Introduction to Mathematical Optimization and Mixed-Integer Programming

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12/02/2019

Overview

- Mathematical optimization
- Mixed-integer programming (MIP)
- MIP examples
    - Traveling salesman problem
    - Job scheduling
    - Resource allocation

Overview of mathematical optimization

Mixed-Integer Programming (MIP)

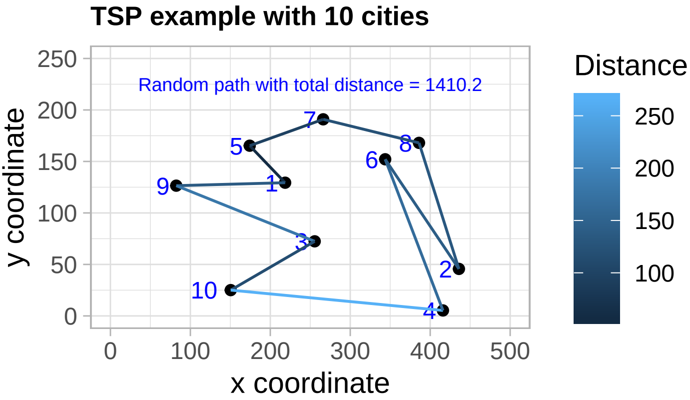
$$
\begin{aligned}
\max \quad & f(x) \\
s.t. \quad & g(x) \leq 0 \\
& x \in \mathbb{R} \times \mathbb{Z}
\end{aligned}
$$

Example 1 - traveling salesman problem (TSP)

- Problem definition (from wikipedia)

The travelling salesman problem (TSP) asks the following question: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Example of a random path



- Finding the optimal path by brute force takes $O(n!)$

Formulate TSP as a MIP

Miller-Tucker-Zemlin formulation

- Label the cities with $1, ..., n$ and define:

$$x_{ij} = \begin{cases} 1 & \text{if a path goes from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

- Use another variable u_i to denote the ordering of the cities.

Formulate TSP as a MIP (continued)

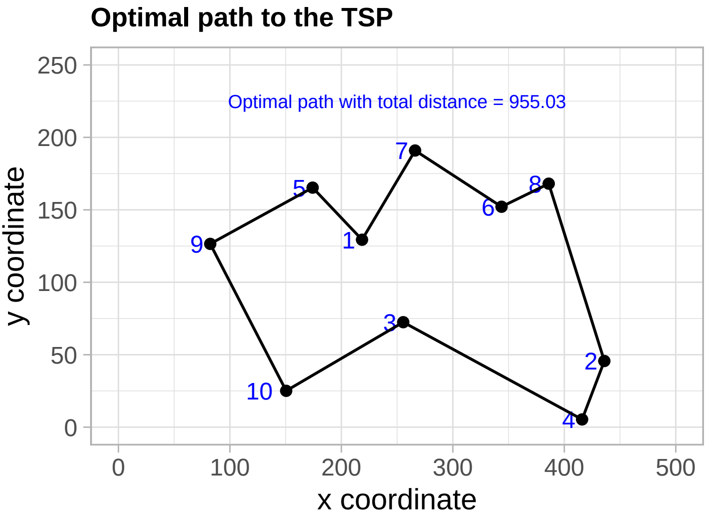
$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} d_{ij} x_{ij} & \\
s.t. & x_{ij} \in \{0,1\} & i,j = 1,...,n \\
& u_i \in \mathbf{Z} & i = 2,...,n \\
& \displaystyle\sum_{i=1, i \neq j}^{n} x_{ij} = 1 & j = 1,...,n \\
& \displaystyle\sum_{j=1, j \neq i}^{n} x_{ij} = 1 & i = 1,...,n \\
& u_i - u_j + n x_{ij} \leq n - 1 & 2 \leq i \neq j \leq n \\
& 0 \leq u_i \leq n - 1 & 2 \leq i \leq n
\end{array}
$$

MIP solution to TSP

- We use the GLPK solver to optimize the MIP formulated above.

trip_id	from (i)	to (j)	distance
1	1	5	57.053
2	5	9	99.740
3	9	10	122.182
4	10	3	115.221
5	3	4	173.915
6	4	2	44.997
7	2	8	132.161
8	8	6	45.231
9	6	7	86.718
10	7	1	77.813

Visualize the optimal path to the TSP

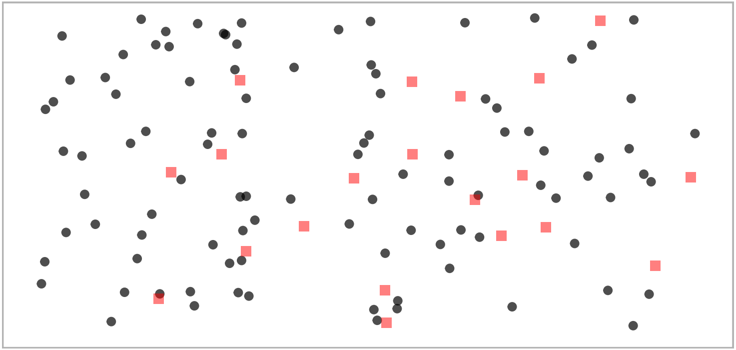


Example 2 - facility location problem (FLP)

- Wikipedia definition: *A branch of operations research concerned with the optimal placement of facilities to minimize transportation cost while considering other factors.*
- We consider a simple warehouse allocation problem, where facilities can be built in m potential locations to serve n customers. The goal is to minimize the total transportation cost and fixed facility cost.

Visualization of a FLP

Warehouse location problem (n=100, m=20)



Black dots represent customers. Red triangles show potential locations to build warehouses.

Formulate FLP as MIP

Warehouse allocation problem

- Let $I = \{1, ..., n\}$ be the set of customers, $J = \{1, ..., m\}$ the set of potential locations.
- Define two set of binary variables:

$$y_j = \begin{cases} 1 & \text{a warehouse is built at location } j \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_{ij} = \begin{cases} 1 & \text{if customer } i \text{ will be served by warehouse at } j \\ 0 & \text{otherwise} \end{cases}$$

Formulate FLP as a MIP

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j}^{m} \textsf{transportation\_cost}_{ij} \cdot x_{ij} + \sum_{j=1}^{m} \textsf{fixed\_cost}_{j} \cdot y_j \\
s.t. \quad & \sum_{j}^{m} x_{ij} = 1, \quad \forall i = 1, ..., n \\
& x_{ij} \leq y_j, \qquad \forall i = 1, ..., n \quad j = 1, ..., m \\
& x_{ij} \in \{0, 1\}, \quad \forall i = 1, ..., n \quad j = 1, ..., m \\
& y_j \in \{0, 1\}, \quad \forall j = 1, ..., m
\end{aligned}
$$

MIP solution to the FLP