The Variation of Some Other Speculative Prices
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THE VARIATION OF SOME OTHER SPECULATIVE PRICES

BENOIT MANDELBROT*

I. INTRODUCTION

The present article continues my earlier work, "The Variation of Certain Speculative Prices" (VCSP) [20]. There, it was argued that the description of time series of prices requires probability models less special than the widely used Gaussian, because the price relatives of *certain* price series have a variance so large that it may in practice be assumed infinite.

Section II of the present work restates the theoretical argument of VCSP, with little mathematics, but stress upon the motivation of my generalization of the Gaussian model. I trust that this will (implicitly) show certain responses to VCSP to have been unwarranted.

Because very similar reservations about VCSP were often voiced by different authors, and because I hope that they will be withdrawn and do not want to preserve them through controversy, the text will name neither the friendly nor the unfriendly commentators of VCSP, though many are listed in the Bibliography.

Section III is devoted to additional empirical evidence in favor of my "stable Paretian" model, relative to wheat, railroad securities, and rates of exchange or of interest. Moreover, some unpublished figures concerning cotton prices are incorporated in Section II. Much of the newly published empirical evidence was already quoted in the larger unpublished work from which VCSP is excerpted [20, n. 9].[1] The evidence now available is so extensive that only a small fraction of it can be reported below.

Section IV is a token contribution to the statistical problems raised by VCSP. The statistics of the stable processes has raised many exciting and very new questions and attracts increasing attention. The practical applicability of the findings of VCSP is naturally much dependent upon the development of statistics.

The preparation of this paper having been very slow, it would by itself give an outdated idea of the status of the theory started in VCSP. Much progress has been made since, in references 9, 10, 22, 23, and 24 and in forthcoming papers by Eugene Fama and myself. Reference 24 touches upon a currently active issue, being devoted to various relations existing between, on the one hand, price changes over fixed time intervals (such as days) and, on the other hand, price changes between successive transactions. Though the distribution of the latter changes is necessarily very short tailed (for institutional and other reasons), the number of transactions within a day is sufficiently variable to account for the long-tailedness of the distribution of daily price changes.

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[1] This original of VCSP appeared as an IBM Research Note in March, 1962.

II. THE STABLE PARETIAN MODEL OF PRICE VARIATION

A. BACHELIER'S THEORY OF SPECULATION

Consider a time series of prices, $Z(t)$, and designate by $L(t,T)$ its logarithmic relative

$$L(t, T) = \log_e Z(t, T) - \log_e Z(t).$$

The basic model of price variation, a modification of one proposed in 1900 in Louis Bachelier's theory of speculation [3], assumes that successive increments $L(t,T)$ have the following properties: They are (*a*) random, (*b*) statistically independent, (*c*) identically distributed, and (*d*) their marginal distribution is Gaussian with zero mean. Such a process is called a "stationary Gaussian random walk" or "Brownian motion."

Although his model continues to be extremely important, all four of Bachelier's assumptions are working approximations that should not be made into dogmas. In fact, writing in 1914,[2] Bachelier himself [4] made no mention of his earlier claims of the existence of empirical evidence in favor of Brownian motion, and noted that his original model diverges from the evidence in at least two ways: First of all, the sample variance of $L(t,T)$ varies in time. He attributed this to variability of the population variance, interpreted the sample histograms as being relative to mixtures, and noted that the tails of the histogram could be expected to be fatter than in the Gaussian case. Second, Bachelier noted that no reasonable mixture of Gaussian distributions could account for the size of the very largest price changes, and he treated them as "contaminators" or "outliers." Thus, he pioneered, not only in stating the oft-rediscovered Gaussian random-walk model, but also in exposing its oft-rediscovered major weaknesses.

However, new advances in the theory of speculation are still best expressed as improvements upon his 1900 model: The approach to price variation, proposed in VCSP, shows that an appropriate generalization of hypothesis (*d*) suffices to "save" (*a*), (*b*), and (*c*) in many cases, and in others greatly postpones the need of amending them. I shall comment upon Bachelier's four hypotheses, then come to the argument of VCSP. Readers acquainted with VCSP may proceed immediately to Section III.

B. RANDOMNESS

I have only a few words about the description of price variation by a random process. To say that a price change is random implies, not that it is irrational, but only that it was unpredictable before the fact *and* is describable by the powerful mathematical theory of probability. Therefore, there are two alternatives to randomness, namely, "predictable behavior" and "haphazard behavior," where I use the latter term as meaning "unpredictable and not subject to probability theory." By treating the largest price changes as "outliers," Bachelier implicitly resorted to this concept of "haphazard." This might have been unavoidable in his time, but the power of probability theory has much increased since and should be used to the fullest.

C. INDEPENDENCE

To assume statistical independence of successive $L(t,T)$ is undoubtedly a simplification of reality.[3] The single but

[2] To my shame, I missed this discussion when sampling this book and privately criticized Bachelier for blind reliance on the Gaussian. Luckily, my criticism was not committed to print.

[3] I was surprised to see VCSP criticized for expressing blind belief in independence. For examples

very strong defense of this hypothesis is in the surprising fact that models making this assumption account for many features of price behavior.

Incidentally, independence implies that no investor can use his knowledge of past data to increase his expected profit. But the converse is *not* true. There exist processes in which the expected profit vanishes, but dependence is of extremely long range, and knowledge of the past may be profitable to those investors whose utility function differs from the market's. See, for example, the "martingale" model of reference 22, which is developed and generalized in reference 23 (the latter paper also touches on various aspects of the spectral analysis of economic time series, also an active topic whose relations with my work have aroused interest; for example, when a time series is non-Gaussian, its spectral whiteness, that is, lack of correlation, is compatible with great departures from the random-walk hypothesis).

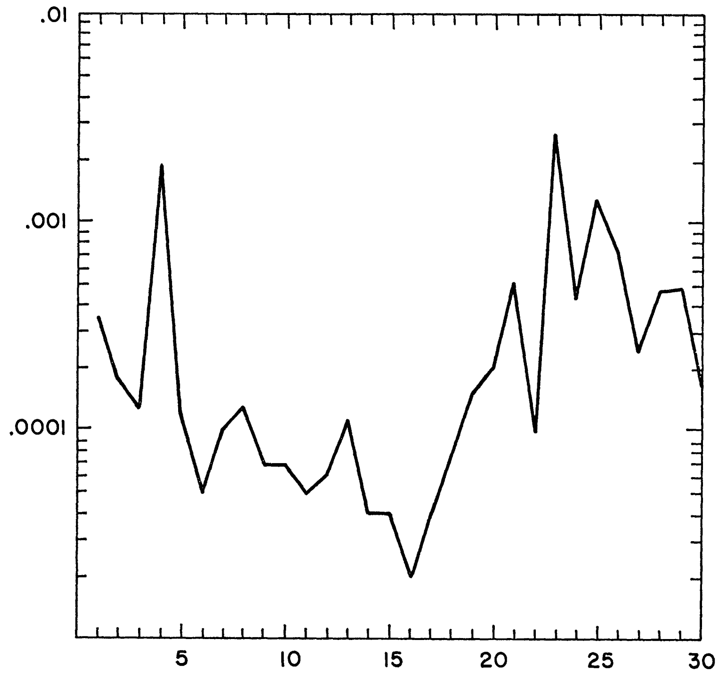

FIG. 1.—Second moment of the daily change of log $Z(t)$, with $Z(t)$ the spot price of cotton. The period 1900–1905 was divided into thirty successive fifty-day samples, and the abscissa designates the number of the sample in chronological order. Logarithmic ordinate. A line joins the sample points to improve legibility.

of reservations on this account, see its Sec. VII as well as the final paragraphs of its Secs. III. E, III. F, and IV. B.

D. STATIONARITY

One implication of stationarity is that sample moments vary little from sample to sample, as long as the sample length is sufficient. In fact, it is notorious that price moments often "misbehave" from this viewpoint (though this fact is understated in the literature, since "negative" results are seldom published; see, however, F. C. Mills [25]).

Figure 1 is an example of the enormous

variability in time of the sample second moment. The points refer to successive fifty-day sample means of $[L(t,1)]^2$ for cotton prices in the period 1900–1905 (recall that $L(t,1)$ is the daily logarithmic price relative). According to Bachelier's model, these sample means should already have stabilized near the population mean. Since no stabilization is in fact observed, we see conclusively that the price of cotton did not follow a Gaussian stationary random walk.

To account for this, it is usual to say that the mechanism of price variation itself changes in time. We shall, loosely speaking, distinguish systematic, random, and haphazard changes of mechanism.

To refer to systematic changes is especially tempting. Indeed, to explain the temporal changes of the statistical parameters of the process of price variation would constitute a worthwhile first step toward an ultimate explanation of price variation itself. An example of systematic change is given by the yearly seasonal effects, which are strong in the case of agricultural commodities. However, in Figure 1 not all ends of season are accompanied by large price changes, and not all large price changes occur at any prescribed time in the growing season.

The most controversial systematic changes are those due to deliberate changes in the policies of the government and of the exchanges. The existence of long-term changes of this type is unquestionable. For example, Section III. D of VCSP concluded that various measures of the scale of $L(t,T)$ (such as the interquartile interval) have varied in the case of cotton prices between 1816 and 1958. One evidence is that lines *1a* and *2a* of Figure 5 in VCSP, relative to the 1900's, clearly differ from lines *1b* and *2b*, relative to the 1950's. This decrease in price variability must, at least in part, be a consequence of the deep changes in economic policy that occurred in the early half of this century. However, precisely because it is so easy to read in the facts a proof of the success or failure of changes in economic policy, the temptation to resort to systematic nonstationarity must be carefully controlled.

An example of "random change" is a random-walk process in which the sizes and probabilities of the steps are chosen by some other process. If this second "master process" is stationary, $Z(t)$ itself is not a random walk but remains a stationary random process.

The final possibility is that the variability of the price mechanism is haphazard, that is, not capable of being treated by probability theory. In practice, it is not very reasonable to resort to the haphazard at this late stage: Indeed, why bother to construct complicated statistical models for the behavior of prices if one expects this behavior to change before the model has had time to unfold? Moreover, and more important, early resort to the haphazard need not be necessary, as is demonstrated by the smoothness and regularity of the graph of Figure 2, which is the histogram of the data of Figure 1.

E. GAUSSIAN HYPOTHESIS

Bachelier's assumption, that the marginal distribution of $L(t,T)$ is Gaussian with vanishing expectation, might be convenient, but virtually every student of the distribution of prices has commented on their leptokurtic (i.e., very long-tailed) character.[4] As mentioned,

[4] For an old but eminent practitioner's opinion, see Mills [25]; for several recent theorists' opinions see Paul Cootner's anthology [5].

Bachelier himself realized this and regarded $L(t, T)$ as a contaminated mixture of Gaussian variables; see reference 24.

F. INFINITE VARIANCE AND THE STABLE PARETIAN DISTRIBUTIONS

Still other approaches were suggested to take account of the failure of Brownian motion to fit data on price variation. A common feature of all these approaches, however, is that each new fact necessitates an addition to the explanation. Since a new set of parameters is thereby added, I don't doubt that reasonable "curve-fitting" is achievable in many cases.

However, this form of "symptomatic medicine" (a new drug for each complaint) could not be the last word! The effectiveness as well as the beauty of science is that it sometimes evolves central assumptions, from which many independently made observations can be shown to be consequences. These observations are thus organized, and fresh facts can be predicted. The ambition of VCSP was to suggest such a central assumption, the infinite-variance hypothesis, and to show that it accounts for substantial features of price series (of various degrees of volatility) without non-stationarity, without mixture, without master processes, without contamination, and with a choice of increasingly accurate assumptions about interdependence of successive price changes.

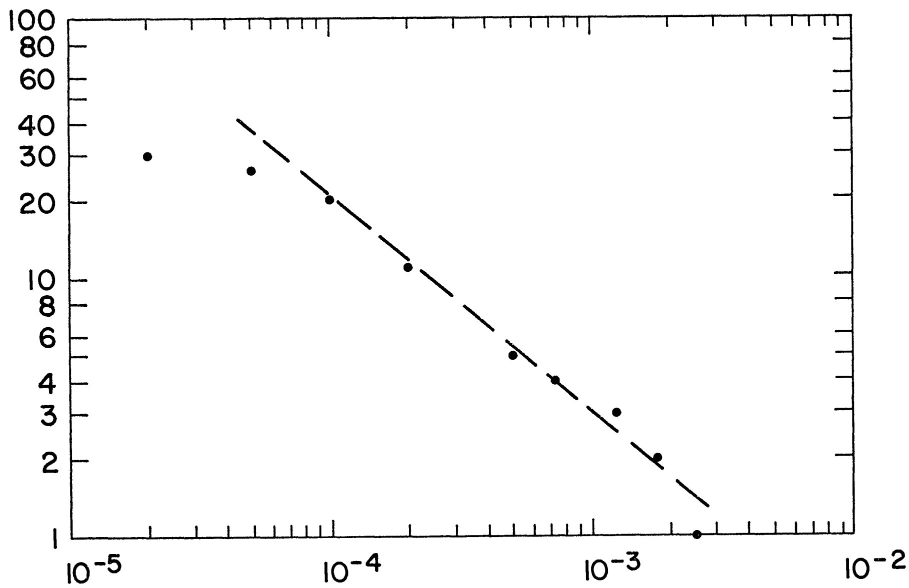

FIG. 2.—Cumulated absolute-frequency distribution for the data of Fig. 1. Abscissa: log of the sample second moment. Ordinate: log of the absolute number of instances where the sample moment marked as abscissa has been exceeded. The stable Paretian model predicts a straight line of slope $\alpha/2 \sim 1.7/2$, which is plotted as a dashed line.

When selecting a family of distributions to implement the infinite-variance hypothesis, one must be led by mathematical convenience (e.g., the existence of a ready-made mathematical theory) and by simplicity. For a probability distribution, one important criterion of simplicity is the variety of the properties of "invariance" that it possesses: For example, it would be most desirable to have the same distribution (up to some—hopefully linear—weighting) apply to daily, monthly, etc., price changes. Another measure of simplicity is the role that a family of distributions plays in central limit theorems of the calculus of probability.

Following these reasons, VCSP proposed to represent the marginal distribution of $L(t,T)$ by an appropriate member of a family of probability laws called "stable,"[5] which measure volatility by a single parameter α[6] ranging between 2 and 0 and whose simplest members are the symmetric probability densities $p_\alpha(u) = (1/\pi)\int_0^\infty \exp(-\gamma s^\alpha) \cos(su)ds$.

Their limit case $\alpha = 2$ is, duly, Gaussian, but my theory also allows non-Gaussian or "stable Paretian"[7] cases $\alpha < 2$, which indeed turn out to represent satisfactorily the data on volatile prices (see VCSP and Sec. III below).

In assessing the realm of applicability of my theory, one should always understand it as including its classical limit. It is therefore impossible to claim that VCSP was "disproved" when one has pointed out price series for which the Gaussian hypothesis may be tenable.

Now to discuss the main fact, that stable variables with $\alpha < 2$ have an infinite[8] population variance (one says sometimes that they have "no variance"). Concern was expressed at the implication of this feature for statistics, and surprise was expressed at the paradoxically discontinuous change that seems to occur when α becomes exactly 2.

This impression of paradox is unfounded. The population variance itself cannot be measured, and every measurable characteristic of a stable distribution behaves continuously near $\alpha = 2$. We shall present an example later on. Consequently, there is no "black and white" contrast between the Paretian case $\alpha < 2$ and the Gaussian $\alpha = 2$, but a continuous shading of gray,[9] which is less desperate but more sensible and more interesting. The fact that the population second moment is discontinuous at $\alpha = 2$ "only" shows that it is not well suited to a study of price variation.[10]

In particular, the applicability of second-order statistical methods is questionable. "Questionable" does not mean "totally inapplicable," because the statistical methods based upon variances suffer no sudden and catastrophic breakdown as α ceases to equal 2. Therefore, to be unduly concerned with a few specks of "gray" in a price series whose α is near 2 may be as inadvisable as to treat very gray series as white. Moreover, statistics would be unduly restricted if its tools were to be used only where justified. (As a matter of fact, the

[5] Throughout the long-awaited second volume of William Feller's "Introduction to Probability" [11], one finds a wealth of facts concerning these laws. However, B. V. Gnedenko and A. N. Kolmogoroff's monograph [12] remains the only up-to-date book discussing these laws in a single chapter. For a compact briefer treatment, see J. Lamperti [15].

[6] α is related to "Pareto's exponent," but it would be extremely dangerous to underestimate the differences between the two concepts [19].

[7] I first proposed for these cases the term of "Pareto-Lévy laws," then tried to withdraw it. I am now resigned to consider "stable Paretian" and "Pareto-Lévy" as synonymous.

[8] I may at this point reassure those who expressed in print the fear that I find $E[L^2]$ to be infinite because I inadvertently took the logarithm of zero.

[9] I do not propose this colorful metaphor as a scientific terminology!

[10] To my knowledge, $\alpha > 1$ for prices, and the first moment is well suited to the study of log $Z(t)$. ($Z(t)$ itself is another matter.) However, the stable laws with $\alpha < 1$ play a central role in economics [21, 23].

quality of a statistical method is partly assessed by its "robustness," i.e., the quality of its performance when used without justification.) However, one should look for other methods. For example, as predicted, least-squares forecasting (as applied to past data) would have often led to very poor inferences; least-sums-of-absolute-deviations forecasting is always at least as good and usually much superior, and its development should be pressed.

This section will end by two remarks concerning second and fourth moments, respectively.

The behavior of sample second moments.—Define $V(\alpha,N)$ as the variance of a sample of N independent random variables $u_1, \ldots u_n, \ldots u_N$, whose common distribution is stable of exponent α. To obtain a balanced view of the practical properties of such variables, it is best *not* to focus upon mathematical expectations and/or infinite sample sizes. One should rather consider quantiles and samples of large but finite size. Let us therefore select a "finite horizon" by choosing a value of N and a quantile threshold q (such that events whose probability is below q will be considered to be "unlikely"). Save for extreme cases contributing to a "tail" of probability q, the values of $V(\alpha,N)$ will be less than some function $V(\alpha,N,q)$ whose behavior summarizes much of what needs to be known about the sample variance.

As mentioned earlier, when N is finite and $q > 0$, the function $V(\alpha,N,q)$ varies smoothly with α. For example, over a wide range of values of N, the derivative of $V(\alpha,N,q)$ at $\alpha = 2$ is very close to zero, so that $V(\alpha,N,q)$ changes very little if $\alpha = 2.00$ is replaced by, say, $\alpha = 1.99$. This insensitivity is due to the fact that $\alpha = 2.00$ and $\alpha = 1.99$ only differ in the sizes that they predict for some outliers, constituting a small proportion of all cases, whose effects were excluded by the definition of $V(\alpha,N,q)$. By increasing N, or by decreasing q, one decreases the range of exponents in which α is approximable by 2.

If one really objects to infinite variance, while being only concerned with meaningful finite-sample problems, one may "truncate" U so as to attribute to its variance a very large finite value depending upon α, N, and q. The resulting theory may have the asset of familiarity, but the specification of the value of the truncated variance will be *useless* because it will tell nothing about the "transient" behavior of $V(\alpha,N,q)$ when N is finite and small. Thus, even when one knows the variance to be finite but very large (as is the case in certain of my more detailed models of price variation; see [23]), the study of the behavior of $V(\alpha,N,q)$ is much simplified if one approximates the distribution with finite but very large variance by a distribution with infinite variance. This feature can be illustrated by the following homely example: It is well known that photography is simplest when the object is at an infinite distance from the camera. Therefore, even if the actual distance is known to be finite, the photographer ought to set the distance at infinity if that distance exceeds some finite threshold, depending upon the quality of the lens and its aperture.

The behavior of sample kurtosis.—Pearson's kurtosis, a measure of the peakedness of a distribution defined by $E(U^4)[E(U^2)]^{-2} - 3$, was discussed in VCSP. But the discussion lacked numerical illustration and was called obscure, and additional detail may be useful. If $E(U^2) = \infty$, the value of the kurtosis is undeterminate. One can, however, show that, as $N \rightarrow \infty$, and if U is stable with $\alpha < 2$, the random variable $\Sigma_{n=1}^{N} U_n^4 [\Sigma_{n=1}^{N} U_n^2]^{-2}$ tends toward a limit that is different from zero. Therefore, the "expected sample kurtosis," defined as

$$E\left\{N \sum_{n=1}^{N} U_n^4 \left[\sum_{n=1}^{N} U_n^2\right]^2 - 3\right\},$$

is asymptotically proportional to N.

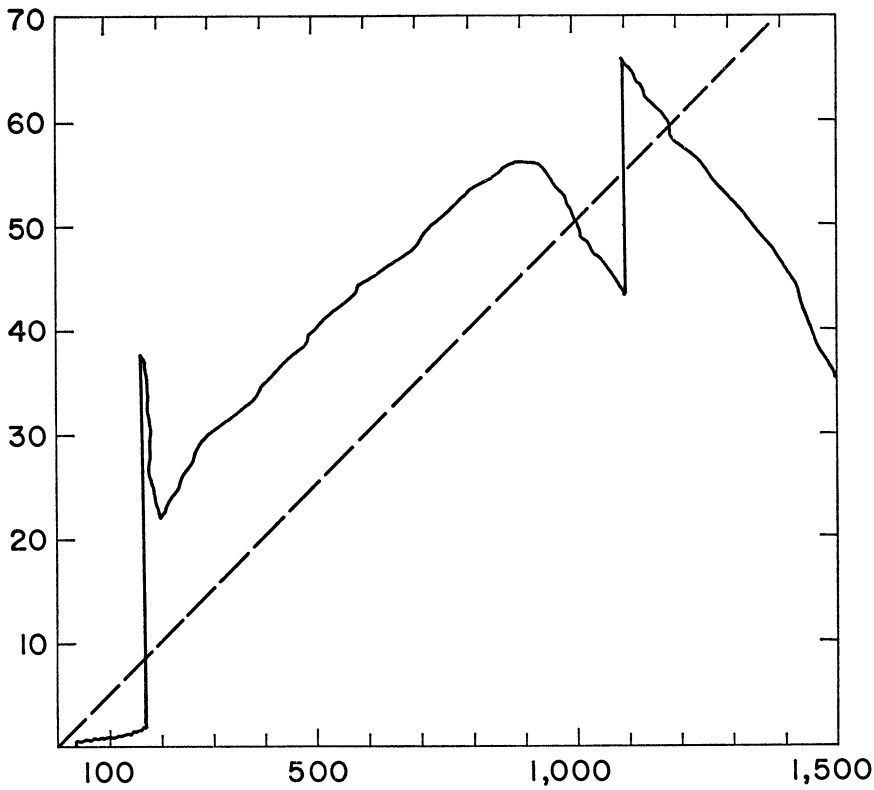

FIG. 3.—Sequential variation of the kurtosis of the daily changes of log $Z(t)$, with $Z(t)$ the spot price of cotton, 1900–1905. The abscissa is the sample size. Linear coordinates.

The kurtosis of $L(t,1)$ was plotted on Figure 3 for the case of cotton, 1900–1905, and is indeed seen to increase steeply with N. Though exact comparison is impossible because the theoretical distribution is not yet tabulated, this kurtosis indeed fluctuates around a line expressing proportionality to sample size. (For samples less than fifty, the kurtosis was negative.)

G. THREE APPROXIMATIONS TO A STABLE DISTRIBUTION: IMPLICATIONS FOR STATISTICS AND FOR THE DESCRIPTION OF THE BEHAVIOR OF PRICES

It is important that there should exist a single theory of prices that subsumes various degrees of volatility. My theory is unfortunately hard to handle, while simple approximations are available in different ranges of values of α. Thus, given a practical problem with a finite time horizon N, it is best to replace the continuous range of degrees of "grayness" by the following trichotomy (where the boundaries between the categories are dependent upon the problem in question).

The Gaussian $\alpha = 2$ is best known and simplest. Here, not having to worry about a long-tailed marginal distribution, one stands a reasonable chance of rapid progress in the study of dependence. For example, one can use spectral methods and other covariance-oriented approaches. In the immediate neighborhood of the Gaussian, Gaussian techniques cannot lead one too far astray.

In the zone far away from $\alpha = 2$, another kind of simplicity reigns. Substantial tails of the stable Paretian law are approximable by Pareto's hyperbolic law, with the same α-exponent ruling both tails. The prime example of this zone was provided by the cotton prices studied in VCSP. In the present paper, we shall examine some other price series of similarly high volatility: the prices of some nineteenth-century rail securities and some exchange and interest rates.

The third and final zone constitutes a transition between the almost Gaussian and the highly Paretian cases and is far more complicated than either. It also provides a test of the meaningfulness and generality of the stable Paretian hypothesis: If that hypothesis holds, the histogram of price changes is expected to plot on bilogarithmic paper as one of a specific family of inverse-S-shaped curves. (Lévy's α-exponent, therefore, is not to be confused with a "Pareto" slope for a straight bilogarithmic plot [18].) If the stable Paretian hypothesis failed, the transition between the almost Gaussian and the highly Paretian cases would be performed in some other way. We shall examine in this light the variation of wheat prices, and we shall find it to conform fully to the stable Paretian prediction concerning the "light gray" zone of low but positive values for $2 - \alpha$ and medium volatility. Section III. A below, where wheat data are examined, is thus seen to have a purpose similar to Fama's 1964 Chicago thesis [9], which was the first to test further the ideas of VCSP. To minimize "volatility" and maximize the contrast with my original data, Fama chose thirty stocks of large and diversified contemporary corporations and found their stable Paretian "grayness" to be unquestionable although less marked than that of cotton.

III. ADDITIONAL DATA CONCERNING LOGARITHMIC RELATIVES OF PRICES

A. THE VARIATION OF THE PRICE OF WHEAT IN CHICAGO, 1883–1936

Introduction.—Contrary to the spot prices of cotton, which refer to standardized qualities, wheat cash prices refer to the variable grades of grain. Hence, at any given time (say, at closing time), one can at best speak of a *span* of cash prices, and the closing spans corresponding to successive days very often overlap. As a result, the week is probably the shortest period for which one can reasonably express "the" change of wheat price by a single number rather than by an interval. In any event, the week was chosen in the present work because it is used in H. Working's classic monograph [26].[11]

The stable Paretian hypothesis for wheat.—It had been suggested [14] that wheat price relatives follow a Gaussian distribution. Indeed, a casual visual inspection of the histograms of these relatives, as plotted on *natural* coordinates, shows them to be nicely "bell shaped." The importance of the "tails" is, however, notoriously underestimated by plotting the data on natural coordinates. It is, on the contrary, stressed by using probability paper. As seen in Figure 4, probability-paper plots of

[11] It follows that, despite the length of the 1883–1936 record, the number of items in the series of wheat prices is not as large as one might have hoped —although it is naturally very long by the standards of economics.

wheat price relatives are definitely S-shaped (though less so than the plots relative to cotton). As the Gaussian corresponds to $\alpha = 2$, and as I found the value $\alpha = 1.7$ for cotton, it is natural to investigate whether wheat is stable Paretian with an α contained between 1.7 and 2.

The evidence of doubly logarithmic graphs.—As seen in Figure 5, reproduced from VCSP, this stable Paretian hypothesis would imply that every doubly logarithmic plot of a histogram of wheat price changes should have a characteristic S-shape. It would end with a "Paretian" straight line of slope near 2, but it would start with a region where the local slope increases with u and even begins by exceeding markedly its asymptotic value [19].

The above conjecture is indeed verified, as seen in Figures 6 and 7. Moreover, by comparing the data relative to successive subsamples of the period

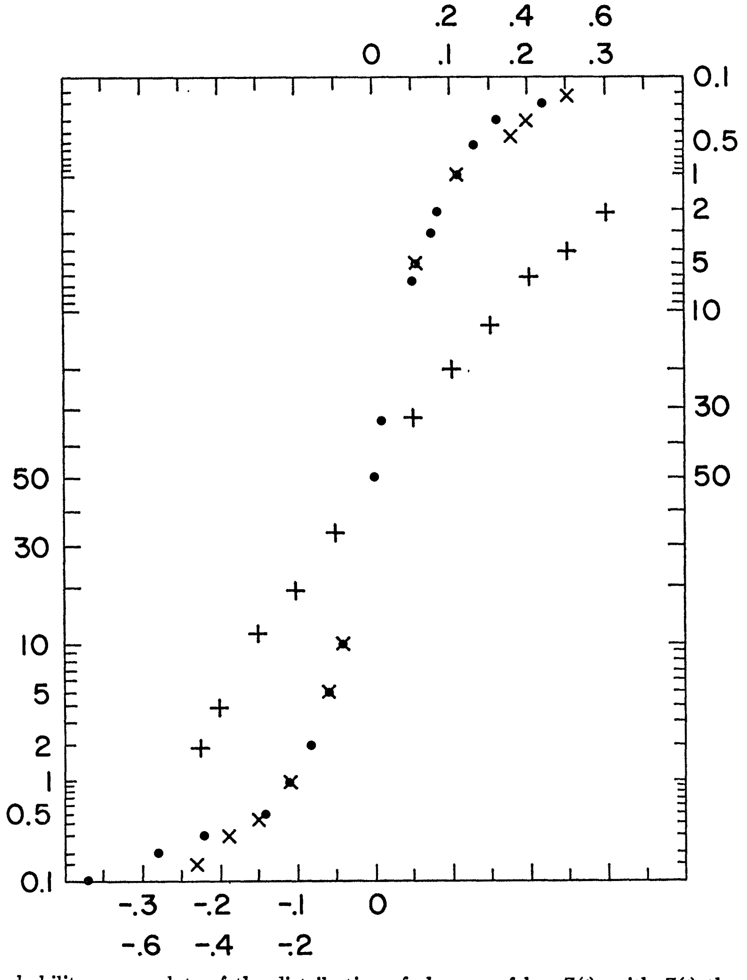

FIG. 4.—Probability-paper plots of the distribution of changes of log $Z(t)$, with $Z(t)$ the spot price of wheat in Chicago, 1883–1934, as reported by Holbrook Working. The scale −0.3, −0.2, −0.1, 0, 0.1, 0.2, 0.3 applies to the weekly changes, marked by dots, and to the yearly charges, marked by crosses. The other scale applies to changes over lunar months, marked by ×.

1883–1936, I found no evidence that the *law* of price variation has changed in kind, despite the erratic behavior of the outliers.

This is evidence that the stable Paretian hypothesis has *predicted* how a price histogram "should" behave, when it was only known to be intermediate between the highly erratic cotton series and the minimally erratic Gaussian limit.

To establish the "goodness of fit" of such an S-shaped graph would unfortunately require an even larger sample of data than in the case of the straight graphs characteristic of cotton, while we know the available sample sizes to be rather smaller. Thus the doubly loga-

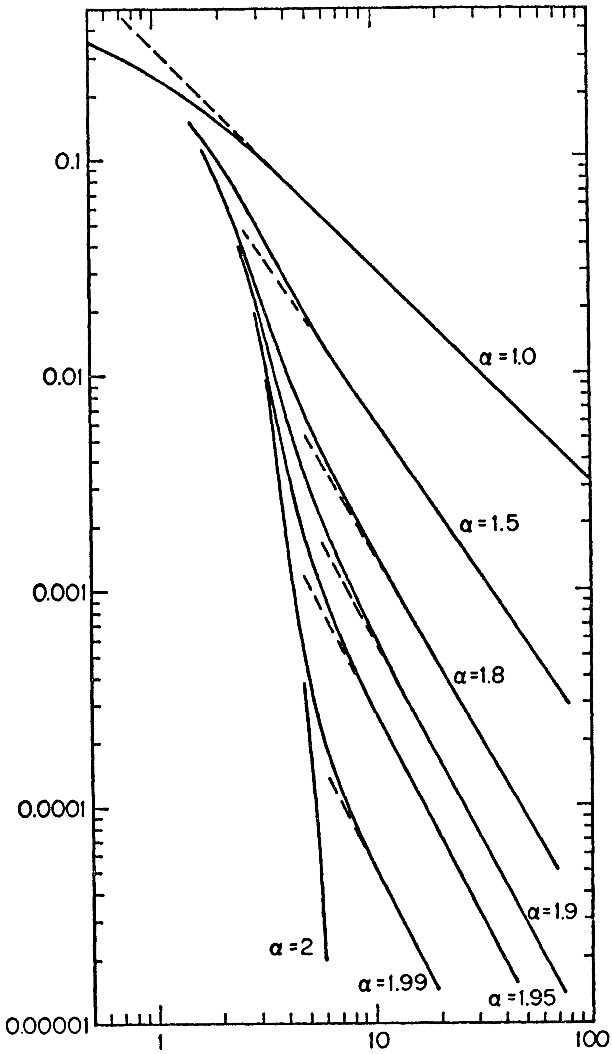

FIG. 5.—Theoretical cumulated probability distribution for the stable Paretian random variables, whose probability density is $\rho_\alpha(u)$ with $\gamma = 1$ (see text). Abscissa: log u; ordinate: log $Pr(U \geq u) = \log Pr(U \leq -u)$. Source: VCSP.

rithmic evidence is *unavoidably* less clear cut than in the case of cotton.

The evidence of sequential variance.—When a series of prices is approximately stationary, a test of whether $\alpha = 2$ or $\alpha < 2$ is provided by the behavior of the sequential sample second moment. If $\alpha < 2$, the median of the distribution of the sample variance increases as $N^{-1+2/\alpha}$ for "large" N, while it tends to a limit when $\alpha = 2$. *More important*, the variation of the sample variance (about its median value) becomes increasingly erratic as α departs from 2. Thus, the cotton second moment increases very erratically, but the wheat second moment should increase more slowly and more regularly. Figure 8 shows that such is indeed the case.

Direct test of the stability.—The term "stable" arose from the fact that, when N "stable" random variables U_n are independent and identically distributed, one has

$$Pr\left[N^{-1/\alpha}\sum_{n=1}^{N} U_n \geq u\right] = Pr[U_n \geq u].$$

The stability of Gaussian variables ($\alpha = 2$) is well known and often used in elementary statistics.

I settled on $N = 4$. When the ran-

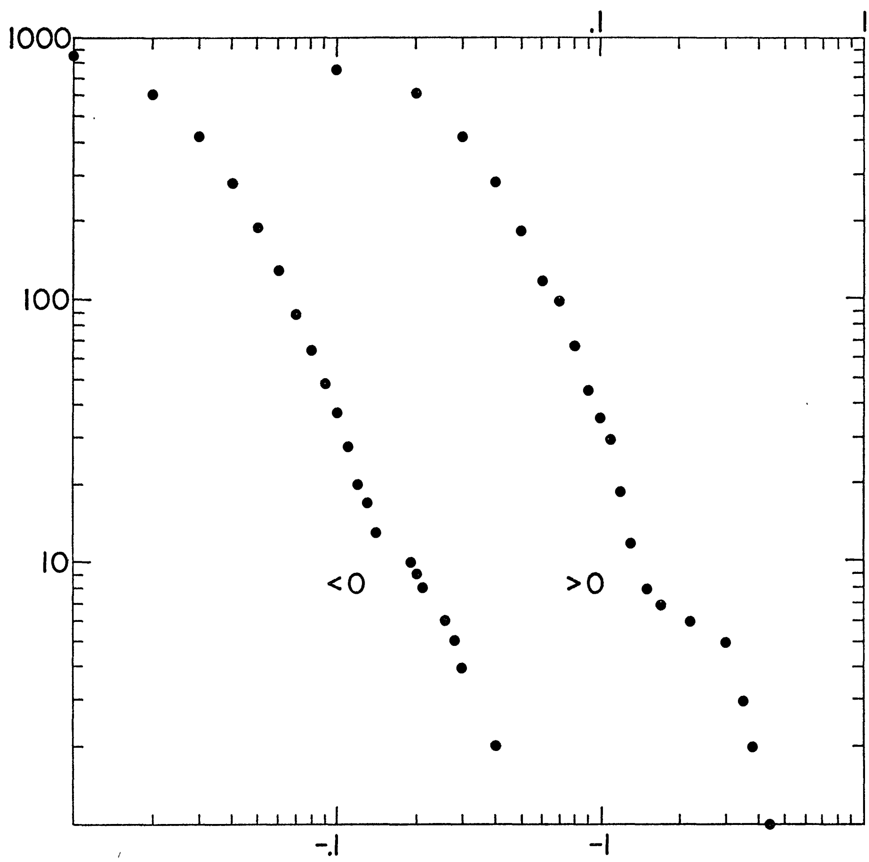

FIG. 6.—Weekly changes of log $Z(t)$, with $Z(t)$ the price of wheat as reported by Working. Ordinate: log of the absolute frequency with which $L \geq u$, respectively $L \leq -u$. Abscissas: the lower scale refers to negative changes, the upper scale to positive changes.

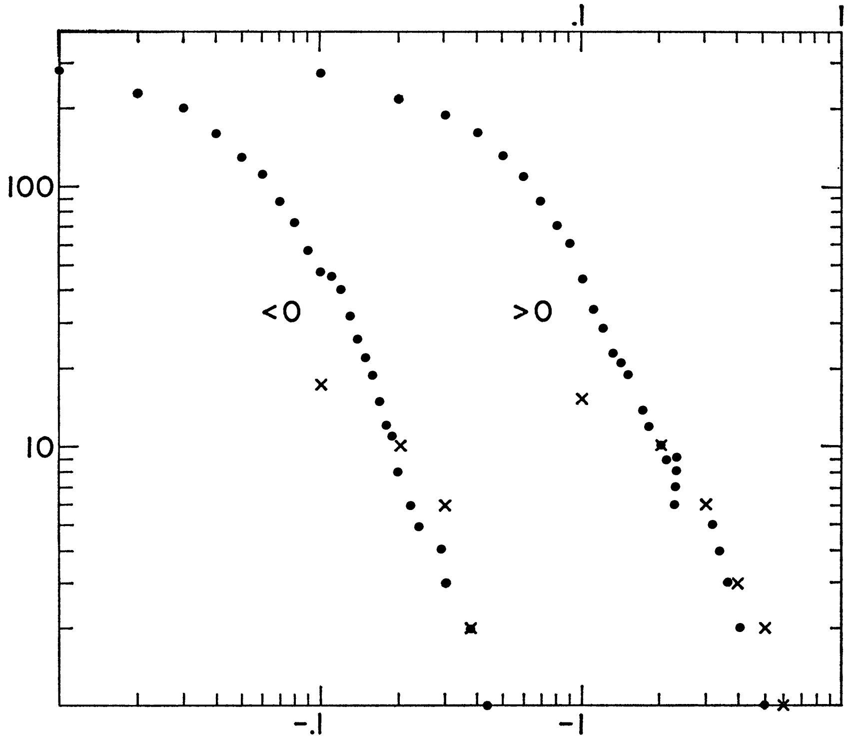

FIG. 7.—Monthly (lunar months) and yearly changes of log $Z(t)$, with $Z(t)$ the price of wheat as reported by Working. Abscissas and ordinates as in Fig. 6.

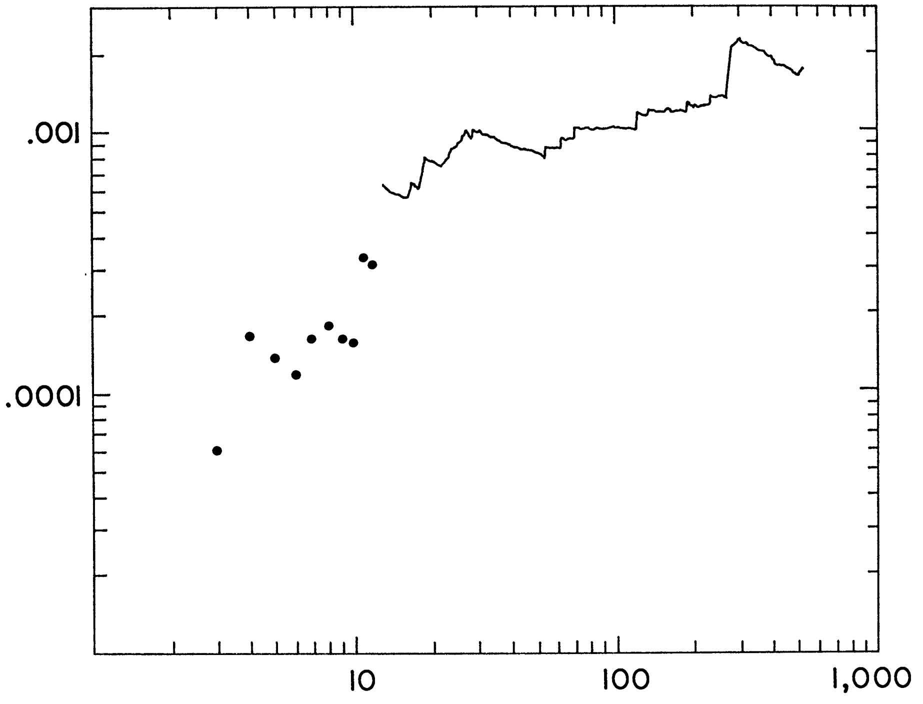

FIG. 8.—Sequential variation of the second moment of the weekly changes of log $Z(t)$, with $Z(t)$ the price of wheat as reported by Working. One thousand weeks beginning in 1896. Bilogarithmic coordintaes.For small values of the sample size, the sample second moments are plotted separately; later on, they are replaced (for the sake of legibility) by a freely drawn continuous line.

dom variables U_n are the weekly price changes, $\Sigma_{n=1}^{4} U_n$ is the price change over a "lunar month" of four weeks. Since α is expected to be near 2, $4^{-1/\alpha}$ will be near $\frac{1}{2}$.

One can see in Figure 4 that weekly price changes indeed have an S-shaped distribution *undistinguishable* from that of one-half of monthly changes.[12] (The bulk of the graph, corresponding to the central bell containing 80% of the cases, was not plotted for the sake of legibility.) Sampling fluctuations are apparent only at the extreme tails and do not appear systematic.

The combination of Figures 6 and 7 provides another test of stability. They were plotted with absolute, not relative, frequency as the ordinate, and the stable Paretian theory predicts that such curves should be superposable in their tails, except, or course, for sampling fluctuations. (Roughly, the reason for this prediction is that, in a stable Paretian universe, a large monthly price change is of the same order of magnitude as the largest among the four weekly changes adding to this monthly change.) Clearly, wheat data pass this second test also.

It should be stressed that the two tests, though using the same data, are *distinct* conceptually: Figure 4 compares one-half of a monthly change to the weekly change of the same frequency; Figures 6 and 7 compare monthly and weekly changes of the same size. Stability is thus doubly striking.

The evidence of yearly price changes.—Working [26] also published a table of average January prices of wheat, and Figure 4 also includes the corresponding changes of log $Z(t)$.

Assuming that successive weekly price changes are independent, the evidence of the yearly changes again favors the stable Paretian hypothesis. It is astonishing that the hypothesis of independence of weekly changes can be consistently carried so far, showing no discernible discontinuity between long-term adjustments to follow supply and demand, which would be the subject matter of economics, and the short-term fluctuations that some economists discuss as "mere effects of speculation."

B. THE VARIATION OF THE PRICES OF RAILROAD STOCKS, 1857–1936

Railroad stocks were pre-eminent among corporation securities that played for nineteenth-century speculators a role comparable to that of the basic commodities. Unfortunately, the convenient book of F. R. Macaulay [17] reports them incompletely: (1) For each of the major stocks, it gives the mean of the highest and lowest quotation during the months of January; (2) for each month, it gives a weighted index of the highest and lowest quotation of every stock.

I began by examining the second series, even though it is averaged too many times for comfort. If one considers that there "should" have been no difference in kind between various nineteenth-century speculations, one would expect railroad stock changes to be stable Paretian, and averaging would bring an increase in the slope of the corresponding doubly logarithmic graphs, similar to what has been observed in the case of cotton price averages (Sec. III. E of VCSP). Indeed, Figure 9, relative to the variation of the monthly averages, yields precisely what one expects for such averages from Paretian processes with an exponent very close to that of cotton.

On yearly data, on the contrary, aver-

[12] The same method was applied by Fama [9] to common-stock price changes, and he also found that it is a favorable test of their stability.

aging has little effect. Figure 10 should be regarded as made of two parts, the first five graphs being relative to companies with less-than-average merger activity, the other to companies with above-the-average merger history.

The first five graphs, in my opinion, are a striking confirmation of the ideas suggested by speculation on cotton. Basically, one sees that the fluctuations of the price of these stocks were all stable Paretian, with the same α-exponent (clearly below the critical value 2). (Moreover, they all had practically the same value of the positive and negative "standard deviations" σ' and σ'', defined in VCSP.)

For the companies with an unusual amount of merger activity, the evidence is similar but more erratic.

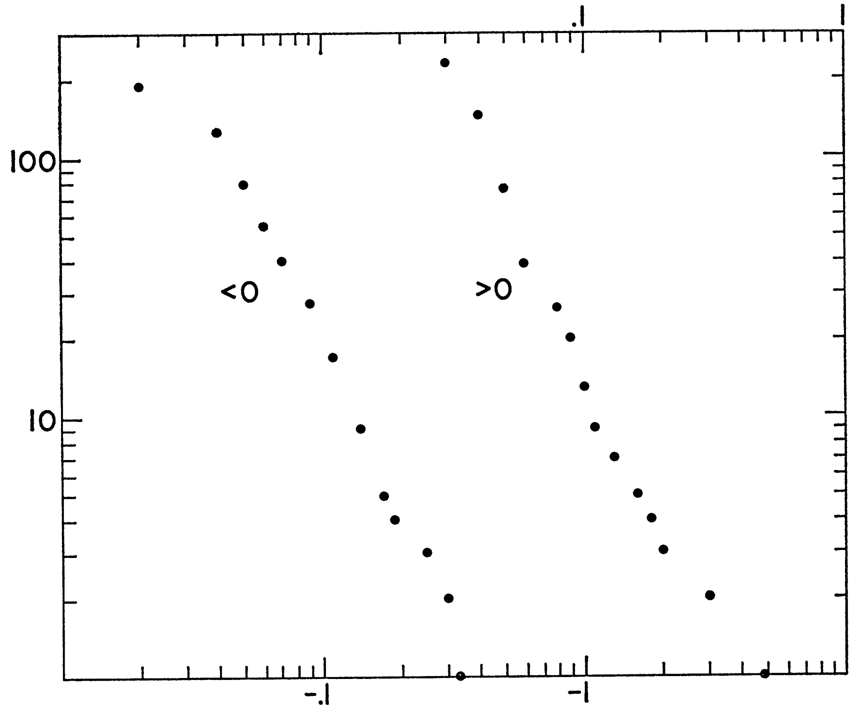

FIG. 9.—Monthly changes of log $Z(t)$, with $Z(t)$ Macaulay's index of rail stock prices. Abscissas and ordinates as in Fig. 6.

C. THE VARIATION OF INTEREST AND EXCHANGE RATES

Introduction.—Various rates of money—and especially the rate of call money in its heyday—are reflections of the overall state of a speculative market. One would therefore expect to find that the behavior of speculative prices and of speculative rates present strong similarities. But one cannot expect them to be ruled by identical processes. For example, one cannot assume (even as rough approximations) that successive changes of a money rate are statistically independent: Such rates would indeed eventually blow up to infinity, or they would vanish. Neither behavior is admissible. As a result, the distribution of $Z(t)$ itself, which is meaningless when Z is a commodity price, is meaningful when it

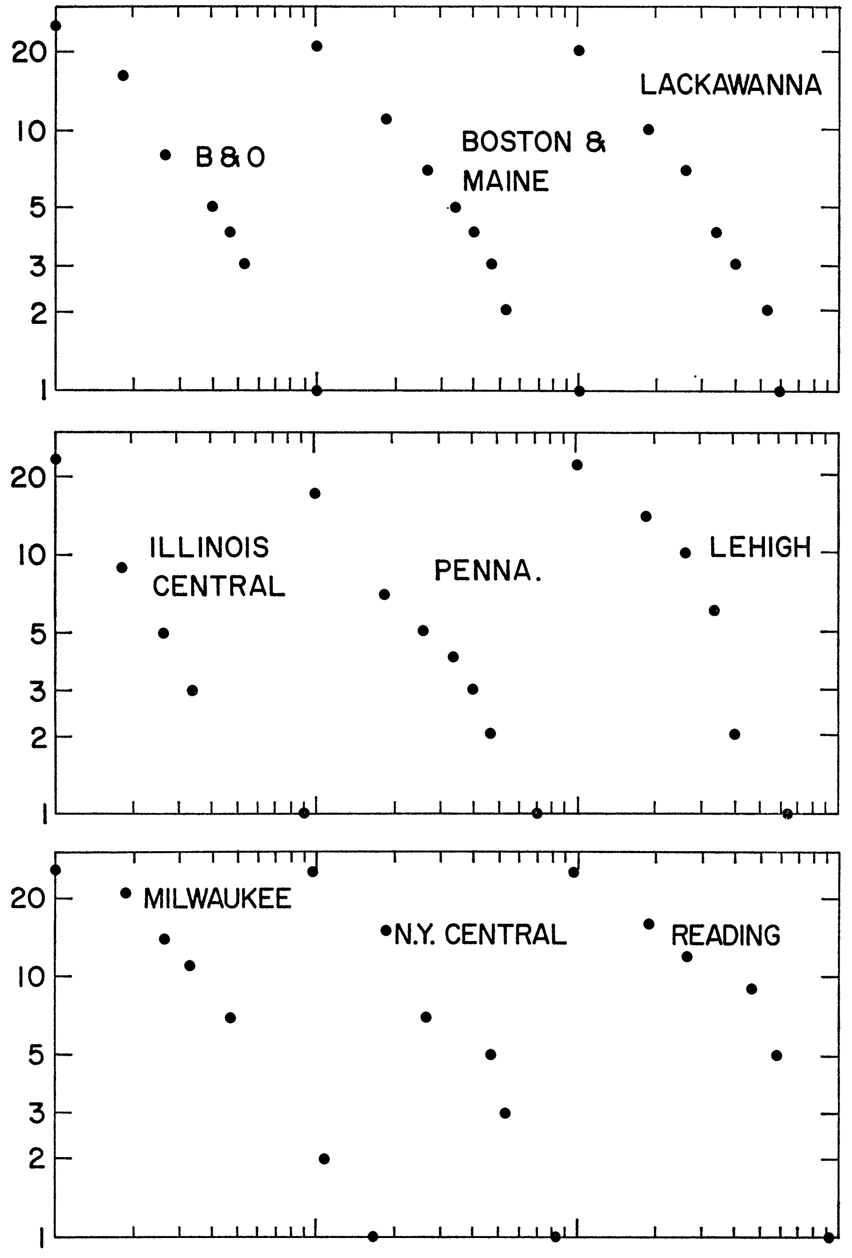

FIG. 10.—Yearly changes of log $Z(t)$, with $Z(t)$ Macaulay's January index of the price of nine selected rail stocks. Ordinates: absolute frequencies. Abscissas are not marked to avoid confusion: for each graph, they vary from 1 to 10.

is a money rate. Moreover, when investigating changes, one will study $Z(t + T) - Z(t)$ rather than $\log Z(t + T) - \log Z(t)$.

The rate of interest on call money.—In Figure 11, the abscissa is the excess of Macaulay's rate of call money [16] over its "typical" value, 6 per cent. I have not even attempted to plot the distribution of the other tail of the difference "rate minus 6 per cent," since that expression is by definition very short tailed, being bounded by 6 per cent, while the positive value of "rate minus 6 per cent" can go sky high (and occasionally did).

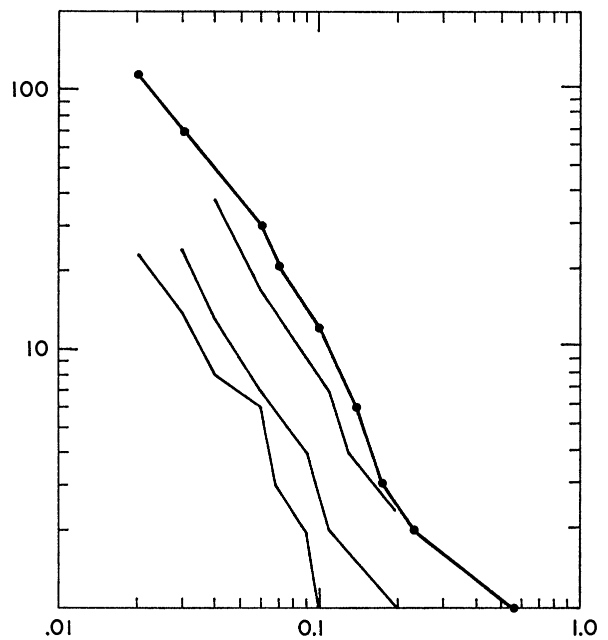

FIG. 11.—The distribution of the excess over 6 per cent of Macaulay's [16] monthly average of call money rates. Ordinate: absolute frequencies. *Bold line:* total sample 1857–1936. *Thin lines,* read from left to right: subsamples 1877–97, 1898–1936, 1857–76. Note that the second subsample is twice as long as the other two. Thus, the general shape of the curves has not changed except for scale, and the scale has steadily decreased in time.

The several lines of Figure 11 correspond, respectively, to the total period 1857–1936 and to three subperiods. They show that call money rates satisfy a single-tailed Paretian law, with an exponent markedly smaller than 2. Scale factors (such as the upper quartile) have changed—a form of non-stationarity—but the exponent α seems to have preserved a constant value, lying within the range in which the law of Pareto is known to be invariant under mixing of data from populations have the same α and different γ [18].

Other rates of interest.—Examine next the distribution of the classic data collected by Erastus B. Bigelow (Fig. 12, dashed line), relative to "street rates of first class paper in Boston" (and New York) at the *end* of each month from January 1836 to December 1860 (Bigelow also reports some rates applicable at the beginnings or middles of the same months, but I disregarded them to avoid the difficulties due to averaging). The dots on Figure 12 again represent the difference between Bigelow's rates and the typical 6 per cent; their behavior is what we would expect if essentially the same stable Paretian law applied to these rates and those of call money.

Examine, finally, a short sample of rates, reported by L. E. Davis [6], on the basis of the records of New England textile mills. These rates remained much closer to 6 per cent than those of Bigelow and were plotted in such a way that the crosses of Figure 12 represent ten times their excess over 6 per cent. The sample is too short for comfort, but, until further notice, it suggests that the two series have differed mostly by their scales.

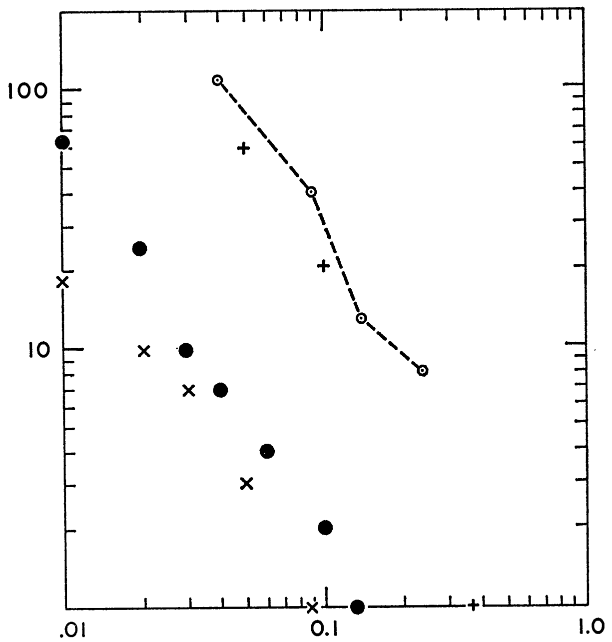

FIG. 12.—Four miscellaneous distributions of interest and exchange rates. Reading from the left: two different series of dollar-sterling premium rates. *Crosses:* ten times the excess over 6 per cent of Davis's textile interest rate data. *Dashed line:* excess over 6 per cent of Bigelow's money rates. The four series, all very short, were chosen haphazardly. The point of the figure is the remarkable similarity between the various curves.

The dollar-sterling exchange in the nineteenth century.—The exchange premium or discount in effect on a currency exchange seems to reflect directly the difference between the various "forces" that condition the variations of the values of the two currencies taken separately. This differential quantity has even an advantage over the changes of rates; indeed, one can consider it without resorting to any kind of economic theory, not even the minimal assumption that price changes are more important than price levels. We have therefore plotted the values of the premium or discount between dollar and sterling between 1803 and 1895, as reported by L. E. Davis and J. R. T. Hughes [7] (Fig. 12). This series is based upon operations which involved credit as well as exchange; in order to eliminate the credit component, the authors used various series of money rates; we also plotted the series based upon Bigelow's rates. One will note that all the graphs of Figure 12 conform strikingly with the expectations generalized from the known behavior of cotton prices.

IV. STATISTICAL ESTIMATION OF α BY MAXIMUM LIKELIHOOD, WHEN α IS NEARLY 2

A. INTRODUCTION

A handicap for the theory of VCSP is that no closed analytic form is known for the stable Paretian distributions, nor is a closed form likely to be ever discovered. Luckily, the cases where the exponent α is near 1.7 can be dealt with on the basis of an approximating hyperbolic distribution. Now let α be very near 2. To estimate $2 - \alpha$ or to test $\alpha = 2$ against $\alpha < 2$ is extremely important, because $\alpha = 2$ corresponds to the Gaussian law and differs "qualitatively" from other values of α. To estimate such an α is very difficult, however, and the estimate will be intrinsically highly dependent upon the number and the "erratic" sizes of the few most "outlying" values of u_n. I hope to show in the present section that simplifying approximations are fortunately available for certain purposes. The main idea is to represent a stable Paretian density as a sum of two easily manageable expressions, one of which concerns the central "bell," while the other concerns the tails.

B. A SQUARE CENTRAL "BELL," WITH HYPERBOLIC SIDE PORTIONS

Consider the following probability density, which can be defined for $\frac{3}{2} < \alpha < 2$:

$p(u) = \alpha - \frac{3}{2}$

if $|u| \leq 1$ (adding up to $2\alpha - 3$);

$p(u) = (2 - \alpha)\alpha|u|^{-(\alpha+1)}$

if $|u| > 1$ (adding up to $4 - 2\alpha$).

When α is near 2, $p(u)$ is a rough first approximation to a stable Paretian density. Its advantage is to lead itself readily to maximum-likelihood estimation.

Let indeed $u_1, \ldots u_n, \ldots, u_N$ be a sample of values of U, ordered by decreasing absolute size, and let M of these have an absolute size greater than 1. Given these sample values u_n, the likelihood of a value of α is defined as being $\Pi p(u_n)$, which equals

$$\left(\alpha - \frac{3}{2}\right)^{N-M}[(2-\alpha)\alpha]^M \times \left[\prod_{n=1}^{M} |u_n|\right]^{-(\alpha+1)}.$$

The logarithm of the likelihood is

$$L(\alpha) = (N-M)\log\left(\alpha - \frac{3}{2}\right) + M\log[(2-\alpha)\alpha] - (\alpha+1)\sum_{n=1}^{M}\log|u_n|.$$

This $L(\alpha)$ is a continuous function of α. If $M = 0$, it is monotone increasing and attains its maximum for $\alpha = 2$. This is a reasonable answer, since $|U| < 1$ for $\alpha = 2$.

If $M > 0$, on the contrary, $L(\alpha)$ tends to $-\infty$ as α tends to either of the ends of its domain of variation, namely, $\frac{3}{2}$ and 2. It has therefore at least one maximum, and the most likely value of α, namely $\hat{\alpha}$, is among the roots of the third-degree algebraic equation.

$$\frac{N-M}{\alpha - \frac{3}{2}} - \frac{M}{2-\alpha} + \frac{M}{\alpha} - \sum_{n=1}^{M}\log|u_n| = 0.$$

Thus, $\hat{\alpha}$ only depends upon M/N and upon $M^{-1}\Sigma_{n=1}^{M} \log |u_n| = V$, the logarithm of the geometric mean of these u_n's whose absolute value exceeds 1.

Let us examine the latter term closer. The random variable $\log |U|$, conditioned by $\log |U| > 1$, has for distribution

$$Pr(\log |U| > u | \log |U| > 0) = \exp(-\alpha u) .$$

Its expected value is $1/\alpha$. Therefore, as the sample sizes M and N tend to infinity, one will have

$$\lim\left[\frac{1}{\alpha}-\frac{1}{M}\sum_{n=1}^{M}\log\,(u_n)\right]=0 .$$

As a result, *in the first approximation*, one can neglect the term

$$\frac{1}{\alpha}-\frac{1}{M}\sum_{n=1}^{M}\log|u_n|$$

when both M and N are large. The equation in $\hat{\alpha}$ simplifies to the first degree and yields

$$\frac{2-\hat{\alpha}}{\hat{\alpha}-\frac{3}{2}}\sim\frac{M}{N-M},$$

that is,

$$\hat{\alpha}=2-M/2N .$$

In particular, $\hat{\alpha}$ no longer depends upon the precise values of the u_n but depends only on their relative numbers in the two categories $|U| < 1$ and $|U| > 1$. The ratio M/N may, incidentally, be interpreted as the relative number of outliers for which $|U| > 1$.

For example, if M/N is very small, $\hat{\alpha}$ is very close to 2. (At the other end, if N/M barely exceeds 1, $\hat{\alpha}$ nears $\frac{3}{2}$. However, this is a range in which $p(u)$ is a very poor approximation to a stable Paretian probability density.)

It may be observed that, knowing N, M/N is an asymptotically Gaussian random variable. We have thus easily proved that $\hat{\alpha}$ is asymptotically normally distributed for all values of α.

In a second approximation, valid for α near 2, one will insert $\alpha = 2$ in computing the value of

$$W=\frac{1}{\alpha}-\frac{1}{M}\sum\log|u_n| .$$

The equation in $\hat{\alpha}$ will thus go down in degree from the third to the second. One of its roots is very large and irrelevant; the other root is such that $\alpha - (2 - M/2N)$ is proportional to W.

C. SCOPE OF THE ESTIMATION PROCEDURE BASED UPON THE COUNTING OF OUTLIERS

The method of Section IV. B, namely, estimation of $\hat{\alpha}$ from M/N, applies without change under a variety of seemingly generalized conditions:

1. Suppose that the tails are asymmetric, that is,

$$p(u) = (2 - \alpha)\alpha p' u^{-(\alpha+1)} \quad \text{if} \quad u > 1$$

$$p(u) = (2 - \alpha)\alpha p'' |u|^{-(\alpha+1)} \text{ if } \quad u < -1 ,$$

where $p' + p'' = 1$. To estimate α, one will naturally concentrate upon the random variable $|U|$, which is the same as in Section B.

2. Further, the results of Section B remain valid if the conditional density of U, given that $|U| < 1$, is non-uniform but independent of α. Suppose, for example, that for $|u| < 1$, $p(u)$ is equal to $(\alpha - \frac{3}{2})$ multiplying the truncated Gaussian density $D \exp(-u^2/2\sigma^2)$, where $1/D(\sigma)$ is defined as equal to $\int_1^{-1} \exp(-s^2/2\sigma^2)ds$. The likelihood of α then equals

$$\left[D(\sigma)\,2^{-1}\left(\alpha-\frac{3}{2}\right)\right]^{N-M}$$

$$\times\exp\left(-\sum_{n=M+1}^{N}\frac{u_n^2}{2\sigma^2}\right)[(2-\alpha)\alpha]^{M}$$

$$\times\left[\prod_{n=1}^{M}|u_n|\right]^{-(\alpha+1)} .$$

The maximum likelihood, considered as a function of the U_n, is unchanged from Section IV. B.

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