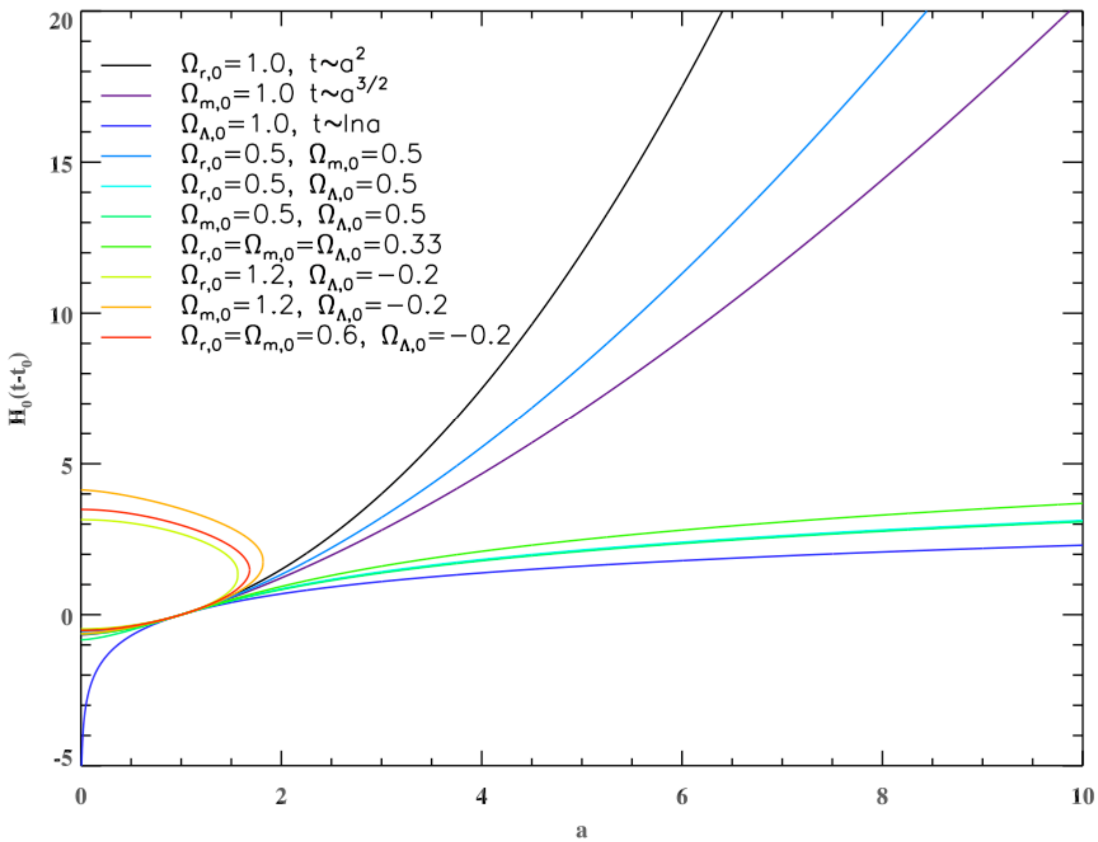
HW6 Solution AST422

6.1



6.2/6.3

$$H_0 t = \int_0^a \frac{da}{[\Omega_0 / a + (1-\Omega_0)]^{1/2}} = \int_0^a \frac{ada}{[\Omega_0 a + (1-\Omega_0) a^2]^{1/2}}$$

$$= \int_0^a \frac{ada}{(1-\Omega_0)^{1/2}[(a + \frac{\Omega_0}{2(1-\Omega_0)})^2 - (\frac{\Omega_0}{2(1-\Omega_0)})^2]^{1/2}}$$

$\Omega_0 > 1$

$$H_0 t = \int_0^a \frac{ada}{(\Omega_0 - 1)^{1/2}[-(a - \frac{\Omega_0}{2(\Omega_0 - 1)})^2 + (\frac{\Omega_0}{2(1-\Omega_0)})^2]^{1/2}}$$

$$a = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta)$$

$$\Rightarrow H_0 t = \int_0^\theta \frac{\frac{\Omega_0}{2(\Omega_0 - 1)}(1-\cos\theta) \cdot \frac{\Omega_0}{2(\Omega_0 - 1)} \sin\theta d\theta}{(\Omega_0 - 1)^{1/2}[-(\frac{\Omega_0}{2(\Omega_0 - 1)})^2 \cos^2\theta + (\frac{\Omega_0}{2(1-\Omega_0)})^2]^{1/2}}$$

$$= \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \int_0^\theta (1-\cos\theta) d\theta = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)$$

$$\Rightarrow t = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)$$

$\Omega_0 < 1$

$$H_0 t = \int_0^a \frac{ada}{(1-\Omega_0)^{1/2}[(a + \frac{\Omega_0}{2(1-\Omega_0)})^2 - (\frac{\Omega_0}{2(1-\Omega_0)})^2]^{1/2}}$$

$$a = \frac{\Omega_0}{2(\Omega_0 - 1)}(\cosh\theta - 1)$$

$$\Rightarrow H_0 t = \int_0^\theta \frac{\frac{\Omega_0}{2(1-\Omega_0)}(\cosh\theta - 1) \cdot \frac{\Omega_0}{2(1-\Omega_0)} \sinh\theta d\theta}{(1-\Omega_0)^{1/2}[(\frac{\Omega_0}{2(\Omega_0 - 1)})^2 \cosh^2\theta - (\frac{\Omega_0}{2(1-\Omega_0)})^2]^{1/2}}$$

$$= \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \int_0^\theta (\cosh\theta - 1) d\theta = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}(\sinh\theta - \theta)$$

$$\Rightarrow t = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(\theta - \sinh\theta)$$

6.4a

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0}) = \frac{\dot{a}^2}{H_0^2 a^2}$$

$$\Rightarrow \frac{da}{H_0 a dt} = [\frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0})]^{1/2}$$

$$\Rightarrow H_0 dt = \frac{da}{a[\frac{\Omega_{m,0}}{a^3} - (\Omega_{m,0}-1)]^{1/2}} = \frac{da}{a(\frac{\Omega_{m,0}}{(\Omega_{m,0}-1)a^3} - 1)^{1/2}(\Omega_{m,0}-1)^{1/2}}$$

$$a^3 = \frac{\Omega_{m,0}}{(\Omega_{m,0}-1)\csc^2\theta} = \frac{a_{\max}{}^3}{\csc^2\theta} \Rightarrow a = \frac{a_{\max}}{\csc^{2/3}\theta} \Rightarrow \theta = \sin^{-1}(\frac{a}{a_{\max}})^{3/2}$$

$$H_0 dt = \frac{a_{\max}(-\frac{2}{3}\csc^{-5/3}\theta)(-\csc\theta\cot\theta)d\theta}{\frac{a_{\max}}{\csc^{2/3}\theta}\cot\theta(\Omega_{m,0}-1)^{1/2}} = \frac{2d\theta}{3(\Omega_{m,0}-1)^{1/2}}$$

$$\Rightarrow H_0 t = \frac{2\theta}{3(\Omega_{m,0}-1)^{1/2}} = \frac{2}{3(\Omega_{m,0}-1)^{1/2}}\sin^{-1}(\frac{a}{a_{\max}})^{3/2}$$

6.4b

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0}) = \frac{\dot{a}^2}{H_0^2 a^2}$$

$$\Rightarrow \frac{da}{H_0 a dt} = [\frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0})]^{1/2}$$

$$\Rightarrow H_0 dt = \frac{da}{a[\frac{\Omega_{m,0}}{a^3} + (1-\Omega_{m,0})]^{1/2}} = \frac{da}{a(\frac{\Omega_{m,0}}{(1-\Omega_{m,0})a^3} + 1)^{1/2}(1-\Omega_{m,0})^{1/2}}$$

$$a^3 = \frac{\Omega_{m,0}}{(1-\Omega_{m,0})\cot^2\theta} = \frac{a_{m\Lambda}{}^3}{\cot^2\theta} \Rightarrow a = \frac{a_{m\Lambda}}{\cot^{2/3}\theta} \Rightarrow \tan\theta = (\frac{a}{a_{\max}})^{3/2}$$

$$H_0 dt = \frac{a_{m\Lambda}(-\frac{2}{3}\cot^{-5/3}\theta)(-\csc^2\theta)d\theta}{\frac{a_{m\Lambda}}{\cot^{2/3}\theta}\csc\theta(1-\Omega_{m,0})^{1/2}} = \frac{2\sec\theta d\theta}{3(1-\Omega_{m,0})^{1/2}}$$

$$\Rightarrow H_0 t = \frac{2}{3(1-\Omega_{m,0})^{1/2}}\int_0^\theta \sec\theta d\theta = \frac{2}{3(1-\Omega_{m,0})^{1/2}}\int_0^\theta \sec\theta\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}d\theta$$

$$H_0 t = \frac{2}{3(1-\Omega_{m,0})^{1/2}}\int_0^\theta \frac{d(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta} = \frac{2}{3(1-\Omega_{m,0})^{1/2}}\ln(\sec\theta+\tan\theta)$$

$$H_0 t = \frac{2}{3(1-\Omega_{m,0})^{1/2}}\ln[(\frac{a}{a_{m\Lambda}})^{3/2} + \sqrt{1+(\frac{a}{a_{m\Lambda}})^3}]$$

$$\Rightarrow t_0 = \frac{2H_0^{-1}}{3(1-\Omega_{m,0})^{1/2}}\ln[(\frac{1-\Omega_{m,0}}{\Omega_{m,0}})^{1/2} + \frac{1}{\Omega_{m,0}{}^{1/2}}] = \frac{2H_0^{-1}}{3(1-\Omega_{m,0})^{1/2}}\ln[\frac{\sqrt{1-\Omega_{m,0}}+1}{\Omega_{m,0}{}^{1/2}}]$$

$$\Omega_{m,0} = 0.27, H_0 = 71 km/s/Mpc$$

$$\Rightarrow t_0 = \frac{2}{71 km/s/Mpc \times 3(1-0.27)^{1/2}}\ln[\frac{\sqrt{0.73}+1}{0.27^{1/2}}] = 13.6 Gyr$$

6.5

$$H_0 t = \frac{ada}{\sqrt{\Omega_{r,0}}}\frac{1}{\sqrt{1+\frac{a}{a_{rm}}}}$$

$$x = \frac{a}{a_{rm}}$$

$$\Rightarrow H_0 t = \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}\frac{xdx}{\sqrt{1+x}}$$

$$x = \tan^2\theta$$

$$\Rightarrow H_0 t = \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}\frac{\tan^2\theta\cdot 2\tan\theta\sec^2\theta d\theta}{\sec\theta} = \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}\tan^2\theta\cdot 2\tan\theta\sec\theta d\theta$$

$$= \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}2\tan^2\theta d(\sec\theta) = \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}2(\sec^2\theta - 1)d(\sec\theta)$$

$$\Rightarrow H_0 t = \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}2(\frac{1}{3}\sec^3\theta - \sec\theta)\Big|_0^\theta = \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}[\frac{4}{3} + 2\sec\theta(\frac{1}{3}\sec^2\theta - 1)]$$

$$= \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}[\frac{4}{3} + 2\sqrt{1+x}(\frac{1}{3}(1+x) - 1)] = \frac{a_{rm}{}^2}{\sqrt{\Omega_{r,0}}}\frac{4}{3}[1 - \sqrt{1+x}(1 - \frac{x}{2})]$$

$$\Rightarrow H_0 t = \frac{4a_{rm}{}^2}{3\sqrt{\Omega_{r,0}}}[1 - (1 - \frac{a}{2a_{rm}})(1 + \frac{a}{a_{rm}})^{1/2}]$$

$$a << a_{rm}$$

$$H_0 t = \frac{4a_{rm}{}^2}{3\sqrt{\Omega_{r,0}}}[1 - (1 - \frac{a}{2a_{rm}})(1 + \frac{a}{a_{rm}})^{1/2}] \approx \frac{4a_{rm}{}^2}{3\sqrt{\Omega_{r,0}}}[1 - (1 - \frac{a}{2a_{rm}})(1 + \frac{a}{2a_{rm}} - \frac{a^2}{8a_{rm}{}^2})]$$

$$= \frac{4a_{rm}{}^2}{3\sqrt{\Omega_{r,0}}}[1 - (1 - \frac{3}{8}(\frac{a}{a_{rm}})^2)] = \frac{a^2}{2\sqrt{\Omega_{r,0}}}$$

$$\Rightarrow a \approx (2\sqrt{\Omega_{r,0}}H_0 t)^{1/2}$$

$a >> a_{rm}$

$$H_0 t = \frac{4{a_{rm}}^2}{3\sqrt{\Omega_{r,0}}}[1-(1-\frac{a}{2a_{rm}})(1+\frac{a}{a_{rm}})^{1/2}] \approx \frac{4{a_{rm}}^2}{3\sqrt{\Omega_{r,0}}}(\frac{a}{2a_{rm}})(\frac{a}{a_{rm}})^{1/2}$$

$$= \frac{2{a_{rm}}^2}{3\sqrt{\Omega_{r,0}}}(\frac{a}{a_{rm}})^{3/2} = \frac{2}{3\sqrt{\Omega_{r,0}/a_{rm}}}a^{3/2} = \frac{2}{3\sqrt{\Omega_{m,0}}}a^{3/2}$$

$$\Rightarrow a \approx (\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t)^{2/3}$$

$a = a_{rm}$

$$\Rightarrow H_0 t_{rm} = \frac{4{a_{rm}}^2}{3\sqrt{\Omega_{r,0}}}[1-(1-\frac{1}{2})(1+1)^{1/2}] = \frac{4{a_{rm}}^2}{3\sqrt{\Omega_{r,0}}}[1-\frac{1}{\sqrt{2}}]$$

$$\Rightarrow t_{rm} \approx 0.39\frac{{\Omega_{r,0}}^{3/2}}{{\Omega_{m,0}}^2}H_0^{-1}$$

$\Omega_{r,0} = 8.4\times 10^{-5}, \Omega_{m,0} = 0.3, H_0^{-1} = 14Gyr$

$$\Rightarrow t_{rm} \approx 0.39\frac{{\Omega_{r,0}}^{3/2}}{{\Omega_{m,0}}^2}H_0^{-1} = 47{,}000\,yr$$