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HW4

4.1 $F = \cancel{m}a = -\frac{GM_s \cancel{m}}{R_s(t)^2} \rightarrow a = \frac{d^2R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2}$

$$\int \frac{dR_s}{dt} \cdot \frac{d^2R_s}{dt^2}dt = \int -\frac{GM_s}{R_s(t)^2}\frac{dR_s}{dt}\,dt$$

$u = \frac{dR_s}{dt} \quad du = \frac{d^2R_s}{dt^2}dt \qquad v = R_s(t) \quad dv = \frac{dR_s}{dt}dt$

$$\int u\,du = \int -\frac{GM_s}{v^2}dv$$

$$\frac{1}{2}u^2 = \frac{GM_s}{v} + U$$

(4.12) $\boxed{\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U}$

4.3 $\left(\frac{\dot{a}}{a}\right)^2 = H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}$

$H(t_0) = H_0$
$\varepsilon(t_0) = \varepsilon_0$
$a(t_0) = 1$

(4.28a) $\boxed{H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2}}$

$\kappa = 0$ represents the scenario where spacetime is flat

for $\kappa = 0$: $H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_0 \qquad \varepsilon_0 = \rho_0 c^2$

$H_0^2 = \frac{8\pi G}{3}\rho_0 \rightarrow \boxed{\rho_0 = \frac{3H_0^2}{8\pi G}}$ (4.28b)

$$\rho_0 = \frac{3(68\text{ km/s/Mpc})^2}{8\pi(6.67\times10^{-11}\,m^3kg^{-1}s^{-2})} = \frac{3}{8\pi(6.67\times10^{-11}\,m^3kg^{-1}s^{-2})}\left(\frac{68\,km}{s\,Mpc}\cdot\frac{1000\,m}{1\,km}\cdot\frac{1\,Mpc}{3.09\times10^{22}\,m}\right)^2$$

$\boxed{\rho_0 = 8.67\times10^{-27}\ kg/m^3}$

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HW4

4.2

$$(4.12)\quad \frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$$

$$(4.15)\quad M_s = \frac{4\pi}{3}\rho(t)R_s(t)^3$$

$$(4.16)\quad R_s(t) = a(t)r_s \rightarrow \frac{dR_s}{dt} = \frac{da}{dt}r_s = r_s\dot{a}$$

$$\frac{1}{2}(r_s\dot{a})^2 = \frac{G\cdot\frac{4\pi}{3}\rho(t)R_s(t)^3}{R_s(t)} + U$$

$$\frac{1}{2}r_s^2\dot{a}^2 = G\cdot\frac{4\pi}{3}R_s(t)^2\rho(t) + U$$

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}G(a(t)r_s)^2\rho(t) + U$$

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Ga(t)^2r_s^2\rho(t) + U$$

$$(4.17)\quad \frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Gr_s^2\rho(t)a(t)^2 + U$$

$$\cancel{\frac{1}{2}}\cancel{r_s^2}\dot{a}^2\left(\frac{\cancel{2}}{\cancel{r_s^2}a^2}\right) = \frac{4\pi}{3}G\cancel{r_s^2}\rho(t)\cancel{a(t)^2}\left(\frac{2}{\cancel{r_s^2}\cancel{a^2}}\right) + \frac{2U}{r_s^2a^2}$$

$$(4.18)\quad \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\cdot\frac{1}{a(t)^2}$$

$$k=0, U=0:\quad \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho(t)$$

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}}\rho(t)^{1/2}$$

$$\frac{da}{dt} = a\rho(t)^{1/2}\sqrt{\frac{8\pi G}{3}}$$

$$\cancel{\int\frac{1}{a}da = \int\rho(t)^{1/2}\sqrt{\frac{8\pi G}{3}}dt}$$

$$\cancel{\ln|a|}$$

$$\rho(t) = \frac{3M_s}{R_s(t)^3\cdot 4\pi}$$

$$\rho(t) = \frac{3M_s}{a^3r_s^3 4\pi}$$

$$\dot{a} = a\sqrt{\frac{\cancel{3}M_s}{a^3r_s^3\cancel{4\pi}}}\sqrt{\frac{\cancel{8}^{2}\cancel{\pi}G}{\cancel{3}}}$$

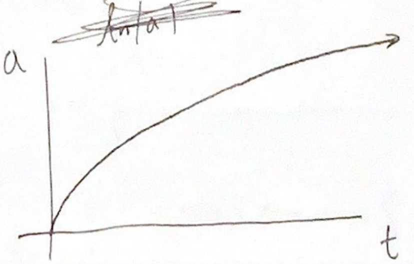
$$da = \frac{1}{\sqrt{a}}\sqrt{\frac{2GM_s}{r_s^3}}dt$$

$$\int a^{1/2}da = \int\sqrt{\frac{2GM_s}{r_s^3}}dt$$

$$\frac{2}{3}a^{3/2}\Big|_{a_0=1}^{a} = t\sqrt{\frac{2GM_s}{r_s^3}}\Big|_{t_0=0}^{t=t_0}$$

$$\frac{2}{3}a^{3/2} - \frac{2}{3} = t\sqrt{\frac{2GM_s}{r_s^3}}$$

$$a = \left(\frac{3}{2}t\sqrt{\frac{2GM_s}{r_s^3}} + 1\right)^{2/3}$$



$\sim t^{2/3}$

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HW4

4.2

$k=0$: $$\frac{da}{dt} = \cancel{\frac{2}{3}}\left(\frac{3}{2}t\sqrt{\frac{2GM_s}{r_s^3}}+1\right)^{-1/3}\left(\cancel{\frac{3}{2}}\sqrt{\frac{2GM_s}{r_s^3}}\right)$$

$$\frac{da}{dt} = \frac{\sqrt{2GM_s/r_s^3}}{\left(\frac{3}{2}t\sqrt{2GM_s/r_s^3}+1\right)^{1/3}}$$

$$\lim_{t\to\infty}\frac{da}{dt} = 0$$ [Since denominator is only thing increasing]

— This universe stops expanding as $t\to\infty$

$k \neq 0$: $$\left(\frac{\dot a}{a}\right)^2 = \frac{\cancel{8\pi}^{\,2} G}{\cancel{3}}\underbrace{\frac{\cancel{3}M_s}{\cancel{4\pi}a^3 r_s^3}}_{\rho(t)} + \frac{2U}{r_s^2a^2} = \frac{2GM_s+2Ur_sa}{a^3r_s^3}$$

$$\left(\frac{da}{dt}\right)^2 = \frac{2GM_s+2Ur_sa}{a^{\cancel{4}}r_s^3} \Longrightarrow 0 = \frac{2GM_s}{a^{\cancel{4}}r_s^3} + \frac{2U}{r_s^2}$$

$$\frac{GM_s}{ar_s^3} = \frac{-U}{r_s^2} \Longrightarrow a = -\frac{GM_s}{Ur_s} = a_{max}$$

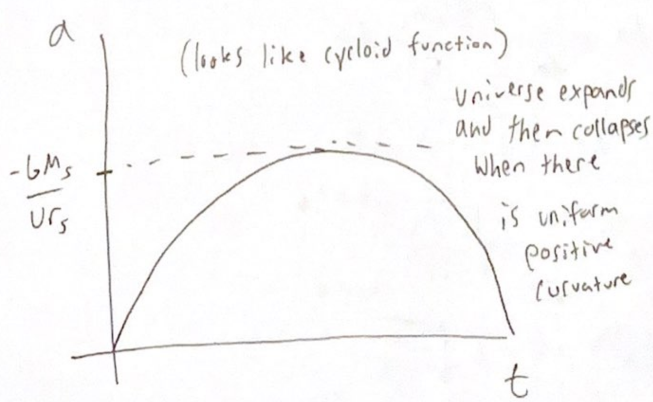
$U=0$: No maximum

$U>0$: Maximum is at $a<0$ (so is irrelevant)

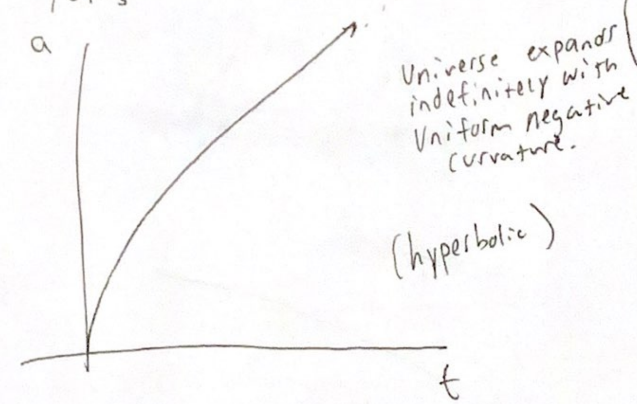
$U<0$: Maximum at $a_{max} = \frac{GM_s}{|U|r_s}$ (expands then contracts)

$$\dot a^2 = \underbrace{\frac{2GM_s}{ar_s^3}}_{\text{Small when } a \text{ is large}} + \underbrace{\frac{2U}{r_s^2}}_{\text{Natural rate}}$$

- When $U>0$, as $a(t)$ increases, the first term gets smaller and smaller while the second is constant.
  → Constant slope for $t\to\infty$
- When $U<0$, the terms cancel at some point.



$k=1 \to U<0$



$k=-1 \to U>0$