HOMEWORK 6.7 – 6.9

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H/W 6.7a: For Omega_0>1, show that (5.90) and (5.91) are a solution of (5.89). Then sketch or plot a(t) for this universe, and show that it results in a big crunch after a time given in (5.91). At what time is the maximum expansion reached?

$$H_0 t = \int_0^a \frac{da}{[\Omega_0/a + (1-\Omega_0)]^{1/2}} \quad \text{— } 5.89$$

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0-1}(1-\cos\theta) \quad \text{— } 5.90$$

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(\theta-\sin\theta) \quad \text{— } 5.91$$

- Showing 5.90 & 5.91 are solutions of 5.89

Using 5.89,

$$\left(\frac{da}{dt}\right)^2 = H_0^2 \cdot \left[\frac{\Omega_0}{a} + (1-\Omega_0)\right] \quad \text{— } 1$$

we know $a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0-1}(1-\cos\theta)$

from this, $\frac{da}{d\theta} = 0 - \frac{1}{2}\frac{\Omega_0}{\Omega_0-1}(-\sin\theta) = \frac{\sin\theta}{2}\frac{\Omega_0}{\Omega_0-1}$

we know $t(\theta) = \frac{1}{2H_0(\Omega_0-1)^{3/2}}\Omega_0(\theta-\sin\theta)$

from this, $\frac{dt}{d\theta} = \frac{1}{2H_0(\Omega_0-1)^{3/2}}\Omega_0(1-\cos\theta)$

$$\left(\frac{da}{d\theta}\right)\left(\frac{dt}{d\theta}\right)^{-1} = \left(\frac{\sin\theta}{\cancel{2}}\frac{\cancel{\Omega_0}}{\Omega_0-1}\right) \times \left(\frac{\cancel{2}H_0(\Omega_0-1)^{3/2}}{\cancel{\Omega_0}(1-\cos\theta)}\right) = \frac{da}{dt}$$

$$\frac{da}{dt} = H_0(\Omega_0-1)^{1/2} \times \frac{\sin\theta}{1-\cos\theta}.$$

$$\left(\frac{da}{dt}\right)^2 = H_0^2(\Omega_0-1) \times \frac{\sin^2\theta}{(1-\cos\theta)^2} \rightarrow *$$

$$* \quad \frac{\sin^2\theta}{(1-\cos\theta)^2} = \frac{1-\cos^2\theta}{(1-\cos\theta)^2} = \frac{(1-\cancel{\cos\theta})(1+\cos\theta)}{(1-\cos\theta)^{\cancel{2}}} = \frac{1+\cos\theta}{1-\cos\theta}$$

$$\Rightarrow \frac{1+\cos\theta+1-1}{1-\cos\theta} = \frac{-(-1-\cos\theta-1+1)}{1-\cos\theta} = -\frac{(1-\cos\theta-2)}{1-\cos\theta}$$

$$= -\left[\frac{1-\cos\theta}{1-\cos\theta} - \frac{2}{1-\cos\theta}\right] = \left[\frac{2}{1-\cos\theta} - 1\right]$$

Substituting this back into the equation,

$$\left(\frac{da}{dt}\right)^2 = H_0^2(\Omega_0-1)\left[\frac{2}{1-\cos\theta} - 1\right]$$

From the equation of $a(\theta) \Rightarrow 1-\cos\theta = \frac{2a(\Omega_0-1)}{\Omega_0}$

$$\left(\frac{da}{dt}\right)^2 = H_0^2(\Omega_0-1)\left[\frac{\cancel{2}\Omega_0}{\cancel{2}a(\Omega_0-1)} - 1\right] = H_0^2\left[\frac{\Omega_0}{a} - (\Omega_0-1)\right]$$

$$= H_0^2\left[\frac{\Omega_0}{a} + (1-\Omega_0)\right]$$

$$\Rightarrow \boxed{\left(\frac{da}{dt}\right)^2 = H_0^2\left[\frac{\Omega_0}{a} + (1-\Omega_0)\right]}$$

We got back the differential equation and this shows 5.90 and 5.91 are solutions of 5.89.

• Plot for a(t)

First, the code for plotting a(t) is shown as mentioned in the hints file. The time was left in the unit of H_0^{-1} and Ω_0 was chosen to be 1.1.

Editor - C:\Users\parin\OneDrive\Documents\MATLAB\AST 322\Hw6_7to6_9.m

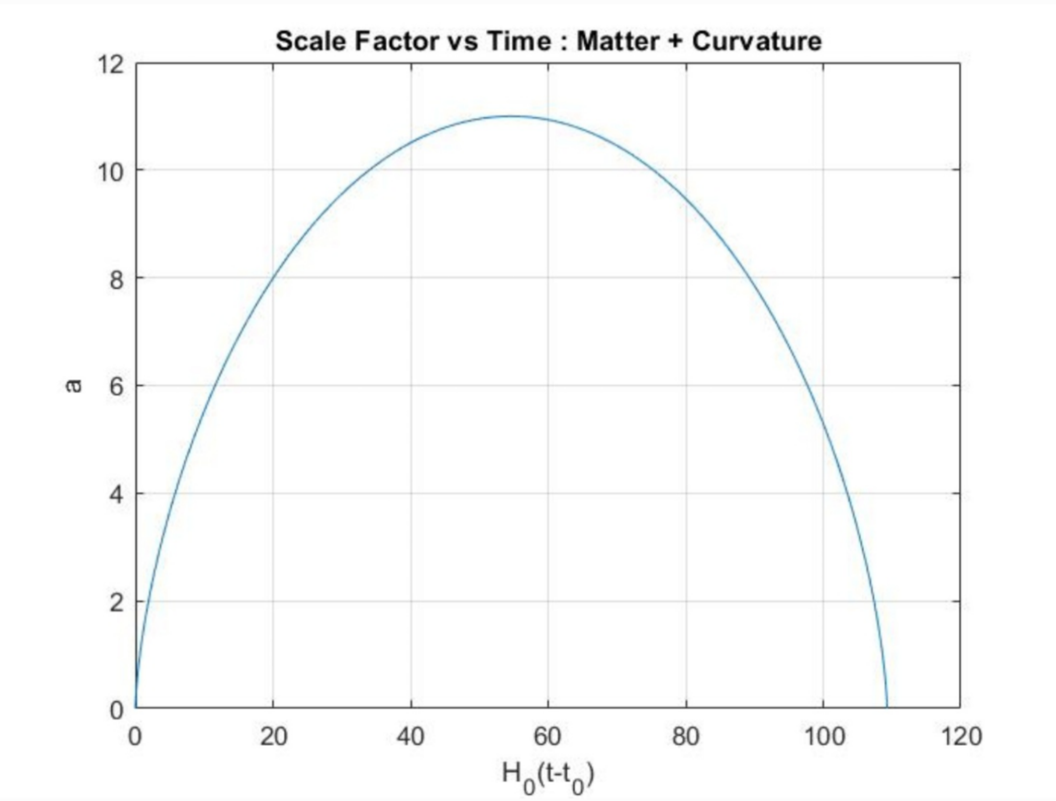
Hw6_7to6_9.m

```
theta = [0:0.01:2*pi] ;
omega_0 = 1.1 ;
H_0 = 1 ;
a_theta = 1/2*omega_0/(omega_0-1)*(1-cos(theta)) ;
t_theta = 1/(2*H_0)*omega_0/(omega_0-1)^(3/2)*(theta-sin(theta)) ;
plot(t_theta,a_theta)
xlabel('H_0(t-t_0)')
ylabel('a')
title('Scale Factor vs Time : Matter + Curvature')
grid on
Max_a_theta = max(a_theta)
index = find(a_theta==Max_a_theta)
t_max = t_theta(index)
```

Command Window

```
Max_a_theta =

   11.0000


index =

   315


t_max =

   54.5848

fx >>
```

a(t) is plotted and shown below.



On seeing the plot, the value of 'a' first increases to a maximum value then decreases to 0. This shows that it results in a big crunch.

On seeing the array of $t(\theta)$ and from the plot, the plot intersects the x-axis at a value of $109.28\ H_0^{-1}$

$\Rightarrow$ From the plot $t_{crunch} = 109.28\ H_0^{-1}$

To verify this from 5.91

According to 5.91, $t(\theta) = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \cdot (\theta - \sin\theta)$

t_{crunch} occurs at $\theta = 2\pi$

$$t_{crunch} = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \left(2\pi - \sin(2\pi)\right) \quad {=}0$$

$$\Rightarrow t_{crunch} = \frac{\pi}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}.$$

$\Omega_0 = 1.1 \Rightarrow t_{crunch} = \frac{\pi}{H_0} \times \frac{1.1}{(1.1-1)^{3/2}} \Rightarrow t_{crunch} = 109.28\ H_0^{-1}$

This is the same value as obtained from the plot.

- Time when maximum expansion was reached.

From the code it is seen that the time was found when the maximum expansion was reached and this time is $54.5848\ H_0^{-1}$.

H/W 6.8b: For Omega_Lambda,0>0, show that (5.100)--(5.103) are a solution of (5.96) in the presence of Lambda. Then sketch or plot a(t) for this universe, and verify that it expands forever. Verify that the resulting (5.105) and (5.106) are true for this universe (this is the Universe we live in!).

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0}) \quad \text{—— } 5.96$$

$$a_{m\lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\lambda,0}}\right) = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \quad \text{—— } 5.100$$

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\lambda}}\right)^3}\right] \quad \text{—— } 5.101$$

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t\right)^{2/3} \quad \text{—— } 5.102$$

$$a(t) \approx a_{m\lambda} \exp\left(\sqrt{1-\Omega_{m,0}}\, H_0 t\right) \quad \text{—— } 5.103$$

Using 5.96,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0})$$

$$H = \frac{\dot{a}}{a} = \frac{da}{dt} \times \frac{1}{a} \implies \left(\frac{da}{dt}\right)^2 \cdot \frac{1}{H_0^2 . a^2} = \frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0})$$

$$\left(\frac{da}{dt}\right)^2 \cdot \frac{1}{H_0^2} = \frac{\Omega_{m,0}}{a} + a^2(1-\Omega_{m,0})$$

$$H_0^2 \cdot dt^2 = da^2 \cdot \frac{1}{\frac{\Omega_{m,0}}{a} + a^2(1-\Omega_{m,0})}$$

$$\implies H_0\, dt = \frac{da}{\left(\frac{\Omega_{m,0}}{a} + a^2(1-\Omega_{m,0})\right)^{1/2}}$$

$$H_0 \int_0^t dt = \int_0^a \frac{da}{\left(\frac{\Omega_{m,0}}{a} + a^2(1-\Omega_{m,0})\right)^{1/2}}$$

$$H_0 t = \int_0^a \frac{da}{\left(\frac{\Omega_{m,0}}{a} + a^2 \cdot \Omega_{\lambda,0}\right)^{1/2}}$$

$$a_{m\lambda}^3 = \frac{\Omega_{m,0}}{1-\Omega_{m,0}} \Rightarrow a_{m\lambda}^3 - a_{m\lambda}^3 \cdot \Omega_{m,0} = \Omega_{m,0}$$

$$\Rightarrow a_{m\lambda}^3 = \Omega_{m0}(1 + a_{m\lambda}^3)$$

$$\Rightarrow \Omega_{m,0} = \frac{a_{m\lambda}^3}{1+a_{m\lambda}^3}$$

using 5.100

$$a_{m\lambda}^3 = \frac{\Omega_{m,0}}{\Omega_{\lambda,0}} \Rightarrow \Omega_{\lambda,0} = \frac{\Omega_{m,0}}{a_{m\lambda}^3} = \frac{a_{m\lambda}^3}{1+a_{m\lambda}^3} \times \frac{1}{a_{m\lambda}^3}$$

$$\Rightarrow \Omega_{\lambda,0} = \frac{1}{1+a_{m\lambda}^3}$$

$$H_0 t = \int_0^a \frac{da}{\left(\frac{1}{a} \cdot \frac{a_{m\lambda}^3}{1+a_{m\lambda}^3} + a^2 \cdot \frac{1}{1+a_{m\lambda}^3}\right)^{1/2}}$$

$$H_0 t = (1+a_{m\lambda}^3)^{1/2} \int_0^a \underbrace{\frac{da}{\left(\frac{a_{m\lambda}^3}{a} + a^2\right)^{1/2}}}_{*} \qquad (1)$$

Solving $*$

$$\int_0^a \frac{da}{\left(\frac{a_{m\lambda}^3}{a} + a^2\right)^{1/2}} = \int_0^a \frac{\sqrt{a}\,.da}{\sqrt{a^3 + a_{m\lambda}^3}}$$

Let $u = \frac{a^{3/2}}{a_{m\lambda}^{3/2}} \Rightarrow \frac{du}{da} = \frac{3\sqrt{a}}{2a_{m\lambda}^{3/2}} \Rightarrow da = \frac{2 \cdot a_{m\lambda}^{3/2}}{3\sqrt{a}} \cdot du$

$\Rightarrow \sqrt{a} \cdot da = \frac{2a_{m\lambda}^{3/2}}{3} \cdot du$

&

$a^{3/2} = u \cdot a_{m\lambda}^{3/2} \Rightarrow a^3 = u^2 a_{m\lambda}^3$

$$\Rightarrow \frac{2}{3} \cdot a_{m\lambda}^{3/2} \cdot \int_0^{\frac{a^{3/2}}{a_{m\lambda}^{3/2}}} \frac{du}{\underbrace{\sqrt{u^2 \cdot a_{m\lambda}^3 + a_{m\lambda}^3}}}$$

taking out $a_{m\lambda}^3$

$$\Rightarrow \frac{2}{3} \int_0^{\frac{a^{3/2}}{a_{m\lambda}^{3/2}}} \frac{du}{\sqrt{u^2+1}}$$

$$= \frac{2}{3} \ln \left(u + \sqrt{u^2+1} \right)_0^{\frac{a^{3/2}}{a_{m\lambda}^{3/2}}}$$

$$= \frac{2}{3} \ln \left(\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\lambda}}\right)^2} \right)$$

Putting this back into (1)

$$H_0 t = \frac{2}{3} (1+a_{m\lambda}^3)^{1/2} \ln \left(\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\lambda}}\right)^2} \right) \quad (2)$$

Now simplifying $(1+a_{m\lambda}^3)^{1/2}$

$$a_{m\lambda}^3 = \frac{\Omega_{m,0}}{1-\Omega_{m,0}}.$$

Putting this in $(1+a_{m\lambda}{}^3)^{1/2}$

$$\left(1+\frac{\Omega_{m_0}}{1-\Omega_{m,0}}\right)^{1/2} = \left(\frac{1-\Omega_{m,0}+\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/2} = \left(\frac{1}{1-\Omega_{m,0}}\right)^{1/2}$$

$$= \left(\left(1-\Omega_{m,0}\right)^{-1}\right)^{1/2} = \left(1-\Omega_{m,0}\right)^{-1/2} = \frac{1}{\sqrt{(1-\Omega_{m,0})}}$$

Putting this in (2)

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\lambda}}\right)^3}\right] \quad \text{— 5.101}$$

It is seen that 5.96 is integrated using expressions from 5.100 and we get 5.101. This shows 5.100 and 5.101 are solutions to 5.96.

- For $a \ll a_{m\lambda}$

Due $a \ll a_{m\lambda} \Rightarrow \frac{a}{a_{m\lambda}} \approx 0$

From 5.101:

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + \sqrt{1+\underbrace{\left(\frac{a}{a_{m\lambda}}\right)^3}_{\approx 0}}\right]$$

Using the approximation,

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + 1\right]$$

In the above equation, $\left(\frac{a}{a_{m\lambda}}\right)^{3/2}$ is small, so we use a taylor series approximation of the form $\ln(1+x) \approx x$

$$\Rightarrow \quad H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \cdot \left(\frac{a}{a_{m\lambda}}\right)^{3/2}$$

$$\frac{3}{2} H_0 \cdot t = a^{3/2} \cdot \frac{1}{\left[(1-\Omega_{m,0}) \cdot a_{m\lambda}^3\right]^{1/2}}$$

From 5.100 $\Rightarrow$ $a_{m\lambda}^3 = \frac{\Omega_{m,0}}{1-\Omega_{m,0}}$ $\Rightarrow$ $a_{m\lambda}^3 (1-\Omega_{m,0}) = \Omega_{m,0}$

$$\Rightarrow \quad \frac{3}{2} H_0 t = \frac{a^{3/2}}{\sqrt{\Omega_{m,0}}} \quad \Rightarrow \quad a^{3/2} = \frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t$$

$$\Rightarrow \quad \boxed{a(t) = \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t\right)^{2/3}}$$

— This is the same as 5.102

- For $a \gg a_{m\lambda}$

Since $a \gg a_{m\lambda}$ $\Rightarrow$ $1 + \frac{a}{a_{m\lambda}} \approx \frac{a}{a_{m\lambda}}$

From 5.101 we have

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\lambda}}\right)^3}\right]$$

Using the approximation, $H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\lambda}}\right)^{3/2} + \left(\frac{a}{a_{m\lambda}}\right)^{3/2}\right]$

$$\Rightarrow H_0 t \cdot \sqrt{1-\Omega_{m,0}} = \frac{2}{3} \times \frac{3}{2} \ln\left[2\left(\frac{a}{a_{m\lambda}}\right)\right]$$

$$H_0 t \sqrt{1-\Omega_{m,0}} = \ln(2) + \ln\left(\frac{a}{a_{m\lambda}}\right)$$

↳ can neglect this since $\frac{a}{a_{m\lambda}} \gg 2$

$$\Rightarrow H_0 t \sqrt{1-\Omega_{m,0}} = \ln\left(\frac{a}{a_{m\lambda}}\right)$$

$$\Rightarrow \frac{a(t)}{a_{m\lambda}} = e^{H_0 t \sqrt{1-\Omega_{m,0}}}$$

$$\Rightarrow \boxed{a(t) = a_{m\lambda} \, e^{H_0 t \sqrt{1-\Omega_{m,0}}}}$$

– This is the same as 5.103

- Plot for a(t)

For the plot, since 5.101 is not separable for 'a', 5.102 and 5.103 were both plotted. Since $\Omega_{\lambda,0} > 0$, $\Omega_{m,0} < 1$. An arbitrary value of $\Omega_{m,0}$ was chosen like 0.31

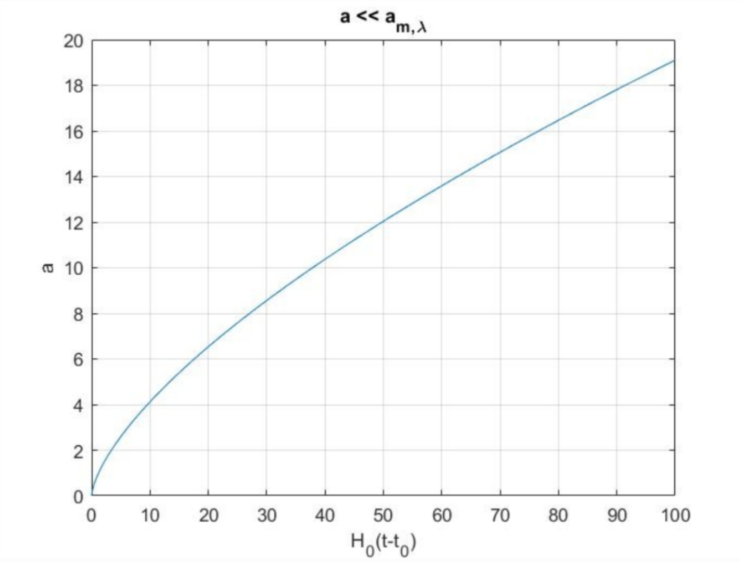
current day value.

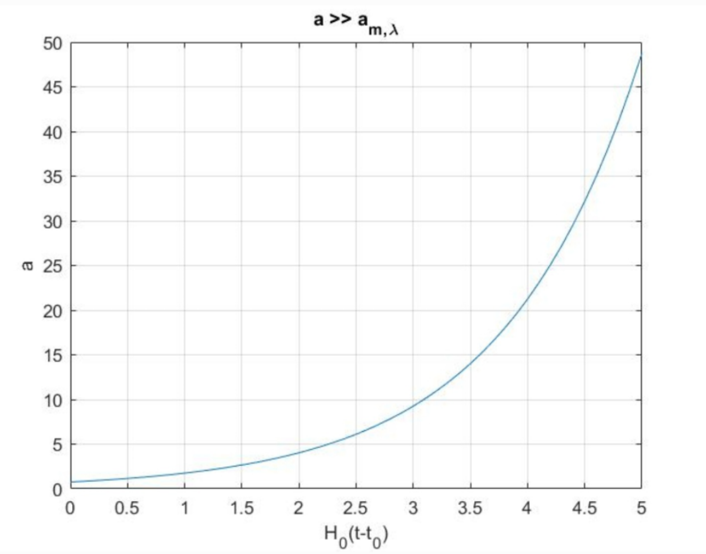
The code for plotting is shown below.

```
omega_m0 = 0.31 ;
H_0 = 1 ;
a_mlambda = (omega_m0/(1-omega_m0))^(1/3) ;
t1 = [0:0.1:100] ;
t2 = [0:0.1:5] ;

% for a<<a_mlambda
a1 = (3/2*sqrt(omega_m0)*H_0*t1).^(2/3) ;
figure;
plot(t1,a1)
xlabel('H_0(t-t_0)')
ylabel('a')
title('a << a_m_,_\lambda')
grid on
```

```
% for a>>a_mlambda
a2 = a_mlambda*exp(sqrt(1-omega_m0)*H_0*t2) ;
figure;
plot(t2,a2)
xlabel('H_0(t-t_0)')
ylabel('a')
title('a >> a_m_,_\lambda')
grid on
```

The two plots are shown below. The scale of the x-axis is different to see the behavior of each plot.





The second plot, where $a \gg a_{m\lambda}$ represents the current universe we live in as it is λ dominated. It is seen that the t is in terms of H_0^{-1}. With increasing time, the scale factor keeps increasing. This shows that the universe expands forever.

- Verifying the value t_0

To verify t_0, we plug in $t = t_0$ and $a = 1$ in 5.101.

$$H_0 t_0 = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{1}{a_{m\lambda}}\right)^{3/2} + \left(1 + \frac{1}{a_{m\lambda}^{3}}\right)^{1/2}\right]$$

From 5.100 $\rightarrow$ $a_{m\lambda} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3}$

Given in the book that $\Omega_{m,0} = 0.31$

$$\Rightarrow a_{m\lambda} = \left(\frac{0.31}{1-0.31}\right)^{1/3} = 0.766$$

$$\Rightarrow t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{1}{0.766}\right)^{3/2} + \left(1 + \frac{1}{0.766^3}\right)^{1/2}\right]$$

$$t_0 = \frac{2}{3\sqrt{0.69}} (1.19) H_0^{-1} \Rightarrow \boxed{t_0 = 0.955 H_0^{-1}}$$ - same as 5.105

Given in the book that $H_0^{-1} = 14.4$ Gyr

$$\Rightarrow t_0 = 0.955 \times 14.4 \Rightarrow \boxed{t_0 = 13.75 \text{ Gyr}}$$

- Verifying the value of $t_{m\lambda}$

To find $t_{m\lambda}$, $a = a_{m\lambda}$

$\Rightarrow$ 5.101 becomes,

$$H_0 t_{m\lambda} = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a_{m\lambda}}{a_{m\lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a_{m\lambda}}{a_{m\lambda}}\right)^3}\right]$$

$$t_{m\lambda} = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln\left[1 + \sqrt{2}\right]$$

Given in the book that $\Omega_{m,0} = 0.31$

$$t_{m\Lambda} = \frac{2 H_0^{-1}}{3\sqrt{0.69}} \ln\left[1+\sqrt{2}\right]$$

$$\Rightarrow \boxed{t_{m\Lambda} = 0.707\, H_0^{-1}}$$ — same as 5.106

Given that $H_0^{-1} = 14.4$ Gyr

$$t_{m\Lambda} = 0.707 \times 14.4 \Rightarrow \boxed{t_{m\Lambda} = 10.18 \text{ Gyr}}$$

```
H/W 6.9: Show that (5.109)--(5.113) are valid solutions for the Friedmann
equation (5.108) in the case of a Radiation+Matter only universe. (This is the
universe we lived in before Lambda took over at 3.5 Gyr ago (at
z_Lambda,m=0.29). Using (5.111)--(5.113), verify the value of z_r,m and t_rm
that we will use a lot later on.
```

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \quad \text{— 5.108}$$

$$H_0 dt = \frac{a\,da}{\Omega_{r,0}^{1/2}} \left[1 + \frac{a}{a_{rm}}\right]^{-1/2} \quad \text{— 5.109}$$

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2}\right] \quad \text{— 5.110}$$

$$a \ll a_{rm}, \quad a \approx \left(2\sqrt{\Omega_{r,0}}\, H_0 t\right)^{1/2} \quad \text{— 5.111}$$

$$a \gg a_{rm}, \quad a = \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t\right)^{2/3} \quad \text{— 5.112}$$

$$t_{rm} = \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}} H_0^{-1} \approx 0.391 \frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2} H_0^{-1} \quad \text{— 5.113}$$

According to 5.108 we know that

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

$$H(t) = \frac{\dot{a}}{a}$$

$$\left(\frac{da}{dt}\right)^2 \cdot \frac{1}{H_0^2 \cdot a^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

$$= \left(\frac{da}{dt}\right)^2 \cdot \frac{1}{H_0^2} = \frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a}$$

$$= \left(\frac{da}{dt}\right)^2 \cdot \frac{1}{H_0^2} = \frac{\Omega_{r,0}}{a}\underbrace{\left(\frac{1}{a} + \frac{\Omega_{m,0}}{\Omega_{r,0}}\right)}_{} \quad \frac{\Omega_{m,0}}{\Omega_{r,0}} \rightarrow \frac{1}{a_{rm}}$$

$$\frac{a\,da^2}{\Omega_{r,0}} \times \frac{1}{\left(\frac{1}{a} + \frac{1}{a_{rm}}\right)} = H_0^2\,dt^2$$

$\left(\frac{1}{a} + \frac{1}{a_{rm}}\right) \longrightarrow$ taking out $\frac{1}{a} \Rightarrow \frac{1}{a}\left(1 + \frac{a}{a_{rm}}\right)$

$$\Rightarrow \frac{a^2 da^2}{\Omega_{r,0}} \times \frac{1}{\left(1 + \frac{a}{a_{r,m}}\right)} = H_0^2 dt^2.$$

Now, taking the square root on both the sides.

$$H_0 dt = \frac{a\,da}{\Omega_{r,0}^{1/2}} \cdot \frac{1}{\left(1 + \frac{a}{a_{r,m}}\right)^{1/2}} \Rightarrow \boxed{H_0 dt = \frac{a\,da}{\Omega_{r,0}^{1/2}}\left(1 + \frac{a}{a_{r,m}}\right)^{-1/2}} \quad -5.109$$

This shows 5.109 is equivalent to 5.108.

- To find if 5.110 is a solution to 5.108 | 5.109

According to 5.110,

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2}\right]$$

Differentiating the above equation with respect to t.

$$H_0 = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[0 - \left[\left(\frac{-1}{2a_{rm}} \cdot \frac{da}{dt}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2} + \left(1 - \frac{a}{2a_{rm}}\right) \cdot \frac{1}{2}\left(1 + \frac{a}{a_{rm}}\right)^{-1/2}\left(\frac{1}{a_{rm}} \cdot \frac{da}{dt}\right)\right]\right]$$

$$H_0 = \frac{\cancel{4}^{\,2}\, a_{rm}^{\cancel{2}}}{3\sqrt{\Omega_{r,0}}} \cdot \frac{da}{dt} \cdot \frac{1}{\cancel{2a_{rm}}}\left[\left(1 + \frac{a}{a_{rm}}\right)^{1/2} - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{-1/2}\right]$$

$$H_0 dt = \frac{da \cdot 2a_{rm}}{3\sqrt{\Omega_{r,0}}} \underbrace{\left[\frac{\left(1+\frac{a}{a_{rm}}\right) - \left(1 - \frac{a}{2a_{rm}}\right)}{\left(1+\frac{a}{a_{rm}}\right)^{1/2}} \right]}$$

$$\downarrow$$

$$\frac{\frac{a}{a_{rm}} + \frac{a}{2a_{rm}}}{\left(1+\frac{a}{a_{rm}}\right)^{1/2}} = \frac{\frac{3a}{2a_{rm}}}{\left(1+\frac{a}{a_{rm}}\right)^{1/2}}$$

$$\Rightarrow \quad H_0 dt = \frac{da \cdot \cancel{2}\cancel{a_{rm}}}{\cancel{3}\sqrt{\Omega_{r,0}}} \cdot \frac{\cancel{3}a}{\cancel{2}\cancel{a_{rm}}\left(1+\frac{a}{a_{rm}}\right)^{1/2}}$$

$$\Rightarrow \quad \boxed{H_0 dt = \frac{a \cdot da}{\Omega_{r,0}^{1/2}} \left(1 + \frac{a}{a_{r,m}}\right)^{-1/2}}$$

This is the same as 5.109 and this shows 5.110 is a solution to 5.108/5.109.

- For $a << a_{rm}$

In this case $\frac{a}{a_{rm}} \to 0$

$$5.109 \to \quad H_0 dt = \frac{a \cdot da}{\Omega_{r,0}^{1/2}} \left[1 + \frac{a}{a_{rm}}\right]^{-1/2}$$

Since $\frac{a}{a_{rm}} \to 0 \Rightarrow H_0 dt = \frac{a \cdot da}{\Omega_{r,0}^{1/2}}$

$$H_0 \int_0^t dt = \frac{1}{\Omega_{r,0}^{1/2}} \int_0^a a\,da \Rightarrow H_0 t = \frac{a^2}{2\Omega_{r,0}^{1/2}}$$

$$\Rightarrow \boxed{a = \left(2\sqrt{\Omega_{r,0}} \cdot H_0 t\right)^{1/2}}$$ This is 5.111

- For $a \gg a_{rm}$

We know 5.110 $\rightarrow$ $$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2}\right]$$

For $a \gg a_{rm}$

$$H_0 t = \frac{\cancel{4}^{\,2} a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[\left(\frac{a}{\cancel{2}a_{rm}}\right)\left(\frac{a}{a_{rm}}\right)^{1/2}\right]$$

$$H_0^2 t^2 = \frac{4\, a_{rm}^4}{9 \cdot \Omega_{r,0}} \times \frac{a^3}{\cancel{a_{rm}^3}}$$

$$H_0^2 t^2 = \frac{4}{9}\frac{a_{rm}}{\Omega_{r,0}} \cdot a^3$$

Here, $\Omega_{r,0} = \Omega_{m,0} \cdot a_{r.m}$ $$\Rightarrow H_0^2 t^2 = \frac{4}{9} \cdot \frac{\cancel{a_{rm}}}{\Omega_{m,0} \cdot \cancel{a_{rm}}} \cdot a^3$$

$$\Rightarrow H_0^2 t^2 = \frac{4}{9}\frac{a^3}{\Omega_{m,0}} \Rightarrow H_0 t = \frac{2}{3}\frac{a^{3/2}}{\sqrt{\Omega_{m,0}}}$$

$$\Rightarrow a^{3/2} = \frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t$$

$$\Rightarrow \boxed{a = \left(\frac{3}{2}\sqrt{\Omega_{m,0}}\, H_0 t\right)^{2/3}}$$ This is 5.112

• Setting $a = a_{rm}$ in 5.110,

$$H_0 t_{rm} = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \left(1 - \frac{1}{2}\right)(1+1)^{1/2}\right]$$

$$H_0 t_{rm} = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \frac{\sqrt{2}}{2}\right]$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Substituting $a_{rm}^2 = \frac{\Omega_{r,0}^2}{\Omega_{m,0}^2}$

$$H_0 t_{rm} = \underbrace{\left(1 - \frac{1}{\sqrt{2}}\right) \cdot \frac{4}{3}}_{=0.391} \cdot \frac{\Omega_{r,0}^2}{\Omega_{m,0}^2} \cdot \frac{1}{\sqrt{\Omega_{r,0}}}$$

$$\Rightarrow \boxed{t_{rm} = 0.391 \frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2} \cdot H_0^{-1}}$$ This is 5.113

• Value of t_{rm}

$$t_{rm} = 0.391 \frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2} \cdot H_0^{-1}$$

as given in the text book

$\Omega_{r,0} = 9 \times 10^{-5}$, $\Omega_{m,0} = 0.31$, $H_0^{-1} = 14.4 \times 10^9$ years

$$t_{rm} = 0.391 \times \frac{(9 \times 10^{-5})^{3/2}}{0.31^2} \times 14.4 \times 10^9$$

$$\Rightarrow \boxed{t_{rm} = 50024.139 \text{ years} \approx 50,000 \text{ years}}$$

Same as the value mentioned in the book.

- Value of z_{rm}

It is given that $a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.9 \times 10^{-4}$

$$a_{rm} = \frac{1}{1+z_{rm}} \Rightarrow z_{rm} = \frac{1}{a_{rm}} - 1$$

$$\Rightarrow z_{rm} = \frac{1}{2.9 \times 10^{-4}} - 1 \Rightarrow \boxed{z_{rm} = 3447.276}$$
