AST 322 - Introduction to Galactic and Extragalactic Astrophysics - HW 6

5.1 (5 pts)
i) Verify that Equation 5.9 is a valid solution of Equation 5.8 by plugging Equation 5.9 into Equation 5.8.
(EXTRA CREDIT) For an additional 1 point of extra credit, derive Equation 5.9 directly from Equation 5.8 by solving the differential equation with the appropriate initial conditions.
ii) Using Equation 5.9, which w component dominates the expansion of the universe for $a \to 0$ ($z \to \infty$)?
iii) Using Equation 5.9, which w component dominates the expansion of the universe for $a \to \infty$ ($t \to \infty$)?
iv) Discuss the cases $w = 0$ and $w = 1/3$ and solve for $\varepsilon(a)$ for each case by plugging in the appropriate w-value into Equation 5.9.

i) We are given the solution of Equation 5.8 to be

$$\varepsilon_i(a) = \varepsilon_{i,0} a^{-3(1+w_i)}$$

and we can calculate the derivative of this solution with respect to a:

$$\frac{d\varepsilon_i}{da} = -3(1+w_i)\varepsilon_{i,0} a^{-3(1+w_i)-1} = -3(1+w_i)\frac{\varepsilon_i}{a}$$

Dividing equation 5.8 by da and multiplying by ε_i, we obtain

$$\frac{d\varepsilon_i}{da} = -3(1+w_i)\frac{\varepsilon_i}{a}$$

which is exactly the same as our derivative that we calculated above. Thus, Equation 5.9 is a valid solution of Equation 5.8.
(EXTRA CREDIT) Starting with Equation 5.8,

$$\frac{d\varepsilon_i}{\varepsilon_i} = -3(1+w_i)\frac{da}{a}$$

we see that this is a first order linear differential equation that has already been separated for us. Thus all that is left to do is integrate, using the initial condition that $\varepsilon_i(a=1) = \varepsilon_{i,0}$:

$$\int_{\varepsilon_{i,0}}^{\varepsilon_i} \frac{d\varepsilon_i}{\varepsilon_i} = -3(1+w_i)\int_1^a \frac{da}{a}$$

$$\ln\left|\frac{\varepsilon_i}{\varepsilon_{i,0}}\right| = -3(1+w_i)\ln\left|\frac{a}{1}\right|$$

$$\varepsilon_i = \varepsilon_{i,0} a^{-3(1+w_i)} \tag{1}$$

ii) As $a \to 0$, the equations of state that cause energy density to explode are given by $-3(1+w_i) < 0$. Solving this yields $w_i > -1$. This means that the most positive w-value, **radiation** ($w = 1/3$), will dominate.
iii) As $a \to \infty$, the equations of state that cause energy density to explode are given by $-3(1+w_i) > 0$. Solving this yields $w_i < -1$. This means that the most negative w-value will dominate. In particular, this means that the **cosmological constant** Λ ($w = -1$) will dominate.

iv) The $w = 0$ case corresponds to matter and results in the following equation for energy density:

$$\varepsilon_m(a) = \varepsilon_{m,0} a^{-3} \tag{2}$$

This means that the matter energy density decreases as volume increases, so it had a high energy density in the past and decreases in density over time.
The $w = 1/3$ case corresponds to radiation and results in the following equation for energy density:

$$\varepsilon_r(a) = \varepsilon_{r,0} a^{-4} \tag{3}$$

This equation shows that radiation energy density decreases as volume increases, just like the matter energy density, but it does so at a faster rate ($\propto a^{-4}$) than matter. This equation shows that radiation dominated the early universe, but quickly died out.

5.2 (4 pts)
i) Using Equation 5.9 and the appropriate w-values, calculate the redshift, $z_{\Lambda m}$, for which $\varepsilon_m = \varepsilon_\Lambda$.
ii) Discuss what this means for the growth of galaxies with time (no more than 1-2 paragraphs).
Hint: In preparation for a possible term project, do a little literature search (use ADS abstract link on Links) on the cosmic star-formation rate vs. redshift SFR(z). (e.g., Madau, P. & Dickinson, M. 2014, Ann. Rev. A&Ap, 52, p. 415). The PDF is available here: `https://arxiv.org/abs/1403.0007` *. Based on their Fig. 9, discuss what impact the expansion and Lambda (> 0) may have had on the growth of galaxies with cosmic time, and whether this is visible in the data.*
[For extra credit: You could also discuss how the galaxy merger rate has changed as a function of redshift, by comparing the mix of spiral and elliptical galaxies in high redshift clusters to those at low redshifts. (Hint: You need to know that galaxy mergers tend to transform spiral galaxies into ellipticals; Do a literature search of how the mix of spiral and elliptical galaxies has changed in galaxy clusters with redshift, as observed e.g. with Hubble).]

i) From Equation 5.9 we can see that with $w = 0$,

$$\varepsilon_m = \varepsilon_{m,0} a^{-3}$$

and with $w = -1$,

$$\varepsilon_\Lambda = \varepsilon_{\Lambda,0}$$

Equating these,

$$\varepsilon_{m,0} a^{-3} = \varepsilon_{\Lambda,0} \Rightarrow a_{\Lambda m} = \sqrt[3]{\frac{\varepsilon_{m,0}}{\varepsilon_{\Lambda,0}}}$$

Using the definition of the density parameter, $\Omega_i = \varepsilon_i / \varepsilon_c$,

$$a_{\Lambda m} = \sqrt[3]{\frac{\varepsilon_{m,0}}{\varepsilon_c} \frac{\varepsilon_c}{\varepsilon_{\Lambda,0}}} = \sqrt[3]{\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}} \approx \sqrt[3]{\frac{0.32}{0.68}} = 0.778$$

Using the definition of redshift, we see

$$z_{\Lambda m} = \frac{1 - a_{\Lambda m}}{a_{\Lambda m}} \approx 0.286 \tag{4}$$

ii) *Example Response:* Given that in the era of $\Lambda > 0$, the scale factor of the universe increases at an increasing rate, one would expect galaxies to grow at slower speeds for $z \leq 0.286$ because the space between them increases more rapidly during this era. P. Madau & M. Dickinson (2014) show with their Figure 9 that the cosmic star formation rate peaks at $z \approx 2$ and then declines for smaller redshifts. This seems to go against what we would expect from the Λ-dominated era, but does not directly contradict the expectation. This is because the cosmological constant is always present, and increases gradually over time; this means that it was active before $z \approx 0.286$. Additionally, the universe was expanding already before Λ-matter equality, so galaxies were already becoming more and more separated. This would naturally allow for a peak in star formation rate at some point. That peak just happens to occur before $z \approx 0.286$.

(EXTRA CREDIT) *Example Response:* Galaxy growth and star formation rate are inherently linked, because galaxy mergers can fuel star formation. Ferreira et al. (2020) found an equation that models galaxy merger rates based on redshift for $0.5 \leq z \leq 3$. Their equation is $R(z) = 0.02 \times (1 + z)^{2.76}$, not including uncertainties. This equation shows that the galaxy merger rate increases with higher redshifts, at least up to $z \approx 3$. Beyond merger rates, we can also examine galaxy morphology, as elliptical galaxies tend to be products of the collisions of more complex galaxies, such as spirals. Elmegreen et al. (2005) classified galaxies in the Hubble Ultra Deep Field according to their morphology, and found that 100 of around 900 galaxies were elliptical. This characterizes the galaxies expected for $3 \leq z \leq 7$. On the other hand, Zhao et al. (2018) studied "625 low-redshift brightest cluster galaxies" and described their morphologies. The results of this study showed that 7% of the galaxies were disk-like, compared to a total of 91% for galaxies ranging from cD (central dominant—essentially large ellipticals) to E (elliptical) classifications. These two studies show that more elliptical galaxies exist now than in the past, indicating that past spiral galaxies merged and formed elliptical galaxies.

5.3 (6 pts)
i) Derive Equation 5.38 from Equation 5.37 by making the ansatz that $a \propto t^q$:

$$q = \frac{2}{3+3w}$$

Discuss the meaning of this equation for $w = 0$ and/or $w = 1/3$.
ii) Derive Equation 5.39 from Equation 5.38 by applying the normalization condition that $a(t_0) = 1$:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$$

Discuss the meaning of this equation for $w = 0$ and/or $w = 1/3$.
iii) Derive Equation 5.41 from Equation 5.39:

$$H_0 = \frac{2}{3(1+w)} t_0^{-1}$$

iv) Discuss the meaning of Equation 5.42 for $w = 0$ and/or $w = 1/3$:

$$t_0 = \frac{2}{3(1+w)} H_0^{-1}$$

i) If we assume that $a = Ct^q$, this means that $\dot{a} = Cqt^{q-1}$. Plugging these assumptions into Equation 5.37,

$$\begin{aligned} \left(Cqt^{q-1}\right)^2 &= \frac{8\pi G}{3c^2}\left(Ct^q\right)^{-(1+3w)} \\ C^2q^2t^{2(q-1)} &= \frac{8\pi G}{3c^2} t^{-q(1+3w)} \end{aligned}$$

Ignoring the coefficients, for this equation to be true, the exponents of each t term must me equal, so

$$2(q-1) = -q(1+3w) \Rightarrow q = \frac{2}{3+3w} \tag{5}$$

For $w = 0$, we have $q = 2/3$ so $a \propto t^{2/3}$. For $w = 1/3$, we have $q = 1/2$, so $a \propto t^{1/2}$. This means that a radiation-dominated universe expands quickly initially, but in the long-term evolves slower than a matter-dominated universe.
ii) From our ansatz, we know that

$$a(t) = Ct^{2/(3+3w)}$$

To normalize this equation, we require that $a(t_0) = 1$, so

$$a(t_0) = 1 = Ct_0^{2/(3+3w)} \Rightarrow C = t_0^{-2/(3+3w)}$$

Thus overall we have

$$a(t) = t_0^{-2/(3+3w)} t^{2/(3+3w)} = \left(\frac{t}{t_0}\right)^{2/(3+3w)} \tag{6}$$

This equation yields $a(t) = (t/t_0)^{2/3}$ for $w = 0$ and $a(t) = (t/t_0)^{1/2}$ for $w = 1/3$. Both universes have the same size at $t = t_0$ and obey the same trends as discussed above.

iii) Differentiating our above equation,

$$\dot{a}(t) = \frac{2}{3+3w}\left(\frac{t}{t_0}\right)^{2/(3+3w)-1}\frac{1}{t_0}$$

Now using the definition of the Hubble Constant, $H(t) \equiv \dot{a}/a|_{t=t_0}$:

$$H_0 = \left(\frac{\dot{a}}{a}\right)\Bigg|_{t=t_0} = \frac{2}{3+3w}\left(\frac{t_0}{t_0}\right)^{2/(3+3w)-1}\frac{1}{t_0}\left(\frac{t_0}{t_0}\right)^{2/(3+3w)}$$

$$H_0 = \frac{2}{3(1+w)}t_0^{-1} \tag{7}$$

iv) We will now discuss the following equation, which is from a simple algebraic manipulation of the previous result:

$$t_0 = \frac{2}{3(1+w)}H_0^{-1} \tag{8}$$

For $w = 0$ we have $t_0 = \frac{2}{3}H_0^{-1}$, which means that the actual age of a universe composed only of matter is 2/3 of the Hubble Time. For $w = 1/3$ we have $t_0 = \frac{1}{2}H_0^{-1}$, which means that the actual age of the universe composed of all radiation is 1/2 of the Hubble Time.

5.4 (5 pts)
i) Derive Equation 5.48:

$$t_e(z) = \left(\frac{2}{3(1+w)H_0}\right)\left(\frac{1}{(1+z)^{3(1+w)/2}}\right)$$

ii) Derive Equation 5.49:

$$d_p(t_0) = ct_0\frac{3(1+w)}{1+3w}\left[1-\left(\frac{t_e}{t_0}\right)^{\frac{1+3w}{3+3w}}\right]$$

iii) Derive Equation 5.50:

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1-(1+z)^{-(1+3w)/2}\right]$$

iv) Plot $d_p(t_0)$ and t_e as a function of redshift for both $w=0$ and $w=1/3$, and discuss their meaning.

i) We are given Equation 5.47,

$$1+z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/(3+3w)}$$

Rearranging this equation, we obtain

$$(1+z)^{(3+3w)/2} = \frac{t_0}{t_e} \Rightarrow t_e = \frac{t_0}{(1+z)^{3(1+w)/2}}$$

Plugging in Equation 5.42 for t_0, we see that

$$t_e = \left(\frac{2}{3(1+w)H_0}\right)\left(\frac{1}{(1+z)^{3(1+w)/2}}\right) \tag{9}$$

ii) From Equation 5.33, we know that

$$d_p(t_0) = c\int_{t_e}^{t_0}\frac{dt}{a(t)}$$

Plugging in Equation 5.39,

$$\begin{aligned}
d_p(t_0) &= c\int_{t_e}^{t_0}\left(\frac{t_0}{t}\right)^{2/(3+3w)} dt \\
&= ct_0^{2/(3+3w)}\left[\frac{3+3w}{1+3w}t^{\frac{1+3w}{3+3w}}\right]_{t_e}^{t_0} \\
&= ct_0^{2/(3+3w)}\frac{3(1+w)}{1+3w}\left[t_0^{\frac{1+3w}{3+3w}} - t_e^{\frac{1+3w}{3+3w}}\right] \\
d_p(t_0) &= ct_0\frac{3(1+w)}{1+3w}\left[1-\left(\frac{t_e}{t_0}\right)^{\frac{1+3w}{3+3w}}\right]
\end{aligned} \tag{10}$$

iii) Plugging in Equation 5.42 and Equation 5.48,

$$d_p(t_0) = c\left(\frac{2}{3(1+w)}H_0^{-1}\right)\frac{3(1+w)}{1+3w}\left\{1-\left[(1+z)^{-(3+3w)/2}\right]^{\frac{1+3w}{3+3w}}\right\}$$

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1-(1+z)^{-(1+3w)/2}\right] \tag{11}$$

iv) We now plot $d_p(t_0)$ and t_e as a function of redshift in Figure 1.

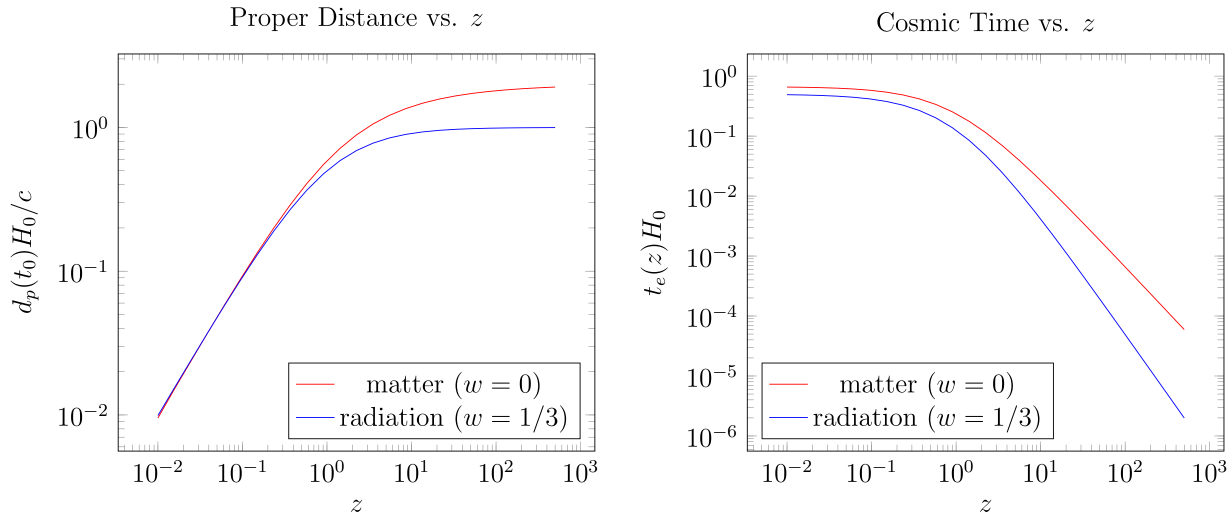

FIGURE 1. Proper distance and cosmic time for radiation– and matter–dominated universes as a function of redshift.

Figure 1 shows that proper distance changes must faster at low redshifts, and reaches an asymptote for high redshifts. At redshifts close to the big bang, a small change in distance requires a large change in redshift. The curves for matter and radiation behave similarly for low redshift, but the radiation–dominated universe reaches an asymptote quicker, reflecting the idea that the radiation energy density dominates at small scale factors, causing the universe to have a smaller overall size at the present epoch.

We can see that cosmic time starts at an asymptote initially for low redshift, then decreases as redshift increases. This means that at low redshifts, nearby galaxies emitted their light not too long ago. At high redshifts, however, the curve has a power law slope, since to get closer to the big bang, redshift increases significantly. The matter–dominated universe has a higher asymptote again, since radiation–dominated universes tend to contract the universe more at high redshifts.