Pmath 651: Measure Theory

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Contents

1 Introductions

1.1 Lebesgue Measure

Definition 1.1. Lebesgue outer measure of $A \in \mathbb{R}$ is $\lambda^*(A) := \inf\{\sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i\}$, where each $I_i \subseteq \mathbb{R}$ is an open interval.

Definition 1.2. If $\forall E \in \mathbb{R}, \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$, then A is Lebesgue measurable, and its Lebesgue measure is defined to be $\lambda(A) := \lambda^*(A)$

Proposition 1.1. $\forall a < b \in \mathbb{R}, \lambda((a, b)) = b - a$

Proposition 1.2. $\forall x \in \mathbb{R}, \lambda(x + A) = \lambda(A)$

Proposition 1.3. *If A_m are $\mathcal{L}-$measurable and pairwise disjoint $(A_m \cap A_n = \emptyset, \forall n \neq m)$, then* $m(\sqcup_{i\geq 1} A_i) = \sum_{i=1}^{\infty} m(A_i)$

Proposition 1.4. *Every Riemann integrable function is Lebesgue integrable.*

2 Measure

2.1 Algebra of Sets

Definition 2.1. Let X be a set and $\mathcal{P}(X) := \{A|A \subseteq X\}$, then an algebra of subsets of X is $\mathcal{A} \subseteq \mathcal{P}(X)$, such that

1. $\emptyset \in \mathcal{A}$
2. If $E \in \mathcal{A}$, then $E^c := X \setminus E \in \mathcal{A}$
3. If $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} E_i \in \mathcal{A}$

Definition 2.2. Let X be a set and $\mathcal{P}(X) := \{A|A \subseteq X\}$, then a $\sigma-$algebra of subsets of X is $\mathcal{M} \subseteq \mathcal{P}(X)$, such that

1. $\emptyset \in \mathcal{M}$
2. If $E \in \mathcal{M}$, then $E^c := X \setminus E \in \mathcal{M}$
3. If $E_1, E_2, \cdots \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$

Definition 2.3. If $\mathcal{M}$ is $\sigma-$algebra, we call $(X, \mathcal{M})$ a measurable space, and a set $E \in \mathcal{M}$ is called $\mathcal{M}$-measurable.

Remark. Every $\sigma-$algebra is an algebra.

Proposition 2.1. *If $\mathcal{A}$ is an algebra, and $E_1, E_2 \in \mathcal{A}$, then $E_1 \cap E_2 \in \mathcal{A}$*

Proof. $(E_1 \cap E_2)^c = E_1^c \cup E_2^c$ is in $\mathcal{A}$ by 2,3. □

Proposition 2.2. *If $\mathcal{A}$ is an algebra, and $E, F \in \mathcal{A}$, then $E \setminus F = E \cap F^c \in \mathcal{A}$.*

Proposition 2.3. *If $\mathcal{A}$ is an algebra, and $E, F \in \mathcal{A}$, then $E \Delta F = (E \setminus F) \cup (F \setminus E) \in \mathcal{A}$.*

Proposition 2.4. *If $\mathcal{M}$ is $\sigma-$algebra, $E_i \in \mathcal{M}$, then we can define $F_i := E_i \setminus \bigcup_{j=1}^{i-1} E_i$, and $\bigcup_{i=1}^{\infty} E_i = \bigsqcup_{i=1}^{\infty} F_i$*

Proposition 2.5. *If $\mathcal{M}$ is $(\sigma-)$algebra, and $E \in \mathcal{M}$, then $A|_E := \{E \cap A | A \in \mathcal{M}\}$ is an $(\sigma-)$algebra.*

Example 2.1.1. $P(X)$ is $\sigma - algebra$, and $\{\emptyset, X\}$ is $\sigma - algebra$.

Example 2.1.2. $\mathcal{A} = \{E \subseteq X : |E| < \infty \vee |E^c| < \infty\}$ is an algebra. However, if X is infinite, then it is not a $\sigma - algebra$

Example 2.1.3. $\mathcal{M} = \{E \subseteq X : |E| \leq \mathcal{N}_0 \vee |E^c| \leq \mathcal{N}_0\}$ is a $\sigma-$algebra.

Example 2.1.4. Let $X = \mathbb{R}$, the collection of all finite union of sets in $\{\mathbb{R}, (-\infty, b], (a, b], (a, \infty)|a, b \in \mathbb{R}\}$ is an algebra but not $\sigma-$algebra.

Proposition 2.6. *Let $\{\mathcal{M}_\alpha\}_{\alpha \in I}$ is a collection of ($\sigma-$)algebras of X, then $\bigcap_{\alpha \in I} \mathcal{M}_\alpha$ is an ($\sigma-$)algebra*

Definition 2.4. Let $\mathcal{C}$ be a collection of subsets of X, then $\sigma(\mathcal{C}) := \bigcap \{\mathcal{M} : \sigma - alg, \mathcal{C} \subseteq \mathcal{M}\}$ is a $\sigma - algebra$ containing $\mathcal{C}$, and is called the $\sigma-$algebra generated by $\mathcal{C}$.

Definition 2.5. Let X be a topological space, and let $\mathcal{G}$ be the collection of all open sets of X, then the Borel algebra is $Bol_X := \sigma(\mathcal{G})$

2.2 Measures

Definition 2.6. A function $\mu : \mathcal{M} \to [0, \infty]$ is called a **positive measure** if it satisfies **countable additivity**. Namely, for any pairwise disjoint sets $E_1, E_2, \dots$ in $\mathcal{M}$, we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

We call $(X, \mathcal{M}, \mu)$ a **measure space**.

Remark. We can similarly define a complex or a signed measure to be a function $\mathcal{M} \to \mathbb{C}$ or $\mathcal{M} \to [-\infty, \infty]$ that satisfies countable additivity. (See later chapter for more about this.)

Definition 2.7. μ is finite if $\mu(X) < \infty$. μ is $\sigma-$finite if $X = \bigcup_{i=1}^{\infty} A_i$, where each $\mu(A_i) < \infty$. μ is semi-finite if $\forall E \in \mathcal{M}$, such that $\mu(E) \neq 0$, there is always $F \in \mathcal{M}, F \subseteq E, 0 < \mu(F) < \infty$

Remark. We will only work with positive measures where it satisfies $\exists A \in \mathcal{M}, \mu(A) < \infty$.

Example 2.2.1. For any X, we can define $\mu : \mathcal{P}(X) \to [0, \infty]$ by $\mu(A) := \begin{cases} |A|, & |A| < \infty \\ \infty, & \text{otherwise} \end{cases}$ is the **counting measure** on X

Example 2.2.2. For any set X and $x \in X$, we can define $\delta_x : \mathcal{P}(X) \to [0, \infty]$ by $\delta_x(A) := \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$ is the **point measure** or **Dirac measure** of x.

Example 2.2.3. Let $X = \mathbb{R}, \mathcal{M} = \mathcal{P}(X)$, let $x_1, x_2, \dots \in \mathbb{R}, a_1, a_2, \dots \geq 0$, then $\mu(E) := \sum_{i|x_i \in E} a_i$ is a measure.

Definition 2.8. A positive measure μ is a **probability measure** if $\mu(X) = 1$. In this case, $(X, \mathcal{M}, \mu)$ is called a probability space.

Proposition 2.7. $\mu(\emptyset) = 0$

Proof. Choose $A \in \mathcal{M}$ with finite measure, take $A_1 = A$, and $A_2 = A_3 = \dots = \emptyset$.
Then $\mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset) = \mu(A) < \infty$, thus we must have $\mu(\emptyset) = 0$ □

Proposition 2.8 (Finite Additivity)**.** *If $E_1, E_2, \dots, E_n \in \mathcal{M}$, then $\mu(\bigsqcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$*

Proof. Take $E_{n+1} = E_{n+2} = \dots = \emptyset$, then $\mu(\bigsqcup_{i=1}^{n} E_i) = \mu(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{n} \mu(E_i) + \sum_{i=n+1}^{\infty} \mu(E_i) = \sum_{i=1}^{n} \mu(E_i)$ □

Remark. This holds for complex measures as well.

Proposition 2.9 (Monotonicity)**.** *If $E, F \in \mathcal{M}, E \subseteq F$, then $\mu(E) \leq \mu(F)$*

Proof. We have $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$ □

Remark. This does not hold for complex measures.

Proposition 2.10 (Subadditivity)**.** *If $E_1, E_2, \cdots \in \mathcal{M}$, then $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$*

Proposition 2.11 (Continuity)**.** *If $E_1, E_2, \cdots \in \mathcal{M}, E_n \subseteq E_{n+1}$, we have that $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_n)$. If $E_1, E_2, \cdots \in \mathcal{M}, E_{n+1} \subseteq E_n, \mu(E_1) < \infty$ we have that $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_n)$.*

Proof. Let $E_0 = \emptyset$, then we can write $\bigcup_{i=1}^{\infty} E_i = \bigsqcup_{i=1}^{\infty}(E_i \setminus E_{i-1})$, and we have $E_n = \bigsqcup_{i=1}^{n}(E_i \setminus E_{i-1})$.

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigsqcup_{i=1}^{\infty}(E_i \setminus E_{i-1})\right) \\
&= \sum_{i=1}^{\infty} \mu(E_i \setminus E_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i \setminus E_{i-1}) \\
&= \lim_{n\to\infty} \mu(\bigsqcup_{i=1}^{n}(E_i \setminus E_{i-1})) \\
&= \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

For the second part, let $A = \bigcap_{i=1}^{\infty} E_i$.

$$\begin{aligned}
\mu(E_1 \setminus A) &= \mu(E_1 \cap A^c) \\
&= \mu\left(E_1 \cap \left(\bigcap_{i=1}^{\infty} E_i\right)^c\right) \\
&= \mu\left(E_1 \cap \bigcup_{i=1}^{\infty} E_i^c\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} E_1 \cap E_i^c\right) \\
&= \lim_{n\to\infty} \mu(E_1 \cap E_n^c) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n)
\end{aligned}$$

By finite additivity, we have that

$$\begin{aligned}
\mu(E_1 \setminus A) + \mu(A) &= \mu(E_1 \setminus A \sqcup A) \\
&= \mu(E_1) \\
&= \lim_{n\to\infty} \mu(E_1) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n \sqcup E_n) \\
&= \lim_{n\to\infty} (\mu(E_1 \setminus E_n) + \mu(E_n)) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n) + \lim_{n\to\infty} \mu(E_n) \\
&= \mu(E_1 \setminus A) + \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

Since $\mu(E_1 \setminus A) \leq \mu(E_1) < \infty$, we have $\mu(A) = \lim_{n\to\infty} \mu(E_n)$ □

Remark. This holds for complex measures as well. However, for the second property, it is essential for $\mu(E_1) < \infty$. Indeed, consider the following example:

Example 2.2.4. Let $X = \mathbb{N}, \mathcal{M} = \mathcal{P}(X), \mu$ be the counting measure. Let $A_n := \{i : i \geq n\}$. Notice that $A_1 \supseteq A_2 \supseteq A_3 \ldots$ and $\lim_{n\to\infty} \mu(A_n) = \infty \neq 0 = \mu(\emptyset)$. However, $\bigcap_{n=1}^{\infty} A_n = \emptyset$

2.3 Measurable Function

Definition 2.9. If $(X, \mathcal{M}_1), (Y, \mathcal{M}_2)$ are measure spaces, then $f : X \to Y$ is a **measurable function** if $\forall B \in \mathcal{M}_2, f^{-1}(B) \in \mathcal{M}_1$.

Definition 2.10. If $(Y, \mathcal{T})$ is a topological space, we say a function $f : X \to Y$ is **Borel measurable** if it is measurable with respect to $\mathcal{M}_2 = Bol_{(Y,\mathcal{T})}$, the Borel σ-algebra.

Proposition 2.12. *For $(B_i) \subseteq Y$, we have*

1. $f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$
2. $f^{-1}(\bigcup_i B_i)) = \bigcup_i f^{-1}(B)$
3. $f^{-1}(\bigcap_i B_i)) = \bigcap_i f^{-1}(B)$.

Proposition 2.13. *If $(Y, \mathcal{T})$ is a topological space, a function $f : X \to Y$ is Borel measurable if and only if $\forall B \in \mathcal{T}$ open, $f^{-1}(B) \in \mathcal{M}_1$.*

Proposition 2.14. *For $f : X \to \mathbb{R}$, the following are equal:*

1. *f is (Borel) measurable*
2. *$\forall a, f^{-1}((-\infty, a))$ is measurable*
3. *$\forall a, f^{-1}((-\infty, a])$ is measurable*
4. *$\forall a, f^{-1}((a, \infty))$ is measurable*
5. *$\forall a, f^{-1}([a, \infty))$ is measurable*
6. *$\forall a < b, f^{-1}((a, b))$ is measurable*

Proposition 2.15. *If $f : X \to Y, g : Y \to Z$ are both measurable, then $f \circ g$ is also measurable.*

Corollary 2.16. *If $f : X \to \mathbb{C}$ is measurable, we have $u = Re(f), v = Im(f), z = |f|$ are all measurable.*

Theorem 2.17. *Let $(X, \mathcal{M})$ is a measurable space, and $u, v : X \to \mathbb{R}$ be measurable, and (Y, τ) is a topological space. If $\Phi : \mathbb{R}^2 \to Y$ is continuous, then $h : X \to Y; x \mapsto \Phi(u(x), v(x))$ is measurable.*

Proof. Let $f : X \to \mathbb{R}^2; x \mapsto (u(x), v(x))$, it suffices to check that f is measurable.
Notice that $Bol_{\mathbb{R}^2}$ is generated by open rectangles $R = (a, b) \times (c, d)$.
Yet $f^{-1}(R) = u^{-1}(a, b) \cap v^{-1}(c, d)$ is measurable. ☐

Corollary 2.18. *If $u, v : X \to \mathbb{R}$ are both measurable, we have $f := u + iv : X \to \mathbb{C}$ is also measurable.*

Proof. Choose $\Phi : \mathbb{R}^2 \to \mathbb{C}; (s, t) \mapsto s + it$. ☐

Corollary 2.19. *If $f, g : X \to \mathbb{R}$ are measurable, then we have $fg, f + g$ are both measurable.*

Proof. choose $\Phi : (s, t) \mapsto st$ or $\Phi : (s, t) \mapsto s + t$. ☐

Corollary 2.20. *If $f, g : X \to \mathbb{C}$ are measurable, then for any $\alpha \in \mathbb{C}$, we have $fg, f + g, \alpha f$ are all measurable.*

Proof. We write $f = u + iv, g = w + iz$. We have that u, v, z, z are all real-valued and measurable, so are $u + w, v + z$, and so are $(u + w) + i(v + z) = f + g$ and $(uw - vz) + i(vw + uz) = fg$.
For αf, it is obvious since $B \in Bol(\mathbb{C}) \iff \alpha B \in Bol(\mathbb{C})$ for $\alpha \neq 0$, and $0f = 0$ is measurable. ☐

Definition 2.11. For **extended real functions** $f : X \to [-\infty, \infty]$, it is measurable if $\forall \alpha \in \mathbb{R}, f^{-1}((\alpha, \infty]) \in \mathcal{M}$, or equivalently, $\forall \alpha \in \mathbb{R}, f^{-1}([-\infty, \alpha)) \in \mathcal{M}$.

Proposition 2.21. *If $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions $X \to [-\infty, \infty]$, we have*

$$g(x) := \sup_n f_n(x), \; h(x) := \limsup_{n\to\infty} f_n(x) = \inf_k \left(\sup_{n\geq k} f_n(x)\right)$$

are also measurable. Similarly for $\inf$ *and* $\liminf$.

Proof. Notice that

$$\begin{aligned} x \in g^{-1}((\alpha, \infty]) &\iff g(x) > \alpha \\ &\iff \exists f_n(x) > \alpha \\ &\iff x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty]), \end{aligned}$$

which is a union of measurable sets. Thus g is measurable. □

Corollary 2.22. *If $f_n : X \to [-\infty, \infty]$ or $f_n : X \to \mathbb{C}$ are measurable functions, and $\forall x \in X, f(x) = \lim_{n\to\infty} f_n(x)$ exists, then f is measurable.*

Corollary 2.23. *If $f, g : X \to [-\infty, \infty]$ are both measurable, then $\max(f, g), \min(f, g)$ are measurable.*

Corollary 2.24. *If $f : X \to [-\infty, \infty]$ is measurable, then $f^+ := \max(f, 0), f^- := \max(-f, 0)$ are both measurable, with $f = f^+ - f^-$.*

Proposition 2.25. *If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.*

Proof. Let $\alpha \in \mathbb{R}$, we need to show that $\{x \in \mathbb{R} | f(x) > \alpha\}$ is a Borel set.
We may assume that f is non-decreasing, if not we take $f \to -f$. If $\{x \in \mathbb{R} | f(x) > \alpha\} \in \{\emptyset, \mathbb{R}\}$, we have nothing to prove.
Now if $\{x \in \mathbb{R} | f(x) > \alpha\} \notin \{\emptyset, \mathbb{R}\}$, we have that $\{x \in \mathbb{R} | f(x) \leq \alpha\}$ is not empty and bounded above since f is increasing. Let $x_0 := \sup\{x \in \mathbb{R} | f(x) \leq \alpha\}$. If $f(x_0) \leq \alpha, \{x \in \mathbb{R} | f(x) > \alpha\} = (x_0, \infty)$, otherwise $\{x \in \mathbb{R} | f(x) > \alpha\} = [x_0, \infty)$, both Borel. □

2.4 Simple Functions

Definition 2.12. Let $(X, \mathcal{M})$ be a measurable space, a **characteristic function** for a subset $E \subseteq X$ is

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

Definition 2.13. Let $(X, \mathcal{M})$ be a measurable space, a function $\phi : X \to [-\infty, \infty]$ is **simple** if $\phi(X)$ is finite.

Proposition 2.26. *Let $(X, \mathcal{M})$ be a measurable space, for any simple function ϕ with $\phi(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we have*

$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

where $E_i = \phi^{-1}(\{\alpha_i\})$ are pairwise disjoint. In this case, ϕ is measurable if and only if $\forall i, E_i \in \mathcal{M}$.

Lemma 2.27. *For any $\alpha \in \mathbb{R}, n \geq 1$, we have that*

$$\alpha - \frac{1}{2^n} < \frac{\lfloor 2^n \alpha \rfloor}{2^n} \leq \alpha$$

Proof.

$$\begin{aligned}\lfloor 2^n\alpha\rfloor &\le 2^n\alpha < \lfloor 2^n\alpha\rfloor + 1\\ 2^n\alpha - 1 &< \lfloor 2^n\alpha\rfloor\\ \alpha - \frac{1}{2^n} &< \frac{\lfloor 2^n\alpha\rfloor}{2^n}\\ \frac{\lfloor 2^n\alpha\rfloor}{2^n} &\le \alpha\end{aligned}$$

□

Lemma 2.28. *Consider* $id : [0,\infty) \to [0,\infty), x \mapsto x$*, then there are simple functions* $s_n : [0,\infty] \to [0,\infty)$ *such that each* s_n *is measurable and* $\begin{cases} 0 \le s_1 \le s_2 \le \cdots \le id, \\ \forall x \in X, \lim_{n\to\infty} s_n(x) = id(x), \\ \forall R > 0, s_n \to id \text{ uniformly on } [0,R] \end{cases}$

Proof. For $n \ge 1, t \in [0,\infty)$, let $s_n(t) := \begin{cases} \frac{\lfloor 2^n t\rfloor}{2^n}, & t \in [0,n] \\ n, & t > n \end{cases}$
Notice that s_n is simple. It is also measurable since it is monotone.
We also have that $0 \le s_1 \le s_2 \le \cdots \le f$, and by squeeze theorem, we have that

$$\lim_{n\to\infty} s_n(x) = x = id(x).$$

In addition, we can check that this convergence is uniform on any $[0,R]$. □

Theorem 2.29. *Let* $f : X \to [0,\infty]$ *be measurable, then there are simple functions* $s_n : X \to [0,\infty)$ *such that each* s_n *is measurable and* $\begin{cases} 0 \le s_1 \le s_2 \le \cdots \le f \\ \forall x \in X, \lim_{n\to\infty} s_n(x) = f(x) \\ \forall R > 0, s_n \to f \text{ uniformly on } E_R := \{x \in X : f(x) \le R\}. \end{cases}$

Proof. Notice that for any simple function s and any arbitrary measurable function f, we have that $s \circ f$ is simple. Thus it suffices to find s'_n that approximates $id : x \mapsto x$, which is done by the above lemma.
Let $s_n := s'_n \circ f$, they are measurable by result on compositions, and

$$0 \le s_1 \le \cdots \le f, \ \lim_{n\to\infty} (s_n \circ f)(x) = f(x).$$

□

Corollary 2.30 (Simple function approximation)**.** *Let* $f : X \to \mathbb{C}$ *be measurable, then there are simple functions* $s_n : X \to [0,\infty)$ *such that each* s_n *is measurable and*

$$\begin{cases} 0 \le |s_1| \le |s_2| \le \cdots \le |f| \\ \forall x \in X, \lim_{n\to\infty} s_n(x) = f(x) \\ \forall R > 0, s_n \to f \text{ uniformly on } E_R := \{x \in X : |f(x)| \le R\}. \end{cases}$$

Corollary 2.31. *For* $f, g : X \to [0.\infty]$ *being measurable, we have that* $f \cdot g$ *is also measurable.*

Proof. One can check that for monotone non-decreasing $(a_n), (b_n) \subseteq [0,\infty)$ with $a_n \to a, b_n \to b$ for $a, b \in [0,\infty]$, then $a_n b_n \to ab$.
Approximate f with simple functions s_n, and g with simple functions t_n, then each of them is measurable, hence so is $s_n \cdot t_n$, hence so is $\lim_{n\to\infty} s_n t_n = fg$ □

3 Integration

3.1 Integration of non-negative functions

Definition 3.1. Let $(X, \mathcal{M}, \mu)$ be a measure space, $s : X \to [0, \infty)$ be a simple measurable function, with $s(X) = \{a_1, \dots, a_n\}$, such that $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, where $A_i := s^{-1}(\{a_i\})$. For $A \in \mathcal{M}$, define the **integral** of s over A to be

$$\int_A s d\mu := \sum_{i=1}^{n} a_i \mu(A_i \cap A).$$

Definition 3.2. For $f : X \to [0, \infty]$ measurable, the **integral** of f over $A \in \mathcal{M}$ is

$$\int_A f d\mu := \sup \int_A s d\mu,$$

where the sup is taken over all measurable simple $s : X \to [0, \infty)$ such that $0 \le s \le f$.

Proposition 3.1. *Let $f, g : X \to [0, \infty]$ be measurable, then*

1. $f \le g \implies \forall\ A \in \mathcal{M}, \int_A f d\mu \le \int_A g d\mu$
2. *For any $A \subseteq B \in \mathcal{M}$, we have that $\int_A f d\mu \le \int_B f d\mu$*
3. $\forall\ c \in [0, \infty), A \in \mathcal{M}$, *we have that* $\int_A c f d\mu = c \int_A f d\mu$
4. *If $\forall\ x \in X, f(x) = 0$, we have that $\forall\ A \in \mathcal{M}, \int_A f d\mu = 0$*
5. *If $\forall\ x \in A \in \mathcal{M}, f(x) = 0$, we have that $\int_A f d\mu = 0$*
6. *If $\mu(A) = 0$ for $A \in \mathcal{M}$, we have that $\int_A f d\mu = 0$*
7. $\int_A f d\mu = \int_X \mathcal{X}_A f d\mu$

Proposition 3.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $s : X \to [0, \infty)$ a measurable simple function. Then $\lambda : \mathcal{M} \to [0, \infty]$ defined by*

$$\lambda(A) := \int_A s d\mu$$

is a measure on (X, M)

Proof. Write $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, and let $C := \bigsqcup_{k=1}^{\infty} C_k$, then

$$\begin{aligned}
\lambda(C) &= \sum_{i=1}^{n} a_i \mu(Ai \cap C) \\
&= \sum_{i=1}^{n} a_i \mu(\bigsqcup_{k=1}^{\infty} (Ai \cap C_k)) \\
&= \sum_{i=1}^{n} a_i \sum_{k=1}^{\infty} \mu(A_i \cap C_k) \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{n} a_i \mu(A_i \cap C_k) \\
&= \sum_{k=1}^{\infty} \lambda(C_k)
\end{aligned}$$

Thus λ satisfies countable additivity, and in addition $\lambda(\emptyset) = \sum_{i=1}^{n} a_i \cancelto{0}{\mu(A_i \cap \emptyset)} = 0$. ☐

Corollary 3.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $s : X \to [0, \infty)$ a measurable simple function, with $C := \bigsqcup_{k=1}^{\infty} C_k$. Then we have*

$$\int_C s d\mu = \sum_{k=1}^{\infty} \int_{C_k} s d\mu.$$

Proof.

$$\begin{aligned}
\int_C s d\mu &= \lambda_s(C) \\
&= \lambda_s\left(\bigsqcup_{k=1}^{\infty} C_k\right) \\
&= \sum_{k=1}^{\infty} \lambda_s(C_k) \\
&= \sum_{k=1}^{\infty} \int_{C_k} s d\mu
\end{aligned}$$

□

Proposition 3.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $s, t : X \to [0, \infty)$ both be measurable simple functions, then*

$$\int_X (s+t) d\mu = \int_X s d\mu + \int_X t d\mu$$

Proof. Write $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, $t = \sum_{j=1}^{m} b_j \mathcal{X}_{B_j}$, and let $C_{ij} = A_i \cap B_j$, then C_{ij} are disjoint, and $\bigsqcup_{ij} C_{ij} = X$

$$\begin{aligned}
\int_{C_{ij}} (s+t) d\mu &= (a_i + b_j)\mu(C_{ij}) \\
&= a_i \mu(C_{ij}) + b_j \mu(C_{ij}) \\
&= \int_{C_{ij}} s d\mu + \int_{C_{ij}} t d\mu \\
\int_X (s+t) d\mu &= \int_{\bigsqcup_{ij} C_{ij}} (s+t) d\mu \\
&= \sum_{ij} \int_{C_{ij}} (s+t) d\mu \\
&= \sum_{ij} \int_{C_{ij}} s d\mu + \sum_{ij} \int_{C_{ij}} t d\mu \\
&= \int_X s d\mu + \int_X t d\mu
\end{aligned}$$

□

Theorem 3.5 (Lebesgue's Monotone Convergence)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to [0, \infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$. Let $f(x) := \lim_{n\to\infty} f_n(x)$, then $f : X \to [0, \infty]$ is measurable, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Proof. Since $f_n \leq f_{n+1}$, we have that $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$, so by monotone convergence theorem,

$$\alpha := \lim_{n\to\infty} \int_X f_n d\mu \in [0, \infty]$$

exists.
As a limit of measurable functions, f is measurable. Also, $\forall n, \int_X f_n d\mu \leq \int_X f d\mu$, and thus $\alpha \leq \int_X f d\mu$.
Consider any $s : X \to [0, \infty)$ be simple and measurable with $0 \leq s \leq f$, and consider any $0 < c < 1$.
For $n \geq 1$, let $A_n := \{x \in X : f_n(x) \geq cs(x)\}$.
Then $X = \bigcup_{n=1}^{\infty} A_n$ since f_n converges point-wise.
In addition, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$
Also each A_n is measurable, since $A_n = \{x \in X : (f_n - cs)(x) \geq 0\} = (f_n - cs)^{-1}([0, \infty])$, and $f_n - cs$ is measurable.
Since $\lambda_s : A \mapsto \int_A s d\mu$ is a measure, so by property of measures,

$$\int_X s d\mu = \lambda_s(X) = \lim_{n\to\infty} \lambda_s(A_n) = \lim_{n\to\infty} \int_{A_n} s d\mu.$$

In addition, we have

$$\begin{aligned}
\int_X f_n d\mu &\geq \int_{A_n} f_n d\mu \\
&\geq \int_{A_n} cs d\mu \\
&= c \int_{A_n} s\mu \\
\alpha = \lim_{n\to\infty} \int_X f_n d\mu \\
&\geq \lim_{n\to\infty} c \int_{A_n} s\mu \\
&= c \int_X s d\mu.
\end{aligned}$$

Now take $c \to 1$, we have that $\alpha \geq \int_X s d\mu$.
Then take sup of all simple $s \leq f$, we have that $\alpha \geq \int_X f d\mu$. ☐

Corollary 3.6. *For a measure space $(X, \mathcal{M}, \mu), A \in \mathcal{M}$, let $f_n : X \to [0, \infty]$ be measurable functions with $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$. Let $f(x) := \lim_{n\to\infty} f_n(x)$. We can consider the restriction $(A, \mathcal{M}' := \{B \cap A : B \in \mathcal{M}\}, \mu|_{\mathcal{M}'})$, and we will have*

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu$$

Corollary 3.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be measurable. Let $s_n : X \to [0, \infty]$ be any measurable simple functions with $0 \leq s_1 \leq s_2 \leq \cdots \leq \infty$ with $f(x) = \lim_{n\to\infty} s_n(x)$. We have*

$$\lim_{n\to\infty} \int_X s_n d\mu = \int_X f d\mu.$$

Proposition 3.8 (finite additivity for positive functions)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g : X \to [0, \infty]$ be measurable functions, then*

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Proof. Approximate f, g by simple functions s_n, t_n, such that $\lim_{n\to\infty} s_n(x) = f(x), \lim_{n\to\infty} t_n(x) = g(x)$ and $0 \leq s_1 \leq \cdots \leq f, 0 \leq t_1 \leq \cdots \leq g$.

Notice that $0 \le s_1 + t_1 \le \cdots \le f + g$, and $\lim_{n\to\infty}(s_n + t_n)(x) = (f + g)(x)$. Thus

$$\begin{aligned}
\int_X (f+g)d\mu &= \lim_{n\to\infty} \int_X (s_n + t_n)d\mu \\
&= \lim_{n\to\infty} \left(\int_X s_n d\mu + \int_X st_n d\mu \right) \\
&= \lim_{n\to\infty} \int_X s_n d\mu + \lim_{n\to\infty} \int_X t_n d\mu \\
&= \int_X f d\mu + \int_X g d\mu.
\end{aligned}$$

$\square$

Corollary 3.9 (countable additivity for positive functions)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to [0, \infty]$ be measurable functions. Then*

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

is measurable and

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n(x) d\mu.$$

Proof. Define $g_n(x) := \sum_{i=1}^{n} f_i(x)$, then $0 \le g_1 \le \cdots \le f$ and $\lim_{n\to\infty} g_n = f$.
By previous proposition and induction,

$$\int_X g_n d\mu = \sum_{i=1}^{n} \int_X f_n d\mu.$$

By LMCT, we have

$$\begin{aligned}
\int_X f d\mu &= \int_X \lim_{n\to\infty} g_n d\mu \\
&= \lim_{n\to\infty} \int_X g_n d\mu \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int_X f_i d\mu \\
&= \sum_{n=1}^{\infty} \int_X f_n d\mu
\end{aligned}$$

$\square$

Theorem 3.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f : X \to [0, \infty]$ a measurable function. Then $\lambda : \mathcal{M} \to [0, \infty]$ defined by*

$$\lambda(A) := \int_A f d\mu$$

is a measure on (X, M). Moreover, for some $g : X \to [0, \infty]$ such that fg is measurable, then

$$\int_X g d\lambda = \int_X g f d\mu.$$

Proof. Let $A = \bigsqcup_{n=1}^{\infty} A_n$ with A_n disjoint measurable subsets of X. We have that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$, and thus

$$\begin{aligned}\lambda(A) &= \int_X \chi_A f d\mu \\ &= \int_X \sum_{n=1}^{\infty} \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \int_X \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \lambda(A_n)\end{aligned}$$

Thus λ is a measure.
In addition, when $g = \chi_A$ for $A \in \mathcal{M}$, we have that $\int_X g d\lambda = \lambda(A) = \int_X \chi_A f d\mu = \int_X g f d\mu$. And thus simple functions, and thus all non-negative measurable functions by LMCT. ☐

3.2 Integration of real and complex functions

Definition 3.3. For $f : X \to \mathbb{R} \cup \{\pm\infty\}$, we define $f^+(x) := \max(f(x), 0), f^-(x) := \max(-f(x), 0)$, and thus $f = f^+ - f^-$, with both $f^+.f^- : X \to [0, \infty]$. We define

$$\int_X f d\mu := \int_X f^+ d\mu - \int_X f^- d\mu$$

when only one of the integrals is ∞.

Definition 3.4. Let $(X, \mathcal{M}, \mu)$ be a measure space, the define

$$\mathcal{L}^1(X, \mu) := \left\{ f : X \to \mathbb{C} \,\middle|\, \int_X |f| d\mu < \infty \right\}$$

be the **set of Lebesgue integrable functions**.

Definition 3.5. For $f = u + iv \in \mathcal{L}^1(X, \mu)$, where $u, v : X \to \mathbb{R}$, then the integral of f is defined as

$$\int_X f d\mu := \int_X u d\mu + i \int_X v d\mu.$$

Proposition 3.11. *The above integral is well-defined.*

Proof. u^+, u^-, v^+, v^- are measurable, and $0 \leq u^+, u^- \leq |u| \leq |f|$, thus each integral is finite. ☐

Proposition 3.12. *For $f = u + iv \in \mathcal{L}^1(X, \mu)$, where $u, v : X \to \mathbb{R}$, we have*

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu.$$

Proof. By definition. ☐

Theorem 3.13. *$\forall f, g \in \mathcal{L}^1(X, \mu), \alpha \in \mathbb{C}$, we have that $\alpha f + g \in \mathcal{L}^1(X, \mu)$ and*

$$\int_X \alpha f + g d\mu = \alpha \int_X f d\mu + \int_X g d\mu.$$

Thus, $\mathcal{L}^1(X, \mu)$ is a vector space over $\mathbb{C}$.

Proof. Clearly $\alpha f + g$ is measurable. In addition,

$$\begin{aligned}
\int_X |\alpha f + g| d\mu &\leq \int_X |\alpha f| + |g| d\mu \\
&= \int_X |\alpha||f| d\mu + \int_X |g| d\mu \\
&= |\alpha| \int_X |f| d\mu + \int_X |g| d\mu \\
&< \infty
\end{aligned}$$

Now we check the addition: Consider ant $f = a + ib, g = c + id : X \to \mathbb{C}$, such that $a, b, c, d : X \to \mathbb{R}$.

$$\begin{aligned}
(a+c)^+ - (a+c)^- &= a + c \\
&= (a^+ - a^-) + (c^+ - c^-) \\
&= (a^+ + c^+) - (a^- + c^-). \\
(a+c)^+ + (a^- + c^-) &= (a+c)^- + (a^+ + c^+),
\end{aligned}$$

where both sides of the equality are sums of two non-negative functions. Thus we have

$$\begin{aligned}
\int_X (a+c)^+ + (a^- + c^-) d\mu &= \int_X (a+c)^- + (a^+ + c^+) d\mu \\
\int_X (a+c)^+ d\mu + \int_X (a^- + c^-) d\mu &= \int_X (a+c)^- d\mu + \int_X (a^+ + c^+) d\mu \\
\int_X (a+c)^+ d\mu - \int_X (a+c)^- d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\
\int_X (a+c) d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\
&= \int_X a^+ d\mu + \int_X c^+ d\mu - \int_X a^- d\mu - \int_X c^- d\mu \\
&= \left(\int_X a^+ d\mu - \int_X a^- d\mu \right) + \left(\int_X c^+ d\mu - \int_X c^- d\mu \right) \\
&= \int_X a d\mu + \int_X c d\mu \\
\int_X (f+g) d\mu &= \int_X (a+c) \mu + i \int_X (b+d) d\mu \\
&= \int_X a d\mu + \int_X c d\mu + i \int_X b d\mu + i \int_X d d\mu \\
&= \left(\int_X a d\mu + i \int_X b d\mu \right) + \left(\int_X c d\mu + i \int_X d d\mu \right) \\
&= \int_X f d\mu + \int_X g d\mu.
\end{aligned}$$

Now we check the scalar multiplication: $\forall \alpha \geq 0$, we have $\int_X \alpha f d\mu = \alpha \int_X f d\mu$ by definition.
We can also check for $\alpha = -1$ and $\alpha = i$, and conclude this holds for all $\alpha \in \mathbb{C}$. □

Theorem 3.14. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f \in \mathcal{L}^1(X, \mu)$, then*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$$

Proof. Let $\alpha := \int_X f d\mu \in \mathbb{C}$, and let $\beta \in \mathbb{C}, |\beta| = 1$, such that $\alpha\beta = |\alpha|$. Take $u = Re(\beta f) : X \to \mathbb{R}$, note

$u \le |\beta f| = |f|$. Now

$$\begin{aligned}\left|\int_X f d\mu\right| &= |\alpha| \\ &= \beta\alpha \\ &= \beta \int_X f d\mu \\ &= \int_X \beta f d\mu \\ &= \int_X u d\mu \\ &\le \int_X |f| d\mu\end{aligned}$$

□

3.3 Lebesgue Dominated Convergence Theorem

Lemma 3.15 (Fatou's)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to [0, \infty]$ be measurable functions. Then*

$$\int_X (\liminf_n f_n) d\mu \le \liminf_n \int_X f_n d\mu$$

Proof. Let $g_n(x) := \inf_{i \ge n} f_i(x)$, then $\liminf_n f_n(x) = \lim_{n\to\infty} g_n(x)$.
Also, $g_n \le f_n$, so $\int_X g_n d\mu \le \int_X f_n d\mu,\ \forall n \ge 1$.
Note g_n is measurable, and $0 \le g_1 \le g_2 \le \cdots$.
By LMCT,

$$\lim_{n\to\infty} \int_X g_n d\mu = \int_X (\lim_{n\to\infty} g_n) d\mu = \int_X (\liminf_n f_n) d\mu.$$

Since the left hand side converges,

$$\int_X (\liminf_n f_n) d\mu = \lim_{n\to\infty} \int_X g_n d\mu = \liminf_n \int_X g_n d\mu \le \liminf_n \int_X f_n d\mu.$$

□

Theorem 3.16 (Lebesgue Dominated Convergence)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{C}$ be measurable functions such that $f(x) := \lim_{n\to\infty} f_n(x)$ exists $\forall x \in X$. If there is $0 \le g(x) \in \mathcal{L}^1(X, \mu)$, such that $\forall x \in X, \forall n \in \mathbb{N},\ |f_n(x)| \le g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n\to\infty} \int_X |f - f_n| d\mu = 0.$$

Proof. Firstly, $\forall n \in \mathbb{N}, x \in X,\ |f_n(x)| \le g(x)$ implies that $\forall x \in X,\ |f(x)| \le g(x)$, and thus $\int_X |f| d\mu \le \int_X g d\mu$. This hows that $f \in \mathcal{L}^1(X, \mu)$.
Notice that $|f_n - f| \le |f_n| + |f| \le 2g$. Thus $2g - |f_n - f| \ge 0$. By Fatou's Lemma, we have that

$\int_X(\liminf(2g-|f_n-f|))d\mu \leq \liminf \int_X 2g-|f_n-f|d\mu$. Thus

$$\begin{aligned}
\int_X 2g d\mu &= \int_X (2g - \liminf(|f_n-f|))d\mu \\
&= \int_X (\liminf(2g-|f_n-f|))d\mu \\
&\leq \liminf \int_X 2g - |f_n-f|d\mu \\
&= \int_X 2gd\mu + \liminf(-\int_X |f_n-f|d\mu) \\
&= \int_X 2gd\mu - \limsup \int_X |f_n-f|d\mu \\
0 &\leq -\limsup \int_X |f_n-f|d\mu
\end{aligned}$$

Thus $0 \leq \liminf \int_X |f_n-f|d\mu \leq \limsup \int_X |f_n-f|d\mu \leq 0$, and thus $\lim_{n\to\infty} \int_X |f_n-f|d\mu = 0$. Finally,

$$\begin{aligned}
\left|\lim_{n\to\infty} \int_X f_n d\mu - \int_X f d\mu\right| &= \left|\lim_{n\to\infty} \int_X (f_n-f)d\mu\right| \\
&\leq \lim_{n\to\infty} \int_X |f_n-f|d\mu \\
&= 0
\end{aligned}$$

□

3.4 Almost Everywhere

Definition 3.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, $A \in \mathcal{M}$, and $P = \{p(x)\}_{x\in A}$ be a family of logical statements, then we say the property P holds or is true μ-everywhere on A, if $\exists N \in \mathcal{M}$, such that $\mu(N) = 0$ and $\forall x \in A \setminus N,\ p(x) = True$.

Definition 3.7. For measurable functions $f, g : X \to Y$, we say that $f = g$ μ-almost everywhere if

$$\mu(\{x \in X | f(x) \neq g(x)\}) = 0.$$

Remark. For some $A \in \mathcal{M}$, we have that $\mu(A \cap N) \leq \mu(N) = 0$, and thus $\int_{A\cap N}(f-g)d\mu = 0$. Thus $\int_A(f-g)d\mu = \int_{A\cap N}(f-g)d\mu + \int_{A\setminus N}(f-g)d\mu = 0$.

Definition 3.8. Let $(f_n)_{n=1}^{\infty}$, we say $f_n \to f$ a.e. if $f_n(x) \to f(x)$ for $\mu-$a.e.$x \in X$.

Proposition 3.17. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *If $f : X \to [0, \infty]$ is measurable, we have $f = 0$ μ-a.e. $\iff \int_E f d\mu = 0\ \forall E \in \mathcal{M}$.*
2. *If $f \in \mathcal{L}^1(X, \mu)$ we have $f = 0$ μ-a.e. $\iff \int_E f d\mu = 0\ \forall E \in \mathcal{M}$.*
3. *If $f \in \mathcal{L}^1(X, \mu)$, and $\left|\int_X f d\mu\right| = \int_X |f|d\mu$, there exist must a constant α such that $\alpha f = |f|$ almost everywhere.*

Proof. 1. Let $N = \{x \in X : f(x) > 0\}$.
Suppose $f = 0$ μ-a.e., then $\mu(N) = 0$.
We have

$$\int_X f d\mu = \int_{X\setminus N} f d\mu + \int_N f d\mu = \int_{X\setminus N} 0 d\mu + 0 = 0.$$

Thus $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$.
Now suppose $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$.
Let $A_n := \left\{x \in X : f(x) > \frac{1}{n}\right\}$, then we have

$$\frac{1}{n}\mu(A_n) = \int_{A_n} \frac{1}{n} d\mu \leq \int_{A_n} f d\mu = 0.$$

Thus $\mu(A_n) = 0$.
Notice that $N = \bigcup_{n=1}^{\infty} A_n, A_1 \subseteq A_2 \subseteq \cdots$, thus $\mu(N) = \lim_{n\to\infty} \mu(A_n) = 0$.

2. Suppose $f = 0$ μ-a.e., we have that $|f| = 0$ μ-a.e., thus $\left|\int_E f d\mu\right| \leq \int_E |f| d\mu = 0$.
Now suppose $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$.
Let $f = u + iv$, then we have $\int_E u d\mu = \int_E v d\mu = 0 \ \forall E \in \mathcal{M}$.
Let $u = u^+ - u^-$, and $E = \{x \in X : u(x) \geq 0\}$.

$$\begin{aligned}\int_X u^+ d\mu &= \int_{X\setminus E} u^+ d\mu + \int_E u^+ d\mu \\ &= \int_{X\setminus E} 0 d\mu + \int_E u d\mu \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Thus $u^+ = 0$ μa.e..
Similarly for u^-, and thus $u = 0$ $\mu -$ a.e..
Similarly for v, and thus $f = 0$ $\mu -$ a.e..

□

Theorem 3.18 (Lebesgue Dominated Convergence - almost everywhere)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{C}$ be measurable functions, defined $\mu-$almost everywhere on X, such that $f(x) := \lim_{n\to\infty} f_n(x)$ is defined $\mu-$almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that for $\mu-$almost everywhere $x \in X, \forall n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$, then $f \in \mathcal{L}^1(X, \mu)$, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n\to\infty} \int_X |f - f_n| d\mu = 0.$$

Proof. Let N_n denote the zero measure set where f_n is not defined. Let N' denote the zero measure set where f is not defined. Then

$$N := N' \cup \{x \in X : \exists n \in \mathbb{N}, \text{ such that } |f_n(x) > g(x)|\} \cup \bigcup_{n=1}^{\infty} N_n$$

is measurable and has zero measure.
Define

$$h_n(x) := \begin{cases} f_n(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \ h(x) := \begin{cases} f(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \ g'(x) := \begin{cases} g(x) & x \in X \setminus N, \\ 0 & x \in N. \end{cases}$$

It is clear $\forall x \in X, h_n(x) \to h(x)$ point-wise, and dominated by $g'(x)$.
Since $g = g' \mu$-a.e. and thus $g' \in \mathcal{L}^1(X, \mu)$, by LDCT, we have

$$\lim_{n\to\infty} \int_X |f - f_n| d\mu = \lim_{n\to\infty} \int_X |g - g_n| d\mu = 0.$$

□

Theorem 3.19 (countable additivity)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{C}$ be measurable functions, defined $\mu-$almost everywhere on X, such that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$. We have that $f(x) := \sum_{n=1}^{\infty} f_n(x)$ exists $\mu-$almost everywhere for $x \in X$, and that $f \in \mathcal{L}^1(X, \mu)$, and that*

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

Proof. For each n, let $D_n \subseteq X$ be the domain of f_n, then by assumption $\mu(X \setminus D_n) = 0$.
Let $D := \bigcap_{n=1}^{\infty} D_n, g(x) := \sum_{n=1}^{\infty} |f_n(x)|$. Note that $\mu(D^c) = \mu((\bigcap_{n=1}^{\infty} D_n)^c) = \mu(\bigcup_{n=1}^{\infty} D_n^c) = 0$.
Thus $g : X \to [0, \infty]$ is defined almost everywhere by Monotone Convergence Theorem.
By countable additivity of positive functions and assumption,

$$\int_X g d\mu = \sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty,$$

so $g \in \mathcal{L}^1$.
Let $A := \{x \in D : g(x) < \infty\}$, then we have $\mu(A^c) = 0$.
By definition, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ absolutely on A. Thus $f \in \mathcal{L}^1(A, \mathcal{M}|_A, \mu|_{\mathcal{M}|_A})$.
Let $h_n = \sum_{i=1}^{n} f_i$ on A, then $|h_n| \leq \sum_{i=1}^{n} |f_n| \leq g$. Also, we have that $h_n(x) \to f(x)$ for any $x \in A$, then by LDCT and linearity, we have

$$\int_A f d\mu = \lim_{n\to\infty} \int_A h_n d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_A f_i d\mu = \sum_{n=1}^{\infty} \int_A f_n d\mu.$$

Since $\mu(A^c) = 0$, we have that $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$. $\square$

3.5 Complete Measure

Theorem 3.20. *Let $(X, \mathcal{M}, \mu)$ be a measure space, let*

$$\mathcal{M}^* := \{A \subseteq X : \exists B, C \in \mathcal{M}, \text{ such that } B \subseteq A \subseteq C, \mu(C \setminus B) = 0\}.$$

Define $\mu^*(A) = \mu(B) = \mu(C)$, then $\mathcal{M}^*$ is a σ-algebra, μ^* is a measure, and $(X, \mathcal{M}^*, \mu^*)$ is a measure space.

Proof. $X \in \mathcal{M}$, and $X \subseteq X \subseteq X$ and $\mu(X \setminus X) = 0$, thus $X \in \mathcal{M}^*$.
Let $A \in \mathcal{M}^*$, then there are $B, C \in \mathcal{M}$ such that $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$. Thus $B^c \supseteq A^c \supseteq C^c$, and $B^c, C^c \in \mathcal{M}$. In addition, $\mu(B^c \setminus C^c) = \mu(B^c \cap C) = \mu(C \setminus B) = 0$. Thus $A^c \in \mathcal{M}^*$.
Let $A_n \in \mathcal{M}^*$ and $A = \bigcup_n A_n$, and $B_n, C_n \in \mathcal{M}$ such that $B_n \subseteq A_n \subseteq C_n, \mu(C_n \setminus B_n) = 0$. Let $B = \bigcup_n B_n, C = \bigcup_n C_n \in \mathcal{M}$, thus $B \subseteq A \subseteq C$. Now $\mu(C \setminus B) = \mu(\bigcup_n C_n \setminus B) \leq \mu(\bigcup_n C_n \setminus B_n) \leq \sum_n \mu(C_n \setminus b_n) = 0$.
Thus $\mathcal{M}^*$ is a σ-algebra.
Now suppose $B, B', C, C' \in \mathcal{M}$, and $A, \in \mathcal{M}^*$, with $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$, and $B' \subseteq A \subseteq C', \mu(C' \setminus B') = 0$.
Thus $B \setminus B' \subseteq A \setminus B' \subseteq C' \setminus B'$, and thus $\mu(B \subset B') \leq \mu(C' \setminus B') = 0$. Thus $\mu(B) = \mu(B \cap B') + \mu(B \subset B') = \mu(B \cap B')$. Similarly, we can show that $\mu(B') = \mu(B \cap B')$. Thus $\mu^* : \mathcal{M}^* \to [0, \infty]$ is well-defined.
Consider A_n a sequence of disjoint sets in $\mathcal{M}^*$, and $B_n, C_n \in \mathcal{M}$ as above. We have $\mu^*(A) = \mu(B) = \mu(\bigsqcup_n b_n) = \sum_n \mu(B_n) = \sum_n \mu(A_n)$. $\square$

Corollary 3.21. *$(X, \mathcal{M}^*, \mu^*)$ has the property that if $N \in \mathcal{M}^*$ has $\mu(N) = 0$, we always have*

$$\forall A \subseteq N, \ A \in \mathcal{M}^*, \mu^*(A) = 0.$$

Proof. Notice that $\forall A \subseteq N$, we have $\mu(N) = \mu(\emptyset) = 0$, with $\emptyset \subseteq A \subseteq N$, so $A \in \mathcal{M}^*, \mu^*(A) = 0$. $\square$

Definition 3.9. $(X, \mathcal{M}^*, \mu^*)$ defined above is called the **completion** of $(X, \mathcal{M}, \mu)$. In addition, we say $(X, \mathcal{M}, \mu)$ is **complete** if $(X, \mathcal{M}, \mu) = (X, \mathcal{M}^*, \mu^*)$

Remark. If there is some $A \in \mathcal{M}$ such that $\mu(A^c) = 0$, then for any measurable $f : A \to Y$, we can extend it to X by $\forall x \in A^c,\ f(x) := 0$. Furthermore, if $(X, \mathcal{M}, \mu)$ is complete, we can extend f to whatever value we want. One can check that $f : X \to Y$ is measurable, and the integral $\int_X f d\mu$ does not depend on the extension.

Proposition 3.22. *If $(X, \mathcal{M}, \mu)$ is a complete measure, we always have that property P holds μ-a.e. iff*

$$\mu(\{x \in A : p(x) = False\}) = 0.$$

Proof. If P holds μ-a.e., there is $\exists N \in \mathcal{M}$, such that $\mu(N) = 0$ and $\forall x \in A \setminus N,\ p(x) = True$. Since $\{x \in A : p(x) = False\} \subseteq A \setminus (A \setminus N) = N$, we have $\mu(\{x \in A : p(x) = False\}) = 0$.
On the other hand, if $\mu(\{x \in A : p(x) = False\}) = 0$, we can just let $N := \mu(\{x \in A : p(x) = False\})$. Notice $\mu(N) = 0$, and $\forall x \in A \setminus N,\ p(x) = True$. □

Proposition 3.23. *Let μ be a complete measure on $(X, \mathcal{M})$, suppose that f is measurable, and $g = f$, a.e., then g is also measurable. Moreover, if (f_n) is a sequence of measurable functions, and $f_n \to f$, μ-a.e., we always have that f is also measurable.*

Proof. Suppose f is measurable, and we consider $D := \{x : X | f(x) \neq g(x)\}, \mu(D) = 0$.
Now let $B \subseteq \mathbb{R}$ be a Borel set, we need to show that $\{x \in X | g(x) \in B\} \in \mathcal{M}$.
Write $\{x \in X | g(x) \in B\} = (\{x \in X | g(x) \in B\} \cap D) \sqcup (\{x \in X | g(x) \in B\} \setminus D)$.
Since μ is complete, we have that $\{x \in X | g(x) \in B\} \cap D \in \mathcal{M}$ and has measure zero. Since f is measurable, we have that $f^{-1}(B) = \{x \in X | f(x) \in B\} \supseteq \{x \in X | f(x) = g(x) \in B\} = \{x \in X | g(x) \in B\} \setminus D$ is measurable.
Since μ is complete, we have that $\{x \in X | g(x) \in B\} \setminus D$ is measurable.
Thus $\{x \in X | g(x) \in B\} \in \mathcal{M}$ is measurable.
For the second part, consider $g = \limsup_{n \to \infty} f_n$. □

4 Construction of Measure

4.1 Caratheodory Theorem

Definition 4.1. Let X be a non-empty set, an **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\mu^*(\emptyset) = 0$
2. Monotone: $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$
3. Countable subadditive: For any $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$, we have that $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$

Proposition 4.1. *Let $\mathcal{C} \subseteq \mathcal{P}(x)$ with $\emptyset, X \in \mathcal{C}$. Let $\tilde{\mu} : \mathcal{C} \to [0, \infty]$ be a function such that $\tilde{\mu}(\emptyset) = 0$. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by $\mu^*(A) := \inf \{\sum_{i=1}^\infty \tilde{\mu}(C_i) : C_i \in \mathcal{C}, A \subseteq \bigcup_{i=1}^\infty C_i\}$. Then μ^* is an outer measure.*

Proof. Clearly $\mu^*(\emptyset) = 0$, since $\emptyset \in \mathcal{C}$.

In addition, $A \subseteq B \subseteq \bigcup_{i=1}^\infty C_i$ for any cover for B, and thus $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

Given any $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$. If $\sum_{n=1}^\infty \mu^*(A_n) = \infty$, then $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$ trivially.

Now assume that $\sum_{n=1}^\infty \mu^*(A_n) < \infty$. Consider any $\epsilon > 0$. For each $n \geq 0$, choose $(C_{n,i})_{i=1}^\infty \subset \mathcal{C}$, such that $A_n \subseteq \bigcup_{i=1}^\infty C_{n,i}$ and $\mu^*(A_n) \leq \sum_{i=1}^\infty \tilde{\mu}(C_{n,i}) < \mu^*(A_n) + \frac{\epsilon}{2^n}$. Thus $\bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty C_{n,i}$, so by construction of the outer measure,

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^\infty A_n\right) &\leq \sum_{n=1}^\infty \sum_{i=1}^\infty \tilde{\mu}(C_{n,i}) \\ &\leq \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) \\ &\leq \sum_{n=1}^\infty \mu^*(A_n) + \epsilon. \end{aligned}$$

Taking $\epsilon \to 0$, we have $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$. □

Remark. Notice that if $X = \mathbb{R}$, and we take $\mathcal{C} := \{(a, b] : a < b \in \mathbb{R}\}$ to be the collection of finite half open intervals, and $\mu((a, b])$ to be the length of the interval $b - a$, then the outer measure is the Lebesgue outer measure.

Definition 4.2. For an outer measure μ^*, we say $A \subseteq X$ is μ^***-measurable**, or satisfies the **Caratheodory condition** if

$$\forall E \subseteq X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Remark. To check $A \in \mathcal{M}$, it suffices to check for $\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(E \cap A^c)$, since $\mu^*(E) \leq \mu^*(A \cap E) + \mu^*(E \cap A^c)$ always holds by subadditivity of μ^*. Further, when $\mu^*(E) = \infty$, the inequality is always true, so it suffices to check

$$\forall E \subseteq X, \text{ such that } \mu^*(E) < \infty,\ \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Lemma 4.2. *Let μ^* be an outer measure on X, and $\mathcal{M}$ be the μ^*-measurable subsets of X, then $\mathcal{M}$ is an algebra, and $\mu := \mu^*|_{\mathcal{M}}$ has finite additivity.*

Proof. When $\mu^*(E) = \infty$, this holds trivially, and thus it suffices to only check for $\mu^*(E) < \infty$.

- Clearly $\emptyset \in \mathcal{M}$.
- $A \in \mathcal{M} \implies A^c \in \mathcal{M}$ since the condition is symmetric.
- Now consider $A, B \in \mathcal{M}$. For any $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*((E \cap A) \cup (E \cap A^c \cap B)) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).\end{aligned}$$

  Thus $A \cup B \in \mathcal{M}$.

Thus $\mathcal{M}$ is an algebra.
To show finite additivity: Let $(A_i)_{i=1}^n$ be disjoint in $\mathcal{M}$, we will use induction on n.
Clearly, it is true for $n = 1$.
Now suppose it holds for n, let $B = \bigsqcup_{i=1}^n A_i$. Since $\mathcal{M}$ is an algebra, we have $B \in \mathcal{M}$. For any $E \subseteq X$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\ &= \mu^*\left(\bigsqcup_{i=1}^n (E \cap A_i)\right) + \mu^*(E \cap B^c) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B^c).\end{aligned}$$

Taking $E = \bigsqcup_{i=1}^{n+1} A_i$, we have

$$\begin{aligned}\mu^*\left(\bigsqcup_{i=1}^{n+1} A_i\right) &= \sum_{i=1}^n \mu^*\left(\left(\bigsqcup_{i=1}^{n+1} A_i\right) \cap A_i\right) + \mu^*\left(\bigsqcup_{i=1}^{n+1} A_i \cap B^c\right) \\ &= \sum_{i=1}^n \mu^*(A_i) + \mu^*(A_{n+1}) \\ &= \sum_{i=1}^{n+1} \mu^*(A_i).\end{aligned}$$

By induction, we have finite additivity for any $n \geq 1$. □

Theorem 4.3 (Caratheodory)**.** *Let μ^* be an outer measure on X, and $\mathcal{M}$ be the μ^*-measurable subsets of X, then $\mathcal{M}$ is a σ-algebra, and $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure.*

Proof. Consider any $\{A_i\} \subset \mathcal{M}, B := \bigcup_{i=1}^{\infty} A_i$. By taking $\tilde{A}_n := A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ we can WLOG assume A_n are pair-wise disjoint, and $B = \bigsqcup_{i=1}^{\infty} A_i$.
For any $E \in X$, we have $\forall n \geq 1, \bigsqcup_{i=1}^{n} A_i \in \mathcal{M}$, and thus

$$\begin{aligned}
\mu^*(E) &= \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)\right) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)^c\right) \\
&= \mu^*\left(\bigsqcup_{i=1}^{n} (E \cap A_i)\right) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)^c\right) \\
&= \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{n} A_i\right)^c\right) \\
&\geq \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{\infty} A_i\right)^c\right) \\
&= \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap B^c).
\end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned}
\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c) \\
&\geq \mu^*\left(\bigsqcup_{i=1}^{\infty} (E \cap A_i)\right) + \mu^*(E \cap B^c) \\
&\geq \mu^*\left(E \cap \bigsqcup_{i=1}^{\infty} A_i\right) + \mu^*(E \cap B^c) \\
&= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\
&\geq \mu^*((E \cap B) \cup (E \cap B^c)) \\
&= \mu^*(E).
\end{aligned}$$

Thus $B \in \mathcal{M}$, and thus $\mathcal{M}$ is a σ-algebra.
In addition, taking $E = B$, we have

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) + \mu^*(B \cap B^c) = \sum_{i=1}^{\infty} \mu^*(A_i) + \mu^*(\emptyset) = \sum_{i=1}^{\infty} \mu^*(A_i),$$

which shows countable additivity, and thus $\mu^*|_{\mathcal{M}}$ is a measure.
To show completeness, suppose $A \subseteq X$ such that $\mu^*(A) = 0$, then for any $E \subseteq X$, we have

$$\begin{aligned}
\mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&\leq \mu^*(A) + \mu^*(E) \\
&= \mu^*(E).
\end{aligned}$$

Thus we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and thus $A \in \mathcal{M}$. $\square$

4.2 Premeasures

Definition 4.3. Recall an algebra of subsets of a set X is a family of subsets that is closed under complements, finite unions, and finite intersections, and contains the empty set.

Definition 4.4. A premeasure on an algebra of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ is a function $\tilde{\mu} : \mathcal{A} \to [0, \infty]$, such that $\tilde{\mu}$ is countably additive. Namely, if $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$ are disjoint, and $\bigsqcup_{i=1}^{\infty} A_i \subseteq \mathcal{A}$, then we have

$$\tilde{\mu}\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \tilde{\mu}(A_i).$$

Remark. If $\mathcal{A}$ is a σ-algebra, a premeasure on $\mathcal{A}$ is always a measure.

Theorem 4.4. *Let $\mathcal{A}$ be an algebra of subsets of X, and $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ be a premeasure. Apply the Caratheodory Theorem 4.3 to the outer measure μ^* by proposition 4.1 gives a complete measure space $(X, \mathcal{M}, \mu)$, such that $\mathcal{A} \subseteq \mathcal{M}$, and $\mu|_{\mathcal{A}} = \tilde{\mu}$.*

Proof. Choose any $A \in \mathcal{A}$. We have

$$\mu^*(A) := \inf\left\{\sum_{i=1}^{\infty} \tilde{\mu}(A_i) : A_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^{\infty} A_i\right\} \leq \tilde{\mu}(A).$$

Choose any $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$, such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$. Let $B_i = (A \cap A_i) \setminus \bigcup_{j=1}^{i-1} A_j$. Notice that $B_i \in \mathcal{A}$, and are pairwise disjoint, and $A = \bigsqcup_{i=1}^{\infty} B_i$. Since $\tilde{\mu}$ is a premeasure, $\tilde{\mu}(A) = \sum_{i=1}^{\infty} \tilde{\mu}(B_i) \leq \sum_{i=1}^{\infty} \tilde{\mu}(A_i)$. Since above holds for any $\bigcup_{i=1}^{\infty} A_i \supseteq A$, we can see that $\mu^*(A) \geq \tilde{\mu}(A)$, which forces

$$\mu^*(A) = \tilde{\mu}(A).$$

Now it remains to show $A \in \mathcal{M}$, which is the same as A is μ^*-measurable. Given any $E \subseteq X$ with $\mu^*(E) < \infty$. Fix any $\epsilon > 0$, there are $(E_i)_{i=1}^{\infty} \subseteq \mathcal{A}$, such that $E \subseteq \bigcup_{i=1}^{\infty} E_i$, and

$$\sum_{i=1}^{\infty} \tilde{\mu}(E_i) < \mu^*(E) + \epsilon.$$

Notice that $E \cap A \subseteq \bigcup_{i=1}^{\infty} E_i \cap A$, and $E \cap A^c \subseteq \bigcup_{i=1}^{\infty} E_i \cap A^c$. Also, each $E_i \cap A, E_i \cap A^c \in \mathcal{A}$, since $\mathcal{A}$ is an algebra. Thus,

$$\begin{aligned}
\mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \sum_{i=1}^{\infty} \mu^*(E_i \cap A) + \sum_{i=1}^{\infty} \mu^*(E_i \cap A^c) \\
&= \sum_{i=1}^{\infty} \tilde{\mu}(E_i \cap A) + \sum_{i=1}^{\infty} \tilde{\mu}(E_i \cap A^c) \\
&= \sum_{i=1}^{\infty} \tilde{\mu}(E_i) \\
&< \mu^*(E) + \epsilon
\end{aligned}$$

Now take $\epsilon \to 0$, we have that

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E).$$

This shows that A is μ^*-measurable, which means $\mathcal{A} \subseteq \mathcal{M}$.

We have shown that $\tilde{\mu} = \mu^*|_A$, but we also know that $\mu = \mu^*|_{\mathcal{M}}$, and $\mathcal{A} \subseteq \mathcal{M}$, so $\mu|_{\mathcal{A}} = \tilde{\mu}$. ☐

Definition 4.5. A premeasure $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ on an algebra $\mathcal{A}$ for X is σ-finite if there are $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$, such that $\tilde{\mu}(A_i) < \infty$ and $\bigcup_{i=1}^{\infty} A_i = X$.

Proposition 4.5. *Let $\mathcal{A}$ be an algebra of sets on X. Let $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ be a premeasure, with the corresponding complete measure space $(X, \mathcal{M}, \mu)$ as in the above theorem. Suppose $(X, \mathcal{N}, \nu)$ is a measure space with $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M}$ and $\nu|_{\mathcal{A}} = \tilde{\mu}$. Then if $\tilde{\mu}$ is σ-finite, we have that*

$$\nu = \mu|_{\mathcal{N}},$$

so $\mu|_{\mathcal{N}}$ is the unique extension of $\tilde{\mu}$ to a measure on $\mathcal{N}$.

4.3 Lebesgue-Stieltjes Measures

Definition 4.6. Let μ be a Borel measure on $\mathbb{R}$, such that $\mu(\mathcal{K}) < \infty$ for any compact $\mathcal{K} \subseteq \mathbb{R}$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \begin{cases} \mu((0,x]) & \text{if } x \geq 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}$

Proposition 4.6. *F is monotone non-decreasing. i.e. If $b \geq a$, then $F(b) - F(a) \geq 0$.*

Proof. For $0 < a < b$, we have that $\mu((a,b]) = \mu((0,b] \setminus (0,a]) = \mu((0,b]) - \mu((0,a]) = F(b) - F(a)$.
For $0 \geq b > a$, we have that $\mu((a,b]) = \mu((a,0] \setminus (b,0]) = \mu((a,0]) - \mu((b,0]) = -F(a) - (-F(b)) = F(b) - F(a)$.
Similarly, we can check for $a < 0 \leq b$. ☐

Proposition 4.7. *F is right continuous.*

Proof. Fix $x \geq 0 \in \mathbb{R}$, and choose any sequence $(x_n) \subseteq \mathbb{R}$ such that $x_n \geq x_{n+1}$ and $x_n \to x$.
Since μ is a measure and $\mu((0,x_1]) \leq \mu([0,x_1]) < \infty$, we have

$$\begin{aligned} F(x) &= \mu((0,x]) \\ &= \lim_{n\to\infty} \mu((0,x_n]) \\ &= \lim_{n\to\infty} F(x_n). \end{aligned}$$

Similar proof for $x < 0 \in \mathbb{R}$. ☐

Example 4.3.1. If $\mu = \delta_c, \delta_c(A) = \begin{cases} 0 & \text{if } c \in A \\ 1 & \text{if } c \notin A \end{cases}$, then F is the (translated) Heaviside function.

Example 4.3.2. If μ is the Lebesgue measure, then F is the identity function $F(x) = x$.

Now given a right-continuous increasing function F, we want to construct a measure.

Proposition 4.8. *Let $\mathcal{A}$ be the collection of sets consisting of all the finite disjoint unions of half-open intervals $(a,b], -\infty \leq a \leq b \leq \infty$. Then $\mathcal{A}$ is an algebra of sets.*

Proof. Firstly notice that for any interval $(a,b] \in \mathcal{A}$, we have that $(a,b]^c = [-\infty, a] \cup (b, \infty] \in \mathcal{A}$.
Also, any finite union of such disjoint unions can be written as a disjoint union. $(a,b] \cup (c,d] = (a,c] \cup (c,b] \cup (b,d]$ for $c < b$. We can show any finite union by induction. ☐

Definition 4.7. Let $\mathcal{A}$ be the algebra of sets consisting of all the finite disjoint unions of half-open intervals $(a,b], -\infty \leq a \leq b \leq \infty$. Let $F : \mathbb{R} \to \mathbb{R}$ be a right-continuous monotone non-decreasing function, we can extend F to $[-\infty, \infty]$ by $F(\pm\infty) := \lim_{x\to\pm\infty} F(x)$, which exists by MCT. Now define $\tilde{\mu}_F : \mathcal{A} \to [0,\infty]$ to be

$$\tilde{\mu}_F\left(\bigsqcup_{i=1}^{n} (a_i, b_i]\right) := \sum_{i=1}^{n} F(b_i) - F(a_i)$$

Lemma 4.9. *$\tilde{\mu}_F$ is a pre-measure*

Proof. We firstly show that $\tilde{\mu}_F$ is well defined.
Consider $I_i = (a_i, b_i]$ and $I = (a,b] = \bigsqcup_1^n I_i$
By reordering, we can WLOG assume that $a = a_1 < b_1 = a_2 < b_2 = \cdots < b_{n-1} = a_n < b_n = b$.
Let $a_{n+1} := b_n = b$, we have that

$$\begin{aligned} \tilde{\mu}_F\left(\bigsqcup_{i=1}^{n} (a_i, b_i]\right) &= \sum_{i=1}^{n} F(b_i) - F(a_i) \\ &= \sum_{i=1}^{\infty} F(a_{i+1}) - F(a_i) \\ &= F(a_{n+1}) - F(a_1) \\ &= F(b) - F(a) \\ &= \tilde{\mu}_F(I). \end{aligned}$$

Thus $\tilde{\mu}_F(I)$ does not depend on the decomposition of I. This extends to finite disjoint unions of half-open intervals. Hence $\tilde{\mu}_F$ is well-defined.
Monotone follows from the fact that F is increasing.
Consider pair-wise disjoint $(A_i)_{i=1}^{\infty} \in \mathcal{A}$, and $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$, we want to show $\tilde{\mu}_F(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \tilde{\mu}_F(A_i)$.
We first assume that each $A_i = (a_i, b_i]$ and $\bigsqcup_1^{\infty} A_i = (a, b]$
Notice that $\forall n \in \mathbb{N}$, we have that $\bigsqcup_{i=1}^{n}(a_i, b_i] \in \mathcal{A}$, and thus $(a, b] \setminus \bigsqcup_{i=1}^{n}(a_i, b_i] \in \mathcal{A}$.

$$\begin{aligned}
\tilde{\mu}_F((a,b]) &= \tilde{\mu}_F(\bigsqcup_{i=1}^{n}(a_i,b_i]) + \tilde{\mu}_F((a,b] \setminus \bigsqcup_{i=1}^{n}(a_i,b_i]) \\
&= \sum_{i=1}^{n} \tilde{\mu}_F((a_i,b_i]) + \tilde{\mu}_F((a,b] \setminus \bigsqcup_{i=1}^{n}(a_i,b_i]) \\
&\geq \sum_{i=1}^{n} \tilde{\mu}_F((a_i,b_i])
\end{aligned}$$

Thus $\tilde{\mu}_F((a,b]) \geq \sum_1^{\infty} \tilde{\mu}_F((a_i,b_i])$.
For the other direction, fix $\epsilon > 0$, by right continuity, $\exists \delta > 0$, such that $F(a+\delta) < F(a) + \epsilon, a + \delta < b$. Now suppose $b \neq \infty$, then $\exists \delta_i > 0, F(b_i + \delta_i) < F(b_i) + 2^{-i}\epsilon$.
Thus $[a_i + \delta, b_i] \subseteq (a_i, b_i + \delta_i)$, and thus $\{(a_i, b_i + \delta_i)\}$ is an open cover for $[a + \delta, b] = \bigcup_{i=1}^{\infty}[a_i + \delta, b_i]$. Since the closed interval is compact, there is a finite sub-cover $\left\{(a_{i_j}, b_{i_j} + \delta_{i_j})\right\}_{j=1}^{n}$. Then

$$\begin{aligned}
\sum_{j=1}^{n}(F(b_{i_j} + \delta_{i_j}) - F(a_{i_j})) &= \sum_{j=1}^{n} \tilde{\mu}_F((a_{i_j}, b_{i_j} + \delta_{i_j})) \\
&\geq \tilde{\mu}_F((a+\delta, b]) \\
&\geq F(b) - F(a+\delta)
\end{aligned}$$

since $\tilde{\mu}_F$ is monotone. Hence

$$\begin{aligned}
\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i,b_i]) &= \sum_{i=1}^{\infty}(F(b_i) - F(a_i)) \\
&\geq \sum_{j=1}^{n}(F(b_{i_j}) - F(a_{i_j})) \\
&\geq \sum_{j=1}^{n}(F(b_{i_j} + \delta_{i_j}) - 2^{-i_j}\epsilon - F(a_{i_j})) \\
&\geq F(b) - F(a+\delta) - \epsilon \\
&\geq F(b) - F(a) - 2\epsilon \\
&= \tilde{\mu}_F((a,b]) - 2\epsilon.
\end{aligned}$$

Take $\epsilon \to 0$, we have $\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i,b_i]) \geq \tilde{\mu}_F((a,b])$
When $b = \infty$, we have that $\forall N \geq a, \sum_{i=1}^{\infty} \tilde{\mu}_F((a_i,b_i]) \geq \tilde{\mu}_F((a,N]) = F(N) - F(a)$.
Hence $\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i,b_i]) \geq F(b) - F(a) = \lim_{N\to b} F(N) - F(a)$
Thus we have shown that $\sum_{i=1}^{\infty} \tilde{\mu}_F((a_i,b_i]) = \tilde{\mu}_F((a,b])$
If $A = \bigsqcup_1^{m}(c_i, d_i]$, we can use finite additivity and the previous case. □

Theorem 4.10. *Let $F : \mathbb{R} \to \mathbb{R}$ is monotone non-decreasing and right-continuous, then there is a complete measure space $(\mathbb{R}, \mathcal{M}, \mu_F)$, which extends $\tilde{\mu}_F$ as $\mu_F|_{\mathcal{A}} = \tilde{\mu}_F$, and the $\sigma-$algebra $\mathcal{M}$ contains $Bor(\mathbb{R})$ and $\mu_F|_{Bor(\mathbb{R})}$ is the unique extension of $\tilde{\mu}_F$, i.e. $\mu_F((a,b]) = F(b) - F(a)$.*

Conversely, given a Borel measure μ on $\mathbb{R}$, such that $\forall K$ compact, $\mu(K) < \infty$, there is a (up to constant) unique non-decreasing right-continuous F with $\mu = \mu_F|_{B_{\mathbb{R}}}$.

Proof. By the previous lemma, $\tilde{\mu}_F$ is a premeasure, so applying Caratheodory gives a complete measure space $(\mathbb{R}, \mathcal{M}, \mu_F)$. We have seen that the σ-algebra generated by $\mathcal{A}$ is $Bor(\mathbb{R})$, so $Bor(\mathbb{R}) \subseteq \mathcal{M}$. The uniqueness follows from the fact that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}(n, n+2]$, $\tilde{\mu}_F((n, n+2]) = F(n+2) - F(n) < \infty$ and thus $\tilde{\mu}_F$ is σ-finite.

Conversely, let F be the function defined at the beginning fo this section. Then we know that $\mu = \mu_F|_{B_\mathbb{R}}$ since μ is σ-finite and agree with the premeasure $\tilde{\mu}_F$ on the algebra $\mathcal{M}$ of all finite disjoint unions of half open interval.

If $G : \mathbb{R} \to \mathbb{R}$ is another monotone non-decreasing right-continuous function, then $\mu_F = \bar{\mu_G} \implies \tilde{\mu}_F((a, b]) = \mu_G((a, b]) \implies F(b) - F(a) = G(b) - G(a)$ for any $a < b$. Thus $\forall a \in \mathbb{R}, F(x) - G(x) = c := F(0) - G(0)$, which is a constant. $\square$

Example 4.3.3. The Lebesgue measure is got by taking $F(x) = x$.

Example 4.3.4. The Dirac measure

$$\delta_c(A) := \begin{cases} 1 & c \in A \\ 0 & c \notin A \end{cases}$$

is got by taking

$$F(x) = H_c(x) = \begin{cases} 1 & x \geq c \\ 0 & x < c \end{cases},$$

the Heaviside function.

Definition 4.8. A μ be a Borel measure on $\mathbb{R}$, such that $\mu(K) < \infty$ for any compact $K \subseteq \mathbb{R}$, is called a **Lebesgue-Stieltjes measure**. For any $F : \mathbb{R} \to \mathbb{R}$ that is monotone non-decreasing and right-continuous, and any Borel measure μ such that $\mu = \mu_F|_{\mathrm{Bor}_\mathbb{R}}$, we call μ the **Lebesgue-Stieltjes measure corresponding to F**.

Proposition 4.11. *Let μ be a Lebesgue-Stieltjes measure corresponding to some $F : \mathbb{R} \to \mathbb{R}$, then $\forall a \in \mathbb{R}$,*

$$\mu(\{a\}) = \mu\left(\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, a]\right) = \lim_{n\to\infty} \mu((a - \frac{1}{n}), a]) = F(a) - \lim_{n\to\infty} F(a - \frac{1}{n}) = F(a) - F(a^-).$$

Thus $\mu(\{a\}) > 0$ if and only if F has a jump discontinuity at a, since every discontinuity of a monotone non-decreasing function is a jump discontinuity.

Corollary 4.12. *If $F(x) = x$ is the identity function, every countable set has measure 0, by subadditivity and that $\forall a \in \mathbb{R}, \mu(\{a\}) = 0$*

Proposition 4.13. *A monotone non-decreasing function F can have at most countably many discontinuities.*

Proof. Choose countably many disjoint points $\{c_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Define a measure $\mu := \sum_{n\geq 1} \frac{1}{2^n}\delta_{c_n}$. This is a Borel measure with $\mu(\mathcal{K}) < \infty$ for any compact $\mathcal{K} \subseteq \mathbb{R}$. Thus μ is a Lebesgue-Stieltjes measure. Note $\mu(\{c_n\}) = \frac{1}{2^n} > 0$, thus each c_n is a jump discontinuity for the corresponding F. Thus F has countably many discontinuities.
In fact, no such F can have uncountably many discontinuities. $\square$

Theorem 4.14. *Lebesgue measure $(\mathbb{R}, \mathcal{L}, \lambda)$ is translation-invariant, meaning*

$$\forall A \in \mathcal{L}, s \in \mathbb{R},\ \lambda(A + s) = \lambda(A).$$

Also,

$$\forall s > 0, A \in \mathcal{L},\ \lambda(sA) = s\lambda(A).$$

Proof. If $A \subseteq \mathbb{R}$ is open, then so is $A + s$. Similarly for closed sets. Hence for $A \in B_\mathbb{R}, A + s \in B_\mathbb{R}$. Define a new measure λ_s on $B_\mathbb{R}$ by $\lambda_s(A) = \lambda(A + s)$. Note that λ and λ_s correspond to the functions

$$F(x) = \begin{cases} \lambda((0, x]) & \text{if } x \geq 0 \\ -\lambda((x, 0]) & \text{if } x < 0 \end{cases},$$

$$G(x) = \begin{cases} \lambda_s((0,x]) & \text{if } x \geq 0 \\ -\lambda_s((x,0]) & \text{if } x < 0 \end{cases}.$$

Yet $\lambda((0,x]) = \lambda((s,x+s]) = \lambda_s((0,x+s])$, and thus $F = G$. Thus $\lambda_s|_{B_\mathbb{R}} = \lambda|_{B_\mathbb{R}}$. By uniqueness for σ-finite Caratheodoy Theorem, we have that they extends to $\lambda = \lambda_s$. $\square$

Definition 4.9. A point $c \in \mathbb{R}$ with $\mu(\{x\}) \neq 0$ is called an **atom** of μ.

Corollary 4.15. *Lebesgue-Stieltjes measures can have at most countably many atoms.*

Definition 4.10. Let X be a topological space, then a $\mathcal{G}_\delta$ set is a countable intersection of open subsets of X, and a $\mathcal{F}_\sigma$ set is a countable union of closed subsets.

Remark. $\mathcal{G}_\delta$ sets and $\mathcal{F}_\sigma$ sets are Borel sets.

Theorem 4.16 (regularity)**.** *Let μ be a Lebesgue-Stieltjes measure with outer measure μ_F^*, and $E \subseteq \mathbb{R}$, the following are equal:*

(1) E is μ-measurable

(2) $\forall \epsilon > 0$, there is some open $O \supseteq E, \mu_F^*(O \setminus E) < \epsilon$ (Outer regularity)

(3) $\forall \epsilon > 0$, there is some closed $C \subseteq E, \mu_F^*(E \setminus C) < \epsilon$ (Inner regularity)

(4) There is a $\mathcal{G}_\delta$ set $G \supseteq E, \mu_F^*(G \setminus E) = 0$

(5) There is a $\mathcal{F}_\sigma$ set $F \subseteq E, \mu_F^*(E \setminus F) = 0$

Proof. Notice that E is μ-measurable means that

$$\forall A \subseteq \mathbb{R}, \mu_F^*(A) = \mu_F^*(E \cup A) + \mu_F^*(E^c \cup A).$$

1. (1) implies (2):
   If E is μ-measurable,

$$\begin{aligned} \mu(E) &= \mu_F^*(E) \\ &= \inf_{B \supseteq E} \mu_F(B), \end{aligned}$$

   where $B = \bigcup_{i=1}^\infty (a_i, b_i] \supseteq E$.
   Firstly, assume that E is bounded, we have $\mu_F^*(B) < \mu(E) + \frac{\epsilon}{2}$ for some $B = \bigsqcup_{i=1}^\infty (a_i, b_i] \supseteq E$.
   Since F is right-continuous, we have that $\forall i, \exists c_i > b_i,$ such that $F(c_i) < F(b_i) + \frac{\epsilon}{2^{i+1}}$.
   Let $O := \bigcup_i (a_i, c_i) \supseteq B \supseteq E$.
   Since E is measurable, we have that $\mu_F^*(B) = \mu_F^*(B \cap E) + \mu_F^*(B \setminus E) = \mu(E) + \mu_F^*(B \setminus E)$, thus

$\mu_F^*(B \setminus E) < \frac{\epsilon}{2}$

$$\begin{aligned}
\mu_F^*(O \setminus B) &= \mu_F^*(\bigcup_i (a_i, c_i) \cap B^c) \\
&\leq \sum_i \mu_F^*((a_i, c_i) \cap B^c) \\
&\leq \sum_i \mu_F^*((b_i, c_i)) \\
&\leq \sum_i \mu_F^*((b_i, c_i]) \\
&= \sum_i F(c_i) - F(b_i) \\
&< \sum_i \frac{\epsilon}{2^{i+1}} \\
&= \frac{\epsilon}{2}. \\
\mu_F^*(O \setminus E) &\leq \mu_F^*(O \cap E^c \cap B) + \mu_F^*(O \cap E^c \setminus B) \\
&= \mu_F^*(B \setminus E) + \mu_F^*(O \setminus B) \\
&< \epsilon.
\end{aligned}$$

This proves the bounded case.
If E is not bounded, we let $E_n = E \cap (n-1, n], n \in \mathbb{Z}$, each is bounded, and we can have open $O_n \supseteq E_n, \mu_F(O_n \setminus E_n) < \frac{\epsilon}{2^{2|n|}+1}$, and take $O = \cup_{n \in \mathbb{Z}} O_n \supseteq A$.

2. (2) implies (4):
For each $n \geq 1$, take open $O_n \supseteq E, \mu_F(O_n \setminus E) < \frac{1}{n}$, and WLOG, take $O_n = O_n \cap O_{n-1}$ so that $O_n \supseteq O_{n-1}$.
Take $G := \bigcap_{n=1}^{\infty} O_n$, which is a $\mathcal{G}_\delta$ set.
We have that $\forall n \geq 1, \mu_F^*(G \setminus E) \leq \mu_F^*(U_n \setminus E) < \frac{1}{n}$.
Thus $\mu_F^*(G \setminus E) = 0$.

3. (4) implies (1):
$G \setminus E$ is measurable since it is a null set, and μ is complete. G is also measurable, thus $E = G \setminus (G \setminus E)$ is also measurable.

4. (1) implies (3):
E is μ-measurable, so is E^c.
By (2), there is some open $O \supseteq E^c$, such that $\mu_F^*(O \setminus E^c) < \epsilon$.
Notice that $C := O^c$ is closed, and $C \subseteq E$, and

$$\begin{aligned}
\mu_F^*(E \setminus C) &= \mu_F^*(E \cap C^c) \\
&= \mu_F^*((E^c)^c \cap O) \\
&= \mu_F^*(O \setminus E^c) \\
&< \epsilon.
\end{aligned}$$

5. (3) implies (5)
For each $n \geq 1$, take closed $C_n \subseteq E, \mu_F(E \setminus C_n) < \frac{1}{n}$, and WLOG, take $C_n = C_n \cap C_{n-1}$ so that $C_n \supseteq C_{n-1}$.
Take $F := \bigcup_{n=1}^{\infty} C_n$, which is a $\mathcal{F}_\sigma$ set.
We have that $\forall n \geq 1, \mu_F^*(E \setminus F) \leq \mu_F^*(E \setminus C_n) < \frac{1}{n}$.
Thus $\mu_F^*(E \setminus F) = 0$.

6. (5) implies (1)
$E \setminus F$ is a measurable set. F is also a measurable set, and thus so is $E = (E \setminus F) \cup F$.

□

Corollary 4.17. *Let μ be a Lebesgue-Stieltjes measure, and A be μ-measurable, we have*

$$\mu(A) = \inf\{\mu(O) : O \supseteq E \text{ is open}\} = \sup\{\mu(C) : C \subseteq A \text{ is compact}\}$$

Proof. The first equality is (2).

For the second equality, if A is bounded, and $C \subseteq A$ is closed, then it is compact. We can use (3) to prove it.
If A is not bounded, let $A_n := A \cap [-n, n]$ for each $N \geq 1$. Thus,

$$\mu(A) = \sup_{n\geq 1} \mu(A_n) = \sup_{n\geq 1} \sup_{C \subseteq A_n \text{ is compact}} \mu(C).$$

□

4.4 Littlewood's Three Principles

Recall Littlewood's Three Principles for Lebesgue Measure:

Theorem 4.18. *Littlewood's first Principle (regularity)*
Every measurable set is almost a finite union of intervals.

Theorem 4.19. *Littlewood's second Principle (Lusin's)*
Every measurable function is almost continuous.

Theorem 4.20. *Littlewood's third Principle (Egorov's)*
A point-wise convergent sequence of measurable functions is almost uniformly convergent.

Theorem 4.21 (Egorov's)**.** *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $f_n : X \to \mathbb{C}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ μ-almost everywhere. Then $\forall \epsilon > 0, \exists A \in \mathcal{M}$, such that $\mu(X \setminus A) < \epsilon$, and $f_n \to f$ uniformly on A.*

Proof. $f_n \to f$ uniformly on A means $\forall m \in \mathbb{N}^+, \exists N_m \geq 1$, such that

$$\forall x \in A, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m}.$$

Let $A_{mN} := \left\{x \in X : \forall n \geq N, |f_n(x) - f(x)| < \frac{1}{m}\right\} = \bigcap_{n\geq N} \left\{x \in X : |f_n(x) - f(x)| < \frac{1}{m}\right\}$, which is intersection of preimages of $(-\frac{2}{m}, \frac{1}{m})$ of measurable functions $f_n - f$, thus measurable.
Note $A_{m1} \subseteq A_{m2} \subseteq \cdots$, and $\bigcup_{n\geq 1} A_{m,n} = X \setminus N$ for some $N \in \mathcal{M}, \mu(N) = 0$ since $f_n \to f$ μ-a.e..

$$\begin{aligned}\mu(X) &= \mu(X \setminus N)\\ &= \mu(\bigcup_{n\geq 1} A_{mn})\\ &= \lim_{n\to\infty} \mu(A_{mn}).\end{aligned}$$

Since $\mu(X) < \infty$, there is $N_m \geq 1$ such that $\mu(A_{m,N_m}) > \mu(X) - \frac{\epsilon}{2^m}$ for any $\epsilon > 0$.
Thus $\mu(X \setminus A_{m,N_m}) < \frac{\epsilon}{2^m}$.
Letting $E := \bigcap_{m\geq 1} A_{m,N_m}$, we have that

$$\begin{aligned}\mu(X \setminus E) &= \mu(\bigcup_{m\geq 1} (X \setminus A_{m,N_m}))\\ &\leq \sum_{m\geq 1} \mu(X \setminus A_{m,N_m})\\ &< \epsilon.\end{aligned}$$

In addition,

$$\begin{aligned} E &= \bigcap_{m\geq 1} A_{m,N_m} \\ &= \left\{ x \in X : \forall m \geq 1, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m} \right\}. \end{aligned}$$

Thus $f_n \to f$ uniformly on E. $\square$

Theorem 4.22 (Lusin's)**.** *Let $f : [a,b] \to \mathbb{C}$ be a Lebesgue-Stieltjes measurable function. For any $\epsilon > 0$, there is a continuous function $g : [a,b] \to \mathbb{C}$ such that*

$$\mu(\{x \in [a,b] : f(x) \neq g(x)\}) < \epsilon.$$

Proof. Consider a simple function $s := \sum_{i=1}^m \alpha_i \chi_{E_i}$, where $\alpha_i \in \mathbb{C}$, and E_i are disjoint and Lebesgue measurable.
Notice that by the regularity theorem, for any $\delta > 0$, there are closed sets $A_i \subseteq E_i$, such that $\mu(E_i \setminus A_i) < \frac{\delta}{m}$ for each i. Thus, $\mu(\bigsqcup_{i=1}^m E_i \setminus A_i) = \mu((\bigsqcup_{i=1}^m E_i) \setminus (\bigsqcup_{i=1}^m A_i)) = \mu([a,b] \setminus \mathcal{K}) < \delta$, for $\mathcal{K} := \bigsqcup_{i=1}^m A_i$.
Notice that $\mathcal{K}$ is closed (thus compact since $[a,b]$ is bounded), and $s|_{\mathcal{K}}$ is continuous since s is locally constant. Indeed, $\forall x \in \mathcal{K}$, there is unique A_i such that $x \in A_i$.
Suppose for contradiction that $\forall \delta_0 > 0$, there is some $y \in (x - \delta_0, x + \delta_0) \cap A_j$ for some $j \neq i$. Let $\mathcal{K}' := \bigsqcup_{j=1, j\neq i}^m A_j$, which is closed. Now we have a sequence $y_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap \mathcal{K}'$. Notice that $y_n \to x$, and since $\mathcal{K}'$ is closed, $x \in \mathcal{K}'$, which is a contradiction.
Thus $\exists \delta_0 > 0$, such that $(x - \delta_0, x + \delta_0) \cap \mathcal{K} \subseteq A_i$; namely, s in constant on $(x - \delta_0, x + \delta_0) \cap \mathcal{K}$. Thus, $\forall \epsilon_0 > 0, y \in \mathcal{K}$, such that $|y - x| < \delta_0, |s(x) - s(y)| = 0 < \epsilon_0$, whcih shows s is continuous around x.
Now given any measurable f, we can choose simple functions $s_n : [a,b] \to \mathbb{C}$ converging point-wise to f. For each n, construct $\mathcal{K}_n$ as above such that $s_n|_{\mathcal{K}_n}$ is continuous and $\mu([a,b] \setminus \mathcal{K}_n) < \frac{\epsilon}{2^{n+1}}$.
Let $\mathcal{K}_0 = \bigcap_{n\geq 1} \mathcal{K}_n$, which is compact. For all n, we have that $s_n|_{\mathcal{K}_n}$ is continuous.
In addition, $\mu([a,b] \setminus \mathcal{K}_0) \leq \sum_{n=1}^\infty \mu([a,b] \setminus \mathcal{K}_n) < \epsilon/2$.
By Egorov's Theorem, there is a measurable $E \subseteq \mathcal{K}_0$, such that $\mu(\mathcal{K}_0 \setminus E) < \epsilon/4$ and $s_n \to f$ uniformly on E.
Applying the regularity theorem again, there is a compact $\mathcal{K} \subseteq E$ such that $\mu(E \setminus \mathcal{K}) < \epsilon/4$. Notice that $s_n \to f$ uniformly on $\mathcal{K}$. Thus $f|_{\mathcal{K}}$ is continuous.
Also, $\mu([a,b] \setminus \mathcal{K}) \leq \mu([a,b] \setminus \mathcal{K}_0) + \mu(\mathcal{K}_0 \setminus E) + \mu(E \setminus \mathcal{K}) = \epsilon$.
By Tietze's Theorem, we can extend $f|_{\mathcal{K}}$ to some continuous $g : [a,b] \to \mathbb{C}$. We thus have

$$\mu(\{x \in [a,b] : f(x) \neq g(x)\}) \leq \mu([a,b] \setminus \mathcal{K}) < \epsilon.$$

$\square$

5 Lebesgue Spaces

5.1 The First Lebesgue Space

Definition 5.1. Given some measure space $(X, \mathcal{M}, \mu)$, define

$$\mathcal{L}^1(\mu) := \mathcal{L}^1(X, \mathcal{M}, \mu) := \left\{ f : X \to \mathbb{C} \,\middle|\, f \text{ is measurable}, \int_X |f| d\mu < \infty \right\}.$$

Proposition 5.1. *$\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space.*

Proof. Clearly $\int_X |0| d\mu = 0$, so the zero function $0 \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, we have

$$\begin{aligned} \int_X |c \cdot f| d\mu &= \int_X |c||f| d\mu \\ &= |c| \int_X |f| d\mu \\ &< \infty. \end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Now for any $f, g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, we have

$$\begin{aligned}\int_X |f+g| d\mu &\leq \int_X |f| + |g| d\mu \\ &= \int_X |f| d\mu + \int_X |g| d\mu \\ &< \infty.\end{aligned}$$

Thus $f + g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Since the set of all functions $\{f : X \to \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a subspace of it. □

Definition 5.2. Let

$$N = \left\{ f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : \int_X |f| d\mu = 0 \right\} = \left\{ f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f = 0 \ \mu - a.e. \right\}.$$

Define

$$L^1(X, \mathcal{M}, \mu) := \mathcal{L}^1(X, \mathcal{M}, \mu)/N,$$

which is the quotient vector space of $\mathcal{L}^1(X, \mathcal{M}, \mu)$ mod N.

Remark. $[f] = \left\{ g \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f - g = 0 \ \mu - \text{a.e.} \right\} \in L^1(X, \mathcal{M}, \mu)$

Definition 5.3. $\|[f]\|_{L^1(X,\mathcal{M},\mu)} := \int_X |f| d\mu$ for any choice of representative $f \in [f]$.

When the context is clear, we might write $L^1(X, \mathcal{M}, \mu)$ as $L^1(\mu)$ or $L^1(X)$. We might also write $\|\cdot\|_{L^1(X,\mathcal{M},\mu)}$ as $\|\cdot\|_{L^1(\mu)}$, $\|\cdot\|_{L^1(\mu)}$, $\|\cdot\|_1$.

Lemma 5.2. *The above definition is well defined.*

Proof. Take any $g, f \in [f]$. Let $K = \{x \in X : f(x) \neq g(x)\}$, we have $\mu(K) = 0$.

$$\begin{aligned}\int_X |f| d\mu &= \int_{X \setminus K} |f| d\mu + \int_K |f| d\mu \\ &= \int_{X \setminus K} |f| d\mu \\ &= \int_{X \setminus K} |g| d\mu \\ &= \int_{X \setminus K} |g| d\mu + \int_K |g| d\mu \\ &= \int_X |g| d\mu\end{aligned}$$

□

Proposition 5.3. $\|\cdot\|_1$ *is a norm on* $L^1(X, \mathcal{M}, \mu)$.

Proof. Consider any $[f], [g] \in L^1(X, \mathcal{M}, \mu)$.

$$\begin{aligned}\|[f] + [g]\|_1 &= \|[f + g]\|_1 \\ &= \int_X |f + g| d\mu \\ &\leq \int_X |f| d\mu + \int_X |g| d\mu \\ &= \|[f]\|_1 + \|[g]\|_1\end{aligned}$$

For any $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}\|\alpha[f]\|_1 &= \|[\alpha f]\|_1 \\ &= \int_X |\alpha f| d\mu \\ &= |\alpha| \int_X |f| d\mu \\ &= |\alpha| \|[f]\|_1\end{aligned}$$

If $\|[f]\|_1 = 0$, we must have $f = 0$ $\mu - a.e.$. Thus $f \in N$, thus $[f] = [0] = 0$. ☐

Theorem 5.4 (Fischer-Riesz)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $\left(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)}\right)$ is a Banach Space.*

Proof. Let $([f_n])_1^\infty$ be a Cauchy sequence in $L^1(X, \mathcal{M}, \mu)$. Then for each $k \in \mathbb{N}^+$, there is some $N_k \geq 1$, such that $\forall m, n \geq N_k, \|[f_m] - [f_n]\|_{L^1(\mu)} < \frac{1}{2^k}$.
WLOG, $\forall k, N_{k+1} \geq N_k$.
Thus $\left\|[f_{N_{k+1}}] - [f_{N_k}]\right\|_{L^1(\mu)} < \frac{1}{2^k}$.
Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} \left|f_{N_{j+1}} - f_{N_j}\right|$, where we fix f_n to be a representative of $[f_n]$.
Notice that $\forall k \geq 1$,

$$\begin{aligned}f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\ |f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\ &\leq |f_{N_1}| + \sum_{j=1}^{k-1} \left|f_{N_{j+1}} - f_{N_j}\right| \\ &= g_k\end{aligned}$$

We have that $\int_X g_k d\mu = \int_X |f_{N_1}| d\mu + \sum_{j=1}^n \int_X \left|f_{N_{j+1}} - f_{N_j}\right| d\mu$.
Let $g = \lim_{k \to \infty} g_k = |f_{N_1}| + \sum_{j=1}^\infty \left|f_{N_{j+1}} - f_{N_j}\right|$.
By LMCT, we have that

$$\begin{aligned}\int_X g d\mu &= \lim_{k \to \infty} \int_X g_k d\mu \\ &= \int_X |f_{N_1}| d\mu + \sum_{j=1}^\infty \int_X \left|f_{N_{j+1}} - f_{N_j}\right| d\mu \\ &= \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^\infty \left\|[f_{N_{j+1}} - f_{N_j}]\right\|_{L^1(\mu)} \\ &= \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^\infty \left\|[f_{N_{j+1}}] - [f_{N_j}]\right\|_{L^1(\mu)} \\ &< \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^\infty \frac{1}{2^j} \\ &< \infty.\end{aligned}$$

Thus $g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Thus $N := \{x \in X : g(x) = \infty\}$ has measure 0.
This implies that $f_{N_1}(x) + \sum_{k=1}^\infty (f_{N_{k+1}}(x) - f_{N_k}(x))$ converges absolutely for $x \in X \setminus N$.

We can thus define

$$
\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty} (f_{N_{k+1}}(x) - f_{N_k}(x)) \\
&= \lim_{k\to\infty} \left(f_{N_1}(x) + \sum_{j=1}^{k} (f_{N_{j+1}}(x) - f_{N_j}(x)) \right) \\
&= \lim_{k\to\infty} f_{N_{k+1}}(x) \\
&= \lim_{k\to\infty} f_{N_k}(x)
\end{aligned}
$$

for $x \in X \setminus N$.
We then extend f to X by $f|_N := 0$.
Then $|f| \le g$, and thus $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Notice that $|f_{N_k}| \le g_k \le g$, thus $|f - f_{N_k}| \le g + g = 2g$.
By LDCT,

$$
\begin{aligned}
\lim_{k\to\infty} \|[f_{N_k}] - [f]\|_{L^1(\mu)} &= \lim_{k\to\infty} \|[f_{N_k} - f]\|_{L^1(\mu)} \\
&= \lim_{k\to\infty} \int_X |f_{N_k} - f| d\mu \\
&= \int_X \lim_{k\to\infty} |f_{N_k} - f| d\mu \\
&= 0.
\end{aligned}
$$

Thus $\lim_{k\to\infty}[f_{N_k}]$ converges to $[f]$. Since this is a subsequence of the Cauchy sequence $([f_n])_1^\infty$, we have that $\lim_{n\to\infty}[f_n] = [f]$.
This shows that $(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)})$ is complete. □

Remark. When we write $f \in L^1(\mu)$, we will mean $[f] \in L^1(\mu)$, and let $f \in \mathcal{L}^1(\mu)$ be any representative of $[f]$ when the context is clear.

5.2 Convex functions

Definition 5.4. A function $\phi : U \to \mathbb{R}$ is **convex** if

$$\forall x, y \in U, \forall \lambda \in [0,1],\ \phi(\lambda x + (1-\lambda)y) \le \lambda\phi(x) + (1-\lambda)\phi(y).$$

Theorem 5.5 (Jensen's Inequality)**.** *If ϕ is convex, we have $\forall x_1, \dots, x_n \in U$, and $\forall 0 \le \lambda_1, \dots, \lambda_n \le 1$ such that $\sum_{i=1}^n \lambda_i = 1$,*

$$\phi\left(\sum_{i=1}^n \lambda_i x_i\right) \le \sum_{i=1}^n \lambda_i \phi(x_i).$$

Proof. The base case is when $n = 1$, which is trivial.
Now suppose this holds for $n - 1 \in \mathbb{N}$.
Given any $\forall x_1, \dots, x_n \in U$, and $0 \le \lambda_1, \dots, \lambda_n \le 1$ such that $\sum_{i=1}^n \lambda_i = 1$. If $\lambda_n = 0$, we can reduce the sum to a $n-1$ sum. If $\lambda_n = 1$, then the other λ_i must be all 0, and we can reduce the sum to only x_n.
Now suppose $0 < \lambda_1 < 1$. Notice that $\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} = \frac{\sum_{i=1}^{n-1}\lambda_i}{1-\lambda_n} = \frac{1-\lambda_n}{1-\lambda_n} = 1$.

We have that

$$
\begin{aligned}
\phi\left(\sum_{i=1}^{n}\lambda_i x_i\right) &= \phi\left(\lambda_n x_n + (1-\lambda_n)\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}x_i\right) \\
&\leq \lambda_n\phi(x_n) + (1-\lambda_n)\phi\left(\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}x_i\right) \\
&\leq \lambda_n\phi(x_n) + (1-\lambda_n)\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}\phi(x_i) \\
&= \lambda_n\phi(x_n) + \sum_{i=1}^{n-1}\lambda_i\phi(x_i) \\
&= \sum_{i=1}^{n}\lambda_i\phi(x_i).
\end{aligned}
$$

By induction, this is true for any $n \geq 1$. $\square$

Theorem 5.6 (Arithmetic Mean Geometric Mean Inequality)**.** *Let $x_1, \ldots, x_n \geq 0$, with $0 \leq \lambda_1, \ldots, \lambda_n \leq 1$ such that $\sum_{i=1}^{n}\lambda_i = 1$. We have that*

$$
\prod_{i=1}^{n} x_i^{\lambda_i} \leq \sum_{i=1}^{n}\lambda_i x_i.
$$

Proof. If any of $x_i = 0$, then the inequality is trivially true.
Now suppose $\forall i, x_i > 0$.
Notice that exp is convex, and we have

$$
\begin{aligned}
\prod_{i=1}^{n} x_i^{\lambda_i} &= \exp\left(\sum_{i=1}^{n}\lambda_i \ln(x_i)\right) \\
&\leq \sum_{i=1}^{n}\lambda_i \exp(\ln(x_i)) \\
&= \sum_{i=1}^{n}\lambda_i x_i.
\end{aligned}
$$

$\square$

Proposition 5.7. *Let $x_1, \ldots, x_n \geq 0$, and $n \in \mathbb{N}^+, p \geq 1$, we have that*

$$
\sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p \leq n^{p-1}\sum_{i=1}^{n} x_i^p.
$$

Proof. For $p \geq 1$, we have $(\cdot)^p$ is convex.

$$
\begin{aligned}
\left(\sum_{i=1}^{n}\frac{1}{n}x_i\right)^p &\leq \sum_{i=1}^{n}\frac{1}{n}x_i^p \\
\frac{1}{n^p}\left(\sum_{i=1}^{n}x_i\right)^p &\leq \frac{1}{n}\sum_{i=1}^{n}x_i^p \\
\left(\sum_{i=1}^{n}x_i\right)^p &\leq n^{p-1}\sum_{i=1}^{n}x_i^p.
\end{aligned}
$$

This proves the second inequality.
Now when $n = 1$, we have the first inequality trivially.
Suppose the first inequality holds for $n \in \mathbb{N}^+$, we have

$$\begin{aligned}\left(\sum_{i=1}^{n+1} x_i\right)^p &= \left(\sum_{i=1}^{n} x_i + x_{n+1}\right)^p \\ &\geq \left(\sum_{i=1}^{n} x_i\right)^p + x_{n+1}^p \\ &\geq \left(\sum_{i=1}^{n} x_i^p\right) + x_{n+1}^p \\ &= \sum_{i=1}^{n+1} x_i^p.\end{aligned}$$

By induction, the first inequality is true for all $n \in \mathbb{N}^+$. □

5.3 L^p Spaces

Definition 5.5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \leq p < \infty$ we define

$$\mathcal{L}^p(\mu) := \left\{f : X \to \mathbb{C}\middle| f^p \in L^1(\mu)\right\} = \left\{f : X \to \mathbb{C}\middle| f \text{ is measurable}, \int_X |f|^p d\mu < \infty\right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p} := \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}.$$

Proposition 5.8. *$\mathcal{L}^p(\mu)$ is a vector space.*

Proof. Clearly $\int_X |0|^p d\mu = 0$, so the zero function $0 \in \mathcal{L}^p(\mu)$.
Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^p(\mu)$, we have

$$\begin{aligned}\int_X |c \cdot f|^p d\mu &= \int_X |c|^p |f|^p d\mu \\ &= |c|^p \int_X |f|^p d\mu \\ &< \infty.\end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^p(\mu)$.
Now for any $f, g \in \mathcal{L}^p(\mu)$, we have

$$\begin{aligned}\int_X |f + g|^p d\mu &\leq \int_X (|f| + |g|)^p d\mu \\ &\leq \int_X 2^{p-1}(|f|^p + |g|^p) d\mu \\ &= 2^{p-1}\left(\int_X |f|^p d\mu + \int_X |g|^p d\mu\right) \\ &< \infty.\end{aligned}$$

Thus $f + g \in \mathcal{L}^p(\mu)$.
Since the set of all functions $\{f : X \to \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^p(\mu)$ is a subspace of it. □

Definition 5.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, the **essential supremum** of a function $f : X \to \mathbb{R}$ is

$$\operatorname{ess\,sup} f := \inf \{M \in \mathbb{R} : \mu(\{x : f(x) > M\}) = 0\}.$$

Proposition 5.9. *For any* $\lambda \geq 0, f : X \to \mathbb{R}$, *we have*

$$\lambda(\operatorname{ess\,sup} f) = \operatorname{ess\,sup}(\lambda f).$$

Proof. It is easy to see this is true for $\lambda = 0$.
Now suppose $\lambda > 0$.

$$\begin{aligned}
\operatorname{ess\,sup}(\lambda f) &= \inf \{M \in \mathbb{R} : \mu(\{x : \lambda f(x) > M\}) = 0\} \\
&= \inf \left\{M \in \mathbb{R} : \mu\left(\left\{x : f(x) > \frac{M}{\lambda}\right\}\right) = 0\right\} \\
&= \inf \{\lambda \cdot N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0\} \\
&= \lambda \inf \{N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0\} \\
&= \lambda(\operatorname{ess\,sup} f).
\end{aligned}$$

□

Definition 5.7. Let $(X, \mathcal{M}, \mu)$ be a measure space, we define

$$\mathcal{L}^\infty(\mu) := \{f : X \to \mathbb{C} | f \text{ is measurable}, \operatorname{ess\,sup}|f| < \infty\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty} := \operatorname{ess\,sup}|f|.$$

Proposition 5.10. $\mathcal{L}^\infty(\mu)$ *is a vector space.*

Proof. Clearly $\operatorname{ess\,sup} 0 = 0$, so the zero function $0 \in \mathcal{L}^\infty(\mu)$.
Also, for any $c \in \mathbb{C}$, and $f \in \mathcal{L}^\infty(\mu)$, we have

$$\begin{aligned}
\|c \cdot f\|_{\mathcal{L}^\infty} &= \operatorname{ess\,sup}|c \cdot f| \\
&= \operatorname{ess\,sup}(|c| \cdot |f|) \\
&= |c| \operatorname{ess\,sup}|f| \\
&= |c| \|f\|_{\mathcal{L}^\infty} \\
&< \infty.
\end{aligned}$$

Thus $c \cdot f \in \mathcal{L}^\infty(\mu)$.
Now for any $f, g \in \mathcal{L}^\infty(\mu)$.
Consider any $L, N \in \mathbb{R}$, such that $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$.
Thus $\mu(\{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}) = 0$.
Now for any $x \in X$, if $|f(x) + g(x)| > L + N$, we must have $|f(x)| + |g(x)| \geq |f(x) + g(x)| > L + N$.
Thus $|f(x)| > L$ or $|g(x)| > N$.
Since this holds for any $x \in X$, we have $\{x : |f(x) + g(x)| > N + L\} \subseteq \{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}$.
Thus $\mu(\{x : |f(x) + g(x)| > N + L\}) = 0$.
By definition, we have

$$\begin{aligned}
\|f + g\|_{\mathcal{L}^\infty} &= \operatorname{ess\,sup}|f + g| \\
&= \inf \{M \in \mathbb{R} : \mu(\{x : |f(x) + g(x)| > M\}) = 0\} \\
&\leq N + L.
\end{aligned}$$

Since this hold for any such $N, L \in \mathbb{R}$, such that $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$, we have

$$\begin{aligned}
\|f + g\|_{\mathcal{L}^\infty} &= \inf \{N + L : \mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0\} \\
&= \inf \{N : \mu(\{x : |f(x)| > N\}) = 0\} + \inf \{L : \mu(\{x : |g(x)| > L\}) = 0\} \\
&= \|f\|_{\mathcal{L}^\infty} + \|g\|_{\mathcal{L}^\infty} \\
&< \infty.
\end{aligned}$$

Thus $f+g \in \mathcal{L}^\infty(\mu)$.
Since the set of all functions $\{f : X \to \mathbb{C}\}$ is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have $\mathcal{L}^\infty(\mu)$ is a subspace of it. $\square$

Proposition 5.11. *For any $1 \le p \le \infty$, we have $\|f-g\|_{\mathcal{L}^p} = 0 \iff f = g$ almost everywhere.*

Proof. For $1 \le p < \infty$,

$$\begin{aligned} & \|f-g\|_{\mathcal{L}^p} = 0 \\ \iff & \int_X |f-g|^p d\mu = 0 \\ \iff & |f-g|^p = 0 \text{ a.e.} \\ \iff & f-g = 0 \text{ a.e.} \\ \iff & f = g \text{ a.e..} \end{aligned}$$

For $p = \infty$,

$$\begin{aligned} & \|f-g\|_{\mathcal{L}^\infty} = 0 \\ \iff & \operatorname{ess\,sup} |f-g| = 0 \\ \iff & f-g = 0 \text{ a.e.} \\ \iff & f = g \text{ a.e..} \end{aligned}$$

$\square$

Definition 5.8. For any $1 \le p \le \infty$, if we identify $f, g \in \mathcal{L}^p(\mu)$ by $f \sim g \iff f = g$ almost everywhere, we get the quotient vector space

$$L^p(\mu) := \mathcal{L}^p(\mu)/_\sim = \{[f] : f \in \mathcal{L}^p(\mu)\}$$

to be the collection of all equivalence classes of functions in $\mathcal{L}^p$.

Definition 5.9. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \le p \le \infty$ we define the norm

$$\|[f]\|_{L^p(\mu)} := \|f\|_{\mathcal{L}^p}$$

for any representative $f \in [f]$.

Lemma 5.12. *The above definition is well-defined.*

Remark. As before, when we write $f \in L^p(\mu)$, we will mean $[f] \in L^p(\mu)$, and let $f \in \mathcal{L}^p(\mu)$ be any representative of $[f]$ when the context is clear.

Theorem 5.13 (Holder's Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \le p \le \infty$. Suppose $\frac{1}{p} + \frac{1}{q} = 1$, then $\forall f \in L^p(\mu), g \in L^q(\mu),\ fg \in L^1(\mu)$ and*

$$\|fg\|_{L^1(\mu)} \le \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Proof. If $p = 1$, then $q = \infty$.
Now $|fg| = |f||g| \le |f| \|g\|_{L^\infty(\mu)}$.
Thus

$$\begin{aligned} \|fg\|_{L^1(\mu)} &= \int_X |fg| d\mu \\ &\le \int_X |f| \|g\|_{L^\infty(\mu)} d\mu \\ &= \|g\|_{L^\infty(\mu)} \int_X |f| d\mu \\ &= \|g\|_{L^\infty(\mu)} \|f\|_{L^1(\mu)}. \end{aligned}$$

Now suppose $1 < p < \infty$. We have $1 < q < \infty$.
If $\|f\|_{L^p(\mu)} = 0$ or $\|g\|_{L^q(\mu)} = 0$, then it is trivial, since this implies $f = 0$ a.e. or $g = 0$ a.e., which means $fg = 0$ a.e..
Now let $F := \frac{|f|}{\|f\|_{L^p(\mu)}}, G := \frac{|g|}{\|g\|_{L^q(\mu)}}$.
By the Arithmetic Mean Geometric Mean Inequality 5.6, we have that

$$\begin{aligned}
F(x)G(x) &= (F(x)^p)^{1/p}(G(x)^q)^{1/q} \\
&\leq \frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q \\
\int_X FG d\mu &\leq \frac{1}{p}\int_X F^p d\mu + \frac{1}{q}\int_X G^q d\mu \\
\frac{\|fg\|_{L^1(\mu)}}{\|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}} &\leq \frac{1}{p}\int_X \frac{|f|^p}{\|f\|^p_{L^p(\mu)}} d\mu + \frac{1}{q}\int_X \frac{|g|^p}{\|g\|^q_{L^q(\mu)}} d\mu \\
&= \frac{1}{p}\frac{\|f\|^p_{L^p(\mu)}}{\|f\|^p_{L^p(\mu)}} + \frac{1}{q}\frac{\|g\|^q_{L^q(\mu)}}{\|g\|^q_{L^q(\mu)}} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

Thus $\|fg\|_{L^1(\mu)} \leq \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}$. $\square$

Theorem 5.14 (Minkowski's Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$. For any $f, g \in L^p(\mu)$, we have*

$$\|f + g\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}.$$

Proof. We have proven for $p = 1$ and $p = \infty$.
Now suppose $p \in (1, \infty)$. Then $q = \frac{p}{p-1} \in (1, \infty)$.
Since $f, g \in L^p(\mu)$, we have $f + g \in L^p(\mu)$, so

$$\begin{aligned}
\left\| |f+g|^{p-1} \right\|^q_{L^q(\mu)} &= \int_X \left(|f+g|^{p-1}\right)^q d\mu \\
&= \int_X \left(|f+g|^{p-1}\right)^{\frac{p}{p-1}} d\mu \\
&= \int_X |f+g|^p d\mu \\
&= \|f+g\|^p_{L^p(\mu)} \\
&< \infty.
\end{aligned}$$

Thus $|f+g|^{p-1} \in L^q(\mu)$. By Holder's Inequality, we have

$$\begin{aligned}
\|f+g\|_{L^p(\mu)}^p &= \int_X |f+g|^p d\mu \\
&= \int_X |f+g||f+g|^{p-1} d\mu \\
&\leq \int_X (|f|+|g|)|f+g|^{p-1} d\mu \\
&\leq \int_X |f| \cdot |f+g|^{p-1} d\mu + \int_X |g| \cdot |f+g|^{p-1} d\mu \\
&\leq \|f\|_{L^p(\mu)} \left\| |f+g|^{p-1} \right\|_{L^q(\mu)} + \|g\|_{L^p(\mu)} \left\| |f+g|^{p-1} \right\|_{L^q(\mu)} \\
&= \left(\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \right) \left\| |f+g|^{p-1} \right\|_{L^q(\mu)} \\
&= \left(\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \right) \|f+g\|_{L^p(\mu)}^{p/q} \\
\|f+g\|_{L^p(\mu)}^{p-p/q} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\
\|f+g\|_{L^p(\mu)}^{p(1-1/q)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\
\|f+g\|_{L^p(\mu)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}.
\end{aligned}$$

□

Corollary 5.15. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$. We have $\|\cdot\|_{L^p(\mu)}$ is a norm over $L^p(\mu)$.*

Proof. The triangle Inequality is done by Minkowski's Inequality.
Consider any $\in L^p(\mu)$.
For any $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\|\alpha f\|_{L^p(\mu)}^p &= \int_X |\alpha f|^p d\mu \\
&= \int_X |\alpha|^p |f|^p d\mu \\
&= |\alpha|^p \int_X |f|^p d\mu \\
&= |\alpha|^p \|f\|_{L^p(\mu)}^p \\
&\Longrightarrow \\
\|\alpha f\|_{L^p(\mu)} &= |\alpha| \|f\|_{L^p(\mu)}.
\end{aligned}$$

In addition, $\|f\|_{L^p(\mu)} = 0$, if and only if $f = 0$ $\mu - a.e.$. □

Theorem 5.16 (Fischer-Riesz)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$. $\left(L^p(\mu), \|\cdot\|_{L^p(\mu)}\right)$ is a Banach Space.*

Proof. 1. We first consider $1 \leq p < \infty$.
Let $(f_n)_1^\infty$ be a Cauchy sequence in $L^p(X, \mathcal{M}, \mu)$. Then for each $k \in \mathbb{N}^+$, there is some $N_k \geq 1$, such that $\forall m, n \geq N_k, \|f_m - f_n\|_{L^p(\mu)} < \frac{1}{2^k}$.
WLOG, $\forall k, N_{k+1} \geq N_k$.
Thus $\left\| f_{N_{k+1}} - f_{N_k} \right\|_{L^p(\mu)} < \frac{1}{2^k}$.
Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} \left| f_{N_{j+1}} - f_{N_j} \right|$, where we fix f_n to be a representative of $[f_n]$.

Notice that $\forall k \geq 1$,

$$\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1}(f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1}(f_{N_{j+1}} - f_{N_j}) \right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1} \left| f_{N_{j+1}} - f_{N_j} \right| \\
&= g_k \\
\|g_k\|_{L^p(\mu)} &= \left\| |f_{N_1}| + \sum_{j=1}^{k-1} \left| f_{N_{j+1}} - f_{N_j} \right| \right\|_{L^p(\mu)} \\
&\leq \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \left\| f_{N_{j+1}} - f_{N_j} \right\|_{L^p(\mu)}.
\end{aligned}$$

Let $g = \lim_{k\to\infty} g_k = |f_{N_1}| + \sum_{j=1}^{\infty} \left| f_{N_{j+1}} - f_{N_j} \right|$.
Notice that g_k are monotone increasing. By LMCT, we have that

$$\begin{aligned}
\|g\|_{L^p(\mu)} &= \int_X |g|^p d\mu \\
&= \int_X g^p d\mu \\
&= \int_X \lim_{k\to\infty} g_k^p d\mu \\
&= \lim_{k\to\infty} \int_X g_k^p d\mu \\
&= \lim_{k\to\infty} \|g_k\|_{L^p(\mu)} \\
&\leq \lim_{k\to\infty} \left(\|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \left\| f_{N_{j+1}} - f_{N_j} \right\|_{L^p(\mu)} \right) \\
&= \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \left\| f_{N_{j+1}} - f_{N_j} \right\|_{L^p(\mu)} \\
&< \|[f_{N_1}]\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&< \infty.
\end{aligned}$$

Thus $g \in \mathcal{L}^p(X, \mathcal{M}, \mu)$, which means $g^p \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ and $N := \{x \in X : g(x) = \infty\}$ has measure 0.
This implies that $f_{N_1}(x) + \sum_{k=1}^{\infty}(f_{N_{k+1}}(x) - f_{N_k}(x))$ converges absolutely for $x \in X \setminus N$.

We can thus define

$$
\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty}(f_{N_{k+1}}(x) - f_{N_k}(x)) \\
&= \lim_{k\to\infty}\left(f_{N_1}(x) + \sum_{j=1}^{k}(f_{N_{j+1}}(x) - f_{N_j}(x))\right) \\
&= \lim_{k\to\infty} f_{N_{k+1}}(x) \\
&= \lim_{k\to\infty} f_{N_k}(x)
\end{aligned}
$$

for $x \in X \setminus N$.
We then extend f to X by $f|_N := 0$.
Then $|f| \leq g \implies |f|^p \leq g^p$, and thus $f \in \mathcal{L}^p(X, \mathcal{M}, \mu)$.
Notice that $|f_{N_k}| \leq g_k \leq g$, thus $|f - f_{N_k}|^p \leq (g+g)^p = 2^p g^p$.
By LDCT,

$$
\begin{aligned}
\lim_{k\to\infty} \|f_{N_k} - f\|_{L^p(\mu)}^p &= \lim_{k\to\infty} \int_X |f_{N_k} - f|^p d\mu \\
&= \int_X \lim_{k\to\infty} |f_{N_k} - f|^p d\mu \\
&= 0.
\end{aligned}
$$

Thus $\lim_{k\to\infty} f_{N_k}$ converges to f.
Since this is a subsequence of the Cauchy sequence $(f_n)_1^\infty$, we have that $\lim_{n\to\infty} f_n = f$.
This shows that $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(\mu)})$ is complete.

2. Now consider $p = \infty$.
Let $(f_n)_1^\infty$ be a Cauchy sequence in $L^\infty(X, \mathcal{M}, \mu)$. As before, we can take some subsequence $(f_{N_k})_{k=1}^\infty$ with $\left\|f_{N_{k+1}} - f_{N_k}\right\|_{L^p(\mu)} < \frac{1}{2^k}$.
Let $g_k = |f_{N_1}| + \sum_{j=1}^{k-1}\left|f_{N_{j+1}} - f_{N_j}\right|$, where we fix f_n to be a representative of $[f_n]$.
Notice that $\forall k \geq 1$,

$$
\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1}(f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left|f_{N_1} + \sum_{j=1}^{k-1}(f_{N_{j+1}} - f_{N_j})\right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1}\left|f_{N_{j+1}} - f_{N_j}\right| \\
&= g_k.
\end{aligned}
$$

□

Theorem 5.17 (Density of simple functions)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$. The simple functions*

$$S := \{\phi \in L^p(\mu) \,|\, \phi \text{ is simple, measurable}\}$$

are dense in $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(X,\mathcal{M},\mu)})$.

Proof. 1. First consider $1 \leq p < \infty$.
Let $f \in L^p(X, \mathcal{M}, \mu), \exists (\phi_n)_{n=1}^\infty$ be simple and measurable functions, such that

$$f(x) = \lim_{n\to\infty} \phi_n(x), \text{a.e. } x \in X,$$

and

$$|\phi_1| \leq |\phi_1| \leq \cdots \leq |f|.$$

Thus

$$f^p(x) = \lim_{n\to\infty} \phi_n^p(x), \text{a.e. } x \in X,$$

and

$$|\phi_1|^p \leq |\phi_1|^p \leq \cdots \leq |f|^p.$$

Since $|f-\phi_n|^p \leq (2|f|)^p = 2^p|f|^p \in L^1(X,\mathcal{M},\mu)$, by LDCT 3.18, we have that

$$\begin{aligned}
\lim_{n\to\infty} \|f-\phi_n\|^p_{L^p(X,\mathcal{M},\mu)} &= \lim_{n\to\infty} \int_X |f-\phi_n|^p d\mu \\
&= \int_X \lim_{n\to\infty} |f-\phi_n|^p d\mu \\
&= \int_X \lim_{n\to\infty} 0 d\mu \\
&= 0.
\end{aligned}$$

2. Now consider $p = \infty$.
Let $f \in L^p(X,\mathcal{M},\mu)$, we know $\mu(N) = 0$ for $N := \left\{x \in X : |f(x)| > \|f\|_{L^\infty(X,\mathcal{M},\mu)}\right\}$.
Let $f' := f\chi_{N^c}$. We notice that f' is measurable and bounded, with $|f'| \leq \|f\|_{L^\infty(X,\mathcal{M},\mu)}, \forall x \in X$.
Thus we can find $(\phi_n)_{n=1}^\infty$ be simple and measurable functions, such that

$$f(x) = \lim_{n\to\infty} \phi_n(x), \text{a.e. } x \in X \text{ uniformly, and } |\phi_1| \leq |\phi_1| \leq \cdots \leq |f|.$$

Now

$$\begin{aligned}
\|f-\phi_n\|_{L^\infty(X,\mathcal{M},\mu)} &= \|f\chi_N + f' - \phi_n\|_{L^\infty(X,\mathcal{M},\mu)} \\
&\leq \|f\chi_N\|_{L^\infty(X,\mathcal{M},\mu)} + \|f'-\phi_n\|_{L^\infty(X,\mathcal{M},\mu)} \\
&= \|f'-\phi_n\|_{L^\infty(X,\mathcal{M},\mu)} \\
&= \operatorname*{ess\,sup}_{x\in X} |f'(x) - \phi(x)| \\
&\to 0.
\end{aligned}$$

□

Remark. For $1 \leq p < \infty$,

$$S = \text{Span}\{\chi_E \,|\mu(E) < \infty\} = \{\phi : X \to \mathbb{C}\,|\phi \text{ is simple, measurable}, \mu(\{x \in X|\phi(x) \neq 0\}) < \infty\}$$

Theorem 5.18 (Density of compactly supported continuous functions)**.** *Given some measure space* $(X,\mathcal{M},\mu)$, *where* μ *is a Randon measure, then* $C_c(X)$ *is dense for* $p < \infty$.

Proof. Given any $\epsilon > 0$.
Consider any measurable E, such that $\mu(E) < \infty$.
By regularity, we can find some compact $K \subset E \subset V$ open , such that $\mu(V \setminus E) < \frac{\epsilon^p}{2^p}$.
Now we take the bump function $K < f < V$ by Urysohn's Lemma 6.9, where $f \in C_c(V) \subseteq C_c(X)$, and $f|_K = 1, f|_{V^c} = 0, 0 \leq f \leq 1$.

Now

$$\begin{aligned}
\|\chi_E - f\|_{L^p(\mu)}^p &= \int_X |\chi_E - f|^p d\mu \\
&= \int_{V\setminus K} |\chi_E - f|^p d\mu \\
&\leq \int_{V\setminus K} 2^p d\mu \\
&= 2^p \mu(V \setminus K) \\
&< \epsilon^p.
\end{aligned}$$

Thus $\chi_E \in \overline{C_c(X)}$.
Since $S = \text{Span}\{\chi_E \,|\mu(E) < \infty\}$ is dense in $L^p(\mu)$, so is $C_c(X)$. $\square$

Remark. This is not true for $p = \infty$. For instance, consider $X = \mathbb{R}$ with Lebesgue measure, or $X = \mathbb{N}$ with counting measure.

Proposition 5.19 ($L^q(\mu) \subseteq L^p(\mu)^*$)**.** *Let $p \in [1, \infty]$, with $\frac{1}{p} + \frac{1}{q} = 1$. Let $g \in L^q(\mu)$, then $\Lambda_g \in L^p(\mu)^*$, where $\Lambda_g(f) = \int_X fgd\mu$. Moreover, $\forall p \in (1, \infty], \|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$. This also holds for $p = 1$ if μ is semi-finite.*

Proof. clearly Λ_g is linear.
By Holder's Inequality, we have

$$\begin{aligned}
|\Lambda_g(f)| &= \left|\int_X fgd\mu\right| \\
&\leq \int_X |fg|d\mu \\
&\leq \|g\|_{L^q(\mu)}\|f\|_{L^p(\mu)}.
\end{aligned}$$

Thus $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f\neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \leq \|g\|_{L^q(\mu)} < \infty$.
Thus Λ_g is bounded and $\Lambda_g \in L^p(\mu)^*$.
We now want to show $\|\Lambda_g\|_{L^p(\mu)^*} \geq \|g\|_{L^q(\mu)}$.
If $\|g\|_{L^q(U)} = 0$, we have that $g = 0$a.e., and $\|\Lambda_g\|_{L^p(\mu)^*} = 0 = \|g\|_{L^q(U)}$.
Now consider $\|g\|_{L^q(U)} \neq 0$. It suffices to find some $\|f\|_{L^q(U)} = 1$, such that $\Lambda_g(f) \geq \|g\|_{L^q(U)}$.

1. $1 < p < \infty$.
   Notice that $p = \frac{1}{1-\frac{1}{q}} = \frac{1}{\frac{q-1}{q}} = \frac{q}{q-1}$.
   Let $f = \overline{\text{sgn}(g)} \frac{|g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}}$, we have that

$$\begin{aligned}
\|f\|_{L^p(\mu)}^p &= \int |f|^p d\mu \\
&= \int \frac{|g|^q}{\|g\|_{L^q(\mu)}^q} d\mu \\
&= \frac{1}{\|g\|_{L^q(\mu)}^q} \int |g|^q d\mu \\
&= \frac{1}{\|g\|_{L^q(\mu)}^q} \|g\|_{L^q(U)}^q \\
&= 1,
\end{aligned}$$

which means that $f \in L^p(\mu)$. In addition,

$$\begin{aligned}
|\Lambda_g(f)| &= \left|\int_X fg d\mu\right| \\
&= \left|\int_X \overline{\mathrm{sgn}(g)} \frac{|g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}} g d\mu\right| \\
&= \frac{1}{\|g\|_{L^q(\mu)}^{q/p}} \left|\int_X |g|^{1+q/p} d\mu\right| \\
&= \frac{1}{\|g\|_{L^q(\mu)}^{q-1}} \left|\int_X |g|^q d\mu\right| \\
&= \|g\|_{L^q(U)}.
\end{aligned}$$

Thus, $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$.

2. $p = \infty, q = 1$.
   Let $f = \overline{\mathrm{sgn}(g)} \in L^\infty(\mu)$. We have $\|f\|_{L^\infty(\mu)} = 1$. In addition,

$$\Lambda_g(f) = \int_X \overline{\mathrm{sgn}(g)} g d\mu = \int_X |g| d\mu = \|g\|_{L^1(\mu)}.$$

   Thus, $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$

3. $p = 1, q = \infty$, and μ is semi-finite.
   Choose $\epsilon \in (0, \|g\|_{L^\infty(\mu)})$.
   Let $A = \left\{x \in X \middle| |g(x)| > \|g\|_{L^\infty(\mu)} - \epsilon\right\}$.
   Notice that $\mu(A) > 0$, otherwise $\|g\|_{L^\infty(\mu)} = \epsilon$.
   Since μ is semi-finite, we can find $E \in \mathcal{M}$, such that $0 < \mu(E) < \infty, E \subseteq A$.
   Let $f = \frac{\chi_E}{\mu(E)} \overline{\mathrm{sgn}(g)}$.
   Notice that

$$\begin{aligned}
\|f\|_{L^1(\mu)} &= \int_X |f| d\mu \\
&= \int_X \left|\frac{\chi_E}{\mu(E)} \overline{\mathrm{sgn}(g)}\right| \\
&= \frac{1}{\mu(E)} \int_X \chi_E d\mu \\
&= 1.
\end{aligned}$$

   Thus $f \in L^1(\mu)$. In addition,

$$\begin{aligned}
\Lambda_g(f) &= \int_X fg d\mu \\
&= \int_X \frac{\chi_E}{\mu(E)} \overline{\mathrm{sgn}(g)} g d\mu \\
&= \int_E \frac{|g|}{\mu(E)} d\mu \\
&\geq \int_E \frac{\|g\|_{L^\infty(\mu)} - \epsilon}{\mu(E)} d\mu \\
&\geq \|g\|_{L^\infty(\mu)} - \epsilon.
\end{aligned}$$

Since this holds for any $\epsilon > 0$, we have that

$$\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f\in L^p(\mu), f\neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \sup_{\epsilon>0} \|g\|_{L^\infty(\mu)} - \epsilon = \|g\|_{L^q(\mu)}.$$

We thus have $\|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$. □

Remark. In the above case, the map $g \mapsto \Lambda_g$ is isometric.

6 Borel Measures on Topological Spaces

6.1 Topological Spaces

See more in my notes for Pmath753 Functional Analysis or the notes of Pmath367 Topology by Professor S. New.

Definition 6.1. Let $X \neq \emptyset$ be a set. A **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X) := 2^X$ = power set of X satisfying

1. $\emptyset, X \in \mathcal{T}$,
2. $\mathcal{T}$ is closed under arbitrary union; namely, $\forall \{A_\alpha\}_{\alpha\in K} \subseteq \mathcal{T},\ \bigcup_{\alpha\in K} A_\alpha \in \mathcal{T}$, and
3. $\mathcal{T}$ is closed under finite intersection; namely, $\forall \{A_i\}_{i=1}^n \subseteq \mathcal{T},\ \bigcup_{i=1}^n A_i \in \mathcal{T}$.

Also, $(X, \mathcal{T})$ is a **topological space** if $\mathcal{T}$ is a topology on X.

Definition 6.2. Let $(X, \mathcal{T})$ be a topological space, then we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$. We say $E \subseteq X$ is **closed** if $E^c \in \mathcal{T}$ is open.

Definition 6.3. For $E \in X$, the **closure** of E is

$$\bar{E} = \bigcap_{F\supseteq E: F \text{ is closed}} F.$$

Definition 6.4. A set $K \subseteq X$ is **compact** if every open cover of K has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha\in A} \text{ be open}, K \subseteq \bigcup_{\alpha\in A} U_\alpha \implies \exists n \in \mathbb{N},\ \alpha_1, \dots, \alpha_n,\ \text{such that } K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

Definition 6.5. An (open) neighborhood of $x \in X$ is some

$$U_x \in \mathcal{T},\ \text{such that } x \in U_x.$$

Definition 6.6. X is **Hausdorff** if

$$\forall x \neq y \in X,\ \exists U_x, U_y \text{ open neighborhoods for } x, y,\ \text{such that } U_x \cap U_y = \emptyset.$$

Example 6.1.1. Every metric space is Hausdorff.

Definition 6.7. X is **locally compact** if $\forall x \in X$, there is a neighborhood U_x such that $\overline{U_x}$ is compact.

Example 6.1.2. $\mathbb{R}^n$ are locally compact by Heinz-Borel theorem.

Proposition 6.1. *A Banach space $(X, \|\cdot\|)$ is locally compact iff* $\dim(X) < \infty$.

Theorem 6.2. *Let $(X, \mathcal{T})$ be a topological space,*

1. *Suppose K is compact, then $\forall F \subseteq K$ that is closed, F is also compact.*

2. *If X is Hausdorff, for any compact $K \subseteq X, x \in X \setminus K$, $\exists$ open neighborhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$.*

Proof. 1. Let $(U_\alpha)_{\alpha \in A}$ be an open cover for F.
Since F is closed, then F^c is open. Thus $\{F^c\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover for K.
Thus there are $U_{\alpha_1}, \ldots, U_{\alpha_n}$, such that $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$. Thus $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ has a finite subcover.

2. Consider any $y \in K$, there is some open neighborhoods $U_y \ni x, W_y \ni y$, such that $U_y \cap W_y = \emptyset$.
Since $K \subseteq \bigcup_{y \in K} W_y$ is compact, we have $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$ for some $y_1, \ldots, y_n$.
Let $U = \bigcap_{i=1}^n U_{y_i}$, we have $x \in U, K \subseteq W, U \cap W = \emptyset$ as required.

$\square$

Corollary 6.3. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, then any compact set K is closed. In addition, for any closed $F \subseteq X$, we have $F \cap K$ is compact.*

Proof. Suppose for contradiction that K is not closed, then there is some $y \in \bar{K} \setminus K$. Thus we can find open neighborhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$. Now $K \subset \bar{K} \setminus U \subsetneq \bar{K}$ is closed, which is a contradiction.
Since K is closed, so is $F \cap K \subseteq K$, and thus it is compact. $\square$

Lemma 6.4. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, and $(K_\alpha)_{\alpha \in A}$ be a collections of compact sets such that*

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

We must have $\alpha_1, \ldots, \alpha_n \in A$, such that

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

Proof. Fix $\alpha_1 \in A$, then $K_{\alpha_0} \subseteq \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$ is compact and has an open cover.
Thus there must be $\alpha_2, \ldots, \alpha_n \in A$, such that $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = \left(\bigcap_{i=2}^n K_{\alpha_i}\right)^c$.
Thus $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$. $\square$

Theorem 6.5. *Let X be a Locally Compact Hausdorff space, and let $K \subseteq U \subseteq X$ be such that K is compact, and U is open. Then there exists some open set V such that $\bar{V}$ is compact, and*

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof. Since X is a Locally Compact Hausdorff space, there are $V_1, \ldots, V_n$, each with $\bar{V}_i$ be compact, such that $K \subseteq \bigcup_{i=1}^n V_i =: G$. Note that $\bar{G}$ is compact, and G is open.
If $U = X$, then $G \subseteq U$, and we are done.
Otherwise, let $C := X \setminus U$ be non-empty and closed.
Consider any $y \in C$, we know that $y \notin K$. Since X is Hausdorff, we can find open $W_y \supset K$, and $U_y \ni y$, such that $W_y \cap U_y = \emptyset$. Then $W_y \subseteq U_y^c$, and thus $\bar{W}_y \subseteq U_y^c$, since U_y^c is closed. Yet $y \notin U_y^c$, thus $y \notin \bar{W}_y$.
Now consider the family $\left\{C \cap \bar{W}_y \cap \bar{G}\right\}_{y \in C}$. Notice that each $C \cap \bar{W}_y \cap \bar{G}$ is compact, since $C, \bar{W}_y$ are closed, and $\bar{G}$ is compact.
Yet $\bigcap_{y \in C} \left(C \cap \bar{W}_y \cap \bar{G}\right) = \emptyset$.
Thus $\exists y_1, \ldots, y_n \in C$, such that $\bigcap_{i=1}^n \left(C \cap \bar{W}_y \cap \bar{G}\right) = \emptyset$.
Now let $V := G \cap \bigcap_{i=1}^n W_{y_i}$.
Clearly V is open, and $K \subseteq V$.
In addition, $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$, yet the intersection of righthand side and C is empty, thus contained in $C^c = U$. $\square$

6.2 Compactly Supported Functions

Definition 6.8. Let $C(X)$ be the collection of functions $f : X \to \mathbb{C}$ that are continuous.

Proposition 6.6. *$C(X)$ is a $\mathbb{C}$ vector space, and also an Algebra over $\mathbb{C}$. It also admits a partial order by $f \geq g \iff \forall x \in X, f(x) \geq g(x)$.*

Definition 6.9. For $f \in C(X)$, the **support** of it is

$$\mathrm{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

Definition 6.10. The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \mathrm{Supp}(f) \text{ is compact}\}.$$

Proposition 6.7. *Suppose every compact set K is Borel-measurable, then $C_c(X)$ is a sub-algebra of $C(X)$.*

Proposition 6.8. *Suppose every compact set K is Borel-measurable, let $\mu : Bor(X) \to [0, \infty]$ be a Borel measure on X, such that $\forall K$ be compact, $\mu(K) < \infty$, then*

$$C_c(X) \subseteq L^1(\mu).$$

Proof. Given any $f \in C_c(X)$. Let $K = \mathrm{Supp}(f)$, then

$$\int_X |f| d\mu = \int_K |f| d\mu \leq \int_K \|f\|_\infty d\mu = \mu(K)\|f\|_\infty < \infty.$$

□

6.3 Partition of Unity

Definition 6.11. Let K be a compact set, and V be an open set of X. Let $f \in C_c(X)$. We say $f < V$ if $0 \leq f \leq 1$, and $\mathrm{Supp}(f) \subseteq V$. We say $K < f$ if $0 \leq f \leq 1$, and $f|_K = 1$. We say $K < f < V$ if $K \subset V, K < f, f < V$.

Remark. f is a "bump" function that approximates χ_K when V shrinks towards K.

Theorem 6.9 (Urysohn's Lemma)**.** *Let X be a Locally Compact Hausdorff space, $K \subseteq V \subseteq X$ be such that K is compact, and V is open. Then there exists $f \in C_c(V)$, such that $K < f < V$.*

Proof. we want to construct a family of open sets $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s$.
By 6.5, we can find $K \subset V_0 \subset \bar{V_0} \subset V$.
Pick an enumeration of $r \in \mathbb{Q} \cap (0, 1]$, i.e. $(r_n)_{n=1}^\infty$. WLOG, we can let $r_1 = 1$.
By 6.5, we can find $K \subset V_1 \subset \bar{V_1} \subset V_0$.
Suppose we have constructed the V_{r_i} for $1 \leq i \leq n$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s \in \{r_i\}_{i=1}^n$.
Let $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$.
Now by 6.5, we can find $\bar{V_t} \subset V_{n+1} \subset \bar{V_{n+1}} \subset V_s$.
For any $r < r_{n+1}$, we have $r \leq s$, and thus $V_{n+1} \subset \bar{V_{n+1}} \subset V_s \subset \bar{V_s} \subseteq V_r$ by induction hypothesis, and similarly for any $r > r_{n+1}$.
Inductively, we can prove there is such a family.
Define $f_r := r\chi_{V_r}$, and $g_r := r\chi_{\bar{V_r}^c} + \chi_{\bar{V_r}}$, and $f := \sup_r f_r, g := \inf_r g_r$.
We can show that f, g are upper and lower continuous, respectively.

In addition, f, g are both 0 outside of V_1, and 1 on K.
Suppose there is some $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, such that $f_r(x) > g_s(x)$. Then we must have $f_r(x) > 0$, and thus $x \in V_r$ and $1 \leq r = f_r(x)$.
Thus $1 > g_s(x)$, and thus $x \in \bar{V_s}^c$ and $f_s = s$.
Since $r > s$, we must have $V_r \subset \bar{V_r} \subset V_s \subset \bar{V_s}$, which is a contradiction to $x \in V_r, x \notin \bar{V_s}$.
Thus for any $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, we must have $f_r(x) \leq g_s(x)$.
Thus we must have $f(x) \leq g(x)$ for any $x \in V$.
Now suppose there is some $x \in X$, such that $f(x) < g(x)$.
There must be some rationals, such that $f(x) < r < s < g(x)$, since $\mathbb{Q}$ is dense.
Thus $\sup_r f_r(x) < r$, and thus $x \notin V_r$.
Also, $\inf_s g_s(x) > s$, and thus $x \in \bar{V_s}$.
Yet $r < s$, we must have $V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r}$, which is a contradiction.
Thus we must have $f = g$, and it forces f to be continuous. $\square$

Definition 6.12. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$.
A collection $(h_i)_{i=1}^n \subset C_c(X)$ is called a **partition of unity** on K subordinate to $(V_i)_{i=1}^n$ if

$$\begin{cases} \forall 1 \leq i \leq n, & h_i < V_i, \\ \forall x \in K, & \sum_{i=1}^n h(x) = 1. \end{cases}$$

Theorem 6.10. *Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$, there always exists a partition of unity on K subordinated to $(V_i)_{i=1}^n$.*

Proof. Since K is compact, we can find some open cover $W_1, \ldots, W_m$, such that for all j, we have $W_j \subset \bar{W}_j \subset V_{i(j)}$ for some $1 \leq i(j) \leq n$.
Let $K_i := \bigcup_{1 \leq j \leq m \text{ such that } W_j \subset V_i} \bar{W}_j \subset V_i$, which is compact.
By Urysohn's lemma, we can find $K_i < g_i < V_i$.
Now let $h_1 := g_1$, and in general, $h_i := g_i \prod_{j<i}(1 - g_j)$.
It is easy to check that $0 \leq h_i \leq 1$, and $h_i \in C_c(X)$.
In addition, $\mathrm{Supp}(h_i) \subseteq \mathrm{Supp}(g_i) \subset V_i$.
Thus $h_i < V_i$. Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_n)$.
For any $x \in K$, there must be some $i \in [n]$ such that $x \in K_i$, and thus $g_i(x) = 1$, and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_n(x)) = 1 - 0 = 1.$$

$\square$

6.4 Linear Functional

Definition 6.13. Let X be a locally compact Hausdorff space. A **linear functional** on $C_c(X)$ is a linear map $\Lambda : C_c(X) \to \mathbb{R}$. A linear functional Λ is **positive** if $\Lambda(f) \geq 0$ for all $f \in C_c(X)$ such that $f \geq 0$.

Remark. If X is a compact Hausdorff space, $C_c(X) = C(X)$.

Proposition 6.11. *Let X be a compact Hausdorff space, then for a Borel measure μ on X,*

1. *If μ is finite, $\Lambda_\mu(f) := \int_X f d\mu$ is a positive linear functional.*
2. *If μ is finite, Λ_μ is bounded and hence continuous. Indeed, $\forall f \in C(X), |\Lambda_\mu(f)| \leq \mu(X)\|f\|_\infty$.*

3. Λ_μ *is a finite-value linear functional iff* $\mu(X) < \infty$.

Proof. 1. By properties fo integral and 6.8.

2. For $f \in C(X)$.

$$\begin{aligned} |\Lambda_\mu(f)| &= \left| \int_X f d\mu \right| \\ &\leq \int_X |f| d\mu \\ &\leq \int_X \|f\|_\infty d\mu \\ &= \mu(X)\|f\|_\infty. \end{aligned}$$

□

6.5 Radon Meaure

Definition 6.14. Let X be a topological space, $\mu : \mathrm{Bor}(X) \to [0, \infty]$ be a Borel measure on X. For $A \in \mathrm{Bor}(X)$, we say μ is **outer regular** if $\mu(A) = \inf \{\mu(U) : \text{open } U \supseteq A\}$. μ is **inner regular** if $\mu(A) = \sup \{\mu(K) : \text{compact } K \subseteq A\}$. μ is **regular** is it is inner and outer regular for any $A \in \mathrm{Bor}(X)$.

Definition 6.15. Let X be a topological space, $\mu : \mathrm{Bor}(X) \to [0, \infty]$ be a Borel measure on X. μ is a **Radon** measure if

1. $\forall$compact $K \subseteq X,\ \mu(K) < \infty$,
2. μ is outer regular on Borel sets,
3. μ is inner regular on open sets.

Remark. We have seen that Lebesgue-Stieltjes Measures are regular and Radon.

Proposition 6.12. *A finite Borel measure on a compact metric space is always regular (hence Radon).*

Proof. Let μ be a finite Borel measure on a compact metric space X. Let $S \subseteq Bol(X)$ on which μ is regular. If $C \subseteq X$ is closed, it is compact. Thus μ is inner regular for C. Since X is a metric space, $C = \bigcap_{n \geq 1} \left\{x \in X : d(x, C) < \frac{1}{n}\right\}$ is G_δ. By continuity from above of μ, it follows that μ is also outer-regular. Thus all the closed sets belong to S.
Since Borel sets are generated by closed sets, it suffices to show S is a σ-algebra.
For $A \in S, \epsilon > 0$, there is compact K and open U such that $K \subseteq A \subseteq U, \mu(U \setminus K) < \epsilon$. Then $U^c \subseteq A^c \subseteq K^c$, where U^c is compact, K^c is open. In addition,

$$\mu(K^c \setminus U^c) = \mu(K^c \cap U) = \mu(U \setminus K) < \epsilon.$$

Thus $A^c \in S$.
Consider $(A_i)_{i=1}^\infty \subseteq S, \epsilon > 0$. Choose compact $K_i \subseteq A_i$ and open $U_i \supseteq A_i$, such that $\mu(U_i \setminus K_i) < \epsilon/2^i$. Let $A = \bigcup_{i=1}^\infty A_i, C_n = \bigcup_{i=1}^n K_i, C = \bigcup_{i=1}^\infty K_i, U = \bigcup_{i=1}^\infty U_i$.
Thus C_n are closed, U is open, and $C_n \subseteq A \subseteq U$. By continuity and finiteness of μ, we have

$$\begin{aligned} \lim_{n \to \infty} \mu(U \setminus C_n) &= \mu(U \setminus C) \\ &\leq \sum_{i=1}^\infty \mu(U_i \setminus K_i) \\ &< \epsilon. \end{aligned}$$

Thus μ is regular on A, and thus $A \in S$, and thus S is closed under countable unions.
Thus $S = Bol(X)$. □

6.6 Extremely Disconnected Spaces

Definition 6.16. A compact space X is **extremely disconnected** if the closure of every open set is open.

Proposition 6.13. *If X is extremely disconnected, then there is a basis of clopen sets.*

Proposition 6.14. *If $A \subseteq X$ is clopen, then $\chi_A \in C(X)$.*

Proposition 6.15. *(Stone–Čech compactification) Let D be a discrete space (thus every subset is open). The* ***Stone–Čech compactification*** *is the unique compact (Hausdorff) space βD with the following universal properties:*

1. *$D \subseteq \beta D$ as topology inclusion.*
2. *For any compact K, and every continuous map $f : D \to K$, there is a unique continuous extension $\beta f : \beta D \to K$.*

Proposition 6.16. $\ell^\infty(D) \cong C(\beta D)$

Proposition 6.17. *βD is the set of ultrafilters on D.*

Proposition 6.18. *βD is extremely disconnected.*

6.7 Riesz-Markov-Kakutani

Lemma 6.19. *Let X be a Locally Compact Hausdorff space, $\Lambda : C(X) \to \mathbb{C}$ a positive linear functional. For any compact $K \subseteq X$, there is $C_K \geq 0$, such that for any f with $\operatorname{Supp}(f) \subseteq K$, we have*

$$|\Lambda(f)| \leq C_K \|f\|_\infty.$$

Proof. By Urysohn's Lemma 6.9, there is some $g \in C_c(X)$, such that $K < g$. Since $g \geq 0$, we have $C_K := \Lambda(g) \geq 0$.

Also, since $g|_K = 1$, we have $\|f\|_\infty g \pm f \geq 0$. Thus,

$$\Lambda(\|f\|_\infty g \pm f) \geq 0 \implies \|f\|_\infty \Lambda(g) \pm \Lambda(f) \geq 0,$$

which means

$$\pm\Lambda(f) \leq \|f\|_\infty \Lambda(g) = C_K \|f\|_\infty.$$

Thus,

$$|\Lambda(f)| \leq C_K \|f\|_\infty.$$

□

Theorem 6.20 (Riesz-Markov-Kakutani)**.** *Let X be a Locally Compact Hausdorff space, $\Lambda : C(X) \to \mathbb{F}$ a positive linear functional. There is a unique Radon measure μ on X, such that $\Lambda(f) = \Lambda_\mu(f) = \int_X f d\mu$. In addition,*

(a) $\forall U \subseteq X$ be open, we have $\mu(U) = \sup\{\Lambda(f) : f < U\}$.

(b) $\forall K \subseteq X$ be compact, we have $\mu(K) = \inf\{\Lambda(f) : K < f\}$.

Proof. 1. We first show the uniqueness. Let μ be any Radon measure such that $\Lambda(f) = \int_X f d\mu$.

Given any open U. Consider any compact K such that $K \subset U$. By Urysohn's lemma, we can find a function f, such that $K < f < U$. Thus $\chi_K \leq f \leq \chi_U \implies \mu(K) \leq \Lambda(f) \leq \mu(U)$. Thus by inner regularity, $\mu(U) = \sup\{\mu(K) : \text{compact } K \subseteq U\} = \sup\{\Lambda(f) : f < U\}$, which is uniquely determined by Λ.

For any Borel set A, we have that by outer regularity, $\mu(A) = \inf\{\mu(U) : \text{open } U \supseteq A\}$, which is uniquely determined.

This shows the uniqueness, now let us check existence.

2. We now construct μ by defining an outer measure and apply Caratheodory.

   Define $\mu^* : \mathcal{T} \to [0, \infty]$ by: for any open $U \subseteq X$,

$$\mu^*(U) := \sup \{\Lambda(f) : f < U\}.$$

   Clearly for any open $V \supseteq U$, we have $\mu^*(U) \leq \mu^*(V)$. Thus we have $\mu^*(U) := \inf \{\mu^*(V) : \text{open } V \subseteq U\}$. Now we extend $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu^*(A) := \inf \{\mu^*(U) : \text{open } U \supseteq A\}.$$

   Notice that

$$\mu^*(A) = \inf \{\mu^*(U) : \text{open } U \supseteq A\} \geq \inf \left\{ \sum_{i=1}^{\infty} \mu^*(U_i) : \text{open } U_i, A \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$

   On the other hand, consider any open U_i, such that $A \subseteq \bigcup_{i=1}^{\infty} U_i$. Notice that $U := \bigcup_{i=1}^{\infty} U_i$ is open, and $U \supseteq A$. Pick any $f < U$, and we have $K := \operatorname{Supp}(f) \subseteq U = \bigcup_{i=1}^{\infty} U_i$ is compact. Thus, there is a finite subcover $\bigcup_{i=1}^{n} U_i \supset K$. By theorem 6.10, there is a partition of unity on K subordinated to $(U_i)_{i=1}^n$, $(h_i)_{i=1}^n \subseteq C_c(X)$, such that each $h_i < U_i$, and $\sum_{i=1}^n h_i = 1$. Now take $f_i := h_i f$ for all $i \in [n]$, and $f_i := 0$ for $i > n$. Clearly each $f_i < U_i$, so $\Lambda(f_i) \leq \mu^*(U_i)$. Also, $\sum_{i=1}^n f_i = f$ on $K = \operatorname{Supp}(f)$, and thus on X. Thus,

$$\begin{aligned}\Lambda(f) &= \sum_{i=1}^{n} \Lambda(f_i) \\ &= \sum_{i=1}^{\infty} \Lambda(f_i) \\ &\leq \sum_{i=1}^{\infty} \mu^*(U_i).\end{aligned}$$

   Since this holds for all such $f < U$, we have that

$$\mu^*(U) \leq \sum_{i=1}^{\infty} \mu^*(U_i).$$

   Since U is open, and $U \supseteq A$, we have $\mu^*(A) \leq \mu^*(U) \leq \sum_{i=1}^{\infty} \mu^*(U_i)$. Since this holds for all such open U_i, such that $A \subseteq \bigcup_{i=1}^{\infty} U_i$, we have

$$\mu^*(A) \leq \inf \left\{ \sum_{i=1}^{\infty} \mu^*(U_i) : \text{open } U_i, A \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$

   This shows $\mu^*(A) = \inf \{\sum_{i=1}^{\infty} \mu^*(U_i) : \text{open } U_i, A \subseteq \bigcup_{i=1}^{\infty} U_i\}$, and clearly we have $\mu^*(\emptyset) = \Lambda(0) = 0$. By proposition 4.1, μ^* is an outer measure. Thus, there is a complete measure space $(X, \mathcal{M}, \mu)$ induced by the Caratheodory Theorem 4.3.

3. We now check any open set $O \in \mathcal{M}$, so μ is a Borel measure. Namely, for all $A \subseteq X$,

$$\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A).$$

   First, suppose A is open, then $O \cap A$ is open. Fix any $\epsilon > 0$. By definition, there is $f < O \cap A$, such that

$$\Lambda(f) \geq \mu^*(A \cap O) - \frac{1}{2}\epsilon.$$

Since $A \setminus \mathrm{Supp}(f)$ is open, there is $g < A \setminus \mathrm{Supp}(f) \subseteq A \setminus (A \cap O) = A \setminus O$, such that

$$\begin{aligned}\Lambda(g) &\geq \mu^*(A \setminus \mathrm{Supp}(f)) - \frac{1}{2}\epsilon \\ &\geq \mu^*(A \setminus O) - \frac{1}{2}\epsilon.\end{aligned}$$

Notice that $0 \leq f + g \leq 1$, since $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$, and$0 \leq f, g \leq 1$. Also, $\mathrm{Supp}(f) \cup \mathrm{Supp}(g) \subseteq (A \cap O) \cup (A \setminus O) = A$. Also, $f + g$ is continuous. This shows $f + g < A$. Thus,

$$\begin{aligned}\mu^*(A) &\geq \Lambda(f + g) \\ &= \Lambda(f) + \Lambda(g) \\ &\geq \mu^*(A \cap O) - \frac{1}{2}\epsilon + \mu^*(A \setminus O) - \frac{1}{2}\epsilon \\ &= \mu^*(O \cap A) + \mu^*(O^c \cap A) - \epsilon.\end{aligned}$$

Since this holds for all $\epsilon > 0$, we have that $\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A)$.

Now for any set A, fix any $\epsilon > 0$. We have that there is some open $U \supseteq A$, such that

$$\mu^*(U) \leq \mu^*(A) + \epsilon.$$

Since U is open, we have

$$\mu^*(U) = \mu^*(O \cap U) + \mu^*(O^c \cap U) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A).$$

Thus, $\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A) - \epsilon$. Since this holds for all $\epsilon > 0$, we have

$$\mu^*(A) \geq \mu^*(O \cap A) + \mu^*(O^c \cap A).$$

This shows that any open $O \in \mathcal{M}$. Since the Borel algebra is generated by the open sets, all Borel sets are in $\mathcal{M}$. Thus, μ (restricted to the Borel algebra) is a Borel Measure.

4. (a) is by definition of μ^*, and that any open U is measurable, so

$$\mu(U) = \mu^*(U) = \sup\{\Lambda(f) : f < U\}.$$

To check (b), we fix any compact K. Consider any function f such that $K < f$. For any $\epsilon > 0$, let $O_\epsilon := \{x \in X : f(x) > 1 - \epsilon\}$, which is open. Clearly $K \subseteq O_\epsilon$, so

$$\mu(K) \leq \mu(O_\epsilon) = \sup\{\Lambda(g) : g < O_\epsilon\}.$$

However, for any such g, we always have $\frac{f}{1-\epsilon} > 1 \geq g$ on O_ϵ, and $\frac{f}{1-\epsilon} \geq 0 = g$ outside of O_ϵ, so $\frac{f}{1-\epsilon} - g$ is positive. Thus, $\Lambda\left(\frac{f}{1-\epsilon}\right) \geq \Lambda(g)$. Since this holds for all such g,

$$\mu(K) \leq \sup\{\Lambda(g) : g < O_\epsilon\} \leq \Lambda\left(\frac{f}{1-\epsilon}\right) = \frac{1}{1-\epsilon}\Lambda(f).$$

Since this holds for all $\epsilon > 0$, we have $\mu(K) \leq \Lambda(f)$. Since this holds for all $K < f$, we have

$$\mu(K) \leq \inf\{\Lambda(f) : K < f\}.$$

On the other hand, for any open $U \supseteq K$, we can find some $K < f < U$. Notice that $\inf\{\Lambda(f) : K < f\} \leq \Lambda(f) \leq \mu(U)$. Since this holds for all open $U \supseteq K$, we have

$$\mu(K) = \inf\{\mu(U) : \text{open } U \supseteq K\} \geq \inf\{\Lambda(f) : K < f\}.$$

This shows (b).

5. We now check that μ is indeed a Radon measure.

   For any compact $K \subseteq X$, by Urysohn's lemma 6.9, there is some function f such that $K < f$. We thus have
$$\mu(K) \leq \Lambda(f) < \infty.$$

   The outer regularity on Borel sets is by definition of μ^*, since
$$\mu(A) = \mu^*(A) = \inf\{\mu^*(U) : \text{open } U \supseteq A\} = \inf\{\mu(U) : \text{open } U \supseteq A\}.$$

   Now consider any open set U. For any $a < \mu(U)$, by (a), there is some $f < U$, such that $\Lambda(f) > a$. Let $K := \operatorname{Supp}(f) \subseteq U$, which is compact. For any function g such that $K < g$, we have that $g \geq 1 \geq f$ on K, and $g \geq 0 = f$ outside of f. Thus, $g - f \geq 0$, so $\Lambda(f) \leq \Lambda(g)$. Since this holds for all g, by (b),
$$a < \Lambda(f) \leq \inf\{\Lambda(g) : K < g\} \leq \mu(K) \leq \sup\{\mu(K) : \text{compact } K \subseteq U\}.$$

   Since this holds for all $a < \mu(U)$, we have
$$\mu(U) \leq \sup\{\mu(K) : \text{compact } K \subseteq U\}.$$

   On the other hand, clearly $\sup\{\mu(K) : \text{compact } K \subseteq U\} \leq \mu(U)$, so we have
$$\mu(U) = \sup\{\mu(K) : \text{compact } K \subseteq U\},$$

   which show outer regularity on open sets.

   Now we have shown that μ is Radon.

6. It is left to show for all $f \in C_c(U)$, $\Lambda(f) = \int_X f d\mu$. We first assume $\operatorname{Im}(f) \in [0,1]$. Fix some $N \geq 1$, let $K_0 := \operatorname{Supp}(f)$, and for each $j \in [N]$, let $K_j := \left\{x : f(x) \geq \frac{j}{N}\right\}$, each of which is closed since f is continuous. Since K_0 is compact, and $K_N \subseteq K_{N-1} \subseteq \cdots K_1 \subseteq K_0$, we have that all of them are compact. Now define
$$f_j(x) := \begin{cases} 0, & x \notin K_{j-1} \\ f(x) - \frac{j-1}{N}, & x \in K_{j-1} \setminus K_j \\ \frac{1}{N}, & x \in K_j. \end{cases}$$

   We have that for any $x \in K_0$, and let n be maximal such that $x \in K_n$, we have that
$$\begin{aligned} \sum_{j=1}^{N} f_j(x) &= \sum_{j=1}^{n} f_j(x) + f_{n+1}(x) \\ &= n\frac{1}{N} + f(x) - \frac{n+1-1}{N} \\ &= f(x). \end{aligned}$$

   Thus $f(x) = \sum_{j=1}^{N} f_j$. Also, for $x \in K_j$, we have $Nf_j(x) = 1 = \chi_{K_j}(x) = \chi_{K_{j-1}}(x)$, and for all $x \notin K_j$, we have $Nf_j(x) = 0 = \chi_{K_j}(x) = \chi_{K_{j-1}}(x)$. Lastly, for all $x \in K_{j-1} \setminus K_j$, we have $\frac{j-1}{N} \leq f(x) < \frac{j}{N}$, so
$$\begin{gathered} Nf_j(x) = N\left(f(x) - \frac{j-1}{N}\right) \\ N\left(\frac{j-1}{N} - \frac{j-i}{N}\right) \leq Nf_j(x) < N\left(\frac{j}{N} - \frac{j-1}{N}\right) \\ 0 \leq Nf_j(x) < 1 \\ \chi_{K_j}(x) \leq Nf_j(x) < \chi_{K_{j-1}}(x) \end{gathered}$$

   Thus,
$$\mu(K_j) = \int_X \chi_{K_j}(x) d\mu \leq \int_X Nf_j(x) d\mu < \int_X \chi_{K_{j-1}}(x) d\mu = \mu(K_{j-1}).$$

Summing up over $j \in [N]$, we have

$$\begin{aligned}\sum_{j=1}^{N} \mu(K_j) &\leq \sum_{j=1}^{N} \int_X N f_j d\mu < \sum_{j=1}^{N} \mu(K_{j-1}) \\ \sum_{j=1}^{N} \mu(K_j) &\leq N \int_X f d\mu < \sum_{j=0}^{N-1} \mu(K_j).\end{aligned}$$

On the other hand, notice that we have $0 \leq N f_j < 1$. Also, each f_j is continuous, and $\mathrm{Supp}(f_j) = K_{j-1}$ is compact, so $f_j \in C_c(X)$. Clearly $K_j < N f_j$, so by (b), $\mu(K_j) \leq \Lambda(N f_j) = N\Lambda(f_j)$. Also, for any open $U \supset K_{j-1}$, we have $\mathrm{Supp}(f_j) = K_{j-1} \subseteq U$, so $N f_j < U$. By (a), $\mu(U) \geq \Lambda(N f_j) = N\Lambda(f_j)$. By outer regularity, $\mu(K_{j-1}) \geq N\Lambda(f_j)$. Thus,

$$\begin{aligned}\sum_{j=1}^{N} \mu(K_j) &\leq \sum_{j=1}^{N} N\Lambda(f_j) < \sum_{j=1}^{N} \mu(K_{j-1}) \\ \sum_{j=1}^{N} \mu(K_j) &\leq N\Lambda(f) < \sum_{j=0}^{N-1} \mu(K_j).\end{aligned}$$

This show that

$$\begin{aligned}N\left|\Lambda(f) - \int_X f d\mu\right| &= \left|N\Lambda(f) - N\int_X f d\mu\right| \\ &\leq \sum_{j=0}^{N-1} \mu(K_j) - \sum_{j=1}^{N} \mu(K_j) \\ &= \mu(K_0) - \mu(K_N) \\ &= \mu(K_0) \\ &< \infty.\end{aligned}$$

Since this holds for all $N \geq 1$, we must have

$$\Lambda(f) = \int_X f d\mu.$$

Now for any $f \in C_c(X)$ such that $\mathrm{Im}(f) \subseteq \mathbb{R}^+$, we can let $\tilde{f} := \frac{f}{\|f\|_\infty}$, which is still in $C_c(X)$, and $\mathrm{Im}(\tilde{f}) \in [0, 1]$. Thus, the previous case applies. By linearity of Λ and the integral, we have

$$\Lambda(f) = \|f\|_\infty \Lambda(\tilde{f}) = \|f\|_\infty \int_X \tilde{f} d\mu = \int_X f d\mu.$$

Now for any $f \in C_c(X)$, such that $\mathrm{Im}(f) \subseteq \mathbb{R}$, we can break it into $f^+ := \max\{0, f\}, f^- := -\min\{0, f\}$, both nonnegative and still in $C_c(X)$. Since $f = f^+ - f^-$, by linearity, we have $\Lambda(f) = \int_X f d\mu$. Lastly, for any $f \in C_c(X)$, we have that $\Re(f), \Im(f) \in C_c(X)$ and are both real-valued, so again by linearity,

$$\Lambda(f) = \int_X f d\mu.$$

□

7 Signed and Complex measures

7.1 Signed measures

Recall that if $(X, \mathcal{M}, \mu)$ is a measure space, and $f : X \to [0, \infty)$ is measurable, then we can set a measure $\mu_f(A) := \int_X \chi_A f d\mu$, and we have $\int_X g d\mu_f = \int g f d\mu$.

Example 7.1.1. Consider the regular Lebesgue measure, and $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, then λ_f gives a probability measure with the standard distribution.

Now we want to generalize this to functions that are not non-negative.

Definition 7.1. Let $(X, \mathcal{M})$ be a measurable space. A function $\nu : \mathcal{M} \to [-\infty, \infty]$ is a **signed measure** if ν only takes at most one of $\pm\infty$ and satisfies countable additivity. Namely, for any pairwise disjoint sets $E_1, E_2, \dots$ in $\mathcal{M}$, we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Proposition 7.1. *Suppose $|\nu(\bigsqcup_{i=1}^{\infty} E_i)| < \infty$, then $\sum_{i=1}^{\infty} \nu(E_i)$ must converge absolutely, since we want $\nu(\bigsqcup_{i=1}^{\infty} E_i)$ to be invariant of the order of union.*

Proposition 7.2. *If $f \in \mathcal{L}^1(X, \mu)$, then $\mu_f(A) := \int_X \chi(A) f d\mu$ is a signed measure.*

Proposition 7.3. *If $f, g \geq 0$ is measurable, and $g \in \mathcal{L}^1(\mu)$, then $\nu(E) := \int_X \chi_E (f - g) d\mu$ is a signed measure.*

Definition 7.2. Suppose ν is a signed measure, then $E \in \mathcal{M}$ is

1. **null** for ν if $\forall F \subseteq E, \nu(F) = 0$.
2. **positive** for ν if $\forall F \subseteq E, \nu(F) \geq 0$.
3. **negative** for ν if $\forall F \subseteq E, \nu(F) \leq 0$.

Lemma 7.4. *Let $E \in \mathcal{M}$, if $0 < \nu(E) < \infty$, then $\exists A \subseteq E, A \in \mathcal{M}$ is positive, and $\nu(A) > 0$.*

Proof. Suppose E is positive, then we are done.
Suppose E is not positive, then $\inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\} < 0$.
Thus, $\frac{1}{2} \inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\} > \inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\}$, and we can choose $B_1 \subseteq E, B_1 \in \mathcal{M}$, such that

$$\nu(B_1) \leq \frac{1}{2} \inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\} \leq \max\left\{-1, \frac{1}{2} \inf\{\nu(B) | B \subseteq E, B \in \mathcal{M}\}\right\}.$$

Recursively choose $B_n \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i$ with $\nu(B_n) \leq \max\left\{-1, \frac{1}{2} \inf\left\{\nu(B) | B \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i, B \in \mathcal{M}\right\}\right\}$.
Now either this sequence terminates (then $A = E \setminus \bigsqcup_{i=1}^{n-1} B_i$ is positive), or we get an infinite sequence.
Set $A := E \setminus \bigsqcup_{i=1}^{\infty} B_i$.
We have $\nu(E) = \nu(A) + \sum_{i=1}^{\infty} \nu(B_i) < \infty$, thus $\sum_{i=1}^{\infty} \nu(B_i)$ converges absolutely.
Thus, $\nu(A) = \nu(E) - \sum_{i=1}^{\infty} \nu(B_i) > \nu(E) > 0$, since each $\nu(B_i) < 0$ by construction.
Notice that $\nu(B_n) \to 0^-$ since the sum converges, so if $B \subseteq A \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i$ has $\nu(B) < 0$, we must have $\nu(B) < 2\nu(B_n)$ for some large n. However, by construction,

$$\inf\left\{\nu(B) | B \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i, B \in \mathcal{M}\right\} \geq 2\nu(B_n),$$

which is a contradiction.
Thus A is positive. □

Lemma 7.5. *If $(A_n)_{n=1}^{\infty}$ is a sequence of positive sets, then $A := \bigcup_{n=1}^{\infty} A_n$ is positive.*

Proof. Consider any $B \subseteq A, B \in \mathcal{M}$. Let $B_n := B \cap (A_n \setminus \bigcup_{i=1}^{n-1} A_i)$, then $(B_n)_{n=1}^{\infty} \subseteq \mathcal{M}$ are pairwise disjoint, with $B = \bigsqcup_{n=1}^{\infty} B_n$.
For any n, since A_n is positive, and $B_n \subseteq A_n$, we have $\nu(B_n) > 0$.
Thus $\nu(B) = \sum_{n=1}^{\infty} \nu(B_n) > 0$. □

Theorem 7.6 (Hahn decomposition). *Let ν be a signed measure on $(X, \mathcal{M})$, there are $P, N \in \mathcal{M}$ such that $X = P \sqcup N$, and P is positive, N is negative. Moreover, this is unique in the sense that if $X = P' \sqcup N'$ is another such decomposition, then the symmetric difference $P \Delta P'$ is null.*

Proof. Existence:
By taking $-\nu$ if necessary, we can WLOG assume ν never takes $+\infty$.
Let $m := \sup\{\nu(A) : A \text{ is positive}\} < \infty$.
Choose positive sets A_n such that $\nu(A_n) \to m$, and let $P := \bigcup_{n=1}^{\infty} A_n$.
Thus P is positive by lemma, so $\nu(P) \leq m$.
Notice that $\forall n,\ \nu(P) = \nu(A_n) + \nu(P \setminus A_n) \geq \nu(A_n)$, since $P \setminus A_n \subseteq P$ must have $\nu(P \setminus A_n) \geq 0$.
Taking the supremum over n, we have $\nu(P) \geq m$.
Thus $\nu(P) = m$.
Let $N := X \setminus P$.
Suppose N is not negative, $\exists E \subseteq N, E \in \mathcal{M}$, such that $\nu(E) > 0$. By lemma, there is positive $A \subseteq E$ with $\nu(A) > 0$. Then $P \sqcup A$ is measurable, positive, and $\nu(P \sqcup A) = \nu(P) + \nu(A) > m$, which is a contradiction.
Uniqueness:
Let

$$A := P \setminus P' = (X \setminus N) \setminus P' = X \cap N^c \cap (P')^c = N' \setminus N.$$

It is both positive and negative, thus null. Similarly, $B := P' \setminus P = N \setminus N'$ is null. Thus, $P \Delta P' = A \cup B$ is null. $\square$

Definition 7.3. Let ν be a signed measure on $(X, \mathcal{M})$, and $P, N \in \mathcal{M}$ be as from Hahn decomposition, the **Jordan decomposition** of it is $\nu = \nu^+ - \nu^-$, where $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$ are positive measures.

Proposition 7.7. *Let ν be a signed measure on $(X, \mathcal{M})$, with Jordan decomposition $\nu = \nu^+ - \nu^-$, then for any other positive $\lambda_1, \lambda_2,$ such that $\nu = \lambda_1 - \lambda_2$, we have $\lambda_1 \geq \nu^+, \lambda_2 \geq \nu^-$.*

Proof. Let $X = P \sqcup N$ be the Hahn decomposition.
Consider any $E \in \mathcal{M}$, we have

$$\begin{aligned}
\lambda_1(E) &\geq \lambda_1(E \cap P) \\
&= \nu(E \cap P) + \lambda_2(E \cap P) \\
&\geq \nu(E \cap P) \\
&= \nu^+(E) \\
\lambda_2(E) &\geq \lambda_2(E \cap N) \\
&= -\nu(E \cap N) + \lambda_1(E \cap N) \\
&\geq -\nu(E \cap N) \\
&= \nu^-(E).
\end{aligned}$$

$\square$

Definition 7.4. Let ν be a signed measure on $(X, \mathcal{M})$, with Jordan decomposition $\nu = \nu^+ - \nu^-$, the **total variation** is

$$|\nu| := \nu^+ + \nu^-.$$

Proposition 7.8. *Let $(X, \mathcal{M})$ be a measure space, ν be a signed measure, we have*

$$\forall E \in \mathcal{M}, |\nu|(E) = \sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}\right\}.$$

Proof. For any $E = \bigsqcup_{i=1}^{\infty} E_i \in \mathcal{M}$, we have $|\nu(E_i)| = |\nu^+(E_i) - \nu^-(E_i)| \leq \nu^+(E_i) + \nu^-(E_i)$, since both $\nu^+(E_i), \nu^-(E_i) \geq 0$ are positive measures. Thus,

$$\begin{aligned}\sum_{i=1}^{\infty} |\nu(E_i)| &\leq \sum_{i=1}^{\infty} \nu^+(E_i) + \nu^-(E_i) \\ &= \nu^+(E) + \nu^-(E) \\ &|\nu|(E).\end{aligned}$$

This shows

$$\sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}\right\} \leq |\nu|(E).$$

Now for $E = (E \cap P) \sqcup (E \cap N)$, we have

$$\begin{aligned}|\nu(E \cap P)| + |\nu(E \cap N)| &= \nu(E \cap P) + \nu(E \cap N) \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E).\end{aligned}$$

This shows

$$\sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}\right\} \geq |\nu|(E).$$

□

Definition 7.5. Let ν be a signed measure with its Jordan decomposition $\nu = \nu^+ - \nu^-$, we define

$$\int_X f d\nu := \int_X f d\nu^+ - \int_X f d\nu^-$$

for any measurable function $f : X \to \mathbb{C}$ which is integrable with respect to each $\nu^\pm$ and the subtraction makes sense.

7.2 Complex measures

Definition 7.6. Let $(X, \mathcal{M})$ be a measurable space, a **complex measure** is a function $\nu : \mathcal{M} \to \mathbb{C}$, such that it satisfies countable additivity. Namely, for any pairwise disjoint sets $E_1, E_2, \ldots$ in $\mathcal{M}$, we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Proposition 7.9. *Suppose $|\nu(\bigsqcup_{i=1}^{\infty} E_i)| < \infty$, then $\sum_{i=1}^{\infty} \nu(E_i)$ must converge absolutely, since we want $\nu(\bigsqcup_{i=1}^{\infty} E_i)$ to be invariant of the order of union.*

Proposition 7.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f \in \mathcal{L}^\infty(\mu)$ with $\|f\|_{L^\infty()} = 1$, then with $\nu(E) := \int_E f d\mu$, we have ν is a complex measure, and $\forall E \in \mathcal{M}$,*

$$\begin{aligned}\mu(E) &= \int_E d\mu \\ &\geq \int_E |f| d\mu \\ &\geq \left|\int_E h d\mu\right| \\ &\geq \left|\int_E d\nu\right| \\ &= |\nu(E)|.\end{aligned}$$

Definition 7.7. Let $(X, \mathcal{M})$ be a measurable space, the **total variation** of a complex measure μ is $|\mu| : \mathcal{M} \to [0, \infty]$, defined by

$$\forall E \in \mathcal{M}, |\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M} \right\}.$$

Proposition 7.11. *Let $(X, \mathcal{M})$ be a measurable space, μ be a complex measure, then $|\mu|$ is a positive measure.*

Proof.

1. $|\mu|(\emptyset) = 0$.
2. $|\mu|(E) \geq 0, \forall E \in \mathcal{M}$.
3. Fix $E = \bigsqcup_{i=1}^{\infty} E_i \in \mathcal{M},\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}$.
   Consider any $(A_j)_{j=1}^{\infty} \subset \mathcal{M}$ such that $E = \bigsqcup_{j=1}^{\infty} A_j$, then $A_j = A_j \cap E = \bigsqcup_{i=1}^{\infty} A_j \cap E_i$.

$$\begin{aligned} \sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)| \\ &\leq \sum_{i=1}^{\infty} |\mu| \left(\bigsqcup_{j=1}^{\infty} (A_j \cap E_i) \right) \\ &= \sum_{i=1}^{\infty} |\mu|(E_i). \end{aligned}$$

   We have that

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(A_j)| : E = \bigsqcup_{j=1}^{\infty} A_j,\ (A_j)_{j=1}^{\infty} \subset \mathcal{M} \right\} \leq \sum_{i=1}^{\infty} |\mu|(E_i).$$

   Now given any $\epsilon > 0$.
   $\forall i$, pick $t_i := |\mu|(E_i) - \frac{\epsilon}{2^i}$ and we can find $E_{ij} \in \mathcal{M}$, such that

$$E_i = \bigsqcup_{j=1}^{\infty} E_{ij},\ \sum_{j=1}^{\infty} |\mu(E_{ij})| > t_i.$$

   We have

$$\begin{aligned} |\mu|(E) &\geq \sum_{i,j=1}^{\infty} |\mu(E_{ij})| \\ &\geq \sum_{i=1}^{\infty} t_i \\ &= \sum_{i=1}^{\infty} |\mu|(E_i) - \epsilon. \end{aligned}$$

   Taking $\epsilon \to 0$, we have $|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(E_i)$.
   Thus $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(E_i)$.

$\square$

Lemma 7.12. *Let $\{z_1, \ldots, z_N\} \subset \mathbb{C}$, then $\exists S \subseteq [N]$, such that*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

Theorem 7.13. *Let $(X, \mathcal{M})$ be a measurable space, μ be a complex measure, then $|\mu|$ is a finite measure. Namely, $\forall E \in \mathcal{M},$ such that $|\mu|(E) < \infty$.*

Proof. Suppose $\exists E \in \mathcal{M},$ such that $|\mu|(E) = \infty$.
Let $B_0 = E$.
Let $t := \pi(1 + |\mu(E)|) \geq \pi$.
Then we can find a partition $E_i \in \mathcal{M}$, such that

$$E = \bigsqcup_{i=1}^{\infty} E_i, \quad \sum_{i=1}^{\infty} |\mu(E_{ij})| > t.$$

Thus there is some $N \in \mathbb{N},$ such that $\sum_{i=1}^{N} |\mu(E_{ij})| > t$.
By lemma, $\exists S \subseteq [N]$, such that

$$\begin{aligned} \left|\mu\left(\bigsqcup_{i \in S} E_i\right)\right| &= \left|\sum_{i \in S} \mu(E_i)\right| \\ &\geq \frac{1}{\pi} \sum_{i=1}^{N} |\mu(E_i)| \\ &> \frac{t}{\pi} \\ &\geq 1. \end{aligned}$$

Now let $A := \bigsqcup_{i \in S} E_i, B = E \setminus A$.
We have

$$\begin{aligned} |\mu(B)| &= |\mu(E) - \mu(A)| \\ &\geq |\mu(A)| - |\mu(E)| \\ &> \frac{t}{\pi} - |\mu(E)| \\ &= 1. \end{aligned}$$

Thus $E = A \sqcup B$, where $|\mu(A)| > 1, |\mu(B)| > 1$.
Since $|\mu|(A \sqcup B) = |\mu|(A) + |\mu|(B) = \infty$, at least one of $|\mu|(A), |\mu|(B)$ is ∞.
WLOG, say $|\mu|(B) = \infty$. We let $A_1 = A, B_1 = B$.
Now apply the above argument on $B_1 = A_2 \sqcup B_2$, where $|\mu(A_2)| > 1, |\mu(B_2)| > 1, |\mu|(B_2) = \infty$.
Repetitively, we construct disjoint $(A_k)_{k=1}^{\infty}$, such that $\forall i \geq 1, |\mu(A_i)| > 1$.
Notice that $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{M}$, and we have $\mu(\bigsqcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$ absolutely.
However, $\sum_{k=1}^{\infty} |\mu(A_k)| \geq \sum_{k=1}^{\infty} 1 = \infty$ diverges. $\square$

Definition 7.8. Let ν be a complex measure on $(X, \mathcal{M})$, with its real-imaginary decomposition $\nu = \nu_{\Re} + i\nu_{\Im}$, where each $\nu_{\Re} := \Re(\nu), \nu_{\Im} := \Im(\nu)$ is a finite signed measure. The **real-imaginary Jordan decomposition** of ν is $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, where each of them is a finite positive measure from the Jordan decompositions of $\nu_{\Re}, \nu_{\Im}$.

Definition 7.9. Let ν be a complex measure with its real-imaginary Jordan decomposition $\nu = \nu_{\Re} + i\nu_{\Im} = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, we define

$$\int_X f d\nu := \int_X f d\nu_{\Re} + i \int_X f d\nu_{\Im} = \int_X f d\nu_1 - \int_X f d\nu_2 + i \int_X f d\nu_3 - i \int_X f d\nu_4$$

for any measurable function f which is integrable with respect to each composition.

8 Radon–Nikodym-Lebesgue Decomposition

Now we want to consider the converse: Given two measures ν, μ, when can we find a f, such that $\nu(E) = \int_E f d\mu$?

8.1 Absolutely Continuous Measures

Definition 8.1. Let μ, ν be two (complex or signed or positive) measures on $(X, \mathcal{M})$, we say ν is **absolutely continuous** with respect to μ if $\mu(A) = 0 \implies \nu(A) = 0$, and is written as $\nu \ll \mu$.

Example 8.1.1. Consider the counting measure μ, then for any other (non trivially infinite) measure ν, we always have $\nu \ll \mu$, since $\mu(E) = 0 \implies E = \emptyset$.

Proposition 8.1. *Let ν be a signed measure with its Jordan decomposition $\nu = \nu^+ + \nu^-$, we have that $\nu^+ \ll |\nu|,\ \nu^- \ll |\nu|$.*

Proof. Since both ν^+, ν^- are positive measures, $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0 \implies \nu^+(E) = \nu^-(E) = 0$. $\square$

Proposition 8.2. *Let ν be a complex measure with its real-imaginary Jordan decomposition, we have that $|\nu_{\Re}|, |\nu_{\Im}| \ll |\nu|$, so each $\nu_i \ll |\nu|$.*

Proof. For any $E = \bigsqcup_{i=1}^{\infty} E_i \in \mathcal{M}$, we have that $|\nu_{\Re}(E_i)|, |\nu_{\Im}(E_i)| \leq |\nu(E_i)|$, so

$$\begin{aligned}
|\nu_{\Re}|(E) &= \sup\left\{\sum_{i=1}^{\infty} |\nu_{\Re}(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}\right\} \\
&\leq \sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}\right\} \\
&= |\nu|(E).
\end{aligned}$$

Thus, $|\nu|(E) = 0 \implies abs\nu_{\Re}(E) = 0$, so $|\nu_{\Re}| \ll |\nu|$. Similarly $|\nu_{\Im}| \ll |\nu|$. We can apply the previous result to get $\nu_1, \nu_2 \ll |\nu_{\Re}| \ll |\nu|, \nu_3, \nu_4 \ll |\nu_{\Im}| \ll |\nu|$. $\square$

8.2 Radon–Nikodym Derivatives

Definition 8.2. Let μ, ν be two (complex or signed or positive) measures on $(X, \mathcal{M})$. A function f is the **Radon–Nikodym derivative**, written as $\frac{d\nu}{d\mu}$, if

$$\forall E \in \mathcal{M},\ \nu(E) = \int_E f d\mu.$$

Proposition 8.3. *If the Radon–Nikodym derivative exists, it is unique μ-a.e..*

Proof. Suppose there are two f, h that satisfies $\forall E \in \mathcal{M},\ \nu(E) = \int_E f d\mu = \int_E h d\mu$, then $\int_E (f - h) d\mu = 0$. Thus $f = h$ μ-a.e.. $\square$

Proposition 8.4. *Suppose μ is a positive measure on a measurable space $(X, \mathcal{M})$, and ν is a (complex or signed or positive) measure such that $\frac{d\nu}{d\mu}$ exists, then $|\nu| \ll \mu$.*

Proof. Consider any $E \in \mathcal{M}$, suppose $\mu(E) = 0$, then $c = \int_E \frac{d\nu}{d\mu} d\mu = 0$. Thus, $\nu \ll \mu$.

Now consider any $E = \bigsqcup_{i=1}^{\infty} E_i$, since each $E_i \subseteq E$, we have $\mu(E) = 0 \implies \mu(E_i) = 0 \implies \nu(E_i) = 0$. Thus, $\sum_{i=1}^{\infty} |\nu(E_i)| = 0$. Since this holds for all decompositions,

$$|\nu|(E) = \sup\left\{\sum_{i=1}^{\infty} |\nu(E_i)| :,\ (E_i)_{i=1}^{\infty} \subset \mathcal{M}\right\} = 0.$$

$\square$

Proposition 8.5. *Suppose μ is a positive measure on a measurable space $(X, \mathcal{M})$, and ν is a (complex or signed or positive) measure such that $\frac{d\nu}{d\mu}$ exists, then for any g integrable with respect to ν, we have*

$$\int_X g d\nu = \int_X g \frac{d\nu}{d\mu} d\mu.$$

Thus, we can also abuse the notation and write $d\nu = \frac{d\nu}{d\mu} d\mu$ when μ is a positive measure.

Proof. 1. Assume μ, ν are both positive measure. First suppose $g = \chi_E$ for some $E \in \mathcal{M}$. Clearly,

$$\int_X g d\nu = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_X \chi_E \frac{d\nu}{d\mu} d\mu = \int_X g \frac{d\nu}{d\mu} d\mu.$$

By linearity, it holds for all simple functions.

By Lebesgue's Monotone Convergence Theorem, this holds for all measurable $g : X \to [0, \infty]$.

For $g : X \to [-\infty, \infty]$, write $g = g^+ - g^-$, where each function is $X \to [0, \infty]$. The result holds by linearity.

For $g \in \mathcal{L}^1(X, \mu)$, write $g = \Re(g) + i\Im(g)$, where each function is $X \to \mathbb{R}$. The result holds by linearity.

2. Now suppose ν is a signed measure. Consider its Jordan decomposition $\nu = \nu^+ - \nu^-$ as in 7.3. We have that for any $E \in \mathcal{M}$,

$$\begin{aligned}\nu^+(E) &= \nu(E \cap P) \\ &= \int_{E\cap P} \frac{d\nu}{d\mu} d\mu \\ &= \int_E \chi_P \frac{d\nu}{d\mu} d\mu.\end{aligned}$$

Thus, $\int_X g d\nu^+ = \int_X g\chi_P \frac{d\nu}{d\mu} d\mu$. Similarly, for any $E \in \mathcal{M}$,

$$\begin{aligned}\nu^-(E) &= -\nu(E \cap N) \\ &= -\int_{E\cap N} \frac{d\nu}{d\mu} d\mu \\ &= \int_E -\chi_N \frac{d\nu}{d\mu} d\mu.\end{aligned}$$

Thus, $\int_X g d\nu^- = \int_X -g\chi_N \frac{d\nu}{d\mu} d\mu$. Now,

$$\begin{aligned}\int_X g d\nu &= \int_X g d\nu^+ - \int_X g d\nu^- \\ &= \int_X g\chi_P \frac{d\nu}{d\mu} d\mu - \int_X -g\chi_N \frac{d\nu}{d\mu} d\mu \\ &= \int_X g(\chi_P + \chi_N) \frac{d\nu}{d\mu} d\mu \\ &= \int_X g \frac{d\nu}{d\mu} d\mu.\end{aligned}$$

3. Now suppose ν is a complex measure. Consider its real-imaginary decompositions $\nu_\Re, \nu_\Im$, both signed measures. We have that for any $E \in \mathcal{M}$,

$$\begin{aligned}\nu_\Re(E) &= \Re(\nu(E)) \\ &= \Re\left(\int_E \frac{d\nu}{d\mu} d\mu\right) \\ &= \int_E \Re\left(\frac{d\nu}{d\mu}\right) d\mu.\end{aligned}$$

Thus, $\int_X g d\nu_{\Re} = \int_X g\Re\left(\frac{d\nu}{d\mu}\right)d\mu$. Similarly, $\int_X g d\nu_{\Im} = \int_X g\Im\left(\frac{d\nu}{d\mu}\right)d\mu$. We have

$$\begin{aligned}
\int_X g d\nu &= \int_X g d\nu_{\Re} + i\int_X g d\nu_{\Im} \\
&= \int_X g\Re\left(\frac{d\nu}{d\mu}\right)d\mu + i\int_X g\Im\left(\frac{d\nu}{d\mu}\right)d\mu \\
&= \int_X g\left(\Re\left(\frac{d\nu}{d\mu}\right) + i\Im\left(\frac{d\nu}{d\mu}\right)\right)d\mu \\
&= \int_X g\frac{d\nu}{d\mu}d\mu.
\end{aligned}$$

□

Proposition 8.6 (Chain Rule)**.** *Suppose μ is a positive measure on a measurable space $(X, \mathcal{M})$, and ν, λ are (complex or signed or positive) measures such that $\frac{d\nu}{d\mu}, \frac{d\lambda}{d\nu}$ exists, then $\frac{d\lambda}{d\mu}$ exists, and*

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}.$$

Proof. For any $E \in \mathcal{M}$, we have

$$\begin{aligned}
\lambda(E) &= \int_E \frac{d\lambda}{d\nu}d\nu \\
&= \int_X \chi_E\frac{d\lambda}{d\nu}d\nu \\
&= \int_X \chi_E\frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}d\mu \\
&= \int_E \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu}d\mu.
\end{aligned}$$

□

8.3 Radon–Nikodym Theorem for Positive Measure

Lemma 8.7. *Let μ be a σ-finite measure on a measurable space $(X, \mathcal{M})$, then there is some $w \in L^1(\mu)$, such that $\forall x \in X, 0 < w(x) < 1$.*

Proof. Write $X = \bigcup_{n=1}^\infty E_n$, where $\forall n \geq 1, \mu(E_n) < \infty$.
Let $w_n := \frac{2^{-n}\chi_{E_n}}{1+\mu(E_n)}, w = \sum_{n=1}^\infty w_n$.
Notice that $0 < w_n(x) < 1$, and

$$\begin{aligned}
\int_X w d\mu &= \sum_{n=1}^\infty \int_X w_n d\mu \\
&= \sum_{n=1}^\infty \int_X \frac{2^{-n}\chi_{E_n}}{1+\mu(E_n)}d\mu \\
&\leq \sum_{n=1}^\infty \frac{2^{-n}\mu(E_n)}{1+\mu(E_n)} \\
&< \sum_{n=1}^\infty 2^{-n} \\
&< \infty.
\end{aligned}$$

□

Lemma 8.8. *Let μ be a σ-finite measure on a measurable space $(X, \mathcal{M})$, and $g \in L^1(\mu)$. Suppose $\forall E \in \mathcal{M}$ such that $\mu(E) > 0$, we have that*

$$\frac{1}{\nu(E)} \int_E g d\mu \in S$$

for some closed $S \subseteq \mathbb{C}$, then

$$g(x) \in S, a.e.\ x \in X.$$

Proof. Assume for contradiction that there is $E =: g^{-1}\big(\bar{B}(t, r)\big)$ such that $\mu(E) > 0, \bar{B}(t, r) \subseteq S^c$.
Then $A_E(g) := \frac{1}{\mu(E)} \int_E g d\mu \in S$, while

$$\begin{aligned}
|A_E(g) - t| &= \left| \frac{1}{\mu(E)} \int_E (g - t) d\mu \right| \\
&\leq \frac{1}{\mu(E)} \int_E |g - t| d\mu \\
&\leq \frac{1}{\mu(E)} \int_E r d\mu \\
&= r.
\end{aligned}$$

Thus $A_E(g) \in \bar{B}(t, r) \subseteq S^c$, which is a contradiction. $\square$

Theorem 8.9 (Radon–Nikodym for finite measures)**.** *Let μ be a σ-finite measure, and ν is a positive finite measure on a measurable space $(X, \mathcal{M})$. Suppose $\nu \ll \mu$, then $\exists h \in L^1(\mu) \cap \mathcal{L}^+$, such that $\forall E \in \mathcal{M}, \nu(E) = \int_E h d\mu$. Namely, $\frac{d\nu}{d\mu}$ exists in $L^1(\mu) \cap \mathcal{L}^+$. Moreover, $\frac{d\nu}{d\mu}$ is unique μ-a.e.*

Proof. (Von Neumann's proof).
Since μ is σ-finite, there is some $w \in L^1(\mu)$, such that $\forall x \in X, 0 < w(x) < 1$.
Define a new measure $d\lambda := d\nu + w d\mu$, namely, $\forall E \in \mathcal{M}, \lambda(E) := \nu(E) + \int_E w d\mu$.

Claim 8.9.1. *There is some measurable g such that $\forall x \in X, g(x) \in [0, 1]$, and for any measurable $f \in L^2(\lambda)$, we have $\int_E f(1 - g) d\nu = \int_E f g w d\mu$.*

Proof. Notice that $\int_X f d\lambda = \int_X f d\lambda + \int_X f w d\lambda$ for any measurable f. Consider any $f \in L^2(\lambda)$,

$$\begin{aligned}
\left| \int_X f d\nu \right| &\leq \int_X |f| d\nu \\
&= \int_X |f| d\lambda - \int_X |f| w d\mu \\
&\leq \int_X |f| d\lambda \\
&\leq \int_X |f| \cdot 1 d\lambda \\
&\leq \|f\|_{L^2(\lambda)} \lambda(X).
\end{aligned}$$

Notice that $\lambda(X) = \nu(X) + \int_X w d\mu < \infty$, so $\Lambda : f \mapsto \int_X f d\nu \in L^2(\lambda)^*$. Since $L^2(\lambda)$ is a Hilbert space, there is a unique $g \in L^2(\lambda)$, such that $\int_X f g d\lambda = \Lambda(f) = \int_X f d\nu,\ \forall f \in L^2(\lambda)$.
Now we know $\int_X f d\nu = \int_X f g d\lambda = \int_X f g d\nu + \int_X f g w d\mu$.

For any $E \in \mathcal{M}, f \in L^2(\lambda)$, we can take $\tilde{f} := f\chi_E \in L^2(\lambda)$, and we get

$$\begin{aligned}
\int_E f(1-g)d\nu &= \int_X \tilde{f}(1-g)d\nu \\
&= \int_X \tilde{f}d\nu - \int_X \tilde{f}gd\nu \\
&= \int_X \tilde{f}gwd\mu + \int_X \tilde{f}gd\nu - \int_X \tilde{f}gd\nu \\
&= \int_X \tilde{f}gwd\mu \\
&= \int_E fgwd\mu.
\end{aligned}$$

In addition, for any $E \in \mathcal{M}$, taking $f = 1$, we have that

$$\nu(E) = \int_E d\nu = \int_E gd\lambda.$$

Thus $0 \leq \int_E gd\lambda \leq \lambda(E)$. Thus $\forall E \in \mathcal{M},$ such that $\lambda(E) > 0$, we have

$$\frac{\int_E gd\lambda}{\lambda(E)} \in [0,1].$$

By the above lemma, we have that $g(x) \in [0,1], \lambda$-a.e. $x \in X$.
WLOG, we can redefine $g(x) = 0$ for any $g(x) \notin [0,1]$. $\square$

Let $A := g^{-1}([0,1)), B := g^{-1}(\{1\})$. Let $f = \chi_B$, we have that

$$\begin{aligned}
\int_X \chi_B(1-g)d\nu &= \int_X \chi_B gwd\mu \\
\int_B (1-g)d\nu &= \int_B wd\mu \\
0 &= \int_B wd\mu.
\end{aligned}$$

Since $w > 0$, we must have $\mu(B) = 0$. Since $\nu \ll \mu, \nu(B) = 0$. Thus $\forall E \in \mathcal{M}, \nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A)$.

Now, let $f_n := \sum_{k=0}^n g^k$, we have that $f_n(1-g) = 1 - g^{n+1}$, so

$$\int_E (1-g^{n+1})d\nu = \int_E f_n(1-g)d\nu = \int_E f_n gwd\mu.$$

Notice that $1 - g^{n+1}(x) \to \begin{cases} 1, & x \in A, \\ 0, & x \in B. \end{cases}$ monotonically. In addition, $(f_n gw)(x)$ is increasing and bounded, so there is some $h(x) := \lim_{n\to\infty}(f_n gw)(x)$. Thus, by LMCT, we have

$$\begin{aligned}
\nu(E) &= \nu(A \cap E) \\
&= \int_{E\cap A} d\nu \\
&= \int_{E\cap A} \lim_{n\to\infty}(1-g^{n+1})d\nu \\
&= \lim_{n\to\infty}\int_{E\cap A}(1-g^{n+1})d\nu \\
&= \lim_{n\to\infty}\int_E f_n gwd\mu \\
&= \int_E hd\mu.
\end{aligned}$$

Since ν is finite, we have $h \in L^1(\mu)$. $\square$

Theorem 8.10. *[Radon–Nikodym] Let μ, ν be two σ-finite measures on a measurable space $(X, \mathcal{M})$. Suppose $\nu \ll \mu$, then $\exists h \in \mathcal{L}^+$, such that $\forall E \in \mathcal{M}, \nu(E) = \int_E h d\mu$. Namely, $\frac{d\nu}{d\mu}$ exists in $\mathcal{L}^+$. Moreover, $\frac{d\nu}{d\mu}$ is unique μ-a.e.*

Proof. Since ν is σ-finite, we have $X = \bigsqcup_{n=1}^{\infty} X_n$, where each $\nu(X_n)$ is finite.

We can apply the above theorem on $\nu_n(E) := \nu(E \cap X_n)$, which are finite measures, and let $h = \sum_{n=1}^{\infty} h_n$. h will be positive and measurable, but not in $L^1(\mu)$. Yet it is in $L^1(\mu|_{X_n})$ for all n. $\square$

Remark. The σ-finiteness is essential. Indeed, consider the following counterexample.

Example 8.3.1. Consider λ to be the Lebesgue measure on $(0, 1)$, and μ to be the counting measure, which is not σ-finite.
Although $\lambda \ll \mu$, it is impossible to find such an $h = \frac{d\lambda}{d\mu}$, because for any $E \in \mathcal{M}$, we will have

$$\begin{aligned} \lambda_a(E) &= \int_E h d\mu \\ &= \sum_{x \in E} h(x), \end{aligned}$$

which is not possible.

8.4 Signed and Complex Measures

Proposition 8.11 (Polar decomposition of signed measure)**.** *Let $\nu = \nu^+ - \nu^-$ be a signed measure on a measurable space $(X, \mathcal{M})$. There exists a measurable function $f : X \to \mathbb{R}$ with $|f| = 1$, and*

$$\forall E \in \mathcal{M},\ \nu(E) = \int_E f d|\nu|.$$

Namely, $f = \frac{d\nu}{d|\nu|}$.

Proof. Let $X = P \sqcup N$ be the Hahn decomposition theorem 7.6. Define $f := \chi_P - \chi_N$, then clearly $|f| = 1$. In addition,

$$\begin{aligned} \int_E f d|\nu| &= \int_E \chi_P - \chi_N d|\nu| \\ &= \int_{E \cap P} d|\nu| - \int_{E \cap N} d|\nu| \\ &= |\nu|(E \cap P) - |\nu|(E \cap N) \\ &= \nu^+(E \cap P) - \nu^-(E \cap N) \\ &= \nu^+(E) - \nu^-(E) \\ &= \nu(E). \end{aligned}$$

$\square$

Corollary 8.12 (Radon–Nikodym for Signed Measure)**.** *Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. Suppose $|\nu|, \mu$ are σ-finite with $|\nu| \ll \mu$, then there is a measurable function $g : X \to [-\infty, \infty]$, with at most one of g^+, g^- takes ∞, such that $\forall E \in \mathcal{M},\ \nu(E) = \int_E g d\mu$.*

Proof. By the Radon–Nikodym Theorem 8.10, there is a unique $h \in \mathcal{L}^+$, such that $h = \frac{d|\nu|}{d\mu}$. Now consider $g^+ := h\chi_P, g^- := h\chi_N$, we have that $g^+, g^- \geq 0$. Also, $\forall E \in \mathcal{M}$,

$$\begin{aligned}
\nu^+(E) &= |\nu|(E \cap P) \\
&= \int_{E\cap P} hd\mu \\
&= \int_E h\chi_P d\mu \\
&= \int_E g^+ d\mu.
\end{aligned}$$

Similarly, $\nu^-(E) = \int_E g^- d\mu$. Since only one of ν^+, ν^- takes infinity, g^+, g^- cannot be both infinity on a non-μ-null space.

Let $g := g^+ - g^-$, we have

$$\begin{aligned}
\nu(E) &= \nu^+(E) - \nu^-(E) \\
&= \int_E g^+ d\mu - \int_E g^- d\mu \\
&= \int_E g d\mu.
\end{aligned}$$

□

Corollary 8.13 (Polar decomposition of complex measures)**.** *Let ν be a complex measure on $(X, \mathcal{M})$. There is a unique measurable function h, such that $|h| = 1$ $|\nu|$-a.e., and $d\nu = hd|\nu|$. Also, for any any integrable function f, we have*

$$\int_X f d\nu = \int_X fhd|\nu|.$$

Proof. Write $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ with Jordan decomposition. Notice that $\nu_i \ll |\nu|$. Applying Randon-Nikodym, we have some $h_i \in \mathcal{L}^+$, where $\forall E \in \mathcal{M}, \nu_i(E) = \int_E h_i d|\nu|$.

Define $h := (h_1 - h_2 + ih_3 - ih_4)$. Since $\nu_i, |\nu|$ are all positive measures, for any measurable f, we have

$$\begin{aligned}
\int_X f d\nu &= \int_X f d\nu_1 - \int_X f d\nu_2 + i\int_X f d\nu_3 - i\int_X f d\nu_4 \\
&= \int_X fh_1 d|\nu| - \int_X fh_2 d|\nu| + i\int_X fh_3 d|\nu| - i\int_X fh_4 d|\nu| \\
&= \int_X f(h_1 - h_2 + ih_3 - ih_4) d|\nu| \\
&= \int_X fhd|\nu|.
\end{aligned}$$

For any $E \in \mathcal{M}$, taking $f = \chi_E$, we have

$$\nu(E) = \int_X \chi_E d\nu = \int_X \chi_E hd|\nu| = \int_E hd|\nu|.$$

For any $E \in \mathcal{M}$, we have

$$1 \geq \frac{|\nu(E)|}{|\nu|(E)} = \left|\frac{1}{|\nu|(E)} \int_E hd|\nu|\right|.$$

Thus $|h(x)| \leq 1$ a.e..

For any $0 < r < 1$, consider $A_r := \{x \in X : |h(x)| < r\} = \bigsqcup_{i=1}^\infty E_i$.

We have

$$\begin{aligned}\sum_{i=1}^{\infty}|\nu(E_i)| &= \sum_{i=1}^{\infty}\left|\int_{E_i} hd|\nu|\right| \\ &\leq \sum_{i=1}^{\infty}\left|\int_{E_i} rd|\nu|\right| \\ &= r\sum_{i=1}^{\infty}|\nu|(E_i) \\ &= r|\nu|(A_r).\end{aligned}$$

Taking sup over all E_i, we have that $|\nu|(A_r) \leq r|\nu|(E_i)$. Since $r < 1$, we have $|\nu|(E_i) = 0$.
Thus $|h(x)| > 1$ a.e.$x \in X$ for all $0 < r < 1$.
Thus $|h| = 1$ a.e.. □

Corollary 8.14. *Suppose μ, ν are (positive or signed or complex) measures on a measurable space $(X, \mathcal{M})$, where $|\mu|$ is σ-finite, and $f = \frac{d\nu}{d\mu}$ exists, then we have $\frac{d|\nu|}{d|\mu|}$ exists and*

$$\frac{d|\nu|}{d|\mu|} = |f|.$$

Proof. Let h_1, h_2 be measurable functions such that $d\mu = h_1 d|\mu|, d\nu = h_2 d|\nu|$, with $|h_1| = |h_2| = 1$ from polar decomposition. Now, $\frac{d\nu}{d|\mu|} = \frac{d\nu}{d\mu}\frac{d\mu}{d|\mu|} = h_1 f$ exists by proposition 8.6 since $|\mu|$ is a positive measure. Thus, $|\nu| \ll |\mu|$ by proposition 8.4.

By Radon–Nikodym theorem 8.10, there is $h \in \mathcal{L}^+$, with $d|\nu| = hd|\mu|$.

$$\begin{aligned}\int_E h_1 f d|\mu| &= \nu(E) \\ &= \int_E h_2 d|\nu| \\ &= \int_E hh_2 d|\mu|.\end{aligned}$$

Thus $hh_2 = h_1 f$ $|\mu|$-a.e.. Since $h \in \mathcal{L}^+$, and $|h_1| = |h_2| = 1$, we must have $h = |f|$ $|\mu|$-a.e.. □

8.5 Lebesgue Decompositions

Definition 8.3. Two positive measures μ, ν on a measurable space $(X, \mathcal{M})$ are said to be **mutually singular**, written as $\mu \perp \nu$, if $X = A \sqcup B$, where A is μ-null and B is ν-null. Namely, for all measurable $E \subseteq A$, $\mu(E) = 0$, and for all measurable $E \subseteq B$, $\nu(E) = 0$.

Proposition 8.15. *Let ν be a signed measure on $(X, \mathcal{M})$, with Jordan decomposition $\nu = \nu^+ - \nu^-$, then $\nu^+ \perp \nu^-$.*

Theorem 8.16 (Lebesgue decomposition)**.** *Let μ, ν be two σ-finite measures on a measurable space $(X, \mathcal{M})$. There is a unique decomposition $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu$ be the absolutely continuous part, and $\nu_s \perp \mu$ be the singular part, both positive measures.*

Proof. Take $\lambda = \mu + \nu$, which is σ-finite, and $\mu, \nu \ll \lambda$. By the Radon-Nikodym theorem 8.10, $\exists f, g \in \mathcal{L}^+$, such that

$$\mu(E) = \int_E f d\lambda,\ \nu(E) = \int_E g d\lambda.$$

Let $A = f^{-1}((0, \infty]), B = f^{-1}(\{0\}), \nu_a(E) = \nu(E \cap A), \nu_s(E) = \nu(E \cap B)$. Since $X = A \sqcup B$, clearly $\nu = \nu_a + \nu_s$. We can see that for any measurable $E \subseteq A$,

$$\nu_s(E) = \nu(E \cap B) = \nu(\emptyset) = 0.$$

On the other hand, for any measurable $E \subseteq B$, $f = 0$ on E, so

$$\mu(E) = \int_E f d\lambda = 0.$$

This shows $\nu_s \perp \mu$.

In addition, suppose $\mu(E) = \int_E f d\lambda = \int_{E\cap A} f d\lambda = 0$, then we must have $\lambda(E \cap A) = 0$, since $f > 0$ on $E \cap A$. Thus,

$$\nu_a(E) = \nu(E \cap A) = \int_{E\cap A} g d\lambda = 0.$$

This shows $\nu_a \ll \mu$. $\square$

Theorem 8.17 (Lebegue-Radon-Nikodym)**.** *Let ν be a (complex or signed or positive) measure on a measurable space $(X, \mathcal{M})$ such that $|\nu|$ is σ-finite. If μ is a σ-finite measure on $(X, \mathcal{M})$, then ν decomposes uniquely as $\nu = \nu_a + \nu_s$, such that $\nu_a \ll \mu$ is the absolutely continuous part and $\nu_s \perp \mu$ is the singular part. Also, $d\nu_a = f d\mu$ for some $f \in \mathcal{L}^1(\mu)$.*

Proof. From the polar decomposition, we have $d\nu = hd|\nu|$ for some $|h| = 1$, and we can use Lebesgue decomposition to write $|\nu| = |\nu|_a + |\nu|_s$, where $|\nu|_a \ll \mu$, and $|\nu|_s \perp \mu$.

By the Randon-Nikodym theorem 8.10, $g = \frac{d|\nu|_a}{d\mu}$ exists, so we have $d\nu = hgd\mu + hd|\nu|_s$. Let $f := hg$, and $d\nu_a := f d\mu, d\nu_s := hd|\nu|_s$, and we have the result.

Now we want to uniqueness: Suppose $\nu = \nu_a + \nu_s = \nu'_a + \nu'_s$ are two decompositions as in the theorem, then $(\nu_a - \nu'_a) + (\nu_s - \nu'_s) = 0$ is the zero measure. Thus, $\mu' := \nu_a - \nu'_a = \nu'_s - \nu_s$, but $\nu_a - \nu'_a \ll \mu$, and $\nu'_s - \nu_s \perp \mu$. Thus, $\mu' = 0$, and $\nu_a = \nu'_a, \nu'_s = \nu_s$. $\square$

9 Dual of Function Spaces

9.1 Dual of L^p Spaces

Theorem 9.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\frac{1}{p} + \frac{1}{q} = 1$ for $p \in (1, \infty)$, we have*

$$L^q(\mu) \cong L^p(\mu)^*,$$

where the isometric isomorphism $L^q(\mu) \xrightarrow{\sim} L^p(\mu)^*$; $g \mapsto \Lambda_g$ is defined to be

$$\forall f \in L^p(\mu), \Lambda_g(f) := \int_X f g dx.$$

In addition, the same is true for $p = 1$ if μ is σ-finite.

Proof. Let $1 \le p < \infty$.

1. By 5.19, we only need to show the subjectivity: $\forall \Lambda \in L^p(\mu)^*, \exists g \in L^q(\mu)$, such that $\Lambda = \Lambda_g$.

2. First we assume $\mu(X) < \infty$ is finite.
   Given any $\Lambda \in L^p(\mu)^*$.
   Consider the mapping $\nu : E \mapsto \Lambda(\chi_E)$ for any measurable $E \in \mathcal{M}$.
   This is well-defined since $\chi_E \in L^\infty(X) \subseteq L^p(X)$.
   Notice that $|\nu(E)| = |\Lambda(\chi_E)| \le \|\Lambda\|_{L^p(\mu)^*} \|\chi_E\|_{L^p(\mu)} < \infty$, thus ν is finite.
   We have $\nu(\emptyset) = \Lambda(0) = 0$ since Λ is linear.
   For any $B = \bigsqcup_{i=0}^\infty A_i$, with $A_i \subseteq U$ be measurable, we have $\chi_B = \sum_{i=0}^\infty \chi_{A_i}$ in $L^p(\mu)$.

Indeed,

$$\begin{aligned}
\left\| \chi_B - \sum_{i=0}^{N} \chi_{A_i} \right\|_{L^p(\mu)}^p &= \left\| \sum_{i=N+1}^{\infty} \chi_{A_i} \right\|_{L^p(\mu)}^p \\
&= \left\| \chi_{\bigsqcup_{i=N+1}^{\infty} A_i} \right\|_{L^p(\mu)}^p \\
&= \mu\left(\bigsqcup_{i=N+1}^{\infty} A_i \right)^p \\
&\to 0.
\end{aligned}$$

Notice that this fails when $p = \infty$!
Thus,

$$\begin{aligned}
\nu(B) &= \Lambda(\chi_B) \\
&= \Lambda\left(\sum_{i=0}^{\infty} \chi_{A_i} \right) \\
&= \Lambda\left(\lim_{n\to\infty} \sum_{i=0}^{n} \chi_{A_i} \right) \\
&= \lim_{n\to\infty} \Lambda\left(\sum_{i=0}^{n} \chi_{A_i} \right) && \text{continuity of } \Lambda \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \Lambda(\chi_{A_i}) && \text{linearity of } \Lambda \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \nu(A_i) \\
&= \sum_{i=0}^{\infty} \nu(A_i),
\end{aligned}$$

which shows countable additivity.

In addition,

$$\begin{aligned}
\sum_{i=0}^{\infty} |\nu(A_i)| &= \lim_{n\to\infty} \sum_{i=0}^{n} |\Lambda(\chi_{A_i})| \\
&\leq \lim_{n\to\infty} \sum_{i=0}^{n} \|\Lambda\|_{L^p(\mu)^*} \|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*} \lim_{n\to\infty} \sum_{i=0}^{n} \|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*} \lim_{n\to\infty} \sum_{i=0}^{n} \mu(A_i)^{1/p} \\
&\leq \|\Lambda\|_{L^p(\mu)^*} \left(\lim_{n\to\infty} \sum_{i=0}^{n} \mu(A_i) \right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*} \mu \left(\bigsqcup_{i=0}^{\infty} A_i \right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*} \mu(B)^{1/p} \\
&\leq \|\Lambda\|_{L^p(\mu)^*} \mu(X)^{1/p} \\
&< \infty,
\end{aligned}$$

which converges absolutely.
Thus ν is a complex measure.
In addition, if $\mu(E) = 0$, we have $\nu(E) = \Lambda(\chi_E) = \Lambda(0) = 0$.
Thus, $\nu \ll \mu$.
By Lebesgue-Radon-Nikodym for complex measures, $\exists! g \in L^1(\mu)$, such that $\Lambda(\chi_E) = \nu(E) = \int_E g d\mu$.
By linearity, $\Lambda(f) = \int_X f g d\mu$ for all simple measurable f.
By uniform simple function approximation, we have $\Lambda(f) = \int_X f g d\mu$ for all $f \in L^\infty(\mu)$.
Indeed, given any $f \in L^\infty(\mu)$, we have a sequence of simple measurable functions $|f_1| \leq |f_2| \leq \cdots \leq |f|$ that converges uniformly with $\|f - f_n\|_{L^\infty(\mu)} \to 0$.
Thus $\|f - f_n\|_{L^p(\mu)} \to 0$.
Thus $|\Lambda(f) - \Lambda(f_n)| \leq \|\Lambda\|_{L^p(\mu)^*} \|f - f_n\|_{L^p(\mu)} \to 0$.

$$\begin{aligned}
\Lambda(f) &= \lim_{n\to\infty} \Lambda(f_n) \\
&= \lim_{n\to\infty} \int_X f_n g d\mu \\
&= \int_X \lim_{n\to\infty} f_n g d\mu \\
&= \int_X f g d\mu.
\end{aligned}$$

(a) $p = 1, q = \infty$.
    Consider ant $E \in \mathcal{M}$, such that $\mu(E) > 0$.

We have

$$
\begin{aligned}
\left|\frac{1}{\mu(E)}\int_E g d\mu\right| &= \left|\frac{1}{\mu(E)}\Lambda(\chi_E)\right| \\
&\le \frac{1}{\mu(E)}\|\Lambda\|_{L^1(\mu)^*}\|\chi_E\|_{L^1(\mu)} \\
&= \frac{1}{\mu(E)}\|\Lambda\|_{L^1(\mu)^*}\mu(E) \\
&= \|\Lambda\|_{L^1(\mu)^*}.
\end{aligned}
$$

Thus $|g(x)| \le \|\Lambda\|_{L^1(\mu)^*}$ a.e..
Thus $g \in L^\infty(\mu)$. Since simple functions are dense in $L^p(\mu)$, we have $L^\infty(\mu)$ is dense in $L^p(\mu)$.
Since Λ, Λ_q are both bounded linear functionals, we have $\Lambda(f) = \int_X fgd\mu$ for all $f \in L^p(\mu)$.

(b) $p > 1$.
Let $E_n := \{x \in X : |g(x)| \le n\}$.
By LMCT, we have $\|g\|_q = \lim_{n\to\infty} \|\chi_{E_n} g\|_q$.
Let $f = \chi_{E_n}\overline{\operatorname{sgn}(g)}|g|^{q-1} \in L^\infty(\mu)$, we have

$$
\begin{aligned}
\|f\|_{L^p(\mu)}^p &= \int_{E_n} |g|^{(q-1)p} d\mu \\
&= \int_{E_n} |g|^q d\mu \\
&= \|\chi_{E_n} g\|_{L^q()}^q \\
\|\chi_{E_n} g\|_q^q &= \int_{E_n} |g|^q d\mu \\
&= \int_X fgd\mu \\
&= |\Lambda(f)| \\
&\le \|\Lambda\|\|f\|_{L^p(\mu)} \\
&\Longrightarrow \\
\|g\chi_{E_n}\|_{L^q(\mu)}^{q-\frac{q}{p}} &\le \|\Lambda\| \\
&\Longrightarrow \\
\|g\|_{L^q(\mu)}^{q-\frac{q}{p}} &\le \|\Lambda\| \\
&< \infty.
\end{aligned}
$$

Thus $g \in L^q(U)$.
Since Λ, Λ_q are both bounded linear functionals, we have $\Lambda(f) = \int_X fgd\mu$ for all $f \in L^p(\mu)$.

3. Now we assume that μ is σ-finite.
We have $X = \bigcup_{n=1}^\infty X_n, \forall n \ge 1,\ X_n \subset X_{n+1}, \mu(X_n) < \infty$.
We can get

$$\forall n \ge 1, g_n \in L^q(X_n, \mu), \text{ such that } \Lambda(f) = \int_X fg_n d\mu, \forall f \in L^p(X_n, \mu).$$

Notice that $L^p(X_n, \mu) \subset L^p(X_{n+1}, \mu)$.
We thus have $\forall n > m, g_n|_{X_m} = g_m$.
Let $g : X \to \mathbb{C};\ x \mapsto g_n(x)$ for $x \in X_n$.
Then $g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} g\chi_n$ in $\|\cdot\|_{L^q(\mu)}$, and thus $g \in L^q(\mu)$.
In addition, for any $f \in L^p(\mu)$, we have $\lim_{n\to\infty} f\chi_{X_n} = f$ in $\|\cdot\|_{L^q(\mu)}$.

We have

$$\begin{aligned}\Lambda(f) &= \Lambda(\lim_{n\to\infty} f\chi_{X_n})\\ &= \lim_{n\to\infty} \Lambda(f\chi_{X_n})\\ &= \lim_{n\to\infty} \int_X f\chi_{X_n} g d\mu\\ &= \int_X fgd\mu.\end{aligned}$$

4. Now suppose μ is not necessarily σ-finite, but $p \in (1, \infty)$.
   $\forall E \subseteq X$ be σ-finite, we have

$$g_E \in L^q(E, \mu), \text{ such that } \Lambda(f) = \int_X fg_E d\mu, \forall f \in L^p(E, \mu)$$

   .
   In addition, $\|g_E\|_{L^q(\mu)} \leq \|\Lambda\|$.
   Let $M := \sup_{E \text{ is } \sigma\text{-finite}} \|g_E\|_{L^q(\mu)} \leq \|\Lambda\|$.
   Choose $(E_n)_{n=1}^{\infty}$ such that $\|g_E\|_{L^q(\mu)} \to M$.
   Then $F := \bigcup_{n=1}^{\infty} E_n$ is σ-finite, and $\|g_F\|_{L^q(\mu)} = M$.
   In addition, for any σ-finite $A \supseteq F$, we have $A \setminus F$ is σ-finite as well. Thus $g_A = g_F + g_{A\setminus F}$.
   We have $g_{A\setminus F} = 0$a.e., which means $g_a = g_F$a.e..
   Let $g := g_F \in L^q(\mu)$.
   Given any $f \in L^p()$, let $A := \{x \in X : f(x) \neq 0\}$, which has to be σ-finite.
   Thus $\Lambda(f) = \int_X g_A f d\mu = \int_X g_X f d\mu = \int_X g f d\mu$.

□

Remark. This is in general not true for $p = \infty$.

Remark. If μ is not σ-finite, it might be the case where $L^1(\mu) = \{0\}$, while $L^\infty(\mu) \neq \{0\}$.

9.2 Complex Regular Measure Space

Definition 9.1. Let $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ be a complex Borel measure on a locally compact Hausdorff space X, with its Jordan decomposition. We say μ is a **complex Radon measure** or **complex regular measure** if all μ_i are finite Radon measures.

Proposition 9.2. *μ is a complex Radon measure if and only if $|\mu|$ is a Radon measure.*

Proof. It follows from $\mu_i \leq |\mu| \leq \mu_1 + \mu_2 + \mu_3 + \mu_4$. □

Definition 9.2. Let X be a locally compact Hausdorff space, we define $M(X) := \{\mu : \text{complex Radon measure}\}$, and $\|\mu\|_{M(X)} := |\mu|(X)$

Proposition 9.3. *$(M(X), \|\cdot\|_{M(X)})$ is a normed vector space over $\mathbb{C}$.*

Definition 9.3. $C_0(X)$ is the closure of $C_c(X)$ in $\|\cdot\|_\infty$.

Theorem 9.4 (Jordan Decomposition for $C_0(X, \mathbb{R})$)**.** *For any $\phi \in C_0(X, \mathbb{R})^*$, we have ϕ^+, ϕ^- positive bounded linear functionals, such that $\phi = \phi^+ - \phi^-$ on $C_0(X, \mathbb{R})$.*

Proof. For any $f \geq 0$, let $\phi^+(f) := \sup (\phi(g) : 0 \leq g \leq f)$.
Notice that if $c \geq 0$, we have $\phi^+(cf) = c\phi^+(f)$.
In addition, $\forall f_1, f_2 \geq 0 \in C_0(\mathbb{R})$, and any $0 \leq g_1 \leq f_1, 0 \leq g_2 \leq f_2$, we have $0 \leq g_1 + g_2 \leq f_1 + f_2$.
Thus $\phi(g_1) + \phi(g_2) = \phi(g_1 + g_2) \leq \phi^+(f_1 + f_2)$. Since g_1, g_2 are arbitrary, we have $\phi^+(f_1) + \phi^+(f_2) \leq \phi^+(f_1 + f_2)$.

On the other hand, if we take any $g \leq f_1 + f_2$, and $g_1 := \min(g, f_1), g_2 := g - g_2$, we have $g_2 \leq g - f_1 \leq f_2$.
Thus $\phi(g) = \phi(g_1) + \phi(g_2) \leq \phi^+(f_1) + \phi^+(f_2)$.
Since g is arbitrary, $\phi^+(f_1 + f_2) \leq \phi^+(f_1) + \phi^+(f_2)$.
Thus,

$$\phi^+(f_1) + \phi^+(f_2) = \phi^+(f_1 + f_2).$$

Now extend ϕ^+ to $C_0(X, \mathbb{R})$ by $\phi^+(f) = \phi^+(f^+) - \phi^+(f^-)$.
This is well-defined. Indeed, if $f = g - h = f^+ - f^-$ for $g, h \geq 0$, we have $g + f^- = h + f^+$, and thus $\phi^+(g) + \phi^+(f^-) = \phi^+(h) + \phi^+(f^+)$.
We can also check that ϕ^+ is linear, and $|\phi^+(f)| \leq \|\phi\|\|f\|$.
Thus ϕ^+ is a bounded positive linear functional.
We will then define $\phi^- := \phi^+ - \phi$, and check it is also a bounded positive linear functional. □

Theorem 9.5. *Let $\Lambda \in C_0(X)^*$, then $\exists!$ complex Radon measure $\mu \in M(X)$, such that*

$$\forall f \in C_0(X), \Lambda(f) = \int_X f d\mu.$$

Moreover, $\|\Lambda\| = \|\mu\|_{M(X)} = |\mu|(X)$.

Proof. We first consider $C_0(X; \mathbb{R})$, which is a real Banach subspace of $C_0(X)$.
Let $\Psi := \Lambda|_{C_0(X;\mathbb{R})}$.
Now let $\Psi_1 := \Re(\Psi), \Psi_2 := \Im(\Psi)$, we have that $\Psi_1, \Psi_2 \in C_0(X, \mathbb{R})^*$ over $\mathbb{R}$, with $\|\Psi_i\|_{C_0(X,\mathbb{R})^*} \leq \|\Lambda\|$.
In addition,

$$\begin{aligned}\Lambda(f) &= \Lambda(\Re(f) + i\Im(f)) \\ &= \Lambda(\Re(f)) + i\Lambda(\Im(f)) \\ &= \Psi_1(\Re(f)) + i\Psi_2(\Re(f)) + i(\Psi_1(\Im(f)) + i\Psi_2(\Im(f)))\end{aligned}$$

is uniquely determined by Ψ_1, Ψ_2.
Yet $\Psi_1 = \Psi_1^+ - \Psi_1^-, \Psi_2 = \Psi_2^+ - \Psi_2^-$, thus by Riesz-Markov-Kakutani, we have $\mu_i^\pm$ being finite Radon measures, such that $\Psi_i^\pm = \int_X f d\mu_i^\pm$.
Let $\mu := (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$, we have the result.
Now the uniqueness:
If $\Lambda = \Lambda_{\mu_1} = \Lambda_{\mu_2}$, we have $\forall f \in C_0(X)$,

$$\begin{aligned}0 &= \int_X f d(\mu_1 - \mu_2) \\ &= \int_X f h d|\mu_1 - \mu_2|.\end{aligned}$$

By density of $C_0(X)$, it is also true for all $f \in L^1(X)$, so $|\mu_1 - \mu_2| = 0$. □

Corollary 9.6. *$(M(X), \|\cdot\|_{M(X)}) \cong C_0(X)^*$ isometrically.*

10 Product Measures

Definition 10.1. Let $(X_i)_{i \in I}$ be a collection of non-empty sets, we define the **product** of the sets to be

$$X := \prod_{i \in I} X_i := \{(x_i)_{i \in I} | \forall i \in I, x_i \in X_i\} = \left\{ f : I \to \bigsqcup_{i \in I} X_i | \forall i \in I, f(i) \in X_i \right\}$$

Definition 10.2. We have a **canonical coordinate projections** $\pi : X \to X_i$ by $(x_j)_{j \in I} \mapsto x_i$.

Definition 10.3. If $(X_i, \mathcal{M}_i)$ are measurable spaces, then the **product measurable** space is

$$\left(\prod_{i\in I} X_i, \bigotimes_{i\in I} \mathcal{M}_i\right),$$

where $\bigotimes_{i\in I} \mathcal{M}_i$ is the σ-algebra generated by the sets $\{\pi_i^{-1}(A) | i \in I, A \in \mathcal{M}_i\}$.

Remark. When I is finite, this is the same as tensor products generated by $A_1 \times A_2 \times \cdots \times A_n$.

Proposition 10.1. *Let $(X_i, d_i)_{i=1}^n$ be separable metric spaces, then*

$$\bigotimes_{i=1}^n Bor(X_i) = Bor\left(\prod_{i=1}^n X_i\right).$$

Proof. Given any open $U_i \subseteq X_i$, we must have $\pi_i^{-1}(U_i) \subseteq X$ is open. Thus $\bigotimes_{i=1}^n Bor(X_i) \subseteq Bor(\prod_{i=1}^n X_i)$.
On the other hand, each X_i is separable, so X is also separable. Thus X is second countable. If $(x_n)_{n=1}^\infty$ is a dense sequence in X, then

$$\{B_r(x_n) | n \in \mathbb{N}, r \in \mathbb{Q}^{++}\}$$

is a basis for the topology. Namely, every open set can be written as a countable union of these open balls.
Setting $x_n^i := \pi_i(x_n)$, we have that $B_r(x_n) = \prod_{i=1}^n B_r(x_N^i)$, which is a subset of $\bigotimes_{i=1}^n Bor(X_i)$.
Since every open set $U \subseteq X$ is a countable union of these sets, so $U \subseteq \bigotimes_{i=1}^n Bor(X_i)$.
Thus $\bigotimes_{i=1}^n Bor(X_i) \supseteq Bor(\prod_{i=1}^n X_i)$. $\square$

Corollary 10.2.

$$Bor(\mathbb{R}^n) = \bigotimes_{i=1}^n Bor(\mathbb{R})$$

Proposition 10.3. *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be measure spaces. Let R be the collection of all finite unions of disjoint rectangles $A \times B$ with $A \in \mathcal{M}, B \in \mathcal{N}$. Then R is an algebra of subsets of $X \times Y$*

Proof.

$$(A \times B)^c = (A^c \times Y) \sqcup (A \times B^c)$$
$$(A_1 \times B_1) \cup (A_2 \times B_2) = (A_1 \times B_1) \sqcup ((A_2 \setminus A_1) \times B_2) \sqcup ((A_2 \setminus A_1) \times (B_1 \setminus B_2))$$

$\square$

Proposition 10.4. *The σ-algebra generated by R is $\mathcal{M} \otimes \mathcal{N}$.*

Definition 10.4. We can define a function $\pi : R \to [0, \infty]$ by $\pi(\bigsqcup_{i=1}^n A_i \times B_i) := \sum_{i=1}^n \mu(A_i)\nu(B_i)$

Lemma 10.5. *π is a premeasure.*

Proof. Firstly, $\pi(\emptyset) = \mu(\emptyset) \times \nu(\emptyset) = 0$.
Secondly, we consider any $A \times B = \bigsqcup_{n\in\mathbb{N}}(A_n \times B_n) \subseteq R$, where $(A_n \times B_n)_{n\in\mathbb{N}} \subseteq R$.
Fix any $y \in Y$, we have that $\chi_A(x)\chi_B(y) = \sum_{n\in\mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y)$, which is a sum of non-negative measurable

functions on X. By LMCT, we have that

$$\begin{aligned}
\mu(A)\chi_B(y) &= \int_X \chi_A(x) d\mu \chi_B(y) \\
&= \int_X \chi_A(x)\chi_B(y) d\mu \\
&= \int_X \sum_{n\in\mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y) d\mu \\
&= \sum_{n\in\mathbb{N}} \int_X \chi_{A_n}(x)\chi_{B_n}(y) d\mu \\
&= \sum_{n\in\mathbb{N}} \int_X \chi_{A_n}(x) d\mu \chi_{B_n}(y) \\
&= \sum_{n\in\mathbb{N}} \mu(A_n)\chi_{B_n}(y).
\end{aligned}$$

In addition, $\sum_{n\in\mathbb{N}} \mu(A_n)\chi_{B_n}(y)$ is a sum of non-negative measurable functions on Y. By LMCT, we again have that

$$\begin{aligned}
\mu(A)\nu(B) &= \mu(A) \int_Y \chi_B(y) d\nu \\
&= \int_Y \mu(A)\chi_B(y) d\nu \\
&= \int_Y \sum_{n\in\mathbb{N}} \mu(A_n)\chi_{B_n}(y) d\nu \\
&= \sum_{n\in\mathbb{N}} \int_Y \mu(A_n)\chi_{B_n}(y) d\nu \\
&= \sum_{n\in\mathbb{N}} \mu(A_n) \int_Y \chi_{B_n}(y) d\nu \\
&= \sum_{n\in\mathbb{N}} \mu(A_n)\nu(B_n).
\end{aligned}$$

This will now extend to any $\bigsqcup_{n\in\mathbb{N}}(A_n \times B_n) \subseteq R$, by finite additivity. □

Theorem 10.6. *There is a complete measure space $(X \times X, \overline{\mathcal{M} \otimes \mathcal{N}}, \mu \times \nu)$, such that $\mu \times \nu(A \times B) = \mu(A) \times \nu(B)$.*

Proof. Apply Caratheodory on the above lemma. □

For the flowing, let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces.

Definition 10.5. Take R a s before, let $R_\sigma := \left\{\bigcup_{n\geq 1} A_n | A_n \in R\right\}, R_{\sigma\delta} := \left\{\bigcap_{n\geq 1} E_n | E_n \in R_\sigma\right\}$

Lemma 10.7. *If $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$, with $\mu \times \nu(E) < \infty$, then $\exists G \in R_{\sigma\delta}$, such that $E \subseteq G, \mu \times \nu(G \setminus E) = 0$*

Proof. We have $\mu \times \nu(E) = \inf\left\{\sum_{i\geq 1} \mu \times \nu(A_i) | A_i \in R, E \subseteq \bigcup_{i\geq 1} A_i\right\}$.
Let $E_j := \bigcup_{i\geq 1} A_{ji} \supseteq E$, with $\mu \times \nu(E_j) < \mu \times \nu(E) + \frac{1}{j}$.
Notice that $E_j \in R_\sigma$ by construction.
Now take $G = \bigcup_{j\geq 1} E_j \in R_{\sigma\delta}$. Then we have that $E \subseteq G$, and $\forall j, \mu \times \nu(G) \leq \mu \times \nu(E_j) < \mu \times \nu(E) + \frac{1}{j}$.
Thus $\mu \times \nu(G) = \mu \times \nu(E)$. □

Lemma 10.8. *Let $E \in R_{\sigma\delta}$, with $\mu \times \nu(E) < \infty$. Let $E_x = \{y \in Y | (x, y) \in E\}, E^y = \{x \in X | (x, y) \in E\}$. Define $g(x) := \nu(E_x), h(y) := \mu(E^y)$. Then we have g is non-negative and μ-measurable, $g \in \mathcal{L}^1(\mu), \int_X g d\mu = \mu \times \nu(E)$. Similarly, h is non-negative and ν-measurable, $h \in \mathcal{L}^1(\nu), \int_Y g d\nu = \mu \times \nu(E)$*

Proof. If $E = A \times B$, with $A \in \mathcal{M}, B \in \mathcal{N}$, then $E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$.
Then $g(x) = \nu(B)\chi_A$ is μ-measurable, and $g \geq 0$. Moreover,

$$\int_X g d\mu = \int_X \nu(B)\chi_A d\mu = \nu(B) \int_X \chi_A d\mu = \mu(A)\nu(B) = \mu \times \nu(A \times B).$$

Now suppose $E = \bigcup_{i\geq 1} A_i \times B_i \in R_\delta$, with $A_i \in \mathcal{M}, B_i \in \mathcal{N}$. WLOG, we can take $E = \bigsqcup_{i\geq 1} A_i \times B_i$.
Let $g_i(x) = \nu(B_i)\chi_{A_i}(x)$, we have $\sum_{i=1}^n g_i(x) = \sum_{i=1}^n \nu(B_i)\chi_{A_i}(x) = \begin{cases} \nu(B_i) = \nu(E_x) & \text{if } x \in A_i \\ 0 & \text{if } x \notin \bigsqcup_{i=1}^n A_n \end{cases}$.
Thus $g(x) = \sum_{i=1}^\infty g_i(x)$ is measurable. By LMCT, $g \in \mathcal{L}^1(\mu)$, and

$$\begin{aligned} \int_X g d\mu &= \sum_{i=1}^\infty \int_X g_i d\mu \\ &= \sum_{i=1}^\infty \int_X \nu(B_i)\chi_{A_i} d\mu \\ &= \sum_{i=1}^\infty \nu(B_i)\mu(A_i) \\ &= \sum_{i=1}^\infty \mu \times \nu(A_i \times B_i) \\ &= \mu \times \nu\left(\bigsqcup_{i=1}^\infty A_i \times B_i\right) \\ &= \mu \times \nu(E). \end{aligned}$$

Now take $E = \bigcap_{i\geq 1} E_i \in R_{\delta\sigma}$ with $E_i \in R_\delta$. WLOG, we can take $E_i \supseteq E_{i+1}$.
Notice that $(E_i)_x = \{y \in Y | (x, y) \in E_i\} \supseteq \{y \in Y | (x, y) \in E_{i+1}\} = (E_{i+1})_x \supseteq \cdots \supseteq E_x$.
Let $g_i(x) = \nu((E_i)_x) = \mu \times \nu(E_i)$, then we have $0 \leq g \leq \cdots \leq g_i \leq \cdots \leq g_1$.
In addition, $E_x = \bigcap_{i\geq 1}(E_i)_x$, and thus $g(x) = \lim_{i\to\infty} g_i(x)$ by continuity of ν.
Thus g is μ-measurable, and since g_i are all dominated by g_1, we can used LDCT to get

$$\begin{aligned} \int_X g d\mu &= \lim_{i\to\infty} \int_X g_i d\mu \\ &= \lim_{i\to\infty} \mu \times \nu(E_i) \\ &= \mu \times \nu(E). \end{aligned}$$

□

Lemma 10.9. *Let $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$ with $\mu \times \nu(E) = 0$, then for μ-a.e. $x \in X$, we have $\nu(E_x) = 0$; for ν-a.e. $y \in Y$, we have $\mu(E_y) = 0$.*

Proof. We have some $G \in R_{\sigma\delta}$, such that $E \subseteq G, \mu \times \nu(G \setminus E) = 0$.
Let $f(x) := \nu(G_x)$, we have $f \in \mathcal{L}^1(\mathcal{M})$ is nonnegative. Yet $\int_X f d\mu = 0$, and thus $f(x) = 0$ for μ-a.e. $x \in X$.
Since $E_X \subseteq G_X$, and that ν is complete, we have that $g(x) = \nu(E_x) = 0$ for μ-a.e. $x \in X$. □

Corollary 10.10. *Let $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$ with $\mu \times \nu(E) < \infty$, then E_x is ν-measurable, for μ-a.e. $x \in X$, and $g(x) = \nu(E_x)$ is μ-measurable, with $g \geq 0, g \in \mathcal{L}^1(\mathcal{M})$, and $\int_X g d\mu = \mu \times \nu(E)$.*

Theorem 10.11 (Fubini's)**.** *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces. Take $f \in \mathcal{L}^1(\mu \times \nu)$, then*

1. *For μ-a.e. $x \in X, f_x := f(x, \cdot) \in \mathcal{L}^1(\nu)$.*
2. *For ν-a.e. $y \in Y, f_y := f(\cdot, y) \in \mathcal{L}^1(\mu)$.*

3. $F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^1(\mu)$.
4. $G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^1(\nu)$.
5. $\int_{X\times Y} f d(\mu\times\nu) = \int_X \left(\int_Y f(x,y) d\nu\right) d\mu = \int_Y \left(\int_X f(x,y) d\mu\right) d\nu$

Proof. Notice that $f^1 \in \mathcal{L}^1$ means that $f = f_1 - f_2 + if_3 - if_4$, where $f_i \geq 0, f_i \in \mathcal{L}^1$.
We first show the theorem holds for $f \geq 0, f \in \mathcal{L}^1$. There are simple functions $0 \leq s_1 \leq \cdots \leq s_n \leq \cdots \leq f$, such that $f(x) = \lim_{n\to\infty} s_n(x)$.
Let $F_n(x) = \int_Y s_n(x,y) d\nu \geq 0$ be measurable and $\mathcal{L}^1$. We have that □

Theorem 10.12 (Tonelli's)**.** *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be complete measure spaces. Take $f \in \mathcal{L}^+(\mu\times\nu)$, and $\mu\times\nu$ is σ-finite, then*

1. *For μ-a.e. $x \in X, f_x := f(x,\cdot) \in \mathcal{L}^+(\nu)$.*
2. *For ν-a.e. $y \in Y, f_y := f(\cdot,y) \in \mathcal{L}^+(\mu)$.*
3. *$F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^+(\mu)$.*
4. *$G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^+(\nu)$.*
5. *$\int_{X\times Y} f d(\mu\times\nu) = \int_X \left(\int_Y f(x,y) d\nu\right) d\mu = \int_Y \left(\int_X f(x,y) d\mu\right) d\nu$*

Proof. $\mu\times\nu$ is σ-finite, thus $\exists C_1 \subseteq C_2 \subseteq \cdots$, with $C_n \in \overline{\mathcal{M}\otimes\mathcal{N}}, X\times Y = \bigcup_{n=1}^\infty C_n$, and $\mu\times\nu(C_n) < \infty$.
Let $f_n(x) = \max\{f(x), n\}\chi_{C_n}(x)$, we have $0 \leq f_n \leq n\chi_{C_n}$, and $f_n \in \mathcal{L}^+ \cap \mathcal{L}^1(\mu\times\nu), \lim_{n\to\infty} f_n(x) = f(x)$.

$$
\begin{aligned}
\int f d\mu\times\nu &= \lim_{n\to\infty}\int f_n d\mu\times\nu \\
&= \lim_{n\to\infty}\int_X\int_Y f_n(x,y) d\nu d\mu \\
&=: \lim_{n\to\infty}\int_X F_n(x) d\mu.
\end{aligned}
$$

Then F_n are measurable, non-negative, and monotone increasing to $F(x) := \int_Y f(x,y) d\nu$. By LMCT, we have F is measurable, and

$$
\begin{aligned}
\int f d\mu\times\nu &= \lim_{n\to\infty}\int_X F_n(x) d\mu \\
&= \int_X F(x) d\mu \\
&= \int_X\int_Y f(x,y) d\nu d\mu
\end{aligned}
$$

□

Remark. If $f \in \mathcal{L}^1$, we get σ-finite by free on $C = \text{Supp}(f)$ if we look at $C_n := \left\{(x,y) : |f(x,y)| \geq \frac{1}{n}\right\}$. Notice that $\mu\times\nu(C_n) \leq n\int |f| d\mu\times\nu < \infty$.

Example 10.0.1. Consider $X = Y = \mathbb{N}, \mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, with the counting measure m_c.

Consider $f(m,n) := \begin{cases} 1 & n = m \\ -1 & n = m+1 \\ 0 & o.w. \end{cases} \sim \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ \vdots & 0 & 1 & -1 & \cdots \\ \vdots & \vdots & \ddots & & \end{pmatrix}$.

However,

$$\begin{aligned}
\int_X \int_Y f(x,y) dm_c(n) dm_c(m) &= \sum_{m \geq 1} \sum_{n \geq 1} f(m,n) \\
&= \sum_{m \geq 1} 0 \\
&= 0, \\
\int_Y \int_X f(x,y) dm_c(m) dm_c(n) &= \sum_{n \geq 1} \sum_{m \geq 1} f(m,n) \\
&= 1 + \sum_{n \geq 2} 0 \\
&= 1.
\end{aligned}$$

This is because $f \notin \mathcal{L}^1$.

Example 10.0.2. Consider $X = Y = [0,1]$, and the Lebesgue measure.
Take $t_n = 1 - \frac{1}{n}$.
Define $g_n : [0,1] \to \mathbb{R}$ by starting at $\frac{2t_n}{3} + \frac{t_{n+1}}{3}$, linear and reach $\frac{t_{n+1} - t_n}{3}$ at mid point, and decrease linearly to 0 at $\frac{t_n}{3} + \frac{2t_{n+1}}{3}$, and 0 outside. We thus have $\int g_n(x)dx = 1$.
Define $f(x,y) = \sum_{i=1}^{\infty} (g_n(x) - g_{n+1}(x)) g_n(y)$, where only one of these summands will be non-zero in each interval of x. Actually $f(x,y)$ is continuous $\forall (x,y) \neq (1,1)$.
However, $\int f(x,y)dx = g_n(y)$, and thus $\int \int f(x,y)dxdy = 1$, while $\int \int f(x,y)dydx = 0$.

Theorem 10.13. *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be (not necessarily complete) measure spaces. Then Fubini and Tonelli still apply with restriction to $\mathcal{M} \otimes \mathcal{N}$.*

10.1 Lebesgue Measure on $\mathbb{R}^n$

Lemma 10.14. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be Lebesgue measurable, then there is a G_δ set $G \subseteq \mathbb{R}^n$, such that $\lambda^n(G) = 0$, and $g = f\chi_{G^c}$ be Borel measurable, and $f = g$ λ^n-a.e..*

Proof. By writing $f = f_1 - f_2 + if_3 - if_4$ for $f_i \geq 0$, we can assume $f \geq 0$. We first consider $n = 1$.
Choose a dense subset $\{r_i\}_{i \in \mathbb{N}}$ of $[0, \infty)$. Let $A_i = f^{-1}([0, r_n])$. Since f is Lebesgue measurable, $A_n \in \mathcal{L}$. By regularity for the Lebesgue measure, there is an F_δ set $F_i \subseteq A_n$ and a null set $N_i = A_i \setminus F_i$. Let $N = \bigcup_{i \in \mathbb{N}} N_i$, then N is a null set.
Applying regularity again, there is a G_δ set $G \supseteq N$ such that $\lambda(G) = 0$.
Let $g = f\chi_{G^c}$. we have $g^{-1}([0, r_i)) = f^{-1}([0, r_i)) \cup G = A_i \cup G = (F_i \cup N_n) \cap G = F_i \cup G$, which is a union of two Borel sets, and thus Borel.
To verify that g is Borel, it surfaces to prove $g^{-1}([0, r))$ is Borel for all $r > 0$. By density of $\{r_i\}$, there is a sequence r_{n_k} such that $r_{n_k} \leq r$ and $r_{n_k} \to r$. Thus $\bigcup_{k \geq 1} [0, r_{n_k}) = [0, r)$, so $g^{-1}([0, r)) = \bigcup_{k \geq 1} ([0, r_{n_k}))$ is a union of Borel sets, and thus Borel.
By construction, G is a null set and $f|_{G^c} = g|_{G^c}$, so $f = g$ λ-a.e..
Now suppose $n \geq 1$. For each i, let f_{x_i} be the function obtained by fixing all but the i^{th} variable x_i. From above we can find $G\delta$ set $G_i \subseteq \mathbb{R}$, such that $f_{x_i} = f_{x_i} \chi_{G_i^c}$ λ-a.e..
Let $G = (G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup (\mathbb{R} \times G_2 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup \cdots (\mathbb{R} \times \cdots \times \mathbb{R} \times G_n)$.
Then $G^c = G_1^c \times G_2^c \times \cdots \times G_n^c$. Let $G_1 = \bigcap_k U_{1k}$, then $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R} = \bigcap_k (U_{1k} \times \mathbb{R} \times \cdots \times \mathbb{R})$, where each is open, since $U_{1k}, \mathbb{R}$ are open. Thus $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}$ is a G_δ set. Thus G is a finite union of G_δ sets, which is G_δ. $\square$

Definition 10.6. For $A \in \mathcal{L}^n, X \in \mathbb{R}^n$, write the **translation** of A by x as $A + x = \{a + x : a \in A\}$.

Definition 10.7. Let GLn be the set of invertible $n \times n$ matrices.

Theorem 10.15. *Consider the Lebesgue measure λ^n in $\mathbb{R}^n$.*

1. *(translation) For $A \in \mathcal{L}^n$ and $x \in \mathbb{R}^n$, we have $A + x \in \mathcal{L}^n, \lambda^n(A + x) = \lambda^n(A)$.*
2. *(scaling) For $T \in GLn, f : \mathbb{R}^n \to \mathbb{C}$ be Lebesgue measurable, $f \circ T$ is Lebesgue measurable, and*
$$\int f d\lambda^n = |\det(T)| \int (f \circ T) d\lambda^n.$$
In particular, for $A \in \mathcal{L}^n$, we have $\lambda^n(T(A)) = |\det(T)|\lambda^n(A)$.
3. *(rotation) For a unitary $U \in GLn$, we have*
$$\int (f \circ U) d\lambda^n = \int f d\lambda^n,$$
and $\forall A \in \mathcal{L}^n, \lambda^n(U(A)) = \lambda(A)$.

Proof. 1.

2. Notice that $x \in T(A) \iff T^{-1}x \in A$, thus $\chi_{T(A)} = \chi_A \circ T^{-1}$. Thus
$$\begin{aligned}\lambda^n(T(A)) &= \int \chi_{T(A)} d\lambda^n \\ &= \int \chi_A \circ T^{-1} d\lambda^n \\ &= \frac{1}{|\det(T^{-1})|} \int \chi_A d\lambda^n \\ &= |\det(T)|\lambda^n(A).\end{aligned}$$

□

11 Convolutions and Fourier Transforms

Definition 11.1. For $y \in \mathbb{R}, f : \mathbb{R} \to \mathbb{C}$, we define the **translation** of f by y to be $L_y f(x) := f(x - y)$.

Proposition 11.1. *We have $L_y : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is linear, isometric, and $\forall f \in L^1(\mathbb{R})$, we have*
$$\lim_{y \to 0} \|L_y f - f\|_1 = 0.$$

Proof. If $f \in C_c(\mathbb{R})$, then it is uniformly continuous, so
$$\lim_{y \to 0} \|L_y f - f\|_\infty = 0.$$
Take compact $K \supseteq \mathrm{Supp}(f)$, we have that
$$\begin{aligned}\|L_y f - f\|_1 &= \int_{K \cup (K+y)} |f(x - y) - f(x)| dx \\ &\leq \lambda(K \cup (K + y)) \|Lxf - f\|_\infty,\end{aligned}$$
where the first term is bounded by $2\lambda(K) < \infty$, and the second term goes to 0.
Now since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, we have the result by triangle inequality. □

Theorem 11.2 (Young's Convolution Inequality)**.** *Consider $X = \mathbb{R}$, with Lebesgue measure λ. Let $f, g \in L^1(\mathbb{R})$, then for a.e. $x \in \mathbb{R}$, the function $y \mapsto f(x - y)g(y)$ is in $L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$, and the **convolution***
$$(f * g)(x) := \int_{\mathbb{R}} f(x - y)g(y) dy$$
is also in $L^1(\mathbb{R})$. In addition, $\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$.

Proof. Consider the function $F : (x, y) \mapsto f(x - y)g(y)$, which is a measurable function on $\mathbb{R} \times \mathbb{R}$ (can show with approximation by $C_c(\mathbb{R})$ functions).
By Tonelli's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| d\lambda^2 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x, y)| dx \right) dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x - y)||g(y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |f(x - y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \|f\|_{L^1(\mathbb{R})} dy \\
&= \|f\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} |g(y)| dy \\
&= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \\
&< \infty.
\end{aligned}$$

Thus, $F \in L^1(\mathbb{R}^2)$.
Now we apply Fubini's Theorem to F, and get $F_x(y) = f(x - y)g(y) \in L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$.
In addition,

$$\begin{aligned}
\|f * g\|_{L^1(U)} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x - y)g(y) dy \right| dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x - y)g(y)| dy dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |F(x, y)| dy dx \\
&= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.
\end{aligned}$$

□

Corollary 11.3. $(L^1(\mathbb{R}), *)$ *defines a communicative associative algebra.*

Definition 11.2. Given $f \in L^1(\mathbb{R})$, its **Fourier Transform** is $\mathcal{F}(f) := \hat{f} : \mathbb{R} \to \mathbb{C}$, where

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-ix\omega} dx.$$

Lemma 11.4 (Riemann-Lebesgue)**.** $\forall f \in L^1(\mathbb{R})$, *we have* $\hat{f} \in C_0(\mathbb{R})$, *and* $\left\|\hat{f}\right\|_\infty \leq \|f\|_{L^1(\mathbb{R})}$. *Namely,* $\mathcal{F}$ *is a contraction map.*

Proof. Consider any sequence $(\omega_n)_{n=1}^\infty \subset \mathbb{R}$ that converges to $\omega \in \mathbb{R}$.
Let $h_n(x) := f(x)(e^{i\omega_n x} - e^{i\omega x})$, we have that $h_n \in L^1(\mathbb{R})$, $h_n(x) \to 0$ pointwise for a.e. $x \in \mathbb{R}$, and $|h_n| \leq |f| \left| e^{i\omega_n x} - e^{i\omega x} \right| \leq 2|f|$.
In addition,

$$\begin{aligned}
\hat{f}(\omega_n) - \hat{f}(\omega) &= \int_{\mathbb{R}} f(x)(e^{i\omega_n x} - e^{i\omega x}) dx \\
&= \int_{\mathbb{R}} h_n(x) dx
\end{aligned}$$

By LDCT, we have that $\lim_{n\to\infty}\left(\hat{f}(\omega_n) - \hat{f}(\omega)\right) = 0$, so $\hat{f}$ is continuous.
In addition,

$$\begin{aligned}
\left|\hat{f}(\omega)\right| &\le \int_{\mathbb{R}} |f(x)||e^{ix\omega}|dx \\
&= \int_{\mathbb{R}} |f(x)|dx \\
&= \|f\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Now

$$\begin{aligned}
\hat{f}(\omega) &= \int_{\mathbb{R}} f(x)e^{-ix\omega}dx \\
&= -\int_{\mathbb{R}} f(x)e^{-ix\omega+\pi i}dx \\
&= -\int_{\mathbb{R}} f(x)e^{-i\omega(x-\pi/\omega)}dx \\
&= -\int_{\mathbb{R}} f(z+\pi/\omega)e^{-i\omega z}dz \\
&= -\int_{\mathbb{R}} L_{-\pi/\omega}f(z)e^{-i\omega z}dz \\
2\hat{f}(\omega) &= \int_{\mathbb{R}} f(x)e^{-ix\omega}dx - \int_{\mathbb{R}} L_{-\pi/\omega}f(z)e^{-i\omega z}dz \\
&= \int_{\mathbb{R}} (f - L_{-\pi/\omega}f)(x)e^{-i\omega x}dx \\
&= \mathcal{F}(f - L_{-\pi/\omega}f)(\omega) \\
2\left|\hat{f}(\omega)\right| &\le \left\|f - L_{-\pi/\omega}f\right\|_{L^1(\mathbb{R})},
\end{aligned}$$

which goes to 0 when $\omega \to \infty$.
Thus, $\hat{f} \in C_0(\mathbb{R})$. □

Theorem 11.5 ($L^1(\mathbb{R})$ Inversion)**.** *If $f, \hat{f} \in L^1(\mathbb{R})$, we have*

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)e^{ix\omega}d\omega$$

for a.e. $x \in \mathbb{R}$.
In particular, such f must be almost everywhere equal to a continuous function.

Proof. Let $\lambda > 0$, and $H_\lambda(\omega) := e^{-\lambda|\omega|}$.
Let

$$\begin{aligned}
h_\lambda(x) &:= \frac{1}{2\pi}\int_{\mathbb{R}} H_\lambda(\omega)e^{ix\omega}d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\omega-\lambda|\omega|}d\omega \\
&= \frac{\lambda}{\pi}\frac{1}{x^2+\lambda^2}.
\end{aligned}$$

Fix $f \in L^1(\mathbb{R})$, we have

$$
\begin{aligned}
(f * h_\lambda)(x) &= \int_{\mathbb{R}} f(x-y)h_\lambda(y)dy \\
&= \int_{\mathbb{R}} f(x-y)\frac{1}{2\pi}\int_{\mathbb{R}} H_\lambda(\omega)e^{iy\omega}d\omega dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)H_\lambda(\omega)e^{iy\omega}d\omega dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)H_\lambda(\omega)e^{iy\omega}dy d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(z)H_\lambda(\omega)e^{i\omega(x-z)}dz d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} H_\lambda(\omega)\hat{f}(\omega)e^{i\omega(x)}dz d\omega.
\end{aligned}
$$

One can show that $H_\lambda(\omega) = 1$ as $\lambda \to 0$, and $f * h_\lambda \to f$.
If $\hat{f} \in L^1(\mathbb{R})$, we can use DCT to get the result. □

Corollary 11.6. *If $f, g \in L^1(\mathbb{R})$, and $\mathcal{F}(f) = \mathcal{F}(g)$, we must have $\mathcal{F}(f-g) = 0 \in L^1(\mathbb{R})$. Thus, $f = g$ a.e.$x \in \mathbb{R}$.*

Remark. Not all $\hat{f} \in L^1(\mathbb{R})$.

Example 11.0.1. If $f = \chi_{[-1,1]}$, we have $\hat{f} = \frac{2\sin(\omega)}{\omega} \in C_0(\mathbb{R}) \setminus L^1(\mathbb{R})$.

Remark. We can equivalently define

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx,$$

then the inverse is given by

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i x\xi}d\xi.$$

Proposition 11.7. *For any $f, g \in L^1(\mathbb{R})$, we always have $\widehat{f * g} = \hat{f}\hat{g}$.*

Example 11.0.2. Consider the heat equation initial problem: $\begin{cases} \frac{\partial}{\partial t}u = \Delta u \\ u(\cdot, 0) = f. \end{cases}$ Taking the Fourier Transform with respect to x, we have $\begin{cases} \frac{\partial}{\partial t}\hat{u} = (2\pi i\xi)^2\hat{u} \\ \hat{u}(\cdot, 0) = \hat{f}, \end{cases}$ with the solution $\hat{u}(\xi, t) = e^{-4\pi^2\xi^2 t}\hat{f(\xi)}$.
Assuming we can apply the inverse formula, we have

$$
\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} \hat{u}(\xi, t)e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} e^{-4\pi^2\xi^2 t}\hat{f(\xi)}e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} e^{-4\pi^2\xi^2 t}\int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}dy e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} e^{-4\pi^2\xi^2 t}e^{-2\pi i y\xi}dy e^{2\pi i\xi x}d\xi \\
&= \int_{\mathbb{R}} f(y)\frac{1}{(4\pi t)^{\frac{1}{2}}}e^{-\frac{(x-y)^2}{4t}}d\xi \\
&= \int_{\mathbb{R}} f(y)H_t(x-y)d\xi \\
&= (H_t * f)(x),
\end{aligned}
$$

where the **heat kernel** is $H_t(x) = \frac{1}{(4\pi t)^2} e^{-\frac{x^2}{4t}}$.
The heat kernel satisfies $\frac{\partial}{\partial t} H_t = \Delta H_t$, $\int_{\mathbb{R}} H_t(x)dx = 1$, and $\int_{|x|\geq \epsilon} H_t(x)dx \to 0$.

Definition 11.3. The **Schwartz class** S is the set of functions $f : \mathbb{R} \to \mathbb{C}$, such that

$$\exists C \geq 0, \text{ such that } \forall \alpha, \beta, \left| x^\alpha \frac{d}{dx^\beta} f \right| \leq C.$$

Proposition 11.8. *$C_c^\infty(\mathbb{R}) \subset S$.*

Proposition 11.9. *Suppose $f \in S$, then $\hat{f} \in S$.*

Theorem 11.10 (Plancherel)**.** *Suppose $f \in S$, then $\|f\|_{L^2(\mathbb{R})} = \left\|\hat{f}\right\|_{L^2(\mathbb{R})}$.*

Proposition 11.11. *S is dense in $L^2(\mathbb{R})$. Namely, $\bar{S} = L^2(\mathbb{R})$.*

Definition 11.4. For $f \in L^2(\mathbb{R})$, with $(f_i)_{i=1}^\infty$ in S such that $f_i \to f$, we define the **Fourier Transform** of f to be

$$\hat{f} := \lim_{i\to\infty} \hat{f}_i.$$

Lemma 11.12. *The above definition is well-defined.*

Proof. Given any $\epsilon > 0$.
Since $f_i \to f$ in $L^2(\mathbb{R})$, there is $N \geq 1$, such that $\forall i \geq N$, $\|f_i - f\|_{L^2(\mathbb{R})} < \epsilon/2$.
Thus, for any $i, j \geq N$,

$$\begin{aligned}
\left\|\hat{f}_i - \hat{f}_j\right\|_{L^2(\mathbb{R})} &= \|f_i - f_j\|_{L^2(\mathbb{R})} \\
&\leq \|f_i - f\|_{L^2(\mathbb{R})} + \|f - f_j\|_{L^2(\mathbb{R})} \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Thus $(\hat{f}_i)_{i=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R})$, so $\lim_{i\to\infty} \hat{f}_i$ exists in $L^2(\mathbb{R})$. We can also see that this is independent of the choice of choice of the sequence. ☐

12 Bochner Spaces

Definition 12.1. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \to B$ is **weakly measurable** if $\forall \Lambda \in B^*$, $\Lambda \circ f : X \to \mathbb{C}$ is measurable.

Definition 12.2. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \to B$ is **Bochner measurable** or **strongly measurable** if $f(x) = g(x)$ for μ-a.e. $x \in X$, for some measurable g, with $\text{Im}(g) \subseteq B$ being separable.

Proposition 12.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, then a function $f : X \to B$ is strongly measurable if $f(x) = \lim_{n\to\infty} f_n(x)$ for μ-a.e. $x \in X$, for some sequence of measurable functions f_n, each with countable range.*

Definition 12.3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $s : X \to [0, \infty)$ be a simple measurable function, with $s(X) = \{a_1, \ldots, a_n\} \subset B$, such that

$$s = \sum_{i=1}^n a_i \mathcal{X}_{A_i},$$

where $A_i := s^{-1}(\{a_i\})$. For $A \in \mathcal{M}$, we say s is integrable over A if $\forall a_i \neq 0, \mu(A_i \cap A) < \infty$, and define the **integral** of s over A to be

$$\int_A s d\mu := \sum_{i=1}^n a_i \mu(A_i \cap A).$$

Definition 12.4. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $f : X \to [0, \infty)$ be a measurable function. If there is a sequence of simple integrable functions $(s_n)_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \int_A \|f - s_n\|_B d\mu = 0,$$

then we say f is **Bochner integrable**, and we define the **Bochner integral** to be

$$\int_A f d\mu := \lim_{n\to\infty} \int_A s_n d\mu.$$

Lemma 12.2. *The right hand side of the above definition always exists, and is independent of the choice of the sequence of simple integrable functions $(s_n)_{n=1}^{\infty}$. Thus, the above definition is well-defined.*

Theorem 12.3 (Bochner)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. A strongly measurable function $f : X \to B$ is Bochner integrable if and only if $x \mapsto \|f(x)\|_B$ is integrable. In this case, $\forall E \in \mathcal{M}$,*

$$\left\| \int_E f(x)dx \right\|_B \leq \int_E \|f(x)\|_B dx,$$
$$\forall \Lambda \in B^*, \; \Lambda\left(\int_E f(x)dx \right) = \int_E \Lambda(f(x))dx.$$

Theorem 12.4 (Dominated Convergence Theorem for Bochner integral)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. Let $f_n : X \to \mathbb{C}$ be measurable functions, defined μ-a.e. on X, such that $f(x) := \lim_{n\to\infty} f_n(x)$ is defined $\mu-$almost everywhere for $x \in X$. If there is $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$, such that for μ-a.e. $x \in X, \forall n \in \mathbb{N}$, $\|f_n(x)\|_B \leq g(x)$, then f is Bochner integrable, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n\to\infty} \int_X \|f - f_n\|_B d\mu = 0.$$

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

Definition 12.5. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, and $1 \leq p < \infty$, we define

$$\mathcal{L}^p(\mu, B) := \left\{ f : X \to B \middle| f \text{ is measurable}, \int_X \|f\|_B^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p(\mu,B)} := \left(\int_X \|f\|_B^p d\mu \right)^{\frac{1}{p}}.$$

Definition 12.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space, we define

$$\mathcal{L}^\infty(\mu, B) := \{ f : X \to B | f \text{ is measurable}, \operatorname{ess\,sup} \|f\|_B < \infty \}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty(\mu,B)} := \operatorname{ess\,sup} \|f\|_B.$$

Definition 12.7. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. For any $p \in [1, \infty]$, we define

$$L^p(\mu, B) := \mathcal{L}^p(\mu, B)/N,$$

where $N := \{ f : X \to B | f \text{ is measurable}, f = 0 \; \mu-\text{a.e.} \}$. Namely, $[f] \in L^p(\mu, B)$ is the equivalence class of all $g = f$ μ-a.e. for $f \in \mathcal{L}^p(\mu, B)$.
In addition, we define

$$\|[f]\|_{L^p(\mu,B)} := \|f\|_{\mathcal{L}^\infty(\mu,B)}$$

for any representative f.

Theorem 12.5 (Fischer-Riesz-Bochner)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(B, \|\cdot\|)$ be a Banach Space. For all $1 \leq p \leq \infty$, we have that $\left(L^p(\mu, B), \|\cdot\|_{L^p(\mu,B)} \right)$ is a Banach Space.*