Jake Summers - 1221695592 - HW7 - AST322 - 3/24/2022

5.5a) Derive Equations 5.53 - 5.57 from 5.36 - 5.52 for $w = 0$; plot $d_p(t_0)$ and $d_p(t_e)$ vs. z. From Equation 5.42, we have the following, where we can plug in $w = 0$:

$$t_0 = \frac{2}{3(1+w)} H_0^{-1} = \boxed{\frac{2}{3H_0}} \tag{1}$$

From Equation 5.52, we have the horizon distance, where we can plug in $w = 0$:

$$d_{hor}(t_0) = ct_0 \frac{3(1+w)}{1+3w} = \frac{c}{H_0}\frac{2}{1+3w} \boxed{= 3ct_0 = \frac{2c}{H_0}} \tag{2}$$

From Equation 5.39, we have an equation for the scale factor as a function of time, into which we can plug $w = 0$:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)} = \boxed{\left(\frac{t}{t_0}\right)^{2/3}} \tag{3}$$

Equation 5.50 gives a formula for the current proper distance, which can be solved for a matter-only universe:

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}[1-(1+z)^{-(1+3w)/2}] \boxed{= \frac{2c}{H_0}[1-(1+z)^{-1/2}] = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]} \tag{4}$$

Finally, we can use Equation 5.36 to derive Equation 5.57:

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} \boxed{= \frac{2c}{H_0(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right]} \tag{5}$$

Figure 1 shows the plots of $d_p(t_0)$ and $d_p(t_e)$ vs. z.

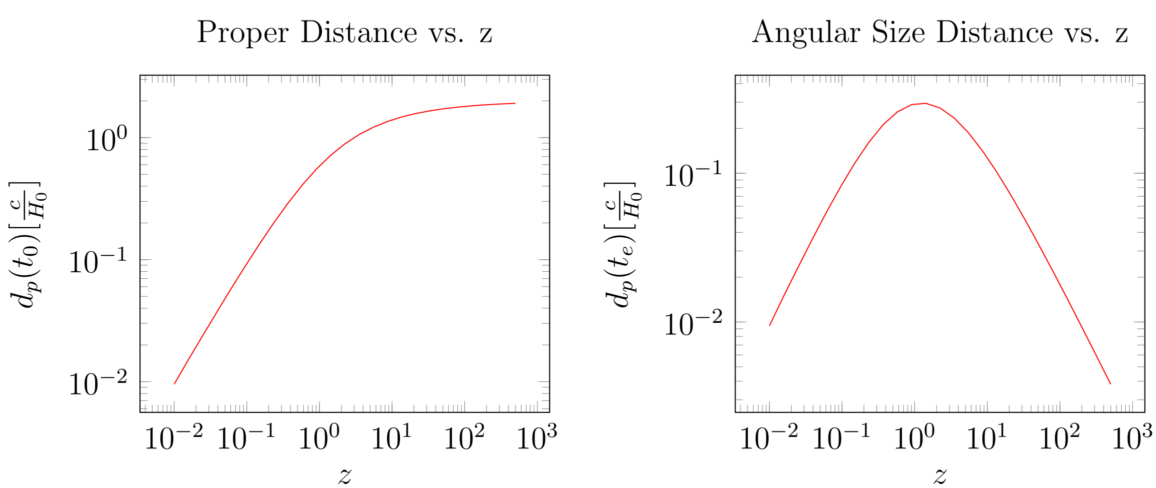

FIGURE 1. Proper distance and Angular Size Distance for a matter-dominated universe as a function of redshift

To find the maximum angular size distance, we can utilize the following equation:

$$d_p(t_e) = \frac{c}{H_0(1+z)}\frac{2}{1+3w}[1-(1+z)^{-(1+3w)/2}] \tag{6}$$

We can then differentiate Equation 6 and set its derivative equal to zero:

$$\begin{aligned}\frac{\partial d_p(t_e)}{\partial z} = 0 &= \frac{2c}{H_0(1+3w)} \cdot \frac{\partial}{\partial z}\left[(1+z)^{-1} - (1+z)^{\frac{-3-3w}{2}}\right] \\ &= \frac{2c}{H_0(1+3w)}\left[\frac{3+3w}{2}(1+z)^{\frac{-5-3w}{2}} - (1+z)^{-2}\right] \\ &= \frac{3+3w}{2}(1+z)^{\frac{-5-3w}{2}} - (1+z)^{-2}\end{aligned}$$

We can then rearrange the above equation and multiply by $(1+z)^2$:

$$1 = \frac{3+3w}{2}(1+z)^{\frac{-5-3w}{2}+2} \Rightarrow \frac{2}{3+3w} = (1+z)^{\frac{-1-3w}{2}} \tag{7}$$

For a matter-only universe, we have the following:

$$\frac{2}{3} = (1+z)^{-1/2} \Rightarrow z_{max} = (2/3)^{-2} - 1 = \boxed{1.25 = z_{max}} \tag{8}$$

From this, we can then calculate the maximum value of $d_p(t_e)$:

$$d_p(t_e)_{max} = \frac{2c}{H_0 \cdot 2.25}\left[1 - \frac{1}{\sqrt{2.25}}\right] \approx \boxed{0.296\frac{c}{H_0}} \tag{9}$$

5.5b) Derive Equations 5.58 - 5.62 from 5.36 - 5.52 for $w = 1/3$; plot $d_p(t_0)$ and $d_p(t_e)$ vs. z.

Equation 5.42 gives the following, into which we can plug $w = 1/3$:

$$t_0 = \frac{2}{3(1+w)}H_0^{-1} = \boxed{\frac{1}{2H_0}} \tag{10}$$

Equation 5.52 gives us the horizon distance, where we can plug in $w = 1/3$:

$$d_{hor}(t_0) = ct_0\frac{3(1+w)}{1+3w} = \frac{c}{H_0}\frac{2}{1+3w}\boxed{= 2ct_0 = \frac{c}{H_0}} \tag{11}$$

From Equation 5.39, we have an equation for the scale factor, where we can plug in $w = 1/3$:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)} = \boxed{\left(\frac{t}{t_0}\right)^{1/2}} \tag{12}$$

From Equation 5.50, we have the current proper distance, where we can plug in $w = 1/3$:

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}[1-(1+z)^{-(1+3w)/2}]\boxed{= \frac{c}{H_0}[1-(1+z)^{-1}] = \frac{c}{H_0}\left(\frac{z}{1+z}\right)} \tag{13}$$

Finally, we can use Equation 5.36 to derive Equation 5.62:

$$d_p(t_e) = \frac{c}{H_0}\frac{\ln(1+z)}{1+z} = \frac{d_p(t_0)}{1+z}\boxed{= \frac{c}{H_0(1+z)}\left(\frac{z}{1+z}\right) = \frac{c}{H_0}\frac{z}{(1+z)^2}} \tag{14}$$

Figure 2 shows the plots of $d_p(t_0)$ and $d_p(t_e)$ vs. z for a radiation-dominated universe.

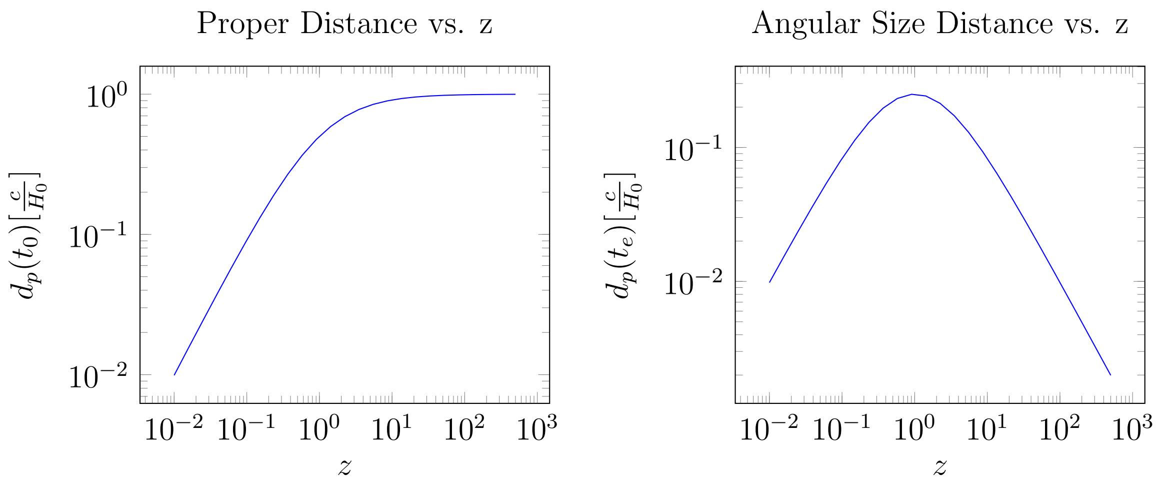

FIGURE 2. Proper distance and Angular Size Distance for a radiation-dominated universe as a function of redshift

We can then use Equation 7 to solve for z_{max} when $w = 1/3$:

$$\frac{1}{2} = (1+z)^{-1} \Rightarrow z_{max} = 2 - 1 = \boxed{1 = z_{max}} \tag{15}$$

We can then calculate the maximum value of $d_p(t_e)$:

$$d_p(t_e)_{max} = \boxed{0.25\frac{c}{H_0}} \tag{16}$$

5.5c) Show Equations 5.70 - 5.74; plot $d_p(t_0)$ and $d_p(t_e)$ vs. z.
Start with the generalized Friedmann equation from Equation 5.25:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\sum_i \varepsilon_{i,0}a^{-1-3w_i} - \frac{\kappa c^2}{R_0^2} \tag{17}$$

We can then plug in $w = -1$ and $\kappa = 0$:

$$\boxed{\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon_{\Lambda,0}a^2 = \frac{8\pi G\varepsilon_\Lambda}{3c^2}a^2} \tag{18}$$

Since $H_0^2 = \frac{8\pi G\varepsilon_\Lambda}{3c^2}$, we can write the following:

$$\boxed{\dot{a} = H_0 a} \tag{19}$$

Equation 19 is true because the simplification is simply from the Friedmann equation for Lambda-only:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2a^2} = \frac{8\pi G}{3c^2}\varepsilon_\Lambda \Rightarrow \boxed{H_0 = \left(\frac{8\pi G\varepsilon_\Lambda}{3c^2}\right)^{1/2}} \tag{20}$$

Solving Equation 19 for a yields the following:

$$\int_1^a \frac{da}{a} = \int_{t_0}^t H_0 dt \Rightarrow \ln a = H_0 t - H_0 t_0 \Rightarrow \boxed{a(t) = e^{H_0(t-t_0)}} \tag{21}$$

Next, we can use Equation 5.33 to solve for proper distance:

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)} = c\int_{t_e}^{t_0} e^{-H_0(t-t_0)}dt = -\frac{c}{H_0}\left[e^{H_0(t_0-t)}\right]_{t_e}^{t_0} = -\frac{c}{H_0}\left[e^0 - e^{H_0(t_0-t_e)}\right] \tag{22}$$

Further simplification yields the following:

$$d_p(t_0) = \frac{c}{H_0}\left[e^{H_0(t_0-t_e)} - 1\right] = \frac{c}{H_0}\left[\frac{1}{a(t)} - 1\right] = \frac{c}{H_0}(1+z-1)\boxed{= \frac{c}{H_0}z} \tag{23}$$

We can then divide Equation 23 by $1+z$ to get an equation for angular size distance:

$$d_p(t_e) = \frac{c}{H_0}\frac{z}{1+z} \tag{24}$$

Figure 3 shows the plots of $d_p(t_0)$ and $d_p(t_e)$ vs. z for a Lambda-dominated universe.

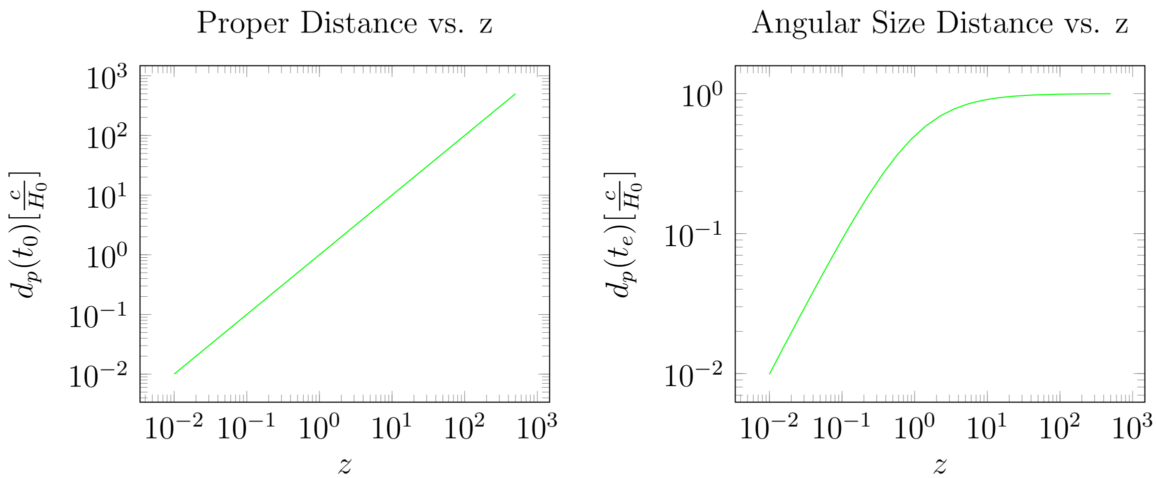

FIGURE 3. Proper distance and Angular Size Distance for a Lambda-dominated universe as a function of redshift

As seen in Figure 3, a Lambda-dominated universe has no maximum for angular size distance. Rather, it approaches 1 as $z \to \infty$. Equations 5.35 and 5.36 describe the distances for an empty universe ($w = -1/3$):

$$d_p(t_0) = \frac{c}{H_0}\ln(1+z) \tag{25}$$

$$d_p(t_e) = \frac{c}{H_0}\frac{\ln(1+z)}{1+z} \tag{26}$$

The maximum value of Equation 26 is where the following is true:

$$\frac{dd_p(t_e)}{dz} = 0 = \frac{1-\ln(1+z)}{(1+z)^2} \Rightarrow 1 - \ln(1+z) = 0 \Rightarrow z = e - 1 \approx 1.718 \tag{27}$$

The maximum angular size distance for an empty universe is then

$$d_p(t_e)_{max} = \frac{c}{H_0}\frac{\ln(1+e-1)}{1+e-1} = \frac{1}{e}\frac{c}{H_0} \approx 0.368\frac{c}{H_0} \tag{28}$$

We can then graph all four universes. Figure 4 shows this, along with vertical lines indicating the z_{max} values found previously and horizontal lines indicating the $d_p(t_e)_{max}$ values found

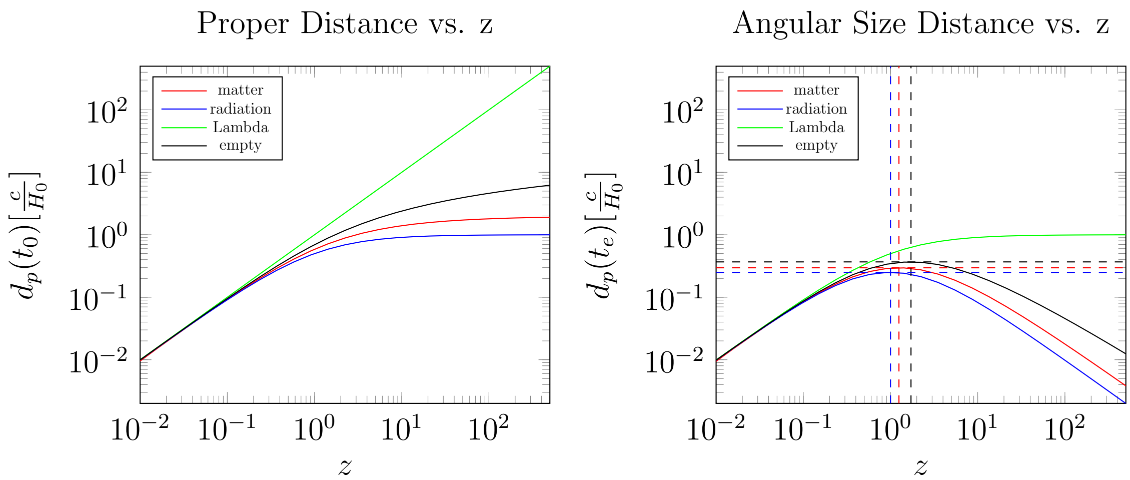

FIGURE 4. Proper distance and Angular Size Distance for simple universes as a function of redshift

previously. The plotted lines for the maxima appear to line up with the maximum values for each curve.

If Congress allowed me to move to any universe using a wormhole, I would go to a radiation-only universe. This is because for a galaxy to have its angular resolution optimized to be as large as possible for high redshifts, $d_p(t_e)$ should be as small as possible at high redshifts. From Figure 4, we can see that a radiation-only universe has the smallest angular size distance for large redshifts. The peak of that curve also occurs at the smallest redshift.

However, a radiation-only universe would have no galaxies, so in reality I would choose to go to a radiation-dominant universe with no cosmological constant. The few galaxies that one would see in such a universe would have a smaller angular size distance (so would appear bigger) and would have a smaller luminosity distance (so would appear brighter).

Of course, this reasoning relies on the idea that surface brightness of galaxies in the various universes is constant. We can verify this by using equations from Ryden Chapter 2:

$$\Omega = \frac{R^2}{4d_A^2} \tag{29}$$

$$f = \frac{L}{4\pi d_L^2} \tag{30}$$

Using Equations 29 and 30, we can then calculate surface brightness:

$$\Sigma = \frac{f}{\Omega} = \frac{L}{4\pi d_L^2} \cdot \frac{4d_A^2}{R^2} = \frac{L}{\pi R^2}\left(\frac{d_A}{d_L}\right)^2 \tag{31}$$

Next, we can use the fact that $d_A = d_p/(1+z)$ and $d_L = d_p(1+z)$:

$$\Sigma = \frac{L}{\pi R^2}\left(\frac{d_p/(1+z)}{d_p(1+z)}\right)^2 = \frac{L}{\pi R^2}(1+z)^{-4} \tag{32}$$

Equation 32 shows that surface brightness does not depend on $d_p(t_0)$, so the decision to visit a radiation-dominated universe stands–such a universe would have high-redshift galaxies that look larger but have the same surface brightness as galaxies in other universes.