Richard Lindeman (1000722418)

AST 322 – HW Ch. 1

1/10/2018

Ry Ch 1.1: Derive from the order-of-magnitude equations for the Planck epoch the energy density during the Planck time: rho_P = c^7 / (h_bar . G^2)

Density: $\rho = \frac{m}{V}$ Energy: $e = mc^2$ Volume: $V = \frac{4}{3}\pi r^3$

Since all the mass is energy we replace m with mc^2. At the Planck time the mass is the Planck mass M_p and the radius is the Planck length l_P. Let's ignore the $\frac{4}{3}\pi$ factor to simplify the math. We are looking at numbers so fantastic the order of magnitude is really all that matters.

Therefore: $\rho_P = \frac{M_p c^2}{l_p^3}$

Planck Mass: $M_p = (\frac{\hbar c}{G})^{1/2}$ Planck Length: $l_P = (\frac{G\hbar}{c^3})^{1/2}$

Plugging these into the density equation above and squaring both sides to make the math easier results in this equation:

$$\rho_P^2 = \frac{\frac{\hbar c}{G} * c^4}{(\frac{G\hbar}{c^3})^3} = \frac{\hbar c}{G} * \frac{c^9}{G^3\hbar^3} * c^4 = \frac{c^{14}}{G^4\hbar^2}$$

Taking the square root gives us the final Planck energy density equation:

$$\rho_P = \frac{c^7}{\hbar G^2} \checkmark$$

3/3

$c = 3.00x10^8$ m/s $G = 6.67x10^{-11}$ m³/kgs² $\hbar = 1.05x10^{-34}$ Js

Therefore: $\rho_P = \frac{(3.00x10^8)^7}{(1.05x10^{-34})(6.67x10^{-11})^2} = 4.7x10^{113}$ kg/ms²

Ry Ch 1.2a: The Schwarzschild radius of a black hole is r_s = 2 G M_BH / c^2. Derive the Schwarzschild radius of the Universe during the Planck time, r_s,P.

Planck Mass: $M_p = 2.18x10^{-8} kg$

Therefore, the Schwarzschild radius of the universe during Planck time was:

$r_s = \frac{2GM}{c^2} = \frac{2(6.67x10^{-11})(2.18x10^{-8})}{(3.00x10^8)^2} = 3.23x10^{-35}m$

2/3

?

Ry Ch 1.2b: How does r_s,P compare to l_P? Describe the fallacy here.

Planck Length: $l_p = 1.62x10^{-35}m$

derive

The Schwarzschild radius of the universe at the Plank time is twice as large as the Plank length.

2/2

$$\frac{r_s}{l_p} = \frac{3.23x10^{-35}}{1.62x10^{-35}} = 2.00 \qquad \checkmark$$

Therefore the whole universe should be a black hole and nothing should ever be able to escape it. Yet, here we are today, outside that radius.

**Ry Ch 1.2c: Briefly describe two possible ways out of this dilemma.**

1) The universe never really was that small. At some point winding the clock backward and watching the mass get hotter and denser and closer together isn't really what happened.
2) Pressures and temperatures were so great that our understanding of basic physics breaks down and under those extreme conditions even gravity acts differently allowing expansion beyond what we understand to be possible today.

2/2

Ry Ch 1.2d (Extra Credit): Later on this semester, we will ask you to compute r_s,U for the universe today, and compare it to the Hubble radius r_H today. Describe the same issue here as in 1.2a--1.2c.

The estimated mass of the universe within the Hubble radius today is much larger than the Planck mass. Not counting dark matter or dark energy, estimates are in the range of 10^{53} kg. This estimate yields a Schwarzschild radius of our current universe of:

$$r_s = \frac{2GM}{c^2} = \frac{2\,(6.67x10^{-11})(10^{53})}{(3.00x10^8)^2} = 1.48x10^{27}m$$

Since the Hubble radius is $1.3x10^{26}$ m the same problem arises. The entire universe should be inside a black hole. If we are inside a black hole, why is it letting us expand, it doesn't make sense.

2/2

11/10
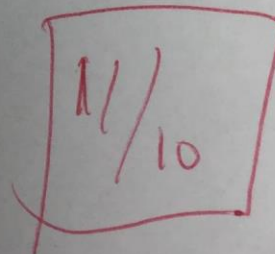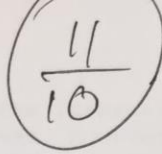
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AST 322 – HW Ch. 2

1/16/2018

Ry Ch 2.1: Assume you are a perfect blackbody at a temperature of T = 310 K. What is the rate, in watts, at which you radiate energy? (For the purposes of this problem, you may assume you are spherical.)

Blackbody radiation power equation (in Watts):

$\frac{3}{3}$

$$P = A\varepsilon\sigma T^4$$

$A = Surface\ Area\ (m^2), \varepsilon = 1\ for\ perfect\ blackbody, T = Temperature\ in\ K, \sigma = 5.67x10^{-8}\ \frac{W}{m^2K^4}$

Assume for simplicity that I am a sphere of radius 1m, therefore:

$$A = 4\pi r^2 = 4\pi \approx 12.6\ m^2$$

$$P = (12.6)(5.67x10^{-8})(310)^4 = 6.60x10^3\ W$$

This is the rate I would radiate my energy given my temperature and surface area if I were not receiving any energy from the surroundings.

Ry Ch 2.3: Suppose that intergalactic space pirates toss you out the airlock of your spacecraft without a spacesuit. Combining the results of the two previous questions, at what rate would your temperature change? (Assume your heat capacity is that of pure water, C = 4200 J kg-1 K-1 .) Would you be most worried about overheating, freezing, or asphyxiating?

Power gained from the microwave background radiation of space is negligible and can be ignored given the temperature of only 2.7K:

$$P = (12.6)(5.67x10^{-8})(2.73)^4 = 3.97x10^{-5}\ W$$

Specific heat capacity of water is $= 4200\ \frac{J}{kg\ K}$. I am mostly water, so let's assume that my total heat capacity is my mass (approx. 100kg) times C. This gives us the number of Joules needed to change my body mass temperature by one K.

$$\Delta T = \frac{\Delta Q}{C}$$

Since Joules per second is Watts and we know that number from above lets solve this equation for the change in temperature for a single second.

$$\Delta T = \frac{6.60x10^3}{(4200)(100)} = .0157\ K$$

This is how many K my temperature will change every second. Multiplying it by 60 gives us .94 K per minute. This means that my body temperature will drop by just under one K per minute floating in space.

The body would lose this temperature disproportionately near the outside of the body, keeping the internal organs warmer longer. Also, it takes several

degrees of temperature loss at the internal organs for several minutes before
any significant damage is observed. However, lack of oxygen to the brain can
cause permanent brain damage and death in 3-5 minutes. Also, you can't "hold
your breath" in space due to the pressure differences, so your body uses up
the oxygen in the blood very quickly and you would pass out. Therefore the
biggest worry would be asphyxiation.

Ry Ch 2.6: Show that for an energy threshold $E_0 \ll kT$, the fraction of blackbody
photons that have energy $hf < E_0$ is:

$$\frac{n(hf < E_0)}{n_\gamma} \approx 0.21 (\frac{E_0}{kT})^2$$

Microwave (and far infrared) photon with a wavelength $\lambda < 3\,cm$ are strongly absorbed by
H_2O and O_2 molecules. What fraction of the photons in today's cosmic microwave
background have $\lambda > 3\,cm$, and thus are capable of passing though the Earth's atmosphere
and being detected on the ground?

equation 2.30 from Book : $n(f)\,df = \frac{E(f)\,df}{hf} = \frac{8\pi}{c^3} \frac{f^2\,df}{e^{hf/kT} - 1}$

mclauren expansion of $e^x = \underbrace{1 + x}_{} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots$

use first 2 terms as simplification / estimation of $e^{hf/kT}$: $n(f)\,df \approx \frac{8\pi}{c^3} \frac{f^2 df}{1 + \frac{hf}{kT} - 1} = \frac{8\pi}{c^3} \frac{f^2\,df\ kT}{hf}$

$$= \frac{8\pi\ kT}{c^3 h} f\,df$$

$E = hf$ so $f = \frac{E}{h}$ replacing f yields: $\frac{8\pi kT}{c^3 h^3} E\,dE$

$dE = h\,df$ so $df = \frac{dE}{h}$ and df

integrating from 0 to E_0 : $n(hf < E_0) \approx \int_0^{E_0} \frac{8\pi kT}{c^3 h^3} E\,dE = \frac{8\pi kT}{c^3 h^3}\int_0^{E_0} E\,dE = \frac{8\pi kT}{c^3 h^3} \frac{E_0^2}{2}$

$$= \frac{4\pi kT E_0^2}{c^3 h^3}$$

equation 2.31 from Book : $n_\gamma = \beta T^3$ $\beta = \frac{2.4041\ k^3}{\pi^2 \hbar^3 c^3}$

$\hbar = \frac{h}{2\pi}$ so $n_\gamma = \frac{2.4041\ k^3 8\pi^3 T^3}{\pi^2 h^3 c^3} = \frac{2.4041\ k^3 8\pi T^3}{h^3 c^3}$

finally solving for the fraction:

$$\frac{n(hf < E_0)}{n_\gamma} \approx \frac{4\pi kT E_0^2}{c^3 h^3} \cdot \frac{h^3 c^3}{2.4041\ k^3 8\pi T^3} = 0.208 \frac{E_0^2}{k^2 T^2}$$

$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{3 \times 10^{-2}} = 1 \times 10^{10}$ Hz $h = $ plancks constant $= 6.626 \times 10^{-34}$ J/s

$K = $ Boltzmann constant $= 1.38 \times 10^{-23}$ J/K

$E_0 = hf = 6.626 \times 10^{-34} (1 \times 10^{10}) = 6.626 \times 10^{-24}$ J

Temperature of microwave background $T = 2.7255$ K

fraction passed $\approx .21 (\frac{E_0}{kT})^2 = .21 (\frac{6.626 \times 10^{-24}}{(1.38 \times 10^{-23})(2.7255)})^2 = .0065$

$$= 0.65\%$$

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AST 322 – HW Ch. 3

1/26/2018

3a: Prove Ryden equations (3.20) and (3.21) from (3.19)

3.19 $(\Delta s')^2 = -c^2(\Delta t')^2 + (\Delta x')^2$

+2

Lorentz transformation computes spacial distance measured in primed frame

$(\Delta x')^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2 = r^2[x_1 - x_2 - v(t_1 - t_2)]^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

time elapsed between events in primed frame is:

$\Delta t' = t_1 - t_2 = r[t_1 - t_2 - \frac{v}{c^2}(x_1 - x_2)]$

Substituting into origional eqaution

$(\Delta s')^2 = -c^2 r^2 [t_1 - t_2 - \frac{v}{c^2}(x_1 - x_2)]^2 + \gamma^2[x_1 - x_2 - v(t_1 - t_2)]^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

taking c inside brackets gets us to equation 3.20

3.20 $(\Delta s')^2 = -r^2[c(t_1 - t_2) - \frac{v}{c}(x_1 - x_2)]^2 + \gamma^2[x_1 - x_2 - v(t_1 - t_2)]^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

$= -r^2[c^2(t_1 - t_2)^2 + \frac{v^2}{c^2}(x_1 - x_2)^2 - 2v(t_1 - t_2)(x_1 - x_2)] + \gamma^2[(x_1 - x_2)^2 + v^2(t_1 - t_2)^2 - 2v(x_1 - x_2)(t_1 - t_2)$

$+ (y_1 - y_2)^2 + (z_1 - z_2)^2$

$= r^2[-c^2(t_1 - t_2)^2 - \frac{v^2}{c^2}(x_1 - x_2)^2 + (x_1 - x_2)^2 + v^2(t_1 - t_2)^2] + (y_1 - y_2)^2 + (z_1 - z_2)^2$

$= \gamma^2[(v^2 - c^2)(t_1 - t_2)^2 - \frac{v^2}{c^2}(x_1 - x_2)^2 + (x_1 - x_2)^2] + (y_1 - y_2)^2 + (z_1 - z_2)^2$

if $v \ll c$ then simplifies to: $\gamma^2 = \frac{1}{1 - v^2/c^2}$ then distribute

3.21 $(\Delta s')^2 = -c^2(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

3b: In the caption for Ryden Figure 3.3, it states that the deflection of a light beam traveling across a box 2 meters wide subjected to an acceleration of g=9.8m/s/s will be about 2*10 -14 m. Is Ryden correct about this? Do the calculation for yourself.

y = deflection

$g = 9.8 \, m/s^2$

$c = 3 \times 10^8 \, m/s$

⊢—2m—⊣
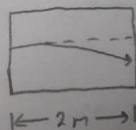
$t = \frac{d}{v} = \frac{2}{3 \times 10^8} = 6.67 \times 10^{-9} \, s$

deflection $y = \frac{1}{2}at^2 = \frac{1}{2}(9.8)(6.67 \times 10^{-9}) = 2.18 \times 10^{-16} \, m$

no, Ryden has a typo. Probably meant $2 \times 10^{-14} \, cm$.

ok!

+3 ~~deceased~~

Suppose you wanted to measure the deflection of such a beam to verify whether or not this deflection actually occurs. To do this, you have a long, perfectly flat room, with a beam emitter aligned perfectly parallel to the ground at one end, and a detector wall at the other end. If you were able to measure the deflection of this beam (relative to a perfectly straight line) with a precision of 10nm, how far away would your detector wall have to be from your emitter?

if you can measure a deflection (y) of 10nm (10×10^{-9} m)   Then:

$$y = \frac{1}{2} a t^2 \qquad t^2 = \frac{2y}{a} \qquad t = \sqrt{\frac{2y}{a}} = \sqrt{\frac{2(10 \times 10^{-9})}{9.8}} = 4.5 \times 10^{-5} \, s$$

$$t = \frac{L}{v} \qquad L = t \cdot v = (4.5 \times 10^{-5})(3 \times 10^{8}) = 1.355 \times 10^{4} \, m$$

The room would need to be 13.6 Km long.

3c: Show that Ryden equation (3.36) can be derived from (3.33), (3.34), and the assumption that x = S$_\kappa$ (r)

3.33 $$d\ell^2 = dr^2 + S_\kappa(r)^2 \, d\Omega^2$$

3.34 $$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$x = S_\kappa(r)$$

Substituting into original equation

$$d\ell^2 = dr^2 + x^2 d\Omega^2$$

$x = R \sin(r/R)$ for K = +1

$$\frac{dx}{dr} = \cos(\frac{r}{R})$$

$$\frac{dx^2}{dr^2} = \cos^2(\frac{r}{R})$$

$$\frac{dx^2}{dr^2} = 1 - \sin^2(\frac{r}{R})$$

$$dr^2 = \frac{dx^2}{1 - \sin^2(\frac{r}{R})}$$

Since $x = R \sin(r/R)$

$$\sin(\frac{r}{R}) = \frac{x}{R}$$

$$\sin^2(\frac{r}{R}) = \frac{x^2}{R^2}$$

Substitutin in

$$dr^2 = \frac{dx^2}{1 - \frac{x^2}{R^2}}$$

So $$d\ell^2 = \frac{dx^2}{1 - \frac{x^2}{R^2}} + x^2 d\Omega^2 \text{ for } K = +1$$

for K = -1

$x = R \sinh(r/R)$

$$\frac{dx}{dr} = \cosh(\frac{r}{R})$$

$$\frac{dx^2}{dr^2} = \cosh^2(\frac{r}{R})$$

$$\frac{dx^2}{dr^2} = 1 + \sinh^2(\frac{r}{R})$$

$$dr^2 = \frac{dx^2}{1 + \sinh^2(\frac{r}{R})}$$

Since $x = R \sinh(r/R)$

$$\sinh(r/R) = \frac{x}{R}$$

$$\sinh^2(r/R) = \frac{x^2}{R^2}$$

Substituting in

$$dr^2 = \frac{dx^2}{1 + \frac{x^2}{R^2}}$$

So $$d\ell^2 = \frac{dx^2}{1 + \frac{x^2}{R^2}} + x^2 d\Omega^2 \text{ for } K = -1$$

for K = 0

$$x = r$$

$$d\ell^2 = dx^2 + x^2 d\Omega^2 \text{ for } K = 0$$

+3

+1

Combine all 3 answers into one equation

3.36 $$d\ell^2 = \frac{dx^2}{1 - K\frac{x^2}{R^2}} + x^2 d\Omega^2$$

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AST 322 – HW Ch. 4

2/12/2018

4a: Prove equations (4.17) and (4.18).

homogeneous, isotropic expansion or contraction

$R_s(t)$ Sphere of radius $R_s(t)$, M_s

Newtons law of gravity $F = -\dfrac{G M_s m}{R_s(t)^2}$

gravitational acceleration at surface of sphere $\dfrac{d^2 R_s}{dt^2} = -\dfrac{G M_s}{R_s(t)^2}$

$$\int \left(\dfrac{d^2 R_s}{dt^2}\right)^2 = \int \dfrac{(G M_s \, d R_s}{R_s(t)^2 \, dt}$$

$$\dfrac{1}{2}\left(\dfrac{d R_s}{dt}\right)^2 = \dfrac{G M_s}{R_s(t)} + U$$

$E_{Kinetic}$ per unit mass $E_{gravitational}$ potential energy per unit mass

$M_s = \dfrac{4\pi}{3} \rho(t) R_s(t)^3$

Since expansion or contraction is isotropic $R_s(t) = a(t) r_s$

Substituting in :

4.17 $\quad \dfrac{1}{2}\dot{a}^2 r_s^2 = \dfrac{4\pi}{3}\rho(t)\dfrac{a(t)^3 r_s^3}{a(t) r_s}G = \dfrac{4\pi}{3}G r_s^2 \rho(t) a(t)^2 + U$ $\overset{+U}{}$

$\dfrac{3}{3}$

multiply by $\dfrac{2}{r_s^2 a^2}$

4.18 $\quad \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi G}{3}\rho(t) + \dfrac{2U}{r_s^2 a(t)^2}$

4b: Use Ryden equations (4.26) and (4.33) to show that $\left(1 - \dfrac{1}{\Omega(t)}\right) = (1+z)^{-1}\left(1 - \dfrac{1}{\Omega_o}\right)$

4.26 $\quad H(t)^2 = \dfrac{8\pi G}{3c^2}\varepsilon(t) - \dfrac{\kappa c^2}{R_o a(t)^2}$ $\qquad H(t) = \dfrac{\dot{a}}{a}$

4.33 $\quad \Omega(t) = \dfrac{\varepsilon(t)}{\varepsilon_c(t)} = \dfrac{8\pi G}{3c^2 H(t)}\varepsilon(t)$

4.26/$H(t)^2$ $\quad 1 = \dfrac{\varepsilon(t)}{\varepsilon_c(t)} - \dfrac{\kappa c^2}{R_o a(t)^2 H(t)^2}$

Substituting in 4.33

4.34 $\quad 1 - \Omega(t) = -\dfrac{\kappa c^2}{R_o^2 H(t)^2 a(t)^2}$ $\qquad 1 - \Omega_o = -\dfrac{\kappa c^2}{R_o^2 H_o^2}$

$1 + z = \dfrac{1}{a(t)}$

Substituting

$$1 - \Omega(t) = \dfrac{(1-\Omega_o)}{a(t)^2} = (1+z)^{-2}(1-\Omega_o)$$

Also prove (4.35) and (4.36). Discuss the implication of these equations, particularly as they relate to the theory of inflation.

Starting with 4.39 as shown above

$$1 - \Omega(t) = \frac{-Kc^2}{R_0^2 H(t)^2 a(t)^2}$$

$$H(t) = \frac{\dot{a}}{a} \qquad so \qquad 1 - \Omega_0 = -\frac{Kc^2}{R_0^2 H_0^2} \qquad when \ time \ is \ now$$

rearranging $\frac{K}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)$

The equations suggest an increasing variance system. meaning that a small variance from a perfect $\Omega_0 = 1$ in the distant past (large red shift) would make the Ω today much farther from 1. which means that if we measure an Ω of 1 to several decimal places of accuracy today, in the very early universe of a redshift of 10,000, the universe would need to have been $\Omega = 1$ with an accuracy of at least 4 more decimal places. And if all things are random and there is an equal chance of having any universe Ω then why would we be in one with such a near perfect $\Omega = 1$ if it isn't exactly 1. So it seems reasonable to assume $\Omega =$ exactly 1 in our universe which translates back into a $K = 0$ and a flat universe.

$$\frac{4}{4}$$

4c: Do Ryden Exercise 4.2 - Consider Einstein's static universe, in which the attractive force of the mater density ρ is exactly balanced by the repulsive force of the cosmological constant, $\Lambda = 4\pi G\rho$. Suppose that some of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Explain your answer.

$$\Lambda = 4\pi G\rho$$

Gives universe a constant energy density $\mathcal{E}_\Lambda$ and pressure $P_\Lambda = -\mathcal{E}_\Lambda$ and no $a(t)$

Einstein's Static universe is Static, but unstable. with Stars converting matter into radiation, radiation pressure increases, the ballance is tipped and the whole Thing goes into runaway acceleration. It would expand and at an increasing rate over time.

$$\frac{2}{3}$$

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AST 322 – HW Ch. 5

2/26/2018

5a: Show equations (5.38), (5.39), (5.41), and (5.42). Use the equations to find the current age of the universe if you assume that w = 0 and the energy density is the current empirically known energy density of our universe. Discuss the physical meaning of the assumptions you must make for these equations to be true.

5b: Show equations (5.53) through (5.57), (5.58) through (5.62), and (5.70) through (5.74), using equations (5.36) through (5.52).

5c: Using a plotting program, and assuming H_0 = 70 km/s/Mpc, make one plot for d $_p$(t $_e$) and one plot for d $_p$(t $_0$) as functions of z, with different curves for w=0, w=⅓, and w=-1. Use a log-log scale, label appropriately, print, and turn these plots in with your homework. You may compare them to Ryden figure 5.3 to check your answers.

5a)

$(5.38) \quad q = \frac{2}{3+3w}$ assume $w = 0$

$(5.39) \quad a(t) = \left(\frac{t}{t_o}\right)^{2/3+3w}$ find t_o

derivations next on post

$(5.41) \quad H_o = \left(\frac{\dot{a}}{a}\right)_{t=t_o} = \frac{2}{3(1+w)} t_o^{-1}$

$(4.31) \quad \varepsilon_{c,o} = \frac{3c^2}{8\pi G} H_o^2 = 7.8 \times 10^{-10} \, J/m^3 = 4870 \, Mev/_{m^3}$

$(4.27) \quad H_o = H(t_o) = \left(\frac{\dot{a}}{a}\right)_{t=t_o} = 68 \, Km/s /Mpc$

$(5.42) \quad t_o = \frac{2}{3(1+w)} H_o^{-1}$

$t_o = \frac{2}{3 \cdot 68} \frac{s \cdot Mpc}{Km} \cdot \frac{3.0887 \times 10^{19} \, Km}{Mpc} = 3.025 \times 10^{17} s \cdot \frac{1 \, yr}{31536000 \, s} = 9.59 \times 10^{9} \, yr$

$\approx 9.6 \, Gyr$

     This estimate of the age of the universe of 9.6 Billion years has several assumptions built in. This universe is flat and all the energy density is in the form of matter (w=0). This is obviously not true. Our universe has matter, but also radiation (w=⅓) and current estimates are that the cosmological constant Λ equates to an $\Omega_{\Lambda,o}$ of .69 wich is even more significient than the matter portion $\Omega_{m,o}$ of .31. The cosmological constant portion of the matter density has a w=-1 so that would significiently change the estimated age of the universe, if that were added into the equations.

5a) Derivations:

(5.25) Friedmann equation

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_i \varepsilon_{i,0}\, a^{-(1+3w_i)} - \frac{Kc^2}{R_0^2}$$

for spacially flat universe $K=0$

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_i \varepsilon_{i,0}\, a^{-(1+3w_i)}$$

for universe with only one component

$$\dot{a}^2 = \frac{8\pi G \varepsilon_0}{3c^2} a^{-(1+3w)}$$

if scale factor follows power law then $a \propto t^{q}$ and $\dot{a} \propto t^{2q-2}$

$$t^{2q-2} = \frac{8\pi G \varepsilon_0}{3c^2} t^{-(1+3w)q}$$

equating powers of t

$$2q - 2 = -(1+3w)q$$

$$\frac{-2}{q} + 2 = -1 - 3w$$

$$\frac{-2}{q} = -1 - 3w - 2$$

$$\frac{-2}{q} = -3w - 3$$

(5.38)
$$q = \frac{2}{3w+3} \qquad \text{for } w \neq -1$$

so: $a \propto t^{\left(\frac{2}{3w+3}\right)}$

However in accelerating flat universe $a = \frac{t}{t_0}$ so to normalize this

(5.39)
$$a(t) = \left(\frac{t}{t_0}\right)^{2/(3w+3)}$$

$$t_0 = \frac{1}{1+w}\left(\frac{c^2}{6\pi G \varepsilon_0}\right)^{1/2} \qquad \varepsilon_0 = \frac{c^2}{6\pi G (1+w)^2} t_0^{-2}$$

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{8\pi G \varepsilon_0 \dot{a}}{3c^2} \cdot \frac{3}{2}\frac{(1+w) t_0}{\dot{a}} = \frac{4\pi G (1+w) t_0 \varepsilon_0}{c^2}$$

(5.41)
$$\frac{4\pi G (1+w) t_0}{c^2} \cdot \frac{c^2}{6\pi G (1+w)^2} t_0^{-2} = \frac{2}{3(1+w)} t_0^{-1}$$

rearrange for t_0

(5.42)
$$t_0 = \frac{2}{3(1+w)} H_0^{-1}$$

with only non-relativistic matter $w = 0$

starting with (5.42) $t_o = \dfrac{2}{3(1+w)H_o}$

(5.33) $t = \dfrac{2}{3H_o}$

for flat universe where $w > -1/3$

(5.52) $d_{hor}(t_o) = ct_o\dfrac{3(1+w)}{1+3w} = \dfrac{c}{H_o}\dfrac{2}{1+3w}$
plugging in $w=0$

(5.54) $d_{hor}(t_o) = \dfrac{2c}{H_o} = 3ct_o$

(5.39) $a(t) = \left(\dfrac{t}{t_o}\right)^{\frac{2}{3+3w}}$
plugging in $w = 0$

(5.55) $a_m(t) = \left(\dfrac{t}{t_o}\right)^{2/3}$

(5.49)
$$d_p(t_o) = c\int_{t_e}^{t_o}\dfrac{dt}{a(t)} = ct_o\dfrac{3(1+w)}{1+3w}\left[1-\left(t_e/t_o\right)^{(1+3w)/(3+3w)}\right]$$
plugging in $w=0$

$$d_p(t_o) = c\int_{t_e}^{t_o}\dfrac{dt}{a(t)} = ct_o(3)\left[1-\left(\dfrac{t_e}{t_o}\right)^{1/3}\right]$$

plugging in 5.53-5.55 and $a = \dfrac{1}{1+z}$ (rearrange 5.55 to $\dfrac{t_e}{t_o}=a^{3/2}$) So $\left(\dfrac{t_e}{t_o}\right)^{1/3}=a^{1/2}=(1+z)^{-1/2}$

(5.56) $d_p(t_o) = c\int_{t_e}^{t_o}\dfrac{dt}{\left(t/t_o\right)^{2/3}} = \dfrac{2c}{H_o}\left[1-(1+z)^{-1/2}\right]$

to get proper distance at time of emission must divide by $(1+z)$

(5.57) $d_p(t_e) = \dfrac{2c}{H_o(1+z)}\left[1-\dfrac{1}{\sqrt{1+z}}\right]$

---

with Radiation only $w = 1/3$

(5.42) $t_o = \dfrac{2}{3(1+w)H_o}$
plugging in $w = 1/3$

(5.58) $t_o = \dfrac{2}{3(1+\frac{1}{3})H_o} = \dfrac{2}{3(\frac{4}{3})H_o} = \dfrac{1}{2H_o}$

$d_{hor}(t_o) = ct_o\dfrac{3(1+w)}{1+3w} = \dfrac{c}{H_o}\dfrac{2}{1+3w}$
plugging in $w = 1/3$

(5.59) $d_{hor}(t_o) = ct_o\dfrac{3(\frac{4}{3})}{2} = 2ct_o = \dfrac{c}{H_o}\dfrac{2}{2} = \dfrac{c}{H_o}$

$a(t) = \left(\dfrac{t}{t_o}\right)^{\frac{2}{3+3w}}$
plugging in $w = 1/3$

(5.60) $a(t) = \left(\dfrac{t}{t_o}\right)^{\frac{2}{4}} = \left(\dfrac{t}{t_o}\right)^{1/2}$

$$d_p(t_o) = c\int_{t_e}^{t_o}\dfrac{dt}{a(t)} = ct_o\dfrac{3(1+w)}{1+3w}\left[1-\left(\dfrac{t_e}{t_o}\right)^{(1+3w)/(3+3w)}\right]$$

plugging in $w = 1/3$

$$d_p(t_o) = C \int_{t_e}^{t_o} \frac{dt}{a(t)} = C t_o \left(\frac{3(4/3)}{2}\right)\left[1 - \left(\frac{t_e}{t_o}\right)^{2/(4/3)}\right]$$

plugging in 5.60, $a = (1+z)^{-1}$, and $\left(\frac{t_e}{t_o}\right)^{1/2} = a = (1+z)^{-1}$

(5.61) $d_p(t_o) = C \int_{t_e}^{t_o} \frac{dt}{(t/t_o)^{1/2}} = 2Ct_o\left[1 - \left(\frac{t_e}{t_o}\right)^{1/2}\right] = \frac{C}{H_o}\frac{z}{1+z}$

$$1 - \frac{1}{1+z} = \frac{1+z-1}{1+z} \nearrow$$

divide by $(1+z)$ to get $d_p(t_e)$

(5.62) $d_p(t_e) = \frac{C}{H_o}\frac{z}{(1+z)^2}$

---

with Λ only $\quad w = -1 \quad$ (flat)

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\sum_i \varepsilon_{i,o}\, a^{-(1+3w)} - \frac{Kc^2}{R_o^2}$$

becomes $w = -1$, $K = 0$

(5.70) $\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon_\Lambda a^2$

if ε_Λ is constant with time

$H_o = \left(\frac{\dot{a}}{a}\right)_{t=t_o}$ so

(5.71) $\dot{a} = H_o a$

so $\dot{a}^2 = H_o^2 a^2$

which means

$$H_o^2 = \frac{8\pi G}{3c^2}\varepsilon_\Lambda$$

$$H(t) = \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}}$$

(5.72) $H_o = \left(\frac{8\pi G \varepsilon_\Lambda}{3c^2}\right)^{1/2}$

$\varepsilon_\Lambda(t) = \varepsilon_\Lambda(t_o)$

$$H_o^2 = \frac{8\pi G}{3c^2}\varepsilon_\Lambda t_o$$

$\frac{da}{dt} = H_o a$ so; $\int \frac{1}{a}da = \int_{t_o}^{t_r} H_o\, dt$

$\ln a = H_o (t - t_o)$

(5.73) $a(t) = e^{H_o(t-t_o)}$

$\swarrow$ cant use because of o's

$$d_p(t_o) = C \int_{t_e}^{t_o} \frac{dt}{a(t)} = Ct_o \frac{3(1+w)}{1+3w}\left[1 - \left(\frac{t_e}{t_o}\right)^{(1+3w)/(3+3w)}\right] = \frac{C}{H_o}\frac{2}{1+3w}\left[1 - (1+z)^{-(1+3w)/2}\right]$$

$$d_p(t_o) = C \int_{t_e}^{t_o} \frac{dt}{e^{H_o(t-t_o)}} = \frac{-C}{H_o}\left[1 - (1+z)^1\right] = -\frac{C}{H_o}z \quad \text{or since its a distance} = \frac{C}{H_o}z$$

since $z+1 = e^{H_o(t_o-t_e)}$

(5.74) $d_p(t_o) = C\int_{t_e}^{t_o} e^{H_o(t_o-t)}dt = \frac{C}{H_o}\left[e^{H_o(t_o-t_e)} - 1\right] = \frac{C}{H_o}z$

5c)

3/3

H0   70 km/s/Mpc
c   3.00E+08 m/s/s

	Observation			Emission		
z	dpt0 w=0	w=-1	w=1/3	dpte w=0	w=-1	w=1/3
0.01	9.9256E-03	1.0000E-02	9.9010E-03	9.8273E-03	9.9010E-03	9.8030E-03
0.1	9.3075E-02	1.0000E-01	9.0909E-02	8.4613E-02	9.0909E-02	8.2645E-02
1	5.8579E-01	1.0000E+00	5.0000E-01	2.9289E-01	5.0000E-01	2.5000E-01
10	1.3970E+00	1.0000E+01	9.0909E-01	1.2700E-01	9.0909E-01	8.2645E-02
100	1.8010E+00	1.0000E+02	9.9010E-01	1.7832E-02	9.9010E-01	9.8030E-03
1000	1.9368E+00	1.0000E+03	9.9900E-01	1.9349E-03	9.9900E-01	9.9800E-04

Measured in units of Hubble distance like in the book

Observation
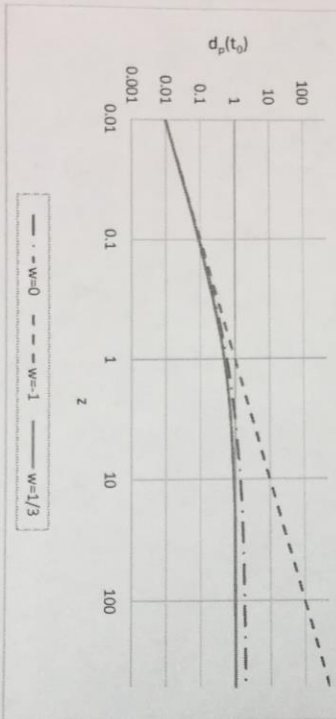
Emission
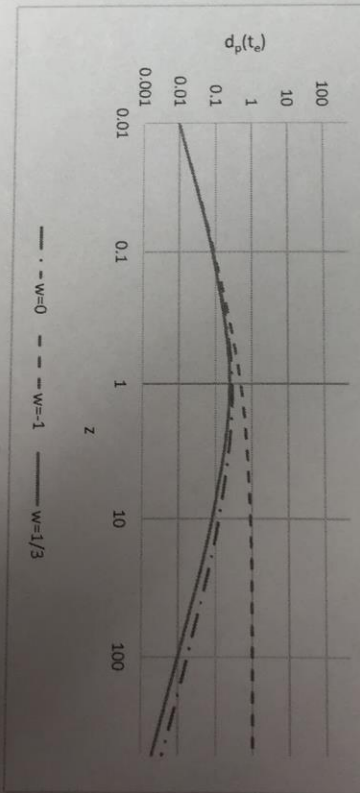
$\frac{11}{10}$ Nice!

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AST 322 – HW 6 (Ch.5)

3/12/2018

6a: Integrate to prove equations (5.90) and (5.91) for a matter + positive curvature universe.

6b: Integrate to prove equation (5.101) for a matter + positive lambda universe. One could argue that, with respect to our modern understanding of cosmology, this is the most important model in this chapter. In a sentence or two, briefly argue for or against this claim.

6c: Using a plotting program, plot a (on the y axis) vs t (on the x axis) for equation (5.101), given H0 = 68km/s/Mpc and Ωm,0 = .32. Use a range of t = 0 to 15 Gigayears. On the plot, include also a line of constant t at the time of lambda-matter equality (that is, the time for which a = amΛ). Label appropriately, print, and turn this plot in with your homework.

$$(5.90) \quad a(\theta) = \frac{1}{2} \frac{\Omega_0}{\Omega_0 - 1} (1 - \cos\theta)$$

$$(5.91) \quad t(\theta) = \frac{1}{2H_0} \cdot \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (\theta - \sin\theta)$$

$$(5.83) \quad \int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + \Omega_{\Lambda,0} a^2 + (1 + \Omega_0)]^{1/2}} = H_0 t$$

for matter + positive curvature universe $\Omega_{m,0} = \Omega_0$

friedmann equation

$$(5.35) \quad \frac{H(t)^2}{H_0^2} = \frac{\Omega_0}{a^3} + \frac{1 - \Omega_0}{a^2}$$

$$(5.38) \quad \frac{H(t)^2 a^2}{H_0^2} = \frac{\Omega_0}{a} + (1 - \Omega_0) = \frac{\dot{a}^2}{H_0^2}$$

$\boxed{\times 1}$

$$\frac{H_0}{\frac{da}{dt}} = \frac{1}{\sqrt{\Omega_0/a + (1 - \Omega_0)}}$$

$$(5.89) \quad H_0 t = \int_0^a \frac{da}{[\Omega_0/a + (1 - \Omega_0)]^{1/2}} \quad \rightarrow \text{ or just use } (5.83) \text{ eliminating factors for radiation and } \Lambda.$$

$$d\theta \equiv \frac{dt}{a} \qquad \left(\frac{da}{dt}\right)^2 = H_0^2\left(\frac{\Omega_0}{a} + 1 - \Omega_0\right)$$

$$H_0 = \frac{\dot{a}}{a}\Big|_{t=t_0}$$

$$a = \frac{dt}{d\theta}$$

$$dt = d\theta \, a$$

$$\left(\frac{da}{dt}\right)^2 = H_0^2\left(\frac{\Omega_0}{a} + 1 - \Omega_0\right)$$

$$\left(\frac{da}{d\theta}\frac{1}{a}\right)^2 = H_0^2 a^2\left(\frac{\Omega_0}{a} + (1-\Omega_0)\right)$$

$$\left(\frac{da}{d\theta}\right)^2 = H_0^2\left(a^2(1-\Omega_0) + a\,\Omega_0\right)$$

$$\frac{da}{d\theta} = H_0\sqrt{a^2(1-\Omega_0) + a\,\Omega_0}$$

$$\theta = \int_0^a \frac{da}{H_0\sqrt{a^2(1-\Omega_0)+a\,\Omega_0}} \qquad\qquad A = 1-\Omega_0$$
$$B = \Omega_0$$

$$\int_0^{a'} \frac{da}{\sqrt{Aa^2 + Ba}} = -\frac{1}{\sqrt{-A}}\left[\arcsin\left[\frac{2Aa'+B}{|B|}\right] - \arcsin\frac{B}{|B|}\right]$$

as long as $\qquad A<0 \quad\text{and}\quad |a'A+B| < |B|$

$$= -\frac{1}{H_0\sqrt{\Omega_0 - 1}}\left[\arcsin\left[\frac{2\left[1-\Omega_0\right]a + \Omega_0}{|\Omega_0|}\right] - \arcsin\frac{\Omega_0}{|\Omega_0|}\right] \qquad +1 \; \text{(good)}$$

when $\Omega_0 > 1$

$$\theta = -\frac{1}{\boxed{H_0(\Omega_0-1)^{1/2}}}\left[\arcsin\left[\frac{2a(1-\Omega_0)+\Omega_0}{\Omega_0}\right] - \frac{\pi}{2}\right]$$

if this were on the top then it would cancel and simplify math

so lets start by making $\qquad d\theta = \dfrac{dt\, H_0}{a\,(\Omega_0-1)^{1/2}} \qquad$ so $\qquad dt = \dfrac{d\theta\, a\,(\Omega_0-1)^{1/2}}{H_0\,(\ )}$

which also makes units match.
plug that into step 2 yields

$$\left(\frac{da\, H_0}{d\theta\, a\,(\Omega_0-1)^{1/2}}\right)^2 = H_0^2\left(\frac{\Omega_0}{a} + (1-\Omega_0)\right)$$

$$\left(\frac{da}{d\theta}\right)^2 = \frac{a^2 H_0^2(\Omega_0-1)}{H_0^2}\left(\frac{\Omega_0}{a} + (1-\Omega_0)\right)$$

$$\left(\frac{da}{d\theta}\right)^2 = (\Omega_0-1)\left((1-\Omega_0)a^2 + \Omega_0 a\right)$$

$$\frac{da}{d\theta} = (\Omega_0-1)^{1/2}\sqrt{a^2(1-\Omega_0) + a\,\Omega_0}$$

$$\theta = (\Omega_0-1)^{1/2}\int_0^a \frac{da}{\sqrt{a^2(1-\Omega_0)+a\,\Omega_0}}$$

again using

$$\int_0^{a'} \frac{da}{\sqrt{Aa^2 + Ba}} = -\frac{1}{\sqrt{-A}} \left[\arcsin\left[\frac{2Aa' + B}{|B|} \right] - \arcsin\frac{B}{|B|} \right]$$

$$A = 1 - \Omega_0$$
$$B = \Omega_0$$

$$\theta = -\frac{(\Omega_0 - 1)^{1/2}}{(\Omega_0 - 1)^{1/2}} \left[\arcsin\left[\frac{2a(1 - \Omega_0) + \Omega_0}{|\Omega_0|} \right] - \arcsin\frac{\Omega_0}{|\Omega_0|} \right]$$

for $\Omega_0 > 0$

$$\theta = -\left[\arcsin\left[\frac{2a(1 - \Omega_0)}{\Omega_0} + 1 \right] - \frac{\pi}{2} \right]$$

$$\theta = \frac{\pi}{2} - \arcsin\left[\frac{2a(1 - \Omega_0)}{\Omega_0} + 1 \right]$$

$$\theta - \frac{\pi}{2} = -\arcsin\left[\frac{2a(1 - \Omega_0)}{\Omega_0} + 1 \right]$$

$$\frac{\pi}{2} - \theta = \arcsin\left[\frac{2a(1 - \Omega_0)}{\Omega_0} + 1 \right]$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{2a(1 - \Omega_0)}{\Omega_0} + 1$$

$$\cos(\theta) - 1 = \frac{2a(1 - \Omega_0)}{\Omega_0}$$

$+1$

$$\frac{\Omega_0(\cos\theta - 1)}{2(1 - \Omega_0)} = a$$

$$(5.90) \quad a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\theta)$$

This is what I used to equate θ to t

This is (5.90)

$$dt = \frac{d\theta \ a \ (\Omega_0 - 1)^{1/2}}{H_0}$$

$$a = \frac{1}{2} \frac{\Omega_0}{(\Omega_0 - 1)} (1 - \cos\theta)$$

combine to get:

$$dt = \frac{d\theta}{2} \frac{\Omega_0}{H_0 (\Omega_0 - 1)^{3/2}} (1 - \cos\theta)$$

$$dt = \frac{\Omega_0}{2 H_0 (\Omega_0 - 1)^{3/2}} (1 - \cos\theta) \ d\theta$$

$$t(\theta) = \frac{\Omega_0}{2 H_0 (\Omega_0 - 1)^{3/2}} \int_0^\theta (1 - \cos\theta) \ d\theta$$

$$\int d\theta \quad - \quad \int \cos\theta$$

$$\theta \quad - \quad \sin\theta$$

(5.91) $$t(\theta) = \frac{1}{2 H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (\theta - \sin\theta)$$

(6b)

$$H_0 t = \int_0^a \frac{da}{[\Omega_{m,0}/a + \Omega_{\Lambda,0} a^2]^{1/2}}$$

$$\Omega_{\Lambda_0} = 1 - \Omega_{m,0}$$

$$\frac{a^{1/2} \ da}{[\Omega_{m,0} + (1 - \Omega_{m,0}) a^3]^{1/2}}$$

$$u = a^{3/2}$$

$$du = \frac{3}{2} a^{1/2} \ da$$

$$\frac{2}{3} \int_0^u \frac{du}{[\Omega_{m,0} + (1 - \Omega_{m,0}) u^2]^{1/2}}$$

$$\frac{2}{3} \frac{1}{(1 - \Omega_{m,0})^{1/2}} \int_0^u \frac{du}{\left[\frac{\Omega_{m,0}}{1 - \Omega_{m,0}} + u^2\right]^{1/2}}$$

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}}\right)^{1/3}$$

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \int_0^u \frac{du}{[a_{m\Lambda}^3 + u^2]^{1/2}}$$

+1

+1

+4

④

$$\int_0^{u'} \frac{du}{\sqrt{A+u^2}} = \ln \frac{u' + \sqrt{A+u^2}}{\sqrt{A}} \qquad A = a_{m\Lambda}^3$$

+0.5

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln \frac{a^{3/2} + \sqrt{a_{m\Lambda}^3 + a^3}}{a_{m\Lambda}^{3/2}}$$

(5.101) $\qquad H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln \left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{m\Lambda}}\right)^3} \right] \qquad$ divide top + bottom by $a_{m\Lambda}^{3/2}$

Our universe is flat and has radiation, matter, and Λ. However, in our current state radiation has very little influence, but on a log scale we are living close to matter / Lambda equality. In other words this Matter + Lambda only version is a very good approximation for our current state of our universe, so I would agree this could be called the most important model in this chapter.

+1.5

+3

6c

H_0	68 km/s/Mpc	2.20371E-18 /s	3.15576E+16 Sec / Gigayear	
$\Omega_{m,0}$	0.32			
$\Omega_{\Lambda,0}$	0.68			
$a_{m\Lambda}$	0.777822			

time of lambda matter equality

a	H0t	t	t (Gigayear)	t @ $a_{m\Lambda}$
0	0	0	0	10.24602
0.1	0.037255	1.691E+16	0.535699054	10.24602
0.2	0.105113	4.77E+16	1.51146002	10.24602
0.3	0.191844	8.705E+16	2.75859733	10.24602
0.4	0.291767	1.324E+17	4.19544 1025	10.24602
0.5	0.40013	1.816E+17	5.75623632	10.24602
0.6	0.512666	2.326E+17	7.371828251	10.24602
0.7	0.625811	2.84E+17	8.998791839	10.24602
0.77822237	0.712548	3.233E+17	10.24601852	10.24602
0.8	0.736908	3.344E+17	10.5962955	10.24602
0.9	0.844217	3.831E+17	12.13934209	10.24602
1	0.94677	4.296E+17	13.61398269	10.24602
1.1	1.044152	4.738E+17	15.01427564	10.24602
1.2	1.136312	5.156E+17	16.33948083	10.24602
1.3	1.223412	5.552E+17	17.5919277	10.24602
1.4	1.305727	5.925E+17	18.7755963	10.24602
1.5	1.383582	6.278E+17	19.8950757	10.24602

+1

+3