HW6

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5.50 an equation for proper distance

5.5 Dividing proper distance by 1+z gives angular size distance: dividing 5.50 by 1+z

$$d_A(z) = \frac{c}{H_0}\frac{2}{(1+3w)(1+z)}\left[1-(1+z)^{-\frac{(1+3w)}{2}}\right]$$

Differentiating with respect to z,

$$\frac{d\,d_A(z)}{dz} = -\frac{c}{H_0}\frac{2}{(1+3w)(1+z)^2} + \frac{c}{H_0}\frac{3+3w}{1+3w}(1+z)^{-\frac{5+3w}{2}}$$

when this equals 0, $d_A(z)$ will be at maximum

$c/H_0 = R_0$

$$-R_0\left(\frac{2}{(1+3w)(1+z)^2} - \frac{3+3w}{1+3w}(1+z)^{-\frac{5+3w}{2}}\right) = 0$$

$$\to \frac{2}{1+3w} - \frac{3+3w}{1+3w}(1+z)^{\frac{-1-3w}{2}} = 0 \to (1+z)^{\frac{-1-3w}{2}} = \frac{2}{1+3w}\cdot\frac{1+3w}{3+3w}$$

$$= \frac{2}{3+3w} = 0 \to z_{max} = \left(\frac{2}{3+3w}\right)^{\frac{2}{-1-3w}} - 1$$ an equation for z when $d_A(z)$ is at maximum.

At $w=0$, $z_{max} = \left(\frac{2}{3}\right)^{-2} - 1 = 1.25 = \frac{5}{4}$

At this z value, when $w=0$,

$$d_A(t_0) = R_0\frac{2}{1+1.25}\left(1-(1+1.25)^{-\frac{1}{2}}\right) = \frac{8}{27}R_0 \approx 0.30R_0$$

At $w=\frac{1}{3}$, $z_{max} = \left(\frac{1}{2}\right)^{-1} - 1 = 1$ At this z, and w

$$d_A(t_0) = R_0\frac{2}{2\cdot 2}\left(1-(1+1)^{-\left(\frac{2}{2}\right)}\right) = 0.25R_0$$

At $w=-1$, the equation for z_{max} becomes undefined. Thus there is no maximum $d_A(t_0)$ when $w=-1$ and it just keep increasing with z. The equation of $d_A(t_0)$ becomes

$$d_A(t_0) = R_0\left(-\frac{1}{1+z}\left(1-(1+z)\right)\right) = R_0\frac{z}{1+z}$$

$\frac{d\,d_A(t_0)}{dz} = \frac{R_0}{(1+z)^2}$ and it's clear that this equation never equals zero as expected. Plotting all 3 $d_A(t_0)$ for the 3 w values, scaled to R_0, on the page attached shows all these maximums hold true.

With a telescope that could be moved to any universe, the $w=\frac{1}{3}$ universe will be the best. In a radiation dominated $w=\frac{1}{3}$ universe, the proper distance or angular size, $d_p(t_e)$, reaches the maximum at the smallest redshift of 1, beyond that for which the angular sizes of objects will continuously increase, being easier to observe objects very far away at high redshifts. In matter dominated for $w=0$, this peak is at $z \approx 1.25$ so that the rate of increase isn't as great, and in Λ dominated universe there's no peak so that objects always decrease in angular size as z increases, making $w=\frac{1}{3}$ the best option.

6.7 a. For $\Omega_0 > 1$, consider $\frac{da}{dt} = \frac{da}{d\theta}\frac{d\theta}{dt} = \frac{da}{d\theta} \Big/ \frac{dt}{d\theta}$

(5.90) $a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos\theta) \rightarrow \frac{da}{d\theta} = \frac{1}{2}\frac{\Omega_0}{\Omega_0-1}\sin\theta$

(5.91) $t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(\theta - \sin\theta) \rightarrow \frac{dt}{d\theta} = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(1-\cos\theta)$

Then, $\dot{a} = H_0(\Omega_0 - 1)^{1/2}\frac{\sin\theta}{1-\cos\theta}$

$\rightarrow \dot{a}^2 = H_0^2(\Omega_0 - 1)\frac{\sin^2\theta}{(1-\cos\theta)^2} = H_0^2(\Omega_0-1)\frac{1-\cos^2\theta}{(1-\cos\theta)^2}$

$= H_0^2(\Omega_0-1)\frac{(1+\cos\theta)(1-\cos\theta)}{(1-\cos\theta)^2} = H_0^2(\Omega_0-1)\frac{1+\cos\theta}{1-\cos\theta} = \dot{a}^2$

$\rightarrow \frac{\dot{a}^2}{H_0^2} = (\Omega_0 - 1)\frac{1+\cos\theta}{1-\cos\theta} \rightarrow \int_0^t H_0 dt = H_0 t = \int_0^a \frac{da}{\sqrt{(\Omega_0-1)\frac{1+\cos\theta}{1-\cos\theta}}}$

An integral of combined ~~derivatives of~~ 5.90 and 5.91

Compare this to (5.89) $H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_0/a + (1-\Omega_0)}}$

From 5.90 $\Omega_0 - 1 = \frac{\Omega_0}{2a}(1-\cos\theta)$

Then, $\frac{\Omega_0}{a} - (\Omega_0 - 1) = (\Omega_0 - 1)\left(\frac{\Omega_0}{a(\Omega_0-1)} - 1\right)$

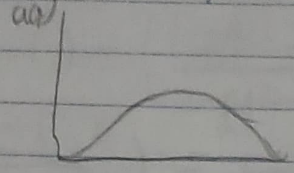
$= (\Omega_0 - 1)\left(\frac{\Omega_0}{a}\frac{2a}{\Omega_0(1-\cos\theta)} - 1\right) = (\Omega_0-1)\left(\frac{2}{1-\cos\theta} - 1\right)$

$= (\Omega_0 - 1)\left(\frac{2-1+\cos\theta}{1-\cos\theta}\right) = (\Omega_0 - 1)\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$

so that $H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_0/a + 1 - \Omega_0}} = \int_0^a \frac{da}{\sqrt{(\Omega_0 - 1)\frac{1+\cos\theta}{1-\cos\theta}}}$

and 5.89 is equal to the derivatives of 5.90 and 5.91, combined, making (integral of the) the two equations a solution to 5.89.

Plotting $a(t)$, as it will go to $1-\cos\theta$ for $0 \le \theta \le 2\pi$, it will from 5.90,



and will reach a maximum when $\theta = \pi$, when $1-\cos\theta$ is at a maximum.

Using 5.91, $t(\pi) = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} = t_{max}$

5.42 gives $t_{crunch} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}$ which from (5.91), is

$t_{crunch} = t(2\pi) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(\pi - \sin 2\pi)$

At $\theta = 2\pi$

$a(2\pi) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos 2\pi) = 0$ so there's indeed a big crunch also seen from the graph at the t_{crunch} given by 5.42

6.9.b (5.93) $a(\eta) = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh\eta - 1)$

(5.94) $t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\eta - \eta)$

$\frac{da}{d\eta} = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\sinh\eta)$, $\frac{dt}{d\eta} = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\cosh\eta - 1)$

$\frac{da}{dt} = \frac{da}{d\eta} / \frac{dt}{d\eta} = \dot{a} = H_0(1-\Omega_0)^{1/2}\frac{\sinh\eta}{(\cosh\eta - 1)}$

$\rightarrow \dot{a}^2 = H_0^2(1-\Omega_0)\frac{(\sinh\eta)^2}{(\cosh\eta-1)^2} = H_0^2(1-\Omega_0)\frac{\cosh^2\eta - 1}{(\cosh\eta - 1)^2}$

$= H_0^2(1-\Omega_0)\frac{(\cosh\eta+1)(\cosh\eta-1)}{(\cosh\eta-1)^2} = H_0^2(1-\Omega_0)\frac{\cosh\eta+1}{\cosh\eta-1}$

and $\rightarrow \frac{\dot{a}^2}{H_0^2} = (1-\Omega_0)\frac{\cosh\eta+1}{\cosh\eta-1} \rightarrow \int_0^t H_0\,dt = H_0 t = \int_0^a \frac{da}{\sqrt{(1-\Omega_0)\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)}}$

Compare to 5.89 $H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_0/a + (1-\Omega_0)}}$

From 5.93, $1-\Omega_0 = \frac{\Omega_0}{2a}(\cosh\eta - 1)$

Then, $\frac{\Omega_0}{a} + (1-\Omega_0) = (1-\Omega_0)\left(\frac{\Omega_0}{a(1-\Omega_0)} + 1\right) =$

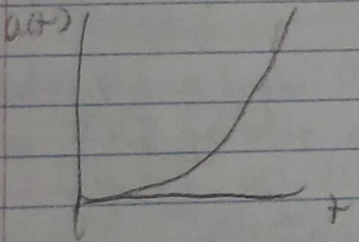
$(1-\Omega_0)\left(\frac{\Omega_0}{a}\frac{2a}{\Omega_0(\cosh\eta-1)} + 1\right) = (1-\Omega_0)\left(\frac{2}{\cosh\eta-1} + 1\right)$

$= (1-\Omega_0)\left(\frac{2}{\cosh\eta-1} + \frac{\cosh\eta-1}{\cosh\eta-1}\right) = (1-\Omega_0)\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)$

so that $H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_0/a + (1-\Omega_0)}} = \int_0^a \frac{da}{\sqrt{(1-\Omega_0)\frac{\cosh\eta+1}{\cosh\eta-1}}}$

and 5.94 is equal to the integral of the combined derivatives of 5.93 and 5.94, making them a solution.

plotting $a(t)$, it will then go as $\cosh\eta - 1$ so that and the universe will expand forever, $\cosh\eta-1$ never reaching 0 but continuously increasing.



6.8 a. $\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0})$ (5.46)

At maximum expansion or a, $H(t)=0$, the slope of the tangent line being 0. Then, in other words $\dot{a}=0$,

$0 = \frac{\Omega_{m,0}}{a_{max}^3} + (1-\Omega_{m,0}) \rightarrow a_{max}^3(1-\Omega_{m,0}) = \Omega_{m,0}$

$\rightarrow a_{max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0}-1}\right)^{1/3}$ and 5.97 is shown as a solution when $H=0$

5.96 can be written as $\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0})$

$\Rightarrow \frac{\dot{a}^2}{H_0^2} = \frac{\Omega_{m,0}}{a} + (1-\Omega_{m,0})a^2 \rightarrow \frac{1}{H_0}\frac{da}{dt} = \sqrt{\frac{\Omega_{m,0}}{a} + (1-\Omega_{m,0})a^2}$

$\rightarrow \int_0^t dt = H_0 t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{m,0}}{a} + (1-\Omega_{m,0})a^2}} = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{m,0}}{a}\left(1 - \frac{(\Omega_{m,0}-1)}{\Omega_{m,0}}a^3\right)}}$

this expression is possible because for $\Omega_{m0} > 1$
$0 < \frac{\Omega_{m0}-1}{\Omega_{m0}} < 1$ so $\sin^{-1}$ is defined

While this seems like a difficult integral, consider

5.99 $H_0 t = \frac{2}{3\sqrt{\Omega_{m,0}-1}} \sin^{-1}\left(\frac{a}{a_{max}}\right)^{3/2} = \frac{2}{3\sqrt{\Omega_{m,0}-1}}\sin^{-1}\sqrt{\frac{\Omega_{m0}-1}{\Omega_{m,0}}}\,a^{3/2}$

Differentiating with respect to a, $= \frac{2}{3\sqrt{\Omega_{m0}-1}}\left(\frac{3}{2}\sqrt{\frac{\Omega_{m,0}-1}{\Omega_{m,0}}}\,a^{\frac{1}{2}}\,\frac{1}{\sqrt{1-\frac{\Omega_{m,0}-1}{\Omega_{m0}}a^3}}\right)$

$= \frac{1}{\sqrt{\frac{\Omega_{m0}}{a}}\sqrt{1-\frac{(\Omega_{m,0}-1)}{\Omega_{m,0}}a^3}}$ which is precisely the integrand from 5.96 on the previous page.

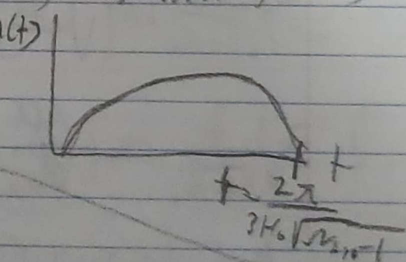
Thus, 5.99 is a solution to 5.96. t_{crunch} is when $a=0$. Substituting this to 5.99, ignoring the zero case as that's the Big Bang,

$\sin^{-1}0 = n\pi$,

$H_0 t_{crunch} = \frac{2}{3\sqrt{\Omega_{m,0}-1}}\sin^{-1}0 \to t_{crunch} = \frac{2\pi}{3H_0\sqrt{\Omega_{m,0}-1}}$ (5.98)

is a solution when $a=0$.

and 5.98 is shown. Plotting $a(t)$, it will go as $(\sin t)^{2/3}$ as seen from 5.99 →



and it was already shown that at $t = \frac{2\pi}{3H_0\sqrt{\Omega_{m,0}-1}}$, $a=0$ or the Big Crunch.

6.3b. From 5.21, $a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}$

In a matter/lambda universe as assumed by 5.96, $\Omega_{\Lambda,0} = 1-\Omega_{m,0}$ and $a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3}$ and 5.100 is shown.
from 6.3a

Also substituting this to 5.96, $\frac{H^2}{H_0^2} = 1-\Omega_{m,0}+1-\Omega_{m,0}$
and $\frac{\Omega_{m0}}{a^3_{m\Lambda}} = (1-\Omega_{m0}) = \Omega_{\Lambda,0}$ and the two contributions of m and Λ are equal as expected and satisfies 5.96.

To solve 5.96 again, the previous solution no longer works as the $\sin^{-1}\frac{a}{a_{max}}$ term isn't defined with $\Omega_\Lambda > 0 \to \Omega_{m,0} < 1$ so that $\frac{1}{a_{max}} = \frac{\Omega_{m,0}-1}{\Omega_{m0}} < 1$. Instead, consider redefining 5.96 as $H_0 t = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{m0}}{a}+(1-\Omega_{m,0})a^2}} = \int_0^a \frac{a^{-1}\,da}{\sqrt{\frac{\Omega_{m0}}{a^3}+(1-\Omega_{m0})}}$

Using derivative calculator

$$= \int_0^a \frac{a^{-1}\,da}{\sqrt{1-\Omega_{m,0}}\sqrt{\frac{\Omega_{m,0}}{(1-\Omega_{m,0})a^3}+1}} \qquad \text{Let } Y = \sqrt{\frac{\Omega_{m,0}}{(1-\Omega_{m,0})a^3}+1}$$

$$dY = \frac{-3\,\Omega_{m,0}\,da}{2(1-\Omega_{m,0})a^4\sqrt{\frac{\Omega_{m,0}}{(1-\Omega_{m,0})a^3}+1}} = \frac{-3\,da}{2Ya}\cdot\frac{\Omega_{m,0}}{(1-\Omega_{m,0})a^3} = -\frac{3(Y^2-1)\,da}{2aY}$$

The integral then can be rewritten $\int_0^Y \frac{-2Ya\,dY}{3a(Y^2-1)\cdot\sqrt{1-\Omega_{m,0}}\,Y}$

using integral calculator

$$= \int_0^Y \frac{-2\,dY}{3\sqrt{1-\Omega_{m,0}}(Y^2-1)} = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\cdot\frac{1}{2}\ln\frac{Y+1}{Y-1} = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\sqrt{\frac{Y+1}{Y-1}}$$

$$= \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\frac{Y+1}{\sqrt{Y^2-1}} = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\left(\frac{1}{\sqrt{Y^2-1}}+\frac{Y}{\sqrt{Y^2-1}}\right)$$ Substituting Y back

$$= \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\left(\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}a^3} + \sqrt{\frac{\Omega_{m,0}}{1-\Omega_{m,0}}a^3+1}\Big/\sqrt{\frac{\Omega_{m,0}}{1-\Omega_{m,0}}a^3}\right)$$

$$= \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\left(\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}a^3} + \sqrt{1+\frac{1-\Omega_{m,0}}{\Omega_{m,0}}a^3}\right)$$ using 5.100

$$= \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\left(\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^3}\right) \quad (5.101)$$

and is a solution to 5.96 when $1-\Omega_{m,0} > 0$ as the ln term is defined in this case.

When $a \ll a_{m\Lambda}$, 5.101 can be rewritten as $\frac{a}{a_{m\Lambda}}$ goes to 0

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\left(\left(\frac{a}{a_{m\Lambda}}\right)^{3/2}+1\right)$$ using the identity $\ln(x+1) \simeq x$ for small x,

$$= \frac{2}{3\sqrt{1-\Omega_{m,0}}}\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\frac{\sqrt{1-\Omega_{m,0}}}{\sqrt{\Omega_{m,0}}}a^{3/2}$$ Solving for a,

$$a(t) = \left(\frac{3}{2}H_0 t\sqrt{\Omega_{m,0}}\right)^{2/3}$$ and 5.102 is shown.

When $a \gg a_{m\Lambda}$, 5.101 can be written as with $\frac{a}{a_{m\Lambda}} \gg 1$,

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ln\left(2\left(\frac{a}{a_{m\Lambda}}\right)^{3/2}\right) = \frac{1}{\sqrt{1-\Omega_{m,0}}}\left(\frac{2}{3}\ln 2 + \ln\frac{a}{a_{m\Lambda}}\right) \approx \frac{1}{\sqrt{1-\Omega_{m,0}}}\ln\frac{a}{a_{m\Lambda}},$$

assuming $\ln 2$ is negligible for large $\frac{a}{a_{m\Lambda}}$

solving for a, $a(t) = a_{m\Lambda} e^{\sqrt{1-\Omega_0} H_0 t}$ and 5.103 is shown.

Solving 5.101 for t,

$$t = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}}\ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^3}\right]$$ for t_0, $a_0 = 1$,

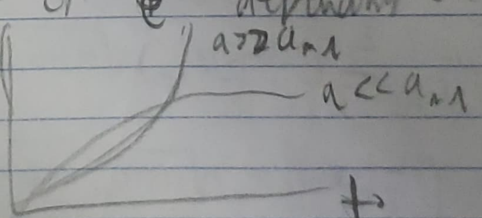
and substituting for $a_{m\Lambda}$,

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}}\ln\left[\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}} + \sqrt{1+\frac{1-\Omega_{m,0}}{\Omega_{m,0}}}\right]$$

$$= \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}}\ln\left[\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}} + \sqrt{\frac{1}{\Omega_{m,0}}}\right] = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}}\ln\left(\frac{\sqrt{1-\Omega_{m,0}}+1}{\sqrt{\Omega_{m,0}}}\right)$$

and 5.104 is shown.

Sketching $a(t)$ from 5.101, from 5.102 and 5.103 it goes as $t^{2/3}$ or e^{t} depending on the current scale factor. In both cases, the universe expands forever as shown.



Substituting current values $\Omega_{m,0} = 0.32$ and $\Omega_{\Lambda,0} = 1-\Omega_{m,0} = 0.68$, into 5.104 to find current age, and $H_0 = 67\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ as given,

$$t_0 = \frac{2}{3\sqrt{0.68}}\ln\left(\frac{\sqrt{0.68}+1}{\sqrt{0.32}}\right)H_0^{-1} = 0.947\,H_0^{-1} = \frac{0.947}{67}\,\frac{\mathrm{s\,Mpc}}{\mathrm{km}} \times \frac{3.1\times10^{19}\ \mathrm{km}}{1\ \mathrm{Mpc}}$$

$= 4.38\times10^{17}\ \mathrm{s} \approx 13.8\ \mathrm{Gyr}$ as given in 5.105 and shown.

To find $t_{m\Lambda}$, again substitute given values of Ω and let $a = a_{m\Lambda}$. 5.109 becomes

$$t_{m\Lambda} = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{\Lambda,0}}}\ln(1+\sqrt{2}) = \frac{2H_0^{-1}}{3\sqrt{0.68}}\ln(1+\sqrt{2})$$

$$= 0.71 H_0^{-1} = \frac{0.71}{67} \times \frac{s \times Mpc}{km} \times \frac{3.1\times10^{19} km}{1 Mpc} = 3.3\times10^{17} s \simeq 10.45 Gyr$$

and 5.106 is shown.

6.4. For $\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$ (5.108)

this can be written as, for $a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

$$\rightarrow \frac{da}{dt}\frac{1}{H_0} = \sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a}} \rightarrow H_0 dt = \frac{da}{\left(\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a}\right)^{1/2}}$$

$$= \frac{a\,da}{\Omega_{r,0}^{1/2}}\left(1 + \frac{a}{a_{rm}}\right)^{-1/2}$$ and 5.109 is shown.

Integrating,

$$\int_0^t H_0 dt = H_0 t = \int_0^a \frac{a\,da}{\Omega_{r,0}^{1/2}}\left(1+\frac{a}{a_{rm}}\right)^{-1/2} = \frac{\sqrt{a_{rm}}}{\Omega_{r,0}^{1/2}}\int_0^a \frac{a\,da}{\sqrt{a_{rm}+a}}$$

From integral table

$$= \frac{\sqrt{a_{rm}}}{\Omega_{r,0}^{1/2}}\left(\frac{2}{3}(a-2a_{rm})\sqrt{a+a_{rm}}\right]_0^a$$

$$= \frac{\sqrt{a_{rm}}}{\Omega_{r,0}^{1/2}}\left(\frac{2}{3}(a-2a_{rm})\sqrt{a+a_{rm}} + \frac{4}{3}a_{rm}\sqrt{a_{rm}}\right)$$

$$= \frac{4}{3}\frac{a_{rm}^2}{\Omega_{r,0}^{1/2}}\left(\frac{1}{2}\frac{(a-2a_{rm})\sqrt{a+a_{rm}}}{a_{rm}^{3/2}} + 1\right)$$

$$= \frac{4}{3}\frac{a_{rm}^2}{\Omega_{r,0}^{1/2}}\left(\frac{a-2a_{rm}}{2a_{rm}}\frac{\sqrt{a+a_{rm}}}{a_{rm}^{1/2}} + 1\right)$$

$$= \frac{4}{3}\frac{a_{rm}^2}{\Omega_{r,0}^{1/2}}\left(\left(\frac{a}{2a_{rm}} - 1\right)\left(1+\frac{a}{a_{rm}}\right)^{1/2} + 1\right)$$

$$= \frac{4}{3}\frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}}\left(1-\left(1-\frac{a}{2a_{rm}}\right)\left(1+\frac{a}{a_{rm}}\right)^{1/2}\right)$$ and 5.110 is shown

If $a \ll a_{rm}$, using the approximation for square roots of small numbers - $\sqrt{1+x} \approx 1+\frac{1}{2}x$ and

$$H_0 t = \frac{4}{3}\frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}}\left(1-\left(1-\frac{a}{2a_{rm}}\right)\left(1+\frac{a}{2a_{rm}}\right)\right) = \frac{4}{3}\frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}}\left(1-1+\frac{a^2}{4a_{rm}^2}\right)$$

$$\to H_0 t = \frac{1}{3}\frac{a^2}{\sqrt{\Omega_{r,0}}} \to a = (3\sqrt{\Omega_{r,0}}\, H_0 t)^{1/2}$$ and 5.111 is shown

If $a \gg a_{rm}$, $1-\frac{a}{2a_{rm}} \approx \frac{-a}{2a_{rm}}$ and $1+\frac{a}{a_{rm}} \approx \frac{a}{a_{rm}}$
Then, 5.110 becomes
1 negligible for large $\frac{a}{a_{rm}}$

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left(1+\frac{a^{3/2}}{2a_{rm}^{3/2}}\right) \approx \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\frac{a^{3/2}}{2a_{rm}^{3/2}} = \frac{2}{3}\frac{a_{rm}^{1/2}}{\sqrt{\Omega_{r,0}}}a^{3/2}$$

$$= \frac{2}{3}\left(\frac{\Omega_{r,0}}{\Omega_{m,0}}\right)^{1/2}\frac{a^{3/2}}{\sqrt{\Omega_{r,0}}} = \frac{2}{3}\frac{a^{3/2}}{\sqrt{\Omega_{m,0}}} \to a \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t\right)^{2/3}$$

and 5.112 is shown
To find t_{rm}, set $a = a_{rm}$ and 5.110 becomes

$$H_0 t_{rm} = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left(1-\left(1-\frac{1}{2}\right)(1+1)^{1/2}\right)$$

$$= \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left(1-\frac{1}{2}\sqrt{2}\right) \to t_{rm} = \frac{4}{3}\left(1-\frac{1}{\sqrt{2}}\right)\frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}}H_0^{-1}$$

$$\approx 0.391\left(\frac{\Omega_{r,0}}{\Omega_{m,0}}\right)^2\frac{H_0^{-1}}{\sqrt{\Omega_{r,0}}} = 0.391\frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2}H_0^{-1}$$ and 5.113 is shown.

With Benchmark model, $\Omega_{r,0} = 9.0\times10^{-5}$ and $\Omega_{m,0} = 0.32$
and $H_0^{-1} = 14.4$ Gyr.
Substituting to find t_{rm},

$$t_{rm} = 0.391\frac{(9.0\times10^{-5})^{3/2}}{0.32^2}(14.4\text{ Gyr}) = 49000\text{ yr}$$ and shown.

At this t_{rm}, using $a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}} = \frac{9.0\times10^{-5}}{0.32} = 2.8\times10^{-4} = \frac{1}{1+z} \to z_{rm} = \frac{1}{2.8\times10^{-4}} - 1$
≈ 3500
and value from 5.24 is shown

To solve for 1 explicitly, consider w=0 case for 5.5 a

5.42 gives $t_0 = \left(\frac{2}{3(1+w)} H_0^{-1}\right)^{1/2}$ at $w=0$

$t_0 = \frac{2}{3} H_0^{-1} = \frac{2}{3H_0}$ and 5.53 is shown.

5.52 gives $d_{hor}(t_0) = \frac{c}{H_0}\frac{2}{1+3w}$ and at $w=0$

$d_{hor} = 2\frac{c}{H_0} = 2R_0$ for $R_0 = \frac{c}{H_0}$ and 5.54 is shown.

5.34 gives $a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$ and at $w=0$

$a_m(t) = \left(\frac{t}{t_0}\right)^{2/3}$ and 5.55 is shown.

5.50 gives $d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left(1-(1+z)^{-(1+3w)/2}\right)$

and at $w=0$

$d_p(t_0) = \frac{2c}{H_0}\left(1-\frac{1}{\sqrt{1+z}}\right)$ and 5.56 is shown.

Angular size distance is $d_p(t_0)$ or 5.56 divided by $(1+z)$,

or $d_A(t_0) = \frac{2c}{H_0(1+z)}\left(1-\frac{1}{\sqrt{1+z}}\right)$ for $w=0$ and 5.57 is shown.

Differentiating and solving for z gives at $d_A = 0$, $z = 1.25$ as before, confirming the solution.

Using $w = \frac{1}{3}$ instead of 0, the previous equations become

$t_0 = \frac{1}{2}H_0^{-1} = \frac{1}{2H_0}$ (5.58), $d_{hor}(t_0) = \frac{c}{H_0} = R_0$ (5.59)

$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}$ (5.60), $d_p(t_0) = \frac{c}{H_0}\left(1-(1+z)^{-1}\right) = \frac{c}{H_0}\left(1-\frac{1}{1+z}\right)$

$= \frac{c}{H_0}\left(\frac{z}{1+z}\right)$ (5.61), and angular size distance $d_A(t_0) = \frac{d_p(t_0)}{1+z} =$

$\frac{c}{H_0}\frac{z}{(1+z)^2}$ (5.62) and all shown. Differentiating also gives $z=1$ at $d_A(t_0)=0$ and confirm the solution.

For $w=-1$, 5.38 gives $\dot{a}^2 = \frac{8\pi G \varepsilon_0}{3c^2} a^{-(1+3w)} \rightarrow \dot{a}^2 = \frac{8\pi G \varepsilon_\Lambda}{3c^2} a^2$ (5.70)

As $H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_c(t_0)$ from 4.29, $\frac{\dot{a}^2}{a} = H_0^2 a^2 \Leftrightarrow \dot{a} = H_0 a$ (5.71) and

$H_0 = \left(\frac{8\pi G \varepsilon_\Lambda}{3c^2}\right)^{1/2}$ (5.72) Solving 5.71 for a, $\frac{da}{dt} = H_0 a \rightarrow$

$\frac{da}{a} = H_0\,dt \rightarrow \ln a = H_0 t + C \rightarrow a = Ce^{H_0 t}$ $a = 1$ at t_0, so

$C = e^{-H_0 t_0}$ so that $a = e^{H_0(t-t_0)}$ (5.73). 5.50 at $w=0$ is

$d_p(t_0) = \frac{c}{H_0}(-1)(1-(1+z)) = \frac{c}{H_0}z$ (5.74). Dividing by $1+z$ gives $d_A(t_0) = R_0\frac{z}{1+z}$, same as before