HOMEWORK 2

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H/W 2.1: Confirm the Delta_rho/rho values in Fig. 2 (a), (b), (c), (d), (e).

Density of the universe, $\rho \rightarrow 2 * M_H = 3.34\times10^{-27}\ kg/m^3$

→ FIG 2 (a)

Average mass of a human $= 62\ kg$

$$\rho_h = \frac{62}{\frac{4}{3}\pi\left(\frac{3}{2}\right)^3} = 4.386\ kg/m^3$$

$$\frac{\rho_h}{\rho} = \frac{4.386}{3.34\times10^{-27}} = 1.313\times10^{-27}$$

→ FIG 2(b)

mass of sun & Earth $\rightarrow 1.989\times10^{30}\ kg + 5.972\times10^{24}\ kg$
$= 1.989\times10^{30}\ kg$

$3\ AU = 4.488\times10^{11}\ m$

$$\rho_{S,E} = \frac{1.989\times10^{30}}{\frac{4}{3}\pi\left(\frac{4.488}{2}\times10^{11}\right)^3} = 4.202\times10^{-5}\ kg/m^3$$

$$\frac{\rho_{S,E}}{\rho} = \frac{4.202\times10^{-5}}{3.34\times10^{-27}} = 1.258\times10^{22}$$

→ FIG 2(c)

mass of M31 $\rightarrow 1.23\times10^{12}\ M_\odot$

mass of milky way $\rightarrow 1.5\times10^{12}\ M_\odot$

Total mass of galaxies $M_g = (1.23\times10^{12} + 1.5\times10^{12})\ M_\odot = 2.73\times10^{12}\ M_\odot$

$3\,Mpc = 9.257 \times 10^{22}\ m$

$$\rho_g = \frac{2.73 \times 10^{12} \times 1.989 \times 10^{30}}{\frac{4}{3}\pi\left(\frac{9.257}{2} \times 10^{22}\right)^3} = 1.307 \times 10^{-26}\ kg/m^3$$

$$\frac{\rho_g}{\rho} = \frac{1.307 \times 10^{-26}}{3.34 \times 10^{-27}} = 0.391 \times 10^{1}$$

→ FIG 2(D)

In this part the mass of the universe is to be found — M_u

$200\,Mpc = 6.171 \times 10^{24}$

$$\rho_u = \frac{M_u}{\frac{4}{3}\pi\left(\frac{6.171}{2} \times 10^{24}\right)^3}$$

$$\frac{\rho_u}{\rho_o} = 1 \Rightarrow \rho_u = \rho_o \Rightarrow \frac{M_u}{\frac{4}{3}\pi\left(\frac{6.171}{2} \times 10^{24}\right)^3} = 3.34 \times 10^{-27}$$

$$\Rightarrow M_u = 4.11 \times 10^{47}\ kg$$

→ PART E

Mass of blackhole → $5M_\odot = 5 \times 1.989 \times 10^{30}\ kg$

$$R_s = \frac{2GM_{BH}}{c^2} = \frac{2 \times 6.673 \times 10^{-11} \times 5 \times 1.989 \times 10^{30}}{(3 \times 10^8)^2} \Rightarrow R_s = 14747.33\ m$$

$$\rho_{BH} = \frac{5 \times 1.989 \times 10^{30}}{\frac{4}{3}\pi\,(14747.33)^3} = 7.403 \times 10^{17}\ kg/m^3$$

$$\rho_{BH}/\rho = 7.403 \times 10^{17} / 3.34 \times 10^{-27} = 2.217 \times 10^{44}$$

H/W 2.2: Given a value of H_o, calculate R_o in Gpc and t_o in years as accurately as you currently can (after Ch 6, you will be given the exact formulae or methods to do so).

Given that $H_0 = 68 \text{ km/s/Mpc}$

→ Calculating t_0

We know that $t_0 = \frac{1}{H_0}$

$$68 \frac{\text{km}}{\text{s.Mpc}} \times \frac{1}{3.08 \times 10^{19}} \frac{\text{Mpc}}{\text{km}} = 2.208 \times 10^{-18} \, s^{-1}$$

There are 3.15×10^{7} seconds in a year

$$2.208 \times 10^{-18} \frac{1}{s} \times 3.15 \times 10^{7} \frac{s}{yr} = 6.955 \times 10^{-11} \, y^{-1}$$

Using the formula $t_0 = 1/H_0$

$$\Rightarrow \frac{1}{6.955 \times 10^{-11}} \Rightarrow t_0 = 1.438 \times 10^{10} \text{ years} = 14.38 \text{ billion years.}$$

→ Calculating R_0

We know $R_0 = \frac{c}{H_0}$

$$R_0 = \frac{3 \times 10^{5} \text{ km/s}}{68 \text{ km/s/Mpc}} = 4411.765 \text{ Mpc}$$

$$\Rightarrow R_0 = 4.412 \text{ Mpc}$$

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H/W 2.3: With what you are given in Ry Chapter 2.4, VERIFY that:

a) Wien's law: lambda_peak = 0.29 cm / T (K).
So what is the ``color'' of the Sun really?

b) Mean energy in Planck curve of temp T: E_mean = 2.7 k T (K).
What is the mean energy of a photon coming from the Sun?
And what is it for a cosmic microwave background photon?
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→ PART A

- Verifying wien's law → $\lambda p = \frac{0.29}{T}$ cm

$E = h\nu_p$, $\nu p = \frac{c}{\lambda p}$

$\Rightarrow \frac{hc}{\lambda p} = 4.97\, KT$

K → boltzmann's constant $= 1.38 \times 10^{-23} \frac{J}{K}$

$$\lambda p = \frac{hc}{4.97\, KT} = \frac{6.626 \times 10^{-34} \times 3 \times 10^{8}}{4.97 \times 1.38 \times 10^{-23}\, T} \Rightarrow \lambda p = \frac{0.0029}{T}\, m$$

$$\Rightarrow \lambda p = \frac{0.29}{T}\, cm$$

Unit Analysis → $\dfrac{\overbrace{kg\, m^2\, s^{-1}}^{h} \times \overbrace{m\, s^{-1}}^{c}}{\underbrace{kg\, m^2\, s^{-2}\, K^{-1}}_{K} \times \underbrace{K}_{T}} = m$

∴ This verifies wien's law.

- To find the color of the sun.

$T = 5778\, K$

$$\lambda p = \frac{0.29}{5778} = 5.019 \times 10^{-5}\, cm = 501.9\, nm.$$

This falls in the wavelength range for green color.

∴ The sun should really be green.

→ PART B

- Verifying mean energy in planck curve → $E_{mean} = 2.7\,KT$

Total energy density for blackbody radiation → $\varepsilon_\gamma = \alpha T^4$

Here, $\alpha = \dfrac{\pi^2 K^4}{15 h^3 c^3}$

Number density of photons in black body radiation → $n_\gamma = \beta T^3$

Here, $\beta = \dfrac{2.4041}{\pi^2} \dfrac{K^3}{h^3 c^3}$

$$E_{mean} = \frac{\varepsilon_\gamma}{n_\gamma} = \frac{\alpha T^4}{\beta T^3} = \frac{\alpha}{\beta} T.$$

Inserting the expression for α & β from above,

$$\frac{\alpha}{\beta} = \frac{\pi^2 K^4}{15 h^3 c^3} \times \frac{\pi^2 h^3 c^3}{2.4041\, K^3} = \frac{\pi^4 K}{15(2.4041)} = 2.7\,K$$

Putting this value of $\frac{\alpha}{\beta}$ in the equation for E_{mean},

$$E_{mean} = 2.7\,KT$$

Unit Analysis

$$E_{mean} = \frac{\alpha}{\beta} T = \frac{Kg\, m^2 s^{-2} \; m^{-3} K^{-4}}{m^{-3}\, K^{-3}} \times K = Kg\, m^2 s^{-2} = \boxed{J}$$

∴ This verifies the mean energy in planck curve.

- Mean energy of photon from the sun

$$E_{mean} = 2.7\left(1.38\times10^{-23}\right)(5778) = 2.153\times10^{-19}\ \text{J}$$
$$= 1.344\ \text{eV}$$

- Mean energy for a cosmic wave background photon.

$$E_{mean} = 2.7\left(1.38\times10^{-23}\right)(2.7255) = 1.016\times10^{-22}\ \text{J}$$
$$= 6.341\times10^{-4}\ \text{eV}$$

(EXTRA CREDIT): If I told you, as we will in Ch 8, that stars half the temperature of the Sun can begin to ionize Hydrogen, AND that both the Sun and the cosmic microwave background are subject to the same Hydrogen (ionization) physics, what does that tell you about the redshift where the cosmic microwave background was generated? Discuss briefly.

For the cosmic microwave background we know that $T_0 = 2.7255\,K$.

Early temperature of the cosmic microwave background $\approx 3000\,K$

To find the redshift where the CMB was generated,

we use the formula, $T(z) = T_0(1+z)$

$$3000\,K = 2.7255(1+z) \Rightarrow z = 1099.716$$

From this we know that the redshift where the CMB was generated was very high.

From the equation, temperature increases by a factor of $(1+z)$ and due to this (increase in temperature) the photons were able to ionize hydrogen.

H/W 2.4: Like at the end of Ch. 2, show that indeed V(z) = (r_0)^3/(1+z)^3, T(z) = T_0 (1+z), and eps_gamma (z) = alpha (T_0)^4 (1+z)^4, with T_0 = 2.7255 K and alpha given in Eq. (2.29).

To show,

$$V(z) = \frac{r_0^3}{(1+z)^3}, \quad T(z) = T_0(1+z), \quad \varepsilon_\gamma(z) = \alpha T_0^4 (1+z)^4$$

From the first law of thermodynamics,

$$dQ = dE + P\,dV$$

For a homogeneous universe, there is no net flow of heat.

$$\Rightarrow dQ = 0$$

$$\Rightarrow \frac{dE}{dt} = -P(t)\frac{dV}{dt} \quad \text{— (1)}$$

we know $E = \varepsilon_\gamma V = \alpha T^4 V$ & $P = P_\gamma = \alpha T^4/3$

$$\frac{dE}{dt} = \alpha\left(4T^3 \frac{dT}{dt} V + T^4 \frac{dV}{dt}\right)$$

Putting this in (1), $\alpha\left(4T^3 \frac{dT}{dt} V + T^4 \frac{dV}{dt}\right) = -\frac{1}{3}\alpha T^4 \frac{dV}{dt}$

multiplying both the sides with $\frac{1}{4\alpha T^4 V}$

$$\alpha\left(\frac{4T^3}{4\alpha T^4 V}\frac{dT}{dt} V + \frac{T^4}{4\alpha T^4 V}\frac{dV}{dt}\right) = -\frac{1}{3}\frac{\alpha T^4}{4\alpha T^4 V}\frac{dV}{dt}$$

$$\alpha\left(\frac{1}{T}\frac{dT}{dt} + \frac{1}{4\alpha V}\frac{dV}{dt}\right) = -\frac{1}{3}\cdot\frac{1}{4V}\frac{dV}{dt}$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{-1}{12\alpha V}\frac{dV}{dt} - \frac{1}{4\alpha V}\frac{dV}{dt} \Rightarrow \frac{1}{T}\frac{dT}{dt} = -\frac{1}{3}V\frac{dV}{dt}$$

we know, $V \propto a(t)^3$, due to this the above equation becomes,

$$\frac{d}{dt}(\ln T) = -\frac{d}{dt}\ln(a)$$

On integrating this,

$$\ln(T) = -\ln(a) + C \Rightarrow T = e^{\ln(a^{-1}) + C} = e^{c}a^{-1}$$

Let $e^{c} = C_1 \rightarrow$ another constant

$$T = C_1 a^{-1}. \quad \text{we know } a = \frac{1}{1+z} \Rightarrow a^{-1} = 1+z$$

$$T(z) = C_1\left(\frac{1}{1+z}\right)^{-1}$$

$$\Rightarrow T(z) = C_1(1+z)$$

At $z=0$, $T = T_0 = 2.7255\,K$, This implies $C_1 = T_0$

$$\Rightarrow \boxed{T(z) = T_0(1+z) \,/\, T(z) = 2.7255(1+z)}$$

we know $V \propto a(t)^3$

Since $a(t) = \frac{1}{1+z} \Rightarrow V \propto \left(\frac{1}{1+z}\right)^3$

To remove the proportionality, a constant is added.

$V(z) = K\left(\frac{1}{1+z}\right)^3$, At $z=0$, Initial volume $= r_0^3$

$\Rightarrow K = r_0^3$

$$\Rightarrow \boxed{V(z) = \frac{r_0^3}{(1+z)^3}}$$

$$\varepsilon_\gamma = \alpha T^4 = \alpha(T_0(1+z))^4 \Rightarrow \boxed{\varepsilon_\gamma = \alpha T_0^4(1+z)^4}$$

$\alpha = 7.566 \times 10^{-16}\,\frac{J}{m^3K^4}$, $T_0 = 2.7255\,K$ $\Rightarrow \boxed{\varepsilon_\gamma = 4.175 \times 10^{-14}(1+z)^4}$