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FROM ITO AND STRATONOVICH TO BLACK-SCHOLES

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Abstract. Options are financial instruments designed to protect investors from the stock market randomness. In 1973 Black, Scholes and Merton proposed a very popular option pricing method using stochastic differential equations within the Ito interpretation. Herein, We have reviewed Black-Scholes theory using Ito calculus, which is standard to mathematical finance. Moreover, the Black-Scholes equation obtained using Stratonovich calculus is the same as the one obtained by means of the Ito calculus. In fact, this is the result we expected in advance because Ito and Stratonovich conventions are just different rules of calculus. The option pricing method obtains the so-called Black-Scholes equation which is a partial differential equation of the same kind as the diffusion equation. In fact, it was this similarity that led Black and Scholes to obtain their option price formula as the solution of the diffusion equation with the initial and boundary conditions given by the option contract terms.

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1. Introduction

The trading of options and their theoretical study have been known for long, but they were relatively unknown up to the early 1970s [10]. The Black-Scholes model [3] represented a triumph for mathematical

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modeling in finance. It has become an indispensable tool in the trading of options and other financial derivatives. As said in [1] the interest in pricing financial derivative - including pricing options - arises from the fact that financial derivatives can be used to minimize losses caused by price fluctuations of the underlying assets. The option price depends on the stock price and this is a random variable evolving with time. The random term in the stochastic equation must be delta-correlated (prices are driven by white noise)[8]. It is known that the relationship between the stock market and the mathematical concept of Brownian motion goes back to Bachelier [2]. As is well known, a Brownian motion corresponds to a process, the increments of which are independent stationary normal random variables. As said in [4] since the stock price can not be negative, Samuelson [9] suggested using this process to represent the return of the stock price, which will make the stock price a geometric (or exponential) Brownian motion. Interestingly, despite having different differential equations as starting point by Ito and Stratonovich concepts, resulting Black-Scholes equation is not different, not being affected by the interpretation of the noise term.

2. Ito and Stratonovich conventions

We follow [8] to summarize Ito and Stratonovich interpretations of stochastic differential equation which describes the price changes in market, deriving the Black-Scholes equation for the option price. According to Ito interpretation, the value of any random process X is the left end point of the interval $(t, t+dt)$, i.e.

$$X = X(t), \tag{2.1}$$

while according to Stratonovich interpretation, the value of X is the mid point of the interval $(t, t+dt)$, i.e.

$$X = X(t + \frac{dt}{2}) = X(t) + \frac{dX(t)}{2}. \tag{2.2}$$

The differential of the random process $X(t)$ is defined by

$$dX(t) = X(t+dt) - X(t) \tag{2.3}$$

In accordance with (2.3), the differential of the product of two random process X and Y is

$$d(XY) = [(X+dX)(Y+dY)] - XY \tag{2.4}$$

The differential in (2.4) can be seen in two ways:

- $d(XY) = XdY + YdX + dXdY$

- $d(XY) = (X + \frac{dX}{2})dY + (Y + \frac{dY}{2})dX$

By denoting $X_S(t) \equiv X(t + \frac{dt}{2}) = X(t) + \frac{dX(t)}{2}$ and $X_I(t) \equiv X(t)$, the first item of above is Ito product rule:

$$d(XY) = X_I dY + Y_I dX + dXdY, \tag{2.5}$$

and the second item is Stratonovich product rule:

$$d(XY) = X_S dY + Y_S dX. \tag{2.6}$$

We can see (2.5) does not agree with the rules of calculus while (2.6) does. The product rules (2.5) and (2.6) for any two functions of the random process X, like $U(X)$ and $V(X)$ are as:

$$d(UV) = U(X)dV(X) + V(X)dU(X) + dU(X)dV(X), \tag{2.7}$$

and

$$d(UV) = U(X_S)dV(X) + V(X_S)dU(X), \tag{2.8}$$

Consider the SDE:

$$dX = f(X)dt + g(X)dW(t). \tag{2.9}$$

which W is the Brownian motion. If $h(X,t)$ be a function of X and t, the differential of h in two interpretations Ito and Stratonovich are respectively [6]:

$$dh = \frac{\partial h}{\partial X}dX + \frac{\partial h}{\partial t}dt + \frac{1}{2}g^2\frac{\partial^2 h}{\partial X^2}dt, \tag{2.10}$$

and

$$dh = \frac{\partial h}{\partial X_S}dX + \frac{\partial h}{\partial t}dt. \tag{2.11}$$

In [7] was proposed a geometric motion for describing the price changes in market. It is a SDE as

$$dR(t) = \mu dt + \sigma dW(t), \tag{2.12}$$

which $R(t)$ is the return rate after a period t. An initial price X_0 after a period t is

$$X(t) = X_0 e^{R(t)}. \tag{2.13}$$

by assuming $h(t, R(t)) = X(t) = X_0 e^{R(t)}$, we have

$$\frac{\partial h}{\partial R} = X_0 e^{R(t)}, \quad \frac{\partial^2 h}{\partial R^2} = X_0 e^{R(t)}, \quad \frac{\partial h}{\partial t} = 0. \tag{2.14}$$

Therefore by (2.10) and (2.11), we obtain dX for Ito and Stratonovich interpretations respectively as:

$$dX = \frac{\partial h}{\partial R}dR + \frac{\partial h}{\partial t}dt + \frac{1}{2}\sigma^2\frac{\partial^2 h}{\partial R^2}dt = (\mu + \frac{\sigma^2}{2})Xdt + \sigma XdW(t), \tag{2.15}$$

and

$$dX = \frac{\partial h}{\partial R}dR + \frac{\partial h}{\partial t}dt = X_0 e^{R(t)}\Big(\mu dt + \sigma dW(t)\Big) = \mu X dt + \sigma X dW(t). \tag{2.16}$$

A portfolio is a collection of different assets for diversifying away financial risk. We consider a portfolio based on one type of share whose price is the random process $X(t)$. In this portfolio, there are Δ shares, Ψ calls and Φ bounds. We have a number of calls having value ΨC, (C is the call price) and we owe $\Delta X + \Phi B$, (B is the bond price). Therefore the value of portfolio P is

$$P = \Psi C - (\Delta X + \Phi B) \tag{2.17}$$

On the other hand, $dB = rBdt$. According to [8], P must be zero for any time t. Therefore

$$C = \frac{\Delta}{\Psi}X + \frac{\Phi}{\Psi}B \tag{2.18}$$

By denoting $\delta \equiv \frac{\Delta}{\Psi}$ and $\phi = \frac{\Phi}{\Psi}$, we have

$$C = \delta X + \phi B. \tag{2.19}$$

As [8], using Ito interpretation, the differential of the call price C is

$$dC = \delta dX + \phi dB. \tag{2.20}$$

In addition, $\phi dB = r(C - \delta X)dt$. Then

$$dC = \delta dX + r(C - \delta X)dt. \tag{2.21}$$

Now, according to (2.10), the call price $C(X, t)$ is:

$$dC = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 C}{\partial X^2}\right)dt + \frac{\partial C}{\partial X}dX. \tag{2.22}$$

By (2.21) and (2.22) we have:

$$(\delta - \frac{\partial C}{\partial X})dX = [\frac{\partial C}{\partial t} - r(C - \delta X) + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 C}{\partial X^2}]dt. \tag{2.23}$$

When $\left(\delta - \frac{\partial C}{\partial X}\right) = 0$, which is famous as "delta hedging", the Black-Scholes equation is obtained

$$\frac{\partial C}{\partial t} = rC - rx\frac{\partial C}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}. \tag{2.24}$$

In Stratonovich interpretation, by $X = X_S - \frac{dX}{2}$, we have

$$\delta(X, t) = \delta(X_S - \frac{dX}{2}, t) = \delta(X_S, t) - \frac{1}{2}\frac{\partial \delta}{\partial X_S}dX \tag{2.25}$$

By (2.21) and (2.25) we have

$$dC = \delta(X_S,t)dX + \Big(rC - rX_S\delta(X_S,t) - \frac{1}{2}\sigma^2 X_S^2 \frac{\partial\delta}{\partial X_S}\Big)dt. \tag{2.26}$$

From (2.11), we obtain dC as

$$dC = \frac{\partial C}{\partial X_S}dX + \frac{\partial C}{\partial t}dt. \tag{2.27}$$

by (2.26) and (2.27), we have

$$\left(\frac{\partial C}{\partial X_S} - \delta(X_S,t)\right)dX = \left(rC - rX_S\delta(X_S,t) - \frac{1}{2}\sigma^2 X_S^2 \frac{\partial\delta}{\partial X_S} - \frac{\partial C}{\partial t}\right)dt \tag{2.28}$$

This equation will be non stochastic if $\frac{\partial C}{\partial X_S} - \delta(X_S,t) = 0$. In this situation we have

$$rC - rX_S\delta(X_S,t) - \frac{1}{2}\sigma^2 X_S^2 \frac{\partial\delta}{\partial X_S} - \frac{\partial C}{\partial t} = 0. \tag{2.29}$$

Because of $\delta(X_S,t) = \frac{\partial C}{\partial X_S}$ and $\frac{\partial\delta}{\partial X_S} = \frac{\partial^2 C}{\partial X_S^2}$, we get the Black-Scholes equation from (2.29)

$$\frac{\partial C}{\partial t} = rC - rx\frac{\partial C}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}. \tag{2.30}$$

3. Conclusion

We have reviewed Black-Scholes theory using Ito calculus, which is standard to mathematical finance. the Black-Scholes equation obtained using Stratonovich calculus is the same as the one obtained by means of the Ito calculus. In fact, this is the result we expected in advance because Ito and Stratonovich conventions are just different rules of calculus.

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