HOMEWORK 5.1 - 5.4

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H/W 5.1: Show from the previous Eqs. that (5.9) is a valid solution of (5.8). Then use this result to answer: What w-component dominates the expansion of the universe for:

a) a ---> 0 (or z ---> infinity); and

b) a ---> infinity (or t ---> infinity).

c) Discuss the cases w=0 and w=1/3.

$$5.9 \rightarrow \varepsilon_i(a) = \varepsilon_{i,0}\, a^{-3(1+w_i)}$$

we know that the fluid equation is,

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

we also know that the equation of state is,

$$P = w\varepsilon$$

If there are N components of the universe then an ith component has an energy of ε_i. The total energy density is $\varepsilon = \sum \varepsilon_i$ and the total pressure is $P = \sum w_i \varepsilon_i$.

Using, the fluid equation becomes $\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(\varepsilon_i + P_i) = 0$ and the equation of state becomes $P_i = w_i \varepsilon_i$

Combining these two equations,

$$\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(\varepsilon_i + w_i\varepsilon_i) = \frac{d\varepsilon_i}{dt} + 3\frac{da}{dt}\cdot\frac{\varepsilon_i}{a}(1+w_i) = 0$$

$$\Rightarrow \frac{d\varepsilon_i}{\cancel{dt}} = -3\frac{da}{\cancel{dt}}\frac{\varepsilon_i}{a}(1+w_i) \Rightarrow \frac{d\varepsilon_i}{\varepsilon_i} = -3(1+w_i)\frac{da}{a}$$

— 5.8

Integrating the above equation

$$\int \frac{d\varepsilon_i}{\varepsilon_i} = -3(1+w_i) \int \frac{da}{a}$$

$$\ln(\varepsilon_i) = -3(1+w_i)\ln(a) + C \Rightarrow \varepsilon_i = e^{-3(1+w_i)\ln(a) + C}$$

$$\varepsilon_i = \underbrace{e^{-3(1+w_i)\ln(a)}}_{e^{\ln(a)^{-3(1+w_i)}}} \cdot \underbrace{e^C}_{C_1 \rightarrow \text{another constant}}$$

$$\Rightarrow \varepsilon_i = C_1 e^{\ln(a)^{-3(1+w_i)}} \Rightarrow \varepsilon_i = C_1 a^{-3(1+w_i)}$$

at $t = t_0$, $\varepsilon_i = \varepsilon_{i,0}$ and $a = 1$

$$\Rightarrow C_1 = \varepsilon_{i,0}$$

Using this value we get

$$\boxed{\varepsilon_i(a) = \varepsilon_{i,0}\, a^{-3(1+w_i)}} \longrightarrow 5.9$$

a) $a \rightarrow 0$ $(z \rightarrow \infty)$

As $a \rightarrow 0$, the largest positive value of w is dominant. This means, as $a \rightarrow 0$, $(w = \frac{1}{3})$ radiation component dominates the expansion of the universe.

b) $a \to \infty$ $(t \to \infty)$

As $a \to \infty$, the component with the smallest value of w is dominant. This means, as $a \to \infty$, $(w = -1)$, the component of the cosmological constant dominates the expansion of the universe.

c) Discussing the cases of $w = 0$ and $w = \frac{1}{3}$.

- $w = 0 \to$ matter dominant

Putting this value of 'w' in equation 9, we get

$$\varepsilon_m(a) = \frac{\varepsilon_{m,0}}{a^{3(1+0)}} = \frac{\varepsilon_{m,0}}{a^3}$$

- $w = \frac{1}{3} \to$ radiation dominant.

Putting this value of 'w' in equation 9, we get,

$$\varepsilon_m(a) = \frac{\varepsilon_{m,0}}{a^{3(1+\frac{1}{3})}} = \frac{\varepsilon_{m,0}}{a^4}$$

From this it is seen that, for nonrelativistic particles $\varepsilon \propto a^{-3}$

For relativistic particles $\varepsilon \propto a^{-4}$. This can also be written in terms of number density 'n' since $\varepsilon = nE$.

nonrelativistic particles $\to n \propto a^{-3}$, relativistic particles $n \propto a^{-4}$

This shows that energy density for relativistic particles drops at a steeper rate (a^{-4}) compared to nonrelativistic matter.

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H/W 5.2: For what value of z_Lambda_m is eps_Lambda = eps_m? Discuss what this
means for the growth of galaxies with time (no more than 1-2 paragraphs).

Hint: In preparation for a possible term project, do a little literature search
(use ADS abstract link on Links) on the cosmic star-formation rate vs. redshift
SFR(z). (e.g., Madau, P. & Dickinson, M. 2014, Ann. Rev. A&Ap, 52, p. 415). The
PDF is available here:  https://arxiv.org/abs/1403.0007 . Based on their Fig. 9,
discuss what impact the expansion and Lambda (> 0) may have had on the growth of
galaxies with cosmic time, and whether this is visible in the data.

[For extra credit: You could also discuss how the galaxy merger rate has changed
as a function of redshift, by comparing the mix of spiral and elliptical galaxies
in high redshift clusters to those at low redshifts. (Hint: You need to know that
galaxy mergers tend to transform spiral galaxies into ellipticals; Do a
literature search of how the mix of spiral and elliptical galaxies has changed in
galaxy clusters with redshift, as observed e.g. with Hubble).]
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The ratio of energy densities for a λ 4 mass dominated universe is $\dfrac{\varepsilon_\Lambda(a)}{\varepsilon_m(a)}$.

From equation 5.9, we know $\varepsilon_i(a) = \varepsilon_{i,0}\, a^{-3(1+w_i)}$.

For a matter dominated universe $(w=0)$.

$$\varepsilon_m(a) = \varepsilon_{m,0}\, a^{-3(1+0)} \Rightarrow \varepsilon_m(a) = \varepsilon_{m,0}\, a^{-3}$$

For a Λ dominated universe $(w=-1)$

$$\varepsilon_\Lambda(a) = \varepsilon_{\Lambda,0}\, a^{-3(1+(-1))} = \varepsilon_{\Lambda,0}$$

$$\Rightarrow \frac{\varepsilon_\Lambda(a)}{\varepsilon_m(a)} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}\, a^{-3}}$$

We know that $\Omega(t) = \dfrac{\varepsilon(t)}{\varepsilon_c(t)}$

$$\Rightarrow \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}\, a^{-3}} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}\, a^{-3}}$$

For matter - Λ equality, $\dfrac{\Omega_{\Lambda,0}}{\Omega_{m,0}\, a^{-3}} = 1$

$$\Rightarrow \quad a^3 = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \quad \Rightarrow \quad a = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}$$

we know that $\Omega_{m,0} = 0.31$ and $\Omega_{\Lambda,0} = 0.69$

$$\Rightarrow a = \left(\frac{0.31}{0.69}\right)^{1/3} = 0.766 \text{ (as given in the text book)}$$

To find the redshift corresponding to this value of 'a', the value of 'a' given in the notes is used which is $a = 0.778$

$$a = \frac{1}{1+z} = 0.778 \Rightarrow z = \frac{1}{0.778} - 1 = 0.285$$

$$\Rightarrow \quad z_{\Lambda,m} = 0.285 \text{ (for matter} - \Lambda \text{ equality)}$$

To discuss what this means for growth of galaxies with time.

From above it was found that the lambda-matter equality happened at a redshift of about 0.285. In question 5.3 it was found that the universe was matter dominated ($w=0$) when the age of universe was $\frac{2}{3}$ of the current age. This could be used to say that before the equality, matter was dominant and currently, the universe was lambda dominant. So when the universe was matter dominant, it could be guessed that more matter was being pulled together to form galaxies. But, when the universe is lambda dominant, the rate of expansion is higher and due to this, the growth of galaxies with time is decreasing.

EXTRA CREDIT

As an extension to the information mentioned above, it is seen that the growth of galaxies is decreasing with time as we are in a lamba dominated universe where the rate of expansion eis higher. At higher redshifts, the universe was matter or radiation dominated and here galaxies were being formed at a higher and the density of the universe was also higher at high redshifts. This means that the galaxy merger rate is higher with high redshifts.

H/W 5.3: Show that (5.38), (5.39), (5.41) and (5.42) follow from the previous math (for w .neq. -1). Discuss the meaning of each of these equations for simple cases like w=0 and w=1/3. E.g., calculate t_o (in units of 1/H_o) for the current value of eps_o if w=0 or w=1/3.

$$5.38 \rightarrow \quad q = 2/(3+3w)$$

$$5.39 \rightarrow \quad a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$$

$$5.40 \rightarrow \quad t_0 = \frac{1}{1+w}\left(\frac{c^2}{6\pi G\varepsilon_0}\right)^{1/2}$$

$$5.41 \rightarrow \quad H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+w)} t_0^{-1}$$

It is given that for a spatially flat, single-component universe, the Friedmann equation is

$$\dot{a}^2 = \frac{8\pi G\varepsilon_0}{3c^2} a^{-(1+3w)}$$

It is also given that $a \propto t^q \Rightarrow a^{-(1+3w)} \propto t^{-q(1+3w)}$ and $\dot{a}^2 \propto t^{2q-2}$.

So for the L.H.S we have $\dot{a}^2 \propto t^{2q-2}$ and for the R.H.S, $a^{-(1+3w)} \propto t^{-q(1+3w)}$.

Equating the powers for these two

$$2q-2 = -q(1+3w) \Rightarrow 2q-2 = -q-3wq$$

$$\Rightarrow 3q+3wq = 2 \Rightarrow q(3+3w) = 2$$

$$\Rightarrow \boxed{q = \frac{2}{3+3w}} \quad \text{— } 5.38$$

we know that $a \propto t^{q}$

$$\Rightarrow a \propto t^{\frac{2}{3+3w}}$$

To remove the proportionality sign, a constant K is added

$$\Rightarrow a = Kt^{\frac{2}{3+3w}}$$

at $t = t_0$, $a = 1$

$$1 = K\, t_0^{\frac{2}{3+3w}} \Rightarrow K = \frac{1}{t_0^{\frac{2}{3+3w}}}$$

On substituting the value of K.

$$a(t) = \frac{t^{\frac{2}{3+3w}}}{t_0^{\frac{2}{3+3w}}} \Rightarrow \boxed{a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3+3w}}} \quad — 5.39$$

It is found that $q = \frac{2}{3+3w} = \frac{2}{3(1+w)}$

we also found that $a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3+3w}} \Rightarrow a(t) = \frac{t^{q}}{t_0^{q}}$

On differentiating this equation,

$$\dot{a}(t) = \frac{q\, t_0^{q-1}}{t_0^{q}}$$

Dividing the expressions $\dot{a}(t)$ and $a(t)$

$$\left(\frac{\dot{a}}{a}\right) = \frac{q t^{q-1}/t_0^{q}}{t^{q}/t_0^{q}} = \frac{q t^{q-1}}{t^{q}} = q t^{-1}$$

we know $H(t) = \left(\frac{\dot{a}}{a}\right) = q\,t^{-1}$

Putting in the expression for q, $\left(\frac{\dot{a}}{a}\right) = \frac{2}{3+3\omega}\; t^{-1}$

at $t = t_0$

$$\boxed{H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+\omega)}\, t_0^{-1}} \quad \text{— } 5.41$$

On multiplying the above equation by t_0 and dividing by H_0,

$$\Rightarrow \boxed{t_0 = \frac{2}{3(1+\omega)}\, H_0^{-1}} \quad \text{— } 5.42$$

Discussing the four equations for the cases $\omega = 0$ & $\omega = \frac{1}{3}$

- For $\omega = 0$

$\rightarrow q = 2/3$

$\rightarrow a(t) = \left(\frac{t}{t_0}\right)^{2/3}$

$\rightarrow H_0 = \left(\frac{\dot{a}}{a}\right) = \frac{2}{3}\, t_0^{-1}$

$\rightarrow t_0 = \frac{2}{3} H_0^{-1}$

- For $\omega = 1/3$

$\rightarrow q = 1/2$

$\rightarrow a(t) = \left(\frac{t}{t_0}\right)^{1/2}$

$\rightarrow H_0 = \left(\frac{\dot{a}}{a}\right) = \frac{1}{2}\, t_0^{-1}$

$\rightarrow t_0 = \frac{1}{2} H_0^{-1}$

Above found are the equations 5.38, 5.39, 5.41 & 5.42 for $\omega = 0$ and $\omega = 1$.

In these equations, 'q' does not really have a symbolic meaning as it is more of a scale factor and this value is different for both the cases of ω. On comparing the equation for $a(t)$, it is seen that when $\omega = 0$ (matter dominated) the expansion rate is faster than compared to when $\omega = \frac{1}{3}$ (radiation dominated). this can be seen by the power of $\frac{2}{3}$ for $\omega = 0$ and power of $\frac{1}{2}$ for $\omega = 1/3$.

The last two equations for each case represents the same thing. So we know that H_0^{-1} gives an estimate to the age of the universe. For $\omega = 0$ we see that $t_0 = \frac{2}{3} H_0^{-1}$. This shows that the universe was matter dominated when the universe was two-thirds of the current age (~ 9.2 billion years old) Similarly for $\omega = \frac{1}{3}$ we have $t_0 = \frac{1}{2} H_0^{-1}$. this gives an estimate that the universe was radiation dominated when its age was half of the current age (~ 6.9 billion years old).

H/W 5.4: Show that (5.48), (5.49), and (5.50) follow from the previous math. Plot d_p(t_o) and t(z) for w=0 and w=1/3, and discuss their meaning. (As before, you may omit the cases --- for now --- where a w-value causes a zero-divide).

$$5.48 \rightarrow t_e = \frac{t_0}{(1+z)^{3(1+w)/2}} = \frac{2}{3(1+w)H_0} \frac{1}{(1+z)^{3(1+w)/2}}$$

$$5.49 \rightarrow d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)} = c t_0 \frac{3(1+w)}{1+3w}\left[1-(t_e/t_0)^{(1+3w)/(3+3w)}\right]$$

$$5.50 \rightarrow d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1-(1+z)^{-(1+3w)/2}\right]$$

As derived in the previous question, $t_0 = \frac{2}{3(1+w)} H_0^{-1}$

5-39 was also derived in the previous question $\Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3+3w}}$

Using the above relation, $a(t_0) = 1^{\frac{2}{3+3w}} = 1$

$$a(t_e) = \left(\frac{t_e}{t_0}\right)^{2/(3+3w)}$$

We also know $a(t_0) = \frac{1}{1+0} = 1$ and $a(t_e) = \frac{1}{1+z}$

On dividing $a(t_0)$ and $a(t_e)$,

$$\frac{1}{1/1+z} = \frac{a(t_0)}{a(t_e)} = \frac{1}{\left(\frac{t_e}{t_0}\right)^{2/(3+3w)}}$$

$$\Rightarrow 1+z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/(3+3w)}$$

Rearranging the above equation for te

$$t_e^{\frac{2}{3(1+w)}} = \frac{t_o^{\frac{2}{3(1+w)}}}{1+z} \Rightarrow t_e = \frac{t_o}{(1+z)^{1/3/3(1+w)}}$$

$$\Rightarrow t_e = \frac{t_o}{(1+z)^{3(1+w)/2}} \quad — (1)$$

As mentioned in the beginning $t_o = \frac{2}{3(1+w)} H_0^{-1}$

Plugging this in (1)

$$t_e = \frac{2}{3(1+w)H_0} \times \frac{1}{(1+z)^{3(1+w)/2}} \quad — (2)$$

From (1) & (2)

$$\boxed{t_e = \frac{t_o}{(1+z)^{3(1+w)/2}} = \frac{2}{3(1+w)H_0} \cdot \frac{1}{(1+z)^{3(1+w)/2}}} \quad — 5.48$$

To find 5.49, the integral for $d_p(t_o) = c\int_{t_e}^{t_o} \frac{dt}{a(t)}$ has to be evaluated.

we know $a(t) = \left(\frac{t}{t_o}\right)^{\frac{2}{3+3w}} = \left(\frac{t}{t_o}\right)^{q}$

Putting this expression for $a(t)$ in the integral,

$$d_p(t_o) = c\int_{t_e}^{t_o} \frac{t_o^{q}}{t^{q}}\, dt$$

$$d_p(t_0) = c\, t_0^{\,q} \int_{t_e}^{t_0} t^{-q}\, dt = c\, t_0^{\,q} \left[\frac{t^{-q+1}}{-q+1} \right]_{t_e}^{t_0}$$

$$d_p(t_0) = \frac{c\, t_0^{\,q}}{1-q} \left(t_0^{\,1-q} - t_e^{\,1-q} \right) = \frac{c\, t_0}{1-q} - \frac{c\, t_0^{\,q}\, t_e^{\,1-q}}{1-q}$$

$$d_p(t_0) = \frac{c\, t_0}{1-q} \left(1 - t_0^{\,q-1}\, t_e^{\,1-q} \right)$$

In this $t_0^{\,q-1} = t_0^{\,-(1-q)}$

$$\Rightarrow \quad d_p(t_0) = \frac{c\, t_0}{1-q} \left(1 - \frac{t_e^{\,1-q}}{t_0^{\,1-q}} \right)$$

substituting $q = \dfrac{2}{3+3w}$

$$d_p(t_0) = \frac{c\, t_0}{1 - \frac{2}{3+3w}} \left(1 - \left(\frac{t_e}{t_0} \right)^{1 - \frac{2}{3+3w}} \right)$$

$$= \frac{c\, t_0}{3+3w-2} (3+3w) \left(1 - \left(\frac{t_e}{t_0} \right)^{\frac{3+3w-2}{3+3w}} \right)$$

$$= c\, t_0\, \frac{3(1+w)}{1+3w} \left(1 - \left(\frac{t_e}{t_0} \right)^{\frac{1+3w}{3+3w}} \right)$$

$$\Rightarrow \quad \boxed{d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} = c\, t_0\, \frac{3(1+w)}{1+3w} \left[1 - \left(\frac{t_e}{t_0} \right)^{\frac{1+3w}{3+3w}} \right]} \quad \text{— } 5.49$$

From equation 5.41, $H_0 = \dfrac{2}{3(1+w)}\, t_0^{-1} \Rightarrow 3(1+w)\, t_0 = \dfrac{2}{H_0}$

Using the above relation,

$\Rightarrow$ the term $\left(t_0 \dfrac{3(1+w)}{1+3w}\right)$ becomes $\dfrac{c}{H_0} \cdot \dfrac{2}{1+3w}$ — (3)

From 5.47 $\left(\dfrac{t_0}{t_e}\right)^{2/(3+3w)} = 1+z$

$$\Rightarrow \frac{t_0}{t_e} = (1+z)^{\frac{3+3w}{2}}$$

$$\frac{t_e}{t_0} = (1+z)^{-\left(\frac{3+3w}{2}\right)}$$

On substituting this expression of $\dfrac{t_e}{t_0}$ in $\left(\dfrac{t_e}{t_0}\right)^{\frac{1+3w}{3+3w}}$ from 5.49,

$$\left(\frac{t_e}{t_0}\right)^{\frac{1+3w}{3+3w}} = (1+z)^{-\left(\frac{3+3w}{2}\right) \times \frac{1+3w}{3+3w}} = (1+z)^{-\frac{(1+3w)}{2}} \quad \text{— (4)}$$

On using (3) & (4) and putting it in 5.49, we get

$$\Rightarrow \boxed{d_p(t_0) = \frac{c}{H_0} \frac{2}{1+3w} \left[1 - (1+z)^{-\frac{(1+3w)}{2}}\right]} \quad \text{— 5.50}$$

Plotting $d_p(t_0)$ and $t(z)$

So the equations were,

$$d_p(t_0) = \frac{c}{H_0} \frac{2}{1+3w} \left[1 - (1+z)^{-\frac{(1+3w)}{2}}\right]$$

$$t_e = \frac{2}{3(1+w)H_0} \frac{1}{(1+z)^{3(1+w)/2}}$$

For $w = 0$, $\quad d_p(t_0) = \dfrac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right]$

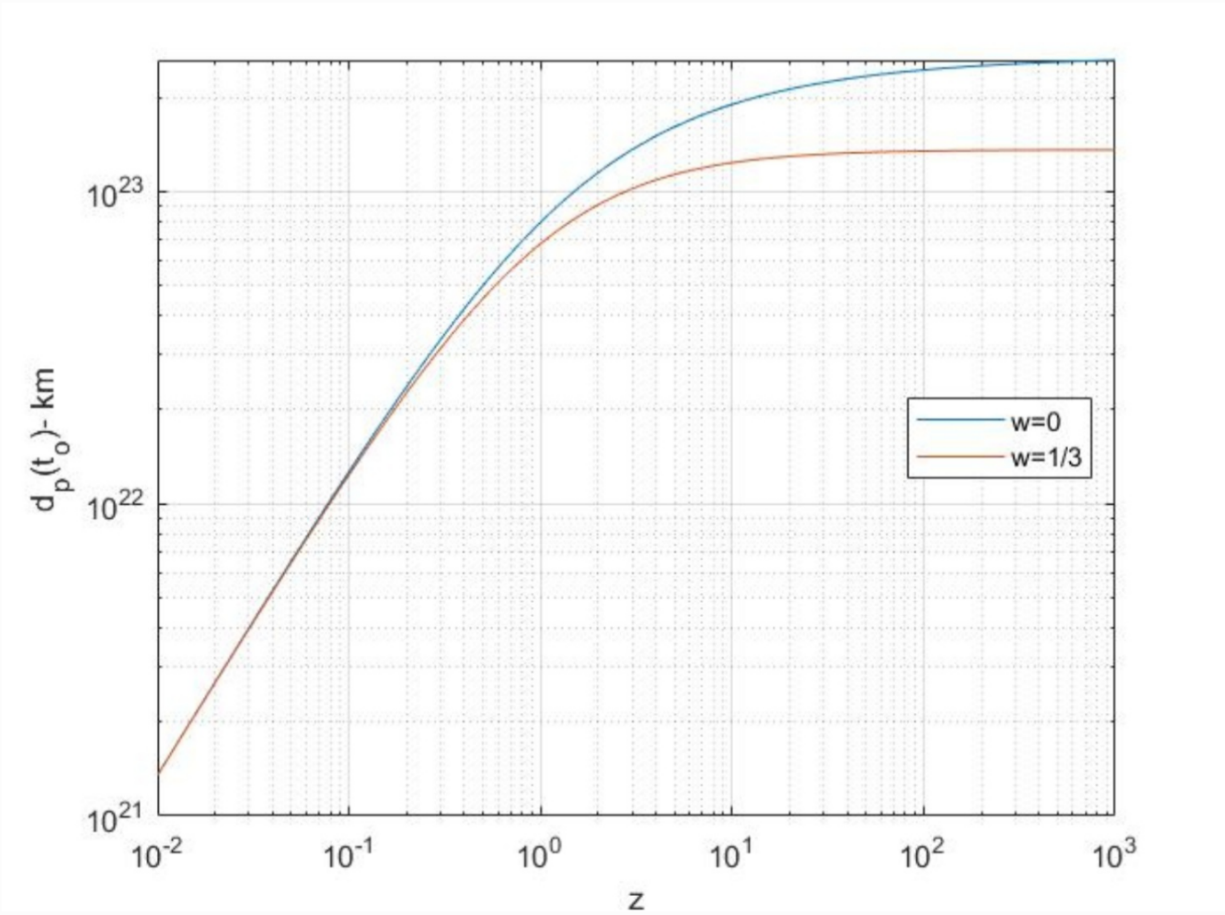
$$t_e = \frac{2}{3H_0}\frac{1}{(1+z)^{3/2}}$$

For $w = 1/3$, $\quad d_p(t_0) = \dfrac{c}{H_0}\left[1 - (1+z)^{-1}\right]$

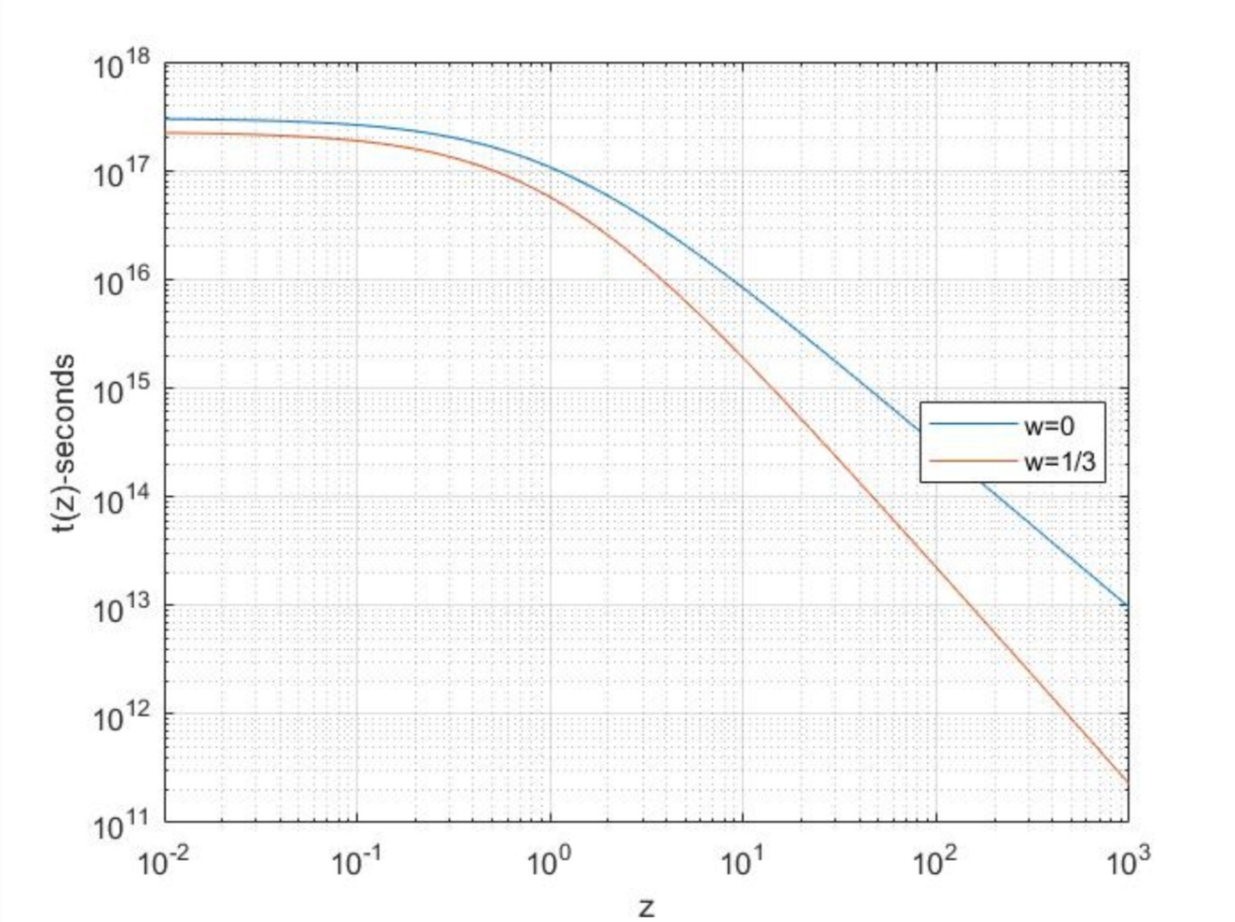
$$t_e = \frac{1}{2H_0}\frac{1}{(1+z)^{2}}$$

To stay consistent with the units $c = 3\times10^5$ km/s and $H_0 = 2.204\times10^{-18}\ s^{-1}$ to get the units for distance in km and time in seconds. Both the plots are plotted in the logarithmic scale.

Plot for $d_p(t_0)$ is shown below.



Plot for t(z) is shown below.



At lower redshifts it is seen that the proper distance for both the cases increases linearly with redshift but for higher redshifts, the proper distance increases more for a matter dominated universe compared to a radiation dominated universe. Since the proper distance is higher for $w=0$, it supports the fact that a matter dominated universe expands faster than a radiation dominated universe. This can also be seen in the t(z) plot where the time at which light from a distant galaxy was emitted is higher for $w=0$ than $w=1/3$.