HOMEWORK - 3

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H/W 3.1: Show that, or derive how Ry 3.20 and 3.21 follow from the previous
equations. We will use these metrics a lot in what follows. [Note the typo in
Ry Eq. 3.20: The two square exponents INSIDE the first set of square brackets
shouldn't be there ...]
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We know that,

$$c^2t^2 = x^2 + y^2 + z^2$$

and for a corresponding time t'

$$c^2(t')^2 = (x')^2 + (y')^2 + (z')^2$$

From the Lorentz transformation

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned}\right\} \text{In this, } \gamma \text{ is the Lor}$$

For an unprimed frame,

At t_1 let the location be (x_1, y_1, z_1)

At t_2 let the location be (x_2, y_2, z_2)

Distance between these $\rightarrow$ $\Delta l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$

time elapsed $\rightarrow$ $\Delta t = t_1 - t_2$

Accounting for the Lorentz transformation,

$$(\Delta l')^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2$$

we know that $x' = \gamma(x - vt)$, $y' = y$, $z' = z$

Putting these in the equation for $(\Delta l')^2$

$$(\Delta l')^2 = \left(\gamma(x_1 - vt_1) - \gamma(x_2 - vt_2)\right)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$= \left(\gamma(x_1 - vt_1 - x_2 + vt_2)\right)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$(\Delta l')^2 = \gamma^2 (x_1 - x_2 - v(t_1 - t_2))^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \quad \text{— (1)}$$

$$\Delta t' = t_1' - t_2'$$

we know that $t' = \gamma(t - vx/c^2)$

Putting this in the equation for $\Delta t'$

$$\Delta t' = \gamma(t_1 - vx_1/c^2) - \gamma(t_2 - vx_2/c^2)$$

$$= \gamma\left(t_1 - \frac{vx_1}{c^2} - t_2 + \frac{vx_2}{c^2}\right)$$

$$\Delta t' = \gamma\left(t_1 - t_2 - \frac{v}{c^2}(x_1 - x_2)\right) \quad \text{— (2)}$$

Considering time as the fourth dimension, space time separation between the two points / events is,

$$(\Delta s)^2 = -c^2\underbrace{(t_1 - t_2)^2}_{\Delta t^2} + \underbrace{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}_{\Delta l^2}$$

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta l)^2$$

For the primed frame,

$$(\Delta s')^2 = -c^2(\Delta t')^2 + (\Delta l')^2$$

On using substituting $\Delta l'$ and $\Delta t'$ from equation (1) and (2), we get

$$(\Delta S')^2 = -c^2\left[\gamma\left(t_1 - t_2 - \frac{v}{c^2}(x_1 - x_2)\right)\right]^2 + \gamma^2\left(x_1 - x_2 - v(t_1 - t_2)\right)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$(\Delta S')^2 = -\gamma^2\left[ct_1 - ct_2 - \frac{vc}{c^2}(x_1 - x_2)\right]^2 + \gamma^2\left(x_1 - x_2 - v(t_1 - t_2)\right)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$(\Delta S')^2 = -\gamma^2\left[c(t_1 - t_2) - \frac{v}{c}(x_1 - x_2)\right]^2 + \gamma^2\left(x_1 - x_2 - v(t_1 - t_2)\right)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

This above equation is equivalent to equation 3.20

$$(\Delta S')^2 = \underbrace{-\gamma^2\left[c(t_1 - t_2) - \frac{v}{c}(x_1 - x_2)\right]^2 + \gamma^2\left(x_1 - x_2 - v(t_1 - t_2)\right)^2}_{} + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$\ast$

Considering the region in red and simplifying it,

$$-\gamma^2\left[c^2(t_1 - t_2)^2 - \cancel{2v(t_1 - t_2)(x_1 - x_2)} + \frac{v^2}{c^2}(x_1 - x_2)^2\right]$$

$$+\gamma^2\left[(x_1 - x_2)^2 - \cancel{2v(x_1 - x_2)(t_1 - t_2)} + v^2(t_1 - t_2)^2\right]$$

$$= -\gamma^2 c^2 (t_1 - t_2)^2 - \frac{\gamma^2 v^2}{c^2}(x_1 - x_2)^2 + \gamma^2(x_1 - x_2)^2 + \gamma^2 v^2 (t_1 - t_2)^2$$

$$= -\gamma^2 c^2 (t_1 - t_2)^2 + \gamma^2 v^2 (t_1 - t_2)^2 + \gamma^2 (x_1 - x_2)^2 - \frac{\gamma^2 v^2}{c^2}(x_1 - x_2)^2$$

$$= \gamma^2\left[(v^2 - c^2)(t_1 - t_2)^2 + \left(1 - \frac{v^2}{c^2}\right)(x_1 - x_2)^2\right]$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} \Rightarrow \gamma^2 = \frac{c^2}{c^2 - v^2}$$

$$= \frac{c^2}{c^2 - v^2} (v^2 - c^2)(t_1 - t_2)^2 + \frac{1 - v^2/c^2}{1 - v^2/c^2} (x_1 - x_2)^2$$

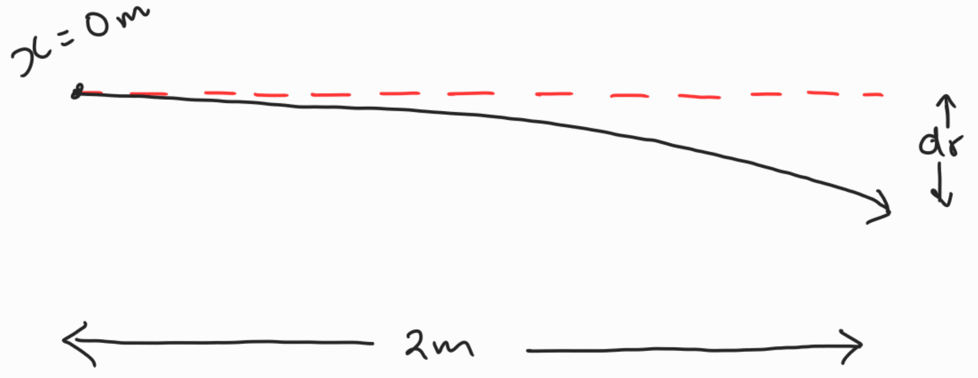
$$= -c^2 (t_1 - t_2)^2 + (x_1 - x_2)^2$$

Putting this simplified part in *

$$(\Delta S')^2 = -c^2 (t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

This is the same as equation 21.

H/W 3.2: See Ry Fig. 3.3. Show that the actual downwards deflection of a light ray --- horizontally emitted in this closed elevator or box --- is ~2x10^-16 m in the presence of Earth's gravity (or an equivalent acceleration of the elevator!) across the box length of 2 m. [Note typo in Fig. 3.2's caption: 2x10^-14 should read 2x10^-16 !]



$c = 3\times10^{8}$ m/s $= V_x$ (all the velocity is in the x-direction)

$V_y = 0$ m/s

$g = 9.8$ m/s^2

To prove $dr \sim 2\times10^{-6}$ m

Treating it as projectile motion

$$x = x_0 + V_x t + \frac{1}{2} a_x t^2$$

For the x-direction $a_x = 0$ m/s^2, $x = 2$m, $x_0 = 0$ m, $V_x = 3\times10^{8}$ m/s

$$2 = (3\times10^{8})\,t \implies t = \frac{2}{3\times10^{8}} \text{ s}$$

For the y direction,

$$y = y_0 + V_y t - \frac{1}{2}(9.8)\,t^2 \quad , y_0 = 0\text{m}, V_y = 0 \text{ m/s}$$

$$dr = -\frac{1}{2}(9.8)\left(\frac{2}{3\times10^{8}}\right)^2 \implies dr = -2.178\times10^{-16}\text{m}$$

-ve sign indicates the vertical displacement is downward

$\implies dr \approx 2\times10^{-16}$ m downward

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H/W 3.3: Show that the metric as written in 3.36 is mathematically equivalent to
that in 3.33+3.34 for all three values of k (=+1, 0 -1). We will use both
notations of these metrics interchangeably in what follows.
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$3.33 \longrightarrow dl^2 = dr^2 + S_k(r)^2\, d\Omega^2$

$3.34 \longrightarrow d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$

$3.36 \longrightarrow dl^2 = \dfrac{dx^2}{1 - k\dfrac{x^2}{R^2}} + x^2 d\Omega^2$

To show that 3.36 is equivalent to 3.33 + 3.34

we know that $S_k(r) = x = \begin{cases} R\sin(r/R) & k = +1 \\ r & k = 0 \\ R\sinh(r/R) & k = -1 \end{cases}$

1. FOR $k = +1$

$x = R\sin(r/R)$

using this,

eqn 3.33 $\longrightarrow dl^2 = dr^2 + R^2\sin^2(r/R)\, d\Omega^2$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$

$x = R\sin(r/R)$, Differentiating this with respect to r,

$\dfrac{dx}{dr} = R\cos(r/R)\cdot\dfrac{1}{R} \Rightarrow dx = \cos(r/R)\, dr$

Putting $x = R\sin(r/R)$, $dx = \cos(r/R)\, dr$ & $k=1$ in eqn 3.36

$$dl^2 = \frac{\cos^2(r/R)\, dr^2}{1 - \dfrac{R^2\sin^2(r/R)}{R^2}} + R^2\sin^2(r/R)\, d\Omega^2$$

$$dl^2 = \frac{\cos^2(r/R)\, dr^2}{1 - \sin^2(r/R)} + R^2\sin^2(r/R)\, d\Omega^2$$

$$1-\sin^2(r/R) = \cos^2(r/R)$$

$$\Rightarrow dl^2 = \frac{\cos^2(r/R)\, dr^2}{\cos^2(r/R)} + R^2\sin^2(r/R)\, d\Omega^2$$

this gives,

$$dl^2 = dr^2 + R^2\sin^2(r/R)\, d\Omega^2$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$

This shows equation 3·36 is equivalent to 3·33 + 3·34.

2. FOR K = 0

$$x = r$$

using this,

equ 3·33 $\longrightarrow$ $dl^2 = dr^2 + r^2 d\Omega^2$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$

$x = r$, Differentiating this with respect to r,

$$\frac{dx}{dr} = 1 \Rightarrow dx = dr$$

Putting $x = r$, $dx = dr$ & $K = 0$ in equ 3·36

$$dl^2 = \frac{dr^2}{1 - 0\left(\frac{r^2}{R^2}\right)} + r^2 d\Omega^2$$

this gives,

$$dl^2 = dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$

This shows equation 3·36 is equivalent to 3·33 + 3·34.

1. FOR $K = -1$

$$x = R\sinh(r/R)$$

using this,

equ 3.33 $\longrightarrow$ $dl^2 = dr^2 + R^2\sinh^2(r/R)\, d\Omega^2$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$

$x = R\sinh(r/R)$, Differentiating this with respect to r,

$$\frac{dx}{dr} = R\cosh(r/R)\cdot\frac{1}{R} \Rightarrow dx = \cosh(r/R)\, dr$$

Putting $x = R\sinh(r/R)$, $dx = \cosh(r/R)\, dr$ + $K=-1$ in eqn 3.36.

$$dl^2 = \frac{\cosh^2(r/R)\, dr^2}{1 - (-1)\frac{R^2\sinh^2(r/R)}{R^2}} + R^2\sinh^2(r/R)\, d\Omega^2$$

$$dl^2 = \frac{\cosh^2(r/R)\, dr^2}{1 + \sinh^2(r/R)} + R^2\sinh^2(r/R)\, d\Omega^2$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh^2(x) - \sinh^2(x) = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{(e^{2x} - 2 + e^{-2x})}{4}$$

$$= \frac{2+2}{4} = 1 \Rightarrow \cosh^2(x) - \sinh^2(x) = 1$$

$$\cosh^2(x) = 1 + \sinh^2(x)$$

using this identity,

$$dl^2 = \frac{\cosh^2(r/R)\, dr^2}{\cosh^2(r/R)} + R^2\sinh^2(r/R)\, d\Omega^2$$

This gives,

$$dl^2 = dr^2 + R^2\sinh^2(r/R)\, d\Omega^2$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$

This shows equation 3.36 is equivalent to 3.33 + 3.34.