HOMEWORK 4.1 - 4.3

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H/W 4.1: Show from the previous eqs. that eq. (4.12) is true.

$$4.12 \rightarrow \frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$$

Consider a sphere that has total mass M_s. (matter is spread homogeneously)

Radius → $R_s(t)$: This is a function of time since the sphere is contracting or expanding isotropically.

Considering an object (test mass) with mass 'm' on the surface of the sphere and by applying Newtons law of gravity,

$$F = -\frac{GM_s m}{R_s(t)^2}$$

Equating this to Newton's second law,

$$F = \cancel{m}a = -\frac{GM_s \cancel{m}}{R_s(t)^2} \Rightarrow a = -\frac{GM_s}{R_s(t)^2}$$

But $a = \ddot{R}_s(t) = \frac{d^2R_s(t)}{dt}$

$$\Rightarrow \frac{d^2R_s(t)}{dt^2} = -\frac{GM_s}{R_s(t)^2}$$

Multiplying each side of the equation by $\frac{dR_s}{dt}$,

$$\frac{d^2R_s(t)}{dt^2} \times \frac{dR_s(t)}{dt} = -\frac{GM_s}{R_s(t)^2} \cdot \frac{dR_s(t)}{dt}$$

$$\frac{dR_s(t)}{dt} = V \quad \text{and} \quad \frac{dv}{dt} = \frac{d^2R_s(t)}{dt^2}$$

$$v\frac{dv}{dt} = -\frac{GM_s}{R_s(t)^2} \frac{dR_s(t)}{dt}$$

Multiplying both the sides by dt, we get

$$v\,dv = -\frac{GM_s}{R_s(t)^2} dR_s(t)$$

$$\int v\,dV = -GM_s \int \frac{dR_s(t)}{R_s(t)^2}$$

$$\frac{V^2}{2} = \frac{GM_s}{R_s(t)} + U$$

Replacing the term 'v' by $\frac{dR_s(t)}{dt}$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{dR_s(t)}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$$

H/W 4.3: Show that (4.28a) and (4.28b) are true. What is the special meaning of the case k=kappa=0? What is the actual value of the critical density of the universe rho_0 that you derive in that case? Assuming the constants given in the text, calculate rho_o from (4.28b). [Do 4.3 before 4.2].

4.28 (a) $\longrightarrow$ $H_0^2 = \frac{8\pi G}{3c^2} \varepsilon_0 - \frac{kc^2}{R_0^2}$

4.28(b) $\longrightarrow$ $\rho_0 = \frac{3H_0^2}{8\pi G}$

The equation $\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s(t)} + U$ was derived in the previous question. — ①

$R_s(t) = a(t)\, r_s$

Density $= \frac{\text{mass}}{\text{volume}}$ $\Rightarrow$ Mass, $M_s = \rho(t) \times \frac{4}{3}\pi R_s(t)^3$

$= \rho(t) \times \frac{4}{3}\pi\, a(t)^3 r_s^3$

Putting the expression for $R_s(t)$ & $\rho(t)$ in ①

$$\frac{1}{2}\left(\frac{d\,a(t)\, r_s}{dt}\right)^2 = \frac{G\rho(t) \times \frac{4}{3}\pi\, a(t)^3 r_s^3}{a(t)\, r_s} + U$$

r_s does not depend on time. $\dot{a} = \frac{da(t)}{dt}$

$$\Rightarrow \quad \frac{1}{2} r_s^2 \dot{a}^2 = \frac{4\pi}{3} G\rho(t)\, r_s^2\, a(t)^2 + U$$

Dividing throughout by $r_s^2 a^2/2$

$$\Rightarrow \quad \frac{2 r_s^2 \dot{a}^2}{2 r_s^2 a^2} = \frac{4\pi}{3} \frac{G\rho(t)\, r_s^2\, a(t)^2 \times 2}{r_s^2\, a(t)^2} + \frac{2U}{r_s^2\, a(t)^2}$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho(t) + \frac{2U}{r_s^2} \frac{1}{a(t)^2}$$

To change this equation to the relativistic form,

$\rho(t) = \frac{\varepsilon(t)}{c^2} \rightarrow$ energy density and $\frac{2U}{r_s^2} = -\frac{Kc^2}{R_0^2}$ where $U = -\frac{1}{2}Kc^2$

making these adjustments in the equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{Kc^2}{R_0^2} \frac{1}{a(t)^2}$$

$$H(t) \equiv \frac{\dot{a}}{a}$$

$$\Rightarrow H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{Kc^2}{R_0^2 a(t)^2}$$

For the present moment

$t = t_0$ and $z = 0$

$$a(t_0) = \frac{1}{1+z} = \frac{1}{1+0} \Rightarrow a(t_0) = 1$$

$$H(t_0) = H_0 \quad , \varepsilon(t_0) = \varepsilon_0$$

$$\Rightarrow \boxed{H_0^2 = \frac{8\pi G}{3c^2} \varepsilon_0 - \frac{Kc^2}{R_0^2}} \qquad \text{— 4.28 (a)}$$

* In special case of $K = 0$, this means that the universe is spatially flat. There is a critical density for a particular value of the hubble parameter.

Putting $K = 0$ in 4.28 (a) $\Rightarrow H_0^2 = \frac{8\pi G}{3c^2} \varepsilon_0$

we know that $\frac{\varepsilon(t)}{c} = \rho(t)$

$\therefore$ for the present time $\frac{\varepsilon_0}{c} = \rho_0$

$$H_0^2 = \frac{8\pi G}{3}\rho_0 \Rightarrow \boxed{\rho_0 = \frac{3H_0^2}{8\pi G}} \quad \text{—— } 4.28(b)$$

To calculate the value of the critical density ρ_0,

$$H_0 = 68\ \text{km/s/Mpc} = 68\ \frac{\text{km}}{\text{sMpc}} = \frac{68(1000)}{3.086\times10^{22}}\ s^{-1}$$

$$H_0 = 2.204\times10^{-18}\ s^{-1}$$

$$G = 6.67\times10^{-11}\ \text{N m}^2/\text{kg}^2$$

$$\rho_0 = \frac{3\,(2.204\times10^{-18})^2}{8\pi\,(6.67\times10^{-11})}$$

$\Rightarrow$ critical density, $\rho_0 = 8.689\times10^{-27}\ \text{kg m}^{-3}$

H/W 4.2: Show from the eqs. before (4.17) that eq. (4.17)+(4.18) are true. Then solve (4.18) for a(t) and sketch a(t) for 3 different values of U, representing a k=+1, k=0, and k=-1 universe, respectively. You may assume reasonable values of U, r_s. [Hint: Solving the case of k=0 is easy, since we already discussed the answer in class, and you just have to show that this solution works. The cases of k=+/-1 are harder to solve (will get to the exact solution in Ch 5), so it will suffice if you just discuss and sketch what the the solutions for k=+/-1 look like].

$$4.17 \rightarrow \frac{1}{2} r_s^2 \dot{a}^2 = \frac{4\pi}{3} G r_s^2 \rho(t) a(t)^2 + U$$

$$4.18 \rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho(t) + \frac{2U}{r_s^2} \frac{1}{a(t)^2}$$

By Newtons law of Gravitation,

$$F = -\frac{G M_s m}{R_s(t)^2}$$

$M_s \rightarrow$ mass of sphere

$m \rightarrow$ mass of test charge

Equating this to Newton's second law,

$$F = \cancel{m} a = -\frac{G M_s \cancel{m}}{R_s(t)^2} \Rightarrow a = -\frac{G M_s}{R_s(t)^2}$$

But $a = \ddot{R}_s(t) = \dfrac{d^2 R_s(t)}{dt}$

$$\Rightarrow \frac{d^2 R_s(t)}{dt^2} = -\frac{G M_s}{R_s(t)^2} \Rightarrow \frac{d^2 R_s(t)}{dt^2} \times \frac{dR_s(t)}{dt} = -\frac{G M_s}{R_s(t)^2} \cdot \frac{dR_s(t)}{dt}$$

$$\frac{dR_s(t)}{dt} = V \text{ and } \frac{dV}{dt} = \frac{d^2 R_s(t)}{dt^2} \Rightarrow V \frac{dV}{dt} = -\frac{G M_s}{R_s(t)^2} \frac{dR_s(t)}{dt}$$

$$\int V dV = -G M_s \int \frac{dR_s(t)}{R_s(t)^2} \Rightarrow \frac{V^2}{2} = \frac{G M_s}{R_s(t)} + U$$

$$\Rightarrow \frac{1}{2}\left(\frac{dR_s(t)}{dt}\right)^2 = \frac{G M_s}{R_s(t)} + U$$

$R_s(t) = a(t)\, r_s$

Density $= \dfrac{\text{mass}}{\text{volume}} \Rightarrow$ Mass, $M_s = \rho(t) \times \frac{4}{3}\pi R_s(t)^3$

$$= \rho(t) \times \frac{4}{3}\pi\, a(t)^3 r_s^3$$

Putting the expression for $R_s(t)$ & $\rho(t)$ in the relation derived

$$\frac{1}{2}\left(\frac{d\,a(t)\, r_s}{dt}\right)^2 = \frac{G\,\rho(t) \times \frac{4}{3}\pi\, a(t)^3 r_s^3}{a(t)\, r_s} + U$$

Here r_s is independent of time $\Rightarrow \dfrac{d\,a(t)\, r_s}{dt} = r_s \dfrac{d\,a(t)}{dt}$

$$\Rightarrow \boxed{\frac{1}{2} r_s^2 \dot{a}^2 = \frac{4\pi}{3} G\,\rho(t)\, r_s^2\, a(t)^2 + U} \quad - 4.17$$

Dividing throughout by $r_s^2 a^2/2$

$$\Rightarrow \frac{2 r_s^2 \dot{a}^2}{2 r_s^2 a^2} = \frac{4\pi}{3} G\,\rho(t)\, r_s^2\, a(t)^2 \times \frac{2}{r_s^2 a(t)^2} + \frac{2U}{r_s^2 a(t)^2}$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}} \quad - 4.18$$

To solve for $a(t)$ when $k=0$,

According to equation $4.18 \rightarrow \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi G}{3}\rho(t) + \dfrac{2U}{r_s^2}\dfrac{1}{a(t)^2}$

we know, $\dfrac{2U}{r_s^2} = -\dfrac{kc^2}{R_0^2}$ where $U = -\frac{1}{2}kc^2$

Using this, we get equation $4.20 \rightarrow \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi G}{3}\rho(t) - \dfrac{kc^2}{R_0^2}\dfrac{1}{a(t)^2}$

For $k=0$, $\rho(t)=\rho_0 \rightarrow$ critical density

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0$$

we know $(\dot{a})^2 = \left(\frac{da}{dt}\right)^2$

$$\Rightarrow \left(\frac{da}{dt}\right)^2 \cdot \frac{1}{a(t)^2} = \frac{8\pi G}{3}\rho_0$$

Taking the square root on both the sides

$$\frac{da}{a(t)} \cdot \frac{1}{dt} = \sqrt{\frac{8\pi G\rho_0}{3}}$$

$$\int \frac{da}{a(t)} = \int \sqrt{\frac{8\pi G\rho_0}{3}}\, dt$$

$$\ln(a(t)) = \sqrt{\frac{8\pi G\rho_0}{3}}\, t + C \Rightarrow a(t) = e^{\sqrt{\frac{8\pi G\rho_0}{3}}\, t + C}$$

$$\Rightarrow a(t) = e^{\sqrt{\frac{8\pi G\rho_0}{3}}\, t} \cdot e^{C}$$, $e^C \rightarrow C_1$ (another constant)

$$a(t) = C_1 e^{\sqrt{\frac{8\pi G\rho_0}{3}}\, t} \qquad \text{—— (1)}$$

Now to find the value of C_1.

At $t=t_0$

$$a(t_0) = \frac{1}{1+z(t_0)} = \frac{1}{1+0} \Rightarrow a(t_0) = 1$$

$$1 = C_1 e^{\sqrt{\frac{8\pi G\rho_0}{3}}\, t_0} \Rightarrow C_1 = \frac{1}{e^{\sqrt{\frac{8\pi G\rho_0}{3}}\, t_0}}$$

Putting this value of c_1 in (1)

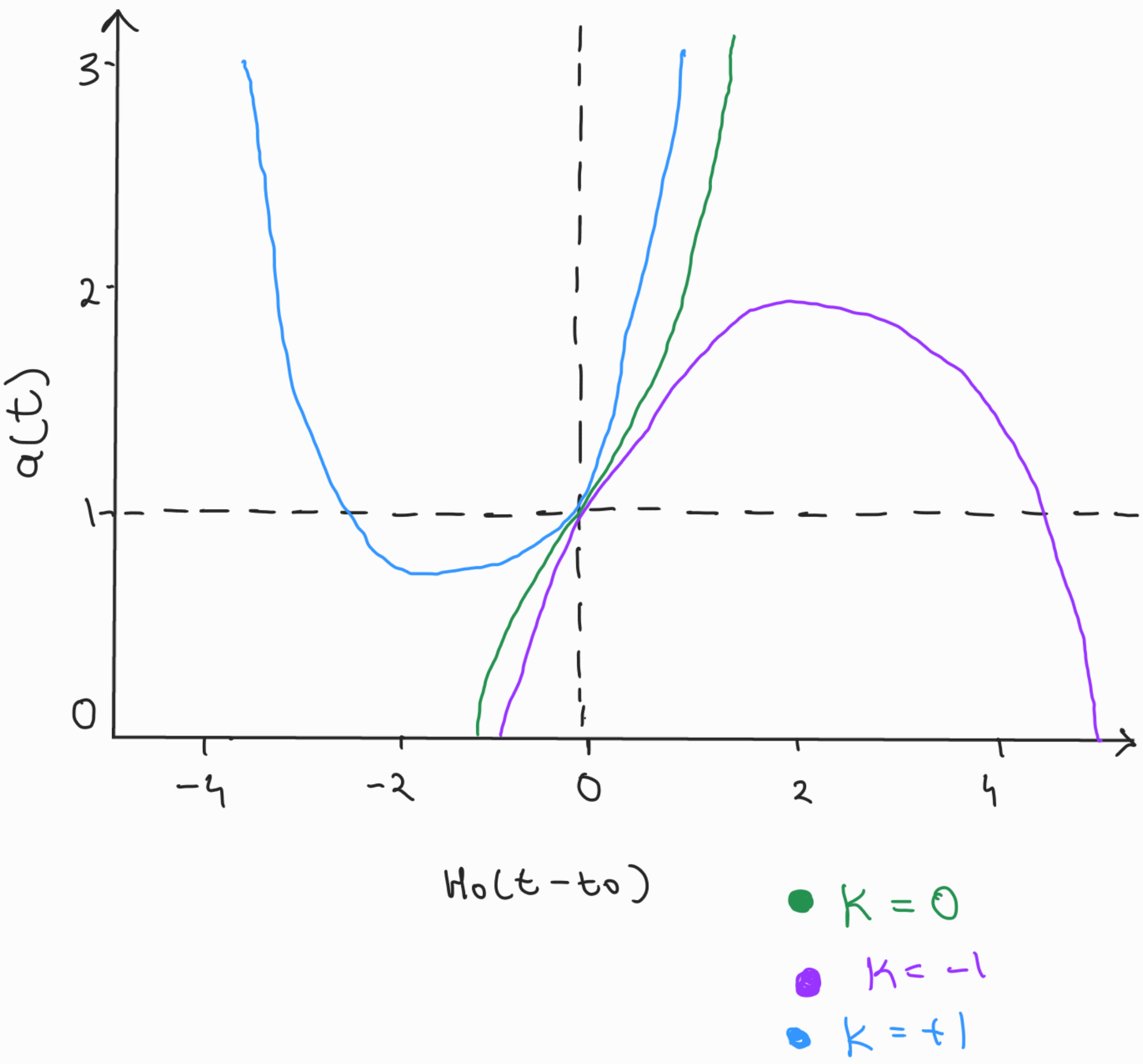
$$a(t) = \frac{1}{e^{\sqrt{\frac{8\pi G \rho_0}{3}}\, t_0}} \times e^{\sqrt{\frac{8\pi G \rho_0}{3}}\, t}$$

on substituting the values for the constants

$$\Rightarrow \quad a(t) = e^{\sqrt{\frac{8\pi G \rho_0}{3}}(t - t_0)} = a(t) = e^{2.205 \times 10^{-18}(t - t_0)}$$

solution for $a(t)$ when $K = 0$

Plotting the solution for $a(t)$, $K = +1, 0, -1$



For $k=0$, the Solution was obtained and using that the solution for it was drawn. As mentioned in the text book, the plot for $k=-1$ would be analogous to a ball being thrown up and it will reach a maximum height where velocity is 0 and it will curve down. Using that, the plot for $k=-1$ was drawn and the opposite to this would be $k=+1$.
