11

Structure Formation: Gravitational Instability

The universe can be approximated as being homogeneous and isotropic only if we smooth it with a filter $\sim 100\,\mathrm{Mpc}$ across. On smaller scales, the universe contains density fluctuations ranging from subatomic quantum fluctuations up to the large superclusters and voids, $\sim 50\,\mathrm{Mpc}$ across, which characterize the distribution of galaxies in space. If we relax the strict assumption of homogeneity and isotropy that underlies the Robertson–Walker metric and the Friedmann equation, we can ask (and, to some extent, answer) the question, "How do density fluctuations in the universe evolve with time?"

The formation of relatively small objects, such as planets, stars, or even galaxies, involves fairly complicated physics. Because of the greater complexity when baryons are involved in structure formation, we will postpone the discussion of galaxies and stars until the next chapter. In this chapter, we will focus on the formation of structures larger than galaxies – clusters, superclusters, and voids. Cosmologists use the term *large scale structure of the universe* to refer to all structures bigger than individual galaxies. A map of the large scale structure of the universe, as traced by the positions of galaxies, can be made by measuring the redshifts of a sample of galaxies and using the Hubble relation, $d = (c/H_0)z$, to estimate their distances. For instance, Figure 11.1 shows two slices through the universe based on data from the 2dF Galaxy Redshift Survey (2dFGRS). From our location near the midplane of our galaxy's dusty disk, we get our best view of distant galaxies when we look perpendicular to the disk, toward what are conventionally called the "north galactic pole" (in the constellation Coma Berenices) and the antipodal "south galactic pole" (in the constellation Sculptor). The 2dFGRS selected two long narrow stripes on the sky, one near the north galactic pole and the other near the south galactic pole. In each stripe, the redshift was measured for $\sim 100\,000$ galaxies. By plotting the redshift of each galaxy as a function of its angular position along the stripe, a pair of two-dimensional slices through the universe were mapped out.

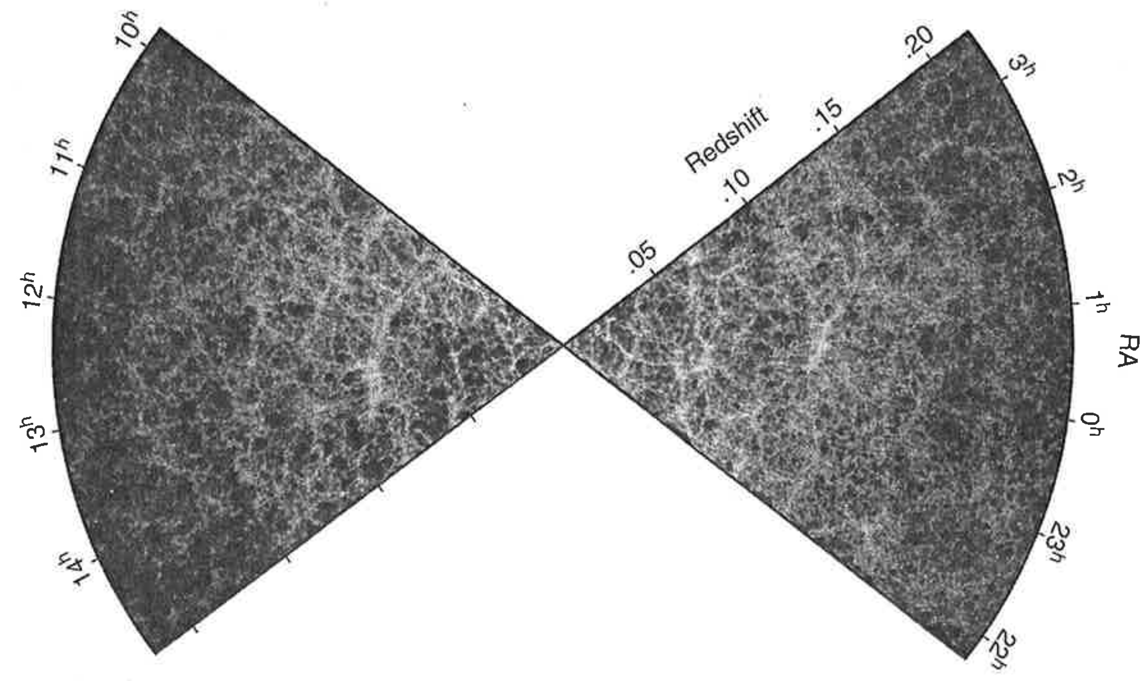

Figure 11.1 Two slices through the universe, based on the redshift distribution of $\sim 200{,}000$ galaxies measured by the 2dF Galaxy Redshift Survey; white regions are higher in density, black regions are lower. Left slice: data from the north stripe. Right slice: data from the south stripe. [van de Weygaert & Schaap 2009, *Lect. Notes Phys.*, **665**, 291]

Figure 11.1 shows the two slices that the 2dF Galaxy Redshift Survey made through the universe. They reach as far as a redshift $z \approx 0.22$, corresponding to a proper distance $d_p(t_0) \sim 1000\,\text{Mpc}$. The white regions in the plot represent regions with a high number density of galaxies. The galaxies obviously do not have a random Poisson distribution, in which the position of each galaxy is uncorrelated with the position of other galaxies. Instead, the galaxies are found primarily in long superclusters separated by lower density voids. Superclusters are objects that are just now collapsing under their own self-gravity. Superclusters are typically flattened (roughly planar) or elongated (roughly linear) structures. A supercluster will contain one or more clusters embedded within it; a cluster is a fully collapsed object that has come to equilibrium, more or less, and hence obeys the virial theorem, as discussed in Section 7.3. In comparison to the flattened or elongated superclusters, the underdense voids are roughly spherical in shape. When gazing at the large scale structure of the universe, as traced by the distribution of galaxies, cosmologists are likely to call it "bubbly" or "spongy," with the low-density voids constituting the bubbles, or the holes of the sponge. Alternatively, if they focus on the high-density superclusters rather than the low-density voids, cosmologists refer to a "cosmic web" of filaments and walls.

Figure 11.2 The northeastern United States and southeastern Canada at night, as seen by the *Suomi-NPP* satellite. [NASA Worldview]

11.1 The Matthew Effect

Being able to describe the distribution of galaxies in space doesn't automatically lead to an understanding of the origin of large scale structure. Consider, as an analogy, the distribution of luminous objects shown in Figure 11.2. The distribution of illuminated cities on the Earth's surface is obviously not random. There are "superclusters" of cities, such as the Boswash supercluster stretching from Boston to Washington. There are "voids" such as the Appalachian void. However, the influences that determine the exact location of cities are often far removed from fundamental physics.[1]

Fortunately, the distribution of galaxies in space is more closely tied to fundamental physics than is the distribution of cities on the Earth. The basic mechanism for growing large structures, such as voids and superclusters, is *gravitational instability*. Suppose that at some time in the past, the density of the universe had slight inhomogeneities. We know, for instance, that such density fluctuations existed at the time of last scattering, since they left their stamp on the cosmic microwave background. When the universe is matter-dominated, the overdense regions expand less rapidly than the universe as a whole; if their density is sufficiently great, they will collapse and become gravitationally bound objects such as clusters. The dense clusters will, in addition, draw matter to themselves from the surrounding underdense regions. The effect of gravity on density fluctuations is sometimes referred to as the Matthew effect: "For whosoever hath, to him shall be given, and he shall have more abundance; but whosoever hath not, from him shall be taken away even that he hath" [Matthew 13:12]. In less biblical language, the rich get richer and the poor get poorer.[2]

1 Consider, for instance, the complicated politics that went into determining the location of Washington, DC.
2 The singer-songwriter Billie Holiday may have summed it up best: "Them that's got shall have / Them that's not shall lose."

To put our study of gravitational instability on a more quantitative basis, consider some component of the universe whose energy density $\varepsilon(\vec{r}, t)$ is a function of position as well as time. At a given time t, the spatially averaged energy density is

FRW assume $\varepsilon(\vec{r}, t) \equiv \varepsilon(t)$!

$$\bar{\varepsilon}(t) = \frac{1}{V} \int_V \varepsilon(\vec{r}, t) d^3 r. \qquad (11.1)$$

To ensure that we have found the true average, the volume V over which we are averaging must be large compared to the size of the biggest structure in the universe. It is useful to define a dimensionless density fluctuation

OVER/UNDER DENSITY
$$\delta(\vec{r}, t) \equiv \frac{\varepsilon(\vec{r}, t) - \bar{\varepsilon}(t)}{\bar{\varepsilon}(t)} = \frac{\delta\varepsilon}{\bar{\varepsilon}} = \frac{\delta\rho c^2}{\rho c^2} \qquad (11.2)$$

The value of δ is thus negative in underdense regions and positive in overdense regions. If the density ε is constrained to be non-negative, as it is for matter and radiation, then the minimum possible value of δ is $\delta = -1$, corresponding to $\varepsilon = 0$. There is no such upper limit on δ, which can have a value $\delta \gg +1$. (A human being has $\delta \simeq 2 \times 10^{30}$!) - Ch 2.

To get a feel for how initially small density contrasts grow with time, consider a particularly simple case. Start with a static, homogeneous, matter-only universe with uniform mass density $\bar{\rho}$. (At this point, we stumble over the inconvenient fact that there's no such thing as a static, homogeneous, matter-only universe. This is the awkward fact that inspired Einstein to introduce the cosmological constant. However, there are conditions under which we can consider some region of the universe to be approximately static and homogeneous. For instance, the air in a closed room is approximately static and homogeneous; it is stabilized by a pressure gradient with a scale length much greater than the height of the ceiling.) In a region of the universe that is *approximately* static and homogeneous, we add a small amount of mass within a sphere of radius R, as seen in Figure 11.3, so that the density within the sphere is $\bar{\rho}(1 + \delta)$, with $\delta \ll 1$. If the density excess

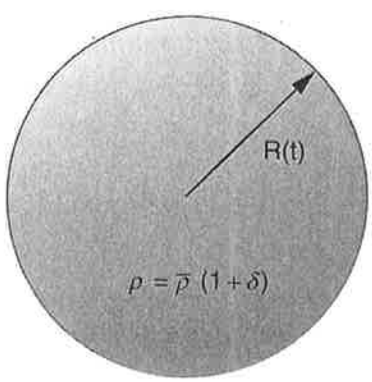

Figure 11.3 A sphere of radius $R(t)$ expanding or contracting under the influence of the density fluctuation $\delta(t)$.

δ is uniform within the sphere, then the gravitational acceleration at the sphere's surface, due to the excess mass, will be

$$\ddot{R} = -\frac{G(\Delta M)}{R^2} = -\frac{G}{R^2}\underbrace{\left(\frac{4\pi}{3}R^3\bar{\rho}\delta\right)}_{\Delta M}, \tag{11.3}$$

or

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t). \Rightarrow \begin{cases} \delta(t)>0 \Rightarrow \ddot{R}<0 \text{ - COLLAPSES} \\ \delta(t)<0 \Rightarrow \ddot{R}>0 \text{ - EXPANDS} \end{cases} \tag{11.4}$$

Thus, a mass excess ($\delta > 0$) will cause the sphere to collapse inward ($\ddot{R} < 0$).

Equation 11.4 contains two unknowns, $R(t)$ and $\delta(t)$. If we want to find an explicit solution for $\delta(t)$, we need a second equation involving $R(t)$ and $\delta(t)$. Conservation of mass tells us that the mass of the sphere,

$$M = \frac{4\pi}{3}\bar{\rho}[1+\delta(t)]R(t)^3, \left[\equiv \frac{4\pi}{3}R_0^3 \cdot \bar{\rho} = \text{CONSTANT if } \delta=0\right] \tag{11.5}$$

at $t=0$

remains constant during the collapse. Thus we can write another relation between $R(t)$ and $\delta(t)$ that must hold true during the collapse:

$$R(t) = R_0[1+\delta(t)]^{-1/3}, \tag{11.6}$$

where

$$(11.5) \Rightarrow \qquad R_0 \equiv \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} = \text{constant}. \tag{11.7}$$

When $\delta \ll 1$, we may make the approximation

$$(11.6) \Rightarrow \qquad R(t) \approx R_0\left[1 - \frac{1}{3}\delta(t)\right]. \tag{11.8}$$

Taking the second time derivative yields

$$\ddot{R} \approx -\frac{1}{3}R_0\ddot{\delta} \approx -\frac{1}{3}R\ddot{\delta}. \tag{11.9}$$

Thus, mass conservation tells us that

$$(11.4+11.9) \Rightarrow \qquad \frac{\ddot{R}}{R} \approx -\frac{1}{3}\ddot{\delta} \overset{(11.4)}{=} -\frac{4\pi G\bar{\rho}}{3}\cdot\delta(t) \tag{11.10}$$

in the limit that $\delta \ll 1$. Combining Equations 11.4 and 11.10, we find a tidy equation that tells us how the small overdensity δ evolves as the sphere collapses:

$$(11.4)+(11.10) \Rightarrow \qquad \boxed{\ddot{\delta} = 4\pi G\bar{\rho}\delta.} \Rightarrow \frac{\partial^2\delta}{\partial t^2} = 4\pi G\bar{\rho}\,\delta \tag{11.11}$$

$\hookrightarrow \partial t = 1/\sqrt{4\pi G\bar{\rho}}$

The most general solution of Equation 11.11 has the form

$$\boxed{\delta(t) = A_1 e^{t/t_{\rm dyn}} + A_2 e^{-t/t_{\rm dyn}},} \qquad (11.13) \tag{11.12}$$

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where the dynamical time for collapse is

$\bar{\rho} = \bar{\varepsilon}/c^2$

$$t_{\rm dyn} = \frac{1}{(4\pi G\bar{\rho})^{1/2}} \approx 9.6\,\text{hours}\left(\frac{\bar{\rho}}{1\,\text{kg}\,\text{m}^{-3}}\right)^{-1/2} = \left(\frac{c^2}{4\pi G\bar{\varepsilon}}\right)^{1/2} \quad (11.13)$$

Note that the dynamical time depends only on $\bar{\rho}$, and not on R. The constants A_1 and A_2 in Equation 11.12 depend on the initial conditions of the sphere. For instance, if the overdense sphere starts at rest, with $\dot{\delta} = 0$ at $t = 0$, then $A_1 = A_2 = \delta(0)/2$. After a few dynamical times, however, only the exponentially growing term of Equation 11.12 is significant. Thus, gravity tends to make small density fluctuations in a static, pressureless medium grow exponentially with time.

11.2 The Jeans Length

The exponential growth of density perturbations is slightly alarming, at first glance. For instance, the density of the air around you is $\bar{\rho} \approx 1\,\text{kg}\,\text{m}^{-3}$, yielding a dynamical time for collapse of $t_{\rm dyn} \sim 10\,\text{hours}$.[3] What keeps small density perturbations in the air from undergoing a runaway collapse over the course of a few days? The answer, of course, is pressure. A nonrelativistic gas, as shown in Section 4.4, has an equation-of-state parameter

(4.55) $P = w\varepsilon$
(4.67) $P = \frac{kT}{\mu c^2}\varepsilon$ $\Longrightarrow$

$$w \approx \frac{kT}{\mu c^2}, \quad (4.57) \qquad (11.14)$$

where T is the temperature of the gas and μ is the mean mass per gas particle. Thus, the pressure of an ideal gas will never totally vanish, but will only approach zero in the limit that the temperature approaches absolute zero.

When a sphere of gas is compressed by its own gravity, a pressure gradient will build up that tends to counter the effects of gravity. In the universe today, a star is the prime example of a sphere of gas in which the inward force of gravity is balanced by the outward force provided by a pressure gradient; the hot intracluster gas of the Coma cluster is also in a balance between gravity and pressure. However, hydrostatic equilibrium, the state in which gravity is exactly balanced by a pressure gradient, cannot always be attained. Consider an overdense sphere with initial radius R. If pressure were not present, it would collapse on a timescale

$\bar{\rho} = \bar{\varepsilon}/c^2$

(11.13) $\Longrightarrow$

$$t_{\rm dyn} \sim \frac{1}{(G\bar{\rho})^{1/2}} \sim \left(\frac{c^2}{G\bar{\varepsilon}}\right)^{1/2}. \qquad (11.15)$$

If the pressure is nonzero, the attempted collapse will be countered by a steepening of the pressure gradient within the perturbation. The steepening of the pressure gradient, however, doesn't occur instantaneously. Any change in pressure

[3] Slightly longer if you are using this book for recreational reading as you climb Mount Everest.

travels at the sound speed.[4] Thus, the time it takes for the pressure gradient to build up in a region of radius R will be

$$t_{\rm pre} \sim \frac{R}{c_s}, \tag{11.16}$$

where c_s is the local sound speed. In a medium with equation-of-state parameter $w > 0$, the sound speed is

for Λ, $w=-1 \Rightarrow$ "$c_s = c\sqrt{w} = c\,i$"

$\Uparrow$ Sound speed meaningless in pure Λ-univ.

$$c_s = c\left(\frac{dP}{d\varepsilon}\right)^{1/2} = \sqrt{w}c \simeq \frac{c}{\sqrt{3}} \text{ for } w = 1/3 \text{ radiation domin.} \tag{11.17}$$

For hydrostatic equilibrium to be attained, the pressure gradient must build up before the overdense region collapses, implying

$$t_{\rm pre} < t_{\rm dyn}. \tag{11.18}$$

Comparing Equation 11.15 with Equation 11.16, we find that for a density perturbation to be stabilized by pressure against collapse, it must be smaller than some reference size λ_J, given by the relation (11.13)'

$$\left.\begin{matrix}(11.13),(11.15)\\(11.16),(11.18)\end{matrix}\right\} \Rightarrow R \simeq c_s t_{\rm pre} \Rightarrow \lambda_J \sim c_s t_{\rm dyn} \sim c_s\left(\frac{c^2}{G\bar{\varepsilon}}\right)^{1/2}. \tag{11.19}$$

The length scale λ_J is known as the *Jeans length*, after the astrophysicist James Jeans, who was among the first to study gravitational instability in a cosmological context. Overdense regions larger than the Jeans length collapse under their own gravity. Overdense regions smaller than the Jeans length merely oscillate in density; they constitute stable standing sound waves.

A more precise derivation of the Jeans length, including all the appropriate factors of π, yields the result

typical timescale for weather changes

$$\lambda_J = c_s\left(\frac{\pi c^2}{G\bar{\varepsilon}}\right)^{1/2} = 2\pi c_s t_{\rm dyn}. \tag{11.20}$$

The Jeans length of the Earth's atmosphere, for instance, where the sound speed is a third of a kilometer per second and the dynamical time is ten hours, is $\lambda_J \sim 10^5$ km, far longer than the scale height of the Earth's atmosphere. You don't have to worry about density fluctuations in the air undergoing a catastrophic collapse.

To consider the behavior of density fluctuations on cosmological scales, consider a spatially flat universe in which the mean density is $\bar{\varepsilon}$, but which contains density fluctuations with amplitude $|\delta| \ll 1$. The characteristic time for expansion of such a universe is the Hubble time,

$$(4.30)\quad \varepsilon_0 = \frac{3c^2H_0^2}{8\pi G} \Rightarrow \quad H^{-1} = \left(\frac{3c^2}{8\pi G\bar{\varepsilon}}\right)^{1/2}. \tag{11.21}$$

[4] What is sound, after all, but a traveling change in pressure?

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Comparison of Equation 11.13 with Equation 11.21 reveals that the Hubble time is comparable in magnitude to the dynamical time $t_{\rm dyn}$ for the collapse of an overdense region:

(11.13), (11.21) $\Rightarrow$

$$H^{-1} = \left(\frac{3}{2}\right)^{1/2} t_{\rm dyn} \approx 1.22 t_{\rm dyn}. \tag{11.22}$$

The Jeans length in an expanding flat universe will then be

(11.20), (11.22) $\Rightarrow$

$$\lambda_J \overset{(11.20)}{=} 2\pi c_s t_{\rm dyn} \overset{(11.22)}{=} 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{c_s}{H}. \tag{11.23}$$

If we focus on one particular component of the universe, with equation-of-state parameter w and sound speed $c_s = \sqrt{w}c$, the Jeans length for that component will be

$$\lambda_J = 2\pi \left(\frac{2}{3}\right)^{1/2} \sqrt{w}\frac{c}{H}. \tag{11.24}$$

Consider, for instance, the "radiation" component of the universe. With $w = 1/3$, the sound speed in a gas of photons or other relativistic particles is $c_s = c/\sqrt{3}$. The Jeans length for radiation in an expanding universe is then

$$\lambda_J = \frac{2\pi\sqrt{2}}{3}\frac{c}{H} \approx 3.0\frac{c}{H} \simeq 3R_H \tag{11.25}$$

Density fluctuations in the radiative component will be pressure-supported if they are smaller than three times the Hubble distance. Although a universe containing nothing but radiation can have density perturbations smaller than $\lambda_J \sim 3c/H$, they will be stable sound waves, and will not collapse under their own gravity.

In order for a universe to have gravitationally collapsed structures much smaller than the Hubble distance, it must have a nonrelativistic component, with $\sqrt{w} \ll 1$. The gravitational collapse of the *baryonic* component of the universe is complicated by the fact that it was coupled to photons until the time of decoupling at a redshift $z_{\rm dec} \approx z_{\rm ls} \approx 1090$. Before this time, the photons, electrons, and baryons were all coupled together to form a single photon–baryon fluid. Since photons were still dominant over baryons at the time of decoupling, with $\varepsilon_\gamma > \varepsilon_{\rm bary}$, we can regard the baryons (with mild exaggeration) as being a dynamically insignificant contaminant in the photon–baryon fluid. Just *before* decoupling, the Jeans length of the photon–baryon fluid was roughly the same as the Jeans length of a pure photon gas:

$z \approx 1090$

$$\boxed{\lambda_J(\text{before}) \approx 3c/H(z_{\rm dec}) \approx 0.66\,\text{Mpc} \approx 2.0 \times 10^{22}\,\text{m}.} \tag{11.26}$$

3×0.22

(8.56), (8.65) $\quad d_{\rm hor}(t_{ls}) \simeq 0.25\,\text{Mpc} \simeq d_s(t_{ls})$

The *baryonic Jeans mass*, M_J, is defined as the mass of baryons contained within a sphere of radius λ_J:

$$\boxed{M_J \equiv \rho_{\text{bary}} \left(\frac{4\pi}{3} \lambda_J^3 \right).} \tag{11.27}$$

Immediately before decoupling, when the baryon density was $\rho_{\text{bary}} \approx 5.6 \times 10^{-19}\,\text{kg}\,\text{m}^{-3}$, the baryonic Jeans mass was

$$M_J(\text{before}) = \rho_{\text{bary}} \frac{4\pi}{3} \lambda_J(\text{before})^3 \approx 2 \times 10^{49}\,\text{kg} \approx 10^{19}\,\text{M}_\odot. \tag{11.28}$$

This is very large compared to the baryonic mass of the Coma cluster ($M_{\text{bary}} \sim 2 \times 10^{14}\,\text{M}_\odot$); it is even large when compared to the baryonic mass of a supercluster ($M_{\text{bary}} \sim 10^{16}\,\text{M}_\odot$).

Now consider what happens to the baryonic Jeans mass immediately after decoupling. Once the photons are decoupled, the photons and baryons form two separate gases, instead of a single photon–baryon fluid. The sound speed in the photon gas is $= c/\sqrt{w}$

$$c_s(\text{photon}) = c/\sqrt{3} \approx 0.58c. \tag{11.29}$$

The sound speed in the baryonic gas, by contrast, is

$$c_s(\text{baryon}) = \left(\frac{kT}{mc^2} \right)^{1/2} c. \tag{11.30}$$

$c = \frac{c}{\sqrt{w}}$ with $w = \frac{mc^2}{kT}$ for a non-relativistic gas

At the time of decoupling, the thermal energy per particle was $kT_{\text{dec}} \approx 0.26\,\text{eV}$, and the mean rest energy of the atoms in the baryonic gas was $mc^2 = 1.22 m_p c^2 \approx 1140\,\text{MeV}$, taking into account the helium mass fraction of $Y_p = 0.24$. Thus, the sound speed of the baryonic gas immediately after decoupling was

$n_n/n_p \approx 0.15$, $Y \sim 0.24$

$$c_s(\text{baryon}) \approx \left(\frac{0.26\,\text{eV}}{1140 \times 10^6\,\text{eV}} \right)^{1/2} c \approx 1.5 \times 10^{-5} c, \tag{11.31}$$

only 5 kilometers per second. Thus, once the baryons were decoupled from the photons, their associated Jeans length decreased by a factor

$$F = \frac{c_s(\text{baryon})}{c_s(\text{photon})} \approx \frac{1.5 \times 10^{-5}}{0.58} \approx 2.6 \times 10^{-5}. \tag{11.32}$$

Decoupling causes the baryonic Jeans mass to decrease by a factor $F^3 \approx 1.8 \times 10^{-14}$, plummeting from $M_J(\text{before}) \approx 10^{19}\,\text{M}_\odot$ to $(2.6 \times 10^{-5})^3 \times 10^{19} M_\odot \approx 1.7\, 10^5 M_\odot$

$$M_J(\text{after}) = F^3 M_J(\text{before}) \approx 2 \times 10^5\,\text{M}_\odot. \tag{11.33}$$

This is very small compared to the baryonic mass of our own galaxy ($M_{\text{bary}} \sim 10^{11}\,\text{M}_\odot$); it is even small when compared to a more modest galaxy like the Small Magellanic Cloud ($M_{\text{bary}} \sim 10^9\,\text{M}_\odot$).

from Longair's book

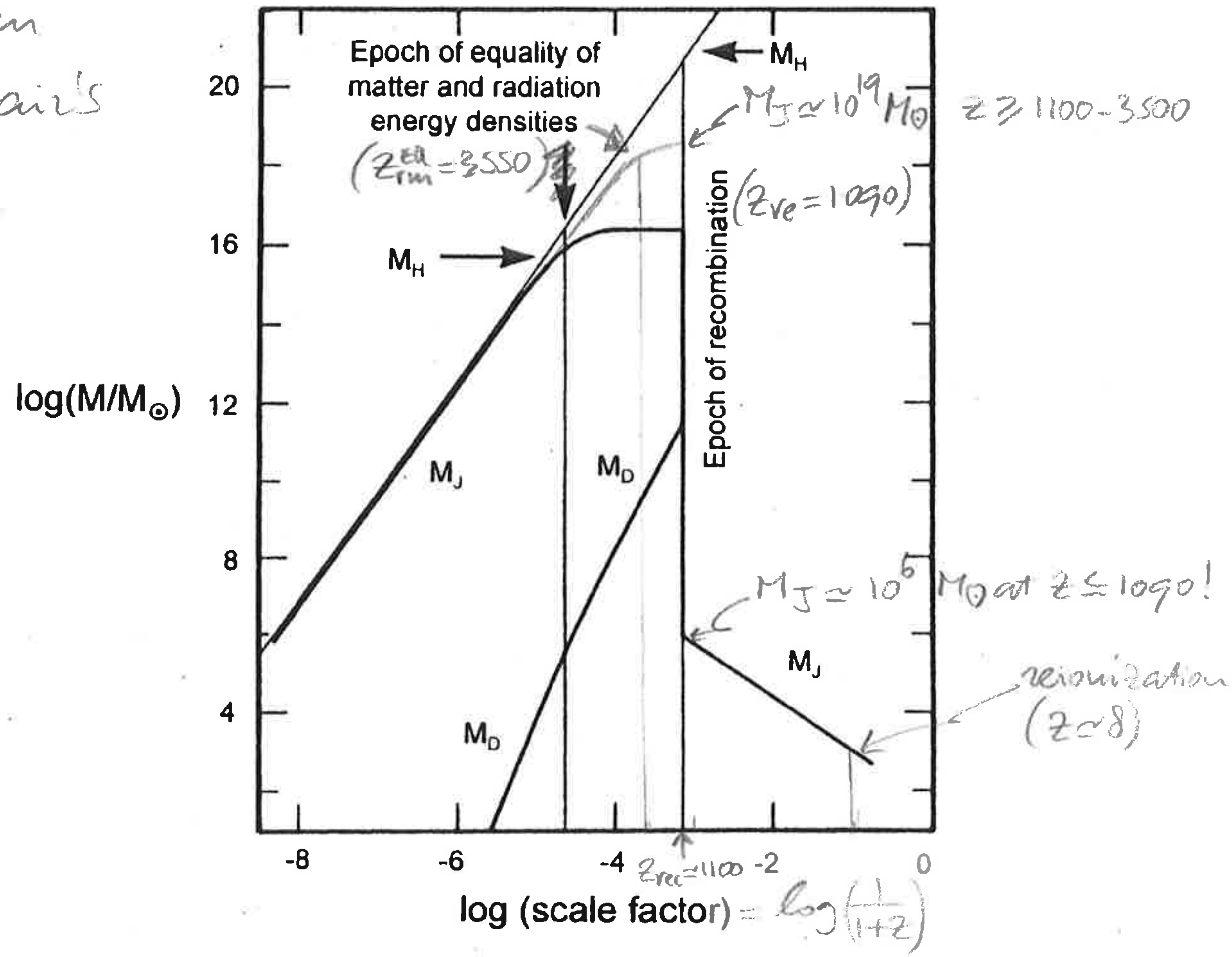

Fig. 12.1. The evolution of the Jeans' mass and the baryonic mass within the particle horizon with scale factor. Also shown is the evolution of the mass scales which are damped by photon diffusion.

era. Using (12.7) for the variation of the particle horizon with scale factor, the baryonic mass within the horizon is

$$M_H = \left(\frac{\pi r_H^3}{6}\right) \varrho_B = \frac{3.0 \times 10^{22}}{(\Omega_0 h^2)^{1/2}} R^{3/2} M_\odot, \tag{12.17}$$

that is, $M_H \propto R^{3/2} \propto t$, so long as $\Omega_0 z \gg 1$. This relation is shown in Fig. 12.1.

To determine the Jeans' length, we need to know the variation of the speed of sound with redshift. As the epoch of equality of the rest mass energies in matter and radiation is approached, the sound speed becomes less than the relativistic sound speed $c/\sqrt{3}$ and is given by (9.32).

$$c_s^2 = \frac{c^2}{3} \frac{4\varrho_{rad}}{4\varrho_{rad} + 3\varrho_m}. \tag{9.32}$$

During this phase, the pressure within the perturbations is provided by the

The abrupt decrease of the baryonic Jeans mass at the time of decoupling marks an important epoch in the history of structure formation. Perturbations in the baryon density, from supercluster scales down to the size of the smallest dwarf galaxies, couldn't grow in amplitude until the time of photon decoupling, when the universe had reached the ripe old age of $t_{\rm dec} \approx 0.37$ Myr. After decoupling, the growth of density perturbations in the baryonic component was off and running.

11.3 Instability in an Expanding Universe

$t_{dyn} \sim \left(\frac{c^2}{G\bar{\varepsilon}}\right)^{1/2} \sim H(t)^{-1}$

Density perturbations smaller than the Hubble distance can grow in amplitude only when they are no longer pressure-supported. However, if the pressure (and hence the Jeans length) of some component becomes negligibly small, this doesn't necessarily imply that the amplitude of density fluctuations is free to grow exponentially with time. The analysis of Section 11.1, which yielded $\delta \propto \exp(t/t_{\rm dyn})$, assumed that the universe was *static* as well as pressureless. In an expanding Friedmann universe, the timescale for the growth of a density perturbation by self-gravity, $t_{\rm dyn} \sim (c^2/G\bar{\varepsilon})^{1/2}$, is comparable to the timescale for expansion, $H^{-1} \sim (c^2/G\bar{\varepsilon})^{1/2}$. Self-gravity, in the absence of global expansion, causes overdense regions to become *more dense* with time. The global expansion of the universe, in the absence of self-gravity, causes overdense regions to become *less dense* with time. Because the timescales for these two competing processes are similar, they must both be taken into account when computing the time evolution of a density perturbation.

To see how small density perturbations in an expanding universe evolve with time, let's do a Newtonian analysis of the problem, similar in spirit to the Newtonian derivation of the Friedmann equation given in Chapter 4. Suppose you are in a universe containing only pressureless matter with mass density $\bar{\rho}(t)$. As the universe expands, the density decreases at the rate $\bar{\rho}(t) \propto a(t)^{-3}$. Within a spherical region of radius R, a small amount of matter is added, or removed, so that the density within the sphere is

$$\rho(t) = \bar{\rho}(t)[1 + \delta(t)], \tag{11.34}$$

with $|\delta| \ll 1$. The total gravitational acceleration at the surface of the sphere will be

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}\rho R^3\right) = -\frac{4\pi}{3}G\bar{\rho}R - \frac{4\pi}{3}G(\bar{\rho}\delta)R = \ddot{R} \tag{11.35}$$

The equation of motion for a point at the surface of the sphere can then be written in the form

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta = -\frac{4\pi}{3}G\bar{\rho}\left(1+\delta(t)\right) \tag{11.36}$$

Mass conservation tells us that the mass inside the sphere,

(11.5) $\Rightarrow$
$$M = \frac{4\pi}{3}\bar{\rho}(t)[1+\delta(t)]R(t)^3 = \text{CONST} \tag{11.37}$$

remains constant as the sphere expands. Thus,

$$R(t) \propto \bar{\rho}(t)^{-1/3}[1+\delta(t)]^{-1/3}, \tag{11.38}$$

or, since $\bar{\rho} \propto a^{-3}$,

$\bar{\rho}^{-1/3} \propto a \Longrightarrow$
$$\boxed{R(t) \propto a(t)[1+\delta(t)]^{-1/3} \simeq a(t)\left[1-\tfrac{1}{3}\delta(t)\right]} \tag{11.39}$$

(11.39)' $\downarrow$ $= R$

That is, if the sphere is slightly overdense, its radius will grow slightly less rapidly than the scale factor $a(t)$. If the sphere is slightly underdense, it will grow slightly more rapidly than the scale factor.

Taking two time derivatives of Equation 11.39 yields $\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}(a-\frac{1}{3}a\delta)\right) = \frac{\partial}{\partial t}(\dot{a}-\frac{1}{3}\dot{a}\delta-\frac{1}{3}a\dot{\delta})$

$= \ddot{a}-\frac{1}{3}\ddot{a}\delta-\frac{1}{3}\dot{a}\dot{\delta}-\frac{1}{3}\dot{a}\dot{\delta}-\frac{1}{3}a\ddot{\delta}$

$= \ddot{a}-\frac{1}{3}\ddot{a}\delta-\frac{2}{3}\dot{a}\dot{\delta}-\frac{1}{3}a\ddot{\delta}$ $[/a] \Rightarrow * =$

$\Rightarrow * = \frac{\ddot{a}}{a}\,a(1-\frac{1}{3}\delta) - \left(\frac{2}{3}\frac{\dot{a}}{a}\dot{\delta}+\frac{1}{3}\ddot{\delta}\right).a(1-\frac{1}{3}\delta)$

$\Rightarrow *^2\ \ddot{R}(t) = \frac{\ddot{a}}{a}R - \left(\frac{2}{3}\frac{\dot{a}}{a}\dot{\delta}+\frac{1}{3}\ddot{\delta}\right).R \Longrightarrow$

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta}, \tag{11.40}$$

when $|\delta| \ll 1$. Combining Equations 11.36 and 11.40, we find

$$\frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta = \frac{-4\pi}{3}G\bar{\rho}(1+\delta(t)) \tag{11.41}$$

When $\delta = 0$, Equation 11.41 reduces to

Acceleration Equation (4.49)

(4.49) $\Rightarrow$
$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\bar{\rho} \simeq \frac{-4\pi G}{3c^2}(\varepsilon+3P) \tag{11.42}$$

with $\bar{\rho}c^2 = \varepsilon + 3P$

which is the correct acceleration equation for a homogeneous, isotropic universe containing nothing but pressureless matter (compare to Equation 4.49). By subtracting Equation 11.42 from Equation 11.41 to leave only the terms linear in the perturbation δ, we find the equation that governs the growth of small amplitude perturbations:

$$\boxed{-\frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho}\delta,} \tag{11.43}$$

or

GROWTH OF FLUCTUATIONS IN EXPANDING UNIVERSE

$$\boxed{\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta,} \quad \text{with } \bar{\rho} = \frac{\bar{\varepsilon}}{c^2} \text{ since } \bar{\varepsilon} = \bar{\rho}c^2 \tag{11.44}$$

remembering that $H \equiv \dot{a}/a$. In a static universe, with $H = 0$, Equation 11.44 reduces to the result we already found in Equation 11.11:

GROWTH OF FLUCTUATIONS IN STATIC UNIVERSE (11.11)

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta. \quad \Leftarrow (11.44) \text{ WITH } H=0! \tag{11.45}$$

The additional term, $\propto H\dot{\delta}$, found in an expanding universe, is a "Hubble friction" term; it acts to slow the growth of density perturbations in an expanding universe.

A fully relativistic calculation for the growth of density perturbations yields the more general result

(4.49)
(11.42) $\Rightarrow$

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G}{c^2}\bar{\varepsilon}_m \delta. \quad \text{with } \bar{\rho} = \frac{\bar{\varepsilon}_m}{c^2} \tag{11.46}$$

This form of the equation can be applied to a universe that contains components with non-negligible pressure, such as radiation ($w = 1/3$) or a cosmological constant ($w = -1$). In multiple-component universes, however, it should be remembered that δ represents the fluctuation in the density of *matter* alone. That is,

$$\boxed{\delta = \frac{\varepsilon_m - \bar{\varepsilon}_m}{\bar{\varepsilon}_m},} \tag{11.47}$$

where $\bar{\varepsilon}_m(t)$, the average matter density, might be only a small fraction of $\bar{\varepsilon}(t)$, the average total density. Rewritten in terms of the density parameter for matter,

$$\frac{3}{2}\Omega_m H^2 = 4\pi G \frac{\bar{\varepsilon}_m}{c^2} \Leftarrow \Omega_m = \frac{\bar{\varepsilon}_m}{\varepsilon_c} = \frac{8\pi G \bar{\varepsilon}_m}{3c^2 H^2}, \quad \text{with } \varepsilon_c = \frac{3c^2H^2}{8\pi G} \tag{11.48}$$

(4.30)

Equation 11.46 takes the form

(11.48)'
(11.46) $\Rightarrow$

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0.} \tag{11.49}$$

During epochs when the universe is not dominated by matter, density perturbations in the matter do not grow rapidly in amplitude. Take, for instance, the early radiation-dominated phase of the universe. During this epoch, $\Omega_m \ll 1$ and $H = 1/(2t)$, meaning that Equation 11.49 takes the form

(5.58): $t_0 = \frac{1}{2} H_0^{-1}$ for radiation dom.:

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} \approx 0, \Rightarrow \ddot{\delta} \simeq -\frac{1}{t}\dot{\delta} \tag{11.50}$$

which has a solution of the form

SLOW LOGARITHMIC GROWTH
IN RADIATION-DOMINATED ERA. $\Leftrightarrow$

$$\delta(t) \approx B_1 + B_2 \ln t. \tag{11.51}$$

During the radiation-dominated epoch, density fluctuations in the dark matter grew only at a logarithmic rate. In the far future, if the universe is indeed dominated by a cosmological constant, the density parameter for matter will again be negligibly small, the Hubble parameter will have the constant value $H = H_\Lambda$, and Equation 11.49 will take the form

$$\ddot{\delta} + 2H_\Lambda \dot{\delta} \approx 0, \Rightarrow \ddot{\delta} = -2H_\Lambda \dot{\delta} \tag{11.52}$$

which has a solution of the form

STRUCTURE IS FROZEN
IN THE Λ-DOMINATED ERA!! $\Leftarrow$

$$\delta(t) \approx C_1 + C_2 e^{-2H_\Lambda t}. \tag{11.53}$$

(since $e^{-2H_\Lambda t} \to 0$ for $t \to \infty$)
(for $t \gg t_{m\Lambda}$)

In a lambda-dominated phase, therefore, fluctuations in the matter density reach a constant fractional amplitude, while the average matter density plummets at the rate $\bar{\varepsilon}_m \propto e^{-3H_\Lambda t}$.

(C_1)

Only when matter dominates the energy density can fluctuations in the matter density grow at a significant rate. In a flat, matter-dominated universe, $\Omega_m = 1$, $H = 2/(3t)$, and Equation 11.49 takes the form

(5.55) Matter dominated: $a(t) \propto t^{2/3}$

$H = \frac{\dot a}{a} = \frac{2/3\, t^{-1/3}}{t^{2/3}} = \frac{2}{3} t^{-1} \Longrightarrow$ (5.53)

$$\ddot\delta + \frac{4}{3t}\dot\delta - \frac{2}{3t^2}\delta = 0. \tag{11.54}$$

If we guess that the solution to the above equation has the power-law form Dt^n, plugging this guess into the equation yields

$\frac{\partial}{\partial t} Dt^n = nDt^{n-1} \Longrightarrow$

$\frac{\partial^2}{\partial t^2} Dt^n = \frac{\partial}{\partial t} nDt^{n-1} = n(n-1)Dt^{n-2} \Rightarrow$

$$n(n-1)Dt^{n-2} + \frac{4}{3t}nDt^{n-1} - \frac{2}{3t^2}Dt^n = 0, \quad (/Dt^{n-2}) \tag{11.55}$$

or

$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0. \quad Dt^{-1} \quad Dt^{2/3} \tag{11.56}$$

The two possible solutions for this quadratic equation are $n = -1$ and $n = 2/3$. Thus, the general solution for the time evolution of density perturbations in a spatially flat, matter-only universe is

Growing mode / Decaying mode

"NORMAL" $t^{2/3}$ GROWTH IN MATTER DOMINATED ERA

$$\delta(t) \approx D_1 t^{2/3} + D_2 t^{-1}. \tag{11.57}$$

The values of D_1 and D_2 are determined by the initial conditions for $\delta(t)$. The decaying mode, $\propto t^{-1}$, eventually becomes negligibly small compared to the growing mode, $\propto t^{2/3}$. When the growing mode is the only survivor, the density perturbations in a flat, matter-only universe grow at the rate

$$\boxed{\delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z},} \tag{11.58}$$

as long as $|\delta| \ll 1$.

If baryonic matter were the only type of nonrelativistic matter in the universe, then density perturbations could have started to grow at $z_{\rm dec} = 1090$, and they could have grown in amplitude by a factor ~ 1090 by the present day. However, the dominant form of nonrelativistic matter is dark matter. The density perturbations in the dark matter started to grow effectively at $z_{rm} = 3440$. At the time of decoupling, the baryons fell into the preexisting gravitational wells of the dark matter. The situation is schematically illustrated in Figure 11.4. Having nonbaryonic dark matter allows the universe to get a "head start" on structure formation; perturbations in the matter density can start growing at $z_{rm} = 3440$ rather than $z_{\rm dec} = 1090$, as they would in a universe without dark matter.

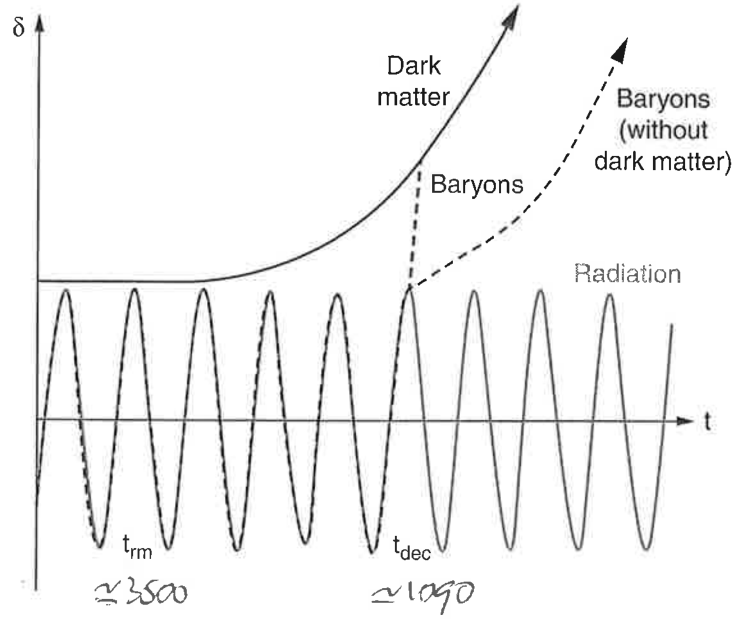

Figure 11.4 A highly schematic drawing of how density fluctuations in different components of the universe evolve with time.

11.4 The Power Spectrum

When deriving Equation 11.44, which determines the growth rate of density perturbations in a Newtonian universe, we assumed that the perturbation was spherically symmetric. In fact, Equation 11.44 and its relativistically correct brother, Equation 11.46, both apply to low-amplitude perturbations of any shape. This is fortunate, since the density perturbations in the real universe are not spherically symmetric. The bubbly structure shown in redshift maps of galaxies, such as Figure 11.1, has grown from the density perturbations that were present when the universe became matter dominated. Great oaks from tiny acorns grow – but then, great pine trees from tiny pinenuts grow. By looking at the current large scale structure (the "tree"), we can deduce the properties of the early, low-amplitude, density fluctuations (the "nut").[5]

Let us consider the properties of the early density fluctuations at some time t_i, when they were still very low in amplitude ($|\delta| \ll 1$). As long as the density fluctuations are small in amplitude, the expansion of the universe is still nearly isotropic, and the geometry of the universe is still well described by the Robertson–Walker metric (Equation 3.41):

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_\kappa(r)^2 d\Omega^2]. \tag{11.59}$$

Under these circumstances, it is useful to set up a comoving coordinate system. Choose some point as the origin. In a universe described by the Robertson–Walker

[5] At the risk of carrying the arboreal analogy too far, I should mention that the temperature fluctuations of the cosmic microwave background, as shown in Figure 8.3, offer us a look at the "sapling."

metric, as shown in Section 3.6, the proper distance of any point from the origin can be written in the form

(3.44)

$$d_p(t_i) = a(t_i)r, \tag{11.60}$$

where the comoving distance r is what the proper distance would be at the present day ($a = 1$) if the expansion continued to be perfectly isotropic. If we label each bit of matter in the universe with its comoving coordinate position $\vec{r} = (r, \theta, \phi)$, then $\vec{r}$ will remain very nearly constant as long as $|\delta| \ll 1$. Thus, when considering the regime where density fluctuations are small, cosmologists typically consider $\delta(\vec{r})$, the density fluctuation at a comoving location $\vec{r}$, at some time t_i. (The exact value of t_i doesn't matter, as long as it's a time after the density perturbations are in place, but before they reach an amplitude $|\delta| \sim 1$. Switching to a different value of t_i, under these restrictions, simply changes the amplitude of $\delta(\vec{r})$, and not its shape.)

When discussing the temperature fluctuations of the cosmic microwave background in Chapter 8, I pointed out that cosmologists weren't interested in the exact pattern of hot and cold spots on the last scattering surface, but rather in the statistical properties of the field $\delta T/T(\theta, \phi)$. Similarly, cosmologists are not interested in the exact locations of the density maxima and minima in the early universe, but rather in the statistical properties of the field $\delta(\vec{r})$. When studying the temperature fluctuations of the CMB, it is useful to expand $\delta T/T(\phi, \theta)$ in spherical harmonics. A similar decomposition of $\delta(\vec{r})$ is also useful. Since δ is defined in three-dimensional space rather than on the surface of a sphere, a useful expansion of δ is in terms of Fourier components.

Within a large box expanding along with the universe, of comoving volume V, the density fluctuation field $\delta(\vec{r})$ can be expressed as

$P_k = \delta_k \cdot e^{-i\vec{k}\cdot\vec{r}} \propto k^n \propto \lambda^{-n}$

$$\delta(\vec{r}) = \frac{V}{(2\pi)^3} \int \delta_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} d^3k, \tag{11.61}$$

$k \equiv \frac{2\pi}{\lambda} \Leftrightarrow \lambda = \frac{2\pi}{k}$

where the individual Fourier components $\delta_{\vec{k}}$ are found by performing the integral

$$\delta_{\vec{k}} = \frac{1}{V} \int \delta(\vec{r}) e^{i\vec{k}\cdot\vec{r}} d^3r. \tag{11.62}$$

In performing the Fourier transform, we are breaking up the function $\delta(\vec{r})$ into an infinite number of sine waves, each with comoving wavenumber $\vec{k}$ and comoving wavelength $\lambda = 2\pi/k$. If we have complete, uncensored knowledge of $\delta(\vec{r})$, we can compute all the Fourier components $\delta_{\vec{k}}$; conversely, if we know all the Fourier components, we can reconstruct the density field $\delta(\vec{r})$.

Each Fourier component is a complex number, which can be written in the form

$$\delta_{\vec{k}} = |\delta_{\vec{k}}| e^{i\phi_{\vec{k}}}. \tag{11.63}$$

When $|\delta_{\vec{k}}| \ll 1$, each Fourier component obeys Equation 11.49,

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - \frac{3}{2}\Omega_m H^2 \delta_{\vec{k}} = 0, \quad \Leftrightarrow \lambda_J \ll a(t)\frac{2\pi}{k} \ll c/H(t) = R(t) \tag{11.64}$$

as long as the proper wavelength, $a(t)2\pi/k$, is large compared to the Jeans length and small compared to the Hubble distance c/H. The phase $\phi_{\vec{k}}$ remains constant as long as the amplitude $|\delta_{\vec{k}}|$ remains small. Even after fluctuations with a short proper wavelength have reached $|\delta_{\vec{k}}| \sim 1$ and collapsed under their own gravity, the growth of the longer wavelength perturbations is still described by Equation 11.64. This means, helpfully enough, that we can use linear perturbation theory to study the growth of very large scale structures even after smaller structures, such as galaxies and clusters of galaxies, have already collapsed.

The mean square amplitude of the Fourier components defines the *power spectrum*:

$$P(k) = \langle |\delta_{\vec{k}}|^2 \rangle, \tag{11.65}$$

where the average is taken over all possible orientations of the wavenumber $\vec{k}$. (If $\delta(\vec{r})$ is isotropic, then no information is lost, statistically speaking, if we average the power spectrum over all angles.) When the phases $\phi_{\vec{k}}$ of the different Fourier components are uncorrelated with each other, then $\delta(\vec{r})$ is called a *Gaussian field*. If a Gaussian field is homogeneous and isotropic, then all its statistical properties are summed up in the power spectrum $P(k)$. If $\delta(\vec{r})$ is a Gaussian field, then the value of δ at a randomly selected point is drawn from the Gaussian probability distribution

$$p(\delta) = \frac{1}{\sqrt{2\pi}\,\sigma_\delta} \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right), \tag{11.66}$$

where the standard deviation σ_δ can be computed from the power spectrum:

(11.61) $P_k = \delta_{\vec{k}} \cdot e^{-i\vec{k}\cdot\vec{r}}$?} $\Rightarrow$

$$\sigma_\delta^2 = \frac{V}{(2\pi)^3}\int P(k)d^3k = \frac{V}{2\pi^2}\int_0^\infty P(k)k^2dk. \tag{11.67}$$

using $d^3k = 4\pi k^2 dk$ $\Downarrow$ $\frac{4\pi}{8\pi^3} = \frac{1}{2\pi^2}$

The study of Gaussian density fields is of particular interest to cosmologists because most inflationary scenarios predict that the density fluctuations created by inflation (Section 10.5) constitute an isotropic, homogeneous Gaussian field. In addition, it is expected that the power spectrum of inflationary density fluctuations can be well described by a power law, with

$$P(k) \propto k^n. \tag{11.68}$$

Theoretically, the value of the power-law index favored by inflation is $n \approx 1$. A power spectrum with $n = 1$ *exactly* is often referred to as a Harrison–Zel'dovich spectrum, after Edward Harrison and Yakov Zel'dovich, who independently proposed it as a physically plausible spectrum for density perturbations as early as 1970 (even before the concept of inflation entered cosmology). Observationally, the value of the power-law index n can be deduced from the temperature fluctuations of the cosmic microwave background on large scales ($\theta > 1.1°$). The temperature correlation function measured by the *Planck* satellite (Figure 8.6) is consistent with $n = 0.97 \pm 0.01$. Thus, the observed inflationary power spectrum is only slightly "tilted" relative to a Harrison–Zel'dovich spectrum.

What does a universe with $P(k) \propto k^n$ look like? Imagine going through such a universe and marking out randomly located spheres of comoving radius r. The mean mass of each sphere (considering only the nonrelativistic matter which it contains) will be

(11.27) $\rho_m = \frac{\varepsilon_m}{c^2}$ $\Rightarrow$

$$\langle M \rangle = \frac{4\pi}{3} r^3 \rho_{m,0} = 1.67 \times 10^{11}\,\mathrm{M}_\odot \left(\frac{r}{1\,\mathrm{Mpc}}\right)^3. \qquad (11.69)$$

However, the actual mass of each sphere will vary; some spheres will be slightly underdense, and others will be slightly overdense. The mean square density fluctuation of the mass inside each sphere is a function of the power spectrum and of the size of the sphere:

$$\left(\frac{\delta M}{M}\right)^2 \simeq \left\langle \left(\frac{M - \langle M \rangle}{\langle M \rangle}\right)^2 \right\rangle = \frac{V}{2\pi^2} \int P(k) \left[\frac{3 j_1(kr)}{kr}\right]^2 k^2 dk$$
$$= \frac{9V}{2\pi^2 r^2} \int P(k) j_1(kr)^2 dk, \qquad (11.70)$$

where $j_1(x) = (\sin x - x \cos x)/x^2$ is a spherical Bessel function of the first kind.

In general, to compute the mean square mass fluctuation, you must know the power spectrum $P(k)$. However, if the power spectrum has the simple form $P(k) \propto k^n$, we can substitute a new variable of integration, $u = kr$, to find

$$\left\langle \left(\frac{M - \langle M \rangle}{\langle M \rangle}\right)^2 \right\rangle = \frac{9V}{2\pi^2} r^{-3-n} \int_0^\infty u^n j_1(u)^2 du. \qquad (11.71)$$

Thus, the root mean square mass fluctuation within spheres of comoving radius r will have the dependence

$r \Rightarrow \lambda$
$r = \frac{2\pi}{k} \Leftrightarrow k = \frac{2\pi}{r}$
$\Leftrightarrow \left\langle \left(\frac{M-\langle M\rangle}{M}\right)^2 \right\rangle \sim k^3 P_k \propto r^{-3} r^{-n}$

$$\frac{\delta M}{M} \equiv \left\langle \left(\frac{M - \langle M \rangle}{\langle M \rangle}\right)^2 \right\rangle^{1/2} \propto r^{-(3+n)/2} \propto M^{-(3+n)/6} \qquad (11.72) \sim M^{-2/3}$$

for $n = 1$

$P_K = k^n \propto r^{-n}$

This can also be expressed in the form $\delta M/M \propto M^{-(3+n)/6}$. (If you scattered point masses randomly throughout the universe, so that they formed a Poisson distribution, you would expect mass fluctuations of amplitude $\delta M/M \propto N^{-1/2}$, where N is the expected number of point masses within the sphere. Since the average mass within a sphere is $M \propto N$, a Poisson distribution has $\delta M/M \propto M^{-1/2}$, or $n = 0$. The Harrison–Zel'dovich spectrum, with $n = 1$, thus will produce more power on small length scales than a Poisson distribution of points.) Note that the potential fluctuations associated with mass fluctuations on a length scale r will have an amplitude $\delta\Phi \sim G\delta M/r \propto \delta M/M^{1/3} \propto M^{(1-n)/6}$. Thus, the Harrison–Zel'dovich spectrum, with $n = 1$, is the only power law that prevents the divergence of the potential fluctuations on both large and small scales. For this reason, the Harrison–Zel'dovich spectrum is also referred to as a *scale invariant* power spectrum of density perturbations.

11.5 Hot versus Cold

Immediately after inflation, the power spectrum for density perturbations has the form $P(k) \propto k^n$, with an index $n \approx 1$. However, the shape of this primordial power spectrum will be modified between the end of inflation at t_f and the time of radiation–matter equality at $t_{rm} \approx 5.0 \times 10^4$ yr. The shape of the power spectrum at t_{rm}, when density perturbations start to grow significantly in amplitude, depends on the properties of the dark matter. More specifically, it depends on whether the dark matter is predominantly *cold dark matter* or *hot dark matter.*

$a_{rm} \simeq 1/3450$
$z_{rm} \simeq 3450$

Saying that dark matter is "cold" or "hot" is a statement about the thermal velocity of the dark matter particles. More specifically, it's a statement about the thermal velocity at a particular time in the history of the universe; the time when the mass within a Hubble volume (that is, a sphere of proper radius c/H) was equal to the total mass of a large galaxy like our own, $M_{\text{gal}} \approx 10^{12}\,\text{M}_\odot$. In the radiation-dominated universe, the Hubble parameter was, from Equation 5.81,

(5.81) = Fr Eq with $\Omega_{r,0} \neq 0$, $\Omega_m = \Omega_\Lambda = 0 = k$

$$H(a) = H_0 \frac{\sqrt{\Omega_{r,0}}}{a^2}. \qquad (11.73)$$

This means the mass within a Hubble volume during the early, radiation-dominated era was

$R_H = c/H$ using (11.73)

$$M_H = \frac{4\pi}{3}\left(\frac{c}{H_0}\frac{a^2}{\sqrt{\Omega_{r,0}}}\right)^3 \frac{\rho_{m,0}}{a^3} = 1.6 \times 10^{28}\,\text{M}_\odot\, a^3, \qquad (11.74)$$

using the parameters of the Benchmark Model. The mass within a Hubble volume was equal to $M_{\text{gal}} \approx 10^{12}\,\text{M}_\odot$ at a scale factor $a \approx 4 \times 10^{-6}$, corresponding to a temperature $kT \approx 60\,\text{eV}$ and a cosmic time $t \approx 8 \times 10^{-10} H_0^{-1} \approx 12\,\text{yr}$. Particles with a rest energy $mc^2 \ll 3kT \sim 180\,\text{eV}$ were highly relativistic at this time, with a particle energy $E \sim 3kT \sim 180\,\text{eV}$. Particles with a rest energy $mc^2 \gg 180\,\text{eV}$ were highly nonrelativistic at this time, with a particle energy E almost entirely contributed by their rest energy mc^2.

Cold dark matter consists of particles that were nonrelativistic at $t \approx 12\,\text{yr}$, either because they were nonrelativistic at the time they decoupled from photons and baryonic matter, or because they have $mc^2 \gg 3kT \sim 180\,\text{eV}$, and thus had cooled to nonrelativistic thermal velocities by $t \approx 12\,\text{yr}$. For instance, WIMPs decoupled at $t \sim 1\,\mathrm{s}$, when the universe had a mean particle energy $\sim 3kT \sim 3\,\mathrm{MeV}$. However, WIMPs are predicted to have a mass $mc^2 \sim 100\,\mathrm{GeV}$. With a thermal energy much smaller than their rest energy at the time of decoupling, WIMPs were already nonrelativistic when they decoupled, and thus qualify as cold dark matter. Axions are a type of elementary particle first proposed by particle physicists for noncosmological purposes. If they exist, however, they would have formed out of equilibrium in the early universe, with very low thermal velocities. Thus, axions would act as cold dark matter as well.

Hot dark matter, by contrast, consists of particles that were *relativistic* at the time they decoupled from the other components of the universe, and that remained relativistic until $t \gg 12\,\mathrm{yr}$, when the mass inside the Hubble volume was large compared to the mass of a galaxy. For instance, neutrinos, like WIMPs, decoupled at $t \sim 1\,\mathrm{s}$, when the mean particle energy was $\sim 3\,\mathrm{MeV}$. Thus, a neutrino with mass $m_\nu c^2 \ll 3\,\mathrm{MeV}$ was hot enough to be relativistic at the time it decoupled. Moreover, a neutrino with mass $5 \times 10^{-4}\,\mathrm{eV} \ll m_\nu c^2 \ll 180\,\mathrm{eV}$ was relativistic at $t \approx 12\,\mathrm{yr}$, but is nonrelativistic today, at $t_0 \approx 13.7\,\mathrm{Gyr}$. Thus, neutrinos in this mass range qualify as hot dark matter.

To see how the existence of hot dark matter modifies the spectrum of density perturbations, consider what would happen in a universe filled with hot dark matter consisting of particles with mass m_h. The initially relativistic hot dark matter particles cool as the universe expands, until their thermal velocities drop to well below c when $3kT \approx m_h c^2$. This happens at a temperature

$$T_h \approx \frac{m_h c^2}{3k} \approx 12\,000\,\mathrm{K}\left(\frac{m_h c^2}{3\,\mathrm{eV}}\right). \qquad (11.75)$$

For a particle mass $m_h c^2 > 2.4\,\mathrm{eV}$, the particles become nonrelativistic when the universe is still radiation-dominated. In a radiation-dominated universe, the temperature of Equation 11.75 corresponds to a cosmic time

$T = T_0(1+z) = T_0/a$

$$t_h \approx 42\,000\,\mathrm{yr}\left(\frac{m_h c^2}{3\,\mathrm{eV}}\right)^{-2} \qquad (11.76)$$

since and $a \propto t^{1/2} \to T \propto t^{-1/2}$

$t \propto T^{-2}$

Prior to the time t_h, the hot dark matter particles move freely in random directions with a speed close to that of light. This motion, called *free streaming*, acts to wipe out any density fluctuations present in the hot dark matter. Thus, the net effect of free streaming in the hot dark matter is to wipe out any density fluctuations whose wavelength is smaller than $\sim ct_h$. When the hot dark matter particles become nonrelativistic, there will be no density fluctuations on scales smaller than the physical scale

$\frac{42{,}000}{3.26} = 13\,\mathrm{K}$

$$d_{\min} \approx ct_h \approx 13\,\mathrm{kpc}\left(\frac{m_h c^2}{3\,\mathrm{eV}}\right)^{-2}, \qquad (11.77)$$

corresponding to a comoving length scale

$$r_{\rm min} = \frac{d_{\rm min}}{a(t_h)} \approx \left(\frac{T_h}{2.7255\,{\rm K}}\right) d_{\rm min} \approx 55\,{\rm Mpc}\left(\frac{m_h c^2}{3\,{\rm eV}}\right)^{-1}. \tag{11.78}$$

The total amount of matter within a sphere of comoving radius $r_{\rm min}$ is

$$M_{\rm min} = \frac{4\pi}{3} r_{\rm min}^3 \rho_{m,0} \approx 2.7 \times 10^{16}\,{\rm M}_\odot \left(\frac{m_h c^2}{3\,{\rm eV}}\right)^{-3}. \tag{11.79}$$

Hot dark matter particles with $m_h c^2 \sim 50\,{\rm eV}$ will smear out fluctuations smaller than the Local Group, particles with $m_h c^2 \sim 8\,{\rm eV}$ will rub out fluctuations smaller than the Coma cluster, and particles with $m_h c^2 \sim 3\,{\rm eV}$ will wipe away fluctuations smaller than superclusters. ⟺ CDM ⟺ HDM

The left panel of Figure 11.5 shows the power spectrum of density fluctuations in hot dark matter, once the hot dark matter particles have cooled enough to become nonrelativistic; the plot assumes that $m_h c^2 = 3\,{\rm eV}$ and thus $r_{\rm min} \approx 55\,{\rm Mpc}$. For wavenumbers $k \ll 2\pi/r_{\rm min} \approx 0.1\,{\rm Mpc}^{-1}$, the power spectrum of hot dark matter (shown as the dotted line) is close to the original $P \propto k$ ($n=1$) spectrum (shown as the dashed line). However, the free streaming of the hot dark matter results in a severe loss of power for wavenumbers $k \gg 2\pi/r_{\rm min}$. The right panel of Figure 11.5 shows the root mean square mass fluctuations, $\delta M/M$, calculated using Equation 11.70. For the original $P \propto k$ spectrum (dashed line), the amplitude of density fluctuations is larger on smaller mass scales, with $\delta M/M \propto M^{-2/3}$. However, for the hot dark matter power spectrum (dotted line),

for $n = 1 = n_{H-Z}$

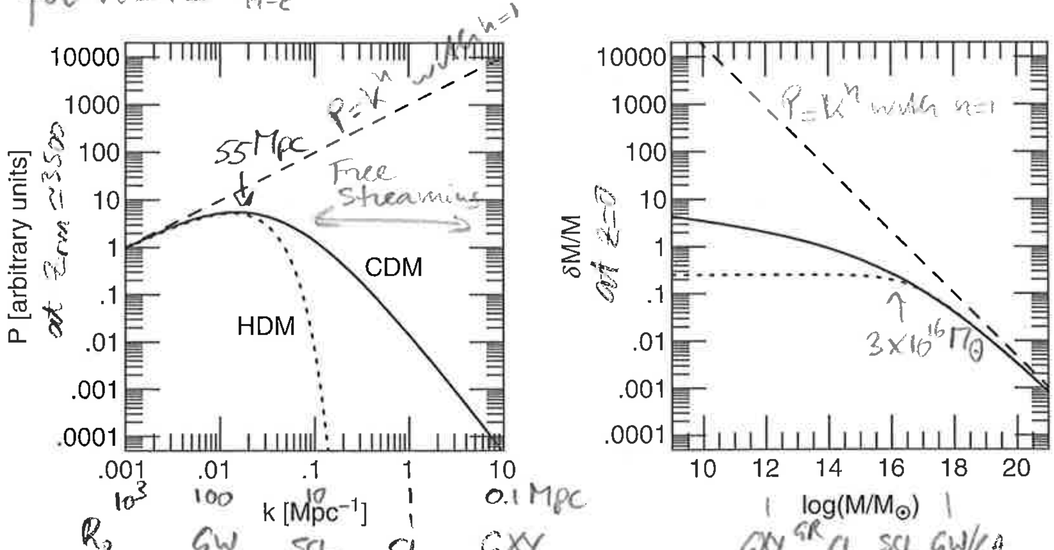

Figure 11.5 Left panel: Power spectrum at the time of radiation–matter equality for cold dark matter (solid line) and for hot dark matter (dotted line). The initial power spectrum produced by inflation (dashed line) is assumed to have the form $P(k) \propto k$. Right panel: The root mean square mass fluctuations $\delta M/M$ are shown as a function of M. Line types are the same as in the left panel.

the value of $\delta M/M$ levels off on scales smaller than $M < M_{\rm min} \sim 3 \times 10^{16}\,{\rm M}_\odot$. In a universe with this hot dark matter power spectrum, the first structures to collapse are regions of mass $M \sim M_{\rm min}$ with nearly uniform overdensity. These regions collapse to form superclusters; smaller structures, such as clusters and galaxies, then form by fragmentation of the superclusters. This scenario, in which the largest observable structures form first, is called the *top-down* scenario.

If most of the dark matter in the universe were hot dark matter with $m_h c^2 \sim 3\,{\rm eV}$, then we would expect the oldest structures in the universe to be superclusters, and that galaxies would be relatively young. In fact, the opposite seems to be true in our universe. Superclusters are just collapsing today, while galaxies have been around since at least $z \sim 10$, when the universe was about half a gigayear old. Thus, most of the dark matter in the universe must be *cold* dark matter, for which free streaming has been negligible.

The evolution of the power spectrum of cold dark matter, given the absence of free streaming, is different from the evolution of the power spectrum for hot dark matter. Remember, when the universe is radiation-dominated, density fluctuations $\delta_{\vec{k}}$ in the dark matter do not grow significantly in amplitude, as long as their proper wavelength $a(t)2\pi/k$ is small compared to the Hubble distance $c/H(t)$. However, when the proper wavelength of a density perturbation is large compared to the Hubble distance, its amplitude will be able to increase, regardless of whether the universe is radiation-dominated or matter-dominated. If the cold dark matter consists of WIMPs, they decouple from the radiation at a time $t_d \sim 1\,{\rm s}$, when $z \simeq 3\times10^9$ the scale factor is $a_d \sim 3 \times 10^{-10}$. At the time of WIMP decoupling, the Hubble distance is $c/H \sim 2ct_d \sim 6 \times 10^8\,{\rm m}$. This corresponds to a comoving length scale

$$r_d = \frac{2ct_d}{a_d} \sim 60\,{\rm pc}, \simeq \frac{2\times10^{18}\,{\rm m}}{3.1\times10^{16}\,{\rm m/pc}} \tag{11.80}$$

a mass scale

$$M_d = \frac{4\pi}{3} r_d^3 \rho_{m,0} \sim 0.05\,{\rm M}_\odot, \tag{11.81}$$

and a comoving wavenumber $k_d \sim 2\pi/r_d \sim 10^5\,{\rm Mpc}^{-1}$. Thus, density fluctuations with a wavenumber $k < k_d$ and a mass $M > M_d$ will have a wavelength greater than the Hubble distance at the time of WIMP decoupling, and will be able to grow freely in amplitude as long as their wavelength remains longer than the Hubble distance.

How the fluctuations grow once their wavelength is smaller than the Hubble distance depends on whether that transition happens before or after the time of radiation–matter equality $t_{rm} = 0.050\,{\rm Myr}$, when the scale factor was $a_{rm} = 2.9 \times 10^{-4}$. $z \simeq 3500$ At the time of radiation–matter equality, the Hubble distance was $c/H \approx 1.8ct_{rm} \approx 0.027\,{\rm Mpc}$. This corresponds to a comoving length scale

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$$r_{rm} = \frac{1.8ct_{rm}}{a_{rm}} \approx 90\,\text{Mpc}, \tag{11.82}$$

a mass scale $M_{rm} \approx 1.3 \times 10^{17}\,\text{M}_\odot$, and a comoving wavenumber $k_{rm} = 2\pi/r_{rm} \approx 0.07\,\text{Mpc}^{-1}$. Density fluctuations with a wavenumber $k < k_{rm}$ and a mass $M > M_{rm}$ will remain larger than the Hubble distance during the entire radiation-dominated era, and will grow steadily in amplitude during all that time. Thus, for wavenumbers $k < k_{rm} \approx 0.07\,\text{Mpc}^{-1}$, the power spectrum for cold dark matter retains the original $P(k) \propto k$ form that it had immediately after inflation (see the left panel of Figure 11.5). By contrast, cold dark matter density perturbations with a wavenumber $k_d > k > k_{rm}$ will grow in amplitude only until their physical wavelength $a(t)r \propto t^{1/2}$ is smaller than the Hubble distance $c/H(t) \propto t$. At that time, their amplitude is frozen until the time t_{rm}, when matter dominates, and density perturbations smaller than the Hubble distance are free to grow again. Thus, for wavenumbers $k > k_{rm}$, the power spectrum for cold dark matter is suppressed in amplitude, with the suppression being greatest for the largest wavenumbers (corresponding to shorter wavelengths, which become smaller than the Hubble distance at an earlier time).

The left panel of Figure 11.5 shows, as the solid line, the power spectrum for cold dark matter (CDM) at the time of radiation–matter equality. Note the broad maximum in the power spectrum at $k \sim k_{rm} \sim 0.1\,\text{Mpc}^{-1}$. The root mean square mass fluctuations in the cold dark matter, $\delta M/M$, are shown as the solid line in the right panel of Figure 11.5. The amplitude of the fluctuations is normalized so that $\delta M/M = 1$ at $M = 10^{14}\,\text{M}_\odot$. This normalization gives agreement with the observed density fluctuations today on very large scales, where $\delta M/M < 1$, and the growth of density perturbations is still in the linear regime. At the time of radiation–matter equality, the amplitude of the density fluctuations was smaller, with $(\delta M/M)_{rm}$ equal to $\sim a_{rm} \sim 3 \times 10^{-4}$ times the value of $\delta M/M$ shown in Figure 11.5. Notice that the mass fluctuations in the CDM scenario are largest in amplitude for the smallest mass scales. This implies that in a universe filled with cold dark matter, the first objects to form are the smallest, with galaxies forming ~~first~~ next, then clusters, then superclusters. This scenario, called the *bottom-up* scenario, is consistent with the observed relative ages of galaxies and superclusters.

$z_{rm} \simeq 3500$

$\frac{\delta M}{M}$	$\frac{M_{TOT}}{M_\odot}$	GROWTH
$\lesssim 1$	$\gtrsim 10^{14}$	LINEAR
$\gg 1$	$\lesssim 10^{14}$	NON-LIN

GALAXIES IN GROUPS

Stars first

Assuming that the dark matter consists of nothing but hot dark matter gives a poor fit to the observed large scale structure of the universe. However, there is strong evidence that neutrinos do have some mass, and thus that the universe contains at least *some* hot dark matter. Cosmologists studying the large scale structure of the universe can adjust the assumed power spectrum of the dark matter, by mixing together hot and cold matter. Comparison of the assumed power spectrum with the observed large scale structure indicates that $\Omega_{\text{hdm},0} \leq 0.007$. If the hot dark matter consists entirely of the three standard flavors of neutrino in the cosmic neutrino background, this implies an upper limit on the sum of the neutrino masses, with (Equation 2.26)

$$[m(\nu_e) + m(\nu_\mu) + m(\nu_\tau)]c^2 \leq 0.3\,\text{eV}. \tag{11.83}$$

If there were more hot dark matter than this amount, free streaming of the hot dark matter particles would make the universe too smooth on small scales. Some like it hot, but most like it cold – the majority of the dark matter in the universe must be *cold* dark matter.

Cosmologists frequently refer to the "ΛCDM model" for the universe. This is the particular variant of the Hot Big Bang model in which the dominant contributors to the energy density today are dark energy in the form of a cosmological constant (this is the "Λ" of ΛCDM) and nonrelativistic matter in the form of cold dark matter (this is the "CDM" of ΛCDM). The basic ΛCDM model is spatially flat. The Greco-Roman abbreviation "ΛCDM" was first used in the mid-1990s; in the years since then, the accumulation of evidence has led the ΛCDM model to be adopted as the standard model of cosmology.

11.6 Baryon Acoustic Oscillations

The cold dark matter power spectrum shown as the solid line in Figure 11.5 assumes that the only density fluctuations are the quantum fluctuations from the inflationary era, as modified by gravitationally-driven growth. This would be an excellent assumption if the only matter in the universe were cold dark matter. Things become complicated, though, when we consider that 15 percent of the matter in the universe is baryonic. As far as gravity is concerned, a kilogram of hydrogen is the same as a kilogram of WIMPs; however, baryonic matter has the additional ability to interact with photons through the electromagnetic force. This ability significantly changes the amplitude of mass fluctuations in the universe. On relatively small mass scales, the ability of baryonic matter to cool by emitting photons permits the formation of high-density galaxies on a mass scale $M < 10^{12}\,\text{M}_\odot$ and the formation of ultra-high-density stars on a mass scale $M < 100\,\text{M}_\odot$. On mass scales bigger than the largest supercluster, $M > 10^{17}\,\text{M}_\odot$, the ability of baryons to interact with photons has also placed its mark on the power spectrum of density perturbations. It has done so through the mechanism of *baryon acoustic oscillations*.

To understand the origin of baryon acoustic oscillations, go back in time to the era when baryons decoupled from photons, at a redshift $z_{\text{dec}} \approx z_{\text{ls}} \approx 1090$. As we noted in Section 8.5, the densest regions in the photon–baryon fluid at the time of last scattering were regions that had just managed to compress themselves to maximum density before rebounding under their own pressure. The size of these high density regions was comparable to the sound horizon distance at the time of last scattering. From Equation 8.65, the sound horizon distance had a physical length[6]

$$d_s(t_{\rm ls}) = 0.145\,\text{Mpc}. \qquad (11.84)$$

After the photons and baryons parted ways, the overdensity of the photons in these high density regions reveals itself as hot spots on the last scattering surface, with a characteristic angular size $\theta \approx \theta_s \approx 0.7°$. (see Fig 8.6)

However, just as the overdense photons left an observable mark on the universe, the overdense baryons left their mark as well. A physical size of $d_s = 0.145$ Mpc at a redshift $z_{\rm ls} = 1090$ corresponds to a comoving length

$$r_s = d_s(t_{\rm ls})(1 + z_{\rm ls}) \approx 160\,\text{Mpc}. \qquad (11.85)$$

This comoving length, called the *acoustic scale*, in turn corresponds to a mass scale

(11.79) ? $m_h c^2 \sim 1$ eV $\Rightarrow$

$$M_s = \frac{4\pi}{3} r_s^3 \rho_{m,0} \approx 7 \times 10^{17}\,\text{M}_\odot. \qquad (11.86)$$

The density fluctuations that are present on a comoving scale $r_s \approx 160$ Mpc are called "baryon acoustic oscillations" since they are the baryonic manifestation of the acoustic oscillations that were present in the photon–baryon fluid just before decoupling. Immediately after decoupling, the baryon overdensity on the scale r_s was small in amplitude. However, the Matthew effect has caused the initially low-amplitude density fluctuations to grow to detectable levels at the present day. (Despite this growth, the density fluctuations on a length scale $r_s \approx 160$ Mpc are still not high enough in amplitude to destroy our assumption that the universe is homogeneous and isotropic on scales larger than 100 Mpc.)

The best way to detect low-amplitude overdensities on a scale as large as 160 Mpc is to look at the *correlation function* of galaxies in space. Suppose that the average number density of galaxies at the present day is $n_{\rm gal}$. Choose a galaxy at random, then look at a small volume dV at a comoving distance r from the chosen galaxy. The correlation function $\xi(r)$ is a way of telling you how many galaxies are likely to be found in that small volume dV. Expressed mathematically, the expected number dN is given by the relation

$$dN = n_{\rm gal}[1 + \xi(r)]dV. \qquad (11.87)$$

If the galaxies have a Poisson distribution, then the correlation function is random $\Leftarrow$ $\xi(r) = 0$; the number of galaxies you expect to find is simply the number density $n_{\rm gal}$ times the volume dV. However, galaxies emphatically do not have a Poisson distribution in space, as Figure 11.1 illustrates; instead, they tend to cluster, giving a correlation function ξ that is positive at small separations.

[6] Given the existence of an inflationary epoch, we must slightly redefine the sound horizon distance as the distance traveled by sound since the universe was refilled with particles at $t \sim t_f \sim 10^{-34}$ s, during the reheating period at the end of inflation.

angular

When discussing the correlation function $C(\theta)$ of temperature fluctuations in the cosmic microwave background, we found it useful to break it down, with the use of spherical harmonics, into different multipole moments l. Similarly, it is useful to break down the correlation function $\xi(r)$ for galaxies, with the use of Fourier transforms, into different wavenumbers k. In particular, we can express the correlation function $\xi(r)$ in terms of the statistical properties of the initial mass fluctuations in the universe. On sufficiently large scales, the galaxy distribution is a mapping of the mass density distribution $\delta(\vec{r})$; that is, the number density of galaxies is highest in regions where the initial overdensity δ was largest. For a Gaussian density field $\delta(\vec{r})$, the correlation function $\xi(r)$ is simply the Fourier transform of the power spectrum $P(k)$:

$\left.\begin{matrix}(11.67)+\\ P_k = \delta_k \cdot e^{-i\vec{k}\cdot\vec{r}}\end{matrix}\right\} \Rightarrow$

$$\xi(r) = \frac{V}{(2\pi)^3}\int P(k)e^{-i\vec{k}\cdot\vec{r}}d^3k. \qquad (11.88)$$

For an isotropic power spectrum $P(k)$, this can be written as

à la $\left.\begin{matrix}(11.67)/\\ (11.70)\end{matrix}\right\} \Rightarrow$

$$\xi(r) = \frac{V}{2\pi^2}\int P(k)j_0(kr)k^2dk, \qquad (11.89)$$

where $j_0(x) = \sin x/x$ is yet another spherical Bessel function. Since the power spectrum $P(k)$ and the correlation function $\xi(r)$ are Fourier transforms of each other, they contain the same information. Which one you chose in a given situation depends on convenience. If you are doing a redshift survey of galaxies, it is generally more convenient to compute the correlation function $\xi(r)$ of the galaxies' distribution in space.

The acoustic scale is $r_s \approx 160\,\text{Mpc} \approx 0.04c/H_0$. Thus, a redshift survey of galaxies that goes to redshifts significantly greater than $z \approx 0.04$ should have the ability, if it contains enough galaxies, to observe an extra bump in the correlation function $\xi(r)$ at a comoving scale r_s, reflecting the presence of baryonic acoustic oscillations in the distribution of matter. Figure 11.6 shows the correlation function $\xi(r)$ for a sample of nearly a million luminous galaxies in a redshift range $0.4 < z < 0.7$, near the end of the matter-dominated era. Notice the enhancement of $\xi(r)$ at a comoving length scale $r \sim r_s \sim 160\,\text{Mpc}$. This "BAO bump" is a modern-day relic of the fluctuations that were present in the photon–baryon fluid at the time of last scattering, $t_{\text{ls}} \approx 0.37\,\text{Myr}$.

Baryon acoustic oscillations provide a useful tool for computing cosmological parameters. Since the acoustic scale is a standard yardstick, whose length is determined by well-understood physical principles, it can be used, for instance, to probe the properties of dark energy. By seeing how the angular size of the BAO bump varies with redshift during the recent dark-energy-dominated era, we can determine (once we have enough data) whether the dark energy is a simple cosmological constant with $w = -1$, or whether it has a different value of w. In addition (once we have enough data), we can determine whether w for dark energy is changing with time.

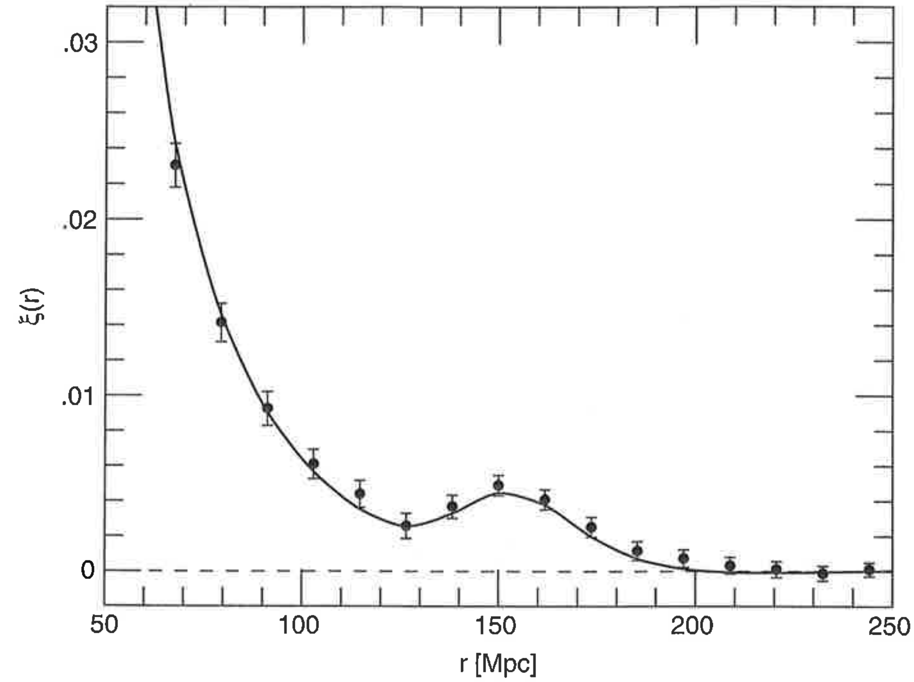

Figure 11.6 The correlation function as a function of comoving radius r. Points with error bars are the data for a large sample of luminous galaxies from the Sloan Digital Sky Survey. The solid line is a model including the effects of baryon acoustic oscillations. [data from Anderson *et al.* 2014, *MNRAS*, **441**, 24]

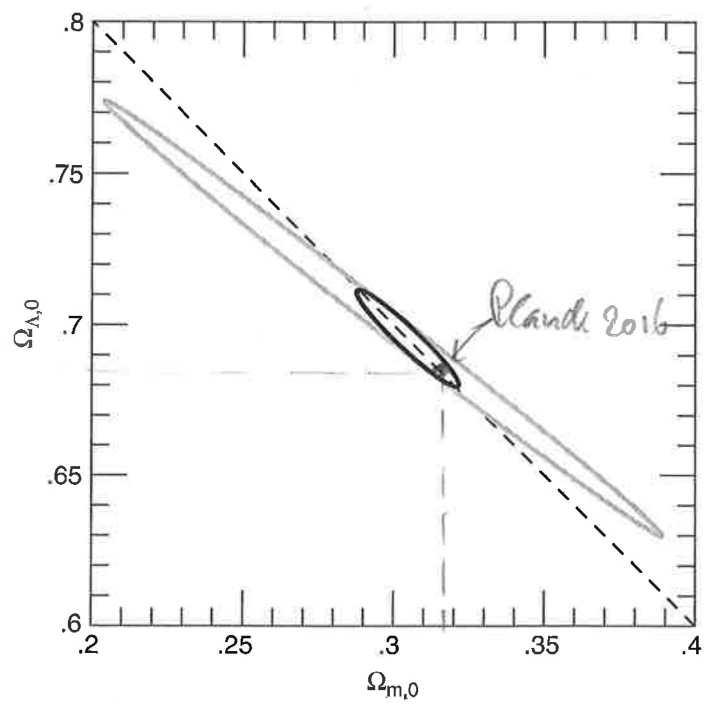

Figure 11.7 The gray elliptical contour gives the 95% confidence interval for $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ from the joint supernova and CMB data (compare with Figure 8.7). The black elliptical contour is the 95% confidence interval when the baryon acoustic oscillation observations are added. [Anže Slosar & José Alberto Vázquez, BNL]

If we assume, for the moment, that the ΛCDM model is correct, and that $w = -1$ for the dark energy, the location and amplitude of the BAO bump in Figure 11.6 enable us to place further constraint on the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$. For instance, in Figure 11.7, the gray contour gives the values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ that are consistent, at the 95% confidence level, with the type Ia supernova results and the CMB temperature fluctuations observed by *Planck*. The smaller black ellipse gives the 95% confidence interval once the BAO measurements are added to the supernova and CMB measurements. The tiny size of the black ellipse (centered on $\Omega_{m,0} = 0.305$ and $\Omega_{\Lambda,0} = 0.696$ for this particular combination of data) is the result of a happy concurrence among the three sources of information, and is the basis for the "Benchmark Model" that we have been using in this text. The fact that the black ellipse clings so closely to the $\kappa = 0$ line is also the source of the $|\Omega_0 - 1| \leq 0.005$ limit that we used when discussing the flatness problem in Section 10.1.

2016 Planck
$\Omega_{m,0} = 0.315$
$\Omega_{\Lambda,0} = 0.685$

Exercises

11.1 Consider a spatially flat, matter-dominated universe ($\Omega = \Omega_m = 1$) that is *contracting* with time. What is the functional form of $\delta(t)$ in such a universe?

11.2 Consider an empty, negatively curved, expanding universe, as described in Section 5.2. If a dynamically insignificant amount of matter ($\Omega_m \ll 1$) is present in such a universe, how do density fluctuations in the matter evolve with time? That is, what is the functional form of $\delta(t)$?

11.3 A volume containing a photon–baryon fluid is adiabatically expanded or compressed. The energy density of the fluid is $\varepsilon = \varepsilon_\gamma + \varepsilon_{\text{bary}}$, and the pressure is $P = P_\gamma = \varepsilon_\gamma/3$. What is $dP/d\varepsilon$ for the photon–baryon fluid? What is the sound speed, c_s? In Equation 11.26, how large an error did we make in our estimate of λ_J(before) by ignoring the effect of the baryons on the sound speed of the photon–baryon fluid?

11.4 Suppose that the stars in a disk galaxy have a constant orbital speed v out to the edge of its spherical dark halo, at a distance R_{halo} from the galaxy's center. If a bound structure, such as a galaxy, forms by gravitational collapse of an initially small density perturbation, the minimum time for collapse is $t_{\text{min}} \approx t_{\text{dyn}} \approx 1/\sqrt{G\bar{\rho}}$. Show that $t_{\text{min}} \approx R_{\text{halo}}/v$ for a disk galaxy. What is t_{min} for our own galaxy? What is the maximum possible redshift at which you would expect to see galaxies comparable in v and R_{halo} to our own galaxy? (Assume the Benchmark Model is correct.)

11.5 Within the Coma cluster, as discussed in Section 7.3, galaxies have a root mean square velocity of $\langle v^2 \rangle^{1/2} \approx 1520\,\text{km}\,\text{s}^{-1}$ relative to the center of mass of the cluster; the half-mass radius of the Coma cluster is $r_h \approx 1.5\,\text{Mpc}$. Using arguments similar to those of the previous problem, compute the