Pmath753 Functional Analysis

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Contents

1 Metric Spaces and Complete Spaces

Definition 1.1. A **metric space** is a set X that has a (distance) metric:

$$
\begin{aligned}
d(\cdot,\cdot) : X \times X \to \mathbb{R}, \text{ such that } \forall x, y, z \in X \\
d(x,x) = 0 \\
\forall x \neq y, d(x,y) > 0 \\
d(x,y) = d(y,x) \\
d(x,z) \leq d(x,y) + d(y,z)
\end{aligned}
$$

Definition 1.2. Given a metric space (X, d), a sequence $(x_n)_{n=1}^{\infty}$ in X has a **limit point** $x \in X$ if $\lim\limits_{n\to\infty} d(x, x_n) = 0$. In this case, we say $(x_n)_{n=1}^{\infty}$ is a **convergent sequence**, and write $x = \lim_{n\to\infty} x_n$.

Definition 1.3. A sequence $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** in a metric space (X, d) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \text{ such that } \forall m, n \in \mathbb{N},\ m, n > N \implies d(x_m, x_n) < \epsilon.$$

Definition 1.4. A metric space X is **complete** if every Cauchy sequence $(x_i)_{i=1}^{\infty}$ converges to a limit point in X. i.e. $\exists x \in X, \lim\limits_{i\to\infty} x_i = x$.

Proposition 1.1. *Let (X, d) be a metric space; then every convergent sequence is Cauchy.*

Proposition 1.2. *Let (X, d) be a metric space. Suppose $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a convergent subsequence such that $\lim_{k\to\infty} x_{n_k} = x \in X$, then $\lim_{n\to\infty} x_n = x$.*

2 Topology

See more on the notes of Pmath367 Topology by S. New.

2.1 Topological Spaces

Definition 2.1. Let $X \neq \emptyset$ be a set. A **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X) := 2^X =$ power set of X, satisfying

1. $\emptyset, X \in \mathcal{T}$,
2. $\mathcal{T}$ is closed under arbitrary union; namely, $\forall \{A_\alpha\}_{\alpha\in I} \subseteq \mathcal{T},\ \bigcup_{\alpha\in I} A_\alpha \in \mathcal{T}$, and
3. $\mathcal{T}$ is closed under finite intersection; namely, $\forall \{A_i\}_{i=1}^{n} \subseteq \mathcal{T},\ \bigcup_{i=1}^{n} A_i \in \mathcal{T}$.

Also, $(X, \mathcal{T})$ is a **topological space** if $\mathcal{T}$ is a topology on X.

Definition 2.2. For any $S \subseteq \mathcal{P}(X)$, we define the **topology generated by** S to be

$$\mathcal{T}_S := \langle S \rangle := \{\emptyset, X, \text{ all unions of finite intersections of elements of } S\},$$

which is the intersection of all topologies on X that contains S, and it is the smallest topology on X containing S.

Proposition 2.1. *Let (X, d) be a metric space, then there is a **metric topology** $\mathcal{T}_d$ that is generated by open balls.*

Definition 2.3. $(X, \leq)$ is a **partially ordered set (poset)** if $\leq$ is

1. anti-symetric: $\forall x, y \in X$, if $x \leq y$ and $y \leq x$, we have $x = y$,
2. reflexive: $\forall x \in X,\ x \leq x$, and

3. transitive: $\forall x, y, z \in X$, if $x \leq y, y \leq z$, we have $x \leq z$.

We can define $\geq, <, >$ by

$$\begin{aligned} x \geq y &\iff y \leq x \\ x < y &\iff x \leq y \wedge x \neq y \\ x > y &\iff y < x. \end{aligned}$$

Definition 2.4. $(X, \leq)$ is a **totally ordered set** if it is a partially ordered set such that $\forall x, y \in X,\ x \leq y$ or $y \leq x$.

Proposition 2.2. *$(X, \leq)$ is a totally ordered set if and only if $<$ satisfies*

1. *$\forall x, y \in X$, exactly one of the following is true: $x < y,\ x = y,\ y < x$.*
2. *$\forall x, y, z \in X$, if $x < y, y < z$, we have $x < z$.*

Definition 2.5. Let $(X, \leq)$ be a totally ordered set, we can define for each $a, b \in X$,

1. $(-\infty, a) := \{x \in X : x < a\}$,
2. $(a, \infty) := \{x \in X : a < x\}$, and
3. $(a, b) := (a, \infty) \cap (-\infty, b)$.

Proposition 2.3. *Let $\mathcal{T}_{\leq}$ be the topology generated by all the sets above, then $\mathcal{T}_{\leq}$ is a topology.*

Definition 2.6. Let $(X, \mathcal{T})$ be a topological space, then we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$. We say $E \subseteq X$ is **closed** if $E^c \in \mathcal{T}$ is open.

Definition 2.7. For $E \in X$, the **closure** of E is

$$\bar{E} = \bigcap_{F \supseteq E : F \text{ is closed}} F,$$

which is the smallest closed set containing E.

Definition 2.8. For $E \in X$, the **interior** of E is

$$E^o = \bigcup_{U \subseteq E : U \text{ is open}} U,$$

which is the largest open set contained in E.

Proposition 2.4. *Closed sets are stable under finite unions and arbitrary intersections.*

Proposition 2.5. *For any set A,*

$$\bar{A} = ((A^c)^o)^c.$$

Proposition 2.6. *For any set A,*

$$x \in \bar{A} \iff (\forall U \, open\ ,\ x \in U \implies U \cap A \neq \emptyset).$$

Definition 2.9. Let $(X, \mathcal{T})$ be a topological space, a $\mathcal{B} = \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{P}(X)$ is said to be a **basis/base** of the topology $\mathcal{T}$ if it is a collection of open sets, and for every $U \in \mathcal{T}$, we have $U = \bigcup_{\alpha \in J} U_\alpha$ for some $J \subseteq I$.

Proposition 2.7. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subseteq \mathcal{T}$ is a basis of $\mathcal{T}$ iff*

$$\forall x \in U \in \mathcal{T}, \exists U_\alpha \in \mathcal{B} \text{ such that } x \in U_\alpha \subseteq U.$$

Proposition 2.8. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis of $\mathcal{T}$, iff*

1. $X = \bigcup_{\alpha\in I} U_\alpha$,

2. *For any* $U_1, U_2 \in \mathcal{B}, x \in U_1 \cap U_2$, *we have* $\exists U_x \in \mathcal{B}, x \in U_x \subseteq U_1 \cap U_2$,

3. $\mathcal{T} = \langle \mathcal{B} \rangle$ *is the topology generated by* $\mathcal{B}$.

Example 2.1.1. Let (X, d) be a metric space, then $\left\{B(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\right\}$ is a base.

Definition 2.10. Let $(X, \mathcal{T})$ be a topological space, a **subbase** is a collection of open sets $S \subseteq \mathcal{T}$ such that

$$\left\{X, \bigcap_{i=1}^{n} S_i : n \in \mathbb{N}^+, S_1, \dots, S_n \in S\right\}$$

forms a base for $\mathcal{T}$.

Definition 2.11. Given $x \in X$, a **neighbourhood** of x is a set $V \ni x$, such that $\exists U \in \mathcal{T}$ with $x \in U \subset V$.

Definition 2.12. We say $S \subseteq X$ is **dense** in a topological space $(X, \mathcal{T})$ if $\forall$open $U \neq \emptyset,\ S \cap U \neq \emptyset$.

Proposition 2.9. *S is dense iff* $\bar{S} = X$.

Definition 2.13. A topological space $(X, \mathcal{T})$ is **separable** if there is a countable subset.

Definition 2.14. A topological space $(X, \mathcal{T})$ is **first countable** if $\forall x \in X$, there is a countable open neighbourhood base $\{U_n\}_{n=1}^{\infty} \subset \mathcal{T}$ at x.

Proposition 2.10. *A topological space* $(X, \mathcal{T})$ *is first countable iff* $\forall x \in X$, *there is a countable open neighbourhood* $\{U_n\}_{n=1}^{\infty} \subset \mathcal{T}$, *such that for any neighbourhood* V *of* x, *there is* $n \in \mathbb{N}$ *such that* $x \in U_n \subseteq V$.

Definition 2.15. A topological space $(X, \mathcal{T})$ is called **2nd countable** if it has a countable basis.

Proposition 2.11. *Every metric space* (X, d) *are first countable.*

Proposition 2.12. *The discrete topology of* X *is separable iff* $|X|$ *is at most countable.*

Definition 2.16 (Axiom of Choice)**.** If $X \neq \emptyset$, then there is a choice function $C : P(X) \setminus \{\emptyset\} \to X$ such that $\forall A \subseteq X$, if $A \neq \emptyset$, we have $C(A) \in A$.

Proposition 2.13 (Axiom of Choice Equivalence)**.** *The Axiom of Choice is equivalent to: Let* $\{X_\alpha\}_{\alpha\in A}$ *be a family of non-empty sets, then*

$$\Pi_{\alpha\in A} X_\alpha := \left\{ f : A \to \bigcup_{\alpha\in A} X_\alpha \,\middle|\, f(\alpha) \in X_\alpha \right\} \neq \emptyset.$$

Proof. Suppose AOC holds, then taking $X = \bigcup_{\alpha\in A} X_\alpha$, we have the choice function C. Now take $f(a) := C(X_a)$.
On the other hand, suppose the latter holds, then consider $\Pi_{X_\alpha \in (P(X)\setminus\{\emptyset\})} X_\alpha$, which is non-empty. Consider any $f \in \Pi_{X_\alpha \in (P(X)\setminus\{\emptyset\})} X_\alpha$, then $C(X_\alpha) := f(a)$ is a choice function. □

Proposition 2.14. *A metric space* (X, d) *is separable iff it is 2nd countable.*

Proof. If $S = \{x_k\}_{k=1}^{\infty}$ is dense, then $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a countable base.
Indeed, consider any $x \in X$ with any open $U \ni x$, we know $\exists r > 0$, such that $x \in B(x, r) \subset U$. Also, there is x_k such that $d(x, x_k) < \frac{r}{2}$.
Now choose some $r' \in \mathbb{Q}$ such that $d(x, x_k) < r' < \frac{r}{2}$, then $x \in B(x_k, r') \subset B(x, r) \subset U$.
Thus $\{B(x_k, r) : r \in \mathbb{Q}, k \in \mathbb{N}\}$ is a base.
On the other hand, suppose X is second countable with a countable base $\{U_n\}_{n=1}^{\infty}$. WLOG, $U_n \neq \emptyset$.
Now for any $n \in \mathbb{N}$, pick $x_n \in U_n$ by the axiom of countable choice. Let $S = \{x_n\}_{n=1}^{\infty}$, then we claim S is dense.
Indeed, for any open $U \neq \emptyset$, we can find some $U_n \subset U$. Thus $x_n \in S \cap U$. □

Proposition 2.15. *If $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ is a set of topologies on X,*

1. *There is a weakest topology $\tau := \left\langle \bigcup_{\alpha \in A} \mathcal{T}_\alpha \right\rangle$ that is stronger than each $\mathcal{T}_\alpha$.*
2. *There is a strongest topology $\delta := \bigcap_{\alpha \in A} \mathcal{T}_\alpha$ that is weaker than each $\mathcal{T}_\alpha$.*

Definition 2.17. A topological space $(X, \mathcal{S})$ is **Hausdorff** if

$$\forall x \neq y \in X,\ \exists S_x, S_y \in \mathcal{S},\ \text{such that } x \in S_x, y \in S_y, S_x \cap S_y = \emptyset.$$

Proposition 2.16. *Any space with its discrete topology is always Hausdorff.*

Example 2.1.2. Every metric space is Hausdorff.

Proposition 2.17. *Any space with more than one element with the trivial topology is not Hausdorff.*

Example 2.1.3. Consider $X := (0,1) \cup \{1^+, 1^-\}$. Let $(0,1)$ have the usual open topology. Also, let $(r,1) \cup \{1^+\}$ and $(r,1) \cup \{1^-\}$ be open for any $0 < r < 1$. The topology generated by this basis will not be Hausdorff.
Indeed, consider $1^+, 1^-$, then for any $U \ni 1^+, V \ni 1^-$, we can find $r_U, r_V \in (0,1)$, such that $(r_U, 1) \cup \{1^+\} \subseteq U, (r_V, 1) \cup \{1^-\} \subseteq V$. Yet $(\max(r_U, r_V), 1) \subseteq U \cap V$, which is not empty.

Proposition 2.18. *If X is Hausdorff, then for any $x \in X$, we have that $\{x\}$ is closed.*

Proof. For any $y \neq x$, we can find open $V_y \ni y, U_y \ni x$, such that $V_y \cap \{x\} \subseteq V_y \cap U_y = \emptyset$.
Thus $X \setminus \{x\} = \bigcup_{y \in X} V_y$, which is open. $\square$

2.2 Nets

Definition 2.18. Let $(X, \mathcal{T})$ be a topological space, say a sequence $(x_i)_{i=1}^\infty$ **converges to** $x \in X$ if $\forall$open $U \ni x$, $\exists N \in \mathbb{N}$ such that $\forall i \geq N,\ x_i \in U$.

Example 2.2.1. Consider $X = \mathbb{N} \times \mathbb{N}$, and the projection $\pi_1 : X \to \mathbb{N}$ by $\pi_1(m,n) := m$.
Let any U be open if $(0,0) \notin U$, or if $\left\{m \in \mathbb{N} : \pi^{-1}(m) \cap U \text{ is co-finite in } \{m\} \times \mathbb{N}\right\}$ is co-finite.
One can show this defines a topology on X, and it is Hausdorff.
Indeed, let $X_0 := X \backslash \{(0,0)\}$. Consider any $(m,n) \neq (m',n') \in X$. If both are in X_0 then $\{(m,n)\}, \{(m',n')\}$ are open and disjoint.
If $(m',n') = (0,0)$, then $\{(m,n)\}, X \setminus \{(m,n)\}$ are open and disjoint.
This shows Hausdorff.
Also, $\bar{X}_0 = X$.
Indeed, consider any open $U \ni (0,0)$, we must have that $U \cap X_0 \neq \emptyset$.
However, there is no sequence in X_0 that converges to $(0,0)$.
Indeed, assume for contradiction that there is such a convergent sequence $(x_k)_{k=1}^\infty$ in X_0.
Write each $x_k = (m^k, n^k)$.
Suppose there is some $M \in \mathbb{N}^+$, such that $\forall k \in \mathbb{N}^+, m_k \leq M$.
Consider $U := \{(m,n) : m > M, n \in \mathbb{N}\} \cup \{(0,0)\}$.
Now for each $m \in \mathbb{N}$,

$$\pi^{-1}(m) \cap U = \begin{cases} (0,0) & \text{if } m = 0 \\ \emptyset & \text{if } 0 < m \leq M \\ \{m\} \times \mathbb{N} & \text{if } m > M. \end{cases}$$

Thus for all $m > M$, $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.
Yet $U \cap \{x_k\}_{k=1}^\infty = \emptyset$.
Now suppose there is no such M, then we can find a subsequence $m_{k_1} < m_{k_2} < m_{k_3} < \cdots$.
Now let $U := X \setminus \{x_{k_i}\}_{i=1}^\infty$.
For each $m \in \mathbb{N}$,

$$\pi^{-1}(m) \cap U = \begin{cases} \{m\} \times \mathbb{N} & \text{if } m \notin \{m_{k_i}\}_{i=1}^\infty \\ \{m\} \times (\mathbb{N} \setminus n_{k_i}) & \text{if } m \in \{m_{k_i}\}_{i=1}^\infty. \end{cases}$$

Notice that there cannot be two $n_{k_i} \neq n_{k_j}$ for any m, since $k_i \neq k_j \implies m_{k_i} \neq m_{k_j}$.
Thus all $\pi^{-1}(m) \cap U$ is co-finite in $\{m\} \times \mathbb{N}$. This shows U is open.
Yet $U \cap \{x_{k_i}\}_{i=1}^{\infty} = \emptyset$.
Thus there cannot be any convergent sequence $(x_k)_{k=1}^{\infty}$ in X_0.

Remark. The above example shows that sequences do not behave as we want in topological spaces.

Definition 2.19. An **upwards directed set** is a poset $(\Lambda, \leq)$ such that if $\lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_0 \in \Lambda$ such that $\lambda_1 \leq \lambda_0, \lambda_2 \leq \lambda_0$.

Definition 2.20. For $X \neq \emptyset$, a **net** in X is a function $j : \Lambda \to X$, where $(\Lambda, \leq)$ is an upwards directed set. Write $x_\lambda := j(\lambda) \in X$, and we can use $(x_\lambda)_{\lambda \in \Lambda}$ to represent a net.

Definition 2.21. Let $(X, \mathcal{T})$ be a topological space, say a net $(x_\lambda)_{\lambda \in \Lambda}$ **converges to** $x \in X$ if $\forall$open $U \ni x$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \geq \lambda_0$, $x_\lambda \in U$. In this case, we say x is a **limit** of the net, and write it as $x = \lim_{\lambda \in \Lambda} x_\lambda$ or $x_\lambda \to x$.

Definition 2.22. Given a net $(x_\lambda)_{\lambda \in \Lambda}$, then a **subnet** of it $(y_\gamma)_{\gamma \in \Gamma}$ is given by an upwards directed set $(\Gamma, \leq)$ and a function $\phi : \Gamma \to \Lambda$ that is **cofinal**, which means $\forall \lambda_0 \in \Lambda$, $\exists \gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0$, $\phi(\gamma) \geq \lambda_0$. Each y_γ is given by $x_{\phi(\gamma)}$.

Example 2.2.2. Notice that if we take $(\mathbb{N}, \leq)$, the net is just a sequence. To get a subsequence, we can take $\Gamma = \mathbb{N}$, and $\phi : \mathbb{N} \to \mathbb{N}$ to be any increasing function. The generated subnet will be a subsequence.

Definition 2.23. Let $(X, \mathcal{T})$ be a topological space, and $x \in X$, define the **system of open neighbourhoods of** x to be $\mathcal{O}(x) := \{U \in \mathcal{T} : x \in U\}$.

Proposition 2.19. *$(\mathcal{O}(x), \supseteq)$ is an upwards directed set.*

Example 2.2.3. For $X = \mathbb{N} \times \mathbb{N}$ and $X_0 = X \setminus \{(0,0)\}$ as above, there is a net in X_0 converging to $(0,0)$.
Indeed, let us enumerate $X_0 = \{x_k\}_{k=1}^{\infty}$ as $(0,1), (1,0), (0,2), (1,1), (2,0), \ldots$.
Now $\Lambda := \mathcal{O}((0,0))$ is an upward directed set by containment.
Then for each $U \in \Lambda$, we can pick $x_U := x_{k_U}$, where k_U is the first $k \in \mathbb{N}$ such that $x_k \in U$.
Claim: $(x_U)_{U \in \Lambda}$ converges to $(0,0)$.
Pick any $U_0 \ni (0,0)$, then for all $U \geq U_0$, it is open and $U \subseteq U_0$. Thus, we must have $x_U \in U \subseteq U_0$.
Indeed, $(x_U)_{U \in \Lambda}$ is a subnet of $\{x_k\}_{k \in \mathbb{N}^+}$ by $\phi(U) := k_U$.

Theorem 2.20. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces, then*

1. *For any $A \subseteq X$, we have $x \in \bar{A}$ if and only if $\exists$ a net $(x_\lambda)_{\lambda \in \Lambda}$ in A, such that $x_\lambda \to x$.*
2. *$f : X \to Y$ is continuous if and only if for any net $(x_\lambda)_{\lambda \in \Lambda} \subset X$ such that $x_\lambda \to x \in X$, we have $f(x_\lambda) \to f(x)$.*

Proof.
1. Consider any $x \in \bar{A}$, then for any open $U \ni x$, we have $U \cap A \neq \emptyset$.
By the Axiom of Choice, we can have $x_U \in U \cap A$ for each open neighbourhood U.
Consider the net $(x_U)_{U \in \mathcal{O}(x)}$.
Given any open $U \ni X$, if $V \geq U$, we must have $V \subseteq U$, and $x_v \in V \subseteq U$.
Thus $x_U \to x$.
On the other hand, for any net $x_\lambda \to x$ in A, consider any open $U \ni x$, there is λ_0, such that
$$\forall \lambda \in \Lambda, \ \lambda_0 \leq \lambda \implies x_\lambda \in U \implies U \cap A \neq \emptyset.$$
Thus $x \in \bar{A}$.
2. Assume f is continuous, and $x_\lambda \to x$. Let $V \in \mathcal{O}(f(x))$, then $U := f^{-1}(V)$ is open and $x \in U$.
Thus there is $\lambda_0 \in \Lambda$, such that $\forall \lambda \geq \lambda_0, x_\lambda \in U$.
Thus $f(x_\lambda) \in V$.
On the other hand, assume for any net $(x_\lambda)_{\lambda \in \Lambda} \subset X$ such that $x_\lambda \to x \in X$, we have $f(x_\lambda) \to f(x)$.
For contradiction, suppose there is open $V \in Y$, with $U := f^{-1}(V)$ is not open in X.

Then U^c is not closed, and $U^c \neq \overline{U^c}$.
Thus, there is $x \in \overline{U^c} \setminus U^c = \overline{U^c} \cap U$.
Since $x \in \overline{U^c}$, by 1., there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in U^c, such that $x_\lambda \to x$.
By assumption, we have $f(x_\lambda) \to f(x)$.
Since each $f(x_\lambda)$ is in $f(U^c) = V^c$, by 1., we have that $f(x) \in \overline{V^c} = V^c$ since V^c is closed (V is open).
However, since $x \in U$, we also have $f(x) \in f(U) = V$, which is a contradiction.

□

2.3 Compactness

Definition 2.24. Let $(X, \mathcal{T})$ be a topological space. A collection of subsets $C = \{U_\alpha \subseteq X\}_{\alpha\in A}$ is called a **cover** for X if $X = \bigcup_{\alpha\in A} U_\alpha$. A cover is called an open cover if every U_α is open in $\mathcal{T}$.

Definition 2.25. Let $(X, \mathcal{T})$ be a topological space. A collection $\{C_\alpha\}_{\alpha\in A}$ of non-empty closed sets is a FIP-family if for any finite $F \subseteq A$, $\bigcap_{\alpha\in F} C_\alpha \neq \emptyset$. X has the finite intersection property (FIP) if for all FIP-familie $\{C_\alpha\}_{\alpha\in A}$, we have $\bigcap_{\alpha\in A} C_\alpha \neq \emptyset$.

Definition 2.26. Let $(X, \mathcal{T})$ be a topological space. A set $K \subseteq X$ is **compact** if every open cover of K has a finite subcover. Namely,

$$\forall (U_\alpha)_{\alpha\in A} \text{ be open}, K \subseteq \bigcup_{\alpha\in A} U_\alpha \implies \exists n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n,\ \text{such that } K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

Theorem 2.21. *Let $(X, \mathcal{T})$ be a topological space, TFAE:*

1. *X is compact.*
2. *X has the finite intersection property.*
3. *For all nets $(x_\lambda)_{\lambda\in\Lambda}$ in X, there is a convergent subnet.*

Proof. (1) $\implies$ (2).
For contradiction, suppose there is some FIP-family such that $\bigcap_{\alpha\in A} C_\alpha = \emptyset$.
We have $X = \bigcup_{\alpha\in A} C_\alpha^c$, which is an open cover for X.
Since X is compact, there is a finite $F \subseteq A$, such that $X = \bigcup_{\alpha\in F} C_\alpha^c$.
Thus $\bigcap_{\alpha\in A} C_\alpha = \emptyset$, which contradict $\{C_\alpha\}_{\alpha\in A}$ being a FIP family.
(2) $\implies$ (1).
Consider any open cover $X = \bigcup_{\alpha\in A} U_\alpha$, then $\bigcap_{\alpha\in A} U_\alpha^c = \emptyset$, and it is not a FIP-family.
Thus there is a finite $F \subseteq A$, such that $\bigcap_{\alpha\in F} U_\alpha^c = \emptyset$.
Thus $X = \bigcup_{\alpha\in F} U_\alpha$ is a finite open cover.
(2) $\implies$ (3).
Let $(x_\lambda)_{\lambda\in\Lambda}$ be any net in X.
Define $C_\lambda := \overline{\{x_\mu : \mu \geq \lambda\}}$. Notice that $C_\lambda \neq \emptyset$ since $x_\lambda \in C_\lambda$.
We claim that $\{C_\lambda\}_{\lambda\in\Lambda}$ is a FIP family.
The closeness is by definition.
Now fix any $\lambda_1, \ldots, \lambda_n \in \Lambda$.
Since Λ is upwards directed, there is $\lambda_0 \in \Lambda$, such that $\forall i \in [n], \lambda_i \leq \lambda_0$.
Thus $\bigcap_{i=1}^{n} C_{\lambda_i} \supseteq C_\lambda \neq \emptyset$.
By FIP, $\bigcap_{\lambda\in\Lambda} C_{\lambda_i} \neq \emptyset$.
Pick any $x \in \bigcap_{\lambda\in\Lambda} C_{\lambda_i}$.
Let $\Gamma := \Lambda \times \mathcal{O}(x)$ with the partial order $(\lambda, U) \leq (\lambda', U')$ if $\lambda \leq \lambda'$ and $U \supseteq U'$.
Fix $(\lambda, U) \in \Gamma$, we know that $x \in C_\lambda = \overline{\{x_\mu : \mu \geq \lambda\}}$.
Thus $U \cap \{x_\mu : \mu \geq \lambda\} \neq \emptyset$.
By the Axiom of Choice, there is $y_{(\lambda,U)} = x_{\phi(\lambda,U)} \in U \cap \{x_\mu : \mu \geq \lambda\}$, where $\phi(\lambda, U) := C(\{\mu \geq \lambda : x_\mu \in U\})$.
For any $\lambda_0 \in \Lambda$, let $\gamma_0 = (\lambda_0, X)$, then for any $\gamma = (\lambda, U) \geq \gamma_0$, we have $\phi(\gamma) \geq \lambda \geq \lambda_0$.
Thus ϕ is cofinal, and $(y_\gamma)_{\gamma\in\Gamma}$ is a subnet.

In addition, given any $U_0 \in \mathcal{O}(x)$, we can pick any $\lambda_0 \in \Lambda$, and let $\gamma_0 := (\lambda_0, U_0)$.
Then for any $(\lambda, U) \geq \gamma_0$, we must have $y_{(\lambda,U)} = x_{\phi(\lambda,U)} \in U \subseteq U_0$.
(3) $\implies$ (2).
Fix any FIP-family $\{C_\alpha\}_{\alpha\in A}$ in X. Then for any finite $F \subseteq A$, by the Axiom of Choice, we can find $x_F \in \bigcap_{\alpha\in F} C_\alpha$.
Now consider the net $(x_F)_{\text{finite } F\subseteq A}$, where $F_1 \leq F_2$ if $F_1 \subseteq F_2$.
By 3., there is a convergent subnet $\phi : \Gamma \to \Lambda$, such that $x_{\phi(\gamma)} \to x \in X$.
Now fix any $\alpha \in A$, then $\{\alpha\} \in \Lambda$.
Thus there is some $\gamma_0 \in \Gamma$, such that $\forall \gamma \geq \gamma_0,\ \phi(\gamma) \supseteq \{\alpha\} \ni \alpha$.
We have $x_{\phi(\gamma)} \in \bigcap_{\beta\in\phi(\gamma)} C_\beta \subseteq C_\alpha$.
Since this holds for all $\gamma \geq \gamma_0$, and $x_{\phi(\gamma)} \to x$, we have that $x \in \bar{C_\alpha} = C_\alpha$.
Since this holds for any $\alpha \in A$, we have that $x \in \bigcap_{\alpha\in F} C_\alpha$. Thus $\bigcap_{\alpha\in F} C_\alpha \neq 0$. □

Proposition 2.22. *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two topological spaces. If X is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact in Y.*

Theorem 2.23. *Let $(X, \mathcal{T})$ be a topological space,*

1. *Suppose K is compact, then $\forall F \subseteq K$ that is closed, F is also compact.*
2. *If X is Hausdorff, for any compact $K \subseteq X, x \in X \setminus K$, $\exists$ open neighborhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$.*

Proof. 1. Let $(U_\alpha)_{\alpha\in A}$ be an open cover for F.
Since F is closed, then F^c is open. Thus $\{F^c\} \cup \{U_\alpha\}_{\alpha\in A}$ is an open cover for K.
Thus there are $U_{\alpha_1}, \ldots, U_{\alpha_n}$, such that $K \subseteq F^c \cup \bigcup_{i=1}^n U_{\alpha_i}$. Thus $F \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ has a finite subcover.

2. Consider any $y \in K$, there is some open neighborhoods $U_y \ni x, W_y \ni y$, such that $U_y \cap W_y = \emptyset$.
Since $K \subseteq \bigcup_{y\in K} W_y$ is compact, we have $K \subseteq \bigcup_{i=1}^n W_{y_i} =: W$ for some $y_1, \ldots, y_n$.
Let $U = \bigcap_{i=1}^n U_{y_i}$, we have $x \in U, K \subseteq W, U \cap W = \emptyset$ as required.

□

Corollary 2.24. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, then any compact set K is closed. In addition, for any closed $F \subseteq X$, we have $F \cap K$ is compact.*

Proof. Suppose for contradiction that K is not closed, then there is some $y \in \bar{K} \setminus K$. Thus we can find open neighbourhood U of x, and open $W \supset K$, such that $W \cap U = \emptyset$. Now $K \subset \bar{K} \setminus U \subsetneq \bar{K}$ is closed, which is a contradiction.
Since K is closed, so is $F \cap K \subseteq K$, and thus it is compact. □

Definition 2.27. X is **locally compact** if $\forall x \in X$, there is an open neighbourhood $U_x \in \mathcal{O}(x)$ such that $\overline{U_x}$ is compact.

Example 2.3.1. $\mathbb{R}^n$ is locally compact by the Heinz-Borel theorem.

Proposition 2.25. *A Banach space $(X, ||\cdot||)$ is locally compact iff* $\dim(X) < \infty$.

Lemma 2.26. *Let $(X, \mathcal{T})$ be a Hausdorff topological space, and $(K_\alpha)_{\alpha\in A}$ be a collections of compact sets such that*
$$\bigcap_{\alpha\in A} K_\alpha = \emptyset.$$
We must have $\alpha_1, \ldots, \alpha_n \in A$, such that
$$\bigcap_{i=1}^n K_{\alpha_i} = \emptyset.$$

Proof. Fix $\alpha_1 \in A$, then $K_{\alpha_1} \subseteq \left(\bigcap_{\alpha\neq\alpha_1} K_\alpha\right)^c = \bigcup_{\alpha\neq\alpha_1} K_\alpha^c$ is compact and has an open cover.
Thus there must be $\alpha_2, \ldots, \alpha_n \in A$, such that $K_{\alpha_1} \subseteq \bigcup_{i=2}^n K_{\alpha_i}^c = (\bigcap_{i=2}^n K_{\alpha_i})^c$.
Thus $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$. □

Theorem 2.27. *Let X be a Locally Compact Hausdorff space, and let $K \subseteq U \subseteq X$ be such that K is compact, and U is open. Then there exists some open set V such that $\bar{V}$ is compact, and*

$$K \subseteq V \subseteq \bar{V} \subseteq U.$$

Proof. Since X is a Locally Compact Hausdorff space, there are $V_1, \ldots, V_n$, each with $\bar{V_i}$ be compact, such that $K \subseteq \bigcup_{i=1}^n V_i =: G$. Note that $\bar{G}$ is compact, and G is open.
If $U = X$, then $G \subseteq U$, and we are done.
Otherwise, let $C := X \setminus U$ be non-empty and closed.
Consider any $y \in C$, we know that $y \notin K$. Since X is Hausdorff, we can find open $W_y \supset K$, and $U_y \ni y$, such that $W_y \cap U_y = \emptyset$. Then $W_y \subseteq U_y^c$, and thus $\bar{W_y} \subseteq U_y^c$, since U_y^c is closed. Yet $y \notin U_y^c$, thus $y \notin \bar{W_y}$.
Now consider the family $\left\{C \cap \bar{W_y} \cap \bar{G}\right\}_{y \in C}$. Notice that each $C \cap \bar{W_y} \cap \bar{G}$ is compact, since $C, \bar{W_y}$ are closed, and $\bar{G}$ is compact.
Yet $\bigcap_{y \in C} \left(C \cap \bar{W_y} \cap \bar{G}\right) = \emptyset$.
Thus $\exists y_1, \ldots, y_n \in C$, such that $\bigcap_{i=1}^n \left(C \cap \bar{W_y} \cap \bar{G}\right) = \emptyset$.
Now let $V := G \cap \bigcap_{i=1}^n W_{y_i}$.
Clearly V is open, and $K \subseteq V$.
In addition, $\bar{V} \subseteq \bar{G} \cap \bigcap_{i=1}^n \bar{W_{y_i}}$, yet the intersection of righthand side and C is empty, thus contained in $C^c = U$. $\square$

2.4 Continuous Functions

Definition 2.28. A function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is **continuous** if

$$\forall U \in \mathcal{S},\ f^{-1}(U) \in \mathcal{T}.$$

Namely, the preimage of any open set is open.

Definition 2.29. A function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is **open** if

$$\forall V \in \mathcal{T},\ f(V) \in \mathcal{S}.$$

Namely, the image of any open set is open.

Definition 2.30. Given two sets X, Y, and their corresponding topology $\mathcal{T}, \mathcal{S}$, a continuous map $f : X \to Y$ is a **homeomorphism** if it is bijective, and its inverse function is also continuous.

Remark. A homeomorphism is a map that preserves the topological structure between two sets.

Definition 2.31. Let $C(X)$ be the collection of functions $f : X \to \mathbb{C}$ that are continuous.

Definition 2.32. Let $C(X, \mathbb{R})$ be the collection of functions $f : X \to \mathbb{R}$ that are continuous.

Definition 2.33. Let $C_b(X)$ be the collection of functions $f \in C(X) : ||f||_\infty < \infty$.

Definition 2.34. Let $C_b(X, \mathbb{R})$ be the collection of functions $f \in C(X, \mathbb{R}) : ||f||_\infty < \infty$.

Proposition 2.28. *If $C(X)$ separates points, so do $C_b(X), C_b(X, \mathbb{R})$. Also, X is Hausdorff.*

Proof. If $x \neq y$, we have $f \in C(X)$ such that $f(x) \neq f(y)$.
WLOG, $\Re(f(x)) \neq \Re(f(y))$, and $\Re(f(x)) < \Re(f(y))$.
Now define $g(z) := \min\left\{\Re(f(y)), \max\left\{\Re(f(x)), \Re(f(y))\right\}\right\}$, which is bounded and continuous. Also $g(x) = f(x), g(y) = f(y)$.
Thus $C_b(X, \mathbb{R})$ separates the points.
Now if $x \neq y$, we can find $f \in C(X)$, such that $|f(x) - f(y)| = r > 0$.
Now let $U := f^{-1}\big(B(f(x), \frac{r}{2})\big), V := f^{-1}\big(B(f(y), \frac{r}{2})\big)$, which are both open. Also, $x \in U, y \in V$, and $U \cap V = f^{-1}(\emptyset) = \emptyset$. $\square$

Definition 2.35. A topological space $(X, \mathcal{T})$ is **normal** if for any disjoint closed sets A, B, we can find open $U \supset A, V \supset B$ such that $U \cap V \neq \emptyset$.

Lemma 2.29 (Urysohn's Lemma for normal spaces)**.** *$(X, \mathcal{T})$ is normal iff for any disjoint closed sets $A, B,\ \exists f : X \to [0, 1]$ continuous, such that $f|_A = 0, f|_B = 1$.*

Corollary 2.30. *If X is normal and Hausdorff, $C_b(X)$ separates the points.*

Definition 2.36. For $f \in C(X)$, the **support** of it is

$$\mathrm{Supp}(f) := \overline{f^{-1}(\mathbb{C} \setminus \{0\})} \subseteq X.$$

Definition 2.37. The set of **compactly supported functions** are

$$C_c(X) := \{f \in C(X) : \mathrm{Supp}(f) \text{ is compact}\}.$$

Definition 2.38. $C_0(X)$ is the closure of $C_c(X)$ in $||\cdot||_\infty$.

Proposition 2.31. *$C_0(X)$ is the set of all continuous functions that vanishes at ∞. $(C_0(X), ||\cdot||_\infty)$ is a Banach Space and a commutative C^*-algebra with the involution $f^*(x) := \overline{f(x)}$.*

Proposition 2.32. *$f \in C_0(X)$ if and only if $\forall \epsilon > 0,\ \exists K \subset\subset X,$ such that $\forall x \in X \setminus K,\ |f(x)| < \epsilon$.*

Theorem 2.33. *Any commutative C^*-algebra $(A, ||\cdot||)$ is isomorphic to $C_0(X)$ for some unique Locally Compact Hausdorff X.*

2.5 Partition of Unity

Definition 2.39. Let K be a compact set, and V be an open set of X. Let $f \in C_c(X)$. We say $f < V$ if $0 \leq f \leq 1$, and $\mathrm{Supp}(f) \subseteq V$. We say $K < f$ if $0 \leq f \leq 1$, and $f|_K = 1$. We say $K < f < V$ if $K \subset V, K < f, f < V$.

Remark. f is a "bump" function that approximates χ_K when V shrinks towards K.

Lemma 2.34 (Urysohn's lemma for Locally Compact Hausdorff Space)**.** *Let X be a Locally Compact Hausdorff space, $K \subseteq V \subseteq X$ be such that K is compact, and V is open. Then there exists $f \in C_c(V)$, such that $K < f < V$.*

Proof. we want to construct a family of open sets $\{V_r\}_{r \in \mathbb{Q} \cap [0,1]}$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s$.
By 2.27, we can find $K \subset V_0 \subset \bar{V_0} \subset V$.
Pick an enumeration of $r \in \mathbb{Q} \cap (0, 1]$, i.e. $(r_n)_{n=1}^\infty$. WLOG, we can let $r_1 = 1$.
By 2.27, we can find $K \subset V_1 \subset \bar{V_1} \subset V_0$.
Suppose we have constructed the V_{r_i} for $1 \leq i \leq n$, such that $\bar{V_r}$ is compact, and

$$K \subset V_1 \subset \bar{V_1} \subset V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r} \subset \cdots \subset V_0 \subset \bar{V_0} \subset V,$$

for $r < s \in \{r_i\}_{i=1}^n$.
Let $s = \max r_i : r_i < r_{n+1}, i \leq n, s = \min r_i : r_i > r_{n+1}, i \leq n$.
Now by 2.27, we can find $\bar{V_t} \subset V_{n+1} \subset \bar{V_{n+1}} \subset V_s$.
For any $r < r_{n+1}$, we have $r \leq s$, and thus $V_{n+1} \subset \bar{V_{n+1}} \subset V_s \subset \bar{V_s} \subseteq V_r$ by induction hypothesis, and similarly for any $r > r_{n+1}$.
Inductively, we can prove there is such a family.
Define $f_r := r\chi_{V_r}$, and $g_r := r\chi_{\bar{V_r}^c} + \chi_{\bar{V_r}}$, and $f := \sup_r f_r, g := \inf_r g_r$.
We can show that f, g are upper and lower continuous, respectively.
In addition, f, g are both 0 outside of V_1, and 1 on K.

Suppose there is some $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, such that $f_r(x) > g_s(x)$. Then we must have $f_r(x) > 0$, and thus $x \in V_r$ and $1 \leq r = f_r(x)$.
Thus $1 > g_s(x)$, and thus $x \in \bar{V_s}^c$ and $f_s = s$.
Since $r > s$, we must have $V_r \subset \bar{V_r} \subset V_s \subset \bar{V_s}$, which is a contradiction to $x \in V_r, x \notin \bar{V_s}$.
Thus for any $x \in X, r, s \in \mathbb{Q} \cap [0,1]$, we must have $f_r(x) \leq g_s(x)$.
Thus we must have $f(x) \leq g(x)$ for any $x \in V$.
Now suppose there is some $x \in X$, such that $f(x) < g(x)$.
There must be some rationals, such that $f(x) < r < s < g(x)$, since $\mathbb{Q}$ is dense.
Thus $\sup_r f_r(x) < r$, and thus $x \notin V_r$.
Also, $\inf_s g_s(x) > s$, and thus $x \in \bar{V_s}$.
Yet $r < s$, we must have $V_s \subset \bar{V_s} \subset V_r \subset \bar{V_r}$, which is a contradiction.
Thus we must have $f = g$, and it forces f to be continuous. □

Definition 2.40. Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$.
A collection $(h_i)_{i=1}^n \subset C_c(X)$ is called a **partition of unity** on K subordinate to $(V_i)_{i=1}^n$ if

$$\begin{cases} \forall 1 \leq i \leq n, & h_i < V_i, \\ \forall x \in K, & \sum_{i=1}^n h(x) = 1. \end{cases}$$

Theorem 2.35. *Let X be a Locally Compact Hausdorff space, $K \subseteq X$ be compact, and some finite open cover $\bigcup_{i=1}^n V_i \supseteq K$, there always exists a partition of unity on K subordinated to $(V_i)_{i=1}^n$.*

Proof. Since K is compact, we can find some open cover $W_1, \ldots, W_m$, such that for all j, we have $W_j \subset \bar{W}_j \subset V_{i(j)}$ for some $1 \leq i(j) \leq n$.
Let $K_i := \bigcup_{1 \leq j \leq m \text{ such that } W_j \subset V_i} \bar{W}_j \subset V_i$, which is compact.
By Urysohn's lemma, we can find $K_i < g_i < V_i$.
Now let $h_1 := g_1$, and in general, $h_i := g_i \prod_{j<i}(1 - g_j)$.
It is easy to check that $0 \leq h_i \leq 1$, and $h_i \in C_c(X)$.
In addition, $\text{Supp}(h_i) \subseteq \text{Supp}(g_i) \subset V_i$.
Thus $h_i < V_i$. Lastly, we can check

$$\begin{aligned} h_1 + h_2 &= g_1 + (1 - g_1)g_2 \\ &= 1 - (1 - g_1) + (1 - g_1)g_2 \\ &= 1 - (1 - g_1)(1 - g_2). \end{aligned}$$

Inductively, we have $\sum_{i=1}^n h_i = 1 - \prod_{i=1}^n (1 - g_n)$.
For any $x \in K$, there must be some $i \in [n]$ such that $x \in K_i$, and thus $g_i(x) = 1$, and thus

$$\sum_{i=1}^n h_i(x) = 1 - \prod_{i=1}^n (1 - g_n(x)) = 1 - 0 = 1.$$

□

2.6 Product Topology

Definition 2.41. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of topological spaces. The **product topology** on $\prod_\alpha X_\alpha$ is the topology generated by the sets

$$\left\{ U_\alpha \times \prod_{\beta \in A, \beta \neq \alpha} X_\beta \middle| \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\} = \left\{ \pi_\alpha^{-1}(U_\alpha) \middle| \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \right\},$$

where the **projection map onto** X_α is $\pi_\alpha : \prod_{\beta \in A} X_\beta \to X_\alpha$ by $(x_\beta)_{\beta \in A} \mapsto x_\alpha$.

Proposition 2.36. *The product topology is the weakest topology in which each π_α is continuous.*

Proposition 2.37. *A net $(x_\lambda)_{\lambda\in\Lambda}$ in $\prod_{\alpha\in A} X_\alpha$ converges to x if and only if $\forall\alpha\in A,\ \pi_\alpha(x_\lambda)\to\pi_\alpha(x)$ in X_α.*

Proof. See A1. $\square$

Theorem 2.38 (Tychonoff)**.** *Let $(X_\alpha,\mathcal{T}_\alpha)_{\alpha\in A}$ be a family of compact topological spaces, then $\prod_\alpha X_\alpha$ is compact under product topology.*

Definition 2.42. Let $(P,\leq)$ be a partially ordered set. We call a totally ordered subset $Q\subseteq P$ a **chain**.

Definition 2.43. Let $(P,\leq)$ be a partially ordered set. We call $\leq$ **inductive** if every chain $Q\subseteq P$ has an upper bound.

Definition 2.44. $\leq$ is called a **well-order** if it is a total order, and for every $\emptyset\neq S\subseteq X$ has a minimal element. $\exists\, x\in P$ such that $\forall y\in P,\ y\leq x \implies x=y$.

Lemma 2.39 (Zorn's)**.** *Every inductive partial order $(P,\leq)$, defined on a nonempty P, has a maximal element. Namely, $\exists\, x\in P$ such that $\forall y\in P,\ x\leq y\implies x=y$.*

Proposition 2.40. *Every vector space V has a basis.*

Proof. Consider $P=\{S\subset V|S \text{ is linearly independent}\}$, with $S\leq S' \iff S\subseteq S'$.
Let Q be a chain in P, then it has an upper bound $\tilde{S}=\bigcup_{S\in Q} S$, which we can check is still linearly independent.
By Zorn's lemma, there is a maximal $S\in P$. $\square$

Theorem 2.41 (Well-Ordering Principle)**.** *Every set X admits a well-ordering.*

Theorem 2.42. *The Following Are Equal:*

1. *Tychonoff's Theorem*
2. *Axiom of Choice*
3. *Zorn's Lemma*
4. *Well-Ordering Principle*

Proof. (1) $\implies$ (2).
Let $(X_\alpha)_{\alpha\in A}$ be a family of non-empty set.
Let $Y_\alpha:=\{p_\alpha\}\sqcup X_\alpha$ for some additional symbol p_α.
Define the topology $\mathcal{T}_\alpha:=\{\emptyset,Y_\alpha,X_\alpha,\{p_\alpha\}\}$.
Then $(Y_\alpha,\mathcal{T}_\alpha)_{\alpha\in A}$ are all compact.
By Tychonoff's Theorem, $\prod_\alpha Y_\alpha$ is also compact.
Now consider $C_\alpha:=X_\alpha\times\prod_{\beta\neq\alpha}Y_\beta$.
Since $C_\alpha^c=\{p_\alpha\}\times\prod_{\beta\neq\alpha}Y_\beta$ is open, we have that C_α is closed in the product topology.
Also, $C_\alpha\neq\emptyset$.
Now for any finite $\bigcap_{i=1}^n C_{\alpha_i}$, we have that $(x_{\alpha_1},x_{\alpha_2},\dots,x_{\alpha_n},p_\alpha,\dots)\in\bigcap_{i=1}^n C_{\alpha_i}$.
Thus $(C_\alpha)_{\alpha\in A}$ is an FIP family.
Since $\prod_\alpha Y_\alpha$ is compact, we have that $\bigcap_{\alpha\in A}U_\alpha\neq\emptyset$.
Now $\prod_{\alpha\in A}X_\alpha=\bigcap_{\alpha\in A}U_\alpha$, and we have seen that it being nonempty is equivalent as the Axiom of Choice.
(4) $\implies$ (2).
Take $C:\mathcal{P}(X)\setminus\emptyset\to X$ to be $C(S):=$ minimal element of S.
(3) $\implies$ (2)
Let $\{X_\alpha\}_{\alpha\in A}$ to be non-empty sets. Let $X:=\bigcup_{\alpha\in A}X_\alpha,\ P=\{f_B:B\to X|B\subseteq A, f_B(\beta)\in X_\beta,\forall\beta\in B\}$.
Clearly $P\neq\emptyset$.
Define the order by $f_B\leq f_{B'}\iff B\subseteq B', f_{B'}|_B=f_B$.
For any chain $Q\subseteq P$, define $\tilde{B}=\bigcup_{B\in Q}B$, and $f_{\tilde{B}}(\beta)=f_B(\beta)$ for $\beta\in B$ of any B.
We can check $f_{\tilde{B}}\in P$ is an upper bound.

By Zorn's Lemma, there is a maximal element $f_B \in P$.
If $B \subsetneq A$, then we can extend the function to contain another point, and send the point to itself, contradicting maximality.
Thus there is some $f_A \in P$, which we have seen is equivalent to the Axiom of Choice.
$(4) + (2) \implies (3)$.
See Pmath432 A1.
$(3) + (2) \implies (1)$.
For contradiction, suppose $X = \prod_{\alpha \in A} X_\alpha$ is not compact.
Let NFS:= $\{\mathcal{C} \subseteq \mathcal{P}(X) : \mathcal{C}$ is a cover with no finite sub cover$\}$.
Define $\mathcal{C}_1 \leq \mathcal{C}_2 \iff \mathcal{C}_1 \subseteq \mathcal{C}_2$.
Take any chain Q in NFS.
Let $\mathcal{C}' := \bigcup_{\mathcal{C} \in Q} \mathcal{C}$, which we can check is an open cover and an upper bound for the chain.
Indeed, suppose $\mathcal{C}' \notin$ NFS, then $\exists U_1, U_2, \ldots, U_n \in \mathcal{C}'$ such that $X = \bigcup_{i=1}^n U_i$.
For all $i \in [n], U_i \in \mathcal{C}_i$ for some $C_i \in Q$.
Since Q is a chain, there is some $i_0 \in [n]$ such that $\forall i \in [n],\ \mathcal{C}_i \subseteq \mathcal{C}_{i_0}$.
Thus $\mathcal{C}_{i_0} \notin$NFS, a contradiction.
Thus $\mathcal{C} \in$NFS.
By Zorn's Lemma, there is a maximal open cover $\mathcal{C}_{max}$ with no subcover.
Notice that if $U \in \mathcal{C}_{max}$, and $V \subseteq U$ is open, then $V \in \mathcal{C}_{max}$ as well, since any finite subcover of $\{V\} \cup \mathcal{C}_{max}$ can give a finite subcover of $\mathcal{C}_{max}$ by replacing V by U.
Also, if $U_1, U_2 \in \mathcal{C}_{max}$, we must have $U_1 \cup U_2 \in \mathcal{C}_{max}$ as well.
Also, suppose $V_1, \ldots, V_n$ are open in X, such that $\bigcap_{i \in [n]} V_i \in \mathcal{C}_{max}$, then $\exists i_0 \in [n]$, such that $V_{i_0} \in \mathcal{C}_{max}$.
Indeed, suppose not, for any $i \in [n]$, there is a finite cover $V_i \cup \bigcup_{j \in [N_i]} U_{i,j}$ for $U_{i,j} \in \mathcal{C}_{max}$. We must have $\left(\bigcap_{i \in [n]} V_i\right) \cup \bigcup_{i \in [n], j \in [N_i]} U_{i,j}$ is a finite sub-cover of $\mathcal{C}_{max}$.
Now let $W_\alpha := \left\{\text{open } U_\alpha \subseteq X_\alpha \middle| \pi^{-1}(U_\alpha) \in \mathcal{C}_{max}\right\}$.
For contradiction, suppose W_α covers X_α, then there is a finite subcover $\{U_i\}_{i \in [n]}$ such that $X_\alpha = \bigcup_{i \in [n]} U_i$.
Thus $X = \bigcup_{i=1}^n \pi_\alpha^{-1}(U_i)$, which is a subcover of $\mathcal{C}_{max}$.
Thus $X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right) \neq \emptyset$.
By the Axiom of Choice, there is $x_\alpha \in X_\alpha \setminus \left(\bigcup_{U \in W_\alpha} U\right)$ for each α.
Let $x \in X$ be $x(\alpha) = x_\alpha$.
Since $\mathcal{C}_{max}$ is a cover for X, there is some open $U \in \mathcal{C}_{max}$ with $x \in U$.
Thus, there must be some $x \in U_1 \times U_2 \times \cdots \times U_n \times \prod_{\beta \in A \setminus \{\alpha_i : i \in [i]\}} = \bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \subseteq U$ for open $U_i \in X_{\alpha_i}$, since such sets forms a basis.
Thus, $\bigcap_{i \in [n]} \pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{max}$ as well.
Thus, there is some $i_0 \in [n]$, such that $\pi_{\alpha_i}^{-1}(U_i) \in \mathcal{C}_{max}$, which means $U_i \in W_{\alpha_i}$.
However, $x_{\alpha_i} \in U_{\alpha_i}$, which is a contradiction to the choice of $x_{\alpha_i} \notin \left(\bigcup_{U \in W_{\alpha_i}} U\right)$. $\square$

3 Banach Spaces

Definition 3.1. A **normed vector space** is a vector space $(X, ||\cdot||)$ that has an norm (length):

$$
\begin{gathered}
||\cdot|| : X \to \mathbb{R}, \text{ such that } \forall x, y \in X, a \in \mathbb{C} \\
||a \cdot x|| = |a| ||x|| \\
||x + y|| \leq ||x|| + ||y|| \\
||x|| \geq 0 \\
||x|| = 0 \iff x = 0.
\end{gathered}
$$

Proposition 3.1. *For every **normed space** with $||\cdot||$, there is a metric $d(x, y) = ||x - y||$.*

Proof.

$$\begin{aligned}
d(x,x) &= ||x-x|| = ||0|| = 0 \\
\forall x \neq y, d(x,y) &= ||x-y|| > 0 \\
d(x,y) &= ||x-y|| = ||-(y-x)|| = |-1|||y-x|| = ||y-x|| = d(y,x) \\
d(x,z) &= ||x-z|| = ||x-y+y-z|| \geq ||x-y|| + ||y-z|| = d(x,y) + d(y,z)
\end{aligned}$$

Thus $d(x,y) = ||x-y||$ is a metric. ∎

Definition 3.2. A normed space is called a **Banach space** if it is complete.

Proposition 3.2. *The Euclidean space* $\mathbb{R}^n$ *or* $\mathbb{C}^n$*, with the Euclidean norm* $||x||_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ *is a Banach space.*

Definition 3.3. For $\mathbb{R}^n$ or $\mathbb{C}^n$, and $p \in [1,\infty)$, the ℓ_p norm is $||x||_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$. For $p = \infty$, the ℓ_∞ norm is $||x||_\infty = \max_{i\in[n]} |x_i|$.

Proposition 3.3. $\mathbb{R}^n$ *or* $\mathbb{C}^n$*, with any* ℓ_p *norm is a Banach space.*

Remark. Notice that $\forall n \in \mathbb{N}^+,\ ||x||_\infty \leq ||x||_2 \leq ||x||_1 \leq n||x||_\infty$.

Proposition 3.4. *If* X *is compact and Hausdorff, we have* $(C(X), ||\cdot||_\infty)$ *is a Banach Space.*

Proof. The Extreme Value Theorem shows it is a normed space.
The convergence in $||\cdot||_\infty$ is uniform convergence, and the uniform limit of continuous functions is continuous.
Thus, $(C(X), ||\cdot||_\infty)$ is complete. ∎

Proposition 3.5. *For a Locally Compact Hausdorff Space* X*,* $(C_b(X), ||\cdot||_\infty)$ *and* $(C_0(X), ||\cdot||_\infty)$ *are both Banach Spaces.*

Example 3.0.1. $C_0(\mathbb{N}) = \{(x_n)_{n=1}^\infty : \lim_{n\to\infty} x_n = 0\}$ with the discrete topology.

Proposition 3.6. $C^k([0,1])$ *is a Banach Space with* $||f||_{C^k([a,b])} := \sum_{i=1}^k \left|\left|f^{(i)}\right|\right|_\infty$.

Proof. It is easy to check that this is a norm.
Now take any Cauchy sequence $(f_n)_{n=1}^\infty$, then for each $i \in [k]$, we have $(f_n^{(i)})_{n=1}^\infty$ is Cauchy in $C([a,b])$ as well.
Since $[a,b]$ is compact and Haudorff, we have $C([a,b])$ is a Banach Space, so there is a $g_i(x) := \lim_{n\to\infty} f_n^{(i)}(x)$.
In addition, since the convergence is uniform, we have $g_i \in C([0,1])$.
By the Fundamental Theorem of Calculus, we have that for all $i \in [k-1], x \in [0,1]$,

$$f_n^{(i)}(x) = f_n^{(i)}(0) + \int_0^x f_n^{(i+1)}(t)dt.$$

Taking the limit of $n \to \infty$, we have that

$$g_i(x) = g_i(0) + \int_0^x g_{i+1}(t)dt,$$

which means $g_i \in C^1([0,1])$ with $g_i' = g_{i+1}$.
Thus $g_0 \in C^k([0,1])$. ∎

3.1 Bounded linear operators

Definition 3.4. Let X, Y be vector spaces, $T : X \to Y$ is a linear operator if $\forall c \in \mathbb{R}, u, v \in X$,

$$T(u + cv) = Tu + cTv.$$

Definition 3.5. Let X, Y be linear normed spaces, the **operator norm** of a linear operator $T : X \to Y$ is

$$||T|| := \sup_{||u||_X \leq 1} ||Tu||_Y = \sup_{||u||_X = 1} ||Tu||_Y = \sup_{u \neq 0 \in X} \frac{||Tu||_Y}{||x||_X}.$$

Definition 3.6. Let X, Y be normed spaces, a linear operator $T : X \to Y$ is **bounded** if $||A|| < \infty$.

Theorem 3.7. *Let X, Y be two normed linear spaces, let $T : X \to Y$ be linear, then the following are equal:*

1. *T is continuous,*
2. *T is continuous at 0,*
3. *T is bounded,*
4. *T is uniformly continuous.*

Proof. (4) $\implies$ (1) $\implies$ (2) trivially.
(3) $\implies$ (4).
Suppose T is bounded, then

$$\begin{aligned} ||T(x) - T(y)|| &= ||T(x - y)|| \\ &\leq ||T||||x - y||. \end{aligned}$$

Thus, T is $||T||$ Lipschitz and so uniformly continuous.
(2) $\implies$ (3).
Suppose T is continuous at 0, and suppose for contradiction that $||T|| = \infty$.
There must be $(x_n)_{n=1}^{\infty}$ in X, such that $||x_n|| \leq 1,\ ||Tx_n|| \geq n^2$ for each $n \geq 1$.
Notice that $\frac{x_n}{n} \to 0$, but $\left|\left|T(\frac{x_n}{n})\right|\right| = \frac{1}{n}||Tx_n|| \geq n$ for each n.
Thus $\lim_{n\to\infty} T(\frac{x_n}{n}) \neq 0 = T(0)$, which contradicts that T is continuous at 0. ∎

Definition 3.7. Let X, Y be normed spaces, we denote

$$B(X, Y) := \{T : X \to Y | T \text{ is a bounded linear operator}\}.$$

Theorem 3.8. *The set $B(X, Y)$ is a normed linear space with the operator norm.*

Proposition 3.9. *Let X, Y, Z be normed spaces, if $T : X \to Y, S : Y \to Z$ are both linear bounded operators, then so is $S \circ T$, with*

$$||S \circ T|| \leq ||S||||T||.$$

Theorem 3.10. *Let X be a normed space, and Y be a Banach Space, then $B(x, y)$ is a Banach Space.*

Proof. Let $(T_n)_{n=1}^{\infty}$ be a Cauchy sequence in $B(X, Y)$.
For any $x \in X$, we have that $(T_n x)_{n=1}^{\infty}$ is Cauchy in Y. Indeed, $||T_n x - T_m x|| = ||(T_n - T_m)x|| \leq ||T_n - T_m||||x||$.
Since Y is complete, there must be a unique $y = \lim_{n\to\infty} T_n x \in Y$.
Define $Tx := \lim_{n\to\infty} T_n x$ for any $x \in X$.
Notice that T is linear.
Given $\epsilon > 0$, we know there must be some $N \in \mathbb{N}$, such that $\forall m, n \geq N,\ ||T_n - T_m|| < \epsilon$.

Consider any $x \in X$.

$$\begin{aligned}
||Tx|| &\le ||(T - T_N)x|| + ||T_N x|| \\
&= \lim_{m\to\infty} ||(T_m - T_N)x|| + ||T_N x|| \\
&\le \limsup_m ||T_m - T_N||||x|| + ||T_N||||x|| \\
&\le \epsilon||x|| + ||T_N||||x||.
\end{aligned}$$

Thus $||T|| \le \epsilon + ||T_N|| < \infty$.
This shows $||T|| \in B(X, Y)$.
Again, for any $x \in X$, $n \ge N$, we have

$$\begin{aligned}
||(T_n - T)x|| &= \lim_{m\to\infty} ||(T_n - T_m)x|| \\
&\le ||T_n - T_m||||x|| \\
&< \epsilon||x||.
\end{aligned}$$

Thus $||T_n - T|| < \epsilon$ for any $n \ge N$, which shows $\lim_{n\to\infty} T_n = T$ in $B(X, Y)$ with the operator norm. $\square$

Definition 3.8. Let X, Y be normed spaces. We call $T : X \to Y$ a **contraction** if $||T|| \le 1$.

Definition 3.9. Let X, Y be normed spaces. We call $T : X \to Y$ an **isometry** if $\forall x \in X,\ ||Tx|| = ||x||$.

Proposition 3.11. *Let X, Y be normed spaces. If a linear operator $T : X \to Y$ is a surjective isometry, it is an isometric isomorphism.*

Definition 3.10. Let X, Y be normed spaces. We say a linear operator $T : X \to Y$ is **bounded below** if $\exists c > 0,$ such that $\forall x \in X,\ ||Tx|| \ge c||x||$.

Proposition 3.12. *Let Y be a Banach space, S be a dense subset of a normed space X. For any bounded linear operator $E : S \to Y$, we can extend it to $\tilde{E} : X \to Y$, such that $\tilde{E}$ is also bounded and linear, with $\left|\left|\tilde{E}\right|\right| = ||E||$, and $\tilde{E}|_S = E$.*

Proof. Consider any $x \in X$.
Since S is dense in X, We know $\forall m \in \mathbb{N}^+, \exists x_m \in S$, such that $||x - x_m||_X \le \frac{1}{m}$.
Since E is linear on S, we have that

$$\begin{aligned}
||Ex_m - Ex_l||_Y &= ||E(x_m - x_l)||_Y \\
&\le ||E||||x_m - x_l||_X \\
&= ||E||||(x_m - x) + (x - x_l)||_X \\
&\le ||E||||x - x_m||_X + ||E||||x - x_l||_X \\
&\le ||E||\left(\frac{1}{m} + \frac{1}{l}\right).
\end{aligned}$$

Thus given any $\epsilon > 0$, for any $m, l \ge \lceil \frac{2\epsilon}{||E||} \rceil$, we can make $||Ex_m - Ex_l||_Y < \epsilon$. Thus $(Ex_m)_{m=1}^{\infty}$ is a Cauchy sequence in Y.
Since Y is a Banach space, $\exists y^* \in Y$, such that $Ex_m \to y^*$ in Y.
We claim that y^* is independent of choice of the sequence $(x_m)_{m=1}^{\infty}$.
Indeed, consider any other sequence $(v_m)_{m=1}^{\infty} \subseteq C^{\infty}(\bar{x})$, such that $\forall m \in \mathbb{N}^+, ||x - x_m||_X \le \frac{1}{m}$,

$$\begin{aligned}
||y^* - Ev_m||_Y &\le ||y^* - Ex_m||_Y + ||Ex_m - Ev_m||_Y \\
&\le ||y^* - Ex_m||_Y + ||E||||x_m - v_m||_X \\
&\le ||y^* - Ex_m||_Y + ||E||||x_m - x||_X + ||E||||x - v_m||_X.
\end{aligned}$$

Since all three terms on the right go to 0 when $m \to \infty$, we have that $Ev_m \to y^*$ in Y.
Thus we can uniquely define $\tilde{E}x := y^*$. In addition,

$$\begin{aligned}\left|\left|\tilde{E}x\right|\right|_Y &= \left|\left|\lim_{m\to\infty} Ex_m\right|\right|_Y \\ &= \lim_{m\to\infty} ||Ex_m||_Y \\ &\leq \lim_{m\to\infty} ||E||||x_m||_X \\ &= ||E||\left|\left|\lim_{m\to\infty} x_m\right|\right|_X \\ &= ||E||||x||_X.\end{aligned}$$

Thus $\left|\left|\tilde{E}\right|\right| = ||E||$. □

3.1.1 Dual Spaces

Definition 3.11. Let X be a normed space over $\mathbb{F}$, a **functional** is an operator that maps into $\mathbb{F}$.

Definition 3.12. Let X be a normed space over $\mathbb{F}$, the **dual space** of X is the collection of bounded linear functionals on X, denoted

$$X^* := B(X, \mathbb{F}) = \{\phi : X \to \mathbb{F} : \phi \text{ is linear and bounded}\}.$$

Definition 3.13. Let X be a normed space, if $v \in X, u^* \in X^*$, we can write $\langle u^*|v\rangle_{X^*,X} := u^*(v)$ as the action of u^* on v.

Definition 3.14. Let X be a normed space, the **dual norm** is defined to be

$$||u^*||_{X^*} := \sup_{||u||\leq 1} \left|\langle u^*|u\rangle_{X^*,X}\right|.$$

Example 3.1.1. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^\infty | \lim_{i\to\infty} x_i = 0\}$, then $C_0(\mathbb{N})^* \cong \ell_1(\mathbb{N})$.
Indeed, let $e_n := (m \mapsto \delta_{nm}) = (\delta_{nm})_{n=1}^\infty \in C_0(\mathbb{N})$.
Given any $\phi \in C_0(\mathbb{N})^*$, we define $a_n := \phi(e_n) \in \mathbb{F}$.
We claim $a = (a_n)_{n=1}^\infty$ completely determines ϕ.
Indeed, consider any $x \in C_0$, let $x^N := \sum_{n=1}^N x_n e_n$.
We have $\left|\left|x - x^N\right|\right|_{c_0} = 0$, so

$$\begin{aligned}\phi(x) &= \lim_{N\to\infty} \phi(x^N) \\ &= \lim_{N\to\infty} \phi\left(\sum_{n=1}^N x_n e_n\right) \\ &= \lim_{N\to\infty} \sum_{n=1}^N x_n \phi(e_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^N x_n a_n.\end{aligned}$$

Now

$$\begin{aligned}\sum_{n=1}^N |a_n| &= \sum_{n=1}^N a_n \operatorname{sgn}(a_n) \\ &= \sum_{n=1}^N \phi(\operatorname{sgn}(a_n)e_n) \\ &= \phi(y_N) \\ &\leq ||\phi||_{C_0^*} ||y_N||_{C_0},\end{aligned}$$

where $y_N = \sum_{i=1}^{N} \operatorname{sgn}(a_n) e_n$.
Since $||y_N|| = 1$, we have $\sum_{n=1}^{N} |a_n| \leq ||\phi||_{C_0^*}$ for any N.
Thus

$$||a||_1 = \sum_{n=1}^{\infty} |a_n| = \lim_{N \to \infty} \sum_{n=1}^{N} |a_n| \leq ||\phi||_{C_0^*} < \infty.$$

Thus $a \in \ell_1(\mathbb{F})$.
Notice that $\Phi : C_0^*(\mathbb{N}) \to \ell_1(\mathbb{F})$ by $\phi \mapsto a$ is linear, contractive, and injective.
Also, given any $a \in \ell_1(\mathbb{F})$, we can set $\phi_a(x) := \sum_{n=1}^{\infty} x_n a_n$.
Thus,

$$\begin{aligned} |\phi_n(x)| &\leq \sum_{n=1}^{\infty} |x_n||a_n| \\ &\leq ||x||_\infty ||a||_1, \end{aligned}$$

which shows $||\phi_a||_{C_0^*} \leq ||a||_1$.
Thus $\phi_a \in C_0^*(\mathbb{F})$. Notice that $\Phi(\phi_a) = a$.
This shows that Φ is surjective, and it is actually an isometry, since $\forall \phi \in C_0^*(\mathbb{F})$, we have

$$||\Phi(\phi)||_1 \leq ||\phi||_{C_0^*} = \left|\left|\Phi^{-1}(\Phi(\phi))\right|\right|_{C_0^*} \leq ||\Phi(\phi)||_1.$$

Example 3.1.2. Consider $C_0(\mathbb{N}) := \{(x_i)_{i=1}^{\infty} | \lim_{i \to \infty} x_i = 0\}$, then

$$B(C_0(\mathbb{N})) = B(C_0(\mathbb{N}), C_0(\mathbb{N})) \cong \{(t_{ij})_{i,j \in \mathbb{N}} : ||(t_{ij})_{i,j \in \mathbb{N}}|| < \infty,\ \forall j \in \mathbb{N},\ (t_{ij})_{i \in \mathbb{N}}\} \in C_0(\mathbb{N}),$$

which are infinite matrices whose rows are uniformly in ℓ_1, and columns are in $C_0(\mathbb{N})$.
In addition, it is an isometry under

$$||(t_{ij})_{i,j \in \mathbb{N}}|| := \sup_{i \in \mathbb{N}} \left|\left|(t_{ij})_{j=1}^{\infty}\right|\right|_1.$$

Indeed, let

$$e_n := (m \mapsto \delta_{nm}) \simeq (\delta_{nm})_{n=1}^{\infty} \in C_0(\mathbb{N}), \delta_i := ((x_j)_{j=1}^{\infty} \mapsto x_i) \in C_0^*(\mathbb{N})$$

with $\Phi(\delta_i) = (\delta_{ij})_{j=1}^{\infty} \in \ell_1(\mathbb{F})$ as in previous example.
Easy to check $||\delta_i|| = 1$ by the above example.
Consider any $T \in B(C_0(\mathbb{N}))$, define $\phi_i := (\delta_i \circ T), t_{ij} := \phi_i(e_j)$.
Notice that $\phi_i : C_0(\mathbb{N}) \to \mathbb{F}$ is linear, and $||\phi_i|| \leq ||\delta_i||||T|| = ||T|| < \infty$, so $\phi_i \in C_0^*(\mathbb{N})$.
Thus, $(t_{ij})_{j=1}^{\infty} = (\phi_i(e_j))_{j=1}^{\infty} = \Phi(\phi_i) \in \ell_1(\mathbb{F})$ as in previous example, with $||\phi_i|| = \left|\left|(t_{ij})_{j=1}^{\infty}\right|\right|_1$.
Since this hold for all $i \in \mathbb{N}$, we have $\sup_{i \in \mathbb{N}} \left|\left|(t_{ij})_{j=1}^{\infty}\right|\right|_1 \leq ||T||$.
In addition, $(t_{ij})_{i \in \mathbb{N}} = ((\delta_i \circ T)(e_j))_{i \in \mathbb{N}} = Te_j \in C_0(\mathbb{N})$.
On the other hand, suppose we have such $(t_{ij})_{i,j \in \mathbb{N}}$ with $\sup_{i \in \mathbb{N}} \left|\left|(t_{ij})_{j=1}^{\infty}\right|\right|_1 < \infty$, we can define $\phi_i := \Phi^{-1}((t_{ij})_{j=1}^{\infty}) \in C_0^*(\mathbb{N})$ with $\phi_i(x) = \sum_{j=1}^{\infty} x_j t_{ij}$ as in previous example.
Let $Tx := \sum_{i=1}^{\infty} \phi_i(x) e_i$ for any $x \in C_0(\mathbb{N})$.
Clearly, T is linear, and we have $(\delta_i \circ T)(x) = \delta_i(\sum_{j=1}^{\infty} \phi_j(x) e_j) = \phi_i(x)$.

$$\begin{aligned} ||Tx||_\infty &= ||(\phi_i(x))_{i=1}^{\infty}||_\infty \\ &\leq \sup_{i \in \mathbb{N}} |\phi_i(x)| \\ &\leq \sup_{i \in \mathbb{N}} ||\phi_i||||x|| \\ ||T|| &\leq \sup_{i \in \mathbb{N}} ||\phi_i|| \\ &= \sup_{i \in \mathbb{N}} \left|\left|(t_{ij})_{j=1}^{\infty}\right|\right| \\ &< \infty. \end{aligned}$$

Thus $T \in B(C_0(\mathbb{N}), \ell_\infty)$.
Now we claim $T(C_0(\mathbb{N})) = C_0(\mathbb{N})$, which will mean $T \in B(C_0(\mathbb{N}))$.
Indeed, for any $x = \sum_{n=1}^{\infty} x_n e_n \in C_0(\mathbb{N})$, we have

$$\begin{aligned} Tx &= T\left(\lim_{N\to\infty} \sum_{n=1}^{N} x_n e_n\right) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} x_n T(e_n). \end{aligned}$$

Since each

$$\begin{aligned} T(e_n) &= \sum_{i=1}^{\infty} \phi_i(x) e_i \\ &= \sum_{i=1}^{\infty} t_{in} e_i \\ &\in C_0(\mathbb{N}), \end{aligned}$$

and $C_0(\mathbb{N})$ is closed, we have $Tx \in C_0(\mathbb{N})$.
In addition, $(\delta_i \circ T)(e_j) = \phi_i(e_j) = \Phi^{-1}((t_{ik})_{k=1}^{\infty})(e_j) = t_{ij}$.
Thus $T \longleftrightarrow (t_{ij})_{i,j\in\mathbb{N}}$ is an isometric bijection.

Example 3.1.3. Consider the **Disk Algebra**

$$A(\mathbb{D}) := \left\{ f \in C(\mathbb{T}) : \forall n \in \mathbb{Z}^{--}, \hat{f}(n) = 0 \right\},$$

where $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C}, |z| = 1\}$ is the unit circle, and

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{it} e^{-int}\right) dt$$

is the n^{th} Fourier Transform of f.
Consider $\phi_n : f \mapsto \hat{f}(n)$, which is clearly in $C^*(\mathbb{T})$.
We notice that $A(\mathbb{D}) = \bigcap_{n<0} \ker(\phi_n)$ is closed in $C(\mathbb{T})$, since each kernel of a continuous functional is closed.
In fact, for $f, g \in C(\mathbb{T})$, we have $\hat{fg}(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\hat{g}(n-k)$.
Thus, for $f, g \in A(\mathbb{D})$, we have $\hat{fg}(n) = \sum_{k\in\mathbb{N}} \hat{f}(k)\hat{g}(n-k) = 0$ for $n < 0$.
This shows $A(\mathbb{D})$ is actually an Algebra.
Also, $A(\mathbb{D})$ is exactly the set of $f \in C(\mathbb{T})$ that admits an extension $F \in C\left(\bar{\mathbb{D}}\right)$ with $F|_{\mathbb{D}}$ being analytic, with $F(z) := \sum_{n=1}^{\infty} \hat{f}(n) z^n$.

Definition 3.15. A Banach space X is **reflexive** if $(X^*)^* \simeq X$. Namely, $\forall u^{**} \in (X^*)^*, \exists! u \in X$ such that

$$\forall v^* \in X^*, \langle u^{**} | v^* \rangle_{(X^*)^*, X^*} = \langle v^* | u \rangle_{X^*, X}.$$

Theorem 3.13. *(Riesz-Frechet Representation theorem)*
Let $\mathcal{H}$ be a Hilbert space, then for each $u^* \in \mathcal{H}, \exists! u \in \mathcal{H}$, such that $\forall v \in \mathcal{H}, \langle u^* | v \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$, and $||u^*||_{\mathcal{H}^*} = ||u||_{\mathcal{H}}$.

Corollary 3.14. *Every Hilbert space is reflexive.*

Corollary 3.15. *Let $\mathcal{H}$ be a Hilbert space, then $\mathcal{H} \cong^* \mathcal{H}$, where the map $\Phi : \mathcal{H} \to \mathcal{H}^*;\ u \mapsto \langle u, \cdot \rangle_{\mathcal{H}}$ is the **canonical bijective isometric antilinear isomorphism**.*

Remark. We thus abuse the notation, and denote canonical bijective isometric antilinear isomorphism by $u^\dagger := \Phi(u)\ \forall u \in \mathcal{H}$, and $(u^*)^\dagger := \Phi^{-1}(u^*)\ \forall u^* \in \mathcal{H}^*$. Notice that by definition

$$(u^\dagger)^\dagger = u, ((u^*)^\dagger)^\dagger = u^*\ \forall u \in \mathcal{H}, u^* \in \mathcal{H}^*.$$

We might further abuse the notation, and write

$$\langle u|v\rangle := \langle u, v\rangle = \langle u^\dagger|v\rangle =: \langle u^\dagger, v\rangle$$

interchangeably instead of $\langle u^\dagger|v\rangle_{\mathcal{H}^*,\mathcal{H}}$ or $\langle u, v\rangle_{\mathcal{H}}$ when the context is clear.

Definition 3.16. Let X be a Banach Space, we say $(u_k)_{k=1}^\infty \subset X$ converges to $u \in X$ weakly, denoted $u_k \rightharpoonup u$, if

$$\forall v^* \in X^*, \langle v^*|u_k\rangle \to \langle v^*|u\rangle$$

as real numbers.

Proposition 3.16. *Let X be a Banach Space, $(u_k)_{k=1}^\infty \subset X$ be a sequence, then*

1. *If $u_k \to u$, we always have $u_k \rightharpoonup u$.*
2. *If $u_k \rightharpoonup u$, we have that u is unique.*
3. *If $u_k \rightharpoonup u$, we have $(u_k)_{k=1}^\infty$ is bounded.*
4. *If $u_k \rightharpoonup u$, every subsequence $(u_{k_j})_{j=1}^\infty$ also converges weakly to u.*

Proof. See A5Q1 for 1. □

Theorem 3.17 (Weakly compact for reflexive Banach Space)**.** *Let X be a reflexive Banach Space, and $(u_k)_{k=1}^\infty \subset X$ be a bounded sequence, then $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.*

Proposition 3.18. *Let $\mathcal{H}$ be a Hilbert space, then $u_k \rightharpoonup u$ if and only if $\forall v \in \mathcal{H}, \langle v, u_k\rangle \to \langle v, u\rangle$ as real numbers.*

Proof. Suppose $u_k \rightharpoonup u$.
Notice that for all $v \in \mathcal{H}$, we have that $v^\dagger \in \mathcal{H}^*$, and thus $\langle v, u_k\rangle = \langle v^\dagger|u_k\rangle \to \langle v^\dagger|u\rangle = \langle v, u\rangle$.
Now suppose $\forall v \in \mathcal{H}, \langle v, u_k\rangle \to \langle v, u\rangle$.
Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 3.13, there is some $f^\dagger \in \mathcal{H}$, such that

$$\langle f|u_{k_j}\rangle = \langle f^\dagger, u_{k_j}\rangle \to \langle f^\dagger, u\rangle = \langle f|u\rangle.$$

Thus, $u_{k_j} \rightharpoonup u$. □

Proposition 3.19. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Tu_k \rightharpoonup \mathcal{H}_2$.*

Proof. Let $y_k := Tu_k, y := Tu \in \mathcal{H}_2$.
Consider any $g \in \mathcal{H}_2^*$, we define $f := g \circ K \in \mathcal{H}_1^*$.
Since $u_k \rightharpoonup u$, we must have

$$\lim_{k\to\infty} f(u_k) = f(u)$$
$$\lim_{k\to\infty} g(Ku_k) = g(Ku)$$
$$\lim_{k\to\infty} g(y_k) = g(y).$$

We thus have $y_k \rightharpoonup y$. □

Proposition 3.20. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $K : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator, and $(u_k)_{k=1}^\infty \subset \mathcal{H}_1$ be a sequence. If $u_k \rightharpoonup u \in \mathcal{H}_1$, then $Ku_k \to \mathcal{H}_2$.*

Proof. Let $y_k := Ku_k, y := Ku \in \mathcal{H}_2$.
Since K is compact, it is bounded, so $y_k \rightharpoonup y$.
Now suppose for contradiction $\lim_{k\to\infty} ||y_k - y|| \neq 0$.
Then there is some $\epsilon > 0$ and a subsequence $(u_{k_j})_{j=1}^\infty$ such that $\forall j \geq 1, ||y_{k_j} - y|| \geq \epsilon$.
Since $u_k \rightharpoonup u \in \mathcal{H}$, we have $(u_k)_{k=1}^\infty$ is bounded, and thus $(u_{k_j})_{j=1}^\infty$ is bounded.
Since K is compact, there is some further subsequence $(u_{k_{j_m}})_{m=1}^\infty$ such that $\lim_{m\to\infty} Ku_{k_{j_m}} = \tilde{y} \in \mathcal{H}_2$.
Thus $Ku_{k_{j_m}} \rightharpoonup \tilde{y}$. Since weak convergence, we must have $\tilde{y} = y$.
Thus $\lim_{m\to\infty} Ku_{k_{j_m}} = y$, which is a contradiction. □

3.2 Quotient Spaces

Definition 3.17. Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. The **quotient space** is $X/Y := \{x+Y : x \in X\}$, with the **quotient map** $Q : X \to X/Y$ by $Q(x) := [x] := x+Y = \{x+y : y \in Y\}$.

Proposition 3.21. *Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. X/Y is always a vector space with $[0] = Y, [x]+[y] = [x+y], c[x] = [cx]$.*

Proposition 3.22. *Let X be a Banach Space, and $Y \subseteq X$ be a closed subspace. $\left(X/Y, ||\cdot||_{X/Y}\right)$ is always a Banach space with $||[x]||_{X/Y} := \inf_{y\in Y} ||x+y||_X$. In addition, Q is isometric if $Y \subsetneq X$.*

Proof. $||[x]||_{X/Y} = 0 \iff \inf_{y\in Y} ||x+y||_X = 0 \iff x \in \bar{Y} \iff x \in Y$.
Scaling is clear.
Also,

$$\begin{aligned} ||[x]+[z]||_{X/Y} &= \inf_{y\in Y} ||x+y+z||_X \\ &= \inf_{y_1,y_2\in Y} ||x+y_1+z+y_2||_X \\ &\le \inf_{y_1\in Y} ||x+y_1||_X + \inf_{y_2\in Y} ||z+y_2||_X \\ &= ||[x]||_{X/Y} + ||[z]||_{X/Y}. \end{aligned}$$

This shows $\left(X/Y, ||||_{X/Y}\right)$ is a normed space.
We note that $||Qx||_{X/Y} = \inf_{y\in Y} ||x+y||_X \le ||x+0||_X = ||x||_X$, so $||Q|| \le 1$.
Now consider any Cauchy sequence $([x_n])_{n=1}^{\infty}$ in X/Y.
We can pick a subsequence $([x_{n_i}])_{i=1}^{\infty}$ such that $\left|\left|[x_{n_{i+1}}] - [x_{n_i}]\right|\right|_{X/Y} < 2^{-i}$.
Pick $z_1 \in X$ such that $[z_1] = [x_{n_i}]$.
Since $||[x_{n_2}] - [x_1]||_{X/Y} = \inf_{y\in Y} ||x_{n_2} - z_1 + y||_X < \frac{1}{2}$, there is $y \in Y$, such that $||x_{n_2} - z_1 + y||_X < \frac{1}{2}$.
Take $z_2 = x_{n_2} + y$, we have $||z_2 - z_1||_X < \frac{1}{2}$.
Inductively, we can pick $(z_i)_{i=1}^{\infty}$, such that $||z_i - z_i||_X < 2^{-i}$.
We can check that this is a Cauchy sequence in X, so it has a limit $z = \lim_{i\to\infty} z_i \in X$.
Now for any $i \in \mathbb{N}^+$, we have

$$\begin{aligned} ||[x_{n_i}] - [z]||_{X/Y} &= ||[z_i] - [z]||_{X/Y} \\ &= ||Q(z_i) - Q(z)||_{X/Y} \\ &= ||Q(z_i - z)||_{X/Y} \\ &\le ||Q||||z_i - z|| \\ &\to 0. \end{aligned}$$

Thus $([x_{n_i}])_{i=1}^{\infty} \to [z]$ is a convergent subsequence, which mean $([x_n])_{n=1}^{\infty}$ is convergent.
This shows $\left(X/Y, ||\cdot||_{X/Y}\right)$ is a Banach space.
Now if $Y \subsetneq X$, then $X/Y \neq \{0\}$, there must be some $[x] \in X/Y$ with $||[x]||_{X/Y} = 1$.
Thus, for all $k \in \mathbb{N}^+$, there is some $y_k \in Y$, such that $||x+y_k||_X \le 1 + \frac{1}{k}$.
Now

$$\left|\left|Q\left(\frac{x+y_k}{||x+y_k||_X}\right)\right|\right|_{X/Y} = \frac{1}{||x+y_k||_X}||Q(x+y_k)|| = \frac{||[x]||_{X/Y}}{||x+y_k||_X} \ge \frac{1}{1+\frac{1}{k}}.$$

Since this is true for any $k \in \mathbb{N}^+$, taking the limit $k \to \infty$, we have $\left|\left|Q\left(\frac{x+y_k}{||x+y_k||_X}\right)\right|\right|_{X/Y} \ge 1$.
Yet $\left|\left|\frac{x+y_k}{||x+y_k||_X}\right|\right|_X = 1$, so $||Q|| \ge 1$.
This shows $||Q|| = 1$. ∎

Example 3.2.1. Consider a compact and Hausdorff X, and consider $(C(X), ||\cdot||_\infty)$. Let $E \subseteq X$ be closed, and $I(E) := \{f \in C(X) : f|_E = 0\}$.
One can check $I(E)$ is closed (ideal), and $C(X)/I(E) \cong C(E)$ with an isometric isomorphism $\tilde{R} : [f] \mapsto f|_E$.
We claim that $\tilde{R}$ is well-defined.
Indeed, if $[f] = [g]$, we must have $f - g \in I(E)$, which means $(f-g)|_E = 0$.
Thus $\tilde{R}([f]) = f|_E = g|_E = \tilde{R}([g])$.
Clearly $\tilde{R}$ is linear.
Also, $\tilde{R}([f]) = 0 \implies f|_E = 0 \implies f \in I(E) \implies [f] = 0$, so $\tilde{R}$ is injective.
By Tietze's Theorem, given any $g \in C(E)$, we can extend it to $f \in C(X)$, such that $f|_E = g$. Thus, $\tilde{R}$ is surjective.
Consider any $f \in C(X), g \in I(E)$, we have

$$\begin{aligned}\left|\left|\tilde{R}([f])\right|\right| &= ||f|_E||_{C(E)} \\ &= \sup_{x\in E} |f(x)| \\ &= \sup_{x\in E} |(f+g)(x)| \\ &\leq \sup_{x\in X} |(f+g)(x)| \\ &= ||f+g||_{C(X)}.\end{aligned}$$

Since this hold for all $g \in I(E)$, we have

$$\left|\left|\tilde{R}([f])\right|\right| \leq \inf_{g\in I(E)} ||f+g||_{C(X)} = ||[f]||.$$

Thus, $\tilde{R}$ is a contraction.
Consider any $f \in C(X)$.
If $f|_E = 0$, we have $||[f]|| = ||[0]|| = 0 = ||f|_E|| = \left|\left|\tilde{R}([f])\right|\right|$.
Now consider $f|_E \neq 0$.
Define the function $k : \mathbb{C} \to \mathbb{C}$ by $k(z) := \begin{cases} z, & |z| \leq ||f|_E||_\infty \\ \frac{z}{|z|}||f|_E||_\infty & |z| \geq ||f|_E||_\infty, \end{cases}$ which is well-defined and continuous.
Let $g := k \circ f \in C(X)$.
For any $x \in E$, we have $|f(x)| \leq ||f|_E||_\infty$, so $g(x) = k(f(x)) = f(x)$.
Thus $g|_E = f|_E$, and there is $h \in I(E)$, such that $g = f + h$.

$$\begin{aligned}||[f]|| &= \inf_{h\in I(E)} ||f+h||_{C(X)} \\ &\leq ||g||_{C(X)} \\ &\leq ||k|| \\ &\leq ||f|_E||_\infty \\ &= \left|\left|\tilde{R}(f)\right|\right|.\end{aligned}$$

This proves $\tilde{R}$ is an isometry.

3.3 Baire Category Theorem

Definition 3.18. Let $(X, \mathcal{T})$ be a topological space, then $A \subseteq X$ is called **nowhere dense** if $(\bar{A})^o = \emptyset$.

Theorem 3.23 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space, then X cannot be written as a countable union of nowhere dense sets.*

Corollary 3.24. *Let $\{U_i\}_{i=1}^\infty$ be a countable set of open dense sets, then $\bigcap_{i=1}^\infty U_i$ is dense.*

Definition 3.19. Let X, Y be Banach spaces, $\mathcal{S} \subseteq B(X,Y)$ is called **pointwise bounded** if $\forall x \in X$, $\mathcal{S}x$ is bounded. Namely, $\exists k_x > 0,$ such that $\forall s \in \mathcal{S},\ ||sx|| \leq k_x$.

Theorem 3.25 (Banach-Steinhaus). *Let X, Y be Banach spaces, suppose $\mathcal{S} \subseteq B(X,Y)$ is pointwise bounded, then $\mathcal{S}$ is bounded in $B(X,Y)$. Namely,* $\sup_{s\in\mathcal{S}} ||s|| < \infty$.

Proof. For each x, let $k_x > 0$ be such $\forall s \in \mathcal{S},\ ||sx|| \leq k_x$.
For each $n \in \mathbb{N}$, let $A_n := \{x \in X : k_x \leq n\}$.
For any $(x_k)_{k\in\mathbb{N}}$ in A_n, with $x = \lim_{k\to\infty} x_k$, we have

$$||sx|| = \lim_{k\to\infty} ||sx_k|| \leq n.$$

Thus, $x \in A_n$, which shows A_n is closed.
Notice that $X = \bigcup_{n\in\mathbb{N}} A_n$, so by Baire Category Theorem, there is $n_0 \in \mathbb{N}$, such that $(\bar{A_{n_0}})^o \neq \emptyset$.
Thus, there is $x_0 \in A_{n_0}, r > 0$, such that $\bar{B}(x_0, r) \subset B(x_0, 2r) \subseteq (\bar{A_{n_0}})^o \subset \bar{A_{n_0}} = A_{n_0}$.
Now for any $s \in \mathcal{S}, y \in X$ such that $||y|| \leq 1$, we have that

$$\begin{aligned} ||sy|| &= \left|\left|\frac{s(x_0) - s(x_0 - ry)}{r}\right|\right| \\ &\leq \frac{||s(x_0)|| + ||s(x_0 - ry)||}{r} \\ &\leq \frac{2n_0}{r}, \end{aligned}$$

since $x_0, x_0 - ry \in \bar{B}(x_0, r) \subset A_{n_0}$.
Thus, $||s|| = \sup_{y\in X \text{ such that } ||y||\leq 1} ||sy|| \leq \frac{2n_0}{r}$.
Since this holds for all $s \in \mathcal{S}$, we have $\sup_{s\in\mathcal{S}} ||s|| = \frac{2n_0}{r} < \infty$. □

Corollary 3.26 (Limit of bounded operators). *Let X, Y be Banach spaces, consider $(T_n)_{n=1}^\infty$ be a sequence of $B(X,Y)$. Suppose $\forall x \in X,\ (T_n x)_{n=1}^\infty$ is convergent, then $\mathcal{S} := \{T_n : n \in \mathbb{N}\}$ is bounded. In addition, for $Tx := \lim_{n\to\infty} T_n x$, we have $T \in B(X,Y)$, and $||T|| \leq \sup_{n\in\mathbb{N}} ||T_n||$.*

Proof. Since $(T_n x)_{n=1}^\infty$ is convergent, it is bounded. This is equivalent to saying $\mathcal{S}$ is pointwise bounded. By the Banach-Steinhaus Theorem, $\mathcal{S}$ is bounded.
Now $||Tx|| = \lim_{n\to\infty} ||T_n x|| = \sup_{n\in\mathbb{N}} ||T_n x|| \leq \sup_{n\in\mathbb{N}} ||T_n|| ||x||$.
Since this holds for all $x \in X$, we have $||T|| \leq \sup_{n\in\mathbb{N}} ||T_n||$. □

Example 3.3.1. Consider $f \in C(\mathbb{T})$, we can define the N^{th} partial sum of its Fourier series $S_N(f)(e^{it}) := \sum_{n=-N}^{N} \hat{f}(n) e^{int}$.
We note that $S_N(f)$ does not necessarily converge to f in $C(\mathbb{T})$, nor pointwise.
Indeed, consider $\phi_N(f) := S_N(f)(1) \in \mathbb{C}$. Note that ϕ_N is linear.
One can show that

$$\begin{aligned} \phi_N(f) &= \sum_{n=-N}^{N} \hat{f}(n) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \left(\sum_{n=-N}^{N} e^{-int} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) D_N(t) dt, \end{aligned}$$

where $D_N(t) = \frac{\sin\left(\left(N+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}$ is the **Dirichlet's Kernel**.
Actually, $\exists C > 0$, such that $||D_N||_1 \geq C \log(N)$.
Also, $||\phi_N||_{C_*(\mathbb{T})} = ||D_N||_1$.
Thus, $(\phi_N)_{N\in\mathbb{N}}$ is not bounded.

By Banach-Steinhaus, $(\phi_N)_{N\in\mathbb{N}}$ is not pointwise bounded.
Thus, there is $f \in C(\mathbb{T})$, such that $|\phi_N(f)| \to \infty$.
Thus, $S_N f$ does not converge to f at 1.

3.3.1 Open Mapping Theorem

Theorem 3.27 (Open Mapping Theorem)**.** *Let X, Y be Banach spaces, suppose $T \in B(X, Y)$ is surjective, then it is **open**. i.e. $\forall$ open $U \subseteq X,\ T(U) \subseteq Y$ is open.*

Proof. We have

$$\begin{aligned} Y &= T(X) \\ &= T\left(\bigcup_{n=1}^{\infty} B^X(0,n)\right) \\ &= T\left(\bigcup_{n=1}^{\infty} nB^X(0,1)\right) \\ &= \bigcup_{n=1}^{\infty} nT(B^X(0,1)). \end{aligned}$$

By the Baire Category Theorem, there is n_0, such that $\left(n_0\overline{T(B^X(0,1))}\right)^o = \left(\overline{n_0 T(B^X(0,1))}\right)^o \neq \emptyset$.
Thus there is $r > 0, y_0 \in Y$, such that $B^Y(y_0, r) \subset n_0\overline{T(B^X(0,1))}$.
Notice that $B^Y(-y_0, r) \subset n_0\overline{T(B^X(0,1))}$ as well, and $n_0\overline{T(B^X(0,1))}$ is convex.
Thus, for any $y \in B^Y(0,r)$, we have $y = \frac{1}{2}(y-y_0)+\frac{1}{2}(y+y_0)$, where $y-y_0 \in B^Y(-y_0, r),\ y+y_0 \in B^Y(y_0, r)$.
By convexity, $y \in n_0\overline{T(B^X(0,1))}$.
Thus $B^Y(0,r) \subset n_0\overline{T(B^X(0,1))}$. ∎

Corollary 3.28 (Bounded inverse Theorem)**.** *Let X, Y be Banach spaces, suppose $A \in B(X, Y)$ is bijective, then A^{-1} is continuous and bounded as well.*

Proof. A is open by the open mapping theorem.
Now for any open $U \subseteq X$, we have that $\left(A^{-1}\right)^{-1}(U) = A(U)$ is open.
Thus A^{-1} is continuous. ∎

Corollary 3.29. *Let X, Y be Banach spaces, suppose $T \in B(X, Y)$ is surjective, then $X/\operatorname{Ker}(T) \cong Y$ as Banach spaces with $\tilde{A} : X/\operatorname{Ker}(T) \to Y$ by $\tilde{T}([x]) := T(x)$.*

Proof. We can check that $\tilde{T}$ is a well-defined bijection.
Also, for any $x \in X, y \in \operatorname{Ker}(T)$, we have

$$\begin{aligned} \left|\left|\tilde{T}([x])\right|\right| &= ||T(x)|| \\ &= ||T(x+y)|| \\ &\leq ||T||||x+y||. \end{aligned}$$

Since this holds for all $y \in \operatorname{Ker}(T)$, we have

$$\left|\left|\tilde{T}([x])\right|\right| \leq \inf_{y\in\operatorname{Ker}(T)} ||T||||x+y|| = ||T||||[x]||.$$

Thus, $\left|\left|\tilde{T}\right|\right| \leq ||T||$, which means $\tilde{T} \in B(X/\operatorname{Ker}(T), Y)$ is continuous.
By the bounded inverse theorem, $\tilde{T}^{-1}$ is continuous as well. ∎

Corollary 3.30. *If X is a vector space that is complete under two different norms $||\cdot||_1, ||\cdot||_2$, and $\exists C > 0,$ such that $\forall x \in X$,*

3.4 Compact Operators

Definition 3.20. Let X, Y be metric spaces, a linear operator $T : X \to Y$ is **compact** if for each bounded subset $S \subseteq X$, we have its image $T(S)$ is pre-compact in Y.

Proposition 3.31. *Let X, Y be metric spaces, a linear operator $T : X \to Y$ is compact if and only if T is bounded, and each bounded sequence $(x_n)_{n=1}^{\infty} \subseteq X$ has some subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $(Tx_{n_k})_{k=1}^{\infty}$ converges to some $y \in Y$.*

Definition 3.21. Let X, Y be Banach spaces and $X \subseteq Y$, then we say X is **compactly embedded** in Y, denoted

$$X \subset\subset Y$$

if the inclusion map $i : X \hookrightarrow Y;\ x \mapsto x$ is compact.
Namely, $\exists C > 0,$ such that $\forall x \in X, ||x||_Y \leq C||x||_X$, and each bounded sequence $(x_n)_{n=1}^{\infty} \subseteq X$ having some subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to some $y \in Y$.

Proposition 3.32. *Let X, Y, Z be Banach spaces and $X \subset\subset Y$, if an operator $T : Z \to X$ is bounded, then $\tilde{T} := i \circ T : Z \to Y$ is compact.*

Proof. Consider any bounded set $S \subseteq Z$, such that $\forall z \in S, ||z||_Z \leq M$.
We have $||Tz||_X \leq ||T||||z||_M \leq M||T|| < \infty$, and thus $T(S)$ is bounded in X.
Yet i is compact, and thus $i(T(S))$ is pre-compact.
This shows $\tilde{T}(S) = (i \circ T)(S)$ is pre-compact for any bounded set $S \subseteq Z$.
Thus $\tilde{T}$ is compact. □

Theorem 3.33 (Spectral theorem for compact operators)**.** *Let $K : \mathcal{H} \to \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space $\mathcal{H}$, then*

1. $0 \in \mathrm{Spec}(K)$.
2. $\mathrm{Spec}(K) \setminus \{0\} = \mathrm{Spec}_p(K) \setminus \{0\}$.
3. $\mathrm{Spec}(K) \setminus \{0\}$ *is finite, or* $\mathrm{Spec}(K) \setminus \{0\} = (\lambda_k)_{k=1}^{\infty}$ *such that* $\lim_{k\to\infty} \lambda_k = 0$.

3.4.1 Adjoint Operator

Definition 3.22. Let X, Y be normed spaces, the **dual operator** of a linear operator $T : X \to Y$ is

$$T^* : Y^* \to X^*;\ f \mapsto (f \circ T).$$

Proposition 3.34. *Let X, Y, Z be normed spaces, $S \in B(X, Y), T \in B(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.*

Proof. Consider any $f \in Z^*$, and any $x \in X$, we have

$$\begin{aligned}(T^* \circ S^*)(f)(x) &= (S^*)(f)(Tx) \\ &= (f)(S(T(x))) \\ &= (f \circ (S \circ T))(x) \\ &= (S \circ T)^*(f)(x).\end{aligned}$$

Thus $(T^* \circ S^*)(f) = (S \circ T)^*(f)$. □

Definition 3.23. Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, the **Hilbert adjoint operator** of T is $T^\dagger : \mathcal{H} \to \mathcal{H}$ such that $\langle x, Ty\rangle = \langle T^\dagger x, y\rangle \forall x, y \in \mathcal{H}$.

Theorem 3.35. *Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, $T^\dagger$ always exists, and is given by $T^\dagger = \Phi^{-1} \circ T^* \circ \Phi$, where $\Phi : \mathcal{H} \to \mathcal{H}^*;\ u \mapsto \langle u, \cdot\rangle_{\mathcal{H}}$ is the canonical bijective isometric antilinear isomorphism, and T^* is the dual operator of T. In addition, $T^\dagger$ is also a bounded linear operator, with $||T^\dagger|| = ||T||$, and $(T^\dagger)^\dagger = T$.*

Proof. $\forall y \in \mathcal{H}$, we have that

$$\begin{aligned}
\langle T^\dagger x, y \rangle &= \langle (\Phi^{-1} \circ T^* \circ \Phi)(x), y \rangle \\
&= ((T^* \circ \Phi)(x))(y) \\
&= (\Phi(x))(Ty) \\
&= \langle x, Ty \rangle.
\end{aligned}$$

Now consider any $x, y, z \in \mathcal{H}, c \in \mathbb{C}$, we have that

$$\begin{aligned}
\langle T^\dagger(x + cz), y \rangle &= \langle x + cz, Ty \rangle \\
&= \langle x, Ty \rangle + \bar{c}\langle z, Ty \rangle \\
&= \langle T^\dagger x, y \rangle + \bar{c}\langle T^\dagger z, y \rangle \\
&= \langle T^\dagger x + cT^\dagger z, y \rangle.
\end{aligned}$$

Since this holds for any $y \in \mathcal{H}$, we have that $T^\dagger(x + cz) = T^\dagger x + cT^\dagger z$, and thus $T^\dagger$ is linear.
Now given any $x \in \mathcal{H}$, we have that

$$\begin{aligned}
||T^\dagger x||^2 &= \langle T^\dagger x, T^\dagger x \rangle \\
&= \langle x, TT^\dagger x \rangle \\
&\leq ||x|| ||TT^\dagger x|| \\
&\leq ||x|| ||T|| ||T^\dagger x|| \\
&\implies \\
||T^\dagger x|| &\leq ||x|| ||T|| \\
&\implies \\
||T^\dagger|| &= \sup_{x \neq 0 \in \mathcal{H}} \frac{||T^\dagger x||}{||x||} \\
&\leq \sup_{x \neq 0 \in \mathcal{H}} \frac{||x|| ||T||}{||x||} \\
&= ||T||.
\end{aligned}$$

Thus $T^\dagger$ is also a bounded linear operator.
Now $\forall x, y \in \mathcal{H}$, $\langle x, T^\dagger y \rangle = \overline{\langle T^\dagger y, x \rangle} = \overline{\langle y, Tx \rangle} = \langle Tx, y \rangle$.
Thus $(T^\dagger)^\dagger = T$. ☐

Remark. $\forall x, y \in \mathcal{H}$, $\langle (Tx)^\dagger | y \rangle = \langle Tx, y \rangle = \langle x, T^\dagger y \rangle = \langle x^\dagger | T^\dagger y \rangle$. We thus abuse the notation, and write $(Tx)^\dagger = \langle x | T^\dagger$

Definition 3.24. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is **delf-adjoint** if $T^\dagger = T$.

Theorem 3.36. *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then $K^\dagger$ is also compact.*

Proof. $K^\dagger$ is bounded by 3.31.
Let $(u_k)_{k=1}^\infty$ be any bounded sequence in $\mathcal{H}$.
By 3.17, we have that $\exists (u_{k_j})_{j=1}^\infty$ a subsequence, and $u \in X$, such that $u_{k_j} \rightharpoonup u$.
Notice that for any $f \in \mathcal{H}^*$, by Riesz-Frechet Representation theorem 3.13, there is some $f^\dagger \in \mathcal{H}$, such that

$$\begin{aligned}
\langle f | K^\dagger(u_{k_j} - u) \rangle &= \langle f^\dagger, K^\dagger(u_{k_j} - u) \rangle \\
&= \langle Kf^\dagger, u_{k_j} - u \rangle \\
&= \langle Kf^\dagger, u \rangle - \langle Kf^\dagger, u \rangle \qquad \to 0,
\end{aligned}$$

since $u_{k_j} \rightharpoonup u$ and by 3.18.
Since $\left\langle f \middle| K^\dagger(u_{k_j} - u)\right\rangle \to 0 = \langle f|0\rangle$ for any $f \in \mathcal{H}^*$, we have that $K^\dagger(u_{k_j} - u) \rightharpoonup 0$.
By 3.20, we have that $KK^\dagger(u - u_{k_j}) \to 0$.

$$\begin{aligned}
\left|\left|K^\dagger u - K^\dagger u_{k_j}\right|\right|^2 &= \left\langle K^\dagger u - K^\dagger u_{k_j}, K^\dagger u - K^\dagger u_{k_j}\right\rangle \\
&= \left\langle K^\dagger(u - u_{k_j}), K^\dagger(u - u_{k_j})\right\rangle \\
&= \left\langle KK^\dagger(u - u_{k_j}), u - u_{k_j}\right\rangle \\
&\le \left|\left|KK^\dagger(u - u_{k_j})\right|\right| \left|\left|u - u_{k_j}\right|\right| \\
&\to 0.
\end{aligned}$$

Thus $K^\dagger u_{k_j} \to K^\dagger u \in \mathcal{H}$,
Since $(u_k)_{k=1}^\infty$ is any bounded sequence, we have that $K^\dagger$ is compact by 3.31. □

Theorem 3.37. *(Fredholm's alternative)*
Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then

1. $\mathrm{Ker}(I - K)$ *is finite dimensional.*
2. $\mathrm{Im}(I - K)$ *is closed.*
3. $\mathrm{Im}(I - K) = \mathrm{Ker}(I - K^\dagger)^\perp$.
4. $\dim(\mathrm{Ker}(I - K)) = \dim(\mathrm{Ker}(I - K^\dagger))$.
5. $\mathrm{Ker}(I - K) = \{0\} \iff \mathrm{Im}(I - K) = \mathcal{H}$.

Corollary 3.38. *Let $\mathcal{H}$ be a Hilbert space, and $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator, then exactly one of the following holds:*

1. $\forall v \in \mathcal{H},\ \exists! u \in \mathcal{H},$ *such that* $(I - K)u = v$.
2. $\exists u \neq 0 \in \mathcal{H},$ *such that* $(I - K)u = 0$.

Proof. When $\mathrm{Ker}(I - K) = \{0\}$, we have that $I - K$ is injective, and $\mathrm{Im}(I - K) = \mathcal{H}$.
Thus $\forall v \in \mathcal{H},\ \exists! u \in \mathcal{H},$ such that $(I - K)u = v$.
On the other hand, if 1. is true, we have that $(I - K)$ is surjective, so $\mathrm{Im}(I - K) = \mathcal{H}$, so $\mathrm{Ker}(I - K) = \{0\}$.
Thus $\mathrm{Ker}(I - K) = \{0\} \iff$ 1..
We also have that $\mathrm{Ker}(I - K) \neq \{0\} \iff \exists u \neq 0 \in \mathrm{Ker}(I - K) \iff$ 2.. □

Theorem 3.39. *(Spectral theorem / Hilbert-Schmidt Theorem)*
Let $T : \mathcal{H} \to \mathcal{H}$ be a compact linear self-adjoint operator on an infinite dimensional complex Hilbert space $\mathcal{H}$, and $n = \dim(\Im(T)) \in \mathbb{N} \cap \{\infty\}$, then

1. *There exists orthonormal eigenvectors $(\phi_k)_{k=1}^n \subset \mathcal{H}$ and eigenvalues $(\lambda_k)_{k=1}^n \subset \mathbb{R}$ such that $|\lambda_1| \ge |\lambda_2| \ge \dots,$ and*
$$T\phi_k = \lambda_k \phi_k, \lambda_k \neq 0,\ \forall 1 \le k \le n,$$
$$\forall v \in \mathcal{H}, Tv = \sum_{k=1}^n \lambda_k \langle \phi_k, v\rangle \phi_k = \sum_{k=1}^n \langle \phi_k, Tv\rangle \phi_k.$$
2. *If $n = \infty$, then $\lim_{k\to\infty} \lambda_k = 0$, and $(\phi_k)_{k=1}^\infty$ is an orthonormal set for $\mathcal{H}$ iff 0 is not an eigenvalue for T.*

4 Hilbert Spaces

Definition 4.1. An **inner product space** is a vector space H that has an inner product: $\langle \cdot, - \rangle : H \times H \to \mathbb{C}$, such that $\forall u, v, w \in H, a, b \in \mathbb{C}$, it satisfies

1. conjugate symmetry; i.e. $\langle v, w \rangle = \overline{\langle w, v \rangle}$,
2. linearity in the first argument; i.e. $\langle au + bw, v \rangle = a\langle u, v \rangle + b\langle w, v \rangle$, and
3. positive definiteness; i.e. if $v \neq 0$, we must have $\langle v, v \rangle > 0$.

Lemma 4.1. *For every inner product space with $\langle -, \cdot \rangle$, and $x, y \in H$, we have*

$$\langle x, y \rangle + \langle y, x \rangle = 2\Re(\langle x, y \rangle) = 2\Re(\langle y, x \rangle),$$

which is twice the real part of $\langle x, y \rangle$. Similarly,

$$\langle x, y \rangle - \langle y, x \rangle = 2\Im(\langle x, y \rangle) = -2\Im(\langle y, x \rangle),$$

which is twice the imaginary part of $\langle x, y \rangle$.
Also, we have

$$\langle x, y \rangle \langle y, x \rangle = |\langle x, y \rangle|^2.$$

Proof.

$$\begin{aligned}
\langle x, y \rangle + \langle y, x \rangle &= \langle x, y \rangle + \overline{\langle x, y \rangle} \\
&= 2\Re(\langle x, y \rangle) \\
\langle x, y \rangle - \langle y, x \rangle &= \langle x, y \rangle - \overline{\langle x, y \rangle} \\
&= 2\Im(\langle x, y \rangle) \\
\langle x, y \rangle \langle y, x \rangle &= \langle x, y \rangle \langle y, x \rangle \\
&= |\langle x, y \rangle|^2
\end{aligned}$$

□

Theorem 4.2 (Cauchy-Schwarz)**.** *For every inner product space H,*

$$\forall u, v \in H, |\langle u, v \rangle| \leq ||u||||v||,$$

where we define $||x|| = \sqrt{\langle x, x \rangle}$ or any $x \in H$.
In particular, when $||u|| \neq 0, ||u||^2||v||^2 - |\langle u, v \rangle|^2 = ||z||^2$, where $z := ||u||v - \frac{\langle u, v \rangle}{||u||}u$.

Proof. Notice that this is trivially true and equality holds to be zero when $u = 0$.
Now we assume $u \neq 0$, then $||u|| = \sqrt{\langle u, u \rangle} > 0$.

$$\begin{aligned}
||z||^2 &= \langle z, z \rangle \\
&= \left\langle ||u||v - \frac{\langle u, v \rangle}{||u||}u, ||u||v - \frac{\langle u, v \rangle}{||u||}u \right\rangle \\
&= ||u||^2\langle v, v \rangle - \langle u, v \rangle\langle u, v \rangle - \overline{\langle u, v \rangle}\langle v, u \rangle + \frac{\langle v, u \rangle\langle u, v \rangle}{\cancel{||u||^2}}\cancelto{||u||^2}{\langle u, u \rangle} \\
&= ||u||^2||v||^2 - |\langle u, v \rangle|^2 - \cancel{|\langle v, u \rangle|^2} + \cancel{|\langle v, u \rangle|^2} \\
&= ||u||^2||v||^2 - |\langle u, v \rangle|^2.
\end{aligned}$$

Now $||z||^2 = \langle z, z \rangle \geq 0$, we have the result.

□

Proposition 4.3. *For every inner product space with $\langle -, \cdot \rangle$, there is a norm $||x|| = \sqrt{\langle x, x \rangle}$.*

Proof. Consider any $x \in H, a \in \mathbb{C}$,

$$\begin{aligned}
||a \cdot x|| &= \sqrt{\langle ax, ax \rangle} = \sqrt{a^* a \langle x, x \rangle} = \sqrt{|a|^2}\sqrt{\langle x, x \rangle} = |a|||x|| \\
\forall x \neq 0, ||x|| &= \sqrt{\langle x, x \rangle} > 0 \\
||0|| &= \sqrt{\langle 0, 0 \rangle} = 0 \\
||x+y||^2 &= \langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \\
&= ||x||^2 + ||y||^2 + 2\Re(\langle x, y \rangle) \\
&\leq ||x||^2 + ||y||^2 + 2|\langle x, y \rangle| \\
&\leq ||x||^2 + ||y||^2 + 2||x||||y|| \\
&\leq (||x|| + ||y||)^2.
\end{aligned}$$

Thus $||x|| = \sqrt{\langle x, x \rangle}$ is a norm. ☐

Corollary 4.4. *For every inner product space, there is a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$*

Proposition 4.5. *If $\forall v, \langle v, u \rangle = 0$, then $u = 0$.*

Proposition 4.6. *For an Inner product space $H, \forall y, x = \lim_{i \to \infty} x_i \in H$, we have*

$$\langle x, y \rangle = \lim_{i \to \infty} \langle x_i, y \rangle.$$

Proof. Given any $\epsilon > 0$, let $\epsilon_0 = \frac{\epsilon}{||y||}$.
Since $x = \lim_{i \to \infty} x_i$, we can find $N > 0$, such that $\forall n > N, ||x - x_n|| < \epsilon_0$,
thus $|\langle x, y \rangle - \langle x_n, y \rangle| = |\langle x - x_n, y \rangle| \leq ||x - x_n||||y|| < \epsilon_0 ||y|| = \epsilon$ ☐

Corollary 4.7. *For an Inner product space $H, \forall y, x = \lim_{i \to \infty} x_i \in H$, we have $\langle y, x \rangle = \lim_{i \to \infty} \langle y, x_i \rangle$.*

Definition 4.2. An inner product space $\mathcal{H}$ is called a Hilbert space if it is complete.

Definition 4.3. Let H be an inner product space. Two vectors $u, v \in H$ are called **orthogonal** if $\langle u, v \rangle = 0$.

Definition 4.4. Let H be an inner product space. A set $\{e_i\}_{i \in I} \subseteq H$ is called an **orthonormal set** if

$$\forall i, j \in I, \langle e_i, e_j \rangle = \delta_{ij}.$$

Definition 4.5. Let H be an inner product space. An orthonormal set $\{e_i\}_{i \in \mathbb{N}} \subseteq H$ is called a **maximal orthonormal set / orthonormal basis / total orthonormal set** if

$$H = \overline{Span(\{e_1, e_2, \ldots\})}.$$

Theorem 4.8. *Let $\mathcal{H}$ be a Hilbert space, and $\{e_i\}_{i \in \mathbb{N}} \subseteq \mathcal{H}$ be an orthonormal set, then TFAE:*

1. *$\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis*
2. *If $\forall i \in \mathbb{N}, \langle x, e_i \rangle = 0$, then $x = 0$.*
3. *$\forall x \in \mathcal{H}, x = \sum_{i \in \mathbb{N}} \langle e_i, x \rangle e_i$. (Fourier series)*
4. *$\forall x \in \mathcal{H}, ||x||^2 = \sum_{i \in \mathbb{N}} |\langle e_i, x \rangle|^2$. (Parseval Identity)*

Theorem 4.9. *$\mathcal{H}$ is a separable Hilbert space, if and only if there is a maximal orthonormal set in $\mathcal{H}$. Moreover, in this case, every maximal orthonormal set is at most countable.*

Definition 4.6. Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$, the subspace **orthogonal** to S is

$$S^{\perp} := \{u \in \mathcal{H} : \langle u, v \rangle = 0, \forall u \in S\}.$$

Lemma 4.10. *Let $\mathcal{H}$ be a Hilbert space, $S \subseteq \mathcal{H}$, we always have $S^{\perp}$ is a subspace of $\mathcal{H}$.*

Definition 4.7. Let V be a vector space, and $U, W \subseteq V$ be two subspaces, we say $V = U \oplus W$, if $\forall v \in V$, it can be uniquely written as $v = u + w$, where $u \in U, w \in W$.

Theorem 4.11. *Let $\mathcal{H}$ be a Hilbert space, if $S \subseteq \mathcal{H}$ is a closed subspace, then*

$$\mathcal{H} = S \oplus S^{\perp}.$$