

# Pmath 651: Measure Theory

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## Contents

<b>1</b>	<b>Introductions</b>	<b>3</b>
1.1	Lebesgue Measure . . . . .	3
<b>2</b>	<b>Measure</b>	<b>3</b>
2.1	Algebra of Sets . . . . .	3
2.2	Measures . . . . .	4
2.3	Measurable Function . . . . .	6
2.4	Simple Functions . . . . .	7
<b>3</b>	<b>Integration</b>	<b>9</b>
3.1	Integration of non-negative functions . . . . .	9
3.2	Integration of complex functions . . . . .	13
3.3	Lebesgue Dominated Convergence Theorem . . . . .	15
3.4	Almost Everywhere . . . . .	16
3.5	Complete Measure . . . . .	18
<b>4</b>	<b>Construction of Measure</b>	<b>19</b>
4.1	Caratheodoy Theorem . . . . .	19
4.2	Premeasures . . . . .	21
4.3	Lebesgue-Stieltjes Measures . . . . .	22
4.4	Littlewood's Three Principles . . . . .	27
<b>5</b>	<b>Borel Measures on Topological Spaces</b>	<b>29</b>
5.1	Topological Spaces . . . . .	29
5.2	Compactly Supported Functions . . . . .	31
5.3	Partition of Unity . . . . .	31
5.4	Linear Functional . . . . .	32
5.5	Radon Meaure . . . . .	33
5.6	Extremely Disconnected Spaces . . . . .	34
5.7	Riesz-Markov-Kakutani . . . . .	34
<b>6</b>	<b>Lebesgue Spaces</b>	<b>34</b>
6.1	The First Lebesgue Space . . . . .	34
6.2	Convex functions . . . . .	37
6.3	L <sup>p</sup> Spaces . . . . .	39
<b>7</b>	<b>Complex measures</b>	<b>49</b>
7.1	Signed measures . . . . .	49
7.2	Complex measures . . . . .	51
7.3	Radon–Nikodym-Lebegue Decomposition . . . . .	53

<b>8 Dual of Function Spaces</b>	<b>58</b>
8.1 Dual of $L_p$ Spaces . . . . .	58
8.2 Complex Regular Measure Space . . . . .	61
<b>9 Product Measures</b>	<b>63</b>
9.1 Lebesgue Measure on $\mathbb{R}^n$ . . . . .	67
<b>10 Convolutions and Fourier Transforms</b>	<b>68</b>
<b>11 Bochner Spaces</b>	<b>72</b>

# 1 Introductions

## 1.1 Lebesgue Measure

**Definition 1.1.** Lebesgue outer measure of  $A \in \mathbb{R}$  is  $\lambda^*(A) := \inf \{\sum_{i=1}^{\infty} l(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i\}$ , where each  $I_i \subseteq \mathbb{R}$  is an open interval.

**Definition 1.2.** If  $\forall E \in \mathbb{R}, \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$ , then  $A$  is Lebesgue measurable, and its Lebesgue measure is defined to be  $\lambda(A) := \lambda^*(A)$

**Proposition 1.1.**  $\forall a < b \in \mathbb{R}, \lambda((a, b)) = b - a$

**Proposition 1.2.**  $\forall x \in \mathbb{R}, \lambda(x + A) = \lambda(A)$

**Proposition 1.3.** If  $A_m$  are  $\mathcal{L}$ -measurable and pairwise disjoint ( $A_m \cap A_n = \emptyset, \forall n \neq m$ ), then  $m(\bigcup_{i \geq 1} A_i) = \sum_{i=1}^{\infty} m(A_i)$

**Proposition 1.4.** Every Riemann integrable function is Lebesgue integrable.

# 2 Measure

## 2.1 Algebra of Sets

**Definition 2.1.** Let  $X$  be a set and  $\mathcal{P}(X) := \{A | A \subseteq X\}$ , then an algebra of subsets of  $X$  is  $\mathcal{A} \subseteq \mathcal{P}(X)$ , such that

1.  $\emptyset \in \mathcal{A}$
2. If  $E \in \mathcal{A}$ , then  $E^c := X \setminus E \in \mathcal{A}$
3. If  $E_1, \dots, E_n \in \mathcal{A}$ , then  $\bigcup_{i=1}^n E_i \in \mathcal{A}$

**Definition 2.2.** Let  $X$  be a set and  $\mathcal{P}(X) := \{A | A \subseteq X\}$ , then a  $\sigma$ -algebra of subsets of  $X$  is  $\mathcal{M} \subseteq \mathcal{P}(X)$ , such that

1.  $\emptyset \in \mathcal{M}$
2. If  $E \in \mathcal{M}$ , then  $E^c := X \setminus E \in \mathcal{M}$
3. If  $E_1, E_2, \dots \in \mathcal{M}$ , then  $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$

**Definition 2.3.** If  $\mathcal{M}$  is  $\sigma$ -algebra, we call  $(X, \mathcal{M})$  a measurable space, and a set  $E \in \mathcal{M}$  is called  $\mathcal{M}$ -measurable.

*Remark.* Every  $\sigma$ -algebra is an algebra.

**Proposition 2.1.** If  $\mathcal{A}$  is an algebra, and  $E_1, E_2 \in \mathcal{A}$ , then  $E_1 \cap E_2 \in \mathcal{A}$

*Proof.*  $(E_1 \cap E_2)^c = E_1^c \cup E_2^c$  is in  $\mathcal{A}$  by 2,3. □

**Proposition 2.2.** If  $\mathcal{A}$  is an algebra, and  $E, F \in \mathcal{A}$ , then  $E \setminus F = E \cap F^c \in \mathcal{A}$ .

**Proposition 2.3.** If  $\mathcal{A}$  is an algebra, and  $E, F \in \mathcal{A}$ , then  $E \Delta F = (E \setminus F) \cup (F \setminus E) \in \mathcal{A}$ .

**Proposition 2.4.** If  $\mathcal{M}$  is  $\sigma$ -algebra,  $E_i \in \mathcal{M}$ , then we can define  $F_i := E_i \setminus \bigcup_{j=1}^{i-1} E_j$ , and  $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$

**Proposition 2.5.** If  $\mathcal{M}$  is  $(\sigma)$ -algebra, and  $E \in \mathcal{M}$ , then  $A|_E := \{E \cap A | A \in \mathcal{M}\}$  is an  $(\sigma)$ -algebra.

**Example 2.1.1.**  $P(X)$  is  $\sigma$ -algebra, and  $\{\emptyset, X\}$  is  $\sigma$ -algebra.

**Example 2.1.2.**  $\mathcal{A} = \{E \subseteq X : |E| < \infty \vee |E^c| < \infty\}$  is an algebra. However, if  $X$  is infinite, then it is not a  $\sigma$ -algebra

**Example 2.1.3.**  $\mathcal{M} = \{E \subseteq X : |E| \leq \mathcal{N}_0 \vee |E^c| \leq \mathcal{N}_0\}$  is a  $\sigma$ -algebra.

**Example 2.1.4.** Let  $X = \mathbb{R}$ , the collection of all finite union of sets in  $\{\mathbb{R}, (-\infty, b], (a, b], (a, \infty) | a, b \in \mathbb{R}\}$  is an algebra but not  $\sigma$ -algebra.

**Proposition 2.6.** Let  $\{\mathcal{M}_\alpha\}_{\alpha \in I}$  is a collection of ( $\sigma$ -)algebras of  $X$ , then  $\bigcap_{\alpha \in I} \mathcal{M}_\alpha$  is an ( $\sigma$ -)algebra

**Definition 2.4.** Let  $\mathcal{C}$  be a collection of subsets of  $X$ , then  $\sigma(\mathcal{C}) := \bigcap \{\mathcal{M} : \sigma\text{-alg}, \mathcal{C} \subseteq \mathcal{M}\}$  is a  $\sigma$ -algebra containing  $\mathcal{C}$ , and is called the  $\sigma$ -algebra generated by  $\mathcal{C}$ .

**Definition 2.5.** Let  $X$  be a topological space, and let  $\mathcal{G}$  be the collection of all open sets of  $X$ , then the Borel algebra is  $Bol_X := \sigma(\mathcal{G})$

## 2.2 Measures

**Definition 2.6.** A function  $\mu : \mathcal{M} \rightarrow [0, \infty]$  is called a **positive measure** if:

1.  $\mu(\emptyset) = 0$
2. If  $E_1, E_2, \dots$  are pairwise disjoint sets in  $\mathcal{M}$ , then  $\mu(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$

We call  $(X, \mathcal{M}, \mu)$  a **measure space**.

**Definition 2.7.**  $\mu$  is finite if  $\mu(X) < \infty$ .  $\mu$  is  $\sigma$ -finite if  $X = \bigcup_{i=1}^{\infty} A_i$ , where each  $\mu(A_i) < \infty$ .  $\mu$  is semi-finite if  $\forall E \in \mathcal{M}$ , such that  $\mu(E) \neq 0$ , there is always  $F \in \mathcal{M}, F \subseteq E, 0 < \mu(F) < \infty$

**Definition 2.8.** A **complex measure** is a function  $\mu : \mathcal{M} \rightarrow \mathbb{C}$  that is countably additive, as above. Similarly, we can define a **real measure**  $\mu : \mathcal{M} \rightarrow \mathbb{R}$

*Remark.* We will only work with positive measures where it satisfies  $\exists A \in \mathcal{M}, \mu(A) < \infty$ .

**Example 2.2.1.** For any  $X$ , we can define  $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$  by  $\mu(A) := \begin{cases} |A|, & |A| < \infty \\ \infty, & \text{otherwise} \end{cases}$  is the **counting measure** on  $X$

**Example 2.2.2.** For any set  $X$  and  $x \in X$ , we can define  $\delta_x : \mathcal{P}(X) \rightarrow [0, \infty]$  by  $\delta_x(A) := \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$  is the **point measure** or **Dirac measure** of  $x$ .

**Example 2.2.3.** Let  $X = \mathbb{R}, \mathcal{M} = \mathcal{P}(X)$ , let  $x_1, x_2, \dots \in \mathbb{R}, a_1, a_2, \dots \geq 0$ , then  $\mu(E) := \sum_{i|x_i \in E} a_i$  is a measure.

**Definition 2.9.** A positive measure  $\mu$  is a **probability measure** if  $\mu(X) = 1$ . In this case  $(X, \mathcal{M}, \mu)$  is called a probability space.

**Proposition 2.7.**  $\mu(\emptyset) = 0$

*Proof.* Choose  $A \in \mathcal{M}$  with finite measure, take  $A_1 = A$ , and  $A_2 = A_3 = \dots = \emptyset$ .

Then  $\mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset) = \mu(A) < \infty$ , thus we must have  $\mu(\emptyset) = 0$  □

*Remark.* This holds for complex measures as well.

**Proposition 2.8. Finite Additivity:** If  $E_1, E_2, \dots, E_n \in \mathcal{M}$ , then  $\mu(\bigsqcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$

*Proof.* Take  $E_{n+1} = E_{n+2} = \dots = \emptyset$ , then  $\mu(\bigsqcup_{i=1}^{\infty} E_i) = \mu(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^n \mu(E_i) + \sum_{i=n+1}^{\infty} \mu(E_i) = \sum_{i=1}^n \mu(E_i)$  □

*Remark.* This holds for complex measures as well.

**Proposition 2.9. Monotonicity:** If  $E, F \in \mathcal{M}, E \subseteq F$ , then  $\mu(E) \leq \mu(F)$

*Proof.* We have  $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$  □

*Remark.* This does not hold for complex measures.

**Proposition 2.10.** *Subadditivity: If  $E_1, E_2, \dots \in \mathcal{M}$ , then  $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$*

**Proposition 2.11.** *Continuity:*

*If  $E_1, E_2, \dots \in \mathcal{M}, E_n \subseteq E_{n+1}$ , we have that  $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{n \rightarrow \infty} \mu(E_n)$ .*

*If  $E_1, E_2, \dots \in \mathcal{M}, E_{n+1} \subseteq E_n, \mu(E_1) < \infty$  we have that  $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{n \rightarrow \infty} \mu(E_n)$ .*

*Proof.* Let  $E_0 = \emptyset$ , then we can write  $\bigcup_{i=1}^{\infty} E_i = \bigsqcup_{i=1}^{\infty} (E_i \setminus E_{i-1})$ , and we have  $E_n = \bigsqcup_{i=1}^n (E_i \setminus E_{i-1})$ .

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigsqcup_{i=1}^{\infty} (E_i \setminus E_{i-1})\right) \\ &= \sum_{i=1}^{\infty} \mu(E_i \setminus E_{i-1}) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i \setminus E_{i-1}) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigsqcup_{i=1}^n (E_i \setminus E_{i-1})\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

For the second part, let  $A = \bigcap_{i=1}^{\infty} E_i$ .

$$\begin{aligned} \mu(E_1 \setminus A) &= \mu(E_1 \cap A^c) \\ &= \mu\left(E_1 \cap \left(\bigcap_{i=1}^{\infty} E_i\right)^c\right) \\ &= \mu\left(E_1 \cap \bigcup_{i=1}^{\infty} E_i^c\right) \\ &= \mu\left(\bigcup_{i=1}^{\infty} E_1 \cap E_i^c\right) \\ &= \lim_{n \rightarrow \infty} \mu(E_1 \cap E_n^c) \\ &= \lim_{n \rightarrow \infty} \mu(E_1 \setminus E_n) \end{aligned}$$

By finite additivity, we have that

$$\begin{aligned} \mu(E_1 \setminus A) + \mu(A) &= \mu(E_1 \setminus A \sqcup A) \\ &= \mu(E_1) \\ &= \lim_{n \rightarrow \infty} \mu(E_1) \\ &= \lim_{n \rightarrow \infty} \mu(E_1 \setminus E_n \sqcup E_n) \\ &= \lim_{n \rightarrow \infty} (\mu(E_1 \setminus E_n) + \mu(E_n)) \\ &= \lim_{n \rightarrow \infty} \mu(E_1 \setminus E_n) + \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \mu(E_1 \setminus A) + \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

Since  $\mu(E_1 \setminus A) \leq \mu(E_1) < \infty$ , we have  $\mu(A) = \lim_{n \rightarrow \infty} \mu(E_n)$   $\square$

*Remark.* This holds for complex measures as well. However, for the second property, it is essential for  $\mu(E_1) < \infty$ . Indeed, consider the following example:

**Example 2.2.4.** Let  $X = \mathbb{N}, \mathcal{M} = \mathcal{P}(X), \mu$  be the counting measure. Let  $A_n := \{i : i \geq n\}$ . Notice that  $A_1 \supseteq A_2 \supseteq A_3 \dots$  and  $\lim_{n \rightarrow \infty} \mu(A_n) = \infty \neq 0 = \mu(\emptyset)$ . However,  $\bigcap_{n=1}^{\infty} A_n = \emptyset$

### 2.3 Measurable Function

**Definition 2.10.** If  $(X, \mathcal{M}_1), (Y, \mathcal{M}_2)$  are measure spaces, then  $f : X \rightarrow Y$  is a **measurable function** if  $\forall B \in \mathcal{M}_2, f^{-1}(B) \in \mathcal{M}_1$ .

**Definition 2.11.** If  $(Y, \mathcal{T})$  is a topological space, we say a function  $f : X \rightarrow Y$  is **Borel measurable** if it is measurable with respect to  $\mathcal{M}_2 = \text{Bol}_{(Y, \mathcal{T})}$ , the Borel  $\sigma$ -algebra.

**Proposition 2.12.** For  $(B_i) \subseteq Y$ , we have

1.  $f^{-1}(B^c) = (f^{-1}(B))^c$
2.  $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$
3.  $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$ .

**Proposition 2.13.** If  $(Y, \mathcal{T})$  is a topological space, a function  $f : X \rightarrow Y$  is Borel measurable if and only if  $\forall B \in \mathcal{T}$  open,  $f^{-1}(B) \in \mathcal{M}_1$ .

**Proposition 2.14.** For  $f : X \rightarrow \mathbb{R}$ , the following are equal:

1.  $f$  is (Borel) measurable
2.  $\forall a, f^{-1}((-\infty, a))$  is measurable
3.  $\forall a, f^{-1}((-\infty, a])$  is measurable
4.  $\forall a, f^{-1}((a, \infty))$  is measurable
5.  $\forall a, f^{-1}([a, \infty))$  is measurable
6.  $\forall a < b, f^{-1}((a, b))$  is measurable

**Proposition 2.15.** If  $f : X \rightarrow Y, g : Y \rightarrow Z$  are both measurable, then  $f \circ g$  is also measurable.

**Corollary 2.16.** If  $f : X \rightarrow \mathbb{C}$  is measurable, we have  $u = \text{Re}(f), v = \text{Im}(f), z = |f|$  are all measurable.

**Theorem 2.17.** Let  $(X, \mathcal{M})$  is a measurable space, and  $u, v : X \rightarrow \mathbb{R}$  be measurable, and  $(Y, \mathcal{T})$  is a topological space. If  $\Phi : \mathbb{R}^2 \rightarrow Y$  is continuous, then  $h : X \rightarrow Y; x \mapsto \Phi(u(x), v(x))$  is measurable.

*Proof.* Let  $f : X \rightarrow \mathbb{R}^2; x \mapsto (u(x), v(x))$ , it suffices to check that  $f$  is measurable.

Notice that  $\text{Bol}_{\mathbb{R}^2}$  is generated by open rectangles  $R = (a, b) \times (c, d)$ .

Yet  $f^{-1}(R) = u^{-1}(a, b) \cap v^{-1}(c, d)$  is measurable.  $\square$

**Corollary 2.18.** If  $u, v : X \rightarrow \mathbb{R}$  are both measurable, we have  $f := u + iv : X \rightarrow \mathbb{C}$  is also measurable.

*Proof.* Choose  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{C}; (s, t) \mapsto s + it$ .  $\square$

**Corollary 2.19.** If  $f, g : X \rightarrow \mathbb{R}$  are measurable, then we have  $fg, f + g$  are both measurable.

*Proof.* choose  $\Phi : (s, t) \mapsto st$  or  $\Phi : (s, t) \mapsto s + t$ .  $\square$

**Corollary 2.20.** If  $f, g : X \rightarrow \mathbb{C}$  are measurable, then for any  $\alpha \in \mathbb{C}$ , we have  $fg, f + g, \alpha f$  are all measurable.

*Proof.* We write  $f = u + iv, g = w + iz$ . We have that  $u, v, z, w$  are all real-valued and measurable, so are  $u + w, v + z$ , and so are  $(u + w) + i(v + z) = f + g$  and  $(uw - vz) + i(vw + uz) = fg$ .

For  $\alpha f$ , it is obvious since  $B \in \text{Bol}(\mathbb{C}) \iff \alpha B \in \text{Bol}(\mathbb{C})$  for  $\alpha \neq 0$ , and  $0f = 0$  is measurable.  $\square$

**Definition 2.12.** For extended real functions  $f : X \rightarrow [-\infty, \infty]$ , it is measurable if  $\forall \alpha \in \mathbb{R}, f^{-1}((\alpha, \infty]) \in \mathcal{M}$ , or equivalently,  $\forall \alpha \in \mathbb{R}, f^{-1}([-\infty, \alpha)) \in \mathcal{M}$ .

**Proposition 2.21.** If  $(f_n)_{n=1}^\infty$  is a sequence of measurable functions  $X \rightarrow [-\infty, \infty]$ , we have

$$g(x) := \sup_n f_n(x), \quad h(x) := \limsup_{n \rightarrow \infty} f_n(x) = \inf_k \left( \sup_{n \geq k} f_n(x) \right)$$

are also measurable. Similarly for  $\inf$  and  $\liminf$ .

*Proof.* Notice that

$$\begin{aligned} x \in g^{-1}((\alpha, \infty]) &\iff g(x) > \alpha \\ &\iff \exists f_n(x) > \alpha \\ &\iff x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty]), \end{aligned}$$

which is a union of measurable sets. Thus  $g$  is measurable.  $\square$

**Corollary 2.22.** If  $f_n : X \rightarrow [-\infty, \infty]$  or  $f_n : X \rightarrow \mathbb{C}$  are measurable functions, and  $\forall x \in X, f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists, then  $f$  is measurable.

**Corollary 2.23.** If  $f, g : X \rightarrow [-\infty, \infty]$  are both measurable, then  $\max(f, g), \min(f, g)$  are measurable.

**Corollary 2.24.** If  $f : X \rightarrow [-\infty, \infty]$  is measurable, then  $f^+ := \max(f, 0), f^- := \max(-f, 0)$  are both measurable, with  $f = f^+ - f^-$ .

**Proposition 2.25.** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is monotone, then  $f$  is Borel measurable.

*Proof.* Let  $\alpha \in \mathbb{R}$ , we need to show that  $\{x \in \mathbb{R} | f(x) > \alpha\}$  is a Borel set.

We may assume that  $f$  is non-decreasing, if not we take  $f \rightarrow -f$ . If  $\{x \in \mathbb{R} | f(x) > \alpha\} \in \{\emptyset, \mathbb{R}\}$ , we have nothing to prove.

Now if  $\{x \in \mathbb{R} | f(x) > \alpha\} \notin \{\emptyset, \mathbb{R}\}$ , we have that  $\{x \in \mathbb{R} | f(x) \leq \alpha\}$  is not empty and bounded above since  $f$  is increasing. Let  $x_0 := \sup \{x \in \mathbb{R} | f(x) \leq \alpha\}$ . If  $f(x_0) \leq \alpha$ ,  $\{x \in \mathbb{R} | f(x) > \alpha\} = (x_0, \infty)$ , otherwise  $\{x \in \mathbb{R} | f(x) > \alpha\} = [x_0, \infty)$ , both Borel.  $\square$

## 2.4 Simple Functions

**Definition 2.13.** Let  $(X, \mathcal{M})$  be a measurable space, a **characteristic function** for a subset  $E \subseteq X$  is

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

**Definition 2.14.** Let  $(X, \mathcal{M})$  be a measurable space, a function  $\phi : X \rightarrow [-\infty, \infty]$  is **simple** if  $\phi(X)$  is finite.

**Proposition 2.26.** Let  $(X, \mathcal{M})$  be a measurable space, for any simple function  $\phi$  with  $\phi(X) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ , we have

$$\phi = \sum_{i=1}^n \alpha_i \chi_{E_i},$$

where  $E_i = \phi^{-1}(\{\alpha_i\})$  are pairwise disjoint. In this case,  $\phi$  is measurable if and only if  $\forall i, E_i \in \mathcal{M}$ .

**Lemma 2.27.** For any  $\alpha \in \mathbb{R}, n \geq 1$ , we have that

$$\alpha - \frac{1}{2^n} < \frac{\lfloor 2^n \alpha \rfloor}{2^n} \leq \alpha$$

*Proof.*

$$\begin{aligned} \lfloor 2^n \alpha \rfloor &\leq 2^n \alpha < \lfloor 2^n \alpha \rfloor + 1 \\ 2^n \alpha - 1 &< \lfloor 2^n \alpha \rfloor \\ \alpha - \frac{1}{2^n} &< \frac{\lfloor 2^n \alpha \rfloor}{2^n} \\ \frac{\lfloor 2^n \alpha \rfloor}{2^n} &\leq \alpha \end{aligned}$$

□

**Lemma 2.28.** Consider  $\text{id} : [0, \infty) \rightarrow [0, \infty)$ ,  $x \mapsto x$ , then there are simple functions  $s_n : [0, \infty] \rightarrow [0, \infty)$

such that each  $s_n$  is measurable and  $\begin{cases} 0 \leq s_1 \leq s_2 \leq \dots \leq \text{id}, \\ \forall x \in X, \lim_{n \rightarrow \infty} s_n(x) = \text{id}(x), \\ \forall R > 0, s_n \rightarrow \text{id} \text{ uniformly on } [0, R] \end{cases}$

*Proof.* For  $n \geq 1, t \in [0, \infty)$ , let  $s_n(t) := \begin{cases} \frac{\lfloor 2^n t \rfloor}{2^n}, & t \in [0, n] \\ n, & t > n \end{cases}$

Notice that  $s_n$  is simple. It is also measurable since it is monotone.

We also have that  $0 \leq s_1 \leq s_2 \leq \dots \leq f$ , and by squeeze theorem, we have that

$$\lim_{n \rightarrow \infty} s_n(x) = x = \text{id}(x).$$

In addition, we can check that this convergence is uniform on any  $[0, R]$ . □

**Theorem 2.29.** Let  $f : X \rightarrow [0, \infty]$  be measurable, then there are simple functions  $s_n : X \rightarrow [0, \infty]$  such

that each  $s_n$  is measurable and  $\begin{cases} 0 \leq s_1 \leq s_2 \leq \dots \leq f \\ \forall x \in X, \lim_{n \rightarrow \infty} s_n(x) = f(x) \\ \forall R > 0, s_n \rightarrow f \text{ uniformly on } E_R := \{x \in X : f(x) \leq R\}. \end{cases}$

*Proof.* Notice that for any simple function  $s$  and any arbitrary measurable function  $f$ , we have that  $s \circ f$  is simple. Thus it suffices to find  $s'_n$  that approximates  $\text{id} : x \mapsto x$ , which is done by the above lemma.

Let  $s_n := s'_n \circ f$ , they are measurable by result on compositions, and

$$0 \leq s_1 \leq \dots \leq f, \quad \lim_{n \rightarrow \infty} (s_n \circ f)(x) = f(x).$$

□

**Corollary 2.30** (Simple function approximation). Let  $f : X \rightarrow \mathbb{C}$  be measurable, then there are simple functions  $s_n : X \rightarrow [0, \infty)$  such that each  $s_n$  is measurable and

$\begin{cases} 0 \leq |s_1| \leq |s_2| \leq \dots \leq |f| \\ \forall x \in X, \lim_{n \rightarrow \infty} s_n(x) = f(x) \\ \forall R > 0, s_n \rightarrow f \text{ uniformly on } E_R := \{x \in X : |f(x)| \leq R\}. \end{cases}$

**Corollary 2.31.** For  $f, g : X \rightarrow [0, \infty]$  being measurable, we have that  $f \cdot g$  is also measurable.

*Proof.* One can check that for monotone non-decreasing  $(a_n), (b_n) \subseteq [0, \infty)$  with  $a_n \rightarrow a, b_n \rightarrow b$  for  $a, b \in [0, \infty]$ , then  $a_n b_n \rightarrow ab$ .

Approximate  $f$  with simple functions  $s_n$ , and  $g$  with simple functions  $t_n$ , then each of them is measurable, hence so is  $s_n \cdot t_n$ , hence so is  $\lim_{n \rightarrow \infty} s_n t_n = fg$  □

### 3 Integration

#### 3.1 Integration of non-negative functions

**Definition 3.1.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $s : X \rightarrow [0, \infty]$  be a simple measurable function, with  $s(X) = \{a_1, \dots, a_n\}$ , such that  $s = \sum_{i=1}^n a_i \chi_{A_i}$ , where  $A_i := s^{-1}(\{a_i\})$ . For  $A \in \mathcal{M}$ , define the **integral** of  $s$  over  $A$  to be

$$\int_A s d\mu := \sum_{i=1}^n a_i \mu(A_i \cap A).$$

**Definition 3.2.** For  $f : X \rightarrow [0, \infty]$  measurable, the **integral** of  $f$  over  $A \in \mathcal{M}$  is

$$\int_A f d\mu := \sup \int_A s d\mu,$$

where the sup is taken over all measurable simple  $s : X \rightarrow [0, \infty)$  such that  $0 \leq s \leq f$ .

**Proposition 3.1.** Let  $f, g : X \rightarrow [0, \infty]$  be measurable, then

1.  $f \leq g \implies \forall A \in \mathcal{M}, \int_A f d\mu \leq \int_A g d\mu$
2. For any  $A \subseteq B \in \mathcal{M}$ , we have that  $\int_A f d\mu \leq \int_B f d\mu$
3.  $\forall c \in [0, \infty), A \in \mathcal{M}$ , we have that  $\int_A c f d\mu = c \int_A f d\mu$
4. If  $\forall x \in X, f(x) = 0$ , we have that  $\forall A \in \mathcal{M}, \int_A f d\mu = 0$
5. If  $\forall x \in A \in \mathcal{M}, f(x) = 0$ , we have that  $\int_A f d\mu = 0$
6. If  $\mu(A) = 0$  for  $A \in \mathcal{M}$ , we have that  $\int_A f d\mu = 0$
7.  $\int_A f d\mu = \int_X \chi_A f d\mu$

**Proposition 3.2.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $s : X \rightarrow [0, \infty)$  a measurable simple function. Then  $\lambda : \mathcal{M} \rightarrow [0, \infty]$  defined by

$$\lambda(A) := \int_A s d\mu$$

is a measure on  $(X, \mathcal{M})$

*Proof.* Write  $s = \sum_{i=1}^n a_i \chi_{A_i}$ , and let  $C := \bigsqcup_{k=1}^{\infty} C_k$ , then

$$\begin{aligned} \lambda(C) &= \sum_{i=1}^n a_i \mu(A_i \cap C) \\ &= \sum_{i=1}^n a_i \mu(\bigsqcup_{k=1}^{\infty} (A_i \cap C_k)) \\ &= \sum_{i=1}^n a_i \sum_{k=1}^{\infty} \mu(A_i \cap C_k) \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^n a_i \mu(A_i \cap C_k) \\ &= \sum_{k=1}^{\infty} \lambda(C_k) \end{aligned}$$

Thus  $\lambda$  satisfies countable additivity, and in addition  $\lambda(\emptyset) = \sum_{i=1}^n a_i \mu(A_i \cap \emptyset) \xrightarrow{0} 0$ .  $\square$

**Corollary 3.3.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $s : X \rightarrow [0, \infty)$  a measurable simple function, with  $C := \bigsqcup_{k=1}^{\infty} C_k$ . Then we have

$$\int_C s d\mu = \sum_{k=1}^{\infty} \int_{C_k} s d\mu.$$

*Proof.*

$$\begin{aligned} \int_C s d\mu &= \lambda_s(C) \\ &= \lambda_s\left(\bigsqcup_{k=1}^{\infty} C_k\right) \\ &= \sum_{k=1}^{\infty} \lambda_s(C_k) \\ &= \sum_{k=1}^{\infty} \int_{C_k} s d\mu \end{aligned}$$

□

**Proposition 3.4.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $s, t : X \rightarrow [0, \infty)$  both be measurable simple functions, then

$$\int_X (s + t) d\mu = \int_X s d\mu + \int_X t d\mu$$

*Proof.* Write  $s = \sum_{i=1}^n a_i \chi_{A_i}, t = \sum_{j=1}^m b_j \chi_{B_j}$ , and let  $C_{ij} = A_i \cap B_j$ , then  $C_{ij}$  are disjoint, and  $\bigsqcup_{ij} C_{ij} = X$

$$\begin{aligned} \int_{C_{ij}} (s + t) d\mu &= (a_i + b_j) \mu(C_{ij}) \\ &= a_i \mu(C_{ij}) + b_j \mu(C_{ij}) \\ &= \int_{C_{ij}} s d\mu + \int_{C_{ij}} t d\mu \\ \int_X (s + t) d\mu &= \int_{\bigsqcup_{ij} C_{ij}} (s + t) d\mu \\ &= \sum_{ij} \int_{C_{ij}} (s + t) d\mu \\ &= \sum_{ij} \int_{C_{ij}} s d\mu + \sum_{ij} \int_{C_{ij}} t d\mu \\ &= \int_X s d\mu + \int_X t d\mu \end{aligned}$$

□

**Theorem 3.5** (Lebesgue's Monotone Convergence). Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f_n : X \rightarrow [0, \infty]$  be measurable functions with  $0 \leq f_1 \leq f_2 \leq \dots \leq \infty$ . Let  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ , then  $f : X \rightarrow [0, \infty]$  is measurable, and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

*Proof.* Since  $f_n \leq f_{n+1}$ , we have that  $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$ , so by monotone convergence theorem,

$$\alpha := \lim_{n \rightarrow \infty} \int_X f_n d\mu \in [0, \infty]$$

exists.

As a limit of measurable functions,  $f$  is measurable. Also,  $\forall n, \int_X f_n d\mu \leq \int_X f d\mu$ , and thus  $\alpha \leq \int_X f d\mu$ . Consider any  $s : X \rightarrow [0, \infty)$  be simple and measurable with  $0 \leq s \leq f$ , and consider any  $0 < c < 1$ .

For  $n \geq 1$ , let  $A_n := \{x \in X : f_n(x) \geq cs(x)\}$ .

Then  $X = \bigcup_{n=1}^{\infty} A_n$  since  $f_n$  converges point-wise.

In addition,  $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

Also each  $A_n$  is measurable, since  $A_n = \{x \in X : (f_n - cs)(x) \geq 0\} = (f_n - cs)^{-1}([0, \infty])$ , and  $f_n - cs$  is measurable.

Since  $\lambda_s : A \mapsto \int_A s d\mu$  is a measure, so by property of measures,

$$\int_X s d\mu = \lambda_s(X) = \lim_{n \rightarrow \infty} \lambda_s(A_n) = \lim_{n \rightarrow \infty} \int_{A_n} s d\mu.$$

In addition, we have

$$\begin{aligned} \int_X f_n d\mu &\geq \int_{A_n} f_n d\mu \\ &\geq \int_{A_n} cs d\mu \\ &= c \int_{A_n} s d\mu \\ \alpha &= \lim_{n \rightarrow \infty} \int_X f_n d\mu \\ &\geq \lim_{n \rightarrow \infty} c \int_{A_n} s d\mu \\ &= c \int_X s d\mu. \end{aligned}$$

Now take  $c \rightarrow 1$ , we have that  $\alpha \geq \int_X s d\mu$ .

Then take sup of all simple  $s \leq f$ , we have that  $\alpha \geq \int_X f d\mu$ .  $\square$

**Corollary 3.6.** For a measure space  $(X, \mathcal{M}, \mu)$ ,  $A \in \mathcal{M}$ , let  $f_n : X \rightarrow [0, \infty]$  be measurable functions with  $0 \leq f_1 \leq f_2 \leq \dots \leq \infty$ . Let  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ . We can consider the restriction  $(A, \mathcal{M}' := \{B \cap A : B \in \mathcal{M}\}, \mu|_{\mathcal{M}'})$ , and we will have

$$\lim_{n \rightarrow \infty} \int_A f_n d\mu = \int_A f d\mu$$

**Corollary 3.7.** Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f : X \rightarrow [0, \infty]$  be measurable. Let  $s_n : X \rightarrow [0, \infty]$  be any measurable simple functions with  $0 \leq s_1 \leq s_2 \leq \dots \leq \infty$  with  $f(x) = \lim_{n \rightarrow \infty} s_n(x)$ . We have

$$\lim_{n \rightarrow \infty} \int_X s_n d\mu = \int_X f d\mu.$$

**Proposition 3.8** (finite additivity for positive functions). Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f, g : X \rightarrow [0, \infty]$  be measurable functions, then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

*Proof.* Approximate  $f, g$  by simple functions  $s_n, t_n$ , such that  $\lim_{n \rightarrow \infty} s_n(x) = f(x), \lim_{n \rightarrow \infty} t_n(x) = g(x)$  and  $0 \leq s_1 \leq \dots \leq f, 0 \leq t_1 \leq \dots \leq g$ .

Notice that  $0 \leq s_1 + t_1 \leq \dots \leq f + g$ , and  $\lim_{n \rightarrow \infty} (s_n + t_n)(x) = (f + g)(x)$ . Thus

$$\begin{aligned}\int_X (f + g)d\mu &= \lim_{n \rightarrow \infty} \int_X (s_n + t_n)d\mu \\ &= \lim_{n \rightarrow \infty} \left( \int_X s_n d\mu + \int_X st_n d\mu \right) \\ &= \lim_{n \rightarrow \infty} \int_X s_n d\mu + \lim_{n \rightarrow \infty} \int_X t_n d\mu \\ &= \int_X f d\mu + \int_X g d\mu.\end{aligned}$$

□

**Corollary 3.9** (countable additivity for positive functions). *Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f_n : X \rightarrow [0, \infty]$  be measurable functions. Then*

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

is measurable and

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n(x) d\mu.$$

*Proof.* Define  $g_n(x) := \sum_{i=1}^n f_i(x)$ , then  $0 \leq g_1 \leq \dots \leq f$  and  $\lim_{n \rightarrow \infty} g_n = f$ . By previous proposition and induction,

$$\int_X g_n d\mu = \sum_{i=1}^n \int_X f_i d\mu.$$

By LMCT, we have

$$\begin{aligned}\int_X f d\mu &= \int_X \lim_{n \rightarrow \infty} g_n d\mu \\ &= \lim_{n \rightarrow \infty} \int_X g_n d\mu \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_X f_i d\mu \\ &= \sum_{n=1}^{\infty} \int_X f_n d\mu\end{aligned}$$

□

**Theorem 3.10.** *Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $f : X \rightarrow [0, \infty]$  a measurable function. Then  $\lambda : \mathcal{M} \rightarrow [0, \infty]$  defined by*

$$\lambda(A) := \int_A f d\mu$$

*is a measure on  $(X, \mathcal{M})$ . Moreover, for some  $g : X \rightarrow [0, \infty]$  such that  $fg$  is measurable, then*

$$\int_X g d\lambda = \int_X g f d\mu.$$

*Proof.* Let  $A = \bigsqcup_{n=1}^{\infty} A_n$  with  $A_n$  disjoint measurable subsets of  $X$ . We have that  $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$ , and thus

$$\begin{aligned}\lambda(A) &= \int_X \chi_A f d\mu \\ &= \int_X \sum_{n=1}^{\infty} \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \int_X \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \lambda(A_n)\end{aligned}$$

Thus  $\lambda$  is a measure.

In addition, when  $g = \chi_A$  for  $A \in \mathcal{M}$ , we have that  $\int_X g d\lambda = 1 * \lambda(A) = \int_X \chi_A f d\mu = \int_X g f d\mu$ . And thus simple functions, and thus all non-negative measurable functions by LMCT.  $\square$

### 3.2 Integration of complex functions

**Definition 3.3.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $|f| : X \rightarrow [0, \infty)$  a measurable function. Let

$$\mathcal{L}^1(X, \mu) := \left\{ f : X \rightarrow \mathbb{C} \mid \int_X |f| d\mu < \infty \right\}$$

be the **set of Lebesgue integrable functions**. For  $f \in \mathcal{L}^1(X, \mu)$ , we define  $\|f\|_1 := \int_X |f| d\mu$ .

**Definition 3.4.** For  $f = u + iv \in \mathcal{L}^1(X, \mu)$ , where  $u, v : X \rightarrow \mathbb{R}$ , then the integral of  $f$  is defined as

$$\int_X f d\mu := \int_X u^+ d\mu - \int_X u^- d\mu + i \int_X v^+ d\mu - i \int_X v^- d\mu,$$

where  $u^+(x) := \max(u(x), 0)$ ,  $u^-(x) := \max(-u(x), 0)$ , and thus  $u = u^+ - u^-$ . Similar definition for  $v^+, v^-$ .

**Proposition 3.11.** *The above integral is well-defined.*

*Proof.*  $u^+, u^-, v^+, v^-$  are measurable, and  $0 \leq u^+, u^- \leq |u| \leq |f|$ , thus the integral is finite.  $\square$

**Proposition 3.12.** *For  $f = u + iv \in \mathcal{L}^1(X, \mu)$ , where  $u, v : X \rightarrow \mathbb{R}$ , we have*

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu.$$

*Proof.* By definition.  $\square$

**Theorem 3.13.**  $\forall f, g \in \mathcal{L}^1(X, \mu), \alpha \in \mathbb{C}$ , we have that  $\alpha f + g \in L^1$  and

$$\int_X \alpha f + g d\mu = \alpha \int_X f d\mu + \int_X g d\mu$$

*Proof.* Clearly  $\alpha f + g$  is measurable. In addition,

$$\begin{aligned}\int_X |\alpha f + g| d\mu &\leq \int_X |\alpha f| + |g| d\mu \\ &= \int_X |\alpha| |f| d\mu + \int_X |g| d\mu \\ &= |\alpha| \int_X |f| d\mu + \int_X |g| d\mu \\ &< \infty\end{aligned}$$

Now we check the addition: Consider ant  $f = a + ib, g = c + id : X \rightarrow \mathbb{C}$ , such that  $a, b, c, d : X \rightarrow \mathbb{R}$ .

$$\begin{aligned}(a+c)^+ - (a+c)^- &= a+c \\&= (a^+ - a^-) + (c^+ - c^-) \\&= (a^+ + c^+) - (a^- + c^-). \\(a+c)^+ + (a^- + c^-) &= (a+c)^- + (a^+ + c^+),\end{aligned}$$

where both sides of the equality are sums of two non-negative functions. Thus we have

$$\begin{aligned}\int_X (a+c)^+ + (a^- + c^-) d\mu &= \int_X (a+c)^- + (a^+ + c^+) d\mu \\ \int_X (a+c)^+ d\mu + \int_X (a^- + c^-) d\mu &= \int_X (a+c)^- d\mu + \int_X (a^+ + c^+) d\mu \\ \int_X (a+c)^+ d\mu - \int_X (a+c)^- d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\ \int_X (a+c) d\mu &= \int_X (a^+ + c^+) d\mu - \int_X (a^- + c^-) d\mu \\&= \int_X a^+ d\mu + \int_X c^+ d\mu - \int_X a^- d\mu - \int_X c^- d\mu \\&= \left( \int_X a^+ d\mu - \int_X a^- d\mu \right) + \left( \int_X c^+ d\mu - \int_X c^- d\mu \right) \\&= \int_X ad\mu + \int_X cd\mu \\ \int_X (f+g) d\mu &= \int_X (a+c)d\mu + i \int_X (b+d) d\mu \\&= \int_X ad\mu + \int_X cd\mu + i \int_X bd\mu + i \int_X dd\mu \\&= \left( \int_X ad\mu + i \int_X bd\mu \right) + \left( \int_X cd\mu + i \int_X dd\mu \right) \\&= \int_X f d\mu + \int_X g d\mu.\end{aligned}$$

Now we check the scalar multiplication:  $\forall \alpha \geq 0$ , we have  $\int_X \alpha f d\mu = \alpha \int_X f d\mu$  by definition. We can also check for  $\alpha = -1$  and  $\alpha = i$ , and conclude this holds for all  $\alpha \in \mathbb{C}$ .  $\square$

**Theorem 3.14.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $f \in \mathcal{L}^1(X, \mu)$ , then

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$$

*Proof.* Let  $\alpha := \int_X f d\mu \in \mathbb{C}$ , and let  $\beta \in \mathbb{C}, |\beta| = 1$ , such that  $\alpha\beta = |\alpha|$ . Take  $u = \operatorname{Re}(\beta f) : X \rightarrow \mathbb{R}$ , note

$u \leq |\beta f| = |f|$ . Now

$$\begin{aligned}
\left| \int_X f d\mu \right| &= |\alpha| \\
&= \beta \alpha \\
&= \beta \int_X f d\mu \\
&= \int_X \beta f d\mu \\
&= \int_X u d\mu \\
&\leq \int_X |f| d\mu
\end{aligned}$$

□

### 3.3 Lebesgue Dominated Convergence Theorem

**Lemma 3.15** (Fatou's). *Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f_n : X \rightarrow [0, \infty]$  be measurable functions. Then*

$$\int_X (\liminf_n f_n) d\mu \leq \liminf_n \int_X f_n d\mu$$

*Proof.* Let  $g_n(x) := \inf_{i \geq n} f_i(x)$ , then  $\liminf_n f_n(x) = \lim_{n \rightarrow \infty} g_n(x)$ .

Also,  $g_n \leq f_n$ , so  $\int_X g_n d\mu \leq \int_X f_n d\mu$ ,  $\forall n \geq 1$ .

Note  $g_n$  is measurable, and  $0 \leq g_1 \leq g_2 \leq \dots$ .

By LMCT,

$$\lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X (\lim_{n \rightarrow \infty} g_n) d\mu = \int_X (\liminf_n f_n) d\mu.$$

Since the left hand side converges,

$$\int_X (\liminf_n f_n) d\mu = \lim_{n \rightarrow \infty} \int_X g_n d\mu = \liminf_n \int_X g_n d\mu \leq \liminf_n \int_X f_n d\mu.$$

□

**Theorem 3.16.** (*Lebesgue Dominated Convergence*)

Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f_n : X \rightarrow \mathbb{C}$  be measurable functions such that  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$  exists  $\forall x \in X$ . If there is  $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$ , such that  $\forall x \in X, \forall n \in \mathbb{N}$ ,  $|f_n(x)| \leq g(x)$ , then  $f \in \mathcal{L}^1(X, \mu)$ , and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \quad \lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = 0.$$

*Proof.* Firstly,  $\forall n \in \mathbb{N}, x \in X$ ,  $|f_n(x)| \leq g(x)$  implies that  $\forall x \in X$ ,  $|f(x)| \leq g(x)$ , and thus  $\int_X |f| d\mu \leq \int_X g d\mu$ . This shows that  $f \in \mathcal{L}^1(X, \mu)$ .

Notice that  $|f_n - f| \leq |f_n| + |f| \leq 2g$ . Thus  $2g - |f_n - f| \geq 0$ . By Fatou's Lemma, we have that

$\int_X (\liminf(2g - |f_n - f|))d\mu \leq \liminf \int_X 2g - |f_n - f|d\mu$ . Thus

$$\begin{aligned} \int_X 2gd\mu &= \int_X (2g - \liminf(|f_n - f|))d\mu \\ &= \int_X (\liminf(2g - |f_n - f|))d\mu \\ &\leq \liminf \int_X 2g - |f_n - f|d\mu \\ &= \int_X 2gd\mu + \liminf(- \int_X |f_n - f|d\mu) \\ &= \int_X 2gd\mu - \limsup \int_X |f_n - f|d\mu \\ 0 &\leq - \limsup \int_X |f_n - f|d\mu \end{aligned}$$

Thus  $0 \leq \liminf \int_X |f_n - f|d\mu \leq \limsup \int_X |f_n - f|d\mu \leq 0$ , and thus  $\lim_{n \rightarrow \infty} \int_X |f_n - f|d\mu = 0$ . Finally,

$$\begin{aligned} \left| \lim_{n \rightarrow \infty} \int_X f_n d\mu - \int_X f d\mu \right| &= \left| \lim_{n \rightarrow \infty} \int_X (f_n - f) d\mu \right| \\ &\leq \lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu \\ &= 0 \end{aligned}$$

□

### 3.4 Almost Everywhere

**Definition 3.5.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $A \in \mathcal{M}$ , and  $P = \{p(x)\}_{x \in A}$  be a family of logical statements, then we say the property  $P$  holds or is true  $\mu$ -everywhere on  $A$ , if  $\exists N \in \mathcal{M}$ , such that  $\mu(N) = 0$  and  $\forall x \in A \setminus N$ ,  $p(x) = True$ .

**Definition 3.6.** For measurable functions  $f, g : X \rightarrow Y$ , we say that  $f = g$   $\mu$ -almost everywhere if

$$\mu(\{x \in X | f(x) \neq g(x)\}) = 0.$$

*Remark.* For some  $A \in \mathcal{M}$ , we have that  $\mu(A \cap N) \leq \mu(N) = 0$ , and thus  $\int_{A \cap N} (f - g) d\mu = 0$ . Thus  $\int_A (f - g) d\mu = \int_{A \cap N} (f - g) d\mu + \int_{A \setminus N} (f - g) d\mu = 0$ .

**Definition 3.7.** Let  $(f_n)_{n=1}^{\infty}$ , we say  $f_n \rightarrow f$  a.e. if  $f_n(x) \rightarrow f(x)$  for  $\mu$ -a.e.  $x \in X$ .

**Proposition 3.17.** Let  $(X, \mathcal{M}, \mu)$  be a measure space.

1. If  $f : X \rightarrow [0, \infty]$  is measurable, we have  $f = 0$   $\mu$ -a.e.  $\iff \int_E f d\mu = 0 \ \forall E \in \mathcal{M}$ .
2. If  $f \in L^1(X, \mu)$  we have  $f = 0$   $\mu$ -a.e.  $\iff \int_E f d\mu = 0 \ \forall E \in \mathcal{M}$ .
3. If  $f \in L^1(X, \mu)$ , and  $|\int_X f d\mu| = \int_X |f| d\mu$ , there exist must a constant  $\alpha$  such that  $\alpha f = |f|$  almost everywhere.

*Proof.* 1. Let  $N = \{x \in X : f(x) > 0\}$ .

Suppose  $f = 0$   $\mu$ -a.e., then  $\mu(N) = 0$ .

We have

$$\int_X f d\mu = \int_{X \setminus N} f d\mu + \int_N f d\mu = \int_{X \setminus N} 0 d\mu + 0 = 0.$$

Thus  $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$ .

Now suppose  $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$ .

Let  $A_n := \{x \in X : f(x) > \frac{1}{n}\}$ , then we have

$$\frac{1}{n} \mu(A_n) = \int_{A_n} \frac{1}{n} d\mu \leq \int_{A_n} f d\mu = 0.$$

Thus  $\mu(A_n) = 0$ .

Notice that  $N = \bigcup_{n=1}^{\infty} A_n, A_1 \subseteq A_2 \subseteq \dots$ , thus  $\mu(N) = \lim_{n \rightarrow \infty} \mu(A_n) = 0$ .

2. Suppose  $f = 0$   $\mu$ -a.e., we have that  $|f| = 0$   $\mu$ -a.e., thus  $|\int_E f d\mu| \leq \int_E |f| d\mu = 0$ .

Now suppose  $\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$ .

Let  $f = u + iv$ , then we have  $\int_E u d\mu = \int_E v d\mu = 0 \ \forall E \in \mathcal{M}$ .

Let  $u = u^+ - u^-$ , and  $E = \{x \in X : u(x) \geq 0\}$ .

$$\begin{aligned} \int_X u^+ d\mu &= \int_{X \setminus E} u^+ d\mu + \int_E u^+ d\mu \\ &= \int_{X \setminus E} 0 d\mu + \int_E u d\mu \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Thus  $u^+ = 0$   $\mu$ a.e..

Similarly for  $u^-$ , and thus  $u = 0$   $\mu$ -a.e..

Similarly for  $v$ , and thus  $f = 0$   $\mu$ -a.e..

□

**Theorem 3.18** (Lebesgue Dominated Convergence - almost everywhere). *Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f_n : X \rightarrow \mathbb{C}$  be measurable functions, defined  $\mu$ -almost everywhere on  $X$ , such that  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$  is defined  $\mu$ -almost everywhere for  $x \in X$ . If there is  $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$ , such that for  $\mu$ -almost everywhere  $x \in X, \forall n \in \mathbb{N}$ ,  $|f_n(x)| \leq g(x)$ , then  $f \in \mathcal{L}^1(X, \mu)$ , and*

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = 0.$$

*Proof.* Let  $N_n$  denote the zero measure set where  $f_n$  is not defined. Let  $N'$  denote the zero measure set where  $f$  is not defined. Then

$$N := N' \cup \{x \in X : \exists n \in \mathbb{N}, \text{ such that } |f_n(x) - g(x)|\} \cup \bigcup_{n=1}^{\infty} N_n$$

is measurable and has zero measure.

Define

$$h_n(x) := \begin{cases} f_n(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \quad h(x) := \begin{cases} f(x) & x \in X \setminus N, \\ 0 & x \in N \end{cases}, \quad g'(x) := \begin{cases} g(x) & x \in X \setminus N, \\ 0 & x \in N. \end{cases}$$

It is clear  $\forall x \in X, h_n(x) \rightarrow h(x)$  point-wise, and dominated by  $g'(x)$ .

Since  $g = g'\mu$ -a.e. and thus  $g' \in \mathcal{L}^1(X, \mu)$ , by LDCT, we have

$$\lim_{n \rightarrow \infty} \int_X |f - f_n| d\mu = \lim_{n \rightarrow \infty} \int_X |g - g_n| d\mu = 0.$$

□

**Theorem 3.19** (countable additivity). *Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f_n : X \rightarrow \mathbb{C}$  be measurable functions, defined  $\mu$ -almost everywhere on  $X$ , such that  $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$ . We have that  $f(x) := \sum_{n=1}^{\infty} f_n(x)$  exists  $\mu$ -almost everywhere for  $x \in X$ , and that  $f \in \mathcal{L}^1(X, \mu)$ , and that*

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

*Proof.* For each  $n$ , let  $D_n \subseteq X$  be the domain of  $f_n$ , then by assumption  $\mu(X \setminus D_n) = 0$ .

Let  $D := \bigcap_{n=1}^{\infty} D_n$ ,  $g(x) := \sum_{n=1}^{\infty} |f_n(x)|$ . Note that  $\mu(D^c) = \mu((\bigcap_{n=1}^{\infty} D_n)^c) = \mu(\bigcup_{n=1}^{\infty} D_n^c) = 0$ .

Thus  $g : X \rightarrow [0, \infty]$  is defined almost everywhere by Monotone Convergence Theorem.

By countable additivity of positive functions and assumption,

$$\int_X g d\mu = \sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty,$$

so  $g \in \mathcal{L}^1$ .

Let  $A := \{x \in D : g(x) < \infty\}$ , then we have  $\mu(A^c) = 0$ .

By definition,  $f(x) = \sum_{n=1}^{\infty} f_n(x)$  absolutely on  $A$ . Thus  $f \in \mathcal{L}^1(A, \mathcal{M}|_A, \mu|_{\mathcal{M}|_A})$ .

Let  $h_n = \sum_{i=1}^n f_i$  on  $A$ , then  $|h_n| \leq \sum_{i=1}^n |f_i| \leq g$ . Also, we have that  $h_n(x) \rightarrow f(x)$  for any  $x \in A$ , then by LDCT and linearity, we have

$$\int_A f d\mu = \lim_{n \rightarrow \infty} \int_A h_n d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_A f_i d\mu = \sum_{n=1}^{\infty} \int_A f_n d\mu.$$

Since  $\mu(A^c) = 0$ , we have that  $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$ . □

### 3.5 Complete Measure

**Theorem 3.20.** *Let  $(X, \mathcal{M}, \mu)$  be a measure space, let*

$$\mathcal{M}^* := \{A \subseteq X : \exists B, C \in \mathcal{M}, \text{ such that } B \subseteq A \subseteq C, \mu(C \setminus B) = 0\}.$$

Define  $\mu^*(A) = \mu(B) = \mu(C)$ , then  $\mathcal{M}^*$  is a  $\sigma$ -algebra,  $\mu^*$  is a measure, and  $(X, \mathcal{M}^*, \mu^*)$  is a measure space.

*Proof.*  $X \in \mathcal{M}$ , and  $X \subseteq X \subseteq X$  and  $\mu(X \setminus X) = 0$ , thus  $X \in \mathcal{M}^*$ .

Let  $A \in \mathcal{M}^*$ , then there are  $B, C \in \mathcal{M}$  such that  $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$ . Thus  $B^c \supseteq A^c \supseteq C^c$ , and  $B^c, C^c \in \mathcal{M}$ . In addition,  $\mu(B^c \setminus C^c) = \mu(B^c \cap C) = \mu(C \setminus B) = 0$ . Thus  $A^c \in \mathcal{M}^*$ .

Let  $A_n \in \mathcal{M}^*$  and  $A = \bigcup_n A_n$ , and  $B_n, C_n \in \mathcal{M}$  such that  $B_n \subseteq A_n \subseteq C_n, \mu(C_n \setminus B_n) = 0$ . Let  $B = \bigcup_n B_n, C = \bigcup_n C_n \in \mathcal{M}$ , thus  $B \subseteq A \subseteq C$ . Now  $\mu(C \setminus B) = \mu(\bigcup_n C_n \setminus B) \leq \mu(\bigcup_n C_n \setminus B_n) \leq \sum_n \mu(C_n \setminus b_n) = 0$ .

Thus  $\mathcal{M}^*$  is a  $\sigma$ -algebra.

Now suppose  $B, B', C, C' \in \mathcal{M}$ , and  $A \in \mathcal{M}^*$ , with  $B \subseteq A \subseteq C, \mu(C \setminus B) = 0$ , and  $B' \subseteq A \subseteq C', \mu(C' \setminus B') = 0$ .

Thus  $B \setminus B' \subseteq A \setminus B' \subseteq C' \setminus B'$ , and thus  $\mu(B \setminus B') \leq \mu(C' \setminus B') = 0$ . Thus  $\mu(B) = \mu(B \cap B') + \mu(B \setminus B') = \mu(B \cap B')$ . Similarly, we can show that  $\mu(B') = \mu(B \cap B')$ . Thus  $\mu^* : \mathcal{M}^* \rightarrow [0, \infty]$  is well-defined.

Consider  $A_n$  a sequence of disjoint sets in  $\mathcal{M}^*$ , and  $B_n, C_n \in \mathcal{M}$  as above. We have  $\mu^*(A) = \mu(B) = \mu(\bigsqcup_n b_n) = \sum_n \mu(B_n) = \sum_n \mu(A_n)$ . □

**Corollary 3.21.**  *$(X, \mathcal{M}^*, \mu^*)$  has the property that if  $N \in \mathcal{M}^*$  has  $\mu(N) = 0$ , we always have*

$$\forall A \subseteq N, A \in \mathcal{M}^*, \mu^*(A) = 0.$$

*Proof.* Notice that  $\forall A \subseteq N$ , we have  $\mu(N) = \mu(\emptyset) = 0$ , with  $\emptyset \subseteq A \subseteq N$ , so  $A \in \mathcal{M}^*, \mu^*(A) = 0$ . □

**Definition 3.8.**  *$(X, \mathcal{M}^*, \mu^*)$  defined above is called the **completion** of  $(X, \mathcal{M}, \mu)$ . In addition, we say  $(X, \mathcal{M}, \mu)$  is **complete** if  $(X, \mathcal{M}, \mu) = (X, \mathcal{M}^*, \mu^*)$*

*Remark.* If there is some  $A \in \mathcal{M}$  such that  $\mu(A^c) = 0$ , then for any measurable  $f : A \rightarrow Y$ , we can extend it to  $X$  by  $\forall x \in A^c, f(x) := 0$ . Furthermore, if  $(X, \mathcal{M}, \mu)$  is complete, we can extend  $f$  to whatever value we want. One can check that  $f : X \rightarrow Y$  is measurable, and the integral  $\int_X f d\mu$  does not depend on the extension.

**Proposition 3.22.** *If  $(X, \mathcal{M}, \mu)$  is a complete measure, we always have that property P holds  $\mu$ -a.e. iff*

$$\mu(\{x \in A : p(x) = \text{False}\}) = 0.$$

*Proof.* If P holds  $\mu$ -a.e., there is  $\exists N \in \mathcal{M}$ , such that  $\mu(N) = 0$  and  $\forall x \in A \setminus N, p(x) = \text{True}$ . Since  $\{x \in A : p(x) = \text{False}\} \subseteq A \setminus (A \setminus N) = N$ , we have  $\mu(\{x \in A : p(x) = \text{False}\}) = 0$ .

On the other hand, if  $\mu(\{x \in A : p(x) = \text{False}\}) = 0$ , we can just let  $N := \mu(\{x \in A : p(x) = \text{False}\})$ . Notice  $\mu(N) = 0$ , and  $\forall x \in A \setminus N, p(x) = \text{True}$ .  $\square$

**Proposition 3.23.** *Let  $\mu$  be a complete measure on  $(X, \mathcal{M})$ , suppose that  $f$  is measurable, and  $g = f$ , a.e., then  $g$  is also measurable. Moreover, if  $(f_n)$  is a sequence of measurable functions, and  $f_n \rightarrow f$ ,  $\mu$ -a.e., we always have that  $f$  is also measurable.*

*Proof.* Suppose  $f$  is measurable, and we consider  $D := \{x : X | f(x) \neq g(x)\}, \mu(D) = 0$ .

Now let  $B \subseteq \mathbb{R}$  be a Borel set, we need to show that  $\{x \in X | g(x) \in B\} \in \mathcal{M}$ .

Write  $\{x \in X | g(x) \in B\} = (\{x \in X | g(x) \in B\} \cap D) \cup (\{x \in X | g(x) \in B\} \setminus D)$ .

Since  $\mu$  is complete, we have that  $\{x \in X | g(x) \in B\} \cap D \in \mathcal{M}$  and has measure zero. Since  $f$  is measurable, we have that  $f^{-1}(B) = \{x \in X | f(x) \in B\} \supseteq \{x \in X | f(x) = g(x) \in B\} = \{x \in X | g(x) \in B\} \setminus D$  is measurable. Since  $\mu$  is complete, we have that  $\{x \in X | g(x) \in B\} \setminus D$  is measurable.

Thus  $\{x \in X | g(x) \in B\} \in \mathcal{M}$  is measurable.

For the second part, consider  $g = \limsup_{n \rightarrow \infty} f_n$ .  $\square$

## 4 Construction of Measure

### 4.1 Caratheodory Theorem

**Definition 4.1.** Let  $X$  be a non-empty set, an **outer measure** on  $X$  is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  such that

1.  $\mu^*(\emptyset) = 0$
2. Monotone:  $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$
3. Countable subadditive: For any  $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$ , we have that  $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$

**Proposition 4.1.** Let  $\mathcal{C} \subseteq \mathcal{P}(X)$  with  $\emptyset, X \in \mathcal{C}$ . Let  $\tilde{\mu} : \mathcal{C} \rightarrow [0, \infty]$  be a function such that  $\tilde{\mu}(\emptyset) = 0$ . Define  $\mu^* : X \rightarrow [0, \infty]$  by  $\mu^*(A) := \inf \{\sum_{i=1}^\infty \tilde{\mu}(C_i) : C_i \in \mathcal{C}, A \subseteq \bigcup_{i=1}^\infty C_i\}$ . Then  $\mu^*$  is an outer measure.

*Proof.* Clearly  $\mu^*(\emptyset) = 0$  by taking  $\forall i, C_i = \emptyset$ .

In addition,  $A \subseteq B \subseteq \bigcup_{i=1}^\infty C_i$  for any cover for  $B$ , and thus  $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$ .

Given any  $(A_n)_{n=1}^\infty \subseteq \mathcal{P}(X)$ . If  $\sum_{n=1}^\infty \mu^*(A_n) = \infty$ , then  $\mu^*(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu^*(A_n)$  trivially.

Now assume that  $\sum_{n=1}^\infty \mu^*(A_n) < \infty$ .

Consider any  $\epsilon > 0$ .

For each  $n \geq 0$ , choose  $(C_{n,i})_{i=1}^\infty \subset \mathcal{C}$ , such that  $A_n \subseteq \bigcup_{i=1}^\infty C_{n,i}$  and  $\mu^*(A_n) \leq \sum_{i=1}^\infty \tilde{\mu}(C_{n,i}) < \mu^*(A_n) + \frac{\epsilon}{2^n}$ . Thus  $\bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty C_{n,i}$ , so by construction of the outer measure,

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^\infty A_n\right) &\leq \sum_{n=1}^\infty \sum_{i=1}^\infty \tilde{\mu}(C_{n,i}) \\ &\leq \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) \\ &\leq \sum_{n=1}^\infty \mu^*(A_n) + \epsilon. \end{aligned}$$

Taking  $\epsilon \rightarrow 0$ , we have  $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$   $\square$

*Remark.* Notice that if  $X = \mathbb{R}$ , and we take  $\mathcal{C} := \{(a, b] : a < b \in \mathbb{R}\}$  to be the collection of finite half open intervals, and  $\mu((a, b])$  to be the length of the interval  $b - a$ , then the outer measure is the Lebesgue outer measure.

**Definition 4.2.** For an outer measure  $\mu^*$ , we say  $A \subseteq X$  is  $\mu^*$ -measurable, or satisfies the **Caratheodoy condition** if

$$\forall E \subseteq X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

**Lemma 4.2.** Let  $\mu^*$  be an outer measure on  $X$ , and  $\mathcal{M}$  be the  $\mu^*$ -measurable subsets of  $X$ , then  $\mathcal{M}$  is an algebra, and  $\mu := \mu^*|_{\mathcal{M}}$  has finite additivity.

*Proof.* To check  $A \in \mathcal{M}$ , it suffices to check for  $\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(E \cap A^c)$ , since  $\mu^*(E) \leq \mu^*(A \cap E) + \mu^*(E \cap A^c)$  holds by subadditivity of  $\mu^*$ .

When  $\mu^*(E) = \infty$ , this holds trivially, and thus it suffices to only check for  $\mu^*(E) < \infty$ .

- Clearly  $\emptyset \in \mathcal{M}$ .
- $A \in \mathcal{M} \implies A^c \in \mathcal{M}$  since the condition is symmetric.
- Now consider  $A, B \in \mathcal{M}$ . For any  $E \subseteq X$ , we have

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*((E \cap A) \cup (E \cap A^c \cap B)) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c). \end{aligned}$$

Thus  $A \cup B \in \mathcal{M}$ .

Thus  $\mathcal{M}$  is an algebra.

To show finite additivity: Let  $(A_i)_{i=1}^n$  be disjoint in  $\mathcal{M}$ , we will use induction on  $n$ .

Clearly it is true for  $n = 1$ .

Now suppose it holds for  $n$ , let  $B = \bigsqcup_{i=1}^n A_i$ . Since  $\mathcal{M}$  is an algebra, we have  $B \in \mathcal{M}$ . For any  $E \subseteq X$ ,

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\ &= \mu^*\left(\bigsqcup_{i=1}^n (E \cap A_i)\right) + \mu^*(E \cap B^c) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B^c). \end{aligned}$$

Taking  $E = \bigsqcup_{i=1}^{n+1} A_i$ , we have

$$\begin{aligned} \mu^*\left(\bigsqcup_{i=1}^{n+1} A_i\right) &= \sum_{i=1}^n \mu^*\left(\left(\bigsqcup_{i=1}^{n+1} A_i\right) \cap A_i\right) + \mu^*\left(\bigsqcup_{i=1}^{n+1} A_i \cap B^c\right) \\ &= \sum_{i=1}^n \mu^*(A_i) + \mu^*(A_{n+1}) \\ &= \sum_{i=1}^{n+1} \mu^*(A_i). \end{aligned}$$

By induction, we have finite additivity for any  $n \geq 1$ .  $\square$

**Theorem 4.3** (Caratheodoy). Let  $\mu^*$  be an outer measure on  $X$ , and  $\mathcal{M}$  be the  $\mu^*$ -measurable subsets of  $X$ , then  $\mathcal{M}$  is a  $\sigma$ -algebra, and  $\mu := \mu^*|_{\mathcal{M}}$  is a complete measure.

*Proof.* Consider any  $\{A_i\} \subset \mathcal{M}$ ,  $B := \bigcup_{i=1}^{\infty} A_i$ . By taking  $\tilde{A}_n := A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$  we can WLOG assume  $A_n$  are pair-wise disjoint, and  $B = \bigcup_{i=1}^{\infty} A_i$ . For any  $E \in X$ , we have  $\forall n \geq 1, \bigcup_{i=1}^n A_i \in \mathcal{M}$ , and thus

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \left(\bigsqcup_{i=1}^n A_i\right)\right) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^n A_i\right)^c\right) \\ &= \mu^*\left(\bigsqcup_{i=1}^n (E \cap A_i)\right) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^n A_i\right)^c\right) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^n A_i\right)^c\right) \\ &\geq \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigsqcup_{i=1}^{\infty} A_i\right)^c\right) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B^c).\end{aligned}$$

Taking  $n \rightarrow \infty$ , we have

$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c) \\ &\geq \mu^*\left(\bigsqcup_{i=1}^{\infty} (E \cap A_i)\right) + \mu^*(E \cap B^c) \\ &\geq \mu^*\left(E \cap \bigsqcup_{i=1}^{\infty} A_i\right) + \mu^*(E \cap B^c) \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\ &\geq \mu^*((E \cap B) \cup (E \cap B^c)) \\ &= \mu^*(E).\end{aligned}$$

Thus  $B \in \mathcal{M}$ , and thus  $\mathcal{M}$  is a  $\sigma$ -algebra.

In addition, taking  $E = B$ , we have

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B \cap A_i) + \mu^*(B \cap B^c) = \sum_{i=1}^{\infty} \mu^*(A_i) + \mu^*(\emptyset) = \sum_{i=1}^{\infty} \mu^*(A_i),$$

which shows countable additivity, and thus  $\mu^*|_{\mathcal{M}}$  is a measure.

To show completeness, suppose  $A \subseteq X$  such that  $\mu^*(A) = 0$ , then for any  $E \subseteq X$ , we have

$$\begin{aligned}\mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(A) + \mu^*(E) \\ &= \mu^*(E).\end{aligned}$$

Thus we have  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ , and thus  $A \in \mathcal{M}$ .  $\square$

## 4.2 Premeasures

**Definition 4.3.** Recall an algebra of subsets of a set  $X$  is a family of subsets that is closed under complements, finite unions, and finite intersections, and contains the empty set.

**Definition 4.4.** A premeasure on an algebra of sets  $\mathcal{A} \subseteq \mathcal{P}(X)$  is a function  $\tilde{\mu} : \mathcal{A} \rightarrow [0, \infty]$ , such that  $\tilde{\mu}$  is countably additive. Namely, if  $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$  are disjoint, and  $\bigsqcup_{i=1}^{\infty} A_i \subseteq \mathcal{A}$ , then we have

$$\tilde{\mu}\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \tilde{\mu}(A_i).$$

*Remark.* If  $\mathcal{A}$  is a  $\sigma$ -algebra, a premeasure on  $\mathcal{A}$  is always a measure.

**Theorem 4.4.** Let  $\mathcal{A}$  be an algebra of subsets of  $X$ , and  $\tilde{\mu} : \mathcal{A} \rightarrow [0, \infty]$  be a premeasure. Apply Caratheodory to the outer measure  $\mu^*$  gives a complete measure space  $(X, \mathcal{M}, \mu)$ , such that  $\mathcal{A} \subseteq \mathcal{M}$ , and  $\mu|_{\mathcal{A}} = \tilde{\mu}$ .

*Proof.* Choose any  $A \in \mathcal{A}$  and  $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$ , such that  $A \subseteq \bigsqcup_{i=1}^{\infty} A_i$ .

Let  $B_i = (A \cap A_i) \setminus \bigsqcup_{j=1}^{i-1} A_j$ .

Notice that  $B_i \in \mathcal{A}$ , and are pairwise disjoint, and  $A = \bigsqcup_{i=1}^{\infty} B_i$ ,

Since  $\tilde{\mu}$  is a premeasure,  $\tilde{\mu}(A) = \sum_{i=1}^{\infty} \tilde{\mu}(B_i) \leq \sum_{i=1}^{\infty} \tilde{\mu}(A_i)$ .

Now  $\mu^*(A) := \inf \{\sum_{i=1}^{\infty} \tilde{\mu}(A_i) : A_i \in \mathcal{A}, A \subseteq \bigsqcup_{i=1}^{\infty} A_i\} \leq \tilde{\mu}(A)$ .

Since above holds for any  $\bigsqcup_{i=1}^{\infty} A_i \supseteq A$ , we can see that  $\mu^*(A) \geq \tilde{\mu}(A)$ , which forces  $\mu^*(A) = \tilde{\mu}(A)$ .

Now it remains to show  $A \in \mathcal{M}$ , which is the same as  $A$  is  $\mu^*$ -measurable.

Choose  $E \subseteq X$  with  $\mu^*(E) < \infty$  and  $\epsilon > 0$ .

There are  $(E_i)_{i=1}^{\infty} \subseteq \mathcal{A}$ , such that  $E \subseteq \bigsqcup_{i=1}^{\infty} E_i$ , and  $\sum_{i=1}^{\infty} \tilde{\mu}(E_i) < \mu^*(E) + \epsilon$ .

Then  $E \cap A \subseteq \bigsqcup_{i=1}^{\infty} E_i \cap A$ , and  $E \cap A^c \subseteq \bigsqcup_{i=1}^{\infty} E_i \cap A^c$ . Thus

$$\begin{aligned} \mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \sum_{i=1}^{\infty} \mu^*(E_i \cap A) + \sum_{i=1}^{\infty} \mu^*(E_i \cap A^c) \\ &= \sum_{i=1}^{\infty} \tilde{\mu}(E_i \cap A) + \sum_{i=1}^{\infty} \tilde{\mu}(E_i \cap A^c) \\ &= \sum_{i=1}^{\infty} \tilde{\mu}(E_i) \\ &< \mu^*(E) + \epsilon \end{aligned}$$

Now take  $\epsilon \rightarrow 0$ , we have that  $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E)$ .

However, by subadditivity of  $\mu^*$ , we have that  $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ , and thus  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$

We have shown that  $\tilde{\mu} = \mu^*|_{\mathcal{A}}$ , but we also know that  $\mu = \mu^*|_{\mathcal{M}}$ , and  $\mathcal{A} \subseteq \mathcal{M}$ , so  $\mu|_{\mathcal{A}} = \tilde{\mu}$ .  $\square$

**Definition 4.5.** A premeasure  $\tilde{\mu} : \mathcal{A} \rightarrow [0, \infty]$  on an algebra  $\mathcal{A}$  for  $X$  is  $\sigma$ -finite if there are  $(A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$ , such that  $\tilde{\mu}(A_i) < \infty$  and  $\bigsqcup_{i=1}^{\infty} A_i = X$ .

**Proposition 4.5.** Let  $\mathcal{A}$  be an algebra of sets on  $X$ . Let  $\tilde{\mu} : \mathcal{A} \rightarrow [0, \infty]$  be a premeasure, with the corresponding complete measure space  $(X, \mathcal{M}, \mu)$  as in the above theorem. Suppose  $(X, \mathcal{N}, \nu)$  is a measure space with  $\mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{M}$  and  $\nu|_{\mathcal{A}} = \tilde{\mu}$ . Then if  $\tilde{\mu}$  is  $\sigma$ -finite, we have that

$$\nu = \mu|_{\mathcal{N}},$$

so  $\mu|_{\mathcal{N}}$  is the unique extension of  $\tilde{\mu}$  to a measure on  $\mathcal{N}$ .

### 4.3 Lebesgue-Stieltjes Measures

**Definition 4.6.** Let  $\mu$  be a Borel measure on  $\mathbb{R}$ , such that  $\mu(\mathcal{K}) < \infty$  for any compact  $\mathcal{K} \subseteq \mathbb{R}$ . Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by  $F(x) = \begin{cases} \mu((0, x]) & \text{if } x \geq 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$

**Proposition 4.6.**  $F$  is monotone non-decreasing. i.e. If  $b \geq a$ , then  $F(b) - F(a) \geq 0$ .

*Proof.* For  $0 < a < b$ , we have that  $\mu((a, b]) = \mu((0, b] \setminus (0, a]) = \mu((0, b]) - \mu((0, a]) = F(b) - F(a)$ . For  $0 \geq b > a$ , we have that  $\mu((a, b]) = \mu((a, 0] \setminus (b, 0]) = \mu((a, 0]) - \mu((b, 0]) = -F(a) - (-F(b)) = F(b) - F(a)$ . Similarly, we can check for  $a < 0 \leq b$ .  $\square$

**Proposition 4.7.**  $F$  is right continuous.

*Proof.* Fix  $x \geq 0 \in \mathbb{R}$ , and choose any sequence  $(x_n) \subseteq \mathbb{R}$  such that  $x_n \geq x_{n+1}$  and  $x_n \rightarrow x$ . Since  $\mu$  is a measure and  $\mu((0, x_1]) \leq \mu([0, x_1]) < \infty$ , we have

$$\begin{aligned} F(x) &= \mu((0, x]) \\ &= \lim_{n \rightarrow \infty} \mu((0, x_n]) \\ &= \lim_{n \rightarrow \infty} F(x_n). \end{aligned}$$

Similar proof for  $x < 0 \in \mathbb{R}$ .  $\square$

**Example 4.3.1.** If  $\mu = \delta_c, \delta_c(A) = \begin{cases} 0 & \text{if } c \in A \\ 1 & \text{if } c \notin A \end{cases}$ , then  $F$  is the (translated) Heaviside function.

**Example 4.3.2.** If  $\mu$  is the Lebesgue measure, then  $F$  is the identity function  $F(x) = x$ .

Now given a right-continuous increasing function  $F$ , we want to construct a measure.

**Proposition 4.8.** Let  $\mathcal{A}$  be the collection of sets consisting of all the finite disjoint unions of half-open intervals  $(a, b], -\infty \leq a \leq b \leq \infty$ . Then  $\mathcal{A}$  is an algebra of sets.

*Proof.* Firstly notice that for any interval  $(a, b] \in \mathcal{A}$ , we have that  $(a, b]^c = [-\infty, a] \cup (b, \infty] \in \mathcal{A}$ . Also, any finite union of such disjoint unions can be written as a disjoint union.  $(a, b] \cup (c, d] = (a, c] \cup (c, b] \cup (b, d]$  for  $c < b$ . We can show any finite union by induction.  $\square$

**Definition 4.7.** Let  $\mathcal{A}$  be the algebra of sets consisting of all the finite disjoint unions of half-open intervals  $(a, b], -\infty \leq a \leq b \leq \infty$ . Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a right-continuous monotone non-decreasing function, we can extend  $F$  to  $[-\infty, \infty]$  by  $F(\pm\infty) := \lim_{x \rightarrow \pm\infty} F(x)$ , which exists by MCT. Now define  $\tilde{\mu}_F : \mathcal{A} \rightarrow [0, \infty]$  to be

$$\tilde{\mu}_F\left(\bigsqcup_{i=1}^n (a_i, b_i]\right) := \sum_{i=1}^n F(b_i) - F(a_i)$$

**Lemma 4.9.**  $\tilde{\mu}_F$  is a pre-measure

*Proof.* We firstly show that  $\tilde{\mu}_F$  is well defined.

Consider  $I_i = (a_i, b_i]$  and  $I = (a, b] = \bigsqcup_1^n I_i$

By reordering, we can WLOG assume that  $a = a_1 < b_1 = a_2 < b_2 = \dots < b_{n-1} = a_n < b_n = b$ .

Let  $a_{n+1} := b_n = b$ , we have that

$$\begin{aligned} \tilde{\mu}_F\left(\bigsqcup_{i=1}^n (a_i, b_i]\right) &= \sum_{i=1}^n F(b_i) - F(a_i) \\ &= \sum_{i=1}^{\infty} F(a_{i+1}) - F(a_i) \\ &= F(a_{n+1}) - F(a_1) \\ &= F(b) - F(a) \\ &= \tilde{\mu}_F(I). \end{aligned}$$

Thus  $\tilde{\mu}_F(I)$  does not depend on the decomposition of  $I$ . This extends to finite disjoint unions of half-open intervals. Hence  $\tilde{\mu}_F$  is well-defined.

Monotone follows from the fact that  $F$  is increasing.

Consider pair-wise disjoint  $(A_i)_{i=1}^{\infty} \in \mathcal{A}$ , and  $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$ , we want to show  $\tilde{\mu}_F(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \tilde{\mu}_F(A_i)$ .

We first assume that each  $A_i = (a_i, b_i]$  and  $\bigsqcup_1^\infty A_i = (a, b]$   
Notice that  $\forall n \in \mathbb{N}$ , we have that  $\bigsqcup_{i=1}^n (a_i, b_i] \in \mathcal{A}$ , and thus  $(a, b] \setminus \bigsqcup_{i=1}^n (a_i, b_i] \in \mathcal{A}$ .

$$\begin{aligned}\tilde{\mu}_F((a, b]) &= \tilde{\mu}_F\left(\bigsqcup_{i=1}^n (a_i, b_i]\right) + \tilde{\mu}_F((a, b] \setminus \bigsqcup_{i=1}^n (a_i, b_i]) \\ &= \sum_{i=1}^n \tilde{\mu}_F((a_i, b_i]) + \tilde{\mu}_F((a, b] \setminus \bigsqcup_{i=1}^n (a_i, b_i]) \\ &\geq \sum_{i=1}^n \tilde{\mu}_F((a_i, b_i])\end{aligned}$$

Thus  $\tilde{\mu}_F((a, b]) \geq \sum_{i=1}^\infty \tilde{\mu}_F((a_i, b_i]).$

For the other direction, fix  $\epsilon > 0$ , by right continuity,  $\exists \delta > 0$ , such that  $F(a + \delta) < F(a) + \epsilon, a + \delta < b$ . Now suppose  $b \neq \infty$ , then  $\exists \delta_i > 0, F(b_i + \delta_i) < F(b_i) + 2^{-i}\epsilon$ .

Thus  $[a_i + \delta, b_i] \subseteq (a_i, b_i + \delta_i)$ , and thus  $\{(a_i, b_i + \delta_i)\}$  is an open cover for  $[a + \delta, b] = \bigcup_{i=1}^\infty [a_i + \delta, b_i]$ . Since the closed interval is compact, there is a finite sub-cover  $\{(a_{i_j}, b_{i_j} + \delta_{i_j})\}_{j=1}^n$ . Then

$$\begin{aligned}\sum_{j=1}^n (F(b_{i_j} + \delta_{i_j}) - F(a_{i_j})) &= \sum_{j=1}^n \tilde{\mu}_F((a_{i_j}, b_{i_j} + \delta_{i_j})) \\ &\geq \tilde{\mu}_F((a + \delta, b]) \\ &\geq F(b) - F(a + \delta)\end{aligned}$$

since  $\tilde{\mu}_F$  is monotone. Hence

$$\begin{aligned}\sum_{i=1}^\infty \tilde{\mu}_F((a_i, b_i]) &= \sum_{i=1}^\infty (F(b_i) - F(a_i)) \\ &\geq \sum_{j=1}^n (F(b_{i_j}) - F(a_{i_j})) \\ &\geq \sum_{j=1}^n (F(b_{i_j} + \delta_{i_j}) - 2^{-i_j}\epsilon - F(a_{i_j})) \\ &\geq F(b) - F(a + \delta) - \epsilon \\ &\geq F(b) - F(a) - 2\epsilon \\ &= \tilde{\mu}_F((a, b]) - 2\epsilon.\end{aligned}$$

Take  $\epsilon \rightarrow 0$ , we have  $\sum_{i=1}^\infty \tilde{\mu}_F((a_i, b_i]) \geq \tilde{\mu}_F((a, b])$

When  $b = \infty$ , we have that  $\forall N \geq a, \sum_{i=1}^\infty \tilde{\mu}_F((a_i, b_i]) \geq \tilde{\mu}_F((a, N]) = F(N) - F(a)$ .

Hence  $\sum_{i=1}^\infty \tilde{\mu}_F((a_i, b_i]) \geq F(b) - F(a) = \lim_{N \rightarrow \infty} F(N) - F(a)$

Thus we have shown that  $\sum_{i=1}^\infty \tilde{\mu}_F((a_i, b_i]) = \tilde{\mu}_F((a, b])$

If  $A = \bigsqcup_1^m (c_i, d_i]$ , we can use finite additivity and the previous case.  $\square$

**Theorem 4.10.** *If  $F : \mathbb{R} \rightarrow \mathbb{R}$  is monotone non-decreasing and right-continuous, then there is a complete measure space  $(\mathbb{R}, \mathcal{M}, \mu_F)$ , which extends  $\tilde{\mu}_F$  as  $\mu_F|_{\mathcal{A}} = \tilde{\mu}_F$ , the  $\sigma$ -algebra  $\mathcal{M}$  contains  $Bor(\mathbb{R})$  and  $\mu_F|_{Bor(\mathbb{R})}$  is the unique extension of  $\tilde{\mu}_F$ , i.e.  $\mu_F((a, b]) = F(b) - F(a)$ . Conversely, given a Borel measure  $\mu$  on  $\mathbb{R}$ , such that  $\forall K$  compact,  $\mu(K) < \infty$ , there is a (up to constant) unique non-decreasing right-continuous  $F$  with  $\mu = \mu_F|_{B_{\mathbb{R}}}$ .*

*Proof.* By the previous lemma,  $\tilde{\mu}_F$  is a premeasure, so applying Caratheodory gives a complete measure space  $(\mathbb{R}, \mathcal{M}, \mu_F)$ . We have seen that the  $\sigma$ -algebra generated by  $\mathcal{A}$  is  $Bor(\mathbb{R})$ , so  $Bor(\mathbb{R}) \subseteq \mathcal{M}$ . The uniqueness follows from the fact that  $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+2], \tilde{\mu}_F((n, n+2]) = F(n+2) - F(n) < \infty$  and thus  $\tilde{\mu}_F$  is  $\sigma$ -finite.

Conversely, let  $F$  be the function defined at the beginning of this section. Then we know that  $\mu = \mu_F|_{B_{\mathbb{R}}}$

since  $\mu$  is  $\sigma$ -finite and agree with the premeasure  $\tilde{\mu}_F$  on the algebra  $\mathcal{M}$  of all finite disjoint unions of half open interval.

If  $G : \mathbb{R} \rightarrow \mathbb{R}$  is another monotone non-decreasing right-continuous function, then  $\mu_F = \mu_G \implies \tilde{\mu}_F((a, b]) = \mu_G((a, b]) \implies F(b) - F(a) = G(b) - G(a)$  for any  $a < b$ . Thus  $\forall a \in \mathbb{R}, F(x) - G(x) = c := F(0) - G(0)$ , which is a constant.  $\square$

**Example 4.3.3.** The Lebesgue measure is got by taking  $F(x) = x$ .

**Example 4.3.4.** The Dirac measure

$$\delta_c(A) := \begin{cases} 1 & c \in A \\ 0 & c \notin A \end{cases}$$

is got by taking

$$F(x) = H_c(x) = \begin{cases} 1 & x \geq c \\ 0 & x < c \end{cases},$$

the Heaviside function.

**Definition 4.8.** A  $\mu$  be a Borel measure on  $\mathbb{R}$ , such that  $\mu(\mathcal{K}) < \infty$  for any compact  $\mathcal{K} \subseteq \mathbb{R}$ , is called a **Lebesgue-Stieltjes measure**. For any  $F : \mathbb{R} \rightarrow \mathbb{R}$  that is monotone non-decreasing and right-continuous, and any Borel measure  $\mu$  such that  $\mu = \mu_F|_{\text{Bol}_{\mathbb{R}}}$ , we call  $\mu$  the **Lebesgue-Stieltjes measure corresponding to  $F$** .

**Proposition 4.11.** Let  $\mu$  be a Lebesgue-Stieltjes measure corresponding to some  $F : \mathbb{R} \rightarrow \mathbb{R}$ , then  $\forall a \in \mathbb{R}$ ,

$$\mu(\{a\}) = \mu\left(\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a]\right) = \lim_{n \rightarrow \infty} \mu((a - \frac{1}{n}, a]) = F(a) - \lim_{n \rightarrow \infty} F(a - \frac{1}{n}) = F(a) - F(a^-).$$

Thus  $\mu(\{a\}) > 0$  if and only if  $F$  has a jump discontinuity at  $a$ , since every discontinuity of a monotone non-decreasing function is a jump discontinuity.

**Corollary 4.12.** If  $F(x) = x$  is the identity function, every countable set has measure 0, by subadditivity and that  $\forall a \in \mathbb{R}, \mu(\{a\}) = 0$

**Proposition 4.13.** A monotone non-decreasing function  $F$  can have at most countably many discontinuities.

*Proof.* Choose countably many disjoint points  $\{c_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ . Define a measure  $\mu := \sum_{n \geq 1} \frac{1}{2^n} \delta_{c_n}$ . This is a Borel measure with  $\mu(\mathcal{K}) < \infty$  for any compact  $\mathcal{K} \subseteq \mathbb{R}$ . Thus  $\mu$  is a Lebesgue-Stieltjes measure. Note  $\mu(\{c_n\}) = \frac{1}{2^n} > 0$ , thus each  $c_n$  is a jump discontinuity for the corresponding  $F$ . Thus  $F$  has countably many discontinuities.

In fact, no such  $F$  can have uncountably many discontinuities.  $\square$

**Theorem 4.14.** Lebesgue measure  $(\mathbb{R}, \mathcal{L}, \lambda)$  is translation-invariant, meaning

$$\forall A \in \mathcal{L}, s \in \mathbb{R}, \lambda(A + s) = \lambda(A).$$

Also,

$$\forall s > 0, A \in \mathcal{L}, \lambda(sA) = s\lambda(A).$$

*Proof.* If  $A \subseteq \mathbb{R}$  is open, then so is  $A + s$ . Similarly for closed sets. Hence for  $A \in \mathcal{B}_{\mathbb{R}}, A + s \in \mathcal{B}_{\mathbb{R}}$ . Define a new measure  $\lambda_s$  on  $\mathcal{B}_{\mathbb{R}}$  by  $\lambda_s(A) = \lambda(A + s)$ . Note that  $\lambda$  and  $\lambda_s$  correspond to the functions

$$F(x) = \begin{cases} \lambda((0, x]) & \text{if } x \geq 0 \\ -\lambda((x, 0]) & \text{if } x < 0 \end{cases},$$

$$G(x) = \begin{cases} \lambda_s((0, x]) & \text{if } x \geq 0 \\ -\lambda_s((x, 0]) & \text{if } x < 0 \end{cases}.$$

Yet  $\lambda((0, x]) = \lambda((s, x + s]) = \lambda_s((0, x + s])$ , and thus  $F = G$ . Thus  $\lambda_s|_{\mathcal{B}_{\mathbb{R}}} = \lambda|_{\mathcal{B}_{\mathbb{R}}}$ . By uniqueness for  $\sigma$ -finite Caratheodoy Theorem, we have that they extends to  $\lambda = \lambda_s$ .  $\square$

**Definition 4.9.** A point  $c \in \mathbb{R}$  with  $\mu(\{c\}) \neq 0$  is called an **atom** of  $\mu$ .

**Corollary 4.15.** Lebesgue-Stieltjes measures can have at most countably many atoms.

**Definition 4.10.** Let  $X$  be a topological space, then a  $\mathcal{G}_\delta$  set is a countable intersection of open subsets of  $X$ , and a  $\mathcal{F}_\sigma$  set is a countable union of closed subsets.

*Remark.*  $\mathcal{G}_\delta$  sets and  $\mathcal{F}_\sigma$  sets are Borel sets.

**Theorem 4.16** (regularity). Let  $\mu$  be a Lebesgue-Stieltjes measure with outer measure  $\mu_F^*$ , and  $E \subseteq \mathbb{R}$ , the following are equal:

- (1)  $E$  is  $\mu$ -measurable
- (2)  $\forall \epsilon > 0$ , there is some open  $O \supseteq E$ ,  $\mu_F^*(O \setminus E) < \epsilon$  (Outer regularity)
- (3)  $\forall \epsilon > 0$ , there is some closed  $C \subseteq E$ ,  $\mu_F^*(E \setminus C) < \epsilon$  (Inner regularity)
- (4) There is a  $\mathcal{G}_\delta$  set  $G \supseteq E$ ,  $\mu_F^*(G \setminus E) = 0$
- (5) There is a  $\mathcal{F}_\sigma$  set  $F \subseteq E$ ,  $\mu_F^*(E \setminus F) = 0$

*Proof.* Notice that  $E$  is  $\mu$ -measurable means that

$$\forall A \subseteq \mathbb{R}, \mu_F^*(A) = \mu_F^*(E \cup A) + \mu_F^*(E^c \cup A).$$

1. (1) implies (2):

If  $E$  is  $\mu$ -measurable,

$$\begin{aligned} \mu(E) &= \mu_F^*(E) \\ &= \inf_{B \supseteq E} \mu_F(B), \end{aligned}$$

where  $B = \bigcup_{i=1}^{\infty} (a_i, b_i] \supseteq E$ .

Firstly, assume that  $E$  is bounded, we have  $\mu_F^*(B) < \mu(E) + \frac{\epsilon}{2}$  for some  $B = \bigcup_{i=1}^{\infty} (a_i, b_i] \supseteq E$ .

Since  $F$  is right-continuous, we have that  $\forall i, \exists c_i > b_i$ , such that  $F(c_i) < F(b_i) + \frac{\epsilon}{2^{i+1}}$ .

Let  $O := \bigcup_i (a_i, c_i) \supseteq B \supseteq E$ .

Since  $E$  is measurable, we have that  $\mu_F^*(B) = \mu_F^*(B \cap E) + \mu_F^*(B \setminus E) = \mu(E) + \mu_F^*(B \setminus E)$ , thus  $\mu_F^*(B \setminus E) < \frac{\epsilon}{2}$

$$\begin{aligned} \mu_F^*(O \setminus B) &= \mu_F^*\left(\bigcup_i (a_i, c_i) \cap B^c\right) \\ &\leq \sum_i \mu_F^*((a_i, c_i) \cap B^c) \\ &\leq \sum_i \mu_F^*((b_i, c_i)) \\ &\leq \sum_i \mu_F^*((b_i, c_i]) \\ &= \sum_i F(c_i) - F(b_i) \\ &< \sum_i \frac{\epsilon}{2^{i+1}} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

$$\begin{aligned} \mu_F^*(O \setminus E) &\leq \mu_F^*(O \cap E^c \cap B) + \mu_F^*(O \cap E^c \setminus B) \\ &= \mu_F^*(B \setminus E) + \mu_F^*(O \setminus B) \\ &< \epsilon. \end{aligned}$$

This proves the bounded case.

If  $E$  is not bounded, we let  $E_n = E \cap (n-1, n]$ ,  $n \in \mathbb{Z}$ , each is bounded, and we can have open  $O_n \supseteq E_n$ ,  $\mu_F(O_n \setminus E_n) < \frac{\epsilon}{2^{2|n|+1}}$ , and take  $O = \bigcup_{n \in \mathbb{Z}} O_n \supseteq E$ .

2. (2) implies (4):

For each  $n \geq 1$ , take open  $O_n \supseteq E$ ,  $\mu_F(O_n \setminus E) < \frac{1}{n}$ , and WLOG, take  $O_n = O_n \cap O_{n-1}$  so that  $O_n \supseteq O_{n-1}$ .

Take  $G := \bigcap_{n=1}^{\infty} O_n$ , which is a  $\mathcal{G}_\delta$  set.

We have that  $\forall n \geq 1$ ,  $\mu_F^*(G \setminus E) \leq \mu_F^*(O_n \setminus E) < \frac{1}{n}$ .

Thus  $\mu_F^*(G \setminus E) = 0$ .

3. (4) implies (1):

$G \setminus E$  is measurable since it is a null set, and  $\mu$  is complete.  $G$  is also measurable, thus  $E = G \setminus (G \setminus E)$  is also measurable.

4. (1) implies (3):

$E$  is  $\mu$ -measurable, so is  $E^c$ .

By (2), there is some open  $O \supseteq E^c$ , such that  $\mu_F^*(O \setminus E^c) < \epsilon$ .

Notice that  $C := O^c$  is closed, and  $C \subseteq E$ , and

$$\begin{aligned}\mu_F^*(E \setminus C) &= \mu_F^*(E \cap C^c) \\ &= \mu_F^*((E^c)^c \cap O) \\ &= \mu_F^*(O \setminus E^c) \\ &< \epsilon.\end{aligned}$$

5. (3) implies (5)

For each  $n \geq 1$ , take closed  $C_n \subseteq E$ ,  $\mu_F(E \setminus C_n) < \frac{1}{n}$ , and WLOG, take  $C_n = C_n \cap C_{n-1}$  so that  $C_n \supseteq C_{n-1}$ .

Take  $F := \bigcup_{n=1}^{\infty} C_n$ , which is a  $\mathcal{F}_\sigma$  set.

We have that  $\forall n \geq 1$ ,  $\mu_F^*(E \setminus F) \leq \mu_F^*(E \setminus C_n) < \frac{1}{n}$ .

Thus  $\mu_F^*(E \setminus F) = 0$ .

6. (5) implies (1)

$E \setminus F$  is a measurable set.  $F$  is also a measurable set, and thus so is  $E = (E \setminus F) \cup F$ .

□

**Corollary 4.17.** *Let  $\mu$  be a Lebesgue-Stieltjes measure, and  $A$  be  $\mu$ -measurable, we have*

$$\mu(A) = \inf \{\mu(O) : O \supseteq A \text{ is open}\} = \sup \{\mu(C) : C \subseteq A \text{ is compact}\}$$

*Proof.* The first equality is (2).

For second equality, if  $A$  is bounded, and  $C \subseteq A$  is closed, then it is compact. We can use (3) to prove it. If  $A$  is not bounded, let  $A_n := A \cap [-n, n]$  for each  $N \geq 1$ . Thus

$$\mu(A) = \sup_{n \geq 1} \mu(A_n) = \sup_{n \geq 1} \sup_{C \subseteq A_n \text{ is compact}} \mu(C).$$

□

## 4.4 Littlewood's Three Principles

Recall Littlewood's Three Principles for Lebesgue Measure:

**Theorem 4.18.** *Littlewood's first Principle (regularity)*

*Every measurable set is almost a finite union of intervals.*

**Theorem 4.19.** *Littlewood's second Principle (Lusin's)*

*Every measurable function is almost continuous.*

**Theorem 4.20.** *Littlewood's third Principle (Egorov's)*

*A point-wise convergent sequence of measurable functions is almost uniformly convergent.*

**Theorem 4.21** (Egorov's). *Let  $(X, \mathcal{M}, \mu)$  be a finite measure space. Suppose  $f_n : X \rightarrow \mathbb{C}$  is a sequence of measurable functions such that  $f_n(x) \rightarrow f(x)$   $\mu$ -almost everywhere. Then  $\forall \epsilon > 0, \exists A \in \mathcal{M}$ , such that  $\mu(X \setminus A) < \epsilon$ , and  $f_n \rightarrow f$  uniformly on  $A$ .*

*Proof.*  $f_n \rightarrow f$  uniformly on  $A$  means  $\forall m \in \mathbb{N}^+, \exists N_m \geq 1$ , such that

$$\forall x \in A, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m}.$$

Let  $A_{mN} := \{x \in X : \forall n \geq N, |f_n(x) - f(x)| < \frac{1}{m}\} = \bigcap_{n \geq N} \{x \in X : |f_n(x) - f(x)| < \frac{1}{m}\}$ , which is intersection of preimages of  $(-\frac{2}{m}, \frac{1}{m})$  of measurable functions  $f_n - f$ , thus measurable.

Note  $A_{m1} \subseteq A_{m2} \subseteq \dots$ , and  $\bigcup_{n \geq 1} A_{m,n} = X \setminus N$  for some  $N \in \mathcal{M}, \mu(N) = 0$  since  $f_n \rightarrow f$   $\mu$ -a.e..

$$\begin{aligned} \mu(X) &= \mu(X \setminus N) \\ &= \mu\left(\bigcup_{n \geq 1} A_{mn}\right) \\ &= \lim_{n \rightarrow \infty} \mu(A_{mn}). \end{aligned}$$

Since  $\mu(X) < \infty$ , there is  $N_m \geq 1$  such that  $\mu(A_{m,N_m}) > \mu(X) - \frac{\epsilon}{2^m}$  for any  $\epsilon > 0$ .

Thus  $\mu(X \setminus A_{m,N_m}) < \frac{\epsilon}{2^m}$ .

Letting  $E := \bigcap_{m \geq 1} A_{m,N_m}$ , we have that

$$\begin{aligned} \mu(X \setminus E) &= \mu\left(\bigcup_{m \geq 1} (X \setminus A_{m,N_m})\right) \\ &\leq \sum_{m \geq 1} \mu(X \setminus A_{m,N_m}) \\ &< \epsilon. \end{aligned}$$

In addition,

$$\begin{aligned} E &= \bigcap_{m \geq 1} A_{m,N_m} \\ &= \left\{x \in X : \forall m \geq 1, \forall n \geq N_m, |f_n(x) - f(x)| < \frac{1}{m}\right\}. \end{aligned}$$

Thus  $f_n \rightarrow f$  uniformly on  $E$ . □

**Theorem 4.22** (Lusin's). *Let  $f : [a, b] \rightarrow \mathbb{C}$  be a Lebesgue-Stieltjes measurable function. For any  $\epsilon > 0$ , there is a continuous function  $g : [a, b] \rightarrow \mathbb{C}$  such that*

$$\mu(\{x \in [a, b] : f(x) \neq g(x)\}) < \epsilon.$$

*Proof.* Consider a simple function  $s := \sum_{i=1}^m \alpha_i \chi_{E_i}$ , where  $\alpha_i \in \mathbb{C}$ , and  $E_i$  are disjoint and Lebesgue measurable.

Notice that by the regularity theorem, for any  $\delta > 0$ , there are closed sets  $A_i \subseteq E_i$ , such that  $\mu(E_i \setminus A_i) < \frac{\delta}{m}$  for each  $i$ . Thus,  $\mu(\bigcup_{i=1}^m E_i \setminus A_i) = \mu((\bigcup_{i=1}^m E_i) \setminus (\bigcup_{i=1}^m A_i)) = \mu([a, b] \setminus \mathcal{K}) < \delta$ , for  $\mathcal{K} := \bigcup_{i=1}^m A_i$ .

Notice that  $\mathcal{K}$  is closed (thus compact since  $[a, b]$  is bounded), and  $s|_{\mathcal{K}}$  is continuous since  $s$  is locally constant. Indeed,  $\forall x \in \mathcal{K}$ , there is unique  $A_i$  such that  $x \in A_i$ .

Suppose for contradiction that  $\forall \delta_0 > 0$ , there is some  $y \in (x - \delta_0, x + \delta_0) \cap A_j$  for some  $j \neq i$ . Let  $\mathcal{K}' := \bigcup_{j=1, j \neq i}^m A_j$ , which is closed. Now we have a sequence  $y_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap \mathcal{K}'$ . Notice that  $y_n \rightarrow x$ , and since  $\mathcal{K}'$  is closed,  $x \in \mathcal{K}'$ , which is a contradiction.

Thus  $\exists \delta_0 > 0$ , such that  $(x - \delta_0, x + \delta_0) \cap \mathcal{K} \subseteq A_i$ ; namely,  $s$  is constant on  $(x - \delta_0, x + \delta_0) \cap \mathcal{K}$ . Thus,  $\forall \epsilon_0 > 0, y \in \mathcal{K}$ , such that  $|y - x| < \delta_0, |s(x) - s(y)| = 0 < \epsilon_0$ , which shows  $s$  is continuous around  $x$ .

Now given any measurable  $f$ , we can choose simple functions  $s_n : [a, b] \rightarrow \mathbb{C}$  converging point-wise to  $f$ . For each  $n$ , construct  $\mathcal{K}_n$  as above such that  $s_n|_{\mathcal{K}_n}$  is continuous and  $\mu([a, b] \setminus \mathcal{K}_n) < \frac{\epsilon}{2^{n+1}}$ .

Let  $\mathcal{K}_0 = \bigcap_{n \geq 1} \mathcal{K}_n$ , which is compact. For all  $n$ , we have that  $s_n|_{\mathcal{K}_n}$  is continuous.

In addition,  $\mu([a, b] \setminus \mathcal{K}_0) \leq \sum_{n=1}^{\infty} \mu([a, b] \setminus \mathcal{K}_n) < \epsilon/2$ .

By Egorov's Theorem, there is a measurable  $E \subseteq \mathcal{K}_0$ , such that  $\mu(\mathcal{K}_0 \setminus E) < \epsilon/4$  and  $s_n \rightarrow f$  uniformly on  $E$ .

Applying the regularity theorem again, there is a compact  $\mathcal{K} \subseteq E$  such that  $\mu(E \setminus \mathcal{K}) < \epsilon/4$ . Notice that  $s_n \rightarrow f$  uniformly on  $\mathcal{K}$ . Thus  $f|_{\mathcal{K}}$  is continuous.

Also,  $\mu([a, b] \setminus \mathcal{K}) \leq \mu([a, b] \setminus \mathcal{K}_0) + \mu(\mathcal{K}_0 \setminus E) + \mu(E \setminus \mathcal{K}) = \epsilon$ .

By Tietze's Theorem, we can extend  $f|_{\mathcal{K}}$  to some continuous  $g : [a, b] \rightarrow \mathbb{C}$ . We thus have

$$\mu(\{x \in [a, b] : f(x) \neq g(x)\}) \leq \mu([a, b] \setminus \mathcal{K}) < \epsilon$$

□

## 5 Borel Measures on Topological Spaces

### 5.1 Topological Spaces

**Definition 5.1.** Let  $X \neq \emptyset$  be a set. A **topology** on  $X$  is a collection  $\mathcal{T} \subseteq \mathcal{P}(X) := 2^X$  = power set of  $X$  satisfying

1.  $\emptyset, X \in \mathcal{T}$ ,
2.  $\mathcal{T}$  is closed under arbitrary union; namely,  $\forall \{A_\alpha\}_{\alpha \in K} \subseteq \mathcal{T}, \bigcup_{\alpha \in K} A_\alpha \in \mathcal{T}$ , and
3.  $\mathcal{T}$  is closed under finite intersection; namely,  $\forall \{A_i\}_{i=1}^n \subseteq \mathcal{T}, \bigcap_{i=1}^n A_i \in \mathcal{T}$ .

Also,  $(X, \mathcal{T})$  is a **topological space** if  $\mathcal{T}$  is a topology on  $X$ .

**Definition 5.2.** Let  $(X, \mathcal{T})$  be a topological space, then we say  $U \subseteq X$  is **open** if  $U \in \mathcal{T}$ . We say  $E \subseteq X$  is **closed** if  $E^c \in \mathcal{T}$  is open.

**Definition 5.3.** For  $E \in X$ , the **closure** of  $E$  is

$$\bar{E} = \bigcap_{F \supseteq E: F \text{ is closed}} F.$$

**Definition 5.4.** A set  $K \subseteq X$  is **compact** if every open cover of  $K$  has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha \in A} \text{ be open, } K \subseteq \bigcup_{\alpha \in A} U_\alpha \implies \exists n \in \mathbb{N}, \alpha_1, \dots, \alpha_n, \text{ such that } K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

**Definition 5.5.** An (open) neighborhood of  $x \in X$  is some

$$U_x \in \mathcal{T}, \text{ such that } x \in U_x.$$

**Definition 5.6.**  $X$  is **Hausdorff** if

$$\forall x \neq y \in X, \exists U_x, U_y \text{ open neighborhoods for } x, y, \text{ such that } U_x \cap U_y = \emptyset.$$

**Example 5.1.1.** Every metric space is Hausdorff.

**Definition 5.7.**  $X$  is **locally compact** if  $\forall x \in X$ , there is a neighborhood  $U_x$  such that  $\overline{U_x}$  is compact.

**Example 5.1.2.**  $\mathbb{R}^n$  are locally compact by Heinz-Borel theorem.

**Proposition 5.1.** A Banach space  $(X, \|\cdot\|)$  is locally compact iff  $\dim(X) < \infty$ .

**Theorem 5.2.** Let  $(X, \mathcal{T})$  be a topological space,

1. Suppose  $K$  is compact, then  $\forall F \subseteq K$  that is closed,  $F$  is also compact.

2. If  $X$  is Hausdorff, for any compact  $K \subseteq X$ ,  $x \in X \setminus K$ ,  $\exists$  open neighborhood  $U$  of  $x$ , and open  $W \supset K$ , such that  $W \cap U = \emptyset$ .

*Proof.* 1. Let  $(U_\alpha)_{\alpha \in A}$  be an open cover for  $F$ .

Since  $F$  is closed, then  $F^c$  is open. Thus  $\{F^c\} \cup \{U_\alpha\}_{\alpha \in A}$  is an open cover for  $K$ .

Thus there are  $U_{\alpha_1}, \dots, U_{\alpha_n}$ , such that  $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$ . Thus  $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$  has a finite subcover.

2. Consider any  $y \in K$ , there is some open neighborhoods  $U_y \ni x, W_y \ni y$ , such that  $U_y \cap W_y = \emptyset$ .

Since  $K \subseteq \bigcup_{y \in K} W_y$  is compact, we have  $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$  for some  $y_1, \dots, y_n$ .

Let  $U = \bigcap_{i=1}^n U_{y_i}$ , we have  $x \in U, K \subseteq W, U \cap W = \emptyset$  as required.  $\square$

**Corollary 5.3.** Let  $(X, \mathcal{T})$  be a Hausdorff topological space, then any compact set  $K$  is closed. In addition, for any closed  $F \subseteq X$ , we have  $F \cap K$  is compact.

*Proof.* Suppose for contradiction that  $K$  is not closed, then there is some  $y \in \bar{K} \setminus K$ . Thus we can find open neighborhood  $U$  of  $x$ , and open  $W \supset K$ , such that  $W \cap U = \emptyset$ . Now  $K \subset \bar{K} \setminus U \subsetneq \bar{K}$  is closed, which is a contradiction.

Since  $K$  is closed, so is  $F \cap K \subseteq K$ , and thus it is compact.  $\square$

**Lemma 5.4.** Let  $(X, \mathcal{T})$  be a Hausdorff topological space, and  $(K_\alpha)_{\alpha \in A}$  be a collections of compact sets such that

$$\bigcap_{\alpha \in A} K_\alpha = \emptyset.$$

We must have  $\alpha_1, \dots, \alpha_n \in A$ , such that

$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

*Proof.* Fix  $\alpha_1 \in A$ , then  $K_{\alpha_0} \subseteq \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$  is compact and has an open cover.

Thus there must be  $\alpha_2, \dots, \alpha_n \in A$ , such that  $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = (\bigcap_{i=2}^n K_{\alpha_i})^c$ .

Thus  $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$ .  $\square$

**Theorem 5.5.** Let  $X$  be a Locally Compact Hausdorff space, and let  $K \subseteq U \subseteq X$  be such that  $K$  is compact, and  $U$  is open. Then there exists some open set  $V$  such that  $\bar{V}$  is compact, and

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

*Proof.* Since  $X$  is a Locally Compact Hausdorff space, there are  $V_1, \dots, V_n$ , each with  $\bar{V}_i$  be compact, such that  $K \subseteq \bigcup_{i=1}^n V_i =: G$ . Note that  $\bar{G}$  is compact, and  $G$  is open.

If  $U = X$ , then  $G \subseteq U$ , and we are done.

Otherwise, let  $C := X \setminus U$  be non-empty and closed.

Consider any  $y \in C$ , we know that  $y \notin K$ . Since  $X$  is Hausdorff, we can find open  $W_y \supset K$ , and  $U_y \ni y$ , such that  $W_y \cap U_y = \emptyset$ . Then  $W_y \subseteq U_y^c$ , and thus  $\bar{W}_y \subseteq U_y^c$ , since  $U_y^c$  is closed. Yet  $y \notin U_y^c$ , thus  $y \notin \bar{W}_y$ .

Now consider the family  $\{C \cap \bar{W}_y \cap \bar{G}\}_{y \in C}$ . Notice that each  $C \cap \bar{W}_y \cap \bar{G}$  is compact, since  $C, \bar{W}_y$  are closed, and  $\bar{G}$  is compact.

Yet  $\bigcap_{y \in C} (C \cap \bar{W}_y \cap \bar{G}) = \emptyset$ .

Thus  $\exists y_1, \dots, y_n \in C$ , such that  $\bigcap_{i=1}^n (C \cap \bar{W}_{y_i} \cap \bar{G}) = \emptyset$ .

Now let  $V := G \cap \bigcap_{i=1}^n W_{y_i}$ .

Clearly  $V$  is open, and  $K \subseteq V$ .

In addition,  $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W}_{y_i}$ , yet the intersection of righthand side and  $C$  is empty, thus contained in  $C^c = U$ .  $\square$

## 5.2 Compactly Supported Functions

**Definition 5.8.** Let  $C(X)$  be the collection of functions  $f : X \rightarrow \mathbb{C}$  that are continuous.

**Proposition 5.6.**  $C(X)$  is a  $\mathbb{C}$  vector space, and also an Algebra over  $\mathbb{C}$ . It also admits a partial order by  $f \geq g \iff \forall x \in X, f(x) \geq g(x)$ .

**Definition 5.9.** For  $f \in C(X)$ , the **support** of it is

$$\text{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

**Definition 5.10.** The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \text{Supp}(f) \text{ is compact}\}.$$

**Proposition 5.7.** Suppose every compact set  $K$  is Borel-measurable, then  $C_c(X)$  is a sub-algebra of  $C(X)$ .

**Proposition 5.8.** Suppose every compact set  $K$  is Borel-measurable, let  $\mu : \text{Bol}(X) \rightarrow [0, \infty]$  be a Borel measure on  $X$ , such that  $\forall K$  be compact,  $\mu(K) < \infty$ , then

$$C_c(X) \subseteq L^1(\mu).$$

*Proof.* Given any  $f \in C_c(X)$ . Let  $K = \text{Supp}(f)$ , then

$$\int_X |f| d\mu = \int_K |f| d\mu \leq \int_K \|f\|_\infty d\mu = \mu(K) \|f\|_\infty < \infty.$$

□

## 5.3 Partition of Unity

**Definition 5.11.** Let  $K$  be a compact set, and  $V$  be an open set of  $X$ . Let  $f \in C_c(X)$ . We say  $f < V$  if  $0 \leq f \leq 1$ , and  $\text{Supp}(f) \subseteq V$ . We say  $K < f$  if  $0 \leq f \leq 1$ , and  $f|_K = 1$ . We say  $K < f < V$  if  $K \subset V, K < f, f < V$ .

*Remark.*  $f$  is a “bump” function that approximates  $\chi_K$  when  $V$  shrinks towards  $K$ .

**Lemma 5.9** (Urysohn’s). Let  $X$  be a Locally Compact Hausdorff space,  $K \subseteq V \subseteq X$  be such that  $K$  is compact, and  $V$  is open. Then there exists  $f \in C_c(V)$ , such that  $K < f < V$ .

*Proof.* we want to construct a family of open sets  $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$ , such that  $\bar{V}_r$  is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for  $r < s$ .

By 5.5, we can find  $K \subset V_0 \subset \bar{V}_0 \subset V$ .

Pick an enumeration of  $r \in \mathbb{Q} \cap (0, 1]$ , i.e.  $(r_n)_{n=1}^\infty$ . WLOG, we can let  $r_1 = 1$ .

By 5.5, we can find  $K \subset V_1 \subset \bar{V}_1 \subset V_0$ .

Suppose we have constructed the  $V_{r_i}$  for  $1 \leq i \leq n$ , such that  $\bar{V}_r$  is compact, and

$$K \subset V_1 \subset \bar{V}_1 \subset V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r \subset \cdots \subset V_0 \subset \bar{V}_0 \subset V,$$

for  $r < s \in \{r_i\}_{i=1}^n$ .

Let  $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$ .

Now by 5.5, we can find  $\bar{V}_t \subset V_{n+1} \subset \bar{V}_{n+1} \subset V_s$ .

For any  $r < r_{n+1}$ , we have  $r \leq s$ , and thus  $V_{n+1} \subset \bar{V}_{n+1} \subset V_s \subset \bar{V}_s \subseteq V_r$  by induction hypothesis, and similarly for any  $r > r_{n+1}$ .

Inductively, we can prove there is such a family.

Define  $f_r := r\chi_{V_r}$ , and  $g_r := r\chi_{\bar{V}_r^c} + \chi_{\bar{V}_r}$ , and  $f := \sup_r f_r, g := \inf_r g_r$ .

We can show that  $f, g$  are upper and lower continuous, respectively.

In addition,  $f, g$  are both 0 outside of  $V_1$ , and 1 on  $K$ .

Suppose there is some  $x \in X, r, s \in \mathbb{Q} \cap [0, 1]$ , such that  $f_r(x) > g_s(x)$ . Then we must have  $f_r(x) > 0$ , and thus  $x \in V_r$  and  $1 \leq r = f_r(x)$ .

Thus  $1 > g_s(x)$ , and thus  $x \in \bar{V}_s^c$  and  $f_s = s$ .

Since  $r > s$ , we must have  $V_r \subset \bar{V}_r \subset V_s \subset \bar{V}_s$ , which is a contradiction to  $x \in V_r, x \notin \bar{V}_s$ .

Thus for any  $x \in X, r, s \in \mathbb{Q} \cap [0, 1]$ , we must have  $f_r(x) \leq g_s(x)$ .

Thus we must have  $f(x) \leq g(x)$  for any  $x \in V$ .

Now suppose there is some  $x \in X$ , such that  $f(x) < g(x)$ .

There must be some rationals, such that  $f(x) < r < s < g(x)$ , since  $\mathbb{Q}$  is dense.

Thus  $\sup_r f_r(x) < r$ , and thus  $x \notin V_r$ .

Also,  $\inf_s g_s(x) > s$ , and thus  $x \in \bar{V}_s$ .

Yet  $r < s$ , we must have  $V_s \subset \bar{V}_s \subset V_r \subset \bar{V}_r$ , which is a contradiction.

Thus we must have  $f = g$ , and it forces  $f$  to be continuous.  $\square$

**Definition 5.12.** Let  $X$  be a Locally Compact Hausdorff space,  $K \subseteq X$  be compact, and some finite open cover  $\bigcup_{i=1}^n V_i \supseteq K$ .

A collection  $(h_i)_{i=1}^n \subset C_c(X)$  is called a **partition of unity** on  $K$  subordinate to  $(V_i)_{i=1}^n$  if

$$\begin{cases} \forall 1 \leq i \leq n, \quad h_i < V_i, \\ \forall x \in K, \quad \sum_{i=1}^n h_i(x) = 1. \end{cases}$$

**Theorem 5.10.** Let  $X$  be a Locally Compact Hausdorff space,  $K \subseteq X$  be compact, and some finite open cover  $\bigcup_{i=1}^n V_i \supseteq K$ , there always exists a partition of unity on  $K$  subordinated to  $(V_i)_{i=1}^n$ .

*Proof.* Since  $K$  is compact, we can find some open cover  $W_1, \dots, W_m$ , such that for all  $j$ , we have  $W_j \subset \bar{W}_j \subset V_{i(j)}$  for some  $1 \leq i(j) \leq n$ .

Let  $K_i := \bigcup_{1 \leq j \leq m} \text{such that } W_j \subset V_{i(j)} \bar{W}_j \subset V_i$ , which is compact.

By Urysohn's lemma, we can find  $K_i < g_i < V_i$ .

Now let  $h_1 := g_1$ , and in general,  $h_i := g_i \prod_{j < i} (1 - g_j)$ .

It is easy to check that  $0 \leq h_i \leq 1$ , and  $h_i \in C_c(X)$ .

In addition,  $\text{Supp}(h_i) \subseteq \text{Supp}(g_i) \subset V_i$ .

Thus  $h_i < V_i$ . Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have  $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_i)$ .

For any  $x \in K$ , there must be some  $i \in [n]$  such that  $x \in K_i$ , and thus  $g_i(x) = 1$ , and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_i(x)) = 1 - 0 = 1.$$

$\square$

## 5.4 Linear Functional

**Definition 5.13.** Let  $X$  be a compact Hausdorff space.

A linear functional on  $C(X)$  is a linear map  $\Lambda : C(X) \rightarrow \mathbb{C}$ .

A linear functional  $\Lambda$  is **positive** if  $\Lambda(f) \geq 0$  for all  $f \in C(X)$  such that  $f \geq 0$ .

**Proposition 5.11.** Let  $X$  be a compact Hausdorff space, then for a Borel measure  $\mu$  on  $X$ ,

1. If  $\mu$  is finite,  $\Lambda_\mu(f) := \int_X f d\mu$  is a positive linear functional.

2. If  $\mu$  is finite,  $\Lambda_\mu$  is bounded and hence continuous. Indeed,  $\forall f \in C(X), |\Lambda_\mu(f)| \leq \mu(X) \|f\|_\infty$ .

3.  $\Lambda_\mu$  is a finite-value linear functional iff  $\mu(X) < \infty$ .

*Proof.* 1. By properties of integral and 5.8.

2. For  $f \in C(X)$ .

$$\begin{aligned} |\Lambda_\mu(f)| &= \left| \int_X f d\mu \right| \\ &\leq \int_X |f| d\mu \\ &\leq \int_X \|f\|_\infty d\mu \\ &= \mu(X) \|f\|_\infty. \end{aligned}$$

□

## 5.5 Radon Measure

**Definition 5.14.** Let  $X$  be a topological space,  $\mu : \text{Bol}(X) \rightarrow [0, \infty]$  be a Borel measure on  $X$ . For  $A \in \text{Bol}(X)$ ,  $\mu$  is **outer regular** if  $\mu(A) = \inf \{\mu(U) : \text{open } U \supseteq A\}$ .  $\mu$  is **inner regular** if  $\mu(A) = \sup \{\mu(K) : \text{compact } K \subseteq A\}$ .  $\mu$  is **regular** if it is inner and outer regular for any  $A \in \text{Bol}(X)$ .

**Definition 5.15.** Let  $X$  be a topological space,  $\mu : \text{Bol}(X) \rightarrow [0, \infty]$  be a Borel measure on  $X$ .  $\mu$  is a **Radon** measure if

1.  $\forall \text{compact } K \subseteq X, \mu(K) < \infty$ ,
2.  $\mu$  is outer regular on Borel sets,
3.  $\mu$  is inner regular on open sets.

*Remark.* We have seen that Lebesgue-Stieltjes Measures are regular and Radon.

**Proposition 5.12.** A finite Borel measure on a compact metric space is always regular (hence Radon).

*Proof.* Let  $\mu$  be a finite Borel measure on a compact metric space  $X$ . Let  $S \subseteq \text{Bol}(X)$  on which  $\mu$  is regular. If  $C \subseteq X$  is closed, it is compact. Thus  $\mu$  is inner regular for  $C$ . Since  $X$  is a metric space,  $C = \bigcap_{n \geq 1} \{x \in X : d(x, C) < \frac{1}{n}\}$  is  $G_\delta$ . By continuity from above of  $\mu$ , it follows that  $\mu$  is also outer-regular. Thus all the closed sets belong to  $S$ .

Since Borel sets are generated by closed sets, it suffices to show  $S$  is a  $\sigma$ -algebra.

For  $A \in S, \epsilon > 0$ , there is compact  $K$  and open  $U$  such that  $K \subseteq A \subseteq U, \mu(U \setminus K) < \epsilon$ . Then  $U^c \subseteq A^c \subseteq K^c$ , where  $U^c$  is compact,  $K^c$  is open. In addition,

$$\mu(K^c \setminus U^c) = \mu(K^c \cap U) = \mu(U \setminus K) < \epsilon.$$

Thus  $A^c \in S$ .

Consider  $(A_i)_{i=1}^\infty \subseteq S, \epsilon > 0$ . Choose compact  $K_i \subseteq A_i$  and open  $U_i \supseteq A_i$ , such that  $\mu(U_i \setminus K_i) < \epsilon/2^i$ . Let  $A = \bigcup_{i=1}^\infty A_i, C_n = \bigcup_{i=1}^n K_i, C = \bigcup_{i=1}^\infty K_i, U = \bigcup_{i=1}^\infty U_i$ .

Thus  $C_n$  are closed,  $U$  is open, and  $C_n \subseteq A \subseteq U$ . By continuity and finiteness of  $\mu$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(U \setminus C_n) &= \mu(U \setminus C) \\ &\leq \sum_{i=1}^\infty \mu(U_i \setminus K_i) \\ &< \epsilon. \end{aligned}$$

Thus  $\mu$  is regular on  $A$ , and thus  $A \in S$ , and thus  $S$  is closed under countable unions.

Thus  $S = \text{Bol}(X)$ . □

## 5.6 Extremely Disconnected Spaces

**Definition 5.16.** A compact space  $X$  is **extremely disconnected** if the closure of every open set is open.

**Proposition 5.13.** If  $X$  is extremely disconnected, then there is a basis of clopen sets.

**Proposition 5.14.** If  $A \subseteq X$  is clopen, then  $\chi_A \in C(X)$ .

**Proposition 5.15.** (Stone–Čech compactification) Let  $D$  be a discrete space (thus every subset is open). The **Stone–Čech compactification** is the unique compact (Hausdorff) space  $\beta D$  with the following universal properties:

1.  $D \subseteq \beta D$  as topology inclusion.
2. For any compact  $K$ , and every continuous map  $f : D \rightarrow K$ , there is a unique continuous extension  $\beta f : \beta D \rightarrow K$ .

**Proposition 5.16.**  $\ell^\infty(D) \cong C(\beta D)$

**Proposition 5.17.**  $\beta D$  is the set of ultrafilters on  $D$ .

**Proposition 5.18.**  $\beta D$  is extremely disconnected.

## 5.7 Riesz-Markov-Kakutani

**Theorem 5.19** (Riesz-Markov-Kakutani). Let  $X$  be a Locally Compact Hausdorff space,  $\Lambda : C(X) \rightarrow \mathbb{C}$  a positive linear functional. Then there is a unique Radon measure  $\mu$  on  $X$ , such that  $\Lambda(f) = \Lambda_\mu(f) = \int_X f d\mu$ . In addition,

1.  $\forall U \subseteq X$  be open, we have  $\mu(U) = \sup \{\Lambda(f) : f < U\}$ .
2.  $\forall K \subseteq X$  be compact, we have  $\mu(K) = \inf \{\Lambda(f) : K < f\}$ .

*Proof.* Define  $\mu^* : \mathcal{T} \rightarrow [0, \infty]$  by: for any open  $U \subseteq X$ ,  $\mu^*(U) := \sup \{\Lambda(f) : f < U\}$ .

Clearly for any  $V \supseteq U$ , we have  $\mu^*(U) \leq \mu^*(V)$ .

Thus we have  $\mu^*(U) := \inf \{\mu^*(f) : \text{open } C \subseteq U\}$ .

Now we extend  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  by  $\mu^*(E) := \inf \{\mu^*(U) : \text{open } U \supseteq E\}$ .

One can check this is an outer measure, and the measure induced by Caratheodory is a Radon measure.

Now let us check the uniqueness.

Suppose there is some Radon measure  $\mu$ , such that  $\Lambda = \Lambda_\mu = \int_X f d\mu$ .

Given any open  $U$ .

Consider any compact  $K$  such that  $K \subset U$ . By Urysohn's lemma, we can find a function  $f$ , such that  $K < f < U$ .

Then  $\chi_K \leq f \leq \chi_U \implies \mu(K) \leq \Lambda(f) \leq \mu(U)$ .

Thus by inner regularity,  $\mu(U) = \sup \{\mu(K) : \text{compact } K \subseteq U\} = \sup \{\Lambda(f) : f < U\}$ , which is uniquely determined.

For any Borel set  $E$ , we have that by outer regularity,  $\mu(E) = \inf \{\mu(U) : \text{open } U \supseteq E\}$ , which is uniquely determined.  $\square$

## 6 Lebesgue Spaces

### 6.1 The First Lebesgue Space

**Definition 6.1.** Given some measure space  $(X, \mathcal{M}, \mu)$ , define

$$\mathcal{L}^1(\mu) := \mathcal{L}^1(X, \mathcal{M}, \mu) := \left\{ f : X \rightarrow \mathbb{C} \mid f \text{ is measurable, } \int_X |f| d\mu < \infty \right\}.$$

**Proposition 6.1.**  $\mathcal{L}^1(X, \mathcal{M}, \mu)$  is a vector space.

*Proof.* Clearly  $\int_X |0|d\mu = 0$ , so the zero function  $0 \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ . Also, for any  $c \in \mathbb{C}$ , and  $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ , we have

$$\begin{aligned}\int_X |c \cdot f|d\mu &= \int_X |c||f|d\mu \\ &= |c| \int_X |f|d\mu \\ &< \infty.\end{aligned}$$

Thus  $c \cdot f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ .

Now for any  $f, g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ , we have

$$\begin{aligned}\int_X |f + g|d\mu &\leq \int_X |f| + |g|d\mu \\ &= \int_X |f|d\mu + \int_X |g|d\mu \\ &< \infty.\end{aligned}$$

Thus  $f + g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ .

Since the set of all functions  $\{f : X \rightarrow \mathbb{C}\}$  is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have  $\mathcal{L}^1(X, \mathcal{M}, \mu)$  is a subspace of it.  $\square$

**Definition 6.2.** Let

$$N = \left\{ f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : \int_X |f|d\mu = 0 \right\} = \{f \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f = 0 \text{ } \mu - \text{a.e.}\}.$$

Define

$$L^1(X, \mathcal{M}, \mu) := \mathcal{L}^1(X, \mathcal{M}, \mu)/N,$$

which is the quotient vector space of  $\mathcal{L}^1(X, \mathcal{M}, \mu)$  mod  $N$ .

*Remark.*  $[f] = \{g \in \mathcal{L}^1(X, \mathcal{M}, \mu) : f - g = 0 \text{ } \mu - \text{a.e.}\} \in L^1(X, \mathcal{M}, \mu)$

**Definition 6.3.**  $\|[f]\|_{L^1(X, \mathcal{M}, \mu)} := \int_X |f|d\mu$  for any choice of representative  $f \in [f]$ .

When the context is clear, we might write  $L^1(X, \mathcal{M}, \mu)$  as  $L^1(\mu)$  or  $L^1(X)$ . We might also write  $\|\cdot\|_{L^1(X, \mathcal{M}, \mu)}$  as  $\|\cdot\|_{L^1(\mu)}$ ,  $\|\cdot\|_{L^1(\mu)}$ ,  $\|\cdot\|_1$ .

**Lemma 6.2.** *The above definition is well defined.*

*Proof.* Take any  $g, f \in [f]$ . Let  $K = \{x \in X : f(x) \neq g(x)\}$ , we have  $\mu(K) = 0$ .

$$\begin{aligned}\int_X |f|d\mu &= \int_{X \setminus K} |f|d\mu + \int_K |f|d\mu \\ &= \int_{X \setminus K} |f|d\mu \\ &= \int_{X \setminus K} |g|d\mu \\ &= \int_{X \setminus K} |g|d\mu + \int_K |g|d\mu \\ &= \int_X |g|d\mu\end{aligned}$$

$\square$

**Proposition 6.3.**  $\|\cdot\|_1$  is a norm on  $L^1(X, \mathcal{M}, \mu)$ .

*Proof.* Consider any  $[f], [g] \in L^1(X, \mathcal{M}, \mu)$ .

$$\begin{aligned} \|[f] + [g]\|_1 &= \|[f + g]\|_1 \\ &= \int_X |f + g| d\mu \\ &\leq \int_X |f| d\mu + \int_X |g| d\mu \\ &= \|[f]\|_1 + \|[g]\|_1 \end{aligned}$$

For any  $\alpha \in \mathbb{C}$ , we have

$$\begin{aligned} \|\alpha[f]\|_1 &= \|\alpha f\|_1 \\ &= \int_X |\alpha f| d\mu \\ &= |\alpha| \int_X |f| d\mu \\ &= |\alpha| \|[f]\|_1 \end{aligned}$$

If  $\|[f]\|_1 = 0$ , we must have  $f = 0 \mu - a.e..$  Thus  $f \in N$ , thus  $[f] = [0] = 0$ .  $\square$

**Theorem 6.4** (Fischer-Riesz). *Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)})$  is a Banach Space.*

*Proof.* Let  $([f_n])_1^\infty$  be a Cauchy sequence in  $L^1(X, \mathcal{M}, \mu)$ . Then for each  $k \in \mathbb{N}^+$ , there is some  $N_k \geq 1$ , such that  $\forall m, n \geq N_k, \|[f_m] - [f_n]\|_{L^1(\mu)} < \frac{1}{2^k}$ .

WLOG,  $\forall k, N_{k+1} \geq N_k$ .

Thus  $\|[f_{N_{k+1}}] - [f_{N_k}]\|_{L^1(\mu)} < \frac{1}{2^k}$ .

Let  $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}|$ , where we fix  $f_n$  to be a representative of  $[f_n]$ . Notice that  $\forall k \geq 1$ ,

$$\begin{aligned} f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\ |f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\ &\leq |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \\ &= g_k \end{aligned}$$

We have that  $\int_X g_k d\mu = \int_X |f_{N_1}| d\mu + \sum_{j=1}^n \int_X |f_{N_{j+1}} - f_{N_j}| d\mu$ .

Let  $g = \lim_{k \rightarrow \infty} g_k = |f_{N_1}| + \sum_{j=1}^\infty |f_{N_{j+1}} - f_{N_j}|$ .

By LMCT, we have that

$$\begin{aligned}
\int_X g d\mu &= \lim_{k \rightarrow \infty} \int_X g_k d\mu \\
&= \int_X |f_{N_1}| d\mu + \sum_{j=1}^{\infty} \int_X |f_{N_{j+1}} - f_{N_j}| d\mu \\
&= \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^{\infty} \|[f_{N_{j+1}} - f_{N_j}]\|_{L^1(\mu)} \\
&= \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^{\infty} \|[f_{N_{j+1}}] - [f_{N_j}]\|_{L^1(\mu)} \\
&< \|[f_{N_1}]\|_{L^1(\mu)} + \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&< \infty.
\end{aligned}$$

Thus  $g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ . Thus  $N := \{x \in X : g(x) = \infty\}$  has measure 0.

This implies that  $f_{N_1}(x) + \sum_{k=1}^{\infty} (f_{N_{k+1}}(x) - f_{N_k}(x))$  converges absolutely for  $x \in X \setminus N$ . We can thus define

$$\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty} (f_{N_{j+1}}(x) - f_{N_j}(x)) \\
&= \lim_{k \rightarrow \infty} \left( f_{N_1}(x) + \sum_{j=1}^k (f_{N_{j+1}}(x) - f_{N_j}(x)) \right) \\
&= \lim_{k \rightarrow \infty} f_{N_{k+1}}(x) \\
&= \lim_{k \rightarrow \infty} f_{N_k}(x)
\end{aligned}$$

for  $x \in X \setminus N$ .

We then extend  $f$  to  $X$  by  $f|_N := 0$ .

Then  $|f| \leq g$ , and thus  $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ .

Notice that  $|f_{N_k}| \leq g_k \leq g$ , thus  $|f - f_{N_k}| \leq g + g = 2g$ .

By LDCT,

$$\begin{aligned}
\lim_{k \rightarrow \infty} \|[f_{N_k}] - [f]\|_{L^1(\mu)} &= \lim_{k \rightarrow \infty} \|[f_{N_k} - f]\|_{L^1(\mu)} \\
&= \lim_{k \rightarrow \infty} \int_X |f_{N_k} - f| d\mu \\
&= \int_X \lim_{k \rightarrow \infty} |f_{N_k} - f| d\mu \\
&= 0.
\end{aligned}$$

Thus  $\lim_{k \rightarrow \infty} [f_{N_k}]$  converges to  $[f]$ . Since this is a subsequence of the Cauchy sequence  $([f_n])_1^\infty$ , we have that  $\lim_{n \rightarrow \infty} [f_n] = [f]$ .

This shows that  $(L^1(X, \mathcal{M}, \mu), \|\cdot\|_{L^1(\mu)})$  is complete.  $\square$

*Remark.* When we write  $f \in L^1(\mu)$ , we will mean  $[f] \in L^1(\mu)$ , and let  $f \in \mathcal{L}^1(\mu)$  be any representative of  $[f]$  when the context is clear.

## 6.2 Convex functions

**Definition 6.4.** A function  $\phi : U \rightarrow \mathbb{R}$  is **convex** if

$$\forall x, y \in U, \forall \lambda \in [0, 1], \phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y).$$

**Theorem 6.5** (Jensen's Inequality). *If  $\phi$  is convex, we have  $\forall x_1, \dots, x_n \in U$ , and  $\forall 0 \leq \lambda_1, \dots, \lambda_n \leq 1$  such that  $\sum_{i=1}^n \lambda_i = 1$ ,*

$$\phi\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i \phi(x_i).$$

*Proof.* The base case is when  $n = 1$ , which is trivial.

Now suppose this holds for  $n - 1 \in \mathbb{N}$ .

Given any  $\forall x_1, \dots, x_n \in U$ , and  $0 \leq \lambda_1, \dots, \lambda_n \leq 1$  such that  $\sum_{i=1}^n \lambda_i = 1$ . If  $\lambda_n = 0$ , we can reduce the sum to a  $n - 1$  sum. If  $\lambda_n = 1$ , then the other  $\lambda_i$  must be all 0, and we can reduce the sum to only  $x_n$ .

Now suppose  $0 < \lambda_1 < 1$ . Notice that  $\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} = \frac{\sum_{i=1}^{n-1} \lambda_i}{1-\lambda_n} = \frac{1-\lambda_n}{1-\lambda_n} = 1$ .

We have that

$$\begin{aligned} \phi\left(\sum_{i=1}^n \lambda_i x_i\right) &= \phi\left(\lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i\right) \\ &\leq \lambda_n \phi(x_n) + (1 - \lambda_n) \phi\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i\right) \\ &\leq \lambda_n \phi(x_n) + (1 - \lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} \phi(x_i) \\ &= \lambda_n \phi(x_n) + \sum_{i=1}^{n-1} \lambda_i \phi(x_i) \\ &= \sum_{i=1}^n \lambda_i \phi(x_i). \end{aligned}$$

By induction, this is true for any  $n \geq 1$ . □

**Theorem 6.6** (Arithmetic Mean Inequality). *Let  $x_1, \dots, x_n \geq 0$ , with  $0 \leq \lambda_1, \dots, \lambda_n \leq 1$  such that  $\sum_{i=1}^n \lambda_i = 1$ . We have that*

$$\prod_{i=1}^n x_i^{\lambda_i} \leq \sum_{i=1}^n \lambda_i x_i.$$

*Proof.* If any of  $x_i = 0$ , then the inequality is trivially true.

Now suppose  $\forall i, x_i > 0$ .

Notice that  $\exp$  is convex, and we have

$$\begin{aligned} \prod_{i=1}^n x_i^{\lambda_i} &= \exp\left(\sum_{i=1}^n \lambda_i \ln(x_i)\right) \\ &\leq \sum_{i=1}^n \lambda_i \exp(\ln(x_i)) \\ &= \sum_{i=1}^n \lambda_i x_i. \end{aligned}$$

□

**Proposition 6.7.** *Let  $x_1, \dots, x_n \geq 0$ , and  $n \in \mathbb{N}^+, p \geq 1$ , we have that*

$$\sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i\right)^p \leq n^{p-1} \sum_{i=1}^n x_i^p.$$

*Proof.* For  $p \geq 1$ , we have  $(\cdot)^p$  is convex.

$$\begin{aligned} \left( \sum_{i=1}^n \frac{1}{n} x_i \right)^p &\leq \sum_{i=1}^n \frac{1}{n} x_i^p \\ \frac{1}{n^p} \left( \sum_{i=1}^n x_i \right)^p &\leq \frac{1}{n} \sum_{i=1}^n x_i^p \\ \left( \sum_{i=1}^n x_i \right)^p &\leq n^{p-1} \sum_{i=1}^n x_i^p. \end{aligned}$$

This proves the second inequality.

Now when  $n = 1$ , we have the first inequality trivially.

Suppose the first inequality holds for  $n \in \mathbb{N}^+$ , we have

$$\begin{aligned} \left( \sum_{i=1}^{n+1} x_i \right)^p &= \left( \sum_{i=1}^n x_i + x_{n+1} \right)^p \\ &\geq \left( \sum_{i=1}^n x_i \right)^p + x_{n+1}^p \\ &\geq \left( \sum_{i=1}^n x_i^p \right) + x_{n+1}^p \\ &= \sum_{i=1}^{n+1} x_i^p. \end{aligned}$$

By induction, the first inequality is true for all  $n \in \mathbb{N}^+$ .  $\square$

### 6.3 $L^p$ Spaces

**Definition 6.5.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $1 \leq p < \infty$  we define

$$\mathcal{L}^p(\mu) := \left\{ f : X \rightarrow \mathbb{C} \mid f^p \in L^1(\mu) \right\} = \left\{ f : X \rightarrow \mathbb{C} \mid f \text{ is measurable, } \int_X |f|^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p} := \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

**Proposition 6.8.**  $\mathcal{L}^p(\mu)$  is a vector space.

*Proof.* Clearly  $\int_X |0|^p d\mu = 0$ , so the zero function  $0 \in \mathcal{L}^p(\mu)$ .

Also, for any  $c \in \mathbb{C}$ , and  $f \in \mathcal{L}^p(\mu)$ , we have

$$\begin{aligned} \int_X |c \cdot f|^p d\mu &= \int_X |c|^p |f|^p d\mu \\ &= |c|^p \int_X |f|^p d\mu \\ &< \infty. \end{aligned}$$

Thus  $c \cdot f \in \mathcal{L}^p(\mu)$ .

Now for any  $f, g \in \mathcal{L}^p(\mu)$ , we have

$$\begin{aligned}\int_X |f + g|^p d\mu &\leq \int_X (|f| + |g|)^p d\mu \\ &\leq \int_X 2^{p-1}(|f|^p + |g|^p) d\mu \\ &= 2^{p-1} \left( \int_X |f|^p d\mu + \int_X |g|^p d\mu \right) \\ &< \infty.\end{aligned}$$

Thus  $f + g \in \mathcal{L}^p(\mu)$ .

Since the set of all functions  $\{f : X \rightarrow \mathbb{C}\}$  is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have  $\mathcal{L}^p(\mu)$  is a subspace of it.  $\square$

**Definition 6.6.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, the **essential supremum** of a function  $f : X \rightarrow \mathbb{R}$  is

$$\text{ess sup } f := \inf \{M \in \mathbb{R} : \mu(\{x : f(x) > M\}) = 0\}.$$

**Proposition 6.9.** For any  $\lambda \geq 0$ ,  $f : X \rightarrow \mathbb{R}$ , we have

$$\lambda(\text{ess sup } f) = \text{ess sup}(\lambda f).$$

*Proof.* It is easy to see this is true for  $\lambda = 0$ .

Now suppose  $\lambda > 0$ .

$$\begin{aligned}\text{ess sup}(\lambda f) &= \inf \{M \in \mathbb{R} : \mu(\{x : \lambda f(x) > M\}) = 0\} \\ &= \inf \left\{ M \in \mathbb{R} : \mu \left( \left\{ x : f(x) > \frac{M}{\lambda} \right\} \right) = 0 \right\} \\ &= \inf \{\lambda \cdot N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0\} \\ &= \lambda \inf \{N \in \mathbb{R} : \mu(\{x : f(x) > N\}) = 0\} \\ &= \lambda(\text{ess sup } f).\end{aligned}$$

$\square$

**Definition 6.7.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, we define

$$\mathcal{L}^\infty(\mu) := \{f : X \rightarrow \mathbb{C} \mid f \text{ is measurable, } \text{ess sup } |f| < \infty\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty} := \text{ess sup } |f|.$$

**Proposition 6.10.**  $\mathcal{L}^\infty(\mu)$  is a vector space.

*Proof.* Clearly  $\text{ess sup } 0 = 0$ , so the zero function  $0 \in \mathcal{L}^\infty(\mu)$ .

Also, for any  $c \in \mathbb{C}$ , and  $f \in \mathcal{L}^\infty(\mu)$ , we have

$$\begin{aligned}\|c \cdot f\|_{\mathcal{L}^\infty} &= \text{ess sup } |c \cdot f| \\ &= \text{ess sup } (|c| \cdot |f|) \\ &= |c| \text{ess sup } |f| \\ &= |c| \|f\|_{\mathcal{L}^\infty} \\ &< \infty.\end{aligned}$$

Thus  $c \cdot f \in \mathcal{L}^\infty(\mu)$ .

Now for any  $f, g \in \mathcal{L}^\infty(\mu)$ .

Consider any  $L, N \in \mathbb{R}$ , such that  $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$ .

Thus  $\mu(\{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}) = 0$ .

Now for any  $x \in X$ , if  $|f(x) + g(x)| > L + N$ , we must have  $|f(x)| + |g(x)| \geq |f(x) + g(x)| > L + N$ .

Thus  $|f(x)| > L$  or  $|g(x)| > N$ .

Since this holds for any  $x \in X$ , we have  $\{x : |f(x) + g(x)| > N + L\} \subseteq \{x : |f(x)| > N\} \cup \{x : |g(x)| > L\}$ .

Thus  $\mu(\{x : |f(x) + g(x)| > N + L\}) = 0$ .

By definition, we have

$$\begin{aligned} \|f + g\|_{\mathcal{L}^\infty} &= \text{ess sup } |f + g| \\ &= \inf \{M \in \mathbb{R} : \mu(\{x : |f(x) + g(x)| > M\}) = 0\} \\ &\leq N + L. \end{aligned}$$

Since this hold for any such  $N, L \in \mathbb{R}$ , such that  $\mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0$ , we have

$$\begin{aligned} \|f + g\|_{\mathcal{L}^\infty} &= \inf \{N + L : \mu(\{x : |f(x)| > N\}) = \mu(\{x : |g(x)| > L\}) = 0\} \\ &= \inf \{N : \mu(\{x : |f(x)| > N\}) = 0\} + \inf \{L : \mu(\{x : |g(x)| > L\}) = 0\} \\ &= \|f\|_{\mathcal{L}^\infty} + \|g\|_{\mathcal{L}^\infty} \\ &< \infty. \end{aligned}$$

Thus  $f + g \in \mathcal{L}^\infty(\mu)$ .

Since the set of all functions  $\{f : X \rightarrow \mathbb{C}\}$  is a vector space with the pointwise multiplication and addition, by the subspace criterion, we have  $\mathcal{L}^\infty(\mu)$  is a subspace of it.  $\square$

**Proposition 6.11.** For any  $1 \leq p \leq \infty$ , we have  $\|f - g\|_{\mathcal{L}^p} = 0 \iff f = g$  almost everywhere.

*Proof.* For  $1 \leq p < \infty$ ,

$$\begin{aligned} \|f - g\|_{\mathcal{L}^p} &= 0 \\ \iff \int_X |f - g|^p d\mu &= 0 \\ \iff |f - g|^p &= 0 \text{ a.e.} \\ \iff f - g &= 0 \text{ a.e.} \\ \iff f &= g \text{ a.e..} \end{aligned}$$

For  $p = \infty$ ,

$$\begin{aligned} \|f - g\|_{\mathcal{L}^\infty} &= 0 \\ \iff \text{ess sup } |f - g| &= 0 \\ \iff f - g &= 0 \text{ a.e.} \\ \iff f &= g \text{ a.e..} \end{aligned}$$

$\square$

**Definition 6.8.** For any  $1 \leq p \leq \infty$ , if we identify  $f, g \in \mathcal{L}^p(\mu)$  by  $f \sim g \iff f = g$  almost everywhere, we get the quotient vector space

$$L^p(\mu) := \mathcal{L}^p(\mu)/\sim = \{[f] : f \in \mathcal{L}^p(\mu)\}$$

to be the collection of all equivalence classes of functions in  $\mathcal{L}^p$ .

**Definition 6.9.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $1 \leq p \leq \infty$  we define the norm

$$\|[f]\|_{L^p(\mu)} := \|f\|_{\mathcal{L}^p}$$

for any representative  $f \in [f]$ .

**Lemma 6.12.** *The above definition is well-defined.*

*Remark.* As before, when we write  $f \in L^p(\mu)$ , we will mean  $[f] \in L^p(\mu)$ , and let  $f \in \mathcal{L}^p(\mu)$  be any representative of  $[f]$  when the context is clear.

**Theorem 6.13** (Holder's Inequality). *Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $1 \leq p \leq \infty$ . Suppose  $\frac{1}{p} + \frac{1}{q} = 1$ , then  $\forall f \in L^p(\mu), g \in L^q(\mu)$ ,  $fg \in L^1(\mu)$  and*

$$\|fg\|_{L^1(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

*Proof.* If  $p = 1$ , then  $q = \infty$ .

Now  $|fg| = |f||g| \leq |f| \|g\|_{L^\infty(\mu)}$ .

Thus

$$\begin{aligned} \|fg\|_{L^1(\mu)} &= \int_X |fg| d\mu \\ &\leq \int_X |f| \|g\|_{L^\infty(\mu)} d\mu \\ &= \|g\|_{L^\infty(\mu)} \int_X |f| d\mu \\ &= \|g\|_{L^\infty(\mu)} \|f\|_{L^1(\mu)}. \end{aligned}$$

Now suppose  $1 < p < \infty$ . We have  $1 < q < \infty$ .

If  $\|f\|_{L^p(\mu)} = 0$  or  $\|g\|_{L^q(\mu)} = 0$ , then it is trivial, since this implies  $f = 0$  a.e. or  $g = 0$  a.e., which means  $fg = 0$  a.e..

Now let  $F := \frac{|f|}{\|f\|_{L^p(\mu)}}$ ,  $G := \frac{|g|}{\|g\|_{L^q(\mu)}}$ .

By the Arithmetic Mean Inequality, we have that

$$\begin{aligned} F(x)G(x) &= (F(x)^p)^{1/p} (G(x)^q)^{1/q} \\ &\leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q \\ \int_X FG d\mu &\leq \frac{1}{p} \int_X F^p d\mu + \frac{1}{q} \int_X G^q d\mu \\ \frac{\|fg\|_{L^1(\mu)}}{\|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}} &\leq \frac{1}{p} \int_X \frac{|f|^p}{\|f\|_{L^p(\mu)}^p} d\mu + \frac{1}{q} \int_X \frac{|g|^q}{\|g\|_{L^q(\mu)}^q} d\mu \\ &= \frac{1}{p} \frac{\|f\|_{L^p(\mu)}^p}{\|f\|_{L^p(\mu)}^p} + \frac{1}{q} \frac{\|g\|_{L^q(\mu)}^q}{\|g\|_{L^q(\mu)}^q} \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

Thus  $\|fg\|_{L^1(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}$ . □

**Theorem 6.14** (Minkowski's Inequality). *Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $1 \leq p \leq \infty$ . For any  $f, g \in L^p(\mu)$ , we have*

$$\|f + g\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}.$$

*Proof.* We have proven for  $p = 1$  and  $p = \infty$ .

Now suppose  $p \in (1, \infty)$ . Then  $q = \frac{p}{p-1} \in (1, \infty)$ .

Since  $f, g \in L^p(\mu)$ , we have  $f + g \in L^p(\mu)$ , so

$$\begin{aligned} \left\| |f + g|^{p-1} \right\|_{L^q(\mu)}^q &= \int_X \left( |f + g|^{p-1} \right)^q d\mu \\ &= \int_X \left( |f + g|^{p-1} \right)^{\frac{p}{p-1}} d\mu \\ &= \int_X |f + g|^p d\mu \\ &= \|f + g\|_{L^p(\mu)}^p \\ &< \infty. \end{aligned}$$

Thus  $|f + g|^{p-1} \in L^q(\mu)$ . By Holder's Inequality, we have

$$\begin{aligned} \|f + g\|_{L^p(\mu)}^p &= \int_X |f + g|^p d\mu \\ &= \int_X |f + g| |f + g|^{p-1} d\mu \\ &\leq \int_X (|f| + |g|) |f + g|^{p-1} d\mu \\ &\leq \int_X |f| \cdot |f + g|^{p-1} d\mu + \int_X |g| \cdot |f + g|^{p-1} d\mu \\ &\leq \|f\|_{L^p(\mu)} \left\| |f + g|^{p-1} \right\|_{L^q(\mu)} + \|g\|_{L^p(\mu)} \left\| |f + g|^{p-1} \right\|_{L^q(\mu)} \\ &= (\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}) \left\| |f + g|^{p-1} \right\|_{L^q(\mu)} \\ &= (\|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}) \|f + g\|_{L^p(\mu)}^{p/q} \\ \|f + g\|_{L^p(\mu)}^{p-p/q} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\ \|f + g\|_{L^p(\mu)}^{p(1-1/q)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)} \\ \|f + g\|_{L^p(\mu)} &\leq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}. \end{aligned}$$

□

**Corollary 6.15.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $1 \leq p \leq \infty$ . We have  $\|\cdot\|_{L^p(\mu)}$  is a norm over  $L^p(\mu)$ .

*Proof.* The triangle Inequality is done by Minkowski's Inequality.

Consider any  $\alpha \in L^p(\mu)$ .

For any  $\alpha \in \mathbb{C}$ , we have

$$\begin{aligned} \|\alpha f\|_{L^p(\mu)}^p &= \int_X |\alpha f|^p d\mu \\ &= \int_X |\alpha|^p |f|^p d\mu \\ &= |\alpha|^p \int_X |f|^p d\mu \\ &= |\alpha|^p \|f\|_{L^p(\mu)}^p \\ &\implies \|\alpha f\|_{L^p(\mu)} = |\alpha| \|f\|_{L^p(\mu)}. \end{aligned}$$

In addition,  $\|f\|_{L^p(\mu)} = 0$ , if and only if  $f = 0 \mu - a.e..$

□

**Theorem 6.16** (Fischer-Riesz). Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $1 \leq p \leq \infty$ .  $\left(L^p(\mu), \|\cdot\|_{L^p(\mu)}\right)$  is a Banach Space.

*Proof.* 1. We first consider  $1 \leq p < \infty$ .

Let  $(f_n)_1^\infty$  be a Cauchy sequence in  $L^p(X, \mathcal{M}, \mu)$ . Then for each  $k \in \mathbb{N}^+$ , there is some  $N_k \geq 1$ , such that  $\forall m, n \geq N_k, \|f_m - f_n\|_{L^p(\mu)} < \frac{1}{2^k}$ .

WLOG,  $\forall k, N_{k+1} \geq N_k$ .

Thus  $\|f_{N_{k+1}} - f_{N_k}\|_{L^p(\mu)} < \frac{1}{2^k}$ .

Let  $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}|$ , where we fix  $f_n$  to be a representative of  $[f_n]$ . Notice that  $\forall k \geq 1$ ,

$$\begin{aligned} f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\ |f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\ &\leq |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \\ &= g_k \\ \|g_k\|_{L^p(\mu)} &= \left\| |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \right\|_{L^p(\mu)} \\ &\leq \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \|f_{N_{j+1}} - f_{N_j}\|_{L^p(\mu)}. \end{aligned}$$

Let  $g = \lim_{k \rightarrow \infty} g_k = |f_{N_1}| + \sum_{j=1}^{\infty} |f_{N_{j+1}} - f_{N_j}|$ .

Notice that  $g_k$  are monotone increasing. By LMCT, we have that

$$\begin{aligned} \|g\|_{L^p(\mu)} &= \int_X |g|^p d\mu \\ &= \int_X g^p d\mu \\ &= \int_X \lim_{k \rightarrow \infty} g_k^p d\mu \\ &= \lim_{k \rightarrow \infty} \int_X g_k^p d\mu \\ &= \lim_{k \rightarrow \infty} \|g_k\|_{L^p(\mu)} \\ &\leq \lim_{k \rightarrow \infty} \left( \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{k-1} \|f_{N_{j+1}} - f_{N_j}\|_{L^p(\mu)} \right) \\ &= \|f_{N_1}\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \|f_{N_{j+1}} - f_{N_j}\|_{L^p(\mu)} \\ &< \|[f_{N_1}]\|_{L^p(\mu)} + \sum_{j=1}^{\infty} \frac{1}{2^j} \\ &< \infty. \end{aligned}$$

Thus  $g \in L^p(X, \mathcal{M}, \mu)$ , which means  $g^p \in L^1(X, \mathcal{M}, \mu)$  and  $N := \{x \in X : g(x) = \infty\}$  has measure 0. This implies that  $f_{N_1}(x) + \sum_{k=1}^{\infty} (f_{N_{k+1}}(x) - f_{N_k}(x))$  converges absolutely for  $x \in X \setminus N$ .

We can thus define

$$\begin{aligned}
f(x) &:= f_{N_1}(x) + \sum_{j=1}^{\infty} (f_{N_{j+1}}(x) - f_{N_j}(x)) \\
&= \lim_{k \rightarrow \infty} \left( f_{N_1}(x) + \sum_{j=1}^k (f_{N_{j+1}}(x) - f_{N_j}(x)) \right) \\
&= \lim_{k \rightarrow \infty} f_{N_{k+1}}(x) \\
&= \lim_{k \rightarrow \infty} f_{N_k}(x)
\end{aligned}$$

for  $x \in X \setminus N$ .

We then extend  $f$  to  $X$  by  $f|_N := 0$ .

Then  $|f| \leq g \implies |f|^p \leq g^p$ , and thus  $f \in L^p(X, \mathcal{M}, \mu)$ .

Notice that  $|f_{N_k}| \leq g_k \leq g$ , thus  $|f - f_{N_k}|^p \leq (g + g)^p = 2^p g^p$ .

By LDCT,

$$\begin{aligned}
\lim_{k \rightarrow \infty} \|f_{N_k} - f\|_{L^p(\mu)}^p &= \lim_{k \rightarrow \infty} \int_X |f_{N_k} - f|^p d\mu \\
&= \int_X \lim_{k \rightarrow \infty} |f_{N_k} - f|^p d\mu \\
&= 0.
\end{aligned}$$

Thus  $\lim_{k \rightarrow \infty} f_{N_k}$  converges to  $f$ .

Since this is a subsequence of the Cauchy sequence  $(f_n)_1^\infty$ , we have that  $\lim_{n \rightarrow \infty} f_n = f$ .

This shows that  $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(\mu)})$  is complete.

2. Now consider  $p = \infty$ .

Let  $(f_n)_1^\infty$  be a Cauchy sequence in  $L^\infty(X, \mathcal{M}, \mu)$ . As before, we can take some subsequence  $(f_{N_k})_{k=1}^\infty$  with  $\|f_{N_{k+1}} - f_{N_k}\|_{L^p(\mu)} < \frac{1}{2^k}$ .

Let  $g_k = |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}|$ , where we fix  $f_n$  to be a representative of  $[f_n]$ . Notice that  $\forall k \geq 1$ ,

$$\begin{aligned}
f_{N_k} &= f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \\
|f_{N_k}| &= \left| f_{N_1} + \sum_{j=1}^{k-1} (f_{N_{j+1}} - f_{N_j}) \right| \\
&\leq |f_{N_1}| + \sum_{j=1}^{k-1} |f_{N_{j+1}} - f_{N_j}| \\
&= g_k.
\end{aligned}$$

□

**Theorem 6.17** (Density of simple functions). *Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $1 \leq p \leq \infty$ . The simple functions*

$$S := \{\phi \in L^p(\mu) \mid \phi \text{ is simple, measurable}\}$$

*are dense in  $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p(X, \mathcal{M}, \mu)})$ .*

*Proof.* 1. First consider  $1 \leq p < \infty$ .

Let  $f \in L^p(X, \mathcal{M}, \mu)$ ,  $\exists (\phi_n)_{n=1}^\infty$  be simple and measurable functions, such that

$$f(x) = \lim_{n \rightarrow \infty} \phi_n(x), \text{a.e. } x \in X,$$

and

$$|\phi_1| \leq |\phi_2| \leq \cdots \leq |f|.$$

Thus

$$f^p(x) = \lim_{n \rightarrow \infty} \phi_n^p(x), \text{ a.e. } x \in X,$$

and

$$|\phi_1|^p \leq |\phi_2|^p \leq \cdots \leq |f|^p.$$

Since  $|f - \phi_n|^p \leq (2|f|)^p = 2^p|f|^p \in L^1(X, \mathcal{M}, \mu)$ , by LDCT 3.18, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|f - \phi_n\|_{L^p(X, \mathcal{M}, \mu)}^p &= \lim_{n \rightarrow \infty} \int_X |f - \phi_n|^p d\mu \\ &= \int_X \lim_{n \rightarrow \infty} |f - \phi_n|^p d\mu \\ &= \int_X \lim_{n \rightarrow \infty} 0 d\mu \\ &= 0. \end{aligned}$$

2. Now consider  $p = \infty$ .

Let  $f \in L^p(X, \mathcal{M}, \mu)$ , we know  $\mu(N) = 0$  for  $N := \{x \in X : |f(x)| > \|f\|_{L^\infty(X, \mathcal{M}, \mu)}\}$ .

Let  $f' := f\chi_{N^c}$ . We notice that  $f'$  is measurable and bounded, with  $|f'| \leq \|f\|_{L^\infty(X, \mathcal{M}, \mu)}$ ,  $\forall x \in X$ . Thus we can find  $(\phi_n)_{n=1}^\infty$  be simple and measurable functions, such that

$$f(x) = \lim_{n \rightarrow \infty} \phi_n(x), \text{ a.e. } x \in X \text{ uniformly, and } |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|.$$

Now

$$\begin{aligned} \|f - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} &= \|f\chi_N + f' - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} \\ &\leq \|f\chi_N\|_{L^\infty(X, \mathcal{M}, \mu)} + \|f' - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} \\ &= \|f' - \phi_n\|_{L^\infty(X, \mathcal{M}, \mu)} \\ &= \operatorname{ess\,sup}_{x \in X} |f'(x) - \phi_n(x)| \\ &\rightarrow 0. \end{aligned}$$

□

*Remark.* For  $1 \leq p < \infty$ ,

$$S = \operatorname{Span} \{\chi_E \mid \mu(E) < \infty\} = \{\phi : X \rightarrow \mathbb{C} \mid \phi \text{ is simple, measurable, } \mu(\{x \in X \mid \phi(x) \neq 0\}) < \infty\}$$

**Theorem 6.18** (Density of compactly supported continuous functions). *Given some measure space  $(X, \mathcal{M}, \mu)$ , where  $\mu$  is a Random measure, then  $C_c(X)$  is dense for  $p < \infty$ .*

*Proof.* Given any  $\epsilon > 0$ .

Consider any measurable  $E$ , such that  $\mu(E) < \infty$ .

By regularity, we can find some compact  $K \subset E \subset V$  open, such that  $\mu(V \setminus E) < \frac{\epsilon^p}{2^p}$ .

Now we take the bump function  $K < f < V$  by Urysohn's Lemma 5.9, where  $f \in C_c(V) \subseteq C_c(X)$ , and  $f|_K = 1$ ,  $f|_{V^c} = 0$ ,  $0 \leq f \leq 1$ .

Now

$$\begin{aligned}
\|\chi_E - f\|_{L^p(\mu)}^p &= \int_X |\chi_E - f|^p d\mu \\
&= \int_{V \setminus K} |\chi_E - f|^p d\mu \\
&\leq \int_{V \setminus K} 2^p d\mu \\
&= 2^p \mu(V \setminus K) \\
&< \epsilon^p.
\end{aligned}$$

Thus  $\chi_E \in \overline{C_c(X)}$ .

Since  $S = \text{Span}\{\chi_E \mid \mu(E) < \infty\}$  is dense in  $L^p(\mu)$ , so is  $C_c(X)$ .  $\square$

*Remark.* This is not true for  $p = \infty$ . For instance, consider  $X = \mathbb{R}$  with Lebesgue measure, or  $X = \mathbb{N}$  with counting measure.

**Proposition 6.19** ( $L^q(\mu) \subseteq L^p(\mu)^*$ ). Let  $p \in [1, \infty]$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $g \in L^q(\mu)$ , then  $\Lambda_g \in L^p(\mu)^*$ , where  $\Lambda_g(f) = \int_X f g d\mu$ . Moreover,  $\forall p \in (1, \infty]$ ,  $\|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$ . This also holds for  $p = 1$  if  $\mu$  is semi-finite.

*Proof.* clearly  $\Lambda_g$  is linear.

By Holder's Inequality, we have

$$\begin{aligned}
|\Lambda_g(f)| &= \left| \int_X f g d\mu \right| \\
&\leq \int_X |f g| d\mu \\
&\leq \|g\|_{L^q(\mu)} \|f\|_{L^p(\mu)}.
\end{aligned}$$

Thus  $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \leq \|g\|_{L^q(\mu)} < \infty$ .

Thus  $\Lambda_g$  is bounded and  $\Lambda_g \in L^p(\mu)^*$ .

We now want to show  $\|\Lambda_g\|_{L^p(\mu)^*} \geq \|g\|_{L^q(\mu)}$ .

If  $\|g\|_{L^q(U)} = 0$ , we have that  $g = 0$ a.e., and  $\|\Lambda_g\|_{L^p(\mu)^*} = 0 = \|g\|_{L^q(U)}$ .

Now consider  $\|g\|_{L^q(U)} \neq 0$ . It suffices to find some  $\|f\|_{L^q(U)} = 1$ , such that  $\Lambda_g(f) \geq \|g\|_{L^q(U)}$ .

1.  $1 < p < \infty$ .

Notice that  $p = \frac{1}{1-\frac{1}{q}} = \frac{1}{\frac{q-1}{q}} = \frac{q}{q-1}$ .

Let  $f = \overline{\text{sgn}(g)} \frac{|g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}}$ , we have that

$$\begin{aligned}
\|f\|_{L^p(\mu)}^p &= \int |f|^p d\mu \\
&= \int \frac{|g|^q}{\|g\|_{L^q(\mu)}^q} d\mu \\
&= \frac{1}{\|g\|_{L^q(\mu)}^q} \int |g|^q d\mu \\
&= \frac{1}{\|g\|_{L^q(\mu)}^q} \|g\|_{L^q(U)}^q \\
&= 1,
\end{aligned}$$

which means that  $f \in L^p(\mu)$ . In addition,

$$\begin{aligned}
|\Lambda_g(f)| &= \left| \int_X f g d\mu \right| \\
&= \left| \int_X \overline{\operatorname{sgn}(g)} \frac{|g|^{q/p}}{\|g\|_{L^q(\mu)}^{q/p}} g d\mu \right| \\
&= \frac{1}{\|g\|_{L^q(\mu)}^{q/p}} \left| \int_X |g|^{1+q/p} d\mu \right| \\
&= \frac{1}{\|g\|_{L^q(\mu)}^{q-1}} \left| \int_X |g|^q d\mu \right| \\
&= \|g\|_{L^q(U)}.
\end{aligned}$$

Thus,  $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$ .

2.  $p = \infty, q = 1$ .

Let  $f = \overline{\operatorname{sgn}(g)} \in L^\infty(\mu)$ . We have  $\|f\|_{L^\infty(\mu)} = 1$ . In addition,

$$\Lambda_g(f) = \int_X \overline{\operatorname{sgn}(g)} g d\mu = \int_X |g| d\mu = \|g\|_{L^1(\mu)}.$$

Thus,  $\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \|g\|_{L^q(\mu)}$

3.  $p = 1, q = \infty$ , and  $\mu$  is semi-finite.

Choose  $\epsilon \in (0, \|g\|_{L^\infty(\mu)})$ .

Let  $A = \left\{ x \in X \mid |g(x)| > \|g\|_{L^\infty(\mu)} - \epsilon \right\}$ .

Notice that  $\mu(A) > 0$ , otherwise  $\|g\|_{L^\infty(\mu)} = \epsilon$ .

Since  $\mu$  is semi-finite, we can find  $E \in \mathcal{M}$ , such that  $0 < \mu(E) < \infty, E \subseteq A$ .

Let  $f = \frac{\chi_E}{\mu(E)} \overline{\operatorname{sgn}(g)}$ .

Notice that

$$\begin{aligned}
\|f\|_{L^1(\mu)} &= \int_X |f| d\mu \\
&= \int_X \left| \frac{\chi_E}{\mu(E)} \overline{\operatorname{sgn}(g)} \right| d\mu \\
&= \frac{1}{\mu(E)} \int_X \chi_E d\mu \\
&= 1.
\end{aligned}$$

Thus  $f \in L^1(\mu)$ . In addition,

$$\begin{aligned}
\Lambda_g(f) &= \int_X f g d\mu \\
&= \int_X \frac{\chi_E}{\mu(E)} \overline{\operatorname{sgn}(g)} g d\mu \\
&= \int_E \frac{|g|}{\mu(E)} d\mu \\
&\geq \int_E \frac{\|g\|_{L^\infty(\mu)} - \epsilon}{\mu(E)} d\mu \\
&\geq \|g\|_{L^\infty(\mu)} - \epsilon.
\end{aligned}$$

Since this holds for any  $\epsilon > 0$ , we have that

$$\|\Lambda_g\|_{L^p(\mu)^*} = \sup_{f \in L^p(\mu), f \neq 0} \frac{|\Lambda_g(f)|}{\|f\|_{L^p(\mu)}} \geq \sup_{\epsilon > 0} \|g\|_{L^\infty(\mu)} - \epsilon = \|g\|_{L^q(\mu)}.$$

We thus have  $\|\Lambda_g\|_{L^p(\mu)^*} = \|g\|_{L^q(\mu)}$ .  $\square$

*Remark.* In the above case, the map  $g \mapsto \Lambda_g$  is isometric.

## 7 Complex measures

### 7.1 Signed measures

Recall that if  $(X, \mathcal{M}, \mu)$  is a measure space, and  $f : X \rightarrow [0, \infty)$  is measurable, then we can set a measure  $\mu_f(A) := \int_X \chi_A f d\mu$ , and we have  $\int_X g d\mu_f = \int g f d\mu$ .

**Example 7.1.1.** Consider the regular Lebesgue measure, and  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ , then  $\lambda_f$  gives a probability measure with the standard distribution.

We want to generalize this to functions that are not non-negative.

**Definition 7.1.** Let  $(X, \mathcal{M})$  be a measurable space. A function  $\nu : \mathcal{M} \rightarrow [-\infty, \infty]$  is a **signed measure** if

$$\begin{aligned} \nu(\emptyset) &= 0 \\ \nu\left(\bigsqcup_{i=1}^{\infty} E_i\right) &= \sum_{i=1}^{\infty} \nu(E_i), \end{aligned}$$

and  $\nu$  only takes at most one of  $\pm\infty$ .

**Proposition 7.1.** Suppose  $|\nu(\bigsqcup_{i=1}^{\infty} E_i)| < \infty$ , then  $\sum_{i=1}^{\infty} \nu(E_i)$  must converge absolutely, since we want  $\nu(\bigsqcup_{i=1}^{\infty} E_i)$  to be invariant of the order of union.

**Proposition 7.2.** If  $f \in \mathcal{L}^1(X, \mu)$ , then  $\mu_f := \int_X \chi(A) f d\mu$  is a signed measure.

**Proposition 7.3.** If  $f, g \geq 0$  is measurable, and  $g \in \mathcal{L}^1(\mu)$ , then  $\nu(E) := \int_X \chi_E(f - g) d\mu$  is a signed measure.

**Definition 7.2.** Suppose  $\nu$  is a signed measure, then  $E \in \mathcal{M}$  is

1. **null** for  $\nu$  if  $\forall F \subseteq E, \nu(F) = 0$ .
2. **positive** for  $\nu$  if  $\forall F \subseteq E, \nu(F) \geq 0$ .
3. **negative** for  $\nu$  if  $\forall F \subseteq E, \nu(F) \leq 0$ .

**Lemma 7.4.** Let  $E \in \mathcal{M}$ , if  $0 < \nu(E) < \infty$ , then  $\exists A \subseteq E, A \in \mathcal{M}$  is positive, and  $\nu(A) > 0$ .

*Proof.* Suppose  $E$  is positive, then we are done.

Suppose  $E$  is not positive, then  $\inf \{\nu(B) | B \subseteq E, B \in \mathcal{M}\} < 0$ .

Thus,  $\frac{1}{2} \inf \{\nu(B) | B \subseteq E, B \in \mathcal{M}\} > \inf \{\nu(B) | B \subseteq E, B \in \mathcal{M}\}$ , and we can choose  $B_1 \subseteq E, B_1 \in \mathcal{M}$ , such that

$$\nu(B_1) \leq \frac{1}{2} \inf \{\nu(B) | B \subseteq E, B \in \mathcal{M}\} \leq \max \left\{ -1, \frac{1}{2} \inf \{\nu(B) | B \subseteq E, B \in \mathcal{M}\} \right\}.$$

Recursively choose  $B_n \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i$  with  $\nu(B_n) \leq \max \left\{ -1, \frac{1}{2} \inf \{\nu(B) | B \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i, B \in \mathcal{M}\} \right\}$ .

Now either this sequence terminates (then  $A = E \setminus \bigsqcup_{i=1}^{n-1} B_i$  is positive), or we get an infinite sequence.

Set  $A := E \setminus \bigsqcup_{i=1}^{\infty} B_i$ .

We have  $\nu(E) = \nu(A) + \sum_{i=1}^{\infty} \nu(B_i) < \infty$ , thus  $\sum_{i=1}^{\infty} \nu(B_i)$  converges absolutely.

Thus,  $\nu(A) = \nu(E) - \sum_{i=1}^{\infty} \nu(B_i) > \nu(E) > 0$ , since each  $\nu(B_i) < 0$  by construction.

Notice that  $\nu(B_n) \rightarrow 0^-$  since the sum converges, so if  $B \subseteq A \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i$  has  $\nu(B) < 0$ , we must have  $\nu(B) < 2\nu(B_n)$  for some large  $n$ . However, by construction,

$$\inf \left\{ \nu(B) \mid B \subseteq E \setminus \bigsqcup_{i=1}^{n-1} B_i, B \in \mathcal{M} \right\} \geq 2\nu(B_n),$$

which is a contradiction.

Thus  $A$  is positive.  $\square$

**Lemma 7.5.** *If  $(A_n)_{n=1}^{\infty}$  is a sequence of positive sets, then  $A := \bigcup_{n=1}^{\infty} A_n$  is positive.*

*Proof.* Consider any  $B \subseteq A, B \in \mathcal{M}$ . Let  $B_n := B \cap (A_n \setminus \bigsqcup_{i=1}^{n-1} A_i)$ , then  $(B_n)_{n=1}^{\infty} \subseteq \mathcal{M}$  are pairwise disjoint, with  $B = \bigsqcup_{n=1}^{\infty} B_n$ .

For any  $n$ , since  $A_n$  is positive, and  $B_n \subseteq A_n$ , we have  $\nu(B_n) > 0$ .

Thus  $\nu(B) = \sum_{n=1}^{\infty} \nu(B_n) > 0$ .  $\square$

**Theorem 7.6** (Hahn decomposition). *Let  $\nu$  be a signed measure on  $(X, \mathcal{M})$ , there are  $P, N \in \mathcal{M}$  such that  $X = P \sqcup N$ , and  $P$  is positive,  $N$  is negative. Moreover, this is unique in the sense that if  $X = P' \sqcup N'$  is another such decomposition, then the symmetric difference  $P \Delta P'$  is null.*

*Proof.* Existence:

By taking  $-\nu$  if necessary, we can WLOG assume  $\nu$  never takes  $+\infty$ .

Let  $m := \sup \{ \nu(A) : A \text{ is positive} \} < \infty$ .

Choose positive sets  $A_n$  such that  $\nu(A_n) \rightarrow m$ , and let  $P := \bigcup_{n=1}^{\infty} A_n$ .

Thus  $P$  is positive by lemma, so  $\nu(P) \leq m$ .

Notice that  $\forall n, \nu(P) = \nu(A_n) + \nu(P \setminus A_n) \geq \nu(A_n)$ , since  $P \setminus A_n \subseteq P$  must have  $\nu(P \setminus A_n) \geq 0$ .

Taking the supremum over  $n$ , we have  $\nu(P) \geq m$ .

Thus  $\nu(P) = m$ .

Let  $N := X \setminus P$ .

Suppose  $N$  is not negative,  $\exists E \subseteq N, E \in \mathcal{M}$ , such that  $\nu(E) > 0$ . By lemma, there is positive  $A \subseteq E$  with  $\nu(A) > 0$ . Then  $P \sqcup A$  is measurable, positive, and  $\nu(P \sqcup A) = \nu(P) + \nu(A) > m$ , which is a contradiction.

Uniqueness:

Let

$$A := P \setminus P' = (X \setminus N) \setminus P' = X \cap N^c \cap (P')^c = N' \setminus N.$$

It is both positive and negative, thus null. Similarly,  $B := P' \setminus P = N \setminus N'$  is null. Thus,  $P \Delta P' = A \cup B$  is null.  $\square$

**Definition 7.3.** Let  $\nu$  be a signed measure on  $(X, \mathcal{M})$ , and  $P, N \in \mathcal{M}$  be as from Hahn decomposition, the **Jordan decomposition** of it is  $\nu = \nu^+ - \nu^-$ , where  $\nu^+(E) := \nu(E \cap P), \nu^-(E) := -\nu(E \cap N)$  are positive measures.

**Definition 7.4.** Let  $\nu$  be a signed measure on  $(X, \mathcal{M})$ , with Jordan decomposition  $\nu = \nu^+ - \nu^-$ , the total variation  $|\nu| = \nu^+ + \nu^-$ .

**Corollary 7.7.** *Let  $\nu = \nu_+ - \nu_-$  be a signed measure with its Jordan decomposition, there exists a measurable function  $f : X \rightarrow \mathbb{R}$  with  $|f| = 1$ , and*

$$\forall E \in \mathcal{M}, \nu(E) = \int_E f d|\nu|,$$

where  $|\nu| = \nu_+ + \nu_-$ .

*Proof.* Let  $X = P \sqcup N$  be the Hahn decomposition. Define  $f(x) = \chi_P - \chi_N$ , then clearly  $|f| = 1$ . In addition,

$$\begin{aligned} \int_E f d|\nu| &= \int_E \chi_P - \chi_N d|\nu| \\ &= \int_{E \cap P} d|\nu| - \int_{E \cap N} d|\nu| \\ &= |\nu|(E \cap P) - |\nu|(E \cap N) \\ &= \nu^+(E \cap P) - \nu^-(E \cap N) \\ &= \nu^+(E) - \nu^-(E) \\ &= \nu(E). \end{aligned}$$

□

**Proposition 7.8.** Let  $\nu$  be a signed measure on  $(X, \mathcal{M})$ , with Jordan decomposition  $\nu = \nu^+ - \nu^-$ , then for any other positive  $\lambda_1, \lambda_2$ , such that  $\nu = \lambda_1 - \lambda_2$ , we have  $\lambda_1 \geq \nu^+, \lambda_2 \geq \nu^-$ .

*Proof.* Let  $X = P \sqcup N$  be the Hahn decomposition.

Consider any  $E \in \mathcal{M}$ , we have

$$\begin{aligned} \lambda_1(E) &\geq \lambda_1(E \cap P) \\ &= \nu(E \cap P) + \lambda_2(E \cap P) \\ &\geq \nu(E \cap P) \\ &= \nu^+(E) \\ \lambda_2(E) &\geq \lambda_2(E \cap N) \\ &= -\nu(E \cap N) + \lambda_1(E \cap N) \\ &\geq -\nu(E \cap N) \\ &= \nu^-(E). \end{aligned}$$

□

## 7.2 Complex measures

**Definition 7.5.** Let  $(X, \mathcal{M})$  be a measurable space, a **complex measure** is a function  $\nu : \mathcal{M} \rightarrow \mathbb{C}$ , such that

1.  $\nu(\emptyset) = 0$
2.  $\nu(\bigsqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$  absolutely.

*Remark.* The absolute convergence is important since we want  $\nu(\bigsqcup_{i=1}^{\infty} E_i)$  to be invariant of the order of union.

**Example 7.2.1.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $f \in \mathcal{L}^{\infty}(\mu)$  with  $\|f\|_{L^{\infty}(\mu)} = 1$ , then with  $\nu(E) := \int_E f d\mu$ , we have  $\nu$  is a complex measure, and  $\forall E \in \mathcal{M}$ ,

$$\begin{aligned} \mu(E) &= \int_E d\mu \\ &\geq \int_E |f| d\mu \\ &\geq \left| \int_E f d\mu \right| \\ &\geq \left| \int_E d\nu \right| \\ &= |\nu(E)|. \end{aligned}$$

**Definition 7.6.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, the **total variation** of a complex measure  $\mu$  is  $|\mu| : \mathcal{M} \rightarrow [0, \infty]$ , defined by

$$\forall E \in \mathcal{M}, |\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : E = \bigsqcup_{i=1}^{\infty} E_i, (E_i)_{i=1}^{\infty} \subset \mathcal{M} \right\}.$$

**Proposition 7.9.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $\mu$  be a complex measure, then  $|\mu|$  is a positive measure.

*Proof.* 1.  $|\mu|(\emptyset) = 0$ .

2.  $|\mu|(E) \geq 0, \forall E \in \mathcal{M}$ .

3. Fix  $E = \bigsqcup_{i=1}^{\infty} E_i \in \mathcal{M}, (E_i)_{i=1}^{\infty} \subset \mathcal{M}$ .

Consider any  $(A_j)_{j=1}^{\infty} \subset \mathcal{M}$  such that  $E = \bigsqcup_{j=1}^{\infty} A_j$ , then  $A_j = A_j \cap E = \bigsqcup_{i=1}^{\infty} A_j \cap E_i$ .

$$\begin{aligned} \sum_{j=1}^{\infty} |\mu(A_j)| &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)| \\ &\leq \sum_{i=1}^{\infty} |\mu| \left( \bigsqcup_{j=1}^{\infty} (A_j \cap E_i) \right) \\ &= \sum_{i=1}^{\infty} |\mu|(E_i). \end{aligned}$$

We have that

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(A_i)| : E = \bigsqcup_{i=1}^{\infty} A_i, (A_i)_{i=1}^{\infty} \subset \mathcal{M} \right\} \leq \sum_{i=1}^{\infty} |\mu|(E_i).$$

Now given any  $\epsilon > 0$ .

$\forall i$ , pick  $t_i := |\mu|(E_i) - \frac{\epsilon}{2^i}$  and we can find  $E_{ij} \in \mathcal{M}$ , such that

$$E_i = \bigsqcup_{j=1}^{\infty} E_{ij}, \quad \sum_{j=1}^{\infty} |\mu(E_{ij})| > t_i.$$

We have

$$\begin{aligned} |\mu|(E) &\geq \sum_{i,j=1}^{\infty} |\mu(E_{ij})| \\ &\geq \sum_{i=1}^{\infty} t_i \\ &= \sum_{i=1}^{\infty} |\mu|(E_i) - \epsilon. \end{aligned}$$

Taking  $\epsilon \rightarrow 0$ , we have  $|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(E_i)$ .

Thus  $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(E_i)$ . □

**Lemma 7.10.** Let  $\{z_1, \dots, z_N\} \subset \mathbb{C}$ , then  $\exists S \subseteq [N]$ , such that

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^N |z_k|.$$

**Theorem 7.11.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $\mu$  be a complex measure, then  $|\mu|$  is a finite measure. Namely,  $\forall E \in \mathcal{M}$ , such that  $|\mu|(E) < \infty$ .

*Proof.* Suppose  $\exists E \in \mathcal{M}$ , such that  $|\mu|(E) = \infty$ .

Let  $B_0 = E$ .

Let  $t := \pi(1 + |\mu(E)|) \geq \pi$ .

Then we can find a partition  $E_i \in \mathcal{M}$ , such that

$$E = \bigsqcup_{i=1}^{\infty} E_i, \quad \sum_{i=1}^{\infty} |\mu(E_{ij})| > t.$$

Thus there is some  $N \in \mathbb{N}$ , such that  $\sum_{i=1}^N |\mu(E_{ij})| > t$ .

By lemma,  $\exists S \subseteq [N]$ , such that

$$\begin{aligned} \left| \mu \left( \bigsqcup_{i \in S} E_i \right) \right| &= \left| \sum_{i \in S} \mu(E_i) \right| \\ &\geq \frac{1}{\pi} \sum_{i=1}^N |\mu(E_i)| \\ &> \frac{t}{\pi} \\ &\geq 1. \end{aligned}$$

Now let  $A := \bigsqcup_{i \in S} E_i, B = E \setminus A$ .

We have

$$\begin{aligned} |\mu(B)| &= |\mu(E) - \mu(A)| \\ &\geq |\mu(A)| - |\mu(E)| \\ &> \frac{t}{\pi} - |\mu(E)| \\ &= 1. \end{aligned}$$

Thus  $E = A \sqcup B$ , where  $|\mu(A)| > 1, |\mu(B)| > 1$ .

Since  $|\mu|(A \sqcup B) = |\mu|(A) + |\mu|(B) = \infty$ , at least one of  $|\mu|(A), |\mu|(B)$  is  $\infty$ .

WLOG, say  $|\mu|(B) = \infty$ . We let  $A_1 = A, B_1 = B$ .

Now apply the above argument on  $B_1 = A_2 \sqcup B_2$ , where  $|\mu(A_2)| > 1, |\mu(B_2)| > 1, |\mu|(B_2) = \infty$ .

Repetitively, we construct disjoint  $(A_k)_{k=1}^{\infty}$ , such that  $\forall i \geq 1, |\mu(A_i)| > 1$ .

Notice that  $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{M}$ , and we have  $\mu(\bigsqcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$  absolutely.

However,  $\sum_{k=1}^{\infty} |\mu(A_k)| \geq \sum_{k=1}^{\infty} 1 = \infty$  diverges. □

**Definition 7.7.** Let  $\mu, \nu$  be two complex measures on  $(X, \mathcal{M})$ . A function  $f$  is the **Radon–Nikodym derivative**, written as  $\frac{d\nu}{d\mu}$ , if  $\forall E \in \mathcal{M}, \nu(E) = \int_E f d\mu$ . We can also abuse the notation and write  $d\nu = f d\mu$ .

**Proposition 7.12.** If the Radon–Nikodym derivative exists, it is unique  $\mu$ -a.e..

*Proof.* Suppose there are two  $f, h$  that satisfies  $\forall E \in \mathcal{M}, \nu(E) = \int_E f d\mu = \int_E h d\mu$ , then  $\int_E (f - h) d\mu = 0$ . Thus  $f = h$   $\mu$ -a.e.. □

### 7.3 Radon–Nikodym–Lebegue Decomposition

**Definition 7.8.** Let  $\mu, \nu$  be two measures on  $(X, \mathcal{M})$ , we say  $\nu$  is **absolutely continuous** with respect to  $\mu$  if  $\mu(A) = 0 \implies \nu(A) = 0$ , and is written as  $\nu << \mu$ .

**Example 7.3.1.** Consider the counting measure  $\mu$ , then for any other (non trivially infinite) measure  $\nu$ , we always have  $\nu << \mu$ , since  $\mu(E) = 0 \implies E = \emptyset$ .

**Lemma 7.13.** Let  $\mu$  be a  $\sigma$ -finite measure on  $(X, \mathcal{M})$ , then there is some  $w \in L^1(\mu)$ , such that  $\forall x \in X, 0 < w(x) < 1$ .

*Proof.* Write  $X = \bigcup_{n=1}^{\infty} E_n$ , where  $\forall n \geq 1, \mu(E_n) < \infty$ .

Let  $w_n := \frac{2^{-n}\chi_{E_n}}{1+\mu(E_n)}$ ,  $w = \sum_{n=1}^{\infty} w_n$ .

Notice that  $0 < w_n(x) < 1$ , and

$$\begin{aligned} \int_X w d\mu &= \sum_{n=1}^{\infty} \int_X w_n d\mu \\ &= \sum_{n=1}^{\infty} \int_X \frac{2^{-n}\chi_{E_n}}{1+\mu(E_n)} d\mu \\ &\leq \sum_{n=1}^{\infty} \frac{2^{-n}\mu(E_n)}{1+\mu(E_n)} \\ &< \sum_{n=1}^{\infty} 2^{-n} \\ &< \infty. \end{aligned}$$

□

**Lemma 7.14.** Let  $\mu$  be a  $\sigma$ -finite measure on  $(X, \mathcal{M})$ , and  $g \in L^1(\mu)$ . Suppose  $\forall E \in \mathcal{M}$  such that  $\mu(E) > 0$ , we have that

$$\frac{1}{\nu(E)} \int_E g d\mu \in S$$

for some closed  $S \subseteq \mathbb{C}$ , then

$$g(x) \in S, \text{ a.e. } x \in X.$$

*Proof.* Assume for contradiction that there is  $E =: g^{-1}(\bar{B}(t, r))$  such that  $\mu(E) > 0, \bar{B}(t, r) \subseteq S^c$ . Then  $A_E(g) := \frac{1}{\mu(E)} \int_E g d\mu \in S$ , while

$$\begin{aligned} |A_E(g) - t| &= \left| \frac{1}{\mu(E)} \int_E (g - t) d\mu \right| \\ &\leq \frac{1}{\mu(E)} \int_E |g - t| d\mu \\ &\leq \frac{1}{\mu(E)} \int_E r d\mu \\ &= r. \end{aligned}$$

Thus  $A_E(g) \in \bar{B}(t, r) \subseteq S^c$ , which is a contradiction. □

**Theorem 7.15** (Radon–Nikodym for finite measure). Let  $\mu$  be a  $\sigma$ -finite measure, and  $\nu$  is a positive finite measure on  $(X, \mathcal{M})$ . Suppose  $\nu \ll \mu$ , then  $\exists h \in L^1(\mu) \cap \mathcal{L}^+$ , such that  $\forall E \in \mathcal{M}, \nu(E) = \int_E h d\mu$ . Moreover,  $h$  is unique  $\mu$ -a.e.

*Proof.* (Von Neumann's proof).

Since  $\mu$  is  $\sigma$ -finite, there is some  $w \in L^1(\mu)$ , such that  $\forall x \in X, 0 < w(x) < 1$ .

Define a new measure  $d\lambda := d\nu + wd\mu$ , namely,  $\forall E \in \mathcal{M}, \lambda(E) := \nu(E) + \int_E w d\mu$ .

**Claim 7.15.1.** There is some measurable  $g$  such that  $\forall x \in X, g(x) \in [0, 1]$ , and for any measurable  $f \in L^2(\lambda)$ , we have  $\int_E f(1 - g) d\nu = \int_E fgw d\mu$ .

*Proof.* Notice that  $\int_X f d\lambda = \int_X f d\lambda + \int_X f w d\lambda$  for any measurable  $f$ . Consider any  $f \in L^2(\lambda)$ ,

$$\begin{aligned} \left| \int_X f d\nu \right| &\leq \int_X |f| d\nu \\ &= \int_X |f| d\lambda - \int_X |f| w d\mu \\ &\leq \int_X |f| d\lambda \\ &\leq \int_X |f| \cdot 1 d\lambda \\ &\leq \|f\|_{L^2(\lambda)} \lambda(X). \end{aligned}$$

Notice that  $\lambda(X) = \nu(X) + \int_X w d\mu < \infty$ , so  $\Lambda : f \mapsto \int_X f d\nu \in L^2(\lambda)^*$ . Since  $L^2(\lambda)$  is a Hilbert space, there is a unique  $g \in L^2(\lambda)$ , such that  $\int_X f g d\lambda = \Lambda(f) = \int_X f d\nu, \forall f \in L^2(\lambda)$ . Now we know  $\int_X f d\nu = \int_X f g d\lambda = \int_X f g d\nu + \int_X f g w d\mu$ . For any  $E \in \mathcal{M}, f \in L^2(\lambda)$ , we can take  $\tilde{f} := f \chi_E \in L^2(\lambda)$ , and we get

$$\begin{aligned} \int_E f(1-g) d\nu &= \int_X \tilde{f}(1-g) d\nu \\ &= \int_X \tilde{f} d\nu - \int_X \tilde{f} g d\nu \\ &= \int_X \tilde{f} g w d\mu + \int_X \tilde{f} g d\nu - \int_X \tilde{f} g d\nu \\ &= \int_X \tilde{f} g w d\mu \\ &= \int_E f g w d\mu. \end{aligned}$$

In addition, for any  $E \in \mathcal{M}$ , taking  $f = 1$ , we have that

$$\nu(E) = \int_E d\nu = \int_E g d\lambda.$$

Thus  $0 \leq \int_E g d\lambda \leq \lambda(E)$ .

Thus  $\forall E \in \mathcal{M}$ , such that  $\lambda(E) > 0$ , we have

$$\frac{\int_E g d\lambda}{\lambda(E)} \in [0, 1].$$

By the above lemma, we have that  $g(x) \in [0, 1], \lambda\text{-a.e. } x \in X$ .

WLOG, we can redefine  $g(x) = 0$  for any  $g(x) \notin [0, 1]$ . □

Let  $A := g^{-1}([0, 1]), B := g^{-1}(\{1\})$ .

Let  $f = \chi_B$ , we have that

$$\begin{aligned} \int_X \chi_B (1-g) d\nu &= \int_X \chi_B g w d\mu \\ \int_B (1-g) d\nu &= \int_B w d\mu \\ 0 &= \int_B w d\mu. \end{aligned}$$

Since  $w > 0$ , we must have  $\mu(B) = 0$ . Since  $\nu \ll \mu$ ,  $\nu(B) = 0$

Thus  $\forall E \in \mathcal{M}$ ,  $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu(E \cap A)$ .

Now, let  $f_n := \sum_{k=0}^n g^k$ , we have that  $f_n(1-g) = 1 - g^{n+1}$ , so

$$\int_E (1 - g^{n+1}) d\nu = \int_E f_n(1-g) d\nu = \int_E f_n g w d\mu.$$

Notice that  $1 - g^{n+1}(x) \rightarrow \begin{cases} 1, & x \in A, \\ 0, & x \in B. \end{cases}$  monotonically.

In addition,  $(f_n g w)(x)$  is increasing and bounded.

So there is some  $h(x) := \lim_{n \rightarrow \infty} (f_n g w)(x)$ .

Thus, by LMCT, we have

$$\begin{aligned} \nu(E) &= \nu(A \cap E) \\ &= \int_{E \cap A} d\nu \\ &= \int_{E \cap A} \lim_{n \rightarrow \infty} (1 - g^{n+1}) d\nu \\ &= \lim_{n \rightarrow \infty} \int_{E \cap A} (1 - g^{n+1}) d\nu \\ &= \lim_{n \rightarrow \infty} \int_E f_n g w d\mu \\ &= \int_E h d\mu. \end{aligned}$$

Since  $\nu$  is finite, we have  $h \in L^1(\mu)$ . □

**Theorem 7.16** (Radon–Nikodym). *Let  $\mu, \nu$  be two  $\sigma$ -finite measures on  $(X, \mathcal{M})$ . Suppose  $\nu \ll \mu$ , then  $\exists f \in \mathcal{L}^+$ , such that  $\forall E \in \mathcal{M}$ ,  $\nu(E) = \int_E f d\mu$ . Moreover,  $f$  is unique  $\mu$ -a.e.*

*Proof.* Since  $\nu$  is  $\sigma$ -finite, we have  $X = \bigsqcup_{n=1}^{\infty} X_n$ , where each  $\nu(X_n)$  is finite.

We can apply the above theorem on  $\nu_n(E) := \nu(E \cap X_n)$ , which are finite measures, and let  $h = \sum_{n=1}^{\infty} h_n$ .  $h$  will be positive and measurable, but not in  $L^1(\mu)$ . Yet it is in  $L^1(\mu|_{X_n})$  for all  $n$ . □

*Remark.* The  $\sigma$ -finiteness is essential. Indeed, consider the following counterexample.

**Example 7.3.2.** Consider  $\lambda$  to be the Lebesgue measure on  $(0, 1)$ , and  $\mu$  to be the counting measure, which is not  $\sigma$ -finite.

Although  $\lambda \ll \mu$ , it is impossible to find such an  $h = \frac{d\lambda}{d\mu}$ , because for any  $E \in \mathcal{M}$ , we will have

$$\begin{aligned} \lambda_a(E) &= \int_E h d\mu \\ &= \sum_{x \in E} h(x), \end{aligned}$$

which is not possible.

**Corollary 7.17.** *Let  $\nu = \nu_+ - \nu_-$  be a signed measure with its Jordan decomposition. Suppose  $|\nu|, \mu$  are  $\sigma$ -finite with  $|\nu| \ll \mu$ , then  $\exists g = g_+ + g_- : X \rightarrow \mathbb{R} \cup \pm\infty$  measurable function, with at most one of  $g_+, g_-$  take  $\infty$ , such that  $\nu(E) = \int_E g d\mu$ .*

**Corollary 7.18** (Polar decomposition of complex measures). *Let  $\nu$  be a complex measure on  $(X, \mathcal{M})$ . There is a unique measurable function  $h$ , such that  $|h| = 1$   $|\nu|$ -a.e., and  $d\nu = hd|\nu|$ .*

*Proof.* Write  $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$  with Jordan decomposition.

Notice that  $\nu_i << |\nu|$  since  $|\nu_i(E)| \leq |\nu(E)| \leq |\nu|(E)$ . Applying Radon-Nikodym, we have some  $f_i \in \mathcal{L}^+$ , where  $\forall E \in \mathcal{M}, \nu_i(E) = \int_E f_i d|\nu|$ . Define  $h := (f_1 - f_2 + if_3 - if_4)$ , and we have

$$\begin{aligned}\nu(E) &= \nu_1(E) - \nu_2(E) + i\nu_3(E) - i\nu_4(E) \\ &= \int_E (f_1 - f_2 + if_3 - if_4) d|\nu| \\ &= \int_E h d|\nu|.\end{aligned}$$

For any  $E \in \mathcal{M}$ , we have

$$1 \geq \frac{|\nu(E)|}{|\nu|(E)} = \left| \frac{1}{|\nu|(E)} \int_E h d|\nu| \right|.$$

Thus  $|h(x)| \leq 1$  a.e..

For any  $0 < r < 1$ , consider  $A_r := \{x \in X : |h(x)| < r\} = \bigsqcup_{i=1}^{\infty} E_i$ .

We have

$$\begin{aligned}\sum_{i=1}^{\infty} |\nu(E_i)| &= \sum_{i=1}^{\infty} \left| \int_{E_i} h d|\nu| \right| \\ &\leq \sum_{i=1}^{\infty} \left| \int_{E_i} r d|\nu| \right| \\ &= r \sum_{i=1}^{\infty} |\nu|(E_i) \\ &= r |\nu|(A_r).\end{aligned}$$

Taking sup over all  $E_i$ , we have that  $|\nu|(A_r) \leq r|\nu|(E_i)$ . Since  $r < 1$ , we have  $|\nu|(E_i) = 0$ .

Thus  $|h(x)| > 1$  a.e.  $x \in X$  for all  $0 < r < 1$ .

Thus  $|h| = 1$  a.e..  $\square$

**Corollary 7.19.** If  $d\lambda = gd\mu$ , then we have  $h|\lambda| = |g|d\mu$ .

**Definition 7.9.** Two measures  $\mu, \nu$  on  $(X, \mathcal{M})$  are said to be **mutually singular**, written as  $\mu \perp \nu$ , if  $X = A \sqcup B$ , where  $A$  is  $\mu$ -null and  $B$  is  $\nu$ -null.

**Theorem 7.20** (Lebesgue decomposition). Let  $\mu, \nu$  be two  $\sigma$ -finite measures on  $(X, \mathcal{M})$ . There is a unique decomposition  $\nu = \nu_a + \nu_s$  with  $\nu_a << \mu$  be the absolutely continuous part, and  $\nu_s \perp \mu$  be the singular part, both positive measures.

*Proof.* Take  $\lambda = \mu + \nu$ , which is  $\sigma$ -finite, and  $\mu, \nu << \lambda$ . By Radon-Nikodym,  $\exists f, g \in \mathcal{L}^+$ , such that

$$\mu(E) = \int_E f d\lambda, \quad \nu(E) = \int_E g d\lambda.$$

Let  $A = f^{-1}((0, \infty]), B = f^{-1}(\{0\}), \nu_a(E) = \nu(E \cap A), \nu_s(E) = \nu(E \cap B)$ .

Since  $X = A \sqcup B$ , clearly  $\nu = \nu_a + \nu_s$ .

We can see that  $\nu_s$  is  $A$ -null.

On the other hand,  $\forall E \subseteq B, \mu(E) = \int_E f d\lambda = 0$ , so  $\mu$  is  $B$ -null.

This shows  $\nu_s \perp \mu$ .

In addition, suppose  $\mu(E) = \int_E f d\lambda = 0$ , then we must have  $\lambda(E \cap A) = 0$ , which implies  $\nu_a(E) = \nu(E \cap A) = \int_{E \cap A} g d\lambda = 0$ . Thus  $\nu_a << \mu$ .  $\square$

**Theorem 7.21** (Lebegue-Radon-Nikodym for complex measures). Let  $\nu$  be a complex measure on  $(X, \mathcal{M})$ . If  $\mu$  is a  $\sigma$ -finite measure on  $(X, \mathcal{M})$ , then  $\nu$  decomposes uniquely as  $\nu = \nu_a + \nu_s$ , such that  $\nu_a << \mu$  is the absolutely continuous part and  $\nu_s \perp \mu$  is the singular part. Also,  $d\nu_a = fd\mu$  for some  $f \in \mathcal{L}^1(\mu)$ .

*Proof.* From the polar decomposition, we have  $d\nu = hd|\nu|$ , and we can use Lebesgue decomposition to write  $|\nu| = |\nu|_a + |\nu|_s$ .

Let  $g = \frac{d|\nu|_a}{d\mu}$  be the Randon-Nikodym derivative, we have  $d\nu = hgd\mu + hd|\nu|_s$ .

Let  $f := hg$ , and  $d\nu_a := fd\mu$ ,  $d\nu_s := hd|\nu|_s$ , and we have the result.

Now we want to uniqueness:

Suppose  $\nu = \nu_a + \nu_s = \nu'_a + \nu'_s$  are two decompositions as in the theorem.

Then  $(\nu_a - \nu'_a) + (\nu_s - \nu'_s) = 0$  is the zero measure.

Thus,  $\mu' := \nu_a - \nu'_a = \nu'_s - \nu_s$ , where  $\nu_a - \nu'_a \ll \mu$ , and  $\nu'_s - \nu_s \perp \mu$ .

Thus,  $\mu' = 0$ , and  $\nu_a = \nu'_a$ ,  $\nu'_s = \nu_s$ .  $\square$

## 8 Dual of Function Spaces

### 8.1 Dual of $L^p$ Spaces

**Theorem 8.1.** Let  $(X, \mathcal{M}, \mu)$  be a measure space, and  $\frac{1}{p} + \frac{1}{q} = 1$  for  $p \in (1, \infty)$ , we have

$$L^q(\mu) \cong L^p(\mu)^*,$$

where the isometric isomorphism  $L^q(\mu) \xrightarrow{\sim} L^p(\mu)^*$ ;  $g \mapsto \Lambda_g$  is defined to be

$$\forall f \in L^p(\mu), \Lambda_g(f) := \int_X f g d\mu.$$

In addition, the same is true for  $p = 1$  if  $\mu$  is  $\sigma$ -finite.

*Proof.* Let  $1 \leq p < \infty$ .

1. By 6.19, we only need to show the subjectivity:  $\forall \Lambda \in L^p(\mu)^*, \exists g \in L^q(\mu)$ , such that  $\Lambda = \Lambda_g$ .

2. First we assume  $\mu(X) < \infty$  is finite.

Given any  $\Lambda \in L^p(\mu)^*$ .

Consider the mapping  $\nu : E \mapsto \Lambda(\chi_E)$  for any measurable  $E \in \mathcal{M}$ .

This is well-defined since  $\chi_E \in L^\infty(X) \subseteq L^p(X)$ .

Notice that  $|\nu(E)| = |\Lambda(\chi_E)| \leq \|\Lambda\|_{L^p(\mu)^*} \|\chi_E\|_{L^p(\mu)} < \infty$ , thus  $\nu$  is finite.

We have  $\nu(\emptyset) = \Lambda(0) = 0$  since  $\Lambda$  is linear.

For any  $B = \bigcup_{i=0}^{\infty} A_i$ , with  $A_i \subseteq U$  be measurable, we have  $\chi_B = \sum_{i=0}^{\infty} \chi_{A_i}$  in  $L^p(\mu)$ .

Indeed,

$$\begin{aligned} \left\| \chi_B - \sum_{i=0}^N \chi_{A_i} \right\|_{L^p(\mu)}^p &= \left\| \sum_{i=N+1}^{\infty} \chi_{A_i} \right\|_{L^p(\mu)}^p \\ &= \left\| \chi_{\bigcup_{i=N+1}^{\infty} A_i} \right\|_{L^p(\mu)}^p \\ &= \mu \left( \bigcup_{i=N+1}^{\infty} A_i \right)^p \\ &\rightarrow 0. \end{aligned}$$

Notice that this fails when  $p = \infty$ !

Thus,

$$\begin{aligned}
\nu(B) &= \Lambda(\chi_B) \\
&= \Lambda\left(\sum_{i=0}^{\infty} \chi_{A_i}\right) \\
&= \Lambda\left(\lim_{n \rightarrow \infty} \sum_{i=0}^n \chi_{A_i}\right) \\
&= \lim_{n \rightarrow \infty} \Lambda\left(\sum_{i=0}^n \chi_{A_i}\right) && \text{continuity of } \Lambda \\
&= \lim_{n \rightarrow \infty} \sum_{i=0}^n \Lambda(\chi_{A_i}) && \text{linearity of } \Lambda \\
&= \lim_{n \rightarrow \infty} \sum_{i=0}^n \nu(A_i) \\
&= \sum_{i=0}^{\infty} \nu(A_i),
\end{aligned}$$

which shows countable additivity.

In addition,

$$\begin{aligned}
\sum_{i=0}^{\infty} |\nu(A_i)| &= \lim_{n \rightarrow \infty} \sum_{i=0}^n |\Lambda(\chi_{A_i})| \\
&\leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \|\Lambda\|_{L^p(\mu)^*} \|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*} \lim_{n \rightarrow \infty} \sum_{i=0}^n \|\chi_{A_i}\|_{L^p(\mu)} \\
&= \|\Lambda\|_{L^p(\mu)^*} \lim_{n \rightarrow \infty} \sum_{i=0}^n \mu(A_i)^{1/p} \\
&\leq \|\Lambda\|_{L^p(\mu)^*} \left( \lim_{n \rightarrow \infty} \sum_{i=0}^n \mu(A_i) \right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*} \mu\left(\bigcup_{i=0}^{\infty} A_i\right)^{1/p} \\
&= \|\Lambda\|_{L^p(\mu)^*} \mu(B)^{1/p} \\
&\leq \|\Lambda\|_{L^p(\mu)^*} \mu(X)^{1/p} \\
&< \infty,
\end{aligned}$$

which converges absolutely.

Thus  $\nu$  is a complex measure.

In addition, if  $\mu(E) = 0$ , we have  $\nu(E) = \Lambda(\chi_E) = \Lambda(0) = 0$ .

Thus,  $\nu \ll \mu$ .

By Lebesgue-Radon-Nikodym for complex measures,  $\exists! g \in L^1(\mu)$ , such that  $\Lambda(\chi_E) = \nu(E) = \int_E g d\mu$ .

By linearity,  $\Lambda(f) = \int_X f g d\mu$  for all simple measurable  $f$ .

By uniform simple function approximation, we have  $\Lambda(f) = \int_X f g d\mu$  for all  $f \in L^\infty(\mu)$ .

Indeed, given any  $f \in L^\infty(\mu)$ , we have a sequence of simple measurable functions  $|f_1| \leq |f_2| \leq \dots \leq |f|$  that converges uniformly with  $\|f - f_n\|_{L^\infty(\mu)} \rightarrow 0$ .

Thus  $\|f - f_n\|_{L^p(\mu)} \rightarrow 0$ .

Thus  $|\Lambda(f) - \Lambda(f_n)| \leq \|\Lambda\|_{L^p(\mu)^*} \|f - f_n\|_{L^p(\mu)} \rightarrow 0$ .

$$\begin{aligned}\Lambda(f) &= \lim_{n \rightarrow \infty} \Lambda(f_n) \\ &= \lim_{n \rightarrow \infty} \int_X f_n g d\mu \\ &= \int_X \lim_{n \rightarrow \infty} f_n g d\mu \\ &= \int_X f g d\mu.\end{aligned}$$

(a)  $p = 1, q = \infty$ .

Consider any  $E \in \mathcal{M}$ , such that  $\mu(E) > 0$ .

We have

$$\begin{aligned}\left| \frac{1}{\mu(E)} \int_E g d\mu \right| &= \left| \frac{1}{\mu(E)} \Lambda(\chi_E) \right| \\ &\leq \frac{1}{\mu(E)} \|\Lambda\|_{L^1(\mu)^*} \|\chi_E\|_{L^1(\mu)} \\ &= \frac{1}{\mu(E)} \|\Lambda\|_{L^1(\mu)^*} \mu(E) \\ &= \|\Lambda\|_{L^1(\mu)^*}.\end{aligned}$$

Thus  $|g(x)| \leq \|\Lambda\|_{L^1(\mu)^*}$  a.e..

Thus  $g \in L^\infty(\mu)$ . Since simple functions are dense in  $L^p(\mu)$ , we have  $L^\infty(\mu)$  is dense in  $L^p(\mu)$ .

Since  $\Lambda, \Lambda_q$  are both bounded linear functionals, we have  $\Lambda(f) = \int_X f g d\mu$  for all  $f \in L^p(\mu)$ .

(b)  $p > 1$ .

Let  $E_n := \{x \in X : |g(x)| \leq n\}$ .

By LMCT, we have  $\|g\|_q = \lim_{n \rightarrow \infty} \|\chi_{E_n} g\|$ .

Let  $f = \chi_{E_n} \overline{\operatorname{sgn}(g)} |g|^{q-1} \in L^\infty(\mu)$ , we have

$$\begin{aligned}\|f\|_{L^p(\mu)}^p &= \int_{E_n} |g|^{(q-1)p} d\mu \\ &= \int_{E_n} |g|^q d\mu \\ &= \|\chi_{E_n} g\|_{L^q(\mu)}^q \\ \|\chi_{E_n} g\|_q^q &= \int_{E_n} |g|^q d\mu \\ &= \int_X f g d\mu \\ &= |\Lambda(f)| \\ &\leq \|\Lambda\| \|f\|_{L^p(\mu)} \\ &\implies \\ \|g \chi_{E_n}\|_{L^q(\mu)}^{q-\frac{q}{p}} &\leq \|\Lambda\| \\ &\implies \\ \|g\|_{L^q(\mu)}^{q-\frac{q}{p}} &\leq \|\Lambda\| \\ &< \infty.\end{aligned}$$

Thus  $g \in L^q(U)$ .

Since  $\Lambda, \Lambda_q$  are both bounded linear functionals, we have  $\Lambda(f) = \int_X f g d\mu$  for all  $f \in L^p(\mu)$ .

3. Now we assume that  $\mu$  is  $\sigma$ -finite.

We have  $X = \bigcup_{n=1}^{\infty} X_n, \forall n \geq 1, X_n \subset X_{n+1}, \mu(X_n) < \infty$ .

We can get

$$\forall n \geq 1, g_n \in L^q(X_n, \mu), \text{ such that } \Lambda(f) = \int_X f g_n d\mu, \forall f \in L^p(X_n, \mu).$$

Notice that  $L^p(X_n, \mu) \subset L^p(X_{n+1}, \mu)$ .

We thus have  $\forall n > m, g_n|_{X_m} = g_m$ .

Let  $g : X \rightarrow \mathbb{C}; x \mapsto g_n(x)$  for  $x \in X_n$ .

Then  $g = \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} g \chi_{X_n}$  in  $\|\cdot\|_{L^q(\mu)}$ , and thus  $g \in L^q(\mu)$ .

In addition, for any  $f \in L^p(\mu)$ , we have  $\lim_{n \rightarrow \infty} f \chi_{X_n} = f$  in  $\|\cdot\|_{L^q(\mu)}$ .

We have

$$\begin{aligned} \Lambda(f) &= \Lambda\left(\lim_{n \rightarrow \infty} f \chi_{X_n}\right) \\ &= \lim_{n \rightarrow \infty} \Lambda(f \chi_{X_n}) \\ &= \lim_{n \rightarrow \infty} \int_X f \chi_{X_n} g d\mu \\ &= \int_X f g d\mu. \end{aligned}$$

4. Now suppose  $\mu$  is not necessarily  $\sigma$ -finite, but  $p \in (1, \infty)$ .

$\forall E \subseteq X$  be  $\sigma$ -finite, we have

$$g_E \in L^q(E, \mu), \text{ such that } \Lambda(f) = \int_X f g_E d\mu, \forall f \in L^p(E, \mu)$$

In addition,  $\|g_E\|_{L^q(\mu)} \leq \|\Lambda\|$ .

Let  $M := \sup_{E \text{ is } \sigma\text{-finite}} \|g_E\|_{L^q(\mu)} \leq \|\Lambda\|$ .

Choose  $(E_n)_{n=1}^{\infty}$  such that  $\|g_{E_n}\|_{L^q(\mu)} \rightarrow M$ .

Then  $F := \bigcup_{n=1}^{\infty} E_n$  is  $\sigma$ -finite, and  $\|g_F\|_{L^q(\mu)} = M$ .

In addition, for any  $\sigma$ -finite  $A \supseteq F$ , we have  $A \setminus F$  is  $\sigma$ -finite as well. Thus  $g_A = g_F + g_{A \setminus F}$ .

We have  $g_{A \setminus F} = 0$ a.e., which means  $g_a = g_F$ a.e..

Let  $g := g_F \in L^q(\mu)$ .

Given any  $f \in L^p()$ , let  $A := \{x \in X : f(x) \neq 0\}$ , which has to be  $\sigma$ -finite.

Thus  $\Lambda(f) = \int_X g_A f d\mu = \int_X g_X f d\mu = \int_X g f d\mu$ .

□

*Remark.* This is in general not true for  $p = \infty$ .

*Remark.* If  $\mu$  is not  $\sigma$ -finite, it might be the case where  $L^1(\mu) = \{0\}$ , while  $L^\infty(\mu) \neq \{0\}$ .

## 8.2 Complex Regular Measure Space

**Definition 8.1.** Let  $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$  be a complex Borel measure on a locally compact Hausdorff space  $X$ , with its Jordan decomposition. We say  $\mu$  is a **complex Radon measure** or **complex regular measure** if all  $\mu_i$  are finite Radon measures.

**Proposition 8.2.**  $\mu$  is a complex Radon measure if and only if  $|\mu|$  is a Radon measure.

*Proof.* It follows  $\mu_i \leq |\mu| \leq \mu_1 + \mu_2 + \mu_3 + \mu_4$ . □

**Definition 8.2.** We define  $M(X) := \{\mu : \text{complex Radon measure}\}$ , and  $\|\mu\|_{M(X)} := |\mu|(X)$

**Proposition 8.3.**  $(M(X), \|\cdot\|_{M(X)})$  is a normed vector space over  $\mathbb{C}$ .

**Definition 8.3.**  $C_0(X)$  is the closure of  $C_c(X)$  in  $\|\cdot\|_\infty$ .

**Definition 8.4.** Let  $\mu$  be a complex measure with  $d\mu = hd|\mu|$ , we define  $\int_X f d\mu := \int_X fh d|\mu|$ .

**Theorem 8.4** (Jordan Decomposition for  $C_0(X, \mathbb{R})$ ). *For any  $\phi \in C_0(X, \mathbb{R})^*$ , we have  $\phi^+, \phi^-$  positive bounded linear functionals, such that  $\phi = \phi^+ - \phi^-$  on  $C_0(X, \mathbb{R})$ .*

*Proof.* For any  $f \geq 0$ , let  $\phi^+(f) := \sup(\phi(g) : 0 \leq g \leq f)$ .

Notice that if  $c \geq 0$ , we have  $\phi^+(cf) = c\phi^+(f)$ .

In addition,  $\forall f_1, f_2 \geq 0 \in C_0(\mathbb{R})$ , and any  $0 \leq g_1 \leq f_1, 0 \leq g_2 \leq f_2$ , we have  $0 \leq g_1 + g_2 \leq f_1 + f_2$ .

Thus  $\phi(g_1) + \phi(g_2) = \phi(g_1 + g_2) \leq \phi^+(f_1 + f_2)$ . Since  $g_1, g_2$  are arbitrary, we have  $\phi^+(f_1) + \phi^+(f_2) \leq \phi^+(f_1 + f_2)$ .

On the other hand, if we take any  $g \leq f_1 + f_2$ , and  $g_1 := \min(g, f_1), g_2 := g - g_1$ , we have  $g_2 \leq g - f_1 \leq f_2$ .

Thus  $\phi(g) = \phi(g_1) + \phi(g_2) \leq \phi^+(f_1) + \phi^+(f_2)$ .

Since  $g$  is arbitrary,  $\phi^+(f_1 + f_2) \leq \phi^+(f_1) + \phi^+(f_2)$ .

Thus,

$$\phi^+(f_1) + \phi^+(f_2) = \phi^+(f_1 + f_2).$$

Now extend  $\phi^+$  to  $C_0(X, \mathbb{R})$  by  $\phi^+(f) = \phi^+(f^+) - \phi^+(f^-)$ .

This is well-defined. Indeed, if  $f = g - h = f^+ - f^-$  for  $g, h \geq 0$ , we have  $g + f^- = h + f^+$ , and thus  $\phi^+(g) + \phi^+(f^-) = \phi^+(h) + \phi^+(f^+)$ .

We can also check that  $\phi^+$  is linear, and  $|\phi^+(f)| \leq \|\phi\| \|f\|$ .

Thus  $\phi^+$  is a bounded positive linear functional.

We will then define  $\phi^- := \phi^+ - \phi$ , and check it is also a bounded positive linear functional.  $\square$

**Theorem 8.5.** *Let  $\Lambda \in C_0(X)^*$ , then  $\exists!$  complex Radon measure  $\mu \in M(X)$ , such that*

$$\forall f \in C_0(X), \Lambda(f) = \int_X f d\mu.$$

Moreover,  $\|\Lambda\| = \|\mu\|_{M(X)} = |\mu|(X)$ .

*Proof.* We first consider  $C_0(X; \mathbb{R})$ , which is a real Banach subspace of  $C_0(X)$ .

Let  $\Psi := \Lambda|_{C_0(X; \mathbb{R})}$ .

Now let  $\Psi_1 := \Re(\Psi), \Psi_2 := \Im(\Psi)$ , we have that  $\Psi_1, \Psi_2 \in C_0(X, \mathbb{R})^*$  over  $\mathbb{R}$ , with  $\|\Psi_i\|_{C_0(X, \mathbb{R})^*} \leq \|\Lambda\|$ .

In addition,

$$\begin{aligned} \Lambda(f) &= \Lambda(\Re(f) + i\Im(f)) \\ &= \Lambda(\Re(f)) + i\Lambda(\Im(f)) \\ &= \Psi_1(\Re(f)) + i\Psi_2(\Re(f)) + i(\Psi_1(\Im(f)) + i\Psi_2(\Im(f))) \end{aligned}$$

is uniquely determined by  $\Psi_1, \Psi_2$ .

Yet  $\Psi_1 = \Psi_1^+ - \Psi_1^-, \Psi_2 = \Psi_2^+ - \Psi_2^-$ , thus by Riesz-Markov-Kakutani, we have  $\mu_i^\pm$  being finite Radon measures, such that  $\Psi_i^\pm = \int_X f d\mu_i^\pm$ .

Let  $\mu := (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$ , we have the result.

Now the uniqueness:

If  $\Lambda = \Lambda_{\mu_1} = \Lambda_{\mu_2}$ , we have  $\forall f \in C_0(X)$ ,

$$\begin{aligned} 0 &= \int_X f d(\mu_1 - \mu_2) \\ &= \int_X fhd|\mu_1 - \mu_2|. \end{aligned}$$

By density of  $C_0(X)$ , it is also true for all  $f \in L^1(X)$ , so  $|\mu_1 - \mu_2| = 0$ .  $\square$

**Corollary 8.6.**  $(M(X), \|\cdot\|_{M(X)}) \cong (C_0(X)^*, \text{isometrically})$ .

## 9 Product Measures

**Definition 9.1.** Let  $(X_i)_{i \in I}$  be a collection of non-empty sets, we define the **product** of the sets to be

$$X := \prod_{i \in I} X_i := \{(x_i)_{i \in I} | \forall i \in I, x_i \in X_i\} = \left\{ f : I \rightarrow \bigsqcup_{i \in I} X_i | \forall i \in I, f(i) \in X_i \right\}$$

**Definition 9.2.** We have a **canonical coordinate projections**  $\pi : X \rightarrow X_i$  by  $(x_j)_{j \in I} \mapsto x_i$ .

**Definition 9.3.** If  $(X_i, \mathcal{M}_i)$  are measurable spaces, then the **product measurable** space is

$$\left( \prod_{i \in I} X_i, \bigotimes_{i \in I} \mathcal{M}_i \right),$$

where  $\bigotimes_{i \in I} \mathcal{M}_i$  is the  $\sigma$ -algebra generated by the sets  $\{\pi_i^{-1}(A) | i \in I, A \in \mathcal{M}_i\}$ .

*Remark.* When  $I$  is finite, this is the same as tensor products generated by  $A_1 \times A_2 \times \cdots \times A_n$ .

**Proposition 9.1.** Let  $(X_i, d_i)_{i=1}^n$  be separable metric spaces, then

$$\bigotimes_{i=1}^n \text{Bor}(X_i) = \text{Bor}\left(\prod_{i=1}^n X_i\right).$$

*Proof.* Given any open  $U_i \subseteq X_i$ , we must have  $\pi_i^{-1}(U_i) \subseteq X$  is open. Thus  $\bigotimes_{i=1}^n \text{Bor}(X_i) \subseteq \text{Bor}(\prod_{i=1}^n X_i)$ . On the other hand, each  $X_i$  is separable, so  $X$  is also separable. Thus  $X$  is second countable. If  $(x_n)_{n=1}^\infty$  is a dense sequence in  $X$ , then

$$\{B_r(x_n) | n \in \mathbb{N}, r \in \mathbb{Q}^{++}\}$$

is a basis for the topology. Namely, every open set can be written as a countable union of these open balls.

Setting  $x_n^i := \pi_i(x_n)$ , we have that  $B_r(x_n) = \prod_{i=1}^n B_r(x_N^i)$ , which is a subset of  $\bigotimes_{i=1}^n \text{Bor}(X_i)$ .

Since every open set  $U \subseteq X$  is a countable union of these sets, so  $U \subseteq \bigotimes_{i=1}^n \text{Bor}(X_i)$ .

Thus  $\bigotimes_{i=1}^n \text{Bor}(X_i) \supseteq \text{Bor}(\prod_{i=1}^n X_i)$ . □

**Corollary 9.2.**

$$\text{Bor}(\mathbb{R}^n) = \bigotimes_{i=1}^n \text{Bor}(\mathbb{R})$$

**Proposition 9.3.** Let  $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  be measure spaces. Let  $R$  be the collection of all finite unions of disjoint rectangles  $A \times B$  with  $A \in \mathcal{M}, B \in \mathcal{N}$ . Then  $R$  is an algebra of subsets of  $X \times Y$

*Proof.*

$$\begin{aligned} (A \times B)^c &= (A^c \times Y) \sqcup (A \times B^c) \\ (A_1 \times B_1) \cup (A_2 \times B_2) &= (A_1 \times B_1) \sqcup ((A_2 \setminus A_1) \times B_2) \sqcup ((A_2 \setminus A_1) \times (B_1 \setminus B_2)) \end{aligned}$$

□

**Proposition 9.4.** The  $\sigma$ -algebra generated by  $R$  is  $\mathcal{M} \otimes \mathcal{N}$ .

**Definition 9.4.** We can define a function  $\pi : R \rightarrow [0, \infty]$  by  $\pi(\bigsqcup_{i=1}^n A_i \times B_i) := \sum_{i=1}^n \mu(A_i)\nu(B_i)$

**Lemma 9.5.**  $\pi$  is a premeasure.

*Proof.* Firstly,  $\pi(\emptyset) = \mu(\emptyset) \times \nu(\emptyset) = 0$ .

Secondly, we consider any  $A \times B = \bigsqcup_{n \in \mathbb{N}} (A_n \times B_n) \subseteq R$ , where  $(A_n \times B_n)_{n \in \mathbb{N}} \subseteq R$ .

Fix any  $y \in Y$ , we have that  $\chi_A(x)\chi_B(y) = \sum_{n \in \mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y)$ , which is a sum of non-negative measurable functions on  $X$ . By LMCT, we have that

$$\begin{aligned} \mu(A)\chi_B(y) &= \int_X \chi_A(x)d\mu\chi_B(y) \\ &= \int_X \chi_A(x)\chi_B(y)d\mu \\ &= \int_X \sum_{n \in \mathbb{N}} \chi_{A_n}(x)\chi_{B_n}(y)d\mu \\ &= \sum_{n \in \mathbb{N}} \int_X \chi_{A_n}(x)\chi_{B_n}(y)d\mu \\ &= \sum_{n \in \mathbb{N}} \int_X \chi_{A_n}(x)d\mu\chi_{B_n}(y) \\ &= \sum_{n \in \mathbb{N}} \mu(A_n)\chi_{B_n}(y). \end{aligned}$$

In addition,  $\sum_{n \in \mathbb{N}} \mu(A_n)\chi_{B_n}(y)$  is a sum of non-negative measurable functions on  $Y$ . By LMCT, we again have that

$$\begin{aligned} \mu(A)\nu(B) &= \mu(A) \int_Y \chi_B(y)d\nu \\ &= \int_Y \mu(A)\chi_B(y)d\nu \\ &= \int_Y \sum_{n \in \mathbb{N}} \mu(A_n)\chi_{B_n}(y)d\nu \\ &= \sum_{n \in \mathbb{N}} \int_Y \mu(A_n)\chi_{B_n}(y)d\nu \\ &= \sum_{n \in \mathbb{N}} \mu(A_n) \int_Y \chi_{B_n}(y)d\nu \\ &= \sum_{n \in \mathbb{N}} \mu(A_n)\nu(B_n). \end{aligned}$$

This will now extend to any  $\bigsqcup_{n \in \mathbb{N}} (A_n \times B_n) \subseteq R$ , by finite additivity.  $\square$

**Theorem 9.6.** *There is a complete measure space  $(X \times X, \overline{\mathcal{M} \otimes \mathcal{N}}, \mu \times \nu)$ , such that  $\mu \times \nu(A \times B) = \mu(A) \times \nu(B)$ .*

*Proof.* Apply Caratheodory on the above lemma.  $\square$

For the flowing, let  $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  be complete measure spaces.

**Definition 9.5.** Take  $R$  as before, let  $R_\sigma := \left\{ \bigcup_{n \geq 1} A_n \mid A_n \in R \right\}$ ,  $R_{\sigma\delta} := \left\{ \bigcap_{n \geq 1} E_n \mid E_n \in R_\sigma \right\}$

**Lemma 9.7.** *If  $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$ , with  $\mu \times \nu(E) < \infty$ , then  $\exists G \in R_{\sigma\delta}$ , such that  $E \subseteq G, \mu \times \nu(G \setminus E) = 0$*

*Proof.* We have  $\mu \times \nu(E) = \inf \left\{ \sum_{i \geq 1} \mu \times \nu(A_i) \mid A_i \in R, E \subseteq \bigcup_{i \geq 1} A_i \right\}$ .

Let  $E_j := \bigcup_{i \geq 1} A_{ji} \supseteq E$ , with  $\mu \times \nu(E_j) < \mu \times \nu(E) + \frac{1}{j}$ .

Notice that  $\bar{E}_j \in R_\sigma$  by construction.

Now take  $G = \bigcup_{j \geq 1} E_j \in R_{\sigma\delta}$ . Then we have that  $E \subseteq G$ , and  $\forall j, \mu \times \nu(G) \leq \mu \times \nu(E_j) < \mu \times \nu(E) + \frac{1}{j}$ . Thus  $\mu \times \nu(G) = \mu \times \nu(E)$ .  $\square$

**Lemma 9.8.** Let  $E \in R_{\sigma\delta}$ , with  $\mu \times \nu(E) < \infty$ . Let  $E_x = \{y \in Y | (x, y) \in E\}$ ,  $E^y = \{x \in X | (x, y) \in E\}$ . Define  $g(x) := \nu(E_x)$ ,  $h(y) := \mu(E^y)$ . Then we have  $g$  is non-negative and  $\mu$ -measurable,  $g \in \mathcal{L}^1(\mu)$ ,  $\int_X g d\mu = \mu \times \nu(E)$ . Similarly,  $h$  is non-negative and  $\nu$ -measurable,  $h \in \mathcal{L}^1(\nu)$ ,  $\int_Y g d\nu = \mu \times \nu(E)$

*Proof.* If  $E = A \times B$ , with  $A \in \mathcal{M}, B \in \mathcal{N}$ , then  $E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$ .

Then  $g(x) = \nu(B)\chi_A$  is  $\mu$ -measurable, and  $g \geq 0$ . Moreover,

$$\int_X g d\mu = \int_X \nu(B)\chi_A d\mu = \nu(B) \int_X \chi_A d\mu = \mu(A)\nu(B) = \mu \times \nu(A \times B).$$

Now suppose  $E = \bigcup_{i \geq 1} A_i \times B_i \in R_\delta$ , with  $A_i \in \mathcal{M}, B_i \in \mathcal{N}$ . WLOG, we can take  $E = \bigsqcup_{i \geq 1} A_i \times B_i$ .

Let  $g_i(x) = \nu(B_i)\chi_{A_i}(x)$ , we have  $\sum_{i=1}^n g_i(x) = \sum_{i=1}^n \nu(B_i)\chi_{A_i}(x) = \begin{cases} \nu(B_i) = \nu(E_x) & \text{if } x \in A_i \\ 0 & \text{if } x \notin \bigsqcup_{i=1}^n A_i \end{cases}$ .

Thus  $g(x) = \sum_{i=1}^\infty g_i(x)$  is measurable. By LMCT,  $g \in \mathcal{L}^1(\mu)$ , and

$$\begin{aligned} \int_X g d\mu &= \sum_{i=1}^\infty \int_X g_i d\mu \\ &= \sum_{i=1}^\infty \int_X \nu(B_i)\chi_{A_i} d\mu \\ &= \sum_{i=1}^\infty \nu(B_i)\mu(A_i) \\ &= \sum_{i=1}^\infty \mu \times \nu(A_i \times B_i) \\ &= \mu \times \nu\left(\bigsqcup_{i=1}^\infty A_i \times B_i\right) \\ &= \mu \times \nu(E). \end{aligned}$$

Now take  $E = \bigcap_{i \geq 1} E_i \in R_{\delta\sigma}$  with  $E_i \in R_\delta$ . WLOG, we can take  $E_i \supseteq E_{i+1}$ .

Notice that  $(E_i)_x = \{y \in Y | (x, y) \in E_i\} \supseteq \{y \in Y | (x, y) \in E_{i+1}\} = (E_{i+1})_x \supseteq \dots \supseteq E_x$ .

Let  $g_i(x) = \nu((E_i)_x) = \mu \times \nu(E_i)$ , then we have  $0 \leq g \leq \dots \leq g_i \leq \dots \leq g_1$ .

In addition,  $E_x = \bigcap_{i \geq 1} (E_i)_x$ , and thus  $g(x) = \lim_{i \rightarrow \infty} g_i(x)$  by continuity of  $\nu$ .

Thus  $g$  is  $\mu$ -measurable, and since  $g_i$  are all dominated by  $g_1$ , we can used LDCT to get

$$\begin{aligned} \int_X g d\mu &= \lim_{i \rightarrow \infty} \int_X g_i d\mu \\ &= \lim_{i \rightarrow \infty} \mu \times \nu(E_i) \\ &= \mu \times \nu(E). \end{aligned}$$

□

**Lemma 9.9.** Let  $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$  with  $\mu \times \nu(E) = 0$ , then for  $\mu$ -a.e.  $x \in X$ , we have  $\nu(E_x) = 0$ ; for  $\nu$ -a.e.  $y \in Y$ , we have  $\mu(E_y) = 0$ .

*Proof.* We have some  $G \in R_{\sigma\delta}$ , such that  $E \subseteq G, \mu \times \nu(G \setminus E) = 0$ .

Let  $f(x) := \nu(G_x)$ , we have  $f \in \mathcal{L}^1(\mathcal{M})$  is nonnegative. Yet  $\int_X f d\mu = 0$ , and thus  $f(x) = 0$  for  $\mu$ -a.e.  $x \in X$ . Since  $E_X \subseteq G_X$ , and that  $\nu$  is complete, we have that  $g(x) = \nu(E_x) = 0$  for  $\mu$ -a.e.  $x \in X$ . □

**Corollary 9.10.** Let  $E \in \overline{\mathcal{M} \otimes \mathcal{N}}$  with  $\mu \times \nu(E) < \infty$ , then  $E_x$  is  $\nu$ -measurable, for  $\mu$ -a.e.  $x \in X$ , and  $g(x) = \nu(E_x)$  is  $\mu$ -measurable, with  $g \geq 0, g \in \mathcal{L}^1(\mathcal{M})$ , and  $\int_X g d\mu = \mu \times \nu(E)$ .

**Theorem 9.11** (Fubini's). *Let  $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  be complete measure spaces. Take  $f \in \mathcal{L}^1(\mu \times \nu)$ , then*

1. For  $\mu$ -a.e.  $x \in X$ ,  $f_x := f(x, \cdot) \in \mathcal{L}^1(\nu)$ .
2. For  $\nu$ -a.e.  $y \in Y$ ,  $f_y := f(\cdot, y) \in \mathcal{L}^1(\mu)$ .
3.  $F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^1(\mu)$ .
4.  $G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^1(\nu)$ .
5.  $\int_{X \times Y} f d(\mu \times \nu) = \int_X (\int_Y f(x, y) d\nu) d\mu = \int_Y (\int_X f(x, y) d\mu) d\nu$

*Proof.* Notice that  $f^1 \in \mathcal{L}^1$  means that  $f = f_1 - f_2 + i f_3 - i f_4$ , where  $f_i \geq 0, f_i \in \mathcal{L}^1$ .

We first show the theorem holds for  $f \geq 0, f \in \mathcal{L}^1$ . There are simple functions  $0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f$ , such that  $f(x) = \lim_{n \rightarrow \infty} s_n(x)$ .

Let  $F_n(x) = \int_Y s_n(x, y) d\nu \geq 0$  be measurable and  $\mathcal{L}^1$ . We have that  $\square$

**Theorem 9.12** (Tonelli's). *Let  $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  be complete measure spaces. Take  $f \in \mathcal{L}^+(\mu \times \nu)$ , and  $\mu \times \nu$  is  $\sigma$ -finite, then*

1. For  $\mu$ -a.e.  $x \in X$ ,  $f_x := f(x, \cdot) \in \mathcal{L}^+(\nu)$ .
2. For  $\nu$ -a.e.  $y \in Y$ ,  $f_y := f(\cdot, y) \in \mathcal{L}^+(\mu)$ .
3.  $F(x) := \int_Y f_x(y) d\nu \in \mathcal{L}^+(\mu)$ .
4.  $G(y) := \int_X f_y(x) d\mu \in \mathcal{L}^+(\nu)$ .
5.  $\int_{X \times Y} f d(\mu \times \nu) = \int_X (\int_Y f(x, y) d\nu) d\mu = \int_Y (\int_X f(x, y) d\mu) d\nu$

*Proof.*  $\mu \times \nu$  is  $\sigma$ -finite, thus  $\exists C_1 \subseteq C_2 \subseteq \dots$ , with  $C_n \in \overline{\mathcal{M} \otimes \mathcal{N}}$ ,  $X \times Y = \bigcup_{n=1}^{\infty} C_n$ , and  $\mu \times \nu(C_n) < \infty$ . Let  $f_n(x) = \max \{f(x), n\} \chi_{C_n}(x)$ , we have  $0 \leq f_n \leq n \chi_{C_n}$ , and  $f_n \in \mathcal{L}^+ \cap \mathcal{L}^1(\mu \times \nu)$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

$$\begin{aligned} \int f d(\mu \times \nu) &= \lim_{n \rightarrow \infty} \int f_n d(\mu \times \nu) \\ &= \lim_{n \rightarrow \infty} \int_X \int_Y f_n(x, y) d\nu d\mu \\ &=: \lim_{n \rightarrow \infty} \int_X F_n(x) d\mu. \end{aligned}$$

Then  $F_n$  are measurable, non-negative, and monotone increasing to  $F(x) := \int_Y f(x, y) d\nu$ . By LMCT, we have  $F$  is measurable, and

$$\begin{aligned} \int f d(\mu \times \nu) &= \lim_{n \rightarrow \infty} \int_X F_n(x) d\mu \\ &= \int_X F(x) d\mu \\ &= \int_X \int_Y f(x, y) d\nu d\mu \end{aligned}$$

$\square$

*Remark.* If  $f \in \mathcal{L}^1$ , we get  $\sigma$ -finite by free on  $C = \text{Supp}(f)$  if we look at  $C_n := \{(x, y) : |f(x, y)| \geq \frac{1}{n}\}$ . Notice that  $\mu \times \nu(C_n) \leq n \int |f| d(\mu \times \nu) < \infty$ .

**Example 9.0.1.** Consider  $X = Y = \mathbb{N}$ ,  $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$ , with the counting measure  $m_c$ .

$$\text{Consider } f(m, n) := \begin{cases} 1 & n = m \\ -1 & n = m + 1 \\ 0 & o.w. \end{cases} \sim \begin{pmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ \vdots & 0 & 1 & -1 & \dots \\ \vdots & \vdots & \ddots & & \end{pmatrix}.$$

However,

$$\begin{aligned} \int_X \int_Y f(x, y) dm_c(n) dm_c(m) &= \sum_{m \geq 1} \sum_{n \geq 1} f(m, n) \\ &= \sum_{m \geq 1} 0 \\ &= 0, \end{aligned}$$

$$\begin{aligned} \int_Y \int_X f(x, y) dm_c(m) dm_c(n) &= \sum_{n \geq 1} \sum_{m \geq 1} f(m, n) \\ &= 1 + \sum_{n \geq 2} 0 \\ &= 1. \end{aligned}$$

This is because  $f \notin \mathcal{L}^1$ .

**Example 9.0.2.** Consider  $X = Y = [0, 1]$ , and the Lebesgue measure.

Take  $t_n = 1 - \frac{1}{n}$ .

Define  $g_n : [0, 1] \rightarrow \mathbb{R}$  by starting at  $\frac{2t_n}{3} + \frac{t_{n+1}}{3}$ , linear and reach  $\frac{t_{n+1}-t_n}{3}$  at mid point, and decrease linearly to 0 at  $\frac{t_n}{3} + \frac{2t_{n+1}}{3}$ , and 0 outside. We thus have  $\int g_n(x) dx = 1$ .

Define  $f(x, y) = \sum_{i=1}^{\infty} (g_n(x) - g_{n+1}(x))g_n(y)$ , where only one of these summands will be non-zero in each interval of  $x$ . Actually  $f(x, y)$  is continuous  $\forall (x, y) \neq (1, 1)$ .

However,  $\int f(x, y) dx = g_n(y)$ , and thus  $\int \int f(x, y) dy dx = 1$ , while  $\int \int f(x, y) dx dy = 0$ .

**Theorem 9.13.** Let  $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  be (not necessarily complete) measure spaces. Then Fubini and Tonelli still apply with restriction to  $\mathcal{M} \otimes \mathcal{N}$ .

## 9.1 Lebesgue Measure on $\mathbb{R}^n$

**Lemma 9.14.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be Lebesgue measurable, then there is a  $G_\delta$  set  $G \subseteq \mathbb{R}^n$ , such that  $\lambda^n(G) = 0$ , and  $g = f\chi_{G^c}$  be Borel measurable, and  $f = g \lambda^n$ -a.e..

*Proof.* By writing  $f = f_1 - f_2 + if_3 - if_4$  for  $f_i \geq 0$ , we can assume  $f \geq 0$ . We first consider  $n = 1$ .

Choose a dense subset  $\{r_i\}_{i \in \mathbb{N}}$  of  $[0, \infty)$ . Let  $A_i = f^{-1}([0, r_i])$ . Since  $f$  is Lebesgue measurable,  $A_n \in \mathcal{L}$ . By regularity for the Lebesgue measure, there is an  $F_\delta$  set  $F_i \subseteq A_i$  and a null set  $N_i = A_i \setminus F_i$ . Let  $N = \bigcup_{i \in \mathbb{N}} N_i$ , then  $N$  is a null set.

Applying regularity again, there is a  $G_\delta$  set  $G \supseteq N$  such that  $\lambda(G) = 0$ .

Let  $g = f\chi_{G^c}$ . we have  $g^{-1}([0, r_i]) = f^{-1}([0, r_i]) \cup G = A_i \cup G = (F_i \cup N_i) \cap G = F_i \cup G$ , which is a union of two Borel sets, and thus Borel.

To verify that  $g$  is Borel, it surfaces to prove  $g^{-1}([0, r])$  is Borel for all  $r > 0$ . By density of  $\{r_i\}$ , there is a sequence  $r_{n_k}$  such that  $r_{n_k} \leq r$  and  $r_{n_k} \rightarrow r$ . Thus  $\bigcup_{k \geq 1} [0, r_{n_k}] = [0, r]$ , so  $g^{-1}([0, r]) = \bigcup_{k \geq 1} (F_{n_k} \cup G)$  is a union of Borel sets, and thus Borel.

By construction,  $G$  is a null set and  $f|_{G^c} = g|_{G^c}$ , so  $f = g \lambda$ -a.e..

Now suppose  $n \geq 1$ . For each  $i$ , let  $f_{x_i}$  be the function obtained by fixing all but the  $i^{th}$  variable  $x_i$ . From above we can find  $G_\delta$  set  $G_i \subseteq \mathbb{R}$ , such that  $f_{x_i} = f_{x_i}\chi_{G_i^c}$   $\lambda$ -a.e..

Let  $G = (G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup (\mathbb{R} \times G_2 \times \mathbb{R} \times \cdots \times \mathbb{R}) \cup \cdots (\mathbb{R} \times \cdots \times \mathbb{R} \times G_n)$ .

Then  $G^c = G_1^c \times G_2^c \times \cdots \times G_n^c$ . Let  $G_1 = \bigcap_k U_{1k}$ , then  $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R} = \bigcap_k (U_{1k} \times \mathbb{R} \times \cdots \times \mathbb{R})$ , where each is open, since  $U_{1k}, \mathbb{R}$  are open. Thus  $G_1 \times \mathbb{R} \times \cdots \times \mathbb{R}$  is a  $G_\delta$  set. Thus  $G$  is a finite union of  $G_\delta$  sets, which is  $G_\delta$ .  $\square$

**Definition 9.6.** For  $A \in \mathcal{L}^n$ ,  $X \in \mathbb{R}^n$ , write the **translation** of  $A$  by  $x$  as  $A + x = \{a + x : a \in A\}$ .

**Definition 9.7.** Let  $GL_n$  be the set of invertible  $n \times n$  matrices.

**Theorem 9.15.** Consider the Lebesgue measure  $\lambda^n$  in  $\mathbb{R}^n$ .

1. (translation) For  $A \in \mathcal{L}^n$  and  $x \in \mathbb{R}^n$ , we have  $A + x \in \mathcal{L}^n$ ,  $\lambda^n(A + x) = \lambda^n(A)$ .
2. (scaling) For  $T \in GL_n$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  be Lebesgue measurable,  $f \circ T$  is Lebesgue measurable, and

$$\int f d\lambda^n = |\det(T)| \int (f \circ T) d\lambda^n.$$

In particular, for  $A \in \mathcal{L}^n$ , we have  $\lambda^n(T(A)) = |\det(T)|\lambda^n(A)$ .

3. (rotation) For a unitary  $U \in GL_n$ , we have

$$\int (f \circ U) d\lambda^n = \int f d\lambda^n,$$

and  $\forall A \in \mathcal{L}^n$ ,  $\lambda^n(U(A)) = \lambda^n(A)$ .

*Proof.* 1.

2. Notice that  $x \in T(A) \iff T^{-1}x \in A$ , thus  $\chi_{T(A)} = \chi_A \circ T^{-1}$ . Thus

$$\begin{aligned} \lambda^n(T(A)) &= \int \chi_{T(A)} d\lambda^n \\ &= \int \chi_A \circ T^{-1} d\lambda^n \\ &= \frac{1}{|\det(T^{-1})|} \int \chi_A d\lambda^n \\ &= |\det(T)|\lambda^n(A). \end{aligned}$$

□

## 10 Convolutions and Fourier Transforms

**Definition 10.1.** For  $y \in \mathbb{R}$ ,  $f : \mathbb{R} \rightarrow \mathbb{C}$ , we define the **translation** of  $f$  by  $y$  to be  $L_y f(x) := f(x - y)$ .

**Proposition 10.1.** We have  $L_y : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  is linear, isometric, and  $\forall f \in L^1(\mathbb{R})$ , we have

$$\lim_{y \rightarrow 0} \|L_y f - f\|_1 = 0.$$

*Proof.* If  $f \in C_c(\mathbb{R})$ , then it is uniformly continuous, so

$$\lim_{y \rightarrow 0} \|L_y f - f\|_\infty = 0.$$

Take compact  $K \supseteq \text{Supp}(f)$ , we have that

$$\begin{aligned} \|L_y f - f\|_1 &= \int_{K \cup (K+y)} |f(x - y) - f(x)| dx \\ &\leq \lambda(K \cup (K+y)) \|L_y f - f\|_\infty, \end{aligned}$$

where the first term is bounded by  $2\lambda(K) < \infty$ , and the second term goes to 0.

Now since  $C_c(\mathbb{R})$  is dense in  $L^1(\mathbb{R})$ , we have the result by triangle inequality. □

**Theorem 10.2** (Young's Convolution Inequality). Consider  $X = \mathbb{R}$ , with Lebesgue measure  $\lambda$ . Let  $f, g \in L^1(\mathbb{R})$ , then for a.e.  $x \in \mathbb{R}$ , the function  $y \mapsto f(x - y)g(y)$  is in  $L^1(\mathbb{R})$  for a.e.  $x \in \mathbb{R}$ , and the **convolution**

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y)g(y)dy$$

is also in  $L^1(\mathbb{R})$ . In addition,  $\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}$ .

*Proof.* Consider the function  $F : (x, y) \mapsto f(x - y)g(y)$ , which is a measurable function on  $\mathbb{R} \times \mathbb{R}$  (can show with approximation by  $C_c(\mathbb{R})$  functions).

By Tonelli's theorem,

$$\begin{aligned} \int_{\mathbb{R}^2} |F| d\lambda^2 &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |F(x, y)| dx \right) dy \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x - y)| |g(y)| dx \right) dy \\ &= \int_{\mathbb{R}} |g(y)| \left( \int_{\mathbb{R}} |f(x - y)| dx \right) dy \\ &= \int_{\mathbb{R}} |g(y)| \|f\|_{L^1(\mathbb{R})} dy \\ &= \|f\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} |g(y)| dy \\ &= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} \\ &< \infty. \end{aligned}$$

Thus,  $F \in L^1(\mathbb{R}^2)$ .

Now we apply Fubini's Theorem to  $F$ , and get  $F_x(y) = f(x - y)g(y) \in L^1(\mathbb{R})$  for a.e.  $x \in \mathbb{R}$ . In addition,

$$\begin{aligned} \|f * g\|_{L^1(U)} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x - y)g(y) dy \right| dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x - y)g(y)| dy dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |F(x, y)| dy dx \\ &= \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}. \end{aligned}$$

□

**Corollary 10.3.**  $(L^1(\mathbb{R}), *)$  defines a communicative associative algebra.

**Definition 10.2.** Given  $f \in L^1(\mathbb{R})$ , its **Fourier Transform** is  $\mathcal{F}(f) := \hat{f} : \mathbb{R} \rightarrow \mathbb{C}$ , where

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-ix\omega} dx.$$

**Lemma 10.4** (Riemann-Lebesgue).  $\forall f \in L^1(\mathbb{R})$ , we have  $\hat{f} \in C_0(\mathbb{R})$ , and  $\|\hat{f}\|_{\infty} \leq \|f\|_{L^1(\mathbb{R})}$ . Namely,  $\mathcal{F}$  is a contraction map.

*Proof.* Consider any sequence  $(\omega_n)_{n=1}^{\infty} \subset \mathbb{R}$  that converges to  $\omega \in \mathbb{R}$ .

Let  $h_n(x) := f(x)(e^{i\omega_n x} - e^{i\omega x})$ , we have that  $h_n \in L^1(\mathbb{R})$ ,  $h_n(x) \rightarrow 0$  pointwise for a.e.  $x \in \mathbb{R}$ , and

$|h_n| \leq |f| |e^{i\omega_n x} - e^{i\omega x}| \leq 2|f|$ .  
In addition,

$$\begin{aligned}\hat{f}(\omega_n) - \hat{f}(\omega) &= \int_{\mathbb{R}} f(x)(e^{i\omega_n x} - e^{i\omega x})dx \\ &= \int_{\mathbb{R}} h_n(x)dx\end{aligned}$$

By LDCT, we have that  $\lim_{n \rightarrow \infty} (\hat{f}(\omega_n) - \hat{f}(\omega)) = 0$ , so  $\hat{f}$  is continuous.  
In addition,

$$\begin{aligned}|\hat{f}(\omega)| &\leq \int_{\mathbb{R}} |f(x)| |e^{ix\omega}| dx \\ &= \int_{\mathbb{R}} |f(x)| dx \\ &= \|f\|_{L^1(\mathbb{R})}.\end{aligned}$$

Now

$$\begin{aligned}\hat{f}(\omega) &= \int_{\mathbb{R}} f(x)e^{-ix\omega} dx \\ &= - \int_{\mathbb{R}} f(x)e^{-ix\omega+\pi i} dx \\ &= - \int_{\mathbb{R}} f(x)e^{-i\omega(x-\pi/\omega)} dx \\ &= - \int_{\mathbb{R}} f(z + \pi/\omega)e^{-i\omega z} dz \\ &= - \int_{\mathbb{R}} L_{-\pi/\omega} f(z)e^{-i\omega z} dz \\ 2\hat{f}(\omega) &= \int_{\mathbb{R}} f(x)e^{-ix\omega} dx - \int_{\mathbb{R}} L_{-\pi/\omega} f(z)e^{-i\omega z} dz \\ &= \int_{\mathbb{R}} (f - L_{-\pi/\omega} f)(x)e^{-ix\omega} dx \\ &= \mathcal{F}(f - L_{-\pi/\omega} f)(\omega) \\ 2|\hat{f}(\omega)| &\leq \|f - L_{-\pi/\omega} f\|_{L^1(\mathbb{R})},\end{aligned}$$

which goes to 0 when  $\omega \rightarrow \infty$ .

Thus,  $\hat{f} \in C_0(\mathbb{R})$ . □

**Theorem 10.5** ( $L^1(\mathbb{R})$  Inversion). *If  $f, \hat{f} \in L^1(\mathbb{R})$ , we have*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{ix\omega} d\omega$$

for a.e.  $x \in \mathbb{R}$ .

In particular, such  $f$  must be almost everywhere equal to a continuous function.

*Proof.* Let  $\lambda > 0$ , and  $H_\lambda(\omega) := e^{-\lambda|\omega|}$ .

Let

$$\begin{aligned}h_\lambda(x) &:= \frac{1}{2\pi} \int_{\mathbb{R}} H_\lambda(\omega) e^{ix\omega} d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\omega - \lambda|\omega|} d\omega \\ &= \frac{\lambda}{\pi} \frac{1}{x^2 + \lambda^2}.\end{aligned}$$

Fix  $f \in L^1(\mathbb{R})$ , we have

$$\begin{aligned}
(f * h_\lambda)(x) &= \int_{\mathbb{R}} f(x-y)h_\lambda(y)dy \\
&= \int_{\mathbb{R}} f(x-y) \frac{1}{2\pi} \int_{\mathbb{R}} H_\lambda(\omega) e^{iy\omega} d\omega dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) H_\lambda(\omega) e^{iy\omega} d\omega dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) H_\lambda(\omega) e^{iy\omega} dy d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) H_\lambda(\omega) e^{i\omega(x-z)} dz d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} H_\lambda(\omega) \hat{f}(\omega) e^{i\omega(x)} dz d\omega.
\end{aligned}$$

One can show that  $H_\lambda(\omega) = 1$  as  $\lambda \rightarrow 0$ , and  $f * h_\lambda \rightarrow f$ .

If  $\hat{f} \in L^1(\mathbb{R})$ , we can use DCT to get the result.  $\square$

**Corollary 10.6.** If  $f, g \in L^1(\mathbb{R})$ , and  $\mathcal{F}(f) = \mathcal{F}(g)$ , we must have  $\mathcal{F}(f - g) = 0 \in L^1(\mathbb{R})$ . Thus,  $f = g$  a.e.  $x \in \mathbb{R}$ .

*Remark.* Not all  $\hat{f} \in L^1(\mathbb{R})$ .

**Example 10.0.1.** If  $f = \chi_{[-1,1]}$ , we have  $\hat{f} = \frac{2 \sin(\omega)}{\omega} \in C_0(\mathbb{R}) \setminus L^1(\mathbb{R})$ .

*Remark.* We can equivalently define

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx,$$

then the inverse is given by

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

**Proposition 10.7.** For any  $f, g \in L^1(\mathbb{R})$ , we always have  $\widehat{f * g} = \hat{f} \hat{g}$ .

**Example 10.0.2.** Consider the heat equation initial problem:  $\begin{cases} \frac{\partial}{\partial t} u = \Delta u \\ u(\cdot, 0) = f. \end{cases}$  Taking the Fourier Transform

with respect to  $x$ , we have  $\begin{cases} \frac{\partial}{\partial t} \hat{u} = (2\pi i \xi)^2 \hat{u} \\ \hat{u}(\cdot, 0) = \hat{f}, \end{cases}$  with the solution  $\hat{u}(\xi, t) = e^{-4\pi^2 \xi^2 t} \hat{f}(\xi)$ .

Assuming we can apply the inverse formula, we have

$$\begin{aligned}
u(x, t) &= \int_{\mathbb{R}} \hat{u}(\xi, t) e^{2\pi i \xi x} d\xi \\
&= \int_{\mathbb{R}} e^{-4\pi^2 \xi^2 t} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \\
&= \int_{\mathbb{R}} e^{-4\pi^2 \xi^2 t} \int_{\mathbb{R}} f(y) e^{-2\pi i y \xi} dy e^{2\pi i \xi x} d\xi \\
&= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} e^{-4\pi^2 \xi^2 t} e^{-2\pi i y \xi} dy e^{2\pi i \xi x} d\xi \\
&= \int_{\mathbb{R}} f(y) \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4t}} d\xi \\
&= \int_{\mathbb{R}} f(y) H_t(x-y) dy \\
&= (H_t * f)(x),
\end{aligned}$$

where the **heat kernel** is  $H_t(x) = \frac{1}{(4\pi t)^2} e^{-\frac{x^2}{4t}}$ .

The heat kernel satisfies  $\frac{\partial}{\partial t} H_t = \Delta H_t$ ,  $\int_{\mathbb{R}} H_t(x) dx = 1$ , and  $\int_{|x| \geq \epsilon} H_t(x) dx \rightarrow 0$ .

**Definition 10.3.** The **Schwartz class**  $S$  is the set of functions  $f : \mathbb{R} \rightarrow \mathbb{C}$ , such that

$$\exists C \geq 0, \text{ such that } \forall \alpha, \beta, \left| x^\alpha \frac{d}{dx^\beta} f \right| \leq C.$$

**Proposition 10.8.**  $C_c^\infty(\mathbb{R}) \subset S$ .

**Proposition 10.9.** Suppose  $f \in S$ , then  $\hat{f} \in S$ .

**Theorem 10.10** (Plancherel). Suppose  $f \in S$ , then  $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$ .

**Proposition 10.11.**  $S$  is dense in  $L^2(\mathbb{R})$ . Namely,  $\bar{S} = L^2(\mathbb{R})$ .

**Definition 10.4.** For  $f \in L^2(\mathbb{R})$ , with  $(f_i)_{i=1}^\infty$  in  $S$  such that  $f_i \rightarrow f$ , we define the **Fourier Transform** of  $f$  to be

$$\hat{f} := \lim_{i \rightarrow \infty} \hat{f}_i.$$

**Lemma 10.12.** The above definition is well-defined.

*Proof.* Given any  $\epsilon > 0$ .

Since  $f_i \rightarrow f$  in  $L^2(\mathbb{R})$ , there is  $N \geq 1$ , such that  $\forall i \geq N$ ,  $\|f_i - f\|_{L^2(\mathbb{R})} < \epsilon/2$ . Thus, for any  $i, j \geq N$ ,

$$\begin{aligned} \|\hat{f}_i - \hat{f}_j\|_{L^2(\mathbb{R})} &= \|f_i - f_j\|_{L^2(\mathbb{R})} \\ &\leq \|f_i - f\|_{L^2(\mathbb{R})} + \|f - f_j\|_{L^2(\mathbb{R})} \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Thus  $(\hat{f}_i)_{i=1}^\infty$  is a Cauchy sequence in  $L^2(\mathbb{R})$ , so  $\lim_{i \rightarrow \infty} \hat{f}_i$  exists in  $L^2(\mathbb{R})$ .  $\square$

## 11 Bochner Spaces

**Definition 11.1.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space, then a function  $f : X \rightarrow B$  is **weakly measurable** if  $\forall \Lambda \in B^*$ ,  $\Lambda \circ f : X \rightarrow \mathbb{C}$  is measurable.

**Definition 11.2.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space, then a function  $f : X \rightarrow B$  is **Bochner measurable or strongly measurable** if  $f(x) = g(x)$  for  $\mu$ -a.e.  $x \in X$ , for some measurable  $g$ , with  $\text{Im}(g) \subseteq B$  being separable.

**Proposition 11.1.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space, then a function  $f : X \rightarrow B$  is **strongly measurable** if  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for  $\mu$ -a.e.  $x \in X$ , for some sequence of measurable functions  $f_n$ , each with countable range.

**Definition 11.3.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space, and  $s : X \rightarrow [0, \infty)$  be a simple measurable function, with  $s(X) = \{a_1, \dots, a_n\} \subset B$ , such that

$$s = \sum_{i=1}^n a_i \chi_{A_i},$$

where  $A_i := s^{-1}(\{a_i\})$ . For  $A \in \mathcal{M}$ , we say  $s$  is integrable over  $A$  if  $\forall a_i \neq 0$ ,  $\mu(A_i \cap A) < \infty$ , and define the **integral** of  $s$  over  $A$  to be

$$\int_A s d\mu := \sum_{i=1}^n a_i \mu(A_i \cap A).$$

**Definition 11.4.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space, and  $f : X \rightarrow [0, \infty)$  be a measurable function. If there is a sequence of simple integrable functions  $(s_n)_{n=1}^{\infty}$  such that

$$\lim_{n \rightarrow \infty} \int_A \|f - s_n\|_B d\mu = 0,$$

then we say  $f$  is **Bochner integrable**, and we define the **Bochner integral** to be

$$\int_A f d\mu := \lim_{n \rightarrow \infty} \int_A s_n d\mu.$$

**Lemma 11.2.** The right hand side of the above definition always exists, and is independent of the choice of the sequence of simple integrable functions  $(s_n)_{n=1}^{\infty}$ . Thus, the above definition is well-defined.

**Theorem 11.3** (Bochner). Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space. A strongly measurable function  $f : X \rightarrow B$  is Bochner integrable if and only if  $x \mapsto \|f(x)\|_B$  is integrable. In this case,  $\forall E \in \mathcal{M}$ ,

$$\begin{aligned} \left\| \int_E f(x) dx \right\|_B &\leq \int_E \|f(x)\|_B dx, \\ \forall \Lambda \in B^*, \quad \Lambda \left( \int_E f(x) dx \right) &= \int_E \Lambda(f(x)) dx. \end{aligned}$$

**Theorem 11.4** (Dominated Convergence Theorem for Bochner integral). Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space. Let  $f_n : X \rightarrow \mathbb{C}$  be measurable functions, defined  $\mu$ -a.e. on  $X$ , such that  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$  is defined  $\mu$ -almost everywhere for  $x \in X$ . If there is  $0 \leq g(x) \in \mathcal{L}^1(X, \mu)$ , such that for  $\mu$ -a.e.  $x \in X, \forall n \in \mathbb{N}, \|f_n(x)\|_B \leq g(x)$ , then  $f$  is Bochner integrable, and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu, \quad \lim_{n \rightarrow \infty} \int_X \|f - f_n\|_B d\mu = 0.$$

Similarly to the Lebesgue Spaces, we can define the Bochner Spaces and Bochner norms:

**Definition 11.5.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space, and  $1 \leq p < \infty$ , we define

$$\mathcal{L}^p(\mu, B) := \left\{ f : X \rightarrow B \mid f \text{ is measurable, } \int_X \|f\|_B^p d\mu < \infty \right\}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^p(\mu, B)} := \left( \int_X \|f\|_B^p d\mu \right)^{\frac{1}{p}}.$$

**Definition 11.6.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space, we define

$$\mathcal{L}^\infty(\mu, B) := \{ f : X \rightarrow B \mid f \text{ is measurable, } \text{ess sup } \|f\|_B < \infty \}.$$

In addition, we define

$$\|f\|_{\mathcal{L}^\infty(\mu, B)} := \text{ess sup } \|f\|_B.$$

**Definition 11.7.** Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space. For any  $p \in [1, \infty]$ , we define

$$L^p(\mu, B) := \mathcal{L}^p(\mu, B)/N,$$

where  $N := \{f : X \rightarrow B \mid f \text{ is measurable, } f = 0 \text{ } \mu - \text{a.e.}\}$ . Namely,  $[f] \in L^p(\mu, B)$  is the equivalence class of all  $g = f \text{ } \mu\text{-a.e. for } f \in \mathcal{L}^p(\mu, B)$ .

In addition, we define

$$\|[f]\|_{L^p(\mu, B)} := \|f\|_{\mathcal{L}^\infty(\mu, B)}$$

for any representative  $f$ .

**Theorem 11.5** (Fischer-Riesz-Bochner). Let  $(X, \mathcal{M}, \mu)$  be a measure space,  $(B, \|\cdot\|)$  be a Banach Space. For all  $1 \leq p \leq \infty$ , we have that  $(L^p(\mu, B), \|\cdot\|_{L^p(\mu, B)})$  is a Banach Space.