HOMEWORK 5.5

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H/W 5.5: By differentiating the appropriate equations in section 5.4, derive for what redshift z_max the proper distance d_p(t_e) reaches a maximum value d_p(t_e)_max (See Fig. 5.3). What is this maximum value d_p(t_e)_max in units of R_o = c/H_o for each of the w-values that we discussed (w=0, 1/3, -1, etc)?

Now discuss the following: If Congress gave you a fixed large budget to build a the best telescope you can build within that budget, but passed a law that you could move --- for ONE time --- to any universe of your liking (using a Hawking-Penrose wormhole that the DoD had been developing), discuss which universe would you move to to make optimal use of that telescope? Motivate your answer well, since you may only move once! (Hint: Consider that you wish to observe galaxies with a telescope of finite resolution in that universe at redshifts that are as large as possible; Ponder Fig. 5.3 and its consequences carefully!).

1) $w = 0$

For a matter dominated universe, the proper distance is given by the following equation.

$$d_p(t_e) = \frac{2c}{H_0(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

Simplifying this equation,

$$d_p(t_e) = \frac{2c}{H_0}\left[\frac{1}{1+z} - \frac{1}{(1+z)\sqrt{1+z}}\right] = \frac{2c}{H_0}\left[\frac{\sqrt{1+z}-1}{(1+z)^{3/2}}\right]$$

To find the maximum value, $\frac{d\,d_p(t_e)}{dz} = 0$

$$\frac{d}{dz}\left(\frac{2c}{H_0}\left(\frac{\sqrt{1+z}-1}{(1+z)^{3/2}}\right)\right) = 0$$

$$\Rightarrow \frac{\frac{2c}{H_0}\left(\frac{1}{2\sqrt{1+z}}(1+z)^{3/2} - (\sqrt{1+z}-1)\cdot\frac{3}{2}(1+z)^{1/2}\right)}{\left((1+z)^{3/2}\right)^2} = 0$$

$$\frac{2c}{H_0} \neq 0$$

$$\Rightarrow \frac{\left(\frac{1}{2\sqrt{1+z}}(1+z)^{3/2} - (\sqrt{1+z}-1)\cdot\frac{3}{2}(1+z)^{1/2}\right)}{\left((1+z)^{3/2}\right)^2} = 0$$

$$\Rightarrow \frac{1}{2\sqrt{1+z}}(1+z)^{3/2} - (\sqrt{1+z}-1)\cdot\frac{3}{2}(1+z)^{1/2} = 0$$

$$\frac{(1+z)}{2} - \frac{3}{2}\left((1+z) - (1+z)^{1/2}\right) = 0$$

$$\frac{1}{2} + \frac{z}{2} - \frac{3}{2} - \frac{3z}{2} + \frac{3}{2}(1+z)^{1/2} = 0$$

$$-1 - z + \frac{3}{2}(1+z)^{1/2} = 0 \Rightarrow \frac{3}{2}(1+z)^{1/2} = (1+z)$$

$$\Rightarrow (1+z)^{1/2} = \frac{2}{3}(1+z)$$

On squaring both the sides,

$$(1+z) = \frac{4}{9}(1+z)^2 \Rightarrow (1+z) = \frac{4}{9}\left(z^2+2z+1\right)$$

$$\Rightarrow 1+z = \frac{4}{9}z^2 + \frac{8z}{9} + \frac{4}{9} \Rightarrow \frac{4}{9}z^2 - \frac{z}{9} - \frac{5}{9} = 0$$

multiplying by 9 throughout,

$$4z^2 - z - 5 = 0$$

On solving this equation, $z = -1, \frac{5}{4}$ (cannot be negative)

$$\Rightarrow z_{max} = \frac{5}{4}$$

For $z_{max} = \frac{5}{4}$

$$d_p(t_e)_{max} = \frac{2c}{H_0\left(1+\frac{5}{4}\right)}\left[1 - \frac{1}{\sqrt{1+\frac{5}{4}}}\right]$$

$$= \frac{8c}{9H_0}\left[1 - \frac{1}{3/2}\right] = \frac{8c}{9H_0}\left[1 - \frac{2}{3}\right] = \frac{8c}{9H_0}\left(\frac{1}{3}\right) = \frac{8c}{27H_0}$$

$$\Rightarrow \boxed{\text{For } w=0,\ z_{max} = \frac{5}{4},\ d_p(t_e)_{max} = \frac{8c}{27H_0}}$$

2) $w = 1/3$

For a radiation dominated universe, the proper distance is given by the following equation.

$$d_p(t_e) = \frac{c}{H_0}\frac{z}{(1+z)^2}$$

To find the maximum value, $\frac{d\,d_p(t_e)}{dz} = 0$

$$\frac{d}{dz}\left(\frac{c}{H_0}\frac{z}{(1+z)^2}\right) = 0$$

$$\Rightarrow \frac{c}{H_0}\left(\frac{(1+z)^2 - z\cdot 2(1+z)}{((1+z)^2)^2}\right) = 0$$

$$\frac{c}{H_0} \neq 0 \Rightarrow \frac{(1+z)^2 - 2z(1+z) = 0}{(1+z)^4}$$

$$\Rightarrow (1+z)^2 - 2z(1+z) = 0$$

$$(1+z)^{\cancel{2}} = 2z\cancel{(1+z)} \Rightarrow (1+z) = 2z$$

$$\Rightarrow z_{max} = 1$$

For $z_{max} = 1$

$$d_p(t_e)_{max} = \frac{c}{H_0} \times \frac{1}{(1+1)^2} = \frac{1}{4}\frac{c}{H_0} = \frac{0.25c}{H_0}$$

$$\Rightarrow \boxed{\text{For } w = \frac{1}{3},\ z_{max} = 1,\ d_p(t_e)_{max} = \frac{0.25c}{H_0}}$$

3) $w = -1$

For the cosmological constant, the proper distance is given by the following equation.

$$d_p(t_e) = \frac{c}{H_0}\frac{z}{1+z}$$

To find the maximum value, $\frac{d}{dz} d_p(t_e) = 0$

$$\frac{d}{dz}\left(\frac{c}{H_0}\frac{z}{1+z}\right) = 0 \Rightarrow \frac{c}{H_0}\left(\frac{(1+z)-z}{(1+z)^2}\right) = 0$$

$$\frac{c}{H_0} \neq 0 \Rightarrow \frac{(1+z)-z}{(1+z)^2} = 0$$

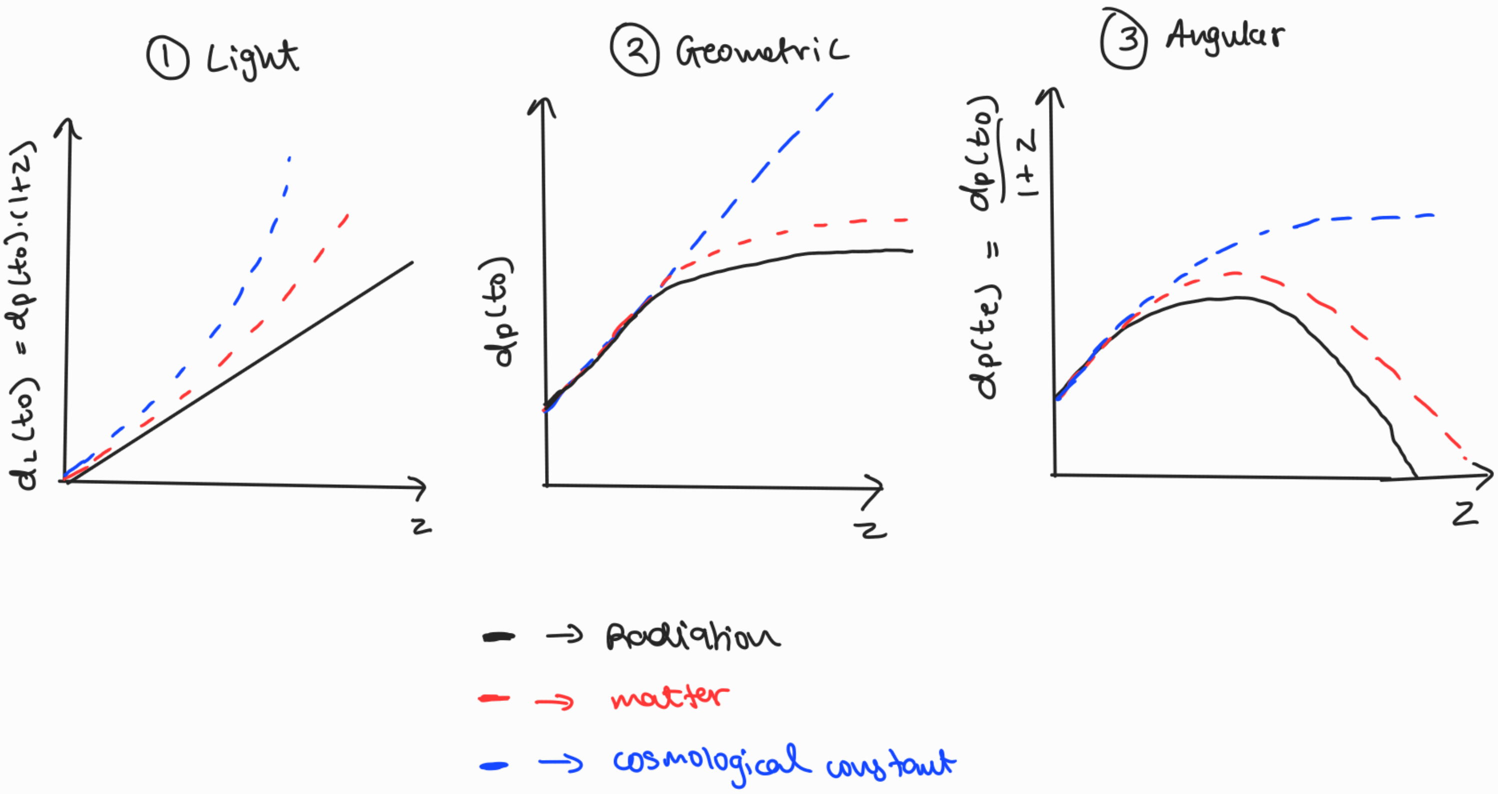
$$\Rightarrow \quad (1+z) - z = 0$$

This shows that, there is no maximum for the curve of $d_p(t_e)$ when $w = -1$. As $z \to \infty$, $d_p(t_e) \to \frac{c}{H_0}$. there is an asymptode at c/H_0.

No maximum value for z & $d_p(t_e)$.

4) Discussion Question

Having a telescope of finite resolution, to observe galaxies at redshifts that are as large as possible, it would be the best if the telescope was moved to a radiation dominated universe. There were 3 plots given. They are shown below.



So the third plot describes the variation of angular position with redshift. As we go to higher redshifts, $d_p(t_e)$ decreases the fastest for radiation dominated and this shows that objects will appear larger at high redshifts. This is not the case for the cosmological constant. For matter-dominated the same trend as radiation dominated takes place but it takes more time. For the second plot as well, the geometric distance appears to reach a constant value at high redshifts for a radiation dominated universe (faster than matter dominated), but for Λ, it does not decrease. For the first plot, it is seen that objects do get dimmer at high z for all the 3 cases but this happens very quickly for matter dominated and Λ. The dimming happens slower for a radiation dominated universe. Due to all these reasons, it would be the best if the telescope was moved to a radiation dominated universe.
