

HW4 Solution

AST422

1.

$$\varepsilon = \sum_{\omega} \varepsilon_{\omega}$$

$$P = \sum_{\omega} P_{\omega} = \sum_{\omega} \omega \varepsilon_{\omega}$$

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

ω :

$$\dot{\varepsilon}_{\omega} + 3\frac{\dot{a}}{a}(\varepsilon_{\omega} + \omega \varepsilon_{\omega}) = 0 = \dot{\varepsilon}_{\omega} + 3\frac{\dot{a}}{a}(1 + \omega)\varepsilon_{\omega} = 0$$

$$\Rightarrow \frac{d\varepsilon_{\omega}}{\varepsilon_{\omega}} = -3(1 + \omega)\frac{da}{a} \Rightarrow \varepsilon_{\omega} = \varepsilon_{\omega,0}a^{-3(1+\omega)}$$

$a \rightarrow 0$, largest $\omega = 1/3$ radiation dominate

$a \rightarrow \infty$, smallest $\omega = -1$ cosmological constant dominate

$$\omega = 0 \Rightarrow \varepsilon_m(a) = \varepsilon_{m,0}a^{-3}$$

$$\omega = 1/3 \Rightarrow \varepsilon_r(a) = \varepsilon_{r,0}a^{-4}$$

$$\varepsilon = nE$$

$$n = n_0a^{-3} \propto a^{-3}$$

$$E_m = mc^2 \Rightarrow \varepsilon_m(a) = \varepsilon_{m,0}a^{-3}$$

$$E_r = hc/\lambda = hc/(\lambda_0 a) \propto a^{-1} \Rightarrow \varepsilon_r(a) = \varepsilon_{r,0}a^{-4}$$

$$E_r = \sigma T^4 \Rightarrow T = T_0/a$$

2.

$$\varepsilon_\Lambda = \varepsilon_{\Lambda,0}$$

$$\frac{\varepsilon_\Lambda}{\varepsilon_m} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}a^{-3}} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}} a^3 = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} a^3 = \frac{0.7}{0.3} a^3 = 2.3(1+z)^{-3} = 1$$

$$\Rightarrow z_{m\Lambda} = 1/3$$

the simulation shows the galaxy merger rate goes with $(1+z)^{0.6-0.8}$ at $0 < z < 2$, the SFR peak at $z \sim 1-2$, and the QSO/AGN peak at $z \sim 2$, little effect of the Λ at high redshift. At recent time, the Λ has an effect of decreasing the merger rate which will also decrease the SFR and the AGN activities.

3.

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_\omega \varepsilon_{\omega,0} a^{-(1+3\omega)} - \frac{\kappa c^2}{R_0^2}$$

curvature, only

$$\dot{a}^2 = -\frac{\kappa c^2}{R_0^2}$$

$\kappa = 0 \Rightarrow \dot{a} = 0$, empty, static, spatially, flat

$\kappa = 1 \Rightarrow \dot{a}^2 < 0 \Rightarrow \dot{a}$, imaginary, forbidden

$$\kappa = -1 \Rightarrow \dot{a} = \pm \frac{c}{R_0} \Rightarrow a(t) = \pm \frac{c}{R_0} t = \frac{t}{t_0}$$

4.

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_{\omega} \varepsilon_{\omega,0} a^{-(1+3\omega)} - \frac{\kappa c^2}{R_0^2}$$

$$\kappa = 0 \Rightarrow \dot{a}^2 = \frac{8\pi G}{3c^2} \sum_{\omega} \varepsilon_{\omega,0} a^{-(1+3\omega)}$$

$\omega :$

$$\dot{a}^2 = \frac{8\pi G \varepsilon_0}{3c^2} a^{-(1+3\omega)}$$

$$a = At^q$$

$$\Rightarrow A^2 q^2 t^{2q-2} = \frac{8\pi G \varepsilon_0}{3c^2} A^{-(1+3\omega)} t^{-(1+3\omega)q} \Rightarrow \begin{cases} q = \frac{2}{3(1+\omega)} \\ A = \left(\frac{8\pi G \varepsilon_0}{3c^2}\right)^{\frac{1}{3(1+\omega)}} \left(\frac{2}{3(1+\omega)}\right)^{-\frac{2}{3(1+\omega)}} \end{cases}$$

$$\Rightarrow a(t) = \left(\frac{2}{3(1+\omega)}\right)^{-\frac{2}{3(1+\omega)}} \left(\frac{8\pi G \varepsilon_0}{3c^2}\right)^{\frac{1}{3(1+\omega)}} t^{\frac{2}{3(1+\omega)}} = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}$$

$$\Rightarrow t_0 = \frac{1}{1+\omega} \left(\frac{c^2}{6\pi G \varepsilon_0}\right)^{1/2}$$

$$\Rightarrow H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3(1+\omega)} t_0^{-1}$$

$$\Rightarrow t_0 = \frac{2}{3(1+\omega)} H_0^{-1}$$

$\omega > -1/3$, $t_0 < H_0^{-1}$, younger than the Hubble time
 $\omega < -1/3$, $t_0 > H_0^{-1}$, older than the Hubble time

5.

$$1+z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/3(1+\omega)}$$

$$t_e = \frac{t_0}{(1+z)^{3(1+\omega)/2}} = \frac{1}{(1+z)^{3(1+\omega)/2}} \frac{2}{3(1+\omega)H_0}$$

$$d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} = c \int_{t_e}^{t_0} \left(\frac{t_0}{t}\right)^{\frac{2}{3(1+\omega)}} dt = ct_0 \frac{3(1+\omega)}{1+3\omega} \left[1 - \left(\frac{t_e}{t_0}\right)^{\frac{1+3\omega}{3(1+\omega)}}\right] = \frac{c}{H_0} \frac{2}{1+3\omega} \left[1 - (1+z)^{-\frac{1+3\omega}{2}}\right]$$

$$(when, w \neq -\frac{1}{3})$$

6.

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}$$

$$t_0 = \frac{2}{3(1+\omega)} H_0^{-1}$$

$$d_p(t_0) = \frac{c}{H_0} \frac{2}{1+3\omega} [1 - (1+z)^{-\frac{1+3\omega}{2}}]$$

a.

matter, dominate

$$\omega = 0$$

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$

$$t_0 = \frac{2}{3(1+\omega)} H_0^{-1} = \frac{2}{3} H_0^{-1}$$

$$d_p(t_0) = \frac{c}{H_0} \frac{2}{1+3\omega} [1 - (1+z)^{-\frac{1+3\omega}{2}}] = \frac{2c}{H_0} [1 - (1+z)^{-\frac{1}{2}}]$$

b.

radiation, dominate

$$\omega = \frac{1}{3}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}} = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}$$

$$t_0 = \frac{2}{3(1+\omega)} H_0^{-1} = \frac{1}{2} H_0^{-1}$$

$$d_p(t_0) = \frac{c}{H_0} \frac{2}{1+3\omega} [1 - (1+z)^{-\frac{1+3\omega}{2}}] = \frac{c}{H_0} [1 - (1+z)^{-1}] = \frac{c}{H_0} \frac{z}{1+z}$$

c.

Λ , dominate

$$\omega = -1$$

$$\dot{a}^2 = \frac{8\pi G \epsilon_0}{3c^2} a^{-(1+3\omega)} = \frac{8\pi G \epsilon_0}{3c^2} a^2 \Rightarrow \frac{da}{a} = \left(\frac{8\pi G \epsilon_0}{3c^2}\right)^{1/2} dt = H_0 dt$$

$$\Rightarrow a(t) = e^{H_0(t-t_0)}$$

$$a \rightarrow 0, t_0 \rightarrow \infty$$

$$d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} = c \int_{t_e}^{t_0} e^{H_0(t_0-t)} dt = \frac{c}{H_0} [e^{H_0(t_0-t_e)} - 1] = \frac{c}{H_0} z$$

7.

$$\begin{aligned}d_p(t_e) &= \frac{2c}{H_0(1+z)} \left[1 - \frac{1}{\sqrt{1+z}} \right] \\ \frac{d}{dz} d_p(t_e) &= -\frac{2c}{H_0(1+z)^2} \left[1 - \frac{1}{\sqrt{1+z}} \right] + \frac{2c}{H_0(1+z)} \frac{1}{2} \frac{1}{(1+z)^{3/2}} = 0 \\ \Rightarrow \frac{1}{\sqrt{1+z}} - 1 + \frac{1}{2} \frac{1}{\sqrt{1+z}} &= \frac{3}{2} \frac{1}{\sqrt{1+z}} - 1 = 0 \Rightarrow \sqrt{1+z} = \frac{3}{2} \Rightarrow z = \frac{5}{4} = 1.25 \\ \frac{d^2}{dz^2} d_p(t_e) &= -\frac{3}{4} \frac{1}{(1+z)^{3/2}} < 0 \Rightarrow \text{max} \\ \Rightarrow \theta &= \frac{D}{d_p(t_e)}, \text{min, at } z = 1.25\end{aligned}$$