

Pmath753 Functional Analysis

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1 Metric Spaces and Complete Spaces

Definition 1.1. A **metric space** is a set X that has a (distance) metric:

$$\begin{aligned} d(\cdot, \cdot) : X \times X &\rightarrow \mathbb{R}, \text{ such that } \forall x, y, z \in X \\ d(x, x) &= 0 \\ \forall x \neq y, d(x, y) &> 0 \\ d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

Definition 1.2. Given a metric space (X, d) , a sequence $(x_n)_{n=1}^{\infty}$ in X has a **limit point** $x \in X$ if $\lim_{n \rightarrow \infty} d(x, x_n) = 0$. In this case, we say $(x_n)_{n=1}^{\infty}$ is a **convergent sequence**, and write $x = \lim_{n \rightarrow \infty} x_n$.

Definition 1.3. A sequence $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** in a metric space (X, d) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m, n \in \mathbb{N}, m, n > N \implies d(x_m, x_n) < \epsilon.$$

Definition 1.4. A metric space X is **complete** if every Cauchy sequence $(x_i)_{i=1}^{\infty}$ converges to a limit point in X . i.e. $\exists x \in X, \lim_{i \rightarrow \infty} x_i = x$.

Proposition 1.1. Let (X, d) be a metric space; then every convergent sequence is Cauchy.

Proposition 1.2. Let (X, d) be a metric space. Suppose $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a convergent subsequence such that $\lim_{k \rightarrow \infty} x_{n_k} = x \in X$, then $\lim_{n \rightarrow \infty} x_n = x$.

2 Topology

See more in the notes of Pmath367 Topology by Professor S. New.

2.1 Topological Spaces

Definition 2.1. Let $X \neq \emptyset$ be a set. A **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X) := 2^X$ = power set of X , satisfying

1. $\emptyset, X \in \mathcal{T}$,
2. \mathcal{T} is closed under arbitrary union; namely, $\forall \{A_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}, \bigcup_{\alpha \in I} A_\alpha \in \mathcal{T}$, and
3. \mathcal{T} is closed under finite intersection; namely, $\forall \{A_i\}_{i=1}^n \subseteq \mathcal{T}, \bigcap_{i=1}^n A_i \in \mathcal{T}$.

Also, (X, \mathcal{T}) is a **topological space** if \mathcal{T} is a topology on X .

Definition 2.2. For any $S \subseteq \mathcal{P}(X)$, we define the **topology generated by S** to be

$$\mathcal{T}_S := \langle S \rangle := \{\emptyset, X, \text{ all unions of finite intersections of elements of } S\},$$

which is the intersection of all topologies on X that contains S , and it is the smallest topology on X containing S .

Proposition 2.1. Let (X, d) be a metric space, then there is a **metric topology** \mathcal{T}_d that is generated by open balls.

Definition 2.3. (X, \leq) is a **partially ordered set (poset)** if \leq is

1. anti-symmetric: $\forall x, y \in X$, if $x \leq y$ and $y \leq x$, we have $x = y$,
2. reflexive: $\forall x \in X, x \leq x$, and

3. transitive: $\forall x, y, z \in X$, if $x \leq y, y \leq z$, we have $x \leq z$.

We can define $\geq, <, >$ by

$$\begin{aligned} x \geq y &\iff y \leq x \\ x < y &\iff x \leq y \wedge x \neq y \\ x > y &\iff y < x. \end{aligned}$$

Definition 2.4. (X, \leq) is a **totally ordered set** if it is a partially ordered set such that $\forall x, y \in X$, $x \leq y$ or $y \leq x$.

Proposition 2.2. (X, \leq) is a totally ordered set if and only if $<$ satisfies

1. $\forall x, y \in X$, exactly one of the following is true: $x < y, x = y, y < x$.
2. $\forall x, y, z \in X$, if $x < y, y < z$, we have $x < z$.

Definition 2.5. Let (X, \leq) be a totally ordered set, we can define for each $a, b \in X$,

1. $(-\infty, a) := \{x \in X : x < a\}$,
2. $(a, \infty) := \{x \in X : a < x\}$, and
3. $(a, b) := (a, \infty) \cap (-\infty, b)$.

Proposition 2.3. Let \mathcal{T}_\leq be the topology generated by all the sets above, then \mathcal{T}_\leq is a topology.

Definition 2.6. Let (X, \mathcal{T}) be a topological space, then we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$. We say $E \subseteq X$ is **closed** if $E^c \in \mathcal{T}$ is open.

Definition 2.7. For $E \in X$, the **closure** of E is

$$\bar{E} = \bigcap_{F \supseteq E: F \text{ is closed}} F,$$

which is the smallest closed set containing E .

Definition 2.8. For $E \in X$, the **interior** of E is

$$E^\circ = \bigcup_{U \subseteq E: U \text{ is open}} U,$$

which is the largest open set contained in E .

Proposition 2.4. Closed sets are stable under finite unions and arbitrary intersections.

Proposition 2.5. For any set A ,

$$\bar{A} = ((A^c)^\circ)^c.$$

Proposition 2.6. For any set A ,

$$x \in \bar{A} \iff (\forall U \text{ open}, x \in U \implies U \cap A \neq \emptyset).$$

Definition 2.9. Let (X, \mathcal{T}) be a topological space, a $\mathcal{B} = \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{P}(X)$ is said to be a **basis/base** of the topology \mathcal{T} if it is a collection of open sets, and for every $U \in \mathcal{T}$, we have $U = \bigcup_{\alpha \in J} U_\alpha$ for some $J \subseteq I$.

Proposition 2.7. Let (X, \mathcal{T}) be a topological space, $\mathcal{B} \subseteq \mathcal{T}$ is a basis of \mathcal{T} if and only if

$$\forall x \in U \in \mathcal{T}, \exists U_\alpha \in \mathcal{B} \text{ such that } x \in U_\alpha \subseteq U.$$

Proposition 2.8. Let (X, \mathcal{T}) be a topological space, $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis of \mathcal{T} , if and only if

1. $X = \bigcup_{\alpha \in I} U_\alpha$,
2. For any $U_1, U_2 \in \mathcal{B}$, $x \in U_1 \cap U_2$, we have $\exists U_x \in \mathcal{B}, x \in U_x \subseteq U_1 \cap U_2$,
3. $\mathcal{T} = \langle \mathcal{B} \rangle$ is the topology generated by \mathcal{B} .

Example 2.1.1. Let (X, d) be a metric space, then $\{B(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\}$ is a base.

Definition 2.10. Let (X, \mathcal{T}) be a topological space, a **subbase** is a collection of open sets $S \subseteq \mathcal{T}$ such that

$$\left\{ X, \bigcap_{i=1}^n S_i : n \in \mathbb{N}^+, S_1, \dots, S_n \in S \right\}$$

forms a base for \mathcal{T} .

Proposition 2.9. Let (X, \mathcal{T}) be a topological space, then any subbase S generates \mathcal{T} . Also, any set S such that $\bigcup S = X$ is always a subbase for the topology \mathcal{T}_S generated by S .

Definition 2.11. Let (X, \mathcal{T}) be a topological space. Given $x \in X$, a **neighbourhood** of x is a set $V \ni x$, such that $\exists U \in \mathcal{T}$ with $x \in U \subset V$.

Definition 2.12. Let (X, \mathcal{T}) be a topological space. Given $x \in X$, a **neighbourhood basis** of x is a set of open neighbourhoods $\mathcal{B}_x \subset \mathcal{T}$, such that for any (open) neighbourhood U of x , there is $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$.

Proposition 2.10. Let (X, \mathcal{T}) be a topological space, $\mathcal{B} \subseteq \mathcal{T}$ is a basis of \mathcal{T} if and only if $\forall x \in X$, \mathcal{B} is a neighbourhood basis of x .

Definition 2.13. We say $S \subseteq X$ is **dense** in a topological space (X, \mathcal{T}) if $\forall \text{open } U \neq \emptyset$, $S \cap U \neq \emptyset$.

Proposition 2.11. S is dense if and only if $\bar{S} = X$.

Definition 2.14. A topological space (X, \mathcal{T}) is **separable** if there is a countable subset.

Definition 2.15. A topological space (X, \mathcal{T}) is **first countable** if $\forall x \in X$, there is a countable open neighbourhood basis $\{B_n\}_{n=1}^\infty \subset \mathcal{T}$ at x . Namely, for any neighbourhood U of x , there is $n \in \mathbb{N}$ such that $x \in B_n \subseteq U$.

Definition 2.16. A topological space (X, \mathcal{T}) is called **2nd countable** if it has a countable basis.

Proposition 2.12. Every metric space (X, d) are first countable.

Proposition 2.13. Every metric space (X, d) is 2nd countable if and only if X is countable.

Proposition 2.14. The discrete topology of X is separable if and only if $|X|$ is at most countable.

Definition 2.17 (Axiom of Choice). If $X \neq \emptyset$, then there is a choice function $C : P(X) \setminus \{\emptyset\} \rightarrow X$ such that $\forall A \subseteq X$, if $A \neq \emptyset$, we have $C(A) \in A$.

Proposition 2.15 (Axiom of Choice Equivalence). The Axiom of Choice is equivalent to: Let $\{X_\alpha\}_{\alpha \in A}$ be a family of non-empty sets, then

$$\Pi_{\alpha \in A} X_\alpha := \left\{ f : A \rightarrow \bigcup_{\alpha \in A} X_\alpha \mid f(\alpha) \in X_\alpha \right\} \neq \emptyset.$$

Proof. Suppose AOC holds, then taking $X = \bigcup_{\alpha \in A} X_\alpha$, we have the choice function C . Now take $f(a) := C(X_a)$.

On the other hand, suppose the latter holds, then consider $\Pi_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$, which is non-empty. Consider any $f \in \Pi_{X_\alpha \in (P(X) \setminus \{\emptyset\})} X_\alpha$, then $C(X_\alpha) := f(a)$ is a choice function. \square

Proposition 2.16. A metric space (X, d) is separable if and only if it is 2nd countable.

Proof. If $S = \{x_k\}_{k=1}^\infty$ is dense, then $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a countable base.

Indeed, consider any $x \in X$ with any open $U \ni x$, we know $\exists r > 0$, such that $x \in B(x, r) \subset U$. Also, there is x_k such that $d(x, x_k) < \frac{r}{2}$.

Now choose some $r' \in \mathbb{Q}$ such that $d(x, x_k) < r' < \frac{r}{2}$, then $x \in B(x_k, r') \subset B(x, r) \subset U$.

Thus $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a base.

On the other hand, suppose X is second countable with a countable base $\{U_n\}_{n=1}^\infty$. WLOG, $U_n \neq \emptyset$.

Now for any $n \in \mathbb{N}$, pick $x_n \in U_n$ by the axiom of countable choice. Let $S = \{x_n\}_{n=1}^\infty$, then we claim S is dense.

Indeed, for any open $U \neq \emptyset$, we can find some $U_n \subset U$. Thus $x_n \in S \cap U$. \square

Proposition 2.17. If $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ is a set of topologies on X ,

1. There is a weakest topology $\tau := \langle \bigcup_{\alpha \in A} \mathcal{T}_\alpha \rangle$ that is stronger than each \mathcal{T}_α .
2. There is a strongest topology $\delta := \bigcap_{\alpha \in A} \mathcal{T}_\alpha$ that is weaker than each \mathcal{T}_α .

Definition 2.18. A topological space (X, \mathcal{S}) is **Hausdorff** if

$$\forall x \neq y \in X, \exists S_x, S_y \in \mathcal{S}, \text{ such that } x \in S_x, y \in S_y, S_x \cap S_y = \emptyset.$$

Proposition 2.18. Any space with its discrete topology is always Hausdorff.

Example 2.1.2. Every metric space is Hausdorff.

Proposition 2.19. Any space with more than one element with the trivial topology is not Hausdorff.

Example 2.1.3. Consider $X := (0, 1) \cup \{1^+, 1^-\}$. Let $(0, 1)$ have the usual open topology. Also, let $(r, 1) \cup \{1^+\}$ and $(r, 1) \cup \{1^-\}$ be open for any $0 < r < 1$. The topology generated by this basis will not be Hausdorff.

Indeed, consider $1^+, 1^-$, then for any $U \ni 1^+, V \ni 1^-$, we can find $r_U, r_V \in (0, 1)$, such that $(r_U, 1) \cup \{1^+\} \subseteq U, (r_V, 1) \cup \{1^-\} \subseteq V$. Yet $(\max(r_U, r_V), 1) \subseteq U \cap V$, which is not empty.

Proposition 2.20. If X is Hausdorff, then for any $x \in X$, we have that $\{x\}$ is closed.

Proof. For any $y \neq x$, we can find open $V_y \ni y, U_y \ni x$, such that $V_y \cap \{x\} \subseteq V_y \cap U_y = \emptyset$. Thus $X \setminus \{x\} = \bigcup_{y \in X} V_y$, which is open. \square

2.2 Continuous Functions

Definition 2.19. A function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ is **continuous** if

$$\forall U \in \mathcal{S}, f^{-1}(U) \in \mathcal{T}.$$

Namely, the preimage of any open set is open.

Definition 2.20. A function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ is **continuous at $x \in X$** if

$$\forall V \in \mathcal{S}, \text{ such that } f(x) \in V \exists U \in \mathcal{T}, \text{ such that } x \in U \subseteq f^{-1}(V).$$

Proposition 2.21. Let $f : X \rightarrow Y$ be a map between topological spaces. Then f is continuous (on X) if and only if f is continuous at every point $x \in X$.

Proof. (\implies):

Assume f is continuous, then for every point $x \in X$ and open $V \ni f(x)$, we have $f^{-1}(V)$ is open. Clearly $x \in f^{-1}(V)$.

(\impliedby):

Assume f is continuous at every point x . Given any open $V \in Y$, and any point $x \in f^{-1}(V)$, we have $f(x) \in V$.

By assumption, there is open U_x , such that $x \in U_x \subseteq f^{-1}(V)$.

Now $\bigcup_{x \in f^{-1}(V)} U_x$ is open, while $\bigcup_{x \in f^{-1}(V)} U_x \supseteq \bigcup_{x \in f^{-1}(V)} \{x\} = f^{-1}(V) \supseteq \bigcup_{x \in f^{-1}(V)} U_x$.

Thus $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is open. \square

Definition 2.21. A function $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ is **open** if

$$\forall V \in \mathcal{T}, f(V) \in \mathcal{S}.$$

Namely, the image of any open set is open.

Definition 2.22. Given two sets X, Y , and their corresponding topology \mathcal{T}, \mathcal{S} , a continuous map $f : X \rightarrow Y$ is a **homeomorphism** if it is bijective, and its inverse function is also continuous.

Remark. A homeomorphism is a map that preserves the topological structure between two sets.

Definition 2.23. Let $C(X)$ be the collection of functions $f : X \rightarrow \mathbb{C}$ that are continuous.

Definition 2.24. Let $C(X, \mathbb{R})$ be the collection of functions $f : X \rightarrow \mathbb{R}$ that are continuous.

Definition 2.25. Let $C_b(X)$ be the collection of functions $f \in C(X) : \|f\|_\infty < \infty$.

Definition 2.26. Let $C_b(X, \mathbb{R})$ be the collection of functions $f \in C(X, \mathbb{R}) : \|f\|_\infty < \infty$.

Proposition 2.22. If $C(X)$ separates points, so do $C_b(X), C_b(X, \mathbb{R})$. Also, X is Hausdorff.

Proof. If $x \neq y$, we have $f \in C(X)$ such that $f(x) \neq f(y)$.

WLOG, $\Re(f(x)) \neq \Re(f(y))$, and $\Re(f(x)) < \Re(f(y))$.

Now define $g(z) := \min \{\Re(f(y)), \max \{\Re(f(x)), \Re(f(y))\}\}$, which is bounded and continuous. Also $g(x) = f(x), g(y) = f(y)$.

Thus $C_b(X, \mathbb{R})$ separates the points.

Now if $x \neq y$, we can find $f \in C(X)$, such that $|f(x) - f(y)| = r > 0$.

Now let $U := f^{-1}(B(f(x), \frac{r}{2})), V := f^{-1}(B(f(y), \frac{r}{2}))$, which are both open. Also, $x \in U, y \in V$, and $U \cap V = f^{-1}(\emptyset) = \emptyset$. \square

Definition 2.27. A topological space (X, \mathcal{T}) is **normal** if for any disjoint closed sets A, B , we can find open $U \supset A, V \supset B$ such that $U \cap V \neq \emptyset$.

Theorem 2.23 (Urysohn's Lemma for normal spaces). (X, \mathcal{T}) is normal if and only if for any disjoint closed sets A, B , $\exists f : X \rightarrow [0, 1]$ continuous, such that $f|_A = 0, f|_B = 1$.

Corollary 2.24. If X is normal and Hausdorff, $C_b(X)$ separates the points.

2.3 Nets

Definition 2.28. Let (X, \mathcal{T}) be a topological space, say a sequence $(x_i)_{i=1}^\infty$ **converges to** $x \in X$ if \forall open $U \ni x, \exists N \in \mathbb{N}$ such that $\forall i \geq N, x_i \in U$.

Example 2.3.1. Consider $X = \mathbb{N} \times \mathbb{N}$, and the projection $\pi_1 : X \rightarrow \mathbb{N}$ by $\pi_1(m, n) := m$.

Let any U be open if $(0, 0) \notin U$, or if $\{m \in \mathbb{N} : \pi_1^{-1}(m) \cap U\}$ is co-finite in $\{m\} \times \mathbb{N}$ is co-finite.

One can show this defines a topology on X , and it is Hausdorff.

Indeed, let $X_0 := X \setminus \{(0, 0)\}$. Consider any $(m, n) \neq (m', n') \in X$. If both are in X_0 then $\{(m, n)\}, \{(m', n')\}$ are open and disjoint.

If $(m', n') = (0, 0)$, then $\{(m, n)\}, X \setminus \{(m, n)\}$ are open and disjoint.

This shows Hausdorff.

Also, $\bar{X}_0 = X$.

Indeed, consider any open $U \ni (0, 0)$, we must have that $U \cap X_0 \neq \emptyset$.

However, there is no sequence in X_0 that converges to $(0, 0)$.

Indeed, assume for contradiction that there is such a convergent sequence $(x_k)_{k=1}^\infty$ in X_0 .

Write each $x_k = (m^k, n^k)$.

Suppose there is some $M \in \mathbb{N}^+$, such that $\forall k \in \mathbb{N}^+, m_k \leq M$.

Consider $U := \{(m, n) : m > M, n \in \mathbb{N}\} \cup \{(0, 0)\}$.

Now for each $m \in \mathbb{N}$,

$$\pi_1^{-1}(m) \cap U = \begin{cases} (0, 0) & \text{if } m = 0 \\ \emptyset & \text{if } 0 < m \leq M \\ \{m\} \times \mathbb{N} & \text{if } m > M. \end{cases}$$

Thus for all $m > M$, $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.

Yet $U \cap \{x_k\}_{k=1}^{\infty} = \emptyset$.

Now suppose there is no such M , then we can find a subsequence $m_{k_1} < m_{k_2} < m_{k_3} < \dots$

Now let $U := X \setminus \{x_{k_i}\}_{i=1}^{\infty}$.

For each $m \in \mathbb{N}$,

$$\pi^{-1}(m) \cap U = \begin{cases} \{m\} \times \mathbb{N} & \text{if } m \notin \{m_{k_i}\}_{i=1}^{\infty} \\ \{m\} \times (\mathbb{N} \setminus n_{k_i}) & \text{if } m \in \{m_{k_i}\}_{i=1}^{\infty}. \end{cases}$$

Notice that there cannot be two $n_{k_i} \neq n_{k_j}$ for any m , since $k_i \neq k_j \implies m_{k_i} \neq m_{k_j}$.

Thus all $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.

Yet $U \cap \{x_{k_i}\}_{i=1}^{\infty} = \emptyset$.

Thus there cannot be any convergent sequence $(x_k)_{k=1}^{\infty}$ in X_0 .

Remark. The above example shows that sequences do not behave as we want in topological spaces.

Definition 2.29. An **upwards directed set** is a poset (Λ, \leq) such that if $\lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_0 \in \Lambda$ such that $\lambda_1 \leq \lambda_0, \lambda_2 \leq \lambda_0$.

Definition 2.30. For $X \neq \emptyset$, a **net** in X is a function $j : \Lambda \rightarrow X$, where (Λ, \leq) is an upwards directed set. Write $x_{\lambda} := j(\lambda) \in X$, and we can use $(x_{\lambda})_{\lambda \in \Lambda}$ to represent a net.

Definition 2.31. Let (X, \mathcal{T}) be a topological space, say a net $(x_{\lambda})_{\lambda \in \Lambda}$ **converges to** $x \in X$ if

$$\forall \text{open } U \ni x, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_{\lambda} \in U.$$

In this case, we say x is a **limit** of the net, and write it as $x = \lim_{\lambda \in \Lambda} x_{\lambda}$ or $x_{\lambda} \rightarrow x$.

Proposition 2.25. Let (X, \mathcal{T}) be a topological space with a neighbourhood basis \mathcal{B} at $x \in X$, then a net $(x_{\lambda})_{\lambda \in \Lambda}$ converges to x if and only if

$$\forall U \in \mathcal{B} \text{ such that } x \in U, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_{\lambda} \in U.$$

Proof. The forward direction is trivial.

Now assume $\forall U \in \mathcal{B}$ such that $x \in U, \exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0, x_{\lambda} \in U$.

Given any open $V \ni x$, since \mathcal{B} is a neighbourhood basis, there is some $U \in \mathcal{B}$, such that $x \in U \subseteq V$.

Thus, there is $\lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0, x_{\lambda} \in U \subseteq V$. \square

Definition 2.32. Given a net $(x_{\lambda})_{\lambda \in \Lambda}$, then a **subnet** of it $(y_{\gamma})_{\gamma \in \Gamma}$ is given by an upwards directed set (Γ, \leq) and a function $\phi : \Gamma \rightarrow \Lambda$ that is **cofinal**, which means $\forall \lambda_0 \in \Lambda, \exists \gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0, \phi(\gamma) \geq \lambda_0$. Each y_{γ} is given by $x_{\phi(\gamma)}$.

Example 2.3.2. Notice that if we take (\mathbb{N}, \leq) , the net is just a sequence. To get a subsequence, we can take $\Gamma = \mathbb{N}$, and $\phi : \mathbb{N} \rightarrow \mathbb{N}$ to be any increasing function. The generated subnet will be a subsequence.

Definition 2.33. Let (X, \mathcal{T}) be a topological space, and $x \in X$, define the **system of open neighbourhoods of x** to be $\mathcal{O}(x) := \{U \in \mathcal{T} : x \in U\}$.

Proposition 2.26. $(\mathcal{O}(x), \supseteq)$ is an upwards directed set.

Example 2.3.3. For $X = \mathbb{N} \times \mathbb{N}$ and $X_0 = X \setminus \{(0, 0)\}$ as above, there is a net in X_0 converging to $(0, 0)$.

Indeed, let us enumerate $X_0 = \{x_k\}_{k=1}^{\infty}$ as $(0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \dots$.

Now $\Lambda := \mathcal{O}((0, 0))$ is an upward directed set by containment.

Then for each $U \in \Lambda$, we can pick $x_U := x_{k_U}$, where k_U is the first $k \in \mathbb{N}$ such that $x_k \in U$.

Claim: $(x_U)_{U \in \Lambda}$ converges to $(0, 0)$.

Pick any $U_0 \ni (0, 0)$, then for all $U \geq U_0$, it is open and $U \subseteq U_0$. Thus, we must have $x_U \in U \subseteq U_0$.

Indeed, $(x_U)_{U \in \Lambda}$ is a subnet of $\{x_k\}_{k \in \mathbb{N}^+}$ by $\phi(U) := k_U$.

Theorem 2.27. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces, then

1. For any $A \subseteq X$, we have $x \in \bar{A}$ if and only if \exists a net $(x_{\lambda})_{\lambda \in \Lambda}$ in A , such that $x_{\lambda} \rightarrow x$.

2. $f : X \rightarrow Y$ is continuous if and only if for any net $(x_\lambda)_{\lambda \in \Lambda} \subset X$ such that $x_\lambda \rightarrow x \in X$, we have $f(x_\lambda) \rightarrow f(x)$.

Proof. 1. Consider any $x \in \bar{A}$, then for any open $U \ni x$, we have $U \cap A \neq \emptyset$.

By the Axiom of Choice, we can have $x_U \in U \cap A$ for each open neighbourhood U .

Consider the net $(x_U)_{U \in \mathcal{O}(x)}$.

Given any open $U \ni X$, if $V \geq U$, we must have $V \subseteq U$, and $x_v \in V \subseteq U$.

Thus $x_U \rightarrow x$.

On the other hand, for any net $x_\lambda \rightarrow x$ in A , consider any open $U \ni x$, there is λ_0 , such that

$$\forall \lambda \in \Lambda, \lambda_0 \leq \lambda \implies x_\lambda \in U \implies U \cap A \neq \emptyset.$$

Thus $x \in \bar{A}$.

2. Assume f is continuous, and $x_\lambda \rightarrow x$. Let $V \in \mathcal{O}(f(x))$, then $U := f^{-1}(V)$ is open and $x \in U$.

Thus there is $\lambda_0 \in \Lambda$, such that $\forall \lambda \geq \lambda_0, x_\lambda \in U$.

Thus $f(x_\lambda) \in V$.

On the other hand, assume for any net $(x_\lambda)_{\lambda \in \Lambda} \subset X$ such that $x_\lambda \rightarrow x \in X$, we have $f(x_\lambda) \rightarrow f(x)$.

For contradiction, suppose there is open $V \in \mathcal{O}(f(x))$, with $U := f^{-1}(V)$ is not open in X .

Then U^c is not closed, and $U^c \neq \overline{U^c}$.

Thus, there is $x \in \overline{U^c} \setminus U^c = \overline{U^c} \cap U$.

Since $x \in \overline{U^c}$, by 1., there is a net $(x_\lambda)_{\lambda \in \Lambda}$ in U^c , such that $x_\lambda \rightarrow x$.

By assumption, we have $f(x_\lambda) \rightarrow f(x)$.

Since each $f(x_\lambda)$ is in $f(U^c) = V^c$, by 1., we have that $f(x) \in \overline{V^c} = V^c$ since V^c is closed (V is open).

However, since $x \in U$, we also have $f(x) \in f(U) = V$, which is a contradiction.

□

2.4 Compactness

Definition 2.34. Let (X, \mathcal{T}) be a topological space. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$ is called a **cover** for X if $X = \bigcup_{\alpha \in A} U_\alpha$. A cover is called an open cover if every U_α is open in \mathcal{T} .

Definition 2.35. Let (X, \mathcal{T}) be a topological space. A collection $\{C_\alpha\}_{\alpha \in A}$ of non-empty closed sets is a FIP-family if for any finite $F \subseteq A$, $\bigcap_{\alpha \in F} C_\alpha \neq \emptyset$. X has the finite intersection property (FIP) if for all FIP-familie $\{C_\alpha\}_{\alpha \in A}$, we have $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$.

Definition 2.36. Let (X, \mathcal{T}) be a topological space. X is **compact** if every open cover of X has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open}, X = \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } X = \bigcup_{i=1}^n U_{\alpha_i}.$$

Definition 2.37. Let (X, \mathcal{T}) be a topological space. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha \in A}$ is called a **cover** for $K \subseteq X$ in X if $K = \bigcup_{\alpha \in A} U_\alpha$. A cover in X is called an open cover in X if every U_α is open in \mathcal{T} .

Definition 2.38. Let (X, \mathcal{T}) be a topological space. A set $K \subseteq X$ is **compact in X** if every open cover of K in X has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open}, K \subseteq \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } K \subseteq \bigcup_{i=1}^n U_{\alpha_i}.$$

Proposition 2.28. Let (X, \mathcal{T}) be a topological space. A set $K \subseteq X$ is compact under the subspace topology if and only if it is compact in X .

Theorem 2.29. Let (X, \mathcal{T}) be a topological space, TFAE:

1. X is compact.

2. X has the finite intersection property.

3. For all nets $(x_\lambda)_{\lambda \in \Lambda}$ in X , there is a convergent subnet.

Proof. (1) \implies (2).

For contradiction, suppose there is some FIP-family such that $\bigcap_{\alpha \in A} C_\alpha = \emptyset$.

We have $X = \bigcup_{\alpha \in A} C_\alpha^c$, which is an open cover for X .

Since X is compact, there is a finite $F \subseteq A$, such that $X = \bigcup_{\alpha \in F} C_\alpha^c$.

Thus $\bigcap_{\alpha \in A} C_\alpha = \emptyset$, which contradict $\{C_\alpha\}_{\alpha \in A}$ being a FIP family.

(2) \implies (1).

Consider any open cover $X = \bigcup_{\alpha \in A} U_\alpha$, then $\bigcap_{\alpha \in A} U_\alpha^c = \emptyset$, and it is not a FIP-family.

Thus there is a finite $F \subseteq A$, such that $\bigcap_{\alpha \in F} U_\alpha^c = \emptyset$.

Thus $X = \bigcup_{\alpha \in F} U_\alpha$ is a finite open cover.

(2) \implies (3).

Let $(x_\lambda)_{\lambda \in \Lambda}$ be any net in X .

Define $C_\lambda := \{x_\mu : \mu \geq \lambda\}$. Notice that $C_\lambda \neq \emptyset$ since $x_\lambda \in C_\lambda$.

We claim that $\{C_\lambda\}_{\lambda \in \Lambda}$ is a FIP family.

The closeness is by definition.

Now fix any $\lambda_1, \dots, \lambda_n \in \Lambda$.

Since Λ is upwards directed, there is $\lambda_0 \in \Lambda$, such that $\forall i \in [n], \lambda_i \leq \lambda_0$.

Thus $\bigcap_{i=1}^n C_{\lambda_i} \supseteq C_{\lambda_0} \neq \emptyset$.

By FIP, $\bigcap_{\lambda \in \Lambda} C_{\lambda_0} \neq \emptyset$.

Pick any $x \in \bigcap_{\lambda \in \Lambda} C_{\lambda_0}$.

Let $\Gamma := \Lambda \times \mathcal{O}(x)$ with the partial order $(\lambda, U) \leq (\lambda', U')$ if $\lambda \leq \lambda'$ and $U \supseteq U'$.

Fix $(\lambda, U) \in \Gamma$, we know that $x \in C_\lambda = \overline{\{x_\mu : \mu \geq \lambda\}}$.

Thus $U \cap \{x_\mu : \mu \geq \lambda\} \neq \emptyset$.

By the Axiom of Choice, there is $y_{(\lambda, U)} = x_{\phi(\lambda, U)} \in U \cap \{x_\mu : \mu \geq \lambda\}$, where $\phi(\lambda, U) := C(\{\mu \geq \lambda : x_\mu \in U\})$.

For any $\lambda_0 \in \Lambda$, let $\gamma_0 = (\lambda_0, X)$, then for any $\gamma = (\lambda, U) \geq \gamma_0$, we have $\phi(\gamma) \geq \lambda \geq \lambda_0$.

Thus ϕ is cofinal, and $(y_\gamma)_{\gamma \in \Gamma}$ is a subnet.

In addition, given any $U_0 \in \mathcal{O}(x)$, we can pick any $\lambda_0 \in \Lambda$, and let $\gamma_0 := (\lambda_0, U_0)$.

Then for any $(\lambda, U) \geq \gamma_0$, we must have $y_{(\lambda, U)} = x_{\phi(\lambda, U)} \in U \subseteq U_0$.

(3) \implies (2).

Fix any FIP-family $\{C_\alpha\}_{\alpha \in A}$ in X . Then for any finite $F \subseteq A$, by the Axiom of Choice, we can find $x_F \in \bigcap_{\alpha \in F} C_\alpha$.

Now consider the net $(x_F)_{\text{finite } F \subseteq A}$, where $F_1 \leq F_2$ if $F_1 \subseteq F_2$.

By 3., there is a convergent subnet $\phi : \Gamma \rightarrow \Lambda$, such that $x_{\phi(\gamma)} \rightarrow x \in X$.

Now fix any $\alpha \in A$, then $\{\alpha\} \in \Lambda$.

Thus there is some $\gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0, \phi(\gamma) \supseteq \{\alpha\} \ni \alpha$.

We have $x_{\phi(\gamma)} \in \bigcap_{\beta \in \phi(\gamma)} C_\beta \subseteq C_\alpha$.

Since this holds for all $\gamma \geq \gamma_0$, and $x_{\phi(\gamma)} \rightarrow x$, we have that $x \in \bar{C}_\alpha = C_\alpha$.

Since this holds for any $\alpha \in A$, we have that $x \in \bigcap_{\alpha \in A} C_\alpha$. Thus $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$. \square

With the above theorem, we can identify the definition of compactness in a metric space via sequences with the definition of compactness with its metric topology.

Proposition 2.30. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces. If X is compact, and $f : X \rightarrow Y$ is continuous, then $f(X)$ is compact in Y .

Theorem 2.31. Let (X, \mathcal{T}) be a topological space,

1. Suppose K is compact, then $\forall F \subseteq K$ that is closed, F is also compact.

2. If X is Hausdorff, for any compact $K \subseteq X, x \in X \setminus K$, \exists open neighbourhood U of x , and open $W \supset K$, such that $W \cap U = \emptyset$.

Proof. 1. Let $(U_\alpha)_{\alpha \in A}$ be an open cover for F .

Since F is closed, then F^c is open. Thus $\{F^c\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover for K .

Thus there are $U_{\alpha_1}, \dots, U_{\alpha_n}$, such that $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$. Thus $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ has a finite subcover.

2. Consider any $y \in K$, there is some open neighborhoods $U_y \ni x, W_y \ni y$, such that $U_y \cap W_y = \emptyset$.

Since $K \subseteq \bigcup_{y \in K} W_y$ is compact, we have $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$ for some y_1, \dots, y_n .

Let $U = \bigcap_{i=1}^n U_{y_i}$, we have $x \in U, K \subseteq W, U \cap W = \emptyset$ as required.

□

Corollary 2.32. Let (X, \mathcal{T}) be a Hausdorff topological space, then any compact set K is closed. In addition, for any closed $F \subseteq X$, we have $F \cap K$ is compact.

Proof. Suppose for contradiction that K is not closed, then there is some $y \in \bar{K} \setminus K$. Thus we can find open neighbourhood U of x , and open $W \supset K$, such that $W \cap U = \emptyset$. Now $K \subset \bar{K} \setminus U \subsetneq \bar{K}$ is closed, which is a contradiction.

Since K is closed, so is $F \cap K \subseteq K$, and thus it is compact.

□

Definition 2.39. X is **locally compact** if $\forall x \in X$, there is an open neighbourhood $U_x \in \mathcal{O}(x)$ such that $\overline{U_x}$ is compact.

Example 2.4.1. \mathbb{R}^n is locally compact by the Heinz-Borel theorem.

Proposition 2.33. A Banach space $(X, \|\cdot\|)$ is locally compact iff $\dim(X) < \infty$.

Lemma 2.34. Let (X, \mathcal{T}) be a Hausdorff topological space, and $(K_\alpha)_{\alpha \in A}$ be a collections of compact sets such that

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

We must have $\alpha_1, \dots, \alpha_n \in A$, such that

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

Proof. Fix $\alpha_1 \in A$, then $K_{\alpha_1} \subseteq \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$ is compact and has an open cover.

Thus there must be $\alpha_2, \dots, \alpha_n \in A$, such that $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = (\bigcap_{i=2}^n K_{\alpha_i})^c$.

Thus $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$.

□

Theorem 2.35. Let X be a Locally Compact Hausdorff space, and let $K \subseteq U \subseteq X$ be such that K is compact, and U is open. Then there exists some open set V such that \bar{V} is compact, and

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof. Since X is a Locally Compact Hausdorff space, there are V_1, \dots, V_n , each with \bar{V}_i be compact, such that $K \subseteq \bigcup_{i=1}^n V_i =: G$. Note that \bar{G} is compact, and G is open.

If $U = X$, then $G \subseteq U$, and we are done.

Otherwise, let $C := X \setminus U$ be non-empty and closed.

Consider any $y \in C$, we know that $y \notin K$. Since X is Hausdorff, we can find open $W_y \supset K$, and $U_y \ni y$, such that $W_y \cap U_y = \emptyset$. Then $W_y \subseteq U_y^c$, and thus $\bar{W}_y \subseteq U_y^c$, since U_y^c is closed. Yet $y \notin U_y^c$, thus $y \notin \bar{W}_y$.

Now consider the family $\{C \cap \bar{W}_y \cap \bar{G}\}_{y \in C}$. Notice that each $C \cap \bar{W}_y \cap \bar{G}$ is compact, since C, \bar{W}_y are closed, and \bar{G} is compact.

Yet $\bigcap_{y \in C} (C \cap \bar{W}_y \cap \bar{G}) = \emptyset$.

Thus $\exists y_1, \dots, y_n \in C$, such that $\bigcap_{i=1}^n (C \cap \bar{W}_{y_i} \cap \bar{G}) = \emptyset$.

Now let $V := G \cap \bigcap_{i=1}^n W_{y_i}$.

Clearly V is open, and $K \subseteq V$.

In addition, $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$, yet the intersection of righthand side and C is empty, thus contained in $C^c = U$.

□

2.5 Compactly Supported Continuous Functions

Definition 2.40. For $f \in C(X)$, the **support** of it is

$$\text{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

Definition 2.41. The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \text{Supp}(f) \text{ is compact}\}.$$

Definition 2.42. $C_0(X)$ is the closure of $C_c(X)$ in $\|\cdot\|_\infty$.

Proposition 2.36. $C_0(X)$ is the set of all continuous functions that vanishes at ∞ . $(C_0(X), \|\cdot\|_\infty)$ is a Banach Space and a commutative C^* -algebra with the involution $f^*(x) := \overline{f(x)}$.

Proposition 2.37. $f \in C_0(X)$ if and only if $\forall \epsilon > 0$, $\exists K \subset\subset X$, such that $\forall x \in X \setminus K$, $|f(x)| < \epsilon$.

Theorem 2.38. Any commutative C^* -algebra $(A, \|\cdot\|)$ is isomorphic to $C_0(X)$ for some unique Locally Compact Hausdorff X .

2.5.1 Partition of Unity

Definition 2.43. Let K be a compact set, and V be an open set of X . Let $f \in C_c(X)$. We say $f < V$ if $0 \leq f \leq 1$, and $\text{Supp}(f) \subseteq V$. We say $K < f$ if $0 \leq f \leq 1$, and $f|_K = 1$. We say $K < f < V$ if $K \subset V, K < f, f < V$.

Remark. f is a “bump” function that approximates χ_K when V shrinks towards K .

Lemma 2.39 (Urysohn’s lemma for Locally Compact Hausdorff Space). *Let X be a Locally Compact Hausdorff space, $K \subseteq V \subseteq X$ be such that K is compact, and V is open. Then there exists $f \in C_c(V)$, such that $K < f < V$.*

Proof. we want to construct a family of open sets $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$, such that \bar{V}_r is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for $r < s$.

By 2.35, we can find $K \subset V_0 \subset \bar{V}_0 \subset V$.

Pick an enumeration of $r \in \mathbb{Q} \cap (0,1]$, i.e. $(r_n)_{n=1}^\infty$. WLOG, we can let $r_1 = 1$.

By 2.35, we can find $K \subset V_1 \subset \bar{V}_1 \subset V_0$.

Suppose we have constructed the V_{r_i} for $1 \leq i \leq n$, such that \bar{V}_r is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for $r < s \in \{r_i\}_{i=1}^n$.

Let $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$.

Now by 2.35, we can find $\bar{V}_t \subset V_{n+1} \subset \bar{V}_{n+1} \subset V_s$.

For any $r < r_{n+1}$, we have $r \leq s$, and thus $V_{n+1} \subset \bar{V}_{n+1} \subset V_s \subset \bar{V}_s \subseteq V_r$ by induction hypothesis, and similarly for any $r > r_{n+1}$.

Inductively, we can prove there is such a family.

Define $f_r := r\chi_{V_r}$, and $g_r := r\chi_{\bar{V}_r^c} + \chi_{\bar{V}_r}$, and $f := \sup_r f_r, g := \inf_r g_r$.

We can show that f, g are upper and lower continuous, respectively.

In addition, f, g are both 0 outside of V_1 , and 1 on K .

Suppose there is some $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, such that $f_r(x) > g_s(x)$. Then we must have $f_r(x) > 0$, and thus $x \in V_r$ and $1 \leq r = f_r(x)$.

Thus $1 > g_s(x)$, and thus $x \in \bar{V}_s^c$ and $f_s = s$.

Since $r > s$, we must have $V_r \subset \bar{V}_r \subset V_s \subset \bar{V}_s$, which is a contradiction to $x \in V_r, x \notin \bar{V}_s$.

Thus for any $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, we must have $f_r(x) \leq g_s(x)$.

Thus we must have $f(x) \leq g(x)$ for any $x \in V$.

Now suppose there is some $x \in X$, such that $f(x) < g(x)$.

There must be some rationals, such that $f(x) < r < s < g(x)$, since \mathbb{Q} is dense.

Thus $\sup_r f_r(x) < r$, and thus $x \notin V_r$.

Also, $\inf_s g_s(x) > s$, and thus $x \in \bar{V}_s$.

Yet $r < s$, we must have $V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r$, which is a contradiction.

Thus we must have $f = g$, and it forces f to be continuous. \square

Definition 2.44. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$.

A collection $(h_i)_{i=1}^n \subset C_c(X)$ is called a **partition of unity** on K subordinate to $(V_i)_{i=1}^n$ if

$$\begin{cases} \forall 1 \leq i \leq n, \quad h_i < V_i, \\ \forall x \in K, \quad \sum_{i=1}^n h_i(x) = 1. \end{cases}$$

Theorem 2.40. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$, there always exists a partition of unity on K subordinated to $(V_i)_{i=1}^n$.

Proof. Since K is compact, we can find some open cover W_1, \dots, W_m , such that for all j , we have $W_j \subset \bar{W}_j \subset V_{i(j)}$ for some $1 \leq i(j) \leq n$.

Let $K_i := \bigcup_{1 \leq j \leq m} \text{such that } W_j \subset V_i \bar{W}_j \subset V_i$, which is compact.

By Urysohn's lemma, we can find $K_i < g_i < V_i$.

Now let $h_1 := g_1$, and in general, $h_i := g_i \prod_{j < i} (1 - g_j)$.

It is easy to check that $0 \leq h_i \leq 1$, and $h_i \in C_c(X)$.

In addition, $\text{Supp}(h_i) \subseteq \text{Supp}(g_i) \subset V_i$.

Thus $h_i < V_i$. Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_i)$.

For any $x \in K$, there must be some $i \in [n]$ such that $x \in K_i$, and thus $g_i(x) = 1$, and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_i(x)) = 1 - 0 = 1.$$

\square

2.6 Product Topology

Definition 2.45. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of topological spaces. The **product topology** on $\prod_\alpha X_\alpha$ is the topology generated by the sets

$$\left\{ U_\alpha \times \prod_{\beta \in A, \beta \neq \alpha} X_\beta \mid \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\} = \left\{ \pi_\alpha^{-1}(U_\alpha) \mid \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\},$$

where the **projection map onto** X_α is $\pi_\alpha : \prod_{\beta \in A} X_\beta \rightarrow X_\alpha$ by $(x_\beta)_{\beta \in A} \mapsto x_\alpha$.

Proposition 2.41. The product topology is the weakest topology in which each π_α is continuous.

Proposition 2.42. A net $(x_\lambda)_{\lambda \in \Lambda}$ in $\prod_{\alpha \in A} X_\alpha$ converges to x if and only if $\forall \alpha \in A$, $\pi_\alpha(x_\lambda) \rightarrow \pi_\alpha(x)$ in X_α .

Proof. See A1. \square

Theorem 2.43 (Tychonoff). Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of compact topological spaces, then $\prod_\alpha X_\alpha$ is compact under product topology.

Definition 2.46. Let (P, \leq) be a partially ordered set. We call a totally ordered subset $Q \subseteq P$ a **chain**.

Definition 2.47. Let (P, \leq) be a partially ordered set. We call \leq **inductive** if every chain $Q \subseteq P$ has an upper bound.

Definition 2.48. \leq is called a **well-order** if it is a total order, and for every $\emptyset \neq S \subseteq X$ has a minimal element. $\exists x \in P$ such that $\forall y \in P, y \leq x \implies x = y$.

Lemma 2.44 (Zorn's). Every inductive partial order (P, \leq) , defined on a nonempty P , has a maximal element. Namely, $\exists x \in P$ such that $\forall y \in P, x \leq y \implies x = y$.

Proposition 2.45. Every vector space V has a basis.

Proof. Consider $P = \{S \subset V | S \text{ is linearly independent}\}$, with $S \leq S' \iff S \subseteq S'$.

Let Q be a chain in P , then it has an upper bound $\tilde{S} = \bigcup_{S \in Q} S$, which we can check is still linearly independent.

By Zorn's lemma, there is a maximal $S \in P$. □

Theorem 2.46 (Well-Ordering Principle). Every set X admits a well-ordering.

Theorem 2.47. The Following Are Equal:

1. Tychonoff's Theorem
2. Axiom of Choice
3. Zorn's Lemma
4. Well-Ordering Principle

Proof. (1) \implies (2).

Let $(X_\alpha)_{\alpha \in A}$ be a family of non-empty set.

Let $Y_\alpha := \{p_\alpha\} \sqcup X_\alpha$ for some additional symbol p_α .

Define the topology $\mathcal{T}_\alpha := \{\emptyset, Y_\alpha, X_\alpha, \{p_\alpha\}\}$.

Then $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ are all compact.

By Tychonoff's Theorem, $\prod_\alpha Y_\alpha$ is also compact.

Now consider $C_\alpha := X_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$.

Since $C_\alpha^c = \{p_\alpha\} \times \prod_{\beta \neq \alpha} Y_\beta$ is open, we have that C_α is closed in the product topology.

Also, $C_\alpha \neq \emptyset$.

Now for any finite $\bigcap_{i=1}^n C_{\alpha_i}$, we have that $(x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_n}, p_\alpha, \dots) \in \bigcap_{i=1}^n C_{\alpha_i}$.

Thus $(C_\alpha)_{\alpha \in A}$ is an FIP family.

Since $\prod_\alpha Y_\alpha$ is compact, we have that $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$.

Now $\bigcap_{\alpha \in A} X_\alpha = \bigcap_{\alpha \in A} U_\alpha$, and we have seen that it being nonempty is equivalent as the Axiom of Choice.
(4) \implies (2).

Take $C : \mathcal{P}(X) \setminus \emptyset \rightarrow X$ to be $C(S) :=$ minimal element of S .

(3) \implies (2)

Let $\{X_\alpha\}_{\alpha \in A}$ to be non-empty sets. Let $X := \bigcup_{\alpha \in A} X_\alpha$, $P = \{f_B : B \rightarrow X | B \subseteq A, f_B(\beta) \in X_\beta, \forall \beta \in B\}$.

Clearly $P \neq \emptyset$.

Define the order by $f_B \leq f_{B'} \iff B \subseteq B', f_{B'}|_B = f_B$.

For any chain $Q \subseteq P$, define $\tilde{B} = \bigcup_{B \in Q} B$, and $f_{\tilde{B}}(\beta) = f_B(\beta)$ for $\beta \in B$ of any B .

We can check $f_{\tilde{B}} \in P$ is an upper bound.

By Zorn's Lemma, there is a maximal element $f_B \in P$.

If $B \subsetneq A$, then we can extend the function to contain another point $a \in A \setminus B$, and send a to any $x \in X_a$, contradicting maximality.

Thus there is some $f_A \in P$, which we have seen is equivalent to the Axiom of Choice.

(4) + (2) \implies (3).

See Pmath432 A1.

(3) + (2) \implies (1).

For contradiction, suppose $X = \prod_{\alpha \in A} X_\alpha$ is not compact.

Let NFS := $\{\mathcal{C} \subseteq \mathcal{P}(X) : \mathcal{C} \text{ is a cover with no finite sub cover}\}$.

Define $\mathcal{C}_1 \leq \mathcal{C}_2 \iff \mathcal{C}_1 \subseteq \mathcal{C}_2$.

Take any chain Q in NFS.

Let $\mathcal{C}' := \bigcup_{\mathcal{C} \in Q} \mathcal{C}$, which we can check is an open cover and an upper bound for the chain.

Indeed, suppose $\mathcal{C}' \notin \text{NFS}$, then $\exists U_1, U_2, \dots, U_n \in \mathcal{C}'$ such that $X = \bigcup_{i=1}^n U_i$.

For all $i \in [n]$, $U_i \in \mathcal{C}_i$ for some $\mathcal{C}_i \in Q$.

Since Q is a chain, there is some $i_0 \in [n]$ such that $\forall i \in [n], \mathcal{C}_i \subseteq \mathcal{C}_{i_0}$.

Thus $\mathcal{C}_{i_0} \notin \text{NFS}$, a contradiction.

Thus $\mathcal{C} \in \text{NFS}$.

By Zorn's Lemma, there is a maximal open cover \mathcal{C}_{max} with no subcover.

Notice that if $U \in \mathcal{C}_{max}$, and $V \subseteq U$ is open, then $V \in \mathcal{C}_{max}$ as well, since any finite subcover of $\{V\} \cup \mathcal{C}_{max}$ can give a finite subcover of \mathcal{C}_{max} by replacing V by U .

Also, if $U_1, U_2 \in \mathcal{C}_{max}$, we must have $U_1 \cup U_2 \in \mathcal{C}_{max}$ as well.

Also, suppose V_1, \dots, V_n are open in X , such that $\bigcap_{i \in [n]} V_i \in \mathcal{C}_{max}$, then $\exists i_0 \in [n]$, such that $V_{i_0} \in \mathcal{C}_{max}$.

Indeed, suppose not, for any $i \in [n]$, there is a finite cover $V_i \cup \bigcup_{j \in [N_i]} U_{i,j}$ for $U_{i,j} \in \mathcal{C}_{max}$. We must have $\left(\bigcap_{i \in [n]} V_i\right) \cup \bigcup_{i \in [n], j \in [N_i]} U_{i,j}$ is a finite sub-cover of \mathcal{C}_{max} .

Now let $W_\alpha := \{\text{open } U_\alpha \subseteq X_\alpha \mid \pi^{-1}(U_\alpha) \in \mathcal{C}_{max}\}$.

For contradiction, suppose W_α covers X_α , then there is a finite subcover $\{U_i\}_{i \in [n]}$ such that $X_\alpha = \bigcup_{i \in [n]} U_i$.

Thus $X = \bigcup_{i=1}^n \pi_\alpha^{-1}(U_i)$, which is a subcover of \mathcal{C}_{max} .

Thus $X_\alpha \setminus (\bigcup_{U \in W_\alpha} U) \neq \emptyset$.

By the Axiom of Choice, there is $x_\alpha \in X_\alpha \setminus (\bigcup_{U \in W_\alpha} U)$ for each α .

Let $x \in X$ be $x(\alpha) = x_\alpha$.

Since \mathcal{C}_{max} is a cover for X , there is some open $U \in \mathcal{C}_{max}$ with $x \in U$.

Thus, there must be some $x \in U_1 \times U_2 \times \dots \times U_n \times \prod_{\beta \in A \setminus \{\alpha_i : i \in [n]\}} = \bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$ for open $U_i \in X_{\alpha_i}$, since such sets forms a basis.

Thus, $\bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{max}$ as well.

Thus, there is some $i_0 \in [n]$, such that $\pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{max}$, which means $U_i \in W_{\alpha_i}$.

However, $x_{\alpha_i} \in U_{\alpha_i}$, which is a contradiction to the choice of $x_{\alpha_i} \notin \left(\bigcup_{U \in W_{\alpha_i}} U\right)$. \square

3 Banach Spaces

Definition 3.1. A **normed vector space** is a vector space $(X, \|\cdot\|)$ over a field \mathbb{F} endowed with a norm (length) function: $\|\cdot\| : X \rightarrow [0, \infty)$, such that $\forall x, y \in X, a \in \mathbb{F}$, it satisfies

1. subadditivity (triangular inequality); i.e. $\|x + y\| \leq \|x\| + \|y\|$,
2. absolute homogeneity; i.e. $\|a \cdot x\| = |a| \|x\|$, and
3. positive definiteness; i.e. if $x \neq 0$, we must have $\|x\| > 0$.

Proposition 3.1. For every **normed space** with $\|\cdot\|$, there is a metric $d(x, y) = \|x - y\|$.

Proof.

$$d(x, x) = \|x - x\| = \|0\| = 0$$

$$\forall x \neq y, d(x, y) = \|x - y\| > 0$$

$$d(x, y) = \|x - y\| = \|-(y - x)\| = |-1| \|y - x\| = \|y - x\| = d(y, x)$$

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \geq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

Thus $d(x, y) = \|x - y\|$ is a metric. \square

Definition 3.2. Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X , then they are called **equivalent** if there are $C_1, C_2 > 0$, such that

$$\forall x \in X, C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1.$$

Definition 3.3. A normed space is called a **Banach space** if it is complete.

Proposition 3.2. The Euclidean space \mathbb{R}^n or \mathbb{C}^n , with the Euclidean norm $\|x\|_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ is a Banach space.

Definition 3.4. For \mathbb{R}^n or \mathbb{C}^n , and $p \in [1, \infty)$, the ℓ^p norm is $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$. For $p = \infty$, the ℓ_∞ norm is $\|x\|_\infty = \max_{i \in [n]} |x_i|$.

Proposition 3.3. \mathbb{R}^n or \mathbb{C}^n , with any ℓ^p norm is a Banach space.

Remark. Notice that $\forall n \in \mathbb{N}^+$, $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty$, so they are equivalent.

Proposition 3.4. If X is compact and Hausdorff, we have $(C(X), \|\cdot\|_\infty)$ is a Banach Space.

Proof. The Extreme Value Theorem shows it is a normed space.

The convergence in $\|\cdot\|_\infty$ is uniform convergence, and the uniform limit of continuous functions is continuous. Thus, $(C(X), \|\cdot\|_\infty)$ is complete. \square

Proposition 3.5. For a Locally Compact Hausdorff Space X , $(C_b(X), \|\cdot\|_\infty)$ and $(C_0(X), \|\cdot\|_\infty)$ are both Banach Spaces.

Example 3.0.1. $C_0(\mathbb{N}) = \{(x_n)_{n=1}^\infty : \lim_{n \rightarrow \infty} x_n = 0\}$ with the discrete topology.

Proposition 3.6. $C^k([0, 1])$ is a Banach Space with $\|f\|_{C^k([a, b])} := \sum_{i=1}^k \|f^{(i)}\|_\infty$.

Proof. It is easy to check that this is a norm.

Now take any Cauchy sequence $(f_n)_{n=1}^\infty$, then for each $i \in [k]$, we have $(f_n^{(i)})_{n=1}^\infty$ is Cauchy in $C([a, b])$ as well.

Since $[a, b]$ is compact and Hausdorff, we have $C([a, b])$ is a Banach Space, so there is a $g_i(x) := \lim_{n \rightarrow \infty} f_n^{(i)}(x)$. In addition, since the convergence is uniform, we have $g_i \in C([0, 1])$.

By the Fundamental Theorem of Calculus, we have that for all $i \in [k-1], x \in [0, 1]$,

$$f_n^{(i)}(x) = f_n^{(i)}(0) + \int_0^x f_n^{(i+1)}(t) dt.$$

Taking the limit of $n \rightarrow \infty$, we have that

$$g_i(x) = g_i(0) + \int_0^x g_{i+1}(t) dt,$$

which means $g_i \in C^1([0, 1])$ with $g'_i = g_{i+1}$.

Thus $g_0 \in C^k([0, 1])$. \square

3.1 Bounded linear operators

Definition 3.5. Let X, Y be vector spaces, $T : X \rightarrow Y$ is a linear operator if $\forall c \in \mathbb{R}, u, v \in X$,

$$T(u + cv) = Tu + cTv.$$

Definition 3.6. Let X, Y be linear normed spaces, the **operator norm** of a linear operator $T : X \rightarrow Y$ is

$$\|T\| := \sup_{\|u\|_X \leq 1} \|Tu\|_Y = \sup_{\|u\|_X = 1} \|Tu\|_Y = \sup_{u \neq 0 \in X} \frac{\|Tu\|_Y}{\|u\|_X}.$$

Definition 3.7. Let X, Y be normed spaces, a linear operator $T : X \rightarrow Y$ is **bounded** if $\|T\| < \infty$.

Theorem 3.7. Let X, Y be two normed linear spaces, let $T : X \rightarrow Y$ be linear, then the following are equal:

1. T is continuous,
2. T is continuous at 0,
3. T is bounded,
4. T is uniformly continuous.

Proof. (4) \implies (1) \implies (2) trivially.

(3) \implies (4).

Suppose T is bounded, then

$$\begin{aligned}\|T(x) - T(y)\| &= \|T(x - y)\| \\ &\leq \|T\| \|x - y\|.\end{aligned}$$

Thus, T is $\|T\|$ Lipschitz and so uniformly continuous.

(2) \implies (3).

Suppose T is continuous at 0, and suppose for contradiction that $\|T\| = \infty$.

There must be $(x_n)_{n=1}^{\infty}$ in X , such that $\|x_n\| \leq 1$, $\|Tx_n\| \geq n^2$ for each $n \geq 1$.

Notice that $\frac{x_n}{n} \rightarrow 0$, but $\|T(\frac{x_n}{n})\| = \frac{1}{n} \|Tx_n\| \geq n$ for each n .

Thus $\lim_{n \rightarrow \infty} T(\frac{x_n}{n}) \neq 0 = T(0)$, which contradicts that T is continuous at 0. \square

Proposition 3.8. Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X , then they are equivalent if and only if they induce the same topology.

Proof. Assume that $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, then there are $C_1, C_2 > 0$, such that

$$\forall x \in X, C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1.$$

Consider the identity function $id : (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$, we see that

$$\|id\| = \sup_{x \neq 0 \in X} \frac{\|x\|_2}{\|x\|_1} \leq \sup_{x \neq 0 \in X} \frac{C_2 \|x\|_1}{\|x\|_1} = C_2,$$

and

$$\|id^{-1}\| = \sup_{x \neq 0 \in X} \frac{\|x\|_1}{\|x\|_2} \leq \sup_{x \neq 0 \in X} \frac{\|x\|_1}{C_1 \|x\|_1} = \frac{1}{C_1}.$$

Thus id is a homeomorphism.

On the other hand, suppose $\|\cdot\|_1, \|\cdot\|_2$ induces the same topology, then id is a homeomorphism.

Thus,

$$\frac{1}{\|id\|} \|x\|_2 = \frac{1}{\|id\|} \|id(x)\|_2 \leq \|x\|_1 = \|id^{-1}(x)\|_1 \leq \|id^{-1}\| \|x\|_2.$$

\square

Definition 3.8. Let X, Y be normed spaces, we denote

$$B(X, Y) := \{T : X \rightarrow Y | T \text{ is a bounded linear operator}\}.$$

Theorem 3.9. The set $B(X, Y)$ is a normed linear space with the operator norm.

Proposition 3.10. Let X, Y, Z be normed spaces, if $T : X \rightarrow Y, S : Y \rightarrow Z$ are both linear bounded operators, then so is $S \circ T$, with

$$\|S \circ T\| \leq \|S\| \|T\|.$$

Theorem 3.11. Let X be a normed space, and Y be a Banach Space, then $B(X, Y)$ is a Banach Space.

Proof. Let $(T_n)_{n=1}^{\infty}$ be a Cauchy sequence in $B(X, Y)$.

For any $x \in X$, we have that $(T_n x)_{n=1}^{\infty}$ is Cauchy in Y .

$$\text{Indeed, } \|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\|.$$

Since Y is complete, there must be a unique $y = \lim_{n \rightarrow \infty} T_n x \in Y$.

Define $Tx := \lim_{n \rightarrow \infty} T_n x$ for any $x \in X$.

Notice that T is linear.

Given $\epsilon > 0$, we know there must be some $N \in \mathbb{N}$, such that $\forall m, n \geq N$, $\|T_n - T_m\| < \epsilon$.

Consider any $x \in X$.

$$\begin{aligned}\|Tx\| &\leq \|(T - T_N)x\| + \|T_N x\| \\ &= \lim_{m \rightarrow \infty} \|(T_m - T_N)x\| + \|T_N x\| \\ &\leq \limsup_m \|T_m - T_N\| \|x\| + \|T_N\| \|x\| \\ &\leq \epsilon \|x\| + \|T_N\| \|x\|.\end{aligned}$$

Thus $\|T\| \leq \epsilon + \|T_N\| < \infty$.

This shows $T \in B(X, Y)$.

Again, for any $x \in X$, $n \geq N$, we have

$$\begin{aligned}\|(T_n - T)x\| &= \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| \\ &\leq \limsup_m \|T_n - T_m\| \|x\| \\ &< \epsilon \|x\|.\end{aligned}$$

Thus $\|T_n - T\| < \epsilon$ for any $n \geq N$, which shows $\lim_{n \rightarrow \infty} T_n = T$ in $B(X, Y)$ with the operator norm. \square

Definition 3.9. Let X, Y be normed spaces. We say a linear operator $T : X \rightarrow Y$ is **bounded below** if $\exists m > 0$, such that $\forall x \in X$, $\|Tx\| \geq m\|x\|$.

Definition 3.10. Let X, Y be normed spaces. We say $T : X \rightarrow Y$ is an **isomorphism between normed spaces** or T is **invertible** if T is bijective and T, T^{-1} are both bounded. i.e. T is a homeomorphism between X, Y .

Theorem 3.12. Let $T : X \rightarrow Y$ be a linear operator between normed linear spaces X and Y . The operator T is invertible, namely, T has a bounded inverse $T^{-1} : \text{Im}(T) \rightarrow X$, if and only if T is bounded below. In this case,

$$\|T^{-1}\| = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

Proof. (\implies) Suppose T has a bounded inverse operator. Let $x \in X$ and $y = Tx \in \text{Im}(T)$. By the assumption,

$$\begin{aligned}\|x\| &= \|T^{-1}y\| \leq \|T^{-1}\| \|y\| = \|T^{-1}\| \|Tx\|, \\ \|Tx\| &\geq \frac{1}{\|T^{-1}\|} \|x\|.\end{aligned}$$

Since the inequality holds for all $x \in X$, T is bounded below.

(\impliedby) Suppose T is bounded below. There exists $m > 0$ such that $\|Tx\| \geq m\|x\|$ for all $x \in X$. Let $z \in \ker T$. We have $0 = \|Tz\| \geq m\|z\| \geq 0$. Therefore, $\|z\| = 0$, which means $z = 0$. Thus $\ker T = \{0\}$ and T is one-to-one. Thus, T is a bijection between $X \longleftrightarrow \text{Im}(T)$, which means $T^{-1} : \text{Im}(T) \rightarrow X$ exists.

Finally, for any $y \in \text{Im}(T)$, $y = Tx$ for some $x \in X$. Then

$$\|T^{-1}y\| = \|x\| \stackrel{T \text{ bounded below}}{\leq} \frac{1}{m} \|Tx\| = \frac{1}{m} \|y\|.$$

Therefore T^{-1} is bounded. In particular,

$$\|T^{-1}\| = \sup_{y \in \text{Im}(T), y \neq 0} \frac{\|T^{-1}y\|}{\|y\|} = \sup_{x \neq 0} \frac{\|x\|}{\|Tx\|} = \frac{1}{\inf_{x \neq 0} \frac{\|Tx\|}{\|x\|}} = \frac{1}{\inf_{\|x\|=1} \|Tx\|}.$$

□

Definition 3.11. Let X, Y be normed spaces. We call $T : X \rightarrow Y$ a **contraction** if $\|T\| \leq 1$.

Definition 3.12. Let X, Y be normed spaces. We call $T : X \rightarrow Y$ an **isometry** if $\forall x \in X$, $\|Tx\| = \|x\|$.

Proposition 3.13. Let X, Y be normed spaces. If a linear operator $T : X \rightarrow Y$ is a surjective isometry, it is an isometric isomorphism between normed spaces.

Proof. T is bounded since $\|T\| = 1$. Suppose $T(x) = 0$, we have $\|Tx\|_Y = 0$. Since T is an isometry and thus bounded below by 1, T has a bounded inverse $T^{-1} : \text{Im}(T) \rightarrow X$ by theorem 3.12. Since T is surjective, $\text{Im}(T) = Y$. □

Proposition 3.14. Let Y be a Banach space, S be a dense subset of a normed space X . For any bounded linear operator $E : S \rightarrow Y$, we can extend it to $\tilde{E} : X \rightarrow Y$, such that \tilde{E} is also bounded and linear, with $\|\tilde{E}\| = \|E\|$, and $\tilde{E}|_S = E$.

Proof. Consider any $x \in X$.

Since S is dense in X , We know $\forall m \in \mathbb{N}^+$, $\exists x_m \in S$, such that $\|x - x_m\|_X \leq \frac{1}{m}$.

Since E is linear on S , we have that

$$\begin{aligned} \|Ex_m - Ex_l\|_Y &= \|E(x_m - x_l)\|_Y \\ &\leq \|E\| \|x_m - x_l\|_X \\ &= \|E\| \|(x_m - x) + (x - x_l)\|_X \\ &\leq \|E\| \|x - x_m\|_X + \|E\| \|x - x_l\|_X \\ &\leq \|E\| \left(\frac{1}{m} + \frac{1}{l} \right). \end{aligned}$$

Thus given any $\epsilon > 0$, for any $m, l \geq \lceil \frac{2\epsilon}{\|E\|} \rceil$, we can make $\|Ex_m - Ex_l\|_Y < \epsilon$. Thus $(Ex_m)_{m=1}^\infty$ is a Cauchy sequence in Y .

Since Y is a Banach space, $\exists y^* \in Y$, such that $Ex_m \rightarrow y^*$ in Y .

We claim that y^* is independent of choice of the sequence $(x_m)_{m=1}^\infty$.

Indeed, consider any other sequence $(v_m)_{m=1}^\infty \subseteq C^\infty(\bar{x})$, such that $\forall m \in \mathbb{N}^+$, $\|x - v_m\|_X \leq \frac{1}{m}$,

$$\begin{aligned} \|y^* - Ev_m\|_Y &\leq \|y^* - Ex_m\|_Y + \|Ex_m - Ev_m\|_Y \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - v_m\|_X \\ &\leq \|y^* - Ex_m\|_Y + \|E\| \|x_m - x\|_X + \|E\| \|x - v_m\|_X. \end{aligned}$$

Since all three terms on the right go to 0 when $m \rightarrow \infty$, we have that $Ev_m \rightarrow y^*$ in Y .

Thus we can uniquely define $\tilde{E}x := y^*$. In addition,

$$\begin{aligned} \|\tilde{E}x\|_Y &= \left\| \lim_{m \rightarrow \infty} Ex_m \right\|_Y \\ &= \lim_{m \rightarrow \infty} \|Ex_m\|_Y \\ &\leq \lim_{m \rightarrow \infty} \|E\| \|x_m\|_X \\ &= \|E\| \left\| \lim_{m \rightarrow \infty} x_m \right\|_X \\ &= \|E\| \|x\|_X. \end{aligned}$$

Thus $\|\tilde{E}\| = \|E\|$. □

3.1.1 Dual Spaces

Definition 3.13. Let X be a normed space over \mathbb{F} , a **functional** is an operator that maps into \mathbb{F} .

Definition 3.14. Let X be a normed space over \mathbb{F} , the **dual space** of X is the collection of bounded linear functionals on X , denoted

$$X^* := B(X, \mathbb{F}) = \{\phi : X \rightarrow \mathbb{F} : \phi \text{ is linear and bounded}\}.$$

Definition 3.15. Let X be a normed space over \mathbb{F} , and a subspace $Y \subseteq X^*$ we define the **duality pairing** to be $\langle \cdot | \cdot \rangle_{X,Y} : X \times Y \rightarrow \mathbb{F}$ to be $x \in X, \phi \in Y$, we can write $\langle x | \phi \rangle_{X,Y} := \phi(x)$ as the **action** of ϕ on x .

Definition 3.16. Let X be a normed space, the **dual norm** is defined to be

$$\|u^*\|_{X^*} := \sup_{\|u\| \leq 1} |\langle u^* | u \rangle_{X^*, X}|.$$

Example 3.1.1. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^\infty \mid \lim_{i \rightarrow \infty} x_i = 0\}$, then $C_0(\mathbb{N})^* \cong \ell^1(\mathbb{N})$.

Indeed, let $e_n := (m \mapsto \delta_{nm}) = (\delta_{nm})_{n=1}^\infty \in C_0(\mathbb{N})$.

Given any $\phi \in C_0(\mathbb{N})^*$, we define $a_n := \phi(e_n) \in \mathbb{F}$.

We claim $a = (a_n)_{n=1}^\infty$ completely determines ϕ .

Indeed, consider any $x \in C_0$, let $x^N := \sum_{n=1}^N x_n e_n$.

We have $\|x - x^N\|_{C_0} = 0$, so

$$\begin{aligned} \phi(x) &= \lim_{N \rightarrow \infty} \phi(x^N) \\ &= \lim_{N \rightarrow \infty} \phi\left(\sum_{n=1}^N x_n e_n\right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n \phi(e_n) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n a_n. \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=1}^N |a_n| &= \sum_{n=1}^N a_n \operatorname{sgn}(a_n) \\ &= \sum_{n=1}^N \phi(\operatorname{sgn}(a_n) e_n) \\ &= \phi(y_N) \\ &\leq \|\phi\|_{C_0^*} \|y_N\|_{C_0}, \end{aligned}$$

where $y_N = \sum_{n=1}^N \operatorname{sgn}(a_n) e_n$.

Since $\|y_N\| = 1$, we have $\sum_{n=1}^N |a_n| \leq \|\phi\|_{C_0^*}$ for any N .

Thus

$$\|a\|_1 = \sum_{n=1}^\infty |a_n| = \lim_{N \rightarrow \infty} \sum_{n=1}^N |a_n| \leq \|\phi\|_{C_0^*} < \infty.$$

Thus $a \in \ell^1(\mathbb{N})$.

Notice that $\Phi : C_0^*(\mathbb{N}) \rightarrow \ell^1(\mathbb{N})$ by $\phi \mapsto a$ is linear, contractive, and injective.

Also, given any $a \in \ell^1(\mathbb{N})$, we can set $\phi_a(x) := \sum_{n=1}^{\infty} x_n a_n$.
Thus,

$$\begin{aligned} |\phi_n(x)| &\leq \sum_{n=1}^{\infty} |x_n| |a_n| \\ &\leq \|x\|_{\infty} \|a\|_1, \end{aligned}$$

which shows $\|\phi_a\|_{C_0^*} \leq \|a\|_1$.

Thus $\phi_a \in C_0^*(\mathbb{F})$. Notice that $\Phi(\phi_a) = a$.

This shows that Φ is surjective, and it is actually an isometry, since $\forall \phi \in C_0^*(\mathbb{F})$, we have

$$\|\Phi(\phi)\|_1 \leq \|\phi\|_{C_0^*} = \|\Phi^{-1}(\Phi(\phi))\|_{C_0^*} \leq \|\Phi(\phi)\|_1.$$

Example 3.1.2. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^{\infty} \mid \lim_{i \rightarrow \infty} x_i = 0\}$, then

$$B(C_0(\mathbb{N})) = B(C_0(\mathbb{N}), C_0(\mathbb{N})) \cong \{(t_{ij})_{i,j \in \mathbb{N}} : \|(t_{ij})_{i,j \in \mathbb{N}}\| < \infty, \forall j \in \mathbb{N}, (t_{ij})_{i \in \mathbb{N}}\} \in C_0(\mathbb{N}),$$

which are infinite matrices whose rows are uniformly in ℓ^1 , and columns are in $C_0(\mathbb{N})$.

In addition, it is an isometry under

$$\|(t_{ij})_{i,j \in \mathbb{N}}\| := \sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1.$$

Indeed, let

$$e_n := (m \mapsto \delta_{nm}) \cong (\delta_{nm})_{n=1}^{\infty} \in C_0(\mathbb{N}), \delta_i := ((x_j)_{j=1}^{\infty} \mapsto x_i) \in C_0^*(\mathbb{N})$$

with $\Phi(\delta_i) = (\delta_{ij})_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ as in previous example.

Easy to check $\|\delta_i\| = 1$ by the above example.

Consider any $T \in B(C_0(\mathbb{N}))$, define $\phi_i := (\delta_i \circ T), t_{ij} := \phi_i(e_j)$.

Notice that $\phi_i : C_0(\mathbb{N}) \rightarrow \mathbb{F}$ is linear, and $\|\phi_i\| \leq \|\delta_i\| \|T\| = \|T\| < \infty$, so $\phi_i \in C_0^*(\mathbb{N})$.

Thus, $(t_{ij})_{j=1}^{\infty} = (\phi_i(e_j))_{j=1}^{\infty} = \Phi(\phi_i) \in \ell^1(\mathbb{N})$ as in previous example, with $\|\phi_i\| = \|(t_{ij})_{j=1}^{\infty}\|_1$.

Since this hold for all $i \in \mathbb{N}$, we have $\sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1 \leq \|T\|$.

In addition, $(t_{ij})_{i \in \mathbb{N}} = ((\delta_i \circ T)(e_j))_{i \in \mathbb{N}} = Te_j \in C_0(\mathbb{N})$.

On the other hand, suppose we have such $(t_{ij})_{i,j \in \mathbb{N}}$ with $\sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\|_1 < \infty$, we can define $\phi_i := \Phi^{-1}((t_{ij})_{j=1}^{\infty}) \in C_0^*(\mathbb{N})$ with $\phi_i(x) = \sum_{j=1}^{\infty} x_j t_{ij}$ as in previous example.

Let $Tx := \sum_{i=1}^{\infty} \phi_i(x) e_i$ for any $x \in C_0(\mathbb{N})$.

Clearly, T is linear, and we have $(\delta_i \circ T)(x) = \delta_i(\sum_{j=1}^{\infty} \phi_j(x) e_j) = \phi_i(x)$.

$$\begin{aligned} \|Tx\|_{\infty} &= \|(\phi_i(x))_{i=1}^{\infty}\|_{\infty} \\ &\leq \sup_{i \in \mathbb{N}} |\phi_i(x)| \\ &\leq \sup_{i \in \mathbb{N}} \|\phi_i\| \|x\| \\ \|T\| &\leq \sup_{i \in \mathbb{N}} \|\phi_i\| \\ &= \sup_{i \in \mathbb{N}} \|(t_{ij})_{j=1}^{\infty}\| \\ &< \infty. \end{aligned}$$

Thus $T \in B(C_0(\mathbb{N}), \ell_{\infty})$.

Now we claim $T(C_0(\mathbb{N})) = C_0(\mathbb{N})$, which will mean $T \in B(C_0(\mathbb{N}))$.

Indeed, for any $x = \sum_{n=1}^{\infty} x_n e_n \in C_0(\mathbb{N})$, we have

$$\begin{aligned} Tx &= T \left(\lim_{N \rightarrow \infty} \sum_{n=1}^N x_n e_n \right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n T(e_n). \end{aligned}$$

Since each

$$\begin{aligned} T(e_n) &= \sum_{i=1}^{\infty} \phi_i(x) e_i \\ &= \sum_{i=1}^{\infty} t_{in} e_i \\ &\in C_0(\mathbb{N}), \end{aligned}$$

and $C_0(\mathbb{N})$ is closed, we have $Tx \in C_0(\mathbb{N})$.

In addition, $(\delta_i \circ T)(e_j) = \phi_i(e_j) = \Phi^{-1}((t_{ik})_{k=1}^{\infty})(e_j) = t_{ij}$.
Thus $T \longleftrightarrow (t_{ij})_{i,j \in \mathbb{N}}$ is an isometric bijection.

Example 3.1.3. Consider the Disk Algebra

$$A(\mathbb{D}) := \left\{ f \in C(\mathbb{T}) : \forall n \in \mathbb{Z}^{--}, \hat{f}(n) = 0 \right\},$$

where $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}$, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C}, |z| = 1\}$ is the unit circle, and

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt$$

is the n^{th} Fourier Transform of f .

Consider $\phi_n : f \mapsto \hat{f}(n)$, which is clearly in $C^*(\mathbb{T})$.

We notice that $A(\mathbb{D}) = \bigcap_{n<0} \ker(\phi_n)$ is closed in $C(\mathbb{T})$, since each kernel of a continuous functional is closed.

In fact, for $f, g \in C(\mathbb{T})$, we have $\hat{f}\hat{g}(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k)\hat{g}(n-k)$.

Thus, for $f, g \in A(\mathbb{D})$, we have $\hat{f}\hat{g}(n) = \sum_{k \in \mathbb{N}} \hat{f}(k)\hat{g}(n-k) = 0$ for $n < 0$.

This shows $A(\mathbb{D})$ is actually an Algebra.

Also, $A(\mathbb{D})$ is exactly the set of $f \in C(\mathbb{T})$ that admits an extension $F \in C(\bar{\mathbb{D}})$ with $F|_{\mathbb{D}}$ being analytic, with $F(z) := \sum_{n=1}^{\infty} \hat{f}(n)z^n$.

3.2 Quotient Spaces

Definition 3.17. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. The **quotient space** is $X/Y := \{x+Y : x \in X\}$, with the **quotient map** $Q : X \rightarrow X/Y$ by $Q(x) := [x] := x+Y = \{x+y : y \in Y\}$.

Proposition 3.15. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. X/Y is always a vector space with $[0] = Y$, $[x] + [y] = [x+y]$, $c[x] = [cx]$.

Proposition 3.16. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. $(X/Y, \|\cdot\|_{X/Y})$ is always a Banach space with $\|[x]\|_{X/Y} := \inf_{y \in Y} \|x+y\|_X$. In addition, Q is isometric if $Y \subsetneq X$.

Proof. $\|[x]\|_{X/Y} = 0 \iff \inf_{y \in Y} \|x+y\|_X = 0 \iff x \in \bar{Y} \iff x \in Y$.

Scaling is clear.

Also,

$$\begin{aligned} \|[x] + [z]\|_{X/Y} &= \inf_{y \in Y} \|x+y+z\|_X \\ &= \inf_{y_1, y_2 \in Y} \|x+y_1+z+y_2\|_X \\ &\leq \inf_{y_1 \in Y} \|x+y_1\|_X + \inf_{y_2 \in Y} \|z+y_2\|_X \\ &= \|[x]\|_{X/Y} + \|[z]\|_{X/Y}. \end{aligned}$$

This shows $(X/Y, \|\cdot\|_{X/Y})$ is a normed space.

We note that $\|Qx\|_{X/Y} = \inf_{y \in Y} \|x+y\|_X \leq \|x+0\|_X = \|x\|_X$, so $\|Q\| \leq 1$.

Now consider any Cauchy sequence $([x_n])_{n=1}^\infty$ in X/Y .

We can pick a subsequence $([x_{n_i}])_{i=1}^\infty$ such that $\|[x_{n_{i+1}}] - [x_{n_i}]\|_{X/Y} < 2^{-i}$.

Pick $z_1 \in X$ such that $[z_1] = [x_{n_1}]$.

Since $\|[x_{n_2}] - [x_1]\|_{X/Y} = \inf_{y \in Y} \|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$, there is $y \in Y$, such that $\|x_{n_2} - z_1 + y\|_X < \frac{1}{2}$.

Take $z_2 = x_{n_2} + y$, we have $\|z_2 - z_1\|_X < \frac{1}{2}$.

Inductively, we can pick $(z_i)_{i=1}^\infty$, such that $\|z_i - z_{i-1}\|_X < 2^{-i}$.

We can check that this is a Cauchy sequence in X , so it has a limit $z = \lim_{i \rightarrow \infty} z_i \in X$.

Now for any $i \in \mathbb{N}^+$, we have

$$\begin{aligned} \|[x_{n_i}] - [z]\|_{X/Y} &= \|[z_i] - [z]\|_{X/Y} \\ &= \|Q(z_i) - Q(z)\|_{X/Y} \\ &= \|Q(z_i - z)\|_{X/Y} \\ &\leq \|Q\| \|z_i - z\| \\ &\rightarrow 0. \end{aligned}$$

Thus $([x_{n_i}])_{i=1}^\infty \rightarrow [z]$ is a convergent subsequence, which mean $([x_n])_{n=1}^\infty$ is convergent.

This shows $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.

Now if $Y \subsetneq X$, then $X/Y \neq \{0\}$, there must be some $[x] \in X/Y$ with $\|[x]\|_{X/Y} = 1$.

Thus, for all $k \in \mathbb{N}^+$, there is some $y_k \in Y$, such that $\|x + y_k\|_X \leq 1 + \frac{1}{k}$.

Now

$$\left\| Q\left(\frac{x + y_k}{\|x + y_k\|_X}\right) \right\|_{X/Y} = \frac{1}{\|x + y_k\|_X} \|Q(x + y_k)\| = \frac{\|[x]\|_{X/Y}}{\|x + y_k\|_X} \geq \frac{1}{1 + \frac{1}{k}}.$$

Since this is true for any $k \in \mathbb{N}^+$, taking the limit $k \rightarrow \infty$, we have $\left\| Q\left(\frac{x + y_k}{\|x + y_k\|_X}\right) \right\|_{X/Y} \geq 1$.

Yet $\left\| \frac{x + y_k}{\|x + y_k\|_X} \right\|_X = 1$, so $\|Q\| \geq 1$.

This shows $\|Q\| = 1$. □

Example 3.2.1. Consider a compact and Hausdorff X , and consider $(C(X), \|\cdot\|_\infty)$. Let $E \subseteq X$ be closed, and $I(E) := \{f \in C(X) : f|_E = 0\}$.

One can check $I(E)$ is closed (ideal), and $C(X)/I(E) \cong C(E)$ with an isometric isomorphism $\tilde{R} : [f] \mapsto f|_E$. We claim that \tilde{R} is well-defined.

Indeed, if $[f] = [g]$, we must have $f - g \in I(E)$, which means $(f - g)|_E = 0$.

Thus $\tilde{R}([f]) = f|_E = g|_E = \tilde{R}([g])$.

Clearly \tilde{R} is linear.

Also, $\tilde{R}([f]) = 0 \implies f|_E = 0 \implies f \in I(E) \implies [f] = 0$, so \tilde{R} is injective.

By Tietze's Theorem, given any $g \in C(E)$, we can extend it to $f \in C(X)$, such that $f|_E = g$. Thus, \tilde{R} is surjective.

Consider any $f \in C(X), g \in I(E)$, we have

$$\begin{aligned} \|\tilde{R}([f])\| &= \|f|_E\|_{C(E)} \\ &= \sup_{x \in E} |f(x)| \\ &= \sup_{x \in E} |(f + g)(x)| \\ &\leq \sup_{x \in X} |(f + g)(x)| \\ &= \|f + g\|_{C(X)}. \end{aligned}$$

Since this hold for all $g \in I(E)$, we have

$$\|\tilde{R}([f])\| \leq \inf_{g \in I(E)} \|f + g\|_{C(X)} = \|[f]\|.$$

Thus, \tilde{R} is a contraction.

Consider any $f \in C(X)$.

If $f|_E = 0$, we have $\|[f]\| = \|[0]\| = 0 = \|f|_E\| = \|\tilde{R}([f])\|$.

Now consider $f|_E \neq 0$.

Define the function $k : \mathbb{C} \rightarrow \mathbb{C}$ by $k(z) := \begin{cases} z, & |z| \leq \|f|_E\|_\infty \\ \frac{z}{|z|}\|f|_E\|_\infty & |z| \geq \|f|_E\|_\infty \end{cases}$, which is well-defined and continuous.

Let $g := k \circ f \in C(X)$.

For any $x \in E$, we have $|f(x)| \leq \|f|_E\|_\infty$, so $g(x) = k(f(x)) = f(x)$.

Thus $g|_E = f|_E$, and there is $h \in I(E)$, such that $g = f + h$.

$$\begin{aligned} \|[f]\| &= \inf_{h \in I(E)} \|f + h\|_{C(X)} \\ &\leq \|g\|_{C(X)} \\ &\leq \|k\| \\ &\leq \|f|_E\|_\infty \\ &= \|\tilde{R}(f)\|. \end{aligned}$$

This proves \tilde{R} is an isometry.

Proposition 3.17. *For any infinite dimensional Banach Space X , it does not have a countable basis. Also, the unit ball is not compact.*

Proof. See A2. □

3.3 Baire Category Theorem

Definition 3.18. Let (X, \mathcal{T}) be a topological space, then $A \subseteq X$ is called **nowhere dense** if $(\bar{A})^o = \emptyset$.

Theorem 3.18 (Baire Category Theorem). *Let (X, d) be a complete metric space, then X cannot be written as a countable union of nowhere dense sets.*

Corollary 3.19. *Let $\{U_i\}_{i=1}^\infty$ be a countable set of open dense sets, then $\bigcap_{i=1}^\infty U_i$ is dense.*

3.3.1 Banach-Steinhaus Theorem

Definition 3.19. Let X, Y be Banach spaces, $\mathcal{S} \subseteq B(X, Y)$ is called **pointwise bounded** if $\forall x \in X$, $\mathcal{S}x$ is bounded. Namely, $\exists k_x > 0$, such that $\forall S \in \mathcal{S}$, $\|Sx\| \leq k_x$.

Theorem 3.20 (Banach-Steinhaus). *Let X, Y be Banach spaces, suppose $\mathcal{S} \subseteq B(X, Y)$ is pointwise bounded, then \mathcal{S} is bounded in $B(X, Y)$. Namely, $\sup_{S \in \mathcal{S}} \|S\| < \infty$.*

Proof. For each x , let $k_x > 0$ be such $\forall S \in \mathcal{S}$, $\|Sx\| \leq k_x$.

For each $n \in \mathbb{N}$, let $A_n := \{x \in X : k_x \leq n\}$.

For any $(x_k)_{k \in \mathbb{N}}$ in A_n , with $x = \lim_{k \rightarrow \infty} x_k$, we have

$$\|Sx\| = \lim_{k \rightarrow \infty} \|Sx_k\| \leq n.$$

Thus, $x \in A_n$, which shows A_n is closed.

Notice that $X = \bigcup_{n \in \mathbb{N}} A_n$, so by Baire Category Theorem, there is $n_0 \in \mathbb{N}$, such that $(\bar{A}_{n_0})^o \neq \emptyset$.

Thus, there is $x_0 \in \bar{A}_{n_0}$, $r > 0$, such that $\bar{B}(x_0, r) \subset B(x_0, 2r) \subseteq (\bar{A}_{n_0})^o \subset \bar{A}_{n_0} = A_{n_0}$.

Now for any $s \in \mathcal{S}, y \in X$ such that $\|y\| \leq 1$, we have that

$$\begin{aligned} \|sy\| &= \left\| \frac{s(x_0) - s(x_0 - ry)}{r} \right\| \\ &\leq \frac{\|s(x_0)\| + \|s(x_0 - ry)\|}{r} \\ &\leq \frac{2n_0}{r}, \end{aligned}$$

since $x_0, x_0 - ry \in \bar{B}(x_0, r) \subset A_{n_0}$.

Thus, $\|s\| = \sup_{y \in X \text{ such that } \|y\| \leq 1} \|sy\| \leq \frac{2n_0}{r}$.

Since this holds for all $s \in S$, we have $\sup_{s \in S} \|s\| = \frac{2n_0}{r} < \infty$. \square

Corollary 3.21 (Limit of bounded operators). *Let X, Y be Banach spaces, consider $(T_n)_{n=1}^{\infty}$ be a sequence of $B(X, Y)$. Suppose $\forall x \in X$, $(T_n x)_{n=1}^{\infty}$ is convergent, then $S := \{T_n : n \in \mathbb{N}\}$ is bounded. In addition, for $Tx := \lim_{n \rightarrow \infty} T_n x$, we have $T \in B(X, Y)$, and $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$.*

Proof. Since $(T_n x)_{n=1}^{\infty}$ is convergent, it is bounded. This is equivalent to saying S is pointwise bounded. By the Banach-Steinhaus Theorem, S is bounded.

Now $\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| = \sup_{n \in \mathbb{N}} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n\| \|x\|$.

Since this holds for all $x \in X$, we have $\|T\| \leq \sup_{n \in \mathbb{N}} \|T_n\|$. \square

Example 3.3.1. Consider $f \in C(\mathbb{T})$, we can define the N^{th} partial sum of its Fourier series $S_N(f)(e^{it}) := \sum_{n=-N}^N \hat{f}(n)e^{int}$.

We note that $S_N(f)$ does not necessarily converge to f in $C(\mathbb{T})$, nor pointwise.

Indeed, consider $\phi_N(f) := S_N(f)(1) \in \mathbb{C}$. Note that ϕ_N is linear.

One can show that

$$\begin{aligned} \phi_N(f) &= \sum_{n=-N}^N \hat{f}(n) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \left(\sum_{n=-N}^N e^{-int} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) D_N(t) dt, \end{aligned}$$

where $D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}$ is the **Dirichlet's Kernel**.

Actually, $\exists C > 0$, such that $\|D_N\|_1 \geq C \log(N)$.

Also, $\|\phi_N\|_{C_*(\mathbb{T})} = \|D_N\|_1$.

Thus, $(\phi_N)_{N \in \mathbb{N}}$ is not bounded.

By Banach-Steinhaus, $(\phi_N)_{N \in \mathbb{N}}$ is not pointwise bounded.

Thus, there is $f \in C(\mathbb{T})$, such that $|\phi_N(f)| \rightarrow \infty$.

Thus, $S_N f$ does not converge to f at 1.

3.3.2 Open Mapping Theorem

Theorem 3.22 (Open Mapping Theorem). *Let X, Y be Banach spaces, suppose $T \in B(X, Y)$ is surjective, then it is **open**. i.e. \forall open $U \subseteq X$, $T(U) \subseteq Y$ is open.*

Proof. We have

$$\begin{aligned} Y &= T(X) \\ &= T\left(\bigcup_{n=1}^{\infty} B^X(0, n)\right) \\ &= T\left(\bigcup_{n=1}^{\infty} nB^X(0, 1)\right) \\ &= \bigcup_{n=1}^{\infty} nT(B^X(0, 1)). \end{aligned}$$

By the Baire Category Theorem, there is n_0 , such that $\left(n_0 \overline{T(B^X(0, 1))}\right)^o = \left(\overline{n_0 T(B^X(0, 1))}\right)^o \neq \emptyset$.

Thus there is $r_0 > 0, y_0 \in Y$, such that $B^Y(y_0, r_0) \subset n_0 \overline{T(B^X(0, 1))}$.

Notice that $B^Y(-y_0, r_0) \subset n_0 \overline{T(B^X(0, 1))}$ as well, and $n_0 \overline{T(B^X(0, 1))}$ is convex.
Thus, for any $y \in B^Y(0, r)$, we have $y = \frac{1}{2}(y-y_0) + \frac{1}{2}(y+y_0)$, where $y-y_0 \in B^Y(-y_0, r_0)$, $y+y_0 \in B^Y(y_0, r_0)$.

By convexity, $y \in n_0 \overline{T(B^X(0, 1))}$.

Thus $B^Y(0, r_0) \subset n_0 \overline{T(B^X(0, 1))}$.

Take $r := \frac{r_0}{n_0}$, we have $B^Y(0, r) \subset \overline{T(B^X(0, 1))}$.

Now we want to show $\overline{T(B^X(0, 1))} \subset T(B^X(0, 2))$.

Let $y \in \overline{T(B^X(0, 1))}$, there is $x_1 \in B^X(0, 1)$, such that $\|y - Tx_1\| < \frac{r}{2}$.

Let $y_1 := y - Tx_1 \in B^Y(0, \frac{r}{2}) \overline{T(B^X(0, \frac{1}{2}))}$.

Thus there is $x_2 \in B^X(0, \frac{1}{2})$, such that $\|y - Tx_1 - Tx_2\| = \|y_1 - Tx_2\| < \frac{r}{4}$.

Recursively, we can find $x_k \in B^X(0, \frac{1}{2^{k-1}})$, such that $\|y_k\| < \frac{r}{2^k}$ for $y_k := y - Tx_1 - \dots - Tx_k$.

Since $\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 2^{-k+1} = 2$, and X is complete, we have $x := \sum_{k=1}^{\infty} x_k$ converges in X , and $x \in B^X(0, 2)$.

Since T is continuous, we have $y = \sum_{k=1}^{\infty} Tx_k = Tx \in T(B^X(0, 2))$.

Thus $B^Y(0, r) \subset \overline{T(B^X(0, 1))} \subset T(B^X(0, 2))$.

Now for any open $U \subseteq X$, there is $\epsilon > 0$, such that $x + \frac{\epsilon}{2}B^X(0, 2) = B^X(x, \epsilon) \subseteq U$.

Thus $B^Y(Tx, \frac{\epsilon}{2}r) = Tx + \frac{\epsilon}{2}B^Y(0, r) \subseteq Tx + \frac{\epsilon}{2}T(B^X(0, 2)) \subseteq T(U)$.

Thus $T(U)$ is open. \square

Theorem 3.23 (Banach Isomorphism Theorem). *Let X, Y be Banach spaces, suppose $A \in B(X, Y)$ is bijective, then A^{-1} is continuous and bounded as well. Namely, A is invertible and is an isomorphism of $X \cong Y$ as Banach Spaces.*

Proof. A is open by the open mapping theorem. Now for any open $U \subseteq X$, we have that $(A^{-1})^{-1}(U) = A(U)$ is open. Thus A^{-1} is continuous. \square

Corollary 3.24. *Let X, Y be Banach spaces, suppose $T \in B(X, Y)$ is surjective, then $X/\ker(T) \cong Y$ as Banach spaces with $\tilde{T} : X/\ker(T) \rightarrow Y$ by $\tilde{T}([x]) := T(x)$.*

Proof. We can check that \tilde{T} is a well-defined bijection.

Also, for any $x \in X, y \in \ker(T)$, we have

$$\begin{aligned}\|\tilde{T}([x])\| &= \|T(x)\| \\ &= \|T(x+y)\| \\ &\leq \|T\|\|x+y\|.\end{aligned}$$

Since this holds for all $y \in \ker(T)$, we have

$$\|\tilde{T}([x])\| \leq \inf_{y \in \ker(T)} \|T\|\|x+y\| = \|T\|\|[x]\|.$$

Thus, $\|\tilde{T}\| \leq \|T\|$, which means $\tilde{T} \in B(X/\ker(T), Y)$ is continuous.

By the Banach Isomorphism Theorem, \tilde{T}^{-1} is continuous as well. \square

Corollary 3.25. *Suppose X is a vector space that is complete under two different norms $\|\cdot\|_1, \|\cdot\|_2$, and $\exists C > 0$, such that $\forall x \in X, \|x\|_1 \leq C\|x\|_2$, then the two norms are equivalent.*

Proof. Consider the map $id : (X, \|\cdot\|_1) \rightarrow (X, \|\cdot\|_2)$, we have that id is bijective and bounded by C . By the Banach Isomorphism Theorem, id^{-1} is bounded as well.

Thus $\forall x \in X$, we have $\|x\|_2 \leq \|id^{-1}\|\|x\|_1$ \square

Corollary 3.26. *Let X be any finite-dimensional linear normed space over \mathbb{F} , then any two norms on X are equivalent and X is complete.*

Proof. Let $\|\cdot\|$ be any norm on X , and let $\|\cdot\|_2$ be the Euclidean norm on \mathbb{F}^n .

Pick any basis $\{e_i\}_{i=1}^n$ for X .

Consider the function $T : (\mathbb{F}^n, \|\cdot\|_2) \rightarrow (X, \|\cdot\|)$ by $T(a) := \sum_{i=1}^n a^i e_i \in X$ for any $a = (a^i)_{i=1}^n \in \mathbb{F}^n$.

It is easy to check that T is linear and bijective. Also,

$$\begin{aligned}\|Ta\| &= \left\| \sum_{i=1}^n a^i e_i \right\| \\ &\leq \sum_{i=1}^n |a^i| \|e_i\| \\ &\leq \|a\|_2 \left(\sum_{i=1}^n \|e_i\|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Thus, $\|T\| \leq \alpha := \left(\sum_{i=1}^n \|e_i\|^2 \right)^{\frac{1}{2}}$, so it is continuous.

Since $S := \{a \in \mathbb{F}^n : \|a\|_2 = 1\}$ is closed and bounded thus compact in $(\mathbb{F}^n, \|\cdot\|_2)$, and $\|\cdot\|$ is continuous, we have $r := \inf_{a \in S} \|T(a)\|$ is achieved in S by the Extreme Value Theorem.

Since $0 \notin S$, and $\|T(a)\| = 0 \implies T(a) = 0 \implies a = 0$, we have $r \neq 0$.

Consider any $a \neq 0 \in \mathbb{F}^n$ such that $\|T(a)\| \leq r$, we must have $\frac{a}{\|a\|_2} \in S$, so

$$\frac{1}{\|a\|_2} \|T(a)\| = \left\| T\left(\frac{a}{\|a\|_2}\right) \right\| \geq r.$$

Thus $\|a\|_2 \leq 1$.

This shows $\forall x \in X$ with $\|x\| \leq 1$, we must have $\|T^{-1}(x)\|_2 \leq \frac{1}{r}$, which show that $\|T^{-1}\| \leq \frac{1}{r} < \infty$ is bounded.

This shows that $T : (\mathbb{F}^n, \|\cdot\|_2) \rightarrow (X, \|\cdot\|)$ is an homeomorphism for any $\|\cdot\|$ on X , which means X is complete and all norms are equivalent. \square

3.3.3 Closed Graph Theorem

Definition 3.20. Let X, Y be Banach spaces, we can consider $X \oplus Y := \{(x, y) : x \in X, y \in Y\}$, which is a vector space under component-wise addition and scale multiplication. For $1 \leq p < \infty$, we define

$$\|(x, y)\|_p := (\|x\|^p + \|y\|^p)^{\frac{1}{p}},$$

and

$$\|(x, y)\|_\infty := \max \{\|x\|, \|y\|\}.$$

Definition 3.21. Let X, Y be Banach spaces, and $D \subseteq X$ is a subspace, the **graph** of a linear map $T : D \rightarrow Y$ is

$$\mathcal{G}(T) := \{(x, Tx) : x \in D\}.$$

We say T is **closed** if $\mathcal{G}(T)$ is closed in $X \oplus_\infty Y$.

Theorem 3.27 (Closed Graph Theorem). *Let X, Y be Banach spaces, a linear map $T : X \rightarrow Y$ is closed if and only if $T \in B(X, Y)$.*

Proof. (\implies).

Consider the projection maps $\pi_1(x, y) := x, \pi_2(x, y) := y$, which are both continuous.

Since $\mathcal{G}(T)$ is closed in $X \oplus Y$, it is a Banach Space.

Since T is defined on entire X , we have that $\pi_1|_{\mathcal{G}(T)}$ is a continuous bijection.

By the Bounded Inverse Theorem, $(\pi_1|_{\mathcal{G}(T)})^{-1} : X \rightarrow \mathcal{G}(T)$ is bounded.

Notice that $T = \pi_2 \circ (\pi_1|_{\mathcal{G}(T)})^{-1}$, which will also be bounded.

(\impliedby).

Consider any sequence $((x_i, Tx_i))_{i \in \mathbb{N}}$ in $\mathcal{G}(T)$, such that $(x_i, Tx_i) \rightarrow (x, y) \in X \oplus_\infty Y$.

Thus $x_i \rightarrow x \in X$ and $Tx_i \rightarrow y \in Y$.

Since T is continuous, $Tx_i \rightarrow Tx$, which means $Tx = y$, so $(x, y) \in \mathcal{G}(T)$. \square

Remark. It is important that T is defined on the entire X .

Example 3.3.2. Consider $X = Y = (C[0, 1], \|\cdot\|_1)$, and $D = C^1([0, 1])$. Let $T := \frac{d}{dx} : D \rightarrow C^1([0, 1])$, which is unbounded but closed.

Corollary 3.28. Let X, Y be Banach spaces, then for any linear map $T : X \rightarrow Y$ $T \in B(X, Y)$ if and only if $\forall (x_n)_{n=1}^\infty$ in X , such that $x_n \rightarrow 0$, $Tx_n \rightarrow y \in Y$, we have $y = 0$.

Proof. (\implies) is easy.

(\impliedby).

Consider any $((x_n, Tx_n))_{n=1}^\infty$ in $\mathcal{G}(T)$, such that $x_n \rightarrow x \in X$, and $Tx_n \rightarrow y \in Y$.

We have that $(x - x_n) \rightarrow 0$, and by linearity of T , we have $T(x - x_n) = Tx - Tx_n \rightarrow Tx - y$.

By assumption, we have $y - Tx = 0$, which means $y = Tx$.

Thus, $(x, y) \in \mathcal{G}(T)$, which means $\mathcal{G}(T)$ is closed. \square

3.4 Hahn-Banach Theorem

Definition 3.22. Let X be a normed linear space. $p : X \rightarrow \mathbb{R}$ is called a **sublinear functional** if $\forall t > 0, \forall x \in X$, $p(tx) = tp(x)$, and $p(x + y) \leq p(x) + p(y)$.

Proposition 3.29. Let X be a normed linear space, then $\|\cdot\|$ is always a sublinear functional.

Proposition 3.30. Let X be a Banach space, then for any $x \in X$, the functional $p_x : B(X) \rightarrow \mathbb{R}$ defined by $p_x(T) := \|Tx\|$ is sublinear.

Proposition 3.31. Let X be a Banach space, then for any $x \in X, \phi \in X^*$, the functional $p_{\phi,x} : B(X) \rightarrow \mathbb{F}$ defined by $p_{\phi,x}(T) := |\phi(Tx)|$ is sublinear.

Theorem 3.32 (Extension). Let X be a linear vector space over \mathbb{R} . Let $M_0 \subseteq X$ be a linear subspace, and $p : X \rightarrow \mathbb{R}$ be a sublinear functional, then for any linear $f_0 : M_0 \rightarrow X$ such that $\forall x \in M_0$, $f_0(x) \leq p(x)$, there is an extension $f : X \rightarrow \mathbb{R}$, such that $f|_{M_0} = f_0$, and $\forall x \in X$, $f(x) \leq p(x)$.

Proof. Consider

$$P := \{(M, f) | M_0 \subseteq M \subseteq X \text{ is a subspace; } f : M \rightarrow \mathbb{R} \text{ is linear, } f|_{M_0} = f_0; \forall x \in M, f(x) \leq p(x)\},$$

with the partial order $(M, f) \leq (M', f')$ if $M \subseteq M'$, $f'|_M = f$.

Consider any chain $\{(M_\alpha, f_\alpha)\}_{\alpha \in A} \subset P$.

Let $M := \bigcup_{\alpha \in A} M_\alpha \subseteq X$, and let $f(x) := f_\alpha(x)$ for any $M_\alpha \ni x$.

We can check that f is well-defined and linear, satisfying the requirement.

Thus, $(M, f) \in P$ is an upper bound for the chain.

By Zorn's lemma, there is a maximal element (M_1, f_1) of P .

Suppose for contradiction that $M_1 \neq X$, then there is some $x \in X \setminus M_1$.

Notice that if we take any $m_1, m_2 \in M_1$, we have that

$$\begin{aligned} f_1(m_1) + f_1(m_2) &= f_1(m_1 + m_2) \\ &\leq p(m_1 + m_2) \\ &\leq p(m_1 - x) + p(m_2 + x). \end{aligned}$$

Thus, $f_1(m_1) - p(m_1 - x) \leq p(m_2 + x) - f_1(m_2)$.

Since this holds for all $m_1, m_2 \in M_1$, we have

$$\sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \leq \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)).$$

Take any $a \in [\sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)), \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2))]$.

Let $M := M_1 \oplus \text{Span}\{x\}$. Since M_1 is a subspace, this is a direct product, i.e., $\forall y \in M_1 \oplus \text{Span}\{x\}$, there is some unique $m \in M_1, t \in \mathbb{R}$, such that $y = m + tx$.

Define $f : M \rightarrow \mathbb{R}$ by $f(m + tx) := f_1(m) + |t|a$ for any $t \in \mathbb{R}$.

We can easily check $f|_{M_1} = f_1$ and that f is linear.

Suppose $t > 0$, we have

$$\begin{aligned} f(m + tx) &= f_1(m) + |t|a \\ &= f_1(m) + ta \\ &\leq f_1(m) + t \inf_{m_2 \in M_1} (p(m_2 + x) - f_1(m_2)) \\ &\leq f_1(m) + t \left(p\left(\frac{m}{t} + x\right) - f_1\left(\frac{m}{t}\right) \right) \\ &= f_1(m) + p\left(t\left(\frac{m}{t} + x\right)\right) - f_1\left(t\frac{m}{t}\right) \\ &= p(m + tx). \end{aligned}$$

Similarly, if $t \leq 0$, we have

$$\begin{aligned} f(m + tx) &= f_1(m) + |t|a \\ &= f_1(m) - ta \\ &\leq f_1(m) - t \sup_{m_1 \in M_1} (f_1(m_1) - p(m_1 - x)) \\ &\leq f_1(m) - t \left(f_1\left(\frac{m}{t}\right) - p\left(\frac{m}{t} - x\right) \right) \\ &= f_1(m) - f_1\left(t\frac{m}{t}\right) + p\left(t\left(\frac{m}{t} + x\right)\right) \\ &= p(m + tx). \end{aligned}$$

This contradicts with the maximality of (M_1, f_1) .

Thus, $M_1 = X$, and $f_1 : X \rightarrow \mathbb{R}$ is the desired extension. \square

Theorem 3.33 (Hahn-Banach). *Let X be a normed linear space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Let $M \subseteq X$ be a linear subspace, and linear $\phi_0 : M \rightarrow \mathbb{F}$ with $\|\phi_0\|_{M^*} < \infty$, then there is a norm preserving extension $\phi \in X^*$, such that $\phi|_M = \phi_0$, and $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$.*

Proof. First, consider $\mathbb{F} = \mathbb{R}$.

We note that $p : x \mapsto \|\phi_0\|_{M^*} \|x\|$ is sublinear, and $\forall x \in M$, $\phi_0(x) \leq \|\phi_0\|_{M^*} \|x\| = p(x)$.

Thus, there is a linear extension $\phi : X \rightarrow \mathbb{R}$, such that $\forall x \in X$, $\phi(x) \leq p(x) = \|\phi_0\|_{M^*} \|x\|$.

Also, $-\phi(x) = \phi(-x) \leq \|\phi_0\|_{M^*} \|-x\| = \|\phi_0\|_{M^*} \|x\|$.

Thus $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$.

Now suppose $\mathbb{F} = \mathbb{C}$.

Consider $g_0 := \Re \phi_0 : M \rightarrow \mathbb{R}$, which is \mathbb{R} -linear, and $\|g_0\|_{M_{\mathbb{R}}^*} \leq \|\phi_0\|_{M^*}$.

Using the real case, we can extend g_0 to $g \in X_{\mathbb{R}}^*$.

Now define $\phi : X \rightarrow \mathbb{C}$ by $\phi(x) := g(x) + ig(-ix)$.

We can see that ϕ is \mathbb{R} linear.

Also,

$$\phi(ix) = g(ix) + ig(x) = i(g(x) - ig(ix)) = i(g(x) + ig(-ix)) = i\phi(x),$$

so ϕ is \mathbb{C} -linear.

For any $m \in M$, we have that

$$\begin{aligned} \phi(x) &= g(m) + ig(-im) \\ &= g_0(m) + ig_0(-im) \\ &= \Re(\phi_0(m)) + i\Re(\phi_0(-im)) \\ &= \Re(\phi_0(m)) + i\Re(-i\phi_0(m)) \\ &= \Re(\phi_0(m)) + i\Im(\phi_0(m)) \\ &= \phi_0(m). \end{aligned}$$

Thus $\phi|_M = \phi_0$.

Now consider any $x \in X$ with $\|x\| \leq 1$.

We have that

$$|\phi(x)| = \lambda\phi(x) = \phi(\lambda x) = g(\lambda x) + ig(-i\lambda x)$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Since g is real-valued, and $|\phi(x)| \in \mathbb{R}$, we must have

$$|\phi(x)| = g(\lambda x) \leq \|g\|_{X^*} \|\lambda x\| = \|g_0\|_{M_{\mathbb{R}}^*} \|\lambda x\| \leq \|\phi_0\|_{M^*}.$$

Thus $\|\phi\|_{X^*} \leq \|\phi_0\|_{M^*}$.

Lastly, we always have

$$\begin{aligned} \|\phi_0\|_{M^*} &= \sup_{x \in M, \|x\| \leq 1} |\phi_0(x)| \\ &= \sup_{x \in M, \|x\| \leq 1} |\phi(x)| \\ &\leq \sup_{x \in X, \|x\| \leq 1} |\phi(x)| \\ &= \|\phi\|_{X^*}. \end{aligned}$$

□

Corollary 3.34. Let X be a normed linear space over \mathbb{R} . Let $M \subseteq X$ be a linear subspace, and linear $\phi_0 : M \rightarrow \mathbb{F}$ with $\|\phi_0\|_{M^*} < \infty$, then for any $x \in X, a \in \mathbb{R}$, there is a Hahn-Banach extension $\phi \in X^*$, such that $\phi|_M = \phi_0$, $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$ and $\phi(x) = a$ if and only if

$$\begin{aligned} \sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) \leq a \leq \inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|) \\ = \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)). \end{aligned}$$

Proof. We first note that since M is a subspace, $m \in M$ if and only if $-m \in M$, and

$$\begin{aligned} \phi_0(m) + \|\phi_0\|_{M^*} \|m - x\| &= -\phi_0(-m) + \|\phi_0\|_{M^*} \| -(-m) - x\| \\ &= \|\phi_0\|_{M^*} \|(-m) + x\| - \phi_0(-m). \end{aligned}$$

Thus,

$$\inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|) = \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)).$$

Consider any Hahn-Banach extension $\phi \in X^*$, such that $\phi|_M = \phi_0$, $\|\phi\|_{X^*} = \|\phi_0\|_{M^*}$. For any $x \in X$, we must satisfy

$$\begin{aligned} |\phi_0(m) - \phi(x)| &= |\phi(m) - \phi(x)| \\ &= |\phi(m - x)| \\ &\leq \|\phi\|_{X^*} \|m - x\| \\ &= \|\phi_0\|_{M^*} \|m - x\|. \end{aligned}$$

Thus,

$$\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\| \leq \phi(x) \leq \phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|.$$

Since this holds for all $m \in M$, we have

$$\sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) \leq \phi(x) \leq \inf_{m \in M} (\phi_0(m) + \|\phi_0\|_{M^*} \|m - x\|).$$

On the other hand, suppose

$$\sup_{m \in M} (\phi_0(m) - \|\phi_0\|_{M^*} \|m - x\|) \leq a \leq \inf_{m \in M} (\|\phi_0\|_{M^*} \|m + x\| - \phi_0(m)).$$

Taking $p : x \mapsto \|\phi_0\|_{M^*} \|x\|$ as in the proof of the Hahn-Banach Theorem, we have that

$$\sup_{m \in M} (\phi_0(m) - p(m-x)) \leq a \leq \inf_{m \in M} (p(m+x) - \phi_0(m)).$$

By the proof of the Hahn-Banach Theorem, we can always extend ϕ_0 on $M \oplus \text{Span}\{x\}$ to

$$\tilde{\phi}(m+tx) := \phi_0(m) + |t|a$$

for any $t \in \mathbb{R}$. Also,

$$\tilde{\phi}(x) = \tilde{\phi}(0 + 1 \cdot x) = \phi_0(0) + 1 \cdot a = a.$$

Now we can take the Hahn-Banach extension of $\tilde{\phi}$ to be ϕ , which will satisfy $\phi(x) = \tilde{\phi}(x) = a$. \square

Corollary 3.35. *Let X be a normed linear space. Given $0 \neq x \in X$, then there is $\phi \in X^*$, such that $\|\phi\| = 1$, and $\phi(x) = \|x\|$. In particular, $\|x\|_X = \sup \{|\phi(x)| : \phi \in X^*, \|\phi\| = 1\}$.*

Proof. Let $M = \text{Span}\{x\}$, and define $\phi_0(\lambda x) := \lambda \|x\|$.

Then we have $\|\phi_0\| = 1$, and by Hahn-Banach Theorem, there is $\phi \in X^*$, such that $\|\phi\| = \|\phi_0\| = 1$, and $\phi(x) = \phi_0(x) = \|x\|$.

In particular, $\|x\| = \phi(x) = |\phi(x)|$, so $\|x\|_X \leq \sup \{|\psi(x)| : \psi \in X^*, \|\psi\| = 1\}$.

Also, for any $\psi \in X^*, \|\psi\| = 1$, we have $|\psi(x)| \leq \|\psi\| \|x\| = \|x\|$. \square

Corollary 3.36. *Let X be a normed linear space, then X^* separates the points of X .*

Proof. For any $x \neq y$, we have $x - y \neq 0$, so there is $\phi \in X^*$, such that

$$\phi(x) - \phi(y) = \phi(x-y) = \|x-y\| \neq 0,$$

which separates x, y . \square

Corollary 3.37. *Let X be a normed linear space, then there is a canonical linear isometric embedding $i : X \hookrightarrow X^{**}$ by $i(x) := \hat{x}$, $\hat{x}(\phi) := \phi(x)$.*

Proof. We have $\|\hat{x}\| = \sup_{\|\phi\|=1} \|\hat{x}(\phi)\| = \sup_{\|\phi\|=1} |\phi(x)| = \|x\|$. \square

Remark. It is not necessarily that $X \cong X^{**}$, Indeed, if $X = C_0(\mathbb{N})$, we have that $X^* \cong \ell^1(\mathbb{N})$, and $X^{**} \cong \ell_\infty(\mathbb{F})$.

Definition 3.23. A Banach space X is **reflexive** if $i(X) = X^{**}$, where $i : X \hookrightarrow X^{**}$ is the canonical linear isometric embedding by $i(x) := \hat{x}$, $\hat{x}(\phi) := \phi(x)$.

Corollary 3.38. *Let X be a normed linear space. For any closed subspace M , and $x \notin M$, there is $f \in X^*$, such that $\|f\| = 1$, $f|_M = 0$, and $f(x) = \text{dist}(x, M)$.*

Proof. Consider the quotient map $Q : X \rightarrow Y$, where $Y := X/M$. Since $x \notin M$, we have $[x] \neq 0$.

Let $\phi \in Y^*$, such that $\|\phi\|_{Y^*} = 1$, and $\phi([x]) = \|[x]\|_Y = \inf_{m \in M} \|x+m\| = \text{dist}(x, M)$.

Let $f := \phi \circ Q$, we have that $f(x) = \text{dist}(x, M)$, and $\forall m \in M$, $f(m) = \phi(m+M) = \phi(0) = 0$.

Also, $\|f\| \leq \|\phi\| \|Q\| \leq \|\phi\| = 1$. \square

Definition 3.24. Let X be a Banach Space. For $Y \subseteq X$, the **annihilator** of Y is

$$Y^\perp := \{\varphi \in X^* : \varphi(Y) = \{0\}\}.$$

For $Z \subseteq X^*$, the **preannihilator** of Z is

$$Z_\perp : \{x \in X : \hat{x}|_Z = 0\} = Z^\perp \cap X.$$

Proposition 3.39. *Let X be a Banach Space, then*

$$(Y^\perp)_\perp = \overline{\text{Span}(Y)}.$$

Proof. Let $M := \overline{\text{Span}(Y)}$.

For any $y \in M, f \in Y^\perp$, we have $\hat{y}(f) = f(y) = 0$, so $y \in (Y^\perp)_\perp$. Thus, $M \subseteq (Y^\perp)_\perp$.

Suppose $x \notin Y$, then there is $f \in X^*$ such that $f|_M = 0$, and $f(x) = \text{dist}(x, M)$. In particular, $f \in Y^\perp$ and $\hat{x}(f) = f(x) \neq 0$.

Thus, $x \notin (Y^\perp)_\perp$.

This shows $(Y^\perp)_\perp \subseteq M$.

Thus,

$$M = (Y^\perp)_\perp.$$

□

4 Hilbert Spaces

Definition 4.1. Let X be a vector space over \mathbb{F} . $M : X \times X \rightarrow \mathbb{F}$ is a **sesquilinear positive semidefinite form** on X if it satisfies

1. conjugate symmetry; i.e. $M[x, y] = \overline{M[y, x]}$.
2. linearity in the second argument; i.e. $M[x, ay + z] = M[x, y] + aM[x, z]$.
3. positive semi-definiteness; i.e. $M[x, x] \geq 0$.

Proposition 4.1. Let X be a vector space over \mathbb{F} , with a sesquilinear positive semidefinite form M . For all $x, y \in H$, we have

1. $M[x, y] + M[y, x] = 2\Re(M[x, y]) = 2\Re(M[y, x])$, which is twice the real part of $M[x, y]$.
2. $M[x, y] - M[y, x] = 2\Im(M[x, y]) = -2\Im(M[y, x])$, which is twice the imaginary part of $M[x, y]$.
3. $M[x, y]M[y, x] = |M[x, y]|^2 = |M[y, x]|^2$.
4. $M[x + y, x + y] = M[x, x] + 2\Re(M[y, x]) + M[y, y]$.
5. $M[x - y, x - y] = M[x, x] - 2\Re(M[x, y]) + M[y, y]$.

Proof. 1. $M[x, y] + M[y, x] = M[x, y] + \overline{M[x, y]} = 2\Re(M[y, x])$.

$$2. M[x, y] - M[y, x] = M[x, y] - \overline{M[x, y]} = 2\Im(M[x, y]).$$

$$3. M[x, y]M[y, x] = M[x, y]\overline{M[y, x]} = |M[x, y]|^2.$$

4.

$$\begin{aligned} M[x + y, x + y] &= M[x + y, x] + M[x + y, y] \\ &= M[x, x] + M[y, x] + M[x, y] + M[y, y] \\ &= M[x, x] + M[y, x] + \overline{M[y, x]} + M[y, y] \\ &= M[x, x] + 2\Re(M[y, x]) + M[y, y]. \end{aligned}$$

5. By 3., we have

$$\begin{aligned} M[x - y, x - y] &= M[x + (-y), x + (-y)] \\ &= M[x, x] + 2\Re(M[-y, x]) + M[-y, -y] \\ &= M[x, x] - 2\Re(M[y, x]) + M[y, y]. \end{aligned}$$

□

Definition 4.2. An **inner product space** is a vector space H that has an inner product: $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$, such that $\forall u, v, w \in H, a, b \in \mathbb{C}$, it satisfies

1. conjugate symmetry; i.e. $\langle v, w \rangle = \overline{\langle w, v \rangle}$,
2. linearity in the second argument; i.e. $\langle v, au + bw \rangle = a\langle v, u \rangle + b\langle v, w \rangle$, and
3. positive definiteness; i.e. $\langle v, v \rangle \geq 0$, and if $v \neq 0$, we must have $\langle v, v \rangle > 0$.

Remark. Notice that the inner product is a sesquilinear positive semidefinite form, with the additional requirement: if $v \neq 0$, we must have $\langle v, v \rangle > 0$.

Proposition 4.2. *For every inner product space with $\langle - , \cdot \rangle$, and for all $x, y \in H$, we have*

1. $\langle x, y \rangle + \langle y, x \rangle = 2\Re(\langle x, y \rangle) = 2\Re(\langle y, x \rangle)$.
2. $\langle x, y \rangle - \langle y, x \rangle = 2\Im(\langle x, y \rangle) = -2\Im(\langle y, x \rangle)$.
3. $\langle x, y \rangle \langle y, x \rangle = |\langle x, y \rangle|^2$.
4. $\langle x + y, x + y \rangle = \langle x, x \rangle + 2\Re(\langle y, x \rangle) + \langle y, y \rangle$.
5. $\langle x - y, x - y \rangle = \langle x, x \rangle - 2\Re(\langle x, y \rangle) + \langle y, y \rangle$.

Theorem 4.3 (Cauchy-Schwarz). *For every inner product space H ,*

$$\forall u, v \in H, |\langle u, v \rangle| \leq \|u\| \|v\|,$$

where we define $\|x\| = \sqrt{\langle x, x \rangle}$ or any $x \in H$.

In particular, when $\|u\| \neq 0$, $\|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 = \|z\|^2$, where $z := \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u$.

Proof. Notice that this is trivially true and equality holds to be zero when $u = 0$.

Now we assume $u \neq 0$, then $\|u\| = \sqrt{\langle u, u \rangle} > 0$.

$$\begin{aligned} \|z\|^2 &= \langle z, z \rangle \\ &= \left\langle \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u, \|u\|v - \frac{\langle u, v \rangle}{\|u\|}u \right\rangle \\ &= \|u\|^2 \langle v, v \rangle - \langle u, v \rangle \langle u, v \rangle - \overline{\langle u, v \rangle} \langle v, u \rangle + \frac{\langle v, u \rangle \langle u, v \rangle}{\|u\|^2} \underbrace{\langle u, u \rangle}_{\|u\|^2} \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 - |\langle v, u \rangle|^2 + |\langle v, u \rangle|^2 \\ &= \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2. \end{aligned}$$

Now $\|z\|^2 = \langle z, z \rangle \geq 0$, we have the result. □

Remark. This proof does not use the positive definiteness of the inner product, so it also works for a sesquilinear positive semidefinite form M . i.e.

$$|M[x, y]| \leq M[x, x]M[y, y].$$

Proposition 4.4. *For every inner product space with $\langle - , \cdot \rangle$, there is a norm $\|x\| = \sqrt{\langle x, x \rangle}$.*

Proof. Consider any $x \in H, a \in \mathbb{C}$,

$$\begin{aligned}\|a \cdot x\| &= \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2} \sqrt{\langle x, x \rangle} = |a| \|x\| \\ \forall x \neq 0, \|x\| &= \sqrt{\langle x, x \rangle} > 0 \\ \|0\| &= \sqrt{\langle 0, 0 \rangle} = 0 \\ \|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \\ &= \|x\|^2 + \|y\|^2 + 2\Re(\langle x, y \rangle) \\ &\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y \rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &\leq (\|x\| + \|y\|)^2.\end{aligned}$$

Thus $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm. \square

Corollary 4.5. *For every inner product space, there is a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$*

Proposition 4.6 (polarization identity). *Let H be an inner product space over \mathbb{F} . The inner product is completely determined by the induced norm. Indeed, for any $x, y \in H$, we have*

$$\langle x, y \rangle = \frac{1}{4} \sum_{\epsilon \in \{\pm i, \pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if $\mathbb{F} = \mathbb{C}$, and

$$\langle x, y \rangle = \frac{1}{2} \sum_{\epsilon \in \{\pm 1\}} \epsilon \|x + \epsilon y\|^2$$

if $\mathbb{F} = \mathbb{R}$.

Proposition 4.7. *If $\forall v, \langle v, u \rangle = 0$, then $u = 0$.*

Proposition 4.8. *For an Inner product space H , $\forall y, x = \lim_{i \rightarrow \infty} x_i \in H$, we have*

$$\langle x, y \rangle = \lim_{i \rightarrow \infty} \langle x_i, y \rangle.$$

Proof. Given any $\epsilon > 0$, let $\epsilon_0 = \frac{\epsilon}{\|y\|}$.

Since $x = \lim_{i \rightarrow \infty} x_i$, we can find $N > 0$, such that $\forall n > N, \|x - x_n\| < \epsilon_0$, thus $|\langle x, y \rangle - \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq \|x - x_n\| \|y\| < \epsilon_0 \|y\| = \epsilon$ \square

Corollary 4.9. *For an Inner product space H , $\forall y, x = \lim_{i \rightarrow \infty} x_i \in H$, we have $\langle y, x \rangle = \lim_{i \rightarrow \infty} \langle y, x_i \rangle$.*

Definition 4.3. An inner product space \mathcal{H} is called a **Hilbert space** if it is complete.

Definition 4.4. Let H be an inner product space. Two vectors $u, v \in H$ are called **orthogonal** if $\langle u, v \rangle = 0$.

Definition 4.5. Let H be an inner product space. A set $\{e_i\}_{i \in I} \subseteq H$ is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij}.$$

Definition 4.6. Let H be an inner product space. An orthonormal set $\{e_i\}_{i \in I} \subseteq H$ is called a **maximal orthonormal set / orthonormal basis / total orthonormal set** if $\text{Span}(\{e_i\}_{i \in I})$ is dense in H . Namely,

$$H = \overline{\text{Span}(\{e_i\}_{i \in I})}.$$

Theorem 4.10 (generalized Fourier series). *Let \mathcal{H} be a Hilbert space, and $\{e_i\}_{i \in \mathbb{N}} \subseteq \mathcal{H}$ be an orthonormal set, then TFAE:*

1. $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis
2. If $\forall i \in \mathbb{N}, \langle x, e_i \rangle = 0$, then $x = 0$.
3. $\forall x \in \mathcal{H}, x = \sum_{i \in \mathbb{N}} \langle e_i, x \rangle e_i$. (Fourier series)
4. $\forall x \in \mathcal{H}, \|x\|^2 = \sum_{i \in \mathbb{N}} |\langle e_i, x \rangle|^2$. (Parseval Identity)

Theorem 4.11. \mathcal{H} is a separable Hilbert space, if and only if there is a maximal orthonormal set in \mathcal{H} . Moreover, in this case, every maximal orthonormal set is at most countable.

Definition 4.7. Let \mathcal{H} be a Hilbert space. $S \subseteq \mathcal{H}$, the subspace **orthogonal** to S is

$$S^\perp := \{u \in \mathcal{H} : \langle u, v \rangle = 0, \forall v \in S\}.$$

Lemma 4.12. Let \mathcal{H} be a Hilbert space, $S \subseteq \mathcal{H}$, we always have S^\perp is a subspace of \mathcal{H} .

Definition 4.8. Let V be a vector space, and $U, W \subseteq V$ be two subspaces, we say $V = U \oplus W$, if $V = U + W = \{v + w : v \in U, w \in W\}$, and $V \cap W = \{0\}$.

Proposition 4.13. Let V be a vector space, and $U, W \subseteq V$ be two subspaces. $V = U \oplus W$, if and only if $\forall v \in V$, it can be uniquely written as $v = u + w$, where $u \in U, w \in W$.

Theorem 4.14. Let \mathcal{H} be a Hilbert space, if $S \subseteq \mathcal{H}$ is a subspace, then

$$\mathcal{H} = \bar{S} \oplus S^\perp.$$

Theorem 4.15. [Riesz-Frechet Representation theorem] Let \mathcal{H} be a Hilbert space, then for each $u^* \in \mathcal{H}^*$, $\exists! u \in \mathcal{H}$, such that $\forall v \in \mathcal{H}, \langle u^* | v \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$, and $\|u^*\|_{\mathcal{H}^*} = \|u\|_{\mathcal{H}}$.

Definition 4.9. Let X be a vector space over \mathbb{F} . We define

$$\bar{X} := \{\bar{x} : x \in X\}$$

to be the **conjugate vector space**. The scalar multiplication is given by $\lambda \bar{x} = \overline{\lambda x}$.

Proposition 4.16. Let X be a normed space over \mathbb{F} . The map $X \rightarrow \bar{X}$ by $x \mapsto \bar{x}$ is an isometric antilinear bijection.

Corollary 4.17. Let \mathcal{H} be a Hilbert space, then $\bar{\mathcal{H}} \cong \mathcal{H}^*$, where the map $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*; u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism.

Corollary 4.18. Every Hilbert space is reflexive.

Definition 4.10. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We say $U \in B(\mathcal{H}_1, \mathcal{H}_2)$ is **unitary** if it is an isometric isomorphism between Banach Spaces.

Proposition 4.19. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. If U is unitary, it preserves the inner product as well. Namely,

$$\forall x, y \in \mathcal{H}_1, \langle x, y \rangle = \langle Ux, Uy \rangle.$$

Proof. Since U is isometric, by the polarization identity proposition 4.6, it preserves the inner product as well. \square

5 Locally Convex Topological Vector Spaces and Weak Topology

5.1 Locally Convex Topological Vector Spaces

Definition 5.1. Let X be a vector space over \mathbb{F} , a **semi-norm** is a map $p : X \rightarrow [0, \infty)$ such that $\forall t \in \mathbb{F}, x, y \in X$,

$$\begin{aligned} p(tx) &= |t|p(x), \\ p(x+y) &\leq p(x) + p(y). \end{aligned}$$

The null space of p is denoted $N_p := \{x \in X : p(x) = 0\}$.

Remark. Notice that p is a norm if and only if $N_p = \{0\}$.

Definition 5.2. A **Locally Convex Topological Vector Space** is a vector space X over \mathbb{F} with a family of semi-norms $P = \{p_\alpha\}_{\alpha \in A}$, such that $\bigcap_{p \in P} \ker(p) = \{0\}$.

\mathcal{T}_P is the topology on X generated by the convex sets

$$U(x, p, r) := \{y \in X : p(y-x) < r\}$$

for $x \in X, r > 0, p \in P$.

Definition 5.3. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, for $r > 0, x_0 \in X$, and a finite subset $F \subseteq P$, we define

$$U_{F,r}(x_0) := \{x \in X : \forall p \in F, p(x-x_0) < r\},$$

and $U_{F,r} := U_{F,r}(0)$.

Proposition 5.1. Each $U_{F,r}(x_0)$ is a finite intersection of $\bigcap_{p \in F} U(x, p, r)$, so it is open. Also, each $U_{F,r}(x_0) = x_0 + U_{F,r}$.

Proposition 5.2. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then for any $x_0 \in X$ the sets $\{U_{F,r}(x_0)\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at x_0 .

Proof. Consider any open $U \ni 0$, then there are $p_1, \dots, p_n \in P, r_1, \dots, r_n > 0, x_1, \dots, x_n \in X$, such that

$$0 \in \bigcap_{i=1}^n U(x_i, p_i, r_i) \subseteq U.$$

Let $r := \min_{i \in [n]} (r_i - p_i(x_i)) > 0$, and let $F := \{p_1, \dots, p_n\}$.

Consider any $x \in U_{F,r}$, we have for any $i \in [n]$,

$$\begin{aligned} p_i(x_i - x) &\leq p_i(x_i) + p_i(x) \\ &\leq p_i(x_i) + r \\ &\leq p_i(x_i) + r_i - p_i(x_i) \\ &= r_i. \end{aligned}$$

Thus $x \in \bigcap_{i=1}^n U(x_i, p_i, r_i) \subseteq U$.

This shows $0 \in U_{F,r} \subseteq U$.

Thus, $\{U_{F,r}\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at 0. By translation, $\{U_{F,r}(x_0)\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at x_0 . \square

Proposition 5.3. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, it is Hausdorff.

Proof. Given any $x \neq y \in X$, then $x - y > 0$.

There is $p \in P$ such that $r = p(x-y) > 0$.

Now $U(x, p, \frac{r}{2}) \ni x$ and $U(y, p, \frac{r}{2}) \ni y$ has empty intersection. \square

Proposition 5.4. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then the addition map $A : X \times X \mapsto X$ and scalar multiplication map $B : \mathbb{F} \times X \rightarrow X$ are continuous.

Proof. Given any open set U , with $x_0 + y_0 \in U$ for some $x_0, y_0 \in X$.

Since $\{U_{F,r}(x_0 + y_0)\}_{\text{finite } F \subseteq P, r > 0}$ form a neighbourhood basis at $x_0 + y_0$, there is some finite $F \subseteq P$ and $r > 0$, such that

$$U_{F,r}(x_0 + y_0) \subseteq U,$$

since they form a neighbourhood basis.

We claim that $A^{-1}(U_{F,r}(x_0 + y_0)) \supseteq (x_0 + U_{F,\frac{r}{2}}) \times (y_0 + U_{F,\frac{r}{2}})$, which is open.

Indeed, take any $x \in x_0 + U_{F,\frac{r}{2}}, y \in y_0 + U_{F,\frac{r}{2}}$ and $p \in F$, we have

$$\begin{aligned} p((x + y) - (x_0 + y_0)) &\leq p(x - x_0) + p(y - y_0) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

Thus, we have find

$$(x_0, y_0) \in (x_0 + U_{F,\frac{r}{2}}) \times (y_0 + U_{F,\frac{r}{2}}) \subseteq A^{-1}(U_{F,r}(x_0 + y_0)) \subseteq A^{-1}(U).$$

Since this hold for all $(x_0, y_0) \in A^{-1}(U)$, we have that $A^{-1}(U)$ is open. \square

Proposition 5.5. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to $x \in X$ if and only if $\forall p \in P$, $p(x - x_\lambda) \rightarrow 0$.

Proof.

$$(x_\lambda)_{\lambda \in \Lambda} \rightarrow x$$

if and only if

$$\forall \text{finite } F \subseteq P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U_{F,r}(x)$$

if and only if

$$\forall p \in P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, x_\lambda \in U(x, p, r)$$

if and only if

$$\forall p \in P, r > 0, \exists \lambda_0 \in \Lambda \text{ such that } \forall \lambda \geq \lambda_0, p(x_\lambda - x) < r$$

if and only if

$$\forall p \in P, p(x_\lambda - x) \rightarrow 0.$$

\square

Proposition 5.6. Let $(X, \|\cdot\|)$ be a normed vector space, then taking $P := \{\|\cdot\|\}$, we have a Locally Convex Topological Vector Space.

5.2 Weak Topology

Proposition 5.7. Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then $P := \{p_\phi\}_{\phi \in Y}$ given by $p_\phi(x) := |\phi(x)|$ gives a Locally Convex Topological Vector Space (X, \mathcal{T}_Y) , where $\mathcal{T}_Y := \mathcal{T}_P$.

Proof. Clearly, $\forall t \in \mathbb{F}, p_\phi(tx) = |\phi(tx)| = |t\phi(x)| = |t||\phi(x)| = |t|p_\phi(x)$.

Also, $p_\phi(x + y) = |\phi(x + y)| = |\phi(x) + \phi(y)| \leq |\phi(x)| + |\phi(y)| = p_\phi(x) + p_\phi(y)$.

Thus, each p_ϕ is a semi-norm.

Suppose $p_\phi(x) = 0$ for all $\phi \in Y$, then $\phi(x) = 0$ for all $\phi \in Y$. Since Y separates the points, we must have $x = 0$. \square

Remark. If Y is not a subspace but just a subset, we can WLOG take $Y' = \text{Span}(Y)$, which will generate the same topology.

Definition 5.4. Let $(X, \|\cdot\|)$ be a normed vector space. The **weak topology on X** $\sigma(X, X^*)$ is (X, \mathcal{T}_{X^*}) , where we take $Y = X^*$, which separates points by the Hahn-Banach Theorem. We say $(x_\lambda)_{\lambda \in \Lambda}$ **converges weakly to $x \in X^*$** if it converges in the weak topology, denoted $x_\lambda \rightharpoonup x$.

Also, the **weak-* topology on X^*** $\sigma(X^*, X)$ is (X^*, \mathcal{T}_X) , where we take $Y = X \subseteq X^{**}$. We say $(\phi_\lambda)_{\lambda \in \Lambda}$ **converges weakly to $\phi \in X^*$** if it converges in the weak-* topology, denoted $\phi_\lambda \rightharpoonup \phi$.

Proposition 5.8. Let $(X, \|\cdot\|)$ be a normed vector space. $x_\lambda \rightharpoonup x$ in X if and only if $\forall \phi \in X^*$, $\phi(x_\lambda) \rightarrow \phi(x)$. Also, $\phi_\lambda \rightharpoonup \phi$ in X^* if and only if $\forall x \in X$, $\phi_\lambda(x) \rightarrow \phi(x)$.

Proposition 5.9. Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then if $x_\lambda \rightarrow x$ in norm, it also converges in (X, \mathcal{T}_Y) .

Proposition 5.10. Let $(X, \|\cdot\|)$ be a normed vector space, $(u_k)_{k=1}^\infty \subset X$ be a sequence, then

1. If $u_k \rightarrow u$, we always have $u_k \rightharpoonup u$.
2. If $u_k \rightharpoonup u$, we have that u is unique.
3. If $u_k \rightharpoonup u$, we have $(u_k)_{k=1}^\infty$ is bounded.
4. If $u_k \rightharpoonup u$, every subsequence $(u_{k_j})_{j=1}^\infty$ also converges weakly to u .

Theorem 5.11. Let X be a reflexive Banach Space, and $(u_k)_{k=1}^\infty \subset X$ be a bounded sequence, then $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.

Proposition 5.12. Let \mathcal{H} be a Hilbert space, then $u_k \rightharpoonup u$ if and only if $\forall v \in \mathcal{H}$, $\langle v, u_k \rangle \rightarrow \langle v, u \rangle$ as real numbers.

Proof. Suppose $u_k \rightharpoonup u$.

Notice that for all $v \in \mathcal{H}$, we have that $v^\dagger \in \mathcal{H}^*$, and thus $\langle v, u_k \rangle = \langle v^\dagger | u_k \rangle \rightarrow \langle v^\dagger | u \rangle = \langle v, u \rangle$.

Now suppose $\forall v \in \mathcal{H}$, $\langle v, u_k \rangle \rightarrow \langle v, u \rangle$.

Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 4.15, there is some $f^\dagger \in \mathcal{H}$, such that

$$\langle f | u_{k_j} \rangle = \langle f^\dagger, u_{k_j} \rangle \rightarrow \langle f^\dagger, u \rangle = \langle f | u \rangle.$$

Thus, $u_{k_j} \rightharpoonup u$. □

Proposition 5.13. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Tu_k \rightharpoonup Tu \in \mathcal{H}_2$.

Proof. Let $y_k := Tu_k, y := Tu \in \mathcal{H}_2$.

Consider any $g \in \mathcal{H}_2^*$, we define $f := g \circ T \in \mathcal{H}_1^*$.

Since $u_k \rightharpoonup u$, we must have

$$\begin{aligned} \lim_{k \rightarrow \infty} f(u_k) &= f(u) \\ \lim_{k \rightarrow \infty} g(Tu_k) &= g(Tu) \\ \lim_{k \rightarrow \infty} g(y_k) &= g(y). \end{aligned}$$

We thus have $y_k \rightharpoonup y$. □

Proposition 5.14. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a compact operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Ku_k \rightharpoonup Ku \in \mathcal{H}_2$.

Proof. Let $y_k := Ku_k, y := Ku \in \mathcal{H}_2$.

Since K is compact, it is bounded, so $y_k \rightharpoonup y$.

Now suppose for contradiction $\lim_{k \rightarrow \infty} \|y_k - y\| \neq 0$.

Then there is some $\epsilon > 0$ and a subsequence $(u_{k_j})_{j=1}^\infty$ such that $\forall j \geq 1$, $\|y_{k_j} - y\| \geq \epsilon$.

Since $u_k \rightharpoonup u \in \mathcal{H}$, we have $(u_k)_{k=1}^\infty$ is bounded, and thus $(u_{k_j})_{j=1}^\infty$ is bounded.

Since K is compact, there is some further subsequence $(u_{k_{j_m}})_{m=1}^\infty$ such that $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$.

Thus $Ku_{k_{j_m}} \rightharpoonup \tilde{y}$. Since weak convergence, we must have $\tilde{y} = y$.

Thus $\lim_{m \rightarrow \infty} Ku_{k_{j_m}} = y$, which is a contradiction. □

Definition 5.5. Let X, Y be two normed vector spaces, then the **weak operator topology** on $B(X, Y)$ is induced by

$$P := \{p_{x,\phi}(T) : x \in X, \phi \in Y^*\},$$

where for all $T \in B(X, Y)$,

$$p_{x,\phi}(T) := |\phi(T(x))|.$$

We say $(T_\lambda)_{\lambda \in \Lambda}$ converges weakly to $T \in B(X, Y)$ if it converges in the weak operator topology, denoted $T_\lambda \rightharpoonup T$.

Remark. Notice that these functions separate points by the Hahn-Banach Theorem. Indeed, $T \neq S$ implies $\exists x \in X$ such that $Tx \neq Tx$, which implies $\exists \phi \in Y^*$ such that $\phi(Tx) \neq \phi(Tx)$.

Proposition 5.15. Let X, Y be two normed vector spaces, then $T_\lambda \rightharpoonup T \in B(X, Y)$ if and only if $\forall \phi \in X^*, x \in X, \phi(T_\lambda x) \rightarrow \phi(Tx)$.

Definition 5.6. Let X, Y be two normed vector spaces, then the **strong operator topology** is the topology induced by

$$P := \{p_x(T) := \|Tx\|_Y : x \in X\}.$$

Proposition 5.16. Let X, Y be two normed vector spaces, then $T_\lambda \rightarrow T \in B(X, Y)$ strongly if and only if $\forall x \in X, T_\lambda x \rightarrow Tx \in Y$.

Proposition 5.17. Let X, Y be two normed vector spaces, then convergence in the operator norm implies convergence in strong operator topology, which implies convergence in weak operator topology.

Proposition 5.18. When $X = Y = \mathcal{H}$ is a Hilbert space, then $T_\lambda \rightharpoonup T$ if and only if

$$\forall \xi, \eta \in \mathcal{H}, \langle T_\lambda \xi, \eta \rangle \rightarrow \langle T\xi, \eta \rangle.$$

Proof. By the Riesz-Representation Theorem, for each $\phi \in \mathcal{H}^*$, there is unique $\eta \in \mathcal{H}$ such that

$$\forall \xi \in \mathcal{H}, \phi(\xi) = \langle \xi, \eta \rangle.$$

Thus $T_\lambda \rightharpoonup T$ if and only if $\forall \phi \in \mathcal{H}^*, \xi \in \mathcal{H}, \phi(T_\lambda \xi) \rightarrow \phi(T\xi)$, if and only if $\forall \xi, \eta \in \mathcal{H}, \langle T_\lambda \xi, \eta \rangle \rightarrow \langle T\xi, \eta \rangle$. \square

5.3 Continuous Functions

Theorem 5.19. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then for a linear $\phi : X \rightarrow \mathbb{F}$, the following are equal:

1. ϕ is continuous,
2. ϕ is continuous at 0,
3. $\ker(\Phi)$ is closed,
4. $\exists p_1, \dots, p_n \in P, \alpha_1, \dots, \alpha_n > 0$, such that

$$\forall x \in X, |\phi(x)| \leq \sum_{i=1}^n \alpha_i p_i(x).$$

Proof. (1) \implies (2) is trivial.

(2) \implies (3):

Let $x_\lambda \in \ker(\phi)$, with $x_\lambda \rightarrow x \in X$.

For any λ , we have $\phi(x) = \phi(x) - \phi(x_\lambda) = \phi(x - x_\lambda)$.

Since $x - x_\lambda \rightarrow 0$, and ϕ is continuous at 0, we have $\phi(x - x_\lambda) \rightarrow 0$, which means $\phi(x) = 0$. Namely, $x \in \ker(\phi)$.

Thus $\ker(\phi)$ is closed.

(3) \implies (4):

Suppose $\phi = 0$, (4) is trivially true.

Otherwise, there is $x_0 \in \ker(\Phi)^c$. WLOG, by taking $x'_0 := \frac{x_0}{\phi(x_0)}$, we can assume $\phi(x_0) = 1$.

Since $\ker(\phi)$ is closed, $\ker(\Phi)^c$ is open.

There must be some finite $F \subseteq P$ and $r > 0$, such that $x_0 + U_{F,r} = U_{F,r}(x_0) \subseteq \ker(\phi)^c$.

Thus, $0 \notin \phi(x_0 + U_{F,r}) = \phi(x_0) + \Phi(U_{F,r}) = 1 + \Phi(U_{F,r})$.

Namely, $-1 \notin \Phi(U_{F,r})$.

Thus $\forall x \in U_{F,r}$, we have $\phi(x) \neq -1$.

Take $\{p_1, \dots, p_n\} = F$, and $\alpha_i = \frac{1}{r}$.

Suppose for contradiction that there is some $x \in X$ with $|\phi(x)| > \sum_{i=1}^n \frac{1}{r} p_i(x)$.

In particular, $|\phi(x)| > \frac{1}{r} p_i(x)$ for all $p_i \in F$.

We must have some $|\lambda|$ such that $\phi(x) = \lambda |\phi(x)|$.

Then for $y := \frac{x}{-\lambda |\phi(x)|}$, we have

$$\phi(y) = \phi\left(\frac{x}{-\lambda |\phi(x)|}\right) = \frac{\phi(x)}{-\lambda |\phi(x)|} = -1.$$

However,

$$p_i(y) = \left| \left(\frac{1}{-\lambda |\phi(x)|} \right) p_i(x) \right| = \frac{p_i(x)}{|\phi(x)|} < \frac{r |\phi(x)|}{|\phi(x)|} = r.$$

Thus, $y \in U_{F,r}$, which is a contradiction.

(4) \implies (1):

If $x_\lambda \rightarrow x$, then $\forall p \in P$, $p(x_\lambda - x) \rightarrow 0$.

Thus, $\sum_{i=1}^n \alpha_i p_i(x_\lambda - x) \rightarrow 0$.

Thus, $|\phi(x_\lambda) - \phi(x)| = |\phi(x_\lambda - x)| \rightarrow 0$, which means $\phi(x_\lambda) \rightarrow \phi(x)$.

This shows ϕ is continuous. \square

Definition 5.7. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then the **continuous dual space** $(X, \mathcal{T}_P)^*$ is the set of continuous linear functions $\phi : X \rightarrow \mathbb{F}$.

Theorem 5.20. Let $(X, \|\cdot\|)$ be a normed vector space, and $Y \subseteq X^*$ be a linear subspace that separates the points, then $\phi \in (X, \mathcal{T}_Y)^*$ if and only if $\phi \in \text{Span}(Y)$.

Proof. $\phi \in (X, \mathcal{T}_Y)^*$ if and only if $\exists \phi_1, \dots, \phi_n \in Y, \alpha_i > 0$, such that $\forall x \in X$, $|\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)|$.

(\Leftarrow):

Suppose $\phi = \sum_{i=1}^n c_i \phi_i$, then taking $\alpha_i := \max(|c_1|, 1)$, we have

$$\begin{aligned} |\phi(x)| &= \left| \sum_{i=1}^n c_i \phi_i(x) \right| \\ &\leq \sum_{i=1}^n |c_i| |\phi_i(x)| \\ &\leq \sum_{i=1}^n \alpha_i |\phi_i(x)|. \end{aligned}$$

(\implies):

Suppose $\{\phi_i\}_{i=1}^n$ is not linearly independent, WLOG, $\phi_n = \sum_{i=1}^{n-1} c_i \phi_i$, then

$$|\phi(x)| \leq \sum_{i=1}^n \alpha_i |\phi_i(x)| \leq \sum_{i=1}^{n-1} (\alpha_i + \alpha_n |c_i|) |\phi_i(x)|.$$

Thus, we can assume $\{\phi_i\}_{i=1}^n$ is linearly independent.

Consider $T : X \rightarrow \mathbb{F}^n$ by $T(x) := (\phi_1(x), \dots, \phi_n(x))$.

Clearly $\ker(T) = \bigcap_{i=1}^n \ker(\phi_i)$.

Since $\{\phi_i\}_{i=1}^n$ is linearly independent, T is surjective.

By a corollary of the Banach Isomorphism Theorem, $X/\bigcap_{i=1}^n \ker(\phi_i) \cong \mathbb{F}^n$ by $\hat{T} : \hat{x} \mapsto (\phi_1(x), \dots, \phi_n(x))$. Since $\ker(\phi) \supseteq \bigcap_{i=1}^n \ker(\phi_i)$, we can define $\tilde{\phi} : \mathbb{F}^n \rightarrow \mathbb{F}$ by

$$\tilde{\phi}(\hat{T}\hat{x}) := \phi(x).$$

This is well-defined, since $\hat{x} = \hat{y} \implies x - y \in \bigcap_{i=1}^n \ker(\phi_i) \implies \phi(x - y) = 0 \implies \phi(x) = \phi(y)$.

Thus for all $i \in [n]$, there is $\beta_i := \hat{\phi}(e_i)$, such that $\tilde{\phi}(z_1, \dots, z_n) = \sum_{i=1}^n \beta_i z_i$.

Thus, $\phi(x) = \tilde{\phi}(\hat{T}\hat{x}) = \tilde{\phi}(Tx) = \sum_{i=1}^n \beta_i \phi_i(x)$, which means $\phi = \sum_{i=1}^n \beta_i \phi_i \in \text{Span}(\{\phi_i\}_{i=1}^n) \subseteq \text{Span}(Y)$. \square

Corollary 5.21. Let $(X, \|\cdot\|)$ be a normed vector space, then $\sigma(X, X^*)^* = (X, \mathcal{T}_{X^*})^* = X^*$ and $\sigma(X^*, X)^* = (X^*, \mathcal{T}_X)^* = X$.

Remark. A convergent net in weak-* topology is not necessarily bounded.

Example 5.3.1. Consider $X = \ell^1 = C_0^*$, with the weak-* topology \mathcal{T}_{C_0} .

For any finite $F \subsetneq C_0 \subset (\ell^1)^*$, we have that $\bigcap_{\phi \in F} \ker(\phi) \neq \{0\}$, so there is a $x_F \neq 0 \in \bigcap_{\phi \in F} \ker(\phi)$. By taking $\tilde{x}_F := \frac{|F|}{\|x_F\|} x_F$, we can have $\tilde{x}_F \in \bigcap_{\phi \in F} \ker(\phi)$, with $\|\tilde{x}_F\| = |F|$.

Clearly $(\tilde{x}_F)_{\text{finite } F \subsetneq C_0}$ is not bounded.

However, consider any finite $F \subseteq C_0, r > 0$, we can pick $F_0 := F$.

For all finite $F' \supseteq F_0$, we have that $\forall \phi \in F, |\phi(\tilde{x}_{F'} - 0)| = 0 < r$, which means $\tilde{x}_{F'} \in U_{F,r}$. Thus $\tilde{x}_F \rightharpoonup 0$.

5.4 Geometric Hahn–Banach Theorems

Definition 5.8. Let X be a vector space, a set $K \subseteq X$ is **convex** if

$$\forall x, y \in K, t \in (0, 1), (1-t)x + ty \in K.$$

Definition 5.9. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, the **Minkowski functional associated to U** is

$$p_U(x) := \inf \{t > 0 : x \in tU\}.$$

Lemma 5.22. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, then the Minkowski functional associated to U is always well-defined.

Proof. Since $0 \in U$, which is open, there is finite $F \subseteq P, r > 0$ such that $U_{F,r} \subseteq U$.

Thus, for $t = \frac{r}{2 \max\{p(x) : p \in F\}} > 0$, we have $\forall p \in P, p(tx) = \frac{r}{2 \max\{p(x) : p \in F\}} p(x) \leq \frac{r}{2} < r$, so $tx \in U_{F,r} \subseteq U$. Thus $\{t > 0 : x \in tU\} \neq \emptyset$. \square

Theorem 5.23. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $U \ni 0$ be a convex open set, then the Minkowski functional $p_U : X \rightarrow [0, \infty)$ is a sublinear functional, and $U = \{x \in X : p_U(x) < 1\}$.

Proof. For any $t > 0$, we have $x \in sU \iff tx \in tsU$.

$$\begin{aligned} p_U(tx) &= \inf \{r > 0 : tx \in rU\} \\ &= \inf \{st > 0 : tx \in tsU\} \\ &= \inf \{st > 0 : x \in sU\} \\ &= t \inf \{s > 0 : x \in sU\} \\ &= tp_U(x). \end{aligned}$$

Also, consider any $x, y \in X$ and any $s, t > 0$ such that $x \in sU, y \in tU$, we have that $\frac{x}{s}, \frac{y}{t} \in U$. By the convexity of U ,

$$\begin{aligned} \frac{s}{s+t} \frac{x}{s} + \frac{t}{s+t} \frac{y}{t} &\in U \\ x + y &= \left(\frac{s}{s+t} \frac{x}{s} + \frac{t}{s+t} \frac{y}{t} \right) (s+t) \\ &\in (s+t)U. \end{aligned}$$

Thus, $p_U(x+y) \leq s+t$. Since this holds for any such s, t , we have

$$p_U(x+y) \leq p_U(x) + p_U(y).$$

This shows P_U is sublinear.

Suppose $p_U(x) < 1$, then there is $t < 1$ such that $x \in tU \implies \frac{x}{t} \in U \implies x = t\frac{x}{t} + (1-t)0 \in U$ by convexity of U .

Now for any $x \in U$, since $t \mapsto tx$ is continuous, and $1x = x \in U$ which is open, we have some $\delta > 0$, such that $(1-\delta, 1+\delta)x \subseteq U$.

Thus $\forall t \in (\frac{1}{1+\delta}, 1)$, $x \in tU$, which means $p_U(x) \leq \frac{1}{1+\delta} < 1$. \square

Theorem 5.24 (First Separation). *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, $A, B \subseteq X$ be disjoint convex sets. Suppose A is open, then $\exists t \in \mathbb{R}$, $\phi \in (X, \mathcal{T}_P)^*$, such that*

$$\forall x \in A, y \in B, \quad \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

Namely, $\phi(A), \phi(B)$ can be separated by a vertical line in \mathbb{C} .

Proof. 1. We first assume $\mathbb{F} = \mathbb{R}$.

Fix $x_0 \in A, y_0 \in B$, let $z_0 := y_0 - x_0 \neq 0$.

Consider $U := z_0 + A - B = \bigcup_{y \in B} (z_0 - y + A)$, which is open and convex. Also, $0 \in U$.

Consider the Minkowski functional $p_U : X \rightarrow [0, \infty)$, which is sublinear.

Notice that $z_0 \notin U$, so $p_U(z_0) \geq 1$.

Let $\phi_0 : \text{Span}\{z_0\} \rightarrow \mathbb{R}$ be $\lambda z_0 \mapsto \lambda$, which is linear.

In addition, $\forall \lambda \geq 0$, we have $\phi_0(\lambda z_0) = \lambda \leq \lambda p_U(z_0) = p_U(\lambda z_0)$.

Also, $\phi_0(-\lambda z_0) = -\lambda \leq 0 \leq p_U(-\lambda z_0)$.

Thus $\phi_0 \leq P_U$ on $\text{Span}\{z_0\}$. By the extension theorem 3.32, there is a linear extension $\phi : X \rightarrow \mathbb{R}$, such that $\phi \leq p_U$.

Let $\epsilon > 0$.

Take any $x \in \epsilon U \cap (-\epsilon U) \in \mathcal{O}(0)$, which is open.

We have that $\pm \frac{x}{\epsilon} \in U$, so $\phi(\pm \frac{x}{\epsilon}) \leq p_U(\pm \frac{x}{\epsilon}) < 1$.

Thus, $|\phi(x)| < \epsilon$.

This shows that ϕ is continuous.

Now take any $x \in A, y \in B$, we have that $z_0 + x - y \in U$, so $1 + \phi(x) - \phi(y) = \phi(z_0 + x - y) \leq p_U(z_0 + x - y) < 1$.

Thus, $\phi(x) < \phi(y)$.

Notice that since A is open, there is $\epsilon_0 > 0$ such that $\forall 0 < \epsilon < \epsilon_0$, $(1 \pm \epsilon)x \in A$. Thus, $(1 \pm \epsilon)\phi(x) \in \phi(A)$, which means $\phi(A)$ is open.

Since A is convex, $\phi(A)$ is also convex, thus connected. Thus, $\phi(A) = (b, t)$ is an interval.

This shows $\forall x \in A, y \in B, \phi(x) < t \leq \phi(y)$.

2. Now assume $\mathbb{F} = \mathbb{C}$.

Consider $(X_{\mathbb{R}}, \mathcal{T}_P)$, we have that A, B are still disjoint convex sets, and A is open.

Thus there is a \mathbb{R} -linear continuous map $\psi \in (X_{\mathbb{R}}, \mathcal{T}_P)^*$, such that $\psi(A) < \psi(B)$.

Now take $\phi(x) := \psi(x) - i\psi(ix)$. \square

However, this is not always true when A is not open.

Example 5.4.1. Consider the weak topology $(\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})$, and $A := \{x = (x_i)_{i=1}^\infty \in \ell^1(\mathbb{N}) : \sum_{i=1}^\infty x_i = 0\}$, $B = \{\delta_1\}$. They are disjoint convex sets.

However, for any $\phi \in C_0(\mathbb{N}) = \text{Span}(C_0(\mathbb{N})) = (\ell^1(\mathbb{N}), \mathcal{T}_{C_0(\mathbb{N})})^*$, we have that $\phi(A) \cap \phi(B) \neq \emptyset$.

Indeed, consider any $\phi = (a_1, a_2, \dots) \in C_0(\mathbb{N})$, there is $m \in \mathbb{N}$ such that $a_m \neq 0$.

Now consider $(\delta_m - \delta_n)_{n=1}^\infty \subset A$, we have that $\phi(\delta_m - \delta_n) = a_m - a_n \rightarrow a_m \neq 0$.

Since $\ker(\phi)$ is closed, A is not contained in $\ker(\phi)$, which means $\phi(A) = \mathbb{C}$.

Lemma 5.25. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, suppose compact $K \subseteq$ open $V \subseteq X$, then there is an open convex neighbourhood U of 0, such that $K + U \subseteq V$.

Proof. For all $x \in K$, since $x \in V$, there is finite $F_x \subseteq P$, and $r_x > 0$, such that $U_{F_x, 2r_x}(x) \subseteq V$.

Since $K \subseteq \bigcup_{x \in K} U_{F_x, r_x}(x)$ is compact, there is a finite subcover $K \subseteq \bigcup_{i=1}^n U_{F_{x_i}, r_{x_i}}(x_i)$.

Now let $F := \bigcup_{i=1}^n F_{x_i}$, which is finite, and $r := \min_{i \in [n]} \{r_{x_i}\} > 0$.

Let $U := U_{F, r}$.

For any $z \in K + U$, there is some $x \in K$ such that $z \in U_{F, r}(x)$.

Also, since $K \subseteq \bigcup_{i=1}^n U_{F_{x_i}, r_{x_i}}(x_i)$, there is some $i \in [n]$ such that $x \in U_{F_{x_i}, r_{x_i}}(x_i)$.

Thus for any $p \in F_{x_i} \subseteq F$, we have

$$\begin{aligned} p(z - x_i) &\leq p(z - x) + p(x - x_i) \\ &< r + r_{x_i} \\ &\leq 2r_{x_i}. \end{aligned}$$

Thus, $z \in U_{F_{x_i}, 2r_{x_i}}(x_i) \subseteq V$. □

Theorem 5.26 (Second Separation). Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, $A, B \subseteq X$ be disjoint convex sets. Suppose A is compact, B is closed, then $\exists t \in \mathbb{R}$, $\phi \in (X, \mathcal{T}_P)^*$, such that

$$\forall x \in A, y \in B, \Re(\phi(x)) < t \leq \Re(\phi(y)).$$

Namely, $\phi(A), \phi(B)$ can be separated by a vertical line in \mathbb{C} .

Proof. Since A is compact, and B^c is open, there is an open convex neighbourhood U of 0, such that $A + U \subseteq B^c$; namely $A + U \cap B = \emptyset$.

By the first separation theorem, there is $t \in \mathbb{R}, \phi \in (X, \mathcal{T}_P)^*$, such that

$$\sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

Since ϕ is continuous and A is compact, by the Extreme Value Theorem, there is $x_0 \in A$, such that $\Re(\phi(x_0)) = \sup_{x \in A} \Re(\phi(x))$.

Notice that $x_0 = x_0 + 0 \in A + U$, so

$$\sup_{x \in A} \Re(\phi(x)) = \Re(\phi(x_0)) \leq \sup_{z \in A+U} \Re(\phi(z)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

□

Corollary 5.27. Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then $(X, \mathcal{T}_P)^*$ separates the points of X .

Proof. Given $x \neq y \in X$.

Take $A := \{x\}$, and $B := \{y\}$, which is closed. They are trivially convex and disjoint. □

Definition 5.10. Let (X, \mathcal{T}) be a topological space, and $A \subseteq X$. The **convex hull** of A is

$$\text{conv}(A) := \left\{ x = \sum_{i=1}^n \lambda_i x_i \mid n \in \mathbb{N}, x_i \in A, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The **closed convex hull** of A is $\overline{\text{conv}(A)}$.

Proposition 5.28. *The convex hull is the smallest convex set containing A , and the closed convex hull is the smallest closed convex set containing A .*

Definition 5.11. Let (X, \mathcal{T}) be a topological space, and $A \subseteq X, \phi \in (X, \mathcal{T})$. The **closed half square containing A under ϕ** is

$$H_{\phi, A} := \{x \in X : \Re(\phi(x)) \leq \alpha_{\phi, A}\},$$

where $\alpha_{\phi, A} := \sup_{x \in A} \Re(\phi(x))$.

Proposition 5.29. *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, then for all $A \subseteq X$, we have*

$$\overline{\text{conv}(A)} = \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}.$$

Proof. For any $\phi \in (X, \mathcal{T}_P)^*$, we have that $H_{\phi, A}$ is convex and closed, and $A \subseteq H_{\phi, A}$, so $\overline{\text{conv}(A)} \subseteq H_{\phi, A}$. Thus,

$$\overline{\text{conv}(A)} \subseteq \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}.$$

On the other hand, suppose for contradiction that $\bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \setminus \overline{\text{conv}(A)} \neq \emptyset$.

Take any $x \in \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \setminus \overline{\text{conv}(A)}$, we have that $\{x\}$ is compact, and $\overline{\text{conv}(A)}$ is closed.

By the Second Separation Theorem, there is $\phi \in (X, \mathcal{T}_P)^*$ such that $\Re(\phi(x)) < \inf_{y \in \overline{\text{conv}(A)}} \Re(\phi(y))$.

We thus have $\Re(-\phi(x)) > \sup_{y \in \overline{\text{conv}(A)}} \Re(-\phi(y)) \geq \sup_{y \in A} \Re(-\phi(y))$, so $x \notin H_{-\phi, A}$, which contradicts $x \in \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A}$. \square

Corollary 5.30. *Suppose X is a vector space with two Locally Convex Topologies $\mathcal{T}_P, \mathcal{T}_{P'}$. Suppose $(X, \mathcal{T}_P)^* = (X, \mathcal{T}_{P'})^*$, then $(X, \mathcal{T}_P), (X, \mathcal{T}_{P'})$ have the same closed convex sets.*

Proof. Suppose A is convex and closed in (X, \mathcal{T}_P) , then

$$\begin{aligned} A &= \overline{\text{conv}(A)}^{\mathcal{T}_P} \\ &= \bigcap_{\phi \in (X, \mathcal{T}_P)^*} H_{\phi, A} \\ &= \bigcap_{\phi \in (X, \mathcal{T}_{P'})^*} H_{\phi, A} \\ &= \overline{\text{conv}(A)}^{\mathcal{T}_{P'}} \\ &= \overline{A}^{\mathcal{T}_{P'}}. \end{aligned}$$

Thus, A is convex and closed in $(X, \mathcal{T}_{P'})$. \square

5.5 Weakly Closeness and Compactness

Proposition 5.31. *Let $(X, \|\cdot\|)$ be a Normed Space. Suppose $A \subseteq X$ is closed in (X, \mathcal{T}_Y) , where $Y \subseteq X$ separates the points, then A is normed closed.*

Proof. Suppose a net $(x_\lambda)_{\lambda \in \Lambda}$ in A converges to $x \in X$, we must have $\phi(x_\lambda) \rightarrow \phi(x)$ for all $\phi \in X^*$.

Thus $x_\lambda \rightarrow x$ in (X, \mathcal{T}_Y) , which means $x \in A$. \square

The converse is not in general true.

Example 5.5.1. Consider $(C[0, 1], \|\cdot\|_\infty)$, with the dual space $M([0, 1])$ (the complex Borel measures on $[0, 1]$).

For $n \geq 1$, let $f_n := \begin{cases} 1 - 2nx, & 0 \leq x \leq \frac{1}{2n} \\ 2nx - 1, & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 1, & x \geq \frac{1}{n} \end{cases}$. We have that $f_n \in C[0, 1]$, and $f_n(x) \rightarrow 1$ pointwise for all $x \in [0, 1]$.

$x \in [0, 1]$.

By Lebesgue's Dominated Convergence Theorem, for any complex Borel measure $\mu \in M([0, 1])$, we have $\int_0^1 f_n d\mu \rightarrow \int_0^1 1 d\mu$.

Thus, $f_n \rightharpoonup f$.

However, for any $n \geq 1$, $\|f_n - 1\|_n = 1$.

Proposition 5.32. *Let $(X, \|\cdot\|)$ be a Normed Space, then $(X, \|\cdot\|)$ and (X, \mathcal{T}_{X^*}) have the same closed convex set.*

Proof. This follows from that $(X, \mathcal{T}_{X^*})^* = X^*$. \square

Proposition 5.33. *Let $(X, \|\cdot\|)$ be a Normed Space, then the closed balls are weak-* closed. Namely, the closed balls in $(X^*, \|\cdot\|_{X^*})$ are closed in (X^*, \mathcal{T}_X) .*

Proof. For any $\phi_0 \in X^*, r > 0$, then

$$\begin{aligned} \bar{B}^{(X^*, \|\cdot\|_{X^*})}(\phi_0, r) &= \{\phi \in X^* : \|\phi - \phi_0\|_{X^*} \leq r\} \\ &= \{\phi \in X^* : |(\phi - \phi_0)(x)| \leq r, \forall x \in X \text{ such that } \|x\| \leq 1\} \\ &= \bigcap_{x \in X \text{ such that } \|x\| \leq 1} \{\phi \in X^* : |\hat{x}(\phi - \phi_0)| \leq r\}, \end{aligned}$$

which is an intersection of weak-* closed sets. \square

Theorem 5.34 (Goldstine). *Let $(X, \|\cdot\|)$ be a Banach Space, then $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weak-* dense in $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$, with the weak-* topology $(X^{**}, \mathcal{T}_{X^*})$. In particular, X is weak-* dense in X^{**} .*

Proof. If $x_\lambda \rightharpoonup \psi \in X^{**}$, with $(x_\lambda)_{\lambda \in \Lambda} \subset \bar{B}^{(X, \|\cdot\|)}(0, 1)$, for any $\phi \in X^*$, we have

$$\begin{aligned} |\psi(\phi)| &= \left| \lim_\lambda \hat{x}_\lambda(\phi) \right| \\ &= \lim_\lambda |\phi(x_\lambda)| \\ &\leq \|\phi\|. \end{aligned}$$

Thus, $\|\psi\| \leq 1$ and $\psi \in \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$.

This shows $\overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}} \subseteq \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$.

Suppose for contradiction that there is $\psi \in \bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) \setminus \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}}$.

Since $\{\psi\}$ is convex and compact, and $\overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}}$ is closed in $(X^{**}, \mathcal{T}_{X^*})$, by the second separation theorem, we can find $\phi \in (X^{**}, \mathcal{T}_{X^*})^* = X^*$, such that

$$\Re(\psi(\phi)) < \inf_{\xi \in \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}}} \Re(\xi(\phi)).$$

We thus have

$$\begin{aligned} \|\phi\| &= \sup_{x \in \bar{B}^{(X, \|\cdot\|)}(0, 1)} \Re(-\phi(x)) \\ &\leq \sup_{\xi \in \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}}} \Re(\xi(\phi)) \\ &< \Re(\psi(-\phi)) \\ &\leq \|\psi\| \|\phi\| \\ &\leq \|\phi\|, \end{aligned}$$

which is a contradiction.

Thus,

$$\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}}$$

\square

Theorem 5.35 (Banach-Alaoglu). Let $(X, \|\cdot\|)$ be a Normed Vector Space, then $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is compact in the weak-* topology.

Proof. Consider the set

$$\begin{aligned} D &:= \{f : X \rightarrow \mathbb{F} \mid |f(x)| \leq \|x\|\} \\ &= \prod_{x \in X} D_x, \end{aligned}$$

where $D_x := \{z \in \mathbb{F} : |z| \leq \|x\|\}$.

By Tychonoff's Theorem, D is compact and Hausdorff in the product topology.

Also, $\bar{B}^{(X^*, \|\cdot\|)}(0, 1) \subseteq D$.

In addition, $f_\lambda \rightarrow f$ in the product topology if and only if $\forall x \in X$, $f_\lambda(x) \rightarrow f(x)$ if and only if $f_\lambda \rightharpoonup f$ in weak-* topology.

Thus, the weak-* topology and the product topology are the same.

Since $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is weak-* closed, it is compact. \square

Corollary 5.36. Let $(X, \|\cdot\|)$ be a Banach Space, then X is reflexive if and only if $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weakly compact in (X, \mathcal{T}_{X^*}) .

Proof. (\implies) :

If $X^{**} = X$, then the weak-* topology \mathcal{T}_{X^*} on X^{**} is the same as the weak topology \mathcal{T}_{X^*} on X .

Since $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1)$ is weakly compact in $(X^{**}, \mathcal{T}_{X^*})$ by the Banach-Alaoglu Theorem, we have that $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weakly compact in (X, \mathcal{T}_{X^*}) .

(\impliedby) :

By the Goldstine Theorem, we have that $\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}}$.

Since $\bar{B}^{(X, \|\cdot\|)}(0, 1)$ is weakly compact in (X, \mathcal{T}_{X^*}) , and $i : (X, \mathcal{T}_{X^*}) \rightarrow (X^{**}, \mathcal{T}_{X^*})$ is continuous, it is also weakly compact in (X, \mathcal{T}_{X^*}) . Thus,

$$\bar{B}^{(X^{**}, \|\cdot\|)}(0, 1) = \overline{\bar{B}^{(X, \|\cdot\|)}(0, 1)}^{\text{weak}} = \bar{B}^{(X, \|\cdot\|)}(0, 1).$$

Thus, $X = X^{**}$. \square

5.6 Extreme Points

Definition 5.12. Let X be a Vector Space, and let $\emptyset \neq A \subseteq X$ be convex. A **face** of A is some convex $\emptyset \neq F \subseteq A$, such that for all $t \in (0, 1), x, y \in A$, if $(1-t)x + ty \in F$, then $x, y \in F$.

If a face $F = \{z\}$, then we call z an **extreme point** of A .

$\text{Ext}(A)$ is the set of extreme points of A .

Proposition 5.37. Suppose F is a face for A , and F' is a face for F , then F' is also a face for A .

Proof. Consider any $x, y \in A, t \in (0, 1)$, with $(1-t)x + ty \in F' \subseteq F$. Since F is a face of A , $x, y \in F$. Since F' is a face of F , $x, y \in F'$. Thus, F' is a face of A . \square

Example 5.6.1. Let $X = L^1([0, 1])$ with the Lebesgue measure, and consider $A := \bar{B}^{(X, \|\cdot\|_1)}(0, 1)$. For any $f \in A$ such that $\|f\|_{L^1([0, 1])} = a \neq 0$, we can pick $t_0 \in (0, 1)$, such that $\int_0^{t_0} |f| dx = \int_{t_0}^1 |f| dx = \frac{1}{2}a$.

Now take $g := 2f\chi_{[0, t_0]}, h := 2f\chi_{[t_0, 1]}$, we have that $f = \frac{1}{2}g + \frac{1}{2}h$, and $g, h \in A$.

Thus, $f \notin \text{Ext}(A)$.

Thus, $\bar{B}(0, 1)$ has no extreme points.

Proposition 5.38. Let (X, \mathcal{T}) be a Topological Vector Space, and $\emptyset \neq K \subseteq X$ be convex and compact, the for any $\phi \in (X, \mathcal{T})^*$,

$$F_\phi := \arg \min_{x \in K} (\Re(\phi(x)))$$

is always a closed face of K .

Proof. Let $\alpha_\phi := \inf_{x \in K} (\Re(\phi(x)))$.

Since K is compact, and ϕ is continuous, α_ϕ is achieved.

Thus $F_\phi = \{x \in K : \Re(\phi(x)) = \alpha_\phi\} \neq \emptyset$.

For any $(x_\lambda)_{\lambda \in \Lambda}$ in F_ϕ , such that $x_\lambda \rightarrow x \in K$, since ϕ is continuous, we have that

$$\phi(x) = \lim_\lambda \phi(x_\lambda) = \lim_\lambda \alpha_\phi = \alpha_\phi.$$

Thus, $x \in F_\phi$, so F_ϕ is closed.

For any $x, y \in F_\phi, t \in (0, 1)$, we have

$$\phi((1-t)x + ty) = (1-t)\phi(x) + t\phi(y) = (1-t)\alpha_\phi + t\alpha_\phi = \alpha_\phi.$$

Thus, $(1-t)x + ty \in F_\phi$, so F_ϕ is convex.

For any $x, y \in K, t \in (0, 1)$, if $\phi((1-t)x + ty) \in F_\phi$, we must have

$$\begin{aligned} \alpha_\phi &= \phi((1-t)x + ty) \\ &= (1-t)\phi(x) + t\phi(y) \\ &\geq (1-t)\alpha_\phi + t\alpha_\phi \\ &= \alpha_\phi. \end{aligned}$$

This forces the inequality to be equality, and $\phi(x) = \phi(y) = \alpha_\phi$. Thus, $x, y \in F_\phi$, so F_ϕ is a face of K . \square

Theorem 5.39 (Krein-Milman). *Let (X, \mathcal{T}_P) be a Locally Convex Topological Vector Space, and $\emptyset \neq K \subseteq X$ be convex and compact, then*

$$K = \overline{\text{conv}(\text{Ext}(K))}.$$

Proof. Since (X, \mathcal{T}_P) is Hausdorff, K is compact means K is closed.

Thus $K \supseteq \text{conv}(\text{Ext}(K))$ since it is a closed convex set containing $\text{Ext}(K)$.

On the other hand, we firstly show that for any closed face $\emptyset \neq F_0 \subseteq K$, we have $\text{Ext}(K) \cap F_0 \neq \emptyset$.

Let $\Lambda := \{F \subseteq F_0 : F \text{ is a closed face of } F_0\}$, with the partial order $F_1 \leq F_2$ if $F_2 \subseteq F_1$.

Let $\mathcal{C} = \{F_\alpha\}_{\alpha \in A}$ be a chain in Λ .

Let $F := \bigcap_{\alpha \in A} F_\alpha$.

Since K is compact, by FIP, $F \neq \emptyset$, and it is closed and convex.

Also, if $x, y \in F_0, t \in (0, 1)$, and $(1-t)x + ty \in F$, we have $(1-t)x + ty \in F_\alpha$ for some $\alpha \in A$.

Since F_α is a face of F_0 , we must have $x, y \in F_\alpha \subseteq F$.

Thus, $F \in \Gamma$, and it's clear that F is an upper bound for \mathcal{C} .

By Zorn's lemma, there is a maximal element F of Γ . Notice that it is also a face of K .

Suppose for contradiction, that there are $x \neq y \in F$, then by the second separation theorem, there is $\phi \in (X, \mathcal{T}_P)^*$, such that $\Re(\phi(x)) \neq \Re(\phi(y))$.

Now let $F_\phi := \arg \min_{x \in F} (\Re(\phi(x)))$.

Since F is a closed subset of compact K , it is compact. By the proposition, F_ϕ is a closed face of F , and thus a closed face of F_0 . Thus $F_\phi \in \Gamma$.

By maximality of F , we must have $F_\phi = F$, which means $\phi(x) = \phi(y) = \min_{x \in F} (\Re(\phi(x)))$, a contradiction with the choice of ϕ .

Thus F only has one point x , so $x \in \text{Ext}(K) \cap F_0 \neq \emptyset$.

In particular, since K is a closed face for itself, $\text{Ext}(K) \neq \emptyset$.

Now suppose for contradiction that there is $x_0 \in K \setminus B$, where $B := \overline{\text{conv}(\text{Ext}(K))}$.

By the Second Separation Theorem, there is $\phi \in (X, \mathcal{T}_P)^*, t \in \mathbb{R}$ such that

$$\Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y)).$$

In particular, $\min_{x \in K} \Re(\phi(x)) < \Re(\phi(x_0)) < t \leq \inf_{y \in B} \Re(\phi(y))$.

Thus, $F_\phi \cap B = \emptyset$ for $F_\phi := \arg \min_{x \in K} (\Re(\phi(x)))$.

However, F_ϕ is a closed face, so $F_\phi \cap \text{Ext}(K) \neq \emptyset$, thus a contradiction. \square

Corollary 5.40. Let $(X, \|\cdot\|)$ be a Normed Vector Space, then $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is the weak-* closed convex hull of its extreme points.

Proof. By the Banach-Alaoglu Theorem, $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is compact in the weak-* topology. The convexity is easy to see. Indeed, for any $f, g \in \bar{B}^{(X^*, \|\cdot\|)}(0, 1)$, $t \in (0, 1)$, we have $\|tf + (1-t)g\| \leq t\|f\| + (1-t)\|g\| \leq t + (1-t) = 1$, so $tf + (1-t)g \in \bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ and $\bar{B}^{(X^*, \|\cdot\|)}(0, 1)$ is convex. \square

Corollary 5.41. $L^1([0, 1])$ with the Lebesgue measure is not a dual space.

5.6.1 Probability Measure

See more about the space of measures in my Pmath651 Measure Theory Notes.

Definition 5.13. Let X be a locally compact Hausdorff space, we define $M(X) := \{\mu : \text{complex Radon measure}\}$, and $\|\mu\|_{M(X)} := |\mu|(X)$, where $|\mu|$ is the total variation.

Theorem 5.42. $(M(X), \|\cdot\|_{M(X)}) \cong (C_0(X), \|\cdot\|_\infty)^*$ isometrically by $\mu(f) := \int_X f d\mu$.

Definition 5.14. The probability measures on X is

$$P(X) := \{\mu \in M(X) | \mu \geq 0, \mu(X) = 1\}.$$

The **Dirac measures** are $\delta_x : f \mapsto f(x)$ for $x \in X, f \in C(X)$.

Notice that $P(X)$ is clearly convex by linearity. Also, when X is compact, $C(X) = C_c(X) = C_0(X)$.

Lemma 5.43. Let X be a compact Hausdorff space, suppose $\mu \in P(X), x \in X$, and $\ker(\delta_x) \supseteq \ker(\mu)$, then $\mu = \delta_x$.

Proof. For any $g \in C(X)$, we have $\mu(g - \mu(g)\mathbb{1}) = \mu(g) - \mu(g)\mu(\mathbb{1}) = 0$, so $g - \mu(g)\mathbb{1} \in \ker(\mu)$. Thus, $\delta_x(g - \mu(g)\mathbb{1}) = 0$, which means $\delta_x(g) = \mu(g) \cdot 1 = \mu(g)$.

Thus $\mu = \delta_x$. \square

Proposition 5.44. Let X be a compact Hausdorff space, then

$$\text{Ext}(P(X)) = \{\delta_x : x \in X\}.$$

Proof. Let $\mu \in \text{Ext}(P(X))$.

Fix any $0 \leq f < 1$ in $C(X)$.

Let $\lambda := \mu(f) \in [0, 1]$.

If $0 < \lambda < 1$, we can define $\mu_1(g) := \frac{1}{\lambda}\mu(fg) = \frac{1}{\lambda} \int_X g f d\mu$, and $\mu_2(g) := \frac{1}{1-\lambda}\mu((1-f)g) = \frac{1}{1-\lambda} \int_X g(1-f) d\mu$. We can check that $\mu_1, \mu_2 \in P(X)$, and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$.

Since $\mu \in \text{Ext}(P(X))$, we must have $\mu = \mu_1 = \mu_2$.

Thus, for any $g \in C(X)$,

$$\begin{aligned} \mu(g) &= \mu_1(g) \\ &= \frac{\mu(fg)}{\lambda} \\ &= \frac{\mu(fg)}{\mu(f)}. \end{aligned}$$

Thus, $\mu(fg) = \mu(f)\mu(g)$.

Now suppose $\mu(f) = 0$, we have

$$\begin{aligned} 0 &\leq |\mu(fg)| \\ &= \int_X fgd\mu \\ &\leq \int_X |fg|d\mu \\ &= \int_X f|g|d\mu \\ &\leq \|g\|_\infty \int_X f|g|d\mu \\ &= 0. \end{aligned}$$

Which means $\mu(fg) = 0 = \mu(f)\mu(g)$.

Thus, $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in C(X)$ such that $0 \leq f < 1$.

Since μ is linear, and $\text{Span}\{f \in C(X) : 0 \leq f < 1\} = C(X)$, we have that for all $f, g \in C(X)$,

$$\mu(fg) = \mu(f)\mu(g).$$

We claim that $\exists x \in X$, such that $\ker(\delta_x) \supseteq \ker(\mu)$.

Indeed, suppose for contradiction that $\forall x \in X$, there is $f_x \in C(X)$, such that $f_x \in \ker(\mu) \setminus \ker(\delta_x)$.

Namely, $f_x(x) \neq 0, \mu(f_x) = 0$.

Thus $X = \bigcup_{x \in X} \{y : f_x(y) \neq 0\}$ is an open cover. Since X is compact, there is a finite subcover

$$X = \bigcup_{i=1}^n \{y : f_{x_i}(y) \neq 0\}.$$

Define $f := \sum_{i=1}^n |f_{x_i}|^2 \in C(X)$. Notice that $f > 0$, which means $\frac{1}{f} \in C(X)$.
Now we have

$$\begin{aligned} 1 &= \mu(\mathbb{1}) \\ &= \mu\left(\frac{f}{f}\right) \\ &= \mu(f)\mu\left(\frac{1}{f}\right) \\ &= \sum_{i=1}^n \mu(|f_{x_i}|^2)\mu\left(\frac{1}{f}\right) \\ &= \sum_{i=1}^n \mu(f_{x_i})\mu(\overline{f_{x_i}})\mu\left(\frac{1}{f}\right) \\ &= 0\mu\left(\frac{1}{f}\right) \\ &= 0, \end{aligned}$$

which is a contradiction.

This proves the claim.

By the lemma,

$$\text{Ext}(P(X)) \subseteq \{\delta_x : x \in X\}.$$

Now given any $x \in X$, suppose $\delta_x = \lambda\mu + (1 - \lambda)\nu$ for some $\mu, \nu \in P(X), \lambda \in (0, 1)$. For any $f \in C(X)$, we

have

$$\begin{aligned}
|\delta_x(f)| &= |f(x)| \\
&= \delta_x(|f|) \\
&= \lambda\mu(|f|) + (1 - \lambda)\nu(|f|) \\
&\geq \lambda\mu(|f|) \\
&= \lambda \int_X |f| d\mu \\
&= \lambda \left| \int_X f d\mu \right| \\
&\geq \lambda |\mu(f)|.
\end{aligned}$$

Suppose $f \in \ker(\delta_x)$, we will have $\lambda|\mu(f)| \leq |\delta_x(f)| = 0$, so $\mu(f) = 0$. Thus $\ker(\delta_x) \subseteq \ker(\mu)$.

By lemma, $\delta_x = \mu$, so $\delta_x = \lambda\delta_x + (1 - \lambda)\nu$, which make $\nu = \delta_x$ as well. Thus $\delta_x \in \text{Ext}(P(X))$. \square

Corollary 5.45. *Let X be a compact Hausdorff space, then*

$$P(X) = \overline{\text{conv} \{ \delta_x : x \in X \}}^{\text{weak}}.$$

Proof. For any $\mu \in P(X)$, we have $\|\mu\| = |\mu|(X) = \mu(X) = 1$, so

$$P(X) \subseteq \bar{B}^{(M(X), \|\cdot\|)}(0, 1).$$

By Banach-Alaoglu, $\bar{B}^{(M(X), \|\cdot\|)}(0, 1)$ is weak-* compact.

Suppose there are $(\mu_\lambda)_{\lambda \in \Lambda}$ in $P(X)$ with $\mu_\lambda \rightharpoonup \mu \in M(X)$, we have that

$$\mu(X) = \mu(\mathbb{1}) = \lim_{\lambda \in \Lambda} \mu_\lambda(\mathbb{1}) = 1,$$

since $\mathbb{1} \in C(X)$. We can also show μ is a positive measure, so $\mu \in P(X)$.

This shows $P(X)$ is weak-* closed, so it is also weak-* compact. The result follows from the Krein-Milman Theorem. \square

Remark. When X is not compact, $P(X)$ is not weak-* closed.

Example 5.6.2. Consider $\delta_n \in P(\mathbb{R})$, then $\delta_n \rightharpoonup 0$ in $\bar{B}^{M(\mathbb{R})}(0, 1)^{\|\cdot\|}$, but $0 \notin P(\mathbb{R})$.

Nevertheless, when X is a Locally Compact Hausdorff space, with a variation of the proof, we still have

$$\text{Ext}(P(X)) = \{ \delta_x | x \in X \},$$

and

$$\text{Ext} \left(\bar{B}^{M(X)}(0, 1)^{\|\cdot\|} \right) = \{ z\delta_x : x \in X, z \in \mathbb{C} \text{ such that } \|z\| = 1 \}.$$

5.6.2 Stone-Weierstrass Theorem

Theorem 5.46. *Let X be a locally compact Hausdorff Space, and $A \subseteq C_0(X, \mathbb{R})$, such that*

1. *A is closed under $\|\cdot\|_\infty$,*
2. *A is a sub-algebra of $C_0(X, \mathbb{R})$,*
3. *A separates points (i.e. $\forall x \neq y \in X, \exists f \in A$ such that $f(x) \neq f(y)$),*
4. *A vanishes nowhere (i.e. $\forall x \in X, \exists f \in A$, such that $f(x) \neq 0$),*

then $A = C_0(X, \mathbb{R})$.

Proof. Suppose for contradiction that $A \subsetneq C_0(X, \mathbb{R})$.

By 1. and a corollary of the Hahn-Banach theorem, there is $\mu \neq 0$ in $A^\perp = \{\mu \in M(X) : \mu(f) = 0 \ \forall f \in A\}$. Thus,

$$K := \bar{B}^{(M(X), \|\cdot\|)}(0, 1) \cap A^\perp$$

is a non-empty weak-* compact convex set. By the Krein-Milman theorem, there is $\mu \in \text{Ext}(K)$.

We claim that

$$\text{Supp}(\mu) := \{x \in X : \forall U \in \mathcal{O}(X), |\mu|(U) > 0\}$$

is a singleton $\{x\}$ for some $x \in X$.

Suppose for contradiction that there are $x \neq y \in \text{Supp}(\mu)$. By 3., there is $f \in A$ such that $f(x) \neq f(y)$. Notice that f is not $|\mu|$ -a.e. constant since it is continuous.

Now find $a, b > 0$ such that $g(x) := a(f(x) + b)$ satisfies $g(x) \geq 0$ and $\int_X g d|\mu| = 1$. Suppose that $\|g\|_\infty \leq 1$, then $0 \leq g \leq 1$ a.e., which means

$$\begin{aligned} \int_X |1 - g| d|\mu| &= \int_X (1 - g) d|\mu| \\ &= |\mu|(X) - \int_X g d|\mu| \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Thus, $1 - g = 0$ $|\mu|$ -a.e., which means $g = 1$ $|\mu|$ -a.e. and it contradicts with f not being constant a.e.. Thus, we must have $\|g\|_\infty > 1$, so

$$\lambda := \frac{1}{\|g\|_\infty} \in (0, 1).$$

Let $\mu_1, \mu_2 \in M(X)$ be as

$$\mu_1(h) := \mu(gh) = \int_X g h d\mu, \quad \mu_2(h) := \mu\left(\frac{1 - \lambda g}{1 - \lambda} h\right) = \int_X \frac{1 - \lambda g}{1 - \lambda} h d\mu.$$

Notice that $\forall h \in A$, $hg = a(f + b)h = afh + abh \in A$, so $\mu_1, \mu_2 \in A^\perp$. We can check that

$$\begin{aligned} |\mu_1|(X) &= \int_X d|\mu_1| \\ &= \int_X g d|\mu| \\ &= 1, \\ |\mu_2|(X) &= \int_X d|\mu_2| \\ &= \int_X \frac{1 - \lambda g}{1 - \lambda} d|\mu| \\ &= \frac{1 - \lambda}{1 - \lambda} \\ &= 1. \end{aligned}$$

Thus, $\mu_1, \mu_2 \in K$, with $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$. Since $\mu \in \text{Ext}(K)$, we must have $\mu_1 = \mu_2 = \mu$, which means $g = 1$ $|\mu|$ -a.e., a contradiction.

This proves that $\text{Supp}(\mu) = \{x\}$ for some $x \in X$. Thus, $\mu = z\delta_x$ for some $z \in \mathbb{C}$. Also, $|z| = |\mu|(X) = 1$. Since $\mu \in A^\perp$, for any $h \in A$, we must have $zh(x) = \int_X h d\mu = \mu(h) = 0$ which means $h(x) = 0$ and A vanishes at x . \square

6 Spectral Theory

6.1 Adjoint Operators

6.1.1 Adjoint Operators on Normed Spaces

Definition 6.1. Let X, Y be normed spaces, the **adjoint operator** or **dual operator** of a linear operator $A : X \rightarrow Y$ is

$$A^* : Y^* \rightarrow X^*; \quad f \mapsto f \circ A,$$

namely,

$$\forall x \in X, f \in Y^*, \quad \langle x | A^* f \rangle := \langle Ax | f \rangle.$$

Proposition 6.1. Let X, Y, Z be normed spaces, $S \in B(X, Y)$, $T \in B(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.

Proof. Consider any $f \in Z^*$, and any $x \in X$, we have

$$\begin{aligned} (T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx) \\ &= (f)(S(T(x))) \\ &= (f \circ (S \circ T))(x) \\ &= (S \circ T)^*(f)(x). \end{aligned}$$

Thus $(T^* \circ S^*)(f) = (S \circ T)^*(f)$. □

Proposition 6.2. Let X, Y be normed spaces, and $T \in B(X, Y)$, then

1. $\|T\| = \|T^*\|$, so $T^* \in B(Y^*, X^*)$;
2. The map $T \mapsto T^*$ is linear;
3. T^* is weak-* continuous; namely, if a net $(\phi_\lambda)_{\lambda \in \Lambda}$ converges to $\phi \in Y$ in weak-* topology on Y^* , we have that $T^* \phi_\lambda$ converges to $T^* \phi$ in weak-* topology on X^* ;
4. For $\mathbb{1}_X : x \mapsto x$, we have $(\mathbb{1}_X)^* = \mathbb{1}_{X^*}$;
5. $T^{**} \in B(X^{**}, Y^{**})$ satisfies $T^{**}|_X = T$.

Proof. 1. We have

$$\begin{aligned} \|T^*\| &= \sup_{\phi \in Y^*, \|\phi\| \leq 1} \|T^* \phi\|_{X^*} \\ &= \sup_{\phi \in Y^*, \|\phi\| \leq 1} \sup_{x \in X, \|x\| \leq 1} |\langle x | T^* \phi \rangle| \\ &= \sup_{x \in X, \|x\| \leq 1} \sup_{\phi \in Y^*, \|\phi\| \leq 1} |\langle Tx | \phi \rangle| \\ &= \sup_{x \in X, \|x\| \leq 1} \|Tx\| \\ &= \|T\|. \end{aligned}$$

2. It is obvious by definition.

3. For net $(\phi_\lambda)_{\lambda \in \Lambda}$ in Y^* that converges to $\phi \in Y$ in weak-* topology, and any $x \in X$, we have

$$\begin{aligned} \langle x | T^* \phi_\lambda \rangle &= \langle Tx | \phi_\lambda \rangle \\ &\rightarrow \langle Tx | \phi \rangle \\ &= \langle x | T^* \phi \rangle. \end{aligned}$$

Thus, $T^* \phi_\lambda \rightharpoonup T^* \phi$ in weak-* topology in X^* .

4. This is obvious.

5. Consider the canonical embeddings $x \mapsto \hat{x}, y \mapsto \hat{y}$, we want to show $T^{**}\hat{x} = \widehat{Tx}$. Indeed, for any $\phi \in Y^*$

$$\begin{aligned}\langle \phi | T^{**}\hat{x} \rangle &= \langle T^*\phi | \hat{x} \rangle \\ &= \langle x | T^*\phi \rangle \\ &= \langle Tx | \phi \rangle \\ &= \langle \phi | \widehat{Tx} \rangle.\end{aligned}$$

□

Theorem 6.3. Let X, Y be normed spaces, and $S \in B(Y^*, X^*)$, then there is $T \in B(X, Y)$ such that $T^* = S$ if and only if S is weak-* weak-* continuous.

Proof. (\implies) is by the previous proposition.

(\impliedby): Assume S is weak-* weak-* continuous.

Consider $S^* \in B(X^{**}, Y^{**})$, and let $T := S^*|_X \in B(X, Y^{**})$. Fix $x \in X$, let $(\phi_\lambda)_{\lambda \in \Lambda}$ be a net that converges in weak-* topology (Y^*, \mathcal{T}_Y) , then

$$\begin{aligned}\langle \phi_\lambda | S^*\hat{x} \rangle &= \langle S\phi_\lambda | \hat{x} \rangle \\ &\rightarrow \langle S\phi | \hat{x} \rangle \\ &= \langle \phi | S^*\hat{x} \rangle.\end{aligned}$$

Thus, $S^*\hat{x}$ is continuous in weak-* topology (Y^*, \mathcal{T}_Y) , so $S^*\hat{x} \in Y$. This shows we can redefine $T \in B(X, Y)$. Now for any $x \in X, \phi \in Y^*$, we have

$$\begin{aligned}\langle x | T^*\phi \rangle &= \langle Tx | \phi \rangle \\ &= \langle \phi | S^*\hat{x} \rangle \\ &= \langle S\phi | \hat{x} \rangle \\ &= \langle x | S\phi \rangle.\end{aligned}$$

□

Theorem 6.4. Let X, Y be Banach spaces, then $T : X \rightarrow Y$ is bounded if and only if T is weak-weak continuous.

Proof. See A4. □

6.1.2 Hilbert Adjoint Operators

Definition 6.2. Let \mathcal{H} be a Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator, the **Hilbert adjoint operator** of T is $T^\dagger : \mathcal{H} \rightarrow \mathcal{H}$ such that $\langle x, Ty \rangle = \langle T^\dagger x, y \rangle \forall x, y \in \mathcal{H}$.

Proposition 6.5. Let \mathcal{H} be a Hilbert space, and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator, we have $\langle Ty, x \rangle = \langle y, T^\dagger x \rangle \forall x, y \in \mathcal{H}$. Thus, $(T^\dagger)^\dagger = T$.

Proof.

$$\begin{aligned}\langle Ty, x \rangle &= \overline{\langle x, Ty \rangle} \\ &= \overline{\langle T^\dagger x, y \rangle} \\ &= \langle y, T^\dagger x \rangle.\end{aligned}$$

□

Theorem 6.6. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. T^\dagger always exists, and is given by $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$, where $\Phi : \mathcal{H} \rightarrow \mathcal{H}^*$; $u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism, and T^* is the dual operator of T . In addition, T^\dagger is also a bounded linear operator, with $\|T^\dagger\| = \|T\|$.

Proof. $\forall y \in \mathcal{H}$, we have that

$$\begin{aligned}\langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\ &= \langle y | (T^* \circ \Phi)(x) \rangle \\ &= \langle Ty | \Phi(x) \rangle \\ &= \langle x, Ty \rangle.\end{aligned}$$

Linearity follows from Φ, Φ^{-1} being anti-linear, and T^* being linear. We can also compute it directly from the definition. Indeed, consider any $x, y, z \in \mathcal{H}, c \in \mathbb{C}$, we have that

$$\begin{aligned}\langle T^\dagger(x + cz), y \rangle &= \langle x + cz, Ty \rangle \\ &= \langle x, Ty \rangle + \bar{c} \langle z, Ty \rangle \\ &= \langle T^\dagger x, y \rangle + \bar{c} \langle T^\dagger z, y \rangle \\ &= \langle T^\dagger x + cT^\dagger z, y \rangle.\end{aligned}$$

Since this holds for any $y \in \mathcal{H}$, we have that $T^\dagger(x + cz) = T^\dagger x + cT^\dagger z$, and thus T^\dagger is linear.

Also,

$$\|T^\dagger\| \leq \|\Phi^{-1}\| \|T\| \|\Phi\| = \|T\|.$$

We can also compute this directly from the definition. Indeed, given any $x \in \mathcal{H}$, we have that

$$\begin{aligned}\|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\ &= \langle x, TT^\dagger x \rangle \\ &\leq \|x\| \|TT^\dagger x\| \\ &\leq \|x\| \|T\| \|T^\dagger x\| \\ &\implies \\ \|T^\dagger x\| &\leq \|x\| \|T\| \\ &\implies \\ \|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\ &\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{\|x\| \|T\|}{\|x\|} \\ &= \|T\|.\end{aligned}$$

Thus T^\dagger is also a bounded linear operator, with

$$\|T\| = \|(T^\dagger)^\dagger\| \leq \|T^\dagger\| \leq \|T\|.$$

□

Remark. $\forall x, y \in \mathcal{H}$, $\langle (Tx)^\dagger | y \rangle = \langle Tx, y \rangle = \langle x, T^\dagger y \rangle = \langle x^\dagger | T^\dagger y \rangle$. We thus abuse the notation, and write $(Tx)^\dagger = \langle x | T^\dagger$

Corollary 6.7. Let \mathcal{H} be a Hilbert space, and $S, T \in B(\mathcal{H})$ be bounded linear operators. We have that $(S \circ T)^\dagger = T^\dagger \circ S^\dagger$. Also, for any $\alpha \in \mathbb{C}$, we have $(\alpha T + S)^\dagger = \bar{\alpha} T^\dagger + S^\dagger$.

Proposition 6.8. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. We have

$$\|T\|^2 = \|TT^\dagger\| = \|T^\dagger T\|.$$

Proof. We have $\|TT^\dagger\| \leq \|T\|\|T^\dagger\| = \|T\|^2$.

Now for any $x \in \mathcal{H}$, we have

$$\begin{aligned} \|T^\dagger x\|^2 &= \langle T^\dagger x, T^\dagger x \rangle \\ &= \langle x, TT^\dagger x \rangle \\ &\leq \|x\| \|TT^\dagger x\| \\ &\leq \|x\| \|TT^\dagger\| \|x\| \\ &= \|TT^\dagger\| \|x\|^2; \\ &\implies \\ \|T^\dagger x\| &\leq \sqrt{\|TT^\dagger\|} \|x\| \\ &\implies \\ \|T^\dagger\| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{\|T^\dagger x\|}{\|x\|} \\ &\leq \sqrt{\|TT^\dagger\|} \\ &\implies \\ \|T^\dagger\|^2 &\leq \|TT^\dagger\|. \end{aligned}$$

□

Corollary 6.9. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a bounded linear operator. We have that $T = 0$ if and only if $T^\dagger T = 0$.

There is a deep analogy between operators $B(\mathcal{H})$ and algebra of the form $(\ell^\infty(X), \|\cdot\|_\infty)$ for a set X , with the pointwise multiplication and involution $f^*(x) := \overline{f(x)}$.

Example 6.1.1. Consider $\pi : \ell^\infty(X) \rightarrow B(\ell^2(X))$ by $f \mapsto M_f$, where the **multiplication operator** $M_f(\phi)(x) := f(x)\phi(x)$ for all $\phi \in \ell^2(X), x \in X$. We claim that it is a unital isometric homomorphism. Indeed, for any $\phi \in \ell^2(X)$, we have

$$\begin{aligned} \pi(\mathbb{1})(\phi)(x) &= 1 \cdot \phi(x) \\ &= \phi(x), \end{aligned}$$

so $\pi(\mathbb{1})(\phi) = \phi$ and $\pi(\mathbb{1}) = \mathbb{1}$. Also, for any $f, g \in \ell^\infty(X)$,

$$\begin{aligned} (\pi(fg)(\phi))(x) &= (fg)(x)\phi(x) \\ &= f(x)g(x)\phi(x) \\ &= f(x)(\pi(g)(\phi))(x) \\ &= \pi(f)(\pi(g)(\phi))(x) \\ &= ((\pi(f) \circ \pi(g))(\phi))(x). \end{aligned}$$

Thus, $\pi(fg) = \pi(f) \circ \pi(g)$. Also, for any $\phi, \eta \in X$, we have

$$\begin{aligned}\langle \pi(f^*)(\phi), \eta \rangle &= \sum_{x \in X} \pi(f^*)(\phi)(x) \overline{\eta(x)} \\ &= \sum_{x \in X} f^*(x) \phi(x) \overline{\eta(x)} \\ &= \sum_{x \in X} \phi(x) \overline{f(x) \eta(x)} \\ &= \sum_{x \in X} \phi(x) \overline{\pi(f)(\eta)(x)} \\ &= \langle \phi, \pi(f)(\eta) \rangle \\ &= \langle \pi(f)^*(\phi), \eta \rangle.\end{aligned}$$

Thus, $\pi(f^*) = \pi(f)^*$. Lastly, we have that

$$\begin{aligned}\|\pi(f)(\phi)\|^2 &= \langle \pi(f)(\phi), \pi(f)(\phi) \rangle \\ &= \sum_{x \in X} |\pi(f)(\phi)(x)|^2 \\ &= \sum_{x \in X} |f(x) \phi(x)|^2 \\ &= \sum_{x \in X} |f(x)|^2 |\phi(x)|^2 \\ &\leq \|f\|_\infty^2 \left(\sum_{x \in X} |\phi(x)|^2 \right) \\ &= \|f\|_\infty^2 \|\phi\|^2.\end{aligned}$$

Thus, $\|\pi(f)\| \leq \|f\|_\infty$. On the other hand, $\delta_x \in \ell^2(X)$ for any x , and

$$\begin{aligned}\|f\|_\infty^2 &= \sup_{x \in X} |f(x)|^2 \\ &= \sup_{x \in X} \langle \pi(f)(\delta_x), \pi(f)(\delta_x) \rangle \\ &= \sup_{x \in X} \|\pi(f)(\delta_x)\|^2 \\ &\leq \sup_{\phi \in \ell^2(X)} \|\pi(f)(\phi)\|^2 \\ &= \|\pi(f)\|^2.\end{aligned}$$

Thus, $\|f\|_\infty = \|\pi(f)\|$.

Indeed, when we take $X = \{1, \dots, n\}$, then $\ell^2(X) = \mathbb{C}^n$, and $\pi : \ell^\infty(X) \rightarrow B(\ell^2(X)) \cong M_n(\mathbb{C})$, and $\pi(\ell^\infty(X)) = D_n$, the diagonal matrices.

Proposition 6.10. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$. We have*

$$\ker(T^\dagger) = \text{Im}(T)^\perp.$$

Proof. For any $x \in \ker(T^\dagger), Ty \in \text{Im}(T)$, we have

$$\begin{aligned}\langle x, Ty \rangle &= \langle T^\dagger x, y \rangle \\ &= \langle 0, y \rangle \\ &= 0.\end{aligned}$$

Thus, $x \in \text{Im}(T)^\perp$, and we have $\ker(T^\dagger) \subseteq \text{Im}(T)^\perp$.

On the other hand, for any $x \in \text{Im}(T)^\perp, y \in \mathcal{H}$, we have

$$\begin{aligned}\langle T^\dagger x, y \rangle &= \langle x, Ty \rangle \\ &= 0.\end{aligned}$$

Thus, $T^\dagger x = 0$, so $x \in \ker(T^\dagger)$. We thus have $\text{Im}(T)^\perp \subseteq \ker(T^\dagger)$. \square

Proposition 6.11. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$. Suppose $T^{-1} \in B(\mathcal{H})$ exists, then $(T^\dagger)^{-1} \in B(\mathcal{H})$ exists, and*

$$(T^\dagger)^{-1} = (T^{-1})^\dagger.$$

Proof.

$$\begin{aligned}TT^{-1} &= \mathbb{1} = T^{-1}T \\ (TT^{-1})^\dagger &= \mathbb{1}^\dagger = (T^{-1}T)^\dagger \\ (T^{-1})^\dagger T^\dagger &= \mathbb{1} = T^\dagger (T^{-1})^\dagger.\end{aligned}$$

Thus, $(T^\dagger)^{-1} = (T^{-1})^\dagger$. \square

Proposition 6.12. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$. The following are equivalent:*

1. T is invertible (i.e. an isomorphism between \mathcal{H} and \mathcal{H} as Banach spaces);
2. T^\dagger is invertible;
3. T is bounded below;
4. T^\dagger is bounded below;
5. Both $\text{Im}(T), \text{Im}(T^\dagger)$ are closed, and T, T^\dagger are injective;
6. $\mathcal{H} = \text{Im}(T) = \text{Im}(T^\dagger)$;
7. T is bijective;
8. T^\dagger is bijective.

Proof. 1 \implies 2 is done above.

2 \implies 1. We have $T = (T^\dagger)^\dagger$, so applying the above result to T^\dagger gives 1.

1 \implies 3. This is by theorem 3.12.

2 \implies 4. This is by theorem 3.12.

3, 4 \implies 5. Let T be bounded below by c . The injectivity is from theorem 3.12. Also, consider any convergent sequence $(Tx_n)_{n=1}^\infty$ in $\text{Im}(T)$. Since it is Cauchy, for any $\epsilon > 0$, there is $N \geq 1$, such that for all $n, m \geq N$,

$$c\|x_n - x_m\| \leq \|Tx_n - Tx_m\| < c\epsilon.$$

Thus, $(x_n)_{n=1}^\infty$ is Cauchy in \mathcal{H} , so there must be some $x \in \mathcal{H}$ such that $x_n \rightarrow x$. Since T is continuous, $Tx_n \rightarrow Tx \in \text{Im}(T)$. This shows $\overline{\text{Im}(T)}$ is closed. Similarly, $\text{Im}(T^\dagger)$ is closed and it is injective.

5 \implies 6. We have that $\overline{\text{Im}(T)} \oplus \text{Im}(T)^\perp = \mathcal{H}$ by theorem 4.14. Since T^\dagger is injective, we have $\text{Im}(T)^\perp = \ker(T^\dagger) = \{0\}$ by proposition 6.10. Thus, $\overline{\text{Im}(T)} = \mathcal{H}$. Since $\text{Im}(T)$ is closed, we have $\text{Im}(T) = \mathcal{H}$. Similarly, $\text{Im}(T^\dagger) \oplus \ker(T) = \mathcal{H}$, so $\text{Im}(T^\dagger) = \mathcal{H}$.

6 \implies 7. We have that $\ker(T) = \text{Im}(T^\dagger)^\perp = (\mathcal{H})^\perp = \{0\}$, so T is injective. Also, T is surjective by assumption.

7 \implies 1. By the Banach Isomorphism theorem 3.23, $T^{-1} \in B(\mathcal{H})$ is bounded.

6 \implies 8. We have that $\ker(T^\dagger) = \text{Im}(T)^\perp = (\mathcal{H})^\perp = \{0\}$, so T^\dagger is injective. Also, T^\dagger is surjective by assumption.

8 \implies 2. By the Banach Isomorphism Theorem, $(T^\dagger)^{-1} \in B(\mathcal{H})$ is bounded. \square

6.1.3 Normal, Self-adjoint and Positive Semidefinite Operators

Definition 6.3. Let \mathcal{H} be a Hilbert space. A bounded linear operator $T \in B(\mathcal{H})$ is **normal** if $T^\dagger T = TT^\dagger$.

Proposition 6.13. Let \mathcal{H} be a Hilbert space. If $T \in B(\mathcal{H})$ is normal, we have that for all $c \in \mathbb{F}$, $c\mathbb{I} - T$ is also normal.

Proof.

$$\begin{aligned}(c\mathbb{I} - T)^\dagger(c\mathbb{I} - T) &= (\bar{c}\mathbb{I} - T^\dagger)(c\mathbb{I} - T) \\&= |c|^2|\mathbb{I} - \bar{c}T - cT^\dagger + T^\dagger T| \\&= |c|^2|\mathbb{I} - cT^\dagger - \bar{c}T + TT^\dagger|, \\(c\mathbb{I} - T)(c\mathbb{I} - T)^\dagger &= (c\mathbb{I} - T)(\bar{c}\mathbb{I} - T^\dagger) \\&= |c|^2 - cT^\dagger - \bar{c}T + TT^\dagger \\&= (c\mathbb{I} - T)^\dagger(c\mathbb{I} - T).\end{aligned}$$

□

Proposition 6.14. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be a normal operator. For $x \in \mathcal{H}$, we have $Tx = \lambda x$ if and only if $T^\dagger x = \bar{\lambda}x$.

Proof. Suppose $Tx = \lambda x$. Let $N := \lambda\mathbb{I} - T$, which is normal, and $Nx = \lambda x - Tx = 0$, $N^\dagger = \bar{\lambda} - T^\dagger$. We have that

$$\begin{aligned}\|N^\dagger x\|^2 &= \langle N^\dagger x, N^\dagger x \rangle \\&= \langle NN^\dagger x, x \rangle \\&= \langle N^\dagger Nx, x \rangle \\&= \langle N^\dagger 0, x \rangle \\&= 0.\end{aligned}$$

Thus $N^\dagger x = 0$, which means $\bar{\lambda}x - T^\dagger x = 0$.

The converse is true by $(T^\dagger)^\dagger = T$.

□

Definition 6.4. Let \mathcal{H} be a Hilbert space. A bounded linear operator $T \in B(\mathcal{H})$ is **self-adjoint** if $T^\dagger = T$. The set of all self-adjoint operators is $B(\mathcal{H})_{sa}$.

Definition 6.5. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$, we define

$$\Re(T) := \frac{T + T^\dagger}{2}, \quad \Im(T) := \frac{T - T^\dagger}{2i}.$$

Proposition 6.15. Let \mathcal{H} be a Hilbert space. $B(\mathcal{H})_{sa}$ is a \mathbb{R} subspace of $B(\mathcal{H})$. We have $\Re(T), \Im(T) \in B(\mathcal{H})_{sa}$, and $\Re(T) + i\Im(T) = T$, so

$$B(\mathcal{H}) = B(\mathcal{H})_{sa} + iB(\mathcal{H})_{sa}.$$

Proposition 6.16. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})_{sa}$. We have $\langle Tx, x \rangle \in \mathbb{R}$ for any $x \in \mathcal{H}$.

Proof. We have

$$\begin{aligned}\langle Tx, x \rangle &= \langle x, T^\dagger x \rangle \\&= \langle x, Tx \rangle \\&= \overline{\langle x, Tx \rangle}.\end{aligned}$$

□

Proposition 6.17. Let \mathcal{H} be a Hilbert space. Any self-adjoint operator T is normal.

Proof. $T^\dagger T = TT = TT^\dagger$. □

Proposition 6.18. Let \mathcal{H} be a Hilbert space, for any $T \in B(\mathcal{H})$, we have that $T^\dagger T, TT^\dagger \in B(\mathcal{H})_{sa}$.

Proof. We have $(T^\dagger T)^\dagger = T^\dagger(T^\dagger)^\dagger = T^\dagger T$, so it is self-adjoint. Similarly, $(TT^\dagger)^\dagger = (T^\dagger)^\dagger T^\dagger = TT^\dagger$. □

Lemma 6.19. Let \mathcal{H} be a Hilbert space, for any $T \in B(\mathcal{H})_{sa}$, we have that $\|T^2\| = \|T\|^2$.

Proof. $\|T^2\| = \|TT^\dagger\| = \|T\|^2$. □

Proposition 6.20. Let \mathcal{H} be a Hilbert space, for any normal $T \in B(\mathcal{H})$, we have that $\|T^2\| = \|T\|^2$.

Proof. Since $T^\dagger T$ is self-adjoint, we have $\|(T^\dagger T)^2\| = \|T^\dagger T\|^2$. Thus,

$$\begin{aligned} \|T^2\| &= \|(T^2)^\dagger T^2\|^{\frac{1}{2}} \\ &= \|T^\dagger T^\dagger TT\|^{\frac{1}{2}} \\ &= \|T^\dagger TT^\dagger T\|^{\frac{1}{2}} \\ &= \|(T^\dagger T)^2\|^{\frac{1}{2}} \\ &= (\|T^\dagger T\|^2)^{\frac{1}{2}} \\ &= \|T^\dagger T\| \\ &= \|T\|^2. \end{aligned}$$

□

Definition 6.6. Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})_{sa}$. We say T is **positive semidefinite**, written as $T \geq 0$, if

$$\forall x \in \mathcal{H}, \langle Tx, x \rangle \geq 0.$$

For $S, T \in B(\mathcal{H})_{sa}$, we say $S \leq T$ if $T - S \geq 0$. We also define $B(\mathcal{H})^+$ to be the set of all positive semidefinite operators.

Proposition 6.21. Let \mathcal{H} be a Hilbert space, for any $T \in B(\mathcal{H})$, we have that $T^\dagger T \geq 0$.

Proof. It is self-adjoint by above proposition. Consider any $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle T^\dagger Tx, x \rangle &= \langle Tx, Tx \rangle \\ &= \|Tx\|^2 \\ &\geq 0. \end{aligned}$$

□

Proposition 6.22. Let \mathcal{H} be a Hilbert space, and $T \geq 0$. We have that $M : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ by $M[x, y] := \langle Tx, y \rangle$ is a positive semidefinite form on \mathcal{H} .

Proof. 1. conjugate symmetry; i.e. $M[x, y] = \langle Tx, y \rangle = \langle x, Ty \rangle = \overline{\langle Ty, x \rangle} = \overline{M[y, x]}$.

2. linearity in the second argument; i.e. $M[x, ay + z] = \langle Tx, ay + z \rangle = a\langle Tx, y \rangle + \langle Tx, z \rangle = M[x, y] + aM[x, z]$.

3. positive semi-definiteness; i.e. $M[x, x] = \langle Tx, x \rangle \geq 0$. □

Proposition 6.23. Let \mathcal{H} be a Hilbert space, and $S, T \in B(\mathcal{H})_{sa}$.

1. Suppose $S \leq T$, then $A^\dagger S A \leq A^\dagger T A$ for all $A \in B(\mathcal{H})$.
2. Suppose $0 \leq S \leq T$, then $\|S\| \leq \|T\|$.
3. $-\|T\|\mathbb{I} \leq T \leq \|T\|\mathbb{I}$.
4. Suppose $-C\mathbb{I} \leq T \leq C\mathbb{I}$ for some $C \geq 0$, then $\|T\| \leq C$.
5. Suppose $0 \leq S$, we have $aS^n \geq 0$ for any non-negative $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$.

Proof. 1. For any $x \in \mathcal{H}$, we have $\langle (T - S)Ax, Ax \rangle \geq 0$, since $T - S \geq 0$. Thus, $\langle A^\dagger(T - S)Ax, x \rangle \geq 0$, which means $A^\dagger TA - A^\dagger SA = A^\dagger(T - S)A \geq 0$.

2. Consider $M[x, y] := \langle Sx, y \rangle$, which defines a positive semidefinite form 4.1 on \mathcal{H} . By Cauchy-Schwartz theorem 4.3, we have

$$\begin{aligned} |\langle Sx, y \rangle|^2 &\leq \langle Sx, x \rangle \langle Sy, y \rangle \\ &\leq \langle Tx, x \rangle \langle Ty, y \rangle \\ &\leq \|T\| \|x\|^2 \|T\| \|y\|^2 \\ &\leq \|T\|^2 \|x\|^2 \|y\|^2. \end{aligned}$$

Thus $|\langle Sx, y \rangle| \leq \|T\|$ for any $\|x\|, \|y\| \leq 1$, which means $\|S\| \leq \|T\|$.

3. Since $|\langle Tx, x \rangle| \leq \|T\| \|x^2\|$, we have

$$\begin{aligned} -\|T\| \|x^2\| &\leq \langle Tx, x \rangle \leq \|T\| \|x^2\| \\ -\langle \|T\| x, x \rangle &\leq \langle Tx, x \rangle \leq \langle \|T\| x, x \rangle. \end{aligned}$$

4. For any $x, y \in \mathcal{H}$, we have $\langle T(x + y), x + y \rangle = \langle Tx, x \rangle + 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$ and $\langle T(x - y), x - y \rangle = \langle Tx, x \rangle - 2\Re(\langle Ty, x \rangle) + \langle Ty, y \rangle$ by proposition 4.1, so

$$\langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle = 4\Re(\langle Tx, y \rangle).$$

Similarly, $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

By assumption, we have

$$\langle T(x + y), x + y \rangle \leq \langle C\mathbb{I}(x + y), x + y \rangle = C\|x + y\|^2$$

since $T \leq C\mathbb{I}$, and

$$\langle T(x - y), x - y \rangle \geq \langle -C\mathbb{I}(x - y), x - y \rangle = -C\|x - y\|^2$$

since $T \geq -C\mathbb{I}$. Thus we have

$$\begin{aligned} \langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle &\leq C\|x + y\|^2 - (-C\|x - y\|^2) \\ &= C(\|x + y\|^2 + \|x - y\|^2) \\ 4\Re(\langle Tx, y \rangle) &\leq 2C(\|x\|^2 + \|y\|^2). \end{aligned}$$

Taking supreme over all $\|x\|, \|y\| \leq 1$, we have $4\|T\| \leq 4C$.

5. It is obviously true when $n = 0, 1$. Now assume $n \geq 2$, and it already holds for $n - 2$. Clearly aS^n is self-adjoint. For any $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle aS^n x, x \rangle &= a \langle SS^{n-2} Sx, x \rangle \\ &= a \langle S^{n-2} Sx, S^\dagger x \rangle \\ &= a \langle S^{n-2}(Sx), Sx \rangle \\ &\geq 0. \end{aligned}$$

By induction, it holds for all $n \in \mathbb{N}$. □

Theorem 6.24 (Dini's). *Let K be compact, and $f \in C(K)$. Suppose there are $(g_n)_{n=1}^\infty \subseteq C(K)$ such that they are monotone increasing, and for all $x \in K$, $g_n(x) \nearrow f(x)$ pointwise, then $g_n \rightarrow f$ uniformly.*

Proof. Let $g_n := f - q_n$, we have that $g_n(x) \searrow 0$ monotonously pointwise. Given any $\epsilon > 0$, for each $n \geq 1$, define

$$U_n := \{x \in K : g_n(x) < \epsilon\},$$

which are open in K . We have that $K = \bigcup_{n=1}^\infty U_n$ is an open cover. Since K is compact, there is $N \geq 1$, such that $K = \bigcup_{n=1}^N U_n$. Since $U_1 \subseteq U_2 \subseteq \dots \subseteq U_N$, we have that $K = U_N$. For any $n \geq N$, we have that $K = U_N \subseteq U_n \subseteq K$, so $K = U_n$. This show that $\forall x \in K$,

$$f(x) - q_n(x) = g_n(x) < \epsilon.$$

□

Lemma 6.25. *For $f \in C([0, 1])$, with $f(t) := 1 - \sqrt{1-t}$, there are polynomials $(p_n)_{n=1}^\infty$, whose coefficients are all non-negative, then $f = \sum_{n=1}^\infty p_n$ uniformly.*

Proof. Let $q_0(t) := 0$, and $q_n(t) := \frac{1}{2}(t + q_{n-1}(t)^2)$. By induction, $0 \leq q_n \leq 1$, and q_n has non-negative coefficients. Also, $q_n \leq q_{n+1}$.

Define $p_n := q_n - q_{n-1}$ for all $n \geq 1$.

$$\begin{aligned} p_n(q_n + q_{n-1}) &= (q_n - q_{n-1})(q_n + q_{n-1}) \\ &= q_n^2 - q_{n-1}^2 \\ &= 2q_{n+1} - t - (1q_n - t) \\ &= 2(q_{n+1} - q_n) \\ &= 2p_{n+1}. \end{aligned}$$

By induction, we have that all p_n have non-negative coefficient.

For any $t \in [0, 1]$, since $q_n(t)$ is monotone increasing, there is some $q(t) := \lim_{n \rightarrow \infty} q_n(t) \leq 1$. Also, $q(t) = \frac{1}{2}(t + q(t)^2)$. Thus, $q(t) = 1 - \sqrt{1-t} = f(t)$. We now have

$$f(t) = \lim_{n \rightarrow \infty} q_n(t) = \sum_{n=1}^\infty p_n(t).$$

This convergence is uniform by Dini's Theorem. □

Theorem 6.26 (Square Root Theorem). *Let \mathcal{H} be a Hilbert space, and $0 \leq T$, then there is a unique $0 \leq S$, such that $S^2 = T$. We write $S := T^{\frac{1}{2}}$, and it commutes with all operators that commute with T .*

Proof. Assume $0 \leq T \leq \mathbb{I}$. Consider $S := \mathbb{I} - T$. Note that $0 \leq S \leq \mathbb{I}$. Let $p_n(t) := \sum_k \alpha_{n,k} t^k$ and $q_n := \sum_{i=1}^n p_i$ be from the above lemma. For any $n \geq 1$, we have

$$\begin{aligned} \|p_n(S)\| &\leq \sum_k \alpha_{n,k} \|S\|^k \\ &\leq \sum_k \alpha_{n,k} \|\mathbb{I}\|^k \\ &= \sum_k \alpha_{n,k} 1^k \\ &= p_n(1). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n=1}^\infty \|p_n(S)\| &\leq \sum_{n=1}^\infty p_n(1) \\ &= f(1) \\ &= 1. \end{aligned}$$

Since $B(\mathcal{H})$ is a Banach space, we can define

$$R := \sum_{n=1}^{\infty} p_n(S) = \lim_{n \rightarrow \infty} q_n(S)$$

by A2Q4. Also, $\|R\| \leq 1$, so $R \leq \mathbb{I}$. Also, since each coefficient in q_n is non-negative, we have that each $q_n(S) \geq 0$. Thus, $R \geq 0$. We now have

$$\begin{aligned} (\mathbb{I} - R)^2 &= (\mathbb{I} - \lim_{n \rightarrow \infty} q_n(S))^2 \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - q_n(S))^2 \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - 2q_n(S) + q_n(S)^2) \\ &= \lim_{n \rightarrow \infty} (\mathbb{I} - 2q_n(S) + 2q_{n+1}(S) - S) \\ &= \mathbb{I} - S - 2 \lim_{n \rightarrow \infty} q_n(S) + 2 \lim_{n \rightarrow \infty} q_{n+1}(S) \\ &= T - 2R + 2R \\ &= T. \end{aligned}$$

We can thus define $T^{\frac{1}{2}} := \mathbb{I} - R$.

For any A that commutes with T , we have

$$SA = A(\mathbb{I} - T) = A - AT = A - TA = (\mathbb{I} - T)A = SA.$$

Also, for any $k \geq 1$, we have

$$S^k A = S^{k-1} AS = S^{k-2} AS^2 = \dots = AS^k.$$

Thus, $Aq_n(S) = q_n(S)A$, so $AR = RA$. This shows

$$AT^{\frac{1}{2}} = A(\mathbb{I} - R) = A - AR = A - RA = (\mathbb{I} - R)A = T^{\frac{1}{2}}A.$$

Now for any general T , we can just take $\tilde{T} := \frac{1}{\|T\|}T$, so we have $\tilde{T} \leq \frac{1}{\|T\|}\|T\|\mathbb{I} = \mathbb{I}$. Now take $T^{\frac{1}{2}} := \sqrt{\|T\|}\tilde{T}^{\frac{1}{2}}$. Clearly, $T^{\frac{1}{2}}$ is self-adjoint, and

$$T^{\frac{1}{2}}T^{\frac{1}{2}} = \sqrt{\|T\|}\tilde{T}^{\frac{1}{2}}\sqrt{\|T\|}\tilde{T}^{\frac{1}{2}} = \|T\|\tilde{T} = T.$$

Also, if $AT = TA$, we will have $A\tilde{T} = \tilde{T}A$, so $A\tilde{T}^{\frac{1}{2}} = \tilde{T}^{\frac{1}{2}}A$ and $AT^{\frac{1}{2}} = T^{\frac{1}{2}}A$.

Suppose there is some $A \geq 0$, such that $A^2 = T$, we have $AT = A^3 = TA$. Since A commutes with T , it also commutes with $T^{\frac{1}{2}}$. We thus have

$$(T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A) = T - A^2 = 0.$$

Now let $Y := \text{Im}(T^{\frac{1}{2}} + A)$. For any $y \in Y$, we have $(T^{\frac{1}{2}} - A)(y) = 0$, since there will be $x \in \mathcal{H}$, such that $(T^{\frac{1}{2}} - A)(x) = y$, so

$$(T^{\frac{1}{2}} - A)(y) = (T^{\frac{1}{2}} - A)(T^{\frac{1}{2}} + A)(x_n) = 0.$$

On the other hand, for any $z \in Y^\perp = \ker((T^{\frac{1}{2}} + A)^\dagger) = \ker(T^{\frac{1}{2}} - A)$ by proposition 6.10, we have that

$$\begin{aligned} \|T^{\frac{1}{4}}z\|^2 &= \langle T^{\frac{1}{4}}z, T^{\frac{1}{4}}z \rangle \\ &= \langle z, T^{\frac{1}{4}}T^{\frac{1}{4}}z \rangle \\ &= \langle z, T^{\frac{1}{2}}z \rangle \\ &\leq \langle z, T^{\frac{1}{2}}z \rangle + \langle z, Az \rangle \\ &= \langle z, (T^{\frac{1}{2}} + A)z \rangle \\ &= 0. \end{aligned}$$

Thus, $T^{\frac{1}{4}}z = 0$, and $T^{\frac{1}{2}}z = T^{\frac{1}{4}}T^{\frac{1}{4}}z = 0$. Similarly,

$$\begin{aligned}\left\|A^{\frac{1}{2}}z\right\|^2 &= \langle A^{\frac{1}{2}}z, A^{\frac{1}{2}}z \rangle \\ &= \langle z, Az \rangle \\ &\leq \langle z, (A + T^{\frac{1}{2}})z \rangle \\ &= 0,\end{aligned}$$

so $A^{\frac{1}{2}}z = 0$ and $Az = 0$. Thus,

$$(T^{\frac{1}{2}} - A)(z) = T^{\frac{1}{2}}z - Az = 0.$$

Now for any $x \in \mathcal{H}$, there is $y \in Y$, and $z \in Y^\perp$, such that $x = y + z$, so we have

$$(A - T^{\frac{1}{2}})(x) = (A - T^{\frac{1}{2}})(y) + (A - T^{\frac{1}{2}})(z) = 0.$$

This shows $T^{\frac{1}{2}} = A$, which means $T^{\frac{1}{2}}$ is unique. \square

Corollary 6.27. *Let \mathcal{H} be a Hilbert space. If $S, T \geq 0$, and $ST = TS$, then $ST \geq 0$.*

Proof. We have $(ST)^\dagger = T^\dagger S^\dagger = TS = ST$, so $ST \in B(\mathcal{H})_{sa}$.

Also, for any $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle STx, x \rangle &= \langle S^{\frac{1}{2}}S^{\frac{1}{2}}Tx, x \rangle \\ &= \langle S^{\frac{1}{2}}Tx, (S^{\frac{1}{2}})^\dagger x \rangle \\ &= \langle TS^{\frac{1}{2}}x, S^{\frac{1}{2}}x \rangle \\ &\geq 0.\end{aligned}$$

\square

Corollary 6.28. *Let \mathcal{H} be a Hilbert space. Suppose $T \in B(\mathcal{H})_{sa}$, and $\langle Tx, x \rangle = 0$ for all $x \in \mathcal{H}$, then $T = 0$.*

Proof. For any $x \in \mathcal{H}$, we have $\langle Tx, x \rangle = 0 \geq 0$, so $T \geq 0$. In addition,

$$\begin{aligned}0 &= \langle Tx, x \rangle \\ &= \langle T^{\frac{1}{2}}T^{\frac{1}{2}}x, x \rangle \\ &= \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x \rangle \\ &= \|T^{\frac{1}{2}}x\|^2.\end{aligned}$$

Thus $T^{\frac{1}{2}}x = 0$, so $Tx = T^{\frac{1}{2}}T^{\frac{1}{2}}x = T^{\frac{1}{2}}0 = 0$. \square

Proposition 6.29. *Let \mathcal{H} be a Hilbert space, and $U \in B(\mathcal{H})$. U is an isometry if and only if $U^\dagger U = \mathbb{I}$. Also, the following are equivalent:*

1. U is unitary,
2. $U^\dagger = U^{-1}$, and
3. U is a normal isometry.

Proof. See A5. \square

6.2 Compact Operators

6.2.1 Compactness

See more in Real Analysis Pmath351 or Applied Functional Analysis Amath731.

Definition 6.7. Let (X, d) be a metric space. We say $S \subseteq X$ is **sequentially compact** if for every bounded sequence $(x_n)_{n=1}^{\infty}$, there is a convergent subsequence $x_{n_k} \rightarrow x \in S$.

Definition 6.8. Let (X, d) be a metric space. We say $S \subseteq X$ is **relatively compact** if for every bounded sequence $(x_n)_{n=1}^{\infty}$, there is a convergent subsequence $x_{n_k} \rightarrow x \in X$.

Definition 6.9. Let X be a metric space. We say $S \subseteq X$ is **totally bounded** if for any $\epsilon > 0$, there exists a finite subset $\{x_i\}_{i=1}^n$, such that $S \subseteq \bigcup_{i=1}^n B(x_i, \epsilon)$. Such a subset is called an ϵ -**net**.

Definition 6.10. Let X be a metric space. We say $S \subseteq X$ is **bounded** if there exists a $r > 0$, such that for all $x, y \in S$, $d(x, y) < r$.

Theorem 6.30. Let S be a subset of a metric space X . The following are equivalent:

1. S is compact.
2. S has the finite intersection property.
3. S is sequentially compact.
4. Every infinite subset $A \subseteq S$ has a limit point.
5. S is complete and totally bounded.
6. S is relatively compact and closed.

Corollary 6.31. Let S be a subset of a metric space X , then S is relatively compact if and only if \bar{S} is compact.

Corollary 6.32. Let S be a subset of a complete space X , then S is relatively compact if and only if S is totally bounded.

Proposition 6.33. Let S be a subset of a metric space X .

1. Suppose S is relatively compact, then S is bounded.
2. Suppose S is totally bounded, then S is bounded.
3. Suppose S is compact, then S is closed and bounded.

Theorem 6.34 (Heine-Borel). Let S be a subset of a finite dimensional normed vector space X , then S is compact if and only if S is closed and bounded.

6.2.2 Compact Operators

Definition 6.11. Let X, Y be normed vector spaces. A linear operator $K : X \rightarrow Y$ is **compact** if $K(B^X(0, 1))$ is compact in Y . Let $\mathcal{K}(X, Y)$ denote the set of compact operators.

Proposition 6.35. Let X, Y be normed vector spaces, and $K : X \rightarrow Y$ be a linear operator, then the following are equivalent:

1. K is compact.
2. $K(B^X(0, 1))$ is relative compact in Y .
3. For any bounded set $S \subseteq X$, $\overline{K(S)}$ is compact in Y .
4. For any bounded set $S \subseteq X$, $K(S)$ is relative compact in Y .

5. Each bounded sequence $(x_n)_{n=1}^\infty \subseteq X$ has some subsequence $(x_{n_k})_{k=1}^\infty$ such that $(Kx_{n_k})_{k=1}^\infty$ converges to some $y \in Y$.

Proposition 6.36. Let X, Y be normed vector spaces. Suppose a linear operator $K : X \rightarrow Y$ is compact, then it is always bounded. Namely, $\mathcal{K}(X, Y) \subseteq B(X, Y)$.

Proposition 6.37. Let X be an infinite-dimensional Banach space, then the identity map $\mathbb{I} : X \rightarrow X$ is not compact.

Proof. $\bar{B}^X(0, 1)$ is not compact by A2. \square

Proposition 6.38. Let X be a Banach space. Suppose $E = E^2 \in B(X)$ is compact, then E is finite rank.

Proof. See A5. \square

Definition 6.12. Let X, Y be vector spaces. Let

$$\mathcal{F}(X, Y) : \{T \in B(x, Y) \mid \text{rank}(T) < \infty\}$$

denote the set of finite rank bounded linear operators.

Proposition 6.39. Let X, Y be normed vector spaces. Suppose a linear operator $T \in B(x, Y)$ is finite-dimensional, then it is always compact. Namely, $\mathcal{F}(X, Y) \subseteq \mathcal{K}(X, Y)$.

Proof. $\text{Im}(K) \subseteq Y$ is finite dimensional. For any bounded $M \subseteq X$, we have that $K(M)$ is bounded. Since $\overline{K(M)} \subseteq \text{Im}(T)$ is closed and bounded, it is compact in $\text{Im}(T)$, thus compact in Y . \square

Theorem 6.40. Let X, Y be Banach Spaces. $\mathcal{K}(X, Y)$ is norm-closed $B(X) - B(Y)$ sub-bi-module of $B(X, Y)$. Namely, for any $a \in B(Y), b \in B(X)$, and $K \in \mathcal{K}(X, Y)$, we have $aKb \in \mathcal{K}(X, Y)$. Also, if $(K_n)_{n=1}^\infty$ is a sequence of compact operators in $\mathcal{K}(X, Y)$, and $K_n \rightarrow K$ in $B(X, Y)$, we have $K \in \mathcal{K}(X, Y)$ as well.

Proof. Clearly $\mathcal{K}(X, Y)$ is a vector space.

Since $b \in B(X)$, we have $b(\bar{B}^X(0, 1)) \subseteq \bar{B}^X(0, \|b\|)$, which is bounded. Since K is compact, $\overline{K(\bar{B}^X(0, \|b\|))}$ is compact. Thus, $\overline{Kb(\bar{B}^X(0, 1))} \subseteq \overline{K(\bar{B}^X(0, \|b\|))}$ is compact since it is closed. Since $a \in B(Y)$ is continuous, we have that $\overline{a(Kb(\bar{B}^X(0, 1)))}$ is compact as well. Thus $\overline{aKb(\bar{B}^X(0, 1))} \subseteq \overline{a(Kb(\bar{B}^X(0, 1)))} = \overline{a(Kb(\bar{B}^X(0, 1)))}$ is compact, since it is closed.

This shows aTb is compact.

Now consider any sequence of compact operators $(K_n)_{n=1}^\infty$ in $\mathcal{K}(X, Y)$, with $K_n \rightarrow K$ in $B(X, Y)$. Given any $\epsilon > 0$, there is $N \geq 1$ such that $\|K - K_N\| < \frac{\epsilon}{3}$. Since K_N is compact, $\overline{K_N(\bar{B}^X(0, 1))}$ is compact and thus totally bounded, so we can find $\{x_1, \dots, x_m\} \subseteq \bar{B}^X(0, 1)$, such that $\overline{K_N(\bar{B}^X(0, 1))} \subseteq \bigcup_{i=1}^m \bar{B}^X(x_i, \frac{\epsilon}{3})$. Now for any $x \in \bar{B}^X(0, 1)$, consider $K_Nx \in \bar{B}^X(x_i, \frac{\epsilon}{3})$, we have

$$\begin{aligned} \|Kx - Kx_i\| &\leq \|Kx - K_Nx\| + \|K_Nx - K_Nx_i\| + \|K_Nx_i - Kx_i\| \\ &< \|K - K_N\|\|x\| + \frac{\epsilon}{3} + \|K_N - K\|\|x_i\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

This shows that $K(\bar{B}^X(0, 1)) \subseteq \bigcup_{i=1}^m \bar{B}^X(x_i, \epsilon)$, so it is totally bounded and thus relatively compact. This proves K is compact, so $\mathcal{K}(X, Y)$ is norm-closed. \square

Proposition 6.41. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We have

$$\overline{\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)} = \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

Proof. Let $K \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, and $\epsilon > 0$. We can find an ϵ -net $\{Kx_1, \dots, Kx_m\}$ for $K(B^{\mathcal{H}_1}(0, 1))$, since it is totally bounded.

Let $K_0 := \text{Span}\{Kx_1, \dots, Kx_m\} \subseteq \mathcal{H}_2$, and consider $P : \mathcal{H}_2 \rightarrow K_0$, the orthogonal projection onto K_0 . For any $x \in B^{\mathcal{H}_1}(0, 1)$, we have

$$\begin{aligned} \|(K - PK)x\| &= \|(\mathbb{I} - P)Kx\| \\ &= \text{dist}(Kx, K_0) \\ &\leq \min_{i \in [m]} \|Kx - Kx_i\| \\ &< \epsilon. \end{aligned}$$

Thus, $\|K - PK\| < \epsilon$, where $\text{rank}(PK) \leq \dim(K_0) \leq m < \infty$. \square

Corollary 6.42. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. For $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, we have $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $T^\dagger \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. By finite-dimensional approximation. \square

Example 6.2.1. Consider $f \in \ell^\infty(\mathbb{N})$, and the multiplication operator $M_f : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$. We have that M_f is compact if and only if $f \in C_0(\mathbb{N})$, namely, $\lim_{i \rightarrow \infty} f(i) = 0$.

Indeed, if $f \in C_0(\mathbb{N})$, we can let $f_N(i) := \begin{cases} f(i), & i \leq N \\ 0, & \text{o.w.} \end{cases}$ We have $\|M_f - M_{f_N}\| = \|M_{f-f_N}\| = \|f - f_N\|_\infty \rightarrow 0$.

On the other hand, if $f \notin C_0(\mathbb{N})$, there is $\epsilon > 0$, and $i_1 < i_2 < i_3 < \dots$, such that $|f(i_k)| \geq \epsilon$. Let $K_0 := \overline{\text{Span}\{e_{i_k}\}} \subseteq \ell^2(\mathbb{N})$, we have $M_f(e_{i_k})(i) = \begin{cases} f(i), & i = i_k \\ 0, & \text{o.w.} \end{cases}$, so $M_f(e_{i_k}) = f(i_k)e_{i_k} \in K_0$. Thus, $K := M_f|_{K_0} : K_0 \rightarrow K_0$ is well defined. In addition, K is bounded below by ϵ . By theorem 3.12, it has a bounded inverse K^{-1} .

Suppose for contradiction that M_f is compact, then $\overline{K(B^{K_0}(0, 1))} \subseteq \overline{M_f(B^{\ell^2(\mathbb{N})}(0, 1))}$. Since K^{-1} is bounded, $K^{-1}\overline{K(B^{K_0}(0, 1))}$ is compact as well. Now

$$\overline{B^{K_0}(0, 1)} = \overline{K^{-1}K(B^{K_0}(0, 1))} \subseteq \overline{K^{-1}\overline{K(B^{K_0}(0, 1))}} = K^{-1}\overline{K(B^{K_0}(0, 1))}$$

is compact. Since K_0 is infinite dimensional, $\overline{B^{K_0}(0, 1)}$ cannot be compact, a contradiction.

6.2.3 Integral Operators

Definition 6.13. Let $k \in L^2([0, 1] \times [0, 1])$, the **integral operator** with **kernel** k is $T_k \in B(L^2(0, 1))$, with

$$(T_k f)(x) := \int_0^1 k(x, y)f(y)dy.$$

Lemma 6.43. *Let $K \in L^2([0, 1] \times [0, 1])$, the integral operator with kernel K is well-defined.*

Proof. Consider any $f \in L^2(0, 1)$,

$$\begin{aligned} \|T_k f\|_2^2 &= \int_0^1 \left| \int_0^1 k(x, y)f(y)dy \right|^2 dx \\ &\leq \int_0^1 \left(\int_0^1 |k(x, y)|^2 dy \right) \|f\|_2^2 dx \\ &= \|f\|_2^2 \|k\|_2^2 \\ &< \infty. \end{aligned}$$

Thus, $T_k f \in L^2(0, 1)$. \square

Proposition 6.44. Let $k \in L^2([0, 1] \times [0, 1])$, then T_k is compact.

Proof. Pick any orthonormal basis $(e_i)_{i=1}^\infty$ for $L^2(0, 1)$, then $e_{ij}(x, y) := e_i(x)e_j(y)$ forms an orthonormal basis for $L^2([0, 1] \times [0, 1])$.

Thus, $k = \sum_{i,j=1}^\infty a^{ij}e_{ij}$, where $\|k\|_2^2 = \sum_{i,j=1}^\infty |a_{ij}|^2$. Let $k_N := \sum_{i,j=1}^N a^{ij}e_{ij}$. Thus,

$$\|T_k - T_{k_N}\|^2 \leq \|k - k_N\|_2^2 = \sum_{i,j=N+1}^\infty |a_{ij}|^2 \rightarrow 0.$$

Now for any $f \in L^2(0, 1)$, we have

$$\begin{aligned} (T_{k_N}f)(x) &= \int_0^1 k_N(x, y)f(y)dy \\ &= \langle \bar{k}_N(x, \cdot), f \rangle \\ &= \left\langle \sum_{i,j=1}^N a^{ij}e_{ij}(x, \cdot), f \right\rangle \\ &= \sum_{i,j=1}^N a^{ij}e_i(x)\langle \bar{e}_j, f \rangle \\ &= \sum_{i,j=1}^N a^{ij}\langle \bar{e}_j, f \rangle e_i(x) \\ T_{k_N}f &= \sum_{i,j=1}^N a_{ij}\langle \bar{e}_j, f \rangle e_i \\ &\in \text{Span}\{e_1, \dots, e_N\}. \end{aligned}$$

Thus, T_{k_N} is finite rank and thus compact, which means T_k is compact as well. \square

Proposition 6.45. Let $k \in L^2([0, 1] \times [0, 1])$, then

$$(T_k^\dagger g)(x) = \int_0^1 k^*(x, y)g(y)dy,$$

where $k^*(x, y) := \overline{k(y, x)}$.

Proof. Consider any $f, g \in L^2(0, 1)$, let $h(x) := \int_0^1 k^*(x, y)g(y)dy$, we have

$$\begin{aligned} \langle V^*g, f \rangle &= \langle g, Vf \rangle \\ &= \int_0^1 g(x)\overline{Vf}(x)dx \\ &= \int_0^1 g(x)\overline{\int_0^1 k(x, y)f(y)dy}dx \\ &= \int_0^1 g(x)\int_0^1 \bar{k}(x, y)\bar{f}(y)dydx \\ &= \int_0^1 \int_0^1 g(x)k^*(y, x)dx\bar{f}(y)dy \\ &= \int_0^1 h(y)\bar{f}(y)dy \\ &= \langle h, f \rangle. \end{aligned}$$

We can swap the order of integration by the Fubini Theorem and the fact that $f, g \in L^2(0, 1)$.

Thus we have $(T_k^\dagger g)(x) = h(x) = \int_0^1 k^*(x, y)g(y)dy$. \square

Example 6.2.2. The **Volterra Operator** is $V : L^2(0, 1) \rightarrow L^2(0, 1)$ by

$$(Vf)(x) := \int_0^x f(y) dy.$$

Its kernel is $k(x, y) := \chi_{y \leq x}(x, y) := \begin{cases} 1, & y \leq x \\ 0, & \text{o.w.} \end{cases}$. We have

$$\begin{aligned} \int_0^1 \int_0^1 |\chi_{y \leq x}(x, y)|^2 dx dy &= \int_0^1 \int_0^y 1 dx dy \\ &= \int_0^1 y dy \\ &= \frac{1}{2}, \end{aligned}$$

so $k \in L^2([0, 1] \times [0, 1])$ by the Fubini Theorem. Also,

$$\begin{aligned} (T_k f)(x) &= \int_0^1 k(x, y) f(y) dy \\ &= \int_0^1 \chi_{y \leq x}(x, y) f(y) dy \\ &= \int_0^x f(y) dy \\ &= (Vf)(x). \end{aligned}$$

See more in A5Q6.

6.3 Spectrum

Definition 6.14. Let X be a Banach space, and $T \in B(X)$. The **spectrum** of T is

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda \mathbb{I} - T \text{ is not invertible}\}.$$

The **point spectrum** of T is

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda \mathbb{I} - T \text{ is not injective}\} = \{\lambda \in \mathbb{C} : \exists v \neq 0 \text{ such that } Tv = \lambda v\} = \{\text{eigenvalues of } T\}.$$

The **resolvent** of T is

$$\rho(T) := \mathbb{C} \setminus \sigma(T).$$

Proposition 6.46. When X is finite dimensional, then

$$\sigma(T) = \{\lambda : \lambda \mathbb{I} - T \text{ is not injective}\} = \sigma_p(T).$$

Example 6.3.1. Consider $f(n) := \frac{1}{n} \in \ell^\infty$, and the multiplication operator M_f we have that $0 \in \sigma(M_f)$, but not an eigenvalue.

Example 6.3.2. Let $f \in C([0, 1])$, and the multiplication operator $M_f : L^2(0, 1) \rightarrow L^2(0, 1)$ by $(M_f(\xi))(x) := f(x)\xi(x)$, where we have $M_f M_g = M_{fg}$, $M_f^\dagger = M_{\bar{f}}$, $\|M_f\| = \|f\|_\infty$.

Suppose $\lambda \notin \text{Im}(f)$, then $\lambda - f \neq 0$ on $[0, 1]$, so $\frac{1}{\lambda - f} \in C([0, 1])$. We thus have

$$M_{\frac{1}{\lambda - f}} = (M_{\lambda - f})^{-1} = (\lambda \mathbb{I} - M_f)^{-1}.$$

This shows $\sigma(M_f) \subseteq \text{Im}(f)$.

On the other hand, suppose $\lambda = f(x_0)$ for some $x_0 \in [0, 1]$, we have that $(\lambda - M_f)(\xi)(x) = (f(x_0) - f(x))\xi(x)$. Now for any $\epsilon > 0$, since f is continuous, we have that

$$A_\epsilon := \{x : |f(x) - f(x_0)| < \epsilon\}$$

has positive measure. Let $\xi_\epsilon := \frac{\chi_{A_\epsilon}}{\mu(A_\epsilon)^{\frac{1}{2}}}$, with $\|\xi_\epsilon\|_2 = 1$, we have that

$$\begin{aligned} \|(\lambda\mathbb{I} - M_f)\xi_\epsilon\|_2^2 &= \int_{A_\epsilon} |\lambda - f(x)|^2 \frac{1}{\mu(A_\epsilon)} dx \\ &< \frac{1}{\mu(A_\epsilon)} \int_{A_\epsilon} \epsilon^2 dx \\ &= \epsilon^2. \end{aligned}$$

Thus, $\lambda\mathbb{I} - M_f$ is not bounded below. By theorem 3.12, $\lambda\mathbb{I} - M_f$ is not invertible, so $\lambda \in \sigma(M_f)$.

Thus, $\sigma(M_f) = \text{Im}(f)$.

However, such a M_f may have no eigenvalue! Let $f(x) = x$, if t is an eigenvalue with eigenvector ξ , we have $x\xi(x) = t\xi(x)$ a.e., so $(x - t)\xi(x) = 0$ a.e., which means $\xi(x) = 0$ a.e..

Theorem 6.47. *Let X be a Banach space, and $T \in B(X)$. We have $\sigma(T) \neq \emptyset$, and it is always compact.*

Proof. For any $|\lambda| > \|T\|$, we have that $\left\|\frac{T}{\lambda}\right\| < 1$, so $(\mathbb{I} - \frac{T}{\lambda})^{-1} = \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$, and

$$(\lambda\mathbb{I} - T)^{-1} = \left(\lambda\left(\mathbb{I} - \frac{T}{\lambda}\right)\right)^{-1} = \lambda^{-1}\left(\mathbb{I} - \frac{T}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}.$$

Thus, $\lambda \in \rho(T)$. This shows that $\forall \lambda \in \sigma(T)$, $|\lambda| \leq \|T\|$, which means $\sigma(T)$ is bounded.

Take any $\phi \in B(\mathcal{H})^*$, we can define $f_\phi : \rho(T) \rightarrow \mathbb{C}$ by $f_\phi(z) := \phi((z\mathbb{I} - T)^{-1})$, for any $z \in \rho(T) = \mathbb{C}$. Fix $z_0 \in \mathbb{C}$, and pick $|h| < \|(Z_0\mathbb{I} - T)^{-1}\|^{-1}$, we have $\|h(z_0\mathbb{I} - T)^{-1}\| < 1$, so $(\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})^{-1} = \sum_{k=0}^{\infty} (h(z_0\mathbb{I} - T)^{-1})^k$.

$$\begin{aligned} f_\phi(z_0 + h) &= \phi((z_0\mathbb{I} + h\mathbb{I} - T)^{-1}) \\ &= \phi(((\mathbb{I} + h(z_0\mathbb{I} - T)^{-1})(z_0\mathbb{I} - T))^{-1}) \\ &= \phi\left((z_0\mathbb{I} - T)^{-1} \sum_{k=0}^{\infty} (h(z_0\mathbb{I} - T)^{-1})^k\right) \\ &= \sum_{k=0}^{\infty} h^k \phi\left((z_0\mathbb{I} - T)^{-1} ((z_0\mathbb{I} - T)^{-1})^k\right) \\ &= \sum_{k=0}^{\infty} h^k \phi((z_0\mathbb{I} - T)^{-1-k}), \end{aligned}$$

which is analytic. Now, suppose for contradiction that $\sigma(T) = \emptyset$, then $f_\phi : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function. By Liouville's theorem, f_ϕ must be a constant function.

Also, we can see that when $|z| \rightarrow \infty$, $f_\phi(z) = \phi((z\mathbb{I} - T)^{-1}) \rightarrow 0$. Thus, f_ϕ must be constantly zero, for any ϕ . However, by Hahn-Banach Theorem, there has to be some $\phi \in B(\mathcal{H})^*$ that separates $\{(z\mathbb{I} - T)^{-1}\} \subseteq B(\mathcal{H})$, a contradiction.

This shows $\sigma(T) \neq \emptyset$.

In addition, the above argument shows that for any $z \in \rho(T)$, $(z_0\mathbb{I} + h\mathbb{I} - T)^{-1}$ is well-defined in a neighbourhood around it, so $\rho(T)$ is open and $\sigma(T)$ is closed. \square

Definition 6.15. Let X be a Banach space, and $T \in B(X)$. The **spectral radius** of T is

$$\text{spr}(T) := \sup \{|\lambda| : \lambda \in \sigma(T)\}.$$

Notice that by the above theorem, we have $\text{spr}(T) = \max \{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$.

Theorem 6.48 (Spectral mapping). *Let X be a Banach space, and $T \in B(X)$. For any polynomial p , we have that $\sigma(p(T)) = p(\sigma(T))$.*

Proof. Suppose $p(x)$ is order n . Let $q(x) = \lambda - p(x) = (\beta_1 - x) \cdots (\beta_n - x)$, where β_i 's are the n roots of q .

Let $\lambda \in \mathbb{C}$, we have $\lambda \in \sigma(p(T))$, if and only if $q(T) = \lambda - p(T) = (\beta_1 \mathbb{I} - T) \cdots (\beta_n \mathbb{I} - T)$ is not invertible, if and only if there is some β_i , such that $\beta_i \mathbb{I} - T$ is not invertible, if and only if $q(x)$ has a root $\beta_i \in \sigma(T)$, if and only if $\lambda = p(\beta_i) \in p(\sigma(T))$. \square

Theorem 6.49 (Spectral Radius Formula). *Let X be a Banach space, and $T \in B(X)$.*

$$\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}.$$

Proof. By the spectral mapping theorem, $\sigma(T^n) = \sigma(T)^n$. Thus, $\|T^n\| \geq \text{spr}(T^n) = \text{spr}(T)^n$, which means $\text{spr}(T) \leq \|T^n\|^{\frac{1}{n}}$. Since this holds for all $n \in \mathbb{N}$, we have

$$\text{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}.$$

On the other hand, consider any $\phi \in B(X)^*$, we have that $f_\phi(\lambda) := \phi((\lambda \mathbb{I} - T)^{-1})$ is analytic on $\rho(T)$ as in the proof of theorem 6.47. In addition, for $|\lambda| > \|T\|$, we have

$$f_\phi(\lambda) = \phi\left(\sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}\right) = \sum_{k=0}^{\infty} \frac{\phi(T^k)}{\lambda^{k+1}}.$$

Since there is no residue between the annulus $\|T\|$ and $\text{spr}(T)$, from complex analysis, this Laurent series has to converge for any $|\lambda| > \text{spr}(T)$.

Thus, for any $|\lambda| > \text{spr}(T)$, $\frac{\phi(T^k)}{\lambda^{k+1}} = \phi\left(\frac{T^k}{\lambda^{k+1}}\right)$ has to be bounded. Since this holds for any $\phi \in B(X)^*$, by the Banach-Steinhaus Theorem 3.20, $\left\{\frac{T^k}{\lambda^{k+1}} : k \geq 0\right\}$ has to be bounded in $B(X)^{**}$, and thus in $B(X)$. Namely, there is $C_\lambda > 0$, such that $\left\|\frac{T^k}{\lambda^{k+1}}\right\| < C_\lambda$ for all $k \geq 0$. Namely, $\|T^k\| \leq C_\lambda |\lambda|^{k+1}$. Thus, we have

$$\|T^k\|^{\frac{1}{k}} \leq C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} \implies \limsup_k \|T^k\|^{\frac{1}{k}} \leq \lim_{k \rightarrow \infty} C_\lambda^{\frac{1}{k}} |\lambda|^{\frac{k+1}{k}} = |\lambda|.$$

Since this holds for all $|\lambda| > \text{spr}(T)$, we have

$$\limsup_k \|T^k\|^{\frac{1}{k}} \leq \text{spr}(T).$$

Since $\limsup_n \|T^n\|^{\frac{1}{n}} \leq \text{spr}(T) \leq \liminf_n \|T^n\|^{\frac{1}{n}}$, we have $\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$. \square

Corollary 6.50. *Let \mathcal{H} be a Hilbert space, and $T \in B(\mathcal{H})$ be normal. We have $\text{spr}(T) = \|T\|$.*

Proof. By proposition 6.20, we have $\|T^2\| = \|T\|^2$, and inductively, we have $\|T^{2^n}\| = \|T\|^{2^n}$. Thus,

$$\text{spr}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|T^{2^n}\|^{\frac{1}{2^n}} = \|T\|.$$

\square

Example 6.3.3. For the Volterra operator, we have that $\|V^n\|^{\frac{1}{n}} \rightarrow 0$, so $\sigma(V) \subseteq \{0\}$, which means $\sigma(V) = \{0\}$.

Theorem 6.51. [Fredholm alternative] *Let X be a Banach space, and $K \in \mathcal{K}(X)$ be a compact linear operator. Suppose $\lambda \neq 0$, then $\ker(\lambda \mathbb{I} - T)$ is finite dimensional and $\text{Im}(\lambda \mathbb{I} - T)$ is finite co-dimensional. Also, either $\lambda \in \rho(K)$, or $\lambda \in \sigma_p(K)$.*

Theorem 6.52 (Structure theorem for compact operators). *Let $K \in \mathcal{K}(X)$ be a compact operator on an infinite-dimensional Banach space X , then*

1. $0 \in \sigma(K)$.
2. $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$.
3. $\sigma(K) \setminus \{0\}$ is finite, or $\sigma(K) \setminus \{0\} = (\lambda_k)_{k=1}^{\infty}$ such that $\lim_{k \rightarrow \infty} \lambda_k = 0$.

Theorem 6.53 (Spectral theorem for normal operators). *Let \mathcal{H} be a Hilbert space, and $K \in \mathcal{K}(\mathcal{H})$ be a compact normal operator, then there exist eigenvectors $(\phi_j)_{j \in J}$, which form an orthonormal basis for \mathcal{H} .*

Proof. By Zorn's Lemma, we can find a maximal set of orthonormal eigenvectors $(\phi_j)_{j \in J}$ of K . Let $M := \overline{\text{Span}\{\phi_j\}_{j \in J}}$, and $P : \mathcal{H} \rightarrow M$ be the projection onto M . Also, let $\lambda_j \in \mathbb{C}$ be such that $K\phi_j = \lambda_j\phi_j$.

Consider any $x \in \mathcal{H}$, we have that

$$\begin{aligned} KPx &= K \sum_{j \in J} \langle \phi_j, x \rangle \phi_j \\ &= \sum_{j \in J} \langle \phi_j, x \rangle K\phi_j \\ &= \sum_{j \in J} \langle \phi_j, x \rangle \lambda_j \phi_j \\ &= \sum_{j \in J} \langle \bar{\lambda}_j \phi_j, x \rangle \phi_j \\ &= \sum_{j \in J} \langle K^\dagger \phi_j, x \rangle \phi_j \\ &= \sum_{j \in J} \langle \phi_j, Kx \rangle \phi_j \\ &= PKx. \end{aligned}$$

Thus, $KP = PK$.

Since $\mathbb{I} - P \in B(\mathcal{H})_{sa}$, we have that $(\mathbb{I} - P)K$ is compact and normal. Suppose $(\mathbb{I} - P)K \neq 0$, then by corollary 6.50, there is $\lambda \in \sigma((\mathbb{I} - P)K)$, such that $|\lambda| = \text{spr}((\mathbb{I} - P)K) = \|(\mathbb{I} - P)K\| > 0$. Thus, by Fredholm alternative 6.51, $\lambda \in \sigma_p((\mathbb{I} - P)K)$, which means there is a $\phi_\lambda \neq 0 \in \mathcal{H}$, such that $(\mathbb{I} - P)K\phi_\lambda = \lambda\phi_\lambda$. WLOG, we may pick $\|\phi_\lambda\| = 1$. Notice that this means $\phi_\lambda \in \text{Im}(I - P) = M^\perp = \ker(P)$, so

$$\begin{aligned} \lambda\phi_\lambda &= (\mathbb{I} - P)K\phi_\lambda \\ &= K\phi_\lambda - PK\phi_\lambda \\ &= K\phi_\lambda - K\phi_\lambda \\ &= K\phi_\lambda - K(0) \\ &= K\phi_\lambda, \end{aligned}$$

which means ϕ_λ is an eigenvector of K . However, $\phi_\lambda \in M^\perp$, so $(\phi_j)_{j \in J} \cup \{\phi_\lambda\} \supsetneq (\phi_j)_{j \in J}$ is still orthonormal, contradicting the maximality of $(\phi_j)_{j \in J}$.

This shows $(\mathbb{I} - P)K = 0$. Now suppose $\mathbb{I} - P \neq 0$, we have $K(\mathbb{I} - P) = K = KP = K - PK = (\mathbb{I} - P)K = 0$, so there is $\phi \neq 0 \in \text{Im}(\mathbb{I} - P) = M^\perp$, such that $K\phi = 0 = 0\phi$, so ϕ is an eigenvector of K , which again contradicts the maximality of $(\phi_j)_{j \in J}$.

Thus, $\mathbb{I} - P = 0$, which means $M = \text{Im}(P) = \text{Im}(\mathbb{I}) = \mathcal{H}$. □

Theorem 6.54 (Spectral theorem for infinite range normal operators). *Let \mathcal{H} be a Hilbert space, and $K \in \mathcal{K}(\mathcal{H})$ be a compact normal operator with infinite range, then there exists countably many eigenvalues $(\lambda_k)_{k=1}^{\infty} \subset \mathbb{C}$ such that $\forall k \geq 1$, $n_k := \dim(\ker(\lambda_k \mathbb{I} - K)) < \infty$, and $\sigma(K) = (\lambda_k)_{k=0}^{\infty}$ for $\lambda_0 = 0 = \lim_{k \rightarrow \infty} \lambda_k$. In addition, there are eigenvectors $(\phi_{k,i})_{k \geq 0, i \in [n_k]}$, which form an orthonormal basis for \mathcal{H} , and $\forall k \geq 0$, $i \in [n_k]$, $K\phi_{k,i} = \lambda_k \phi_{k,i}$.*

Proof. □