HW3 Solution AST 422

1. Derive Eq (3.19) from (3.16)

$$ds^2 = dr^2 + S_k(r)^2 d\Omega^2$$

$$x = S_k(r) = \begin{cases} R\sin(r/R), (\kappa = +1) \\ r, (\kappa = 0) \\ R\sinh(r/R), (\kappa = -1) \end{cases}$$

$$\kappa = +1, dx = \cos(r/R)dr \Rightarrow dr = \frac{dx}{\cos(r/R)} = \frac{dx}{\sqrt{1-\sin^2(r/R)}} = \frac{dx}{\sqrt{1-\frac{x^2}{R^2}}} = \frac{dx}{\sqrt{1-\kappa\frac{x^2}{R^2}}}$$

$$\kappa = 0, dx = dr = \frac{dx}{\sqrt{1-\kappa\frac{x^2}{R^2}}}$$

$$\kappa = -1, dx = \cosh(r/R)dr \Rightarrow dr = \frac{dx}{\cosh(r/R)} = \frac{dx}{\sqrt{1+\sinh^2(r/R)}} = \frac{dx}{\sqrt{1+\frac{x^2}{R^2}}} = \frac{dx}{\sqrt{1-\kappa\frac{x^2}{R^2}}}$$

$$\Rightarrow ds^2 = \frac{dx^2}{1-\kappa x^2/R^2} + x^2 d\Omega^2$$

2. Show how Eq (4.5) follows from (4.3)-(4.4)

$$F = -\frac{GM_s m}{R_s(t)^2} = m\frac{d^2R_s}{dt^2}$$

$$\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}$$

$$\frac{dR_s}{dt}\frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}\frac{dR_s}{dt}$$

$$\int \frac{dR_s}{dt}\frac{d}{dt}(\frac{dR_s}{dt}) = \int -\frac{GM_s}{R_s^2}\frac{dR_s}{dt}$$

$$\Rightarrow \frac{1}{2}(\frac{dR_s}{dt})^2 = \frac{GM_s}{R_s} + U$$

3. Prove Eq (4.11) from what is given before. Can you solve Eq (4.11)

$$\frac{1}{2}\left(\frac{dR_s(t)}{dt}\right)^2 = \frac{GM_s}{R_s} + U$$

$$M_s = \frac{4\pi}{3}\rho(t)R_s(t)^3$$

$$R_s(t) = a(t)r_s$$

$$\Rightarrow \frac{1}{2}r_s^2\left(\frac{da(t)}{dt}\right)^2 = \frac{G}{R_s}\frac{4\pi}{3}\rho(t)R_s(t)^3 + U$$

$$\Rightarrow \frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}G\rho(t)a^2r_s^2 + U$$

$$\Rightarrow \dot{a}^2 = \frac{8\pi}{3}G\rho(t)a^2 + \frac{2U}{r_s^2}$$

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G\rho(t) + \frac{2U}{r_s^2a^2}$$

4. Show that Eq (4.21) is true from the previous.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2a^2}$$

$$v = Hd,$$

$$v = \dot{a}r_s, d = r_sa$$

$$\Rightarrow H = \frac{\dot{a}}{a}$$

$$t = t_0, H = H_0, a = a_0 = 1$$

$$\Rightarrow H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2a_0^2} = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2}$$

5. Show that Eq (4.31) from the previous stuff
Discuss what (4.31) means: VERY IMPORTANT

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^{\ 2}a^2}$$

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G}H(t)^2$$

$$\Rightarrow \frac{3c^2}{8\pi G}H(t)^2 = \varepsilon_c(t) = \varepsilon(t) - \frac{\kappa c^2}{R_0^{\ 2}a^2}\frac{3c^2}{8\pi G}$$

$$\Rightarrow 1 - \frac{\varepsilon(t)}{\varepsilon_c(t)} = -\frac{\kappa c^2}{R_0^{\ 2}a^2}\frac{3c^2}{8\pi G}\frac{1}{\varepsilon_c(t)}$$

$$\Rightarrow 1 - \Omega(t) = -\frac{\kappa c^2}{R_0^{\ 2}a^2 H(t)^2}$$

$$t = t_0, a_0 = 1$$

$$1 - \Omega_0 = -\frac{\kappa c^2}{R_0^{\ 2}H_0^{\ 2}}$$

$$i.e. \frac{\kappa c^2}{R_0^{\ 2}H_0^{\ 2}} = \Omega_0 - 1$$

when $\Omega_0 > 1$, $\kappa > 0$, the universe is positively curved
when $\Omega_0 = 1$, $\kappa = 0$, the universe is flat
when $\Omega_0 < 1$, $\kappa < 0$, the universe is negatively curved

6. Show that Eq (4.55) follows from the previous

Discuss why ω < 1 (what does this mean)

ω ~ 0 ⇔ non-relativistic, give example

ω = 1/3 ⇔ relativistic, give example

ω < -1/3 ⇔ accelerating, give example

ω = -1 ⇔ Λ, give example

$$P = P(\varepsilon) = \omega\varepsilon$$

$$non-relativistic, massive, particles$$

$$v << c$$

$$P = \frac{\rho}{\mu} kT$$

$$\varepsilon \approx \rho c^2$$

$$3kT = \mu < v^2 >$$

$$\Rightarrow P = \omega\varepsilon \approx \frac{\varepsilon}{\mu c^2} kT = \frac{\mu < v^2 >}{3\mu c^2}\varepsilon = \frac{< v^2 >}{3c^2}\varepsilon$$

$$\Rightarrow \omega \approx \frac{< v^2 >}{3c^2} << 1$$

For small perturbations in a substance with P, the sound speed c_s cannot travel faster than the speed of light c, i. e. $c_s \leq c$

$$c_s^{\ 2} = c^2(\frac{dP}{d\varepsilon}) = \omega c^2 \leq c^2 \Rightarrow \omega \leq 1$$

ω ~ 0: non-relativistic, matter

ω = 1/3: relativistic, photons

ω < - 1/3: accelerating, dark energy

ω = - 1: Λ, cosmological constant

7. Prove (4.59), (4.60) and show that $\Lambda = 4\pi G\rho$, Discuss this result.

$$\nabla^2\Phi = 4\pi G\rho$$

$$Gauss's, Theorem$$

$$\Phi = \Phi(r)$$

$$\int_V \nabla \bullet \nabla\Phi dV = \oint_S \nabla_r \Phi dS = \nabla_r \Phi \times 4\pi r^2 = \int_V 4\pi G\rho dV = 4\pi GM$$

$$\Rightarrow \vec{a} = -\frac{GM}{r^2}\hat{r} = -\nabla\Phi$$

static universe

$$\vec{a} = 0 = -\nabla\Phi$$

$$\Rightarrow \Phi = const$$

$$\Rightarrow \nabla^2\Phi = 4\pi G\rho = 0$$

$$\Rightarrow \rho = 0$$

The static universe is an empty universe.
For a matter-filled universe that is initially static, the gravity will cause it to collapse;
For a matter-filled universe that is initially expanding,
It will expand forever if $U > 0$;
It will expand to a maximum and then collapse if $U < 0$.

Since the universe contains matter, Einstein modified the equation to

$$\nabla^2\Phi + \Lambda = 4\pi G\rho$$

$$static \Rightarrow \nabla^2\Phi = 0$$

$$\Rightarrow \Lambda = 4\pi G\rho$$