AST 322 - Introduction to Galactic and Extragalactic Astrophysics - HW 2

2.1) Confirm the $\Delta\rho/\rho$ values in Fig. 2.2 (a), (b), (c), (d), (e). Fig. 2e refers to Dr. Windhorst's handwritten notes (they can be found on the AST 322 website or in Files here on Canvas) in Ch. 2 referring to the mass density of a 5 solar-mass black hole, where the radius is the Schwarzschild radius. For (a) through (d) you can start with the ρ values quoted in Ryden. You can use a density of $\rho = 6 \times 10^{-27}$ kg m^{-3} for Fig 2.2 (d).

(a) (1 pt)

$$\frac{\Delta\rho}{\rho} = \frac{\rho - \rho_0}{\rho_0} \approx \frac{\rho}{\rho_0} = \frac{100 \text{ kg m}^{-3}}{2.7 \times 10^{-27} \text{ kg m}^{-3}} = 3.7 \times 10^{28} \approx 10^{28}$$

(b) (1 pt)

$$\frac{\Delta\rho}{\rho} = \frac{\rho - \rho_0}{\rho_0} \approx \frac{\rho}{\rho_0} = \frac{4 \times 10^{-5} \text{ kg m}^{-3}}{2.7 \times 10^{-27} \text{ kg m}^{-3}} = 1.5 \times 10^{22} \approx 10^{22}$$

(c) (1 pt)

$$\frac{\Delta\rho}{\rho} = \frac{\rho - \rho_0}{\rho_0} = \frac{3 \times 10^{-26} \text{ kg m}^{-3} - 2.7 \times 10^{-27} \text{ kg m}^{-3}}{2.7 \times 10^{-27} \text{ kg m}^{-3}} = 10.1 \approx 10$$

(d) (1 pt)

$$\frac{\Delta\rho}{\rho} = \frac{\rho - \rho_0}{\rho_0} = \frac{6 \times 10^{-27} \text{ kg m}^{-3} - 2.7 \times 10^{-27} \text{ kg m}^{-3}}{2.7 \times 10^{-27} \text{ kg m}^{-3}} = 1.2 \approx 1$$

(e) (2 pts)

$$R_S = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(5 \times 1.99 \times 10^{30} \text{ kg})}{(3 \times 10^8 \text{ m s}^{-1})^2} = 1.47 \times 10^4 \text{ m}$$

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R_S^3} = \frac{3(5 \times 1.99 \times 10^{30} \text{ kg})}{4\pi(1.47 \times 10^4 \text{ m})^3} = 7.41 \times 10^{17} \text{ kg m}^{-3}$$

$$\frac{\Delta\rho}{\rho} = \frac{\rho - \rho_0}{\rho_0} \approx \frac{\rho}{\rho_0} = \frac{7.41 \times 10^{17} \text{ kg m}^{-3}}{2.7 \times 10^{-27} \text{ kg m}^{-3}} = 2.7 \times 10^{44} \approx 10^{44}$$

2.2) Given a value of H_0, calculate R_0 in Gpc and t_0 in years as accurately as you currently can (after Ch 6, you will be given the exact formulae or methods to do so).

Using $H_0 = 68$ km s^{-1} Mpc^{-1}
(2 pts)

$$R_0 = \frac{c}{H_0} = \frac{2.99 \times 10^5 \text{ km s}^{-1}}{68 \text{ km s}^{-1} \text{ Mpc}^{-1}} = 4.40 \times 10^3 \text{ Mpc} = 4.40 \text{ Gpc}$$

(2 pts)

$$t_0 = \frac{1}{H_0} = \frac{1}{68 \text{ km s}^{-1} \text{ Mpc}^{-1}} \cdot \frac{3.09 \times 10^{19} \text{ km}}{1 \text{ Mpc}} \cdot \frac{1 \text{ yr}}{3.16 \times 10^7 \text{ s}} = 1.44 \times 10^{10} \text{ yr}$$

2.3) With what you are given in Ryden Chapter 2.4, VERIFY that:
a) *Wien's law:* $\lambda_{\text{peak}} = (0.29 \text{ cm K})/T$. So what is the "color" of the Sun really?
b) *Mean energy in Planck curve of temp T:* $E_{\text{mean}} = 2.7kT$. What is the mean energy of a photon coming from the Sun? And what is it for a cosmic microwave background photon?

(a) (2 pts) Starting with $h\nu_{\text{peak}} = 4.97kT$,

$$\frac{hc}{\lambda_{\text{peak}}} = 4.97kT \Rightarrow \lambda_{\text{peak}} = \frac{hc}{4.97kT} = \frac{(6.626 \times 10^{-34} \text{ J s})(2.99 \times 10^8 \text{ m s}^{-1})}{(4.97)(1.38 \times 10^{-23} \text{ J K}^{-1})T} = \frac{2.9 \times 10^{-3} \text{ m K}}{T}$$

$$\Rightarrow \lambda_{\text{peak}} = \frac{0.29 \text{ m K}}{T}$$

The Sun's temperature is ~ 5800 K, so

$$\lambda_{p,\odot} = \frac{0.29 \text{ cm K}}{5800 \text{ K}} = 5 \times 10^{-5} \text{ cm} = 500 \text{ nm}$$

This means that the Sun's color is green.

(b) (3 pts) Using Eq. 2.28 and Eq. 2.31,

$$E_{\text{mean}} = \frac{\varepsilon_\gamma}{n_\gamma} = \frac{\alpha T^4}{\beta T^3} = \frac{\pi^2}{15} \frac{k^4}{\hbar^3 c^3} \frac{\pi^2}{2.4041} \frac{\hbar^3 c^3}{k^3} T = \frac{\pi^4}{15 \cdot 2.4041} kT = 2.7kT$$

For the Sun, with $T = 5800$ K,

$$E_{\text{mean}} = 2.7(1.38 \times 10^{-23} \text{ J K}^{-1})(5800 \text{ K}) = 2.16 \times 10^{-19} \text{ J}$$

For the CMB, $T = 2.7255$ K:

$$E_{\text{mean}} = 2.7(1.38 \times 10^{-23} \text{ J K}^{-1})(2.7255 \text{ K}) = 1.02 \times 10^{-22} \text{ J}$$

(EXTRA CREDIT) Derive Wien's Law $[\lambda_{\text{peak}} = (0.29 \text{ cm K})/T]$ starting from Planck's Law (Eq. 2.27). (Hint: you'll have to make a change of variables to write Planck's law in terms of wavelength)

(1 pt) Starting from Planck's Law in frequency space:

$$\varepsilon(f)df = \frac{8\pi h}{c^3} \frac{f^3 df}{\exp(hf/kT) - 1}$$

We apply the change of variable $f = c/\lambda$ and $df = -c/\lambda^2 d\lambda$ to the blackbody function, $\varepsilon(f)df$, to transform it into $\varepsilon(\lambda)d\lambda$,

$$\varepsilon(\lambda)d\lambda = \frac{8\pi h}{c^3} \frac{c^3/\lambda^3(-c/\lambda^2 d\lambda)}{\exp(hc/\lambda kT) - 1} = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{\exp(hc/\lambda kT) - 1}$$

Where the negative sign goes away when you swap the order of integration. Differentiating to find the peak:

$$\frac{d\varepsilon}{d\lambda} = \frac{-40\pi hc}{\lambda^6} \frac{1}{\exp(hc/\lambda kT) - 1} + \frac{8\pi hc}{\lambda^5} \frac{\exp(hc/\lambda kT) \cdot hc/(\lambda^2 kT)}{[\exp(hc/\lambda kT) - 1]^2} = 0$$

Simplifying,

$$\frac{5\lambda kT}{hc} = \frac{\exp(hc/\lambda kT)}{\exp(hc/\lambda kT) - 1}$$

Letting $u = hc/(\lambda kT)$,

$$5e^u = ue^u + 5$$

Using the Mathematica command

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NSolve[5*Exp[u] == u*Exp[u] + 5, u]
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results in $u = 4.96511$, leaving us with Wien's Law:

$$\lambda_{\text{peak}} = \frac{hc}{ukT} = \frac{(6.626 \times 10^{-34} \text{ J s})(2.99 \times 10^8 \text{ m s}^{-1})}{(4.96511)(1.38 \times 10^{-23} \text{ J K}^{-1})T} = \frac{2.9 \times 10^{-3} \text{ m K}}{T}$$

> EXTRA CREDIT) If I told you, as we will in Ch 8, that stars half the temperature of the Sun can begin to ionize Hydrogen, AND that both the Sun and the cosmic microwave background are subject to the same Hydrogen (ionization) physics, what does that tell you about the redshift where the cosmic microwave background was generated? Discuss briefly.

(1 pt) We are given that the temperature of a H-ionizing star in the CMB era is $T_{\text{ionization}} = T_\odot/2 = 5800 \text{ K}/2 = 2900 \text{ K}$. This means that

$$\lambda_{\text{p,ionization}} = \frac{0.29 \text{ cm}}{2900 \text{ K}} = 10^{-4} \text{ cm}$$

We can also calculate the peak wavelength of the redshift 0 CMB:

$$\lambda_{\text{p,CMB}} = \frac{0.29 \text{ cm}}{2.7255 \text{ K}} = 1.06 \times 10^{-1} \text{ cm}$$

Using these to estimate the redshift of the CMB,

$$z_{\text{CMB}} = \frac{\Delta\lambda}{\lambda} = \frac{\lambda_{\text{p,CMB}} - \lambda_{\text{p,ion}}}{\lambda_{\text{p,ion}}} = \frac{1.06 \times 10^{-1} \text{ cm} - 10^{-4} \text{ cm}}{10^{-4} \text{ cm}} = 1063 \approx 1100$$

Alternatively, one could use the relation between temperature and redshift at the end of Chapter 2.

> 2.4) Like at the end of Ch. 2, show that indeed $V(z) = r_0^3/(1+z)^3$, $T(z) = T_0(1+z)$, and $\varepsilon_\gamma(z) = \alpha T_0^4(1+z)^4$, with $T_0 = 2.7255$ K and α given in Eq. (2.29).

(a) (3 pts) First we will solve for $T(z)$. From Ryden Eq. 2.37, we know that

$$dQ = dE + PdV$$

Since there is no net flow of heat, we know that $dQ = 0$

$$0 = dE + PdV \Rightarrow dE = -PdV$$

Plugging in $E = \varepsilon_\gamma V = \alpha T^4 V$ and $P = \varepsilon_\gamma/3 = \alpha T^4/3$,

$$d\left(\alpha T^4 V\right) = -\left(\frac{\alpha T^4}{3}\right)dV$$

$$4\alpha T^3 V dT + \alpha T^4 dV = -\frac{\alpha T^4}{3}dV$$

$$4T^3 V dT = \left(-T^4 - \frac{T^4}{3}\right)dV \Rightarrow \frac{1}{T}dT = -\frac{1}{3V}dV$$

Solving this differential equation,

$$\int \frac{dT}{T} = -\frac{1}{3} \int \frac{dV}{V} \Rightarrow \ln(T) = -\frac{1}{3}\ln(V) + C \Rightarrow T = e^C V^{-1/3}$$

Since $a \propto V^{1/3}$, we can redefine our constant of proportionality:

$$T = T_0/a = T_0(1 + z)$$

(b) (1 pt) For $V(z)$, we can recognize that $V \propto a^3$, so $V = r_0^3 a^3 = r_0^3(1 + z)^{-3}$, where r_0^3 is the constant of proportionality.

(c) (1 pt) For $\varepsilon_\gamma(z)$, we can use the result from (a).

$$\varepsilon_\gamma = \alpha T^4 = \alpha \left[T_0(1 + z)\right]^4 = \alpha T_0^4(1 + z)^4$$

Optionally, one could plug in numerical values.