Jake Summers - 1221695592 - HW5 - AST322 - 3/3/2022

4.4a) Start with the Friedmann Equation with the Hubble Parameter:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2} \tag{1}$$

We know that $\varepsilon_c(t) \equiv \frac{3c^2}{8\pi G}H(t)^2$ and that $\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)}$, so it follows that $\Omega(t) = \frac{8\pi G}{3c^2 H(t)^2}\varepsilon(t)$. Dividing the Friedmann equation by $H(t)^2$ yields

$$1 = \frac{8\pi G}{3c^2 H(t)^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2} \tag{2}$$

Which can be rewritten as

$$1 = \Omega(t) - \frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2} \tag{3}$$

So we have

$$1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2} \tag{4}$$

For the present day, we have $\Omega(t) = \Omega_0$, $a(t) = 1$, and $H(t) = H_0$, turning the above equation into

$$1 - \Omega_0 = -\frac{\kappa c^2}{R_0^2 H_0^2} \tag{5}$$

Rearranging this equation yields

$$\Omega_0 - 1 = \frac{\kappa c^2}{R_0^2 H_0^2} \Rightarrow \frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1) \tag{6}$$

4.4b) From this, I will form an equation in terms of z, $\Omega(t)$, and Ω_0. Plugging (6) into (3) yields

$$1 - \Omega(t) = -\frac{H_0^2}{c^2}(\Omega_0 - 1)\frac{c^2}{a(t)^2 H(t)^2} \tag{7}$$

Canceling the c^2s and dividing by $-\Omega(t)$ gives us

$$1 - \frac{1}{\Omega(t)} = \frac{1}{\Omega(t)}\frac{H_0^2(\Omega_0 - 1)}{a(t)^2 H(t)^2} \tag{8}$$

Factor Ω_0 out of $(\Omega_0 - 1)$ and rearrange:

$$1 - \frac{1}{\Omega(t)} = \frac{H_0^2}{a(t)^2 H(t)^2}\frac{\Omega_0}{\Omega(t)}\left(1 - \frac{1}{\Omega_0}\right) \tag{9}$$

Next, we can isolate the part of (9) that should be written in terms of z.

$$1 - \frac{1}{\Omega(t)} = \left(1 - \frac{1}{\Omega_0}\right)\frac{H_0^2 \Omega_0}{a(t)^2 H(t)^2 \Omega(t)} \tag{10}$$

We can then substitute $\Omega_0 = \frac{\varepsilon_0}{\varepsilon_{c,0}}$ and $\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}$:

$$1 - \frac{1}{\Omega(t)} = \left(1 - \frac{1}{\Omega_0}\right)\frac{H_0^2 \varepsilon_0 \varepsilon_c(t)}{\varepsilon_{c,0} a(t)^2 H(t)^2 \varepsilon(t)} \tag{11}$$

Next we can use the definition of critical energy density:

$$1 - \frac{1}{\Omega(t)} = \left(1 - \frac{1}{\Omega_0}\right) \frac{H_0^2 \varepsilon_0 3c^2 H(t)^2 8\pi G}{8\pi G 3c^2 H_0^2 H(t)^2 \varepsilon(t) a(t)^2} = \left(1 - \frac{1}{\Omega_0}\right) \frac{\varepsilon_0}{\varepsilon(t) a(t)^2} \tag{12}$$

Next, $\varepsilon = \frac{E}{V}$.

$$1 - \frac{1}{\Omega(t)} = \left(1 - \frac{1}{\Omega_0}\right) \frac{EV(t)}{a(t)^2 EV_0} = \left(1 - \frac{1}{\Omega_0}\right) \frac{\frac{4}{3}\pi R(t)^3}{\frac{4}{3}\pi R_0^3 a(t)^2} = \left(1 - \frac{1}{\Omega_0}\right) \frac{R_0^3 a(t)^3}{R_0^3 a(t)^2} \tag{13}$$

Finally, using $a = \frac{1}{1+z}$, we can write

$$1 - \frac{1}{\Omega(t)} = \left(1 - \frac{1}{\Omega_0}\right) a(t) \Rightarrow 1 - \frac{1}{\Omega(t)} = (1+z)^{-1} \left(1 - \frac{1}{\Omega_0}\right) \tag{14}$$

4.4a) The significance of (14) is that if $1 - \frac{1}{\Omega_0}$ is ± 0.005, as it is in our universe, then $1 - \frac{1}{\Omega(t)}$ would approximately be that number multiplied by $(1+z)^{-1}$. Around the Planck epoch, at $z \approx 10^{31}$, the uncertainty in the density parameter before inflation would be $10^{-31}(\pm 0.005) \approx 10^{-33}$. This means that to measure $\Omega_0 \approx 1.001$, the density parameter before inflation would have to be $\Omega(t_p) \approx 1.000000000000000000000000000000001$, which is absurdly close to 1. Essentially, any minor variation in $\Omega(t)$ from 1 before inflation would become much larger in the current era. These equations also imply that if $\Omega_0 \equiv 1$ then $\Omega(t) \equiv 1$, if $\Omega_0 > 1$ then $\Omega(t) > 1$, and if $\Omega_0 < 1$ then $\Omega(t) < 1$.

4.5) We are given the following equation to start:

$$3kT = \mu \langle v^2 \rangle \tag{15}$$

$$P_{nonrel} = w \varepsilon_{nonrel} \tag{16}$$

$$w \approx \frac{\langle v^2 \rangle}{3c^2} \ll 1 \tag{17}$$

$$P_{rel} = \frac{1}{3} \varepsilon_{rel} \tag{18}$$

$w \approx 0$**:** Such an equation of state requires that $\langle v^2 \rangle \approx 0$, and so represents non-relativistic matter. In most cases, this would be baryonic matter and dark matter.

$w = \frac{1}{3}$**:** Using (17), we must have $\langle v^2 \rangle = c^2$ to have $w = \frac{1}{3}$. This means that the corresponding equation of state represents relativistic matter such as photons. Neutrinos can be described with approximately the same equation since they travel relativisticly as well. Essentially, this w represents radiation.

$w \leq -\frac{1}{3}$**:** From the acceleration equation, the term $(\varepsilon + 3P)$ becomes negative when $P < -\frac{1}{3}\varepsilon$. This describes an equation of state with $w < -\frac{1}{3}$. Hence, such a component in the universe would bring about a positive acceleration, rather than a negative acceleration associated with our normal matter and energy. By causing a positive acceleration, such components are called 'dark energy.'

The case of $w = -\frac{1}{3}$ represents a case where acceleration is zero. In this case, $\dot{a}$ can be nonzero, such as the component $-\frac{\kappa c^2}{R_0^2}$ in the Friedmann equation multiplied by a^2. Hence, this equation of state can correspond to curvature.

$w = -1$: A component with $w = -1$ would have $P = -\varepsilon$ and yield an acceleration equation of

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(-2\varepsilon) = \frac{8\pi G}{3c^2}\varepsilon \tag{19}$$

The corresponding fluid equation would be:

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) \Rightarrow \dot{\varepsilon} = 0 \tag{20}$$

Since (20) proves that the energy density of a component with $w = -1$ is constant, the component of the Friedmann equation would be constant, meaning that this component with $w = -1$ corresponds to the Cosmological Constant, Λ. Ryden later reveals that $\varepsilon_\Lambda \equiv \frac{c^2}{8\pi G}\Lambda$, meaning that the right side of (19) is equal to $\frac{\Lambda}{3}$; the same expression appears in the Friedmann equation.