Pmath753 Functional Analysis

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1 Metric Spaces and Complete Spaces

Definition 1.1. A **metric space** is a set X that has a (distance) metric:

$$\begin{aligned} d(\cdot,\cdot): X\times X \to \mathbb{R}, \text{ such that } \forall x,y,z \in X \\ d(x,x)=0 \\ \forall x\neq y, d(x,y)>0 \\ d(x,y)=d(y,x) \\ d(x,z)\leq d(x,y)+d(y,z) \end{aligned}$$

Definition 1.2. Given a metric space (X,d), a sequence $(x_n)_{n=1}^{\infty}$ in X has a **limit point** $x\in X$ if $\lim\limits_{n\to\infty} d(x,x_n)=0$. In this case, we say $(x_n)_{n=1}^{\infty}$ is a **convergent sequence**, and write $x=\lim_{n\to\infty} x_n$.

Definition 1.3. A sequence $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** in a metric space (X,d) if

$$\forall \epsilon>0, \exists N\in\mathbb{N}^+, \text{ such that } \forall m,n\in\mathbb{N},\ m,n>N \implies d(x_m,x_n)<\epsilon.$$

Definition 1.4. A metric space X is **complete** if every Cauchy sequence $(x_i)_{i=1}^{\infty}$ converges to a limit point in X. i.e. $\exists x\in X, \lim\limits_{i\to\infty} x_i = x$.

Proposition 1.1. *Let (X,d) be a metric space; then every convergent sequence is Cauchy.*

Proposition 1.2. *Let (X,d) be a metric space. Suppose $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a convergent subsequence such that $\lim_{k\to\infty} x_{n_k} = x \in X$, then $\lim_{n\to\infty} x_n = x$.*

2 Topology

See more in the notes of Pmath367 Topology by Professor S. New.

2.1 Topological Spaces

Definition 2.1. Let $X\neq\emptyset$ be a set. A **topology** on X is a collection $\mathcal{T}\subseteq\mathcal{P}(X):=2^X=$ power set of X, satisfying

1. $\emptyset, X\in\mathcal{T}$,
2. $\mathcal{T}$ is closed under arbitrary union; namely, $\forall\{A_\alpha\}_{\alpha\in I}\subseteq\mathcal{T},\ \bigcup_{\alpha\in I} A_\alpha\in\mathcal{T}$, and
3. $\mathcal{T}$ is closed under finite intersection; namely, $\forall\{A_i\}_{i=1}^{n}\subseteq\mathcal{T},\ \bigcup_{i=1}^{n} A_i\in\mathcal{T}$.

Also, $(X,\mathcal{T})$ is a **topological space** if $\mathcal{T}$ is a topology on X.

Definition 2.2. For any $S\subseteq\mathcal{P}(X)$, we define the **topology generated by** S to be

$$\mathcal{T}_S := \langle S\rangle := \{\emptyset, X, \text{ all unions of finite intersections of elements of } S\},$$

which is the intersection of all topologies on X that contains S, and it is the smallest topology on X containing S.

Proposition 2.1. *Let (X,d) be a metric space, then there is a **metric topology** $\mathcal{T}_d$ that is generated by open balls.*

Definition 2.3. $(X,\leq)$ is a **partially ordered set (poset)** if $\leq$ is

1. anti-symetric: $\forall x,y\in X$, if $x\leq y$ and $y\leq x$, we have $x=y$,
2. reflexive: $\forall x\in X,\ x\leq x$, and

3. transitive: $\forall x, y, z \in X$, if $x \leq y, y \leq z$, we have $x \leq z$.

We can define $\geq, <, >$ by

$$\begin{aligned} x \geq y &\iff y \leq x \\ x < y &\iff x \leq y \wedge x \neq y \\ x > y &\iff y < x. \end{aligned}$$

Definition 2.4. $(X, \leq)$ is a **totally ordered set** if it is a partially ordered set such that $\forall x, y \in X,\ x \leq y$ or $y \leq x$.

Proposition 2.2. *$(X, \leq)$ is a totally ordered set if and only if $<$ satisfies*

1. *$\forall x, y \in X$, exactly one of the following is true: $x < y,\ x = y,\ y < x$.*
2. *$\forall x, y, z \in X$, if $x < y, y < z$, we have $x < z$.*

Definition 2.5. Let $(X, \leq)$ be a totally ordered set, we can define for each $a, b \in X$,

1. $(-\infty, a) := \{x \in X : x < a\}$,
2. $(a, \infty) := \{x \in X : a < x\}$, and
3. $(a, b) := (a, \infty) \cap (-\infty, b)$.

Proposition 2.3. *Let $\mathcal{T}_{\leq}$ be the topology generated by all the sets above, then $\mathcal{T}_{\leq}$ is a topology.*

Definition 2.6. Let $(X, \mathcal{T})$ be a topological space, then we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$. We say $E \subseteq X$ is **closed** if $E^c \in \mathcal{T}$ is open.

Definition 2.7. For $E \in X$, the **closure** of E is

$$\bar{E} = \bigcap_{F \supseteq E : F \text{ is closed}} F,$$

which is the smallest closed set containing E.

Definition 2.8. For $E \in X$, the **interior** of E is

$$E^o = \bigcup_{U \subseteq E : U \text{ is open}} U,$$

which is the largest open set contained in E.

Proposition 2.4. *Closed sets are stable under finite unions and arbitrary intersections.*

Proposition 2.5. *For any set A,*

$$\bar{A} = ((A^c)^o)^c.$$

Proposition 2.6. *For any set A,*

$$x \in \bar{A} \iff (\forall U \, open\ ,\ x \in U \implies U \cap A \neq \emptyset).$$

Definition 2.9. Let $(X, \mathcal{T})$ be a topological space, a $\mathcal{B} = \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{P}(X)$ is said to be a **basis/base** of the topology $\mathcal{T}$ if it is a collection of open sets, and for every $U \in \mathcal{T}$, we have $U = \bigcup_{\alpha \in J} U_\alpha$ for some $J \subseteq I$.

Proposition 2.7. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subseteq \mathcal{T}$ is a basis of $\mathcal{T}$ if and only if*

$$\forall x \in U \in \mathcal{T}, \exists U_\alpha \in \mathcal{B} \text{ such that } x \in U_\alpha \subseteq U.$$

Proposition 2.8. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis of $\mathcal{T}$, if and only if*

1. $X = \bigcup_{\alpha \in I} U_\alpha$,
2. *For any* $U_1, U_2 \in \mathcal{B}, x \in U_1 \cap U_2$, *we have* $\exists U_x \in \mathcal{B}, x \in U_x \subseteq U_1 \cap U_2$,
3. $\mathcal{T} = \langle \mathcal{B} \rangle$ *is the topology generated by* $\mathcal{B}$.

Example 2.1.1. Let (X, d) be a metric space, then $\left\{B(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\right\}$ is a base.

Definition 2.10. Let $(X, \mathcal{T})$ be a topological space, a **subbase** is a collection of open sets $S \subseteq \mathcal{T}$ such that

$$\left\{X, \bigcap_{i=1}^{n} S_i : n \in \mathbb{N}^+, S_1, \ldots, S_n \in S\right\}$$

forms a base for $\mathcal{T}$.

Proposition 2.9. *Let* $(X, \mathcal{T})$ *be a topological space, then any subbase* S *generates* $\mathcal{T}$. *Also, any set* S *such that* $\bigcup S = X$ *is always a subbase for the topology* $\mathcal{T}_S$ *generated by* S.

Definition 2.11. Let $(X, \mathcal{T})$ be a topological space. Given $x \in X$, a **neighbourhood** of x is a set $V \ni x$, such that $\exists U \in \mathcal{T}$ with $x \in U \subset V$.

Definition 2.12. Let $(X, \mathcal{T})$ be a topological space. Given $x \in X$, a **neighbourhood basis** of x is a set of open neighbourhoods $\mathcal{B}_x \subset \mathcal{T}$, such that for any (open) neighbourhood U of x, there is $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$.

Proposition 2.10. *Let* $(X, \mathcal{T})$ *be a topological space,* $\mathcal{B} \subseteq \mathcal{T}$ *is a basis of* $\mathcal{T}$ *if and only if* $\forall x \in X$, $\mathcal{B}$ *is a neighbourhood basis of* x.

Definition 2.13. We say $S \subseteq X$ is **dense** in a topological space $(X, \mathcal{T})$ if $\forall$open $U \neq \emptyset$, $S \cap U \neq \emptyset$.

Proposition 2.11. S *is dense if and only if* $\bar{S} = X$.

Definition 2.14. A topological space $(X, \mathcal{T})$ is **separable** if there is a countable subset.

Definition 2.15. A topological space $(X, \mathcal{T})$ is **first countable** if $\forall x \in X$, there is a countable open neighbourhood basis $\{B_n\}_{n=1}^{\infty} \subset \mathcal{T}$ at x. Namely, for any neighbourhood U of x, there is $n \in \mathbb{N}$ such that $x \in B_n \subseteq U$.

Definition 2.16. A topological space $(X, \mathcal{T})$ is called **2nd countable** if it has a countable basis.

Proposition 2.12. *Every metric space* (X, d) *are first countable.*

Proposition 2.13. *Every metric space* (X, d) *is 2nd countable if and only if* X *is countable.*

Proposition 2.14. *The discrete topology of* X *is separable if and only if* $|X|$ *is at most countable.*

Definition 2.17 (Axiom of Choice)**.** If $X \neq \emptyset$, then there is a choice function $C : P(X) \setminus \{\emptyset\} \to X$ such that $\forall A \subseteq X$, if $A \neq \emptyset$, we have $C(A) \in A$.

Proposition 2.15 (Axiom of Choice Equivalence)**.** *The Axiom of Choice is equivalent to: Let* $\{X_\alpha\}_{\alpha \in A}$ *be a family of non-empty sets, then*

$$\Pi_{\alpha \in A} X_\alpha := \left\{f : A \to \bigcup_{\alpha \in A} X_\alpha \,\middle|\, f(\alpha) \in X_\alpha\right\} \neq \emptyset.$$

Proof. Suppose AOC holds, then taking $X = \bigcup_{\alpha \in A} X_\alpha$, we have the choice function C. Now take $f(a) := C(X_a)$.
On the other hand, suppose the latter holds, then consider $\Pi_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$, which is non-empty. Consider any $f \in \Pi_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$, then $C(X_\alpha) := f(a)$ is a choice function. $\square$

Proposition 2.16. *A metric space* (X, d) *is separable if and only if it is 2nd countable.*

Proof. If $S = \{x_k\}_{k=1}^{\infty}$ is dense, then $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a countable base.
Indeed, consider any $x \in X$ with any open $U \ni x$, we know $\exists r > 0,$ such that $x \in B(x, r) \subset U$. Also, there is x_k such that $d(x, x_k) < \frac{r}{2}$.
Now choose some $r' \in \mathbb{Q}$ such that $d(x, x_k) < r' < \frac{r}{2}$, then $x \in B(x_k, r') \subset B(x, r) \subset U$.
Thus $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a base.
On the other hand, suppose X is second countable with a countable base $\{U_n\}_{n=1}^{\infty}$. WLOG, $U_n \neq \emptyset$.
Now for any $n \in \mathbb{N}$, pick $x_n \in U_n$ by the axiom of countable choice. Let $S = \{x_n\}_{n=1}^{\infty}$, then we claim S is dense.
Indeed, for any open $U \neq \emptyset$, we can find some $U_n \subset U$. Thus $x_n \in S \cap U$. □

Proposition 2.17. *If $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ is a set of topologies on X,*

1. *There is a weakest topology $\tau := \left\langle \bigcup_{\alpha \in A} \mathcal{T}_\alpha \right\rangle$ that is stronger than each $\mathcal{T}_\alpha$.*
2. *There is a strongest topology $\delta := \bigcap_{\alpha \in A} \mathcal{T}_\alpha$ that is weaker than each $\mathcal{T}_\alpha$.*

Definition 2.18. A topological space $(X, \mathcal{S})$ is **Hausdorff** if

$$\forall x \neq y \in X,\ \exists S_x, S_y \in \mathcal{S},\ \text{such that } x \in S_x, y \in S_y, S_x \cap S_y = \emptyset.$$

Proposition 2.18. *Any space with its discrete topology is always Hausdorff.*

Example 2.1.2. Every metric space is Hausdorff.

Proposition 2.19. *Any space with more than one element with the trivial topology is not Hausdorff.*

Example 2.1.3. Consider $X := (0, 1) \cup \{1^+, 1^-\}$. Let $(0, 1)$ have the usual open topology. Also, let $(r, 1) \cup \{1^+\}$ and $(r, 1) \cup \{1^-\}$ be open for any $0 < r < 1$. The topology generated by this basis will not be Hausdorff.
Indeed, consider $1^+, 1^-$, then for any $U \ni 1^+, V \ni 1^-$, we can find $r_U, r_V \in (0, 1)$, such that $(r_U, 1) \cup \{1^+\} \subseteq U, (r_V, 1) \cup \{1^-\} \subseteq V$. Yet $(\max(r_U, r_V), 1) \subseteq U \cap V$, which is not empty.

Proposition 2.20. *If X is Hausdorff, then for any $x \in X$, we have that $\{x\}$ is closed.*

Proof. For any $y \neq x$, we can find open $V_y \ni y, U_y \ni x$, such that $V_y \cap \{x\} \subseteq V_y \cap U_y = \emptyset$.
Thus $X \setminus \{x\} = \bigcup_{y \in X} V_y$, which is open. □

2.2 Continuous Functions

Definition 2.19. A function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is **continuous** if

$$\forall U \in \mathcal{S},\ f^{-1}(U) \in \mathcal{T}.$$

Namely, the preimage of any open set is open.

Definition 2.20. A function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is **continuous at** $x \in X$ if

$$\forall V \in \mathcal{S},\ \text{such that } f(x) \in V\ \exists U \in \mathcal{T},\ \text{such that } x \in U \subseteq f^{-1}(V).$$

Proposition 2.21. *Let $f : X \to Y$ be a map between topological spaces. Then f is continuous (on X) if and only if f is continuous at every point $x \in X$.*

Proof. ($\implies$):
Assume f is continuous, then for every point $x \in X$ and open $V \ni f(x)$, we have $f^{-1}(V)$ is open. Clearly $x \in f^{-1}(V)$.
($\impliedby$):
Assume f is continuous at every point x. Given any open $V \in Y$, and any point $x \in f^{-1}(V)$, we have $f(x) \in V$.
By assumption, there is open U_x, such that $x \in U_x \subseteq f^{-1}(V)$.
Now $\bigcup_{x \in f^{-1}(V)} U_x$ is open, while $\bigcup_{x \in f^{-1}(V)} U_x \supseteq \bigcup_{x \in f^{-1}(V)} \{x\} = f^{-1}(V) \supseteq \bigcup_{x \in f^{-1}(V)} U_x$.
Thus $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is open. □

Definition 2.21. A function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is **open** if

$$\forall V \in \mathcal{T},\ f(V) \in \mathcal{S}.$$

Namely, the image of any open set is open.

Definition 2.22. Given two sets X, Y, and their corresponding topology $\mathcal{T}, \mathcal{S}$, a continuous map $f : X \to Y$ is a **homeomorphism** if it is bijective, and its inverse function is also continuous.

Remark. A homeomorphism is a map that preserves the topological structure between two sets.

Definition 2.23. Let $C(X)$ be the collection of functions $f : X \to \mathbb{C}$ that are continuous.

Definition 2.24. Let $C(X, \mathbb{R})$ be the collection of functions $f : X \to \mathbb{R}$ that are continuous.

Definition 2.25. Let $C_b(X)$ be the collection of functions $f \in C(X) : \|f\|_\infty < \infty$.

Definition 2.26. Let $C_b(X, \mathbb{R})$ be the collection of functions $f \in C(X, \mathbb{R}) : \|f\|_\infty < \infty$.

Proposition 2.22. *If $C(X)$ separates points, so do $C_b(X), C_b(X, \mathbb{R})$. Also, X is Hausdorff.*

Proof. If $x \neq y$, we have $f \in C(X)$ such that $f(x) \neq f(y)$.
WLOG, $\Re(f(x)) \neq \Re(f(y))$, and $\Re(f(x)) < \Re(f(y))$.
Now define $g(z) := \min\{\Re(f(y)), \max\{\Re(f(x)), \Re(f(y))\}\}$, which is bounded and continuous. Also $g(x) = f(x), g(y) = f(y)$.
Thus $C_b(X, \mathbb{R})$ separates the points.
Now if $x \neq y$, we can find $f \in C(X)$, such that $|f(x) - f(y)| = r > 0$.
Now let $U := f^{-1}\big(B(f(x), \frac{r}{2})\big), V := f^{-1}\big(B(f(y), \frac{r}{2})\big)$, which are both open. Also, $x \in U, y \in V$, and $U \cap V = f^{-1}(\emptyset) = \emptyset$. ☐

Definition 2.27. A topological space $(X, \mathcal{T})$ is **normal** if for any disjoint closed sets A, B, we can find open $U \supset A, V \supset B$ such that $U \cap V \neq \emptyset$.

Theorem 2.23 (Urysohn's Lemma for normal spaces)**.** *$(X, \mathcal{T})$ is normal if and only if for any disjoint closed sets A, B, $\exists f : X \to [0, 1]$ continuous, such that $f|_A = 0, f|_B = 1$.*

Corollary 2.24. *If X is normal and Hausdorff, $C_b(X)$ separates the points.*

2.3 Nets

Definition 2.28. Let $(X, \mathcal{T})$ be a topological space, say a sequence $(x_i)_{i=1}^\infty$ **converges to** $x \in X$ if $\forall$open $U \ni x$, $\exists N \in \mathbb{N}$ such that $\forall i \geq N,\ x_i \in U$.

Example 2.3.1. Consider $X = \mathbb{N} \times \mathbb{N}$, and the projection $\pi_1 : X \to \mathbb{N}$ by $\pi_1(m, n) := m$.
Let any U be open if $(0, 0) \notin U$, or if $\big\{m \in \mathbb{N} : \pi^{-1}(m) \cap U \text{ is co-finite in } \{m\} \times \mathbb{N}\big\}$ is co-finite.
One can show this defines a topology on X, and it is Hausdorff.
Indeed, let $X_0 := X \backslash \{(0, 0)\}$. Consider any $(m, n) \neq (m', n') \in X$. If both are in X_0 then $\{(m, n)\}, \{(m', n')\}$ are open and disjoint.
If $(m', n') = (0, 0)$, then $\{(m, n)\}, X \setminus \{(m, n)\}$ are open and disjoint.
This shows Hausdorff.
Also, $\bar{X_0} = X$.
Indeed, consider any open $U \ni (0, 0)$, we must have that $U \cap X_0 \neq \emptyset$.
However, there is no sequence in X_0 that converges to $(0, 0)$.
Indeed, assume for contradiction that there is such a convergent sequence $(x_k)_{k=1}^\infty$ in X_0.
Write each $x_k = (m^k, n^k)$.
Suppose there is some $M \in \mathbb{N}^+$, such that $\forall k \in \mathbb{N}^+, m_k \leq M$.
Consider $U := \{(m, n) : m > M, n \in \mathbb{N}\} \cup \{(0, 0)\}$.
Now for each $m \in \mathbb{N}$,

$$\pi^{-1}(m) \cap U = \begin{cases} (0, 0) & \text{if } m = 0 \\ \emptyset & \text{if } 0 < m \leq M \\ \{m\} \times \mathbb{N} & \text{if } m > M. \end{cases}$$

Thus for all $m > M$, $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.
Yet $U \cap \{x_k\}_{k=1}^{\infty} = \emptyset$.
Now suppose there is no such M, then we can find a subsequence $m_{k_1} < m_{k_2} < m_{k_3} < \cdots$.
Now let $U := X \setminus \{x_{k_i}\}_{i=1}^{\infty}$.
For each $m \in \mathbb{N}$,

$$\pi^{-1}(m) \cap U = \begin{cases} \{m\} \times \mathbb{N} & \text{if } m \notin \{m_{k_i}\}_{i=1}^{\infty} \\ \{m\} \times (\mathbb{N} \setminus n_{k_i}) & \text{if } m \in \{m_{k_i}\}_{i=1}^{\infty} \end{cases}.$$

Notice that there cannot be two $n_{k_i} \neq n_{k_j}$ for any m, since $k_i \neq k_j \implies m_{k_i} \neq m_{k_j}$.
Thus all $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.
Yet $U \cap \{x_{k_i}\}_{i=1}^{\infty} = \emptyset$.
Thus there cannot be any convergent sequence $(x_k)_{k=1}^{\infty}$ in X_0.

Remark. The above example shows that sequences do not behave as we want in topological spaces.

Definition 2.29. An **upwards directed set** is a poset $(\Lambda, \leq)$ such that if $\lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_0 \in \Lambda$ such that $\lambda_1 \leq \lambda_0, \lambda_2 \leq \lambda_0$.

Definition 2.30. For $X \neq \emptyset$, a **net** in X is a function $j : \Lambda \to X$, where $(\Lambda, \leq)$ is an upwards directed set. Write $x_\lambda := j(\lambda) \in X$, and we can use $(x_\lambda)_{\lambda \in \Lambda}$ to represent a net.

Definition 2.31. Let $(X, \mathcal{T})$ be a topological space, say a net $(x_\lambda)_{\lambda \in \Lambda}$ **converges to** $x \in X$ if

$$\forall \text{open } U \ni x,\ \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0,\ x_\lambda \in U.$$

In this case, we say x is a **limit** of the net, and write it as $x = \lim_{\lambda \in \Lambda} x_\lambda$ or $x_\lambda \to x$.

Proposition 2.25. *Let $(X, \mathcal{T})$ be a topological space with a neighbourhood basis $\mathcal{B}$ at $x \in X$, then a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to x if and only if*

$$\forall U \in \mathcal{B} \text{ such that } x \in U,\ \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0,\ x_\lambda \in U.$$

Proof. The forward direction is trivial.
Now assume $\forall U \in \mathcal{B}$ such that $x \in U,\ \exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0,\ x_\lambda \in U$.
Given any open $V \ni x$, since $\mathcal{B}$ is a neighbourhood basis, there is some $U \in \mathcal{B}$, such that $x \in U \subseteq V$.
Thus, there is $\lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0,\ x_\lambda \in U \subseteq V$. □

Definition 2.32. Given a net $(x_\lambda)_{\lambda \in \Lambda}$, then a **subnet** of it $(y_\gamma)_{\gamma \in \Gamma}$ is given by an upwards directed set $(\Gamma, \leq)$ and a function $\phi : \Gamma \to \Lambda$ that is **cofinal**, which means $\forall \lambda_0 \in \Lambda,\ \exists \gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0,\ \phi(\gamma) \geq \lambda_0$. Each y_γ is given by $x_{\phi(\gamma)}$.

Example 2.3.2. Notice that if we take $(\mathbb{N}, \leq)$, the net is just a sequence. To get a subsequence, we can take $\Gamma = \mathbb{N}$, and $\phi : \mathbb{N} \to \mathbb{N}$ to be any increasing function. The generated subnet will be a subsequence.

Definition 2.33. Let $(X, \mathcal{T})$ be a topological space, and $x \in X$, define the **system of open neighbourhoods of** x to be $\mathcal{O}(x) := \{U \in \mathcal{T} : x \in U\}$.

Proposition 2.26. *$(\mathcal{O}(x), \supseteq)$ is an upwards directed set.*

Example 2.3.3. For $X = \mathbb{N} \times \mathbb{N}$ and $X_0 = X \setminus \{(0,0)\}$ as above, there is a net in X_0 converging to $(0,0)$.
Indeed, let us enumerate $X_0 = \{x_k\}_{k=1}^{\infty}$ as $(0,1), (1,0), (0,2), (1,1), (2,0), \ldots$.
Now $\Lambda := \mathcal{O}((0,0))$ is an upward directed set by containment.
Then for each $U \in \Lambda$, we can pick $x_U := x_{k_U}$, where k_U is the first $k \in \mathbb{N}$ such that $x_k \in U$.
Claim: $(x_U)_{U \in \Lambda}$ converges to $(0,0)$.
Pick any $U_0 \ni (0,0)$, then for all $U \geq U_0$, it is open and $U \subseteq U_0$. Thus, we must have $x_U \in U \subseteq U_0$.
Indeed, $(x_U)_{U \in \Lambda}$ is a subnet of $\{x_k\}_{k \in \mathbb{N}^+}$ by $\phi(U) := k_U$.

Theorem 2.27. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces, then*

1. *For any $A \subseteq X$, we have $x \in \bar{A}$ if and only if $\exists$ a net $(x_\lambda)_{\lambda \in \Lambda}$ in A, such that $x_\lambda \to x$.*

2. $f : X \to Y$ *is continuous if and only if for any net* $(x_\lambda)_{\lambda\in\Lambda} \subset X$ *such that* $x_\lambda \to x \in X$*, we have* $f(x_\lambda) \to f(x)$.

Proof. 1. Consider any $x \in \bar{A}$, then for any open $U \ni x$, we have $U \cap A \neq \emptyset$.
By the Axiom of Choice, we can have $x_U \in U \cap A$ for each open neighbourhood U.
Consider the net $(x_U)_{U\in\mathcal{O}(x)}$.
Given any open $U \ni X$, if $V \geq U$, we must have $V \subseteq U$, and $x_v \in V \subseteq U$.
Thus $x_U \to x$.
On the other hand, for any net $x_\lambda \to x$ in A, consider any open $U \ni x$, there is λ_0, such that

$$\forall\lambda \in \Lambda,\ \lambda_0 \leq \lambda \implies x_\lambda \in U \implies U \cap A \neq \emptyset.$$

Thus $x \in \bar{A}$.

2. Assume f is continuous, and $x_\lambda \to x$. Let $V \in \mathcal{O}(f(x))$, then $U := f^{-1}(V)$ is open and $x \in U$.
Thus there is $\lambda_0 \in \Lambda$, such that $\forall\lambda \geq \lambda_0, x_\lambda \in U$.
Thus $f(x_\lambda) \in V$.
On the other hand, assume for any net $(x_\lambda)_{\lambda\in\Lambda} \subset X$ such that $x_\lambda \to x \in X$, we have $f(x_\lambda) \to f(x)$.
For contradiction, suppose there is open $V \in Y$, with $U := f^{-1}(V)$ is not open in X.
Then U^c is not closed, and $U^c \neq \overline{U^c}$.
Thus, there is $x \in \overline{U^c} \setminus U^c = \overline{U^c} \cap U$.
Since $x \in \overline{U^c}$, by 1., there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in U^c, such that $x_\lambda \to x$.
By assumption, we have $f(x_\lambda) \to f(x)$.
Since each $f(x_\lambda)$ is in $f(U^c) = V^c$, by 1., we have that $f(x) \in \overline{V^c} = V^c$ since V^c is closed (V is open).
However, since $x \in U$, we also have $f(x) \in f(U) = V$, which is a contradiction.

□

2.4 Compactness

Definition 2.34. Let $(X, \mathcal{T})$ be a topological space. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha\in A}$ is called a **cover** for X if $X = \bigcup_{\alpha\in A} U_\alpha$. A cover is called an open cover if every U_α is open in $\mathcal{T}$.

Definition 2.35. Let $(X, \mathcal{T})$ be a topological space. A collection $\{C_\alpha\}_{\alpha\in A}$ of non-empty closed sets is a FIP-family if for any finite $F \subseteq A$, $\bigcap_{\alpha\in F} C_\alpha \neq \emptyset$. X has the finite intersection property (FIP) if for all FIP-familie $\{C_\alpha\}_{\alpha\in A}$, we have $\bigcap_{\alpha\in A} C_\alpha \neq \emptyset$.

Definition 2.36. Let $(X, \mathcal{T})$ be a topological space. X is **compact** if every open cover of X has a finite subcover. Namely,

$$\forall(U_\alpha)_{\alpha\in A} \text{ be open}, X = \bigcup_{\alpha\in A} U_\alpha \implies \exists n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n,\ \text{such that } X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Definition 2.37. Let $(X, \mathcal{T})$ be a topological space. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha\in A}$ is called a **cover** for $K \subseteq X$ **in** X if $K = \bigcup_{\alpha\in A} U_\alpha$. A cover in X is called an open cover in X if every U_α is open in $\mathcal{T}$.

Definition 2.38. Let $(X, \mathcal{T})$ be a topological space. A set $K \subseteq X$ is **compact in** X if every open cover of K in X has a finite subcover. Namely,

$$\forall(U_\alpha)_{\alpha\in A} \text{ be open}, K \subseteq \bigcup_{\alpha\in A} U_\alpha \implies \exists n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n,\ \text{such that } K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Proposition 2.28. *Let* $(X, \mathcal{T})$ *be a topological space. A set* $K \subseteq X$ *is compact under the subspace topology if and only if it is compact in* X.

Theorem 2.29. *Let* $(X, \mathcal{T})$ *be a topological space, TFAE:*

1. *X is compact.*

2. *X has the finite intersection property.*

3. *For all nets $(x_\lambda)_{\lambda\in\Lambda}$ in X, there is a convergent subnet.*

Proof. (1) $\implies$ (2).
For contradiction, suppose there is some FIP-family such that $\bigcap_{\alpha\in A} C_\alpha = \emptyset$.
We have $X = \bigcup_{\alpha\in A} C_\alpha^c$, which is an open cover for X.
Since X is compact, there is a finite $F \subseteq A$, such that $X = \bigcup_{\alpha\in F} C_\alpha^c$.
Thus $\bigcap_{\alpha\in A} C_\alpha = \emptyset$, which contradict $\{C_\alpha\}_{\alpha\in A}$ being a FIP family.
(2) $\implies$ (1).
Consider any open cover $X = \bigcup_{\alpha\in A} U_\alpha$, then $\bigcap_{\alpha\in A} U_\alpha^c = \emptyset$, and it is not a FIP-family.
Thus there is a finite $F \subseteq A$, such that $\bigcap_{\alpha\in F} U_\alpha^c = \emptyset$.
Thus $X = \bigcup_{\alpha\in F} U_\alpha$ is a finite open cover.
(2) $\implies$ (3).
Let $(x_\lambda)_{\lambda\in\Lambda}$ be any net in X.
Define $C_\lambda := \overline{\{x_\mu : \mu \geq \lambda\}}$. Notice that $C_\lambda \neq \emptyset$ since $x_\lambda \in C_\lambda$.
We claim that $\{C_\lambda\}_{\lambda\in\Lambda}$ is a FIP family.
The closeness is by definition.
Now fix any $\lambda_1, \dots, \lambda_n \in \Lambda$.
Since Λ is upwards directed, there is $\lambda_0 \in \Lambda$, such that $\forall i \in [n], \lambda_i \leq \lambda_0$.
Thus $\bigcap_{i=1}^n C_{\lambda_i} \supseteq C_\lambda \neq \emptyset$.
By FIP, $\bigcap_{\lambda\in\Lambda} C_{\lambda_i} \neq \emptyset$.
Pick any $x \in \bigcap_{\lambda\in\Lambda} C_{\lambda_i}$.
Let $\Gamma := \Lambda \times \mathcal{O}(x)$ with the partial order $(\lambda, U) \leq (\lambda', U')$ if $\lambda \leq \lambda'$ and $U \supseteq U'$.
Fix $(\lambda, U) \in \Gamma$, we know that $x \in C_\lambda = \overline{\{x_\mu : \mu \geq \lambda\}}$.
Thus $U \cap \{x_\mu : \mu \geq \lambda\} \neq \emptyset$.
By the Axiom of Choice, there is $y_{(\lambda,U)} = x_{\phi(\lambda,U)} \in U \cap \{x_\mu : \mu \geq \lambda\}$, where $\phi(\lambda, U) := C(\{\mu \geq \lambda : x_\mu \in U\})$.
For any $\lambda_0 \in \Lambda$, let $\gamma_0 = (\lambda_0, X)$, then for any $\gamma = (\lambda, U) \geq \gamma_0$, we have $\phi(\gamma) \geq \lambda \geq \lambda_0$.
Thus ϕ is cofinal, and $(y_\gamma)_{\gamma\in\Gamma}$ is a subnet.
In addition, given any $U_0 \in \mathcal{O}(x)$, we can pick any $\lambda_0 \in \Lambda$, and let $\gamma_0 := (\lambda_0, U_0)$.
Then for any $(\lambda, U) \geq \gamma_0$, we must have $y_{(\lambda,U)} = x_{\phi(\lambda,U)} \in U \subseteq U_0$.
(3) $\implies$ (2).
Fix any FIP-family $\{C_\alpha\}_{\alpha\in A}$ in X. Then for any finite $F \subseteq A$, by the Axiom of Choice, we can find $x_F \in \bigcap_{\alpha\in F} C_\alpha$.
Now consider the net $(x_F)_{\text{finite } F\subseteq A}$, where $F_1 \leq F_2$ if $F_1 \subseteq F_2$.
By 3., there is a convergent subnet $\phi : \Gamma \to \Lambda$, such that $x_{\phi(\gamma)} \to x \in X$.
Now fix any $\alpha \in A$, then $\{\alpha\} \in \Lambda$.
Thus there is some $\gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0,\ \phi(\gamma) \supseteq \{\alpha\} \ni \alpha$.
We have $x_{\phi(\gamma)} \in \bigcap_{\beta\in\phi(\gamma)} C_\beta \subseteq C_\alpha$.
Since this holds for all $\gamma \geq \gamma_0$, and $x_{\phi(\gamma)} \to x$, we have that $x \in \bar{C_\alpha} = C_\alpha$.
Since this holds for any $\alpha \in A$, we have that $x \in \bigcap_{\alpha\in F} C_\alpha$. Thus $\bigcap_{\alpha\in F} C_\alpha \neq 0$. $\square$

With the above theorem, we can identify the definition of compactness in a metric space via sequences with the definition of compactness with its metric topology.

Proposition 2.30. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces. If X is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact in Y.*

Theorem 2.31. *Let $(X, \mathcal{T})$ be a topological space,*

1. *Suppose K is compact, then $\forall F \subseteq K$ that is closed, F is also compact.*

2. *If X is Hausdorff, for any compact $K \subseteq X, x \in X \setminus K$, $\exists$ open neighbourhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$.*

Proof. 1. Let $(U_\alpha)_{\alpha\in A}$ be an open cover for F.
Since F is closed, then F^c is open. Thus $\{F^c\} \cup \{U_\alpha\}_{\alpha\in A}$ is an open cover for K.
Thus there are $U_{\alpha_1}, \dots, U_{\alpha_n}$, such that $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$. Thus $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ has a finite subcover.

2. Consider any $y \in K$, there is some open neighborhoods $U_y \ni x, W_y \ni y,$ such that $U_y \cap W_y = \emptyset$.
   Since $K \subseteq \bigcup_{y \in K} W_y$ is compact, we have $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$ for some $y_1, \ldots, y_n$.
   Let $U = \bigcap_{i=1}^n U_{y_i}$, we have $x \in U, K \subseteq W, U \cap W = \emptyset$ as required.

□

Corollary 2.32. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, then any compact set K is closed. In addition, for any closed $F \subseteq X$, we have $F \cap K$ is compact.*

Proof. Suppose for contradiction that K is not closed, then there is some $y \in \bar{K} \setminus K$. Thus we can find open neighbourhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$. Now $K \subset \bar{K} \setminus U \subsetneq \bar{K}$ is closed, which is a contradiction.
Since K is closed, so is $F \cap K \subseteq K$, and thus it is compact. □

Definition 2.39. X is **locally compact** if $\forall x \in X$, there is an open neighbourhood $U_x \in \mathcal{O}(x)$ such that $\overline{U_x}$ is compact.

Example 2.4.1. $\mathbb{R}^n$ is locally compact by the Heinz-Borel theorem.

Proposition 2.33. *A Banach space $(X, \|\cdot\|)$ is locally compact iff $\dim(X) < \infty$.*

Lemma 2.34. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, and $(K_\alpha)_{\alpha \in A}$ be a collections of compact sets such that*

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

We must have $\alpha_1, \ldots, \alpha_n \in A$, such that

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

Proof. Fix $\alpha_1 \in A$, then $K_{\alpha_1} \subseteq \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$ is compact and has an open cover.
Thus there must be $\alpha_2, \ldots, \alpha_n \in A$, such that $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = \left(\bigcap_{i=2}^n K_{\alpha_i}\right)^c$.
Thus $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$. □

Theorem 2.35. *Let X be a Locally Compact Hausdorff space, and let $K \subseteq U \subseteq X$ be such that K is compact, and U is open. Then there exists some open set V such that $\bar{V}$ is compact, and*

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof. Since X is a Locally Compact Hausdorff space, there are $V_1, \ldots, V_n$, each with $\bar{V_i}$ be compact, such that $K \subseteq \bigcup_{i=1}^n V_i =: G$. Note that $\bar{G}$ is compact, and G is open.
If $U = X$, then $G \subseteq U$, and we are done.
Otherwise, let $C := X \setminus U$ be non-empty and closed.
Consider any $y \in C$, we know that $y \notin K$. Since X is Hausdorff, we can find open $W_y \supset K$, and $U_y \ni y$, such that $W_y \cap U_y = \emptyset$. Then $W_y \subseteq U_y^c$, and thus $\bar{W}_y \subseteq U_y^c$, since U_y^c is closed. Yet $y \notin U_y^c$, thus $y \notin \bar{W}_y$.
Now consider the family $\left\{C \cap \bar{W}_y \cap \bar{G}\right\}_{y \in C}$. Notice that each $C \cap \bar{W}_y \cap \bar{G}$ is compact, since $C, \bar{W}_y$ are closed, and $\bar{G}$ is compact.
Yet $\bigcap_{y \in C} \left(C \cap \bar{W}_y \cap \bar{G}\right) = \emptyset$.
Thus $\exists y_1, \ldots, y_n \in C$, such that $\bigcap_{i=1}^n \left(C \cap \bar{W}_y \cap \bar{G}\right) = \emptyset$.
Now let $V := G \cap \bigcap_{i=1}^n W_{y_i}$.
Clearly V is open, and $K \subseteq V$.
In addition, $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$, yet the intersection of righthand side and C is empty, thus contained in $C^c = U$. □

2.5 Compactly Supported Continuous Functions

Definition 2.40. For $f \in C(X)$, the **support** of it is

$$\text{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

Definition 2.41. The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \text{Supp}(f) \text{ is compact}\}.$$

Definition 2.42. $C_0(X)$ is the closure of $C_c(X)$ in $\|\cdot\|_\infty$.

Proposition 2.36. *$C_0(X)$ is the set of all continuous functions that vanishes at ∞. $(C_0(X), \|\cdot\|_\infty)$ is a Banach Space and a commutative C^*-algebra with the involution $f^*(x) := \overline{f(x)}$.*

Proposition 2.37. *$f \in C_0(X)$ if and only if $\forall \epsilon > 0,\ \exists K \subset\subset X$, such that $\forall x \in X \setminus K,\ |f(x)| < \epsilon$.*

Theorem 2.38. *Any commutative C^*-algebra $(A, \|\cdot\|)$ is isomorphic to $C_0(X)$ for some unique Locally Compact Hausdorff X.*

2.5.1 Partition of Unity

Definition 2.43. Let K be a compact set, and V be an open set of X. Let $f \in C_c(X)$. We say $f < V$ if $0 \leq f \leq 1$, and $\text{Supp}(f) \subseteq V$. We say $K < f$ if $0 \leq f \leq 1$, and $f|_K = 1$. We say $K < f < V$ if $K \subset V, K < f, f < V$.

Remark. f is a "bump" function that approximates χ_K when V shrinks towards K.

Lemma 2.39 (Urysohn's lemma for Locally Compact Hausdorff Space)**.** *Let X be a Locally Compact Hausdorff space, $K \subseteq V \subseteq X$ be such that K is compact, and V is open. Then there exists $f \in C_c(V)$, such that $K < f < V$.*

Proof. we want to construct a family of open sets $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s$.
By 2.35, we can find $K \subset V_0 \subset \bar{V_0} \subset V$.
Pick an enumeration of $r \in \mathbb{Q} \cap (0,1]$, i.e. $(r_n)_{n=1}^\infty$. WLOG, we can let $r_1 = 1$.
By 2.35, we can find $K \subset V_1 \subset \bar{V_1} \subset V_0$.
Suppose we have constructed the V_{r_i} for $1 \leq i \leq n$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s \in \{r_i\}_{i=1}^n$.
Let $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$.
Now by 2.35, we can find $\bar{V_t} \subset V_{n+1} \subset \bar{V_{n+1}} \subset V_s$.
For any $r < r_{n+1}$, we have $r \leq s$, and thus $V_{n+1} \subset \bar{V_{n+1}} \subset V_s \subset \bar{V_s} \subseteq V_r$ by induction hypothesis, and similarly for any $r > r_{n+1}$.
Inductively, we can prove there is such a family.
Define $f_r := r\chi_{V_r}$, and $g_r := r\chi_{\bar{V_r}^c} + \chi_{\bar{V_r}}$, and $f := \sup_r f_r, g := \inf_r g_r$.
We can show that f, g are upper and lower continuous, respectively.
In addition, f, g are both 0 outside of V_1, and 1 on K.
Suppose there is some $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, such that $f_r(x) > g_s(x)$. Then we must have $f_r(x) > 0$, and thus $x \in V_r$ and $1 \leq r = f_r(x)$.
Thus $1 > g_s(x)$, and thus $x \in \bar{V_s}^c$ and $f_s = s$.
Since $r > s$, we must have $V_r \subset \bar{V_r} \subset V_s \subset \bar{V_s}$, which is a contradiction to $x \in V_r, x \notin \bar{V_s}$.
Thus for any $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, we must have $f_r(x) \leq g_s(x)$.
Thus we must have $f(x) \leq g(x)$ for any $x \in V$.

Now suppose there is some $x \in X$, such that $f(x) < g(x)$.
There must be some rationals, such that $f(x) < r < s < g(x)$, since $\mathbb{Q}$ is dense.
Thus $\sup_r f_r(x) < r$, and thus $x \notin V_r$.
Also, $\inf_s g_s(x) > s$, and thus $x \in \bar{V}_s$.
Yet $r < s$, we must have $V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r$, which is a contradiction.
Thus we must have $f = g$, and it forces f to be continuous. ☐

Definition 2.44. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$.
A collection $(h_i)_{i=1}^n \subset C_c(X)$ is called a **partition of unity** on K subordinate to $(V_i)_{i=1}^n$ if

$$\begin{cases} \forall 1 \leq i \leq n, & h_i < V_i, \\ \forall x \in K, & \sum_{i=1}^n h(x) = 1. \end{cases}$$

Theorem 2.40. *Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$, there always exists a partition of unity on K subordinated to $(V_i)_{i=1}^n$.*

Proof. Since K is compact, we can find some open cover $W_1, \ldots, W_m$, such that for all j, we have $W_j \subset \bar{W}_j \subset V_{i(j)}$ for some $1 \leq i(j) \leq n$.
Let $K_i := \bigcup_{1 \leq j \leq m \text{ such that } W_j \subset V_i} \bar{W}_j \subset V_i$, which is compact.
By Urysohn's lemma, we can find $K_i < g_i < V_i$.
Now let $h_1 := g_1$, and in general, $h_i := g_i \prod_{j<i}(1 - g_j)$.
It is easy to check that $0 \leq h_i \leq 1$, and $h_i \in C_c(X)$.
In addition, $\mathrm{Supp}(h_i) \subseteq \mathrm{Supp}(g_i) \subset V_i$.
Thus $h_i < V_i$. Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_n)$.
For any $x \in K$, there must be some $i \in [n]$ such that $x \in K_i$, and thus $g_i(x) = 1$, and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_n(x)) = 1 - 0 = 1.$$

☐

2.6 Product Topology

Definition 2.45. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of topological spaces. The **product topology** on $\prod_\alpha X_\alpha$ is the topology generated by the sets

$$\left\{ U_\alpha \times \prod_{\beta \in A, \beta \neq \alpha} X_\beta \middle| \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\} = \left\{ \pi_\alpha^{-1}(U_\alpha) \middle| \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\},$$

where the **projection map onto** X_α is $\pi_\alpha : \prod_{\beta \in A} X_\beta \to X_\alpha$ by $(x_\beta)_{\beta \in A} \mapsto x_\alpha$.

Proposition 2.41. *The product topology is the weakest topology in which each π_α is continuous.*

Proposition 2.42. *A net $(x_\lambda)_{\lambda \in \Lambda}$ in $\prod_{\alpha \in A} X_\alpha$ converges to x if and only if $\forall \alpha \in A,\ \pi_\alpha(x_\lambda) \to \pi_\alpha(x)$ in X_α.*

Proof. See A1. ☐

Theorem 2.43 (Tychonoff)**.** *Let* $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in A}$ *be a family of compact topological spaces, then* $\prod_\alpha X_\alpha$ *is compact under product topology.*

Definition 2.46. Let $(P, \leq)$ be a partially ordered set. We call a totally ordered subset $Q \subseteq P$ a **chain**.

Definition 2.47. Let $(P, \leq)$ be a partially ordered set. We call $\leq$ **inductive** if every chain $Q \subseteq P$ has an upper bound.

Definition 2.48. $\leq$ is called a **well-order** if it is a total order, and for every $\emptyset \neq S \subseteq X$ has a minimal element. $\exists\, x \in P$ such that $\forall y \in P,\ y \leq x \implies x = y$.

Lemma 2.44 (Zorn's)**.** *Every inductive partial order* $(P, \leq)$*, defined on a nonempty* P*, has a maximal element. Namely,* $\exists\, x \in P$ *such that* $\forall y \in P,\ x \leq y \implies x = y$*.*

Proposition 2.45. *Every vector space* V *has a basis.*

Proof. Consider $P = \{S \subset V | S \text{ is linearly independent}\}$, with $S \leq S' \iff S \subseteq S'$.
Let Q be a chain in P, then it has an upper bound $\tilde{S} = \bigcup_{S\in Q} S$, which we can check is still linearly independent.
By Zorn's lemma, there is a maximal $S \in P$. □

Theorem 2.46 (Well-Ordering Principle)**.** *Every set* X *admits a well-ordering.*

Theorem 2.47. *The Following Are Equal:*

1. *Tychonoff's Theorem*
2. *Axiom of Choice*
3. *Zorn's Lemma*
4. *Well-Ordering Principle*

Proof. (1) $\implies$ (2).
Let $(X_\alpha)_{\alpha\in A}$ be a family of non-empty set.
Let $Y_\alpha := \{p_\alpha\} \sqcup X_\alpha$ for some additional symbol p_α.
Define the topology $\mathcal{T}_\alpha := \{\emptyset, Y_\alpha, X_\alpha, \{p_\alpha\}\}$.
Then $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha\in A}$ are all compact.
By Tychonoff's Theorem, $\prod_\alpha Y_\alpha$ is also compact.
Now consider $C_\alpha := X_\alpha \times \prod_{\beta\neq\alpha} Y_\beta$.
Since $C_\alpha^c = \{p_\alpha\} \times \prod_{\beta\neq\alpha} Y_\beta$ is open, we have that C_α is closed in the product topology.
Also, $C_\alpha \neq \emptyset$.
Now for any finite $\bigcap_{i=1}^n C_{\alpha_i}$, we have that $(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}, p_\alpha, \ldots) \in \bigcap_{i=1}^n C_{\alpha_i}$.
Thus $(C_\alpha)_{\alpha\in A}$ is an FIP family.
Since $\prod_\alpha Y_\alpha$ is compact, we have that $\bigcap_{\alpha\in A} U_\alpha \neq \emptyset$.
Now $\prod_{\alpha\in A} X_\alpha = \bigcap_{\alpha\in A} U_\alpha$, and we have seen that it being nonempty is equivalent as the Axiom of Choice.
(4) $\implies$ (2).
Take $C : \mathcal{P}(X) \setminus \emptyset \to X$ to be $C(S) :=$ minimal element of S.
(3) $\implies$ (2)
Let $\{X_\alpha\}_{\alpha\in A}$ to be non-empty sets. Let $X := \bigcup_{\alpha\in A} X_\alpha,\ P = \{f_B : B \to X | B \subseteq A, f_B(\beta) \in X_\beta, \forall \beta \in B\}$.
Clearly $P \neq \emptyset$.
Define the order by $f_B \leq f_{B'} \iff B \subseteq B', f_{B'}|_B = f_B$.
For any chain $Q \subseteq P$, define $\tilde{B} = \bigcup_{B\in Q} B$, and $f_{\tilde{B}}(\beta) = f_B(\beta)$ for $\beta \in B$ of any B.
We can check $f_{\tilde{B}} \in P$ is an upper bound.
By Zorn's Lemma, there is a maximal element $f_B \in P$.
If $B \subsetneq A$, then we can extend the function to contain another point $a \in A \setminus B$, and send a to any $x \in X_a$, contradicting maximality.
Thus there is some $f_A \in P$, which we have seen is equivalent to the Axiom of Choice.
$(4) + (2) \implies (3)$.

See Pmath432 A1.
$(3) + (2) \implies (1)$.
For contradiction, suppose $X = \prod_{\alpha \in A} X_\alpha$ is not compact.
Let NFS:$= \{\mathcal{C} \subseteq \mathcal{P}(X) : \mathcal{C}$ is a cover with no finite sub cover$\}$.
Define $\mathcal{C}_1 \leq \mathcal{C}_2 \iff \mathcal{C}_1 \subseteq \mathcal{C}_2$.
Take any chain Q in NFS.
Let $\mathcal{C}' := \bigcup_{\mathcal{C} \in Q} \mathcal{C}$, which we can check is an open cover and an upper bound for the chain.
Indeed, suppose $\mathcal{C}' \notin$ NFS, then $\exists U_1, U_2, \ldots, U_n \in \mathcal{C}'$ such that $X = \bigcup_{i=1}^n U_i$.
For all $i \in [n], U_i \in \mathcal{C}_i$ for some $C_i \in Q$.
Since Q is a chain, there is some $i_0 \in [n]$ such that $\forall i \in [n],\ \mathcal{C}_i \subseteq \mathcal{C}_{i_0}$.
Thus $\mathcal{C}_{i_0} \notin$NFS, a contradiction.
Thus $\mathcal{C} \in$NFS.
By Zorn's Lemma, there is a maximal open cover $\mathcal{C}_{max}$ with no subcover.
Notice that if $U \in \mathcal{C}_{max}$, and $V \subseteq U$ is open, then $V \in \mathcal{C}_{max}$ as well, since any finite subcover of $\{V\} \cup \mathcal{C}_{max}$ can give a finite subcover of $\mathcal{C}_{max}$ by replacing V by U.
Also, if $U_1, U_2 \in \mathcal{C}_{max}$, we must have $U_1 \cup U_2 \in \mathcal{C}_{max}$ as well.
Also, suppose $V_1, \ldots, V_n$ are open in X, such that $\bigcap_{i \in [n]} V_i \in \mathcal{C}_{max}$, then $\exists i_0 \in [n]$, such that $V_{i_0} \in \mathcal{C}_{max}$.
Indeed, suppose not, for any $i \in [n]$, there is a finite cover $V_i \cup \bigcup_{j \in [N_i]} U_{i,j}$ for $U_{i,j} \in \mathcal{C}_{max}$. We must have $\left(\bigcap_{i \in [n]} V_i\right) \cup \bigcup_{i \in [n], j \in [N_i]} U_{i,j}$ is a finite sub-cover of $\mathcal{C}_{max}$.
Now let $W_\alpha := \left\{\text{open } U_\alpha \subseteq X_\alpha \middle| \pi^{-1}(U_\alpha) \in \mathcal{C}_{max}\right\}$.
For contradiction, suppose W_α covers X_α, then there is a finite subcover $\{U_i\}_{i \in [n]}$ such that $X_\alpha = \bigcup_{i \in [n]} U_i$.
Thus $X = \bigcup_{i=1}^n \pi_\alpha^{-1}(U_i)$, which is a subcover of $\mathcal{C}_{max}$.
Thus $X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right) \neq \emptyset$.
By the Axiom of Choice, there is $x_\alpha \in X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right)$ for each α.
Let $x \in X$ be $x(\alpha) = x_\alpha$.
Since $\mathcal{C}_{max}$ is a cover for X, there is some open $U \in \mathcal{C}_{max}$ with $x \in U$.
Thus, there must be some $x \in U_1 \times U_2 \times \cdots \times U_n \times \prod_{\beta \in A \setminus \{\alpha_i : i \in [i]\}} = \bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$ for open $U_i \in X_{\alpha_i}$, since such sets forms a basis.
Thus, $\bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{max}$ as well.
Thus, there is some $i_0 \in [n]$, such that $\pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{max}$, which means $U_i \in W_{\alpha_i}$.
However, $x_{\alpha_i} \in U_{\alpha_i}$, which is a contradiction to the choice of $x_{\alpha_i} \notin \left(\bigcup_{U \in W_{\alpha_i}} U\right)$. □

3 Banach Spaces

Definition 3.1. A **normed vector space** is a vector space $(X, \|\cdot\|)$ over a field $\mathbb{F}$ endowed with a norm (length) function: $\|\cdot\| : X \to [0, \infty)$, such that $\forall x, y \in X, a \in \mathbb{F}$, it satisfies

1. subadditivity (triangular inequality); i.e. $\|x + y\| \leq \|x\| + \|y\|$,
2. absolute homogeneity; i.e. $\|a \cdot x\| = |a|\|x\|$, and
3. positive definiteness; i.e. if $x \neq 0$, we must have $\|x\| > 0$.

Proposition 3.1. *For every* ***normed space*** *with $\|\cdot\|$, there is a metric $d(x, y) = \|x - y\|$.*

Proof.

$$\begin{aligned}
d(x,x) &= \|x - x\| = \|0\| = 0 \\
\forall x \neq y, d(x,y) &= \|x - y\| > 0 \\
d(x,y) &= \|x - y\| = \|-(y - x)\| = |-1|\|y - x\| = \|y - x\| = d(y,x) \\
d(x,z) &= \|x - z\| = \|x - y + y - z\| \geq \|x - y\| + \|y - z\| = d(x,y) + d(y,z)
\end{aligned}$$

Thus $d(x, y) = \|x - y\|$ is a metric. □

Definition 3.2. Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X, then they are called **equivalent** if there are $C_1, C_2 > 0$, such that

$$\forall x \in X,\ C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Definition 3.3. A normed space is called a **Banach space** if it is complete.

Proposition 3.2. *The Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$, with the Euclidean norm $\|x\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ is a Banach space.*

Definition 3.4. For $\mathbb{R}^n$ or $\mathbb{C}^n$, and $p \in [1, \infty)$, the ℓ^p norm is $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$. For $p = \infty$, the ℓ_∞ norm is $\|x\|_\infty = \max_{i \in [n]} |x_i|$.

Proposition 3.3. *$\mathbb{R}^n$ or $\mathbb{C}^n$, with any ℓ^p norm is a Banach space.*

Remark. Notice that $\forall n \in \mathbb{N}^+,\ \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty$, so they are equivalent.

Proposition 3.4. *If X is compact and Hausdorff, we have $(C(X), \|\cdot\|_\infty)$ is a Banach Space.*

Proof. The Extreme Value Theorem shows it is a normed space.
The convergence in $\|\cdot\|_\infty$ is uniform convergence, and the uniform limit of continuous functions is continuous.
Thus, $(C(X), \|\cdot\|_\infty)$ is complete. □

Proposition 3.5. *For a Locally Compact Hausdorff Space X, $(C_b(X), \|\cdot\|_\infty)$ and $(C_0(X), \|\cdot\|_\infty)$ are both Banach Spaces.*

Example 3.0.1. $C_0(\mathbb{N}) = \{(x_n)_{n=1}^\infty : \lim_{n\to\infty} x_n = 0\}$ with the discrete topology.

Proposition 3.6. *$C^k([0, 1])$ is a Banach Space with $\|f\|_{C^k([a,b])} := \sum_{i=1}^k \left\|f^{(i)}\right\|_\infty$.*

Proof. It is easy to check that this is a norm.
Now take any Cauchy sequence $(f_n)_{n=1}^\infty$, then for each $i \in [k]$, we have $(f_n^{(i)})_{n=1}^\infty$ is Cauchy in $C([a, b])$ as well.
Since $[a, b]$ is compact and Haudorff, we have $C([a, b])$ is a Banach Space, so there is a $g_i(x) := \lim_{n\to\infty} f_n^{(i)}(x)$.
In addition, since the convergence is uniform, we have $g_i \in C([0, 1])$.
By the Fundamental Theorem of Calculus, we have that for all $i \in [k - 1], x \in [0, 1]$,

$$f_n^{(i)}(x) = f_n^{(i)}(0) + \int_0^x f_n^{(i+1)}(t)dt.$$

Taking the limit of $n \to \infty$, we have that

$$g_i(x) = g_i(0) + \int_0^x g_{i+1}(t)dt,$$

which means $g_i \in C^1([0, 1])$ with $g_i' = g_{i+1}$.
Thus $g_0 \in C^k([0, 1])$. □

3.1 Bounded linear operators

Definition 3.5. Let X, Y be vector spaces, $T : X \to Y$ is a linear operator if $\forall c \in \mathbb{R}, u, v \in X$,

$$T(u + cv) = Tu + cTv.$$

Definition 3.6. Let X, Y be linear normed spaces, the **operator norm** of a linear operator $T : X \to Y$ is

$$\|T\| := \sup_{\|u\|_X \leq 1} \|Tu\|_Y = \sup_{\|u\|_X < 1} \|Tu\|_Y = \sup_{\|u\|_X = 1} \|Tu\|_Y = \sup_{u \neq 0 \in X} \frac{\|Tu\|_Y}{\|x\|_X}.$$

Definition 3.7. Let X, Y be normed spaces, a linear operator $T : X \to Y$ is **bounded** if $\|A\| < \infty$.

Theorem 3.7. *Let X, Y be two normed linear spaces, let $T : X \to Y$ be linear, then the following are equal:*

1. *T is continuous,*
2. *T is continuous at 0,*
3. *T is bounded,*
4. *T is uniformly continuous.*

Proof. (4) $\implies$ (1) $\implies$ (2) trivially.
(3) $\implies$ (4).
Suppose T is bounded, then

$$\begin{aligned}\|T(x) - T(y)\| &= \|T(x - y)\| \\ &\leq \|T\| \|x - y\|.\end{aligned}$$

Thus, T is $\|T\|$ Lipschitz, so it is uniformly continuous.
(2) $\implies$ (3).
Suppose T is continuous at 0, and suppose for contradiction that $\|T\| = \infty$.
There must be $(x_n)_{n=1}^{\infty}$ in X, such that $\|x_n\| \leq 1,\ \|Tx_n\| \geq n^2$ for each $n \geq 1$.
Notice that $\frac{x_n}{n} \to 0$, but $\left\|T(\frac{x_n}{n})\right\| = \frac{1}{n}\|Tx_n\| \geq n$ for each n.
Thus $\lim_{n\to\infty} T(\frac{x_n}{n}) \neq 0 = T(0)$, which contradicts that T is continuous at 0. □

Proposition 3.8. *Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X, then they are equivalent if and only if they induce the same topology.*

Proof. Assume that $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, then there are $C_1, C_2 > 0$, such that

$$\forall x \in X,\ C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

Consider the identity function $id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$, we see that

$$\|id\| = \sup_{x\neq 0\in X} \frac{\|x\|_2}{\|x\|_1} \leq \sup_{x\neq 0\in X} \frac{C_2\|x\|_1}{\|x\|_1} = C_2,$$

and

$$\left\|id^{-1}\right\| = \sup_{x\neq 0\in X} \frac{\|x\|_1}{\|x\|_2} \leq \sup_{x\neq 0\in X} \frac{\|x\|_1}{C_1\|x\|_1} = \frac{1}{C_1}.$$

Thus id is a homeomorphism.
On the other hand, suppose $\|\cdot\|_1, \|\cdot\|_2$ induces the same topology, then id is a homeomorphism.
Thus,

$$\frac{1}{\|id\|}\|x\|_2 = \frac{1}{\|id\|}\|id(x)\|_2 \leq \|x\|_1 = \left\|id^{-1}(x)\right\|_1 \leq \left\|id^{-1}\right\|\|x\|_2.$$

□

Definition 3.8. Let X, Y be normed spaces, we denote

$$B(X, Y) := \{T : X \to Y | T \text{ is a bounded linear operator}\}.$$

Theorem 3.9. *The set $B(X, Y)$ is a normed linear space with the operator norm.*

Proposition 3.10. *Let X, Y, Z be normed spaces, if $T : X \to Y, S : Y \to Z$ are both linear bounded operators, then so is $S \circ T$, with*

$$\|S \circ T\| \leq \|S\|\|T\|.$$

Theorem 3.11. *Let X be a normed space, and Y be a Banach Space, then $B(X, Y)$ is a Banach Space.*

Proof. Let $(T_n)_{n=1}^{\infty}$ be a Cauchy sequence in $B(X, Y)$.
For any $x \in X$, we have that $(T_n x)_{n=1}^{\infty}$ is Cauchy in Y.
Indeed, $\|T_n x - T_m x\| = \|(T_n - T_m)x\| \le \|T_n - T_m\|\|x\|$.
Since Y is complete, there must be a unique $y = \lim_{n\to\infty} T_n x \in Y$.
Define $Tx := \lim_{n\to\infty} T_n x$ for any $x \in X$.
Notice that T is linear.
Given $\epsilon > 0$, we know there must be some $N \in \mathbb{N}$, such that $\forall m, n \ge N,\ \|T_n - T_m\| < \epsilon$.
Consider any $x \in X$.

$$\begin{aligned}
\|Tx\| &\le \|(T - T_N)x\| + \|T_N x\| \\
&= \lim_{m\to\infty} \|(T_m - T_N)x\| + \|T_N x\| \\
&\le \limsup_m \|T_m - T_N\|\|x\| + \|T_N\|\|x\| \\
&\le \epsilon\|x\| + \|T_N\|\|x\|.
\end{aligned}$$

Thus $\|T\| \le \epsilon + \|T_N\| < \infty$.
This shows $T \in B(X, Y)$.
Again, for any $x \in X$, $n \ge N$, we have

$$\begin{aligned}
\|(T_n - T)x\| &= \lim_{m\to\infty} \|(T_n - T_m)x\| \\
&\le \limsup_m \|T_n - T_m\|\|x\| \\
&< \epsilon\|x\|.
\end{aligned}$$

Thus $\|T_n - T\| < \epsilon$ for any $n \ge N$, which shows $\lim_{n\to\infty} T_n = T$ in $B(X, Y)$ with the operator norm. ☐

Definition 3.9. Let X, Y be normed spaces. We say a linear operator $T : X \to Y$ is **bounded below** if $\exists m > 0,$ such that $\forall x \in X,\ \|Tx\| \ge m\|x\|$.

Definition 3.10. Let X, Y be normed spaces. We say $T : X \to Y$ is an **isomorphism between normed spaces** or T is **invertible** if T is bijective and T, T^{-1} are both bounded. i.e. T is a homeomorphism between X, Y.

Theorem 3.12. *Let $T : X \to Y$ be a linear operator between normed linear spaces X and Y. The operator T has a bounded inverse $T^{-1} : \mathrm{Im}(T) \to X$ if and only if T is bounded below. In this case, T is injective, and*

$$\|T^{-1}\| = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

Proof. ($\Longrightarrow$) Suppose T has a bounded inverse operator. Let $x \in X$ and $y = Tx \in \mathrm{Im}(T)$. By the assumption,

$$\|x\| = \|T^{-1}y\| \le \left\|T^{-1}\right\|\|y\| = \left\|T^{-1}\right\|\|Tx\|,$$

$$\|Tx\| \ge \frac{1}{\|T^{-1}\|}\|x\|.$$

Since the inequality holds for all $x \in X$, T is bounded below.

($\Longleftarrow$) Suppose T is bounded below. There exists $m > 0$ such that $\|Tx\| \ge m\|x\|$ for all $x \in X$. Let $z \in \ker T$. We have $0 = \|Tz\| \ge m\|z\| \ge 0$. Therefore, $\|z\| = 0$, which means $z = 0$. Thus $\ker T = \{0\}$ and T is one-to-one. Thus, T is a bijection between $X \longleftrightarrow \mathrm{Im}(T)$, which means $T^{-1} : \mathrm{Im}(T) \to X$ exists.

Finally, for any $y \in \mathrm{Im}(T)$, $y = Tx$ for some $x \in X$. Then

$$\|T^{-1}y\| = \|x\| \overset{T \text{ bounded below}}{\le} \frac{1}{m}\|Tx\| = \frac{1}{m}\|y\|.$$

Therefore T^{-1} is bounded. In particular,

$$\|T^{-1}\| = \sup_{y\in \mathrm{Im}(T),\, y\neq 0} \frac{\|T^{-1}y\|}{\|y\|} = \sup_{x\neq 0} \frac{\|x\|}{\|Tx\|} = \frac{1}{\inf_{x\neq 0} \frac{\|Tx\|}{\|x\|}} = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

$\square$

Definition 3.11. Let X, Y be normed spaces. We call $T : X \to Y$ a **contraction** if $\|T\| \leq 1$.

Definition 3.12. Let X, Y be normed spaces. We call $T : X \to Y$ an **isometry** if $\forall x \in X,\ \|Tx\| = \|x\|$.

Proposition 3.13. *Let X, Y be normed spaces. If a linear operator $T : X \to Y$ is a surjective isometry, it is an isometric isomorphism between normed spaces.*

Proof. T is bounded since $\|T\| = 1$. Suppose $T(x) = 0$, we have $\|Tx\|_Y = 0$. Since T is an isometry and thus bounded below by 1, T has a bounded inverse $T^{-1} : \mathrm{Im}(T) \to X$ by theorem 3.12. Since T is surjective, $\mathrm{Im}(T) = Y$. $\square$

Proposition 3.14. *Let Y be a Banach space, S be a dense subset of a normed space X. For any bounded linear operator $E : S \to Y$, we can extend it to $\tilde{E} : X \to Y$, such that $\tilde{E}$ is also bounded and linear, with $\left\|\tilde{E}\right\| = \|E\|$, and $\tilde{E}|_S = E$.*

Proof. Consider any $x \in X$.
Since S is dense in X, We know $\forall m \in \mathbb{N}^+, \exists x_m \in S$, such that $\|x - x_m\|_X \leq \frac{1}{m}$.
Since E is linear on S, we have that

$$\begin{aligned}
\|Ex_m - Ex_l\|_Y &= \|E(x_m - x_l)\|_Y \\
&\leq \|E\|\|x_m - x_l\|_X \\
&= \|E\|\|(x_m - x) + (x - x_l)\|_X \\
&\leq \|E\|\|x - x_m\|_X + \|E\|\|x - x_l\|_X \\
&\leq \|E\|\left(\frac{1}{m} + \frac{1}{l}\right).
\end{aligned}$$

Thus given any $\epsilon > 0$, for any $m, l \geq \lceil \frac{2\epsilon}{\|E\|} \rceil$, we can make $\|Ex_m - Ex_l\|_Y < \epsilon$. Thus $(Ex_m)_{m=1}^{\infty}$ is a Cauchy sequence in Y.
Since Y is a Banach space, $\exists y^* \in Y$, such that $Ex_m \to y^*$ in Y.
We claim that y^* is independent of choice of the sequence $(x_m)_{m=1}^{\infty}$.
Indeed, consider any other sequence $(v_m)_{m=1}^{\infty} \subseteq C^{\infty}(\bar{x})$, such that $\forall m \in \mathbb{N}^+, \|x - x_m\|_X \leq \frac{1}{m}$,

$$\begin{aligned}
\|y^* - Ev_m\|_Y &\leq \|y^* - Ex_m\|_Y + \|Ex_m - Ev_m\|_Y \\
&\leq \|y^* - Ex_m\|_Y + \|E\|\|x_m - v_m\|_X \\
&\leq \|y^* - Ex_m\|_Y + \|E\|\|x_m - x\|_X + \|E\|\|x - v_m\|_X.
\end{aligned}$$

Since all three terms on the right go to 0 when $m \to \infty$, we have that $Ev_m \to y^*$ in Y.
Thus we can uniquely define $\tilde{E}x := y^*$. In addition,

$$\begin{aligned}
\left\|\tilde{E}x\right\|_Y &= \left\|\lim_{m\to\infty} Ex_m\right\|_Y \\
&= \lim_{m\to\infty} \|Ex_m\|_Y \\
&\leq \lim_{m\to\infty} \|E\|\|x_m\|_X \\
&= \|E\|\left\|\lim_{m\to\infty} x_m\right\|_X \\
&= \|E\|\|x\|_X.
\end{aligned}$$

Thus $\left\|\tilde{E}\right\| = \|E\|$. $\square$

3.1.1 Dual Spaces

Definition 3.13. Let X be a normed space over $\mathbb{F}$, a **functional** is an operator that maps into $\mathbb{F}$.

Definition 3.14. Let X be a normed space over $\mathbb{F}$, the **dual space** of X is the collection of bounded linear functionals on X, denoted

$$X^* := B(X, \mathbb{F}) = \{\phi : X \to \mathbb{F} : \phi \text{ is linear and bounded}\}.$$

Definition 3.15. Let X be a normed space over $\mathbb{F}$, and a subspace $Y \subseteq X^*$ we define the **duality pairing** to be $\langle \cdot | \cdot \rangle_{Y,X} : Y \times X \to \mathbb{F}$ by $\langle \phi | x \rangle_{Y,X} := \phi(x)$, the **action** of ϕ on x.

Definition 3.16. Let X be a normed space, the **dual norm** is defined to be

$$\|u^*\|_{X^*} := \sup_{\|u\| \leq 1} \left| \langle u^* | u \rangle_{X^*,X} \right|.$$

Example 3.1.1. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^{\infty} | \lim_{i\to\infty} x_i = 0\}$, then $C_0(\mathbb{N})^* \cong \ell^1(\mathbb{N})$.
Indeed, let $e_n := (m \mapsto \delta_{nm}) = (\delta_{nm})_{n=1}^{\infty} \in C_0(\mathbb{N})$.
Given any $\phi \in C_0(\mathbb{N})^*$, we define $a_n := \phi(e_n) \in \mathbb{F}$.
We claim $a = (a_n)_{n=1}^{\infty}$ completely determines ϕ.
Indeed, consider any $x \in C_0$, let $x^N := \sum_{n=1}^{N} x_n e_n$.
We have $\left\|x - x^N\right\|_{c_0} = 0$, so

$$\begin{aligned}
\phi(x) &= \lim_{N\to\infty} \phi(x^N) \\
&= \lim_{N\to\infty} \phi\left(\sum_{n=1}^{N} x_n e_n\right) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} x_n \phi(e_n) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} x_n a_n.
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{n=1}^{N} |a_n| &= \sum_{n=1}^{N} a_n \operatorname{sgn}(a_n) \\
&= \sum_{n=1}^{N} \phi(\operatorname{sgn}(a_n) e_n) \\
&= \phi(y_N) \\
&\leq \|\phi\|_{C_0^*} \|y_N\|_{C_0},
\end{aligned}$$

where $y_N = \sum_{i=1}^{N} \operatorname{sgn}(a_n) e_n$.
Since $\|y_N\| = 1$, we have $\sum_{n=1}^{N} |a_n| \leq \|\phi\|_{C_0^*}$ for any N.
Thus

$$\|a\|_1 = \sum_{n=1}^{\infty} |a_n| = \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| \leq \|\phi\|_{C_0^*} < \infty.$$

Thus $a \in \ell^1(\mathbb{N})$.
Notice that $\Phi : C_0^*(\mathbb{N}) \to \ell^1(\mathbb{N})$ by $\phi \mapsto a$ is linear, contractive, and injective.

Also, given any $a \in \ell^1(\mathbb{N})$, we can set $\phi_a(x) := \sum_{n=1}^\infty x_n a_n$.
Thus,

$$
\begin{aligned}
|\phi_n(x)| &\le \sum_{n=1}^\infty |x_n||a_n| \\
&\le \|x\|_\infty \|a\|_1,
\end{aligned}
$$

which shows $\|\phi_a\|_{C_0^*} \le \|a\|_1$.
Thus $\phi_a \in C_0^*(\mathbb{F})$. Notice that $\Phi(\phi_a) = a$.
This shows that Φ is surjective, and it is actually an isometry, since $\forall \phi \in C_0^*(\mathbb{F})$, we have

$$
\|\Phi(\phi)\|_1 \le \|\phi\|_{C_0^*} = \left\|\Phi^{-1}(\Phi(\phi))\right\|_{C_0^*} \le \|\Phi(\phi)\|_1.
$$

Example 3.1.2. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^\infty | \lim_{i\to\infty} x_i = 0\}$, then

$$
B(C_0(\mathbb{N})) = B(C_0(\mathbb{N}), C_0(\mathbb{N})) \cong \{(t_{ij})_{i,j\in\mathbb{N}} : \|(t_{ij})_{i,j\in\mathbb{N}}\| < \infty,\ \forall j \in \mathbb{N},\ (t_{ij})_{i\in\mathbb{N}}\} \in C_0(\mathbb{N}),
$$

which are infinite matrices whose rows are uniformly in ℓ^1, and columns are in $C_0(\mathbb{N})$.
In addition, it is an isometry under

$$
\|(t_{ij})_{i,j\in\mathbb{N}}\| := \sup_{i\in\mathbb{N}} \left\|(t_{ij})_{j=1}^\infty\right\|_1.
$$

Indeed, let

$$
e_n := (m \mapsto \delta_{nm}) \cong (\delta_{nm})_{n=1}^\infty \in C_0(\mathbb{N}), \delta_i := ((x_j)_{j=1}^\infty \mapsto x_i) \in C_0^*(\mathbb{N})
$$

with $\Phi(\delta_i) = (\delta_{ij})_{j=1}^\infty \in \ell^1(\mathbb{N})$ as in previous example.
Easy to check $\|\delta_i\| = 1$ by the above example.
Consider any $T \in B(C_0(\mathbb{N}))$, define $\phi_i := (\delta_i \circ T), t_{ij} := \phi_i(e_j)$.
Notice that $\phi_i : C_0(\mathbb{N}) \to \mathbb{F}$ is linear, and $\|\phi_i\| \le \|\delta_i\|\|T\| = \|T\| < \infty$, so $\phi_i \in C_0^*(\mathbb{N})$.
Thus, $(t_{ij})_{j=1}^\infty = (\phi_i(e_j))_{j=1}^\infty = \Phi(\phi_i) \in \ell^1(\mathbb{N})$ as in previous example, with $\|\phi_i\| = \left\|(t_{ij})_{j=1}^\infty\right\|_1$.
Since this hold for all $i \in \mathbb{N}$, we have $\sup_{i\in\mathbb{N}} \left\|(t_{ij})_{j=1}^\infty\right\|_1 \le \|T\|$.
In addition, $(t_{ij})_{i\in\mathbb{N}} = ((\delta_i \circ T)(e_j))_{i\in\mathbb{N}} = Te_j \in C_0(\mathbb{N})$.
On the other hand, suppose we have such $(t_{ij})_{i,j\in\mathbb{N}}$ with $\sup_{i\in\mathbb{N}} \left\|(t_{ij})_{j=1}^\infty\right\|_1 < \infty$, we can define $\phi_i := \Phi^{-1}((t_{ij})_{j=1}^\infty) \in C_0^*(\mathbb{N})$ with $\phi_i(x) = \sum_{j=1}^\infty x_j t_{ij}$ as in previous example.
Let $Tx := \sum_{i=1}^\infty \phi_i(x) e_i$ for any $x \in C_0(\mathbb{N})$.
Clearly, T is linear, and we have $(\delta_i \circ T)(x) = \delta_i(\sum_{j=1}^\infty \phi_j(x) e_j) = \phi_i(x)$.

$$
\begin{aligned}
\|Tx\|_\infty &= \|(\phi_i(x))_{i=1}^\infty\|_\infty \\
&\le \sup_{i\in\mathbb{N}} |\phi_i(x)| \\
&\le \sup_{i\in\mathbb{N}} \|\phi_i\|\|x\| \\
\|T\| &\le \sup_{i\in\mathbb{N}} \|\phi_i\| \\
&= \sup_{i\in\mathbb{N}} \left\|(t_{ij})_{j=1}^\infty\right\| \\
&< \infty.
\end{aligned}
$$

Thus $T \in B(C_0(\mathbb{N}), \ell_\infty)$.
Now we claim $T(C_0(\mathbb{N})) = C_0(\mathbb{N})$, which will mean $T \in B(C_0(\mathbb{N}))$.
Indeed, for any $x = \sum_{n=1}^\infty x_n e_n \in C_0(\mathbb{N})$, we have

$$
\begin{aligned}
Tx &= T\left(\lim_{N\to\infty} \sum_{n=1}^N x_n e_n\right) \\
&= \lim_{N\to\infty} \sum_{n=1}^N x_n T(e_n).
\end{aligned}
$$

Since each

$$\begin{aligned}T(e_n) &= \sum_{i=1}^{\infty} \phi_i(x)e_i \\ &= \sum_{i=1}^{\infty} t_{in}e_i \\ &\in C_0(\mathbb{N}),\end{aligned}$$

and $C_0(\mathbb{N})$ is closed, we have $Tx \in C_0(\mathbb{N})$.
In addition, $(\delta_i \circ T)(e_j) = \phi_i(e_j) = \Phi^{-1}((t_{ik})_{k=1}^{\infty})(e_j) = t_{ij}$.
Thus $T \longleftrightarrow (t_{ij})_{i,j\in\mathbb{N}}$ is an isometric bijection.

Example 3.1.3. Consider the **Disk Algebra**

$$A(\mathbb{D}) := \left\{ f \in C(\mathbb{T}) : \forall n \in \mathbb{Z}^{--}, \hat{f}(n) = 0 \right\},$$

where $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C}, |z| = 1\}$ is the unit circle, and

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it}\right) e^{-int} dt$$

is the n^{th} Fourier Transform of f.
Consider $\phi_n : f \mapsto \hat{f}(n)$, which is clearly in $C^*(\mathbb{T})$.
We notice that $A(\mathbb{D}) = \bigcap_{n<0} \ker(\phi_n)$ is closed in $C(\mathbb{T})$, since each kernel of a continuous functional is closed.
In fact, for $f, g \in C(\mathbb{T})$, we have $\hat{fg}(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\hat{g}(n-k)$.
Thus, for $f, g \in A(\mathbb{D})$, we have $\hat{fg}(n) = \sum_{k\in\mathbb{N}} \hat{f}(k)\hat{g}(n-k) = 0$ for $n < 0$.
This shows $A(\mathbb{D})$ is actually an Algebra.
Also, $A(\mathbb{D})$ is exactly the set of $f \in C(\mathbb{T})$ that admits an extension $F \in C\left(\bar{\mathbb{D}}\right)$ with $F|_{\mathbb{D}}$ being analytic, with $F(z) := \sum_{n=1}^{\infty} \hat{f}(n)z^n$.

3.2 Quotient Spaces

Definition 3.17. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. The **quotient space** is $X/Y := \{x + Y : x \in X\}$, with the **quotient map** $Q : X \to X/Y$ by $Q(x) := [x] := x+Y = \{x + y : y \in Y\}$.

Proposition 3.15. *Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. X/Y is always a vector space with $[0] = Y, [x] + [y] = [x + y], c[x] = [cx]$.*

Proposition 3.16. *Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. $\left(X/Y, \|\cdot\|_{X/Y}\right)$ is always a Banach space with $\|[x]\|_{X/Y} := \inf_{y\in Y} \|x + y\|_X$. In addition, Q is isometric if $Y \subsetneq X$.*

Proof. $\|[x]\|_{X/Y} = 0 \iff \inf_{y\in Y} \|x + y\|_X = 0 \iff x \in \bar{Y} \iff x \in Y$.
Scaling is clear.
Also,

$$\begin{aligned}\|[x] + [z]\|_{X/Y} &= \inf_{y\in Y} \|x + y + z\|_X \\ &= \inf_{y_1,y_2\in Y} \|x + y_1 + z + y_2\|_X \\ &\leq \inf_{y_1\in Y} \|x + y_1\|_X + \inf_{y_2\in Y} \|z + y_2\|_X \\ &= \|[x]\|_{X/Y} + \|[z]\|_{X/Y}.\end{aligned}$$

This shows $\left(X/Y, \|\|_{X/Y}\right)$ is a normed space.
We note that $\|Qx\|_{X/Y} = \inf_{y\in Y} \|x + y\|_X \leq \|x + 0\|_X = \|x\|_X$, so $\|Q\| \leq 1$.

Now consider any Cauchy sequence $([x_n])_{n=1}^{\infty}$ in X/Y.
We can pick a subsequence $([x_{n_i}])_{i=1}^{\infty}$ such that $\left\|[x_{n_{i+1}}] - [x_{n_i}]\right\|_{X/Y} < 2^{-i}$.
Pick $z_1 \in X$ such that $[z_1] = [x_{n_i}]$.
Since $\|[x_{n_2}] - [x_1]\|_{X/Y} = \inf_{y \in Y} \|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$, there is $y \in Y$, such that $\|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$.
Take $z_2 = x_{n_2} + y$, we have $\|z_2 - z_1\|_X < \frac{1}{2}$.
Inductively, we can pick $(z_i)_{i=1}^{\infty}$, such that $\|z_i - z_i\|_X < 2^{-i}$.
We can check that this is a Cauchy sequence in X, so it has a limit $z = \lim_{i\to\infty} z_i \in X$.
Now for any $i \in \mathbb{N}^+$, we have

$$\begin{aligned}
\|[x_{n_i}] - [z]\|_{X/Y} &= \|[z_i] - [z]\|_{X/Y} \\
&= \|Q(z_i) - Q(z)\|_{X/Y} \\
&= \|Q(z_i - z)\|_{X/Y} \\
&\leq \|Q\|\|z_i - z\| \\
&\to 0.
\end{aligned}$$

Thus $([x_{n_i}])_{i=1}^{\infty} \to [z]$ is a convergent subsequence, which mean $([x_n])_{n=1}^{\infty}$ is convergent.
This shows $\left(X/Y, \|\cdot\|_{X/Y}\right)$ is a Banach space.
Now if $Y \subsetneq X$, then $X/Y \neq \{0\}$, there must be some $[x] \in X/Y$ with $\|[x]\|_{X/Y} = 1$.
Thus, for all $k \in \mathbb{N}^+$, there is some $y_k \in Y$, such that $\|x + y_k\|_X \leq 1 + \frac{1}{k}$.
Now

$$\left\|Q\left(\frac{x + y_k}{\|x + y_k\|_X}\right)\right\|_{X/Y} = \frac{1}{\|x + y_k\|_X}\|Q(x + y_k)\| = \frac{\|[x]\|_{X/Y}}{\|x + y_k\|_X} \geq \frac{1}{1 + \frac{1}{k}}.$$

Since this is true for any $k \in \mathbb{N}^+$, taking the limit $k \to \infty$, we have $\left\|Q\left(\frac{x+y_k}{\|x+y_k\|_X}\right)\right\|_{X/Y} \geq 1$.
Yet $\left\|\frac{x+y_k}{\|x+y_k\|_X}\right\|_X = 1$, so $\|Q\| \geq 1$.
This shows $\|Q\| = 1$. □

Example 3.2.1. Consider a compact and Hausdorff X, and consider $(C(X), \|\cdot\|_\infty)$. Let $E \subseteq X$ be closed, and $I(E) := \{f \in C(X) : f|_E = 0\}$.
One can check $I(E)$ is closed (ideal), and $C(X)/I(E) \cong C(E)$ with an isometric isomorphism $\tilde{R} : [f] \mapsto f|_E$.
We claim that $\tilde{R}$ is well-defined.
Indeed, if $[f] = [g]$, we must have $f - g \in I(E)$, which means $(f - g)|_E = 0$.
Thus $\tilde{R}([f]) = f|_E = g|_E = \tilde{R}([g])$.
Clearly $\tilde{R}$ is linear.
Also, $\tilde{R}([f]) = 0 \implies f|_E = 0 \implies f \in I(E) \implies [f] = 0$, so $\tilde{R}$ is injective.
By Tietze's Theorem, given any $g \in C(E)$, we can extend it to $f \in C(X)$, such that $f|_E = g$. Thus, $\tilde{R}$ is surjective.
Consider any $f \in C(X), g \in I(E)$, we have

$$\begin{aligned}
\left\|\tilde{R}([f])\right\| &= \|f|_E\|_{C(E)} \\
&= \sup_{x\in E} |f(x)| \\
&= \sup_{x\in E} |(f + g)(x)| \\
&\leq \sup_{x\in X} |(f + g)(x)| \\
&= \|f + g\|_{C(X)}.
\end{aligned}$$

Since this hold for all $g \in I(E)$, we have

$$\left\|\tilde{R}([f])\right\| \leq \inf_{g\in I(E)} \|f + g\|_{C(X)} = \|[f]\|.$$

Thus, $\tilde{R}$ is a contraction.
Consider any $f \in C(X)$.
If $f|_E = 0$, we have $\|[f]\| = \|[0]\| = 0 = \|f|_E\| = \left\|\tilde{R}([f])\right\|$.
Now consider $f|_E \neq 0$.
Define the function $k : \mathbb{C} \to \mathbb{C}$ by $k(z) := \begin{cases} z, & |z| \leq \|f|_E\|_\infty \\ \frac{z}{|z|}\|f|_E\|_\infty & |z| \geq \|f|_E\|_\infty, \end{cases}$ which is well-defined and continuous.
Let $g := k \circ f \in C(X)$.
For any $x \in E$, we have $|f(x)| \leq \|f|_E\|_\infty$, so $g(x) = k(f(x)) = f(x)$.
Thus $g|_E = f|_E$, and there is $h \in I(E)$, such that $g = f + h$.

$$\begin{aligned}\|[f]\| &= \inf_{h \in I(E)} \|f + h\|_{C(X)} \\ &\leq \|g\|_{C(X)} \\ &\leq \|k\| \\ &\leq \|f|_E\|_\infty \\ &= \left\|\tilde{R}(f)\right\|.\end{aligned}$$

This proves $\tilde{R}$ is an isometry.

Proposition 3.17. *For any infinite dimensional Banach Space X, it does not have a countable basis. Also, the unit ball is not compact.*

Proof. See A2. ☐

3.3 Baire Category Theorem

Definition 3.18. Let $(X, \mathcal{T})$ be a topological space, then $A \subseteq X$ is called **nowhere dense** if $(\bar{A})^o = \emptyset$.

Theorem 3.18 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space, then X cannot be written as a countable union of nowhere dense sets.*

Corollary 3.19. *Let $\{U_i\}_{i=1}^\infty$ be a countable set of open dense sets, then $\bigcap_{i=1}^\infty U_i$ is dense.*

3.3.1 Banach-Steinhaus Theorem

Definition 3.19. Let X, Y be normed vector spaces, $\mathcal{S} \subseteq B(X, Y)$ is called **pointwise bounded** if $\forall x \in X$, $\mathcal{S}x$ is bounded. Namely, $\exists k_x > 0$, such that $\forall S \in \mathcal{S},\ \|Sx\| \leq k_x$.

Theorem 3.20 (Banach-Steinhaus)**.** *Let X be a Banach space, and Y be a normed vector space. Suppose $\mathcal{S} \subseteq B(X, Y)$ is pointwise bounded, then $\mathcal{S}$ is bounded in $B(X, Y)$. Namely, $\sup_{S \in \mathcal{S}} \|s\| < \infty$.*

Proof. For each x, let $k_x > 0$ be such $\forall S \in \mathcal{S},\ \|Sx\| \leq k_x$.
For each $n \in \mathbb{N}$, let $A_n := \{x \in X : k_x \leq n\}$.
For any $(x_k)_{k \in \mathbb{N}}$ in A_n, with $x = \lim_{k \to \infty} x_k$, we have

$$\|Sx\| = \lim_{k \to \infty} \|Sx_k\| \leq n.$$

Thus, $x \in A_n$, which shows A_n is closed.
Notice that $X = \bigcup_{n \in \mathbb{N}} A_n$, so by Baire Category Theorem, there is $n_0 \in \mathbb{N}$, such that $(\bar{A_{n_0}})^o \neq \emptyset$.
Thus, there is $x_0 \in A_{n_0}, r > 0$, such that $\bar{B}(x_0, r) \subset B(x_0, 2r) \subseteq (\bar{A_{n_0}})^o \subset \bar{A_{n_0}} = A_{n_0}$.
Now for any $s \in \mathcal{S}, y \in X$ such that $\|y\| \leq 1$, we have that

$$\begin{aligned}\|sy\| &= \left\|\frac{s(x_0) - s(x_0 - ry)}{r}\right\| \\ &\leq \frac{\|s(x_0)\| + \|s(x_0 - ry)\|}{r} \\ &\leq \frac{2n_0}{r},\end{aligned}$$

since $x_0, x_0 - ry \in \bar{B}(x_0, r) \subset A_{n_0}$.
Thus, $\|s\| = \sup_{y \in X \text{ such that } \|y\| \leq 1} \|sy\| \leq \frac{2n_0}{r}$.
Since this holds for all $s \in \mathcal{S}$, we have $\sup_{s \in \mathcal{S}} \|s\| = \frac{2n_0}{r} < \infty$. $\square$

Corollary 3.21 (Limit of bounded operators)**.** *Let X be a Banach space, and Y be a normed vector space. Consider any sequence $(T_n)_{n=1}^{\infty}$ in $B(X,Y)$. Suppose $\forall x \in X$, $(T_n x)_{n=1}^{\infty}$ is convergent, then $\mathcal{S} := \{T_n : n \in \mathbb{N}\}$ is bounded. In addition, for $Tx := \lim_{n \to \infty} T_n x$, we have $T \in B(X,Y)$, and $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$.*

Proof. Since $(T_n x)_{n=1}^{\infty}$ is convergent, it is bounded. This is equivalent to saying $\mathcal{S}$ is pointwise bounded. By the Banach-Steinhaus Theorem, $\mathcal{S}$ is bounded.
Now $\|Tx\| = \lim_{n \to \infty} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n\| \|x\|$.
Since this holds for all $x \in X$, we have $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$. $\square$

Example 3.3.1. Consider $f \in C(\mathbb{T})$, we can define the N^{th} partial sum of its Fourier series $S_N(f)(e^{it}) := \sum_{n=-N}^{N} \hat{f}(n) e^{int}$.
We note that $S_N(f)$ does not necessarily converge to f in $C(\mathbb{T})$, nor pointwise.
Indeed, consider $\phi_N(f) := S_N(f)(1) \in \mathbb{C}$. Note that ϕ_N is linear.
One can show that

$$\begin{aligned}\phi_N(f) &= \sum_{n=-N}^{N} \hat{f}(n) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \left(\sum_{n=-N}^{N} e^{-int} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) D_N(t) dt,\end{aligned}$$

where $D_N(t) = \frac{\sin\left(\left(N+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}$ is the **Dirichlet's Kernel**.
Actually, $\exists C > 0$, such that $\|D_N\|_1 \geq C \log(N)$.
Also, $\|\phi_N\|_{C_*(\mathbb{T})} = \|D_N\|_1$.
Thus, $(\phi_N)_{N \in \mathbb{N}}$ is not bounded.
By Banach-Steinhaus, $(\phi_N)_{N \in \mathbb{N}}$ is not pointwise bounded.
Thus, there is $f \in C(\mathbb{T})$, such that $|\phi_N(f)| \to \infty$.
Thus, $S_N f$ does not converge to f at 1.

3.3.2 Open Mapping Theorem

Theorem 3.22 (Open Mapping Theorem)**.** *Let X, Y be Banach spaces, suppose $T \in B(X,Y)$ is surjective, then it is **open**. i.e. $\forall$ open $U \subseteq X$, $T(U) \subseteq Y$ is open.*

Proof. We have

$$\begin{aligned}Y &= T(X) \\ &= T\left(\bigcup_{n=1}^{\infty} B^X(0,n) \right) \\ &= T\left(\bigcup_{n=1}^{\infty} nB^X(0,1) \right) \\ &= \bigcup_{n=1}^{\infty} nT(B^X(0,1)).\end{aligned}$$

By the Baire Category Theorem, there is n_0, such that $\left(n_0\overline{T(B^X(0,1))}\right)^o = \left(\overline{n_0 T(B^X(0,1))}\right)^o \neq \emptyset$.
Thus there is $r_0 > 0, y_0 \in Y$, such that $B^Y(y_0, r_0) \subset n_0\overline{T(B^X(0,1))}$.
Notice that $B^Y(-y_0, r_0) \subset n_0\overline{T(B^X(0,1))}$ as well, and $n_0\overline{T(B^X(0,1))}$ is convex.
Thus, for any $y \in B^Y(0,r)$, we have $y = \frac{1}{2}(y-y_0)+\frac{1}{2}(y+y_0)$, where $y-y_0 \in B^Y(-y_0, r_0),\ y+y_0 \in B^Y(y_0, r_0)$.
By convexity, $y \in n_0\overline{T(B^X(0,1))}$.
Thus $B^Y(0, r_0) \subset n_0\overline{T(B^X(0,1))}$.
Take $r := \frac{r_0}{n_0}$, we have $B^Y(0,r) \subset \overline{T(B^X(0,1))}$.
Now we want to show $\overline{T(B^X(0,1))} \subset T(B^X(0,2))$.
Let $y \in \overline{T(B^X(0,1))}$, there is $x_1 \in B^X(0,1)$, such that $\|y - Tx_1\| < \frac{r}{2}$.
Let $y_1 := y - Tx_1 \in B^Y(0, \frac{r}{2})\overline{T(B^X(0, \frac{1}{2}))}$.
Thus there is $x_2 \in B^X(0, \frac{1}{2})$, such that $\|y - Tx_1 - Tx_2\| = \|y_1 - Tx_2\| < \frac{r}{4}$.
Recursively, we can find $x_k \in B^X(0, \frac{1}{2^{k-1}})$, such that $\|y_k\| < \frac{r}{2^k}$ for $y_k := y - Tx_1 - \cdots - Tx_k$.
Since $\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k+1} = 2$, and X is complete, we have $x := \sum_{k=1}^{\infty} x_k$ converges in X, and $x \in B^X(0,2)$.
Since T is continuous, we have $y = \sum_{k=1}^{\infty} Tx_k = Tx \in T(B^X(0,2))$.
Thus $B^Y(0,r) \subset \overline{T(B^X(0,1))} \subset T(B^X(0,2))$.
Now for any open $U \subseteq X$, there is $\epsilon > 0$, such that $x + \frac{\epsilon}{2}B^X(0,2) = B^X(x,\epsilon) \subseteq U$.
Thus $B^Y(Tx, \frac{\epsilon}{2}r) = Tx + \frac{\epsilon}{2}B^Y(0,r) \subseteq Tx + \frac{\epsilon}{2}T(B^X(0,2)) \subseteq T(U)$.
Thus $T(U)$ is open. □

Theorem 3.23 (Banach Isomorphism Theorem)**.** *Let X, Y be Banach spaces, suppose $A \in B(X,Y)$ is bijective, then A^{-1} is continuous and bounded as well. Namely, A is invertible and is an isomorphism of $X \cong Y$ as Banach Spaces.*

Proof. A is open by the open mapping theorem. Now for any open $U \subseteq X$, we have that $\left(A^{-1}\right)^{-1}(U) = A(U)$ is open. Thus A^{-1} is continuous. □

Corollary 3.24. *Let X, Y be Banach spaces, suppose $T \in B(X,Y)$ is surjective, then $X/\ker(T) \cong Y$ as Banach spaces with $\tilde{T} : X/\ker(T) \to Y$ by $\tilde{T}([x]) := T(x)$.*

Proof. We can check that $\tilde{T}$ is a well-defined bijection.
Also, for any $x \in X, y \in \ker(T)$, we have

$$\begin{aligned} \left\|\tilde{T}([x])\right\| &= \|T(x)\| \\ &= \|T(x+y)\| \\ &\leq \|T\|\|x+y\|. \end{aligned}$$

Since this holds for all $y \in \ker(T)$, we have

$$\left\|\tilde{T}([x])\right\| \leq \inf_{y \in \ker(T)} \|T\|\|x+y\| = \|T\|\|[x]\|.$$

Thus, $\left\|\tilde{T}\right\| \leq \|T\|$, which means $\tilde{T} \in B(X/\ker(T), Y)$ is continuous.
By the Banach Isomorphism Theorem, $\tilde{T}^{-1}$ is continuous as well. □

Corollary 3.25. *Suppose X is a vector space that is complete under two different norms $\|\cdot\|_1, \|\cdot\|_2$, and $\exists C > 0,$ such that $\forall x \in X,\ \|x\|_1 \leq C\|x\|_2$, then the two norms are equivalent.*

Proof. Consider the map $id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$, we have that id is bijective and bounded by C. By the Banach Isomorphism Theorem, id^{-1} is bounded as well.
Thus $\forall x \in X$, we have $\|x\|_2 \leq \left\|id^{-1}\right\|\|x\|_1$ □

Corollary 3.26. *Let X be any finite-dimensional linear normed space over $\mathbb{F}$, then any two norms on X are equivalent and X is complete.*

Proof. Let $\|\cdot\|$ be any norm on X, and let $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{F}^n$.
Pick any basis $\{e_i\}_{i=1}^n$ for X.
Consider the function $T : (\mathbb{F}^n, \|\cdot\|_2) \to (X, \|\cdot\|)$ by $T(a) := \sum_{i=1}^n a^i e_i \in X$ for any $a = (a^i)_{i=1}^n \in \mathbb{F}^n$.
It is easy to check that T is linear and bijective. Also,

$$\begin{aligned}\|Ta\| &= \left\|\sum_{i=1}^n a^i e_i\right\| \\ &\leq \sum_{i=1}^n \left|a^i\right| \|e_i\| \\ &\leq \|a\|_2 \left(\sum_{i=1}^n \|e_i\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, $\|T\| \leq \alpha := \left(\sum_{i=1}^n \|e_i\|^2\right)^{\frac{1}{2}}$, so it is continuous.
Since $S := \{a \in \mathbb{F}^n : \|a\|_2 = 1\}$ is closed and bounded thus compact in $(\mathbb{F}^n, \|\cdot\|_2)$, and $\|\cdot\|$ is continuous, we have $r := \inf_{a\in S} \|T(a)\|$ is achieved in S by the Extreme Value Theorem.
Since $0 \notin S$, and $\|T(a)\| = 0 \implies T(a) = 0 \implies a = 0$, we have $r \neq 0$.
Consider any $a \neq 0 \in \mathbb{F}^n$ such that $\|T(a)\| \leq r$, we must have $\frac{a}{\|a\|_2} \in S$, so

$$\frac{1}{\|a\|_2}\|T(a)\| = \left\|T\left(\frac{a}{\|a\|_2}\right)\right\| \geq r.$$

Thus $\|a\|_2 \leq 1$.
This shows $\forall x \in X$ with $\|x\| \leq 1$, we must have $\left\|T^{-1}(x)\right\|_2 \leq \frac{1}{r}$, which show that $\left\|T^{-1}\right\| \leq \frac{1}{r} < \infty$ is bounded.
This shows that $T : (\mathbb{F}^n, \|\cdot\|_2) \to (X, \|\cdot\|)$ is an homeomorphism for any $\|\cdot\|$ on X, which means X is complete and all norms are equivalent. ☐

3.3.3 Closed Graph Theorem

Definition 3.20. Let X, Y be Banach spaces, we can consider $X \oplus Y := \{(x, y) : x \in X, y \in Y\}$, which is a vector space under component-wise addition and scale multiplication. For $1 \leq p < \infty$, we define

$$\|(x, y)\|_p := (\|x\|^p + \|y\|^p)^{\frac{1}{p}},$$

and

$$\|(x, y)\|_\infty := \max\{\|x\|, \|y\|\}.$$

Definition 3.21. Let X, Y be Banach spaces, and $D \subseteq X$ is a subspace, the **graph** of a linear map $T : D \to Y$ is

$$\mathcal{G}(T) := \{(x, Tx) : x \in D\}.$$

We say T is **closed** if $\mathcal{G}(T)$ is closed in $X \oplus_\infty Y$.

Theorem 3.27 (Closed Graph Theorem)**.** *Let X, Y be Banach spaces, a linear map $T : X \to Y$ is closed if and only if $T \in B(X, Y)$.*

Proof. $(\implies)$.
Consider the projection maps $\pi_1(x, y) := x, \pi_2(x, y) := y$, which are both continuous. Since $\mathcal{G}(T)$ is closed in $X \oplus Y$, it is a Banach Space. Since T is defined on entire X, we have that $\pi_1|_{\mathcal{G}(T)}$ is a continuous bijection. By the Banach Isomorphism Theorem, $(\pi_1|_{\mathcal{G}(T)})^{-1} : X \to \mathcal{G}(T)$ is bounded. Notice that $T = \pi_2 \circ (\pi_1|_{\mathcal{G}(T)})^{-1}$, which will also be bounded.

$(\impliedby)$.
Consider any sequence $((x_i, Tx_i))_{i\in\mathbb{N}}$ in $\mathcal{G}(T)$, such that $(x_i, Tx_i) \to (x, y) \in X \oplus_\infty Y$.
Thus $x_i \to x \in X$ and $Tx_i \to y \in Y$.
Since T is continuous, $Tx_i \to Tx$, which means $Tx = y$, so $(x, y) \in \mathcal{G}(T)$. ☐

Remark. It is important that T is defined on the entire X.

Example 3.3.2. Consider $X = Y = (C[0,1], \|\cdot\|_1)$, and $D = C^1([0,1])$. Let $T := \frac{d}{dx} : D \to C^1([0,1])$, which is unbounded but closed.

Corollary 3.28. *Let X, Y be Banach spaces, then for any linear map $T : X \to Y$ $T \in B(X,Y)$ if and only if $\forall (x_n)_{n=1}^{\infty}$ in X, such that $x_n \to 0$, $Tx_n \to y \in Y$, we have $y = 0$.*

Proof. ($\implies$) is easy.
($\impliedby$).
Consider any $((x_n, Tx_n))_{n=1}^{\infty}$ in $\mathcal{G}(T)$, such that $x_n \to x \in X$, and $Tx_n \to y \in Y$.
We have that $(x - x_n) \to 0$, and by linearity of T, we have $T(x - x_n) = Tx - Tx_n \to Tx - y$.
By assumption, we have $y - Tx = 0$, which means $y = Tx$.
Thus, $(x, y) \in \mathcal{G}(T)$, which means $\mathcal{G}(T)$ is closed. $\square$

3.4 Hahn-Banach Theorem

Definition 3.22. Let X be a normed linear space. $p : X \to \mathbb{R}$ is called a **sublinear functional** if $\forall t > 0, \forall x \in X,\ p(tx) = tp(x)$, and $p(x+y) \le p(x) + p(y)$.

Proposition 3.29. *Let X be a normed linear space, then $\|\cdot\|$ is always a sublinear functional.*

Proposition 3.30. *Let X be a Banach space, then for any $x \in X$, the functional $p_x : B(X) \to \mathbb{R}$ defined by $p_x(T) := \|Tx\|$ is sublinear.*

Proposition 3.31. *Let X be a Banach space, then for any $x \in X, \phi \in X^*$, the functional $p_{\phi,x} : B(X) \to \mathbb{F}$ defined by $p_{\phi,x}(T) := |\phi(Tx)|$ is sublinear.*

Theorem 3.32 (Extension)**.** *Let X be a linear vector space over $\mathbb{R}$. Let $M_0 \subseteq X$ be a linear subspace, and $p : X \to \mathbb{R}$ be a sublinear functional, then for any linear $f_0 : M_0 \to X$ such that $\forall x \in M_0,\ f_0(x) \le p(x)$, there is an extension $f : X \to \mathbb{R}$, such that $f|_{M_0} = f_0$, and $\forall x \in X,\ f(x) \le p(x)$.*

Proof. Consider

$$P := \{(M, f) | M_0 \subseteq M \subseteq X \text{ is a subspace; } f : M \to \mathbb{R} \text{ is linear, } f|_{M_0} = f_0;\ \forall x \in M,\ f(x) \le p(x)\},$$

with the partial order $(M, f) \le (M', f')$ if $M \subseteq M', f'|_M = f$.
Consider any chain $\{(M_\alpha, f_\alpha)\}_{\alpha \in A} \subset P$.
Let $M := \bigcup_{\alpha \in A} M_\alpha \subseteq X$, and let $f(x) := f_\alpha(x)$ for any $M_\alpha \ni x$.
We can check that f is well-defined and linear, satisfying the requirement.
Thus, $(M, f) \in P$ is an upper bound for the chain.
By Zorn's lemma, there is a maximal element (M_1, f_1) of P.
Suppose for contradiction that $M_1 \neq X$, then there is some $x \in X \setminus M_1$.
Notice that if we take any $m_1, m_2 \in M_1$, we have that

$$\begin{aligned} f_1(m_1) + f_1(m_2) &= f_1(m_1 + m_2) \\ &\le p(m_1 + m_2) \\ &\le p(m_1 - x) + p(m_2 + x). \end{aligned}$$

Thus, $f_1(m_1) - p(m_1 - x) \le p(m_2 + x) - f_1(m_2)$.
Since this holds for all $m_1, m_2 \in M_1$, we have

$$\sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \le \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)).$$

Take any $a \in \left[\sup_{m_1 \in M_1}(f_1(m_1) - p(m_1 - x)), \inf_{m_2 \in M_1}(p(m_2 + x) - f_1(m_2))\right]$.
Let $M := M_1 \oplus \operatorname{Span}\{x\}$. Since M_1 is a subspace, this is a direct sum, i.e, $\forall y \in M_1 \oplus \operatorname{Span}\{x\}$, there is some unique $m \in M_1,\ t \in \mathbb{R}$, such that $y = m + tx$.

Define $f : M \to \mathbb{R}$ by $f(m + tx) := f_1(m) + |t|a$ for any $t \in \mathbb{R}$.
We can easily check $f|_{M_1} = f_1$ and that f is linear.
Suppose $t > 0$, we have

$$\begin{aligned} f(m+tx) &= f_1(m) + |t|a \\ &= f_1(m) + ta \\ &\leq f_1(m) + t \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)) \\ &\leq f_1(m) + t\Big(p\Big(\frac{m}{t} + x\Big) - f_1\Big(\frac{m}{t}\Big)\Big) \\ &= f_1(m) + p\Big(t\Big(\frac{m}{t} + x\Big)\Big) - f_1\Big(t\frac{m}{t}\Big) \\ &= p(m+tx). \end{aligned}$$

Similarly, if $t \leq 0$, we have

$$\begin{aligned} f(m+tx) &= f_1(m) + |t|a \\ &= f_1(m) - ta \\ &\leq f_1(m) - t \sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \\ &\leq f_1(m) - t\Big(f_1\Big(\frac{m}{t}\Big) - p\Big(\frac{m}{t} - x\Big)\Big) \\ &= f_1(m) - f_1\Big(t\frac{m}{t}\Big) + p\Big(t\Big(\frac{m}{t} + x\Big)\Big) \\ &= p(m+tx). \end{aligned}$$

This contradicts with the maximality of (M_1, f_1).
Thus, $M_1 = X$, and $f_1 : X \to \mathbb{R}$ is the desired extension. $\square$

Theorem 3.33 (Hahn-Banach)**.** *Let X be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $M \subseteq X$ be a linear subspace, and linear $\phi_0 : M \to \mathbb{F}$ with $\|\phi_0\|_{M^*} < \infty$, then there is a norm preserving extension $\phi \in X^*$, such that $\phi|_M = \phi_0$, and $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$.*

Proof. First, consider $\mathbb{F} = \mathbb{R}$.
We note that $p : x \mapsto \|\phi_0\|_{M^*}\|x\|$ is sublinear, and $\forall x \in M,\ \phi_0(x) \leq \|\phi_0\|_{M^*}\|x\| = p(x)$.
Thus, there is a linear extension $\phi : X \to \mathbb{R}$, such that $\forall x \in X,\ \phi(x) \leq p(x) = \|\phi_0\|_{M^*}\|x\|$.
Also, $-\phi(x) = \phi(-x) \leq \|\phi_0\|_{M^*}\|-x\| = \|\phi_0\|_{M^*}\|x\|$.
Thus $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$.
Now suppose $\mathbb{F} = \mathbb{C}$.
Consider $g_0 := \Re\phi_0 : M \to \mathbb{R}$, which is $\mathbb{R}$-linear, and $\|g_0\|_{M^*_{\mathbb{R}}} \leq \|\phi_0\|_{M^*}$.
Using the real case, we can extend g_0 to $g \in X^*_{\mathbb{R}}$.
Now define $\phi : X \to \mathbb{C}$ by $\phi(x) := g(x) + ig(-ix)$.
We can see that ϕ is $\mathbb{R}$ linear.
Also,

$$\phi(ix) = g(ix) + ig(x) = i(g(x) - ig(ix)) = i(g(x) + ig(-ix)) = i\phi(x),$$

so ϕ is $\mathbb{C}$-linear.
For any $m \in M$, we have that

$$\begin{aligned} \phi(x) &= g(m) + ig(-im) \\ &= g_0(m) + ig_0(-im) \\ &= \Re(\phi_0(m)) + i\Re(\phi_0(-im)) \\ &= \Re(\phi_0(m)) + i\Re(-i\phi_0(m)) \\ &= \Re(\phi_0(m)) + i\Im(\phi_0(m)) \\ &= \phi_0(m). \end{aligned}$$

Thus $\phi|_M = \phi_0$.
Now consider any $x \in X$ with $\|x\| \leq 1$.
We have that

$$|\phi(x)| = \lambda\phi(x) = \phi(\lambda x) = g(\lambda x) + ig(-i\lambda x)$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.
Since g is real-valued, and $|\phi(x)| \in \mathbb{R}$, we must have

$$|\phi(x)| = g(\lambda x) \leq \|g\|_{X^*}\|\lambda x\| = \|g_0\|_{M^*_{\mathbb{R}}}\|x\| \leq \|\phi_0\|_{M^*}.$$

Thus $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$.
Lastly, we always have

$$\begin{aligned}\|\phi_0\|_{M^*} &= \sup_{x\in M,\ \|x\|\leq 1} |\phi_0(x)| \\ &= \sup_{x\in M,\ \|x\|\leq 1} |\phi(x)| \\ &\leq \sup_{x\in X,\ \|x\|\leq 1} |\phi(x)| \\ &= \|\phi\|_{X^*}.\end{aligned}$$

□

Corollary 3.34. *Let X be a normed linear space over $\mathbb{R}$. Let $M \subseteq X$ be a linear subspace, and linear $\phi_0 : M \to \mathbb{F}$ with $\|\phi_0\|_{M^*} < \infty$, then for any $x \in X, a \in \mathbb{R}$, there is a Hahn-Banach extension $\phi \in X^*$, such that $\phi|_M = \phi_0,\ \|\phi\|_{X^*} = \|\phi_0\|_{M^*}$ and $\phi(x) = a$ if and only if*

$$\begin{aligned}\sup_{m\in M}(\phi_0(m) - \|\phi_0\|_{M^*}\|m - x\|) \leq a &\leq \inf_{m\in M}(\phi_0(m) + \|\phi_0\|_{M^*}\|m - x\|) \\ &= \inf_{m\in M}(\|\phi_0\|_{M^*}\|m + x\| - \phi_0(m)).\end{aligned}$$

Proof. We first note that since M is a subspace, $m \in M$ if and only if $-m \in M$, and

$$\begin{aligned}\phi_0(m) + \|\phi_0\|_{M^*}\|m - x\| &= -\phi_0(-m) + \|\phi_0\|_{M^*}\|-(-m) - x\| \\ &= \|\phi_0\|_{M^*}\|(-m) + x\| - \phi_0(-m).\end{aligned}$$

Thus,

$$\inf_{m\in M}(\phi_0(m) + \|\phi_0\|_{M^*}\|m - x\|) = \inf_{m\in M}(\|\phi_0\|_{M^*}\|m + x\| - \phi_0(m)).$$

Consider any Hahn-Banach extension $\phi \in X^*$, such that $\phi|_M = \phi_0,\ \|\phi\|_{X^*} = \|\phi_0\|_{M^*}$.
For any $x \in X$, we must satisfy

$$\begin{aligned}|\phi_0(m) - \phi(x)| &= |\phi(m) - \phi(x)| \\ &= |\phi(m - x)| \\ &\leq \|\phi\|_{X^*}\|m - x\| \\ &= \|\phi_0\|_{M^*}\|m - x\|.\end{aligned}$$

Thus,

$$\phi_0(m) - \|\phi_0\|_{M^*}\|m - x\| \leq \phi(x) \leq \phi_0(m) + \|\phi_0\|_{M^*}\|m - x\|.$$

Since this holds for all $m \in M$, we have

$$\sup_{m\in M}(\phi_0(m) - \|\phi_0\|_{M^*}\|m - x\|) \leq \phi(x) \leq \inf_{m\in M}(\phi_0(m) + \|\phi_0\|_{M^*}\|m - x\|).$$

On the other hand, suppose

$$\sup_{m\in M}(\phi_0(m) - \|\phi_0\|_{M^*}\|m - x\|) \leq a \leq \inf_{m\in M}(\|\phi_0\|_{M^*}\|m + x\| - \phi_0(m)).$$

Taking $p : x \mapsto \|\phi_0\|_{M^*}\|x\|$ as in the proof of the Hahn-Banach Theorem, we have that

$$\sup_{m\in M}(\phi_0(m) - p(m-x)) \leq a \leq \inf_{m\in M}(p(m+x) - \phi_0(m)).$$

By the proof of the Hahn-Banach Theorem, we can always extend ϕ_0 on $M \oplus \mathrm{Span}\{x\}$ to

$$\tilde{\phi}(m+tx) := \phi_0(m) + |t|a$$

for any $t \in \mathbb{R}$. Also,

$$\tilde{\phi}(x) = \tilde{\phi}(0 + 1\cdot x) = \phi_0(0) + 1\cdot a = a.$$

Now we can take the Hahn-Banach extension of $\tilde{\phi}$ to be ϕ, which will satisfy $\phi(x) = \tilde{\phi}(x) = a$. □

Corollary 3.35. *Let X be a normed linear space. Given $0 \neq x \in X$, then there is $\phi \in X^*$, such that $\|\phi\| = 1$, and $\phi(x) = \|x\|$. In particular, $\|x\|_X = \sup\{|\phi(x)| : \phi \in X^*, \|\phi\| = 1\}$.*

Proof. Let $M = \mathrm{Span}\{x\}$, and define $\phi_0(\lambda x) := \lambda\|x\|$.
Then we have $\|\phi_0\| = 1$, and by Hahn-Banach Theorem, there is $\phi \in X^*$, such that $\|\phi\| = \|\phi_0\| = 1$, and $\phi(x) = \phi_0(x) = \|x\|$.
In particular, $\|x\| = \phi(x) = |\phi(x)|$, so $\|x\|_X \leq \sup\{|\psi(x)| : \psi \in X^*, \|\psi\| = 1\}$.
Also, for any $\psi \in X^*, \|\psi\| = 1$, we have $|\psi(x)| \leq \|\psi\|\|x\| = \|x\|$. □

Corollary 3.36. *Let X be a normed linear space, then X^* separates the points of X.*

Proof. For any $x \neq y$, we have $x - y \neq 0$, so there is $\phi \in X^*$, such that

$$\phi(x) - \phi(y) = \phi(x-y) = \|x-y\| \neq 0,$$

which separates x, y. □

Corollary 3.37. *Let X be a normed linear space, then there is a canonical linear isometric embedding $i : X \hookrightarrow X^{**}$ by $i(x) := \hat{x},\ \hat{x}(\phi) := \phi(x)$.*

Proof. We have $\|\hat{x}\| = \sup_{\|\phi\|=1}\|\hat{x}(\phi)\| = \sup_{\|\phi\|=1}|\phi(x)| = \|x\|$. □

Corollary 3.38. *Let X be a Banach space, and $A \subseteq X$. A is bounded if and only if for all $\phi \in X^*$, $\phi(A)$ is bounded.*

Proof. ($\Longrightarrow$): Assume A is bounded. Namely, there is $M > 0$ such that for all $x \in A$, $\|x\| \leq M$. Now for any $\phi \in X^*, x \in A$, we have

$$|\phi(x)| \leq \|\phi\|\|x\| \leq M\|\phi\|.$$

($\Longleftarrow$): Assume $\phi(A)$ is bounded for each ϕ. Namely, there is $M_\phi > 0$ such that for all $x \in A$, $|\phi(x)| \leq M_\phi$. Consider $i(A) \subseteq X^{**}$. For any $\hat{x} \in i(A)$, we have that $|\hat{x}(\phi)| = |\phi(x)| \leq M_\phi$. Thus, $i(A)$ is pointwise bounded. Since X^* is a Banach space, by the Banach-Steinhaus theorem 3.20, $i(A)$ is bounded. Since $\|x\| = \|\hat{x}\|$, we have that A is bounded as well. □

Remark. It is not necessarily that $X \cong X^{**}$, Indeed, if $X = C_0(\mathbb{N})$, we have that $X^* \cong \ell^1(\mathbb{N})$, and $X^{**} \cong \ell_\infty(\mathbb{F})$.

Definition 3.23. A Banach space X is **reflexive** if $i(X) = X^{**}$, where $i : X \hookrightarrow X^{**}$ is the canonical linear isometric embedding by $i(x) := \hat{x},\ \hat{x}(\phi) := \phi(x)$.

Corollary 3.39. *Let X be a normed linear space. For any closed subspace M, and $x \notin M$, there is $f \in X^*$, such that $\|f\| = 1, f|_M = 0$, and $f(x) = \mathrm{dist}(x, M)$.*

Proof. Consider the quotient map $Q : X \to Y$, where $Y := X/M$. Since $x \notin M$, we have $[x] \neq 0$.
Let $\phi \in Y^*$, such that $\|\phi\|_{Y^*} = 1$, and $\phi([x]) = \|[x]\|_Y = \inf_{m\in M}\|x+m\| = \mathrm{dist}(x, M)$.
Let $f := \phi \circ Q$, we have that $f(x) = \mathrm{dist}(x, M)$, and $\forall m \in M,\ f(m) = \phi(m+M) = \phi(0) = 0$.
Also, $\|f\| \leq \|\phi\|\|Q\| \leq \|\phi\| = 1$. □

Definition 3.24. Let X be a Banach Space. For $Y \subseteq X$, the **annihilator** of Y is

$$Y^{\perp} := \{\varphi \in X^* : \varphi(Y) = \{0\}\}.$$

For $Z \subseteq X^*$, the **preannihilator** of Z is

$$Z_{\perp} : \{x \in X : \hat{x}|_Z = 0\} = Z^{\perp} \cap X.$$

Proposition 3.40. *Let X be a Banach Space, then*

$$(Y^{\perp})_{\perp} = \overline{\mathrm{Span}(Y)}.$$

Proof. Let $M := \overline{\mathrm{Span}(Y)}$.
For any $y \in M, f \in Y^{\perp}$, we have $\hat{y}(f) = f(y) = 0$, so $y \in (Y^{\perp})_{\perp}$. Thus, $M \subseteq (Y^{\perp})_{\perp}$.
Suppose $x \notin Y$, then there is $f \in X^*$ such that $f|_M = 0$, and $f(x) = \mathrm{dist}(x, M)$. In particular, $f \in Y^{\perp}$ and $\hat{x}(f) = f(x) \neq 0$.
Thus, $x \notin (Y^{\perp})_{\perp}$.
This shows $(Y^{\perp})_{\perp} \subseteq M$.
Thus,

$$M = (Y^{\perp})_{\perp}.$$

□

4 Hilbert Spaces

See more in Prof Tran's notes for Amath731.

In this section, we will always assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

4.1 Sesquilinear Forms

Definition 4.1. Let X be a vector space over $\mathbb{F}$. A **bilinear form** on X is a map $B : X \times X \to \mathbb{F}$ that is linear in both entries. Namely, for all $x, y, z \in X, a \in \mathbb{F}$,

$$B[x, ay + z] = B[x, y] + aB[x, z],\ B[x + ay, z] = B[x, z] + aB[y, z].$$

A **sesquilinear form** on X is a map $M : X \times X \to \mathbb{F}$ that is linear in the second entry, conjugate linear in the first entry. Namely,

$$B[x, ay + z] = B[x, y] + aB[x, z],\ B[x + ay, z] = B[x, z] + \bar{a}B[y, z].$$

When $\mathbb{F} = \mathbb{R}$, considering $\bar{a} = a$, then conjugate linear and linear are the same, and sesquilinear and bilinear are the same.

Definition 4.2. Let X be a vector space over $\mathbb{F}$. A bilinear form B is **symmetric** if

$$\forall x, y \in X,\ B[x, y] = B[y, x].$$

A sesquilinear form M is **conjugate symmetric** or **Hermitian** if

$$\forall x, y \in X,\ B[x, y] = \overline{B[y, x]}.$$

When $\mathbb{F} = \mathbb{R}$, considering $\bar{a} = a$, then conjugate symmetry and symmetry are the same.

Remark. It is sufficient to check (conjugate) linearity in the second entry and (conjugate) symmetry to show B or M is a (conjugate) symmetric (sesquilinear) bilinear form.

Proposition 4.1. *Let X be a vector space over $\mathbb{F}$, with a conjugate symmetric sesquilinear form M. For all $x, y \in H$, we have*

1. $M[x,y] + M[y,x] = 2\Re(M[x,y]) = 2\Re(M[y,x])$*, which is twice the real part of* $M[x,y]$*.*

2. $M[x,y] - M[y,x] = 2\Im(M[x,y]) = -2\Im(M[y,x])$*, which is twice the imaginary part of* $M[x,y]$*.*

3. $M[x,x] \in \mathbb{R}$*.*

4. $M[x,y]M[y,x] = |M[x,y]|^2 = |M[y,x]|^2$*.*

5. $M[x+y, x+y] = M[x,x] + 2\Re(M[y,x]) + M[y,y]$*.*

6. $M[x-y, x-y] = M[x,x] - 2\Re(M[x,y]) + M[y,y]$*.*

Proof. 1. $M[x,y] + M[y,x] = M[x,y] + \overline{M[x,y]} = 2\Re(M[y,x])$.

2. $M[x,y] - M[y,x] = M[x,y] - \overline{M[x,y]} = 2\Im(M[x,y])$.

3. $2\Im(M[x,y]) = M[x,x] - M[x,x] = 0$.

4. $M[x,y]M[y,x] = M[x,y]\overline{M[x,y]} = |M[x,y]|^2$.

5.

$$\begin{aligned} M[x+y, x+y] &= M[x+y, x] + M[x+y, y] \\ &= M[x,x] + M[y,x] + M[x,y] + M[y,y] \\ &= M[x,x] + M[y,x] + \overline{M[y,x]} + M[y,y] \\ &= M[x,x] + 2\Re(M[y,x]) + M[y,y]. \end{aligned}$$

6. By 4., we have

$$\begin{aligned} M[x-y, x-y] &= M[x+(-y), x+(-y)] \\ &= M[x,x] + 2\Re(M[-y,x]) + M[-y,-y] \\ &= M[x,x] - 2\Re(M[y,x]) + M[y,y]. \end{aligned}$$

□

Definition 4.3. Let X be a vector space over $\mathbb{F}$. A (conjugate) symmetric (sesquilinear) bilinear form $M : X \times X \to \mathbb{F}$ is **positive semidefinite** on X if it satisfies

$$\forall x \in X,\ M[x,x] \geq 0.$$

A (conjugate) symmetric (sesquilinear) bilinear form $M : X \times X \to \mathbb{F}$ is **positive definite** on X if it satisfies

$$\forall x \neq 0 \in X,\ M[x,x] > 0.$$

Notice that if M is positive definite, it is always positive semidefinite.

4.2 Inner Product Spaces

Definition 4.4. An **inner product space** is a vector space H that has an inner product $\langle -, \cdot \rangle : H \times H \to \mathbb{F}$, which is a positive definite (conjugate) symmetric (sesquilinear) bilinear form. Namely, $\forall u, v, w \in H, a, b \in \mathbb{F}$, it satisfies

1. conjugate symmetry; i.e. $\langle v, w \rangle = \overline{\langle w, v \rangle}$,

2. linearity in the second argument; i.e. $\langle v, au + bw \rangle = a\langle v, u \rangle + b\langle v, w \rangle$, and

3. positive definiteness; i.e. $\langle v, v \rangle \geq 0$, and if $v \neq 0$, we must have $\langle v, v \rangle > 0$.

Proposition 4.2. *For every inner product space with* $\langle -, \cdot \rangle$*, and for all* $x, y \in H$*, we have*

1. $\langle x, y\rangle + \langle y, x\rangle = 2\Re(\langle x, y\rangle) = 2\Re(\langle y, x\rangle)$.
2. $\langle x, y\rangle - \langle y, x\rangle = 2\Im(\langle x, y\rangle) = -2\Im(\langle y, x\rangle)$.
3. $\langle x, y\rangle\langle y, x\rangle = |\langle x, y\rangle|^2$.
4. $\langle x + y, x + y\rangle = \langle x, x\rangle + 2\Re(\langle y, x]\rangle + \langle y, y\rangle$.
5. $\langle x - y, x - y\rangle = \langle x, x\rangle - 2\Re(\langle x, y\rangle) + \langle y, y\rangle$.

Theorem 4.3 (Cauchy-Schwarz)**.** *For every inner product space H,*

$$\forall u, v \in H, |\langle u, v\rangle| \leq \|u\|\|v\|,$$

where we define $\|x\| = \sqrt{\langle x, x\rangle}$ or any $x \in H$.
In particular, when $\|u\| \neq 0, \|u\|^2\|v\|^2 - |\langle u, v\rangle|^2 = \|z\|^2$, where $z := \|u\|v - \frac{\langle u,v\rangle}{\|u\|}u$.

Proof. Notice that this is trivially true and equality holds to be zero when $u = 0$.
Now we assume $u \neq 0$, then $\|u\| = \sqrt{\langle u, u\rangle} > 0$.

$$\begin{aligned}
\|z\|^2 &= \langle z, z\rangle \\
&= \left\langle \|u\|v - \frac{\langle u, v\rangle}{\|u\|}u, \|u\|v - \frac{\langle u, v\rangle}{\|u\|}u \right\rangle \\
&= \|u\|^2\langle v, v\rangle - \langle u, v\rangle\langle u, v\rangle - \overline{\langle u, v\rangle}\langle v, u\rangle + \frac{\langle v, u\rangle\langle u, v\rangle}{\cancel{\|u\|^2}}\cancelto{\|u\|^2}{\langle u, u\rangle} \\
&= \|u\|^2\|v\|^2 - |\langle u, v\rangle|^2 - \cancel{|\langle v, u\rangle|^2} + \cancel{|\langle v, u\rangle|^2} \\
&= \|u\|^2\|v\|^2 - |\langle u, v\rangle|^2.
\end{aligned}$$

Now $\|z\|^2 = \langle z, z\rangle \geq 0$, we have the result.

$\square$

Remark. This proof does not use the positive definiteness of the inner product, so it also works for a sesquilinear positive semidefinite form M. i.e.

$$|M[x, y]| \leq M[x, x]M[y, y].$$

Proposition 4.4. *For every inner product space with $\langle -, \cdot\rangle$, there is a norm $\|x\| = \sqrt{\langle x, x\rangle}$.*

Proof. Consider any $x \in H, a \in \mathbb{C}$,

$$\begin{aligned}
\|a \cdot x\| &= \sqrt{\langle ax, ax\rangle} = \sqrt{a^* a\langle x, x\rangle} = \sqrt{|a|^2}\sqrt{\langle x, x\rangle} = |a|\|x\| \\
\forall x \neq 0, \|x\| &= \sqrt{\langle x, x\rangle} > 0 \\
\|0\| &= \sqrt{\langle 0, 0\rangle} = 0 \\
\|x + y\|^2 &= \langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\Re(\langle x, y\rangle) \\
&\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\
&\leq (\|x\| + \|y\|)^2.
\end{aligned}$$

Thus $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm. $\square$

Corollary 4.5. *For every inner product space, there is a metric $d(x, y) = \sqrt{\langle x - y, x - y\rangle}$*

Proposition 4.6 (polarization identity)**.** *Let H be an inner product space over $\mathbb{F}$. The inner product is completely determined by the induced norm. Indeed, for any $x, y \in H$, we have*

$$\langle x, y\rangle = \frac{1}{4} \sum_{\epsilon \in \{\pm i, \pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if $\mathbb{F} = \mathbb{C}$, and

$$\langle x, y\rangle = \frac{1}{2} \sum_{\epsilon \in \{\pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if $\mathbb{F} = \mathbb{R}$.

Proposition 4.7. *If $\forall v, \langle v, u\rangle = 0$, then $u = 0$.*

Proposition 4.8. *For an Inner product space $H, \forall y, x = \lim\limits_{i\to\infty} x_i \in H$, we have*

$$\langle x, y\rangle = \lim_{i\to\infty} \langle x_i, y\rangle.$$

Proof. Given any $\epsilon > 0$, let $\epsilon_0 = \frac{\epsilon}{\|y\|}$.
Since $x = \lim_{i\to\infty} x_i$, we can find $N > 0$, such that $\forall n > N, \|x - x_n\| < \epsilon_0$,
thus $|\langle x, y\rangle - \langle x_n, y\rangle| = |\langle x - x_n, y\rangle| \leq \|x - x_n\|\|y\| < \epsilon_0\|y\| = \epsilon$ ∎

Corollary 4.9. *For an Inner product space $H, \forall y, x = \lim\limits_{i\to\infty} x_i \in H$, we have $\langle y, x\rangle = \lim\limits_{i\to\infty} \langle y, x_i\rangle$.*

Definition 4.5. An inner product space $\mathcal{H}$ is called a **Hilbert space** if it is complete.

4.3 Orthornormal Basis

Definition 4.6. Let H be an inner product space. Two vectors $u, v \in H$ are called **orthogonal** if $\langle u, v\rangle = 0$.

Definition 4.7. Let H be an inner product space. A set $\{e_i\}_{i\in I} \subseteq H$ is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j\rangle = \delta_{ij}.$$

Definition 4.8. Let H be an inner product space. An orthonormal set $\{e_i\}_{i\in I} \subseteq H$ is called a **maximal orthonormal set / orthonormal basis / total orthonormal set** if $\mathrm{Span}(\{e_i\}_{i\in I})$ is dense in H. Namely,

$$H = \overline{\mathrm{Span}(\{e_i\}_{i\in I})}.$$

Definition 4.9. Let $\mathcal{H}$ be a Hilbert space. $S \subseteq \mathcal{H}$, the subspace **orthogonal** to S is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v\rangle = 0, \forall u \in S\}.$$

Lemma 4.10. *Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$, we always have $S^\perp$ is a subspace of $\mathcal{H}$.*

Definition 4.10. Let V be a vector space, and $U, W \subseteq V$ be two subspaces, we say $V = U \oplus W$, if $V = U + W = \{v + w : v \in V, w \in W\}$, and $V \cap W = \{0\}$.

Proposition 4.11. *Let V be a vector space, and $U, W \subseteq V$ be two subspaces. $V = U \oplus W$, if and only if $\forall v \in V$, it can be uniquely written as $v = u + w$, where $u \in U, w \in W$.*

Theorem 4.12. *Let $\mathcal{H}$ be a Hilbert space, if $S \subseteq \mathcal{H}$ is a subspace, then*

$$\mathcal{H} = \bar{S} \oplus S^\perp.$$

Theorem 4.13. *Let $\mathcal{H}$ be a Hilbert space, and $\{e_i\}_{i\in I} \subseteq \mathcal{H}$ be an orthonormal set, with $M := \overline{\mathrm{Span}\{e_i\}_{i\in I}}$, then*

1. *The set* $\{e_i\}_{i\in I}$ *is linearly independent, and*

$$\mathrm{dist}(e_i, \overline{\mathrm{Span}\,\{e_j\}_{j\in I, j\neq i}}) = 1.$$

2. *(Bessel'sinequality) For any* $x \in \mathcal{H}$*, we have*

$$\sum_{i\in I} |\langle e_i, x\rangle|^2 \leq \|x\|^2.$$

3. *If* $\{a_i\}_{i\in I} \subseteq \mathbb{F}, \sum_{i\in I} |a_i|^2 = L^2$*, we have* $\sum_{i\in I} a_i e_i$ *converges unconditionally to* $x \in \mathcal{H}$ *with* $\|x\| = L$*. Moreover, in this case,* $\langle e_i, x\rangle = a_i$*.*

4. *We have* $P : \mathcal{H} \to \mathcal{H}$ *defined by*

$$P(x) := \sum_{i\in I} \langle e_i, x\rangle e_i$$

is the projection onto M*. Namely,* $P \in B(\mathcal{H}), \|P\| = 1, P^2 = P, \mathrm{Im}(P) = M, \ker(P) = M^\perp$*, and* $Px = x$ *if and only if* $x \in M$*.*

5. *(Parseval Identity)* $\forall x \in M, \|x\|^2 = \sum_{i\in I} |\langle e_i, x\rangle|^2$*.*

Theorem 4.14 (generalized Fourier series)**.** *Let* $\mathcal{H}$ *be a Hilbert space, and* $\{e_i\}_{i\in I} \subseteq \mathcal{H}$ *be an orthonormal set, then the following are equivalent:*

1. $\{e_i\}_{i\in I}$ *is an orthonormal basis*

2. *If* $\forall i \in I, \langle x, e_i\rangle = 0$*, then* $x = 0$*.*

3. *(Fourier series)* $\forall x \in \mathcal{H}, x = \sum_{i\in I} \langle e_i, x\rangle e_i$*.*

4. *(Parseval Identity)*

$$\forall x \in \mathcal{H},\ \|x\|^2 = \sum_{i\in I} |\langle e_i, x\rangle|^2.$$

Theorem 4.15. $\mathcal{H}$ *is a separable Hilbert space, if and only if there is a maximal orthonormal set in* $\mathcal{H}$*. Moreover, in this case, every maximal orthonormal set is at most countable.*

4.4 Dual of Hilbert Space

Theorem 4.16. *[Riesz-Frechet Representation theorem] Let* $\mathcal{H}$ *be a Hilbert space, then for each* $u^* \in \mathcal{H}, \exists! u \in \mathcal{H}$*, such that* $\forall v \in \mathcal{H}, \langle u^*|v\rangle_{\mathcal{H}^*,\mathcal{H}} = \langle u, v\rangle_{\mathcal{H}}$*, and* $\|u^*\|_{\mathcal{H}^*} = \|u\|_{\mathcal{H}}$*.*

Definition 4.11. Let X be a vector space over $\mathbb{F}$. We define

$$\bar{X} := \{\bar{x} : x \in X\}$$

to be the **conjugate vector space**. The scalar multiplication is given by $\lambda\bar{x} = \overline{\bar{\lambda}x}$.

Proposition 4.17. *Let* X *be a normed space over* $\mathbb{F}$*. The map* $X \to \bar{X}$ *by* $x \mapsto \bar{x}$ *is an isometric antilinear bijection.*

Corollary 4.18. *Let* $\mathcal{H}$ *be a Hilbert space, then* $\bar{\mathcal{H}} \cong \mathcal{H}^*$*, where the map* $\Phi : \mathcal{H} \to \mathcal{H}^*;\ u \mapsto \langle u, \cdot\rangle_{\mathcal{H}}$ *is the* ***canonical bijective isometric antilinear isomorphism****.*

Corollary 4.19. *Every Hilbert space is reflexive.*

4.5 Bounded Sesquilinear Forms

Definition 4.12. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle -, \cdot \rangle$. A **bounded sesquilinear form** on $\mathcal{H}$ is a sesquilinear form 4.1 that satisfies

$$\|B\| := \sup_{\|x\|=\|y\|=1} |B(x, y)| < \infty.$$

Lemma 4.20. *The quantity $\|B\|$ is called the **norm** of B, and the bounded sesquilinear forms become a normed vector space with this norm.*

Theorem 4.21. *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle -, \cdot \rangle$. $B(\mathcal{H})$ is isometrically isomorphic to the space of bounded sesquilinear forms, with $B_T[x, y] := \langle x, Ty \rangle$ for $T \in B(\mathcal{H}), x, y \in \mathcal{H}$.*

Proof. Firstly, given any $T \in B(\mathcal{H})$. Clearly $B_T[x, y]$ is sesquilinear. Indeed,

$$\begin{aligned}
B[x, y + az] &= \langle x, T(y + az) \rangle \\
&= \langle x, Ty + aTz \rangle \\
&= a\langle x, Ty \rangle + a\langle x, Tz \rangle \\
&= B[x, y] + aB[x, z], \\
B[x + ay, z] &= \langle x + ay, Tz \rangle \\
&= \langle x, z \rangle + \bar{a}\langle x, Tz \rangle \\
&= B[x, y] + \bar{a}B[x, z].
\end{aligned}$$

Now, for any $\|x\| = \|y\| = 1$, we have

$$\begin{aligned}
|B_T[x, y]| &= |\langle x, Ty \rangle| \\
&\leq \|x\|\|Ty\| \\
&\leq \|x\|\|T\|\|y\| \\
&= \|T\|.
\end{aligned}$$

Thus, $\|B_T\| \leq \|T\|$. Also, for any $\epsilon > 0$, we can find $\|y\| = 1$, such that $r := \|Ty\| > \sup_{\|y\|=1} \|Ty\| - \epsilon = \|T\| - \epsilon$. Take $x := \frac{Ty}{r}$ with $\|x\| = 1$, we have

$$\begin{aligned}
\|B_T\| &\geq |B_T[x, y]| \\
&= |\langle x, Ty \rangle| \\
&= \left| \left\langle \frac{Ty}{r}, Ty \right\rangle \right| \\
&= \frac{1}{r} |\langle Ty, Ty \rangle| \\
&= \frac{1}{r} \|Ty\|^2 \\
&= \|Ty\| \\
&> \|T\| - \epsilon.
\end{aligned}$$

Since this holds for any $\epsilon > 0$, we must have $\|B_T\| \geq \|T\|$. This proves

$$\|B_T\| = \|T\|.$$

On the other hand, given any bounded sesquilinear form B. Fix any $y \in \mathcal{H}$. Consider $\phi : \mathcal{H} \to \mathbb{F}$ by $\phi(x) := \overline{B[x, y]}$. Notice that

$$\begin{aligned}
\phi(ax + z) &= \overline{B[ax + z, y]} \\
&= \overline{\bar{a}B[x, y] + B[z, y]} \\
&= a\overline{B[x, y]} + \overline{B[x, y]} \\
&= a\phi(x) + \phi(z),
\end{aligned}$$

so ϕ is linear. Also, for any $\|x\| = 1$, we have

$$\begin{aligned}
|\phi(x)| &= \left|\overline{B[x,y]}\right| \\
&= |B[x,y]| \\
&= \left|\|y\|B\left[x, \frac{y}{\|y\|}\right]\right| \\
&= \|y\|\left|B\left[x, \frac{y}{\|y\|}\right]\right| \\
&\leq \|y\|\|B\| \\
&< \infty,
\end{aligned}$$

so $\phi \in \mathcal{H}^*$. By the Riesz-Frechet Representation theorem 4.16, there is a unique $z \in \mathcal{H}$, such that

$$\langle z, x\rangle = \phi(x) = \overline{B(x,y)}.$$

Now define $T(y) := z$, we have

$$B[x,y] = \overline{\langle z, x\rangle} = \langle x, z\rangle = \langle x, Ty\rangle.$$

In addition, for any $x, y, z \in \mathcal{H}$, and $a \in \mathbb{F}$, we have

$$\begin{aligned}
\langle x, T(y+az)\rangle &= B[x, y+az] \\
&= B[x,y] + aB[x,z] \\
&= \langle x, Ty\rangle + a\langle x, Tz\rangle \\
&= \langle x, Ty + aTz\rangle.
\end{aligned}$$

Thus, T is linear. Also, since $\|Ty\| = \|z\| = \|\phi\| \leq \|y\|\|B\|$, we have that $\|T\| \leq \|B\|$, so $T \in B(\mathcal{H})$. This shows that $T \mapsto B_T$ is surjective.

Since $T \mapsto B_T$ is clearly linear, and we have shown it is linear and isometric, it is an isometric isomorphism. □

5 Locally Convex Topological Vector Spaces and Weak Topology

5.1 Locally Convex Topological Vector Spaces

Definition 5.1. Let X be a vector space over $\mathbb{F}$, a **semi-norm** is a map $p : X \to [0, \infty)$ such that $\forall t \in \mathbb{F}, x, y \in X$,

$$p(tx) = |t|p(x),$$
$$p(x+y) \leq p(x) + p(y).$$

The null space of p is denoted $N_p := \{x \in X : p(x) = 0\}$.

Remark. Notice that p is a norm if and only if $N_p = \{0\}$.

Definition 5.2. A **Locally Convex Topological Vector Space** is a vector space X over $\mathbb{F}$ with a family of semi-norms $P = \{p_\alpha\}_{\alpha \in A}$, such that $\bigcap_{p \in P} \ker(p) = \{0\}$.
$\mathcal{T}_P$ is the topology on X generated by the convex sets

$$U(x,p,r) := \{y \in X : p(y-x) < r\}$$

for $x \in X, r > 0, p \in P$.

Definition 5.3. Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, for $r > 0, x_0 \in X$, and a finite subset $F \subseteq P$, we define

$$U_{F,r}(x_0) := \{x \in X : \forall p \in F,\ p(x - x_0) < r\},$$

and $U_{F,r} := U_{F,r}(0)$.

Proposition 5.1. *Each $U_{F,r}(x_0)$ is a finite intersection of $\bigcap_{p\in F} U(x,p,r)$, so it is open. Also, each $U_{F,r}(x_0) = x_0 + U_{F,r}$.*

Proposition 5.2. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, then for any $x_0 \in X$ the sets $\{U_{F,r}(x_0)\}_{\text{finite } F\subseteq P, r>0}$ form a neighbourhood basis at x_0.*

Proof. Consider any open $U \ni 0$, then there are $p_1, \ldots, p_n \in P,\ r_1, \ldots, r_n > 0,\ x_1, \ldots, x_n \in X$, such that

$$0 \in \bigcap_{i=1}^{n} U(x_i, p_i, r_i) \subseteq U.$$

Let $r := \min_{i\in[n]}(r_i - p_i(x_i)) > 0$, and let $F := \{p_1, \ldots, p_n\}$.
Consider any $x \in U_{F,r}$, we have for any $i \in [n]$,

$$\begin{aligned} p_i(x_i - x) &\leq p_i(x_i) + p_i(x) \\ &\leq p_i(x_i) + r \\ &\leq p_i(x_i) + r_i - p_i(x_i) \\ &= r_i. \end{aligned}$$

Thus $x \in \bigcap_{i=1}^{n} U(x_i, p_i, r_i) \subseteq U$.
This shows $0 \in U_{F,r} \subseteq U$.
Thus, $\{U_{F,r}\}_{\text{finite } F\subseteq P, r>0}$ form a neighbourhood basis at 0. By translation, $\{U_{F,r}(x_0)\}_{\text{finite } F\subseteq P, r>0}$ form a neighbourhood basis at x_0. □

Proposition 5.3. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, it is Hausdorff.*

Proof. Given any $x \neq y \in X$, then $x - y > 0$.
There is $p \in P$ such that $r = p(x - y) > 0$.
Now $U(x, p, \frac{r}{2}) \ni x$ and $U(y, p, \frac{r}{2}) \ni y$ has empty intersection. □

Proposition 5.4. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, then the addition map $A : X \times X \mapsto X$ and scalar multiplication map $B : \mathbb{F} \times X \to X$ are continuous.*

Proof. Given any open set U, with $x_0 + y_0 \in U$ for some $x_0, y_0 \in X$.
Since $\{U_{F,r}(x_0 + y_0)\}_{\text{finite } F\subseteq P, r>0}$ form a neighbourhood basis at $x_0 + y_0$, there is some finite $F \subseteq P$ and $r > 0$, such that

$$U_{F,r}(x_0 + y_0) \subseteq U,$$

since they form a neighbourhood basis.
We claim that $A^{-1}(U_{F,r}(x_0 + y_0)) \supseteq (x_0 + U_{F,\frac{r}{2}}) \times (y_0 + U_{F,\frac{r}{2}})$, which is open.
Indeed, take any $x \in x_0 + U_{F,\frac{r}{2}}, y \in y_0 + U_{F,\frac{r}{2}}$ and $p \in F$, we have

$$\begin{aligned} p((x + y) - (x_0 + y_0)) &\leq p(x - x_0) + p(y - y_0) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

Thus, we have find

$$(x_0, y_0) \in (x_0 + U_{F,\frac{r}{2}}) \times (y_0 + U_{F,\frac{r}{2}}) \subseteq A^{-1}(U_{F,r}(x_0 + y_0)) \subseteq A^{-1}(U).$$

Since this hold for all $(x_0, y_0) \in A^{-1}(U)$, we have that $A^{-1}(U)$ is open. □

Proposition 5.5. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, a net $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x \in X$ if and only if $\forall p \in P,\ p(x - x_\lambda) \to 0$.*

Proof.

$$(x_\lambda)_{\lambda\in\Lambda} \to x$$

if and only if

$$\forall \text{finite } F \subseteq P, r > 0,\ \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0,\ x_\lambda \in U_{F,r}(x)$$

if and only if

$$\forall p \in P, r > 0,\ \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0,\ x_\lambda \in U(x, p, r)$$

if and only if

$$\forall p \in P, r > 0,\ \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0,\ p(x_\lambda - x) < r$$

if and only if

$$\forall p \in P,\ p(x_\lambda - x) \to 0.$$

□

Proposition 5.6. *Let $(X, \|\cdot\|)$ be a normed vector space, then taking $P := \{\|\cdot\|\}$, we have a Locally Convex Topological Vector Space.*

5.2 Weak Topology

Proposition 5.7. *Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then $P := \{p_\phi\}_{\phi\in Y}$ given by $p_\phi(x) := |\phi(x)|$ gives a Locally Convex Topological Vector Space $(X, \mathcal{T}_Y)$, where $\mathcal{T}_Y := \mathcal{T}_P$.*

Proof. Clearly, $\forall t \in \mathbb{F}$, $p_\phi(tx) = |\phi(tx)| = |t\phi(x)| = |t||\phi(x)| = |t|p_\phi(x)$.
Also, $p_\phi(x + y) = |\phi(x + y)| = |\phi(x) + \phi(y)| \leq |\phi(x)| + |\phi(y)| = p_\phi(x) + p_\phi(y)$.
Thus, each p_ϕ is a semi-norm.
Suppose $p_\phi(x) = 0$ for all $\phi \in Y$, then $\phi(x) = 0$ for all $\phi \in Y$. Since Y separates the points, we must have $x = 0$. □

Remark. If Y is not a subspace but just a subset, we can WLOG take $Y' = \mathrm{Span}(Y)$, which will generate the same topology.

Definition 5.4. Let $(X, \|\cdot\|)$ be a normed vector space. The **weak topology on X** $\sigma(X, X^*)$ is $(X, \mathcal{T}_{X^*})$, where we take $Y = X^*$, which separates points by the Hahn-Banach Theorem. We say $(x_\lambda)_{\lambda\in\Lambda}$ **converges weakly to** $x \in X^*$ if it converges in the weak topology, denoted $x_\lambda \rightharpoonup x$.
Also, the **weak-* topology on** X^* $\sigma(X^*, X)$ is $(X^*, \mathcal{T}_X)$, where we take $Y = X \subseteq X^{**}$. We say $(\phi_\lambda)_{\lambda\in\Lambda}$ **converges weakly to** $\phi \in X^*$ if it converges in the weak-* topology, denoted $\phi_\lambda \rightharpoonup \phi$.

Proposition 5.8. *Let $(X, \|\cdot\|)$ be a normed vector space. $x_\lambda \rightharpoonup x$ in X if and only if $\forall \phi \in X^*,\ \phi(x_\lambda) \to \phi(x)$. Also, $\phi_\lambda \rightharpoonup \phi$ in X^* if and only if $\forall x \in X,\ \phi_\lambda(x) \to \phi(x)$.*

Proposition 5.9. *Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then if $x_\lambda \to x$ in norm, it also converges in $(X, \mathcal{T}_Y)$.*

Proposition 5.10. *Let $(X, \|\cdot\|)$ be a normed vector space, $(u_k)_{k=1}^\infty \subset X$ be a sequence, then*

1. *If $u_k \to u$, we always have $u_k \rightharpoonup u$.*
2. *If $u_k \rightharpoonup u$, we have that u is unique.*
3. *If $u_k \rightharpoonup u$, we have $(u_k)_{k=1}^\infty$ is bounded.*
4. *If $u_k \rightharpoonup u$, every subsequence $(u_{k_j})_{j=1}^\infty$ also converges weakly to u.*

Theorem 5.11. *Let X be a reflexive Banach Space, and $(u_k)_{k=1}^\infty \subset X$ be a bounded sequence, then $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.*

Proposition 5.12. *Let $\mathcal{H}$ be a Hilbert space, then $u_k \rightharpoonup u$ if and only if $\forall v \in \mathcal{H}, \langle v, u_k\rangle \to \langle v, u\rangle$ as real numbers.*

Proof. Suppose $u_k \rightharpoonup u$.
Notice that for all $v \in \mathcal{H}$, we have that $v^\dagger \in \mathcal{H}^*$, and thus $\langle v, u_k \rangle = \langle v^\dagger | u_k \rangle \to \langle v^\dagger | u \rangle = \langle v, u \rangle$.
Now suppose $\forall v \in \mathcal{H}, \langle v, u_k \rangle \to \langle v, u \rangle$.
Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 4.16, there is some $f^\dagger \in \mathcal{H}$, such that

$$\langle f | u_{k_j} \rangle = \langle f^\dagger, u_{k_j} \rangle \to \langle f^\dagger, u \rangle = \langle f | u \rangle.$$

Thus, $u_{k_j} \rightharpoonup u$. □

Proposition 5.13. *Let $(X, \|\cdot\|)$ be a Banach space, and $Y \subseteq X^*$ be a linear subspace that separates the points. Suppose Y norms X, namely, $\forall x \in X,\ \|x\| = \sup_{\phi \in Y, \|\phi\| \le 1} |\phi(x)|$, then every convergent sequence $(x_n)_{n=1}^\infty$ in $(X, \mathcal{T}_Y)$ is bounded.*

Proof. Let $(x_n)_{n=1}^\infty$ be a sequence in X that converges to $x \in X$ in $(X, \mathcal{T}_Y)$.
Consider $\mathcal{S} := (\hat{x}_n)_{n=1}^\infty \subseteq X^{**} \subseteq Y^*$. Since Y norms x, we have

$$||x|| = \sup_{\phi \in Y, ||\phi|| \le 1} |\phi(x)| = \sup_{\phi \in Y, ||\phi|| \le 1} |\hat{x}(\phi)| = \|\hat{x}\|.$$

Thus, it is sufficient to show that $\mathcal{S}$ is bounded.
Indeed, for any $\phi \in Y$, we have $\hat{x}_n(\phi) = \phi(x_n) \to \phi(x)$ in $\mathbb{R}$, so $(\hat{x}_n(\phi))_{n=1}^\infty$ is pointwise bounded for each ϕ. Since Y is closed in a Banach space X^*, it is a Banach space itself, and by Banach-Steinhaus theorem 3.20, $\mathcal{S} = (\hat{x}_n)_{n=1}^\infty$ is bounded as required. □

Remark. In particular, a convergent sequence in weak topology or weak-*topology is bounded. However, this does not in general hold for nets. Indeed, a convergent net in weak-* topology is not necessarily bounded.

Example 5.2.1. Consider $X = \ell^1 = C_0^*$, with the weak-* topology $\mathcal{T}_{C_0}$.
For any finite $F \subsetneq C_0 \subset (\ell^1)^*$, we have that $\bigcap_{\phi \in F} \ker(\phi) \neq \{0\}$, so there is a $x_F \neq 0 \in \bigcap_{\phi \in F} \ker(\phi)$. By taking $\tilde{x}_F := \frac{|F|}{\|x_F\|} x_F$, we can have $\tilde{x}_F \in \bigcap_{\phi \in F} \ker(\phi)$, with $\|\tilde{x}_F\| = |F|$.
Clearly $(\tilde{x}_F)_{\text{finite } F \subsetneq C_0}$ is not bounded.
However, consider any finite $F \subseteq C_0, r > 0$, we can pick $F_0 := F$.
For all finite $F' \supseteq F_0$, we have that $\forall \phi \in F,\ |\phi(\tilde{x}_{F'} - 0)| = 0 < r$, which means $\tilde{x}_{F'} \in U_{F,r}$.
Thus $\tilde{x}_F \rightharpoonup 0$.

Definition 5.5. Let X, Y be normed spaces, a linear operator $T : X \to Y$ is **weak-weak continuous** if it is continuous with respect to $\sigma(X, X^*) \to \sigma(Y, Y^*)$. Namely, for any $x_\lambda \rightharpoonup x \in X$, we have $T(x_\lambda) \rightharpoonup T(x) \in Y$.

Theorem 5.14. *Let X, Y be normed spaces, a linear operator $T : X \to Y$ is weak-weak continuous if and only if $T \in B(X, Y)$.*

Proof. ($\Longrightarrow$) : Suppose $T : X \to Y$ is weak-weak continuous, and suppose for contradiction that T is not bounded. Thus, for each $n \ge 1$, we can find $x_n \in X$ with $\|x_n\| \le 1, \|T(x_n)\| \ge n^2$. We have $\left\|\frac{x_n}{n}\right\| \to 0$, so $\frac{x_n}{n} \to 0$, which means $\frac{x_n}{n} \rightharpoonup 0$. Since T is weak-weak continuous, $T(\frac{x_n}{n}) \rightharpoonup T(0) = 0$. Thus, $(T(\frac{x_n}{n}))_{n=1}^\infty$ is weakly convergent, and thus bounded. However, $\left\|T(\frac{x_n}{n})\right\| = \left\|\frac{T(x_n)}{n}\right\| = \frac{\|T(x_n)\|}{n} \ge n$, a contradiction. Thus, T must be bounded.
($\Longleftarrow$) : Suppose T is bounded and thus continuous, consider any $x_\lambda \rightharpoonup x \in \sigma(X, X^*)$. Consider any $g \in Y^*$, we define $f := g \circ T : X \to \mathbb{F}$. Since $\|f\| \le \|g\|\|T\| < \infty$, we have $f \in X^*$. Since $x_\lambda \rightharpoonup x$, we must have

$$\lim_\lambda f(x_\lambda) = f(x)$$
$$\lim_\lambda g(T(x_\lambda)) = g(T(x)).$$

Since this holds for all $g \in Y^*$, we have that $T(x_\lambda) \rightharpoonup T(x) \in \sigma(Y, Y^*)$, which means T is weak-weak continuous. □

Definition 5.6. Let X, Y be two normed vector spaces, then the **weak operator topology** on $B(X, Y)$ is induced by

$$P := \{p_{x,\phi}(T) : x \in X, \phi \in Y^*\},$$

where for all $T \in B(X, Y)$,

$$p_{x,\phi}(T) := |\phi(T(x))|.$$

We say $(T_\lambda)_{\lambda \in \Lambda}$ **converges weakly to** $T \in B(X, Y)$ if it converges in the weak operator topology, denoted $T_\lambda \rightharpoonup T$.

Remark. Notice that these functions separate points by the Hahn-Banach Theorem. Indeed, $T \neq S$ implies $\exists x \in X$ such that $Tx \neq Ts$, which implies $\exists \phi \in Y^*$ such that $\phi(Tx) \neq \phi(Ts)$.

Proposition 5.15. *Let X, Y be two normed vector spaces, then $T_\lambda \rightharpoonup T \in B(X, Y)$ if and only if $\forall \phi \in X^*, x \in X,\ \phi(T_\lambda x) \to \phi(Tx)$.*

Definition 5.7. Let X, Y be two normed vector spaces, then the **strong operator topology** is the topology induced by

$$P := \{p_x(T) := \|Tx\|_Y : x \in X\}.$$

Proposition 5.16. *Let X, Y be two normed vector spaces, then $T_\lambda \to T \in B(X, Y)$ in the strong operator topology if and only if $\forall x \in X,\ T_\lambda x \to Tx \in Y$.*

Proposition 5.17. *Let X, Y be two normed vector spaces, then convergence in the operator norm implies convergence in strong operator topology, which implies convergence in weak operator topology.*

Proposition 5.18. *When $X = Y = \mathcal{H}$ is a Hilbert space, then $T_\lambda \rightharpoonup T$ if and only if*

$$\forall \xi, \eta \in \mathcal{H},\ \langle T_\lambda \xi, \eta \rangle \to \langle T\xi, \eta \rangle.$$

Proof. By the Riesz-Representation Theorem, for each $\phi \in \mathcal{H}^*$, there is unique $\eta \in H$ such that

$$\forall\ \xi \in \mathcal{H},\ \phi(\xi) = \langle \xi, \eta \rangle.$$

Thus $T_\lambda \rightharpoonup T$ if and only if $\forall \phi \in \mathcal{H}^*, \xi \in \eta,\ \phi(T_\lambda \xi) \to \phi(T\xi)$, if and only if $\forall \xi, \eta \in \mathcal{H},\ \langle T_\lambda \xi, \eta \rangle \to \langle T\xi, \eta \rangle$. ☐

5.3 Continuous Functionals

Theorem 5.19. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, then for a linear $\phi : X \to \mathbb{F}$, the following are equal:*

1. *ϕ is continuous,*
2. *ϕ is continuous at 0,*
3. *$\ker(\Phi)$ is closed,*
4. *$\exists p_1, \ldots, p_n \in P, \alpha_1, \ldots, \alpha_n > 0$, such that*

$$\forall x \in X,\ |\phi(x)| \leq \sum_{i=1}^{n} \alpha_i p_i(x).$$

Proof. (1) $\implies$ (2) is trivial.
(2) $\implies$ (3):
Let $x_\lambda \in \ker(\phi)$, with $x_\lambda \to x \in X$.
For any λ, we have $\phi(x) = \phi(x) - \phi(x_\lambda) = \phi(x - x_\lambda)$.
Since $x - x_\lambda \to 0$, and ϕ is continuous at 0, we have $\phi(x - x_\lambda) \to 0$, which means $\phi(x) = 0$. Namely, $x \in \ker(\phi)$.
Thus $\ker(\phi)$ is closed.
(3) $\implies$ (4):

Suppose $\phi = 0$, (4) is trivially true.
Otherwise, there is $x_0 \in \ker(\Phi)^c$. WLOG, by taking $x_0' := \frac{x_0}{\phi(x_0)}$, we can assume $\phi(x_0) = 1$.
Since $\ker(\phi)$ is closed, $\ker(\Phi)^c$ is open.
There must be some finite $F \subseteq P$ and $r > 0$, such that $x_0 + U_{F,r} = U_{F,r}(x_0) \subseteq \ker(\phi)^c$.
Thus, $0 \notin \phi(x_0 + U_{F,r}) = \phi(x_0) + \Phi(U_{F,r}) = 1 + \Phi(U_{F,r})$.
Namely, $-1 \notin \Phi(U_{F,r})$.
Thus $\forall x \in U_{F,r}$, we have $\phi(x) \neq -1$.
Take $\{p_1, \dots, p_n\} = F$, and $\alpha_i = \frac{1}{r}$.
Suppose for contradiction that there is some $x \in X$ with $|\phi(x)| > \sum_{i=1}^n \frac{1}{r} p_i(x)$.
In particular, $|\phi(x)| > \frac{1}{r} p_i(x)$ for all $p_i \in F$.
We must have some $|\lambda|$ such that $\phi(x) = \lambda|\phi(x)|$.
Then for $y := \frac{x}{-\lambda|\phi(x)|}$, we have

$$\phi(y) = \phi\left(\frac{x}{-\lambda|\phi(x)|}\right) = \frac{\phi(x)}{-\lambda|\phi(x)|} = -1.$$

However,

$$p_i(y) = \left|\left(\frac{1}{-\lambda|\phi(x)|}\right)\right| p_i(x) = \frac{p_i(x)}{|\phi(x)|} < \frac{r|\phi(x)|}{|\phi(x)|} = r.$$

Thus, $y \in U_{F,r}$, which is a contradiction.
(4) $\implies$ (1):
If $x_\lambda \to x$, then $\forall p \in P,\ p(x_\lambda - x) \to 0$.
Thus, $\sum_{i=1}^n \alpha_i p_i(x_\lambda - x) \to 0$.
Thus, $|\phi(x_\lambda) - \phi(x)| = |\phi(x_\lambda - x)| \to 0$, which means $\phi(x_\lambda) \to \phi(x)$.
This shows ϕ is continuous. □

Definition 5.8. Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, then the **continuous dual space** $(X, \mathcal{T}_P)^*$ is the set of continuous linear functions $\phi : X \to \mathbb{F}$.

Theorem 5.20. *Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then $\phi \in (X, \mathcal{T}_Y)^*$ if and only if $\phi \in \operatorname{Span}(Y)$.*

Proof. $\phi \in (X, \mathcal{T}_Y)^*$ if and only if $\exists \phi_1, \dots, \phi_n \in Y, \alpha_i > 0$, such that $\forall x \in X,\ |\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)|$.
($\impliedby$):
Suppose $\phi = \sum_{i=1}^n c_i \phi_i$, then taking $\alpha_i := \max(|c_1|, 1)$, we have

$$\begin{aligned} |\phi(x)| &= \left|\sum_{i=1}^n c_i \phi_i(x)\right| \\ &\leq \sum_{i=1}^n |c_i||\phi_i(x)| \\ &\leq \sum_{i=1}^n \alpha_i |\phi_i(x)|. \end{aligned}$$

($\implies$):
Suppose $\{\phi_i\}_{i=1}^n$ is not linearly independent, WLOG, $\phi_n = \sum_{i=1}^{n-1} c_i \phi_i$, then

$$|\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)| \leq \sum_{i=1}^{n-1} (\alpha_i + \alpha_n |c_i|) |\phi_i(x)|.$$

Thus, we can assume $\{\phi_i\}_{i=1}^n$ is linearly independent.
Consider $T : X \to \mathbb{F}^n$ by $T(x) := (\phi_1(x), \dots, \phi_n(x))$.
Clearly $\ker(T) = \bigcap_{i=1}^n \ker(\phi_i)$.
Since $\{\phi_i\}_{i=1}^n$ is linearly independent, T is surjective.

By a corollary of the Banach Isomorphism Theorem, $X/\bigcap_{i=1}^n \ker(\phi_i) \cong \mathbb{F}^n$ by $\hat{T}: \hat{x} \mapsto (\phi_1(x), \dots, \phi_n(x))$.
Since $\ker(\phi) \supseteq \bigcap_{i=1}^n \ker(\phi_i)$, we can define $\tilde{\phi}: \mathbb{F}^n \to \mathbb{F}$ by

$$\tilde{\phi}(\hat{T}\hat{x}) := \phi(x).$$

This is well-defined, since $\hat{x} = \hat{y} \implies x - y \in \bigcap_{i=1}^n \ker(\phi_i) \implies \phi(x-y) = 0 \implies \phi(x) = \phi(y)$.
Thus for all $i \in [n]$, there is $\beta_i := \hat{\phi}(e_i)$, such that $\tilde{\phi}(z_1, \dots, z_n) = \sum_{i=1}^n \beta_i z_i$.
Thus, $\phi(x) = \tilde{\phi}(\hat{T}\hat{x}) = \tilde{\phi}(Tx) = \sum_{i=1}^n \beta_i \phi_i(x)$, which means $\phi = \sum_{i=1}^n \beta_i \phi_i \in \mathrm{Span}\left(\{\phi_i\}_{i=1}^n\right) \subseteq \mathrm{Span}(Y)$. $\square$

Corollary 5.21. *Let $(X, \|\cdot\|)$ be a normed vector space, then $\sigma(X, X^*)^* = (X, \mathcal{T}_{X^*})^* = X^*$ and $\sigma(X^*, X)^* = (X^*, \mathcal{T}_X)^* = X$.*

5.4 Geometric Hahn–Banach Theorems

Definition 5.9. Let X be a vector space, a set $K \subseteq X$ is **convex** if

$$\forall x, y \in K, t \in (0,1),\ (1-t)x + ty \in K.$$

Definition 5.10. Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, the **Minkowski functional associated to** U is

$$p_U(x) := \inf\{t > 0 : x \in tU\}.$$

Lemma 5.22. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, then the Minkowski functional associated to U is always well-defined.*

Proof. Since $0 \in U$, which is open, there is finite $F \subseteq P, r > 0$ such that $U_{F,r} \subseteq U$.
Thus, for $t = \frac{r}{2\max\{p(x):p\in F\}} > 0$, we have $\forall p \in P,\ p(tx) = \frac{r}{2\max\{p(x):p\in F\}} p(x) \leq \frac{r}{2} < r$, so $tx \in U_{F,r} \subseteq U$.
Thus $\{t > 0 : x \in tU\} \neq \emptyset$. $\square$

Theorem 5.23. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, then the Minkowski functional $p_U : X \to [0, \infty)$ is a sublinear functional, and $U = \{x \in X : P_U(x) < 1\}$.*

Proof. For any $t > 0$, we have $x \in sU \iff tx \in tsU$.

$$\begin{aligned} p_U(tx) &= \inf\{r > 0 : tx \in rU\} \\ &= \inf\{st > 0 : tx \in tsU\} \\ &= \inf\{st > 0 : x \in sU\} \\ &= t \inf\{s > 0 : x \in sU\} \\ &= t p_U(x). \end{aligned}$$

Also, consider any $x, y \in X$ and any $s, t > 0$ such that $x \in sU, y \in tU$, we have that $\frac{x}{s}, \frac{y}{t} \in U$.
By the convexity of U,

$$\begin{aligned} \frac{s}{s+t}\frac{x}{s} + \frac{t}{s+t}\frac{y}{t} &\in U \\ x + y &= \left(\frac{s}{s+t}\frac{x}{s} + \frac{t}{s+t}\frac{y}{t}\right)(s+t) \\ &\in (s+t)U. \end{aligned}$$

Thus, $p_U(x+y) \leq s + t$. Since this holds for any such s, t, we have

$$p_U(x+y) \leq p_U(x) + p_U(y).$$

This shows P_U is sublinear.
Suppose $p_U(x) < 1$, then there is $t < 1$ such that $x \in tU \implies \frac{x}{t} \in U \implies x = t\frac{x}{t} + (1-t)0 \in U$ by

convexity of U.
Now for any $x \in U$, since $t \mapsto tx$ is continuous, and $1x = x \in U$ which is open, we have some $\delta > 0$, such that $(1-\delta, 1+\delta)x \subseteq U$.
Thus $\forall t \in \left(\frac{1}{1+\delta}, 1\right)$, $x \in tU$, which means $p_U(x) \leq \frac{1}{1+\delta} < 1$. $\square$

Theorem 5.24 (First Separation)**.** *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, $A, B \subseteq X$ be disjoint convex sets. Suppose A is open, then $\exists t \in \mathbb{R}$, $\phi \in (X, \mathcal{T}_P)^*$, such that*

$$\forall x \in A, y \in B, \ \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

Namely, $\phi(A), \phi(B)$ can be separated by a vertical line in $\mathbb{C}$.

Proof.

1. We first assume $\mathbb{F} = \mathbb{R}$.
   Fix $x_0 \in A, y_0 \in B$, let $z_0 := y_0 - x_0 \neq 0$.
   Consider $U := z_0 + A - B = \bigcup_{y \in B}(z_0 - y + A)$, which is open and convex. Also, $0 \in U$.
   Consider the Minkowski functional $p_U : X \to [0, \infty)$, which is sublinear.
   Notice that $z_0 \notin U$, so $p_U(z_0) \geq 1$.
   Let $\phi_0 : \operatorname{Span}\{z_0\} \to \mathbb{R}$ be $\lambda z_0 \mapsto \lambda$, which is linear.
   In addition, $\forall \lambda \geq 0$, we have $\phi_0(\lambda z_0) = \lambda \leq \lambda p_U(z_0) = p_U(\lambda z_0)$.
   Also, $\phi_0(-\lambda z_0) = -\lambda \leq 0 \leq p_U(-\lambda z_0)$.
   Thus $\phi_0 \leq P_U$ on $\operatorname{Span}\{z_0\}$. By the extension theorem 3.32, there is a linear extension $\phi : X \to \mathbb{R}$, such that $\phi \leq p_U$.
   Let $\epsilon > 0$.
   Take any $x \in \epsilon U \cap (-\epsilon U) \in \mathcal{O}(0)$, which is open.
   We have that $\pm\frac{x}{\epsilon} \in U$, so $\phi\left(\pm\frac{x}{\epsilon}\right) \leq p_U\left(\pm\frac{x}{\epsilon}\right) < 1$.
   Thus, $|\phi(x)| < \epsilon$.
   This shows that ϕ is continuous.
   Now take any $x \in A, y \in B$, we have that $z_0 + x - y \in U$, so $1 + \phi(x) - \phi(y) = \phi(z_0 + x - y) \leq p_U(z_0 + x - y) < 1$.
   Thus, $\phi(x) < \phi(y)$.
   Notice that since A is open, there is $\epsilon_0 > 0$ such that $\forall 0 < \epsilon < \epsilon_0$, $(1 \pm \epsilon)x \in A$. Thus, $(1 \pm \epsilon)\phi(x) \in \phi(A)$, which means $\phi(A)$ is open.
   Since A is convex, $\phi(A)$ is also convex, thus connected. Thus, $\phi(A) = (b, t)$ is an interval.
   This shows $\forall x \in A, y \in B$, $\phi(x) < t \leq \phi(y)$.

2. Now assume $\mathbb{F} = \mathbb{C}$.
   Consider $(X_{\mathbb{R}}, \mathcal{T}_P)$, we have that A, B are still disjoint convex sets, and A is open.
   Thus there is a $\mathbb{R}$-linear continuous map $\psi \in (X_{\mathbb{R}}, \mathcal{T}_P)^*$, such that $\psi(A) < \psi(B)$.
   Now take $\phi(x) := \psi(x) - i\psi(ix)$.

$\square$

However, this is not always true when A is not open.

Example 5.4.1. Consider the weak topology $(\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})$, and $A := \left\{x = (x_i)_{i=1}^\infty \in \ell^1(\mathbb{N}) : \sum_{i=1}^\infty x_i = 0\right\}$, $B = \{\delta_1\}$. They are disjoint convex sets.
However, for any $\phi \in C_0(\mathbb{N}) = \operatorname{Span}(C_0(\mathbb{N})) = (\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})^*$, we have that $\phi(A) \cap \phi(B) \neq \emptyset$.
Indeed, consider any $\phi = (a_1, a_2, \dots) \in C_0(\mathbb{N})$, there is $m \in \mathbb{N}$ such that $a_m \neq 0$.
Now consider $(\delta_m - \delta_n)_{n=1}^\infty \subset A$, we have that $\phi(\delta_m - \delta_n) = a_m - a_n \to a_m \neq 0$.
Since $\ker(\phi)$ is closed, A is not contained in $\ker(\phi)$, which means $\phi(A) = \mathbb{C}$.

Lemma 5.25. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, suppose compact $K \subseteq$ open $V \subseteq X$, then there is an open convex neighbourhood U of 0, such that $K + U \subseteq V$.*

Proof. For all $x \in K$, since $x \in V$, there is finite $F_x \subseteq P$, and $r_x > 0$, such that $U_{F_x, 2r_x}(x) \subseteq V$.
Since $K \subseteq \bigcup_{x \in K} U_{F_x, r_x}(x)$ is compact, there is a finite subcover $K \subseteq \bigcup_{i=1}^n U_{F_{x_i}, r_{x_i}}(x_i)$.
Now let $F := \bigcup_{i=1}^n F_{x_i}$, which is finite, and $r := \min_{i \in [n]}\{r_{x_i}\} > 0$.

Let $U := U_{F,r}$.
For any $z \in K + U$, there is some $x \in K$ such that $z \in U_{F,r}(x)$.
Also, since $K \subseteq \bigcup_{i=1}^n U_{F_{x_1},r_{x_i}}(x_i)$, there is some $i \in [n]$ such that $x \in U_{F_{x_i},r_{x_i}}(x_i)$.
Thus for any $p \in F_{x_i} \subseteq F$, we have

$$\begin{aligned} p(z - x_i) &\leq p(z - x) + p(x - x_i) \\ &< r + r_{x_i} \\ &\leq 2r_{x_i}. \end{aligned}$$

Thus, $z \in U_{F_{x_i},2r_{x_i}}(x_i) \subseteq V$. □

Theorem 5.26 (Second Separation). *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, $A, B \subseteq X$ be disjoint convex sets. Suppose A is compact, B is closed, then $\exists t \in \mathbb{R},\ \phi \in (X, \mathcal{T}_P)^*$, such that*

$$\forall x \in A, y \in B,\ \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

Namely, $\phi(A), \phi(B)$ can be separated by a vertical line in $\mathbb{C}$.

Proof. Since A is compact, and B^c is open, there is an open convex neighbourhood U of 0, such that $A + U \subseteq B^c$; namely $A + U \cap B = \emptyset$.
By the first separation theorem, there is $t \in \mathbb{R}, \phi \in (X, \mathcal{T}_P)^*$, such that

$$\sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

Since ϕ is continuous and A is compact, by the Extreme Value Theorem, there is $x_0 \in A$, such that $\Re(\phi(x_0)) = \sup_{x \in A} \Re(\phi(x))$.
Notice that $x_0 = x_0 + 0 \in A + U$, so

$$\sup_{x \in A} \Re(\phi(x)) = \Re(\phi(x_0)) \leq \sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

□

Corollary 5.27. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, then $(X, \mathcal{T}_P)^*$ separates the points of X.*

Proof. Given $x \neq y \in X$.
Take $A := \{x\}$, and $B := \{y\}$, which is closed. They are trivially convex and disjoint. □

Definition 5.11. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$. The **convex hull** of A is

$$\operatorname{conv}(A) := \left\{ x = \sum_{i=1}^n \lambda_i x_i \,\middle|\, n \in \mathbb{N}, x_i \in A, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The **closed convex hull** of A is $\overline{\operatorname{conv}(A)}$.

Proposition 5.28. *The convex hull is the smallest convex set containing A, and the closed convex hull is the smallest closed convex set containing A.*

Definition 5.12. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X, \phi \in (X, \mathcal{T})$. The **closed half square containing** A under ϕ is

$$H_{\phi,A} := \{x \in X : \Re(\phi(x)) \leq \alpha_{\phi,A}\},$$

where $\alpha_{\phi,A} := \sup_{x \in A} \Re(\phi(x))$.

Proposition 5.29. *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, then for all $A \subseteq X$, we have*

$$\overline{\operatorname{conv}(A)} = \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi,A}.$$

Proof. For any $\phi \in (X, \mathcal{T}_P)^*$, we have that $H_{\phi,A}$ is convex and closed, and $A \subseteq H_{\phi,A}$, so $\overline{\text{conv}(A)} \subseteq H_{\phi,A}$. Thus,

$$\overline{\text{conv}(A)} \subseteq \bigcap_{\phi \in (X,\mathcal{T}_P)^*} H_{\phi,A}.$$

On the other hand, suppose for contradiction that $\bigcap_{\phi \in (X,\mathcal{T}_P)^*} H_{\phi,A} \setminus \overline{\text{conv}(A)} \neq \emptyset$.
Take any $x \in \bigcap_{\phi \in (X,\mathcal{T}_P)^*} H_{\phi,A} \setminus \overline{\text{conv}(A)}$, we have that $\{x\}$ is compact, and $\overline{\text{conv}(A)}$ is closed.
By the Second Separation Theorem, there is $\phi \in (X, \mathcal{T}_P)^*$ such that $\Re(\phi(x)) < \inf_{y \in \overline{\text{conv}(A)}} \Re(\phi(y))$.
We thus have $\Re(-\phi(x)) > \sup_{y \in \overline{\text{conv}(A)}} \Re(-\phi(y)) \geq \sup_{y \in A} \Re(-\phi(y))$, so $x \notin H_{-\phi,A}$, which contradicts $x \in \bigcap_{\phi \in (X,\mathcal{T}_P)^*} H_{\phi,A}$. □

Corollary 5.30. *Suppose X is a vector space with two Locally Convex Topologies $\mathcal{T}_P, \mathcal{T}_{P'}$. Suppose $(X, \mathcal{T}_P)^* = (X, \mathcal{T}_{P'})^*$, then $(X, \mathcal{T}_P), (X, \mathcal{T}_{P'})$ have the same closed convex sets.*

Proof. Suppose A is convex and closed in $(X, \mathcal{T}_P)$, then

$$\begin{aligned} A &= \overline{\text{conv}(A)}^{\mathcal{T}_P} \\ &= \bigcap_{\phi \in (X,\mathcal{T}_P)^*} H_{\phi,A} \\ &= \bigcap_{\phi \in (X,\mathcal{T}_{P'})^*} H_{\phi,A} \\ &= \overline{\text{conv}(A)}^{\mathcal{T}_{P'}} \\ &= \overline{A}^{\mathcal{T}_{P'}}. \end{aligned}$$

Thus, A is convex and closed in $(X, \mathcal{T}_{P'})$. □

5.5 Weakly Closeness and Compactness

Proposition 5.31. *Let $(X, \|\cdot\|)$ be a Normed Space. Suppose $A \subseteq X$ is closed in $(X, \mathcal{T}_Y)$, where $Y \subseteq X$ separates the points, then A is normed closed.*

Proof. Suppose a net $(x_\lambda)_{\lambda \in \Lambda}$ in A converges to $x \in X$, we must have $\phi(x_\lambda) \to \phi(x)$ for all $\phi \in X^*$.
Thus $x_\lambda \to x$ in $(X, \mathcal{T}_Y)$, which means $x \in A$. □

The converse is not in general true.

Example 5.5.1. Consider $(C[0,1], \|\cdot\|_\infty)$, with the dual space $M([0,1])$ (the complex Borel measures on $[0,1]$).
For $n \geq 1$, let $f_n := \begin{cases} 1 - 2nx, & 0 \leq x \leq \frac{1}{2n} \\ 2nx - 1, & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 1, & x \geq \frac{1}{n} \end{cases}$. We have that $f_n \in C[0,1]$, and $f_n(x) \to 1$ pointwise for all $x \in [0,1]$.
By Lebesgue's Dominated Convergence Theorem, for any complex Borel measure $\mu \in M([0,1])$, we have $\int_0^1 f_n d\mu \to \int_0^1 1 d\mu$.
Thus, $f_n \rightharpoonup f$.
However, for any $n \geq 1$, $\|f_n - 1\|_n = 1$.

Proposition 5.32. *Let $(X, \|\cdot\|)$ be a Normed Space, then $(X, \|\cdot\|)$ and $(X, \mathcal{T}_{X^*})$ have the same closed convex set.*

Proof. This follows from that $(X, \mathcal{T}_{X^*})^* = X^*$. □

Proposition 5.33. *Let $(X, \|\cdot\|)$ be a Normed Space, then the closed balls are weak-* closed. Namely, the closed balls in $(X^*, \|\cdot\|_{X^*})$ are closed in $(X^*, \mathcal{T}_X)$.*

Proof. For any $\phi_0 \in X^*, r > 0$, then

$$\begin{aligned}
\bar{B}^{(X^*,\|\cdot\|_{X^*})}(\phi_0, r) &= \{\phi \in X^* : \|\phi - \phi_0\|_{X^*} \le r\} \\
&= \{\phi \in X^* : |(\phi - \phi_0)(x)| \le r,\ \forall x \in X \text{ such that } \|x\| \le 1\} \\
&= \bigcap_{x \in X \text{ such that } \|x\| \le 1} \{\phi \in X^* : |\hat{x}(\phi - \phi_0)| \le r\},
\end{aligned}$$

which is an intersection of weak-* closed sets. □

Theorem 5.34 (Goldstine)**.** *Let $(X, \|\cdot\|)$ be a Banach Space, then $\bar{B}^{(X,\|\cdot\|)}(0,1)$ is weak-* dense in $\bar{B}^{(X^{**},\|\cdot\|)}(0,1)$, with the weak-* topology $(X^{**}, \mathcal{T}_{X^*})$. In particular, X is weak-* dense in X^{**}.*

Proof. If $x_\lambda \rightharpoonup \psi \in X^{**}$, with $(x_\lambda)_{\lambda\in\Lambda} \subset \bar{B}^{(X,\|\cdot\|)}(0,1)$, for any $\phi \in X^*$, we have

$$\begin{aligned}
|\psi(\phi)| &= \left|\lim_\lambda \hat{x}_\lambda(\phi)\right| \\
&= \lim_\lambda |\phi(x_\lambda)| \\
&\le \|\phi\|.
\end{aligned}$$

Thus, $\|\psi\| \le 1$ and $\psi \in \bar{B}^{(X^{**},\|\cdot\|)}(0,1)$.
This shows $\overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak} \subseteq \bar{B}^{(X^{**},\|\cdot\|)}(0,1)$.
Suppose for contradiction that there is $\psi \in \bar{B}^{(X^{**},\|\cdot\|)}(0,1) \setminus \overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak}$.
Since $\{\psi\}$ is convex and compact, and $\overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak}$ is closed in $(X^{**}, \mathcal{T}_{X^*})$, by the second separation theorem, we can find $\phi \in (X^{**}, \mathcal{T}_{X^*})^* = X^*$, such that

$$\Re(\psi(\phi)) < \inf_{\xi \in \overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak}} \Re(\xi(\phi)).$$

We thus have

$$\begin{aligned}
\|\phi\| &= \sup_{x \in \bar{B}^{(X,\|\cdot\|)}(0,1)} \Re(-\phi(x)) \\
&\le \sup_{\xi \in \overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak}} \Re(\xi(\phi)) \\
&< \Re(\psi(-\phi)) \\
&\le \|\psi\|\|\phi\| \\
&\le \|\phi\|,
\end{aligned}$$

which is a contradiction.
Thus,

$$\bar{B}^{(X^{**},\|\cdot\|)}(0,1) = \overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak}$$

□

Theorem 5.35 (Banach-Alaoglu)**.** *Let $(X, \|\cdot\|)$ be a Normed Vector Space, then $\bar{B}^{(X^*,\|\cdot\|)}(0,1)$ is compact in the weak-* topology.*

Proof. Consider the set

$$\begin{aligned}
D &:= \{f : X \to \mathbb{F} : |f(x)| \le \|x\|\} \\
&= \prod_{x \in X} \bar{B}(0, \|x\|),
\end{aligned}$$

where $\bar{B}(0, \|x\|) := \{z \in \mathbb{F} : |z| \le \|x\|\}$ is compact. By Tychonoff's Theorem, D is compact and Hausdorff in the product topology. Also, $\bar{B}^{(X^*,\|\cdot\|)}(0,1) \subseteq D$.

In addition, $f_\lambda \to f$ in the product topology if and only if $\forall x \in X,\ f_\lambda(x) \to f(x)$ if and only if $f_\lambda \rightharpoonup f$ in weak-* topology. Thus, the weak-* topology and the product topology are the same. Since $\bar{B}^{(X^*,\|\cdot\|)}(0,1)$ is weak-* closed, it is compact. $\square$

Corollary 5.36. *Let $(X, \|\cdot\|)$ be a Banach Space, then X is reflexive if and only if $\bar{B}^{(X,\|\cdot\|)}(0,1)$ is weakly compact in $(X, \mathcal{T}_{X^*})$.*

Proof. $(\implies)$:
If $X^{**} = X$, then the weak-* topology $\mathcal{T}_{X^*}$ on X^{**} is the same as the weak topology $\mathcal{T}_{X^*}$ on X.
Since $\bar{B}^{(X^{**},\|\cdot\|)}(0,1)$ is weakly compact in $(X^{**}, \mathcal{T}_{X^*})$ by the Banach-Alaoglu Theorem, we have that $\bar{B}^{(X,\|\cdot\|)}(0,1)$ is weakly compact in $(X, \mathcal{T}_{X^*})$.
$(\impliedby)$:
By the Goldstine Theorem, we have that $\bar{B}^{(X^{**},\|\cdot\|)}(0,1) = \overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak}$.
Since $\bar{B}^{(X,\|\cdot\|)}(0,1)$ is weakly compact in $(X, \mathcal{T}_{X^*})$, and $i : (X, \mathcal{T}_{X^*}) \to (X^{**}, \mathcal{T}_{X^*})$ is continuous, it is also weakly compact in $(X, \mathcal{T}_{X^*})$. Thus,

$$\bar{B}^{(X^{**},\|\cdot\|)}(0,1) = \overline{\bar{B}^{(X,\|\cdot\|)}(0,1)}^{weak} = \bar{B}^{(X,\|\cdot\|)}(0,1).$$

Thus, $X = X^{**}$. $\square$

5.6 Extreme Points

Definition 5.13. Let X be a Vector Space, and let $\emptyset \neq A \subseteq X$ be convex. A **face** of A is some convex $\emptyset \neq F \subseteq A$, such that for all $t \in (0,1), x, y \in A$, if $(1-t)x + ty \in F$, then $x, y \in F$.
If a face $F = \{z\}$, then we call z an **extreme point** of A.
$\text{Ext}(A)$ is the set of extreme points of A.

Proposition 5.37. *Suppose F is a face for A, and F' is a face for F, then F' is also a face for A.*

Proof. Consider any $x, y \in A, t \in (0,1)$, with $(1-t)x + ty \in F' \subseteq F$. Since F is a face of A, $x, y \in F$. Since F' is a face of F, $x, y \in F'$. Thus, F' is a face of A. $\square$

Example 5.6.1. Let $X = L^1([0,1])$ with the Lebesgue measure, and consider $A := \bar{B}^{(X,\|\cdot\|_1)}(0,1)$. For any $f \in A$ such that $\|f\|_{L^1([0,1])} = a \neq 0$, we can pick $t_0 \in (0,1)$, such that $\int_0^{t_0} |f| dx = \int_{t_0}^1 |f| dx = \frac{1}{2}a$.
Now take $g := 2f\chi_{[0,t_0]}, h := 2f\chi_{[t_0,1]}$, we have that $f = \frac{1}{2}g + \frac{1}{2}h$, and $g, h \in A$.
Thus, $f \notin \text{Ext}(A)$.
Thus, $\bar{B}(0,1)$ has no extreme points.

Proposition 5.38. *Let $(X, \mathcal{T})$ be a Topological Vector Space, and $\emptyset \neq K \subseteq X$ be convex and compact, the for any $\phi \in (X, \mathcal{T})^*$,*

$$F_\phi := \arg\min_{x \in K} (\Re(\phi(x)))$$

is always a closed face of K.

Proof. Let $\alpha_\phi := \inf_{x\in K} (\Re(\phi(x)))$.
Since K is compact, and ϕ is continuous, α_ϕ is achieved.
Thus $F_\phi = \{x \in K : \Re(\phi(x)) = \alpha_\phi\} \neq \emptyset$.
For any $(x_\lambda)_{\lambda\in\Lambda}$ in F_ϕ, such that $x_\lambda \to x \in K$, since ϕ is continuous, we have that

$$\phi(x) = \lim_\lambda \phi(x_\lambda) = \lim_\lambda \alpha_\phi = \alpha_\phi.$$

Thus, $x \in F_\phi$, so F_ϕ is closed.
For any $x, y \in F_\phi, t \in (0,1)$, we have

$$\phi((1-t)x + ty) = (1-t)\phi(x) + t\phi(y) = (1-t)\alpha_\phi + t\alpha_\phi = \alpha_\phi.$$

Thus, $(1-t)x+ty \in F_\phi$, so F_ϕ is convex.
For any $x, y \in K, t \in (0,1)$, if $\phi((1-t)x+ty) \in F_\phi$, we must have

$$\begin{aligned}\alpha_\phi &= \phi((1-t)x+ty) \\ &= (1-t)\phi(x)+t\phi(y) \\ &\geq (1-t)\alpha_\phi + t\alpha_\phi \\ &= \alpha_\phi.\end{aligned}$$

This forces the inequality to be equality, and $\phi(x) = \phi(y) = \alpha_\phi$. Thus, $x, y \in F_\phi$, so F_ϕ is a face of K. $\square$

Theorem 5.39 (Krein-Milman). *Let $(X, \mathcal{T}_P)$ be a Locally Convex Topological Vector Space, and $\emptyset \neq K \subseteq X$ be convex and compact, then*

$$K = \overline{\operatorname{conv}(\operatorname{Ext}(K))}.$$

Proof. Since $(X, \mathcal{T}_P)$ is Hausdorff, K is compact means K is closed.
Thus $K \supseteq \overline{\operatorname{conv}(\operatorname{Ext}(K))}$ since it is a closed convex set containing $\operatorname{Ext}(K)$.
On the other hand, we firstly show that for any closed face $\emptyset \neq F_0 \subseteq K$, we have $\operatorname{Ext}(K) \cap F_0 \neq \emptyset$.
Let $\Lambda := \{F \subseteq F_0 : F \text{ is a closed face of } F_0\}$, with the partial order $F_1 \leq F_2$ if $F_2 \subseteq F_1$.
Let $\mathcal{C} = \{F_\alpha\}_{\alpha \in A}$ be a chain in Λ.
Let $F := \bigcap_{\alpha \in A} F_\alpha$.
Since K is compact, by FIP, $F \neq \emptyset$, and it is closed and convex.
Also, if $x, y \in F_0, t \in (0,1)$, and $(1-t)x+ty \in F$, we have $(1-t)x+ty \in F_\alpha$ for some $\alpha \in A$.
Since F_α is a face of F_0, we must have $x, y \in F_\alpha \subseteq F$.
Thus, $F \in \Gamma$, and it's clear that F is an upper bound for $\mathcal{C}$.
By Zorn's lemma, there is a maximal element F of Γ. Notice that it is also a face of K.
Suppose for contradiction, that there are $x \neq y \in F$, then by the second separation theorem, there is $\phi \in (X, \mathcal{T}_P)^*$, such that $\Re(\phi(x)) \neq \Re(\phi(y))$.
Now let $F_\phi := \arg\min_{x \in F} (\Re(\phi(x)))$.
Since F is a closed subset of compact K, it is compact. By the proposition, F_ϕ is a closed face of F, and thus a closed face of F_0. Thus $F_\phi \in \Gamma$.
By maximality of F, we must have $F_\phi = F$, which means $\phi(x) = \phi(y) = \min_{x \in F} (\Re(\phi(x)))$, a contradiction with the choice of ϕ.
Thus F only has one point x, so $x \in \operatorname{Ext}(K) \cap F_0 \neq \emptyset$.
In particular, since K is a closed face for itself, $\operatorname{Ext}(K) \neq \emptyset$.
Now suppose for contradiction that there is $x_0 \in K \setminus B$, where $B := \overline{\operatorname{conv}(\operatorname{Ext}(K))}$.
By the Second Separation Theorem, there is $\phi \in (X, \mathcal{T}_P)^*, t \in \mathbb{R}$ such that

$$\Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

In particular, $\min_{x \in K} \Re(\phi(x)) < \Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y))$.
Thus, $F_\phi \cap B = \emptyset$ for $F_\phi := \arg\min_{x \in K} (\Re(\phi(x)))$.
However, F_ϕ is a closed face, so $F_\phi \cap \operatorname{Ext}(K) \neq \emptyset$, thus a contradiction. $\square$

Corollary 5.40. *Let $(X, \|\cdot\|)$ be a Normed Vector Space, then $\bar{B}^{(X^*, \|\cdot\|)}(0,1)$ is the weak-* closed convex hull of its extreme points.*

Proof. By the Banach-Alaoglu Theorem, $\bar{B}^{(X^*, \|\cdot\|)}(0,1)$ is compact in the weak-* topology. The convexity is easy to see. Indeed, for any $f, g \in \bar{B}^{(X^*, \|\cdot\|)}(0,1)$, $t \in (0,1)$, we have $\|tf + (1-t)g\| \leq t\|f\| + (1-t)\|g\| \leq t + (1-t) = 1$, so $tf + (1-t)g \in \bar{B}^{(X^*, \|\cdot\|)}(0,1)$ and $\bar{B}^{(X^*, \|\cdot\|)}(0,1)$ is convex. $\square$

Corollary 5.41. *$L^1([0,1])$ with the Lebesgue measure is not a dual space.*

5.6.1 Probability Measure

See more about the space of measures in my Pmath651 Measure Theory Notes.

Definition 5.14. Let X be a locally compact Hausdorff space, we define $M(X) := \{\mu : \text{complex Radon measure}\}$, and $\|\mu\|_{M(X)} := |\mu|(X)$, where $|\mu|$ is the total variation.

Theorem 5.42. *$(M(X), \|\cdot\|_{M(X)}) \cong (C_0(X), \|\cdot\|_\infty)^*$ isometrically by $\mu(f) := \int_X f d\mu$.*

Definition 5.15. The **probability measures** on X is

$$P(X) := \{\mu \in M(X) | \mu \geq 0, \mu(X) = 1\}.$$

The **Dirac measures** are $\delta_x : f \mapsto f(x)$ for $x \in X, f \in C(X)$.

Notice that $P(X)$ is clearly convex by linearity. Also, when X is compact, $C(X) = C_c(X) = C_0(X)$.

Lemma 5.43. *Let X be a compact Hausdorff space, suppose $\mu \in P(X), x \in X$, and $\ker(\delta_x) \supseteq \ker(\mu)$, then $\mu = \delta_x$.*

Proof. For any $g \in C(X)$, we have $\mu(g - \mu(g)\mathbb{1}) = \mu(g) - \mu(g)\mu(\mathbb{1}) = 0$, so $g - \mu(g)\mathbb{1} \in \ker(\mu)$.
Thus, $\delta_x(g - \mu(g)\mathbb{1}) = 0$, which means $\delta_x(g) = \mu(g) \cdot 1 = \mu(g)$.
Thus $\mu = \delta_x$. $\square$

Proposition 5.44. *Let X be a compact Hausdorff space, then*

$$\operatorname{Ext}(P(X)) = \{\delta_x : x \in X\}.$$

Proof. Let $\mu \in \operatorname{Ext}(P(X))$.
Fix any $0 \leq f < 1$ in $C(X)$.
Let $\lambda := \mu(f) \in [0, 1]$.
If $0 < \lambda < 1$, we can define $\mu_1(g) := \frac{1}{\lambda}\mu(fg) = \frac{1}{\lambda}\int_X gfd\mu$, and $\mu_2(g) := \frac{1}{1-\lambda}\mu((1-f)g) = \frac{1}{1-\lambda}\int_X g(1-f)d\mu$.
We can check that $\mu_1, \mu_2 \in P(X)$, and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$.
Since $\mu \in \operatorname{Ext}(P(X))$, we must have $\mu = \mu_1 = \mu_2$.
Thus, for any $g \in C(X)$,

$$\begin{aligned} \mu(g) &= \mu_1(g) \\ &= \frac{\mu(fg)}{\lambda} \\ &= \frac{\mu(fg)}{\mu(f)}. \end{aligned}$$

Thus, $\mu(fg) = \mu(f)\mu(g)$.
Now suppose $\mu(f) = 0$, we have

$$\begin{aligned} 0 &\leq |\mu(fg)| \\ &= \int_X fgd\mu \\ &\leq \int_X |fg|d\mu \\ &= \int_X f|g|d\mu \\ &\leq \|g\|_\infty \int_X f|g|d\mu \\ &= 0. \end{aligned}$$

Which means $\mu(fg) = 0 = \mu(f)\mu(g)$.
Thus, $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in C(X)$ such that $0 \leq f < 1$.
Since μ is linear, and $\mathrm{Span}\{f \in C(X) : 0 \leq f < 1\} = C(X)$, we have that for all $f, g \in C(X)$,

$$\mu(fg) = \mu(f)\mu(g).$$

We claim that $\exists x \in X$, such that $\ker(\delta_x) \supseteq \ker(\mu)$.
Indeed, suppose for contradiction that $\forall x \in X$, there is $f_x \in C(X)$, such that $f_x \in \ker(\mu) \setminus \ker(\delta_x)$.
Namely, $f_x(x) \neq 0, \mu(f_x) = 0$.
Thus $X = \bigcup_{x \in X} \{y : f_x(y) \neq 0\}$ is an open cover. Since X is compact, there is a finite subcover

$$X = \bigcup_{i=1}^{n} \{y : f_{x_i}(y) \neq 0\}.$$

Define $f := \sum_{i=1}^{n} |f_{x_i}|^2 \in C(X)$. Notice that $f > 0$, which means $\frac{1}{f} \in C(X)$.
Now we have

$$\begin{aligned}
1 &= \mu(\mathbb{1}) \\
&= \mu\left(\frac{f}{f}\right) \\
&= \mu(f)\mu\left(\frac{1}{f}\right) \\
&= \sum_{i=1}^{n} \mu(|f_{x_i}|^2)\mu\left(\frac{1}{f}\right) \\
&= \sum_{i=1}^{n} \mu(f_{x_i})\mu(\overline{f_{x_i}})\mu\left(\frac{1}{f}\right) \\
&= 0\mu\left(\frac{1}{f}\right) \\
&= 0,
\end{aligned}$$

which is a contradiction.
This proves the claim.
By the lemma,

$$\mathrm{Ext}(P(X)) \subseteq \{\delta_x : x \in X\}.$$

Now given any $x \in X$, suppose $\delta_x = \lambda\mu + (1-\lambda)\nu$ for some $\mu, \nu \in P(X), \lambda \in (0,1)$. For any $f \in C(X)$, we have

$$\begin{aligned}
|\delta_x(f)| &= |f(x)| \\
&= \delta_x(|f|) \\
&= \lambda\mu(|f|) + (1-\lambda)\nu(|f|) \\
&\geq \lambda\mu(|f|) \\
&= \lambda \int_X |f| d\mu \\
&= \lambda \left| \int_X f d\mu \right| \\
&\geq \lambda|\mu(f)|.
\end{aligned}$$

Suppose $f \in \ker(\delta_x)$, we will have $\lambda|\mu(f)| \leq |\delta_x(f)| = 0$, so $\mu(f) = 0$.
Thus $\ker(\delta_x) \subseteq \ker(\mu)$.
By lemma, $\delta_x = \mu$, so $\delta_x = \lambda\delta_x + (1-\lambda)\nu$, which make $\nu = \delta_x$ as well.
Thus $\delta_x \in \mathrm{Ext}(P(X))$. $\square$

Corollary 5.45. *Let X be a compact Hausdorff space, then*

$$P(X) = \overline{\operatorname{conv}\{\delta_x : x \in X\}}^{weak}.$$

Proof. For any $\mu \in P(X)$, we have $\|\mu\| = |\mu|(X) = \mu(X) = 1$, so

$$P(X) \subseteq \bar{B}^{(M(X),\|\cdot\|)}(0,1).$$

By Banach-Alaoglu, $\bar{B}^{(M(X),\|\cdot\|)}(0,1)$ is weak-* compact.

Suppose there are $(\mu_\lambda)_{\lambda\in\Lambda}$ in $P(X)$ with $\mu_\lambda \rightharpoonup \mu \in M(X)$, we have that

$$\mu(X) = \mu(\mathbb{1}) = \lim_{\lambda\in\Lambda} \mu_\lambda(\mathbb{1}) = 1,$$

since $\mathbb{1} \in C(X)$. We can also show μ is a positive measure, so $\mu \in P(X)$.

This shows $P(X)$ is weak-* closed, so it is also weak-* compact. The result follows from the Krein-Milman Theorem. □

Remark. When X is not compact, $P(X)$ is not weak-* closed.

Example 5.6.2. Consider $\delta_n \in P(\mathbb{R})$, then $\delta_n \rightharpoonup 0$ in $\bar{B}^{M(\mathbb{R})}(0,1)^{\|\cdot\|}$, but $0 \notin P(\mathbb{R})$.

Nevertheless, when X is a Locally Compact Hausdorff space, with a variation of the proof, we still have

$$\operatorname{Ext}(P(X)) = \{\delta_x | x \in X\},$$

and

$$\operatorname{Ext}\left(\bar{B}^{M(X)}(0,1)^{\|\cdot\|}\right) = \{z\delta_x : x \in X, z \in \mathbb{C} \text{ such that } \|z\| = 1\}.$$

5.6.2 Stone-Weierstrass Theorem

Theorem 5.46. *Let X be a locally compact Hausdorff Space, and $A \subseteq C_0(X,\mathbb{R})$, such that*

1. *A is closed under $\|\cdot\|_\infty$,*
2. *A is a sub-algebra of $C_0(X,\mathbb{R})$,*
3. *A separates points (i.e. $\forall x \neq y \in X$, $\exists f \in A$ such that $f(x) \neq f(y)$),*
4. *A vanishes nowhere (i.e. $\forall x \in X, \exists f \in A$, such that $f(x) \neq 0$),*

then $A = C_0(X,\mathbb{R})$.

Proof. Suppose for contradiction that $A \subsetneq C_0(X,\mathbb{R})$.

By 1. and a corollary of the Hahn-Banach theorem, there is $\mu \neq 0$ in $A^\perp = \{\mu \in M(X) : \mu(f) = 0 \ \forall f \in A\}$. Thus,

$$K := \bar{B}^{(M(X),\|\cdot\|)}(0,1) \cap A^\perp$$

is a non-empty weak-* compact convex set. By the Krein-Milman theorem, there is $\mu \in \operatorname{Ext}(K)$.

We claim that

$$\operatorname{Supp}(\mu) := \{x \in X : \forall U \in \mathcal{O}(X), |\mu|(U) > 0\}$$

is a singleton $\{x\}$ for some $x \in X$.

Suppose for contradiction that there are $x \neq y \in \operatorname{Supp}(\mu)$. By 3., there is $f \in A$ such that $f(x) \neq f(y)$. Notice that f is not $|\mu|$ a.e. constant since it is continuous.

Now find $a, b > 0$ such that $g(x) := a(f(x) + b)$ satisfies $g(x) \geq 0$ and $\int_X gd|\mu| = 1$. Suppose that $\|g\|_\infty \leq 1$, then $0 \leq g \leq 1$ a.e., which means

$$\begin{aligned}\int_X |1-g|d|\mu| &= \int_X (1-g)d|\mu| \\ &= |\mu|(X) - \int_X gd|\mu| \\ &= 1 - 1 \\ &= 0.\end{aligned}$$

Thus, $1 - g = 0$ $|\mu|$-a.e., which means $g = 1$ $|\mu|$-a.e. and it contradicts with f not being constant a.e.. Thus, we must have $\|g\|_\infty > 1$, so

$$\lambda := \frac{1}{\|g\|_\infty} \in (0, 1).$$

Let $\mu_1, \mu_2 \in M(X)$ be as

$$\mu_1(h) := \mu(gh) = \int_X ghd\mu,\ \ \mu_2(h) := \mu\left(\frac{1-\lambda g}{1-\lambda}h\right) = \int_X \frac{1-\lambda g}{1-\lambda}hd\mu.$$

Notice that $\forall h \in A,\ hg = a(f+b)h = afh + abh \in A$, so $\mu_1, \mu_2 \in A^\perp$. We can check that

$$\begin{aligned}|\mu_1|(X) &= \int_X d|\mu_1| \\ &= \int_X gd|\mu| \\ &= 1, \\ |\mu_2|(X) &= \int_X d|\mu_2| \\ &= \int_X \frac{1-\lambda g}{1-\lambda}d|\mu| \\ &= \frac{1-\lambda}{1-\lambda} \\ &= 1.\end{aligned}$$

Thus, $\mu_1, \mu_2 \in K$, with $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. Since $\mu \in \mathrm{Ext}(K)$, we must have $\mu_1 = \mu_2 = \mu$, which means $g = 1$ $|\mu|$-a.e., a contradiction.

This proves that $\mathrm{Supp}(\mu) = \{x\}$ for some $x \in X$. Thus, $\mu = z\delta_x$ for some $z \in \mathbb{C}$. Also, $|z| = |\mu|(X) = 1$. Since $\mu \in A^\perp$, for any $h \in A$, we must have $zh(x) = \int_X hd\mu = \mu(h) = 0$ which means $h(x) = 0$ and A vanishes at x. $\square$

6 Adjoint Operators

6.1 Adjoint Operators on Normed Spaces

Definition 6.1. Let X, Y be normed spaces, the **adjoint operator** or **dual operator** of a linear operator $A : X \to Y$ is

$$A^* : Y^* \to X^*; \quad f \mapsto f \circ A,$$

namely,

$$\forall x \in X, f \in Y^*,\ \langle A^*f|x\rangle := \langle f|Ax\rangle.$$

Proposition 6.1. *Let X, Y, Z be normed spaces, $S \in B(X,Y), T \in B(Y,Z)$, then $(S \circ T)^* = T^* \circ S^*$.*

Proof. Consider any $f \in Z^*$, and any $x \in X$, we have

$$\begin{aligned}(T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx)\\ &= (f)(S(T(x)))\\ &= (f \circ (S \circ T))(x)\\ &= (S \circ T)^*(f)(x).\end{aligned}$$

Thus $(T^* \circ S^*)(f) = (S \circ T)^*(f)$. □

Proposition 6.2. *Let X, Y be normed spaces, and $T \in B(X,Y)$, then*

1. *$\|T\| = \|T^*\|$, so $T^* \in B(Y^*, X^*)$;*
2. *The map $T \mapsto T^*$ is linear;*
3. *T^* is weak-* weak-* continuous; namely, if a net $(\phi_\lambda)_{\lambda\in\Lambda}$ converges to $\phi \in Y$ in weak-* topology on Y^*, we have that $T^*\phi_\lambda$ converges to $T^*\phi$ in weak-* topology on X^*;*
4. *For $\mathbb{1}_X : x \mapsto x$, we have $(\mathbb{1}_X)^* = \mathbb{1}_{X^*}$;*
5. *$T^{**} \in B(X^{**}, Y^{**})$ satisfies $T^{**}|_X = T$.*

Proof. 1. We have

$$\begin{aligned}\|T^*\| &= \sup_{\phi\in Y^*, \|\phi\|\le 1} \|T^* y\|_{X^*}\\ &= \sup_{\phi\in Y^*, \|\phi\|\le 1} \sup_{x\in X, \|x\|\le 1} |\langle T^*\phi|x\rangle|\\ &= \sup_{x\in X, \|x\|\le 1} \sup_{\phi\in Y^*, \|\phi\|\le 1} |\langle \phi|Tx\rangle|\\ &= \sup_{x\in X, \|x\|\le 1} \|Tx\|\\ &= \|T\|.\end{aligned}$$

2. It is obvious by definition.

3. For net $(\phi_\lambda)_{\lambda\in\Lambda}$ in Y^* that converges to $\phi \in Y$ in weak-* topology, and any $x \in X$, we have

$$\begin{aligned}\langle T^*\phi_\lambda|x\rangle &= \langle \phi_\lambda|Tx\rangle\\ &\to \langle \phi|Tx\rangle\\ &= \langle T^*\phi|x\rangle.\end{aligned}$$

   Thus, $T^*\phi_\lambda \rightharpoonup T^*\phi$ in weak-* topology in X^*.

4. This is obvious.

5. Consider the canonical embeddings $x \mapsto \hat{x}, y \mapsto \hat{y}$, we want to show $T^{**}\hat{x} = \widehat{Tx}$. Indeed, for any $\phi \in Y^*$

$$\begin{aligned}\langle T^{**}\hat{x}|\phi\rangle &= \langle \hat{x}|T^*\phi\rangle\\ &= \langle T^*\phi|x\rangle\\ &= \langle \phi|Tx\rangle\\ &= \left\langle \widehat{Tx}\middle|\phi\right\rangle.\end{aligned}$$

□

Theorem 6.3. *Let X, Y be normed spaces, and $S \in B(Y^*, X^*)$, then there is $T \in B(X,Y)$ such that $T^* = S$ if and only if S is weak-* weak-* continuous.*

Proof. ($\implies$) is by the previous proposition.

($\impliedby$): Assume S is weak-* weak-* continuous.

Consider $S^* \in B(X^{**}, Y^{**})$, and let $T := S^*|_X \in B(X, Y^{**})$. Fix $x \in X$, let $(\phi_\lambda)_{\lambda \in \Lambda}$ be a net that converges in weak-* topology $(Y^*, \mathcal{T}_Y)$, then

$$\begin{aligned} \langle S^* \hat{x} | \phi_\lambda \rangle &= \langle \hat{x} | S\phi_\lambda \rangle \\ &\to \langle \hat{x} | S\phi \rangle \\ &= \langle S^* \hat{x} | \phi \rangle. \end{aligned}$$

Thus, $S^*\hat{x}$ is continuous in weak-* topology $(Y^*, \mathcal{T}_Y)$, so $S^*\hat{x} \in Y$. This shows we can redefine $T \in B(X, Y)$. Now for any $x \in X, \phi \in Y^*$, we have

$$\begin{aligned} \langle T^* \phi | x \rangle &= \langle \phi | Tx \rangle \\ &= \langle S^* \hat{x} | \phi \rangle \\ &= \langle \hat{x} | S\phi \rangle \\ &= \langle S\phi | x \rangle. \end{aligned}$$

□

6.2 Hilbert Adjoint Operators

Definition 6.2. Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, the **Hilbert adjoint operator** of T is $T^\dagger : \mathcal{H} \to \mathcal{H}$ such that $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \ \forall x, y \in \mathcal{H}$.

Proposition 6.4. *Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, we have $\langle Ty, x \rangle = \langle y, T^\dagger x \rangle \ \forall x, y \in \mathcal{H}$. Thus, $(T^\dagger)^\dagger = T$.*

Proof.

$$\begin{aligned} \langle Ty, x \rangle &= \overline{\langle x, Ty \rangle} \\ &= \overline{\langle T^\dagger x, y \rangle} \\ &= \langle y, T^\dagger x \rangle. \end{aligned}$$

□

Theorem 6.5. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. $T^\dagger$ always exists, and is given by $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$, where $\Phi : \mathcal{H} \to \mathcal{H}^*$; $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism, and T^* is the dual operator of T. In addition, $T^\dagger$ is also a bounded linear operator, with $\|T^\dagger\| = \|T\|$.*

Proof. $\forall y \in \mathcal{H}$, we have that

$$\begin{aligned} \langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\ &= \langle (T^* \circ \Phi)(x) | y \rangle \\ &= \langle \Phi(x) | Ty \rangle \\ &= \langle x, Ty \rangle. \end{aligned}$$

Linearity follows from Φ, Φ^{-1} being anti-linear, and T^* being linear. We can also compute it directly from the definition. Indeed, consider any $x, y, z \in \mathcal{H}, c \in \mathbb{C}$, we have that

$$\begin{aligned} \langle T^\dagger (x + cz), y \rangle &= \langle x + cz, Ty \rangle \\ &= \langle x, Ty \rangle + \bar{c} \langle z, Ty \rangle \\ &= \langle T^\dagger x, y \rangle + \bar{c} \langle T^\dagger z, y \rangle \\ &= \langle T^\dagger x + cT^\dagger z, y \rangle. \end{aligned}$$

Since this holds for any $y \in \mathcal{H}$, we have that $T^\dagger(x+cz) = T^\dagger x + cT^\dagger z$, and thus $T^\dagger$ is linear.
Also,

$$\left\|T^\dagger\right\| \leq \left\|\Phi^{-1}\right\| \|T\| \|\Phi\| = \|T\|.$$

We can also compute this directly from the definition. Indeed, given any $x \in \mathcal{H}$, we have that

$$\begin{aligned}
\left\|T^\dagger x\right\|^2 &= \left\langle T^\dagger x, T^\dagger x\right\rangle \\
&= \left\langle x, TT^\dagger x\right\rangle \\
&\leq \|x\| \left\|TT^\dagger x\right\| \\
&\leq \|x\| \|T\| \left\|T^\dagger x\right\| \\
&\implies \\
\left\|T^\dagger x\right\| &\leq \|x\| \|T\| \\
&\implies \\
\left\|T^\dagger\right\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\left\|T^\dagger x\right\|}{\|x\|} \\
&\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{\|x\| \|T\|}{\|x\|} \\
&= \|T\|.
\end{aligned}$$

Thus $T^\dagger$ is also a bounded linear operator, with

$$\|T\| = \left\|(T^\dagger)^\dagger\right\| \leq \left\|T^\dagger\right\| \leq \|T\|.$$

□

Remark. $\forall x, y \in \mathcal{H}$, $\left\langle (Tx)^\dagger \middle| y\right\rangle = \langle Tx, y\rangle = \left\langle x, T^\dagger y\right\rangle = \left\langle x^\dagger \middle| T^\dagger y\right\rangle$. We thus abuse the notation, and write $(Tx)^\dagger = \langle x| T^\dagger$

Corollary 6.6. *Let $\mathcal{H}$ be a Hilbert space, and $S, T \in B(\mathcal{H})$ be bounded linear operators. We have that $(S \circ T)^\dagger = T^\dagger \circ S^\dagger$. Also, for any $\alpha \in \mathbb{C}$, we have $(\alpha T + S)^\dagger = \bar{\alpha} T^\dagger + S^\dagger$.*

Proposition 6.7. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. We have*

$$\|T\|^2 = \left\|TT^\dagger\right\| = \left\|T^\dagger T\right\|.$$

Proof. We have $\left\|TT^\dagger\right\| \leq \|T\| \left\|T^\dagger\right\| = \|T\|^2$.

Now for any $x \in \mathcal{H}$, we have

$$\begin{aligned}
\left\|T^\dagger x\right\|^2 &= \left\langle T^\dagger x, T^\dagger x\right\rangle \\
&= \left\langle x, TT^\dagger x\right\rangle \\
&\leq \|x\| \left\|TT^\dagger x\right\| \\
&\leq \|x\| \left\|TT^\dagger\right\| \|x\| \\
&= \left\|TT^\dagger\right\| \|x\|^2; \\
&\implies \\
\left\|T^\dagger x\right\| &\leq \sqrt{\|TT^\dagger\|} \|x\| \\
&\implies \\
\left\|T^\dagger\right\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\left\|T^\dagger x\right\|}{\|x\|} \\
&\leq \sqrt{\|TT^\dagger\|} \\
&\implies \\
\left\|T^\dagger\right\|^2 &\leq \left\|TT^\dagger\right\|.
\end{aligned}$$

$\square$

Corollary 6.8. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. We have that $T = 0$ if and only if $T^\dagger T = 0$.*

There is a deep analogy between operators $B(\mathcal{H})$ and algebra of the form $(\ell^\infty(X), \|\cdot\|_\infty)$ for a set X, with the pointwise multiplication and involution $f^*(x) := \overline{f(x)}$.

Example 6.2.1. Consider $\pi : \ell^\infty(X) \to B(\ell^2(X))$ by $f \mapsto M_f$, where the **multiplication operator** $M_f(\phi)(x) := f(x)\phi(x)$ for all $\phi \in \ell^2(X), x \in X$. We claim that it is a unital isometric homomorphism. Indeed, for any $\phi \in \ell^2(X)$, we have

$$\begin{aligned}\pi(\mathbb{1})(\phi)(x) &= 1 \cdot \phi(x) \\ &= \phi(x),\end{aligned}$$

so $\pi(\mathbb{1})(\phi) = \phi$ and $\pi(\mathbb{1}) = \mathbb{1}$. Also, for any $f, g \in \ell^\infty(X)$,

$$\begin{aligned}(\pi(fg)(\phi))(x) &= (fg)(x)\phi(x) \\ &= f(x)g(x)\phi(x) \\ &= f(x)(\pi(g)(\phi))(x) \\ &= \pi(f)(\pi(g)(\phi))(x) \\ &= ((\pi(f) \circ \pi(g))(\phi))(x).\end{aligned}$$

Thus, $\pi(fg) = \pi(f) \circ \pi(g)$. Also, for any $\phi, \eta \in X$, we have

$$\begin{aligned}\langle \pi(f^*)(\phi), \eta\rangle &= \sum_{x\in X} \pi(f^*)(\phi)(x)\overline{\eta(x)} \\ &= \sum_{x\in X} f^*(x)\phi(x)\overline{\eta(x)} \\ &= \sum_{x\in X} \phi(x)\overline{f(x)\eta(x)} \\ &= \sum_{x\in X} \phi(x)\overline{\pi(f)(\eta)(x)} \\ &= \langle \phi, \pi(f)(\eta)\rangle \\ &= \langle \pi(f)^*(\phi), \eta\rangle.\end{aligned}$$

Thus, $\pi(f^*) = \pi(f)^*$. Lastly, we have that

$$\begin{aligned}\|\pi(f)(\phi)\|^2 &= \langle \pi(f)(\phi), \pi(f)(\phi)\rangle \\ &= \sum_{x\in X} |\pi(f)(\phi)(x)|^2 \\ &= \sum_{x\in X} |f(x)\phi(x)|^2 \\ &= \sum_{x\in X} |f(x)|^2|\phi(x)|^2 \\ &\le \|f\|_\infty^2 \left(\sum_{x\in X} |\phi(x)|^2\right) \\ &= \|f\|_\infty^2 \|\phi\|^2.\end{aligned}$$

Thus, $\|\pi(f)\| \leq \|f\|_\infty$. On the other hand, $\delta_x \in \ell^2(X)$ for any x, and

$$\begin{aligned}
\|f\|_\infty^2 &= \sup_{x\in X} |f(x)|^2 \\
&= \sup_{x\in X} \langle \pi(f)(\delta_x), \pi(f)(\delta_x)\rangle \\
&= \sup_{x\in X} \|\pi(f)(\delta_x)\|^2 \\
&\leq \sup_{\phi\in\ell^2(X)} \|\pi(f)(\phi)\|^2 \\
&= \|\pi(f)\|^2.
\end{aligned}$$

Thus, $\|f\|_\infty = \|\pi(f)\|$.

Indeed, when we take $X = \{1, \dots, n\}$, then $\ell^2(X) = \mathbb{C}^n$, and $\pi : \ell^\infty(X) \to B(\ell^2(X)) \cong M_n(\mathbb{C})$, and $\pi(\ell^\infty(X)) = D_n$, the diagonal matrices.

Proposition 6.9. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$. We have*

$$\ker(T^\dagger) = \mathrm{Im}(T)^\perp.$$

Proof. For any $x \in \ker(T^\dagger), Ty \in \mathrm{Im}(T)$, we have

$$\begin{aligned}
\langle x, Ty\rangle &= \left\langle T^\dagger x, y\right\rangle \\
&= \langle 0, y\rangle \\
&= 0.
\end{aligned}$$

Thus, $x \in \mathrm{Im}(T)^\perp$, and we have $\ker(T^\dagger) \subseteq \mathrm{Im}(T)^\perp$.

On the other hand, for any $x \in \mathrm{Im}(T)^\perp, y \in \mathcal{H}$, we have

$$\begin{aligned}
\left\langle T^\dagger x, y\right\rangle &= \langle x, Ty\rangle \\
&= 0.
\end{aligned}$$

Thus, $T^\dagger x = 0$, so $x \in \ker(T^\dagger)$. We thus have $\mathrm{Im}(T)^\perp \subseteq \ker(T^\dagger)$. ☐

Proposition 6.10. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$. Suppose $T^{-1} \in B(\mathcal{H})$ exists, then $(T^\dagger)^{-1} \in B(\mathcal{H})$ exists, and*

$$(T^\dagger)^{-1} = (T^{-1})^\dagger.$$

Proof.

$$\begin{aligned}
TT^{-1} &= \mathbb{1} = T^{-1}T \\
(TT^{-1})^\dagger &= \mathbb{1}^\dagger = (T^{-1}T)^\dagger \\
(T^{-1})^\dagger T^\dagger &= \mathbb{1} = T^\dagger (T^{-1})^\dagger.
\end{aligned}$$

Thus, $(T^\dagger)^{-1} = (T^{-1})^\dagger$. ☐

Proposition 6.11. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$. The following are equivalent:*

1. *T is invertible (i.e. an isomorphism between $\mathcal{H}$ and $\mathcal{H}$ as Banach spaces);*
2. *$T^\dagger$ is invertible;*
3. *$T, T^\dagger$ are both bounded below;*
4. *Both $\mathrm{Im}(T), \mathrm{Im}(T^\dagger)$ are closed, and $T, T^\dagger$ are injective;*
5. *$\mathcal{H} = \mathrm{Im}(T) = \mathrm{Im}(T^\dagger)$;*

6. *T is bijective;*

7. *$T^\dagger$ is bijective.*

Proof. $1 \implies 2$ is done above.

$2 \implies 1$. We have $T = (T^\dagger)^\dagger$, so applying the above result to $T^\dagger$ gives 1.

$1, 2 \implies 3$. This is by theorem 3.12.

$3 \implies 4$. Let T be bounded below by c. The injectivity is from theorem 3.12. Also, consider any convergent sequence $(Tx_n)_{n=1}^\infty$ in $\mathrm{Im}(T)$. Since it is Cauchy, for any $\epsilon > 0$, there is $N \geq 1$, such that for all $n, m \geq N$,

$$c\|x_n - x_m\| \leq \|Tx_n - Tx_m\| < c\epsilon.$$

Thus, $(x_n)_{n=1}^\infty$ is Cauchy in $\mathcal{H}$, so there must be some $x \in \mathcal{H}$ such that $x_n \to x$. Since T is continuous, $Tx_n \to Tx \in \mathrm{Im}(T)$. This shows $\overline{\mathrm{Im}(T)}$ is closed. Similarly, $\mathrm{Im}(T^\dagger)$ is closed and it is injective.

$4 \implies 5$. We have that $\overline{\mathrm{Im}(T)} \oplus \mathrm{Im}(T)^\perp = \mathcal{H}$ by theorem 4.12. Since $T^\dagger$ is injective, we have $\mathrm{Im}(T)^\perp = \ker(T^\dagger) = \{0\}$ by proposition 6.9. Thus, $\overline{\mathrm{Im}(T)} = \mathcal{H}$. Since $\mathrm{Im}(T)$ is closed, we have $\mathrm{Im}(T) = \mathcal{H}$. Similarly, $\mathrm{Im}(T^\dagger) \oplus \ker(T) = \mathcal{H}$, so $\mathrm{Im}(T^\dagger) = \mathcal{H}$.

$5 \implies 6$. We have that $\ker(T) = \mathrm{Im}(T^\dagger)^\perp = (\mathcal{H})^\perp = \{0\}$, so T is injective. Also, T is surjective by assumption.

$6 \implies 1$. By the Banach Isomorphism theorem 3.23.

$5 \implies 7$. We have that $\ker(T^\dagger) = \mathrm{Im}(T)^\perp = (\mathcal{H})^\perp = \{0\}$, so $T^\dagger$ is injective. Also, $T^\dagger$ is surjective by assumption.

$7 \implies 1$. By the Banach Isomorphism theorem. □

Lemma 6.12. *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle -, \cdot \rangle$. A bounded linear operator $T \in B(\mathcal{H})$ is bounded below by $b > 0$ if $\forall x \in \mathcal{H}, |\langle x, Tx \rangle| = |B[x,x]| \geq b\|x\|^2$.*

Proof. Suppose for contradiction that T is not bounded below by b, then there is $x \in \mathcal{H}$, such that $\|Tx\| < b\|x\|$. Now,

$$\begin{aligned} b\|x\|^2 &\leq |\langle x, Tx \rangle| \\ &\leq \|x\|\|Tx\| \\ &< \|x\|b\|x\| \\ &= b\|x\|^2, \end{aligned}$$

which means $\|x\|^2 = 0$ and $x = 0$. That contradicts $0 = \|Tx\| < b\|x\| = 0$. □

Corollary 6.13. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$. Suppose there is $b > 0$, such that $\forall x \in \mathcal{H}, |\langle x, Tx \rangle| = |B[x,x]| \geq b\|x\|^2$, then T is invertible.*

Proof. By above lemma, T is bounded below. We also have $b\|x\|^2 \leq |\langle x, Tx \rangle| = \left|\langle T^\dagger x, x \rangle\right| = \left|\langle x, T^\dagger x \rangle\right|$, so $T^\dagger$ is also bounded below. By proposition 6.11, T is invertible. □

6.3 Normal, Self-adjoint and Positive Semidefinite Operators

Definition 6.3. Let $\mathcal{H}$ be a Hilbert space. A bounded linear operator $T \in B(\mathcal{H})$ is **normal** if $T^\dagger T = TT^\dagger$.

Definition 6.4. Let $\mathcal{H}$ be a Hilbert space. A bounded linear operator $T \in B(\mathcal{H})$ is **self-adjoint** if $T^\dagger = T$. The set of all self-adjoint operators is $B(\mathcal{H})_{sa}$.

Definition 6.5. Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$, we define

$$\Re(T) := \frac{T + T^\dagger}{2}, \ \Im(T) := \frac{T - T^\dagger}{2i}.$$

Proposition 6.14. *Let $\mathcal{H}$ be a Hilbert space. $B(\mathcal{H})_{sa}$ is a $\mathbb{R}$ subspace of $B(\mathcal{H})$. We have $\Re(T), \Im(T) \in B(\mathcal{H})_{sa}$, and $\Re(T) + i\Im(T) = T$, so*

$$B(\mathcal{H}) = B(\mathcal{H})_{sa} + iB(\mathcal{H})_{sa}.$$

Proposition 6.15. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(X)$. Consider the bounded sesquilinear form $M : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ by $M[x, y] := \langle x, Ty \rangle$ as in theorem 4.21. It is conjugate symmetric or Hermitian if and only if T is self-adjoint.*

Proof. ($\Longrightarrow$): Assume M is Hermitian, then $\langle Tx, y \rangle = \overline{\langle y, Tx \rangle} = \overline{M[y, x]} = M[x, y] = \langle x, Ty \rangle$.
($\Longleftarrow$): Assume $T \in B(\mathcal{H})_{sa}$, then $M[x, y] = \langle x, Ty \rangle = \langle Tx, y \rangle = \overline{\langle y, Tx \rangle} = \overline{M[y, x]}$. □

Corollary 6.16. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})_{sa}$. We have $\langle x, Tx \rangle = \langle Tx, x \rangle \in \mathbb{R}$ for any $x \in \mathcal{H}$.*

Proof. By proposition 4.1.3. □

Proposition 6.17. *Let $\mathcal{H}$ be a Hilbert space. Any self-adjoint operator T is normal.*

Proof. $T^\dagger T = TT = TT^\dagger$. □

Lemma 6.18. *Let $\mathcal{H}$ be a Hilbert space, for any $T \in B(\mathcal{H})$, we have that $T^\dagger T, TT^\dagger \in B(\mathcal{H})_{sa}$.*

Proof. We have $(T^\dagger T)^\dagger = T^\dagger (T^\dagger)^\dagger = T^\dagger T$, so it is self-adjoint. Similarly, $(TT^\dagger)^\dagger = (T^\dagger)^\dagger T^\dagger = TT^\dagger$. □

Lemma 6.19. *Let $\mathcal{H}$ be a Hilbert space, for any $T \in B(\mathcal{H})_{sa}$, we have that $\left\|T^2\right\| = \|T\|^2$.*

Proof. $\left\|T^2\right\| = \left\|TT^\dagger\right\| = \|T\|^2$. □

Proposition 6.20. *Let $\mathcal{H}$ be a Hilbert space, for any normal $T \in B(\mathcal{H})$, we have that $\left\|T^2\right\| = \|T\|^2$.*

Proof. Since $T^\dagger T$ is self-adjoint, we have $\left\|(T^\dagger T)^2\right\| = \left\|T^\dagger T\right\|^2$. Thus,

$$
\begin{aligned}
\left\|T^2\right\| &= \left\|(T^2)^\dagger T^2\right\|^{\frac{1}{2}} \\
&= \left\|T^\dagger T^\dagger T T\right\|^{\frac{1}{2}} \\
&= \left\|T^\dagger T T^\dagger T\right\|^{\frac{1}{2}} \\
&= \left\|(T^\dagger T)^2\right\|^{\frac{1}{2}} \\
&= (\left\|T^\dagger T\right\|^2)^{\frac{1}{2}} \\
&= \left\|T^\dagger T\right\| \\
&= \|T\|^2.
\end{aligned}
$$

□

Definition 6.6. Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})_{sa}$. We say T is **positive semidefinite**, written as $T \geq 0$, if

$$\forall x \in \mathcal{H},\ \langle Tx, x \rangle = \langle x, Tx \rangle \geq 0.$$

For $S, T \in B(\mathcal{H})_{sa}$, we say $S \leq T$ if $T - S \geq 0$. We also define $B(\mathcal{H})^+$ to be the set of all positive semidefinite operators.

Proposition 6.21. *Let $\mathcal{H}$ be a Hilbert space, for any $T \in B(\mathcal{H})$, we have that $T^\dagger T, TT^\dagger \geq 0$.*

Proof. They are self-adjoint by the above lemma. Consider any $x \in \mathcal{H}$, we have

$$
\begin{aligned}
\left\langle T^\dagger T x, x \right\rangle &= \langle Tx, Tx \rangle \\
&= \|Tx\|^2 \\
&\geq 0, \\
\left\langle TT^\dagger x, x \right\rangle &= \left\langle T^\dagger x, T^\dagger x \right\rangle \\
&= \left\|T^\dagger x\right\|^2 \\
&\geq 0.
\end{aligned}
$$

□

Proposition 6.22. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(X)$. Consider the bounded sesquilinear form $M : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ by $M[x, y] := \langle x, Ty \rangle$ as in theorem 4.21. It is positive semidefinite if and only if T is positive semidefinite.*

Proposition 6.23. *Let $\mathcal{H}$ be a Hilbert space, and $S, T \in B(\mathcal{H})_{sa}$.*

1. *Suppose $S \leq T$, then $A^\dagger S A \leq A^\dagger T A$ for all $A \in B(\mathcal{H})$.*
2. *Suppose $0 \leq S \leq T$, then $\|S\| \leq \|T\|$.*
3. *$-\|T\|\mathbb{I} \leq T \leq \|T\|\mathbb{I}$.*
4. *Suppose $-C\mathbb{I} \leq T \leq C\mathbb{I}$ for some $C \geq 0$, then $\|T\| \leq C$.*
5. *Suppose $0 \leq S$, we have $aS^n \geq 0$ for any non-negative $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$.*

Proof.

1. For any $x \in \mathcal{H}$, we have $\langle (T - S)Ax, Ax \rangle \geq 0$, since $T - S \geq 0$. Thus, $\left\langle A^\dagger (T - S)Ax, x \right\rangle \geq 0$, which means $A^\dagger T A - A^\dagger S A = A^\dagger (T - S)A \geq 0$.

2. Consider $M[x, y] := \langle x, Sy \rangle = \langle Sx, y \rangle$, which defines a positive semidefinite form 4.3 on $\mathcal{H}$. By Cauchy-Schwartz theorem 4.3, we have

$$\begin{aligned} |\langle Sx, y \rangle|^2 &\leq \langle Sx, x \rangle \langle Sy, y \rangle \\ &\leq \langle Tx, x \rangle \langle Ty, y \rangle \\ &\leq \|T\| \|x\|^2 \|T\| \|y\|^2 \\ &\leq \|T\|^2 \|x\|^2 \|y\|^2. \end{aligned}$$

Thus $|\langle Sx, y \rangle| \leq \|T\|$ for any $\|x\|, \|y\| \leq 1$, which means $\|S\| \leq \|T\|$.

3. Since $|\langle Tx, x \rangle| \leq \|T\| \|x\|^2$, we have

$$\begin{aligned} -\|T\| \|x\|^2 \leq \langle Tx, x \rangle \leq \|T\| \|x\|^2 \\ -\langle \|T\| x, x \rangle \leq \langle Tx, x \rangle \leq \langle \|T\| x, x \rangle. \end{aligned}$$

4. Consider $M[x, y] := \langle x, Ty \rangle = \langle Tx, y \rangle$, which defines a conjugate symmetric sesquilinear form on $\mathcal{H}$. For any $x, y \in \mathcal{H}$, we have $\langle T(x + y), x + y \rangle = \langle Tx, x \rangle + 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$ and $\langle T(x - y), x - y \rangle = \langle Tx, x \rangle - 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$ by proposition 4.1, so

$$\langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle = 4\Re(\langle Tx, y \rangle).$$

Similarly, $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

By assumption, we have

$$\langle T(x + y), x + y \rangle \leq \langle C\mathbb{I}(x + y), x + y \rangle = C\|x + y\|^2$$

since $T \leq C\mathbb{I}$, and

$$\langle T(x - y), x - y \rangle \geq \langle -C\mathbb{I}(x - y), x - y \rangle = -C\|x - y\|^2$$

since $T \geq -C\mathbb{I}$. Thus we have

$$\begin{aligned} \langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle &\leq C\|x + y\|^2 - (-C\|x - y\|^2) \\ &= C\Big(\|x + y\|^2 + \|x - y\|^2\Big) \\ 4\Re(\langle Tx, y \rangle) &\leq 2C(\|x\|^2 + \|y\|^2). \end{aligned}$$

Taking supreme over all $\|x\|, \|y\| \leq 1$, we have $4\|T\| \leq 4C$.

5. It is obviously true when $n = 0, 1$. Now assume $n \geq 2$, and it already holds for $n - 2$. Clearly aS^n is self-adjoint. For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle aS^n x, x\rangle &= a\big\langle SS^{n-2}Sx, x\big\rangle \\ &= a\big\langle S^{n-2}Sx, S^\dagger x\big\rangle \\ &= a\big\langle S^{n-2}(Sx), Sx\big\rangle \\ &\geq 0.\end{aligned}$$

By induction, it holds for all $n \in \mathbb{N}$.

□

Theorem 6.24 (Dini's)**.** *Let K be compact, and $f \in C(K)$. Suppose there are $(g_n)_{n=1}^\infty \subseteq C(K)$ such that they are monotone increasing, and for all $x \in K$, $g_n(x) \nearrow f(x)$ pointwise, then $g_n \to f$ uniformly.*

Proof. Let $g_n := f - q_n$, we have that $g_n(x) \searrow 0$ monotonously pointwise. Given any $\epsilon > 0$, for each $n \geq 1$, define

$$U_n := \{x \in K : g_n(x) < \epsilon\},$$

which are open in K. We have that $K = \bigcup_{n=1}^\infty U_n$ is an open cover. Since K is compact, there is $N \geq 1$, such that $K = \bigcup_{n=1}^N U_n$. Since $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_N$, we have that $K = U_N$. For any $n \geq N$, we have that $K = U_N \subseteq U_n \subseteq K$, so $K = U_n$. This show that $\forall x \in K$,

$$f(x) - q_n(x) = g_n(x) < \epsilon.$$

□

Lemma 6.25. *For $f \in C([0,1])$, with $f(t) := 1 - \sqrt{1-t}$, there are polynomials $(p_n)_{n=1}^\infty$, whose coefficients are all non-negative, then $f = \sum_{n=1}^\infty p_n$ uniformly.*

Proof. Let $q_0(t) := 0$, and $q_n(t) := \frac{1}{2}(t + q_{n-1}(t)^2)$. By induction, $0 \leq q_n \leq 1$, and q_n has non-negative coefficients. Also, $q_n \leq q_{n+1}$.

Define $p_n := q_n - q_{n-1}$ for all $n \geq 1$.

$$\begin{aligned}p_n(q_n + q_{n-1}) &= (q_n - q_{n-1})(q_n + q_{n-1}) \\ &= q_n^2 - q_{n-1}^2 \\ &= 2q_{n+1} - t - (1q_n - t) \\ &= 2(q_{n+1} - q_n) \\ &= 2p_{n+1}.\end{aligned}$$

By induction, we have that all p_n have non-negative coefficient.

For any $t \in [0,1]$, since $q_n(t)$ is monotone increasing, there is some $q(t) := \lim_{n\to\infty} q_n(t) \leq 1$. Also, $q(t) = \frac{1}{2}(t + q(t)^2)$. Thus, $q(t) = 1 - \sqrt{1-t} = f(t)$. We now have

$$f(t) = \lim_{n\to\infty} q_n(t) = \sum_{n=1}^\infty p_n(t).$$

This convergence is uniform by Dini's Theorem. □

Theorem 6.26 (Square Root Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space, and $0 \leq T$, then there is a unique $0 \leq S$, such that $S^2 = T$. We write $S := T^{\frac{1}{2}}$, and it commutes with all operators that commute with T.*

Proof. Assume $0 \leq T \leq \mathbb{I}$. Consider $S := \mathbb{I} - T$. Note that $0 \leq S \leq \mathbb{I}$. Let $p_n(t) := \sum_k \alpha_{n,k} t^k$ and $q_n := \sum_{i=1}^n p_n$ be from the above lemma. For any $n \geq 1$, we have

$$\begin{aligned}
\|p_n(S)\| &\leq \sum_k \alpha_{n,k} \|S\|^k \\
&\leq \sum_k \alpha_{n,k} \|\mathbb{I}\|^k \\
&= \sum_k \alpha_{n,k} 1^k \\
&= p_n(1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} \|p_n(S)\| &\leq \sum_{n=1}^{\infty} p_n(1) \\
&= f(1) \\
&= 1.
\end{aligned}$$

Since $B(\mathcal{H})$ is a Banach space, we can define

$$R := \sum_{n=1}^{\infty} p_n(S) = \lim_{n \to \infty} q_n(S)$$

by A2Q4. Also, $\|R\| \leq 1$, so $R \leq \mathbb{I}$. Also, since each coefficient in q_n is non-negative, we have that each $q_n(S) \geq 0$. Thus, $R \geq 0$. We now have

$$\begin{aligned}
(\mathbb{I} - R)^2 &= (\mathbb{I} - \lim_{n \to \infty} q_n(S))^2 \\
&= \lim_{n \to \infty} (\mathbb{I} - q_n(S))^2 \\
&= \lim_{n \to \infty} (\mathbb{I} - 2q_n(S) + q_n(S)^2) \\
&= \lim_{n \to \infty} (\mathbb{I} - 2q_n(S) + 2q_{n+1}(S) - S) \\
&= \mathbb{I} - S - 2 \lim_{n \to \infty} q_n(S) + 2 \lim_{n \to \infty} q_{n+1}(S) \\
&= T - 2R + 2R \\
&= T.
\end{aligned}$$

We can thus define $T^{\frac{1}{2}} := \mathbb{I} - R$.

For any A that commutes with T, we have

$$SA = A(\mathbb{I} - T) = A - AT = A - TA = (\mathbb{I} - T)A = SA.$$

Also, for any $k \geq 1$, we have

$$S^k A = S^{k-1} A S = S^{k-2} A S^2 = \cdots = A S^k.$$

Thus, $Aq_n(S) = q_n(S)A$, so $AR = RA$. This shows

$$AT^{\frac{1}{2}} = A(\mathbb{I} - R) = A - AR = A - RA = (\mathbb{I} - R)A = T^{\frac{1}{2}} A.$$

Now for any general T, we can just take $\tilde{T} := \frac{1}{\|T\|} T$, so we have $\tilde{T} \leq \frac{1}{\|T\|} \|T\| \mathbb{I} = \mathbb{I}$. Now take $T^{\frac{1}{2}} := \sqrt{\|T\|} \tilde{T}^{\frac{1}{2}}$. Clearly, $T^{\frac{1}{2}}$ is self-adjoint, and

$$T^{\frac{1}{2}} T^{\frac{1}{2}} = \sqrt{\|T\|} \tilde{T}^{\frac{1}{2}} \sqrt{\|T\|} \tilde{T}^{\frac{1}{2}} = \|T\| \tilde{T} = T.$$

Also, if $AT = TA$, we will have $A\tilde{T} = \tilde{T}A$, so $A\tilde{T}^{\frac{1}{2}} = \tilde{T}^{\frac{1}{2}}A$ and $AT^{\frac{1}{2}} = T^{\frac{1}{2}}A$.

Suppose there is some other $A \geq 0$, such that $A^2 = T$, we have $AT = A^3 = TA$. Since A commutes with T, it also commutes with $T^{\frac{1}{2}}$. We thus have

$$(T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A) = T - A^2 = 0.$$

Now let $Y := \mathrm{Im}(T^{\frac{1}{2}} + A)$. For any $y \in Y$, we have $(T^{\frac{1}{2}} - A)(y) = 0$, since there will be $x \in \mathcal{H}$, such that $(T^{\frac{1}{2}} - A)(x) = y$, so

$$(T^{\frac{1}{2}} - A)(y) = (T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A)(x_n) = 0.$$

On the other hand, for any $z \in Y^{\perp} = \ker((T^{\frac{1}{2}} + A)^{\dagger}) = \ker(T^{\frac{1}{2}} - A)$ by proposition 6.9, we have that

$$\begin{aligned}\left\|T^{\frac{1}{4}}z\right\|^2 &= \left\langle T^{\frac{1}{4}}z, T^{\frac{1}{4}}z\right\rangle \\ &= \left\langle z, T^{\frac{1}{4}}T^{\frac{1}{4}}z\right\rangle \\ &= \left\langle z, T^{\frac{1}{2}}z\right\rangle \\ &\leq \left\langle z, T^{\frac{1}{2}}z\right\rangle + \langle z, Az\rangle \\ &= \left\langle z, (T^{\frac{1}{2}} + A)z\right\rangle \\ &= 0.\end{aligned}$$

Thus, $T^{\frac{1}{4}}z = 0$, and $T^{\frac{1}{2}}z = T^{\frac{1}{4}}T^{\frac{1}{4}}z = 0$. Similarly,

$$\begin{aligned}\left\|A^{\frac{1}{2}}z\right\|^2 &= \left\langle A^{\frac{1}{2}}z, A^{\frac{1}{2}}z\right\rangle \\ &= \langle z, Az\rangle \\ &\leq \left\langle z, (A + T^{\frac{1}{2}})z\right\rangle \\ &= 0,\end{aligned}$$

so $A^{\frac{1}{2}}z = 0$ and $Az = 0$. Thus,

$$(T^{\frac{1}{2}} - A)(z) = T^{\frac{1}{2}}z - Az = 0.$$

Now for any $x \in \mathcal{H}$, there is $y \in Y$, and $z \in Y^{\perp}$, such that $x = y + z$, so we have

$$(A - T^{\frac{1}{2}})(x) = (A - T^{\frac{1}{2}})(y) + (A - T^{\frac{1}{2}})(z) = 0.$$

This shows $T^{\frac{1}{2}} = A$, which means $T^{\frac{1}{2}}$ is unique. □

Corollary 6.27. *Let $\mathcal{H}$ be a Hilbert space. Suppose $S, T \geq 0$, and $ST = TS$, then $ST \geq 0$.*

Proof. We have $(ST)^{\dagger} = T^{\dagger}S^{\dagger} = TS = ST$, so $ST \in B(\mathcal{H})_{sa}$.

Also, for any $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle STx, x\rangle &= \left\langle S^{\frac{1}{2}}S^{\frac{1}{2}}Tx, x\right\rangle \\ &= \left\langle S^{\frac{1}{2}}Tx, (S^{\frac{1}{2}})^{\dagger}x\right\rangle \\ &= \left\langle TS^{\frac{1}{2}}x, S^{\frac{1}{2}}x\right\rangle \\ &\geq 0.\end{aligned}$$

□

Corollary 6.28. *Let $\mathcal{H}$ be a Hilbert space. Suppose $T \in B(\mathcal{H})_{sa}$, and $\langle Tx, x\rangle = 0$ for all $x \in \mathcal{H}$, then $T = 0$.*

Proof. For any $x \in \mathcal{H}$, we have $\langle Tx, x\rangle = 0 \geq 0$, so $T \geq 0$. In addition,

$$\begin{aligned}0 &= \langle Tx, x\rangle \\ &= \left\langle T^{\frac{1}{2}}T^{\frac{1}{2}}x, x\right\rangle \\ &= \left\langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\right\rangle \\ &= \left\|T^{\frac{1}{2}}x\right\|^2.\end{aligned}$$

Thus $T^{\frac{1}{2}}x = 0$, so $Tx = T^{\frac{1}{2}}T^{\frac{1}{2}}x = T^{\frac{1}{2}}0 = 0$. □

Proposition 6.29. *Let $\mathcal{H}$ be a Hilbert space. Suppose $T \geq 0$ is invertible, then $T^{-1} \geq 0$. Thus, there is $T^{-\frac{1}{2}} \geq 0$, such that $T^{-\frac{1}{2}} = (T^{\frac{1}{2}})^{-1}, (T^{-\frac{1}{2}})^2 = T^{-1}$.*

Proof. For any $x \in \mathcal{H}$, let $y := T^{-1}x$, then we have

$$\begin{aligned}\left\langle x, T^{-1}x\right\rangle &= \langle Ty, y\rangle \\ &= \langle y, Ty\rangle \\ &\geq 0.\end{aligned}$$

Thus, there is $T^{-\frac{1}{2}} \geq 0$, such that $(T^{-\frac{1}{2}})^2 = T^{-1}$. Also, since $TT^{-1} = T^{-1}T$ commutes, $T^{\frac{1}{2}}T^{-1} = T^{-1}T^{\frac{1}{2}}$ commutes, and $T^{-\frac{1}{2}}T^{\frac{1}{2}} = T^{\frac{1}{2}}T^{-\frac{1}{2}}$ commutes. By corollary 6.27, $T^{\frac{1}{2}}T^{-\frac{1}{2}} \geq 0$. Now,

$$\begin{aligned}(T^{\frac{1}{2}}T^{-\frac{1}{2}})^2 &= T^{\frac{1}{2}}T^{-\frac{1}{2}}T^{\frac{1}{2}}T^{-\frac{1}{2}} \\ &= T^{\frac{1}{2}}T^{\frac{1}{2}}T^{-\frac{1}{2}}T^{-\frac{1}{2}} \\ &= TT{-}1 \\ &= \mathbb{I}.\end{aligned}$$

Since $\mathbb{I} \geq 0$ with $\mathbb{I}^2 = \mathbb{I}$, we must have $T^{\frac{1}{2}}T^{-\frac{1}{2}} = \mathbb{I}^{\frac{1}{2}} = \mathbb{I}$. □

Lemma 6.30. *Let $\mathcal{H}$ be a Hilbert space. Suppose $T \geq \mathbb{I} \geq 0$ is invertible, then $\mathbb{I} \geq T^{-1} \geq 0$.*

Proof. Since $T(T - \mathbb{I}) = T^2 - T = (T - \mathbb{I})T$, we have that $T(T - \mathbb{I}) \geq 0$ by corollary 6.27. For any $x \in \mathcal{H}$, define $y := T^{-1}x$. We have

$$\begin{aligned}\left\langle x, \left(\mathbb{I} - T^{-1}\right)x\right\rangle &= \langle x, x\rangle - \left\langle x, T^{-1}x\right\rangle \\ &= \langle Ty, Ty\rangle - \langle Ty, y\rangle \\ &= \langle Ty, Ty - y\rangle \\ &= \langle y, T(T - \mathbb{I})y\rangle \\ &\geq 0.\end{aligned}$$

□

Proposition 6.31. *Let $\mathcal{H}$ be a Hilbert space. Suppose $T \geq S \geq 0$ are both invertible, then $S^{-1} \geq T^{-1} \geq 0$.*

Proof. Since $S^{-\frac{1}{2}} = (S^{-\frac{1}{2}})^\dagger$, by proposition 6.23, we have that

$$S^{-\frac{1}{2}}TS^{-\frac{1}{2}} \geq S^{-\frac{1}{2}}SS^{-\frac{1}{2}} = S^{-\frac{1}{2}}S^{\frac{1}{2}}S^{\frac{1}{2}}S^{-\frac{1}{2}} = \mathbb{I},$$

Thus, $\mathbb{I} \geq (S^{-\frac{1}{2}}TS^{-\frac{1}{2}})^{-1} = S^{\frac{1}{2}}T^{-1}S^{\frac{1}{2}}$, and

$$S^{-1} = S^{-\frac{1}{2}}\mathbb{I}S^{-\frac{1}{2}} \geq S^{-\frac{1}{2}}S^{\frac{1}{2}}T^{-1}S^{\frac{1}{2}}S^{-\frac{1}{2}} = T^{-1}.$$

□

Proposition 6.32. *Let $\mathcal{H}$ be a Hilbert space. Suppose $T \in B(\mathcal{H})_{sa}$, then T is invertible if and only if T is bounded below. Suppose $T \geq 0$, then T is invertible if and only if $T \geq \epsilon\mathbb{I}$ for some $\epsilon > 0$.*

Proof. The first claim is by proposition 6.11, and $T = T^\dagger$.

Now suppose $T \geq \epsilon\mathbb{I}$ for some $\epsilon > 0$, then for any $x \in \mathcal{H}$, we have $\langle x, Tx\rangle \geq \langle x, \epsilon x\rangle = \epsilon\|x\|^2$. By corollary 6.13,$T$ is invertible.

On the other hand, suppose T is invertible, we have that $T^{-1} \geq 0$. Since $T^{-1} \leq \left\|T^{-1}\right\|\mathbb{I}$ by proposition 6.23, we have $\frac{1}{\|T^{-1}\|}\mathbb{I} = (\left\|T^{-1}\right\|\mathbb{I})^{-1} \leq (T^{-1})^{-1} = T$. ☐

6.4 Unitary Operators

Definition 6.7. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We say $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ is **unitary** if it is invertible, namely, it is an isometric isomorphism between Banach Spaces.

Proposition 6.33. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ is isometric if and only if it preserves the inner product. Namely,*

$$\forall x, y \in \mathcal{H}_1,\ \langle x, y\rangle = \langle Ux, Uy\rangle.$$

Proof. ($\Longrightarrow$): Suppose U is isometric, then by the polarization identity proposition 4.6, it preserves the inner product as well.

($\Longleftarrow$): Suppose U preserves the inner product, then $\forall x \in \mathcal{H}_1,\ \|x\|^2 = \langle x, x\rangle = \langle Ux, Ux\rangle = \|Ux\|^2$. ☐

Proposition 6.34. *Let $\mathcal{H}$ be a Hilbert space, and $U \in B(\mathcal{H})$. U is an isometry if and only if $U^\dagger U = \mathbb{I}$. Also, the following are equivalent:*

1. *U is unitary,*
2. *U is invertible and preserves the inner product,*
3. *$U^\dagger = U^{-1}$, and*
4. *U is a normal isometry.*

Proof. See A5. ☐

7 Compact Operators

7.1 Compactness

See more in Real Analysis Pmath351 or Applied Functional Analysis Amath731.

Definition 7.1. Let (X, d) be a metric space. We say $S \subseteq X$ is **sequentially compact** if for every bounded sequence $(x_n)_{n=1}^\infty$, there is a convergent subsequence $x_{n_k} \to x \in S$.

Definition 7.2. Let (X, d) be a metric space. We say $S \subseteq X$ is **relatively compact** if for every bounded sequence $(x_n)_{n=1}^\infty$, there is a convergent subsequence $x_{n_k} \to x \in X$.

Definition 7.3. Let X be a metric space. We say $S \subseteq X$ is **totally bounded** if for any $\epsilon > 0$, there exists a finite subset $\{x_i\}_{i=1}^n$, such that $S \subseteq \bigcup_{i=1}^n B(a_i, \epsilon)$. Such a subset is called an **ϵ-net**.

Definition 7.4. Let X be a metric space. We say $S \subseteq X$ is **bounded** if there exists a $r > 0$, such that for all $x, y \in X,\ d(x, y) < r$.

Theorem 7.1. *Let S be a subset of a metric space X. The following are equivalent:*

1. *S is compact.*
2. *S has the finite intersection property.*
3. *S is sequentially compact.*

4. *Every infinite subset $A \subseteq S$ has a limit point.*

5. *S is complete and totally bounded.*

6. *S is relatively compact and closed.*

Corollary 7.2. *Let S be a subset of a metric space X, then S is relatively compact if and only if $\bar{S}$ is compact.*

Corollary 7.3. *Let S be a subset of a complete space X, then S is relatively compact if and only if S is totally bounded.*

Proposition 7.4. *Let S be a subset of a metric space X.*

1. *Suppose S is relatively compact, then S is bounded.*

2. *Suppose S is totally bounded, then S is bounded.*

3. *Suppose S is compact, then S is closed and bounded.*

Theorem 7.5 (Heine-Borel)**.** *Let S be a subset of a finite dimensional normed vector space X, then S is compact if and only if S is closed and bounded.*

7.2 Compact Operators

Definition 7.5. Let X, Y be normed vector spaces. A linear operator $K : X \to Y$ is **compact** if $\overline{K(B^X(0,1))}$ is compact in Y. Let $\mathcal{K}(X, Y)$ denote the set of compact operators.

Proposition 7.6. *Let X, Y be normed vector spaces, and $K : X \to Y$ be a linear operator, then the following are equivalent:*

1. *K is compact.*

2. *$K(B^X(0,1))$ is relative compact in Y.*

3. *For any bounded set $S \subseteq X$, $\overline{K(F)}$ is compact in Y.*

4. *For any bounded set $S \subseteq X$, $K(F)$ is relative compact in Y.*

5. *Each bounded sequence $(x_n)_{n=1}^{\infty} \subseteq X$ has some subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $(Kx_{n_k})_{k=1}^{\infty}$ converges to some $y \in Y$.*

Proposition 7.7. *Let X, Y be normed vector spaces. Suppose a linear operator $K : X \to Y$ is compact, then it is always bounded. Namely, $\mathcal{K}(X, Y) \subseteq B(X, Y)$.*

Proposition 7.8. *Let X be an infinite-dimensional Banach space, then the identity map $\mathbb{I} : X \to X$ is not compact.*

Proof. $\bar{B}^X(0,1)$ is not compact by A2. □

Proposition 7.9. *Let X be a Banach space. Suppose $E = E^2 \in B(X)$ is compact, then E is finite rank.*

Proof. See A5. □

Definition 7.6. Let X, Y be vector spaces. Let

$$\mathcal{F}(X, Y) : \{T \in B(x, Y) | \operatorname{rank}(T) < \infty\}$$

denote the set of finite rank bounded linear operators.

Proposition 7.10. *Let X, Y be normed vector spaces. Suppose a linear operator $T \in B(x, Y)$ is finite-dimensional, then it is always compact. Namely, $\mathcal{F}(X, Y) \subseteq \mathcal{K}(X, Y)$.*

Proof. $\mathrm{Im}(K) \subseteq Y$ is finite dimensional. For any bounded $M \subseteq X$, we have that $K(M)$ is bounded. Since $\overline{K(M)} \subseteq \mathrm{Im}(T)$ is closed and bounded, it is compact in $\mathrm{Im}(T)$, thus compact in Y. □

Theorem 7.11. *Let X, Y be Banach Spaces. $\mathcal{K}(X, Y)$ is norm-closed $B(X) - B(Y)$ sub-bi-module of $B(X, Y)$. Namely, for any $a \in B(Y), b \in B(X)$, and $K \in \mathcal{K}(X, Y)$, we have $aKb \in \mathcal{K}(X, Y)$. Also, if $(K_n)_{n=1}^{\infty}$ is a sequence of compact operators in $\mathcal{K}(X, Y)$, and $K_n \to K$ in $B(X, Y)$, we have $K \in \mathcal{K}(X, Y)$ as well.*

Proof. Clearly $\mathcal{K}(X, Y)$ is a vector space.

Since $b \in B(X)$, we have $\overline{b(B^X(0,1))} \subseteq \overline{B^X(0, \|b\|)}$, which is bounded. Since K is compact, $\overline{K(B^X(0, \|b\|))}$ is compact. Thus, $\overline{Kb(B^X(0,1))} \subseteq \overline{K(B^X(0, \|b\|))}$ is compact since it is closed. Since $a \in B(Y)$ is continuous, we have that $a(\overline{Kb(B^X(0,1))})$ is compact as well. Thus $\overline{aKb(B^X(0,1))} \subseteq \overline{a(\overline{Kb(B^X(0,1))})} = a(\overline{Kb(B^X(0,1))})$ is compact, since it is closed.

This shows aTb is compact.

Now consider any sequence of compact operators $(K_n)_{n=1}^{\infty}$ in $\mathcal{K}(X, Y)$, with $K_n \to K$ in $B(X, Y)$. Given any $\epsilon > 0$, there is $N \geq 1$ such that $\|K - K_N\| < \frac{\epsilon}{3}$. Since K_N is compact, $\overline{K_N(B^X(0,1))}$ is compact and thus totally bounded, so we can find $\{x_1, \ldots, x_m\} \subseteq B^X(0,1)$, such that $\overline{K_N(B^X(0,1))} \subseteq \bigcup_{i=1}^m B^X\left(x_i, \frac{\epsilon}{3}\right)$. Now for any $x \in B^X(0,1)$, consider $K_N x \in B^X\left(x_i, \frac{\epsilon}{3}\right)$, we have

$$\begin{aligned}
\|Kx - Kx_i\| &\leq \|Kx - K_N x\| + \|K_N x - K_N x_i\| + \|K_N x_i - Kx_i\| \\
&< \|K - K_N\|\|x\| + \frac{\epsilon}{3} + \|K_N - K\|\|x_i\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

This shows that $K(B^X(0,1)) \subseteq \bigcup_{i=1}^m B^X(x_i, \epsilon)$, so it is totally bounded and thus relatively compact. This proves K is compact, so $\mathcal{K}(X, Y)$ is norm-closed. □

Proposition 7.12. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We have*

$$\overline{\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)} = \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

Proof. Let $K \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, and $\epsilon > 0$. We can find an ϵ-net $\{Kx_1, \cdots, Kx_m\}$ for $K(B^{\mathcal{H}_1}(0,1))$, since it is totally bounded.

Let $K_0 := \mathrm{Span}\{Kx_1, \cdots, Kx_m\} \subseteq \mathcal{H}_2$, and consider $P : \mathcal{H}_2 \to K_0$, the orthogonal projection onto K_0. For any $x \in B^{\mathcal{H}_1}(0,1)$, we have

$$\begin{aligned}
\|(K - PK)x\| &= \|(\mathbb{I} - P)Kx\| \\
&= \mathrm{dist}(Kx, K_0) \\
&\leq \min_{i \in [m]} \|Kx - Kx_i\| \\
&< \epsilon.
\end{aligned}$$

Thus, $\|K - PK\| < \epsilon$, where $\mathrm{rank}(PK) \leq \dim(K_0) \leq m < \infty$. □

Corollary 7.13. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. For $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, we have $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $T^\dagger \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. By finite-dimensional approximation. □

Example 7.2.1. Consider $f \in \ell^\infty(\mathbb{N})$, and the multiplication operator $M_f : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$. We have that M_f is compact if and only if $f \in C_0(\mathbb{N})$, namely, $\lim_{i \to \infty} f(i) = 0$.

Indeed, if $f \in C_0(\mathbb{N})$, we can let $f_N(i) := \begin{cases} f(i), & i \leq N \\ 0, & \text{o.w.} \end{cases}$. We have $\|M_f - M_{f_N}\| = \|M_{f - f_N}\| = \|f - f_N\|_\infty \to 0$.

On the other hand, if $f \notin C_0(\mathbb{N})$, there is $\epsilon > 0$, and $i_1 < i_2 < i_3 < \cdots$, such that $|f(i_k)| \geq \epsilon$. Let $K_0 := \overline{\operatorname{Span}\{e_{i_k}\}} \subseteq \ell^2(\mathbb{N})$, we have $M_f(e_{i_k})(i) = \begin{cases} f(i), & i = i_k \\ 0, & \text{o.w.} \end{cases}$, so $M_f(e_{i_k}) = f(i_k)e_{i_k} \in K_0$. Thus, $K := M_f|_{K_0} : K_0 \to K_0$ is well defined. In addition, K is bounded below by ϵ. By theorem 3.12, it has a bounded inverse K^{-1}.

Suppose for contradiction that M_f is compact, then $\overline{K(B^{K_0}(0,1))} \subseteq \overline{M_f(B^{\ell^2(\mathbb{N})}(0,1))}$. Since K^{-1} is bounded, $K^{-1}\overline{K(B^{K_0}(0,1))}$ is compact as well. Now

$$\overline{B^{K_0}(0,1)} = \overline{K^{-1}K(B^{K_0}(0,1))} \subseteq \overline{K^{-1}\overline{K(B^{K_0}(0,1))}} = K^{-1}\overline{K(B^{K_0}(0,1))}$$

is compact. Since K_0 is infinite dimensional, $\overline{B^{K_0}(0,1)}$ cannot be compact, a contradiction.

7.3 Integral Operators

Definition 7.7. Let $k \in L^2([0,1]\times[0,1])$, the **integral operator** with **kernel** k is $T_k \in B(L^2(0,1))$, with

$$(T_k f)(x) := \int_0^1 k(x,y)f(y)dy.$$

Lemma 7.14. *Let $k \in L^2([0,1]\times[0,1])$, the integral operator with kernel k is well-defined.*

Proof. Consider any $f \in L^2(0,1)$,

$$\begin{aligned}
\|T_k f\|_2^2 &= \int_0^1 \left| \int_0^1 k(x,y)f(y)dy \right|^2 dx \\
&\leq \int_0^1 \left(\int_0^1 |k(x,y)|^2 dy \right) \|f\|_2^2 dx \\
&= \|f\|_2^2 \|k\|_2^2 \\
&< \infty.
\end{aligned}$$

Thus, $T_k f \in L^2(0,1)$. $\square$

Proposition 7.15. *Let $k \in L^2([0,1]\times[0,1])$, then T_k is compact.*

Proof. Pick any orthonormal basis $(e_i)_{i=1}^\infty$ for $L^2(0,1)$, then $e_{ij}(x,y) := e_i(x)e_j(y)$ forms an orthornormal basis for $L^2([0,1]\times[0,1])$.

Thus, $k = \sum_{i,j=1}^\infty a^{ij}e_{ij}$, where $\|k\|_2^2 = \sum_{i,j=1}^\infty |a_{ij}|^2$. Let $k_N := \sum_{i,j=1}^N a^{ij}e_{ij}$. Thus,

$$\|T_k - T_{k_N}\|^2 \leq \|k - k_N\|_2^2 = \sum_{i,j=N+1}^\infty |a_{ij}|^2 \to 0.$$

Now for any $f \in L^2(0,1)$, we have

$$\begin{aligned}
(T_{k_N} f)(x) &= \int_0^1 k_N(x,y) f(y) dy \\
&= \langle \bar{k_N}(x,\cdot), f \rangle \\
&= \left\langle \sum_{i,j=1}^{N} \bar{a^{ij}} \bar{e_{ij}}(x,\cdot), f \right\rangle \\
&= \sum_{i,j=1}^{N} a^{ij} e_i(x) \langle \bar{e_j}, f \rangle \\
&= \sum_{i,j=1}^{N} a^{ij} \langle \bar{e_j}, f \rangle e_i(x) \\
T_{k_N} f &= \sum_{i,j=1}^{N} a_{ij} \langle \bar{e_j}, f \rangle e_i \\
&\in \mathrm{Span}\left\{ e_1, \ldots, e_N \right\}.
\end{aligned}$$

Thus, T_{k_N} is finite rank and thus compact, which means T_k is compact as well. □

Proposition 7.16. *Let $k \in L^2([0,1] \times [0,1])$, then*

$$(T_k^\dagger g)(x) = \int_0^1 k^*(x,y) g(y) dy,$$

where $k^*(x,y) := \overline{k(y,x)}$.

Proof. Consider any $f, g \in L^2(0,1)$, let $h(x) := \int_0^1 k^*(x,y) g(y) dy$, we have

$$\begin{aligned}
\langle V^* g, f \rangle &= \langle g, V f \rangle \\
&= \int_0^1 g(x) \overline{V f}(x) dx \\
&= \int_0^1 g(x) \overline{\int_0^1 k(x,y) f(y) dy dx} \\
&= \int_0^1 g(x) \int_0^1 \overline{k}(x,y) \overline{f}(y) dy dx \\
&= \int_0^1 \int_0^1 g(x) k^*(y,x) dx \overline{f}(y) dy \\
&= \int_0^1 h(y) \bar{f}(y) dy \\
&= \langle h, f \rangle.
\end{aligned}$$

We can swap the order of integration by the Fubini Theorem and the fact that $f, g \in L^2(0,1)$.

Thus we have $(T_k^\dagger g)(x) = h(x) = \int_0^1 k^*(x,y) g(y) dy$. □

Example 7.3.1. The **Volterra Operator** is $V : L^2(0,1) \to L^2(0,1)$ by

$$(V f)(x) := \int_0^x f(y) dy.$$

Its kernel is $k(x,y) := \chi_{y\le x}(x,y) := \begin{cases} 1, & y \le x \\ 0, & \text{o.w.}. \end{cases}$. We have

$$\begin{aligned}
\int_0^1 \int_0^1 |\chi_{y\le x}(x,y)|^2 dxdy &= \int_0^1 \int_0^y 1dxdy \\
&= \int_0^1 ydy \\
&= \frac{1}{2},
\end{aligned}$$

so $k \in L^2([0,1]\times[0,1])$ by the Fubini Theorem. Also,

$$\begin{aligned}
(T_k f)(x) &= \int_0^1 k(x,y)f(y)dy \\
&= \int_0^1 \chi_{y\le x}(x,y)f(y)dy \\
&= \int_0^x f(y)dy \\
&= (Vf)(x).
\end{aligned}$$

See more in A5Q6.

8 Spectral Theory

8.1 Spectrum

Definition 8.1. Let X be a Banach space, and $T \in B(X)$. The **spectrum** of T is

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda\mathbb{I} - T \text{ is not invertible.}\}.$$

The **point spectrum** of T is

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda\mathbb{I} - T \text{ is not injective.}\} = \{\lambda \in \mathbb{C} : \exists v \neq 0 \text{ such that } Tv = \lambda v\} = \{\text{eigenvalues of } T\}.$$

The **resolvent** of T is

$$\rho(T) := \mathbb{C} \setminus \sigma(T).$$

Proposition 8.1. *When X is finite dimensional, then*

$$\sigma(T) = \{\lambda : \lambda\mathbb{I} - T \textit{ is not injective}\} = \sigma_p(T).$$

Example 8.1.1. Consider $f(n) := \frac{1}{n} \in \ell^\infty$, and the multiplication operator M_f we have that $0 \in \sigma(M_f)$, but not an eigenvalue.

Example 8.1.2. Let $f \in C([0,1])$, and the multiplication operator $M_f : L^2(0,1) \to L^2(0,1)$ by $(M_f(\xi))(x) := f(x)\xi(x)$, where we have $M_f M_g = M_{fg}, M_f^\dagger = M_{\bar{f}}, \|M_f\| = \|f\|_\infty$.

Suppose $\lambda \notin \mathrm{Im}(f)$, then $\lambda - f \neq 0$ on $[0,1]$, so $\frac{1}{\lambda - f} \in C([0,1])$. We thus have

$$M_{\frac{1}{\lambda-f}} = (M_{\lambda-f})^{-1} = (\lambda\mathbb{I} - M_f)^{-1}.$$

This shows $\sigma(M_f) \subseteq \mathrm{Im}(f)$.

On the other hand, suppose $\lambda = f(x_0)$ for some $x_0 \in [0,1]$, we have that $(\lambda - M_f)(\xi)(x) = (f(x_0) - f(x))\xi(x)$. Now for any $\epsilon > 0$, since f is continuous, we have that

$$A_\epsilon := \{x : |f(x) - f(x_0)| < \epsilon\}$$

has positive measure. Let $\xi_\epsilon := \frac{\chi_{A_\epsilon}}{\mu(A_\epsilon)^{\frac{1}{2}}}$, with $\|\xi_\epsilon\|_2 = 1$, we have that

$$\begin{aligned}\|(\lambda\mathbb{I} - M_f)\xi_\epsilon\|_2^2 &= \int_{A_\epsilon} |\lambda - f(x)|^2 \frac{1}{\mu(A_\epsilon)} dx \\ &< \frac{1}{\mu(A_\epsilon)} \int_{A_\epsilon} \epsilon^2 dx \\ &= \epsilon^2.\end{aligned}$$

Thus, $\lambda\mathbb{I} - M_f$ is not bounded below. By theorem 3.12, $\lambda\mathbb{I} - M_f$ is not invertible, so $\lambda \in \sigma(M_f)$.

Thus, $\sigma(M_f) = \mathrm{Im}(f)$.

However, such a M_f may have no eigenvalue! Let $f(x) = x$, if t is an eigenvalue with eigenvector ξ, we have $x\xi(x) = t\xi(x)$ a.e., so $(x - t)\xi(x) = 0$ a.e., which means $\xi(x) = 0$ a.e..

Theorem 8.2. *Let X be a Banach space, and $T \in B(X)$. We have $\sigma(T) \neq \emptyset$, and it is always compact and $\sigma(T) \subseteq \|T\|\mathbb{D}$.*

Proof. For any$|\lambda| > \|T\|$, we have that $\left\|\frac{T}{\lambda}\right\| < 1$, so $(\mathbb{I} - \frac{T}{\lambda})^{-1} = \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$, and

$$(\lambda\mathbb{I} - T)^{-1} = \left(\lambda\left(\mathbb{I} - \frac{T}{\lambda}\right)\right)^{-1} = \lambda^{-1}\left(\mathbb{I} - \frac{T}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}.$$

Thus, $\lambda \in \rho(T)$. This shows that $\forall \lambda \in \sigma(T),\ |\lambda| \leq \|T\|$, which means $\sigma(T) \subseteq \|T\|\mathbb{D}$ is bounded.

Take any $\phi \in B(\mathcal{H})^*$, we can define $f_\phi : \rho(T) \to \mathbb{C}$ by $f_\phi(z) := \phi\big((z\mathbb{I} - T)^{-1}\big)$, for any $z \in \rho(T) = \mathbb{C}$. Fix $z_0 \in \mathbb{C}$, and pick $|h| < \left\|(Z_0\mathbb{I} - T)^{-1}\right\|^{-1}$, we have $\left\|h(z_0\mathbb{I} - T)^{-1}\right\| < 1$, so $(\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})^{-1} = \sum_{k=0}^{\infty} \left(h(z_0\mathbb{I} - T)^{-1}\right)^k$.

$$\begin{aligned}f_\phi(z_0 + h) &= \phi\big((z_0\mathbb{I} + h\mathbb{I} - T)^{-1}\big) \\ &= \phi\big(((\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})(z_0\mathbb{I} - T))^{-1}\big) \\ &= \phi\left((z_0\mathbb{I} - T)^{-1} \sum_{k=0}^{\infty} \left(h(z_0\mathbb{I} - T)^{-1}\right)^k\right) \\ &= \sum_{k=0}^{\infty} h^k \phi\Big((z_0\mathbb{I} - T)^{-1}\big((z_0\mathbb{I} - T)^{-1}\big)^k\Big) \\ &= \sum_{k=0}^{\infty} h^k \phi\big((z_0\mathbb{I} - T)^{-1-k}\big),\end{aligned}$$

which is analytic. Now, suppose for contradiction that $\sigma(T) = \emptyset$, then $f_\phi : \mathbb{C} \to \mathbb{C}$ is an entire function. By Liouville's theorem, f_ϕ must be a constant function.

Also, we can see that when $|z| \to \infty$, $f_\phi(z) = \phi\big((z\mathbb{I} - T)^{-1}\big) \to 0$. Thus, f_ϕ must be constantly zero, for any ϕ. However, by Hahn-Banach Theorem, there has to be some $\phi \in B(\mathcal{H})^*$ that separates $\left\{(z\mathbb{I} - T)^{-1}\right\} \subseteq B(\mathcal{H})$, a contradiction.

This shows $\sigma(T) \neq \emptyset$.

In addition, the above argument shows that for any $z \in \rho(T)$, $(z_0\mathbb{I} + h\mathbb{I} - T)^{-1}$ is well-defined in a neighbourhood around it, so $\rho(T)$ is open and $\sigma(T)$ is closed. $\square$

Definition 8.2. Let X be a Banach space, and $T \in B(X)$. The **spectral radius** of T is

$$\mathrm{spr}(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Notice that by the above theorem, we have $\mathrm{spr}(T) = \max\{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$.

Theorem 8.3 (Spectral mapping)**.** *Let X be a Banach space, and $T \in B(X)$. For any polynomial p, we have that $\sigma(p(T)) = p(\sigma(T))$.*

Proof. Suppose $p(x)$ is order n. Let $q(x) = \lambda - p(x) = (\beta_1 - x) \cdots (\beta_n - x)$, where β_i's are the n roots of q.

Let $\lambda \in \mathbb{C}$, we have $\lambda \in \sigma(p(T))$, if and only if $q(T) = \lambda - p(T) = (\beta_1 \mathbb{I} - T) \cdots (\beta_n \mathbb{I} - T)$ is not invertible, if and only if there is some β_i, such that $\beta_i \mathbb{I} - T$ is not invertible, if and only if $q(x)$ has a root $\beta_i \in \sigma(T)$, if and only if $\lambda = p(\beta_i) \in p(\sigma(T))$. □

Theorem 8.4 (Spectral Radius Formula)**.** *Let X be a Banach space, and $T \in B(X)$.*

$$spr(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Proof. By the spectral mapping theorem, $\sigma(T^n) = \sigma(T)^n$. Thus, $\|T^n\| \geq \operatorname{spr}(T^n) = \operatorname{spr}(T)^n$, which means $\operatorname{spr}(T) \leq \|T^n\|^{\frac{1}{n}}$. Since this holds for all $n \in \mathbb{N}$, we have

$$\operatorname{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}.$$

On the other hand, consider any $\phi \in B(X)^*$, we have that $f_\phi(\lambda) := \phi((\lambda\mathbb{I} - T)^{-1})$ is analytic on $\rho(T)$ as in the proof of theorem 8.2. In addition, for $|\lambda| > \|T\|$, we have

$$f_\phi(\lambda) = \phi\left(\sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}\right) = \sum_{k=0}^{\infty} \frac{\phi(T^k)}{\lambda^{k+1}}.$$

Since there is no residue between the annulus $\|T\|$ and $\operatorname{spr}(T)$, from complex analysis, this Laurent series has to converge for any $|\lambda| > \operatorname{spr}(T)$.

Thus, for any $|\lambda| > \operatorname{spr}(T)$, $\frac{\phi(T^k)}{\lambda^{k+1}} = \phi\left(\frac{T^k}{\lambda^{k+1}}\right)$ has to be bounded. Since this holds for any $\phi \in B(X)^*$, by the Banach-Steinhaus Theorem 3.20, $\left\{\frac{T^k}{\lambda^{k+1}} : k \geq 0\right\}$ has to be bounded in $B(X)^{**}$, and thus in $B(X)$. Namely, there is $C_\lambda > 0$, such that $\left\|\frac{T^k}{\lambda^{k+1}}\right\| < C_\lambda$ for all $k \geq 0$. Namely, $\left\|T^k\right\| \leq C_\lambda \left|\lambda^{k+1}\right|$. Thus, we have

$$\left\|T^k\right\|^{\frac{1}{k}} \leq C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} \implies \limsup_k \left\|T^k\right\|^{\frac{1}{k}} \leq \lim_{k\to\infty} C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} = |\lambda|.$$

Since this holds for all $|\lambda| > \operatorname{spr}(T)$, we have

$$\limsup_k \left\|T^k\right\|^{\frac{1}{k}} \leq \operatorname{spr}(T).$$

Since $\limsup_n \|T^n\|^{\frac{1}{n}} \leq \operatorname{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}$, we have $\operatorname{spr}(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. □

Corollary 8.5. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$ be normal. We have $spr(T) = \|T\|$.*

Proof. By proposition 6.20, we have $\left\|T^2\right\| = \|T\|^2$, and inductively, we have $\left\|T^{2^n}\right\| = \|T\|^{2n}$. Thus,

$$\operatorname{spr}(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|T^{2^n}\right\|^{\frac{1}{2^n}} = \|T\|.$$

□

Example 8.1.3. For the Volterra operator, we have that $\|V^n\|^{\frac{1}{n}} \to 0$, so $\sigma(V) \subseteq \{0\}$, which means $\sigma(V) = \{0\}$.

8.2 Adjoint Operators

Proposition 8.6. *Let $\mathcal{H}$ be a Hilbert space. If $T \in B(\mathcal{H})$ is normal, we have that for all $c \in \mathbb{F}$, $c\mathbb{I} - T$ is also normal.*

Proof.

$$\begin{aligned}(c\mathbb{I}-T)^{\dagger}(c\mathbb{I}-T) &= (\bar{c}\mathbb{I}-T^{\dagger})(c\mathbb{I}-T)\\ &= \left|c^2\right|\mathbb{I}-\bar{c}T-cT^{\dagger}+T^{\dagger}T\\ &= \left|c^2\right|\mathbb{I}-cT^{\dagger}-\bar{c}T+TT^{\dagger},\\ (c\mathbb{I}-T)(c\mathbb{I}-T)^{\dagger} &= (c\mathbb{I}-T)(\bar{c}\mathbb{I}-T^{\dagger})\\ &= \left|c\right|^2-cT^{\dagger}-\bar{c}T+TT^{\dagger}\\ &= (c\mathbb{I}-T)^{\dagger}(c\mathbb{I}-T).\end{aligned}$$

□

Proposition 8.7. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$ be a normal operator. For $x \in \mathcal{H}$, we have $Tx = \lambda x$ if and only if $T^{\dagger}x = \bar{\lambda}x$.*

Proof. Suppose $Tx = \lambda x$. Let $N := \lambda\mathbb{I} - T$, which is normal, and $Nx = \lambda x - Tx = 0$, $N^{\dagger} = \bar{\lambda} - T^{\dagger}$. We have that

$$\begin{aligned}\left\|N^{\dagger}x\right\|^2 &= \left\langle N^{\dagger}x, N^{\dagger}x\right\rangle\\ &= \left\langle NN^{\dagger}x, x\right\rangle\\ &= \left\langle N^{\dagger}Nx, x\right\rangle\\ &= \left\langle N^{\dagger}0, x\right\rangle\\ &= 0.\end{aligned}$$

Thus $N^{\dagger}x = 0$, which means $\bar{\lambda}x - T^{\dagger}x = 0$.

The converse is true by $(T^{\dagger})^{\dagger} = T$. □

Proposition 8.8. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$. We have that $\sigma(T^{\dagger}) = \left\{\bar{\lambda} : \lambda \in \sigma(T)\right\}$.*

Proof. $\lambda \in \sigma(T^{\dagger})$, if and only if $\lambda\mathbb{I} - T^{\dagger}$ is not invertible, if and only if $\lambda\mathbb{I} - T^{\dagger}$ is not injective or not surjective (by Banach Isomorphism theorem 3.23), if and only if $\ker(\lambda\mathbb{I} - T^{\dagger}) \neq \{0\}$ or $\operatorname{Im}(\lambda\mathbb{I} - T^{\dagger}) \neq \mathcal{H}$, if and only if $\operatorname{Im}((\lambda\mathbb{I} - T^{\dagger})^{\dagger}) \neq \{0\}^{\perp} = \mathcal{H}$ or $\ker((\lambda\mathbb{I} - T^{\dagger})^{\dagger}) \neq \mathcal{H}^{\perp} = \{0\}$ by proposition 6.9, if and only if $\bar{\lambda}\mathbb{I} - T = (\lambda\mathbb{I} - T^{\dagger})^{\dagger}$ is not surjective or injective, if and only if $\bar{\lambda}\mathbb{I} - T$ is not invertible, if and only if $\bar{\lambda} \in \sigma(T)$. □

Proposition 8.9. *Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})_{sa}$. We have that $\sigma(T) \subset \mathbb{R}$.*

Proof. Since $T \in B(\mathcal{H})_{sa}$, we have $\operatorname{Im}(\langle x, Tx\rangle) = 0$. Suppose $\lambda = a + ib$ for some $b \neq 0$. For any $x \in \mathcal{H}$, we have

$$\begin{aligned}b\|x\|^2 &= \langle x, bx\rangle\\ &= \operatorname{Im}(\langle x, \lambda x\rangle)\\ &= \operatorname{Im}(\langle x, \lambda x\rangle) - \operatorname{Im}(\langle x, Tx\rangle)\\ &= \operatorname{Im}(\langle x, (\lambda\mathbb{I} - T)x\rangle).\end{aligned}$$

Thus, $|b|\|x\|^2 \leq |\langle x, (\lambda\mathbb{I} - T)x\rangle|$. By corollary 6.13, $\lambda\mathbb{I} - T$ is invertible, and $\lambda \notin \sigma(T)$. □

Proposition 8.10. *Let $\mathcal{H}$ be a Hilbert space, and $T \geq 0$. We have that $\sigma(T) \subseteq [0, \|T\|]$.*

Proof. $\sigma(T) \subseteq \mathbb{R}$ since $T \in B(\mathcal{H})_{sa}$. Consider any $\lambda < 0$, we have that $-(\lambda\mathbb{I} - T) = -\lambda\mathbb{I} + T \geq -\lambda\mathbb{I} \geq 0$, since $T \geq 0$. By proposition 6.32, $-(\lambda\mathbb{I} - T)$ is invertible, so $\lambda\mathbb{I} - T$ is invertible and $\lambda \notin \sigma(T)$.

For any $\lambda > \|T\|$, we have that $\lambda\mathbb{I} - T \geq \|T\|\mathbb{I} \geq 0$. Suppose $\|T\| > 0$, then $\lambda\mathbb{I} - T$ is invertible. Suppose $\|T\| = 0$, then $T = 0$ and $\lambda\mathbb{I} - T = \lambda\mathbb{I}$ is invertible. In either case, $\lambda \notin \sigma(T)$. □

Proposition 8.11. *Let $\mathcal{H}$ be a Hilbert space, and $U \in B(\mathcal{H})$ be a unitary operator. We have that $\sigma(U) \subseteq \mathbb{T}$.*

Proof. By theorem 8.2, $\sigma(U) \subseteq \|U\|\mathbb{T} = \mathbb{T}$. Since U is invertible with $U^{-1} = U^\dagger$, $0 \notin \sigma(U)$.

Now consider any $0 < |\lambda| < 1$. Suppose for contradiction that $\lambda\mathbb{I} - U = -\lambda U(\frac{1}{\lambda}\mathbb{I} - U^\dagger)$ is invertible. Since $-\lambda U$ is invertible, $\frac{1}{\lambda}\mathbb{I} - U^\dagger$ is invertible. Thus, $\frac{1}{\lambda} \in \sigma(U^\dagger) \subseteq \mathbb{T}$, which means $\left|\frac{1}{\lambda}\right| \leq 1$, which contradicts with $|\lambda| < 1$.

Thus, $|\lambda| = 1$ for any $\lambda \in \sigma(U)$. □

8.3 Compact Operators

Theorem 8.12. *[Fredholm alternative] Let X be a Banach space, and $K \in \mathcal{K}(X)$ be a compact linear operator. Suppose $\lambda \neq 0$, then $\ker(\lambda\mathbb{I} - T)$ is finite dimensional and $\operatorname{Im}(\lambda\mathbb{I} - T)$ is finite co-dimensional. Also, either $\lambda \in \rho(K)$, or $\lambda \in \sigma_p(K)$.*

Theorem 8.13 (Structure theorem for compact operators)**.** *Let $K \in \mathcal{K}(X)$ be a compact operator on an infinite-dimensional Banach space X, then*

1. $0 \in \sigma(K)$.
2. $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$.
3. $\sigma(K) \setminus \{0\}$ *is finite, or* $\sigma(K) \setminus \{0\} = (\lambda_k)_{k=1}^\infty$ *such that* $\lim_{k\to\infty} \lambda_k = 0$.

Theorem 8.14 (Spectral theorem for normal operators)**.** *Let $\mathcal{H}$ be a Hilbert space, and $K \in \mathcal{K}(\mathcal{H})$ be a compact normal operator, then there exist eigenvectors $(\phi_j)_{j\in J}$, which form an orthonormal basis for $\mathcal{H}$.*

Proof. By Zorn's Lemma, we can find a maximal set of orthornormal eigenvectors $(\phi_j)_{j\in J}$ of K. Let $M := \overline{\operatorname{Span}\{\phi_j\}_{j\in J}}$, and $P : \mathcal{H} \to M$ be the projection onto M. Also, let $\lambda_j \in \mathbb{C}$ be such that $K\phi_j = \lambda_j\phi_j$.

Consider any $x \in \mathcal{H}$, we have that

$$\begin{aligned}
KPx &= K\sum_{j\in J} \langle \phi_j, x\rangle \phi_j \\
&= \sum_{j\in J} \langle \phi_j, x\rangle K\phi_j \\
&= \sum_{j\in J} \langle \phi_j, x\rangle \lambda_j\phi_j \\
&= \sum_{j\in J} \left\langle \bar{\lambda}_j\phi_j, x\right\rangle \phi_j \\
&= \sum_{j\in J} \left\langle K^\dagger\phi_j, x\right\rangle \phi_j \quad 8.7 \\
&= \sum_{j\in J} \langle \phi_j, Kx\rangle \phi_j \\
&= PKx.
\end{aligned}$$

Thus, $KP = PK$.

Since $\mathbb{I} - P \in B(\mathcal{H})_{sa}$, we have that $(\mathbb{I} - P)K$ is compact and normal. Suppose $(\mathbb{I} - P)K \neq 0$, then by corollary 8.5, there is $\lambda \in \sigma((\mathbb{I} - P)K)$, such that $|\lambda| = \operatorname{spr}((\mathbb{I} - P)K) = \|(\mathbb{I} - P)K\| > 0$. Thus, by Fredholm alternative 8.12, $\lambda \in \sigma_p((\mathbb{I} - P)K)$, which means there is a $\phi_\lambda \neq 0 \in \mathcal{H}$, such that $(\mathbb{I} - P)K\phi_\lambda = \lambda\phi_\lambda$. WLOG, we may pick $\|\phi_\lambda\| = 1$. Notice that this means $\phi_\lambda \in \operatorname{Im}(I - P) = M^\perp = \ker(P)$, so

$$\begin{aligned}
\lambda\phi_\lambda &= (\mathbb{I} - P)K\phi_\lambda \\
&= K\phi_\lambda - PK\phi_\lambda \\
&= K\phi_\lambda - KP\phi_\lambda \\
&= K\phi_\lambda - K(0) \\
&= K\phi_\lambda,
\end{aligned}$$

which means ϕ_λ is an eigenvector of K. However, $\phi_\lambda \in M^\perp$, so $(\phi_j)_{j\in J} \cup \{\phi_\lambda\} \supsetneq (\phi_j)_{j\in J}$ is still orthornormal, contradicting the maximality of $(\phi_j)_{j\in J}$.

This shows $(\mathbb{I} - P)K = 0$. Now suppose $\mathbb{I} - P \neq 0$, we have $K(\mathbb{I} - P) = K = KP = K - PK = (\mathbb{I} - P)K = 0$, so there is $\phi \neq 0 \in \operatorname{Im}(\mathbb{I} - P) = M^\perp$, such that $K\phi = 0 = 0\phi$, so ϕ is an eigenvector of K, which again contradicts the maximality of $(\phi_j)_{j\in J}$.

Thus, $\mathbb{I} - P = 0$, which means $M = \operatorname{Im}(P) = \operatorname{Im}(\mathbb{I}) = \mathcal{H}$. □

Theorem 8.15 (Spectral theorem for infinite range normal operators). *Let $\mathcal{H}$ be a Hilbert space, and $K \in \mathcal{K}(\mathcal{H})$ be a compact normal operator with infinite range, then there exists countably many eigenvalues $(\lambda_k)_{k=1}^\infty \subset \mathbb{C}$ such that $\forall k \geq 1,\ n_k := \dim(\ker(\lambda_k\mathbb{I} - K)) < \infty$, and $\sigma(K) = (\lambda_k)_{k=0}^\infty$ for $\lambda_0 = 0 = \lim_{k\to\infty} \lambda_k$. In addition, there are eigenvectors $(\phi_{k_i})_{k\geq 0, i\in[n_k]}$, which form an orthonormal basis for $\mathcal{H}$, and $\forall k \geq 0,\ i \in [n_k],\ K\phi_{k_i} = \lambda_k \phi_{k_i}$.*