Amath753 Advanced PDEs

Yixing Gu

June 18, 2025

This note is based on the book "Partial Differential Equations" by Lawrence C. Evans and AMATH752 taught by Prof. Sander Rhebergen in Winter 2025.

Contents

1 Preliminaries

See more in Section 2.9 of AMATH731-Functional Analysis Notes from Prof. Giang Tran, and my PMATH651-Measure Theory Notes.

1.1 Introduction

Definition 1.1. We will use the following notations:

- C means a positive constant.

- $U \subset \mathbb{R}^n$ is open.

- If $u : U \to \mathbb{R}$ is a function, we write $u(x) := u(x^1, \ldots, x^n)$ for $x = (x^1, \ldots, x^n) \in U$.

- A function u is **smooth** if $u \in C^\infty(U)$.

- For $1 \le i \le n$, we write $\partial_i u := u_{x^i} := u_i := D_i u := \frac{\partial}{\partial x^i} u := \frac{\partial u}{\partial x^i}$.

- Let $\alpha = (\alpha_1 \ldots, \alpha_n) \in \mathbb{N}^n$, we let $|\alpha| := \sum_{i=1}^n \alpha_i$, and

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n}} = \partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n} u.$$

- If $k \in \mathbb{N}$, we let $D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}$

- When $k = 1$, we write $Du := D_x u := (u_{x^1}, \ldots, u_{x^n})^T = \nabla u$ to be the **gradient**.

- When $k = 2$, we write $D^2 u := \begin{pmatrix} u_{x^1,x^1} & \cdots & u_{x^1,x^n} \\ \vdots & & \vdots \\ u_{x^n,x^1} & \cdots & u_{x^n,x^n} \end{pmatrix}$ to be the **Hessian** matrix.

- $\Delta u := \sum_{i=1}^n u_{x^i,x^i} = div \ Du = tr(D^2 u)$ is the **Laplacian** of u.

Example 1.1.1. Consider a body $U \subset \mathbb{R}^3$ and let $U_0 \subseteq U$ with boundary ∂U_0, which does not change over time.

The Conservation of Energy states that the rate of change of total energy in U_0 is the inflow of heat through the boundaries plus heat produced by the source in U_0.

Let $e(x, t) \in \mathbb{R}$ be the density of internal energy, then the total energy is $\int_{U_0} e dx$.

Let $j(x, t) \in \mathbb{R}^3$ be the heat flux (vector pointing at the direction that heat is flowing).

Let n denote the exterior unit normal on ∂U_0.

The net outflow of the heat through ∂U_0 is $\int_{\partial U_0} j \cdot n ds$.

Let $p(x, t) \in \mathbb{R}$ be the power density of the source. Heat production in U_0 is $\int_{U_0} p dx$.

Thus we have

$$\frac{d}{dx} \int_{U_0} e dx = -\int_{\partial U_0} j \cdot n ds + \int_{U_0} p dx.$$

By divergence theorem, we have $\int_{\partial U_0} j \cdot n ds = \int_{U_0} div \ j dx$.

Thus we have

$$\int_{U_0} (\partial_t e + div \ j - p) dx = 0.$$

Since U_0 is arbitrary, we must have

$$\partial_t e + div \ j - p = 0.$$

Assume that e depends linearly on temperature T as $e = e_0 + \sigma u$, where e_0 is a constant reference internal energy, and $u = T - T_0$, where T_0 is a constant reference temperature, and σ is the specific heat capacity.

A generalized form of Fourier's law states that:

- Heat flow is proportional to the temperature gradient.

- Heat is transformed by convection with heat flux be, where $b(x,t) \in \mathbb{R}^3$ is a given convection velocity.

Namely, $j = -aDu + be$, where $a(x)$ is a known heat conductivity.
Thus we have

$$\sigma \partial_t u + div(b\sigma u) - div(aDu) = p - div(be_0).$$

Definition 1.2. We consider the operator

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij} u_{x^i} \right)_{x^j} + \sum_{i=1}^{n} b^i u_{x^i} + cu,$$

for given coefficients a^{ij}, b^i, c.

- The second-order elliptic boundary-value problems are $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

- The second-order parabolic boundary-value problems are $\begin{cases} u_t + Lu = f & x \in U, t \in (0,T] \\ u = 0 & \text{on } \partial U, t \in (0,T] \\ u = u_0 & \text{on } \partial U, t = 0 \end{cases}$

Example 1.1.2. Some special cases are

- Laplace equation: $-\Delta u = 0$

- Poisson's equation: $-\Delta u = f$

- Heat equation: $u_t - \Delta u = 0$

1.2   Metric Spaces and Complete Spaces

Definition 1.3. A **metric space** is a vector space $\mathcal{V}$ that has a (distance) metric:

$$d(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}, \text{ such that } \forall x, y, z \in \mathcal{V}$$
$$d(x,x) = 0$$
$$\forall x \neq y, d(x,y) > 0$$
$$d(x,y) = d(y,x)$$
$$d(x,z) \geq d(x,y) + d(y,z)$$

Definition 1.4. Given a metric d, a sequence $(x_n)_{n=1}^{\infty}$ has a **limit point** $x = \lim_{n\to\infty} x_n$ if $\lim_{n\to\infty} d(x, x_n) = 0$. In this case, we say $(x_n)_{n=1}^{\infty}$ is a **convergent sequence**.

Definition 1.5. A sequence $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** in a metric space with metric d if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m, n > N \in \mathbb{N}, d(x_m, x_n) < \epsilon.$$

Definition 1.6. A metric space $\mathcal{V}$ is **complete** if every Cauchy sequence $(x_i)_{i=1}^{\infty}$ converges to a limit point in $\mathcal{V}$. i.e. $\exists x \in \mathcal{V}, \lim_{i\to\infty} x_i = x$.

Proposition 1.1. *Let $(\mathcal{V}, d(\cdot, \cdot))$ be a metric space, then every convergent sequence is Cauchy.*

Proposition 1.2. *Let $(\mathcal{V}, d(\cdot, \cdot))$ be a metric space. If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a convergent subsequence such that $\lim_{k\to\infty} x_{n_k} = x \in \mathcal{V}$, then $\lim_{n\to\infty} x_n = x$.*

1.2.1 Compactness

Remark. See the definition of compactness and more in Section 2.9 of AMATH731 Notes from Prof. Tran.

Definition 1.7. Let $(\mathcal{V}, d(\cdot, \cdot))$ be a metric space. A set $S \subseteq \mathcal{V}$ is **relatively compact**, or **pre-compact** if its closure $\overline{S}$ is compact in $\mathcal{V}$.

Proposition 1.3. *Let $(\mathcal{V}, d(\cdot, \cdot))$ be a metric space, then $S \subseteq \mathcal{V}$ is relatively compact iff for any sequence $(x_n)_{n=1}^{\infty} \subseteq S$, it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$, such that $x_{n_k} \to x$ for some $x \in \mathcal{V}$.*

1.3 Banach Spaces

Definition 1.8. A **normed vector space** is a vector space $(X, ||\cdot||)$ that has an norm (length):

$$||\cdot|| : X \to \mathbb{R}, \text{ such that } \forall x, y \in X, a \in \mathbb{C}$$
$$||a \cdot x|| = |a|\,||x||$$
$$||x + y|| \le ||x|| + ||y||$$
$$||x|| \ge 0$$
$$||x|| = 0 \iff x = 0.$$

Proposition 1.4. *For every **normed space** with $||\cdot||$, there is a metric $d(x, y) = ||x - y||$.*

Proof.

$$d(x, x) = ||x - x|| = ||0|| = 0$$
$$\forall x \ne y, d(x, y) = ||x - y|| > 0$$
$$d(x, y) = ||x - y|| = ||-(y - x)|| = |-1|\,||y - x|| = ||y - x|| = d(y, x)$$
$$d(x, z) = ||x - z|| = ||x - y + y - z|| \ge ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$$

Thus $d(x, y) = ||x - y||$ is a metric. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 1.9. A normed space is called a **Banach space** if it is complete.

Definition 1.10. Let $(X, ||\cdot||)$ be a Banach space, a subset $A \subseteq X$ is **dense** in X if the closure $\bar{A} = X$.

Definition 1.11. A Banach space is **separable** if there is a dense countable subset of it.

1.4 Hilbert Spaces

Definition 1.12. An **inner product space** is a vector space H that has an inner product:

$$\langle \cdot, - \rangle : H \times H \to \mathbb{C}, \text{ such that } \forall u, v, w \in H, a, b \in \mathbb{C}$$
$$\langle v, au + bw \rangle = a\langle v, u \rangle + b\langle v, w \rangle$$
$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$
$$\langle v, v \rangle = 0 \iff v = 0.$$

Remark. The conventional mathematical definition of an inner product is linear in the first entry. We are using the current definition to make the "bra-ket" notaion easier to understand.

Proposition 1.5. *For every inner product space with $\langle \cdot, - \rangle$, there is a norm $||x|| = \sqrt{\langle x, x \rangle}$.*

Proof.

$$||a \cdot x|| = \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2}\sqrt{\langle x, x \rangle} = |a|||x||$$

$$||x + y||^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle$$

$$\leq ||x||^2 + ||y||^2 + 2||x||||y||$$

$$\leq (||x|| + ||y||)^2$$

$$\forall x \neq 0, ||x|| = \sqrt{\langle x, x \rangle} > 0$$

$$||0|| = \sqrt{\langle 0, 0 \rangle} = 0$$

Thus $||x|| = \sqrt{\langle x, x \rangle}$ is a norm. $\square$

Corollary 1.6. *For every inner product space, there is a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$*

Theorem 1.7. *Cauchy-Schwarz: For every inner product space H,*

$$\forall u, v \in H, |\langle u, v \rangle| \leq ||u||||v||.$$

In particular, when $||u|| \neq 0, ||u||^2||v||^2 - |\langle u, v \rangle|^2 = ||z||^2$, where $z := ||u||v - \frac{\langle u, v \rangle}{||u||}u$.

Proof. Notice that this is trivially true and equality holds to be zero when $u = 0$.
Now we assume $||u|| \neq 0$, then

$$||z||^2 = \langle z, z \rangle$$

$$= \left\langle ||u||v - \frac{\langle u, v \rangle}{||u||}u, ||u||v - \frac{\langle u, v \rangle}{||u||}u \right\rangle$$

$$= ||u||^2\langle v, v \rangle - \langle u, v \rangle\langle u, v \rangle - \overline{\langle u, v \rangle}\langle v, u \rangle + \frac{\langle v, u \rangle\langle u, v \rangle}{\cancel{||u||^2}}\cancel{\langle u, u \rangle}^{||u||^2}$$

$$= ||u||^2||v||^2 - |\langle u, v \rangle|^2 - \cancel{|\langle v, u \rangle|^2} + \cancel{|\langle v, u \rangle|^2}$$

$$= ||u||^2||v||^2 - |\langle u, v \rangle|^2.$$

$\square$

Proposition 1.8. *If $\forall v, \langle v, u \rangle = 0$, then $u = 0$.*

Proposition 1.9. *For an Inner product space $H, \forall y, x = \lim_{i \to \infty} x_i \in H$, we have*

$$\langle x, y \rangle = \lim_{i \to \infty} \langle x_i, y \rangle.$$

Proof. Given any $\epsilon > 0$, let $\epsilon_0 = \frac{\epsilon}{||y||}$.
Since $x = \lim_{i \to \infty} x_i$, we can find $N > 0$, such that $\forall n > N, ||x - x_n|| < \epsilon_0$,
thus $|\langle x, y \rangle - \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq ||x - x_n||||y|| < \epsilon_0||y|| = \epsilon$ $\square$

Corollary 1.10. *For an Inner product space $H, \forall y, x = \lim_{i \to \infty} x_i \in H$, we have $\langle y, x \rangle = \lim_{i \to \infty} \langle y, x_i \rangle$.*

Definition 1.13. An inner product space $\mathcal{H}$ is called a Hilbert space if it is complete.

Definition 1.14. Let H be an inner product space. Two vectors $u, v \in H$ are called **orthogonal** if $\langle u, v \rangle = 0$.

Definition 1.15. Let H be an inner product space. A set $\{e_i\}_{i \in I} \subseteq H$ is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij}.$$

Definition 1.16. Let H be an inner product space. An orthonormal set $\{e_i\}_{i\in\mathbb{N}} \subseteq H$ is called a **maximal orthonormal set / orthonormal basis / total orthonormal set** if

$$H = \overline{Span(\{e_1, e_2, \ldots\})}.$$

Theorem 1.11. *Let $\mathcal{H}$ be a Hilbert space, and $\{e_i\}_{i\in\mathbb{N}} \subseteq \mathcal{H}$ be an orthonormal set, then TFAE:*

1. $\{e_i\}_{i\in\mathbb{N}}$ *is an orthonormal basis*

2. *If $\forall i \in \mathbb{N}, \langle x, e_i \rangle = 0$, then $x = 0$.*

3. $\forall x \in \mathcal{H}, x = \sum_{i\in\mathbb{N}} \langle e_i, x \rangle e_i$. *(Fourier series)*

4. $\forall x \in \mathcal{H}, ||x||^2 = \sum_{i\in\mathbb{N}} |\langle e_i, x \rangle|^2$. *(Parseval Identity)*

Theorem 1.12. *$\mathcal{H}$ is a separable Hilbert space, if and only if there is a maximal orthonormal set in $\mathcal{H}$. Moreover, in this case, every maximal orthonormal set is at most countable.*

Definition 1.17. Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$, the subspace **orthogonal** to S is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v \rangle = 0, \forall u \in S\}.$$

Lemma 1.13. *Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$, we always have $S^\perp$ is a subspace of $\mathcal{H}$.*

Definition 1.18. Let V be a vector space, and $U, W \subseteq V$ be two subspaces, we say $V = U \oplus W$, if $\forall v \in V$, it can be uniquely written as $v = u + w$, where $u \in U, w \in W$.

Theorem 1.14. *Let $\mathcal{H}$ be a Hilbert space, if $S \subseteq \mathcal{H}$ is a closed subspace, then*

$$\mathcal{H} = S \oplus S^\perp.$$

1.5 Bounded linear operators

Definition 1.19. Let X, Y be vector spaces, $A : X \to Y$ is a linear operator if $\forall c \in \mathbb{R}, u, v \in X$,

$$A(u + cv) = Au + cAv.$$

Definition 1.20. Let X, Y be normed spaces, the **operator norm** of a linear operator $A : X \to Y$ is

$$||A|| := \sup_{||u||_X \leq 1} ||Au||_Y = \sup_{||u||_X = 1} ||Au||_Y = \sup_{u \neq 0 \in X} \frac{||Au||_Y}{||x||_X}.$$

Definition 1.21. Let X, Y be normed spaces, a linear operator $A : X \to Y$ is **bounded** if $||A|| < \infty$.

Definition 1.22. Let X, Y be normed spaces, we denote

$$B(X, Y) := \{A : X \to Y | A \text{ is bounded linear opeartor}\}.$$

Theorem 1.15. *The set $B(X, Y)$ is a normed linear space with the operator norm.*

Proposition 1.16. *Let X, Y, Z be normed spaces, if $A : X \to Y, B : Y \to Z$ are both linear bounded operators, then so is $B \circ A$, with*
$$||B \circ A|| \leq ||B||||A||.$$

Theorem 1.17. *Let X, Y be normed spaces, a linear operator $A : X \to Y$ is bounded if and only if it is continuous.*

Definition 1.23. Let X, Y be normed spaces, a linear operator $A : X \to Y$ is **closed** if $\forall u_k \to u$ in X and $Au_k \to v$ in Y, we have $Au = v$.

Theorem 1.18. *(closed graph) Let X, Y be Banach spaces, if a linear operator $A : X \to Y$ is closed, it is also bounded.*

Theorem 1.19. *(Bounded inverse Theorem) Let X, Y be normed spaces, if a bounded linear operator $A : X \to Y$ is bijective, then A^{-1} is continuous and bounded as well.*

Proposition 1.20. *Let Y be a Banach space, S be a dense subset of a normed space X. For any bounded linear operator $E : S \to Y$, we can extend it to $\tilde{E} : X \to Y$, such that $\tilde{E}$ is also bounded and linear, with $\left\| \tilde{E} \right\| = \|E\|$, and $\tilde{E}|_S = E$.*

Proof. Consider any $x \in X$.

Since S is dense in X, We know $\forall m \in \mathbb{N}^+, \exists x_m \in S$, such that $\|x - x_m\|_X \leq \frac{1}{m}$.

Since E is linear on S, we have that

$$
\begin{aligned}
\|Ex_m - Ex_l\|_Y &= \|E(x_m - x_l)\|_Y \\
&\leq \|E\| \|x_m - x_l\|_X \\
&= \|E\| \|(x_m - x) + (x - x_l)\|_X \\
&\leq \|E\| \|x - x_m\|_X + \|E\| \|x - x_l\|_X \\
&\leq \|E\| \left(\frac{1}{m} + \frac{1}{l} \right).
\end{aligned}
$$

Thus given any $\epsilon > 0$, for any $m, l \geq \lceil \frac{2\epsilon}{\|E\|} \rceil$, we can make $\|Ex_m - Ex_l\|_Y < \epsilon$. Thus $(Ex_m)_{m=1}^\infty$ is a Cauchy sequence in Y.

Since Y is a Banach space, $\exists y^* \in Y$, such that $Ex_m \to y^*$ in Y.

We claim that y^* is independent of choice of the sequence $(x_m)_{m=1}^\infty$.

Indeed, consider any other sequence $(v_m)_{m=1}^\infty \subseteq C^\infty(\bar{x})$, such that $\forall m \in \mathbb{N}^+, \|x - x_m\|_X \leq \frac{1}{m}$,

$$
\begin{aligned}
\|y^* - Ev_m\|_Y &\leq \|y^* - Ex_m\|_Y + \|Ex_m - Ev_m\|_Y \\
&\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - v_m\|_X \\
&\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - x\|_X + \|E\| \|x - v_m\|_X.
\end{aligned}
$$

Since all three terms on the right go to 0 when $m \to \infty$, we have that $Ev_m \to y^*$ in Y.

Thus we can uniquely define $\tilde{E}x := y^*$. In addition,

$$
\begin{aligned}
\left\| \tilde{E}x \right\|_Y &= \left\| \lim_{m \to \infty} Ex_m \right\|_Y \\
&= \lim_{m \to \infty} \|Ex_m\|_Y \\
&\leq \lim_{m \to \infty} \|E\| \|x_m\|_X \\
&= \|E\| \left\| \lim_{m \to \infty} x_m \right\|_X \\
&= \|E\| \|x\|_X.
\end{aligned}
$$

Thus $\left\| \tilde{E} \right\| = \|E\|$. $\qquad\square$

1.5.1 Compact Operators

Definition 1.24. *Let X, Y be metric spaces, a linear operator $A : X \to Y$ is **compact** if for each bounded subset $S \subseteq X$, we have its image $A(S)$ is pre-compact in Y.*

Proposition 1.21. *Let X, Y be metric spaces, a linear operator $A : X \to Y$ is compact if and only if A is bounded, and each bounded sequence $(x_n)_{n=1}^\infty \subseteq X$ has some subsequence $(x_{n_k})_{k=1}^\infty$ such that $(Ax_{n_k})_{k=1}^\infty$ converges to some $y \in Y$.*

Definition 1.25. Let X, Y be Banach spaces and $X \subseteq Y$, then we say X is **compactly embedded** in Y, denoted

$$X \subset\subset Y$$

if the inclusion map $i : X \hookrightarrow Y; \ x \mapsto x$ is compact.

Namely, $\exists C > 0$, such that $\forall x \in X, ||x||_Y \leq C||x||_X$, and each bounded sequence $(x_n)_{n=1}^\infty \subseteq X$ having some subsequence $(x_{n_k})_{k=1}^\infty$ that converges to some $y \in Y$.

Proposition 1.22. *Let X, Y, Z be Banach spaces and $X \subset\subset Y$, if an operator $T : Z \to X$ is bounded, then $\tilde{T} := i \circ T : Z \to Y$ is compact.*

Proof. Consider any bounded set $S \subseteq Z$, such that $\forall z \in S, ||z||_Z \leq M$.

We have $||Tz||_X \leq ||T|| ||z||_M \leq M ||T|| < \infty$, and thus $T(S)$ is bounded in X.

Yet i is compact, and thus $i(T(S))$ is pre-compact.

This shows $\tilde{T}(S) = (i \circ T)(S)$ is pre-compact for any bounded set $S \subseteq Z$.

Thus $\tilde{T}$ is compact. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 1.23. *(Spectral theorem for compact operators)*
Let $K : \mathcal{H} \to \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space $\mathcal{H}$, then

1. $0 \in \mathrm{Spec}(K)$.

2. $\mathrm{Spec}(K) \setminus \{0\} = \mathrm{Spec}_p(K) \setminus \{0\}$.

3. $\mathrm{Spec}(K) \setminus \{0\}$ *is finite, or* $\mathrm{Spec}(K) \setminus \{0\} = (\lambda_k)_{k=1}^\infty$ *such that* $\lim_{k \to \infty} \lambda_k = 0$.

1.5.2 Dual Space

Definition 1.26. Let X be a normed space over $\mathbb{F}$, a **functional** is an operator that maps into $\mathbb{F}$.

Definition 1.27. Let X be a normed space over $\mathbb{F}$, the **dual space** of X is the collection of bounded linear functionals on X, denoted

$$X^* := B(X, \mathbb{F}).$$

Definition 1.28. Let X be a normed space, if $v \in X, u^* \in X^*$, we can write $\langle u^* | v \rangle_{X^*, X} := u^*(v)$ as the action of u^* on v.

Definition 1.29. Let X be a normed space, the **dual norm** is defined to be

$$||u^*||_{X^*} := \sup_{||u|| \leq 1} \left| \langle u^* | u \rangle_{X^*, X} \right|.$$

Definition 1.30. A Banach space X is **reflexive** if $(X^*)^* \simeq X$. Namely, $\forall u^{**} \in (X^*)^*, \exists! u \in X$ such that

$$\forall v^* \in X^*, \langle u^{**} | v^* \rangle_{(X^*)^*, X^*} = \langle v^* | u \rangle_{X^*, X}.$$

Theorem 1.24. *(Riesz-Frechet Representation theorem)*
Let $\mathcal{H}$ be a Hilbert space, then for each $u^* \in \mathcal{H}, \exists! u \in \mathcal{H}$, such that $\forall v \in \mathcal{H}, \langle u^* | v \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$, and $||u^*||_{\mathcal{H}^*} = ||u||_{\mathcal{H}}$.

Corollary 1.25. *Every Hilbert space is reflexive.*

Corollary 1.26. *Let $\mathcal{H}$ be a Hilbert space, then $\mathcal{H} \cong^* \mathcal{H}$, where the map $\Phi : \mathcal{H} \to \mathcal{H}^*; \ u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the* **canonical bijective isometric antilinear isomorphism.**

Remark. We thus abuse the notation, and denote canonical bijective isometric antilinear isomorphism by $u^\dagger := \Phi(u) \; \forall u \in \mathcal{H}$, and $(u^*)^\dagger := \Phi^{-1}(u^*) \; \forall u^* \in \mathcal{H}^*$. Notice that by definition

$$(u^\dagger)^\dagger = u, ((u^*)^\dagger)^\dagger = u^* \; \forall u \in \mathcal{H}, u^* \in \mathcal{H}^*.$$

We might further abuse the notation, and write

$$\langle u|v\rangle := \langle u, v\rangle = \langle u^\dagger | v\rangle =: \langle u^\dagger, v\rangle$$

interchangeably instead of $\langle u^\dagger | v\rangle_{\mathcal{H}^*, \mathcal{H}}$ or $\langle u, v\rangle_{\mathcal{H}}$ when the context is clear.

Definition 1.31. Let X be a Banach Space, we say $(u_k)_{k=1}^\infty \subset X$ converges to $u \in X$ weakly, denoted $u_k \rightharpoonup u$, if
$$\forall v^* \in X^*, \langle v^*|u_k\rangle \to \langle v^*|u\rangle$$

as real numbers.

Proposition 1.27. *Let X be a Banach Space, $(u_k)_{k=1}^\infty \subset X$ be a sequence, then*

1. *If $u_k \to u$, we always have $u_k \rightharpoonup u$.*

2. *If $u_k \rightharpoonup u$, we have that u is unique.*

3. *If $u_k \rightharpoonup u$, we have $(u_k)_{k=1}^\infty$ is bounded.*

4. *If $u_k \rightharpoonup u$, every subsequence $(u_{k_j})_{j=1}^\infty$ also converges weakly to u.*

Proof. See A5Q1 for 1. $\qquad\square$

Theorem 1.28. *(Weakly compact for reflexive Banach Space)*
Let X be a reflexive Banach Space, and $(u_k)_{k=1}^\infty \subset X$ be a bounded sequence, then $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.

Proposition 1.29. *Let $\mathcal{H}$ be a Hilbert space, then $u_k \rightharpoonup u$ if and only if $\forall v \in \mathcal{H}, \langle v, u_k\rangle \to \langle v, u\rangle$ as real numbers.*

Proof. Suppose $u_k \rightharpoonup u$.
Notice that for all $v \in \mathcal{H}$, we have that $v^\dagger \in \mathcal{H}^*$, and thus $\langle v, u_k\rangle = \langle v^\dagger | u_k\rangle \to \langle v^\dagger | u\rangle = \langle v, u\rangle$.
Now suppose $\forall v \in \mathcal{H}, \langle v, u_k\rangle \to \langle v, u\rangle$.
Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 1.24, there is some $f^\dagger \in \mathcal{H}$, such that

$$\langle f | u_{k_j}\rangle = \langle f^\dagger, u_{k_j}\rangle \to \langle f^\dagger, u\rangle = \langle f|u\rangle.$$

Thus, $u_{k_j} \rightharpoonup u$. $\qquad\square$

Proposition 1.30. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Tu_k \rightharpoonup \mathcal{H}_2$.*

Proof. Let $y_k := Tu_k, y := Tu \in \mathcal{H}_2$.
Consider any $g \in \mathcal{H}_2^*$, we define $f := g \circ K \in \mathcal{H}_1^*$.
Since $u_k \rightharpoonup u$, we must have

$$\lim_{k\to\infty} f(u_k) = f(u)$$
$$\lim_{k\to\infty} g(Ku_k) = g(Ku)$$
$$\lim_{k\to\infty} g(y_k) = g(y).$$

We thus have $y_k \rightharpoonup y$. $\qquad\square$

Proposition 1.31. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $K : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Ku_k \to \mathcal{H}_2$.*

Proof. Let $y_k := Ku_k, y := Ku \in \mathcal{H}_2$.

Since K is compact, it is bounded, so $y_k \rightharpoonup y$.

Now suppose for contradiction $\lim_{k\to\infty} ||y_k - y|| \neq 0$.

Then there is some $\epsilon > 0$ and a subsequence $(u_{k_j})_{j=1}^\infty$ such that $\forall j \geq 1, ||y_{k_j} - y|| \geq \epsilon$.

Since $u_k \rightharpoonup u \in \mathcal{H}$, we have $(u_k)_{k=1}^\infty$ is bounded, and thus $(u_{k_j})_{j=1}^\infty$ is bounded.

Since K is compact, there is some further subsequence $(u_{k_{j_m}})_{m=1}^\infty$ such that $\lim_{m\to\infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$.

Thus $Ku_{k_{j_m}} \rightharpoonup \tilde{y}$. Since weak convergence, we must have $\tilde{y} = y$.

Thus $\lim_{m\to\infty} Ku_{k_{j_m}} = y$, which is a contradiction. $\qquad\square$

1.5.3 Adjoint Operator

Definition 1.32. Let X, Y be normed spaces, the **dual operator** of a linear operator $A : X \to Y$ is

$$A^* : Y^* \to X^*; \ f \mapsto f \circ A.$$

Proposition 1.32. *Let X, Y, Z be normed spaces, $S \in B(X, Y), T \in B(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.*

Proof. Consider any $f \in Z^*$, and any $x \in X$, we have

$$\begin{aligned}
(T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx) \\
&= (f)(S(T(x))) \\
&= (f \circ (S \circ T))(x) \\
&= (S \circ T)^*(f)(x).
\end{aligned}$$

Thus $(T^* \circ S^*)(f) = (S \circ T)^*(f)$. $\qquad\square$

Definition 1.33. Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, the **Hilbert adjoint operator** of T is $T^\dagger : \mathcal{H} \to \mathcal{H}$ such that $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \forall x, y \in \mathcal{H}$.

Theorem 1.33. *Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, $T^\dagger$ always exists, and is given by $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$, where $\Phi : \mathcal{H} \to \mathcal{H}^*; \ u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism, and T^* is the dual operator of T. In addition, $T^\dagger$ is also a bounded linear operator, with $||T^\dagger|| = ||T||$, and $(T^\dagger)^\dagger = T$.*

Proof. $\forall y \in \mathcal{H}$, we have that

$$\begin{aligned}
\langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\
&= ((T^* \circ \Phi)(x))(y) \\
&= (\Phi(x))(Ty) \\
&= \langle x, Ty \rangle.
\end{aligned}$$

Now consider any $x, y, z \in \mathcal{H}, c \in \mathbb{C}$, we have that

$$\begin{aligned}
\langle T^\dagger(x + cz), y \rangle &= \langle x + cz, Ty \rangle \\
&= \langle x, Ty \rangle + \bar{c}\langle z, Ty \rangle \\
&= \langle T^\dagger x, y \rangle + \bar{c}\langle T^\dagger z, y \rangle \\
&= \langle T^\dagger x + cT^\dagger z, y \rangle.
\end{aligned}$$

Since this holds for any $y \in \mathcal{H}$, we have that $T^\dagger(x + cz) = T^\dagger x + cT^\dagger z$, and thus $T^\dagger$ is linear.

Now given any $x \in \mathcal{H}$, we have that

$$\begin{aligned}
\left|\left|T^\dagger x\right|\right|^2 &= \left\langle T^\dagger x, T^\dagger x\right\rangle \\
&= \left\langle x, TT^\dagger x\right\rangle \\
&\leq ||x||\left|\left|TT^\dagger x\right|\right| \\
&\leq ||x||||T||\left|\left|T^\dagger x\right|\right| \\
&\implies \\
\left|\left|T^\dagger x\right|\right| &\leq ||x||||T|| \\
&\implies \\
\left|\left|T^\dagger\right|\right| = \sup_{x \neq 0 \in \mathcal{H}} &\frac{\left|\left|T^\dagger x\right|\right|}{||x||} \\
&\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{||x||||T||}{||x||} \\
&= ||T||.
\end{aligned}$$

Thus $T^\dagger$ is also a bounded linear operator.
Now $\forall x, y \in \mathcal{H}$, $\left\langle x, T^\dagger y\right\rangle = \overline{\langle T^\dagger y, x\rangle} = \overline{\langle y, Tx\rangle} = \langle Tx, y\rangle$.
Thus $(T^\dagger)^\dagger = T$. $\qquad\square$

Remark. $\forall x, y \in \mathcal{H}$, $\left\langle (Tx)^\dagger \big| y\right\rangle = \langle Tx, y\rangle = \left\langle x, T^\dagger y\right\rangle = \left\langle x^\dagger \big| T^\dagger y\right\rangle$. We thus abuse the notation, and write $(Tx)^\dagger = \langle x | T^\dagger$

Definition 1.34. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is **delf-adjoint** if $T^\dagger = T$.

Theorem 1.34. *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then $K^\dagger$ is also compact.*

Proof. $K^\dagger$ is bounded by 1.21.
Let $(u_k)_{k=1}^\infty$ be any bounded sequence in $\mathcal{H}$.
By 1.28, we have that $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.
Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 1.24, there is some $f^\dagger \in \mathcal{H}$, such that

$$\begin{aligned}
\left\langle f \big| K^\dagger(u_{k_j} - u)\right\rangle &= \left\langle f^\dagger, K^\dagger(u_{k_j} - u)\right\rangle \\
&= \left\langle Kf^\dagger, u_{k_j} - u\right\rangle \\
&= \left\langle Kf^\dagger, u\right\rangle - \left\langle Kf^\dagger, u\right\rangle \qquad\qquad \to 0,
\end{aligned}$$

since $u_{k_j} \rightharpoonup u$ and by 1.29.
Since $\left\langle f \big| K^\dagger(u_{k_j} - u)\right\rangle \to 0 = \langle f | 0\rangle$ for any $f \in \mathcal{H}^*$, we have that $K^\dagger(u_{k_j} - u) \rightharpoonup 0$.
By 1.31, we have that $KK^\dagger(u - u_{k_j}) \to 0$.

$$\begin{aligned}
\left|\left|K^\dagger u - K^\dagger u_{k_j}\right|\right|^2 &= \left\langle K^\dagger u - K^\dagger u_{k_j}, K^\dagger u - K^\dagger u_{k_j}\right\rangle \\
&= \left\langle K^\dagger(u - u_{k_j}), K^\dagger(u - u_{k_j})\right\rangle \\
&= \left\langle KK^\dagger(u - u_{k_j}), u - u_{k_j}\right\rangle \\
&\leq \left|\left|KK^\dagger(u - u_{k_j})\right|\right|\left|\left|u - u_{k_j}\right|\right| \\
&\to 0.
\end{aligned}$$

Thus $K^\dagger u_{k_j} \to K^\dagger u \in \mathcal{H}$,
Since $(u_k)_{k=1}^\infty$ is any bounded sequence, we have that $K^\dagger$ is compact by 1.21. $\qquad\square$

Theorem 1.35. *(Fredholm's alternative)*
Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then

1. $\mathrm{Ker}(I - K)$ *is finite dimensional.*

2. $\mathrm{Im}(I - K)$ *is closed.*

3. $\mathrm{Im}(I - K) = \mathrm{Ker}(I - K^\dagger)^\perp$.

4. $\dim(\mathrm{Ker}(I - K)) = \dim(\mathrm{Ker}(I - K^\dagger))$.

5. $\mathrm{Ker}(I - K) = \{0\} \iff \mathrm{Im}(I - K) = \mathcal{H}$.

Corollary 1.36. *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then exactly one of the following holds:*

1. $\forall v \in \mathcal{H}$, $\exists! u \in \mathcal{H}$, *such that* $(I - K)u = v$.

2. $\exists u \neq 0 \in \mathcal{H}$, *such that* $(I - K)u = 0$.

Proof. When $\mathrm{Ker}(I - K) = \{0\}$, we have that $I - K$ is injective, and $\mathrm{Im}(I - K) = \mathcal{H}$.
Thus $\forall v \in \mathcal{H}$, $\exists! u \in \mathcal{H}$, such that $(I - K)u = v$.
On the other hand, if 1. is true, we have that $(I - K)$ is surjective, so $\mathrm{Im}(I - K) = \mathcal{H}$, so $\mathrm{Ker}(I - K) = \{0\}$.
Thus $\mathrm{Ker}(I - K) = \{0\} \iff 1.$.
We also have that $\mathrm{Ker}(I - K) \neq \{0\} \iff \exists u \neq 0 \in \mathrm{Ker}(I - K) \iff 2.$. $\qquad\square$

Theorem 1.37. *(Spectral theorem / Hilbert-Schmidt Theorem)*
Let $T : \mathcal{H} \to \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space $\mathcal{H}$, and $n = \dim(\Im(T)) \in \mathbb{N} \cap \{\infty\}$, then

1. *There exists orthonormal eigenvectors $(\phi_k)_{k=1}^n \subset \mathcal{H}$ and eigenvalues $(\lambda_k)_{k=1}^n \subset \mathbb{R}$ such that $|\lambda_1| \geq |\lambda_2| \geq \ldots$, and*
$$T\phi_k = \lambda_k \phi_k, \lambda_k \neq 0, \ \forall 1 \leq k \leq n,$$
$$\forall v \in \mathcal{H}, Tv = \sum_{k=1}^n \lambda_k \langle \phi_k, v \rangle \phi_k = \sum_{k=1}^n \langle \phi_k, Tv \rangle \phi_k.$$

2. *If $n = \infty$, then $\lim_{k \to \infty} \lambda_k = 0$, and $(\phi_k)_{k=1}^\infty$ is an orthonormal set for $\mathcal{H}$ iff 0 is not an eigenvalue for T.*

1.6 Function Spaces

1.6.1 Continuous functions

Definition 1.35. $u : U \to \mathbb{R}$ is **continuous** at $x \in U$ if
$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in U, ||x - y|| < \delta \implies |u(x) - u(y) < \epsilon|.$$

A function u is continuous if it is continuous at all $x \in U$.

- $C(U) := \{u : U \to \mathbb{R} : u \text{ is continuous}\}$

- $C^k(U) := \{u : U \to \mathbb{R} : u \text{ is k-times continuously differentiable}\}$

- $C^\infty(U) := \{u : U \to \mathbb{R} : u \text{ has continuous derivatives of all orders}\}$

Definition 1.36. $u : U \to \mathbb{R}$ is **uniformly continuous** if
$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in U, ||x - y|| < \delta \implies |u(x) - u(y) < \epsilon|.$$

- $C(\bar{U}) := \{u : U \to \mathbb{R} : u \text{ is uniformly continuous on bounded subsets of } U\}$

- $C^k(\bar{U}) := \{u : U \to \mathbb{R} : \forall |\alpha| \leq k, D^\alpha u \text{ is uniformly continuous on bounded subsets of } U, \}$

- If $u \in C^k(\bar{U})$, then we can extend $D^\alpha u$ continuously to $\bar{U}$.

Definition 1.37. The support of $u : U \to \mathbb{R}$ is

$$\text{Supp}(u) := \overline{\{x \in U : u(x) \neq 0\}}.$$

Definition 1.38. $u : U \to \mathbb{R}$ has compact support if $\text{Supp}(u)$ is a compact subset of U.

Definition 1.39. We denote the functions in $C(U)$ and $C^k(U)$ with compact support by $C_c(U), C_c^k(U)$.

Definition 1.40. Consider a sequence of functions $\{u_m\}_1^\infty$ with $u_m : U \to \mathbb{R}$ and a function $u : U \to \mathbb{R}$, we have

- $u_m \to u$ point-wise on U if

$$\forall x \in U, \ \delta > 0, \ \exists M \in \mathbb{N}, \quad \text{such that } m > M \implies |u_m(x) - u(x)| < \delta.$$

- $u_m \to u$ uniformly on U if

$$\forall \delta > 0, \ \exists M \in \mathbb{N}, \quad \text{such that } \forall x \in U, \ m > M \implies |u_m(x) - u(x)| < \delta.$$

1.6.2 Lebesgue Spaces

Definition 1.41. We denote the Lebesgue measure by λ on $\mathbb{R}^n$. We denote $\int_A f d\lambda$ by $\int_A f(x) dx$ for any measurable set $A \subseteq \mathbb{R}^n$.

Definition 1.42. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, we define

$$\mathcal{L}^1(\Omega) := \left\{ f : \Omega \to \mathbb{R} \Big| \int_\Omega |f(x)| dx < \infty \right\}.$$

Definition 1.43. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, and $1 \leq p < \infty$ we define

$$\mathcal{L}^p(\Omega) := \left\{ f : \Omega \to \mathbb{R} | f^p \in L^1(\Omega) \right\} = \left\{ f : \Omega \to \mathbb{R} \Big| \int_\Omega |f(x)|^p dx < \infty \right\}.$$

In addition, we define the norm

$$||f||_p := \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Definition 1.44. The **essential supremum** of a function $u : U \to \mathbb{R}$ is

$$\text{ess}\sup f := \inf \left\{ M \in \mathbb{R} : |\{x : f(x) > M\}| = 0 \right\}.$$

Definition 1.45. Let $\Omega \subseteq \mathbb{R}^n$ be Lebesgue measurable, we define

$$\mathcal{L}^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} | \text{ess}\sup f < \infty \right\}.$$

In addition, we define the norm

$$||f||_\infty := \text{ess}\sup f.$$

Definition 1.46. Two measurable functions $f, g : \Omega \to \mathbb{R}$ are said to be equal almost everywhere if $\{x \in \Omega : f(x) \neq g(x)\}$ has measure zero.

Proposition 1.38. *For any $1 \leq p \leq \infty$, we have $||f - g||_p = 0 \iff f = g$ almost everywhere.*

Definition 1.47. For any $1 \leq p \leq \infty$, if we identify $f, g \in \mathcal{L}^p(\Omega)$ by $f \sim g \iff f = g$ almost everywhere, we get the quotient space

$$L^p := \mathcal{L}^p/_\sim = \{[f] : f \in L^p(\Omega)\}$$

to be the collection of all equivalence classes of functions in $\mathcal{L}^p$.

Theorem 1.39. *(completeness of L^p)*
For any $1 \leq p \leq \infty$, we have the space $(L^p, ||\cdot||_p)$ is a Banach space, where $||[f]||_p := ||f||_p$ for any representative $f \in [f]$. One can check this norm is well-defined.

Theorem 1.40. *For any $1 \leq p < \infty$,*

- $C_c(U)$ *is dense in* $L^p(U)$.

- $C(\bar{U})$ *is dense in* $L^p(U)$.

Definition 1.48. Let $U, V \subseteq \mathbb{R}^n$ be open, we say that V is **compactly contained** in U if $V \subseteq \bar{V} \subseteq U$, and $\bar{V}$ is compact. We write this as $V \subset\subset U$.

Definition 1.49. The **locally summable spaces** are

$$L^p_{loc}(U) := \{f : U \to \mathbb{R} : \forall V \subset\subset U, u \in L^p(V)\}.$$

Definition 1.50. We say some property holds for $L^p_{loc}(U)$, if it holds $\forall L^p(V)$ such that $V \subset\subset U$.
For instance, let $(f_n)_{n=1}^\infty \subseteq L^p_{loc}(U)$, then $f_n \to f$ in $L^p_{loc}(U)$ if $f_n \to f$ in $L^p(V)$, $\forall V \subset\subset U$.

Proposition 1.41. *For any $1 \leq p \leq \infty$, we have*

$$L^p(U) \subseteq L^1_{loc}(U).$$

Example 1.6.1. Let $u(x) = \frac{1}{x}$ on $U = (0, 1)$.
We have $\int_0^1 |u| dx = \infty$, and thus $u \notin L^1(U)$. However, $u \in L^1_{loc}(U)$.

Theorem 1.42. *(Holder's Inequality)*
Assume $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.
If $u \in L^p(U), v \in L^q(U)$, we have

$$\int_U |uv| dx \leq ||u||_p ||v||_q.$$

For $a, b \in \mathbb{R}^n$, we have

$$\left| \sum_{k=1}^n a_k b_k \right| \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \left(\sum_{k=1}^n |b_k|^q \right)^{1/q}$$

Theorem 1.43. *(Minkowski's Inequality)*
Assume $1 \leq p \leq \infty$.
Let $u, v \in L^p(U)$, we have

$$||u + v||_p \leq ||u||_p + ||v||_p.$$

For $a, b \in \mathbb{R}^n$, we have

$$\left(\sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p \right)^{1/p}$$

Theorem 1.44. *(Lebesgue Monotone Convergence)*
Let $f_n : X \to [0, \infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$. Let $f(x) := \lim_{n \to \infty} f_n(x)$, then $f : X \to [0, \infty]$ is measurable, and

$$\lim_{n \to \infty} \int_X f_n dx = \int_X f dx.$$

Theorem 1.45. *(Lebesgue Dominated Convergence)*
Let $f_n : X \to \mathbb{C}$ be measurable functions, defined almost everywhere on X, such that $f(x) := \lim_{n \to \infty} f_n(x)$ is defined almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that for almost everywhere $x \in X, \forall n \in \mathbb{N}, |f_n(x)| \leq g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n \to \infty} \int_X |f - f_n| d\mu = 0.$$

Theorem 1.46. *We have that*
$$L^q(U) \simeq L^p(U)^*,$$
where $\frac{1}{p} + \frac{1}{q} = 1$, and the isometric isomorphism $L^q(U) \xrightarrow{\sim} L^p(U)^*$; $u \mapsto u^*$ is defined to be

$$\forall v \in L^p(U), \langle u^*|v \rangle := \int_U uv dx.$$

Proof. Consider any $u \in L^q(U)$, and define $u^* : L^p(U)^* \to \mathbb{R}$ by $\forall v \in L^p(U), u^*(v) := \int_U uv dx$.
clearly u^* is linear.
By Holder's Inequality, we have

$$|u^*(v)| = \left| \int_U uv dx \right|$$

$$\leq \int_U |uv| dx$$

$$\leq ||u||_{L^q(U)} ||v||_{L^q(U)}.$$

Thus $||u^*||_{L^p(U)^*} = \sup_{v \in L^p(U), v \neq 0} \frac{|u^*(v)|}{||v||_{L^q(U)}} \leq ||u||_{L^q(U)} < \infty.$
Thus u^* is bounded and $u^* \in L^p(U)^*$.
In addition, if we pick $v = \mathrm{sgn}(u)|u|^{q/p}$, we have that

$$||v||_{L^p(U)}^p = \int |v|^p dx$$

$$= \int |u|^q dx$$

$$= ||u||_{L^q(U)}^q$$

$$< \infty,$$

which means that $v \in L^p(U)$.
In addition, if $v = 0$a.e., we must have $u = 0$a.e., and $||u^*||_{L^p(U)^*} = 0 = ||u||_{L^q(U)}$.
Now suppose v is not 0.

$$|u^*(v)| = \left| \int_U uv dx \right|$$

$$= \left| \int_U u \, \mathrm{sgn}(u)|u|^{q/p} dx \right|$$

$$= \left| \int_U |u|^{1+q/p} dx \right|$$

$$= \left| \int_U |u|^{1+q(1-1/q)} dx \right|$$

$$= \left| \int_U |u|^q dx \right|$$

$$= ||u||_{L^q(U)}^q,$$

$$\frac{|u^*(v)|}{||v||_{L^p(U)}} = \frac{||u||_{L^q(U)}^q}{\left(||u||_{L^q(U)}^q \right)^{1/p}}$$

$$= \left(||u||_{L^q(U)}^q \right)^{1-1/p}$$

$$= \left(||u||_{L^q(U)}^q \right)^{1/q}$$

$$= ||u||_{L^q(U)}.$$

Thus $||u^*||_{L^p(U)^*} = \sup_{v \in L^p(U), v \neq 0} \frac{|u^*(v)|}{||v||_{L^q(U)}} = ||u||_{L^q(U)}$, and the mapping $u \mapsto u^*$ is isometric.

If $u^* = 0$, we must have $\forall v \in L^p(U), \int_U uv dx = 0$. Picking v as above, we have that $u = 0$ a.e.. This shows that the mapping $u \mapsto u^*$ is injective.

Suppose for now U is bounded. Consider the mapping $\nu : E \mapsto u^*(\chi_E)$ for any measurable $E \subseteq U$.

Notice that $|\nu(E)| = |u^*(1)| \leq ||u^*||_{L^p(U)^*}||\chi_E||_{L^p(U)} = ||u^*||_{L^p(U)^*}||1||_{L^p(U)} < \infty$, thus ν is finite.

We have $\nu(\emptyset) = u^*(0) = 0$ since u^* is linear.

For any $B = \bigsqcup_{i=0}^\infty A_i$, with $A_i \subseteq U$ be measurable, we have

$$
\begin{aligned}
\nu(B) &= u^*(\chi_B) \\
&= u^*\left(\sum_{i=0}^\infty \chi_{A_i}\right) \\
&= u^*\left(\lim_{n\to\infty} \sum_{i=0}^n \chi_{A_i}\right) \\
&= \lim_{n\to\infty} u^*\left(\sum_{i=0}^n \chi_{A_i}\right) && \text{continuity of } u^* \\
&= \lim_{n\to\infty} \sum_{i=0}^n u^*(\chi_{A_i}) && \text{linearity of } u^* \\
&= \lim_{n\to\infty} \sum_{i=0}^n \nu(A_i) \\
&= \sum_{i=0}^\infty \nu(A_i),
\end{aligned}
$$

which shows countable additivity.

In addition,

$$
\begin{aligned}
\sum_{i=0}^\infty |\nu(A_i)| &= \lim_{n\to\infty} \sum_{i=0}^n |u^*(\chi_{A_i})| \\
&\leq \lim_{n\to\infty} \sum_{i=0}^n ||u^*||_{L^p(U)^*}||\chi_{A_i}||_{L^p(U)} \\
&= ||u^*||_{L^p(U)^*} \lim_{n\to\infty} \sum_{i=0}^n ||\chi_{A_i}||_{L^p(U)} \\
&= ||u^*||_{L^p(U)^*} \lim_{n\to\infty} \sum_{i=0}^n |A_i|^{1/p} \\
&\leq ||u^*||_{L^p(U)^*} \left(\lim_{n\to\infty} \sum_{i=0}^n |A_i|\right)^{1/p} \\
&= ||u^*||_{L^p(U)^*} \left|\bigsqcup_{i=0}^\infty A_i\right|^{1/p} \\
&= ||u^*||_{L^p(U)^*} |B|^{1/p} \\
&\leq ||u^*||_{L^p(U)^*} |U|^{1/p} \\
&< \infty,
\end{aligned}
$$

which converges absolutely.

Thus ν is a signed measure.

This proof can be generalized to any σ-finite U, and since $\mathbb{R}^n$ is σ-finite, it works for all U.

By Radon-Nikodym Theorem, we can find $u = \frac{d}{dx}\nu$ to be the Radon-Nikodym derivative.

In addition, we can also check that $\int_U vu dx = \int_U v d\nu = \langle u^*|v\rangle$ and $u \in L^q(U)$.

Thus the mapping is subjective and thus an isometric isomorphism. $\qquad\square$

Remark. We will abuse the notation, and write $\langle u|v\rangle := \int_U uv dx$ with $u \in L^q(U)$ instead of $u^* \in L^p(U)^*$.

Corollary 1.47. *In particular, $L^2(U) \simeq L^2(U)^*$, with the isometric isomorphism $L^2(U) \to L^2(U)^*$; $u \mapsto u^*$ is defined to be*

$$\forall v \in L^2(U), \langle u^*|v\rangle = \int_U uv dx = \langle u, v\rangle_{L^2(U)}.$$

Definition 1.51. For $f : U \to \mathbb{R}^m$, we define

$$\|f\|_{L^p(U)} := \left\|\|f\|_p\right\|_{L^p(U)}.$$

2 Sobolev Spaces

This section follows Chapter 5 in Evan's book.

2.1 Holder Spaces

Definition 2.1. For $u : U \to \mathbb{R}$ be bounded and continuous, we write

$$||u||_{C(\bar{U})} := \sup_{x \in U} |u(x)|.$$

Definition 2.2. A function $u : U \to \mathbb{R}$ is **Holder continuous** with $0 < \gamma \leq 1$ if

$$\exists C, \text{ such that } \forall x, y \in U, \ |u(x) - u(y)| \leq C||x - y||^{\gamma}.$$

Definition 2.3. The γth-**Holder semi-norm** of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{x,y \in U, x \neq y} \left(\frac{|u(x) - u(y)|}{||x - y||^{\gamma}} \right).$$

The γth-**Holder norm** of $u : U \to \mathbb{R}$ is

$$||u||_{C^{0,\gamma}(\bar{U})} := [u]_{C^{0,\gamma}(\bar{U})} + ||u||_{C(\bar{U})}.$$

Definition 2.4. For $k \in \mathbb{N}, u \in C^k(\bar{U})$ we define

$$||u||_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} ||D^{\alpha}u||_{C(\bar{U})} + \sum_{|\alpha|=k} [D^{\alpha}u]_{C^{0,\gamma}(\bar{U})}.$$

The **Holder Space** is

$$C^{k,\gamma}(\bar{U}) := \left\{ u \in C^k(\bar{U}) : ||u||_{C^{k,\gamma}(\bar{U})} < \infty \right\}.$$

Theorem 2.1.

$$\left(C^{k,\gamma}(\bar{U}), ||\cdot||_{C^{k,\gamma}(\bar{U})} \right)$$

is a Banach Space.

2.2 Weak derivative and Sobolev Spaces

Theorem 2.2. *For $u \in C^k(U), \phi \in C_c^{\infty}(U), |\alpha| = k$, integration by parts gives:*

$$\int_U u D^{\alpha} \phi \, dx = (-1)^{|\alpha|} \int_U D^{\alpha} u \phi \, dx.$$

Definition 2.5. Suppose $u, v \in L_{loc}^1(U)$, then v is the $\alpha^{th}-$**weak derivative** of u if

$$\forall \phi \in C_c^{\infty}(U), \int_U u D^{\alpha} \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx.$$

If v exists, we say that $D^{\alpha}u = v$ in the weak sense. Otherwise, u does not possess a αth weak derivative.

Theorem 2.3. *Suppose $v \in L_{loc}^1(U)$ be such that*

$$\forall \phi \in C_c^{\infty}(U), \int_U \phi v \, dx = 0,$$

we must have $v = 0 a.e..$

Proposition 2.4. *If $D^{\alpha}u$ exists, it is uniquely defined up to a set of measure zero.*

Proof. Suppose $v, \tilde{v}$ are both $D^\alpha u$, then $\forall \phi \in C_c^\infty(U)$,

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi \, dx.$$

Thus $\forall \phi \in C_c^\infty(U), \int_U (v - \tilde{v}) \phi \, dx = 0$.
By the previous theorem, we have that $v = \tilde{v}$ a.e.. $\qquad \square$

Definition 2.6. Let $k \in \mathbb{N}, 1 \leq p \leq \infty, u \in L_{loc}^1(U)$, suppose $D^\alpha u$ exists in the weak sense for each $|\alpha| \leq k$. The **Sobolev norm** is

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(U)}^p \right)^{1/p}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_{x \in U} |D^\alpha u(x)| \simeq \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(U)}, & p = \infty \end{cases}.$$

Definition 2.7. For $k = 1$, we write

$$\|Du\|_{L^p(U)}^p := \int_U \|Du\|_p^p \, dx = \int_U \sum_{i=1}^n |\partial_i u|^p \, dx = \sum_{i=1}^n \|\partial_i u\|_{L^p(U)}^p$$

for $1 \leq p < \infty$, and

$$\|Du\|_{L^\infty(U)} := \operatorname*{ess\,sup}_{x \in U} \|Du(x)\|_1 = \operatorname*{ess\,sup}_{x \in U} \sum_{i=1}^n |\partial_i u(x)| = \sum_{i=1}^n \|\partial_i u\|_{L^\infty(U)}$$

for $p = \infty$.
In this case,

$$\|u\|_{W^{1,p}(U)} = \begin{cases} \left(\|u\|_{L^p(U)}^p + \|Du\|_{L^p(U)}^p \right)^{1/p} & 1 \leq p < \infty \\ \|u\|_{L^\infty(U)} + \|Du\|_{L^\infty(U)} & p = \infty. \end{cases}.$$

Proposition 2.5. *Let $k \in \mathbb{N}, 1 \leq p \leq \infty, u \in L_{loc}^1(U)$, we have*

$$\forall |\alpha| \leq k, \|u\|_{W^{k,p}(U)} \geq \|D^\alpha u\|_{L^p(U)}.$$

Definition 2.8. The Sobolev space is defined as

$$W^{k,p}(U) := \left\{ v \in L_{loc}^1(U) : \|v\|_{W^{k,p}(U)} < \infty \right\}.$$

Definition 2.9.
$$H^k(U) := W^{k,2}(U).$$

Remark.
$$W^{0,1}(U) = H^0(U) = L^2(U).$$

Definition 2.10. Let $(u_m)_{m=1}^\infty, u \in W^{k,p}(U)$, then

- $u_m \to u$ in $W^{k,p}(U)$ if $\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(U)} = 0$.

- $u_m \to u$ in $W_{loc}^{k,p}(U)$ if $u_m \to u$ in $W^{k,p}(V)$ for all $V \subset\subset U$.

Definition 2.11.

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)} = \left\{ u \in W^{k,p}(U) : \exists (u_m)_{m=1}^\infty \subset C_c^\infty(U) \text{ such that } u_m \to u \text{ in } W^{k,p}(U) \right\}.$$

$$H_0^k(U) = W_0^{k,2}.$$

Remark. $W_0^{k,p}(U)$ are those $u \in W^{k,p}(U)$ such that $D^\alpha u = 0$ on ∂U.

Theorem 2.6. *Assume* $u, v \in W^{k,p}(U), |\alpha| \le k$, *then*

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$.

2. $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u, \forall \alpha, \beta$ *such that* $|\alpha| + |\beta| \le k$.

3. $\lambda u + v \in W^{k,p}(U), D^\alpha(\lambda u + v) = \lambda D^\alpha u + D^\alpha v, \forall \lambda \in \mathbb{R}$.

4. $\forall V \subseteq U$ *be open,* $u \in W^{k,p}(U)$.

Proof. 1. This is by definition.

2. Consider any $\phi \in C_c^\infty(U)$, we have

$$\int_U D^\alpha(D^\beta u)\phi dx = (-1)^{|\alpha|} \int_U D^\beta u D^\alpha \phi dx$$

$$= (-1)^{|\alpha|}(-1)^{|\beta|} \int_U u D^\beta(D^\alpha \phi)dx$$

$$= (-1)^{|\alpha+\beta|} \int_U u D^{\alpha+\beta}\phi dx$$

$$= \int_U D^{\alpha+\beta} u \phi dx.$$

Thus $D^{\alpha+\beta}u = D^\alpha(D^\beta u)$. Similarly, $D^{\alpha+\beta}u = D^\beta(D^\alpha u)$.

3. See A2.

4. See A2.

$\square$

Proposition 2.7. *Assume* $u \in W^{k,p}(U), |\alpha| \le k$. *If* $\xi \in C_c^\infty(U), \xi u \in W^{k,p}(U)$, *and the Leibniz formula holds:*

$$D^\alpha(\xi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} u,$$

where $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, *and* $\alpha! := \alpha_1! \cdots \alpha_n!$.

Proof. We have $\forall \phi \in C_c^\infty(U), \int_U \xi u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha(\xi u)\phi dx$.
We prove by induction:
The base case is $|\alpha| = 1$, we have

$$D^\alpha(\xi \phi) = \xi D^\alpha \phi + \phi D^\alpha \xi$$

$$\int_U \xi u D^\alpha \phi dx = \int_U u(D^\alpha(\xi \phi) - \phi D^\alpha \xi)dx$$

$$= \int_U u D^\alpha(\xi \phi)dx - \int_U u\phi D^\alpha \xi dx$$

$$= -\int_U \xi \phi D^\alpha u dx - \int_U u\phi D^\alpha \xi dx$$

$$= -\int_U \phi(\xi D^\alpha u + u D^\alpha \xi)dx.$$

Since this hold for any $\phi \in C_c^\infty(U)$, we have

$$\xi D^\alpha u + u D^\alpha \xi = D^\alpha(u\xi).$$

Now suppose $l < k$ and the result holds $\forall |\beta| \leq l$.
Consider any $|\alpha| = l+1$, we have $\alpha = \beta + \gamma$ where $|\beta| = l, |\gamma| = 1$.

$$
\begin{aligned}
\int_U \xi u D^\alpha \phi \, dx &= \int_U \xi u D^\beta (D^\gamma \phi) \, dx \\
&= (-1)^{|\beta|} \int_U D^\beta (\xi u) D^\gamma \phi \, dx \\
&= (-1)^{|\beta|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\beta - \sigma} u D^\gamma \phi \, dx \\
&= (-1)^{|\beta|}(-1)^{|\gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\gamma (D^\sigma \xi D^{\beta - \sigma} u) \phi \, dx \\
&= (-1)^{|\beta + \gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} \left(D^\sigma \xi D^\gamma D^{\beta - \sigma} u + D^{\beta - \sigma} u D^\gamma D^\sigma \xi \right) \phi \, dx \\
&= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} \left(D^\sigma \xi D^{\gamma + \beta - \sigma} u + D^{\beta - \sigma} u D^{\gamma + \sigma} \xi \right) \phi \, dx \\
&= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} \left(D^\sigma \xi D^{\alpha - \sigma} u + D^{\alpha - (\gamma + \sigma)} u D^{\gamma + \sigma} \xi \right) \phi \, dx \\
&= (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \xi D^{\alpha - \sigma} u + \sum_{\rho \leq \alpha, \rho_j \geq 1} \binom{\beta}{\rho - \gamma} D^{\alpha - \rho} u D^\rho \xi \right) \phi \, dx \\
&= (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \leq \alpha} \left(\mathbb{1}_{\sigma_j \leq \alpha_j - 1} \binom{\beta}{\sigma} + \mathbb{1}_{\sigma_j \geq 1} \binom{\beta}{\sigma - \gamma} \right) D^\sigma \xi D^{\alpha - \sigma} u \right) \phi \, dx,
\end{aligned}
$$

where $\gamma_i = \delta_{ij}$.
Now consider any $\sigma \leq \alpha$.
If $\sigma_j = 0$, we have

$$
\begin{aligned}
\binom{\beta}{\sigma} &= \frac{\beta!}{\sigma!(\beta - \sigma)!} \\
&= \frac{\beta!(\beta_j + 1)}{\sigma!(\beta - \sigma)!(\beta_j + \sigma_j + 1)} \\
&= \frac{(\beta + \gamma)!}{\sigma!(\beta - \sigma + \gamma)!} \\
&= \frac{\alpha!}{\sigma!(\alpha - \sigma)!} \\
&= \binom{\alpha}{\sigma}.
\end{aligned}
$$

If $\sigma_j = \alpha_j$, we have

$$\binom{\beta}{\sigma - \gamma} = \frac{\beta!}{(\sigma - \gamma)!(\beta - \sigma + \gamma)!}$$
$$= \frac{\beta!\alpha_j}{\alpha_j(\sigma - \gamma)!(\alpha - \sigma)!}$$
$$= \frac{\beta!(\beta_j + 1)}{\sigma_j(\sigma - \gamma)!(\alpha - \sigma)!}$$
$$= \frac{(\beta + \gamma)!}{(\sigma - \gamma + \gamma)!(\alpha - \sigma)!}$$
$$= \frac{\alpha!}{\sigma!(\alpha - \sigma)!}$$
$$= \binom{\alpha}{\sigma}.$$

If $1 \le \sigma_j \le \alpha_j - 1$, we have

$$\binom{\beta}{\sigma} + \binom{\beta}{\sigma - \gamma} = \frac{\beta!}{\sigma!(\beta - \sigma)!} + \frac{\beta!}{(\sigma - \gamma)!(\beta - \sigma + \gamma)!}$$
$$= \frac{\beta!(\beta_j - \sigma_j + 1)}{\sigma!(\beta - \sigma)!(\beta_j - \sigma_j + 1)} + \frac{\beta!\sigma_j}{\sigma_j(\sigma - \gamma)!(\beta - \sigma + \gamma)!}$$
$$= \frac{\beta!(\beta_j - \sigma_j + 1) + \beta!\sigma_j}{\sigma!(\beta - \sigma + \gamma)!}$$
$$= \frac{\beta!(\beta_j + 1)}{\sigma!(\beta - \sigma + \gamma)!}$$
$$= \frac{(\beta + \gamma)!}{\sigma!(\beta - \sigma + \gamma)!}$$
$$= \frac{\alpha!}{\sigma!(\alpha - \sigma)!}$$
$$= \binom{\alpha}{\sigma}.$$

Thus we can see that

$$\int_U \xi u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U \left(\sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha - \sigma} u \right) \phi \, dx.$$

Since ϕ is arbitrary, we have that

$$D^\alpha(\xi u) = \sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^\sigma \xi D^{\alpha - \sigma} u.$$

Inductively, we can prove this for any $|\alpha| = n \ge 1$. $\qquad \square$

Theorem 2.8. $\left(W^{k,p}(U), ||\cdot||_{W^{k,p}(U)} \right)$ *is a Banach space for* $k \in \mathbb{N}, 1 \le p \le \infty$.

Proof. See A2 for the proof of $||\cdot||_{W^{1,\infty}(U)}$ is a norm.
Now for $1 \le p < \infty$, we want to check:

1. If $||u||_{W^{k,p}(U)} = 0$, then $||u||_{L^p(U)} = 0$, and thus $u = 0$ a.e. on U.

2. If $u = 0$ a.e. on U, then $\forall \phi \in C_c^\infty(U)$, we have

$$\int_U D^\alpha u \phi \, dx = (-1)^{|\alpha|} \int_U u D^\alpha \phi \, dx = 0.$$

Thus $D^\alpha u = 0$ a.e. for any $|\alpha| \le k$.
Thus $||u||_{W^{k,p}(U)} = 0$.

3. Let $\lambda \in \mathbb{R}$, we have

$$
\begin{aligned}
||\lambda u||_{W^{k,p}(U)} &= \left(\sum_{|\alpha| \le k} ||D^\alpha(\lambda u)||_{L^p(U)}^p \right)^{1/p} \\
&= \left(\sum_{|\alpha| \le k} ||\lambda D^\alpha(u)||_{L^p(U)}^p \right)^{1/p} \\
&= \left(\sum_{|\alpha| \le k} |\lambda|^p ||D^\alpha(u)||_{L^p(U)}^p \right)^{1/p} \\
&= |\lambda| \left(\sum_{|\alpha| \le k} ||D^\alpha(u)||_{L^p(U)}^p \right)^{1/p} \\
&= |\lambda| ||u||_{W^{k,p}(U)}.
\end{aligned}
$$

4. Consider any $u, v \in W^{k,p}(U)$,

$$
\begin{aligned}
||u + v||_{W^{k,p}(U)} &= \left(\sum_{|\alpha| \le k} ||D^\alpha(u+v)||_{L^p(U)}^p \right)^{1/p} \\
&\le \left(\sum_{|\alpha| \le k} \left(||D^\alpha u||_{L^p(U)} + ||D^\alpha v||_{L^p(U)} \right)^p \right)^{1/p} \\
&\le \left(\sum_{|\alpha| \le k} ||D^\alpha u||_{L^p(U)}^p \right)^{1/p} + \left(\sum_{|\alpha| \le k} ||D^\alpha u||_{L^p(U)}^p \right)^{1/p} \\
&= ||u||_{W^{k,p}(U)} + ||v||_{W^{k,p}(U)}.
\end{aligned}
$$

Thus $||\cdot||_{W^{k,p}(U)}$ is a norm.
Consider any Cauchy sequence $(u_m)_{m=1}^\infty$.
Given any $\epsilon > 0, \exists N \ge 1$, such that $\forall n, m \ge N, ||u_m - u_n||_{W^{k,p}(U)} < \epsilon$.
Consider any $|\alpha| \le k$, we have

$$
||D^\alpha u_m - D^\alpha u_n||_{L^p(U)} = ||u||_{W^{k,p}(U)} \ge ||D^\alpha(u_m - u_n)||_{L^p(U)} \le ||u_m - u_n||_{W^{k,p}(U)} < \epsilon.
$$

Thus $(D^\alpha u_n)_{n=1}^\infty$ must be a Cauchy sequence in $(L^p(U), ||\cdot||_{L^p(U)})$ for any $|\alpha| \le k$.
Since $(L^p(U), ||\cdot||_{L^p(U)})$ is complete, there must be some

$$
u_\alpha \in L^p(U) \text{ such that } \lim_{n \to \infty} ||u_\alpha - D^\alpha u_n||_{L^p(U)} = 0.
$$

In particular, we have $u \in L^p(U)$, such that $\lim_{n \to \infty} ||u - u_n||_{L^p(U)} = 0$.
Now consider any $|\alpha| \le k$.

Given any $\phi \in C_c^\infty(U)$, we have

$$\left| \int_U u D^\alpha \phi dx - \int_U u_n D^\alpha \phi dx \right| = \left| \int_U (u - u_n) D^\alpha \phi dx \right|$$

$$\leq \int_U |(u - u_n) D^\alpha \phi| dx$$

$$\leq ||u - u_n||_{L^p(U)} ||D^\alpha \phi||_{L^{\frac{p}{p-1}}(U)},$$

$$\left| \int_U u_\alpha \phi dx - \int_U D^\alpha u_n \phi dx \right| = \left| \int_U (u_\alpha - D^\alpha u_n) \phi dx \right|$$

$$\leq \int_U |(u_\alpha - D^\alpha u_n) \phi| dx$$

$$\leq ||u_\alpha - D^\alpha u_n||_{L^p(U)} ||\phi||_{L^{\frac{p}{p-1}}(U)}.$$

Since $u_n \to u, D^\alpha u_n \to u_\alpha$ in $L^p(U)$, and $||\phi||_{L^{\frac{p}{p-1}}(U)}, ||D^\alpha \phi||_{L^{\frac{p}{p-1}}(U)} < \infty$, those two limits converges to 0. Thus we have

$$\int_U u D^\alpha \phi dx = \lim_{n \to \infty} \int_U u_n D^\alpha \phi dx$$

$$= \lim_{n \to \infty} (-1)^{|\alpha|} \int_U D^\alpha u_n \phi dx$$

$$= (-1)^{|\alpha|} \int_U u_\alpha \phi dx.$$

Since this is true for any $\phi \in C_c^\infty(U)$, we have that $D^\alpha u = u_\alpha = \lim_{n \to \infty} D^\alpha u_n$ in $L^p(U)$. Since this is true for any $|\alpha| \leq k$, we have that $u_n \to u$ in $W^{k,p}(U)$. $\qquad \square$

Proposition 2.9. *For any $1 \leq s \leq r < \infty, k \geq 1$, and bounded U, we have some constant $C := |U|^{\frac{1}{s} - \frac{1}{r}} m^{\frac{1}{s} - \frac{1}{r}}$, where $m = |\{\beta \in \mathbb{N}^n : |\beta| \leq k\}|$, such that*

$$\forall u \in W^{k,r}(U), \ ||u||_{W^{k,s}(U)} \leq C||u||_{W^{k,r}(U)}, u \in W^{k,s}(U).$$

Proof. We have

$$||u||_{W^{1,s}(U)}^s = \sum_{|\beta| \leq 1} ||D^\beta u||_{L^s(U)}^s$$

$$\leq \sum_{|\beta| \leq 1} \left(|U|^{\frac{1}{s} - \frac{1}{r}} ||D^\beta u||_{L^r(U)} \right)^s$$

$$= \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s \sum_{|\beta| \leq 1} ||D^\beta u||_{L^r(U)}^{r * \frac{s}{r}}$$

$$\leq \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \left(\sum_{|\beta| \leq 1} ||D^\beta u||_{L^r(U)}^r \right)^{\frac{s}{r}}$$

$$= \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} \left(||D^\alpha u||_{W^{1,r}(U)}^r \right)^{\frac{s}{r}}$$

$$= \left(|U|^{\frac{1}{s} - \frac{1}{r}} \right)^s m^{1 - \frac{s}{r}} ||D^\alpha u||_{W^{1,r}(U)}^s$$

$$\implies$$

$$||u||_{W^{1,s}(U)} \leq |U|^{\frac{1}{s} - \frac{1}{r}} m^{\frac{1}{s} - \frac{1}{r}} ||D^\alpha u||_{W^{1,r}(U)}.$$

$\square$

2.2.1 Weak and Normal Derivatives

Proposition 2.10. *If $u, v \in C(U)$ are both continuous, and $u = v$ a.e., then $\forall x \in U, u(x) = v(x)$.*

Proof. Consider any $x \in U$.

Since U is open, we can find some $r > 0$, such that $B(x, r) \subseteq U$.

For any $i \geq \lceil \frac{1}{r} \rceil$, we must have some $x_i \in B(x, \frac{1}{i}) \subseteq U$, such that $u(x_i) = v(x_i)$.

Otherwise $\{x \in U : u(x) \neq v(x)\} \supseteq B(x, \frac{1}{i}) \cap U = B(x, \frac{1}{i})$ does not have measure 0.

Thus $\lim_{i \to \infty} x_i = x$.

Since u, v are both continuous, we have that

$$
\begin{aligned}
u(x) &= u\left(\lim_{i \to \infty} x_i\right) \\
&= \lim_{i \to \infty} u(x_i) \\
&= \lim_{i \to \infty} v(x_i) \\
&= v\left(\lim_{i \to \infty} x_i\right) \\
&= v(x).
\end{aligned}
$$

This is true for any $x \in U$, which completes the proof. $\qquad \square$

Remark. For the following part in this subsection, we will use D^α to denote the αth normal derivative of u, and $\bar{D}^\alpha$ to be the αth weak derivative of u to avoid confusion.

Proposition 2.11. *Given any $\alpha \in \mathbb{N}^n$. $\forall u$ such that its αth normal derivative $D^\alpha u$ exists and is continuous, and $v = u$ a.e., we have that $D^\alpha u$ is an αth weak derivative of v. Namely, $\bar{D}^\alpha v = D^\alpha u$ a.e..*

Proof. Consider any $\phi \in C_c^\infty(U)$, we have that

$$
\begin{aligned}
\int_U v D^\alpha \phi \, dx &= \int_U u D^\alpha \phi \, dx \\
&= (-1)^{|\alpha|} \int_U D^\alpha u \phi \, dx,
\end{aligned}
$$

where the second equality follows from integration by part over some $\mathrm{Supp}(\phi) \subseteq V \subseteq U$ with Lipschitz boundary. $\qquad \square$

Definition 2.12. A domain U is **path-connected** if $\forall x, y$, there is some continuous path $\gamma : [0,1] \to U$, such that $\gamma(0) = x, \gamma(1) = y$.

Proposition 2.12. *Let $U \subseteq \mathbb{R}^n$ be open and connected, and $1 \leq p \leq \infty$, $u \in W^{1,p}(U)$, then $\bar{D}u = 0$ a.e. $\iff u$ is a constant a.e..*

Lemma 2.13. *Consider $1 \leq p \leq \infty$, and $U = (a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n$ be an open rectangle. Let $1 \leq i \leq n$, suppose $u \in W^{1,p}(U)$ has a continuous representative $u^* \in C(U)$, and its ith weak derivative $\bar{\partial}_i u$ has a continuous representative $(\bar{\partial}_i u)^* \in C(U)$, then the regular i^{th} partial derivative*

$$
\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \ \forall x \in U
$$

exists and is continuous.

Proof. Pick some $s \in (a_i, b_i)$, let $S := \{x \in U : x^i = s\}$ be the slice of U.

By FTC, there is a unique v, defined by

$$
v(x^1, \ldots, x^n) := u^*(x^1, \ldots, x^{i-1}, s, x^{i+1}, \ldots x^n) + \int_s^{x^i} (\bar{\partial}_i u)^*(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n) dt,
$$

such that $v|_S = u^*|_S$, and the ith normal partial derivative

$$\partial_i v(x) = (\bar{\partial}_i u)^*(x) \; \forall x \in U.$$

We notice that $\bar{\partial}_i v = \partial_i v$ a.e. by 2.11.
Thus the weak derivative $\bar{\partial}_i(u^* - v) = \bar{\partial}_i(u^*) - \bar{\partial}_i v = \bar{\partial}_i u - \partial_i v = \bar{\partial}_i u - (\bar{\partial}_i u)^* = 0$ a.e..
Fix any $(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots x^n)$, and denote $w : (a_i, b_i) \to \mathbb{R}$ by

$$w(t) := (u^* - v)(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n).$$

We have that $\bar{D}w = \bar{\partial}_i(u^* - v) = 0$ with respect to $t \in (a_i, b_i)$ a.e..
By 2.12, $w(t) = C$ a.e. $t \in (a_i, b_i)$ form some constant C, since (a_i, b_i) is clearly connected.
Notice that w is continuous, since both u^*, v are continuous on the x^i direction.
Since both w, C are continuous, we have $\forall t \in (a_i, b_i), w(t) = C$.
Since $v|_S = u^*|_S$, we must have $C = w(s) = 0$ and thus

$$\forall t \in (a_i, b_i), u^*(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n) = v(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots x^n).$$

Since this holds for all $(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots x^n)$, we must have $u^*(x) = v(x) \; \forall x \in U$.
By construction of v, we have that

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \; \forall x \in U.$$

$\square$

Lemma 2.14. *Consider $1 \le p \le \infty$, and $U \subseteq \mathbb{R}^n$ be open. If $u \in W^{1,p}(U)$ has a continuous representative $u^* \in C(U)$, and its weak derivative $\partial_i u$ has a continuous representative $(\bar{\partial}_i u)^* \in C(U)$, then the regular i^{th} partial derivative*

$$\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) \; \forall x \in U$$

exists and is continuous.

Proof. Notice that any open $U \subseteq \mathbb{R}^n$ can be written as $\bigcup_{j=1}^{\infty} R_j$, where each R_j is an open rectangle.
Fix any $x \in U$, there must be some $R_j \ni x$.
By previous lemma, $\partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x)$.
Since this holds for any $x \in U$, we have our result. $\square$

Proposition 2.15. *Consider $1 \le p \le \infty, k \ge 0$, and $U \subseteq \mathbb{R}^n$ be open. If $u \in W^{k,p}(U)$ has a continuous representative $u^* \in C(U)$, and all of its weak derivatives $D^\alpha u$ have continuous representatives $(D^\alpha u)^* \in C(U)$ for any $|\alpha| \le k$, then*

$$u^* \in C^k(U), \; D^\alpha(u^*)(x) = (\bar{D}^\alpha u)^*(x) \; \forall x \in U, \forall |\alpha| \le k.$$

Proof. We will use induction on k.
The base case is $k = 0$.
Since $|\alpha| = 0$, we trivially have $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$.
Now, suppose this holds for $k - 1$.
Consider any $u \in W^{k,p}(U)$.
If $|\alpha| = 0$, we trivially have $D^\alpha(u^*)(x) = u^*(x) = (\bar{D}^\alpha u)^*(x)$ as before.
Now consider any $|\gamma| = 1$. We know $\gamma = e_i$ for some $1 \le i \le n$.
By previous lemma, we have that

$$D^\gamma(u^*)(x) = \partial_i(u^*)(x) = (\bar{\partial}_i u)^*(x) = (\bar{D}^\gamma u)^*(x) \; \forall x \in U.$$

Notice that $\bar{D}^\gamma u \in W^{k-1,p}(U)$, and all of its weak derivatives $\bar{D}^\beta \bar{D}^\gamma u = \bar{D}^{\beta+\gamma} u$ have continuous representatives $(\bar{D}^{\beta+\gamma} u)^* \in C(U)$ for any $|\beta| \le k - 1$. By the induction hypothesis, we have that

$$D^\beta((\bar{D}^\gamma u)^*)(x) = (\bar{D}^{\beta+\gamma} u)^*(x) \; \forall x \in U, \forall |\beta| \le k - 1.$$

27

For any $1 \leq |\alpha| \leq k$, we can have $\alpha = \beta + \gamma$, where $|\beta| \leq k - 1, |\gamma| = 1$.
Now we have $\forall x \in U$,

$$
\begin{aligned}
(\bar{D}^\alpha u)^*(x) &= (\bar{D}^{\beta+\gamma} u)^*(x) \\
&= D^\beta((\bar{D}^\gamma u)^*)(x) \\
&= D^\beta(D^\gamma(u^*))(x) \\
&= D^{\beta+\gamma}(u^*)(x) \\
&= D^\alpha(u^*)(x).
\end{aligned}
$$

We have thus proven the result for any $|\alpha| \leq k$.

Since all of its αth derivatives exists and are continuous, we further have that $u^* \in C^k(U)$. $\qquad \square$

2.3 Convolution and Mollification

Definition 2.13. For $f, g : \mathbb{R}^n \to \mathbb{R}$, we define the **convolution** $f * g : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ to be

$$
(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y)dy.
$$

Proposition 2.16.

$$
\mathrm{Supp}(f * g) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(g).
$$

Proof. Let $f^x(y) := f(x - y)$, we have $f * g(x) = \int_{\mathbb{R}^n} f^x(y)g(y)dy$.
Suppose $\mathrm{Supp}(f^x) \cap \mathrm{Supp}(g) = \emptyset$, then we have $(f * g)(x) = 0$.
In addition,

$$
\begin{aligned}
&\mathrm{Supp}(f^x) \cap \mathrm{Supp}(g) \neq \emptyset \\
\iff\;& \exists y, x - y \in \mathrm{Supp}(f), y \in \mathrm{Supp}(g) \\
\iff\;& x \in \mathrm{Supp}(f) + \mathrm{Supp}(g).
\end{aligned}
$$

Thus $\mathrm{Supp}(f * g) \subseteq \{x \in \mathbb{R}^n : \mathrm{Supp}(f^x) \cap \mathrm{Supp}(g) \neq \emptyset\} = \mathrm{Supp}(f) + \mathrm{Supp}(g)$. $\qquad \square$

Proposition 2.17. *(Young's Convolution Inequality)*
Let $f \in L^1(\mathbb{R}^n), g \in L^p(\mathbb{R}^n)$ *with* $1 \leq p \leq \infty$*, then for a.e.* $x \in \mathbb{R}^n$*, the function* $f(x - y)g(y)$ *is integrable.*
Thus $f * g : \mathbb{R}^n \to \mathbb{R}$ *is well-defined a.e.. In addition,* $f * g \in L^p(\mathbb{R}^n)$*, and*

$$
||f * g||_{L^p(\mathbb{R}^n)} \leq ||f||_{L^1(\mathbb{R}^n)} ||g||_{L^p(\mathbb{R}^n)}.
$$

Definition 2.14.

$$
\bar{B}(x, r) = \{y \in \mathbb{R}^n : ||x - y|| \leq r\}
$$

is the closed ball around x of radius r, and

$$
B(x, r) = \{y \in \mathbb{R}^n : ||x - y|| < r\}
$$

is the closed ball around x of radius r.

Definition 2.15. For $\epsilon > 0$,

$$
U_\epsilon := \{x \in U : \mathrm{dist}(x, \partial u) > \epsilon\}.
$$

Remark. This definition does not require U to be bounded.

Definition 2.16. The **standard mollifier** $\eta(x) \in C^\infty(\mathbb{R}^n)$ is defined as

$$
\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|-1}\right), & |x| < 1 \\ 0, & o.w. \end{cases},
$$

with C such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.
For each $\epsilon > 0$,

$$
\eta_\epsilon := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).
$$

Proposition 2.18. $\forall \epsilon > 0$, we have

1. $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$,

2. $\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1$,

3. $\text{Supp}(\eta_\epsilon) \subseteq \bar{B}(0, \epsilon)$.

Definition 2.17. Let $f \in L^1_{loc}(U), \epsilon > 0$, its **mollification** $f^\epsilon : U_\epsilon \to \mathbb{R}$ is defined as

$$f^\epsilon(x) := \eta_\epsilon * f = \int_U \eta_\epsilon(x - y)f(y)dy = \int_{\bar{B}(0,\epsilon)} f(x - z)\eta_\epsilon(z)dz.$$

Theorem 2.19. Let f^ϵ be defined as above, we have:

1. $f^\epsilon \in C^\infty(U_\epsilon)$,

2. $D^\alpha(f^\epsilon) = (D^\alpha \eta_\epsilon) * f$ on U_ϵ,

3. $f^\epsilon \to f$ a.e., as $\epsilon \to 0$,

4. If $f \in C(U)$, we have $f^\epsilon \to f$ uniformly on compact subsets of U,

5. If $1 \leq p < \infty, f \in L^p_{loc}(U)$, we have $f^\epsilon \to f$ in $L^p_{loc}(U)$. Namely, $f^\epsilon \to f$ in $L^p(V), \forall V \subset\subset U$.

6. $\text{Supp}(f^\epsilon) \subseteq \text{Supp}(f) + \text{Supp}(\eta_\epsilon) = \text{Supp}(f) + \bar{B}(0, \epsilon)$.

Proposition 2.20. For any $u \in W^{k,p}(U)$, and $|\alpha| \leq k, \epsilon > 0$, we have that

$$D^\alpha u^\epsilon|_{U_\epsilon} = (\eta_\epsilon * D^\alpha u)|_{U_\epsilon}.$$

Proof. Fix any $x \in U_\epsilon$, we have

$$\begin{aligned}
D^\alpha u^\epsilon(x) &= D^\alpha(\eta_\epsilon * u)(x) \\
&= (D^\alpha \eta_\epsilon * u)(x) \qquad\qquad 2.19 \\
&= \int_U D^\alpha \eta_\epsilon(x - y)u(y)dy,
\end{aligned}$$

Consider $\eta_{\epsilon,x}(y) := \eta_\epsilon(x - y)$, we can see $\forall i \in [n], \ \partial_i \eta_{\epsilon,x}(y) = -\partial_i \eta_\epsilon(x - y)$, thus we have

$$\begin{aligned}
D^\alpha u^\epsilon(x) &= \int_U D^\alpha \eta_\epsilon(x - y)u(y)dy \\
&= (-1)^{|\alpha|} \int_U D^\alpha \eta_{\epsilon,x}(y)u(y)dy \\
&= \int_U \eta_{\epsilon,x}(y)D^\alpha u(y)dy \\
&= \int_U \eta_\epsilon(x - y)D^\alpha u(y)dy \\
&= (\eta_\epsilon * D^\alpha u)(x).
\end{aligned}$$

Since this holds for any $x \in U_\epsilon$, we proved our result. $\qquad\square$

Proposition 2.21. Let $1 \leq p \leq \infty$. Let $u \in L^p(\mathbb{R}^n)$, then for any $\epsilon > 0$, $V \supseteq \text{Supp}(u^\epsilon) \supseteq \text{Supp}(u)$, we have

$$||u^\epsilon||_{L^p(V)} \leq ||u||_{L^p(V)}.$$

Proof. By 2.17, we have

$$\|u^\epsilon\|_{L^p(V)} = \|\eta_\epsilon * u\|_{L^p(V)}$$
$$= \|\eta_\epsilon * u\|_{L^p(\mathbb{R}^n)}$$
$$\leq \|\eta_\epsilon\|_{L^1(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R}^n)}$$
$$= \left(\int_{\mathbb{R}^n} |\eta_\epsilon(x)|dx\right)\|u\|_{L^p(V)}$$
$$= \|u\|_{L^p(V)}.$$

$\square$

Corollary 2.22. *Let $1 \leq p \leq \infty, k \geq 1$. Let $u \in W^{k,p}(\mathbb{R}^n)$, then for any $\epsilon > 0$, $V \supseteq \text{Supp}(u) + \bar{B}(0,\epsilon)$, we have that*

$$\|u^\epsilon\|_{W^{k,p}(V)} \leq \|u\|_{W^{k,p}(V)}.$$

Proof. By 2.20, $\forall |\alpha| \leq k$, we have $D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$ on the entire $\mathbb{R}^n$.

$$\|u^\epsilon\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\leq k} \|D^\alpha u^\epsilon\|_{L^p(V)}$$
$$= \sum_{|\alpha|\leq k} \|\eta_\epsilon * D^\alpha u\|_{L^p(V)}$$
$$\leq \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p(V)}$$
$$= \|u\|^p_{W^{k,p}(V)},$$

since $\forall |\alpha| \leq k, \text{Supp}(D^\alpha u) \subseteq \text{Supp}(u)$, and thus $\text{Supp}(\eta_\epsilon * D^\alpha u) \subseteq \text{Supp}(u) + \bar{B}(0,\epsilon) \subseteq V$. $\square$

2.4 Smooth Approximation

Theorem 2.23. *(Local Smooth Approximation)*
Let $1 \leq p < \infty, k \geq 1$. Suppose U is open, and $u \in W^{k,p}(U)$, we have that

1. *$\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$,*

2. *$u^\epsilon \to u$ in $W^{k,p}_{loc}(U)$ as $\epsilon \to 0$.*

Proof. $\forall \epsilon > 0, u^\epsilon \in C^\infty(U_\epsilon)$ by 2.19.1.
Given any $V \subset\subset U$, we can find some $\epsilon_V > 0$ such that $V \subset\subset U^{\epsilon_V}$.
Consider any $|\alpha| \leq k$.
We have $D^\alpha u \in L^p(U) \subseteq L^p_{loc}(U)$.
By 2.19.5, we have that $\eta_\epsilon * D^\alpha u \to D^\alpha u$ in $L^p_{loc}(U)$ as $\epsilon \to 0$, and thus $\eta_\epsilon * D^\alpha u \to D^\alpha u$ in $L^p(V)$.
In addition, by 2.20, $\forall \epsilon > 0$, $D^\alpha(u^\epsilon) = \eta_\epsilon * D^\alpha u$ in U^ϵ.
Now $\forall 0 < \epsilon < \epsilon_V$, $V \subset\subset U^{\epsilon_V} \subseteq U^\epsilon$, and thus $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$ in V.
Thus $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$.
Since this is true $\forall |\alpha| \leq k$, we have $u^\epsilon \to u$ in $W^{k,p}(V)$.
Since this holds for any $V \subset\subset U$, $u^\epsilon \to u$ in $W^{k,p}_{loc}(U)$. $\square$

Corollary 2.24. *Suppose U is open, and $u \in W^{k,p}(U)$ is compactly supported in U, then $u \in W^{k,p}_0(U)$.*

Proof. Since $\text{Supp}(u) \subset U$ is compact, we must have $r := \frac{1}{2}\text{dist}(\text{Supp}(u), \partial U) > 0$.
For $n \in \mathbb{N}^+$, let $u_n := u^{\frac{r}{n}}$.
We have that $u_n \to u$ in $W^{k,p}_{loc}(U)$ as $n \to \infty$.
Let $W := \overline{\text{Supp}(u) + \bar{B}(0,r/2)} \subset U$. Notice that it is compact, and $\forall n \in \mathbb{N}^+, \text{Supp}(u_m) \subseteq \text{Supp}(u) + \bar{B}(0,\frac{r}{n}) \subseteq W$, which means $u_m \in C^\infty_c(U)$.

Now there is some $W \subset V \subset\subset U$, so $u_m \to u$ in $W^{k,p}(V)$.
In addition,

$$||u - u_m||^p_{W^{k,p}(U)} = \int_U \sum_{|\alpha| \leq k} |D^\alpha (u - u_m)|^p dx$$

$$= \int_V \sum_{|\alpha| \leq k} |D^\alpha (u - u_m)|^p dx$$

$$= ||u - u_m||^p_{W^{k,p}(V)}.$$

Thus $\lim_{m \to \infty} ||u - u_m||_{W^{k,p}(V)} = \lim_{m \to \infty} ||u - u_m||_{W^{k,p}(U)} = 0$.
Since each $u_m \in C_c^\infty(U)$, we have $u \in \overline{C_c^\infty(U)} = W_0^{k,p}(U)$. $\qquad\square$

Theorem 2.25. *(Meyer-Serrin)*
Let $1 \leq p < \infty, k \geq 1$. Suppose U is open and bounded, and $u \in W^{k,p}(U)$. There exists $(u_m)_{m=1}^\infty \subseteq C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.

Proof. Let $\delta > 0$ be given.
Let $U_i := U_{\frac{1}{i}} = \left\{ x \in U : \operatorname{dist}(x, \partial u) > \frac{1}{i} \right\}$ for $i \in \mathbb{N}^+$.
We have $U_1 \subseteq \bar{U}_1 \subseteq U_2 \subseteq \bar{U}_2 \subseteq U_3 \subseteq \cdots U$.
Indeed, for some $x \in \bar{U}_i$, we know that $\forall y \in \partial U, ||x - y|| \geq \frac{1}{i} > \frac{1}{i+1} \implies x \in U_{i+1}$.
Since U is open, for any $x \in U$, we can find some $i \geq 1$, such that $B(x, \frac{1}{i}) \subseteq U$, which means $\operatorname{dist}(x, \partial U) \geq \frac{1}{i}$, and thus $x \in \bar{U}_i \subseteq U_{i+1}$. Thus we have $U = \bigcup_{i=1}^\infty U_i$.
Let $V_i := U_{i+3} \setminus \bar{U}_{i+1}$ for $i \in \mathbb{N}^+$. Since U is bounded, we can choose $V_0 \subset\subset U$ with $V_0 \supset \bar{U}_2$, we claim that $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$.
It is easy to see $\bigcup_{i=0}^n V_i \subseteq U_{i+3}$. For the other direction, we will prove by induction.
The base case $n = 1$, we can see that $V_0 \cup V_1 \supset \bar{U}_2 \cup (U_4 \setminus \bar{U}_2) = U_4$.
Now suppose $n > 1$, and it holds for $n - 1$, we have that

$$\bigcup_{i=0}^n V_i = \left(\bigcup_{i=0}^{n-1} V_i \right) \cup (V_n)$$

$$= U_{n-1+3} \cup \left(U_{n+3} \setminus \bar{U}_{n+1} \right)$$

$$\supset U_{n+2} \cup (U_{n+3} \setminus U_{n+2})$$

$$= U_{n+3}.$$

By induction, we have that $\forall n \geq 1, \bigcup_{i=0}^n V_i = U_{n+3}$.
Notice that $\forall x \in U = \bigcup_{n=1}^\infty U_n, \exists n \geq 1$, such that $x \in U_n \subseteq U_{n+3} \subseteq \bigcup_{i=0}^n V_i \implies \exists i \geq 0$, such that $x \in V_i$.
Thus

$$U = \bigcup_{i=0}^\infty V_i.$$

Now let $W_i := U_{i+4} \setminus \bar{U}_i$ for $i \in \mathbb{N}^+$.
Since each $U_{i+4} \subseteq \bar{U}_{i+4} \subseteq U_{i+5} \subseteq U$, we also have $U_{i+4} \subset\subset U$ and thus

$$W_i \subset\subset U.$$

Notice that $\forall x, y \in U$,

$$\operatorname{dist}(x, \partial U) = \inf \{||z - x|| : z \in \partial U\}$$

$$= \inf \{||z - y + y - x|| : z \in \partial U\}$$

$$\leq \inf \{||z - y|| + ||y - x|| : z \in \partial U\}$$

$$= \inf \{||z - y|| : z \in \partial U\} + ||y - x||$$

$$= \operatorname{dist}(y, \partial U) + ||y - x||.$$

Similarly, $\text{dist}(y, \partial U) \leq \text{dist}(x, \partial U) + ||y - x||$. Thus we have

$$\text{dist}(y, \partial U) - ||y - x|| \leq \text{dist}(x, \partial U) \leq \text{dist}(y, \partial U) + ||y - x||.$$

Consider any $0 < \epsilon < \frac{1}{i+3} - \frac{1}{i+4} < \frac{1}{i} - \frac{1}{i+1}$, we have that

$$
\begin{aligned}
x \in \bar{B}(0, \epsilon) + V_i &\implies \exists y \in U_{i+3} \setminus \overline{U_{i+1}} \text{ such that } ||x - y|| \leq \epsilon \\
&\implies \exists y \in U \text{ such that } \frac{1}{i+3} < \text{dist}(y, \partial U) < \frac{1}{i+1}, ||x - y|| \leq \epsilon \\
&\implies \exists y \in U \text{ such that } \frac{1}{i+3} - ||x - y|| < \text{dist}(x, \partial U) < \frac{1}{i+1} + ||x - y||, ||x - y|| \leq \epsilon \\
&\implies \frac{1}{i+3} - \epsilon < \text{dist}(x, \partial U) < \frac{1}{i+1} + \epsilon \\
&\implies \frac{1}{i+4} < \text{dist}(x, \partial U) < \frac{1}{i} \\
&\implies x \in W_i.
\end{aligned}
$$

Thus we have

$$\forall 0 < \epsilon < \frac{1}{i+3}, \ \bar{B}(0, \epsilon) + V_i \subseteq W_i.$$

Finally, since $V_0 \subset\subset U$, we can choose $V_0 \subset\subset W_0 \subset\subset U$, such that $V_0 + B(0, \epsilon_0'') \subseteq W_i$ for some $\epsilon_0' > 0$.
Let $(\zeta_i)_{i=0}^\infty$ be a **smooth partition of unity** such that

$$\forall x \in U \ \sum_{i=0}^\infty \zeta_i(x) = 1, \ \forall i \geq 0, \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^\infty(U), \\ \text{Supp}(\zeta_i) \subseteq V_i. \end{cases}$$

Notice that $\forall u \in W^{k,p}(U)$, we have $\zeta_i u \in W^{k,p}(U)$ as well. Moreover, $\text{Supp}(\zeta_i u) \subseteq V_i$.
Let $u_i^\epsilon := \eta_\epsilon * (\zeta_i u) \ \forall \epsilon > 0$.
By previous theorem, we have that $u_i^\epsilon \to \zeta_i u$ in $W_{loc}^{k,p}(U)$.
Thus for $W_i \subset\subset U$, we can find $\epsilon_i' > 0$ such that $\forall \epsilon < \epsilon_i'$, $||u_i^\epsilon - \zeta_i u||_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}$.
Now pick $\epsilon_0 = \min(\epsilon_0'', \epsilon_0'), \forall i \in \mathbb{N}^+, \epsilon_i = \frac{1}{2} \min\left(\frac{1}{i+3} - \frac{1}{i+4}, \epsilon_i'\right) > 0$.
We have that by 2.19,

$$\text{Supp}(u_i^{\epsilon_i}) \subseteq \text{Supp}(\eta_{\epsilon_i}) + \text{Supp}(\zeta_i u) \subseteq \bar{B}(0, \epsilon_i) + V_i \subseteq W_i,$$

and

$$||u_i^{\epsilon_i} - \zeta_i u||_{W^{k,p}(U)} = ||u_i^{\epsilon_i} - \zeta_i u||_{W^{k,p}(W_i)} < \frac{\delta}{2^{i+1}}.$$

Now let $v := \sum_{i=0}^\infty u_i^{\epsilon_i}$.
Notice that $\forall x \in U, \exists V \subset\subset U_{\epsilon_i}$ be open, such that $x \in V$. Since $V \cap W_i \neq \emptyset$ for only finitely many i, and $\text{Supp}(u_i^{\epsilon_i}) \subseteq W_i$, we must have $v = \sum_{i=0}^k u_i^{\epsilon_i}$ on V for some finite k.
In addition, by 2.19, each $u_i^{\epsilon_i} \in C^\infty(U_{\epsilon_i})$, thus infinitely differentiable at x.
Thus $v = \sum_{i=0}^k u_i^{\epsilon_i}$ is infinitely differentiable at x.
Since $x \in U$ is arbitrary, we have that $v \in C^\infty(U)$.
In addition,

$$\forall x \in U, \ u(x) = \sum_{i=0}^\infty \zeta_i(x) u(x) = \sum_{i=0}^\infty (\zeta_i u)(x)$$

by definition of partition of unity. Thus

$$u(x) - v(x) = \sum_{i=0}^\infty (\zeta_i u)(x) - \sum_{i=0}^\infty u_i^{\epsilon_i}(x) = \sum_{i=0}^\infty (\zeta_i u)(x) - u_i^{\epsilon_i}(x) = \sum_{i=0}^\infty (\zeta_i u - u_i^{\epsilon_i})(x).$$

Since this holds for all $x \in U$, we have that

$$u - v = \sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i}.$$

Now we have

$$
\begin{aligned}
||v - u||_{W^{k,p}(U)} &= \left|\left| \sum_{i=0}^{\infty} \zeta_i u - u_i^{\epsilon_i} \right|\right|_{W^{k,p}(U)} \\
&\leq \sum_{i=0}^{\infty} ||\zeta_i u - u_i^{\epsilon_i}||_{W^{k,p}(U)} \\
&< \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} \\
&= \delta.
\end{aligned}
$$

$\square$

Definition 2.18. Let $U \subseteq \mathbb{R}^n$ be open and bounded, then ∂U is C^k if $\forall z \in \partial U, \exists r > 0, \gamma \in C^k(\mathbb{R}^{n-1})$, such that

$$U \cap \bar{B}(z,r) = \left\{ x \in B(z,r) : x^n > \gamma(x^1, \ldots, x^{n-1}) \right\}.$$

Theorem 2.26. *Let U be bounded, and ∂U is C^1, then $\forall u \in W^{k,p}(U)$ for $1 \leq p < \infty$, there exists functions $u_m \in C^{\infty}(\bar{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.*

Proof. See 5.3.3 in Eavan's book. $\square$

2.5 Extensions

Proposition 2.27. *Let $U \subseteq \mathbb{R}^n$ be open and bounded, with ∂U be C^k. Then $\forall z \in \partial U, \exists r > 0, \Phi \in C^k(B(z,r), \mathbb{R}^n)$ a diffeomorphism, such that $\Phi(\partial U \cap B(z,r))$ is in a flat hyperplane, and $\det(D\Phi) = \det(D\Psi) = 1$, for $\Psi := \Phi^{-1}$.*

Proof. Let

$$\Phi^i(x) := x^i \ \forall i \in [n-1], \Phi^n(x) := x^n - \gamma(x^1, \ldots, x^{n-1}),$$

and let

$$\Phi^i(y) := y^i \ \forall i \in [n-1], \Phi^n(y) := y^n - \gamma(y^1, \ldots, y^{n-1}).$$

$\square$

Theorem 2.28. *(Sobolev Norm Equivalence Under Diffeomorphism)*
Let $W \subseteq \mathbb{R}^n$ and Φ is a $C^1(W)$ diffeomorphism, i.e., it has inverse $\Psi \in C^1(W)$. Let $v := u \circ \Psi$, then

$$\exists C_0, C_1 \text{ such that } C_0 ||u||_{W^{1,p}(W)} \leq ||v||_{W^{1,p}(\Phi(W))} \leq C_1 ||u||_{W^{1,p}(W)}.$$

Lemma 2.29. *Let $1 \leq p < \infty$. Assume U is bounded, with ∂U be C^1. Let V be open and bounded, with $U \subset\subset V$, then there exists a bounded linear operator $E : C^1(\bar{U}) \to W^{1,p}(\mathbb{R}^n)$, such that $\forall u \in C^1(\bar{U})$:*

1. *$Eu = u$ in U,*

2. *$\text{Supp}(Eu) \subseteq V$,*

3. *$\exists C > 0$, such that $||Eu||_{W^{1,p}(\mathbb{R}^n)} \leq C ||u||_{W^{1,p}(U)}.$*

Proof. Fix $z \in \partial U$.

In addition, we assume ∂U is flat around z on the plane $\{x^n = 0\}$.

Then there exists an open ball $B := B(z, r)$, such that

$$B^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B^- := B \cap \{x^n \leq 0\} \subseteq \mathbb{R}^n \setminus U.$$

Let $\bar{u}_z(x) := \begin{cases} u(x) & x \in B^+ \\ -3u(x^1, \ldots, x^{n-1}, -x^n) + 4u(x^1, \ldots, x^{n-1}, -\frac{1}{2}x^n) & x \in B^-. \end{cases}$

Then we claim $\bar{u}_z \in C^1(B)$.

Indeed, let $u^- := \bar{u}_z|_{B^-}, u^+ := \bar{u}_z|_{B^+}$.

$$u^-|_{x^n=0} = -3 + 4u|_{x^n=0}$$
$$= u|_{x^n=0}$$
$$= u^+|_{x^n=0};$$
$$\forall i \in [n-1],$$
$$\partial_i u^-|_{x^n=0} = -3\partial_i u|_{x^n=0} + 4\partial_i u|_{x^n=0}$$
$$= \partial_i u|_{x^n=0}$$
$$= \partial_i u^+|_{x^n=0}$$
$$\partial_n u^-|_{x^n=0} = 3\partial_n u|_{x^n=0} - 4\frac{1}{2}\partial_n u|_{x^n=0}$$
$$= u|_{x^n=0}$$
$$= \partial_n u^+|_{x^n=0}.$$

Thus $\bar{u} \in C^1(B)$.

By A2, we have

$$\exists C > 0, \text{ such that } ||\bar{u}_z||_{W^{1,p}(B)} \leq C||u||_{W^{1,p}(B^+)}.$$

Now suppose ∂U is not flat around z, we can find $r_1 > 0, \Phi \in C^1(B(z, r_1), \mathbb{R}^n)$, such that $\Phi(\partial U \cap B(z, r_1))$ is in a flat hyperplane, WLOG $\{y_n = 0\}$, and $\det(D\Phi) = \det(D\Psi) = 1$, for $\Psi := \Phi^{-1}$.

Notice that we can find $B(z, r_2) \subset\subset V$ since V is open and $z \in \bar{U} \subseteq V$.

By setting $r = \min(r_1, r_2) > 0$, we can WLOG work with $B(z, r) \subset\subset V$.

Let $z' := \Phi(z), v := u \circ \Psi \in C^1(\Phi(\bar{U})) = C^1(\overline{\Phi(U)})$.

Since $\Phi(B(z, r))$ is open, we can choose some open ball $B := B(z', r') \subseteq \Phi(B(z, r))$. Let $W_z := \Psi(B)$.

Since $\Phi(\partial U \cap B(z, r))$ is in the plane $\{y_n = 0\}$, we have

$$B^+ := B \cap \{y^n \geq 0\} = \Phi(W_z \cap \bar{U}), B^- := B \cap \{y^n \leq 0\} = \Phi(W_z \setminus U).$$

Now we can extend v form B^+ to B with

$$||\bar{v}||_{W^{1,p}(B)} \leq C||v||_{W^{1,p}(B^+)}.$$

Now let $\bar{u}_z := \bar{v} \circ \Psi$, we have that $B = \Phi(W_z), \bar{v} = \bar{u}_z \circ \Phi$, and by 2.28, we have

$$||\bar{u}_z||_{W^{1,p}(W_z)} \leq C_1||\bar{v}||_{W^{1,p}(\Phi(W_z))}$$
$$= C_1||\bar{v}||_{W^{1,p}(B)}$$
$$\leq C_2||v||_{W^{1,p}(B^+)}$$
$$\leq C_3||u||_{W^{1,p}(\Psi(B^+))}$$
$$\leq C_3||u||_{W^{1,p}(U)}$$

Notice that $\forall z, \Phi(z) \in B \implies z \in W_z$, thus $\{W_z\}_{z \in \partial U}$ forms an open cover for ∂U.

Since ∂U is compact, we can find a finite subcover $\{W_i\}_{i=1}^N$.

Notice that $\left(\bar{U} \setminus \bigcup_{i=1}^{N} W_i\right) \subset U$ is closed, and U is bounded, so we can find $\left(\bar{U} \setminus \bigcup_{i=1}^{N} W_i\right) \subseteq W_0 \subset\subset U$.
We then have $\bigcup_{i=0}^{N} W_i = U$.
Now let $(\zeta_i)_{i=0}^{N}$ be a partition of unity subordinate to W_i, such that

$$\forall x \in U \ \sum_{i=0}^{N} \zeta_i(x) = 1, \ \forall i \geq 0, \begin{cases} 0 \leq \zeta_i \leq 1, \\ \zeta_i \in C_c^{\infty}(\mathbb{R}^n), \\ \mathrm{Supp}(\zeta_i) \subseteq W_i. \end{cases}$$

Let $\bar{u} := \sum_{i=0}^{N} \zeta_i \bar{u}_i$, with $\bar{u}_0 := u$. We have that

$$\begin{aligned}
||\bar{u}||_{W^{1,p}(\mathbb{R}^n)} &\leq \sum_{i=0}^{N} ||\zeta_i \bar{u}_i||_{W^{1,p}(\mathbb{R}^n)} \\
&= \sum_{i=0}^{N} ||\zeta_i \bar{u}_i||_{W^{1,p}(W_i)} \\
&\leq C_4 \sum_{i=0}^{N} ||\bar{u}_i||_{W^{1,p}(W_i)} \\
&= C_5 ||u||_{W^{1,p}(W_i)},
\end{aligned}$$

since each term is bounded, and we have a finite sum.
We thus define $Eu := \bar{u}$.
We can check that E is linear and bounded. $\qquad\square$

Theorem 2.30. *Let $1 \leq p < \infty$. Assume U is bounded, with ∂U be C^1. Let V be open and bounded, with $U \subset\subset V$, then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$, such that $\forall u \in W^{1,p}(U)$:*

1. *$Eu = u \, a.e. \ in \ U$,*

2. *$\mathrm{Supp}(Eu) \subseteq V$,*

3. *$\exists C > 0, \ such \ that \ ||Eu||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)}.$*

Proof. By 2.26, we know $C^{\infty}(\bar{U}) \subseteq C^1(\bar{U})$ is dense in $W^{1,p}(U)$, and thus $C^1(\bar{U})$ is also dense in $W^{1,p}(U)$.
By 1.20, we can extend the result in the above lemma to get $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$.
In addition, since $Eu = \lim_{m \to \infty} Eu_m$ for some $u_m \to u$ in $W^{1,p}(U)$, we also have $Eu = \lim_{m \to \infty} Eu_m = Eu = \lim_{m \to \infty} u_m = u$, a.e..
Also, $\mathrm{Supp}(Eu) \subseteq \bigcup_{m=1}^{\infty} \mathrm{Supp}(Eu_m) \subseteq V$. $\qquad\square$

2.6 Traces

Proposition 2.31. *(Young's inequality)*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \forall a, b > 0, \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.32. *Let U be bounded, and ∂U is C^1, and $1 \leq p < \infty$. Then there exists a bounded linear operator $T : C^1(\bar{U}) \to L^p(\partial U); \ u \mapsto u|_{\partial U}$ and a constant $C > 0$, such that*

$$\forall u \in C^1(\bar{U}), ||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(U)}.$$

Proof. Consider $z \in \partial U$.
Assume ∂U is flat near z in the hyperplane $\{x^n = 0\}$.
Then there exists an open ball $B_z := B(z, r)$, such that

$$B_z^+ := B \cap \{x^n > 0\} \subseteq \bar{U}, B_z^- := B \cap \{x^n \leq 0\} \subseteq \mathbb{R}^n \setminus U.$$

Since u is C^1 and thus continuous, WLOG, we can take r small enough, such that u does not change sign in B_z. Namely, $|u| = u \operatorname{sgn}(u(z))$ in B_z.

Let $\hat{B}_z := B(z, \frac{r}{2})$, and let $\xi \in C_c^\infty(B_z)$ such that $\xi \geq 0$ in B_z, and $\xi = 1$ in $\hat{B}_z$.

Let $\Gamma_z := \hat{B}_z \cap \partial U$, then we have $\operatorname{Supp}(\xi u) \subseteq B_z^+$, and $\xi u = u$ on T.

Let $x' := (x^1, \ldots, x^{n-1})$, by Fundamental Theorem of Calculus, we have

$$\int_0^\infty (\xi |u|^p)(x', t) dt = -(\xi |u|^p)(x', 0).$$

In addition, we have

$$
\begin{aligned}
||u||_{L^p(\Gamma_z)}^p &= \int_{\Gamma_z} |u|^p (x', 0) dx' \\
&\leq \int_{\mathbb{R}^{n-1}} (\xi |u|^p)(x', 0) dx' \\
&= -\int_0^\infty \int_{\mathbb{R}^{n-1}} (\xi |u|^p)(x', t) dx' dt \\
&= -\int_{B_z^+} (\xi |u|^p)(x) dx \\
&= -\int_{B_z^+} \xi_{x_n} |u|^p + \xi p |u|^{p-1}(\operatorname{sgn} u(z)) u_{x_n} dx \\
&\leq \int_{B_z^+} |\xi_{x_n}||u|^p + \xi p |u|^{p-1}|u_{x_n}| dx \\
&\leq \int_{B_z^+} |\xi_{x_n}||u|^p + \xi p \left(\frac{(|u|^{p-1})^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{|u_{x_n}|^p}{p} \right) dx \\
&= \int_{B_z^+} |\xi_{x_n}||u|^p + \xi(p-1)|u|^p + \xi |u_{x_n}|^p dx \\
&\leq \int_{B_z^+} (|\xi_{x_n}| + \xi(p-1))|u|^p + \xi |Du|^p dx.
\end{aligned}
$$

Since $\xi \in C_c^\infty(B_z)$, by EVT, $|\xi_{x_n}|, \xi$ are all bounded. Thus $\exists C > 0$, such that $|\xi_{x_n}| + \xi(p-1), \xi \leq C$ in B_z. Thus

$$||u||_{L^p(\Gamma_z)}^p \leq \int_{B_z^+} C|u|^p + C|Du|^p dx = C||u||_{W^{1,p}(B_z^+)}^p \leq C||u||_{W^{1,p}(U)}^p.$$

Now if ∂U is not flat near z, we can find a C^1 diffeomorphism Φ to make it flat. We still have

$$||u||_{L^p(\Gamma_z)} \leq C||u||_{W^{1,p}(U)},$$

by the equivalence of Sobolev norms under diffeomorphism 2.28.

Since $\{B_z\}_{z \in \partial U}$ form an open cover for ∂U, and ∂U is compact, we can find a finite subcover $\{B_i : x_i \in \partial U\}_{i=1}^N$, and their corresponding Γ_i.

For each $i \in [N]$, we have that

$$||u||_{L^p(\Gamma_i)} \leq C_i ||u||_{W^{1,p}(U)}^p.$$

We have that

$$\begin{aligned}
||Tu||^p_{L^p(\partial U)} &= \int_{\partial U} |u|^p dx \\
&\leq \sum_{i=1}^{N} \int_{\Gamma_i} |u|^p dx \\
&= \sum_{i=1}^{N} ||u||^p_{L^p(\Gamma_i)} \\
&\leq \sum_{i=1}^{N} C_i ||u||^p_{W^{1,p}(U)} \\
&= C||u||^p_{W^{1,p}(U)},
\end{aligned}$$

by taking $C := \sum_{i=1}^{N} C_i$. $\square$

Theorem 2.33. *Let U be bounded, and ∂U is C^1, and $1 \leq p < \infty$. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ and a constant $C > 0$, such that*

$$\forall u \in W^{1,p}(U) \cap C(\bar{U}), Tu = u|_{\partial U},$$

and

$$\forall u \in W^{1,p}(U), ||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(U)}.$$

Proof. By 2.26, we know $C^\infty(\bar{U}) \subseteq C^1(\bar{U})$ is dense in $W^{1,p}(U)$, and thus $C^1(\bar{U})$ is also dense in $W^{1,p}(U)$. By 1.20, we can extend the result in the above lemma to get $T : W^{1,p}(U) \to L^p(\partial U)$. $\square$

Theorem 2.34. *Let U be bounded, and ∂U is C^1, then for any $u \in W^{1,p}(U)$, we have that*

$$u \in W_0^{1,p}(U) \iff Tu = 0 \text{ on } \partial U.$$

2.7   Sobolev Inequalities

Definition 2.19. For $1 \leq p < n$, the **Sobolev conjugate** of p is $p^* := \frac{np}{n-p}$, with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Theorem 2.35. *(Gagliardo–Nirenberg–Sobolev)*
Let $1 \leq p < n$, then

$$\exists C > 0, \text{ such that } \forall u \in C_c^1(\mathbb{R}^n), \ ||u||_{L^{p^*}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)}.$$

Corollary 2.36. *Let $1 \leq p < n$, and $U \subseteq \mathbb{R}^n$, then*

$$\exists C > 0, \text{ such that } \forall u \in C_c^1(U), \ ||u||_{L^{p^*}(U)} \leq C||Du||_{L^p(U)}.$$

Proof. By Gagliardo–Nirenberg–Sobolev's Inequality, there is some $C > 0$, such that

$$\forall v \in C_c^1(\mathbb{R}^n), ||v||_{L^{p^*}(\mathbb{R}^n)} \leq C||Dv||_{L^p(\mathbb{R}^n)}.$$

Notice that for each $u \in C_c^1(U)$, we can extend it by $v(x) := \begin{cases} u(x) & x \in U \\ 0 & x \in \mathbb{R}^n \setminus U \end{cases}$.

Notice that $\text{Supp}(v) \subseteq U$, and $v = u$ on U.
Thus $||v||_{L^{p^*}(\mathbb{R}^n)} = ||v||_{L^{p^*}(U)} = ||u||_{L^{p^*}(U)}$, and $||Dv||_{L^p(\mathbb{R}^n)} = ||Du||_{L^p(U)}$.
In addition, we have that $\lim_{x \to \partial U} D^\alpha u(x) = 0 = \lim_{x \to \partial U} D^\alpha 0, \forall |\alpha| \leq 1$.
Thus this extension is smooth. i.e. $v \in C_c^1(\mathbb{R}^n)$.
We thus have

$$||u||_{L^{p^*}(U)} = ||v||_{L^{p^*}(\mathbb{R}^n)} \leq C||Dv||_{L^p(\mathbb{R}^n)} = C||Du||_{L^p(U)}.$$

$\square$

Theorem 2.37. *($W^{1,p}$ embedding into L^{p^*}, with $1 \leq p < n$)*
Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded. If ∂U is C^1, then

$$\exists C > 0, \ such \ that \ \forall u \in W^{1,p}(U), \ ||u||_{L^{p^*}(U)} \leq C||u||_{W^{1,p}(U)}.$$

In addition, since U is bounded, $\forall q \in [1, p^*]$, we have

$$\exists C > 0, \forall u \in W^{1,p}(U), \ ||u||_{L^q(U)} \leq C||u||_{W^{1,p}(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q1. $\qquad\square$

Theorem 2.38. *(Poincaré's Inequality)*
Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded, then

$$\forall q \in [1, p^*], \exists C \geq 0, \ such \ that \ \forall u \in W_0^{1,p}(U), \ ||u||_{L^q(U)} \leq C||Du||_{L^p(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q2. $\qquad\square$

Corollary 2.39. *Let $1 \leq p < n$, $U \subseteq \mathbb{R}^n$ be open and bounded, then $||Du||_{L^p(U)}$ and $||u||_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$.*

Theorem 2.40. *Let $1 \leq p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, then*

$$\exists C \geq 0, \ such \ that \ \forall u \in W_0^{1,p}(U), \ ||u||_{L^p(U)} \leq C||Du||_{L^p(U)}.$$

Proof. See Theorem 5.6-2 of Evans and A3Q2. $\qquad\square$

Corollary 2.41. *Let $1 \leq p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, then $||Du||_{L^p(U)}$ and $||u||_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$.*

Proof. See A3Q2. $\qquad\square$

Theorem 2.42. *($W^{1,p}(U)$ embedding into $C^{0,\gamma}(\bar{U})$, with $n < p \leq \infty$, Morrey's)*
Let $n < p \leq \infty$, $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1 Then there is some constant $C \geq 0$ such that
$$\forall u \in W^{1,p}(U), \exists u^* \in C^{0,\gamma}(\bar{U}), \ such \ that \ ||u^*||_{C^{0,\gamma}(\bar{U})} \leq C||u||_{W^{1,p}(U)},$$

where $\gamma := 1 - \frac{n}{p}$, and $u^* \in [u]$ is a representative of the equivalence class $[u] \in W^{1,p}(U)$.

Remark. If $p = \infty$, then $\gamma = 1$, and u^* is Lipschitz.

Theorem 2.43. *(Sobolev Inequalities)*
Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1. Let $u \in W^{k,p}(U)$, we have

1. *If $k < \frac{n}{p}$, we define q by $\frac{1}{q} := \frac{1}{p} - \frac{k}{n}$, then*

$$||u||_{L^q(U)} \leq C||u||_{W^{k,p}(U)}.$$

2. *If $k > \frac{n}{p}$, we define $t := k - \lfloor \frac{n}{p} \rfloor - 1$, then we have a representative $u^* \in C^{t,\gamma}(\bar{U})$, such that*

$$||u^*||_{C^{t,\gamma}(\bar{U})} \leq C||u||_{W^{k,p}(U)},$$

where $\gamma = \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}$ if $\frac{n}{p} \notin \mathbb{Z}$, and γ can be any integer if $\frac{n}{p} \in \mathbb{Z}$.

Proof. See Theorem 5.6-6 of Evans and A3Q3. $\qquad\square$

2.8   Compactness

Definition 2.20. Let $(f_k)_{k=1}^\infty$ be a sequence of real-valued functions on $\mathbb{R}^n$. It is **uniformly bounded** if

$$\exists M > 0, \text{ such that } |f_k(x)| \leq M, \ \forall k \in \mathbb{N}^+, x \in \mathbb{R}^n$$

Definition 2.21. Let $(f_k)_{k=1}^\infty$ be a sequence of real-valued functions on $\mathbb{R}^n$. It is **equicontinuous** if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x, y \in \mathbb{R}^n, \ ||x - y|| < \delta \implies |f_k(x) - f_k(y)| < \epsilon, \ \forall k \in \mathbb{N}^+$$

Theorem 2.44. *(Arzela-Ascoli Compact criterion)*
Let $(f_k)_{k=1}^\infty$ be a sequence of real-valued functions on $\mathbb{R}^n$ such that it is uniformly bounded and equicontinuous, then there exists a subsequence $(f_{k_j})_{j=1}^\infty$ and a continuous function f such that $f_{k_j} \to f$ uniformly on compact subsets of $\mathbb{R}^n$.

Proposition 2.45. *(interpolation) Assume $1 \leq s \leq r \leq t \leq \infty$, and $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$ with $0 \leq \theta \leq 1$. Suppose $u \in L^s(U) \cap L^t(U)$, then $u \in L^r(U)$ and*

$$||u||_{L^r(U)} \leq ||u||_{L^s(U)}^\theta ||u||_{L^t(U)}^{1-\theta}.$$

Proof. See AMATH731 A2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 2.46. *Let $V \subseteq \mathbb{R}^n$ be open and bounded. Let $1 \leq p < n$, and $(u_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$ be any bounded sequence with $\mathrm{Supp}(u_m) \subseteq V$. For $u_m^\epsilon := \eta_\epsilon * u_m$, we have that for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$.*

Proof.

Claim 2.46.1. *The sequence $(u_m^\epsilon)_{m=1}^\infty$ is uniformly bounded.*

Proof. Since $(u_m)_{m=1}^\infty$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, ||\hat{u}_m||_{W^{1,p}(\mathbb{R}^n)} \leq M$. Consider any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
|u_m^\epsilon(x)| &= \left| \int_{\mathbb{R}^n} \eta_\epsilon(x - y) u_m(y) dy \right| \\
&\leq ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\
&\leq ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |u_m(y)| dy \\
&= ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} ||u_m||_{L^1(\mathbb{R}^n)} \\
&= ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} ||u_m||_{L^p(\mathbb{R}^n)} \\
&= ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} ||u_m||_{W^{1,p}(\mathbb{R}^n)} \\
&\leq ||\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
&= \frac{1}{\epsilon^n} ||\eta||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
&\leq \frac{C}{\epsilon^n} |V|^{1-\frac{1}{p}} M.
\end{aligned}$$

Since $\frac{C}{\epsilon^n} |V|^{1-\frac{1}{p}} M < \infty$ is independent of m, we have that the sequence $(u_m^\epsilon)_{m=1}^\infty$ is uniformly bounded. $\quad \square$

Claim 2.46.2. *The sequence $(u_m^\epsilon)_{m=1}^\infty$ is equicontinuous.*

Proof. Since $(u_m)_{m=1}^\infty$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, ||\hat{u}_m||_{W^{1,p}(\mathbb{R}^n)} \leq M$.

By 2.19.2, we have that $\partial_i u_m^\epsilon = (\partial_i \eta_\epsilon) * u_m$.

Thus for any $x \in \mathbb{R}^n, 1 \leq i \leq n$, we have

$$
\begin{aligned}
|\partial_i u_m^\epsilon(x)| = & \left| \int_{\mathbb{R}^n} (\partial_i \eta_\epsilon)(x-y) u_m(y) dy \right| \\
\leq & ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} u_m(y) dy \right| \\
\leq & ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} ||u_m||_{L^1(\mathbb{R}^n)} \\
\leq & ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
||Du_m^\epsilon(x)||_1 \leq & \sum_{i=1}^n ||\partial_i \eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
= & ||D\eta_\epsilon||_{L^\infty(\mathbb{R}^n)} |V|^{1-\frac{1}{p}} M \\
= & ||D\eta_\epsilon||_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M \\
< & \infty.
\end{aligned}
$$

Since $||D\eta_\epsilon||_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M$ is independent of x, m, we have that

$$
C := \sup_{m \geq 1} ||Du_m^\epsilon||_{L^\infty(U)} \leq ||D\eta_\epsilon||_{L^\infty(B(0,\epsilon))} |V|^{1-\frac{1}{p}} M < \infty.
$$

Since $u_m^\epsilon \in C_c^\infty(\mathbb{R}^n)$ by 2.19.1, we have each u_m^ϵ is Lipschitz with Lipschitz-constant C.

Given any $\delta > 0$, we can let $\delta_0 = \frac{\delta}{C}$.

Thus $\forall x, y \in \mathbb{R}^n$, such that $||x - y|| < \delta_0$, we have

$$
|u_m^\epsilon(x) - u_m^\epsilon(y)| \leq C||x - y|| < \delta, \ \forall m \in \mathbb{N}^+.
$$

Thus the sequence $(u_m^\epsilon)_{m=1}^\infty$ is equicontinuous. $\qquad \square$

By the above two lemmas and Arzela-Ascoli Compact criterion 2.44, we know for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ and a continuous function u^ϵ such that $u_{m_j}^\epsilon \to f$ uniformly on compact subsets of $\mathbb{R}^n$.

Since V is bounded, $\bar{V}$ is compact, we have that $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges uniformly on $\bar{V}$.

Thus $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges in $L^\infty(V)$.

Thus $(u_{m_j}^\epsilon)_{j=1}^\infty$ converges in $L^q(V)$. $\qquad \square$

Lemma 2.47. *Let $V \subseteq \mathbb{R}^n$ be open and bounded, such that ∂V is C^1. Let $1 \leq p < n$, and $(u_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$ be any bounded sequence with $\mathrm{Supp}(u_m) \subseteq V$. For $u_m^\epsilon := \eta_\epsilon * u_m$, we have that $u_m^\epsilon \to u_m$ uniformly in $L^q(V)$ as $\epsilon \to 0$.*

Proof. By taking V' to be $V + B(0,1)$ and WLOG consider $\epsilon < 1$, we assume the support of u_m^ϵ is in V.

Since $(u_m)_{m=1}^\infty$ is bounded, there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, ||u_m||_{W^{1,p}(\mathbb{R}^n)} = ||u_m||_{W^{1,p}(V)} \leq M$.

Claim 2.47.1. *If u_m are smooth, then $||u_m^\epsilon - u_m||_{L^1(V)} \leq \epsilon |V|^{1-\frac{1}{p}} M$ for any $\epsilon > 0$.*

Proof.

$$
\begin{aligned}
u_m^\epsilon(x) - u_m(x) = & (\eta_\epsilon * u_m)(x) - u_m(x) \\
= & \int_{B(0,\epsilon)} \eta_\epsilon(y) u_m(x-y) dy - u_m(x) \int_{B(0,\epsilon)} \eta_\epsilon(y) dy \\
= & \int_{B(0,\epsilon)} \eta_\epsilon(y)(u_m(x-y) - u_m(x)) dy
\end{aligned}
$$

40

Let $z := \frac{y}{\epsilon}$, we have $dy = \epsilon^n dz$. Recall $\eta_\epsilon = \frac{1}{\epsilon^n}\eta\left(\frac{y}{\epsilon}\right)$. We thus have

$$u_m^\epsilon(x) - u_m(x) = \int_{B(0,\epsilon)} \eta_\epsilon(y)(u_m(x-y) - u_m(x))dy$$

$$= \int_{B(0,1)} \frac{\eta(z)}{\epsilon^n} - (u_m(x-\epsilon z)u_m(x))(\epsilon^n dz)$$

$$= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))dy$$

$$= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x-\epsilon yt)dtdy$$

$$= \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\epsilon yt)\cdot(-\epsilon y)dtdy$$

$$|u_m^\epsilon(x) - u_m(x)| \le \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x-\epsilon yt)\cdot(-\epsilon y)|dtdy$$

$$= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x-\epsilon yt)\cdot y|dtdy$$

$$\le \epsilon \int_{B(0,1)} \eta(y) \int_0^1 ||Du_m(x-\epsilon yt)||_1 dtdy,$$

since $||y||_2 < \epsilon < 1$. Thus

$$||u_m^\epsilon - u_m||_{L^1(V)} = \int_V |u_m^\epsilon(x) - u_m(x)|dx$$

$$\le \int_V \epsilon \int_{B(0,1)} \eta(y) \int_0^1 ||Du_m(x-\epsilon yt)||_1 dtdydx$$

$$= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V ||Du_m(x-\epsilon yt)||_1 dxdtdy$$

$$= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} ||Du_m(x-\epsilon yt)||_1 dxdtdy$$

$$= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_{\mathbb{R}^n} ||Du_m(z)||_1 dzdtdy$$

$$= \epsilon\left(\int_{B(0,\epsilon)} \eta(y)dy\right)\left(\int_0^1 dt\right)\left(\int_{\mathbb{R}^n} ||Du_m(z)||_1 dz\right)$$

$$= \epsilon \int_{\mathbb{R}^n} ||Du_m(z)||_1 dz$$

$$= \epsilon \int_V ||Du_m(z)||_1 dz$$

$$= \epsilon \sum_{i=1}^n ||\partial_i u_m||_{L^1(V)}$$

$$\le \epsilon \sum_{i=1}^n |V|^{1-\frac{1}{p}} ||\partial_i u_m||_{L^p(V)}$$

$$\le \epsilon |V|^{1-\frac{1}{p}} ||u_m||_{W^{1,p}(V)}$$

$$\le \epsilon |V|^{1-\frac{1}{p}} M.$$

Notice that this is true for any $\epsilon > 0$. $\qquad\square$

Let $\delta > 0$ be given. Since $C^\infty(\bar{V})$ is dense in $W^{1,p}(V)$ by 2.26, we can find some $\bar{u}_m \in W^{1,p}(V)$, such that $||\bar{u}_m - u_m||_{W^{1,p}(V)} < \frac{\delta}{3|V|^{1-\frac{1}{p}}}$.

Notice that $\forall m, ||\bar{u}_m||_{W^{1,p}(V)} \le M + \frac{\delta}{3|V|^{1-\frac{1}{p}}}$ is bounded. From the claim above, we can find

$$\epsilon_0 := \frac{\delta}{3\left(M + \frac{\delta}{3|V|^{1-\frac{1}{p}}}\right)|V|^{1-\frac{1}{p}}} > 0,$$

such that $\forall 0 < \epsilon < \epsilon_0$, we have

$$||\bar{u}_m^\epsilon - \bar{u}_m||_{L^1(V)} < \frac{\delta}{3}, \ \forall m \in \mathbb{N}^+.$$

Now $||u_m - \bar{u}_m||_{L^1(V)} \le |V|^{1-\frac{1}{p}}||u_m - \bar{u}_m||_{L^p(V)} \le |V|^{1-\frac{1}{p}}||u_m - \bar{u}_m||_{W^1(V)} < \frac{\delta}{3}$.

In addition, by 2.21, we have

$$\begin{aligned}
||u_m^\epsilon - \bar{u}_m^\epsilon||_{L^1(V)} &= ||\eta_\epsilon * u_m - \eta_\epsilon * \bar{u}_m||_{L^1(V)} \\
&= ||\eta_\epsilon * (u_m - \bar{u}_m)||_{L^1(V)} \\
&\le ||u_m - \bar{u}_m||_{L^1(V)} \\
&< \frac{\delta}{3}.
\end{aligned}$$

Now we have

$$||u_m^\epsilon - u_m||_{L^1(V)} \le ||u_m^\epsilon - \bar{u}_m^\epsilon||_{L^1(V)} + ||\bar{u}_m^\epsilon - \bar{u}_m||_{L^1(V)} + ||\bar{u}_m - u_m||_{L^1(V)} < \delta.$$

Notice that this holds for all $\epsilon < \epsilon_0, m \in \mathbb{N}^+$, where the choice of ϵ_0 does not depend on m, and thus $||u_m^\epsilon - u_m||_{L^1(V)} \to 0$ uniformly when $\epsilon \to 0$.

Now $1 \le q \le p^*$, by letting $s = 1, r = q, t = p^*$, we have

$$\begin{aligned}
||u_m^\epsilon - u_m||_{L^q(V)} &\le ||u_m^\epsilon - u_m||_{L^1(V)}^\theta ||u_m^\epsilon - u_m||_{L^{p^*}(V)}^{1-\theta} && 2.45 \\
&\le ||u_m^\epsilon - u_m||_{L^1(V)}^\theta C^{1-\theta} ||u_m^\epsilon - u_m||_{W^{1,p}(V)}^{1-\theta} && 2.43 \\
&\le ||u_m^\epsilon - u_m||_{L^1(V)}^\theta C^{1-\theta} \left(||u_m^\epsilon||_{W^{1,p}(V)} + ||u_m||_{W^{1,p}(V)}\right)^{1-\theta} \\
&\le ||u_m^\epsilon - u_m||_{L^1(V)}^\theta C^{1-\theta} \left(2||u_m||_{W^{1,p}(V)}\right)^{1-\theta} && 2.22 \\
&\le ||u_m^\epsilon - u_m||_{L^1(V)}^\theta (2CM)^{1-\theta}.
\end{aligned}$$

Given any $\delta > 0$, since $||u_m^\epsilon - u_m||_{L^1(V)} \to 0$ uniformly when $\epsilon \to 0$, we can always find some $\epsilon_0 > 0$, such that

$$\forall \epsilon < \epsilon_0, m \in \mathbb{N}^+, \ ||u_m^\epsilon - u_m||_{L^1(V)} < \left(\frac{\delta}{(2CM)^{1-\theta}}\right)^{1/\theta}.$$

Now for any $m \in \mathbb{N}^+$, we have

$$||u_m^\epsilon - u_m||_{L^q(V)} \le ||u_m^\epsilon - u_m||_{L^1(V)}^\theta (2CM)^{1-\theta} < \delta.$$

This proves that $u_m^\epsilon \to u_m$ uniformly in $L^q(V)$ as $\epsilon \to 0$. $\qquad\square$

Theorem 2.48. *(Rellich-Kondrachov Compactness)*
Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1. Let $1 \le p < n$, then

$$W^{1,p}(U) \subset\subset L^q(U)$$

for any $1 \le q < p^*$.

Proof. The continuous embedding is done before in 2.43.

Now consider any bounded sequence $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(U)$.

Thus there is some $M > 0$, such that $\forall m \in \mathbb{N}^+, \|\hat{u}_m\|_{W^{1,p}(U)} \leq M$.

By extension theorem, we may assume $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(\mathbb{R}^n)$, with $u_m|_U = \hat{u}_m$, and there is some V such that $U \subset\subset V$ and $\forall m \in \mathbb{N}^+, \text{Supp}(u_m) \subseteq V$. In addition,

$$\sup \|u_m\|_{W^{1,p}(\mathbb{R}^n)} = \sup \|u_m\|_{W^{1,p}(V)} \leq \sup C \|\hat{u}_m\|_{W^{1,p}(U)} \leq CM.$$

Thus $(u_m)_{m=1}^\infty$ is bounded.

WLOG, we can take V to have ∂V being C^1.

Let $u_m^\epsilon := \eta_\epsilon * u_m$.

By the above lemmas, we know that

1. for each $\epsilon > 0$, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$, and

2. $u_m^\epsilon \to u_m$ uniformly in $L^q(V)$ as $\epsilon \to 0$.

Now given any $\delta > 0$.

By 2, we can find some $\epsilon_0 > 0$, such that $\forall 0 < \epsilon < \epsilon_0$, we have $\forall m \in \mathbb{N}^+, \|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{3}$.

Now fix some $0 < \epsilon < \epsilon_0$.

By 1, there exists a subsequence $(u_{m_j}^\epsilon)_{j=1}^\infty$ that converges in $L^q(V)$.

In particular, it is Cauchy, and we can find some $N \in \mathbb{N}^+$, such that $\forall i, j \geq N, \left\|u_{m_j}^\epsilon - u_{m_i}^\epsilon\right\|_{L^q(V)} < \frac{\delta}{3}$.

Now for any $i, j \geq N$, we have that

$$\left\|u_{m_i} - u_{m_j}\right\|_{L^q(V)} = \left\|u_{m_i} - u_{m_i}^\epsilon + u_{m_i}^\epsilon - u_{m_j}^\epsilon + u_{m_i}^\epsilon - u_{m_j}\right\|_{L^q(V)}$$

$$\leq \left\|u_{m_i} - u_{m_i}^\epsilon\right\|_{L^q(V)} + \left\|u_{m_i}^\epsilon - u_{m_j}^\epsilon\right\|_{L^q(V)} + \left\|u_{m_i}^\epsilon - u_{m_j}\right\|_{L^q(V)}$$

$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3}$$

$$= \delta.$$

Thus $(u_{m_j})_{j=1}^\infty$ is a Cauchy sequence in $L^q(V)$.

Since $L^q(V)$ is complete, there is some $u \in L^q(V)$, such that $\lim_{j\to\infty} \left\|u_{m_j} - u\right\|_{L^q(V)} = 0$.

Since $U \subseteq V$, we have that $\lim_{j\to\infty} \left\|u_{m_j} - u\right\|_{L^q(U)} = 0$.

Since $u + m|_U = \hat{u}_m$, we also have that $\lim_{j\to\infty} \left\|\hat{u}_{m_j} - u\right\|_{L^q(U)} = 0$.

Thus the subsequence $\hat{u}_{m_j}$ converges to some $u \in L^q(V) \subseteq L^q(U)$.

Since $(\hat{u}_m)_{m=1}^\infty \subset W^{1,p}(U)$ is any bounded sequence, we have that any bounded subset of $W^{1,p}(U)$ is relative compact in $L^q(U)$. $\square$

Theorem 2.49. *Let $U \subseteq \mathbb{R}^n$ be open and bounded, such that ∂U is C^1. Let $1 \leq p \leq \infty$, then*

$$W^{1,p}(U) \subset\subset L^p(U).$$

Theorem 2.50. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $1 \leq p \leq \infty$, then*

$$W_0^{1,p}(U) \subset\subset L^p(U).$$

2.9   Poincare Inequalities

Definition 2.22. For a bounded domain $U \subset \mathbb{R}^n$, we denote the average of u over U by

$$(u)U := \frac{1}{|U|} \int_U u dx.$$

43

Theorem 2.51. *(Poincaré–Wirtinger's Inequality)*
Let $U \subset \mathbb{R}^n$ be open, bounded, and connected, such that ∂U is C^1. For any $1 \leq p \leq \infty$, $\exists C > 0$, such that

$$\forall u \in W^{1,p}(U), \ ||u - (u)_U||_{L^p(U)} \leq C||Du||_{L^p(U)}.$$

Proof. Suppose for contradiction it is not true.
Then $\forall k \in \mathbb{N}, \exists u_k \in W^{1,p}(U)$, such that $||u_k - (u_k)_U||_{L^p(U)} > k||Du||_{L^p(U)}$.
Let $v_k := \frac{u_k - (u_k)_U}{||u_k - (u_k)_U||_{L^p(U)}}$.
Notice that

$$\forall k \in \mathbb{N}^+, \ ||v_k||_{L^p(U)} = 1, (v_k)_U = 0, Dv_k = \frac{Du_k}{||u_k - (u_k)_U||_{L^p(U)}}.$$

Thus $||Dv_k||_{L^p(U)} = \frac{||Du_k||_{L^p(U)}}{||u_k - (u_k)_U||_{L^p(U)}} < \frac{1}{k}$.
Which means $||v_k||_{W^{1,p}(U)}^p = ||v_k||_{L^p(U)}^p + ||Dv_k||_{L^p(U)}^p < 1 + \frac{1}{k^p} \leq 2$.
Since this is true for any $k \in \mathbb{N}^+$, we have that $(v_k)_{k=1}^\infty$ is bounded in $W^{1,p}(U)^p$.
Since $W^{1,p}(U)^p \subset\subset L^p(U)$, there is a subsequence $(v_{k_j})_{j=1}^\infty$ and some $v \in L^p(U)$, such that

$$\lim_{j \to \infty} ||v_{k_j} - v||_{L^p(U)} = 0.$$

Now consider any $1 \leq i \leq k$, and any $\phi \in C_c^\infty(U)$.

$$||v_{k_j}\partial_i\phi - v\partial_i\phi||_{L^p(U)}^p = \int_U |v_{k_j}\partial_i\phi - v\partial_i\phi|^p dx$$
$$= \int_U |\partial_i\phi|^p |v_{k_j} - v|^p dx$$
$$\leq ||\partial_i\phi||_{L^\infty(U)}^p ||v_{k_j} - v||_{L^p(U)}^p.$$

Since $\phi \in C_c^\infty(U)$, we have that $||\partial_i\phi||_{L^\infty(U)}^p$ is bounded by some $M > 0$.
Since $\lim_{j\to\infty} ||v_{k_j} - v||_{L^p(U)} = 0$, we also have $\lim_{j\to\infty} ||v_{k_j}\partial_i\phi - v\partial_i\phi||_{L^p(U)}^p = 0$.
In addition, $\lim_{j\to\infty} ||v_{k_j}\partial_i\phi - v\partial_i\phi||_{L^1(U)} \leq \lim_{j\to\infty} |U|^{1-\frac{1}{p}} ||v_{k_j}\partial_i\phi - v\partial_i\phi||_{L^p(U)} = 0$.
We have

$$\lim_{j \to \infty} \int_U |v_{k_j}\partial_i\phi - v\partial_i\phi| dx = 0$$
$$\implies \lim_{j \to \infty} \int_U (v_{k_j}\partial_i\phi - v\partial_i\phi) dx = 0$$
$$\implies \lim_{j \to \infty} \int_U v_{k_j}\partial_i\phi dx = \lim_{j \to \infty} \int_U v\partial_i\phi dx$$
$$\implies -\lim_{j \to \infty} \int_U \partial_i v_{k_j}\phi dx = \int_U v\partial_i\phi dx.$$

Yet

$$\left|\lim_{j \to \infty} \int_U \partial_i v_{k_j}\phi dx\right| \leq \lim_{j \to \infty} \int_U |\partial_i v_{k_j}\phi| dx$$
$$\leq \lim_{j \to \infty} ||\partial_i v_{k_j}||_{L^p(U)} ||\phi||_{L^{\frac{p}{p-1}}(U)}$$
$$\leq \lim_{j \to \infty} ||Dv_{k_j}||_{L^p(U)} ||\phi||_{L^{\frac{p}{p-1}}(U)}$$
$$\leq \lim_{j \to \infty} \frac{1}{k_j} ||\phi||_{L^{\frac{p}{p-1}}(U)}$$
$$= 0,$$

since $\phi \in C_c^\infty(U)$ and U is bounded, which implies $||\phi||_{L^{\frac{p}{p-1}}(U)} < \infty$. Thus

$$\int_U v \partial_i \phi \, dx = -\lim_{j \to \infty} \int_U \partial_i v_{k_j} \phi \, dx = 0 = -\int_U 0 \phi \, dx.$$

Since this holds for any $\phi \in C_c^\infty(U)$, we must have $\partial_i v = 0$a.e. for any $1 \le i \le n$.
Thus $v \in W^{1,p}(U)$, with $Dv = 0$a.e..
Since U is connected, v is a constant.
Since $(v)_U = 0$, we must have $v = 0$a.e..
However, this contradicts with $||v||_{L^p(U)} = 1$. $\qquad\qquad\square$

2.10 H^{-1} Spaces

Definition 2.23. The dual space to $H_0^1(U)$ is $H^{-1}(U)$.

Theorem 2.52. *Consider any $f \in H^{-1}(U)$.*

1. *There is a tuple $(f^0, \ldots, f^n)$ of functions in $L^2(U)$, such that*

$$\forall v \in H_0^1(H), \ \langle f|v \rangle_{H^{-1}(H), H_0^1(H)} = \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f^i, \partial_i v \rangle_{L^2(U)}.$$

   In this case, we write $f = f^0 - \sum_{i=1}^n f_{x^i}^i$.

2.

$$||f||_{H^{-1}(U)} = \inf \left\{ \left(\sum_{i=0}^n ||f^i||_{L^2(U)}^2 \right)^{\frac{1}{2}} : (f^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

Proof. 1. Let $f \in H^{-1}(U)$, by the Riesz-Frechet Representation theorem1.24, $\exists! u \in H_0^1(U)$, such that

$$\forall v \in H_0^1(U), \langle f|v \rangle_{H^{-1}(U), H_0^1(U)} = \langle u, v \rangle_{H_0^1(U)},$$

and $||f||_{H_0^{-1}(U)} = ||u||_{H_0^1(U)}$.
Let $f^0 = u, \forall 1 \le n, f^i := \partial_i u$. we have

$$\langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle f^i, \partial_i v \rangle_{L^2(U)} = \langle u, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle \partial_i u, \partial_i v \rangle_{L^2(U)}$$

$$= \langle u, v \rangle_{H_0^1(U)}$$

$$= \langle f|v \rangle_{H^{-1}(U), H_0^1(U)}.$$

2. Consider any $f \in H^{-1}(U)$, from 1, we know that there is such $f^0 = u, \forall 1 \le n, f^i := \partial_i u$, satisfying 1, with

$$||f||_{H_0^{-1}(U)} = ||u||_{H_0^1(U)} = \left(\sum_{i=0}^n ||f^i||_{L^2(U)}^2 \right)^{\frac{1}{2}} \ge \inf \left\{ \left(\sum_{i=0}^n ||g^i||_{L^2(U)}^2 \right)^{\frac{1}{2}} : (g^i)_{i=0}^n \text{ satisfies 1.} \right\}.$$

Now consider any $g^0, \ldots, g^n \in L^2(U)$, such that they satisfies

$$\langle f|v \rangle = \langle g^0, v \rangle_{L^2(U)} + \sum_{i=1}^n \langle g^i, v \rangle_{L^2(U)}.$$

45

For any $v \in H^1_0(U)$, we have

$$|\langle f|v\rangle| = \left| \langle g^0, v\rangle_{L^2(U)} + \sum_{i=1}^{n} \langle g^i, \partial_i v\rangle_{L^2(U)} \right|$$

$$\leq \left| \langle g^0, v\rangle_{L^2(U)} \right| + \sum_{i=1}^{n} \left| \langle g^i, \partial_i v\rangle_{L^2(U)} \right|$$

$$\leq ||g^0||_{L^2(U)} ||v||_{L^2(U)} + \sum_{i=1}^{n} ||g^i||_{L^2(U)} ||\partial_i v||_{L^2(U)}$$

$$\leq \left(\sum_{i=0}^{n} ||g^i||^2_{L^2(U)} \right)^{\frac{1}{2}} \left(||v||^2_{L^2(U)} + \sum_{i=1}^{n} ||\partial_i v||^2_{L^2(U)} \right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=0}^{n} ||g^i||^2_{L^2(U)} \right)^{\frac{1}{2}} ||v||_{H^1_0(U)}.$$

Thus we know

$$||f||_{H_0^{-1}(U)} = \sup_{v \in H^1_0(U), v \neq 0} \frac{|\langle f|v\rangle|}{||v||_{H^1_0(U)}} \leq \inf \left\{ \left(\sum_{i=0}^{n} ||g^i||^2_{L^2(U)} \right)^{\frac{1}{2}} : (g^i)_{i=0}^{n} \text{ satisfies 1.} \right\}.$$

$\square$

Corollary 2.53. *For any $v^* \in L^2(U)^* \subset L(L^2(U), \mathbb{R}) \subset L(H^1_0(U), \mathbb{R})$, with v^* identified with $v \in L^2(U)$, and any $u \in H^1_0(U) \subseteq L^2(U)$, we have*

$$\langle v^*|u\rangle_{H^{-1}(U), H^1_0(U)} = \langle v^*|u\rangle_{L^2(U)^*, L^2(U)} = \langle v, u\rangle_{L^2(U)}.$$

In addition, $v^* \in H^{-1}(U)$, and has a representation $(v, 0, \ldots, 0)$ as in above theorem, with

$$||v^*||_{H^{-1}(U)} \leq ||v||_{L^2(U)}.$$

Proof. The first equality is by definition and the second equality is by 1.47.
Thus, for any $||u||_{H^1_0(U)} = 1$, we have that

$$\left| \langle v^*|u\rangle_{H^{-1}(U), H^1_0(U)} \right| = \left| \langle v, u\rangle_{L^2(U)} \right|$$

$$\leq ||v||_{L^2(U)} ||u||_{L^2(U)}$$

$$\leq ||v||_{L^2(U)} ||u||_{H^1_0(U)}$$

$$= ||v||_{L^2(U)}.$$

Since this holds for any unitary $u \in H^1_0(U)$, we have that

$$||v^*||_{H^{-1}(U)} = \sup_{||u||_{H^1_0(U)} = 1} \left| \langle v^*|u\rangle_{H^{-1}(U), H^1_0(U)} \right| \leq ||v||_{L^2(U)} < \infty,$$

which proves $v^* \in H^{-1}(U)$.
In addition, $\langle u, v\rangle_{L^2(U)} + \sum_{i=1}^{n} \langle 0, \partial_i v\rangle_{L^2(U)} = \langle u, v\rangle_{L^2(U)} = \langle v^*|u\rangle_{H^{-1}(U), H^1_0(U)}.$ $\square$

Corollary 2.54. $\forall v \in H^1_0(U)$, *we have $v^* := \langle v, \cdot\rangle_{L^2(U)} \in H^{-1}(U)$, with*

$$||v^*||_{H^{-1}(U)} \leq ||v||_{H^1_0(U)}.$$

Proof. Since $v \in H^1_0(U) \subseteq L^2(U)$, by the above corollary, $v^* \in H^{-1}(U)$, and has

$$||v^*||_{H^{-1}(U)} \leq ||v||_{L^2(U)} \leq ||v||_{H^1_0(U)}.$$

$\square$

2.11 Difference Quotients

Definition 2.24. Let $U \subset \mathbb{R}^n$ be open, $u \in L^1_{loc}(U), V \subset\subset U$, then for $|h| \in (0, \text{dist}(V, \partial U)), x \in V$, we define:

1. For $i \in [n]$, $u_i^h(x) := u(x + he_i)$

2. For $i \in [n]$, the ith **difference quotient** of size h at x is

$$D_i^h u(x) = \frac{u_i^h(x) - u(x)}{h} = \frac{u(x + he_i) - u(x)}{h}.$$

3.

$$D^h u(x) := (D_1^h u(x), \ldots, D_n^h u(x)).$$

Proposition 2.55. Let $U \subset \mathbb{R}^n$ be open, $u \in L^1_{loc}(U)$, then $\forall i \in [n], |h| > 0$, we have

$$\text{Supp}(D_i^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|).$$

Thus,

$$\text{Supp}(D^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|).$$

Proposition 2.56. Let $U \subset \mathbb{R}^n$ be open, $u, v \in L^1_{loc}(U), V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \text{dist}(V, \partial U))$, we have

$$D_i^h(uv) = v_i^h D_i^h u + u D_i^h v.$$

Proof. We have

$$
\begin{aligned}
v_i^h D_i^h u + u D_i^h v &= v_i^h \frac{u_i^h - u}{h} + u \frac{v_i^h - v}{h} \\
&= \frac{v_i^h u_i^h - v_i^h u + u v_i^h - uv}{h} \\
&= \frac{v_i^h u_i^h - uv}{h} \\
&= \frac{(uv)_i^h - uv}{h} \\
&= D_i^h(uv).
\end{aligned}
$$

$\square$

Proposition 2.57. Let $U \subset \mathbb{R}^n$ be open, $u, v \in L^1_{loc}(U), \text{Supp}(u) \subset V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \frac{1}{3} \text{dist}(V, \partial U))$, we have

$$\int_U v D_i^{-h} u \, dx = -\int_U u D_i^h v \, dx.$$

Proof. Notice that $\text{Supp}(D_i^h u) \subseteq \text{Supp}(u) + \bar{B}(0, |h|) \subseteq V + \bar{B}(0, |h|) \subseteq \overline{V + B(0, |h|)}$.
Since $\text{dist}(\overline{V + B(0, |h|)}, \partial U) \geq 2|h|$, we can find $\overline{V + B(0, |h|)} \subset W \subset\subset V$, with $|h| < \text{dist}(W, \partial U)$, where $D_i^{-h} u$ is well-defined in W.
In addition, $\text{Supp}(u) \subset V \subset W$, so we can view the integrals as over W, by extending $D_i^{-h} u$ to be zero

outside of W.

$$\int_W vD_i^{-h}u\,dx = \int_V vD_i^{-h}u\,dx$$

$$= \int_V v(x)\frac{u(x-he_i)-u(x)}{-h}\,dx$$

$$= -\int_V \frac{v(x)u(x-he_i)-v(x)u(x)}{h}\,dx$$

$$= -\left(\int_V \frac{v(x-he_i+he_i)u(x-he_i)}{h}\,dx - \int_V \frac{v(x)u(x)}{h}\,dx\right)$$

$$= -\left(\int_{V-he_i} \frac{v(y+he_i)u(y)}{h}\,dy - \int_V \frac{v(x)u(x)}{h}\,dx\right)$$

$$= -\left(\int_W \frac{v_i^h(y)u(y)}{h}\,dy - \int_W \frac{v(x)u(x)}{h}\,dx\right)$$

$$= -\int_W \frac{v_i^h(x)u(x)-v(x)u(x)}{h}\,dx$$

$$= -\int_W u\frac{v_i^h-v}{h}\,dx$$

$$= -\int_W uD_i^h v\,dx.$$

$\square$

Proposition 2.58. *Let $U \subset \mathbb{R}^n$ be open, $u, D^\alpha u \in L_{loc}^p(U), V \subset\subset U$, then $\forall i \in [n], |h| \in (0, \text{dist}(V, \partial U))$, we have*

$$D^\alpha(u_i^h) = (D^\alpha u)_i^h, \ D^\alpha(D_i^h u) = D_i^h(D^\alpha u) \ in \ V.$$

In addition, if $u \in W^{k,p}(U)$, we have $u_i^h, D_i^h u \in W^{k,p}(V)$.

Proof. Given any $i \in [n], |h| \in (0, \text{dist}(V, \partial U))$.
$\forall \phi \in C_c^\infty(V)$, we have $\phi_i^{-h} \in C_c^\infty(V+he_i) \subseteq C_c^\infty(U)$, with $D^\alpha\phi(x) = D^\alpha\phi_i^{-h}(x+he_i)$.

$$\int_V u_i^h(x)D^\alpha\phi(x)\,dx = \int_V u(x+he_i)D^\alpha\phi_i^{-h}(x+he_i)\,dx$$

$$= \int_{V+he_i} u(y)D^\alpha\phi_i^{-h}(y)\,dy$$

$$= \int_U u(y)D^\alpha\phi_i^{-h}(y)\,dy$$

$$= (-1)^{|\alpha|}\int_U D^\alpha u(y)\phi_i^{-h}(y)\,dy$$

$$= (-1)^{|\alpha|}\int_{V+he_i} D^\alpha u(y)\phi_i^{-h}(y)\,dy$$

$$= (-1)^{|\alpha|}\int_V D^\alpha u(x+he_i)\phi_i^{-h}(x+he_i)\,dx$$

$$= (-1)^{|\alpha|}\int_V (D^\alpha u)_i^h(x)\phi(x)\,dx.$$

Since this holds for all $\phi \in C_c^\infty(V)$, we must have $D^\alpha(u_i^h) = (D^\alpha u)_i^h$.

In addition,

$$D^\alpha(D_i^h u) = D^\alpha\left(\frac{u_i^h - u}{h}\right)$$
$$= \frac{D^\alpha(u_i^h) - D^\alpha u}{h}$$
$$= \frac{(D^\alpha u)_i^h - D^\alpha u}{h}$$
$$= D_i^h(D^\alpha u).$$

Now suppose $u \in W^{k,p}(U)$.

$$||u_i^h||_{W^{k,p}(V)}^p = \int_V \sum_{|\alpha|\leq k} \left|(D^\alpha(u_i^h))(x)\right|^p dx$$
$$= \int_V \sum_{|\alpha|\leq k} \left|(D^\alpha u)_i^h(x)\right|^p dx$$
$$= \int_V \sum_{|\alpha|\leq k} |D^\alpha u(x + he_i)|^p dx$$
$$= \int_{V+he_i} \sum_{|\alpha|\leq k} |D^\alpha u(y)|^p dy$$
$$\leq \int_U \sum_{|\alpha|\leq k} |D^\alpha u(y)|^p dy$$
$$= ||u||_{W^{k,p}(U)}^p.$$

Thus $u_i^h \in W^{k,p}(V)$.
Clearly $u \in W^{k,p}(V)$, so a linear combination of them $D_i^h u \in W^{k,p}(V)$. □

Theorem 2.59. *Let $U \subset \mathbb{R}^n$ be open, we have:*

1. For $p \in [1,\infty)$, and $\forall V \subset\subset U, \exists C > 0$, such that

$$\left||D^h u\right||_{L^p(V)} \leq C||Du||_{L^p(U)}, \ \forall u \in W^{1,p}(U), \forall |h| \in (0, \mathrm{dist}(V, \partial U)).$$

2. For $p \in (1,\infty), V \subset\subset U, u \in L^p(V)$, if $\exists C, \delta > 0$, such that $\left||D^h u\right||_{L^p(V)} \leq C, \ \forall |h| \in (0,\delta)$, then

$$u \in W^{1,p}(V), \ ||Du||_{L^p(V)} \leq C.$$

Theorem 2.60. *Let $U \subset \mathbb{R}^n$ be open and bounded, with ∂U being C^1, then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$.*

3 Elliptic PDEs

3.1 Weak Solutions

We will consider the model problem: $U \in \mathbb{R}^n$ be open and bounded, with some $f : U \to \mathbb{R}$ be given. We want to find $u : \bar{U} \to \mathbb{R}$, such that $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$.

Definition 3.1. A **second order differential operator** is

$$Lu := -\sum_{i,j=1}^{n} \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^{n} b^i(x)\partial_i u + c(x)u.$$

Definition 3.2. A **symmetric (uniformly) elliptic second order differential operator** is an L such that $a^{ij} = a^{ji}$, and

$$\exists \theta > 0, \text{ such that } \sum_{i,j=1}^{n} a^{ij}(x)\xi_i \xi_j \geq \theta \|\xi\|_2^2$$

for $x \in U$ a.e., $\forall \xi \in \mathbb{R}^n$.

Remark. The above definition is equivalent to saying $A = (a^{ij}) \in \mathbb{R}^{n*n}$ is symmetric positive definite.

Example 3.1.1. If we take $a^{ij} = C\delta_{ij}$, we have $Lu = -C\Delta u + b \cdot Du + cu$.

Definition 3.3. The **bilinear form associated with** L is given by:

$$B[u,v] := \int_U \left(\sum_{i,j=1}^{n} a^{ij}\partial_i u \partial_j v + \sum_{i=1}^{n} b^i \partial_i uv + cuv \right) dx, \ \forall u,v \in H_0^1(U).$$

Definition 3.4. Consider $f = f^0 - \sum_{i=1}^{n} f_{x^i}^i \in H^{-1}(U)$ as in 2.52.

$u \in H_0^1(U)$ is called a **weak solution** to the BVP $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if u satisfies the **weak formulation:**

$$\forall v \in H_0^1(U), B[u,v] = \langle f|v \rangle = \langle f^0, v \rangle_{L^2(U)} + \sum_{i=1}^{n} \langle f^i, \partial_i v \rangle_{L^2(U)}.$$

Definition 3.5. For $f \in L^2(U)$, we have the special case:

$u \in H_0^1(U)$ is called a **weak solution** to the BVP $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if u satisfies the **weak formulation:**

$$\forall v \in H_0^1(U), B[u,v] = \langle f, v \rangle_{L^2(U)}.$$

Proposition 3.1. *If a classical solution u exists, i.e u is smooth, and $Lu = f, u|_{\partial U} = 0$, then u is always a weak solution.*

Proof. Firstly consider any $v \in C_c^\infty(U)$, we have

$$\langle f | v \rangle = \langle Lu, v \rangle$$

$$= \int_U Luvdx$$

$$= \int_U \left(-\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u) + \sum_{i=1}^n b^i \partial_i u + cu \right) vdx$$

$$= -\sum_{i,j=1}^n \int_U \partial_j(a^{ij}\partial_i u)vdx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx$$

$$= -\sum_{i,j=1}^n \int_{\partial U} a^{ij}\partial_i u \overset{0}{\cancel{v}} e_i \cdot \hat{n} dx + \sum_{i,j=1}^n \int_U a^{ij}\partial_i u \partial_j vdx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx$$

$$= \int_U \sum_{i,j=1}^n a^{ij}\partial_i u \partial_j vdx + \int_U \left(\sum_{i=1}^n b^i \partial_i uv + cuv \right) dx$$

$$= \int_U \left(\sum_{i,j=1}^n a^{ij}\partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i uv + cuv \right) dx$$

$$= B[u, v].$$

Since $H_0^1(U) = \overline{C_c^\infty(v)}$, this holds for any $v \in H_0^1(U)$. $\qquad \square$

3.2 Existence of weak solution

3.2.1 First Existence Theorem

Theorem 3.2. *(Lax-Milgram)*
Consider a real Hilbert space $\mathcal{H}$ with $\langle \cdot, \cdot \rangle$ and action $\langle \cdot | \cdot \rangle_{\mathcal{H}^*, \mathcal{H}}$. Assume $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bilinear form such that $\exists a, b > 0$ such that $\forall u, v \in \mathcal{H}$,

$$|B[u, v]| \le a||u||||v||$$
$$B[u, u] \ge b||u||^2.$$

Then $\forall f \in \mathcal{H}^*, \exists! u \in \mathcal{H}$ such that $\forall v \in \mathcal{H}, \ B[u, v] = \langle f | v \rangle$.

Proof. For each $u \in \mathcal{H}$, we define the operator $T_u : v \mapsto B[u, v]$.
$|T_u v| = |B[u, v]| \le a||u||||v||$, and thus $||T_u||_{\mathcal{H}^*} \le a||u|| < \infty$ is bounded. Thus $T_u \in \mathcal{H}^*$.
By Riesz-Frechet Representation theorem 1.24, we have that $\exists! w \in \mathcal{H}$, such that $\forall v \in \mathcal{H}, T_u v = \langle w, v \rangle_{\mathcal{H}}$, and $||T_u||_{\mathcal{H}^*} = ||w||_{\mathcal{H}}$.
Now define $A : \mathcal{H} \to \mathcal{H}$ by $u \mapsto w$ in the above setting, such that $\forall v \in \mathcal{H}, \langle Au, v \rangle = B[u, v]$.

Claim 3.2.1. *For any $u \in \mathcal{H}$, we have that*

$$b||u|| \le ||Au|| \le a||u||.$$

Proof. We have
$$||Au||^2 = \langle Au, Au \rangle = B[u, Au] \le a||u||||Au||.$$

If $||Au|| = 0$, clearly $||Au|| \le a||u||$.
Otherwise we can divide both side by $||Au||$, and get $||Au|| \le a||u||$.
On the other hand, we have
$$b||u||^2 \le B[u, u] = \langle Au, u \rangle \le ||Au||||u||.$$

If $||u|| = 0$, clearly $b||u|| \le ||Au||$.
Otherwise we can divide both side by $||u||$, and get $b||u|| \le ||Au||$. $\qquad \square$

Claim 3.2.2. *We have $A \in \mathcal{H}^*$.*

Proof. For any $u_1, u_2, v \in \mathcal{H}, c \in \mathbb{R}$, we have that

$$
\begin{aligned}
\langle A(u_1 + cu_2), v \rangle &= B[u_1 + cu_2, v] \\
&= B[u_1, v] + cB[u_2, v] \\
&= \langle Au_1, v \rangle + c\langle Au_2, v \rangle \\
&= \langle Au_1 + cAu_2, v \rangle.
\end{aligned}
$$

Since this holds for all $v \in \mathcal{H}$, we have $A(u_1 + cu_2) = Au_1 + cAu_2$, and thus A is linear.
In addition, we have

$$
\|A\|_{\mathcal{H}^*} = \sup_{u \in \mathcal{H}, u \neq 0} \frac{\|Au\|}{\|u\|} \leq \sup_{u \in \mathcal{H}, u \neq 0} \frac{a\|u\|}{\|u\|} = a < \infty.
$$

This shows A is bounded, and thus $A \in \mathcal{H}^*$. $\qquad\square$

Claim 3.2.3. *A is bijective.*

Proof. Suppose $Au = 0$, we have that
$$
b\|u\| \leq \|Au\| = 0,
$$
which means that $u = 0$. Thus A is injective.
Consider any sequence $(y_j)_{j=1}^{\infty} \subset \text{Im}(A)$, such that $\lim_{j \to \infty} y_j = y \in \mathcal{H}$.
We can find $(x_j)_{j=1}^{\infty} \subset \mathcal{H}$, such that $\forall j \geq 1, Ax_j = y_j$.
Since $(y_j)_{j=1}^{\infty}$ is convergent and thus Cauchy, given any $\epsilon > 0$, we can find some $N \geq 1$, such that $\forall i, j \geq N, \|y_j - y_i\| < b\epsilon$.
Now

$$
\begin{aligned}
\|x_j - x_i\| &\leq \frac{1}{b}\|A(x_j - x_i)\| \\
&= \frac{1}{b}\|Ax_j - Ax_i\| \\
&= \frac{1}{b}\|y_j - y_i\| \\
&< \frac{1}{b}b\epsilon \\
&< \epsilon.
\end{aligned}
$$

Thus $(x_j)_{j=1}^{\infty}$ is Cauchy.
Since $\mathcal{H}$ is complete, there is some $x \in \mathcal{H}$, such that $\lim_{j \to \infty} x_j = x$.
Since A is bounded and thus continuous, we have that

$$
\begin{aligned}
Ax &= A\left(\lim_{j \to \infty} x_j\right) \\
&= \lim_{j \to \infty} Ax_j \\
&= \lim_{j \to \infty} y_j \\
&= y.
\end{aligned}
$$

Thus $y \in \text{Im}(A)$.
This proves that $\text{Im}(A)$ is closed.
Since A is linear, $\text{Im}(A)$ is a closed subspace of $\mathcal{H}$, and thus $\mathcal{H} = \text{Im}(A) \oplus \text{Im}(A)^{\perp}$.
Consider any $w \in \text{Im}(A)^{\perp}$, we must have

$$
b\|w\|^2 \leq B[w, w] = \langle Aw, w \rangle = 0.
$$

Thus $\text{Im}(A)^{\perp} = \{0\}$, and thus $\text{Im}(A) = \mathcal{H}$.
Thus A is surjective. $\qquad\square$

Now by the Bounded inverse Theorem, A^{-1} exists and is bounded.

By Riesz-Frechet Representation theorem1.24, given any $f \in \mathcal{H}^*$, we have

$$\exists! w \in \mathcal{H}, \text{ such that } \langle f|v \rangle = \langle w, v \rangle \ \forall v \in \mathcal{H}.$$

Let $u = A^{-1}w$, we have that

$$\forall v \in \mathcal{H}, \ B[u, v] = \langle Au, v \rangle = \langle w, v \rangle = \langle f|v \rangle.$$

This proves the existence.

Now suppose there is some $\hat{u}$ such that $\forall v \in \mathcal{H}, \ B[\hat{u}, v] = \langle f|v \rangle = B[u, v]$.

We must have $B[u - \hat{u}, v] = 0, \ \forall v \in \mathcal{H}$. Thus

$$b||u - \hat{u}|| \leq B[u - \hat{u}, u - \hat{u}] = 0,$$

and thus $\hat{u} = u$ is unique. $\qquad \square$

Proposition 3.3. *(Cauchy's inequality) For any $a, b, \epsilon > 0$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

Theorem 3.4. *(Energy estimates) Let $U \subseteq \mathbb{R}^n$ be bounded and open, and $a^{ij}, b^i, c \in L^\infty(U)$, such that (a^{ij}) is symmetric positive definite. For the bilinear form defined in 3.3, there exists constants $\alpha, \beta > 0, \gamma \geq 0$, such that $\forall u, v \in H_0^1(U)$,*

$$|B[u, v]| \leq \alpha ||u||_{H^1(U)} ||v||_{H^1(U)} \tag{1}$$

$$\beta ||u||_{H^1(U)}^2 \leq B[u, u] + \gamma ||u||_{L^2(U)}^2. \tag{2}$$

Proof. We have

$$|B[u, v]| = \left| \int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cuv \right) dx \right|$$

$$\leq \sum_{i,j=1}^n ||a^{ij}||_{L^\infty(U)} \int_U |\partial_i u||\partial_j v| dx + \sum_{i=1}^n ||b^i||_{L^\infty(U)} \int_U |\partial_i u||v| dx + ||c||_{L^\infty(U)} \int_U |u||v| dx$$

$$\leq \sum_{i,j=1}^n ||a^{ij}||_{L^\infty(U)} ||\partial_i u||_{L^2(U)} ||\partial_j v||_{L^2(U)} + \sum_{i=1}^n ||b^i||_{L^\infty(U)} ||\partial_i u||_{L^2(U)} ||v||_{L^2(U)}$$

$$+ ||c||_{L^\infty(U)} ||u||_{L^2(U)} ||v||_{L^2(U)}$$

$$\leq \sum_{i,j=1}^n ||a^{ij}||_{L^\infty(U)} ||u||_{H^1(U)} ||v||_{H^1(U)} + \sum_{i=1}^n ||b^i||_{L^\infty(U)} ||u||_{H^1(U)} ||v||_{H^1(U)}$$

$$+ ||c||_{L^\infty(U)} ||u||_{H^1(U)} ||v||_{H^1(U)}$$

$$= \left(\sum_{i,j=1}^n ||a^{ij}||_{L^\infty(U)} + \sum_{i=1}^n ||b^i||_{L^\infty(U)} + ||c||_{L^\infty(U)} \right) ||u||_{H^1(U)} ||v||_{H^1(U)}.$$

Taking $\alpha := \sum_{i,j=1}^n ||a^{ij}||_{L^\infty(U)} + \sum_{i=1}^n ||b^i||_{L^\infty(U)} + ||c||_{L^\infty(U)}$, we notice that $\alpha \geq 0$, and $\alpha = 0 \implies \forall i, j, \ a^{ij} = 0$, which contradicts (a_{ij}) is positive definite. Thus $\alpha > 0$, and $|B[u, v]| \leq \alpha ||u||_{H^1(U)} ||v||_{H^1(U)}$.

On the other hand, consider $\xi = Du \in \mathbb{R}^n$.

We have that

$$\theta ||Du||_2^2 \leq \sum_{i,j=1}^n a^{ij} \partial_i u \partial_j u.$$

53

Thus

$$\theta||Du||_{L^2(U)}^2 = \theta \int_U ||Du||_2^2 dx$$

$$\leq \int_U \sum_{i,j=1}^n a^{ij}\partial_i u \partial_j u \, dx$$

$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i \partial_i u u + cuu\right)dx$$

$$\leq B[u,u] + \sum_{i=1}^n ||b^i||_{L^\infty(U)}||\partial_i u||_{L^2(U)}||u||_{L^2(U)} + ||c||_{L^\infty(U)}||u||_{L^2(U)}^2$$

$$\leq B[u,u] + \sum_{i=1}^n ||b^i||_{L^\infty(U)}\left(\epsilon||\partial_i u||_{L^2(U)}^2 + \frac{1}{4\epsilon}||u||_{L^2(U)}^2\right) + ||c||_{L^\infty(U)}||u||_{L^2(U)}^2$$

$$= B[u,u] + \epsilon\sum_{i=1}^n ||b^i||_{L^\infty(U)}||\partial_i u||_{L^2(U)}^2 + \left(\frac{1}{4\epsilon}\sum_{i=1}^n ||b^i||_{L^\infty(U)} + ||c||_{L^\infty(U)}\right)||u||_{L^2(U)}^2$$

$$\leq B[u,u] + \epsilon\sum_{i=1}^n ||b^i||_{L^\infty(U)}||Du||_{L^2(U)}^2 + \left(\frac{1}{4\epsilon}\sum_{i=1}^n ||b^i||_{L^\infty(U)} + ||c||_{L^\infty(U)}\right)||u||_{L^2(U)}^2.$$

If $\sum_{i=1}^n ||b^i||_{L^\infty(U)} = 0$, pick any $\epsilon > 0$.

Otherwise choose $\epsilon := \frac{\theta}{2\sum_{i=1}^n ||b^i||_{L^\infty(U)}} > 0$, and $\gamma := \frac{1}{4\epsilon}\sum_{i=1}^n ||b^i||_{L^\infty(U)} + ||c||_{L^\infty(U)}$, we have

$$\frac{\theta}{2}||Du||_{L^2(U)}^2 \leq B[u,u] + \gamma||u||_{L^2(U)}^2.$$

Since $||Du||_{L^p(U)}$ and $||u||_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}(U)$ by 2.39, we have that

$$\exists C > 0, \text{ such that } \forall u \in H_0^1(U), \ ||u||_{H^1(U)}^2 \leq C||Du||_{L^p(U)}^2.$$

Taking $\beta := \frac{\theta}{2C} > 0$, we have

$$\beta||u||_{H^1(U)}^2 \leq \frac{\theta}{2}||Du||_{L^2(U)}^2 \leq B[u,u] + \gamma||u||_{L^2(U)}^2.$$

$\square$

Definition 3.6. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\mu \in \mathbb{R}$, we define the operator L_μ by

$$L_\mu u := Lu + \mu u.$$

We define the bilinear form associated to L_μ to be B_μ.

Proposition 3.5.
$$B_\mu[u,v] = B[u,v] + \int_U \mu uv \, dx = B[u,v] + \mu\langle u,v\rangle_{L^2(U)}.$$

Theorem 3.6. *(First Existence Theorem)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$ and $\forall f \in H^{-1}(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP: $\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

54

Proof. By Energy estimates, we have that $\forall u, v \in H_0^1(U)$,

$$
\begin{aligned}
|B_\mu[u,v]| &\leq |B[u,v]| + \mu \left| \langle u, v \rangle_{L^2(U)} \right| \\
&\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} + \mu \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\
&\leq (\alpha + \mu) \|u\|_{H^1(U)} \|v\|_{H^1(U)} \\
B_\mu[u,u] &= B[u,u] + \mu \langle u, u \rangle_{L^2(U)} \\
&= B[u,u] + \mu \|u\|_{L^2(U)} \\
&\geq \beta \|u\|_{H^1(U)}^2 + (\mu - \gamma) \|u\|_{L^2(U)}^2 \\
&\geq \beta \|u\|_{H^1(U)}^2.
\end{aligned}
$$

By Lax–Milgram Theorem, for any $f \in H^{-1}(U)$, there is a unique $u \in H_0^1(U)$, such that

$$
\forall v \in H_0^1(U), B_\mu[u,v] = \langle f | v \rangle.
$$

$\square$

Corollary 3.7. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$ and $\forall f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP:* $\begin{cases} Lu + \mu u = f & in\ U, \\ u = 0 & on\ \partial U. \end{cases}$

3.2.2 More Existence Theorems

Definition 3.7. Consider $Lu := -\sum_{i,j=1}^n \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^n b^i(x)\partial_i u + c(x)u$, we define its **formal adjoint**

$$
L^\dagger v := -\sum_{i,j=1}^n \partial_i(a^{ij}(x)\partial_j v) + \sum_{i=1}^n b^i(x)\partial_i v + c(x)v.
$$

For $f \in H^{-1}(U)$, the **adjoint problem** is $\begin{cases} L^\dagger v = f & in\ U, \\ v = 0 & on\ \partial U \end{cases}$, and the bilinear form associated with it is $B^*[u,v]$.

Notice that $v \in H_0^1(U)$ is a weak solution of the adjoint problem if v satisfies $\forall u \in H_0^1(U), B^*[u,v] = \langle f | u \rangle$.

Proposition 3.8.

$$
B^*[u,v] := B[v,u].
$$

Remark. Since L is not bounded, $L^\dagger$ is not its usual adjoint operator. However, when u, v are both smooth, we have that $\langle Lu, v \rangle_{L^2(U)} = B[u,v] = B^*[v,u] = \langle v, L^\dagger u \rangle$.

Definition 3.8. For $\mu \in \mathbb{R}$, we can similarly define $L_\mu^\dagger u := L^\dagger u + \mu u$, and the bilinear form associated with it is $B_\mu^*[u,v]$.

Proposition 3.9.

$$
B_\mu^*[u,v] = B^*[u,v] + \mu \langle u, v \rangle_{L^2(U)} = B[v,u] + \mu \langle v, u \rangle_{L^2(U)} = B_\mu[v,u].
$$

Proposition 3.10. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$ and $\forall f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ of the BVP:* $\begin{cases} L^\dagger u + \mu u = f & in\ U, \\ u = 0 & on\ \partial U. \end{cases}$ *Namely,*

$$
\exists! u \in H_0^1(U), \ such\ that\ \forall v \in H_0^1(U), B_\mu^*[u,v] = \langle f, v \rangle_{L^2(U)}.
$$

Proof. For $\alpha, \beta > 0, \gamma \geq 0$ from Energy Estimate 3.4, we have that $\forall u, v \in H_0^1(U)$,

$$|B^*[v, u]| = |B[u, v]| \tag{3}$$

$$\leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} \tag{4}$$

$$\beta \|u\|_{H^1(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2 \tag{5}$$

$$= B^*[u, u] + \gamma \|u\|_{L^2(U)}^2. \tag{6}$$

Thus B and B^* have the same energy estimate. By First Existence Theorem 3.6, we have the result. $\square$

Definition 3.9. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $\mu \geq \gamma$, we define $L_\mu^{-1} : L^2(U) \to H_0^1(U)$ by $f \mapsto u$, where u is the unique solution to

$$\forall v \in H_0^1(U), B_\mu[u, v] = \langle f, v \rangle_{L^2(U)}$$

given by the First Existence Theorem 3.6.
We can also define $(L_\mu^\dagger)^{-1} : L^2(U) \to H_0^1(U)$ by $f \mapsto u$, where u is the unique solution to

$$\forall v \in H_0^1(U), B_\mu^*[u, v] = \langle f, v \rangle_{L^2(U)}.$$

Remark. We notice that by definition $B_\mu[L_\mu^{-1} f, v] = \langle f, v \rangle_{L^2(U)}, \forall v \in H_0^1(U), \forall f \in L^2(U), \forall \mu \geq \gamma$.

Lemma 3.11. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. Then for any $\mu \geq \gamma$, if we let $K = \mu L_\mu^{-1}$, we have that $K : L^2(U) \to H_0^1(U) \subseteq L^2(U)$ is compact.*

Proof. Consider any $g \in L^2(U)$, we have that

$$\beta \left\| L_\mu^{-1} g \right\|_{H^1(U)}^2 \leq B[L_\mu^{-1} g, L_\mu^{-1} g] + \gamma \left\| L_\mu^{-1} g \right\|_{L^2(U)}^2$$

$$\leq B[L_\mu^{-1} g, L_\mu^{-1} g] + \mu \left\| L_\mu^{-1} g \right\|_{L^2(U)}^2$$

$$= B_\mu[L_\mu^{-1} g, L_\mu^{-1} g]$$

$$= \left\langle g, L_\mu^{-1} g \right\rangle_{L^2(U)}$$

$$\leq \|g\|_{L^2(U)} \left\| L_\mu^{-1} g \right\|_{L^2(U)}$$

$$\leq \|g\|_{L^2(U)} \left\| L_\mu^{-1} g \right\|_{H^1(U)}$$

$$\implies$$

$$\left\| L_\mu^{-1} g \right\|_{H^1(U)} \leq \frac{1}{\beta} \|g\|_{L^2(U)}$$

$$\implies$$

$$\|Kg\|_{H^1(U)} \leq \frac{\mu}{\beta} \|g\|_{L^2(U)}.$$

Thus, $K : L^2(U) \to H_0^1(U)$ is bounded.
Since $H_0^1(U) \subset\subset L^2(U)$, by 1.22, we have that $K : L^2(U) \to L^2(U)$ is compact. $\square$

Lemma 3.12. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4. For any $f \in L^2(U)$, if we let $h := L_\gamma^{-1} f, K = \gamma L_\gamma^{-1}$, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ if and only if u solves $(I - K)u = h$.*

Proof. We will firstly show that u solves $\forall v \in H_0^1(U),\ B_\gamma[u,v] = \langle f + \gamma u, v \rangle_{L^2(U)}$, if and only if u solves $u = L_\gamma^{-1}(f + \gamma u)$.

Suppose $\forall v \in H_0^1(U),\ B_\gamma[u,v] = \langle f + \gamma u, v \rangle_{L^2(U)}$.

we have that $u' := L_\gamma^{-1}(f + \gamma u) \in H_0^1(U)$ is the unique solution, such that

$$B_\gamma[u', v] = \langle f + \gamma u, v \rangle,\ \forall v \in H_0^1(U).$$

Thus $u = u' = L_\gamma^{-1}(f + \gamma u)$.

On the other hand, suppose $u = L_\gamma^{-1}(f + \gamma u)$, then we have that

$$\forall v \in H_0^1(U),\ B_\gamma[u,v] = B_\gamma[L_\gamma^{-1}(f + \gamma u), v] = \langle f + \gamma u, v \rangle_{L^2(U)}.$$

Thus, $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if and only if

u solves $\forall v \in H_0^1(U),\ B[u,v] = \langle f, v \rangle_{L^2(U)}$, if and only if

u solves $\forall v \in H_0^1(U),\ B[u,v] + \gamma\langle u,v\rangle_{L^2(U)} = \langle f,v\rangle_{L^2(U)} + \gamma\langle u,v\rangle_{L^2(U)}$, if and only if

u solves $\forall v \in H_0^1(U),\ B_\gamma[u,v] = \langle f + \gamma u, v\rangle_{L^2(U)}$, if and only if

u solves $u = L_\gamma^{-1}(f + \gamma u)$, if and only if

u solves $u = L_\gamma^{-1}f + \gamma L_\gamma^{-1}u$, if and only if

u solves $Iu = h + Ku$, if and only if

u solves $(I - K)u = h$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Theorem 3.13. *(Second Existence Theorem)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator.

1. Precisely one of the following must be true:

   (a) $\forall f \in L^2(U), \exists! u \in H_0^1(U)$, a unique weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$

   (b) There is a weak solution $u \neq 0 \in H_0^1(U)$ to the homogeneous problem $\begin{cases} Lu = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U. \end{cases}$

2. Let $N \subset H_0^1(U)$ be the solution space of weak solutions to $\begin{cases} Lu = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$, and let $N^* \subset H_0^1(U)$ be the solution space of weak solutions to $\begin{cases} L^\dagger u = 0, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$, then $\dim(N) = \dim(N^*) < \infty$.

3. The problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a weak solution if and only if $f \in (N^*)^\perp \subseteq L^2(U)$.

Proof. Take $\mu = \gamma$.

From the above lemma, we know that for any $f \in L^2(U)$, if we let $K = \gamma L_\gamma^{-1}$, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$ if and only if u solves $(I - K)u = L_\gamma^{-1}f$.

We also have shown that $K : L^2(U) \to H_0^1(U) \subseteq L^2(U)$ is compact.

1. By 1.35, we have that exactly one of the following holds:

   (a) $\forall v \in L^2(U), \exists! u \in L^2(U)$, such that $(I - K)u = v$.
   In this case, for any $f \in L^2(U), \exists! u \in L^2(U)$, such that $(I - K)u = L_\gamma^{-1}f$.
   In addition, since $L_\gamma^{-1}f \in H_0^1(U), Ku = \gamma L_\gamma^{-1}u \in H_0^1(U)$, we must have $u = L_\gamma^{-1}f + Ku \in H_0^1(U)$.

   Thus u is the unique weak solution to $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$.

57

(b) $\exists u \neq 0 \in L^2(U)$, such that $(I - K)u = 0 = L_\gamma^{-1}0$.

Similarly, we can see that $u = Ku = \gamma L_\gamma^{-1}u \in H_0^1(U)$.

Thus u is a non-trivial solution to $\begin{cases} Lu = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$.

2. By the above lemma, $N = \text{Ker}(I - K)$.

By 1.35, we have that $\dim(N) = \dim(\text{Ker}(I - K^\dagger)) < \infty$. Let $L_\gamma^\dagger u := L^\dagger u + \gamma u$.

Consider any $g, h \in L^2(U)$, we have that

$$\langle h, K^\dagger g \rangle = \langle Kh, g \rangle$$
$$= \langle g, Kh \rangle_{L^2(U)}$$
$$= \gamma \langle g, L_\gamma^{-1}h \rangle_{L^2(U)}$$
$$= \gamma B_\gamma^*[(L_\gamma^\dagger)^{-1}g, L_\gamma^{-1}h]$$
$$= \gamma B_\gamma[L_\gamma^{-1}h, (L_\gamma^\dagger)^{-1}g]$$
$$= \gamma \langle h, (L_\gamma^\dagger)^{-1}g \rangle_{L^2(U)}$$
$$= \langle h, \gamma(L_\gamma^\dagger)^{-1}g \rangle_{L^2(U)}.$$

Since this holds for all $g, h \in L^2(U)$, we have that $K^\dagger = \gamma(L_\gamma^\dagger)^{-1}$.

By the above lemma, we have that $u \in H_0^1(U)$ is a weak solution to $\begin{cases} L^\dagger u = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases}$, if and only if

u solves $(I - K^\dagger)u = 0$, if and only if $u \in \text{Ker}(I - K^\dagger)$.

Thus $N^* = \text{Ker}(I - K^\dagger)$.

3. (a) $\gamma = 0$.

Notice that $K = 0$, and thus $N^* = \ker(I - K^\dagger) = \ker(I) = \{0\}$.

Thus $(N^*)^\perp = L^2(U)$.

In addition, $N = \ker(I - K) = \ker(I) = \{0\}$, so we must be in case (a).

Thus $\forall f \in (N^*)^\perp$, the problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a (unique) weak solution.

The other direction is trivial since $(N^*)^\perp = L^2(U)$ is the whole space.

(b) $\gamma \neq 0$.

By 1.35, we have that $\text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$.

By the above lemma, we have that the problem $\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$ has a weak solution, if and

only if,

there is some u that solves $(I - K)u = L_\gamma^{-1}f$, if and only if,

$L_\gamma^{-1}f \in \text{Im}(I - K) = \text{Ker}(I - K^\dagger)^\perp$, if and only if,

$\forall v \in \text{Ker}(I - K^\dagger) = N^*$,

$$\langle L_\gamma^{-1}f, v \rangle = 0$$
$$\frac{1}{\gamma}\langle Kf, v \rangle = 0$$
$$\frac{1}{\gamma}\langle f, K^\dagger v \rangle = 0$$
$$\frac{1}{\gamma}\langle f, K^\dagger v + (I - K^\dagger)v \rangle = 0$$
$$\frac{1}{\gamma}\langle f, v \rangle = 0$$
$$\langle f, v \rangle = 0,$$

if and only if $f \in (N^*)^\perp$.

$\square$

Definition 3.10. Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. The **spectrum** of L is defined to be

$$\Sigma := \mathbb{R} \setminus \left\{ \lambda \in \mathbb{R} : \forall f \in L^2(U), \exists! u \in H_0^1(U), \text{ such that } \forall v \in H_0^1(U), B_{-\lambda}[u,v] = \langle f, v \rangle_{L^2(U)} \right\}.$$

Proposition 3.14. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let Σ be the spectrum of L.*

1. $\lambda \notin \Sigma$ if and only if $\begin{cases} Lu = \lambda u + f & in \ U, \\ u = 0, & on \ \partial U \end{cases}$ has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$.

2. $\lambda \in \Sigma$ if and only if $\begin{cases} Lu = \lambda u & in \ U, \\ u = 0, & on \ \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$.

Proof. 1. This is by definition.

2. By Second Existence Theorem 3.13 on $L_{-\lambda}$.

$\square$

Lemma 3.15. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let $\gamma \geq 0$ be the same as in Energy Estimate 3.4, and Σ be the spectrum of L, we always have $\Sigma \subseteq (-\gamma, \infty)$.*

Proof. If $\lambda \leq -\gamma$, we have that $-\lambda \geq \gamma$, and by First Existence Theorem 3.6, we have that the problem has a unique weak solution, and thus $\lambda \notin \Sigma$. $\square$

Theorem 3.16. *(Third existence theorem)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator. Let Σ be the spectrum of L.

1. Σ is at most countable.

2. If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$ can be arranged in non-decreasing sequence with $\lim_{k\to\infty} \lambda_k = \infty$.

Proof. Let $\gamma' \geq 0$ be the same as in Energy Estimate 3.4, we have $\Sigma \subseteq (-\gamma', \infty) \subseteq (-\gamma, \infty)$ for any $\gamma \geq \gamma'$. We will take some $\gamma > 0$, and consider $\lambda > -\gamma$.
$\lambda \in \Sigma$, if and only if $\begin{cases} Lu = \lambda u & in \ U, \\ u = 0, & on \ \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$,

if and only if $\begin{cases} Lu + \gamma u = (\lambda + \gamma)u, & in \ U, \\ u = 0, & on \ \partial U \end{cases}$ has a non-trivial weak solution $u \neq 0 \in H_0^1(U)$.

Suppose $\lambda \in \Sigma$, then let $g = (\lambda + \gamma)u$. By First Existence Theorem 3.6, there is a unique weak solution $(L_\gamma)^{-1}((\lambda + \gamma)u) = \frac{\lambda+\gamma}{\gamma}Ku$ to the problem $\begin{cases} Lu + \gamma u = g, & in \ U, \\ u = 0, & on \ \partial U \end{cases}$.

Since $u \neq 0 \in H_0^1(U)$ is a weak solution to the problem, we have

$$u = \frac{\lambda + \gamma}{\gamma} Ku.$$

Thus $u \neq 0 \in L^2(U)$ is an eigen-vector for K, with corresponding eigenvalue $\frac{\gamma}{\lambda+\gamma}$.
Notice that $\frac{\gamma}{\lambda+\gamma} > 0$, since $\gamma > 0, \lambda > -\gamma$, and thus $\frac{\gamma}{\lambda+\gamma} \in \text{Spec}_p(K) \setminus (\infty, 0]$.
Since this holds for any $\lambda \in \Sigma$, we have $\left\{ \frac{\gamma}{\lambda+\gamma} : \lambda \in \Sigma \right\} \subseteq \text{Spec}_p(K) \setminus (\infty, 0]$.

On the other hand, $\forall \mu \in \text{Spec}_p(K) \setminus \{0\}$, we have that $\lambda' := \frac{\gamma(1-\mu)}{\mu} = -\gamma + \frac{\gamma}{\mu}$ satisfies $\mu = \frac{\gamma}{\lambda'+\gamma}$.
Pick any eigen-vector $u \neq 0 \in L^2(U)$ corresponds to μ, we have that $\frac{\gamma}{\lambda+\gamma}u = Ku$.

Thus $u = (L_\gamma)^{-1}((\lambda'+\gamma)u) \neq 0 \in H_0^1(U)$ is a weak solution to the problem $\begin{cases} Lu + \gamma u = (\lambda'+\gamma)u, & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$.

If $\lambda' > -\gamma \iff \frac{\gamma}{\mu} > 0 \iff \mu > 0$, we have that $\lambda' \in \Sigma$.

Thus, we have $\left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \text{Spec}_p(K) \setminus (\infty, 0] \right\} \subseteq \Sigma$.

We have shown that

$$\Sigma = \left\{ \frac{\gamma(1-\mu)}{\mu} : \mu \in \text{Spec}_p(K) \setminus (\infty, 0] \right\}.$$

Since K is compact, by the Spectral theorem 1.23, we have that either

1. $\text{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^N$ is finite, which means $\Sigma \subseteq \left(\lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^N$ is finite.

2. $\text{Spec}_p(K) \setminus \{0\} = \{\mu_k\}_{k=1}^\infty$ is countable, and $\lim_{k \to \infty} \mu_k = 0$, which means that $\Sigma \subseteq \left(\lambda_k = \frac{\gamma(1-\mu_k)}{\mu_k} \right)_{k=1}^\infty$ is at most countable.
   In addition, if Σ is infinite, it must be $(\lambda_{k_j})_{j=1}^\infty \subseteq (\lambda_k)_{k=1}^\infty$.

$$\lim_{k \to \infty} |\lambda_k| = \lim_{k \to \infty} \left| \frac{\gamma(1-\mu_k)}{\mu_k} \right| = \lim_{k \to \infty} \left| \frac{\gamma}{\mu_k} \right| = \infty.$$

   Thus $\lim_{k \to \infty} \left| \lambda_{k_j} \right| = \lim_{k \to \infty} |\lambda_k| = \infty$.
   Since we have $\forall j, \lambda_{k_j} > -\gamma$, we must have $\lim_{j \to \infty} \lambda_{k_j} = \infty$.

$\square$

Theorem 3.17. *(Boundedness of inverse)*
Let Σ be the spectrum of L, and $\lambda \notin \Sigma$. Then there is a constant $C > 0$ such that for all $f \in L^2(U)$ and the unique weak solution $u \in H_0^1(U)$ to $\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0, & \text{on } \partial U \end{cases}$, we always have

$$\|u\|_{L^2(U)} \leq C \|f\|_{L^2(U)}.$$

Proof. Consider any $\lambda \notin \Sigma$.
Suppose for contradiction, we can find $(\tilde{u}_k)_{k=1}^\infty \subset H_0^1(U), (\tilde{f}_k)_{k=1}^\infty \subset L^2(U)$, such that $\forall k \geq 1$,

$$\begin{cases} L\tilde{u}_k = \lambda\tilde{u}_k + \tilde{f}_k & \text{in } U, \\ \tilde{u}_k = 0, & \text{on } \partial U \end{cases}$$

and

$$\|\bar{u}\|_{L^2(U)} > k \|\bar{f}\|_{L^2(U)}.$$

Let $u_k := \frac{\tilde{u}_k}{\|\tilde{u}_k\|_{L^2(U)}}, f_k := \frac{\tilde{f}_k}{\|\tilde{u}_k\|_{L^2(U)}}$.

Notice that $\forall k \geq 1, \|u_k\|_{L^2(U)} = 1$, and $\|f_k\|_{L^2(U)} = \frac{\|\tilde{f}_k\|_{L^2(U)}}{\|\tilde{u}_k\|_{L^2(U)}} < \frac{1}{k}$.
In addition, $\forall v \in H_0^1(U)$,

$$B[u_k, v] = \frac{1}{\|\tilde{u}_k\|_{L^2(U)}} B[\tilde{u}_k, v]$$

$$= \frac{1}{\|\tilde{u}_k\|_{L^2(U)}} \left(\left\langle \tilde{f}_k, v \right\rangle_{L^2(U)} + \lambda \langle \tilde{u}_k, v \rangle_{L^2(U)} \right)$$

$$= \left\langle \frac{\tilde{f}_k}{\|\tilde{u}_k\|_{L^2(U)}}, v \right\rangle_{L^2(U)} + \lambda \left\langle \frac{\tilde{u}_k}{\|\tilde{u}_k\|_{L^2(U)}}, v \right\rangle_{L^2(U)}$$

$$= \langle f_k, v \rangle_{L^2(U)} + \lambda \langle u_k, v \rangle_{L^2(U)}.$$

60

By Energy Estimate 3.4, we have that

$$\beta||u_k||^2_{H^1(U)} \le B[u_k, u_k] + \gamma||u_k||^2_{L^2(U)}$$

$$= \langle f_k, u_k \rangle_{L^2(U)} + \lambda\langle u_k, u_k \rangle_{L^2(U)} + \gamma||u_k||^2_{L^2(U)}$$

$$\le ||f_k||_{L^2(U)}||u_k||_{L^2(U)} + (\lambda + \gamma)||u_k||^2_{L^2(U)}$$

$$= ||f_k||_{L^2(U)} + \lambda + \gamma$$

$$< \lambda + \gamma + \frac{1}{k}$$

$$\le \lambda + \gamma + 1.$$

$$||u_k||_{H^1(U)} \le \sqrt{\frac{\lambda + \gamma + 1}{\beta}}$$

Thus $(u_k)_{k=1}^{\infty}$ is a bounded sequence in $H_0^1(U)$.

Since $H_0^1(U)$ is a Hilbert space, and thus reflexive, by 1.28, there $\exists (u_{k_j})_{j=1}^{\infty}$ a subsequence, and $u \in H_0^1(U)$, such that $u_{k_j} \rightharpoonup u$.

Also, since $H_0^1(U) \subset\subset L^2(U)$, by 1.31, we have that $u_{k_j} \to u$ in $L^2(U)$. Thus,

$$||u||_{L^2(U)} = \lim_{j\to\infty} ||u_{k_j}||_{L^2(U)} = 1.$$

Now consider any $v \in H_0^1(U)$, we have that the map $w \mapsto B[w, v]$ is a linear bounded operator, so by weak convergence of $(u_{k_j})_{j=1}^{\infty}$, we have that

$$B[u, v] = \lim_{j\to\infty} B[u_{k_j}, v]$$

$$= \lim_{j\to\infty} \left(\langle f_{k_j}, v \rangle_{L^2(U)} + \lambda\langle u_{k_j}, v \rangle_{L^2(U)} \right)$$

$$= \lim_{j\to\infty} \langle f_{k_j}, v \rangle_{L^2(U)} + \lambda\left\langle \lim_{j\to\infty} u_{k_j}, v \right\rangle_{L^2(U)}$$

$$\le \lim_{j\to\infty} ||f_{k_j}||_{L^2(U)}||v||_{L^2(U)} + \lambda\langle u, v \rangle_{L^2(U)}$$

$$\le \lim_{j\to\infty} \frac{1}{k_j}||v||_{L^2(U)} + \lambda\langle u, v \rangle_{L^2(U)}$$

$$= \lambda\langle u, v \rangle_{L^2(U)}.$$

Namely, $\hat{u} = u$ satisfies $\forall v \in H_0^1(U),\ B_{-\lambda}[\hat{u}, v] = 0 = \langle 0, v \rangle_{L^2(0)}$.

Yet since $\lambda \notin \Sigma$, by definition, we know there is a unique $\hat{u}$ that satisfies the above condition.

Clearly $\hat{u} = 0$ satisfies, so by the uniqueness of weak solution, $u = 0$.

This contradicts with $||u||_{L^2(U)} = 1$. □

3.3 Regularity

Theorem 3.18. *(Interior H^2 regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(U), b^i, c \in L^{\infty}(U), \forall i, j \in [n]$. $\forall V \subset\subset U, \exists C > 0$, such that for all $f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U, namely,

$$\forall v \in H_0^1(U), B[u, v] = \langle f, v \rangle_{L^2(U)},$$

then

$$||u||_{H^2(V)} \le C\Big(||f||_{L^2(U)} + ||u||_{L^2(U)}\Big).$$

Thus $u \in H_{loc}^2(U)$.

Proof. Let $V \subset\subset U$ be given.

The idea is to choose a particular v, then repeatedly bound all $||D_k^h u||$ from the product rule by $||Du||$. The only leftover term will be either $D_k^h(Du)$, or part of $\langle f, v \rangle_{L^2(U)}$ or $||u||_H^1(U)$. We thus achieve a bound on $||D_k^h(Du)||$, which allows us to say $u \in H_{loc}^2(U)$.

1. We now fix some $f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U. We have $\forall v \in H_0^1(U)$,

$$B[u,v] = \langle f, v \rangle_{L^2(U)}$$

$$\int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + cuv \right) dx = \int_U fv\, dx$$

$$\int_U \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v \right) dx = \int_U \tilde{f} v\, dx$$

$$\sum_{i,j=1}^n \int_U \left(a^{ij} \partial_i u \partial_j v \right) dx = \left\langle \tilde{f}, v \right\rangle_{L^2(U)},$$

where $\tilde{f} := f - \sum_{i=1}^n b^i \partial_i u - cu \in L^2(U)$, since $f, \partial_i u, u \in L^2(U), b^i, c \in L^\infty(U)$.

2. Since $V \subset\subset U$, we can choose some $V \subset\subset W \subset\subset U$, and $\zeta \in C_c^\infty(U)$ such that $V < \zeta < W$.
   Choose $|h| > 0$ such that $\operatorname{dist}(V, \partial U) > 8|h|, \operatorname{dist}(W, \partial U) > 6|h|$.
   WLOG, we assume $h > 0$.
   Fix some $k \in [n]$.
   Let $Z := U_{2h} := \{x \in U : \operatorname{dist}(x, \partial U) > 2h\}$ be open.
   Since U is bounded, we have that $Z \subset\subset U, \operatorname{dist}(Z, \partial U) = 2h > |h|$.
   Let $v(x) := \begin{cases} -D_k^{-h}(\zeta^2 D_k^h u)(x) & x \in Z \\ 0 & x \in U \setminus Z \end{cases}$.

   Remark. For $x \in V$, we have that

$$v(x) = -D_k^{-h}(D_k^h u)(x)$$

$$= -D_k^{-h}\left(\frac{u(x+he_k) - u(x)}{h} \right)$$

$$= -\frac{\frac{u(x+he_k-he_k) - u(x-he_k)}{h} - \frac{u(x+he_k) - u(x)}{h}}{h}$$

$$= -\frac{2u(x) - u(x+he_k) - u(x-he_k)}{h^2}$$

$$= \frac{u(x+he_k) - 2u(x) + u(x-he_k)}{h^2},$$

which is an approximation to $\partial_k^2 u$ if u is smooth.

Since $u \in H^1(U)$, we have $D_k^h u \in H^1(Z)$.
Since $\operatorname{Supp}(\zeta) \subset W \subset\subset Z$ is compact, we have $\zeta \in C_c^\infty(Z)$, so $\zeta^2 D_k^h u \in H^1(Z)$.
Since $U_{4h} \subset\subset Z, \operatorname{dist}(U_{4h}, \partial Z) = 2h > |h|$, we have that $v \in H^1(U_{4h})$.
In addition, $\operatorname{Supp}(v) \subset \operatorname{Supp}(\zeta^2 D_k^h u) + \bar{B}(0, h) \subseteq W + \bar{B}(0, h) \subseteq U_{6h} + \bar{B}(0, h) \subset U_{4h}$.
Since $v \in H^1(U_{4h})$ and $\operatorname{Supp}(v) \subset U_{4h}$, we must have $v \in H_0^1(U)$.

3. Now we have

$$\sum_{i,j=1}^{n} \int_{U} \left(a^{ij} \partial_i u \partial_j v \right) dx = \sum_{i,j=1}^{n} \int_{Z} \left(a^{ij} \partial_i u \partial_j v \right) dx$$

$$= sum_{i,j=1}^{n} \int_{Z} \left(a^{ij} \partial_i u \partial_j \left(-D_k^{-h}(\zeta^2 D_k^h u) \right) \right) dx$$

$$= - \sum_{i,j=1}^{n} \int_{Z} \left(a^{ij} \partial_i u D_k^{-h} \left(\partial_j (\zeta^2 D_k^h u) \right) \right) dx$$

$$= \sum_{i,j=1}^{n} \int_{Z} D_k^h (a^{ij} \partial_i u) \left(\partial_j (\zeta^2 D_k^h u) \right) dx$$

$$= \sum_{i,j=1}^{n} \int_{Z} \left(D_k^h(a^{ij}) \partial_i u + a^{ij} D_k^h(\partial_i u) \right) \left(\partial_j (\zeta^2) D_k^h u + \zeta^2 \partial_j (D_k^h u) \right) dx$$

$$= A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 := \sum_{i,j=1}^{n} \int_{Z} a^{ij} D_k^h(\partial_i u) \zeta^2 \partial_j (D_k^h u) dx,$$

$$A_2 := \sum_{i,j=1}^{n} \int_{Z} a^{ij} D_k^h(\partial_i u) \partial_j (\zeta^2) D_k^h u \, dx$$

$$= \sum_{i,j=1}^{n} \int_{Z} a^{ij} D_k^h(\partial_i u) 2\zeta(\partial_j \zeta) D_k^h u \, dx$$

$$A_3 := \sum_{i,j=1}^{n} \int_{Z} D_k^h(a^{ij}) \partial_i u \zeta^2 \partial_j (D_k^h u) dx$$

$$A_4 := \sum_{i,j=1}^{n} \int_{Z} D_k^h(a^{ij}) \partial_i u \partial_j (\zeta^2) D_k^h u \, dx$$

$$= \sum_{i,j=1}^{n} \int_{Z} D_k^h(a^{ij}) \partial_i u 2\zeta(\partial_j \zeta) D_k^h u \, dx$$

Now we will examine each term.

$$A_1 = \sum_{i,j=1}^{n} \int_{Z} a^{ij} D_k^h(\partial_i u) \zeta^2 \partial_j (D_k^h u) dx$$

$$= \int_{Z} \zeta^2 \sum_{i,j=1}^{n} a^{ij} \partial_i (D_k^h u) \partial_j (D_k^h u) dx$$

$$\geq \int_{Z} \zeta^2 \theta \left| \left| D(D_k^h u) \right| \right|_2^2 dx$$

$$= \theta \int_{Z} \zeta^2 \left| \left| D(D_k^h u) \right| \right|_2^2 dx.$$

We also have

$$|A_2| \leq \sum_{i,j=1}^{n} \int_Z \left| a^{ij} D_k^h(\partial_i u) 2\zeta(\partial_j \zeta) D_k^h u \right| dx$$

$$\leq \sum_{i,j=1}^{n} \int_Z \left|\left|a^{ij}\right|\right|_{L^\infty(U)} \left|\left|\partial_j \zeta\right|\right|_{L^\infty(U)} \left| D_k^h(\partial_i u) 2\zeta D_k^h u \right| dx$$

$$= 2 \sum_{i,j=1}^{n} \left|\left|a^{ij}\right|\right|_{L^\infty(U)} \left|\left|\partial_j \zeta\right|\right|_{L^\infty(U)} \int_Z \left| D_k^h(\partial_i u) \zeta D_k^h u \right| dx$$

$$\leq 2 \sum_{i,j=1}^{n} \left|\left|a^{ij}\right|\right|_{L^\infty(U)} \left|\left|\partial_j \zeta\right|\right|_{L^\infty(U)} \int_Z \epsilon \left| D_k^h(\partial_i u)\right|^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx$$

$$= C_1 \int_Z \epsilon \left| D_k^h(\partial_i u)\right|^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx$$

$$= C_1 \int_Z \epsilon \left| \partial_i (D_k^h u)\right|^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx$$

$$\leq C_1 \int_Z \epsilon \left|\left| D(D_k^h u)\right|\right|_2^2 \zeta^2 + \frac{1}{4\epsilon} \left|D_k^h u\right|^2 dx,$$

since $a^{ij} \in L^\infty(U)$, and $\zeta \in C^c(U)$, we have $C_1 := 2 \sum_{i,j=1}^{n} \left|\left|a^{ij}\right|\right|_{L^\infty(U)} \left|\left|\partial_j \zeta\right|\right|_{L^\infty(U)} \in (0, \infty)$.
Similarly,

$$|A_3| \leq \sum_{i,j=1}^{n} \int_Z \left| D_k^h(a^{ij}) \partial_i u \zeta^2 \partial_j (D_k^h u)\right| dx$$

$$\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z \left|\partial_i u \partial_j (D_k^h u)\right| \zeta^2 dx$$

$$\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z \left|\left|Du\right|\right|_2 \left|\left|D(D_k^h u)\right|\right|_2 \zeta^2 dx$$

$$\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z \left|\left|Du\right|\right|_2 \left|\left|D(D_k^h u)\right|\right|_2 \zeta dx$$

$$\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} \int_Z \frac{1}{4\epsilon} \left|\left|Du\right|\right|_2^2 + \epsilon \zeta^2 \left|\left|D(D_k^h u)\right|\right|_2^2 dx$$

$$= C_2 \int_Z \frac{1}{4\epsilon} \left|\left|Du\right|\right|_2^2 + \epsilon \zeta^2 \left|\left|D(D_k^h u)\right|\right|_2^2 dx,$$

where

$$C_2 := \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)}$$

$$\leq \frac{1}{h} \sum_{i,j=1}^{n} \left(\left|\left|a^{ij}\right|\right|_{L^\infty(Z)} + \left|\left|a^{ij}\right|\right|_{L^\infty(Z+he_k)} \right)$$

$$\leq \frac{1}{h} \sum_{i,j=1}^{n} \left(\left|\left|a^{ij}\right|\right|_{L^\infty(U)} + \left|\left|a^{ij}\right|\right|_{L^\infty(U)} \right)$$

$$\in (0, \infty).$$

Lastly,

$$|A_4| \leq \sum_{i,j=1}^{n} \int_Z \left|D_k^h(a^{ij})\partial_i u 2\zeta(\partial_j\zeta)D_k^h u\right| dx$$

$$\leq 2\sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} ||\partial_j\zeta||_{L^\infty(U)} \int_Z |\partial_i u D_k^h u| dx$$

$$\leq \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} ||\partial_j\zeta||_{L^\infty(U)} \int_Z |\partial_i u|^2 + \left|D_k^h u\right|^2 dx$$

$$\leq C_3 \int_Z ||Du||_2^2 + \left|D_k^h u\right|^2 dx,$$

where $C_3 := \sum_{i,j=1}^{n} \left|\left|D_k^h(a^{ij})\right|\right|_{L^\infty(Z)} ||\partial_j\zeta||_{L^\infty(U)} \in (0,\infty)y$ as argued before.
Now

$$|A_2 + A_3 + A_4|$$
$$\leq |A_1| + |A_2| + |A_3|$$
$$\leq \int_Z \epsilon C_1 \left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 + \frac{C_1}{4\epsilon}\left|D_k^h u\right|^2 + \frac{C_2}{4\epsilon}||Du||_2^2 + C_2\epsilon\zeta^2\left|\left|D(D_k^h u)\right|\right|_2^2 + C_3||Du||_2^2 + C_3\left|D_k^h u\right|^2 dx$$
$$= \int_Z (C_1 + C_2)\epsilon\left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 + \left(\frac{C_1}{4\epsilon} + C_3\right)\left|D_k^h u\right|^2 + \left(\frac{C_2}{4\epsilon} + C_3\right)||Du||_2^2 dx$$
$$\leq \int_Z (C_1 + C_2)\epsilon\left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 + \left(\frac{C_1}{4\epsilon} + C_3\right)\left|\left|D^h u\right|\right|_2^2 + \left(\frac{C_2}{4\epsilon} + C_3\right)||Du||_2^2 dx$$
$$= (C_1 + C_2)\epsilon \int_Z \left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 dx + \left(\frac{C_1}{4\epsilon} + C_3\right)\left|\left|D^h u\right|\right|_{L^2(U)}^2 + \left(\frac{C_2}{4\epsilon} + C_3\right)||Du||_{L^2(U)}^2.$$

We know there $\exists C_4 > 0$, such that

$$\left|\left|D^h u\right|\right|_{L^2(Z)} \leq C_4 ||Du||_{L^2(U)}, \forall |h| \in (0, \text{dist}(Z, \partial U)), \forall u \in H_0^1(U).$$

Thus

$$|A_2 + A_3 + A_4| \leq (C_1 + C_2)\epsilon \int_Z \left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 dx + \left(\frac{C_2}{4\epsilon} + C_3 + \left(\frac{C_1}{4\epsilon} + C_3\right)C_4^2\right)||Du||_{L^2(U)}^2.$$

Taking $\epsilon := \frac{\theta}{2(C_1 + C_2)}, C_5(\epsilon) := \frac{C_2}{4\epsilon} + C_3 + \left(\frac{C_1}{4\epsilon} + C_3\right)C_4^2 \in (0,\infty)$, we have

$$\sum_{i,j=1}^{n} \int_U \left(a^{ij}\partial_i u\partial_j v\right)dx = A_1 + A_2 + A_3 + A_4$$

$$\geq A_1 - |A_2 + A_3 + A_4|$$
$$\geq \theta \int_Z \zeta^2 \left|\left|D(D_k^h u)\right|\right|_2^2 dx - \frac{\theta}{2}\int_Z \left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 dx - C_5||Du||_{L^2(U)}^2$$
$$= \frac{\theta}{2}\int_Z \left|\left|D(D_k^h u)\right|\right|_2^2 \zeta^2 dx - C_5||Du||_{L^2(U)}^2.$$

65

4. On the other hand,

$$\left|\left\langle \tilde{f}, v\right\rangle_{L^2(U)}\right| = \int_U \left|f - \sum_{i=1}^n b^i \partial_i u - cu\right| |v|dx$$

$$= \int_U \left(|f| + \sum_{i=1}^n |b^i \partial_i u| + |cu|\right)|v|dx$$

$$\leq \int_U \left(|f| + \sum_{i=1}^n ||b^i||_{L^\infty(U)}|\partial_i u| + ||c||_{L^\infty(U)}|u|\right)|v|dx$$

$$= \int_U |f||v|dx + \sum_{i=1}^n ||b^i||_{L^\infty(U)} \int_U |\partial_i u||v|dx + ||c||_{L^\infty(U)}\int_U |u||v|dx$$

$$\leq C_6\left(\int_U |f||v|dx + \int_U |\partial_i u||v|dx + \int_U |u||v|dx\right)$$

$$\leq C_6\left(\int_U \frac{1}{4\epsilon}|f|^2 + \epsilon|v|^2 dx + \int_U \frac{1}{4\epsilon}|\partial_i u|^2 + \epsilon|v|^2 dx + \int_U \frac{1}{4\epsilon}|u|^2 + \epsilon|v|^2 dx\right)$$

$$\leq C_6 \int_U \frac{1}{4\epsilon}\left(|f|^2 + |\partial_i u|^2 + |u|^2\right) + 3\epsilon|v|^2 dx$$

$$\leq \frac{C_6}{4\epsilon}\int_U |f|^2 + ||Du||_2^2 + |u|^2 dx + 3C_6\epsilon \int_U |v|^2 dx,$$

where $C_6 := \max\left(1, \sum_{i=1}^n ||b^i||_{L^\infty(U)}, ||c||_{L^\infty(U)}\right) \in (0, \infty)$.

We have shown in step 2 that $\zeta^2 D_k^h u \in H^1(Z), \text{Supp}(\zeta^2 D_k^h u) \subset Z \subset\subset U$, thus $\zeta^2 D_k^h u \in H_0^1(U)$.

$$\int_U |v|^2 dx = \int_Z |v|^2 dx$$

$$= \int_Z \left|-D_k^{-h}(\zeta^2 D_k^h u)\right|^2 dx$$

$$\leq \int_Z \left|D^{-h}(\zeta^2 D_k^h u)\right|^2 dx$$

$$\leq C_4^2 \int_U \left|D(\zeta^2 D_k^h u)\right|^2 dx$$

$$= C_4^2 \int_W \left|D(\zeta^2 D_k^h u)\right|^2 dx$$

$$= C_4^2 \int_W \left|D(\zeta^2) D_k^h u + D(D_k^h u)\zeta^2\right|^2 dx$$

$$\leq 2C_4^2 \int_W D(\zeta^2)^2 |D_k^h u|^2 + \left|D(D_k^h u)\right|^2 \zeta^4 dx$$

$$\leq 2C_4^2 \int_W \left|\left|D(\zeta^2)^2\right|\right|_{L^\infty(U)}|D_k^h u|^2 + \left|D(D_k^h u)\right|^2 \zeta^2 dx$$

$$\leq 2C_4^2 \left|\left|D(\zeta^2)^2\right|\right|_{L^\infty(U)} \int_W |D_k^h u|^2 dx + 2C_4^2 \int_W \left|D(D_k^h u)\right|^2 \zeta^2 dx$$

$$\leq 2C_4^4 \left|\left|D(\zeta^2)^2\right|\right|_{L^\infty(U)} \int_U ||Du||_2^2 dx + 2C_4^2 \int_U \left|D(D_k^h u)\right|^2 \zeta^2 dx$$

$$\leq C_7 \int_U ||Du||_2^2 + \left|D(D_k^h u)\right|^2 \zeta^2 dx,$$

where $C_7 := 2C_4^2 \max\left(C_4^2 ||D(\zeta^2)^2||_{L^\infty(U)}, 1\right) \in (0, \infty)$. Thus we have

$$\left|\left\langle \tilde{f}, v \right\rangle_{L^2(U)}\right| \leq \frac{C_6}{4\epsilon} \int_U |f|^2 + ||Du||_2^2 + |u|^2 dx + 3C_6\epsilon \int_U |v|^2 dx$$

$$\leq \frac{C_6}{4\epsilon} \int_U |f|^2 + ||Du||_2^2 + |u|^2 dx + 3C_6C_7\epsilon \int_U ||Du||_2^2 + \left|D(D_k^h u)\right|^2 \zeta^2 dx$$

$$\leq \left(\frac{C_6}{4\epsilon} + 3C_6C_7\epsilon\right)\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) + 3C_6C_7\epsilon \int_U \left|D(D_k^h u)\right|^2 \zeta^2 dx.$$

5. Taking $\epsilon := \frac{\theta}{12C_6C_7} > 0, C_8 := \frac{C_6}{4\epsilon} + 3C_6C_7\epsilon > 0$, we have

$$\sum_{i,j=1}^n \int_U \left(a^{ij}\partial_i u \partial_j v\right) dx = \left\langle \tilde{f}, v \right\rangle_{L^2(U)}$$

$$\leq \left|\left\langle \tilde{f}, v \right\rangle_{L^2(U)}\right|$$

$$\leq C_8\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) + \frac{\theta}{4} \int_U \left|D(D_k^h u)\right|^2 \zeta^2 dx$$

$$= C_8\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right) + \frac{\theta}{4} \int_Z \left|D(D_k^h u)\right|^2 \zeta^2 dx$$

$$\sum_{i,j=1}^n \int_U \left(a^{ij}\partial_i u \partial_j v\right) dx \geq \frac{\theta}{2} \int_Z ||D(D_k^h u)||_2^2 \zeta^2 dx - C_5||Du||_{L^2(U)}^2$$

$$\frac{\theta}{4} \int_Z ||D(D_k^h u)||_2^2 \zeta^2 dx \leq (C_5 + C_8)\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right)$$

$$\frac{\theta}{4} \int_V ||D(D_k^h u)||_2^2 dx \leq (C_5 + C_8)\left(||f||_{L^2(U)}^2 + ||u||_{L^2(U)}^2 + ||Du||_{L^2(U)}^2\right)$$

$$\int_V ||D_k^h(Du)||_2^2 dx \leq C_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right),$$

where $C_9 := \frac{4(C_5 + C_8)}{\theta} \in (0, \infty)$.
Notice that for all $j \in [n]$, we have $\partial_j u \in L^2(U)$, and

$$\int_V ||D_k^h(\partial_j u)||_2^2 dx \leq \int_V ||D_k^h(Du)||_2^2 dx \leq C_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right),$$

and this holds for all $k \in [n]$. Thus,

$$||D^h(\partial_j u)||_{L^2(V)}^2 = \int_V ||D^h(\partial_j u)||_2^2 dx$$

$$= \int_V \sum_{k=1}^n ||D_k^h(\partial_j u)||_2^2 dx$$

$$= \sum_{k=1}^n \int_V ||D_k^h(\partial_j u)||_2^2 dx$$

$$\leq \sum_{k=1}^n C_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right)$$

$$= nC_9\left(||f||_{L^2(U)}^2 + ||u||_{H^1(U)}^2\right).$$

$$||D^h(\partial_j u)||_{L^2(V)} \leq \sqrt{nC_9}\left(||f||_{L^2(U)} + ||u||_{H^1(U)}\right)$$

$$< \infty.$$

Since this holds for all $|h| > 0$ such that $\operatorname{dist}(V, \partial U) > 8|h|, \operatorname{dist}(W, \partial U) > 6|h|$, we have $\partial_j u \in H^1(U)$, with

$$||D(\partial_j u)||_{L^2(V)} \leq \sqrt{nC_9}\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big).$$

Since this holds for all $j \in [n]$, we have $u \in H^2(V)$, and

$$\begin{aligned}
||D^2 u||_{L^2(V)}^2 &= \int_V ||D^2 u||_2^2 dx \\
&= \int_V \sum_{j=1}^n ||\partial_j(Du)||_2^2 dx \\
&= \sum_{j=1}^n \int_V ||D(\partial_j u)||_2^2 dx \\
&\leq \sum_{j=1}^n nC_9\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2 \\
&= n^2 C_9\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2 \\
&\implies
\end{aligned}$$

$$\begin{aligned}
||u||_{H^2(V)}^2 &= ||D^2 u||_{L^2(V)}^2 + ||u||_{H^1(V)}^2 \\
&\leq n^2 C_9\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2 + ||u||_{H^1(V)}^2 \\
&\leq (n^2 C_9 + 1)\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)^2.
\end{aligned}$$

Thus we have found $C := \sqrt{n^2 C_9 + 1} \in (0, \infty)$, such that $||u||_{H^2(V)} \leq C\Big(||f||_{L^2(U)} + ||u||_{H^1(U)}\Big)$. Since V is arbitrary, we have that $u \in H^2_{loc}(U)$.

6. Notice that the above estimate holds as long as $V \subset\subset U$ and $u \in H^1(U)$. Since $u \in H^1(W)$, we can find some constant C', such that $||u||_{H^2(V)}^2 \leq C'\Big(||f||_{L^2(W)} + ||u||_{H^1(W)}\Big)$.

   Now consider $v := \xi^2 u \in H^1_0(U)$, we can find $||Du||_{L^2(W)} \leq C''||u||_{L^2(U)}$ for some $C'' > 0$. Plugging in will give us

$$||u||_{H^2(V)}^2 \leq C\Big(||f||_{L^2(U)} + ||u||_{L^2(U)}\Big).$$

$\square$

Definition 3.11. If $Lu(x) = f(x)$ a.e.$x \in U$, we say u is a **strong solution** to the problem $Lu = f$ in U.

Corollary 3.19. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(U), b^i, c \in L^\infty(U), \forall i, j \in [n]$. If $f \in L^2(U)$, and $u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then u is a strong solution.*

Proof. We have that $u \in H^2_{loc}(U)$.
Consider any $V \subset\subset U$, since $a^{ij} \in C^1$, we have $a^{ij}u \in H^2(V)$.

Consider any $v \in C_c^\infty(V)$, we must have

$$\langle f, v \rangle_{L^2(V)} = B[u, v]$$

$$= \int_V \left(\sum_{i,j=1}^n a^{ij} \partial_i u \partial_j v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx$$

$$= \int_V \left(\sum_{i,j=1}^n a^{ij} \partial_j (\partial_i u) v + \sum_{i=1}^n b^i \partial_i u v + c u v \right) dx$$

$$= \int_V \left(\sum_{i,j=1}^n \partial_j (a^{ij} \partial_i u) + \sum_{i=1}^n b^i \partial_i u + c u \right) v dx$$

$$= \int_V (Lu) v dx$$

$$= \langle Lu, v \rangle_{L^2(V)}.$$

Since this holds for all $v \in C_c^\infty(V)$, we must have $Lu(x) = f(x)$a.e. $x \in V$.
Since this hold for all $V \subset\subset U$, we have that $Lu(x) = f(x)$a.e. $x \in U$. $\qquad \square$

Theorem 3.20. *(Higher Interior regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(U), \forall i, j \in [n]$ for some $m \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in H_{loc}^{m+2}(U)$. In addition, $\forall V \subset\subset U, \exists C > 0$, such that $\forall f \in L^2(U)$, and $u \in H^1(U)$ being a weak solution to $Lu = f$ in U, we have

$$||u||_{H^{m+2}(U)} \leq C \Big(||f||_{H^m(U)} + ||u||_{L^2(U)} \Big).$$

Corollary 3.21. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(U), \forall i, j \in [n]$ for some $m > \frac{n}{2} - 2 \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in C^l(U)$, where $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$.*

Theorem 3.22. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^\infty(U), \forall i, j \in [n]$. If $f \in C^\infty(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in C^\infty(U)$.*

Theorem 3.23. *(Boundary H^2 regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^2, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U), \forall i, j \in [n]$. Then $\exists C > 0$, such that $\forall f \in L^2(U)$ and

$u \in H_0^1(U)$ *being a weak solution to* $\begin{cases} Lu = f & in\ U, \\ u = 0, & on\ \partial U \end{cases}$*, we have*

$$||u||_{H^2(U)} \leq C \Big(||f||_{L^2(U)} + ||u||_{L^2(U)} \Big),$$

and thus $u \in H^2(U)$.

Proof. 1. First prove the case if the boundary is locally flat:

$$U = B(0, 1) \cap \{x : x^n > 0\}, V = B(0, \frac{1}{2}) \cap \{x : x^n > 0\}.$$

Similar to the proof of Interior H^2 regularity, we first use difference quotients to obtain a bound for derivatives that are not normal to the flat boundary:

$$\sum_{k,l=1, k+l<2n}^n ||\partial_k \partial_l u||_{L^2(V)} \leq C \Big(||f||_{L^2(U)} + ||u||_{H^1(U)} \Big),$$

where we can transform $||u||_{H^1(U)}$ to $||u||_{L^2(U)}$.

For the derivative that is normal to the flat boundary $\partial_n \partial_n$, we write the PDE in non divergence form, and use ellipticity to note that $a^{nn} > \theta > 0$ to find:

$$|\partial_n \partial_n| \leq C\left(\sum_{k,l=1,k+l<2n}^{n} |\partial_k \partial_l u| + ||Du||_2 + |u| + |f|\right) \text{a.e.} x \in U.$$

Thus

$$||\partial_n \partial_n||_{L^2(U)} \leq C\left(\sum_{k,l=1,k+l<2n}^{n} ||\partial_k \partial_l u||_{L^2(U)} + ||Du||_{L^2(U)} + ||u||_{L^2(U)} + ||f||_{L^2(U)}\right).$$

This leads to

$$||u||_{H^2(V)} \leq C\left(||f||_{L^2(U)} + ||u||_{L^2(U)}\right)$$

2. Take any $x_0 \in \partial U$, let $y = \Phi(x)$ be a C^2 straightening map on $B(x_0, r)$ with a C^2 inverse $x = \Psi(y)$. Pick some small enough s, such that

$$U' = B(0,s) \cap \{y : y^n > 0\} \subseteq \Phi(U \cap B(x_0, r)), V' = B(0, \frac{1}{2}s) \cap \{y : y^n > 0\}.$$

We check the weak formulation is well-defined on U' and that L' satisfies the assumptions of L. Apply step 1 to get

$$||u'||_{H^2(V')} \leq C\left(||f'||_{L^2(U')} + ||u'||_{L^2(U')}\right).$$

Transform back using Ψ.

3. Use compactness to find $V_1, \ldots, V_N$ to cover ∂U. Find $V_0 \subset\subset U$ such that $U = \bigcup_{i=0}^{N} V_i$. Use interior result on V_0.
Combine them together.

$\square$

Remark. When the solution is unique, we can throw away the $||u||_{L^2(U)}$ by boundedness of inverse in the last section.

Theorem 3.24. *(Higher boundary regularity)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^{m+2}, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \forall i, j \in [n]$. Then $\exists C > 0$, such that $\forall f \in H^m(U)$ and $u \in H_0^1(U)$ being a weak solution to $\begin{cases} Lu = f & in \ U, \\ u = 0, & on \ \partial U \end{cases}$, we have

$$||u||_{H^{m+2}(U)} \leq C\left(||f||_{H^m(U)} + ||u||_{L^2(U)}\right),$$

and thus $u \in H^{m+2}(U)$.

Corollary 3.25. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^{m+2}, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \forall i, j \in [n]$ for some $m > \frac{n}{2} - 2 \in \mathbb{N}$. If $f \in H^m(U), u \in H^1(U)$ is a weak solution to $Lu = f$ in U, then $u \in C^l(U)$, where $l = m + 2 - \lfloor \frac{n}{2} \rfloor - 1$.*

Theorem 3.26. *(Infinite differentiability up to the boundary)*
Let $U \subseteq \mathbb{R}^n$ be bounded and open, with ∂U being C^∞, and L be a symmetric (uniformly) elliptic second order differential operator, with $a^{ij}, b^i, c \in C^\infty(\bar{U}), \forall i, j \in [n]$. Then $\forall f \in H^\infty(U)$ and $u \in H_0^1(U)$ being a weak solution to $\begin{cases} Lu = f & in \ U, \\ u = 0, & on \ \partial U \end{cases}$, we have $u \in C^\infty(\bar{U})$.

4 Parabolic PDEs

4.1 Spaces Involving Time

Definition 4.1. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, a function $u : [0, T] \to X$ is **continuous** at a point $t \in (0, T)$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall s \in [0, T], |s - t| < \delta \implies ||u(s) - u(t) < \epsilon||.$$

A function u is continuous if it is continuous at all $t \in (0, 1)$.
$||u||_{C([0,T];X)} := \sup_{t \in (0, T)} ||u(t)||$.

Theorem 4.1. $(C([0, T]; X), ||u||_{C([0,T];X)})$ *is a Banach Space.*

See the definition of Bochner integrable functions in notes of Measure Theory. We will still consider the Lebesgue measure on $[0, T]$.

Theorem 4.2 (Bochner). *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, a strongly measurable function $f : [0, T] \to X$ is Bochner integrable if and only if $t \mapsto ||f(t)||_X$ is integrable. In this case,*

$$\left|\left| \int_0^T f(t) dt \right|\right|_X \leq \int_0^T ||f(t)||_X dt,$$

$$\forall u^* \in X^*, \left\langle u^* \middle| \int_0^T f(t) dt \right\rangle = \int_0^T \langle u^* | f(t) \rangle dt.$$

Theorem 4.3. *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, then Dominated Convergence Theorem, Holder's Inequality, and Minkowski's Inequality still work with the Bochner integral.*

Theorem 4.4. *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, then for any Bochner integrable $f : [0, T] \to X$, we have $\int_s^t f(\tau) d\tau$ is continuous in both $s, t \in [0, T]$.*

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

Definition 4.2. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, and $1 \leq p < \infty$, we define

$$\mathcal{L}^p([0, T]; X) := \left\{ f : [0, T] \to X \middle| f \text{ is measurable}, \int_X ||f||_X^p d\mu < \infty \right\}.$$

In addition, we define

$$||f||_{\mathcal{L}^p([0,T];X)} := \left(\int_X ||f||_X^p d\mu \right)^{\frac{1}{p}}.$$

Definition 4.3. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, $(B, ||\cdot||)$ be a Banach Space, we define

$$\mathcal{L}^\infty([0, T]; X) := \{ f : X \to B | f \text{ is measurable}, \text{ess sup} ||f||_X < \infty \}.$$

In addition, we define

$$||f||_{\mathcal{L}^\infty([0,T];X)} := \text{ess sup} ||f||_B.$$

Definition 4.4. Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space. For any $p \in [1, \infty]$, we define

$$L^p([0, T]; X) := \mathcal{L}^p([0, T]; X) / N,$$

where $N := \{ f : X \to B | f \text{ is measurable}, f = 0 \ \mu - \text{a.e.} \}$. Namely, $[f] \in L^p([0, T]; X)$ is the equivalence class of all $g = f$ μ-a.e. for $f \in \mathcal{L}^p([0, T]; X)$.
In addition, we define

$$||[f]||_{L^p([0,T];X)} := ||f||_{\mathcal{L}^\infty([0,T];X)}$$

for any representative f.

Theorem 4.5 (Fischer-Riesz-Bochner)**.** *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space. For all $1 \le p \le \infty$, we have that $\left(L^p([0,T]; X), ||\cdot||_{L^p([0,T];X)} \right)$ is a Banach Space.*

Similarly, we can also define $L^p_{loc}(0, T; X), W^{k,p}(0, T; X), H^k(0, T; X)$ and weak derivatives where the test functions are $\phi \in C^\infty_c(0, T; \mathbb{R})$.

We can similarly define the mollification of $f \in L^1_{loc}(0, T; X)$ to be

$$f^\epsilon := \eta_\epsilon * f : (\epsilon, T - \epsilon) \to X; \ t \mapsto \int_{t-\epsilon}^{t+\epsilon} \eta_\epsilon(t - \tau) f(\tau) d\tau.$$

Similarly, we have

Theorem 4.6. *Let f^ϵ be defined as above, we have:*

1. *$f^\epsilon \in C^\infty(\epsilon, T - \epsilon)$,*

2. *$\partial_t^k(f^\epsilon) = (\partial_t^k \eta_\epsilon) * f$ on $(\epsilon, T - \epsilon)$,*

3. *$f^\epsilon \to f$ a.e.t $\in (0, T)$, as $\epsilon \to 0$,*

4. *If $f \in C(0, T; X)$, we have $f^\epsilon \to f$ uniformly on compact subsets of U,*

5. *If $1 \le p < \infty, f \in L^p_{loc}(0, T; X)$, we have $f^\epsilon \to f$ in $L^p_{loc}(0, T; X)$. Namely, $f^\epsilon \to f$ in $L^p(V), \forall V \subset\subset (0, T)$.*

Theorem 4.7. *Let $T > 0$ and $(X, ||\cdot||)$ be a Banach Space, $p \in [1, \infty]$, and $u \in W^{1,p}(0, T; X)$, then*

1. *$u(t) = u(s) + \int_s^t u'(\tau) d\tau$ for a.e. $0 \le s \le t \le T$.*

2. *There is a representative $u^* \in C([0, T], X)$ of u. In particular, $u^*(t) = u^*(s) + \int_s^t u'(\tau) d\tau$ for any $0 \le s \le t \le T$.*

3. *$\exists C > 0$ such that $\forall u \in W^{1,p}(0, T; X), \sup_{t \in [0,T]} ||u(t)||_X \le C ||u||_{W^{1,p}(0,T;X)}$.*

Proof. We will prove for $p \in [1, \infty)$.

1. Let $u^\epsilon := \eta_\epsilon * u$, we have that $u^\epsilon \in C^\infty(\epsilon, T - \epsilon)$, and $\partial_t(u^\epsilon) = (\partial_t \eta_\epsilon) * u$ on $(\epsilon, T - \epsilon)$.
   We also have $f^\epsilon(t) \to f(t)$ a.e. $t \in (0, T)$.
   Similar to 2.20, we can show that $\partial_t(u^\epsilon) = \eta_\epsilon * \partial_t u = (\partial_t u)^\epsilon$ on $(\epsilon, T - \epsilon)$.
   Since $u \in W^{1,p}(0, T; X)$, we know that $\partial_t u \in L^p_{loc}(0, T; X)$, so $(\partial_t u)^\epsilon \to \partial_t u$ in $L^p_{loc}(0, T; X)$.
   Since $|(0, T)| = T < \infty$, we have that $\partial_t(u^\epsilon) \to \partial_t u$ in $L^1_{loc}(0, T; X)$, which means

$$\forall [s, t] \subset (0, T), \ \lim_{\epsilon \to 0} \int_s^t ||(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)||_X d\tau = 0.$$

We have that $\left|\left| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right|\right|_X \le \int_s^t ||(\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau)||_X d\tau$ for any fixed $[s, t] \subset (0, T)$ and $\epsilon < \min(s, T - t)$. Thus

$$\lim_{\epsilon \to 0} \left|\left| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right|\right|_X = 0$$

for any $[s, t] \subset (0, T)$.
Now $u^\epsilon(t) = u^\epsilon(s) + \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau$ for any $[s, t] \subset (\epsilon, T - \epsilon)$ by FTC, since $u^\epsilon \in C^\infty(\epsilon, T - \epsilon)$.

We have

$$\left\| -u(t) + u(s) + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X$$

$$= \left\| u^\epsilon(t) - u(t) - u^\epsilon(s) + u(s) - \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X$$

$$\leq \|u^\epsilon(t) - u(t)\|_X + \|u^\epsilon(s) - u(s)\|_X + \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) d\tau - \int_s^t (\partial_t u)(\tau) d\tau \right\|_X$$

$$\leq \|u^\epsilon(t) - u(t)\|_X + \|u^\epsilon(s) - u(s)\|_X + \left\| \int_s^t (\partial_t(u^\epsilon))(\tau) - (\partial_t u)(\tau) d\tau \right\|_X$$

for any s, t, ϵ such that $[s, t] \subset (\epsilon, T - \epsilon)$.

Since each term goes to 0 as $\epsilon \to 0$ for a.e. $0 \leq s \leq t \leq T$, we must have

$$\left\| -u(t) + u(s) + \int_s^t (\partial_t u)(\tau) d\tau \right\|_X = 0$$

for a.e. $0 \leq s \leq t \leq T$.

We thus have

$$u(t) = u(s) + \int_s^t (\partial_t u)(\tau) d\tau$$

for a.e. $0 \leq s \leq t \leq T$.

2. Fix any representative for u.

   Notice that the set N where the above property does not hold has measure 0.

   Now fix some point $s \in [0, T] \setminus N$, we define

   $$u^*(t) := \begin{cases} u(s) - \int_t^s u'(\tau) d\tau & t < s \\ u(s) + \int_s^t u'(\tau) d\tau & t \geq s \end{cases}.$$

   For any $t \in [0, T] \setminus N$, we have that

   $$u(t) := u(s) + \int_s^t u'(\tau) d\tau = u^*(t)$$

   if $t \geq s$, and

   $$u(s) = u(t) + \int_t^s u'(\tau) d\tau \implies u(t) = u(s) - \int_t^s u'(\tau) d\tau = u(s)$$

   if $t < s$.

   Thus $u^* = u$ a.e. $t \in [0, T]$, which means u^* is a representative of u.

   In addition, $u^*(t)$ is continuous since $\int_t^s u'(\tau) d\tau$ and $\int_s^t u'(\tau) d\tau$ are both continuous in t, and

   $$\lim_{t \to s^-} u^*(t) = \lim_{t \to s^-} \left(u^*(t) u(s) - \int_t^s u'(\tau) d\tau \right) = u(s) = u(s) + \int_s^s u'(\tau) d\tau = u^*(s).$$

3. See A5Q2.

$\square$

Definition 4.5. $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** if $\forall \epsilon > 0, \exists \delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),

$$\sum_{k=1}^n (b_k - a_k) < \delta \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

Theorem 4.8 (Royden-Fitzpatrick)**.** $f : [a, b] \to \mathbb{R}$ *is absolutely continuous if and only if there is a Lebesgue integrable function g, such that $\forall x \in [a, b]$, $f(x) = f(a) + \int_a^x g(t)dt$.*

Proposition 4.9. *Suppose $u, v \in L^2\big(0, T; H_0^1(U)\big), u', v' \in L^2\big(0, T; H^{-1}(U)\big)$, then*

$$\frac{d}{dt}\langle u(t), v(t)\rangle_{L^2(U)} = \langle u'(t)|v(t)\rangle + \langle v'(t)|u(t)\rangle,$$

where we identify $u'(t), v'(t) \in L^2(U)$ with the action $\langle u'(t), \cdot\rangle_{L^2(U)}, \langle v'(t), \cdot\rangle_{L^2(U)} \in L^2(U)^* \subset H^{-1}(U)$ as usual.

Theorem 4.10. *Suppose $u \in L^2\big(0, T; H_0^1(U)\big), u' \in L^2\big(0, T; H^{-1}(U)\big)$, then*

1. *There is a representative $u^* \in C\big([0, T]; L^2(U)\big)$ of u.*

2. *The mapping $t \mapsto \|u^*(t)\|_{L^2(U)}^2$ is absolutely continuous, and*

$$\frac{d}{dt}\|u^*(t)\|_{L^2(U)}^2 = 2\langle u'(t)|u(t)\rangle \ t \in [0, T]a.e..$$

3. *$\exists C > 0$, such that $\forall u \in L^2\big(0, T; H_0^1(U)\big), u' \in L^2\big(0, T; H^{-1}(U)\big)$,*

$$\sup_{t \in [0,T]} \|u^*(t)\|_{L^2(U)} \leq C\Big(\|u\|_{L^2\big(0, T; H_0^1(U)\big)} + \|u'\|_{L^2\big(0, T; H^{-1}(U)\big)}\Big).$$

Proof. 1. We can extend u to $[-\sigma, T + \sigma]$ for an $\delta > 0$ by reflection and cut off.
Now for any $\epsilon, \delta \in (0, \sigma)$, we can define $u^\epsilon := \eta_\epsilon * u, u^\delta := \eta_\delta * u$, both well-defined on $[0, T]$.

2.

3. Integrate for $0 \leq s \leq T$, we have

$$\int_0^T \|u(t)\|_{L^2(U)}^2 ds = \int_0^T \|u(s)\|_{L^2(U)}^2 ds + 2\int_0^T \int_s^t \langle u'(\tau)|u(\tau)\rangle d\tau ds$$

$$T\|u(t)\|_{L^2(U)}^2 = \int_0^T \|u(s)\|_{L^2(U)}^2 ds + 2\int_0^T \int_s^t \langle u'(\tau)|u(\tau)\rangle d\tau ds$$

$\square$

4.2 Second Order Parabolic Equations

Definition 4.6. Let $U \subseteq \mathbb{R}^n$ be open and bounded, we define $U_T := U \times (0, T]$ for $T > 0$.

Definition 4.7. An **initial boundary value problem** is: given $f : U_t \to \mathbb{R}, g : U \to \mathbb{R}$, we want to find $u(x, t) : \bar{U}_T \to \mathbb{R}$, such that

$$\begin{cases} u_t + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases},$$

where

$$Lu := -\sum_{i,j=1}^n \partial_j(a^{ij}(x, t)\partial_i u) + \sum_{i=1}^n b^i(x, t)\partial_i u + c(x, t)u.$$

A **symmetric (uniformly) parabolic second order differential operator** is an $\partial_t + L$ such that $a^{ij} = a^{ji}$, and

$$\exists \theta > 0, \text{ such that } \sum_{i,j=1}^n a^{ij}(x, t)\xi_i\xi_j \geq \theta\|\xi\|_2^2$$

for $(x, t) \in U$a.e., $\forall \xi \in \mathbb{R}^n$.

Definition 4.8. The **parabolic assumptions** are:

1. $U \subseteq \mathbb{R}^n$ is bounded and open

2. $T > 0$

3. $a^{ij}, b^i, c \in L^\infty(U_T)$

4. $f \in L^2(U_T), g \in L^2(U)$

5. $\partial_t + L$ is a symmetric (uniformly) parabolic second order differential operator.

Definition 4.9. Given a function $u : U_T \to \mathbb{R}$, we want to consider $\mathbf{u} : t \mapsto u(\cdot, t)$, for any $t \in [0, T]$.

Proposition 4.11. *Let $U \subseteq \mathbb{R}^n$ be bounded and open, $T > 0$, then $\forall f \in L^2(U_T)$, $\boldsymbol{f} \in L^2\big(0, T; L^2(U)\big)$.*

Proof. We have

$$\|\mathbf{f}\|^2_{L^2(0,T;L^2(U))} = \int_0^T \|\mathbf{f}(t)\|^2_{L^2(U)} dt$$
$$= \int_0^T \int_U |f(x,t)|^2 dx dt$$
$$= \|f\|_{L^2(U_T)}.$$

$\square$

Definition 4.10. $\mathbf{u} \in L^2\big(0, T; H_0^1(U)\big)$ with $\mathbf{u}' \in L^2\big(0, T; H^{-1}(U)\big)$ is a weak solution of the IBVP if

$$\forall v \in H_0^1(U), \; \langle \mathbf{u}'(t) | v \rangle_{H^{-1}(U), H_0^1(U)} + B[\mathbf{u}, v; t] = \langle \mathbf{f}(t), v \rangle, \text{a.e. } t \in (0, T),$$
$$\mathbf{u}(0) = g,$$

where bilinear form associated to the above problem is

$$B[\mathbf{u}, v; t] := \int_U \left(\sum_{i,j=1}^n a^{ij(x,t)} \partial_i \mathbf{u}(t) \partial_j v + \sum_{i=1}^n b^i(x,t) \partial_i \mathbf{u}(t) v + c\mathbf{u}(t) v \right) dx.$$

4.3   Galerkin Method

Definition 4.11. Let $(w_k)_{k=1}^\infty$ be an orthogonal basis of $H_0^1(U)$, and also an orthonormal basis of $L^2(U)$. For $m \in \mathbb{N}^+$, we define $V_m := \text{Span}\left(\{w_j\}_{j=1}^m \right) \subset H_0^1(U)$ be a space. A function $\mathbf{u}_m(t) = \sum_{k=1}^m d_m^k w_k \in L^2(V_m)$ is a weak solution of the problem in V_m if $\forall v \in V_m$,

$$\left\langle \sum_{k=1}^m d_m^{k'}(t) w_k, v \right\rangle_{L^2(U)} + B\left[\sum_{k=1}^m d_m^k w_k, v; t \right] = \langle \mathbf{f}(t), v \rangle,$$
$$\langle \mathbf{u}_m(0), v \rangle_{L^2(U)} = \langle g, v \rangle_{L^2(U)}$$

We also define $e_j^k(t) := B[w_k, w_j; t], f^j(t) := \langle \mathbf{f}(t), w_j \rangle$.

Definition 4.12. We define the **ODE system associated to the IBVP** to be $\forall j \in [m]$,

$$d_m^{j'}(t) = -\sum_{k=1}^m e_i^j d_m^k(t) + f^j(t),$$
$$d_m^{j'}(0) = \langle g, w_j \rangle_{L^2(U)}$$

Proposition 4.12. $\boldsymbol{u}_m(t) = \sum_{k=1}^m d_m^k w_k$ *is a weak solution in* V_m *if and only if* $\vec{d}_m$ *is a solution to the ODE system.*

Proof. Since $(w_k)_{k=1}^\infty$ is an orthonormal basis of V_m in $\langle \cdot, \cdot \rangle L^2(U)$, we have

$$\left\langle \sum_{k=1}^m {d_m^k}'(t) w_k, v \right\rangle_{L^2(U)} + B\left[\sum_{k=1}^m d_m^k w_k, v; t \right] = \langle \mathbf{f}(t), v \rangle, \qquad \forall v \in V_m$$

$$\Longleftrightarrow$$

$$\left\langle \sum_{k=1}^m {d_m^k}'(t) w_k, w_j \right\rangle_{L^2(U)} + B\left[\sum_{k=1}^m d_m^k w_k, w_j; t \right] = \langle \mathbf{f}(t), w_j \rangle, \qquad \forall j \in [m]$$

$\square$

Theorem 4.13. *Since* f^i, e_j^k *are locally integrable, there is a unique absolutely continuous solution* $\vec{d}_m$ *to the ODE system.*

Theorem 4.14. *(Gronwall's inequality)*
Let $\eta : [0, T] \to \mathbb{R}$ *be nonnegative and absolutely continuous,* ϕ, ψ *both nonnegative summable functions. If*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t) \, a.e. \ t \in [0, T],$$

then

$$\eta(t) \leq \exp\left(\int_0^t \phi(s) ds \right) \left(\eta(0) + \int_0^t \psi(s) ds \right), \ \forall t \in [0, T].$$

Theorem 4.15. *(Energy Estimate)*
There exists $C > 0$ *such that* $\forall m \in \mathbb{N}^+$,

$$\sup_{0 \leq t \leq T} ||\boldsymbol{u}_m(t)||_{L^2(U)} + ||\boldsymbol{u}_m||_{L^2\left(0, T; H_0^1(U)\right)} + ||\boldsymbol{u}_m'||_{L^2(0, T; H^{-1}(U))} \leq C\left(||\boldsymbol{f}||_{L^2(0, T; L^2(U))} + ||g||_{L^2(U)} \right).$$

Proof.

Now fix any function $v \in H_0^1(U)$ with $||v||_{H_0^1(U)}$, write $v = \sum_{j=1}^\infty \hat{v}_j w_j = v_1 + v_2$, where $v_1 := \sum_{j=1}^m \hat{v}_j w_j \in V_m, v_2 := \sum_{j=m+1}^\infty \hat{v}_j w_j \in V_m^\perp$.
Notice that $||v||_{H^1(U)}^2 = ||v_1||_{H^1(U)}^2 + ||v_2||_{H^1(U)}^2$.
Thus $||v_1||_{H^1(U)} \leq 1$.
There exists a unique $\mathbf{u}_m$ such that $\forall k \in [m], t \in [0, T]$,

$$\langle \mathbf{u}_m'(t) | w_k \rangle + B[\mathbf{u}_m, w_k; t] = \langle \mathbf{f}(t), w_k \rangle.$$

Since $v_1 \in V_m$, we have $\langle \mathbf{u}_m'(t) | v_1 \rangle + B[\mathbf{u}_m, v_1; t] = \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)}$.
Now

$$\langle \mathbf{u}_m'(t) | v \rangle = \langle \mathbf{u}_m'(t) | v_1 \rangle + \langle \mathbf{u}_m'(t) | v_2 \rangle$$

$$= \langle \mathbf{u}_m'(t) | v_1 \rangle + \left\langle \sum_{k=1}^m (d_m^k)'(t) w_k, v_2 \right\rangle_{L^2(U)}$$

$$= \langle \mathbf{u}_m'(t) | v_1 \rangle$$

$$= \langle \mathbf{f}(t), v_1 \rangle_{L^2(U)} - B[\mathbf{u}_m, w_k; t].$$

$\square$

Theorem 4.16. *There is a weak solution to the IBVP, namely, $\exists \boldsymbol{u}$, such that*

$$\langle \boldsymbol{u}'(t)|v\rangle + B[\boldsymbol{u}, v; t] = \langle \boldsymbol{f}(t), v\rangle_{L^2(U)} \ \forall v \in H_0^1(U), t \in [0, T] a.e.$$

$$\boldsymbol{u}(0) = g$$

Proof. By energy estimate, we have that $(\mathbf{u}_m)_{m=1}^\infty$ is bounded in $L^2\left(0, T; H_0^1(U)\right)$.

Consider $\mathbf{v} \in C^1\left([0, T]; H_0^1(U)\right)$ of the form $\mathbf{v}(t) = \sum_{k=1}^N d^k(t) w_k$, where $N > 0$ is an integer, $(d^k(t))_{k=1}^N$ are smooth functions, and $(w_k)_{k=1}^\infty$ be a basis as before.
We can show that these $\mathbf{v}$ are dense in $L^2\left(0, T; H_0^1(U)\right)$.
Choose any $m \geq N$, we have $\vec{u}_m$ satisfies

$$\langle \mathbf{u}'_m(t)|w_k\rangle + B[\mathbf{u}_m, w_k; t] = \langle \mathbf{f}(t), w_k\rangle_{L^2(U)}, \ \forall k \in [m], t \in [0, T].$$

Multiplying by $d^k(t)$ and summing over k, we have that

$$\langle \mathbf{u}'_m(t)|\mathbf{v}(t)\rangle + B[\mathbf{u}_m, \mathbf{v}(t); t] = \langle \mathbf{f}(t), \mathbf{v}(t)\rangle_{L^2(U)}, \ \forall t \in [0, T].$$

Integrating over $t \in [0, T]$, we have

$$\int_0^T \langle \mathbf{u}'_m(t)|\mathbf{v}(t)\rangle dt + \int_0^T B[\mathbf{u}_m, \mathbf{v}(t); t] dt = \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t)\rangle_{L^2(U)} dt.$$

Since $\mathbf{v} \in C^1\left([0, T]; H_0^1(U)\right) \subset L^2\left([0, T]; H_0^1(U)\right)$, and $\mathbf{u}'_m \rightharpoonup \mathbf{v}$, we have
By IBP, we have

$$\int_0^T \langle \mathbf{u}'|v\rangle dt = -\int_0^T \langle \mathbf{v}'|\mathbf{u}\rangle dt + \langle \mathbf{u}(T), \mathbf{v}(T)\rangle_{L^2(U)} - \langle \mathbf{u}(0), \mathbf{v}(0)\rangle_{L^2(U)}.$$

$\square$

Theorem 4.17. *A weak solution to our IBVP is unique.*

Proof. Assume $\mathbf{u}_1, \mathbf{u}_2$ are both weak solutions to our IBVP.
Then

$$\langle \mathbf{u}'_1(t)|v\rangle + B[\mathbf{u}_1, v; t] = \langle \mathbf{u}'_2(t)|v\rangle + B[\mathbf{u}_2, v; t] = \langle \mathbf{f}(t), v\rangle_{L^2(U)}, \ \forall v \in H_0^1(U), t \in [0, T] a.e.,$$

and

$$\mathbf{u}_1(0) = g = \mathbf{u}_2(0).$$

Let $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$, we have that

$$\langle \mathbf{u}'(t)|v\rangle + B[\mathbf{u}, v; t] = 0, \ \forall v \in H_0^1(U), t \in [0, T] a.e.,$$

and

$$\mathbf{u}(0) = 0.$$

Choosing $v = \mathbf{u}(t) \in H_0^1(U)$, we have that

$$\langle \mathbf{u}'(t)|\mathbf{u}(t)\rangle + B[\mathbf{u}, \mathbf{u}; t] = 0.$$

$$\frac{d}{dt}\|\mathbf{u}(t)\|_{L^2(U)}^2 = 2\langle \mathbf{u}'(t)|\mathbf{u}(t)\rangle$$

$\square$