AST 322 - Introduction to Galactic and Extragalactic Astrophysics - HW 4

4.1 (5 pts) Derive from Equation 4.10, combined with Newton's 2nd Law, that Equation 4.12 is true:

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$$

Where U is a constant of integration.

We can see from Equation 4.10 that

$$F = -\frac{GM_s m}{R_s(t)^2} = ma = m\frac{d^2R_s}{dt^2}$$

Because in this case the radius of the sphere, $R_s(t)$, determines the gravitational force, and the mass, m, also happens to be at a distance $R_s(t)$ (on the surface of the sphere). This logic provides us with Equation 4.11:

$$\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}$$

If we multiply both sides by $\dot{R}_s$, we obtain

$$\frac{dR_s}{dt}\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}\frac{dR_s}{dt}$$

Multiplying by dt and integrating, we see

$$\int \frac{dR_s}{dt} d\left(\frac{dR_s}{dt}\right) = -\int \frac{GM_s}{R_s(t)^2} dR_s$$

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$$

4.3 (7 pts)
i) Show that Equation 4.28a is true (from 4.20):

$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2}$$

ii) What is the special meaning of the case $\kappa = 0$?
iii) What is the actual value of the critical density of the universe ρ_0 that you derive in that case? i.e., show that Equation 4.28b is true:

$$\rho_0 = \frac{3H_0^2}{8\pi G}$$

iv) Assuming the constants given in the text, calculate the numerical value of ρ_o from 4.28b.
[Do 4.3 before 4.2]

i) We start with Equation 4.20:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

plugging in $t = t_0$ to this equation,

$$\left(\frac{\dot{a}}{a}\right)^2\Bigg|_{t=t_0} = \frac{8\pi G}{3c^2}\varepsilon(t)\Bigg|_{t=t_0} - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}\Bigg|_{t=t_0}$$

$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2}$$

Because $(\dot{a}/a)|_{t=t_0} \equiv H_0$, $\varepsilon(t_0) \equiv \varepsilon_0$, and $a(t_0) = 1$.
ii) $\kappa = 0$ corresponds to the case where spacetime is flat.
iii) Plugging $\kappa = 0$ into Equation 4.28 and recognizing that $\varepsilon_0 = \rho_0 c^2$,

$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 = \frac{8\pi G}{3}\rho_0$$

Rearranging,

$$\rho_0 = \frac{3H_0^2}{8\pi G}$$

iv) Using the constants given in Ryden,

$$\rho_0 = \frac{3(68\text{ km s}^{-1}\text{ Mpc}^{-1})^2}{8\pi(6.67\times10^{-11}\text{ m}^3\text{ kg}^{-1}\text{ s}^{-2})}\cdot\left(\frac{1\text{ Mpc}}{3.09\times10^{19}\text{ km}}\right)^2 = 8.67\times10^{-27}\text{ kg m}^{-3}$$

4.2 (8 pts)
i) Derive Equation 4.17 from the equations that come before it:

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Gr_s^2\rho(t)a(t)^2 + U$$

Then derive Equation 4.18 from that:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}$$

ii) Solve Equation 4.18 for $a(t)$ in the case of $\kappa = 0$, i.e., $U = 0$. Then sketch this solution.
iii) Discuss and sketch the solutions for the case of $\kappa = +1$ ($U < 0$) and $\kappa = -1$ ($U > 0$).

i) We start with Equation 4.12:

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$$

But we know from Equation 4.15 that $M_s = 4\pi\rho(t)R_s(t)^3/3$ and from Equation 4.16 that $R_s(t) = a(t)r_s$. Thus,

$$\begin{aligned}
\frac{1}{2}\left(\frac{d}{dt}a(t)r_s\right)^2 &= \frac{G[4\pi\rho(t)R_s(t)^3/3]}{R_s(t)} + U \\
\frac{1}{2}\dot{a}^2r_s^2 &= \frac{4\pi G\rho(t)R_s(t)^2}{3} + U \\
&= \frac{4\pi G\rho(t)}{3}\left(a(t)r_s\right)^2 + U \\
\frac{1}{2}r_s^2\dot{a}^2 &= \frac{4\pi}{3}Gr_s^2\rho(t)a(t)^2 + U
\end{aligned}$$

Multiplying both sides by $2/(r_s^2a(t)^2)$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}$$

ii) Setting $U = 0$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t)$$

We can then recognize that

$$\rho(t) = \frac{3M_s}{4\pi R_s(t)^3} = \frac{3M_s}{4\pi r_s^3}\frac{1}{a(t)^3}$$

Using this expression for $\rho(t)$,

$$\dot{a}^2 = \left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\frac{3M_s}{4\pi r_s^3}\frac{a^2}{a^3}$$

Square rooting both sides and multiplying by dt,

$$\sqrt{a}da = \sqrt{\frac{2GM_s}{r_s^3}}dt$$

Now, integrating (with a reference point of $t = t_0$, $a = 1$,

$$\int_1^a a^{1/2} da = \sqrt{\frac{2GM_s}{r_s^3}} \int_{t_0}^t dt$$

$$\frac{2}{3}a^{3/2} - \frac{2}{3} = (t - t_0)\sqrt{\frac{2GM_s}{r_s^3}}$$

Solving for a,

$$a(t) = \left[1 + \frac{3}{2}(t - t_0)\sqrt{\frac{2GM_s}{r_s^3}}\right]^{2/3}$$

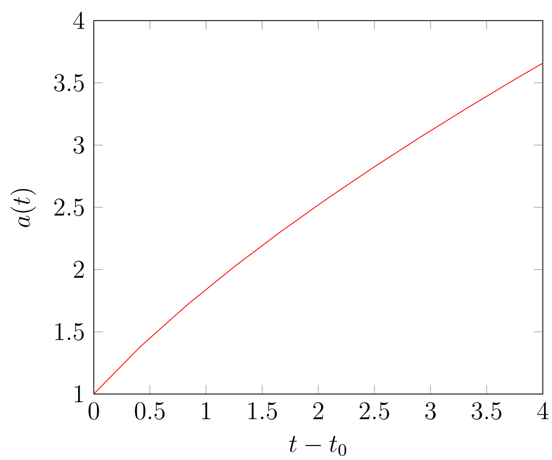

FIGURE 1. Scale factor vs. time for a classical $\kappa = 0$ universe (arbitrary units).

iii) With $\kappa = +1$, the curvature is positive so the universe will collapse inward on itself, resulting in a "big crunch". With $\kappa = -1$, the universe will have hyperbolic geometry and will expand indefinitely, resulting in the "big chill".

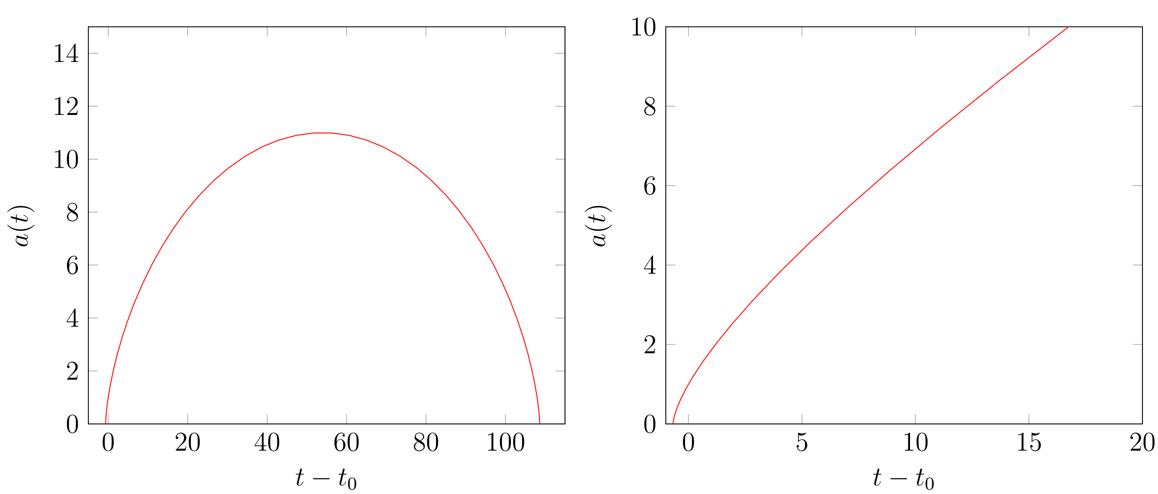

FIGURE 2. Scale factor vs. time for a (left) $\kappa = +1$ universe and a (right) $\kappa = -1$ universe. (arbitrary units).

(EXTRA CREDIT)
i) (1 pt) What is the end behavior of your $a(t)$ for the case of $\kappa = 0$? What does this mean physically? Hint: calculate

$$\lim_{t\to\infty} \frac{da}{dt}$$

ii) (2 pts) Derive from Equation 4.18 that $a(t)$ has critical points (minima or maxima) determined by Equation 4.19:

$$a_{\max} = -\frac{GM_s}{Ur_s}$$

Discuss what this means for the case of $\kappa > 0$ and $\kappa < 0$.

i) Recall from the previous problem that

$$a(t) = \left[1 + \frac{3}{2}(t - t_0)\sqrt{\frac{2GM_s}{r_s^3}}\right]^{2/3}$$

Then

$$\frac{da}{dt} = \frac{2}{3}\left[1 + \frac{3}{2}(t - t_0)\sqrt{\frac{2GM_s}{r_s^3}}\right]^{-1/3} \sqrt{\frac{2GM_s}{r_s^3}} \cdot \frac{3}{2}$$

So we see

$$\lim_{t\to\infty} \frac{da}{dt} = 0$$

Since the only t variable appears to the $-1/3$ power. This means that as time goes on, the universe will begin to stabilize at some constant size.
ii) When $\dot{a} = 0$,

$$0 = \frac{8\pi G}{3}\frac{3M_s}{4\pi r_s^3}\frac{1}{a} + \frac{2U}{r_s^2} = \frac{2GM_S}{ar_s^3} + \frac{2U}{r_s^2}$$

Solving for a,

$$a_{\max} = -\frac{GM_s}{Ur_s}$$

When $\kappa > 0$, $U < 0$ so we have a positive maximum. The universe will reach a maximum size at some positive time, then collapse back down. When $\kappa < 0$, $U > 0$ so the maximum is negative. This is unphysical and represents a relative minimum in the size of the universe. This means that the universe came in from infinite size, "bounced" at some $t < t_0$, and is expanding now at an alarming rate.