ESG Documentation Pack

Stochastic Volatility Jump Diffusion

Model Definition

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Contents

Introduction

This note describes the Stochastic Volatility Jump Diffusion (SVJD) model, its implementation, its parameters, the ESG inputs/outputs and the ESG test analysis.

Overview of SVJD Model

Key features

The SVJD model consists of two models: Heston's Stochastic Volatility model and Merton's Jump model. The Heston's model consists of two stochastic differential equations which describe the evolution of the asset price process and the variance process. The Merton's model describes asset price evolution through a compound Poisson process with independent log-normally distributed jumps.

Market consistent (MC) calibrations fit the model to market forward-at-the-money equity option implied volatilities. Real world (RW) calibrations fit the model to equity return distribution targets.

Benefits

The SVJD model is a fairly realistic and yet parsimonious model. It can capture the skew and term structure of the implied volatility surface, both at long and short maturities. It can also capture skew and kurtosis in return distributions. Compared to other simpler B&H equity models, it is more suitable for valuing multiple guarantees where the characteristics vary by term and moneyness.

Limitations

Jumps in different equity assets are uncorrelated and therefore dilute the correlation of returns. To compensate for this effect a bespoke calibration with increased correlation targets is required.

The SVJD Model

Equity returns are modelled in excess of the nominal risk-free rate. This means that if $S(t)$ is the stock total return index and $C(t)$ is the cash total return index, we specify the dynamics for $S^{\text{XS}}(t) = \frac{S(t)}{C(t)}$. In this section we describe the stochastic differential equations underlying the SVJD model and focus on the correlation matrix approach for equity correlations (i.e. we assume that no factors are used). This is the setup that is typically used in a market consistent framework.

The Heston stochastic differential equations

The log excess stock price $\ln S^{\text{XS}}$ follows both a continuous and discontinuous process. The continuous process is a standard Heston stochastic volatility model:

Equation 1

$$d\ln S^{SV}(t) = \left(\mu - \frac{v(t)}{2}\right)dt + \sqrt{v(t)}dW_1$$

where μ is the risk premium, which vanishes in the risk neutral world, and $dW_1$is a random Wiener process. The variance process v follows a Cox-Ingersoll-Ross process:

Equation 2

$$dv(t) = \alpha(\theta - v(t))dt + \xi\sqrt{v(t)}dW_2,$$

where θ is the mean reversion level, α is the speed of mean reversion, ξ is the volatility of variance and dW_2 is another random Wiener process. The two Wiener processes are correlated with correlation value ρ.

The Merton Jump stochastic differential equations

The discontinuous process is a Merton jump model:

Equation 3

$$d\ln S^{JD}(t) = -\lambda\bar{\mu}dt + \ln J\, dN(t), \quad \text{where} \quad dN(t) \equiv N(t+dt) - N(t),$$

where $\bar{\mu} = \exp\left(\mu_J + \frac{1}{2}\sigma_J^2\right) - 1$ and J is a log-normally distributed random jump value such that $\ln J \sim N(\mu_J, \sigma_J^2)$. The number of random jumps $N(t)$ that occur over the period $[0, t]$ is determined by a Poisson process with parameterλt, where λ is the jump arrival rate (the expected number of jumps per unit time). The Merton jump model can also be written in the form:

Equation 4

$$\frac{dS^{JD}(t)}{S^{JD}(t_{-})} = -\lambda\bar{\mu}dt + (J-1)dN(t)$$

where $S^{JD}(t_{-}) = \lim_{t\to t_{-}} S^{JD}(t)$. The two models are combined assuming $S^{SV}(0) = S_0$ and $S^{JD}(0) = 1$ to give:

Equation 5

$$S^{\text{XS}}(t) = S^{SV}(t)S^{JD}(t), \quad S^{\text{XS}}(0) = S_0.$$

Dividend Yield

The dividend yield process $y(t)$ satisfies the stochastic differential equation

Equation 6

$$d \ln y(t) = \alpha_y \big(\mu_y - \ln y(t)\big) dt + \sigma_y dZ$$

where $E[dW_1 dZ] = \rho_y dt$ and σ_y is equal to total return equity volatility. More details about dividend yield modelling and calibration can be found in [1997-68] and [2010-1727].

The SVJD Model Implementation

The continuous part S^{SV} is simulated using an approximate method, the biased Euler type discretisation:

Equation 7

$$S_{t+\Delta t}^{SV} = S_t^{SV} \exp\left\{\left(\mu - \frac{(v_t)_+}{2}\right)\Delta t + \sqrt{(v_t)_+}\sqrt{\Delta t}Z^{(1)}\right\}, \text{ where } (x)_+ = \max(x, 0)$$

$$v_{t+\Delta t} = v_t + \alpha(\theta - (v_t)_+)\Delta t + \xi\sqrt{(v_t)_+}\sqrt{\Delta t}Z^{(2)}$$

$$\begin{pmatrix} Z^{(1)} \\ Z^{(2)} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

Although v_t is allowed to become negative, in which case the second equation reduces to $v_{t+\Delta t} = v_t + \alpha\theta\Delta t$, numerical experiments suggest that this simple scheme performs well in most practical circumstances of interest.

The discontinuous part S^{JD} is simulated using the following scheme:

Equation 8

$$S_{t+\Delta t}^{JD} = S_t^{JD} \exp(-\lambda\bar{\mu}\Delta t) \prod_u^{N(\Delta t)} J_u$$

$$N(\Delta t) \sim \text{Poisson}(\lambda\Delta t)$$

$$\ln J_u \sim N(\mu_J, \sigma_J^2)$$

Finally, we simulate S by combining Equations 7 and 8:

Equation 9

$$S_t^{\text{XS}} = S_t^{SV} S_t^{JD}$$

The ESG implementation uses a Brownian Bridge Algorithm (see [2008-297]) that allows the user to select a time step smaller than the ESG time step in order to improve accuracy.

Valuing a European equity option

To value a European option on the equity total return index, the SVJD calibration tool uses the COS method (see [2013-2532]), whereas the ESG up to version 7.4 uses the methodology described in [2008-1260]. The tool prices put options. Call options can be calculated using put-call parity.

The pricing formula for a put option maturing at time T, assuming deterministic interest rates, is given by

Equation 10

$$v(x_0, 0) = \widetilde{K} \sum_{k=0}^{N-1} q_k \mathrm{Re}[\phi(\omega_k) e^{i\omega_k a}] \ U_k$$

where $\widetilde{K} = Ke^{-RT}$ is the forward strike price, R is the continuously compounded interest rate between time 0 and maturity T, $q_0 = 1/2$, $q_k = 1$ for $k \geq 1$, $\phi(\omega)$ is the characteristic function of

Equation 11

$$X_T = \ln \frac{S_T}{K} = \ln \frac{S_T^{\mathrm{XS}}}{\widetilde{K}}$$

and $\omega_k = k\pi/(b-a)$. The coefficient U_k is the cosine Fourier transform of the payoff function $v(x, T)$ divided by the strike K. An explicit expression for U_k for a European put option is given in Equation 16 of [2013-2532]. The choice of integration interval $[a, b]$ is discussed below.

The SVJD characteristic function

The characteristic function can be written as the product of the characteristic functions for the Heston model and for the Merton Jump model:

Equation 12

$$\begin{aligned}
\phi(\omega) &= E[e^{i\omega X_T}] \\
&= E\left[e^{i\omega \ln \frac{S_T^{SV}}{\widetilde{K}}} e^{i\omega \ln S_T^{JD}}\right] \\
&= E\left[e^{i\omega \ln \frac{S_T^{SV}}{\widetilde{K}}}\right] E\left[e^{i\omega \ln S_T^{JD}}\right] \\
&= \ \phi^{SV}(\omega)\, \phi^{JD}(\omega)
\end{aligned}$$

The Heston characteristic function is given by

Equation 13

$$\phi^{SV}(\omega) = e^{i\omega \tilde{x}_0 + A + BV_0}$$

where $\tilde{x}_0 = \ln \frac{S_0}{K}$ and

Equation 14

$$\begin{aligned}
A &= \frac{\alpha\theta}{\xi^2}\left((\beta - D)T - 2\log\left(\frac{1 - Ge^{-DT}}{1 - G}\right)\right) \\
B &= \frac{1}{\xi^2} \frac{1 - e^{-DT}}{1 - Ge^{-DT}} (\beta - D) \\
G &= \frac{\beta - D}{\beta + D} \\
D &= \sqrt{\beta^2 + (\omega^2 + i\omega)\xi^2} \\
\beta &= \alpha - i\rho\xi\omega
\end{aligned}$$

For a derivation see Appendix A.

The Merton jump characteristic function is given by

Equation 15

$$\phi^{JD}(\omega) = \exp\left(-i\omega\lambda\bar{\mu}T + \lambda T\left(e^{i\omega\mu_J - \frac{1}{2}\omega^2\sigma_J^2} - 1\right)\right)$$

For a derivation see Appendix B.

It is possible to decompose the characteristic function into the product

Equation 16

$$\phi(\omega) = e^{i\omega\tilde{x}_0}\varphi(\omega)$$

With this decomposition $\varphi(\omega)$ has no dependence on the strike and options with different strikes can be prices simultaneously[1] by caching the values of $\varphi(\omega)$. We are currently not taking advantage of this technique, but might do so in the future.

Effect of nominal interest rates

Close inspection of Equations 10-13 reveals that, when working in terms of forward strikes $\widetilde{K}$, the explicit dependence on (deterministic) nominal interest rates is effectively removed.

If nominal interest rates are stochastic, which is typically the case, equity option prices are affected. This effect is not always negligible. Although it is monitored via the ESG validation, it is ignored by the calibration tool. We might review this in the future.

Choice of *N* and integration interval

Our default choice for the number of cosine terms is $N = 128$. In general, for a given level of accuracy, the SVJD model requires more points than the Black-Scholes model due to the lower decay rate of the characteristic function. The choice of interval used in the COS pricing method, $[a, b]$, is consistent with that described in Appendix A of [2013-2532]. In particular, we set L consistently with the optimal choice for the Black-Scholes model

Equation 17

$$L = \sqrt{\frac{\pi N}{2}}$$

This choice of interval is unlikely to be the most efficient one for the SVJD model. We might therefore review this in the future.

[1] *F. Kilin, Accelerating the calibration of stochastic volatility models, MPRA Paper No. 2975, 2006.*

The SVJD Model Parameters and ESG Inputs

The SVJD model parameters, their input location in the ESG and the calibration tool model parameters are summarised in the table below.

Parameter	ESG input location[2]	Calibration tool parameters (MC)	Description
μ	.MeanReturn	N/A	The equity return in excess of the risk free rate. Set to zero in MC simulations.
α	.Variance.MeanReversionRate	Reversion Speed	The rate of mean reversion of the variance process.
θ	.Variance.MeanReversionLevel	Reversion Level	The mean reversion level of the variance process.
ξ	.Variance.Volatility	Vol Variance	The volatility of the variance process.
$v(0)$	.Variance.InitialValue	Initial Variance	The initial value of the variance process.
ρ	.Variance.Correlation	Correlation	The correlation between the variance shock and the equity return shock.
λ	.Jump.ArrivalRate	Jump Intensity	The expected number of jumps per unit time.
μ_J	.Jump.JumpMean	Jump Mean	The mean of the log jump.
σ_J	.Jump.JumpVolatility	Jump Volatility	The volatility of the log jump.
$\hat{\alpha}$	.Variance.RiskNeutralMeanReversionRate	N/A	The mean reversion rate of the variance process under the risk neutral measure. Used exclusively for implied volatility modeling when the UseRiskNeutralValuation is False.
$\hat{\theta}$	.Variance.RiskNeutralMeanReversionLevel	N/A	The mean reversion level of the variance process under the risk neutral measure. Used exclusively for implied volatility modeling when the UseRiskNeutralValuation is False.

[2] *In the table the ESG inputs may be found under ESG.Assets.EquityAssets.EquityAssetFactors.[Factor i].Volatility if the SVJD is a sub-model of factor i, ESG.Assets.EquityAssets.[ParentEquityAsset].Sigma if the SVJD is a sub-model of ParentEquityAsset and ESG.Assets.EquityAssets.[ChildEquityAsset].Sigma if the SVJD is a sub-model of ChildEquityAsset.*

The ESG Outputs

The ESG Outputs are described in the table below.

ESG output location[3]	Description
.TotalReturn	The total return earned on the assed over the time step.
.TotalReturnIndex	The total return index (TRI) of the asset which is given by $$TRI(t) = TRI(t - \Delta t)(1 + TotalReturn(t))$$ with $$TRI(0) = 1.$$
.RescaledTRI (StartVal)	The total return index corresponding to the asset, rescaled to the designated starting value.
.CapitalChange	Price changes within the equity model are described by the output CapitalChange. This is given by $$\begin{aligned} CapitalChange(t) &= \frac{Price(t)}{Price(t - \Delta t)} - 1 \\ &= \frac{1 + TotalReturn(t)}{1 + DividendYield(t) * \Delta t} - 1 \end{aligned}$$ The output at time zero is meaningless and a default value of 0 is given.
.CapitalIndex	The capital index (CI) of the asset is calculated as $$CI(t) = CI(t - \Delta t) * CI(1 + CapitalChange(t))$$ where $CI(0) = 1$
.IncomeReturn	The income return (IR) is the difference between the total return and the capital change: $$IR(t) = TotalReturn(t) - CapitalChange(t)$$
.IncomeIndex	The income index is calculated from the capital index and the dividend yield as $$IncomeIndex(t) = CapitalIndex(t) * DividendYield(t).$$
.EarningsYield	The earnings yield of an asset is calculated by $$EarningsYield(t) = \frac{DividendYield(t)}{DividendPayout(t)}.$$
.EarningsIndex	The earnings index of an asset is given by $$\begin{aligned} EarningsIndex(t) &= \frac{IncomeIndex(t)}{DividendPayout(t)} \\ &= CapitalIndex(t) * EarningsYield(t). \end{aligned}$$
.Volatility	The total volatility of the equity asset.
.ZScore	A normal approximation of the StandardisedShock.
.StandardisedShock	The standardised return of the asset.
.EquityOptionImpliedVolatility (Time to maturity, Strike type, Strike Rate)	This output is used for modelling equity option implied volatility in real world simulations. The output can be selected when the ParentEquityAsset and up to one of the equity asset factors are modelled using the SVJD model.

[3] *In the table the ESG inputs may be found under ESG.Assets.EquityAssets.[ParentEquityAsset] if the SVJD is a sub-model of ParentEquityAsset and ESG.Assets.EquityAssets.[ChildEquityAsset] if the SVJD is a sub-model of ChildEquityAsset.*

ESG output location[4]	Description
.Volatility	The square root of .Variance.Value $\sqrt{v(t)}$, i.e. the instantaneous value of the volatility.
.JumpReturn	The Jump return is the return on the asset due to the jump process alone. It is given by $$\exp\left\{\sum_{i=1}^{n}(\mu_J+\sigma_J Z_i)-\bar{\mu}\right\}-1,$$ where μ_J is the JumpMean parameter, σ_J is the JumpVolatility parameter, Z_i is a standard normal shock, and n is the NumberOfJumps over the time-step.
.StandardisedVolatility	Equal to Sigma.Variance.StandardisedVolatility (see below).
.StandardisedShock	Equal to Sigma.Variance.StandardisedShock (see below).
.Jump.NumberOfJumps	The number of jumps that have occurred over the time-step Δt. These are realisations of a Poisson random variable with parameter $\lambda\Delta t$.
.Jump.NumberOfMicroJumps(Index)	The number of jumps occurring in each subdivision of the time step. It is generated using a Brownian Bridge algorithm, which is controlled by the NumberOfSubdivisions parameter. Each micro-jump is a realisation of a Poisson random variable.
.Variance.Value	The instantaneous value of the variance, $v(t)$.
.Variance.StandardisedVolatility	The Standardised Volatility is equal to $$\sqrt{\frac{1}{N}\sum_{i=1}^{N}V_{i-1}},$$ where N is the number of subdivisions, and V_i is the instantaneous variance of the SVJD process at micro-step i.
.Variance.StandardisedShock	The Standardised Shock is equal to $$\sum_{i=1}^{N}\sqrt{V_{i-1}}\,Z_i \Big/ \sqrt{\sum_{i=1}^{N}V_{i-1}},$$ where Z_i are the micro-shocks generated using the Brownian bridge algorithm.

[4] *In the table the ESG inputs may be found under ESG.Assets.EquityAssets.EquityAssetFactors.[Factor i].Volatility if the SVJD is a sub-model of factor i, ESG.Assets.EquityAssets.[ParentEquityAsset].Sigma if the SVJD is a sub-model of ParentEquityAsset and ESG.Assets.EquityAssets.[ChildEquityAsset].Sigma if the SVJD is a sub-model of ChildEquityAsset.*

ESG Analysis Tests

Asset Martingale Test

The Asset Martingale Tests assess the arbitrage-free nature of the model. Under the risk-neutral probability measure Q, the expected discounted future value of any asset is equal to its current value. If $S(T)$ is the total return index of an asset and $C(T)$ is the cash total return index then, for any projection time T,

Equation 18

$$E_Q\left[\frac{S(T)}{C(T)}\right] = 1$$

In the ESG test, the left hand side of Equation 18 is estimated as the sample mean of $\boldsymbol{S_i(T)/C_i(T)}$ for a set of $\boldsymbol{N}$ Monte Carlo trials,

Equation 19

$$S_{est}(0,T) = \frac{1}{N}\sum_{i=1}^{N}\frac{S_i(T)}{C_i(T)}$$

The standard error of the estimate $SE(T)$ is given by the standard deviation of $S_i(T)/C_i(T)$ divided by the square root of the number of trials. If antithetic variables are used and the number of trials is $2N$, then the sample mean and the standard deviation are calculated from the sequence

Equation 20

$$\frac{1}{2}\left(\frac{S_i(T)}{C_i(T)} + \frac{S_i^{\text{anti}}(T)}{C_i^{\text{anti}}(T)}\right)$$

for $i = 1, \dots, N$. The standard error can be used to generate confidence intervals. For sufficiently large N there is a 95% probability that

Equation 21

$$S_{\text{est}}(0,T) - 1.96 \times SE(T) < 1 < S_{\text{est}}(0,T) + 1.96 \times SE(T)$$, where $1.96 = \Phi^{-1}(0.975)$.

Equity Option Implied Volatility Test

The Equity Option Implied Volatility Test estimates the option prices implied by the simulation output. These estimates can be compared with the market prices to test the validity of the model and its calibration. To estimate call option prices from the Monte Carlo simulation, we calculate the sample mean of the discounted payoff across all trials:

Equation 22

$$\frac{1}{C_i(T)}\max(S_i(T) - F(T), 0)$$

where $F(T) = 1/P(0,T)$ is the strike price (which is set equal to the forward price of the equity total return index) and $P(0,T)$ is the price at time 0 of a zero coupon bond maturing at time T.

The standard error of the estimate is given by the standard deviation of the discounted payoff divided by the square root of the number of trials. This can be used to calculate confidence intervals for the equity option price. Antithetic variables may be used as a variance reduction technique.

The prices generated by the model are converted into implied volatilities using the Black-Scholes equation. These volatilities can then be compared to market implied volatilities. The upper and lower implied volatilities are the values of the volatility parameter in the Black-Scholes equation that reproduce the upper and lower bounds of the 95% confidence interval of the call price.

Further reading

More details about the SVJD model and calibration are available in the Knowledge Base at
http://www.barrhibb.com/knowledge_base/category/stochastic_volatility_jump_diffusion_model/

Appendix A - Derivation of the Heston characteristic function

In this appendix we derive the Heston characteristic function. We start by rewriting the Heston stochastic differential equations in terms of $X_t = \ln S_t$:

Equation A1

$$dX_t = \left(\mu - \frac{v(t)}{2}\right)dt + \sqrt{v(t)}\,dW_1,$$

$$dv(t) = \alpha\big(\theta - v(t)\big)dt + \xi\sqrt{v(t)}\,dW_2$$

For any sufficiently smooth function $u(x)$, the expectation of $u(X_T)$ given the history of the process up to time t

Equation A2

$$g(t, x_t, v_t) = E[u(X_T)|\mathcal{F}_t]$$

is a martingale. This follows from the Tower Law

$$E[g(\tau, x_\tau, v_\tau)|\mathcal{F}_t] = E[E[u(X_T)|\mathcal{F}_\tau|\mathcal{F}_t] = E[u(X_T)|\mathcal{F}_t] = g(t, x_t, v_t)$$

where $t < \tau < T$.

If we choose $u(X_T) = e^{i\omega X_T}$, we have

Equation A3

$$g(t, x_t, v_t) = E[e^{i\omega X_T}|\mathcal{F}_t] = \phi(\omega; t, x_t, v_t)$$

i.e. $g(t, x_t, v_t)$ corresponds to the characteristic function of $X_T|\mathcal{F}_t$ and is a martingale. From Ito's lemma, the characteristic function satisfies the stochastic differential equation

Equation A4

$$d\phi = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial v}dv + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}dx^2 + \frac{\partial^2\phi}{\partial x\,\partial v}dxdv + \frac{1}{2}\frac{\partial^2\phi}{\partial v^2}dv^2$$

$$= \left[\frac{\partial\phi}{\partial t} + \left(\mu - \frac{v(t)}{2}\right)\frac{\partial\phi}{\partial x} + \alpha\big(\theta - v(t)\big)\frac{\partial\phi}{\partial v} + \frac{v(t)}{2}\frac{\partial^2\phi}{\partial x^2} + \rho\xi v(t)\frac{\partial^2\phi}{\partial x\,\partial v} + \xi^2\frac{v(t)}{2}\frac{\partial^2\phi}{\partial v^2}\right]dt$$

$$+ \sqrt{v(t)}\frac{\partial\phi}{\partial x}dW_1 + \xi\sqrt{v(t)}\frac{\partial\phi}{\partial v}dW_2$$

Setting the drift to zero leads to the following final value problem

Equation A5

$$0 = \frac{\partial\phi}{\partial t} + \left(\mu - \frac{v(t)}{2}\right)\frac{\partial\phi}{\partial x} + \alpha\big(\theta - v(t)\big)\frac{\partial\phi}{\partial v} + \frac{v(t)}{2}\frac{\partial^2\phi}{\partial x^2} + \rho\xi v(t)\frac{\partial^2\phi}{\partial x\,\partial v} + \frac{1}{2}\xi^2 v\frac{\partial^2\phi}{\partial v^2}$$

$$\phi(\omega; T, x_T, v_T) = e^{ix_0\omega}$$

where $0 \le t \le T$. Working in terms of time to maturity $\tau = T - t$, we obtain the initial value problem

Equation A6

$$\frac{\partial\phi}{\partial\tau} = \left(\mu - \frac{v}{2}\right)\frac{\partial\phi}{\partial x} + \alpha(\theta - v)\frac{\partial\phi}{\partial v} + \frac{v}{2}\frac{\partial^2\phi}{\partial x^2} + \rho\xi v\frac{\partial^2\phi}{\partial x\,\partial v} + \frac{1}{2}\xi^2 v\frac{\partial^2\phi}{\partial v^2}$$

$$\phi(\omega; 0, x_0, v_0) = e^{ix_0\omega}$$

We look for solutions of the form $\phi(\omega;\tau,x,v)=e^{i\omega x+A(\omega,\tau)+B(\omega,\tau)v}$. From the first equation in A6 we get

Equation A7

$$\left(\frac{\partial A}{\partial\tau}+\frac{\partial B}{\partial\tau}\,v\right)\phi=\left(\mu-\frac{v}{2}\right)i\omega\phi+\alpha(\theta-v)B\phi-\frac{1}{2}\,v\omega^2\phi+i\omega\rho\xi vB\phi+\frac{1}{2}\,\xi^2vB^2\phi.$$

The second equation in A6 leads to $A(\omega,0)=B(\omega,0)=0$.

We can rewrite the equation A7 as

Equation A8

$$\left(-\frac{\partial B}{\partial\tau}-\frac{i}{2}\,\omega-\frac{\omega^2}{2}+\frac{\xi^2}{2}\,B^2-\alpha B+i\omega\rho\xi\,B\right)v\,+\left(-\frac{\partial A}{\partial\tau}+\alpha\theta B+\,i\omega\mu\right)=0.$$

Since this equation must be true for all $0\le\tau\le T$, x and v, the coefficient of the stochastic process v and the constant term must both vanish. Hence

Equation A9

$$\frac{dB}{d\tau}=\frac{\xi^2}{2}\,B^2-(\alpha-\ i\omega\rho\xi)B-\frac{\omega}{2}(\,\omega+\,i)$$

$$\frac{dA}{d\tau}=\alpha\theta B+i\omega\mu$$

The first equation is a Riccati ordinary differential equation with constant coefficients. Introducing

Equation A10

$$\lambda=\frac{\omega}{2}(\,\omega+\,i)$$

$$\beta=\alpha-\ i\omega\rho\xi$$

$$\gamma=\ \xi^2/2$$

we can express the first equation as

Equation A11

$$\frac{dB}{d\tau}=\gamma B^2-\beta B+\lambda$$

$$=\gamma(B-B_+)(B-B_-)$$

where $B_\pm=\frac{(\beta\pm D)}{\xi^2}$ and $D=\sqrt{\beta^2+(\omega^2+i\omega)\xi^2}$.

The substitution

Equation A12

$$B(\tau)=B_-+\frac{1}{w(\tau)}$$

transforms Equation A11 into a first order constant coefficient ordinary differential equation

Equation A13

$$\frac{dw}{d\tau}=Dw-\gamma$$

$$w(0)=-\frac{1}{B_-}$$

The solution is

Equation A14

$$w(T) = -\frac{e^{DT}}{B_-} - \frac{\gamma}{D}(e^{DT} - 1)$$

After inserting the expression for $w(T)$ into Equation A12 and a little algebraic manipulation we obtain

Equation A15

$$B(T) = \frac{\beta - D}{\xi^2} \frac{1 - e^{-DT}}{1 - Ge^{-DT}}$$

where

Equation A16

$$G = \frac{B_-}{B_+} = \frac{\beta - D}{\beta + D}$$

We can now solve the second ordinary differential equation for $A(\tau)$

Equation A17

$$\frac{dA}{d\tau} = \alpha\theta B + i\omega\mu$$

and solve it by direct integration. We have

Equation A18

$$A(T) = \alpha\theta \int_0^T B ds + i\omega\mu T$$

where

Equation A19

$$\begin{aligned}
\int_0^T B\, ds &= \frac{\beta - D}{\xi^2} \int_0^T \frac{1 - e^{-Ds}}{(1 - G\, e^{-Ds})} ds \\
&= \frac{\beta - D}{\xi^2 D} \int_G^{Ge^{-DT}} \frac{y/G - 1}{y(1-y)} dy \\
&= \frac{\beta - D}{\xi^2 D} \left[\frac{G-1}{G} \ln \frac{G\, e^{-DT} - 1}{G - 1} - \ln \frac{G\, e^{-DT}}{G} \right] \\
&= \xi^{-2} \left[(\beta - D)T - 2 \ln \frac{G\, e^{-DT} - 1}{G - 1} \right]
\end{aligned}$$

This leads to

Equation A20

$$A = \frac{\alpha\theta}{\xi^2} \left[(\beta - D)T - 2 \ln \frac{Ge^{-DT} - 1}{G - 1} \right] + i\omega\mu T\,.$$

To recover Equation 14 we need to allow for the change of phase due to the strike in $X_t = \ln\frac{S_t}{K}$ by replacing $i\omega\mu T$ with $i\omega(\mu T + x_0)$, where $x_0 = \ln\frac{S_0}{K}$ and $\mu = R$.

Appendix B - Derivation of the Merton jump characteristic function

The dynamics of the Merton Jump model is described in [2008-1260]. In this appendix we derive the characteristic function for the Merton Jump model. According to this model, the price of the stock at time T, assuming that the stock price at time 0 is 1, is given by

Equation B1

$$S_T = \prod_{u=1}^{N(T)} J_u \, e^{-\lambda\bar{\mu}T}$$

where $\ln J_u \sim N(\mu_J, \sigma_J^2)$, $N(T) \sim \text{Poisson}(\lambda T)$, $\bar{\mu} = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$. Taking natural logs leads to

Equation B2

$$\ln S_T = \sum_{u=1}^{N(T)} \ln J_u - \lambda\bar{\mu}T$$

The characteristic function of the log of the stock price can be calculated directly

Equation B3

$$\begin{aligned}
\phi(\omega) &= E\left[e^{i\omega \ln S_T}\right] = E\left[\exp\left(i\omega \sum_{u=1}^{N(T)} \ln J_u - i\omega\lambda\bar{\mu}T\right)\right] \\
&= e^{-i\omega\lambda\bar{\mu}T} E\left[\exp\left(i\omega \sum_{u=1}^{N(T)} \ln J_u\right)\right] = e^{-i\omega\lambda\bar{\mu}T} E\left[E\left[\exp\left(i\omega \sum_{u=1}^{N(T)} \ln J_u\right) | N(T)\right]\right] \\
&= e^{-i\omega\lambda\bar{\mu}T} E\left[\prod_{u=1}^{N(T)} E\left[\exp(i\omega \ln J_u) \,|N(T)\right]\right] = e^{-i\omega\lambda\bar{\mu}T} E\left[\prod_{u=1}^{N(T)} \exp\left(i\omega\mu_J - \frac{1}{2}\omega^2\sigma_J^2\right)\right] \\
&= e^{-i\omega\lambda\bar{\mu}T} E\left[\exp\left(N(T)\left(i\omega\mu_J - \frac{1}{2}\omega^2\sigma_J^2\right)\right)\right] = e^{-i\omega\lambda\bar{\mu}T} \sum_{n=0}^{\infty} \exp\left(n\left(i\omega\mu_J - \frac{1}{2}\omega^2\sigma_J^2\right)\right) e^{-\lambda T}\frac{(\lambda T)^n}{n!} \\
&= e^{-i\omega\lambda\bar{\mu}T} e^{-\lambda T} \exp\left(\lambda T e^{i\omega\mu_J - \frac{1}{2}\omega^2\sigma_J^2}\right) = \exp\left(-i\omega\lambda\bar{\mu}T + \lambda T\left(e^{i\omega\mu_J - \frac{1}{2}\omega^2\sigma_J^2} - 1\right)\right)
\end{aligned}$$

In the derivation we used the following properties:

Equation B4

$$\begin{aligned}
&E[X] = E\left[E[X|Y]\right] \\
&E\left[e^{N(\mu,\sigma^2)}\right] = e^{\mu + \frac{\sigma^2}{2}} \\
&X \sim \text{Poisson}(\lambda) \;\Rightarrow\; \text{Prob}(X = x) \;=\; \frac{e^{-\lambda}\lambda^x}{x!} \\
&E[f(x)] = \sum_x f(x)\,\text{Prob}(X = x)
\end{aligned}$$

The value of $\bar{\mu}$ is set by imposing the martingale condition:

Equation B5

$$1 = E[S_T] = E\left[e^{\ln S_T}\right] = \phi(-i) = \exp\left(-\lambda\bar{\mu}T + \lambda T\left(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\right)\right) \;\Rightarrow\; \bar{\mu} = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$$

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