AST 322
HW 6
Due date: March 28, 2019

These are some instructions for homework 6 which might help you solve the given problems. This homework is quite a bit more involved than the previous homeworks so I suggest you get started as soon as you can.
Homework problems in this assignment: 6.7a, 6.8b, and 6.9. Total points: 45

Some generic suggestions: (i) Please read this document in its entirety. (ii) Do not skip steps while doing the math. I can't give you credit if I cannot see/follow what you're doing. (iii) Do not skip any problems. If you get stuck then come see us in GWC 591. (iv) While verifying a numerical value, please make sure to explicitly show what numbers you plugged in. Be careful with the units and conversions!

6.7a problem statement (15 points):
(a) For $\Omega_0 > 1$, show that (5.90) and (5.91) are a solution of (5.89).
(b) Then plot $a(t)$ for this universe.
(c) Show that it results in a big crunch after a time given in (5.92).
(d) At what time is the maximum expansion reached?

Suggestions for problem 6.7a:
(a) Begin with (5.89) which is:

$$H_0\, t = \int_0^a \frac{da}{\left[\Omega_0/a + (1-\Omega_0)\right]^{1/2}}, \tag{1}$$

and convert this equation to a differential equation. Do not attempt to do this integral! Your differential equation will be of the form:

$$\left(\frac{da}{dt}\right)^2 = ... \tag{2}$$

Now work with the equations (5.90) and (5.91), which are:

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos\theta), \tag{3}$$

and

$$t(\theta) = \frac{1}{2\,H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(\theta - \sin\theta), \tag{4}$$

and the following trigonometric identities:

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2), \tag{5}$$

and

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 1 - 2\sin^2(\theta/2) = 2\cos^2(\theta/2) - 1. \tag{6}$$

These equations will allow you to solve for da/dt. Keep the following rule for derivatives in mind:

$$\left(\frac{da}{dt}\right) = \left(\frac{da}{d\theta}\right)\left(\frac{dt}{d\theta}\right)^{-1}. \tag{7}$$

Now you can solve for da/dt and show that it leads to the same differential equation you got from (5.89).

(b) For this plot, I suggest you use a programming language (like Python for example). Create an array for θ which goes from 0 to 2π. Use this array and equations (5.90) and (5.91) which will give you the y and x values for the plot, i.e, a and t , respectively.
Do not simply sketch this. I would like to see a computer generated plot. Please attach your code and the plot to your homework.

(c) This can be done in a couple ways. You can either simply read off the value of t at the end of your plot, i.e., corresponding to $\theta = 2\pi$ or you can plug in $\theta = 2\pi$ into (5.91) and solve for $t_{\rm crunch}$.

(d) This should be easy once you do part (c) (because the curve of $a(t)$ is symmetric about $t(a_{\rm max})$).

6.8b problem statement (15 points):
(a) For $\Omega_{\Lambda,0} > 1$, show that (5.101)–(5.103) are a solution of (5.96) in the presence of Lambda.
(b) Then plot $a(t)$ for this universe, and verify that it expands forever.
(c) Verify that the resulting (5.105) and (5.106) are true for this universe (this is the Universe we live in!).

Suggestions for problem 6.8b:
(a) This is an integral that can actually be done analytically. Although, since it is a difficult integral to do, I will give you extra credit (5 points) if you can show me the analytical solution. The other, slightly easier, way to do this problem is to differentiate (5.101) and show that it gives you the differential equation in (5.96). You will also have to take (5.100) into account.

To verify (5.102) and (5.103), consider $a \ll a_{m\Lambda}$ and $a \gg a_{m\Lambda}$, respectively, in equation (5.101). You might have to Taylor expand (5.101) to first order. For example, the Taylor series approximation for $\ln(1+x)$ for small values of x is given by:

$$\ln(1+x) \approx x;\ \text{for } x \approx 0 \tag{8}$$

(b) This is similar to problem 6.7a part (b). Again you need to plot a vs. t, i.e., you need to plot equation (5.101). Check my suggestions from earlier. Attach your code and plot.

(c) To verify (5.105), which is the age of the Universe, you need to use the fact that at $t = t_0$, $a = 1$ in (5.101). This should lead you to (5.104). Plug in the measured values – $\Omega_{m,0} = 0.32$ and $H_0 = 67\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ – and this should give you the age in (5.105). See Dr. Windhorst's class notes on the website to check the updated age value.

To verify (5.106), which is the age of the Universe at matter-Λ equality, you need to use (5.101) with $a = a_{m,\Lambda}$ which will lead you to (5.106).

6.9 problem statement (15 points):
(a) Show that (5.110)–(5.113) are valid solutions for the Friedmann equation (5.108) in the case of a Radiation+Matter only universe. (This is the universe we lived in before Lambda took over at 3.5 Gyr ago (at $z_{\Lambda m} = 0.29$).
(b) Verify the value of z_{rm} and t_{rm} that we will use a lot later on.

Suggestions for problem 6.9: This problem is very similar to the previous two problems, so if you get the previous two problems, this one should be very easy to get through.
(a) To verify that (5.110) is a valid solution to (5.109) and also (5.108) which is the same equation, you can simply differentiate (5.110) and show that you can recover equation (5.109).

To show that (5.111) and (5.112) are also solutions, again plug in the appropriate conditions in (5.110), as shown in the book, i.e., either $a \ll a_{rm}$ or $a \gg a_{rm}$. See my suggestions above for the similar case in 6.8b part (a). To verify (5.113), you can simply set $a = a_{rm}$ in (5.110) and solve for t_{rm}. Plug in the values for the Benchmark model (see class notes) and verify that you get the value in (5.114), i.e, 47000 years.

(b) To find z_{rm}, you should use the fact that $a_{rm} = \Omega_{r,0}/\Omega_{m,0} \approx 2.8 \times 10^{-4}$ and the relation between a and z.