of S .

(3) \implies (2)

Let $\{X_\alpha\}_{\alpha \in A}$ to be non-empty sets. Let $X := \bigcup_{\alpha \in A} X_\alpha$, $P = \{f_B : B \rightarrow X | B \subseteq A, f_B(\beta) \in X_\beta, \forall \beta \in B\}$.

Clearly $P \neq \emptyset$.

Define the order by $f_B \leq f_{B'} \iff B \subseteq B', f_{B'}|_B = f_B$.

For any chain $Q \subseteq P$, define $\tilde{B} = \bigcup_{B \in Q} B$, and $f_{\tilde{B}}(\beta) = f_B(\beta)$ for $\beta \in B$ of any B .

We can check $f_{\tilde{B}} \in P$ is an upper bound.

By Zorn's Lemma, there is a maximal element $f_B \in P$.

If $B \subsetneq A$, then we can extend the function to contain another point $a \in A \setminus B$, and send a to any $x \in X_a$, contradicting maximality.

Thus there is some $f_A \in P$, which we have seen is equivalent to the Axiom of Choice.

(4) + (2) \implies (3).

See Pmath432 A1.

(3) + (2) \implies (1).

For contradiction, suppose $X = \prod_{\alpha \in A} X_\alpha$ is not compact.

Let $\text{NFS} := \{\mathcal{C} \subseteq \mathcal{P}(X) : \mathcal{C} \text{ is a cover with no finite sub cover}\}$.

Define $\mathcal{C}_1 \leq \mathcal{C}_2 \iff \mathcal{C}_1 \subseteq \mathcal{C}_2$.

Take any chain Q in NFS.

Let $\mathcal{C}' := \bigcup_{\mathcal{C} \in Q} \mathcal{C}$, which we can check is an open cover and an upper bound for the chain.

Indeed, suppose $\mathcal{C}' \notin \text{NFS}$, then $\exists U_1, U_2, \dots, U_n \in \mathcal{C}'$ such that $X = \bigcup_{i=1}^n U_i$.

For all $i \in [n]$, $U_i \in \mathcal{C}_i$ for some $\mathcal{C}_i \in Q$.

Since Q is a chain, there is some $i_0 \in [n]$ such that $\forall i \in [n]$, $\mathcal{C}_i \subseteq \mathcal{C}_{i_0}$.

Thus $\mathcal{C}_{i_0} \notin \text{NFS}$, a contradiction.

Thus $\mathcal{C} \in \text{NFS}$.

By Zorn's Lemma, there is a maximal open cover \mathcal{C}_{\max} with no subcover.

Notice that if $U \in \mathcal{C}_{\max}$, and $V \subseteq U$ is open, then $V \in \mathcal{C}_{\max}$ as well, since any finite subcover of $\{V\} \cup \mathcal{C}_{\max}$ can give a finite subcover of \mathcal{C}_{\max} by replacing V by U .

Also, if $U_1, U_2 \in \mathcal{C}_{\max}$, we must have $U_1 \cup U_2 \in \mathcal{C}_{\max}$ as well.

Also, suppose V_1, \dots, V_n are open in X , such that $\bigcap_{i \in [n]} V_i \in \mathcal{C}_{\max}$, then $\exists i_0 \in [n]$, such that $V_{i_0} \in \mathcal{C}_{\max}$.

Indeed, suppose not, for any $i \in [n]$, there is a finite cover $V_i \cup \bigcup_{j \in [N_i]} U_{i,j}$ for $U_{i,j} \in \mathcal{C}_{\max}$. We must have

$\left(\bigcap_{i \in [n]} V_i\right) \cup \bigcup_{i \in [n], j \in [N_i]} U_{i,j}$ is a finite sub-cover of \mathcal{C}_{\max} .

Now let $W_\alpha := \{\text{open } U_\alpha \subseteq X_\alpha \mid \pi_\alpha^{-1}(U_\alpha) \in \mathcal{C}_{\max}\}$.

For contradiction, suppose W_α covers X_α , then there is a finite subcover $\{U_i\}_{i \in [n]}$ such that $X_\alpha = \bigcup_{i \in [n]} U_i$.

Thus $X = \bigcup_{i=1}^n \pi_\alpha^{-1}(U_i)$, which is a subcover of \mathcal{C}_{\max} .

Thus $X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right) \neq \emptyset$.

By the Axiom of Choice, there is $x_\alpha \in X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right)$ for each α .

Let $x \in X$ be $x(\alpha) = x_\alpha$.

Since \mathcal{C}_{\max} is a cover for X , there is some open $U \in \mathcal{C}_{\max}$ with $x \in U$.

Thus, there must be some $x \in U_1 \times U_2 \times \dots \times U_n \times \prod_{\beta \in A \setminus \{\alpha_i : i \in [i]\}} = \bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$ for open $U_i \in X_{\alpha_i}$, since such sets forms a basis.

Thus, $\bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{\max}$ as well.

Thus, there is some $i_0 \in [n]$, such that $\pi_{\alpha_{i_0}}^{-1}(U_{i_0}) \in \mathcal{C}_{\max}$, which means $U_{i_0} \in W_{\alpha_{i_0}}$.

However, $x_{\alpha_{i_0}} \in U_{\alpha_{i_0}}$, which is a contradiction to the choice of $x_{\alpha_{i_0}} \notin \left(\bigcup_{U \in W_{\alpha_{i_0}}} U\right)$. □

3 Banach Spaces

Definition 3.1. A **normed vector space** is a vector space $(X, \|\cdot\|)$ over a field \mathbb{F} endowed with a norm (length) function: $\|\cdot\| : X \rightarrow [0, \infty)$, such that $\forall x, y \in X, a \in \mathbb{F}$, it satisfies

1. subadditivity (triangular inequality); i.e. $\|x + y\| \leq \|x\| + \|y\|$,
2. absolute homogeneity; i.e. $\|a \cdot x\| = |a| \|x\|$, and
3. positive definiteness; i.e. if $x \neq 0$, we must have $\|x\| > 0$.

Proposition 3.1. For every **normed space** with $\|\cdot\|$, there is a metric $d(x, y) = \|x - y\|$.

Proof.

$$d(x, x) = \|x - x\| = \|0\| = 0$$

$$\forall x \neq y, d(x, y) = \|x - y\| > 0$$

$$d(x, y) = \|x - y\| = \|-(y - x)\| = |-1| \|y - x\| = \|y - x\| = d(y, x)$$

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \geq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

Thus $d(x, y) = \|x - y\|$ is a metric. □

Definition 3.2. Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X , then they are called **equivalent** if there are $C_1, C_2 > 0$, such that

$$\forall x \in X, C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Definition 3.3. A normed space is called a **Banach space** if it is complete.

Proposition 3.2. The Euclidean space \mathbb{R}^n or \mathbb{C}^n , with the Euclidean norm $\|x\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ is a Banach space.

Definition 3.4. For \mathbb{R}^n or \mathbb{C}^n , and $p \in [1, \infty)$, the ℓ^p norm is $\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$. For $p = \infty$, the ℓ_∞ norm is $\|x\|_\infty = \max_{i \in [n]} |x_i|$.

Proposition 3.3. \mathbb{R}^n or \mathbb{C}^n , with any ℓ^p norm is a Banach space.

Remark. Notice that $\forall n \in \mathbb{N}^+, \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty$, so they are equivalent.

Proposition 3.4. If X is compact and Hausdorff, we have $(C(X), \|\cdot\|_\infty)$ is a Banach Space.

Proof. The Extreme Value Theorem shows it is a normed space.

The convergence in $\|\cdot\|_\infty$ is uniform convergence, and the uniform limit of continuous functions is continuous. Thus, $(C(X), \|\cdot\|_\infty)$ is complete. \square

Proposition 3.5. For a Locally Compact Hausdorff Space X , $(C_b(X), \|\cdot\|_\infty)$ and $(C_0(X), \|\cdot\|_\infty)$ are both Banach Spaces.

Example 3.0.1. $C_0(\mathbb{N}) = \{(x_n)_{n=1}^\infty : \lim_{n \rightarrow \infty} x_n = 0\}$ with the discrete topology.

Proposition 3.6. $C^k([0, 1])$ is a Banach Space with $\|f\|_{C^k([a, b])} := \sum_{i=1}^k \|f^{(i)}\|_\infty$.

Proof. It is easy to check that this is a norm.

Now take any Cauchy sequence $(f_n)_{n=1}^\infty$, then for each $i \in [k]$, we have $(f_n^{(i)})_{n=1}^\infty$ is Cauchy in $C([a, b])$ as well.

Since $[a, b]$ is compact and Hausdorff, we have $C([a, b])$ is a Banach Space, so there is a $g_i(x) := \lim_{n \rightarrow \infty} f_n^{(i)}(x)$. In addition, since the convergence is uniform, we have $g_i \in C([0, 1])$.

By the Fundamental Theorem of Calculus, we have that for all $i \in [k-1], x \in [0, 1]$,

$$f_n^{(i)}(x) = f_n^{(i)}(0) + \int_0^x f_n^{(i+1)}(t) dt.$$

Taking the limit of $n \rightarrow \infty$, we have that

$$g_i(x) = g_i(0) + \int_0^x g_{i+1}(t) dt,$$

which means $g_i \in C^1([0, 1])$ with $g'_i = g_{i+1}$.

Thus $g_0 \in C^k([0, 1])$. \square

3.1 Bounded linear operators

Definition 3.5. Let X, Y be vector spaces, $T : X \rightarrow Y$ is a linear operator if $\forall c \in \mathbb{R}, u, v \in X$,

$$T(u + cv) = Tu + cTv.$$

Definition 3.6. Let X, Y be linear normed spaces, the **operator norm** of a linear operator $T : X \rightarrow Y$ is

$$\|T\| := \sup_{\|u\|_X \leq 1} \|Tu\|_Y = \sup_{\|u\|_X = 1} \|Tu\|_Y = \sup_{u \neq 0 \in X} \frac{\|Tu\|_Y}{\|u\|_X}.$$

Definition 3.7. Let X, Y be normed spaces, a linear operator $T : X \rightarrow Y$ is **bounded** if $\|A\| < \infty$.

Theorem 3.7. Let X, Y be two normed linear spaces, let $T : X \rightarrow Y$ be linear, then the following are equal:

1. T is continuous,
2. T is continuous at 0,
3. T is bounded,
4. T is uniformly continuous.

Proof. (4) \implies (1) \implies (2) trivially.

(3) \implies (4).

Suppose T is bounded, then

$$\begin{aligned}\|T(x) - T(y)\| &= \|T(x - y)\| \\ &\leq \|T\| \|x - y\|.\end{aligned}$$

Thus, T is $\|T\|$ Lipschitz and so uniformly continuous.

(2) \implies (3).

Suppose T is continuous at 0, and suppose for contradiction that $\|T\| = \infty$.

There must be $(x_n)_{n=1}^\infty$ in X , such that $\|x_n\| \leq 1$, $\|Tx_n\| \geq n^2$ for each $n \geq 1$.

Notice that $\frac{x_n}{n} \rightarrow 0$, but $\|T(\frac{x_n}{n})\| = \frac{1}{n} \|Tx_n\| \geq n$ for each n .

Thus $\lim_{n \rightarrow \infty} T(\frac{x_n}{n}) \neq 0 = T(0)$, which contradicts that T is continuous at 0. \square

Proposition 3.8. Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X , then they are equivalent if and only if they induce the same topology.

Proof. Assume that $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, then there are $C_1, C_2 > 0$, such that

$$\forall x \in X, C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1.$$

Consider the identity function $id : (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$, we see that

$$\|id\| = \sup_{x \neq 0 \in X} \frac{\|x\|_2}{\|x\|_1} \leq \sup_{x \neq 0 \in X} \frac{C_2 \|x\|_1}{\|x\|_1} = C_2,$$

and

$$\|id^{-1}\| = \sup_{x \neq 0 \in X} \frac{\|x\|_1}{\|x\|_2} \leq \sup_{x \neq 0 \in X} \frac{\|x\|_1}{C_1 \|x\|_1} = \frac{1}{C_1}.$$

Thus id is a homeomorphism.

On the other hand, suppose $\|\cdot\|_1, \|\cdot\|_2$ induces the same topology, then id is a homeomorphism.

Thus,

$$\frac{1}{\|id\|} \|x\|_2 = \frac{1}{\|id\|} \|id(x)\|_2 \leq \|x\|_1 = \|id^{-1}(x)\|_1 \leq \|id^{-1}\| \|x\|_2.$$

\square

Definition 3.8. Let X, Y be normed spaces, we denote

$$B(X, Y) := \{T : X \rightarrow Y \mid T \text{ is a bounded linear operator}\}.$$

Theorem 3.9. The set $B(X, Y)$ is a normed linear space with the operator norm.

Proposition 3.10. Let X, Y, Z be normed spaces, if $T : X \rightarrow Y, S : Y \rightarrow Z$ are both linear bounded operators, then so is $S \circ T$, with

$$\|S \circ T\| \leq \|S\| \|T\|.$$

Theorem 3.11. Let X be a normed space, and Y be a Banach Space, then $B(X, Y)$ is a Banach Space.

Proof. Let $(T_n)_{n=1}^\infty$ be a Cauchy sequence in $B(X, Y)$.
For any $x \in X$, we have that $(T_n x)_{n=1}^\infty$ is Cauchy in Y .
Indeed, $\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\|$.
Since Y is complete, there must be a unique $y = \lim_{n \rightarrow \infty} T_n x \in Y$.
Define $Tx := \lim_{n \rightarrow \infty} T_n x$ for any $x \in X$.

Notice that T is linear.

Given $\epsilon > 0$, we know there must be some $N \in \mathbb{N}$, such that $\forall m, n \geq N$, $\|T_n - T_m\| < \epsilon$.
Consider any $x \in X$.

$$\begin{aligned} \|Tx\| &\leq \|(T - T_N)x\| + \|T_N x\| \\ &= \lim_{m \rightarrow \infty} \|(T_m - T_N)x\| + \|T_N x\| \\ &\leq \limsup_m \|T_m - T_N\| \|x\| + \|T_N\| \|x\| \\ &\leq \epsilon \|x\| + \|T_N\| \|x\|. \end{aligned}$$

Thus $\|T\| \leq \epsilon + \|T_N\| < \infty$.

This shows $T \in B(X, Y)$.

Again, for any $x \in X$, $n \geq N$, we have

$$\begin{aligned} \|(T_n - T)x\| &= \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| \\ &\leq \limsup_m \|T_n - T_m\| \|x\| \\ &< \epsilon \|x\|. \end{aligned}$$

Thus $\|T_n - T\| < \epsilon$ for any $n \geq N$, which shows $\lim_{n \rightarrow \infty} T_n = T$ in $B(X, Y)$ with the operator norm. \square

Definition 3.9. Let X, Y be normed spaces. We say a linear operator $T : X \rightarrow Y$ is **bounded below** if $\exists m > 0$, such that $\forall x \in X$, $\|Tx\| \geq m\|x\|$.

Definition 3.10. Let X, Y be normed spaces. We say $T : X \rightarrow Y$ is an **isomorphism between normed spaces** or T is **invertible** if T is bijective and T, T^{-1} are both bounded. i.e. T is a homeomorphism between X, Y .

Theorem 3.12. Let $T : X \rightarrow Y$ be a linear operator between normed linear spaces X and Y . The operator T is invertible, namely, T has a bounded inverse $T^{-1} : \text{Im}(T) \rightarrow X$, if and only if T is bounded below. In this case,

$$\|T^{-1}\| = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

Proof. (\implies) Suppose T has a bounded inverse operator. Let $x \in X$ and $y = Tx \in \text{Im}(T)$. By the assumption,

$$\begin{aligned} \|x\| &= \|T^{-1}y\| \leq \|T^{-1}\| \|y\| = \|T^{-1}\| \|Tx\|, \\ \|Tx\| &\geq \frac{1}{\|T^{-1}\|} \|x\|. \end{aligned}$$

Since the inequality holds for all $x \in X$, T is bounded below.

(\impliedby) Suppose T is bounded below. There exists $m > 0$ such that $\|Tx\| \geq m\|x\|$ for all $x \in X$. Let $z \in \ker T$. We have $0 = \|Tz\| \geq m\|z\| \geq 0$. Therefore, $\|z\| = 0$, which means $z = 0$. Thus $\ker T = \{0\}$ and T is one-to-one. Thus, T is a bijection between $X \longleftrightarrow \text{Im}(T)$, which means $T^{-1} : \text{Im}(T) \rightarrow X$ exists.

Finally, for any $y \in \text{Im}(T)$, $y = Tx$ for some $x \in X$. Then

$$\|T^{-1}y\| = \|x\| \stackrel{T \text{ bounded below}}{\leq} \frac{1}{m} \|Tx\| = \frac{1}{m} \|y\|.$$

Therefore T^{-1} is bounded. In particular,

$$\|T^{-1}\| = \sup_{y \in \text{Im}(T), y \neq 0} \frac{\|T^{-1}y\|}{\|y\|} = \sup_{x \neq 0} \frac{\|x\|}{\|Tx\|} = \frac{1}{\inf_{x \neq 0} \frac{\|Tx\|}{\|x\|}} = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

□

Definition 3.11. Let X, Y be normed spaces. We call $T : X \rightarrow Y$ a **contraction** if $\|T\| \leq 1$.

Definition 3.12. Let X, Y be normed spaces. We call $T : X \rightarrow Y$ an **isometry** if $\forall x \in X, \|Tx\| = \|x\|$.

Proposition 3.13. Let X, Y be normed spaces. If a linear operator $T : X \rightarrow Y$ is a surjective isometry, it is an isometric isomorphism between normed spaces.

Proof. T is bounded since $\|T\| = 1$. Suppose $T(x) = 0$, we have $\|Tx\|_Y = 0$. Since T is an isometry and thus bounded below by 1, T has a bounded inverse $T^{-1} : \text{Im}(T) \rightarrow X$ by theorem 3.12. Since T is surjective, $\text{Im}(T) = Y$. □

Proposition 3.14. Let Y be a Banach space, S be a dense subset of a normed space X . For any bounded linear operator $E : S \rightarrow Y$, we can extend it to $\tilde{E} : X \rightarrow Y$, such that \tilde{E} is also bounded and linear, with $\|\tilde{E}\| = \|E\|$, and $\tilde{E}|_S = E$.

Proof. Consider any $x \in X$.

Since S is dense in X , We know $\forall m \in \mathbb{N}^+, \exists x_m \in S$, such that $\|x - x_m\|_X \leq \frac{1}{m}$. Since E is linear on S , we have that

$$\begin{aligned} \|Ex_m - Ex_l\|_Y &= \|E(x_m - x_l)\|_Y \\ &\leq \|E\| \|x_m - x_l\|_X \\ &= \|E\| \|(x_m - x) + (x - x_l)\|_X \\ &\leq \|E\| \|x - x_m\|_X + \|E\| \|x - x_l\|_X \\ &\leq \|E\| \left(\frac{1}{m} + \frac{1}{l} \right). \end{aligned}$$

Thus given any $\epsilon > 0$, for any $m, l \geq \lceil \frac{2\epsilon}{\|E\|} \rceil$, we can make $\|Ex_m - Ex_l\|_Y < \epsilon$. Thus $(Ex_m)_{m=1}^\infty$ is a Cauchy sequence in Y .

Since Y is a Banach space, $\exists y^* \in Y$, such that $Ex_m \rightarrow y^*$ in Y .

We claim that y^* is independent of choice of the sequence $(x_m)_{m=1}^\infty$.

Indeed, consider any other sequence $(v_m)_{m=1}^\infty \subset C^\infty(\bar{x})$, such that $\forall m \in \mathbb{N}^+, \|x - v_m\|_X \leq \frac{1}{m}$,

$$\begin{aligned} \|y^* - Ev_m\|_Y &\leq \|y^* - Ex_m\|_Y + \|Ex_m - Ev_m\|_Y \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - v_m\|_X \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - x\|_X + \|E\| \|x - v_m\|_X. \end{aligned}$$

Since all three terms on the right go to 0 when $m \rightarrow \infty$, we have that $Ev_m \rightarrow y^*$ in Y .

Thus we can uniquely define $\tilde{E}x := y^*$. In addition,

$$\begin{aligned} \|\tilde{E}x\|_Y &= \left\| \lim_{m \rightarrow \infty} Ex_m \right\|_Y \\ &= \lim_{m \rightarrow \infty} \|Ex_m\|_Y \\ &\leq \lim_{m \rightarrow \infty} \|E\| \|x_m\|_X \\ &= \|E\| \left\| \lim_{m \rightarrow \infty} x_m \right\|_X \\ &= \|E\| \|x\|_X. \end{aligned}$$

Thus $\|\tilde{E}\| = \|E\|$. □

3.1.1 Dual Spaces

Definition 3.13. Let X be a normed space over \mathbb{F} , a **functional** is an operator that maps into \mathbb{F} .

Definition 3.14. Let X be a normed space over \mathbb{F} , the **dual space** of X is the collection of bounded linear functionals on X , denoted

$$X^* := B(X, \mathbb{F}) = \{\phi : X \rightarrow \mathbb{F} : \phi \text{ is linear and bounded}\}.$$

Definition 3.15. Let X be a normed space over \mathbb{F} , and a subspace $Y \subseteq X^*$ we define the **duality pairing** to be $\langle \cdot | \cdot \rangle_{X,Y} : X \times Y \rightarrow \mathbb{F}$ to be $x \in X, \phi \in Y$, we can write $\langle x | \phi \rangle_{X,Y} := \phi(x)$ as the **action** of ϕ on x .

Definition 3.16. Let X be a normed space, the **dual norm** is defined to be

$$\|u^*\|_{X^*} := \sup_{\|u\| \leq 1} |\langle u^* | u \rangle_{X^*, X}|.$$

Example 3.1.1. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^\infty | \lim_{i \rightarrow \infty} x_i = 0\}$, then $C_0(\mathbb{N})^* \cong \ell^1(\mathbb{N})$.

Indeed, let $e_n := (m \mapsto \delta_{nm}) = (\delta_{nm})_{n=1}^\infty \in C_0(\mathbb{N})$.

Given any $\phi \in C_0(\mathbb{N})^*$, we define $a_n := \phi(e_n) \in \mathbb{F}$.

We claim $a = (a_n)_{n=1}^\infty$ completely determines ϕ .

Indeed, consider any $x \in C_0$, let $x^N := \sum_{n=1}^N x_n e_n$.

We have $\|x - x^N\|_{C_0} = 0$, so

$$\begin{aligned} \phi(x) &= \lim_{N \rightarrow \infty} \phi(x^N) \\ &= \lim_{N \rightarrow \infty} \phi\left(\sum_{n=1}^N x_n e_n\right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n \phi(e_n) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n a_n. \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=1}^N |a_n| &= \sum_{n=1}^N a_n \operatorname{sgn}(a_n) \\ &= \sum_{n=1}^N \phi(\operatorname{sgn}(a_n) e_n) \\ &= \phi(y_N) \\ &\leq \|\phi\|_{C_0^*} \|y_N\|_{C_0}, \end{aligned}$$

where $y_N = \sum_{i=1}^N \operatorname{sgn}(a_i) e_i$.

Since $\|y_N\| = 1$, we have $\sum_{n=1}^N |a_n| \leq \|\phi\|_{C_0^*}$ for any N .

Thus

$$\|a\|_1 = \sum_{n=1}^\infty |a_n| = \lim_{N \rightarrow \infty} \sum_{n=1}^N |a_n| \leq \|\phi\|_{C_0^*} < \infty.$$

Thus $a \in \ell^1(\mathbb{N})$.

Notice that $\Phi : C_0^*(\mathbb{N}) \rightarrow \ell^1(\mathbb{N})$ by $\phi \mapsto a$ is linear, contractive, and injective.

Also, given any $a \in \ell^1(\mathbb{N})$, we can set $\phi_a(x) := \sum_{n=1}^{\infty} x_n a_n$.
Thus,

$$\begin{aligned} |\phi_n(x)| &\leq \sum_{n=1}^{\infty} |x_n| |a_n| \\ &\leq \|x\|_{\infty} \|a\|_1, \end{aligned}$$

which shows $\|\phi_a\|_{C_0^*} \leq \|a\|_1$.

Thus $\phi_a \in C_0^*(\mathbb{F})$. Notice that $\Phi(\phi_a) = a$.

This shows that Φ is surjective, and it is actually an isometry, since $\forall \phi \in C_0^*(\mathbb{F})$, we have

$$\|\Phi(\phi)\|_1 \leq \|\phi\|_{C_0^*} = \|\Phi^{-1}(\Phi(\phi))\|_{C_0^*} \leq \|\Phi(\phi)\|_1.$$

Example 3.1.2. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^{\infty} \mid \lim_{i \rightarrow \infty} x_i = 0\}$, then

$$B(C_0(\mathbb{N})) = B(C_0(\mathbb{N}), C_0(\mathbb{N})) \cong \{(t_{ij})_{i,j \in \mathbb{N}} : \|(t_{ij})_{i,j \in \mathbb{N}}\| < \infty, \forall j \in \mathbb{N}, (t_{ij})_{i \in \mathbb{N}} \in C_0(\mathbb{N}),$$

which are infinite matrices whose rows are uniformly in ℓ^1 , and columns are in $C_0(\mathbb{N})$.

In addition, it is an isometry under

$$\|(t_{ij})_{i,j \in \mathbb{N}}\| := \sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1.$$

Indeed, let

$$e_n := (m \mapsto \delta_{nm}) \cong (\delta_{nm})_{n=1}^{\infty} \in C_0(\mathbb{N}), \delta_i := ((x_j)_{j=1}^{\infty} \mapsto x_i) \in C_0^*(\mathbb{N})$$

with $\Phi(\delta_i) = (\delta_{ij})_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ as in previous example.

Easy to check $\|\delta_i\| = 1$ by the above example.

Consider any $T \in B(C_0(\mathbb{N}))$, define $\phi_i := (\delta_i \circ T), t_{ij} := \phi_i(e_j)$.

Notice that $\phi_i : C_0(\mathbb{N}) \rightarrow \mathbb{F}$ is linear, and $\|\phi_i\| \leq \|\delta_i\| \|T\| = \|T\| < \infty$, so $\phi_i \in C_0^*(\mathbb{N})$.

Thus, $(t_{ij})_{j=1}^{\infty} = (\phi_i(e_j))_{j=1}^{\infty} = \Phi(\phi_i) \in \ell^1(\mathbb{N})$ as in previous example, with $\|\phi_i\| = \|(t_{ij})_{j=1}^{\infty}\|_1$.

Since this hold for all $i \in \mathbb{N}$, we have $\sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1 \leq \|T\|$.

In addition, $(t_{ij})_{i \in \mathbb{N}} = ((\delta_i \circ T)(e_j))_{i \in \mathbb{N}} = T e_j \in C_0(\mathbb{N})$.

On the other hand, suppose we have such $(t_{ij})_{i,j \in \mathbb{N}}$ with $\sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1 < \infty$, we can define $\phi_i := \Phi^{-1}((t_{ij})_{j=1}^{\infty}) \in C_0^*(\mathbb{N})$ with $\phi_i(x) = \sum_{j=1}^{\infty} x_j t_{ij}$ as in previous example.

Let $Tx := \sum_{i=1}^{\infty} \phi_i(x) e_i$ for any $x \in C_0(\mathbb{N})$.

Clearly, T is linear, and we have $(\delta_i \circ T)(x) = \delta_i(\sum_{j=1}^{\infty} \phi_j(x) e_j) = \phi_i(x)$.

$$\begin{aligned} \|Tx\|_{\infty} &= \|(\phi_i(x))_{i=1}^{\infty}\|_{\infty} \\ &\leq \sup_{i \in \mathbb{N}} |\phi_i(x)| \\ &\leq \sup_{i \in \mathbb{N}} \|\phi_i\| \|x\| \\ \|T\| &\leq \sup_{i \in \mathbb{N}} \|\phi_i\| \\ &= \sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\| \\ &< \infty. \end{aligned}$$

Thus $T \in B(C_0(\mathbb{N}), \ell_{\infty})$.

Now we claim $T(C_0(\mathbb{N})) = C_0(\mathbb{N})$, which will mean $T \in B(C_0(\mathbb{N}))$.

Indeed, for any $x = \sum_{n=1}^{\infty} x_n e_n \in C_0(\mathbb{N})$, we have

$$\begin{aligned} Tx &= T \left(\lim_{N \rightarrow \infty} \sum_{n=1}^N x_n e_n \right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n T(e_n). \end{aligned}$$

Since each

$$\begin{aligned} T(e_n) &= \sum_{i=1}^{\infty} \phi_i(x) e_i \\ &= \sum_{i=1}^{\infty} t_{in} e_i \\ &\in C_0(\mathbb{N}), \end{aligned}$$

and $C_0(\mathbb{N})$ is closed, we have $Tx \in C_0(\mathbb{N})$.

In addition, $(\delta_i \circ T)(e_j) = \phi_i(e_j) = \Phi^{-1}((t_{ik})_{k=1}^{\infty})(e_j) = t_{ij}$.

Thus $T \longleftrightarrow (t_{ij})_{i,j \in \mathbb{N}}$ is an isometric bijection.

Example 3.1.3. Consider the **Disk Algebra**

$$A(\mathbb{D}) := \left\{ f \in C(\mathbb{T}) : \forall n \in \mathbb{Z}^{--}, \hat{f}(n) = 0 \right\},$$

where $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}$, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C}, |z| = 1\}$ is the unit circle, and

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt$$

is the n^{th} Fourier Transform of f .

Consider $\phi_n : f \mapsto \hat{f}(n)$, which is clearly in $C^*(\mathbb{T})$.

We notice that $A(\mathbb{D}) = \bigcap_{n < 0} \ker(\phi_n)$ is closed in $C(\mathbb{T})$, since each kernel of a continuous functional is closed.

In fact, for $f, g \in C(\mathbb{T})$, we have $\hat{f}g(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \hat{g}(n-k)$.

Thus, for $f, g \in A(\mathbb{D})$, we have $\hat{f}g(n) = \sum_{k \in \mathbb{N}} \hat{f}(k) \hat{g}(n-k) = 0$ for $n < 0$.

This shows $A(\mathbb{D})$ is actually an Algebra.

Also, $A(\mathbb{D})$ is exactly the set of $f \in C(\mathbb{T})$ that admits an extension $F \in C(\bar{\mathbb{D}})$ with $F|_{\mathbb{D}}$ being analytic, with $F(z) := \sum_{n=1}^{\infty} \hat{f}(n) z^n$.

3.2 Quotient Spaces

Definition 3.17. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. The **quotient space** is $X/Y := \{x + Y : x \in X\}$, with the **quotient map** $Q : X \rightarrow X/Y$ by $Q(x) := [x] := x + Y = \{x + y : y \in Y\}$.

Proposition 3.15. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. X/Y is always a vector space with $[0] = Y$, $[x] + [y] = [x + y]$, $c[x] = [cx]$.

Proposition 3.16. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. $(X/Y, \|\cdot\|_{X/Y})$ is always a Banach space with $\|[x]\|_{X/Y} := \inf_{y \in Y} \|x + y\|_X$. In addition, Q is isometric if $Y \subsetneq X$.

Proof. $\|[x]\|_{X/Y} = 0 \iff \inf_{y \in Y} \|x + y\|_X = 0 \iff x \in \bar{Y} \iff x \in Y$.

Scaling is clear.

Also,

$$\begin{aligned} \|[x] + [z]\|_{X/Y} &= \inf_{y \in Y} \|x + y + z\|_X \\ &= \inf_{y_1, y_2 \in Y} \|x + y_1 + z + y_2\|_X \\ &\leq \inf_{y_1 \in Y} \|x + y_1\|_X + \inf_{y_2 \in Y} \|z + y_2\|_X \\ &= \|[x]\|_{X/Y} + \|[z]\|_{X/Y}. \end{aligned}$$

This shows $(X/Y, \|\cdot\|_{X/Y})$ is a normed space.

We note that $\|Qx\|_{X/Y} = \inf_{y \in Y} \|x + y\|_X \leq \|x + 0\|_X = \|x\|_X$, so $\|Q\| \leq 1$.

Now consider any Cauchy sequence $([x_n])_{n=1}^\infty$ in X/Y .

We can pick a subsequence $([x_{n_i}])_{i=1}^\infty$ such that $\|[x_{n_{i+1}}] - [x_{n_i}]\|_{X/Y} < 2^{-i}$.

Pick $z_1 \in X$ such that $[z_1] = [x_{n_1}]$.

Since $\|[x_{n_2}] - [x_1]\|_{X/Y} = \inf_{y \in Y} \|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$, there is $y \in Y$, such that $\|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$.

Take $z_2 = x_{n_2} + y$, we have $\|z_2 - z_1\|_X < \frac{1}{2}$.

Inductively, we can pick $(z_i)_{i=1}^\infty$, such that $\|z_i - z_{i-1}\|_X < 2^{-i}$.

We can check that this is a Cauchy sequence in X , so it has a limit $z = \lim_{i \rightarrow \infty} z_i \in X$.

Now for any $i \in \mathbb{N}^+$, we have

$$\begin{aligned} \|[x_{n_i}] - [z]\|_{X/Y} &= \|[z_i] - [z]\|_{X/Y} \\ &= \|Q(z_i) - Q(z)\|_{X/Y} \\ &= \|Q(z_i - z)\|_{X/Y} \\ &\leq \|Q\| \|z_i - z\| \\ &\rightarrow 0. \end{aligned}$$

Thus $([x_{n_i}])_{i=1}^\infty \rightarrow [z]$ is a convergent subsequence, which mean $([x_n])_{n=1}^\infty$ is convergent.

This shows $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.

Now if $Y \subsetneq X$, then $X/Y \neq \{0\}$, there must be some $[x] \in X/Y$ with $\|[x]\|_{X/Y} = 1$.

Thus, for all $k \in \mathbb{N}^+$, there is some $y_k \in Y$, such that $\|x + y_k\|_X \leq 1 + \frac{1}{k}$.

Now

$$\left\| Q\left(\frac{x + y_k}{\|x + y_k\|_X}\right) \right\|_{X/Y} = \frac{1}{\|x + y_k\|_X} \|Q(x + y_k)\| = \frac{\|[x]\|_{X/Y}}{\|x + y_k\|_X} \geq \frac{1}{1 + \frac{1}{k}}.$$

Since this is true for any $k \in \mathbb{N}^+$, taking the limit $k \rightarrow \infty$, we have $\left\| Q\left(\frac{x + y_k}{\|x + y_k\|_X}\right) \right\|_{X/Y} \geq 1$.

Yet $\left\| \frac{x + y_k}{\|x + y_k\|_X} \right\|_X = 1$, so $\|Q\| \geq 1$.

This shows $\|Q\| = 1$. □

Example 3.2.1. Consider a compact and Hausdorff X , and consider $(C(X), \|\cdot\|_\infty)$. Let $E \subseteq X$ be closed, and $I(E) := \{f \in C(X) : f|_E = 0\}$.

One can check $I(E)$ is closed (ideal), and $C(X)/I(E) \cong C(E)$ with an isometric isomorphism $\tilde{R} : [f] \mapsto f|_E$.

We claim that \tilde{R} is well-defined.

Indeed, if $[f] = [g]$, we must have $f - g \in I(E)$, which means $(f - g)|_E = 0$.

Thus $\tilde{R}([f]) = f|_E = g|_E = \tilde{R}([g])$.

Clearly \tilde{R} is linear.

Also, $\tilde{R}([f]) = 0 \implies f|_E = 0 \implies f \in I(E) \implies [f] = 0$, so \tilde{R} is injective.

By Tietze's Theorem, given any $g \in C(E)$, we can extend it to $f \in C(X)$, such that $f|_E = g$. Thus, \tilde{R} is surjective.

Consider any $f \in C(X), g \in I(E)$, we have

$$\begin{aligned} \|\tilde{R}([f])\| &= \|f|_E\|_{C(E)} \\ &= \sup_{x \in E} |f(x)| \\ &= \sup_{x \in E} |(f + g)(x)| \\ &\leq \sup_{x \in X} |(f + g)(x)| \\ &= \|f + g\|_{C(X)}. \end{aligned}$$

Since this hold for all $g \in I(E)$, we have

$$\|\tilde{R}([f])\| \leq \inf_{g \in I(E)} \|f + g\|_{C(X)} = \|[f]\|.$$

Thus, \tilde{R} is a contraction.

Consider any $f \in C(X)$.

If $f|_E = 0$, we have $\|[f]\| = \|[0]\| = 0 = \|f|_E\| = \|\tilde{R}([f])\|$.

Now consider $f|_E \neq 0$.

Define the function $k : \mathbb{C} \rightarrow \mathbb{C}$ by $k(z) := \begin{cases} z, & |z| \leq \|f|_E\|_\infty \\ \frac{z}{|z|} \|f|_E\|_\infty, & |z| \geq \|f|_E\|_\infty, \end{cases}$ which is well-defined and continuous.

Let $g := k \circ f \in C(X)$.

For any $x \in E$, we have $|f(x)| \leq \|f|_E\|_\infty$, so $g(x) = k(f(x)) = f(x)$.

Thus $g|_E = f|_E$, and there is $h \in I(E)$, such that $g = f + h$.

$$\begin{aligned} \|[f]\| &= \inf_{h \in I(E)} \|f + h\|_{C(X)} \\ &\leq \|g\|_{C(X)} \\ &\leq \|k\| \\ &\leq \|f|_E\|_\infty \\ &= \|\tilde{R}(f)\|. \end{aligned}$$

This proves \tilde{R} is an isometry.

Proposition 3.17. *For any infinite dimensional Banach Space X , it does not have a countable basis. Also, the unit ball is not compact.*

Proof. See A2. □

3.3 Baire Category Theorem

Definition 3.18. Let (X, \mathcal{T}) be a topological space, then $A \subseteq X$ is called **nowhere dense** if $(\bar{A})^\circ = \emptyset$.

Theorem 3.18 (Baire Category Theorem). *Let (X, d) be a complete metric space, then X cannot be written as a countable union of nowhere dense sets.*

Corollary 3.19. *Let $\{U_i\}_{i=1}^\infty$ be a countable set of open dense sets, then $\bigcap_{i=1}^\infty U_i$ is dense.*

3.3.1 Banach-Steinhaus Theorem

Definition 3.19. Let X, Y be Banach spaces, $\mathcal{S} \subseteq B(X, Y)$ is called **pointwise bounded** if $\forall x \in X$, $\mathcal{S}x$ is bounded. Namely, $\exists k_x > 0$, such that $\forall S \in \mathcal{S}$, $\|Sx\| \leq k_x$.

Theorem 3.20 (Banach-Steinhaus). *Let X, Y be Banach spaces, suppose $\mathcal{S} \subseteq B(X, Y)$ is pointwise bounded, then \mathcal{S} is bounded in $B(X, Y)$. Namely, $\sup_{S \in \mathcal{S}} \|S\| < \infty$.*

Proof. For each x , let $k_x > 0$ be such $\forall S \in \mathcal{S}$, $\|Sx\| \leq k_x$.

For each $n \in \mathbb{N}$, let $A_n := \{x \in X : k_x \leq n\}$.

For any $(x_k)_{k \in \mathbb{N}}$ in A_n , with $x = \lim_{k \rightarrow \infty} x_k$, we have

$$\|Sx\| = \lim_{k \rightarrow \infty} \|Sx_k\| \leq n.$$

Thus, $x \in A_n$, which shows A_n is closed.

Notice that $X = \bigcup_{n \in \mathbb{N}} A_n$, so by Baire Category Theorem, there is $n_0 \in \mathbb{N}$, such that $(A_{n_0})^\circ \neq \emptyset$.

Thus, there is $x_0 \in A_{n_0}$, $r > 0$, such that $\bar{B}(x_0, r) \subset B(x_0, 2r) \subseteq (A_{n_0})^\circ \subset A_{n_0} = A_{n_0}$.

Now for any $s \in \mathcal{S}$, $y \in X$ such that $\|y\| \leq 1$, we have that

$$\begin{aligned} \|sy\| &= \left\| \frac{s(x_0) - s(x_0 - ry)}{r} \right\| \\ &\leq \frac{\|s(x_0)\| + \|s(x_0 - ry)\|}{r} \\ &\leq \frac{2n_0}{r}, \end{aligned}$$

since $x_0, x_0 - ry \in \bar{B}(x_0, r) \subset A_{n_0}$.

Thus, $\|s\| = \sup_{y \in X \text{ such that } \|y\| \leq 1} \|sy\| \leq \frac{2n_0}{r}$.

Since this holds for all $s \in \mathcal{S}$, we have $\sup_{s \in \mathcal{S}} \|s\| = \frac{2n_0}{r} < \infty$. \square

Corollary 3.21 (Limit of bounded operators). *Let X, Y be Banach spaces, consider $(T_n)_{n=1}^\infty$ be a sequence of $B(X, Y)$. Suppose $\forall x \in X$, $(T_n x)_{n=1}^\infty$ is convergent, then $\mathcal{S} := \{T_n : n \in \mathbb{N}\}$ is bounded. In addition, for $Tx := \lim_{n \rightarrow \infty} T_n x$, we have $T \in B(X, Y)$, and $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$.*

Proof. Since $(T_n x)_{n=1}^\infty$ is convergent, it is bounded. This is equivalent to saying \mathcal{S} is pointwise bounded. By the Banach-Steinhaus Theorem, \mathcal{S} is bounded.

Now $\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| = \sup_{n \in \mathbb{N}} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n\| \|x\|$.

Since this holds for all $x \in X$, we have $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$. \square

Example 3.3.1. Consider $f \in C(\mathbb{T})$, we can define the N^{th} partial sum of its Fourier series $S_N(f)(e^{it}) := \sum_{n=-N}^N \hat{f}(n) e^{int}$.

We note that $S_N(f)$ does not necessarily converge to f in $C(\mathbb{T})$, nor pointwise.

Indeed, consider $\phi_N(f) := S_N(f)(1) \in \mathbb{C}$. Note that ϕ_N is linear.

One can show that

$$\begin{aligned} \phi_N(f) &= \sum_{n=-N}^N \hat{f}(n) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \left(\sum_{n=-N}^N e^{-int} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) D_N(t) dt, \end{aligned}$$

where $D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}$ is the **Dirichlet's Kernel**.

Actually, $\exists C > 0$, such that $\|D_N\|_1 \geq C \log(N)$.

Also, $\|\phi_N\|_{C_*(\mathbb{T})} = \|D_N\|_1$.

Thus, $(\phi_N)_{N \in \mathbb{N}}$ is not bounded.

By Banach-Steinhaus, $(\phi_N)_{N \in \mathbb{N}}$ is not pointwise bounded.

Thus, there is $f \in C(\mathbb{T})$, such that $|\phi_N(f)| \rightarrow \infty$.

Thus, $S_N f$ does not converge to f at 1.

3.3.2 Open Mapping Theorem

Theorem 3.22 (Open Mapping Theorem). *Let X, Y be Banach spaces, suppose $T \in B(X, Y)$ is surjective, then it is **open**. i.e. \forall open $U \subseteq X$, $T(U) \subseteq Y$ is open.*

Proof. We have

$$\begin{aligned} Y &= T(X) \\ &= T\left(\bigcup_{n=1}^{\infty} B^X(0, n)\right) \\ &= T\left(\bigcup_{n=1}^{\infty} nB^X(0, 1)\right) \\ &= \bigcup_{n=1}^{\infty} nT(B^X(0, 1)). \end{aligned}$$

By the Baire Category Theorem, there is n_0 , such that $\left(n_0 \overline{T(B^X(0, 1))}\right)^o = \left(\overline{n_0 T(B^X(0, 1))}\right)^o \neq \emptyset$.

Thus there is $r_0 > 0, y_0 \in Y$, such that $B^Y(y_0, r_0) \subset n_0 \overline{T(B^X(0, 1))}$.

Notice that $B^Y(-y_0, r_0) \subset \overline{n_0 T(B^X(0, 1))}$ as well, and $\overline{n_0 T(B^X(0, 1))}$ is convex.

Thus, for any $y \in B^Y(0, r)$, we have $y = \frac{1}{2}(y - y_0) + \frac{1}{2}(y + y_0)$, where $y - y_0 \in B^Y(-y_0, r_0)$, $y + y_0 \in B^Y(y_0, r_0)$.

By convexity, $y \in \overline{n_0 T(B^X(0, 1))}$.

Thus $B^Y(0, r_0) \subset \overline{n_0 T(B^X(0, 1))}$.

Take $r := \frac{r_0}{n_0}$, we have $B^Y(0, r) \subset \overline{T(B^X(0, 1))}$.

Now we want to show $\overline{T(B^X(0, 1))} \subset T(B^X(0, 2))$.

Let $y \in \overline{T(B^X(0, 1))}$, there is $x_1 \in B^X(0, 1)$, such that $\|y - Tx_1\| < \frac{r}{2}$.

Let $y_1 := y - Tx_1 \in B^Y(0, \frac{r}{2}) \subset \overline{T(B^X(0, \frac{1}{2}))}$.

Thus there is $x_2 \in B^X(0, \frac{1}{2})$, such that $\|y_1 - Tx_2\| = \|y - Tx_1 - Tx_2\| < \frac{r}{4}$.

Recursively, we can find $x_k \in B^X(0, \frac{1}{2^{k-1}})$, such that $\|y_k\| < \frac{r}{2^k}$ for $y_k := y - Tx_1 - \dots - Tx_k$.

Since $\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k+1} = 2$, and X is complete, we have $x := \sum_{k=1}^{\infty} x_k$ converges in X , and $x \in B^X(0, 2)$.

Since T is continuous, we have $y = \sum_{k=1}^{\infty} Tx_k = Tx \in T(B^X(0, 2))$.

Thus $B^Y(0, r) \subset \overline{T(B^X(0, 1))} \subset T(B^X(0, 2))$.

Now for any open $U \subseteq X$, there is $\epsilon > 0$, such that $x + \frac{\epsilon}{2}B^X(0, 2) = B^X(x, \epsilon) \subseteq U$.

Thus $B^Y(Tx, \frac{\epsilon}{2}r) = Tx + \frac{\epsilon}{2}B^Y(0, r) \subseteq Tx + \frac{\epsilon}{2}T(B^X(0, 2)) \subseteq T(U)$.

Thus $T(U)$ is open. \square

Theorem 3.23 (Banach Isomorphism Theorem). *Let X, Y be Banach spaces, suppose $A \in B(X, Y)$ is bijective, then A^{-1} is continuous and bounded as well. Namely, A is invertible and is an isomorphism of $X \cong Y$ as Banach Spaces.*

Proof. A is open by the open mapping theorem. Now for any open $U \subseteq X$, we have that $(A^{-1})^{-1}(U) = A(U)$ is open. Thus A^{-1} is continuous. \square

Corollary 3.24. *Let X, Y be Banach spaces, suppose $T \in B(X, Y)$ is surjective, then $X/\ker(T) \cong Y$ as Banach spaces with $\tilde{T} : X/\ker(T) \rightarrow Y$ by $\tilde{T}([x]) := T(x)$.*

Proof. We can check that \tilde{T} is a well-defined bijection.

Also, for any $x \in X, y \in \ker(T)$, we have

$$\begin{aligned} \|\tilde{T}([x])\| &= \|T(x)\| \\ &= \|T(x + y)\| \\ &\leq \|T\| \|x + y\|. \end{aligned}$$

Since this holds for all $y \in \ker(T)$, we have

$$\|\tilde{T}([x])\| \leq \inf_{y \in \ker(T)} \|T\| \|x + y\| = \|T\| \|x\|.$$

Thus, $\|\tilde{T}\| \leq \|T\|$, which means $\tilde{T} \in B(X/\ker(T), Y)$ is continuous.

By the Banach Isomorphism Theorem, \tilde{T}^{-1} is continuous as well. \square

Corollary 3.25. *Suppose X is a vector space that is complete under two different norms $\|\cdot\|_1, \|\cdot\|_2$, and $\exists C > 0$, such that $\forall x \in X, \|x\|_1 \leq C\|x\|_2$, then the two norms are equivalent.*

Proof. Consider the map $id : (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$, we have that id is bijective and bounded by C . By the Banach Isomorphism Theorem, id^{-1} is bounded as well.

Thus $\forall x \in X$, we have $\|x\|_2 \leq \|id^{-1}\| \|x\|_1$. \square

Corollary 3.26. *Let X be any finite-dimensional linear normed space over \mathbb{F} , then any two norms on X are equivalent and X is complete.*

Proof. Let $\|\cdot\|$ be any norm on X , and let $\|\cdot\|_2$ be the Euclidean norm on \mathbb{F}^n .

Pick any basis $\{e_i\}_{i=1}^n$ for X .

Consider the function $T : (\mathbb{F}^n, \|\cdot\|_2) \rightarrow (X, \|\cdot\|)$ by $T(a) := \sum_{i=1}^n a^i e_i \in X$ for any $a = (a^i)_{i=1}^n \in \mathbb{F}^n$.

It is easy to check that T is linear and bijective. Also,

$$\begin{aligned} \|Ta\| &= \left\| \sum_{i=1}^n a^i e_i \right\| \\ &\leq \sum_{i=1}^n |a^i| \|e_i\| \\ &\leq \|a\|_2 \left(\sum_{i=1}^n \|e_i\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, $\|T\| \leq \alpha := \left(\sum_{i=1}^n \|e_i\|^2 \right)^{\frac{1}{2}}$, so it is continuous.

Since $S := \{a \in \mathbb{F}^n : \|a\|_2 = 1\}$ is closed and bounded thus compact in $(\mathbb{F}^n, \|\cdot\|_2)$, and $\|\cdot\|$ is continuous, we have $r := \inf_{a \in S} \|T(a)\|$ is achieved in S by the Extreme Value Theorem.

Since $0 \notin S$, and $\|T(a)\| = 0 \implies T(a) = 0 \implies a = 0$, we have $r \neq 0$.

Consider any $a \neq 0 \in \mathbb{F}^n$ such that $\|T(a)\| \leq r$, we must have $\frac{a}{\|a\|_2} \in S$, so

$$\frac{1}{\|a\|_2} \|T(a)\| = \left\| T\left(\frac{a}{\|a\|_2}\right) \right\| \geq r.$$

Thus $\|a\|_2 \leq 1$.

This shows $\forall x \in X$ with $\|x\| \leq 1$, we must have $\|T^{-1}(x)\|_2 \leq \frac{1}{r}$, which show that $\|T^{-1}\| \leq \frac{1}{r} < \infty$ is bounded.

This shows that $T : (\mathbb{F}^n, \|\cdot\|_2) \rightarrow (X, \|\cdot\|)$ is an homeomorphism for any $\|\cdot\|$ on X , which means X is complete and all norms are equivalent. \square

3.3.3 Closed Graph Theorem

Definition 3.20. Let X, Y be Banach spaces, we can consider $X \oplus Y := \{(x, y) : x \in X, y \in Y\}$, which is a vector space under component-wise addition and scale multiplication. For $1 \leq p < \infty$, we define

$$\|(x, y)\|_p := (\|x\|^p + \|y\|^p)^{\frac{1}{p}},$$

and

$$\|(x, y)\|_\infty := \max\{\|x\|, \|y\|\}.$$

Definition 3.21. Let X, Y be Banach spaces, and $D \subseteq X$ is a subspace, the **graph** of a linear map $T : D \rightarrow Y$ is

$$\mathcal{G}(T) := \{(x, Tx) : x \in D\}.$$

We say T is **closed** if $\mathcal{G}(T)$ is closed in $X \oplus_\infty Y$.

Theorem 3.27 (Closed Graph Theorem). *Let X, Y be Banach spaces, a linear map $T : X \rightarrow Y$ is closed if and only if $T \in B(X, Y)$.*

Proof. (\implies).

Consider the projection maps $\pi_1(x, y) := x, \pi_2(x, y) := y$, which are both continuous.

Since $\mathcal{G}(T)$ is closed in $X \oplus Y$, it is a Banach Space.

Since T is defined on entire X , we have that $\pi_1|_{\mathcal{G}(T)}$ is a continuous bijection.

By the Bounded Inverse Theorem, $(\pi_1|_{\mathcal{G}(T)})^{-1} : X \rightarrow \mathcal{G}(T)$ is bounded.

Notice that $T = \pi_2 \circ (\pi_1|_{\mathcal{G}(T)})^{-1}$, which will also be bounded.

(\impliedby).

Consider any sequence $((x_i, Tx_i))_{i \in \mathbb{N}}$ in $\mathcal{G}(T)$, such that $(x_i, Tx_i) \rightarrow (x, y) \in X \oplus_\infty Y$.

Thus $x_i \rightarrow x \in X$ and $Tx_i \rightarrow y \in Y$.

Since T is continuous, $Tx_i \rightarrow Tx$, which means $Tx = y$, so $(x, y) \in \mathcal{G}(T)$. \square

Remark. It is important that T is defined on the entire X .

Example 3.3.2. Consider $X = Y = (C[0, 1], \|\cdot\|_1)$, and $D = C^1([0, 1])$. Let $T := \frac{d}{dx} : D \rightarrow C^1([0, 1])$, which is unbounded but closed.

Corollary 3.28. Let X, Y be Banach spaces, then for any linear map $T : X \rightarrow Y$ $T \in B(X, Y)$ if and only if $\forall (x_n)_{n=1}^\infty$ in X , such that $x_n \rightarrow 0$, $Tx_n \rightarrow y \in Y$, we have $y = 0$.

Proof. (\Rightarrow) is easy.

(\Leftarrow).

Consider any $((x_n, Tx_n))_{n=1}^\infty$ in $\mathcal{G}(T)$, such that $x_n \rightarrow x \in X$, and $Tx_n \rightarrow y \in Y$.

We have that $(x - x_n) \rightarrow 0$, and by linearity of T , we have $T(x - x_n) = Tx - Tx_n \rightarrow Tx - y$.

By assumption, we have $y - Tx = 0$, which means $y = Tx$.

Thus, $(x, y) \in \mathcal{G}(T)$, which means $\mathcal{G}(T)$ is closed. \square

3.4 Hahn-Banach Theorem

Definition 3.22. Let X be a normed linear space. $p : X \rightarrow \mathbb{R}$ is called a **sublinear functional** if $\forall t > 0, \forall x \in X$, $p(tx) = tp(x)$, and $p(x + y) \leq p(x) + p(y)$.

Proposition 3.29. Let X be a normed linear space, then $\|\cdot\|$ is always a sublinear functional.

Proposition 3.30. Let X be a Banach space, then for any $x \in X$, the functional $p_x : B(X) \rightarrow \mathbb{R}$ defined by $p_x(T) := \|Tx\|$ is sublinear.

Proposition 3.31. Let X be a Banach space, then for any $x \in X, \phi \in X^*$, the functional $p_{\phi, x} : B(X) \rightarrow \mathbb{R}$ defined by $p_{\phi, x}(T) := |\phi(Tx)|$ is sublinear.

Theorem 3.32 (Extension). Let X be a linear vector space over \mathbb{R} . Let $M_0 \subseteq X$ be a linear subspace, and $p : X \rightarrow \mathbb{R}$ be a sublinear functional, then for any linear $f_0 : M_0 \rightarrow \mathbb{R}$ such that $\forall x \in M_0, f_0(x) \leq p(x)$, there is an extension $f : X \rightarrow \mathbb{R}$, such that $f|_{M_0} = f_0$, and $\forall x \in X, f(x) \leq p(x)$.

Proof. Consider

$$P := \{(M, f) | M_0 \subseteq M \subseteq X \text{ is a subspace; } f : M \rightarrow \mathbb{R} \text{ is linear, } f|_{M_0} = f_0; \forall x \in M, f(x) \leq p(x)\},$$

with the partial order $(M, f) \leq (M', f')$ if $M \subseteq M', f'|_M = f$.

Consider any chain $\{(M_\alpha, f_\alpha)\}_{\alpha \in A} \subset P$.

Let $M := \bigcup_{\alpha \in A} M_\alpha \subseteq X$, and let $f(x) := f_\alpha(x)$ for any $M_\alpha \ni x$.

We can check that f is well-defined and linear, satisfying the requirement.

Thus, $(M, f) \in P$ is an upper bound for the chain.

By Zorn's lemma, there is a maximal element (M_1, f_1) of P .

Suppose for contradiction that $M_1 \neq X$, then there is some $x \in X \setminus M_1$.

Notice that if we take any $m_1, m_2 \in M_1$, we have that

$$\begin{aligned} f_1(m_1) + f_1(m_2) &= f_1(m_1 + m_2) \\ &\leq p(m_1 + m_2) \\ &\leq p(m_1 - x) + p(m_2 + x). \end{aligned}$$

Thus, $f_1(m_1) - p(m_1 - x) \leq p(m_2 + x) - f_1(m_2)$.

Since this holds for all $m_1, m_2 \in M_1$, we have

$$\sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \leq \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)).$$

Take any $a \in [\sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)), \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2))]$.

Let $M := M_1 \oplus \text{Span}\{x\}$. Since M_1 is a subspace, this is a direct product, i.e., $\forall y \in M_1 \oplus \text{Span}\{x\}$, there is some unique $m \in M_1, t \in \mathbb{R}$, such that $y = m + tx$.

Define $f : M \rightarrow \mathbb{R}$ by $f(m + tx) := f_1(m) + |t|a$ for any $t \in \mathbb{R}$.

We can easily check $f|_{M_1} = f_1$ and that f is linear.

Suppose $t > 0$, we have

$$\begin{aligned}
f(m + tx) &= f_1(m) + |t|a \\
&= f_1(m) + ta \\
&\leq f_1(m) + t \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)) \\
&\leq f_1(m) + t \left(p\left(\frac{m}{t} + x\right) - f_1\left(\frac{m}{t}\right) \right) \\
&= f_1(m) + p\left(t\left(\frac{m}{t} + x\right)\right) - f_1\left(t\frac{m}{t}\right) \\
&= p(m + tx).
\end{aligned}$$

Similarly, if $t \leq 0$, we have

$$\begin{aligned}
f(m + tx) &= f_1(m) + |t|a \\
&= f_1(m) - ta \\
&\leq f_1(m) - t \sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \\
&\leq f_1(m) - t \left(f_1\left(\frac{m}{t}\right) - p\left(\frac{m}{t} - x\right) \right) \\
&= f_1(m) - f_1\left(t\frac{m}{t}\right) + p\left(t\left(\frac{m}{t} + x\right)\right) \\
&= p(m + tx).
\end{aligned}$$

This contradicts with the maximality of (M_1, f_1) .

Thus, $M_1 = X$, and $f_1 : X \rightarrow \mathbb{R}$ is the desired extension. \square

Theorem 3.33 (Hahn-Banach). *Let X be a normed linear space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Let $M \subseteq X$ be a linear subspace, and linear $\phi_0 : M \rightarrow \mathbb{F}$ with $\|\phi_0\|_{M^*} < \infty$, then there is a norm preserving extension $\phi \in X^*$, such that $\phi|_M = \phi_0$, and $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$.*

Proof. First, consider $\mathbb{F} = \mathbb{R}$.

We note that $p : x \mapsto \|\phi_0\|_{M^*} \|x\|$ is sublinear, and $\forall x \in M$, $\phi_0(x) \leq \|\phi_0\|_{M^*} \|x\| = p(x)$.

Thus, there is a linear extension $\phi : X \rightarrow \mathbb{R}$, such that $\forall x \in X$, $\phi(x) \leq p(x) = \|\phi_0\|_{M^*} \|x\|$.

Also, $-\phi(x) = \phi(-x) \leq \|\phi_0\|_{M^*} \|-x\| = \|\phi_0\|_{M^*} \|x\|$.

Thus $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$.

Now suppose $\mathbb{F} = \mathbb{C}$.

Consider $g_0 := \Re \phi_0 : M \rightarrow \mathbb{R}$, which is \mathbb{R} -linear, and $\|g_0\|_{M_{\mathbb{R}}^*} \leq \|\phi_0\|_{M^*}$.

Using the real case, we can extend g_0 to $g \in X_{\mathbb{R}}^*$.

Now define $\phi : X \rightarrow \mathbb{C}$ by $\phi(x) := g(x) + ig(-ix)$.

We can see that ϕ is \mathbb{R} linear.

Also,

$$\phi(ix) = g(ix) + ig(x) = i(g(x) - ig(ix)) = i(g(x) + ig(-ix)) = i\phi(x),$$

so ϕ is \mathbb{C} -linear.

For any $m \in M$, we have that

$$\begin{aligned}
\phi(x) &= g(m) + ig(-im) \\
&= g_0(m) + ig_0(-im) \\
&= \Re(\phi_0(m)) + i\Re(\phi_0(-im)) \\
&= \Re(\phi_0(m)) + i\Re(-i\phi_0(m)) \\
&= \Re(\phi_0(m)) + i\Im(\phi_0(m)) \\
&= \phi_0(m).
\end{aligned}$$

Thus $\phi|_M = \phi_0$.

Now consider any $x \in X$ with $\|x\| \leq 1$.

We have that

$$|\phi(x)| = \lambda \phi(x) = \phi(\lambda x) = g(\lambda x) + ig(-i\lambda x)$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Since g is real-valued, and $|\phi(x)| \in \mathbb{R}$, we must have

$$|\phi(x)| = g(\lambda x) \leq \|g\|_{X^*} \|\lambda x\| = \|g_0\|_{M_{\mathbb{R}}^*} \|x\| \leq \|\phi_0\|_{M^*}.$$

Thus $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$.

Lastly, we always have

$$\begin{aligned} \|\phi_0\|_{M^*} &= \sup_{x \in M, \|x\| \leq 1} |\phi_0(x)| \\ &= \sup_{x \in M, \|x\| \leq 1} |\phi(x)| \\ &\leq \sup_{x \in X, \|x\| \leq 1} |\phi(x)| \\ &= \|\phi\|_{X^*}. \end{aligned}$$

□

Corollary 3.34. *Let X be a normed linear space over \mathbb{R} . Let $M \subseteq X$ be a linear subspace, and linear $\phi_0 : M \rightarrow \mathbb{F}$ with $\|\phi_0\|_{M^*} < \infty$, then for any $x \in X, a \in \mathbb{R}$, there is a Hahn-Banach extension $\phi \in X^*$, such that $\phi|_M = \phi_0$, $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$ and $\phi(x) = a$ if and only if*

$$\begin{aligned} \sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) &\leq a \leq \inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|) \\ &= \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)). \end{aligned}$$

Proof. We first note that since M is a subspace, $m \in M$ if and only if $-m \in M$, and

$$\begin{aligned} \phi_0(m) + \|\phi_0\|_{M^*} \|m - x\| &= -\phi_0(-m) + \|\phi_0\|_{M^*} \|(-m) - x\| \\ &= \|\phi_0\|_{M^*} \|(-m) + x\| - \phi_0(-m). \end{aligned}$$

Thus,

$$\inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|) = \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)).$$

Consider any Hahn-Banach extension $\phi \in X^*$, such that $\phi|_M = \phi_0$, $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$.

For any $x \in X$, we must satisfy

$$\begin{aligned} |\phi_0(m) - \phi(x)| &= |\phi(m) - \phi(x)| \\ &= |\phi(m - x)| \\ &\leq \|\phi\|_{X^*} \|m - x\| \\ &= \|\phi_0\|_{M^*} \|m - x\|. \end{aligned}$$

Thus,

$$\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\| \leq \phi(x) \leq \phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|.$$

Since this holds for all $m \in M$, we have

$$\sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) \leq \phi(x) \leq \inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|).$$

On the other hand, suppose

$$\sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) \leq a \leq \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)).$$

Taking $p : x \mapsto \|\phi_0\|_{M^*} \|x\|$ as in the proof of the Hahn-Banach Theorem, we have that

$$\sup_{m \in M} (\phi_0(m) - p(m - x)) \leq a \leq \inf_{m \in M} (p(m + x) - \phi_0(m)).$$

By the proof of the Hahn-Banach Theorem, we can always extend ϕ_0 on $M \oplus \text{Span}\{x\}$ to

$$\tilde{\phi}(m + tx) := \phi_0(m) + |t|a$$

for any $t \in \mathbb{R}$. Also,

$$\tilde{\phi}(x) = \tilde{\phi}(0 + 1 \cdot x) = \phi_0(0) + 1 \cdot a = a.$$

Now we can take the Hahn-Banach extension of $\tilde{\phi}$ to be ϕ , which will satisfy $\phi(x) = \tilde{\phi}(x) = a$. \square

Corollary 3.35. *Let X be a normed linear space. Given $0 \neq x \in X$, then there is $\phi \in X^*$, such that $\|\phi\| = 1$, and $\phi(x) = \|x\|$. In particular, $\|x\|_X = \sup \{|\phi(x)| : \phi \in X^*, \|\phi\| = 1\}$.*

Proof. Let $M = \text{Span}\{x\}$, and define $\phi_0(\lambda x) := \lambda\|x\|$.

Then we have $\|\phi_0\| = 1$, and by Hahn-Banach Theorem, there is $\phi \in X^*$, such that $\|\phi\| = \|\phi_0\| = 1$, and $\phi(x) = \phi_0(x) = \|x\|$.

In particular, $\|x\| = \phi(x) = |\phi(x)|$, so $\|x\|_X \leq \sup \{|\psi(x)| : \psi \in X^*, \|\psi\| = 1\}$.

Also, for any $\psi \in X^*$, $\|\psi\| = 1$, we have $|\psi(x)| \leq \|\psi\| \|x\| = \|x\|$. \square

Corollary 3.36. *Let X be a normed linear space, then X^* separates the points of X .*

Proof. For any $x \neq y$, we have $x - y \neq 0$, so there is $\phi \in X^*$, such that

$$\phi(x) - \phi(y) = \phi(x - y) = \|x - y\| \neq 0,$$

which separates x, y . \square

Corollary 3.37. *Let X be a normed linear space, then there is a canonical linear isometric embedding $i : X \hookrightarrow X^{**}$ by $i(x) := \hat{x}$, $\hat{x}(\phi) := \phi(x)$.*

Proof. We have $\|\hat{x}\| = \sup_{\|\phi\|=1} \|\hat{x}(\phi)\| = \sup_{\|\phi\|=1} |\phi(x)| = \|x\|$. \square

Remark. It is not necessarily that $X \cong X^{**}$. Indeed, if $X = C_0(\mathbb{N})$, we have that $X^* \cong \ell^1(\mathbb{N})$, and $X^{**} \cong \ell_\infty(\mathbb{N})$.

Definition 3.23. A Banach space X is **reflexive** if $i(X) = X^{**}$, where $i : X \hookrightarrow X^{**}$ is the canonical linear isometric embedding by $i(x) := \hat{x}$, $\hat{x}(\phi) := \phi(x)$.

Corollary 3.38. *Let X be a normed linear space. For any closed subspace M , and $x \notin M$, there is $f \in X^*$, such that $\|f\| = 1$, $f|_M = 0$, and $f(x) = \text{dist}(x, M)$.*

Proof. Consider the quotient map $Q : X \rightarrow Y$, where $Y := X/M$. Since $x \notin M$, we have $[x] \neq 0$.

Let $\phi \in Y^*$, such that $\|\phi\|_{Y^*} = 1$, and $\phi([x]) = \|[x]\|_Y = \inf_{m \in M} \|x + m\| = \text{dist}(x, M)$.

Let $f := \phi \circ Q$, we have that $f(x) = \text{dist}(x, M)$, and $\forall m \in M$, $f(m) = \phi(m + M) = \phi(0) = 0$.

Also, $\|f\| \leq \|\phi\| \|Q\| \leq \|\phi\| = 1$. \square

Definition 3.24. Let X be a Banach Space. For $Y \subseteq X$, the **annihilator** of Y is

$$Y^\perp := \{\varphi \in X^* : \varphi(Y) = \{0\}\}.$$

For $Z \subseteq X^*$, the **preannihilator** of Z is

$$Z_\perp := \{x \in X : \hat{x}|_Z = 0\} = Z^\perp \cap X.$$

Proposition 3.39. *Let X be a Banach Space, then*

$$(Y^\perp)_\perp = \overline{\text{Span}(Y)}.$$

Proof. Let $M := \overline{\text{Span}(Y)}$.

For any $y \in M$, $f \in Y^\perp$, we have $\hat{y}(f) = f(y) = 0$, so $y \in (Y^\perp)^\perp$. Thus, $M \subseteq (Y^\perp)^\perp$.

Suppose $x \notin Y$, then there is $f \in X^*$ such that $f|_M = 0$, and $f(x) = \text{dist}(x, M)$. In particular, $f \in Y^\perp$ and $\hat{x}(f) = f(x) \neq 0$.

Thus, $x \notin (Y^\perp)^\perp$.

This shows $(Y^\perp)^\perp \subseteq M$.

Thus,

$$M = (Y^\perp)^\perp.$$

□

4 Hilbert Spaces

See more in Prof Tran's notes for Amath731.

Definition 4.1. Let X be a vector space over \mathbb{F} . $M : X \times X \rightarrow \mathbb{F}$ is a **sesquilinear positive semidefinite form** on X if it satisfies

1. conjugate symmetry; i.e. $M[x, y] = \overline{M[y, x]}$.
2. linearity in the second argument; i.e. $M[x, ay + z] = M[x, y] + aM[x, z]$.
3. positive semi-definiteness; i.e. $M[x, x] \geq 0$.

Proposition 4.1. Let X be a vector space over \mathbb{F} , with a sesquilinear positive semidefinite form M . For all $x, y \in H$, we have

1. $M[x, y] + M[y, x] = 2\Re(M[x, y]) = 2\Re(M[y, x])$, which is twice the real part of $M[x, y]$.
2. $M[x, y] - M[y, x] = 2\Im(M[x, y]) = -2\Im(M[y, x])$, which is twice the imaginary part of $M[x, y]$.
3. $M[x, y]M[y, x] = |M[x, y]|^2 = |M[y, x]|^2$.
4. $M[x + y, x + y] = M[x, x] + 2\Re(M[y, x]) + M[y, y]$.
5. $M[x - y, x - y] = M[x, x] - 2\Re(M[y, x]) + M[y, y]$.

Proof. 1. $M[x, y] + M[y, x] = M[x, y] + \overline{M[x, y]} = 2\Re(M[x, y])$.

2. $M[x, y] - M[y, x] = M[x, y] - \overline{M[x, y]} = 2\Im(M[x, y])$.

3. $M[x, y]M[y, x] = M[x, y]\overline{M[x, y]} = |M[x, y]|^2$.

4.

$$\begin{aligned} M[x + y, x + y] &= M[x + y, x] + M[x + y, y] \\ &= M[x, x] + M[y, x] + M[x, y] + M[y, y] \\ &= M[x, x] + M[y, x] + \overline{M[y, x]} + M[y, y] \\ &= M[x, x] + 2\Re(M[y, x]) + M[y, y]. \end{aligned}$$

5. By 3., we have

$$\begin{aligned} M[x - y, x - y] &= M[x + (-y), x + (-y)] \\ &= M[x, x] + 2\Re(M[-y, x]) + M[-y, -y] \\ &= M[x, x] - 2\Re(M[y, x]) + M[y, y]. \end{aligned}$$

□

Definition 4.2. An **inner product space** is a vector space H that has an inner product: $\langle -, \cdot \rangle : H \times H \rightarrow \mathbb{C}$, such that $\forall u, v, w \in H, a, b \in \mathbb{C}$, it satisfies

1. conjugate symmetry; i.e. $\langle v, w \rangle = \overline{\langle w, v \rangle}$,
2. linearity in the second argument; i.e. $\langle v, au + bw \rangle = a\langle v, u \rangle + b\langle v, w \rangle$, and
3. positive definiteness; i.e. $\langle v, v \rangle \geq 0$, and if $v \neq 0$, we must have $\langle v, v \rangle > 0$.

Remark. Notice that the inner product is a sesquilinear positive semidefinite form, with the additional requirement: if $v \neq 0$, we must have $\langle v, v \rangle > 0$.

Proposition 4.2. For every inner product space with $\langle -, \cdot \rangle$, and for all $x, y \in H$, we have

1. $\langle x, y \rangle + \langle y, x \rangle = 2\Re(\langle x, y \rangle) = 2\Re(\langle y, x \rangle)$.
2. $\langle x, y \rangle - \langle y, x \rangle = 2\Im(\langle x, y \rangle) = -2\Im(\langle y, x \rangle)$.
3. $\langle x, y \rangle \langle y, x \rangle = |\langle x, y \rangle|^2$.
4. $\langle x + y, x + y \rangle = \langle x, x \rangle + 2\Re(\langle y, x \rangle) + \langle y, y \rangle$.
5. $\langle x - y, x - y \rangle = \langle x, x \rangle - 2\Re(\langle x, y \rangle) + \langle y, y \rangle$.

Theorem 4.3 (Cauchy-Schwarz). For every inner product space H ,

$$\forall u, v \in H, |\langle u, v \rangle| \leq \|u\| \|v\|,$$

where we define $\|x\| = \sqrt{\langle x, x \rangle}$ or any $x \in H$.

In particular, when $\|u\| \neq 0$, $\|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 = \|z\|^2$, where $z := \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u$.

Proof. Notice that this is trivially true and equality holds to be zero when $u = 0$. Now we assume $u \neq 0$, then $\|u\| = \sqrt{\langle u, u \rangle} > 0$.

$$\begin{aligned} \|z\|^2 &= \langle z, z \rangle \\ &= \left\langle \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u, \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u \right\rangle \\ &= \|u\|^2 \langle v, v \rangle - \langle u, v \rangle \langle u, v \rangle - \overline{\langle u, v \rangle} \langle v, u \rangle + \frac{\langle v, u \rangle \langle u, v \rangle}{\|u\|^2} \langle u, u \rangle \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 - \cancel{|\langle v, u \rangle|^2} + \cancel{|\langle v, u \rangle|^2} \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2. \end{aligned}$$

Now $\|z\|^2 = \langle z, z \rangle \geq 0$, we have the result. □

Remark. This proof does not use the positive definiteness of the inner product, so it also works for a sesquilinear positive semidefinite form M . i.e.

$$|M[x, y]| \leq M[x, x]M[y, y].$$

Proposition 4.4. For every inner product space with $\langle -, \cdot \rangle$, there is a norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Proof. Consider any $x \in H, a \in \mathbb{C}$,

$$\begin{aligned}
\|a \cdot x\| &= \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2} \sqrt{\langle x, x \rangle} = |a| \|x\| \\
\forall x \neq 0, \|x\| &= \sqrt{\langle x, x \rangle} > 0 \\
\|0\| &= \sqrt{\langle 0, 0 \rangle} = 0 \\
\|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \\
&= \|x\|^2 + \|y\|^2 + 2\Re(\langle x, y \rangle) \\
&\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y \rangle| \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\
&\leq (\|x\| + \|y\|)^2.
\end{aligned}$$

Thus $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm. □

Corollary 4.5. For every inner product space, there is a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$

Proposition 4.6 (polarization identity). Let H be an inner product space over \mathbb{F} . The inner product is completely determined by the induced norm. Indeed, for any $x, y \in H$, we have

$$\langle x, y \rangle = \frac{1}{4} \sum_{\epsilon \in \{\pm i, \pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if $\mathbb{F} = \mathbb{C}$, and

$$\langle x, y \rangle = \frac{1}{2} \sum_{\epsilon \in \{\pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if $\mathbb{F} = \mathbb{R}$.

Proposition 4.7. If $\forall v, \langle v, u \rangle = 0$, then $u = 0$.

Proposition 4.8. For an Inner product space $H, \forall y, x = \lim_{i \rightarrow \infty} x_i \in H$, we have

$$\langle x, y \rangle = \lim_{i \rightarrow \infty} \langle x_i, y \rangle.$$

Proof. Given any $\epsilon > 0$, let $\epsilon_0 = \frac{\epsilon}{\|y\|}$.

Since $x = \lim_{i \rightarrow \infty} x_i$, we can find $N > 0$, such that $\forall n > N, \|x - x_n\| < \epsilon_0$,

thus $|\langle x, y \rangle - \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq \|x - x_n\| \|y\| < \epsilon_0 \|y\| = \epsilon$ □

Corollary 4.9. For an Inner product space $H, \forall y, x = \lim_{i \rightarrow \infty} x_i \in H$, we have $\langle y, x \rangle = \lim_{i \rightarrow \infty} \langle y, x_i \rangle$.

Definition 4.3. An inner product space \mathcal{H} is called a **Hilbert space** if it is complete.

Definition 4.4. Let H be an inner product space. Two vectors $u, v \in H$ are called **orthogonal** if $\langle u, v \rangle = 0$.

Definition 4.5. Let H be an inner product space. A set $\{e_i\}_{i \in I} \subseteq H$ is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij}.$$

Definition 4.6. Let H be an inner product space. An orthonormal set $\{e_i\}_{i \in I} \subseteq H$ is called a **maximal orthonormal set** / **orthonormal basis** / **total orthonormal set** if $\text{Span}(\{e_i\}_{i \in I})$ is dense in H . Namely,

$$H = \overline{\text{Span}(\{e_i\}_{i \in I})}.$$

Definition 4.7. Let \mathcal{H} be a Hilbert space. $S \subseteq \mathcal{H}$, the subspace **orthogonal** to S is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v \rangle = 0, \forall v \in S\}.$$

Lemma 4.10. Let \mathcal{H} be a Hilbert space, $S \subseteq \mathcal{H}$, we always have S^\perp is a subspace of \mathcal{H} .

Definition 4.8. Let V be a vector space, and $U, W \subseteq V$ be two subspaces, we say $V = U \oplus W$, if $V = U + W = \{v + w : v \in U, w \in W\}$, and $V \cap W = \{0\}$.

Proposition 4.11. Let V be a vector space, and $U, W \subseteq V$ be two subspaces. $V = U \oplus W$, if and only if $\forall v \in V$, it can be uniquely written as $v = u + w$, where $u \in U, w \in W$.

Theorem 4.12. Let \mathcal{H} be a Hilbert space, if $S \subseteq \mathcal{H}$ is a subspace, then

$$\mathcal{H} = \bar{S} \oplus S^\perp.$$

Theorem 4.13. Let \mathcal{H} be a Hilbert space, and $\{e_i\}_{i \in I} \subseteq \mathcal{H}$ be an orthonormal set, with $M := \overline{\text{Span}\{e_i\}_{i \in I}}$, then

1. The set $\{e_i\}_{i \in I}$ is linearly independent, and

$$\text{dist}(e_i, \overline{\text{Span}\{e_j\}_{j \in I, j \neq i}}) = 1.$$

2. (Bessel's inequality) For any $x \in \mathcal{H}$, we have

$$\sum_{i \in I} |\langle e_i, x \rangle|^2 \leq \|x\|^2.$$

3. If $\{a_i\}_{i \in I} \subseteq \mathbb{F}$, $\sum_{i \in I} |a_i|^2 = L^2$, we have $\sum_{i \in I} a_i e_i$ converges unconditionally to $x \in \mathcal{H}$ with $\|x\| = L$. Moreover, in this case, $\langle e_i, x \rangle = a_i$.

4. We have $P : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$P(x) := \sum_{i \in I} \langle e_i, x \rangle e_i$$

is the projection onto M . Namely, $P \in B(\mathcal{H})$, $\|P\| = 1$, $P^2 = P$, $\text{Im}(P) = M$, $\ker(P) = M^\perp$, and $Px = x$ if and only if $x \in M$.

5. (Parseval Identity) $\forall x \in M$, $\|x\|^2 = \sum_{i \in I} |\langle e_i, x \rangle|^2$.

Theorem 4.14 (generalized Fourier series). Let \mathcal{H} be a Hilbert space, and $\{e_i\}_{i \in I} \subseteq \mathcal{H}$ be an orthonormal set, then the following are equivalent:

1. $\{e_i\}_{i \in I}$ is an orthonormal basis
2. If $\forall i \in I$, $\langle x, e_i \rangle = 0$, then $x = 0$.
3. (Fourier series) $\forall x \in \mathcal{H}$, $x = \sum_{i \in I} \langle e_i, x \rangle e_i$.
4. (Parseval Identity)

$$\forall x \in \mathcal{H}, \|x\|^2 = \sum_{i \in I} |\langle e_i, x \rangle|^2.$$

Theorem 4.15. \mathcal{H} is a separable Hilbert space, if and only if there is a maximal orthonormal set in \mathcal{H} . Moreover, in this case, every maximal orthonormal set is at most countable.

Theorem 4.16. [Riesz-Frechet Representation theorem] Let \mathcal{H} be a Hilbert space, then for each $u^* \in \mathcal{H}$, $\exists! u \in \mathcal{H}$, such that $\forall v \in \mathcal{H}$, $\langle u^*, v \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$, and $\|u^*\|_{\mathcal{H}^*} = \|u\|_{\mathcal{H}}$.

Definition 4.9. Let X be a vector space over \mathbb{F} . We define

$$\bar{X} := \{\bar{x} : x \in X\}$$

to be the **conjugate vector space**. The scalar multiplication is given by $\lambda \bar{x} = \overline{\lambda x}$.

Proposition 4.17. *Let X be a normed space over \mathbb{F} . The map $X \rightarrow \bar{X}$ by $x \mapsto \bar{x}$ is an isometric antilinear bijection.*

Corollary 4.18. *Let \mathcal{H} be a Hilbert space, then $\bar{\mathcal{H}} \cong \mathcal{H}^*$, where the map $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$; $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism.*

Corollary 4.19. *Every Hilbert space is reflexive.*

Definition 4.10. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We say $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ is **unitary** if it is an isometric isomorphism between Banach Spaces.

Proposition 4.20. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. If U is unitary, it preserves the inner product as well. Namely,*

$$\forall x, y \in \mathcal{H}_1, \quad \langle x, y \rangle = \langle Ux, Uy \rangle.$$

Proof. Since U is isometric, by the polarization identity proposition 4.6, it preserves the inner product as well. \square

5 Locally Convex Topological Vector Spaces and Weak Topology

5.1 Locally Convex Topological Vector Spaces

Definition 5.1. Let X be a vector space over \mathbb{F} , a **semi-norm** is a map $p : X \rightarrow [0, \infty)$ such that $\forall t \in \mathbb{F}, x, y \in X$,

$$\begin{aligned} p(tx) &= |t|p(x), \\ p(x+y) &\leq p(x) + p(y). \end{aligned}$$

The null space of p is denoted $N_p := \{x \in X : p(x) = 0\}$.

Remark. Notice that p is a norm if and only if $N_p = \{0\}$.

Definition 5.2. A **Locally Convex Topological Vector Space** is a vector space X over \mathbb{F} with a family of semi-norms $P = \{p_\alpha\}_{\alpha \in A}$, such that $\bigcap_{p \in P} \ker(p) = \{0\}$.

\mathcal{T}_P is the topology on X generated by the convex sets

$$U(x, p, r) := \{y \in X : p(y - x) < r\}$$

for $x \in X, r > 0, p \in P$.

Definition 5.3. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, for $r > 0, x_0 \in X$, and a finite subset $F \subseteq P$, we define

$$U_{F,r}(x_0) := \{x \in X : \forall p \in F, p(x - x_0) < r\},$$

and $U_{F,r} := U_{F,r}(0)$.

Proposition 5.1. *Each $U_{F,r}(x_0)$ is a finite intersection of $\bigcap_{p \in F} U(x, p, r)$, so it is open. Also, each $U_{F,r}(x_0) = x_0 + U_{F,r}$.*

Proposition 5.2. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then for any $x_0 \in X$ the sets $\{U_{F,r}(x_0)\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at x_0 .*

Proof. Consider any open $U \ni 0$, then there are $p_1, \dots, p_n \in P, r_1, \dots, r_n > 0, x_1, \dots, x_n \in X$, such that

$$0 \in \bigcap_{i=1}^n U(x_i, p_i, r_i) \subseteq U.$$

Let $r := \min_{i \in [n]}(r_i - p_i(x_i)) > 0$, and let $F := \{p_1, \dots, p_n\}$.
Consider any $x \in U_{F,r}$, we have for any $i \in [n]$,

$$\begin{aligned} p_i(x_i - x) &\leq p_i(x_i) + p_i(x) \\ &\leq p_i(x_i) + r \\ &\leq p_i(x_i) + r_i - p_i(x_i) \\ &= r_i. \end{aligned}$$

Thus $x \in \bigcap_{i=1}^n U(x_i, p_i, r_i) \subseteq U$.

This shows $0 \in U_{F,r} \subseteq U$.

Thus, $\{U_{F,r}\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at 0. By translation, $\{U_{F,r}(x_0)\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at x_0 . \square

Proposition 5.3. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, it is Hausdorff.*

Proof. Given any $x \neq y \in X$, then $x - y > 0$.

There is $p \in P$ such that $r = p(x - y) > 0$.

Now $U(x, p, \frac{r}{2}) \ni x$ and $U(y, p, \frac{r}{2}) \ni y$ has empty intersection. \square

Proposition 5.4. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then the addition map $A : X \times X \rightarrow X$ and scalar multiplication map $B : \mathbb{F} \times X \rightarrow X$ are continuous.*

Proof. Given any open set U , with $x_0 + y_0 \in U$ for some $x_0, y_0 \in X$.

Since $\{U_{F,r}(x_0 + y_0)\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at $x_0 + y_0$, there is some finite $F \subseteq P$ and $r > 0$, such that

$$U_{F,r}(x_0 + y_0) \subseteq U,$$

since they form a neighbourhood basis.

We claim that $A^{-1}(U_{F,r}(x_0 + y_0)) \supseteq (x_0 + U_{F,\frac{r}{2}}) \times (y_0 + U_{F,\frac{r}{2}})$, which is open.

Indeed, take any $x \in x_0 + U_{F,\frac{r}{2}}$, $y \in y_0 + U_{F,\frac{r}{2}}$ and $p \in F$, we have

$$\begin{aligned} p((x + y) - (x_0 + y_0)) &\leq p(x - x_0) + p(y - y_0) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

Thus, we have find

$$(x_0, y_0) \in (x_0 + U_{F,\frac{r}{2}}) \times (y_0 + U_{F,\frac{r}{2}}) \subseteq A^{-1}(U_{F,r}(x_0 + y_0)) \subseteq A^{-1}(U).$$

Since this hold for all $(x_0, y_0) \in A^{-1}(U)$, we have that $A^{-1}(U)$ is open. \square

Proposition 5.5. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to $x \in X$ if and only if $\forall p \in P, p(x - x_\lambda) \rightarrow 0$.*

Proof.

$$(x_\lambda)_{\lambda \in \Lambda} \rightarrow x$$

if and only if

$$\forall \text{finite } F \subseteq P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U_{F,r}(x)$$

if and only if

$$\forall p \in P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U(x, p, r)$$

if and only if

$$\forall p \in P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, p(x_\lambda - x) < r$$

if and only if

$$\forall p \in P, p(x_\lambda - x) \rightarrow 0.$$

\square

Proposition 5.6. *Let $(X, \|\cdot\|)$ be a normed vector space, then taking $P := \{\|\cdot\|\}$, we have a Locally Convex Topological Vector Space.*

5.2 Weak Topology

Proposition 5.7. Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then $P := \{p_\phi\}_{\phi \in Y}$ given by $p_\phi(x) := |\phi(x)|$ gives a Locally Convex Topological Vector Space (X, \mathcal{T}_Y) , where $\mathcal{T}_Y := \mathcal{T}_P$.

Proof. Clearly, $\forall t \in \mathbb{F}$, $p_\phi(tx) = |\phi(tx)| = |t\phi(x)| = |t||\phi(x)| = |t|p_\phi(x)$.

Also, $p_\phi(x+y) = |\phi(x+y)| = |\phi(x) + \phi(y)| \leq |\phi(x)| + |\phi(y)| = p_\phi(x) + p_\phi(y)$.

Thus, each p_ϕ is a semi-norm.

Suppose $p_\phi(x) = 0$ for all $\phi \in Y$, then $\phi(x) = 0$ for all $\phi \in Y$. Since Y separates the points, we must have $x = 0$. \square

Remark. If Y is not a subspace but just a subset, we can WLOG take $Y' = \text{Span}(Y)$, which will generate the same topology.

Definition 5.4. Let $(X, \|\cdot\|)$ be a normed vector space. The **weak topology on X** $\sigma(X, X^*)$ is (X, \mathcal{T}_{X^*}) , where we take $Y = X^*$, which separates points by the Hahn-Banach Theorem. We say $(x_\lambda)_{\lambda \in \Lambda}$ **converges weakly to** $x \in X^*$ if it converges in the weak topology, denoted $x_\lambda \rightharpoonup x$.

Also, the **weak-* topology on X^*** $\sigma(X^*, X)$ is (X^*, \mathcal{T}_X) , where we take $Y = X \subseteq X^{**}$. We say $(\phi_\lambda)_{\lambda \in \Lambda}$ **converges weakly to** $\phi \in X^*$ if it converges in the weak-* topology, denoted $\phi_\lambda \rightharpoonup^* \phi$.

Proposition 5.8. Let $(X, \|\cdot\|)$ be a normed vector space. $x_\lambda \rightharpoonup x$ in X if and only if $\forall \phi \in X^*$, $\phi(x_\lambda) \rightarrow \phi(x)$. Also, $\phi_\lambda \rightharpoonup^* \phi$ in X^* if and only if $\forall x \in X$, $\phi_\lambda(x) \rightarrow \phi(x)$.

Proposition 5.9. Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then if $x_\lambda \rightarrow x$ in norm, it also converges in (X, \mathcal{T}_Y) .

Proposition 5.10. Let $(X, \|\cdot\|)$ be a normed vector space, $(u_k)_{k=1}^\infty \subset X$ be a sequence, then

1. If $u_k \rightarrow u$, we always have $u_k \rightharpoonup u$.
2. If $u_k \rightharpoonup u$, we have that u is unique.
3. If $u_k \rightharpoonup u$, we have $(u_k)_{k=1}^\infty$ is bounded.
4. If $u_k \rightharpoonup u$, every subsequence $(u_{k_j})_{j=1}^\infty$ also converges weakly to u .

Theorem 5.11. Let X be a reflexive Banach Space, and $(u_k)_{k=1}^\infty \subset X$ be a bounded sequence, then $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.

Proposition 5.12. Let \mathcal{H} be a Hilbert space, then $u_k \rightharpoonup u$ if and only if $\forall v \in \mathcal{H}$, $\langle v, u_k \rangle \rightarrow \langle v, u \rangle$ as real numbers.

Proof. Suppose $u_k \rightharpoonup u$.

Notice that for all $v \in \mathcal{H}$, we have that $v^\dagger \in \mathcal{H}^*$, and thus $\langle v, u_k \rangle = \langle v^\dagger | u_k \rangle \rightarrow \langle v^\dagger | u \rangle = \langle v, u \rangle$.

Now suppose $\forall v \in \mathcal{H}$, $\langle v, u_k \rangle \rightarrow \langle v, u \rangle$.

Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 4.16, there is some $f^\dagger \in \mathcal{H}$, such that

$$\langle f | u_{k_j} \rangle = \langle f^\dagger, u_{k_j} \rangle \rightarrow \langle f^\dagger, u \rangle = \langle f | u \rangle.$$

Thus, $u_{k_j} \rightharpoonup u$. \square

Proposition 5.13. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Tu_k \rightharpoonup Tu \in \mathcal{H}_2$.

Proof. Let $y_k := Tu_k, y := Tu \in \mathcal{H}_2$.
Consider any $g \in \mathcal{H}_2^*$, we define $f := g \circ K \in \mathcal{H}_1^*$.
Since $u_k \rightharpoonup u$, we must have

$$\begin{aligned}\lim_{k \rightarrow \infty} f(u_k) &= f(u) \\ \lim_{k \rightarrow \infty} g(Ku_k) &= g(Ku) \\ \lim_{k \rightarrow \infty} g(y_k) &= g(y).\end{aligned}$$

We thus have $y_k \rightharpoonup y$. □

Proposition 5.14. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a compact operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Ku_k \rightarrow \mathcal{H}_2$.*

Proof. Let $y_k := Ku_k, y := Ku \in \mathcal{H}_2$.

Since K is compact, it is bounded, so $y_k \rightharpoonup y$.

Now suppose for contradiction $\lim_{k \rightarrow \infty} \|y_k - y\| \neq 0$.

Then there is some $\epsilon > 0$ and a subsequence $(u_{k_j})_{j=1}^\infty$ such that $\forall j \geq 1, \|y_{k_j} - y\| \geq \epsilon$.

Since $u_k \rightharpoonup u \in \mathcal{H}$, we have $(u_k)_{k=1}^\infty$ is bounded, and thus $(u_{k_j})_{j=1}^\infty$ is bounded.

Since K is compact, there is some further subsequence $(u_{k_{j_m}})_{m=1}^\infty$ such that $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$.

Thus $Ku_{k_{j_m}} \rightharpoonup \tilde{y}$. Since weak convergence, we must have $\tilde{y} = y$.

Thus $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = y$, which is a contradiction. □

Definition 5.5. Let X, Y be two normed vector spaces, then the **weak operator topology** on $B(X, Y)$ is induced by

$$P := \{p_{x, \phi}(T) : x \in X, \phi \in Y^*\},$$

where for all $T \in B(X, Y)$,

$$p_{x, \phi}(T) := |\phi(T(x))|.$$

We say $(T_\lambda)_{\lambda \in \Lambda}$ **converges weakly to** $T \in B(X, Y)$ if it converges in the weak operator topology, denoted $T_\lambda \rightharpoonup T$.

Remark. Notice that these functions separate points by the Hahn-Banach Theorem. Indeed, $T \neq S$ implies $\exists x \in X$ such that $Tx \neq Sx$, which implies $\exists \phi \in Y^*$ such that $\phi(Tx) \neq \phi(Sx)$.

Proposition 5.15. *Let X, Y be two normed vector spaces, then $T_\lambda \rightharpoonup T \in B(X, Y)$ if and only if $\forall \phi \in X^*, x \in X, \phi(T_\lambda x) \rightarrow \phi(Tx)$.*

Definition 5.6. Let X, Y be two normed vector spaces, then the **strong operator topology** is the topology induced by

$$P := \{p_x(T) := \|Tx\|_Y : x \in X\}.$$

Proposition 5.16. *Let X, Y be two normed vector spaces, then $T_\lambda \rightarrow T \in B(X, Y)$ strongly if and only if $\forall x \in X, T_\lambda x \rightarrow Tx \in Y$.*

Proposition 5.17. *Let X, Y be two normed vector spaces, then convergence in the operator norm implies convergence in strong operator topology, which implies convergence in weak operator topology.*

Proposition 5.18. *When $X = Y = \mathcal{H}$ is a Hilbert space, then $T_\lambda \rightharpoonup T$ if and only if*

$$\forall \xi, \eta \in \mathcal{H}, \langle T_\lambda \xi, \eta \rangle \rightarrow \langle T\xi, \eta \rangle.$$

Proof. By the Riesz-Representation Theorem, for each $\phi \in \mathcal{H}^*$, there is unique $\eta \in \mathcal{H}$ such that

$$\forall \xi \in \mathcal{H}, \phi(\xi) = \langle \xi, \eta \rangle.$$

Thus $T_\lambda \rightharpoonup T$ if and only if $\forall \phi \in \mathcal{H}^*, \xi \in \mathcal{H}, \phi(T_\lambda \xi) \rightarrow \phi(T\xi)$, if and only if $\forall \xi, \eta \in \mathcal{H}, \langle T_\lambda \xi, \eta \rangle \rightarrow \langle T\xi, \eta \rangle$. □

5.3 Continuous Functions

Theorem 5.19. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then for a linear $\phi : X \rightarrow \mathbb{F}$, the following are equal:*

1. ϕ is continuous,
2. ϕ is continuous at 0,
3. $\ker(\phi)$ is closed,
4. $\exists p_1, \dots, p_n \in P, \alpha_1, \dots, \alpha_n > 0$, such that

$$\forall x \in X, |\phi(x)| \leq \sum_{i=1}^n \alpha_i p_i(x).$$

Proof. (1) \implies (2) is trivial.

(2) \implies (3):

Let $x_\lambda \in \ker(\phi)$, with $x_\lambda \rightarrow x \in X$.

For any λ , we have $\phi(x) = \phi(x) - \phi(x_\lambda) = \phi(x - x_\lambda)$.

Since $x - x_\lambda \rightarrow 0$, and ϕ is continuous at 0, we have $\phi(x - x_\lambda) \rightarrow 0$, which means $\phi(x) = 0$. Namely, $x \in \ker(\phi)$.

Thus $\ker(\phi)$ is closed.

(3) \implies (4):

Suppose $\phi = 0$, (4) is trivially true.

Otherwise, there is $x_0 \in \ker(\phi)^c$. WLOG, by taking $x'_0 := \frac{x_0}{\phi(x_0)}$, we can assume $\phi(x_0) = 1$.

Since $\ker(\phi)$ is closed, $\ker(\phi)^c$ is open.

There must be some finite $F \subseteq P$ and $r > 0$, such that $x_0 + U_{F,r} = U_{F,r}(x_0) \subseteq \ker(\phi)^c$.

Thus, $0 \notin \phi(x_0 + U_{F,r}) = \phi(x_0) + \Phi(U_{F,r}) = 1 + \Phi(U_{F,r})$.

Namely, $-1 \notin \Phi(U_{F,r})$.

Thus $\forall x \in U_{F,r}$, we have $\phi(x) \neq -1$.

Take $\{p_1, \dots, p_n\} = F$, and $\alpha_i = \frac{1}{r}$.

Suppose for contradiction that there is some $x \in X$ with $|\phi(x)| > \sum_{i=1}^n \frac{1}{r} p_i(x)$.

In particular, $|\phi(x)| > \frac{1}{r} p_i(x)$ for all $p_i \in F$.

We must have some $|\lambda|$ such that $\phi(x) = \lambda |\phi(x)|$.

Then for $y := \frac{x}{-\lambda |\phi(x)|}$, we have

$$\phi(y) = \phi\left(\frac{x}{-\lambda |\phi(x)|}\right) = \frac{\phi(x)}{-\lambda |\phi(x)|} = -1.$$

However,

$$p_i(y) = \left| \left(\frac{1}{-\lambda |\phi(x)|} \right) \right| p_i(x) = \frac{p_i(x)}{|\phi(x)|} < \frac{r |\phi(x)|}{|\phi(x)|} = r.$$

Thus, $y \in U_{F,r}$, which is a contradiction.

(4) \implies (1):

If $x_\lambda \rightarrow x$, then $\forall p \in P, p(x_\lambda - x) \rightarrow 0$.

Thus, $\sum_{i=1}^n \alpha_i p_i(x_\lambda - x) \rightarrow 0$.

Thus, $|\phi(x_\lambda) - \phi(x)| = |\phi(x_\lambda - x)| \rightarrow 0$, which means $\phi(x_\lambda) \rightarrow \phi(x)$.

This shows ϕ is continuous. □

Definition 5.7. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then the **continuous dual space** $(X, \mathcal{T}_P)^*$ is the set of continuous linear functions $\phi : X \rightarrow \mathbb{F}$.

Theorem 5.20. *Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then $\phi \in (X, \mathcal{T}_Y)^*$ if and only if $\phi \in \text{Span}(Y)$.*

Proof. $\phi \in (X, \mathcal{T}_Y)^*$ if and only if $\exists \phi_1, \dots, \phi_n \in Y, \alpha_i > 0$, such that $\forall x \in X, |\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)|$.
 (\Leftarrow) :

Suppose $\phi = \sum_{i=1}^n c_i \phi_i$, then taking $\alpha_i := \max(|c_i|, 1)$, we have

$$\begin{aligned} |\phi(x)| &= \left| \sum_{i=1}^n c_i \phi_i(x) \right| \\ &\leq \sum_{i=1}^n |c_i| |\phi_i(x)| \\ &\leq \sum_{i=1}^n \alpha_i |\phi_i(x)|. \end{aligned}$$

(\Rightarrow) :

Suppose $\{\phi_i\}_{i=1}^n$ is not linearly independent, WLOG, $\phi_n = \sum_{i=1}^{n-1} c_i \phi_i$, then

$$|\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)| \leq \sum_{i=1}^{n-1} (\alpha_i + \alpha_n |c_i|) |\phi_i(x)|.$$

Thus, we can assume $\{\phi_i\}_{i=1}^n$ is linearly independent.

Consider $T : X \rightarrow \mathbb{F}^n$ by $T(x) := (\phi_1(x), \dots, \phi_n(x))$.

Clearly $\ker(T) = \bigcap_{i=1}^n \ker(\phi_i)$.

Since $\{\phi_i\}_{i=1}^n$ is linearly independent, T is surjective.

By a corollary of the Banach Isomorphism Theorem, $X / \bigcap_{i=1}^n \ker(\phi_i) \cong \mathbb{F}^n$ by $\hat{T} : \hat{x} \mapsto (\phi_1(x), \dots, \phi_n(x))$.

Since $\ker(\phi) \supseteq \bigcap_{i=1}^n \ker(\phi_i)$, we can define $\tilde{\phi} : \mathbb{F}^n \rightarrow \mathbb{F}$ by

$$\tilde{\phi}(\hat{T}\hat{x}) := \phi(x).$$

This is well-defined, since $\hat{x} = \hat{y} \implies x - y \in \bigcap_{i=1}^n \ker(\phi_i) \implies \phi(x - y) = 0 \implies \phi(x) = \phi(y)$.

Thus for all $i \in [n]$, there is $\beta_i := \tilde{\phi}(e_i)$, such that $\tilde{\phi}(z_1, \dots, z_n) = \sum_{i=1}^n \beta_i z_i$.

Thus, $\phi(x) = \tilde{\phi}(\hat{T}\hat{x}) = \tilde{\phi}(Tx) = \sum_{i=1}^n \beta_i \phi_i(x)$, which means $\phi = \sum_{i=1}^n \beta_i \phi_i \in \text{Span}(\{\phi_i\}_{i=1}^n) \subseteq \text{Span}(Y)$. \square

Corollary 5.21. *Let $(X, \|\cdot\|)$ be a normed vector space, then $\sigma(X, X^*)^* = (X, \mathcal{T}_{X^*})^* = X^*$ and $\sigma(X^*, X)^* = (X^*, \mathcal{T}_X)^* = X$.*

Remark. A convergent net in weak-* topology is not necessarily bounded.

Example 5.3.1. Consider $X = \ell^1 = C_0^*$, with the weak-* topology \mathcal{T}_{C_0} .

For any finite $F \subsetneq C_0 \subset (\ell^1)^*$, we have that $\bigcap_{\phi \in F} \ker(\phi) \neq \{0\}$, so there is a $x_F \neq 0 \in \bigcap_{\phi \in F} \ker(\phi)$. By

taking $\tilde{x}_F := \frac{|F|}{\|x_F\|} x_F$, we can have $\tilde{x}_F \in \bigcap_{\phi \in F} \ker(\phi)$, with $\|\tilde{x}_F\| = |F|$.

Clearly $(\tilde{x}_F)_{\text{finite } F \subsetneq C_0}$ is not bounded.

However, consider any finite $F \subseteq C_0, r > 0$, we can pick $F_0 := F$.

For all finite $F' \supseteq F_0$, we have that $\forall \phi \in F, |\phi(\tilde{x}_{F'} - 0)| = 0 < r$, which means $\tilde{x}_{F'} \in U_{F,r}$.

Thus $\tilde{x}_F \rightharpoonup 0$.

5.4 Geometric Hahn–Banach Theorems

Definition 5.8. Let X be a vector space, a set $K \subseteq X$ is **convex** if

$$\forall x, y \in K, t \in (0, 1), (1 - t)x + ty \in K.$$

Definition 5.9. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, the **Minkowski functional associated to U** is

$$p_U(x) := \inf \{t > 0 : x \in tU\}.$$

Lemma 5.22. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, then the Minkowski functional associated to U is always well-defined.*

Proof. Since $0 \in U$, which is open, there is finite $F \subseteq P, r > 0$ such that $U_{F,r} \subseteq U$.

Thus, for $t = \frac{r}{2 \max\{p(x) : p \in F\}} > 0$, we have $\forall p \in P, p(tx) = \frac{r}{2 \max\{p(x) : p \in F\}} p(x) \leq \frac{r}{2} < r$, so $tx \in U_{F,r} \subseteq U$.
Thus $\{t > 0 : x \in tU\} \neq \emptyset$. \square

Theorem 5.23. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, then the Minkowski functional $p_U : X \rightarrow [0, \infty)$ is a sublinear functional, and $U = \{x \in X : p_U(x) < 1\}$.*

Proof. For any $t > 0$, we have $x \in sU \iff tx \in tsU$.

$$\begin{aligned} p_U(tx) &= \inf \{r > 0 : tx \in rU\} \\ &= \inf \{st > 0 : tx \in tsU\} \\ &= \inf \{st > 0 : x \in sU\} \\ &= t \inf \{s > 0 : x \in sU\} \\ &= tp_U(x). \end{aligned}$$

Also, consider any $x, y \in X$ and any $s, t > 0$ such that $x \in sU, y \in tU$, we have that $\frac{x}{s}, \frac{y}{t} \in U$.

By the convexity of U ,

$$\begin{aligned} \frac{s}{s+t} \frac{x}{s} + \frac{t}{s+t} \frac{y}{t} &\in U \\ x + y &= \left(\frac{s}{s+t} \frac{x}{s} + \frac{t}{s+t} \frac{y}{t} \right) (s+t) \\ &\in (s+t)U. \end{aligned}$$

Thus, $p_U(x+y) \leq s+t$. Since this holds for any such s, t , we have

$$p_U(x+y) \leq p_U(x) + p_U(y).$$

This shows p_U is sublinear.

Suppose $p_U(x) < 1$, then there is $t < 1$ such that $x \in tU \implies \frac{x}{t} \in U \implies x = t \frac{x}{t} + (1-t)0 \in U$ by convexity of U .

Now for any $x \in U$, since $t \mapsto tx$ is continuous, and $1x = x \in U$ which is open, we have some $\delta > 0$, such that $(1-\delta, 1+\delta)x \subseteq U$.

Thus $\forall t \in \left(\frac{1}{1+\delta}, 1\right)$, $x \in tU$, which means $p_U(x) \leq \frac{1}{1+\delta} < 1$. \square

Theorem 5.24 (First Separation). *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, $A, B \subseteq X$ be disjoint convex sets. Suppose A is open, then $\exists t \in \mathbb{R}, \phi \in (X, \mathcal{T}_P)^*$, such that*

$$\forall x \in A, y \in B, \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

Namely, $\phi(A), \phi(B)$ can be separated by a vertical line in \mathbb{C} .

Proof. 1. We first assume $\mathbb{F} = \mathbb{R}$.

Fix $x_0 \in A, y_0 \in B$, let $z_0 := y_0 - x_0 \neq 0$.

Consider $U := z_0 + A - B = \bigcup_{y \in B} (z_0 - y + A)$, which is open and convex. Also, $0 \in U$.

Consider the Minkowski functional $p_U : X \rightarrow [0, \infty)$, which is sublinear.

Notice that $z_0 \notin U$, so $p_U(z_0) \geq 1$.

Let $\phi_0 : \text{Span}\{z_0\} \rightarrow \mathbb{R}$ be $\lambda z_0 \mapsto \lambda$, which is linear.

In addition, $\forall \lambda \geq 0$, we have $\phi_0(\lambda z_0) = \lambda \leq \lambda p_U(z_0) = p_U(\lambda z_0)$.

Also, $\phi_0(-\lambda z_0) = -\lambda \leq 0 \leq p_U(-\lambda z_0)$.

Thus $\phi_0 \leq p_U$ on $\text{Span}\{z_0\}$. By the extension theorem 3.32, there is a linear extension $\phi : X \rightarrow \mathbb{R}$, such that $\phi \leq p_U$.

Let $\epsilon > 0$.

Take any $x \in \epsilon U \cap (-\epsilon U) \in \mathcal{O}(0)$, which is open.

We have that $\pm \frac{x}{\epsilon} \in U$, so $\phi(\pm \frac{x}{\epsilon}) \leq p_U(\pm \frac{x}{\epsilon}) < 1$.

Thus, $|\phi(x)| < \epsilon$.

This shows that ϕ is continuous.

Now take any $x \in A, y \in B$, we have that $z_0 + x - y \in U$, so $1 + \phi(x) - \phi(y) = \phi(z_0 + x - y) \leq p_U(z_0 + x - y) < 1$.

Thus, $\phi(x) < \phi(y)$.

Notice that since A is open, there is $\epsilon_0 > 0$ such that $\forall 0 < \epsilon < \epsilon_0$, $(1 \pm \epsilon)x \in A$. Thus, $(1 \pm \epsilon)\phi(x) \in \phi(A)$, which means $\phi(A)$ is open.

Since A is convex, $\phi(A)$ is also convex, thus connected. Thus, $\phi(A) = (b, t)$ is an interval.

This shows $\forall x \in A, y \in B$, $\phi(x) < t \leq \phi(y)$.

2. Now assume $\mathbb{F} = \mathbb{C}$.

Consider $(X_{\mathbb{R}}, \mathcal{T}_P)$, we have that A, B are still disjoint convex sets, and A is open.

Thus there is a \mathbb{R} -linear continuous map $\psi \in (X_{\mathbb{R}}, \mathcal{T}_P)^*$, such that $\psi(A) < \psi(B)$.

Now take $\phi(x) := \psi(x) - i\psi(ix)$.

□

However, this is not always true when A is not open.

Example 5.4.1. Consider the weak topology $(\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})$, and $A := \{x = (x_i)_{i=1}^{\infty} \in \ell^1(\mathbb{N}) : \sum_{i=1}^{\infty} x_i = 0\}$, $B = \{\delta_1\}$. They are disjoint convex sets.

However, for any $\phi \in C_0(\mathbb{N}) = \text{Span}(C_0(\mathbb{N})) = (\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})^*$, we have that $\phi(A) \cap \phi(B) \neq \emptyset$.

Indeed, consider any $\phi = (a_1, a_2, \dots) \in C_0(\mathbb{N})$, there is $m \in \mathbb{N}$ such that $a_m \neq 0$.

Now consider $(\delta_m - \delta_n)_{n=1}^{\infty} \subset A$, we have that $\phi(\delta_m - \delta_n) = a_m - a_n \rightarrow a_m \neq 0$.

Since $\ker(\phi)$ is closed, A is not contained in $\ker(\phi)$, which means $\phi(A) = \mathbb{C}$.

Lemma 5.25. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, suppose compact $K \subseteq$ open $V \subseteq X$, then there is an open convex neighbourhood U of 0, such that $K + U \subseteq V$.

Proof. For all $x \in K$, since $x \in V$, there is finite $F_x \subseteq P$, and $r_x > 0$, such that $U_{F_x, 2r_x}(x) \subseteq V$.

Since $K \subseteq \bigcup_{x \in K} U_{F_x, r_x}(x)$ is compact, there is a finite subcover $K \subseteq \bigcup_{i=1}^n U_{F_{x_i}, r_{x_i}}(x_i)$.

Now let $F := \bigcup_{i=1}^n F_{x_i}$, which is finite, and $r := \min_{i \in [n]} \{r_{x_i}\} > 0$.

Let $U := U_{F, r}$.

For any $z \in K + U$, there is some $x \in K$ such that $z \in U_{F, r}(x)$.

Also, since $K \subseteq \bigcup_{i=1}^n U_{F_{x_i}, r_{x_i}}(x_i)$, there is some $i \in [n]$ such that $x \in U_{F_{x_i}, r_{x_i}}(x_i)$.

Thus for any $p \in F_{x_i} \subseteq F$, we have

$$\begin{aligned} p(z - x_i) &\leq p(z - x) + p(x - x_i) \\ &< r + r_{x_i} \\ &\leq 2r_{x_i}. \end{aligned}$$

Thus, $z \in U_{F_{x_i}, 2r_{x_i}}(x_i) \subseteq V$.

□

Theorem 5.26 (Second Separation). Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, $A, B \subseteq X$ be disjoint convex sets. Suppose A is compact, B is closed, then $\exists t \in \mathbb{R}$, $\phi \in (X, \mathcal{T}_P)^*$, such that

$$\forall x \in A, y \in B, \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

Namely, $\phi(A), \phi(B)$ can be separated by a vertical line in \mathbb{C} .

Proof. Since A is compact, and B^c is open, there is an open convex neighbourhood U of 0, such that $A + U \subseteq B^c$; namely $A + U \cap B = \emptyset$.

By the first separation theorem, there is $t \in \mathbb{R}, \phi \in (X, \mathcal{T}_P)^*$, such that

$$\sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

Since ϕ is continuous and A is compact, by the Extreme Value Theorem, there is $x_0 \in A$, such that $\Re(\phi(x_0)) = \sup_{x \in A} \Re(\phi(x))$.

Notice that $x_0 = x_0 + 0 \in A + U$, so

$$\sup_{x \in A} \Re(\phi(x)) = \Re(\phi(x_0)) \leq \sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

□

Corollary 5.27. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then $(X, \mathcal{T}_P)^*$ separates the points of X .*

Proof. Given $x \neq y \in X$.

Take $A := \{x\}$, and $B := \{y\}$, which is closed. They are trivially convex and disjoint. □

Definition 5.10. Let (X, \mathcal{T}) be a topological space, and $A \subseteq X$. The **convex hull** of A is

$$\text{conv}(A) := \left\{ x = \sum_{i=1}^n \lambda_i x_i \mid n \in \mathbb{N}, x_i \in A, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The **closed convex hull** of A is $\overline{\text{conv}(A)}$.

Proposition 5.28. *The convex hull is the smallest convex set containing A , and the closed convex hull is the smallest closed convex set containing A .*

Definition 5.11. Let (X, \mathcal{T}) be a topological space, and $A \subseteq X, \phi \in (X, \mathcal{T})$. The **closed half square containing A** under ϕ is

$$H_{\phi, A} := \{x \in X : \Re(\phi(x)) \leq \alpha_{\phi, A}\},$$

where $\alpha_{\phi, A} := \sup_{x \in A} \Re(\phi(x))$.

Proposition 5.29. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then for all $A \subseteq X$, we have*

$$\overline{\text{conv}(A)} = \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}.$$

Proof. For any $\phi \in (X, \mathcal{T}_P)^*$, we have that $H_{\phi, A}$ is convex and closed, and $A \subseteq H_{\phi, A}$, so $\overline{\text{conv}(A)} \subseteq H_{\phi, A}$. Thus,

$$\overline{\text{conv}(A)} \subseteq \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}.$$

On the other hand, suppose for contradiction that $\bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \setminus \overline{\text{conv}(A)} \neq \emptyset$.

Take any $x \in \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \setminus \overline{\text{conv}(A)}$, we have that $\{x\}$ is compact, and $\overline{\text{conv}(A)}$ is closed.

By the Second Separation Theorem, there is $\phi \in (X, \mathcal{T}_P)^*$ such that $\Re(\phi(x)) < \inf_{y \in \overline{\text{conv}(A)}} \Re(\phi(y))$.

We thus have $\Re(-\phi(x)) > \sup_{y \in \overline{\text{conv}(A)}} \Re(-\phi(y)) \geq \sup_{y \in A} \Re(-\phi(y))$, so $x \notin H_{-\phi, A}$, which contradicts $x \in \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}$. □

Corollary 5.30. *Suppose X is a vector space with two Locally Convex Topologies $\mathcal{T}_P, \mathcal{T}_{P'}$. Suppose $(X, \mathcal{T}_P)^* = (X, \mathcal{T}_{P'})^*$, then $(X, \mathcal{T}_P), (X, \mathcal{T}_{P'})$ have the same closed convex sets.*

Proof. Suppose A is convex and closed in (X, \mathcal{T}_P) , then

$$\begin{aligned} A &= \overline{\text{conv}(A)}^{\mathcal{T}_P} \\ &= \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \\ &= \bigcap_{\phi \in (X, \mathcal{T}_{P'})^*} H_{\phi, A} \\ &= \overline{\text{conv}(A)}^{\mathcal{T}_{P'}} \\ &= \overline{A}^{\mathcal{T}_{P'}}. \end{aligned}$$

Thus, A is convex and closed in $(X, \mathcal{T}_{P'})$. \square

5.5 Weakly Closeness and Compactness

Proposition 5.31. *Let $(X, \|\cdot\|)$ be a Normed Space. Suppose $A \subseteq X$ is closed in (X, \mathcal{T}_Y) , where $Y \subseteq X$ separates the points, then A is normed closed.*

Proof. Suppose a net $(x_\lambda)_{\lambda \in \Lambda}$ in A converges to $x \in X$, we must have $\phi(x_\lambda) \rightarrow \phi(x)$ for all $\phi \in X^*$. Thus $x_\lambda \rightarrow x$ in (X, \mathcal{T}_Y) , which means $x \in A$. \square

The converse is not in general true.

Example 5.5.1. Consider $(C[0, 1], \|\cdot\|_\infty)$, with the dual space $M([0, 1])$ (the complex Borel measures on $[0, 1]$).

For $n \geq 1$, let $f_n := \begin{cases} 1 - 2nx, & 0 \leq x \leq \frac{1}{2n} \\ 2nx - 1, & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 1, & x \geq \frac{1}{n} \end{cases}$. We have that $f_n \in C[0, 1]$, and $f_n(x) \rightarrow 1$ pointwise for all

$x \in [0, 1]$.

By Lebesgue's Dominated Convergence Theorem, for any complex Borel measure $\mu \in M([0, 1])$, we have $\int_0^1 f_n d\mu \rightarrow \int_0^1 1 d\mu$.

Thus, $f_n \rightharpoonup f$.

However, for any $n \geq 1$, $\|f_n - 1\|_n = 1$.

Proposition 5.32. *Let $(X, \|\cdot\|)$ be a Normed Space, then $(X, \|\cdot\|)$ and (X, \mathcal{T}_{X^*}) have the same closed convex set.*

Proof. This follows from that $(X, \mathcal{T}_{X^*})^* = X^*$. \square

Proposition 5.33. *Let $(X, \|\cdot\|)$ be a Normed Space, then the closed balls are weak-* closed. Namely, the closed balls in $(X^*, \|\cdot\|_{X^*})$ are closed in (X^*, \mathcal{T}_X) .*

Proof. For any $\phi_0 \in X^*$, $r > 0$, then

$$\begin{aligned} \bar{B}^{(X^*, \|\cdot\|_{X^*})}(\phi_0, r) &= \{\phi \in X^* : \|\phi - \phi_0\|_{X^*} \leq r\} \\ &= \{\phi \in X^* : |(\phi - \phi_0)(x)| \leq r, \forall x \in X \text{ such that } \|x\| \leq 1\} \\ &= \bigcap_{x \in X \text{ such that } \|x\| \leq 1} \{\phi \in X^* : |\hat{x}(\phi - \phi_0)| \leq r\}, \end{aligned}$$

which is an intersection of weak-* closed sets. \square

Theorem 5.34 (Goldstine). *Let $(X, \|\cdot\|)$ be a Banach Space, then $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weak-* dense in $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$, with the weak-* topology $(X^{**}, \mathcal{T}_{X^*})$. In particular, X is weak-* dense in X^{**} .*

Proof. If $x_\lambda \rightharpoonup \psi \in X^{**}$, with $(x_\lambda)_{\lambda \in \Lambda} \subset \bar{B}^{(X, \|\cdot\|)}(0, 1)$, for any $\phi \in X^*$, we have

$$\begin{aligned} |\psi(\phi)| &= \left| \lim_{\lambda} \hat{x}_\lambda(\phi) \right| \\ &= \lim_{\lambda} |\phi(x_\lambda)| \\ &\leq \|\phi\|. \end{aligned}$$

Thus, $\|\psi\| \leq 1$ and $\psi \in \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$.

This shows $\overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak} \subseteq \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$.

Suppose for contradiction that there is $\psi \in \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) \setminus \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$.

Since $\{\psi\}$ is convex and compact, and $\overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$ is closed in $(X^{**}, \mathcal{T}_{X^*})$, by the second separation theorem, we can find $\phi \in (X^{**}, \mathcal{T}_{X^*})^* = X^*$, such that

$$\Re(\psi(\phi)) < \inf_{\xi \in \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}} \Re(\xi(\phi)).$$

We thus have

$$\begin{aligned} \|\phi\| &= \sup_{x \in \bar{B}^{(X, \|\cdot\|)}(0, 1)} \Re(-\phi(x)) \\ &\leq \sup_{\xi \in \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}} \Re(\xi(\phi)) \\ &< \Re(\psi(-\phi)) \\ &\leq \|\psi\| \|\phi\| \\ &\leq \|\phi\|, \end{aligned}$$

which is a contradiction.

Thus,

$$\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$$

□

Theorem 5.35 (Banach-Alaoglu). *Let $(X, \|\cdot\|)$ be a Normed Vector Space, then $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is compact in the weak-* topology.*

Proof. Consider the set

$$\begin{aligned} D &:= \{f : X \rightarrow \mathbb{F} \mid |f(x)| \leq \|x\|\} \\ &= \prod_{x \in X} D_x, \end{aligned}$$

where $D_x := \{x \in \mathbb{F} \mid |z| \leq \|x\|\}$.

By Tychonoff's Theorem, D is compact and Hausdorff in the product topology.

Also, $\bar{B}^{(X^*, \|\cdot\|)}(0, 1) \subseteq D$.

In addition, $f_\lambda \rightarrow f$ in the product topology if and only if $\forall x \in X, f_\lambda(x) \rightarrow f(x)$ if and only if $f_\lambda \rightharpoonup f$ in weak-* topology.

Thus, the weak-* topology and the product topology are the same.

Since $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is weak-* closed, it is compact.

□

Corollary 5.36. *Let $(X, \|\cdot\|)$ be a Banach Space, then X is reflexive if and only if $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weakly compact in (X, \mathcal{T}_{X^*}) .*

Proof. (\implies) :

If $X^{**} = X$, then the weak-* topology \mathcal{T}_{X^*} on X^{**} is the same as the weak topology \mathcal{T}_{X^*} on X .

Since $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$ is weakly compact in $(X^{**}, \mathcal{T}_{X^*})$ by the Banach-Alaoglu Theorem, we have that $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weakly compact in (X, \mathcal{T}_{X^*}) .

(\impliedby) :

By the Goldstine Theorem, we have that $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak}$.

Since $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weakly compact in (X, \mathcal{T}_{X^*}) , and $i : (X, \mathcal{T}_{X^*}) \rightarrow (X^{**}, \mathcal{T}_{X^*})$ is continuous, it is also weakly compact in (X, \mathcal{T}_{X^*}) . Thus,

$$\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{weak} = \bar{B}^{(X, \|\cdot\|)}(0, 1).$$

Thus, $X = X^{**}$.

□

5.6 Extreme Points

Definition 5.12. Let X be a Vector Space, and let $\emptyset \neq A \subseteq X$ be convex. A **face** of A is some convex $\emptyset \neq F \subseteq A$, such that for all $t \in (0, 1)$, $x, y \in A$, if $(1 - t)x + ty \in F$, then $x, y \in F$.

If a face $F = \{z\}$, then we call z an **extreme point** of A .

$\text{Ext}(A)$ is the set of extreme points of A .

Proposition 5.37. Suppose F is a face for A , and F' is a face for F , then F' is also a face for A .

Proof. Consider any $x, y \in A, t \in (0, 1)$, with $(1 - t)x + ty \in F' \subseteq F$. Since F is a face of A , $x, y \in F$. Since F' is a face of F , $x, y \in F'$. Thus, F' is a face of A . \square

Example 5.6.1. Let $X = L^1([0, 1])$ with the Lebesgue measure, and consider $A := \bar{B}^{(X, \|\cdot\|_1)}(0, 1)$. For any $f \in A$ such that $\|f\|_{L^1([0, 1])} = a \neq 0$, we can pick $t_0 \in (0, 1)$, such that $\int_0^{t_0} |f| dx = \int_{t_0}^1 |f| dx = \frac{1}{2}a$.

Now take $g := 2f\chi_{[0, t_0]}$, $h := 2f\chi_{[t_0, 1]}$, we have that $f = \frac{1}{2}g + \frac{1}{2}h$, and $g, h \in A$.

Thus, $f \notin \text{Ext}(A)$.

Thus, $\bar{B}(0, 1)$ has no extreme points.

Proposition 5.38. Let (X, \mathcal{T}) be a Topological Vector Space, and $\emptyset \neq K \subseteq X$ be convex and compact, the for any $\phi \in (X, \mathcal{T})^*$,

$$F_\phi := \arg \min_{x \in K} (\Re(\phi(x)))$$

is always a closed face of K .

Proof. Let $\alpha_\phi := \inf_{x \in K} (\Re(\phi(x)))$.

Since K is compact, and ϕ is continuous, α_ϕ is achieved.

Thus $F_\phi = \{x \in K : \Re(\phi(x)) = \alpha_\phi\} \neq \emptyset$.

For any $(x_\lambda)_{\lambda \in \Lambda}$ in F_ϕ , such that $x_\lambda \rightarrow x \in K$, since ϕ is continuous, we have that

$$\phi(x) = \lim_{\lambda} \phi(x_\lambda) = \lim_{\lambda} \alpha_\phi = \alpha_\phi.$$

Thus, $x \in F_\phi$, so F_ϕ is closed.

For any $x, y \in F_\phi, t \in (0, 1)$, we have

$$\phi((1 - t)x + ty) = (1 - t)\phi(x) + t\phi(y) = (1 - t)\alpha_\phi + t\alpha_\phi = \alpha_\phi.$$

Thus, $(1 - t)x + ty \in F_\phi$, so F_ϕ is convex.

For any $x, y \in K, t \in (0, 1)$, if $\phi((1 - t)x + ty) \in F_\phi$, we must have

$$\begin{aligned} \alpha_\phi &= \phi((1 - t)x + ty) \\ &= (1 - t)\phi(x) + t\phi(y) \\ &\geq (1 - t)\alpha_\phi + t\alpha_\phi \\ &= \alpha_\phi. \end{aligned}$$

This forces the inequality to be equality, and $\phi(x) = \phi(y) = \alpha_\phi$. Thus, $x, y \in F_\phi$, so F_ϕ is a face of K . \square

Theorem 5.39 (Krein-Milman). Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $\emptyset \neq K \subseteq X$ be convex and compact, then

$$K = \overline{\text{conv}(\text{Ext}(K))}.$$

Proof. Since (X, \mathcal{T}_P) is Hausdorff, K is compact means K is closed.

Thus $K \supseteq \overline{\text{conv}(\text{Ext}(K))}$ since it is a closed convex set containing $\text{Ext}(K)$.

On the other hand, we firstly show that for any closed face $\emptyset \neq F_0 \subseteq K$, we have $\text{Ext}(K) \cap F_0 \neq \emptyset$.

Let $\Lambda := \{F \subseteq F_0 : F \text{ is a closed face of } F_0\}$, with the partial order $F_1 \leq F_2$ if $F_2 \subseteq F_1$.

Let $\mathcal{C} = \{F_\alpha\}_{\alpha \in A}$ be a chain in Λ .

Let $F := \bigcap_{\alpha \in A} F_\alpha$.

Since K is compact, by FIP, $F \neq \emptyset$, and it is closed and convex.

Also, if $x, y \in F_0, t \in (0, 1)$, and $(1 - t)x + ty \in F$, we have $(1 - t)x + ty \in F_\alpha$ for some $\alpha \in A$.

Since F_α is a face of F_0 , we must have $x, y \in F_\alpha \subseteq F$.

Thus, $F \in \Gamma$, and it's clear that F is an upper bound for \mathcal{C} .

By Zorn's lemma, there is a maximal element F of Γ . Notice that it is also a face of K .

Suppose for contradiction, that there are $x \neq y \in F$, then by the second separation theorem, there is $\phi \in (X, \mathcal{T}_P)^*$, such that $\Re(\phi(x)) \neq \Re(\phi(y))$.

Now let $F_\phi := \arg \min_{x \in F} (\Re(\phi(x)))$.

Since F is a closed subset of compact K , it is compact. By the proposition, F_ϕ is a closed face of F , and thus a closed face of F_0 . Thus $F_\phi \in \Gamma$.

By maximality of F , we must have $F_\phi = F$, which means $\phi(x) = \phi(y) = \min_{x \in F} (\Re(\phi(x)))$, a contradiction with the choice of ϕ .

Thus F only has one point x , so $x \in \text{Ext}(K) \cap F_0 \neq \emptyset$.

In particular, since K is a closed face for itself, $\text{Ext}(K) \neq \emptyset$.

Now suppose for contradiction that there is $x_0 \in K \setminus B$, where $B := \overline{\text{conv}(\text{Ext}(K))}$.

By the Second Separation Theorem, there is $\phi \in (X, \mathcal{T}_P)^*, t \in \mathbb{R}$ such that

$$\Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

In particular, $\min_{x \in K} \Re(\phi(x)) < \Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y))$.

Thus, $F_\phi \cap B = \emptyset$ for $F_\phi := \arg \min_{x \in K} (\Re(\phi(x)))$.

However, F_ϕ is a closed face, so $F_\phi \cap \text{Ext}(K) \neq \emptyset$, thus a contradiction. \square

Corollary 5.40. *Let $(X, \|\cdot\|)$ be a Normed Vector Space, then $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is the weak-* closed convex hull of its extreme points.*

Proof. By the Banach-Alaoglu Theorem, $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is compact in the weak-* topology. The convexity is easy to see. Indeed, for any $f, g \in \bar{B}^{(X^*, \|\cdot\|)}(0, 1)$, $t \in (0, 1)$, we have $\|tf + (1 - t)g\| \leq t\|f\| + (1 - t)\|g\| \leq t + (1 - t) = 1$, so $tf + (1 - t)g \in \bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ and $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is convex. \square

Corollary 5.41. *$L^1([0, 1])$ with the Lebesgue measure is not a dual space.*

5.6.1 Probability Measure

See more about the space of measures in my Pmath651 Measure Theory Notes.

Definition 5.13. Let X be a locally compact Hausdorff space, we define $M(X) := \{\mu : \text{complex Radon measure}\}$, and $\|\mu\|_{M(X)} := |\mu|(X)$, where $|\mu|$ is the total variation.

Theorem 5.42. $(M(X), \|\cdot\|_{M(X)}) \cong (C_0(X), \|\cdot\|_\infty)^*$ isometrically by $\mu(f) := \int_X f d\mu$.

Definition 5.14. The **probability measures** on X is

$$P(X) := \{\mu \in M(X) | \mu \geq 0, \mu(X) = 1\}.$$

The **Dirac measures** are $\delta_x : f \mapsto f(x)$ for $x \in X, f \in C(X)$.

Notice that $P(X)$ is clearly convex by linearity. Also, when X is compact, $C(X) = C_c(X) = C_0(X)$.

Lemma 5.43. *Let X be a compact Hausdorff space, suppose $\mu \in P(X), x \in X$, and $\ker(\delta_x) \supseteq \ker(\mu)$, then $\mu = \delta_x$.*

Proof. For any $g \in C(X)$, we have $\mu(g - \mu(g)\mathbb{1}) = \mu(g) - \mu(g)\mu(\mathbb{1}) = 0$, so $g - \mu(g)\mathbb{1} \in \ker(\mu)$.

Thus, $\delta_x(g - \mu(g)\mathbb{1}) = 0$, which means $\delta_x(g) = \mu(g) \cdot 1 = \mu(g)$.

Thus $\mu = \delta_x$. \square

Proposition 5.44. *Let X be a compact Hausdorff space, then*

$$\text{Ext}(P(X)) = \{\delta_x : x \in X\}.$$

Proof. Let $\mu \in \text{Ext}(P(X))$.

Fix any $0 \leq f < 1$ in $C(X)$.

Let $\lambda := \mu(f) \in [0, 1]$.

If $0 < \lambda < 1$, we can define $\mu_1(g) := \frac{1}{\lambda}\mu(fg) = \frac{1}{\lambda} \int_X fg d\mu$, and $\mu_2(g) := \frac{1}{1-\lambda}\mu((1-f)g) = \frac{1}{1-\lambda} \int_X g(1-f) d\mu$.

We can check that $\mu_1, \mu_2 \in P(X)$, and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$.

Since $\mu \in \text{Ext}(P(X))$, we must have $\mu = \mu_1 = \mu_2$.

Thus, for any $g \in C(X)$,

$$\begin{aligned} \mu(g) &= \mu_1(g) \\ &= \frac{\mu(fg)}{\lambda} \\ &= \frac{\mu(fg)}{\mu(f)}. \end{aligned}$$

Thus, $\mu(fg) = \mu(f)\mu(g)$.

Now suppose $\mu(f) = 0$, we have

$$\begin{aligned} 0 &\leq |\mu(fg)| \\ &= \left| \int_X fg d\mu \right| \\ &\leq \int_X |fg| d\mu \\ &= \int_X f|g| d\mu \\ &\leq \|g\|_\infty \int_X f|g| d\mu \\ &= 0. \end{aligned}$$

Which means $\mu(fg) = 0 = \mu(f)\mu(g)$.

Thus, $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in C(X)$ such that $0 \leq f < 1$.

Since μ is linear, and $\text{Span}\{f \in C(X) : 0 \leq f < 1\} = C(X)$, we have that for all $f, g \in C(X)$,

$$\mu(fg) = \mu(f)\mu(g).$$

We claim that $\exists x \in X$, such that $\ker(\delta_x) \supseteq \ker(\mu)$.

Indeed, suppose for contradiction that $\forall x \in X$, there is $f_x \in C(X)$, such that $f_x \in \ker(\mu) \setminus \ker(\delta_x)$.

Namely, $f_x(x) \neq 0, \mu(f_x) = 0$.

Thus $X = \bigcup_{x \in X} \{y : f_x(y) \neq 0\}$ is an open cover. Since X is compact, there is a finite subcover

$$X = \bigcup_{i=1}^n \{y : f_{x_i}(y) \neq 0\}.$$

Define $f := \sum_{i=1}^n |f_{x_i}|^2 \in C(X)$. Notice that $f > 0$, which means $\frac{1}{f} \in C(X)$.

Now we have

$$\begin{aligned}
1 &= \mu(\mathbb{1}) \\
&= \mu\left(\frac{f}{f}\right) \\
&= \mu(f)\mu\left(\frac{1}{f}\right) \\
&= \sum_{i=1}^n \mu(|f_{x_i}|^2)\mu\left(\frac{1}{f}\right) \\
&= \sum_{i=1}^n \mu(f_{x_i})\mu(\overline{f_{x_i}})\mu\left(\frac{1}{f}\right) \\
&= 0\mu\left(\frac{1}{f}\right) \\
&= 0,
\end{aligned}$$

which is a contradiction.

This proves the claim.

By the lemma,

$$\text{Ext}(P(X)) \subseteq \{\delta_x : x \in X\}.$$

Now given any $x \in X$, suppose $\delta_x = \lambda\mu + (1 - \lambda)\nu$ for some $\mu, \nu \in P(X)$, $\lambda \in (0, 1)$. For any $f \in C(X)$, we have

$$\begin{aligned}
|\delta_x(f)| &= |f(x)| \\
&= \delta_x(|f|) \\
&= \lambda\mu(|f|) + (1 - \lambda)\nu(|f|) \\
&\geq \lambda\mu(|f|) \\
&= \lambda \int_X |f| d\mu \\
&= \lambda \left| \int_X f d\mu \right| \\
&\geq \lambda |\mu(f)|.
\end{aligned}$$

Suppose $f \in \ker(\delta_x)$, we will have $\lambda|\mu(f)| \leq |\delta_x(f)| = 0$, so $\mu(f) = 0$.

Thus $\ker(\delta_x) \subseteq \ker(\mu)$.

By lemma, $\delta_x = \mu$, so $\delta_x = \lambda\delta_x + (1 - \lambda)\nu$, which make $\nu = \delta_x$ as well.

Thus $\delta_x \in \text{Ext}(P(X))$. □

Corollary 5.45. *Let X be a compact Hausdorff space, then*

$$P(X) = \overline{\text{conv}\{\delta_x : x \in X\}}^{\text{weak}}.$$

Proof. For any $\mu \in P(X)$, we have $\|\mu\| = |\mu|(X) = \mu(X) = 1$, so

$$P(X) \subseteq \bar{B}^{(M(X), \|\cdot\|)}(0, 1).$$

By Banach-Alaoglu, $\bar{B}^{(M(X), \|\cdot\|)}(0, 1)$ is weak-* compact.

Suppose there are $(\mu_\lambda)_{\lambda \in \Lambda}$ in $P(X)$ with $\mu_\lambda \rightharpoonup \mu \in M(X)$, we have that

$$\mu(X) = \mu(\mathbb{1}) = \lim_{\lambda \in \Lambda} \mu_\lambda(\mathbb{1}) = 1,$$

since $\mathbb{1} \in C(X)$. We can also show μ is a positive measure, so $\mu \in P(X)$.

This shows $P(X)$ is weak-* closed, so it is also weak-* compact. The result follows from the Krein-Milman Theorem. □

Remark. When X is not compact, $P(X)$ is not weak-* closed.

Example 5.6.2. Consider $\delta_n \in P(\mathbb{R})$, then $\delta_n \rightharpoonup 0$ in $\bar{B}^{M(\mathbb{R})}(0, 1)^{\|\cdot\|}$, but $0 \notin P(\mathbb{R})$.

Nevertheless, when X is a Locally Compact Hausdorff space, with a variation of the proof, we still have

$$\text{Ext}(P(X)) = \{\delta_x | x \in X\},$$

and

$$\text{Ext}\left(\bar{B}^{M(X)}(0, 1)^{\|\cdot\|}\right) = \{z\delta_x : x \in X, z \in \mathbb{C} \text{ such that } \|z\| = 1\}.$$

5.6.2 Stone-Weierstrass Theorem

Theorem 5.46. Let X be a locally compact Hausdorff Space, and $A \subseteq C_0(X, \mathbb{R})$, such that

1. A is closed under $\|\cdot\|_\infty$,
2. A is a sub-algebra of $C_0(X, \mathbb{R})$,
3. A separates points (i.e. $\forall x \neq y \in X, \exists f \in A$ such that $f(x) \neq f(y)$),
4. A vanishes nowhere (i.e. $\forall x \in X, \exists f \in A$, such that $f(x) \neq 0$),

then $A = C_0(X, \mathbb{R})$.

Proof. Suppose for contradiction that $A \subsetneq C_0(X, \mathbb{R})$.

By 1. and a corollary of the Hahn-Banach theorem, there is $\mu \neq 0$ in $A^\perp = \{\mu \in M(X) : \mu(f) = 0 \forall f \in A\}$. Thus,

$$K := \bar{B}^{(M(X), \|\cdot\|)}(0, 1) \cap A^\perp$$

is a non-empty weak-* compact convex set. By the Krein-Milman theorem, there is $\mu \in \text{Ext}(K)$.

We claim that

$$\text{Supp}(\mu) := \{x \in X : \forall U \in \mathcal{O}(X), |\mu|(U) > 0\}$$

is a singleton $\{x\}$ for some $x \in X$.

Suppose for contradiction that there are $x \neq y \in \text{Supp}(\mu)$. By 3., there is $f \in A$ such that $f(x) \neq f(y)$. Notice that f is not $|\mu|$ a.e. constant since it is continuous.

Now find $a, b > 0$ such that $g(x) := a(f(x) + b)$ satisfies $g(x) \geq 0$ and $\int_X g d|\mu| = 1$. Suppose that $\|g\|_\infty \leq 1$, then $0 \leq g \leq 1$ a.e., which means

$$\begin{aligned} \int_X |1 - g| d|\mu| &= \int_X (1 - g) d|\mu| \\ &= |\mu|(X) - \int_X g d|\mu| \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Thus, $1 - g = 0$ $|\mu|$ -a.e., which means $g = 1$ $|\mu|$ -a.e. and it contradicts with f not being constant a.e.. Thus, we must have $\|g\|_\infty > 1$, so

$$\lambda := \frac{1}{\|g\|_\infty} \in (0, 1).$$

Let $\mu_1, \mu_2 \in M(X)$ be as

$$\mu_1(h) := \mu(gh) = \int_X gh d\mu, \quad \mu_2(h) := \mu\left(\frac{1 - \lambda g}{1 - \lambda} h\right) = \int_X \frac{1 - \lambda g}{1 - \lambda} h d\mu.$$

Notice that $\forall h \in A$, $hg = a(f+b)h = afh + abh \in A$, so $\mu_1, \mu_2 \in A^\perp$. We can check that

$$\begin{aligned} |\mu_1|(X) &= \int_X d|\mu_1| \\ &= \int_X g d|\mu| \\ &= 1, \\ |\mu_2|(X) &= \int_X d|\mu_2| \\ &= \int_X \frac{1-\lambda g}{1-\lambda} d|\mu| \\ &= \frac{1-\lambda}{1-\lambda} \\ &= 1. \end{aligned}$$

Thus, $\mu_1, \mu_2 \in K$, with $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. Since $\mu \in \text{Ext}(K)$, we must have $\mu_1 = \mu_2 = \mu$, which means $g = 1$ $|\mu|$ -a.e., a contradiction.

This proves that $\text{Supp}(\mu) = \{x\}$ for some $x \in X$. Thus, $\mu = z\delta_x$ for some $z \in \mathbb{C}$. Also, $|z| = |\mu|(X) = 1$. Since $\mu \in A^\perp$, for any $h \in A$, we must have $zh(x) = \int_X h d\mu = \mu(h) = 0$ which means $h(x) = 0$ and A vanishes at x . \square

6 Spectral Theory

6.1 Adjoint Operators

6.1.1 Adjoint Operators on Normed Spaces

Definition 6.1. Let X, Y be normed spaces, the **adjoint operator** or **dual operator** of a linear operator $A : X \rightarrow Y$ is

$$A^* : Y^* \rightarrow X^*; \quad f \mapsto f \circ A,$$

namely,

$$\forall x \in X, f \in Y^*, \quad \langle x | A^* f \rangle := \langle Ax | f \rangle.$$

Proposition 6.1. Let X, Y, Z be normed spaces, $S \in B(X, Y), T \in B(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.

Proof. Consider any $f \in Z^*$, and any $x \in X$, we have

$$\begin{aligned} (T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx) \\ &= (f)(S(T(x))) \\ &= (f \circ (S \circ T))(x) \\ &= (S \circ T)^*(f)(x). \end{aligned}$$

Thus $(T^* \circ S^*)(f) = (S \circ T)^*(f)$. \square

Proposition 6.2. Let X, Y be normed spaces, and $T \in B(X, Y)$, then

1. $\|T\| = \|T^*\|$, so $T^* \in B(Y^*, X^*)$;
2. The map $T \mapsto T^*$ is linear;
3. T^* is weak-* weak-* continuous; namely, if a net $(\phi_\lambda)_{\lambda \in \Lambda}$ converges to $\phi \in Y$ in weak-* topology on Y^* , we have that $T^*\phi_\lambda$ converges to $T^*\phi$ in weak-* topology on X^* ;
4. For $\mathbb{1}_X : x \mapsto x$, we have $(\mathbb{1}_X)^* = \mathbb{1}_{X^*}$;
5. $T^{**} \in B(X^{**}, Y^{**})$ satisfies $T^{**}|_X = T$.

Proof. 1. We have

$$\begin{aligned}
\|T^*\| &= \sup_{\phi \in Y^*, \|\phi\| \leq 1} \|T^*\phi\|_{X^*} \\
&= \sup_{\phi \in Y^*, \|\phi\| \leq 1} \sup_{x \in X, \|x\| \leq 1} |\langle x | T^*\phi \rangle| \\
&= \sup_{x \in X, \|x\| \leq 1} \sup_{\phi \in Y^*, \|\phi\| \leq 1} |\langle Tx | \phi \rangle| \\
&= \sup_{x \in X, \|x\| \leq 1} \|Tx\| \\
&= \|T\|.
\end{aligned}$$

2. It is obvious by definition.

3. For net $(\phi_\lambda)_{\lambda \in \Lambda}$ in Y^* that converges to $\phi \in Y$ in weak-* topology, and any $x \in X$, we have

$$\begin{aligned}
\langle x | T^*\phi_\lambda \rangle &= \langle Tx | \phi_\lambda \rangle \\
&\rightarrow \langle Tx | \phi \rangle \\
&= \langle x | T^*\phi \rangle.
\end{aligned}$$

Thus, $T^*\phi_\lambda \rightarrow T^*\phi$ in weak-* topology in X^* .

4. This is obvious.

5. Consider the canonical embeddings $x \mapsto \hat{x}, y \mapsto \hat{y}$, we want to show $T^{**}\hat{x} = \widehat{Tx}$. Indeed, for any $\phi \in Y^*$

$$\begin{aligned}
\langle \phi | T^{**}\hat{x} \rangle &= \langle T^*\phi | \hat{x} \rangle \\
&= \langle x | T^*\phi \rangle \\
&= \langle Tx | \phi \rangle \\
&= \langle \phi | \widehat{Tx} \rangle.
\end{aligned}$$

□

Theorem 6.3. *Let X, Y be normed spaces, and $S \in B(Y^*, X^*)$, then there is $T \in B(X, Y)$ such that $T^* = S$ if and only if S is weak-* weak-* continuous.*

Proof. (\implies) is by the previous proposition.

(\impliedby): Assume S is weak-* weak-* continuous.

Consider $S^* \in B(X^{**}, Y^{**})$, and let $T := S^*|_X \in B(X, Y^{**})$. Fix $x \in X$, let $(\phi_\lambda)_{\lambda \in \Lambda}$ be a net that converges in weak-* topology (Y^*, \mathcal{T}_Y) , then

$$\begin{aligned}
\langle \phi_\lambda | S^*\hat{x} \rangle &= \langle S\phi_\lambda | \hat{x} \rangle \\
&\rightarrow \langle S\phi | \hat{x} \rangle \\
&= \langle \phi | S^*\hat{x} \rangle.
\end{aligned}$$

Thus, $S^*\hat{x}$ is continuous in weak-* topology (Y^*, \mathcal{T}_Y) , so $S^*\hat{x} \in Y$. This shows we can redefine $T \in B(X, Y)$. Now for any $x \in X, \phi \in Y^*$, we have

$$\begin{aligned}
\langle x | T^*\phi \rangle &= \langle Tx | \phi \rangle \\
&= \langle \phi | S^*\hat{x} \rangle \\
&= \langle S\phi | \hat{x} \rangle \\
&= \langle x | S\phi \rangle.
\end{aligned}$$

□

Theorem 6.4. *Let X, Y be Banach spaces, then $T : X \rightarrow Y$ is bounded if and only if T is weak-weak continuous.*

Proof. See A4.

□

6.1.2 Hilbert Adjoint Operators

Definition 6.2. Let \mathcal{H} be a Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator, the **Hilbert adjoint operator** of T is $T^\dagger : \mathcal{H} \rightarrow \mathcal{H}$ such that $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \forall x, y \in \mathcal{H}$.

Proposition 6.5. Let \mathcal{H} be a Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator, we have $\langle Ty, x \rangle = \langle y, T^\dagger x \rangle \forall x, y \in \mathcal{H}$. Thus, $(T^\dagger)^\dagger = T$.

Proof.

$$\begin{aligned} \langle Ty, x \rangle &= \overline{\langle x, Ty \rangle} \\ &= \overline{\langle T^\dagger x, y \rangle} \\ &= \langle y, T^\dagger x \rangle. \end{aligned}$$

□

Theorem 6.6. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. T^\dagger always exists, and is given by $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$, where $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$; $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism, and T^* is the dual operator of T . In addition, T^\dagger is also a bounded linear operator, with $\|T^\dagger\| = \|T\|$.

Proof. $\forall y \in \mathcal{H}$, we have that

$$\begin{aligned} \langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\ &= \langle y | (T^* \circ \Phi)(x) \rangle \\ &= \langle Ty | \Phi(x) \rangle \\ &= \langle x, Ty \rangle. \end{aligned}$$

Linearity follows from Φ, Φ^{-1} being anti-linear, and T^* being linear. We can also compute it directly from the definition. Indeed, consider any $x, y, z \in \mathcal{H}, c \in \mathbb{C}$, we have that

$$\begin{aligned} \langle T^\dagger(x + cz), y \rangle &= \langle x + cz, Ty \rangle \\ &= \langle x, Ty \rangle + \bar{c} \langle z, Ty \rangle \\ &= \langle T^\dagger x, y \rangle + \bar{c} \langle T^\dagger z, y \rangle \\ &= \langle T^\dagger x + cT^\dagger z, y \rangle. \end{aligned}$$

Since this holds for any $y \in \mathcal{H}$, we have that $T^\dagger(x + cz) = T^\dagger x + cT^\dagger z$, and thus T^\dagger is linear. Also,

$$\|T^\dagger\| \leq \|\Phi^{-1}\| \|T\| \|\Phi\| = \|T\|.$$

We can also compute this directly from the definition. Indeed, given any $x \in \mathcal{H}$, we have that

$$\begin{aligned} \|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\ &= \langle x, TT^\dagger x \rangle \\ &\leq \|x\| \|TT^\dagger x\| \\ &\leq \|x\| \|T\| \|T^\dagger x\| \\ &\implies \\ \|T^\dagger x\| &\leq \|x\| \|T\| \\ &\implies \\ \|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\ &\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{\|x\| \|T\|}{\|x\|} \\ &= \|T\|. \end{aligned}$$

Thus T^\dagger is also a bounded linear operator, with

$$\|T\| = \|(T^\dagger)^\dagger\| \leq \|T^\dagger\| \leq \|T\|.$$

□

Remark. $\forall x, y \in \mathcal{H}$, $\langle (Tx)^\dagger | y \rangle = \langle Tx, y \rangle = \langle x, T^\dagger y \rangle = \langle x^\dagger | T^\dagger y \rangle$. We thus abuse the notation, and write $(Tx)^\dagger = \langle x | T^\dagger$

Corollary 6.7. *Let \mathcal{H} be a Hilbert space, and $S, T \in B(\mathcal{H})$ be bounded linear operators. We have that $(S \circ T)^\dagger = T^\dagger \circ S^\dagger$. Also, for any $\alpha \in \mathbb{C}$, we have $(\alpha T + S)^\dagger = \bar{\alpha} T^\dagger + S^\dagger$.*

Proposition 6.8. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. We have*

$$\|T\|^2 = \|TT^\dagger\| = \|T^\dagger T\|.$$

Proof. We have $\|TT^\dagger\| \leq \|T\|\|T^\dagger\| = \|T\|^2$.

Now for any $x \in \mathcal{H}$, we have

$$\begin{aligned} \|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\ &= \langle x, TT^\dagger x \rangle \\ &\leq \|x\| \|TT^\dagger x\| \\ &\leq \|x\| \|TT^\dagger\| \|x\| \\ &= \|TT^\dagger\| \|x\|^2; \\ &\implies \\ \|T^\dagger x\| &\leq \sqrt{\|TT^\dagger\|} \|x\| \\ &\implies \\ \|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\ &\leq \sqrt{\|TT^\dagger\|} \\ &\implies \\ \|T^\dagger\|^2 &\leq \|TT^\dagger\|. \end{aligned}$$

□

Corollary 6.9. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. We have that $T = 0$ if and only if $T^\dagger T = 0$.*

There is a deep analogy between operators $B(\mathcal{H})$ and algebra of the form $(\ell^\infty(X), \|\cdot\|_\infty)$ for a set X , with the pointwise multiplication and involution $f^*(x) := \overline{f(x)}$.

Example 6.1.1. Consider $\pi : \ell^\infty(X) \rightarrow B(\ell^2(X))$ by $f \mapsto M_f$, where the **multiplication operator** $M_f(\phi)(x) := f(x)\phi(x)$ for all $\phi \in \ell^2(X), x \in X$. We claim that it is a unital isometric homomorphism. Indeed, for any $\phi \in \ell^2(X)$, we have

$$\begin{aligned} \pi(\mathbb{1})(\phi)(x) &= 1 \cdot \phi(x) \\ &= \phi(x), \end{aligned}$$

so $\pi(\mathbb{1})(\phi) = \phi$ and $\pi(\mathbb{1}) = \mathbb{1}$. Also, for any $f, g \in \ell^\infty(X)$,

$$\begin{aligned} (\pi(fg)(\phi))(x) &= (fg)(x)\phi(x) \\ &= f(x)g(x)\phi(x) \\ &= f(x)(\pi(g)(\phi))(x) \\ &= \pi(f)(\pi(g)(\phi))(x) \\ &= ((\pi(f) \circ \pi(g))(\phi))(x). \end{aligned}$$

Thus, $\pi(fg) = \pi(f) \circ \pi(g)$. Also, for any $\phi, \eta \in X$, we have

$$\begin{aligned}
\langle \pi(f^*)(\phi), \eta \rangle &= \sum_{x \in X} \pi(f^*)(\phi)(x) \overline{\eta(x)} \\
&= \sum_{x \in X} f^*(x) \phi(x) \overline{\eta(x)} \\
&= \sum_{x \in X} \phi(x) \overline{f(x) \eta(x)} \\
&= \sum_{x \in X} \phi(x) \overline{\pi(f)(\eta)(x)} \\
&= \langle \phi, \pi(f)(\eta) \rangle \\
&= \langle \pi(f)^*(\phi), \eta \rangle.
\end{aligned}$$

Thus, $\pi(f^*) = \pi(f)^*$. Lastly, we have that

$$\begin{aligned}
\|\pi(f)(\phi)\|^2 &= \langle \pi(f)(\phi), \pi(f)(\phi) \rangle \\
&= \sum_{x \in X} |\pi(f)(\phi)(x)|^2 \\
&= \sum_{x \in X} |f(x)\phi(x)|^2 \\
&= \sum_{x \in X} |f(x)|^2 |\phi(x)|^2 \\
&\leq \|f\|_\infty^2 \left(\sum_{x \in X} |\phi(x)|^2 \right) \\
&= \|f\|_\infty^2 \|\phi\|^2.
\end{aligned}$$

Thus, $\|\pi(f)\| \leq \|f\|_\infty$. On the other hand, $\delta_x \in \ell^2(X)$ for any x , and

$$\begin{aligned}
\|f\|_\infty^2 &= \sup_{x \in X} |f(x)|^2 \\
&= \sup_{x \in X} \langle \pi(f)(\delta_x), \pi(f)(\delta_x) \rangle \\
&= \sup_{x \in X} \|\pi(f)(\delta_x)\|^2 \\
&\leq \sup_{\phi \in \ell^2(X)} \|\pi(f)(\phi)\|^2 \\
&= \|\pi(f)\|^2.
\end{aligned}$$

Thus, $\|f\|_\infty = \|\pi(f)\|$.

Indeed, when we take $X = \{1, \dots, n\}$, then $\ell^2(X) = \mathbb{C}^n$, and $\pi : \ell^\infty(X) \rightarrow B(\ell^2(X)) \cong M_n(\mathbb{C})$, and $\pi(\ell^\infty(X)) = D_n$, the diagonal matrices.

Proposition 6.10. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$. We have*

$$\ker(T^\dagger) = \text{Im}(T)^\perp.$$

Proof. For any $x \in \ker(T^\dagger)$, $Ty \in \text{Im}(T)$, we have

$$\begin{aligned}
\langle x, Ty \rangle &= \langle T^\dagger x, y \rangle \\
&= \langle 0, y \rangle \\
&= 0.
\end{aligned}$$

Thus, $x \in \text{Im}(T)^\perp$, and we have $\ker(T^\dagger) \subseteq \text{Im}(T)^\perp$.

On the other hand, for any $x \in \text{Im}(T)^\perp, y \in \mathcal{H}$, we have

$$\begin{aligned}\langle T^\dagger x, y \rangle &= \langle x, Ty \rangle \\ &= 0.\end{aligned}$$

Thus, $T^\dagger x = 0$, so $x \in \ker(T^\dagger)$. We thus have $\text{Im}(T)^\perp \subseteq \ker(T^\dagger)$. \square

Proposition 6.11. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$. Suppose $T^{-1} \in B(\mathcal{H})$ exists, then $(T^\dagger)^{-1} \in B(\mathcal{H})$ exists, and*

$$(T^\dagger)^{-1} = (T^{-1})^\dagger.$$

Proof.

$$\begin{aligned}TT^{-1} &= \mathbb{1} = T^{-1}T \\ (TT^{-1})^\dagger &= \mathbb{1}^\dagger = (T^{-1}T)^\dagger \\ (T^{-1})^\dagger T^\dagger &= \mathbb{1} = T^\dagger (T^{-1})^\dagger.\end{aligned}$$

Thus, $(T^\dagger)^{-1} = (T^{-1})^\dagger$. \square

Proposition 6.12. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$. The following are equivalent:*

1. *T is invertible (i.e. an isomorphism between \mathcal{H} and \mathcal{H} as Banach spaces);*
2. *T^\dagger is invertible;*
3. *T is bounded below;*
4. *T^\dagger is bounded below;*
5. *Both $\text{Im}(T), \text{Im}(T^\dagger)$ are closed, and T, T^\dagger are injective;*
6. *$\mathcal{H} = \text{Im}(T) = \text{Im}(T^\dagger)$;*
7. *T is bijective;*
8. *T^\dagger is bijective.*

Proof. 1 \implies 2 is done above.

2 \implies 1. We have $T = (T^\dagger)^\dagger$, so applying the above result to T^\dagger gives 1.

1 \implies 3. This is by theorem 3.12.

2 \implies 4. This is by theorem 3.12.

3, 4 \implies 5. Let T be bounded below by c . The injectivity is from theorem 3.12. Also, consider any convergent sequence $(Tx_n)_{n=1}^\infty$ in $\text{Im}(T)$. Since it is Cauchy, for any $\epsilon > 0$, there is $N \geq 1$, such that for all $n, m \geq N$,

$$c\|x_n - x_m\| \leq \|Tx_n - Tx_m\| < c\epsilon.$$

Thus, $(x_n)_{n=1}^\infty$ is Cauchy in \mathcal{H} , so there must be some $x \in \mathcal{H}$ such that $x_n \rightarrow x$. Since T is continuous, $Tx_n \rightarrow Tx \in \text{Im}(T)$. This shows $\text{Im}(T)$ is closed. Similarly, $\text{Im}(T^\dagger)$ is closed and it is injective.

5 \implies 6. We have that $\overline{\text{Im}(T)} \oplus \text{Im}(T)^\perp = \mathcal{H}$ by theorem 4.12. Since T^\dagger is injective, we have $\text{Im}(T)^\perp = \ker(T^\dagger) = \{0\}$ by proposition 6.10. Thus, $\overline{\text{Im}(T)} = \mathcal{H}$. Since $\text{Im}(T)$ is closed, we have $\text{Im}(T) = \mathcal{H}$. Similarly, $\text{Im}(T^\dagger) \oplus \ker(T) = \mathcal{H}$, so $\text{Im}(T^\dagger) = \mathcal{H}$.

6 \implies 7. We have that $\ker(T) = \text{Im}(T^\dagger)^\perp = (\mathcal{H})^\perp = \{0\}$, so T is injective. Also, T is surjective by assumption.

7 \implies 1. By the Banach Isomorphism theorem 3.23, $T^{-1} \in B(\mathcal{H})$ is bounded.

6 \implies 8. We have that $\ker(T^\dagger) = \text{Im}(T)^\perp = (\mathcal{H})^\perp = \{0\}$, so T^\dagger is injective. Also, T^\dagger is surjective by assumption.

8 \implies 2. By the Banach Isomorphism Theorem, $(T^\dagger)^{-1} \in B(\mathcal{H})$ is bounded. \square

6.1.3 Normal, Self-adjoint and Positive Semidefinite Operators

Definition 6.3. Let \mathcal{H} be a Hilbert space. A bounded linear operator $T \in B(\mathcal{H})$ is **normal** if $T^\dagger T = TT^\dagger$.

Proposition 6.13. Let \mathcal{H} be a Hilbert space. If $T \in B(\mathcal{H})$ is normal, we have that for all $c \in \mathbb{F}$, $c\mathbb{I} - T$ is also normal.

Proof.

$$\begin{aligned} (c\mathbb{I} - T)^\dagger (c\mathbb{I} - T) &= (\bar{c}\mathbb{I} - T^\dagger)(c\mathbb{I} - T) \\ &= |c|^2 \mathbb{I} - \bar{c}T - cT^\dagger + T^\dagger T \\ &= |c|^2 \mathbb{I} - cT^\dagger - \bar{c}T + TT^\dagger, \\ (c\mathbb{I} - T)(c\mathbb{I} - T)^\dagger &= (c\mathbb{I} - T)(\bar{c}\mathbb{I} - T^\dagger) \\ &= |c|^2 \mathbb{I} - cT^\dagger - \bar{c}T + TT^\dagger \\ &= (c\mathbb{I} - T)^\dagger (c\mathbb{I} - T). \end{aligned}$$

□

Proposition 6.14. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a normal operator. For $x \in \mathcal{H}$, we have $Tx = \lambda x$ if and only if $T^\dagger x = \bar{\lambda}x$.

Proof. Suppose $Tx = \lambda x$. Let $N := \lambda\mathbb{I} - T$, which is normal, and $Nx = \lambda x - Tx = 0$, $N^\dagger = \bar{\lambda}\mathbb{I} - T^\dagger$. We have that

$$\begin{aligned} \|N^\dagger x\|^2 &= \langle N^\dagger x, N^\dagger x \rangle \\ &= \langle NN^\dagger x, x \rangle \\ &= \langle N^\dagger Nx, x \rangle \\ &= \langle N^\dagger 0, x \rangle \\ &= 0. \end{aligned}$$

Thus $N^\dagger x = 0$, which means $\bar{\lambda}x - T^\dagger x = 0$.

The converse is true by $(T^\dagger)^\dagger = T$.

□

Definition 6.4. Let \mathcal{H} be a Hilbert space. A bounded linear operator $T \in B(\mathcal{H})$ is **self-adjoint** if $T^\dagger = T$. The set of all self-adjoint operators is $B(\mathcal{H})_{sa}$.

Definition 6.5. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$, we define

$$\Re(T) := \frac{T + T^\dagger}{2}, \quad \Im(T) := \frac{T - T^\dagger}{2i}.$$

Proposition 6.15. Let \mathcal{H} be a Hilbert space. $B(\mathcal{H})_{sa}$ is a \mathbb{R} subspace of $B(\mathcal{H})$. We have $\Re(T), \Im(T) \in B(\mathcal{H})_{sa}$, and $\Re(T) + i\Im(T) = T$, so

$$B(\mathcal{H}) = B(\mathcal{H})_{sa} + iB(\mathcal{H})_{sa}.$$

Proposition 6.16. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})_{sa}$. We have $\langle Tx, x \rangle \in \mathbb{R}$ for any $x \in \mathcal{H}$.

Proof. We have

$$\begin{aligned} \langle Tx, x \rangle &= \langle x, T^\dagger x \rangle \\ &= \langle x, Tx \rangle \\ &= \overline{\langle x, Tx \rangle}. \end{aligned}$$

□

Proposition 6.17. Let \mathcal{H} be a Hilbert space. Any self-adjoint operator T is normal.

Proof. $T^\dagger T = TT = TT^\dagger$. □

Proposition 6.18. Let \mathcal{H} be a Hilbert space, for any $T \in B(\mathcal{H})$, we have that $T^\dagger T, TT^\dagger \in B(\mathcal{H})_{sa}$.

Proof. We have $(T^\dagger T)^\dagger = T^\dagger (T^\dagger)^\dagger = T^\dagger T$, so it is self-adjoint. Similarly, $(TT^\dagger)^\dagger = (T^\dagger)^\dagger T^\dagger = TT^\dagger$. □

Lemma 6.19. Let \mathcal{H} be a Hilbert space, for any $T \in B(\mathcal{H})_{sa}$, we have that $\|T^2\| = \|T\|^2$.

Proof. $\|T^2\| = \|TT^\dagger\| = \|T\|^2$. □

Proposition 6.20. Let \mathcal{H} be a Hilbert space, for any normal $T \in B(\mathcal{H})$, we have that $\|T^2\| = \|T\|^2$.

Proof. Since $T^\dagger T$ is self-adjoint, we have $\|(T^\dagger T)^2\| = \|T^\dagger T\|^2$. Thus,

$$\begin{aligned} \|T^2\| &= \|(T^2)^\dagger T^2\|^{\frac{1}{2}} \\ &= \|T^\dagger T^\dagger T T\|^{\frac{1}{2}} \\ &= \|T^\dagger T T^\dagger T\|^{\frac{1}{2}} \\ &= \|(T^\dagger T)^2\|^{\frac{1}{2}} \\ &= (\|T^\dagger T\|^2)^{\frac{1}{2}} \\ &= \|T^\dagger T\| \\ &= \|T\|^2. \end{aligned}$$

□

Definition 6.6. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})_{sa}$. We say T is **positive semidefinite**, written as $T \geq 0$, if

$$\forall x \in \mathcal{H}, \langle Tx, x \rangle \geq 0.$$

For $S, T \in B(\mathcal{H})_{sa}$, we say $S \leq T$ if $T - S \geq 0$. We also define $B(\mathcal{H})^+$ to be the set of all positive semidefinite operators.

Proposition 6.21. Let \mathcal{H} be a Hilbert space, for any $T \in B(\mathcal{H})$, we have that $T^\dagger T \geq 0$.

Proof. It is self-adjoint by above proposition. Consider any $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle T^\dagger T x, x \rangle &= \langle T x, T x \rangle \\ &= \|T x\|^2 \\ &\geq 0. \end{aligned}$$

□

Proposition 6.22. Let \mathcal{H} be a Hilbert space, and $T \geq 0$. We have that $M : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ by $M[x, y] := \langle T x, y \rangle$ is a positive semidefinite form on \mathcal{H} .

Proof. 1. conjugate symmetry; i.e. $M[x, y] = \langle T x, y \rangle = \langle x, T y \rangle = \overline{\langle T y, x \rangle} = \overline{M[y, x]}$.

2. linearity in the second argument; i.e. $M[x, ay + z] = \langle T x, ay + z \rangle = a \langle T x, y \rangle + \langle T x, z \rangle = M[x, y] + aM[x, z]$.

3. positive semi-definiteness; i.e. $M[x, x] = \langle T x, x \rangle \geq 0$. □

Proposition 6.23. Let \mathcal{H} be a Hilbert space, and $S, T \in B(\mathcal{H})_{sa}$.

1. Suppose $S \leq T$, then $A^\dagger SA \leq A^\dagger TA$ for all $A \in B(\mathcal{H})$.
2. Suppose $0 \leq S \leq T$, then $\|S\| \leq \|T\|$.
3. $-\|T\|\mathbb{I} \leq T \leq \|T\|\mathbb{I}$.
4. Suppose $-C\mathbb{I} \leq T \leq C\mathbb{I}$ for some $C \geq 0$, then $\|T\| \leq C$.
5. Suppose $0 \leq S$, we have $aS^n \geq 0$ for any non-negative $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$.

Proof. 1. For any $x \in \mathcal{H}$, we have $\langle (T - S)Ax, Ax \rangle \geq 0$, since $T - S \geq 0$. Thus, $\langle A^\dagger(T - S)Ax, x \rangle \geq 0$, which means $A^\dagger TA - A^\dagger SA = A^\dagger(T - S)A \geq 0$.

2. Consider $M[x, y] := \langle Sx, y \rangle$, which defines a positive semidefinite form 4.1 on \mathcal{H} . By Cauchy-Schwartz theorem 4.3, we have

$$\begin{aligned} |\langle Sx, y \rangle|^2 &\leq \langle Sx, x \rangle \langle Sy, y \rangle \\ &\leq \langle Tx, x \rangle \langle Ty, y \rangle \\ &\leq \|T\| \|x\|^2 \|T\| \|y\|^2 \\ &\leq \|T\|^2 \|x\|^2 \|y\|^2. \end{aligned}$$

Thus $|\langle Sx, y \rangle| \leq \|T\|$ for any $\|x\|, \|y\| \leq 1$, which means $\|S\| \leq \|T\|$.

3. Since $|\langle Tx, x \rangle| \leq \|T\| \|x\|^2$, we have

$$\begin{aligned} -\|T\| \|x\|^2 &\leq \langle Tx, x \rangle \leq \|T\| \|x\|^2 \\ -\|T\| \|x\| &\leq \langle Tx, x \rangle \leq \|T\| \|x\|. \end{aligned}$$

4. For any $x, y \in \mathcal{H}$, we have $\langle T(x + y), x + y \rangle = \langle Tx, x \rangle + 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$ and $\langle T(x - y), x - y \rangle = \langle Tx, x \rangle - 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$ by proposition 4.1, so

$$\langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle = 4\Re(\langle Ty, x \rangle).$$

Similarly, $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

By assumption, we have

$$\langle T(x + y), x + y \rangle \leq \langle C\mathbb{I}(x + y), x + y \rangle = C\|x + y\|^2$$

since $T \leq C\mathbb{I}$, and

$$\langle T(x - y), x - y \rangle \geq \langle -C\mathbb{I}(x - y), x - y \rangle = -C\|x - y\|^2$$

since $T \geq -C\mathbb{I}$. Thus we have

$$\begin{aligned} \langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle &\leq C\|x + y\|^2 - (-C\|x - y\|^2) \\ &= C(\|x + y\|^2 + \|x - y\|^2) \\ 4\Re(\langle Ty, x \rangle) &\leq 2C(\|x\|^2 + \|y\|^2). \end{aligned}$$

Taking supreme over all $\|x\|, \|y\| \leq 1$, we have $4\|T\| \leq 4C$.

5. It is obviously true when $n = 0, 1$. Now assume $n \geq 2$, and it already holds for $n - 2$. Clearly aS^n is self-adjoint. For any $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle aS^n x, x \rangle &= a \langle S^{n-2} Sx, x \rangle \\ &= a \langle S^{n-2} Sx, S^\dagger x \rangle \\ &= a \langle S^{n-2}(Sx), Sx \rangle \\ &\geq 0. \end{aligned}$$

By induction, it holds for all $n \in \mathbb{N}$.

□

Theorem 6.24 (Dini's). *Let K be compact, and $f \in C(K)$. Suppose there are $(g_n)_{n=1}^\infty \subseteq C(K)$ such that they are monotone increasing, and for all $x \in K$, $g_n(x) \nearrow f(x)$ pointwise, then $g_n \rightarrow f$ uniformly.*

Proof. Let $g_n := f - q_n$, we have that $g_n(x) \searrow 0$ monotonously pointwise. Given any $\epsilon > 0$, for each $n \geq 1$, define

$$U_n := \{x \in K : g_n(x) < \epsilon\},$$

which are open in K . We have that $K = \bigcup_{n=1}^\infty U_n$ is an open cover. Since K is compact, there is $N \geq 1$, such that $K = \bigcup_{n=1}^N U_n$. Since $U_1 \subseteq U_2 \subseteq \dots \subseteq U_N$, we have that $K = U_N$. For any $n \geq N$, we have that $K = U_N \subseteq U_n \subseteq K$, so $K = U_n$. This show that $\forall x \in K$,

$$f(x) - q_n(x) = g_n(x) < \epsilon.$$

□

Lemma 6.25. *For $f \in C([0, 1])$, with $f(t) := 1 - \sqrt{1-t}$, there are polynomials $(p_n)_{n=1}^\infty$, whose coefficients are all non-negative, then $f = \sum_{n=1}^\infty p_n$ uniformly.*

Proof. Let $q_0(t) := 0$, and $q_n(t) := \frac{1}{2}(t + q_{n-1}(t)^2)$. By induction, $0 \leq q_n \leq 1$, and q_n has non-negative coefficients. Also, $q_n \leq q_{n+1}$.

Define $p_n := q_n - q_{n-1}$ for all $n \geq 1$.

$$\begin{aligned} p_n(q_n + q_{n-1}) &= (q_n - q_{n-1})(q_n + q_{n-1}) \\ &= q_n^2 - q_{n-1}^2 \\ &= 2q_{n+1} - t - (1q_n - t) \\ &= 2(q_{n+1} - q_n) \\ &= 2p_{n+1}. \end{aligned}$$

By induction, we have that all p_n have non-negative coefficient.

For any $t \in [0, 1]$, since $q_n(t)$ is monotone increasing, there is some $q(t) := \lim_{n \rightarrow \infty} q_n(t) \leq 1$. Also, $q(t) = \frac{1}{2}(t + q(t)^2)$. Thus, $q(t) = 1 - \sqrt{1-t} = f(t)$. We now have

$$f(t) = \lim_{n \rightarrow \infty} q_n(t) = \sum_{n=1}^\infty p_n(t).$$

This convergence is uniform by Dini's Theorem. □

Theorem 6.26 (Square Root Theorem). *Let \mathcal{H} be a Hilbert space, and $0 \leq T$, then there is a unique $0 \leq S$, such that $S^2 = T$. We write $S := T^{\frac{1}{2}}$, and it commutes with all operators that commute with T .*

Proof. Assume $0 \leq T \leq \mathbb{I}$. Consider $S := \mathbb{I} - T$. Note that $0 \leq S \leq \mathbb{I}$. Let $p_n(t) := \sum_k \alpha_{n,k} t^k$ and $q_n := \sum_{i=1}^n p_n$ be from the above lemma. For any $n \geq 1$, we have

$$\begin{aligned} \|p_n(S)\| &\leq \sum_k \alpha_{n,k} \|S\|^k \\ &\leq \sum_k \alpha_{n,k} \|\mathbb{I}\|^k \\ &= \sum_k \alpha_{n,k} 1^k \\ &= p_n(1). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n=1}^\infty \|p_n(S)\| &\leq \sum_{n=1}^\infty p_n(1) \\ &= f(1) \\ &= 1. \end{aligned}$$

Since $B(\mathcal{H})$ is a Banach space, we can define

$$R := \sum_{n=1}^{\infty} p_n(S) = \lim_{n \rightarrow \infty} q_n(S)$$

by A2Q4. Also, $\|R\| \leq 1$, so $R \leq \mathbb{I}$. Also, since each coefficient in q_n is non-negative, we have that each $q_n(S) \geq 0$. Thus, $R \geq 0$. We now have

$$\begin{aligned} (\mathbb{I} - R)^2 &= (\mathbb{I} - \lim_{n \rightarrow \infty} q_n(S))^2 \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - q_n(S))^2 \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - 2q_n(S) + q_n(S)^2) \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - 2q_n(S) + 2q_{n+1}(S) - S) \\ &= \mathbb{I} - S - 2 \lim_{n \rightarrow \infty} q_n(S) + 2 \lim_{n \rightarrow \infty} q_{n+1}(S) \\ &= T - 2R + 2R \\ &= T. \end{aligned}$$

We can thus define $T^{\frac{1}{2}} := \mathbb{I} - R$.

For any A that commutes with T , we have

$$SA = A(\mathbb{I} - T) = A - AT = A - TA = (\mathbb{I} - T)A = SA.$$

Also, for any $k \geq 1$, we have

$$S^k A = S^{k-1} AS = S^{k-2} AS^2 = \dots = AS^k.$$

Thus, $Aq_n(S) = q_n(S)A$, so $AR = RA$. This shows

$$AT^{\frac{1}{2}} = A(\mathbb{I} - R) = A - AR = A - RA = (\mathbb{I} - R)A = T^{\frac{1}{2}}A.$$

Now for any general T , we can just take $\tilde{T} := \frac{1}{\|T\|}T$, so we have $\tilde{T} \leq \frac{1}{\|T\|}\|T\|\mathbb{I} = \mathbb{I}$. Now take $T^{\frac{1}{2}} := \sqrt{\|T\|}\tilde{T}^{\frac{1}{2}}$. Clearly, $T^{\frac{1}{2}}$ is self-adjoint, and

$$T^{\frac{1}{2}}T^{\frac{1}{2}} = \sqrt{\|T\|}\tilde{T}^{\frac{1}{2}}\sqrt{\|T\|}\tilde{T}^{\frac{1}{2}} = \|T\|\tilde{T} = T.$$

Also, if $AT = TA$, we will have $A\tilde{T} = \tilde{T}A$, so $A\tilde{T}^{\frac{1}{2}} = \tilde{T}^{\frac{1}{2}}A$ and $AT^{\frac{1}{2}} = T^{\frac{1}{2}}A$.

Suppose there is some $A \geq 0$, such that $A^2 = T$, we have $AT = A^3 = TA$. Since A commutes with T , it also commutes with $T^{\frac{1}{2}}$. We thus have

$$(T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A) = T - A^2 = 0.$$

Now let $Y := \text{Im}(T^{\frac{1}{2}} + A)$. For any $y \in Y$, we have $(T^{\frac{1}{2}} - A)(y) = 0$, since there will be $x \in \mathcal{H}$, such that $(T^{\frac{1}{2}} - A)(x) = y$, so

$$(T^{\frac{1}{2}} - A)(y) = (T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A)(x) = 0.$$

On the other hand, for any $z \in Y^\perp = \ker((T^{\frac{1}{2}} + A)^\dagger) = \ker(T^{\frac{1}{2}} - A)$ by proposition 6.10, we have that

$$\begin{aligned} \|T^{\frac{1}{4}}z\|^2 &= \langle T^{\frac{1}{4}}z, T^{\frac{1}{4}}z \rangle \\ &= \langle z, T^{\frac{1}{4}}T^{\frac{1}{4}}z \rangle \\ &= \langle z, T^{\frac{1}{2}}z \rangle \\ &\leq \langle z, T^{\frac{1}{2}}z \rangle + \langle z, Az \rangle \\ &= \langle z, (T^{\frac{1}{2}} + A)z \rangle \\ &= 0. \end{aligned}$$

Thus, $T^{\frac{1}{4}}z = 0$, and $T^{\frac{1}{2}}z = T^{\frac{1}{4}}T^{\frac{1}{4}}z = 0$. Similarly,

$$\begin{aligned}\|A^{\frac{1}{2}}z\|^2 &= \langle A^{\frac{1}{2}}z, A^{\frac{1}{2}}z \rangle \\ &= \langle z, Az \rangle \\ &\leq \langle z, (A + T^{\frac{1}{2}})z \rangle \\ &= 0,\end{aligned}$$

so $A^{\frac{1}{2}}z = 0$ and $Az = 0$. Thus,

$$(T^{\frac{1}{2}} - A)(z) = T^{\frac{1}{2}}z - Az = 0.$$

Now for any $x \in \mathcal{H}$, there is $y \in Y$, and $z \in Y^\perp$, such that $x = y + z$, so we have

$$(A - T^{\frac{1}{2}})(x) = (A - T^{\frac{1}{2}})(y) + (A - T^{\frac{1}{2}})(z) = 0.$$

This shows $T^{\frac{1}{2}} = A$, which means $T^{\frac{1}{2}}$ is unique. □

Corollary 6.27. *Let \mathcal{H} be a Hilbert space. If $S, T \geq 0$, and $ST = TS$, then $ST \geq 0$.*

Proof. We have $(ST)^\dagger = T^\dagger S^\dagger = TS = ST$, so $ST \in B(\mathcal{H})_{sa}$.

Also, for any $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle STx, x \rangle &= \langle S^{\frac{1}{2}}S^{\frac{1}{2}}Tx, x \rangle \\ &= \langle S^{\frac{1}{2}}Tx, (S^{\frac{1}{2}})^\dagger x \rangle \\ &= \langle TS^{\frac{1}{2}}x, S^{\frac{1}{2}}x \rangle \\ &\geq 0.\end{aligned}$$

□

Corollary 6.28. *Let \mathcal{H} be a Hilbert space. Suppose $T \in B(\mathcal{H})_{sa}$, and $\langle Tx, x \rangle = 0$ for all $x \in \mathcal{H}$, then $T = 0$.*

Proof. For any $x \in \mathcal{H}$, we have $\langle Tx, x \rangle = 0 \geq 0$, so $T \geq 0$. In addition,

$$\begin{aligned}0 &= \langle Tx, x \rangle \\ &= \langle T^{\frac{1}{2}}T^{\frac{1}{2}}x, x \rangle \\ &= \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x \rangle \\ &= \|T^{\frac{1}{2}}x\|^2.\end{aligned}$$

Thus $T^{\frac{1}{2}}x = 0$, so $Tx = T^{\frac{1}{2}}T^{\frac{1}{2}}x = T^{\frac{1}{2}}0 = 0$. □

Proposition 6.29. *Let \mathcal{H} be a Hilbert space, and $U \in B(\mathcal{H})$. U is an isometry if and only if $U^\dagger U = \mathbb{I}$. Also, the following are equivalent:*

1. U is unitary,
2. $U^\dagger = U^{-1}$, and
3. U is a normal isometry.

Proof. See A5. □

6.2 Compact Operators

6.2.1 Compactness

See more in Real Analysis Pmath351 or Applied Functional Analysis Amath731.

Definition 6.7. Let (X, d) be a metric space. We say $S \subseteq X$ is **sequentially compact** if for every bounded sequence $(x_n)_{n=1}^\infty$, there is a convergent subsequence $x_{n_k} \rightarrow x \in S$.

Definition 6.8. Let (X, d) be a metric space. We say $S \subseteq X$ is **relatively compact** if for every bounded sequence $(x_n)_{n=1}^\infty$, there is a convergent subsequence $x_{n_k} \rightarrow x \in X$.

Definition 6.9. Let X be a metric space. We say $S \subseteq X$ is **totally bounded** if for any $\epsilon > 0$, there exists a finite subset $\{x_i\}_{i=1}^n$, such that $S \subseteq \bigcup_{i=1}^n B(a_i, \epsilon)$. Such a subset is called an ϵ -**net**.

Definition 6.10. Let X be a metric space. We say $S \subseteq X$ is **bounded** if there exists a $r > 0$, such that for all $x, y \in X$, $d(x, y) < r$.

Theorem 6.30. Let S be a subset of a metric space X . The following are equivalent:

1. S is compact.
2. S has the finite intersection property.
3. S is sequentially compact.
4. Every infinite subset $A \subseteq S$ has a limit point.
5. S is complete and totally bounded.
6. S is relatively compact and closed.

Corollary 6.31. Let S be a subset of a metric space X , then S is relatively compact if and only if \bar{S} is compact.

Corollary 6.32. Let S be a subset of a complete space X , then S is relatively compact if and only if S is totally bounded.

Proposition 6.33. Let S be a subset of a metric space X .

1. Suppose S is relatively compact, then S is bounded.
2. Suppose S is totally bounded, then S is bounded.
3. Suppose S is compact, then S is closed and bounded.

Theorem 6.34 (Heine-Borel). Let S be a subset of a finite dimensional normed vector space X , then S is compact if and only if S is closed and bounded.

6.2.2 Compact Operators

Definition 6.11. Let X, Y be normed vector spaces. A linear operator $K : X \rightarrow Y$ is **compact** if $\overline{K(B^X(0, 1))}$ is compact in Y . Let $\mathcal{K}(X, Y)$ denote the set of compact operators.

Proposition 6.35. Let X, Y be normed vector spaces, and $K : X \rightarrow Y$ be a linear operator, then the following are equivalent:

1. K is compact.
2. $K(B^X(0, 1))$ is relative compact in Y .
3. For any bounded set $S \subseteq X$, $\overline{K(S)}$ is compact in Y .
4. For any bounded set $S \subseteq X$, $K(S)$ is relative compact in Y .

5. Each bounded sequence $(x_n)_{n=1}^\infty \subseteq X$ has some subsequence $(x_{n_k})_{k=1}^\infty$ such that $(Kx_{n_k})_{k=1}^\infty$ converges to some $y \in Y$.

Proposition 6.36. Let X, Y be normed vector spaces. Suppose a linear operator $K : X \rightarrow Y$ is compact, then it is always bounded. Namely, $\mathcal{K}(X, Y) \subseteq B(X, Y)$.

Proposition 6.37. Let X be an infinite-dimensional Banach space, then the identity map $\mathbb{I} : X \rightarrow X$ is not compact.

Proof. $\bar{B}^X(0, 1)$ is not compact by A2. □

Proposition 6.38. Let X be a Banach space. Suppose $E = E^2 \in B(X)$ is compact, then E is finite rank.

Proof. See A5. □

Definition 6.12. Let X, Y be vector spaces. Let

$$\mathcal{F}(X, Y) : \{T \in B(X, Y) \mid \text{rank}(T) < \infty\}$$

denote the set of finite rank bounded linear operators.

Proposition 6.39. Let X, Y be normed vector spaces. Suppose a linear operator $T \in B(X, Y)$ is finite-dimensional, then it is always compact. Namely, $\mathcal{F}(X, Y) \subseteq \mathcal{K}(X, Y)$.

Proof. $\text{Im}(K) \subseteq Y$ is finite dimensional. For any bounded $M \subseteq X$, we have that $K(M)$ is bounded. Since $\overline{K(M)} \subseteq \text{Im}(K)$ is closed and bounded, it is compact in $\text{Im}(K)$, thus compact in Y . □

Theorem 6.40. Let X, Y be Banach Spaces. $\mathcal{K}(X, Y)$ is norm-closed $B(X) - B(Y)$ sub-bi-module of $B(X, Y)$. Namely, for any $a \in B(Y), b \in B(X)$, and $K \in \mathcal{K}(X, Y)$, we have $aKb \in \mathcal{K}(X, Y)$. Also, if $(K_n)_{n=1}^\infty$ is a sequence of compact operators in $\mathcal{K}(X, Y)$, and $K_n \rightarrow K$ in $B(X, Y)$, we have $K \in \mathcal{K}(X, Y)$ as well.

Proof. Clearly $\mathcal{K}(X, Y)$ is a vector space.

Since $b \in B(X)$, we have $b(B^X(0, 1)) \subseteq B^X(0, \|b\|)$, which is bounded. Since K is compact, $\overline{K(b(B^X(0, 1)))}$ is compact. Thus, $\overline{Kb(B^X(0, 1))} \subseteq \overline{K(B^X(0, \|b\|))}$ is compact since it is closed. Since $a \in B(Y)$ is continuous, we have that $a(\overline{Kb(B^X(0, 1))})$ is compact as well. Thus $\overline{aKb(B^X(0, 1))} \subseteq \overline{a(\overline{Kb(B^X(0, 1))})} = \overline{a(Kb(B^X(0, 1)))}$ is compact, since it is closed.

This shows aTb is compact.

Now consider any sequence of compact operators $(K_n)_{n=1}^\infty$ in $\mathcal{K}(X, Y)$, with $K_n \rightarrow K$ in $B(X, Y)$. Given any $\epsilon > 0$, there is $N \geq 1$ such that $\|K - K_N\| < \frac{\epsilon}{3}$. Since K_N is compact, $\overline{K_N(B^X(0, 1))}$ is compact and thus totally bounded, so we can find $\{x_1, \dots, x_m\} \subseteq B^X(0, 1)$, such that $\overline{K_N(B^X(0, 1))} \subseteq \bigcup_{i=1}^m B^X(x_i, \frac{\epsilon}{3})$. Now for any $x \in B^X(0, 1)$, consider $K_N x \in B^X(x_i, \frac{\epsilon}{3})$, we have

$$\begin{aligned} \|Kx - Kx_i\| &\leq \|Kx - K_N x\| + \|K_N x - K_N x_i\| + \|K_N x_i - Kx_i\| \\ &< \|K - K_N\| \|x\| + \frac{\epsilon}{3} + \|K_N - K\| \|x_i\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

This shows that $K(B^X(0, 1)) \subseteq \bigcup_{i=1}^m B^X(x_i, \epsilon)$, so it is totally bounded and thus relatively compact. This proves K is compact, so $\mathcal{K}(X, Y)$ is norm-closed. □

Proposition 6.41. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We have

$$\overline{\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)} = \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

Proof. Let $K \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, and $\epsilon > 0$. We can find an ϵ -net $\{Kx_1, \dots, Kx_m\}$ for $K(B^{\mathcal{H}_1}(0, 1))$, since it is totally bounded.

Let $K_0 := \text{Span}\{Kx_1, \dots, Kx_m\} \subseteq \mathcal{H}_2$, and consider $P : \mathcal{H}_2 \rightarrow K_0$, the orthogonal projection onto K_0 . For any $x \in B^{\mathcal{H}_1}(0, 1)$, we have

$$\begin{aligned} \|(K - PK)x\| &= \|(\mathbb{I} - P)Kx\| \\ &= \text{dist}(Kx, K_0) \\ &\leq \min_{i \in [m]} \|Kx - Kx_i\| \\ &< \epsilon. \end{aligned}$$

Thus, $\|K - PK\| < \epsilon$, where $\text{rank}(PK) \leq \dim(K_0) \leq m < \infty$. \square

Corollary 6.42. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. For $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, we have $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $T^\dagger \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. By finite-dimensional approximation. \square

Example 6.2.1. Consider $f \in \ell^\infty(\mathbb{N})$, and the multiplication operator $M_f : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$. We have that M_f is compact if and only if $f \in C_0(\mathbb{N})$, namely, $\lim_{i \rightarrow \infty} f(i) = 0$.

Indeed, if $f \in C_0(\mathbb{N})$, we can let $f_N(i) := \begin{cases} f(i), & i \leq N \\ 0, & \text{o.w.} \end{cases}$. We have $\|M_f - M_{f_N}\| = \|M_{f-f_N}\| = \|f - f_N\|_\infty \rightarrow 0$.

On the other hand, if $f \notin C_0(\mathbb{N})$, there is $\epsilon > 0$, and $i_1 < i_2 < i_3 < \dots$, such that $|f(i_k)| \geq \epsilon$. Let $K_0 := \overline{\text{Span}\{e_{i_k}\}} \subseteq \ell^2(\mathbb{N})$, we have $M_f(e_{i_k})(i) = \begin{cases} f(i), & i = i_k \\ 0, & \text{o.w.} \end{cases}$, so $M_f(e_{i_k}) = f(i_k)e_{i_k} \in K_0$. Thus, $K := M_f|_{K_0} : K_0 \rightarrow K_0$ is well defined. In addition, K is bounded below by ϵ . By theorem 3.12, it has a bounded inverse K^{-1} .

Suppose for contradiction that M_f is compact, then $\overline{K(B^{K_0}(0, 1))} \subseteq \overline{M_f(B^{\ell^2(\mathbb{N})}(0, 1))}$. Since K^{-1} is bounded, $K^{-1}\overline{K(B^{K_0}(0, 1))}$ is compact as well. Now

$$\overline{B^{K_0}(0, 1)} = \overline{K^{-1}K(B^{K_0}(0, 1))} \subseteq \overline{K^{-1}\overline{K(B^{K_0}(0, 1))}} = \overline{K^{-1}K(B^{K_0}(0, 1))}$$

is compact. Since K_0 is infinite dimensional, $\overline{B^{K_0}(0, 1)}$ cannot be compact, a contradiction.

6.2.3 Integral Operators

Definition 6.13. Let $k \in L^2([0, 1] \times [0, 1])$, the **integral operator** with **kernel** k is $T_k \in B(L^2(0, 1))$, with

$$(T_k f)(x) := \int_0^1 k(x, y)f(y)dy.$$

Lemma 6.43. *Let $K \in L^2([0, 1] \times [0, 1])$, the integral operator with kernel K is well-defined.*

Proof. Consider any $f \in L^2(0, 1)$,

$$\begin{aligned} \|T_k f\|_2^2 &= \int_0^1 \left| \int_0^1 k(x, y)f(y)dy \right|^2 dx \\ &\leq \int_0^1 \left(\int_0^1 |k(x, y)|^2 dy \right) \|f\|_2^2 dx \\ &= \|f\|_2^2 \|k\|_2^2 \\ &< \infty. \end{aligned}$$

Thus, $T_k f \in L^2(0, 1)$. \square

Proposition 6.44. *Let $k \in L^2([0, 1] \times [0, 1])$, then T_k is compact.*

Proof. Pick any orthonormal basis $(e_i)_{i=1}^\infty$ for $L^2(0, 1)$, then $e_{ij}(x, y) := e_i(x)e_j(y)$ forms an orthonormal basis for $L^2([0, 1] \times [0, 1])$.

Thus, $k = \sum_{i,j=1}^\infty a^{ij} e_{ij}$, where $\|k\|_2^2 = \sum_{i,j=1}^\infty |a_{ij}|^2$. Let $k_N := \sum_{i,j=1}^N a^{ij} e_{ij}$. Thus,

$$\|T_k - T_{k_N}\|^2 \leq \|k - k_N\|_2^2 = \sum_{i,j=N+1}^\infty |a_{ij}|^2 \rightarrow 0.$$

Now for any $f \in L^2(0, 1)$, we have

$$\begin{aligned} (T_{k_N} f)(x) &= \int_0^1 k_N(x, y) f(y) dy \\ &= \langle \bar{k}_N(x, \cdot), f \rangle \\ &= \left\langle \sum_{i,j=1}^N a^{ij} \bar{e}_{ij}(x, \cdot), f \right\rangle \\ &= \sum_{i,j=1}^N a^{ij} e_i(x) \langle \bar{e}_j, f \rangle \\ &= \sum_{i,j=1}^N a^{ij} \langle \bar{e}_j, f \rangle e_i(x) \\ T_{k_N} f &= \sum_{i,j=1}^N a_{ij} \langle \bar{e}_j, f \rangle e_i \\ &\in \text{Span} \{e_1, \dots, e_N\}. \end{aligned}$$

Thus, T_{k_N} is finite rank and thus compact, which means T_k is compact as well. □

Proposition 6.45. *Let $k \in L^2([0, 1] \times [0, 1])$, then*

$$(T_k^\dagger g)(x) = \int_0^1 k^*(x, y) g(y) dy,$$

where $k^*(x, y) := \overline{k(y, x)}$.

Proof. Consider any $f, g \in L^2(0, 1)$, let $h(x) := \int_0^1 k^*(x, y) g(y) dy$, we have

$$\begin{aligned} \langle V^* g, f \rangle &= \langle g, V f \rangle \\ &= \int_0^1 g(x) \overline{V f(x)} dx \\ &= \int_0^1 g(x) \overline{\int_0^1 k(x, y) f(y) dy} dx \\ &= \int_0^1 g(x) \int_0^1 \overline{k(x, y)} \bar{f}(y) dy dx \\ &= \int_0^1 \int_0^1 g(x) k^*(y, x) dx \bar{f}(y) dy \\ &= \int_0^1 h(y) \bar{f}(y) dy \\ &= \langle h, f \rangle. \end{aligned}$$

We can swap the order of integration by the Fubini Theorem and the fact that $f, g \in L^2(0, 1)$.

Thus we have $(T_k^\dagger g)(x) = h(x) = \int_0^1 k^*(x, y) g(y) dy$. □

Example 6.2.2. The **Volterra Operator** is $V : L^2(0, 1) \rightarrow L^2(0, 1)$ by

$$(Vf)(x) := \int_0^x f(y)dy.$$

Its kernel is $k(x, y) := \chi_{y \leq x}(x, y) := \begin{cases} 1, & y \leq x \\ 0, & \text{o.w.} \end{cases}$. We have

$$\begin{aligned} \int_0^1 \int_0^1 |\chi_{y \leq x}(x, y)|^2 dx dy &= \int_0^1 \int_0^y 1 dx dy \\ &= \int_0^1 y dy \\ &= \frac{1}{2}, \end{aligned}$$

so $k \in L^2([0, 1] \times [0, 1])$ by the Fubini Theorem. Also,

$$\begin{aligned} (T_k f)(x) &= \int_0^1 k(x, y) f(y) dy \\ &= \int_0^1 \chi_{y \leq x}(x, y) f(y) dy \\ &= \int_0^x f(y) dy \\ &= (Vf)(x). \end{aligned}$$

See more in A5Q6.

6.3 Spectrum

Definition 6.14. Let X be a Banach space, and $T \in B(X)$. The **spectrum** of T is

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda \mathbb{I} - T \text{ is not invertible.}\}.$$

The **point spectrum** of T is

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda \mathbb{I} - T \text{ is not injective.}\} = \{\lambda \in \mathbb{C} : \exists v \neq 0 \text{ such that } Tv = \lambda v\} = \{\text{eigenvalues of } T\}.$$

The **resolvent** of T is

$$\rho(T) := \mathbb{C} \setminus \sigma(T).$$

Proposition 6.46. When X is finite dimensional, then

$$\sigma(T) = \{\lambda : \lambda \mathbb{I} - T \text{ is not injective}\} = \sigma_p(T).$$

Example 6.3.1. Consider $f(n) := \frac{1}{n} \in \ell^\infty$, and the multiplication operator M_f we have that $0 \in \sigma(M_f)$, but not an eigenvalue.

Example 6.3.2. Let $f \in C([0, 1])$, and the multiplication operator $M_f : L^2(0, 1) \rightarrow L^2(0, 1)$ by $(M_f(\xi))(x) := f(x)\xi(x)$, where we have $M_f M_g = M_{fg}$, $M_f^\dagger = M_{\bar{f}}$, $\|M_f\| = \|f\|_\infty$.

Suppose $\lambda \notin \text{Im}(f)$, then $\lambda - f \neq 0$ on $[0, 1]$, so $\frac{1}{\lambda - f} \in C([0, 1])$. We thus have

$$M_{\frac{1}{\lambda - f}} = (M_{\lambda - f})^{-1} = (\lambda \mathbb{I} - M_f)^{-1}.$$

This shows $\sigma(M_f) \subseteq \text{Im}(f)$.

On the other hand, suppose $\lambda = f(x_0)$ for some $x_0 \in [0, 1]$, we have that $(\lambda - M_f)(\xi)(x) = (f(x_0) - f(x))\xi(x)$. Now for any $\epsilon > 0$, since f is continuous, we have that

$$A_\epsilon := \{x : |f(x) - f(x_0)| < \epsilon\}$$

has positive measure. Let $\xi_\epsilon := \frac{\chi_{A_\epsilon}}{\mu(A_\epsilon)^{\frac{1}{2}}}$, with $\|\xi_\epsilon\|_2 = 1$, we have that

$$\begin{aligned} \|(\lambda \mathbb{I} - M_f)\xi_\epsilon\|_2^2 &= \int_{A_\epsilon} |\lambda - f(x)|^2 \frac{1}{\mu(A_\epsilon)} dx \\ &< \frac{1}{\mu(A_\epsilon)} \int_{A_\epsilon} \epsilon^2 dx \\ &= \epsilon^2. \end{aligned}$$

Thus, $\lambda \mathbb{I} - M_f$ is not bounded below. By theorem 3.12, $\lambda \mathbb{I} - M_f$ is not invertible, so $\lambda \in \sigma(M_f)$.

Thus, $\sigma(M_f) = \text{Im}(f)$.

However, such a M_f may have no eigenvalue! Let $f(x) = x$, if t is an eigenvalue with eigenvector ξ , we have $x\xi(x) = t\xi(x)$ a.e., so $(x - t)\xi(x) = 0$ a.e., which means $\xi(x) = 0$ a.e..

Theorem 6.47. *Let X be a Banach space, and $T \in B(X)$. We have $\sigma(T) \neq \emptyset$, and it is always compact.*

Proof. For any $|\lambda| > \|T\|$, we have that $\|\frac{T}{\lambda}\| < 1$, so $(\mathbb{I} - \frac{T}{\lambda})^{-1} = \sum_{k=0}^{\infty} (\frac{T}{\lambda})^k$, and

$$(\lambda \mathbb{I} - T)^{-1} = \left(\lambda \left(\mathbb{I} - \frac{T}{\lambda} \right) \right)^{-1} = \lambda^{-1} \left(\mathbb{I} - \frac{T}{\lambda} \right)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}.$$

Thus, $\lambda \in \rho(T)$. This shows that $\forall \lambda \in \sigma(T)$, $|\lambda| \leq \|T\|$, which means $\sigma(T)$ is bounded.

Take any $\phi \in B(\mathcal{H})^*$, we can define $f_\phi : \rho(T) \rightarrow \mathbb{C}$ by $f_\phi(z) := \phi((z\mathbb{I} - T)^{-1})$, for any $z \in \rho(T) = \mathbb{C}$. Fix $z_0 \in \mathbb{C}$, and pick $|h| < \|(Z_0\mathbb{I} - T)^{-1}\|^{-1}$, we have $\|h(z_0\mathbb{I} - T)^{-1}\| < 1$, so $(\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})^{-1} = \sum_{k=0}^{\infty} (h(z_0\mathbb{I} - T)^{-1})^k$.

$$\begin{aligned} f_\phi(z_0 + h) &= \phi((z_0\mathbb{I} + h\mathbb{I} - T)^{-1}) \\ &= \phi(((\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})(z_0\mathbb{I} - T))^{-1}) \\ &= \phi\left((z_0\mathbb{I} - T)^{-1} \sum_{k=0}^{\infty} (h(z_0\mathbb{I} - T)^{-1})^k\right) \\ &= \sum_{k=0}^{\infty} h^k \phi((z_0\mathbb{I} - T)^{-1} ((z_0\mathbb{I} - T)^{-1})^k) \\ &= \sum_{k=0}^{\infty} h^k \phi((z_0\mathbb{I} - T)^{-1-k}), \end{aligned}$$

which is analytic. Now, suppose for contradiction that $\sigma(T) = \emptyset$, then $f_\phi : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function. By Liouville's theorem, f_ϕ must be a constant function.

Also, we can see that when $|z| \rightarrow \infty$, $f_\phi(z) = \phi((z\mathbb{I} - T)^{-1}) \rightarrow 0$. Thus, f_ϕ must be constantly zero, for any ϕ . However, by Hahn-Banach Theorem, there has to be some $\phi \in B(\mathcal{H})^*$ that separates $\{(z\mathbb{I} - T)^{-1}\} \subseteq B(\mathcal{H})$, a contradiction.

This shows $\sigma(T) \neq \emptyset$.

In addition, the above argument shows that for any $z \in \rho(T)$, $(z_0\mathbb{I} + h\mathbb{I} - T)^{-1}$ is well-defined in a neighbourhood around it, so $\rho(T)$ is open and $\sigma(T)$ is closed. \square

Definition 6.15. Let X be a Banach space, and $T \in B(X)$. The **spectral radius** of T is

$$\text{spr}(T) := \sup \{|\lambda| : \lambda \in \sigma(T)\}.$$

Notice that by the above theorem, we have $\text{spr}(T) = \max \{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$.

Theorem 6.48 (Spectral mapping). *Let X be a Banach space, and $T \in B(X)$. For any polynomial p , we have that $\sigma(p(T)) = p(\sigma(T))$.*

Proof. Suppose $p(x)$ is order n . Let $q(x) = \lambda - p(x) = (\beta_1 - x) \cdots (\beta_n - x)$, where β_i 's are the n roots of q .

Let $\lambda \in \mathbb{C}$, we have $\lambda \in \sigma(p(T))$, if and only if $q(T) = \lambda - p(T) = (\beta_1 \mathbb{I} - T) \cdots (\beta_n \mathbb{I} - T)$ is not invertible, if and only if there is some β_i , such that $\beta_i \mathbb{I} - T$ is not invertible, if and only if $q(x)$ has a root $\beta_i \in \sigma(T)$, if and only if $\lambda = p(\beta_i) \in p(\sigma(T))$. \square

Theorem 6.49 (Spectral Radius Formula). *Let X be a Banach space, and $T \in B(X)$.*

$$\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}.$$

Proof. By the spectral mapping theorem, $\sigma(T^n) = \sigma(T)^n$. Thus, $\|T^n\| \geq \text{spr}(T^n) = \text{spr}(T)^n$, which means $\text{spr}(T) \leq \|T^n\|^{\frac{1}{n}}$. Since this holds for all $n \in \mathbb{N}$, we have

$$\text{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}.$$

On the other hand, consider any $\phi \in B(X)^*$, we have that $f_\phi(\lambda) := \phi((\lambda \mathbb{I} - T)^{-1})$ is analytic on $\rho(T)$ as in the proof of theorem 6.47. In addition, for $|\lambda| > \|T\|$, we have

$$f_\phi(\lambda) = \phi\left(\sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}\right) = \sum_{k=0}^{\infty} \frac{\phi(T^k)}{\lambda^{k+1}}.$$

Since there is no residue between the annulus $\|T\|$ and $\text{spr}(T)$, from complex analysis, this Laurent series has to converge for any $|\lambda| > \text{spr}(T)$.

Thus, for any $|\lambda| > \text{spr}(T)$, $\frac{\phi(T^k)}{\lambda^{k+1}} = \phi\left(\frac{T^k}{\lambda^{k+1}}\right)$ has to be bounded. Since this holds for any $\phi \in B(X)^*$, by the Banach-Steinhaus Theorem 3.20, $\left\{\frac{T^k}{\lambda^{k+1}} : k \geq 0\right\}$ has to be bounded in $B(X)^{**}$, and thus in $B(X)$. Namely, there is $C_\lambda > 0$, such that $\left\|\frac{T^k}{\lambda^{k+1}}\right\| < C_\lambda$ for all $k \geq 0$. Namely, $\|T^k\| \leq C_\lambda |\lambda|^{k+1}$. Thus, we have

$$\|T^k\|^{\frac{1}{k}} \leq C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} \implies \limsup_k \|T^k\|^{\frac{1}{k}} \leq \lim_{k \rightarrow \infty} C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} = |\lambda|.$$

Since this holds for all $|\lambda| > \text{spr}(T)$, we have

$$\limsup_k \|T^k\|^{\frac{1}{k}} \leq \text{spr}(T).$$

Since $\limsup_n \|T^n\|^{\frac{1}{n}} \leq \text{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}$, we have $\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$. \square

Corollary 6.50. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be normal. We have $\text{spr}(T) = \|T\|$.*

Proof. By proposition 6.20, we have $\|T^2\| = \|T\|^2$, and inductively, we have $\|T^{2^n}\| = \|T\|^{2^n}$. Thus,

$$\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left\|T^{2^n}\right\|^{\frac{1}{2^n}} = \|T\|.$$

\square

Example 6.3.3. For the Volterra operator, we have that $\|V^n\|^{\frac{1}{n}} \rightarrow 0$, so $\sigma(V) \subseteq \{0\}$, which means $\sigma(V) = \{0\}$.

Theorem 6.51. [Fredholm alternative] *Let X be a Banach space, and $K \in \mathcal{K}(X)$ be a compact linear operator. Suppose $\lambda \neq 0$, then $\ker(\lambda \mathbb{I} - T)$ is finite dimensional and $\text{Im}(\lambda \mathbb{I} - T)$ is finite co-dimensional. Also, either $\lambda \in \rho(K)$, or $\lambda \in \sigma_p(K)$.*

Theorem 6.52 (Structure theorem for compact operators). *Let $K \in \mathcal{K}(X)$ be a compact operator on an infinite-dimensional Banach space X , then*

1. $0 \in \sigma(K)$.
2. $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$.
3. $\sigma(K) \setminus \{0\}$ is finite, or $\sigma(K) \setminus \{0\} = (\lambda_k)_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} \lambda_k = 0$.

Theorem 6.53 (Spectral theorem for normal operators). *Let \mathcal{H} be a Hilbert space, and $K \in \mathcal{K}(\mathcal{H})$ be a compact normal operator, then there exist eigenvectors $(\phi_j)_{j \in J}$, which form an orthonormal basis for \mathcal{H} .*

Proof. By Zorn's Lemma, we can find a maximal set of orthonormal eigenvectors $(\phi_j)_{j \in J}$ of K . Let $M := \overline{\text{Span}\{\phi_j\}_{j \in J}}$, and $P : \mathcal{H} \rightarrow M$ be the projection onto M . Also, let $\lambda_j \in \mathbb{C}$ be such that $K\phi_j = \lambda_j\phi_j$.

Consider any $x \in \mathcal{H}$, we have that

$$\begin{aligned}
 KPx &= K \sum_{j \in J} \langle \phi_j, x \rangle \phi_j \\
 &= \sum_{j \in J} \langle \phi_j, x \rangle K\phi_j \\
 &= \sum_{j \in J} \langle \phi_j, x \rangle \lambda_j \phi_j \\
 &= \sum_{j \in J} \langle \bar{\lambda}_j \phi_j, x \rangle \phi_j \\
 &= \sum_{j \in J} \langle K^\dagger \phi_j, x \rangle \phi_j \\
 &= \sum_{j \in J} \langle \phi_j, Kx \rangle \phi_j \\
 &= PKx.
 \end{aligned}$$

Thus, $KP = PK$.

Since $\mathbb{I} - P \in B(\mathcal{H})_{sa}$, we have that $(\mathbb{I} - P)K$ is compact and normal. Suppose $(\mathbb{I} - P)K \neq 0$, then by corollary 6.50, there is $\lambda \in \sigma((\mathbb{I} - P)K)$, such that $|\lambda| = \text{spr}((\mathbb{I} - P)K) = \|(\mathbb{I} - P)K\| > 0$. Thus, by Fredholm alternative 6.51, $\lambda \in \sigma_p((\mathbb{I} - P)K)$, which means there is a $\phi_\lambda \neq 0 \in \mathcal{H}$, such that $(\mathbb{I} - P)K\phi_\lambda = \lambda\phi_\lambda$. WLOG, we may pick $\|\phi_\lambda\| = 1$. Notice that this means $\phi_\lambda \in \text{Im}(I - P) = M^\perp = \ker(P)$, so

$$\begin{aligned}
 \lambda\phi_\lambda &= (\mathbb{I} - P)K\phi_\lambda \\
 &= K\phi_\lambda - PK\phi_\lambda \\
 &= K\phi_\lambda - KP\phi_\lambda \\
 &= K\phi_\lambda - K(0) \\
 &= K\phi_\lambda,
 \end{aligned}$$

which means ϕ_λ is an eigenvector of K . However, $\phi_\lambda \in M^\perp$, so $(\phi_j)_{j \in J \cup \{\phi_\lambda\}} \supsetneq (\phi_j)_{j \in J}$ is still orthonormal, contradicting the maximality of $(\phi_j)_{j \in J}$.

This shows $(\mathbb{I} - P)K = 0$. Now suppose $\mathbb{I} - P \neq 0$, we have $K(\mathbb{I} - P) = K = KP = K - PK = (\mathbb{I} - P)K = 0$, so there is $\phi \neq 0 \in \text{Im}(\mathbb{I} - P) = M^\perp$, such that $K\phi = 0 = 0\phi$, so ϕ is an eigenvector of K , which again contradicts the maximality of $(\phi_j)_{j \in J}$.

Thus, $\mathbb{I} - P = 0$, which means $M = \text{Im}(P) = \text{Im}(\mathbb{I}) = \mathcal{H}$. □

Theorem 6.54 (Spectral theorem for infinite range normal operators). *Let \mathcal{H} be a Hilbert space, and $K \in \mathcal{K}(\mathcal{H})$ be a compact normal operator with infinite range, then there exists countably many eigenvalues $(\lambda_k)_{k=1}^\infty \subset \mathbb{C}$ such that $\forall k \geq 1$, $n_k := \dim(\ker(\lambda_k \mathbb{I} - K)) < \infty$, and $\sigma(K) = (\lambda_k)_{k=0}^\infty$ for $\lambda_0 = 0 = \lim_{k \rightarrow \infty} \lambda_k$. In addition, there are eigenvectors $(\phi_{k_i})_{k \geq 0, i \in [n_k]}$, which form an orthonormal basis for \mathcal{H} , and $\forall k \geq 0$, $i \in [n_k]$, $K\phi_{k_i} = \lambda_k \phi_{k_i}$.*

Proof. □