AST 322 - Introduction to Galactic and Extragalactic Astrophysics - HW 8

Homework 8 is due in class and on Canvas on Thursday, April 4th. Or, for a point of extra credit, you can turn it in on Tuesday, April 2nd, in class (and Canvas too, but only physical submissions will receive extra credit). This homework is worth 20 points, but it has 10 extra credit points available. Therefore the maximum score, including early-bird extra credit, is 31/20.

6.7a - Matter + Curvature Universe, $\Omega_0 > 1$ (6 pts)

i) Show that Equation 5.89 can be rewritten as

$$\frac{da}{dt} = H_0\sqrt{\Omega_0/a + (1 - \Omega_0)}$$

ii) Show that the set of parametric equations, $a(\theta)$ and $t(\theta)$, (Equation 5.90+5.91, for $\Omega_0 > 1$) are a solution to the above equation.

Hint: You can write

$$\frac{da}{dt} = \frac{da}{d\theta}\frac{d\theta}{dt}$$

and then simplify using a handful of trigonometric identities.

iii) Plot $a(t)$ as a function of $H_0(t - t_0)$ for this matter-dominated universe for $\Omega_0 = 1.2$ (don't just sketch).

iv) By plugging $\theta = 2\pi$ into Equation 5.91, show that this universe has a crunch time of

$$t_{\text{crunch}} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}$$

v) At what time do these universes reach their maximum size?

6.7b - Matter + Curvature Universe, $\Omega_0 < 1$ (EXTRA CREDIT - 4 pts)

i) Show that Equations 5.93+5.94 are a solution to the following equation:

$$\frac{da}{dt} = H_0\sqrt{\Omega_0/a + (1 - \Omega_0)}$$

ii) Plot $a(t)$ as a function of $H_0(t-t_0)$ for this matter-dominated universe for $\Omega_0 = 0.8$ (don't just sketch).

iii) Show that these universes expand forever, by taking the limit as $\eta \to \infty$ on Equation 5.93+5.94.

6.8a - Flat Matter + Λ Universe, $\Omega_{m,0} > 1$ (EXTRA CREDIT - 4 pts)

i) Using the Friedmann equation for a "matter plus lambda" universe (Equation 5.96), derive Equation 5.97:

$$a_{\text{max}} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\right)^{1/3}$$

ii) Show that Equation 5.99 is a valid solution of Equation 5.96.

iii) Show that this universe has a crunch time of

$$t_{\text{crunch}} = \frac{2\pi}{3H_0}\frac{1}{\sqrt{\Omega_{m,0} - 1}}$$

iv) Plot $a(t)$ for this universe, as a function of $H_0(t - t_0)$, for $\Omega_{m,0} = 1.2$ (don't just sketch).

6.8b - Flat Matter + Λ Universe, $\Omega_{m,0} < 1$ (7 pts)
The following problems will guide you through a derivation of Equation 5.101. If you can find an easier way to show that Equation 5.96 implies Equation 5.101, feel free to do that in replacement of parts i-iv (simply verifying that 5.101 is a solution to 5.96 would be insufficient).
i) Starting from the Friedmann Equation for this universe,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

Show that this equation can be rewritten as

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{m,0}/a + (1 - \Omega_{m,0})a^2}}$$

ii) Using Equation 5.97, show that the above equation can be turned into

$$H_0 t = \frac{1}{\sqrt{1 - \Omega_{m,0}}} \int_0^a \frac{da}{a_{m\Lambda}^{3/2} a^{-1/2} \sqrt{1 + (a/a_{m\Lambda})^3}}$$

iii) **(EXTRA CREDIT)** By performing a u-substitution of $u = \sqrt{1 + (a/a_{m\Lambda})^3}$, show that this turns into

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \int_1^{\sqrt{1+(a/a_{m\Lambda})^3}} \frac{du}{\sqrt{u^2 - 1}}$$

iv) (**EXTRA CREDIT**) By performing a trigonometric substitution of $\sec\theta = u/1$, derive Equation 5.101:

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{m\Lambda}}\right)^3}\right]$$

v) By using the Taylor series approximation that $\ln(1 + x) \approx x$ when x is small, derive Equation 5.102:

$$a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}} H_0 t\right)^{2/3}$$

vi) By using the approximation that $\sqrt{x + 1} \approx \sqrt{x}$ when x is large, derive Equation 5.103 (note the slightly different coefficient in front):

$$a(t) \approx 2^{-2/3} a_{m\Lambda} \exp\left(\sqrt{1 - \Omega_{m,0}} H_0 t\right)$$

vii) By setting $a = 1$ in our solution, derive Equation 5.104. Then calculate t_0 by plugging in $\Omega_{m,0} = 0.32$ and $H_0^{-1} = 14.60$ Gyr.

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1 - \Omega_{m,0}}} \ln\left[\frac{\sqrt{1 - \Omega_{m,0}} + 1}{\sqrt{\Omega_{m,0}}}\right]$$

viii) Derive Equation 5.106, and calculate $t_{m\Lambda}$ by plugging in $\Omega_{m,0} = 0.32$ and $H_0^{-1} = 14.60$ Gyr:

$$t_{m\Lambda} = \frac{2H_0^{-1}}{3\sqrt{1 - \Omega_{m,0}}} \ln\left(1 + \sqrt{2}\right)$$

ix) Plot $a(t)$ for this universe, as a function of $H_0(t - t_0)$, for $\Omega_{m,0} = 0.32$ (don't just sketch). This is the universe we live in!

6.9 - Matter + Radiation Universe (7 pts)

i) Starting from the Friedmann Equation for this universe,

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

Using the fact that $a_{rm} = \Omega_{r,0}/\Omega_{m,0}$, show that this can be made into

$$H_0 dt = \frac{a da}{\Omega_{r,0}^{1/2}} \left[1 + \frac{a}{a_{rm}}\right]^{-1/2}$$

ii) The solution of this differential equation is given to be the following:

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2}\right]$$

By differentiating this equation, verify that it is the solution to Equation 1.109 (the second equation in part i).

iii) By expanding the right-hand-side of Equation 5.110 as a Taylor series centered about $a = 0$, show that for $a \ll a_{rm}$, our solution can be approximated as

$$a \approx \left(2\sqrt{\Omega_{r,0}} H_0 t\right)^{1/2}$$

Hint: Recall that the formula for this sort of Taylor series is $f(x) \approx f(0) + f'(0)a + f''(0)a^2/2$, *where* $f(x)$ *is the RHS of Equation 5.110. Also remember that you already differentiated this function for part ii, so you should only have to differentiate once more.*

iv) By using the approximation that $\sqrt{x+1} \approx \sqrt{x}$ when x is large, show that in the case of $a \gg a_{rm}$,

$$a \approx \left(\frac{3}{2}\sqrt{\Omega_{rm}} H_0 t\right)^{2/3}$$

v) By setting $a = a_{rm}$ in Equation 5.110, derive that

$$t_{rm} = \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}} H_0^{-1}$$

vi) Calculate t_{rm} for $\Omega_{r,0} = 9.0 \times 10^{-5}$, $\Omega_{m,0} = 0.32$, and $H_0^{-1} = 14.60$ Gyr.

vii) Calculate z_{rm}, the redshift of radiation-matter equality.