HOMEWORK 4.4a + 4.5

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H/W 4.4a Show that (4.35) and (4.36) are true. Discuss the critical implications of the equation that follows from this:
[1 - 1/Omega(t)] = [1 - 1/Omega_0(t)]/(1+z) [link your finding to Inflation].

$$4.35 \rightarrow 1 - \Omega_0 = \frac{-kc^2}{R_0^2 H_0^2}$$

$$4.36 \rightarrow \frac{k}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)$$

To prove the above equations are true.

we know that the Friedmann Equation in the Newtonian form is,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}$$

To change this equation to the relativistic form,

$$\rho(t) = \frac{\varepsilon(t)}{c^2} \rightarrow \text{energy density} \quad \text{and} \quad \frac{2U}{r_s^2} = -\frac{kc^2}{R_0^2} \quad \text{where } U = -\frac{1}{2}kc^2$$

making these adjustments in the equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2}$$

$$H(t) \equiv \frac{\dot{a}}{a}$$

$$\Rightarrow H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2 a(t)^2} \quad \text{——— } (1)$$

Putting $K=0 \Rightarrow H(t)^2 = \frac{8\pi G \, \varepsilon(t)}{3c^2}$

For a given value of hubble parameter, on re-arranging the above equation, we get a critical density

$$\varepsilon_c(t) = \frac{H(t)^2 3c^2}{8\pi G} \quad \text{—— } 2$$

From equation 4.33, $\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)} \Rightarrow \varepsilon(t) = \Omega(t)\varepsilon_c(t)$

Putting 2 in the above expression,

$$\varepsilon(t) = \Omega(t) \cdot \frac{H(t)^2 3c^2}{8\pi G}$$

Putting this value of $\varepsilon(t)$ in 1

$$H(t)^2 = \frac{\cancel{8\pi G}}{\cancel{3c^2}} \times \frac{\Omega(t) H(t)^2 \cancel{3c^2}}{\cancel{8\pi G}} - \frac{Kc^2}{R_0^2} \cdot \frac{1}{a(t)^2}$$

$$H(t)^2 = \Omega(t) H(t)^2 - \frac{Kc^2}{R_0^2} \cdot \frac{1}{a(t)^2}$$

Dividing throughout by $H(t)^2$

$$1 = \Omega(t) - \frac{Kc^2}{R_0^2 H(t)^2} \cdot \frac{1}{a(t)^2}$$

At $t = t_0$

$$a(t) = \frac{1}{1+z} \Rightarrow a(t_0) = \frac{1}{1+0} = 1$$

$$\Omega(t_0) = \Omega_0$$

$$\Rightarrow \quad 1 = \Omega_0 - \frac{kc^2}{R_0^2 H_0^2}$$

$$\Rightarrow \quad \boxed{1 - \Omega_0 = \frac{-kc^2}{R_0^2 H_0^2}} \quad \text{———} \quad 4.35$$

On multiplying both the sides by $\frac{H_0^2}{c^2}$

$$\frac{H_0^2}{c^2}(1-\Omega_0) = \frac{-K}{R_0^2}$$

$$\Rightarrow \quad \boxed{\frac{K}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)} \quad \text{———} \quad 4.36$$

- critical implications of $1 - \frac{1}{\Omega(t)} = \left[1 - \frac{1}{\Omega_0}\right]\frac{1}{(1+z)}$

This equation can be written as $\frac{\Omega(t)-1}{\Omega(t)} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)\frac{1}{(1+z)}$

Here it is seen that the difference between the density parameter and 1, is inversely proportional to the redshift at all times.

This means that at high redshifts (earlier in the universe), the value of the density parameter was much closer to one and at the present time, this difference is larger.

This relation also shows that, if the value of Ω in the past was less than or greater than 1, then as time passes, the value of Ω will continue to be less than or greater than 1 respectively but the difference from 1 will increase as this difference is inversely proportional to redshift.

To relate this to inflation, we know that there was a time when the universe expanded a lot faster in the beginning and then this rate of expansion decreased. This is period of expansion was known as inflation

we know $\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)} = \frac{8\pi G}{3c^2 H(t)^2} \cdot \varepsilon(t)$

In this $\frac{8\pi G}{3c^2}$ is a constant. So $\Omega(t)$ depends on $\frac{\varepsilon(t)}{H(t)^2}$

After inflation we know that the universe was expanding and with this $\varepsilon(t)$ also decreases. $H(t)$ depends on the expansion rate, and we know the rate of expansion decreases with time.

Since both these parameters decrease, the value for $\Omega(t)$ will be near 1. Since this value is near 1, we could predict from inflation that the universe is flat.

H/W 4.5: Re. (4.58) and following eqs., discuss the physical meaning of the following cases of the expanding universe's content, all for w<1:

w~=0

w=1/3

w=-1/3 or smaller

w=-1

Particles can be relativistic and non relativistic.
we know the equation $P_{nonrel} = w\, \mathcal{E}_{nonrel}$

For the case of $w=0$, we get $p=0$ and this. This refers to as the pressureless matter and corresponds to non relativistic matter. This component of the universe is termed as matter. Since the speed of nonrelativistic matter is much smaller than the speed of light, using the equation $w \approx \frac{\langle v^2 \rangle}{3c^2}$, we see that $w \approx 0$ for $v << c$ (non relativistic matter).

$w = 1/3$ indicates that the universe contains relativistic matter. Using this value of w, $P_{rel} = \frac{1}{3} \mathcal{E}_{rel}$. An example for this case are the photons. Their speed is nearly equal to the speed of light and they have momentum so they exert pressure. This component of the universe is termed as radiation.

$$w \simeq \frac{\langle v^2 \rangle}{3c^2} \approx \frac{c^2}{3c^2} \Rightarrow w \simeq \frac{1}{3} \text{ (relativistic matter)}$$

$w = -1/3$ or smaller → This component of the universe refers to as dark energy. When $w < -\frac{1}{3}$, we see that in equation $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\mathcal{E} + 3P)$, P takes a negative value smaller than $\mathcal{E}$ and this makes the equation positive. This provides a positive acceleration (expansion).

$w = -1 \rightarrow$ The component of the universe that has $w = -1$ is the cosmological constant. This is a special form of dark energy. Since $w = -1$,

$P = -\varepsilon$ and $\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}(2\varepsilon)$. Here the universe keeps expanding.

4.4b This part for extra credit only: [Harder; TAs may give you further hints!]: Use (4.26)+(4.33) to derive Omega(t) as a function of (1+z), and then prove that: [1 - 1/Omega(t)] = [1 - 1/Omega_0(t)]/(1+z) . This equation may be assumed, but is needed to give the discussion in 4.4a.

$$4.26 \rightarrow \quad H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{R_0^2 a(t)^2} \quad \text{—— } (1)$$

$$4.33 \rightarrow \quad \Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)} \quad \text{—— } (2)$$

we know that for $k=0$, there exists a critical density,

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2$$

Putting this in (2)

$$\Omega(t) = \frac{\varepsilon(t) \cdot 8\pi G}{3c^2 H(t)^2} \Rightarrow H(t)^2 = \frac{\varepsilon(t)\, 8\pi G}{\Omega(t)\, 3c^2} \quad \text{—— } (3)$$

$$\Rightarrow \frac{8\pi G}{3c^2} = \frac{H(t)^2 \Omega(t)}{\varepsilon(t)} \quad \text{—— } (4)$$

Equating (1) and (3)

$$\frac{\varepsilon(t)\, 8\pi G}{\Omega(t)\, 3c^2} = \frac{8\pi G\, \varepsilon(t)}{3c^2} - \frac{kc^2}{R_0^2 a(t)^2}$$

Dividing throughout by $\frac{8\pi G\, \varepsilon(t)}{3c^2}$

$$\frac{1}{\Omega(t)} = 1 - \frac{kc^2}{R_0^2 a(t)^2} \times \frac{3c^2}{8\pi G\, \varepsilon(t)} \quad \text{—— } (5)$$

we found above that $\Omega(t) = \frac{\varepsilon(t)\, 8\pi G}{3c^2 H(t)^2}$

$$\Rightarrow \frac{3c^2}{8\pi G} = \underbrace{\frac{\varepsilon(t)}{\Omega(t) H(t)^2}} \; ; \quad \text{at } t = t_0 \quad \frac{3c^2}{8\pi G} = \frac{\varepsilon_0}{\Omega_0 H_0^2}$$

Since this value is a constant as seen.

Putting the above obtained expression in 5

$$1 - \frac{kc^2}{R_0^2 a(t)^2} \cdot \frac{\varepsilon_0}{\Omega_0 H_0^2 \varepsilon(t)} = \frac{1}{\Omega(t)}$$

$$1 - \underbrace{\frac{kc^2}{R_0^2 H_0^2}} \cdot \frac{\varepsilon_0}{a(t)^2 \Omega_0 \varepsilon(t)} = \frac{1}{\Omega(t)}$$

According to 4.35, $\frac{kc^2}{R_0^2 H_0^2} = -(1-\Omega_0) = (\Omega_0 - 1)$

$$1 - (\Omega_0 - 1)\frac{\varepsilon_0}{a(t)^2 \Omega_0 \varepsilon(t)} = \frac{1}{\Omega(t)} \Rightarrow 1 - \frac{1}{\Omega(t)} = \frac{(\Omega_0 - 1)\,\varepsilon_0}{a(t)^2 \Omega_0\, \varepsilon(t)} \quad \text{— 6}$$

Only considering the R.H.S of 6, $\frac{(\Omega_0 - 1)}{a(t)^2 \Omega_0} \cdot \frac{\varepsilon_0}{\varepsilon(t)}$

From 4.15 and 4.16, the mass of an expanding sphere is given by the equation,

$$M_s = \frac{4\pi}{3} \rho(t)\, a(t)^3 r_s^3 = \frac{4\pi}{3} \frac{\varepsilon(t)}{c^2} a(t)^3 r_s^3$$

$$\Rightarrow \varepsilon(t) = \frac{3c^2 M_s}{4\pi a(t)^3 r_s^3}$$, At $t = t_0$, $a(t) = 1$

$$\Rightarrow \varepsilon_0 = \frac{3c^2 M_s}{4\pi r_s^3}$$

Putting these expressions for $\varepsilon(t)$ and ε_0 in the R.H.S term,

$$R.H.S = \frac{(\Omega_0 - 1)}{\cancel{a(t)^2}\, \Omega_0} \cdot \frac{\cancel{3c^2 M_s}}{\cancel{4\pi r_s^3}} \cdot \frac{\cancel{4\pi}\, a(t)^{\cancel{3}}\, \cancel{r_s^3}}{\cancel{3c^2 M_s}} = \frac{(\Omega_0 - 1)\, a(t)}{\Omega_0}$$

we know $a(t) = \frac{1}{1+z}$ $\Rightarrow R.H.S = \frac{(\Omega_0 - 1)}{\Omega_0} \frac{1}{(1+z)}$

Putting the R.H.S expression in 6

$$1 - \frac{1}{\Omega(t)} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right) \frac{1}{1+z}$$

$$\Rightarrow \quad \boxed{1 - \frac{1}{\Omega(t)} = \left(1 - \frac{1}{\Omega_0}\right) \cdot \frac{1}{(1+z)}}$$