# Lecture 5: Transforms, Fourier and Wavelets


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# Outline

- Talk involves matrices and vector spaces
  - You will not be tested on it
- What are Transforms

  = change of basis

  Linear or non-linear (will focus on linear)
- Fourier Transforms
- Wavelet Transforms
- Why??
  - Because a transform or a change in basis may allow you to see things differently, see things that couldn't be seen before, to get a different "perspective"

# What are transforms?

- Vectorize a signal (ECG, MR image, …) into vector x
- A linear transform on this vector is defined as a matrix operation

$$y = Tx$$

  - Linearity: $T(x_1 + x_2) = T x_1 + T x_2$
- Matrix examples
- T is generally a square, full-rank matrix
- If T is a "wide" matrix, then the transform does not have a unique inverse
  - Also known as overcomplete transform

- T is orthogonal if $T^t T$ = diagonal matrix
- T is orthonormal if $T^t T$ = Identity matrix
- Orthonormal transforms retain signal energy

$$||Tx|| = ||x||$$

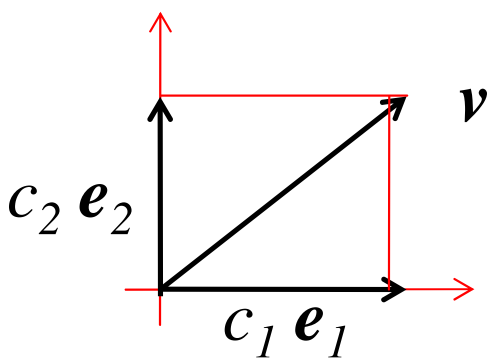
# Transforms

- Examples:
  - Fourier transform is an orthonormal transform
  - Wavelet transform is generally overcomplete, but there also exist orthonormal wavelet transforms
- A good property of a transform is invertibility
  - Both Fourier and wavelet transforms are invertible
- Many other image-based processes are not invertible
  - E.g. Distance transform, JPEG compression, edge detection, blurring

# Transforms

- A transform (with full rank T) is a change of basis
- Definition: A basis on a vector space is a set of linearly independent vectors that are able to express any other vector of the space as a linear combination of them.
  - Example: standard basis of $R^n$ is simply the set of vectors $e_1 \ldots e_n$ where $e_k = [0 \ldots 0\ 1\ 0 \ldots 0]^t$

kth position




$$v = c_1 e_1 + c_2 e_2$$

# Basis and basis vectors

- A set of such vectors can be expressed as a matrix
  $T = [t_1 \mid t_2 \mid \ldots t_n]$, where each $t_k$ is a column vector

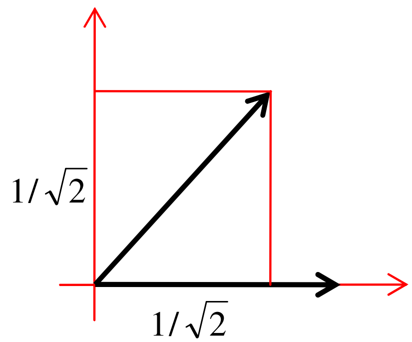
*Basis matrix*

*Basis vectors*

- Any subset of the vectors $T_K = \{t_k\}_{k=1:K}$ then defines a subspace of $R^n$
  - Technically we say the subspace is spanned by the vectors $\{t_k\}$ or equivalently by the matrix $T_K$
- Many basis matrices can span the same space

# Basis

- Example in $R^2$:
  - a pair of vectors rotated by 45 degree:




$$R = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad R^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

  - If you want to change any vector in standard space to this new basis, just left-multiply it by $R^{-1}$
  - Note: R is orthonormal: $R^{-1} = R^t$

# Transform = change of basis

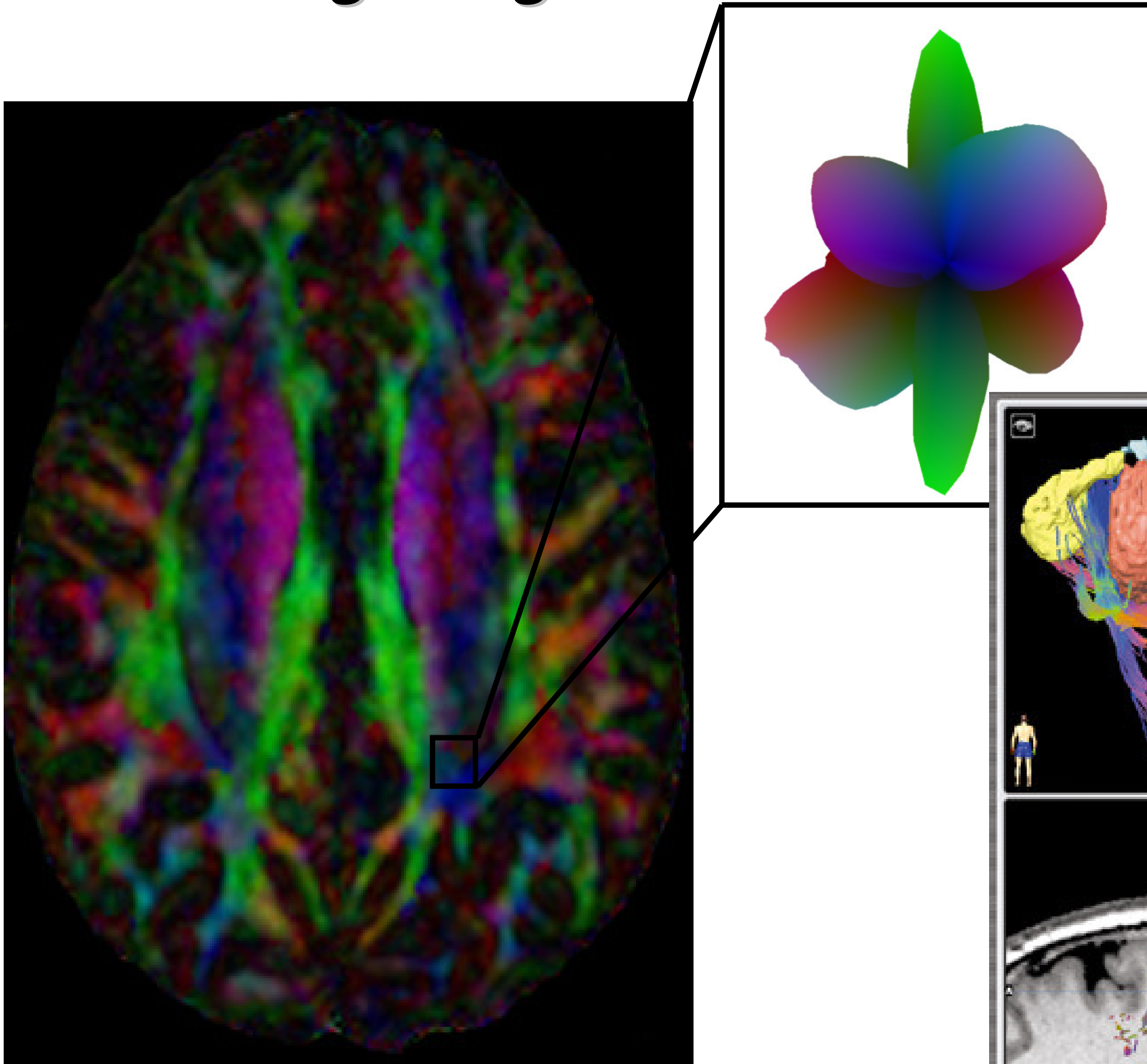
- T is a basis if its columns are linearly independent
  - i.e. it is a full rank matrix
- Orthonormality simply makes everything easier

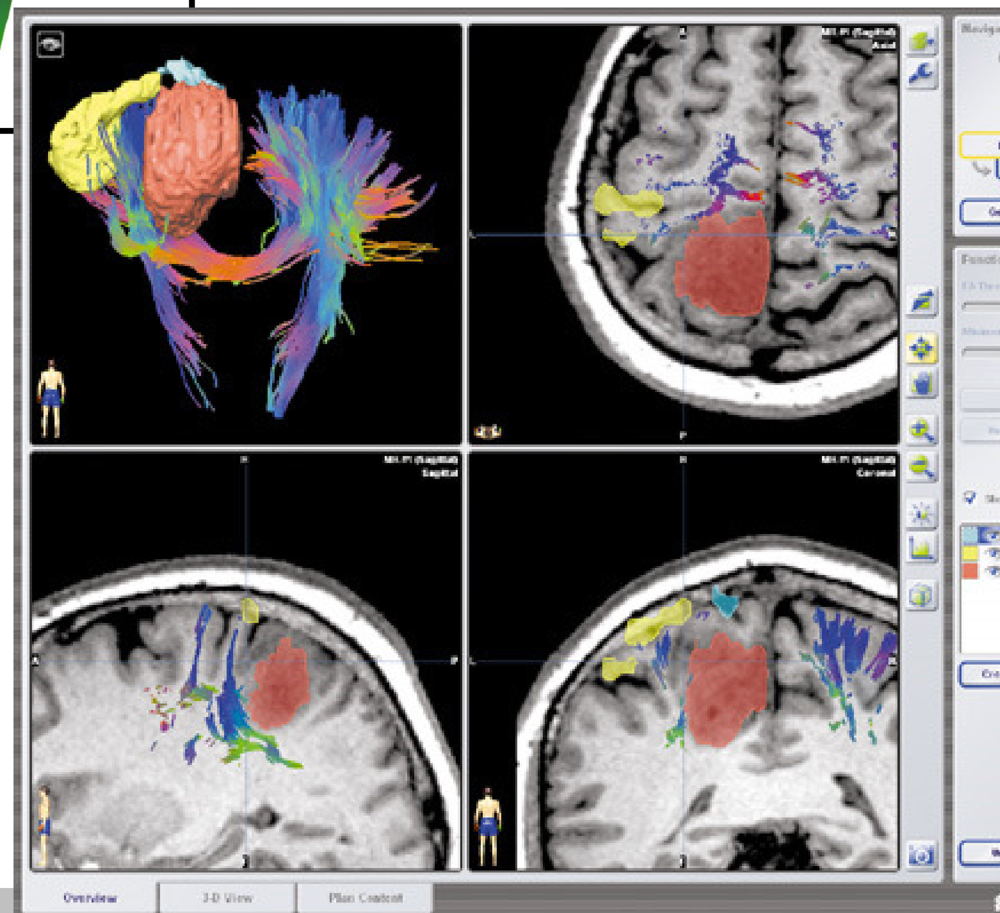
  $T^{-1} = T^{t}$ or for complex vectors, $T^{-1} = T^{H}$
- So computing the inverse transform is very easy:
- If $y = Tx$, then $x = T^{H} y$
- This is why search for (useful) orthonormal transforms is such a huge deal

# Other examples in medical imaging

- **Radon** transform widely used to turn raw CT data into CT images
  - X-ray absorption is a line integral
- **Funk-Radon** is an extension of it, and is used to reconstruct orientation distribution function (ODF) from diffusion MRI data
- Another transform (**spherical harmonic transform**) is used to clean up ODF

# High Angular Resolution Diffusion Imaging

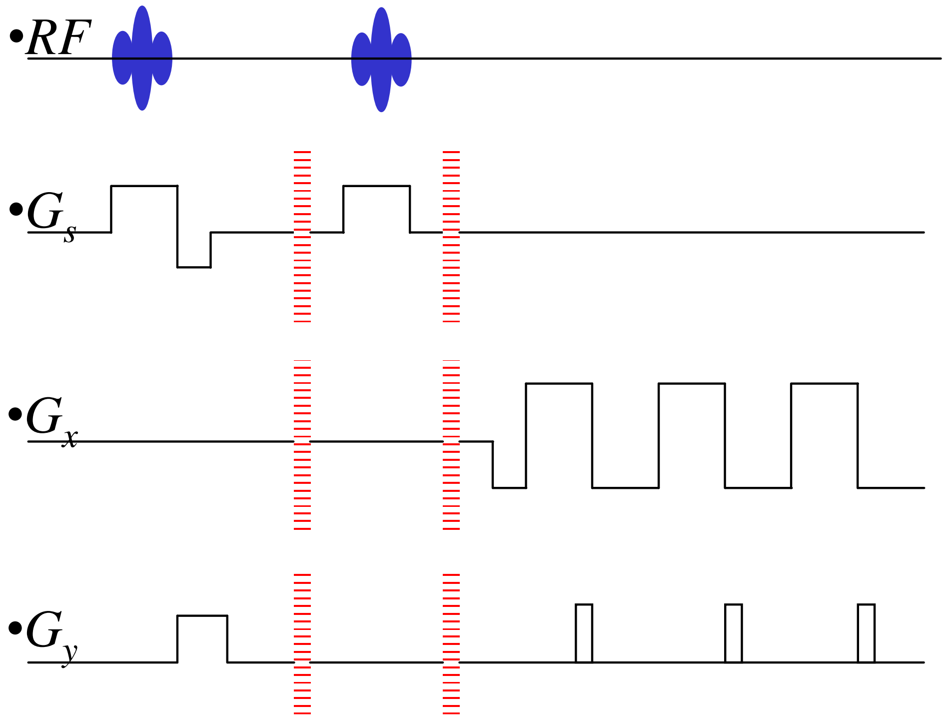


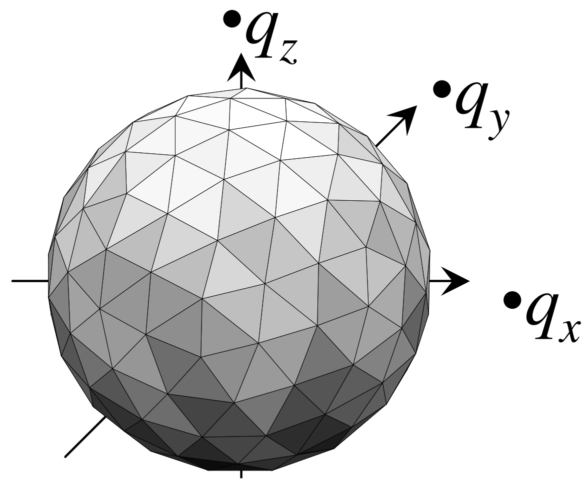


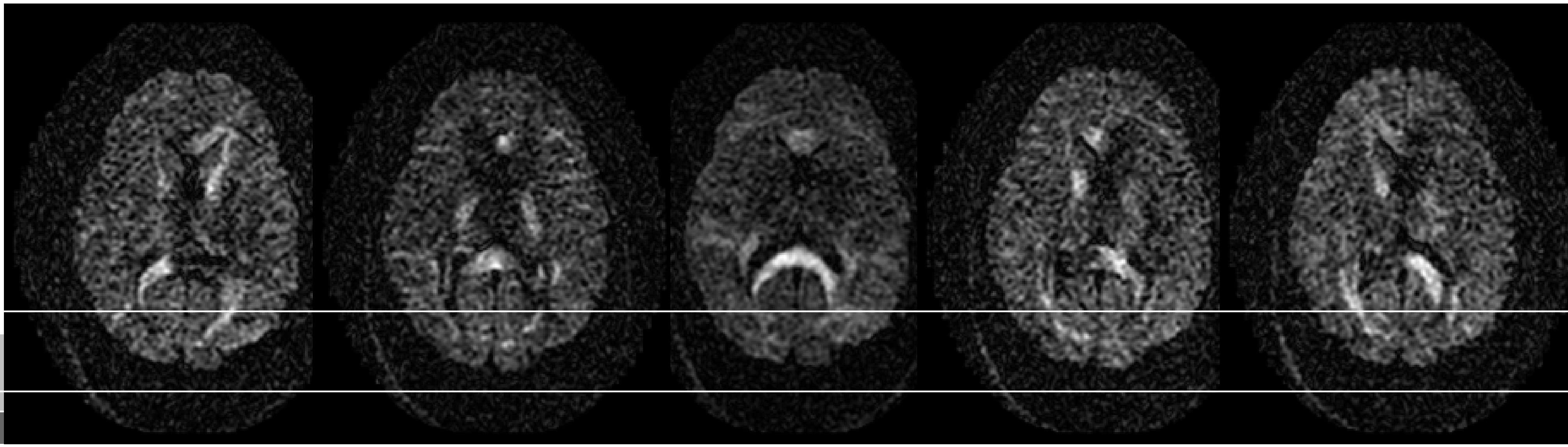
# MR Diffusion Imaging

- Diffusion MRI measures the directionally varying diffusion properties of water in tissue
- D-MRI involves taking several directional diffusion imaging measurements
- Then we fit a 3D shape to these measurements

# •Data Acquisition Strategy

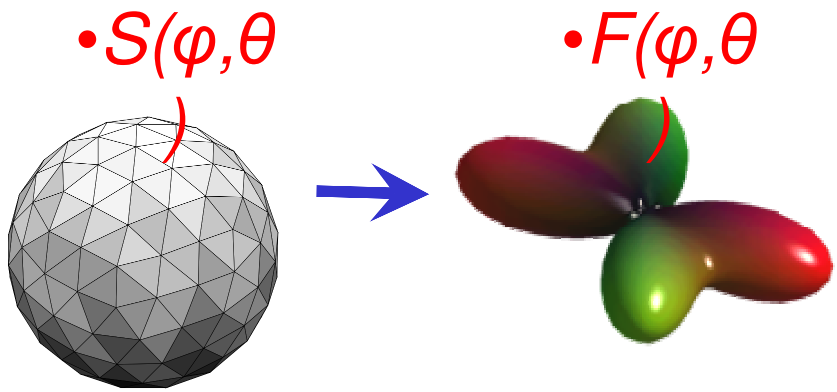


# Reconstruction Problem

Basic Approach Construct a function on the unit sphere characterizing the angular structure of diffusion in each voxel.




Recon using spherical harmonic basis Let f and s be vectors representing functions S(.) and F(.). Then

$$f = [\text{SH transform}]\ [\text{F-R transform}]\ s$$

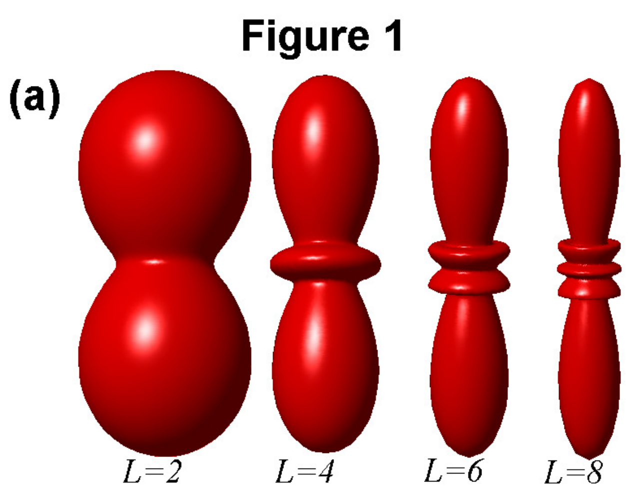
*Represents ODF as linear mix of spherical harmonics*

*Transforms raw MR data to function on unit sphere*

# •High Angular Resolution Diffusion Imaging:

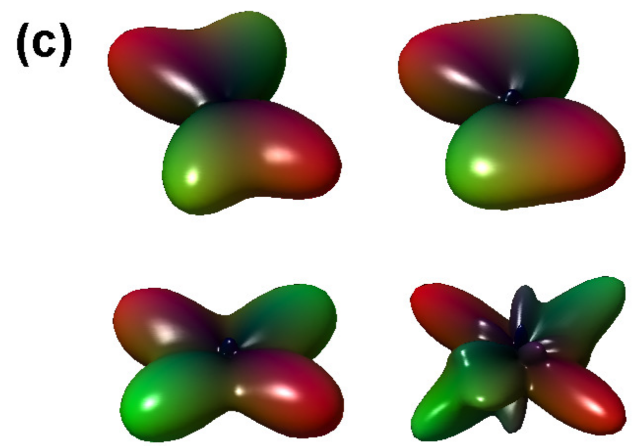
# Spherical Harmonic Transform

*Point Spread Functions*
*= Basis functions or vectors*

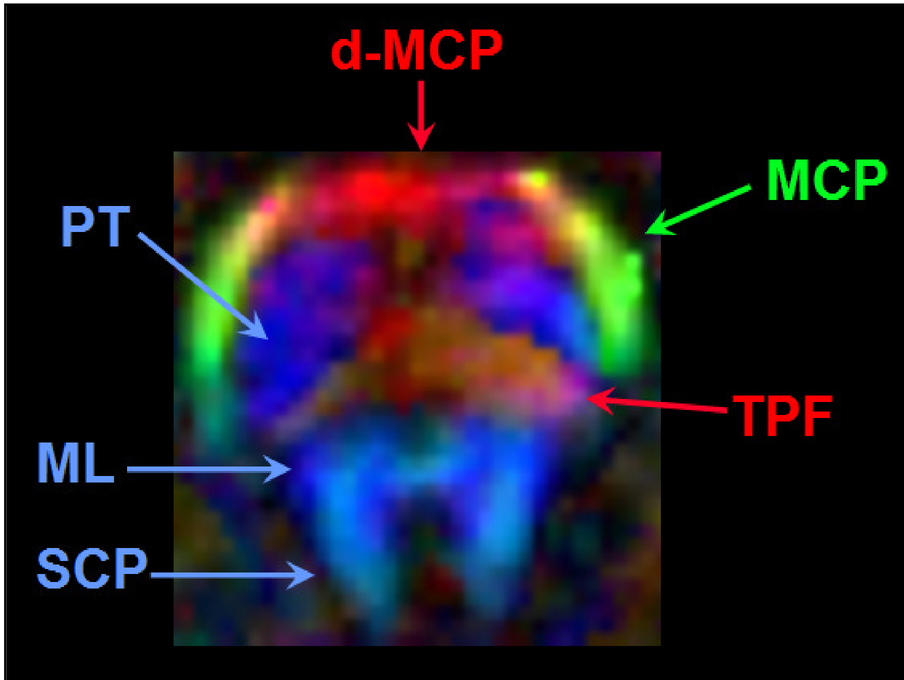


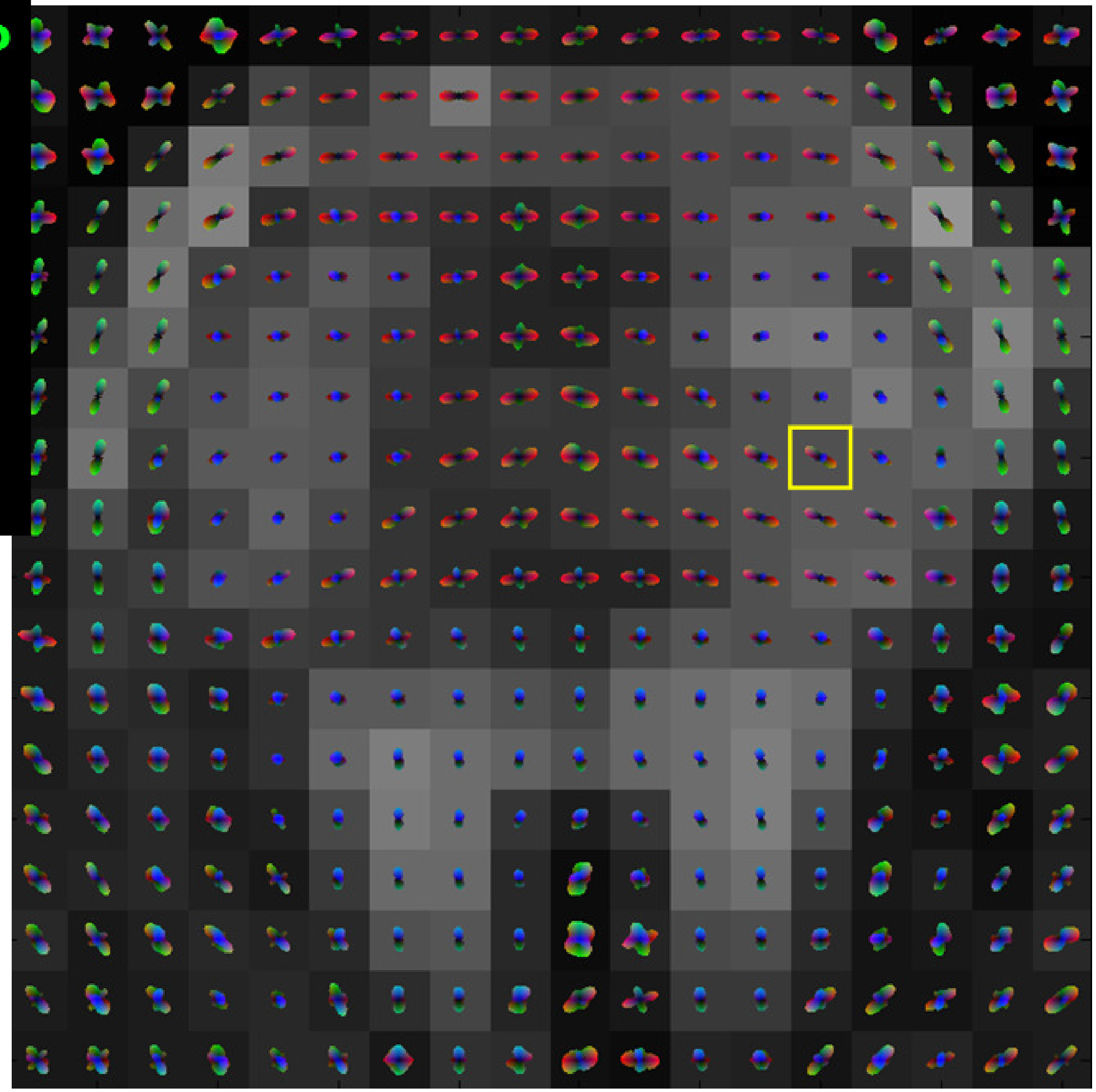

$t_1$ $t_2$ $t_3$ $t_4$

*Example ODFs*

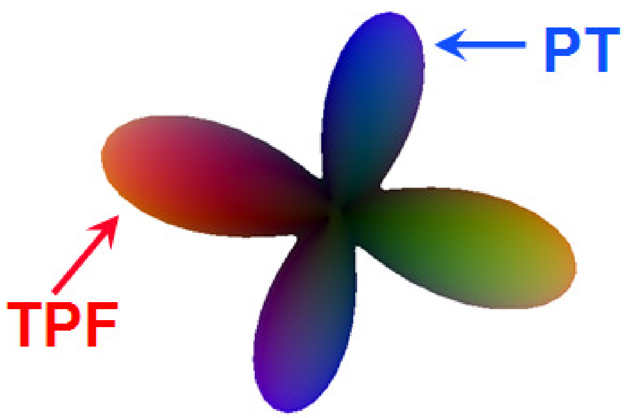



*A linear combination of these (and their rotated versions) can construct an arbitrary ODF on the unit sphere*



- Middle cerebellar peduncle (MCP)
- Superior cerebellar peduncle (SCP)
- Pyramidal tract (PT)
- Trans pontocerebellar fibers (TPF)




Hess CP, Mukherjee P, Han ET, Xu D, Vigneron DB
Magn Reson Med 2006; 56:104-117

# Fourier Transforms - Audio example

- Audio signals like music have various frequencies
  - You can change the contribution of various frequency bands by using a band equalizer
- Each frequency band is represented by a pure tone at frequency k Hz
- Lets say its captured in a vector $e_k$
- The contribution of this band = the dot product

$$s_k = \langle e_k, x \rangle = e_k^H x$$

# Fourier Transforms (FT)

Now do this for each k, and collect the $s_k$'s in vector s, we have:

$$s = F^H x$$

This is called the Fourier Transform

$$F = \left(\mathbf{e}_1 \mid \mathbf{e}_2 \mid \ldots \mathbf{e}_n\right)$$

*Basis vectors*

$$\text{where} \quad \mathbf{e}_k = \begin{pmatrix} \exp(-\frac{i2\pi}{n}k.1) \\ \vdots \\ \exp(-\frac{i2\pi}{n}k.n) \end{pmatrix}$$

*Complex exponentials (like sinusoids)*

- Class task: show that F is an orthonormal basis
- Therefore, FT is easily invertible:

$$x = F X$$

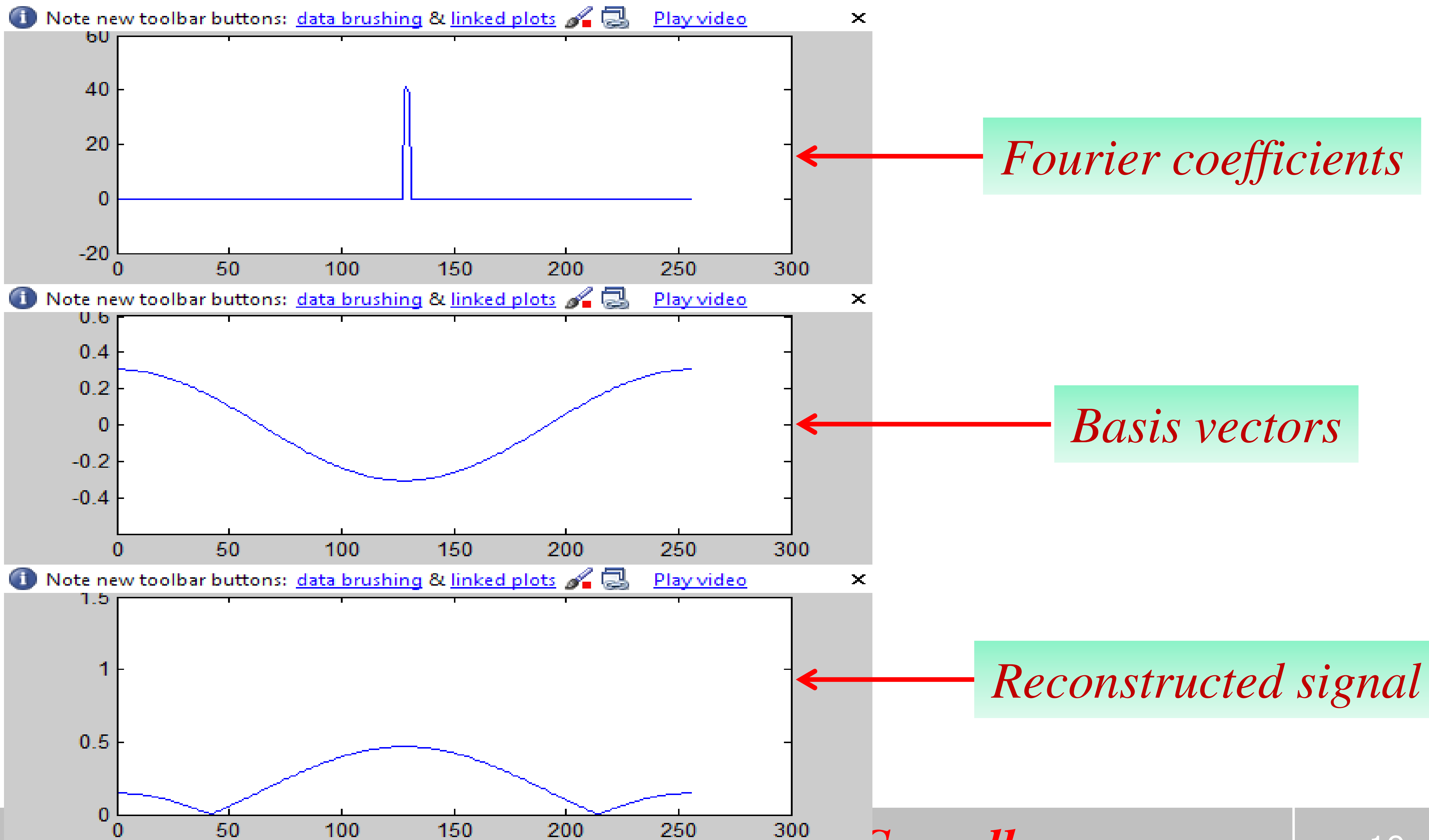
- FT = change of basis from standard space to Fourier space

# What is Fourier space and why does anyone need it?

- Fourier basis is a collection of harmonics
  - Note that complex exponentials are simply sines and cosines
- Therefore the FT simply decomposes a signal into its harmonic components
- FT gives direct information about the sharpness and oscillations present in the data
- An "alternate view" of the data

# FT Demo

- See how a unit pulse signal is constructed from Fourier basis vectors



# Fast Fourier Transform

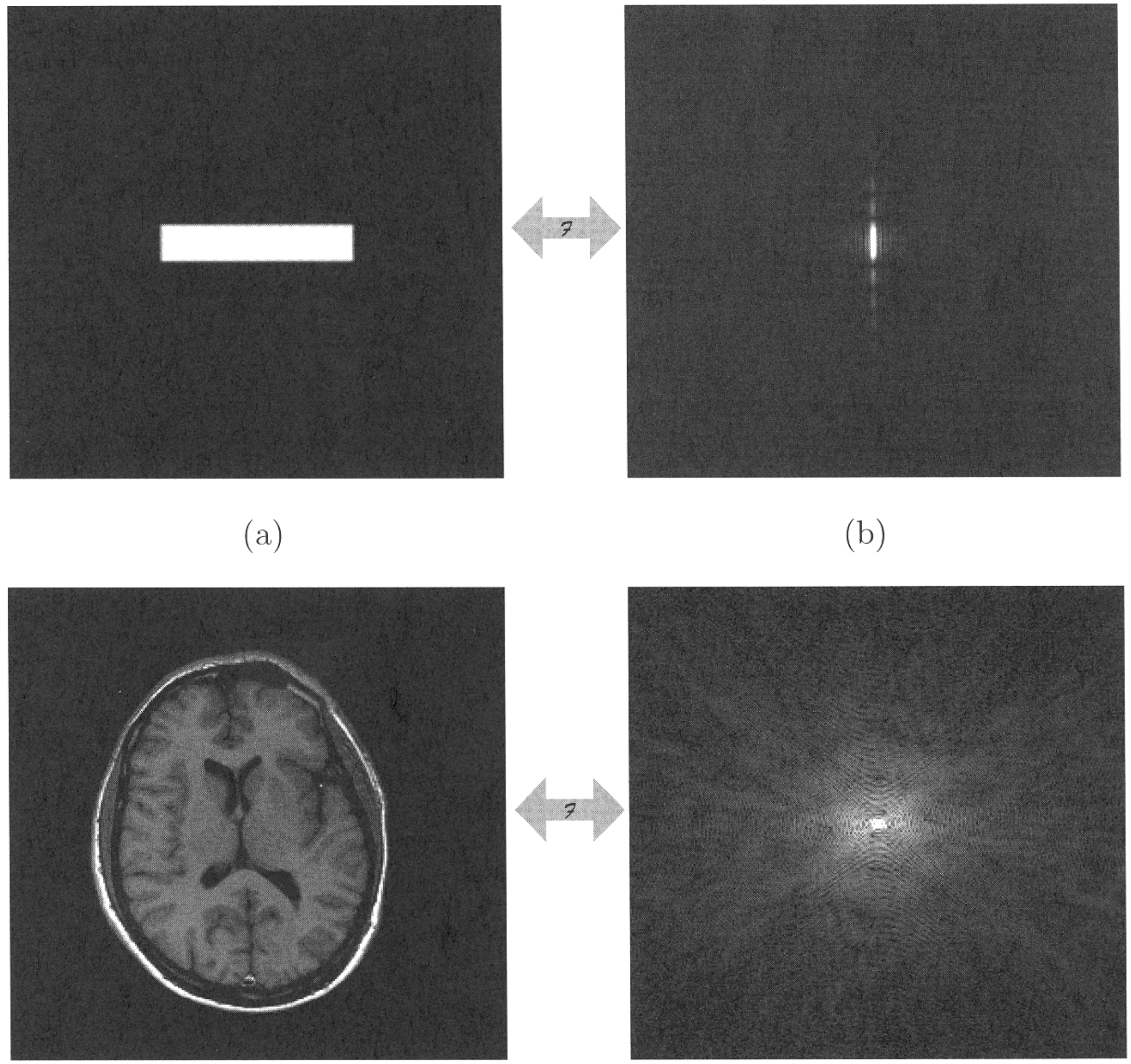
- The computation of FT requires $n^2$ mult operations
- Can get pretty expensive for large n
- An efficient algorithm exists which can do the job in *O(n log(n))* operations
- This is extremely fast vs original FT for large n
- Most programming languages have a FFT library
  - In C++ use FFTW, in MATLAB, built-in function fft.m

# FT in images

- FT is defined on 1D, 2D or nD data.
- In 2D for instance you do FT along image rows, then do FT along columns
- Again, the FT coefficients are dot products of the image with complex exponential basis vectors
- Basis vectors represent frequency (spatial frequency, or how "sharp things are)
- The FT coefficients represent the contribution of each spatial frequency
- Fourier space in 2D is sometimes called "k-space"

# k-space and image-space

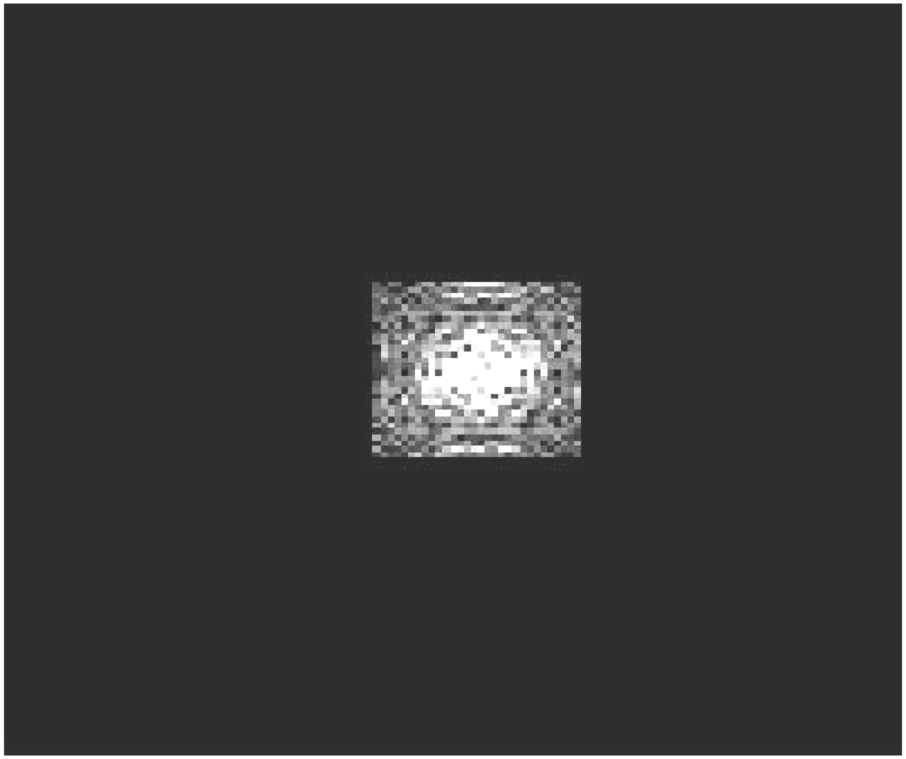
k-space & image-space are related by the 2D FT

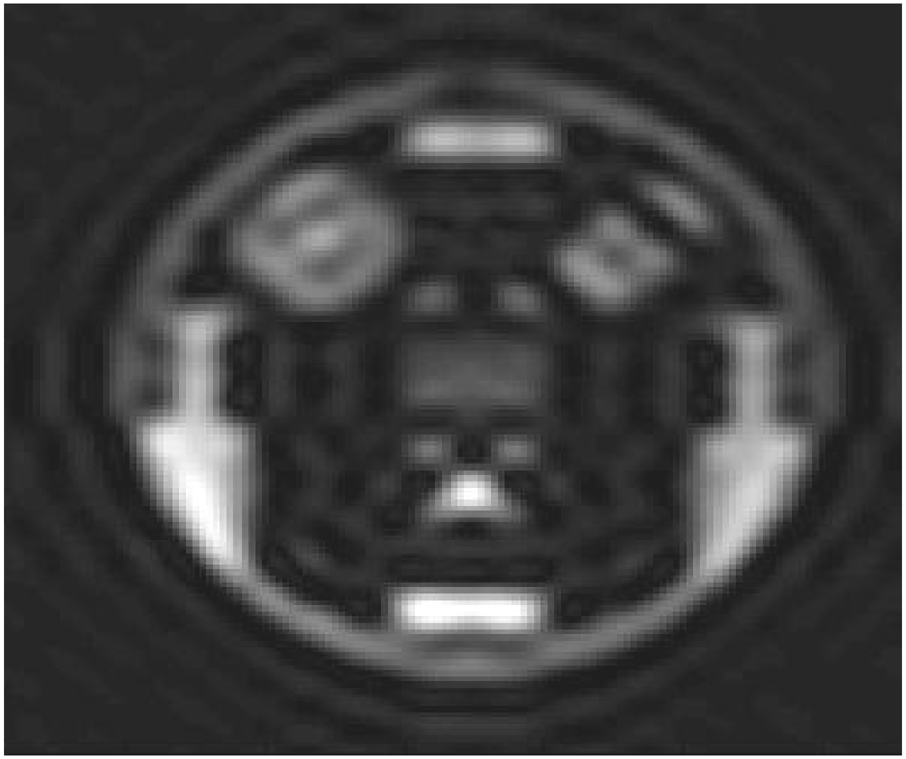


(a) (b) (c) (d)

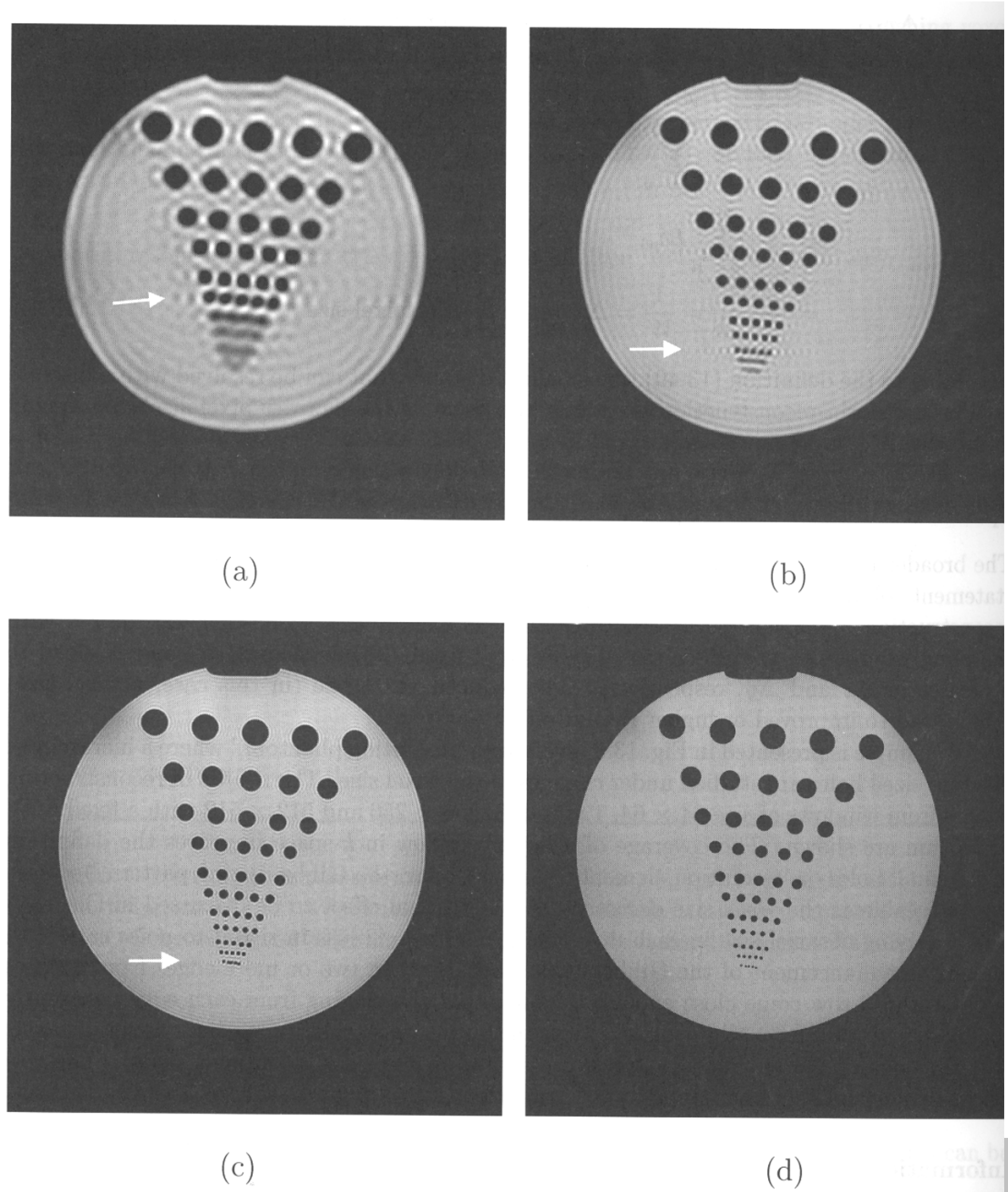
# Truncation

- Just as the number of frequency bands determines the highest pitch in an audio signal, the number of k-space points determines the sharpest features of an image
- Truncation = sampling central part of k-space





# Ringing Example



# Further Reading

- Lots of reading material online
- A great set of lectures given by Dr Kathy Davis at UT Austin
- Homework: do at least a couple exercises from below

- Basis change: http://www.ma.utexas.edu/users/davis/reu/ch2/basis/basis.pdf
- Fourier Transform: http://www.ma.utexas.edu/users/davis/reu/ch2/exponentials/exponentials.pdf
- FFT: http://www.ma.utexas.edu/users/davis/reu/ch2/fft/fft.pdf
- FFT application to heartbeat analysis: http://www.ma.utexas.edu/users/davis/reu/ch2/heart/heart.pdf

# Wavelet Transform

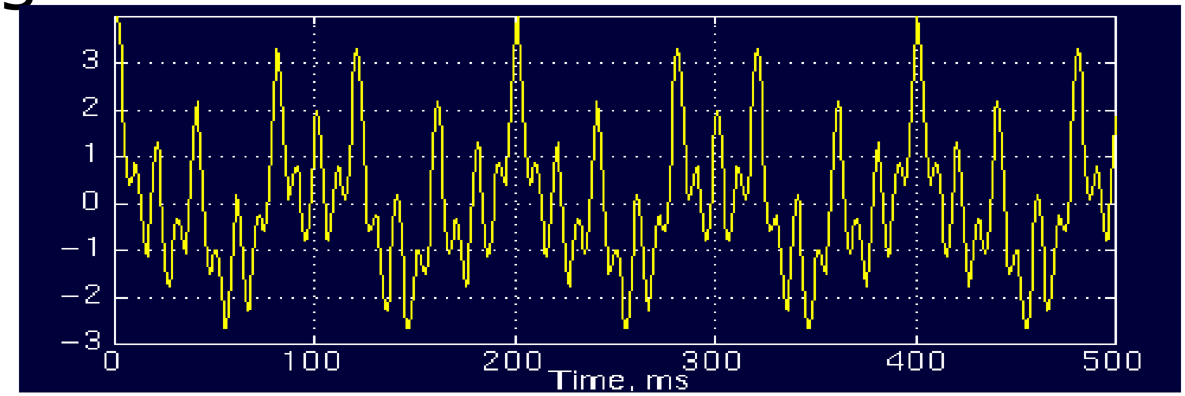
- FTs are great, but they capture global features
  - Harmonic components of the entire signal
  - They are obtained by dot-producting the WHOLE signal
- Problem1: local features can get lost
- Problem2: if signal is not stationary (features change with time or in space) then this is not captured by FT
- Therefore need a transform that provides frequency information LOCALLY

# Wavelet Transforms - motivation

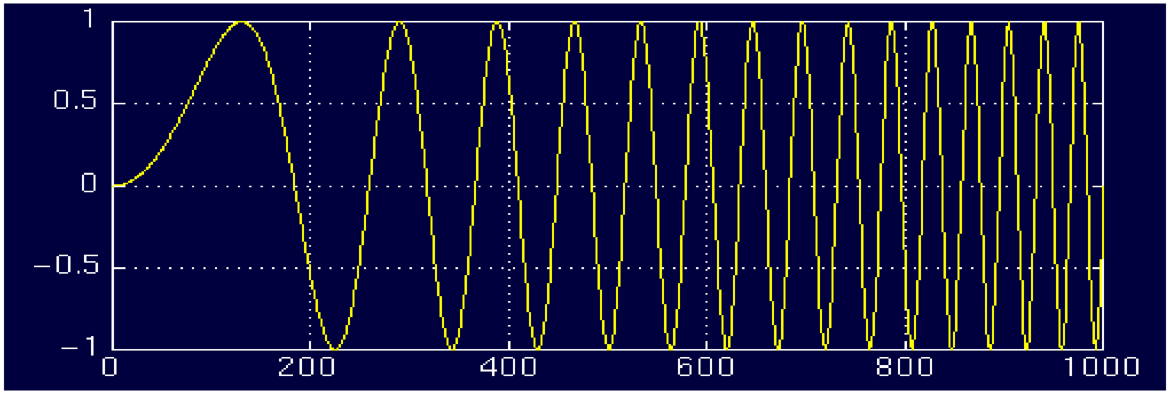
- For example consider the following signal

x(t)=cos(2*pi*10*t)+cos(2*pi*25*t)+cos(2*pi*50*t)+cos(2*pi*100*t)

- Has frequencies 10, 25, 50, and 100 Hz at any given time instant




*stationary signal, FT can provide full info*

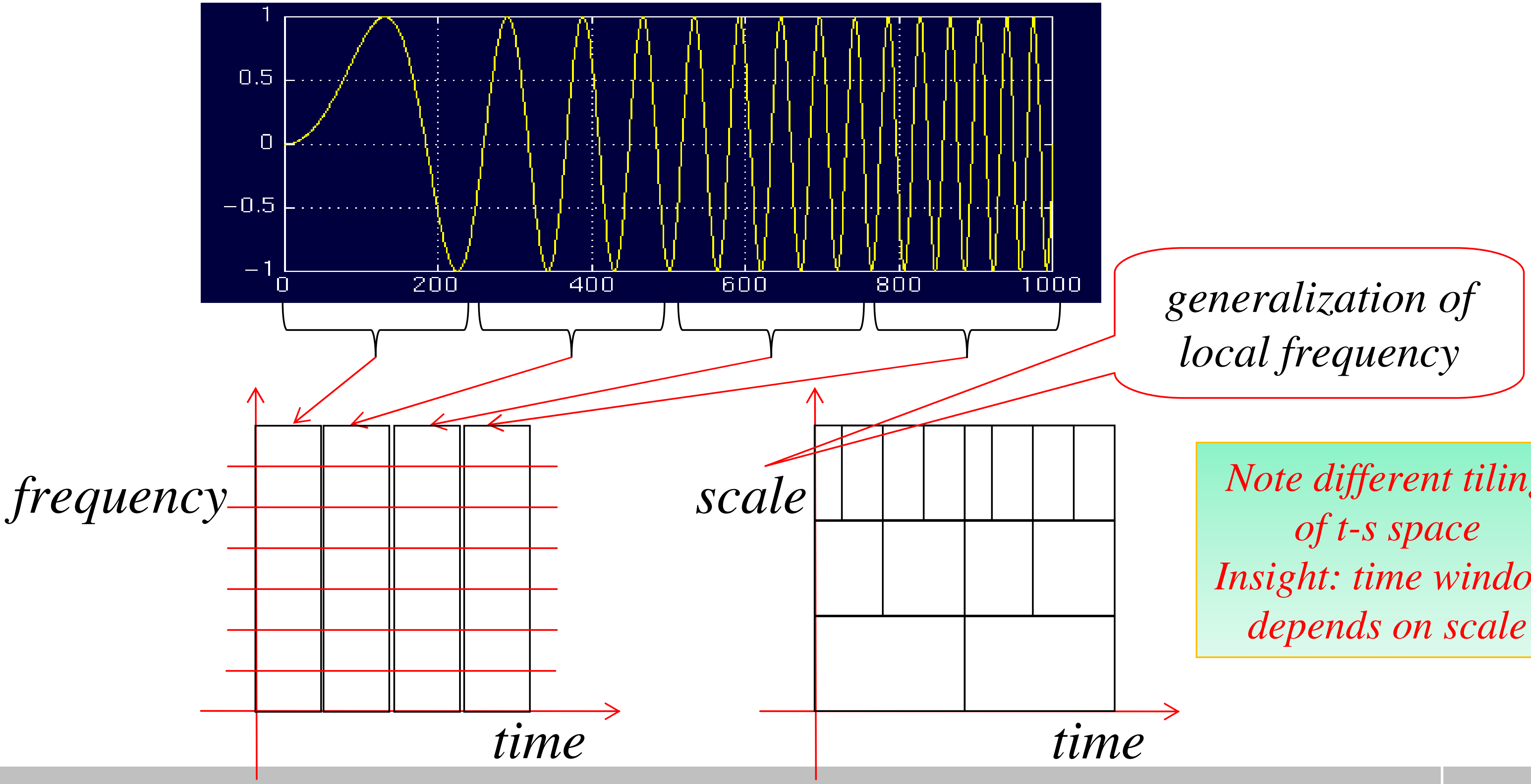



*Non-stationary, frequency content changes with time FT CANNOT provide full info*

[*http://users.rowan.edu/~polikar/WAVELETS/WTpart1.html*](http://users.rowan.edu/~polikar/WAVELETS/WTpart1.html)

# Time-frequency, time-scale analysis

- What we need is a time-frequency analysis
- Do FT in a local time window

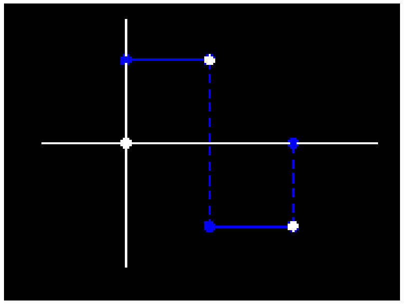



*Note different tiling of t-s space*
*Insight: time window depends on scale*

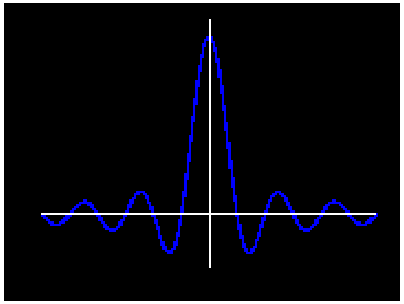
# Basis functions in WT

- Basis functions are called "wavelets"
- Important wavelet property:
- All basis functions are scaled, shifted copies of the same mother wavelet
- By clever construction of mother wavelet, these scaled, shifted copies can be made either orthonormal, or at least linearly independent
- Wavelets form a complete basis, and wavelet transforms are designed to be easily invertible
- Online wavelet tutorial:

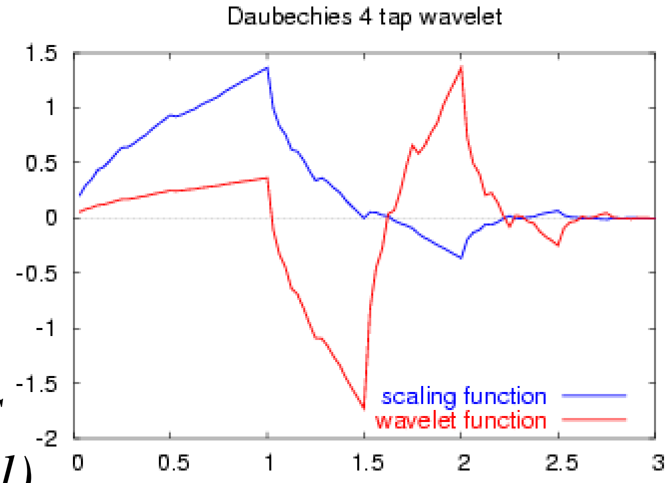
http://cnx.org/content/m10764/latest/



*Haar*



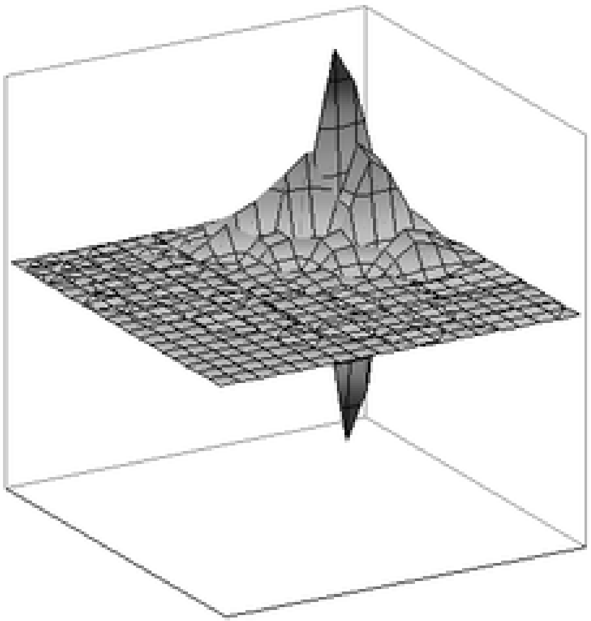
*Mexican Hat*




*Daubechies (orthonormal)*

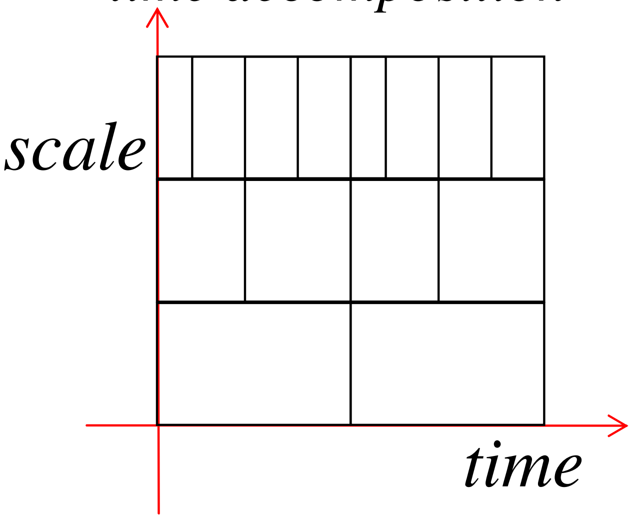
# WT in images

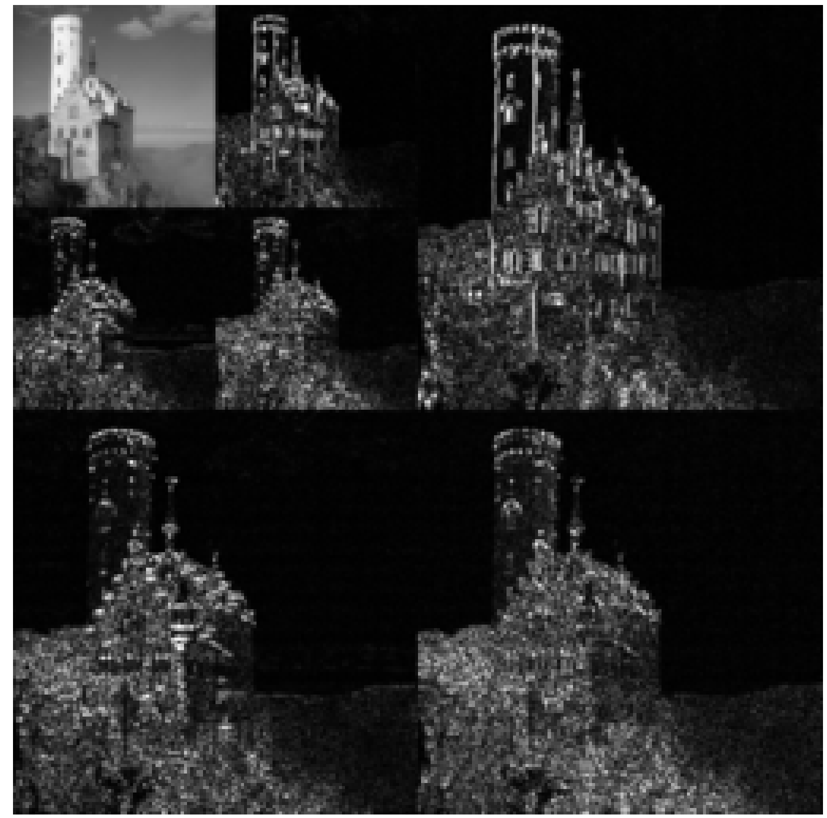
- Images are piecewise smooth or piecewise constant
- Stationarity is even rarer than in 1D signals
- FT even less useful (nnd WT more attractive)
- 2D wavelet transforms are simple extensions of 1D WT, generally performing 1D WT along rows, then columns etc
- Sometimes we use 2D wavelets directly, e.g. orthonormal Daubechies 2D wavelet



# WT on images

*2D generalization of scale-time decomposition*




Successive application of dot product with wavelet of increasing width.

Forms a natural pyramid structure. At each scale:

H = dot product of image rows with wavelet

V = dot product of image rows with wavelet

H-V = dot product of image rows then columns with wavelet

# Wavelet Applications

- Many, many applications!
- Audio, image and video compression
- New JPEG standard includes wavelet compression
- FBI's fingerprints database saved as wavelet-compressed
- Signal denoising, interpolation, image zooming, texture analysis, time-scale feature extraction
- In our context, WT will be used primarily as a feature extraction tool
- Remember, WT is just a change of basis, in order to extract useful information which might otherwise not be easily seen

# WT in MATLAB

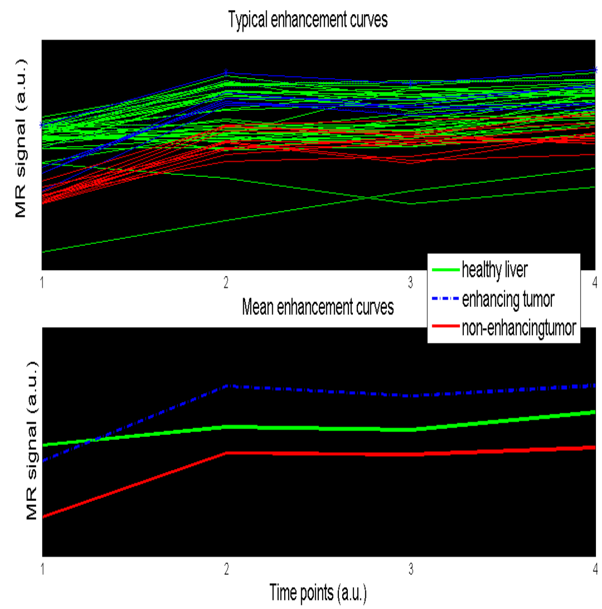
- MATLAB has an extensive wavelet toolbox
- Type help wavelet in MATLAB command window
- Look at their wavelet demo
- Play with Haar, Mexican hat and Daubechies wavelets

# Project Ideas

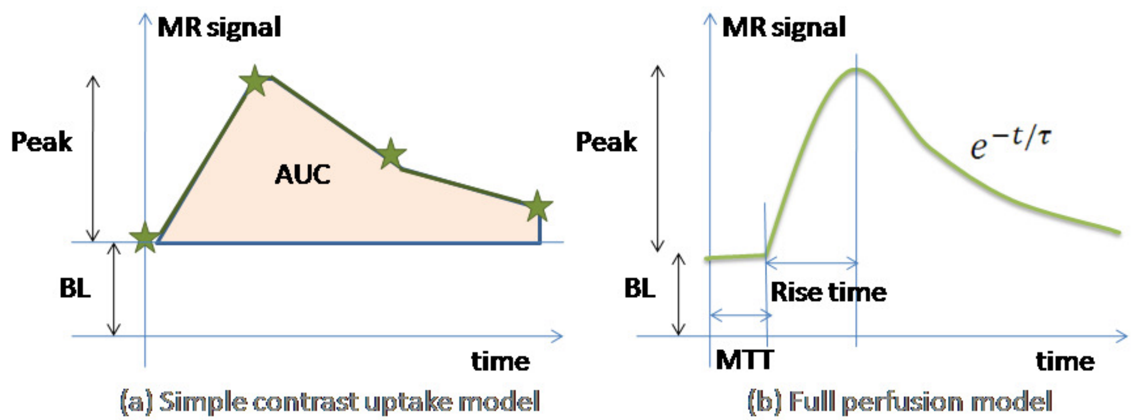
- Idea 1: use WT to extract features from ECG data
  - use these features for classification
- Idea 2: use 2D WT to extract spatio-temporal features from 3D+time MRI data
  - to detect tumors / classify benign vs malignant tumors
- Idea 3: use 2D WT to denoise a given image

# Idea 3: Voxel labeling from contrast-enhanced MRI

- Can segment according to time profile of 3D+time contrast enhanced MR data of liver / mammography



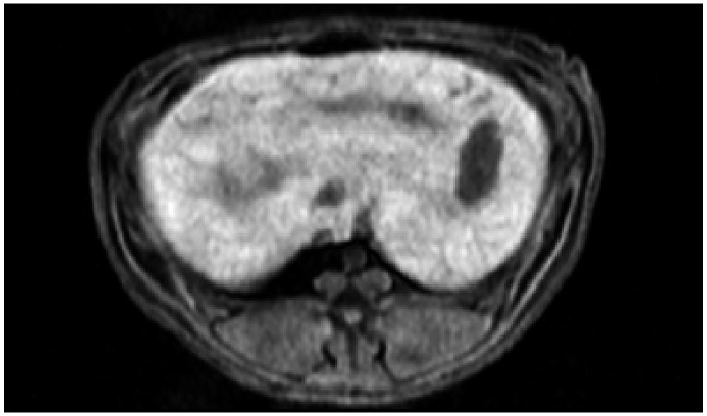

Typical plot of time-resolved MR signal of various tissue classes



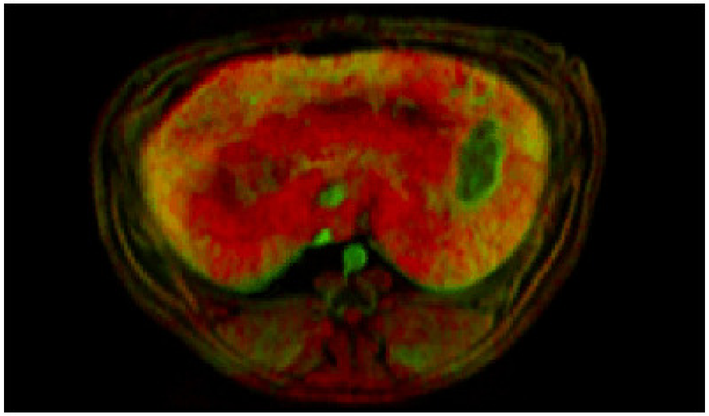

Temporal models used to extract features

Instead of such a simple temporal model, wavelet decomposition could provide spatio-temporal features that you can use for clustering

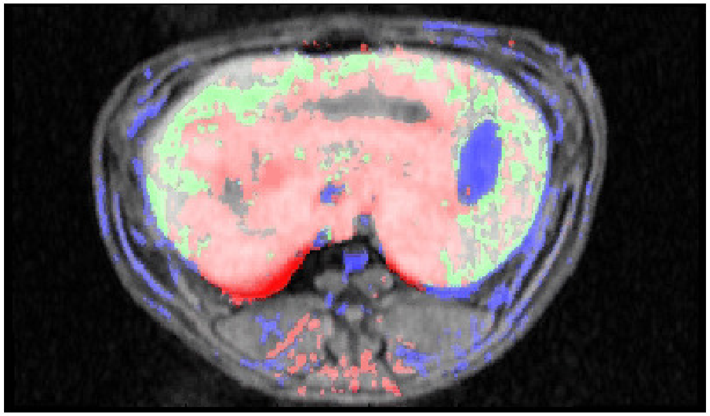
# Liver tumour quantification from DCE-MRI



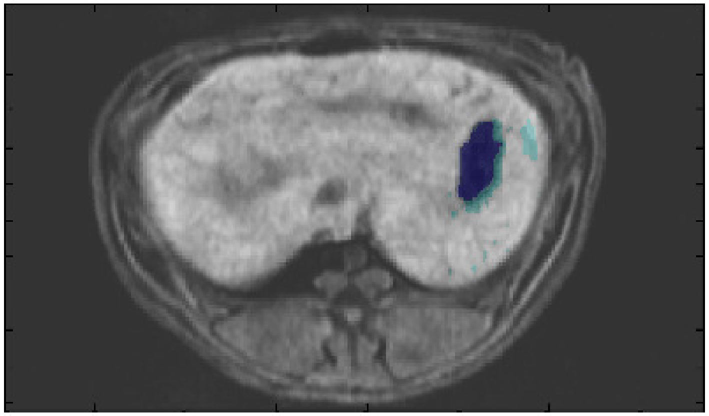
baseline MR image



dynamic parameter map



initial 5-way clustering



final tumor segmentation

# Further Reading on Wavelets

- A Linear Algebra view of wavelet transform

http://www.bearcave.com/misl/misl_tech/wavelets/matrix/index.html

- Wavelet tutorial
- http://users.rowan.edu/~polikar/WAVELETS/WTpart1.html
- http://users.rowan.edu/~polikar/WAVELETS/WTpart2.html
- http://users.rowan.edu/~polikar/WAVELETS/WTpart3.html
- Wavelets application to EKG R wave detection: http://www.ma.utexas.edu/users/davis/reu/ch3/wavelets/wavelets.pdf

# Lecture 5: Transforms, Fourier and Wavelets

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