Lecture 2. Images and Transformations

2D transformations

Juan Carlos Niebles and Jiajun Wu

CS131 Computer Vision: Foundations and Applications

What will we learn today?

- 2D transformations
  - Transformation Matrices
  - Homogeneous coordinates
  - Translation
  - Scaling
  - Rotation

Transformation Matrices

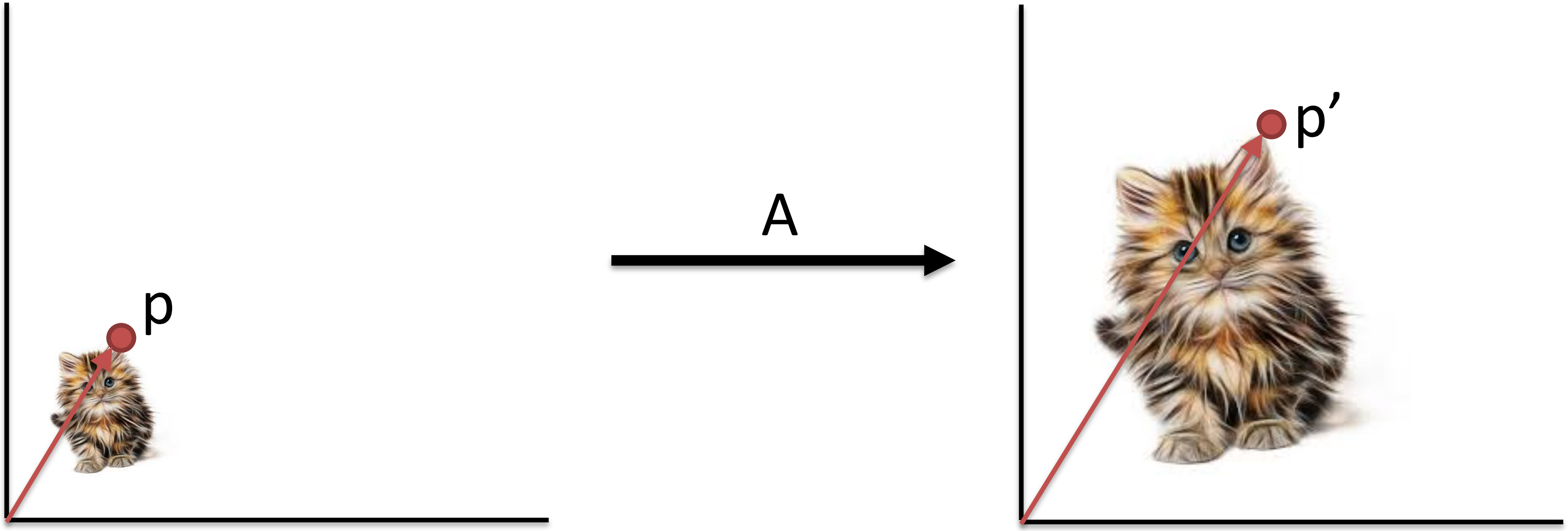
- Matrices can be used to transform vectors in useful ways, through multiplication: p'= A p
- Simplest is scaling:

$$\underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{p} = \underbrace{\begin{bmatrix} s_x x \\ s_y y \end{bmatrix}}_{p'}$$

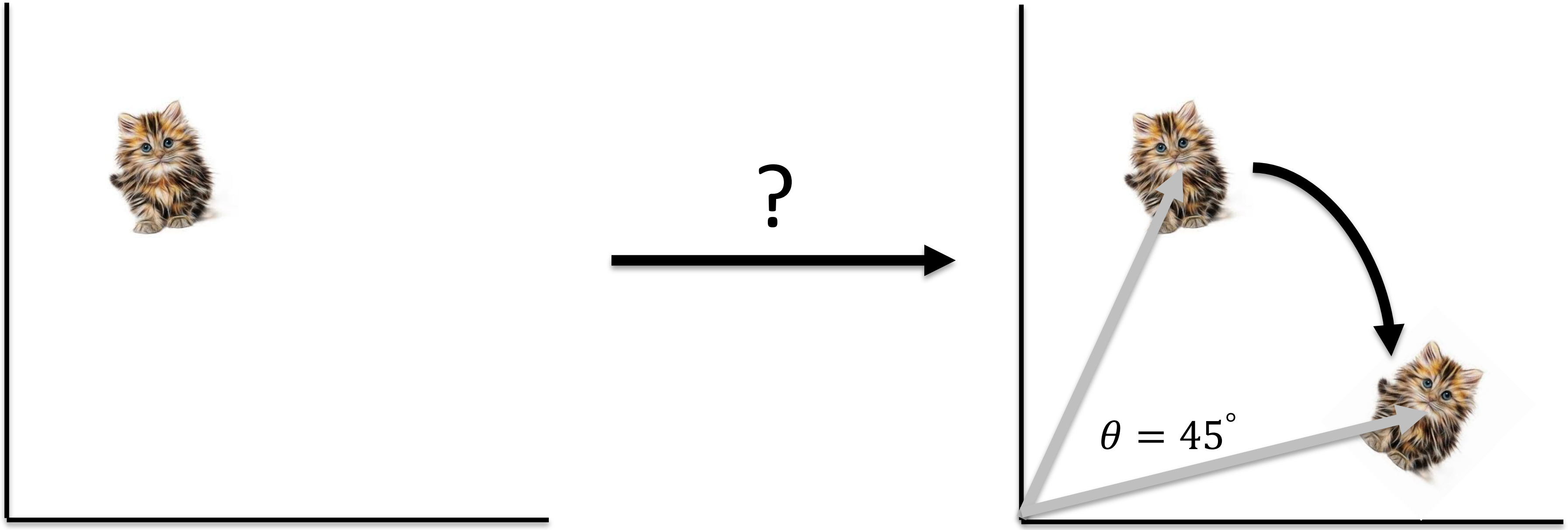
(Verify to yourself that the matrix multiplication works out this way)

Transformation Matrices

$$\underset{A}{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}} \times \underset{p}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{p'}{\begin{bmatrix} s_x x \\ s_y y \end{bmatrix}}$$



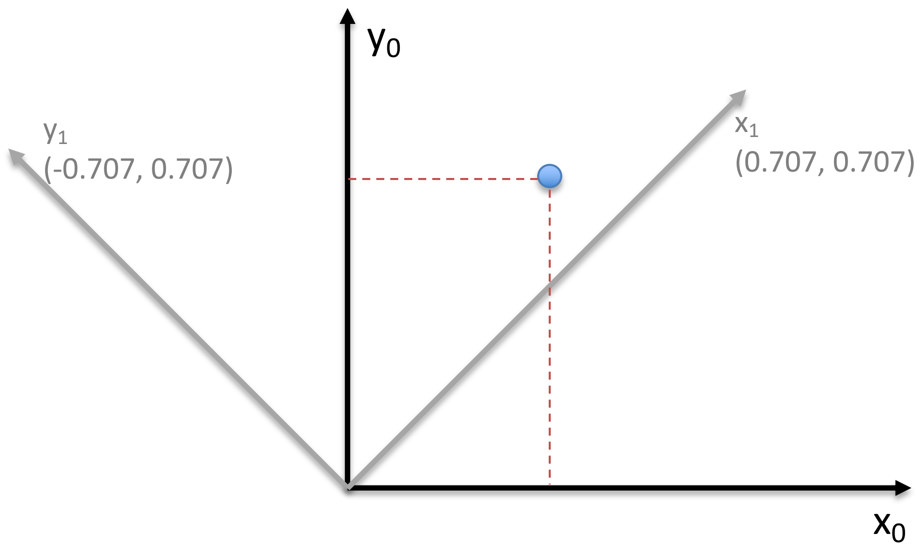
Rotation



Rotation

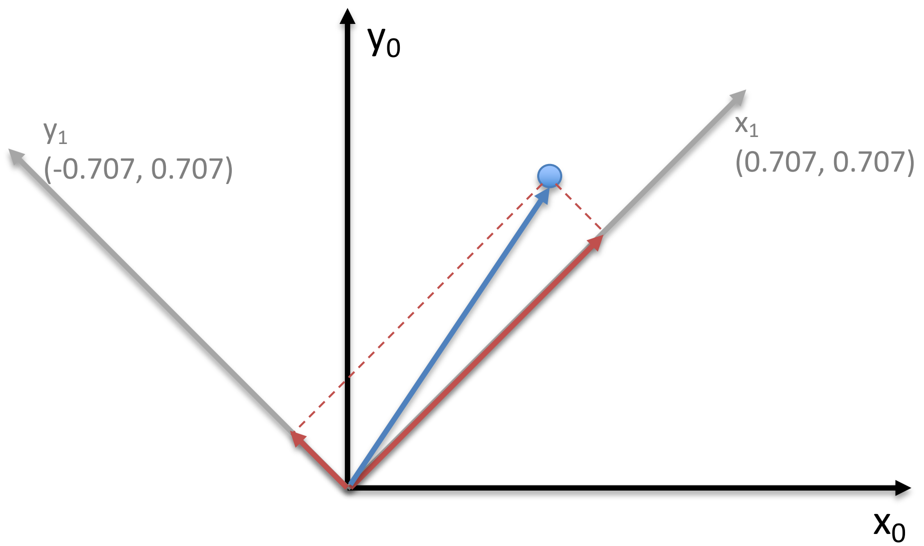
- How can you convert a vector represented in the coordinate frame “0” to a new, rotated coordinate frame “1”?
- Remember what a vector is:

[component in direction of the frame’s x axis, component in direction of y axis]



Rotation

- So to rotate it we must produce this vector:
  [component in direction of **new** x axis, component in direction of **new** y axis]
- We can do this easily with dot products!
- New x coordinate is [the new x axis] **dot** [original vector]
- New y coordinate is [the new y axis] **dot** [original vector]

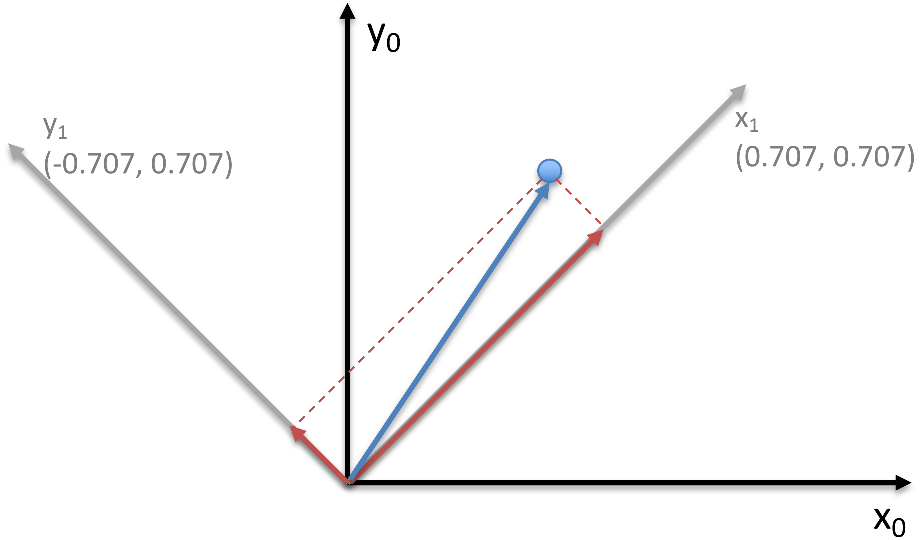


Rotation

- Insight: something similar happens in a matrix*vector multiplication!
- The resulting x coordinate, x', is: [matrix row 1] **dot** [original vector]

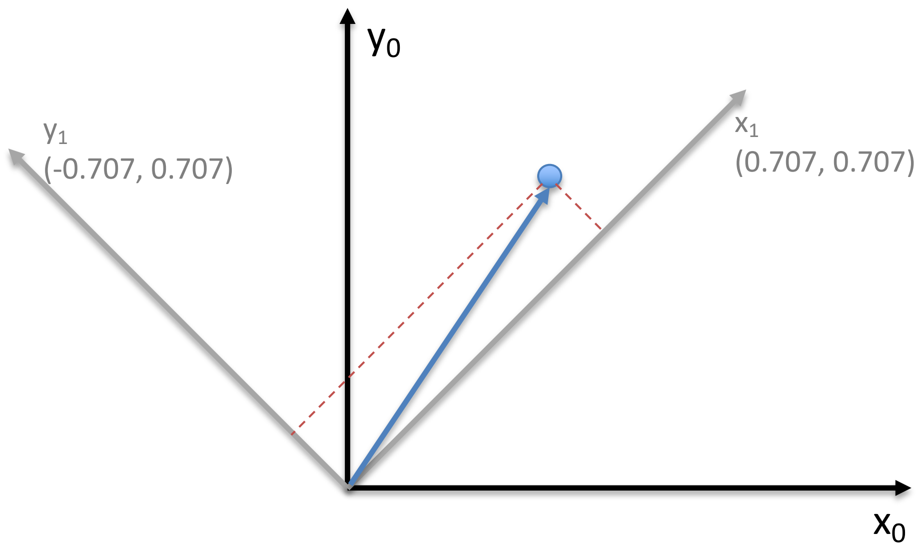
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- The matrix multiplication Rp = p' produces the coordinates in the new frame.



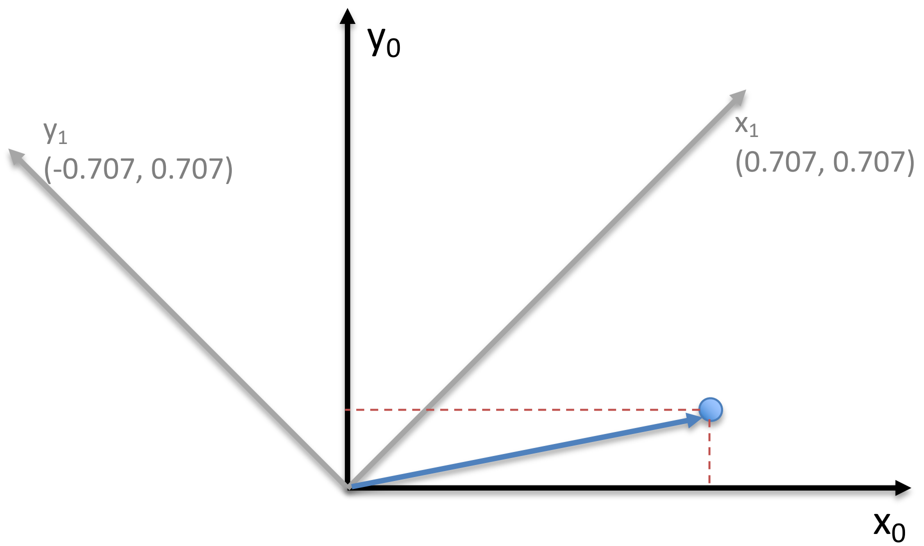
Rotation

- Now we have our point in the new coordinate system which is rotated left
- If we plot the result in the **original** coordinate system, we have rotated the point right



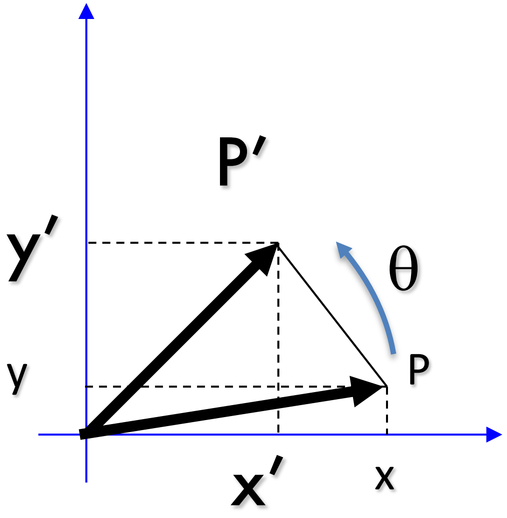
Rotation

- Now we have our point in the new coordinate system which is rotated left
- If we plot the result in the **original** coordinate system, we have rotated the point right
  - Thus, rotation matrices can be used to rotate vectors. We'll usually think of them in that sense-- as operators to rotate vectors



2D Rotation Matrix Formula

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\, x - \sin\theta\, y$$
$$y' = \cos\theta\, y + \sin\theta\, x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p' = Rp$$

Transformation Matrices

- Multiple transformation matrices can be used to transform a point:
$$p' = R_2 R_1 S p$$
- The effect of this is to apply their transformations one after the other, from **right to left**.
- In the example above, the result is equivalent to
$$p' = R_2(R_1(Sp))$$
- The result is exactly the same if we multiply the matrices first, to form a single transformation matrix:
$$p' = (R_2 R_1 S)p$$

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Homogeneous coordinates

- In general, a matrix multiplication lets us linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- This is sufficient for scaling, rotating, and skewing transformations.
- But notice, we can't add a constant! ☹
- That means, we cannot produce a new (translated) vector $\begin{bmatrix} x + k \\ y + k \end{bmatrix}$.

Homogeneous coordinates

- The (somewhat hacky) solution? Stick a "1" at the end of every vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Now we can rotate, scale, and skew like before, AND translate (note how the multiplication works out, above)
- This is called "homogeneous coordinates"

Homogeneous coordinates

- In homogeneous coordinates, the multiplication works out so the rightmost column of the matrix is a vector that gets added.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Generally, a homogeneous transformation matrix will have a bottom row of [0 0 1], so that the result has a “1” at the bottom too.

Homogeneous coordinates

- One more thing we might want: to divide the result by something
  - For example, we may want to divide by a coordinate, to make things scale down as they get farther away in an image
  - Matrix multiplication can't actually divide
  - So, **by convention**, in homogeneous coordinates, we'll divide the result by its last coordinate after doing a matrix multiplication

$$\begin{bmatrix} x \\ y \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x/7 \\ y/7 \\ 1 \end{bmatrix}$$

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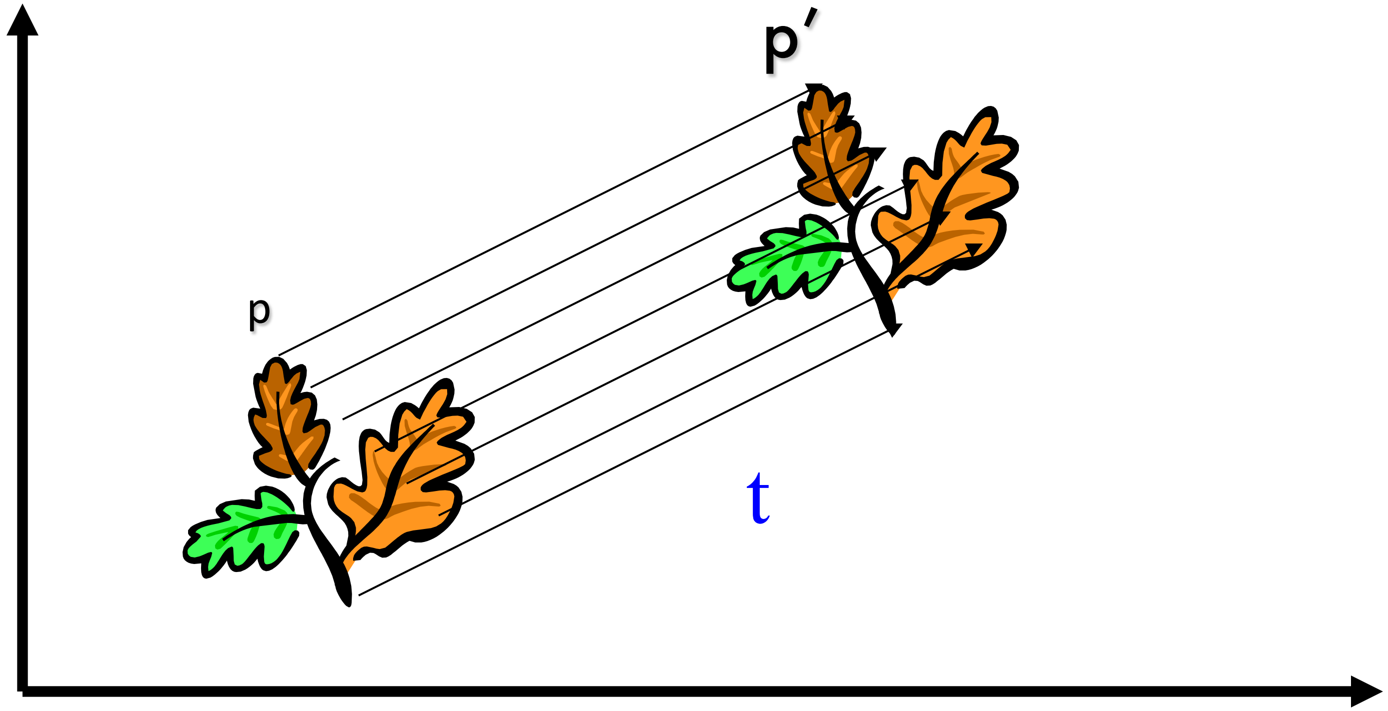
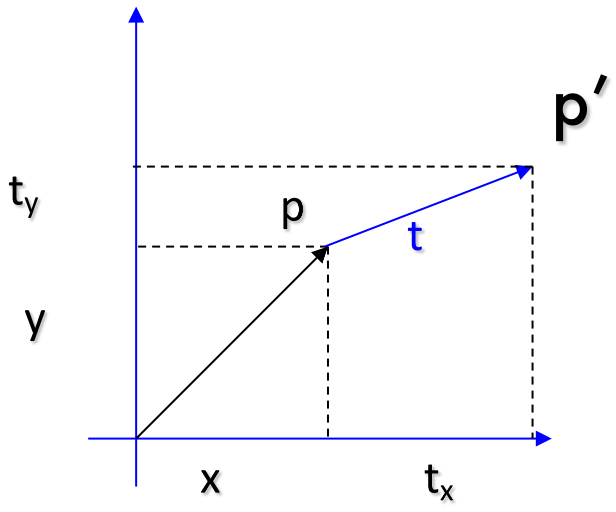
2D Translation



2D Translation using Homogeneous Coordinates



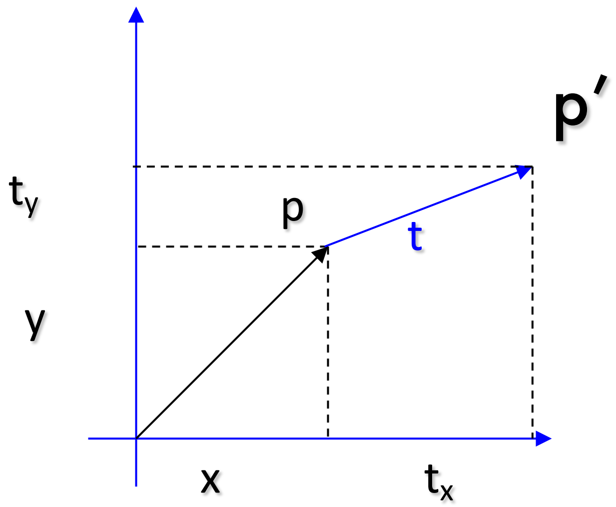
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \rightarrow \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Translation using Homogeneous Coordinates



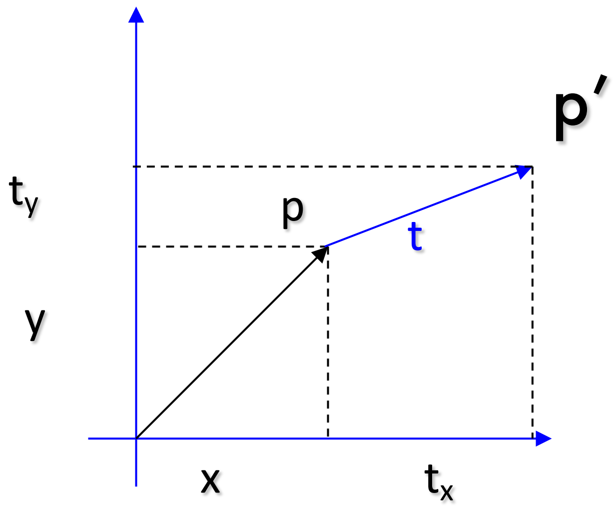
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \rightarrow \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Translation using Homogeneous Coordinates



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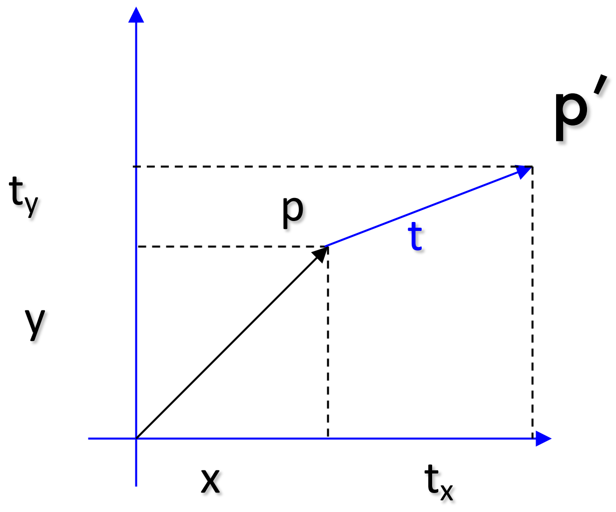
$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Translation using Homogeneous Coordinates



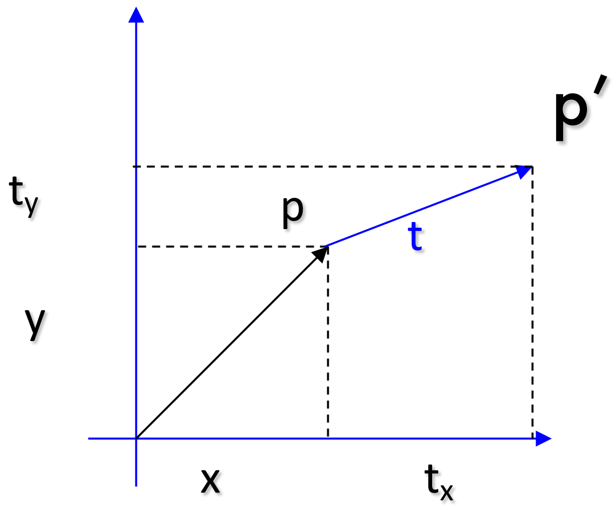
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \rightarrow \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Translation using Homogeneous Coordinates



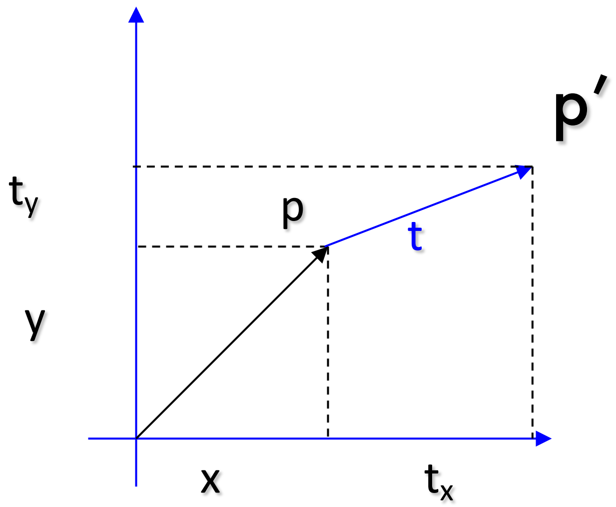
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \rightarrow \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Translation using Homogeneous Coordinates



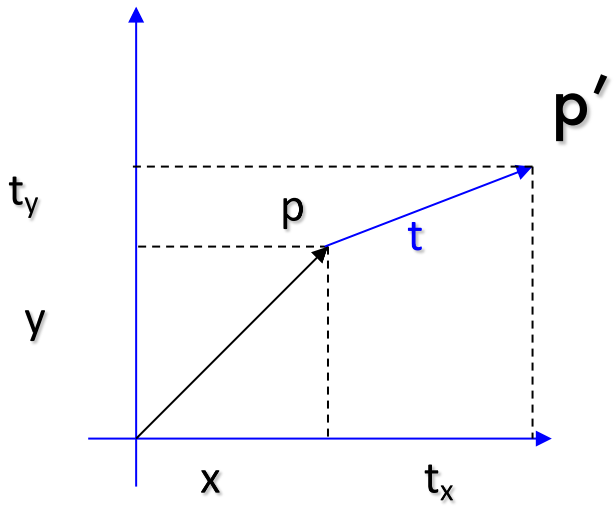
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \rightarrow \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Translation using Homogeneous Coordinates



$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix} p = Tp$$

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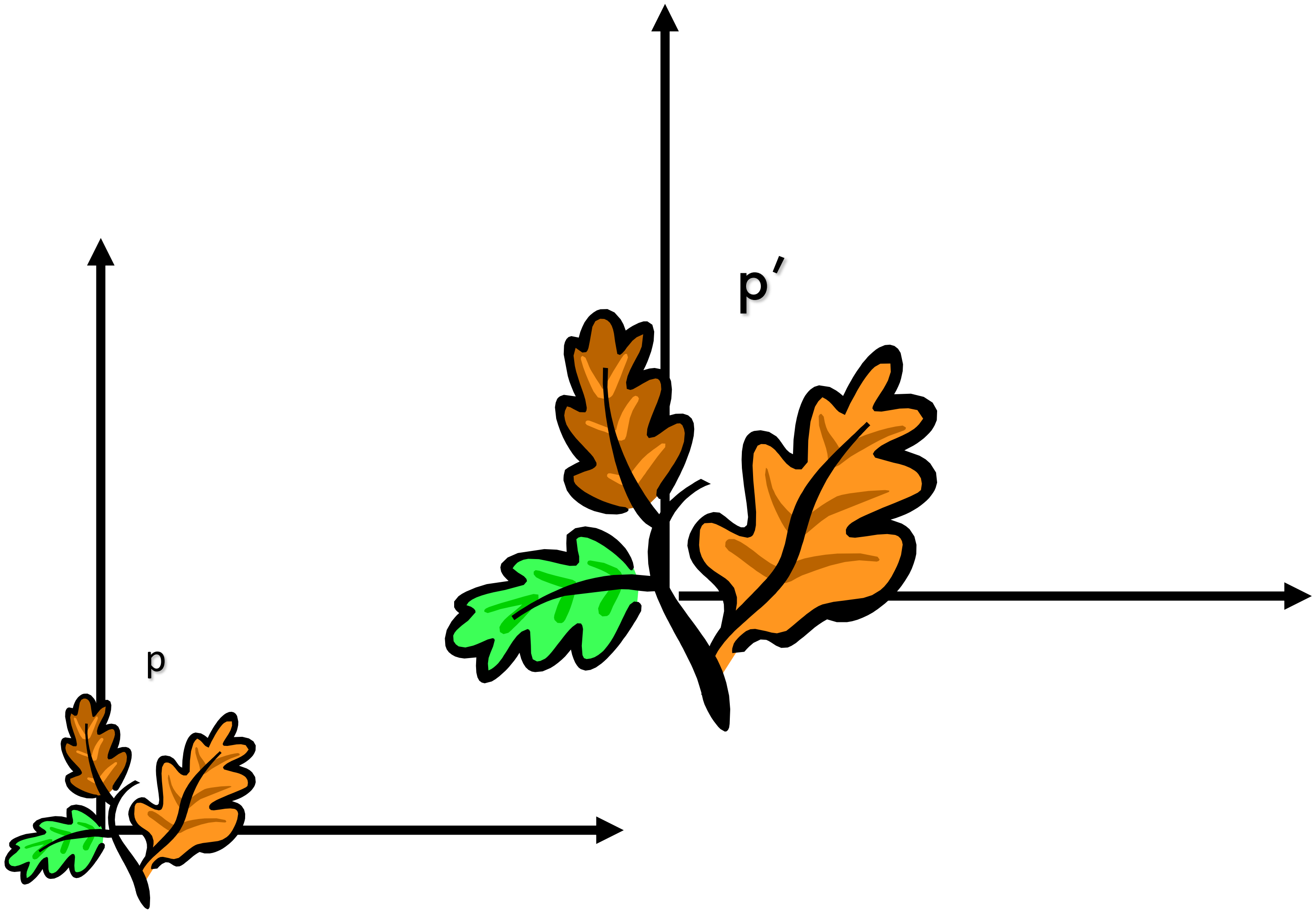
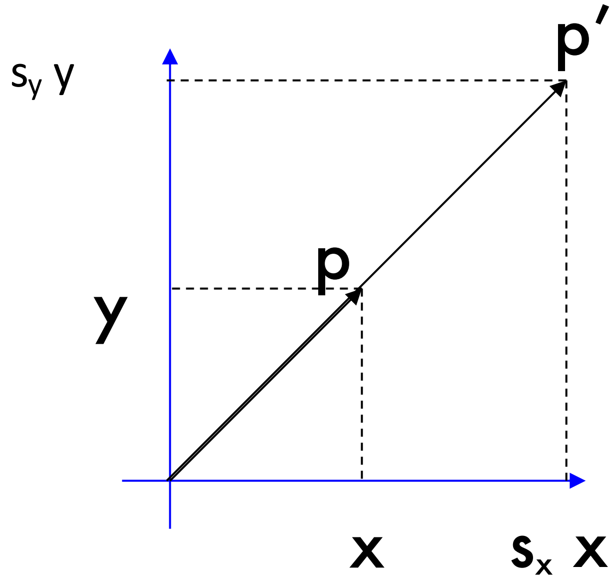
Scaling



Scaling Equation



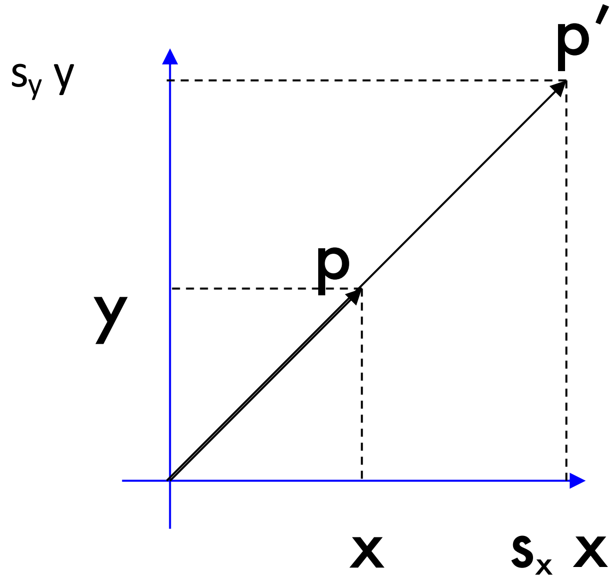
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p' = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix}$$

$$p' = Sp$$

$$p' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling Equation



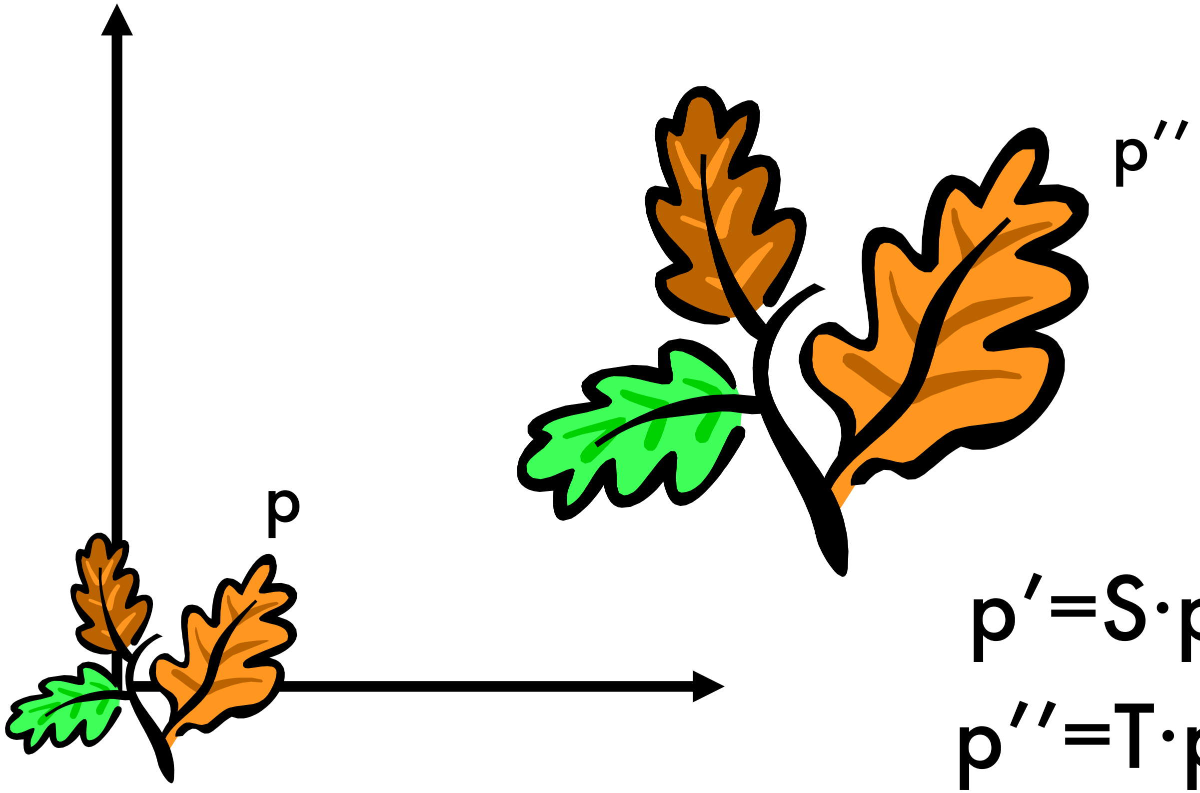
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p' = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix}$$

$$p' = Sp$$

$$p' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} S' & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} p = Sp$$

Scaling & Translating



$p' = S \cdot p$

$p'' = T \cdot p'$

$p'' = T \cdot p' = T \cdot (S \cdot p) = T \cdot S \cdot p$

Scaling & Translating

$$p'' = TSp = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} S' & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + tx \\ s_y y + ty \\ 1 \end{bmatrix}$$

Scaling & Translating != Translating & Scaling

$$p'' = TSp = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + tx \\ s_y y + ty \\ 1 \end{bmatrix}$$

$$p''' = STp = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

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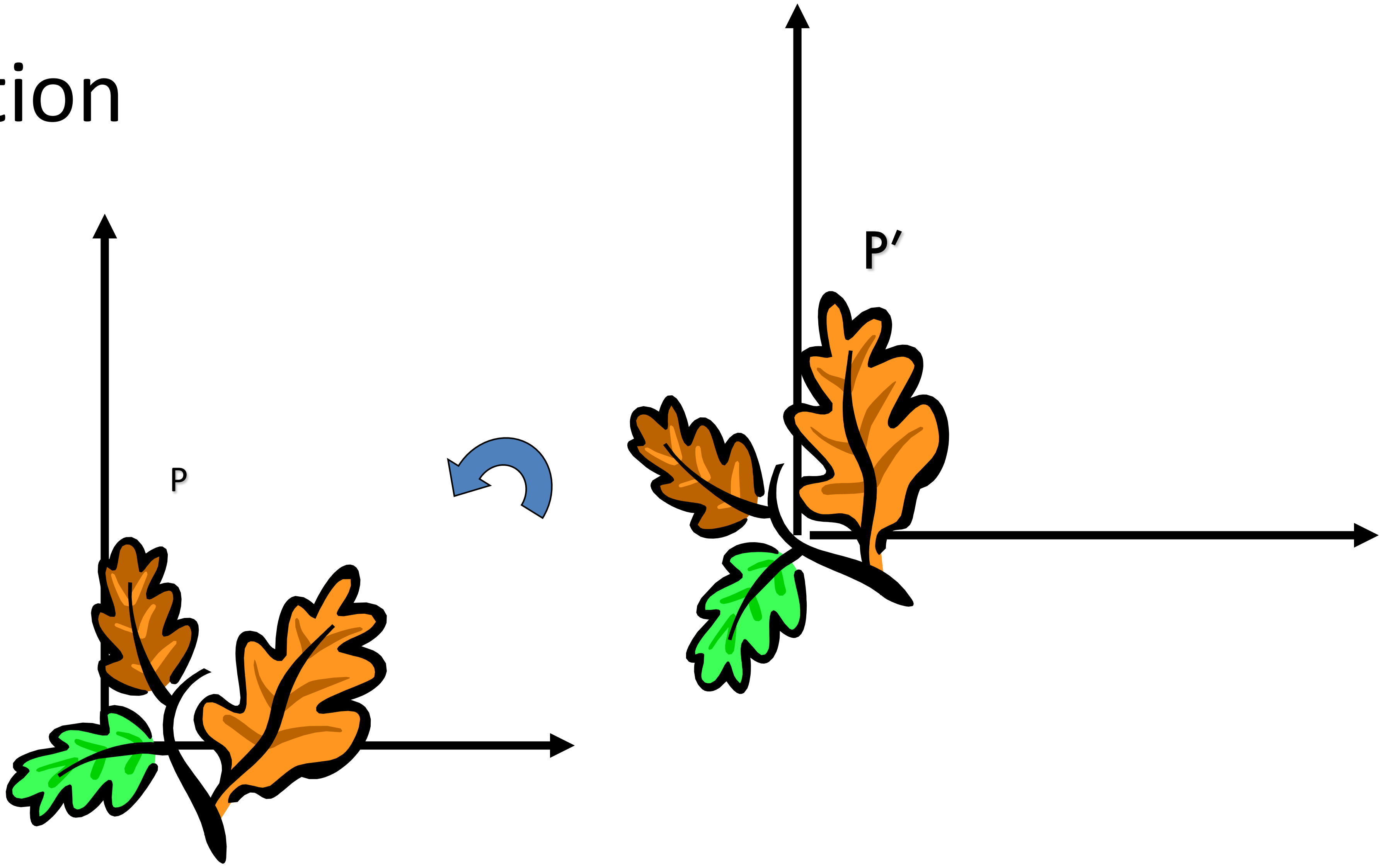
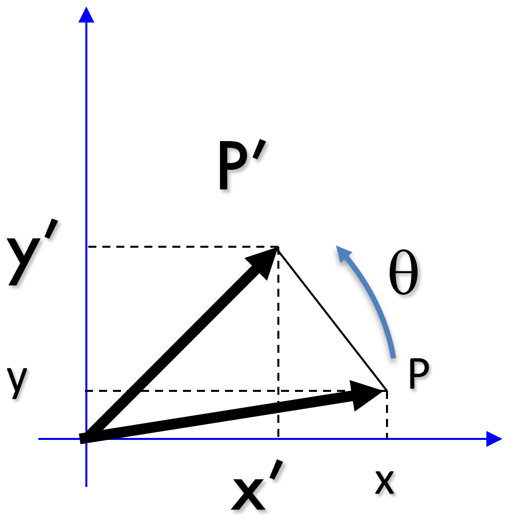
Rotation



2D Rotation Matrix Formula

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\, x - \sin\theta\, y$$

$$y' = \cos\theta\, y + \sin\theta\, x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p' = Rp$$

Rotation Matrix Properties

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A 2D rotation matrix is 2x2

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Rotation Matrix Properties

- Transpose of a rotation matrix produces a rotation in the opposite direction

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

- The rows of a rotation matrix are always mutually perpendicular (a.k.a. orthogonal) unit vectors
  – (and so are its columns)

Scaling + Rotation + Translation

p'= (T R S) p

$$p' = TRSp = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} RS & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is the form of the general-purpose transformation matrix

Summary

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