PHYS704: Assignment 6

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Numerical estimates

1.a

assume room temperature is $300K$ the heat capcity of Fermi gas is $\frac{C_{elec}}{Nk_B} = \frac{\pi^2}{2}(\frac{T}{T_F}) \approx 0.029$, and for Boson it is

$$\frac{C_V}{Nk_B} = 9(\frac{T}{T_D})^3 \int_0^{T_D/T} \frac{x^4}{(e^x-1)^2} dx$$

for phonon gas in iron, $T_D = 470K, T = 300K$, we calculate the above integral numerically, we have

$$\frac{C_V}{Nk_B} = 0.87$$

thus $C_e/C_p = 0.032$

1.b

the thermal wavelength is given by

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}} = \frac{6.67 \times 10^{-34} Js}{\sqrt{2\pi \cdot 1.67 \times 10^{-27} kg \cdot 1.38 \times 10^{-23} JK^{-1} 300K}} \approx 1e{-}10m \approx 1\mathring{A}$$

the minimum wavelength of a phonon in a typical criystal is $0.01m$

1.c

$$n\lambda^3 = \frac{nh^3}{(2\pi m k_B T)^{3/2}}$$

assume they are ideal gas, we have $P = nk_BT$, then

$$n\lambda^3 = \frac{P}{(k_BT)^{5/2}} \frac{h^3}{(2\pi m)^{3/2}}$$

thus, for protons

$$(n\lambda^3)_{\text{proton}} = \frac{10^-5}{(4.1\times10^{-21})^{5/2}}\frac{(6.7\times10^{-34})^3}{(2\pi\cdot1.7\times10^{-27})^{3/2}} = 2\times10^{-5}$$

for other particles we have

$$\frac{n\lambda^3}{(n\lambda^3)_{\text{proton}}} = \frac{m_{\text{proton}}^{3/2}}{m^{3/2}}$$

thus we have

$$(n\lambda^3)_{\text{hydrogen}} = 2\times10^{-5}$$
$$(n\lambda^3)_{\text{helium}} = 3.12\times10^{-6}$$
$$(n\lambda^3)_{\text{oxygen}} = 1.37\times10^{-7}$$

quantum effects become more important when $n\lambda^3 \geq 1$, thus we can have the temperature to be, x can be hydrogen, helium or oxygen calculated above

$$T = 300K\cdot(n\lambda)^{3/2}_{\text{x at room temperature}}$$

1.d

since in 3 dimension, the experimental $C_V \propto T^3$, the energy excitation spectrum should have form $\epsilon(k) = \hbar c_s k$, then at low temperature limit, we have the heat capcity

$$\frac{C_V}{Nk_B} = \frac{12\pi^4}{5}(\frac{T}{T_D})^3, T_D = \frac{\hbar c_s}{k_B}(\frac{6\pi^2 N}{V})^{1/3}$$

thus

$$C_V = Nk_B\frac{12\pi^4}{5}(\frac{T}{T_D})^3 = 20.4T^3$$

thus

$$\epsilon = \hbar ck = k_B(\frac{2\pi^2 k_B V}{5}\frac{T^3}{C_V})^{1/3}k = (3.16\times10^{-34}Jm)k$$

Solar interior

2.a

for massive particles

$$\lambda = \frac{h}{\sqrt{2\pi mk_BT}}$$

and $T = 1.6\times10^7K$ thus

$$\lambda_{electron} = \frac{6.7 \times 10^{-34} J/s}{\sqrt{2\pi \times (9.1 \times 10^{-31} Kg) \cdot (1.4 \times 10^{-23} J/K) \cdot (1.6 \times 10^{7} K)}} \approx 1.9 \times 10^{-11} m$$
$$\lambda_{proton} = \frac{6.7 \times 10^{-34} J/s}{\sqrt{2\pi \times (1.7 \times 10^{-31} Kg) \cdot (1.4 \times 10^{-23} J/K) \cdot (1.6 \times 10^{7} K)}} \approx 4.3 \times 10^{-13} m$$

and $\lambda_\alpha = \frac{1}{2}\lambda_{proton} \approx 2.2 \times 10^{-13} m$

2.b

we can get the corresponding number density n from their density

$$n_H = 3.5 \times 10^{31} m^{-3}$$
$$n_{He} = 1.5 \times 10^{31} m^{-3}$$
$$n_e = 2n_{He} + n_H = 8.5 \times 10^{31} m^{-3}$$

thus we have the criterion for degeneracy

$$n_H \cdot \lambda_H^3 \approx 2.8 \times 10^{-6} \ll 1$$
$$n_{He} \cdot \lambda_{He}^3 \approx 1.6 \times 10^{-7} \ll 1$$
$$n_e \cdot \lambda_e^3 \approx 0.58$$

thus electrons are weakly degenerate, nuclei are not.

2.c

using the ideal gas law, we have

$$P = (n_H + n_{He} + n_e) k_B T = 13.5 \times 10^{31} \cdot 1.38 \times 10^{-23} \cdot 1.6 \times 10^{7} = 3 \times 10^{16} N/m^2$$

2.d

The radiation pressure can be calculated using the black body radiation, which is

$$P = \frac{1}{3}\frac{U}{V} = \frac{4\sigma T^4}{3c} = \frac{4 \cdot 5.7 \times 10^{-8} \cdot 1.6 \times 10^{7}}{3 \cdot 3 \times 10^{8}} = 1.7 \times 10^{13} N/m^2$$

Freezing of He^4

3.a

the energy can be written as

$$
\begin{aligned}
E &= \sum_{\vec{k}} \frac{\epsilon(\vec{k})}{e^{\beta\epsilon(\vec{k})}-1} \\
&= \sum_{\vec{k}} \frac{\hbar ck}{e^{\beta\hbar ck}-1} \\
&= V\int \frac{d^3k}{(2\pi)^3}\frac{\hbar ck}{e^{\beta\hbar ck}-1} \\
&= V\int \frac{4\pi k^2dk}{(2\pi)^3}\frac{\hbar ck}{e^{\beta\hbar ck}-1} \quad (x=\beta\hbar ck) \\
&= \frac{V}{2\pi^2}\hbar c(\frac{k_BT}{\hbar c})^4\int_0^\infty dx\frac{x^3}{e^x-1} \\
&= \frac{\pi^2}{30}V\hbar c(\frac{k_BT}{\hbar c})^4
\end{aligned}
$$

thus the head capcity is

$$C_V = \frac{dE}{dT} = \frac{2\pi^2}{15}Vk_B(\frac{k_BT}{\hbar c})^3$$

and

$$\frac{C_V}{N} = \frac{2\pi^2}{15}\frac{V}{N}k_B(\frac{k_BT}{\hbar c})^3$$

3.b

the contribution of each mode in head capcity is $\frac{2\pi^2}{15}\frac{V}{N}k_B(\frac{k_BT}{\hbar})^3\frac{1}{c^3}$, thus the total is

$$\frac{C_V^s}{N} = \frac{2\pi^2}{15}k_B\frac{V}{N}(\frac{k_BT}{\hbar})^3(\frac{2}{c_T^3}+\frac{1}{c_L^3})$$

3.c

$$s_l(T) = \int_0^T \frac{C_V(T)dT}{T} = \frac{2\pi^2}{45}k_Bv_l(\frac{k_BT}{\hbar c})^3$$

$$s_s(T) = \int_0^T \frac{C_V(T)dT}{T} = \frac{2\pi^2}{45}k_Bv_l(\frac{k_BT}{\hbar})^3(\frac{2}{c_T^3}+\frac{1}{c_L^3})$$

since we have $c \approx c_L \approx c_T$ and $v_l \approx v_s \approx v$, we have

$$s_l - s_s = -\frac{4\pi^2}{45}k_Bv(\frac{k_BT}{\hbar c})^3 < 0$$

thus the solid phase has higher entropy.

3.d

since we have

$$d\mu_l = v_l dP - s_l dT = v_s dP - s_s dT = d\mu_s$$

thus

$$\frac{dP}{dT} = \frac{s_l - s_s}{v_l - v_s} = -\frac{4\pi^2}{45} k_B \frac{v}{\delta v} (\frac{k_B T}{\hbar c})^3$$

thus we have the melting curve as

$$P(T) = P(0) - \frac{\pi^2}{45} k_B \frac{v}{\delta v} (\frac{k_B T}{\hbar c})^3 T$$

in order to decrease the pressure, we should have $\delta v > 0$, which means solid phase must have higher density.

Relativistic Bose gas in d dimensions

4.a

$$\begin{aligned}
\mathcal{Q} &= \sum_{N=0}^{\infty} e^{N\beta\mu} \sum \exp(\beta - \sum_i n_i \epsilon_i) \\
&= \sum_n \prod_i \exp[-\beta(\epsilon_i - \mu) n_i] \\
&= \prod_i \frac{1}{1 - \exp(-\beta(\epsilon_i - \mu))}
\end{aligned}$$

thus

$$
\begin{aligned}
\mathcal{Q} &= \sum_i \ln(1-\exp(\beta(\mu-\epsilon_i))) \\
&= \int V\frac{d^dk}{(2\pi)^d}\ln(1-\exp(\beta(\mu-\epsilon_i))) \\
&= \frac{V}{(d/2-1)!(2\pi)^{d/2}}\int k^{d-1}dk\ln(1-z\exp(-\beta\hbar ck)) \\
&= -\frac{V}{(d/2-1)!(2\pi)^{d/2}}(\frac{k_BT}{\hbar c})^d\int_0^\infty x^{d-1}dx\ln(1-ze^{-x}), \quad x=\beta\hbar ck \\
&= -\frac{V}{(d/2-1)!(2\pi)^{d/2}}(\frac{k_BT}{\hbar c})^d\frac{1}{d}(x^d\ln(1-ze^{-x})\big|_0^\infty-\int x^ddx\frac{ze^{-x}}{1-ze^{-x}}) \\
&= \frac{V}{(d/2-1)!(2\pi)^{d/2}}(\frac{k_BT}{\hbar c})^d\frac{1}{d}\int x^ddx\frac{ze^{-x}}{1-ze^{-x}} \\
&= \frac{V}{(d/2-1)!(2\pi)^{d/2}}(\frac{k_BT}{\hbar c})^d\frac{1}{d}d!f^+_{d+1}(z) \\
&= \frac{V(2\pi)^{d/2}}{(d/2-1)!}(\frac{k_BT}{hc})^d(d-1)!f^+_{d+1}(z)
\end{aligned}
$$

thus

$$
\mathcal{G} = -k_BT\ln(\mathcal{Q}) = -\frac{V(2\pi)^{d/2}}{(d/2-1)!}(\frac{k_BT}{hc})^d(d-1)!k_BTf^+_{d+1}(z)
$$

$$
\begin{aligned}
N &= -\frac{\partial\mathcal{G}}{\partial\mu} = -\beta z\frac{\partial\mathcal{G}}{\partial z} \\
&= \frac{V(2\pi)^{d/2}}{(d/2-1)!}(\frac{k_BT}{hc})^d(d-1)!f^+_d(z)
\end{aligned}
$$

thus density is

$$
n = \frac{(2\pi)^{d/2}}{(d/2-1)!}(\frac{k_BT}{hc})^d(d-1)!f^+_d(z)
$$

4.b

$$
PV = -\mathcal{G}
$$

and

$$
E = -\frac{\partial\ln(\mathcal{Q})}{\partial\beta} = d\frac{\ln(\mathcal{Q})}{\beta} = -d\mathcal{G}
$$

thus $E/(PV) = d$, same as classical.

4.c

the critical temperature is given by $z = 1$, thus

$$n = \frac{(2\pi)^{d/2}}{(d/2-1)!}(\frac{k_B T_c}{hc})^d (d-1)!\zeta_d$$

thus

$$T_c = \frac{hc}{k_B}(\frac{n(d/2-1)!}{(2\pi)^{d/2}(d-1)!\zeta_d})^{1/d}$$

zeta is finite only when $d > 1$, thus, transition exists only for $d > 1$

4.d

$$\begin{aligned} C(T) = \left.\frac{\partial E}{\partial T}\right|_{z=1} &= -d\frac{\partial \mathcal{G}}{\partial T} \\ &= d(d+1)\frac{V(2\pi)^{d/2}}{(d/2-1)!}(\frac{k_B T}{hc})^d (d-1)! k_B \zeta_{d+1} \end{aligned}$$

4.e

$$\begin{aligned} C(T_c) &= d(d+1)\frac{V(2\pi)^{d/2}}{(d/2-1)!}(\frac{k_B}{hc})^d (d-1)! k_B \zeta_{d+1} (\frac{hc}{k_B})^d (\frac{n(d/2-1)!}{(2\pi)^{d/2}(d-1)!\zeta_d}) \\ &= d(d+1) N k_B \frac{\zeta_{d+1}}{\zeta_d} \end{aligned}$$

thus

$$C(T_c)/(Nk_B) = \frac{d(d+1)\zeta_{d+1}}{\zeta_d}$$