

Amath753 Advanced PDEs

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1 Preliminaries

See more in AMATH731-Functional Analysis Notes from Prof. Giang Tran, and my PMATH651-Measure Theory Notes.

1.1 Introduction

Definition 1.1. We will use the following notations:

- C means a positive constant.
- $U \subset \mathbb{R}^n$ is open.
- If $u : U \rightarrow \mathbb{R}$ is a function, we write $u(x) := u(x^1, \dots, x^n)$ for $x = (x^1, \dots, x^n) \in U$.
- A function u is **smooth** if $u \in C^\infty(U)$.
- For $1 \leq i \leq n$, we write $\partial_i u := u_{x^i} := u_i := D_i u := \frac{\partial}{\partial x^i} u := \frac{\partial u}{\partial x^i}$.
- Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we let $|\alpha| := \sum_{i=1}^n \alpha_i$, and

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial_{x^1}^{\alpha_1} \dots \partial_{x^n}^{\alpha_n}} = \partial_{x^1}^{\alpha_1} \dots \partial_{x^n}^{\alpha_n} u.$$

- If $k \in \mathbb{N}$, we let $D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}$
- When $k = 1$, we write $Du := D_x u := (u_{x^1}, \dots, u_{x^n})^T = \nabla u$ to be the **gradient**.
- When $k = 2$, we write $D^2 u := \begin{pmatrix} u_{x^1, x^1} & \dots & u_{x^1, x^n} \\ \vdots & & \vdots \\ u_{x^n, x^1} & \dots & u_{x^n, x^n} \end{pmatrix}$ to be the **Hessian** matrix.
- $\Delta u := \sum_{i=1}^n u_{x^i, x^i} = \operatorname{div} Du = \operatorname{tr}(D^2 u)$ is the **Laplacian** of u .

Example 1.1.1. Consider a body $U \subset \mathbb{R}^3$ and let $U_0 \subseteq U$ with boundary ∂U_0 , which does not change over time.

The Conservation of Energy states that the rate of change of total energy in U_0 is the inflow of heat through the boundaries plus heat produced by the source in U_0 .

Let $e(x, t) \in \mathbb{R}$ be the density of internal energy, then the total energy is $\int_{U_0} e dx$.

Let $j(x, t) \in \mathbb{R}^3$ be the heat flux (vector pointing at the direction that heat is flowing).

Let n denote the exterior unit normal on ∂U_0 .

The net outflow of the heat through ∂U_0 is $\int_{\partial U_0} j \cdot n ds$.

Let $p(x, t) \in \mathbb{R}$ be the power density of the source. Heat production in U_0 is $\int_{U_0} p dx$.

Thus we have

$$\frac{d}{dt} \int_{U_0} e dx = - \int_{\partial U_0} j \cdot n ds + \int_{U_0} p dx.$$

By divergence theorem, we have $\int_{\partial U_0} j \cdot n ds = \int_{U_0} \operatorname{div} j dx$.

Thus we have

$$\int_{U_0} (\partial_t e + \operatorname{div} j - p) dx = 0.$$

Since U_0 is arbitrary, we must have

$$\partial_t e + \operatorname{div} j - p = 0.$$

Assume that e depends linearly on temperature T as $e = e_0 + \sigma u$, where e_0 is a constant reference internal energy, and $u = T - T_0$, where T_0 is a constant reference temperature, and σ is the specific heat capacity.

A generalized form of Fourier's law states that:

- Heat flow is proportional to the temperature gradient.
- Heat is transformed by convection with heat flux be , where $b(x, t) \in \mathbb{R}^3$ is a given convection velocity.

Namely, $j = -aDu + be$, where $a(x)$ is a known heat conductivity.

Thus we have

$$\sigma \partial_t u + \operatorname{div}(b\sigma u) - \operatorname{div}(aDu) = p - \operatorname{div}(be_0).$$

Definition 1.2. We consider the operator

$$Lu := - \sum_{i,j=1}^n (a^{ij} u_{x^i})_{x^j} + \sum_{i=1}^n b^i u_{x^i} + cu,$$

for given coefficients a^{ij}, b^i, c .

- The second-order elliptic boundary-value problems are $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$
- The second-order parabolic boundary-value problems are $\begin{cases} u_t + Lu = f & x \in U, t \in (0, T] \\ u = 0 & \text{on } \partial U, t \in (0, T] \\ u = u_0 & \text{on } \partial U, t = 0 \end{cases}$

Example 1.1.2. Some special cases are

- Laplace equation: $-\Delta u = 0$
- Poisson's equation: $-\Delta u = f$
- Heat equation: $u_t - \Delta u = 0$

1.2 Metric Spaces and Complete Spaces

Definition 1.3. A **metric space** is a set X that has a (distance) metric:

$$\begin{aligned} d(\cdot, \cdot) : X \times X &\rightarrow \mathbb{R}, \text{ such that } \forall x, y, z \in X \\ d(x, x) &= 0 \\ \forall x \neq y, d(x, y) &> 0 \\ d(x, y) &= d(y, x) \\ d(x, z) &\geq d(x, y) + d(y, z) \end{aligned}$$

Definition 1.4. Given a metric space (X, d) , a sequence $(x_n)_{n=1}^\infty$ in X has a **limit point** $x \in X$ if $\lim_{n \rightarrow \infty} d(x, x_n) = 0$. In this case, we say $(x_n)_{n=1}^\infty$ is a **convergent sequence**, and write $x = \lim_{n \rightarrow \infty} x_n$.

Definition 1.5. A sequence $(x_n)_{n=1}^\infty$ is a **Cauchy sequence** in a metric space (X, d) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m, n \in \mathbb{N}, m, n > N \implies d(x_m, x_n) < \epsilon.$$

Definition 1.6. A metric space X is **complete** if every Cauchy sequence $(x_i)_{i=1}^\infty$ converges to a limit point in X . i.e. $\exists x \in X, \lim_{i \rightarrow \infty} x_i = x$.

Proposition 1.1. Let (X, d) be a metric space, then every convergent sequence is Cauchy.

Proposition 1.2. Let (X, d) be a metric space. If $(x_n)_{n=1}^\infty$ is a Cauchy sequence and has a convergent subsequence such that $\lim_{k \rightarrow \infty} x_{n_k} = x \in X$, then $\lim_{n \rightarrow \infty} x_n = x$.

1.2.1 Compactness

Remark. See the definition of compactness and more in Section 2.9 of AMATH731 Notes from Prof. Tran.

Definition 1.7. Let (X, d) be a metric space. A set $S \subseteq X$ is **sequentially compact** if every sequence $(x_i)_{i=1}^{\infty}$ in S has a convergent subsequence whose limit is in S . Namely, $\exists x \in S$, such that $x = \lim_{j \rightarrow \infty} x_{i_j}$ for some choice of i_j 's.

Remark. In metric spaces, sequentially compact and compact are equivalent, so we will just use this as the definition for compactness.

Definition 1.8. Let (X, d) be a metric space. A set $S \subseteq X$ is **relatively compact**, or **pre-compact** if its closure \bar{S} is compact in X .

Proposition 1.3. Let (X, d) be a metric space, then $S \subseteq X$ is relatively compact iff for any sequence $(x_n)_{n=1}^{\infty} \subseteq S$, it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$, such that $x_{n_k} \rightarrow x$ for some $x \in X$.

1.3 Banach Spaces

Definition 1.9. A **normed vector space** is a vector space $(X, \|\cdot\|)$ that has an norm (length):

$$\begin{aligned} \|\cdot\| : X &\rightarrow \mathbb{R}, \text{ such that } \forall x, y \in X, a \in \mathbb{C} \\ \|a \cdot x\| &= |a| \|x\| \\ \|x + y\| &\leq \|x\| + \|y\| \\ \|x\| &\geq 0 \\ \|x\| = 0 &\iff x = 0. \end{aligned}$$

Proposition 1.4. For every **normed space** with $\|\cdot\|$, there is a metric $d(x, y) = \|x - y\|$.

Proof.

$$\begin{aligned} d(x, x) &= \|x - x\| = \|0\| = 0 \\ \forall x \neq y, d(x, y) &= \|x - y\| > 0 \\ d(x, y) &= \|x - y\| = \|-(y - x)\| = |-1| \|y - x\| = \|y - x\| = d(y, x) \\ d(x, z) &= \|x - z\| = \|x - y + y - z\| \geq \|x - y\| + \|y - z\| = d(x, y) + d(y, z) \end{aligned}$$

Thus $d(x, y) = \|x - y\|$ is a metric. □

Definition 1.10. A normed space is called a **Banach space** if it is complete.

Definition 1.11. Let $(X, \|\cdot\|)$ be a Banach space, a subset $A \subseteq X$ is **dense** in X if the closure $\bar{A} = X$.

Definition 1.12. A Banach space is **separable** if there is a dense countable subset of it.

1.4 Hilbert Spaces

Definition 1.13. An **inner product space** is a vector space H that has an inner product:

$$\begin{aligned} \langle \cdot, \cdot \rangle : H \times H &\rightarrow \mathbb{C}, \text{ such that } \forall u, v, w \in H, a, b \in \mathbb{C} \\ \langle v, au + bw \rangle &= a \langle v, u \rangle + b \langle v, w \rangle \\ \langle v, w \rangle &= \overline{\langle w, v \rangle} \\ \langle v, v \rangle &= 0 \iff v = 0. \end{aligned}$$

Remark. The conventional mathematical definition of an inner product is linear in the first entry. We are using the current definition to make the “bra-ket” notation easier to understand.

Proposition 1.5. For every inner product space with $\langle \cdot, \cdot \rangle$, there is a norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Proof.

$$\begin{aligned}
||a \cdot x|| &= \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2 \langle x, x \rangle} = |a| ||x|| \\
||x + y||^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \\
&\leq ||x||^2 + ||y||^2 + 2||x||||y|| \\
&\leq (||x|| + ||y||)^2 \\
\forall x \neq 0, ||x|| &= \sqrt{\langle x, x \rangle} > 0 \\
||0|| &= \sqrt{\langle 0, 0 \rangle} = 0
\end{aligned}$$

Thus $||x|| = \sqrt{\langle x, x \rangle}$ is a norm. □

Corollary 1.6. For every inner product space, there is a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$

Theorem 1.7. Cauchy-Schwarz: For every inner product space H ,

$$\forall u, v \in H, |\langle u, v \rangle| \leq ||u|| ||v||.$$

In particular, when $||u|| \neq 0$, $||u||^2 ||v||^2 - |\langle u, v \rangle|^2 = ||z||^2$, where $z := ||u||v - \frac{\langle u, v \rangle}{||u||}u$.

Proof. Notice that this is trivially true and equality holds to be zero when $u = 0$. Now we assume $||u|| \neq 0$, then

$$\begin{aligned}
||z||^2 &= \langle z, z \rangle \\
&= \left\langle ||u||v - \frac{\langle u, v \rangle}{||u||}u, ||u||v - \frac{\langle u, v \rangle}{||u||}u \right\rangle \\
&= ||u||^2 \langle v, v \rangle - \langle u, v \rangle \langle u, v \rangle - \overline{\langle u, v \rangle} \langle v, u \rangle + \frac{\langle v, u \rangle \langle u, v \rangle}{||u||^2} \langle u, u \rangle \\
&= ||u||^2 ||v||^2 - |\langle u, v \rangle|^2 - \cancel{|\langle v, u \rangle|^2} + \cancel{|\langle v, u \rangle|^2} \\
&= ||u||^2 ||v||^2 - |\langle u, v \rangle|^2.
\end{aligned}$$

□

Proposition 1.8. If $\forall v, \langle v, u \rangle = 0$, then $u = 0$.

Proposition 1.9. For an Inner product space H , $\forall y, x = \lim_{i \rightarrow \infty} x_i \in H$, we have

$$\langle x, y \rangle = \lim_{i \rightarrow \infty} \langle x_i, y \rangle.$$

Proof. Given any $\epsilon > 0$, let $\epsilon_0 = \frac{\epsilon}{||y||}$.

Since $x = \lim_{i \rightarrow \infty} x_i$, we can find $N > 0$, such that $\forall n > N, ||x - x_n|| < \epsilon_0$,

thus $|\langle x, y \rangle - \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq ||x - x_n|| ||y|| < \epsilon_0 ||y|| = \epsilon$ □

Corollary 1.10. For an Inner product space H , $\forall y, x = \lim_{i \rightarrow \infty} x_i \in H$, we have $\langle y, x \rangle = \lim_{i \rightarrow \infty} \langle y, x_i \rangle$.

Definition 1.14. An inner product space \mathcal{H} is called a Hilbert space if it is complete.

Definition 1.15. Let H be an inner product space. Two vectors $u, v \in H$ are called **orthogonal** if $\langle u, v \rangle = 0$.

Definition 1.16. Let H be an inner product space. A set $\{e_i\}_{i \in I} \subseteq H$ is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij}.$$

Definition 1.17. Let H be an inner product space. An orthonormal set $\{e_i\}_{i \in \mathbb{N}} \subseteq H$ is called a **maximal orthonormal set** / **orthonormal basis** / **total orthonormal set** if

$$H = \overline{\text{Span}(\{e_1, e_2, \dots\})}.$$

Theorem 1.11. Let \mathcal{H} be a Hilbert space, and $\{e_i\}_{i \in \mathbb{N}} \subseteq \mathcal{H}$ be an orthonormal set, then TFAE:

1. $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis
2. If $\forall i \in \mathbb{N}, \langle x, e_i \rangle = 0$, then $x = 0$.
3. $\forall x \in \mathcal{H}, x = \sum_{i \in \mathbb{N}} \langle e_i, x \rangle e_i$. (Fourier series)
4. $\forall x \in \mathcal{H}, \|x\|^2 = \sum_{i \in \mathbb{N}} |\langle e_i, x \rangle|^2$. (Parseval Identity)

Theorem 1.12. \mathcal{H} is a separable Hilbert space, if and only if there is a maximal orthonormal set in \mathcal{H} . Moreover, in this case, every maximal orthonormal set is at most countable.

Definition 1.18. Let \mathcal{H} be a Hilbert space, $S \subseteq \mathcal{H}$, the subspace **orthogonal** to S is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v \rangle = 0, \forall v \in S\}.$$

Lemma 1.13. Let \mathcal{H} be a Hilbert space, $S \subseteq \mathcal{H}$, we always have S^\perp is a subspace of \mathcal{H} .

Definition 1.19. Let V be a vector space, and $U, W \subseteq V$ be two subspaces, we say $V = U \oplus W$, if $\forall v \in V$, it can be uniquely written as $v = u + w$, where $u \in U, w \in W$.

Theorem 1.14. Let \mathcal{H} be a Hilbert space, if $S \subseteq \mathcal{H}$ is a closed subspace, then

$$\mathcal{H} = S \oplus S^\perp.$$

1.5 Bounded linear operators

Definition 1.20. Let X, Y be vector spaces, $A : X \rightarrow Y$ is a linear operator if $\forall c \in \mathbb{R}, u, v \in X$,

$$A(u + cv) = Au + cAv.$$

Definition 1.21. Let X, Y be normed spaces, the **operator norm** of a linear operator $A : X \rightarrow Y$ is

$$\|A\| := \sup_{\|u\|_X \leq 1} \|Au\|_Y = \sup_{\|u\|_X = 1} \|Au\|_Y = \sup_{u \neq 0 \in X} \frac{\|Au\|_Y}{\|u\|_X}.$$

Definition 1.22. Let X, Y be normed spaces, a linear operator $A : X \rightarrow Y$ is **bounded** if $\|A\| < \infty$.

Definition 1.23. Let X, Y be normed spaces, we denote

$$B(X, Y) := \{A : X \rightarrow Y | A \text{ is bounded linear operator}\}.$$

Theorem 1.15. The set $B(X, Y)$ is a normed linear space with the operator norm.

Proposition 1.16. Let X, Y, Z be normed spaces, if $A : X \rightarrow Y, B : Y \rightarrow Z$ are both linear bounded operators, then so is $B \circ A$, with

$$\|B \circ A\| \leq \|B\| \|A\|.$$

Theorem 1.17. Let X, Y be normed spaces, a linear operator $A : X \rightarrow Y$ is bounded if and only if it is continuous.

Definition 1.24. Let X, Y be normed spaces, a linear operator $A : X \rightarrow Y$ is **closed** if $\forall u_k \rightarrow u$ in X and $Au_k \rightarrow v$ in Y , we have $Au = v$.

Theorem 1.18. (*closed graph*) Let X, Y be Banach spaces, if a linear operator $A : X \rightarrow Y$ is closed, it is also bounded.

Theorem 1.19. (*Bounded inverse Theorem*) Let X, Y be normed spaces, if a bounded linear operator $A : X \rightarrow Y$ is bijective, then A^{-1} is continuous and bounded as well.

Proposition 1.20. Let Y be a Banach space, S be a dense subset of a normed space X . For any bounded linear operator $E : S \rightarrow Y$, we can extend it to $\tilde{E} : X \rightarrow Y$, such that \tilde{E} is also bounded and linear, with $\|\tilde{E}\| = \|E\|$, and $\tilde{E}|_S = E$.

Proof. Consider any $x \in X$.

Since S is dense in X , We know $\forall m \in \mathbb{N}^+, \exists x_m \in S$, such that $\|x - x_m\|_X \leq \frac{1}{m}$.

Since E is linear on S , we have that

$$\begin{aligned} \|Ex_m - Ex_l\|_Y &= \|E(x_m - x_l)\|_Y \\ &\leq \|E\| \|x_m - x_l\|_X \\ &= \|E\| \|(x_m - x) + (x - x_l)\|_X \\ &\leq \|E\| \|x - x_m\|_X + \|E\| \|x - x_l\|_X \\ &\leq \|E\| \left(\frac{1}{m} + \frac{1}{l} \right). \end{aligned}$$

Thus given any $\epsilon > 0$, for any $m, l \geq \lceil \frac{2\epsilon}{\|E\|} \rceil$, we can make $\|Ex_m - Ex_l\|_Y < \epsilon$. Thus $(Ex_m)_{m=1}^\infty$ is a Cauchy sequence in Y .

Since Y is a Banach space, $\exists y^* \in Y$, such that $Ex_m \rightarrow y^*$ in Y .

We claim that y^* is independent of choice of the sequence $(x_m)_{m=1}^\infty$.

Indeed, consider any other sequence $(v_m)_{m=1}^\infty \subseteq C^\infty(\bar{x})$, such that $\forall m \in \mathbb{N}^+, \|x - v_m\|_X \leq \frac{1}{m}$,

$$\begin{aligned} \|y^* - Ev_m\|_Y &\leq \|y^* - Ex_m\|_Y + \|Ex_m - Ev_m\|_Y \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - v_m\|_X \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - x\|_X + \|E\| \|x - v_m\|_X. \end{aligned}$$

Since all three terms on the right go to 0 when $m \rightarrow \infty$, we have that $Ev_m \rightarrow y^*$ in Y .

Thus we can uniquely define $\tilde{E}x := y^*$. In addition,

$$\begin{aligned} \|\tilde{E}x\|_Y &= \left\| \lim_{m \rightarrow \infty} Ex_m \right\|_Y \\ &= \lim_{m \rightarrow \infty} \|Ex_m\|_Y \\ &\leq \lim_{m \rightarrow \infty} \|E\| \|x_m\|_X \\ &= \|E\| \left\| \lim_{m \rightarrow \infty} x_m \right\|_X \\ &= \|E\| \|x\|_X. \end{aligned}$$

Thus $\|\tilde{E}\| = \|E\|$. □

1.5.1 Compact Operators

Definition 1.25. Let X, Y be metric spaces, a linear operator $A : X \rightarrow Y$ is **compact** if for each bounded subset $S \subseteq X$, we have its image $A(S)$ is pre-compact in Y .

Proposition 1.21. Let X, Y be metric spaces, a linear operator $A : X \rightarrow Y$ is compact if and only if A is bounded, and each bounded sequence $(x_n)_{n=1}^\infty \subseteq X$ has some subsequence $(x_{n_k})_{k=1}^\infty$ such that $(Ax_{n_k})_{k=1}^\infty$ converges to some $y \in Y$.

Definition 1.26. Let X, Y be Banach spaces and $X \subseteq Y$, then we say X is **compactly embedded** in Y , denoted

$$X \subset\subset Y$$

if the inclusion map $i : X \hookrightarrow Y$; $x \mapsto x$ is compact.

Namely, $\exists C > 0$, such that $\forall x \in X, \|x\|_Y \leq C\|x\|_X$, and each bounded sequence $(x_n)_{n=1}^\infty \subseteq X$ having some subsequence $(x_{n_k})_{k=1}^\infty$ that converges to some $y \in Y$.

Proposition 1.22. Let X, Y, Z be Banach spaces and $X \subset\subset Y$, if an operator $T : Z \rightarrow X$ is bounded, then $\tilde{T} := i \circ T : Z \rightarrow Y$ is compact.

Proof. Consider any bounded set $S \subseteq Z$, such that $\forall z \in S, \|z\|_Z \leq M$.

We have $\|Tz\|_X \leq \|T\|\|z\|_Z \leq M\|T\| < \infty$, and thus $T(S)$ is bounded in X .

Yet i is compact, and thus $i(T(S))$ is pre-compact.

This shows $\tilde{T}(S) = (i \circ T)(S)$ is pre-compact for any bounded set $S \subseteq Z$.

Thus \tilde{T} is compact. \square

Theorem 1.23 (Spectral theorem for compact operators). Let $K : \mathcal{H} \rightarrow \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space \mathcal{H} , then

1. $0 \in \text{Spec}(K)$.
2. $\text{Spec}(K) \setminus \{0\} = \text{Spec}_p(K) \setminus \{0\}$.
3. $\text{Spec}(K) \setminus \{0\}$ is finite, or $\text{Spec}(K) \setminus \{0\} = (\lambda_k)_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} \lambda_k = 0$.

1.5.2 Dual Space

Definition 1.27. Let X be a normed space over \mathbb{F} , a **functional** is an operator that maps into \mathbb{F} .

Definition 1.28. Let X be a normed space over \mathbb{F} , the **dual space** of X is the collection of bounded linear functionals on X , denoted

$$X^* := B(X, \mathbb{F}).$$

Definition 1.29. Let X be a normed space, if $v \in X, u^* \in X^*$, we can write $\langle u^* | v \rangle_{X^*, X} := u^*(v)$ as the action of u^* on v .

Definition 1.30. Let X be a normed space, the **dual norm** is defined to be

$$\|u^*\|_{X^*} := \sup_{\|u\| \leq 1} |\langle u^* | u \rangle_{X^*, X}|.$$

Definition 1.31. A Banach space X is **reflexive** if $(X^*)^* \simeq X$. Namely, $\forall u^{**} \in (X^*)^*, \exists! u \in X$ such that

$$\forall v^* \in X^*, \langle u^{**} | v^* \rangle_{(X^*)^*, X^*} = \langle v^* | u \rangle_{X^*, X}.$$

Theorem 1.24. (Riesz-Frechet Representation theorem)

Let \mathcal{H} be a Hilbert space, then for each $u^* \in \mathcal{H}$, $\exists! u \in \mathcal{H}$, such that $\forall v \in \mathcal{H}, \langle u^* | v \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$, and $\|u^*\|_{\mathcal{H}^*} = \|u\|_{\mathcal{H}}$.

Corollary 1.25. Every Hilbert space is reflexive.

Corollary 1.26. Let \mathcal{H} be a Hilbert space, then $\mathcal{H} \cong^* \mathcal{H}$, where the map $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$; $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism.

Remark. We thus abuse the notation, and denote canonical bijective isometric antilinear isomorphism by $u^\dagger := \Phi(u) \forall u \in \mathcal{H}$, and $(u^*)^\dagger := \Phi^{-1}(u^*) \forall u^* \in \mathcal{H}^*$. Notice that by definition

$$(u^\dagger)^\dagger = u, ((u^*)^\dagger)^\dagger = u^* \forall u \in \mathcal{H}, u^* \in \mathcal{H}^*.$$

We might further abuse the notation, and write

$$\langle u | v \rangle := \langle u, v \rangle = \langle u^\dagger | v \rangle =: \langle u^\dagger, v \rangle$$

interchangeably instead of $\langle u^\dagger | v \rangle_{\mathcal{H}^*, \mathcal{H}}$ or $\langle u, v \rangle_{\mathcal{H}}$ when the context is clear.

Definition 1.32. Let X be a Banach Space, we say $(u_k)_{k=1}^\infty \subset X$ converges to $u \in X$ weakly, denoted $u_k \rightharpoonup u$, if

$$\forall v^* \in X^*, \langle v^* | u_k \rangle \rightarrow \langle v^* | u \rangle$$

as real numbers.

Proposition 1.27. Let X be a Banach Space, $(u_k)_{k=1}^\infty \subset X$ be a sequence, then

1. If $u_k \rightarrow u$, we always have $u_k \rightharpoonup u$.
2. If $u_k \rightharpoonup u$, we have that u is unique.
3. If $u_k \rightharpoonup u$, we have $(u_k)_{k=1}^\infty$ is bounded.
4. If $u_k \rightharpoonup u$, every subsequence $(u_{k_j})_{j=1}^\infty$ also converges weakly to u .

Proof. See A5Q1 for 1. □

Theorem 1.28 (Weakly compact for reflexive Banach Space). Let X be a reflexive Banach Space, and $(u_k)_{k=1}^\infty \subset X$ be a bounded sequence, then $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.

Proposition 1.29. Let \mathcal{H} be a Hilbert space, then $u_k \rightharpoonup u$ if and only if $\forall v \in \mathcal{H}, \langle v, u_k \rangle \rightarrow \langle v, u \rangle$ as real numbers.

Proof. Suppose $u_k \rightharpoonup u$.

Notice that for all $v \in \mathcal{H}$, we have that $v^\dagger \in \mathcal{H}^*$, and thus $\langle v, u_k \rangle = \langle v^\dagger | u_k \rangle \rightarrow \langle v^\dagger | u \rangle = \langle v, u \rangle$.

Now suppose $\forall v \in \mathcal{H}, \langle v, u_k \rangle \rightarrow \langle v, u \rangle$.

Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 1.24, there is some $f^\dagger \in \mathcal{H}$, such that

$$\langle f | u_{k_j} \rangle = \langle f^\dagger, u_{k_j} \rangle \rightarrow \langle f^\dagger, u \rangle = \langle f | u \rangle.$$

Thus, $u_{k_j} \rightharpoonup u$. □

Proposition 1.30. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Tu_k \rightharpoonup Tu$.

Proof. Let $y_k := Tu_k, y := Tu \in \mathcal{H}_2$.

Consider any $g \in \mathcal{H}_2^*$, we define $f := g \circ T \in \mathcal{H}_1^*$.

Since $u_k \rightharpoonup u$, we must have

$$\begin{aligned} \lim_{k \rightarrow \infty} f(u_k) &= f(u) \\ \lim_{k \rightarrow \infty} g(Tu_k) &= g(Tu) \\ \lim_{k \rightarrow \infty} g(y_k) &= g(y). \end{aligned}$$

We thus have $y_k \rightharpoonup y$. □

Proposition 1.31. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a compact operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Ku_k \rightarrow Ku$.

Proof. Let $y_k := Ku_k, y := Ku \in \mathcal{H}_2$.

Since K is compact, it is bounded, so $y_k \rightharpoonup y$.

Now suppose for contradiction $\lim_{k \rightarrow \infty} \|y_k - y\| \neq 0$.

Then there is some $\epsilon > 0$ and a subsequence $(u_{k_j})_{j=1}^\infty$ such that $\forall j \geq 1, \|y_{k_j} - y\| \geq \epsilon$.

Since $u_k \rightharpoonup u \in \mathcal{H}_1$, we have $(u_k)_{k=1}^\infty$ is bounded, and thus $(u_{k_j})_{j=1}^\infty$ is bounded.

Since K is compact, there is some further subsequence $(u_{k_{j_m}})_{m=1}^\infty$ such that $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$.

Thus $Ku_{k_{j_m}} \rightarrow \tilde{y}$. Since weak convergence, we must have $\tilde{y} = y$.

Thus $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = y$, which is a contradiction. □

1.5.3 Adjoint Operator

Definition 1.33. Let X, Y be normed spaces, the **dual operator** of a linear operator $A : X \rightarrow Y$ is

$$A^* : Y^* \rightarrow X^*; f \mapsto f \circ A.$$

Proposition 1.32. Let X, Y, Z be normed spaces, $S \in B(X, Y), T \in B(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.

Proof. Consider any $f \in Z^*$, and any $x \in X$, we have

$$\begin{aligned} (T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx) \\ &= (f)(S(T(x))) \\ &= (f \circ (S \circ T))(x) \\ &= (S \circ T)^*(f)(x). \end{aligned}$$

Thus $(T^* \circ S^*)(f) = (S \circ T)^*(f)$. □

Definition 1.34. Let \mathcal{H} be a Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator, the **Hilbert adjoint operator** of T is $T^\dagger : \mathcal{H} \rightarrow \mathcal{H}$ such that $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \forall x, y \in \mathcal{H}$.

Theorem 1.33. Let \mathcal{H} be a Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator, T^\dagger always exists, and is given by $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$, where $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$; $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism, and T^* is the dual operator of T . In addition, T^\dagger is also a bounded linear operator, with $\|T^\dagger\| = \|T\|$, and $(T^\dagger)^\dagger = T$.

Proof. $\forall y \in \mathcal{H}$, we have that

$$\begin{aligned} \langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\ &= ((T^* \circ \Phi)(x))(y) \\ &= (\Phi(x))(Ty) \\ &= \langle x, Ty \rangle. \end{aligned}$$

Now consider any $x, y, z \in \mathcal{H}, c \in \mathbb{C}$, we have that

$$\begin{aligned} \langle T^\dagger(x + cz), y \rangle &= \langle x + cz, Ty \rangle \\ &= \langle x, Ty \rangle + \bar{c} \langle z, Ty \rangle \\ &= \langle T^\dagger x, y \rangle + \bar{c} \langle T^\dagger z, y \rangle \\ &= \langle T^\dagger x + cT^\dagger z, y \rangle. \end{aligned}$$

Since this holds for any $y \in \mathcal{H}$, we have that $T^\dagger(x + cz) = T^\dagger x + cT^\dagger z$, and thus T^\dagger is linear.

Now given any $x \in \mathcal{H}$, we have that

$$\begin{aligned} \|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\ &= \langle x, TT^\dagger x \rangle \\ &\leq \|x\| \|TT^\dagger x\| \\ &\leq \|x\| \|T\| \|T^\dagger x\| \\ &\implies \\ \|T^\dagger x\| &\leq \|x\| \|T\| \\ &\implies \\ \|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\ &\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{\|x\| \|T\|}{\|x\|} \\ &= \|T\|. \end{aligned}$$

Thus T^\dagger is also a bounded linear operator.

Now $\forall x, y \in \mathcal{H}$, $\langle x, T^\dagger y \rangle = \overline{\langle T^\dagger y, x \rangle} = \overline{\langle y, Tx \rangle} = \langle Tx, y \rangle$.

Thus $(T^\dagger)^\dagger = T$. □

Remark. $\forall x, y \in \mathcal{H}$, $\langle (Tx)^\dagger | y \rangle = \langle Tx, y \rangle = \langle x, T^\dagger y \rangle = \langle x^\dagger | T^\dagger y \rangle$. We thus abuse the notation, and write $(Tx)^\dagger = \langle x | T^\dagger$

Definition 1.35. A bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is **delf-adjoint** if $T^\dagger = T$.

Theorem 1.34. Let \mathcal{H} be a Hilbert space, and $K : \mathcal{H} \rightarrow \mathcal{H}$ be a compact linear operator, then K^\dagger is also compact.

Proof. K^\dagger is bounded by 1.21.

Let $(u_k)_{k=1}^\infty$ be any bounded sequence in \mathcal{H} .

By 1.28, we have that $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in \mathcal{H}$, such that $u_{k_j} \rightharpoonup u$.

Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 1.24, there is some $f^\dagger \in \mathcal{H}$, such that

$$\begin{aligned} \langle f | K^\dagger(u_{k_j} - u) \rangle &= \langle f^\dagger, K^\dagger(u_{k_j} - u) \rangle \\ &= \langle K f^\dagger, u_{k_j} - u \rangle \\ &= \langle K f^\dagger, u \rangle - \langle K f^\dagger, u_{k_j} \rangle \end{aligned} \quad \rightarrow 0,$$

since $u_{k_j} \rightharpoonup u$ and by 1.29.

Since $\langle f | K^\dagger(u_{k_j} - u) \rangle \rightarrow 0 = \langle f | 0 \rangle$ for any $f \in \mathcal{H}^*$, we have that $K^\dagger(u_{k_j} - u) \rightharpoonup 0$.

By 1.31, we have that $KK^\dagger(u - u_{k_j}) \rightarrow 0$.

$$\begin{aligned} \|K^\dagger u - K^\dagger u_{k_j}\|^2 &= \langle K^\dagger u - K^\dagger u_{k_j}, K^\dagger u - K^\dagger u_{k_j} \rangle \\ &= \langle K^\dagger(u - u_{k_j}), K^\dagger(u - u_{k_j}) \rangle \\ &= \langle KK^\dagger(u - u_{k_j}), u - u_{k_j} \rangle \\ &\leq \|KK^\dagger(u - u_{k_j})\| \|u - u_{k_j}\| \\ &\rightarrow 0. \end{aligned}$$

Thus $K^\dagger u_{k_j} \rightarrow K^\dagger u \in \mathcal{H}$,

Since $(u_k)_{k=1}^\infty$ is any bounded sequence, we have that K^\dagger is compact by 1.21. □

Theorem 1.35. (Fredholm's alternative)

Let \mathcal{H} be a Hilbert space, and $K : \mathcal{H} \rightarrow \mathcal{H}$ be a compact linear operator, then

1. $\text{Ker}(I - K)$ is finite dimensional.
2. $\text{Im}(I - K)$ is closed.
3. $\text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$.
4. $\dim(\text{Ker}(I - K)) = \dim(\text{Ker}(I - K^\dagger))$.
5. $\text{Ker}(I - K) = \{0\} \iff \text{Im}(I - K) = \mathcal{H}$.

Corollary 1.36. Let \mathcal{H} be a Hilbert space, and $K : \mathcal{H} \rightarrow \mathcal{H}$ be a compact linear operator, then exactly one of the following holds:

1. $\forall v \in \mathcal{H}$, $\exists! u \in \mathcal{H}$, such that $(I - K)u = v$.
2. $\exists u \neq 0 \in \mathcal{H}$, such that $(I - K)u = 0$.

Proof. When $\text{Ker}(I - K) = \{0\}$, we have that $I - K$ is injective, and $\text{Im}(I - K) = \mathcal{H}$.

Thus $\forall v \in \mathcal{H}$, $\exists! u \in \mathcal{H}$, such that $(I - K)u = v$.

On the other hand, if 1. is true, we have that $(I - K)$ is surjective, so $\text{Im}(I - K) = \mathcal{H}$, so $\text{Ker}(I - K) = \{0\}$.

Thus $\text{Ker}(I - K) = \{0\} \iff 1..$

We also have that $\text{Ker}(I - K) \neq \{0\} \iff \exists u \neq 0 \in \text{Ker}(I - K) \iff 2..$ □

Theorem 1.37. (Spectral theorem / Hilbert-Schmidt Theorem)

Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space \mathcal{H} , and $n = \dim(\mathfrak{S}(T)) \in \mathbb{N} \cap \{\infty\}$, then

1. There exists orthonormal eigenvectors $(\phi_k)_{k=1}^n \subset \mathcal{H}$ and eigenvalues $(\lambda_k)_{k=1}^n \subset \mathbb{R}$ such that $|\lambda_1| \geq |\lambda_2| \geq \dots$, and

$$T\phi_k = \lambda_k\phi_k, \lambda_k \neq 0, \forall 1 \leq k \leq n,$$

$$\forall v \in \mathcal{H}, Tv = \sum_{k=1}^n \lambda_k \langle \phi_k, v \rangle \phi_k = \sum_{k=1}^n \langle \phi_k, Tv \rangle \phi_k.$$

2. If $n = \infty$, then $\lim_{k \rightarrow \infty} \lambda_k = 0$, and $(\phi_k)_{k=1}^\infty$ is an orthonormal set for \mathcal{H} iff 0 is not an eigenvalue for T .

1.6 Function Spaces

1.6.1 Continuous functions

Definition 1.36. $u : U \rightarrow \mathbb{R}$ is **continuous** at $x \in U$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in U, \|x - y\| < \delta \implies |u(x) - u(y)| < \epsilon.$$

A function u is continuous if it is continuous at all $x \in U$.

- $C(U) := \{u : U \rightarrow \mathbb{R} : u \text{ is continuous}\}$
- $C^k(U) := \{u : U \rightarrow \mathbb{R} : u \text{ is } k\text{-times continuously differentiable}\}$
- $C^\infty(U) := \{u : U \rightarrow \mathbb{R} : u \text{ has continuous derivatives of all orders}\}$

Definition 1.37. $u : U \rightarrow \mathbb{R}$ is **uniformly continuous** if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in U, \|x - y\| < \delta \implies |u(x) - u(y)| < \epsilon.$$

- $C(\bar{U}) := \{u : U \rightarrow \mathbb{R} : u \text{ is uniformly continuous on bounded subsets of } U\}$
- $C^k(\bar{U}) := \{u : U \rightarrow \mathbb{R} : \forall |\alpha| \leq k, D^\alpha u \text{ is uniformly continuous on bounded subsets of } U, \}$
- If $u \in C^k(\bar{U})$, then we can extend $D^\alpha u$ continuously to \bar{U} .

Definition 1.38. The support of $u : U \rightarrow \mathbb{R}$ is

$$\text{Supp}(u) := \overline{\{x \in U : u(x) \neq 0\}}.$$

Definition 1.39. $u : U \rightarrow \mathbb{R}$ has compact support if $\text{Supp}(u)$ is a compact subset of U .

Definition 1.40. We denote the functions in $C(U)$ and $C^k(U)$ with compact support by $C_c(U), C_c^k(U)$.

Definition 1.41. Consider a sequence of functions $\{u_m\}_1^\infty$ with $u_m : U \rightarrow \mathbb{R}$ and a function $u : U \rightarrow \mathbb{R}$, we have

- $u_m \rightarrow u$ point-wise on U if

$$\forall x \in U, \delta > 0, \exists M \in \mathbb{N}, \text{ such that } m > M \implies |u_m(x) - u(x)| < \delta.$$

- $u_m \rightarrow u$ uniformly on U if

$$\forall \delta > 0, \exists M \in \mathbb{N}, \text{ such that } \forall x \in U, m > M \implies |u_m(x) - u(x)| < \delta.$$

Definition 1.42. $f : [a, b] \rightarrow \mathbb{R}$ is **absolutely continuous** if $\forall \epsilon > 0, \exists \delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) ,

$$\sum_{k=1}^n (b_k - a_k) < \delta \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

1.6.2 Lebesgue Spaces

See more in my Measure Theory Notes.

Definition 1.43. We denote the Lebesgue measure by λ on \mathbb{R}^n . We denote $\int_A f d\lambda$ by $\int_A f(x) dx$ for any measurable set $A \subseteq \mathbb{R}^n$.

Definition 1.44. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, we define

$$\mathcal{L}^1(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} |f(x)| dx < \infty \right\}.$$

Definition 1.45. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, and $1 \leq p < \infty$ we define

$$\mathcal{L}^p(\Omega) := \{ f : \Omega \rightarrow \mathbb{R} \mid f^p \in \mathcal{L}^1(\Omega) \} = \left\{ f : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} |f(x)|^p dx < \infty \right\}.$$

In addition, we define the norm

$$\|f\|_p := \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Definition 1.46. The **essential supremum** of a function $u : U \rightarrow \mathbb{R}$ is

$$\text{ess sup } f := \inf \{ M \in \mathbb{R} : |\{x : f(x) > M\}| = 0 \}.$$

Definition 1.47. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, we define

$$\mathcal{L}^\infty(\Omega) := \{ f : \Omega \rightarrow \mathbb{R} \mid \text{ess sup } f < \infty \}.$$

In addition, we define the norm

$$\|f\|_\infty := \text{ess sup } f.$$

Definition 1.48. Two measurable functions $f, g : \Omega \rightarrow \mathbb{R}$ are said to be equal almost everywhere if $\{x \in \Omega : f(x) \neq g(x)\}$ has measure zero.

Proposition 1.38. For any $1 \leq p \leq \infty$, we have $\|f - g\|_p = 0 \iff f = g$ almost everywhere.

Definition 1.49. For any $1 \leq p \leq \infty$, if we identify $f, g \in \mathcal{L}^p(\Omega)$ by $f \sim g \iff f = g$ almost everywhere, we get the quotient space

$$L^p := \mathcal{L}^p / \sim = \{[f] : f \in \mathcal{L}^p(\Omega)\}$$

to be the collection of all equivalence classes of functions in \mathcal{L}^p .

Theorem 1.39. (completeness of L^p)

For any $1 \leq p \leq \infty$, we have the space $(L^p, \|\cdot\|_p)$ is a Banach space, where $\|[f]\|_p := \|f\|_p$ for any representative $f \in [f]$. One can check this norm is well-defined.

Theorem 1.40. For any $1 \leq p < \infty$,

- $C_c(U)$ is dense in $L^p(U)$.
- $C(\bar{U})$ is dense in $L^p(U)$.

Definition 1.50. Let $U, V \subseteq \mathbb{R}^n$ be open, we say that V is **compactly contained** in U if $V \subseteq \bar{V} \subseteq U$, and \bar{V} is compact. We write this as $V \subset\subset U$.

Definition 1.51. The **locally summable spaces** are

$$L^p_{loc}(U) := \{f : U \rightarrow \mathbb{R} : \forall V \subset\subset U, u \in L^p(V)\}.$$

Definition 1.52. We say some property holds for $L^p_{loc}(U)$, if it holds $\forall L^p(V)$ such that $V \subset\subset U$. For instance, let $(f_n)_{n=1}^\infty \subseteq L^p_{loc}(U)$, then $f_n \rightarrow f$ in $L^p_{loc}(U)$ if $f_n \rightarrow f$ in $L^p(V)$, $\forall V \subset\subset U$.

Proposition 1.41. For any $1 \leq p \leq \infty$, we have

$$L^p(U) \subseteq L^1_{loc}(U).$$

Example 1.6.1. Let $u(x) = \frac{1}{x}$ on $U = (0, 1)$.

We have $\int_0^1 |u| dx = \infty$, and thus $u \notin L^1(U)$. However, $u \in L^1_{loc}(U)$.

Theorem 1.42. (Holder's Inequality)

Assume $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

If $u \in L^p(U)$, $v \in L^q(U)$, we have

$$\int_U |uv| dx \leq \|u\|_p \|v\|_q.$$

For $a, b \in \mathbb{R}^n$, we have

$$\left| \sum_{k=1}^n a_k b_k \right| \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \left(\sum_{k=1}^n |b_k|^q \right)^{1/q}$$

Theorem 1.43. (Minkowski's Inequality)

Assume $1 \leq p \leq \infty$.

Let $u, v \in L^p(U)$, we have

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

For $a, b \in \mathbb{R}^n$, we have

$$\left(\sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p \right)^{1/p}$$

Theorem 1.44. (Lebesgue Monotone Convergence)

Let $f_n : X \rightarrow [0, \infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \dots \leq \infty$. Let $f(x) := \lim_{n \rightarrow \infty} f_n(x)$, then $f : X \rightarrow [0, \infty]$ is measurable, and

$$\lim_{n \rightarrow \infty} \int_X f_n dx = \int_X f dx.$$

Theorem 1.45. (Lebesgue Dominated Convergence)

Let $f_n : X \rightarrow \mathbb{C}$ be measurable functions, defined almost everywhere on X , such that $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ is defined almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that for almost everywhere $x \in X$, $\forall n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = 0.$$

Theorem 1.46. We have that

$$L^q(U) \simeq L^p(U)^*,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and the isometric isomorphism $L^q(U) \xrightarrow{\sim} L^p(U)^*$; $u \mapsto u^*$ is defined to be

$$\forall v \in L^p(U), \langle u^* | v \rangle := \int_U u v dx.$$

Remark. We will abuse the notation, and write $\langle u | v \rangle := \int_U u v dx$ with $u \in L^q(U)$ instead of $u^* \in L^p(U)^*$.

Corollary 1.47. In particular, $L^2(U) \simeq L^2(U)^*$, with the isometric isomorphism $L^2(U) \rightarrow L^2(U)^*$; $u \mapsto u^*$ is defined to be

$$\forall v \in L^2(U), \langle u^* | v \rangle = \int_U u v dx = \langle u, v \rangle_{L^2(U)}.$$

Definition 1.53. For $f : U \rightarrow \mathbb{R}^m$, we define

$$\|f\|_{L^p(U)} := \left\| \|f\|_p \right\|_{L^p(U)}.$$

2 Sobolev Spaces

This section follows Chapter 5 in Evan's book.

2.1 Holder Spaces

Definition 2.1. For $u : U \rightarrow \mathbb{R}$ be bounded and continuous, we write

$$\|u\|_C(\bar{U}) := \sup_{x \in \bar{U}} |u(x)|.$$

Definition 2.2. A function $u : U \rightarrow \mathbb{R}$ is **Holder continuous** with $0 < \gamma \leq 1$ if

$$\exists C, \text{ such that } \forall x, y \in U, |u(x) - u(y)| \leq C \|x - y\|^\gamma.$$

Definition 2.3. The γ^{th} -**Holder semi-norm** of $u : U \rightarrow \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{x,y \in \bar{U}, x \neq y} \left(\frac{|u(x) - u(y)|}{\|x - y\|^\gamma} \right).$$

The γ^{th} -**Holder norm** of $u : U \rightarrow \mathbb{R}$ is

$$\|u\|_{C^{0,\gamma}(\bar{U})} := [u]_{C^{0,\gamma}(\bar{U})} + \|u\|_C(\bar{U}).$$

Definition 2.4. For $k \in \mathbb{N}, u \in C^k(\bar{U})$ we define

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_C(\bar{U}) + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}.$$

The **Holder Space** is

$$C^{k,\gamma}(\bar{U}) := \left\{ u \in C^k(\bar{U}) : \|u\|_{C^{k,\gamma}(\bar{U})} < \infty \right\}.$$

Theorem 2.1.

$$(C^{k,\gamma}(\bar{U}), \|\cdot\|_{C^{k,\gamma}(\bar{U})})$$

is a Banach Space.

2.2 Convolution and Mollification

Definition 2.5. For $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$, we define the **convolution** $f * g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ to be

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$

Proposition 2.2. For $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$, we have $f * g = g * f$.

Proof. Take $z := x - y$, we have $y = x - z$, and $d(z^i) = -d(y^i)$. We have that for any $x \in \mathbb{R}$,

$$\begin{aligned} (f * g)(x) &= \int_{\mathbb{R}^n} f(x - y)g(y)dy \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x - y)g(y)d(y^1) \cdots d(y^n) \\ &= (-1)^n \int_{\infty}^{-\infty} \cdots \int_{\infty}^{-\infty} f(z)g(x - z)d(z^1) \cdots d(z^n) \\ &= (-1)^n (-1)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z)g(x - z)dz \\ &= \int_{\mathbb{R}^n} f(z)g(x - z)dz \\ &= (g * f)(x). \end{aligned}$$

□

Proposition 2.3.

$$\text{Supp}(f * g) \subseteq \text{Supp}(f) + \text{Supp}(g).$$

Proof. Let $f^x(y) := f(x - y)$, we have $f * g(x) = \int_{\mathbb{R}^n} f^x(y)g(y)dy$.

Suppose $\text{Supp}(f^x) \cap \text{Supp}(g) = \emptyset$, then we have $(f * g)(x) = 0$.

In addition,

$$\begin{aligned} & \text{Supp}(f^x) \cap \text{Supp}(g) \neq \emptyset \\ \iff & \exists y, x - y \in \text{Supp}(f), y \in \text{Supp}(g) \\ \iff & x \in \text{Supp}(f) + \text{Supp}(g). \end{aligned}$$

Thus $\text{Supp}(f * g) \subseteq \{x \in \mathbb{R}^n : \text{Supp}(f^x) \cap \text{Supp}(g) \neq \emptyset\} = \text{Supp}(f) + \text{Supp}(g)$. \square

Proposition 2.4 (Young's Convolution Inequality). *Let $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, then for a.e. $x \in \mathbb{R}^n$, the function $f(x - y)g(y)$ is integrable. Thus $f * g : \mathbb{R}^n \rightarrow \mathbb{R}$ is well-defined a.e.. In addition, $f * g \in L^p(\mathbb{R}^n)$, and*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

Definition 2.6.

$$\bar{B}(x, r) = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$$

is the closed ball around x of radius r , and

$$B(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$$

is the closed ball around x of radius r .

Definition 2.7. For $\epsilon > 0$,

$$U_\epsilon := \{x \in U : \text{dist}(x, \partial U) > \epsilon\}.$$

Remark. This definition does not require U to be bounded.

Definition 2.8. The **standard mollifier** $\eta(x) \in C^\infty(\mathbb{R}^n)$ is defined as

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|-1}\right), & |x| < 1 \\ 0, & \text{o.w.} \end{cases},$$

with C such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.

For each $\epsilon > 0$,

$$\eta_\epsilon := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$

Proposition 2.5. $\forall \epsilon > 0$, we have

1. $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$,
2. $\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1$,
3. $\text{Supp}(\eta_\epsilon) \subseteq \bar{B}(0, \epsilon)$.

Definition 2.9. Let $f \in L^1_{loc}(U)$, $\epsilon > 0$, its **mollification** $f^\epsilon : U_\epsilon \rightarrow \mathbb{R}$ is defined as

$$f^\epsilon(x) := \eta_\epsilon * f := \int_U \eta_\epsilon(x - y)f(y)dy = \int_{\bar{B}(0, \epsilon)} f(x - z)\eta_\epsilon(z)dz.$$

Remark. When $U \subsetneq \mathbb{R}^n$, the mollification $\eta_\epsilon * f$ is not using the formal definition of convolution, but we will soon see the abuse of notation makes sense.

Proposition 2.6. Let f^ϵ be defined as above, if we zero-extend f outside of U to be

$$\bar{f}(x) := \begin{cases} f(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases},$$

we have $\forall x \in U_\epsilon$,

$$\begin{aligned} (\eta_\epsilon * \bar{f})(x) &= \int_{\mathbb{R}^n} \eta_\epsilon(x-y) \bar{f}(y) dy \\ &= \int_U \eta_\epsilon(x-y) f(y) dy \\ &= f^\epsilon(x) \end{aligned}$$

Theorem 2.7. Let f^ϵ be defined as above, we have:

1. $f^\epsilon \in C^\infty(U_\epsilon)$,
2. $D^\alpha(f^\epsilon) = (D^\alpha \eta_\epsilon) * f$ on U_ϵ ,
3. $f^\epsilon \rightarrow f$ a.e., as $\epsilon \rightarrow 0$,
4. If $f \in C(U)$, we have $f^\epsilon \rightarrow f$ uniformly on compact subsets of U ,
5. If $1 \leq p < \infty$, $f \in L^p_{loc}(U)$, we have $f^\epsilon \rightarrow f$ in $L^p_{loc}(U)$. Namely, $f^\epsilon \rightarrow f$ in $L^p(V)$, $\forall V \subset\subset U$.
6. $\text{Supp}(f^\epsilon) \subseteq \text{Supp}(f) + \text{Supp}(\eta_\epsilon) = \text{Supp}(f) + \bar{B}(0, \epsilon)$.

Proposition 2.8. Let $1 \leq p \leq \infty$. Let $u \in L^p(U)$, then for any $\epsilon > 0$, $U_\epsilon \supseteq V \supseteq \text{Supp}(u^\epsilon)$, we have

$$\|u^\epsilon\|_{L^p(V)} \leq \|u\|_{L^p(U)}.$$

Proof. Notice that $u^\epsilon \in C^\infty(U_\epsilon) \subseteq L^1(U_\epsilon)$.

If we zero-extend u outside of U to be $\bar{u}(x) := \begin{cases} u(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$, we have

$$\begin{aligned} \|\bar{u}\|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\bar{u}(x)|^p dx \\ &= \int_U |u(x)|^p dx + 0 \\ &= \|u\|_{L^p(U)}^p \\ &< \infty. \end{aligned}$$

Thus, $\bar{u} \in L^p(\mathbb{R}^n)$.

By 2.4, we have

$$\begin{aligned} \|u^\epsilon\|_{L^p(V)} &= \|\eta_\epsilon * \bar{u}\|_{L^p(V)} \\ &\leq \|\eta_\epsilon * \bar{u}\|_{L^p(\mathbb{R}^n)} \\ &\leq \|\eta_\epsilon\|_{L^1(\mathbb{R}^n)} \|\bar{u}\|_{L^p(\mathbb{R}^n)} \\ &= \left(\int_{\mathbb{R}^n} |\eta_\epsilon(x)| dx \right) \|u\|_{L^p(U)} \\ &= \|u\|_{L^p(U)}. \end{aligned}$$

□

2.3 Weak derivative and Sobolev Spaces

Theorem 2.9. For $u \in C^k(U)$, $\phi \in C_c^\infty(U)$, $|\alpha| = k$, integration by parts gives:

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx.$$

Definition 2.10. Suppose $u, v \in L_{loc}^1(U)$, then v is the α^{th} -**weak derivative** of u if

$$\forall \phi \in C_c^\infty(U), \int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx.$$

If v exists, we say that $D^\alpha u = v$ in the weak sense. Otherwise, u does not possess a α^{th} weak derivative.

Theorem 2.10. Suppose $v \in L_{loc}^1(U)$ be such that

$$\forall \phi \in C_c^\infty(U), \int_U \phi v dx = 0,$$

we must have $v = 0$ a.e..

Proof. By 2.7, we have $v^\epsilon \rightarrow v$ a.e., as $\epsilon \rightarrow 0$.

Noe pick any such $y \in U$ where $v^\epsilon(y) \rightarrow v(y)$. Since U is open, we can find $r > 0$, such that $\bar{B}(y, r) \subset U$.

Now we define the function $\phi_{y,\epsilon}(x) := \eta_\epsilon(y - x)$ for each $\epsilon \in (0, r)$.

Since $\text{Supp}(\eta_\epsilon) \subseteq \bar{B}(0, \epsilon)$, we have $\text{Supp}(\phi_{y,\epsilon}) \subseteq \bar{B}(y, \epsilon) \subset U$ is compactly contained in U .

Also, $\phi_{y,\epsilon} \in C^\infty(\mathbb{R}^n) \subset C^\infty(U)$.

This shows that $\phi_{y,\epsilon} \in C_c^\infty(U)$.

Now we have

$$\begin{aligned} 0 &= \int_U \phi_{y,\epsilon} v dx \\ &= \int_U \eta_\epsilon(y - x) v(x) dx \\ &= v^\epsilon(y). \end{aligned}$$

Since this holds for all $\epsilon \in (0, r)$, we must have $v(y) = \lim_{\epsilon \rightarrow 0} v^\epsilon(y) = 0$.

Since this holds for a.e. $y \in U$, we have that $v = 0$ a.e.. □

Proposition 2.11. If $D^\alpha u$ exists, it is uniquely defined up to a set of measure zero.

Proof. Suppose v, \tilde{v} are both $D^\alpha u$, then $\forall \phi \in C_c^\infty(U)$,

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi dx.$$

Thus $\forall \phi \in C_c^\infty(U)$, $\int_U (v - \tilde{v}) \phi dx = 0$.

By the previous theorem, we have that $v = \tilde{v}$ a.e.. □

Definition 2.11. Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$, $u \in L_{loc}^1(U)$, suppose $D^\alpha u$ exists in the weak sense for each $|\alpha| \leq k$. The **Sobolev norm** is

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(U)}^p \right)^{1/p}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \text{ess sup}_{x \in U} |D^\alpha u(x)| \simeq \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(U)}, & p = \infty \end{cases}.$$

Definition 2.12. For $k = 1$, we write

$$\|Du\|_{L^p(U)}^p := \int_U \|Du\|_p^p dx = \int_U \sum_{i=1}^n |\partial_i u|^p dx = \sum_{i=1}^n \|\partial_i u\|_{L^p(U)}^p$$

for $1 \leq p < \infty$, and

$$\|Du\|_{L^\infty(U)} := \operatorname{ess\,sup}_{x \in U} \|Du(x)\|_1 = \operatorname{ess\,sup}_{x \in U} \sum_{i=1}^n |\partial_i u(x)| = \sum_{i=1}^n \|\partial_i u\|_{L^\infty(U)}$$

for $p = \infty$.

In this case,

$$\|u\|_{W^{1,p}(U)} = \begin{cases} \left(\|u\|_{L^p(U)}^p + \|Du\|_{L^p(U)}^p \right)^{1/p} & 1 \leq p < \infty \\ \|u\|_{L^\infty(U)} + \|Du\|_{L^\infty(U)} & p = \infty. \end{cases}$$

Proposition 2.12. Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$, $u \in L_{loc}^1(U)$, we have

$$\forall |\alpha| \leq k, \|u\|_{W^{k,p}(U)} \geq \|D^\alpha u\|_{L^p(U)}.$$

Definition 2.13. The Sobolev space is defined as

$$W^{k,p}(U) := \left\{ v \in L_{loc}^1(U) : \|v\|_{W^{k,p}(U)} < \infty \right\}.$$

Definition 2.14.

$$H^k(U) := W^{k,2}(U).$$

Remark.

$$W^{0,1}(U) = H^0(U) = L^2(U).$$

Definition 2.15. Let $(u_m)_{m=1}^\infty, u \in W^{k,p}(U)$, then

- $u_m \rightarrow u$ in $W^{k,p}(U)$ if $\lim_{m \rightarrow \infty} \|u_m - u\|_{W^{k,p}(U)} = 0$.
- $u_m \rightarrow u$ in $W_{loc}^{k,p}(U)$ if $u_m \rightarrow u$ in $W^{k,p}(V)$ for all $V \subset\subset U$.

Definition 2.16.

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)} = \left\{ u \in W^{k,p}(U) : \exists (u_m)_{m=1}^\infty \subset C_c^\infty(U) \text{ such that } u_m \rightarrow u \text{ in } W^{k,p}(U) \right\}.$$

$$H_0^k(U) = W_0^{k,2}.$$

Remark. $W_0^{k,p}(U)$ are those $u \in W^{k,p}(U)$ such that $D^\alpha u = 0$ on ∂U .

Theorem 2.13. Assume $u, v \in W^{k,p}(U)$, $|\alpha| \leq k$, then

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$.
2. $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u, \forall \alpha, \beta$ such that $|\alpha| + |\beta| \leq k$.
3. $\lambda u + v \in W^{k,p}(U), D^\alpha(\lambda u + v) = \lambda D^\alpha u + D^\alpha v, \forall \lambda \in \mathbb{R}$.
4. $\forall V \subseteq U$ be open, $u \in W^{k,p}(U)$.

Proof. 1. This is by definition.

2. Consider any $\phi \in C_c^\infty(U)$, we have

$$\begin{aligned} \int_U D^\alpha(D^\beta u)\phi dx &= (-1)^{|\alpha|} \int_U D^\beta u D^\alpha \phi dx \\ &= (-1)^{|\alpha|} (-1)^{|\beta|} \int_U u D^\beta(D^\alpha \phi) dx \\ &= (-1)^{|\alpha+\beta|} \int_U u D^{\alpha+\beta} \phi dx \\ &= \int_U D^{\alpha+\beta} u \phi dx. \end{aligned}$$

Thus $D^{\alpha+\beta}u = D^\alpha(D^\beta u)$. Similarly, $D^{\alpha+\beta}u = D^\beta(D^\alpha u)$.

3. See A2.

4. See A2.

□

Proposition 2.14 (Leibniz rule for weak derivatives). *Assume $u \in W^{k,p}(U)$, $|\alpha| \leq k$. If $\xi \in C_c^\infty(U)$, $\xi u \in W^{k,p}(U)$, and the Leibniz formula holds:*

$$D^\alpha(\xi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u,$$

where $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, and $\alpha! := \alpha_1! \cdots \alpha_n!$.

Proof. We have $\forall \phi \in C_c^\infty(U)$, $\int_U \xi u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha(\xi u) \phi dx$.

We prove by induction:

The base case is $|\alpha| = 1$, we have by Leibniz rule on regular derivatives:

$$\begin{aligned} D^\alpha(\xi \phi) &= \xi D^\alpha \phi + \phi D^\alpha \xi \\ \int_U \xi u D^\alpha \phi dx &= \int_U u(D^\alpha(\xi \phi) - \phi D^\alpha \xi) dx \\ &= \int_U u D^\alpha(\xi \phi) dx - \int_U u \phi D^\alpha \xi dx \\ &= - \int_U \xi \phi D^\alpha u dx - \int_U u \phi D^\alpha \xi dx \\ &= - \int_U \phi(\xi D^\alpha u + u D^\alpha \xi) dx. \end{aligned}$$

Since this hold for any $\phi \in C_c^\infty(U)$, we have

$$\xi D^\alpha u + u D^\alpha \xi = D^\alpha(u \xi).$$

Now suppose $l < k$ and the result holds $\forall |\beta| \leq l$.

Consider any $|\alpha| = l + 1$, we have $\alpha = \beta + \gamma$ where $|\beta| = l, |\gamma| = 1$.

$$\begin{aligned}
\int_U \xi u D^\alpha \phi dx &= \int_U \xi u D^\beta (D^\gamma \phi) dx \\
&= (-1)^{|\beta|} \int_U D^\beta (\xi u) D^\gamma \phi dx \\
&= (-1)^{|\beta|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\beta-\sigma} u D^\gamma \phi dx \\
&= (-1)^{|\beta|} (-1)^{|\gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\gamma (D^\sigma \xi D^{\beta-\sigma} u) \phi dx \\
&= (-1)^{|\beta+\gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} (D^\sigma \xi D^\gamma D^{\beta-\sigma} u + D^{\beta-\sigma} u D^\gamma D^\sigma \xi) \phi dx \\
&= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} (D^\sigma \xi D^{\gamma+\beta-\sigma} u + D^{\beta-\sigma} u D^{\gamma+\sigma} \xi) \phi dx \\
&= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} (D^\sigma \xi D^{\alpha-\sigma} u + D^{\alpha-(\gamma+\sigma)} u D^{\gamma+\sigma} \xi) \phi dx \\
&= (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\alpha-\sigma} u + \sum_{\rho \leq \alpha, \rho_j \geq 1} \binom{\beta}{\rho - \gamma} D^{\alpha-\rho} u D^\rho \xi \right) \phi dx \\
&= (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \leq \alpha} \left(\mathbb{1}_{\sigma_j \leq \alpha_j - 1} \binom{\beta}{\sigma} + \mathbb{1}_{\sigma_j \geq 1} \binom{\beta}{\sigma - \gamma} \right) D^\sigma \xi D^{\alpha-\sigma} u \right) \phi dx,
\end{aligned}$$

where $\gamma_i = \delta_{ij}$.

Now consider any $\sigma \leq \alpha$.

If $\sigma_j = 0$, we have

$$\begin{aligned}
\binom{\beta}{\sigma} &= \frac{\beta!}{\sigma!(\beta - \sigma)!} \\
&= \frac{\beta!(\beta_j + 1)}{\sigma!(\beta - \sigma)!(\beta_j + \sigma_j + 1)} \\
&= \frac{(\beta + \gamma)!}{\sigma!(\beta - \sigma + \gamma)!} \\
&= \frac{\alpha!}{\sigma!(\alpha - \sigma)!} \\
&= \binom{\alpha}{\sigma}.
\end{aligned}$$

If $\sigma_j = \alpha_j$, we have

$$\begin{aligned}
\binom{\beta}{\sigma - \gamma} &= \frac{\beta!}{(\sigma - \gamma)!(\beta - \sigma + \gamma)!} \\
&= \frac{\beta! \alpha_j}{\alpha_j (\sigma - \gamma)!(\alpha - \sigma)!} \\
&= \frac{\beta! (\beta_j + 1)}{\sigma_j (\sigma - \gamma)!(\alpha - \sigma)!} \\
&= \frac{(\beta + \gamma)!}{(\sigma - \gamma + \gamma)!(\alpha - \sigma)!} \\
&= \frac{\alpha!}{\sigma! (\alpha - \sigma)!} \\
&= \binom{\alpha}{\sigma}.
\end{aligned}$$

If $1 \leq \sigma_j \leq \alpha_j - 1$, we have

$$\begin{aligned}
\binom{\beta}{\sigma} + \binom{\beta}{\sigma - \gamma} &= \frac{\beta!}{\sigma! (\beta - \sigma)!} + \frac{\beta!}{(\sigma - \gamma)!(\beta - \sigma + \gamma)!} \\
&= \frac{\beta! (\beta_j - \sigma_j + 1)}{\sigma! (\beta - \sigma)! (\beta_j - \sigma_j + 1)} + \frac{\beta! \sigma_j}{\sigma_j (\sigma - \gamma)!(\beta - \sigma + \gamma)!} \\
&= \frac{\beta! (\beta_j - \sigma_j + 1) + \beta! \sigma_j}{\sigma! (\beta - \sigma + \gamma)!} \\
&= \frac{\beta! (\beta_j + 1)}{\sigma! (\beta - \sigma + \gamma)!} \\
&= \frac{(\beta + \gamma)!}{\sigma! (\beta - \sigma + \gamma)!} \\
&= \frac{\alpha!}{\sigma! (\alpha - \sigma)!} \\
&= \binom{\alpha}{\sigma}.
\end{aligned}$$

Thus we can see that

$$\int_U \xi u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha - \sigma} u \right) \phi dx.$$

Since ϕ is arbitrary, we have that

$$D^\alpha (\xi u) = \sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha - \sigma} u.$$

Inductively, we can prove this for any $|\alpha| = n \geq 1$. □

Theorem 2.15. $(W^{k,p}(U), \|\cdot\|_{W^{k,p}(U)})$ is a Banach space for $k \in \mathbb{N}, 1 \leq p \leq \infty$.

Proof. See A2 for the proof of $\|\cdot\|_{W^{1,\infty}(U)}$ is a norm.

Now for $1 \leq p < \infty$, we want to check:

1. If $\|u\|_{W^{k,p}(U)} = 0$, then $\|u\|_{L^p(U)} = 0$, and thus $u = 0$ a.e. on U .
2. If $u = 0$ a.e. on U , then $\forall \phi \in C_c^\infty(U)$, we have

$$\int_U D^\alpha u \phi dx = (-1)^{|\alpha|} \int_U u D^\alpha \phi dx = 0.$$

Thus $D^\alpha u = 0$ a.e. for any $|\alpha| \leq k$.

Thus $\|u\|_{W^{k,p}(U)} = 0$.

3. Let $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
\|\lambda u\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha| \leq k} \|D^\alpha(\lambda u)\|_{L^p(U)}^p \right)^{1/p} \\
&= \left(\sum_{|\alpha| \leq k} \|\lambda D^\alpha(u)\|_{L^p(U)}^p \right)^{1/p} \\
&= \left(\sum_{|\alpha| \leq k} |\lambda|^p \|D^\alpha(u)\|_{L^p(U)}^p \right)^{1/p} \\
&= |\lambda| \left(\sum_{|\alpha| \leq k} \|D^\alpha(u)\|_{L^p(U)}^p \right)^{1/p} \\
&= |\lambda| \|u\|_{W^{k,p}(U)}.
\end{aligned}$$

4. Consider any $u, v \in W^{k,p}(U)$,

$$\begin{aligned}
\|u + v\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha| \leq k} \|D^\alpha(u + v)\|_{L^p(U)}^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha| \leq k} \left(\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)} \right)^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(U)}^p \right)^{1/p} + \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p(U)}^p \right)^{1/p} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}.
\end{aligned}$$

Thus $\|\cdot\|_{W^{k,p}(U)}$ is a norm.

Consider any Cauchy sequence $(u_m)_{m=1}^\infty$.

Given any $\epsilon > 0$, $\exists N \geq 1$, such that $\forall n, m \geq N$, $\|u_m - u_n\|_{W^{k,p}(U)} < \epsilon$.

Consider any $|\alpha| \leq k$, we have

$$\|D^\alpha u_m - D^\alpha u_n\|_{L^p(U)} = \|u_m - u_n\|_{W^{k,p}(U)} \geq \|D^\alpha(u_m - u_n)\|_{L^p(U)} \leq \|u_m - u_n\|_{W^{k,p}(U)} < \epsilon.$$

Thus $(D^\alpha u_n)_{n=1}^\infty$ must be a Cauchy sequence in $(L^p(U), \|\cdot\|_{L^p(U)})$ for any $|\alpha| \leq k$.

Since $(L^p(U), \|\cdot\|_{L^p(U)})$ is complete, there must be some

$$u_\alpha \in L^p(U) \text{ such that } \lim_{n \rightarrow \infty} \|u_\alpha - D^\alpha u_n\|_{L^p(U)} = 0.$$

In particular, we have $u \in L^p(U)$, such that $\lim_{n \rightarrow \infty} \|u - u_n\|_{L^p(U)} = 0$.

Now consider any $|\alpha| \leq k$.

Given any $\phi \in C_c^\infty(U)$, we have

$$\begin{aligned}
\left| \int_U u D^\alpha \phi dx - \int_U u_n D^\alpha \phi dx \right| &= \left| \int_U (u - u_n) D^\alpha \phi dx \right| \\
&\leq \int_U |(u - u_n) D^\alpha \phi| dx \\
&\leq \|u - u_n\|_{L^p(U)} \|D^\alpha \phi\|_{L^{\frac{p}{p-1}}(U)}, \\
\left| \int_U u_\alpha \phi dx - \int_U D^\alpha u_n \phi dx \right| &= \left| \int_U (u_\alpha - D^\alpha u_n) \phi dx \right| \\
&\leq \int_U |(u_\alpha - D^\alpha u_n) \phi| dx \\
&\leq \|u_\alpha - D^\alpha u_n\|_{L^p(U)} \|\phi\|_{L^{\frac{p}{p-1}}(U)}.
\end{aligned}$$

Since $u_n \rightarrow u$, $D^\alpha u_n \rightarrow u_\alpha$ in $L^p(U)$, and $\|\phi\|_{L^{\frac{p}{p-1}}(U)}, \|D^\alpha \phi\|_{L^{\frac{p}{p-1}}(U)} < \infty$, those two limits converges to 0. Thus we have

$$\begin{aligned}
\int_U u D^\alpha \phi dx &= \lim_{n \rightarrow \infty} \int_U u_n D^\alpha \phi dx \\
&= \lim_{n \rightarrow \infty} (-1)^{|\alpha|} \int_U D^\alpha u_n \phi dx \\
&= (-1)^{|\alpha|} \int_U u_\alpha \phi dx.
\end{aligned}$$

Since this is true for any $\phi \in C_c^\infty(U)$, we have that $D^\alpha u = u_\alpha = \lim_{n \rightarrow \infty} D^\alpha u_n$ in $L^p(U)$.

Since this is true for any $|\alpha| \leq k$, we have that $u_n \rightarrow u$ in $W^{k,p}(U)$. \square

Proposition 2.16. *For any $1 \leq s \leq r < \infty, k \geq 1$, and bounded U , we have some constant $C := |U|^{\frac{1}{s} - \frac{1}{r}} m^{\frac{1}{s} - \frac{1}{r}}$, where $m = |\{\beta \in \mathbb{N}^n : |\beta| \leq k\}|$, such that*

$$\forall u \in W^{k,r}(U), \|u\|_{W^{k,s}(U)} \leq C \|u\|_{W^{k,r}(U)}, u \in W^{k,s}(U).$$

Proof. We have

$$\begin{aligned}
\|u\|_{W^{1,s}(U)}^s &= \sum_{|\beta| \leq 1} \|D^\beta u\|_{L^s(U)}^s \\
&\leq \sum_{|\beta| \leq 1} \left(|U|^{\frac{1}{s} - \frac{1}{r}} \|D^\beta u\|_{L^r(U)} \right)^s \\
&= \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s \sum_{|\beta| \leq 1} \|D^\beta u\|_{L^r(U)}^{r \frac{s}{r}} \\
&\leq \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \left(\sum_{|\beta| \leq 1} \|D^\beta u\|_{L^r(U)}^r \right)^{\frac{s}{r}} \\
&= \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \left(\|D^\alpha u\|_{W^{1,r}(U)}^r \right)^{\frac{s}{r}} \\
&= \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \|D^\alpha u\|_{W^{1,r}(U)}^s \\
&\implies \\
\|u\|_{W^{1,s}(U)} &\leq |U|^{\frac{1}{s} - \frac{1}{r}} m^{\frac{1}{s} - \frac{1}{r}} \|D^\alpha u\|_{W^{1,r}(U)}.
\end{aligned}$$

\square

Proposition 2.17. For any $u \in W^{k,p}(U)$, and $|\alpha| \leq k, \epsilon > 0$, we have that

$$D^\alpha u^\epsilon|_{U_\epsilon} = (\eta_\epsilon * D^\alpha u)|_{U_\epsilon}.$$

Proof. Fix any $x \in U_\epsilon$, we have

$$\begin{aligned} D^\alpha u^\epsilon(x) &= D^\alpha(\eta_\epsilon * u)(x) \\ &= (D^\alpha \eta_\epsilon * u)(x) \\ &= \int_U D^\alpha \eta_\epsilon(x-y)u(y)dy, \end{aligned} \tag{2.7}$$

Consider $\eta_{\epsilon,x}(y) := \eta_\epsilon(x-y)$, we can see $\forall i \in [n], \partial_i \eta_{\epsilon,x}(y) = -\partial_i \eta_\epsilon(x-y)$, thus we have

$$\begin{aligned} D^\alpha u^\epsilon(x) &= \int_U D^\alpha \eta_\epsilon(x-y)u(y)dy \\ &= (-1)^{|\alpha|} \int_U D^\alpha \eta_{\epsilon,x}(y)u(y)dy \\ &= \int_U \eta_{\epsilon,x}(y)D^\alpha u(y)dy \\ &= \int_U \eta_\epsilon(x-y)D^\alpha u(y)dy \\ &= (\eta_\epsilon * D^\alpha u)(x). \end{aligned}$$

Since this holds for any $x \in U_\epsilon$, we proved our result. \square

Corollary 2.18. Let $1 \leq p \leq \infty, k \geq 1$. Let $u \in W^{k,p}(U)$, then for any $\epsilon > 0$, $U_\epsilon \subseteq V \supseteq \text{Supp}(u) + \bar{B}(0, \epsilon)$, we have that

$$\|u^\epsilon\|_{W^{k,p}(V)} \leq \|u\|_{W^{k,p}(V)}.$$

Proof. By 2.17, $\forall |\alpha| \leq k$, we have $D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$ on the entire U_ϵ and thus on V . Since $\forall |\alpha| \leq k$, $\text{Supp}(D^\alpha u) \subseteq \text{Supp}(u)$, we have $\text{Supp}(\eta_\epsilon * D^\alpha u) \subseteq \text{Supp}(u) + \bar{B}(0, \epsilon) \subseteq V$. By 2.8,

$$\begin{aligned} \|u^\epsilon\|_{W^{k,p}(V)}^p &= \sum_{|\alpha| \leq k} \|D^\alpha u^\epsilon\|_{L^p(V)}^p \\ &= \sum_{|\alpha| \leq k} \|\eta_\epsilon * D^\alpha u\|_{L^p(V)}^p \\ &\leq \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(V)}^p \\ &= \|u\|_{W^{k,p}(V)}^p. \end{aligned}$$

\square

2.4 Smooth Approximation

Theorem 2.19. (Local Smooth Approximation)

Let $1 \leq p < \infty, k \geq 1$. Suppose U is open, and $u \in W^{k,p}(U)$, we have that

1. $\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$,
2. $u^\epsilon \rightarrow u$ in $W_{loc}^{k,p}(U)$ as $\epsilon \rightarrow 0$.

Proof. $\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$ by 2.7.1.

Given any $V \subset\subset U$, we can find some $\epsilon_V > 0$ such that $V \subset\subset U^{\epsilon_V}$.

Consider any $|\alpha| \leq k$.

We have $D^\alpha u \in L^p(U) \subseteq L_{loc}^p(U)$.

By 2.7.5, we have that $\eta_\epsilon * D^\alpha u \rightarrow D^\alpha u$ in $L^p_{loc}(U)$ as $\epsilon \rightarrow 0$, and thus $\eta_\epsilon * D^\alpha u \rightarrow D^\alpha u$ in $L^p(V)$.

In addition, by 2.17, $\forall \epsilon > 0$, $D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$ in U^ϵ .

Now $\forall 0 < \epsilon < \epsilon_V$, $V \subset \subset U^{\epsilon_V} \subseteq U^\epsilon$, and thus $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$ in V .

Thus $D^\alpha u^\epsilon \rightarrow D^\alpha u$ in $L^p(V)$ as $\epsilon \rightarrow 0$.

Since this is true $\forall |\alpha| \leq k$, we have $u^\epsilon \rightarrow u$ in $W^{k,p}(V)$.

Since this holds for any $V \subset \subset U$, $u^\epsilon \rightarrow u$ in $W^{k,p}_{loc}(U)$. \square

Corollary 2.20. *Suppose U is open, and $u \in W^{k,p}(U)$ is compactly supported in U , then $u \in W^{k,p}_0(U)$.*

Proof. Since $\text{Supp}(u) \subset U$ is compact, we must have $r := \frac{1}{2} \text{dist}(\text{Supp}(u), \partial U) > 0$.

For $n \in \mathbb{N}^+$, let $u_n := u \frac{r}{n}$.

We have that $u_n \rightarrow u$ in $W^{k,p}_{loc}(U)$ as $n \rightarrow \infty$.

Let $W := \overline{\text{Supp}(u) + \bar{B}(0, r/2)} \subset U$. Notice that it is compact, and $\forall n \in \mathbb{N}^+$, $\text{Supp}(u_n) \subseteq \text{Supp}(u) + \bar{B}(0, \frac{r}{n}) \subseteq W$, which means $u_n \in C_c^\infty(U)$.

Now there is some $W \subset V \subset \subset U$, so $u_n \rightarrow u$ in $W^{k,p}(V)$.

In addition,

$$\begin{aligned} \|u - u_n\|_{W^{k,p}(U)}^p &= \int_U \sum_{|\alpha| \leq k} |D^\alpha(u - u_n)|^p dx \\ &= \int_V \sum_{|\alpha| \leq k} |D^\alpha(u - u_n)|^p dx \\ &= \|u - u_n\|_{W^{k,p}(V)}^p. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \|u - u_n\|_{W^{k,p}(V)} = \lim_{n \rightarrow \infty} \|u - u_n\|_{W^{k,p}(U)} = 0$.

Since each $u_n \in C_c^\infty(U)$, we have $u \in \overline{C_c^\infty(U)} = W^{k,p}_0(U)$. \square

Theorem 2.21. (Meyer-Serrin)

Let $1 \leq p < \infty, k \geq 1$. Suppose U is open and bounded, and $u \in W^{k,p}(U)$. There exists $(u_m)_{m=1}^\infty \subseteq C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \rightarrow u$ in $W^{k,p}(U)$.

Proof. Let $\delta > 0$ be given.

Let $U_i := U_{\frac{1}{i}} = \{x \in U : \text{dist}(x, \partial U) > \frac{1}{i}\}$ for $i \in \mathbb{N}^+$.

We have $U_1 \subseteq \bar{U}_1 \subseteq U_2 \subseteq \bar{U}_2 \subseteq U_3 \subseteq \dots \subseteq U$.

Indeed, for some $x \in \bar{U}_i$, we know that $\forall y \in \partial U, \|x - y\| \geq \frac{1}{i} > \frac{1}{i+1} \implies x \in U_{i+1}$.

Since U is open, for any $x \in U$, we can find some $i \geq 1$, such that $B(x, \frac{1}{i}) \subseteq U$, which means $\text{dist}(x, \partial U) \geq \frac{1}{i}$, and thus $x \in \bar{U}_i \subseteq U_{i+1}$. Thus we have $U = \bigcup_{i=1}^\infty U_i$.

Let $V_i := U_{i+3} \setminus \bar{U}_{i+1}$ for $i \in \mathbb{N}^+$. Since U is bounded, we can choose $V_0 \subset \subset U$ with $V_0 \supset \bar{U}_2$, we claim that $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$.

It is easy to see $\bigcup_{i=0}^n V_i \subseteq U_{n+3}$. For the other direction, we will prove by induction.

The base case $n = 1$, we can see that $V_0 \cup V_1 \supset \bar{U}_2 \cup (U_4 \setminus \bar{U}_2) = U_4$.

Now suppose $n > 1$, and it holds for $n - 1$, we have that

$$\begin{aligned} \bigcup_{i=0}^n V_i &= \left(\bigcup_{i=0}^{n-1} V_i \right) \cup (V_n) \\ &= U_{n-1+3} \cup (U_{n+3} \setminus \bar{U}_{n+1}) \\ &\supset U_{n+2} \cup (U_{n+3} \setminus \bar{U}_{n+2}) \\ &= U_{n+3}. \end{aligned}$$

By induction, we have that $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$.

Notice that $\forall x \in U = \bigcup_{n=1}^\infty U_n, \exists n \geq 1$, such that $x \in U_n \subseteq U_{n+3} \subseteq \bigcup_{i=0}^n V_i \implies \exists i \geq 0$, such that $x \in V_i$.

Thus

$$U = \bigcup_{i=0}^\infty V_i.$$

Now let $W_i := U_{i+4} \setminus \bar{U}_i$ for $i \in \mathbb{N}^+$.

Since each $U_{i+4} \subseteq U_{i+4} \subseteq U_{i+5} \subseteq U$, we also have $U_{i+4} \subset\subset U$ and thus

$$W_i \subset\subset U.$$

Notice that $\forall x, y \in U$,

$$\begin{aligned} \text{dist}(x, \partial U) &= \inf \{ \|z - x\| : z \in \partial U \} \\ &= \inf \{ \|z - y + y - x\| : z \in \partial U \} \\ &\leq \inf \{ \|z - y\| + \|y - x\| : z \in \partial U \} \\ &= \inf \{ \|z - y\| : z \in \partial U \} + \|y - x\| \\ &= \text{dist}(y, \partial U) + \|y - x\|. \end{aligned}$$

Similarly, $\text{dist}(y, \partial U) \leq \text{dist}(x, \partial U) + \|y - x\|$. Thus we have

$$\text{dist}(y, \partial U) - \|y - x\| \leq \text{dist}(x, \partial U) \leq \text{dist}(y, \partial U) + \|y - x\|.$$

Consider any $0 < \epsilon < \frac{1}{i+3} - \frac{1}{i+4} < \frac{1}{i} - \frac{1}{i+1}$, we have that

$$\begin{aligned} x \in \bar{B}(0, \epsilon) + V_i &\implies \exists y \in U_{i+3} \setminus U_{i+1}^- \text{ such that } \|x - y\| \leq \epsilon \\ &\implies \exists y \in U \text{ such that } \frac{1}{i+3} < \text{dist}(y, \partial U) < \frac{1}{i+1}, \|x - y\| \leq \epsilon \\ &\implies \exists y \in U \text{ such that } \frac{1}{i+3} - \|x - y\| < \text{dist}(x, \partial U) < \frac{1}{i+1} + \|x - y\|, \|x - y\| \leq \epsilon \\ &\implies \frac{1}{i+3} - \epsilon < \text{dist}(x, \partial U) < \frac{1}{i+1} + \epsilon \\ &\implies \frac{1}{i+4} < \text{dist}(x, \partial U) < \frac{1}{i} \\ &\implies x \in W_i. \end{aligned}$$

Thus we have

$$\forall 0 < \epsilon < \frac{1}{i+3}, \bar{B}(0, \epsilon) + V_i \subseteq W_i.$$

Finally, since $V_0 \subset\subset U$, we can choose $V_0 \subset\subset W_0 \subset\subset U$, such that $V_0 + B(0, \epsilon_0'') \subseteq W_i$ for some $\epsilon_0' > 0$.

Let $(\zeta_i)_{i=0}^\infty$ be a **smooth partition of unity** such that

$$\forall x \in U \sum_{i=0}^\infty \zeta_i(x) = 1, \forall i \geq 0, \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^\infty(U), \\ \text{Supp}(\zeta_i) \subseteq V_i. \end{cases}$$

Notice that $\forall u \in W^{k,p}(U)$, we have $\zeta_i u \in W^{k,p}(U)$ as well. Moreover, $\text{Supp}(\zeta_i u) \subseteq V_i$.

Let $u_i^\epsilon := \eta_\epsilon * (\zeta_i u) \forall \epsilon > 0$.

By previous theorem, we have that $u_i^\epsilon \rightarrow \zeta_i u$ in $W_{loc}^{k,p}(U)$.

Thus for $W_i \subset\subset U$, we can find $\epsilon_i' > 0$ such that $\forall \epsilon < \epsilon_i', \|u_i^\epsilon - \zeta_i u\|_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}$.

Now pick $\epsilon_0 = \min(\epsilon_0'', \epsilon_0'), \forall i \in \mathbb{N}^+, \epsilon_i = \frac{1}{2} \min\left(\frac{1}{i+3} - \frac{1}{i+4}, \epsilon_i'\right) > 0$.

We have that by 2.7,

$$\text{Supp}(u_i^{\epsilon_i}) \subseteq \text{Supp}(\eta_{\epsilon_i}) + \text{Supp}(\zeta_i u) \subseteq \bar{B}(0, \epsilon_i) + V_i \subseteq W_i,$$

and

$$\|u_i^{\epsilon_i} - \zeta_i u\|_{W^{k,p}(U)} = \|u_i^{\epsilon_i} - \zeta_i u\|_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}.$$

Now let $v := \sum_{i=0}^\infty u_i^{\epsilon_i}$.

Notice that $\forall x \in U, \exists V \subset\subset U_{\epsilon_i}$ be open, such that $x \in V$. Since $V \cap W_i \neq \emptyset$ for only finitely many i , and

$\text{Supp}(u_i^{\epsilon_i}) \subseteq W_i$, we must have $v = \sum_{i=0}^k u_i^{\epsilon_i}$ on V for some finite k .
In addition, by 2.7, each $u_i^{\epsilon_i} \in C^\infty(U_{\epsilon_i})$, thus infinitely differentiable at x .
Thus $v = \sum_{i=0}^k u_i^{\epsilon_i}$ is infinitely differentiable at x .
Since $x \in U$ is arbitrary, we have that $v \in C^\infty(U)$.
In addition,

$$\forall x \in U, u(x) = \sum_{i=0}^{\infty} \zeta_i(x)u(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x)$$

by definition of partition of unity. Thus

$$u(x) - v(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x) - \sum_{i=0}^{\infty} u_i^{\epsilon_i}(x) = \sum_{i=0}^{\infty} (\zeta_i u)(x) - u_i^{\epsilon_i}(x) = \sum_{i=0}^{\infty} (\zeta_i u - u_i^{\epsilon_i})(x).$$

Since this holds for all $x \in U$, we have that

$$u - v = \sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i}.$$

Now we have

$$\begin{aligned} \|v - u\|_{W^{k,p}(U)} &= \left\| \sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i} \right\|_{W^{k,p}(U)} \\ &\leq \sum_{i=0}^{\infty} \|\zeta_i u - u_i^{\epsilon_i}\|_{W^{k,p}(U)} \\ &< \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} \\ &= \delta. \end{aligned}$$

□

Definition 2.17. Let $U \subseteq \mathbb{R}^n$ be open and bounded, then ∂U is C^k if $\forall z \in \partial U, \exists r > 0, \gamma \in C^k(\mathbb{R}^{n-1})$, such that

$$U \cap \bar{B}(z, r) = \{x \in B(z, r) : x^n > \gamma(x^1, \dots, x^{n-1})\}.$$

Theorem 2.22. Let U be bounded, and ∂U is C^1 , then $\forall u \in W^{k,p}(U)$ for $1 \leq p < \infty$, there exists functions $u_m \in C^\infty(\bar{U})$ such that $u_m \rightarrow u$ in $W^{k,p}(U)$.

Proof. See 5.3.3 in Eavan's book. □

2.5 Extensions

Proposition 2.23. Let $U \subseteq \mathbb{R}^n$ be open and bounded, with ∂U be C^k . Then $\forall z \in \partial U, \exists r > 0, \Phi \in C^k(B(z, r), \mathbb{R}^n)$ a diffeomorphism, such that $\Phi(\partial U \cap B(z, r))$ is in a flat hyperplane, and $\det(D\Phi) = \det(D\Psi) = 1$, for $\Psi := \Phi^{-1}$.

Proof. Let

$$\Phi^i(x) := x^i \quad \forall i \in [n-1], \Phi^n(x) := x^n - \gamma(x^1, \dots, x^{n-1}),$$

and let

$$\Phi^i(y) := y^i \quad \forall i \in [n-1], \Phi^n(y) := y^n - \gamma(y^1, \dots, y^{n-1}).$$

□

Theorem 2.24. (Sobolev Norm Equivalence Under Diffeomorphism)

Let $W \subseteq \mathbb{R}^n$ and Φ is a $C^1(W)$ diffeomorphism, i.e., it has inverse $\Psi \in C^1(W)$. Let $v := u \circ \Psi$, then

$$\exists C_0, C_1 \text{ such that } C_0 \|u\|_{W^{1,p}(W)} \leq \|v\|_{W^{1,p}(\Phi(W))} \leq C_1 \|u\|_{W^{1,p}(W)}.$$

Lemma 2.25. *Let $1 \leq p < \infty$. Assume U is bounded, with ∂U be C^1 . Let V be open and bounded, with $U \subset \subset V$, then there exists a bounded linear operator $E : C^1(\bar{U}) \rightarrow W^{1,p}(\mathbb{R}^n)$, such that $\forall u \in C^1(\bar{U})$:*

1. $Eu = u$ in U ,
2. $\text{Supp}(Eu) \subseteq V$,
3. $\exists C > 0$, such that $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$.

Proof. Fix $z \in \partial U$.

In addition, we assume ∂U is flat around z on the plane $\{x^n = 0\}$.

Then there exists an open ball $B := B(z, r)$, such that

$$B^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B^- := B \cap \{x^n \leq 0\} \subseteq \mathbb{R}^n \setminus U.$$

$$\text{Let } \bar{u}_z(x) := \begin{cases} u(x) & x \in B^+ \\ -3u(x^1, \dots, x^{n-1}, -x^n) + 4u(x^1, \dots, x^{n-1}, -\frac{1}{2}x^n) & x \in B^-. \end{cases}$$

Then we claim $\bar{u}_z \in C^1(B)$.

Indeed, let $u^- := \bar{u}_z|_{B^-}$, $u^+ := \bar{u}_z|_{B^+}$.

$$\begin{aligned} u^-|_{x^n=0} &= -3 + 4u|_{x^n=0} \\ &= u|_{x^n=0} \\ &= u^+|_{x^n=0}; \\ \forall i \in [n-1], \\ \partial_i u^-|_{x^n=0} &= -3\partial_i u|_{x^n=0} + 4\partial_i u|_{x^n=0} \\ &= \partial_i u|_{x^n=0} \\ &= \partial_i u^+|_{x^n=0} \\ \partial_n u^-|_{x^n=0} &= 3\partial_n u|_{x^n=0} - 4\frac{1}{2}\partial_n u|_{x^n=0} \\ &= u|_{x^n=0} \\ &= \partial_n u^+|_{x^n=0}. \end{aligned}$$

Thus $\bar{u} \in C^1(B)$.

By A2, we have

$$\exists C > 0, \text{ such that } \|\bar{u}_z\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}.$$

Now suppose ∂U is not flat around z , we can find $r_1 > 0$, $\Phi \in C^1(B(z, r_1), \mathbb{R}^n)$, such that $\Phi(\partial U \cap B(z, r_1))$ is in a flat hyperplane, WLOG $\{y_n = 0\}$, and $\det(D\Phi) = \det(D\Psi) = 1$, for $\Psi := \Phi^{-1}$.

Notice that we can find $B(z, r_2) \subset \subset V$ since V is open and $z \in \bar{U} \subseteq V$.

By setting $r = \min(r_1, r_2) > 0$, we can WLOG work with $B(z, r) \subset \subset V$.

Let $z' := \Phi(z)$, $v := u \circ \Psi \in C^1(\Phi(\bar{U})) = C^1(\overline{\Phi(U)})$.

Since $\Phi(B(z, r))$ is open, we can choose some open ball $B := B(z', r') \subseteq \Phi(B(z, r))$. Let $W_z := \Psi(B)$.

Since $\Phi(\partial U \cap B(z, r))$ is in the plane $\{y_n = 0\}$, we have

$$B^+ := B \cap \{y^n \geq 0\} = \Phi(W_z \cap \bar{U}), B^- := B \cap \{y^n \leq 0\} = \Phi(W_z \setminus U).$$

Now we can extend v form B^+ to B with

$$\|\bar{v}\|_{W^{1,p}(B)} \leq C\|v\|_{W^{1,p}(B^+)}.$$

Now let $\bar{u}_z := \bar{v} \circ \Psi$, we have that $B = \Phi(W_z)$, $\bar{v} = \bar{u}_z \circ \Phi$, and by 2.24, we have

$$\begin{aligned} \|\bar{u}_z\|_{W^{1,p}(W_z)} &\leq C_1\|\bar{v}\|_{W^{1,p}(\Phi(W_z))} \\ &= C_1\|\bar{v}\|_{W^{1,p}(B)} \\ &\leq C_2\|v\|_{W^{1,p}(B^+)} \\ &\leq C_3\|u\|_{W^{1,p}(\Psi(B^+))} \\ &\leq C_3\|u\|_{W^{1,p}(U)} \end{aligned}$$

Notice that $\forall z, \Phi(z) \in B \implies z \in W_z$, thus $\{W_z\}_{z \in \partial U}$ forms an open cover for ∂U .

Since ∂U is compact, we can find a finite subcover $\{W_i\}_{i=1}^N$.

Notice that $(\bar{U} \setminus \bigcup_{i=1}^N W_i) \subset U$ is closed, and U is bounded, so we can find $(\bar{U} \setminus \bigcup_{i=1}^N W_i) \subseteq W_0 \subset\subset U$.

We then have $\bigcup_{i=0}^N W_i = U$.

Now let $(\zeta_i)_{i=0}^N$ be a partition of unity subordinate to W_i , such that

$$\forall x \in U \quad \sum_{i=0}^N \zeta_i(x) = 1, \quad \forall i \geq 0, \quad \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^\infty(\mathbb{R}^n), \\ \text{Supp}(\zeta_i) \subseteq W_i. \end{cases}$$

Let $\bar{u} := \sum_{i=0}^N \zeta_i \bar{u}_i$, with $\bar{u}_0 := u$. We have that

$$\begin{aligned} \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} &\leq \sum_{i=0}^N \|\zeta_i \bar{u}_i\|_{W^{1,p}(\mathbb{R}^n)} \\ &= \sum_{i=0}^N \|\zeta_i \bar{u}_i\|_{W^{1,p}(W_i)} \\ &\leq C_4 \sum_{i=0}^N \|\bar{u}_i\|_{W^{1,p}(W_i)} \\ &= C_5 \|u\|_{W^{1,p}(W_i)}, \end{aligned}$$

since each term is bounded, and we have a finite sum.

We thus define $Eu := \bar{u}$.

We can check that E is linear and bounded. □

Theorem 2.26. *Let $1 \leq p < \infty$. Assume U is bounded, with ∂U be C^1 . Let V be open and bounded, with $U \subset\subset V$, then there exists a bounded linear operator $E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$, such that $\forall u \in W^{1,p}(U)$:*

1. $Eu = u$ a.e. in U ,
2. $\text{Supp}(Eu) \subseteq V$,
3. $\exists C > 0$, such that $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)}$.

Proof. By 2.22, we know $C^\infty(\bar{U}) \subseteq C^1(\bar{U})$ is dense in $W^{1,p}(U)$, and thus $C^1(\bar{U})$ is also dense in $W^{1,p}(U)$. By 1.20, we can extend the result in the above lemma to get $E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$.

In addition, since $Eu = \lim_{m \rightarrow \infty} Eu_m$ for some $u_m \rightarrow u$ in $W^{1,p}(U)$, we also have $Eu = \lim_{m \rightarrow \infty} Eu_m = Eu = \lim_{m \rightarrow \infty} u_m = u$, a.e..

Also, $\text{Supp}(Eu) \subseteq \bigcup_{m=1}^\infty \text{Supp}(Eu_m) \subseteq V$. □

2.6 Traces

Proposition 2.27. *(Young's inequality)*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b > 0, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.28. *Let U be bounded, and ∂U is C^1 , and $1 \leq p < \infty$. Then there exists a bounded linear operator $T : C^1(\bar{U}) \rightarrow L^p(\partial U)$; $u \mapsto u|_{\partial U}$ and a constant $C > 0$, such that*

$$\forall u \in C^1(\bar{U}), \quad \|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)}.$$

Proof. Consider $z \in \partial U$.

Assume ∂U is flat near z in the hyperplane $\{x^n = 0\}$.

Then there exists an open ball $B_z := B(z, r)$, such that

$$B_z^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B_z^- := B \cap \{x^n \leq 0\} \subseteq \mathbb{R}^n \setminus U.$$

Since u is C^1 and thus continuous, WLOG, we can take r small enough, such that u does not change sign in B_z . Namely, $|u| = u \operatorname{sgn}(u(z))$ in B_z .

Let $\hat{B}_z := B(z, \frac{r}{2})$, and let $\xi \in C_c^\infty(B_z)$ such that $\xi \geq 0$ in B_z , and $\xi = 1$ in \hat{B}_z .

Let $\Gamma_z := \hat{B}_z \cap \partial U$, then we have $\operatorname{Supp}(\xi u) \subseteq B_z^+$, and $\xi u = u$ on T .

Let $x' := (x^1, \dots, x^{n-1})$, by Fundamental Theorem of Calculus, we have

$$\int_0^\infty (\xi |u|^p)(x', t) dt = -(\xi |u|^p)(x', 0).$$

In addition, we have

$$\begin{aligned} \|u\|_{L^p(\Gamma_z)}^p &= \int_{\Gamma_z} |u|^p(x', 0) dx' \\ &\leq \int_{\mathbb{R}^{n-1}} (\xi |u|^p)(x', 0) dx' \\ &= - \int_0^\infty \int_{\mathbb{R}^{n-1}} (\xi |u|^p)(x', t) dx' dt \\ &= - \int_{B_z^+} (\xi |u|^p)(x) dx \\ &= - \int_{B_z^+} \xi_{x_n} |u|^p + \xi p |u|^{p-1} (\operatorname{sgn} u(z)) u_{x_n} dx \\ &\leq \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi p |u|^{p-1} |u_{x_n}| dx \\ &\leq \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi p \left(\frac{(|u|^{p-1})^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{|u_{x_n}|^p}{p} \right) dx \\ &= \int_{B_z^+} |\xi_{x_n}| |u|^p + \xi(p-1) |u|^p + \xi |u_{x_n}|^p dx \\ &\leq \int_{B_z^+} (|\xi_{x_n}| + \xi(p-1)) |u|^p + \xi |Du|^p dx. \end{aligned}$$

Since $\xi \in C_c^\infty(B_z)$, by EVT, $|\xi_{x_n}|, \xi$ are all bounded. Thus $\exists C > 0$, such that $|\xi_{x_n}| + \xi(p-1), \xi \leq C$ in B_z . Thus

$$\|u\|_{L^p(\Gamma_z)}^p \leq \int_{B_z^+} C |u|^p + C |Du|^p dx = C \|u\|_{W^{1,p}(B_z^+)}^p \leq C \|u\|_{W^{1,p}(U)}^p.$$

Now if ∂U is not flat near z , we can find a C^1 diffeomorphism Φ to make it flat. We still have

$$\|u\|_{L^p(\Gamma_z)} \leq C \|u\|_{W^{1,p}(U)},$$

by the equivalence of Sobolev norms under diffeomorphism 2.24.

Since $\{B_z\}_{z \in \partial U}$ form an open cover for ∂U , and ∂U is compact, we can find a finite subcover $\{B_i : x_i \in \partial U\}_{i=1}^N$, and their corresponding Γ_i .

For each $i \in [N]$, we have that

$$\|u\|_{L^p(\Gamma_i)} \leq C_i \|u\|_{W^{1,p}(U)}.$$

We have that

$$\begin{aligned}
\|Tu\|_{L^p(\partial U)}^p &= \int_{\partial U} |u|^p dx \\
&\leq \sum_{i=1}^N \int_{\Gamma_i} |u|^p dx \\
&= \sum_{i=1}^N \|u\|_{L^p(\Gamma_i)}^p \\
&\leq \sum_{i=1}^N C_i \|u\|_{W^{1,p}(U)}^p \\
&= C \|u\|_{W^{1,p}(U)}^p,
\end{aligned}$$

by taking $C := \sum_{i=1}^N C_i$. □

Theorem 2.29. *Let U be bounded, and ∂U is C^1 , and $1 \leq p < \infty$. Then there exists a bounded linear operator $T : W^{1,p}(U) \rightarrow L^p(\partial U)$ and a constant $C > 0$, such that*

$$\forall u \in W^{1,p}(U) \cap C(\bar{U}), Tu = u|_{\partial U},$$

and

$$\forall u \in W^{1,p}(U), \|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)}.$$

Proof. By 2.22, we know $C^\infty(\bar{U}) \subseteq C^1(\bar{U})$ is dense in $W^{1,p}(U)$, and thus $C^1(\bar{U})$ is also dense in $W^{1,p}(U)$. By 1.20, we can extend the result in the above lemma to get $T : W^{1,p}(U) \rightarrow L^p(\partial U)$. □

Theorem 2.30. *Let U be bounded, and ∂U is C^1 , then for any $u \in W^{1,p}(U)$, we have that*

$$u \in W_0^{1,p}(U) \iff Tu = 0 \text{ on } \partial U.$$

2.7 Weak and Normal Derivatives

Proposition 2.31. *If $u, v \in C(U)$ are both continuous, and $u = v$ a.e., then $\forall x \in U, u(x) = v(x)$.*

Proof. Consider any $x \in U$.

Since U is open, we can find some $r > 0$, such that $B(x, r) \subseteq U$.

For any $i \geq \lceil \frac{1}{r} \rceil$, we must have some $x_i \in B(x, \frac{1}{i}) \subseteq U$, such that $u(x_i) = v(x_i)$.

Otherwise $\{x \in U : u(x) \neq v(x)\} \supseteq B(x, \frac{1}{i}) \cap U = B(x, \frac{1}{i})$ does not have measure 0.

Thus $\lim_{i \rightarrow \infty} x_i = x$.

Since u, v are both continuous, we have that

$$\begin{aligned}
u(x) &= u\left(\lim_{i \rightarrow \infty} x_i\right) \\
&= \lim_{i \rightarrow \infty} u(x_i) \\
&= \lim_{i \rightarrow \infty} v(x_i) \\
&= v\left(\lim_{i \rightarrow \infty} x_i\right) \\
&= v(x).
\end{aligned}$$

This is true for any $x \in U$, which completes the proof. □

Remark. For the following part in this subsection, we will use D^α to denote the α^{th} normal derivative of u , and \bar{D}^α to be the α^{th} weak derivative of u to avoid confusion.

Proposition 2.32. *Given any $\alpha \in \mathbb{N}^n$. $\forall u$ such that its α^{th} normal derivative $D^\alpha u$ exists and is continuous, and any $v = u$ a.e., we have that $D^\alpha u$ is an α^{th} weak derivative of v . Namely, $\bar{D}^\alpha v = D^\alpha u$ a.e..*

Proof. Consider any $\phi \in C_c^\infty(U)$, we have that

$$\begin{aligned} \int_U v D^\alpha \phi dx &= \int_U u D^\alpha \phi dx \\ &= (-1)^{|\alpha|} \int_U D^\alpha u \phi dx, \end{aligned}$$

where the second equality follows from integration by part over some $\text{Supp}(\phi) \subseteq V \subseteq U$ with Lipschitz boundary. \square

Definition 2.18. A domain U is **path-connected** if $\forall x, y$, there is some continuous path $\gamma : [0, 1] \rightarrow U$, such that $\gamma(0) = x, \gamma(1) = y$.

Proposition 2.33. *Let $U \subseteq \mathbb{R}^n$ be open and connected, and $1 \leq p \leq \infty$, $u \in W^{1,p}(U)$, then*

$$\bar{D}u = 0 \text{ a.e.} \iff u \text{ is a constant a.e..}$$

Proof. Suppose u is a constant a.e., then it means u has a version $\tilde{u}(x) = C$, $\forall x \in U$ for some constant C . Clearly \tilde{u} is differentiable, and has continuous normal derivative $D\tilde{u}(x) = 0$, $\forall x \in U$.

By 2.32, and since $\tilde{u} = u$ a.e., we have that $D\tilde{u}$ is an weak derivative of v . Since the weak derivative is unique a.e., we must have $\bar{D}u = 0$ a.e..

On the other hand, suppose $\bar{D}u = 0$ a.e..

We know that by 2.17, for any $\epsilon > 0, x \in U_\epsilon$, and any direction $i \in [n]$,

$$\begin{aligned} (\partial_i u^\epsilon)(x) &= (\eta_\epsilon * \partial_i u)(x) \\ &= \int_U \eta_\epsilon(x) \partial_i u(y) dy \\ &= \int_U \eta_\epsilon(x) 0 dy \\ &= 0, \end{aligned}$$

since $\partial_i u(y) = 0$ for a.e. $y \in U$.

Since this holds for all $i \in [n]$, we must have $\forall x \in U$, $Du^\epsilon = 0$.

Notice that by 2.7, $u^\epsilon \in C^\infty(U_\epsilon)$, which by normal calculus means that $\forall x \in U_\epsilon$, $u_\epsilon(x) = C_\epsilon$ for some constant C_ϵ that does not depend on x .

Again by 2.7, $u^\epsilon \rightarrow u$ a.e. as $\epsilon \rightarrow 0$. Pick any such x with $u^\epsilon(x) \rightarrow u(x)$.

Notice that we can find $\delta > 0$, such that $\forall \epsilon \in (0, \delta), x \in U_\epsilon$.

Thus we have $u(x) = \lim_{\epsilon \rightarrow 0} u_\epsilon(x) = \lim_{\epsilon \rightarrow 0} C_\epsilon$.

Since $\lim_{\epsilon \rightarrow 0} C_\epsilon$ converges, we can call it $C := \lim_{\epsilon \rightarrow 0} C_\epsilon$, which is a constant that is independent of x, ϵ .

Now any such x satisfies $u(x) = \lim_{\epsilon \rightarrow 0} C_\epsilon = C$, and they are by choice a.e.. \square

Lemma 2.34. *Consider $1 \leq p \leq \infty$, and $U = (a_1, b_1) \times \dots \times (a_n, b_n) \subseteq \mathbb{R}^n$ be an open rectangle. Let $1 \leq i \leq n$, suppose $u \in W^{1,p}(U)$ has a continuous representative $u^* \in C(U)$, and its i^{th} weak derivative $\bar{\partial}_i u$ has a continuous representative $(\bar{\partial}_i u)^* \in C(U)$, then the regular i^{th} partial derivative*

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U$$

exists and is continuous.

Proof. Pick some $s \in (a_i, b_i)$, let $S := \{x \in U : x^i = s\}$ be the slice of U .

By FTC, there is a unique v , defined by

$$v(x^1, \dots, x^n) := u^*(x^1, \dots, x^{i-1}, s, x^{i+1}, \dots, x^n) + \int_s^{x^i} (\bar{\partial}_i u)^*(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n) dt,$$

such that $v|_S = u^*|_S$, and the i^{th} normal partial derivative

$$\partial_i v(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U.$$

We notice that $\bar{\partial}_i v = \partial_i v$ a.e. by 2.32.

Thus the weak derivative $\bar{\partial}_i(u^* - v) = \bar{\partial}_i(u^*) - \bar{\partial}_i v = \bar{\partial}_i u - \partial_i v = \bar{\partial}_i u - (\bar{\partial}_i u)^* = 0$ a.e..

Fix any $(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$, and denote $w : (a_i, b_i) \rightarrow \mathbb{R}$ by

$$w(t) := (u^* - v)(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n).$$

We have that $\bar{D}w = \bar{\partial}_i(u^* - v) = 0$ with respect to $t \in (a_i, b_i)$ a.e..

By 2.33, $w(t) = C$ a.e. $t \in (a_i, b_i)$ form some constant C , since (a_i, b_i) is clearly connected.

Notice that w is continuous, since both u^*, v are continuous on the x^i direction.

Since both w, C are continuous, we have $\forall t \in (a_i, b_i), w(t) = C$.

Since $v|_S = u^*|_S$, we must have $C = w(s) = 0$ and thus

$$\forall t \in (a_i, b_i), u^*(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n) = v(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n).$$

Since this holds for all $(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$, we must have $u^*(x) = v(x) \quad \forall x \in U$.

By construction of v , we have that

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U.$$

□

Lemma 2.35. *Consider $1 \leq p \leq \infty$, and $U \subseteq \mathbb{R}^n$ be open. If $u \in W^{1,p}(U)$ has a continuous representative $u^* \in C(U)$, and its weak derivative $\partial_i u$ has a continuous representative $(\bar{\partial}_i u)^* \in C(U)$, then the regular i^{th} partial derivative*

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \quad \forall x \in U$$

exists and is continuous.

Proof. Notice that any open $U \subseteq \mathbb{R}^n$ can be written as $\bigcup_{j=1}^{\infty} R_j$, where each R_j is an open rectangle.

Fix any $x \in U$, there must be some $R_j \ni x$.

By previous lemma, $\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x)$.

Since this holds for any $x \in U$, we have our result. □

Proposition 2.36. *Consider $1 \leq p \leq \infty, k \geq 0$, and $U \subseteq \mathbb{R}^n$ be open. If $u \in W^{k,p}(U)$ has a continuous representative $u^* \in C(U)$, and all of its weak derivatives $D^\alpha u$ have continuous representatives $(\bar{D}^\alpha u)^* \in C(U)$ for any $|\alpha| \leq k$, then*

$$u^* \in C^k(U), \quad D^\alpha(u^*)(x) = (\bar{D}^\alpha u)^*(x) \quad \forall x \in U, \forall |\alpha| \leq k.$$

Proof. We will use induction on k .

The base case is $k = 0$.

Since $|\alpha| = 0$, we trivially have $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$.

Now, suppose this holds for $k - 1$.

Consider any $u \in W^{k,p}(U)$.

If $|\alpha| = 0$, we trivially have $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$ as before.

Now consider any $|\gamma| = 1$. We know $\gamma = e_i$ for some $1 \leq i \leq n$.

By previous lemma, we have that

$$D^\gamma(u^*)(x) = \partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) = (\bar{D}^\gamma u)^*(x) \quad \forall x \in U.$$

Notice that $\bar{D}^\gamma u \in W^{k-1,p}(U)$, and all of its weak derivatives $\bar{D}^\beta \bar{D}^\gamma u = \bar{D}^{\beta+\gamma} u$ have continuous representatives $(\bar{D}^{\beta+\gamma} u)^* \in C(U)$ for any $|\beta| \leq k - 1$. By the induction hypothesis, we have that

$$D^\beta((\bar{D}^\gamma u)^*)(x) = (\bar{D}^{\beta+\gamma} u)^*(x) \quad \forall x \in U, \forall |\beta| \leq k - 1.$$

For any $1 \leq |\alpha| \leq k$, we can have $\alpha = \beta + \gamma$, where $|\beta| \leq k-1, |\gamma| = 1$.
Now we have $\forall x \in U$,

$$\begin{aligned} (\bar{D}^\alpha u)^*(x) &= (\bar{D}^{\beta+\gamma} u)^*(x) \\ &= D^\beta((\bar{D}^\gamma u)^*)(x) \\ &= D^\beta(D^\gamma(u^*))(x) \\ &= D^{\beta+\gamma}(u^*)(x) \\ &= D^\alpha(u^*)(x). \end{aligned}$$

We have thus proven the result for any $|\alpha| \leq k$.

Since all of its α^{th} derivatives exists and are continuous, we further have that $u^* \in C^k(U)$. \square

Theorem 2.37 (Differentiability almost everywhere). (*Theorem 5.8.5 in Eavan's*)

Consider $n \leq p \leq \infty$, and $U \subseteq \mathbb{R}^n$ be open. Assume $u \in W_{loc}^{1,p}(U)$, then u is differentiable a.e. in U , and its gradient $Du(x)$ equals its weak gradient $\bar{D}u(x)$ for a.e. $x \in U$.

2.8 Sobolev Inequalities

Definition 2.19. For $1 \leq p < n$, the **Sobolev conjugate** of p is $p^* := \frac{np}{n-p}$, with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Theorem 2.38. (Gagliardo–Nirenberg–Sobolev)

Let $1 \leq p < n$, then

$$\exists C > 0, \text{ such that } \forall u \in C_c^1(\mathbb{R}^n), \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}.$$

Corollary 2.39. Let $1 \leq p < n$, and $U \subseteq \mathbb{R}^n$, then

$$\exists C > 0, \text{ such that } \forall u \in C_c^1(U), \|u\|_{L^{p^*}(U)} \leq C \|Du\|_{L^p(U)}.$$

Proof. By Gagliardo–Nirenberg–Sobolev's Inequality, there is some $C > 0$, such that

$$\forall v \in C_c^1(\mathbb{R}^n), \|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Dv\|_{L^p(\mathbb{R}^n)}.$$

Notice that for each $u \in C_c^1(U)$, we can extend it by $v(x) := \begin{cases} u(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$.

Notice that $\text{Supp}(v) \subseteq U$, and $v = u$ on U .

Thus $\|v\|_{L^{p^*}(\mathbb{R}^n)} = \|v\|_{L^{p^*}(U)} = \|u\|_{L^{p^*}(U)}$, and $\|Dv\|_{L^p(\mathbb{R}^n)} = \|Du\|_{L^p(U)}$.

In addition, we have that $\lim_{x \rightarrow \partial U} D^\alpha u(x) = 0 = \lim_{x \rightarrow \partial U} D^\alpha 0, \forall |\alpha| \leq 1$.

Thus this extension is smooth. i.e. $v \in C_c^1(\mathbb{R}^n)$.

We thus have

$$\|u\|_{L^{p^*}(U)} = \|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Dv\|_{L^p(\mathbb{R}^n)} = C \|Du\|_{L^p(U)}.$$

\square

Theorem 2.40. ($W^{1,p}$ embedding into L^{p^*} , with $1 \leq p < n$)

Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded. If ∂U is C^1 , then

$$\exists C > 0, \text{ such that } \forall u \in W^{1,p}(U), \|u\|_{L^{p^*}(U)} \leq C \|u\|_{W^{1,p}(U)}.$$

In addition, since U is bounded, $\forall q \in [1, p^*]$, we have

$$\exists C > 0, \forall u \in W^{1,p}(U), \|u\|_{L^q(U)} \leq C \|u\|_{W^{1,p}(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q1. \square

Theorem 2.41. (*Poincaré's Inequality*)

Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded, then

$$\forall q \in [1, p^*], \exists C \geq 0, \text{ such that } \forall u \in W_0^{1,p}(U), \|u\|_{L^q(U)} \leq C \|Du\|_{L^p(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q2. □

Corollary 2.42. Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded, then $\|Du\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$.

Theorem 2.43. Let $1 \leq p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, then

$$\exists C \geq 0, \text{ such that } \forall u \in W_0^{1,p}(U), \|u\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q2. □

Corollary 2.44. Let $1 \leq p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, then $\|Du\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$.

Proof. See A3Q2. □

Theorem 2.45. ($W^{1,p}(U)$ embedding into $C^{0,\gamma}(\bar{U})$, with $n < p \leq \infty$, Morrey's)

Let $n < p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1 . Then there is some constant $C \geq 0$ such that

$$\forall u \in W^{1,p}(U), \exists \tilde{u} \in C^{0,\gamma}(\bar{U}), \text{ such that } \|\tilde{u}\|_{C^{0,\gamma}(\bar{U})} \leq C \|u\|_{W^{1,p}(U)},$$

where $\gamma := 1 - \frac{n}{p}$, and $\tilde{u} \in [u]$ is a representative of the equivalence class $[u] \in W^{1,p}(U)$.

Remark. If $p = \infty$, then $\gamma = 1$, and u^* is Lipschitz.

Theorem 2.46 (Sobolev Inequalities). Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1 . Let $u \in W^{k,p}(U)$, we have

1. If $k < \frac{n}{p}$, we define q by $\frac{1}{q} := \frac{1}{p} - \frac{k}{n}$, then

$$\|u\|_{L^q(U)} \leq C \|u\|_{W^{k,p}(U)}.$$

2. If $k > \frac{n}{p}$, we define $t := k - \lfloor \frac{n}{p} \rfloor - 1$, then we have a representative $\tilde{u} \in C^{t,\gamma}(\bar{U})$, such that

$$\|u^*\|_{C^{t,\gamma}(\bar{U})} \leq C \|u\|_{W^{k,p}(U)},$$

where $\gamma = \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}$ if $\frac{n}{p} \notin \mathbb{Z}$, and γ can be any integer if $\frac{n}{p} \in \mathbb{Z}$.

Proof. See Theorem 5.6-6 of Evans and A3Q3. □

2.9 Compactness

Definition 2.20. Let $(f_k)_{k=1}^\infty$ be a sequence of real-valued functions on \mathbb{R}^n . It is **uniformly bounded** if

$$\exists M > 0, \text{ such that } |f_k(x)| \leq M, \forall k \in \mathbb{N}^+, x \in \mathbb{R}^n$$

Definition 2.21. Let $(f_k)_{k=1}^\infty$ be a sequence of real-valued functions on \mathbb{R}^n . It is **equicontinuous** if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x, y \in \mathbb{R}^n, \|x - y\| < \delta \implies |f_k(x) - f_k(y)| < \epsilon, \forall k \in \mathbb{N}^+$$

Theorem 2.47. (*Arzela-Ascoli Compact criterion*)

Let $(f_k)_{k=1}^\infty$ be a sequence of real-valued functions on \mathbb{R}^n such that it is uniformly bounded and equicontinuous, then there exists a subsequence $(f_{k_j})_{j=1}^\infty$ and a continuous function f such that $f_{k_j} \rightarrow f$ uniformly on compact subsets of \mathbb{R}^n .

Proposition 2.48. (interpolation) Assume $1 \leq s \leq r \leq t \leq \infty$, and $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$ with $0 \leq \theta \leq 1$. Suppose $u \in L^s(U) \cap L^t(U)$, then $u \in L^r(U)$ and

$$\|u\|_{L^r(U)} \leq \|u\|_{L^s(U)}^\theta \|u\|_{L^t(U)}^{1-\theta}.$$

Proof. See AMATH731 A2. □

Lemma 2.49. Let $V \subseteq \mathbb{R}^n$ be open and bounded. Let $1 \leq p < n$, and $(u_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$ be any bounded sequence with $\text{Supp}(u_m) \subseteq V$. For $u_m^\epsilon := \eta_\epsilon * u_m$, we have that for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$.

Proof.

Claim 2.49.1. The sequence $(u_m^\epsilon)_{m=1}^\infty$ is uniformly bounded.

Proof. Since $(u_m)_{m=1}^\infty$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+$, $\|\hat{u}_m\|_{W^{1,p}(\mathbb{R}^n)} \leq M$. Consider any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |u_m^\epsilon(x)| &= \left| \int_{\mathbb{R}^n} \eta_\epsilon(x-y) u_m(y) dy \right| \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |u_m(y)| dy \\ &= \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(\mathbb{R}^n)} \\ &= \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} \|u_m\|_{L^p(\mathbb{R}^n)} \\ &= \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} \|u_m\|_{W^{1,p}(\mathbb{R}^n)} \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &= \frac{1}{\epsilon^n} \|\eta\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &\leq \frac{C}{\epsilon^n} |V|^{1-\frac{1}{p}} M. \end{aligned}$$

Since $\frac{C}{\epsilon^n} |V|^{1-\frac{1}{p}} M < \infty$ is independent of m , we have that the sequence $(u_m^\epsilon)_{m=1}^\infty$ is uniformly bounded. □

Claim 2.49.2. The sequence $(u_m^\epsilon)_{m=1}^\infty$ is equicontinuous.

Proof. Since $(u_m)_{m=1}^\infty$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+$, $\|\hat{u}_m\|_{W^{1,p}(\mathbb{R}^n)} \leq M$.

By 2.7.2, we have that $\partial_i u_m^\epsilon = (\partial_i \eta_\epsilon) * u_m$.

Thus for any $x \in \mathbb{R}^n$, $1 \leq i \leq n$, we have

$$\begin{aligned} |\partial_i u_m^\epsilon(x)| &= \left| \int_{\mathbb{R}^n} (\partial_i \eta_\epsilon)(x-y) u_m(y) dy \right| \\ &\leq \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\ &\leq \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(\mathbb{R}^n)} \\ &\leq \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ \|Du_m^\epsilon(x)\|_1 &\leq \sum_{i=1}^n \|\partial_i \eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &= \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\ &= \|D\eta_\epsilon\|_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M \\ &< \infty. \end{aligned}$$

Since $\|D\eta_\epsilon\|_{L^\infty(B(0,\epsilon))}|V|^{1-\frac{1}{p}}M$ is independent of x, m , we have that

$$C := \sup_{m \geq 1} \|Du_m^\epsilon\|_{L^\infty(U)} \leq \|D\eta_\epsilon\|_{L^\infty(B(0,\epsilon))}|V|^{1-\frac{1}{p}}M < \infty.$$

Since $u_m^\epsilon \in C_c^\infty(\mathbb{R}^n)$ by 2.7.1, we have each u_m^ϵ is Lipschitz with Lipschitz-constant C .

Given any $\delta > 0$, we can let $\delta_0 = \frac{\delta}{C}$.

Thus $\forall x, y \in \mathbb{R}^n$, such that $\|x - y\| < \delta_0$, we have

$$|u_m^\epsilon(x) - u_m^\epsilon(y)| \leq C\|x - y\| < \delta, \quad \forall m \in \mathbb{N}^+.$$

Thus the sequence $(u_m^\epsilon)_{m=1}^\infty$ is equicontinuous. \square

By the above two lemmas and Arzela-Ascoli Compact criterion 2.47, we know for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ and a continuous function u^ϵ such that $u_{m_j}^\epsilon \rightarrow u^\epsilon$ uniformly on compact subsets of \mathbb{R}^n .

Since V is bounded, \bar{V} is compact, we have that $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges uniformly on \bar{V} .

Thus $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges in $L^\infty(V)$.

Thus $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges in $L^q(V)$. \square

Lemma 2.50. *Let $V \subseteq \mathbb{R}^n$ be open and bounded, such that ∂V is C^1 . Let $1 \leq p < n$, and $(u_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$ be any bounded sequence with $\text{Supp}(u_m) \subseteq V$. For $u_m^\epsilon := \eta_\epsilon * u_m$, we have that $u_m^\epsilon \rightarrow u_m$ uniformly in $L^q(V)$ as $\epsilon \rightarrow 0$.*

Proof. By taking V' to be $V + B(0, 1)$ and WLOG consider $\epsilon < 1$, we assume the support of u_m^ϵ is in V . Since $(u_m)_{m=1}^\infty$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, \|u_m\|_{W^{1,p}(\mathbb{R}^n)} = \|u_m\|_{W^{1,p}(V)} \leq M$.

Claim 2.50.1. *If u_m are smooth, then $\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon|V|^{1-\frac{1}{p}}M$ for any $\epsilon > 0$.*

Proof.

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= (\eta_\epsilon * u_m)(x) - u_m(x) \\ &= \int_{B(0,\epsilon)} \eta_\epsilon(y) u_m(x-y) dy - u_m(x) \int_{B(0,\epsilon)} \eta_\epsilon(y) dy \\ &= \int_{B(0,\epsilon)} \eta_\epsilon(y) (u_m(x-y) - u_m(x)) dy \end{aligned}$$

Let $z := \frac{y}{\epsilon}$, we have $dy = \epsilon^n dz$. Recall $\eta_\epsilon = \frac{1}{\epsilon^n} \eta(\frac{y}{\epsilon})$. We thus have

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= \int_{B(0,\epsilon)} \eta_\epsilon(y) (u_m(x-y) - u_m(x)) dy \\ &= \int_{B(0,1)} \frac{\eta(z)}{\epsilon^n} - (u_m(x - \epsilon z) u_m(x)) (\epsilon^n dz) \\ &= \int_{B(0,1)} \eta(y) (u_m(x - \epsilon y) - u_m(x)) dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x - \epsilon y t) dt dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x - \epsilon y t) \cdot (-\epsilon y) dt dy \\ |u_m^\epsilon(x) - u_m(x)| &\leq \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x - \epsilon y t) \cdot (-\epsilon y)| dt dy \\ &= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x - \epsilon y t) \cdot y| dt dy \\ &\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \|Du_m(x - \epsilon y t)\|_1 dt dy, \end{aligned}$$

since $\|y\|_2 < \epsilon < 1$. Thus

$$\begin{aligned}
\|u_m^\epsilon - u_m\|_{L^1(V)} &= \int_V |u_m^\epsilon(x) - u_m(x)| dx \\
&\leq \int_V \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \|Du_m(x - \epsilon y t)\|_1 dt dy dx \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V \|Du_m(x - \epsilon y t)\|_1 dx dt dy \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} \|Du_m(x - \epsilon y t)\|_1 dx dt dy \\
&= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} \|Du_m(z)\|_1 dz dt dy \\
&= \epsilon \left(\int_{B(0,\epsilon)} \eta(y) dy \right) \left(\int_0^1 dt \right) \left(\int_{\mathbb{R}^n} \|Du_m(z)\|_1 dz \right) \\
&= \epsilon \int_{\mathbb{R}^n} \|Du_m(z)\|_1 dz \\
&= \epsilon \int_V \|Du_m(z)\|_1 dz \\
&= \epsilon \sum_{i=1}^n \|\partial_i u_m\|_{L^1(V)} \\
&\leq \epsilon \sum_{i=1}^n |V|^{1-\frac{1}{p}} \|\partial_i u_m\|_{L^p(V)} \\
&\leq \epsilon |V|^{1-\frac{1}{p}} \|u_m\|_{W^{1,p}(V)} \\
&\leq \epsilon |V|^{1-\frac{1}{p}} M.
\end{aligned}$$

Notice that this is true for any $\epsilon > 0$. □

Let $\delta > 0$ be given. Since $C^\infty(\bar{V})$ is dense in $W^{1,p}(V)$ by 2.22, we can find some $\bar{u}_m \in W^{1,p}(V)$, such that $\|\bar{u}_m - u_m\|_{W^{1,p}(V)} < \frac{\delta}{3|V|^{1-\frac{1}{p}}}$.

Notice that $\forall m, \|\bar{u}_m\|_{W^{1,p}(V)} \leq M + \frac{\delta}{3|V|^{1-\frac{1}{p}}}$ is bounded. From the claim above, we can find

$$\epsilon_0 := \frac{\delta}{3 \left(M + \frac{\delta}{3|V|^{1-\frac{1}{p}}} \right) |V|^{1-\frac{1}{p}}} > 0,$$

such that $\forall 0 < \epsilon < \epsilon_0$, we have

$$\|\bar{u}_m^\epsilon - \bar{u}_m\|_{L^1(V)} < \frac{\delta}{3}, \quad \forall m \in \mathbb{N}^+.$$

Now $\|u_m - \bar{u}_m\|_{L^1(V)} \leq |V|^{1-\frac{1}{p}} \|u_m - \bar{u}_m\|_{L^p(V)} \leq |V|^{1-\frac{1}{p}} \|u_m - \bar{u}_m\|_{W^1(V)} < \frac{\delta}{3}$.
In addition, by 2.8, we have

$$\begin{aligned}
\|u_m^\epsilon - \bar{u}_m^\epsilon\|_{L^1(V)} &= \|\eta_\epsilon * u_m - \eta_\epsilon * \bar{u}_m\|_{L^1(V)} \\
&= \|\eta_\epsilon * (u_m - \bar{u}_m)\|_{L^1(V)} \\
&\leq \|u_m - \bar{u}_m\|_{L^1(V)} \\
&< \frac{\delta}{3}.
\end{aligned}$$

Now we have

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \|u_m^\epsilon - \bar{u}_m^\epsilon\|_{L^1(V)} + \|\bar{u}_m^\epsilon - \bar{u}_m\|_{L^1(V)} + \|\bar{u}_m - u_m\|_{L^1(V)} < \delta.$$

Notice that this holds for all $\epsilon < \epsilon_0, m \in \mathbb{N}^+$, where the choice of ϵ_0 does not depend on m , and thus $\|u_m^\epsilon - u_m\|_{L^1(V)} \rightarrow 0$ uniformly when $\epsilon \rightarrow 0$.

Now $1 \leq q \leq p^*$, by letting $s = 1, r = q, t = p^*$, we have

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta} \quad 2.48$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta C^{1-\theta} \|u_m^\epsilon - u_m\|_{W^{1,p}(V)}^{1-\theta} \quad 2.46$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta C^{1-\theta} \left(\|u_m^\epsilon\|_{W^{1,p}(V)} + \|u_m\|_{W^{1,p}(V)} \right)^{1-\theta}$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta C^{1-\theta} \left(2\|u_m\|_{W^{1,p}(V)} \right)^{1-\theta} \quad 2.18$$

$$\leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta (2CM)^{1-\theta}.$$

Given any $\delta > 0$, since $\|u_m^\epsilon - u_m\|_{L^1(V)} \rightarrow 0$ uniformly when $\epsilon \rightarrow 0$, we can always find some $\epsilon_0 > 0$, such that

$$\forall \epsilon < \epsilon_0, m \in \mathbb{N}^+, \|u_m^\epsilon - u_m\|_{L^1(V)} < \left(\frac{\delta}{(2CM)^{1-\theta}} \right)^{1/\theta}.$$

Now for any $m \in \mathbb{N}^+$, we have

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta (2CM)^{1-\theta} < \delta.$$

This proves that $u_m^\epsilon \rightarrow u_m$ uniformly in $L^q(V)$ as $\epsilon \rightarrow 0$. \square

Theorem 2.51 (Rellich-Kondrachov Compactness). *Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1 . Let $1 \leq p < n$, then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for any $1 \leq q < p^*$.

Proof. The continuous embedding is done before in 2.46.

Now consider any bounded sequence $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(U)$.

Thus there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, \|\hat{u}_m\|_{W^{1,p}(U)} \leq M$.

By extension theorem, we may assume $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$, with $u_m|_U = \hat{u}_m$, and there is some V such that $U \subset\subset V$ and $\forall m \in \mathbb{N}^+, \text{Supp}(u_m) \subseteq V$. In addition,

$$\sup \|u_m\|_{W^{1,p}(\mathbb{R}^n)} = \sup \|u_m\|_{W^{1,p}(V)} \leq \sup C \|\hat{u}_m\|_{W^{1,p}(U)} \leq CM.$$

Thus $(u_m)_{m=1}^\infty$ is bounded.

WLOG, we can take V to have ∂V being C^1 .

Let $u_m^\epsilon := \eta_\epsilon * u_m$.

By the above lemmas, we know that

1. for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$, and
2. $u_m^\epsilon \rightarrow u_m$ uniformly in $L^q(V)$ as $\epsilon \rightarrow 0$.

Now given any $\delta > 0$.

By 2, we can find some $\epsilon_0 > 0$, such that $\forall 0 < \epsilon < \epsilon_0$, we have $\forall m \in \mathbb{N}^+, \|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{3}$.

Now fix some $0 < \epsilon < \epsilon_0$.

By 1, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$.

In particular, it is Cauchy, and we can find some $N \in \mathbb{N}^+$, such that $\forall i, j \geq N$, $\|u_{m_j}^\epsilon - u_{m_i}^\epsilon\|_{L^q(V)} < \frac{\delta}{3}$.
Now for any $i, j \geq N$, we have that

$$\begin{aligned} \|u_{m_i} - u_{m_j}\|_{L^q(V)} &= \|u_{m_i} - u_{m_i}^\epsilon + u_{m_i}^\epsilon - u_{m_j}^\epsilon + u_{m_j}^\epsilon - u_{m_j}\|_{L^q(V)} \\ &\leq \|u_{m_i} - u_{m_i}^\epsilon\|_{L^q(V)} + \|u_{m_i}^\epsilon - u_{m_j}^\epsilon\|_{L^q(V)} + \|u_{m_j}^\epsilon - u_{m_j}\|_{L^q(V)} \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} \\ &= \delta. \end{aligned}$$

Thus $(u_{m_j})_{j=1}^\infty$ is a Cauchy sequence in $L^q(V)$.

Since $L^q(V)$ is complete, there is some $u \in L^q(V)$, such that $\lim_{j \rightarrow \infty} \|u_{m_j} - u\|_{L^q(V)} = 0$.

Since $U \subseteq V$, we have that $\lim_{j \rightarrow \infty} \|u_{m_j} - u\|_{L^q(U)} = 0$.

Since $u + m|_U = \hat{u}_m$, we also have that $\lim_{j \rightarrow \infty} \|\hat{u}_{m_j} - u\|_{L^q(U)} = 0$.

Thus the subsequence \hat{u}_{m_j} converges to some $u \in L^q(V) \subseteq L^q(U)$.

Since $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(U)$ is any bounded sequence, we have that any bounded subset of $W^{1,p}(U)$ is relative compact in $L^q(U)$. \square

Theorem 2.52. *Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1 . Let $1 \leq p \leq \infty$, then*

$$W^{1,p}(U) \subset\subset L^p(U).$$

Theorem 2.53. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $1 \leq p \leq \infty$, then*

$$W_0^{1,p}(U) \subset\subset L^p(U).$$

2.10 Poincare Inequalities

Definition 2.22. For a bounded domain $U \subset \mathbb{R}^n$, we denote the average of u over U by

$$(u)_U := \frac{1}{|U|} \int_U u dx.$$

Theorem 2.54. (*Poincaré–Wirtinger’s Inequality*)

Let $U \subset \mathbb{R}^n$ be open, bounded, and connected, such that ∂U is C^1 . For any $1 \leq p \leq \infty$, $\exists C > 0$, such that

$$\forall u \in W^{1,p}(U), \|u - (u)_U\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}.$$

Proof. Suppose for contradiction it is not true.

Then $\forall k \in \mathbb{N}, \exists u_k \in W^{1,p}(U)$, such that $\|u_k - (u_k)_U\|_{L^p(U)} > k \|Du_k\|_{L^p(U)}$.

Let $v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}$.

Notice that

$$\forall k \in \mathbb{N}^+, \|v_k\|_{L^p(U)} = 1, (v_k)_U = 0, Dv_k = \frac{Du_k}{\|u_k - (u_k)_U\|_{L^p(U)}}.$$

Thus $\|Dv_k\|_{L^p(U)} = \frac{\|Du_k\|_{L^p(U)}}{\|u_k - (u_k)_U\|_{L^p(U)}} < \frac{1}{k}$.

Which means $\|v_k\|_{W^{1,p}(U)}^p = \|v_k\|_{L^p(U)}^p + \|Dv_k\|_{L^p(U)}^p < 1 + \frac{1}{k^p} \leq 2$.

Since this is true for any $k \in \mathbb{N}^+$, we have that $(v_k)_{k=1}^\infty$ is bounded in $W^{1,p}(U)^p$.

Since $W^{1,p}(U)^p \subset\subset L^p(U)$, there is a subsequence $(v_{k_j})_{j=1}^\infty$ and some $v \in L^p(U)$, such that

$$\lim_{j \rightarrow \infty} \|v_{k_j} - v\|_{L^p(U)} = 0.$$

Now consider any $1 \leq i \leq k$, and any $\phi \in C_c^\infty(U)$.

$$\begin{aligned} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^p(U)}^p &= \int_U |v_{k_j} \partial_i \phi - v \partial_i \phi|^p dx \\ &= \int_U |\partial_i \phi|^p |v_{k_j} - v|^p dx \\ &\leq \|\partial_i \phi\|_{L^\infty(U)}^p \|v_{k_j} - v\|_{L^p(U)}^p. \end{aligned}$$

Since $\phi \in C_c^\infty(U)$, we have that $\|\partial_i \phi\|_{L^\infty(U)}^p$ is bounded by some $M > 0$.

Since $\lim_{j \rightarrow \infty} \|v_{k_j} - v\|_{L^p(U)} = 0$, we also have $\lim_{j \rightarrow \infty} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^p(U)}^p = 0$.

In addition, $\lim_{j \rightarrow \infty} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^1(U)} \leq \lim_{j \rightarrow \infty} |U|^{1-\frac{1}{p}} \|v_{k_j} \partial_i \phi - v \partial_i \phi\|_{L^p(U)} = 0$.

We have

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_U |v_{k_j} \partial_i \phi - v \partial_i \phi| dx &= 0 \\ \implies \lim_{j \rightarrow \infty} \int_U (v_{k_j} \partial_i \phi - v \partial_i \phi) dx &= 0 \\ \implies \lim_{j \rightarrow \infty} \int_U v_{k_j} \partial_i \phi dx &= \lim_{j \rightarrow \infty} \int_U v \partial_i \phi dx \\ \implies - \lim_{j \rightarrow \infty} \int_U \partial_i v_{k_j} \phi dx &= \int_U v \partial_i \phi dx. \end{aligned}$$

Yet

$$\begin{aligned} \left| \lim_{j \rightarrow \infty} \int_U \partial_i v_{k_j} \phi dx \right| &\leq \lim_{j \rightarrow \infty} \int_U |\partial_i v_{k_j} \phi| dx \\ &\leq \lim_{j \rightarrow \infty} \|\partial_i v_{k_j}\|_{L^p(U)} \|\phi\|_{L^{\frac{p}{p-1}}(U)} \\ &\leq \lim_{j \rightarrow \infty} \|Dv_{k_j}\|_{L^p(U)} \|\phi\|_{L^{\frac{p}{p-1}}(U)} \\ &\leq \lim_{j \rightarrow \infty} \frac{1}{k_j} \|\phi\|_{L^{\frac{p}{p-1}}(U)} \\ &= 0, \end{aligned}$$

since $\phi \in C_c^\infty(U)$ and U is bounded, which implies $\|\phi\|_{L^{\frac{p}{p-1}}(U)} < \infty$. Thus

$$\int_U v \partial_i \phi dx = - \lim_{j \rightarrow \infty} \int_U \partial_i v_{k_j} \phi dx = 0 = - \int_U 0 \phi dx.$$

Since this holds for any $\phi \in C_c^\infty(U)$, we must have $\partial_i v = 0$ a.e. for any $1 \leq i \leq n$.

Thus $v \in W^{1,p}(U)$, with $Dv = 0$ a.e..

Since U is connected, v is a constant by 2.33.

Since $(v)_U = 0$, we must have $v = 0$ a.e..

However, this contradicts with $\|v\|_{L^p(U)} = 1$. □

2.11 H^{-1} Spaces

Definition 2.23. The dual space to $H_0^1(U)$ is $H^{-1}(U)$.

Theorem 2.55. Consider any $f \in H^{-1}(U)$.

1. There is a tuple (f^0, \dots, f^n) of functions in $L^2(U)$, such that

$$\forall v \in H_0^1(U), \langle f|v \rangle_{H^{-1}(U), H_0^1(U)} = \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f^i, \partial_i v \rangle_{L^2(U)}.$$

In this case, we write $f = f^0 - \sum_{i=1}^n f_{x^i}^i$.

2.

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \left(\sum_{i=0}^n \|f^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} : (f^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

Proof. 1. Let $f \in H^{-1}(U)$, by the Riesz-Frechet Representation theorem 1.24, $\exists! u \in H_0^1(U)$, such that

$$\forall v \in H_0^1(U), \langle f|v \rangle_{H^{-1}(U), H_0^1(U)} = \langle u, v \rangle_{H_0^1(U)},$$

and $\|f\|_{H_0^{-1}(U)} = \|u\|_{H_0^1(U)}$.

Let $f^0 = u, \forall 1 \leq n, f^i := \partial_i u$. we have

$$\begin{aligned} \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f^i, \partial_i v \rangle_{L^2(U)} &= \langle u, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle \partial_i u, \partial_i v \rangle_{L^2(U)} \\ &= \langle u, v \rangle_{H_0^1(U)} \\ &= \langle f|v \rangle_{H^{-1}(U), H_0^1(U)}. \end{aligned}$$

2. Consider any $f \in H^{-1}(U)$, from 1, we know that there is such $f^0 = u, \forall 1 \leq n, f^i := \partial_i u$, satisfying 1, with

$$\|f\|_{H_0^{-1}(U)} = \|u\|_{H_0^1(U)} = \left(\sum_{i=0}^n \|f^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \geq \inf \left\{ \left(\sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} : (g^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

Now consider any $g^0, \dots, g^n \in L^2(U)$, such that they satisfies

$$\langle f|v \rangle = \langle g^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle g^i, v \rangle_{L^2(U)}.$$

For any $v \in H_0^1(U)$, we have

$$\begin{aligned} |\langle f|v \rangle| &= \left| \langle g^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle g^i, \partial_i v \rangle_{L^2(U)} \right| \\ &\leq \left| \langle g^0, v \rangle_{L^2(U)} \right| + \sum_{i=1}^n \left| \langle g^i, \partial_i v \rangle_{L^2(U)} \right| \\ &\leq \|g^0\|_{L^2(U)} \|v\|_{L^2(U)} + \sum_{i=1}^n \|g^i\|_{L^2(U)} \|\partial_i v\|_{L^2(U)} \\ &\leq \left(\sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \left(\|v\|_{L^2(U)}^2 + \sum_{i=1}^n \|\partial_i v\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \|v\|_{H_0^1(U)}. \end{aligned}$$

Thus we know

$$\|f\|_{H_0^{-1}(U)} = \sup_{v \in H_0^1(U), v \neq 0} \frac{|\langle f|v \rangle|}{\|v\|_{H_0^1(U)}} \leq \inf \left\{ \left(\sum_{i=0}^n \|g^i\|_{L^2(U)}^2 \right)^{\frac{1}{2}} : (g^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

□

Corollary 2.56. For any $v^* \in L^2(U)^* \subset L(L^2(U), \mathbb{R}) \subset L(H_0^1(U), \mathbb{R})$, with v^* identified with $v \in L^2(U)$, and any $u \in H_0^1(U) \subseteq L^2(U)$, we have

$$\langle v^* | u \rangle_{H^{-1}(U), H_0^1(U)} = \langle v^* | u \rangle_{L^2(U)^*, L^2(U)} = \langle v, u \rangle_{L^2(U)}.$$

In addition, $v^* \in H^{-1}(U)$, and has a representation $(v, 0, \dots, 0)$ as in above theorem, with

$$\|v^*\|_{H^{-1}(U)} \leq \|v\|_{L^2(U)}.$$

Proof. The first equality is by definition and the second equality is by 1.47.

Thus, for any $\|u\|_{H_0^1(U)} = 1$, we have that

$$\begin{aligned} \left| \langle v^* | u \rangle_{H^{-1}(U), H_0^1(U)} \right| &= \left| \langle v, u \rangle_{L^2(U)} \right| \\ &\leq \|v\|_{L^2(U)} \|u\|_{L^2(U)} \\ &\leq \|v\|_{L^2(U)} \|u\|_{H_0^1(U)} \\ &= \|v\|_{L^2(U)}. \end{aligned}$$

Since this holds for any unitary $u \in H_0^1(U)$, we have that

$$\|v^*\|_{H^{-1}(U)} = \sup_{\|u\|_{H_0^1(U)}=1} \left| \langle v^* | u \rangle_{H^{-1}(U), H_0^1(U)} \right| \leq \|v\|_{L^2(U)} < \infty,$$

which proves $v^* \in H^{-1}(U)$.

In addition, $\langle u, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle 0, \partial_i v \rangle_{L^2(U)} = \langle u, v \rangle_{L^2(U)} = \langle v^* | u \rangle_{H^{-1}(U), H_0^1(U)}$. □

Corollary 2.57. $\forall v \in H_0^1(U) \subset L^2(U)$, we have $v^* := \langle v, \cdot \rangle_{L^2(U)} \in H^{-1}(U)$, with

$$\|v^*\|_{H^{-1}(U)} \leq \|v\|_{L^2(U)} \leq \|v\|_{H_0^1(U)}.$$

In other words, if we identify v with v^* , then $H_0^1(U) \subset L^2(U) \subset H^{-1}(U)$ are continuous embeddings.

Proof. Since $v \in H_0^1(U) \subseteq L^2(U)$, by the above corollary, $v^* \in H^{-1}(U)$, and has

$$\|v^*\|_{H^{-1}(U)} \leq \|v\|_{L^2(U)} \leq \|v\|_{H_0^1(U)}.$$

□

2.12 Difference Quotients

Definition 2.24. Let $U \subset \mathbb{R}^n$ be open, $u \in L_{loc}^1(U)$, $V \subset\subset U$, then for $|h| \in (0, \text{dist}(V, \partial U))$, $x \in V$, we define:

1. For $i \in [n]$, $u_i^h(x) := u(x + he_i)$
2. For $i \in [n]$, the i^{th} **difference quotient** of size h at x is

$$D_i^h u(x) = \frac{u_i^h(x) - u(x)}{h} = \frac{u(x + he_i) - u(x)}{h}.$$

- 3.

$$D^h u(x) := (D_1^h u(x), \dots, D_n^h u(x)).$$

Proposition 2.58. Let $U \subset \mathbb{R}^n$ be open, $u \in L_{loc}^1(U)$, then $\forall i \in [n], |h| > 0$, we have

$$\text{Supp}(D_i^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|).$$

Thus,

$$\text{Supp}(D^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|).$$

Proposition 2.59. Let $U \subset \mathbb{R}^n$ be open, $u, v \in L_{loc}^1(U)$, $V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \text{dist}(V, \partial U))$, we have

$$D_i^h(uv) = v_i^h D_i^h u + u D_i^h v.$$

Proof. We have

$$\begin{aligned} v_i^h D_i^h u + u D_i^h v &= v_i^h \frac{u_i^h - u}{h} + u \frac{v_i^h - v}{h} \\ &= \frac{v_i^h u_i^h - v_i^h u + u v_i^h - uv}{h} \\ &= \frac{v_i^h u_i^h - uv}{h} \\ &= \frac{(uv)_i^h - uv}{h} \\ &= D_i^h(uv). \end{aligned}$$

□

Proposition 2.60. Let $U \subset \mathbb{R}^n$ be open, $u, v \in L_{loc}^1(U)$, $\text{Supp}(u) \subset V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \frac{1}{3} \text{dist}(V, \partial U))$, we have

$$\int_U v D_i^{-h} u dx = - \int_U u D_i^h v dx.$$

Proof. Notice that $\text{Supp}(D_i^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|) \subseteq V + \bar{B}(0, |h|) \subseteq \overline{V + B(0, |h|)}$.

Since $\text{dist}(\overline{V + B(0, |h|)}, \partial U) \geq 2|h|$, we can find $V + \bar{B}(0, |h|) \subset W \subset\subset V$, with $|h| < \text{dist}(W, \partial U)$, where $D_i^{-h} u$ is well-defined in W .

In addition, $\text{Supp}(u) \subset V \subset W$, so we can view the integrals as over W , by extending $D_i^{-h} u$ to be zero outside of W .

$$\begin{aligned} \int_W v D_i^{-h} u dx &= \int_V v D_i^{-h} u dx \\ &= \int_V v(x) \frac{u(x - h e_i) - u(x)}{-h} dx \\ &= - \int_V \frac{v(x) u(x - h e_i) - v(x) u(x)}{h} dx \\ &= - \left(\int_V \frac{v(x - h e_i + h e_i) u(x - h e_i)}{h} dx - \int_V \frac{v(x) u(x)}{h} dx \right) \\ &= - \left(\int_{V - h e_i} \frac{v(y + h e_i) u(y)}{h} dy - \int_V \frac{v(x) u(x)}{h} dx \right) \\ &= - \left(\int_W \frac{v_i^h(y) u(y)}{h} dy - \int_W \frac{v(x) u(x)}{h} dx \right) \\ &= - \int_W \frac{v_i^h(x) u(x) - v(x) u(x)}{h} dx \\ &= - \int_W u \frac{v_i^h - v}{h} dx \\ &= - \int_W u D_i^h v dx. \end{aligned}$$

□

Proposition 2.61. Let $U \subset \mathbb{R}^n$ be open, $u, D^\alpha u \in L_{loc}^p(U)$, $V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \text{dist}(V, \partial U))$, we have

$$D^\alpha(u_i^h) = (D^\alpha u)_i^h, \quad D^\alpha(D_i^h u) = D_i^h(D^\alpha u) \text{ in } V.$$

In addition, if $u \in W^{k,p}(U)$, we have $u_i^h, D_i^h u \in W^{k,p}(V)$.

Proof. Given any $i \in [n]$, $|h| \in (0, \text{dist}(V, \partial U))$.

$\forall \phi \in C_c^\infty(V)$, we have $\phi_i^{-h} \in C_c^\infty(V + he_i) \subseteq C_c^\infty(U)$, with $D^\alpha \phi(x) = D^\alpha \phi_i^{-h}(x + he_i)$.

$$\begin{aligned}
\int_V u_i^h(x) D^\alpha \phi(x) dx &= \int_V u(x + he_i) D^\alpha \phi_i^{-h}(x + he_i) dx \\
&= \int_{V+he_i} u(y) D^\alpha \phi_i^{-h}(y) dy \\
&= \int_U u(y) D^\alpha \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_U D^\alpha u(y) \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_{V+he_i} D^\alpha u(y) \phi_i^{-h}(y) dy \\
&= (-1)^{|\alpha|} \int_V D^\alpha u(x + he_i) \phi_i^{-h}(x + he_i) dx \\
&= (-1)^{|\alpha|} \int_V (D^\alpha u)_i^h(x) \phi(x) dx.
\end{aligned}$$

Since this holds for all $\phi \in C_c^\infty(V)$, we must have $D^\alpha(u_i^h) = (D^\alpha u)_i^h$.

In addition,

$$\begin{aligned}
D^\alpha(D_i^h u) &= D^\alpha\left(\frac{u_i^h - u}{h}\right) \\
&= \frac{D^\alpha(u_i^h) - D^\alpha u}{h} \\
&= \frac{(D^\alpha u)_i^h - D^\alpha u}{h} \\
&= D_i^h(D^\alpha u).
\end{aligned}$$

Now suppose $u \in W^{k,p}(U)$.

$$\begin{aligned}
\|u_i^h\|_{W^{k,p}(V)}^p &= \int_V \sum_{|\alpha| \leq k} |(D^\alpha(u_i^h))(x)|^p dx \\
&= \int_V \sum_{|\alpha| \leq k} |(D^\alpha u)_i^h(x)|^p dx \\
&= \int_V \sum_{|\alpha| \leq k} |D^\alpha u(x + he_i)|^p dx \\
&= \int_{V+he_i} \sum_{|\alpha| \leq k} |D^\alpha u(y)|^p dy \\
&\leq \int_U \sum_{|\alpha| \leq k} |D^\alpha u(y)|^p dy \\
&= \|u\|_{W^{k,p}(U)}^p.
\end{aligned}$$

Thus $u_i^h \in W^{k,p}(V)$.

Clearly $u \in W^{k,p}(V)$, so a linear combination of them $D_i^h u \in W^{k,p}(V)$. □

Theorem 2.62. Let $U \subset \mathbb{R}^n$ be open, we have:

1. For $p \in [1, \infty)$, and $\forall V \subset \subset U, \exists C > 0$, such that

$$\|D^h u\|_{L^p(V)} \leq C \|Du\|_{L^p(U)}, \quad \forall u \in W^{1,p}(U), \forall |h| \in (0, \text{dist}(V, \partial U)).$$

2. For $p \in (1, \infty)$, $V \subset\subset U$, $u \in L^p(V)$, if $\exists C, \delta > 0$, such that $\|D^h u\|_{L^p(V)} \leq C$, $\forall |h| \in (0, \delta)$, then

$$u \in W^{1,p}(V), \quad \|Du\|_{L^p(V)} \leq C.$$

Theorem 2.63. Let $U \subset \mathbb{R}^n$ be open and bounded, with ∂U being C^1 , then $u : U \rightarrow \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$.

3 Elliptic PDEs

3.1 Weak Solutions

We will consider the model problem: $U \in \mathbb{R}^n$ be open and bounded, with some $f : U \rightarrow \mathbb{R}$ be given. We want to find $u : \bar{U} \rightarrow \mathbb{R}$, such that
$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Definition 3.1. A second order differential operator is

$$Lu := - \sum_{i,j=1}^n \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^n b^i(x)\partial_i u + c(x)u.$$

Definition 3.2. A symmetric (uniformly) elliptic second order differential operator is an L such that $a^{ij} = a^{ji}$, and $\exists \theta > 0$, such that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta \|\xi\|_2^2 \text{ for a.e. } x \in U, \forall \xi \in \mathbb{R}^n.$$

Remark. The above definition is equivalent to saying $A(x) = (a^{ij}(x)) \in \mathbb{R}^{n \times n}$ is symmetric positive definite for a.e. $x \in U$, with a uniform positive lower bound $\theta > 0$ on their eigenvalues.

Example 3.1.1. If we take $a^{ij} = C\delta_{ij}$, we have $Lu = -C\Delta u + b \cdot Du + cu$.

Definition 3.3. The bilinear form associated with L is given by:

$$B[u, v] := \int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i uv + cuv \right) dx, \quad \forall u, v \in H_0^1(U).$$

Definition 3.4. Consider $f = f^0 - \sum_{i=1}^n f_{x^i}^i \in H^{-1}(U)$ as in 2.55.

$u \in H_0^1(U)$ is called a **weak solution** to the BVP
$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$
 if u satisfies the **weak formulation**:

$$\forall v \in H_0^1(U), B[u, v] = \langle f|v \rangle = \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f^i, \partial_i v \rangle_{L^2(U)}.$$

Definition 3.5. For $f \in L^2(U)$, we have the special case:

$u \in H_0^1(U)$ is called a **weak solution** to the BVP
$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$
 if u satisfies the **weak formulation**:

$$\forall v \in H_0^1(U), B[u, v] = \langle f, v \rangle_{L^2(U)}.$$

Proposition 3.1. If a classical solution u exists, i.e u is smooth, and $Lu = f, u|_{\partial U} = 0$, then u is always a weak solution.

Proof. Firstly consider any $v \in C_c^\infty(U)$, we have

$$\begin{aligned}
\langle f|v \rangle &= \langle Lu, v \rangle \\
&= \int_U Luv dx \\
&= \int_U \left(- \sum_{i,j=1}^n \partial_j (a^{ij} \partial_i u) + \sum_{i=1}^n b^i \partial_i u + cu \right) v dx \\
&= - \sum_{i,j=1}^n \int_U \partial_j (a^{ij} \partial_i u) v dx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= - \sum_{i,j=1}^n \int_U a^{ij} \partial_i u \partial_j v dx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= \int_U \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v dx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= \int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i uv + cuv \right) dx \\
&= B[u, v].
\end{aligned}$$

Since $H_0^1(U) = \overline{C_c^\infty(v)}$, this holds for any $v \in H_0^1(U)$. \square

3.2 Existence of weak solution

3.2.1 First Existence Theorem

Theorem 3.2. (*Lax-Milgram*)

Consider a real Hilbert space \mathcal{H} with $\langle \cdot, \cdot \rangle$ and action $\langle \cdot | \cdot \rangle_{\mathcal{H}^*, \mathcal{H}}$. Assume $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ is a bilinear form such that $\exists a, b > 0$ such that $\forall u, v \in \mathcal{H}$,

$$\begin{aligned}
|B[u, v]| &\leq a \|u\| \|v\| \\
B[u, u] &\geq b \|u\|^2.
\end{aligned}$$

Then $\forall f \in \mathcal{H}^*$, $\exists! u \in \mathcal{H}$ such that $\forall v \in \mathcal{H}$, $B[u, v] = \langle f|v \rangle$.

Proof. For each $u \in \mathcal{H}$, we define the operator $T_u : v \mapsto B[u, v]$.

$|T_u v| = |B[u, v]| \leq a \|u\| \|v\|$, and thus $\|T_u\|_{\mathcal{H}^*} \leq a \|u\| < \infty$ is bounded. Thus $T_u \in \mathcal{H}^*$.

By Riesz-Frechet Representation theorem 1.24, we have that $\exists! w \in \mathcal{H}$, such that $\forall v \in \mathcal{H}$, $T_u v = \langle w, v \rangle_{\mathcal{H}}$, and $\|T_u\|_{\mathcal{H}^*} = \|w\|_{\mathcal{H}}$.

Now define $A : \mathcal{H} \rightarrow \mathcal{H}$ by $u \mapsto w$ in the above setting, such that $\forall v \in \mathcal{H}$, $\langle Au, v \rangle = B[u, v]$.

Claim 3.2.1. For any $u \in \mathcal{H}$, we have that

$$b \|u\| \leq \|Au\| \leq a \|u\|.$$

Proof. We have

$$\|Au\|^2 = \langle Au, Au \rangle = B[u, Au] \leq a \|u\| \|Au\|.$$

If $\|Au\| = 0$, clearly $\|Au\| \leq a \|u\|$.

Otherwise we can divide both side by $\|Au\|$, and get $\|Au\| \leq a \|u\|$.

On the other hand, we have

$$b \|u\|^2 \leq B[u, u] = \langle Au, u \rangle \leq \|Au\| \|u\|.$$

If $\|u\| = 0$, clearly $b \|u\| \leq \|Au\|$.

Otherwise we can divide both side by $\|u\|$, and get $b \|u\| \leq \|Au\|$. \square

Claim 3.2.2. *We have $A \in \mathcal{H}^*$.*

Proof. For any $u_1, u_2, v \in \mathcal{H}, c \in \mathbb{R}$, we have that

$$\begin{aligned}\langle A(u_1 + cu_2), v \rangle &= B[u_1 + cu_2, v] \\ &= B[u_1, v] + cB[u_2, v] \\ &= \langle Au_1, v \rangle + c\langle Au_2, v \rangle \\ &= \langle Au_1 + cAu_2, v \rangle.\end{aligned}$$

Since this holds for all $v \in \mathcal{H}$, we have $A(u_1 + cu_2) = Au_1 + cAu_2$, and thus A is linear. In addition, we have

$$\|A\|_{\mathcal{H}^*} = \sup_{u \in \mathcal{H}, u \neq 0} \frac{\|Au\|}{\|u\|} \leq \sup_{u \in \mathcal{H}, u \neq 0} \frac{a\|u\|}{\|u\|} = a < \infty.$$

This shows A is bounded, and thus $A \in \mathcal{H}^*$. □

Claim 3.2.3. *A is bijective.*

Proof. Suppose $Au = 0$, we have that

$$b\|u\| \leq \|Au\| = 0,$$

which means that $u = 0$. Thus A is injective.

Consider any sequence $(y_j)_{j=1}^\infty \subset \text{Im}(A)$, such that $\lim_{j \rightarrow \infty} y_j = y \in \mathcal{H}$.

We can find $(x_j)_{j=1}^\infty \subset \mathcal{H}$, such that $\forall j \geq 1, Ax_j = y_j$.

Since $(y_j)_{j=1}^\infty$ is convergent and thus Cauchy, given any $\epsilon > 0$, we can find some $N \geq 1$, such that $\forall i, j \geq N, \|y_j - y_i\| < b\epsilon$.

Now

$$\begin{aligned}\|x_j - x_i\| &\leq \frac{1}{b} \|A(x_j - x_i)\| \\ &= \frac{1}{b} \|Ax_j - Ax_i\| \\ &= \frac{1}{b} \|y_j - y_i\| \\ &< \frac{1}{b} b\epsilon \\ &< \epsilon.\end{aligned}$$

Thus $(x_j)_{j=1}^\infty$ is Cauchy.

Since \mathcal{H} is complete, there is some $x \in \mathcal{H}$, such that $\lim_{j \rightarrow \infty} x_j = x$.

Since A is bounded and thus continuous, we have that

$$\begin{aligned}Ax &= A\left(\lim_{j \rightarrow \infty} x_j\right) \\ &= \lim_{j \rightarrow \infty} Ax_j \\ &= \lim_{j \rightarrow \infty} y_j \\ &= y.\end{aligned}$$

Thus $y \in \text{Im}(A)$.

This proves that $\text{Im}(A)$ is closed.

Since A is linear, $\text{Im}(A)$ is a closed subspace of \mathcal{H} , and thus $\mathcal{H} = \text{Im}(A) \oplus \text{Im}(A)^\perp$.

Consider any $w \in \text{Im}(A)^\perp$, we must have

$$b\|w\|^2 \leq B[w, w] = \langle Aw, w \rangle = 0.$$

Thus $\text{Im}(A)^\perp = \{0\}$, and thus $\text{Im}(A) = \mathcal{H}$.

Thus A is surjective. □

Now by the Bounded inverse Theorem, A^{-1} exists and is bounded.
By Riesz-Frechet Representation theorem 1.24, given any $f \in \mathcal{H}^*$, we have

$$\exists! w \in \mathcal{H}, \text{ such that } \langle f|v \rangle = \langle w, v \rangle \quad \forall v \in \mathcal{H}.$$

Let $u = A^{-1}w$, we have that

$$\forall v \in \mathcal{H}, \quad B[u, v] = \langle Au, v \rangle = \langle w, v \rangle = \langle f|v \rangle.$$

This proves the existence.

Now suppose there is some \hat{u} such that $\forall v \in \mathcal{H}, \quad B[\hat{u}, v] = \langle f|v \rangle = B[u, v]$.

We must have $B[u - \hat{u}, v] = 0, \quad \forall v \in \mathcal{H}$. Thus

$$b\|u - \hat{u}\| \leq B[u - \hat{u}, u - \hat{u}] = 0,$$

and thus $\hat{u} = u$ is unique. □

Proposition 3.3 (Cauchy's inequality). *For any $a, b, \epsilon > 0$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

Theorem 3.4 (Energy estimates). *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and $a^{ij}, b^i, c \in L^\infty(U)$, such that (a^{ij}) is symmetric positive definite. For the bilinear form defined in 3.3, there exists constants $\alpha, \beta > 0, \gamma \geq 0$, such that $\forall u, v \in H_0^1(U)$,*

$$|B[u, v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} \tag{1}$$

$$\beta \|u\|_{H^1(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2. \tag{2}$$

Proof. We have

$$\begin{aligned} |B[u, v]| &= \left| \int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx \right| \\ &\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \int_U |\partial_i u| |\partial_j v| dx + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \int_U |\partial_i u| |v| dx + \|c\|_{L^\infty(U)} \int_U |u| |v| dx \\ &\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|\partial_j v\|_{L^2(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|v\|_{L^2(U)} \\ &\quad + \|c\|_{L^\infty(U)} \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\ &\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|u\|_{H^1(U)} \|v\|_{H^1(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|u\|_{H^1(U)} \|v\|_{H^1(U)} \\ &\quad + \|c\|_{L^\infty(U)} \|u\|_{H^1(U)} \|v\|_{H^1(U)} \\ &= \left(\sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right) \|u\|_{H^1(U)} \|v\|_{H^1(U)}. \end{aligned}$$

Taking $\alpha := \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)}$, we notice that $\alpha \geq 0$, and $\alpha = 0 \implies \forall i, j, \quad a^{ij} = 0$, which contradicts (a_{ij}) is positive definite. Thus $\alpha > 0$, and $|B[u, v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}$.
On the other hand, consider $\xi = Du \in \mathbb{R}^n$.

We have that

$$\theta \|Du\|_2^2 \leq \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u.$$

Thus

$$\begin{aligned}
\theta \|Du\|_{L^2(U)}^2 &= \theta \int_U \|Du\|_2^2 dx \\
&\leq \int_U \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u dx \\
&= B[u, u] - \int_U \left(\sum_{i=1}^n b^i \partial_i u u + c u u \right) dx \\
&\leq B[u, u] + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)} \|u\|_{L^2(U)} + \|c\|_{L^\infty(U)} \|u\|_{L^2(U)}^2 \\
&\leq B[u, u] + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \left(\epsilon \|\partial_i u\|_{L^2(U)}^2 + \frac{1}{4\epsilon} \|u\|_{L^2(U)}^2 \right) + \|c\|_{L^\infty(U)} \|u\|_{L^2(U)}^2 \\
&= B[u, u] + \epsilon \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|\partial_i u\|_{L^2(U)}^2 + \left(\frac{1}{4\epsilon} \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right) \|u\|_{L^2(U)}^2 \\
&\leq B[u, u] + \epsilon \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \|Du\|_{L^2(U)}^2 + \left(\frac{1}{4\epsilon} \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right) \|u\|_{L^2(U)}^2.
\end{aligned}$$

If $\sum_{i=1}^n \|b^i\|_{L^\infty(U)} = 0$, pick any $\epsilon > 0$.

Otherwise choose $\epsilon := \frac{\theta}{2 \sum_{i=1}^n \|b^i\|_{L^\infty(U)}} > 0$, and $\gamma := \frac{1}{4\epsilon} \sum_{i=1}^n \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)}$, we have

$$\frac{\theta}{2} \|Du\|_{L^2(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2.$$

Since $\|Du\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$ by 2.42, we have that

$$\exists C > 0, \text{ such that } \forall u \in H_0^1(U), \|u\|_{H^1(U)}^2 \leq C \|Du\|_{L^p(U)}^2.$$

Taking $\beta := \frac{\theta}{2C} > 0$, we have

$$\beta \|u\|_{H^1(U)}^2 \leq \frac{\theta}{2} \|Du\|_{L^2(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2.$$

□

Definition 3.6. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\mu \in \mathbb{R}$, we define the operator L_μ by

$$L_\mu u := Lu + \mu u.$$

We define the bilinear form associated to L_μ to be B_μ .

Proposition 3.5.

$$B_\mu[u, v] = B[u, v] + \int_U \mu u v dx = B[u, v] + \mu \langle u, v \rangle_{L^2(U)}.$$

Theorem 3.6. (First Existence Theorem)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$ and $\forall f \in H^{-1}(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP:
$$\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Proof. By Energy estimates, we have that $\forall u, v \in H_0^1(U)$,

$$\begin{aligned}
|B_\mu[u, v]| &\leq |B[u, v]| + \mu \left| \langle u, v \rangle_{L^2(U)} \right| \\
&\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} + \mu \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\
&\leq (\alpha + \mu) \|u\|_{H^1(U)} \|v\|_{H^1(U)} \\
B_\mu[u, u] &= B[u, u] + \mu \langle u, u \rangle_{L^2(U)} \\
&= B[u, u] + \mu \|u\|_{L^2(U)}^2 \\
&\geq \beta \|u\|_{H^1(U)}^2 + (\mu - \gamma) \|u\|_{L^2(U)}^2 \\
&\geq \beta \|u\|_{H^1(U)}^2.
\end{aligned}$$

By Lax–Milgram Theorem, for any $f \in H^{-1}(U)$, there is a unique $u \in H_0^1(U)$, such that

$$\forall v \in H_0^1(U), B_\mu[u, v] = \langle f, v \rangle.$$

□

Corollary 3.7. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$ and $\forall f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP: $\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$*

3.2.2 More Existence Theorems

Definition 3.7. Consider $Lu := -\sum_{i,j=1}^n \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^n b^i(x)\partial_i u + c(x)u$, we define its **formal adjoint**

$$L^\dagger v := -\sum_{i,j=1}^n \partial_i(a^{ij}(x)\partial_j v) + \sum_{i=1}^n b^i(x)\partial_i v + c(x)v.$$

For $f \in H^{-1}(U)$, the **adjoint problem** is $\begin{cases} L^\dagger v = f & \text{in } U, \\ v = 0 & \text{on } \partial U, \end{cases}$ and the bilinear form associated with it is $B^*[u, v]$.

Notice that $v \in H_0^1(U)$ is a weak solution of the adjoint problem if v satisfies $\forall u \in H_0^1(U), B^*[u, v] = \langle f, u \rangle$.

Proposition 3.8.

$$B^*[u, v] := B[v, u].$$

Remark. Since L is not bounded, L^\dagger is not its usual adjoint operator. However, when u, v are both smooth, we have that $\langle Lu, v \rangle_{L^2(U)} = B[u, v] = B^*[v, u] = \langle v, L^\dagger u \rangle$.

Definition 3.8. For $\mu \in \mathbb{R}$, we can similarly define $L_\mu^\dagger u := L^\dagger u + \mu u$, and the bilinear form associated with it is $B_\mu^*[u, v]$.

Proposition 3.9.

$$B_\mu^*[u, v] = B^*[u, v] + \mu \langle u, v \rangle_{L^2(U)} = B[v, u] + \mu \langle v, u \rangle_{L^2(U)} = B_\mu[v, u].$$

Proposition 3.10. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$ and $\forall f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP: $\begin{cases} L^\dagger u + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ Namely,*

$$\exists! u \in H_0^1(U), \text{ such that } \forall v \in H_0^1(U), B_\mu^*[u, v] = \langle f, v \rangle_{L^2(U)}.$$

Proof. For $\alpha, \beta > 0, \gamma \geq 0$ from Energy Estimate 3.4, we have that $\forall u, v \in H_0^1(U)$,

$$|B^*[v, u]| = |B[u, v]| \quad (3)$$

$$\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} \quad (4)$$

$$\beta \|u\|_{H^1(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2 \quad (5)$$

$$= B^*[u, u] + \gamma \|u\|_{L^2(U)}^2. \quad (6)$$

Thus B and B^* have the same energy estimate. By First Existence Theorem 3.6, we have the result. \square

Definition 3.9. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$, we define $L_\mu^{-1} : L^2(U) \rightarrow H_0^1(U)$ by $f \mapsto u$, where u is the unique solution to

$$\forall v \in H_0^1(U), B_\mu[u, v] = \langle f, v \rangle_{L^2(U)}$$

given by the First Existence Theorem 3.6.

We can also define $(L_\mu^\dagger)^{-1} : L^2(U) \rightarrow H_0^1(U)$ by $f \mapsto u$, where u is the unique solution to

$$\forall v \in H_0^1(U), B_\mu^*[u, v] = \langle f, v \rangle_{L^2(U)}.$$

Remark. We notice that by definition $B_\mu[L_\mu^{-1}f, v] = \langle f, v \rangle_{L^2(U)}, \forall v \in H_0^1(U), \forall f \in L^2(U), \forall \mu \geq \gamma$.

Lemma 3.11. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. Then for any $\mu \geq \gamma$, if we let $K = \mu L_\mu^{-1}$, we have that $K : L^2(U) \rightarrow H_0^1(U) \subseteq L^2(U)$ is compact.

Proof. Consider any $g \in L^2(U)$, we have that

$$\begin{aligned} \beta \|L_\mu^{-1}g\|_{H^1(U)}^2 &\leq B[L_\mu^{-1}g, L_\mu^{-1}g] + \gamma \|L_\mu^{-1}g\|_{L^2(U)}^2 \\ &\leq B[L_\mu^{-1}g, L_\mu^{-1}g] + \mu \|L_\mu^{-1}g\|_{L^2(U)}^2 \\ &= B_\mu[L_\mu^{-1}g, L_\mu^{-1}g] \\ &= \langle g, L_\mu^{-1}g \rangle_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|L_\mu^{-1}g\|_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|L_\mu^{-1}g\|_{H^1(U)} \\ &\implies \\ \|L_\mu^{-1}g\|_{H^1(U)} &\leq \frac{1}{\beta} \|g\|_{L^2(U)} \\ &\implies \\ \|Kg\|_{H^1(U)} &\leq \frac{\mu}{\beta} \|g\|_{L^2(U)}. \end{aligned}$$

Thus, $K : L^2(U) \rightarrow H_0^1(U)$ is bounded.

Since $H_0^1(U) \subset \subset L^2(U)$, by 1.22, we have that $K : L^2(U) \rightarrow L^2(U)$ is compact. \square

Lemma 3.12. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $f \in L^2(U)$, if we let $h := L_\gamma^{-1}f, K = \gamma L_\gamma^{-1}$, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ if and only if u solves $(I - K)u = h$.

Proof. We will firstly show that u solves $\forall v \in H_0^1(U)$, $B_\gamma[u, v] = \langle f + \gamma u, v \rangle_{L^2(U)}$, if and only if u solves $u = L_\gamma^{-1}(f + \gamma u)$.

Suppose $\forall v \in H_0^1(U)$, $B_\gamma[u, v] = \langle f + \gamma u, v \rangle_{L^2(U)}$.

we have that $u' := L_\gamma^{-1}(f + \gamma u) \in H_0^1(U)$ is the unique solution, such that

$$B_\gamma[u', v] = \langle f + \gamma u, v \rangle, \quad \forall v \in H_0^1(U).$$

Thus $u = u' = L_\gamma^{-1}(f + \gamma u)$.

On the other hand, suppose $u = L_\gamma^{-1}(f + \gamma u)$, then we have that

$$\forall v \in H_0^1(U), \quad B_\gamma[u, v] = B_\gamma[L_\gamma^{-1}(f + \gamma u), v] = \langle f + \gamma u, v \rangle_{L^2(U)}.$$

Thus, $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$ if and only if

u solves $\forall v \in H_0^1(U)$, $B[u, v] = \langle f, v \rangle_{L^2(U)}$, if and only if

u solves $\forall v \in H_0^1(U)$, $B[u, v] + \gamma \langle u, v \rangle_{L^2(U)} = \langle f, v \rangle_{L^2(U)} + \gamma \langle u, v \rangle_{L^2(U)}$, if and only if

u solves $\forall v \in H_0^1(U)$, $B_\gamma[u, v] = \langle f + \gamma u, v \rangle_{L^2(U)}$, if and only if

u solves $u = L_\gamma^{-1}(f + \gamma u)$, if and only if

u solves $u = L_\gamma^{-1}f + \gamma L_\gamma^{-1}u$, if and only if

u solves $Iu = h + Ku$, if and only if

u solves $(I - K)u = h$. □

Theorem 3.13. (Second Existence Theorem)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator.

1. Precisely one of the following must be true:

(a) $\forall f \in L^2(U)$, $\exists! u \in H_0^1(U)$, a unique weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

(b) There is a weak solution $u \neq 0 \in H_0^1(U)$ to the homogeneous problem $\begin{cases} Lu = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U. \end{cases}$

2. Let $N \subset H_0^1(U)$ be the solution space of weak solutions to $\begin{cases} Lu = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$ and let $N^* \subset H_0^1(U)$ be the solution space of weak solutions to $\begin{cases} L^\dagger u = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$ then $\dim(N) = \dim(N^*) < \infty$.

3. The problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a weak solution if and only if $f \in (N^*)^\perp \subseteq L^2(U)$.

Proof. Take $\mu = \gamma$.

From the above lemma, we know that for any $f \in L^2(U)$, if we let $K = \gamma L_\gamma^{-1}$, we have that $u \in H_0^1(U)$ is a

weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ if and only if u solves $(I - K)u = L_\gamma^{-1}f$.

We also have shown that $K : L^2(U) \rightarrow H_0^1(U) \subseteq L^2(U)$ is compact.

1. By 1.35, we have that exactly one of the following holds:

(a) $\forall v \in L^2(U)$, $\exists! u \in L^2(U)$, such that $(I - K)u = v$.

In this case, for any $f \in L^2(U)$, $\exists! u \in L^2(U)$, such that $(I - K)u = L_\gamma^{-1}f$.

In addition, since $L_\gamma^{-1}f \in H_0^1(U)$, $Ku = \gamma L_\gamma^{-1}u \in H_0^1(U)$, we must have $u = L_\gamma^{-1}f + Ku \in H_0^1(U)$.

Thus u is the unique weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

- (b) $\exists u \neq 0 \in L^2(U)$, such that $(I - K)u = 0 = L_\gamma^{-1}0$.
Similarly, we can see that $u = Ku = \gamma L_\gamma^{-1}u \in H_0^1(U)$.

Thus u is a non-trivial solution to $\begin{cases} Lu = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

2. By the above lemma, $N = \text{Ker}(I - K)$.

By 1.35, we have that $\dim(N) = \dim(\text{Ker}(I - K^\dagger)) < \infty$. Let $L_\gamma^\dagger u := L^\dagger u + \gamma u$.

Consider any $g, h \in L^2(U)$, we have that

$$\begin{aligned} \langle h, K^\dagger g \rangle &= \langle Kh, g \rangle \\ &= \langle g, Kh \rangle_{L^2(U)} \\ &= \gamma \langle g, L_\gamma^{-1}h \rangle_{L^2(U)} \\ &= \gamma B_\gamma^*[(L_\gamma^\dagger)^{-1}g, L_\gamma^{-1}h] \\ &= \gamma B_\gamma[L_\gamma^{-1}h, (L_\gamma^\dagger)^{-1}g] \\ &= \gamma \langle h, (L_\gamma^\dagger)^{-1}g \rangle_{L^2(U)} \\ &= \langle h, \gamma(L_\gamma^\dagger)^{-1}g \rangle_{L^2(U)}. \end{aligned}$$

Since this holds for all $g, h \in L^2(U)$, we have that $K^\dagger = \gamma(L_\gamma^\dagger)^{-1}$.

By the above lemma, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} L^\dagger u = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$ if and only if

u solves $(I - K^\dagger)u = 0$, if and only if $u \in \text{Ker}(I - K^\dagger)$.

Thus $N^* = \text{Ker}(I - K^\dagger)$.

3. (a) $\gamma = 0$.

Notice that $K = 0$, and thus $N^* = \text{ker}(I - K^\dagger) = \text{ker}(I) = \{0\}$.

Thus $(N^*)^\perp = L^2(U)$.

In addition, $N = \text{ker}(I - K) = \text{ker}(I) = \{0\}$, so we must be in case (a).

Thus $\forall f \in (N^*)^\perp$, the problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a (unique) weak solution.

The other direction is trivial since $(N^*)^\perp = L^2(U)$ is the whole space.

- (b) $\gamma \neq 0$.

By 1.35, we have that $\text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$.

By the above lemma, we have that the problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a weak solution, if and

only if,

there is some u that solves $(I - K)u = L_\gamma^{-1}f$, if and only if,

$L_\gamma^{-1}f \in \text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$, if and only if,

$\forall v \in \text{Ker}(I - K^\dagger) = N^*$,

$$\begin{aligned} \langle L_\gamma^{-1}f, v \rangle &= 0 \\ \frac{1}{\gamma} \langle Kf, v \rangle &= 0 \\ \frac{1}{\gamma} \langle f, K^\dagger v \rangle &= 0 \\ \frac{1}{\gamma} \langle f, K^\dagger v + (I - K^\dagger)v \rangle &= 0 \\ \frac{1}{\gamma} \langle f, v \rangle &= 0 \\ \langle f, v \rangle &= 0, \end{aligned}$$

if and only if $f \in (N^*)^\perp$.

□

Definition 3.10. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. The **spectrum** of L is defined to be

$$\Sigma := \mathbb{R} \setminus \left\{ \lambda \in \mathbb{R} : \forall f \in L^2(U), \exists! u \in H_0^1(U), \text{ such that } \forall v \in H_0^1(U), B_{-\lambda}[u, v] = \langle f, v \rangle_{L^2(U)} \right\}.$$

Proposition 3.14. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let Σ be the spectrum of L .

1. $\lambda \notin \Sigma$ if and only if $\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$.
2. $\lambda \in \Sigma$ if and only if $\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$.

Proof. 1. This is by definition.

2. By Second Existence Theorem 3.13 on $L_{-\lambda}$.

□

Lemma 3.15. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4, and Σ be the spectrum of L , we always have $\Sigma \subseteq (-\gamma, \infty)$.

Proof. If $\lambda \leq -\gamma$, we have that $-\lambda \geq \gamma$, and by First Existence Theorem 3.6, we have that the problem has a unique weak solution, and thus $\lambda \notin \Sigma$. □

Theorem 3.16. (Third existence theorem)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let Σ be the spectrum of L .

1. Σ is at most countable.

2. If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$ can be arranged in non-decreasing sequence with $\lim_{k \rightarrow \infty} \lambda_k = \infty$.

Proof. Let $\gamma' \geq 0$ be the same as in Energy Estimate 3.4, we have $\Sigma \subseteq (-\gamma', \infty) \subseteq (-\gamma, \infty)$ for any $\gamma \geq \gamma'$. We will take some $\gamma > 0$, and consider $\lambda > -\gamma$.

$\lambda \in \Sigma$, if and only if $\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$,

if and only if $\begin{cases} Lu + \gamma u = (\lambda + \gamma)u, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$.

Suppose $\lambda \in \Sigma$, then let $g = (\lambda + \gamma)u$. By First Existence Theorem 3.6, there is a unique weak solution

$$(L_\gamma)^{-1}((\lambda + \gamma)u) = \frac{\lambda + \gamma}{\gamma} Ku \text{ to the problem } \begin{cases} Lu + \gamma u = g, & \text{in } U, \\ u = 0, & \text{on } \partial U. \end{cases}$$

Since $u \neq 0 \in H_0^1(U)$ is a weak solution to the problem, we have

$$u = \frac{\lambda + \gamma}{\gamma} Ku.$$

Thus $u \neq 0 \in L^2(U)$ is an eigen-vector for K , with corresponding eigenvalue $\frac{\gamma}{\lambda + \gamma}$.

Notice that $\frac{\gamma}{\lambda + \gamma} > 0$, since $\gamma > 0, \lambda > -\gamma$, and thus $\frac{\gamma}{\lambda + \gamma} \in \text{Spec}_p(K) \setminus (\infty, 0]$.

Since this holds for any $\lambda \in \Sigma$, we have $\left\{ \frac{\gamma}{\lambda + \gamma} : \lambda \in \Sigma \right\} \subseteq \text{Spec}_p(K) \setminus (\infty, 0]$.

On the other hand, $\forall \mu \in \text{Spec}_p(K) \setminus \{0\}$, we have that $\lambda' := \frac{\gamma(1-\mu)}{\mu} = -\gamma + \frac{\gamma}{\mu}$ satisfies $\mu = \frac{\gamma}{\lambda' + \gamma}$. Pick any eigen-vector $u \neq 0 \in L^2(U)$ corresponds to μ , we have that $\frac{\gamma}{\lambda' + \gamma}u = Ku$.

Thus $u = (L_\gamma)^{-1}((\lambda' + \gamma)u) \neq 0 \in H_0^1(U)$ is a weak solution to the problem $\begin{cases} Lu + \gamma u = (\lambda' + \gamma)u, & \text{in } U, \\ u = 0, & \text{on } \partial U. \end{cases}$

If $\lambda' > -\gamma \iff \frac{\gamma}{\mu} > 0 \iff \mu > 0$, we have that $\lambda' \in \Sigma$.

Thus, we have $\left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \text{Spec}_p(K) \setminus (\infty, 0] \right\} \subseteq \Sigma$.

We have shown that

$$\Sigma = \left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \text{Spec}_p(K) \setminus (\infty, 0] \right\}.$$

Since K is compact, by the Spectral theorem 1.23, we have that either

1. $\text{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^N$ is finite, which means $\Sigma \subseteq \left(\lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^N$ is finite.
 2. $\text{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^\infty$ is countable, and $\lim_{k \rightarrow \infty} \mu_k = 0$, which means that $\Sigma \subseteq \left(\lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^\infty$ is at most countable.
- In addition, if Σ is infinite, it must be $(\lambda_{k_j})_{j=1}^\infty \subseteq (\lambda_k)_{k=1}^\infty$.

$$\lim_{k \rightarrow \infty} |\lambda_k| = \lim_{k \rightarrow \infty} \left| \frac{\gamma(1-\mu_k)}{\mu_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\gamma}{\mu_k} \right| = \infty.$$

Thus $\lim_{k \rightarrow \infty} |\lambda_{k_j}| = \lim_{k \rightarrow \infty} |\lambda_k| = \infty$.

Since we have $\forall j, \lambda_{k_j} > -\gamma$, we must have $\lim_{j \rightarrow \infty} \lambda_{k_j} = \infty$.

□

Theorem 3.17. (Boundedness of inverse)

Let Σ be the spectrum of L , and $\lambda \notin \Sigma$. Then there is a constant $C > 0$ such that for all $f \in L^2(U)$ and the

unique weak solution $u \in H_0^1(U)$ to $\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$ we always have

$$\|u\|_{L^2(U)} \leq C \|f\|_{L^2(U)}.$$

Proof. Consider any $\lambda \notin \Sigma$.

Suppose for contradiction, we can find $(\tilde{u}_k)_{k=1}^\infty \subset H_0^1(U)$, $(\tilde{f}_k)_{k=1}^\infty \subset L^2(U)$, such that $\forall k \geq 1$,

$$\begin{cases} L\tilde{u}_k = \lambda\tilde{u}_k + \tilde{f}_k & \text{in } U, \\ \tilde{u}_k = 0, & \text{on } \partial U \end{cases}$$

and

$$\|\tilde{u}\|_{L^2(U)} > k \|\tilde{f}\|_{L^2(U)}.$$

Let $u_k := \frac{\tilde{u}_k}{\|\tilde{u}_k\|_{L^2(U)}}$, $f_k := \frac{\tilde{f}_k}{\|\tilde{u}_k\|_{L^2(U)}}$.

Notice that $\forall k \geq 1$, $\|u_k\|_{L^2(U)} = 1$, and $\|f_k\|_{L^2(U)} = \frac{\|\tilde{f}_k\|_{L^2(U)}}{\|\tilde{u}_k\|_{L^2(U)}} < \frac{1}{k}$.

In addition, $\forall v \in H_0^1(U)$,

$$\begin{aligned} B[u_k, v] &= \frac{1}{\|\tilde{u}_k\|_{L^2(U)}} B[\tilde{u}_k, v] \\ &= \frac{1}{\|\tilde{u}_k\|_{L^2(U)}} \left(\left\langle \tilde{f}_k, v \right\rangle_{L^2(U)} + \lambda \left\langle \tilde{u}_k, v \right\rangle_{L^2(U)} \right) \\ &= \left\langle \frac{\tilde{f}_k}{\|\tilde{u}_k\|_{L^2(U)}}, v \right\rangle_{L^2(U)} + \lambda \left\langle \frac{\tilde{u}_k}{\|\tilde{u}_k\|_{L^2(U)}}, v \right\rangle_{L^2(U)} \\ &= \langle f_k, v \rangle_{L^2(U)} + \lambda \langle u_k, v \rangle_{L^2(U)}. \end{aligned}$$

By Energy Estimate 3.4, we have that

$$\begin{aligned}
\beta \|u_k\|_{H^1(U)}^2 &\leq B[u_k, u_k] + \gamma \|u_k\|_{L^2(U)}^2 \\
&= \langle f_k, u_k \rangle_{L^2(U)} + \lambda \langle u_k, u_k \rangle_{L^2(U)} + \gamma \|u_k\|_{L^2(U)}^2 \\
&\leq \|f_k\|_{L^2(U)} \|u_k\|_{L^2(U)} + (\lambda + \gamma) \|u_k\|_{L^2(U)}^2 \\
&= \|f_k\|_{L^2(U)} + \lambda + \gamma \\
&< \lambda + \gamma + \frac{1}{k} \\
&\leq \lambda + \gamma + 1. \\
\|u_k\|_{H^1(U)} &\leq \sqrt{\frac{\lambda + \gamma + 1}{\beta}}
\end{aligned}$$

Thus $(u_k)_{k=1}^\infty$ is a bounded sequence in $H_0^1(U)$.

Since $H_0^1(U)$ is a Hilbert space, and thus reflexive, by 1.28, there $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in H_0^1(U)$, such that $u_{k_j} \rightharpoonup u$.

Also, since $H_0^1(U) \subset \subset L^2(U)$, by 1.31, we have that $u_{k_j} \rightarrow u$ in $L^2(U)$. Thus,

$$\|u\|_{L^2(U)} = \lim_{j \rightarrow \infty} \|u_{k_j}\|_{L^2(U)} = 1.$$

Now consider any $v \in H_0^1(U)$, we have that the map $w \mapsto B[w, v]$ is a linear bounded operator, so by weak convergence of $(u_{k_j})_{j=1}^\infty$, we have that

$$\begin{aligned}
B[u, v] &= \lim_{j \rightarrow \infty} B[u_{k_j}, v] \\
&= \lim_{j \rightarrow \infty} \left(\langle f_{k_j}, v \rangle_{L^2(U)} + \lambda \langle u_{k_j}, v \rangle_{L^2(U)} \right) \\
&= \lim_{j \rightarrow \infty} \langle f_{k_j}, v \rangle_{L^2(U)} + \lambda \left\langle \lim_{j \rightarrow \infty} u_{k_j}, v \right\rangle_{L^2(U)} \\
&\leq \lim_{j \rightarrow \infty} \|f_{k_j}\|_{L^2(U)} \|v\|_{L^2(U)} + \lambda \langle u, v \rangle_{L^2(U)} \\
&\leq \lim_{j \rightarrow \infty} \frac{1}{k_j} \|v\|_{L^2(U)} + \lambda \langle u, v \rangle_{L^2(U)} \\
&= \lambda \langle u, v \rangle_{L^2(U)}.
\end{aligned}$$

Namely, $\hat{u} = u$ satisfies $\forall v \in H_0^1(U)$, $B_{-\lambda}[\hat{u}, v] = 0 = \langle 0, v \rangle_{L^2(0)}$.

Yet since $\lambda \notin \Sigma$, by definition, we know there is a unique \hat{u} that satisfies the above condition.

Clearly $\hat{u} = 0$ satisfies, so by the uniqueness of weak solution, $u = 0$.

This contradicts with $\|u\|_{L^2(U)} = 1$. □

3.3 Regularity

Theorem 3.18. (*Interior H^2 regularity*)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, $\forall i, j \in [n]$. $\forall V \subset \subset U$, $\exists C > 0$, such that for all $f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U , namely,

$$\forall v \in H_0^1(U), B[u, v] = \langle f, v \rangle_{L^2(U)},$$

then

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right).$$

Thus $u \in H_{loc}^2(U)$.

Proof. Let $V \subset\subset U$ be given.

The idea is to choose a particular v , then repeatedly bound all $\|D_k^h u\|$ from the product rule by $\|Du\|$. The only leftover term will be either $D_k^h(Du)$, or part of $\langle f, v \rangle_{L^2(U)}$ or $\|u\|_H^1(U)$. We thus achieve a bound on $\|D_k^h(Du)\|$, which allows us to say $u \in H_{loc}^2(U)$.

1. We now fix some $f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U . We have $\forall v \in H_0^1(U)$,

$$\begin{aligned} B[u, v] &= \langle f, v \rangle_{L^2(U)} \\ \int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cuv \right) dx &= \int_U f v dx \\ \int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v \right) dx &= \int_U \tilde{f} v dx \\ \sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= \langle \tilde{f}, v \rangle_{L^2(U)}, \end{aligned}$$

where $\tilde{f} := f - \sum_{i=1}^n b^i \partial_i u - cu \in L^2(U)$, since $f, \partial_i u, u \in L^2(U)$, $b^i, c \in L^\infty(U)$.

2. Since $V \subset\subset U$, we can choose some $V \subset\subset W \subset\subset U$, and $\zeta \in C_c^\infty(U)$ such that $V < \zeta < W$. Choose $|h| > 0$ such that $\text{dist}(V, \partial U) > 8|h|$, $\text{dist}(W, \partial U) > 6|h|$.

WLOG, we assume $h > 0$.

Fix some $k \in [n]$.

Let $Z := U_{2h} := \{x \in U : \text{dist}(x, \partial U) > 2h\}$ be open.

Since U is bounded, we have that $Z \subset\subset U$, $\text{dist}(Z, \partial U) = 2h > |h|$.

$$\text{Let } v(x) := \begin{cases} -D_k^{-h}(\zeta^2 D_k^h u)(x) & x \in Z \\ 0 & x \in U \setminus Z \end{cases}.$$

Remark. For $x \in V$, we have that

$$\begin{aligned} v(x) &= -D_k^{-h}(D_k^h u)(x) \\ &= -D_k^{-h} \left(\frac{u(x + he_k) - u(x)}{h} \right) \\ &= - \frac{\frac{u(x + he_k - he_k) - u(x - he_k)}{h} - \frac{u(x + he_k) - u(x)}{h}}{h} \\ &= - \frac{2u(x) - u(x + he_k) - u(x - he_k)}{h^2} \\ &= \frac{u(x + he_k) - 2u(x) + u(x - he_k)}{h^2}, \end{aligned}$$

which is an approximation to $\partial_k^2 u$ if u is smooth.

Since $u \in H^1(U)$, we have $D_k^h u \in H^1(Z)$.

Since $\text{Supp}(\zeta) \subset W \subset\subset Z$ is compact, we have $\zeta \in C_c^\infty(Z)$, so $\zeta^2 D_k^h u \in H^1(Z)$.

Since $U_{4h} \subset\subset Z$, $\text{dist}(U_{4h}, \partial Z) = 2h > |h|$, we have that $v \in H^1(U_{4h})$.

In addition, $\text{Supp}(v) \subset \text{Supp}(\zeta^2 D_k^h u) + \bar{B}(0, h) \subseteq W + \bar{B}(0, h) \subseteq U_{6h} + \bar{B}(0, h) \subset U_{4h}$.

Since $v \in H^1(U_{4h})$ and $\text{Supp}(v) \subset U_{4h}$, we must have $v \in H_0^1(U)$.

3. Now we have

$$\begin{aligned}
\sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= \sum_{i,j=1}^n \int_Z (a^{ij} \partial_i u \partial_j v) dx \\
&= \sum_{i,j=1}^n \int_Z (a^{ij} \partial_i u \partial_j (-D_k^{-h}(\zeta^2 D_k^h u))) dx \\
&= - \sum_{i,j=1}^n \int_Z (a^{ij} \partial_i u D_k^{-h}(\partial_j(\zeta^2 D_k^h u))) dx \\
&= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij} \partial_i u) (\partial_j(\zeta^2 D_k^h u)) dx \\
&= \sum_{i,j=1}^n \int_Z (D_k^h(a^{ij}) \partial_i u + a^{ij} D_k^h(\partial_i u)) (\partial_j(\zeta^2) D_k^h u + \zeta^2 \partial_j(D_k^h u)) dx \\
&= A_1 + A_2 + A_3 + A_4,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &:= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) \zeta^2 \partial_j(D_k^h u) dx, \\
A_2 &:= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) \partial_j(\zeta^2) D_k^h u dx \\
&= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) 2\zeta(\partial_j \zeta) D_k^h u dx \\
A_3 &:= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij}) \partial_i u \zeta^2 \partial_j(D_k^h u) dx \\
A_4 &:= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij}) \partial_i u \partial_j(\zeta^2) D_k^h u dx \\
&= \sum_{i,j=1}^n \int_Z D_k^h(a^{ij}) \partial_i u 2\zeta(\partial_j \zeta) D_k^h u dx
\end{aligned}$$

Now we will examine each term.

$$\begin{aligned}
A_1 &= \sum_{i,j=1}^n \int_Z a^{ij} D_k^h(\partial_i u) \zeta^2 \partial_j(D_k^h u) dx \\
&= \int_Z \zeta^2 \sum_{i,j=1}^n a^{ij} \partial_i(D_k^h u) \partial_j(D_k^h u) dx \\
&\geq \int_Z \zeta^2 \theta ||D(D_k^h u)||_2^2 dx \\
&= \theta \int_Z \zeta^2 ||D(D_k^h u)||_2^2 dx.
\end{aligned}$$

We also have

$$\begin{aligned}
|A_2| &\leq \sum_{i,j=1}^n \int_Z |a^{ij} D_k^h(\partial_i u) 2\zeta(\partial_j \zeta) D_k^h u| dx \\
&\leq \sum_{i,j=1}^n \int_Z \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} |D_k^h(\partial_i u) 2\zeta D_k^h u| dx \\
&= 2 \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z |D_k^h(\partial_i u) \zeta D_k^h u| dx \\
&\leq 2 \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z \epsilon |D_k^h(\partial_i u)|^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx \\
&= C_1 \int_Z \epsilon |D_k^h(\partial_i u)|^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx \\
&= C_1 \int_Z \epsilon |\partial_i(D_k^h u)|^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx \\
&\leq C_1 \int_Z \epsilon \|D(D_k^h u)\|_2^2 \zeta^2 + \frac{1}{4\epsilon} |D_k^h u|^2 dx,
\end{aligned}$$

since $a^{ij} \in L^\infty(U)$, and $\zeta \in C^c(U)$, we have $C_1 := 2 \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)} \|\partial_j \zeta\|_{L^\infty(U)} \in (0, \infty)$. Similarly,

$$\begin{aligned}
|A_3| &\leq \sum_{i,j=1}^n \int_Z |D_k^h(a^{ij}) \partial_i u \zeta^2 \partial_j(D_k^h u)| dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z |\partial_i u \partial_j(D_k^h u)| \zeta^2 dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z \|Du\|_2 \|D(D_k^h u)\|_2 \zeta^2 dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z \|Du\|_2 \|D(D_k^h u)\|_2 \zeta dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \int_Z \frac{1}{4\epsilon} \|Du\|_2^2 + \epsilon \zeta^2 \|D(D_k^h u)\|_2^2 dx \\
&= C_2 \int_Z \frac{1}{4\epsilon} \|Du\|_2^2 + \epsilon \zeta^2 \|D(D_k^h u)\|_2^2 dx,
\end{aligned}$$

where

$$\begin{aligned}
C_2 &:= \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \\
&\leq \frac{1}{h} \sum_{i,j=1}^n \left(\|a^{ij}\|_{L^\infty(Z)} + \|a^{ij}\|_{L^\infty(Z+he_k)} \right) \\
&\leq \frac{1}{h} \sum_{i,j=1}^n \left(\|a^{ij}\|_{L^\infty(U)} + \|a^{ij}\|_{L^\infty(U)} \right) \\
&\in (0, \infty).
\end{aligned}$$

Lastly,

$$\begin{aligned}
|A_4| &\leq \sum_{i,j=1}^n \int_Z |D_k^h(a^{ij}) \partial_i u 2\zeta(\partial_j \zeta) D_k^h u| dx \\
&\leq 2 \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z |\partial_i u D_k^h u| dx \\
&\leq \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \|\partial_j \zeta\|_{L^\infty(U)} \int_Z |\partial_i u|^2 + |D_k^h u|^2 dx \\
&\leq C_3 \int_Z \|Du\|_2^2 + |D_k^h u|^2 dx,
\end{aligned}$$

where $C_3 := \sum_{i,j=1}^n \|D_k^h(a^{ij})\|_{L^\infty(Z)} \|\partial_j \zeta\|_{L^\infty(U)} \in (0, \infty)$ as argued before.

Now

$$\begin{aligned}
&|A_2 + A_3 + A_4| \\
&\leq |A_1| + |A_2| + |A_3| \\
&\leq \int_Z \epsilon C_1 \|D(D_k^h u)\|_2^2 \zeta^2 + \frac{C_1}{4\epsilon} |D_k^h u|^2 + \frac{C_2}{4\epsilon} \|Du\|_2^2 + C_2 \epsilon \zeta^2 \|D(D_k^h u)\|_2^2 + C_3 \|Du\|_2^2 + C_3 |D_k^h u|^2 dx \\
&= \int_Z (C_1 + C_2) \epsilon \|D(D_k^h u)\|_2^2 \zeta^2 + \left(\frac{C_1}{4\epsilon} + C_3\right) |D_k^h u|^2 + \left(\frac{C_2}{4\epsilon} + C_3\right) \|Du\|_2^2 dx \\
&\leq \int_Z (C_1 + C_2) \epsilon \|D(D_k^h u)\|_2^2 \zeta^2 + \left(\frac{C_1}{4\epsilon} + C_3\right) |D_k^h u|^2 + \left(\frac{C_2}{4\epsilon} + C_3\right) \|Du\|_2^2 dx \\
&= (C_1 + C_2) \epsilon \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx + \left(\frac{C_1}{4\epsilon} + C_3\right) \|D_k^h u\|_{L^2(U)}^2 + \left(\frac{C_2}{4\epsilon} + C_3\right) \|Du\|_{L^2(U)}^2.
\end{aligned}$$

We know there $\exists C_4 > 0$, such that

$$\|D^h u\|_{L^2(Z)} \leq C_4 \|Du\|_{L^2(U)}, \forall |h| \in (0, \text{dist}(Z, \partial U)), \forall u \in H_0^1(U).$$

Thus

$$|A_2 + A_3 + A_4| \leq (C_1 + C_2) \epsilon \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx + \left(\frac{C_2}{4\epsilon} + C_3 + \left(\frac{C_1}{4\epsilon} + C_3\right) C_4^2\right) \|Du\|_{L^2(U)}^2.$$

Taking $\epsilon := \frac{\theta}{2(C_1 + C_2)}$, $C_5(\epsilon) := \frac{C_2}{4\epsilon} + C_3 + \left(\frac{C_1}{4\epsilon} + C_3\right) C_4^2 \in (0, \infty)$, we have

$$\begin{aligned}
\sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= A_1 + A_2 + A_3 + A_4 \\
&\geq A_1 - |A_2 + A_3 + A_4| \\
&\geq \theta \int_Z \zeta^2 \|D(D_k^h u)\|_2^2 dx - \frac{\theta}{2} \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx - C_5 \|Du\|_{L^2(U)}^2 \\
&= \frac{\theta}{2} \int_Z \|D(D_k^h u)\|_2^2 \zeta^2 dx - C_5 \|Du\|_{L^2(U)}^2.
\end{aligned}$$

4. On the other hand,

$$\begin{aligned}
\left| \langle \tilde{f}, v \rangle_{L^2(U)} \right| &= \int_U \left| f - \sum_{i=1}^n b^i \partial_i u - cu \right| |v| dx \\
&= \int_U \left(|f| + \sum_{i=1}^n |b^i \partial_i u| + |cu| \right) |v| dx \\
&\leq \int_U \left(|f| + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} |\partial_i u| + \|c\|_{L^\infty(U)} |u| \right) |v| dx \\
&= \int_U |f| |v| dx + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \int_U |\partial_i u| |v| dx + \|c\|_{L^\infty(U)} \int_U |u| |v| dx \\
&\leq C_6 \left(\int_U |f| |v| dx + \int_U |\partial_i u| |v| dx + \int_U |u| |v| dx \right) \\
&\leq C_6 \left(\int_U \frac{1}{4\epsilon} |f|^2 + \epsilon |v|^2 dx + \int_U \frac{1}{4\epsilon} |\partial_i u|^2 + \epsilon |v|^2 dx + \int_U \frac{1}{4\epsilon} |u|^2 + \epsilon |v|^2 dx \right) \\
&\leq C_6 \int_U \frac{1}{4\epsilon} (|f|^2 + |\partial_i u|^2 + |u|^2) + 3\epsilon |v|^2 dx \\
&\leq \frac{C_6}{4\epsilon} \int_U |f|^2 + \|Du\|_2^2 + |u|^2 dx + 3C_6\epsilon \int_U |v|^2 dx,
\end{aligned}$$

where $C_6 := \max \left(1, \sum_{i=1}^n \|b^i\|_{L^\infty(U)}, \|c\|_{L^\infty(U)} \right) \in (0, \infty)$.

We have shown in step 2 that $\zeta^2 D_k^h u \in H^1(Z)$, $\text{Supp}(\zeta^2 D_k^h u) \subset Z \subset\subset U$, thus $\zeta^2 D_k^h u \in H_0^1(U)$.

$$\begin{aligned}
\int_U |v|^2 dx &= \int_Z |v|^2 dx \\
&= \int_Z |-D_k^{-h}(\zeta^2 D_k^h u)|^2 dx \\
&\leq \int_Z |D^{-h}(\zeta^2 D_k^h u)|^2 dx \\
&\leq C_4^2 \int_U |D(\zeta^2 D_k^h u)|^2 dx \\
&= C_4^2 \int_W |D(\zeta^2 D_k^h u)|^2 dx \\
&= C_4^2 \int_W |D(\zeta^2) D_k^h u + D(D_k^h u) \zeta^2|^2 dx \\
&\leq 2C_4^2 \int_W |D(\zeta^2)|^2 |D_k^h u|^2 + |D(D_k^h u)|^2 \zeta^4 dx \\
&\leq 2C_4^2 \int_W \|D(\zeta^2)\|_{L^\infty(U)} |D_k^h u|^2 + |D(D_k^h u)|^2 \zeta^2 dx \\
&\leq 2C_4^2 \|D(\zeta^2)\|_{L^\infty(U)} \int_W |D_k^h u|^2 dx + 2C_4^2 \int_W |D(D_k^h u)|^2 \zeta^2 dx \\
&\leq 2C_4^4 \|D(\zeta^2)\|_{L^\infty(U)} \int_U \|Du\|_2^2 dx + 2C_4^2 \int_U |D(D_k^h u)|^2 \zeta^2 dx \\
&\leq C_7 \int_U \|Du\|_2^2 + |D(D_k^h u)|^2 \zeta^2 dx,
\end{aligned}$$

where $C_7 := 2C_4^2 \max \left(C_4^2 \|D(\zeta^2)^2\|_{L^\infty(U)}, 1 \right) \in (0, \infty)$. Thus we have

$$\begin{aligned} \left| \left\langle \tilde{f}, v \right\rangle_{L^2(U)} \right| &\leq \frac{C_6}{4\epsilon} \int_U |f|^2 + \|Du\|_2^2 + |u|^2 dx + 3C_6\epsilon \int_U |v|^2 dx \\ &\leq \frac{C_6}{4\epsilon} \int_U |f|^2 + \|Du\|_2^2 + |u|^2 dx + 3C_6C_7\epsilon \int_U \|Du\|_2^2 + |D(D_k^h u)|^2 \zeta^2 dx \\ &\leq \left(\frac{C_6}{4\epsilon} + 3C_6C_7\epsilon \right) \left(\|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) + 3C_6C_7\epsilon \int_U |D(D_k^h u)|^2 \zeta^2 dx. \end{aligned}$$

5. Taking $\epsilon := \frac{\theta}{12C_6C_7} > 0$, $C_8 := \frac{C_6}{4\epsilon} + 3C_6C_7\epsilon > 0$, we have

$$\begin{aligned} \sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &= \left\langle \tilde{f}, v \right\rangle_{L^2(U)} \\ &\leq \left| \left\langle \tilde{f}, v \right\rangle_{L^2(U)} \right| \\ &\leq C_8 \left(\|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) + \frac{\theta}{4} \int_U |D(D_k^h u)|^2 \zeta^2 dx \\ &= C_8 \left(\|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) + \frac{\theta}{4} \int_Z |D(D_k^h u)|^2 \zeta^2 dx \\ \sum_{i,j=1}^n \int_U (a^{ij} \partial_i u \partial_j v) dx &\geq \frac{\theta}{2} \int_Z |D(D_k^h u)|^2 \zeta^2 dx - C_5 \|Du\|_{L^2(U)}^2 \\ \frac{\theta}{4} \int_Z |D(D_k^h u)|^2 \zeta^2 dx &\leq (C_5 + C_8) \left(\|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) \\ \frac{\theta}{4} \int_V |D(D_k^h u)|^2 dx &\leq (C_5 + C_8) \left(\|f\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) \\ \int_V |D_k^h(Du)|^2 dx &\leq C_9 \left(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right), \end{aligned}$$

where $C_9 := \frac{4(C_5+C_8)}{\theta} \in (0, \infty)$.

Notice that for all $j \in [n]$, we have $\partial_j u \in L^2(U)$, and

$$\int_V |D_k^h(\partial_j u)|^2 dx \leq \int_V |D_k^h(Du)|^2 dx \leq C_9 \left(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right),$$

and this holds for all $k \in [n]$. Thus,

$$\begin{aligned} \|D^h(\partial_j u)\|_{L^2(V)}^2 &= \int_V |D^h(\partial_j u)|^2 dx \\ &= \int_V \sum_{k=1}^n |D_k^h(\partial_j u)|^2 dx \\ &= \sum_{k=1}^n \int_V |D_k^h(\partial_j u)|^2 dx \\ &\leq \sum_{k=1}^n C_9 \left(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right) \\ &= nC_9 \left(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2 \right). \\ \|D^h(\partial_j u)\|_{L^2(V)} &\leq \sqrt{nC_9} \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right) \\ &< \infty. \end{aligned}$$

Since this holds for all $|h| > 0$ such that $\text{dist}(V, \partial U) > 8|h|$, $\text{dist}(W, \partial U) > 6|h|$, we have $\partial_j u \in H^1(U)$, with

$$\|D(\partial_j u)\|_{L^2(V)} \leq \sqrt{nC_9} \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right).$$

Since this holds for all $j \in [n]$, we have $u \in H^2(V)$, and

$$\begin{aligned} \|D^2 u\|_{L^2(V)}^2 &= \int_V \|D^2 u\|_2^2 dx \\ &= \int_V \sum_{j=1}^n \|\partial_j(Du)\|_2^2 dx \\ &= \sum_{j=1}^n \int_V \|D(\partial_j u)\|_2^2 dx \\ &\leq \sum_{j=1}^n nC_9 \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right)^2 \\ &= n^2 C_9 \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right)^2 \\ &\implies \|u\|_{H^2(V)}^2 = \|D^2 u\|_{L^2(V)}^2 + \|u\|_{H^1(V)}^2 \\ &\leq n^2 C_9 \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right)^2 + \|u\|_{H^1(V)}^2 \\ &\leq (n^2 C_9 + 1) \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right)^2. \end{aligned}$$

Thus we have found $C := \sqrt{n^2 C_9 + 1} \in (0, \infty)$, such that $\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right)$. Since V is arbitrary, we have that $u \in H_{loc}^2(U)$.

6. Notice that the above estimate holds as long as $V \subset\subset U$ and $u \in H^1(U)$. Since $u \in H^1(W)$, we can find some constant C' , such that $\|u\|_{H^2(V)}^2 \leq C' \left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)} \right)$.

Now consider $v := \xi^2 u \in H_0^1(U)$, we can find $\|Du\|_{L^2(W)} \leq C'' \|u\|_{L^2(U)}$ for some $C'' > 0$.

Plugging in will give us

$$\|u\|_{H^2(V)}^2 \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right).$$

□

Definition 3.11. If $Lu(x) = f(x)$ a.e. $x \in U$, we say u is a **strong solution** to the problem $Lu = f$ in U .

Corollary 3.19. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, $\forall i, j \in [n]$. If $f \in L^2(U)$, and $u \in H^1(U)$ is a weak solution to $Lu = f$ in U , then u is a strong solution.

Proof. We have that $u \in H_{loc}^2(U)$.

Consider any $V \subset\subset U$, since $a^{ij} \in C^1$, we have $a^{ij}u \in H^2(V)$.

Consider any $v \in C_c^\infty(V)$, we must have

$$\begin{aligned}
\langle f, v \rangle_{L^2(V)} &= B[u, v] \\
&= \int_V \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx \\
&= \int_V \left(\sum_{i,j=1}^n a^{ij} \partial_j (\partial_i u) v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx \\
&= \int_V \left(\sum_{i,j=1}^n \partial_j (a^{ij} \partial_i u) + \sum_{i=1}^n b^i \partial_i u + c u \right) v dx \\
&= \int_V (Lu) v dx \\
&= \langle Lu, v \rangle_{L^2(V)}.
\end{aligned}$$

Since this holds for all $v \in C_c^\infty(V)$, we must have $Lu(x) = f(x)$ a.e. $x \in V$.

Since this hold for all $V \subset\subset U$, we have that $Lu(x) = f(x)$ a.e. $x \in U$. \square

Theorem 3.20. (*Higher Interior regularity*)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(U), \forall i, j \in [n]$ for some $m \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U , then $u \in H_{loc}^{m+2}(U)$. In addition, $\forall V \subset\subset U, \exists C > 0$, such that $\forall f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U , we have

$$\|u\|_{H^{m+2}(U)} \leq C \left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)} \right).$$

Corollary 3.21. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(U), \forall i, j \in [n]$ for some $m > \frac{n}{2} - 2 \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U , then $u \in C^l(U)$, where $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$.

Theorem 3.22. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^\infty(U), \forall i, j \in [n]$. If $f \in C^\infty(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U , then $u \in C^\infty(U)$.

Theorem 3.23. (*Boundary H^2 regularity*)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^2 , and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U), \forall i, j \in [n]$. Then $\exists C > 0$, such that $\forall f \in L^2(U)$ and

$u \in H_0^1(U)$ being a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$ we have

$$\|u\|_{H^2(U)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right),$$

and thus $u \in H^2(U)$.

Proof. 1. First prove the case if the boundary is locally flat:

$$U = B(0, 1) \cap \{x : x^n > 0\}, V = B(0, \frac{1}{2}) \cap \{x : x^n > 0\}.$$

Similar to the proof of Interior H^2 regularity, we first use difference quotients to obtain a bound for derivatives that are not normal to the flat boundary:

$$\sum_{k,l=1, k+l < 2n}^n \|\partial_k \partial_l u\|_{L^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right),$$

where we can transform $\|u\|_{H^1(U)}$ to $\|u\|_{L^2(U)}$.

For the derivative that is normal to the flat boundary $\partial_n \partial_n$, we write the PDE in non divergence form, and use ellipticity to note that $a^{nn} > \theta > 0$ to find:

$$|\partial_n \partial_n| \leq C \left(\sum_{k,l=1, k+l < 2n}^n |\partial_k \partial_l u| + \|Du\|_2 + |u| + |f| \right) \text{ a.e. } x \in U.$$

Thus

$$\|\partial_n \partial_n\|_{L^2(U)} \leq C \left(\sum_{k,l=1, k+l < 2n}^n \|\partial_k \partial_l u\|_{L^2(U)} + \|Du\|_{L^2(U)} + \|u\|_{L^2(U)} + \|f\|_{L^2(U)} \right).$$

This leads to

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right)$$

2. Take any $x_0 \in \partial U$, let $y = \Phi(x)$ be a C^2 straightening map on $B(x_0, r)$ with a C^2 inverse $x = \Psi(y)$. Pick some small enough s , such that

$$U' = B(0, s) \cap \{y : y^n > 0\} \subseteq \Phi(U \cap B(x_0, r)), V' = B(0, \frac{1}{2}s) \cap \{y : y^n > 0\}.$$

We check the weak formulation is well-defined on U' and that L' satisfies the assumptions of L . Apply step 1 to get

$$\|u'\|_{H^2(V')} \leq C \left(\|f'\|_{L^2(U')} + \|u'\|_{L^2(U')} \right).$$

Transform back using Ψ .

3. Use compactness to find V_1, \dots, V_N to cover ∂U . Find $V_0 \subset\subset U$ such that $U = \bigcup_{i=0}^N V_i$. Use interior result on V_0 . Combine them together.

□

Remark. When the solution is unique, we can throw away the $\|u\|_{L^2(U)}$ by boundedness of inverse in the last section.

Theorem 3.24. (*Higher boundary regularity*)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^{m+2} , and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \forall i, j \in [n]$. Then $\exists C > 0$, such that $\forall f \in H^m(U)$ and

$u \in H_0^1(U)$ being a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$ we have

$$\|u\|_{H^{m+2}(U)} \leq C \left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)} \right),$$

and thus $u \in H^{m+2}(U)$.

Corollary 3.25. Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^{m+2} , and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \forall i, j \in [n]$ for some $m > \frac{n}{2} - 2 \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U , then $u \in C^l(U)$, where $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$.

Theorem 3.26. (*Infinite differentiability up to the boundary*)

Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^∞ , and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^\infty(\bar{U}), \forall i, j \in [n]$. Then $\forall f \in H^\infty(U)$ and $u \in H_0^1(U)$ being a

weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases}$ we have $u \in C^\infty(\bar{U})$.

4 Parabolic PDEs

4.1 Spaces Involving Time

4.1.1 Bochner Spaces

See more about Bochner Spaces in my Measure Theory Notes.

Definition 4.1. Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space, a function $u : [0, T] \rightarrow X$ is **continuous** at a point $t \in (0, T)$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall s, t \in [0, T], |s - t| < \delta \implies \|u(s) - u(t)\| < \epsilon.$$

A function u is continuous if it is continuous at all $t \in (0, 1)$.

$$\|u\|_{C([0, T]; X)} := \sup_{t \in (0, T)} \|u(t)\|.$$

Theorem 4.1. $(C([0, T]; X), \|u\|_{C([0, T]; X)})$ is a Banach Space.

See the definition of Bochner integrable functions in notes of Measure Theory. We will still consider the Lebesgue measure on $[0, T]$.

Theorem 4.2 (Bochner). Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space, a strongly measurable function $f : [0, T] \rightarrow X$ is Bochner integrable if and only if $t \mapsto \|f(t)\|_X$ is integrable. In this case,

$$\begin{aligned} \left\| \int_0^T f(t) dt \right\|_X &\leq \int_0^T \|f(t)\|_X dt, \\ \forall u^* \in X^*, \left\langle u^* \left| \int_0^T f(t) dt \right. \right\rangle &= \int_0^T \langle u^* | f(t) \rangle dt. \end{aligned}$$

Theorem 4.3. Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space, then Dominated Convergence Theorem, Holder's Inequality, and Minkowski's Inequality still work with the Bochner integral.

Theorem 4.4. Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space, then for any Bochner integrable $f : [0, T] \rightarrow X$, we have $\int_s^t f(\tau) d\tau$ is continuous in both $s, t \in [0, T]$.

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

Definition 4.2. Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space, and $1 \leq p < \infty$, we define

$$\mathcal{L}^p([0, T]; X) := \left\{ f : [0, T] \rightarrow X \mid f \text{ is measurable, } \int_X \|f\|_X^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p([0, T]; X)} := \left(\int_X \|f\|_X^p d\mu \right)^{\frac{1}{p}}.$$

Definition 4.3. Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space, $(B, \|\cdot\|)$ be a Banach Space, we define

$$\mathcal{L}^\infty([0, T]; X) := \{f : X \rightarrow B \mid f \text{ is measurable, } \text{ess sup } \|f\|_X < \infty\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty([0, T]; X)} := \text{ess sup } \|f\|_B.$$

Definition 4.4. Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space. For any $p \in [1, \infty]$, we define

$$L^p([0, T]; X) := \mathcal{L}^p([0, T]; X)/N,$$

where $N := \{f : X \rightarrow B \mid f \text{ is measurable, } f = 0 \mu - \text{a.e.}\}$. Namely, $[f] \in L^p([0, T]; X)$ is the equivalence class of all $g = f \mu - \text{a.e.}$ for $f \in \mathcal{L}^p([0, T]; X)$.

In addition, we define

$$\|[f]\|_{L^p([0, T]; X)} := \|f\|_{\mathcal{L}^p([0, T]; X)}$$

for any representative f .

Theorem 4.5 (Fischer-Riesz-Bochner). *Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space. For all $1 \leq p \leq \infty$, we have that $(L^p([0, T]; X), \|\cdot\|_{L^p([0, T]; X)})$ is a Banach Space.*

Similarly, we can also define $L_{loc}^p(0, T; X)$, $W^{k,p}(0, T; X)$, $H^k(0, T; X)$ and weak derivatives where the test functions are $\phi \in C_c^\infty(0, T; \mathbb{R})$.

We can similarly define the mollification of $f \in L_{loc}^1(0, T; X)$ to be

$$f^\epsilon := \eta_\epsilon * f : (\epsilon, T - \epsilon) \rightarrow X; \quad t \mapsto \int_{t-\epsilon}^{t+\epsilon} \eta_\epsilon(t - \tau) f(\tau) d\tau.$$

Similarly, we have

Theorem 4.6. *Let f^ϵ be defined as above, we have:*

1. $f^\epsilon \in C^\infty((\epsilon, T - \epsilon); X)$,
2. $\partial_t^k(f^\epsilon) = (\partial_t^k \eta_\epsilon) * f$ on $(\epsilon, T - \epsilon)$,
3. $f^\epsilon \rightarrow f$ a.e. $t \in (0, T)$, as $\epsilon \rightarrow 0$,
4. If $f \in C(0, T; X)$, we have $f^\epsilon \rightarrow f$ uniformly on compact subsets of U ,
5. If $1 \leq p < \infty$, $f \in L_{loc}^p(0, T; X)$, we have $f^\epsilon \rightarrow f$ in $L_{loc}^p(0, T; X)$. Namely, $f^\epsilon \rightarrow f$ in $L^p(V)$, $\forall V \subset \subset (0, T)$.

Theorem 4.7. *Let $T > 0$ and $(X, \|\cdot\|)$ be a Banach Space, $p \in [1, \infty]$, and $u \in W^{1,p}(0, T; X)$, then*

1. $u(t) = u(s) + \int_s^t u'(\tau) d\tau$ for a.e. $0 \leq s \leq t \leq T$.
2. There is a representative $\tilde{u} \in C([0, T], X)$ of u . In particular, $\tilde{u}(t) = \tilde{u}(s) + \int_s^t u'(\tau) d\tau$ for any $0 \leq s \leq t \leq T$.
3. $\exists C > 0$ such that $\forall u \in W^{1,p}(0, T; X)$, $\sup_{t \in [0, T]} \|u(t)\|_X \leq C \|u\|_{W^{1,p}(0, T; X)}$.

Proof. We will prove for $p \in [1, \infty)$.

1. Let $u^\epsilon := \eta_\epsilon * u$, we have that $u^\epsilon \in C^\infty((\epsilon, T - \epsilon); X)$, and $\partial_t(u^\epsilon) = (\partial_t \eta_\epsilon) * u$ on $(\epsilon, T - \epsilon)$. We also have $f^\epsilon(t) \rightarrow f(t)$ a.e. $t \in (0, T)$. Similar to 2.17, we can show that $\partial_t(u^\epsilon) = \eta_\epsilon * \partial_t u = (\partial_t u)^\epsilon$ on $(\epsilon, T - \epsilon)$. Since $u \in W^{1,p}(0, T; X)$, we know that $\partial_t u \in L_{loc}^p(0, T; X)$, so $(\partial_t u)^\epsilon \rightarrow \partial_t u$ in $L_{loc}^p(0, T; X)$. Since $|(0, T)| = T < \infty$, we have that $\partial_t(u^\epsilon) \rightarrow \partial_t u$ in $L_{loc}^1(0, T; X)$, which means

$$\forall [s, t] \subset (0, T), \quad \lim_{\epsilon \rightarrow 0} \int_s^t \|(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)\|_X d\tau = 0.$$

We have that $\left\| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right\|_X \leq \int_s^t \|(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)\|_X d\tau$ for any fixed $[s, t] \subset (0, T)$ and $\epsilon < \min(s, T - t)$. Thus

$$\lim_{\epsilon \rightarrow 0} \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right\|_X = 0$$

for any $[s, t] \subset (0, T)$.

Now $u^\epsilon(t) = u^\epsilon(s) + \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau$ for any $[s, t] \subset (\epsilon, T - \epsilon)$ by FTC, since $u^\epsilon \in C^\infty((\epsilon, T - \epsilon); X)$.

We have

$$\begin{aligned}
& \left\| -u(t) + u(s) + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X \\
&= \left\| u^\epsilon(t) - u(t) - u^\epsilon(s) + u(s) - \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X \\
&\leq \|u^\epsilon(t) - u(t)\|_X + \|u^\epsilon(s) - u(s)\|_X + \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau - \int_s^t (\partial_t u)(\tau) d\tau \right\|_X \\
&\leq \|u^\epsilon(t) - u(t)\|_X + \|u^\epsilon(s) - u(s)\|_X + \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right\|_X
\end{aligned}$$

for any s, t, ϵ such that $[s, t] \subset (\epsilon, T - \epsilon)$.

Since each term goes to 0 as $\epsilon \rightarrow 0$ for a.e. $0 \leq s \leq t \leq T$, we must have

$$\left\| -u(t) + u(s) + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X = 0$$

for a.e. $0 \leq s \leq t \leq T$.

We thus have

$$u(t) = u(s) + \int_s^t (\partial_t u)(\tau) d\tau$$

for a.e. $0 \leq s \leq t \leq T$.

2. Fix any representative for u .

Notice that the set N where the above property does not hold has measure 0.

Now fix some point $s \in [0, T] \setminus N$, we define

$$\tilde{u}(t) := \begin{cases} u(s) - \int_t^s u'(\tau) d\tau & t < s \\ u(s) + \int_s^t u'(\tau) d\tau & t \geq s \end{cases}.$$

For any $t \in [0, T] \setminus N$, we have that

$$u(t) := u(s) + \int_s^t u'(\tau) d\tau = \tilde{u}(t)$$

if $t \geq s$, and

$$u(s) = u(t) + \int_t^s u'(\tau) d\tau \implies u(t) = u(s) - \int_t^s u'(\tau) d\tau = u(s)$$

if $t < s$.

Thus $\tilde{u} = u$ a.e. $t \in [0, T]$, which means \tilde{u} is a representative of u .

In addition, $\tilde{u}(t)$ is continuous since $\int_t^s u'(\tau) d\tau$ and $\int_s^t u'(\tau) d\tau$ are both continuous in t , and

$$\lim_{t \rightarrow s^-} \tilde{u}(t) = \lim_{t \rightarrow s^-} \left(\tilde{u}(t)u(s) - \int_t^s u'(\tau) d\tau \right) = u(s) = u(s) + \int_s^s u'(\tau) d\tau = \tilde{u}(s).$$

3. See A5Q2.

□

Proposition 4.8. Suppose \mathcal{H} is a Hilbert Space, and $u, v \in C^1(0, T; \mathcal{H})$, then we have

$$\forall t \in [0, T], \quad \frac{d}{dt} \langle u(t), v(t) \rangle_{\mathcal{H}} = \langle u'(t), v(t) \rangle_{\mathcal{H}} + \langle v'(t), u(t) \rangle_{\mathcal{H}},$$

where $u'(t) := \lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h}$ is the normal derivative in X .

Proof. We have

$$\begin{aligned}
\frac{d}{dt} \langle u(t), v(t) \rangle_{\mathcal{H}} &= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) \rangle_{\mathcal{H}} - \langle u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) \rangle_{\mathcal{H}} - \langle u(t+h), v(t) \rangle_{\mathcal{H}} + \langle u(t+h), v(t) \rangle_{\mathcal{H}} - \langle u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) \rangle_{\mathcal{H}} - \langle u(t+h), v(t) \rangle_{\mathcal{H}}}{h} + \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t) \rangle_{\mathcal{H}} - \langle u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\langle u(t+h), v(t+h) - v(t) \rangle_{\mathcal{H}}}{h} + \lim_{h \rightarrow 0} \frac{\langle u(t+h) - u(t), v(t) \rangle_{\mathcal{H}}}{h} \\
&= \left\langle \lim_{h \rightarrow 0} u(t+h), \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h} \right\rangle_{\mathcal{H}} + \left\langle \lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h}, v(t) \right\rangle_{\mathcal{H}} \\
&= \langle u(t), v'(t) \rangle_{\mathcal{H}} + \langle u'(t), v(t) \rangle_{\mathcal{H}}.
\end{aligned}$$

□

4.1.2 Sobolev Spaces In Time

Now we consider the cases where X might be any of the tuple $H_0^1(U) \subset L^2(U) \cong L^2(U)^* \subset H^{-1}(U)$, and see what is the relationship between the time weak derivatives in each space.

Lemma 4.9. *Suppose $\mathbf{u}, \mathbf{u}' \in L^1(0, T; H_0^1(U))$, then we must have \mathbf{u}' is also the time weak derivative of \mathbf{u} in $L^1(0, T; L^2(U))$.*

Proof. Firstly, for each $t \in [0, T]$, we have $\mathbf{u}(t), \mathbf{u}'(t) \in H_0^1(U) \subset L^2(U)$, so \mathbf{u}, \mathbf{u}' are indeed functions $[0, T] \rightarrow L^2(U)$.

In addition,

$$\begin{aligned}
\int_0^T \|\mathbf{u}(t)\|_{L^2(U)} dt &\leq \int_0^T \|\mathbf{u}(t)\|_{H^2(U)} dt \\
&= \|\mathbf{u}\|_{L^1(0, T; H_0^1(U))} \\
&< \infty.
\end{aligned}$$

Similarly for $\int_0^T \|\mathbf{u}'(t)\|_{L^2(U)} dt < \infty$.

Thus $\mathbf{u}, \mathbf{u}' \in L^1(0, T; L^2(U))$.

Now $\forall \phi \in C_c^\infty(0, T)$, we have

$$\begin{aligned}
0 &\leq \left\| \int_0^T \phi'(t) \mathbf{u}(t) dt + \int_0^T \phi(t) \mathbf{u}'(t) dt \right\|_{L^2(U)} \\
&\leq \left\| \int_0^T \phi'(t) \mathbf{u}(t) dt + \int_0^T \phi(t) \mathbf{u}'(t) dt \right\|_{H_0^1(U)} \\
&= 0.
\end{aligned}$$

Thus $\int_0^T \phi'(t) \mathbf{u}(t) dt = - \int_0^T \phi(t) \mathbf{u}'(t) dt$ in $L^2(U)$ for any $\phi \in C_c^\infty(0, T)$, which shows that \mathbf{u}' is also the time weak derivative of \mathbf{u} in $L^1(0, T; L^2(U))$. □

Lemma 4.10. *Let $\mathbf{u} \in L^1(0, T; H_0^1(U))$, $\mathbf{v} \in L^1(0, T; H^{-1}(U))$, we have $\mathbf{v} = (\mathbf{u}^*)' \iff$*

$$\forall \phi \in C_c^\infty(0, T), w \in H_0^1(U), \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt = - \int_0^T \phi(t) \langle \mathbf{v}(t), w \rangle_{H^{-1}(U), H_0^1(U)} dt,$$

where $\mathbf{u}^*(t) := \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)} \in H^{-1}(U)$ as usual.

Proof. $(\mathbf{u}^*)' = \mathbf{v}$ by definition means

$$\forall \phi \in C_c^\infty(0, T), \quad \int_0^T \phi'(t) \mathbf{u}^*(t) dt = - \int_0^T \phi(t) \mathbf{v}(t) dt$$

in $H^{-1}(U)$.

Consider any $w \in H_0^1(U)$, we have $\langle \cdot | w \rangle_{H^{-1}(U), H_0^1(U)} \in (H^{-1}(U))^*$.

By Bochner's Theorem 4.2 and linearity of duality pairing, we have

$$\begin{aligned} \left\langle \int_0^T \phi'(t) \mathbf{u}^*(t) dt \middle| w \right\rangle_{H^{-1}(U), H_0^1(U)} &= \left\langle - \int_0^T \phi(t) \mathbf{v}(t) dt \middle| w \right\rangle_{H^{-1}(U), H_0^1(U)} \\ \int_0^T \langle \phi'(t) \mathbf{u}^*(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt &= - \int_0^T \langle \phi(t) \mathbf{v}(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}^*(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt &= - \int_0^T \phi(t) \langle \mathbf{v}(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \phi(t) \langle \mathbf{v}(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt. \end{aligned}$$

□

Lemma 4.11. Suppose $\mathbf{u} \in L^1(0, T; H_0^1(U))$, and \mathbf{u}' is its time weak derivative in $L^1(0, T; L^2(U))$, then we must have the action function

$$(\mathbf{u}')^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$$

is the time weak derivative of $\mathbf{u}^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$ in $L^1(0, T; H^{-1}(U))$. Namely,

$$(\mathbf{u}')^* = (\mathbf{u}^*)'.$$

Proof. Consider any $\phi \in C_c^\infty(0, T)$, by definition of weak derivative, we have

$$\int_0^T \phi'(t) \mathbf{u}(t) dt = - \int_0^T \phi(t) \mathbf{u}'(t) dt.$$

in $L^2(U)$.

Now for any $w \in H_0^1(U) \subset L^2(U)$, we have $\langle \cdot, w \rangle_{L^2(U)} \in (L^2(U))^*$.

By Bochner's Theorem 4.2 and linearity of the inner product, we have

$$\begin{aligned} \left\langle \int_0^T \phi'(t) \mathbf{u}(t) dt, w \right\rangle_{L^2(U)} &= \left\langle - \int_0^T \phi(t) \mathbf{u}'(t) dt, w \right\rangle_{L^2(U)} \\ \int_0^T \langle \phi'(t) \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \langle \phi(t) \mathbf{u}'(t), w \rangle_{L^2(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \phi(t) \langle \mathbf{u}'(t), w \rangle_{L^2(U)} dt \\ \int_0^T \phi'(t) \langle \mathbf{u}(t), w \rangle_{L^2(U)} dt &= - \int_0^T \phi(t) \langle (\mathbf{u}')^*(t) | w \rangle_{H^{-1}(U), H_0^1(U)} dt. \end{aligned}$$

Thus $(\mathbf{u}')^* = (\mathbf{u}^*)'$ from the above lemma. □

Corollary 4.12. Suppose $\mathbf{u}, \mathbf{u}' \in L^1(0, T; H_0^1(U))$, then we must have the action function

$$(\mathbf{u}')^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$$

is the time weak derivative of $\mathbf{u}^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$ in $L^1(0, T; H^{-1}(U))$.

Remark. Recall $H_0^1(U) \subset L^2(U) \cong L^2(U)^* \subset H^{-1}(U)$, and we can identify $\mathbf{u}(t) \in H_0^1(U)$ with $\mathbf{u}^*(t) := \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)} \in H^{-1}(U)$. The above lemmas allow us to further abuse this notation and identify \mathbf{u}' with $(\mathbf{u}')^* = (\mathbf{u}^*)'$.

Definition 4.5. Suppose $\mathbf{u} \in L^1(0, T; H_0^1(U))$, we abuse the notation and denote

$$\mathbf{u}' := \mathbf{v} \in L^1(0, T; H^{-1}(U))$$

to be the time weak derivative of \mathbf{u} , if $\mathbf{v} = (\mathbf{u}^*)'$ is the time weak derivative of the action function

$$\mathbf{u}^* := t \mapsto \langle \mathbf{u}(t), \cdot \rangle_{L^2(U)}$$

in $L^1(0, T; H^{-1}(U))$.

Remark. This is a further extension of the original definition of the weak derivative, since \mathbf{u}' may not exist in $L^1(0, T; H_0^1(U))$ even if such a $\mathbf{v} = (\mathbf{u}^*)'$ exists in $L^1(0, T; H^{-1}(U))$ or even $L^1(0, T; L^2(U)^*)$.

We also have the following results:

Theorem 4.13. *The dual space of $L^2(0, T; H_0^1(U))$ is $L^2(0, T; H^{-1}(U))$, and the dual space of $L^2(0, T; H^{-1}(U))$ is $L^2(0, T; H_0^1(U))$. The contraction map is defined to be $\forall \mathbf{u} \in L^2(0, T; H_0^1(U)), \mathbf{v} \in L^2(0, T; H^{-1}(U))$,*

$$\langle \mathbf{u} | \mathbf{v} \rangle_{L^2(0, T; H_0^1(U)), L^2(0, T; H^{-1}(U))} := \langle \mathbf{v} | \mathbf{u} \rangle_{L^2(0, T; H^{-1}(U)), L^2(0, T; H_0^1(U))} := \int_0^T \langle \mathbf{v}(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt.$$

Proof. We can quickly show one side of inclusion:

$$\begin{aligned} \int_0^T \langle \mathbf{v}(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt &\leq \int_0^T \|\mathbf{v}(t)\|_{H^{-1}(U)} \|\mathbf{u}(t)\|_{H_0^1(U)} dt \\ &\leq \left(\int_0^T \|\mathbf{v}(t)\|_{H^{-1}(U)}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|\mathbf{u}(t)\|_{H_0^1(U)}^2 dt \right)^{\frac{1}{2}} \\ &= \|\mathbf{u}\|_{L^2(0, T; H_0^1(U))} \|\mathbf{v}\|_{L^2(0, T; H^{-1}(U))}, \end{aligned}$$

which shows $L^2(0, T; H_0^1(U)) \subseteq L^2(0, T; H^{-1}(U))^*$, and $L^2(0, T; H^{-1}(U)) \subseteq L^2(0, T; H_0^1(U))^*$. \square

Theorem 4.14 (Royden-Fitzpatrick). *$f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous if and only if there is a Lebesgue integrable function g , such that $\forall x \in [a, b]$, $f(x) = f(a) + \int_a^x g(t) dt$. In this case, f is differentiable a.e., and $f'(x) = g(x)$ for a.e. $x \in [a, b]$.*

Lemma 4.15. *Suppose $\mathbf{u} \in L^1(0, T; H_0^1(U))$, $\mathbf{u}' \in L^1(0, T; H^{-1}(U))$, then $\forall \epsilon > 0$,*

$$\mathbf{u}'^\epsilon = \left(t \mapsto \langle (\mathbf{u}^\epsilon)'(t), \cdot \rangle_{L^2(U)} \right),$$

where $(\mathbf{u}^\epsilon)'$ is the normal derivative of \mathbf{u}^ϵ in $C^\infty(\epsilon, T - \epsilon; H_0^1(U))$.

Proof. Since \mathbf{u}' is the time weak derivative of \mathbf{u}^* in $L^2((0, T); H^{-1}(U))$, we can show that

$$\mathbf{u}'^\epsilon = \eta_\epsilon * \mathbf{u}' = (\eta_\epsilon * \mathbf{u}^*)'$$

in $L^2((0, T); H^{-1}(U))$ similarly as in 2.17.

Also, one can see that $\forall t \in [\epsilon, T - \epsilon], v \in H_0^1(U)$,

$$\begin{aligned}
\langle (\eta_\epsilon * \mathbf{u}^*)(t) | v \rangle_{H^{-1}(U), H_0^1(U)} &= \left\langle \int_0^T \eta_\epsilon(t - \tau) \mathbf{u}^*(\tau) d\tau \middle| v \right\rangle_{H^{-1}(U), H_0^1(U)} \\
&= \int_0^T \eta_\epsilon(t - \tau) \langle \mathbf{u}^*(\tau) | v \rangle_{H^{-1}(U), H_0^1(U)} d\tau \\
&= \int_0^T \eta_\epsilon(t - \tau) \langle \mathbf{u}(\tau), v \rangle_{L^2(U)} d\tau \\
&= \left\langle \int_0^T \eta_\epsilon(t - \tau) \mathbf{u}(\tau) d\tau, v \right\rangle_{L^2(U)} \\
&= \langle \mathbf{u}^\epsilon(t), v \rangle_{L^2(U)}.
\end{aligned}$$

Since $(\mathbf{u}^\epsilon)'$ exists as the weak derivative of \mathbf{u}^ϵ , we have

$$(\eta_\epsilon * \mathbf{u}^*)' = \left(t \mapsto \langle (\mathbf{u}^\epsilon)'(t), \cdot \rangle_{L^2(U)} \right)$$

in $L^2((0, T); H^{-1}(U))$. □

Theorem 4.16. Suppose $\mathbf{u} \in L^2(0, T; H_0^1(U))$, $\mathbf{u}' \in L^2(0, T; H^{-1}(U))$, then

1. There is a representative $\tilde{\mathbf{u}} \in C([0, T]; L^2(U))$ of \mathbf{u} .
2. For any $\mathbf{v} \in L^2(0, T; H_0^1(U))$, $\mathbf{v}' \in L^2(0, T; H^{-1}(U))$, the mapping $t \mapsto \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)}$ is absolutely continuous, and for a.e. $t \in [0, T]$, we have

$$\frac{d}{dt} \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} = \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)}.$$

3. The mapping $t \mapsto \|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2$ is absolutely continuous, and for a.e. $t \in [0, T]$, we have

$$\frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(U)}^2 = 2 \langle \mathbf{u}'(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)}.$$

4. $\exists C > 0$, such that $\forall \mathbf{u} \in L^2(0, T; H_0^1(U))$, $\mathbf{u}' \in L^2(0, T; H^{-1}(U))$,

$$\sup_{t \in [0, T]} \|\tilde{\mathbf{u}}(t)\|_{L^2(U)} \leq C \left(\|\mathbf{u}\|_{L^2(0, T; H_0^1(U))} + \|\mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} \right),$$

where the constant C only depends on T .

Proof. 1. We can extend \mathbf{u} to $[-\sigma, T + \sigma]$ for an $\delta > 0$ by reflection and cut off as done in 2.26. Now for any $\epsilon, \delta \in (0, \sigma)$, we can define $\mathbf{u}^\epsilon := \eta_\epsilon * \mathbf{u}$, $\mathbf{u}^\delta := \eta_\delta * \mathbf{u}$, both well-defined on $[0, T]$. By 4.6, $\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty([0, T]; H_0^1(U))$, so we have

$$\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty([0, T]; H_0^1(U)) \subset C^\infty([0, T]; L^2(U)).$$

Now for any $t \in [0, T]$, we have that

$$\begin{aligned}
&\frac{d}{dt} \|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\|_{L^2(U)} \\
&= \frac{d}{dt} \langle \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \rangle_{L^2(U)} \\
&= \langle (\mathbf{u}^\epsilon - \mathbf{u}^\delta)'(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \rangle_{L^2(U)} + \langle \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t), (\mathbf{u}^\epsilon - \mathbf{u}^\delta)'(t) \rangle_{L^2(U)} \\
&= 2 \langle (\mathbf{u}^\epsilon)'(t) - (\mathbf{u}^\delta)'(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t) \rangle_{L^2(U)},
\end{aligned}$$

where $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)'$ are the normal derivatives as functions $[0, T] \rightarrow L^2(U)$.

Also, since $\mathbf{u}^\epsilon, \mathbf{u}^\delta \in C^\infty([0, T]; H_0^1(U))$, we have that their weak derivatives exist in $L^2((0, T); H_0^1(U)) \subset L^1((0, T); H_0^1(U))$, and by above lemma, are also weak derivatives in $L^1((0, T); L^2(U))$.

Since any weak derivative is a.e. equal to the normal derivative if the latter exists, we will just use $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)'$ to represent the weak derivatives in $L^2((0, T); H_0^1(U))$, and the above equality still holds for a.e. $t \in [0, T]$.

Integrating both sides on any $[s, t] \subseteq [0, T]$, we get

$$\begin{aligned}
& \|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\|_{L^2(U)} - \|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\|_{L^2(U)} \\
&= \int_s^t \frac{d}{d\tau} \|\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\|_{L^2(U)} d\tau \\
&= \int_s^t 2 \langle (\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau), \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) \rangle_{L^2(U)} d\tau \\
&= \int_s^t 2 \langle (\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau), \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \\
&\leq \int_s^t 2 \|(\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\|_{H_0^1(U)} d\tau \\
&\leq \int_s^t \|(\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau)\|_{H^{-1}(U)}^2 + \|\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)\|_{H_0^1(U)}^2 d\tau \\
&= \|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\|_{L^2(0, T; H^{-1}(U))}^2 + \|\mathbf{u}^\epsilon - \mathbf{u}^\delta\|_{L^2(0, T; H_0^1(U))}^2,
\end{aligned}$$

where we again identify $(\mathbf{u}^\epsilon)'(\tau), (\mathbf{u}^\delta)'(\tau) \in H_0^1(U)$ with $\langle (\mathbf{u}^\epsilon)'(\tau), \cdot \rangle_{L^2(U)}, \langle (\mathbf{u}^\delta)'(\tau), \cdot \rangle_{L^2(U)} \in H^{-1}(U)$ as usual.

By 4.6, we have $\mathbf{u}^\epsilon, \mathbf{u}^\delta \rightarrow \mathbf{u}$ in $L^2((0, T); H_0^1(U))$, so

$$\begin{aligned}
\|\mathbf{u}^\epsilon - \mathbf{u}^\delta\|_{L^2(0, T; H_0^1(U))} &= \|\mathbf{u}^\epsilon - \mathbf{u} + \mathbf{u} - \mathbf{u}^\delta\|_{L^2(0, T; H_0^1(U))} \\
&\leq \|\mathbf{u}^\epsilon - \mathbf{u}\|_{L^2(0, T; H_0^1(U))} + \|\mathbf{u} - \mathbf{u}^\delta\|_{L^2(0, T; H_0^1(U))} \\
&\rightarrow 0
\end{aligned}$$

as $\epsilon, \delta \rightarrow 0$.

Since \mathbf{u}' is the time weak derivative of \mathbf{u}^* in $L^2((0, T); H^{-1}(U))$, we can again show that $(\eta_\epsilon * \mathbf{u}^*)' = \eta_\epsilon * \mathbf{u}'$ in $L^2((0, T); H^{-1}(U))$.

By 4.6 and lemma, we have $(\mathbf{u}^\epsilon)', (\mathbf{u}^\delta)' \rightarrow \mathbf{u}'$ in $L^2(0, T; H^{-1}(U))$.

Similar as above, we have $\|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\|_{L^2(0, T; H^{-1}(U))} \rightarrow 0$ as $\epsilon, \delta \rightarrow 0$.

In addition, for a.e. $s \in [0, T]$, we must have $\|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\|_{L^2(U)} \rightarrow 0$.

Pick any of these s , we have

$$\|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\|_{L^2(U)} \leq \|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\|_{L^2(U)} + \|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\|_{L^2(0, T; H^{-1}(U))}^2 + \|\mathbf{u}^\epsilon - \mathbf{u}^\delta\|_{L^2(0, T; H_0^1(U))}^2.$$

Since this holds for any $t \in [0, T]$, we have

$$\begin{aligned}
& \|\mathbf{u}^\epsilon - \mathbf{u}^\delta\|_{C([0, T]; L^2(U))} \\
&= \sup_{t \in [0, T]; L^2(U)} \|\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)\|_{L^2(U)} \\
&\leq \|\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)\|_{L^2(U)} + \|(\mathbf{u}^\epsilon)' - (\mathbf{u}^\delta)'\|_{L^2(0, T; H^{-1}(U))}^2 + \|\mathbf{u}^\epsilon - \mathbf{u}^\delta\|_{L^2(0, T; H_0^1(U))}^2 \\
&\rightarrow 0.
\end{aligned}$$

as $\epsilon, \delta \rightarrow 0$, since each term goes to 0.

This shows that \mathbf{u}^ϵ is a Cauchy sequence in $C([0, T]; L^2(U))$, and since it is a Banach Space, there

must be some

$$\tilde{\mathbf{u}} := \lim_{\epsilon \rightarrow 0} \mathbf{u}^\epsilon \in C([0, T]; L^2(U)).$$

Now since for a.e. $t \in [0, T]$, $\mathbf{u}(t) = \lim_{\epsilon \rightarrow 0} \mathbf{u}^\epsilon(t)$, and $\forall t \in [0, T]$, $\tilde{\mathbf{u}}(t) = \lim_{\epsilon \rightarrow 0} \mathbf{u}^\epsilon(t)$, we have that $\tilde{\mathbf{u}} = \mathbf{u}$ for a.e. $t \in [0, T]$ is a representative of \mathbf{u} .

2. Similar as above, we can show that for a.e. $t \in [0, T]$, we have

$$\frac{d}{dt} \langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} = \langle (\mathbf{u}^\epsilon)'(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(t), \mathbf{u}^\epsilon(t) \rangle_{L^2(U)}.$$

Integrating over any $(s, t) \subset [0, T]$ gives

$$\langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} = \langle \mathbf{u}^\epsilon(s), \mathbf{v}^\epsilon(s) \rangle_{L^2(U)} + \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau.$$

Now

$$\begin{aligned} & \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &= \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &\leq \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau - \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &\quad + \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &= \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| + \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| \\ &\leq \int_s^t \|(\mathbf{u}^\epsilon)'(\tau)\|_{H^{-1}(U)} \|\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)\|_{H_0^1(U)} d\tau + \int_s^t \|(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{v}(\tau)\|_{H_0^1(U)} d\tau \\ &\leq \sqrt{\int_s^t \|(\mathbf{u}^\epsilon)'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_s^t \|\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &\quad + \sqrt{\int_s^t \|(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_s^t \|\mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &\leq \sqrt{\int_0^T \|(\mathbf{u}^\epsilon)'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \|\mathbf{v}^\epsilon(\tau) - \mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &\quad + \sqrt{\int_0^T \|(\mathbf{u}^\epsilon)'(\tau) - \mathbf{u}'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \|\mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ &= \|(\mathbf{u}^\epsilon)'\|_{L^2(0, T; H^{-1}(U))} \|\mathbf{v}^\epsilon - \mathbf{v}\|_{L^2(0, T; H_0^1(U))} + \|(\mathbf{u}^\epsilon)' - \mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} \|\mathbf{v}\|_{L^2(0, T; H_0^1(U))}, \end{aligned}$$

by Holder's Inequality.

Notice that

$$\|(\mathbf{u}^\epsilon)'\|_{L^2(0, T; H^{-1}(U))} \rightarrow \|\mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} < \infty, \|(\mathbf{u}^\epsilon)' - \mathbf{u}'\|_{L^2(0, T; H^{-1}(U))} \rightarrow 0,$$

since we have shown $(\mathbf{u}^\epsilon)' \rightarrow \mathbf{u}'$ in $L^2(0, T; H^{-1}(U))$.

Also,

$$\|\mathbf{v}^\epsilon - \mathbf{v}\|_{L^2(0, T; H_0^1(U))} \rightarrow 0,$$

since $\mathbf{v}^\epsilon \rightarrow \mathbf{v}$ in $L^2(0, T; H_0^1(U))$.

Thus, $\lim_{\epsilon \rightarrow 0} \left| \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau - \int_s^t \langle \mathbf{u}'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \right| = 0$, which means

$$\lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau = \int_s^t \langle \mathbf{u}'(\tau), \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Similarly, we also have

$$\lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau = \int_s^t \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Taking the limit of $\epsilon \rightarrow 0$, we have

$$\begin{aligned} & \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} \\ &= \lim_{\epsilon \rightarrow 0} \langle \mathbf{u}^\epsilon(t), \mathbf{v}^\epsilon(t) \rangle_{L^2(U)} \\ &= \lim_{\epsilon \rightarrow 0} \left(\langle \mathbf{u}^\epsilon(s), \mathbf{v}^\epsilon(s) \rangle_{L^2(U)} + \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} + \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau \right) \\ &= \lim_{\epsilon \rightarrow 0} \langle \mathbf{u}^\epsilon(s), \mathbf{v}^\epsilon(s) \rangle_{L^2(U)} + \lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{u}^\epsilon)'(\tau), \mathbf{v}^\epsilon(\tau) \rangle_{L^2(U)} d\tau + \lim_{\epsilon \rightarrow 0} \int_s^t \langle (\mathbf{v}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) \rangle_{L^2(U)} d\tau \\ &= \langle \tilde{\mathbf{u}}(s), \tilde{\mathbf{v}}(s) \rangle_{L^2(U)} + \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau + \int_s^t \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau \\ &= \langle \tilde{\mathbf{u}}(s), \tilde{\mathbf{v}}(s) \rangle_{L^2(U)} + \int_s^t \left(\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau. \end{aligned}$$

In particular, if we take $s = 0$, we have

$$\langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} = \langle \tilde{\mathbf{u}}(0), \tilde{\mathbf{v}}(0) \rangle_{L^2(U)} + \int_0^t \left(\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau.$$

Also, by Holder's Inequality,

$$\begin{aligned} & \int_0^T \left| \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right| d\tau \\ & \leq \int_0^T \left| \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right| d\tau + \int_0^T \left| \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right| d\tau \\ & \leq \int_0^T \|\mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{v}(\tau)\|_{H_0^1(U)} d\tau + \int_0^T \|\mathbf{v}'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}(\tau)\|_{H_0^1(U)} d\tau \\ & \leq \sqrt{\int_0^T \|\mathbf{u}'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \|\mathbf{v}(\tau)\|_{H_0^1(U)}^2 d\tau} + \sqrt{\int_0^T \|\mathbf{v}'(\tau)\|_{H^{-1}(U)}^2 d\tau} \sqrt{\int_0^T \|\mathbf{u}(\tau)\|_{H_0^1(U)}^2 d\tau} \\ & = \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \|\mathbf{v}\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{v}'\|_{L^2(0,T;H^{-1}(U))} \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} \\ & < \infty. \end{aligned}$$

We have shown that $\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)}$ is Lebesgue integrable.

By Royden-Fitzpatrick's Theorem, we have that $t \mapsto \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)}$ is absolutely continuous, and for a.e. $t \in [0, T]$,

$$\frac{d}{dt} \langle \tilde{\mathbf{u}}(t), \tilde{\mathbf{v}}(t) \rangle_{L^2(U)} = \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(t) | \mathbf{u}(t) \rangle_{H^{-1}(U), H_0^1(U)}.$$

Since $\tilde{\mathbf{u}}(t) = \mathbf{u}(t)$ for a.e. $t \in [0, T]$, we have the result.

3. Take $\mathbf{v} = \mathbf{u}$ in 2.

4. Integrate $\|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2 = \|\tilde{\mathbf{u}}(s)\|_{L^2(U)}^2 + \int_s^t 2\langle \mathbf{u}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau$ for $0 \leq s \leq T$, we have

$$\begin{aligned}
T\|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2 &= \int_0^T \|\mathbf{u}(t)\|_{L^2(U)}^2 ds \\
&= \int_0^T \|\tilde{\mathbf{u}}(s)\|_{L^2(U)}^2 ds + 2 \int_0^T \int_s^t \langle \mathbf{u}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau ds \\
&\leq \int_0^T \|\tilde{\mathbf{u}}(s)\|_{H_0^1(U)}^2 ds + 2 \int_0^T \int_s^t \|\mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}(\tau)\|_{H_0^1(U)} d\tau ds \\
&\leq \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + 2 \int_0^T \int_0^t \|\mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}(\tau)\|_{H_0^1(U)} d\tau ds \\
&= \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + 2T \int_0^T \|\mathbf{u}'(\tau)\|_{H^{-1}(U)} \|\mathbf{u}(\tau)\|_{H_0^1(U)} d\tau \\
&\leq \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + T \int_0^T \|\mathbf{u}'(\tau)\|_{H^{-1}(U)}^2 + \|\mathbf{u}(\tau)\|_{H_0^1(U)}^2 d\tau \\
&= (T+1)\|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + T\|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 \\
\|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2 &\leq \frac{T+1}{T} \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 \\
&\leq C^2 \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + C^2 \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 \\
&\leq C^2 \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))}^2 + C^2 \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}^2 + 2C^2 \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \\
&= \left(C \|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} + C \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \right)^2 \\
\|\tilde{\mathbf{u}}(t)\|_{L^2(U)}^2 &\leq C \left(\|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \right),
\end{aligned}$$

where we take $C^2 := \max(1, \frac{T+1}{T}) > 1$, which is independent of \mathbf{u} and \mathbf{u}' .
Since this holds for any $t \in [0, T]$, we have

$$\sup_{t \in [0, T]} \|\tilde{\mathbf{u}}(t)\|_{L^2(U)} \leq C \left(\|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))} \right).$$

□

4.2 Second Order Parabolic Equations

Definition 4.6. Let $U \subseteq \mathbb{R}^n$ be open and bounded, we define $U_T := U \times (0, T]$ for $T > 0$.

Definition 4.7. An **initial boundary value problem** is: given $f : U_T \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}$, we want to find $u(x, t) : \bar{U}_T \rightarrow \mathbb{R}$, such that

$$\begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases},$$

where

$$Lu := - \sum_{i,j=1}^n \partial_j (a^{ij}(\cdot, t) \partial_i u) + \sum_{i=1}^n b^i(\cdot, t) \partial_i u + c(\cdot, t) u$$

for some $a^{ij}, b^i, c : U_T \rightarrow \mathbb{R}$.

We say that the partial differential operator $\partial_t + L$ is an **symmetric (uniformly) parabolic second order differential operator** if $a^{ij} = a^{ji}$, and $\exists \theta > 0$, such that

$$\sum_{i,j=1}^n a^{ij}(x, t) \xi_i \xi_j \geq \theta \|\xi\|_2^2, \quad \forall (x, t) \in U_T, \xi \in \mathbb{R}^n.$$

Definition 4.8. The **parabolic assumptions** are:

1. $U \subseteq \mathbb{R}^n$ is bounded and open
2. $T > 0$
3. $a^{ij}, b^i, c \in L^\infty(U_T)$
4. $f \in L^2(U_T), g \in L^2(U)$
5. $\partial_t + L$ is a symmetric (uniformly) parabolic second order differential operator.

Definition 4.9. Given a function $u : U_T \rightarrow \mathbb{R}$, we want to consider $\mathbf{u} : t \mapsto u(\cdot, t)$, for any $t \in [0, T]$.

Proposition 4.17. Let $U \subseteq \mathbb{R}^n$ be bounded and open, $T > 0$, then $f \in L^2(U_T) \iff \mathbf{f} \in L^2(0, T; L^2(U))$.

Proof. We have

$$\begin{aligned} \|\mathbf{f}\|_{L^2(0, T; L^2(U))}^2 &= \int_0^T \|\mathbf{f}(t)\|_{L^2(U)}^2 dt \\ &= \int_0^T \int_U |f(x, t)|^2 dx dt \\ &= \|f\|_{L^2(U_T)}^2. \end{aligned}$$

□

Definition 4.10. $\mathbf{u} \in L^2(0, T; H_0^1(U))$, identified with its continuous representative $\tilde{\mathbf{u}} \in C([0, T]; L^2(U))$ as in 4.16.1, with the time weak derivative $\mathbf{u}' \in L^2(0, T; H^{-1}(U))$, is a **weak solution** of the IBVP if

$$\begin{aligned} \forall v \in H_0^1(U), \langle \mathbf{u}'(t)|v \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ a.e. } t \in [0, T], \\ \mathbf{u}(0) &= g, \end{aligned}$$

where bilinear form associated to the above problem is

$$\forall w, v \in H_0^1(U), B[w, v; t] := \int_U \left(\sum_{i,j=1}^n a^{ij}(\cdot, t) \partial_i w \partial_j v + \sum_{i=1}^n b^i(\cdot, t) \partial_i w v + c(\cdot, t) w v \right) dx.$$

4.3 Galerkin Method

Definition 4.11. Let $(w_k)_{k=1}^\infty$ be an orthogonal basis of $H_0^1(U)$, and also an orthonormal basis of $L^2(U)$. For $m \in \mathbb{N}^+$, we define $V_m := \text{Span}(\{w_j\}_{j=1}^m) \subset H_0^1(U)$ be a subspace. A function $\mathbf{u}_m := t \mapsto \sum_{k=1}^m d_m^k(t) w_k$ is a **weak solution of the problem in V_m** if $\forall v \in V_m$,

$$\begin{aligned} \left\langle \sum_{k=1}^m d_m^k{}'(t) w_k, v \right\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T], \\ \langle \mathbf{u}_m(0), v \rangle_{L^2(U)} &= \langle g, v \rangle_{L^2(U)}. \end{aligned}$$

Definition 4.12. We define the **ODE system associated to the problem** to be: $\forall j \in [m]$,

1. $d_m^j : [0, T] \rightarrow \mathbb{R}$ is absolutely continuous.
2. For a.e. $t \in [0, T]$, $d_m^j{}'(t) = -\sum_{k=1}^m e_k^j(t) d_m^k(t) + f^j(t)$
3. $d_m^j(0) = \langle g, w_j \rangle_{L^2(U)}$,

where $e_k^j(t) := B[w_k, w_j; t]$, $f^j(t) := \langle \mathbf{f}(t), w_j \rangle_{L^2(U)}$.

Proposition 4.18. $\mathbf{u}_m(t) = \sum_{k=1}^m d_m^k(t)w_k$ is a weak solution in V_m if and only if \vec{d}_m is a solution to the ODE system.

Proof. Since $(w_k)_{k=1}^\infty$ is an orthonormal basis of V_m in $\langle \cdot, \cdot \rangle_{L^2(U)}$, we have

$$\begin{aligned}
\left\langle \sum_{k=1}^m d_m^k{}'(t)w_k, v \right\rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v \rangle, & \forall v \in V_m \\
&\iff \\
\left\langle \sum_{k=1}^m d_m^k{}'(t)w_k, v \right\rangle_{L^2(U)} + B\left[\sum_{k=1}^m d_m^k(t)w_k, v; t\right] &= \langle \mathbf{f}(t), v \rangle, & \forall v \in V_m \\
&\iff \\
\left\langle \sum_{k=1}^m d_m^k{}'(t)w_k, w_j \right\rangle_{L^2(U)} + B\left[\sum_{k=1}^m d_m^k(t)w_k, w_j; t\right] &= \langle \mathbf{f}(t), w_j \rangle, & \forall j \in [m] \\
&\iff \\
\sum_{k=1}^m d_m^k{}'(t)\langle w_k, w_j \rangle_{L^2(U)} + \sum_{k=1}^m d_m^k(t)B[w_k, w_j; t] &= \langle \mathbf{f}(t), w_j \rangle, & \forall j \in [m] \\
&\iff \\
\sum_{k=1}^m d_m^k{}'(t)\delta_k^j + \sum_{k=1}^m d_m^k(t)e_k^j(t) &= f^j(t), & \forall j \in [m] \\
&\iff \\
d_m^j{}'(t) + \sum_{k=1}^m d_m^k(t)e_k^j(t) &= f^j(t), & \forall j \in [m].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \mathbf{u}_m(0), v \rangle_{L^2(U)} &= \langle g, v \rangle_{L^2(U)}, & v \in V_m \\
&\iff \\
\langle \mathbf{u}_m(0), w_j \rangle_{L^2(U)} &= \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
&\iff \\
\left\langle \sum_{k=1}^m d_m^k(0)w_k, w_j \right\rangle_{L^2(U)} &= \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
&\iff \\
\sum_{k=1}^m d_m^k(0)\langle w_k, w_j \rangle_{L^2(U)} &= \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
&\iff \\
\sum_{k=1}^m d_m^k(0)\delta_k^j &= \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m] \\
&\iff \\
d_m^j(0) &= \langle g, w_j \rangle_{L^2(U)}, & \forall j \in [m].
\end{aligned}$$

□

Theorem 4.19. Since f^i, e_j^k are locally integrable, there is a unique absolutely continuous solution \vec{d}_m to the ODE system.

Corollary 4.20. For each $m \in \mathbb{N}^+$, there is a unique weak solution \mathbf{u}_m of the form $t \mapsto \sum_{k=1}^m d_m^k(t)w_k$ of the problem in V_m .

Proposition 4.21. The weak solution \mathbf{u}_m also satisfies $\forall v \in V_m$,

$$\begin{aligned} \langle \mathbf{u}_m'(t), v \rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T], \\ \langle \mathbf{u}_m(0), v \rangle_{L^2(U)} &= \langle g, v \rangle_{L^2(U)}. \end{aligned}$$

In addition,

$$\|\mathbf{u}_m(0)\|_{L^2(U)} \leq \|g\|_{L^2(U)},$$

and

$$\lim_{m \rightarrow \infty} \mathbf{u}_m(0) = g$$

in $L^2(U)$.

Proof. Consider any $\phi \in C_c^\infty(0, T)$. By linearity of the Bochner integral, we have

$$\begin{aligned} \int_0^T \phi'(t) \mathbf{u}_m(t) dt &= \int_0^T \left(\phi'(t) \sum_{k=1}^m d_m^k(t) w_k \right) dt \\ &= \sum_{k=1}^m \left(\int_0^T \phi'(t) d_m^k(t) w_k dt \right) \\ &= \sum_{k=1}^m \left(\int_0^T \phi'(t) d_m^k(t) dt \right) w_k \\ &= \sum_{k=1}^m \left(\int_0^T \phi(t) d_m^{k'}(t) dt \right) w_k \\ &= \int_0^T \phi(t) \left(\sum_{k=1}^m d_m^{k'}(t) w_k \right) dt. \end{aligned}$$

Thus it has a weak derivative $\mathbf{u}_m'(t) = \sum_{k=1}^m d_m^{k'}(t) w_k$.

We then plug that in the definition of weak solution of the problem in V_m .

On the other hand, since $\mathbf{u}_m(0) \in V_m$, we have that

$$\begin{aligned} \|\mathbf{u}_m(0)\|_{L^2(U)}^2 &= \langle \mathbf{u}_m(0), \mathbf{u}_m(0) \rangle_{L^2(U)} \\ &= \langle g, \mathbf{u}_m(0) \rangle_{L^2(U)} \\ &\leq \|g\|_{L^2(U)} \|\mathbf{u}_m(0)\|_{L^2(U)}. \end{aligned}$$

Since $\|\mathbf{u}_m(0)\|_{L^2(U)} \geq 0$, we have $\|\mathbf{u}_m(0)\|_{L^2(U)} \leq \|g\|_{L^2(U)}$.

Also, since $(w_k)_{k=1}^\infty$ is an orthonormal basis of $L^2(U)$, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbf{u}_m(0) &= \lim_{m \rightarrow \infty} \sum_{k=1}^m d_m^k(t) w_k \\ &= \lim_{m \rightarrow \infty} \sum_{k=1}^m \langle g, w_j \rangle_{L^2(U)} w_k \\ &= g. \end{aligned}$$

□

Proposition 4.22. The weak solution $\mathbf{u}_m \in L^2(0, T; H_0^1(U))$.

Proof.

$$\begin{aligned}
\int_0^T \|\mathbf{u}_m\|_{H_0^1(U)}^2 dt &= \int_0^T \left\| \sum_{k=1}^m d_m^k(t) w_k \right\|_{H_0^1(U)}^2 dt \\
&= \int_0^T \left\langle \sum_{k=1}^m d_m^k(t) w_k, \sum_{j=1}^m d_m^j(t) w_j \right\rangle_{H_0^1(U)} dt \\
&= \int_0^T \sum_{k=1}^m \sum_{j=1}^m d_m^k(t) d_m^j(t) \langle w_k, w_j \rangle_{H_0^1(U)} dt \\
&= \int_0^T \sum_{k=1}^m d_m^k{}^2(t) \|w_k\|_{H_0^1(U)}^2 dt \\
&= \sum_{k=1}^m \|w_k\|_{H_0^1(U)}^2 \int_0^T d_m^k{}^2(t) dt \\
&= \sum_{k=1}^m \|w_k\|_{H_0^1(U)}^2 \|d_m^k\|_{L^2(0,T)}^2 \\
&\leq \infty,
\end{aligned}$$

since each $w_k \in H_0^1(U)$, and each d_m^k is absolutely continuous (thus continuous, thus in $L^2(0, T)$). \square

Proposition 4.23. *The weak solution $\mathbf{u}_m \in C([0, T]; L^2(U))$.*

Proof. Given any $\epsilon > 0$.

For any $1 \leq k \leq m$, since d_m^k is absolutely continuous (thus continuous), we can find $\delta_k > 0$, such that

$$\forall s, t \in [0, T], |t - s| < \delta_k \implies |d_m^k(t) - d_m^k(s)| < \frac{\epsilon}{\sqrt{m}}.$$

Now take $\delta := \min_{k \in [m]} \delta_k > 0$, we have that $\forall s, t \in [0, T]$, such that $|t - s| < \delta$,

$$\begin{aligned}
\|\mathbf{u}_m(t) - \mathbf{u}_m(s)\|_{L^2(U)}^2 &= \left\| \sum_{k=1}^m d_m^k{}'(t) w_k - \sum_{k=1}^m d_m^k{}'(s) w_k \right\|_{L^2(U)}^2 \\
&= \left\| \sum_{k=1}^m (d_m^k{}'(t) - d_m^k{}'(s)) w_k \right\|_{L^2(U)}^2 \\
&= \sum_{k=1}^m (d_m^k{}'(t) - d_m^k{}'(s))^2 \\
&< \sum_{k=1}^m \frac{\epsilon^2}{m} \\
&= \epsilon^2,
\end{aligned}$$

since $(w_k)_{k=1}^\infty$ is an orthonormal basis for $L^2(U)$.

Thus $\|\mathbf{u}_m(t) - \mathbf{u}_m(s)\|_{L^2(U)} < \epsilon$ and since this works for any $\epsilon > 0$, we have that $\mathbf{u}_m \in C([0, T]; L^2(U))$. \square

Notice that the above two propositions says that \mathbf{u}_m is the representative \tilde{u}_m as in 4.16, and we thus have the following result:

Corollary 4.24. *The weak solution \mathbf{u}_m satisfies that the mapping $t \mapsto \|\mathbf{u}_m(t)\|_{L^2(U)}^2$ is absolutely continuous, and for a.e. $t \in [0, T]$, we have*

$$\frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 = 2 \langle \mathbf{u}_m'(t), \mathbf{u}_m(t) \rangle_{H^{-1}(U), H_0^1(U)} = 2 \langle \mathbf{u}_m'(t), \mathbf{u}_m(t) \rangle_{L^2(U)}.$$

Theorem 4.25 (Gronwall's inequality). *Let $\eta : [0, T] \rightarrow \mathbb{R}$ be nonnegative and absolutely continuous, ϕ, ψ both nonnegative summable functions. If*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t) \text{ a.e. } t \in [0, T],$$

then

$$\eta(t) \leq \exp\left(\int_0^t \phi(s)ds\right)\left(\eta(0) + \int_0^t \psi(s)ds\right), \quad \forall t \in [0, T].$$

Theorem 4.26 (Energy Estimate). *Let $U \subseteq \mathbb{R}^n$ be bounded and open, $T > 0$, $a^{ij}, b^i, c \in L^\infty(U_T)$, and $\partial_t + L$ be a symmetric (uniformly) parabolic second order differential operator. There exists $C > 0$ that only depends on U, T, L , such that $\forall f \in L^2(U_T)$, $g \in L^2(U)$, $m \in \mathbb{N}^+$,*

$$\sup_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(U)} + \|\mathbf{u}_m\|_{L^2(0, T; H_0^1(U))} + \|\mathbf{u}_m'\|_{L^2(0, T; H^{-1}(U))} \leq C\left(\|\mathbf{f}\|_{L^2(0, T; L^2(U))} + \|g\|_{L^2(U)}\right),$$

where \mathbf{u}_m are the weak solutions in V_m as in above.

Proof. We will bound each term on the left hand side.

1. Consider any $m \in \mathbb{N}^+$, we have that the \mathbf{u}_m satisfies $\forall v \in V_m$,

$$\begin{aligned} \langle \mathbf{u}_m'(t), v \rangle_{L^2(U)} + B[\mathbf{u}_m(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \text{ for a.e. } t \in [0, T], \\ \langle \mathbf{u}_m(0), v \rangle_{L^2(U)} &= \langle g, v \rangle_{L^2(U)}, \end{aligned}$$

In particular, $\mathbf{u}_m(t) = \sum_{k=1}^m d_m^k(t) w_k \in V_m$.

Thus for a.e. $t \in [0, T]$, we have

$$\langle \mathbf{u}_m'(t), \mathbf{u}_m(t) \rangle_{L^2(U)} + B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] = \langle \mathbf{f}(t), \mathbf{u}_m(t) \rangle_{L^2(U)}.$$

By a similar proof as in 3.4, there exists constants $\beta > 0, \gamma \geq 0$ that only depends on U and the coefficients of L , such that $\forall u \in H_0^1(U)$, and for a.e. $t \in [0, T]$,

$$\beta \|u\|_{H^1(U)}^2 \leq B[u, u; t] + \gamma \|u\|_{L^2(U)}^2.$$

We thus have

$$\begin{aligned} \langle \mathbf{u}_m'(t), \mathbf{u}_m(t) \rangle_{L^2(U)} + B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] &\geq \langle \mathbf{u}_m'(t), \mathbf{u}_m(t) \rangle_{L^2(U)} + \beta \|\mathbf{u}_m(t)\|_{H^1(U)}^2 - \gamma \|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ &= \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + \beta \|\mathbf{u}_m(t)\|_{H^1(U)}^2 - \gamma \|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ \langle \mathbf{f}(t), \mathbf{u}_m(t) \rangle_{L^2(U)} &\leq \|\mathbf{f}(t)\|_{L^2(U)} \|\mathbf{u}_m(t)\|_{L^2(U)} \\ &\leq \frac{1}{2} \|\mathbf{f}(t)\|_{L^2(U)}^2 + \frac{1}{2} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + 2\beta \|\mathbf{u}_m(t)\|_{H^1(U)}^2 &\leq \|\mathbf{f}(t)\|_{L^2(U)}^2 + \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + 2\gamma \|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\ &= \|\mathbf{f}(t)\|_{L^2(U)}^2 + (1 + 2\gamma) \|\mathbf{u}_m(t)\|_{L^2(U)}^2. \end{aligned}$$

Notice that this inequality hold for any $f \in L^2(U_T)$, $g \in L^2(U)$, $m \in \mathbb{N}^+$, and the corresponding weak solutions \mathbf{u}_m in V_m .

2. Since $2\beta \|\mathbf{u}_m(t)\|_{H^1(U)}^2 \geq 0$, we have

$$\frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 \leq \|\mathbf{f}(t)\|_{L^2(U)}^2 + (1 + 2\gamma) \|\mathbf{u}_m(t)\|_{L^2(U)}^2.$$

Take $\eta(t) := \|\mathbf{u}_m(t)\|_{L^2(U)}^2$, which is nonnegative and absolutely continuous.

Also, take $\psi(t) := \|\mathbf{f}(t)\|_{L^2(U)}^2$, $\phi(t) := 1 + 2\gamma$, which are both nonnegative and summable.

By Gronwall's inequality, we have that $\forall t \in [0, T]$,

$$\begin{aligned} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 &\leq \exp\left(\int_0^t (1 + 2\gamma)ds\right) \left(\|\mathbf{u}_m(0)\|_{L^2(U)}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2(U)}^2 ds\right) \\ &= \exp(t(1 + 2\gamma)) \left(\|\mathbf{u}_m(0)\|_{L^2(U)}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2(U)}^2 ds\right) \\ &\leq \exp(T(1 + 2\gamma)) \left(\|g\|_{L^2(U)}^2 + \int_0^T \|\mathbf{f}(s)\|_{L^2(U)}^2 ds\right) \\ &= C_1 \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2\right) \\ &\leq C_1 \left(\|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}\right)^2, \end{aligned}$$

where we take $C_1 := \exp(T(1 + 2\gamma)) > 0$ that only depends on T, γ .

We thus have shown that $\forall t \in [0, T]$, $\|\mathbf{u}_m(t)\|_{L^2(U)} \leq \sqrt{C_1} \left(\|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}\right)$.

Thus,

$$\sup_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(U)} \leq \sqrt{C_1} \left(\|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}\right),$$

which bounds the first term in what we want.

3. From step 1, we have that for a.e. $t \in [0, T]$,

$$\frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 + 2\beta \|\mathbf{u}_m(t)\|_{H^1(U)}^2 \leq \|\mathbf{f}(t)\|_{L^2(U)}^2 + (1 + 2\gamma) \|\mathbf{u}_m(t)\|_{L^2(U)}^2.$$

Integrating over $[0, T]$ gives

$$\begin{aligned} \int_0^T \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt + 2\beta \int_0^T \|\mathbf{u}_m(t)\|_{H^1(U)}^2 dt &\leq \int_0^T \|\mathbf{f}(t)\|_{L^2(U)}^2 dt + (2\gamma + 1) \int_0^T \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt \\ \|\mathbf{u}_m(T)\|_{L^2(U)}^2 - \|\mathbf{u}_m(0)\|_{L^2(U)}^2 + 2\beta \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 &\leq \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma + 1) \int_0^T \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt. \end{aligned}$$

Notice that $\|\mathbf{u}_m(T)\|_{L^2(U)}^2 \geq 0$, and in step 2, we have shown that for a.e. $t \in [0, T]$,

$$\|\mathbf{u}_m(t)\|_{L^2(U)}^2 \leq C_1 \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2\right).$$

$$\begin{aligned} 2\beta \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 &\leq \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma + 1) \int_0^T \|\mathbf{u}_m(t)\|_{L^2(U)}^2 dt + \|\mathbf{u}_m(0)\|_{L^2(U)}^2 \\ 2\beta \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 &\leq \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma + 1) \int_0^T C_1 \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2\right) dt + \|g\|_{L^2(U)}^2 \\ &= \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + (2\gamma + 1)TC_1 \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2\right) + \|g\|_{L^2(U)}^2 \\ &= ((2\gamma + 1)TC_1 + 1) \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2\right) \\ \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 &\leq C_2 \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2\right) \\ &\leq C_2 \left(\|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}\right)^2, \end{aligned}$$

where $C_2 := \frac{(2\gamma+1)TC_1+1}{\beta} > 0$ only depends on β, γ, T .

We thus have bounded the second term

$$\|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))} \leq \sqrt{C_2} \left(\|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}\right).$$

4. Now fix any function $v \in H_0^1(U)$ with $\|v\|_{H_0^1(U)} = 1$, write $v = \sum_{j=1}^{\infty} \hat{v}_j w_j = v_1 + v_2$, where $v_1 := \sum_{j=1}^m \hat{v}_j w_j \in V_m$, $v_2 := \sum_{j=m+1}^{\infty} \hat{v}_j w_j \in V_m^\perp$ is the unique decomposition of v in $H_0^1(U) = V_m \oplus V_m^\perp$. Notice that $\|v\|_{H^1(U)}^2 = \|v_1\|_{H^1(U)}^2 + \|v_2\|_{H^1(U)}^2$.

Thus $\|v_1\|_{H^1(U)} \leq 1$.

Since $v_1 \in V_m$, we have for a.e. $t \in [0, T]$, $\langle \mathbf{u}'_m(t), v_1 \rangle_{L^2(U)} + B[\mathbf{u}_m(t), v_1; t] = \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)}$.

Now since $\mathbf{u}'_m(t) \in V_m$, we have $\langle \mathbf{u}'_m(t), v_2 \rangle_{L^2(U)} = 0$, so

$$\begin{aligned} \langle \mathbf{u}'_m(t) | v \rangle_{H^{-1}(U), H_0^1(U)} &= \langle \mathbf{u}'_m(t), v \rangle_{L^2(U)} \\ &= \langle \mathbf{u}'_m(t), v_1 \rangle_{L^2(U)} + \langle \mathbf{u}'_m(t), v_2 \rangle_{L^2(U)} \\ &= \langle \mathbf{u}'_m(t), v_1 \rangle_{L^2(U)} \\ &= \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)} - B[\mathbf{u}_m(t), v_1; t]. \end{aligned}$$

Again by a similar proof as in 3.4, there exists constants $\alpha > 0$ that only depends on U and the coefficients of L , such that $\forall u, v \in H_0^1(U)$, and for a.e. $t \in [0, T]$,

$$|B[u, v; t]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}.$$

We thus have

$$\begin{aligned} \langle \mathbf{u}'_m(t) | v \rangle_{H^{-1}(U), H_0^1(U)} &= \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)} - B[\mathbf{u}_m(t), v_1; t] \\ &\leq \|\mathbf{f}(t)\|_{L^2(U)} \|v_1\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \|v_1\|_{H^1(U)} \\ &\leq \|\mathbf{f}(t)\|_{L^2(U)} \|v_1\|_{H^1(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \|v_1\|_{H^1(U)} \\ &\leq \|\mathbf{f}(t)\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)}. \end{aligned}$$

Since this holds for any $v \in H_0^1(U)$ with $\|v\|_{H_0^1(U)} = 1$, we have

$$\|\mathbf{u}'_m(t)\|_{H^{-1}(U)} = \sup_{v \in H_0^1(U) \text{ such that } \|v\|_{H_0^1(U)}=1} \langle \mathbf{u}'_m(t) | v \rangle_{H^{-1}(U), H_0^1(U)} \leq \|\mathbf{f}(t)\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)}.$$

Squaring and integrating this over $[0, T]$, we have

$$\begin{aligned} \int_0^T \|\mathbf{u}'_m(t)\|_{H^{-1}(U)}^2 dt &\leq \int_0^T \left(\|\mathbf{f}(t)\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \right)^2 dt \\ \|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))}^2 &\leq \int_0^T 2 \left(\|\mathbf{f}(t)\|_{L^2(U)}^2 + \alpha^2 \|\mathbf{u}_m(t)\|_{H^1(U)}^2 \right) dt \\ &\leq 2 \int_0^T \|\mathbf{f}(t)\|_{L^2(U)}^2 dt + 2\alpha^2 \int_0^T \|\mathbf{u}_m(t)\|_{H^1(U)}^2 dt \\ &= 2\|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + 2\alpha^2 \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2 \\ &\leq 2\|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 + 2\alpha^2 C_2 \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) \\ &\leq C_3 \left(\|g\|_{L^2(U)}^2 + \|\mathbf{f}\|_{L^2(0,T;L^2(U))}^2 \right) \\ &\leq C_3 \left(\|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right)^2, \end{aligned}$$

where we take $C_3 := 2\alpha^2 C_2 + 1 > 0$ that only depends on β, γ, α, T .

We thus have bounded the third term $\|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))} \leq \sqrt{C_3} \left(\|g\|_{L^2(U)} + \|\mathbf{f}\|_{L^2(0,T;L^2(U))} \right)$.

Now let us take $C := \sqrt{\max\{C_1, C_2, C_3\}} > 0$, which only depends on U, T, L , we have that

$$\sup_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(U)} + \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))} \leq C \left(\|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)} \right).$$

□

Theorem 4.27. *There is a weak solution to the IBVP, namely, $\exists \mathbf{u} \in L^2(0, T; H_0^1(U))$, identified with its continuous representative $\tilde{\mathbf{u}} \in C([0, T]; L^2(U))$, such that*

$$\begin{aligned} \langle \mathbf{u}'(t) | v \rangle + B[\mathbf{u}(t), v; t] &= \langle \mathbf{f}(t), v \rangle_{L^2(U)}, \quad \forall v \in H_0^1(U), \quad \text{a.e. } t \in [0, T] \\ \mathbf{u}(0) &= g \end{aligned}$$

Proof. By energy estimate, we have that $(\mathbf{u}_m)_{m=1}^\infty$ is bounded in $L^2(0, T; H_0^1(U))$.

By 1.28, there is a subsequence $(\mathbf{u}_{m_j})_{j=1}^\infty$ and $\mathbf{u} \in L^2(0, T; H_0^1(U))$ such that $\mathbf{u}_{m_j} \rightharpoonup \mathbf{u}$. WLOG, we will consider its continuous representative $\mathbf{u} \in C(0, T; L^2(U))$ by 4.16.1.

Similarly, $(\mathbf{u}'_m)_{m=1}^\infty$ is bounded in $L^2(0, T; H^{-1}(U))$, so is $(\mathbf{u}'_{m_j})_{j=1}^\infty$. Thus there is a subsequence $(\mathbf{u}'_{m_{j_l}})_{l=1}^\infty$ and $\mathbf{w} \in L^2(0, T; H^{-1}(U))$ such that $\mathbf{u}'_{m_{j_l}} \rightharpoonup \mathbf{w}$.

Since $\mathbf{u}_{m_j} \rightharpoonup \mathbf{u}$, we must have $\mathbf{u}_{m_{j_l}} \rightharpoonup \mathbf{u}$ as well. By A5Q3, $\mathbf{w} = \mathbf{u}'$.

Now we would like to show that \mathbf{u} is indeed a weak solution to the IBVP.

Consider $\mathbf{v} \in C^1([0, T]; H_0^1(U))$ of the form $\mathbf{v}(t) = \sum_{k=1}^N d^k(t) w_k$, where $N > 0$ is an integer, $(d^k(t))_{k=1}^N$ are smooth functions, and $(w_k)_{k=1}^\infty$ be a basis as before.

We can show that these \mathbf{v} are dense in $L^2(0, T; H_0^1(U))$.

For any such \mathbf{v} , if we choose any $m \geq N$, we have the weak solution $\tilde{\mathbf{u}}_m$ in V_m satisfies

$$\langle \mathbf{u}'_m(t) | w_k \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}_m(t), w_k; t] = \langle \mathbf{f}(t), w_k \rangle_{L^2(U)}, \quad \forall k \in [m], \quad \text{for a.e. } t \in [0, T].$$

Multiplying by $d^k(t)$ and summing over $k \in [N]$, we have that

$$\langle \mathbf{u}'_m(t) | \mathbf{v}(t) \rangle + B[\mathbf{u}_m(t), \mathbf{v}(t); t] = \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)}, \quad \text{for a.e. } t \in [0, T].$$

Integrating over $t \in [0, T]$, we have

$$\int_0^T \langle \mathbf{u}'_m(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}_m(t), \mathbf{v}(t); t] dt = \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt.$$

Since $\mathbf{v} \in C^1([0, T]; H_0^1(U)) \subset L^2(0, T; H_0^1(U)) \cong (L^2(0, T; H^{-1}(U)))^*$, and $\mathbf{u}'_{m_{j_l}} \rightharpoonup \mathbf{v}$, we have

$$\int_0^T \langle \mathbf{u}'_{m_{j_l}}(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt \rightarrow \int_0^T \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt.$$

Also, if we consider the operator $T_v : \mathbf{w} \mapsto \int_0^T B[\mathbf{w}(t), \mathbf{v}(t); t] dt$ for any $\mathbf{w} \in L^2(0, T; H_0^1(U))$, we can see that

$$\begin{aligned} |T_v \mathbf{w}| &= \left| \int_0^T B[\mathbf{w}_m(t), \mathbf{v}(t); t] dt \right| \\ &\leq \int_0^T |B[\mathbf{w}_m(t), \mathbf{v}(t); t]| dt \\ &\leq \int_0^T \alpha \|\mathbf{w}_m(t)\|_{H^1(U)} \|\mathbf{v}(t)\|_{H^1(U)} dt \\ &\leq \alpha \left(\int_0^T \|\mathbf{w}_m(t)\|_{H^1(U)}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|\mathbf{v}(t)\|_{H^1(U)}^2 dt \right)^{\frac{1}{2}} \\ &= \alpha \|\mathbf{w}_m\|_{L^2(H^1(U))} \|\mathbf{v}\|_{L^2(H^1(U))}. \end{aligned}$$

For some $\alpha > 0$ that only depends on U, L .

Thus $\|T_v\|_{(L^2(0, T; H_0^1(U)))^*} \leq \alpha \|\mathbf{v}\|_{L^2(H^1(U))}$, so $T_v \in (L^2(0, T; H_0^1(U)))^*$.

Since $\mathbf{u}_{m_{j_l}} \rightharpoonup \mathbf{u}$, we have that

$$\int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \rightarrow \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt.$$

We now have

$$\begin{aligned}
\int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt &= \lim_{l \rightarrow \infty} \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt \\
&= \lim_{l \rightarrow \infty} \left(\int_0^T \langle \mathbf{u}'_{m_{j_l}}(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \right) \\
&= \lim_{l \rightarrow \infty} \int_0^T \langle \mathbf{u}'_{m_{j_l}}(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \lim_{l \rightarrow \infty} \int_0^T B[\mathbf{u}_{m_{j_l}}(t), \mathbf{v}(t); t] dt \\
&= \int_0^T \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt
\end{aligned}$$

Since such \mathbf{v} are dense in $L^2(0, T; H_0^1(U))$ and both sides of the above equality are continuous, we can extend it so that $\forall \mathbf{v} \in L^2(0, T; H_0^1(U))$,

$$\int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{L^2(U)} dt = \int_0^T \langle \mathbf{u}'(t) | \mathbf{v}(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), \mathbf{v}(t); t] dt.$$

Now consider any $v \in H_0^1(U)$ and any $\phi \in C_c^\infty(0, T)$, we always have $v\phi \in C(0, T; H_0^1(U)) \subset L^2(0, T; H_0^1(U))$. Thus we have

$$\begin{aligned}
\int_0^T \langle \mathbf{f}(t), v\phi(t) \rangle_{L^2(U)} dt &= \int_0^T \langle \mathbf{u}'(t) | v\phi(t) \rangle_{H^{-1}(U), H_0^1(U)} dt + \int_0^T B[\mathbf{u}(t), v\phi(t); t] dt \\
\int_0^T \phi(t) \langle \mathbf{f}(t), v \rangle_{L^2(U)} dt &= \int_0^T \phi(t) \left(\langle \mathbf{u}'(t) | v \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t] \right) dt
\end{aligned}$$

Since this works for all $\phi \in C_c^\infty(0, T)$, we must have for a.e. $t \in [0, T]$,

$$\langle \mathbf{f}(t), v \rangle_{L^2(U)} = \langle \mathbf{u}'(t) | v \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}(t), v; t].$$

Now consider any $\mathbf{v} \in C^1(0, T; H_0^1(U))$ such that $\mathbf{v}(T) = 0$.

By IBP 4.16.2, we have

$$\begin{aligned}
0 &= \langle \mathbf{u}(T), \mathbf{v}(T) \rangle_{L^2(U)} \\
&= \langle \mathbf{u}(0), \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left(\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau \\
0 &= \langle \mathbf{u}_{m_{j_l}}(T), \mathbf{v}(T) \rangle_{L^2(U)} \\
&= \langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left(\langle \mathbf{u}'_{m_{j_l}}(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}_{m_{j_l}}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau.
\end{aligned}$$

Since $\mathbf{v} \in L^2(0, T; H_0^1(U)) \cong L^2(0, T; H^{-1}(U))^*$ and $\mathbf{u}'_{m_{j_l}} \rightharpoonup \mathbf{u}'$ in $L^2(0, T; H^{-1}(U))$, we have

$$\lim_{l \rightarrow \infty} \int_0^T \langle \mathbf{u}'_{m_{j_l}}(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau = \int_0^T \langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Similarly,

$$\lim_{l \rightarrow \infty} \int_0^T \langle \mathbf{v}'(\tau) | \mathbf{u}_{m_{j_l}}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau = \int_0^T \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} d\tau.$$

Also, since $\lim_{m \rightarrow \infty} \mathbf{u}_m(0) = g$ in $L^2(U)$, we have that the subsequence $\lim_{l \rightarrow \infty} \mathbf{u}_{m_{j_l}}(0) = g$ as well.

Thus

$$\lim_{l \rightarrow \infty} \langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \rangle_{L^2(U)} = \langle \mathbf{u}(T), \mathbf{v}(T) \rangle_{L^2(U)}.$$

Thus we have

$$\begin{aligned}
0 &= \langle \mathbf{u}(0), \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left(\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau \\
&= \lim_{l \rightarrow \infty} \left(\langle \mathbf{u}_{m_{j_l}}(0), \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left(\langle \mathbf{u}'_{m_{j_l}}(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}_{m_{j_l}}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau \right) \\
&= \langle g, \mathbf{v}(0) \rangle_{L^2(U)} + \int_0^T \left(\langle \mathbf{u}'(\tau) | \mathbf{v}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} + \langle \mathbf{v}'(\tau) | \mathbf{u}(\tau) \rangle_{H^{-1}(U), H_0^1(U)} \right) d\tau.
\end{aligned}$$

We thus have $\langle \mathbf{u}(0), \mathbf{v}(0) \rangle_{L^2(U)} = \langle g, \mathbf{v}(0) \rangle_{L^2(U)}$ for any such \mathbf{v} .

Notice that for any $v \in H_0^1(U)$, we can simply let $\mathbf{v}(t) := \frac{T-t}{T}v$, which satisfies the requirement, and $\mathbf{v}(0) = v$. Thus $\langle \mathbf{u}(0), v \rangle_{L^2(U)} = \langle g, v \rangle_{L^2(U)}$ for any $v \in H_0^1(U)$.

Since $H_0^1(U)$ is dense in $L^2(U)$, we have that $\langle \mathbf{u}(0), v \rangle_{L^2(U)} = \langle g, v \rangle_{L^2(U)}$ for any $v \in L^2(U)$.

This proves

$$\mathbf{u}(0) = g.$$

□

Theorem 4.28. *A weak solution to our IBVP is unique.*

Proof. Assume $\mathbf{u}_1, \mathbf{u}_2$ are both weak solutions to our IBVP.

Then $\forall v \in H_0^1(U)$, for a.e. $t \in [0, T]$,

$$\langle \mathbf{u}'_1(t) | v \rangle + B[\mathbf{u}_1, v; t] = \langle \mathbf{u}'_2(t) | v \rangle + B[\mathbf{u}_2, v; t] = \langle \mathbf{f}(t), v \rangle_{L^2(U)},$$

and

$$\mathbf{u}_1(0) = g = \mathbf{u}_2(0).$$

Let $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$, we have that

$$\langle \mathbf{u}'(t) | v \rangle + B[\mathbf{u}, v; t] = 0, \quad \forall v \in H_0^1(U), \text{ for a.e. } t \in [0, T],$$

and

$$\mathbf{u}(0) = 0.$$

Choosing $v = \mathbf{u}(t) \in H_0^1(U)$, we have that

$$\langle \mathbf{u}'(t) | \mathbf{u}(t) \rangle + B[\mathbf{u}(t), \mathbf{u}(t); t] = 0.$$

By 4.16.3, we have that for a.e. $t \in [0, T]$,

$$\begin{aligned}
\frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(U)}^2 &= 2\langle \mathbf{u}'(t) | \mathbf{u}(t) \rangle \\
&= -2B[\mathbf{u}(t), \mathbf{u}(t); t] \\
&\leq 2\gamma \|\mathbf{u}(t)\|_{L^2(U)}^2 - 2\beta \|\mathbf{u}(t)\|_{H^1(U)}^2 \\
&\leq 2\gamma \|\mathbf{u}(t)\|_{L^2(U)}^2,
\end{aligned}$$

where $\gamma \geq 0, \beta > 0$ are constants similar in 3.4.

Take $\eta(t) := \|\mathbf{u}(t)\|_{L^2(U)}^2$, by Gronwall's inequality, we have that $\forall t \in [0, T]$,

$$\|\mathbf{u}(t)\|_{L^2(U)}^2 \leq \exp(2t) \|\mathbf{u}(0)\|_{L^2(U)}^2 = 0.$$

Thus $\mathbf{u}(t) = 0$ and so $\mathbf{u}_1 = \mathbf{u}_2$ is unique.

□