Jake Summers - 1221695592 - HW8 - AST322 - 3/31/2022

6.7a) We are given Equation 5.89:

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_0/a + (1-\Omega_0)}} \tag{1}$$

Equation 1 can be returned back to its differential equation state:

$$H_0 dt = \frac{da}{\sqrt{\Omega_0/a + (1-\Omega_0)}} \Rightarrow \frac{da}{dt} = H_0\sqrt{\Omega_0/a + (1-\Omega_0)} \tag{2}$$

The solution to Equation 1 is given to be the following:

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{(\Omega_0 - 1)}(1-\cos\theta) \tag{3}$$

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(\theta - \sin\theta) \tag{4}$$

Equations 3 and 4 can then be differentiated:

$$\frac{da}{d\theta} = \frac{\Omega_0 \sin\theta}{2(\Omega_0 - 1)} \tag{5}$$

$$\frac{dt}{d\theta} = \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}(1-\cos\theta) \tag{6}$$

We can then use the chain rule to solve for $\frac{da}{dt}$:

$$\frac{da}{dt} = \frac{da}{d\theta}\left(\frac{dt}{d\theta}\right)^{-1} = \frac{\Omega_0\sin\theta}{2(\Omega_0-1)}\frac{2H_0(\Omega_0-1)^{3/2}}{\Omega_0}\frac{1}{1-\cos\theta} \tag{7}$$

Equation 7 can then be simplified:

$$\frac{da}{dt} = H_0\sqrt{\Omega_0 - 1}\frac{\sin\theta}{1-\cos\theta} \tag{8}$$

To eliminate the θs from Equation 8, we can first solve Equation 3 for $1-\cos\theta$, and then use trigonometric identities to solve for $\cos\theta$. Solving for $1-\cos\theta$ yields the following:

$$1 - \cos\theta = \frac{2a(\Omega_0-1)}{\Omega_0} \tag{9}$$

We can then use the identity $\cos\theta = 1 - 2\sin^2(\theta/2)$ in Equation 3:

$$a = \frac{\Omega_0}{2(\Omega_0-1)}(2\sin^2(\theta/2)) \Rightarrow \sin(\theta/2) = \sqrt{\frac{a(\Omega_0-1)}{\Omega_0}} \tag{10}$$

Next, we can do the same, but with the identity $\cos\theta = 2\cos^2(\theta/2) - 1$:

$$a = \frac{\Omega_0}{2(\Omega_0-1)}(2 - 2cos^2(\theta/2)) \Rightarrow \cos(\theta/2) = \sqrt{1 - \frac{a(\Omega_0-1)}{\Omega_0}} \tag{11}$$

We can combine Equations 10 and 11 by using the identity $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.

$$\sin\theta = 2\sqrt{\frac{a(\Omega_0-1)}{\Omega_0}}\sqrt{1-\frac{a(\Omega_0-1)}{\Omega_0}} = 2\sqrt{\frac{a(\Omega_0-1)}{\Omega_0}\left(1-\frac{a(\Omega_0-1}{\Omega_0}\right)} \tag{12}$$

Now, we can use Equation 9 and Equation 12 to finish solving Equation 8.

$$\frac{da}{dt} = H_0\sqrt{\Omega_0 - 1}\frac{\Omega_0}{2a(\Omega_0 - 1)} \cdot 2\sqrt{\frac{a(\Omega_0 - 1)}{\Omega_0}\left(1 - \frac{a(\Omega_0 - 1}{\Omega_0}\right)} \tag{13}$$

Simplifying yields the following (Equation 2):

$$\frac{da}{dt} = H_0\sqrt{\frac{\Omega_0}{a}\left(1 - \frac{a(\Omega_0 - 1)}{\Omega_0}\right)} = \boxed{H_0\sqrt{\Omega_0/a + (1 - \Omega_0)}} \tag{14}$$

The scale factor for this universe as a function of time is plotted in Figure 1.

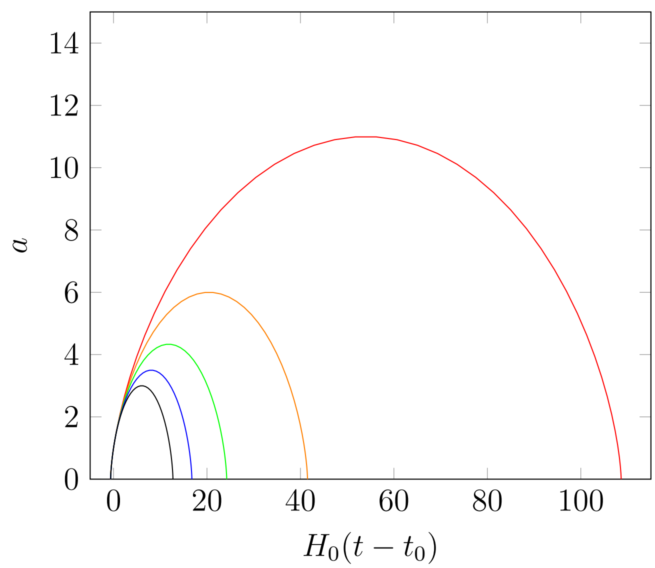

FIGURE 1. Scale Factor vs. Time for universes with $\Omega_0 = 1.1$ (red), 1.2 (orange), 1.3 (green), 1.4 (blue), and 1.5 (black)

Figure 1 shows that for larger values of Ω_0, the maximum scale factor and crunch time decrease. This graph, and all future graphs, were plotted using pgfplots in LaTeX. The H_0t_0 component was calculated with the following formula for each universe and subtracted from $H_0t(\theta)$.

$$H_0t_0 = \frac{1}{2}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left[\cos^{-1}\left(1 - \frac{2(\Omega_0 - 1)}{\Omega_0}\right) - \sin\cos^{-1}\left(1 - \frac{2(\Omega_0 - 1)}{\Omega_0}\right)\right] \tag{15}$$

By plugging in $\theta = 2\pi$ into Equation 3 and 4, we get $a(\theta) = 0$ and the following for $t(\theta)$:

$$t(2\pi) = \boxed{t_{crunch} = \pi H_0^{-1}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}} \tag{16}$$

By plugging in $\Omega_0 = 1.1$, Equation 16 equals $109.28H_0^{-1}$, which appears to agree with Figure 1; the horizontal shift in the plot should be kept in mind here. The time of maximum expansion for this universe is half of t_{crunch}, as the graph is symmetric:

$$\boxed{t_{max} = \frac{1}{2}t_{crunch} = \frac{\pi}{2}H_0^{-1}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}} \tag{17}$$

6.7b) The goal of this exercise is to return to Equation 2 from the following solution:

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)}(\cosh\eta - 1) \tag{18}$$

$$t(\eta) = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\eta - \eta) \tag{19}$$

The process is similar to that of 6.7a. First differentiate Equations 18 and 19:

$$\frac{da}{d\eta} = \frac{\Omega_0}{2(1-\Omega_0)}\sinh\eta \tag{20}$$

$$\frac{dt}{d\eta} = \frac{\Omega_0}{2H_0(1-\Omega_0)^{3/2}}(\cosh\eta - 1) \tag{21}$$

Again, we can use the chain rule:

$$\frac{da}{dt} = \frac{da}{d\eta}\left(\frac{dt}{d\eta}\right)^{-1} = \frac{2H_0\Omega_0(1-\Omega_0)^{3/2}}{2\Omega_0(1-\Omega_0)}\frac{\sinh\eta}{(\cosh\eta - 1)} = H_0\sqrt{1-\Omega_0}\frac{\sinh\eta}{\cosh\eta - 1} \tag{22}$$

To obtain $\cosh\eta - 1$, we can simply use Equation 18:

$$\cosh\eta - 1 = \frac{2a(1-\Omega_0)}{\Omega_0} \tag{23}$$

To find $\sinh\eta$, we again must use the fact that $\sinh\eta = 2\sinh^2(\eta/2)\cosh^2(\eta/2)$. First use Equation 18 and the fact that $\cosh\eta = 1 + 2\sinh^2(\eta/2)$:

$$a = \frac{\Omega_0}{2(1-\Omega_0)}(2\sinh^2(\eta/2)) \Rightarrow \sinh(\eta/2) = \sqrt{\frac{a(1-\Omega_0)}{\Omega_0}} \tag{24}$$

We can then use the identity $\cosh\eta = 2\cosh^2(\eta/2) - 1$ in Equation 18:

$$a = \frac{\Omega_0}{2(1-\Omega_0)}(2\cosh^2(\eta/2) - 2) \Rightarrow \cosh(\eta/2) = \sqrt{1 + \frac{a(1+\Omega_0)}{\Omega_0}} \tag{25}$$

We can then solve for $\sinh\eta$:

$$\sinh\eta = 2\sinh^2(\eta/2)\cosh^2(\eta/2) = 2\sqrt{\frac{a(1-\Omega_0)}{\Omega_0}\left(1 + \frac{a(1+\Omega_0)}{\Omega_0}\right)} \tag{26}$$

Next, we can plug $\cosh\eta - 1$ and $\sinh\eta$ into Equation 22:

$$\frac{da}{dt} = H_0\sqrt{1-\Omega_0}\cdot 2\sqrt{\frac{a(1-\Omega_0)}{\Omega_0}\left(1 + \frac{a(1+\Omega_0)}{\Omega_0}\right)}\left(\frac{\Omega_0}{2a(1-\Omega_0)}\right) \tag{27}$$

We can then simplify to get back to Equation 2:

$$\frac{da}{dt} = H_0\sqrt{\frac{\Omega_0}{a}\left(1 + \frac{a(1+\Omega_0)}{\Omega_0}\right)} = \boxed{H_0\sqrt{\Omega_0/a + (1-\Omega_0)}} \tag{28}$$

Equations 18 and 19 can then be plotted:

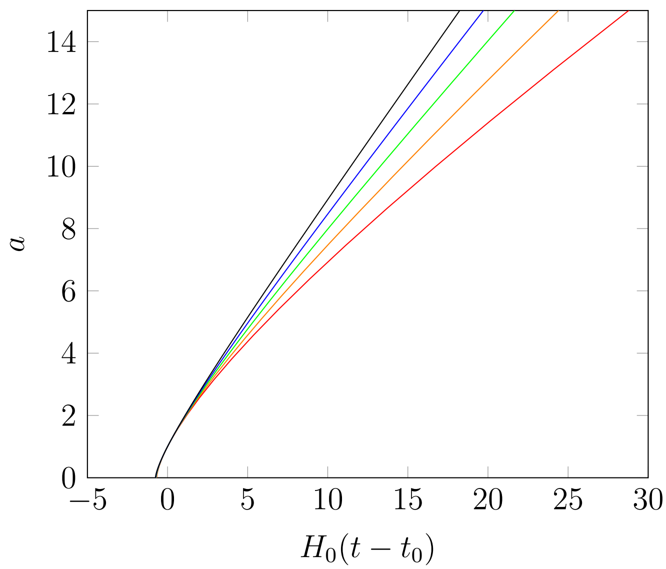

FIGURE 2. Scale Factor vs. Time for universes with $\Omega_0 = 0.9$ (red), 0.8 (orange), 0.7 (green), 0.6 (blue), and 0.5 (black)

Figure 2 shows various universes with $\Omega_0 < 1$. Universes with less matter expand at a faster rate (so are more to the left). Figure 2 was plotted using pgfplots, with the following value of H_0t_0 subtracted from H_0t:

$$H_0t_0 = \frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\left[\sinh\cosh^{-1}\left(\frac{2(1-\Omega_0)}{\Omega_0}\right) - \cosh^{-1}\left(\frac{2(1-\Omega_0)}{\Omega_0}\right)\right] \tag{29}$$

It is clear that these universes expand forever, as Equation 18 shows that as $\eta \to \infty$, $\cosh\eta \to \infty$ so $a(\eta) \to \infty$. Equation 19 also approaches infinity because as $\eta \to \infty$, $\sinh\eta - \eta \to \infty$ so $t(\eta) \to \infty$.

6.8a) The Friedmann Equation in the presence of matter and Lambda takes the following form:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0}) \tag{30}$$

Multiplying by a^2 (using the fact that $H^2 = \dot{a}^2/a^2$) and setting $\dot{a} = 0$ yields the following:

$$0 = \frac{\Omega_{m,0}}{a} + a^2 - \Omega_{m,0}a^2 \tag{31}$$

We can then solve for $a = a_{max}$ after multiplying by a:

$$0 = \Omega_{m,0} - a_{max}^3(\Omega_{m,0} - 1) \Rightarrow a_{max}^3(\Omega_{m,0} - 1) = \Omega_{m,0} \Rightarrow \boxed{a_{max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\right)^{1/3}} \tag{32}$$

Before showing that Equation 5.98 is true, I will verify 5.99 and then use that to confirm 5.98. Equation 5.99 is the following:

$$H_0t = \frac{2}{3\sqrt{\Omega_{m,0} - 1}}\sin^{-1}\left[\left(\frac{a}{a_{max}}\right)^{3/2}\right] \tag{33}$$

Differentiating both sides with respect to t results in

$$H_0 = \frac{2}{3\sqrt{\Omega_{m,0}-1}} \cdot \frac{1}{\sqrt{1-(a/a_{max})^3}} \cdot \frac{3}{2}\left(\frac{a}{a_{max}}\right)^{1/2} \frac{da}{dt} a_{max}^{-1/2} \tag{34}$$

We can then use Equation 32 to substitute for a_{max}, plug in $\dot{a}$ using a variation of Equation 30, and also simplify:

$$\begin{aligned}
H_0 &= \frac{\sqrt{a}}{\sqrt{\Omega_{m,0}-1}} \cdot \frac{1}{\sqrt{1-a^3(\Omega_{m,0}-1)/\Omega_{m,0}}} \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}}\right)^{1/6} H_0\sqrt{\Omega_{m,0}/a+(1-\Omega_{m,0})a^2} \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}}\right)^{1/3} \\
&= \frac{\sqrt{a}}{\sqrt{\Omega_{m,0}-1}} \cdot \frac{H_0\sqrt{\Omega_{m,0}/a+(1-\Omega_{m,0})a^2}}{\sqrt{a/\Omega_{m,0}}\sqrt{\Omega_{m,0}/a+(1-\Omega_{m,0})a^2}} \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}}\right)^{1/6} \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}}\right)^{1/3} \\
&= \sqrt{\frac{\Omega_{m,0}}{\Omega_{m,0}-1}} \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}}\right)^{1/6} \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}}\right)^{1/3} H_0 \\
&= \left(\frac{\Omega_{m,0}-1}{\Omega_{m,0}}\right)^{-1/2+1/6+1/3} H_0 \\
&= \boxed{H_0}
\end{aligned}$$

Hence, substituting da/dt into Equation 34 yields a true statement, so Equation 33 is a valid solution of the Friedman Equation. To verify Equation 5.98, we can use 5.99 and plug in $a = a_{max}$ and $t = t_{max}$.

$$H_0 t_{max} = \frac{2}{3\sqrt{\Omega_{m,0}-1}} \sin^{-1}\left[\left(\frac{a_{max}}{a_{max}}\right)^{3/2}\right] = \frac{2}{3\sqrt{\Omega_{m,0}-1}} \sin^{-1}(1) = \frac{\pi}{3\sqrt{\Omega_{m,0}-1}} \tag{35}$$

We can then solve for t_{crunch} using the fact that $t_{crunch} = 2t_{max}$:

$$2t_{max} = \boxed{t_{crunch} = \frac{2\pi}{3H_0\sqrt{\Omega_{m,0}-1}}} \tag{36}$$

Equation 33 can be plotted:

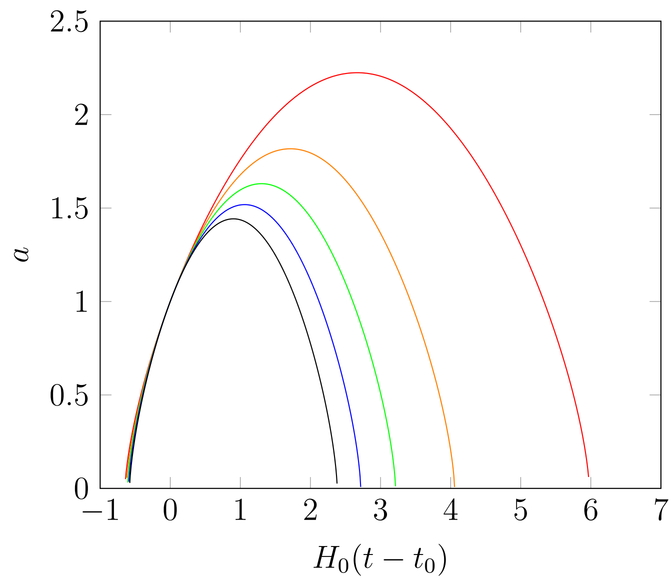

FIGURE 3. Scale Factor vs. Time for universes with $\Omega_{m,0} = 1.1$ (red), 1.2 (orange), 1.3 (green), 1.4 (blue), and 1.5 (black)

Figure 3 shows universes with $\Omega_{m,0} > 1$ and $\Omega_{\Lambda,0} < 0$. The larger values of $\Omega_{m,0}$ have a faster crunch time (so $\Omega_{m,0}$ increases to the left). It was plotted using pgfplots and shifted to the left according to the following equation:

$$H_0 t_0 = \frac{2}{3\sqrt{\Omega_{m,0} - 1}} \sin^{-1}\left(\frac{\Omega_{m,0} - 1}{\Omega_{m,0}}\right)^{1/2} \tag{37}$$

Equation 36 appears to agree with Figure 3. The largest big crunch universe ($\Omega_{m,0} = 1.1$) has $t_{crunch} = 6.623 H_0^{-1}$ and $H_0 t_0 = 0.646$ (from Equation 37) so it has $H_0(t - t_0) = 5.977$.
6.8b) For $\Omega_{m,0} < 1$ and $\Omega_{\Lambda,0} > 0$ we can again start with Equation 30:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0}) \tag{38}$$

By multiplying by $H_0^2 a^2$ and taking the square root, we end up with the following:

$$\frac{da}{dt} = H_0 \sqrt{\Omega_{m,0}/a + (1 - \Omega_{m,0}) a^2} \tag{39}$$

Rearranging Equation 39 yields the following:

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{m,0}/a + (1 - \Omega_{m,0}) a^2}} \tag{40}$$

Before moving on, Equation 5.100 can be proven in the following way:

$$\frac{\Omega_{m,0}}{a_{m\Lambda}^3} = \Omega_{\Lambda,0} \Rightarrow \boxed{a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}}\right)^{1/3}} \tag{41}$$

Back to Equation 40, we can factor and substitute for $a_{m\Lambda}$:

$$H_0 t = \int_0^a \frac{da}{\sqrt{(1-\Omega_{m,0})\left(\frac{\Omega_{m,0}}{(1-\Omega_{m,0})a} + a^2\right)}} = \frac{1}{\sqrt{1-\Omega_{m,0}}} \int_0^a \frac{da}{\sqrt{a_{m\Lambda}^3/a + a^2}} \tag{42}$$

We can then factor again:

$$H_0 t = \frac{1}{\sqrt{1-\Omega_{m,0}}} \int_0^a \frac{da}{a_{m\Lambda}^{3/2}/a^{1/2}\sqrt{1+(a/a_{m\Lambda})^3}} \tag{43}$$

Next, we can perform u-substitution with $u = \sqrt{1+(a/a_{m\Lambda})^3}$, $du = \frac{1}{2}(1+(a/a_{m\Lambda})^3)^{-1/2} \cdot \frac{3a^2}{a_{m\Lambda}^3} da$, so $a = [a_{m\Lambda}^3(u^2-1)]^{1/3}$ and $da = \frac{2a_{m\Lambda}^3}{3a^2}\sqrt{1+(a/a_{m\Lambda})^3}du$. Additionally, the new limits become $a = 0 \to u = 1$ and $a = a \to u = u$. The top limit will be substituted later (it is set as u for simplicity). It is actually $u = \sqrt{1+(a/a_{m\Lambda})^3}$.

$$H_0 t = \frac{1}{\sqrt{1-\Omega_{m,0}}} \int_1^u \frac{a^{1/2}}{a_{m\Lambda}^{3/2}} \cdot \frac{2a_{m\Lambda}^3}{3a^2} \cdot \frac{\sqrt{1+(a/a_{m\Lambda})^3}du}{\sqrt{1+(a/a_{m\Lambda})^3}} \tag{44}$$

We can then complete the substitution:

$$H_0 t = \frac{1}{\sqrt{1-\Omega_{m,0}}} \int_1^u \frac{2a_{m\Lambda}^{3/2}}{3a^{3/2}} du = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \int_1^u \frac{a_{m\Lambda}^{3/2}}{\sqrt{a_{m\Lambda}^3(u^2-1)}} du \tag{45}$$

Equation 45 can be simplified:

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \int_1^u \frac{1}{\sqrt{u^2-1}} du \tag{46}$$

To solve Equation 46, we can use trigonometric substitution. To do this we can let $u = \sec\theta$, so $du = \sec\theta\tan\theta d\theta$. Also, the substitution allows for us to say that $\tan\theta = \sqrt{u^2-1}$. The limits become $u = 1 \to \theta = 0$ and $u = u \to \theta = \sec^{-1} u$.

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \int_0^{\sec^{-1} u} \frac{\sec\theta\tan\theta}{\tan\theta} d\theta = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \int_0^{\sec^{-1} u} \sec\theta d\theta \tag{47}$$

We can then solve the integral:

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \int_0^{\sec^{-1} u} \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} d\theta = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \left[\ln|\sec\theta + \tan\theta|\right]_0^{\sec^{-1} u} \tag{48}$$

Creating a triangle with the angle $\sec\theta = u$ results in $u = \sqrt{1+(a/a_{m\Lambda})^3}$ being the hypotenuse, with 1 the adjacent side and $(a/a_{m\Lambda})^{3/2}$ the opposite side. This means that $\tan(\sec^{-1} u) = (a/a_{m\Lambda})^{3/2}$. We can plug these results into Equation 48.

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln|u + \tan(\sec^{-1} u)| = \boxed{\frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^3} + \left(\frac{a}{a_{m\Lambda}}\right)^{3/2}\right]} \tag{49}$$

For $a \ll a_{m\Lambda}$, we can assume $\sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^3} \approx 1$ because the a component is negligible compared to 1. We can then use the Taylor series approximation $\ln(1+x) \approx x$. This gives the following approximation:

$$H_0 t \approx \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[1+\left(\frac{a}{a_{m\Lambda}}\right)^{3/2}\right] \approx \frac{2}{3\sqrt{1-\Omega_{m,0}}} \left(\frac{a}{a_{m\Lambda}}\right)^{3/2} \tag{50}$$

We can then solve for $a(t)$:

$$a(t) \approx \left(\frac{3H_0 t\sqrt{1-\Omega_{m,0}}}{2a_{m\Lambda}^{3/2}}\right)^{2/3} = \left[\frac{3}{2}H_0 t(1-\Omega_{m,0})^{1/2}\left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/2}\right]^{2/3} \tag{51}$$

Further simplification yields the following solution:

$$\boxed{a(t) \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t\right)^{2/3}} \tag{52}$$

In the case of $a \gg a_{m\Lambda}$, the 1 in the square root term from Equation 49 becomes insignificant compared to the a component.

$$H_0 t \approx \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[2\left(\frac{a}{a_{m\Lambda}}\right)^{3/2}\right] \tag{53}$$

We can then exponentiate to solve for $a(t)$:

$$\exp\left(3H_0 t\sqrt{1-\Omega_{m,0}}/2\right) = 2\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} \Rightarrow \boxed{a(t) \approx 2^{-2/3} a_{m\Lambda} \exp\left(\sqrt{1-\Omega_{m,0}}H_0 t\right)} \tag{54}$$

Equation 54 is slightly different from Equation 5.103–it is shifted vertically. However, it matches Equation 49 when graphed so I will keep the $2^{-2/3}$ factor. To verify Equation 5.104, we can set $a = 1$ in Equation 49:

$$\begin{aligned} H_0 t &= \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left(1/a_{m\Lambda}^{3/2} + \sqrt{1+a_{m\Lambda}^{-3}}\right) \\ &= \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/2} + \left(\frac{\Omega_{m,0}+1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/2}\right] \\ &= \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left(\frac{\sqrt{1-\Omega_{m,0}}+1}{\sqrt{\Omega_{m,0}}}\right) \end{aligned}$$

Solving for $t = t_0$ yields

$$\boxed{t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln\left(\frac{\sqrt{1-\Omega_{m,0}}+1}{\sqrt{\Omega_{m,0}}}\right)} \tag{55}$$

We can then plug in $\Omega_{m,0} = 0.32$ and $H_0^{-1} = 14.60$ Gyr (from $H_0 = 67 \pm 2$ km/s/Mpc) to approximate t_0 for our universe:

$$t_0 \approx \frac{2(H_0^{-1})}{3\sqrt{1-0.32}} \ln\left(\frac{\sqrt{1-0.32}+1}{\sqrt{0.32}}\right) = 0.9468 H_0^{-1} = \boxed{13.822 \pm 0.401\,\text{Gyr}} \tag{56}$$

$t_{m\Lambda}$ is the time when $a = a_{m\Lambda}$ and can be found by making the corresponding substitution:

$$t_{m\Lambda} = \frac{3H_0^{-1}}{2\sqrt{1-\Omega_{m,0}}} \ln(\sqrt{2}+1) \approx \frac{2(H_0^{-1})}{3\sqrt{1-0.32}} \ln(1+\sqrt{2}) = \boxed{0.7125 H_0^{-1} = 10.400 \pm 0.297\,\text{Gyr}} \tag{57}$$

Equation 49 can be plotted:

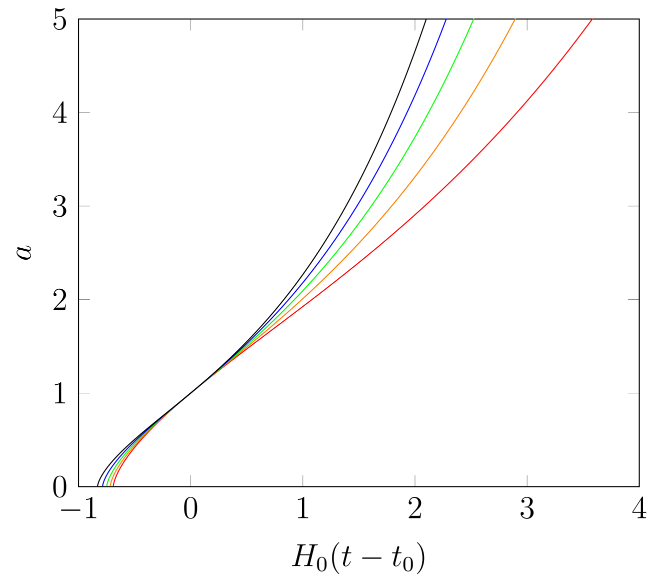

FIGURE 4. Scale Factor vs. Time for universes with $\Omega_{m,0} = 0.9$ (red), 0.8 (orange), 0.7 (green), 0.6 (blue), and 0.5 (black)

Figure 4 shows universes with $\Omega_{m,0} < 1$. The universes that accelerate faster have lower $\Omega_{m,0}$, so $\Omega_{m,0}$ increases to the right. This equation was plotted parametrically, and the x component was shifted to the left according to

$$H_0 t_0 = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/2} + \sqrt{1+\left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)}\right] \tag{58}$$

These universes expand forever, as both terms within the natural log increase with a, so as $a \to \infty$, $H_0 t \to \infty$.

6.9) We are given the following formulation of the Friedmann Equation:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \tag{59}$$

Equation 59 can then be solved for $\dot{a}$:

$$\frac{da}{dt} = H_0\sqrt{\Omega_{r,0}/a^2 + \Omega_{m,0}/a} = \frac{H_0}{a}\sqrt{\Omega_{r,0}\left(1+\frac{\Omega_{m,0}}{\Omega_{r,0}}a\right)} \tag{60}$$

Before moving on, we can make an expression for a_{rm}:

$$\frac{\Omega_{r,0}}{a_{rm}^4} = \frac{\Omega_{m,0}}{a_{rm}^3} \Rightarrow a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \tag{61}$$

Substituting Equation 61 into Equation 60 gives the following:

$$\frac{da}{dt} = \frac{H_0\sqrt{\Omega_{r,0}}}{a}\sqrt{1+a/a_{rm}} \Rightarrow \boxed{H_0 dt = \frac{a da}{\Omega_{r,0}^{1/2}}\left[1+\frac{a}{a_{rm}}\right]^{-1/2}} \tag{62}$$

Equation 62 is the outcome we should get by differentiating 5.110, which is as follows:

$$H_0 t = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1-\left(1-\frac{a}{2a_{rm}}\right)\left(1+\frac{a}{a_{rm}}\right)^{1/2}\right] \tag{63}$$

The following should be true:

$$\frac{a}{\Omega_{r,0}^{1/2}}\left[1+\frac{a}{a_{rm}}\right]^{-1/2} = \frac{d}{da}\left[\frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1-\left(1-\frac{a}{2a_{rm}}\right)\left(1+\frac{a}{a_{rm}}\right)^{1/2}\right]\right] \tag{64}$$

We can then go through with the differentiation:

$$\begin{aligned}
&= \frac{-4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[\left(\frac{-1}{2a_{rm}}\right)\left(1+\frac{a}{a_{rm}}\right)^{1/2} + \left(1-\frac{a}{2a_{rm}}\right)\cdot\frac{1}{2}\left(1+\frac{a}{a_{rm}}\right)^{-1/2}\cdot\frac{1}{a_{rm}}\right] \\
&= \frac{-4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[\frac{-1/(2a_{rm}) - a/(2a_{rm}^2) + 1/(2a_{rm}) - a/(4a_{rm}^2)}{\sqrt{1+a/a_{rm}}}\right] \\
&= \frac{-4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left(\frac{-3a}{4a_{rm}^2}\right)\left[1+\frac{a}{a_{rm}}\right]^{-1/2}
\end{aligned}$$

Finally, we are left with the following statement, confirming the fact that 5.110 is a solution of 5.108:

$$\boxed{\frac{a}{\Omega_{r,0}^{1/2}}\left[1+\frac{a}{a_{rm}}\right]^{-1/2} = \frac{a}{\Omega_{r,0}^{1/2}}\left[1+\frac{a}{a_{rm}}\right]^{-1/2}} \tag{65}$$

For $a \ll a_{rm}$, we can approximate Equation 63 by using the third-order Taylor series expansion of the part of the equation in brackets. The approximation will work the following way, if we say that $f(a) = H_0 t$:

$$H_0 t \approx f(0) + f'(0)a + \frac{1}{2}f''(0)a^2 \tag{66}$$

First, we can solve for $f(0)$ by plugging in $a = 0$ to Equation 63.

$$f(0) = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1-\left(1-\frac{0}{2a_{rm}}\right)\left(1+\frac{0}{a_{rm}}\right)^{1/2}\right] = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}(1-1) = 0 \tag{67}$$

Next, we can solve for $f'(0)$ by differentiating Equation 63, which was already done to obtain Equation 65, so we can simply plug in $a = 0$ to that:

$$f'(a) = \frac{a}{\Omega_{r,0}^{1/2}}\left[1+\frac{a}{a_{rm}}\right]^{-1/2} \Rightarrow f'(0) = \frac{0}{\Omega_{r,0}^{1/2}}\left[1+\frac{0}{a_{rm}}\right]^{-1/2} = 0 \tag{68}$$

Next, we can find $f''(0)$ by differentiating Equation 65 and plugging in $a = 0$.

$$f''(a) = \frac{1}{\Omega_{r,0}^{1/2}}\left[1+\frac{a}{a_{rm}}\right]^{-1/2} - \frac{a}{2\Omega_{r,0}^{1/2}}\left[1+\frac{a}{a_{rm}}\right]^{-3/2}\left(\frac{1}{a_{rm}}\right) \Rightarrow f''(0) = \frac{1}{\Omega_{rm}^{1/2}} \tag{69}$$

Now, we can do the Taylor approximation:

$$H_0 t \approx \frac{a^2}{2\Omega_{r,0}^{1/2}} \Rightarrow \boxed{a \approx (2\sqrt{\Omega_{r,0}}H_0 t)^{1/2}} \tag{70}$$

In the case of $a \gg a_{rm}$, we can approximate Equation 63 by getting rid of all of the 1s, as they are negligible compared to the terms that depend on a:

$$H_0 t \approx \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left(\frac{a}{2a_{rm}}\right)\left(\frac{a}{a_{rm}}\right)^{1/2} = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left(\frac{a^{3/2}}{2a_{rm}^{3/2}}\right) \tag{71}$$

Further simplification yields the following:

$$H_0 t \approx \frac{2a_{rm}^{1/2}a^{3/2}}{3\sqrt{\Omega_{r,0}}} \Rightarrow a \approx \left(\frac{3}{2}H_0 t\Omega_{r,0}^{1/2}\frac{\Omega_{m,0}^{1/2}}{\Omega_{r,0}^{1/2}}\right)^{2/3} \tag{72}$$

We are then left with

$$\boxed{a \approx \left(\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t\right)^{2/3}} \tag{73}$$

To find the time where radiation and matter are equal, we can set $a = a_{rm}$ in Equation 63.

$$H_0 t_{rm} = \frac{4a_{rm}^2}{3\sqrt{\Omega_{r,0}}}\left[1 - \left(1 - \frac{1}{2}\right)(1+1)^{1/2}\right] \Rightarrow \boxed{t_{rm} = \frac{4}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\frac{a_{rm}^2}{\sqrt{\Omega_{r,0}}}H_0^{-1}} \tag{74}$$

Equation 74 is approximately equal to

$$t_{rm} \approx 0.3905\frac{\Omega_{r,0}^2}{\Omega_{m,0}^2\Omega_{r,0}^{1/2}}H_0^{-1} = \boxed{0.3905\frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2}H_0^{-1}} \tag{75}$$

For our universe, Equation 75 is equal to the following (with $H_0^{-1} = 14.60$ Gyr, $\Omega_{m,0} = 0.32$, and $\Omega_{r,0} = 9.0 \times 10^{-5}$):

$$t_{rm} \approx 0.3905\frac{(9.0 \times 10^{-5})^{3/2}}{0.32^2}H_0^{-1} = \boxed{3.256 \times 10^{-6}H_0^{-1} = 47526 \pm 1457\,\mathrm{yr}} \tag{76}$$

z_{rm} can be calculated with Equation 61:

$$a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}} = \frac{9.0 \times 10^{-5}}{0.32} \approx 2.813 \times 10^{-4} \Rightarrow \boxed{z_{rm} = \frac{1 - a_{rm}}{a_{rm}} \approx 3555} \tag{77}$$